NONCOMMUTATIVE SYMMETRIC FUNCTIONS AND 
THE INVERSION PROBLEM

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Abstract. Let $K$ be any unital commutative $\mathbb{Q}$-algebra and $z = (z_1, z_2, \cdots, z_n)$ commutative or noncommutative variables. Let $t$ be a formal central parameter and $K[[t]][\langle\langle z\rangle\rangle]$ the formal power series algebra of $z$ over $K[[t]]$. In [Z6], for each automorphism $F_t(z) = z - H_t(z)$ of $K[[t]][\langle\langle z\rangle\rangle]$ with $H_{t=0}(z) = 0$ and $o(H_t(z)) \geq 1$, a NCS (noncommutative symmetric) system $\Omega_{F_t}$ has been constructed. Consequently, we get a Hopf algebra homomorphism $S_{F_t}: \text{NSym} \to D\langle\langle z\rangle\rangle$ from the Hopf algebra $\text{NSym}$ ($\text{GKLLRT}$) of NCSF’s (noncommutative symmetric functions). In this paper, we first give a list for the identities between any two sequences of differential operators in the NCS system $\Omega_{F_t}$ by using some identities of NCSF’s derived in $\text{GKLLRT}$ and the homomorphism $S_{F_t}$. Secondly, we apply these identities to derive some formulas in terms of differential operator in the system $\Omega_{F_t}$ for the Taylor series expansions of $u(F_t)$ and $u(F_{t}^{-1})$ ($u(z) \in K[[t]][\langle\langle z\rangle\rangle]$); the D-Log and the formal flow of $F_t$ and inversion formulas for the inverse map of $F_t$. Finally, we discuss a connection of the well-known Jacobian conjecture with NCSF’s.

1. Introduction

Let $K$ be any unital commutative $\mathbb{Q}$-algebra and $z = (z_1, z_2, \cdots, z_n)$ commutative or noncommutative variables. Let $t$ be a formal central parameter, i.e. a formal variable which commutes with $z$ and elements of $K$. To keep notation simple, we use the notations for noncommutative variables uniformly for both commutative and noncommutative variables $z$. Let $K\langle\langle z\rangle\rangle$ (resp. $K[[t]][\langle\langle z\rangle\rangle]$) the algebra of formal power series in $z$ over $K$ (resp. $K[[t]]$). For any $\alpha \geq 1$, let $D^{[\alpha]}\langle\langle z\rangle\rangle$ be the unital algebra generated by the differential operators of $K\langle\langle z\rangle\rangle$ which increase the degree in $z$ by at least $\alpha - 1$ and $A_t^{[\alpha]}\langle\langle z\rangle\rangle$ the group of automorphisms $F_t(z) = z - H_t(z)$ of $K[[t]][\langle\langle z\rangle\rangle]$ with $o(H_t(z)) \geq \alpha$ and $2000$ Mathematics Subject Classification. 05E05, 14R10, 14C15. 

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In [Z6], for each automorphism \( F_t(z) \in A[[z]] \), a NCS (noncommutative symmetric) system \( (\Omega_\Pi) \) over the Hopf algebra \( NSym \) of NCSF’s (noncommutative symmetric functions) \( (\text{GKLLRT}) \), we have a Hopf algebra homomorphism \( S_{F_t} : NSym \to D[[z]] \). Consequently, as pointed out in [Z5] as one of the main motivations for the introduction of the NCS systems, by applying the homomorphism \( S_{F_t} \) to the identities of the NCSF in the NCS system \( \Pi \), we get a host of identities for the corresponding differential operators in the NCS system \( \Omega_{F_t} \).

In this paper, we first give a list of the identities for any two sequences of differential operators in the NCS system \( \Omega_{F_t} \). These identities either come directly from the identities of the corresponding NCSF’s derived in [GKLLRT] by applying the homomorphism \( S_{F_t} \), or can be derived easily from those identities of NCSF’s by some simple arguments. Secondly, by using these identities for the differential operators in \( \Omega_{F_t} \) and the special forms of certain differential operators in \( \Omega_{F_t} \) when \( F_t(z) = z - tH(z) \) for some \( H(z) \in K[[z]] \), we derive some formulas in terms of differential operator in the system \( \Omega_{F_t} \), the Taylor series expansions of \( u(F_t) \) and \( u(F_t^{-1}) \), the D-Log and the formal flow of \( F_t \), and more importantly, some inversion formulas for the inverse maps of \( F_t \). Finally, we discuss a connection of the well-known Jacobian conjecture with NCSF’s.

Note that, the NCSF’s were first introduced and studied in the seminal paper [GKLLRT] in 1994. NCS systems over associative algebras were first formulated in [Z5], but mainly motivated by the introduction of the NCSF’s in [GKLLRT] (see Definition 2.1). Actually, in some sense, a NCS system \( \Omega \) over an associative \( K \)-algebra \( A \) can be viewed as a system of analogs of the NCSF’s in \( A \) defined by Eqs. (2.1)–(2.5) over \( A \), which formally are same as the defining equations of certain NCSF’s over the free \( K \)-algebra \( NSym \) generated by a sequence of noncommutative free variables \( \Lambda_m (m \geq 1) \). For some general discussions on the NCS systems, see [Z5]. For more studies on NCSF’s, see [IT], [KLT], [DKKT], [KT1], [KT2] and [DFT].

While, on the other hand, the inversion problem, which is mainly to study various properties of the inverse maps of analytic maps, has much longer history. Since as early as 1770 when L. Lagrange proved the so-called Lagrange inversion formula, there have been numerous papers devoted to find various inversion formulas, i.e. formulas for inverse maps (see [WZ], [Z2], [Z4] and references there). The study on the inversion problem was greatly intensified since O. H. Keller [Ke]
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in 1939 proposed the well-known Jacobian conjecture which says, any polynomial map $F(z) : K^{\times n} \to K^{\times n}$ with $j(F) := \det(\frac{\partial F}{\partial z}) = 1$ must be an automorphism of $K^{\times n}$ and its inverse map $G(z) = F^{-1}(z)$ must also be a polynomial map. Despite enormous efforts from mathematicians in more than sixty years, the conjecture is still open even for the case $n = 2$. In 1998, S. Smale included the Jacobian conjecture in his list of 18 important mathematical problems for 21st century. For more history and known results on the Jacobian conjecture, see [BCW], [E4] and references there. For some very recent developments on the conjecture, see [BE], [Me] and [Z3].

The arrangement of this paper is as follows. In Section 2, we mainly fix some notation and recall some results from [Z6] that will be needed throughout this paper. In Subsection 2.1, we briefly recall the NCS systems in general and the universal NCS system $(NSym, II)$ formed by the generating functions of certain NCSF’s introduced in [GKLLRT]. In Subsection 2.2, we recall the NCS systems $\Omega_{F_t}$ ($F_t \in \mathbb{A}^{[a]}_{\langle \langle z \rangle \rangle}$) and the corresponding differential operator specialization $S_{F_t} : D^{[a]}_{\langle \langle z \rangle \rangle}$ constructed in [Z6]. In Section 3, we focus on the special automorphism $F_t \in \mathbb{A}^{[a]}_{\langle \langle z \rangle \rangle}$ such that $F_t(z) = z - tH(z)$ for some $H(z) \in K^{\langle \langle z \rangle \rangle^{\times n}}$. We recall some results from [Z4] and [Z6] which show that, in this case, certain differential operators have some simple forms. Together with the specialization $S_{F_t}$, the results in this section will be crucial for most of the formulas that will be derived in Subsections 5.3 and 5.4 and also later a connection of NCSF’s with the Jacobian conjecture in Subsection 5.5. In Section 4, we mainly derive some identities for the NCSF’s in the universal NCS system $(NSym, II)$, which will be needed in next section. Note that, by the explicit correspondence in Corollary 2.9, applying the specialization $S_{F_t} : NSym \to D^{[a]}_{\langle \langle z \rangle \rangle}$ ($F_t \in \mathbb{A}^{[a]}_{\langle \langle z \rangle \rangle}$) or simply changing the up case letters to the lower case letters, all the identities given in this section will become the identities of the corresponding differential operators in the NCS system $(D^{[a]}_{\langle z \rangle}, \Omega_{F_t})$. In Subsection 4.1, we give a list of the identities between any two sequences of the NCSF’s in the universal NCS system $(NSym, II)$. In Subsection 4.2, we let $u$ be another formal central parameter and derive the formulas for $e^{-u\Phi(t)}$ in terms of the NCSF’s in $II$. In Section 5, we mainly apply the identities of NCSF’s derived in the previous section and the specialization $S_{F_t}$ ($F_t \in \mathbb{A}^{[a]}_{\langle \langle z \rangle \rangle}$) in Theorem 2.8 to derive some formulas for the inverse maps, the D-Log’s and the formal flow of the automorphism $F_t \in \mathbb{A}^{[a]}_{\langle \langle z \rangle \rangle}$. In Subsections 5.1 and 5.2, we derive formulas for the D-Log of and the formal flow
generated by $F_t \in A_t[\langle z \rangle]$, respectively, in terms of the differential operators in the NCS system $\Omega_{F_t}$. In Subsection 5.3, we mainly apply the results in the previous two subsections to the special automorphisms $F_t(z) \in E_t[\langle z \rangle]$ to derive some inversion formulas. In Subsection 5.4, motivated by the fact that $C_m(z) \in K[\langle z \rangle]^n$ ($m \geq 1$) in the commutative case capture the nilpotence of the Jacobian matrix $JH$, we give formulas for $C_m(z)$'s in terms of the differential operators in $\Omega_{F_t}$ other than $\psi_m$'s. Finally, in Subsection 5.5, we discuss a connection of NCSF’s with the well-known Jacobian conjecture.

2. Differential Operator Specializations of NCSF’s

Let $K$ be any unital commutative $\mathbb{Q}$-algebra and $A$ any unital associative but not necessarily commutative $K$-algebra. Let $t$ be a formal central parameter, i.e. it commutes with all elements of $A$, and $A[[t]]$ the $K$-algebra of formal power series in $t$ with coefficients in $A$. First let us recall the following notion formulated in [Z5].

**Definition 2.1.** For any unital associative $K$-algebra $A$, a 5-tuple $\Omega = (f(t), g(t), d(t), h(t), m(t)) \in A[[t]]^5$ is said to be a NCS (Noncommutative Symmetric) system over $A$ if the following equations are satisfied.

\begin{align}
(2.1) & \quad f(0) = 1 \\
(2.2) & \quad f(-t)g(t) = g(t)f(-t) = 1, \\
(2.3) & \quad e^{d(t)} = g(t), \\
(2.4) & \quad \frac{dg(t)}{dt} = g(t)h(t), \\
(2.5) & \quad \frac{dg(t)}{dt} = m(t)g(t).
\end{align}

When the base algebra $K$ is clear in the context, we also call the ordered pair $(A, \Omega)$ a NCS system. Since NCS systems often come from generating functions of certain elements of $A$ that are under concern, the components of $\Omega$ will also be referred as the generating functions of their coefficients.

In this section, we mainly fix necessary notations and recall some results from [Z5] and [Z6] that will be needed later. In Subsection 2.1, we briefly recall the NCS system (NSym, II) formed by generating functions of some of the NCSF's defined in [GKLLRT] and its universal property (see Theorem 2.4). In Subsection 2.2, we recall the NCS systems (Z6) over differential operator algebras and the resulted
correspondence between NCSF’s and the differential operators in the system.

2.1. The Universal NCS System from Noncommutative Symmetric Functions. Let \( \Lambda = \{ \Lambda_m \mid m \geq 1 \} \) be a sequence of noncommutative free variables and \( N\text{Sym} \) the free associative algebra generated by \( \Lambda \) over \( K \). For convenience, we also set \( \Lambda_0 = 1 \). We denote by \( \lambda(t) \) the generating function of \( \Lambda_m \) \((m \geq 0)\), i.e. we set

\[
\lambda(t) := \sum_{m \geq 0} t^m \Lambda_m = 1 + \sum_{k \geq 1} t^k \Lambda_k.
\] (2.6)

In the theory of NCSF’s ([GKLLRT]), \( \Lambda_m \) \((m \geq 0)\) is the noncommutative analog of the \( m \)th classical (commutative) elementary symmetric function and is called the \( m \)th (noncommutative) elementary symmetric function.

To define some other NCSF’s, we consider Eqs. (2.2)–(2.5) over the free \( K \)-algebra \( N\text{Sym} \) with \( f(t) = \lambda(t) \). The solutions for \( g(t) \), \( d(t) \), \( h(t) \), \( m(t) \) exist and are unique, whose coefficients will be the NCSF’s that we are going to define. Following the notation in [GKLLRT] and [Z5], we denote the resulted 5-tuple by

\[
\Pi = (\lambda(t), \sigma(t), \Phi(t), \psi(t), \xi(t))
\] (2.7)

and write the last four generating functions of \( \Pi \) explicitly as follows.

\[
\sigma(t) = \sum_{m \geq 0} t^m S_m,
\] (2.8)

\[
\Phi(t) = \sum_{m \geq 1} t^m \frac{\Phi_m}{m}
\] (2.9)

\[
\psi(t) = \sum_{m \geq 1} t^{m-1} \Psi_m,
\] (2.10)

\[
\xi(t) = \sum_{m \geq 1} t^{m-1} \Xi_m.
\] (2.11)

Following [GKLLRT], we call \( S_m \) \((m \geq 1)\) the \( m \)th (noncommutative) complete homogeneous symmetric function and \( \Phi_m \) (resp. \( \Psi_m \)) the \( m \)th power sum symmetric function of the second (resp. first) kind. Following [Z5], we call \( \Xi_m \in N\text{Sym} \) \((m \geq 1)\) the \( m \)th (noncommutative) power sum symmetric function of the third kind.

The following proposition proved in [GKLLRT] and [KLT] will be very useful for our later arguments.
Proposition 2.2. Let $\omega_\Lambda$ be the anti-involution of $NSym$ which fixes $\Lambda_m$ ($m \geq 1$). Then, for any $m \geq 1$, we have
\begin{align}
\omega_\Lambda(S_m) &= S_m, \\
\omega_\Lambda(\Phi_m) &= \Phi_m, \\
\omega_\Lambda(\Psi_m) &= \Xi_m.
\end{align}

By applying Proposition 2.10 in [Z5] to the universal NCS system $(NSym, \Pi)$, we have the following proposition.

Proposition 2.3. Let $\tau$ be the involution of $NSym$ such that $\tau(\Phi_m) = -\Phi_m$ for any $m \geq 1$. Then, we have
\begin{align}
\tau(\Lambda_m) &= (-1)^m S_m, \\
\tau(S_m) &= (-1)^m \Lambda_m, \\
\tau(\Psi_m) &= -\Xi_m, \\
\tau(\Xi_m) &= -\Psi_m.
\end{align}

Next, let us recall the following $K$-Hopf algebra structure of $NSym$. It has been shown in [GKLLRT] that $NSym$ is the universal enveloping algebra of the free Lie algebra generated by $\Psi_m$ ($m \geq 1$). Hence, it has a Hopf $K$-algebra structure as all other universal enveloping algebras of Lie algebras do. Its co-unit $\epsilon : NSym \to K$, co-product $\Delta$ and antipode $S$ are uniquely determined by
\begin{align}
\epsilon(\Psi_m) &= 0, \\
\Delta(\Psi_m) &= 1 \otimes \Psi_m + \Psi_m \otimes 1, \\
S(\Psi_m) &= -\Psi_m,
\end{align}
for any $m \geq 1$.

From the definitions of the NCSF’s above, we see that $(NSym, \Pi)$ obviously forms a NCS system. More importantly, as shown in Theorem 2.1 in [Z5], we have the following important theorem on the NCS system $(NSym, \Pi)$.

Theorem 2.4. Let $A$ be a $K$-algebra and $\Omega$ a NCS system over $A$. Then, There exists a unique $K$-algebra homomorphism $S : NSym \to A$ such that $S^{\times 5}(\Pi) = \Omega$.

Note that, when $A$ is further a $K$-bialgebra (resp. $K$-Hopf algebra) some sufficient conditions for the homomorphism $S : NSym \to A$ in the theorem above to be a homomorphism of $K$-bialgebras (resp. $K$-Hopf algebras) were also given in Theorem 2.1 in [Z5].
Remark 2.5. By taking the quotient over the two-sided ideal generated by the commutators of $\Lambda_m$’s or applying a similar argument for proof of Theorem 2.4, it is easy to see that, over the category of commutative $K$-algebras, the universal NCS system is given by the generating functions of the corresponding classical (commutative) symmetric functions $\Lambda$.

2.2. NCS Systems over Differential Operator Algebras. In this subsection, we briefly recall the NCS systems constructed in [Z6] over the differential operator algebras in commutative or noncommutative free variables. First, let us fix the following notation.

Let $K$ be any unital commutative $\mathbb{Q}$-algebra as before and $z = (z_1, z_2, ..., z_n)$ commutative or noncommutative free variables. Let $t$ be a formal central parameter, i.e. it commutes with $z$ and elements of $K$. We denote by $K\langle\langle z \rangle\rangle$ and $K[[t]]\langle\langle z \rangle\rangle$ the $K$-algebras of formal power series in $z$ over $K$ and $K[[t]]$, respectively.

By a $K$-derivation or simply derivation of $K\langle\langle z \rangle\rangle$, we mean a $K$-linear $\delta : K\langle\langle z \rangle\rangle \to K\langle\langle z \rangle\rangle$ that satisfies the Leibniz rule, i.e. for any $f, g \in K\langle\langle z \rangle\rangle, we have

\begin{equation}
\delta(fg) = (\delta f)g + f(\delta g).
\end{equation}

We will denote by $\mathcal{D}er_K\langle\langle z \rangle\rangle$ or $\mathcal{D}er\langle\langle z \rangle\rangle$, when the base algebra $K$ is clear from the context, the set of all $K$-derivations of $K\langle\langle z \rangle\rangle$. The unital subalgebra of $\text{End}_K(K\langle\langle z \rangle\rangle)$ generated by all $K$-derivations of $K\langle\langle z \rangle\rangle$ will be denoted by $\mathcal{D}er_K\langle\langle z \rangle\rangle$ or $\mathcal{D}\langle\langle z \rangle\rangle$. Elements of $\mathcal{D}er_K\langle\langle z \rangle\rangle$ will be called (formal) differential operators in the commutative and noncommutative variables $z$.

For any $\alpha \geq 1$, we denote by $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$ the set of the $K$-derivations of $K\langle\langle z \rangle\rangle$ which increase the degree in $z$ by at least $\alpha - 1$. The unital subalgebra of $\mathcal{D}\langle\langle z \rangle\rangle$ generated by elements of $\mathcal{D}er^{[\alpha]}\langle\langle z \rangle\rangle$ will be denoted by $\mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle$. Note that, by the definitions above, the operators of scalar multiplications are also in $\mathcal{D}\langle\langle z \rangle\rangle$ and $\mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle$. When the base algebra is $K[[t]]$ instead of $K$ itself, the notation $\mathcal{D}er_t\langle\langle z \rangle\rangle$, $\mathcal{D}er_t^{[\alpha]}\langle\langle z \rangle\rangle$, and $\mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle$ will be denoted by $\mathcal{D}er_t\langle\langle z \rangle\rangle$, $\mathcal{D}_t\langle\langle z \rangle\rangle$, $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle$, respectively. For example, $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle$ stands for the set of all $K[[t]]$-derivations of $K[[t]]\langle\langle z \rangle\rangle$ which increase the degree in $z$ by at least $\alpha - 1$. Note that, $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle = \mathcal{D}er_t^{[\alpha]}\langle\langle z \rangle\rangle[[t]]$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z \rangle\rangle = \mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle[[t]]$.

\footnote{Since most of the results as well as their proofs in this paper do not depend on the commutativity of the free variables $z$, we will not distinguish the commutative and the noncommutative case, unless stated otherwise, and adapt the notations for noncommutative variables uniformly for the both cases.}
For any $1 \leq i \leq n$ and $u(z) \in K\langle \langle z \rangle \rangle$, we denote by $[ u(z) \frac{\partial}{\partial z_i} ]$ the $K$-derivation which maps $z_i$ to $u(z)$ and $z_j$ to 0 for any $j \neq i$. For any $\vec{u} = (u_1, u_2, \ldots, u_n) \in K\langle \langle z \rangle \rangle^{\times n}$, we set

\begin{equation}
[ \vec{u} \frac{\partial}{\partial z} ] := \sum_{i=1}^{n} [ u_i \frac{\partial}{\partial z_i} ].
\end{equation}

Note that, in the noncommutative case, we in general do not have $\left[ u(z) \frac{\partial}{\partial z_i} \right] g(z) = u(z) \frac{\partial}{\partial z_i} g(z)$ for all $u(z), g(z) \in K\langle \langle z \rangle \rangle$. This is the reason why we put a bracket $[\cdot]$ in the notation above for the $K$-derivations. With this notation, it is easy to see that any $K$-derivations $\delta$ of $K\langle \langle z \rangle \rangle$ can be written uniquely as $\sum_{i=1}^{n} \left[ f_i(z) \frac{\partial}{\partial z_i} \right]$ with $f_i(z) = \delta z_i \in K\langle \langle z \rangle \rangle$ ($1 \leq i \leq n$).

With the commutator bracket, $\mathcal{D}er^{[\alpha]}\langle \langle z \rangle \rangle$ ($\alpha \geq 1$) forms a Lie algebra and its universal enveloping algebra is exactly the differential operator algebra $\mathbb{D}^{[\alpha]}\langle \langle z \rangle \rangle$. Consequently, $\mathbb{D}^{[\alpha]}\langle \langle z \rangle \rangle$ ($\alpha \geq 1$) has a Hopf algebra structure as all other enveloping algebras of Lie algebras do. In particular, Its coproduct $\Delta$, antipode $S$ and co-unit $\epsilon$ are uniquely determined by the properties

\begin{align}
\Delta(\delta) &= 1 \otimes \delta + \delta \otimes 1, \\
S(\delta) &= -\delta, \\
\epsilon(\delta) &= \delta \cdot 1,
\end{align}

respectively, for any $\delta \in \mathcal{D}er\langle \langle z \rangle \rangle$.

For any $\alpha \geq 1$, let $A^{[\alpha]}_t\langle \langle z \rangle \rangle$ be the set of all the automorphism $F_t(z)$ of $K[[t]]\langle \langle z \rangle \rangle$ over $K[[t]]$, which have the form $F(z) = z - H_t(z)$ for some $H_t(z) \in K[[t]]\langle \langle z \rangle \rangle^{\times n}$ with $o(H_t(z)) \geq \alpha$ and $H_t|_{z=0}(z) = 0$. Note that, for any $F_t \in A^{[\alpha]}_t\langle \langle z \rangle \rangle$ as above, its inverse map $G_t := F_t^{-1}$ can always be written uniquely as $G_t(z) = z + M_t(z)$ for some $M_t(z) \in K[[t]]\langle \langle z \rangle \rangle^{\times n}$ with $o(M_t(z)) \geq \alpha$ and $M_t|_{z=0}(z) = 0$.

Now we recall the NCS systems constructed in [Z6] over the differential operator algebras $\mathbb{D}^{[\alpha]}\langle \langle z \rangle \rangle$ ($\alpha \geq 1$). We fix an $\alpha \geq 1$ and an arbitrary $F_t \in A^{[\alpha]}_t\langle \langle z \rangle \rangle$. We will always let $H_t(z), G_t(z)$ and $M_t(z)$ be determined as above. The NCS system

\begin{equation}
\Omega_{F_t} = (f(t), g(t), d(t), h(t), m(t)) \in \mathbb{D}^{[\alpha]}\langle \langle z \rangle \rangle[[t]]^{\times 5}.
\end{equation}

is determined as follows.

The last two components are given directly by the following two $K[[t]]$-derivations of the $K[[t]]$-algebra $K[[t]]\langle \langle z \rangle \rangle$. 

\( h(t) := \left[ \frac{\partial M_t}{\partial t}(F_t) \frac{\partial}{\partial z} \right], \)

\( m(t) := \left[ \frac{\partial H_t}{\partial t}(G_t) \frac{\partial}{\partial z} \right]. \)

The first three components are given by the following proposition which was proved in Section 3.2 in [Z6].

**Proposition 2.6.** There exist unique \( f(t), g(t), d(t) \in D[\alpha]_{\{z\}} \) with \( f(0) = 1 \) and \( d(0) = 0 \) such that, for any \( u_t(z) \in K[[t]]_{\{z\}} \), we have

\( f(-t)u_t(z) = u_t(F_t), \)

\( g(t)u_t(z) = u_t(G_t), \)

\( e^{d(t)}u_t(z) = u_t(G_t), \)

where, as usual, the exponential in Eq. (2.32) is given by

\( e^{d(t)} = \sum_{m \geq 0} \frac{d(t)^m}{m!}. \)

By using the Taylor series expansions of \( u(F_t) \) and \( u(G_t) \), it is easy to show that the differential operators \( f(t) \) and \( g(t) \) can be given as follows.

**Lemma 2.7.**

\( f(t) = \sum_{m \geq 0} \frac{1}{m!} \left[ H_t(w) \frac{\partial}{\partial z} \right]^m \bigg|_{w=z}, \)

\( g(t) = \sum_{m \geq 0} \frac{1}{m!} \left[ M_t(w) \frac{\partial}{\partial z} \right]^m \bigg|_{w=z}, \)

where \( w = (w_1, \cdots, w_n) \) are free variables that are independent with \( z \), and the notation \( |_{w=z} \) simply means that, after applying the differential operator before \( |_{w=z} \) and then replacing \( w \) back by \( z \).

Note that, when we write \( d(t) \) above as \( d(t) = -\left[ a_t(z) \frac{\partial}{\partial z} \right] \) for some \( a_t(z) \in tK[[t]]_{\{z\}} \), then we get the so-called D-Log \( a_t(z) \) of the automorphism \( F_t(z) \in A_t[\alpha]_{\{z\}} \), which has been studied in [E1]–[E3], [Z1] and [WZ] for the commutative case.

We define five sequences of differential operators by writing the components of \( \Omega_{F_t} \) as follows.

\( f(t) = \sum_{m=0}^{\infty} t^m \lambda_m, \)
Throughout this paper, we will also denote each sequence of the differential operators above by the corresponding letter without sub-index. For example, $\lambda$ denotes the sequence of the differential operator \{\lambda_m \mid m \geq 0\} defined in Eq. (2.36).

Let $S_{F_t} : NSym \rightarrow D^{[\alpha]}\langle\langle z\rangle\rangle$ be the unique $K$-algebra homomorphism that maps $\Lambda_m$ to $\lambda_m$ for any $m \geq 1$. Note that, since $NSym$ is freely generated by $\lambda_m$ ($m \geq 1$), the map $S_{F_t}$ is well-defined. The main result we need later is the following theorem proved in [Z6].

**Theorem 2.8.** For any $\alpha \geq 1$ and $F_t(z) \in A^{[\alpha]}\langle\langle z\rangle\rangle$, we have,

(a) the 5-tuple $\Omega_{F_t}$ defined in Eq. (2.27) forms a NCS system over the differential operator algebra $D^{[\alpha]}\langle\langle z\rangle\rangle$.

(b) $S_{F_t} : NSym \rightarrow D^{[\alpha]}\langle\langle z\rangle\rangle$ defined above is the unique $K$-Hopf algebras homomorphism such that $S_{F_t}(\Pi) = \Omega_{F_t}$.

**Corollary 2.9.** For any $m \geq 1$, we have the following differential operator realizations for the corresponding NCSF’s.

\begin{align*}
(2.41) & \quad S_{F_t}(\Lambda_m) = \lambda_m, \\
(2.42) & \quad S_{F_t}(S_m) = s_m, \\
(2.43) & \quad S_{F_t}(\Psi_m) = \psi_m, \\
(2.44) & \quad S_{F_t}(\Phi_m) = \phi_m, \\
(2.45) & \quad S_{F_t}(\Xi_m) = \xi_m.
\end{align*}

For more properties of the specialization $S_{F_t} : NSym \rightarrow D^{[\alpha]}\langle\langle z\rangle\rangle$, see [Z6] and [Z8]. Finally, let us point out the following result in [Z8] that will be helpful to our later arguments.

For any $z$ and $\alpha \geq 1$ as before, we let $B^{[\alpha]}_t\langle\langle z\rangle\rangle$ be the set of automorphisms $F_t = z - H_t(z)$ of the polynomial algebra $K[t]\langle\langle z\rangle\rangle$ over $K[t]$ such that the following conditions are satisfied.

- $H_{t=0}(z) = 0$. 

• $H_t(z)$ is homogeneous in $z$ of degree $d \geq \alpha$.
• With a proper permutation of the free variables $z_i$’s, the Jacobian matrix $JH_t(z)$ becomes strictly lower triangular.

Theorem 2.10. In both commutative and noncommutative cases, the following statement holds.

For any fixed $\alpha \geq 1$ and non-zero $P \in N\text{Sym}$, there exist $n \geq 1$ (the number of the free variable $z_i$’s) and $F_t(z) \in \mathbb{B}_t^{[\alpha]}\langle z \rangle$ such that $S_{F_t}(P) \neq 0$.

3. A Family of Special Automorphisms $F_t(z)$

Let $K$, $z$, $t$ and $A_t^{[\alpha]}\langle \langle \langle z \rangle \rangle \rangle$ ($\alpha \geq 1$) as fixed in Subsection 2.2. We will also freely use other notations fixed in the earlier sections. First, let us set $\mathbb{E}_t^{[\alpha]}\langle \langle \langle z \rangle \rangle \rangle$ be the set of all automorphisms $F_t \in A_t^{[\alpha]}\langle \langle \langle z \rangle \rangle \rangle$ such that $F_t(z) = z - tH(z)$ for some $H(z) \in K\langle \langle \langle z \rangle \rangle \rangle^{\times n}$.

Note that, the automorphisms $F_t(z) \in \mathbb{E}_t^{[\alpha]}\langle \langle \langle z \rangle \rangle \rangle$ can be viewed as special deformations of the automorphisms $F_t(z) = z - H(z)$ of the $K$-algebra $K\langle \langle \langle z \rangle \rangle \rangle$. They have been studied in [Z2] for the commutative case and later in [Z4] for the noncommutative case.

In this section, we mainly recall some results from [Z5], which show that, for the automorphisms $F_t(z) \in \mathbb{E}_t^{[\alpha]}\langle \langle \langle z \rangle \rangle \rangle$, the differential operators $\lambda_m$’s, $\psi_m$’s and $\xi_m$’s in the NCS system $\Omega_{F_t}$ have some special forms. Together with the correspondence in Corollary 2.9, the results in this section are responsible for most of the formulas that will be derived in Subsections 5.3 and 5.4, and also later a connection of NCSF’s with the Jacobian conjecture in Subsection 5.5.

First, let us fix a $F_t \in \mathbb{E}_t^{[\alpha]}\langle \langle \langle z \rangle \rangle \rangle$ and always write the formal inverse map $G_t(z) := F_t^{-1}(z)$ as $G_t(z) = z + tN_t(z)$ with $N_t(z) \in K[t]\langle \langle \langle z \rangle \rangle \rangle^{\times n}$. Note that, in terms of the notation in Section 2.2 we have

\begin{align*}
H_t(z) &= tH(z), \quad (3.1) \\
M_t(z) &= tN_t(z). \quad (3.2)
\end{align*}

Furthermore, we define a sequence \{$N_{[m]}(z) \in K\langle \langle z \rangle \rangle^{\times n} | m \geq 0$\} by writing

\begin{equation}
N_t(z) = \sum_{m=0}^{+\infty} t^{m-1} N_{[m]}(z). \quad (3.3)
\end{equation}

By Lemma 4.1 in [Z4], the differential operators defined in Eqs. (2.40) and (2.39) have the following special forms.
Lemma 3.1.

\begin{align*}
(3.4) \quad m(t) &= \left[ N_t(z) \frac{\partial}{\partial z} \right], \\
(3.5) \quad h(t) &= \sum_{m \geq 1} t^{m-1} \left[ C_m(z) \frac{\partial}{\partial z} \right],
\end{align*}

where \( C_m(z) \in K\langle\langle z\rangle\rangle^m \) \((m \geq 1)\) are defined recursively by

\begin{align*}
(3.6) \quad C_1(z) &= H(z), \\
(3.7) \quad C_m(z) &= \left[ C_{m-1}(z) \frac{\partial}{\partial z} \right] H,
\end{align*}

for any \( m \geq 2 \).

Note that, by the mathematical induction, it is easy to show that, when \( z \) are commutative variables, we further have

\begin{equation}
(3.8) \quad C_m(z) = (JH)^{m-1} H
\end{equation}

for any \( m \geq 1 \).

Consequently, the \( K \)-derivations \( \psi_m \) and \( \xi_m \) \((m \geq 1)\) defined in Eqs. (2.40) and (2.39) have the following simple forms.

**Corollary 3.2.** For any \( m \geq 1 \), we have

\begin{align*}
(3.9) \quad \psi_m &= \left[ C_m(z) \frac{\partial}{\partial z} \right], \\
(3.10) \quad \xi_m &= \left[ N_m(z) \frac{\partial}{\partial z} \right].
\end{align*}

When \( z \) are commutative variables, we further have

\begin{equation}
(3.11) \quad \psi_m = \left[ (JH)^{m-1} H \right] \frac{\partial}{\partial z}
\end{equation}

Note that, by applying Eq. (2.34) to \( F_t \in \mathcal{E}_t^{[n]} \langle\langle z\rangle\rangle \), we have the following special form for the differential operators \( \lambda_m \)’s.

**Lemma 3.3.** For any \( m \geq 1 \) and \( u(z) \in K\langle\langle z\rangle\rangle \), we have

\begin{equation}
(3.12) \quad \lambda_m u(z) = \frac{1}{m!} \left[ H(w) \frac{\partial}{\partial z} \right]^m u(z) \bigg|_{w=z}
\end{equation}

In a short notation as in Lemma 2.7 we have,

\begin{equation}
(3.13) \quad \lambda_m = \frac{1}{m!} \left[ H(w) \frac{\partial}{\partial z} \right]^m \bigg|_{w=z}
\end{equation}
In particular, when \( z \) are commutative free variables, we have

\[
\lambda_m = \sum_{I \in \mathcal{N}, |I| = m} \frac{1}{I!} H^I(z) \frac{\partial^m}{\partial z^I}.
\]

(3.14)

4. Identities of Noncommutative Symmetric Functions

In this section, we mainly derive some identities for the NCSF’s in the universal NCS system \((NSym, \Pi)\), which will be needed in next section. Note that, by the explicit correspondence in Corollary 2.9, applying the specialization \( S_{F_t} : NSym \to D[[z]] (F_t \in A_t[[z]]) \) or simply changing the up case letters to the lower case letters, all the identities given in this section will become the identities of the corresponding differential operators in the NCS system \((D[[z]], \Omega_F)\).

In Subsection 4.1, we give a list of the identities between any two sequences of the NCSF’s in the universal NCS system \((NSym, \Pi)\). In Subsection 4.2, we let \( u \) be another formal central parameter and derive the formulas for \( e^{-u\Phi(t)} \) in terms of the NCSF’s in \( \Pi \). These formulas will be used in Subsection 5.2 to derive certain formulas for the formal flow generated by \( F_t \in A[[z]] \).

First, let us fix the following notations.

Notation:

(i) For any composition \( I \), i.e. an ordered finite sequence \( I = (i_1, i_2, ..., i_m) \) of positive integers, we define the length \( \ell(I) \) of \( I \) to be \( m \) and the weight \( |I| \) of \( I \) to be \( \sum_{j=1}^{m} i_j \). We denote by \( \mathcal{C} \) (resp. \( \mathcal{C}_m (m \geq 1) \)) the set of all compositions \( I \) (resp. with \( |I| = m \)).

(ii) Let \( I \) as given in (i) and \( \{w_m \mid m \geq 1\} \) a sequence of letters or elements of a \( K \)-algebra, we set \( w^I = w_{i_1}w_{i_2} \cdots w_{i_m} \).

(iii) Let \( I \) as given in (i), we define the first part \( lp(I) \) (resp. the last part \( lp(I) \)) of \( I \) to be \( i_1 \) (resp. \( i_m \)). Furthermore, we also set

\[
\pi(I) = \prod_{j=1}^{m} i_j,
\]

(4.1)

\[
\pi_u(I) = \prod_{j=1}^{m} (i_1 + \cdots + i_j),
\]

(4.2)

\[
sp(I) = \ell(I)! \pi(I).
\]

(4.3)
(iv) For any fixed composition as in (i), we define the mirror image \( \bar{I} \) of \( I \) to be the composition obtained by reversing the ordered sequence \( I \), i.e. \( \bar{I} := (i_m, i_{m-1}, \ldots, i_1) \).

(v) For any compositions \( I, J \in \mathcal{C} \), we denote by \( I \cdot J \) the concatenation product of \( I \) and \( J \). For example, if \( I = (3, 2, 4) \) and \( J = (5, 7) \), then \( I \cdot J = (3, 2, 4, 5, 7) \).

(vi) Let \( I \) be as in (i) and \( J = (j_1, j_2, \ldots, j_k) \) another composition. We say \( J \) is a refinement of \( I \), denoted by \( J \trianglerighteq I \) or \( I \trianglelefteq J \), if there exist \( d_0 = 1 < d_1 < d_2 < \cdots < d_m = k + 1 \) such that, for any \( 1 \leq a \leq m \), we have

\[
j_{d_a-1} + j_{d_a-1+1} + \cdots + j_{d_a-1} = i_a.
\]

For example, \((4, 2, 5, 2, 1) \trianglerighteq (6, 5, 3) \) and \((4, 1, 7) \trianglelefteq (3, 1, 1, 5, 2) \).

(vii) Let \( I \) and \( J \) be any two composition with \( J \trianglerighteq I \). With notation fixed in (vi), we set, for any \( 1 \leq a \leq m \),

\[
J_a := \{ j_{d_a-1}, j_{d_a-1+1}, \ldots, j_{d_a-1} \}.
\]

Then we further set

\[
4.1. \textbf{Identities of the NCSF's in the NCS System } \Pi. \text{ In this subsection, we give a list of the identities of the NCSF's in the NCS System } \Pi. \text{ Note that, by simply applying the specialization } S_{F_1} : NSym \to \mathcal{D}[t] \langle \langle z \rangle \rangle \text{ in Theorem 2.8 or just replacing the up case letters by the lower case letters, these identities will become the identities of the differential operators in the NCS system } \Omega_{F_1}.
\]

First, we fix a composition \( I = (i_1, i_2, \ldots, i_m) \) and start with the following five pairs of the identities of NCSF's, which have been derived in §4 of [GKLLRT].
• The relations between $\Lambda$ and $S$:
  \begin{align}
  S^I &= \sum_{J \geq I} (-1)^{\ell(J)-|I|} \Lambda^J, \\
  \Lambda^I &= \sum_{J \geq I} (-1)^{\ell(J)-|I|} S^J.
  \end{align}

• The relations between $\Lambda$ and $\Psi$:
  \begin{align}
  \Lambda^I &= (-1)^{|I|} \sum_{J \geq I} \frac{(-1)^{\ell(J)}}{\pi_u(J, I)} \Psi^J, \\
  \Psi^I &= (-1)^{|I|} \sum_{J \geq I} (-1)^{\ell(J)} \pi p(J, I) \Lambda^J.
  \end{align}

• The relations between $S$ and $\Psi$:
  \begin{align}
  S^I &= \sum_{J \geq I} \frac{1}{\pi_u(J, I)} \Psi^J, \\
  \Psi^I &= (-1)^{\ell(I)} \sum_{J \geq I} (-1)^{\ell(J)} \pi p(J, I) S^J.
  \end{align}

• The relations between $\Lambda$ and $\Phi$:
  \begin{align}
  \Lambda^I &= (-1)^{|I|} \sum_{J \geq I} \frac{(-1)^{\ell(J)}}{\pi p(J, I)} \Phi^J, \\
  \Phi^I &= (-1)^{|I|} \sum_{J \geq I} \frac{(-1)^{\ell(J)}}{\ell(J, I)} \pi(I) \Lambda^J.
  \end{align}

• The relations between $S$ and $\Phi$:
  \begin{align}
  S^I &= \sum_{J \geq I} \frac{1}{\pi p(J, I)} \Phi^J, \\
  \Phi^I &= (-1)^{\ell(I)} \sum_{J \geq I} \frac{(-1)^{\ell(J)}}{\ell(J, I)} \pi(I) S^J.
  \end{align}

• The relations between $\Psi$ and $\Phi$:
  \begin{align}
  \Phi^I &= (-1)^{\ell(I)} \sum_{J \geq I} \left( \sum_{J \geq K \geq I} \frac{(-1)^{\ell(K)}}{\pi_u(J, K) \ell(K, I)} \right) \Psi^J, \\
  \Psi^I &= (-1)^{\ell(I)} \sum_{J \geq I} \left( \sum_{J \geq K \geq I} \frac{(-1)^{\ell(K)}}{\pi p(K, I))} \right) \Phi^J.
  \end{align}
The last two identities were not given explicitly in [GKLLRT], but can be easily derived as follows.

Proof: First, by combining Eq. (4.14) with Eq. (4.13), we have

$$
\Phi^I = (-1)^{\ell(I)} \sum_{K \succeq I} (-1)^{\ell(K)} \frac{\pi(I)}{\ell(K, I)} S^K
$$

$$
= (-1)^{\ell(I)} \sum_{K \succeq I} (-1)^{\ell(K)} \frac{\pi(I)}{\ell(K, I)} \sum_{J \succeq K} 1 \pi_u(J, K) \Psi^J.
$$

Hence, we get Eq. (4.19). By a similar argument, (4.20) follows by combining Eq. (4.14) with Eq. (4.17).

Proof: By applying the anti-involution $\omega^\Lambda$ in Proposition 2.2 to Eq. (4.12), and then, by Eq. (2.14) in the same proposition, we get

$$
\Xi^I = (-1)^{||I||} \sum_{J \succeq I} (-1)^{\ell(J)} \frac{\pi_u(J, I)}{\ell(J, I)} \Lambda^J.
$$

Note that, for any compositions $I$ and $J$, $J \succ I$ iff $\bar{J} \succ \bar{I}$. By replacing $\bar{I}$ by $I$ and $\bar{J}$ by $J$ in the equation above, we get

$$
\Xi^I = (-1)^{\ell(I)} \sum_{J \succeq I} (-1)^{\ell(J)} \frac{fp(J, I)}{\ell(J, I)} \Lambda^J.
$$

Hence, we get Eq. (4.22). Eq. (4.21) can be proved similarly by applying $\omega^\Lambda$ to Eq. (4.11).
The next two pairs of identities can also be proved similarly as above.

- The relations between $S$ and $\Xi$:

$S^I = \sum_{J \geq I} \frac{1}{\pi_u(J, I)} \Xi^J$,  \hspace{1cm} (4.23)

$\Xi^I = (-1)^{\ell(I)} \sum_{J \geq I} (-1)^{\ell(J)} fp(J, I) S^J$.  \hspace{1cm} (4.24)

- The relations between $\Xi$ and $\Phi$:

$\Phi^I = (-1)^{\ell(I)} \sum_{J \geq I} \left( \sum_{J \geq K \geq I} \frac{(-1)^{\ell(K)} \pi(I)}{\pi_u(J, K) \ell(K, I)} \right) \Xi^J$, \hspace{1cm} (4.25)

$\Xi^I = (-1)^{\ell(I)} \sum_{J \geq I} \left( \sum_{J \geq K \geq I} \frac{(-1)^{\ell(K)} fp(K, I)}{sp(J, K)} \right) \Phi^J$. \hspace{1cm} (4.26)

Finally, let us consider the relations between $\Psi$ and $\Xi$.

**Lemma 4.1.** For any composition $I$, we have

$\Xi^I = \sum_{K \succ I} c_{I,K} \Psi^K$, \hspace{1cm} (4.27)

$\Psi^I = \sum_{K \succ I} c_{I,K} \Xi^K$, \hspace{1cm} (4.28)

where, for any composition $K \succ I$,

$c_{I,K} := \sum_{K \succ J \succ I} (-1)^{\ell(J) - \ell(I)} \frac{fp(J, I)}{\pi_u(K, J)}$. \hspace{1cm} (4.29)

**Proof:** Combining Eq. (4.24) with Eq. (4.13), we get

$\Xi^I = \sum_{J \geq I} (-1)^{\ell(J) - \ell(I)} fp(J, I) \sum_{K \succ J} \frac{1}{\pi_u(K, J)} \Psi^K$

$= \sum_{K \succ I} \left[ \sum_{K \succ J \succ I} (-1)^{\ell(J) - \ell(I)} \frac{fp(J, I)}{\pi_u(K, J)} \right] \Psi^K$.

Hence we get Eq. (4.27). Eq. (4.28) can be easily proved by applying the anti-involution $\omega_\Lambda$ in Proposition 2.2 to Eq. (4.27) and then applying Eq. (2.14). $\square$

Note that, we can also apply the involution $\tau$ in Proposition 2.3, instead of the anti-involution $\omega_\Lambda$, to Eq. (4.27) to get another formula for $\Psi^I$ ($I \in \mathcal{C}$) in terms of $\Xi^J$ ($J \in \mathcal{C}$).
Corollary 4.2. For any $I \in \mathcal{C}$, we have

\begin{equation}
\Psi^I = \sum_{K \succeq I} (-1)^{\ell(I) - \ell(K)} c_{I,K} \Xi^K.
\end{equation}

Furthermore, by comparing Eqs. (4.28), (4.30) and noting that the monomials $\Xi^K (K \in \mathcal{C})$ are free in the $K$-algebra $\mathcal{NSym}$, we get the following identity for the coefficient $c_{I,K}$ for any $I, K \in \mathcal{C}$ with $K \succeq I$.

\begin{equation}
c_{\bar{I}, \bar{K}} = (-1)^{\ell(I) - \ell(K)} c_{I,K}.
\end{equation}

4.2. Formulas for $e^{-u\Phi(t)}$. Let $u$ be another central parameter, i.e. it commutes with $t$ and any NCSF’s in $\mathcal{NSym}$. In this section, we derive some formulas for $e^{-u\Phi(t)}$ in terms of the NCSF’s in the universal NCS system ($\mathcal{NSym}, \Pi$). These formulas later will be needed in Subsection 5.2 for the study of the formal flows generated by $F_t \in \mathcal{A}_t^{[\alpha]}(z)$.

Let us start with the following two lemmas.

**Lemma 4.3.**

\begin{equation}
e^{-u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} \frac{(-u)^{\ell(I)|I|}}{sp(I)} \Phi^I.
\end{equation}

\begin{equation}
e^{-u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} (-1)^{\ell(I)-|I|} \left( \sum_{I \geq J} \frac{(-1)^{\ell(J)} u^{\ell(J)}}{\ell(J)! \ell(I,J)} \right) \Lambda^I,
\end{equation}

\begin{equation}
e^{-u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} \frac{(-u)^{\ell(I)|I|}}{sp(I)} \left( \sum_{I \geq J} \frac{u^{\ell(J)}}{\ell(J)! \ell(I,J)} \right) S^I,
\end{equation}

\begin{equation}
e^{-u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} \frac{(-1)^{\ell(I)} t^{\ell(I)|I|}}{\ell(I)!} \left( \sum_{I \geq K \geq J} \frac{(-1)^{\ell(K)} u^{\ell(J)}}{\pi_u(I,K) \ell(K,J)!} \right) \Psi^I.
\end{equation}

\begin{equation}
e^{-u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} \frac{(-1)^{\ell(I)} t^{\ell(I)|I|}}{\ell(I)!} \left( \sum_{I \geq K \geq J} \frac{(-1)^{\ell(K)} u^{\ell(J)}}{\pi_u(I,K) \ell(K,J)!} \right) \Xi^I.
\end{equation}

**Proof:** First, by Eq. (2.9), we have

\begin{equation}
e^{-u\Phi(t)} = 1 + \sum_{m \geq 1} \frac{(-u)^m}{m!} \left( \sum_{k \geq 1} \frac{t_k \Phi_k}{k} \right)^m.
\end{equation}
\[= 1 + \sum_{m \geq 1} \frac{(-u)^m}{m!} \sum_{I \in \mathcal{C}, \ell(I) = m} \frac{t^{\ell(I)}}{\pi(I)} \Phi^I\]

\[= 1 + \sum_{I \in \mathcal{C}} \frac{(-u)^{\ell(I)} t^{\ell(I)}}{\ell(I) \pi(I)} \Phi^I\]

\[= 1 + \sum_{I \in \mathcal{C}} \frac{(-u)^{\ell(I)} t^{\ell(I)}}{sp(I)} \Phi^I.\]

Therefore we get Eq. (4.32). All other formulas in the lemma follow from Eq. (4.32) and the relations of involved NCSF’s with \(\Phi\). As one example, we give a proof for Eq. (4.33). The proofs for Eqs. (4.34)–(4.36) are similar.

First, by Eqs. (4.32) and (4.16), we have

\[e^{-\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} \frac{(-1)^{|I| - \ell(I)} u^{\ell(I)}}{sp(I)} \sum_{J \geq I} \frac{(-1)^{\ell(J)} \pi(I)}{\ell(J, I)} \Lambda^J\]

Switching the order of the summations and noting that \(|I| = |J|:\)

\[= 1 + \sum_{J \in \mathcal{C}} (-1)^{|J| - \ell(J)} t^{\ell(J)} \left( \sum_{J \geq I} \frac{(-1)^{\ell(I)} u^{\ell(I)}}{\ell(I)! \ell(J, I)} \right) \Lambda^J\]

Switching the summation indices \(I\) and \(J:\)

\[= 1 + \sum_{I \in \mathcal{C}} (-1)^{|I| - \ell(I)} t^{\ell(I)} \left( \sum_{J \geq I} \frac{(-1)^{\ell(J)} u^{\ell(J)}}{\ell(J, I)} \right) \Lambda^I.\]

\[\square\]

**Lemma 4.4.**

\[(4.38) \quad e^{-\Phi(t)} = 1 + \sum_{m \geq 1} (-1)^m t^m \Lambda_m,\]

\[(4.39) \quad e^{\Phi(t)} = 1 + \sum_{m \geq 1} t^m \sum_{I \in \mathcal{C}_m} \frac{1}{\pi_u(I)} \Psi^I,\]

**Proof:** Note that, by Eqs. (2.2), (2.3) for the universal NCS system \((\text{NSym}, \Pi)\) and Eqs. (2.6) and (2.8), we have

\[(4.40) \quad e^{-\Phi(t)} = \sigma(t)^{-1} = \lambda(-t) = 1 + \sum_{m \geq 1} (-1)^m t^m \Lambda_m,\]
\( e^{\Phi(t)} = \sigma(t) = \lambda(-t) = 1 + \sum_{m \geq 1} t^m S_m. \)

Therefore, we have Eq. (4.38). Furthermore, Eq. (4.13) with \( I = \{m\} \), we have

\[ S_m = \sum_{I \in \mathcal{C}_m} \frac{1}{\pi_u(I)}. \]

Combining the equation above with Eq. (4.41), we get Eq. (4.39). \( \square \)

Now we can formulate the main result of this subsection as follows.

**Proposition 4.5.**

\[
\begin{align*}
\exp(-u \Phi(t)) &= 1 + \sum_{I \in \mathcal{C}} (-1)^{|I|} t^{|I|} \left( \frac{u}{\ell(I)} \right) \Lambda^I, \\
\exp(-u \Phi(t)) &= 1 + \sum_{I \in \mathcal{C}} t^{|I|} \left( \frac{-u}{\ell(I)} \right) S^I, \\
\exp(-u \Phi(t)) &= 1 + \sum_{I \in \mathcal{C}} t^{|I|} \left( \sum_{I \geq J} \frac{(-u)}{\pi_u(I,J)} \right) \Psi^I, \\
\exp(-u \Phi(t)) &= 1 + \sum_{I \in \mathcal{C}} (-1)^{\ell(I)} t^{|I|} \left( \sum_{I \geq J} \frac{u}{\pi_u(I,J)} \right) \Xi^I.
\end{align*}
\]

**Proof:** Let us first show Eq. (4.43). From Eq. (4.33), we see that there exist (unique) polynomials \( \varphi_{\Lambda}(I,u) \in K[u] \) \((I \in \mathcal{C})\) of degree \( d \leq \ell(I) \) such that

\[
\exp(-u \Phi(t)) = 1 + \sum_{I \in \mathcal{C}} t^{|I|} \varphi_{\Lambda}(I,u) \Lambda^I,
\]

\[
\varphi_{\Lambda}(I,0) = 0,
\]

for any \( I \in \mathcal{C}. \)

By the fact \( \exp(-(u+1) \Phi(t)) = \exp(-\Phi(t)) \exp(-u \Phi(t)) \) and Eq. (4.47) above, we have

\[
1 + \sum_{I \in \mathcal{C}} t^{|I|} \varphi_{\Lambda}(I,u+1) \Lambda^I \\
= (1 + \sum_{I \in \mathcal{C}} t^{|I|} \varphi_{\Lambda}(I,1) \Lambda^I)(1 + \sum_{I \in \mathcal{C}} t^{|I|} \varphi_{\Lambda}(I,u) \Lambda^I).
\]

By comparing the coefficients of \( \Lambda^I \) \((I \in \mathcal{C})\) and noting that they are free in the \( K \)-algebra NSym, it is easy to see that, for any \( I \in \mathcal{C}, \) we
have
\begin{equation}
\Delta(\varphi_\Lambda(I,u)) = \varphi_\Lambda(I,1) + \sum_{J,K \in C} \varphi_\Lambda(J,1) \varphi_\Lambda(K,u),
\end{equation}
where \( \Delta : K[u] \rightarrow K[u] \) is the difference operator which maps any \( q(u) \in K[u] \) to \( q(u + 1) - q(u) \).

Recall that, we have the following well-known facts. First, for any \( m \geq 0 \), we have
\begin{equation}
\Delta \left( \binom{u}{m} \right) = \binom{u}{m-1}.
\end{equation}
Secondly, for any polynomial \( q(u) \in K[u] \) of degree \( d \geq 0 \), we have
\begin{equation}
q(u) = \sum_{m=0}^{d} a_m \binom{u}{m},
\end{equation}
where \( a_m = (\Delta^m q)(0) \) for any \( 0 \leq m \leq d \).

Now, we apply the facts above to the polynomials \( \varphi_\Lambda(I,u) \in K[u] \) \((I \in C)\). By comparing Eq. (4.38) with Eq. (4.47) with \( u = 1 \), and again noting that the monomials \( \Lambda^I \) \((I \in C)\) are free in \( N\text{Sym} \), we have
\[
\varphi_\Lambda(I,1) = \begin{cases} 
(-1)^{|I|} & \text{if } \ell(I) = 1, \\
0 & \text{if } \ell(I) \geq 2.
\end{cases}
\]

By the equation above and Eqs. (4.48), (4.49), it is easy to check that, for any \( I \in C \) and \( m \geq 0 \), we have
\[
\Delta^m \varphi_\Lambda(I,u) \big|_{u=0} = \begin{cases} 
0 & \text{if } m \neq \ell(I), \\
(-1)^{|I|} & \text{if } m = \ell(I).
\end{cases}
\]
Then, by the general fact given by Eq. (4.51), we have, for any \( I \in C \),
\[
\varphi_\Lambda(I,u) = (-1)^{|I|} \binom{u}{\ell(I)}.
\]
Combining the equation above with Eq. (4.47), we get Eq. (4.43).

To show Eq. (4.44), we first apply the involution \( \tau : N\text{Sym} \rightarrow N\text{Sym} \) in Proposition 2.3 to Eq. (4.43). By Eq. (2.15), we have
\[
e^{u\Phi(t)} = 1 + \sum_{I \in C} t^{|I|} \binom{u}{\ell(I)} S^I
\]
Then, replacing \( u \) by \( -u \) in the equation above, we get Eq. (4.44).

Next, we show Eq. (4.45). Note that, by applying the involution \( \tau \) in Proposition 2.3 to Eq. (4.45), we will get Eq. (4.46).
First, by Eq. (4.35) with \( u \) replaced by \(-u\), there exist (unique) polynomials \( \varphi(I, u) \in K[u] \ (I \in \mathcal{C}) \) of degree \( d \leq \ell(I) \) such that

\[
e^{u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} t^{\ell(I)} \varphi(I, u) \Lambda^{I},
\]

(4.52)

\[
\varphi(I, 0) = 0,
\]

(4.53)

for any \( I \in \mathcal{C} \).

By the fact \( e^{(u+1)\Phi(t)} = e^{\Phi(t)} e^{u\Phi(t)} \) and Eq. (4.52) above, we see that the polynomials \( \varphi(I, u) \in K[u] \ (I \in \mathcal{C}) \) also satisfy Eq. (4.49). On the other hand, by Eqs. (4.39) and (4.52) with \( u = 1 \), and also the freeness of the monomials \( \Psi^{I} \ (I \in \mathcal{C}) \), we have, for any \( I \in \mathcal{C} \),

\[
\varphi(I, 1) = \frac{1}{\pi_u(I)}.
\]

(4.54)

Then, by Eqs. (4.53), (4.54) and Eq. (4.49) for \( \varphi(I, u) \), it is easy to see that, for any \( I \in \mathcal{C} \) and \( m \geq 0 \), we have

\[
\Delta^{m} \varphi(I, u) |_{u=0} = \sum_{I \trianglerighteq J, \ell(J)=m} \frac{1}{\pi_u(I, J)}.
\]

By the general fact given by Eq. (4.51), we have, for any \( I \in \mathcal{C} \),

\[
\varphi(I, u) = \sum_{I \trianglerighteq J} \binom{u}{\ell(J)} \pi_u(I, J).
\]

Hence, we have

\[
e^{u\Phi(t)} = 1 + \sum_{I \in \mathcal{C}} t^{\ell(I)} \sum_{I \trianglerighteq J} \frac{\binom{u}{\ell(J)}}{\pi_u(I, J)} \Psi^{I}.
\]

Replacing \( u \) by \(-u\) in the equation above, we get Eq. (4.55). \( \square \)

Finally, by comparing the formulas Eqs. (4.38) and (4.49) and using the freeness of the NCSF’s \( \Lambda^{K} \ (K \in \mathcal{C}) \), it is easy to see that we have the following identity for composition.

**Corollary 4.6.** For any composition \( I \in \mathcal{C} \) and a free variable \( u \), we have

\[
\sum_{I \trianglerighteq J} \frac{(-1)^{\ell(J)} u^{\ell(J)}}{\ell(J)! \ell(I, J)} = (-1)^{\ell(I)} \binom{u}{\ell(I)}.
\]

(4.55)
5. Applications to the Inversion Problem

In this section, we mainly apply the identities of NCSF’s derived in the previous section and the specialization \( S_{F_i} (F_t \in A_t^{[a]} \langle \langle z \rangle \rangle) \) in Theorem 2.8 to derive some formulas for the inverse maps, the D-Log’s and the formal flow of the automorphism \( F_t \in A_t^{[a]} \langle \langle z \rangle \rangle \). In Subsections 5.1 and 5.2, we derive formulas for the D-Log of and the formal flow generated by \( F_t \in A_t^{[a]} \langle \langle z \rangle \rangle \), respectively, in terms of the differential operators in the NCS system \( \Omega_{F_i} \). In Subsection 5.3, we mainly apply the results in the previous two subsections to the special automorphisms \( F_t(z) \in E_t^{[a]} \langle \langle z \rangle \rangle \) to derive some inversion formulas. In Subsection 5.4, motivated by the fact that \( C_m(z) \in K \langle \langle z \rangle \rangle \times_n (m \geq 1) \) in the commutative case capture the nilpotence of the Jacobian matrix \( JH \), we give formulas for \( C_m(z) \)'s in terms of the differential operators in \( \Omega_{F_i} \) other than \( \psi_m \)'s. Finally, in Subsection 5.5, we discuss a connection of NCSF’s with the well-known Jacobian conjecture.

5.1. D-Log’s in Terms of Other Differential Operators in \( \Omega_{F_i} \).

Considering the important role played by the D-Log’s in the inversion problem (see [E1]–[E3], [Z1] and [WZ] for more discussions in the commutative case), we consider the expressions of the D-Log \( a_t(z) \) of \( F_t(z) \) (see page 9) in terms of other differential operators in the NCS system \( \Omega_{F_i} \). Note that, the problem to express \( \phi \) in terms of the derivations \( \psi \) can be viewed as a special case of the so-called the problem of continuous Baker-Campbell-Hausdorff exponents in the mathematical physics (see §4.10 of [GKLLRT] and the references given there).

Recall that, by Eq. (2.38) and the relation of \( d(t) \) with the D-Log (see page 9), we have

\[
d(t) := - \left[ a_t(z) \frac{\partial}{\partial z} \right] = \sum_{m=1}^{\infty} \frac{t^m}{m} \phi_m.
\]

Proposition 5.1.

\[
a_t(z) = \sum_{I \in \mathcal{C}} \left( \prod_{j \in J} (-1)^{\ell(I)} \psi_{\lambda_j I z} \right) \frac{t^{\|I\|} \pi_v(I, J) \ell(J)}{\ell(I)},
\]

\[
a_t(z) = \sum_{I \in \mathcal{C}} \left( \prod_{j \in J} (-1)^{\ell(I)} \psi_{\lambda_j I z} \right) \frac{t^{\|I\|} \pi_v(I, J) \ell(J)}{\ell(I)},
\]

\[
a_t(z) = \sum_{I \in \mathcal{C}} \left( \prod_{j \in J} (-1)^{\ell(I)} \psi_{\lambda_j I z} \right) \frac{t^{\|I\|} \pi_v(I, J) \ell(J)}{\ell(I)},
\]
(5.5) \[ a_t(z) = \sum_{I \in \mathcal{C}} \left( \sum_{I \geq J} \frac{(-1)^{\ell(J)}}{\pi_u(I, J) \ell(J)} \right) t^{|I|} \xi^I z. \]

Proof: All the formulas above follow directly from the identities of \( \phi \) with the corresponding differential operators. For example, by Eq. (4.25) with \( I = \{m\} \), we have, for any \( m \geq 1 \),

\[ \Phi_m = (-1) \sum_{J \in \mathcal{C}_m} \left( \sum_{J \geq K} \frac{(-1)^{\ell(K)m}}{\pi_u(J, K) \ell(K)} \right) \Xi^J = (-1) \sum_{I \in \mathcal{C}_m} \left( \sum_{I \geq J} \frac{(-1)^{\ell(J)m}}{\pi_u(I, J) \ell(J)} \right) \Xi^I. \]

Then, applying the specialization \( S_F \) to the equation above, and by Eq. (5.1),

\[ \left[ a_t(z) \frac{\partial}{\partial z} \right] = -d(t) = \sum_{I \in \mathcal{C}} \left( \sum_{I \geq J} \frac{(-1)^{\ell(J)}}{\pi_u(I, J) \ell(J)} \right) t^{|I|} \xi^I. \]

Applying the equation above to \( z \), we get Eq. (5.5). \( \square \)

Applying the specialization \( S_F \) to Eq. (71) in [GKLLRT] and then applying Corollary 2.9, we get the following improved formula for \( d(t) \).

**Theorem 5.2.**

(5.6) \[ d(t) = \sum_{r \geq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{r-1}} dt_r \sum_{\sigma \in S_r} \frac{(-1)^{d(\sigma)}}{r} \left( \frac{r-1}{d(\sigma)} \right)^{-1} h(t_{\sigma(r)}) \cdots h(t_{\sigma(1)}), \]

where \( S_r \) is the symmetric group of degree \( r \) and, for any \( \sigma \in S_r \), \( d(\sigma) \) is the numbers of the descents of \( \sigma \).

Note that, by applying the equation above to \(-z\), we get another formula for the D-Log \( a_t(z) \) of \( F_t \).

### 5.2. Formal Flows in Terms of the Differential Operators in \( \Omega_{F_t} \).

In this subsection, we consider the expressions of the formal flows, which has been studied in [E1]–[E3] and [WZ] for the commutative case, in terms of the differential operators in the NCS system \( \Omega_{F_t} \).

Let \( u \) be another central parameter, i.e. it commutes with \( z \) and \( t \). We define

(5.7) \[ F_t(z, u) := e^{u[a_t(z)] \frac{d}{dz}} z = e^{-u d(t)} z. \]
Note that, since $d(0) = 0$, the exponential above is always well-defined. Actually it is easy to see $F_t(z, u) \in (K[[t]])^n$. Therefore, for any $u_0 \in K$, $F_t(z, u_0)$ makes sense.

Following its analog in [E1]–[E3] and [WZ] in the commutative case, we call $F_t(z, u)$ the formal flow generated by $F_t(z)$ or simply the formal flow of $F_t(z)$.

Two remarks on the formal flows defined above are as follows.

First, it is well-known that the exponential of a derivation of any $K$-algebra $A$, when it makes sense, is always an automorphism of the algebra, so in our case, for any $u_0 \in K$, $e^{u_0[a_t(z)\frac{\partial}{\partial z}]}$ is also an automorphism of $K[[t]]\langle\langle z \rangle\rangle$ over $K[[t]]$ which maps $z$ to $F_t(z, u_0)$. From Eq. (5.7), it is clear that this automorphism also lies in $A_t^{[\alpha]}\langle\langle z \rangle\rangle$ since $a_t(z) \geq \alpha$.

Secondly, by Eq. (5.7) and the remark above, the formal flow $F_t(z, u)$ has the following properties:

\begin{align}
F_t(z, 0) &= z, \\
F_t(z, 1) &= F_t(z), \\
F_t(F_t(z, u_2), u_1) &= F_t(z, u_1 + u_2),
\end{align}

for any $u_1, u_2 \in K$.

In other words, $F_t(z, u)$ forms an one-parameter subgroup of the group $A_t^{[\alpha]}\langle\langle z \rangle\rangle$. Therefore, for any integer $m \in \mathbb{Z}$, $F_t(z, m)$ gives the $m^{\text{th}}$ (composing) power of $F_t$ as an element of the group $A_t^{[\alpha]}\langle\langle z \rangle\rangle$. In particular, by setting $m = -1$, we get the inverse map $G_t$ of $F_t$, i.e. $F_t(z, -1) = G_t(z)$.

The main result of this subsection is the following proposition which expresses the D-Log of $F_t \in A_t^{[\alpha]}\langle\langle z \rangle\rangle$ in terms of the differential operators in the NCS $\Omega_{F_t}$.

**Proposition 5.3.**

\begin{align}
F_t(z, u) &= z + \sum_{I \in \mathcal{E}} \frac{(-u)^{\ell(I)}}{sp(I)} t^{|I|} \phi^I z, \\
F_t(z, u) &= z + \sum_{I \in \mathcal{E}} (-1)^{|I|} \left( \frac{u}{\ell(I)} \right) t^{|I|} \chi^I z, \\
F_t(z, u) &= z + \sum_{I \in \mathcal{E}} \left( \frac{-u}{\ell(I)} \right) t^{|I|} s^I z, \\
F_t(z, u) &= z + \sum_{I \in \mathcal{E}} \left( \sum_{I \geq J} \frac{(-u)^{\ell(J)}}{\pi_u(I, J)} \right) t^{|I|} \psi^I z,
\end{align}
\[ F_t(z, u) = z + \sum_{I \in \mathcal{C}} (-1)^{\ell(I)} \left( \sum_{I \supseteq J} \frac{(\ell(J))}{\pi_u(I, J)} \right) t^{|I|} \xi^I z. \]

**Proof:** By the expression of \( d(t) \) given in Eq. (5.1) and the correspondence in Corollary 2.9, we see that the formal flow is given by

\[ F_t(z, u) = S F_t(e^{-u\Phi(t)}) z. \]

Then, by the equation above, it is easy to see that, Eq. (5.11) follows directly from Eq. (4.32) and Eqs. (5.12)–(5.15) follows respectively from Eqs. (4.43)–(4.46). 

\[ \square \]

5.3. Some Inversion Formulas. In this subsection, we mainly apply the identities in Subsection 4.1 to derive some inversion formulas.

First, let us consider the Taylor series expansions of \( u(F_t(z)) \) and \( u(G_t(z)) \) for any \( F_t \in \mathbb{A}_t^{[\alpha]}(\langle \langle z \rangle \rangle) \) and \( u(z) \in K\langle \langle z \rangle \rangle \) in terms of the differential operators \( \lambda \) and \( \psi \).

**Proposition 5.4.** For any \( F_t \in \mathbb{A}_t^{[\alpha]}(\langle \langle z \rangle \rangle) \) (\( \alpha \geq 1 \)) and \( u(z) \in K\langle \langle z \rangle \rangle \), we have

\[ u(F_t(z)) = u(z) + \sum_{m \geq 1} (-1)^m t^m \lambda_m u(z), \]

\[ u(F_t(z)) = u(z) + \sum_{m \geq 1} t^m \sum_{I \in \mathcal{C}_m} \frac{(-1)^{\ell(I)}}{\pi_u(I)} \psi^I u(z), \]

\[ u(G_t(z)) = u(z) + \sum_{m \geq 1} t^m \sum_{I \in \mathcal{C}_m} (-1)^{\ell(I)-m} \lambda^I u(z), \]

\[ u(G_t(z)) = u(z) + \sum_{m \geq 1} t^m \sum_{I \in \mathcal{C}_m} \frac{1}{\pi_u(I)} \psi^I u(z). \]

**Proof:** Eq. (5.17) follows from the definition of the differential operators \( \lambda_m \)'s (see Eqs. (2.30) and (2.36)). Eq. (5.18) follows from Eqs. (5.17) and (4.11). Eq. (5.19) follows from Eqs. (2.31), (2.37) and (4.9). Finally, Eq. (5.20) follows from Eqs. (2.31), (2.37) and (4.13). 

\[ \square \]

From now on and throughout the rest of this paper, we will assume \( F_t(z) \in \mathbb{E}_t^{[\alpha]}(\langle \langle z \rangle \rangle) \), i.e. \( F_t(z) \) is an automorphism of the form \( F_t(z) = z - tH(z) \) for some \( H(z) \in K\langle \langle z \rangle \rangle^{\times n} \) with \( o(H(z)) \geq \alpha \). We will also freely use the notations fixed in Section 3.

First, by Eq. (5.19) with \( u(z) = z \), we get the following inversion formula for \( F_t(z) \).
Proposition 5.5. For any \( m \geq 1 \), we have
\[
N_{\lfloor m \rfloor}(z) = (-1)^m \sum_{I \in C_m} (-1)^{\ell(I)} \lambda^I z,
\]
(5.21)
where \( \lambda = \{ \lambda_k \,|\, k \geq 1 \} \) are given by Eq. (3.13) in general and by Eq. (3.14) when \( z \) are commutative free variables.

Applying Eq. (5.20) with \( u(z) = z \), we get the following inversion formula in terms of \( \psi \).

Proposition 5.6. For any \( m \geq 1 \), we have
\[
N_{\lfloor m \rfloor}(z) = \sum_{I \in C_m} \frac{1}{\pi_u(I)} \psi^I z,
\]
(5.22)
where \( \psi = \{ \psi_k \,|\, k \geq 1 \} \) are given by Eq. (3.9) in general and by Eq. (3.11) when \( z \) are commutative free variables.

By using the identities between the NCSF’s \( \Xi \) and \( \Psi \), we can get another inversion formula in terms of \( \psi \) as follows.

Proposition 5.7. For any \( m \geq 1 \), we have
\[
N_{\lfloor m \rfloor}(z) = \sum_{I \in C_m} c_I \psi^I z,
\]
(5.23)
where, for any composition \( I \), \( c_I \) is given by
\[
c_I = \sum_{I \triangleright J} (-1)^{\ell(J)-1} \frac{fp(J)}{\pi_u(I,J)}.
\]
(5.24)

Proof: First, for any fixed \( m \geq 1 \), let \( I \) be the composition \( \{ m \} \) (of length 1). For any composition \( J \), we have, \( J \triangleright I \iff |J| = m \), and in this case, \( fp(J, I) = fp(J) \) by Eq. (4.18). With these observations and Eq. (4.22), it is easy to see that \( c_{I,K} = c_K \) for any \( K \in C_m \) and Eq. (4.27) becomes
\[
\Xi_m = \sum_{K \in C_m} c_K \Psi^K = \sum_{I \in C_m} c_I \Psi^I.
\]

Applying \( S_{F_1} \) to the equation above and then applying the resulted equation to \( z \), by Eq. (3.10), we get Eq. (5.23). \(\square\)

Next, let us derive the following recurrent inversion formula.

Proposition 5.8. We have the following recurrent inversion formula.
\[
N_{\lfloor 1 \rfloor}(z) = H(z),
\]
(5.25)
\(N_{[m]}(z) = \frac{m}{m-1} \sum_{\ell(I) \geq 2} \frac{1}{\pi_u(I)} \left[ N_{[i_1]} \frac{\partial}{\partial z} \right] \cdots \left[ N_{[i_{k-1}]} \frac{\partial}{\partial z} \right] N_{[i_k]}(z), \)

for any \(m \geq 2\).

For a different but more effective recurrent inversion formula, see Theorem 5.5 in [Z4].

Proof: First, Eq. (5.25) follows easily from Eqs. (5.22), (3.9) with \(m = 1\) and Eq. (3.6).

To show Eq. (5.26), for any \(m \geq 1\), by Eq. (4.23) with \(I = \{m\}\), we have

\[
S_m = \sum_{I \in \mathcal{E}_m} \frac{1}{\pi_u(I)} \xi^I.
\]

Then, we apply the specialization \(S_{F_I} \) to Eq. (5.27) above and, by Corollary 2.9, we get

\[
s_m = \sum_{I \in \mathcal{E}_m} \frac{1}{\pi_u(I)} \xi^I.
\]

Note that, by Eq. (2.31) with \(u(z) = z\) and Eqs. (2.37), (3.10), we have \(s_mz = \xi_m \cdot z = N_{[m]}(z)\) for any \(m \geq 1\). Then, by applying both sides of Eq. (5.28) to \(z\), we get

\[
N_{[m]}(z) = \sum_{I \in \mathcal{E}_m} \frac{1}{\pi_u(I)} \xi^I z,
\]

\[
N_{[m]}(z) = \frac{1}{m} N_{[m]}(z) + \sum_{\ell(I) \geq 2} \frac{1}{\pi_u(I)} \xi^I z,
\]

\[
\frac{m-1}{m} N_{[m]}(z) = \sum_{\ell(I) \geq 2} \frac{1}{\pi_u(I)} \xi^I z.
\]

Therefore, we have

\[
N_{[m]}(z) = \frac{m}{m-1} \sum_{\ell(I) \geq 2} \frac{1}{\pi_u(I)} \xi^I z
\]

Applying Eq. (3.10):

\[
= \frac{m}{m-1} \sum_{\ell(I) \geq 2} \frac{1}{\pi_u(I)} \left[ N_{[i_1]} \frac{\partial}{\partial z} \right] \cdots \left[ N_{[i_{k-1}]} \frac{\partial}{\partial z} \right] N_{[i_k]}(z).
\]
Finally, let us end this subsection with the following identity of differential operators, which does not seem to be obvious.

**Proposition 5.9.** For any $m \geq 1$ and $H(z) \in K\langle\langle z\rangle\rangle^{\times n}$, let $\psi_m$ be the derivation given by Eq. (3.14). Then, we have

$$\frac{1}{m!} \left[ H(w) \frac{\partial}{\partial z} \right]^m \bigg|_{w=z} = \sum_{I \in \mathcal{C}_m} \frac{(-1)^{\ell(I)}}{\pi_u(I)} \psi^I. \quad (5.29)$$

In particular, when $z$ are commutative free variables, we have

$$\psi_m = ((JH)^{m-1}H) \frac{\partial}{\partial z}, \quad (5.30)$$

$$\sum_{I \in \mathbb{N}^n} \frac{1}{I!} H^I(z) \frac{\partial^m}{\partial z^I} = \sum_{I \in \mathcal{C}_m} \frac{(-1)^{\ell(I)}}{\pi_u(I)} \psi^I. \quad (5.31)$$

**Proof:** First, we assume $o(H(z)) \geq 2$ and let $F_t(z) := z - tH(z)$ as before. By applying $S_{F_t}$ to Eq. (4.11) with $I = \{m\}$, we have

$$\lambda_m = (-1)^m \sum_{I \in \mathcal{C}_m} \frac{(-1)^{\ell(I)}}{\pi_u(I)} \psi^I. \quad (5.32)$$

Then, combining the equation above with Eq. (3.13), we get Eq. (5.29). Furthermore, by Eqs. (3.11) and (3.14), we get Eqs. (5.30) and (5.31).

Now, consider the case that $o(H(z)) < 2$. Let $u$ be another free variable which is independent with $z$. Define $\tilde{H}(z,u) := (u^2H(z),0) \in K\langle\langle z,u\rangle\rangle^{n+1}$. Then, by applying Eqs. (5.29)–(5.31) to $\tilde{H}(z,u)$ and then setting $u = 1$, we get the identities in the proposition for $H(z)$ itself. \hfill \Box

5.4. $C_m(z)$'s in terms of other differential operators in $\Omega_{F_t}$. Motivated by the homogeneous Jacobian conjecture that will be discussed in next subsection, below we give formulas for $C_m(z)$ ($m \geq 1$) defined in Lemma 3.1 which is $(JH)^{m-1}H$ in the commutative case.

**Proposition 5.10.** For any $m \geq 1$, we have

$$C_m(z) = (-1)^m \sum_{I \in \mathcal{C}_m} (-1)^{\ell(I)} f p(I) \lambda^I z, \quad (5.32)$$

$$C_m(z) = (-1)^m \sum_{I \in \mathcal{C}_m} (-1)^{\ell(I)} s p(I) \ s^I z, \quad (5.33)$$

$$C_m(z) = (-1)^m \sum_{I \in \mathcal{C}_m} \left( \sum_{J \geq I} (-1)^{\ell(J)} \frac{lp(J)}{sp(I,J)} \right) \phi^I z, \quad (5.34)$$
In particular, when $z$ are commutative variables, we have

$$
(JH)^{m-1} H = (-1)^m \sum_{I \in \mathbb{E}_m} (-1)^{\ell(I)} f_p(I) \lambda^I z,
$$

$$
(JH)^{m-1} H = (-1)^m \sum_{I \in \mathbb{E}_m} (-1)^{\ell(I)} l_p(I) s^I z,
$$

$$
(JH)^{m-1} H = (-1)^m \sum_{I \in \mathbb{E}_m} \left( \sum_{I \geq J} (-1)^{\ell(J)} \ell(J) \frac{l_p(J)}{sp(I, J)} \right) \phi^I z,
$$

$$
(JH)^{m-1} H = \sum_{I \in \mathbb{E}_m} \left( \sum_{I \geq J} (-1)^{\ell(J)-1} \ell(J) \frac{l_p(J)}{\pi_u(I, J)} \right) \xi^I z.
$$

All the formulas above follow directly from the identities of $\psi$ with the corresponding differential operators. For example, by Eq. (4.28) with $I = \{m\}$, we have

$$
\Psi_m = \sum_{K \in \mathbb{E}_m} \left( \sum_{K \geq J} (-1)^{\ell(J)-1} \frac{f_p(J)}{\pi_u(K, J)} \right) \Xi^K
$$

Changing the summation index $J$ by $\bar{J}$ and $K$ by $I$:

$$
= \sum_{I \in \mathbb{E}_m} \left( \sum_{I \geq J} (-1)^{\ell(J)-1} \frac{l_p(J)}{\pi_u(I, J)} \right) \Xi^I.
$$

Note that, by Eq. (4.30), we have $\psi_m z = C_m(z)$. By applying $S_{F_t}$ to the equation above and then applying the resulted equation to $z$, we get Eq. (5.35).

5.5. A Connection of the Jacobian Conjecture with NCSF’s.

In this subsection, we consider the following connection of the well-known Jacobian conjecture with NCSF’s.

Let $K$ be any unital commutative $\mathbb{Q}$-algebra and $z = (z_1, z_2, \cdots, z_n)$ be commutative free variables. We fix a homogeneous $H(z) \in k[z]^n$ of degree $d \geq 2$ and $F_t(z) \in \mathbb{E}[d][\langle\langle z\rangle\rangle]$, $G_t(z)$ and $N_t(z)$ as fixed in Section 3. Denote by $\mathcal{D}[d][z]$ the unital algebra of the differential operators of the polynomial algebra $K[z]$, which increase the degree by at least $d \geq 2$. Let $S_{F_t}$ be the specialization in Theorem 2.8. As one can easily check that, for any $F_t(z) \in \mathbb{E}[d][\langle\langle z\rangle\rangle]$, $S_{F_t}$ actually is a $K$-Hopf algebra homomorphism from $\mathbb{N}Sym$ to $\mathcal{D}[d][z]$. We denote by $J_H$ the kernel
of $S_F$. Since $S_F$ is a homomorphism of $K$-algebras, $I_H$ is a two-sided ideal of the free $K$-algebra $NSym$.

By the homogeneous reduction in [BCW] and [Y] on the Jacobian conjecture, it is easy to see that the Jacobian conjecture is equivalent to the following conjecture.

**Conjecture 5.11.** For any homogeneous $H(z) \in K[z]^{\times n}$ of degree $d \geq 2$, or equivalently, $d = 3$, let $F_t(z) := z - tH(z)$ and $G_t(z) := F_t^{-1}(z)$. Assume that the Jacobian matrix $JH$ is nilpotent. Then the inverse map $G_t(z)$ is also a polynomial map of $z$ over $K[t]$.

Note that, by Euler’s lemma, we have $JH^{m-1}H = \frac{1}{d}(JH)^m z$. Hence the nilpotence of $JH$ implies $(JH^{m-1}) H = 0$ for any $m \geq n$. By Eqs. (3.9) and (3.8), this is same as saying that the derivations $\psi_m = 0$ for any $m \geq n$, or equivalently, $\Psi_m \in I_H$ for any $m \geq n$. On the other hand, by Eq. (3.10), we see that, $G_t(z)$ is a polynomial map iff the derivations $\xi_m = 0$ for $m >> 0$, or equivalently, $\Xi_m \in I_H$ for $m >> 0$. Therefore, by the observations above and the equivalence of the Jacobian conjecture and Conjecture 5.11, we see that the Jacobian conjecture is equivalent to the following conjecture.

**Conjecture 5.12.** For any homogeneous $H(z) \in K[z]^{\times n}$ of degree $d = 3$, suppose that $\Psi_m \in I_H$ for any $m \geq n$. Then, $\Xi_m \in I_H$ for $m >> 0$.

Note that, by the remarkable symmetric reduction on the Jacobian conjecture achieved recently in [BE] and [Me], we may further assume that $H(z)$ is the gradient of a homogeneous polynomial $P(z)$ of degree 4, i.e. $H(z) = (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \cdots, \frac{\partial P}{\partial z_n})$.

Therefore, from the point view of the Jacobian conjecture, we see the following open problem becomes interesting and important.

**Problem 5.13.** For any homogeneous $H(z) \in K[z]^{\times n}$ with the Jacobian matrix $JH$ nilpotent, find the relations among NCSF’s, which decide the two-sided ideal $I_H$.

Note that, by Theorem 2.10, there are no deciding relations for the ideal $I_H$, which are independent of the choices of $n \geq 1$ and $H \in K[z]^{\times n}$. Considering the fact that the classical symmetric functions have been well studied and the fact mentioned in Remark 2.5, a good starting point to approach Problem 5.13 above might be to consider the case when all differential operators in the NCS system $\Omega_F$ commute with each other.
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