SCATTERING FOR THE ONE-DIMENSIONAL KLEIN-GORDON EQUATION WITH EXPONENTIAL NONLINEARITY

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ABSTRACT. We consider the asymptotic behavior of solutions to the Cauchy problem for the defocusing nonlinear Klein-Gordon equation (NLKG) with exponential nonlinearity in the one spatial dimension with data in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. We prove that any energy solution has a global bound of the $L^6_{t,x}$ space-time norm, and hence scatters in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ as $t \to \pm\infty$. The proof is based on the argument by Killip-Stovall-Visan (Trans. Amer. Math. Soc. 364 (2012), no. 3, 1571–1631). However, since well-posedness in $H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ for NLKG with the exponential nonlinearity holds only for small initial data, we use the $L^6_t W_s^{1/2,-1/2}_x$-norm for some $s > \frac{1}{2}$ instead of the $L^6_{t,x}$-norm, where $W_s^{p,q}$ denotes the $s$-th order $L^p$-based Sobolev space.

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1. INTRODUCTION

We consider the asymptotic behavior of solutions to the Cauchy problem to the Klein-Gordon equation with exponential nonlinearity

\begin{equation}
 u_{tt} - u_{xx} + u + \mathcal{N}(u) = 0,
\end{equation}

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ is an unknown function and $\mathcal{N} : \mathbb{R} \to \mathbb{R}$ is defined by

$\mathcal{N}(u) := \{\exp(|u|^2) - 1 - |u|^2\} u.$
In this paper, we assume that the initial data belong to the energy space, i.e. 
\( u(0) \in H^1(\mathbb{R}) \) and \( u_t(0) \in L^2(\mathbb{R}) \). Here the energy space is the set of data for which the energy of the solution 
\[ (1.2) \]
\[ E(u(t), u_t(t)) = \int_{\mathbb{R}} \left( \frac{1}{2} \| u_t(t, x) \|^2 + \frac{1}{2} |u_x(t, x)|^2 + \frac{1}{2} |u(t, x)|^2 + \frac{1}{2} \tilde{N}(u(t, x)) \right) dx \]
is finite, where \( \tilde{N} : \mathbb{R} \to \mathbb{R} \) is defined by
\[ \tilde{N}(u) := \exp(|u|^2) - 1 - |u|^2 - \frac{1}{2} |u|^4. \]
It is known that the energy of the solution to \( (1.1) \) is conserved. We note that the identity \( \tilde{N}^*(u) = 2N(u) \) holds, or equivalently the equation \( \tilde{N}(u) = N(u)u \) is valid for any \( u \in \mathbb{R} \). We also note that as the nonlinearity \( \tilde{N}(u) \) belongs to the energy-subcritical and defocusing \( (\tilde{N}(u) \geq 0) \) case, global well-posedness in the energy space \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) is a simple consequence of the energy conservation law. However the asymptotic behavior of the solution is not clear at all.

Klein-Gordon equations are fundamental hyperbolic ones and their nonlinear problems are intensively studied by many researchers (see for example \[ 15 \] \[ 16 \] \[ 11 \] \[ 12 \] and references therein). Especially, Nakamura and Ozawa \[ 15 \] first studied well-posedness for the Klein-Gordon equation with exponential-type nonlinearity. We note that the exponential-type nonlinearity is corresponding to energy-critical in two space dimensions (see for example \[ 14 \] \[ 5 \]). Ibrahim, Masmoudi and Nakanishi \[ 8 \] (see also \[ 7 \]) studied long-time behavior of solutions in the focusing case below the ground state. The scattering threshold for the focusing energy-critical nonlinear Klein-Gordon equation is given by that for the massless stationary equation.

On the other hand, Nakanishi \[ 16 \] proved that space-time bounds for the focusing mass-critical Klein-Gordon equation imply space-time bounds for the corresponding nonlinear Schrödinger equation. This is a reflection of the fact that the Klein-Gordon equation degenerates to the Schrödinger equation in the nonrelativistic limit. By using this fact, Killip, Stovall and Visan \[ 11 \] proved that any energy solution tends to a free solution in the energy space \( \tilde{H}^{1/2}(\mathbb{R}) \), which the energy of the solution \( \tilde{N}(u) = N(u)u \) is finite, where \( \tilde{N} : \mathbb{R} \to \mathbb{R} \) is the energy space is the set of data for which the energy of the solution
\[ (1.2) \]
\[ E(u(t), u_t(t)) = \int_{\mathbb{R}} \left( \frac{1}{2} \| u_t(t, x) \|^2 + \frac{1}{2} |u_x(t, x)|^2 + \frac{1}{2} |u(t, x)|^2 + \frac{1}{2} \tilde{N}(u(t, x)) \right) dx \]
is finite, where \( \tilde{N} : \mathbb{R} \to \mathbb{R} \) is defined by
\[ \tilde{N}(u) := \exp(|u|^2) - 1 - |u|^2 - \frac{1}{2} |u|^4. \]
It is known that the energy of the solution to \( (1.1) \) is conserved. We note that the identity \( \tilde{N}^*(u) = 2N(u) \) holds, or equivalently the equation \( \tilde{N}(u) = N(u)u \) is valid for any \( u \in \mathbb{R} \). We also note that as the nonlinearity \( \tilde{N}(u) \) belongs to the energy-subcritical and defocusing \( (\tilde{N}(u) \geq 0) \) case, global well-posedness in the energy space \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) is a simple consequence of the energy conservation law. However the asymptotic behavior of the solution is not clear at all.

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The equation \( (1.1) \) is regarded as mass-critical because the nonlinearity \( \tilde{N}(u) \) contains the quintic part \( \frac{1}{2} |u|^4 u \) by the Taylor expansion. However, there is no energy-critical nonlinearity in one spatial dimension, and thus \( (1.1) \) belongs to the energy-subcritical case.

From the viewpoint of Trudinger-Moser’s inequality, the growth rate as \( \exp(|u|^2) \) at infinity seems to be optimal at the level of \( H^{1/2}(\mathbb{R}) \). We note that the \( L^\infty(\mathbb{R}) \)-norm is out of control of the \( H^{1/2}(\mathbb{R}) \)-norm even when the latter is small. Accordingly, well-posedness in \( H^{1/2}(\mathbb{R}) \) for the exponential-type nonlinearity requires the smallness of initial data (see \[ 15 \] for more detail). In order to treat large initial data, it needs to assume \( (u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) for \( s > \frac{1}{2} \). This fact causes several technical difficulties in our analysis.
Our main goal in this paper is to prove that global strong solutions exist and have finite space-time norms.

**Theorem 1.1.** Let \((u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\) and \(s \in (\frac{1}{3}, \frac{5}{6})\). Then, there exists a unique global solution \(u\) to \((1.1)\) with initial data \((u(0), u_t(0)) = (u_0, u_1)\). Moreover, this solution obeys the global space-time bounds

\[
\|u\|_{L^\infty_t H^s_x} + \|u_t\|_{L^\infty_t L^2_x} + \|\partial_x^s u\|_{L^2_t L^6_x} \leq C(E(u_0, u_1)).
\]

As a consequence, the solution scatters both forward and backward in time, that is, there exist scattering states \((u^+, u^-_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\) such that

\[
\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = \begin{pmatrix} \cos((\partial_x)t) & \langle \partial_x \rangle^{-1}\sin((\partial_x)t) \\ -\langle \partial_x \rangle \sin((\partial_x)t) & \cos((\partial_x)t) \end{pmatrix} \begin{pmatrix} u_0^+ \\ u_1^- \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

in \(H^1(\mathbb{R}) \times L^2(\mathbb{R})\) as \(t \to \pm \infty\), where double-sign corresponds.

The main point in the theorem is to derive the \(L^6_{t,x}\)-spacetime bound and concomitant proof of scattering.

On one hand, the lower bound of \(s\) \((s > \frac{1}{2})\) comes from the well-posedness result (Proposition 3.1) as mentioned above. We note that if the exponential-type nonlinearity \(\mathcal{N}(u)\) is replaced with the power-type nonlinearity \(|u|^4u\), which belongs to mass-critical case, the limiting case \(s = \frac{1}{2}\) is allowed (see Corollary 1.2 below more precisely). Since we rely on the well-posedness in \(H^s(\mathbb{R})\) with \(s > \frac{1}{2}\) to obtain the theorem, the stability theory (Proposition 5.2), which is a variant of well-posedness, requires some regularity. Accordingly, we have to treat the inverse Strichartz estimate (Theorem 4.5) (or the linear profile decomposition (Theorem 5.1)) and the nonlinear decoupling (Proposition 5.3) with some regularity. On the other hand, the upper bound of \(s\) \((s < \frac{11}{6})\) is needed to show the inverse Strichartz estimate (Theorem 1.3). More precisely, we use this gap between the upper bound and 1 when we apply the Littlewood-Paley decoupling (Lemma 1.1). However, since it is enough to set \(s\) slightly larger than \(\frac{5}{6}\), this does not cause any problem.

For the Klein-Gordon equation with the focusing exponential-type nonlinearity \(-\mathcal{N}(u)\), we can expect that the dichotomy between scattering and blowup for initial data with energy less than that of the ground state holds. However it is not known existence of the ground state to the equation. Moreover we do not have Trudinger-Moser type inequality in one spatial dimension, which is useful in two spatial dimensions. We hence only consider the defocusing case.

The exponential-type nonlinearity \(\mathcal{N}(u)\) contains the quintic nonlinearity \(\frac{1}{2}|u|^4u = \frac{1}{2}u^5\), which belongs to the mass-critical case in one spatial dimension, and an infinite sum of higher-order nonlinearities, which belong to the mass-supercritical and energy-subcritical case. By neglecting the higher-order part, the similar result as Theorem 1.1 holds for the Klein-Gordon equation with the single power-type nonlinearity:

\[
(1.3) \quad u_{tt} - u_{xx} + u + \mu \frac{1}{2} u^5 = 0,
\]

where \(\mu = +1\), which is known as the defocusing equation. When \(\mu = -1\), the nonlinearity is said to be focusing. With a slight abuse of notation, we define the energy of \(1.3\) by

\[
E(u(t), u_t(t)) = \int_\mathbb{R} \left\{ \frac{1}{2} |u_t(t,x)|^2 + \frac{1}{2} |u_x(t,x)|^2 + \frac{1}{2} |u(t,x)|^2 + \frac{\mu}{12} |u(t,x)|^6 \right\} dx.
\]
Since the local well-posedness in $H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ to (1.3) is valid for arbitrary initial data in $H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, $s = \frac{1}{2}$ in the corresponding theorem for (1.3) of Theorem 1.1 is allowed. Different from the case with the exponential-type nonlinearity, we can treat (1.3) with the focusing nonlinearity ($\mu = -1$). In the focusing case, a static solution $u(t, x) = \sqrt{2}Q(x)$ is expected as the threshold that the dichotomy between scattering and blowup holds. Here, $Q$ is the unique positive even Schwartz solution to the elliptic equation

\begin{equation}
Q'' + Q^5 = Q, \quad \text{in } H^1(\mathbb{R}).
\end{equation}

Namely, $Q$ can be written as

\begin{equation}
Q(x) = \frac{\sqrt{3}}{ \sqrt{ \cosh 2x }},
\end{equation}

It is not appropriate to measure the size of the initial data purely in terms of the energy because of the negative sign appearing in front of the potential energy term.

For this reason, we introduce a second notion of size, namely, the mass, which is defined by

\begin{equation}
M(u(t)) := \int_{\mathbb{R}} |u(t, x)|^2 dx.
\end{equation}

Unlike the energy, this is not conserved. We note that $E(\sqrt{2}Q, 0) = \frac{1}{2}M(\sqrt{2}Q)$ holds. We can obtain the following theorem for (1.3):

**Corollary 1.2.** Let $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and in the focusing case ($\mu = -1$) assume also that $M(u_0) < \sqrt{2}M(Q)$ and $E(u_0, u_1) < E(\sqrt{2}Q, 0)$. Then, there exists a unique global solution $u$ to (1.3) with initial data $(u(0), u_t(0)) = (u_0, u_1)$. Moreover, this solution obeys the global space-time bounds

\begin{equation}
\|u\|_{L^\infty_t H^1_x} + \|u_t\|_{L^\infty_t L^2_x} + \|u\|_{L^6_{t,x}} \leq C(E(u_0, u_1)).
\end{equation}

As a consequence, the solution scatters both forward and backward in time, that is, there exist $(u_0^+, u_1^+) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that

\begin{equation}
\begin{pmatrix}
u(t) \\ u_t(t) \end{pmatrix} = \begin{pmatrix} \cos((\partial_x) t) & (\partial_x)^{-1} \sin((\partial_x) t) \\ -\partial_x \sin((\partial_x) t) & \cos((\partial_x) t) \end{pmatrix} \begin{pmatrix} u_0^+ \\ u_1^+ \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{equation}

in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ as $t \to \pm \infty$, where double-sign corresponds.

We mention a remark on the proof of this corollary in 6.1. Because the remaining argument is almost the same as that in 11, we omit the details in this paper.

1.1. Notation. Let $\mathbb{N}_0$ denote the set of nonnegative integers $\mathbb{N} \cup \{0\}$.

We denote the Fourier transform of $f$ by $\hat{f}$, namely

\begin{equation}
\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.
\end{equation}

We denote the characteristic function of an interval $I$ by $\mathbf{1}_I$. We abbreviate $\mathbf{1}_{(0,\infty)}$ to $\mathbf{1}_{>0}$. For an interval $I \subset \mathbb{R}$, we set

\begin{equation}
S_I(u) := \left( \int_I \int_{\mathbb{R}} |u(t, x)|^6 dx dt \right)^{1/6}.
\end{equation}
Let $\sigma \in C_c^{\infty}(\mathbb{R})$ with $\sigma(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases}$ For $N \in 2^{N_0}$, we define

$$\hat{P}_N f(\xi) := \begin{cases} \sigma(\xi) \hat{f}(\xi), & \text{if } N = 1, \\ \left(\sigma\left(\frac{\xi}{N}\right) - \sigma\left(\frac{2\xi}{N}\right)\right) \hat{f}(\xi), & \text{otherwise}. \end{cases}$$

We denote the dual pair of a Banach space $X$ and its dual space $X'$ by $\langle \cdot, \cdot \rangle_{X,X'}$. We denote the inner product in a Hilbert space $H$ by $\langle \cdot, \cdot \rangle_H$. In estimates, we use $C$ to denote a positive constant that can change from line to line. If $C$ is absolute or depends only on parameters that are considered fixed, then we often write $X \lesssim Y$, which means $X \leq CY$. When an implicit constant depends on a parameter $a$, we sometimes write $X \lesssim a Y$. We define $X \ll Y$ to mean $X \leq C^{-1}Y$ and $X \sim Y$ to mean $C^{-1}Y \leq X \leq CY$.

2. Fundamental tools

It is more convenient for us to recast Klein-Gordon equations as a first-order equation for a complex-valued function $v$ via the map $(u, u_t) \mapsto v := u + i(\partial_x)^{-1}u_t$. This is easily seen to be a bijection between real-valued solutions of (1.1) and complex-valued solutions of

$$-iv_t + (\partial_x) v + (\partial_x)^{-1}N(\Re v) = 0.$$ 

As such, local or global theories for the two equations are equivalent. We will consistently use the letter $u$ to denote solutions to (1.1) and $v$ the corresponding solutions to (2.1). Note that the energy and the scattering norm to (2.1) are written as

$$E(v(t)) := \int_{\mathbb{R}} \left\{ \frac{1}{2} |(\partial_x) v(t,x)|^2 + \frac{1}{2} N(\Re v(t,x)) \right\} dx,$$

$$S_I(u) = S_I(v) = \left( \int_I \int_{\mathbb{R}} |\Re v(t,x)|^6 dx dt \right)^{1/6}.$$ 

We note that Sobolev’s embedding in one dimension implies that

$$\|v\|_{L^\infty_x} \leq \|v\|_{H^1_x}.$$ 

Hence, the energy is finite if $v(0) \in H^1(\mathbb{R})$.

2.1. Strichartz estimates. Since we will use the Strichartz estimates with the scaling parameter $\lambda$, for the reader’s convenience, we record them (see for example [4]).

**Lemma 2.1 (Dispersive estimate).** For $t \neq 0$ and $2 \leq p < \infty$,

$$\|e^{-i\lambda^2(\lambda^{-1}\partial_x)} f\|_{L^p_x} \lesssim |t|^{1/p - 1/2} \|\lambda^{-1}\partial_x \lambda^{1/2 - 1/p} f\|_{L^p_x}.$$ 

Here, the implicit constant is independent of $\lambda$ and $p' := \frac{p}{p-1}$.

We call a pair $(q, r)$ admissible if $4 < q \leq \infty$, $2 \leq r < \infty$, and $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. 
Lemma 2.2 (Strichartz estimate). For \( \lambda > 0 \) and any admissible pairs \((q,r)\) and \((\tilde{q}, \tilde{r})\), we have

\[
\|e^{-i\lambda^2t(\lambda^{-1}\partial_x)}f\|_{L^q_tL^r_x} \lesssim \|\langle \lambda^{-1}\partial_x \rangle^{3(q-2)/4} f\|_{L^q_x},
\]
\[
\left\| \int_0^t e^{-i\lambda^2(t-s)(\lambda^{-1}\partial_x)}F(s)ds \right\|_{L^q_tL^r_x} \lesssim \|\langle \lambda^{-1}\partial_x \rangle^{3(1/r'-1/r)}/2 F\|_{L^q_xL^r'_x}.
\]

Here, the implicit constants are independent of \( \lambda \) and \( \tilde{q}' := \frac{q}{q-1}, \tilde{r}' := \frac{r}{r-1} \).

2.2. Symmetries. The following symmetries play an important role in our analysis. We define the following operators and observe these properties:

- Translations: for any \( y \in \mathbb{R} \), we define \([T_yf](x) := f(x - y)\).
- Lorentz boosts: For any \( \nu \in \mathbb{R}\setminus\{0\} \), we define
  \[
  L_\nu(t,x) := (\langle \nu \rangle t - \nu x, \langle \nu \rangle x - \nu t).
  \]
  - \( L_\nu \) preserves spacetime volume.
  - \( L^{-1}_\nu = L_{-\nu} \) holds true.
  - \( u \) is a solution to (1.1) if and only if \( u \circ L^{-1}_\nu \) is a solution to (1.1).
  Indeed, by setting \((\bar{t}, \bar{x}) = (\langle \nu \rangle t - \nu x, \langle \nu \rangle x - \nu t)\), we have
  \[
  \left| \frac{\partial(\bar{t}, \bar{x})}{\partial(t,x)} \right| = \det \begin{pmatrix} \langle \nu \rangle & -\nu \\ -\nu & \langle \nu \rangle \end{pmatrix} = \langle \nu \rangle^2 - \nu^2 = 1.
  \]

From
\[
\partial_t^2 = \langle \nu \rangle^2 \partial_t^2 + 2\nu\langle \nu \rangle \partial_t \partial_x + \nu^2 \partial_x^2, \quad \partial_x^2 = \nu^2 \partial_t^2 + 2\nu\langle \nu \rangle \partial_t \partial_x + \langle \nu \rangle^2 \partial_x^2,
\]
we have \( \partial_t^2 - \partial_x^2 = \partial_t^2 - \partial_x^2 \). Moreover, \( L_{-\nu}(\bar{t}, \bar{x}) = (t,x) \) holds.

- \( |L_{\nu}f(x) := [e^{-i\langle \nu \rangle \partial_x}f] \circ L_{\nu}(0,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\langle \nu \rangle x} e^{i\nu x} \hat{f}(\xi) d\xi. \)
- Scaling (the scaling is useful although the Klein-Gordon equation does not possess scaling-invariant): For any \( \lambda > 0 \), \([D_{\lambda}f](x) := \lambda^{-1/2} f(\frac{x}{\lambda})\).

The action of \( L_\nu \) is easily understood on the Fourier side. In particular, we have the following:

Lemma 2.3.

(i) \( L^{-1}_\nu = L_{-\nu} \).

(ii) \( \mathcal{F}[L_{\nu}^{-1}f](\xi) = \frac{L_{\nu}^{-1}(\xi)}{\xi}(\hat{\xi}, \xi) = \nu(\xi) - \nu(\xi) \).

(iii) For \( \nu, y \in \mathbb{R} \), \( L_{\nu}^{-1}T_ye^{i\tau(\partial_x)} = T_ye^{i\tau(\partial_x)L_{\nu}^{-1}} \), where \( (\tau, y) = L_\nu(\tau, y) \).

(iv) For any \( s \in \mathbb{R} \),
\[
\langle L_{\nu}^{-1}f, g \rangle_{H^s} = \langle f, L_\nu(-i\partial_x)L_{\nu}g \rangle_{H^s}, \quad \text{(namely} \ L_\nu \text{ is unitary in } H^{1/2}(\mathbb{R}) \text{)}
\]
where \( m_\nu(\xi) := \left| \frac{\xi}{\langle \nu \rangle} \right|^{2s-1} \). Moreover, \( \|m_\nu\|_{L^\infty} + \|m_\nu^{-1}\|_{L^\infty} \lesssim \langle \nu \rangle^{2s-1} \).

(v) \( [e^{-it(\partial_x)}L_{\nu}^{-1}f](x) = [e^{-it(\partial_x)}f] \circ L_{\nu}^{-1}(t,x), \quad [e^{it(\partial_x)}L_{\nu}^{-1}f](x) = [e^{it(\partial_x)}f] \circ L_{\nu}(t,x) \).

(vi) For \( \mu, \nu \in \mathbb{R} \), \( L_{\mu}^{-1}L_{\nu}^{-1} = L_{\mu(\nu) + (\mu)} \).

Proof: A direct calculation shows that
\[
\langle \xi \rangle = \langle \nu \rangle \langle \xi \rangle - \nu \xi
\]
and \( \xi^{-1} = L_{-\nu} \). From
\[
\frac{d\xi}{d\xi} = \langle \nu \rangle - \nu \xi \langle \xi \rangle = \frac{\xi}{\xi} = \frac{\xi}{\xi},
\]
we have
\[
\mathbf{L}_{\nu} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} e^{-i\nu x} \widehat{\mathcal{F}}(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi
\]
(2.3)
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) \frac{\xi}{\xi} d\xi = \mathcal{F}^{-1} \left[ \frac{\langle l_{-\nu}(\xi) \rangle}{\langle \xi \rangle} \widehat{f}(l_{-\nu}(\xi)) \right](x)
\]
Similarly,
\[
\mathcal{F}[\mathbf{L}_{\nu} f](\xi) = \frac{\xi}{\xi} \widehat{f}(\xi),
\]
which shows (i). Accordingly, combining (2.3) with (i), we get (ii).

By (2.2) and 
\[
\mathcal{F}[\mathbf{L}_{\nu}^{-1} T_{\nu} e^{i\tau (\partial_x)} f](\xi) = \frac{\xi}{\xi} e^{-i\xi} \mathcal{F}[\mathbf{L}_{\nu}^{-1} f](\xi) = e^{-i\xi} \mathcal{F}[\mathbf{L}_{\nu}^{-1} f](\xi)
\]
which concludes (iii).

From (ii) and (2.4),
\[
\langle \mathbf{L}_{\nu}^{-1} f, g \rangle_{H^s} = \int_{\mathbb{R}} \langle \xi \rangle^{2s} \widehat{\mathbf{L}_{\nu}^{-1} f}(\xi) \widehat{g}(\xi) d\xi = \int_{\mathbb{R}} \langle \xi \rangle^{2s} \widehat{f}(\xi) \bar{\widehat{g}}(\xi) d\xi
\]
\[
= \int_{\mathbb{R}} \langle \xi \rangle^{2s} \widehat{f}(\xi) \mathbf{L}_{\nu} g(\xi) d\xi = \langle f, m_s(-i\partial_x) \mathbf{L}_{\nu} g \rangle_{H^s}
\]
By (2.2), we have
\[
\frac{1}{\langle \nu \rangle + |\nu|} = \langle \nu \rangle - |\nu| \leq \frac{\langle \xi \rangle}{\langle \xi \rangle} = \langle \nu \rangle - i\frac{\xi}{\xi} \leq \langle \nu \rangle + |\nu|,
\]
which implies
\[
||m_s||_{L^\infty} + ||m_s^{-1}||_{L^\infty} \lesssim \langle \nu \rangle^{2s-1}.
\]
Hence, we obtain (iv).

The claim (v) follows from (ii) and (2.2).
\[
[e^{\pm i(\partial_x)}] \circ L_{\pm \nu}(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i((\nu) \pm \nu t)x} e^{\mp i((\nu) \pm \nu t)\xi} \widehat{f}(\xi) d\xi
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} e^{\mp i\xi} \widehat{f}(\xi) d\xi
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \mathcal{F}[e^{\mp i(\partial_x)} L_{\nu}^{-1} f](\xi) d\xi
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \mathcal{F}[e^{\mp i(\partial_x)} L_{\nu}^{-1} f](\xi) d\xi
\]
\[
= e^{\mp i(\partial_x)} L_{\nu}^{-1} f](x).
\]
Owing to (ii) we have
\[
\mathcal{F}[L_{\nu}^{-1} L_{\nu}^{-1} f](l_{\mu}(l_{\nu}(\xi))) = \frac{\langle l_{\nu}(\xi) \rangle}{\langle l_{\nu}(l_{\nu}(\xi)) \rangle} \mathcal{F}[L_{\nu}^{-1} f](l_{\nu}(\xi)) = \frac{\langle \xi \rangle}{\langle l_{\nu}(l_{\nu}(\xi)) \rangle} \widehat{f}(\xi).
\]
Here, (2.2) yields
\[ l_{\mu}(l_{\nu}(\xi)) = \langle l_{\nu}(\xi) \rangle - \mu(l_{\nu}(\xi)) = (\langle l_{\nu}(\xi) \rangle + \mu \nu) - (\langle l_{\nu}(\xi) \rangle + \langle l_{\nu}(\xi) \rangle) = l_{\mu(\nu)+(\nu)\nu}(\xi), \]
which shows (vi). \( \square \)

2.3. Useful lemmas. In this subsection, we observe certain manipulations of symmetries that we will need in the proof of the inverse Strichartz inequality, Theorem 4.5.

Lemma 2.4. Fix \( h \in L^2(\mathbb{R}) \) and \( B > 0 \). Then, the set
\[ \mathcal{K} := \{ D_\lambda^{-1}L_e^{-1}m_0(-i\partial_x)^{-1}e^{i\nu x}D_\lambda h: |\nu| \leq B, B^{-1} \leq \lambda < \infty \} \]
is precompact in \( L^2(\mathbb{R}) \). Moreover, the closure \( \overline{\mathcal{K}} \) of \( \mathcal{K} \) in \( L^2(\mathbb{R}) \) does not contain 0 unless \( h \equiv 0 \).

If \( h \) is the characteristic function of \([-1, 1]\), then
\[ \text{supp } \hat{\mu} \subset \{|\xi| \lesssim \langle B \rangle\} \]
all uniformly for \( g \in \mathcal{K} \).

Since the proof of this Lemma is same as that of Lemma 2.8 in [11], we omit the details here.

Lemma 2.5. \( \text{(i)} \) Suppose \( g_n \rightharpoonup g \) weakly in \( H^1(\mathbb{R}) \) and \( \lambda_n \to \lambda \in (0, \infty) \).

Then, there is a subsequence so that
\[ [\langle \lambda_n^{-1}\partial_x \rangle^{1/2} e^{-i\mu^2 t(\lambda_n^{-1}\partial_x)} g_n](x) \to [\langle \lambda^{-1}\partial_x \rangle^{1/2} e^{-i\mu^2 t(\lambda^{-1}\partial_x)} g](x) \]
for almost every \((t, x) \in \mathbb{R} \times \mathbb{R}\).

\( \text{(ii)} \) For \( \lambda_n \to \lambda \in (0, \infty) \) and fixed \( g \in H^1(\mathbb{R}) \),
\[ \lim_{n \to \infty} \|\langle \lambda_n^{-1}\partial_x \rangle^{1/2} e^{-i\mu^2 t(\lambda_n^{-1}\partial_x)} g - \langle \lambda^{-1}\partial_x \rangle^{1/2} e^{-i\mu^2 t(\lambda^{-1}\partial_x)} g\|_{L^2_{t,x}} = 0. \]

\( \text{(iii)} \) Let \( s \geq \frac{1}{2} \). Fix \( \theta \in (0, \frac{1}{2}) \) and suppose \( g_n \rightharpoonup g \) weakly in \( L^2(\mathbb{R}) \) and \( \lambda_n \to \infty \).

Then, there is a subsequence so that
\[ [\langle \lambda_n^{-1}\partial_x \rangle^{s-1/2} e^{-i\mu^2 t(\lambda_n^{-1}\partial_x)-1} P_{\leq \lambda_n^s} g_n](x) \to [e^{it\partial_x^2/2} g](x) \]
for almost every \((t, x) \in \mathbb{R} \times \mathbb{R}\).

\( \text{(iv)} \) Let \( s \geq \frac{1}{2} \). For \( \lambda_n \to \infty \) and fixed \( g \in L^2(\mathbb{R}) \),
\[ \lim_{n \to \infty} \|\langle \lambda_n^{-1}\partial_x \rangle^{s-1/2} e^{-i\mu^2 t(\lambda_n^{-1}\partial_x)-1} P_{\leq \lambda_n^s} g - e^{it\partial_x^2/2} g\|_{L^2_{t,x}} = 0. \]

Proof. Firstly we show that \( \text{[1]} \) Owing to Cantor’s diagonal argument, it is enough to consider almost everywhere convergence in \((t, x) \in [-L, L]^2\) for any \( L > 0 \). A simple calculation yields
\[ \limsup_{n \to \infty} \|\langle \partial_x \rangle^{1/4} \langle \partial_x \rangle^{1/4} \langle \lambda_n^{-1}\partial_x \rangle^{1/2} e^{-i\mu^2 t(\lambda_n^{-1}\partial_x)} g_n\|_{L^2_{t,x}([-L, L]^2)} \leq \lambda \limsup_{n \to \infty} \|\langle \partial_x \rangle g_n\|_{L^2_t([-L, L]^2)} \leq \lambda L^{1/2} \sup_{n \in \mathbb{N}} \|g_n\|_{H^1}. \]
Here, we note that the (space-time) Fourier support of \( w := e^{-i\mu^2 t(\lambda_n^{-1}\partial_x)} g_n \) is contained in \( \{t, \xi \in \mathbb{R}^2: |t| + \lambda_n^2(\lambda_n^{-1}\xi) = 0 \} \). Namely, \( |\partial_x^2 w| = \lambda_n^2(\lambda_n^{-1}\partial_x) w \) holds. Combining this with Rellich’s theorem, that is compactness of the embedding \( H^{1/4}([-L, L]^2) \hookrightarrow L^2([-L, L]^2) \), we obtain an \( L^2_{t,x} \) convergent subsequence which...
is also almost everywhere convergence. We confirm that the limit is independent of $L$. For all $F \in C_c^\infty(\mathbb{R} \times \mathbb{R})$, 

$$
\lim_{n \to \infty} \langle (\lambda_n^{-1} \partial_x)^{1/2} e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} g_n, F \rangle_{L^2_T}\n
= \lim_{n \to \infty} \left\langle g_n, \int_{\mathbb{R}} (\lambda_n^{-1} \partial_x)^{1/2} e^{i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} F dt \right\rangle_{L^2_T} 

= \left\langle g, \int_{\mathbb{R}} (\lambda_n^{-1} \partial_x)^{1/2} e^{i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} F dt \right\rangle_{L^2_T} 

= \langle (\lambda^{-1} \partial_x)^{1/2} e^{-i\lambda^2 t(\lambda^{-1} \partial_x)} g, F \rangle_{L^2_T}.
$$

For the proof of (ii), we may assume that $
\partial
$ $g$ is a Schwartz function because of Lemma 2.2. Lemma 2.1 implies 

$$
\| (\lambda_n^{-1} \partial_x)^{1/2} e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} g \|_{L^6_T} \lesssim T^{-1/6} \| (\lambda_n^{-1} \partial_x)^{3/2} g \|_{L^6_T}, 

\| (\lambda^{-1} \partial_x)^{1/2} e^{-i\lambda^2 t(\lambda^{-1} \partial_x)} g \|_{L^6_T} \lesssim T^{-1/6} \| (\lambda^{-1} \partial_x)^{3/2} g \|_{L^6_T}
$$

for all $T > 0$. Thus, we are left to control the region $|t| \leq T$. Since 

$$
\langle \lambda_n^{-1} \xi \rangle^{1/2} - \langle \lambda^{-1} \xi \rangle^{1/2} = \frac{\langle \lambda_n^{-1} \xi \rangle - \langle \lambda^{-1} \xi \rangle}{\langle \lambda_n^{-1} \xi \rangle^{1/2} + \langle \lambda^{-1} \xi \rangle^{1/2}} - \frac{\lambda_n - \lambda)(\lambda_n + \lambda)}{\lambda_n^2 \langle \lambda_n^{-1} \xi \rangle + \lambda^2 \langle \lambda^{-1} \xi \rangle},
$$

we have 

$$
\| (\lambda_n^{-1} \partial_x)^{1/2} e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} g - (\lambda^{-1} \partial_x)^{1/2} e^{-i\lambda^2 t(\lambda^{-1} \partial_x)} g \|_{L^6_T} 

\leq \| (\lambda_n^{-1} \partial_x)^{1/2} - (\lambda^{-1} \partial_x)^{1/2} \|_{L^6_T} \| e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} g \|_{L^6_T} 

+ \| (\lambda^{-1} \partial_x)^{1/2} (e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} - e^{-i\lambda^2 t(\lambda^{-1} \partial_x)}) g \|_{L^6_T} 

\lesssim \frac{\lambda_n - \lambda)(\lambda_n + \lambda)}{\lambda_n^2 \lambda^2} \| g \|_{H^2} + \frac{\lambda_n - \lambda)(\lambda_n + \lambda)}{\lambda_n^2 + \lambda^2} \{ (\lambda_n^2 + \lambda^2) \| g \|_{L^2} + \| g \|_{H^2} \}
$$

On the other hand, by Lemma 2.2 

$$
\| (\lambda_n^{-1} \partial_x)^{1/2} e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} g \|_{L^6_T L^{10}_x} + \| (\lambda^{-1} \partial_x)^{1/2} e^{-i\lambda^2 t(\lambda^{-1} \partial_x)} g \|_{L^6_T L^{10}_x} \lesssim \lambda \| g \|_{H^{11/10}}.
$$

Interpolating those two estimates, by $(L^n_T L^2_x, L^n_T L^{10}_x)^{1/6} = L^n_{T,x}$, we obtain 

$$
\lim_{n \to \infty} \| (\lambda_n^{-1} \partial_x)^{1/2} e^{-i\lambda_n^2 t(\lambda_n^{-1} \partial_x)} g - (\lambda^{-1} \partial_x)^{1/2} e^{-i\lambda^2 t(\lambda^{-1} \partial_x)} g \|_{L^6_T L^{10}_T} = 0
$$

for each fixed $T > 0$.

We consider the case (iii). As in the proof of (i) we may restrict the range of $(t,x)$ to $[-L,L]^2$. Since 

$$
\langle \xi \rangle^{s-1/2} - 1 = \left( s - \frac{1}{2} \right) \xi^2 \int_0^1 a(\xi)^{s-5/2} da,
$$
owing to Lemma 2.2, we have
\[
\|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} \left( (\lambda_n^{-1} \partial_x)^{s-1/2} - 1 \right) g_n \|_{L^2_{t,x}} \leq \lambda_n^{2(\theta-1)} \|g_n\|_{L^2} \to 0
\]  
(2.5)
as \( n \to \infty \). It suffices to show that there is a subsequence so that
\[
[e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} g_n](x) \to [e^{it\partial_x^2/2}g](x)
\]
for almost every \((t, x) \in [-L, L]^2\). By
\[
\lambda_n^2[(\lambda_n^{-1} \xi) - 1] = \frac{\xi^2}{(\lambda_n^{-1} \xi) + 1} = \frac{1}{2} \xi^2 + \frac{\xi^2}{2((\lambda_n^{-1} \xi) + 1)} = \frac{1}{2} \xi^2 - \frac{\lambda_n^{-2} \xi^4}{2((\lambda_n^{-1} \xi) + 1)^2},
\]
we deduce that for \( \theta < \frac{1}{2} \),
\[
\|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} - e^{it\partial_x^2/2}P_{\leq \lambda_n^6}\|_{L^2_{t} \to L^2_{t}} \to 0
\]
as \( n \to \infty \). Thus, it reduces to observe that an \( L^2([-L, L]^2) \)-convergence of a subsequence of \( e^{it\partial_x^2/2}P_{\lambda_n^6} g_n \). We recall the local smoothing estimate for the Schrödinger equation (see [1, 17, 20]):
\[
\int_{\mathbb{R}} \int_{[-L, L]} \|((\partial_x)^{1/2} e^{it\partial_x^2/2} f)(x)\|^2_{L^2_2} dx dt \lesssim L \|f\|^2_{L^2_2},
\]
which implies
\[
\|\langle \partial_x \rangle^{1/8} \langle \partial_x \rangle^{1/4} e^{it\partial_x^2/2} g_n\|_{L^2_{t,x}([-L, L]^2)} \lesssim L^{1/2} \|g_n\|_{L^2_t}.
\]
Rellich’s theorem, or compactness of the embedding \( H^{1/8}([-L, L]^2) \hookrightarrow L^2([-L, L]^2) \), we obtain an \( L^2_{t,x} \) convergent subsequence.

Finally, we prove [iv]. By [46], it suffices to show that
\[
\lim_{n \to \infty} \|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} g - e^{it\partial_x^2/2} g\|_{L^2_{t,x}} = 0
\]
By Lemma 2.2 and the Strichartz estimates for the Schrödinger equation, it suffices to treat the case where \( g \) is a Schwartz function. Lemma 2.1 and the dispersive estimate for the Schrödinger equation imply
\[
\|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} g\|_{L^2_{t,x}(|t| \geq T)} + \|e^{it\partial_x^2/2} g\|_{L^6_{t,x}(|t| \geq T)} \lesssim T^{-1/6} \|g\|_{L^6_{t,x}}
\]
for all \( T > 0 \). Thus we are left to control the region \(|t| \leq T\). Note that from (2.6)
\[
\|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} g - e^{it\partial_x^2/2} g\|_{L^6_{t,x}(|t| \leq T)} \leq \|(e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)} - e^{it\partial_x^2/2})P_{\leq \lambda_n^6} g\|_{L^6_{t} L^2_{x}(|t| \leq T)} + \|e^{it\partial_x^2/2} P_{\geq \lambda_n^6} g\|_{L^6_{t} L^2_{x}(|t| \leq T)} \lesssim (\lambda_n^{-2} T + \lambda_n^{-4}) \|g\|_{H^2_1}.
\]
On the other hand, by Lemma 2.2 and the Strichartz estimates for the Schrödinger equation,
\[
\|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} g\|_{L^6_{t} L^{10}_{x}} + \|e^{it\partial_x^2/2} g\|_{L^6_{t} L^{10}_{x}} \lesssim \|g\|_{H^{11/10}_{t}}.
\]
Interpolating those two estimates, by \( (L^6_{t} L^2_{x}, L^6_{t} L^{10}_{x})_{5/6} = L^6_{t,x} \), we obtain
\[
\lim_{n \to \infty} \|e^{-i\lambda_n^2 t((\lambda_n^{-1} \partial_x) - 1)}P_{\leq \lambda_n^6} g - e^{it\partial_x^2/2} g\|_{L^6_{t,x}(|t| \leq T)} = 0
\]
for each fixed \( T > 0 \). \qed
3. Local theory

We summarize a well-posedness result for (1.1), which is equivalent to (2.1).

**Proposition 3.1.** Let \( s \geq s_0 > \frac{1}{2} \) and \( v_0 \in H^s(\mathbb{R}) \). Then there exists a unique maximal lifespan solution \( v : I \times \mathbb{R} \to \mathbb{C} \) to (2.1) with \( v(0) = v_0 \). Furthermore, the following hold:

- If \( T = \sup I \) is finite, then \( \|v(t)\|_{H^{s_0}} \to \infty \) as \( t \to T \).
- The energy and momentum are finite and conserved if \( s \geq 1 \).
- If \( \|v_0\|_{H^s} \) is sufficiently small, then \( v \) is global and
\[
\|\langle \partial_x \rangle^{s+3/2r-3/4} v\|_{L^2_t(\mathbb{R};L^4_x(\mathbb{R}))} \lesssim \|v_0\|_{H^s}
\]
for any admissible pair \((q,r)\). Moreover, there exists \( v_+ \in H^s(\mathbb{R}) \) such that
\[
\lim_{t \to \infty} \|v(t) - e^{-it\langle \partial_x \rangle} v_+\|_{H^s} = 0.
\]

For each \( v \in H^s(\mathbb{R}) \), there exists a unique solution \( v \) to (2.1) in a neighborhood of \(+\infty\) satisfying (3.1). In either case,
\[
E(v(t)) = \frac{1}{2} \|v\|_{H^1}^2,
\]
provided that \( s \geq 1 \). Similar statements holds backward in time.

- Let \( J \subset \mathbb{R} \) and assume that \( \|\Re v\|_{L^\infty_t(J;L^\infty_x(\mathbb{R}))} + S_J(\langle \partial_x \rangle^{s-1/2} v) < L \). We have
\[
\|\langle \partial_x \rangle^{s+3/2r-3/4} v\|_{L^2_t(J;L^4_x(\mathbb{R}))} \lesssim_{L,s,q,r} \|v_0\|_{H^s}
\]
for any admissible pair \((q,r)\).

This proposition is essentially proved by Nakamura and Ozawa [15]. Hence, we omit the proof here.

We use the following stability theory, which is a consequence of the well-posedness result (Proposition 3.1). The proof follows from minor modifications of that for the nonlinear Schrödinger equation, and hence we omit the details. For the proof, see [10] [11] [19] for example.

**Proposition 3.2.** Let \( s > \frac{1}{2} \) and let \( I \) be an interval and \( \tilde{v} \) be a solution to
\[
-\tilde{\partial}_t \tilde{v} + \langle \partial_x \rangle \tilde{v} + \langle \partial_x \rangle^{-1} \tilde{N}(\Re \tilde{v}) + e_1 + e_2 = 0.
\]
Assume that
\[
\|\tilde{v}\|_{L^\infty_t H^s_x(I \times \mathbb{R})} \leq M \quad \text{and} \quad \|\langle \partial_x \rangle^{s-1/2} \Re \tilde{v}\|_{L^6_t(I \times \mathbb{R})} \leq L
\]
for some positive constants \( M \) and \( L \). Let \( t_0 \in I \) and let \( v_0 \) satisfy the condition
\[
\|v_0 - \tilde{v}(t_0)\|_{H^s_x} \leq M'
\]
for some positive constant \( M' \). If
\[
\|\langle \partial_x \rangle^{s-1/2} e^{-i(t-t_0)\langle \partial_x \rangle} (v_0 - \tilde{v}(t_0))\|_{L^6_t(I \times \mathbb{R})} \leq \varepsilon,
\]
\[
\|\langle \partial_x \rangle^{s+1/2} e_1\|_{L^6_t(I \times \mathbb{R})} + \|e_2\|_{L^1_t H^s_x(I \times \mathbb{R})} \leq \varepsilon
\]
for \( 0 < \varepsilon < \varepsilon_1 = \varepsilon_1(L,M,M') \), then there exists a unique solution \( v \) to (2.1) with initial data \( v_0 \) at time \( t_0 \).

Furthermore, the solution \( v \) satisfies
\[
\|\langle \partial_x \rangle^{s-1/2} (v-\tilde{v})\|_{L^6_t(I \times \mathbb{R})} \leq \varepsilon C(L,M,M'), \quad \|v-\tilde{v}\|_{L^\infty_t H^s_x(I \times \mathbb{R})} \leq M'C(L,M,M').
\]
Next, we observe behaviors of the boosted solutions $u \circ L_{\nu}$.

**Lemma 3.3.** Given $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, there is an $\epsilon > 0$ and a local solution $u$ to (1.1) with $(u(0), u_t(0)) = (u_0, u_1)$ and defined in the space time region

$$\Omega := \{(t, x) : |t| - \epsilon|x| < \epsilon\}. $$

Moreover,

$$\|u\|_{L^1_t L^\infty_x(\Omega)} := \|1_{\Omega} u\|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R})} < \infty \quad \text{for each admissible pair } (q, r),$$

(3.2)

$$\|u\|_{L^\infty_t L^2_x(\Omega)} := \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} 1_{\Omega}(t, x) \left\{|u_t(t, x)|^2 + |u_x(t, x)|^2 + |u(t, x)|^2\right\} \mathrm{d}x < \infty,$$

(3.3)

$$\lim_{R \to \infty} \sup_{|t| < \epsilon R} \int_{|x| > R} \left\{|u_t(t, x)|^2 + |u_x(t, x)|^2 + |u(t, x)|^2\right\} \mathrm{d}x = 0.$$

The solution $u$ with these properties is unique.

Although the proof for this lemma follows from a minor modification of that of Corollary 3.5 in [11], we will use a similar argument later in the proof of Proposition 3.1, hence we give a proof this lemma.

**Proof.** Proposition 3.4 shows that there exist $T_0 = T_0(u_0, u_1) > 0$ and a unique local solution $u$ to (1.1) having finite Strichartz norms on $[-T_0, T_0] \times \mathbb{R}$. Let $\varphi$ denote a smooth cutoff function with $\varphi(x) = 1$ for $|x| \geq 1$ and $\varphi(x) = 0$ for $|x| \leq \frac{1}{2}$.

There exists $R_0 > 0$ such that

$$\int_{\mathbb{R}} \left\{\varphi\left(\frac{x}{R_0}\right) u_1(x) \right\}^2 + \left|\partial_x \left[\varphi\left(\frac{x}{R_0}\right) u_0(x)\right]\right|^2 + \left|\varphi\left(\frac{v}{R_0}\right) u_0(x)\right|^2 \mathrm{d}x$$

is sufficiently small. Then, by Proposition 3.4, there is a global solution $\tilde{u}$ to (1.1) with $(\tilde{u}(0, x), \tilde{u}_t(0, x)) = (\varphi(\frac{v}{R_0}) u_0(x), \varphi(\frac{v}{R_0}) u_1(x))$. From uniqueness and finite speed of propagation, $\tilde{u}$ is an extension of $u$ to $\{(t, x) : |t| - |x| > R_0\}$. Choosing $\epsilon < \frac{T_0}{1 + T_0 + T_0}$, we get a solution $u$ to (1.1) on $\Omega$.

In what follows, we show (3.3). Let $\tilde{u}^{\tilde{R}}$ be a solution (1.1) with $(\tilde{u}^{\tilde{R}}(0, x), \tilde{u}_t^{\tilde{R}}(0, x)) = (\varphi(\frac{v}{\tilde{R}}) u_0(x), \varphi(\frac{v}{\tilde{R}}) u_1(x))$. Then, from small data theory in Proposition 3.1,

$$\lim_{R \to \infty} \left(\|\tilde{u}^{\tilde{R}}\|_{L^\infty_t H^1_x} + \|\partial_t \tilde{u}^{\tilde{R}}\|_{L^2_t L^2_x}\right) = 0.$$

We note that $u$ and $\tilde{u}^{\tilde{R}}$ agree on $|x| - |t| > \tilde{R}$. By taking $\tilde{R} < (1 - \epsilon) R$ for $R > 0$, we get $\{t| < \epsilon R, |x| > R\} \subset \{|x| - |t| > \tilde{R}\}$ and (3.3).

**Remark 3.4.** From the above proof, we find the following:

- Since a global solution exists, we may take any $0 < \epsilon < 1$.
- We can construct a solution $u$ to the linear Klein-Gordon equation with initial data $(u_0, u_1)$ satisfying (3.3).

Let $P(u(t))$ denote the momentum defined by

$$P(u(t)) := - \int_{\mathbb{R}} \partial_t u(t, x) \partial_x u(t, x) \mathrm{d}x.$$

**Corollary 3.5.** Under the same assumption as in Lemma 3.3, for $\frac{|v|}{|\nu|} < \epsilon$, we have the following:

(i) $(u \circ L_{\nu})(t, x)$ is a strong solution to (1.1) on $(-\epsilon, \epsilon) \times \mathbb{R}$.
(ii) $\nu \mapsto (u \circ L_\nu, \partial_t[u \circ L_\nu])(0, \cdot)$ is continuous with values in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

(iii) Einstein’s relation holds:

$$\{E(u \circ L_\nu, \partial_t[u \circ L_\nu]), P(u \circ L_\nu)\} = L_\nu^{-1}(E(u, u_t), P(u)).$$

In particular,

$$E(u \circ L_\nu, \partial_t[u \circ L_\nu])^2 - P(u \circ L_\nu)^2$$

is independent of $t$ and $\nu$.

Proof. Let $(t, x) = L_\nu(t, x) = (\langle \nu \rangle t - \nu x, \langle \nu \rangle x - vt)$. From $t = \frac{\nu x}{\langle \nu \rangle}$, for $\frac{|\nu|}{\langle \nu \rangle} < \varepsilon$, a calculation shows

$$|t| - \varepsilon|x| < (\langle \nu \rangle t - \nu x) - \left| \nu x - \frac{\nu^2}{\langle \nu \rangle} t \right| \leq |t|.$$ 

Thus, we have

$$L_\nu((-\varepsilon, \varepsilon) \times \mathbb{R}) \subseteq \Omega.$$ 

Since $L_\nu$ preserves spacetime volume, by Lemma 3.3 and Sobolev’s embedding,

$$\int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}} N(u \circ L_\nu)(t, x) dx dt \leq \int_{\Omega} N(u(t, x)) dx dt$$

$$\leq \|u\|_{L^6_t L^2_x(\Omega)}^6 \sum_{l=3}^{\infty} \delta_l \|u\|_{L^{2(l-6)}_t H^l_x(\Omega)} < \infty.$$ 

From $\partial_t^2 (u \circ L_\nu) - \partial_\nu^2 (u \circ L_\nu) = (u_{tt} - u_{xx}) \circ L_\nu$, this shows that $u \circ L_\nu$ is a distribution solution to (1.1) in $(-\varepsilon, \varepsilon) \times \mathbb{R}$.

We will prove that

$$(t, \nu) \mapsto (u \circ L_\nu, \partial_t[u \circ L_\nu])(t, \cdot)$$

is a continuous function from $\left\{(t, \nu) : |t| < \varepsilon, \frac{|\nu|}{\langle \nu \rangle} < \varepsilon \right\}$ to $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. If we obtain this continuity, then $u \circ L_\nu$ belongs to $C((-\varepsilon, \varepsilon); H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, i.e., it is a strong solution, and we get (i) and (ii).

Let $u^{\text{lin}}$ denote the linear solution to the Klein-Gordon equation with the initial data $(u_0, u_1)$ at zero, namely

$$u^{\text{lin}}(t, x) := \cos(t \langle \partial_x \rangle)u_0(x) + \langle \partial_x \rangle^{-1} \sin(t \langle \partial_x \rangle)u_1(x).$$

Let $\tilde{u} = u - u^{\text{lin}}$ denote the difference. From (v) in Lemma 2.3

$$u^{\text{lin}} \circ L_\nu = \frac{1}{2} \left\{ e^{it \langle \partial_x \rangle} L_\nu^{-1} + e^{-it \langle \partial_x \rangle} L_\nu \right\} u_0 + \frac{1}{2} \left\{ e^{it \langle \partial_x \rangle} L_\nu^{-1} - e^{-it \langle \partial_x \rangle} L_\nu \right\} \langle \partial_x \rangle^{-1} u_1.$$ 

By (iii) in Lemma 2.3 the mapping

$$(t, \nu) \mapsto (u^{\text{lin}} \circ L_\nu, \partial_t[u^{\text{lin}} \circ L_\nu])(t, \cdot)$$

is continuous from $\left\{(t, \nu) : |t| < \varepsilon, \frac{|\nu|}{\langle \nu \rangle} < \varepsilon \right\}$ to $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Next, we consider the effect of the Lorentz boosts on $\tilde{u}$. Firstly, we note that $\tilde{u}$ satisfies

$$\tilde{u}_{tt} - \tilde{u}_{xx} + \tilde{u} = -\Delta \tilde{u}, \quad \tilde{u}(0, x) = \tilde{u}_t(0, x) = 0.$$ 

By Lemma 2.2, Lemma 3.3 and $\tilde{u} = u - u^{\text{lin}}$, we get

$$\|\tilde{u}\|_{L^6_t L^2_x(\Omega)} + \|\partial_t \tilde{u}\|_{L^{2(l-6)}_t L^l_x(\Omega)} + \|\partial_x \tilde{u}\|_{L^{2(l-6)}_t L^l_x(\Omega)} < \infty.$$
for each admissible pair \((q, r)\). From Lemma 3.3 and Remark 3.4, we also have

\[
\lim_{R \to \infty} \sup_{|t| < R} \int_{|x| > R} \left\{ |\tilde{u}_t(t, x)|^2 + |\tilde{u}_x(t, x)|^2 + |\tilde{u}(t, x)|^2 \right\} dx = 0.
\]

Let \(\tilde{T}\) be the stress-energy tensor for the linear Klein-Gordon equation with respect to \(\tilde{u}\), which has the components

\[
\tilde{T}^{00} := \frac{1}{2}|\tilde{u}_t|^2 + \frac{1}{2}|\tilde{u}_x|^2 + \frac{1}{2}|\tilde{u}|^2, \quad \tilde{T}^{11} := \tilde{T}^{00} - |\tilde{u}_t|^2 = \frac{1}{2}|\tilde{u}_t|^2 + \frac{1}{2}|\tilde{u}_x|^2 - \frac{1}{2}|\tilde{u}|^2.
\]

We also define \(\tilde{p} = (\tilde{p}^0, \tilde{p}^1)\) by

\[
\tilde{p}^0 := \langle \nu \rangle \tilde{T}^{00} + \nu \tilde{T}^{10}, \quad \tilde{p}^1 := \langle \nu \rangle \tilde{T}^{01} + \nu \tilde{T}^{11}.
\]

From \(\partial_t = \langle \nu \rangle \partial_t - \nu \partial_x\) and \(\partial_x = \langle \nu \rangle \partial_x - \nu \partial_t\),

\[
(\langle \nu \rangle \tilde{p}^0 + \nu \tilde{p}^1) \circ L_{\nu} = (\langle \nu \rangle)^2 \tilde{T}^{00} \circ L_{\nu} + 2 \langle \nu \rangle \nu \tilde{T}^{01} \circ L_{\nu} + \nu^2 \tilde{T}^{11} \circ L_{\nu} = \frac{1}{2} \left[ [t \circ \tilde{u}_t \circ L_{\nu} + \langle \nu \rangle \tilde{u}_t \circ L_{\nu}]^2 + [t \circ \tilde{u}_x \circ L_{\nu} + \langle \nu \rangle \tilde{u}_x \circ L_{\nu}]^2 \right] dx.
\]

Since

\[
L_{\nu}(t, \mathbb{R}) = \{ (\langle \nu \rangle t - \nu x, \langle \nu \rangle x - \nu t) : x \in \mathbb{R} \} = \left\{ \left( \frac{t - \nu y}{\langle \nu \rangle}, y \right) : y \in \mathbb{R} \right\},
\]

a tangent vector of \(L_{\nu}(t, \mathbb{R})\) is \((-\nu, \langle \nu \rangle)\). Hence, (3.7)

\[
\int_{L_{\nu}(t, \mathbb{R})} (\tilde{p}^1 dx + \tilde{p}^0 dy) = \int_{\mathbb{R}} (\langle \nu \rangle \tilde{p}^0 + \nu \tilde{p}^1) \circ L_{\nu}(t, x) dx = \frac{1}{2} \int_{\mathbb{R}} \left[ |\partial_t(\tilde{u} \circ L_{\nu})|^2 + |\partial_x(\tilde{u} \circ L_{\nu})|^2 + |\tilde{u} \circ L_{\nu}|^2 \right] (t, x) dx.
\]

For fixed \((t_0, \nu_0)\) with \(|t_0| < \varepsilon\) and \(\frac{|s|}{\langle \nu \rangle} < \varepsilon\), we set

\[
\Omega_{t, \nu} := \left\{ (s, y) : \frac{t_0 - \nu_0 y}{\langle \nu \rangle} < s < \frac{t - \nu y}{\langle \nu \rangle} \right\} \cup \left\{ (s, y) : \frac{t - \nu y}{\langle \nu \rangle} < s < \frac{t_0 - \nu_0 y}{\langle \nu \rangle} \right\}
\]

for \((t, \nu) \in \mathbb{R} \times \mathbb{R}\). Then,

\[
\partial \Omega_{t, \nu} = L_{\nu}(t, \mathbb{R}) \cup L_{\nu_0}(t_0, \mathbb{R}).
\]

From \(|t_0| < \varepsilon\) and \(\frac{|s|}{\langle \nu \rangle} < \varepsilon\), for \(|t - t_0| < 1\) and \(|\nu_0 - \nu| < 1\), we have \(\Omega_{t, \nu} \subset \Omega\). Indeed, for \((s, y) \in \Omega_{t, \nu}\),

\[
|s| - \varepsilon |y| < \max \left\{ \frac{|t_0 - \nu_0 y| - \nu_0 |y|}{\langle \nu \rangle}, \frac{|t - \nu y| - \nu |y|}{\langle \nu \rangle} \right\} < \max(|t_0|, |t|) < \varepsilon.
\]

By using the mollification technique, we may assume that \(\tilde{p}\) is smooth. We apply Green’s theorem to \(\tilde{p}\) on the region \(\Omega_{t, \nu}\). For \(R > 0\), let \(\psi_R(s, y) = \sigma\left( \frac{|s| + |y|}{R} \right)\), where
\[ \sigma \] is the cut-off function defined in (1.1). Then, by (3.6) and (3.7),
\[
\left| \frac{1}{2} \int_R \left[ |\partial_t (\bar{u} \circ L_v)|^2 + |\partial_x (\bar{u} \circ L_v)|^2 + |\bar{u} \circ L_v|^2 \right] (t, x) dx \right|
\]
\[
- \frac{1}{2} \int_R \left[ |\partial_t (\bar{u} \circ L_v)|^2 + |\partial_x (\bar{u} \circ L_v)|^2 + |\bar{u} \circ L_v|^2 \right] (t_0, x) dx \right|
\]
\[
= \left| \int_{L_v(t, R)} (-\bar{p}^1 dx + \bar{p}^0 dy) - \int_{L_v(t_0, R)} (-\bar{p}^1 dx + \bar{p}^0 dy) \right|
\]
\[
= \left| \int_{\partial \Omega_{t, v}} (-\bar{p}^1 \nu dx + \bar{p}^0 \nu dy) \right|
\]
\[
= \lim_{R \to \infty} \int_{\partial \Omega_{t, v}} (-\bar{p}^1 \nu_R dx + \bar{p}^0 \psi_R dy)
\]
\[
\leq \limsup_{R \to \infty} \int_{\Omega_{t, v}} \{ |\psi_R \nabla_{s, y} \cdot \bar{p}||(s, y) + |\nu \cdot \nabla_{s, y} \psi_R||(s, y) \} dy ds
\]
\[
\leq \int_{\Omega_{t, v}} |\langle \nabla_{s, y} \cdot \bar{p} ||\psi ||s, y \rangle| dy ds + \limsup_{R \to \infty} \frac{1}{R} \int_{-\varepsilon R}^{\varepsilon R} \int_{|y| \sim R} |\langle \nabla_{s, y} \psi ||s, y \rangle| dy ds
\]
\[
\leq \int_{\Omega_{t, v}} |\langle \nabla_{s, y} \cdot \bar{p} ||\psi ||s, y \rangle| dy ds.
\]

Since
\[
\nabla_{t, x} \cdot \bar{p} = \partial_t \bar{p}^0 + \partial_x \bar{p}^1
\]
\[
= \langle \nu \rangle (\bar{u}_{tt} \bar{u}_t + \bar{u}_{tx} \bar{u}_x + \bar{u}_t \bar{u}_x) - \nu \langle \bar{u}_{tt} \bar{u}_x + \bar{u}_{tx} \bar{u}_t \rangle
\]
\[
+ \nu \langle \bar{u}_{tx} \bar{u}_t + \bar{u}_{xx} \bar{u}_x - \bar{u}_x \bar{u}_t \rangle
\]
\[
= -\mathcal{N}(u) (\nu \bar{u}_t - \nu \bar{u}_x),
\]

by Hölder’s inequality, we get
\[
\int_{\Omega_{t, v}} |\nabla_{s, y} \cdot \bar{p}| dy ds \leq \int_{\Omega_{t, v}} |\mathcal{N}(u(s, y)) \nabla_{s, y} \bar{u}(s, y)| dy ds
\]
\[
\leq \langle \nu \rangle ||u||_{L_2(L^6_y(\Omega_{t, v}))} ||\nabla_{s, y} \bar{u}||_{L_\infty^2(\Omega)} \sum_{l=2}^{\infty} \frac{1}{l!} ||u||_{L_\infty^2(\Omega)}^{2l-4}.
\]

From Lemma 3.3 (3.8), and Lebesgue’s dominated convergence theorem, this quantity on the right hand side as above goes to zero as \((t, \nu) \to (t_0, \nu_0)\) because \(\Omega_{t, v} \to \emptyset\). This shows the desired continuity.

For the proof of (iii), we use the stress-energy tensor associated to the nonlinear Klein-Gordon equation:
\[
\mathcal{T}^{00} := \frac{1}{2} |u_t|^2 + \frac{1}{2} |u_x|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} \nabla(u), \quad \mathcal{T}^{01} := \mathcal{T}^{10} = -u_t u_x,
\]
\[
\mathcal{T}^{11} := u_x^2 - \mathcal{T}^{00} + |u_t|^2 = \frac{1}{2} |u_t|^2 + \frac{1}{2} |u_x|^2 - \frac{1}{2} |u|^2 - \frac{1}{2} \nabla(u).
\]

Since \(u\) is a solution to (1.1),
\[
\partial_t \mathcal{T}^{00} + \partial_x \mathcal{T}^{01} = 0
\]
for all \( \alpha \in \{0, 1\} \). We also define \( p = (p^0, p^1) \) and \( q = (q^0, q^1) \) by
\[
  p^0 := \langle \nu \rangle T^{00} + \nu T^{10}, \quad p^1 := \langle \nu \rangle T^{01} + \nu T^{11},
\]
\[
  q^0 := \nu T^{00} + \langle \nu \rangle T^{10}, \quad q^1 := \nu T^{01} + \langle \nu \rangle T^{11}.
\]
Then,
\[
  \nabla_{t,x} \cdot p = (\nu)(\partial_t T^{00} + \partial_x T^{01}) + \nu(\partial_t T^{10} + \partial_x T^{11}) = 0,
\]
\[
  \nabla_{t,x} \cdot q = \nu(\partial_t T^{00} + \partial_x T^{01}) + \langle \nu \rangle(\partial_t T^{10} + \partial_x T^{11}) = 0.
\]
Put
\[
  \Xi_{t,\nu} := \left\{(s,y) : 0 < s < \frac{t - \nu y}{\nu} \right\} \cup \left\{(s,y) : \frac{t - \nu y}{\nu} < s < 0 \right\}.
\]
Then, \( \Xi_{t,\nu} \subset \Omega \) for \( |t| < \varepsilon \) and \( \frac{|\nu|}{\nu} < \varepsilon \),
\[
  \partial \Xi_{t,\nu} = L_\nu(t, \mathbb{R}) \cup (\{0\} \times \mathbb{R}).
\]
Hence, the same calculation as in (3.7) yields
\[
  \int_{\partial \Xi_{t,\nu}} (-p^1 dx + p^0 dy) = \int_{\mathbb{R}} \left( (\nu)p^0 + \nu p^1 \right) \circ L_\nu(t, x) dx - \int_{\mathbb{R}} p^0(t, x) dx
\]
\[
  = E(u \circ L_\nu, \partial_t[u \circ L_\nu]) - \{\nu\} E(u, u_t) + \nu P(u),
\]
\[
  \int_{\partial \Xi_{t,\nu}} (-q^1 dx + q^0 dy) = \int_{\mathbb{R}} \left( (\nu)q^0 + \nu q^1 \right) \circ L_\nu(t, x) dx - \int_{\mathbb{R}} q^0(t, x) dx
\]
\[
  = P(u \circ L_\nu) - \{\nu\} E(u, u_t) + \langle \nu \rangle P(u).
\]
Applying Green’s theorem, by (3.3) and (3.8), we obtain
\[
  \left| E(u \circ L_\nu, \partial_t[u \circ L_\nu]) - \{\nu\} E(u, u_t) + \nu P(u) \right|
\]
\[
  = \lim_{R \to \infty} \int_{\partial \Xi_{t,\nu}} (-p^1 \psi_R dx + p^0 \psi_R dy) \leq \lim_{R \to \infty} \sup \int_{\Xi_{t,\nu}} |p \cdot \nabla_{s,y} \psi_R|(s, y) dy ds = 0.
\]
Similarly, we have
\[
  P(u \circ L_\nu) = \nu E(u, u_t) + \langle \nu \rangle P(u).
\]
This completes the proof. \( \square \)

4. Refinements of the Strichartz inequality

The goal of this section is to show an inverse Strichartz inequality (Theorem 4.3), which is an essential ingredient in the concentration compactness argument. For the exponential-type nonlinearity, we have to consider the initial data in \( H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) with \( s > \frac{1}{2} \). Accordingly, we need to consider \( S_{\mathbb{R}}((\partial_x)^{s-1/2}f) \) instead of \( S_{\mathbb{R}}(f) \).

**Lemma 4.1.** For \( f \in H^{1/2}(\mathbb{R}) \),
\[
  \|e^{-it(\partial_x)} f\|_{L^2_{t,x}}^2 \lesssim \sup_{N \in 2^{\mathbb{N}}} \|e^{-it(\partial_x)} P_N f\|_{L^2_{t,x}} \|f\|_{H^s_{t,x}}.
\]
Proof. We apply the Littlewood-Paley square function estimate and Hölder’s inequality to have
\[
\|e^{-it(\partial_x)} f\|_{L^6_t L^6_x}^6 \\
\sim \left\| \left( \sum_{\mathbb{N}^3 \in 2^0} |e^{-it(\partial_x)} P_N f|^2 \right)^{1/2} \right\|_{L^6_t L^6_x}^6 \\
\lesssim \sum_{N_1, N_2, N_3 \in 2^0} \int \int R \prod_{j=1}^3 |e^{-it(\partial_x)} P_{N_j} f|^2 dx dt \\
\lesssim \sum_{N_1, N_2, N_3 \in 2^0} \left( \prod_{j=1}^3 \|e^{-it(\partial_x)} P_{N_j} f\|_{L^6_t L^6_x} \right) \|e^{-it(\partial_x)} P_{N_1} f\|_{L^6_t L^6_x} \\
\times \prod_{k=2, 3} \|e^{-it(\partial_x)} P_{N_k} f\|_{L^{20/3}_t L^{5}_x}.
\]
By the Strichartz estimate (Lemma 2.22 and Schur’s test (see for example Theorem 275 in [3]), we get
\[
\|e^{-it(\partial_x)} f\|_{L^6_t}^6 \\
\lesssim \sup_{K \in 2^0} \|e^{-it(\partial_x)} P_K f\|_{L^6_t}^3 \sum_{N_1, N_2, N_3 \in 2^0, N_1 \leq N_2 \leq N_3} \|P_{N_1} f\|_{H^{3/5}_x} \|P_{N_2} f\|_{H^{9/20}_x} \|P_{N_3} f\|_{H^{9/20}_x} \\
\lesssim \sup_{K \in 2^0} \|e^{-it(\partial_x)} P_K f\|_{L^6_t}^3 \|f\|_{H^{1/2}_x} \sum_{N_1, N_2 \in 2^0, N_1 \leq N_2} \frac{N_1^{1/10}}{N_2^{1/10}} \|P_{N_1} f\|_{H^{1/2}_x} \|P_{N_2} f\|_{H^{1/2}_x} \\
\lesssim \sup_{K \in 2^0} \|e^{-it(\partial_x)} P_K f\|_{L^6_t}^3 \|f\|_{H^{1/2}_x}^3.
\]
We need to divide each dyadic interval into positive and negative intervals.

For \( N \in 2^1 \), we define by \( P^+_N \) and \( P^-_N \) the Fourier multiplier with the symbol \( \sigma(\xi) - \sigma(\frac{2^0}{\xi}) \) \( 1_{>0}(\xi) \) and \( \left( \sigma(\frac{\xi}{2}) - \sigma(\frac{2^0}{\xi}) \right) 1_{>0}(-\xi) \) respectively. For convenience, we set \( P^+_1 \) as \( P_1 \).

Lemma 4.2.
\[
\|e^{-it(\partial_x)} P_N f\|_{L^6_t}^6 \lesssim N^{5/2} \sup_{\pm} \|e^{-it(\partial_x)} P^\pm_N f\|_{L^6_t} \|P_N f\|_{L^5_x}.
\]

Proof. The Strichartz estimate (Lemma 2.22) yields that
\[
\|e^{-it(\partial_x)} P_N f\|_{L^6_t}^6 = \|e^{-it(\partial_x)} P_N f\|_{L^{1+5}_t L^{5}_x}^{1+5} \\
\lesssim \sup_{\pm} \|e^{-it(\partial_x)} P^\pm_N f\|_{L^6_t} \times \left( N^{1/2} \|P_N f\|_{L^2} \right)^5 \\
\lesssim N^{5/2} \sup_{\pm} \|e^{-it(\partial_x)} P^\pm_N f\|_{L^6_t} \|P_N f\|_{L^5_x}^5.
\]
\]
By taking $\nu$ appropriately, the Fourier support of $L_\nu^{-1}P_\nu^+ f$ lies inside an interval centered at the origin.

**Lemma 4.3.** For $N \in 2^{10}$, let $\nu = \frac{5}{7} N$. Then,

$$\|L_\nu^{-1}P_\nu^+ f\|_{L^6} + \|L_\nu^{-1}P_\nu^- f\|_{L^6} \lesssim \|P_{\leq 2}\nu^{-1} f\|_{L^6}.$$

**Proof.** We only consider the estimate for $P_\nu^+$ and $N \in 2^N$ because the remaining cases are similarly handled. Set $\xi := \langle \nu \rangle \xi - \nu \langle \xi \rangle$. By supp $P_\nu^+ f \subset \left[ \frac{5}{7}, 2N \right]$ and

$$\bar{\xi} = \langle \nu \rangle \xi - \nu \langle \xi \rangle = \frac{(\xi + \nu)(\xi - \nu)}{\langle \nu \rangle \xi + \nu \langle \xi \rangle}$$

we get

$$|\bar{\xi}| \leq \frac{13}{2} \times \frac{3}{2} = \frac{39}{20} < 2$$

for $\xi \in \text{supp} \widehat{P_\nu^+} f$. Lemma 2.3(iii) yields that

$$\mathcal{F}[L_\nu^{-1}P_\nu^+ f](\xi) = \left( \sigma \left( \frac{\xi}{N} \right) - \sigma \left( \frac{2\xi}{N} \right) \right) 1_{\geq 0}(\xi) \widehat{f}(\xi)$$

$$= \left( \sigma \left( \frac{\xi}{N} \right) - \sigma \left( \frac{2\xi}{N} \right) \right) 1_{\geq 0}(\xi) \mathcal{F}[P_{\leq 2}\nu^{-1} f](\bar{\xi}).$$

We set

$$\rho(\bar{\xi}) = \left( \sigma \left( \frac{\xi}{N} \right) - \sigma \left( \frac{2\xi}{N} \right) \right) 1_{\geq 0}(\xi).$$

Then, a computation shows that

$$\frac{d^\alpha \rho}{d\xi^\alpha}(\bar{\xi}) = N^{-\alpha} \left( \frac{d^\alpha \sigma}{d\xi^\alpha} \left( \frac{\xi}{N} \right) - 2^\alpha \frac{d^\alpha \sigma}{d\xi^\alpha} \left( \frac{2\xi}{N} \right) \right) 1_{\geq 0}(\xi) \left( \frac{\xi}{N} \right)^\alpha$$

and

$$\sup_{\xi \in [N/2, 2N]} \left| \frac{d^\alpha \rho}{d\xi^\alpha}(\bar{\xi}) \right| \lesssim \alpha$$

for all $\alpha \in \mathbb{N}_0$. Thus, $\mathcal{F}^{-1}[\rho] \in L^1$. From $L_\nu^{-1}P_\nu^+ f = \mathcal{F}^{-1}[\rho] \ast (P_{\leq 2}\nu^{-1} f)$ and Young’s inequality, we have

$$\|L_\nu^{-1}P_\nu^+ f\|_{L^6} = \|\mathcal{F}^{-1}[\rho] \ast (P_{\leq 2}\nu^{-1} f)\|_{L^6} \leq \|\mathcal{F}^{-1}[\rho]\|_{L^1} \|P_{\leq 2}\nu^{-1} f\|_{L^6}$$

$$\lesssim \|P_{\leq 2}\nu^{-1} f\|_{L^6}.$$

We employ the following further decoupling, which is a consequence of the bilinear estimate proved by Tao [18] (see also [11, 12]).

**Lemma 4.4.** Assume that $f \in L^2(\mathbb{R})$ and supp $\widehat{f} \subset \{ \xi \in \mathbb{R} : |\xi| \leq 4 \}$. Then

$$\|e^{-it(|\partial_x|^2)} f\|_{L^6}^3 \lesssim \sup_Q \left( |Q|^{-1/5} \|e^{-it(|\partial_x|^2)} P_Q f\|_{L^{1.9}_x} \right) \|f\|_{L^2}^2.$$

Here, the supremum is take over all dyadic intervals with the length no more than eight and $P_Q f$ denotes the restriction operator (in $\xi$-space) of $f$ to $Q$. 

Theorem 4.5 (Inverse Strichartz inequality). Let \( \{f_n\} \in H^1(\mathbb{R}) \) and let \( s \in \left( \frac{1}{2}, \frac{11}{12} \right) \). Suppose that

\[
\lim_{n \to \infty} \|f_n\|_{H^s} = A \quad \text{and} \quad \lim_{n \to \infty} \| (\partial_x)^{s-1/2} e^{-it(\partial_x)} f_n \|_{L^6_{t,x}} = \varepsilon > 0.
\]

Then, by passing to a subsequence, there exist \( \phi \in L^2(\mathbb{R}) \), \( \{\lambda_n\} \subseteq \left[ \frac{1}{5}, \infty \right) \), \( \{\nu_n\} \subseteq \mathbb{R} \), and \( \{(t_n, x_n)\} \subseteq \mathbb{R} \times \mathbb{R} \) so that we have the following:

- \( \lambda_n \to \lambda_\infty \in \left[ \frac{1}{5}, \infty \right] \) and \( \nu_n \to \nu \) in \( \mathbb{R} \).
- \( \lambda_\infty < \infty \Rightarrow \phi \in H^1(\mathbb{R}) \).

By setting

\[
\phi_n := \begin{cases} 
T_{x_n} e^{it_n(\partial_x)} L_{\nu_n} D_{\lambda_n} \phi, & \text{if } \lambda_\infty < \infty, \\
T_{x_n} e^{it_n(\partial_x)} L_{\nu_n} P_{\leq \lambda_n^s} \phi, & \text{if } \lambda_\infty = \infty,
\end{cases}
\]

the following hold:

\[
\begin{align*}
(4.1) & \quad \lim_{n \to \infty} \left( \|f_n\|_{H^s}^2 - \|f_n - \phi_n\|_{H^s}^2 - \|\phi_n\|_{H^s}^2 \right) = 0, \\
(4.2) & \quad \lim_{n \to \infty} \|\phi_n\|_{H^1} \gtrsim \varepsilon \left( \frac{\varepsilon}{A} \right)^{\frac{11}{13}}, \\
(4.3) & \quad \lim_{n \to \infty} \sup \| (\partial_x)^{s-1/2} e^{-it(\partial_x)} (f_n - \phi_n) \|_{L^6_{t,x}} \leq \varepsilon \left[ 1 - c \left( \frac{\varepsilon}{A} \right) \right]^{1/6}, \\
(4.4) & \quad D_{\lambda_n}^{-1} L_{\nu_n}^{-1} T_{x_n} e^{it_n(\partial_x)} f_n \to \phi \text{ weakly in } \begin{cases} H^1(\mathbb{R}), & \text{if } \lambda_\infty < \infty, \\
L^2(\mathbb{R}), & \text{if } \lambda_\infty = \infty.
\end{cases}
\end{align*}
\]

Here, \( c \) and \( C \) are positive constants.

Proof. We write a subsequence with the same subscript as the original sequence. By Lemma 4.4, we can find dyadic numbers \( N_n \in 2^{\mathbb{N}_0} \) satisfying

\[
e^{2A} A^{-1} \lesssim \| (\partial_x)^{s-1/2} e^{-it(\partial_x)} P_N f_n \|_{L^6_{t,x}}.
\]

Applying Lemma 4.22, we have that there exists a sing \( \pm_n \in \{+, -\} \) such that

\[
e^{2A} A^{-1} \lesssim N_n^{5/2} \| (\partial_x)^{s-1/2} e^{-it(\partial_x)} P_{N_n} f_n \|_{L^6_{t,x}} (N_n^{s-3/2} A)^{A} \]

\[
\sim N_n^{6s-11/2} A^{5} \| e^{-it(\partial_x)} P_{N_n} f_n \|_{L^6_{t,x}}.
\]

Owing to \( s < \frac{11}{12} \), we get

\[
\varepsilon^{12} A^{-11} \lesssim \lim_{n \to \infty} \| e^{-it(\partial_x)} P_{N_n} f_n \|_{L^6_{t,x}}.
\]

On the other hand, Lemma 2.22 implies

\[
\| e^{-it(\partial_x)} P_{N_n} f_n \|_{L^6_{t,x}} \lesssim N_n^{-1/2} A,
\]

which concludes that \( N_n \gtrsim \left( \frac{1}{A} \right)^{24} \).

Set \( \tilde{\nu}_n := \pm_n \frac{5}{7} N_n \). Since the Lorentz boosts preserve the volume, by (v) in Lemmas 2.23, Lemma 4.3, and (4.3), we have

\[
e^{12A} A^{-11} \lesssim \lim_{n \to \infty} \| e^{-it(\partial_x)} P_{\tilde{\nu}_n} f_n \|_{L^6_{t,x}}
\]

\[
= \lim_{n \to \infty} \| e^{-it(\partial_x)} L_{\tilde{\nu}_n}^{-1} P_{\tilde{\nu}_n} f_n \|_{L^6_{t,x}}
\]

\[
\lesssim \lim_{n \to \infty} \| e^{-it(\partial_x)} P_{\leq 2 \tilde{\nu}_n} f_n \|_{L^6_{t,x}}.
\]

From (iv) in Lemma 2.3

\[
\| P_{\leq 2 \tilde{\nu}_n} f_n \|_{L^2} \lesssim \| L_{\tilde{\nu}_n}^{-1} f_n \|_{H^s} \lesssim \| f_n \|_{H^s} < A.
\]

\[
\| f_n \|_{H^s} < A.
\]
From (4.6), (4.7), and Lemma 4.4, there exists an interval $Q_n$ with the length no more than eight such that

$$ (\varepsilon^{12}A^{-11})^{3} \lesssim A^{2} \nu_{n}^{1/5} \| e^{-it(\partial_{x})} P_{Q_{n}} P_{\leq 2} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L_{t,x}^{\infty}}, $$

where $\nu_{n}^{-1} \in (0,8)$ is the length of $Q_{n}$ and $Q_{n}$ are in $|\xi| \leq 4$. By the $L^{p}$-boundedness of $P_{\leq 2}$, Lemma 2.3 and (4.7), we have

$$ \| e^{-it(\partial_{x})} P_{Q_{n}} P_{\leq 2} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L_{t,x}^{\infty}} \lesssim \| e^{-it(\partial_{x})} P_{Q_{n}} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L_{t,x}^{\infty}}^{3/5} \| e^{-it(\partial_{x})} P_{Q_{n}} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L_{t,x}^{2}}^{2/5} \lesssim A^{3/5} \| e^{-it(\partial_{x})} P_{Q_{n}} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L_{t,x}^{2}}^{2/5}. $$

Combining it with (4.8), we get

$$ \lambda_{n}^{-1/2} \varepsilon^{90} A^{-89} \lesssim \| e^{-it(\partial_{x})} P_{Q_{n}} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L_{t,x}^{\infty}}. $$

Therefore, there exists $(\tilde{\nu}_{n}, \tilde{\tau}_{n}) \in \mathbb{R} \times \mathbb{R}$ so that

$$ \lambda_{n}^{-1/2} \varepsilon^{90} A^{-89} \lesssim \| P_{Q_{n}} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{(-\tilde{\tau}_{n})}. $$

Let $\xi_{n}$ be the center of $Q_{n}$. Owing to $|\lambda_{n}^{-1}| \leq 8$ and $|\xi_{n}| \lesssim 1$, by passing to a subsequence, we have the limits $\lambda_{\infty} \in [1/\varepsilon, \infty]$ and $\xi_{\infty} \in \mathbb{R}$ respectively. By Lemma 2.3 (iv) and (vi),

$$ \| D_{\lambda_{n}}^{-1} L_{\xi_{n}}^{-1} T_{\tilde{\tau}_{n}}^{-1} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L^{2}} = \| L_{\xi_{n}}^{-1} T_{\tilde{\tau}_{n}}^{-1} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L^{2}} \lesssim \| \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{L^{2}} \lesssim \langle \tilde{\nu}_{n} \rangle \| f_{n} \|_{L^{2}} \lesssim \left( \frac{A}{\varepsilon} \right)^{24} A. $$

If $\lambda_{\infty} < \infty$, this sequence is also $H^{1}$-bounded because of $\| D_{\lambda_{n}}^{-1} f \|_{H^{1}} \lesssim \langle \lambda_{\infty} \rangle \| f \|_{H^{1}}$.

By passing to a subsequence, there exists $\phi \in L^{2}(\mathbb{R})$ satisfying

$$ \text{weak-limit } D_{\lambda_{n}}^{-1} L_{\xi_{n}}^{-1} T_{\tilde{\tau}_{n}}^{-1} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} = \phi \in \begin{cases} H^{1}(\mathbb{R}), & \text{if } \lambda_{\infty} < \infty, \\ L^{2}(\mathbb{R}), & \text{if } \lambda_{\infty} = \infty. \end{cases} $$

Let $h := \mathcal{F}^{-1} [1_{[-1/2,1/2]}]$. Lemma 2.3 (iv) implies that

$$ \lambda_{n}^{1/2} \| P_{Q_{n}} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{(-\tilde{\tau}_{n})} $$

$$ \lambda_{n}^{1/2} \| P_{\xi_{n} + [1/2,1]} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{(-\tilde{\tau}_{n})} $$

$$ \lambda_{n}^{1/2} \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi \xi_{n}} \tilde{h}(\lambda_{n}(\xi - \xi_{n})) \mathcal{F}[e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}} f_{n}](\xi) d\xi \right) $$

$$ \lambda_{n}^{1/2} \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi \xi_{n}} \tilde{h}(\lambda_{n}(\xi - \xi_{n})) \mathcal{F}[e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n}](\xi) d\xi \right) $$

(4.11)

Then, $\| h_{n} \|_{L^{2}} \lesssim 1$. From Lemma 2.4 by passing to a subsequence, we have the strong limit $h_{\infty} \in L^{2}(\mathbb{R}) \setminus \{0\}$. Hence, it follows from (4.9), (4.10), (4.11), and a duality argument that

$$ \| \phi \|_{L^{2}} \gtrsim \lim_{n \to \infty} \| (h_{n}, \phi)_{L^{2}} \| = \lim_{n \to \infty} \lambda_{n}^{1/2} \| P_{Q_{n}} e^{-i\tilde{\nu}_{n}(\partial_{x})} \tilde{L}_{\nu_{n}}^{-1} f_{n} \|_{(-\tilde{\tau}_{n})} \gtrsim \varepsilon \left( \frac{A}{\varepsilon} \right)^{89}. $$
which leads to $\phi \neq 0$. By \([\text{iii}]\) and \([\text{vi}]\) in Lemma \ref{lem:223}
\[
D_{\lambda_n}^{-1}L_{\xi_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}L_{\nu_n}^{-1} = D_{\lambda_n}^{-1}L_{\xi_n}^{-1}L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)} = D_{\lambda_n}^{-1}L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}
\]
where $(-t_n, -x_n) = L_{\nu_n}(-t_n, -x_n)$ and $\nu_n = \xi_n(\tilde{\nu}_n) + (\xi_n)\tilde{\nu}_n$. By \([\text{4.10}]\), we obtain \([\text{4.4}]\). Moreover, from $|\nu_n| \lesssim (\xi_n)(\tilde{\nu}_n) \lesssim N_n \lesssim (\frac{2}{\xi})^{24}$, by passing to a subsequence, we have the limit $\nu \in \mathbb{R}$ of $\{\nu_n\}$.

In the sequel, we only consider the case $\lambda_\infty = \infty$ because the case $\lambda_\infty < \infty$ is similarly handled. By \([\text{iv}]\) in Lemma \ref{lem:223}
\[
||\phi_n||_{H^1}^2 = ||L_{\nu_n}D_{\lambda_n}P_{\leq \lambda_n^6}\phi||_{H^1} \gtrsim (\nu_n)^{-1}||D_{\lambda_n}P_{\leq \lambda_n^6}\phi||_{H^1} \gtrsim (\nu_n)^{-1}||P_{\leq \lambda_n^6}\phi||_{L^2}.
\]
By \([\text{4.12}]\) and $|\nu_n| \lesssim (\frac{2}{\xi})^{24}$, we get
\[
\liminf_{n \to \infty} ||\phi_n||_{H^1} \gtrsim \left(\frac{\xi}{A}\right)^{24}||\phi||_{L^2} \gtrsim \xi \left(\frac{\xi}{A}\right)^{113},
\]
which shows \([\text{4.2}]\).

Next, we show \([\text{4.1}]\). A direct calculation yields
\[
||f_n||_{H^1}^2 - ||f_n - \phi_n||_{H^1}^2 - ||\phi_n||_{H^1}^2 = 2\langle f_n - \phi_n, \phi_n \rangle_{H^1},
\]
By \([\text{iv}]\) in Lemma \ref{lem:223}
\[
\langle f_n - \phi_n, \phi_n \rangle_{H^1} = \langle T_{x_n}^{-1}e^{-it_n(\partial_x)}f_n - L_{\nu_n}D_{\lambda_n}P_{\leq \lambda_n^6}\phi, L_{\nu_n}D_{\lambda_n}P_{\leq \lambda_n^6}\phi \rangle_{H^1}
\]
\[
= \langle L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}f_n - D_{\lambda_n}P_{\leq \lambda_n^6}\phi, m_1(-i\partial_x; -\nu_n)^{-1}D_{\lambda_n}P_{\leq \lambda_n^6}\phi \rangle_{H^1}
\]
\[
= \langle D_{\lambda_n}^{-1}L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}f_n - P_{\leq \lambda_n^6}\phi, \langle \lambda_n^{-1}\partial_x \rangle^2 m_1(-i\lambda_n^{-1}\partial_x; -\nu_n)^{-1}P_{\leq \lambda_n^6}\phi \rangle_{L^2}.
\]
Here, $m_1(\xi; \nu) = \frac{(\nu(\xi))}{(\xi)} = \frac{(\nu(\xi) - \nu(\xi))}{(\xi)}$ and $P_{\leq \lambda_n^6}\phi \to \phi$. \(\langle \lambda_n^{-1}\partial_x \rangle^2 m_1(-i\lambda_n^{-1}\partial_x; -\nu_n)^{-1}P_{\leq \lambda_n^6}\phi \to \langle \nu_\infty \rangle^{-1}\phi \) in $L^2(\mathbb{R})$ as $n \to \infty$. Therefore, by \([\text{4.4}]\), we get \([\text{4.1}]\).

Finally, we show \([\text{vii}]\). We note that \([\text{vii}]\) in Lemma \ref{lem:223} implies
\[
||e^{-it(\partial_x)}L_{\nu}\beta||_{L^\beta_{t,x}} = ||e^{-it(\partial_x)}f||_{L^\beta_{t,x}} = ||e^{-it(\partial_x)}f||_{L^\beta_{t,x}}.
\]
Making the change of variable $t = \lambda_n^2t$, $x = \lambda_n^2x$, we have
\[
||e^{-it(\partial_x)}f_n||_{L^\beta_{t,x}} = ||e^{-it(\partial_x)}f_n||_{L^\beta_{t,x}}
\]
\[
= ||(\lambda_n^{-1}I_{n}^{-1}e^{-it(\partial_x)}f_n - P_{\leq \lambda_n^6}\phi)||_{L^\beta_{t,x}}
\]
\[
= ||D_{\lambda_n}(\lambda_n^{-1}\partial_x)^{s-1/2}e^{-it(\partial_x)}(D_{\lambda_n}^{-1}L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}f_n - P_{\leq \lambda_n^6}\phi)||_{L^\beta_{t,x}}
\]
\[
= ||(\lambda_n^{-1}\partial_x)^{s-1/2}e^{-it(\partial_x)}(D_{\lambda_n}^{-1}L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}f_n - P_{\leq \lambda_n^6}\phi)||_{L^\beta_{t,x}}.
\]
Set
\[
g_n := D_{\lambda_n}^{-1}L_{\nu_n}^{-1}T_{x_n}^{-1}e^{-it_n(\partial_x)}f_n.
\]
We claim that the high frequency parts tends to zero.

**Claim 1.** By passing to a subsequence,
\[
\langle \lambda_n^{-1}\partial_x \rangle^{s-1/2}e^{-i\lambda_n^2t(\lambda_n^{-1}\partial_x)^{-1}}P_{\geq \lambda_n^6}g_n \to 0
\]
for almost every $((t, x)) \in \mathbb{R} \times \mathbb{R}$. 


Proof. From $\lambda_n \to \infty$, we may assume $\lambda_n > n^{4s/\theta}$ by passing to a subsequence. By Lemma 2.2 and Lemma 2.3 [iv], more precisely $|\{l - \nu_n(\xi)\}| \sim_{\varepsilon, A} (\xi)$,

$$\|\langle \lambda_n^{-1}\partial_t\rangle^{s-1/2} e^{-i\lambda_n^2 t/\{(\lambda_n^{-1}\partial_t)^{-1}\}} P_{\xi} \leq n^{1/\theta} g_n \|_{L^2_{t,x}} \lesssim \|\langle \lambda_n^{-1}\partial_t\rangle^s P_{\xi} \leq n^{1/\theta} g_n \|_{L^2_{t,x}} \lesssim_{\varepsilon, A} \|\partial_t^s P_{\xi} f_n \|_{L^2_t} \lesssim n^{s-1} A \to 0$$

as $n \to \infty$. Next, we focus on the middle frequency case $P_{\xi} \lesssim n^{1/\theta} g_n$. As in the proof of Lemma 2.5, we may restrict the range of $(t, x)$ to $[-L, L]^2$. The local smoothing estimate for the Klein-Gordon equation (see, for example, [13] Theorem 1.9) shows that

$$\begin{aligned}
\int \int_{[-L, L]^2} |\langle \lambda_n^{-1}\partial_t\rangle^{s-1/2} e^{-i\lambda_n^2 t/\{(\lambda_n^{-1}\partial_t)^{-1}\}} P_{\xi} \leq n^{1/\theta} g_n |^2 dt dt \\
\lesssim L \|\partial_t^s P_{\xi} \lesssim n^{1/\theta} g_n \|_{L^2_{t,x}}^2 \lesssim_{\varepsilon, A} L n^{s-1/\theta} \|f_n \|_{L^2_{t,x}}^2 \to 0
\end{aligned}$$

as $n \to \infty$. Therefore, by passing to a subsequence, we obtain the almost everywhere convergence. □

From (iii) and (iv) in Lemma 2.3 [iii], and (1.13), the inverse Fatou lemma (see for example [13] Theorem 1.9) or [11] Lemma 2.10) shows that

$$\lim_{n \to \infty} \sup \|\langle \partial_t \rangle^{s-1/2} e^{-it/\partial_t} (f_n - \phi_n) \|_{L^6_{t,x}}^6 \leq \lim_{n \to \infty} \sup \|\langle \lambda_n^{-1}\partial_t\rangle^{s-1/2} e^{-i\lambda_n^2 t/\{(\lambda_n^{-1}\partial_t)^{-1}\}} g_n \|_{L^6_{t,x}}^6 - \|e^{it\partial_t^2/2} \phi \|_{L^6_{t,x}}^6 = \lim_{n \to \infty} \sup \|\langle \partial_t \rangle^{s-1/2} e^{-it/\partial_t} f_n \|_{L^6_{t,x}}^6 - \|e^{it\partial_t^2/2} \phi \|_{L^6_{t,x}}^6$$

Thus, it suffices to show that

$$(4.14) \quad \|e^{it\partial_t^2/2} \phi \|_{L^6_{t,x}}^6 \gtrsim \varepsilon \left(\frac{l}{A}\right)^C.$$ 

From (4.12), we have

$$|\langle h_\infty, \phi \rangle| \gtrsim \varepsilon \left(\frac{l}{A}\right)^{89}.$$ 

By Lemma 2.4 and the definition of $h_\infty$, there exists $C_1 > 0$ such that $\text{supp } h_\infty \subset \{|x| \leq \frac{l}{2^{C_1}}\}$. Accordingly, we have $h_\infty = P_M h_\infty$ and

$$|\langle h_\infty, P_M \phi \rangle| \gtrsim \varepsilon \left(\frac{l}{A}\right)^{89},$$

where $M := 2\left(\frac{l}{2^{C_1}}\right)^2$. Let $\chi_r$ denote a smooth cut-off to $\{|x| \leq r\}$. Since $\chi_r \to 1$ as $r \to \infty$, there exists $C_2 > 0$ so that

$$\|\langle h, \phi \rangle\|_{L^2} \gtrsim \varepsilon \left(\frac{l}{A}\right)^{89},$$

where $r := \left(\frac{l}{2^{C_1}}\right)^2$ and $h := P_M \chi_r h_\infty$. Hence, the proof of (4.14) is reduced to prove

$$(4.15) \quad \sup_{|t| \leq 1} \|e^{it\partial_t^2/2} \tilde{h}\|_{L^6_{t,x}} \lesssim \left(\frac{A}{\varepsilon}\right)^{C'}.$$
Indeed, if (4.15) holds, we obtain
\[
\|e^{i\xi^2/2\phi}\|_{L^6} > \left\|\|e^{i\xi^2/2\phi}\|_{L^6}\right\|_{L^6((-1,1))} > \left(\frac{\varepsilon}{A}\right)^C \left\|\|e^{i\xi^2/2h, e^{i\xi^2/2\phi}}\|_{L^6}^6\right\|_{L^6((-1,1))} = \left(\frac{\varepsilon}{A}\right)^C \left\|\|\hat{\phi}, \phi\|_{L^6((-1,1))}\right\| \geq \varepsilon \left(\frac{\varepsilon}{A}\right)^{C + 89}.
\]
In the sequel, we show (4.15). Setting \(\phi(\xi) := e^{-i\xi^2/2\sigma}\left(\frac{\xi}{M}\right)\), we write
\[
e^{i\xi^2/2h} = F^{-1}[\phi] * (\chi_{\mathbb{R}}h_{\infty}).
\]
Since
\[
sup_{|t| \leq 1, \xi \in \mathbb{R}} \left|\frac{d^3 \theta}{d\xi^3}(\xi)\right| \lesssim M^\alpha C_3
\]
for all \(\alpha \in \mathbb{N}_0\), we get
\[
\|F^{-1}[\phi]\|_{L^1} \lesssim \|\cdot\|_{L^1} \|\phi\|_{L^2} \lesssim \|\phi\|_{H^1} \lesssim M^{3/2}.
\]
Thus, by Young’s inequality, we obtain
\[
\sup_{|t| \leq 1} \|e^{i\xi^2/2h}\|_{L^6}^{6/5} = \sup_{|t| \leq 1} \|F^{-1}[\phi] * (\chi_{\mathbb{R}}h_{\infty})\|_{L^6}^{6/5} \lesssim \sup_{|t| \leq 1} \|F^{-1}[\phi]\|_{L^1} \|\chi_{\mathbb{R}}h_{\infty}\|_{L^6}^{6/5} \leq \sup_{|t| \leq 1} \|F^{-1}[\phi]\|_{L^1} \|\chi_{\mathbb{R}}h_{\infty}\|_{L^6} \lesssim M^{3/2} \sqrt[3]{3}
\]
which concludes the proof. \(\square\)

**Remark 4.6.** After passing to a further subsequence, we can take the parameters in the conclusion of Theorem 4.5 to satisfy the following:

- If \(\lambda_n\) does not converge to \(+\infty\), then \(\lambda_n \equiv 1\) and \(\nu_n \equiv 0\).
- Irrespective of the behavior of \(\lambda_n\), we have either \(t_n/\lambda_n \to \pm \infty\) or \(t_n \equiv 0\).

This fact follows from the same argument as in Corollary 4.10 in \([11]\). 

5. **Linear profile decomposition**

In this section, at first, we state the linear profile decomposition as follows.

**Theorem 5.1** (Linear profile decomposition). Let \(\{\nu_n\} \subset H^1(\mathbb{R})\) be bounded and let \(s \in \left[\frac{1}{2}, \frac{11}{12}\right]\). Then after passing to a subsequence, there exists \(J_0 \in [1, \infty)\) such that for any integer \(j \in [1, J_0]\), there also exist the following:

- A function \(\phi_j \in L^2(\mathbb{R}) \setminus \{0\}\),
- A sequence \(\{\lambda_n^j\} \subset [1, \infty)\) such that \(\lambda_n^j \to \infty\) or \(\lambda_n^j \equiv 1\),
- A sequence \(\{\nu_n^j\} \subset \mathbb{R}\) such that \(\nu_n^j \to \nu^j \in \mathbb{R}\), which is identically \(0\) if \(\lambda_n^j \equiv 1\),
- A sequence \(\{t_n^j, x_n^j\} \subset \mathbb{R} \times \mathbb{R}\) such that \(t_n^j, x_n^j \to \pm \infty\) or \(t_n^j \equiv 0\).

Let \(P_n^j\) denote the projections defined by

\[
P_n^j \phi_j := \begin{cases} \phi_j \in H^1(\mathbb{R}), & \text{if } \lambda_n^j \equiv 1, \\ P_{\leq (\lambda_n^j)\theta} \phi_j, & \text{if } \lambda_n^j \to \infty, \text{ with } \theta = \frac{1}{100}. \end{cases}
\]
Then for any $J \in [1, J_0)$, we have a decomposition
\begin{equation}
    v_n = \sum_{j=1}^{J} T_{x_n} e^{it_n \langle \partial_x \rangle} L_{\nu_n} P_j \phi^j + w_n^J,
\end{equation}
satisfying
\begin{equation}
    \lim_{j \to \infty} \limsup_{n \to \infty} \| (\partial_x)^{s-1/2} e^{-it \langle \partial_x \rangle} w_n^J \|_{L^s_t L^2_x(\mathbb{R} \times \mathbb{R})} = 0
\end{equation}
\begin{equation}
    \lim_{n \to \infty} \left\{ \| v_n \|^2_{H^1} - \sum_{j=1}^{J} \| T_{x_n} e^{it_n \langle \partial_x \rangle} L_{\nu_n} D_{\lambda_n} P_j \phi^j \|^2_{H^1} - \| w_n^J \|^2_{H^1} \right\} = 0,
\end{equation}
\begin{equation}
    D_{\lambda_n}^{-1} L_{\nu_n}^{-1} T_{x_n}^{-1} e^{-it_n \langle \partial_x \rangle} w_n^J \to 0 \text{ weakly in } L^2_x(\mathbb{R}) \text{ for any } j \leq J.
\end{equation}

Finally, we have the following asymptotic orthogonality condition: for any $j \neq j'$,
\begin{equation}
    \lim_{n \to -\infty} \left\{ \frac{\lambda_j^2}{\lambda_n^2} + \frac{\lambda_j^\prime}{\lambda_n^\prime} + \lambda_n |P_j^\prime - P_n^J| + \frac{|s_{j,n}^J|}{(\lambda_n^2)^2} + \frac{|y_{j,n}^J|}{\lambda_n^\prime} \right\} = 0,
\end{equation}
where $(-s_{j,n}^J, y_{j,n}^J) := L_{\nu_n} (t_n^J - t_n^J, x_n - x_n^J)$.

Theorem 5.1 follows from the inverse Strichartz inequality (Theorem 4.5) and a straightforward modification of the proof of Theorem 5.1 in [11]. Hence, we omit the details of the proof here.

We will use the following energy decoupling in [11]. Because our energy has the exponential-type term, we need some modifications of the proof of Proposition 5.3 in [11].

**Proposition 5.2** (Energy decoupling). Let $\{v_n\}_{n=0}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R})$. Then after passing to a subsequence, the linear profile decomposition (5.1) satisfies the following: for any $J < J_0$,
\begin{equation}
    \lim_{n \to -\infty} \left\{ E(v_n) - \sum_{j=1}^{J} E \left( T_{x_n} e^{it_n \langle \partial_x \rangle} L_{\nu_n} D_{\lambda_n} P_j \phi^j \right) - E(w_n^J) \right\} = 0.
\end{equation}

**Proof.** We will prove that the energy decouples in the inverse Strichartz theorem (Theorem 4.5), that is, in the case $J = 1$. The general case follows by induction. Moreover, we proved (5.3). Thus it suffices to prove that
\begin{equation}
    \lim_{n \to -\infty} \left\{ \int_{\mathbb{R}} \tilde{N}(\Re v_n) dx - \int_{\mathbb{R}} \tilde{N}(\Re \phi_n) dx - \int_{\mathbb{R}} \tilde{N}(\Re w_n) dx \right\} = 0,
\end{equation}
where
\[ \phi_n := T_{x_n} e^{it_n \langle \partial_x \rangle} L_{\nu_n} D_{\lambda_n} P_1 \phi. \]
with $P_n = 1$ if $\lambda_n = 1$ and $P_n = P_{< \lambda_n}$ if $\lambda_n \to \infty$. In order to prove (5.7), it suffices to prove that for any $l \in \mathbb{N}$ with $l \geq 3$
\begin{equation}
    \lim_{n \to -\infty} \left\{ \| \Re v_n \|_{L^2_{t,x}^l}^2 - \| \Re \phi_n \|_{L^2_{t,x}^l}^2 - \| \Re w_n \|_{L^2_{t,x}^l}^2 \right\} = 0.
\end{equation}
Indeed, since \( \{v_n\} \) is bounded in \( H^1(\mathbb{R}) \), where we set \( A := \sup_n \|v_n\|_{H^1} \), by the Sobolev embedding \( H^1(\mathbb{R}) \subset L^{2\ell}(\mathbb{R}) \), we have

\[
\sum_{l=3}^{\infty} \frac{1}{l!} \left( \|\Re v_n\|_{L^{2\ell}}^2 - \|\Re \phi_n\|_{L^{2\ell}}^2 - \|\Re w_n\|_{L^{2\ell}}^2 \right) \\
\leq \sum_{l=3}^{\infty} \frac{1}{l!} \left( \|\Re v_n\|_{L^2}^2 + \|\Re \phi_n\|_{L^2}^2 + \|\Re w_n\|_{L^2}^2 \right) \\
\leq 3 \sum_{l=3}^{\infty} \frac{A^{2l}}{l!} < 3 \exp(A^2) < \infty,
\]

which enables us to apply the dominated convergence theorem, to get (5.7).

The proof of (5.8) is same as the proof of (5.11) in Proposition 5.3 in [11]. So we omit the details, which completes the proof of the proposition. \( \square \)

**Proposition 5.3** (Decoupling of nonlinear profiles). Let \( \psi^j \) and \( \psi^j' \) be in \( C_0^{\infty}(\mathbb{R} \times \mathbb{R}) \). Let \( v_n^j, v_n^{j'}, (t_n^j, x_n^j), (t_n^{j'}, x_n^{j'}) \), \( \lambda_n^j, \lambda_n^{j'} \) be parameters given in Theorem 5.1. We define \( \Phi_n^j \) by

\[
\Phi_n^j(t, x) = e^{-it} \frac{x}{\sqrt{\lambda_n^j}} \psi^j \left( \frac{t}{\lambda_n^j}, \frac{x}{\lambda_n^j} \right)
\]

with \( \psi_n^j \) defined in the similar manner. Then under the orthogonality condition (5.9), we have

\[
\lim_{n \to \infty} \|\psi_n^j \psi_n^{j'}\|_{L^2_{t,x}} = 0.
\]

**Proof.** This proposition can be proved in a similar manner as Proposition 5.5 in [11]. \( \square \)

6. **Isolating NLS inside the nonlinear Klein-Gordon equation**

We recall the result obtained by Dodson [3] for the mass-critical nonlinear Schrödinger equation:

\[
\left( i\partial_t + \frac{1}{2} \partial_x^2 \right) w = \frac{5}{32} |w|^4 w.
\]

**Theorem 6.1** ([3]). Let \( w_0 \in L^2(\mathbb{R}) \). Then, there exists a unique global solution \( w \in C(\mathbb{R}; L^2(\mathbb{R})) \) to (6.1) with \( w(0) = w_0 \). Furthermore, the solution \( w \) satisfies the following estimate:

\[
\|w\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R})} \leq C(M(w_0)),
\]

As a consequence, \( w \) scatters as \( t \to \pm \infty \) in \( L^2(\mathbb{R}) \), that is, there exists \( w_\pm \in L^2(\mathbb{R}) \) such that

\[
\lim_{t \to \pm \infty} \|w(t) - e^{it\partial_x^2/2} w_\pm\|_{L^2_x} = 0,
\]

where the double-sign corresponds. Conversely, for any \( w_\pm \in L^2(\mathbb{R}) \), there exists a unique global solution \( w \) to (6.1) so that the above holds.

**Remark 6.2.** The coefficient on the right hand side of (6.1) is needed to extract \(|v|^4 v\) on the nonlinearity of (2.1). Indeed, we will use the following equality:

\[
(\Re z)^5 = \frac{1}{16} (10 |z|^4 (\Re z + 5 |z|^2 \Re z^3 + \Re z^5) = \frac{1}{16} (5 |z|^4 (z + \overline{z}) + 5 |z|^2 \Re z^3 + \Re z^5)
\]
for $z \in \mathbb{C}$. When $z = re^{i\theta}$ in polar form, (6.2) is equivalent to
\[
\cos^5 \theta = \frac{1}{16}(10 \cos \theta + 5 \cos 3\theta + \cos 5\theta).
\]

The goal in this section is to prove the following theorem.

**Theorem 6.3.** Let $v_n \to \nu \in \mathbb{R}$, $\lambda_n \to \infty$, and $\{t_n\}, \{x_n\} \subset \mathbb{R}$ be given. Assume that either $t_n \equiv 0$ or $t_n/\lambda_n^2 \to \pm \infty$. Let $\phi \in L^2(\mathbb{R})$. If we define
\[
\phi_n := T_{x_n} e^{it_n(\partial_x)} L_{v_n} D_{\lambda_n} P_{\leq \lambda_n^6} \phi
\]
for $\theta = \frac{1}{100}$, then for each $n$ sufficiently large, there exists a unique global solution $v_n$ to (2.1) with initial data $v_n(0) = \phi_n$, which satisfies
\[
\|v_n\|_{L^\infty_t(\mathbb{R}; H^1_x(\mathbb{R}))} + S_{\mathbb{R}}(|(\partial_x)^{1/2}v_n|) \lesssim_{M(\phi)} 1.
\]
Furthermore, for any $s \in \left[\frac{3}{2}, 1\right]$ and $\varepsilon > 0$, there exist $N_\varepsilon \in \mathbb{N}$ and a function $\psi_\varepsilon \in C^\infty_c(\mathbb{R} \times \mathbb{R})$ such that for all $n > N_\varepsilon$,
\[
\left\| \mathbb{R} \left\{ \left( (\partial_x)^{s-1/2}v_n \right) \circ L_{\nu_n}^{-1}(t + t_n, x + x_n) - \frac{e^{-it}}{\lambda_n^{1/2}} \psi_\varepsilon \left( \frac{t}{\lambda_n}, \frac{x}{\lambda_n} \right) \right\} \right\|_{L^s_t L^2_x} < \varepsilon,
\]
where $(t_n, x_n) := L_{\nu_n}(t_n, x_n)$.

**Proof.** First, we consider the case $\nu_n \equiv 0$. Then, since the Klein-Gordon equation is invariant under space translations, we may assume $x_n \equiv 0$. That is, $\phi_n = e^{it_n(\partial_x)} D_{\lambda_n} P_{\leq \lambda_n^6} \phi$ and we will show that
\[
\left\| \left( (\partial_x)^{s-1/2}v_n \right)(t + t_n, x) - \frac{e^{-it}}{\lambda_n^{1/2}} \psi_\varepsilon \left( \frac{t}{\lambda_n}, \frac{x}{\lambda_n} \right) \right\|_{L^s_t L^2_x} < \varepsilon.
\]

- In the case $t_n \equiv 0$, we let $w_n$ and $w_\infty$ be the solutions to (6.1) with $w_n(0) = P_{\leq \lambda_n^6} \phi$ and $w_\infty(0) = \phi$, respectively.
- In the case $t_n/\lambda_n^2 \to -\infty (+\infty)$, we let $w_n$ and $w_\infty$ be the solutions to (6.1) that scatter forward (backward) to $e^{it\partial_x^2/2} P_{\leq \lambda_n^6} \phi$ and $e^{it\partial_x^2/2} \phi$, respectively.

Theorem 6.3 implies
\[
(6.3) \quad S_{\mathbb{R}}(w_n) + S_{\mathbb{R}}(w_\infty) \lesssim_{M(\phi)} 1.
\]
From the construction of $w_n$, we get the following.

**Lemma 6.4.** For $s \geq 0$, we have
\[
(6.4) \quad \| |\partial_x|^s w_n\|_{L^\infty_t L^2_x} + \| |\partial_x|^s w_n\|_{L^s_t L^2_x} \lesssim_{M(\phi)} \lambda_n^d,
\]
\[
\| |\partial_x|^{s+2} \partial_t w_n\|_{L^s_t L^2_x} \lesssim_{M(\phi)} \lambda_n^{(s+2)d}.
\]
Furthermore, the identity is valid:
\[
(6.5) \quad \lim_{n \to \infty} \left\{ \| w_n - w_\infty \|_{L^\infty_t L^2_x} + \| w_n - w_\infty \|_{L^s_t L^2_x} + \| D_{\lambda_n} \left( w_n - P_{\leq \lambda_n^6} w_\infty \right) \|_{L^\infty_t H^2_x} \right\} = 0.
\]

**Proof.** Since
\[
\| |\partial_x|^{s+2} P_{\leq \lambda_n^6} \phi\|_{L^2} \lesssim \lambda_n^d \phi\|_{L^2},
\]
(6.4) follows from a corollary of the local well-posedness.
By the fractional Leibniz rule and Sobolev embeddings $W^{2/15,6}(\mathbb{R}) \hookrightarrow L^{30}(\mathbb{R})$, $H^{7/15}(\mathbb{R}) \hookrightarrow L^{30}(\mathbb{R})$, we have

$$
\| (\partial_x)^s \partial_t w_n \|_{L^6_{t,x}} \lesssim \| (\partial_x)^s \partial^2_x w_n \|_{L^6_{t,x}} + \| (\partial_x)^s w_n \|_{L^6_{t}L^{20}_{x}} \| w_n \|_{L^\infty_t L^{30}_x}^{4} \\
\lesssim_{M(\phi)} \lambda_n^{(s+2)\theta} + \| (\partial_x)^{s+2/15} w_n \|_{L^6_{t,x}} \| (\partial_x)^{7/15} w_n \|_{L^6_{t}L^{2}_{x}}^{4} \\
\lesssim_{M(\phi)} \lambda_n^{(s+2)\theta}.
$$

The estimates for the first and second parts on the left hand side of (6.5) follows from the stability theory for the mass-critical Schrödinger equation (see [19], [12]). We consider the third part on the left hand side of (6.5). From

$$
w_n - P_{\leq \lambda^6} w_\infty = P_{\geq \lambda_n} w_n + P_{\leq \lambda_n} (w_n - w_\infty) + P_{\lambda_n^6 \leq \cdot \leq \lambda_n} w_\infty,
$$

we get

$$
\| D_{\lambda_n} (w_n - P_{\leq \lambda_n^6} w_\infty) \|_{L^\infty_t H^s_x} \\
= \| (\lambda_n^{-1} \partial_x)^s (w_n - P_{\leq \lambda_n^6} w_\infty) \|_{L^\infty_t L^2_x} \\
\lesssim \lambda_n^{-s} \| \partial_x|^{s} w_n \|_{L^\infty_t L^2_x} + \| w_n - w_\infty \|_{L^\infty_t L^2_x} + \| P_{\geq \lambda_n^6} w_\infty \|_{L^\infty_t L^2_x}.
$$

Since

$$
\| P_{\geq \lambda_n^6} w_\infty \|_{L^\infty_t L^2_x(T;\mathbb{R})} \lesssim \| w_\infty - e^{it \partial_x^2/2} w_+ \|_{L^\infty_t L^2_x(T;\mathbb{R})} + \| P_{\geq \lambda_n^6} w_\infty \|_{L^\infty_t L_x^2},
$$

choosing $T$ sufficiently large, we can make the first part on the right hand side small. Then, letting $\lambda_n \to \infty$, we have

$$
\lim_{n \to \infty} \| P_{\geq \lambda_n^6} w_\infty \|_{L^\infty_t L^2_x([-T,T] \times \mathbb{R})} = 0,
$$

which shows the desired bound.

Let $T \gg 1$ to be determined later. We define

$$
\tilde{v}_n(t) = \begin{cases} 
    e^{-it D_{\lambda_n} w_n(t/\lambda_n^2)}, & \text{if } |t| \leq T \lambda_n^2, \\
    e^{-it (t-T \lambda_n^2) \partial_x} \tilde{v}_n(T \lambda_n^2), & \text{if } t > T \lambda_n^2, \\
    e^{-it (t+T \lambda_n^2) \partial_x} \tilde{v}_n(-T \lambda_n^2), & \text{if } t < -T \lambda_n^2.
\end{cases}
$$

By the Strichartz estimate (Lemma 2.2) and Lemma 6.3, we have (6.6)

$$
\| \partial_x^s \tilde{v}_n \|_{L^6_{t,x}} \lesssim \| \partial_x^s D_{\lambda_n} w_n(t/\lambda_n^2) \|_{L^6_{t,x}} + \| \partial_x^s e^{-it (t-T \lambda_n^2) \partial_x} D_{\lambda_n} w_n(T) \|_{L^6_{t,x}(-T \lambda_n^2, \infty)} \\
+ \| \partial_x^s e^{-it (t+T \lambda_n^2) \partial_x} D_{\lambda_n} w_n(T) \|_{L^6_{t,x}((-\infty, -T \lambda_n^2, \infty))} \\
\lesssim \lambda_n^{-s} \| \partial_x^s w_n \|_{L^6_{t,x}} + \lambda_n^{-s} \| \partial_x^s (\lambda_n^{-1} \partial_x)^{1/2} w_n \|_{L^6_{t,x}} \\
\lesssim \lambda_n^{-s (1-\theta)}
$$

for any $s \geq 0$. Moreover, (6.3) implies that

$$
\| \tilde{v}_n \|_{L^\infty_t H^s_x} + \| (\partial_x)^{3/2} \tilde{v}_n \|_{L^6_{t,x}} \lesssim \| D_{\lambda_n} w_n \|_{L^\infty_t H^s_x} + \| \tilde{v}_n \|_{L^6_{t,x}} + \| (\partial_x)^{3/2} \tilde{v}_n \|_{L^6_{t,x}} \\
\lesssim_{M(\phi)} 1 + \lambda_n^{-2} \| \partial_x^2 w_n \|_{L^\infty_t L^2_x} + \lambda_n^{-3 (1-\theta)/2} \\
\lesssim_{M(\phi)} 1.
$$

Here, we note that

$$
\lim_{T \to \infty} \limsup_{n \to \infty} \| \tilde{v}_n (-t_n) - \phi_n \|_{H^2} = 0.
$$
Indeed, in the case $t_n \equiv 0$, (6.8) holds because $\tilde{v}_n(-t_n) = D_{\lambda_n} P_{\leq \lambda_n^2} \phi = \phi_n$. On the other hand, in the case $t_n/\lambda_n^2 \to -\infty$, for any $T > 0$ and $n \gg 1$, we have $-t_n > T\lambda_n^2$. Hence,
\[
\|\tilde{v}_n(-t_n) - \phi_n\|_{H^2} = \|D_{\lambda_n} w_n(T) - e^{-iT\lambda_n^2((\partial_x) - 1)} D_{\lambda_n} P_{\leq \lambda_n^2} \phi\|_{H^2}
\leq \|D_{\lambda_n} [w_n(T) - P_{\leq \lambda_n^2} w_{\infty}(T)]\|_{H^2} + \|D_{\lambda_n} P_{\leq \lambda_n^2} [w_{\infty}(T) - e^{-iT\lambda_n^2((\lambda_n^{-1} \partial_x) - 1) - \lambda_n^2 T}]\|_{H^2}
\]
Thanks to Lemma 6.3, the first part goes to zero. The second part is estimated as follows.
\[
\|D_{\lambda_n} P_{\leq \lambda_n^2} [w_{\infty}(T) - e^{-iT\lambda_n^2((\lambda_n^{-1} \partial_x) - 1) - \lambda_n^2 T}]\|_{H^2}
\leq \|D_{\lambda_n} P_{\leq \lambda_n^2} [w_{\infty}(T) - e^{-iT\lambda_n^2((\lambda_n^{-1} \partial_x) - 1) - \lambda_n^2 T}]\|_{L^2}
\leq \|w_{\infty}(T) - e^{iT\lambda_n^2/2}\|_{L^2} + \|1_{|\xi|\leq 2\lambda_n^2} [1 - e^{-iT((\lambda_n^{-1} \xi) - 1) - \lambda_n^2 T}]\|_{L^2}.
\]
By (2.7) and Lebesgue’s dominated convergence theorem, (6.8) holds.

**Proposition 6.5** (Large time intervals). With the notation above,
\[
\lim_{T \to \infty} \lim_{n \to \infty} \|\langle \partial_x \rangle^{3/2} \tilde{v}_n\|_{L^p_{t,x}((T\lambda_n^2, \infty) \times \mathbb{R})} = 0
\]
and analogously on the time interval $(-\infty, -T\lambda_n^2)$.

**Proof.** From Theorem 6.1, there exists $w_+ \in L^2(\mathbb{R})$ such that
\[
\lim_{t \to \infty} \|w_{\infty}(t) - e^{iT\lambda_n^2/2} w_+\|_{L^2} = 0.
\]
We decompose the integrand as follows: For $t > T\lambda_n^2$,
\[
\tilde{v}_n(t) = e^{-i(t-T\lambda_n^2)(\partial_x)} \tilde{v}_n(T\lambda_n^2)
\]
\[
= e^{-i(t-T\lambda_n^2)(\partial_x)} \left[ \tilde{v}_n(T\lambda_n^2) - e^{-iT\lambda_n^2} D_{\lambda_n} e^{iT\lambda_n^2/2} P_{\leq \lambda_n^2} w_+ \right] + e^{-i(t-T\lambda_n^2)(\partial_x)} e^{-iT\lambda_n^2} D_{\lambda_n} e^{iT\lambda_n^2/2} P_{\leq \lambda_n^2} w_+
\]
\[
= h_1(t) + h_2(t).
\]
The Strichartz estimate (Lemma 2.2) implies
\[
\|\langle \partial_x \rangle^{3/2} h_1\|_{L^p_{t,x}((T\lambda_n^2, \infty) \times \mathbb{R})} \leq \left\| \tilde{v}_n(T\lambda_n^2) - e^{-iT\lambda_n^2} D_{\lambda_n} e^{iT\lambda_n^2/2} P_{\leq \lambda_n^2} w_+\right\|_{H^2}
\leq \|D_{\lambda_n} [w_n(T) - P_{\leq \lambda_n^2} w_{\infty}(T)]\|_{H^2} + \|D_{\lambda_n} P_{\leq \lambda_n^2} [w_{\infty}(T) - e^{iT\lambda_n^2/2} w_+]\|_{H^2}.
\]
By Lemma 6.3, we can estimate the first part. On the estimate of the second part,
\[
\|D_{\lambda_n} P_{\leq \lambda_n^2} [w_{\infty}(T) - e^{iT\lambda_n^2/2} w_+]\|_{H^2} \leq \|w_{\infty}(T) - e^{iT\lambda_n^2/2} w_+\|_{L^2} + \lambda_n^{-2} \|\langle \partial_x \rangle^2 P_{\leq \lambda_n^2} [w_{\infty}(T) - e^{iT\lambda_n^2/2} w_+]\|_{L^2} \to 0 \quad (T \to \infty).
\]
Next, we estimate the Strichartz norm of $h_2$. By the Strichartz estimate and

$$
\| (\partial_x)^{1/2} D_{\lambda_n} P_{\leq \lambda_n^\theta} (f - g) \|_{L^2_t L^2_x} \lesssim \| f - g \|_{L^2},
$$

we may assume that $w_+$ is a Schwartz function with compact frequency support. Then, we have

$$
\| (\partial_x)^{3/2} h_2 \|_{L^6_t L^\infty_x ((T\lambda_n^2, \infty) \times \mathbb{R})} = \| D_{\lambda_n} (\lambda_n^{-2} \partial_x)^{3/2} e^{-i(t - \lambda_n^2 T) (\partial_x / \lambda_n)} e^{-iT\lambda_n^2 e^{iT\partial_x^2/2}} P_{\leq \lambda_n^\theta} w_+ \|_{L^6_t L^\infty_x ((T\lambda_n^2, \infty) \times \mathbb{R})} \lesssim \lambda_n^{-1/3} \| e^{-i(t - \lambda_n^2 T) (\partial_x / \lambda_n)} e^{-iT\lambda_n^2 e^{iT\partial_x^2/2}} P_{\leq \lambda_n^\theta} w_+ \|_{L^6_t L^\infty_x ((T\lambda_n^2, \infty) \times \mathbb{R})}.
$$

We write

$$
e^{-i(t - \lambda_n^2 T) (\partial_x / \lambda_n)} e^{-iT\lambda_n^2 e^{iT\partial_x^2/2}} P_{\leq \lambda_n^\theta} w_+ = \zeta \ast w_+,
$$

where

$$
\zeta(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ik(x)} \sigma \left( \frac{\xi}{\lambda_n^\theta} \right) d\xi, \quad k(\xi) := x\xi - (t - T\lambda_n^2) (\lambda_n^{-1} \xi) - T\lambda_n^2 - T\xi^2/2.
$$

We note that

$$
k'(\xi) = x - (\lambda_n^{-2} t - T) \frac{\xi}{(\lambda_n^{-1} \xi)} - T, \\
k''(\xi) = -(\lambda_n^{-2} t - T) \frac{1}{(\lambda_n^{-1} \xi)^3} - T.
$$

Since $k'$ is a strictly decreasing function and $k'(\xi) \to \mp \infty$ as $\xi \to \pm \infty$, there exists $\xi_0 \in \mathbb{R}$ such that $k'(\xi_0) = 0$. Moreover,

$$
|k''(\xi)| \sim \lambda_n^{-2} t
$$

for $|\xi| \leq 2\lambda_n^\theta$ and $t \geq T\lambda_n^2$. Indeed, for $t \geq 2T\lambda_n^2$,

$$
\frac{1}{2} \lambda_n^{-2} t \leq \lambda_n^{-2} t - T \leq |k''(\xi)| \leq |\lambda_n^{-2} t - T| + T = \lambda_n^{-2} t.
$$

On the other hand, for $T\lambda_n^2 \leq t \leq 2T\lambda_n^2$,

$$
\frac{1}{2} \lambda_n^{-2} t \leq T \leq |k''(\xi)| \leq |\lambda_n^{-2} t - T| + T = \lambda_n^{-2} t.
$$

Let $\varepsilon > 0$ be a positive constant to be chosen later. If $|\xi_0| \leq 4\lambda_n^\theta$, we get

$$
|k'(\xi)| = |k'(\xi) - k'(\xi_0)| = \left| (\xi - \xi_0) \int_0^1 k''(s\xi + (1-s)\xi_0) ds \right| \geq |\xi - \xi_0| \lambda_n^{-2} t
$$

for $|\xi| \leq 2\lambda_n^\theta$. On the other hand, if $\xi_0 > 4\lambda_n^\theta$, noting that $k'(4\lambda_n^\theta) > 0$ because $k'$ is decreasing, we get

$$
|k'(\xi)| = k'(\xi) - k'(4\lambda_n^\theta) = (\xi - 4\lambda_n^\theta) \int_0^1 k''(s\xi + (1-s)4\lambda_n^\theta) ds \geq |\xi - 4\lambda_n^\theta| \lambda_n^{-2} t
$$

for $|\xi| \leq 2\lambda_n^\theta$. The case $\xi_0 < -4\lambda_n^\theta$ is similarly handled. Therefore, by the integration by parts, we have

$$
\left| \int_{[-2\lambda_n^\theta, 2\lambda_n^\theta] \setminus [\xi_0 - \varepsilon, \xi_0 + \varepsilon]} e^{ik(\xi)} \sigma \left( \frac{\xi}{\lambda_n^\theta} \right) d\xi \right| \lesssim \frac{1}{\varepsilon \lambda_n^2 t} + \int_{[-2\lambda_n^\theta, 2\lambda_n^\theta] \setminus [\xi_0 - \varepsilon, \xi_0 + \varepsilon]} \left( \frac{|k''(\xi)|}{k'(\xi)^2} + \frac{\lambda_n^\theta}{k'(\xi)} \right) d\xi \lesssim \frac{\lambda_n^2}{t \varepsilon}.
$$
Thus, we obtain
\[ |\zeta(x)| \lesssim \left| \int_{[-2\lambda_n^2, 2\lambda_n^2] \setminus |\xi_0 - \varepsilon, \xi_0 + \varepsilon|} e^{ik(s)} \sigma \left( \frac{\xi}{\lambda_n} \right) dk \right| + \left| \int_{[\xi_0 - \varepsilon, \xi_0 + \varepsilon]} e^{ik(s)} \sigma \left( \frac{\xi}{\lambda_n} \right) d\xi \right| \lesssim \frac{\lambda_n^2}{t\varepsilon} + \varepsilon. \]

Here, taking \( \varepsilon = \lambda_n/t^{1/2} \), we get
\[ \|\zeta\|_{L^\infty} \lesssim \frac{\lambda_n}{t^{1/2}} \]
for \( t \geq \lambda_n T \).

Hence, we have
\[ \|\zeta \ast w_+\|_{L^2} \lesssim \|\zeta\|_{L^\infty} \|w_+\|_{L^\infty} \lesssim \frac{\lambda_n}{t^{1/2}}. \]

By (6.10) and interpolating this estimate with the trivial \( L^2 \) bound, we obtain
\[ \|e^{-i(t-T\lambda_n^2)(\partial_x/\lambda_n)} e^{-iT\lambda_n^2} e^{iT\theta_n^2/2} P_{\leq \lambda_n^2} w_+\|_{L^6} \lesssim \left( \frac{\lambda_n}{t^{1/2}} \right)^{2/3}. \]

Integrating with respect to time, we have
\[ \text{R.H.S. of (6.9)} \lesssim T^{-1/6}, \]
which concludes the proof. \( \square \)

On the middle interval \( I_n := [-T\lambda_n^2, T\lambda_n^2] \), because \( w_n \) is a solution to (6.1), a direct calculation shows that
\[-i\partial_t \bar{v}_n \]
\[ = e^{-it} \left( -D_{\lambda_n} w_n \left( \frac{t}{\lambda_n^2} \right) - i\lambda_n^{-2} D_{\lambda_n} (\partial_x w_n) \left( \frac{t}{\lambda_n^2} \right) \right) \]
\[ = e^{-it} \left( -D_{\lambda_n} w_n \left( \frac{t}{\lambda_n^2} \right) + \frac{1}{2\lambda_n^2} D_{\lambda_n} (\partial_x^2 w_n) \left( \frac{t}{\lambda_n^2} \right) - \frac{5}{32} \lambda_n^{-2} D_{\lambda_n} (|w_n|^4 w_n) \left( \frac{t}{\lambda_n^2} \right) \right) \]
\[ = e^{-it} D_{\lambda_n} \left( -w_n \left( \frac{t}{\lambda_n^2} \right) + \frac{1}{2\lambda_n^2} (\partial_x^2 w_n) \left( \frac{t}{\lambda_n^2} \right) \right) - \frac{5}{32} |\bar{v}_n|^4 \bar{v}_n. \]

Moreover, (6.2) implies that
\[ (\Re \bar{v}_n)^5 - \frac{5}{16} |\bar{v}_n|^4 \bar{v}_n = \frac{5}{16} |\bar{v}_n|^4 \bar{v}_n + \frac{5}{16} |\bar{v}_n|^2 \Re \bar{v}_n^3 + \frac{1}{16} \Re \bar{v}_n^5. \]

Hence, \( \bar{v}_n \) on \( I_n \) satisfies
\[ (-i\partial_t + (\partial_x)\bar{v}_n + (\partial_x)^{-1}(\exp(\Re \bar{v}_n^2) - 1 - |\Re \bar{v}_n|^2)\Re \bar{v}_n = e_1 + e_2 + e_3 + e_4 + e_5 + e_6, \]
where
\[
e_1 := e^{-it} D_\lambda \left\{ \left[ \langle \lambda_n^{-1} \partial_x \rangle - 1 + \frac{\partial_x^2}{2 \lambda_n^2} \right] w_n \left( \frac{t}{\lambda_n^2} \right) \right\},
\]
\[
e_2 := \langle \partial_x \rangle^{-1} \left( \exp(\|R\nabla_n\|) - 1 - |\nabla_n|^2 - \frac{|\nabla_n|^4}{2} \right) \nabla \nabla_n,
\]
\[
e_3 := \frac{1}{2} [\langle \partial_x \rangle^{-1} - 1] (\nabla \nabla_n)^5,
\]
\[
e_4 := \frac{1}{32} \Re \left\{ e^{-5it} \left[ D_\lambda w_n \left( \frac{t}{\lambda_n^2} \right) \right]^5 \right\},
\]
\[
e_5 := \frac{5}{32} \left| D_\lambda w_n \left( \frac{t}{\lambda_n^2} \right) \right|^2 \Re \left\{ e^{-3it} \left[ D_\lambda w_n \left( \frac{t}{\lambda_n^2} \right) \right]^3 \right\},
\]
\[
e_6 := \frac{5}{32} e^{it} \left| D_\lambda w_n \left( \frac{t}{\lambda_n^2} \right) \right|^4 \frac{D_\lambda w_n \left( \frac{t}{\lambda_n^2} \right)}{D_\lambda w_n \left( \frac{t}{\lambda_n^2} \right)}.
\]

We can treat \( e_1, e_2, \) and \( e_3 \) as errors in Proposition 3.2. In fact, the equation
\[
\langle \lambda_n \xi \rangle - 1 - \frac{\| \xi \|^2}{2 \lambda_n^2} = \frac{\| \xi \|^2}{\lambda_n^2 \left( 1 + \langle \lambda_n \xi \rangle \right)} - \frac{\| \xi \|^2}{2 \lambda_n^2}
\]
\[
= \frac{\| \xi \|^2}{2 \lambda_n^2 \left( 1 + \langle \lambda_n \xi \rangle \right)} = \frac{\| \xi \|^4}{2 \lambda_n^4 \left( 1 + \langle \lambda_n \xi \rangle \right)^2}
\]
and Lemma 6.4 imply that
\[(6.11) \| e_1 \|_{L^2_t H^2_x(I_n \times \mathbb{R})} \lesssim T \lambda_n^{-2} \left( \| \partial_x^4 w_n \|_{L^\infty_t L^2_x} + \lambda_n^{-2} \| \partial_x^6 w_n \|_{L^\infty_t L^2_x} \right) \lesssim T \lambda_n^{-2+4\theta}.
\]
The Taylor expansion and Lemma 6.4 yield
\[(6.12) \| \langle \partial_x \rangle^{5/2} e_2 \|_{L^{6/5}_t H^{3/2}_x(I_n \times \mathbb{R})}
\]
\[\lesssim \sum_{l=3}^\infty \frac{1}{l!} \| \langle \partial_x \rangle^{3/2} R \nabla_n \|_{L^{6/5}_t H^{3/2}_x(I_n \times \mathbb{R})} \]
\[\lesssim \sum_{l=3}^\infty \frac{1}{(l-2)!} \| \langle \partial_x \rangle^{3/2} R \nabla_n \|_{L^6_t H^{3/2}_x(I_n \times \mathbb{R})} \| \nabla \nabla_n \|^4_{L^6_t H^{1/2}_x(I_n \times \mathbb{R})} \| \nabla \nabla_n \|_{L^{2l-4}_t H^{1/2}(I_n \times \mathbb{R})}^{2l-4}
\]
\[\lesssim \| \langle \lambda_n \partial_x \rangle^{3/2} w_n \|_{L^5_x} \sum_{l=3}^\infty \frac{1}{(l-2)!} \left( \lambda_n^{-1/2} \| w_n \|_{L^\infty_t H^1_x([-T,T] \times \mathbb{R})} \right)^{2l-4}
\]
\[\lesssim \sum_{l=3}^\infty \frac{1}{(l-2)!} \left( \lambda_n^{-1/2+\theta} \right)^{2l-4}
\]
\[\lesssim \lambda_n^{-1+2\theta}.
\]
Moreover, by
\[\langle \xi \rangle (\langle \xi \rangle^{-1} - 1) = -\xi \frac{\xi}{1 + \langle \xi \rangle}.
\]
the fractional Leibniz rule, and Lemma 6.4 we have

\begin{align}
\| (\partial_x)^{5/2} e_3 \|_{L_t^{6/5}(I_n \times \mathbb{R})} & \lesssim \| (\partial_x)^{3/2} \partial_x (\varpi_n)^5 \|_{L_t^{6/5}(I_n \times \mathbb{R})} \\
& \lesssim \nu_n \| D_{\nu_n} (\lambda_n^{-1} (\lambda_n^{-1} \partial_x)^{3/2} (\partial_x w_n)) (\frac{t}{\nu_n}) \|_{L_t^6} \| D_{\nu_n} w_n (\frac{t}{\nu_n}) \|_{L_t^6}^4 \\
& \lesssim \nu_n \| (\partial_x w_n)_{L_t^6} + \lambda_n^{-3/2} \| (\partial_x)^{5/2} w_n \|_{L_t^6} \| w_n \|_{L_t^6}^4 \\
& \lesssim \lambda_n^{-1+\theta}.
\end{align}

Unfortunately, the parts $e_4, e_5, e_6$ are not small in either of the spaces $L_t^1 \dot{H}_x^2$ or $L_t^{6/5} W_x^{5/2, 6/5}$. However, they oscillate in space-time like $e^{it}$, and this allows us to modify $\nu_n$, which approximately solves (6.14).

**Lemma 6.6.** For $j = 4, 5, 6$, let $f_{n,j}$ solve

\begin{equation}
(-i \partial_t + (\partial_x)) f_{n,j} = e_j, \quad f_{n,j}(0) = 0.
\end{equation}

Then,

\begin{equation}
\| f_{n,j} \|_{L_t^\infty \dot{H}_x^2(I_n \times \mathbb{R})} + \| (\partial_x)^{3/2} f_{n,j} \|_{L_t^{6/5}(I_n \times \mathbb{R})} \lesssim \lambda_n^{-2+2\theta}.
\end{equation}

**Proof.** We will prove the lemma for $j = 6$. The argument for $j = 4, 5$ is almost identical. We compute

\begin{align}
(-i \partial_t + (\partial_x)) \left( f_{n,6} - \frac{1}{2} e_6 \right) & = \frac{5}{64} \left( \frac{e^{it} \partial_t (w_n^2 w_n^{-2})}{\lambda_n^{9/2}} \left( \frac{t}{\lambda_n^2}, \frac{x}{\lambda_n} \right) - \frac{e^{it} \partial_t (\lambda_n^{-1} \partial_x - 1)(w_n^2 w_n^{-2})}{\lambda_n^{9/2}} \right) \left( \frac{t}{\lambda_n^2}, \frac{x}{\lambda_n} \right).
\end{align}

Lemma 6.4 and the fractional Leibniz rule yield that

\begin{align}
\| (\partial_x)^{5/2} \left( \frac{e^{it} \partial_t (w_n^2 w_n^{-2})}{\lambda_n^{9/2}} \left( \frac{t}{\lambda_n^2}, \frac{x}{\lambda_n} \right) - \frac{e^{it} \partial_t (\lambda_n^{-1} \partial_x - 1)(w_n^2 w_n^{-2})}{\lambda_n^{9/2}} \right) \|_{L_t^{6/5}(I_n \times \mathbb{R})} \\
& \lesssim \lambda_n^{-2} \| (\lambda_n^{-1} \partial_x)^{5/2} \partial_t (w_n^2 w_n^{-2}) \|_{L_t^{6/5}} \\
& \lesssim \lambda_n^{-2} \left\{ \| (\lambda_n^{-1} \partial_x)^{5/2} \partial_t w_n \|_{L_t^6} \| w_n \|_{L_t^6}^3 + \| \partial_x w_n \|_{L_t^6} \| (\lambda_n^{-1} \partial_x)^{5/2} w_n \|_{L_t^6} \| w_n \|_{L_t^6}^3 \right\} \\
& \lesssim \lambda_n^{-2+2\theta}.
\end{align}

On the other hand, from

\begin{equation}
(\partial_x)((\partial_x) - 1) = -\partial_x^2 \frac{1}{1 + (\partial_x)^{-1}},
\end{equation}

and Lemma 6.4 we get

\begin{align}
\| (\partial_x)^{5/2} \left( \frac{e^{it} \partial_t (\lambda_n^{-1} \partial_x - 1)(w_n^2 w_n^{-2})}{\lambda_n^{9/2}} \right) \left( \frac{t}{\lambda_n^2}, \frac{x}{\lambda_n} \right) \|_{L_t^{6/5}(I_n \times \mathbb{R})} \\
& \lesssim \lambda_n^{-2} \| (\lambda_n^{-1} \partial_x)^{3/2} \partial_t^2 (w_n^2 w_n^{-2}) \|_{L_t^{6/5}} \\
& \lesssim \lambda_n^{-2} \left\{ \| \partial_x^2 w_n \|_{L_t^6} \| w_n \|_{L_t^6}^4 + \| \partial_x w_n \|_{L_t^6}^2 \| w \|_{L_t^6}^3 \\
& \quad + \lambda_n^{-2} \| \partial_x^4 w_n \|_{L_t^6} \| w_n \|_{L_t^6}^6 + \| \partial_x w_n \|_{L_t^6}^4 \| w \|_{L_t^6} \right\} \\
& \lesssim \lambda_n^{-2+2\theta}.
\end{align}
By $\dot{H}^{2/5}(\mathbb{R}) \to L^{10}(\mathbb{R})$, and Lemma [6.3] we have

$$
\|e_6\|_{L^{\infty}_t H^2(I_n \times \mathbb{R})} \lesssim \lambda_n^{-2} \|\langle \lambda_n^{-1} \partial_x \rangle^2 (w_n^2 \overline{w_n})\|_{L^{\infty}_t L^2_x} \\
\lesssim \lambda_n^{-2} \|\langle \lambda_n^{-1} \partial_x \rangle^2 w_n\|_{L^{\infty}_t L^\infty_x} \|w_n\|_{L^{4}_t L^{20}_x} \\
\lesssim \lambda_n^{-2} \|\partial_x\|^{2/5} \|\langle \lambda_n^{-1} \partial_x \rangle^2 w_n\|_{L^{\infty}_t L^2_x} \|\partial_x\|^{2/5} w_n\|_{L^{\infty}_t L^2_x} \\
\lesssim \lambda_n^{-2+2\theta}.
$$

By the fractional Leibniz rule, $\dot{H}^{2/15}(\mathbb{R}) \hookrightarrow L^{30}(\mathbb{R})$, $\dot{H}^{7/15}(\mathbb{R}) \hookrightarrow L^{30}(\mathbb{R})$ and Lemma [6.4] we get

$$
\|\langle \partial_x \rangle^{3/2} e_6\|_{L^{6/5}_t L^{6}_x(I_n \times \mathbb{R})} \lesssim \lambda_n^{-2} \|\langle \lambda_n^{-1} \partial_x \rangle^{3/2} (w_n^2 \overline{w_n})\|_{L^{6/5}_t L^{6}_x} \\
\lesssim \lambda_n^{-2} \|\langle \lambda_n^{-1} \partial_x \rangle^{3/2} w_n\|_{L^{6/5}_t L^\infty_x} \|w_n\|_{L^{4}_t L^{20}_x} \\
\lesssim \lambda_n^{-2} \|\partial_x\|^{2/15} \|\langle \lambda_n^{-1} \partial_x \rangle^{3/2} w_n\|_{L^{6/5}_t L^{6}_x} \|\partial_x\|^{7/15} w_n\|_{L^{\infty}_t L^2_x} \\
\lesssim \lambda_n^{-2+2\theta}.
$$

Combining the Strichartz estimate with estimates above, we obtain the desired bound. $\square$

By using this lemma, we modify $\widetilde{v}_n$ as follows:

$$
\widetilde{\nu}_n(t) := \begin{cases} 
\nu_n(t) - f_n,4(t) - f_n,5(t) - f_n,6(t), & \text{if } |t| \leq T\lambda_n^2, \\
e^{-i(t-T\lambda_n^2)\langle \partial_x \rangle} \nu_n(T\lambda_n^2), & \text{if } t > T\lambda_n^2, \\
e^{-i(t+T\lambda_n^2)\langle \partial_x \rangle} \nu_n(-T\lambda_n^2), & \text{if } t < -T\lambda_n^2.
\end{cases}
$$

Proposition 6.7. For each $\varepsilon > 0$, there exists $T > 0$ and $N \in \mathbb{N}$ such that for each $n \geq N$,

$$
(-i\partial_t + \langle \partial_x \rangle) \nu_n + \langle \partial_x \rangle^{-1} \mathcal{N}(\Re \nu_n) = \nu_1 + \nu_2 + \nu_3,
$$

with

$$
\|\nu_1\|_{L^1_t H^2_x} + \|\langle \partial_x \rangle^{5/2} (\nu_2 + \nu_3)\|_{L^{5/3}_t} \leq \varepsilon.
$$

Moreover,

$$
\|\nu_n - \nu_1\|_{L^\infty_t H^2_x} + \|\langle \partial_x \rangle^{3/2} (\nu_n - \nu_1)\|_{L^{6/5}_t} \leq \varepsilon.
$$

Proof. Put

$$
\nu_1 = \begin{cases} 
\epsilon_1, & \text{if } t \in I_n, \\
0, & \text{otherwise,}
\end{cases} \quad \nu_2 = \begin{cases}
\epsilon_2 + \epsilon_3, & \text{if } t \in I_n, \\
0, & \text{otherwise,}
\end{cases} \quad \nu_3 = \begin{cases}
\langle \partial_x \rangle^{-1} \left[ \mathcal{N}(\Re \nu_n) - \mathcal{N}(\Re \nu_1) \right], & \text{if } t \in I_n, \\
\langle \partial_x \rangle^{-1} \mathcal{N}(\Re \nu_1), & \text{otherwise.}
\end{cases}
$$

By (6.11), (6.12), and (6.13),

$$
\|\nu_1\|_{L^1_t H^2_x} + \|\langle \partial_x \rangle^{5/2} \nu_2\|_{L^{5/3}_t} \lesssim T\lambda_n^{-2+2\theta} + \lambda_n^{-1+2\theta}.
$$
By the fractional Leibniz rule, \((6.7)\), and Lemma \(6.6\)
\[
\| (\partial_x)^{5/2} \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \\
\leq \sum_{l=2}^{\infty} \frac{1}{l!} \left\| (\partial_x)^{3/2} \left( (\tilde{\mathbf{r}} v) n \right) ^{2l+1} - (\tilde{\mathbf{r}} v) n \right\|_{L^6_s(I_n \times \mathbb{R})} \\
\lesssim \left\{ \left\| (\partial_x)^{3/2} (\tilde{v}_n - v) \|_{L^6_s(I_n \times \mathbb{R})} \right\| L^6_s(I_n \times \mathbb{R}) + \left\| v \|_{L^6_s(I_n \times \mathbb{R})} + \left\| \left( \partial_x \right)^{3/2} \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \right\| L^6_s(I_n \times \mathbb{R}) \right\} ^4 \\
\times \sum_{l=2}^{\infty} \frac{1}{(l - 2)!} \left( \left\| \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} + \| \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \right\| ^{2l-4} \right) \\
\lesssim \lambda_n ^{2 + 20}.
\]
Thus, we obtain the desired bound on \(I_n\). For the complementary interval, by Sobolev’s embedding \(L^\infty (\mathbb{R}) \rightarrow H^1 (\mathbb{R})\), the Strichartz estimate (Lemma \(2.2\), Lemma \(6.6\) and \(6.7\), we get
\[
\| \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \\
\leq \| \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} + \| f(n,4) \|_{L^6_s(I_n \times \mathbb{R})} + \| f(n,5) \|_{L^6_s(I_n \times \mathbb{R})} + \| f(n,6) \|_{L^6_s(I_n \times \mathbb{R})} \\
\lesssim_{M(\phi)} 1.
\]
Therefore, from the Taylor expansion, Proposition \(6.5\) and Lemma \(6.6\) taking \(T\) and \(n\) sufficiently large, we have
\[
\| (\partial_x)^{3/2} \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \lesssim \| (\partial_x)^{3/2} \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \sum_{l=2}^{\infty} \frac{1}{l!} \left\| \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} ^{2l-4} \right\| L^6_s(I_n \times \mathbb{R}) \\
\lesssim_{M(\phi)} \| (\partial_x)^{3/2} \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} + \sum_{j=4}^{6} \| f(n,j) \|_{L^6_s(I_n \times \mathbb{R})} \\
< \frac{1}{2} \varepsilon.
\]
Finally, from Lemmas \(6.6\) and \(6.7\) we have
\[
\| \tilde{v}_n - v \|_{L^6_s(I_n \times \mathbb{R})} \\
\leq \| f(n,4) + f(n,5) + f(n,6) \|_{L^6_s(I_n \times \mathbb{R})} + \| e^{-i(t - T\lambda_n^2)}(T\lambda_n) \|_{L^6_s(I_n \times \mathbb{R})} \\
+ \| e^{-i(t + T\lambda_n^2)}(T\lambda_n) \|_{L^6_s(I_n \times \mathbb{R})} \\
\lesssim \| f(n,4) + f(n,5) + f(n,6) \|_{L^6_s(I_n \times \mathbb{R})} + \| e^{-i(t + T\lambda_n^2)}(T\lambda_n) \|_{L^6_s(I_n \times \mathbb{R})} \\
\lesssim \lambda_n ^{2 + 20}.
\]
\begin{flushright}
\Box
\end{flushright}

We are ready to show Theorem \(6.3\) with \(v_0 \equiv 0\). By \((6.7)\) and Lemma \(6.6\)
\[
\| \tilde{v}_n \|_{L^6_s(I_n \times \mathbb{R})} \lesssim_{M(\phi)} 1.
\]
Thus, by \((6.8)\) and Proposition \(6.7\) Proposition \(3.2\) with \(s = 2\) is applicable to \(v_n\). Namely, there exists a solution \(v_n\) to \(2.1\) with \(v_n(0) = \phi_n\) satisfying
By spatial translation invariance, we may choose $x_n \in (\mathbb{R}^d, \mathbb{H}^2(\mathbb{R})) \lesssim M(\phi) 1$. In addition, Proposition 6.3 implies that

$$\lim_{n \to \infty} \left\{ \left\| v_n(t) - \bar{v}_n(t - t_n) \right\|_{L^\infty_x H^2} + \left\| (\partial_x)^{3/2} (v_n(t) - \bar{v}_n(t - t_n)) \right\|_{L^6_t x} \right\} = 0.$$ 

(6.14) From (6.14), (6.15), and (6.16), we obtain

Moreover, it obeys

$$\| e^{-it} D_{\lambda_n}[\psi_x(\lambda_n^{-2}t) - w_\infty(\lambda_n^{-2}t)]\|_{L^6_t x} = \| \psi_x - w_\infty \|_{L^6_t x} < \frac{1}{2}.$$ 

(6.15) By the triangle inequality, Lemma 6.4, Proposition 6.5, and Lebesgue’s dominated convergence theorem,

$$\| \tilde{v}_n - e^{-it} D_{\lambda_n} w_\infty(\lambda_n^{-2}t) \|_{L^6_t x} \lesssim \| \tilde{v}_n \|_{L^6_t x} + \| w_\infty \|_{L^6_t x} + \| w_\infty \|_{L^6_t x} ; L^6 t \times \mathbb{R} \to 0.$$ 

(6.16) From (6.14), (6.15), and (6.16), we obtain

$$\| v_n(t + t_n) - e^{-it} D_{\lambda_n} \psi_x(\lambda_n^{-2}t) \|_{L^6_t x} \leq \| v_n(t + t_n) - \bar{v}_n(t) \|_{L^6_t x} + \| \bar{v}_n - e^{-it} D_{\lambda_n} w_\infty(\lambda_n^{-2}t) \|_{L^6_t x} + \| e^{-it} D_{\lambda_n}[w_\infty(\lambda_n^{-2}t) - \psi_x(\lambda_n^{-2}t)] \|_{L^6_t x} \to 0,$$

which concludes the proof of Theorem 6.3 with $\nu_n \equiv 0$ and $s = \frac{1}{2}$. Moreover, (6.6) yields that

$$\| (\partial_x^s) v_n(t) \|_{L^6_t x} \lesssim \| (\partial_x^s) (v_n(t) - \bar{v}_n(t - t_n)) \|_{L^6_t x} + \| (\partial_x^s) \bar{v}_n \|_{L^6_t x} \to 0$$

for any $s > 0$. In addition, since $v_n$ satisfies (2.1), we use $W^{2/15, 6}(\mathbb{R}) \hookrightarrow L^{30}(\mathbb{R})$ to obtain

$$\| (\partial_t + i) v_n(t) \|_{L^6_t x} \leq \| (\partial_x) - 1 \| v_n(t) \|_{L^6_t x} + \| (\partial_x)^{-1} N(\mathcal{R}v_n) \|_{L^6_t x} \lesssim \| (\partial_x^s) v_n(t) \|_{L^6_t x} + \sum_{l=2}^\infty \frac{1}{l!} \| \mathcal{R}v_n \|_{L^6_t L^p} \| \mathcal{R}v_n \|_{L^6_t L^p} \| \mathcal{R}v_n \|_{L^6_t x} \lesssim \| (\partial_x^s) v_n(t) \|_{L^6_t x} + \| (\partial_x^s) v_n \|_{L^6_t x} \sum_{l=2}^\infty \frac{1}{l!} \| v_n \|_{L^6_t H^2} \to 0.$$ 

Second, we consider the general case $\nu_n \neq 0$. From (iii) in Lemma 2.3

$$\phi_n = T_{x_n} e^{it_n (\partial_x)} L_{\nu_n} D_{\lambda_n} P_{\leq \lambda_n^0} \phi = L_{\nu_n} T_{x_n} e^{it_n (\partial_x)} D_{\lambda_n} P_{\leq \lambda_n^0} \phi.$$ 

By spatial translation invariance, we may choose $x_n = \frac{x_n}{(\partial_{t_n})} t_n$, which implies $\bar{x}_n = 0$ and $\bar{t}_n = \frac{t_n}{(\partial_{t_n})}$. By the case $\nu_n \equiv 0$, for sufficiently large $n$, there is a global solution $v_n^0$ to (2.1) with initial data

$$v_n^0(0) = e^{it_n (\partial_x)} D_{\lambda_n} P_{\leq \lambda_n^0} \phi.$$ 

Moreover, it obeys

$$\lim_{n \to \infty} \left( \| (\partial_x v_n^0) \|_{L^6_t x} + \| (\partial_t + i) v_n^0 \|_{L^6_t x} \right) = 0,$$

(6.17) which establishes the scattering.
and for each $\varepsilon > 0$, there exists $\psi_n^0 \in C^\infty(\mathbb{R} \times \mathbb{R})$ and $N_n^0 \in \mathbb{N}$ such that

$$
\left\| \Re \left\{ \psi_n^0(t + \bar{t}_n, x) - e^{-it/\lambda_n^2} \psi_n^0 \left( \frac{t}{\lambda_n^2}, \frac{x}{\lambda_n} \right) \right\} \right\|_{L^6_{t,x}} < \varepsilon
$$

whenever $n \geq N_n^0$. Since $\psi_n^0$ solves (2.1), $\psi_n^0 := \Re v_n^0$ solves (1.1). Thus, by the Lorentz invariance, $u_n^1 := u_n^0 \circ L_{\nu_n}$ also solves (1.1) and

$$
v_n^1 := (1 + i(\partial_x)^{-1}\partial_t)u_n^1 = (1 + i(\partial_x)^{-1}\partial_t)(\Re v_n^0 \circ L_{\nu_n})
$$
solves (2.1).

**Proposition 6.8.** For $n$ sufficiently large, $v_n^1$ is a global solution to (2.1). Moreover,

$$
\sup_{n \in \mathbb{N}} \left\{ \|\Re v_n^1\|_{L^\infty_{t,x}} + S_\mathbb{R}(\{\partial_x\}^{1/2}v_n^1) \right\} \lesssim_{M(\phi)} 1,
\lim_{n \to \infty} \|v_n^1(0) - \phi_n\|_{H^1} = 0.
$$

**Proof.** By Corollary 3.5, $u_n^1$ is a strong solution to (1.1). Hence, $v_n^1$ is a strong solution to (2.1). By the definition, we have

$$
\|\Re v_n^1\|_{L^\infty_{t,x}} = \|\Re v_n^0\|_{L^\infty_{t,x}} \leq \|v_n^0\|_{L^\infty_x H^1_t} \lesssim_{M(\phi)} 1.
$$

Since, by $v_n^0 = (1 + i(\partial_x)^{-1}\partial_t)u_n^0$ and $u_n^0 = \Re v_n^0$,

$$
\|\partial_t u_n^0\|_{L^6_{t,x}} \leq \|\partial_x v_n^1\|_{L^6_{t,x}} + \|\partial_x u_n^0\|_{L^6_{t,x}} \lesssim \|\partial_x v_n^1\|_{L^6_{t,x}} \lesssim_{M(\phi)} 1,
$$

we have

$$
S_\mathbb{R}(\{\partial_x\}^{1/2}v_n^1) \lesssim \|\partial_x v_n^1\|_{L^6_{t,x}} \lesssim \|v_n^0\|_{L^6_{t,x}} + \langle \nu_n \rangle \|\partial_x u_n^0\|_{L^6_{t,x}} + \|\partial_x u_n^0\|_{L^6_{t,x}} \lesssim_{M(\phi)} 1.
$$

We decompose

$$
u_n^0 = u_n^0 + \bar{u}_n^0,
$$

where $u_n^0$ solves the linear Klein-Gordon equation with initial data

$$
(1 + i(\partial_x)^{-1}\partial_t)u_n^0(0,t) = v_n^0(0) = \mathbf{L}^{-1}_{\nu_n}\phi_n.
$$

Since (2.1) implies

$$
(1 + i(\partial_x)^{-1}\partial_t)[u_n^0 \circ L_{\nu_n}](t) = e^{-it(\partial_x)}\mathbf{L}_{\nu_n}[v_n^0(0)] = e^{-it(\partial_x)}\phi_n,
$$

we get

$$
\|v_n^1(0) - \phi_n\|_{H^1} \leq \|\mathbf{L}_{\nu_n}[v_n^0(0)]\|_{H^1} + \|\partial_t[u_n^0 \circ L_{\nu_n}](0,\cdot)\|_{L^2}.
$$

Let

$$
\Omega_n := \{(t, x) : 0 < \langle \nu_n \rangle t < -\nu_n x \} \cup \{(t, x) : -\nu_n x < \langle \nu_n \rangle t < 0\}.
$$

Then, $\partial \Omega_n = L_{\nu_n}(0, \mathbb{R}) \cup \{0\} \times \mathbb{R}$. By the same argument as in the proof of Corollary 3.5, $\bar{u}_n^0(0) = 0$, and the convergence of $\{\nu_n\}$, we get

$$
\lim_{n \to \infty} \sup_{n \in \mathbb{N}} \left( \frac{1}{2} \|\mathbf{L}_{\nu_n}[v_n^0(0, \cdot)]\|_{H^1} + \frac{1}{2} \|\partial_t[u_n^0 \circ L_{\nu_n}](0, \cdot)\|_{L^2} \right)
$$

$$
\leq \lim_{n \to \infty} \langle \nu_n \rangle \int_{\Omega_n} \mathcal{N}(u_n^0(t, x)) \nabla_{t,x} \bar{u}_n^0(t, x) \, dx \, dt
$$

$$
\lesssim \lim_{n \to \infty} \|u_n^0\|_{L^6_{t,x}(\Omega_n)} \|\nabla_{t,x} \bar{u}_n^0\|_{L^6_{t,x}(\mathbb{R} \times \mathbb{R})} \sum_{i=2}^{\infty} \frac{1}{i!} \|u_n^0\|_{L^6_{t,x}(\mathbb{R} \times \mathbb{R})}^{2i-4}.
$$
By (6.19), it suffices to show that
\[ \| \nabla_{t,x} \tilde{v}_n^0 \|_{L^1_{t,x}} \lesssim M(\phi) 1 \]
for \( n \) sufficiently large and
\[ (6.20) \lim_{n \to \infty} \| u_n^0 \|_{L^6_{t,x}(\Omega_n)} = 0. \]
By the triangle inequality, Lemma 2.2, and Proposition 3.1,
\[ \| \nabla_{t,x} \tilde{v}_n^0 \|_{L^1_{t,x}} \leq \| \nabla_{t,x} u_n^0 \|_{L^6_{t,x}} + \| \tilde{v}_n^0 \|_{H^{3/2}} + \| (\partial_x)^{3/2} D\lambda_n P_{\lambda_n^6} \phi \|_{L^2} \]
\[ \lesssim M(\phi) \| u_n^0(0) \|_{H^{3/2}} + \| (\partial_x)^{3/2} D\lambda_n P_{\lambda_n^6} \phi \|_{L^2} \]
\[ \lesssim M(\phi) \| D\lambda_n P_{\lambda_n^6} \phi \|_{H^{3/2}} \lesssim M(\phi) 1. \]

Thanks to (6.18), the estimate (6.20) follows from
\[ \lim_{n \to \infty} \int_{\Omega_n} \lambda_n^{-3} \left| \psi \left( \frac{t - \bar{t}_n}{\lambda_n^2}, \frac{x}{\lambda_n} \right) \right|^6 \, dx \, dt = 0 \]
for every \( \psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}) \). If \( (t, x) \in \Omega_n \) and \( (\frac{t - \bar{t}_n}{\lambda_n^2}, \frac{x}{\lambda_n}) \in \text{supp} \psi \), then we have \( |x| \lesssim \psi \lambda_n \) and \( |t| < |x| \). Therefore,
\[ \lim_{n \to \infty} \int_{\Omega_n} \lambda_n^{-3} \left| \psi \left( \frac{t - \bar{t}_n}{\lambda_n^2}, \frac{x}{\lambda_n} \right) \right|^6 \, dx \, dt \lesssim \psi \lim_{n \to \infty} \lambda_n^{-3} \lambda_n^2 = \lim_{n \to \infty} \lambda_n^{-1} = 0. \]
This completes the proof. \( \square \)

From Propositions 3.1, 2.2 and 6.8, we obtain that for sufficiently large \( n \), there exists a global solution \( v_n \) to (2.1) with \( v_n(0) = \phi_n \) and \( \| v_n \|_{L^\infty_t(R; H^1_x)} + S_R((\partial_x)^{1/2} v_n) \lesssim M(\phi) 1. \) Moreover,
\[ \lim_{n \to \infty} \| (\partial_x)^{1/2} \text{Re}\{ v_n - v_n^1 \} \|_{L^6_{t,x}} = 0. \]

Let \( s \in [\frac{1}{2}, 1] \). From \( \text{Re} v_n^1 = \text{Re} v_n^0 \circ L_{\nu_n} \) and (6.17), we have
\[
\left\| \left( (\partial_x)^{s-1/2} \nu \right) v_n^0 \right\|_{L^6_{t,x}} \leq \left\| \left( (\partial_x)^{s-1/2} \nu \right) v_n^0 \circ L_{\nu_n} \right\|_{L^6_{t,x}} + \| (\nu)^{s-1/2} \|_{L^6_{t,x}} \lesssim \| (\partial_x - i\nu_n) (\text{Re} v_n^0 \circ L_{\nu_n}) \|_{L^6_{t,x}} + |\nu_n - \nu| \| \text{Re} v_n^0 \|_{L^6_{t,x}} \lesssim |\nu_n| \| (\partial_x + i) \text{Re} v_n^0 \|_{L^6_{t,x}} + |\nu_n - \nu| \| \text{Re} v_n^0 \|_{L^6_{t,x}} \to 0,
\]
as \( n \to \infty \). We therefore obtain
\[
\left\| \text{Re} \left( (\partial_x)^{s-1/2} v_n \circ L_{\nu_n}(t + \bar{t}_n, x + \bar{x}_n) - (\nu)^{s-1/2} e^{-it \lambda_n^2/2} \psi \left( \frac{t}{\lambda_n^2}, \frac{x}{\lambda_n} \right) \right) \right\|_{L^6_{t,x}} \lesssim \| (\partial_x)^{1/2} \text{Re} \{ v_n - v_n^1 \} \|_{L^6_{t,x}} + \| (\partial_x)^{s-1/2} \nu \|_{L^6_{t,x}} \to 0,
\]
which concludes the proof of Theorem 6.3. \( \square \)
6.1. Remark on the focusing cases. The arguments in [10] are also valid for the focusing cases if we assume that the initial data are below the corresponding static solutions. More precisely, we can treat the following focusing equation

\( -iv_t + (\partial_x)v - (\partial_x)^{-1}N(Re v) = 0 \)  

or

\( -iv_t + (\partial_x)v - \frac{1}{2}(\partial_x)^{-1}|Re v|^4 Re v = 0. \)

We recall the scattering result obtained by Dodson [2] for the focusing mass-critical nonlinear Schrödinger equation:

\( (i\partial_t + \frac{1}{2}\partial_x^2)w = -\frac{5}{32}|w|^4w. \)

The standing wave solution \( w_Q \) associated to (6.23) is

\[ w_Q(t,x) := e^{it\sqrt{\frac{2}{5}}Q(\sqrt{2}x)}, \]

where \( Q \) is the ground state, which is defined (1.4). Note that

\[ M(w_Q) = \frac{4}{\sqrt{5}}M(Q). \]

**Theorem 6.9 ([2]).** Let \( w_0 \in L^2(\mathbb{R}) \) and assume that \( M(w_0) < \frac{4}{\sqrt{5}}M(Q) \). Then, there exists a unique global solution \( w \in C(\mathbb{R}; L^2_x(\mathbb{R})) \) to (6.1) with \( w(0) = w_0 \). Furthermore, the solution \( w \) satisfies

\[ \|w\|_{L^6_t(L^2_x(\mathbb{R} \times \mathbb{R}))} \leq C(M(u_0)) \]

As a consequence, \( w \) scatters as \( t \to \pm \infty \) in \( L^2_x(\mathbb{R}) \), that is, there exists \( w_\pm \in L^2(\mathbb{R}) \) such that the identity holds:

\[ \lim_{t \to \pm \infty} \|w(t) - e^{it\sqrt{\frac{2}{5}}}w_\pm\|_{L^2_x} = 0. \]

Conversely, for any \( w_\pm \in L^2(\mathbb{R}) \), there exists a unique global solution \( w \) to (6.23) so that the above holds.

**Corollary 6.10.** Let \( \phi \in L^2(\mathbb{R}) \), and assume also that \( M(\phi) < \frac{4}{\sqrt{5}}M(Q) \). Then, the same statement as in Theorem 6.9 holds true for (6.21) or (6.22).

7. Minimal-energy blowup solutions

In this section, we construct so-called minimal-energy blowup solutions in the contradiction argument. First we introduce the definition of almost periodicity modulo translations.

**Definition (Almost periodicity modulo translations).** We say that a global solution \( u \) to (1.1) is almost periodic modulo translations (in \( H^1_x \times L^2_x \)) if there exist functions \( x : \mathbb{R} \to \mathbb{R} \) and \( C : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for every \( t \in \mathbb{R} \) and \( \eta > 0 \), we have

\[ \int_{|x-x(t)|> C(\eta)} (|u(t,x)|^2 + |u_x(t,x)|^2 + |u_t(t,x)|^2) \, dx < \eta \]

\[ \int_{|\xi|> C(\eta)} (|\hat{u}(t,\xi)|^2 + |\hat{u}_x(t,\xi)|^2) \, d\xi < \eta. \]

We refer to \( x(t) \) as the spatial center function and to \( C \) as the compactness modulus function.
We suppose that Theorem 1.1 fails. Then, there exists a critical energy $E_c > 0$ such that if $u$ is a real-valued solution to (1.1) with $E(u, u_t) < E_c$, then the solution $u$ scatters. The goal of this section is to prove the following.

**Theorem 7.1.** Suppose that Theorem 1.1 fails. Then there exists a global solution $u$ to (1.1) with $E(u, u_t) = E_c$. Moreover, $u$ is almost periodic modulo translations and blows up (that is, possesses infinite scattering size) both forward and backward in time.

**Remark 7.2.** The global solution $u$ constructed in Theorem 7.1 has zero momentum, i.e., $P(u) = 0$. Indeed, if $P(u) \neq 0$, the Cauchy-Schwarz inequality yields that

$$|P(u)| \leq \|u_x\|_{L^2} \|u_t\|_{L^2} \leq \frac{1}{2} \|u_x\|^2_{L^2} + \frac{1}{2} \|u_t\|^2_{L^2} \leq E(u, u_t).$$

Here, equality in the first inequality would imply that $u$ is a solution to a transport equation, namely $u(t, x) = u_0(x - kt)$ for some $k \in \mathbb{R}$, which is inconsistent with the fact that $u$ is a solution to (1.1) and $P(u) \neq 0$. By setting $
u := -\frac{P(u)}{\sqrt{E(u, u_t)^2 - P(u)^2}}$, it follows from Corollary 3.5 that $u_\nu := u \circ L_\nu$ is a global solution to (1.1) with $P(u_\nu) = 0$ and

$$E(u_\nu, u_\nu_t) = \sqrt{E(u, u_t)^2 - P(u)^2} < E(u, u_t) = E_c.$$  

This contradicts the criticality of $E_c$.

Once we get the following Proposition, Theorem 7.1 follows from the same argument as in the proof of Theorem 7.19 in [11]. Hence, we only prove Proposition 7.3.

**Proposition 7.3.** Suppose that Theorem 1.1 fails. Let $s \in \left(\frac{1}{2}, \frac{11}{12}\right)$ and let $E_c$ be the critical energy. Assume that $\{u_n\}$ is a sequence of global solutions to (1.1) satisfying

$$E(u_n) \leq E_c \text{ and } \lim_{n \to \infty} E(u_n) = E_c,$$

$$\lim_{n \to \infty} S_{\leq 0}(\partial_x)^{s-1/2} u_n = \lim_{n \to \infty} S_{\geq 0}(\partial_x)^{s-1/2} u_n = \infty.$$  

Then, after passing to a subsequence, $(u_n(0), \partial_t u_n(0))$ converges in $H^1 \times L^2$, modulo translation.

Let $v_n := u_n + i(\partial_x)^{-1} \partial_t u_n$ to work with the first-order version of our equation. Since we are considering defocusing case, we have

$$\|v_n(0)\|^2_{H^1} \lesssim E(v_n) \lesssim E_c < \infty,$$

which implies that $\{v_n(0)\}$ is a bounded sequence in $H^1$. By applying the linear profile decomposition (Theorem 5.1),

$$v_n(0) = \sum_{j=1}^{J} \phi_n^j(w_n^j, 1 \leq J < J_0,$$
where \( \phi^j_n := T_{x_n} e^{it_n^j \langle \partial_x \rangle} L_{v_n} D_{x_n} P_{J_n} \phi^j \). We have \( J_0 \geq 2 \) since \((7.4)\) is not compatible with \((5.2)\) when \( J_0 = 1 \). Passing to a subsequence, we may assume that \( E(\phi^j_n) \) converges for each \( j \in [1, J_0] \). By the energy decoupling \((7.6)\), we have

\[
\lim_{n \to \infty} \left\{ \sum_{j=1}^J E(\phi^j_n) + E(w_n^j) \right\} = \lim_{n \to \infty} E(v_n) = E_c,
\]

for each \( J \in [1, J_0] \). One of the following scenarios occurs by the positivity of the energy.

**Case I.** There is only a single profile and it satisfies

\[
\lim_{n \to \infty} E(\phi^1_n) = E_c.
\]

**Case II.** There exists \( \delta > 0 \) such that for every \( 1 \leq j < J_0 \),

\[
\lim_{n \to \infty} E(\phi^j_n) < E_c - \delta.
\]

In Case I, we have \( v_n \to 0 \) in \( H^1 \) as \( n \to \infty \). Moreover we divide the following three cases in this case:

**Case I-A.** \( \lambda^1_n \to \infty \).

**Case I-B.** \( \lambda^1_n \equiv 1 \) and \( t^1_n \to \pm \infty \).

**Case I-C.** \( \lambda^j_n \equiv 1 \) and \( t^j_n \equiv 1 \).

We will observe that both of the first and second cases do not occur. Note that Case I-C implies the conclusion of the proposition.

In Case I-A, by using Theorem \( \[6.3\] \) the stability theory (Proposition \( \[3.2\] \), and the fact \( w_n^1 \to 0 \) in \( H^1 \), we get a contradiction.

Next we consider Case I-B. Suppose that \( \lambda^1_n \equiv 1 \) for all \( n \in \mathbb{N} \) and \( t^1_n \to \pm \infty \) as \( n \to \infty \). We only treat the case \( t^1_n \to -\infty \) since the other case can be treated in a similar manner. By the Strichartz inequality (Lemma \( \[2.2\] \), we see that \( \langle \partial_x \rangle s^{-1/2} e^{-it \langle \partial_x \rangle} \phi^1 \in L^{p_s}_{t,x}(\mathbb{R} \times \mathbb{R}) \), and so

\[
\| \langle \partial_x \rangle s^{-1/2} e^{-it \langle \partial_x \rangle} \phi^1 \|_{L^{p_s}_{t,x}(\mathbb{R} \times \mathbb{R})} \to 0 \quad \text{as } n \to \infty.
\]

Hence by the local well-posedness (Proposition \( \[3.1\] \), and see also Proposition \( \[3.2\] \), if \( v^1_n \) is the unique solution to \((2.1)\) with initial data \( v^1_n(0) = \phi^1_n \), then for \( n \) sufficiently large, \( S_{\geq 0}(\langle \partial_x \rangle) v^1_n < \infty \). As in Case I-A, we can now use the stability lemma (Proposition \( \[4.2\] \) to conclude that for sufficiently large \( n \), we see \( S_{\geq 0}(\langle \partial_x \rangle) v^1_n < \infty \), which derives a contradiction. Hence we consider Case II.

Case II: We will show that this case is inconsistent with \((7.4)\) by using the identity \((7.6)\) to produce a nonlinear profile decomposition of the \( v_n \) and then applying the stability theory (Proposition \( \[3.2\] \). We begin by introducing nonlinear profiles \( \phi^j_n \), whose definition depends on the behavior of the parameters \( \lambda^j_n \).

First assume that \( j \) is such that \( \lambda^j_n \equiv 1 \) for all \( n \in \mathbb{N} \). Then we see that \( \phi^j \in H^1_x \) and

\[
\phi^j_n = T_{x_n} e^{it_n^j \langle \partial_x \rangle} \phi^j.
\]

If, in addition, \( t^j_n \equiv 0 \) for all \( n \in \mathbb{N} \), then we set \( v^j \) be the maximal-lifespan solution to \((2.1)\) with \( v^j(0) = \phi^j \). If \( t^j_n \to -\infty \) (respectively \( t^j_n \to \infty \)), then we set \( v^j \) be the maximal-lifespan solution to \((2.1)\) which scatters forward (respectively backward) in time to \( e^{-it \langle \partial_x \rangle} \phi^j \).

**Lemma 7.4.** In Case II, if \( \lambda^j_n \equiv 1 \) for some \( j \), then \( v^j \) defined as above is global and scatters.
Proof of Lemma 7.4. This follows from the well-posedness result (Proposition 3.1). If \( t_n^J \equiv 0 \), then \( E(\phi_n^j) = E(\phi^j) = E(v^j) \) by the conservation of the energy. If \( t_n^J \to \pm \infty \), then by using the dispersive estimate (Lemma 2.1) and approximating \( \phi^j \) in \( H_x^2 \) by Schwartz functions, we see that
\[
\lim_{n \to \infty} E(\phi_n^j) = \lim_{n \to \infty} \frac{1}{2}||\phi_n^j||^2_{H_x^1} = \frac{1}{2}||\phi^j||^2_{H_x^1} = E(v^j).
\]
Since we are considering Case II, we have \( E(v^j) < E_c \), which implies that \( v^j \) scatters as \( t \to \pm \infty \).

If \( \lambda_n^j \equiv 1 \) for all \( n \in \mathbb{N} \), we may define nonlinear profiles by
\[
v_n^j(t,x) := v^j(t-t_n^j, x-x_n^j).
\]

Next, we consider the case where \( \lim_{n \to \infty} \lambda_n^j = \infty \). Then by Theorem 6.3, for sufficiently large \( n \), we define \( v_n^j \) as the unique solution to (2.1) with initial data \( v_n^j(0) = \phi_n^j \).

Lemma 7.5. In Case II, for each \( j \) (regardless of the behavior of the \( \lambda_n^j \)) we have
\[
\lim_{n \to \infty} E(v_n^j) = \lim_{n \to \infty} E(\phi_n^j),
\]
(7.10)
\[
\lim_{n \to \infty} \|\mathbb{R}(\langle \partial_x \rangle^{s-1/2} v_n^j)\|_{L_{t,x}^6} \lesssim \lim_{n \to \infty} E(v_n^j).
\]
Furthermore, for each \( j \) and \( \varepsilon > 0 \), there exists \( \psi^j = \psi_{\varepsilon}^j \in C^\infty_c(\mathbb{R} \times \mathbb{R}) \) and \( N_{j,\varepsilon} \) such that if \( \psi_n^j \) is defined as in Theorem 6.3 and \( n > N_{j,\varepsilon} \), then we have
\[
\|\mathbb{R}(\psi_n^j - \langle \partial_x \rangle^{s-1/2} v_n^j)\|_{L_{t,x}^6} < \varepsilon.
\]
(7.11)

Proof of Lemma 7.5. Equality (7.9) is a tautology if \( \lambda_n^j \to \infty \) as \( n \to \infty \) or \( \lambda_n^j \equiv 1 \) and \( t_n^j \equiv 0 \) for all \( n \in \mathbb{N} \), since in these cases \( v_n^j(0) = \phi_n^j \). If \( \lambda_n^j \equiv 1 \) for all \( n \in \mathbb{N} \) and \( t_n^j \to \pm \infty \) as \( n \to \infty \), then by the definition of \( v_n^j \) and the local well-posedness result (Proposition 3.1) (see small data scattering statement) we have
\[
E(\phi_n^j) = E(v^j) = \frac{1}{2}||\phi^j||^2_{H_x^1} = \lim_{n \to \infty} E(\phi_n^j),
\]
where for the last equality, we have used the dispersive estimate (Lemma 2.1) as in the proof of Lemma 7.4.

When the value \( \lim_{n \to \infty} E(\phi_n^j) \) is below the small data scattering threshold, (7.10) follows from the well-posedness result (Proposition 3.1). On the other hand, by the identities (7.7), the limiting energy can only exceed this small data scattering threshold for finitely many values of \( j \). For these cases, we invoke the estimate (7.8) and the definition of the critical energy \( E_c \). As we are invoking the contradiction hypothesis here, there is no hope of being explicit about the constant in (7.10) other than that it is independent of \( j \).

As for (7.11), in the case \( \lambda_n^j \equiv 1 \) for all \( n \in \mathbb{N} \), this follows from the fact that \( v_n^j \) is just a translation of \( \langle \partial_x \rangle^{s-1/2} v^j \in L_{t,x}^6(\mathbb{R} \times \mathbb{R}^2) \). In the case \( \lambda_n^j \to \infty \) as \( n \to \infty \), this approximation follows from Theorem 6.3. This completes the proof of the lemma.

We continue to prove Proposition 7.6. For \( 1 \leq J < J_0 \), we set
\[
V_n^J(t) := \sum_{j=1}^J v_n^j(t) + e^{-it\langle \partial_x \rangle} u_n^J,
\]
which is defined globally in time for sufficiently large \( n \) (depending on \( J \)). Our immediate goal is to show that \( V^J_n(t) \) is a good approximation to \( v_n(t) \) when \( n \) and \( J \) are sufficiently large by the stability theory (Proposition 3.2).

**Lemma 7.6.** We have the following spacetime bounds on \( V^J_n \)

\[
\limsup_{J \to \infty} \limsup_{n \to \infty} \left\{ \| \langle \partial_x \rangle^{s-1/2} R V^J_n \|_{L^6_{t,x}} + \| V^J_n \|_{L^\infty_t H^s_x} \right\} < \infty.
\]

The \( V^J_n \) are approximate solutions to (2.1) in the sense that

\[
(-i\partial_t + \langle \partial_x \rangle) V^J_n + \langle \partial_x \rangle^{-1} N(\Re V^J_n) = E^J_n,
\]

where

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \| \langle \partial_x \rangle^{s+1/2} E^J_n \|_{L^{6/5}_{t,x}} = 0.
\]

Furthermore, for each \( J \), we have

\[
\lim_{n \to \infty} \| v_n(0) - V^J_n(0) \|_{H^s_x} = 0.
\]

**Proof of Lemma 7.6.** First we prove (7.14). By the triangle inequality and the definitions of \( v^J_n \),

\[
\lim_{n \to \infty} \| v_n(0) - V^J_n(0) \|_{H^s_x} \leq \lim_{n \to \infty} \sum_{j=1}^J \| v^J_n(0) - \phi^J_n \|_{H^s_x} = 0.
\]

We note that combining (7.10) and (7.7) yields

\[
\limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J \| \langle \partial_x \rangle^{s-1/2} R v^J_n \|_{L^6_{t,x}} \lesssim \lim_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J E(v^J_n) \leq \tilde{E}_c.
\]

We now bound the \( L^6_t W^{s-1/2}_x \) term in (7.12). By (7.11) and Proposition 5.3, the nonlinear profiles decouple in the sense that whenever \( j \neq j' \), we have

\[
\lim_{n \to \infty} \| (\langle \partial_x \rangle)^{s-1/2} R v^J_n (\langle \partial_x \rangle)^{s-1/2} R v^{J'}_n \|_{L^6_{t,x}} = 0.
\]

Combining this with (5.2) and then using (7.15) show

\[
\limsup_{J \to \infty} \limsup_{n \to \infty} \| \langle \partial_x \rangle^{s-1/2} R V^J_n \|_{L^6_{t,x}} \leq \limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J \| \langle \partial_x \rangle^{s-1/2} R v^J_n \|_{L^6_{t,x}} \lesssim E^J_c.
\]

Thus we get the first estimate in (7.12).

Next, we prove (7.13). A simple computation shows that

\[
E^J_n(t) = \langle \partial_x \rangle^{-1} N \left( \Re \sum_{j=1}^J v^J_n(t) + \Re e^{-it\langle \partial_x \rangle} w^J_n \right) - \langle \partial_x \rangle^{-1} \sum_{j=1}^J N(\Re v^J_n),
\]
and so, by the triangle inequality, it is enough to show

\[(7.18)\]
\[
\lim_{J \to \infty} \lim_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \left\{ \mathcal{N} \left( \sum_{j=1}^{J} \Re v_j^n + \Re e^{-it(\partial_x)} w_j^n \right) - \mathcal{N} \left( \sum_{j=1}^{J} \Re v_j^n \right) \right\} \right\|_{L^{6/5}_{x,t}} = 0,
\]

\[(7.19)\]
\[
\lim_{J \to \infty} \lim_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \left\{ \mathcal{N} \left( \sum_{j=1}^{J} \Re v_j^n \right) - \sum_{j=1}^{J} \mathcal{N} \left( \Re v_j^n \right) \right\} \right\|_{L^{6/5}_{x,t}} = 0.
\]

For simplicity, we set \( A := \sum_{j=1}^{J} \Re v_j^n \) and \( B := \Re e^{-it(\partial_x)} w_j^n \). Since \( \mathcal{N}(u) = \sum_{l=2}^{\infty} \frac{u^{2l+1}}{l^{2l+1}} \), (7.18) can be estimated as follows.

\[
\left\| \langle \partial_x \rangle^{s-1/2} \left\{ \mathcal{N}(A + B) - \mathcal{N}(A) \right\} \right\|_{L^{6/5}_{x,t}} \lesssim \sum_{l=2}^{\infty} \frac{1}{l!} \left\| \langle \partial_x \rangle^{s-1/2} \left\{ (A + B)^{2l+1} - A^{2l+1} \right\} \right\|_{L^{6/5}_{x,t}}.
\]

Since we have

\[
(A + B)^{2l+1} - A^{2l+1} = \sum_{m=1}^{2l+1} \binom{2l+1}{m} A^{2l+1-m} B^m,
\]
we get

\[(7.20)\]
\[
\sum_{l=2}^{\infty} \frac{1}{l!} \left\| \langle \partial_x \rangle^{s-1/2} \left\{ (A + B)^{2l+1} - A^{2l+1} \right\} \right\|_{L^{6/5}_{x,t}} \lesssim \sum_{l=2}^{\infty} \frac{1}{l!} \sum_{m=1}^{2l+1} \binom{2l+1}{m} \left\| \langle \partial_x \rangle^{s-1/2} (A^{2l+1-m} B^m) \right\|_{L^{6/5}_{x,t}}.
\]

If \( 1 \leq m \leq 4 \), then, by the fractional Leibniz rule, we have

\[
\left\| \langle \partial_x \rangle^{s-1/2} (A^{2l+1-m} B^m) \right\|_{L^{6/5}_{x,t}} \lesssim \int \left\| \langle \partial_x \rangle^{s-1/2} A \right\|_{L^6_{x,t}} \left\| A \right\|_{L^{2l+3}_{x,t}} \left\| A \right\|_{L^6_{x,t}} \left\| B \right\|_{L^m_{x,t}}^m + \left\| \langle \partial_x \rangle^{s-1/2} B \right\|_{L^6_{x,t}} \left\| A \right\|_{L^{2l+3}_{x,t}} \left\| A \right\|_{L^6_{x,t}} \left\| B \right\|_{L^{m-1}_{x,t}}^{m-1}.
\]

If \( 5 \leq m \leq 2l \), we have

\[
\left\| \langle \partial_x \rangle^{s-1/2} (A^{2l+1-m} B^m) \right\|_{L^{6/5}_{x,t}} \lesssim \int \left\| \langle \partial_x \rangle^{s-1/2} A \right\|_{L^6_{x,t}} \left\| A \right\|_{L^{2l-m}_{x,t}} \left\| B \right\|_{L^{4}_{x,t}}^4 \left\| B \right\|_{L^{m-4}_{x,t}}^{m-4} + \left\| \langle \partial_x \rangle^{s-1/2} B \right\|_{L^6_{x,t}} \left\| A \right\|_{L^{2l-m+1}_{x,t}} \left\| A \right\|_{L^6_{x,t}}^4 \left\| B \right\|_{L^{m-5}_{x,t}}^{m-5}.
\]

If \( m = 2l + 1 \), we have

\[
\left\| \langle \partial_x \rangle^{s-1/2} (A^{2l+1-m} B^m) \right\|_{L^{6/5}_{x,t}} \lesssim \int \left\| \langle \partial_x \rangle^{s-1/2} A \right\|_{L^6_{x,t}} \left\| B \right\|_{L^{4}_{x,t}}^4 \left\| B \right\|_{L^{2l-4}_{x,t}}^{2l-4}.
\]

We note that \( \left\| \langle \partial_x \rangle^{s-1/2} A \right\|_{L^6_{x,t}} + \left\| \langle \partial_x \rangle^{s-1/2} B \right\|_{L^6_{x,t}} \leq C \) holds by (7.2), (7.15), and (7.16), where \( C = C(E_c) \) is a positive constant independent of \( n \) ant \( J \). Moreover, we also have \( \left\| A \right\|_{L^\infty_{x,t} H^2_x} + \left\| B \right\|_{L^\infty_{x,t} H^2_x} \leq C \). While \( \left\| B \right\|_{L^\infty_{x,t} H^2_x} \leq C \) is trivial,
Since \((7.9)\), \((7.10)\), and \((7.7)\) yield that \((\text{Lemma } 2.2)\) and the fractional Leibniz rule to obtain

\[
\|\mathcal{R}n,J\|_{L^6} \leq C
\]

Thus, by \((5.2)\) and the Lebesgue dominated convergence theorem, we obtain

\[
\|\mathcal{R}n,J\|_{L^6} \leq \limsup_{n \to \infty} \left( \sum_{j=1}^{J} \frac{1}{\gamma} \left( \sum_{m=1}^{M} \left( \frac{1}{\gamma} \right)^{\frac{j(j-1)}{2}} \right)^{\frac{1}{\gamma}} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right) = 0.
\]

as \(n, J \to \infty\). Thus we obtain \((7.18)\).

Here, we show \(\|A\|_{L^T^\infty H_x^s} \leq C\). Note that \(A = \sum_{j=1}^{J} \mathcal{R}n,J\) satisfies the following equation.

\[
(-i\partial_t + \langle \partial_x \rangle) A + \langle \partial_x \rangle^{-1} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) = 0.
\]

Since \((7.20)\), \((7.11)\), and \((7.7)\) yield that \(\|\mathcal{R}n,J\|_{L^6} \leq \|\langle \partial_x \rangle^{-1/2} (\mathcal{R}n,J)\|_{L^6} \leq C\) and \(\|\mathcal{R}n,J\|_{L^6} \leq \mathcal{E}(\mathcal{R}n,J)^{1/2} \leq C\), we apply the Strichartz estimate (Lemma \((2.2)\)) and the fractional Leibniz rule to obtain

\[
\limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|_{L^1_x L^{6/5}_t} \leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|_{L^1_x L^{6/5}_t}
\]

\[
\leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|_{L^{6/5}_x}
\]

\[
\leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|_{L^{6/5}_x}
\]

\[
\leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|_{L^{6/5}_x}
\]

\[
\leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|_{L^{6/5}_x}
\]

\[
\leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|^{2}_{L^{6/5}_x}
\]

\[
\leq \limsup_{n \to \infty} \left\| \langle \partial_x \rangle^{s-1/2} \sum_{j=1}^{J} \mathcal{N}(\mathcal{R}n,J) \right\|^{2}_{L^{6/5}_x}
\]
\begin{align*}
\lesssim & \sum_{l=2}^{\infty} \frac{C^{2l-1}}{(l-1)!} \limsup_{n \to \infty} E(\phi_n^l) \\
\lesssim & C^3 \sum_{l=1}^{\infty} \frac{C^{2l}}{l!} \\
\leq & C^3 \exp(C^2).
\end{align*}

By (7.14) and (7.5), we get

\begin{align*}
\limsup_{n \to \infty} \| A \|_{L^\infty_t H^s_x} \\
\lesssim & \limsup_{n \to \infty} \left\{ \| v_n(0) \|_{H^1_x} + \| w_n^J \|_{H^2_x} + \left\| (\partial_x)^s \int_0^t e^{-i(t-s)}(\partial_x)^{-1} \sum_{j=1}^{J} \mathcal{N}(\Re v_n^j) ds \right\|_{L^\infty_t L^2_x} \right\} \\
\lesssim & \mathcal{E}_n 1.
\end{align*}

Next, we consider (7.19). We observe that

\begin{align*}
\left\| (\partial_x)^{-1/2} \left\{ \sum_{j=1}^{J} \mathcal{N}(\Re v_n^j) - \sum_{j=1}^{J} \mathcal{N}(\Re v_n^j) \right\} \right\|_{L^{6/5}_{t,x}} \\
\lesssim & \sum_{l=2}^{\infty} \frac{1}{l!} \left\| (\partial_x)^{-1/2} \left\{ \left( \sum_{j=1}^{J} \mathcal{N}(\Re v_n^j) \right)^{2l+1} - \sum_{j=1}^{J} (\Re v_n^j)^{2l+1} \right\} \right\|_{L^{6/5}_{t,x}}.
\end{align*}

Since we have

\begin{align*}
\left( \sum_{j=1}^{J} \mathcal{N}(\Re v_n^j) \right)^{2l+1} - \sum_{j=1}^{J} (\Re v_n^j)^{2l+1} \\
= \sum_{0 \leq m_j < 2l} \frac{(2l + 1)!}{m_1! m_2! \cdots m_J!} \left( \mathcal{N}(\Re v_n^1)^{m_1} \left( \mathcal{N}(\Re v_n^2)^{m_2} \cdots (\Re v_n^J)^{m_J} \right) \right).
\end{align*}

we get

\begin{align*}
\sum_{l=2}^{\infty} \frac{1}{l!} \left\| (\partial_x)^{-1/2} \left\{ \left( \sum_{j=1}^{J} \mathcal{N}(\Re v_n^j) \right)^{2l+1} - \sum_{j=1}^{J} (\Re v_n^j)^{2l+1} \right\} \right\|_{L^{6/5}_{t,x}} \\
\lesssim \sum_{l=2}^{\infty} \frac{1}{l!} \left\| \sum_{0 \leq m_j < 2l} \frac{(2l + 1)!}{m_1! m_2! \cdots m_J!} \left\| (\partial_x)^{-1/2} \left\{ \left( \mathcal{N}(\Re v_n^1)^{m_1} \left( \mathcal{N}(\Re v_n^2)^{m_2} \cdots (\Re v_n^J)^{m_J} \right) \right) \right\|_{L^{6/5}_{t,x}}.
\end{align*}

Now, by the fractional Leibniz rule, we have

\begin{align*}
\left\| (\partial_x)^{-1/2} \left\{ \mathcal{N}(\Re v_n^1)^{m_1} \left( \mathcal{N}(\Re v_n^2)^{m_2} \cdots (\Re v_n^J)^{m_J} \right) \right\} \right\|_{L^{6/5}_{t,x}} \\
\lesssim J! C^{2l-1} \left( \sum_{j \neq k} \left\| (\partial_x)^{-1/2} \mathcal{N}(\Re v_n^j) \right\|_{L^2_{t,x}} + \sum_{j \neq k} \left\| \mathcal{N}(\Re v_n^j) \cdot \Re v_n^k \right\|_{L^2_{t,x}} \right) \\
\leq J! C^{2l+1}.
\end{align*}
Note that \( \| (\partial_x)^{s-1/2} (\Re v_n^j \cdot \Re v_n^k) \|_{L^2_{t,x}} \) and \( \| \Re v_n^j \cdot \Re v_n^k \|_{L^2_{t,x}} \) tend to 0 as \( n \to \infty \) for any \( j \neq k \). Indeed, it follows that \( \| \Re v_n^j \cdot \Re v_n^k \|_{L^2_{t,x}} \to 0 \) by an approximation argument and, by the interpolation, we have

\[
\begin{align*}
&\| (\partial_x)^{s-1/2} (\Re v_n^j \cdot \Re v_n^k) \|_{L^2_{t,x}} \\
&\leq \| \Re v_n^j \cdot \Re v_n^k \|_{L^2_{t,x}}^{2(1-s)} \| (\partial_x)^{1/2} (\Re v_n^j \cdot \Re v_n^k) \|_{L^2_{t,x}}^{2s-1} \\
&\lesssim \| \Re v_n^j \cdot \Re v_n^k \|_{L^2_{t,x}}^{2(1-s)} \left( \| (\partial_x)^{1/2} \Re v_n^j \|_{L^6} \| \Re v_n^k \|_{L^6} + \| (\partial_x)^{1/2} \Re v_n^k \|_{L^6} \| \Re v_n^j \|_{L^6} \right)^{2s-1} \\
&\to 0
\end{align*}
\]

since \( \| (\partial_x)^{1/2} \Re v_n^j \|_{L^6}, \| \Re v_n^j \|_{L^6} \) are bounded. By the Lebesgue dominated convergence theorem, we obtain for all \( J \),

\[
(7.21) \quad \limsup_{n \to \infty} \| (\partial_x)^{s-1/2} \left\{ N \left( \sum_{j=1}^J \Re v_n^j \right) - \sum_{j=1}^J N \left( \Re v_n^j \right) \right\} \|_{L^6_{t,x}} = 0.
\]

Thus, we get (7.19).

Finally, we complete the proof of (7.12) by bounding the \( L^6_{t} H^s_x \) norm. By the Strichartz inequality, (7.14), and then (7.15), (7.17), and (7.13),

\[
\begin{align*}
&\limsup_{J \to \infty} \limsup_{n \to \infty} \| V_n^J \|_{L^6_{t} H^s_x} \\
&\lesssim \limsup_{J \to \infty} \limsup_{n \to \infty} \left\{ \| v_n(0) \|_{H^s} + C' + \| (\partial_x)^{s-1/2} E_n^J \|_{L^6_{t,x}} \right\} < \infty,
\end{align*}
\]

where we used that

\[
\begin{align*}
&\left\| (\partial_x)^s \int_0^t e^{-i(t-s)(\partial_x)^{-1} N(\Re V_n^J)} ds \right\|_{L^6_{t} L^2_x} \\
&\lesssim \left\| (\partial_x)^{s-1/2} N(\Re V_n^J) \right\|_{L^6_{t,x}} \\
&\lesssim \sum_{l=0}^{\infty} \frac{1}{l!} \left\| (\partial_x)^{s-1/2} (\Re V_n^J)^{2l+1} \right\|_{L^6_{t,x}} \\
&\lesssim \sum_{l=0}^{\infty} \frac{1}{l!} (2l+1) \left\| (\partial_x)^{s-1/2} (\Re V_n^J) \right\|_{L^6_{t,x}} \left\| \Re V_n^J \right\|_{L^6_{t,x}}^4 \left\| \Re V_n^J \right\|_{L^6_{t,x}}^{2l-4} \\
&\lesssim \sum_{l=0}^{\infty} \frac{1}{(l-1)!} E_{c}^5 C^{2l-4} \\
&\leq C',
\end{align*}
\]

where we used \( \| \Re V_n^J \|_{L^6_{t,x}} \lesssim \| \Re V_n^J \|_{L^6_{t} H^s_x} \leq A \| L^6_{t} H^s_x + \| B \|_{L^6_{t} H^s_x} \leq C < \infty \). This completes the proof of (7.12) and so also the lemma. \( \Box \)

By Lemma 7.6, we may apply the stability theorem (Proposition 3.2) to conclude that in Case II, \( v_n \) is defined globally in time and \( S_\delta(v_n) \lesssim E_{c} \) for sufficiently large \( n \). This contradicts (7.31) and so Case II cannot occur. Tracing back, we see that the only possibility is Case I-C, and so Proposition 7.3 is proved.
8. Death of a solution

In this section, we prove the following theorem (Theorem 8.1), which completes the proof of the main result (Theorem 1.1).

**Theorem 8.1.** There is no non-scattering solution to (1.1) with almost periodicity modulo translation and zero momentum.

In order to prove the theorem, we need the following lemmas (Lemma 8.2 and Lemma 8.3).

**Lemma 8.2** (controlling $x$). Let a global solution $u$ be almost periodic modulo translation and its momentum is zero, i.e. $P(u) = 0$. Then, for sufficiently small $\eta > 0$, there exists $R = R(\eta) \gtrsim C(\eta)$ such that

$$|x(t) - x(0)| \leq R - C(\eta)$$

for any $t \in [0, t_0]$ where $t_0 > cR/\eta$, where $c$ is a positive constant.

This lemma means that $|x(t)| = o(|t|)$ as $|t| \to \infty$.

**Proof.** We may assume that $x(0) = 0$. Let $R > 3C(\eta)$ and

$$t_0 := \inf\{t > 0 : |x(t)| \leq R - C(\eta)\}.$$

By the finite speed of propagation, we have $t_0 > 0$ (see the proof of Lemma 7.4 in [8]). Note that we have $|x - x(t)| \geq C(\eta)$ if $|x| \geq R$ since $|x(t)| \leq R - C(\eta)$ for $t \in (0, t_0)$. Let $\phi \in C^\infty_0(\mathbb{R})$ be an even function with $0 \leq \phi \leq 1$ and

$$\phi(x) = \begin{cases} 
1 & \text{for } |x| \leq 1, \\
0 & \text{for } |x| \geq 2.
\end{cases}$$

We define

$$X_R(t) := \int_\mathbb{R} x\phi(\frac{x}{R}) e_u(t, x) dx,$$

where $e_u := \frac{1}{2}|u|^2 + \frac{1}{2}|ux|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{2}N(u)$. Since $u$ satisfies (1.1), we have

$$\frac{d}{dt}X_R(t) = -\int_\mathbb{R} \phi(\frac{x}{R}) u_xu_t dx - \frac{1}{R} \int_\mathbb{R} x\phi(\frac{x}{R}) u_xu_t dx$$

$$= \int_\mathbb{R} \left(1 - \phi(\frac{x}{R})\right) u_xu_t dx - \frac{1}{R} \int_\mathbb{R} x\phi(\frac{x}{R}) u_xu_t dx$$

$$= \int_{|x| \geq R} \left(1 - \phi(\frac{x}{R})\right) u_xu_t dx - \frac{1}{R} \int_{R \leq |x| \leq 2R} x\phi(\frac{x}{R}) u_xu_t dx,$$

where we have used $P(u) = 0$ in the second equality. This implies that

$$\left|\frac{d}{dt}X_R(t)\right| \leq c\eta,$$

where $c$ is a positive constant. By the triangle inequality, we have

$$|X_R(0)| \leq \int_{|x| \leq C(\eta)} |x|\phi(\frac{x}{R}) e_u dx + \int_{C(\eta) \leq |x| \leq 2R} |x|\phi(\frac{x}{R}) e_u dx$$

$$\leq C(\eta)E(u, u_t) + R\eta.$$
Moreover, by the triangle inequality, we have
\[
|X_R(t)| \
\geq |x(t)| \int_R e_u(t)dx - \left| \int_R (x - x(t)) \phi \left( \frac{x}{R} \right) e_u(t)dx \right| \
- |x(t)| \left| \left( 1 - \phi \left( \frac{x}{R} \right) \right) \int_R e_u(t)dx \right| \
\geq |x(t)| \int_R e_u(t)dx - \left| \int_{|x - x(t)| < C(\eta)} (x - x(t)) \phi \left( \frac{x}{R} \right) e_u(t)dx \right|
\]
\[
- \left| \int_{|x - x(t)| \geq C(\eta)} (x - x(t)) \phi \left( \frac{x}{R} \right) e_u(t)dx \right| - |x(t)| \left| \int_R \left( 1 - \phi \left( \frac{x}{R} \right) \right) e_u(t)dx \right|
\]
\[=: I - II - III - IV.\]

Then, for \( t \in (0, t_0) \), we have
\[
I = |x(t)| E(u, u_t), \\
II \leq C(\eta) E(u, u_t), \\
III \leq \int_{|x - x(t)| \geq C(\eta), |x| \leq 2R} |x - x(t)| \phi \left( \frac{x}{R} \right) e_u(t)dx \\
\leq \int_{C(\eta) \leq |x - x(t)| \leq 2R + |x(t)|} |x - x(t)| \phi \left( \frac{x}{R} \right) e_u(t)dx \\
\leq (2R + |x(t)|) \int_{C(\eta) \leq |x - x(t)|} e_u(t)dx \\
\leq \left( R + \frac{1}{2} |x(t)| \right) \eta, \\
IV \leq |x(t)| \int_{|x| \geq R} e_u(t)dx \\
\leq |x(t)| \int_{|x - x(t)| \geq C(\eta)} e_u(t)dx \\
\leq \frac{1}{2} |x(t)| \eta.
\]

Thus, we obtain
\[
(8.3) \quad |X_R(t)| \geq |x(t)| (E(u, u_t) - \eta) - R\eta - C(\eta) E(u, u_t).
\]

By (8.1), (8.2), and (8.3), for \( \tau \in (0, t_0) \), we get
\[
c_0 \tau \geq \left| \int_0^\tau \frac{d}{dt} X_R(t) dt \right| \\
\geq |X_R(\tau)| - |X_R(0)| \\
\geq |x(t)| (E(u, u_t) - \eta) - R\eta - C(\eta) E(u, u_t) - (C(\eta) E(u, u_t) + R\eta).
\]

Letting \( \eta < \frac{1}{2} E(u, u_t) \), then we have
\[
\frac{1}{2} E(u, u_t)|x(t)| \leq c_0 \tau + 2R\eta + 2C(\eta) E(u, u_t),
\]
and thus we get
\[ |x(t)| \leq \frac{2c}{E(u,u_t)} \eta \tau + \frac{4R \eta}{E(u,u_t)} + 4C(\eta) \]

Taking \( \tau \to t_0 \), then we obtain
\[ R - C(\eta) \leq \frac{2c}{E(u,u_t)} \eta t_0 + \frac{4R \eta}{E(u,u_t)} + 4C(\eta) \]

Letting \( \eta < \frac{1}{8} E(u,u_t) \) and \( R > 20C(\eta) \), we have
\[ \frac{1}{4} R < \frac{1}{2} R - 5C(\eta) \leq \frac{2c}{E(u,u_t)} \eta t_0. \]

This means that
\[ \frac{E(u,u_t) R}{8c} \leq \eta \leq t_0. \]

\[ \square \]

**Lemma 8.3.** Let a global solution \( u \) be almost periodic modulo translation. For any \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that
\[ \|u(t)\|_{L^2}^2 \leq \varepsilon + C(\varepsilon) \|u_x(t)\|_{L^2}^2 \]
for any \( t \in \mathbb{R} \).

**Proof.** Since \( u \) is almost periodic modulo translation, for any \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that
\[ \int_{|\xi| < \frac{1}{\sqrt{c(\varepsilon)}}} |\hat{u}(t,\xi)|^2 d\xi < \varepsilon. \]

By the Plancherel equality, we have
\[ \|u(t)\|_{L^2}^2 = \|\hat{u}(t)\|_{L^2}^2 \]
\[ \leq \varepsilon + \int_{|\xi| \geq \sqrt{c(\varepsilon)}} |\hat{u}(t,\xi)|^2 d\xi \]
\[ \leq \varepsilon + C(\varepsilon) \int_{|\xi| \geq \sqrt{c(\varepsilon)}} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi \]
\[ \leq \varepsilon + C(\varepsilon) \|u_x(t)\|_{L^2}^2. \]

\[ \square \]

We define the localized virial identity as follows.
\[ V_R(t) := \int_{\mathbb{R}} \phi \left( \frac{x}{R} \right) u_t(u + 2xu_x) dx. \]
A direct calculation gives that
\[ V_R'(t) = -2 \| u_x \|_{L^2}^2 - \int_{\mathbb{R}} N(u)u\,dx + \int_{\mathbb{R}} \tilde{N}(u)\,dx \]
\[ + 2 \int_{\mathbb{R}} \left( 1 - \phi \left( \frac{x}{R} \right) \right) |u_x|^2 \,dx + \int_{\mathbb{R}} \left( 1 - \phi \left( \frac{x}{R} \right) \right) N(u)u\,dx \]
\[ - \int_{\mathbb{R}} \left( 1 - \phi \left( \frac{x}{R} \right) \right) \tilde{N}(u)\,dx + \frac{1}{2R^2} \int_{\mathbb{R}} \phi'' \left( \frac{x}{R} \right) |u|^2 \,dx \]
\[ - \int_{\mathbb{R}} \frac{x}{R} \phi' \left( \frac{x}{R} \right) (|u_x|^2 - |u|^2 - \tilde{N}(u) + |u_t|^2) \,dx. \]

**Proof of Theorem 8.1.** We suppose that there exists a non-scattering solution to (1.1) with almost periodicity modulo translation and zero momentum. We may assume that \( x(0) = 0 \) by the translation invariance of the equation (1.1). Since \( 3 \tilde{N}(u) \leq N(u)u \) and we have Lemma 8.2, we find
\[ V_R'(t) \leq -2 \| u_x \|_{L^2}^2 - \frac{2}{3} \int_{\mathbb{R}} N(u)u\,dx + \eta, \]
for \( t \in (0, t_0) \), where \( R \) and \( t_0 \) satisfy the assumption in Lemma 8.2. Therefore, we get
\[ V_R(0) - V_R(t_0) = - \int_0^{t_0} V_R'(t) \,dt \]
\[ \geq 2 \int_0^{t_0} \| u_x \|_{L^2}^2 \,dt + \frac{2}{3} \int_0^{t_0} \int_{\mathbb{R}} N(u)u\,dx\,dt - \eta t_0. \]

On the other hand, since Lemma 8.3 gives us that
\[ \frac{d}{dt} \int_{\mathbb{R}} uu_t \,dx = \| u_t \|_{L^2}^2 - \| u_x \|_{L^2}^2 - \| u \|_{L^2}^2 - \int_{\mathbb{R}} N(u)u\,dx \]
\[ \geq \| u_t \|_{L^2}^2 - \| u_x \|_{L^2}^2 - \epsilon - C(\epsilon) \| u_x \|_{L^2} - \int_{\mathbb{R}} N(u)u\,dx, \]
we obtain
\[ 4E(u, u_t) \geq \int_0^{t_0} \frac{d}{dt} \int_{\mathbb{R}} uu_t \,dx\,dt \]
\[ \geq \int_0^{t_0} \| u_t \|_{L^2}^2 \,dt - \epsilon t_0 - (C(\epsilon) + 1) \int_0^{t_0} \| u_x \|_{L^2}^2 \,dt - \int_0^{t_0} \int_{\mathbb{R}} N(u)u\,dx\,dt, \]
where \( \epsilon \) is chosen later, which is independent of \( \eta \) and \( R \). This implies that
\[ (8.5) \quad \{ C(\epsilon) + 2 \} \int_0^{t_0} \| u_x \|_{L^2}^2 \,dt + \int_0^{t_0} \int_{\mathbb{R}} N(u)u\,dx\,dt \]
\[ \geq \int_0^{t_0} \| u_t \|_{L^2}^2 \,dt + \int_0^{t_0} \| u_x \|_{L^2}^2 \,dt - \epsilon t_0 - 4E(u, u_t). \]
Combining (8.4) with (8.5), we obtain
\[
\frac{C(\varepsilon) + 3}{2} (V_R(0) - V_R(t)) \geq \{C(\varepsilon) + 3\} \int_0^t \|u_x\|^2_{L^2} dt + \frac{C(\varepsilon) + 3}{3} \int_0^t \int_R N(u) u dx dt \\
- \frac{C(\varepsilon) + 3}{2} \eta t_0 \\
\geq \{C(\varepsilon) + 2\} \int_0^t \|u_x\|^2_{L^2} dt + \int_0^t \int_R N(u) u dx dt \\
- \frac{C(\varepsilon) + 3}{2} \eta t_0 \\
\geq \int_0^t (\|u_t\|^2_{L^2} + \|u_x\|^2_{L^2}) dt - \varepsilon t_0 - 4E(u,u_t) - \frac{C(\varepsilon) + 3}{2} \eta t_0.
\]

We find that there exists a positive constant \(\delta\) such that \(\|u_t\|^2_{L^2} + \|u_x\|^2_{L^2} > \delta\) for any \(t \in \mathbb{R}\). Indeed, we have \(\delta_{\text{scat}} > 0\) satisfying that the solution scatters if \(\|u\|^2_{H^1} + \|u_t\|^2_{L^2} < \delta_{\text{scat}}\) by the small data scattering result. Taking \(\varepsilon = \delta_{\text{scat}}/4\) in Lemma 8.3 and assuming \(\|u_t(t)\|^2_{L^2} + \|u_x(t)\|^2_{L^2} < \min\{\frac{\delta_{\text{scat}}}{4}, \frac{\delta_{\text{scat}}}{4}\}\) \(\varepsilon\) \(\delta_{\text{scat}}\) \(\delta_{\text{scat}}\) \(\delta_{\text{scat}}\) \(\delta_{\text{scat}}\), we find that \(u\) scatters by Lemma 8.3 and the small data scattering and thus get a contradiction. Thus, there exists a positive constant \(\delta\) such that \(\|u_t\|^2_{L^2} + \|u_x\|^2_{L^2} > \delta\) for any \(t \in \mathbb{R}\). Fix \(\varepsilon = \delta/2\). We get
\[
\frac{C(\varepsilon) + 3}{2} (V_R(0) - V_R(t)) \geq \delta t_0 - \varepsilon t_0 - 4E(u,u_t) - \frac{C(\varepsilon) + 3}{2} \eta t_0 \\
= \frac{\delta}{2} t_0 - 4E(u,u_t) - \frac{C(\delta/2) + 3}{2} \eta t_0 \\
\geq \frac{C(\delta/2) + 3}{2} \frac{R(\eta)}{\eta} - 4E(u,u_t).
\]

On the other hand, we have \(V_R(0) - V_R(t) \leq R(\eta)\). Therefore, for sufficiently small \(\eta\), we obtain a contradiction since
\[
C_3 R(\eta) < C\left\{\frac{\delta}{2} - \frac{C(\delta/2) + 3}{2} \frac{R(\eta)}{\eta}\right\} - 4E(u,u_t).
\]

\[\square\]

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