Periodically forced ferrofluid pendulum: effect of polydispersity

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Abstract

We investigate a torsional pendulum containing a ferrofluid that is forced periodically to undergo small-amplitude oscillations. A homogeneous magnetic field is applied perpendicular to the pendulum axis. We give an analytical formula for the ferrofluid-induced “selfenergy” in the pendulum’s dynamic response function for monodisperse as well as for polydisperse ferrofluids.
I. INTRODUCTION

Real ferrofluids [1] contain magnetic particles of different sizes [2]. This polydispersity strongly influences the macroscopic magnetic properties of the ferrofluid. We investigate here the effect of polydispersity on the dynamic response of a ferrofluid pendulum.

A torsional pendulum containing a ferrofluid is forced periodically in a homogeneous magnetic field $H_{ext}$ that is applied perpendicular to the pendulum axis $e_z$ (see fig. 1). Such a ferrofluid pendulum is used for measuring the rotational viscosity [3]. The cylindrical ferrofluid container is here of sufficiently large length to be approximated as an infinite long cylinder. We consider rigid-body rotation of the ferrofluid with the time dependent angular velocity $\Omega = \varphi e_z$ as can be realized with the set-up of [3]. The fields $H$ and $M$ inside the cylinder are spatially homogeneous and oscillating in time.

II. EQUATIONS

First, the Maxwell equations demand that the fields $H$ and $M$ within the ferrofluid are related to each other via

$$H + NM = H_{ext}$$

(2.1)

with $N = 1/2$ for the infinitely long cylinder. Then we have the torque balance

$$\ddot{\varphi} = -\omega_0^2 \varphi - \Gamma_0 \dot{\varphi} - \frac{T}{\Theta} + f(t)$$

(2.2)

with the eigenfrequency $\omega_0$ and the damping rate $\Gamma_0$ of the pendulum without field and the total moment of inertia $\Theta$. The magnetic torque reads

$$T = -\mu_0 \int dV (M \times H)_z = -\mu_0 V (M \times H_{ext})_z,$$

(2.3)

and $f(t)$ is the external mechanical forcing.

Finally, we need an equation describing the magnetization dynamics. Here, we consider the polydisperse ferrofluid as a mixture of ideal monodisperse paramagnetic fluids. Then the resulting magnetization is given by $M = \sum M_j$, where $M_j$ denotes the magnetization of the particles with diameter $d_j$. We assume that each $M_j$ obeys a simple Debye relaxation dynamics described by

$$d_t M_j - \Omega \times M_j = -\frac{1}{\tau_j} [M_j - M_{eq}^j(H)]$$

(2.4)
We take the equilibrium magnetization to be given by a Langevin function

\[ \mathbf{M}_j^{eq}(\mathbf{H}) = \chi_j(H)\mathbf{H} = w_j \mathcal{L} \left( \frac{\mu_0 \pi M_{\text{mat}} d_j^3 H}{6 k_B T} \right) \frac{\mathbf{H}}{H} \] (2.5)

with the saturation magnetization of the material \(M_{\text{mat}}\) and the magnetization distribution \(w_j(d_j)\). Note that the magnetization equations (2.4) for the different particle sizes are coupled by the internal field \(H = H^{\text{ext}} - N \mathbf{M}\). As relaxation rate we combine Brownian and Néel relaxation

\[ \tau_j^{-1} = \frac{1}{\tau_B^-} + \frac{1}{\tau_N^-}. \]

The relaxation times depend on the particle size by \(\tau_B^- = \frac{\pi \eta}{2 k_B T} (d_j + 2s)^3\) and \(\tau_N^- = f_0^{-1} \exp \left( \frac{\pi K d_j^4}{6 k_B T} \right)\) with \(\eta\) the viscosity, \(s\) the thickness of the nonmagnetic particle layer, and \(K\) the anisotropy constant.

Altogether we use the following system of equations:

\[ \phi = \Omega \] (2.6)
\[ \dot{\Omega} = -\omega_0^2 \phi - \Gamma_0 \Omega - \mu_0 \frac{V}{\mathcal{Q}} H^{\text{ext}} M_y + f(t) \] (2.7)
\[ \dot{M}_x^j = -\Omega M_y^j - \frac{1}{\tau_j^-} \left[ M_x^j - \chi_j(H)(H^{\text{ext}} - N M_x) \right] \] (2.8)
\[ \dot{M}_y^j = \Omega M_x^j - \frac{1}{\tau_j^-} M_y^j - \frac{1}{\tau_j^-} N \chi_j(H) M_y. \] (2.9)

### III. LINEAR RESPONSE ANALYSIS

For the equilibrium situation of the unforced pendulum at rest that we denote in the following by an index 0 one has \(\phi_0 = \Omega_0 = M_y^{j0} = 0\) and \(M_x^{j0} = M_x^{eq}(H_0)\). Furthermore, \(M_0 = \sum M_x^{eq}(H_0)\) with \(H_0\) solving the equation \(H_0 = H^{\text{ext}} - N M_0(H_0)\).

External forcing with small \(|f|\) leads to small deviations of \(\phi\), of \(\Omega\), and of \(\delta \mathbf{H} = \mathbf{H} - H_0 = -N(\mathbf{M} - M_0) = -N\delta \mathbf{M}/2\) from the above described equilibrium state. We expand each \(\chi_j(H)\) up to linear order in \(\delta \mathbf{H}\)

\[ \chi_j(|H_0 + \delta \mathbf{H}|) = \chi_{j0} - \chi'_{j0} N \delta M_x + \mathcal{O}(\delta \mathbf{H})^2. \] (3.1)

Here, \(\chi_{j0} = \chi_j(H_0)\) and \(\chi'_{j0}\) is the derivative of \(\chi_{j0}\). Then we get the linearized equations

\[ \dot{\phi} = \Omega \] (3.2)
\[ \dot{\Omega} = -\omega_0^2 \phi - \Gamma_0 \Omega - \kappa y + f(t) \] (3.3)
\[ \dot{x}_j = -\frac{1}{\tau_j^-} x_j - \frac{1}{\tau_j^-} N(\chi_{j0} + \chi'_{j0} H_0)x \] (3.4)
\[ \dot{y}_j = \Omega x_j^0 - \frac{1}{\tau_j^-} y_j - \frac{1}{\tau_j^-} N \chi_{j0} y. \] (3.5)
We have introduced the abbreviations \( x_j = \delta M_x / M_0, x_j^0 = M_x^0 / M_0, y_j = \delta M_y / M_0 \) and \( x = \sum_j x_j, y = \sum_j y_j \). The strength of the coupling constant between the mechanical degrees of freedom \( \varphi, \Omega \) and the magnetic ones is \( \kappa = \mu_0 H_{\text{ext}} M_0 V / \Theta \).

For periodic forcing \( f(t) = \hat{f} e^{-i \omega t} \) we look for solutions in the form

\[
\begin{pmatrix}
\varphi(t) \\
\Omega(t) \\
x_j(t) \\
y_j(t)
\end{pmatrix} =
\begin{pmatrix}
\hat{\varphi} \\
\hat{\Omega} \\
\hat{x}_j \\
\hat{y}_j
\end{pmatrix} e^{-i \omega t}.
\]

(3.6)

Inserting the ansatz (3.6) into the linearized equations (3.2) – (3.5) yields

\[
\hat{\Omega} = -i \omega \hat{\varphi}
\]

(3.7)

\[
\hat{x} = 0 = \hat{x}_j
\]

(3.8)

\[
\dot{\hat{y}}_j = - \left[ \frac{i \omega \tau_j}{1 - i \omega \tau_j} x_j^0 \right] \frac{N \chi_j}{1 - i \omega \tau_j} \frac{\omega}{\kappa} \hat{\varphi}
\]

(3.9)

\[
\hat{y} = \frac{\omega}{\kappa} \Sigma \hat{\varphi}
\]

(3.10)

and

\[
\hat{\varphi} = G \hat{f} = \left[ \omega_0^2 - \omega^2 - i \omega \Gamma_0 - \omega \Sigma \right]^{-1} \hat{f}.
\]

(3.11)

The ferrofluid-induced selfenergy \( \Sigma(\omega) \) in the expression for the dynamical response function \( G(\omega) \) of the torsional pendulum is

\[
\Sigma(\omega) = i \kappa \left( 1 + N \sum_j \frac{\chi_j}{1 - i \omega \tau_j} \right)^{-1} \sum_j \frac{\tau_j x_j^0}{1 - i \omega \tau_j}.
\]

(3.12)

Its imaginary part changes the damping rate \( \Gamma_0 \) of the pendulum for \( \kappa = 0 \), i.e., in zero field. The real part shifts the resonance frequency of the pendulum. In the special case of a monodisperse ferrofluid on has

\[
\Sigma(\omega) = \frac{i \kappa \tau}{1 - i \omega \tau + N \chi_0}
\]

(3.13)

IV. RESULTS

We evaluated the linear response function \( G(\omega) = \hat{\varphi}(\omega) / \hat{f} \) of the pendulum’s angular deviation amplitude \( \hat{\varphi}(\omega) \) to the applied forcing amplitude \( \hat{f} \) and the selfenergy \( \Sigma(\omega) \) for
some experimental parameters from [3]: $\omega_0/2\pi = 32.7\,Hz$, $\Gamma_0 = 0.178\,Hz$, $V/\Theta = 20\,m/kg$. The cylinder is filled with the ferrofluid APG 933 of FERROTEC. Therefore, we used in equation (3.13) an experimental $\tau = 0.6\,ms$ and the experimental $M_{eq}(H)$ shown in fig. 2. These monodisperse results were compared with the expression (3.24) for the polydisperse case for the typical parameter values $M_{mat} = 456kA/m$, $\eta = 0.5\,Pa\cdot s$, $s = 2\,nm$, $K = 44\,kJ/m^3$ and $f_0 = 10^9\,Hz$. The contributions $w(d_j)$ that enter into the formulas (2.5) for the susceptibilities $\chi_j$ are given by a lognormal distribution [2]:

$$w(d_j) = M_{sat} \frac{g(d_j) d_j}{\sum_{k=1}^{30} g(d_k) d_k} \quad \text{with} \quad g(d_j) = \frac{1}{\sqrt{2\pi d_j \ln \sigma}} \exp \left( -\frac{\ln^2(d_j/d_0)}{2 \ln^2 \sigma} \right) \quad (4.1)$$

Fitting the experimental $M_{eq}(H)$ with a sum of Langevin functions (2.5) yields $M_{sat} = 18149A/m$, $d_0 = 7\,nm$ and $\sigma = 1.47$ (see fig. 2). We used here 30 different particle sizes from $d_1 = 1\,nm$ to $d_{30} = 30\,nm$ (see fig. 3).

The calculations show the additional damping rate caused by the interaction between ferrofluid and external field. An increasing magnetic field leads to smaller amplitudes; in polydisperse ferrofluids the amplitude decreases faster [fig. 4 and 5 (a)]. Furthermore, one can see a shift of the peak position to higher frequencies $\omega_{max}$, which is stronger in polydisperse ferrofluids [fig. 4 and 5 (b)].

Acknowledgments

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FIG. 1: Schematic plot of the system

FIG. 2: Experimental equilibrium magnetization $M_{eq}(H)$ used as input for the monodisperse calculations; full line: fit with lognormal contribution.
FIG. 3: lognormal contribution $w(d_i)$ ($d_1 = 1\text{nm}\ldots d_{30} = 30\text{nm}$) used as input for the polydisperse calculations.

FIG. 4: $|G|$ near the resonance $\omega_0$: $x$ $H^{ext} = 0\text{kA/m}$, squares $H^{ext} = 5\text{kA/m}$, circles $H^{ext} = 10\text{kA/m}$; filled symbols: polydisperse.
FIG. 5: Maximum value $\max|G|$ (a) and peak position $\omega_{\text{max}}$ (b) as a function of external field $H^{\text{ext}}$; full line monodisperse, dashed line polydisperse.
FIG. 6: $Re(\Sigma)$ (a) and $Im(\Sigma)$ (b) at $\omega = \omega_{\text{max}}$; full line monodisperse, dashed line polydisperse.