Spontaneous compactification in 2D induced quantum gravity

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Abstract

Spontaneous compactification —on a $R^1 \times S^1$ background— in 2D induced quantum gravity (considered as a toy model for more fundamental quantum gravity) is analyzed in the gauge-independent effective action formalism. It is shown that such compactification is stable, in contradistinction to multidimensional quantum gravity on a $R^D \times S^1$ ($D > 2$) background —which is known to be one-loop unstable.

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1 Introduction

The exact solution of 2D induced quantum gravity in the light-cone gauge or in the conformal gauge [1] has originated a number of works dealing with this theory [2-4] (and further references therein). Different approaches, based on conformal field theory [1], on random matrix models [2], and on topological field theory [3], have been developed. Presently, we see that there is some agreement among the exact results obtained from these different approaches. Thus, we have an exactly solvable 2D quantum gravity, which can be considered as a toy model for the much more complicated 4D quantum gravity —where so far only the perturbative viewpoint is quite developed. For this reason, it would be useful to formulate the perturbative approach to 2D induced gravity [5-7]. Some interesting results on this line have been obtained. For instance, the one-loop calculation of the counterterms in 2D induced gravity has been done in different gauges [5-7], and renormalizability has been found for some models. The covariant gauge, in which the counterterms disappear, has also been found [5]. Thus, developing the perturbative approach to 2D induced gravity, one can expect to find an indication (at least qualitative) of some new phenomena which would also take place in 4D induced gravity, in the well-known language of usual, perturbative field theory. In other words, 2D quantum gravity can provide us with important information on general properties of quantum gravity (a good example are the quantum corrections to 2D black holes [8]).

Some years ago, in the spirit of the Kaluza-Klein approach, the quantum spontaneous-compactification program (also called self-consistent dimensional reduction) for multidimensional quantum gravity was developed (for a comprehensive introduction and general review see [9]). The gauge-independent Vilkovisky-De Witt effective action formalism [10] (see [11] for a general review) has been very useful in the investigation of gauge-independent spontaneous compactification in multidimensional gravity [12,13] based on the Einstein- or $\mathcal{R}^2$-gravity action. It would be of interest to discuss the same question for the still unknown $D \geq 4$ induced gravity (some properties of 4D induced quantum gravity have been studied in refs. [14]). As a first step in this direction, we here address the essential question: does the quantum spontaneous compactification program actually
work for 2D induced gravity on an $R^1 \times S^1$ background? Identifying $S^1$ with the time coordinate we discover another motivation for the present study, namely 2D induced gravity at non-zero temperature. As we shall prove, the answer to the compactification question will be positive.

The organization of the paper is as follows. In section 2 we review the procedure of construction of the gauge invariant effective action. This is particularized in section 3 to the case of 2D induced gravity at one loop order. In the gauge independent effective action, only a linear explicit dependence on a constant parameter, $a$, remains. The actual calculation of the traces involved in the expression for the action is carried out in section 4. A non-trivial minimum is found. Finally, section 5 is devoted to conclusions.

## 2 The gauge-independent effective action

In this section we construct the one-loop gauge-invariant effective action for 2D induced gravity. Let us remember that the one-loop conventional (i.e., gauge dependent) effective action is given by

$$\Gamma^{(1)}_{\text{conv}}(\phi) = S(\phi) + \frac{1}{2} \text{Sp} \ln S_{ij}(\phi), \quad (1)$$

where $S(\phi)$ is the classical action, $\phi$ is the background field, and euclidean notation is assumed everywhere. In the background field method it is not necessary for $\phi$ to be a solution of the equations of motion.

According to Vilkovisky and De Witt [10], the gauge-independent effective action can be obtained by the method of replacing the ordinary functional derivative by the covariant functional derivative (in one-loop approximation)

$$S_{ij} \rightarrow S_{ij} = S_{ij} - \Gamma^k_{ij} S_{ik}, \quad (2)$$

where the condensed notation has been used and $\Gamma^k_{ij}$ is the connection in the space of fields. The term $S_{V,ij} = -\Gamma^k_{ij} S_{ik}$ is sometimes called the Vilkovisky correction. It is very convenient to construct the connection using the metric $\gamma_{ij}$ in the space of fields (configuration-space metric). For non-gauge theories, it can be constructed as the Christoffel connection
\[ \Gamma_{jk}^i = \left\{ i_{jk} \right\} = \frac{1}{2} \gamma^{il} (\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}). \]

The rule to define \( \gamma_{ij} \) has been given by Vilkovisky [10].

In gauge theories, the construction of the connection is more complicated. The physical field space is in gauge theories different from the naive field space, because of the local gauge symmetry.

Let \( \gamma_{ij} \) be the metric of naive field space
\[ ds^2 = \gamma_{ij} \delta \phi^i \delta \phi^j. \] (4)

By projecting \( \delta \phi^i \) onto the physical field space, we get
\[ \delta \phi^i_\perp \equiv \Pi^i_j \delta \phi^j, \quad \Pi^i_j \equiv \delta^i_j - R^i_{\alpha} N^{\alpha\beta} R^k_\beta \gamma_{kj}, \] (5)

where \( R^i_\alpha \) is the generator of the gauge symmetry (i.e., \( \delta \phi^i = R^i_\alpha \epsilon^\alpha \)), and we have \( \Pi^i_j \Pi^j_i = 0 \), \( N_{\alpha\beta} = \gamma_{ij} R^i_\alpha R^j_\beta \), \( N^{\alpha\beta} \) being the inverse of \( N_{\alpha\beta} \).

Taking this into account, the metric on the physical field space is
\[ ds^2_\perp = \gamma_{ik} \delta \phi^i_\perp \delta \phi^j_\perp = \gamma_{ik} \Pi^i_j \delta \phi^i_\perp \delta \phi^j_\perp. \] (6)

Using the new metric \( \gamma_{ik} \Pi^i_j \), the connection \( \Gamma^k_{ij} \) for the physical field space is then given by [10,11]
\[ \Gamma^k_{ij} = \left\{ k_{ij} \right\} + T^k_{ij}, \] (7)

where \( T^k_{ij} = -2B^\alpha_i D^\alpha_j R^k_\alpha + R^\alpha_m D^\alpha_i R^k_\beta B^\sigma_j B^\tau_j \), \( B^\alpha_i = \gamma_{ij} N^{\alpha\beta} R^j_\beta \), \( D^\alpha_i R^k_\alpha \equiv R^k_{\alpha j} \).

Finally, for gauge theories the one-loop gauge-independent effective action is given by [10,11]
\[ \Gamma^{(1)} = S(\phi) + \frac{1}{2} \text{Sp} \ln \left[ S_{ij}(\phi) + S_{GF,ij}(\phi) \right] - \Gamma^k_{ij}(\phi) S_{k}(\phi) - \text{Sp} \ln \left[ R^\mu_i(\phi) \chi_{i}^\alpha(\phi) \right], \] (8)

where \( \chi^\alpha \) is the linear gauge condition, \( S_{GF} = \frac{1}{2} \beta_{\mu\nu} \chi^\mu \chi^\nu \), and \( \beta_{\mu\nu} \) is background-field independent (for more details see [10,11]). As it can be checked explicitly, the one-loop effective action (8) is parametrization invariant, gauge invariant, and gauge-fixing independent.
3 One-loop action for induced 2D gravity

Let us now consider induced 2D gravity, with the action

\[ S = \int d^2x \sqrt{g} \left( R \frac{1}{\Delta} R + \Lambda \right), \]  

(9)
on the background \( R^1 \times S^1 \). On such a background, which is not the solution of the classical equations of motion, the convenient effective action is always gauge dependent. However, the \( S \)-matrix (the effective action \textit{on shell}, i.e., at the stationary points) is independent on the gauge condition choice. Actually, we are working in the loop expansion, what leads to explicit gauge dependence even on shell (perturbatively). This is why we prefer to work with the gauge-independent effective action.

In accordance with the standard background field method, we split the metric as

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}, \]  

(10)
where \( g_{\mu\nu} \) is the metric of flat space \( R^1 \times S^1 \) and \( h_{\mu\nu} \) is the quantum gravitational field.

The gauge fixing action is chosen as

\[ S_{GF} = \frac{1}{\alpha} \int d^2x \sqrt{g} \left( \nabla_\mu h^\mu_\rho - \beta \nabla_\rho h \right)^2, \]  

(11)
where \( \alpha \) and \( \beta \) are the gauge parameters and \( h = h^\mu_\mu \).

Now, one can calculate the following terms of (8) for the present case

\[ S_{ij}(\phi) + S_{GF,ij}(\phi) \equiv \frac{\delta^2(S + S_{GF})}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} = \frac{\nabla^\mu \nabla^\nu \nabla_\rho \nabla_\sigma}{\Delta} + 2 \left( \frac{\beta}{\alpha} - 1 \right) \delta^{\rho\sigma} \nabla^\mu \nabla^\nu + \left( \frac{\beta^2}{\alpha} - 1 \right) \delta^{\rho\sigma} \nabla^\mu \nabla^\nu \Delta - \frac{\Lambda}{4} \delta^{\mu\rho} \delta^{\nu\sigma} - \frac{\Lambda}{\alpha} \delta^{\mu\sigma} \nabla^\nu h + \frac{\Lambda}{4} \delta^{\mu\nu} \delta^{\rho\sigma}. \]  

(12)
In the rhs of (12), the symmetrization \((\rho\sigma) \leftrightarrow (\mu\nu)\) is understood.

The problem now is to define the configuration-space metric for the theory (3). In accordance with ref. [10], the configuration-space metric for quantum gravity is given by

\[ \gamma_{ij} \equiv \gamma_{g_{\mu\nu} g_{\nu\beta}} = \frac{1}{2} \sqrt{g} \left( g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - ag^{\mu\nu} g^{\alpha\beta} \right), \]  

(13)
where $a$ is a constant parameter. The $a$-dependence of the gauge independent effective action (so-called configuration-space metric dependence) has been discussed in refs. [15]. It is interesting to notice that 2D induced gravity is related with topological gravity [3]. The fact that the gauge invariant effective action in this theory depends on the configuration-space metric probably means that the off-shell quantum field theory under investigation does actually depend on this metric. Different choices for this configuration-space metric can probably lead to inequivalent quantum theories off shell [3,16].

It is not difficult to find the connection $\Gamma^{i}_{jk}$ (7) using the configuration-space metric (13). On the other hand, the term corresponding to the Vilkovisky correction is here

$$S_{V} = -\Gamma^{g_{\rho\sigma}(z)}_{g_{\mu\nu}(x)g_{\alpha\beta}(y)}h^{\mu\nu}(x)h^{\alpha\beta}(y) \frac{\delta S}{\delta g_{\rho\sigma}(z)},$$

(14)

and it can be written (according to (8)), as

$$\frac{\delta^{2}S_{V}}{\delta h_{\alpha\beta}h_{\rho\sigma}} = \frac{\Lambda}{4} \left[ \frac{3-2a}{8(a-1)}g^{\alpha(\rho}g^{\sigma)\beta} + \frac{-a^{2}+2a-2}{2(a-1)(2-a)}g^{\rho\sigma}g^{\alpha\beta} ight.$$

$$- \frac{1}{8(a-1)\Delta} \left( g^{\alpha(\rho}\nabla^{\sigma)\nabla^{\beta} + g^{\beta(\rho}\nabla^{\sigma)\nabla^{\alpha}} \right)$$

$$+ \frac{a}{2(a-1)(2-a)} \left( g^{\rho\sigma}\nabla^{\alpha}\nabla^{\beta} \Delta + g^{\alpha\beta}\nabla^{\rho}\nabla^{\sigma} \Delta \right) - \frac{1}{2-a} \frac{\nabla^{\rho}\nabla^{\sigma}\nabla^{\alpha}\nabla^{\beta}}{\Delta^{2}} \right].$$

(15)

The ghost operator in (8) (the last term in that expression) has now the following form

$$\delta_{\nu}^{\mu}\Delta + (1-2\beta)\nabla^{\mu}\nabla_{\nu}.$$  

(16)

Finally, collecting (12) and (15), taking into account (16), and expressing $\text{Sp ln}$ for the non-minimal operators in terms of $\text{Sp ln}$ for the minimal operators (see the method developed in refs. [17]), we get the following result for the one-loop action

$$\Gamma^{(1)} = 2\pi R \Lambda + \frac{1}{2} \left[ \text{Sp ln} \left( \Delta + \frac{\Lambda}{4(2-a)} \right) - 2 \text{Sp ln} \Delta \right].$$  

(17)

Here, $2\pi R$ is the length of the compactified dimension, while $S = \int dx$ is the “volume” of the space $R^{1}$. As we see explicitly, in eq. (17) the dependence on the gauge parameters $\alpha$ and $\beta$ has disappeared. However, an explicit dependence on the parameter $a$ remains in this gauge-independent action (17).
4 Calculation of the traces and non-trivial minimum

The trace calculations involved in expression (17) for the one-loop effective action are somehow involved. Non-trivial commutations of series have to be carried out. Using the techniques which have been developed in [18] for the derivation of zeta functions corresponding to partial differential operators, e.g. (already specified to $R^1 \times S^1$)

\[
\zeta_{-\Delta+m^2}\left(\frac{s}{2}\right) = -S \int_0^\infty \frac{dk}{\pi} \sum_{n=-\infty}^{+\infty} \left[ k^2 + \left(\frac{2\pi n}{\beta}\right) + m^2 \right]^{-s/2}
\]

\[
= -\frac{S}{\sqrt{\pi}} m^{1-s} \left\{ \frac{-\Gamma\left(\frac{s-1}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)} + \frac{\beta m}{2\sqrt{\pi}} \frac{1}{s-2} + \frac{(\beta m)^{(s-1)/2}}{\Gamma\left(\frac{s}{2}\right)} \sum_{k=0}^{\infty} \frac{(16\pi)^{-k}}{k!} \right\} \times \left(\frac{2\pi}{\beta m}\right)^k \prod_{j=1}^{k} \left[ (s-2)^2 - (2j-1)^2 \right] \sum_{n=1}^{\infty} n^{-k} e^{-\beta mn}\right\},
\]

we get

\[
V = \frac{\Gamma^{(1)}}{S} = 2\pi RA + \frac{RA}{32(2-a)} \left[ 1 - \ln \left(\frac{\Lambda}{4(2-a)}\right) \right] - \frac{1}{8\sqrt{2\alpha - a} + \frac{1}{24R}}
\]

\[
- \frac{1}{4\pi\sqrt{2R}} \left(\frac{\Lambda}{2-a}\right)^{1/4} \sum_{k=0}^{\infty} \frac{(16\pi)^{-k}}{k!} \left(\frac{R}{2}\sqrt{\frac{\Lambda}{2-a}}\right)^{-k}
\]

\[
\times \prod_{j=1}^{k} \left[ 4 - (2j-1)^2 \right] \sum_{n=1}^{\infty} n^{-(k+3/2)} \exp \left(-\pi R\sqrt{\frac{\Lambda}{2-a}} n\right) .
\]

This expression can be very much simplified if we look for the basic variables of the problem. They are

\[
x = \frac{\Lambda}{4(2-a)}, \quad y = R\sqrt{x} = \frac{R}{2}\sqrt{\frac{\Lambda}{2-a}}.
\]

We have in terms of them

\[
V = \sqrt{x} \left[ 8\pi(2-a)y + \frac{y}{8}(1-\ln x) - \frac{1}{4} + \frac{1}{24y} - F(y) \right],
\]

where the function $F(y)$ is given by

\[
F(y) = \frac{1}{4\pi} \sum_{k=0}^{\infty} \frac{(16\pi)^{-k}}{k!} \prod_{j=1}^{k} \left[ 4 - (2j-1)^2 \right] \sum_{n=1}^{\infty} n^{-(k+3/2)} e^{-2\pi ny}.
\]

It is now clear that all the dependence of the action on $R$, $\Lambda$ and $a$ comes through the specific combination given by variable $y$, but for a global factor, $\sqrt{x}$, and for the first term, which is just linear in $a$. 

7
To proceed with the compactification program, we are ready to impose (as is done in multidimensional gravity) that

\[
\begin{align*}
V(R, \Lambda, a) &= 0, \\
\frac{\partial V(R, \Lambda, a)}{\partial R} &= 0.
\end{align*}
\]  

(23)

The explicit \( a \) dependence can be readily eliminated, and we get

\[
\sqrt{x} \left[ F(y) - yF'(y) - \frac{1}{12y} + \frac{1}{4} \right] = 0.
\]  

(24)

This transcendent equation involves an asymptotic series, and must be solved approximately. Fortunately, the decreasing exponentials come to rescue and, after an explicit calculation one gets the expected result:

\[ y_1 = 0.33. \]  

(25)

This is the non-trivial stationary point of the effective action. The trivial one is obtained for

\[ x_0 = 0. \]  

(26)

As for the second derivative, we get

\[
\frac{\partial^2 V}{\partial y^2} = \sqrt{x} \left[ \frac{1}{12y^3} - F''(y) \right],
\]  

(27)

where the explicit \( a \)-dependence has disappeared. Hence, this second derivative has a definite sign (independent of \( a \)) at the stationary point

\[
\frac{\partial^2 V}{\partial y^2} \bigg|_{y=y_1} \simeq 2 > 0.
\]  

(28)

Thus, the point is clearly a minimum, that we obtain for the following combination of parameters:

\[
\frac{\Lambda R^2}{2 - a} \simeq \left( \frac{2}{3} \right)^2.
\]  

(29)
5 Conclusions

We have calculated the gauge-independent effective action in 2D quantum gravity on the background $R^1 \times S^1$. Considering this theory as a toy model for $D \geq 4$ induced gravity, we have analyzed the spontaneous compactification conditions and have found that the minimum of the effective potential is attained when the spontaneous compactification conditions are fulfilled. We should note that this fact is a general one. For example, if we use the convenient effective action the minimum is attained too (however, in that case the radius of spontaneous compactification depends on the gauge parameters).

In multidimensional quantum gravity models on the background $R^{D-1} \times S^1$, $D > 2$, it has been found [12,13] that quantum spontaneous compactification is unstable (a maximum of the effective potential is reached). On the contrary, in 2D induced quantum gravity we have found that spontaneous compactification is stable, on such a simple background as $R^1 \times S^1$. It would be of interest to understand the origin of this good property of the theory. Maybe, one can guess that $D > 2$ induced gravity (if it exists) should be realized as a Kaluza-Klein type theory.

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