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Bose-Einstein condensate soliton qubit states for metrological applications

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ABSTRACT

We propose a novel platform for quantum metrology based on qubit states of two Bose-Einstein condensate solitons, optically manipulated, trapped in a double-well potential, and coupled through nonlinear Josephson effect. We describe steady-state solutions in different scenarios and perform a phase space analysis in the terms of population imbalance - phase difference variables to demonstrate macroscopic quantum self-trapping regimes. Schrödinger-cat states, maximally path-entangled \((N00N)\) states, and macroscopic soliton qubits are predicted and exploited to distinguish the obtained macroscopic states in the framework of binary (non-orthogonal) state discrimination problem. For arbitrary phase estimation in the framework of linear quantum metrology approach, these macroscopic soliton states are revealed to have a scaling up to the Heisenberg limit (HL). The examples illustrate the HL estimation of angular frequency between the ground and first excited macroscopic states of the condensate, which opens new perspectives for current frequency standard technologies.

Introduction

Nowadays, the formation and interaction of nonlinear collective modes in Kerr-like medium represent an indispensable platform for various practical applications in time and frequency metrology¹,², spectroscopy³,⁴, absolute frequency synthesis⁵, and distance ranging⁶. In photonic systems, frequency combs are proposed for these purposes⁷. The combs occur due to the nonlinear mode mixing in special (ring) microcavities, which possess some certain eigenmodes. Notably, bright soliton formation emerges with vital phenomena accompanying micro-comb generation⁸. Physically, such a soliton arises due to the purely nonlinear effect of temporal self-organization pattern occurring in an open (driven-dissipative) photonic system. However, because of the high level of various noises such systems can be hardly explored for purely quantum metrological purposes.

On the other hand, atomic optics, which operates with Bose-Einstein condensates (BECs) at low temperatures, provides a suitable platform for various quantum devices that may be useful for metrology andensing tasks⁹. In particular, so-called Bosonic Josephson junction (BJJ) systems, established through two weakly linked and trapped atomic condensates, are at the heart of the current quantum technologies in atomtronics, which considers atom condensates and aims to design (on-chip) quantum devices. Condensates in this case represent low dimensional systems and may be manipulated by magnetic and laser field combinations. Thus they represent an alternative to optical analogues.

The BJBs are intensively discussed and examined both in theory and experiment¹⁰–¹⁴. The quantum properties of the BJBs are also widely studied¹⁵–²² including spin-squeezing and entanglement phenomena²³–²⁵, as well as the capability of generating \(N00W\)-states²⁶,²⁷ to go beyond the standard quantum limit²⁷. Physically, the BJBs possess interesting features connected with the interplay between quantum tunneling of the atoms and their nonlinear properties evoked by atom-atom interaction²⁸,²⁹.

With Kerr-like nonlinearities, solitons naturally emerge from atomic condensates in low dimensions³⁰–³⁴. Especially, the bright atomic solitons observed in lithium condensate possessing a negative scattering length³²–³⁴ are worth noticing. Atomic gap solitons are also observed in condensates with repulsive inter-particle interaction³¹. Based on soliton modes, we recently proposed the quantum soliton Josephson junction (SJJ) device with the novel concept to improve the quantum properties of the effectively coupled two-mode system³⁵–³⁷. The SJJ-device consists of two weakly-coupled condensates trapped in a double-well potential and elongated in one dimension. BECs with such a geometry were studied in³⁸. We demonstrated that quantum solitons may be explored for the improvement of phase measurement and estimation up to the Heisenberg limit (HL) and beyond³⁵. In the framework of nonlinear quantum metrology, we also showed that solitons permit a Super-Heisenberg (SH) scaling \((∝ N^{-3/2})\) even with coherent probes³⁶. On the other hand, steady-states of coupled solitons can be useful for effective
formation of Schrödinger-cat (SC) superposition state and maximally path-entangled $N00N$-states, which can be applied for the phase estimation purposes. It is important that such superposition states arise only for soliton-shape condensate wave functions and occur due to the existence of certain steady-states in the phase difference - population imbalance phase plane.

Remarkably, macroscopic states, like SC-states, play an essential role for current information and metrology. In quantum optics, various strategies are proposed for the creation of photonic SC-states and relevant (continuous variable) macroscopic qubits. Special (projective) measurement and detection techniques are also important here. The condensate environment, dealing with matter waves, is potentially promising for macroscopic qubits implementation due to the minimally accessible thermal noises it provides.

In this work, we propose metrological applications for two soliton superposition states as macroscopic qubits. The interaction between these solitons comes from the nonlinear mode mixing in an atomic condensate trapped in a double-well potential. In particular, we reveal the SC-states formation and their implementation for arbitrary phase measurement prior HL and beyond. Since SC-states are non-orthogonal states, a special measurement procedure is applied by so-called sigma operators, as it enables us to estimate the unknown phase parameter. On the other hand, our approach can be also useful in the framework of discrimination of binary coherent (non-orthogonal) states in quantum information and communication. The non-orthogonality of these states leads to so-called Helstrom bound for the quantum error probability that simply indicates the impossibility for a receiver to identify the transmitted state without some errors. In quantum metrology, by means of various regimes of condensate soliton interaction, we deal with a set of quantum states, which may be prepared before the measurement. Our results show that these SC-states approach the soliton $N00N$-states to minimize the quantum error probability.

1 Two-soliton model

1.1 Coupled-mode theory approach

We start with the mean-field description of coupled mode-theory approach to an elongated BEC trapped in $V = V_H + V(x)$ potential, where $V_H$ is a 3D harmonic trapping potential; while $V(x)$ is responsible for the double-well confinement in one ($X$) dimension. The (rescaled) condensate wave function (mean field amplitude) $\Psi(x)$ obeys the familiar 1D Gross-Pitaevskii equation (GPE), cf.:

$$i \frac{\partial}{\partial t} \Psi = - \frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi - uN|\Psi|^2 \Psi + V(x)\Psi,$$

(1)

where $u = 4\pi|a_{sc}|/a_\perp$ characterizes a Kerr-like (focusing) nonlinearity, $a_{sc} < 0$ is the s-wave scattering length that appears due to atom-atom scattering in Born-approximation, $a_\perp = \sqrt{\hbar/m\omega_\perp}$ characterizes the trap scale, and $m$ is the particle mass. To be more specific, we only consider condensates possessing a negative scattering length. In Eqs. (1) we also propose rescaled (dimension-less) spatial and time variables, which are $x, y, z \rightarrow x/a_\perp, y/a_\perp, z/a_\perp$, and $t \rightarrow \omega_\perp t$, cf.35,36,38.

The nonlinear coupled-mode theory admits a solution of Eq. (1) that simply represents a quantum-mechanical superposition

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t),$$

(2)

where the wave functions $\Psi_1(x)$ and $\Psi_2(x)$ characterize the condensate in two wells. For weakly interacting atoms one can assume that

$$\Psi_{1,2}(x,t) = C_{1,2}(t)\Phi_{1,2}(x)e^{i\beta_{1,2}t},$$

(3)

where $\Phi_1(x)$ and $\Phi_2(x)$ are ground- and first-order excited states, with the corresponding wave functions possessing energies $\beta_1$ and $\beta_2$, respectively; $C_1(t)$ and $C_2(t)$ are time-dependent functions. If the particle number is not too large, Eq. (1) may be integrated in spatial dimension, leaving only two condensate variables $C_{1,2}(t)$. In particular, $\Phi_1(x)$ and $\Phi_2(x)$ may be time-independent Gaussian-shape wave functions obeying different symmetry. Practically, this two-mode approximation is valid for the condensates of several hundreds of particles. The condensate in this limit is effectively described by two macroscopically populated modes as a result.

1.2 Quantization of coupled solitons

The sketch in Fig. 1 explains the two-soliton system described in our work. If trapping potential $V(x)$ is weak enough and the interaction among condensed particles is not so weak, the ansatz solution (3) is no longer suitable. For condensates with a negative scattering length, a bright soliton solution is admitted for $\Psi_{1,2}(x,t)$ in Eq. (2). In fact, in this case one can speak about two-soliton solution problem, which is well known in classical theory of solitons.
where we suppose $N$.

Applying variational field theory approach based on the ansatz $a_j(x,t) = \hat{a}_1 + \hat{a}_2$, where $\hat{a}_{1,2}(x,t)$ are field operators corresponding to mean-field amplitudes $\Psi_{1,2}(x,t)$. We assume that experimental conditions allow the formation of atomic bright solitons in each of the double-well potential. In particular, these conditions may be realized by means of laser field manipulation with weakly trapping potential $V(x)$. Experimentally, this manipulation may be performed by a dipole trap and laser field.

Then, considering linear superposition state, one can write down the total Hamiltonian $\hat{H}$ for two BEC solitons in the second quantization form as

$$\hat{H} = \int_{-\infty}^{\infty} \sum_{j=1}^{2} \left( \hat{a}_j^\dagger \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} \right) \hat{a}_j \right) \, dx - \frac{u}{2} \int_{-\infty}^{\infty} \left( \hat{a}_1^\dagger + \hat{a}_2^\dagger \right)^2 \left( \hat{a}_1 + \hat{a}_2 \right)^2 \, dx. \quad (4)$$

The annihilation (creation) operators of bosonic fields, denoted as $\hat{a}_j \,(\hat{a}_j^\dagger)$ with $j = 1,2$, obey the commutation relations:

$$[\hat{a}_i(x), \hat{a}_j^\dagger(x')] = \delta(x-x') \delta_{ij}; \quad i, j = 1,2. \quad (5)$$

In the Hartree approximation for a large particle number, $N >> 1$, one can assume that the quantum $N$-particle two-soliton state is the product of $N$ two-soliton states and can be written as

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}} \int_{-\infty}^{\infty} \left( \Psi_{1}(x,t) \hat{a}_1^\dagger e^{-i\beta_1 t} + \Psi_2(x,t) \hat{a}_2^\dagger e^{-i\beta_2 t} \right) \, dx \, |0\rangle, \quad (6)$$

where $\Psi_{j}(x,t)$ is the unknown wave functions, and $|0\rangle \equiv |0\rangle_1 |0\rangle_2$ denotes a two-mode vacuum state. The state given in Eq. (6) is normalized as $\langle \Psi_N | \Psi_N \rangle = 1$, and the bosonic field-operators $\hat{a}_j$ act on it as

$$\hat{a}_j |\Psi_N\rangle = \sqrt{N} \Psi_{j}(x,t) e^{-i\beta_j t} |\Psi_{N-1}\rangle. \quad (7)$$

Applying variational field theory approach based on the ansatz $\Psi_{j}(x,t)$, one can obtain the Lagrangian density in the form:

$$L_0 = \frac{1}{2} \sum_{j=1}^{2} \left( i [\Psi_{j} \Psi_{j}^\dagger - \frac{\partial \Psi_{j}^\dagger}{\partial x}] - \frac{uN}{2} \left( \Psi_1^2 e^{\beta_1 t} + \Psi_2^2 e^{\beta_2 t} \right) \right)^{-2} \left( \Psi_1 e^{-i\beta_1 t} + \Psi_2 e^{-i\beta_2 t} \right)^2 \, dx, \quad (8)$$

where we suppose $N - 1 \approx N$ and omit common term $N$.

Noteworthy, from Eq. (8), one can obtain the coupled GPEs for $\Psi_{j}$-functions as
\[ \frac{i}{\partial t} \Psi_1 = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi_1 - uN \left( |\Psi_2|^2 + 2 |\Psi_1|^2 \right) \Psi_1 - uN \left( |\Psi_2|^2 + 2 |\Psi_1|^2 \right) \Psi_2 e^{-i\beta t} - uN \Psi_1^* \Psi_2 e^{-2i\beta t} - uN \Psi_1^* \Psi_2^2 e^{i\beta t}; \]  
(9a)

\[ \frac{i}{\partial t} \Psi_2 = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi_2 - uN \left( |\Psi_2|^2 + 2 |\Psi_1|^2 \right) \Psi_2 - uN \left( |\Psi_2|^2 + 2 |\Psi_1|^2 \right) \Psi_1 e^{i\beta t} - uN \Psi_1^* \Psi_2^2 e^{2i\beta t} - uN \Psi_1^* \Psi_2^* e^{-i\beta t}; \]  
(9b)

where \( \beta = \beta_2 - \beta_1 \) is the energy (frequency) spacing.

The set of Eqs. (9) leads to the known problem for transitions between two lowest self-trapped states of condensates in the nonlinear coupled mode approach if we account Eq. (3) for the representation of condensate wave functions \( \Psi_j(x,t) \). In particular, in accordance with Karpman’s approach we can find in Eq. (9) the terms proportional to \( \epsilon_{jk} = \Psi_j \Psi_k^2 + 2 |\Psi_j|^2 \Psi_k \), \( j, k = 1, 2, j \neq k \), as perturbations for two fundamental bright soliton solutions. Physically, \( \epsilon_{jk} \) implies the nonlinear Josephson coupling between the solitons.

In this work we establish a variational approach for the solution of Eqs. (9), cf.\( ^{16} \). For the weakly coupled condensate states, i.e. for \( \epsilon_{jk} \approx 0 \), the set of Eqs. (9) can be reduced to two independent GPEs:

\[ \frac{i}{\partial t} \Psi_j = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi_j - uN |\Psi_j|^2 \Psi_j, \]  
(10)

which possess bright (non-moving) soliton solutions

\[ \Psi_j(x,t) = \frac{N_j}{2} \sqrt{\frac{u}{N}} \text{ sech} \left[ \frac{uN_j}{2} x \right] e^{i\beta t}; \]  
(11)

In the case of \( \epsilon_{jk} \neq 0 \) and non-zero inter-soliton distance \( \delta \), we examine ansatzes for \( \Psi_j(x,t) \) in the form

\[ \Psi_1(x,t) = \frac{N_1}{2} \sqrt{\frac{u}{N}} \text{ sech} \left[ \frac{uN_1}{2} (x - \delta) \right] e^{i\theta_1}; \]  
(12a)

\[ \Psi_2(x,t) = \frac{N_2}{2} \sqrt{\frac{u}{N}} \text{ sech} \left[ \frac{uN_2}{2} (x + \delta) \right] e^{i\theta_2}. \]  
(12b)

In particular, our approach presumes the existence of two well distinguished solitons (separated by the small distance \( \delta \), with the shape preserved) interacting through dynamical variation of the particle numbers, \( N_j \equiv N_j(t) \), and phases, \( \theta_j \equiv \theta_j(t) \), which occurs in the presence of weak coupling between the solitons. In other words, \( N_j \) and \( \theta_j \) should be considered as time-dependent (variational) parameters.

By substituting Eqs. (12) into Eq. (8) we obtain (up to the constant factor and term)

\[ L = \int_{-\infty}^{\infty} L_0 dx \approx -z \Theta + \Lambda z^2 + \frac{\Lambda}{2} (1 - z^2)^2 I(z, \Delta) (\cos[2\Theta] + 2) + \Lambda (1 - z^2) J(z, \Delta) \cos[\Theta], \]  
(13)

where \( z = (N_2 - N_1)/N (N_{1,2} = \frac{N}{2} (1 \mp z)) \) is the particle number population imbalance; \( \Theta = \theta_2 - \theta_1 - (\beta_2 - \beta_1) t \equiv \theta - \Omega t \) is an effective time-dependent phase-shift between the solitons.

Physically, \( \Omega \) is an angular frequency spacing between the ground and first excited macroscopically states of the condensate; it represents a vital (measured) parameter for metrological purposes in this work. In Eq. (13), we also introduce the notation \( \Lambda = N^2 u^2/16 \) and define the functionals

\[ I = I(z, \Delta) = \int_{-\infty}^{\infty} \text{sech}^2 [(1 - z) (x - \Delta)] \text{sech}^2 [(1 + z) (x + \Delta)] \]  
(14a)

\[ J = J(z, \Delta) = \sum_{k=2}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \text{sech}^2 [(1 + sz) (x + s\Delta)] \text{sech} [(1 - sz) (x - s\Delta)] \right), \]  
(14b)

where \( \Delta = \frac{N_2 - N_1}{N} \) is a normalized distance between solitons.
Finally, by using Eq. (13) for the population imbalance and phase-shift difference, $z$ and $\Theta$, we obtain the set of equations

$$
\xi = (1-z^2)\left\{ (1-z^2)I\sin[2\Theta] + J\sin[\Theta] \right\};
$$  \hfill (15a)

$$
\dot{\Theta} = -\frac{\Omega}{\Lambda} + 2z + \frac{d}{dz}\left\{ \frac{1}{2} (1-z^2)^2 I(\cos[2\Theta] + 2) + (1-z^2) J\cos[\Theta] \right\},
$$  \hfill (15b)

where dots denote the derivatives with respect to the renormalized time $\tau = \Lambda t$.

In contrast to the problem with coupled Gaussian-shape condensates, the solutions of Eqs. (15) crucially depend on the features of governing functionals $I(z, \Delta)$ and $J(z, \Delta)$, cf.\cite{28,29}. In Appendix we represent some analytical approximations for $I(z, \Delta)$ and $J(z, \Delta)$, in order to give a clear illustration.

2 Steady-state (SS) solutions

2.1 Steady-state solution for $z^2 = 1$

The steady-state (SS) solutions of Eq. (15) play a crucial role for metrological purposes with coupled solitons\cite{35}. We start from the SS solution $z^2 = 1$ of Eq. (15a) by setting the time-derivatives to zero. As seen from Eq. (14), in the limit of maximal population imbalance, $z^2 = 1$, $I$ and $J$ are independent on $\Delta$ and approach

$$
I(z, \Delta) = 1;
$$  \hfill (16a)

$$
J(z, \Delta) = \pi.
$$  \hfill (16b)

Substituting $z^2 = 1$ and Eq. (16) into Eq. (15b), we obtain

$$
z^2 = 1;
$$  \hfill (17a)

$$
\Theta = \arccos\left[ \frac{2\Lambda - \text{sign}[z]\Omega}{2\pi\Lambda} \right].
$$  \hfill (17b)

Notably, in the quantum domain the SS solutions shown in Eq. (17) admit the existence of quantum states with maximal population imbalance $z = \pm 1$ and phase difference. The latter depends on the frequency spacing $\Omega$, which is the object of precise measurement with maximally path-entangled $N00N$-states in this paper.

Below we perform the analysis of the SS solutions of Eqs. (15) in two limiting cases $\Omega \neq 0$, $\Delta \sim 0$ and $\Omega \sim 0$, $\Delta \neq 0$.

2.2 SS solutions for $\Theta = 0, \pi$ and $\Delta \sim 0$

To find the SS solutions we rewrite Eq. (15b) as

$$
\frac{\Omega}{\Lambda} = 2z - 6z (1-z^2) I + \frac{3}{2} (1-z^2)^2 \frac{\partial I}{\partial z} - 2z J + (1-z^2) \frac{\partial J}{\partial z}
$$  \hfill (18)

for $\Theta = 0$ and

$$
\frac{\Omega}{\Lambda} = 2z - 6z (1-z^2) I + \frac{3}{2} (1-z^2)^2 \frac{\partial I}{\partial z} + 2z J - (1-z^2) \frac{\partial J}{\partial z}
$$  \hfill (19)

for $\Theta = \pi$, respectively.

In Appendix we represent a polynomial approximation for $I, J$ functionals given in Eq. (14). Since the equations obtained from Eqs. (18) and (19) are quite cumbersome, here we just briefly analyze the results.

In the limit of closely spaced solitons and $\Theta = 0$, the population imbalance $z$ at equilibrium depends only on $\Omega$ and obeys

$$
\frac{\Omega}{\Lambda} = 1.27 - 8z^5 + 15z^3 - 12.5z.
$$  \hfill (20)

Similarly, for fixed soliton phase difference $\Theta = \pi$ we have

$$
\frac{\Omega}{\Lambda} = 1.27 - 3.2z^5 + 12.3z^3 - 2z.
$$  \hfill (21)

We plot the graphical solutions of Eqs. (20) and (21) in Fig. 2; the blue and red curves characterize the right parts of Eqs. (20) and (21), respectively. The straight lines in Fig. 2 correspond to different values of the $\Omega/\Lambda$ ratio. These lines cross the curves in the points indicating the solutions of Eqs. (20) and (21). Notice that the solid blue and red curves denote the values of $\Omega/\Lambda$ and $z$ corresponding to the stable SS solutions; while the dotted ones describe parametric unstable solutions. As seen from Fig. 2, at phase difference $\Theta = 0$ there exists one stable SS solution for any $z \in [-0.7; 0.7]$ and only unstable solutions for $|z| > 0.7$. At $|\Omega/\Lambda| > 1.55\pi$, no SS solutions exist.

On the other hand, at $\Theta = \pi$ there exists a tiny region $-0.1\pi \leq \Omega/\Lambda \leq 0.1\pi$ possessing two SS solutions simultaneously. One stable SS solution exists within the domain $0.1\pi < |\Omega/\Lambda| \leq 2.64\pi$. 

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2.3 SS solutions for $\Theta = 0, \pi$ and $\Omega \approx 0$

At $\Omega = 0$ Eqs. (20), (21) admit the SS solutions, which look like:

\begin{align}
  z &= 0, \quad \Theta = 0; \quad (22a) \\
  z &= 0, \quad \Theta = \pi; \quad (22b) \\
  z^2 &\approx 0.17, \quad \Theta = \pi. \quad (22c)
\end{align}

As seen from Eq. (22b), at relative phase $\Theta = \pi$ Eq. (19) possesses three solutions: a parametrically unstable solution occurs at $z = 0$ and two degenerate SS solutions appear for $z = \pm z_0$. Here, $z_0$ varies from 0.41 at $\Delta \approx 0$ to 0.64 at $\Delta \approx 2.8$ for non-zero soliton inter-distance, respectively. For $\Delta > 2.8$ these SS solutions do not exist.

In Fig. 3 we represent a more general analysis of SS solutions for $\Theta = 0$ as functions of inter-soliton distance $\Delta$ for different $\Omega$. For that we exploit the sixth-order polynomial approximation, see Appendix. In particular, at $\Omega \approx 0$ there exists one solution at $z = 0$, stable at $\Delta \leq \Delta_c \approx 0.5867$. For $\Delta > \Delta_c$ this solution becomes parametrically unstable.

On the other hand, for $\Delta > \Delta_c$ Eqs. (18) possess the degenerate SS solutions similar to the ones at $\Theta = \pi$. The bifurcation for population imbalance $z$ occurs at $\Delta = \Delta_c$; in Fig. 3 the $z_+$ (upper, positive) and $z_-$ (lower, negative) branches characterize this bifurcation. In the vicinity of $\Delta_c$ we can consider $z_\pm = \pm z_0$, where

\begin{equation}
  z_0 = 1.2\sqrt{\Delta - \Delta_c}. \quad (23)
\end{equation}

At $\Omega \neq 0$, the behavior of SS solutions become complicated with respect to the distance $\Delta$ - see the green curves in Fig. 3. The solid curves correspond to SS solutions for different $\Delta$, while the dotted ones describe the unstable solutions. As clearly seen from Fig. 3, for $|\Omega| > 0$ there is no bifurcation for population imbalance $z$ and two stationary solution branches $z_\pm$ occur with $|z_-| > |z_+|$. On the other hand, at a relatively large values of parameter $\Omega/\Lambda$, only one SS solution exists - see the red curve in Fig. 3.
3 Mean-field dynamics

3.1 Small amplitude oscillations

We start our analysis here from small amplitude oscillations close to SS solutions given in Eq. (22). For that we linearize Eqs. (15) in the vicinity of the solution in Eq. (22), assuming \( 0 \leq \Delta < 0.6 \) and \( \Omega \ll 1 \). The first assumption allows us to use the approximation of \( I,J \)-functionals by the fourth-degree polynomial, see Appendix.

For zero-phase oscillations, i.e. for \( \Theta \approx 0 \) (\( \cos[\Theta] \approx 1 \), \( \sin[\Theta] \approx \Theta \)), from Eq. (15) we obtain

\[
\ddot{z} + \omega_0^2(\Delta)z = f_0(\Delta)\Omega
\]

with the solution

\[
z(\tau) = A \cos[\omega_0\tau] - \Omega \frac{f_0}{\omega_0^2},
\]

where \( A \) and \( \omega_0(\Delta) = 13.4\sqrt{0.37 - \Delta^2 - 0.25\Delta} \) are the amplitude and angular frequency of oscillations, respectively. The last term in Eq. (25) with \( f_0(\Delta) = 5.36 - 0.8\Delta - 4.22\Delta^2 \) plays a role of constant “external downward displacement force” that vanishes at \( \Omega \approx 0 \). Notably, at \( \Delta > 0.5 \), the oscillations become anharmonic and \( z(\tau) \) diverges at \( \Delta > 0.5867 \). For \( \Delta = 0 \) the frequency of oscillations approaches \( \omega_0 \approx 8.15 \), that agrees with the numerical solution of Eq. (15).

At \( \Delta = \Delta_c \approx 0.5867 \) SS solution given in Eq. (22) splits into two degenerate solutions with \( z = \pm z_0 \) and \( z_0 \) determined by Eq. (23), see Fig. 3. Near these points the equation, similar to Eq. (24), has a form

\[
\ddot{z} + \omega^2z = -18\Delta_+ \sqrt{\Delta^2 - f(\Delta_-)}\Omega
\]
that implies a solution
\[
z(\tau) = \pm \left(2 - \frac{18\Delta_+}{\omega^2}\right) \sqrt{\Delta_+} + A\cos[\omega\tau] - \frac{f(\Delta_+)}{\omega^2},
\]
(27)
where \(\Delta_+ \equiv \Delta - \Delta_c\), \(\omega = 14.53\sqrt{\Delta_- - 4.48\Delta_-^2 + 17.8\Delta_-^4 - 53.5\Delta_-^6}\) is the angular frequency of oscillations, and \(f = 3.4 - 7.26\Delta_+ + 11\Delta_+^2\) is the “external” force. A relative error for Eq. (27) is less than 5%.

In the vicinity of SS points determined by Eq. (22c), we obtain \(\pi\)-phase oscillations characterized by
\[
z(\tau) = \pm z_0 + A\cos[\omega_\pi\tau] + \frac{f_\pi}{\omega^2}\Omega,
\]
(28)
with \(\omega_\pi = \sqrt{2 - 0.9\Delta_-^2 - 0.3\Delta}\), \(f_\pi = 0.1(\Delta_-^2 + 0.38\Delta + 5.5)\), and \(z_0\) determined in Eq. (22c). For \(\Omega \approx 0\) and \(\Delta = 0\), the angular frequency is \(\omega_\pi \approx 1.42\), which is much smaller than that in the zero-phase regime.

The analysis of Eq. (15) in the vicinity of Eq. (22b) reveals that this solution is parametrically unstable, and highly nonlinear behavior is expected. Indeed, a direct numerical simulation demonstrates anharmonic dynamics plotted in Fig. 4. For \(0 < |z| < 0.5\) the nonlinear regime of self-trapping is observed; while it turns into nonlinear oscillations at \(|z| > 0.5\).

The analysis of SS solution (17) reveals a strong sensitivity to \(z\)-perturbation, when condition \(z^2 = 1\) is violated, the high-amplitude nonlinear oscillations occur. On the other hand, solution (17) is robust to phase-perturbations, which is an important property for metrology.

### 3.2 Large separation limit, \(\Delta >> 1\)

For a very large distance \(\Delta\) between the solitons, i.e., \(\Delta >> 1\), the atom tunneling between them vanishes, and the solitons become independent. Strictly speaking, in the limit of \(\Delta \to \infty\) the functionals \(I, J \to 0\), and Eqs. (15) look like
\[
\dot{z} = 0; \quad \dot{\Theta} = -\Omega + 2z,
\]
(29a)
(29b)
i.e., the population imbalance is a constant in time and the running-phase regime establishes.

For large but finite \(\Delta\), SS solution having \(z = \pm z_0\) with \(z_0 \to 1\) exists for the zero-phase regime, \(\Theta = 0\); for example, for \(\Delta = 10\) the SS population imbalance is \(z_0 \approx 0.96\).

### 3.3 Phase-space analysis

The dynamical behavior of the coupled soliton system can be generalized in terms of a phase portrait of two dynamical variables \(z\) and \(\Theta\), as shown in Figs. 5 and 6.
In Fig. 5 we represent \( z - \Theta \) phase-plane for \( \Omega = 0 \) and for different (increasing) values of distance \( \Delta \). We distinguish three different dynamic regimes. The solid curves correspond to the oscillation regime when \( z(\tau) \) and \( \Theta(\tau) \) are some periodic functions of normalized time, see Eq. \((25)\) and the red curve in Fig. 4. The dashed curves in Fig. 5 indicate the self-trapping regime when \( z(\tau) \) is periodic and the sign of \( z \) does not change, see Eq. \((28)\) and the blue curve in Fig. 4. Physically, this is the macroscopic quantum self-trapping (MQST) regime characterized by a nonzero average population imbalance when the most of the particles are “trapped” within one of the solitons. At the same time, the behavior of phase \( \Theta(\tau) \) may be quite complicated but periodic in time. On the other hand, for the running-phase regime depicted by the dashed-dotted curves, \( \Theta(\tau) \) grows infinitely, see the green curve in Fig. 5(b). Due to the symmetry that takes place at \( \Omega = 0 \), the running-phase can be achieved only with the MQST regime, see Fig. 5.

As seen from Fig. 5, the central area of nonlinear Rabi-like oscillations between the ground and first excited macroscopic states occur for a relatively small inter-soliton distance \( \Delta \) and are inherent to zero-phase oscillations, see Fig. 5(a). As discussed before, at \( \Delta = \Delta_c \approx 0.5867 \) this area splits into two regions characterized by the MQST regimes, Fig. 5(a). This splitting occurs due to the bifurcation of population imbalance, see the black curve in Fig. 3. These regions are moving away from each other with growing \( \Delta \), see Fig. 5(c-f). Notably, the bifurcation effect and MQST states, which are the features of the coupled solitons (Fig. 1) at the zero-phase regime, do not occur for the condensates described by Gaussian states\(^{28,29}\).

The phase trajectories inherent to \( \pi \)-phase region \( \frac{\pi}{2} < \Theta < \frac{3\pi}{2} \) stay weakly perturbed until the second critical value \( \Delta \approx 2 \), when the MQST regime in Fig. 5(d) changes to Rabi-like oscillations in Fig. 5(e), then, approaches the running-phase at \( \Delta \approx 6 \), see Fig. 5(f).

At large enough \( \Delta \), the particle tunneling vanishes and the zero-phase MQST domains arise in the vicinity of population imbalance \( z = \pm 1 \), Fig. 5(f). The phase dynamics corresponds to the running-phase regime with \( z = \text{const} \), see Fig. 5(f) and Eq. \((29)\).

For non-zero \( \Omega \), the phase portrait becomes asymmetric, see Fig. 6. To elucidate the role of \( \Omega \), we study the soliton interaction for a given inter-soliton distance \( \Delta = 0.75 > \Delta_c \) that corresponds to the one after the bifurcation. As seen from Fig. 6(a), the phase portrait does not change significantly for small \( \Omega \), cf. Fig. 5(b).

One of the SS solutions for zero and \( \pi \)-phase regimes disappears with increasing \( \Omega \); then the running-phase regime establishes, see Fig. 6(b). Further increasing of \( \Omega \) leads to vanishing the SS solution for zero-phase, Fig. 6(c).

Thus, phase portraits in Figs. 5 and 6 demonstrate the existence of degenerate SSs for coupled solitons by varying \( \Delta \) and \( \Omega \). Such solutions, as we show below, may be exploited for the macroscopic superposition soliton states formation in the quantum approach.
4 Quantum metrology with two-soliton states.

4.1 Phase estimation with macroscopic qubit states.

Suppose that some quantum system is prepared in state $|\psi\rangle$, which carries information about some parameter $\Gamma$ that we would like to estimate. In this work we are interested in fundamental bound for a positive operator valued measurement (POVM) and consider pure states of the quantum system.

In quantum metrology the sensitivity of some parameter $\Gamma$ estimation is described by the error propagation formula given as

$$\sigma_\Gamma = \sqrt{\langle \psi | (\Delta \hat{\Pi})^2 | \psi \rangle - \langle \psi | \hat{\Pi} | \psi \rangle^2},$$

where $\langle \psi | (\Delta \hat{\Pi})^2 | \psi \rangle = \langle \psi | \hat{\Pi}^2 | \psi \rangle - \langle \psi | \hat{\Pi} | \psi \rangle^2$ is the variance of fluctuations of some operator $\hat{\Pi}$ that corresponds to the measurement procedure. Typically, such procedures are based on appropriate interferometric schemes and use quantum superpositions, which contain required information about estimated parameter $\Gamma$. In the case of SC-states, which presume macroscopic (non-orthogonal in general) states, the measurement procedure requires some specification. In particular, we assume the quantum system may be prepared in state $|\psi\rangle$ that we represent as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\pi_0\rangle + e^{-i\phi_\eta} |\pi_1\rangle).$$

(31)

In Eq. (31) $\phi_\eta$ is a relative (estimated) phase between states $|\pi_0\rangle$ and $|\pi_1\rangle$, which are defined as

$$|\pi_0\rangle = c_1 |\Phi_1\rangle - c_2 |\Phi_2\rangle;$$

$$|\pi_1\rangle = c_2 |\Phi_1\rangle - c_1 |\Phi_2\rangle,$$

(32a)

(32b)

where $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are two macroscopic states representing two “halves” of the SC-states. In particular, operators $\hat{\Pi}_i = |\pi_i\rangle \langle \pi_i|$ realize a projection onto the superposition of states $|\Phi_{1,2}\rangle$, which generally are not orthogonal to each other obeying the condition

$$\langle \Phi_1 | \Phi_2 \rangle = \eta.$$

(33)

Simultaneously, we require the states in Eqs. (32) to meet the normalization condition

$$\langle \pi_i | \pi_j \rangle = \delta_{ij}, i, j = 0, 1.$$

(34)

Now, we are able to determine the coefficients $c_{1,2}$, which fulfill Eqs. (33) and (34) and look like

$$c_{1,2} = \frac{1 \pm \sqrt{1 - \eta^2}}{2(1 - \eta^2)}.$$

(35)

In Eqs. (33) and (35), the parameter $\eta$ defines the distinguishability of states $|\Phi_{1,2}\rangle$. The case of $\eta = 1$ corresponds to completely indistinguishable states $|\Phi_{1,2}\rangle$. In this case one can assume that $|\Phi_1\rangle$ and $|\Phi_2\rangle$ represent the same state.
On the other hand, the case of $\eta = 0$ characterizes completely orthogonal states $|\Phi_{1,2}\rangle$; that becomes possible if $|\Phi_{1,2}\rangle$ approach two-mode Fock states. In other words, this is a limit of the $N00N$-state, for which the coupled solitons are examined.

Then, we define a complete set of operators $\hat{\Sigma}_j$, $j = 1, 2, 3$ (cf.\,$49$)

$$\begin{align*}
\hat{\Sigma}_0 &= |\pi_0\rangle \langle \pi_0| + |\pi_1\rangle \langle \pi_1|, \\
\hat{\Sigma}_1 &= |\pi_1\rangle \langle \pi_1| - |\pi_0\rangle \langle \pi_0|, \\
\hat{\Sigma}_2 &= |\pi_0\rangle \langle \pi_0| + |\pi_1\rangle \langle \pi_1|, \\
\hat{\Sigma}_3 &= i(\pi_0) \langle \pi_1| - |\pi_1\rangle \langle \pi_0|),
\end{align*}$$

which obey the $SU(2)$ algebra commutation relation.

The meaning of sigma-operators is evident from their definitions given in Eq. (36). Due to the properties shown in Eq. (34), the states $|\pi_i\rangle$ are suitable candidates for the macroscopic qubit states, which we can define by mapping $|\pi_0\rangle \rightarrow |0\rangle$ and $|\pi_1\rangle \rightarrow |1\rangle$, respectively\,$42,59$. In this form we can use them for POVM measurements defined with operators\,$59$

$$E_1 = \frac{1}{1 + \sqrt{2}} |1\rangle \langle 1| = \frac{1}{\sqrt{2}(1 + \sqrt{2})} (\hat{\Sigma}_0 + \hat{\Sigma}_1);$$

$$E_2 = \frac{1}{\sqrt{2}(1 + \sqrt{2})} (|0\rangle - |1\rangle)(|0\rangle + |1\rangle) = -\frac{1}{\sqrt{2}(1 + \sqrt{2})} (\hat{\Sigma}_1 + i\hat{\Sigma}_3);$$

$$E_3 = I - E_1 - E_2.$$

Importantly, current quantum (photonic) technologies permit POVM tomography\,$43$.

Average values of sigma-operators in Eq. (36) can be obtained with the help of Eqs. (31) and (34), resulting in

$$\begin{align*}
\langle \hat{\Sigma}_1 \rangle &= 0, \\
\langle \hat{\Sigma}_2 \rangle &= \cos[\phi_N], \\
\langle \hat{\Sigma}_3 \rangle &= \sin[\phi_N].
\end{align*}$$

From Eqs. (38) it follows that only $\langle \hat{\Sigma}_{2,3} \rangle$ contain the information about the desired phase $\phi_N$.

To estimate the sensitivity of phase measurement, we can assume that $\phi_N = N\Gamma$ and use Eq. (30) with the measured operator $\hat{\Pi} \equiv \hat{\Sigma}_2$. Taking into account $\langle \hat{\Sigma}_2^2 \rangle = 1$ for the variance of fluctuations $\langle (\Delta \hat{\Sigma}_2)^2 \rangle$, we obtain

$$\langle (\Delta \hat{\Sigma}_2)^2 \rangle = \sin^2[N\Gamma].$$

Finally, from Eqs. (30) and (39) for the phase error propagation we obtain

$$\sigma_\Gamma = \frac{1}{N\Gamma},$$

that clearly corresponds to the HL of arbitrary (linearly $N$-dependent) phase estimation and explores the sigma-operator measurement procedure. Notice this procedure can be mapped onto the parity measurement\,$35,49$.

4.2 Soliton SC-qubit states.

The phase estimation procedure described above enables us to use two-soliton quantum states given in Eq. (6) for the frequency quantum metrology purposes. It is instructive to represent soliton wave functions shown in Eq. (12) in the form of

$$\begin{align*}
\Psi_1 &= \sqrt{uN} \frac{1}{4} (1 - z) \sech \left[ (1 - z) \left( \frac{uN}{4} x - \Delta \right) \right] e^{-ix}, \\
\Psi_2 &= \sqrt{uN} \frac{1}{4} (1 + z) \sech \left[ (1 + z) \left( \frac{uN}{4} x + \Delta \right) \right] e^{ix}.
\end{align*}$$

The degenerate SS solutions of the population imbalance obtained above, see Eq. (17) and Fig. 3, enable one to prepare various superposition soliton states for quantum metrology purposes. In particular, for $\Theta = 0$ from Eq. (6) we obtain

$$\begin{align*}
|\Phi_1\rangle &= \left[ \int_{-\infty}^{0} \left( \Psi_1(+)a_1^\dagger + \Psi_2(+)a_2^\dagger \right) dx \right]^N |0\rangle, \\
|\Phi_2\rangle &= \left[ \int_{-\infty}^{0} \left( \Psi_1(-)a_1^\dagger + \Psi_2(-)a_2^\dagger \right) dx \right]^N |0\rangle.
\end{align*}$$
for two “halves” of the SC-state, where

\[ \Psi_{1}^{(\pm)} = \frac{\sqrt{uN}}{4} (1 - z_{\pm}) \text{sech} \left[ \left(1 - z_{\pm}\right) \left( \frac{uN}{4} x - \Delta \right) \right]; \tag{43a} \]

\[ \Psi_{2}^{(\pm)} = \frac{\sqrt{uN}}{4} (1 + z_{\pm}) \text{sech} \left[ \left(1 + z_{\pm}\right) \left( \frac{uN}{4} x + \Delta \right) \right]. \tag{43b} \]

In Eqs. (43), \( z_{+} \) and \( z_{-} \) are two SS solutions corresponding to the upper and lower branches in Fig. 3, respectively. In Eq. (43), we also omit the common unimportant term \( e^{-iN(\theta/2 + \beta_t \tau)} \). In particular, for \( \Omega \approx 0 \), we have \( z_{\pm} \rightarrow \pm z_0 \).

The scalar product for state given in Eq. (42) is

\[ \eta \equiv \left\langle \Phi_1 | \Phi_2 \right\rangle = \left[ \int_{-\infty}^{\infty} \left( \Psi_{1}^{(+)} \Psi_{1}^{(-)} + \Psi_{2}^{(+)} \Psi_{2}^{(-)} \right) dx \right]^N = e^N, \tag{44} \]

where \( \varepsilon \) characterizes solitons wave functions overlapping. Assuming non-zero and positive \( \Omega \) for \( \varepsilon \), one can obtain

\[ \varepsilon = \frac{1}{2} \left( 1 - \frac{z_{+} + z_{-}}{2} \right) (1 - z_{+})(1 - z_{-}) \left( 1 - 0.21 \left[ \frac{z_{+} - z_{-}}{2 - (z_{+} + z_{-})} \right]^2 \right) \]

\[ + \left( 1 + \frac{z_{+} + z_{-}}{2} \right) (1 + z_{+})(1 + z_{-}) \left( 1 - 0.21 \left[ \frac{z_{+} - z_{-}}{2 + (z_{+} + z_{-})} \right]^2 \right). \tag{45} \]

In Fig. 7, we establish the principal features of coefficients shown in Eq. (35) and parameter \( \varepsilon \), see the inset in Fig. 7, as functions of \( \Delta \). The value of \( \Omega \) plays a significant role in the distinguishability problem for states \( |\Phi_1\rangle \) and \( |\Phi_2\rangle \). In particular, for \( \Omega = 0 \) at the bifurcation point \( \Delta = \Delta_c = 0.5867 \), we have \( \varepsilon = 1 \) that implies indistinguishable states \( |\Phi_1\rangle \) and \( |\Phi_2\rangle \), see the red curve in the inset of Fig. 7. In this limit, the coefficients \( c_{1,2} \rightarrow \infty \).

However, even for the small (positive) \( \Omega \), it follows from Eq. (45) that \( \varepsilon \neq 1 \) for any \( \Delta > \Delta_c \), and states \( |\Phi_1\rangle \), \( |\Phi_2\rangle \) are always distinguishable. In particular, it follows from zero-phase solution given in Eq. (27) that \( z_{\pm} = \pm \left( 1.2 - \frac{18\lambda}{\omega_0^2} \right) \sqrt{\Delta - \Omega \frac{f(\Delta_{c})}{\omega_0^2}} \) and \( |z_{+}| \neq |z_{-}| \). This is displayed by the green curves in Fig. 7. The SS solutions possess \( c_1 = 1.057 \), \( c_2 = 0.203 \) for \( c_{1,2} \) that correspond to \( \Delta \approx 0.647 \), \( \varepsilon \approx 0.9056 \) for \( \Omega = 0.05\pi \).

From Fig. 7, it is evident that coefficients \( c_{1,2} \) rapidly approach (due to the factor \( N \)) levels \( c_1 = 1 \), \( c_2 = 0 \) (completely distinguishable macroscopic SC soliton states), when \( \Delta \) increases. In this limit, as seen from Fig. 3, \( z_{\pm} \) approaches \( \pm z_0 \), and from Eq. (45) we obtain

\[ \varepsilon \approx (1 - z_0^2)(1 - 0.21z_0^2). \tag{46} \]

Practically, in this limit the red and green curves coincide in Fig. 7.

Thus, we can exploit states shown in Eq. (42) for metrological measurement purposes for arbitrary phase \( \phi_N \) estimation that we describe in Sec. 5.1. Phase \( \phi_N \) may be created after soliton SC-state formation by some additional soliton interaction or collisions.

### 4.3 Frequency measurement, \( \Gamma \equiv \Omega \).

Now, we represent a particularly important case of angular frequency \( \Omega \) measurement that characterizes energy spacing between the ground and first excited macroscopic states. The SS solutions given in (17), which correspond to the maximal population imbalance, \( z^2 = 1 \), allow us to prepare the maximally path-entangled superposition state, a.k.a. \( N00N \)-state. As seen from Eq. (17b), the solution with \( z = 1 \) exists when \( -2(\pi - 1) \leq \Omega/\Lambda \leq 2(\pi + 1) \). Similarly, the domain of solution \( z = -1 \) is \( -2(\pi + 1) \leq \Omega/\Lambda \leq 2(\pi - 1) \). To achieve the superposition \( N00N \)-state formation, we require both solutions to exist simultaneously. This restricts the domain of \( \Omega \) as \( -2(\pi - 1) \leq \Omega/\Lambda \leq 2(\pi - 1) \).

Substituting \( z = \pm z_0 \) into Eq. (41) we obtain

\[ \Psi_1 = \frac{\sqrt{uN}}{2} \text{sech} \left[ \left( \frac{uN}{2} x - 2\Delta \right) \right] e^{-i\beta_t \tau}, \tag{47a} \]

\[ \Psi_2 = \frac{\sqrt{uN}}{2} \text{sech} \left[ \left( \frac{uN}{2} x + 2\Delta \right) \right] e^{i\beta_t \tau}, \tag{47b} \]
Figure 7. Coefficients $c_{1,2}$ versus the normalized inter-soliton distance $\Delta$ for $N = 10$ and different $\Omega$. The inset demonstrates the behavior of $\varepsilon$ for different $\Delta$.

which are relevant to the $N00N$-state’s two “halves” defined as

$$
|0N\rangle = \frac{1}{\sqrt{N!}} \left[ \int_{-\infty}^{\infty} \Psi_1^2 \Psi_1^{\dagger} dx \right]^N |0\rangle ,
$$

(48a)

$$
|N0\rangle = \frac{1}{\sqrt{N!}} \left[ \int_{-\infty}^{\infty} \Psi_2^2 \Psi_2^{\dagger} dx \right]^N |0\rangle .
$$

(48b)

Considering the superposition of states shown in Eq. (48) and omitting unimportant common phase $e^{iN(0.5\theta^+ - \beta t)}$, we arrive at

$$
|N00N\rangle = \frac{1}{\sqrt{2}} \left( |N0\rangle + e^{-iN\Theta'} |0N\rangle \right),
$$

(49)

that represents the $N00N$-state of coupled BEC solitons for our problem. Here,

$$
\Theta' = \frac{\Theta^+ + \Theta^-}{2} = \frac{1}{2} \left( \arccos \left[ \frac{2\Lambda - \Omega}{2\pi \Lambda} \right] + \arccos \left[ \frac{2\Lambda + \Omega}{2\pi \Lambda} \right] \right)
$$

(50)

is the phase shift that contains the $\Omega$-parameter required for estimation. Comparing Eq. (50) with Eq. (31), we can conclude that the $N00N$-state’s “halves” $|N0\rangle$ and $|0N\rangle$ in Eq. (50) may be associated with states $|\pi_0\rangle$ and $|\pi_1\rangle$, respectively. To estimate the sensitivity of the $\Omega$-measurement, we use Eq. (30) with measured operator $\hat{\Pi} = \hat{\Sigma}$ defined as

$$
\hat{\Sigma} = |N0\rangle \langle 0N| + |0N\rangle \langle N0| .
$$

(51)

Since states shown in Eq. (48) are orthogonal, the mean value of Eq. (51) is

$$
\langle \hat{\Sigma} \rangle = \cos[N\Theta'].
$$

(52)

Fig. 8 demonstrates $\langle \hat{\Sigma} \rangle$ as a function of $\Omega/\Lambda$. Notice, the interference pattern in Fig. 8 exhibits an essentially nonlinear behavior for measured $\langle \hat{\Sigma} \rangle$.

The variance of fluctuations $\langle (\Delta \hat{\Sigma})^2 \rangle$ for the measured sigma-operator reads as

$$
\langle (\Delta \hat{\Sigma})^2 \rangle = \sin^2[N\Theta'] .
$$

(53)
Now, by using Eqs. (30) and (53) we can easily find the propagation error for the \( \Omega \)-estimation as

\[
\sigma_\Omega = \frac{2\Lambda}{N} \left| \frac{\sqrt{4\pi^2 - (2 + \Omega/\Lambda)^2} - \sqrt{4\pi^2 - (2 - \Omega/\Lambda)^2}}{\sqrt{4\pi^2 - (2 + \Omega/\Lambda)^2} - \sqrt{4\pi^2 - (2 - \Omega/\Lambda)^2}} \right|^{1/2}.
\]  

Equation (54) is non-applicable for \( \Omega = 0 \) since the denominator in Eq. (54) turns to zero. We choose the optimal estimation area for \( \Omega \), with the best sensitivity reached, in the vicinity of the domain border at \( \Omega/\Lambda \to 2(\pi - 1) \). In this limit, Eq. (54)

\[
\sigma_\Omega \approx \frac{10\Lambda}{N} \frac{1.65 \sqrt{4.28 - \Omega/\Lambda}}{1.65 \sqrt{4.28 - \Omega/\Lambda}}.
\]  

Equation (55) exhibits one of the important results of this paper: for a given \( \Lambda \), that characterizes atomic condensate peculiarities, Eq. (55) demonstrates Heisenberg scaling for frequency measurement sensitivity.

Conclusion

In summary, we have considered the problem of two-soliton formation for 1D BECs trapped effectively in a double-well potential. The analytical solution of these soliton Josephson junctions and corresponding phase portraits exhibit the occurrence of novel macroscopic quantum self-trapping (MQST) phases in contrast to the condensates with only Gaussian wave functions. With these soliton states, we have also explored the formation of the Schrödinger-cat (SC) state in the framework of the Hartree approximation. In particular, we have analyzed the distinguishability problem for binary (non-orthogonal) macroscopic states. Compared to the known results, finite frequency spacing \( \Omega \) leads to distinguishable macroscopic states for condensate solitons. This circumstance may be important for the experimental design of the SC-states.

The important part of this work is devoted to the applicability of predicted states for quantum metrology. By utilizing the macroscopic qubits problem with the interacting BEC solitons, one can apply the sigma-operators to elucidate the measurement and subsequent estimation of an arbitrary phase, that linearly depends on the particle number, up to the HL. Notably, the sigma-operators relate to the POVM detection tomography procedure. On the other hand, the phase estimation procedure for the phase-dependent sigma-operator can be realized by means of the parity measurement technique that produces the same accuracy for phase estimation. We have shown that in the limit of soliton state solution with the population imbalance \( |\varepsilon| = \pm 1 \) the coupled soliton system admits the maximally path-entangled \( N00N \)-state formation. The feasibility of frequency \( \Omega \) estimation at the Heisenberg level is also demonstrated.

In this paper, we have not examined the losses and decoherence effects for the quantum soliton system depicted in Fig. 1. Previously, in [37], we examined this problem for quantum solitons possessing simple Josephson coupling. From the experimental point of view, the recent BEC soliton experiments with lithium condensates demonstrated that collisions may be recognized as...
one of the most detrimental effects\textsuperscript{34}. However, as we established in\textsuperscript{37}, the three-body and one-body losses may be unimportant at the time scales of few tens of milliseconds, which is relevant to experimental conditions in\textsuperscript{34}. Moreover, the purely quantum analysis of the problem demonstrated that the superposition of Fock states, occurring at some specific parameters of the system, behaves robust to few particle losses. We will publish the detailed analysis of this problem for interaction of the solitons depicted in Fig. 1 elsewhere.

Appendix: Approximation of functionals $I$ and $J$.

Solutions of Eqs. (15) strictly depend on functionals $I$ and $J$ and their derivatives $I' \equiv dI/dz$ and $J' \equiv dJ/dz$ defined in Eq. (14). In Fig. 9, we represent $I$ and $J$ as two-dimension surfaces given in $z - \Delta$ plane. From Fig. 9 (a,b), it is seen that $I$ and $J$ approach zero for a large $\Delta$, excluding the edge domains where $|z| \simeq 1$. As for small $\Delta$ inherent to $0 \leq \Delta < 1.5$, we apply the polynomial approximations of $I$ and $J$ to illustrate their numerical estimations.

![Surfaces of I and J](image)

**Figure 9.** (a-b) The $I, J$ functionals and (c-d) their derivatives $I', J'$, versus $z$ and $\Delta$.

In particular, within domain $0 \leq \Delta < 0.6$, $I, J(z)$ can be effectively approximated by the forth-order polynomials as follows

\begin{align*}
I(z, \Delta) & \approx a_I(\Delta) z^4 + b_I(\Delta) z^2 + c_I(\Delta); \quad (56a) \\
J(z, \Delta) & \approx a_J(\Delta) z^4 + b_J(\Delta) z^2 + c_J(\Delta), \quad (56b)
\end{align*}
where the coefficients are the polynomials themselves:

\[
\begin{align*}
    a_I &= -\Delta^2 - 0.52\Delta + 0.1; \\
    b_I &= 2\Delta^2 + 0.76\Delta - 0.42; \\
    c_I &= -1.16\Delta^2 - 0.24\Delta + 1.33; \\
    a_f &= -2\Delta^2 - 0.72\Delta + 0.4; \\
    b_f &= 3.9\Delta^2 + 1.03\Delta + 0.07; \\
    c_f &= -1.9\Delta^2 - 0.32\Delta + 2.7.
\end{align*}
\]

At \(0.6 < \Delta \leq 1.5\), in Eq. (56) the sixth-order polynomial approximation is required:

\[
\begin{align*}
    I(z, \Delta) &\approx a_I(\Delta)z^6 + b_I(\Delta)z^4 + c_I(\Delta)z^2 + d_I(\Delta); \\
    J(z, \Delta) &\approx a_f(\Delta)z^6 + b_f(\Delta)z^4 + c_f(\Delta)z^2 + d_f(\Delta),
\end{align*}
\]

where

\[
\begin{align*}
    a_I &= 0.31\Delta^2 - 2.57\Delta + 1.43; \\
    b_I &= 0.9\Delta^2 + 1.24\Delta - 1.6; \\
    c_I &= -1.9\Delta^2 + 3.5\Delta - 0.67; \\
    d_I &= 0.69\Delta^2 - 2.21\Delta + 1.85; \\
    a_f &= -1.5\Delta^2 - 0.13\Delta + 0.89; \\
    b_f &= 4.62\Delta^2 - 4.78\Delta + 0.15; \\
    c_f &= -4\Delta^2 + 8.4\Delta - 1.45; \\
    d_f &= 0.94\Delta^2 - 3.52\Delta + 3.56.
\end{align*}
\]

With theses approximations, an error less than 4% for any \(-1 \leq z \leq 1\) and \(\Delta\) in the mentioned domains can be achieved.

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**Competing interests**

The authors declare that they have no competing interests.
Figure 1
Sketch of the probability density distribution $|Y|^2$ versus spatial coordinates $X$ and $Y$, as a 2D projection of the 3D coupled condensates trapped in a double-well (dashed green curve) and harmonic (dashed magenta curve) potentials, respectively. Shadow regions display 1D condensate wave packets projections; they represent a secant-shape in $X$-direction and Gaussian-shape in the transverse directions.
Figure 2

Normalized frequency spacing $W=L$ (dashed lines) versus reduced population imbalance $z$ for Eq. (20) (blue line) and Eq. (21) (red line), respectively. Dashed regions correspond to unstable solutions.
Figure 3

Population imbalance $z$ versus $D$ for $Q = 0$ and different $W$. The solid curves denote SS solutions of Eq. (18) and the dotted line represents (parametrically) unstable solutions.
Figure 4

Please see the manuscript file for the full caption.

Figure 5

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Figure 6

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Figure 7

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Figure 8

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Figure 9

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