Surface subgroups of graph products of groups

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Abstract. A graph product kernel means the kernel of the natural surjection from a graph product to the corresponding direct product. We prove that a graph product kernel of countable groups is special, and a graph product of finite or cyclic groups is virtually cocompact special in the sense of Haglund and Wise. The proof of this yields conditions for a graph over which the graph product of arbitrary nontrivial groups (or some cyclic groups, or some finite groups) contains a hyperbolic surface group. In particular, the graph product of arbitrary nontrivial groups over a cycle of length at least five, or over its opposite graph, contains a hyperbolic surface group. For the case when the defining graphs have at most seven vertices, we completely characterize right-angled Coxeter groups with hyperbolic surface subgroups.

1. Introduction

By a graph, we mean a simplicial 1–complex. Throughout this paper, we will let $\Gamma$ be a finite graph. The vertex set and the edge set of $\Gamma$ are denoted as $V(\Gamma)$ and $E(\Gamma)$, respectively. Suppose $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$ is a collection of groups indexed by $V(\Gamma)$. We define $GP(\Gamma, \mathcal{G})$ to be the free product of the groups in $\mathcal{G}$ quotient by the normal closure of the set $\{[g,h] : g \in G_u, h \in G_v \text{ for some } \{u,v\} \in E(\Gamma)\}$. We call $GP(\Gamma, \mathcal{G})$ as the graph product of the groups in $\mathcal{G}$ over $\Gamma$, and each $G_v$ as a vertex group of $GP(\Gamma, \mathcal{G})$. The kernel of the natural surjection $GP(\Gamma, \mathcal{G}) \to \prod_{v \in V(\Gamma)} G_v$ is called as the graph product kernel of $\mathcal{G}$ over $\Gamma$ and denoted as $KP_0(\Gamma, \mathcal{G})$.

By a hyperbolic surface group, we mean the fundamental group of a closed hyperbolic surface. For abbreviation, we let $S$ be the class of groups that contain hyperbolic surface groups. Our main question is the following.

**Question 1.** For which graph $\Gamma$ and which collection of groups $\mathcal{G}$, is $GP(\Gamma, \mathcal{G})$ in $S$?

Let us briefly explain some motivation for Question 1. Gromov asked the following intriguing question [21, p.277].

**Question 2.** Is every one-ended word-hyperbolic group in $S$?

Question 2 has been answered for only a few cases, all affirmatively. These include graphs of free groups with cyclic edge groups with nontrivial second rational homology [5], doubles of rank-two free groups symmetrically amalgamated along cyclic edge groups [19, 34, 33], and most remarkably, the fundamental groups of closed hyperbolic 3–manifolds [29]. We note that these groups are all virtually cocompact special in the sense that each one is virtually the fundamental group of a compact special cube complex [24, 25] see Definition [10]. So very broadly, we may ask under which conditions a one-ended, virtually cocompact special group belongs to $S$. On the other hand,

**Theorem 3.**

1. Graph product kernels of countable groups are special.
2. Graph products of finite or cyclic groups are virtually cocompact special.
The proof of Theorem 3 will reveal inclusion relations between certain subgroups of graph products, and so, provide an important tool for this paper. In some sense, a graph product kernel will “remember” only the order of each vertex group, while “forgetting” the group structure of it.

For $2 \leq m \leq \infty$, we let $GP_m(\Gamma)$ denote the graph product of cyclic groups of order $m$ over $\Gamma$. We write $A(\Gamma) = GP_\infty(\Gamma)$ and $C(\Gamma) = GP_2(\Gamma)$. We will call $A(\Gamma)$ and $C(\Gamma)$ as a right-angled Artin group and a right-angled Coxeter group on $\Gamma$, respectively [2]. Question 1 has a close relation to the question of whether $A(\Gamma) \in S$ or $C(\Gamma) \in S$ as described below.

**Theorem 4.** (1) We have $C(\Gamma) \in S$ if and only if the graph product of arbitrary nontrivial groups over $\Gamma$ is in $S$. (2) We have $A(\Gamma) \in S$ if and only if the graph product of some cyclic groups over $\Gamma$ is in $S$. (3) We have $[A(\Gamma), A(\Gamma)] \in S$ if and only if the graph product of some finite groups over $\Gamma$ is in $S$, if and only if $GP_m(\Gamma) \in S$ for some $2 \leq m < \infty$.

We denote by $C_m$ the cycle of length $m$. The opposite graph $\Gamma^{opp}$ of $\Gamma$ is defined by $V(\Gamma^{opp}) = V(\Gamma)$ and $E(\Gamma^{opp}) = \{(u, v) : u$ and $v$ are non-adjacent vertices of $\Gamma\}$. If there is a finite sequence of edge-contractions [12, p.20] from $\Gamma_1^{opp}$ to $\Gamma_2^{opp}$, we say $\Gamma_1$ co-contracts onto $\Gamma_2$. In [31], it was shown that a co-contraction $\Gamma_1 \rightarrow \Gamma_2$ induces an embedding $A(\Gamma_2) \hookrightarrow A(\Gamma_1)$.

**Theorem 5.** Suppose $\Gamma_1$ and $\Gamma_2$ are finite graphs such that $\Gamma_1$ co-contracts onto $\Gamma_2$. If $2 \leq m \leq \infty$, then $GP_m(\Gamma_2)$ embeds into $GP_m(\Gamma_1)$.

It is well-known that $C(C_m)$ and $A(C_m)$ are in $S$ for $m \geq 5$; see [37]. Also, it was shown that $A(C_m^{opp}) \in S$ for $m \geq 5$ in [31, 9]; see [3] for an alternative proof. Using Theorem 5 we generalize these results.

**Corollary 6 ([30, 27], cf. [15]).** For $m \geq 5$, the graph product of arbitrary nontrivial groups over $C_m$ is in $S$.

**Corollary 7.** For $m \geq 5$, the graph product of arbitrary nontrivial groups over $C_m^{opp}$ is in $S$.

Suppose $X \subseteq V(\Gamma)$. The induced subgraph of $\Gamma$ on $X$ is the maximal subgraph of $\Gamma$ whose vertex set is $X$. If $\Lambda$ is isomorphic to an induced subgraph of $\Gamma$, we simply write $\Lambda \leq \Gamma$ and say that $\Lambda$ has an induced $\Lambda$. We also use the notation $H \leq G$ for two groups $G$ and $H$, if there exists an embedding from $H$ into $G$. We say $\Gamma$ is weakly chordal if $\Gamma$ does not contain an induced $C_m$ or $C_m^{opp}$ for $m \geq 5$. For each finite graph $\Gamma_1$, there exists a (algorithmically constructible) graph $\Gamma_2 \supseteq \Gamma_1$, such that $|C(\Gamma_2) : A(\Gamma_1)| < \infty$ [11]. In particular, $A(\Gamma_1) \in S$ if and only if $C(\Gamma_2) \in S$. Hence, the classification of $\Gamma$ satisfying $C(\Gamma) \in S$ is presumably “harder” than that of $\Gamma$ satisfying $A(\Gamma) \in S$. Complete classification of the graphs $\Gamma$ with $|V(\Gamma)| \leq 8$ and $A(\Gamma) \in S$ is given in [9]. We will classify all the graphs $\Gamma$ with $|V(\Gamma)| \leq 7$ and $C(\Gamma) \in S$.

**Theorem 8.** Suppose $\Gamma$ has at most seven vertices. Then $C(\Gamma) \in S$ if and only if $\Gamma$ is not weakly chordal.

In particular, the proof of Theorem 3 will exhibit graphs $\Gamma$ such that $A(\Gamma) \in S$ and $C(\Gamma) \not\in S$. When $\Gamma$ has more than seven vertices, $C(\Gamma) \in S$ does not necessarily imply that $\Gamma \geq C_m$ or $\Gamma \geq C_m^{opp}$ for some $m \geq 5$; see Remark 5. Lastly, we will make an observation that the class of finitely generated groups that “conform” to an affirmative answer to Question 2 is closed under graph products.

Here is the organization of this paper. In Section 2 we summarize basic facts on cube complexes and label-reading maps. We describe two special cube complexes whose fundamental groups are specific subgroups of graph products and use these complexes to prove Theorems 3, 4 and Corollary 5 in Section 3. Section 4 introduces a general, combinatorial group theoretic lemma, which yields nontrivial embeddings between graph products. Theorem 5 and Corollary 6 will follow. In Section 5 we investigate seven-vertex graphs and prove Theorem 8. We discuss a role of graph products in relation to Question 2 in Section 6.
Note on the literature. (1) Haglund has shown that a graph product of finite groups is virtually cocompact special [23] by considering a certain cube complex for its graph product kernel; see also [8]. The cube complex discussed in Section 3 is not a generalized version of his complex.

(2) While it is unknown whether Coxeter groups are virtually cocompact special, they are virtually special [25, Problem 9.2, Theorem 1.2]. This already implies that graph products of finite or cyclic groups are virtually special, since these graph products embed into Coxeter groups. Note that for Coxeter groups, Question 2 has an affirmative answer as well [18].

(3) Holt and Rees constructed a complex \( Z \) for a graph product kernel of cyclic groups [27, Theorem 3.1]. Their complex \( Z \) is different from ours in that \( Z \) is not cubical and not necessarily aspherical. Theorem 3(1) can also be proved using the construction of Holt and Rees, combined with Droms’ description of a complex for \([C(\Gamma), C(\Gamma)]\); see [14].

(4) Corollary 3 was first proved in the author’s Ph.D thesis [30, Theorem 3.6], but never published by the author. After that, the same result was proved again by Futer and Thomas (for \( m \geq 6 \)) [15, Corollary 1.3] and by Holt and Rees [27, Theorem 3.1].

(5) Theorem 5 and Corollary 7 were proved in the author’s Ph.D thesis [30, Corollaries 4.11 and 4.12], but never published anywhere. We will give new accounts of these results.

The methods presented in this article do not depend on the above mentioned works.

Acknowledgement. I am grateful to Andrew Casson for his guidance. I thank Frédéric Haglund for an inspirational conversation.

2. Preliminary on cube complexes and label-reading maps

2.1. Local isometries and special cube complexes. By a cube complex, we mean a CW-complex obtained from unit Euclidean cubes of various dimensions by isometrically gluing some of the faces. A flag complex is a simplicial complex \( L \) such that each complete subgraph in \( L^{(1)} \) is the 1–skeleton of some simplex in \( L \). We say a cube complex \( X \) is nonpositively curved, or simply NPC, if the link of each vertex is a flag complex; this is equivalent to saying that the piecewise Euclidean length metric induced on the universal cover of \( X \) is \( \text{CAT}(0) \) [21].

We denote by \( X_{\Gamma} \) the Salvetti complex of \( A(\Gamma) \) [7]. This means that \( X_{\Gamma}^{(2)} \) is the presentation 2-complex of \( A(\Gamma) \), and for each maximal complete subgraph \( K \) of \( \Gamma \) with \( k \) vertices, a \( k \)-torus \( T \) is glued to \( X_{\Gamma}^{(2)} \) so that the 1–skeleton of \( T \) is the bouquet of the circles corresponding to the vertices of \( K \). Note that \( X_{\Gamma} \) is an NPC cube complex such that \( \pi_1(X_{\Gamma}) \cong A(\Gamma) \); see [7].

If \( X \) is a cube complex and \( v \) is a vertex of \( X \), we denote the link of \( X \) at \( v \) by \( \text{Lk}(X; v) \). Let us consider a combinatorial map \( f : X \to Y \) between cube complexes \( X \) and \( Y \). The map \( f \) induces a simplicial map \( \text{Lk}(f; v) : \text{Lk}(X; v) \to \text{Lk}(Y; f(v)) \) for each vertex \( v \) of \( X \). Following [6], we call \( f \) as a local isometry if

(i) \( \text{Lk}(f; v) \) is injective, and

(ii) the image of \( \text{Lk}(f; v) \) is a full subcomplex of \( \text{Lk}(Y; f(v)) \).

Lemma 9 [6, 11]. Suppose \( X \) and \( Y \) are cube complexes and \( f : X \to Y \) is a combinatorial map. If \( Y \) is NPC and \( f \) is a local isometry, then \( X \) is also NPC and \( f \) is \( \pi_1 \)-injective.

Definition 10 [21, 25].

(1) A cube complex \( X \) is called special if \( X \) combinatorially maps to a Salvetti complex by a local isometry.

(2) A group \( G \) is special if \( G \cong \pi_1(X) \) for some special cube complex \( X \). Furthermore, if \( X \) can be chosen to be compact, then we say \( G \) is cocompact special.

We remark that Definition 10 (1) is different from, but equivalent to, the original definition in [24]; see [25, Proposition 3.2]. For a group theoretic property \( P \), we say a group \( G \) is virtually \( P \) if a finite-index subgroup of \( G \) is \( P \). Virtually special groups are of particular interest in 3–manifold theory [1].
2.2. **Label-reading maps.** By a *curve* on a surface, we will mean a simple closed curve or a properly embedded arc. Let $S$ be a compact surface possibly with boundary. Suppose $\mathcal{V}$ is a finite set of transversely intersecting curves on $S$ and $\lambda: \mathcal{V} \to V(\Gamma)$ is a map such that two curves $\alpha$ and $\beta$ in $\mathcal{V}$ are intersecting only if $\{\lambda(\alpha), \lambda(\beta)\} \in E(\Gamma)$. Following [10], we say that $(\mathcal{V}, \lambda)$ is a label-reading pair on $S$ with the underlying graph $\Gamma$; and for each $\alpha \in \mathcal{V}$, we call $\lambda(\alpha)$ as the label of $\alpha$. If an arc $\alpha$ is labeled by $a \in V(\Gamma)$, we say $\alpha$ is an $a$–arc. For each oriented path $\gamma$ transverse to $\mathcal{V}$, we follow $\gamma$ and read off the labels of the curves in $\mathcal{V}$ that intersect $\gamma$. The word $w(\gamma)$ thus obtained will be called the label-reading of $\gamma$ with respect to $(\mathcal{V}, \lambda)$. The word $w(\gamma)$ represents an element of $C(\Gamma)$. If there exists a group homomorphism $\phi: \pi_1(S) \to C(\Gamma)$ satisfying that $\phi([\gamma]) = w(\gamma)$ for each $[\gamma] \in \pi_1(S)$, we call $\phi$ as a label-reading map with respect to $(\mathcal{V}, \lambda)$.

Recall that a word $w$ representing an element in $C(\Gamma)$ is reduced if no shorter word represents the same element. It is cyclically reduced if $w$ and each of its cyclic conjugations are reduced. If a curve $\gamma$ on a compact surface $S$ is homotopic to a subset of $\partial S$ by a homotopy fixing $\partial \gamma$, then we say $\gamma$ is homotopic into $\partial S$.

Crisp and Wiest proved that the fundamental group of a closed hyperbolic surface $S$ embeds into some right-angled Artin group if and only if $\chi(S) \neq -1$ [10]. A critical tool for the proof was the realization of an arbitrary group homomorphism $\phi: \pi_1(S) \to A(\Gamma)$ as a label-reading map (using $A(\Gamma)$ instead of $C(\Gamma)$). The following is a simple variation of the results in [10] combined with [32].

**Theorem 11 ([10, 32]).** Let $S$ be a compact surface.

1. Suppose $(\mathcal{V}, \lambda)$ is a label-reading pair on $S$ with the underlying graph $\Gamma$. Then for each choice of the base point of $S$, there uniquely exists a label-reading map $\phi: \pi_1(S) \to C(\Gamma)$ with respect to $(\mathcal{V}, \lambda)$.
2. Conversely, every group homomorphism $\phi: \pi_1(S) \to C(\Gamma)$ can be realized as a label-reading map with respect to some label-reading pair $(\mathcal{V}, \lambda)$ that has the underlying graph $\Gamma$.
3. Possibly after composing $\phi$ with an inner automorphism of $C(\Gamma)$, we can choose $(\mathcal{V}, \lambda)$ in (2) further satisfying the following:
   i. curves in $\mathcal{V}$ are minimally intersecting;
   ii. curves in $\mathcal{V}$ are neither null-homotopic nor homotopic into $\partial S$;
   iii. for each component $\partial_i S$ of $\partial S$, the label-reading $w(\partial_i S)$ is cyclically reduced.

**Proof.** (1) and (2) are proved in [10] for right-angled Artin groups. The proofs for right-angled Coxeter groups are very similar, except that we now allow $\mathcal{V}$ to contain orientation-reversing closed curves and also that curves in $\mathcal{V}$ are not assigned with transverse orientations. (3) is obtained by lexicographically minimizing the complexity ($\vert \{\cup \mathcal{V} \cap \partial S\}, \mathcal{V}, \sum_{\alpha \neq \beta \in \mathcal{V}} \vert \alpha \cap \beta \vert$), possibly after changing $\phi$ by $\text{Inn}(C(\Gamma))$; see [10] and [32] for discussion on the same technique.

3. **Special cube complexes for certain subgroups of graph products**

In this section, we write $V(\Gamma) = \{1, 2, \ldots, n\}$ and assume $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a collection of groups indexed by $V(\Gamma)$. Choose $k$ such that $|G_i|$ is finite if and only if $1 \leq i \leq k$. Set $p_i: GP(\Gamma, \mathcal{G}) \to G_i$ to be the natural projection map for $i = 1, 2, \ldots, n$. Recall that we have defined the graph product kernel as $KP_0(\Gamma, \mathcal{G}) = \cap_{1 \leq i \leq n} \ker p_i$. We also define $KP_f(\Gamma, \mathcal{G}) = \cap_{1 \leq i \leq k} \ker p_i$. Note that $KP_0(\Gamma, \mathcal{G}) \leq KP_f(\Gamma, \mathcal{G}) \leq GP(\Gamma, \mathcal{G})$ and that $GP(\Gamma, \mathcal{G})/KP_f(\Gamma, \mathcal{G}) \cong \prod_{1 \leq i \leq k} G_i$ is finite. If all the groups in $\mathcal{G}$ are abelian, $KP_0(\Gamma, \mathcal{G})$ is the commutator subgroup of $GP(\Gamma, \mathcal{G})$.

Let us regard $\mathbb{R}^n$ as a cube complex whose vertices are the lattice points and whose 1–skeleton consists of the grid lines. We set $e_i$ to be the $i$–th standard basis vector. Following [37], $Y_{\Gamma} \leq \mathbb{R}^n$ is defined to be the lift of $X_{\Gamma} \subseteq (S^1)^n$ with respect to the covering $\mathbb{R}^n \to (S^1)^n$. Concretely, $(Y_{\Gamma})^{(1)} = (\mathbb{R}^n)^{(1)}$ and for each complete subgraph of $\Gamma$ having the vertex set $\{i_1, \ldots, i_k\}$, the
following collection of the unit $k$-cubes is contained in $Y_{\Gamma}$:

$$\left\{ \sum_{j=1}^{k} t_j e_{i_j} : t_j \in [0, 1] \right\} + \mathbb{Z}^n.$$ 

We define $Z_0(\Gamma, \mathcal{G}) = Y_{\Gamma} \cap \left( \prod_{i=1}^{k} [0, |G_i| - 1] \times \mathbb{R}^{n-k} \right) \subseteq \mathbb{R}^n$. We let $Z_{\Gamma}$ denote the preimage of $X_{\Gamma} \subseteq (S^1)^n$ with respect to the covering $\mathbb{R}^k \times (S^1)^{n-k} \to (S^1)^n$ and put $Z_f(\Gamma, \mathcal{G}) = Z_{\Gamma} \cap \left( \prod_{i=1}^{k} [0, |G_i| - 1] \times (S^1)^{n-k} \right)$. See Figure 1.

$$Z_0(\Gamma, \mathcal{G}) = Y_{\Gamma} \cap \left( \prod_{i=1}^{k} [0, |G_i| - 1] \times \mathbb{R}^{n-k} \right) \xrightarrow{Y_{\Gamma}} \mathbb{R}^n$$

$Z_f(\Gamma, \mathcal{G}) = Z_{\Gamma} \cap \left( \prod_{i=1}^{k} [0, |G_i| - 1] \times (S^1)^{n-k} \right) \xrightarrow{Z_{\Gamma}} \mathbb{R}^k \times (S^1)^{n-k}$

$$X_{\Gamma} \xrightarrow{} (S^1)^n$$

**Figure 1.** Cube complexes in Theorem 12. The horizontal maps are inclusions and the vertical maps are coverings.

It is well-known that $\pi_1(Y_{\Gamma}) = [A(\Gamma), A(\Gamma)]$ and that $\pi_1(Y_{\Gamma} \cap [0, 1]^n) \cong [C(\Gamma), C(\Gamma)]$; see [37, 4]. We generalize these observations as follows.

**Theorem 12.**

1. If $\mathcal{G}$ consists of countable groups, then $\pi_1(Z_0(\Gamma, \mathcal{G})) \cong KP_0(\Gamma, \mathcal{G})$.

2. If $\mathcal{G}$ consists of finite or cyclic groups, then $\pi_1(Z_f(\Gamma, \mathcal{G})) \cong KP_f(\Gamma, \mathcal{G})$.

**Proof.** (1) We use the notation described so far in this section. Choose the origin $O \in \mathbb{R}^n$ as the base point. Let $q_j : \mathbb{R}^n \to \mathbb{R}$ denotes the natural projection onto the $j$-th component. Enumerate $G_j = \{g_0^{(j)} = 1, g_1^{(j)}, g_2^{(j)}, \ldots, g_{|G_i|-1}^{(j)}\}$ for each $j \leq k$, and $G_j = \{\ldots, g_{-1}^{(j)}, g_0^{(j)} = 1, g_1^{(j)}, \ldots\}$ for each $j > k$. We let $\mathcal{H}$ be the homotopy classes of the edge-paths in $Z_0(\Gamma, \mathcal{G})$ starting from $O$.

We will first define a map $\Phi : \mathcal{H} \to GP(\Gamma, \mathcal{G})$. Let us consider an edge-path $\gamma : [0, l] \to Z_0(\Gamma, \mathcal{G})$ such that $\gamma(0) = O$ and $\gamma^{-1}(Z_0(\Gamma, \mathcal{G})) = Z \cap [0, l]$. For each $i = 1, 2, \ldots, l$, there uniquely exists $j$ such that $\gamma[i, i+1]$ is parallel to $\mathbb{R}e_j$; then we put $k = q_j \circ \gamma(i)$, $k' = q_j \circ \gamma(i + 1)$ and $x_i = (g_k^{(j)})^{-1}g_{k'}^{(j)} \in G_j$. Note that $|k - k'| = 1$. If $\gamma[i, i+1]$ and $\gamma[i + 1, i + 2]$ span a 2-cell in $Z_0(\Gamma, \mathcal{G})$, then the groups containing $x_i$ and $x_{i+1}$ commute; that is, $x_ix_{i+1} = x_{i+1}x_i$. So we can define a map $\Phi : \mathcal{H} \to GP(\Gamma, \mathcal{G})$ by setting $\Phi(\gamma) = x_1x_2\ldots x_l$.

Conversely, suppose $1 \neq g \in GP(\Gamma, \mathcal{G})$ is given. The normal form theorem for graph products [28] implies that we can write $g = g_{k_1}^{(j_1)}g_{k_2}^{(j_2)}\ldots g_{k_l}^{(j_l)}$ such that:

1. $k_i \neq 0$ for each $i = 1, 2, \ldots, l$;
2. $j_i \neq j_{i+1}$ for each $i = 1, 2, \ldots, l - 1$;
3. if $j_i = j_{i'}$, for some $i < i'$, then there exists $i < i'' < i'$ such that $\{j_i, j_{i''}\} \notin E(\Gamma)$.

Let us fix $i \in \{1, 2, \ldots, l\}$ and put $j = j_i$. There exist $k$ and $k'$ such that

$$g_k^{(j)} = \prod_{1 \leq t < i \text{ and } j_t = j} g_{k_t}^{(j)} \quad \text{and} \quad g_k^{(j)} = \prod_{1 \leq t \leq i \text{ and } j_t = j} g_{k_t}^{(j)} g_{k_t}^{(j)}.$$ 

We inductively define $\gamma_i$ to be the edge-path in $\mathbb{R}^n$ starting from the end point of $\gamma_{i-1}$ and changing only its $j$-th coordinate from $k$ to $k'$. We set $O$ as the initial point of $\gamma_1$. By defining $\Psi(g) =$
Let us use the notations in the proof of Theorem 12. Note that \( \Psi \) is well-defined since two normal forms differ only by a finite sequence of swapping certain consecutive terms, which can also be realized as a homotopy in \( \pi_0(\Gamma, G) \). It is clear that \( \Psi \) is the (set-theoretic) inverse of \( \Phi \).

Now if \( 1 \neq g \in KF_0(\Gamma, G) \), then we further have:

(i) \( \Pi_{1 \leq t \leq l} g_{k_t}^{j_t} \) is trivial in \( G_j \) for each \( j = 1, 2, \ldots, n \).

It is clear that \( \Psi \) restricts to a group isomorphism from \( KF_0(\Gamma, G) \) onto \( \pi_1(\pi_0(\Gamma, G)) \).

Proof. (2) In the case when \( 1 \neq g \in KF_1(\Gamma, G) \), we have the following condition instead of (iv) above:

(iv) \( \Pi_{1 \leq t \leq l} g_{k_t}^{j_t} \) is trivial in \( G_j \) for each \( j = 1, 2, \ldots, k \).

Hence, the covering \( Y_{\Gamma} \to Z_{\Gamma} \) projects \( \Psi(g) \) to a homotopy class of a loop in \( Z_f(\Gamma, G) \). We can check that the restriction of \( \psi \) onto \( KF(\Gamma, G) \) is a group isomorphism onto \( \pi_1(Z_f(\Gamma, G)) \). \( \square \)

Remark. The \( \mathcal{H} \) in the above proof depends only on the orders of the groups in \( G \). Hence, \( \Phi \) determines a bijection between \( GP(\Gamma, (\{Z_{|G_i|}: i = 1, 2, \ldots, n\}) \) and \( GP(\Gamma, G) \). In particular, if all the groups in \( G \) are infinite, then \( \Phi \) induces a bijection between \( A(\Gamma) \) and \( GP(\Gamma, G) \).

Example 13. Let us consider \( G = (\mathbb{Z}_3 * \mathbb{Z}_4) \times \mathbb{Z} \), which is regarded as the graph product in Figure 2 (a). Then \( KF_0(\Gamma, G) = [G, G] \) since the vertex groups are abelian. Also, \( KF_1(\Gamma, G) = \{\mathbb{Z}_3 * \mathbb{Z}_4, \mathbb{Z}_3 * \mathbb{Z}_4 \times \mathbb{Z} \} \) is a subgroup of \( G \) with index \( |\mathbb{Z}_3| * |\mathbb{Z}_4| = 12 \). If \( \Lambda \) is the graph shown in Figure 2 (b), then we have \( Z_0(\Gamma, \mathcal{G}) = \Lambda \times \mathbb{R} \) and \( Z_f(\Gamma, \mathcal{G}) = \Lambda \times S^1 \).

\[ \begin{array}{ccc} \mathbb{Z}_3 & \mathbb{Z} & \mathbb{Z}_4 \\ \end{array} \]

(a) \( G \) \hspace{1cm} (b) \( \Lambda \)

Figure 2. Example 13

Proof of Theorem 13. Let us use the notations in the proof of Theorem 12. Note that the compositions \( Z_0(\Gamma, \mathcal{G}) \to Y_{\Gamma} \to X_{\Gamma} \) and \( Z_f(\Gamma, \mathcal{G}) \to Z_{\Gamma} \to X_{\Gamma} \) are local isometries, and that \( Z_f(\Gamma, \mathcal{G}) \) is compact. Moreover, \( [GP(\Gamma, \mathcal{G}) : KF(\Gamma, \mathcal{G})] = \prod_{i=1}^k |G_i| \) is finite. \( \square \)

If \( C \subseteq D \) are (possibly infinite) rectangular boxes in \( \mathbb{R}^n \) whose vertices are lattice points, then the inclusion \( Y_{\Gamma} \cap C \to Y_{\Gamma} \cap D \) is a local isometry. We note two immediate corollaries of Theorem 12.

Corollary 14. (1) The graph product kernel of countable groups embeds into \([A(\Gamma), A(\Gamma)]\).

(2) The graph product of finite or cyclic groups virtually embeds into \( A(\Gamma) \).

Corollary 15. Let \( \mathcal{G} = \{G_v : v \in V(\Gamma)\} \) and \( \mathcal{G}' = \{G'_v : v \in V(\Gamma)\} \) be collection of countable groups.

(1) If \( |G_v| \leq |G'_v| \leq \infty \) for each vertex \( v \), then \( KF_0(\Gamma, \mathcal{G}) \) embeds into \( KF_0(\Gamma, \mathcal{G}') \). If we further assume that \( \mathcal{G} \) and \( \mathcal{G}' \) consist of finite or cyclic groups, then \( KF_1(\Gamma, \mathcal{G}) \) embeds into \( KF_1(\Gamma, \mathcal{G}') \).

(2) If \( |G_v| = |G'_v| \leq \infty \) for each vertex \( v \), then \( KF_0(\Gamma, \mathcal{G}) \cong KF_0(\Gamma, \mathcal{G}') \).

Lemma 16. Each finitely generated subgroup of \([A(\Gamma), A(\Gamma)] \) embeds into \([GP_m(\Gamma), GP_m(\Gamma)]\) for some \( 2 \leq m < \infty \).

Proof. Let \( H \) be a finitely generated subgroup of \([A(\Gamma), A(\Gamma)] \) and \( B \) be the union of edge-paths in \( Y_{\Gamma} \) that correspond to the generators of \( H \). For \( n = |V(\Gamma)| \), there exists \( 2 \leq m < \infty \) such that \( B \) is contained in \([0, m - 1]^n \). Then \( H \to [A(\Gamma), A(\Gamma)] \) factors as \( H \to \pi_1(Y_{\Gamma} \cap [0, m - 1]^n) = [GP_m(\Gamma), GP_m(\Gamma)] \to \pi_1(Y_{\Gamma}) = [A(\Gamma), A(\Gamma)] \). \( \square \)
Proof of Theorem 4. For each of (1) and (2), only one direction of the assertion is not obvious. We will follow the notations in the proof of Theorem 12.

(1) Suppose $C(\Gamma) \in S$. If $\mathcal{G}$ consists of nontrivial groups, Corollary 13 implies that $[C(\Gamma), C(\Gamma)]$ embeds into $KP_0(\Gamma, \mathcal{G})$. Note that $[C(\Gamma), C(\Gamma)]$ is in $S$ since $[C(\Gamma), C(\Gamma)] \leq \infty$.

(2) Suppose $\mathcal{G}$ consists of cyclic groups and $GP(\Gamma, \mathcal{G}) \in S$. Then $KP_f(\Gamma, \mathcal{G})$ is in $S$ since it is a finite-index subgroup of $GP(\Gamma, \mathcal{G})$. The conclusion follows since $KP_f(\Gamma, \mathcal{G}) \leq \pi_1(Z_T) \leq A(\Gamma)$.

(3) By Lemma 16, $[A(\Gamma), A(\Gamma)] \in S$ if and only if $GP_m(\Gamma) \in S$ for sufficiently large $m$. Also note that if $\mathcal{G}$ consists of finite groups, then $KP_0(\Gamma, \mathcal{G})$ embeds into $GP_m(\Gamma)$ for sufficiently large $m$.

Proof of Corollary 6 (1). Since $C(C_m)$ is a cocompact Fuchsian group, it contains a hyperbolic surface group. Apply Theorem 4 (1).

4. Doubles and co-contractions

Suppose $A$ and $B$ are groups. For an isomorphism $\psi: C \to D$ where $C \leq A$ and $D \leq B$, we let $A*_\psi B$ denote the free product of $A$ and $B$ amalgamated along $\psi$. If $\psi: C \to C'$ is an isomorphism for some $C, C' \leq A$, then the HNN extension of $A$ along $\psi$ is denoted as $A*_\psi$.

Lemma 17. Suppose $G$ is a group, $\psi: H \to H$ is an isomorphism for some $H \leq G$ and $2 \leq k < \infty$. Let $G_k = G*_\psi \cdots *_\psi G$ where there are $k$ copies of $G$. We denote the stable generator of $G*_\psi$ by $t$.

(1) The group $G_k$ embeds into $G*_\psi / \langle \langle t^k \rangle \rangle$ as a subgroup of index $k$.

(2) The group $G*_\psi / \langle \langle t^k \rangle \rangle$ virtually embeds into $G*_\psi$.

Proof. Let $L_k = G_k*_\psi$, whose stable generator is denoted by $s$. The groups in the first line of Figure 2 are illustrated as graphs of groups, where each vertex corresponds to $G$ and each directed edge corresponds to $\psi: H \to H$; (b) shows the case when $k = 5$ as an example. The number $1 \times k$ in (e) means that $G*_\psi / \langle \langle t^k \rangle \rangle$ is obtained from $G*_\psi$ by gluing (to a classifying space of $G*_\psi$) $D^2$ such that $\partial D^2$ and $t^k$ are identified. Similarly, $G_k \cong L_k / \langle \langle s \rangle \rangle$ in (d) is obtained from $L_k$ by attaching $k$ copies of $D^2$, whose boundary curves are all identified with $s$; this is described as $k \times 1$. The figure shows a commutative diagram, where $p_1$ and $p_2$ are induced by covering maps. Now for (1), note that $p_2$ is injective. (2) follows from that $G_k$ embeds into $L_k$ and that $p_1$ is injective.

The following is a special case of Lemma 17 (2).

Corollary 18. Let $G$ and $K$ be graph products of groups such that $K$ is obtained from $G$ by replacing a finite cyclic vertex group of $G$ by $Z$. Then $G$ virtually embeds into $K$.

Example 19. (1) Consider the graph product $G$ shown in Figure 2 (a). Suppose $K$ is obtained from $G$ by substituting $Z$ for $Z_4$. Then $G$ virtually embeds into $K$.

(2) We can apply Lemma 17 (2) repeatedly to see that $C(\Gamma)$ virtually embeds into $A(\Gamma)$. This also follows from the well-known fact that $[C(\Gamma), C(\Gamma)] \leq [A(\Gamma), A(\Gamma)]$; see [13].

We let $G*_H G$ denote the free product of two copies of $G$ amalgamated along $id_H$. The HNN extension of $G$ along $id_H$ is denoted as $G*_H$.

Lemma 20. Let $G$ be a group, $H \leq G$ and $k \in Z \setminus \{1, -1\}$. Then $G*_H G$ embeds into $G*_H / \langle \langle t^k \rangle \rangle$.

Proof. The case when $k \neq 0$ follows from Lemma 17 (1). For $k = 0$, consider the first line of Figure 3. We remark that $k = 0$ case is already well-known; see [35, p.187].

Let us consider a graph isomorphism $\mu: \Gamma \to \Gamma'$. Fix a vertex $t \in V(\Gamma)$ and put $\Lambda$ the subgraph of $\Gamma$ induced by $Lk(t)$. Define $\Gamma''$ as the graph obtained from the union of $\Gamma \setminus St(t)$ and $\Gamma' \setminus St(\mu(t))$ after identifying $\Lambda$ and $\mu(\Lambda)$ by $\mu$. Here, $St(v)$ denotes the open star of a vertex $v$. We call $\Gamma''$ as the double of $\Gamma \setminus St(t)$ along the link of $t$. There is a natural projection map $\rho: \Gamma'' \to \Gamma \setminus St(t)$. 
Let a group \( \langle \langle \{ G_v : v \in V(\Gamma) \} \rangle \rangle \) be given. We say \((\Gamma, \mathcal{G})\) embeds into \( GP(\Gamma, \mathcal{G}) \). Since \( GP \) embeds into \( H \), the pair \((\Gamma, \mathcal{G})\) embeds into \( GP(\Gamma, \mathcal{G}) \).\( \Box \)

For \( e \in E(\Gamma) \), we let \( \Gamma/e \) denote the contraction of \( e \) \([12, p.20]\).

**Corollary 22.** Suppose \( e = \{x, t\} \) is an edge of \( \Gamma^{opp} \). We put \( \Gamma_0^{opp} = \Gamma^{opp}/e \) and denote by \( y \in V(\Gamma_0) = V(\Gamma_0^{opp}) \) the contracted vertex of \( e \). If \( \mathcal{G} = \{ G_v : v \in V(\Gamma) \} \) is a collection of nontrivial groups and \( G_y = G_x \), then \( GP(\Gamma_0, \langle \mathcal{G} \rangle \langle \{ G_x, G_1 \} \rangle \cup \{ G_y \}) \) embeds into \( GP(\Gamma, \mathcal{G}) \).

**Proof.** As in Figure 3(a), we partition \( V(\Gamma) : \{x, t\} \) into \( P, Q, R, S \) where \( P = \text{Lk}(x) \cap \text{Lk}(t), Q = \text{Lk}(t) \setminus \text{Lk}(x), R = \text{Lk}(x) \setminus \text{Lk}(t) \) and \( S = V(\Gamma) \setminus (P \cup Q \cup R \cup \{x, t\}) \). Let \( \Gamma_1 \) be the double of \( \Gamma \setminus \text{St}(t) \) along the link of \( t \) as shown in Figure 3(c); we write \( R', S' \) and \( x' \) for the copies of \( R, S \) and \( x \). Figure 3(b) shows that \( \Gamma_1 \) is isomorphic to the subgraph of \( \Gamma_1 \) induced by \( P \cup Q \cup R \cup S \cup \{x'\} \).\( \Box \)

Theorem 5 now follows from Corollary 22. Since there is a finite sequence of edge-contractation from \( C_m \) onto \( C_5 \) for each \( m \geq 5 \), we have an embedding from \( C(C_5) \cong C(C_5^{opp}) \) into \( C(C_5^{opp}) \). Hence, Corollary 7 also follows.

**Remark.** A special case of Corollary 22 is when all the vertex groups are infinite cyclic, and was proved in [31]. Bell gave a shorter proof of this case using the same decomposition of a graph as shown in Figure 4 [3], independently from this writing.

5. **Right-angled Coxeter groups on seven vertex graphs**

For a group \( G \) and \( H \leq G \), we say that \( g \in G \) is *conjugate into \( H \) if \( g \) is conjugate to an element of \( H \).

**Definition 23.** Let a group \( G \) and its subgroup \( H \) be given. We say \((G, H)\) is *big* if there exists a compact hyperbolic surface \( S \) and a monomorphism \( \phi : \pi_1(S) \to G \) such that \( \phi(\gamma) \) is conjugate into \( H \) whenever \( \gamma \) is homotopic into \( \partial S \). The pair \((G, H)\) will be called *small* if it is not big.
Lemma 24. \( \text{(1)} \) Let \( A \) and \( B \) be groups and \( \psi: C \to D \) be an isomorphism where \( C \leq A \) and \( D \leq B \). If \( A \ast \psi \ B \in S \), then either \( (A,C) \) or \( (B,D) \) is big.

\( \text{(2)} \) If \( G \in S \), then \( (G,H) \) is big for each \( H \leq G \).

Remark. (1) If \( (G,H) \) is big and \( H \leq K \leq G \), then \( (G,K) \) is also big.

(2) If \( G \in S \), then \( (G,H) \) is big for each \( H \leq G \).

(3) If a group \( G \) does not contain \( F_2 \), then \( (G,H) \) is small for each \( H \leq G \).

The following lemma is obvious from typical transversality arguments.

Lemma 25. Suppose \( a,b \in V(\Gamma) \) and \( \text{Lk}(b) \subseteq \text{Lk}(a) \). If \( C(\Gamma) \notin S \), then \( (C(\Gamma), \langle a,b \rangle) \) is small.

Proof. Suppose \( S \) is a compact hyperbolic surface and \( \phi: \pi_1(S) \to C(\Gamma) \) is a monomorphism such that \( \phi([\gamma]) \) is conjugate into \( \langle a,b \rangle \) whenever \( \gamma \) is homotopic into \( \partial S \). Since \( C(\Gamma) \notin S \), we have \( \partial S \neq \emptyset \). Let \( \partial_1 S, \partial_2 S, \ldots \) be the boundary components of \( S \). Since \( \langle a,b \rangle \) contains \( \mathbb{Z} \), we see that \( a \) and \( b \) are distinct and non-adjacent in \( \Gamma \). We realize \( \phi \) as a label-reading map with respect to a label-reading pair \( (\mathcal{V},\lambda) \) satisfying the three conditions in Theorem 11(3). Then each \( \partial_i S \) intersects with both \( a \)-arcs and \( b \)-arcs, and no arcs with labels other than \( a \) or \( b \) intersect \( \partial S \). Choose a \( b \)-arc \( \beta \) joining say, \( \partial_1 S \) and \( \partial_2 S \). These two boundary components may coincide; however, \( \beta \) is never homotopic into \( \partial S \). With suitable choices of the base point and the orientations, \( \phi([\beta \cdot \partial_j S \cdot \beta^{-1}]) = w(ab)^lw^{-1} \) for some \( l \neq 0 \) where \( w \) is the label-reading of \( \beta \). Since \( w \in \langle Lk(b) \rangle = \langle Lk(a) \cap Lk(b) \rangle \), we have \( \phi([\beta \cdot \partial_j S \cdot \beta^{-1}]) = (ab)^l \). Also, \( \phi([\partial_i S]) = (ab)^m \) for some \( m \neq 0 \). Since \( [\beta \cdot \partial_j S \cdot \beta^{-1}] \) and \( [\partial_i S] \) do not commute, we have a contradiction. \[ \square \]

Lemma 26. Let \( H,G_1,G_2 \) be groups such that \( H \) is torsion-free word-hyperbolic. Denote by \( p_i: G_1 \times G_2 \to G_i \) the natural projection for \( i = 1,2 \). If \( \phi: H \to G_1 \times G_2 \) is injective, then \( p_1 \circ \phi \) or \( p_2 \circ \phi \) is also injective.

Proof. Suppose \( 1 \neq x_1 \in \ker(p_2 \circ \phi) \) and \( 1 \neq x_2 \in \ker(p_1 \circ \phi) \) so that \( \phi(x_1) \in G_i \) for \( i = 1,2 \). Since \( \phi([x_1,x_2]) = [\phi(x_1),\phi(x_2)] = 1 \), we have \( x_1^M = x_2^N \) for some \( M,N \neq 0 \) [4 Corollary 3.10]. So, \( \phi(x_1^M) = \phi(x_2^N) \in G_1 \cap G_2 = 1 \). \[ \square \]

A repeated application of Lemma 26 easily implies the following.
Lemma 27. If \( k > 0 \) and \( (G_i, H_i) \) is small for \( i = 1, 2, \ldots, k \), then \( \prod_{i=1}^{k} G_i, \prod_{i=1}^{k} H_i \) is small.

Example 28. Label the vertices of \( P_3 \coprod P_3 \) as Figure 5 (a) and let \( \Lambda_0 = (P_3 \coprod P_3)^{opp} \). Let us consider subgroups of \( C(\Lambda_0) = \langle a, b, c, e, f, g \rangle \) from now on. In the subgraph of \( \Lambda_0 \) induced by \( \{a, b, c\} \), we have \( \text{Lk}(b) = \emptyset \) and \( \text{Lk}(a) = c \). Lemma 25 implies that \( \langle \langle a, b, c \rangle, \langle a, b \rangle \rangle \) is small. By Lemma 27 \( \langle \langle a, b, c, e, f, g \rangle, \langle a, b, f, g \rangle \rangle = \langle \langle a, b, c \rangle \times \langle c, f, g \rangle, \langle a, b \rangle \times \langle f, g \rangle \rangle \) is also small.

\[
\begin{align*}
\text{(a)} & \quad \Lambda_0^{opp} = P_3 \coprod P_3 \\
\text{(b)} & \quad P_6 \\
\text{(c)} & \quad P_7
\end{align*}
\]

Figure 5. Some six and seven vertex graphs

Lemma 29. If the free product of two groups \( A \) and \( B \) amalgamated along a finite subgroup is in \( S \), then either \( A \) or \( B \) is in \( S \).

Proof. Suppose \( H \) is a hyperbolic surface group and \( C \) is a finite subgroup of both \( A \) and \( B \) such that \( H \trianglelefteq A *_C B \). There is a graph of groups decomposition for \( H \) such that each edge group embeds into \( C \). Since \( H \) is torsion-free and one-ended, this decomposition should essentially have only one vertex group. \( \square \)

We denote by \( P_n \) the path on \( n \) vertices. Let us recall from [9] the graphs \( P_1(7) \) and \( P_2(7) \), whose opposite graphs are drawn in Figure 6. The result in [9] implies that when \( \Gamma \) at most seven vertices, \( A(\Gamma) \in S \) if and only if \( \Gamma \) contains \( C_{6}^{opp}, F_{6}^{opp}, P_1(7), P_2(7) \) or \( C_m \) for some \( m \geq 5 \) as an induced subgraph.

\[
\begin{align*}
\text{(a)} & \quad P_1(7)^{opp} \\
\text{(b)} & \quad P_2(7)^{opp}
\end{align*}
\]

Figure 6. Graphs from [9].

We now consider some eight-vertex graphs.

Lemma 30. Let \( \Phi_1, \Phi_2, \ldots, \Phi_5 \) be the graphs whose opposite graphs are shown in Figure 7 (a) through (e). Then \( C(\Phi_i) \not\subseteq S \) for each \( i = 1, 2, \ldots, 5 \).

Proof. We use the vertex labels shown in Figure 7. We also set \( H = \langle a, b, f, g \rangle \leq G = \langle a, b, c, e, f, g \rangle \leq C(P_7^{opp}) = \langle a, b, c, a, d, e, f, g \rangle \) as considered in Figure 5 (c). In Example 28 we have seen that \( (G, H) \) is small.

(Case \( \Phi_1 \)) We can write \( C(\Phi_1) = \langle \langle a, b, c \rangle \times \langle e, f, g \rangle \times \langle t \rangle \rangle *_{\langle a, b \rangle \times \langle f, g \rangle} \langle \langle a, b \rangle \times \langle d \rangle \times \langle f, g \rangle \rangle \). By Lemma 27 and Example 28 \( \langle \langle a, b, c \rangle \times \langle e, f, g \rangle \times \langle t \rangle, \langle a, b \rangle \times \langle f, g \rangle \rangle \) is small. Since \( \langle a, b \rangle \times \langle d \rangle \times \langle f, g \rangle \) is virtually abelian, it contains no \( F_2 \) and hence, \( \langle \langle a, b \rangle \times \langle d \rangle \times \langle f, g \rangle, \langle a, b \rangle \times \langle f, g \rangle \rangle \) is small. Lemma 24 (1) implies that \( C(\Phi_1) \not\subseteq S \).

(Case \( \Phi_2 \)) We note that \( C(P_7^{opp}) \leq C(\Phi_1) \) and so, \( C(P_7^{opp}) \not\subseteq S \). Moreover, we have \( C(\Phi_1) = C(P_7^{opp}) *_G \langle t^2 \rangle \) where \( t \) is the stable generator. By Lemma 24 \( C(\Phi_2) = \langle a, b, c, d, e, f, g \rangle *_{\langle a, b, c, e, f, g \rangle} \langle a, b, c, d', e, f, g \rangle \leq C(\Phi_1) \).
(Case $\Phi_3$) Note that $C(P_7^{opp}) = G \ast H / \langle d^2 \rangle$ where $d$ is the stable generator. Hence, $C(\Phi_3) = \langle a,b,c,e,f,g \rangle \ast \langle a,b,c,e',f',g' \rangle \leq C(P_7^{opp})$.

(Case $\Phi_4$) Let us consider $\psi: H \to H$ defined by $\psi(a) = a, \psi(b) = b, \psi(f) = g$ and $\psi(g) = f$. We see that $G \ast \psi: H \to H$ is virtually abelian, we have $(a,b,c,e,f,g) \ast \psi (a,b,c',e',f' = g,g' = f) = C(\Phi_4)$. Lemma 22(1) and Example 23 imply that $C(\Phi_4) \not\leq S$.

(Case $\Phi_5$) Similarly, define $\psi: H \to H$ by $\psi(a) = b, \psi(b) = a, \psi(f) = g$ and $\psi(g) = f$. We again see that $G \ast \psi: H \to H$ is virtually abelian, we have $(a,b,c,e,f,g) \ast \psi (a' = b,b' = a,c',e',f' = g,g' = f) = C(\Phi_5) \not\leq S$. □

![Figure 7. Some eight-vertex graphs in Lemma 30.](image)

**Proof of Theorem 8.** We use notations from Lemma 30 and Figure 7. The backward direction is a restatement of Corollaries 6 and 7.

For the forward direction, suppose $\Gamma$ is weakly chordal and $C(\Gamma) \in S$. Since $[C(\Gamma), C(\Gamma)] \leq [A(\Gamma), A(\Gamma)]$, we see that $A(\Gamma) \in S$. By the result in Section 7, $\Gamma$ contains $P_6^{opp}, P_7(1)$ or $P_2(7)$ as an induced subgraph; see Figure 6. Since $P_1(7) \not\leq \Phi_1$ and $P_2(7) \not\leq \Phi_2, \text{Lemma 30}$ implies that $C(P_1(7)) \not\leq S$ and $C(P_2(7)) \not\leq S$. Hence $P_6^{opp} \not\leq \Gamma$. Moreover, $\vert V(\Gamma) \vert = 7$ since $C(P_6^{opp}) \leq C(P_7^{opp}) \leq C(\Phi_1) \not\leq S$.

We fix the vertex labels of $P_6 \leq \Gamma^{opp}$ as in Figure 5(b) and let $V(\Gamma) \cap V(P_6^{opp}) = \{t\}$. By val($t$), we will mean the valence of $t$ in $\Gamma$.

**Case** val($t$) = 0 or 1 Note $C(\Gamma) = C(P_6^{opp}) \ast Z_2$ or $C(\Gamma) = C(P_6^{opp}) \ast_{Z_2} (Z_2)^2$. By applying Lemma 24, we obtain a contradiction that $C(\Gamma) \not\leq S$.

**Case** val($t$) = 2 In $\Gamma$, the vertex $t$ is joined to two vertices, say $x, y$ of $P_6^{opp}$. We may write $C(\Gamma) = C(P_6^{opp}) \ast \langle x, y, t \rangle$. If $x$ and $y$ are adjacent in $P_6^{opp}$, then $C(\Gamma) = C(P_6^{opp}) \ast (Z_2)^3$; then, Lemma 28 implies that $C(\Gamma) \not\leq S$. So, $x$ and $y$ are adjacent in $P_6 \leq \Gamma^{opp}$. Since $\Gamma^{opp}$ has no induced $C_5$, we should have $\Gamma \cong \Lambda_1$; see Figure 8(a). We have $C(\Lambda_1) = (\langle a,b \rangle \times \langle d,e,f,t \rangle) \ast_{\langle a,e,f \rangle} (\langle a \rangle \times \langle c \rangle \times \langle e,f \rangle)$. By Lemmas 25 and 27, $(\langle a,b \rangle \times \langle d,e,f,t \rangle, \langle a \rangle \times \langle e,f \rangle)$ is small. Since $\langle a, c, e, f \rangle$ is virtually abelian, we have $C(\Lambda_1) \not\leq S$ by Lemma 24(1). This is a contradiction.

**Case** $\Gamma = \Lambda_2$; see Figure 8(b) By Lemma 21, $C(\Lambda_2) = (\langle a, b, c, d, e, f \rangle \ast_{\langle b,c,d,e,f \rangle} \langle t, b, c, d, e, f \rangle)$ embeds into $(\langle a, b, c, d, e, f \rangle \ast_{\langle b,c,d,e,f \rangle} \langle t, x, y, z \rangle) / \langle s^2 \rangle \cong C(P_7^{opp}) \not\leq S$, where $s$ denotes the stable generator.

**Case** $\Gamma = \Lambda_3$; see Figure 8(c) Using the vertex labels of $P_6^{opp}$ in Figure 5(b), we see that $C(\Lambda_3) \cong GP(P_6^{opp}, \{G_a = Z_2 \times Z_2, G_b = G_c = \cdots = G_f = Z_2\})$. Corollary 15 implies that $C(\Lambda_3)$ virtually embeds into $GP(P_6^{opp}, \{G_a = Z_2 \times Z_2, G_b = G_c = \cdots = G_f = Z_2\}) \cong C(\Lambda_2) \not\leq S$.

**Case** val($t$) = 3 If $a$ and $f$ are both adjacent to $t$ in $\Gamma^{opp}$, then only one of $b, c, d, e, f$ are adjacent to $t$ in $\Gamma^{opp}$. This implies that $\Gamma^{opp}$ contains an induced $C_5$, and hence a contradiction. So, we may assume $a$ is not adjacent to $t$ in $\Gamma^{opp}$. Let us say $x$ and $y$ are the other two vertices of $P_6 \leq \Gamma^{opp}$ that are non-adjacent to $t$. If $(a, x, y)$ are pairwise non-adjacent in $P_6$, then $C(\Gamma) = C(P_6^{opp}) \ast_{\langle x,y,z \rangle} \langle t, x, y, z \rangle = C(P_6^{opp}) \ast (Z_2)^4 \not\leq S$. So there exist at least two vertices in $\{a, x, y\}$ that are adjacent in $P_6$. If $b \in \{x, y\}$, then $\Gamma \cong \Lambda_i$ for $i = 4, 5, 6, 7$; see Figure 8(d).
through (g). If \( b \not\in \{x, y\} \), then \( x \) and \( y \) must be adjacent in \( P_6 \). Since \( C_5 \nleq \Gamma^{\text{opp}} \), we would have a graph isomorphism \( \Gamma \cong \Lambda_5 \).

If \( \Gamma \cong \Lambda_4 \), then \( C(\Gamma) = \langle a, b, c, d, e, f \rangle \ast_{\langle a, b, c, d, f \rangle} \langle a, b, c, d, t, f \rangle \leq \langle a, b, c, d, e, f \rangle \ast_{\langle a, b, c, d, f \rangle} \langle a, b, c, d, t, f \rangle / \langle s^2 \rangle \cong C(\Lambda_3) \) where \( s \) is the stable generator. Similarly if \( \Gamma = \Lambda_5 \), \( C(\Gamma) = \langle a, b, c, d, e, f \rangle \ast_{\langle a, b, c, e, f \rangle} \langle a, b, c, d, t, e, f \rangle \leq \langle a, b, c, d, e, f \rangle \ast_{\langle a, b, c, e, f \rangle} / \langle s^2 \rangle \cong C(\Lambda) \) where \( s \) is the stable generator and \( \Lambda^{\text{opp}} \) is the subgraph of \( \Phi^{\text{opp}}_1 \) induced by \( \{a, b, c, d, e, f, t\} \); see Figure 8 (a).

Suppose \( \Gamma = \Lambda_6 \). Then \( C(\Gamma) = \langle (a, b, c) \times (e, f) \rangle \ast_{\langle a, b \rangle \times \langle f \rangle} \langle (a, b) \times \langle d \rangle \times \langle t, f \rangle \rangle \). Since \( \langle a, b \rangle \times \langle d \rangle \times \langle t, f \rangle \) is virtually abelian, Example 28 implies that \( C(\Gamma) \not\in S \).

Consider the case \( \Gamma = \Lambda_7 \). Then \( \Gamma^{\text{opp}} \) is obtained from \( \Phi^{\text{opp}}_1 \) in Figure 7 (d) by contracting \( \{c, c'\} \) to a vertex. By Theorem 5, \( C(\Gamma) \) embeds into \( C(\Phi_4) \).

(\textbf{Case} \( \text{val}(t) = 4 \)) Let \( x \) and \( y \) be the two vertices adjacent to \( t \) in \( \Gamma^{\text{opp}} \). Since \( C_5 \nleq \Gamma^{\text{opp}} \), we see that \( d(x, y) \leq 2 \) in \( P_6 \). So \( \Gamma \cong \Lambda_i \) for \( i = 2, 3, 4, 5, 6 \); see Figure 8.

By Corollary 15, \( C(\Lambda_8) \) and \( C(\Lambda_9) \) virtually embed into \( C(\Lambda_4) \) and \( C(\Lambda_5) \), respectively. If \( \Gamma = \Lambda_9 \), write \( C(\Lambda_9) = \langle (a, b, t) \times \langle d, e, f \rangle \rangle \ast_{\langle a \rangle \times \langle e, f \rangle} \langle (a) \times \langle c \rangle \times \langle e, f \rangle \rangle \not\in S \). We see that \( \Phi^{\text{opp}}_2 \) contracts onto \( \Lambda^{\text{opp}}_{10} \); see Figure 7 (b) and Figure 8 (k). So, \( C(\Lambda_{12}) \leq C(\Phi_2) \).

(\textbf{Case} \( \text{val}(t) = 5 \)) Either \( \Gamma \leq \Phi_1 \) or \( \Gamma \cong \Lambda_3 \).

(\textbf{Case} \( \text{val}(t) = 6 \)) Note \( C(\Gamma) = C(P^{\text{opp}}_6) \times \langle t \rangle \).

Remark. \( \text{(1) Theorem 8} \) is not true if \( \Gamma \) has more than seven vertices. That is, there exists a weakly chordal graph \( \Gamma \) such that \( C(\Gamma) \in S \). For example, let \( \Gamma \) be the graph whose opposite graph is shown in Figure 9. By 11, \( A(P^{\text{opp}}_6) \) is an index-64 subgroup of \( C(\Gamma) \) and so, \( C(\Gamma) \in S \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Seven-vertex graphs in Theorem 8}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{The graph \( \Gamma^{\text{opp}} \) in Proposition 5}
\end{figure}

(2) There exists a graph \( \Gamma \) such that \( A(\Gamma) \in S \) and \( C(\Gamma) \not\in S \). For instance, we may set \( \Gamma \) as one of the graphs \( P^{\text{opp}}_6, P_7(7) \) or \( P_2(7) \).

\textbf{Problem 31.} \( \text{(1) Does there exist a graph} \ \Gamma \ \text{such that} \ [A(\Gamma), A(\Gamma)] \not\in S \ \text{while} \ A(\Gamma) \in S ? \)

\( \text{(2) Does there exist a graph} \ \Gamma \ \text{such that} \ [A(\Gamma), A(\Gamma)] \in S \ \text{while} \ C(\Gamma) \not\in S ?} \)
6. Closure under graph products

A group $G$ is periodic if every element of $G$ has a torsion. The following is well-known.

**Lemma 32** ([10]). A word-hyperbolic group does not have an infinite periodic subgroup.

Let us denote by $\mathcal{X}$ the class of finitely generated groups that are either

(i) not one-ended, or

(ii) not word-hyperbolic, or

(iii) containing hyperbolic surface groups.

An affirmative answer to Question 2 is equivalent to saying that every finitely generated group is in $\mathcal{X}$. How large do we know $\mathcal{X}$ is? We prove that $\mathcal{X}$ is closed under graph products.

**Theorem 33.** If $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$ is a collection of groups in $\mathcal{X}$, then $GP(\Gamma, \mathcal{G})$ is in $\mathcal{X}$.

**Proof.** Suppose that $G = GP(\Gamma, \mathcal{G})$ is one-ended and word-hyperbolic. We may assume that each $G_v$ is nontrivial. If $\Gamma$ contains an induced cycle of length at least five, Corollary 6 implies that $G \in \mathcal{S}$. If $\Gamma$ contains an induced square, whose vertices are denoted as $a, b, c$ and $d$ cyclically, then $G$ would contain $(G_a \ast G_c) \times (G_b \ast G_d) \geq \mathbb{Z} \times \mathbb{Z}$. So from now on, we will assume that $C_n \not\subseteq \Gamma$ for every $n \geq 4$; namely, $\Gamma$ is a chordal graph [17].

Suppose that $\Gamma$ is complete. Then $G$ is the direct product of its vertex groups. Since $G$ is one-ended, at least one vertex group, say $G_a$, must be infinite. By Lemma 32, each infinite vertex group of $G$ contains $\mathbb{Z}$. As $G$ does not contain $\mathbb{Z} \times \mathbb{Z}$, exactly one vertex group is infinite. Then $G$ is virtually $G_a$, and so, $G_a$ is one-ended hyperbolic. Since $G_a \in \mathcal{X}$, we have $G_a \in \mathcal{S}$.

Now, assume that $\Gamma$ is not complete. Since $\Gamma$ is chordal, $\Gamma$ can be written as $\Gamma = \Gamma_1 \cup \Gamma_2$ for some induced subgraphs $\Gamma_1, \Gamma_2$ such that $\Gamma_0 = \Gamma_1 \cap \Gamma_2$ is complete [13]. We choose a minimal such $\Gamma_0$. If all the vertex groups of $\Gamma_0$ are finite, then $G$ splits over a finite group, and hence $G$ has more than one ends. So $G_a$ is infinite for some $a \in V(\Gamma_0)$. By minimality of $\Gamma_0$, we can find $a_i \in \Gamma_1 \setminus \Gamma_0$ such that $a_i$ is adjacent to $a$ for $i = 1, 2$. This implies that $G$ contains $G_a \times (G_{a_1} \ast G_{a_2})$, and hence, $\mathbb{Z} \times \mathbb{Z}$. This is a contradiction. □

**Remark.** (1) Several other classes of groups are known to be closed under the graph product operation. These classes include residually finite groups [20], semihyperbolic groups [2], automatic groups [20] and diagram groups [22]. Meier characterized exactly when a graph product of word-hyperbolic groups is word-hyperbolic [36].

(2) Every 3–manifold group is in $\mathcal{X}$. To see this, suppose $M$ is a 3–manifold such that $\pi_1(M)$ is one-ended and word-hyperbolic. We may assume $M$ is orientable by taking a double cover if necessary. By the Loop Theorem, either $M$ has a hyperbolic incompressible boundary component or $M$ is closed possibly after capping off spherical boundary components. If $M$ is closed, Perelman’s geometrization theorem implies that $M$ is a closed hyperbolic 3–manifold; then, the work of Kahn and Markovic [29] implies that $\pi_1(M) \in \mathcal{S}$.

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