TOPOLOGICAL RIGIDITY OF NON-COMPACT ORIENTABLE SURFACES

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Abstract. We show that a proper homotopy equivalence between two compact orientable surfaces is properly homotopic to a homeomorphism. Thus, non-compact orientable surfaces are topologically rigid in the strong sense in the category of topological spaces with proper maps.

1. Introduction

Throughout this note, all surfaces will be assumed as connected and orientable. An $n$-manifold $M$ is called topologically rigid if every $n$-manifold homotopically equivalent to $M$ is also homeomorphic to $M$. A fundamental question in topology is that if two closed $n$-manifolds are homotopy equivalent, are they homeomorphic? Though it is not generally true in all dimensions, it has a positive answer in dimension two. The Borel Conjecture asks a stronger question (which is also called topological rigidity): is every homotopy equivalence between two closed aspherical $n$-manifolds homotopic to a homomorphism? Fortunately, almost all (except sphere and projective plane) surfaces are aspherical, and the Dehn-Nielsen-Baer theorem says that Borel Conjecture in dimension two has an affirmative answer.

In other words, closed surfaces are determined, up to homeomorphism, by their fundamental groups. But, this is false in general for non-compact surfaces. For example the once punctured torus and thrice punctured sphere are homotopy equivalent but are not homeomorphic as homomorphism preserves the cardinality of the puncture sets (which is the set of ends) as well as the genus. In fact, any non-compact surface is homotopically equivalent to a wedge sum of at most countably many circles. So, the fundamental group of a non-compact surface is always free. If the fundamental group of a surface is finitely generated, the surface is called a finite-type surface, otherwise it is called an infinite-type surface. Any two infinite-type surfaces are homotopy equivalent (as the fundamental groups are free groups on a countably set of generators), but they may not be homeomorphic; for example, Loch Ness Monster and Jacob Ladder surfaces are non-homeomorphic as the former has one end and the latter has two ends.

Now, the best way to deal with non-compact manifolds is to use tools of the proper category (here, each map, including each homotopy, is a proper map). So, one may ask a similar question here, namely: does every proper homotopy equivalence determine a non-compact surface up to homeomorphism? Stating this question outside of the proper category gives the obvious false answer, as we noticed in the above example. Indeed, a good candidate for preserving puncture sets and the genus is a proper homotopy equivalence, and we will prove this in Theorem 3.1. Next, one can ask more generally whether every proper homotopy equivalence between two non-compact surfaces is properly homotopic to a homeomorphism? Again this has positive answer, see Theorem 3.5. Therefore, the Borel conjecture in dimension two, when stated in the proper category, is true.

Finally, a theorem of Edmonds [Edm79] says that any $\pi_1$-injective map of degree one between two closed surfaces is homotopic to a homeomorphism. The analogous fact for non-compact surfaces is Theorem 3.4, which actually follows from Theorem 3.2: If there is a proper map $f : \Sigma' \to \Sigma$ of non-zero degree between two non-compact surfaces, where $\Sigma'$ is not homeomorphic to $\mathbb{R}^2$ and $f$ is an ordinary homotopy equivalence; then these two surfaces are homeomorphic and $f$ is properly homotopic to a homeomorphism. Note that any degree-one self-map of the plane is properly homotopic to a homeomorphism, see Theorem 3.3.

One comment on the arrangement of theorems: (Theorem 3.2 + Theorem 3.3) $\implies$ Theorem 3.4 $\implies$ Theorem 3.5.

We will end up this section sketching the proof of our main theorem, Theorem 3.2. See below.
• Decompose co-domain $\Sigma$ into pair of pants and punctured disks using an (at most) countable collection $\mathcal{P}$ of embedded circles and then properly homotope $f$ to make it transverse to $\mathcal{P}$. If $\mathcal{C}$ is a component of $\mathcal{P}$, this proper homotopy says that the non-zero degree remains invariant which further implies $f^{-1}(\mathcal{C})$ is a non-empty compact 1-manifold. Now, a component of $f^{-1}(\mathcal{P})$ either bounds a disk or represents a primitive element of the fundamental group, see Step 1.

• Unlike the finite-type surface case, here, infinitely many disk-bounding components may appear. However, an arbitrarily large disk bounded by a component of the locally-finite collection $f^{-1}(\mathcal{P})$ is not possible as our surface is not the plane. So, we can talk about an “outermost disk $D$” bounded by some component of $f^{-1}(\mathcal{P})$. Now, all disk bounding components of $f^{-1}(\mathcal{P})$ can be removed by a proper homotopy considering all outermost disks simultaneously, see Step 2.

• Thereafter, we homotope properly so that $f$ maps each (primitive) component of $f^{-1}(\mathcal{P})$ onto a component of $\mathcal{P}$ homeomorphically; this is where we first-time use $f$ as a homotopy equivalence, see Step 3.

• Now, again using $f$ as a homotopy equivalence, we show that any two components of $f^{-1}(\mathcal{P})$ co-bound an annulus if and only if their $f$-images are the same, i.e., arbitrarily large annulus co-bound by two components of $f^{-1}(\mathcal{P})$ is impossible. So, considering all “outermost annuli” simultaneously one can properly homotope $f$ so that for each component of $\mathcal{C}$ of $\mathcal{P}$, $f^{-1}(\mathcal{C})$ is a single circle mapped homeomorphically onto $\mathcal{C}$, see Step 4.

• At this point, $\mathcal{P}$ and $f^{-1}(\mathcal{P})$ break down $\Sigma, \Sigma'$ into pair of pants and punctured disks, respectively; and there is a shape-preserving bijective-correspondence between these two collections of broken parts due to the homotopy equivalence $f$. Now, a proper allowable self-map of a punctured disk or pair of pants, which induces a homeomorphism on the boundary, is properly homotopic to a homomorphism by a proper homotopy relative to the boundary. Finally, gluing all these boundary-relative homotopies, we complete the proof, see Step 5.

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2. Background

2.1. Construction of Non-Compact Surfaces.

Theorem 2.1. [Ric63, Theorem 3] Every non-compact surface $\Sigma$ is obtained by adding (suitably) at most countably many handles to $S^2 \setminus \mathcal{E}$; where $\mathcal{E}$ is a closed, totally disconnected subset of $S^2$.

The space $\mathcal{E}$ is homeomorphic to the space $\text{Ends} (\Sigma)$ of ends. Note that an infinite set of handles may accumulate to some point of $\mathcal{E}$ (but can’t accumulate to any point of $S^2 \setminus \mathcal{E}$), and the collection $\mathcal{E}'$ of all such points of $\mathcal{E}$ is homeomorphic to the space of all non-planar ends, denoted by $\text{Ends}_{np}(\Sigma)$. Notice that $\mathcal{E}'$ is a closed subset of $\mathcal{E}$, hence $\text{Ends}_{np}(\Sigma)$ is also closed in $\text{Ends}(\Sigma)$. We define the space $\text{Ends}_{np}(\Sigma)$ of planar ends as $\text{Ends}_{np}(\Sigma) := \text{Ends}(\Sigma) \setminus \text{Ends}_{np}(\Sigma)$. Let’s note another fact: Since $\mathcal{E} \cong \text{Ends}(\Sigma)$ is a totally disconnected, compact, metrizable space, it is homeomorphic to a subspace of the standard Cantor set. See Figure 1.

Goldman [Gol71, Section 2.6.] gave an inductive procedure of constructing a non-compact surface $\Sigma$ using four compact bordered surfaces: disk, annulus, pair of pants, and torus with two holes. Start with the disk; to get $K_{i+1}$ from $K_i$ (compact subsurface of $\Sigma$ after the $i$-th step of induction), consider one of the last three compact bordered surfaces, say $S$; finally, suitably identify one boundary circle of $S$ with a boundary circle of $K_i$. This process gives all non-compact surfaces. See Figure 2.

2.2. Classification of Non-Compact Surfaces. Let $\Sigma$ be a non-compact surface $\Sigma$. Define the genus $g(\Sigma)$ as the supremum of $\{\text{genus of } S : S \text{ is a compact bordered subsurface of } \Sigma\}$.

Theorem 2.2. [Ric63, Theorem 1] Let $\Sigma, \Sigma'$ be two non-compact orientable surfaces. Then $\Sigma$ is homeomorphic to $\Sigma'$ if and only if
Figure 1: Construction of a non-compact surface having one non-planar end and three planar ends

- \( g(\Sigma) = g(\Sigma') \).
- There is a homeomorphism \( \Phi : \text{Ends}(\Sigma) \to \text{Ends}(\Sigma') \) with \( \Phi(\text{Ends}_{np}(\Sigma)) = \text{Ends}_{np}(\Sigma') \).

Now, we consider the algebraic version of this classification.

**Proposition 2.1.** [Gol71, Sections 1.3., 1.5.] There are cup products \( \smile : H^p_c(\Sigma; \mathbb{Z}_2) \otimes H^q_c(\Sigma; \mathbb{Z}_2) \to H^{p+q}_c(\Sigma; \mathbb{Z}_2) \) and \( \smile : H^p_c(\Sigma; \mathbb{Z}_2) \otimes H^q_e(\Sigma; \mathbb{Z}_2) \to H^{p+q}_e(\Sigma; \mathbb{Z}_2) \) for integers \( p, q \geq 0 \). Here, \( H^*_c, H^*_e \) stand for singular, compactly supported, and end cohomology, respectively.

**Proposition 2.2.** [Gol71, Sections 1.5., 2.1.] Let \( i : H^1_c(\Sigma; \mathbb{Z}_2) \to H^1(\Sigma; \mathbb{Z}_2) \) be induced by the inclusion of cochains. Then \( \ker i = \{ x \in H^1_c(\Sigma; \mathbb{Z}_2) : x \smile H^1(\Sigma; \mathbb{Z}_2) = 0 \} \) and \( 2 \cdot g(\Sigma) = \mathbb{Z}_2\text{-rank of the quotient vector space } H^1(\Sigma; \mathbb{Z}_2) / \ker i \).

**Proposition 2.3.** [Gol71, Sections 3.2., 4.3.] Consider ideals \( A := \{ y \in H^1(\Sigma; \mathbb{Z}_2) : y \smile \ker i = 0 \} \) and \( J := \{ x \in H^0_e(\Sigma; \mathbb{Z}_2) : x \smile A = 0 \} \). Then \( J \) corresponds to the space \( \text{Ends}(\Sigma) \backslash \text{Ends}_{np}(\Sigma) \) of planar ends.

**Theorem 2.3.** [Gol71, Theorem 4.1.] Two non-compact surfaces \( \Sigma, \Sigma' \) are homeomorphic if and only if \( g(\Sigma) = g(\Sigma') \); and there is a ring isomorphism \( \theta : H^0_e(\Sigma; \mathbb{Z}_2) \to H^0_e(\Sigma'; \mathbb{Z}_2) \) with \( \theta(J) = J' \), where \( J, J' \) are the ideals given in **Proposition 2.3** corresponding to the planar ends of \( \Sigma \) and \( \Sigma' \).

### 3. Topological Rigidity of Non-Compact Surfaces

Here we will prove that every non-compact surface is topologically rigid, i.e., every proper self-homotopy equivalence is properly homotopic to a self-homeomorphism.

**Definition.** Let \( X', X \) be two spaces and \( f_0, f_1 : X' \to X \) be two proper maps. We say \( f_0, f_1 \) are properly homotopic if there is a proper map \( F : X' \times [0, 1] \to X \) such that \( F(\cdot, 0) = f_0 \) and \( F(\cdot, 1) = f_1 \).
Remark. Note that homotopy through proper maps is a weaker notion than proper homotopy. For example, consider the homotopy \( H : \mathbb{C} \times [0, 1] \to \mathbb{C} \) given by \( H(z, t) := tz^2 - z \). Being a polynomial, each \( H(-, t) \) is a proper map. But, \( H \) itself is not proper as \( H(n, 1/n) = 0 \) for all integers \( n \geq 1 \).

Definition. Let \( X, X' \) be two spaces and \( f : X' \to X \) and \( g : X \to X' \) be two proper maps. Suppose we have two proper maps \( H : X \times [0, 1] \to X \) and \( H' : X' \times [0, 1] \to X' \) with \( H'(-, 0) = g \circ f \), \( H'(-, 1) = \text{Id}_{X'} \) and \( H(-, 0) = f \circ g \), \( H(-, 1) = \text{Id}_X \). Then \( f \) is called a proper homotopy equivalence, and \( g \) is called its proper homotopy inverse.

Theorem 3.1 shows that a proper homotopy equivalence between two non-compact surfaces preserves the genus and the space of planar ends. Though the same result is also obtained from Theorem 3.2 as a particular case, Theorem 3.1 solves it algebraically.

**Theorem 3.1.** Let \( \Sigma, \Sigma' \) be two non-compact surfaces and \( f : \Sigma \to \Sigma' \) be a proper homotopy equivalence. Then \( \Sigma \) is homeomorphic to \( \Sigma' \).

Proof. We use Theorem 2.3 to prove this. If \( g : \Sigma' \to \Sigma \) is a proper homotopy inverse of \( f \) then \( g^* : H^q_c(\Sigma; \mathbb{Z}_2) \to H^q_c(\Sigma'; \mathbb{Z}_2) \) and \( g^* : H^q_g(\Sigma; \mathbb{Z}_2) \to H^q_g(\Sigma'; \mathbb{Z}_2) \) are isomorphisms for each \( q \geq 0 \) due to proper homotopies \( f \circ g \approx_{\text{proper}} \text{Id}_{\Sigma'} \) and \( g \circ f \approx_{\text{proper}} \text{Id}_{\Sigma} \). Since proper homotopy is also a homotopy the induced map \( g : H^q(\Sigma; \mathbb{Z}_2) \to H^q(\Sigma; \mathbb{Z}_2) \) is also an isomorphism for each \( q \geq 0 \).

Note that the induced maps of \( g \) on different cohomologies preserve the all three cup products given in Proposition 2.1. Now consider the facts below with Theorem 2.3 to conclude the proof. We will use the notations \( i', A', J' \) which can be defined similarly for the surface \( \Sigma' \) as we have defined \( i, A, J \) for surface \( \Sigma \).

- To prove \( g^*(\ker i) \subseteq \ker i' \), notice that \( g^* : H^1_c(\Sigma; \mathbb{Z}_2) \to H^1_c(\Sigma'; \mathbb{Z}_2) \) surjective; and to prove \( g^*(\ker i) \supseteq \ker i' \), note that \( g^* : H^2_c(\Sigma; \mathbb{Z}_2) \to H^2_c(\Sigma'; \mathbb{Z}_2) \) is injective. So, \( g^*(\ker i) = \ker i' \). This
with the fact that $g: H^1_c(\Sigma; \mathbb{Z}_2) \to H^1_c(\Sigma'; \mathbb{Z}_2)$ is an isomorphism, gives us $g(\Sigma) = g(\Sigma')$.

- The fact that induced maps preserve cup products is enough to show $g^*(A) \subseteq A'$ and $g^*(J) \subseteq J'$. Now, both $g^*: H^2_c(\Sigma; \mathbb{Z}_2) \to H^2_c(\Sigma'; \mathbb{Z}_2)$ and $g^*: H^1_c(\Sigma; \mathbb{Z}_2) \to H^1_c(\Sigma'; \mathbb{Z}_2)$ are injective imply $g^*(A) \supseteq A'$ and $g^*(J) \supseteq J'$, respectively. So, $g^*(A) = A'$ and $g^*(J) = J'$.

- Finally, $g^*: H^0_c(\Sigma; \mathbb{Z}_2) \to H^0_c(\Sigma'; \mathbb{Z}_2)$ is the required ring isomorphism sending $J$ onto $J'$. □

We are now ready to prove our main results. The idea of this [FM12, First proof of Theorem 8.9, Page 237] inspires the proof of Theorem 3.2.

**Theorem 3.2.** Let $\Sigma, \Sigma'$ be two non-compact surfaces and $f: \Sigma' \to \Sigma$ be a proper map of non-zero degree. Suppose $\Sigma'$ is not homeomorphic to $\mathbb{R}^2$. If $f$ is a homotopy equivalence, then $\Sigma'$ is homeomorphic to $\Sigma$ and $f$ is properly homotopic to a homeomorphism.

**Proof.** We will complete the proof using several steps and sub-steps given below. Each step will start with the combined rough idea of all its sub-steps.

**Step 1: Decomposition and transversality**

We consider a collection $\mathcal{P}$ of smoothly embedded non-contractible circles inside $\Sigma$ so that $\mathcal{P}$ breaks down $\Sigma$ into a punctured disk and pair of pants. Now, homotope $f$ properly to make it smooth as well as transverse to $\mathcal{P}$. So, $f^{-1}(\mathcal{P})$ is a locally-finite collection of at most countably many pairwise-disjoint embedded circles in $\Sigma'$, and each component of $f^{-1}(\mathcal{P})$ either bounds a disk or represents a primitive element of $\pi_1(\Sigma')$.

**Claim 1.1.** Without loss of generality we may assume $f: \Sigma' \to \Sigma$ is a smooth proper map as every continuous proper map $\Sigma' \to \Sigma$ is properly homotopic to a smooth proper map $\Sigma' \to \Sigma$.

**Proof.** This is a special case of Theorem 4.3. Note that every surface has a $C^\infty$-smooth structure and $f$ still has non-zero degree after proper homotopy. □

**Claim 1.2.** The surface $\Sigma$ is the union (with pairwise disjoint interiors) of punctured disk and pair of pants. In particular, there is a locally finite collection $\mathcal{P}$ of embedded circles in $\Sigma$ such that $\mathcal{P}$ decomposes $\Sigma$ into bordered sub-surfaces with one or three boundary components.

**Proof.** Note that $f$ is homotopy equivalence implies $\Sigma$ is not homeomorphic to $\mathbb{R}^2$. Consider the inductive procedure of constructing all non-compact surfaces given in Section 2.1. Now, we can include the disk (from where we start the construction) into the interior of pair of pants or torus with two holes by pushing it; this is possible, as the surface obtained by this procedure using only disk and annulus, is $\mathbb{R}^2$. Also, note that torus with two holes is the union of two pair of pants and punctured disk is the countable union of annulus. □

**Remark 1.3.** Note that no component of $\mathcal{P}$ bounds a disk, and no two distinct components of $\mathcal{P}$ together bound an annulus. For a different proof of Claim 1.2, see [AR04, Theorem 1.1].
Claim 1.4. Without loss of generality, we may assume \( \mathcal{P} \) is a smoothly embedded 1-dimensional submanifold of \( \Sigma \). Also, \( \mathcal{P} \) is closed in \( \Sigma \) as a subset of \( \Sigma \).

**Proof.** This follows once we consider the fact (attaching smooth 2-dimensional manifolds along their boundaries) below with the inductive procedure of constructing all non-compact surfaces given in Section 2.1.

- Let \( M, N \) be two oriented smooth manifolds (possibly disconnected) of the dimension two with \( \partial M \neq \emptyset \neq \partial N \). Consider the induced orientations for \( \partial M, \partial N \) obtained from orientations of \( M, N \), respectively. Let \( \varphi: \partial M \to \partial N \) be an orientation-reversing diffeomorphism. Then, the adjunction space \( M \cup_{\varphi} N \) has a smooth structure such that there are smooth, properly embedded codimension zero submanifolds \( M, N \) (each has non-empty boundary) of \( M \cup_{\varphi} N \) diffeomorphic to \( M, N \), respectively, and satisfying \( M \cup N = M \cup_{\varphi} N \) with \( \partial N = M \cap N = \partial M \). Also, the 2-manifold \( M \cup_{\varphi} N \) is orientable as \( \varphi \) is orientation-reversing.

Note that at first we have to split up the complete set of building blocks for \( \Sigma \) into two parts, one for \( M \), and the other for \( N \). Here, building block means a pair of pants or a punctured disc. Keeping in mind the arrangement of these building blocks inside \( \Sigma \), we can construct \( \varphi \). Now, to check \( M \cup_{\varphi} N \) is homeomorphic to \( \Sigma \) we need to use Theorem 2.2. Note that \( \mathcal{P} = M \cap N \). Since \( M, N \to M \cup_{\varphi} N \cong \Sigma \) are proper maps; \( \mathcal{P} \) is closed in \( \Sigma \). □

Claim 1.5. We can properly homotope \( f \) to make it transverse to \( \mathcal{P} \).

**Proof.** This is a special case of Theorem 4.4. Again, note that the property of having non-zero degree remains unchanged after proper homotopy. □

Claim 1.6. Let \( C \) be a component of \( \mathcal{P} \). Then \( f^{-1}(C) \) is a non-empty collection of finitely many pairwise disjoint smoothly embedded circles in \( \Sigma' \). Therefore, \( f^{-1}(\mathcal{P}) \) is a locally finite collection of at most countably many pairwise disjoint, smoothly embedded, simple closed curves in \( \Sigma' \).

**Proof.** We first show \( f \) is surjective, which will imply \( f^{-1}(C) \neq \emptyset \). The map \( f \) is proper implies \( \text{im}(f) \) is closed in \( \Sigma \). Therefore, for \( \text{im}(f) \neq \Sigma \) we would have \( \text{im}(f) \subseteq \Sigma \setminus B \) for some small open ball \( B \), and then \( f \) could be thought as the composition \( \Sigma' \xrightarrow{f} \Sigma \setminus B \xrightarrow{i} \Sigma \); so finally \( H^2(f) : H^2(\Sigma; \mathbb{Z}) = \mathbb{Z} \to 0 = H^2(\Sigma \setminus B; \mathbb{Z}) \) would give \( \deg(f) = 0 \), a contradiction to the hypothesis.

Now, \( f \not\equiv \mathcal{P} \) implies \( f \not\equiv C \). So, \( f^{-1}(C) \) is a non-empty compact 1-dimensional smoothly embedded boundary-less submanifold of \( \Sigma' \) as \( f \) is proper. Hence, \( f^{-1}(C) \) has finitely many components, and each component of \( f^{-1}(C) \) is diffeomorphic to \( S^1 \). Finally, \( f^{-1}(\mathcal{P}) \) is a closed subset of \( \Sigma' \) as well as an embedded submanifold of \( \Sigma' \) implies \( f^{-1}(\mathcal{P}) \) is a locally finite collection. □

Claim 1.7. Consider the embedded submanifold \( f^{-1}(\mathcal{P}) \) of \( \Sigma' \). A component of \( f^{-1}(\mathcal{P}) \) represents a trivial element of \( \pi_1(\Sigma') \) if it bounds a disk, otherwise it represents a primitive element of \( \pi_1(\Sigma') \).

**Proof.** The former part is easy as we can use the embedded disk to construct a null-homotopy. For the latter part, we will use the [FM12, Proposition 1.4], where it is proved that on any hyperbolic surface a non null-homotopic simple closed curve represents a primitive element of the fundamental group. So, what we need to prove is that any non-compact surface has a hyperbolic structure.

Note that every non-compact surface covers any closed hyperbolic surface, see [Gol71, Theorem 8.1.]. In particular, we have a covering map \( p \) from \( \Sigma' \) to \( S_{2,0} \) (closed surface of genus 2). Now, \( S_{2,0} \) has a complete Riemannian metric with constant sectional curvature \(-1\). Pulling back the smooth and metric structure of \( S_{2,0} \) on \( \Sigma' \) w.r.t. \( p \), we can now say \( p: \Sigma' \to S_{2,0} \) is smooth covering as well as local isometry. So, \( \Sigma' \) is complete with this pullback metric. Since local isometry preserves sectional curvature, the universal covering of the \( \Sigma' \) is hyperbolic plane. □

**Step 2: Disk removal**

In Step 1, we properly homotope \( f \) so that \( f^{-1}(\mathcal{P}) \) gives a 1-dimensional submanifold of \( \Sigma' \). Here we will properly homotope \( f \) to get a map \( f_1 \) such that \( f_1^{-1}(\mathcal{P}) \) is the sub-collection of all primitive circles of \( f^{-1}(\mathcal{P}) \). For that, we first ensure that an arbitrarily large disk
doesn’t exist whose boundary is a component of the locally finite collection \( f^{-1}(P) \). Now, consider an outermost disk \( D' \) whose boundary is a component of \( f^{-1}(P) \), and using transversality, find a slightly larger disk \( D'_2 \) containing \( D' \). Finally, by a proper homotopy, push \( f(D'_2) \) in \( \Sigma \setminus P \). If we consider all outermost disks, all pushing can be done simultaneously by a single proper homotopy, not one by one; and that’s why we need to consider outermost disks.

**Lemma 2.1.** Let \( X \) be a surface and \( \{C_i : i \in \mathbb{N}\} \) be a locally finite collection of simple closed curves on \( X \) such that for each \( i \) the circle \( C_i \) bounds a disk \( D_i \) with \( C_i \subseteq \text{int}(D_{i+1}) \). Then, \( X \) is homeomorphic to \( \mathbb{R}^2 \).

**Proof.** Recall that all surfaces under consideration are connected and orientable. Enough to show, 

\[
\text{Let } \delta \text{ doesn’t exist whose boundary is a component of the locally finite collection } \{ \text{outermost disk } D \text{ of trivial components of } f^{-1}(P) \} \text{ disks of trivial components of } f^{-1}(P); \text{ but } D' \text{ is not contained in the interior of a disk bounded by any other trivial component of } f^{-1}(P).
\]

Now, each point \( p \) have an open neighborhood \( U_p \) such that \( U_p \subseteq \text{int}(D_i) \) for some \( i \in \mathbb{N} \). Therefore, each closed curve \( \gamma \) is contained in some \( D_i \), i.e., each closed curve is null-homotopic. \( \square \)

**Claim 2.2.** There is a proper homotopy from \( f : \Sigma' \to \Sigma \) to some other proper map, say \( f_1 : \Sigma' \to \Sigma \), such that \( f_1^{-1}(P) \) is the sub-collection of all primitive circles of \( f^{-1}(P) \).

**Proof.** Using Lemma 2.1 and Claim 1.6, we can say, there doesn’t exist infinitely many components \( C'_1, C'_2, \ldots \) of \( f^{-1}(P) \) bounding the disks \( D'_1, D'_2, \ldots \), respectively such that \( C'_n \) is contained in the interior of \( D'_{n+1} \) for each \( n \). So, if \( f^{-1}(P) \) has a trivial component, then there is at least one outermost disk \( D' \), i.e., \( \partial D' \) is a component of \( f^{-1}(P) \), and the interior of \( D' \) may contain some other (of course, only finitely many as \( f^{-1}(P) \) is a locally finite collection) disks of trivial components of \( f^{-1}(P) \); but \( D' \) is not contained in the interior of a disk bounded by any other trivial component of \( f^{-1}(P) \).

Consider such an outer-most disk \( D' \). Let \( \delta \) be the unique component of \( P \) such that \( f(\partial D') \subseteq \delta \). Now, Theorem 4.1 says \( f \) is \( \delta \) for all circle \( \delta \) sufficiently near to \( C \). Next, using Theorem 4.2, we have a component \( C'_\delta \) of \( f^{-1}(P) \) such that \( C'_\delta \) bounds a disk \( D'_\delta \) so that \( \text{int}(D'_\delta) \supseteq D' \). Furthermore, we may choose \( C'_\delta \) so near to \( \partial D' \) that \( D'_\delta \cap f^{-1}(P) = D \cap f^{-1}(P) \) and \( (D'_\delta \setminus D') \cap f^{-1}(C) = C'_\delta \) hold. This is possible as \( f^{-1}(P) \) is a locally finite collection. One fact is worth noting that \( C'_\delta \) doesn’t bound a disk as it is freely homotopic to the non-trivial simple closed curve \( C \). In other words, \( C'_\delta \) represents a primitive element of \( \pi_1(\Sigma) \). See the figure on the next page.

**Before moving on, let’s fix up a notation for brevity.** Let \( X \) be a topological space, and \( \{A_n\} \) be a sequence of subsets of \( X \). We write, \( A_n \to \infty \) if we are given any compact subset \( K \) of \( X \), then \( A_n \cap K = \emptyset \) for all but finitely many \( n \).

We will assume the number of distinct outermost disks is infinite (when these are finite in number, the case can be handled analogously, just ignoring the notion of infinity-convergence). So, let \( \{D'_n\} \) be the sequence of all outermost disks of \( \Sigma' \) such that \( \bigcap_{n \neq m} D'_n \cap D'_m = \emptyset \) when \( n \neq m \). Then \( D'_n \to \infty \) as \( f^{-1}(P) \) is a locally finite collection. Also, \( f(D'_n) \to \infty \) as \( f \) is a proper map. Let \( C_n \) be the unique component of \( P \) such that \( f(\partial D'_n) \subseteq C_n \). Now, \( C_n \to \infty \) as \( P \) is a locally-finite collection and \( f^{-1}(C_n) \) has only finitely many components for any \( n \). Hence, for each \( f(D'_n) \) we have a compact bordered sub-surface \( S_n \) of \( \Sigma \) so that \( C_n \setminus f(D'_n) \subseteq \text{int}(S_n) \) and \( S_n \to \infty \).

We don’t need to introduce the notations \( C_{\delta,n} \), \( C'_{\delta,n} \), \( D'_{\delta,n} \); instead, it is obvious from the previous paragraphs what we mean by this notations. For example, we can choose, \( C_{\delta,n} \) close enough to \( C_n \) so that \( C_{\delta,n} \subseteq \text{int}(S_n) \) for each \( n \in \mathbb{N} \).

Fix \( n \in \mathbb{N} \) and let’s consider the long exact sequence of homotopy groups

\[
\cdots \to \pi_2(S_n) \to \pi_2(S_n, C_{\delta,n}) \to \pi_1(C_{\delta,n}) \to \pi_1(S_n) \to \cdots
\]

Since, \( S_n \) has contractible universal cover, \( \pi_2(S_n) = 0 \). Now, the inclusion induced map \( \pi_1(C_{\delta,n}) \to \pi_1(S_n) \) in the above sequence is injective as \( C_{\delta,n} \) represents a primitive element of \( \pi_1(S_n) \). Hence,
The figure shows \( f(D'_\delta) \subseteq \text{int}(S) \), where \( S \) is a compact bordered subsurface of \( \Sigma \).

A black circle denotes a component of either \( P \) or a component of \( f^{-1}(P) \).

\[ \pi_2(S_n, C_{\delta,n}) = 0 \] by the exactness of the above sequence. Now, \( f|\partial(D'_{\delta,n}, C'_{\delta,n}) \to (S_n, C_{\delta,n}) \) represents an element of the trivial group \( \pi_2(S_n, C_{\delta,n}) \). Thus, we have a homotopy \( H_n : D'_{\delta,n} \times [0,1] \to S_n \) such that \( H_n(-,0) = f|\partial D'_{\delta,n}, H_n(z,t) = f(z) \) for all \((z,t) \in C'_{\delta,n} \times [0,1]\) and \( H_n(D'_{\delta,n}, 1) \subseteq C_{\delta,n} \).

Considering all these relative homotopies \( \{H_n : n \in \mathbb{N}\} \), we have a homotopy \( H : \Sigma' \times [0,1] \to \Sigma \) from \( f \) to a map \( f_1 \) such that for any \( t \in [0,1] \), we have \( H(p,t) = f(p) \) when \( p \in \Sigma' \setminus \bigcup_{n \in \mathbb{N}} \text{int}(D'_{\delta,n}) \).

Clearly, \( f_1^{-1}(P) \) is the sub-collection of all primitive circles of \( f^{-1}(P) \).

Finally we show, \( H \) is a proper map. Take a compact subset \( \mathcal{K} \) of \( \Sigma \). So, \( \text{im}(H_n) \cap \mathcal{K} \subseteq S_n \cap \mathcal{K} = \emptyset \) for all but finitely many \( n \). Notice that outside of \( \bigcup_{n \in \mathbb{N}} D'_{\delta,n} \) the map \( H \) is defined by the proper map \( f \). Hence, \( H^{-1}(\mathcal{K}) \) is compact, and so, we are done. \( \square \)

**Remark 2.3.** Notice that the proof of Claim 2.2 wouldn’t work if our co-domain were \( \mathbb{R}^2 \) as there is no primitive circle in the plane. And this is one of the main reasons for discarding \( \mathbb{R}^2 \) from the hypothesis of Theorem 3.2.

**Step 3: Homotope degree-one map to a homeomorphism**

In Step 2, a proper homotopy of \( f \) which is defined by \( f \) near each primitive circle \( C'_p \) of \( f^{-1}(P) \), gives a map \( f_1 \) such that \( f_1^{-1}(P) \) is the sub-collection of all primitive circles of \( f^{-1}(P) \). Here we will construct a map \( f_2 \) having the following properties: (i) \( f_2 \) is properly homotopic to \( f_1 \), (ii) \( f_2^{-1}(P) = f_1^{-1}(P) \), (iii) \( f_2 \) maps each component of \( f_2^{-1}(P) \) homeomorphically onto a component of \( P \). The idea is the following: Induced map of a homotopy equivalence on \( \pi_1 \) is an isomorphism, so \( f_1(C'_p) \to C := f_1(C'_p) \) is a deg. \( \pm 1 \) map. Thus we can homotope this restriction map to a homeomorphism \( h : C'_p \to C \). Transversality gives small one-sided tubular neighborhoods \( T', T \) of \( C'_p, C \), respectively such that \( f_2^{-1}(\partial T) \cap T' = \partial T' \) and \( f_2^{-1}(C) \cap T' = C'_p \). Now, the homotopy extension property applied on \( T' \) to get \( h \) may end up with adding something extra in \( f_1^{-1}(C) \).
Still, we can use it after pushing the map \( f_1 \mid C_p' \to C \) slightly towards \( f_1 \mid (\partial T' \setminus C_p') \to (\partial T \setminus C) \) and mapping the part of \( T' \) near \( C_p' \) onto the part of \( T \) near \( C \) by \( f_1 \mid C_p' \to C \) in a label-preserving fashion. Applying this process on each side of \( C_p' \), we conclude Step 3.

**Lemma 3.1.** Consider the annulus \( A := S^1 \times [0, 2] \) and a map \( \varphi_0 : A \to A \) such that \( \varphi_0^{-1}(S^1 \times 2) = S^1 \times 2 \), \( \varphi_0^{-1}(S^1 \times 0) = S^1 \times 0 \). Let \( p_1 : A \to S^1 \) and \( p_2 : A \to [0, 2] \) be the projections. Then, there is homotopy \( \varphi_1 : A \to A \), \( t \in [0, 1] \) such that

\[
\varphi_1(e^{i\theta}, r) = \left(p_1 \circ \varphi_0(e^{i\theta}, 2r), \frac{1}{2}p_2 \circ \varphi_0(e^{i\theta}, 2r)\right) \quad \text{for } 0 \leq r \leq 1, \ \theta \in \mathbb{R};
\]

\[
\varphi_1(e^{i\theta}, r) = (p_1 \circ \varphi_0(e^{i\theta}, 2), r) \quad \text{for } 1 \leq r \leq 2, \ \theta \in \mathbb{R}
\]

and \( \varphi_1(e^{i\theta}, 0) = \varphi_0(e^{i\theta}, 0) \) for \( 0 \leq t \leq 1, \ \theta \in \mathbb{R} \).

**Proof.** The Figure above demonstrates the homotopy \( \varphi_1 : A \to A, t \in [0, 1] \). Both in domain and co-domain, as time goes, the blue annulus \( S^1 \times [0, 2] \) gets shorter till it reaches the shape \( S^1 \times [0, 1] \), and the red annulus \( S^1 \times \{2\} \) gets larger till it comes to the shape \( S^1 \times [1, 2] \). Also, at any deformation stage, we describe \( \varphi_0 : A \to A \) by shaping the domain and co-domain on blue annuli. Further, the homotopy maps each circle of the red annulus of the domain into the corresponding circle of the red annulus of the co-domain using the map \( \varphi_0 | S^1 \times 2 \to S^1 \times 2 \). Notice that we are not changing \( \varphi_0 | S^1 \times 0 \to S^1 \times 0 \) as time goes. \( \square \)

**Remark 3.2.** The Lemma 3.1 says, after homotopy, we can realize the map \( \varphi_0 \) in a half-domain and half-co-domain so that rest of the portion of the domain can be mapped into the remaining portion of the co-domain using the restriction \( \varphi_0 | S^1 \times 2 \to S^1 \times 2 \). And the whole process can be performed without changing \( \varphi_0 | S^1 \times 0 \to S^1 \times 0 \), i.e., \( \varphi_0 \simeq \varphi_1 \) relative to \( S^1 \times \{0\} \).

**Lemma 3.3.** Consider the annulus \( A := S^1 \times [0, 2] \) and a map \( \varphi_0 : A \to A \) such that \( \varphi_0^{-1}(S^1 \times 2) = S^1 \times 2 \), \( \varphi_0^{-1}(S^1 \times 0) = S^1 \times 0 \). Let \( p_1 : A \to S^1 \) and \( p_2 : A \to [0, 2] \) be the projections. Suppose, \( \varphi_0 | S^1 \times 2 \to S^1 \times 2 \) is a map of degree \( \pm 1 \). Then, there is homotopy \( \varphi_1 : A \to A, t \in [0, 1] \) such that \( \varphi_1(e^{i\theta}, 0) = \varphi_0(e^{i\theta}, 0) \) for all \( t \in [0, 1], \ \theta \in \mathbb{R} \) and \( \varphi_1^{-1}(S^1 \times 2) = S^1 \times 2 \) so that \( \varphi_1 | S^1 \times 2 \to S^1 \times 2 \) is a homeomorphism.

**Proof.** Let \( h : S^1 \times 2 \to S^1 \times 2 \) be a homotopy from \( \varphi_0 | S^1 \times 2 \to S^1 \times 2 \) to a homeomorphism \( S^1 \times 2 \to S^1 \times 2 \). Using Lemma 3.1 after a homotopy (relative to \( S^1 \times 0 \)), we may assume that \( \varphi_0(e^{i\theta}, r) = (p_1 \circ \varphi_0(e^{i\theta}, 2), r) \) for \( 1 \leq r \leq 2 \) and \( \theta \in \mathbb{R} \). So, \( h \) gives a homotopy \( h_r : S^1 \times r \times [0, 1] \to S^1 \times r \) from \( \varphi_0 | S^1 \times r \to S^1 \times r \) to a homeomorphism \( S^1 \times r \to S^1 \times r \) for each
r satisfying \(1 \leq r \leq 2\).

By \(h_1\), define a homotopy \(G\) on \(S^1 \times \{0, 1\}\), which is stationary on \(S^1 \times 0\). Now, homotopy extension property of the CW-pair \((S^1 \times [0, 1], S^1 \times \{0, 1\})\) gives a homotopy \(G\) on \(S^1 \times [0, 1]\). Pasting \(G\) with the collection \(h_r\), \(1 \leq r \leq 2\) gives the result. □

**Claim 3.4.** There is a proper homotopy from \(f_1: \Sigma' \to \Sigma\) to some other proper map, say \(f_2: \Sigma' \to \Sigma\) so that these happen: (i) \(f_2^{-1}(P) = f_1^{-1}(P)\); (ii) \(f_2|_{C'_p} \to C\) is a homeomorphism for any component \(C\) of \(P\) and any component \(C'_p\) of \(f_1^{-1}(P)\).

**Proof.** Consider any component \(C\) of \(P\) and any component \(C'_p\) of \(f_1^{-1}(P)\). Note that \(f_1^{-1}(P)\) is the same as the collection of primitive components of \(f^{-1}(P)\), and we obtain \(f_1\) from \(f\) by a proper homotopy defined by \(f\) near each primitive component of \(f^{-1}(P)\).

Now, the restriction \(f_1|_{C'_p} \to C\) can be thought as a degree one map of \(S^1\) as the isomorphism \(\pi_1(f) = \pi_1(f_1) : \pi_1(\Sigma') \to \pi_1(\Sigma)\) preserves primitiveness. Using Theorem 4.2 and continuity(nearness) of \(f_1\) there are one-sided tubular neighborhoods \(M', N' \subseteq \Sigma'\) of \(C'_p\) and one-sided tubular neighborhoods \(M, N \subseteq \Sigma\) of \(C\) so that the following hold: (1) \(M' \cup N'\) gives a two-sided tubular neighborhood of \(C'_p\) (2) \(f_1(M') = M\), \(f_1(N') = N\) (3) \(f_1^{-1}(C) \cap M' = C'_p = f_1^{-1}(C) \cap N'\) (4) \(M' \cap f_1^{-1}(\partial M) = \partial M', N' \cap f_1^{-1}(\partial N) = \partial N'\). Note that choosing small enough tubular neighborhoods, we may further assume that \(M \cap P = C = N \cap P\) and \((M' \cup N') \cap f_1^{-1}(P) = C'_p\) as \(P\) and \(f_1^{-1}(P)\) are locally-finite collections.

Notice that the existence of the small enough tubular neighborhoods \(M', N'\) of \(C'_p\) is already provided by \(f_1\) before getting \(f_1\) from \(f\) by a proper homotopy, and while performing this proper homotopy, \(f|M', f|N'\) has never been changed, and that’s why we are still able to use the above properties of \(M', N'\).

Now, think \((M', C'_p) \cong (S^1 \times [0, 2], S^1 \times 2) \cong (M, C)\). A similar thinking for \((N', C'_p)\) and \((N, C)\). Applying Lemma 3.3 to \(f_1|M' \to M\) and \(f_1|N' \to N\) we have two homotopies and then pasting these homotopies we have the following: There exists a homotopy (we will say it’s a local homotopy) \(H: (M' \cup N') \times [0, 1] \to M \cup N\) such that \(H(z, t) = f_1(z)\) for all \((z, t) \in \partial(M' \cup N') \times [0, 1]\) (2) \((H(-1), -1))^{-1}(C) = C'_p\) (3) \(H(-1)|C'_p \to C\) is a homeomorphism.

\[
\begin{array}{c}
0 & 1 & 2 \\
0 & 1 & 2 \\
\hline
\end{array}
\]

Homotope \(f_1|M' \to M\) relative to the left side of \(M'\) to \(f_2|M' \to M\) so that the following hold: (1) \(f_2^{-1}(C) \cap M' = C'_p\) (2) \(f_2(-, r) = \theta(-, r)\) for \(r \in [1, 2]\) and some homeomorphism \(\theta: C'_p \to C\).

Considering all components of \(P\) and all components of \(f_1^{-1}(P)\), we have a collection of local homotopies having properties as given in the previous line. The first property (stationary on the boundary) of each of these local homotopies allows us to define a global homotopy \(H: \Sigma' \times [0, 1] \to \Sigma\) using the map \(f_1\); i.e., if \(x \in \Sigma'\) doesn’t lie inside any of these small tubular neighborhoods, then
\[ \overline{H}(x,t) = f_1(x) \text{ for all } t \in [0,1]. \] Therefore, \( \overline{H} \) is a proper map: A compact subset \( K \) of \( \Sigma \) can hit the images of a finite number of local homotopies and the inverse image of the range of a local homotopy under the global homotopy is always compact. Now, take \( f_2 := \overline{H}(-1, \cdot) \).

**Remark 3.5.** A closer look at the proof of Lemma 3.3 tells that \( \varphi_1(e^{i\theta}, r) = (\psi(e^{i\theta}), r) \) for all \( 1 \leq r \leq 2 \), where \( \psi: S^1 \to S^1 \) is the homeomorphism defined by \( \psi(-) = h(-, 2, 1) \). Applying this on the map \( f_2|\mathcal{M}' \to \mathcal{M} \) gives the following: Each boundary parallel circle of \( \mathcal{M}' \) near to \( C'_p \) is mapped onto the corresponding boundary parallel circle of \( \mathcal{M} \) near to \( C \), and this mapping can be realized by the homeomorphism \( f_2|C'_p \to C \). Similar reasoning for \( f_2|N' \to N' \).

**Step 4: Annulus removal**

In Step 3, we properly homotope \( f_1 \) to a map \( f_2 \) so that for each (primitive) component \( C'_p \) of \( f_2^{-1}(\mathcal{P}) \to f_1^{-1}(\mathcal{P}) \), the map \( f_2 \) sends \( C'_p \to C' \) onto a component \( C' \) of \( \mathcal{P} \) homeomorphically. Here we will properly homotope \( f_2 \) to get a map \( f_3 \) such that for each component \( C' \) of \( \mathcal{P} \), \( f_3^{-1}(C') \) is a single circle mapped homeomorphically onto \( C' \) by \( f_3 \). After all, we show that two components of \( f_2^{-1}(\mathcal{P}) \) are freely homotopic (i.e., co-bound an annulus) if and only if their \( f_2 \)-images are the same. So, any component \( C' \) of \( \mathcal{P} \) gives an outermost annulus \( A' \) bounded by two components of \( f_2^{-1}(C') \). Take such an outermost annulus \( A \) and now homotope \( f_2 \) relative to \( \partial A \) so that after homotopy, \( f_3 \) maps \( A' \) onto \( C' \). Finally, considering a one-sided thickened annulus \( A'_i \) for \( A' \), we homotope \( f_2 \) relative to \( \partial A'_i \) to a map \( f_3 \) so that \( f_3^{-1}(C') \) has exactly one component and \( f_3|f_3^{-1}(C') = \varphi \). This two-step process can be done by a proper homotopy that performs considering all outermost annuli obtained from different components of \( \mathcal{P} \) simultaneously.

At first, we quote the Annulus Embedding theorem. So, let \( X \) be connected 2-dimensional smooth manifold. Recall that there is a bijective correspondence between conjugacy classes in \( \pi_1(X, *) \) and homotopy classes of maps \( S^1 \to X \).

**Annulus Embedding.** [Eps66a, Lemma 2.4.] Let \( \ell_0, \ell_1: S^1 \to X \) be two continuous embeddings such that the image \( \ell_0(S^1) \) is a smoothly embedded submanifold of \( X \). Suppose \( \ell_0 \) and \( \ell_1 \) represent the same conjugacy class, then there is a continuous embedding \( \mathcal{L}: S^1 \times [0,1] \to X \) so that \( \mathcal{L}(-,0) = \ell_0 \) and \( \mathcal{L}(-,1) = \ell_1 \).

**Claim 4.1.** Let \( i_0, i_1: S^1 \to C_p^0, C_p^1 \) be embeddings of two components of \( f_2^{-1}(\mathcal{P}) \). Suppose there is a free homotopy \( S^1 \times [0,1] \to \Sigma' \) from \( i_0 \) to \( i_1 \). Then, \( \text{im}(f_2 \circ i_0) = \text{im}(f_2 \circ i_1) \).

**Proof.** Let \( C_0 := f_2(C_p^0) \) and \( C_1 := f_2(C_p^1) \). So, \( C_0, C_1 \) are components of \( \mathcal{P} \) and Claim 3.4 says that \( f_2|C_p^0 \to C_0, f_2|C_p^1 \to C_1 \) are homeomorphisms. Hence, there are embeddings \( j_0, j_1: S^1 \to C_0, C_1 \) so that \( f_2 \circ i_0(z) = j_0(z) \) and \( f_2 \circ i_1(z) = j_1(z) \) for all \( z \in S^1 \).

Let \( g_0, g_1 \in \pi_1(\Sigma, *) \) be two representatives of the conjugacy classes corresponding to the homotopy classes of \( j_0, j_1 \), respectively. Certainly, the conjugacy classes represented by \( g_0 \) and \( g_1 \) are not trivial since no component of \( \mathcal{P} \) bounds a disk, see Remark 1.3. Also, \( hgh^{-1} = g_1 \) for some \( h \in \pi_1(\Sigma, *) \) since the hypothesis gives a free homotopy from \( f_2 \circ i_0 \) to \( f_2 \circ i_1 \). Now, Annulus Embedding and Remark 1.3 give \( C_0 = \text{im}(j_0) = \text{im}(j_1) = C_1 \).

**Claim 4.2.** Suppose \( f_2^{-1}(C) \) has at least two components for some component \( C \) of \( \mathcal{P} \). Then any two distinct components of \( f_2^{-1}(C) \) together bound an annulus.

**Proof.** Let \( C_p^a, C_p^b \) be any two distinct components of \( f_2^{-1}(C) \). Since, \( f_2|C_p^a \to C \) and \( f_2|C_p^b \to C \) both are homeomorphisms (see Claim 3.4): for \( j: S^1 \to C \) there are \( i_a: S^1 \to C_p^a \) and \( i_b: S^1 \to C_p^b \) such that \( f_2 \circ i_a = j = f_2 \circ i_b \). Let \( g_2: \Sigma \to \Sigma' \) be a homotopy inverse of \( f_2 \). So,

\[
i_a \simeq g_2 \circ f_2 \circ i_a = g_2 \circ j = g_2 \circ f_2 \circ i_b \simeq i_b.
\]

So, \( i_a, i_b \) represent the same non-trivial conjugacy class of \( \pi_1(\Sigma') \). Hence, \( C_p^a \) and \( C_p^b \) bounds an annulus (see Annulus Embedding), i.e., there exists an embedding \( \varphi: S^1 \times [0,1] \to \Sigma' \) such that \( \varphi(-,0) = i_a \) and \( \varphi(-,1) = i_b \). \( \square \)
Lemma 4.3. Let $A := S^1 \times [0, 1]$ and $\varphi: A \to A$ be a map such that $\varphi(-, 0) = (\theta(-), 0) = \varphi(-, 1)$ for some $\theta: S^1 \to S^1$. Then, there is a homotopy relative to $\partial A$ from $\varphi$ to a map $\Phi: A \to A$ so that $\Phi(A) \subseteq S^1 \times 0$.

Proof. Let $\varphi_1, \varphi_2: A \to S^1, [0, 1]$ be the components of $\varphi$. Consider the homotopy $h: A \times [0, 1] \to A$ given by

$$h(z, r, t) := (\varphi_1(z, r), (1 - t)\varphi_2(z, r))$$

for $0 \leq r, t \leq 1$ and $z \in S^1$.

Define $\Phi(z, r) := h(z, r, 1)$ for $0 \leq r < 1$ and $z \in S^1$. □

Lemma 4.4. Let $A := S^1 \times [0, 1]$ and $\varphi: A \to S_{0,3}$ be a map from annulus to pair of pants such that $\varphi(\text{int}(A)) \subseteq \text{int}(S_{0,3})$ and there is a boundary component $C$ of $S_{0,3}$ for which $\varphi(-, 0), \varphi(-, 1): S^1 \to C$ are the same homeomorphisms. Then there exists a homotopy relative to $\partial A$ from $\varphi$ to a map $\Phi: A \to S_{0,3}$ so that $\Phi(A) = C$.

Proof. Write $\partial_0 A = S^1 \times 0$ and $\partial_1 A = S^1 \times 1$. Embed $S_{0,3}$ into thrice punctured sphere $S_{0,3}$ so that each components of $\partial S_{0,3}$ bounds a puncture in $S_{0,3}$. In particular, $C$ now represents a primitive circle in $S_{0,3}$, and we can think $\varphi$ as map from $A$ into $S_{0,3}$. Let $p: X \to S_{0,3}$ be the covering corresponding to the non-trivial subgroup $H := \text{im}(\pi_1(\varphi))$ of $G := \pi_1(S_{0,3})$. Then, $X$ is a non-compact (as $[G : H] = \infty$) surface having infinite cyclic fundamental group, i.e., $X$ is homeomorphic to punctured plane.

Let’s consider a lift $\tilde{\varphi}: A \to X$ of $\varphi$ and define $C_0 := \tilde{\varphi}(\partial_0 A)$, $C_1 := \tilde{\varphi}(\partial_1 A)$. Notice that $\tilde{\varphi}|\partial_0 A \to C_0$ is injective as $p\tilde{\varphi} = \varphi$ sends $\partial_0 A$ homeomorphically onto $C$. Hence, $C_0$ is an embedded circle in $X$. Note that $C_0$ can’t bound a disk, otherwise the injective map $\pi_1(\varphi) = \pi_1(p)(\tilde{\varphi})$ would send the non-trivial element $[\partial_0 A]$ to the trivial element of $\pi_1(S_{0,3})$. In other words, $C_0$ bounds the puncture of $X$. Similarly, $C_1$ bounds the puncture of $X$. Since, $p$ is a local homeomorphism, either $C_0 \cap C_1 = \varnothing$ or $C_0 = C_1$. We will show that the former one is not possible.

On the contrary, assume $C_0 \cap C_1 = \varnothing$. Let $N: C \to [\varnothing, 1] \to S_{0,3}$ be a two-sided tubular neighborhood of $C$, where $N(C, 0) = C$. Choose one point from each of $C_0, C_1$ so that these two points are on the same fiber of $p$; and then consider two lifts $\tilde{N}_0$ and $\tilde{N}_1$ of $N$ corresponding to these points. Notice that $\tilde{N}_0, \tilde{N}_1$ are embeddings and $p$ maps their images homeomorphically onto $\text{im}(N)$. By uniqueness of lifting, $\tilde{N}_0(C, 0) = \tilde{C}_0$ and $\tilde{N}_1(C, 0) = \tilde{C}_1$. Since, $\tilde{C}_0 \cap \tilde{C}_1 = \varnothing$, choosing the tubular neighborhood $N$ small enough, we may assume $\text{im}(\tilde{N}_0) \cap \text{im}(\tilde{N}_1) = \varnothing$. Now, orient $S_{0,3}$ and give pull-back the orientation on $X$ via the covering $p$ so that $p$ is orientation preserving. In particular, $C$ has an orientation consistent with the orientation of $S_{0,3}$; and $\tilde{C}_0, \tilde{C}_1$ have orientations consistent with the orientation of $X$ so that $p$ preserves these orientations.

This gives the following observations:

- If $\tilde{C}_0$ and $\tilde{C}_1$ co-bound the annulus $\mathcal{A}$, then identifying $\tilde{C}_0$ and $\tilde{C}_1$ compatibly with the homeomorphisms $p|\tilde{C}_0 \to C$, $p|\tilde{C}_1 \to C$; we get a torus $\mathcal{T}$ from $\mathcal{A}$ and a map $\mathcal{P}: \mathcal{T} \to S_{0,3}$ from $p|\mathcal{A} \to S_{0,3}$. Let $\mathcal{C}$ be the circle in $\mathcal{T}$ obtained from $\tilde{C}_0$ and $\tilde{C}_1$.

- Small neighborhoods of $\tilde{C}_0, \tilde{C}_1$ in $\mathcal{A}$ map to opposite sides of $C$ in $S_{0,3}$ homeomorphically by $p$. That is, a small neighborhood of $\mathcal{C}$ in $\mathcal{T}$ maps onto a neighborhood of $C$ in $S_{0,3}$ homeomorphically by $\mathcal{P}$.

Consider the simple arc $\lambda(t) := (1, t), t \in [0, 1]$ connecting two boundary components of $A$. Now, recall that we assume $\varphi(\text{int}(A)) \subseteq \text{int}(S_{0,3})$ in the statement of Lemma 4.4. So, $\tilde{\varphi}$ gives a loop $\tilde{\mathcal{C}}'$ in $\mathcal{T}$ such that $[\mathcal{C}], [\mathcal{C}']$ together generate $\pi_1(\mathcal{T}) = \mathbb{Z} \times \mathbb{Z}$; and $p\tilde{\varphi}\lambda(t) \in C$ if and only if $t = 0, 1$. Also, the loop $p\tilde{\varphi}\lambda$ maps $(0, \varepsilon)$ and $(1 - \varepsilon, 1)$ into the opposite sides of $C$ for small enough $\varepsilon > 0$ by $\mathcal{H}$.

Therefore, $\mathcal{P}: \mathcal{T} \to S_{0,3}$ induces a $\pi_1$-injective map, i.e., $\mathbb{Z} \times \mathbb{Z}$ is a subgroup of $\mathbb{Z} \ast \mathbb{Z}$, which is impossible as subgroup of a free group is free. So, the assumption $\tilde{C}_0 \cap \tilde{C}_1 = \varnothing$ is not tenable.

From the previous paragraph, we can say $\tilde{C}_0 = \tilde{C}_1$, in any case. Now, there is a strong deformation retract $\tilde{H}: X \times [0, 1] \to X$ of the punctured plane $X$ onto the puncture bounding circle $\tilde{C}_0 = \tilde{C}_1$.

Define, $H(a, t) := p \circ H(\tilde{\varphi}(a), t)$ for $a \in A, t \in [0, 1]$. Then, $H: A \times [0, 1] \to X$ is a homotopy relative to $\partial A$ such that $\Phi := H(-, 1)$ sends $A$ onto $C$. □
Lemma 4.5. Consider a compact bordered surface $S$ and a finite collection $X$ of pairwise-disjoint embedded circles in $\text{int}(S)$ such that $X \cup \partial S$ breaks down $S$ into pair of pants and annuli. Let $C$ be a component of $X$, $n \in \mathbb{N}$ and $A' := \mathbb{S}^1 \times [0, n]$. Suppose we are given a map $\varphi: A' \to S$ with $\varphi^{-1}(X \cup \partial S) = \varphi^{-1}(C) = \mathbb{S}^1 \times \{0, \ldots, n\}$ and $\varphi(-, k): \mathbb{S}^1 \xrightarrow{\cong} C$ represents the same parametrization of $C$ for any $k = 0, \ldots, n$. Then there is a homotopy relative to $\partial A'$ from $\varphi$ to a map $\Phi$ such that $\text{im}(\Phi) = C$.

Proof. Let $A_k := \mathbb{S}^1 \times [k - 1, k]$ for $k = 1, \ldots, n$ and $S_1, \ldots, S_m$ be the compact sub-surfaces of $S$ obtained from the decomposition of $S$ by $X$. The continuity of $\varphi|A_k \setminus \varphi^{-1}(X) \to S \setminus X$ gives that $\varphi|A_k$ is contained in some $S_k$ such that $\varphi(\text{int}(A_k)) \subseteq \text{int}(S_k)$ and $\varphi(-, k - 1), \varphi(-, k): \mathbb{S}^1 \xrightarrow{\cong} C \subseteq \partial S_k$ are the same homeomorphisms for each $k = 1, \ldots, n$. Now, apply Lemma 4.3 and Lemma 4.4. □

One observation is worth noting: The image of $\varphi$ is contained in the union of at most two sub-surfaces from the list $S_1, \ldots, S_m$, and these two sub-surfaces share $C$ as a common boundary.

Lemma 4.6. Let $\psi: \mathbb{S}^1 \times [-2, 1] \to \mathbb{S}^1 \times [-1, 0]$ be a map and $\psi_1: \mathbb{S}^1 \times [-2, 1] \to \mathbb{S}^1, [-1, 0]$ be its components. Suppose,

$$
\psi_2(-, r) = \begin{cases} 
  r + 1 & \text{if } -2 \leq r \leq -1, \\
  0 & \text{if } -1 \leq r \leq 1.
\end{cases}
$$

Then there is a homotopy from $\psi$ to a map $\Psi$ relative to $\mathbb{S}^1 \times \{-2, 1\}$ such that $\Psi^{-1}(\mathbb{S}^1 \times 0) = \mathbb{S}^1 \times 1$.

Proof. Consider a homeomorphism $\ell: [-2, 1] \to [-1, 0]$ with $\ell(-2) = -1$ and $\ell(1) = 0$. Define

$$
H_t(z, r) := \psi_1(z, r) + t \cdot \ell(r)
$$

for $t \in [0, 1]$, $r \in [-2, 1]$, and $z \in \mathbb{S}^1$.

Let $\Psi(z, r) := H_t(z, r)$ for $z \in \mathbb{S}^1$ and $r \in [-2, 1]$. □

Claim 4.7. There is a proper homotopy from $f_2: \Sigma' \to \Sigma$ to some other proper map, say $f_3: \Sigma' \to \Sigma$, such that $f_3^{-1}(C)$ has exactly one component and $f_3|f_3^{-1}(C) \to C$ is a homeomorphism, for any component $C$ of $\mathcal{P}$.

Proof. Suppose $f_2^{-1}(C) = f_1^{-1}(C)$ has at least two components for some component $C$ of $\mathcal{P}$. Using Claim 4.1 and Claim 4.2, there is at least one outer-most annulus $A'$, that is, $\partial A'$ is the union of two distinct components of $f_2^{-1}(P)$ and the interior of $A'$ may contain some other (of course, finitely many as $f_2^{-1}(P)$ is a locally finite collection) annuli bounded by two components of $f_2^{-1}(P)$; but $A'$ is not contained in the interior of an annulus bounded by any two components of $f_2^{-1}(P)$.

Consider such an outermost annulus $A'$ and let $\partial A' = C'_P \cup C'_\beta$. From Claim 4.1, $f_2(C'_P) = f_2(C'_\beta) = C$, say. Notice that $A' \cap f_2^{-1}(P) = f_2^{-1}(C)$ and $f_2|C'_P \to C$ is a homeomorphism for each component $C'_P$ of $f_2^{-1}(C)$. Fix a parametrization $j: \mathbb{S}^1 \xrightarrow{\cong} C$ and find the corresponding parametrization of each component of $f_2^{-1}(C)$ as given in the proof of Claim 4.2. Now, using Anulus Embedding one can find parametrization of each annulus bounded by two components of $f_2^{-1}(C)$. Considering few of these annulus parametrizations (and discarding others); and pasting altogether we have a global parametrization $\tau: \mathbb{S}^1 \times [0, n] \xrightarrow{\cong} A'$ for some integer $n \geq 1$ so that $\tau(\mathbb{S}^1 \times \{0, \ldots, n\}) = f_2^{-1}(C)$ and $f_2 \circ \tau(-, k): \mathbb{S}^1 \xrightarrow{\cong} C$ represents the same homeomorphism of $C$ for each $k = 0, \ldots, n$.

Now, recall how we defined the notion of convergence to $\infty$ in the proof of Claim 2.2. We will assume the number of distinct outermost annuli is infinite (when these are finite in number, the case can be handled analogously, just ignoring the notion of infinity-convergence). So, let $\{A'_n\}$ be the sequence of all outermost annuli of $\Sigma'$ such that $A'_n \cap A'_m = \emptyset$ when $n \neq m$. Write $C_n = f_2(\partial A'_n)$. Now, Claim 4.1 says $f_2^{-1}(C_m) \cap A'_n = \emptyset$ if $m \neq n$. Also, $f_2^{-1}(C_n)$ breaks down the larger annulus $A'_n$ into smaller annuli. Note that we decompose $\Sigma$ into pair of pants and punctured disks by $\mathcal{P}$. Like the previous observation, we can now say similarly that $f_2(A'_n)$ is contained in the union of at most two sub-surfaces of $\Sigma$, obtained from the decomposition of $\Sigma$ by $\mathcal{P}$. 

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Notice that \( P \) and \( f_2^{-1}(P) \) are locally-finite collections imply \( C_n \to \infty \) and \( \mathcal{A}'_n \to \infty \), respectively. Also, \( f_2(\mathcal{A}'_n) \to \infty \) as \( f_2 \) is proper. At this point, we want to apply Lemma 4.5, i.e., for each \( \mathcal{A}'_n \) we want a compact bordered sub-surface \( S_n \) of \( \Sigma \) such that the following hold:

1. \( f_2(\mathcal{A}'_n) \subseteq S_n \).
2. \( S_n \to \infty \).
3. There is a finite collection \( \mathcal{X}_n \) of pairwise disjoint circles in \( \text{int}(S_n) \) such that \( \mathcal{X}_n \cup \partial S_n \) decomposes \( S_n \) into pair of pants and annuli so that \( C_n \) is a component of \( \mathcal{X}_n \). \( f_2^{-1}(\mathcal{X}_n \cup \partial S_n) \cap \mathcal{A}'_n = f_2^{-1}(C_n) \) and all components of \( f_2^{-1}(C_n) \) maps onto \( C_n \) by \( f_2 \) in similar fashion. To satisfy all these three conditions, we need to notice that \( f_2(\mathcal{A}'_n) \to \infty \) and \( \mathcal{C}_n \to \infty \); also, to construct \( S_n \), we may need to split up punctured disks (obtained from the decomposition of \( \Sigma \) by \( P \)) into annuli.

Once we get all these, using Lemma 4.5 we have a collection of homotopies \( \mathcal{H}_n : \mathcal{A}'_n \times [0, 1] \to S_n \) relative to \( \partial \mathcal{A}'_n \) from \( f_2(\mathcal{A}'_n) \to S_n \) to the map \( \mathcal{H}_n(-, 1) \) such that \( \mathcal{H}_n(\mathcal{A}'_n, 1) = C_n \). Considering all these relative homotopies, we have a homotopy \( \mathcal{H} : \Sigma' \times [0, 1] \to \Sigma \) from \( f_2 \) to another map, which will be again denoted by \( f_2 \) such that after homotopy, \( f_2 \) maps each outermost annulus \( \mathcal{A}'_n \) onto \( C_n \). Note that we are defining \( \mathcal{H}(p, t) = f_2(p) \) for all \( p \notin \bigcup_{n \in \mathbb{N}} \mathcal{A}'_n \) and all \( t \in [0, 1] \). Now, we show \( \mathcal{H} \) is proper. So, take a compact subset \( \mathcal{K} \) of \( \Sigma \), and observe that \( \text{im}(\mathcal{H}_n) \cap \mathcal{K} \subseteq S_n \cap \mathcal{K} = \emptyset \) for all but finitely many \( n \). Also, notice that outside of \( \bigcup_{n \in \mathbb{N}} \mathcal{A}'_n \) the map \( \mathcal{H} \) is defined by the proper map \( f_2 \). Hence, \( \mathcal{H}^{-1}(\mathcal{K}) \) is compact, and so, we are done.

Next, consider an outermost annulus \( \mathcal{A}' \) (omit subscript for simplicity), and write \( \partial \mathcal{A}' = C'_{p\alpha} \cup C'_{p\beta} \). Let \( \mathcal{C} \) be the unique component of \( P \) corresponding to \( \mathcal{A}' \). Use Remark 3.5 to find one-sided tubular neighborhoods \( \mathcal{M}'_\alpha \) of \( C'_{p\alpha} \) and \( \mathcal{M}'_\alpha \) of \( \mathcal{C} \) so that the following hold:

1. \( \mathcal{M}'_\alpha \cap \mathcal{A}' = C'_{p\alpha} \), i.e., the slightly larger annulus \( \mathcal{A}'_\alpha := \mathcal{M}'_\alpha \cup \mathcal{A}' \) shares only one boundary component with \( \mathcal{A}' \) and the other boundary component of \( \mathcal{A}'_\alpha \), namely \( C'_{p\alpha} \), is contained in the interior of \( \mathcal{A}'_\alpha \).
2. \( \mathcal{A}'_\alpha \cap f_2^{-1}(P) = f_2^{-1}(C) \), this is possible if we choose \( \mathcal{M}'_\alpha \) small enough as \( f_2^{-1}(P) \) is a locally finite collection.
3. \( f_2(\mathcal{M}'_\alpha) = \mathcal{M}_\alpha \) and \( \mathcal{M}_\alpha \cap P = \mathcal{C} \), this is possible if we choose \( \mathcal{M}_\alpha \) small enough as \( P \) is a locally finite collection.
4. Each boundary parallel circle of \( \mathcal{M}'_\alpha \) is mapped onto the corresponding boundary parallel circle of \( \mathcal{M}_\alpha \), and this mapping can be realized by the homeomorphism \( f_2|_{\mathcal{C}'_{p\beta}} : C'_{p\beta} \to C \).

Using all these, think \( (\mathcal{A}'_\alpha, C'_{p\beta}) \cong (S^1 \times [-2, 1], S^1 \times 1) \), \( (\mathcal{M}_\alpha, \mathcal{C}) \cong (S^1 \times [-1, 0], S^1 \times 0) \); and then apply Lemma 4.6. See the figure on the next page. That is to say, we have a homotopy (we
say it’s a local homotopy) \( H: \mathcal{A}'_\varepsilon \times [0, 1] \to \mathcal{M}_\alpha \) relative to \( \partial \mathcal{A}'_\varepsilon \) from \( f_2|\mathcal{A}'_\varepsilon \to \mathcal{M}_\alpha \) to the map \( H(-, 1) \) so that \((H(-, 1))^{-1}(\mathcal{C}) = \mathcal{C}'_{p\beta} \) and \( H(-, 1)|\mathcal{C}'_{p\beta} \to \mathcal{C} \) is a homeomorphism.

The map \( f_2 \) sends \( \mathcal{A}' \) onto \( \mathcal{C} \) in such a way that each black circle, in particular, each of \( \mathcal{C}'_{p\alpha}, \mathcal{C}'_{p\beta} \), maps homeomorphically onto \( \mathcal{C} \). Also, \( f_2|\mathcal{M}'_\alpha \to \mathcal{M}_\alpha \) can be realized as \((z, r) \mapsto (\theta(z), r + 1)\) for \( z \in \mathbb{S}^1 \) and \( r \in [-2, 1] \), where \( \theta: \mathbb{S}^1 \to \mathbb{S}^1 \) is homeomorphism provided by \( f_2|\mathcal{C}'_{p\alpha} \to \mathcal{C} \). Now, we can homotope \( f_2|\mathcal{A}'_\varepsilon \to \mathcal{M}_\alpha \) relative to \( \partial \mathcal{A}'_\varepsilon \) to the map \( f_3|\mathcal{A}'_\varepsilon \to \mathcal{M}_\alpha \) so that \( f_3^{-1}(\mathcal{C}) = \mathcal{C}'_{p\beta} \).

Considering all these local homotopies there is a global homotopy \( \overline{H}: \Sigma \times [0, 1] \to \Sigma \) such that \( \overline{H}(x, t) = f_2(x) \) for all \( t \in [0, 1] \) and all \( x \in \Sigma \setminus \bigcup_{n \in \mathbb{N}} \mathcal{A}'_{\varepsilon, n} \). Now, \( \overline{H} \) is a proper map: A compact subset \( \mathcal{K} \) of \( \Sigma \) can hit the images of a finite number of local homotopies and the inverse image of the range of a local homotopy under the global homotopy is always compact. Finally, define \( f_3 := \overline{H}(-1, 1) \). That is, we can properly homotopy \( f_2 \) to a map \( f_3 \), such that \( f_3^{-1}(\mathcal{C}) \) has exactly one component and \( f_3|f_3^{-1}(\mathcal{C}) \to \mathcal{C} \) is a homeomorphism, for any component \( \mathcal{C} \) of \( \mathcal{P} \).

**Step 5: Reduction to the finite-type surfaces to find homeomorphism**

Previous step shows that one can properly homotopy \( f_2 \) to a map \( f_3 \) so that \( f_3^{-1}(\mathcal{C}) \) has exactly one component and \( f_3|f_3^{-1}(\mathcal{C}) \to \mathcal{C} \) is a homeomorphism for any component \( \mathcal{C} \) of \( \mathcal{P} \). Here, in this last step, we will show \( f_3^{-1}(\mathcal{P}) \) breaks down \( \Sigma' \) into punctured disks and pair of pants. Recall also that \( \mathcal{P} \) breaks down \( \Sigma \) into punctured disks and pair of pants. And there is a shape-preserving bijective-correspondence between these two collections of broken parts due to the homotopy equivalence \( f_3 \). Now, a proper allowable self-map of punctured disk or pair of pants, which induces a homeomorphism on the boundary, is properly homotopic to a homeomorphism by a proper homotopy relative to the boundary. Finally, gluing all these boundary-relative homotopies, we will complete the Theorem 3.2.

**Claim 5.1.** The sub-collection \( f_3^{-1}(\mathcal{P}) \) of \( f^{-1}(\mathcal{P}) \) decomposes \( \Sigma' \) into punctured disks and pair of pants.

**Proof.** Using continuity of \( f_3 \) and Claim 4.7, the map \( f_3 \) sends a component of \( \Sigma \setminus f_3^{-1}(\mathcal{P}) \) into a component of \( \Sigma \setminus \mathcal{P} \). Also, every component of \( \Sigma \setminus \mathcal{P} \) has non-empty pre-image, as \( \deg(f_3) = \deg(f) \neq 0 \) implies \( f_3 \) is surjective, see Claim 1.6. Let \( \mathcal{S}' \) be a bordered sub-surface of \( \Sigma' \) obtained from the decomposition of \( \Sigma' \) by \( f_3^{-1}(\mathcal{P}) \). Therefore, \( \mathcal{S} := f_3(\mathcal{S}') \) is a bordered sub-surface of \( \Sigma \) obtained from the decomposition of \( \Sigma \) by \( \mathcal{P} \) such that \( \mathcal{S} \) compact if and only if \( \mathcal{S}' \) is compact, as \( f_3 \) is proper. Notice that \( f_3|\partial \mathcal{S}' \to \partial \mathcal{S} \) is a homeomorphism (see Claim 4.7) implies \( \partial \mathcal{S}' \) has either one or three components as \( \mathcal{S} \) can be either a punctured disk or a pair of pants.

In case, \( \mathcal{S} \) is punctured disk, \( \mathcal{S}' \) is a non-compact bordered surface with one boundary component. Now, \( \pi_1(f_3): \pi_1(\mathcal{S}') \to \pi_1(\Sigma) \) is injective implies \( f_3|\mathcal{S}' \to \mathcal{S} \) induces an injective map \( \pi_1(\mathcal{S}') \to \pi_1(\mathcal{S}) \).
$\pi_1(S)$: for this, one must note that $\partial S'$ is a primitive circle of $\Sigma'$, and each end of $S'$ is also an end of $\Sigma'$. Since $\pi_1(S')$ is a free group of rank at least 1 ($\partial S'$ doesn’t bound a disk), the injection $\pi_1(S') \to \pi_1(S) \cong \mathbb{Z}$ implies $S'$ is also a punctured disk.

On the other hand, if $S$ is pair of pants, $S'$ is a compact bordered surface with three boundary components. Again, $\pi_1(S') \to \pi_1(S)$ is injective and each component of $\partial S'$ doesn’t bound a disk imply $f_3|S' \to S$ induces an injective map $\pi_1(S') \to \pi_1(S)$. We now claim that this restriction-induced map between the fundamental groups is surjective. Assuming this, we can say that the compact bordered surface $S'$ has the fundamental group $\mathbb{Z} \ast \mathbb{Z}$ and three boundary components, i.e., $S'$ is pair of pants. Now to prove the claim, it is enough to show $f_3(S', \partial S') \to (S, \partial S)$ is a map of degree $\pm 1$, see [Eps66b, Corollary 3.4.]. So, we need to prove degree is $\pm 1$. For this, note that the connecting homomorphism $H_2(S', \partial S') \to H_1(\partial S')$ (think as a map from $Z$ to $Z \oplus Z \oplus Z$) sends fundamental class to (sum) fundamental class. Similar observation for the other connecting homomorphism $H_2(S, \partial S) \to H_1(\partial S)$. Finally, the naturality of the long exact sequence and $f_3|S' \to S$ is a homeomorphism imply degree is $\pm 1$. □

**Lemma 5.2.** Let $D^* := \{z \in \mathbb{C} : 0 < |z| \leq 1\}$. Consider a proper map $\varphi: D^* \to D^*$ such that $\varphi^{-1}(\partial D^*) = \partial D^*$ and $\varphi|\partial D^* \to \partial D^*$ is a homeomorphism. Then $\varphi$ is properly homotopic to a homeomorphism $D^* \to D^*$ relative to the boundary $\partial D^*$.

**Proof.** Define $\mathcal{H}: D^* \times [0,1] \to D^*$ by

$$
\mathcal{H}(z,t) := \begin{cases}
(1-t) \cdot \varphi \left( \frac{z}{1-t} \right) & \text{if } 0 < |z| \leq 1-t, \\
|z| \cdot \varphi \left( \frac{z}{|z|} \right) & \text{if } 1-t < |z| \leq 1.
\end{cases}
$$

Notice that $\varphi \simeq \mathcal{H}(-1)$ relative to $\partial D^*$ where $\mathcal{H}(-1): D^* \to D^*$ is a homeomorphism. Also, $\mathcal{H}(z_n, t_n) \to 0$ if and only if $z_n \to 0$, where $\{z_n\} \subseteq D^*$ and $\{t_n\} \subseteq [0,1]$. □

**Claim 5.3.** The two non-compact surfaces $\Sigma, \Sigma'$ are homeomorphic and the proper map $f_3: \Sigma' \to \Sigma$ is properly homotopic to a homeomorphism $\Sigma' \to \Sigma$.

**Proof.** Define $S'$ and $S = f_3(S')$ similarly as in the proof of Claim 5.1. Note that $f_3^{-1}(\partial S) = \partial S'$ and $f_3|\partial S' \to \partial S$ is a homeomorphism. Note also that boundary components of pair of pants and punctured disk determine its fundamental group. Combining all these three facts, $f_3$ induces a surjective (hence bijective) map on $\pi_1(S') \to \pi_1(S)$. In other words, $f_3(S') \to S$ is a homotopy equivalence.

When $S'$ is a punctured disk, $f_3|S' \to S$ is properly homotopic to a homeomorphism relative to $\partial S'$ by Lemma 5.2.

When $S'$ is a pair of pants, $f_3|S' \to S$ is an allowable map of degree $\pm 1$, and hence homotopic to a homeomorphism $S' \to S$ relative to $\partial S'$ by [Edm79, Theorem 3.1.].

Paste all these boundary-relative homotopies to get a proper homotopy $\Sigma' \times [0,1] \to \Sigma$ from $f_3$ to a homeomorphism $\Sigma' \to \Sigma$. □

Since, $f$ is properly homotopic to $f_3$ we are done by Claim 5.3. Here we complete the proof of Theorem 3.2. □

**Theorem 3.3.** Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a proper map of degree $\pm 1$. Then $f$ is properly homotopic to a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$.

**Proof.** Using [Eps66b, Theorems 3.1., 4.1.] we have a proper map $g: \mathbb{R}^2 \to \mathbb{R}^2$ which is properly homotopic to $f$ and a small closed disk $D \subseteq \mathbb{R}^2$ such that $D' := g^{-1}(D)$ is also a closed disk, mapped homeomorphically onto $D$ by $g$. Note that either of $D$ or $D'$ may not be circular; still, one can treat them as $D := \{z \in \mathbb{C} : |z| \leq 1\}$ using Schönflies theorem.

Now, the proper map $g|\mathbb{R}^2 \setminus D' \to \mathbb{R}^2 \setminus D$ can be thought as the map given in Lemma 5.2. Therefore, $g|\mathbb{R}^2 \setminus D' \to \mathbb{R}^2 \setminus D$ properly homotopic to a homeomorphism relative to the boundary $\partial \mathbb{R}^2 \setminus D' = \partial D'$. Also, $g|D' \to D$ is a homeomorphism. Combining these two facts, $g$, hence $f$ is properly homotopic to a homeomorphism. □

**Theorem 3.4.** Let $\Sigma, \Sigma'$ be any two non-compact surfaces. Suppose there exists a $\pi_1$-injective map $f: \Sigma' \to \Sigma$ of degree $\pm 1$. Then $\Sigma$ is homeomorphic to $\Sigma'$ and $f$ is properly homotopic to a homeomorphism.
Proof. Note that \( \deg(f) = \pm 1 \) implies \( \pi_1(f) : \pi_1(\Sigma') \to \pi_1(\Sigma) \) is surjective, see [Eps66b, Corollary 3.4]. From hypothesis \( \pi_1(f) \) is bijective. Since, any non-compact surface is homotopy equivalent to a wedge of at most countably many circles, all higher-homotopy groups of a non-compact surface are trivial. In other words, \( f \) is a weak homotopy equivalence between two CW-complexes; hence itself is a homotopy equivalence. Now, combine Theorem 3.2 and Theorem 3.3. \( \square \)

**Theorem 3.5.** Every proper homotopy equivalence between two non-compact surfaces is properly homotopic to a homeomorphism.

Proof. Proper homotopy equivalence is a \( \pi_1 \)-injective map of degree \( \pm 1 \). Now, Theorem 3.4 gives the desired result. \( \square \)

4. Section 4

**Lemma 4.1.** Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be a smooth map and \( x_n \to x \) in \( \mathbb{R}^2 \) with \( r_n := |g(x_n)| \to 1 \). Write \( S_r := \{ z \in \mathbb{R}^2 : |z| = r \} \) and assume \( \text{im}(dx_n) = T_{g(x_n)}(S_{r_n}) \) for all \( n \). If \( dx_x \neq 0 \), then \( \text{im}(dx_z) = T_{g(x)}(S_1) \).

Proof. The derivative map \( dg : \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2) \) is continuous implies \( dg_{x_n} \to dg_x \), and this convergence can be thought as convergence of \( 2 \times 2 \)-matrices. In particular, if \( i, j \in \mathbb{R}^2 \) are two perpendicular unit vectors, then \( dg_{x_n}(i) \to dg_i(i) \) and \( dg_{x_n}(j) \to dg_j(j) \). Recall that the tangent space at any point of a circle is the vector space of all points perpendicular to the point. So, \( \langle dg_{x_n}(i), g(x_n) \rangle = 0 = \langle dg_{x}(i), g(x) \rangle \) by hypothesis, and now \( \langle dg_{x_n}(j), g(x_n) \rangle = 0 = \langle dg_{x}(j), g(x) \rangle \) by the convergence of inner-product. Hence, \( \text{im}(dx_z) \subseteq T_{g(x)}(S_1) \). Since, \( dx_x \neq 0 \) and \( \dim T_{g(x)}(S_1) = 1 \), we are done. \( \square \)

**Theorem 4.1.** Let \( f : X \to Y \) be a smooth proper map between 2-dimensional boundaryless manifolds. Suppose, \( C \) is a smoothly embedded circle in \( Y \) and \( f \) is transverse to \( C \), i.e., \( \text{im}(df_x) + T_{f(x)}(C) = T_{f(x)}(Y) \) for all \( x \in f^{-1}(C) \). Let \( \phi : C \times [-1, 1] \to Y \) be a smooth embedding with \( \phi(C, 0) = C \). Then there exists small enough \( \varepsilon > 0 \) such that \( f \) is transverse to the circle \( C_{\varepsilon} := \phi(C, \varepsilon) \) for each \( \delta \in (\varepsilon, \varepsilon) \).

Proof. Suppose not. Then we have a sequence \( \delta_n \to 0 \) and points \( x_n \in f^{-1}(C_{\delta_n}) \) such that \( \text{im}(df_{x_n}) + T_{f(x_n)}(C_{\delta_n}) \neq T_{f(x_n)}(Y) \), i.e., \( \text{im}(df_{x_n}) \subseteq T_{f(x_n)}(C_{\delta_n}) \). Since, \( \{x_n\} \) is contained in the compact set \( f^{-1}(\text{im}(\phi)) \), passing to the sub-sequence, if needed, assume \( x_n \to x \in f^{-1}(C) \). So, the sequence \( df_{x_n} \) of linear maps converges to \( df_x \) as the derivative map \( df : X \to L(\mathbb{R}^2, \mathbb{R}^2) \) is continuous. Note that discarding first few terms we may assume \( df_{x_n} \neq 0 \) for all \( n \), otherwise \( df_x \) would be zero, i.e., \( f \) would not be transverse to \( C \). Hence, \( \text{im}(df_{x_n}) = T_{f(x_n)}(C_{\delta_n}) \) for all \( n \).

We will show that \( \text{im}(df_x) = T_{f(x)}(C) \), and then it will contradict the hypothesis that \( f \not\simeq C \). But, considering a chart at \( x \) and thinking \( C \) as a diffeomorphic copy \( S^1 \) this reduces to the above Lemma 4.1. So, we are done. \( \square \)

**Remark 4.1.** For \( \delta \in (\varepsilon, \varepsilon) \), Theorem 4.1 says that \( f^{-1}(C_{\delta}) \) is a compact(as \( f \) is proper) boundary-less smoothly embedded 1-dimensional submanifold of \( X \), i.e., \( f^{-1}(C_{\delta}) \) consists of finitely many pairwise disjoint circles.

**Theorem 4.2.** Let \( C' \) be a component of \( f^{-1}(C) \), where \( f, C \) are as in Theorem 4.1. Suppose we are given an one-sided closed tubular neighborhood \( U' \) of \( C' \) so that \( U' \cap f^{-1}(C) = C' \). Then there exists \( \delta \in (\varepsilon, 0) \cup (0, \varepsilon) \) such that at least one component of \( f^{-1}(C_{\delta}) \) is contained in \( \text{int}(U') \).

Proof. Suppose not. Then there is an one-sided closed tubular neighborhood \( U' \) of \( C' \) having the following properties: 1. \( U' \cap f^{-1}(C) = C' \) 2. Given any sequence \( \{\delta_n\} \subseteq (\varepsilon, 0) \cup (0, \varepsilon) \) of distinct reals converging to 0 such that for any \( n \in \mathbb{N} \), no component of \( f^{-1}(C_{\delta_n}) \) is contained in \( \text{int}(U') \).

Choose a sequence \( \{a_n\} \subseteq \text{int}(U') \) converging to some point \( a \in C' \). Then \( f(a_n) \to f(a) \in C \). By Remark 4.1, we may assume \( f(a_n) \neq f(a_m) \) if \( n \neq m \). Define \( \delta_n \) using \( f(a_n) \in C_{\delta_n} \). Let \( C'_{\delta_n} \) be the component of \( f^{-1}(C_{\delta_n}) \) passing through \( a_n \). By (2), there is a point \( b_n \in \partial U \cap C'_{\delta_n} \). Actually, \( b_n \in \partial U \cap C' \) passing through to some point \( a \in C' \). Then \( f(b_n) \to f(a) \in C \). By Remark 4.1, we may assume \( f(a_n) \neq f(a_m) \) if \( n \neq m \). Define \( \delta_n \) using \( f(a_n) \in C_{\delta_n} \). Let \( C'_{\delta_n} \) be the component of \( f^{-1}(C_{\delta_n}) \) passing through \( a_n \). By (2), there is a point \( b_n \in \partial U \cap C'_{\delta_n} \). Actually, \( b_n \in \partial U \cap C' \) passing through to some point \( a \in C' \). Therefore, \( b \in (\partial U \cap C') \cap f^{-1}(C) = \emptyset \), a contradiction. \( \square \)
The two theorems below are the modifications of [Lee12, Theorems 6.26, 6.36] in the proper category. At first, we will recall the notions. All manifolds and sub-manifolds will be assumed as boundary-less.

Let $M \hookrightarrow \mathbb{R}^n$ be a properly embedded smooth submanifold of dimension $m$. For each $x \in M$ define normal space $N_x M := \{ v \in T_x \mathbb{R}^n | v \perp T_x M \}$ and define normal bundle of $M$ as $NM := \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M, v \in N_x M \}$. So, $NM$ is an $n$-dimensional embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. Consider $\pi : NM \ni (x,v) \mapsto x \in M$ and $E : NM \ni (x,v) \mapsto x + v \in \mathbb{R}^n$.

Then at any point $(x,0) \in NM$ the map $dE$ is non-singular, i.e., for each $x \in M$ we have $\delta > 0$ such that $E$ maps diffeomorphically $V_\delta(x) := \{(x',v') \in NM : |x-x'| < \delta, |v'| < \delta \}$ onto an open neighborhood of $x$ in $\mathbb{R}^n$. Now, consider the continuous map $\rho : M \to (0,1]$ defined by

$$\rho(x) := \sup \{ \delta \leq 1 : E \text{ maps } V_\delta(x) \text{ diffeomorphically onto an open neighborhood of } x \text{ in } \mathbb{R}^n \}.$$  

Next, define $V := \{(x,v) \in NM : |v| < \frac{1}{2} \rho(x) \}$. Then, $E$ maps diffeomorphically $V$ onto an open subset $U$ of $\mathbb{R}^n$ with $M \subseteq U$. Finally, the map $r : U \to M$ defined by $r := \pi \circ E^{-1}$ is a retraction and submersion.

**Lemma 4.2.** For $|y-y'| < \epsilon$ we have $|r(y) - r(y')| \leq \epsilon + 1$

**Proof.** Let $y,y' \in U$ with $|y-y'| < \epsilon$ for some $\epsilon > 0$. Write, $E^{-1}(y) = (x,v) \in V$ and $E^{-1}(y') = (x',v') \in V$, i.e., $r(y) + v = x + v = y$ and $r(y') + v' = x' + v' = y'$. So,

$$|x-x'| = |(y-y') - (v-v')| \leq |y-y'| + |v-v'| \leq \epsilon + |v| + |v'| \leq \epsilon + \frac{1}{2} \rho(x) + \frac{1}{2} \rho(x') \leq \epsilon + 1$$

as $\rho \leq 1$. Therefore, $|y-y'| < \epsilon \implies |r(y) - r(y')| \leq \epsilon + 1.$

**Theorem 4.3.** Let $M,N$ be two smooth manifolds and $F : N \to M$ be a continuous proper map. Then, there is a proper map $\mathcal{H} : N \times [0,1] \to M$ such that $\mathcal{H}(-,0) = F$ and $\mathcal{H}(-,1) : N \to M$ is a smooth proper map.

**Proof.** Let $\delta(x) := \sup \{ \epsilon \leq 1 : B_\epsilon(x) \subseteq U \}$ for each $x \in M$. Using a similar argument of showing that $\rho$ is continuous, one can prove $\delta : M \to (0,1]$ is continuous. Let $\delta := \delta \circ F$ and using Whitney Approximation theorem choose a smooth function $\tilde{F} : N \to \mathbb{R}^n$ such that $|\tilde{F}(y) - F(y)| < \delta(y)$ for each $y \in M$. In particular, $|\tilde{F} - F| \leq 1$. Define, $\mathcal{H} : N \times [0,1] \to M$ as $\mathcal{H}(p,t) := r \left( (1-t)F(p) + t\tilde{F}(p) \right)$ for $(p,t) \in N \times [0,1]$.

This is well-defined because $|\tilde{F}(p) - F(p)| < \delta(F(p))$, i.e., $\tilde{F}(p) \in B_{\delta(F(p))}(F(p)) \subseteq U$.

For $(p,t) \in N \times [0,1]$ we have

$$| (1-t)F(p) + t\tilde{F}(p) - F(p) | \leq t \cdot |\tilde{F}(p) - F(p)| \leq 1$$

$\implies |\mathcal{H}(p,t) - r \circ F(p)| = |\mathcal{H}(p,t) - F(p)| \leq 1$ by Lemma 4.2.

Now, consider a proper embedding $N \hookrightarrow \mathbb{R}^k$ for some $k$. Note that any proper map from a topological space to a locally compact, Hausdorff space is a closed map. Hence, we can think $N$ as a closed subset of $\mathbb{R}^k$. With induced metric on $N$ from $\mathbb{R}^k$ and with usual distance metric on $[0,1]$, consider the product metric on $N \times [0,1]$. So, a closed and bounded subset of $N \times [0,1] \subseteq_{\text{closed}} \mathbb{R}^{k+1}$ is compact by Heine–Borel theorem. Similarly, considering the proper embedding $M \hookrightarrow \mathbb{R}^n$ and giving induced metric on $M$ from $\mathbb{R}^n$, we can say any closed-bounded subset of $M$ is compact.

So, to prove $\mathcal{H}$ is proper, it is enough to show $\mathcal{H}^{-1}(C)$ is closed and bounded in $N \times [0,1]$ for any compact set $C$ is $M$. Let $C$ be a compact subset of $M$. Now, $\mathcal{H}^{-1}(C)$ is closed in $N \times [0,1]$ as $\mathcal{H}$ is continuous and $C$ is closed in $M$. Hence, we only need to show $\mathcal{H}^{-1}(C)$ is bounded subset of $N \times [0,1]$. Now, notice that $\{(p_n,t_n)\} \subseteq N \times [0,1]$ is bounded if and only if $\{p_n\} \subseteq N$ is bounded. Also, $F$ is proper implies $\{F(p_n)\}$ is bounded if and only if $\{p_n\}$ is bounded, for any sequence
unbounded sequence

Theorem 4.4. For a smooth proper map \( f: N \rightarrow M \) and an embedded submanifold \( X \) of \( M \) we have another smooth proper map \( g: N \rightarrow M \) properly homotopic to \( f \) and transverse to \( X \).

Proof. As usual \( M \rightarrow \mathbb{R}^n \) is a properly embedded smooth submanifold of dimension \( m \). Define \( \delta \) as in the proof of Theorem 4.3. Consider a smooth function \( e: N \rightarrow (0, \infty) \) with \( 0 < e(p) < \delta(f(p)) \) for all \( p \in N \). Let \( \mathbb{B}^n := \{ x \in \mathbb{R}^n : |x| < 1 \} \) and define \( F: N \times \mathbb{B}^n \rightarrow M \) as

\[
F(p, s) := r(f(p) + e(p)s) \quad \text{for} \quad (p, s) \in N \times \mathbb{B}^n.
\]

Notice that \( |e(p) \cdot s| < e(p) < \delta(f(p)) \), which implies that \( f(p) + e(p) \cdot s \in U \), so \( F \) is well-defined. Clearly, \( F \) is smooth, and \( F(-, 0) = f \) as \( r \) is a retraction. For each \( p \in N \), the restriction of \( F \) to \( \{p\} \times \mathbb{B}^n \) is the composition of the local diffeomorphism \( s \mapsto f(p) + e(p) \cdot s \) followed by the smooth submersion \( r \), so \( F \) is a smooth submersion and hence transverse to \( X \).

Now, by Parametric Transversality theorem [Lee12, Theorems 6.35], \( F(\cdot, s_0) \) is transverse to \( X \) for some \( s_0 \in \mathbb{R}^n \). So, our required homotopy \( \mathcal{H}: N \times [0, 1] \rightarrow M \) is given by

\[
\mathcal{H}(p, t) := r(f(p) + te(p) \cdot s_0) \quad \text{for} \quad (p, t) \in N \times [0, 1].
\]

For \( (p, t) \in N \times [0, 1] \) we have

\[
|\langle f(p) + te(p) \cdot s_0, f(p) \rangle - f(p) \cdot s_0| < \delta(f(p)) \leq 1
\]

\[
\implies |\mathcal{H}(p, t) - r \circ f(p)| = |\mathcal{H}(p, t) - f(p)| \leq 2 \quad \text{by Lemma 4.2}.
\]

Now, note that \( f \) is a proper map. So, a similar argument as given the last two paragraphs of the proof Theorem 4.3, says that \( \mathcal{H} \) is a proper map. \( \square \)

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