Bootstrap percolation  
on a graph with random and local connections. 

Tatyana S. Turova *  Thomas Vallier †

Abstract

Let $G^1_{n,p}$ be a superposition of the random graph $G_{n,p}$ and a one-dimensional lattice: the $n$ vertices are set to be on a ring with fixed edges between the consecutive vertices, and with random independent edges given with probability $p$ between any pair of vertices. Bootstrap percolation on a random graph is a process of spread of "activation" on a given realisation of the graph with a given number of initially active nodes. At each step those vertices which have not been active but have at least $r \geq 2$ active neighbours become active as well. We study the size of the final active set in the limit when $n \to \infty$. The parameters of the model are $n$, the size $A_0 = A_0(n)$ of the initially active set and the probability $p = p(n)$ of the edges in the graph.

Bootstrap percolation process on $G_{n,p}$ was studied earlier. Here we show that the addition of $n$ local connections to the graph $G_{n,p}$ leads to a more narrow critical window for the phase transition, preserving however, the critical scaling of parameters known for the model on $G_{n,p}$. We discover a range of parameters which yields percolation on $G^1_{n,p}$ but not on $G_{n,p}$.

MSC2010 subject classifications. 05C80, 60K35, 60C05.

Key words and phrases. Bootstrap percolation, random graph, phase transition.

1 Introduction

Bootstrap percolation was introduced on a lattice by Chalupa, Leath and Reich [6] to model some magnetic systems. Also, models of neuronal activity have very similar basic features. (Use of percolation models in neuronal sciences was predicted already by Harris [7].) Bootstrap percolation is defined as a process of spread of activation or infection on a graph

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*Mathematical Centre, University of Lund, Box 118, Lund S-221 00, Sweden.
†Department of Mathematics and Statistics, Box 68, FI-00014 University of Helsinki, Finland
$G = (V,E)$ in the following way. Assume $G$ has a finite set of vertices, call it $V$. There is an initial set $A_0 \subset V$ of active vertices. For a given threshold $r \geq 2$, each inactive vertex that has at least $r$ active neighbours (i.e., the vertices connected to it by the graph edges) becomes active and can spread the activation along its edges. This is repeated until no more vertex becomes active. Active vertices never become inactive, so the set of active vertices grows monotonously. Let $A^*$ denote the final active set. We say that (a sequence of) $A_0$ percolates (completely) if $|A^*| = A^* = n$ and that $A_0$ almost percolates if the number of vertices that remain inactive is $o(n)$, i.e., if $A^* = n - o(n)$.

Bootstrap percolation has been extensively studied on varieties of graphs, as e.g., $d$-dimensional grid (see recent results by Balogh, Bollobás, Duminil-Copin and Morris [2] and Uzzell [17]), hypercube (Balogh and Bollobás [1]), infinite trees (Balogh, Peres and Pete [3]), random regular graphs (Balogh and Pittel [4], Janson [10]), Erdös–Rényi random graph $G_{n,p}$ (Janson, Luczak, Turova and Vallier [11]), Galton-Watson trees (Bollobás, Gunderson, Holmgren, Janson, and Przykucki [5]).

We study here a graph with both local and global links for its connections with the models of neuronal activity ([12], [14], [15]). The geometry of the graph is the same as the small world network model of Newman and Watts [13]. It has a structure typical of the real world networks which usually have both local and global connections. Despite a long history of the subject and even very detailed results both for the $d$-dimensional grid and for the random graph (see the citations above), still only a few theoretical results are available for the graphs with mixed connections.

The graph has two types of edges: random and non-random. We start with a regular one dimensional lattice. The vertices $V = \{1, \ldots, n\}$ are ordered on a ring $R_n$ and have an edge with their two nearest neighbours. We add random connections. The random edges are given independently for each pair of vertices with the same probability $p$. Hence, there might be at most two edges between the vertices in the model, and if there are two edges between a pair of vertices, the edges are necessarily of two types: one from the random graph and another one from the lattice. In such a case, we merge the edges. The subgraph on $V$ with the random edges only is a random graph $G_{n,p}$. Similarly, replacing a ring by a $d$-dimensional torus with $n$ vertices one can define a graph $G^d_{n,p}$ for all $d > 1$. One can also study bootstrap percolation on a 1-dimensional lattice where a vertex has a link with the vertices at distance at most $k$.

We consider a bootstrap percolation on $G^1_{n,p}$ with the threshold $r = 2$ and $p = p(n)$. We assume, that an initial set $A(0)$ consists of a given number $A_0 = A_0(n) \geq 2$ of vertices chosen uniformly at random from the set $\{1, \ldots, n\}$. We study here the process with the threshold $r = 2$ for simplicity and clarity. The results can be extended to general $r$ by introducing some heavier notations.

Typically, a bootstrap percolation process exhibits a threshold phenomenon: either $o(n)$
number of vertices become active, or, on the contrary, \( n - o(n) \) vertices become active. The main question here is how the superposition of different structures affects the phase transition. In particular, is it possible to get a complete percolation combining two subcritical systems? In the case of an ordinary percolation model, a superposition of two subcritical graphs (one being a grid with randomly removed edges, bond percolation, and another one being an Erdős-Rényi random graph, each of which has the largest connected component of order at most \( \log n \)) may have a component of order \( n \) \([16]\). In this case superposition of the graphs produces new critical values on a phase diagram (see \([16]\)). We shall see here that the bootstrap percolation process exhibits different properties (at least in dimension 1).

2 Results

Let us recall some notations and results from \([11]\) which we need here.

2.1 Notations

Let \( 1 \leq i < j \leq n \), the distance between the vertices \( i \) and \( j \) is defined as

\[
d(i, j) = \min \{ j - i, n + i - j \}.
\]

The distance of a vertex \( u \) to a set \( S \) is defined as

\[
d(v, S) = \inf \{ d(u, v) \mid v \in S \}.
\]

We denote \( \partial_{1}S \) the outer boundary of a vertex set \( S \) in the cycle \( R_n \).

\[
\partial_{1}S = \{ v \in R_n \setminus S \mid d(v, S) = 1 \}.
\]

We use the notation \( f(n) = \Theta(g(n)) \) as \( c_1g(n) \leq f(n) \leq c_2g(n) \) for \( c_1, c_2 > 0 \) and as \( n \to \infty \). We use \( O_p \) and \( o_p \) in the standard sense (see e.g., \([9]\)) and we use w.h.p. (with high probability) for events with probability tending to 1 as \( n \to \infty \). Note that, for example, ‘\( = o(1) \) w.h.p.’ is equivalent to ‘\( = o_p(1) \)’ and to ‘\( \to 0 \)’.

We use the notations \( O_{L^k} \) and \( o_{L^k} \) in the same setting as in \([9]\). Let \( a_n \) be some sequence of real numbers, \( X_n = O_{L^k}(a_n) \iff \mathbb{E}((X_n)^k) = O((a_n)^k) \). In particular \( X_n = O_{L^2}(a_n) \Rightarrow X_n = O_{L^1}(a_n) \Rightarrow X_n = O_p(a_n) \).

All unspecified limits are as \( n \to \infty \). For given \( n \) and \( p \) define

\[
a_c := \frac{1}{2} t_c = \frac{1}{2} np^2.
\]
The term \(a_c\) is the first order term of the critical threshold \(a_c^*(n,p)\) for bootstrap percolation on the random graph \(G_{n,p}\). The term \(a_c^*(n,p)\) is defined in [11] as follows:

Let

\[
\tilde{\pi}(t) := \mathbb{P}(\text{Po}(tp) \geq 2) = \psi(tp) := \sum_{j=2}^{\infty} \frac{(pt)^j}{j!} e^{-pt},
\]

where \(\text{Po}(tp)\) denotes a Poisson random variable with mean \(tp\). Then set

\[
a_c^* := -\min_{t \leq 3t_c} \left\{ \frac{n \tilde{\pi}(t) - t}{1 - \tilde{\pi}(t)} \right\},
\]

and let \(t_c^* \in [0, 3t_c]\) be the point where the minimum is attained. Notice that \(t_c = \frac{1}{np^2}\) is also the first order term of \(t_c^*\).

We recall in the following section the technique developed in [11] to explore the set of active vertices. One of the key points is to prove that the process of activation can have more than \((1 + \epsilon)t_c\) active vertices, for some \(\epsilon > 0\). It is shown in [11] that w.h.p. the process either stops before activating \((1 + \epsilon)t_c\) vertices or almost percolates. Therefore, we often refer to \(t_c\) as the ”bottleneck”.

Let here \(A_0^*\) denote the final set of vertices activated due to a bootstrap percolation on a random graph \(G_{n,p}\) starting with \(A_0\) active vertices, i.e., we do not take into account the local edges from the ring \(R_n\). It is clear that there is a coupling of these two models (with and without the short edges) such that

\[
A_0^* \subseteq A^*.
\]

2.2 Results

Note that in [11] the bootstrap percolation on a random graph \(G_{n,p}\) was studied for a general case \(r \geq 2\). The following theorem (which is a particular case when \(r = 2\) of the theorem proved in [11]) describes the phase transitions in the value \(|A_0^*|\) depending on the initial condition \(A_0\). Let \(b_c = npne - p^n\), \(b_c\) is the expected number of vertices of degree 1 in \(G_{n,p}\).

Theorem (Theorem 3.1 [11], case \(r = 2\)). Suppose that \(n^{-1} \ll p \ll n^{-1/2}\). Let \(A_0^* = |A_0^*|\) be the total number of vertices activated due to a bootstrap percolation on a random graph \(G_{n,p}\) starting with \(A_0\) active vertices.

(i) If \(A_0/a_c \to \alpha < 1\), then

\[
A_0^* = (\varphi(\alpha) + o_p(1))t_c,
\]

where \(\varphi(\alpha) = 1 - \sqrt{1 - \alpha} \) with \(\lim_{\alpha \to 1} \varphi(\alpha) = \varphi(1) = 1\).
(ii) If \( A_0/a_c \geq 1 + \delta \), for some \( \delta > 0 \), then \( A^*_0 = n - O_p(b_c) = n - o_p(n) \); in other words, we have w.h.p. almost percolation.

Due to the observation (2.1), if the initial set \( A_0 \) percolates on \( G_{n,p} \), then the same set percolates on \( G^{1}_{n,p} \) which contains \( G_{n,p} \). Therefore, we are interested here in the initial conditions \( A_0 \) which do not yield a percolation on \( G_{n,p} \), i.e., when \( \alpha \leq 1 \) in the above theorem. The following theorem tells us that adding to graph \( G_{n,p} \) the edges between the nearest neighbours (in dimension one) does not change much the subcritical regime, at least when \( p \geq \frac{\log n}{n} \).

**Theorem 2.1.** Suppose that \( n^{-1} \ll p \ll n^{-1/2} \). Let \( A^* = |A^*| \) be the total number of vertices activated due to a bootstrap percolation on a random graph \( G^{1}_{n,p} \) starting with \( A_0 \) active vertices.

(i) If \( p \gg \frac{\log n}{n \log(pn)} \) and \( A_0/a_c \to \alpha < 1 \), then

\[
A^* = (\varphi(\alpha) + o_p(1))t_c.
\]

(ii) If \( A_0/a_c \geq 1 + \delta \), for some \( \delta > 0 \), then w.h.p. \( A^* = n - o(n) \);

**Remark 2.1.** The condition \( p \gg \frac{\log n}{n \log(pn)} \) in the Theorem 2.1 is satisfied, e.g., if \( p \geq \frac{\log n}{n} \).

Theorem 2.1 does not describe the case \( \frac{1}{n} \ll p \ll \frac{\log n}{n} \). (Notice that for \( p \) of order \( 1/n \), addition of \( n \) edges changes the graph properties.) What follows from our proof is that the subcritical phase for very small \( p \) may have a large number of steps before the process stops. Also, it is exactly due to the fact that under the conditions of Theorem 2.1 the value \( pn \) is large enough to yield complete percolation (in the supercritical phase) when the short edges are included.

It turns out that it is the critical case, i.e., when \( \alpha = 1 \), which is affected most by the presence of the local connections. First we recall the situation with \( G_{n,p} \).

**Theorem** (Theorem 3.6 [11]). Suppose that \( n^{-1} \ll p \ll n^{-1/2} \). Let \( A^*_0 \) be the total number of vertices activated due to a bootstrap percolation (with threshold \( r = 2 \)) on a random graph \( G_{n,p} \) starting with \( A_0 \) active vertices.

(i) If \( (A_0 - a^*_c)/\sqrt{a_c} \to -\infty \), then for every \( \varepsilon > 0 \), w.h.p. \( A^*_0 \leq t^*_c \leq t_c(1 + \varepsilon) \). If further \( A_0/a^*_c \to 1 \), then \( A^*_0 = (1 + o_p(1))t_c \).

(ii) If \( (A_0 - a^*_c)/\sqrt{a_c} \to +\infty \), then \( A^*_0 = n - O_p(b_c) \).
(iii) If \( (A_0 - a^*_c)/\sqrt{a_c} \rightarrow y \in (-\infty, \infty) \), then for every \( \varepsilon > 0 \) and every \( b^* \gg b_c \) with \( b^* = o(n) \),

\[
\mathbb{P}(A_0^* > n - b^*) \rightarrow \Phi(y),
\]
\[
\mathbb{P}(A_0^* \in [(1-\varepsilon)t_c, (1+\varepsilon)t_c]) \rightarrow 1 - \Phi(y).
\]

In the following, we show that when \( p \) is small enough, including short edges into the model may lead to percolation even when there is no percolation in \( G_{n,p} \) with the same parameters.

**Theorem 2.2.** Let \( A^* \) be the total number of vertices activated due to a bootstrap percolation on a random graph \( G_{1,n,p} \) starting with \( A_0 \) active vertices.

1. If \( n^{-1} \ll p \ll n^{-3/4} \) and
   \[
a^*_c - A_0 < (1-\varepsilon)2pt^2_c
   \]
   or equivalently
   \[
   \frac{a^*_c - A_0}{\sqrt{a_c}} < (1-\varepsilon)2\sqrt{2}pt^2_c;
   \]
   then w.h.p. \( A^* = n - o(n) \).

2. If \( n^{-2/3} \ll p \ll n^{-1/2} \) and
   \[
   \frac{A_0 - a^*_c}{\sqrt{a_c}} \rightarrow -\infty,
   \]
   then for every \( \varepsilon > 0 \), w.h.p. \( A^* \leq t^*_c \leq t_c(1+\varepsilon) \).

Theorem 2.2 part (i) describes the case when the addition of local edges even in dimension 1 changes the phase diagram. Indeed, condition (2.2) tells us that almost percolation can happen even when \( A_0 < a^*_c \), if \( A_0 \) deviates from \( a^*_c \) by at most \( (1-\varepsilon)2pt^2_c \). Under the assumption \( n^{-1} \ll p \ll n^{-3/4} \), we have \( (1-\varepsilon)2pt^2_c \gg \sqrt{a_c} \). Therefore, w.h.p. percolation does not occur on the edges of \( G_{n,p} \) only but will occur on \( G_{1,n,p}^1 \). Part (ii) tells us that the critical window does not change for "large" \( p \), i.e., when \( n^{-2/3} \ll p \ll n^{-1/2} \). (Observe that it does not lead to a contradiction, since \( a_c = a_c(p) \) changes accordingly.)

The condition \( n^{-2/3} \ll p \) in (ii) is needed such that the contribution of the short edges is negligible. That behaviour can be explained by the fact that the critical parameter \( a^*_c(n,p) \) for bootstrap percolation on \( G_{n,p} \) decreases as \( p(n) \) increases. Therefore, for large \( p \), the set of active vertices is very sparse. The activation using the short edges is restricted to the nearest (direct) neighbours of \( \mathcal{A} \). There are less than \( 2|\mathcal{A}| \) vertices that can potentially become activated by mixed or short activation which is too few.
Finally, we indicate the critical probability for transition from almost percolation ($A^* = n - o_p(n)$) to complete percolation (w.h.p. $A^* = n$). Notice that the subgraph $G_{n,p}$ does not need to be connected to have complete percolation on $G_{n,p}$.

**Theorem 2.3.** Suppose that the process is supercritical (i.e. $A^* = n - o_p(n)$)

(i) If $np - \frac{1}{2} \log n \to -\infty$ then $\lim_{n \to \infty} \mathbb{P}\{A^* \leq n - 2\} = 1$.

(ii) If $np - \frac{1}{2} \log n \to +\infty$ then $\lim_{n \to \infty} \mathbb{P}\{A^* = n\} = 1$.

(iii) If $np - \frac{1}{2} \log n = -\log c$ then $\lim_{n \to \infty} \mathbb{P}\{A^* = n\} = e^{-c^2}$.

3 Discussion on higher dimensions

We show that adding the structure of the one-dimensional grid makes the phase transition even sharper by decreasing the critical window.

The challenge remains to study a bootstrap percolation process on $G^d_{n,p}$ with $d > 1$. In this case the effect of the local connections from the $d$-dimensional grid will be substantial, as one can readily see in the following calculations. Consider for simplicity a two-dimensional discrete torus $T = [1, \ldots, N]^2$ with $n = N^2$ vertices and all edges between these vertices inherited from the two-dimensional lattice. Assume also that with a probability $p$ there is an edge between any pair of vertices, independent for different pairs. Denote the corresponding graph $G^2_{n,p}$. Assume that with probability $q = q(n)$ each vertex is set initially to be active independently of the rest and consider a bootstrap percolation with threshold $r = 2$ as in [8]. It is known (Holroyd [8]) that a complete percolation on torus $T$ with local edges only, will happen with probability at least $1/2$ if $q(n)/q_c(n, 2, 2) > 1$, where

$$q_c(n, 2, 2) := \frac{\pi^2}{18 \log n} (1 + o(1)) =: \frac{c_0}{\log n} (1 + o(1)).$$

Otherwise, if $q(n)/q_c(n, 2, 2) < 1$, the complete percolation will not occur with probability at least $1/2$.

Consider now a bootstrap percolation process on $G^2_{n,p}$ with

$$A_0 = an \frac{c_0}{\log n} =: \alpha a_c = an q_c (1 + o(1))$$

initially active vertices, and with

$$p = \frac{1}{\sqrt{2na_c}}.$$
Let $0 < \alpha < 1$, and therefore
\[
\frac{A_0}{nq_c(n, 2, 2)} = \frac{\alpha}{1 + o(1)} < 1.
\]

Using results \cite{8} one concludes that on the subgraph $T = [1, \ldots, N]^2$ of $G_{n,p}^2$ induced by the local connections only, a complete percolation will not occur with probability tending to 1 as $n \to \infty$ (or equivalently as $N \to \infty$). Also, by the Theorem 3.1 \cite{11} (see above) on the subgraph of $G_{n,p}$ of $G_{n,p}^2$ bootstrap percolation process with a high probability ends with only
\[
A_0^* = (1 - \sqrt{1 - \alpha})^2 a_c = o(n)
\]
active vertices. Hence, neither short edges nor random edges alone may yield with a high probability a complete percolation on $G_{n,p}^2$ with the given parameters. However, one can choose $0 < \alpha < 1$ so that
\[
\frac{A_0^*}{np_c(n, 2, 2)} = \frac{2(1 - \sqrt{1 - \alpha})}{1 + o(1)} > 1.
\]
Then starting with $A_0^*$ vertices one can argue using again results \cite{8} that a complete percolation will happen with a high probability on the graph $G_{n,p}^2$. This confirms that a superposition of two subcritical systems can lead to almost percolation.

A complete analysis of this problem should be the subject of a separate work.

4 Proofs

4.1 Useful reformulation

We shall distinguish the following three types of activation of a vertex depending on the type of connections which caused this activation: the long range activation, the short range activation and the mixed activation. The long range activation uses only random edges, we also call it ”$G_{n,p}$ activation”. The short range activation uses only local edges: if the vertices $i - 1$ and $i + 1$ are active then the vertex $i$ becomes active as well. We say that the activation of a vertex is mixed if it is caused by two edges of each type. See figure 1.

In order to analyse the bootstrap percolation process on $G_{n,p}^1$, we split the process of activation in two distinct phases depending on the type of activation.

4.1.1 First Exploration Phase.

Consider activation through the long (random) connections only.
We say that the vertices are *neighbours* if there is at least one edge between them. For any subgraph $G$ of $G_{n,p}^1$, we shall say that two vertices are *$G$-neighbours* if there is an edge from the subgraph $G$ between them.

We follow the algorithm for revealing the activated vertices as described in [11]. First, we change the time scale: we consider at each time step the activations from one vertex only.

Given a set $A_0$ define $A_1(0) = A_0$. Choose $u_1 \in A_1(0)$ and give each of its neighbours a *mark*; we then say that $u_1$ is used, and let $Z_1(1) := \{u_1\}$ be the set of used vertices at time 1. We continue recursively: At time $t > 1$, choose a vertex $u_t \in A_1(t - 1) \setminus Z_1(t - 1)$. We give each neighbour of $u_t$ a new mark. Let $\Delta A_1(t)$ be the set of inactive vertices with 2 marks; these now become active and we let $A_1(t) = A_1(t - 1) \cup \Delta A_1(t)$ be the set of active vertices at time $t$. We finally set $Z_1(t) = Z_1(t - 1) \cup \{u_t\} = \{u_s : s \leq t\}$, the set of used vertices. (We start with $Z_1(0) = \emptyset$, and note that necessarily $\Delta A_1(t) = \emptyset$ for $t < 2$.)

The process stops when $A_1(t) \setminus Z_1(t) = \emptyset$, i.e., when all active vertices are used. We denote this time by $T_1$:

$$T_1 = \min\{t \geq 0 : A_1(t) \setminus Z_1(t) = \emptyset\} = \min\{t \geq 0 : |A_1(t)| = t\}.$$

For $v \notin Z_1(i)$, let $\xi_{u_i,v}$ be the indicator that there is an edge between the vertices $u_i$ and $v$, $Z_1(i) \in \text{Be}(p)$. It is also the indicator function that $v$ receives a mark at time $i$. Therefore, by independence of the random connections on $G_{n,p}$, we find that the number of marks that a vertex receives is distributed as a binomial random variable. Denote $M_v(t)$, the number of marks of the vertex $v \in V \setminus A(0)$ at time $t$.

$$\pi_1(t) = \mathbb{P}\{v \in A_1(t)\} = \mathbb{P}\{M_v(t) \geq 2\} = \mathbb{P}\left\{\sum_{i=1}^{t} \xi_{u_i,v} \geq 2\right\} = \mathbb{P}\{\text{Bin}(t, p) \geq 2\} \quad (4.3)$$

Figure 1: The 3 different types of activation.
where the last equality follows from the independence of the connections on $G_{n,p}$.

Let for $t > 0$

$$S_1(t) = \{ v \notin A(0), M_v(t) \geq 2 \} = \left\{ v \notin A(0) : \sum_{i=1}^{t} \xi_{u,v} \geq 2 \right\}.$$

Observe that the vertices of set $S_1(t)$ (more precisely, the labels of those vertices) are distributed uniformly over the set $\{1, \ldots, n\}$ (drawing $|S_1(t)|$ points without replacement). Using again the independence of the connections, we find that the random variable $S_1(t) := |S_1(t)|$ is a sum of $n - A_0$ i.i.d Bernoulli random variables. Thus

$$S_1(t) := |S_1(t)| \overset{d}{=} \text{Bin}(n - A_0, \pi_1(t)),$$

where $\pi_1(t)$ is defined in equation (4.3),

$$\pi_1(t) = \mathbb{P}\{\text{Bin}(t,p) \geq 2\}. \quad (4.4)$$

Define now the set of active vertices at time $t > 0$ by

$$A_1(t) = A_0 \cup S_1(t). \quad (4.5)$$

We call this phase an "exploration" phase as we explore the long range connections of the vertices. The total number of active vertices at the end of this phase is denoted $|A_1(T_1)| = T_1$.

4.1.2 First Expansion Phase.

Now we take into account the structure of the local connections. Let us denote $R_n$ the corresponding subgraph (which forms a Hamiltonian cycle on $V$).

After the 1-st exploration phase we have a random set $A_1(T_1)$ of active vertices on $R_n$. Hence, we may represent the set of inactive vertices as a collection of paths on $R_n$. (A path on $R_n$ has a structure inherited from $R_n$: the consecutive points are pairwise connected.)

During the "expansion" phase, the set of active vertices $A_1(T_1)$ may expand to its neighbours, or, in other words the paths of inactive vertices may become only shorter. More precisely, we define the expansion phase in 3 different steps.

1. Any vertex which has two active (i.e., belonging to the set $A_1(T_1)$) neighbours on $R_n$ becomes active. This means that all the paths of inactive vertices which consist of a single vertex become active.

After this step, we are left with the paths of inactive vertices which contain at least two vertices. Each of these vertices may have at most one mark assigned during the exploration phase.
2. Any vertex (in any inactive path of length at least two) which has a mark and which is either an endpoint or is connected to an endpoint only through vertices each of which also has a mark, becomes active.

3. After the second step there may be again paths of inactive vertices which contain a single vertex. Then step 1 is repeated, i.e., again any vertex which has two active neighbours on $R_n$ becomes active.

The third step completes the expansion phase.

After the expansion phase, we may represent the set of inactive vertices as a collection of paths on $R_n$ each of which has the following properties:

(i) any path has at least two vertices,

(ii) the endpoints do not bare a mark but all the other vertices of the intervals may have at most one mark (assigned during the exploration phase).

Let us denote $D_1$ the set of vertices activated during the 1-st expansion phase.

At the end of the first expansion phase, we have $T_1 + |D_1|$ active vertices: $T_1$ of them have been used and the set $D_1$ is still unused.
4.1.3 Alternating the phases.

Having completed the 1st expansion phase, we shall alternate exploration and expansion phases. We shall denote \( \mathcal{A}_k(T_k) \) and \( \mathcal{D}_k \) the sets of vertices acquired in the \( k \)-th exploration and expansion phases, correspondingly, \( k \geq 1 \). Notice that the sets \( \mathcal{A}_k(T_k) \) and \( \mathcal{D}_k \) are disjoint.

We assume that after the \( k \)-th exploration phase we have used all vertices in \( \mathcal{A}_k(T_k) \) so that \( |\mathcal{A}_k(T_k)| = T_k \). Let

\[
\mathcal{A}^k := \bigcup_{i=1}^k \mathcal{A}_i(T_i),
\]

which is the set of all used vertices. Still we have the set \( \mathcal{D}_k \) (assuming \( \mathcal{D}_k \) is not empty) of active vertices to explore: the ones which were activated during the \( k \)-th expansion phase.

Given the sets \( \mathcal{A}_i(T_i), i \leq k \), and \( \mathcal{D}_k \), let us define the \( k+1 \)-st exploration phase similar to the first one: we restart the process, setting again time \( t = 0 \), but now on vertices \( V \setminus \mathcal{A}^k \), among which the set of initially active vertices is

\[
\mathcal{A}_{k+1}(0) := \mathcal{D}_k.
\]

This set plays the same role as \( \mathcal{A}_1(0) \) in the description of the first exploration phase. Notice also that

\[
|V \setminus \mathcal{A}^k| = n - \sum_{i=1}^k T_i.
\]

We explore the vertices (i.e., assign marks to their \( G_{n,p} \)-neighbours) of \( \mathcal{D}_k \) one at a time, calling them again \( u_1, u_2, \ldots \). Observe, however, that some of the vertices may have one mark from set \( \mathcal{A}^k \) and this makes the difference with the first exploration phase. More precisely, we have two types of vertices: the vertices on the boundary of set \( \mathcal{A}^k \cup \mathcal{D}_k \) which do not have any long-type edge to \( \mathcal{A}^k \), and the rest of vertices (i.e., the ones in \( V \setminus (\mathcal{A}^k \cup \mathcal{D}_k \cup \partial_1(\mathcal{A}^k \cup \mathcal{D}_k)) \)) which may have at most one long-type edge to the set \( \mathcal{A}^k \) (i.e., have a mark). Recall that \( \partial_1(\mathcal{A}^k \cup \mathcal{D}_k) \) is the outer boundary of \( \mathcal{A}^k \cup \mathcal{D}_k \), see (2.1). Then the set of vertices activated during the first \( t \) steps of the \( k+1 \)-st exploration phase is

\[
S_{k+1}(t) := \left\{ v \in \partial_1(\mathcal{A}^k \cup \mathcal{D}_k) : \sum_{i=1}^t \xi_{u_i v} \geq 2 \right\}
\cup \left\{ v \not\in \mathcal{A}^k \cup \mathcal{D}_k \cup \partial_1(\mathcal{D}_k \cup \mathcal{A}^k) : \sum_{i=1}^t \xi_{u_i v} + \xi_v(k) \geq 2 \right\},
\]

where \( \xi_v(k) \overset{d}{=} \xi(k) \) is an independent Bernoulli random variable which equals one with the probability that an inactive vertex \( v \) has precisely one mark after the \( k \)-th exploration phase,
i.e.,
\[ P\{\xi_v(k) = 1\} = P\left\{ \text{Bin}(|A^k|, p) = 1 | \text{Bin}(|A^k|, p) < 2 \right\} = \frac{|A^k|p}{1 + (|A^k| - 1)p}. \] (4.6)

Let us define now
\[ \pi_{k+1}(t) = P\{\text{Bin}(t, p) + \xi(k) \geq 2\}, \]
where \( \xi(k) \) and the binomial random variable are independent. Notice that
\[ \pi_{k+1}(t) = P\{\text{Bin}(t, p) \geq 2\} + P\{\text{Bin}(t, p) = 1\} P\{\xi(k) = 1\} = \pi_1(t) + |A^k|p(1 - p)^{t-1}pt, \]
with \( \pi_1(t) \) defined by (4.4). Then the distribution of \( S_{k+1}(t) := |S_{k+1}(t)| \) is
\[ S_{k+1}(t) \overset{d}{=} \text{Bin} \left( n_{k+1} - |D_k| - |\partial_1(D_k \cup A^k)|, \pi_{k+1}(t) \right) + \text{Bin} \left( |\partial_1(D_k \cup A^k)|, \pi_1(t) \right), \] (4.7)
where the binomial variables are independent. Define also (as in (4.5))) for \( t > 0 \)
\[ A_{k+1}(t) = D_k \cup S_{k+1}(t), \]
which is the set of active vertices at the step \( t \) of the \( k + 1\)-st exploration phase. Then, assuming \( D_k \neq \emptyset \), the moment
\[ T_{k+1} := \min\{t > 0 : |A_{k+1}(t)| = t\} \]
is the first time when all the available active vertices are explored, i.e., we have found all the \( G_{n,p}\)-neighbours of active vertices. This completes the \( k + 1\)-st exploration phase.

The \( k + 1\)-st expansion phase is similar to the first one. Recall that after the \( k + 1\)-st exploration phase we may represent the set of all remaining inactive vertices as a collection of intervals on \( R_n \). Each of the vertices of these intervals may have at most one mark (assigned during any of the previous exploration phases). Then at the \( k + 1\)-st expansion phase any vertex which either has two active \( R_n\)-neighbours, or it has a mark and it is connected to an endpoint with a mark through the vertices each of which has also a mark, becomes active. Finish the phase with step 3 by activating the vertices that have two active nearest neighbours on \( R_n \). We denote \( D_{k+1} \) the set of all vertices activated during this phase.

Let us now define the process of bootstrap percolation on \( G_{n,p}^k \) as
\[ A(t) = \bigcup_{i=1}^{k-1} A_i(T_i) \cup A_k \left( t - \sum_{i=1}^{k-1} T_i \right), \quad \sum_{i=1}^{k-1} T_i \leq t < \sum_{i=1}^{k} T_i, \quad k \geq 1. \]
The process of bootstrap percolation on $G_{n,p}$ stops at time $T$ which is
\[ T = \min\{t : A(t) = t\}. \] (4.8)
It follows then that
\[ T = \sum_{k=1}^{K} T_k, \] (4.9)
where
\[ K = \min\{k : D_k = \emptyset\}, \] (4.10)
meaning that no vertex is activated during the $k$-th expansion phase. We shall denote
\[ \mathcal{A}^* := A(T). \]
Notice that by (4.8) and (4.9) we have
\[ |\mathcal{A}^*| = \sum_{k=1}^{K} T_k. \]

**Remark 4.1.** By changing the time and considering the activation in different order, we do not change the limiting set $\mathcal{A}^*$ of activated vertices which depends only on the initial set $\mathcal{A}$.

### 4.2 The number of vertices activated in an expansion phase.

We begin with the first expansion phase, namely, we shall study the set $\mathcal{D}_1$.

**Lemma 4.1.** Let $\mathcal{A}_1(T_1)$ be a set of vertices uniformly distributed on $V = \{1, \ldots, n\}$, and assume that $|\mathcal{A}_1(T_1)| = T_1 \leq \frac{2}{np^2}$, where $n^{-1} \ll p \ll n^{-1/2}$. Then
\[ |\mathcal{D}_1| = D_1 = \begin{cases} 2pT_1^2 + O_{L,1} \left( pT_1^2(T_1/n)^{1/3} \right) + O_{L,1}(T_1^2/n), & \text{if } pT_1^2 \to \infty, \\
2pT_1^2 + O_{L,1} \left( (pT_1^2)^{1/2} \right) + O_{L,1}(T_1^2/n), & \text{otherwise}. \end{cases} \]

**Remark 4.2.** Notice that the random variable $T_1 = A_0^*$ is described by Theorems 3.1 [11] and 3.6 [11] cited above.

**Proof of Lemma 4.1.** For simplicity of the notations let us set here $T_1 = k$. Given a subset $\mathcal{A}_1(T_1) = \{i_1, \ldots, i_k\}$ (assume that $i_1 < \ldots < i_k$) define sets (maybe empty)
\[ I_1 = \{i_k + 1, \ldots, n, 1, \ldots, i_1 - 1\}, I_j = \{i_{j-1} + 1, \ldots, i_j - 1\}, j = 2, \ldots, k. \]
These are the paths (i.e. consecutively connected vertices) on $R_n$ consisting of vertices which remain inactive after the 1-st exploration phase. Hence,

$$A_1(T_1) \cup (\bigcup_{j=1}^{k} I_j) = \{1, \ldots, n\},$$

and

$$k + \sum_{j=1}^{k} |I_j| = n.$$

Define also

$$N_l = \# \{j \geq 1 : |I_j| = l\}, \quad l \geq 0.$$

Assuming the uniform distribution of the set $A_1(T_1)$, we derive for all $l$ such that $l \leq n - k$

$$\mathbb{P}\{|I_j| = l\} = \frac{\binom{n-2-l}{k-2}}{\binom{n-1}{k-1}}. \quad (4.11)$$

In particular, this yields

$$\mathbb{P}\{|I_j| = 1\} = \frac{n-k}{n-1} \frac{k-1}{n-2},$$

and

$$\mathbb{P}\{|I_j| \leq 2\} = 3 \frac{k}{n} + o\left(\frac{k}{n}\right), \quad (4.12)$$

when $k = o(n)$. We have $k = T_1 \leq \frac{2}{np} = o\left(\frac{1}{p}\right) = o(n)$ since $p \gg \frac{1}{n}$.

Recall that any vertex of any $I_j$ has one mark with probability defined by (4.6)

$$p_1 := \frac{kp}{1 + (k-1)p}, \quad (4.13)$$

independent of the other vertices.

For all $l > 1$ and $j \geq 1$, given that $|I_j| = l$, let $M_j(l)$ be the (random) number of vertices in $I_j$ which have a mark and which are either the endpoints of $I_j$ or they are connected in $R_n$ (i.e., through the short-type edges) to the endpoints of $I_j$ through vertices with marks. Observe that only in the case when $M_j(l) = l - 1$, the remaining inactive vertex of the path $I_j$ has 2 active $R_n$-neighbours and it will become active as well by the end of the expansion.
phase, by step 3 of the phase. This leads to the following representation of the number of vertices in the set $D_1$:

$$|D_1| = N_1 + \sum_{l>1} \sum_{j\geq 1} 1\{|I_j| = l\}(M_j(l) + 1\{M_j(l) = l - 1\}). \quad (4.14)$$

Note that the distribution of $M_j(l)$ does not depend on $j$; we set $M(l) \overset{d}{=} M_j(l)$. It is straightforward to derive for all $l \geq 2$

$$\mathbb{P}\{M(l) \geq l - 1\} = p_1^l + l(1-p_1)p_1^{l-1},$$

and for all $0 < m \leq l - 2$

$$\mathbb{P}\{M(l) = m\} = (m + 1)p_1^m(1-p_1)^2.$$

We shall also define a random variable $M_j(|I_j|)$ which, conditionally on $|I_j| = l$, has the same distribution as $M_j(l)$. In particular,

$$\mathbb{P}\{M_j(|I_j|) = 1\} = 2p_1(1-p_1)^2 \mathbb{P}\{|I_j| > 2\} + \mathbb{P}\{|I_j| = 1\}. \quad (4.15)$$

Now we can rewrite (4.14) as

$$|D_1| = \sum_{j\geq 1} 1\{\{M_j(|I_j|) = 1\} \cap \{|I_j| > 2\}\} + R = D + R, \quad (4.16)$$

where

$$R = N_1 + \sum_{l>1} \sum_{j\geq 1} 1\{|I_j| = l\}(M_j(l) + 1\{M_j(l) = l - 1\})1\{M_j(l) > 1\}. $$
Compute
\[
\mathbb{E}\{R \mid N_1, \ldots\} = N_1 + 2N_2 \mathbb{P}\{M(2) > 0\}
\]
\[
+ \sum_{l \geq 3} N_l \left( \sum_{m=2}^{l-2} m \mathbb{P}\{M(l) = m\} + l \mathbb{P}\{M(l) \geq l - 1\} \right)
\]
\[
= N_1 + 2N_2(2p_1 - p_1^2) + 3N_3 \mathbb{P}\{M(3) \geq 2\}
\]
\[
+ \sum_{l \geq 4} N_l \left( (1 - p_1)^2 \sum_{m=2}^{l-2} m(m+1)p_1^m + l(l-1)p_1^{l-1} + p_1^l \right)
\]
\[
= N_1 + 2N_2(2p_1 - p_1^2) + 3N_3(p_1^3 + 3(1 - p_1)p_1^2)
\]
\[
+ \sum_{l \geq 4} lN_l(1 - p_1)^2 (l(1 - p_1)p_1^{l-1} + p_1^l) + \sum_{l \geq 4} N_l(1 - p_1)^2 \sum_{m=2}^{l-2} m(m+1)p_1^m
\]
\[
\leq N_1 + 4N_2p_1 + 9N_3p_1^2 + \left( \max_{l \geq 4} l^2p_1^{l-1} \right) \sum_{l \geq 4} N_l + (6p_1^2 + O(p_1^3)) \sum_{l \geq 4} N_l.
\]

Since \(p_1 = o(1)\), and \(\sum_{l \geq 1} N_l \leq k\), we derive from (4.17) with a help of (4.11):
\[
\mathbb{E}\{R\} \leq O(k^2/n) + O(kp_1^2) = O(k^2/n) + O(k^3p^2) = O(k^2/n) = o(pk^2).
\] (4.18)

Therefore, we have \(R = O_{L_1}(pk^2)\) and thus \(R = o_{p}(pk^2)\). Consider now the main term in (4.16). Let
\[
N_{>2} = \sum_{i=1}^{k} 1\{|I_i| > 2\} = k - \sum_{i=1}^{k} 1\{|I_i| \leq 2\},
\]
and let \(\eta_i, i \geq 1\), be independent copies of the Bernoulli random variable \(\eta\) such that
\[
\mathbb{P}\{\eta = 1\} = \mathbb{P}\{M_j(|I_j|) = 1 \mid |I_j| > 2\} = 2p_1(1 - p_1)^2,
\] (4.19)
as defined in (4.15). Then we have the following equality in distribution:
\[
D := \sum_{j \geq 1} 1\{\{M_j(|I_j|) = 1\} \cap \{|I_j| > 2\}\} \overset{d}{=} \sum_{i=1}^{N_{>2}} \eta_i.
\]

With the help of (4.18), we deduce that
\[
\mathbb{E}(|D_1|) = \mathbb{E}(D) + \mathbb{E}(R) = 2k^2p(1 + o(1))
\]

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Thus we have $|D_1| = O_L(k^2p)$, moreover, if $k^2p \to \infty$ then $|D_1| = 2k^2p(1 + o_L(1))$. It is straightforward to compute, taking into account (4.19) and (4.12), that

$$\mathbb{E} D = \mathbb{E} \eta \mathbb{E} N_{>2} = 2kp_1(1 - p_1)^2(1 - O(k/n))$$

$$= 2k^2p(1 + o(1)),$$

and

$$\text{Var}(D) = \mathbb{E} (\text{Var}(D | N_{>2}) + \text{Var}(\mathbb{E}(D | N_{>2}))$$

$$= \text{Var} \eta \mathbb{E} N_{>2} + (\mathbb{E} \eta)^2 \text{Var} N_{>2} \leq 2p_1k + (2p_1)^2k^2O\left(\frac{k}{n}\right).$$

The last bound under assumption $k \leq 2/(np^2)$ and $p \geq n^{-1}$ yields

$$\text{Var}(D) = \begin{cases} 
  o((p_1k)^2) = o((pk^2)^2), & \text{if } p_1k \to \infty, \\
  O(p_1k), & \text{otherwise}.
\end{cases}$$

This together with (4.16), (4.18) and (4.20) confirms that

$$|D_1| = \begin{cases} 
  2pk^2 + o_p(pk^2), & \text{if } pk^2 \to \infty, \\
  O_L(pk^2), & \text{otherwise}.
\end{cases}$$

Corollary 4.1. Let $A_1(T_1)$ be a set of vertices uniformly distributed on $V = \{1, \ldots, n\}$. Given that $|A_1(T_1)| = k = \Theta\left(\frac{1}{np^2}\right)$, the following holds:

(i) if $n^{-1} \ll p \ll n^{-2/3}$, then

$$|D_1| = 2pk^2 \left(1 + O_L\left((k/n)^{1/3} + p/n\right)\right) = 2pk^2 \left(1 + o_L(1)\right);$$

(ii) if $pn^{2/3} \to \text{const} > 0$, then

$$|D_1| = O_L(1).$$

(iii) if $n^{-2/3} \ll p \ll n^{-1/2}$, then

$$|D_1| = o_L(1);$$
Remark 4.3. Notice that $|D_1| = o_{L}(1)$ in corollary $4.1 \[ \text{iii} \] implies that $|D_1| = o_p(1)$ or in other words

$$\lim_{n \to \infty} \mathbb{P} \{ D_1 = \emptyset \} = 1.$$  \hspace{1cm} (4.21)

Thus, by definition of $K$ in equation (4.10), equation (4.21) implies $K = 1$ w.h.p.

We can immediately prove Theorem 2.2(ii). If $n^{-2/3} \ll p \ll n^{-1/2}$ and $\frac{A_n - a^*}{\sqrt{c}} \to -\infty$, the system is subcritical and for any $\epsilon > 0$, we have w.h.p. $A^* = A(T_1) \leq t_{c}(1 + \epsilon)$.

For the remaining (the second and further on) expansion phases we will need only the upper bounds for the number of activated vertices in the subcritical case.

Lemma 4.2. Let $n^{-1} \ll p \ll n^{-2/3}$. Then for any $k > 1$ conditionally on $\sum_{i=1}^{k} T_i < \frac{\beta}{np}$ where $\beta < 1$, one has

$$|D_k| \leq \begin{cases} 4p \mathbb{E} \left( T_k \sum_{i=1}^{k} T_i \right) \left( 1 + \left( \frac{1}{np} \right)^{1/3} O_{L^1} \left( 1 \right) \right), & \text{if } p \mathbb{E} \left( T_k \sum_{i=1}^{k} T_i \right) \to \infty, \\ 4p \mathbb{E} \left( T_k \sum_{i=1}^{k} T_i \right) + O_{L^1} \left( \left( p T_k \sum_{i=1}^{k} T_i \right)^{1/2} + T_k \sum_{i=1}^{k} T_i / n \right), & \text{otherwise}. \end{cases}$$

Proof of Lemma 4.2. Assume we are given the sets $A_1(T_1), \ldots, A_k(T_k)$. Recall that after the $k$-th expansion phase, the set of remaining inactive vertices forms intervals on $R_n$ with the following properties: the end points of each interval do not have marks from the sets $A_1(T_1), \ldots, A_{k-1}(T_{k-1})$ but may have at most one mark from the set $A_k(T_k)$ and the rest of the points of the intervals may have only one mark from the sets $A_1(T_1), \ldots, A_k(T_k)$.

Notice, that $A_k(T_k)$ is distributed uniformly on the remaining $n - (T_1 + \cdots + T_{k-1})$ vertices, and $|A_k(T_k)| = T_k$. Hence, there are at most $2T_k$ vertices on the boundary of $A_k(T_k)$ denoted $\partial (A_k(T_k))$ and each of these may have at most one mark with a probability at most $p \sum_{i=1}^{k} T_i$. Denote $D_k^1$ the number of the nodes on the outer boundary of $A_k(T_k)$ which have one mark. Furthermore, there are at most $2 \sum_{i=1}^{k-1} T_i$ vertices on the boundary of $\bigcup_{i=1}^{k-1} A_i(T_i)$, each of which may have at most one mark (from the set $A_k(T_k)$) with a probability at most $pT_k$. Denote $D_k^2$ the number of the nodes on the boundary of $A_k(T_k)$ which have one mark.

In order to get an upper bound for $|D_k| = |D_k^1| + |D_k^2|$, we may now almost repeat the proof of Lemma 3.1 twice to get the bounds for each $|D_k^1|$ and $|D_k^2|$ separately: first time we replace $p_1$ in (4.13) by $p \sum_{i=1}^{k} T_i$ and $T_1$ by $T_k$ and the second time, we replace $p_1$ in (4.13) by $pT_k$ and $T_1$ by $\sum_{i=1}^{k-1} T_i$. This gives us Lemma 4.2. \hfill \square
4.3 The number of vertices activated in an exploration phase.

Let us fix $k \geq 1$ arbitrarily. The $k$-th expansion phase leaves us with the set $A^k$ of $T_1 + \ldots + T_k$ used active vertices and a set $D_k$ of unused active vertices. We shall consider here only the values

$$T_1 + \ldots + T_k \leq 3t_c = \frac{3}{np^2}. \quad (4.22)$$

(Observe that if (4.22) does not hold then almost percolation happens even on the edges of $G_{n, p}$ only, see [11]). Also, we shall assume that $n^{-1} \ll p \ll n^{-2/3}$, which by the Corollary 4.1 implies that $|D_1|$ is large w.h.p.

Consider now the $k + 1$-st exploration phase. By the definition (4.7) we have

$$|A_{k+1}(t)| = |D_k| + S_{k+1}(t) \quad (4.23)$$

where

$$\pi_1(t) = \mathbb{P}\{\text{Bin}(t, p) \geq 2\},$$

and

$$\pi_{k+1}(t) = \pi_1(t) + \frac{p \sum_{l=1}^k T_l}{1 + (\sum_{l=1}^k T_l - 1)p} (1 - p)^{t-1}pt =: \pi_1(t) + \pi_+(t). \quad (4.24)$$

Given $T_1, \ldots, T_k$, $D_k$ and set $\partial_1(D_k \cup A^k)$ we shall approximate the terms in (4.23) separately. Let us define two processes

$$S^{(1)}(t) := \text{Bin}(K_1, \pi_1(t)), \quad S^{(k+1)}(t) := \text{Bin}(K_{k+1}, \pi_{k+1}(t)),$$

where

$$K_1 := |\partial_1(D_k \cup A^k)|, \quad K_{k+1} = n - \sum_{l=1}^k T_l - |\partial_1(D_k \cup A^k)|.$$

**Proposition 4.1.** The processes

$$\frac{S^{(1)}(t) - \mathbb{E}S^{(1)}(t)}{1 - \pi_1(t)}, \quad \frac{S^{(k+1)}(t) - \mathbb{E}S^{(k+1)}(t)}{1 - \pi_{k+1}(t)},$$

$t = 0, 1, \ldots$, are martingales.
Proof of Proposition 4.1. For the process $S^{(1)}(t)$ the proof is the same as for Lemma 7.2 in [11]. It is practically the same for the process $S^{(k+1)}(t)$ as well, which we explain now. Note that $S^{(k+1)}(t)$ is a sum of i.i.d. processes so that

$$S^{(k+1)}(t) = \sum_{v=1}^{K_1} 1\{\xi_v + \sum_{j=1}^{t} \xi_{jv} \geq 2\},$$

where $\xi_v, \xi_{jv}, v \geq 1, j \geq 1$ are independent Bernoulli random variables, such that $\xi_v \in Be(p_+)$ with

$$p_+ := \frac{\sum_{l=1}^{k} T_l p}{1 + (\sum_{l=1}^{k} T_l - 1) p},$$

and $\xi_{jv} \in Be(p)$. Then it is straightforward to check that

$$X_v(t) := \frac{1\{\xi_v + \sum_{j=1}^{t} \xi_{jv} \geq 2\}}{1 - \pi_{k+1}(t)} - \pi_{k+1}(t)$$

is a martingale, taking also into account that

$$\pi_{k+1}(t) = \mathbb{P}\{\xi_v + \sum_{j=1}^{t} \xi_{jv} \geq 2\} = \mathbb{P}\{X_v(t) = 1\}.$$

Then

$$\frac{S^{(k+1)}(t) - \mathbb{E}S^{(k+1)}(t)}{1 - \pi_{k+1}(t)} = \sum_{v=1}^{K_1} X_v(t)$$

is also a martingale. \hfill \Box

We make use of the properties of martingales to get immediately the following bounds.

**Corollary 4.2.** (Lemma 7.3 [11]) For any $t_0$,

$$\mathbb{E}\left(\sup_{t \leq t_0} |S^{(1)}(t) - \mathbb{E}S^{(1)}(t)|\right)^2 \leq 4 \frac{\mathbb{E}(K_1)\pi_1(t_0)}{1 - \pi_1(t_0)},$$

$$\mathbb{E}\left(\sup_{t \leq t_0} |S^{(2)}(t) - \mathbb{E}S^{(2)}(t)|\right)^2 \leq 4 \frac{\mathbb{E}(K_{k+1})\pi_{k+1}(t_0)}{1 - \pi_{k+1}(t_0)}.$$
For all $t \leq t_0 \leq o(1/p)$, when in particular, $\pi_i(t) = o(1)$, the bounds from Corollary 4.2 yield the following approximation

$$S_{k+1}(t) = S^{(1)}(t) + S^{(k+1)}(t)$$

$$= \mathbb{E} S^{(1)}(t) + \mathbb{E} S^{(k+1)}(t) + O_{L^2} \left( \sqrt{\mathbb{E}(K_1) \pi_1(t_0)} + \sqrt{\mathbb{E}(K_{k+1}) \pi_{k+1}(t_0)} \right)$$

$$= \mathbb{E} S^{(1)}(t) + \mathbb{E} S^{(k+1)}(t) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right).$$

Combining this with (4.23) we obtain for all $t \leq t_0 \leq o(1/p)$

$$|A_{k+1}(t)| = |D_1| + \mathbb{E} S^{(1)}(t) + \mathbb{E} S^{(k+1)}(t) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right). \quad (4.25)$$

We begin with asymptotic of the number of activated vertices in the second exploration phase. As we will see, under the conditions of Theorem 2.2 (i), the system (almost) percolates during the second exploration phase. Therefore, we concentrate on this phase and prove Theorem 2.2 (i).

Lemma 4.3. Let $n^{-1} \ll p \ll n^{-1/2}$. Then given $T_1 \leq 3t_c$

$$|A_2(t)| = (n - E T_1) \frac{(tp)^2}{2} \left( 1 + o(1) \right) + E (n - 3T_1) p^2 T_1 (1 + o(1)) + |D_1|$$

$$- n \left( tp^2 + \frac{(tp)^3}{3} \right) \left( 1 + o(1) \right) + R_{T_1}(t_0),$$

where for all $t \leq t_0 = O(t_c)$, $R_{T_1}(t_0) = O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right)$.

Proof of Lemma 4.3. From equation (4.25), we have

$$|A_2(t)| = |D_1| + \mathbb{E} (S^{(1)}(t) + S^{(2)}(t)) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right)$$

$$= |D_1| + E (K_1 \pi_1(t) + E (K_2 \pi_2(t)) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right) \quad (4.26)$$

$$= |D_1| + \mathbb{E} (|\partial_1 (D_1 \cup A_1)|) \pi_1(t) + \mathbb{E} ((n - T_1 - |\partial_1 (D_1 \cup A_1)|) \pi_2(t))$$

$$+ O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right).$$

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Notice that the probability $\pi_1(t)$ is independent of the random variable $T_1$ but that $\pi_2(t)$ depends on $T_1$,

$$\pi_2(t) = \pi_1(t) + \pi_+(t) = \pi_1(t) + \frac{pT_1}{1 + (T_1 - 1)p}(1 - p)^{t-1}pt.$$ 

Since the vertices of $D_1$ are connected to the boundary of $A^1$, we have $|\partial_1(D_1 \cup A^1)| \leq |\partial_1(A^1)|$. When an entire interval of inactive vertices becomes active after the 1-st expansion phase, the boundary of the active set loses exactly 2 vertices if this interval has at least 2 vertices, otherwise, it loses 1 vertex. Hence, using again sets $I_i$, $i = 1, \ldots, T_1$, defined in the proof of Lemma 4.1, we get the following representation

$$|\partial_1(D_1 \cup A^1)| = |\partial_1(A^1)| - N_1 - 2 \sum_{j:|I_j| \geq 2} 1\{M_j(|I_j|) \geq |I_j| - 1\}.$$ 

Since

$$|\partial_1(A^1)| = N_1 + 2 \sum_{j \geq 1} 1\{||I_j| \geq 2\} = N_1 + 2(T_1 - N_1 - N_0),$$

we derive

$$|\partial_1(D_1 \cup A^1)| = 2T_1 - 2N_1 - 2N_0 - 2 \sum_{j:|I_j| \geq 2} 1\{M_j(|I_j|) \geq l - 1\}.$$ 

With the same argument as we derived Lemma 4.1 we get from here

$$|\partial_1(D_1 \cup A^1)| = 2T_1(1 + o_L(1)).$$

Equation (4.26) becomes

$$|A_2(t)| = |D_1| + 2E(T_1)\pi_1(t)(1 + o(1)) + E(n - 3T_1(1 + o(1)) \pi_2(t)) + O_L \left(\sqrt{np^2t_0^3} + t_0\right).$$

Using also approximations

$$\pi_1(t) = P\{\text{Bin}(t,p) \geq 2\} = \frac{(tp)^2}{2} - \left(\frac{tp^2}{2} + \frac{(tp)^3}{3}\right)(1 + o(1)), \quad (4.27)$$

and (see (4.21) with $k = 1$)

$$\pi_+(t) = p^2t(1-p)^{t-1}T_1 \frac{1}{1 + (T_1 - 1)p}.$$ 

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we derive from (4.26) that
\[ |A_2(t)| = |D_1| + n\pi_1(t) - \mathbb{E} T_1 \pi_1(t) (1 + o(1)) + np^2 t (1 - p)^{t-1} \mathbb{E} \left( \frac{T_1}{1 + (T_1 - 1)p} \right) - 3 \mathbb{E} (T_1) \pi_1(t) (1 + o(1)) - 3p^2 t (1 - p)^{t-1} \mathbb{E} \left( \frac{T_1^2}{1 + (T_1 - 1)p} \right) (1 + o(1)) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right). \]

Under the assumption \( T_1 \leq \frac{3}{np} = o(p) \) and using the approximation (4.27) with \( t = \Theta \left( \frac{1}{np^2} \right) \), we have
\[
A_2(t) = |D_1| + n \left( \frac{(tp)^2}{2} - \left( \frac{tp^2}{2} + \frac{(tp)^3}{3} \right) (1 + o(1)) \right) - \mathbb{E} (T_1) \frac{(tp)^2}{2} (1 + o(1)) + np^2 t (1 - p)^{t-1} \mathbb{E} (1 + o(1)) - 3p^2 t \mathbb{E} (T_1^2) (1 + o(1)) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right)
\]
\[ = |D_1| + \left( n - \mathbb{E} (T_1) (1 + o(1)) \right) \frac{(tp)^2}{2} t^2 + \left( np^2 \mathbb{E} (T_1) - 3p^2 \mathbb{E} (T_1^2) \right) (1 + o(1)) t - n \left( \frac{(tp)^3}{3} + \frac{tp^2}{2} \right) (1 + o(1)) + O_{L^2} \left( \sqrt{np^2 t_0^2 + t_0} \right). \]

This yields the statement of the Lemma. \qed

Using the result of Lemma 4.3, consider now the function
\[
A_2(t) - t = \left( \left( n - \mathbb{E} (T_1) \right) \frac{(tp)^2}{2} t^2 + \left( \frac{1}{t_c} \mathbb{E} (T_1) - 1 - 3p^2 \mathbb{E} (T_1^2) \right) t + 2p \mathbb{E} (T_1^2) \right) t - n \left( \frac{(tp)^3}{3} + \frac{tp^2}{2} \right) (1 + o(1)) + O_{R_{T_1}} (t_0)
\]
\[ = g_{T_1} (t) + O_{R_{T_1}} (t_0). \quad (4.28) \]

We shall study the minimal value of \( g_{T_1} (t) \).

### 4.4 Critical case: proof of Theorem 2.2

Let us recall one more result from [11] which describes the critical case of bootstrap percolation on \( G_{n,p} \).

**Theorem** (Theorem 3.8 [11]). Suppose that \( n^{-1} \ll p \ll n^{-1/2} \). Let \( A_0^* \) be the total number of vertices activated due to a bootstrap percolation (with threshold \( r = 2 \)) on a random graph \( G_{n,p} \) starting with \( A_0 \) active vertices.

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If $A_0/a_c \to 1$ and also $(A_0 - a_c^*)/\sqrt{a_c} \to -\infty$, then $A^*_0$ is asymptotically normal with the following parameters

$$A^*_0 \in \text{AsN} \left( t_*, \frac{t_c}{2(1 - A_0/a_c^*)} \right),$$

where $t_* = t_c + pt_c^2(1 + o(1)) - \sqrt{2t_c(a_c^* - A_0)(1 + o(1))}$.

Therefore, we have

$$\mathbb{E}(T_1) = t_c \left( 1 + pt_c(1 + o(1)) - \sqrt{2\frac{a_c^* - A_0}{t_c}}(1 + o(1)) \right), \quad (4.29)$$

and

$$\mathbb{E}(T_1^2) = \text{Var}(T_1) + \mathbb{E}(T_1)^2$$

$$= \frac{t_c a_c^*}{2(a_c^* - A_0)} + t_c^2 \left( 1 + pt_c(1 + o(1)) - \sqrt{2\frac{a_c^* - A_0}{t_c}}(1 + o(1)) \right)^2$$

$$= t_c^2 \left( \frac{1}{a_c^* - A_0}(1 + o(1)) + \left( 1 + pt_c(1 + o(1)) - 2\sqrt{\frac{a_c^* - A_0}{t_c}}(1 + o(1)) \right)^2 \right)$$

$$= t_c^2(1 + o(1)). \quad (4.30)$$

Using (4.29) and (4.30), we rewrite $g_{T_1}(t)$ defined in (4.28) as

$$g_{T_1}(t) = \frac{1}{2}t^2(1 + o(1)) + \left( pt_c(1 + o(1)) - \sqrt{2\frac{a_c^* - A_0}{t_c}}(1 + o(1)) - 3p^2t_c^2 - \frac{1}{2t_c} \right) t$$

$$+ 2pt_c^2 - n\left(\frac{pt_c}{3}\right)^3(1 + o(1)). \quad (4.31)$$

Using the condition $t_c = o\left(\frac{1}{p}\right)$, we derive that $3p^2t_c^2 = o(pt_c)$. Moreover, for $t \leq 3t_c$, we have $n\left(\frac{tp}{3}\right)^3 = \frac{1}{3}pt_c^2 - o\left(\frac{1}{2t_c^2}\right)$. Also, we have $\sqrt{a_c^* - A_0} = o(t_c)$, which implies that $\frac{1}{2t_c} = o\left(\sqrt{2\frac{a_c^* - A_0}{t_c}}\right)$. Thus formula (4.31) simplifies to

$$g_{T_1}(t) = \frac{1}{2}t^2(1 + o(1)) + \left( pt_c - \sqrt{2\frac{a_c^* - A_0}{t_c}} \right) t(1 + o(1)) + 2pt_c.$$
The function $f_{T_1}(t) = \frac{1}{2}t^2 + (pt_c - \sqrt{2\frac{a_c^*-A_0}{t_c}})t + 2pt_c^2$ has a minimum at

$$t_{\text{min}} = -}\frac{pt_c + \sqrt{2\frac{a_c^*-A_0}{t_c}}}{\frac{1}{t_c}} = -pt_c^2 + \sqrt{2(a_c^*-A_0)t_c}$$

with the value

$$f_{T_1}(t_{\text{min}}) = 2pt_c^2 - \frac{(pt_c - \sqrt{2\frac{a_c^*-A_0}{t_c}})^2}{2/t_c} = 2pt_c^2 - \frac{1}{2}p^2t_c^3 + pt_c\sqrt{2(a_c^*-A_0)t_c} - (a_c^*-A_0).$$

Using again that $t_c = \frac{1}{2}p^2t_c^3 = o(2pt_c^2)$. We also use that $(a_c^*-A_0) = o(t_c)$ to derive that $pt_c\sqrt{2(a_c^*-A_0)t_c} = o(2pt_c^2)$. Therefore, we have

$$f_{T_1}(t_{\text{min}}) = 2pt_c^2(1 + o(1)) - (a_c^*-A_0),$$

yielding $g_{T_1}(t_{\text{min}}) = 2pt_c^2(1 + o(1)) - (a_c^*-A_0)$.

Finally, for the remaining term $R_{T_1}(t_0)$ in (4.28) with $t_0 = 3t_c$, we have

$$R_{T_1}(t_0) = O_{L^2} \left( \sqrt{np^2t_c^2 + t_0 + t_0^2n} \right) = O_{L^2} \left( \sqrt{t_c} \right).$$

The error term is negligible if $\sqrt{t_c} = o(pt_c^2)$ which happens if and only if $p \gg n^{-3/4}$.

Therefore, if $p \gg n^{-3/4}$ and $a_c^*-A_0 > (1+\varepsilon)2pt_c^2$, w.h.p. the system does not percolate at the second exploration phase.

If $p \gg n^{-3/4}$ and $a_c^*-A_0 < (1-\varepsilon)2pt_c^2$ then for any $t \leq 3t_c$, we have

$$|A_2(t)| - t \geq \varepsilon 2pt_c^2(1 + o_p(1)).$$

Thus, we have w.h.p. that $|A_2^*| \geq 3t_c$. Using the results of (11), we conclude that $|A_2^*| = n(1 + o_p(1))$. Assume now that $pn^{2/3} \to \infty$. The statement (iii) of Theorem 2.2 follows by the assertion (ii) of Corollary 4.1.

4.5 Subcritical case: proof of Theorem 2.1.

Lemma 4.4. Let $n^{-1} \ll p \ll n^{-1/2}$, and let $k > 1$ be fixed arbitrarily. Under assumption that $T_1, \ldots, T_k$ are given such that $\sum_{i=1}^{k} T_i < \beta t_c$ for some $\beta < 1$ we have the following.

(i) If $pT_k \sum_{i=1}^{k} T_i = O(1)$, then $T_{k+1} = O_L(1)$. 

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(ii) otherwise, if \( pT_k \sum_{l=1}^{k} T_l \rightarrow \infty \), then

\[
T_{k+1} \leq \frac{10}{1 - \beta} pT_k \sum_{l=1}^{k} T_l \tag{4.32}
\]

with probability at least

\[
1 - O \left( \left( \frac{1}{np} \right)^{8/3} \right) + O \left( \left( pT_k \sum_{l=1}^{k} T_l \right)^{-1} \right) \tag{4.33}
\]

**Proof of Lemma 4.4.** First we derive from (4.25), taking into account (4.24) that for all \( t \leq t_0 = o(1/p) \)

\[
A_{k+1}(t) = |A_{k+1}(t)| \leq |D_k| + n(\pi_1(t) + \pi_+(t)) + O_{L^2} \left( \sqrt{K_1 \pi_1(t_0) + \sqrt{K_2 \pi_{k+1}(t_0)}} \right) \tag{4.34}
\]

This together with (4.27) and (4.24) gives us for all \( t \leq t_0 = o(1/p) \)

\[
A_{k+1}(t) \leq |D_k| + n \left( \frac{(tp)^2}{2} \right) + tnp^2 \sum_{l=1}^{k} T_l + O_{L^2} \left( p \sqrt{nt_0^2 + nt_0 \sum_{l=1}^{k} T_l} \right) \tag{4.35}
\]

Assume first that \( T_k \sum_{l=1}^{k} T_l = O(1/p) \), which by (4.2) yields \( |D_k| = O_{L^1}(1) \). Then by (4.35) and under the assumption \( \sum_{l=1}^{k} T_l < \beta t_c \) we have for all \( t \leq t_0 \leq O(t_c) = o(1/p) \)

\[
A_{k+1}(t) - t \leq O_{L^1}(1) + n \left( \frac{(tp)^2}{2} \right) - t (1 - \beta) + O_{L^2} \left( \sqrt{t_0} \right) \tag{4.36}
\]

Now it is straightforward to compute (solving the quadratic equation) that (at least) for all

\[
0 \leq t \leq \frac{1 - \beta}{2np^2} \tag{4.37}
\]

we have in (4.36)

\[
A_{k+1}(t) - t \leq - \frac{1 - \beta}{2} t + O_{L^1}(1) + O_{L^2} \left( \sqrt{t_0} \right) \tag{4.38}
\]

Hence, choosing here \( t_0 = O(1) \) (we do not use (4.34) for \( t \gg O(1) \)), will imply that \( A_{k+1}(t) < t \) for some

\[
t = T_{k+1} := O_{L^1}(1) \tag{4.39}
\]
which, notice, also satisfies (4.37). This yields statement (i) of the Lemma.

When \( p T_k \sum_{i=1}^{k} T_i \to \infty \) and \( \sum_{i=1}^{k} T_i < \beta t_c \) we shall use bound (4.2) for \(|D_k|\) to derive from (4.35) for all \( t \leq t_0 = o(1/p) \)

\[
A_{k+1}(t) - t \leq n \frac{(tp)^2}{2} - (1 - \beta)t + 4pT_k \sum_{i=1}^{k} T_i + O_{L^1} \left( T_k \sum_{i=1}^{k} T_i \right) + O_{L^2} \left( p \sqrt{nT_0^2 + nt_0 \sum_{i=1}^{k} T_i} \right)
\]

Then we derive solving the quadratic equation, that (at least) for all

\[
\frac{9}{1 - \beta} p T_k \sum_{i=1}^{k} T_i \leq t \leq \frac{1}{n t_0^2} \leq \frac{1 - \beta}{n p^2}
\]

we have

\[
n \frac{(tp)^2}{2} - (1 - \beta)t + 4pT_k \sum_{i=1}^{k} T_i < -\frac{1}{2} p T_k \sum_{i=1}^{k} T_i,
\]

and therefore by (4.40) with \( t_0 = \frac{10}{1 - \beta} p T_k \sum_{i=1}^{k} T_i \)

\[
A_{k+1}(t) - t \leq -\frac{1}{2} p T_k \sum_{i=1}^{k} T_i + O_{L^1} \left( T_k \right) \left( \frac{1}{np} \right)^{5/3} + O_{L^2} \left( \sqrt{p T_k \sum_{i=1}^{k} T_i} \right).
\]

Hence, for some

\[
t_{k+1} := \frac{9}{1 - \beta} p T_k \sum_{i=1}^{k} T_i + O_{L^1} \left( T_k \right) \left( \frac{1}{np} \right)^{5/3} + O_{L^2} \left( \sqrt{p T_k \sum_{i=1}^{k} T_i} \right)
\]

we have

\[
\mathbb{P}\{ A_{k+1}(t_{k+1}) - t_{k+1} \leq 0 \} \geq 1 - P_k,
\]

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where $P_k$ is a probability that $t_{k+1}$ does not satisfy condition (4.41). The latter we can bound using the Chebyshev’s inequality together with the assumption that $\sum_{l=1}^{k} T_l \leq \beta t_c$

$$P_k := \mathbb{P}\left\{ t_{k+1} > \frac{10}{1-\beta} p T_k \sum_{l=1}^{k} T_l \mid T_k \sum_{l=1}^{k} T_l \right\}$$

$$= O\left( p \sum_{l=1}^{k} T_l \left( \frac{1}{np} \right)^{5/3} \right) + O\left( \left( p T_k \sum_{l=1}^{k} T_l \right)^{-1} \right).$$

We conclude that the $k+1$-st exploration phase will stop with probability at least $1 - P_k$ at $T_{k+1} \leq t_{k+1}$ conditional on $p T_k \sum_{l=1}^{k} T_l \to \infty$. This yields statement ?? of the Lemma

Consider now the relation (4.32). First we study a similar deterministic system, i.e., as if we ignore the random $O_L$ terms in (4.32).

**Lemma 4.5.** For given $c > 0$ and $t_1 > 0$ such that $ct_1 < 1$ define for $k \geq 1$

$$t_{k+1} = c t_k \sum_{l=1}^{k} t_l. \quad (4.43)$$

Then for any $0 < \alpha < 1$ which satisfy

$$(1 - \alpha) \alpha > ct_1, \quad (4.44)$$

one has $t_k \leq \alpha^{k-1} t_1, \ k \geq 1$, and, hence,

$$\sum_{l=1}^{\infty} t_l \leq t_1 \frac{1}{1-\alpha}.$$ 

**Proof of Lemma 4.5.** Write here $S_k := \sum_{l=1}^{k} t_l$. We shall show first that under condition (4.44) one has $c S_k < \alpha$ for all $k \geq 1$.

Assume, on the contrary, that

$$L = L(\alpha) := \max\{k : c S_k \leq \alpha\} < \infty. \quad (4.45)$$

By the definition (4.43)

$$S_{L+1} = \sum_{l=1}^{L+1} t_l = t_1 + \sum_{k=1}^{L} ct_k S_k,$$
where by the definition of $L$ for all $k \leq L$
\[ t_{k+1} = ct_k S_k \leq \alpha t_k \leq \alpha^k t_1. \tag{4.46} \]
Hence,
\[ S_{L+1} \leq t_1 + \sum_{i=1}^{L} \alpha^k t_1 \leq t_1 \frac{1}{1-\alpha}, \]
which under condition (4.44) yields $cS_{L+1} < \alpha$ and thus contradicts (4.45). Therefore $cS_k \leq \alpha$ for all $k \geq 1$. This yields (4.46) for all $k \geq 1$, and the statement of the Lemma follows.

We shall prove now statement (i) of Theorem 2.1.
Assume, that $T_1 = \beta t_c$ for some $\beta < 1$. Hence, $pT_1^2 = \beta^2 \frac{1}{n^2p^3}$. Consider then two cases.
If $\frac{1}{n^2p^3} = O(1)$, then by Lemma 4.4 we have $T_2 = O_L(1)$, which by Lemma 4.2 implies that $E|D_2| = O\left(\frac{1}{\sqrt{np}}\right)$. Hence, w.h.p. the bootstrap percolation stops after the second expansion phase.
Assume now that $\frac{1}{n^2p^3} \to \infty$, i.e.,
\[ \frac{1}{n} \ll p \ll \frac{1}{n^{2/3}}. \tag{4.47} \]
Let $h = h(n) = o(np)$ be an arbitrarily fixed function such that $h(n) \to \infty$. Notice that under assumption (4.47) condition $h(n) = o(np)$ yields
\[ h(n) = o(t_c). \tag{4.48} \]
Define a random time
\[ \tau = \min\{k \geq 1 : pT_k \sum_{l=1}^{k} T_l < h\}. \]
If $pT_1^2 = \beta^2 \frac{1}{n^2p^3} < h$ then $\tau = 1$, in which case by Lemma 4.4 we have $T_3 = O_L(1)$, and then $E|D_3| = o(1)$ which w.h.p. yields a termination of the bootstrap percolation after the 3rd expansion phase.
Assume, that $\beta^2 \frac{1}{n^2p^3} \geq h$, and therefore $\tau > 1$. We shall get first an upper bound in probability for $\tau$.

**Proposition 4.2.** Assume that $\beta^2 \frac{1}{n^2p^3} \geq h$ and $(np)^{np} \gg n$. One can choose an unbounded function $h$ so that $h = o(np)$, and for some $K_0 = o(h)$
\[ \mathbb{P}\{\tau \leq K_0 + 1\} \geq 1 - K_0 O(h^{-1}). \]
Proof. Recall that by Lemma 4.4 for all \( k \geq 1 \) given \( T_k \sum_{l=1}^{k} T_l > h \) we have
\[
T_{k+1} \leq \frac{10}{1 - \beta} p T_k \sum_{l=1}^{k} T_l \tag{4.49}
\]
with probability at least \( 1 - O((np)^{-8/3} + h^{-1}) \). Hence, for any \( K_0 \) we have
\[
\mathbb{P} \left\{ T_{k+1} \leq \frac{10}{1 - \beta} p T_k \sum_{l=1}^{k} T_l, 1 \leq k \leq K_0 \mid \tau > K_0 \right\} \geq 1 - K_0 O((np)^{-8/3} + h^{-1})
= 1 - K_0 O(h^{-1}), \tag{4.50}
\]
where the last equality is due to the assumption that \( h = o(np) \).

Let us now choose \( K_0 \) as follows. Assume that the relation (4.49) holds for all \( 1 \leq k \leq K_0 \). By our assumptions we also have here
\[
\frac{10}{1 - \beta} p T_1 = O \left( \frac{1}{np} \right) = o(1).
\]
Hence, by Lemma 4.5 with \( c = \frac{10}{1 - \beta} p \) we have (conditionally on (4.49) for all \( 1 \leq l \leq k \))
\[
T_k \leq \alpha^{k-1} T_1 \tag{4.51}
\]
for some \( \alpha \) which satisfies condition \((1 - \alpha)\alpha > cT_1 \) (see (4.44)). Notice, that here we can choose
\[
\alpha < 2cT_1 = 2 \frac{10}{1 - \beta} \frac{1}{pn} = \frac{c_1}{pn},
\]
which together with (4.51) yields
\[
T_k \leq \left( \frac{c_1}{pn} \right)^{k-1} T_1 = \beta \left( \frac{c_1}{pn} \right)^{k-1} \frac{1}{p^2n}, \tag{4.52}
\]
as well as
\[
\sum_{l=1}^{k} T_l < \frac{T_1}{1 - (c_1/pn)}. \tag{4.53}
\]
This implies
\[
p T_k \sum_{l=1}^{k} T_l \leq p \beta \left( \frac{c_1}{pn} \right)^{k-1} \frac{1}{p^2n} \frac{T_1}{1 - (c_1/pn)} < \beta^2 \left( \frac{c_1}{pn} \right)^{k} \frac{1}{p^2n}. \tag{4.54}
\]
Setting now

\[ K_0 := \min \left\{ k : \beta^2 \left( \frac{c_1}{pn} \right)^{k+1} \frac{1}{p^2 n} < h \right\}, \tag{4.55} \]

we have by (4.54)

\[ pT_{K_0+1} \sum_{l=1}^{K_0+1} T_l < h. \tag{4.56} \]

**Claim.** One can choose unbounded function \( h \) so that \( h = o(np) \) and

\[ K_0 = o(h) \tag{4.57} \]

if and only if \( n = o((np)^{np}) \).

**Proof of the Claim.** Assume, some function \( h \to \infty \) satisfies (4.57), which by the definition (4.55) is equivalent to

\[ (pn)^{o(h)} = \frac{n}{h}. \]

Under assumption \( h = o(np) \) and \( pn, h \to \infty \), this holds if and only if

\[ o(h) \log(pn) = \log(n) - \log(h). \tag{4.58} \]

Again under the condition that \( h = o(np) \) relation (4.58) is equivalent to

\[ o(h) = \frac{\log n}{\log(pn)}. \tag{4.59} \]

Finally, the last equality is satisfied for some \( h = o(np) \) if and only if

\[ \frac{\log n}{\log(pn)} = o(np). \]

The assertion of the claim follows. \( \square \)

**Proof of Proposition 4.2.** Observe that for \( K_0 \) which is chosen according to (4.55) and, hence, (4.56), we have

\[ \mathbb{P} \left\{ T_{k+1} \leq \frac{10}{1 - \beta} pT_k \sum_{l=1}^{k} T_l, 1 \leq k \leq K_0 \mid \tau > K_0 \right\} \]

\[ = \mathbb{P} \left\{ \left( T_{k+1} \leq \frac{10}{1 - \beta} pT_k \sum_{l=1}^{k} T_l, 1 \leq k \leq K_0 \right) \cap (\tau = K_0 + 1) \mid \tau > K_0 \right\} \]

\[ \leq \mathbb{P} \{ \tau = K_0 + 1 \mid \tau > K_0 \} = \frac{\mathbb{P} \{ \tau = K_0 + 1 \}}{\mathbb{P} \{ \tau > K_0 \}}. \]

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Combining this with (4.50) we get
\[ P \{ \tau > K_0 \} - P \{ \tau > K_0 + 1 \} = P \{ \tau = K_0 + 1 \} \geq (1 - K_0 O(h^{-1})) P \{ \tau > K_0 \}, \]
which yields
\[ P \{ \tau \leq K_0 + 1 \} = 1 - P \{ \tau > K_0 + 1 \} \geq 1 - K_0 O(h^{-1}) P \{ \tau > K_0 \} \geq 1 - K_0 O(h^{-1}). \]

This together with the assertion of the claim completes the proof of the proposition. \(\Box\)

We shall finish now the proof of the assertion (1) of Theorem 2.1.

**Proof of Theorem 2.1 (1).** Using the representation (4.1.3) consider for an arbitrarily fixed \(0 < \varepsilon < 1 - \beta\)

\[ P \{|A^*| < (\beta + \varepsilon)t_c | T_1 = \beta t_c\} \]
\[ = P \left\{ \sum_{l \leq K} T_l < (\beta + \varepsilon)t_c | T_1 = \beta t_c \right\} \]
\[ \geq P \left\{ \left( \sum_{l \leq K} T_l < (\beta + \varepsilon)t_c \right) \cap \left( \sum_{l \leq \tau} T_l < (\beta + \frac{\varepsilon}{2})t_c \right) | T_1 = \beta t_c \right\} \]
\[ \geq P \left\{ \left( \sum_{l \leq K} T_l < (\beta + \varepsilon)t_c \right) \cap (K = \tau + 1) | \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right)t_c, T_1 = \beta t_c \right\} \]
\[ P \left\{ \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right)t_c | T_1 = \beta t_c \right\}. \]

Notice that if \(\sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right)t_c\), then by (4.49) with probability at least \(1 - O(h^{-1})\) we have
\[ T_{\tau+1} \leq \frac{10}{1 - \beta} h, \]
in which case for \(K = \tau + 1\)
\[ \sum_{l \leq K} T_l < \left( \beta + \frac{\varepsilon}{2} \right)t_c + O(h) < (\beta + \varepsilon)t_c, \]
for any \(\varepsilon > 0\) when \(h = o(t_c) < \frac{\varepsilon}{2} t_c\). This yields
\[
\mathbb{P}\left\{ \left( \sum_{l \leq K} T_l < (\beta + \varepsilon) t_c \right) \cap (K = \tau + 1) | \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c, T_1 = \beta t_c \right\} \tag{4.63}
\]

\[
= \mathbb{P}\left\{ K = \tau + 1 | \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c, T_1 = \beta t_c \right\} - O(h^{-1})
\]

\[
= \mathbb{P}\left\{ |D_{\tau+1}| = 0 | \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c, T_1 = \beta t_c \right\} - o(1).
\]

By (4.62) we have \( T_{\tau+1} = O(h) = o(np) \) with probability at least \( 1 - O(h^{-1}) \). Then under condition \( \sum_{l \leq \tau} T_l = O(t_c) \) we have

\[
pT_{\tau+1} \sum_{l \leq \tau+1} T_l = pO(h(t_c + h)) = o(1) \quad \text{and} \quad T_{\tau+1} \sum_{l \leq \tau+1} T_l/n = o(1),
\]

which by Lemma 4.2 yields

\[
\mathbb{P}\left\{ |D_{\tau+1}| = 0 | \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c, T_1 = \beta t_c \right\} = 1 + o(1).
\]

Combining the last bound with (4.63) and substituting the result into (4.61) we get

\[
\mathbb{P}\left\{ \sum_{l \leq K} T_l < (\beta + \varepsilon) t_c | T_1 = \beta t_c \right\} \geq (1 - o(1))\mathbb{P}\left\{ \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c | T_1 = \beta t_c \right\}. \tag{4.64}
\]

Consider now

\[
\mathbb{P}\left\{ \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c | T_1 = \beta t_c \right\}
\]

\[
\geq \mathbb{P}\left\{ \left( \sum_{l \leq \tau} T_l < \left( \beta + \frac{\varepsilon}{2} \right) t_c \right) \cap \left( T_{k+1} \leq \frac{10}{1 - \beta} pT_k \sum_{l=1}^{k} T_l, 1 \leq k \leq \tau \right) | T_1 = \beta t_c, \tau \right\}
\]

\[
= \mathbb{P}\left\{ T_{k+1} \leq \frac{10}{1 - \beta} pT_k \sum_{l=1}^{k} T_l, 1 \leq k \leq \tau | T_1 = \beta t_c, \tau \right\}, \tag{4.65}
\]

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where the last equality is due to (1.53) and the fact that
\[
\frac{T_1}{1 - (c_1/pn)} = \frac{\beta}{1 - (c_1/pn)} t_c < \left(\beta + \frac{\varepsilon}{2}\right) t_c
\]
for any fixed \(\varepsilon > 0\). Now using the same argument as in (4.50) we derive from (4.65)
\[
P\left\{ \sum_{l \leq \tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c \mid T_1 = \beta t_c \right\} \geq E \left( (1 - \tau O(h^{-1})) 1\{\tau < K_0\} \right).
\]
Making use of Proposition 4.2 we derive from here
\[
P\left\{ \sum_{l \leq \tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c \mid T_1 = \beta t_c \right\} \geq (1 - o(1))(1 - K_0 O(h^{-1}))^2 = 1 - o(1),
\]
where the last equation is due to the property (4.57). This proves statement (i) of Theorem 2.1.

Let us turn to the statement (ii) of Theorem 2.1. Under the conditions of Theorem 2.1 (ii), the process restricted to the subgraph \(G_{n,p}\) percolates by the corresponding part (ii) of Theorem 3.1, [11] cited above. That implies \(A^* = n - o(n)\). That proves (ii).

This completes the proof of Theorem 2.1.

4.6 Proof of Theorem 2.3

We do not give here the whole proof of Theorem 2.3 but only a sketch of it with the necessary lemmas.

Suppose that the process almost percolates and denote \(A^*\) the final set of activated vertices. We have then \(A^* = |A^*| = n - o(n)\). Let \(A^\uparrow\) be the maximal set of vertices activated through \(G_{n,p}\) edges only
\[
A^\uparrow = A^* \cap \{v : M_v(|A^\uparrow|) \geq 2\}.
\]
If we denote \(A_0^*\) the final set of active vertices for the process restricted to \(G_{n,p}\) then we have \(A_0^* \subseteq A^\uparrow \subseteq A^*\). We have the advantage that \(A^\uparrow\) is uniformly distributed on \(\{1, \ldots, n\}\).
Similarly as in the proof of Lemma 4.1, the set $\mathcal{B} = V \setminus \mathcal{A}^\dagger$ is a union of clusters of vertices with at most one mark from $\mathcal{A}^\dagger$, and with endpoints without marks. Define

\[ C_0(i) = \{ M_i(|\mathcal{A}^\dagger|) = 0 \} \cap \{ M_{i+1}(|\mathcal{A}^\dagger|) = 0 \}, \quad (4.66) \]

the cluster of length 2 made of two consecutive inactive vertices with no marks and for $l \geq 1$, let $C_l(i)$ be the cluster of $l+2$ vertices with $M_i(|\mathcal{A}^\dagger|) = 0 = M_{i+l+1}(|\mathcal{A}^\dagger|)$ and $M_{i+j}(|\mathcal{A}^\dagger|) \leq 1$ for $1 \leq j \leq l$,

\[ C_l(i) = \{ M_i(|\mathcal{A}^\dagger|) = 0 \} \cap \bigcap_{j=1}^{l} \{ M_{i+j}(|\mathcal{A}^\dagger|) \leq 1 \} \cap \{ M_{i+l+1}(|\mathcal{A}^\dagger|) = 0 \}. \quad (4.67) \]

Lemma 4.6 shows that if there remains an interval of inactive vertices then it is with high probability made of 2 consecutive vertices with 0 mark as defined in (4.66).

**Lemma 4.6.** Let $\frac{1}{n} \ll p \leq \frac{3 \log n}{4}$. Let $C_l(i)$ (resp. $C_0(i)$) denote the interval defined in (4.67) (resp. (4.66))

\[ \mathbb{P} \left( \bigcup_{l=1}^{\left|\mathcal{B}\right|} C_l(i) \right) = o \left( \mathbb{P} \{ C_0(i) \} \right). \]

This is an easy consequence of the fact that the vertices are uniformly and independently distributed on the line and that $\mathcal{B} = o(n)$.

Next, we show in Lemma 4.7 that the intervals of two consecutive vertices with 0 mark consist with high probability of two vertices with degree 0.

**Lemma 4.7.** Let $\frac{1}{n} \ll p \leq \frac{3 \log n}{4} \frac{n}{n}$ then

\[ \mathbb{P} \{ C_0(i) \} = \mathbb{P} \{ \{ d(i) = 0 \} \cap \{ d(i+1) = 0 \} \} \left( 1 + o(1) \right). \]

In order to prove Lemma 4.7 we need an estimate of $d(\mathcal{B})$, the sum of the degrees of the vertices of $\mathcal{B}$

\[ d(\mathcal{B}) = \sum_{v \in \mathcal{B}} d(v). \quad (4.68) \]

We need that estimate as the vertices with 0 mark which do not have degree 0 necessarily share an edge with a vertex of $\mathcal{B}$.

**Lemma 4.8.** Let $d(\mathcal{B})$ be defined by (4.68) then under the conditions of Lemma 4.7

\[ d(\mathcal{B}) = n(np)e^{-np}(1 + o_p(1)). \]
Finally, the problem remains to finding the critical probability for bootstrap percolation in dimension 1 with threshold \( r = 2 \). We find the probability that there are not 2 consecutive inactive vertices. The idea is to consider pairs of vertices and notice that the state of vertices in disjoint pairs are independent and deal with the different cases of intersecting pairs.

**Lemma 4.9.** Let the vertices \( \{1, \ldots, n\} \) be ordered on a circle. Denote \( q(n) \) the probability that a vertex is set as active. Consider the bootstrap percolation with threshold \( r = 2 \), then

(i) if \( 1 - q(n) = o \left( \frac{1}{\sqrt{n}} \right) \) then

\[
\lim_{n \to \infty} \mathbb{P} \{ A^* = n \} = 1,
\]

(ii) if \( 1 - q(n) \gg \frac{1}{\sqrt{n}} \) then

\[
\lim_{n \to \infty} \mathbb{P} \{ A^* < n - 1 \} = 1,
\]

(iii) if \( 1 - q(n) = \frac{c}{\sqrt{n}} (1 + o(1)) \) then

\[
\mathbb{P} \{ A^* = n \} = e^{-c^2}.
\]

Using these results, one can prove Theorem 2.3

**Proof of Theorem 2.3.** By Lemma 4.7 w.h.p. the remaining clusters are made of 2 consecutive vertices of degree 0. We use Lemma 4.9 with \( 1 - q(n) = \frac{E(\varepsilon_0)}{n} \), where \( \varepsilon_0 \) denotes the number of vertices of degree 0. For example in Lemma 4.9 (iii) the condition \( 1 - q(n) = \frac{c}{\sqrt{n}} \) is equivalent to \( ne^{-pn} = c\sqrt{n} \) which gives the condition \( p = \frac{1}{2} \log \frac{2}{n} - \frac{\log c}{n} \). We find the results of Theorem 2.3

For \( p \geq \frac{3}{4} \log \frac{n}{n} \), a coupling argument shows that we have complete percolation. \( \square \)

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