SHARP TUNNELING ESTIMATES FOR A DOUBLE-WELL MODEL IN INFINITE DIMENSION

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Abstract. We consider the stochastic quantization of a quartic double-well energy functional in the semiclassical regime and derive optimal asymptotics for the exponentially small splitting of the ground state energy. Our result provides an infinite-dimensional version of some sharp tunneling estimates known in finite dimensions for semiclassical Witten Laplacians in degree zero. From a stochastic point of view it proves that the $L^2$ spectral gap of the stochastic one-dimensional Allen-Cahn equation in finite volume satisfies a Kramers-type formula in the limit of vanishing noise. We work with finite-dimensional lattice approximations and establish semiclassical estimates which are uniform in the dimension. Our key estimate shows that the constant separating the two exponentially small eigenvalues from the rest of the spectrum can be taken independently of the dimension.

1. Introduction

The study of the semiclassical eigenvalue splitting due to tunneling effects in multiwell systems has a long history dating back to the beginnings of quantum mechanics. In the original setting one deals with the Schrödinger operator in finite dimensions

$$H_h = -h^2 \Delta + V,$$

and with the semiclassical approximation $h \to 0$, describing the transition from quantum to classical mechanics [27, 50]. If $V : \mathbb{R}^N \to \mathbb{R}$ is a confining symmetric double-well potential, the difference $E_h^{(1)} - E_h^{(0)}$ between the two smallest eigenvalues of $H_h$ turns out to be exponentially small in $h$, and the exponential decay rate of this eigenvalue splitting is determined by the so-called Agmon distance [44, 29, 27].

Besides the original motivation of the semiclassical approximation for quantum systems, the analysis of the eigenvalue splitting for Schrödinger operators has been proven to be fruitful in a large number of different situations. These include problems in statistical mechanics, following Kac’s early ideas [32] on eigenvalue degeneracy as ultimate characteristic of first order phase transitions; and in particular the problem of metastability, an example of a dynamical phase transition [25, 40, 14]. Other applications can be found in Differential Topology, more specifically regarding Morse Homology [13], following the pioneering paper of Witten [49].

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In both types of applications, a multiwell potential $V$ naturally appears. However, the relevant Schrödinger operator turns out to be not (1), but rather of the form

$$\tilde{H}_h = -\hbar^2 \Delta + W_h,$$

where $W_h := \frac{1}{4} |\nabla V|^2 - \frac{h^2}{2} \Delta V$. (2)

This operator is sometimes referred to as Witten Laplacian or, more precisely, Witten Laplacian in degree zero, since it is the restriction on the level of 0-forms (i.e functions) of the full-fledged Witten Laplacian acting on the exterior algebra of differential forms. The peculiar form of $\tilde{H}_h$ might be best understood by observing that, up to a factor $\hbar$, it is unitarily equivalent via ground state transformation to the diffusion operator

$$L_h = -\hbar \Delta + \nabla V \cdot \nabla.$$ (3)

The latter acts as selfadjoint operator on the weighted space $L^2(e^{-V/h} dx)$ and its associated quadratic form is given by

$$E_h[f] = h \int_{\mathbb{R}^N} |\nabla f|^2 e^{-V/h} dx.$$ (4)

Moreover $L_h$ is the $L^2$-generator of the stochastic Langevin dynamics

$$\dot{X} = -\nabla V(X) + \sqrt{2\hbar} \eta,$$ (5)

where $t \mapsto X(t)$ is a stochastic process in $\mathbb{R}^N$ and $\eta$ is an $N$-dimensional white noise in time. From this stochastic point of view the semiclassical asymptotic $\hbar \to 0$ turns out to be a small noise asymptotic for a reversible diffusion process. The procedure which, starting from $V$, constructs the essentially equivalent objects (2)-(5) and thus establishes a connection between the formalism of Quantum mechanics and diffusion processes is also called stochastic quantization [39, 5, 41, 31].

There exists a large literature concerning the semiclassical eigenvalue splitting for the stochastic quantization (2)-(5), see e.g. [25, 17, 30, 34, 35, 38] and references therein. Sharpest possible results have been obtained on the asymptotic behaviour of all the exponentially small eigenvalues in the case of rather general multiwell potentials in [28, 15, 22, 48, 37]. For example, in the case of a potential $V$ with nondegenerate critical points, exactly two quadratic minima and growing sufficiently at infinity, the two smallest eigenvalues $E_h^{(0)}, E_h^{(1)}$ of (2) satisfy $E_h^{(0)} = 0$ and, in the limit $\hbar \to 0$,

$$E_h^{(1)} = hA \exp(-B/\hbar) (1 + o(1)).$$ (6)

Here $A, B > 0$ are constants which can be computed explicitly from the potential $V$. Note that $\lambda_1(\hbar) := \frac{E_h^{(1)}}{\hbar}$ is nothing but the $L^2$-spectral gap of (5).

**Semiclassical tunneling in infinite dimensions.** This paper concerns the problem of obtaining sharp asymptotics of the type (6) in infinite-dimensional models. In the presence of spatially extended systems with infinite degrees of freedom, as occurring in statistical mechanics and quantum field theory, the underlying finite-dimensional manifold of states is replaced by a suitable infinite-dimensional topological space. A typical energy
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functional for a system described by a field $\xi : \Lambda \to \mathbb{R}$ then takes the form

$$V(\xi) = \int_\Lambda F(\xi(s)) \, ds + \frac{1}{2} \int_\Lambda |\nabla \xi(s)|^2 \, ds,$$

(7)

where $\Lambda$ is some region in $\mathbb{R}^d$, $J > 0$ is a constant and $F : \mathbb{R} \to \mathbb{R}$ is a local potential. Analogous functionals appear in topological applications when considering e.g. infinite-dimensional Riemannian manifolds of loops, as Witten already had in mind (see Section 4 of [49] and e.g. [2, 18]).

In most situations of interest the energy landscape determined by $V$ is rather complex and in particular $V$ might have several distinct local minima. In analogy to the finite-dimensional case one expects then that exponential eigenvalue splitting occurs for the corresponding stochastic quantization of $V$ in the semiclassical regime.

Our aim is to put forward a general strategy for extending (6) to infinite-dimensional situations. We illustrate this strategy by giving a complete proof of (6) for a special and relatively simple instance of (7), where $\Lambda$ is the one-dimensional torus and $F$ is a symmetric quartic double-well. More specifically we restrict to the case

$$F(\xi) = \frac{1}{4} \xi^4 - \frac{1}{2} \xi^2.$$

For simplicity we also assume that $J$ is large enough (specifically $J > \frac{4\pi}{2}$), so that $V$ admits exactly two minima, given by the constant states $\pm 1$. The resulting double-well functional $V$ is sometimes referred to as Ginzburg-Landau or Allen-Cahn energy functional. While infinite-dimensional versions of the Schrödinger Operator (2) are generally ill-defined, it is well-known that mathematically sound interpretations of (3)-(5) can be given in the case of the Ginzburg-Landau functional considered here, see e.g. [19]. In particular the Langevin dynamics (5) becomes now the semilinear stochastic partial differential equation

$$\partial_t u = J \partial_x^2 u - u^3 + u + \sqrt{2h} \eta,$$

where $t \mapsto u(t)$ is a stochastic process taking values in a space of functions on $\mathbb{T}$ and $\eta$ is a space-time white noise [23].

Our main result, Theorem 2.3 below, states that for this infinite-dimensional quartic model the asymptotic relation (6) holds true with $B = \frac{1}{4}$ (the height of the barrier separating the wells), an explicit prefactor $A$, expressed for notational convenience in terms of $\mu := \frac{\pi^2 J}{4}$, and with the $o(1)$ remainder term of order $O(h)$. We emphasize that the difficult part of this result concerns the lower bound. The upper bound follows indeed rather easily from a suitable choice of test functions already introduced in [21].

Previous results. Several studies have been devoted to questions of semiclassical analysis in large and infinite dimension. Early contributions are [45, 46, 20] and also [36] for semiclassical estimates with uniform bounds on the dimension, mainly restricted to one-well situations. Further we mention Aida’s extensive work on infinite-dimensional Schrödinger operators, see [4] for an overview, and in particular his paper [3] on semiclassical tunneling. The latter concerns an infinite-dimensional version of (1) with a
renormalized polynomial potential $V$ and shows one-sided estimates on the exponential decay rate in terms of a suitable Agmon estimate.

To our knowledge no rigorous results on spectral asymptotics comparable in precision with (6) have been established so far even in simple double-well situations. There exist at least three possible methods to show (6) in finite dimension: 1) the potential-theoretic approach which goes through the computation of hitting times for the stochastic dynamics (5) and exploits variational principles for capacities [14]; 2) the semiclassical approach à la Helffer-Klein-Nier based on WKB expansions and supersymmetry arguments [28]; 3) the approach in [37] exploiting variance decompositions and optimal transport techniques.

The first method has been used also in infinite-dimensional settings, both for generalizations of the quartic model over $\mathbb{T}$ treated here [9, 8, 12] and for the corresponding renormalized problem over the two-dimensional torus $\mathbb{T}^2$ [11]. These papers yield a result for the average time needed for the stochastic process to pass from one well to the other. A result of this type is commonly called Kramers’ Law and comparable in precision with (6). The deduction of sharp eigenvalue asymptotics from an infinite-dimensional Kramers Law is however missing.

A first attempt to generalize approach 2) to the infinite-dimensional model considered here was made in [21]. The authors consider finite-dimensional lattice approximations and show that (6) holds uniformly in the dimension if the two exponentially small eigenvalues are well separated from the rest of the spectrum [21, Theorem 1.2]. We shall refer to the latter property as a rough spectral separation. As explained in [21] the usual finite-dimensional estimates fail to produce a rough spectral separation that holds uniformly in the dimension. This is indeed the major issue when trying to lift tunneling calculations to infinite dimensions. A very similar problem arises with approach 3): here, a rough spectral separation for each basin of attraction is needed, and the method based on Lyapounov functionals employed in [37] does not produce uniform estimates.

Methods. The main technical contribution of this paper is to solve the above mentioned problem and to show that the rough spectral separation holds uniformly in the dimension (see Theorem 2.1 below).

One part of the proof consists in a suitable landscape decomposition and a careful choice of reference potentials for the localized problems. These lead via ground state transformations to infinite-dimensional Schrödinger operators replacing the ill-defined infinite-dimensional version of (2). A second part provides the estimates for the localized problems. The main ingredient here is the NGS bound [26, 42], which is well-known from quantum field theory and has already been used in [1] in a semiclassical context, see also [24]. It provides a quantitative operator bound in case of a singular Schrödinger potential. The NGS bound follows from (and is indeed equivalent to) the
regularizing effect of a logarithmic Sobolev inequality. To obtain the necessary hypercontractive properties we exploit the convexity properties of the reference potentials and the Bakry-Émery criterion.

All our computations are performed for finite-dimensional lattice approximations with uniform estimates in the approximation. The analogous computations could, however, also be done directly in infinite dimension. Our approach has the advantage to include also uniform estimates for the lattice approximation. The latter, we believe, has its own interest, both as a physical model and for numerical schemes. Moreover our approach substantially reduces the technical prerequisites for the proof: the infinite-dimensional objects enter only in the final limiting procedure.

Outline of the paper. In Section 2 we present our main results: the crucial estimate is stated in Theorem 2.1, which provides the rough spectral separation for the lattice approximation of the model. Theorems 2.2 and 2.3 provide the sharp spectral gap asymptotics respectively for the lattice approximation and the infinite-dimensional model. While the former is a direct consequence of Theorem 2.1, for the latter some additional approximation results are needed. These are straightforward and are discussed in the final Section 6 for completeness. The core of the paper is given by the Sections 3-5, where we prove Theorem 2.1. More precisely, in Section 3 we reduce the problem to localized problems around and off the diagonal. In Section 4 we recall some abstract auxiliary tools, in particular the NGS Bound, in the form needed for the subsequent analysis of the localized problems. In Section 5 we prove all the necessary local estimates which permit to conclude the proof of Theorem 2.1.

2. Results

As in [21] we fix $\mu > 1$ and consider for every dimension $N \in \mathbb{N}$ the function $V_N : \mathbb{R}^N \to [0, \infty)$ defined by

$$V_N(x) := \sum_{k=1}^{N} \left( \frac{x_k^2}{4} - 1 \right)^2 + \frac{\mu}{8 \sin^2(\pi N^2)} \sum_{k=1}^{N} (x_k - x_{k+1})^2,$$

where $x_{N+1} := x_1$. We shall refer to $V_N$ as the energy. It is straightforward to show (see e.g. [21, Lemma 2.1]) that, due to the assumption $\mu > 1$, for every $N \in \mathbb{N}$ the function $V_N$ admits exactly three critical points: the two global minimum points given by the constant states $I_+ = (1, \ldots, 1)$ and $I_- = (-1, \ldots, -1)$ and the critical point of index one given by the origin $O = (0, \ldots, 0)$. One might think of $\frac{1}{N} V_N$ as a lattice approximation of the double-well functional $V : H^1(\mathbb{T}) \to [0, \infty)$ defined by

$$V(\xi) := \int_{\mathbb{T}} \frac{1}{4} (\xi^2(s) - 1)^2 ds + \frac{\mu}{8 \pi^2} \int_{\mathbb{T}} |\xi'(s)|^2 ds,$$

where $\mathbb{T}$ is the one-dimensional torus $\mathbb{R}/\mathbb{Z}$.

Let $C_0^\infty(\mathbb{R}^N)$ be the space of smooth real functions on $\mathbb{R}^N$ which are bounded together with all their derivatives. For each $N \in \mathbb{N}$ and $h > 0$ we denote by
$\mathcal{E}_{h,N}: C_0^\infty(\mathbb{R}^N) \to \mathbb{R}$ the quadratic form defined by
\[
\mathcal{E}_{h,N}[f] := hN \int_{\mathbb{R}^N} |\nabla f(x)|^2 e^{-\frac{V_N(x)}{hN}} \, dx,
\]
define for each finite-dimensional linear subspace $S \subset C_0^\infty(\mathbb{R}^N)$
\[
\kappa_{h,N}^{(S)} := \sup \left\{ \mathcal{E}_{h,N}[f] : f \in S \text{ and } \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N(x)}{hN}} \, dx = 1 \right\},
\]
and finally consider for each $j \in \mathbb{N}_0$
\[
\lambda_{h,N}^{(j)} := \inf \left\{ \kappa_{h,N}^{(S)} : S \subset C_0^\infty(\mathbb{R}^N) \text{ and } \dim S = j + 1 \right\}. \tag{10}
\]
It follows from standard arguments that for each $h > 0, N \in \mathbb{N}$ the set $\{\lambda_{h,N}^{(j)}\}_{j \in \mathbb{N}_0}$ gives counting multiplicities both the spectrum of the closure in $L^2(e^{-V_N/hN} \, dx)$ of the diffusion-type differential operator
\[
f \mapsto -hN \Delta f + \nabla V_N \cdot \nabla f, \quad f \in C_0^\infty(\mathbb{R}^N),
\]
and the spectrum of the closure in $L^2(dx)$ of the Schrödinger-type differential operator
\[
f \mapsto -hN \Delta f + \frac{1}{2} |\nabla V_N|^2 - \frac{1}{2} \Delta V_N \, f, \quad f \in C_c^\infty(\mathbb{R}^N), \tag{11}
\]
where $C_c^\infty(\mathbb{R}^N)$ is the space of smooth compactly supported real functions on $\mathbb{R}^N$. The differential operator (11) is also known as (the restriction on 0-forms of) the Witten Laplacian corresponding to the energy $V_N$ [49].

Note that $\lambda_{h,N}^{(0)} = 0$ and $\lambda_{h,N}^{(1)} > 0$ for each $h > 0$ and $N \in \mathbb{N}$. Moreover, considering a suitable test function for the upper bound and rough perturbation arguments for the lower bound, one can show that $\lambda_{h,N}^{(1)}$ is exponentially small in the regime $h \ll 1$, uniformly in the dimension $N$. More precisely [21] there exist $C, C' > 0$ such that for all $h \in (0, 1]$ we have
\[
e^{-\frac{C}{h}} \leq \lambda_{h,N}^{(1)} \leq e^{-\frac{C'}{h}}.
\]
The main technical result of the present paper is to prove that $\lambda_{h,N}^{(2)}$ is bounded from below, uniformly both in $h$ and $N$.

**Theorem 2.1.** There exist constants $C, h_0 > 0$ such that for every $h \in (0, h_0]$ and every $N \in \mathbb{N}$ we have
\[
\lambda_{h,N}^{(2)} \geq C.
\]
Thus, in the semiclassical regime $h \to 0$, the spectrum separates sharply into a “low-lying spectrum” consisting of the two exponentially close eigenvalues $\lambda_{h,N}^{(0)}, \lambda_{h,N}^{(1)}$ and the rest of the spectrum, which is uniformly bounded from below by a strictly positive constant. A statement of this type is well known to hold for a general class of energies $V$ when the dimension $N$ is fixed. In general one finds indeed a cluster $m_0$ of exponentially small eigenvalues, where $m_0$ is the number of local minima of $V$; then there is a large gap, with the rest of the spectrum being bounded away from zero, uniformly in $h$. However the usual arguments, based on suitable Harmonic approximations of the Schrödinger operator (11) [43, 29, 16, 33], do not permit to get bounds...
uniform in $N$ when applied to the sequence of quartic energies $V_N$ defined in (8).

As a corollary of Theorem 2.1 we are able to compute the precise asymptotic behaviour in the limit $h \to 0$ of the spectral gap $\lambda_{h,N}^{(1)}$, with uniform control in the dimension $N$: The exponential decay rate of $\lambda_{h,N}^{(1)}$ equals the height of the barrier $\frac{1}{N} V_N(0) - \frac{1}{N} V_N(I_+) = \frac{1}{4}$ separating the two wells, with an explicit $h$-independent pre-exponential factor given by

$$p(N) := \frac{1}{\pi} \frac{\det \text{Hess} V_N(I_+)}{\det \text{Hess} V_N(0)^{\frac{1}{2}}}.$$  \hspace{1cm} (12)

More precisely we obtain as immediate application of Theorem 2.1 and [21, Theorem 1.2] the following uniform Kramers Law, which, together with its infinite-dimensional version stated below, is the main result of our paper.

**Theorem 2.2.** Let $p(N)$ be given by (12). Then there exist $h_0, C > 0$ such that the error term $(h,N) \mapsto \epsilon(h,N)$ defined for $h > 0$ and $N \in \mathbb{N}$ by

$$\lambda_{h,N}^{(1)} = p(N) e^{-\frac{4}{\pi} \left(1 + \epsilon(h,N)\right)},$$

satisfies for all $h \in (0,h_0]$ and $N \in \mathbb{N}$ the bound $|\epsilon(h,N)| \leq Ch$.

As already noted in [47] for $N \to \infty$ the prefactor converges:

$$p(N) \to \frac{\sinh(\sqrt{2}h^{-1})}{\sqrt{2}h^{-1}}.$$  \hspace{1cm} (13)

Since our bounds are uniform in $N$ we can thus pass to the limit $N \to \infty$ and get the corresponding infinite-dimensional version of Theorem 2.2. To formulate the latter, we shall introduce the following notation. We fix a mass $m > 0$ and consider the trace class operator $G : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined as the inverse of the selfadjoint operator on $L^2(\mathbb{T})$ given by $H^2(\mathbb{T}) \ni x \mapsto mx - \frac{1}{(2\pi)^2} x''$. We denote by $\gamma_h$ the centered Gaussian measure on $L^2(\mathbb{T})$ with covariance operator $hG$ and define $U : L^2(\mathbb{T}) \to \mathbb{R} \cup \{+\infty\}$ by

$$U(\xi) := \int_\pi \left(\frac{1}{4} \xi^4(s) - \frac{1}{2} (m+1) \xi^2(s)\right) ds + \frac{1}{4}.$$  \hspace{1cm} (14)

Finally we define for every $h > 0$

$$\lambda_h^{(1)} := \inf_F \left\{ h \int_{L^2(\mathbb{T})} |DF|^2_{L^2(\mathbb{T})} e^{-\frac{1}{2} U} d\gamma_h \right\},$$

where the infimum is taken over all $F \in FC_0^\infty(L^2(\mathbb{T}))$ satisfying the constraints $\int_{L^2(\mathbb{T})} |F|^2 e^{-\frac{1}{2U}} d\gamma_h = 1$ and $\int_{L^2(\mathbb{T})} F e^{-\frac{1}{2} U} d\gamma_h = 0$ and where $DF$ is the gradient of $F$ . Here $F \in FC_0^\infty(L^2(\mathbb{T}))$ means that $F$ is a cylindrical test function on $L^2(\mathbb{T})$, i.e. there exist $n \in \mathbb{N}$, $y_1, \ldots, y_n \in L^2(\mathbb{T})$ and $f \in C_0^\infty(\mathbb{R}^n)$ such that $F(x) = f((x,y_1), \ldots, (x,y_n))$ for every $x \in L^2(\mathbb{T})$.

**Theorem 2.3.** There exist $h_0, C > 0$ such that the error term $h \mapsto \epsilon(h)$ defined for $h > 0$ by

$$\lambda_h^{(1)} = \frac{\sinh(\sqrt{2}h^{-1})}{\sqrt{2}h^{-1}} e^{-\frac{4}{\pi} \left(1 + \epsilon(h)\right)},$$

satisfies for all $h \in (0,h_0]$ the bound $|\epsilon(h)| \leq Ch$.  


Analogously to the finite-dimensional case, we have now that $\lambda^{(1)}_h$ equals the smallest non-zero eigenvalue of the closure in $L^2(e^{-U/h} \gamma_h)$ of the infinite-dimensional diffusion-type differential operator

$$F \mapsto -L_h F + (DU, DF)_{L^2(\mathbb{T})}, \quad F \in \mathcal{F}C_c^\infty(L^2(\mathbb{T})), \quad (15)$$

where $L_h$ denotes the Ornstein-Uhlenbeck operator which has $\gamma_h$ as invariant measure. Note that, after suitable unitary transformation, one might think of (15) also as a rigorous version of the infinite-dimensional Schrödinger-type differential operator in $L^2(\gamma_h)$ formally given by

$$F \mapsto -L_h F + \left(\frac{1}{\gamma_h}||DU||^2_{L^2(\mathbb{T})} - \frac{1}{2\gamma_h}L_h U\right)F. \quad (16)$$

We remark explicitly that (15) is the $L^2$-generator of the following nonlinear stochastic heat equation, which is known under various names as e.g. stochastic Allen-Cahn, or Chafee-Infante equation:

$$\partial_t u = \frac{u^3}{4\pi} \partial^2_{xx} u - u^4 + u + \sqrt{2h} \xi,$$

On the other hand the operator (16) might be seen as an infinite-dimensional Witten Laplacian (restricted to 0-forms).

3. **Reduction to local problems around and off the diagonal**

We denote by $\overline{x} := \frac{1}{N} \sum_{k=1}^N x_k$ the average of $x \in \mathbb{R}^N$. The first step of the proof of Theorem 2.1 consists in decomposing the problem into two pieces: one localized in a small neighborhood of the space of constant states, i.e. satisfying $x - \overline{x} = 0$, and one in the complementary set, which will turn out to be negligible for the low-lying spectrum in the $h \to 0$ limit. Since we want to analyze a quadratic form, the decomposition is most conveniently realized via a smooth quadratic partition of unity:

We fix a $\chi \in C_c^\infty(\mathbb{R}; [0, 1])$ such that $\chi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\chi \equiv 0$ in $\mathbb{R} \setminus [-1, 1]$. Further, for each $N \in \mathbb{N}$ and $R > 0$ we define $\theta_{N, R}, \tilde{\theta}_{N, R} : \mathbb{R}^N \to [0, 1]$ by setting

$$\theta_{N, R}(x) := \chi \left(\frac{1}{N \overline{x}^2} \sum_k (x_k - \overline{x})^2 \right), \quad \tilde{\theta}_{N, R}(x) := (1 - \theta^2_{N, R}(x))^\frac{1}{2}, \quad (17)$$

so that $\theta^2_{N, R} + \tilde{\theta}^2_{N, R} \equiv 1$. Note that $\theta_{N, R}, \tilde{\theta}_{N, R} \in C_0^\infty(\mathbb{R}^N)$ for all $N \in \mathbb{N}$, $R > 0$.

From a straightforward computation of commutators one gets for all $N \in \mathbb{N}$, $h, R > 0$ and all $f \in C_b^\infty(\mathbb{R}^N)$ the identity (in general also known as IMS localization formula, see [16, Theorem 3.2])

$$\mathcal{E}_{h, N}[f] = \mathcal{E}_{h, N}[\theta_{N, R} f] + \mathcal{E}_{h, N}[\tilde{\theta}_{N, R} f] + h \mathcal{F}_{h, N, R}[f], \quad (18)$$

where the localization error $\mathcal{F}_{h, N, R}[f]$ is given by

$$\mathcal{F}_{h, N, R}[f] := -N \int_{\mathbb{R}^N} \left( |\nabla \theta_{N, R}|^2 + |\nabla \tilde{\theta}_{N, R}|^2 \right) f^2 e^{-\frac{\gamma_h}{\gamma_{\infty}}} dx.$$  

Since for each $R > 0$ there exists a constant $c(R)$ such that for every $N \in \mathbb{N}$

$$N \left( |\nabla \theta_{N, R}|^2 + |\nabla \tilde{\theta}_{N, R}|^2 \right) \leq c(R), \quad (19)$$
one obtains immediately for all \( N \in \mathbb{N}, h, R > 0 \) and \( f \in C_b^\infty(\mathbb{R}^N) \)
\[
|\mathcal{F}_{h,N,R}[f]| \leq c(R) \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} dx.
\] (20)
As we show below at the end of this section, Theorem 2.1 will then be an easy consequence of the following two propositions. The first one concerns the term \( \mathcal{E}_{h,N}[\hat{\theta}_{N,R} f] \) on the right hand side of (18). Indeed it implies that away from the diagonal the quadratic form \( \mathcal{E}_{h,N} \) is large in \( h \) and therefore does not contribute to the low-lying spectrum.

**Proposition 3.1.** For every \( R > 0 \) there exist constants \( C = C(R) > 0 \) and \( h_0 = h_0(R) > 0 \) such that for all \( h \in (0, h_0], N \in \mathbb{N} \) and \( f \in C_b^\infty(\mathbb{R}^N) \) with \( \text{supp} \ f \subset \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_k (x_k - \overline{x})^2 \leq R^2 \right\} \) we have
\[
\mathcal{E}_{h,N}[f] \geq h^{-1} C \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} dx.
\]
The second proposition concerns the term \( \mathcal{E}_{h,N}[\theta_{N,R} f] \) on the right hand side of (18). It states that, when restricting to a sufficiently small neighbourhood of the diagonal, \( \mathcal{E}_{h,N} \) is larger than a constant except on a linear subspace of dimension at most 2.

**Proposition 3.2.** There exist constants \( R_0, h_0, C > 0 \) and, for every \( h > 0, N \in \mathbb{N}, \) there exist functions \( \phi_{h,N}^+, \phi_{h,N}^- \in C_b^\infty(\mathbb{R}^N) \) such that for all \( h \in (0, h_0], N \in \mathbb{N} \) and \( f \in C_b^\infty(\mathbb{R}^N) \) with \( \text{supp} \ f \subset \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_k (x_k - \overline{x})^2 \leq R_0^2 \right\} \) we have
\[
\mathcal{E}_{h,N}[f] \geq C \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} dx - \left( \int_{\mathbb{R}^N} f \phi_{h,N}^+ e^{-\frac{V_N}{hN}} dx \right)^2 - \left( \int_{\mathbb{R}^N} f \phi_{h,N}^- e^{-\frac{V_N}{hN}} dx \right)^2.
\] (21)
The proofs of Proposition 3.1 and Proposition 3.2 will be given respectively in Section 5.1 and Section 5.2.

**Proof of Theorem 2.1.** Thanks to Proposition 3.2 we can fix \( R_0, h'_0, C' > 0 \) and, for every \( h > 0, N \in \mathbb{N}, \) functions \( \phi_{h,N}^+, \phi_{h,N}^- \in C_b^\infty(\mathbb{R}^N) \) such that (21) holds true for all \( h \in (0, h'_0], N \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^N) \) with \( \text{supp} \ f \subset \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_k (x_k - \overline{x})^2 \leq R_0^2 \right\} \) and with \( C' \) instead of \( C \). In particular, denoting by \( S_N \) a generic 3-dimensional linear subspace of \( C_b^\infty(\mathbb{R}^N) \) and picking for every \( N \in \mathbb{N}, h > 0 \) a function \( f_{h,N} \in S_N \) which in \( L^2(e^{-V/hN} dx) \) has norm one and is orthogonal to both \( \theta_{N,R_0} \phi_{h,N}^+ \) and \( \theta_{N,R_0} \phi_{h,N}^- \), one obtains
\[
\mathcal{E}_{h,N}[\theta_{N,R_0} f_{h,N}^*] \geq C' \int_{\mathbb{R}^N} |\theta_{N,R_0} f_{h,N}^*|^2 e^{-\frac{V_N}{hN}} dx \quad \forall h \in (0, h'_0], N \in \mathbb{N}.
\] (22)
Moreover it follows from Proposition (3.1) that there exist \( C'', h''_0 > 0 \) such that for all \( h \in (0, h''_0], N \in \mathbb{N} \) it holds
\[
\mathcal{E}_{h,N}[\hat{\theta}_{N,R_0} f_{h,N}^*] \geq h^{-1} C'' \int_{\mathbb{R}^N} |\hat{\theta}_{N,R_0} f_{h,N}^*|^2 e^{-\frac{V_N}{hN}} dx.
\] (23)
Recalling the definition (9) and putting together the decomposition (18), the estimates (22), (23) and the bound (20) on the localization error one gets for every $N \in \mathbb{N}$ and $h \in (0, \min\{1, h'_0, h''_0\})$ the lower bound

$$
\kappa^{(S_N)}_{h,N} \geq C' \int_{\mathbb{R}^N} |\tilde{\theta}_{N,R_0} f_{h,N}^*| e^{-\frac{V_N}{hN}} dx + C'' \int_{\mathbb{R}^N} |\tilde{\theta}_{N,R_0} f_{h,N}^*| e^{-\frac{V_N}{hN}} dx - hc(R_0) \geq \min\{C', C''\} - hc(R_0).
$$

Taking $C := \frac{1}{2} \min\{C', C''\}$, $h_0 := \min\{1, h'_0, h''_0, \frac{c(R_0)}{C}\}$ and recalling Definition (10) finishes the proof. \hfill \Box

4. Auxiliary Tools

In this section we review briefly some auxiliary tools which will be used in the remainder of the paper. The key ingredient in the proofs of Proposition 3.1 and Proposition 3.2 is the so-called NGS Bound [26, 42], which we recall below for the sake of the reader. We shall use the following standard conventions. We say that a probability measure $m$ on $\mathbb{R}^d$ satisfies a logarithmic Sobolev inequality with constant $\rho > 0$ if for all $f \in C^\infty_b(\mathbb{R}^d)$ the inequality

$$
\int_{\mathbb{R}^d} |\nabla f|^2 dm \geq \frac{\rho}{2} \text{Ent}_m[f^2]
$$

holds true, where we have denoted by

$$
\text{Ent}_m[f^2] := \int_{\mathbb{R}^d} f^2 \log f^2 dm - \int_{\mathbb{R}^d} f^2 dm \log \int_{\mathbb{R}^d} f^2 dm
$$

the entropy of $f^2$ with respect to $m$.

**Proposition 4.1** (NGS Bound). Let $m$ be a probability measure on $\mathbb{R}^d$ and let $M : \mathbb{R}^d \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$ be respectively a continuous function and an open set. If there exist constants $\rho, \Lambda > 0$ such that $m$ satisfies a logarithmic Sobolev inequality with constant $\rho$ and such that $\int_{\Omega} e^{-\frac{2M}{\rho}} dm \in (0, \Lambda]$, then

$$
\int_{\mathbb{R}^d} |\nabla f|^2 dm + \int_{\mathbb{R}^d} M |f|^2 dm \geq C \int_{\mathbb{R}^d} |f|^2 dm, \quad \forall f \in C^\infty_b(\mathbb{R}^d) : \text{supp } f \subset \Omega,
$$

where the constant $C \in \mathbb{R}$ is given by

$$
C := -\frac{\rho}{\Lambda} \log \Lambda.
$$

Note that in case that $M$ is bounded from below by a constant $M_0$ one recovers the trivial bound $C = M_0$. But crucially, for the finiteness of the integral $\int_{\Omega} e^{-\frac{2M}{\rho}} dm$, the boundedness of $M$ from below is not needed.

**Remark 4.2.** The NGS Bound is usually stated with $\Omega = \mathbb{R}^d$. The slightly more general variant presented in Proposition 4.1 can be easily proven by repeating the original proof given in [26] and inserting suitable indicator functions of $\Omega$.

The NGS Bound will be used in combination with the following two facts. The first one follows from a simple computation which permits to transform by a unitary transformation the quadratic form $\mathcal{E}_{h,N}$ into an equivalent quadratic form where a 0-order term appears as in the left hand side of (25).
This transformation is also called ground state transformation. More precisely the following holds.

**Lemma 4.3 (Ground State Transformation).** Let \( U, W \in C^2(\mathbb{R}^d) \). Then for every \( f \in C_c^\infty(\mathbb{R}^d) \), defining \( g := \exp(-U - W) \), one has

\[
\int_{\mathbb{R}^d} |\nabla f|^2 e^{-U} \, dx = \int_{\mathbb{R}^d} |\nabla g|^2 e^{-W} \, dx + \int_{\mathbb{R}^d} M g^2 e^{-W} \, dx,
\]

where \( M : \mathbb{R}^d \to \mathbb{R} \) is given by

\[
M := \frac{1}{4} \left( |\nabla U|^2 - |\nabla W|^2 \right) - \frac{1}{2} \Delta (U - W).
\]

The second important ingredient in order to exploit the NGS-Bound is the well-known Bakry-Émery criterion [7]. The latter permits to give a quantitative bound on the logarithmic Sobolev constant in case of a uniformly convex potential (see also [10] for an extension to singular nonconvex potentials). For the sake of the reader we recall the precise statement of the Bakry-Émery criterion, which we use in the present paper.

**Proposition 4.4 (Bakry-Émery criterion).** Let \( U \in C^2(\mathbb{R}^d) \) and assume that there exists a \( C > 0 \) such that \( \text{Hess} U(x) \geq C \) for all \( x \in \mathbb{R}^d \). Then the probability measure \( m(dx) \) on \( \mathbb{R}^d \) proportional to \( e^{-U} \, dx \) satisfies a logarithmic Sobolev inequality with constant \( C \).

We shall use also the following fact, which might be deduced as a simple corollary of the Bakry-Émery criterion: if \( U \in C^2(\mathbb{R}^d) \) and if there exists a \( C > 0 \) such that \( \text{Hess} U(x) \geq C \) for all \( x \in \mathbb{R}^d \), then the probability measure \( m(dx) \) on \( \mathbb{R}^d \) proportional to \( e^{-U} \, dx \) satisfies the following Poincaré inequality:

\[
\int_{\mathbb{R}^d} |\nabla f|^2 \, dm \geq C \left( \int_{\mathbb{R}^d} f^2 \, dm - \left( \int_{\mathbb{R}^d} f \, dm \right)^2 \right), \quad \forall f \in C^\infty_b(\mathbb{R}^d). \tag{26}
\]

5. Proofs of the local estimates

This section is devoted to the proofs of Proposition 3.1 and Proposition 3.2. We first introduce some notation and discuss basic properties of the interaction part in the energy \( V_N \).

Following the notation of [21] we denote by \( K = K_N : \mathbb{R}^N \to \mathbb{R}^N \) the normalised discrete Laplacian defined by setting for \( x \in \mathbb{R}^N \) and \( k \in \{1, \ldots, N\} \)

\[
(Kx)_k := \frac{\mu}{4 \sin^2(\frac{\pi}{N})} (2x_k - x_{k+1} - x_{k-1}),
\]

with the conventions \( x_{N+1} := x_1 \) and \( x_0 := x_N \). The interaction term in the energy \( V_N \) can then be written more compactly in terms of \( K \), since

\[
\frac{\mu}{8 \sin^2(\frac{\pi}{N})} \sum_{k=1}^N (x_k - x_{k+1})^2 = \frac{1}{2} \langle x, Kx \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard scalar product in \( \mathbb{R}^N \). The operator \( K \) is diagonalised through the discrete Fourier transform \( \hat{x} \in \mathbb{R}^N \) of \( x \in \mathbb{R}^N \), defined

\[
\hat{x} = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{2\pi i (k-1)(x_k)}.
\]
by
\[ \hat{x}_k := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} x_j e^{-i2\pi \frac{j}{N} k}. \]

More precisely we have for every \( k \in \{0, \ldots, N-1\} \),
\[ (Kx)_k = \nu_k \hat{x}_k, \quad \text{where} \quad \nu_k = \nu_{k,N} := \frac{\mu \sin^2 \left( \frac{k\pi}{N} \right)}{\sin^2 \left( \frac{\pi}{N} \right)}. \] (27)

Note that \( \nu_k \sim k^2 \) for large \( N \). As a consequence, due to the convergence of the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \), the resolvent of \( K \) is uniformly trace class in \( N \). More precisely we shall use the following fact, whose proof is elementary.

**Lemma 5.1.** For every \( \alpha > -\mu \) there exists a \( \gamma(\alpha) > 0 \) such that for every \( N \in \mathbb{N} \) we have \( \sum_{k=1}^{N-1} \frac{1}{\nu_k + \alpha} \leq \gamma(\alpha) \).

Observe also that \( \nu_0 = 0 \) is a simple eigenvalue of \( K \) corresponding to the eigenspace of constant states and that its smallest non-zero eigenvalue equals \( \mu \) for every \( N \in \mathbb{N}, N \geq 2 \). This implies immediately the following Poincaré inequality, where we recall that \( x = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \hat{x}_k \):
\[ \langle x, Kx \rangle \geq \mu \sum_{k} (x_k - \bar{x})^2, \quad \forall x \in \mathbb{R}^N. \] (28)

We note also the following Sobolev-type inequality, which, at the cost of lowering the constant \( \mu \), allows to substitute on the right hand side of (28) the Euclidean norm with a supremum norm. This will be useful later in analysing the regions of convexity of \( V_N \).

**Lemma 5.2.** Let \( \gamma(0) > 0 \) as in Lemma 5.1. Then
\[ \langle x, Kx \rangle \geq \frac{1}{\gamma(0)} N \sup_k |x_k - \bar{x}|^2, \quad \forall x \in \mathbb{R}^N. \]

**Proof.** Let \( x \in \mathbb{R}^N \) and assume that \( \bar{x} = 0 \). Then
\[
\sup_k |x_k| \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |\hat{x}_k| = \frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} \sqrt{\nu_k} |\hat{x}_k| \sqrt{\nu_k} \leq \frac{1}{\sqrt{N}} \left( \sum_{k=1}^{N-1} \nu_k |\hat{x}_k|^2 \right)^{\frac{1}{2}} \sqrt{\gamma(0)} = \frac{1}{\sqrt{N}} \langle x, Kx \rangle \sqrt{\gamma(0)},
\]
which finishes the proof in this case. If \( \bar{x} \neq 0 \) one can apply the argument to \( x - \bar{x} \). \( \Box \)

### 5.1. Away from the diagonal: Proof of Proposition 3.1.

In order to prove Proposition 3.1 we consider the decomposition \( \mathbb{R}^N = \mathcal{C} \oplus \mathcal{C}^\perp \), where \( \mathcal{C} := \{ x \in \mathbb{R}^N : x - \bar{x} = 0 \} \) (the space of constant states) and \( \mathcal{C}^\perp := \{ x \in \mathbb{R}^N : \bar{x} = 0 \} \) (the space of mean zero states). On \( \mathcal{C} \) we consider the coordinate \( \xi \) with respect to the (non-normalized) basis vector \((1, \ldots, 1)\). On \( \mathcal{C}^\perp \) we fix an arbitrary orthonormal basis, denote by \( y = (y_1, \ldots, y_{N-1}) \) the corresponding coordinates and by \( A \) the \( N \times N - 1 \) matrix so that each \( x \in \mathbb{R}^N \) is uniquely determined by its coordinates \((\xi, y) \in (\mathbb{R}, \mathbb{R}^{N-1})\) via
\[ x = \xi (1, \ldots, 1) + Ay. \]
Note that in the \((\xi, y)\) coordinates we have
\[
\tilde{V}_N(\xi, y) := V_N(\xi(1, \ldots, 1) + Ay) = F_N^{(\xi)}(y) + \frac{N}{4}(\xi^2 - 1)^2,
\]
where for each \(\xi \in \mathbb{R}\) and \(N \in \mathbb{N}\) we have defined the function \(F_N^{(\xi)} : \mathbb{R}^{N-1} \to \mathbb{R}\) by
\[
F_N^{(\xi)}(y) := \frac{1}{4} P_4(y) + \xi P_3(y) + \frac{3}{2} \xi^2 P_2(y) + \frac{1}{2} Q(y),
\]
with
\[
Q(y) := \frac{\mu}{4 \sin^2\left(\frac{\pi}{N}\right)} \sum_{k=1}^{N} ((Ay)_k - (Ay)_{k+1})^2 - P_2(y),
\]
and with \(P_m(y) := \sum_{k=1}^{N} (Ay)_k^m\) for \(m = 2, 3, 4\). Note that \(P_2(y) = \sum_{k=1}^{N} \nu_k^2\) for every \(y \in \mathbb{R}^{N-1}\), since we have chosen the coordinates \(y\) to be orthonormal.

**Proposition 5.3.** Fix \(R > 0\). Then for every \(\xi \in \mathbb{R}\), \(N \in \mathbb{N}\) and for every \(\phi \in C_c^\infty(\mathbb{R}^{N-1})\) such that \(\text{supp } \phi \subset \{y \in \mathbb{R}^{N-1} : P_2(y) \geq NR^2\}\) we have
\[
hN \int_{\mathbb{R}^{N-1}} |\nabla \phi(y)|^2 \exp\left(-\frac{F_N^{(\xi)}(y)}{hN}\right) dy \geq C(h, N, \xi) \int_{\mathbb{R}^{N-1}} |\phi(y)|^2 \exp\left(-\frac{F_N^{(\xi)}(y)}{hN}\right) dy,
\]
where
\[
C(h, N, \xi) := \frac{\mu - 1}{2} \left(\log \int_{\mathbb{R}^{N-1}} \exp\left(-\frac{F_N^{(\xi)}(y)}{hN}\right) dy - \log \int_{\{P_2 \geq NR^2\}} \exp\left(-\frac{F_N^{(\xi)}(y)}{hN}\right) dy\right).
\]
Proof. It follows from the Poincaré inequality (28) that the Hessian of \(y \mapsto F_N^{(\xi)}(y)\) is strictly positive, uniformly in \(y\) and \(\xi\). More precisely,
\[
\text{Hess } F_N^{(\xi)}(y) \geq \mu - 1 \quad \forall \xi \in \mathbb{R}, y \in \mathbb{R}^{N-1}.
\]
It follows then from the Bakry-Émery criterion (Proposition 4.4) that for each \(\xi \in \mathbb{R}\) the probability measure on \(\mathbb{R}^{N-1}\) with density proportional to \(\exp\left(-\frac{F_N^{(\xi)}(y)}{hN}\right)\) satisfies a logarithmic Sobolev inequality with constant \(\frac{\mu - 1}{hN}\). Thus the claim follows by applying for each \(\xi \in \mathbb{R}\) to this probability measure the NGS Bound (Proposition 4.1) with \(M = 0\) and \(\Omega = \{P_2 \geq NR^2\}\). \(\square\)

In order to estimate the constant \(C(h, N, \xi)\) defined in Prop. 5.3 we shall use the following two simple lemmata, which give uniform estimates on suitable Gaussian integrals. We define for \(t > - (\mu - 1), h > 0\) and \(N \in \mathbb{N}\)
\[
Z_{h,N}(t) := \int_{\mathbb{R}^{N-1}} e^{-\frac{tP_2(y) + Q(y)}{2hN}} dy = h^{\frac{N-1}{2}} Z_{1,N}(t),
\]
and write for short \(Z_N(t) := Z_{1,N}(t)\).

**Lemma 5.4.** Fix \(t_0 > - (\mu - 1)\). Then there exists a constant \(C = C(t_0) > 0\) such that for all \(t \geq t_0\) and for all \(N \in \mathbb{N}\) it holds
\[
\frac{1}{Z_N(t)} \int_{\mathbb{R}^{N-1}} \frac{1}{N} P_4(y) e^{-\frac{(P_2(y) + Q(y))}{2N}} dy \leq C.
\]
Proof. Fix \(t_0 > - (\mu - 1)\) and let \(\nu_0, \ldots, \nu_{N-1}\) be the eigenvalues of \(K\) given by (27). We denote for short for each \(t \geq t_0\) by \(g_t(dy)\) the Gaussian measure
on \( \mathbb{R}^{N-1} \) given by \( \frac{1}{Z_N(t)} e^{-\frac{tP_2(y)+Q(y)}{2N}} dy \). Then for each \( t \geq t_0 \) and \( N \in \mathbb{N} \) an explicit computation of Gaussian 4th order moments gives

\[
3 \left( 1 + \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t} \right)^2 = \frac{1}{Z_N(t) \sqrt{2\pi N}} \int_{\mathbb{R}^N} \frac{1}{N} \sum_k x_k^4 e^{-\frac{((K-1+t)x, x)}{2N}} \frac{2-(\nu^2)}{d} dx =
\]

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^{N-1}} \frac{1}{N} \sum_k (\xi + y_k)^4 g_k(dy) \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} d\xi =
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi^4 e^{-\frac{\xi^2}{2}} d\xi + \frac{6}{\sqrt{2\pi N(t)}} \int_{\mathbb{R}^{N-1}} \frac{1}{N} P_2(y) g_t(dy) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi^2 e^{-\frac{\xi^2}{2}} d\xi + \int_{\mathbb{R}^{N-1}} \frac{1}{N} P_4(y) g_t(dy) =
\]

\[
3 + 6 \int_{\mathbb{R}^{N-1}} \frac{1}{N} P_2(y) g_t(dy) + \int_{\mathbb{R}^{N-1}} \frac{1}{N} P_4(y) g_t(dy).
\]

Since

\[
\int_{\mathbb{R}^{N-1}} \frac{1}{N} P_2(y) g_t(dy) = \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t},
\]

we obtain

\[
\int_{\mathbb{R}^{N-1}} \frac{1}{N} P_4(y) g_t(dy) = 3 \left( 1 + \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t} \right)^2 - 3 - 6 \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t} \leq
\]

\[
3 \left( 1 + \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t_0} \right)^2,
\]

which finishes the proof by Lemma 5.1.

\[\square\]

**Lemma 5.5.** For every \( t_0 > -(\mu - 1) \) there exists a \( \gamma(t_0) > 0 \) such that for all \( N \in \mathbb{N} \) and for all \( t > -(\mu - 1) \) it holds

\[
\frac{Z_N(t)}{Z_N(t_0)} \geq e^{\frac{\gamma(t_0)(t-t_0)}{2}}.
\]

**Proof.** Fix \( t_0 > -(\mu - 1) \) and let \( \nu_0, \ldots, \nu_{N-1} \) be the eigenvalues of \( K \) given by (27). Then, using \( \log(1 + x) \leq x \) for \( x > -1 \), one gets for all \( N \in \mathbb{N} \) and for all \( t > -(\mu - 1) \)

\[
2 \log \left( \frac{Z_N(t)}{Z_N(t_0)} \right) = -\sum_{k=1}^{N-1} \log \left( \frac{\nu_k - 1 + t}{\nu_k - 1 + t_0} \right) =
\]

\[
-\sum_{k=1}^{N-1} \log \left( 1 + \frac{t - t_0}{\nu_k - 1 + t_0} \right) \geq -(t - t_0) \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t_0} \geq -|t - t_0| \gamma(t_0),
\]

where \( \gamma(t_0) \) satisfies \( \sum_{k=1}^{N-1} \frac{1}{\nu_k - 1 + t_0} \leq \gamma(t_0) \) for all \( N \in \mathbb{N} \) (see Lemma 5.1).

\[\square\]

The next proposition provides an estimate on the constant \( C(h, N, \xi) \) defined in Prop. 5.3.
Proposition 5.6. Fix $R > 0$. Then there exist constants $C, h_0 > 0$ such that for every $h \in (0, h_0]$, $N \in \mathbb{N}$ and $\xi \in \mathbb{R}$ it holds
\[
\log \int_{\mathbb{R}^{N-1}} e^{-\frac{F_N^{(\xi)}(y)}{hN}} dy - \log \int_{\{h \geq NR^2\}} e^{-\frac{F_N^{(\xi)}(y)}{hN}} dy \geq \frac{C}{h}.
\]

Proof. Fix $R > 0$. We first show that there exists a constant $C' > 0$ such that for all $h > 0, N \in \mathbb{N}$ and $\xi \in \mathbb{R}$ it holds
\[
\int_{\mathbb{R}^{N-1}} e^{-\frac{F_N^{(\xi)}(y)}{hN}} dy \geq (1 - hC') Z_{h,N}(3\xi^2). \tag{29}
\]
In order to do so we write for short
\[
\tilde{Z}_{h,N}(3\xi^2) := \int_{\mathbb{R}^{N-1}} e^{-\frac{F_N^{(\xi)}(y) - \xi P_h(y)}{hN}} dy,
\]
and $\mathbb{P}_{h,N}^{(\xi)}$ for the probability measure on $\mathbb{R}^{N-1}$ with density $e^{-\frac{F_N^{(\xi)}(y) - \xi P_h(y)}{hN}}/\tilde{Z}_{h,N}(3\xi^2)$. It follows then from Jensen’s inequality, the symmetry of $\mathbb{P}_{h,N}$ and the antisymmetry of $P_h$ that
\[
\int_{\mathbb{R}^{N-1}} e^{-\frac{F_N^{(\xi)}(y)}{hN}} dy = \tilde{Z}_{h,N}(3\xi^2) \int_{\mathbb{R}^{N-1}} e^{-\frac{\xi P_h(y)}{hN}} \mathbb{P}_{h,N}(dy) \geq \tilde{Z}_{h,N}(3\xi^2) e^{-\frac{\xi^2}{hN}} \int_{\mathbb{R}^{N-1}} P_h(y)^2 \mathbb{P}_{h,N}(dy) = \tilde{Z}_{h,N}(3\xi^2).
\]
Moreover, using for the quartic term the inequality $e^{-t} \geq 1 - t$, $t \in \mathbb{R}$, one gets
\[
\tilde{Z}_{h,N}(3\xi^2) = \int_{\mathbb{R}^{N-1}} e^{-\frac{P_h(y)}{4hN} - \frac{3\xi^2 P_h(y) + Q(y)}{2hN}} dy \geq \int_{\mathbb{R}^{N-1}} e^{-\frac{3\xi^2 P_h(y) + Q(y)}{2hN}} dy - \frac{1}{4hN} \int_{\mathbb{R}^{N-1}} P_h(y) e^{-\frac{3\xi^2 P_h(y) + Q(y)}{2hN}} dy = Z_{h,N}(3\xi^2) \left( 1 - \frac{1}{4hN} \int_{\mathbb{R}^{N-1}} P_h(y) e^{-\frac{3\xi^2 P_h(y) + Q(y)}{2hN}} Z_{1,N}^{-1}(3\xi^2) dy \right).
\]
The claim (29) follows then by applying Lemma 5.4 with $t_0 = 0$ and $t = 3\xi^2$.

As second step we pick a $\delta \in (0, \mu - 1)$ and show that for all $\xi \in \mathbb{R}$, $h > 0$ and $N \in \mathbb{N}$ the estimate
\[
\int_{\{h \geq NR^2\}} e^{-\frac{F_N^{(\xi)}(y)}{hN}} dy \leq e^{-\frac{\xi^2 + \delta}{2h} R^2} Z_{h,N}(-\delta) \tag{30}
\]
holds true. Indeed, using with $\alpha = 2$ the inequality $\xi^3 \leq \frac{\alpha \xi^2 t}{2} + \frac{t^2}{2\alpha}$, valid for every $\xi, t \in \mathbb{R}$ and $\alpha > 0$, one gets for every $N \in \mathbb{N}, \xi \in \mathbb{R}$ and $y \in \mathbb{R}^{N-1}$ the lower bound
\[
F_N^{(\xi)}(y) \geq \frac{1}{2}\xi^2 P_2(y) + \frac{1}{2}Q(y).
\]
It follows that for all $\xi \in \mathbb{R}$, $h > 0$ and $N \in \mathbb{N}$ it holds

$$\int_{\{p_2 \geq NR^2\}} e^{\frac{F(\xi(y))}{hN}} \, dy \leq \int_{\{p_2 \geq NR^2\}} e^{-\frac{(\xi^2 + \delta)p_2(y)}{2hN}} e^{-\frac{Q(y) - \delta p_2(y)}{2hN}} \, dy \leq e^{-\frac{\xi^2 + \delta}{2h}} Z_{h,N}(-\delta),$$

i.e. (30) is proven.

To finish the proof of the proposition we observe that it follows from (29), (30) and Lemma 5.5 that there exists constants $C', \delta' > 0$ such that, choosing $h' \in (0, \frac{1}{2C'})$, for every $h \in (0, h'_0]$, $N \in \mathbb{N}$ and $\xi \in \mathbb{R}$ it holds

$$\log \int_{\mathbb{R}^{N-1}} e^{-\frac{F(\xi(y))}{hN}} \, dy - \log \int_{\{p_2(y) \geq NR^2\}} e^{-\frac{F(\xi(y))}{hN}} \, dy \geq \log (1 - hC') + \frac{\xi_2 + \delta}{2h} (3\xi^2 + \delta) \geq \frac{1}{2} \left( \frac{R^2}{h} - 3\delta' \right) \xi^2 + \frac{1}{h} \left( \frac{R^2}{2} - h\delta (\frac{R}{2} - \frac{\log 2}{2}) \right).$$

It follows that, fixing any $C < \frac{R^2}{2}$, we can find an $h_0 \in (0, h'_0]$ such that the statement of the proposition holds. \(\square\)

We can now easily complete the proof of Proposition 3.1 by integrating over the diagonal:

**Proof of Proposition 3.1.** Fix $R > 0$. It follows from Proposition 5.3 and Proposition 5.6 that there exists constants $C, h_0 > 0$ such that for every $N \in \mathbb{N}$, every $h \in (0, h_0]$ and every $f \in C_c(\mathbb{R}^N)$ with the property that $\text{supp } f \subset \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_i (x_i - \bar{x})^2 \geq R^2 \right\}$, denoting for each $\xi \in \mathbb{R}$ by $f^{(\xi)}$ the function $y \mapsto f(x(\xi, y)) \in C^0_c(\mathbb{R}^{N-1})$ and by $\nabla f^{(\xi)}$ its gradient, it holds

$$\mathcal{E}_{h,N}[f] \geq hN \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{N-1}} |\nabla f^{(\xi)}(y)|^2 e^{\frac{F(\xi(y))}{hN}} \, dy \right) e^{-\frac{1}{4h}(\xi^2 - 1)^2} \sqrt{N} d\xi \geq \mathcal{G} \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{N-1}} |f^{(\xi)}(y)|^2 e^{\frac{F(\xi(y))}{hN}} \, dy \right) e^{-\frac{1}{4h}(\xi^2 - 1)^2} \sqrt{N} d\xi = \mathcal{G} \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} \, dx. \quad \square$$

5.2. **Around the diagonal: Proof of Proposition 3.2.** In order to prove Proposition 3.2 we shall consider another quadratic partition of unity, which permits to isolate and treat separately various contributions to $\mathcal{E}_{h,N}$: the contribution coming from a large convexity region of $V_N$ (containing the two local minima $I_+, I_-$), see Proposition 5.10 below; the contribution due to a neighbourhood of the saddle point, see Proposition 5.11 below; and finally the contribution coming from the small remaining region, see Proposition 5.12 below.
We consider in the sequel for every \( N \in \mathbb{N} \) and \( R > 0 \) the strip around the diagonal given by
\[
S_N(R) := \left\{ x \in \mathbb{R}^N : \frac{1}{N} \sum_k (x_k - \bar{x})^2 \leq R^2 \right\}.
\]
Moreover we define for all \( N \in \mathbb{N}, R > 0 \) and \( r \geq 0 \) the sets\[
\Omega_N^{\min}(R, r) := S_N(R) \cap \left\{ x \in \mathbb{R}^N : |\bar{x}| \geq r \right\},
\]
which, for \( 0 \leq r < 1 \), are neighbourhoods of the two minima \( \pm I \) and the sets\[
\Omega_N^0(R, r) := S_N(R) \cap \left\{ x \in \mathbb{R}^N : |\bar{x}| \leq r \right\},
\]
which, for \( r > 0 \), are neighbourhoods of the saddle point 0.

As a preliminary step we show in Lemma 5.7 below that the energy \( V_N \) is uniformly convex on suitable sectors containing the two global minima (and thus in particular on \( \Omega_N^{\min}(R, r) \) if \( R \) is small enough and \( r \) is large enough). This is a rather straightforward consequence of the Sobolev inequality given in Lemma 5.2.

**Lemma 5.7** (Convexity around \( I_+, I_- \)). For every \( r > \frac{1}{\sqrt{3}} \) there exist constants \( \alpha_0 = \alpha_0(r), C = C(r) > 0 \) such that for all \( N \in \mathbb{N} \) and for all \( x \in \mathbb{R}^N \) such that \( |\bar{x}| \geq r \) and \( \sum_k (x_k - \bar{x})^2 \leq N\alpha_0^2|\bar{x}|^2 \) one has the bound\[
\text{Hess} V_N(x) \geq C.
\]
In particular there exists an \( R_0 > 0 \) such that for every \( r > \frac{1}{\sqrt{3}} \) there exists a constant \( C(r) > 0 \) such that\[
\text{Hess} V_N(x) \geq C(r), \quad \forall N \in \mathbb{N} \text{ and } \forall x \in \Omega_N^{\min}(R_0, r).
\]

**Proof.** We note first that for all \( N \in \mathbb{N}, x \in \mathbb{R}^N \) and \( \omega \in \mathbb{R}^N \) such that \( \sum_k \omega_k^2 = 1 \) it holds\[
\langle \text{Hess} V_N(x) \omega, \omega \rangle = 3 \sum_k x_k^2 \omega_k^2 - 1 + \langle K \omega, \omega \rangle.
\]
We fix \( r > \frac{1}{\sqrt{3}}, \varepsilon > 0 \) and let \( N \in \mathbb{N} \) and \( \omega \in \mathbb{R}^N \) with \( \sum_k \omega_k^2 = 1 \). If \( \langle K \omega, \omega \rangle \geq 1 + \varepsilon \) we use \( 3 \sum_k x_k^2 \omega_k^2 \geq 0 \) and conclude that \( \langle \text{Hess} V_N(x) \omega, \omega \rangle \geq \varepsilon \) for all \( x \in \mathbb{R}^N \). Thus we can assume in the sequel that \( \langle K \omega, \omega \rangle \leq 1 + \varepsilon \).

In this case, using \( \langle K \omega, \omega \rangle \geq 0 \) and the decomposition \( x = x + (x - \bar{x}) \), one gets for every \( x \in \mathbb{R}^N \) the estimate\[
\langle \text{Hess} V_N(x) \omega, \omega \rangle \geq 3|\bar{x}|^2 - 1 + 6 \varepsilon \sum_k (x_k - \bar{x}) \omega_k^2 + 3 \sum_k (x_k - \bar{x})^2 \omega_k^2 \geq
\]
\[
\geq 3|\bar{x}|^2 - 1 + 6 \varepsilon \sum_k (x_k - \bar{x}) \omega_k^2.
\]
It follows then from the decomposition \( \omega = \bar{\omega} + (\omega - \bar{\omega}) \) and the Cauchy-Schwarz inequality that, for every \( \alpha > 0 \) and every \( x \in \mathbb{R}^N \) with \( \sum_k (x_k - \bar{x})^2 \leq \alpha^2 \sum_k (x_k - \bar{x}) \omega_k^2 \leq \varepsilon \) for every \( \omega \in \mathbb{R}^N \).
$$\overline{\omega}^2 \leq N\alpha^2|\overline{\omega}|^2,$$

$$\langle \text{Hess } V_N(x)\omega,\omega \rangle \geq 3|\overline{\omega}|^2 - 1 - 12\alpha|\overline{\omega}|^2 - 6\alpha^2 \left(N \sum_k (\omega_k - \overline{\omega})^4 \right)^{\frac{1}{4}}. \quad (31)$$

The Sobolev inequality of Lemma 5.2 implies that there exists a constant \(\gamma > 0\) such that

$$\left(N \sum_k (\omega_k - \overline{\omega})^4 \right)^{\frac{1}{4}} \leq \gamma \langle K\omega,\omega \rangle \leq \gamma (1 + \varepsilon). \quad (32)$$

Thus (31) implies that, for every \(\alpha > 0\) and every \(x \in \mathbb{R}^N\) with \(\sum_k (x_k - \overline{x})^2 \leq N\alpha^2|\overline{\omega}|^2\) and \(|\overline{\omega}| > r\), defining for short \(\delta := 3\alpha^2 - 1 > 0\) and \(t := 6(2 - \gamma(1 + \varepsilon)) \in \mathbb{R}\),

$$\langle \text{Hess } V_N(x)\omega,\omega \rangle \geq 3|\overline{\omega}|^2 - 1 - 12\alpha|\overline{\omega}|^2 - 6\alpha^2 \left(N \sum_k (\omega_k - \overline{\omega})^4 \right)^{\frac{1}{4}} = [1 - \alpha t] (1 + \delta) - 1 \geq \delta(1 - \alpha|t|) - \alpha|t|.$$

We conclude that, taking \(\alpha_0 > 0\) such that \(\alpha_0|t| \leq \min\{\frac{1}{2}, \frac{\delta}{4}\}\), the estimate

$$\langle \text{Hess } V_N(x)\omega,\omega \rangle \geq \frac{\delta}{4}$$
holds for every \(x \in \mathbb{R}^N\) with \(\sum_k (x_k - \overline{x})^2 \leq N\alpha_0^2|\overline{\omega}|^2\) and \(|\overline{\omega}| > r\). \(\square\)

**Remark 5.8** (Convex modifications of \(V_N\)). It follows from Lemma 5.7 that there exists \(R_0 > 0\) such that for every \(r > \frac{1}{\sqrt{3}}, 1\) there exist a constant \(C(r) > 0\) and, for every \(N \in \mathbb{N}\), functions \(V^+_N, V^-_N \in C^\infty(\mathbb{R}^N)\) with the following property:

(i) \(\text{Hess } V^+_N(x) \geq C(r)\) for all \(N \in \mathbb{N}\) and \(x \in \mathbb{R}^N\),

(ii) \(V^+_N(x) = V_N(x)\) for all \(N \in \mathbb{N}\) and \(x \in \Omega^\text{min}_N(R_0, r) \cap \{\pm \overline{\omega} \geq 0\}\).

**Remark 5.9.** Note that it would be enough to have a Sobolev inequality for the 4-norm appearing on the left hand side of (32) instead of the stronger \(\infty\)-norm statement of Lemma 5.2. This remark permits to extend our arguments to space dimension two and three.

In regions where \(V_N\) is uniformly convex we can now estimate \(E_{h, N}\) from below by the Bakry-Émery criterion for the spectral gap given by (26):

**Proposition 5.10** (Estimates in convex regions around the minima). There exists an \(R_0 > 0\) such that for every \(r \in \left(\frac{1}{\sqrt{3}}, 1\right)\) the following holds: there exist constants \(C = C(r) > 0\) and, for every \(h > 0, N \in \mathbb{N}\), there exist functions \(\phi^+_{h, N} = \phi^+_{h, N, r}, \phi^-_{h, N} = \phi^-_{h, N, r} \in C^\infty_b(\mathbb{R}^N)\) such that for all \(h > 0\), \(N \in \mathbb{N}\) and \(f \in C^\infty_b(\mathbb{R}^N)\) with \(\text{supp } f \subset \Omega^\text{min}_N(R_0, r)\) it holds

$$E_{h, N}[f] \geq C \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} dx - \left( \int_{\mathbb{R}^N} f \phi^+_{h, N} e^{-\frac{V_N}{hN}} dx \right)^2 - \left( \int_{\mathbb{R}^N} f \phi^-_{h, N} e^{-\frac{V_N}{hN}} dx \right)^2.$$

**Proof.** Take \(R_0 > 0\) as in Remark 5.8, let \(r \in \left(\frac{1}{\sqrt{3}}, 1\right)\) and consider the constant \(C = C(r)\) and the functions \(V^+_N := V^+_N, V^-_N := V^-_N\) as in Remark 5.8. Moreover define for shortness \(Z^+_{h, N} := \int_{\mathbb{R}^N} \exp(-\frac{V_N}{hN}) dx\) and take \(\chi^+, \chi^- \in C^\infty_b(\mathbb{R}^N)\) such that \(\chi^\pm \equiv 1\) on \(\{\pm \overline{\omega} \geq \frac{3}{2}\}\) and \(\chi^\pm \equiv 0\) on \(\{\pm \overline{\omega} \leq 0\}\).
Then for all $h > 0$, $N \in \mathbb{N}$ and $f \in C^\infty_b(\mathbb{R}^N)$ with $\text{supp} f \subset \Omega_{h,N}^{\min}(R_0, r)$ it follows from (26) that
\[
\mathcal{E}_{h,N}[f] = hN \int_{\mathbb{R}^N} |\nabla (+f)|^2 e^{-V_N^+/(hN)} dx + hN \int_{\mathbb{R}^N} |\nabla (-f)|^2 e^{-V_N^-/(hN)} dx \geq \]
\[
\geq C \left( \int_{\mathbb{R}^N} |+f|^2 e^{-V_N^+/(hN)} dx - \left( \frac{1}{\sqrt{Z_{h,N}}} \int |+f| e^{-V_N^+/(hN)} dx \right)^2 \right) + \]
\[
\quad + C \left( \int_{\mathbb{R}^N} |-f|^2 e^{-V_N^-/(hN)} dx - \left( \frac{1}{\sqrt{Z_{h,N}}} \int |-f| e^{-V_N^-/(hN)} dx \right)^2 \right) = \]
\[
= C \int_{\mathbb{R}^N} f^2 e^{-V_N/(hN)} dx - \left( \int f \sqrt{C \sqrt{Z_{h,N}}} |+f| e^{-V_N^+/(hN)} dx \right)^2 - \left( \int f \sqrt{C \sqrt{Z_{h,N}}} |-f| e^{-V_N^-/(hN)} dx \right)^2. \]
Thus the functions $\phi_{h,N}^\pm := \sqrt{C \sqrt{Z_{h,N}}} \chi^\pm$ satisfy the statement of the proposition. \hfill \Box

The next proposition gives a lower bound on $\mathcal{E}_{h,N}$ on $\Omega_{h,N}^0(R, r)$ for $R, r$ small enough. The proof is elementary if one first bounds the norm of the full gradient of $f$ from below by the norm of the directional derivative in direction of the constants and then performs a one-dimensional version of the ground state transformation (Lemma 4.3).

**Proposition 5.11** (Estimate around the saddle point). Let $\delta, C > 0$ with $C < \delta^2 < \frac{1}{2}$ and define $r(\delta) := \sqrt{\frac{(1-2\delta^2)}{\delta}} > 0$ and $R(\delta) := \sqrt{\frac{2}{3}(\delta^2 - C)} > 0$. Then the following holds: for every $h > 0$, $N \in \mathbb{N}$ and $f \in C^\infty_b(\mathbb{R}^N)$ with $\text{supp} f \subset \Omega_{h,N}^0(R(\delta), r(\delta))$
\[
\mathcal{E}_{h,N}[f] \geq C \int_{\mathbb{R}^N} f^2 e^{-V_N/(hN)} dx. \]

**Proof.** Let $f \in C_b(\mathbb{R}^N)$. Setting $g := e^{-V_N/(hN)} f$ we get
\[
\mathcal{E}_{h,N}[f] \geq hN \int_{\mathbb{R}^N} |\nabla f(x) \cdot \frac{(1-|x|^2)}{\sqrt{N}}|^2 e^{-V_N/(hN)} dx = h \int_{\mathbb{R}^N} |\sum_k \partial_k f|^2 e^{-V_N/(hN)} dx = \]
\[
= h \int_{\mathbb{R}^N} \left\{ \left| \sum_k \partial_k g \right|^2 + \frac{1}{4h^2N} \sum_k \partial_k V_N \right\} \sum_k g^2 - \frac{1}{2hN} \sum_{k,j} \partial^2_{k,j} V_N g^2 \right\} dx \geq \]
\[
- \frac{1}{2hN} \int_{\mathbb{R}^N} \sum_{k,j} \partial^2_{k,j} V_N f^2 e^{-V_N/(hN)} dx. \tag{33} \]

Moreover for every $x \in \mathbb{R}^N$
\[
- \frac{1}{2hN} \sum_{k,j} \partial^2_{k,j} V_N(x) = - \frac{1}{2hN} \sum_k (3x^2 - 1) = \frac{1}{2} - \frac{3}{2} x^2 - \frac{3}{2N} \sum_k (x_k - \bar{x})^2. \]
Thus for every $x \in \Omega_{h,N}^0(R(\delta)r(\delta))$
\[
- \frac{1}{2hN} \sum_{k,j} \partial^2_{k,j} V_N(x) \geq \frac{1}{2} - \frac{3}{2} r(\delta)^2 - \frac{3}{2} R(\delta)^2 \geq C. \]
This gives the desired result by taking \( f \) with \( \text{supp} \, f \subset \Omega_{N}(R(\delta), r(\delta)) \) in (33).

Note that the domains covered by Proposition 5.10 and Proposition 5.11 do not match. In order to fill the gap we consider now domains in \( \mathbb{R}^N \) of the form

\[
\tilde{\Omega}_N(R, r_1, r_2) := S_N(R) \cap \left\{ x \in \mathbb{R}^N : r_1 \leq |x| \leq r_2 \right\},
\]

where \( r_1, r_2 \in (0, 1) \), \( r_1 < r_2 \) and \( R > 0 \). One expects the restriction of the Dirichlet form \( \mathcal{E} \) on functions supported in such a \( \tilde{\Omega}_N(r_1, r_2, R) \) to be bounded from below by a term of order \( h^{-1} \), uniformly in \( N \). The next proposition shows that this is the case, at least if \( (r_1, r_2) \subset (0, \frac{1}{\sqrt{2}}) \) and \( R \) is sufficiently small. This will be enough for our purposes, since for our main proof we will need only to cover the case in which \((r_1, r_2)\) is an arbitrarily small neighbourhood of the inflection point \( \frac{1}{\sqrt{2}} \) and \( R \) is arbitrarily small.

**Proposition 5.12** (Estimates around inflection points). For every \( \delta \in (0, \frac{1}{\sqrt{2}}) \) there exist constants \( C = C(\delta), R_0 = R_0(\delta), h_0 = h_0(\delta) > 0 \) such that for all \( h \in (0, h_0) \), \( N \in \mathbb{N} \) and \( f \in C_b^\infty(\mathbb{R}^N) \) with \( \text{supp} \, f \subset \tilde{\Omega}_N(R_0, \sqrt{\delta}, \frac{1}{\sqrt{2}} \sqrt{1-\delta}) \) we have

\[
\mathcal{E}_{h,N}[f] \geq C h \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} \, dx.
\]

**Proof.** Fix \( \delta \in (0, \frac{1}{\sqrt{2}}) \). We set for short \( \kappa := \min\{\frac{\delta}{2}, \mu - 1\} \) and consider for each \( N \in \mathbb{N} \) the function \( U_N : \mathbb{R}^N \to \mathbb{R} \) given by

\[
U_N(x) := V_N(x) + \frac{1+N}{2} |x|^2 - \frac{N}{4} = \frac{1}{4} \sum_k x_k^2 + Q_N(x),
\]

with \( Q_N(x) := \frac{\delta}{2} |x|^2 + \frac{1}{2} (x - \bar{\mathcal{Q}}, (K - 1)x - \bar{\mathcal{Q}}) \). Observe that \( \text{Hess} \, U_N(x) \geq \kappa > 0 \) for every \( N \in \mathbb{N} \) and every \( x \in \mathbb{R}^N \). Further, given \( f \in C_b^\infty(\mathbb{R}^N) \), according to Remark 4.3 we compute

\[
\mathcal{E}_{h,N}(f) = \int_{\mathbb{R}^N} [hN |\nabla g|^2 + \left( \frac{\delta}{4hN} |\nabla V_N|^2 - \frac{1}{4hN} |\nabla U_N|^2 - \frac{1}{h} \Delta(V_N - U_N) \right) g^2] e^{-\frac{V_N}{hN}} \, dx,
\]

where \( g = e^{-\frac{V_N + U_N}{2hN}} f \). It follows then from the NGS Bound (Prop. 4.1), the Bakry-Emery criterion (Prop. 4.4) and the estimate \( \Delta(V_N - U_N)(x) = -(1 + \kappa) \leq 0 \) that for every \( R > 0 \), \( N \in \mathbb{N} \) and \( f \in C_b^\infty(\mathbb{R}^N) \) with \( \text{supp} \, f \subset \Omega_{N,\delta,R} := \tilde{\Omega}_N(R, \sqrt{\delta}, \frac{1}{\sqrt{2}} \sqrt{1-\delta}) \) we have

\[
\mathcal{E}_{h,N}(f) \geq C(h, N, R) \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN}} \, dx,
\]

where

\[
C(h, N, R) := \frac{\kappa}{2} \left( \log \int_{\mathbb{R}^N} e^{-\frac{U_N(x)}{hN}} \, dx - \log \int_{\Omega_{N,h,R}} e^{-\frac{G_N(x)}{hN}} \, dx \right),
\]

with

\[
G_N := \frac{1}{2\kappa} (|V_N|^2 - |U_N|^2) + U_N = \frac{1}{2\kappa} |\nabla(V_N - U_N)|^2 + \frac{1}{\kappa} \nabla U_N \cdot \nabla(V_N - U_N) + U_N.
\]
A straightforward computation (see below for details) shows that there exists a constant \( c \in \mathbb{R} \) such that for every \( N \in \mathbb{N} \) and every \( R > 0 \) we have the lower bound
\[
G_N(x) \geq N \left( \frac{\xi^2}{4\kappa} - cR^2 \right) + Q_N(x), \quad \forall x \in \Omega_{N,\delta,R}. \tag{36}
\]
Thus, letting \( \tilde{C}(R) := \frac{\xi^2}{4\kappa} - cR^2 \) and denoting for short by \( \gamma_N(dx) \) the Gaussian probability measure on \( \mathbb{R}^N \) with density \( e^{-\frac{Q_N(x)}{N}} \), one gets for all \( h, R > 0, \forall N \in \mathbb{N} \)
\[
C(h, N, R) \geq \frac{\kappa}{2} \left( \log \int_{\mathbb{R}^N} e^{-\frac{U_N(x)}{hN}} dx - \log \int_{\mathbb{R}^N} e^{-\frac{Q_N(x)}{hN}} dx \right) + \frac{\kappa \tilde{C}(R)}{2h} =
\]
\[
= \frac{\kappa}{2} \log \int_{\mathbb{R}^N} e^{-\frac{h \sum_k x_k^2}{4N}} \gamma_N(dx) + \frac{\kappa \tilde{C}(R)}{2h} \geq -h \frac{\kappa}{4N} \sum_k \int_{\mathbb{R}^N} x_k^4 \gamma_N(dx) + \frac{\kappa \tilde{C}(R)}{2h}.
\]
An explicit computation of the Gaussian integrals \( \int_{\mathbb{R}^N} x_k^4 \gamma_N(dx) \) (see also [21, Lemma 2.2] for details) shows that there exists a constant \( M > 0 \) such that
\[
\frac{1}{N} \sum_k \int_{\mathbb{R}^N} x_k^4 \gamma_N(dx) < M, \quad \forall N \in \mathbb{N}.
\]
Thus, choosing \( R_0 > 0 \) small enough, such that \( \tilde{C}(R_0) = \frac{\xi^2}{4\kappa} - cR_0^2 > 0 \) and \( h_0 = \sqrt{\frac{2\tilde{C}(R_0)}{M}} \), one gets for every \( N \in \mathbb{N} \) and every \( h \in (0, h_0] \)
\[
C(h, N, R_0) \geq \frac{\kappa \tilde{C}(R_0)}{4h}.
\]
Together with (35) this provides the final estimate (34) with \( C := \frac{\kappa \tilde{C}(R_0)}{4} \).

In order to complete the proof, we give now the details on how the estimate (36) can be proven. First, to obtain a more explicit formula for \( G_N \), we observe that \( [\nabla ( V_N - U_N) (x)]_j = -(1 + \kappa)\bar{x} \) for every \( j = 1, \ldots, N \) and that
\[
\sum_j [\nabla U_N(x)]_j = \sum_j (x_j^3 - x_j + [Kx]_j) + (1 + \kappa) N \bar{x} = \sum_j x_j^3 + \kappa N \bar{x} = N \bar{x}^3 + \kappa N \bar{x} + 3\bar{x} \sum_j (x_j - \bar{x})^2 + \sum_j (x_j - \bar{x})^3.
\]
It follows that
\[
G_N(x) = \frac{(1 + \kappa)^2}{2\kappa} N \bar{x}^2 - \frac{1 + \kappa}{\kappa} \left( N \bar{x}^4 + \kappa N \bar{x}^2 + 3\bar{x} \sum_j (x_j - \bar{x})^2 + \bar{x} \sum_j (x_j - \bar{x})^3 \right) + \frac{1}{\kappa} \sum_j x_j^4 + Q_N(x) = \frac{N \theta_N(\bar{x})}{\kappa^2} \bar{x}^2 + H_N(x) + Q_N(x),
\]
with \( \theta_N(\xi) := [1 - \kappa^2 - \frac{1}{2}(4 + 3\kappa)\xi^2] \).
and
\[ H_N(x) := \frac{1}{4} \sum_j (x_j - \overline{x})^4 - \frac{1}{2} \sum_j (x_j - \overline{x})^3 - \frac{3}{2} (x_j^2 \sum_j (x_j - \overline{x})^2. \]

Note that for \( \overline{x}^2 \leq \frac{1}{2}(1 - \delta) \) one has \( \theta_N(\overline{x}) \geq \delta - (\kappa^2 + \frac{A}{2}) \geq \frac{\delta}{2} \), where for the last inequality we have used \( \delta < \frac{\delta}{2} \) and \( \kappa \leq \frac{\delta}{2} \). Moreover, the inequality \( ab \leq \alpha a^2 + \frac{1}{4}b^2 \) with \( a = (x_k - \overline{x})^2, b = (x_k - \overline{x}) \) and \( \alpha = \frac{1}{2} \) yields
\[ H_N(x) \geq -c\overline{x}^2 \sum_j (x_j - \overline{x})^2, \quad \text{with} \quad c := \frac{3a}{2} + \frac{1}{2}. \]

It follows that
\[ G_N(x) \geq \left( \frac{\delta^2}{4\kappa} - \frac{\delta}{2}(1 - \delta)R^2 \right) N + Q_N(x), \quad \forall x \in \Omega_{N,\delta,R}, \]
i.e. (36) with \( c = \frac{\delta}{2}(1 - \delta) \).

We can now wrap up all the estimates of this section and prove Proposition 3.2.

Proof of Proposition 3.2. We fix \( a_0, b_0, a_{\min}, b_{\min} > 0 \) such that
\[ a_0 < b_0 < \frac{1}{\sqrt{3}} < a_{\min} < b_{\min} < \frac{1}{\sqrt{2}}, \]
and two functions \( \chi_{\min}, \chi_0 \in C_b^\infty(\mathbb{R}; [0, 1]) \) such that \( \chi_{\min} \equiv 1 \) on \( (-\infty, b_{\min}] \cup [b_{\min}, \infty) \), \( \chi_{\min} \equiv 0 \) on \( [-a_{\min}, a_{\min}] \), \( \chi^0 \equiv 1 \) on \( [-a_0, a_0] \), \( \chi^0 \equiv 0 \) on \( \mathbb{R} \setminus [-b_0, b_0] \). We then define \( \zeta_N^\min, \zeta_N^0, \zeta_N : \mathbb{R}^N \rightarrow [0, 1] \) by setting
\[ \zeta_N^\min(x) := \chi_{\min}(\overline{x}), \quad \zeta_N^0(x) := \chi_0(\overline{x}), \quad \tilde{\zeta}_N(x) := \left( 1 - [\zeta_N^\min]^2(x) - [\zeta_N^0]^2(x) \right)^{\frac{1}{2}}, \]
so that \( [\zeta_N^\min]^2 + [\zeta_N^0]^2 + \tilde{\zeta}_N^2 = 1 \). Note that \( \zeta_N^\min, \zeta_N^0, \tilde{\zeta}_N \in C^\infty(\mathbb{R}^N) \) for all \( N \in \mathbb{N} \).

It follows from the IMS localization formula that for all \( N \in \mathbb{N}, h > 0 \) and all \( f \in C_b^\infty(\mathbb{R}^N) \)
\[ E_{h,N}[f] = E_{h,N}[\zeta_N^\min f] + E_{h,N}[\zeta_N^0 f] + E_{h,N}[\tilde{\zeta}_N f] + hG_{h,N}[f], \quad (38) \]
where the localization error \( G_{h,N}[f] \) is given by
\[ G_{h,N}[f] := -N \int_{\mathbb{R}^N} \left( |\nabla \zeta_N^\min|^2 + |\nabla \zeta_N^0|^2 + |\nabla \tilde{\zeta}_N|^2 \right) f^2 e^{-\frac{\chi_N^2}{\delta}} dx. \]
Computing the gradients in this formula shows that there exists a constant \( c > 0 \) such that for all \( N \in \mathbb{N}, h > 0 \) and \( f \in C_b^\infty(\mathbb{R}^N) \)
\[ |G_{h,N}[f]| \leq c \int_{\mathbb{R}^N} f^2 e^{-\frac{\chi_N^2}{\delta}} dx. \]
Using for the first, second and third term on the right hand side of (38) respectively Proposition 5.10, Proposition 5.11 and Proposition 5.12 gives then the following estimate: there exist constants \( R_0, h_0, C_{\min}, C_0, \tilde{C} > 0 \) and, for every \( h > 0, N \in \mathbb{N} \), there exist functions \( \varphi_{h,N}^+, \varphi_{h,N}^- \in C_b^\infty(\mathbb{R}^N) \)
such that for all \( h \in (0, \tilde{h}_0), N \in \mathbb{N} \) and \( f \in C_b^\infty(\mathbb{R}^N) \) with \( \text{supp} \ f \subset S_N(R_0) \) it holds
\[
\mathcal{E}_{h,N}[f] \geq C_{\text{min}} \int_{\mathbb{R}^N} [\zeta_N^{\text{min}} f]^2 e^{-\frac{V_N}{hN} dx} - \left( \int_{\mathbb{R}^N} f \zeta_N^{\text{min}} \varphi_{h,N}^+ e^{-\frac{V_N}{hN} dx} \right)^2 - \left( \int_{\mathbb{R}^N} f \zeta_N^{\text{min}} \varphi_{h,N}^- e^{-\frac{V_N}{hN} dx} \right)^2 + C_0 \int_{\mathbb{R}^N} [\zeta_N^0 f]^2 e^{-\frac{V_N}{hN} dx} + \int_{\mathbb{R}^N} \zeta_N f^2 e^{-\frac{V_N}{hN} dx} - ch \int_{\mathbb{R}^N} f^2 e^{-\frac{V_N}{hN} dx}.
\]
Taking \( C := \frac{1}{2} \min\{C_{\text{min}}, C_0, \tilde{C}\}, h_0 := \min\{\tilde{h}_0, 1, \frac{C}{e}\} \) and \( \phi_{h,N}^\pm := \zeta_N^{\text{min}} \varphi_{h,N}^\pm \) concludes the proof. \( \square \)

6. Spectral Gap Asymptotics in infinite dimension

In this section we provide the proof of Theorem 2.3. The latter is split into two parts: The lower bound will be shown with Proposition 6.4 below, and the upper Bound with Proposition 6.5 below. Both lower and upper bound are obtained by rather straightforward approximation procedures starting from the corresponding uniform result on the lattice. Thus the lower bound given in Proposition 6.4 will be essentially a corollary of Theorem 2.2.

We start by introducing our notation. We fix \( m > 0 \) and consider for each \( h > 0 \) the centered Gaussian measure \( \gamma_h \) on \( L^2(\mathbb{T}) \) introduced after Theorem 2.2. The Fourier transform of \( \xi \in L^2(\mathbb{T}) \) is denoted by \( k \mapsto \xi(k) \). For the approximation procedure it will be enough and notionally convenient to consider only odd \( N \). We shall then denote by \( \mathbb{N}' := 2\mathbb{N} - 1 \) the set of odd natural numbers and label generic elements of \( \mathbb{R}^N \) for \( N \in \mathbb{N}' \) by \( x = (x_k)_{k \in \mathbb{Z} : |k| \leq \frac{N-1}{2}} \). We still assume periodic boundary conditions, i.e. \( x_{\frac{1}{2}(N+1)} := x_{\frac{N-1}{2}} \).

For each \( N \in \mathbb{N}' \) and \( h > 0 \) we consider the centered Gaussian probability measure \( \gamma_{h,N} \) on \( \mathbb{R}^N \) defined by
\[
\gamma_{h,N}(dx) := Z_{h,N}^{-1} \exp \left( -\frac{m}{2} \sum_k x_k^2 - \frac{\mu}{8 \sin^2(\frac{\pi k}{N})} \sum_k (x_k - x_{k+1})^2 \right) dx,
\]
where \( Z_{h,N} \) is the normalization constant. Note that for every \( N \in \mathbb{N}', h > 0 \) and \( k, j \in \mathbb{Z} \) with \( |k|, |j| \leq \frac{1}{2}(N-1) \) the covariances are given by
\[
\int_{\mathbb{R}^N} x_k x_j \gamma_{h,N}(dx) = h \sum_{|\ell| \leq \frac{1}{2}(N-1)} \frac{1}{m + \nu_{\ell,N}} \exp \left( -i2\pi \frac{k}{N} (k - j) \right), \quad (39)
\]
where the \( \nu_{\ell,N} \)'s are the eigenvalues of \( K_N \) as introduced in (27). In order to relate \( \gamma_h \) and \( \gamma_{h,N} \) we introduce for each \( N \in \mathbb{N}' \) a random vector \( X^{(N)} = (X_k^{(N)})_{k \in \mathbb{Z} : |k| \leq \frac{N-1}{2}} \) on the probability space \( (L^2(\mathbb{T}), \gamma_h) \) which has the property to be distributed according to \( \gamma_{h,N} \). This can be done as follows [42]. We define for each \( N \in \mathbb{N}' \) the function \( \sigma := \sigma_N : \mathbb{Z} \rightarrow \mathbb{R} \) by
setting
\[\sigma_N(k) := \begin{cases} \left(\frac{m+k^2}{m+n_{k,N}}\right)^{\frac{1}{2}} & \text{if } |k| = 0, \ldots, \frac{N-1}{2}, \\ 0 & \text{otherwise}. \end{cases} \tag{40}\]

Note that \(\lim_{N \to \infty} \sigma_N(k) \to 1\) for every \(k \in \mathbb{Z}\) and that there exists a \(C > 0\) such that \(|\sigma_N(k)| \leq C\) for all \(k \in \mathbb{Z}\) and \(N \in \mathbb{N}'\). Further let \(X^{(N)} = (X_k^{(N)})_{k \in \mathbb{Z}, |k| \leq \frac{N-1}{2}} : L^2(T) \to \mathbb{R}^N\) be the continuous linear map given by
\[X_k^{(N)}(\xi) := \int T \xi(s) \sum_{\ell \in \mathbb{Z}} \sigma_N(\ell) e^{-i2\pi \ell(s-k)} ds, \quad \forall \xi \in L^2(T), |k| \leq \frac{N-1}{2}. \tag{41}\]

Then \(X^{(N)}\) is a centered Gaussian random vector on the probability space \((L^2(T), \gamma_h)\) with covariance given by
\[\int_{L^2(T)} X_k^{(N)} X_j^{(N)} \gamma_h(d\xi) = h \sum_{|\ell| \leq \frac{1}{2}(N-1)} \frac{1}{m + \nu_{\ell,N}} \exp \left(-i2\pi \frac{k}{N} (k - j)\right). \]

Thus, it follows from (39) that \(X^{(N)}\) has Law \(\gamma_{h,N}\) as desired. Furthermore, we introduce in analogy to [DaPrato, 11.2], point evaluation on continuous functions on \(T\) and \(\xi \in L^2(T)\) as \(\xi(s) := \delta_s(\xi)\), where \(\delta_s\) is defined as the limit in \(L^2(d\gamma_h)\) of
\[\delta_{s,N}(\xi) := \sum_{|\ell| \leq \frac{1}{2}(N-1)} \hat{\xi}(\ell)e^{i2\pi \ell s}. \tag{42}\]

Next, recalling the functional \(U\) introduced in (14), we consider for each \(N \in \mathbb{N}'\) its discrete version \(U_N : \mathbb{R}^N \to \mathbb{R}\) given by
\[U_N(x) := \sum_k \left(\frac{1}{4}x_k^4 - \frac{1}{2}(1 + m)x_k^2 + \frac{1}{4}\right). \tag{43}\]

We shall consider for each \(h > 0\) and \(N \in \mathbb{N}'\) the perturbed probability measures \(\tilde{\gamma}_h\) and \(\tilde{\gamma}_{h,N}\), defined respectively on the Borel sets of \(L^2(T)\) and \(\mathbb{R}^N\), and given by
\[\tilde{\gamma}_h(d\xi) := \tilde{Z}_h^{-1} e^{-\frac{U_N(x)}{h}} \gamma_h(dx), \quad \tilde{\gamma}_{h,N}(dx) := \tilde{Z}_{h,N}^{-1} e^{-\frac{U_N(x)}{hN}} \gamma_{h,N}(dx), \]
where \(\tilde{Z}_h\) and \(\tilde{Z}_{h,N}\) are normalization constants.

**Remark 6.1.** An explicit Gaussian computation shows that \(\xi \mapsto \int_T \xi^4(s) ds\) is in \(L^1(d\gamma_h)\) for every \(h > 0\). This implies, by Jensen’s inequality, \(\tilde{Z}_h > 0\) and thus \(\tilde{\gamma}_h\) is well-defined. It is well-known that not only \(\gamma_h(L^2(T)) = 1\), but even \(\gamma_h(C^\alpha(T)) = 1\) for \(\alpha < \frac{1}{2}\), where \(C^\alpha(T)\) is the space of \(\alpha\)-Hölder continuous functions on \(T\).

Now fix \(F \in \mathcal{FC}_b^\infty(L^2(T))\). Then by definition of \(\mathcal{FC}_b^\infty(L^2(T))\) there exist \(n \in \mathbb{N}, y = (y_1, \ldots, y_n)\) with \(y_i \in L^2(T)\) and \(f \in C_b^\infty(\mathbb{R}^n)\) such that \(F(\xi) = f([\xi, y])\) for every \(\xi \in L^2(T)\), where we have set for short
\[\langle \xi, y \rangle := \langle (\xi, y_1)_{L^2(T)}, \ldots, (\xi, y_n)_{L^2(T)} \rangle.\]
In particular $F \in C^1(L^2(\mathbb{T}))$ and, denoting by $DF : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ the gradient of $F$, we have $\forall \xi \in L^2(\mathbb{T})$

$$\|DF(\xi)\|_{L^2(\mathbb{T})}^2 = \sum_{j,t=1}^n \partial_j f([\xi,y]) \partial_t f([\xi,y]) \langle y_j,y_t \rangle_{L^2(\mathbb{T})}. \quad (44)$$

In order to perform the approximation procedure it will be convenient to introduce for each $N \in \mathbb{N}$ the function $\tilde{f}_N : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\tilde{f}_N(x) := F(\eta^N(x)) = f([\eta^N(x),y]), \quad (45)$$

where $\eta^N : \mathbb{R}^N \to L^2(\mathbb{T})$ and $\eta^N(x)$ is given by

$$s \mapsto \frac{1}{N} \sum_{|j|,|t| \leq \frac{1}{N}(N-1)} xe^{i2\pi j(s-t)\frac{N}{N}}. \quad (46)$$

Before we can prove the following approximation result, we introduce the covariance functions

$$Q(s,t) := \int_{L^2(\mathbb{T})} \xi(s)\xi(t) \gamma_h(d\xi),$$

$$Q^N(s,t) := \int_{L^2(\mathbb{T})} X^{(N)}_{k_s}(\xi)X^{(N)}_{k_t}(\xi) \gamma_h(d\xi),$$

$$\tilde{Q}^N(s,t) := \int_{L^2(\mathbb{T})} X^{(N)}_{k_s}(\xi)\xi(t) \gamma_h(d\xi),$$

defined for all $s,t \in \mathbb{T}$ where $k_s := \text{int}(N \cdot s)$ is the next lattice point to $s$. Note that $Q^N$ is just the step function associated to the discrete covariance $\int_{L^2(\mathbb{T})} X^{(N)}_{k_s}(\xi)X^{(N)}_{k_t}(\xi) \gamma_h(d\xi)$. Using the boundedness and convergence of $\sigma_N$, a simple argument shows that $Q^N, \tilde{Q}^N$ are uniformly bounded and $Q = \lim_{N \to \infty} Q^N = \lim_{N \to \infty} \tilde{Q}^N$ pointwise. Furthermore, we recall that we can express higher momenta of Gaussian random variables via a polynomial of their covariances. Particularly, there exists a polynomial $p(x,y,z)$ such that

$$\int_{L^2(\mathbb{T})} \big(\frac{1}{4}\phi^4 - \frac{1}{2}(m+1)\phi^2 \big) \big(\frac{1}{4}\psi^4 - \frac{1}{2}(m+1)\psi^2 \big) \gamma_h(d\xi)$$

can be expressed by

$$p \left( \int_{L^2(\mathbb{T})} \phi \cdot \gamma_h(d\xi), \int_{L^2(\mathbb{T})} \psi \cdot \gamma_h(d\xi), \int_{L^2(\mathbb{T})} \phi \cdot \psi \cdot \gamma_h(d\xi) \right),$$

for all Gaussian distributed random variables $\phi, \psi$.

**Lemma 6.2.** Let $U_N$, $X^{(N)}$ and $\eta^{(N)}$ be defined respectively as in (43), (42) and (46). Then, in the limit $N \to \infty$, we have

(i) $\frac{U_N X^{(N)}}{N} \to U(\xi) \text{ in } L^2(d\gamma_h)$,

(ii) $\langle \eta^{(N)}(X^{(N)}(\xi)), y \rangle_{L^2(\mathbb{T})} \to \langle \xi, y \rangle_{L^2(\mathbb{T})}$ for all $\xi, y \in L^2(\mathbb{T})$.

**Proof.** For a series $\{a_k : |k| \leq \frac{1}{N}(N-1)\}$, we have the identity $\frac{1}{N} \sum_{|k| \leq \frac{1}{N}(N-1)} a_k = \int_{\mathbb{T}} a_k d\xi$, hence we obtain

$$\int_{L^2(\mathbb{T})} U(\xi) - \frac{U_N(X^{(N)}(\xi))}{N} \gamma_h(d\xi) =$$

$$= \int_{L^2(\mathbb{T})} \left| \int_{\mathbb{T}} \frac{1}{4} \xi^4(s) - \frac{1}{2}(m+1)\xi^2(s) - \frac{1}{4}(X^{(N)}_{k_s}(\xi))^4 + \frac{1}{2}(m+1)(X^{(N)}_{k_s}(\xi))^2 ds \right|^2 \gamma_h(d\xi)$$
Rewriting this expression in terms of the covariances $Q, Q^N, \tilde{Q}^N$ and polynomial $p(x, y, z)$, yields
\[
\int_{L^2(\mathbb{T})} \int_{L^2(\mathbb{T})} \left( p(Q(s, s), Q(t, t), Q(s, t)) - 2p(Q(s, s), Q^N(t, t), \tilde{Q}^N(s, t)) 
+ p(Q^N(s, s), Q^N(t, t), Q^N(s, t)) \right) ds \, dt,
\]
where we have expanded the square and changed the order of integration. Because of $Q = \lim_{N \to \infty} Q^N = \lim_{N \to \infty} \tilde{Q}^N$, statement (i) follows by dominated convergence. In order to prove (ii) note that
\[
\langle \eta^{(N)}(X^{(N)}(\xi)), y \rangle_{L^2(\mathbb{T})} = \frac{1}{N} \int_{\mathbb{T}} \sum_{|j|, |l| \leq \frac{1}{2}(N-1)} X^{(N)}_\ell(\xi) e^{i2\pi j(s-\xi)} y(s) ds = 
\frac{1}{N} \sum_{|j|, |l|, |p| \leq \frac{1}{2}(N-1)} e^{-i2\pi \frac{\ell}{N}(j+p)} \sigma_N(p) \tilde{\xi}(\xi, p) \tilde{y}(j) = 
\sum_{|j| \leq \frac{1}{2}(N-1)} \sigma_N(j) \tilde{\xi}(j) \tilde{y}(j) \xrightarrow{N \to \infty} \sum_{j \in \mathbb{Z}} \tilde{\xi}(j) \tilde{y}(j) = \langle \xi, y \rangle_{L^2(\mathbb{T})}.
\]

The next lemma provides for fixed $h > 0$ the crucial approximation properties for the objects appearing in the functional inequality which defines the spectral gap $\lambda^{(1)}_h$.

**Lemma 6.3.** Let $F \in FC_b^\infty(L^2(\mathbb{T}))$ and consider for each $N \in \mathbb{N}$ the corresponding $\tilde{f}_N \in C_b^\infty(\mathbb{R}^N)$ defined in (45). Then for each $h > 0$
\[
\lim_{N \to \infty} \left[ \int_{L^2(\mathbb{T})} \|DF(\xi)\|^2_{L^2(\mathbb{T})} \gamma_h(d\xi) - N \int_{\mathbb{R}^N} |\nabla \tilde{f}_N(x)|^2 \gamma_{h,N}(dx) \right] = 0. \tag{47}
\]
Moreover, for each $h > 0$, we have
\[
\lim_{N \to \infty} \left[ \int_{L^2(\mathbb{T})} |F(\xi)|^2 \gamma_h(d\xi) - \int_{\mathbb{R}^N} |\tilde{f}_N(x)|^2 \gamma_{h,N}(dx) \right] = 0,
\]
\[
\lim_{N \to \infty} \left[ \int_{L^2(\mathbb{T})} F(\xi) \gamma_h(d\xi) - \int_{\mathbb{R}^N} \tilde{f}_N(x) \gamma_{h,N}(dx) \right] = 0.
\]
**Proof.** We shall prove only (47), since the statements (i) and (ii) are proven similarly, with even some simplifications. Fix $h > 0$ and $F \in FC_b^\infty(L^2(\mathbb{T}))$. Then
\[
\int_{L^2(\mathbb{T})} \|DF(\xi)\|^2_{L^2(\mathbb{T})} \gamma_h(d\xi) - N \int_{\mathbb{R}^N} |\nabla \tilde{f}_N(x)|^2 \gamma_{h,N}(dx) = 
\mathcal{R}_1(N) + \mathcal{R}_2(N),
\]
where
\[
\mathcal{R}_1(N) := \int_{L^2(\mathbb{T})} \left[ \tilde{Z}_h^{-1} \|DF(\xi)\|^2_{L^2(\mathbb{T})} - \tilde{Z}_h^{-1} \gamma_{h,N} |\nabla \tilde{f}_N(X^{(N)}(\xi))| \right] e^{-\frac{U(\xi)}{h}} \gamma_h(d\xi),
\]
\[
\mathcal{R}_2(N) := \tilde{Z}_h^{-1} \int_{L^2(\mathbb{T})} \gamma_{h,N} |\nabla \tilde{f}_N(X^{(N)}(\xi))| \left[ e^{-\frac{U(\xi)}{h}} - e^{-\frac{U(N)(X^{(N)}(\xi))}{h}} \right] \gamma_h(d\xi).
\]
In order to prove the convergence of the first term, we shall show that
\[ \lim_{N \to \infty} N|\nabla \tilde{f}_N(X^{(N)}(\xi))|^2 = \|DF(\xi)\|^2_{L^2(\mathbb{T})} \quad \text{for all } \xi \in L^2(\mathbb{T}). \] (48)

The convergence \( \mathcal{R}_1(N) \to 0 \) will then follow by dominated convergence. To prove (48) observe that for every \( x \in \mathbb{R}^N \) we have
\[ |\nabla \tilde{f}_N(x)|^2 = \sum_{j,\ell=1}^n \partial_j f([\eta^{(N)}(x),y]) \partial_\ell f([\eta^{(N)}(x),y]) \frac{1}{N^2} \sum_{|k| \leq \frac{1}{2}(N-1)} \tilde{y}_j(k) \tilde{y}_\ell(k), \]
where for generic \( y \in L^2(\mathbb{T}) \), every \( N \in \mathbb{N} \) and \( k \in \mathbb{Z} : |k| \leq \frac{1}{2}(N-1) \) we define \( \tilde{y}(k) = \tilde{y}^{(N)}(k) := \sum_{|p| \leq \frac{1}{2}(N-1)} \int_{\mathbb{T}} y(s) e^{i2\pi p(s-k/N)} ds \). It follows that for every \( \xi \in L^2(\mathbb{T}) \) we have
\[ N|\nabla \tilde{f}_N(X^{(N)}(\xi))|^2 = \sum_{j,\ell=1}^n \partial_j f([\eta^{(N)}(X^{(N)}(\xi)),y]) \partial_\ell f([\eta^{(N)}(X^{(N)}(\xi)),y]) \frac{1}{N} \sum_{|k| \leq \frac{1}{2}(N-1)} \tilde{y}_j(k) \tilde{y}_\ell(k). \]

Statement (48) follows then from the convergence result of Lemma 6.2 (ii), the representation of \( \|DF(\xi)\|^2_{L^2(\mathbb{T})} \) given in (44) and the following computation:
\[ \frac{1}{N} \sum_{|k| \leq N-1} \tilde{y}_j(k) \tilde{y}_\ell(k) = \frac{1}{N} \sum_{|k| \leq \frac{1}{2}(N-1)} e^{i2\pi \frac{k}{N}(q+p)} \sum_{|p| \leq \frac{1}{2}(N-1)} \tilde{y}_j(-p) \tilde{y}_\ell(-q) = \sum_{|p| \leq \frac{1}{2}(N-1)} \tilde{y}_j(p) \tilde{y}_\ell(-p) \xrightarrow{N \to \infty} \sum_{p \in \mathbb{Z}} \tilde{y}_j(p) \tilde{y}_\ell(-p) = \int_{\mathbb{T}} y_j(s)y_\ell(s) ds. \]

Note that \( N|\nabla \tilde{f}_N(X^{(N)}(\xi))|^2 \) is uniformly bounded for \( F \in FC_b^\infty(L^2(\mathbb{T})) \). Regarding the convergence of the second term \( \mathcal{R}_2(N) \), we use this fact together with Lemma 6.2 (i) to obtain \( \tilde{Z}_{h,N} \to \tilde{Z}_h \) as well as \( \mathcal{R}_2(N) \to 0 \).

We can now prove the lower bound in the statement of Theorem 2.3.

**Proposition 6.4** (Asymptotic lower bound on \( \lambda_h^{(1)} \)). There exist \( h_0, C > 0 \) such that for every \( h \in (0, h_0] \) and every \( F \in FC_b^\infty(L^2(\mathbb{T})) \) satisfying \( \int_{L^2(\mathbb{T})} |F(\xi)|^2 \tilde{\gamma}_h(d\xi) = 1 \) and \( \int_{L^2(\mathbb{T})} F(\xi) \tilde{\gamma}_h(d\xi) = 0 \), we have
\[ h \int_{L^2(\mathbb{T})} \|DF(\xi)\|^2_{L^2(\mathbb{T})} \tilde{\gamma}_h(d\xi) \geq \frac{\sinh(\pi \sqrt{2\mu-1})}{\pi \sin(\pi \sqrt{\mu-1})} e^{\frac{\pi}{4\sqrt{\mu}}} (1 - Ch). \]

**Proof.** Take \( h_0 > 0 \) as in the statement of Proposition 2.2. Fix \( h \in (0, h_0] \) and \( F \in FC_b^\infty(L^2(\mathbb{T})) \) such that \( \int_{L^2(\mathbb{T})} |F(\xi)|^2 \tilde{\gamma}_h(d\xi) = 1 \) and \( \int_{L^2(\mathbb{T})} F(\xi) \tilde{\gamma}_h(d\xi) = 0 \). It follows then from Lemma 6.3 that
\[ h \int_{L^2(\mathbb{T})} \|DF(\xi)\|^2_{L^2(\mathbb{T})} \tilde{\gamma}_h(d\xi) = hN \int_{\mathbb{R}^N} |\nabla \tilde{f}(x)|^2 \tilde{\gamma}_{h,N}(dx) + o(1) \geq \lambda_h^{(1)}(1 + o(1)) + o(1), \]

where
\[ \lambda_h^{(1)} = \frac{\sinh(\pi \sqrt{2\mu-1})}{\pi \sin(\pi \sqrt{\mu-1})} e^{\frac{\pi}{4\sqrt{\mu}}} (1 - Ch). \]
where \( o(1) \) denotes a possibly on \( h \) dependent sequence which vanishes in the limit \( N \to \infty \). Proposition 2.2 implies that there exists \( C > 0 \) such that for all \( N \in \mathbb{N}' \)

\[
\lambda^{(1)}_{h,N} \geq p(N) e^{-\frac{1}{hN}} (1 - Ch),
\]

which together with (49) gives

\[
h \int_{L^2(\mathbb{T})} \|DF(\xi)\|^2_{L^2(\mathbb{T})} \tilde{\gamma}_h(d\xi) \geq p(N) e^{-\frac{1}{hN}} (1 - Ch) (1 + o(1)) + o(1). \tag{50}
\]

Passing to the limit \( N \to \infty \) on the right hand side of (50) and recalling (13) finishes the proof.

The proof of Theorem 2.3 is completed by the following proposition.

**Proposition 6.5** (Asymptotic upper bound on \( \lambda^{(1)}_h \)). There exist \( h_0, C > 0 \) such that for every \( h \in (0, h_0] \) we have

\[
\lambda^{(1)}_h \leq \frac{\sinh(\pi \sqrt{2\mu - 1})}{\pi \sinh(\pi \mu^{-1})} e^{-\frac{1}{hN}} (1 + Ch).
\]

**Proof.** We consider for each \( h > 0 \) and \( N \in \mathbb{N}' \) the \( FC_k^\infty(L^2(\mathbb{T})) \) function

\[
\chi_h(\xi) := \chi_h,0(\int_T \xi(s) ds),
\]

where we define

\[
\chi_h,0(x) := \frac{2}{\sqrt{2\pi h}} \int_0^x \frac{\gamma_2}{e^{\frac{1}{2}}} ds.
\]

Because of symmetry, we obtain \( \int_{L^2(\mathbb{T})} \chi_h(\xi) \tilde{\gamma}_h(d\xi) = 0 \), and therefore the proof is complete, if we can show

\[
h \int_{L^2(\mathbb{T})} \|D\chi_h(\xi)\|^2_{L^2(\mathbb{T})} \tilde{\gamma}_h(d\xi) \leq \frac{\sinh(\pi \sqrt{2\mu - 1})}{\pi \sinh(\pi \mu^{-1})} e^{-\frac{1}{hN}} (1 + Ch) \int_{L^2(\mathbb{T})} |\chi_h(\xi)|^2 \tilde{\gamma}_h(d\xi).
\]

A straightforward computation yields \( \tilde{\chi_h} := \chi_h(\eta^{(N)}(x)) = \chi_h,0(x) \), which are exactly the approximate eigenfunctions \( \chi_{h,N} \) introduced in Definition 3.10 in [21]. Moreover it was shown in [21, Lemma 3.12 and Prop. 3.13] by computing the relevant Laplace asymptotics that for all \( N \in \mathbb{N}' \) and \( h \in (0, 1] \)

\[
hN \int_{\mathbb{R}^N} |\nabla \chi_{h,N}(x)|^2 \tilde{\gamma}_{h,N}(dx) \leq p(N) e^{-\frac{1}{hN}} (1 + Ch) \int_{\mathbb{R}^N} |\chi_{h,N}(x)|^2 \tilde{\gamma}_{h,N}(dx).
\]

Together with (13) and Lemma 6.3, this gives us the desired statement. \( \square \)

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