ASYMPTOTIC BEHAVIORS OF SOLUTION TO PARTIAL DIFFERENTIAL EQUATION WITH CAPUTO–HADAMARD DERIVATIVE AND FRACTIONAL LAPLACIAN: HYPERBOLIC CASE

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Abstract. This paper is concerned with the asymptotic behaviors of solution to time-space fractional partial differential equation with Caputo–Hadamard derivative (in time) and fractional Laplacian (in space) in the hyperbolic case, that is, the Caputo–Hadamard derivative order \( \alpha \) lies in \( 1 < \alpha < 2 \). In view of the technique of integral transforms, the fundamental solutions and the exact solution of the considered equation are derived. Furthermore, the fundamental solutions are estimated and asymptotic behaviors of its analytical solution is established in \( L^p(\mathbb{R}^d) \) and \( L^{p,\infty}(\mathbb{R}^d) \). We finally investigate gradient estimates and large time behavior for the solution.

1. Introduction. In this paper we focus on studying the following Cauchy problem of time-space fractional partial differential equation with \( \alpha \in (1, 2) \) and \( s \in (0, 1) \),

\[
\begin{cases}
\mathcal{C}_H D_{\alpha, t}^\alpha u(x, t) + (-\Delta)^s u(x, t) = f(x, t), & x \in \mathbb{R}^d, \ t > a > 0, \\
u(x, a) = \varphi_a(x), & x \in \mathbb{R}^d, \\
\delta u(x, a) = \psi_a(x), & x \in \mathbb{R}^d,
\end{cases}
\]

in which \( \mathcal{C}_H D_{\alpha, t}^\alpha \) is the Caputo–Hadamard derivative operator and \( (-\Delta)^s \) is the fractional Laplace operator. The initial values \( \varphi_a(x), \psi_a(x) \) and the source term \( f(x, t) \) are known functions and the symbol \( \delta = t^{\frac{d}{\alpha}} \) denotes the Delta derivative operator.

The Caputo–Hadamard derivative with order \( \alpha \) \((n - 1 < \alpha < n \in \mathbb{N})\) for a given function \( g(t) \) is defined below [16],

\[
\mathcal{C}_H D_{\alpha, t}^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \left( \log \frac{t}{\tau} \right)^{n-\alpha-1} \delta^n g(\tau) \frac{d\tau}{\tau}, \ t > a > 0,
\]

where \( \Gamma(\cdot) \) is the usual Gamma function and \( \delta^n g(\tau) = \left( \tau^{\frac{d}{\alpha}} \right)^n g(\tau) \).

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The fractional Laplacian \((-\Delta)^s\) is defined as follows [8],
\[
(-\Delta)^s v(x) = C(d, s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x - y|^{d+2s}} \, dy, \quad x \in \mathbb{R}^d, \tag{3}
\]
where P.V. denotes the “principle value sense” and \(C(d, s) = \left( \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{d+2s}} \, dy \right)^{-1}\) with \(y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d\). It is well known that the definition (3) given by integral form can be defined in the following form by using Fourier transform [8],
\[
(-\Delta)^s u(x) = \mathcal{F}^{-1}[\omega(\omega - 2s), x], \quad \forall \, x \in \mathbb{R}^d, \, s \in (0, 1), \tag{4}
\]
where \(\hat{u}\) denotes the Fourier transform of the function \(u\), see also Definition 2.4 below.

A large number of studies on fractional ordinary/partial differential equation with Riemann–Liouville or Caputo derivative have been developed by many scientific researchers [4, 5, 18, 24, 25, 30, 32]. Fractional Laplacian has been attracted much attention in recent years. As far as we know, there exist several kinds of fractional Laplacians, such as the integral definition, the spectral definition, and their variants. For more details, see the review article [23]. And the related studies can be referred to [2, 6, 9, 14, 15, 25, 26, 29, 31] and references cited therein. We are mainly interested in the integral definition of fractional Laplacian in the paper.

Several authors have considered the asymptotic behavior for the evolutionary equation. In 2013, Li et al. [28] showed the asymptotics of solutions to fractional diffusion equation with the standard Laplace operator, where the order \(\alpha\) of Riemann–Liouville or Caputo derivative (in time) belongs to \((0, 1)\) and \((1, 2)\). In 2016, Kim et al. [20] estimated asymptotic behavior of the fundamental solution to time–space fractional partial differential equation. Meanwhile, Kemppainen et al. [21] discussed asymptotic behavior with detailed results, in which the time derivative is the Caputo one with derivative order \(\alpha \in (0, 1)\) and the spatial derivative is still Laplace operator. Shortly after, They extended these results to a more general equation [22] with fractional Laplacian in space, where Caputo derivative is used in the time direction and the derivative order \(\alpha\) lies in \((0, 1)\). Very recently, Djida et al. [7] investigated the model in [22] and derived interesting results, where the Caputo derivative order \(\alpha \in (1, 2]\).

Recently, Hadamard–type fractional derivative proposed early in 1892 [13] has attracted increasing interest due to its potential applications in mechanics and engineering such as the fracture analysis or both planar and three-dimensional elasticities [1, 10, 11, 16, 17, 27]. In [26], Li and Li studied asymptotic behaviors of solution to Caputo–Hadamard fractional partial differential equation with fractional Laplacian, where the order \(\alpha\) of Caputo–Hadamard derivative belongs to \((0, 1)\). That is actually a time–space fractional parabolic equation. The current work extends to \(\alpha \in (1, 2]\) and it can be regarded as a time–space fractional hyperbolic equation. More specifically, in Section 2, we first introduce several integral transforms to solve Equ. (1), where the fundamental solutions are expressed by Fox \(H\)-function and the exact solution is given in a convolution form. Then, according to properties and asymptotic expansion of \(H\)-function at infinity or zero, we can obtain asymptotic estimates of the fundamental solutions. And we further estimate these fundamental solutions in the sense of \(L^p(\mathbb{R}^d)\) and \(L^{p, \infty}(\mathbb{R}^d)\). With the help of Young’s inequalities, asymptotic behaviors of the exact solution to Equ. (1) are established which are our main result in this paper. Section 3 deals with gradient estimates for the
fundamental solutions and the exact solution, and investigates also large time behavior of the exact solution. The conclusion is presented in Section 4 and Appendix collects some elementary knowledge related to this paper such as Young’s inequality and Fox $H$-function. Throughout the paper, $C$ denotes a positive constant which may be dependent of the parameters $a$, $\alpha$, $s$, and $d$, but not necessarily the same at different situations.

2. Asymptotic behaviors of solution to Equ. (1). In this section, we investigate asymptotic behaviors of solution to fractional partial differential equation with Caputo–Hadamard derivative and fractional Laplacian in the hyperbolic case. By using the method of integral transforms, we obtain representation formulas for the fundamental solutions which are some special function called as Fox $H$-function. Then we give the estimates of these fundamental solutions. Finally, based on Young’s inequality, we establish asymptotic estimates of the solution to Equ. (1).

2.1. Several transforms. We first introduce some integral transforms which are needed in the deduction of the fundamental solution.

**Definition 2.1.** ([26]) The amended Laplace transform of a given function $f(t)$ with $t \in (a, \infty)$ ($a > 0$) is defined by

$$\mathcal{F}(\lambda) = \mathcal{L}_a[f(t), \lambda] := \int_a^\infty e^{-\frac{\lambda}{a} \log \frac{t}{a}} f(t) \frac{dt}{t}, \quad 0 \neq \lambda \in \mathbb{C}.$$ 

The inverse amended Laplace transform is given by

$$f(t) = \mathcal{L}_a^{-1}[\mathcal{F}(\lambda), t] := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{\lambda}{a} \log \frac{t}{a}} \frac{\mathcal{F}(\lambda)}{\lambda} d\lambda, \quad c = \text{Re}(\lambda), \quad t \in (a, \infty), \quad a > 0.$$ 

Here, we take principal value branch in the logarithmic function $\log \frac{t}{a}$ since the variable $\lambda$ is a complex number.

**Definition 2.2.** ([26]) Suppose that functions $f(t)$ and $g(t)$ are defined on $[a, +\infty)$ ($a > 0$). The integral $\int_a^t f(a\frac{t}{\tau})g(\tau) \frac{d\tau}{\tau}$ is called the convolution of $f(t)$ and $g(t)$, that is,

$$f(t) \ast g(t) = \int_a^t f(a\frac{t}{\tau})g(\tau) \frac{d\tau}{\tau}.$$ 

From [26], we can get the amended Laplace transform of Caputo–Hadamard derivative (2) below

$$\mathcal{L}_a[CHD_{a,t}^\alpha f(t), \lambda] = \left(\log \frac{\lambda}{a}\right)^\alpha \mathcal{F}(\lambda) - \sum_{k=0}^{n-1} \left(\log \frac{\lambda}{a}\right)^{\alpha-k-1} \delta^k f(a), \quad (5)$$

and the amended Laplace transform of Mittag–Leffler function is given by

$$\int_a^\infty e^{-\frac{\lambda}{a} \log \frac{w}{a}} \left(\log \frac{w}{a}\right)^{\alpha-k-1} E_{\alpha,\beta}^{(k)} \left(\pm \xi \left(\log \frac{w}{a}\right)^\alpha \right) \frac{dw}{w} \bigg|_{w=a}^{w=\infty} = \frac{k!(\log \frac{\lambda}{a})^{\alpha-\beta}}{((\log \frac{\lambda}{a})^\alpha \pm \xi)^{k+1}}, \quad \text{Re}(\lambda) > \frac{\lambda}{a}.$$ 

An amended Mellin transform is defined by the following way.
Definition 2.3. ([26]) The amended Mellin transform of a known function \( f(t) \) with \( t \in (a, \infty) \) is defined by
\[
\tilde{f}(\xi) = \mathcal{M}_a[f(t), \xi] := \int_a^\infty \left( \log \frac{t}{a} \right)^{\xi-1} f(t) \frac{dt}{t}, \quad a > 0, \gamma_1 < \text{Re}(\xi) < \gamma_2.
\]
The inverse amended Mellin transform is given by
\[
f(t) = \mathcal{M}_a^{-1}[\tilde{f}(\xi), \xi] := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \log \frac{t}{a} \right)^{-\xi} \tilde{f}(\xi) d\xi, \quad t > a > 0, \ c = \text{Re}(\xi).
\]
By means of Definitions (2.1) and (2.3), we can easily obtain
\[
\mathcal{M}_a[f(t), \xi] = \frac{1}{\Gamma(1-\xi)} \mathcal{M}_a[\mathcal{L}_a[f(t), \lambda], 1-\xi]. \quad (7)
\]

Definition 2.4. ([18]) The \( d \)-dimensional Fourier transform of a given function \( f(x) \) is defined by
\[
\hat{f}(\omega) = \mathcal{F}[f(x), \omega] := \int_{\mathbb{R}^d} e^{i\omega \cdot x} f(x) dx, \quad \omega \in \mathbb{R}^d,
\]
while the corresponding inverse Fourier transform is given by
\[
f(x) = \mathcal{F}^{-1}[\hat{f}(\omega), x] := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\omega \cdot x} \hat{f}(\omega) d\omega, \quad x \in \mathbb{R}^d.
\]

Definition 2.5. ([18, 26]) The Bessel function of the first kind \( J_\zeta(z) \) is defined by
\[
J_\zeta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\zeta}}{k! \Gamma(\zeta + k + 1)},
\]
where \( z \in \mathbb{C} \setminus (-\infty, 0] \) and \( \zeta \in \mathbb{C} \).

The following relation holds with the Bessel function of the first kind [18]
\[
\int_{\mathbb{R}^d} e^{i\omega \cdot x} \varphi(|\omega|) d\omega = \frac{(2\pi)^d}{|\omega|^{d/2}} \int_0^\infty \varphi(\rho) \rho^{d/2} J_{d/2-1}(\rho |x|) d\rho, \quad (8)
\]
for the function \( \varphi(\rho) \) such that the integral in the right hand side is convergent.

2.2. The fundamental solution. We now deduce the fundamental solution of Equ. (1). By using the Fourier and amended Laplace transforms for Equ. (1), and noting equalities (4) and (5), one gets
\[
\left( \log \frac{\lambda}{a} \right)^\alpha \bar{\varphi}_a(\omega) - \left( \log \frac{\lambda}{a} \right)^{\alpha-1} \bar{\psi}_a(\omega) - \left( \log \frac{\lambda}{a} \right)^{\alpha-2} \bar{\psi}_a(\omega) + |\omega|^{2s} \bar{\bar{u}}(\omega, \lambda) = \bar{f}(\omega, \lambda),
\]
\[
\bar{u}(\omega, \lambda) = \frac{\left( \log \frac{\lambda}{a} \right)^{\alpha-1}}{(\log \frac{\lambda}{a})^\alpha + |\omega|^{2s}} \bar{\varphi}_a(\omega) + \frac{\left( \log \frac{\lambda}{a} \right)^{\alpha-2}}{(\log \frac{\lambda}{a})^\alpha + |\omega|^{2s}} \bar{\psi}_a(\omega) + \frac{1}{(\log \frac{\lambda}{a})^\alpha + |\omega|^{2s}} \bar{f}(\omega, \lambda),
\]
\[
:= \bar{G}_\varphi(\omega, \lambda) \bar{\varphi}_a(\omega) + \bar{G}_\psi(\omega, \lambda) \bar{\psi}_a(\omega) + \bar{G}_f(\omega, \lambda) \bar{f}(\omega, \lambda). \quad (9)
\]

Using the inverse Fourier transform and the inverse amended Laplace transform arrives at
\[
u(x, t) = G_\varphi(x, t) \ast \varphi_a(x) + G_\psi(x, t) \ast \psi_a(x) + G_f(x, t) \ast f(x, t)
\]
\[
= \int_{\mathbb{R}^d} G_\varphi(x - y, t) \varphi_a(y) dy + \int_{\mathbb{R}^d} G_\psi(x - y, t) \psi_a(y) dy
\]
\[ + \int_t^s \int_{\mathbb{R}^d} G_f(x - y, a \frac{t - \tau}{\tau})f(y, \tau) \, dy \, \frac{d\tau}{\tau}, \]  

(10)

where \( * \) is the spatial convolution and \( \ast \) denotes the convolution for time and space simultaneously.

Based on the same derivation process as the fundamental solutions (18) and (19) (corresponding to \( 0 < \alpha < 1 \)) given by [26], it is not difficult to obtain the expressions of \( G_\varphi(x, t) \) and \( G_f(x, t) \) with \( 1 < \alpha < 2 \) below

\[
G_\varphi(x, t) = \frac{1}{|x|^{d - \frac{\alpha}{2}}} H_{\frac{23}{23}}^\alpha \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^\alpha} \right) \left( 1, 1 ; (1, \alpha) \right),
\]

(11)

\[
G_f(x, t) = \frac{(\log \frac{1}{a})^{\alpha - 1}}{|x|^{d - \frac{\alpha}{2}}} H_{\frac{23}{23}}^\alpha \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^\alpha} \right) \left( 1, 1 ; (\alpha, \alpha) \right),
\]

(12)

where \( H_{\frac{23}{23}}^\alpha(z) \) represents the Fox \( H \)-function which is given by (60) in Appendix.

In the following, we shall find the expression of \( G_\varphi(x, t) \). Applying the Fourier and amended Mellin transforms to \( G_\varphi(x, t) \) yields

\[
\tilde{\mathcal{G}}_\varphi(\omega, \xi) = \mathcal{M}_\omega[\tilde{\mathcal{G}}_\varphi(\omega, t), \xi] = \frac{1}{\Gamma(1 - \xi)} \mathcal{M}_\omega[\mathcal{L}_a[\tilde{\mathcal{G}}_\varphi(\omega, t), \lambda], 1 - \xi]
\]

\[
= \frac{1}{\Gamma(1 - \xi)} \mathcal{M}_\omega[\mathcal{G}_\varphi(\omega, \lambda), 1 - \xi]
\]

\[
= \frac{1}{\Gamma(1 - \xi)} \mathcal{M}_\omega \left[ \frac{(\log \frac{\lambda}{a})^{\alpha - 2}}{(\log \frac{\lambda}{a})^\alpha + |\omega|^{2s}} \right] 1 - \xi
\]

\[
= \frac{1}{\Gamma(1 - \xi)} \int_0^\infty \frac{(\log \frac{\lambda}{a})^{\alpha - 2}}{(\log \frac{\lambda}{a})^\alpha + |\omega|^{2s}} \left( \log \frac{\lambda}{a} \right)^{1 - \xi - 1} \frac{d\lambda}{\lambda}
\]

\[
= \frac{\alpha \Gamma(1 - \xi)}{|\omega|^{2s}} \frac{1 + (\alpha - \xi - 1)}{\alpha} \Gamma \left( \frac{\xi + 1}{\alpha} \right).
\]

According to the inverse Fourier transform and formula (8), one has

\[
\tilde{\mathcal{G}}_\varphi(x, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \tilde{\mathcal{G}}_\varphi(\omega, \xi) e^{-i\omega \cdot x} \, d\omega
\]

\[
= \frac{1}{(2\pi)^d} \frac{1}{\alpha \Gamma(1 - \xi)} \Gamma \left( \frac{\alpha - \xi - 1}{\alpha} \right) \Gamma \left( \frac{\xi + 1}{\alpha} \right) \int_{\mathbb{R}^d} (|\omega|^{2s})^{-\frac{\alpha - \xi}{\alpha}} e^{-i\omega \cdot x} \, d\omega
\]

\[
= \frac{1}{(2\pi)^d} \frac{1}{\alpha \Gamma(1 - \xi)} \Gamma \left( \frac{\alpha - \xi - 1}{\alpha} \right) \Gamma \left( \frac{\xi + 1}{\alpha} \right) \frac{2\pi}{|x|^{\frac{\alpha}{2}}}
\]

\[
\times \int_0^\infty (\rho^{2s})^{-\frac{\alpha - 1}{\alpha}} \rho^\frac{d}{2} J_{\frac{d}{2} - 1}(\rho |x|) \, d\rho.
\]

Note that

\[
\int_0^\infty (\rho^{2s})^{-\frac{\alpha - 1}{\alpha}} \rho^\frac{d}{2} J_{\frac{d}{2} - 1}(\rho |x|) \, d\rho
\]

\[
= \int_0^\infty (\rho |x|)^{\frac{d}{2} - \frac{2d + (\xi + 1)}{\alpha}} J_{\frac{d}{2} - 1}(\rho |x|) \, d(\rho |x|)
\]
Therefore one gets

\[
\tilde{G}_\psi(x, \xi) = \frac{|x|^{2s(\xi+1)} \Gamma(1 - \frac{s+1}{\alpha}) \Gamma(\frac{s+1}{\alpha})}{\Gamma(s^{\xi+1}) \Gamma(1 - \xi)}.
\]

It follows from the inverse amended Mellin transform that

\[
G_\psi(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{G}_\psi(x, \xi) \left( \log \frac{t}{a} \right)^{-\xi} d\xi
\]

\[
= \frac{1}{|x|^{d-\frac{d}{2}}} \frac{1}{\Gamma(1 - \frac{s+1}{\alpha}) \Gamma(\frac{s+1}{\alpha}) \Gamma(1 - \xi)}
\times \left( \log \frac{t}{a} \right)^{-\xi}
\]

\[
\times \frac{\Gamma(1 - \frac{s+1}{\alpha}) \Gamma(\frac{s+1}{\alpha}) \Gamma(1 - \xi)}{(2 + \alpha(-\frac{s+1}{\alpha}))(1 - 1 - s(\frac{s+1}{\alpha}))}
\times \frac{|x|^{2s}}{2^{2s} \left( \log \frac{t}{a} \right)^{\alpha}}
\]

\[
= \frac{\log^2 \frac{t}{a}}{|x|^{d-\frac{d}{2}}} H^{2s}_{2s} \left( \frac{|x|^{2s}}{2^{2s} \left( \log \frac{t}{a} \right)^{\alpha}} \right),
\]

that is,

\[
G_\psi(x, t) = \frac{\log^2 \frac{t}{a}}{|x|^{d-\frac{d}{2}}} H^{2s}_{2s} \left( \frac{|x|^{2s}}{2^{2s} \left( \log \frac{t}{a} \right)^{\alpha}} \right).
\]

Next, we study the asymptotic behaviors of the fundamental solutions \(G_\psi(x, t)\), \(G_\phi(x, t)\) and \(G_\xi(x, t)\). Following the almost same method of the proof as [26], we only prove the estimation of \(G_\psi(x, t)\) and directly give the estimation results for \(G_\phi(x, t)\) and \(G_\xi(x, t)\) whose proofs are omitted. Here and after, we shall mainly consider the proof about \(G_\psi(x, t)\). We denote \(R = (\log \frac{t}{a})^{-\alpha}|x|^{2s}\) with \(\alpha \in (1, 2)\) and \(s \in (0, 1)\).

**Theorem 2.6.** ([26]) Let \(d \in \mathbb{N}, 1 < \alpha < 2\), and \(0 < s < 1\). The fundamental solution \(G_\psi(x, t)\) given by (11) satisfies the following estimations:
(1) If $R > 1$, then
\[ |G_\varphi(x,t)| \leq C \left( \log \frac{t}{a} \right)^\alpha |x|^{-2s}, \text{ for } d \geq 1, \ 0 < s < 1. \quad (14) \]

(2) If $R \leq 1$, then
\[ |G_\varphi(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha} |x|^{-d+2s}, \text{ for } d > 2s, \quad (15) \]
\[ |G_\varphi(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha} \left( 1 + \left| \log \left( \frac{1}{2} \left( \log \frac{t}{a} \right)^{-\alpha} |x| \right) \right) \right), \text{ for } d = 2s, \quad (16) \]
\[ |G_\varphi(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{2s}}, \text{ for } d < 2s. \quad (17) \]

(3) If $R > 1$, then
\[ |\nabla G_\varphi(x,t)| \leq C \left( \log \frac{t}{a} \right)^\alpha |x|^{-(d+1)-2s}, \text{ for } d \geq 1, \ 0 < s < 1. \quad (18) \]

(4) If $R \leq 1$, then
\[ |\nabla G_\varphi(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha} |x|^{-(d+1)+2s}, \text{ for } d \geq 1, \ 0 < s < 1. \quad (19) \]

**Theorem 2.7.** ([26]) Let $d \in \mathbb{N}$, $1 < \alpha < 2$, and $0 < s < 1$. The fundamental solution $G_f(x,t)$ given by (12) has the asymptotic behavior:

(1) If $R > 1$, then
\[ |G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{2\alpha-1} |x|^{-d-2s}. \quad (20) \]

(2) If $R \leq 1$, then
\[ |G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha-1} |x|^{-d+4s}, \text{ for } d > 4s, \quad (21) \]
\[ |G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha-1} \left( 1 + \left| \log \left( \left( \frac{|x|}{2} \right)^{2s} \left( \log \frac{t}{a} \right)^{-\alpha} \right) \right) \right), \text{ for } d = 4s, \quad (22) \]
\[ |G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{\alpha-1-\frac{2d}{s}}, \text{ for } d < 4s. \quad (23) \]

(3) If $R > 1$, then
\[ |\nabla G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{2\alpha-1} |x|^{-(d+1)-2s}, \text{ for } d \geq 1, \ 0 < s < 1. \quad (24) \]

(4) If $R \leq 1$, then
\[ |\nabla G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha-1} |x|^{-(d+1)+4s}, \text{ for } d + 2 > 4s, \quad (25) \]
\[ |\nabla G_f(x,t)| \leq C \left( \log \frac{t}{a} \right)^{-\alpha-1} |x| \left( 1 + \left| \log \left( \left( \frac{|x|}{2} \right)^{2s} \left( \log \frac{t}{a} \right)^{-\alpha} \right) \right) \right), \quad \text{for } d + 2 = 4s, \quad (26) \]
\[
\| \nabla G_f(x, t) \| \leq C |x| \left( \log \frac{t}{a} \right)^{\alpha - \frac{\alpha(d+2)}{2s}}, \quad \text{for } d + 2 < 4s. \tag{27}
\]

(5) If \( R > 1 \), then
\[
| \delta G_f(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{2\alpha - 2} |x|^{-d - 2s}, \quad \text{for } d \geq 1, 0 < s < 1. \tag{28}
\]

(6) If \( R \leq 1 \), then
\[
| \delta G_f(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\alpha - 2} |x|^{-d + 4s}, \quad \text{for } d > 4s, \tag{29}
\]
\[
| \delta G_f(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\alpha - 2} \left( 1 + \left| \log \left( \frac{|x|}{2} \right)^{2s} \left( \log \frac{t}{a} \right)^{-\alpha} \right) \right), \quad \text{for } d = 4s, \tag{30}
\]
\[
| \delta G_f(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\alpha - 2 - \frac{4\alpha}{2s}}, \quad \text{for } d < 4s. \tag{31}
\]

We now present the asymptotic behavior of the fundamental solution \( G_\psi(x, t) \).

**Theorem 2.8.** Let \( d \in \mathbb{N} \), \( 1 < \alpha < 2 \), and \( 0 < s < 1 \). For the fundamental solution \( G_\psi(x, t) \) given by (13), the following estimates hold.

(1) If \( R > 1 \), then
\[
| G_\psi(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{\alpha + 1} |x|^{-d - 2s}, \quad \text{for } d \geq 1, 0 < s < 1. \tag{32}
\]

(2) If \( R \leq 1 \), then
\[
| G_\psi(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\alpha + 1} |x|^{-d + 2s}, \quad \text{for } d > 2s, \tag{33}
\]
\[
| G_\psi(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\alpha + 1} \left( 1 + \left| \log \left( \frac{1}{2} \left( \log \frac{t}{a} \right)^{-\alpha} |x| \right) \right) \right), \quad \text{for } d = 2s, \tag{34}
\]
\[
| G_\psi(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2} + 1}, \quad \text{for } d < 2s. \tag{35}
\]

(3) If \( R > 1 \), then
\[
| \nabla G_\psi(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{\alpha + 1} |x|^{-(d+1) - 2s}, \quad \text{for } d \geq 1, 0 < s < 1. \tag{36}
\]

(4) If \( R \leq 1 \), then
\[
| \nabla G_\psi(x, t) \| \leq C \left( \log \frac{t}{a} \right)^{-\alpha + 1} |x|^{-(d+1) + 2s}, \quad \text{for } d \geq 1, 0 < s < 1. \tag{37}
\]

**Proof.** (1) If \( R > 1 \), we consider the Fox \( H \)-function in \( G_\psi(x, t) \) below,
\[
H_{23}^{21} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{t}{a})^\alpha} \right) = H_{23}^{21} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{t}{a})^\alpha} \right) (1, 1); (2, \alpha) (1, \frac{1}{2}, s); (1, s) , \quad x \neq 0.
\]

In this case, \( \alpha^* = 2 - \alpha > 0 \). It follows from equality (69) that
\[
H_{23}^{21} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{t}{a})^\alpha} \right) = \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} h_{lk} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{t}{a})^\alpha} \right)^{l-\frac{1}{2}+k} = \sum_{k=0}^{\infty} \frac{1}{h_{lk}} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{t}{a})^\alpha} \right)^{-k}.
\]
Note that
\[ h_{10} = \frac{\Gamma(1)\Gamma\left(\frac{d}{a}\right)}{\Gamma(2)\Gamma(0)} = 0, \quad h_{11} = -\frac{\Gamma(2)\Gamma\left(\frac{d}{a} + s\right)}{\Gamma(2 + \alpha)\Gamma(-s)} > 0, \]
one has
\[ H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right) = h_{11}\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right)^{-1} + o\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right)^{-1}, \]
which gives
\[ |G_\phi(x, t)| \leq C\left(\log \frac{t}{a}\right)^{\alpha + 1} |x|^{-d - 2s}, \quad R \to \infty. \]

So there exists a constant \( K \) \((K > 1)\) such that
\[ |G_\psi(x, t)| \leq C\left(\log \frac{t}{a}\right)^{\alpha + 1} |x|^{-d - 2s}, \quad R > K. \tag{38} \]

It is easy to see that the H-function \( H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right) \) is bounded in \( 1 < R \leq K \) due to its analyticity for \( a^* > 0 \) and \( x \neq 0 \). Then one obtains
\[ |G_\psi(x, t)| \leq C\left(\log \frac{t}{a}\right)^{\alpha + 1} |x|^{-d - 2s}, \quad 1 < R \leq K. \tag{39} \]

Based on (38) and (39), it holds that
\[ |G_\psi(x, t)| \leq C\left(\log \frac{t}{a}\right)^{\alpha + 1} |x|^{-d - 2s}, \text{ if } R > 1, d \geq 1, 0 < s < 1. \]

(2) Let us show the case for \( R \leq 1 \). If \( d > 2s \), then \( b_{1,0} = -\frac{1-s}{\sigma} = -(\sigma + 1) \), \( b_{2k} = -\frac{d}{2 + k} \), and \( b_{1,0} \neq b_{2k} \) with \( \sigma, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Therefore, \( b_{1,0} \) is a simple pole while the poles \( b_{1,\sigma} \) (\( \sigma = 1, 2, \ldots \)) and \( b_{2k} \) \((k = 0, 1, \cdots)\) may coincide. It follows from formula (70) that
\[ H_{23}^{21}\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right) = h_1^*\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right) + o\left(\frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha}\right), \quad \frac{|x|^{2s}}{2^{2s}(\log \frac{x}{a})^\alpha} \to 0, \]
where \( h_1^* = \frac{\Gamma\left(\frac{d - s}{(2 - \alpha)\Gamma(\sigma)}\right)}{\Gamma\left(\frac{d}{2 - \alpha}\right)} > 0 \). As a result,
\[ |G_\psi(x, t)| \leq C\left(\log \frac{t}{a}\right)^{\alpha + 1} |x|^{-d + 2s}, \quad R \to 0. \]

Thus there exists a constant \( \varepsilon_0 \) \((0 < \varepsilon_0 < 1)\) such that
\[ |G_\psi(x, t)| \leq C\left(\log \frac{t}{a}\right)^{-\alpha + 1} |x|^{-d + 2s}, \quad R < \varepsilon_0. \tag{40} \]
Note that $H_{23}^{21} \left( \frac{|x|}{2^{s}(\log \frac{1}{a})^{\alpha}} \right)$ is bounded in $\delta_0 \leq R \leq 1$ due to the fact that the $H$-function is analytic as $a^* > 0$ and $x \neq 0$, it holds that

$$|G_\psi(x, t)| \leq C \left( \frac{t}{a} \right)^{-\frac{d}{2}} |x|^{-d} \leq C \left( \frac{t}{a} \right)^{-\alpha + 1} |x|^{-d + 2s}, \ v_0 \leq R \leq 1. \quad (41)$$

Combining inequalities (40) and (41) gives

$$|G_\psi(x, t)| \leq C \left( \frac{t}{a} \right)^{-\alpha + 1} |x|^{-d + 2s}, \ if \ R \leq 1, \ d \geq 2s, \ 0 < s < 1.$$

If $d = 2s$ (in this situation $x = x \in \mathbb{R}$), then

$$H_{23}^{21} \left( \frac{|x|}{2(\log \frac{1}{a})^{\alpha}} \right) = H_{23}^{21} \left( \frac{|x|}{2(\log \frac{1}{a})^{\alpha}} \right) \left( 1, 1; (2, \alpha) \right) \left( 1, 1, \left( \frac{1}{2}, \frac{1}{2} \right); (1, \frac{1}{2}) \right), \ x \neq 0.$$

We find that the poles $b_{10} = b_{20} = -1$ are coincided. Thus formula (70) yields that

$$H_{23}^{21} \left( \frac{|x|}{2(\log \frac{1}{a})^{\alpha}} \right) = H_1^*( \left( \frac{|x|}{2(\log \frac{1}{a})^{\alpha}} \right) \log \left( \frac{|x|}{2(\log \frac{1}{a})^{\alpha}} \right)$$

$$= C \left( \frac{t}{a} \right)^{-\alpha + 1} \left( 1 + \log \left( \frac{1}{2} \left( \frac{t}{a} \right)^{-\alpha} \right) \right), \ R \to 0.$$

A similar argument as above analysis produces

$$|G_\psi(x, t)| \leq C \left( \frac{t}{a} \right)^{-\alpha + 1} \left( 1 + \log \left( \frac{1}{2} \left( \frac{t}{a} \right)^{-\alpha} \right) \right), \ if \ R \leq 1, \ d = 2s.$$

If $d < 2s$ (in this case $x = x \in \mathbb{R}$), since $b_{10} = -1$ and $b_{20} = -\frac{1}{2s}$ are both simple poles, using formula (70) again yields that

$$H_{23}^{21} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^{\alpha}} \right) = h_2^* \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^{\alpha}} \right)^{\frac{1}{2s}} + o \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^{\alpha}} \right)^{\frac{1}{2s}}, \ R \to 0.$$

where $h_2^* = \frac{r(1 - \frac{s}{2})}{4(2 - 2s) \Gamma(\frac{1}{2})} > 0$. Consequently,

$$|G_\psi(x, t)| \leq C \left( \frac{t}{a} \right)^{-\frac{1}{2s}} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^{\alpha}} \right)^{\frac{1}{2s}} \leq C \left( \frac{t}{a} \right)^{-\frac{1}{2s} + 1}, \ R \to 0.$$

Using the same technique as before implies

$$|G_\psi(x, t)| \leq C \left( \frac{t}{a} \right)^{-\frac{1}{2s} + 1}, \ if \ R \leq 1, \ d < 2s.$$

(3) Now we estimate $\nabla G_\psi(x, t)$ with $R > 1$. By formula (66), one has

$$\nabla G_\psi(x, t) = -\pi^{-\frac{d}{2}} |x|^{-d - 1} \log t \left( H_{23}^{21} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{a})^{\alpha}} \right) \right) \left( 1, 1; (2, \alpha), (d, 2s) \right) \left( x_1, x_2, \ldots, x_d \right) \left( \frac{x_1}{|x|}, \frac{x_2}{|x|}, \ldots, \frac{x_d}{|x|} \right).$$
\[ -\frac{\log t}{a} H_{34}^{H} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right) \left( (1, 1); (2, \alpha), (d, 2s) \right) \left( (1, 1), (\frac{d}{2}, s), (d + 1, 2s); (1, s) \right) \left( \frac{x_1}{|x|}, \frac{x_2}{|x|}, \ldots, \frac{x_d}{|x|} \right), x \neq 0. \]

Furthermore, it holds that
\[
|\nabla G_\psi(x, t)| = \frac{\log t}{a} H_{34}^{H} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right) \left( (1, 1); (2, \alpha), (d, 2s) \right) \left( (1, 1), (\frac{d}{2}, s), (d + 1, 2s); (1, s) \right), \quad x \neq 0, \quad (42)
\]
and \( a^* = 2 - \alpha > 0. \)

It is not difficult to derive from equality (69) that
\[
H_{34}^{H} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right) = \sum_{k=0}^{\infty} h_{1k} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right)^{-k}.
\]

A simple calculus shows
\[
h_{10} = \frac{\Gamma(1)\Gamma(d+1)\Gamma(\frac{d}{2})}{\Gamma(2)\Gamma(d)\Gamma(0)} = 0, \quad h_{11} = -\frac{\Gamma(2)\Gamma(\frac{d}{2}+s)\Gamma(d+1+2s)}{\Gamma(2+\alpha)\Gamma(d+2s)\Gamma(-s)} > 0.
\]

Thus one obtains
\[
H_{34}^{H} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right) = h_{11} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right)^{-1} + o \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right)^{-1}, \quad \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \to \infty,
\]
and
\[
|\nabla G_\psi(x, t)| \leq C \left( \frac{\log t}{a} \right) \pi^{-\frac{d}{2}} |x|^{-d-1} h_{11} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right)^{-1}
\]
\[
\leq C \left( \frac{\log t}{a} \right)^{\alpha+1} |x|^{-(d+1)-2s}, \quad R \to \infty.
\]

So there holds
\[
|\nabla G_\psi(x, t)| \leq C \left( \frac{\log t}{a} \right)^{\alpha+1} |x|^{-(d+1)-2s}, \quad \text{if } R > 1, \ d \geq 1, \ 0 < s < 1.
\]

Finally, we prove (37). When \( d > 2s, \) since \( b_{10} = -1 \) is a simple pole, applying formula (70) indicates that
\[
H_{34}^{H} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right) = h_{11}^{*} \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right) + o \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \right), \quad \frac{|x|^{2s}}{2^{2s}(\log \frac{1}{t})^{\alpha}} \to 0,
\]
with \( h_{11}^{*} = \frac{\Gamma(\frac{d}{2}-s)\Gamma(d+1-2s)\Gamma(1)}{\Gamma(2-\alpha)\Gamma(d-2s)\Gamma(s)} > 0. \) Therefore,
\[
|\nabla G_\psi(x, t)| \leq C \left( \frac{\log t}{a} \right)^{-\alpha+1} |x|^{-(d+1)+2s}, \quad R \to 0.
\]

Hence one has
\[
|\nabla G_\psi(x, t)| \leq C \left( \frac{\log t}{a} \right)^{-\alpha+1} |x|^{-(d+1)+2s}, \quad \text{if } R \leq 1, \ d > 2s, \ 0 < s < 1. \quad (43)
\]

When \( d = 2s, \) applying (65) to equation (42) yields
\[
H_{34}^{H} \left( \frac{|x|}{2(\log \frac{1}{t})^{\alpha}} \right) = H_{34}^{H} \left( \frac{|x|}{2(\log \frac{1}{t})^{\alpha}} \right) \left( (1, 1); (2, \alpha), (1, 1) \right) \left( (1, 1), (\frac{3}{2}, \frac{1}{2}), (2, 1); (1, \frac{1}{2}) \right)
\]
Therefore it follows that
\[ \text{Theorem 2.9.} \]

Observe that \( b_{10} = -1 \) and \( b_{20} = -2 \) are simple poles, formula (70) gives
\[
H_{34}^{\frac{|x|}{2(\log \frac{|x|}{a})^\alpha}} = h_1^\ast \left( \frac{|x|}{2(\log \frac{|x|}{a})^\alpha} \right) + o \left( \frac{|x|}{2(\log \frac{|x|}{a})^\alpha} \right), \quad \frac{|x|}{2(\log \frac{|x|}{a})^\alpha} \to 0,
\]
where \( h_1^\ast = \frac{2}{\Gamma(2-\alpha)\Gamma(\frac{1}{2})} > 0 \). So there holds
\[
|\nabla G_\psi(x,t)| \leq C \left( \frac{\log t}{a} \right)^{-\frac{2}{\alpha}+\frac{1}{2}} \left| \nabla \right|^{-2} h_1^\ast \left( \frac{|x|}{2(\log \frac{|x|}{a})^\alpha} \right) \leq C \left( \frac{\log t}{a} \right)^{-\alpha+1} |x|^{-1}, \quad R \to 0.
\]
Based on the same reason as before, one has
\[
|\nabla G_\psi(x,t)| \leq C \left( \frac{\log t}{a} \right)^{-\alpha+1} |x|^{-1}, \quad \text{if } R \leq 1, d = 2s. \quad (44)
\]
When \( d < 2s \), we see that \( b_{10} = -1 \) and \( b_{20} = -\frac{1}{2s} \) are simple poles, while
\[
h_2^\ast = \frac{\Gamma(1-\frac{1}{2s})\Gamma(\frac{1}{2})}{\frac{1}{2s}\Gamma(\frac{2-\alpha}{2s})\Gamma(0)\Gamma(\frac{1}{2})} = 0, \quad h_1^\ast = \frac{\Gamma(\frac{1}{2} - s)\Gamma(2 - 2s)\Gamma(1)}{\Gamma(2-\alpha)\Gamma(1-2s)\Gamma(s)} > 0.
\]
Once again formula (70) yields
\[
H_{34}^{\frac{|x|^{2s}}{2^{2s}(\log \frac{|x|}{a})^\alpha}} = h_1^\ast \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{|x|}{a})^\alpha} \right) + o \left( \frac{|x|^{2s}}{2^{2s}(\log \frac{|x|}{a})^\alpha} \right), \quad \frac{|x|^{2s}}{2^{2s}(\log \frac{|x|}{a})^\alpha} \to 0,
\]
which implies
\[
|\nabla G_\psi(x,t)| \leq C \left( \frac{\log t}{a} \right)^{-\alpha+1} |x|^{-2+2s}, \quad \text{if } R \leq 1, d < 2s. \quad (45)
\]
Therefore it follows that
\[
|\nabla G_\psi(x,t)| \leq C \left( \frac{\log t}{a} \right)^{-\alpha+1} |x|^{-2+2s}, \quad \text{if } R \leq 1, d \geq 1, 0 < s < 1. \quad (46)
\]
The proof is thus ended. \( \square \)

In order to study asymptotic behavior of the solution to Equ. (1), we need the following estimations of the fundamental solutions \( G_\varphi(x,t), G_\varphi(X,t) \) and \( G_f(x,t) \) in \( L^p(\mathbb{R}^d) \) and \( L^{p,\infty}(\mathbb{R}^d) \). For the sake of convenience, we denote \( || \cdot ||_p = || \cdot ||_{L^p(\mathbb{R}^d)}, || \cdot ||_{p,\infty} = || \cdot ||_{L^{p,\infty}(\mathbb{R}^d)}, \) and let
\[
\kappa(d,s) = \begin{cases} \frac{d}{2s-2s}, & d > 2s, \\ \infty, & d \leq 2s. \end{cases}
\]

**Theorem 2.9.** Let \( d \in \mathbb{N}, 1 < \alpha < 2, \) and \( 0 < s < 1. \)

1. Assume \( 1 \leq p < \kappa(d,s). \) Then \( G_\varphi(x,t) \in L^p(\mathbb{R}^d) \) for any \( t > a \) and
\[
||G_\varphi(x,t)||_p \leq C \left( \frac{\log \frac{t}{a}}{a} \right)^{-\frac{d}{2s}(1-\frac{1}{d})}, \quad t > a. \quad (46)
\]
(2) If \( p = \frac{d}{d-2s} \) and \( d > 2s \). Then \( G_\varphi(x,t) \in L^{\frac{2s}{d-2s}}(\mathbb{R}^d) \) and there holds
\[
\|G_\varphi(x,t)\|_{\frac{2s}{d-2s},\infty} \leq C \left( \frac{t}{a} \right)^{-\alpha}, \quad t > a. \tag{47}
\]

**Remark 1.** In the case \( d < 2s \), notice that Theorem 2.6, we can obtain \( G_\varphi(\cdot,t) \in L^\infty(\mathbb{R}) \) and \( \|G_\varphi(x,t)\|_{\infty} \leq C \left( \frac{1}{a} \right)^{\frac{1}{d}} \) for any \( t > a \).

**Theorem 2.10.** Let \( d \in \mathbb{N}, \ 1 < \alpha < 2 \) and \( 0 < s < 1 \).

1. Suppose \( 1 \leq p < \kappa(d,s) \). Then \( G_\psi(x,t) \in L^p(\mathbb{R}^d) \) for all \( t > a \) and
\[
\|G_\psi(x,t)\|_p \leq C \left( \frac{t}{a} \right)^{-\frac{d}{p}(1-\frac{1}{p})+1}, \quad t > a. \tag{48}
\]

2. If \( p = \frac{d}{d-2s} \) and \( d > 2s \). Then one has \( G_\psi(x,t) \in L^{\frac{2s}{d-2s}}(\mathbb{R}^d) \) and
\[
\|G_\psi(x,t)\|_{\frac{2s}{d-2s},\infty} \leq C \left( \frac{t}{a} \right)^{-\alpha+1}, \quad t > a. \tag{49}
\]

**Proof.** (1) We decompose the integral of \( G_\psi(x,t) \) in \( L^p(\mathbb{R}^d) \) as
\[
\|G_\psi(x,t)\|_p = \int_{R>1} |G_\psi(x,t)|^p dx + \int_{R \leq 1} |G_\psi(x,t)|^p dx.
\]
By (32), one has
\[
\int_{R>1} |G_\psi(x,t)|^p dx \leq C \int_{R>1} \left( \log \frac{t}{a} \right)^{\alpha p + p} |x|^{-dp - 2sp} dx
\]
\[
\leq C \left( \log \frac{t}{a} \right)^{\alpha p + p} \int_0^\infty \rho^{-dp - 2sp} \rho^{d-1} d\rho
\]
\[
\leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{p}(1-\frac{1}{p})+1}.
\]
That is,
\[
\left( \int_{R>1} |G_\psi(x,t)|^p dx \right)^{\frac{1}{p}} \leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{p}(1-\frac{1}{p})+1}, \quad d \geq 1, \ 1 \leq p < \infty. \tag{50}
\]

In the case with \( d > 2s \), from inequality (33), there holds
\[
\int_{R \leq 1} |G_\psi(x,t)|^p dx \leq C \int_{R \leq 1} \left( \log \frac{t}{a} \right)^{-\alpha p + p} |x|^{-dp + 2sp} dx
\]
\[
\leq C \left( \log \frac{t}{a} \right)^{-\alpha p + p} \int_0^{(\log \frac{t}{a})^{\frac{1}{d}}} \rho^{(2s-d)p} \rho^{d-1} d\rho
\]
\[
\leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{p}p + \frac{2s}{d}p + p},
\]
where the condition \( p < \kappa(d,s) \) is used. Hence one gets
\[
\left( \int_{R \leq 1} |G_\psi(x,t)|^p dx \right)^{\frac{1}{p}} \leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{p}(1-\frac{1}{p})+1}, \quad 1 \leq p < \kappa(d,s).
\]
If $d = 2s$, using (34) implies
\[
\int_{R \leq 1} |G_\psi(x, t)|^p dx \leq C \int_0^{(\log \frac{t}{a})^\alpha} \left( \log \frac{t}{a} \right)^{-\alpha p + p} \left( 1 + |\log \frac{t}{a} (\log \frac{t}{a})^{-\alpha} \right)^p dx
\]
\[
= C \left( \log \frac{t}{a} \right)^{-\alpha p + \alpha} \int_0^\frac{t}{a} (1 + |\log \eta|)^p d\eta
\]
\[
= C \Gamma(p+1) \left( \log \frac{t}{a} \right)^{-\alpha p + \alpha + p}.
\]
Thus,
\[
\left( \int_{R \leq 1} |G_\psi(x, t)|^p dx \right)^\frac{1}{p} \leq C \left( \log \frac{t}{a} \right)^{-\alpha (1 - \frac{d}{2p}) + 1}, \quad 1 \leq p < \infty.
\]
For case $d < 2s$, from (35) one has
\[
\int_{R \leq 1} |G_\psi(x, t)|^p dx \leq C \int_{R \leq 1} \left( \log \frac{t}{a} \right)^{-\frac{d s}{2p} + p} dx
\]
\[
\leq C \int_0^{(\log \frac{t}{a})^\alpha} \left( \log \frac{t}{a} \right)^{-\frac{d s}{2p} + p} dx \leq C \left( \log \frac{t}{a} \right)^{-\frac{d s}{2p} + \frac{1}{2} + p},
\]
i.e.,
\[
\left( \int_{R \leq 1} |G_\psi(x, t)|^p dx \right)^\frac{1}{p} \leq C \left( \log \frac{t}{a} \right)^{-\frac{d s}{2p} (1 - \frac{1}{p}) + 1}, \quad 1 \leq p < \infty.
\]
Combining the above estimates, we obtain for all $1 \leq p < \kappa(d, s)$ with $d \geq 1$ and $0 < s < 1$ that
\[
||G_\psi(x, t)||_p = \left( \int_{R > 1} |G_\psi(x, t)|^p dx + \int_{R \leq 1} |G_\psi(x, t)|^p dx \right)^\frac{1}{p}
\]
\[
\leq \left( \int_{R > 1} |G_\psi(x, t)|^p dx \right)^\frac{1}{p} + \left( \int_{R \leq 1} |G_\psi(x, t)|^p dx \right)^\frac{1}{p}
\]
\[
\leq C \left( \log \frac{t}{a} \right)^{-\frac{d s}{2p} (1 - \frac{1}{p}) + 1},
\]
which proves (48).

(2) Now we come to estimate (49). Let $R = (\log \frac{t}{a})^{-\alpha} |x|^2 s$ and $p = \frac{d}{a - 2s}$, then
\[
||G_\psi(x, t)||_{p, \infty} = (||G_\psi(x, t)\chi_{\{R > 1\}}(t) + G_\psi(x, t)\chi_{\{R \leq 1\}}(t)||_{p, \infty})
\]
\[
\leq 2(||G_\psi(x, t)\chi_{\{R > 1\}}(t)||_{p, \infty} + ||G_\psi(x, t)\chi_{\{R \leq 1\}}(t)||_{p, \infty}),
\]
in which $\chi_{\{E\}}(\xi)$ represents the characteristic function below
\[
\chi_{\{E\}}(\xi) = \begin{cases} 1, & \xi \in E, \\ 0, & \xi \not\in E. \end{cases}
\]
Employing (50) implies
\[
||G_\psi(x, t)\chi_{\{R > 1\}}(t)||_{p, \infty} \leq ||G_\psi(x, t)\chi_{\{R > 1\}}(t)||_p
\]
\[
\leq C \left( \log \frac{t}{a} \right)^{-\frac{d s}{2p} (1 - \frac{1}{p}) + 1} = C \left( \log \frac{t}{a} \right)^{-\alpha + 1}.
\]
We continue to estimate \( \|G_\psi(x, t)\chi_{\{R \leq 1\}}(t)\|_{p, \infty} \). In view of the definition of (56) and inequality (33), it holds that

\[
d_{G_\psi(x, t)\chi_{\{R \leq 1\}}(t)}(\gamma) = \varrho\{\{x \in \mathbb{R}^d : |G_\psi(x, t)| > \gamma \text{ and } R \leq 1\}\}
\]

\[
\leq \varrho\left(\left\{x \in \mathbb{R}^d : \gamma < C \left(\log \frac{t}{a}\right)^{-\alpha + 1} |x|^{-2s - d}\right\}\right)
\]

\[
= \varrho\left(\left\{x \in \mathbb{R}^d : |x|^{d-2s} < C \left(\log \frac{t}{a}\right)^{-\alpha + 1} \gamma^{-1}\right\}\right)
\]

\[
= \varrho\left(\left\{x \in \mathbb{R}^d : |x| < C \left(\log \frac{t}{a}\right)^{-\alpha + 1} \gamma^{-1}\right\}\right)
\]

\[
\leq C \left(\log \frac{t}{a}\right)^{-\alpha + 1} \gamma^{-1} \chi_{\{d > 4s\}} \leq C \left(\log \frac{t}{a}\right)^{-\alpha + 1}
\]

where \( \varrho \) is the measure on \( \mathbb{R}^d \). Furthermore one gets

\[
\gamma(d_{G_\psi(x, t)\chi_{\{R \leq 1\}}}(\gamma))^\frac{1}{p} \leq C \left(\log \frac{t}{a}\right)^{-\alpha + 1}
\]

and thus

\[
\|G_\psi(x, t)\chi_{\{R \leq 1\}}(t)\|_{p, \infty} \leq C \left(\log \frac{t}{a}\right)^{-\alpha + 1}
\]

as required and this yields the second claim.

\[ \square \]

**Remark 2.** When \( d < 2s \), from Theorem 2.8, it follows that \( G_\psi(\cdot, t) \in L^\infty(\mathbb{R}) \) and \( \|G_\psi(x, t)\|_{\infty} \leq C \left(\log \frac{t}{a}\right)^{-\frac{d}{2s} + 1} \) for any \( t > a \).

We can also derive the estimates for the fundamental solution \( G_f(x, t) \) by similar argument as above. Denote

\[
\varpi(d, s) = \begin{cases} \frac{d}{\pi^{d/2}}, & d > 4s, \\ \infty, & d \leq 4s. \end{cases}
\]

**Theorem 2.11.** Let \( d \in \mathbb{N}, 1 < \alpha < 2, \) and \( 0 < s < 1 \).

1. If \( 1 \leq p < \varpi(d, s) \), then \( G_f(x, t) \in L^p(\mathbb{R}^d) \) for all \( t > a \) and

\[
\|G_f(x, t)\|_p \leq C \left(\log \frac{t}{a}\right)^{\alpha - 1 - \frac{d}{2s}(1 - \frac{d}{2})}, \quad t > a.
\]

(51)

2. If \( p = \varpi(d, s) \) and \( d > 4s \), then \( G_f(x, t) \in L^{\frac{d}{2s}w, \infty}(\mathbb{R}^d) \) and it holds that

\[
\|G_f(x, t)\|_{\frac{d}{2s}w, \infty} \leq C \left(\log \frac{t}{a}\right)^{-\alpha - 1}, \quad t > a.
\]

**Remark 3.** If \( d < 4s \), by Theorem 2.7, we easily know that both \( G_f(\cdot, t) \in L^\infty(\mathbb{R}^d) \) and \( \|G_f(x, t)\|_{\infty} \leq C \left(\log \frac{t}{a}\right)^{\alpha - 1 - \frac{d}{2s}} \) hold for all \( t > a \).

Next, we present asymptotic behavior of the solution to Eqn. (1) with \( f \equiv 0 \), which is our main theorem. The basic tool used in the proof is Young’s inequality given by Appendix.
Theorem 2.12. Let $d \in \mathbb{N}$, $1 < \alpha < 2$, $0 < s < 1$, and $1 \leq p \leq \infty$ and $f \equiv 0$. Then $u(x, t) = G_{\varphi}(x, t) * \varphi_a(x) + G_{\psi}(x, t) * \psi_a(x)$ has the following asymptotic estimates.
(1) If $1 \leq p < \infty$, then:
(i) When $p < \kappa(d, s)$, for $1 \leq q, r \leq \infty$, $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $\varphi_a(x), \psi_a(x) \in L^q(\mathbb{R}^d)$, one has
$$||u(x, t)||_r \leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}(1-\frac{1}{r})} ||\varphi_a(x)||_q + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}(1-\frac{1}{r})+1} ||\psi_a(x)||_q, \quad t > a.$$
(ii) When $p = \kappa(d, s) < \infty$, one has:
(a) Assume that $1 < q, r < \infty$, $1 + \frac{1}{r} \leq \frac{1}{q}$, and $\varphi_a(x), \psi_a(x) \in L^q(\mathbb{R}^d)$. Then there holds
$$||u(x, t)||_r \leq C \left( \log \frac{t}{a} \right)^{-\alpha} ||\varphi_a(x)||_q + C \left( \log \frac{t}{a} \right)^{-\alpha+1} ||\psi_a(x)||_q, \quad t > a.$$
(b) Assume that $\varphi_a(x), \psi_a(x) \in L^1(\mathbb{R}^d)$. Then there holds
$$||u(x, t)||_{s, \infty} \leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}} ||\varphi_a(x)||_1 + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}+1} ||\psi_a(x)||_1, \quad t > a.$$
(2) If $p = \infty$, then
$$||u(x, t)||_{\infty} \leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}} ||\varphi_a(x)||_1 + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}+1} ||\psi_a(x)||_1, \quad t > a,$$
holds for $d < 2s$.

Proof. (1) We first consider (i). Applying Young’s inequality (57), Theorems 2.9 and 2.10 yields
$$||u(x, t)||_r = ||G_{\varphi}(x, t) * \varphi_a(x) + G_{\psi}(x, t) * \psi_a(x)||_r \leq ||G_{\varphi}(x, t) * \varphi_a(x)||_r + ||G_{\psi}(x, t) * \psi_a(x)||_r \leq ||G_{\varphi}(x, t)||_p \cdot ||\varphi_a(x)||_q + ||G_{\psi}(x, t)||_p \cdot ||\psi_a(x)||_q.$$
$$\leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}(1-\frac{1}{r})} ||\varphi_a(x)||_q + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{2}(1-\frac{1}{r})+1} ||\psi_a(x)||_q.$$

We now prove (ii). For case (a), by using the inequality (59), Theorems 2.9 and 2.10, it holds that
$$||u(x, t)||_r \leq ||G_{\varphi}(x, t) * \varphi_a(x)||_r + ||G_{\psi}(x, t) * \psi_a(x)||_r \leq C||G_{\varphi}(x, t)||_{p, \infty} \cdot ||\varphi_a(x)||_q + C||G_{\psi}(x, t)||_{p, \infty} \cdot ||\psi_a(x)||_q \leq C \left( \log \frac{t}{a} \right)^{-\alpha} ||\varphi_a(x)||_q + C \left( \log \frac{t}{a} \right)^{-\alpha+1} ||\psi_a(x)||_q.$$

For case (b), due to the inequality (58), Theorems 2.9 and 2.10, one obtains
$$||u(x, t)||_{p, \infty} = ||G_{\varphi}(x, t) * \varphi_a(x) + G_{\psi}(x, t) * \psi_a(x)||_{p, \infty} \leq 2||G_{\varphi}(x, t) * \varphi_a(x)||_{p, \infty} + 2||G_{\psi}(x, t) * \psi_a(x)||_{p, \infty} \leq 2C||G_{\varphi}(x, t)||_{p, \infty} \cdot ||\varphi_a(x)||_1 + 2C||G_{\psi}(x, t)||_{p, \infty} \cdot ||\psi_a(x)||_1 \leq C \left( \log \frac{t}{a} \right)^{-\alpha} ||\varphi_a(x)||_1 + C \left( \log \frac{t}{a} \right)^{-\alpha+1} ||\psi_a(x)||_1.$$

(2) By virtue of Young’s inequality (57), Remarks 1 and 2, one has
$$||u(x, t)||_\infty \leq ||G_{\varphi}(x, t) * \varphi_a(x)||_\infty + ||G_{\psi}(x, t) * \psi_a(x)||_\infty.$$
$\leq ||G_\varphi(x,t)||_\infty \cdot ||\varphi_0(x)||_1 + ||G_\psi(x,t)||_\infty \cdot ||\psi_0(x)||_1$

$\leq C \left( \log \frac{t}{a} \right)^{-\frac{q}{p}} ||\varphi_0(x)||_1 + C \left( \log \frac{t}{a} \right)^{-\frac{q}{p} + 1} ||\psi_0(x)||_1.$

All this completes the proof. $\square$

The following assertion provides asymptotic behavior of the solution to Equ. (1) for the case with $\varphi_0 = \psi_0 \equiv 0$, whose proof is similar to that of Theorem 3.9 in [26] and the details are omitted.

**Theorem 2.13.** Let $d \in \mathbb{N}$, $1 < \alpha < 2$, $0 < s < 1$, and $\varphi_0 = \psi_0 \equiv 0$. Let $1 \leq p < \kappa(d,s)$, $1 \leq q,r < \infty$, and $1 + \frac{1}{r} = \frac{1}{2} + \frac{1}{q}$. Assume that $f(\cdot, t) \in L^q(\mathbb{R}^d)$ for each $t \geq a$ and for some $1 \leq q < \infty$. Assume further that $f(x,t)$ satisfies the condition

$$||f(x,t)||_q \leq C \left( 1 + \log \frac{t}{a} \right)^{-\gamma}, \quad t > a,$$

for some $\gamma > 0$. Then for $u(x,t) = G_f(x,t) \ast f(x,t)$, there holds

$$||u(x,t)||_r \leq C \left( \log \frac{t}{a} \right)^{\alpha - \min(1,\gamma) - \frac{q}{p}(1 - \frac{1}{r})}, \quad t > a,$$

if $\gamma \neq 1$, and

$$||u(x,t)||_r \leq C \left( \log \frac{t}{a} \right)^{\alpha - 1 - \frac{q}{p}(1 - \frac{1}{r})} \log \left( 1 + \log \frac{t}{a} \right), \quad t > a,$$

if $\gamma = 1$.

3. **Gradient estimates and large time behavior.** In the present section, we investigate the spatial gradient of $G_\varphi(x,t)$, $G_\psi(x,t)$ and $G_f(x,t)$ and estimate asymptotic behaviors of $\nabla u(x,t)$. Then the large time behavior of the solution $u(x,t)$ is also provided. We begin with the estimation of $G_\varphi(x,t)$.

**Theorem 3.1.** Let $d \in \mathbb{N}$, $1 < \alpha < 2$, and $0 < s < 1$.

1. Assume $1 \leq p < \kappa^*(d,s) = \frac{d}{d+1-2s}$. Then $\nabla G_\varphi(x,t) \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ for all $t > a$ and

$$||\nabla G_\varphi(x,t)||_p \leq C \left( \log \frac{t}{a} \right)^{-\frac{q}{p} - \frac{q}{p}(1 - \frac{1}{r})}, \quad t > a.$$

2. If $p = \kappa^*(d,s) = \frac{d}{d+1-2s}$, then $\nabla G_\varphi(x,t) \in L^{\frac{d}{d+1-2s}; \infty}(\mathbb{R}^d; \mathbb{R}^d)$, and the following holds

$$||\nabla G_\varphi(x,t)||_{\frac{d}{d+1-2s}; \infty} \leq C \left( \log \frac{t}{a} \right)^{-\alpha}, \quad t > a.$$

**Theorem 3.2.** Let $d \in \mathbb{N}$, $1 < \alpha < 2$, and $0 < s < 1$.

1. Assume $1 \leq p < \kappa^*(d,s)$. Then $\nabla G_\psi(x,t) \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ for all $t > a$ and

$$||\nabla G_\psi(x,t)||_p \leq C \left( \log \frac{t}{a} \right)^{-\frac{q}{p} - \frac{q}{p}(1 - \frac{1}{r}) + 1}, \quad t > a.$$

2. If $p = \kappa^*(d,s) = \frac{d}{d+1-2s}$, then $\nabla G_\psi(x,t) \in L^{\frac{d}{d+1-2s}; \infty}(\mathbb{R}^d; \mathbb{R}^d)$, and there holds

$$||\nabla G_\psi(x,t)||_{\frac{d}{d+1-2s}; \infty} \leq C \left( \log \frac{t}{a} \right)^{-\alpha + 1}, \quad t > a.$$
Let
\[
\varpi^*(d,s) = \begin{cases} 
\frac{d}{d+1-4s}, & d + 2 > 4s, \\
\infty, & d + 2 \leq 4s.
\end{cases}
\]

**Theorem 3.3.** Let \(d \in \mathbb{N}, 1 < \alpha < 2, \) and \(0 < s < 1.\)
(1) Assume \(1 \leq p < \varpi^*(d,s).\) Then \(\nabla G_f(x,t) \in L^p(\mathbb{R}^d;\mathbb{R}^d)\) for all \(t > a\) and
\[
||\nabla G_f(x,t)||_p \leq C \left( \log \frac{t}{a} \right)^{\alpha-1-\frac{d}{2}-\frac{d}{q}(1-\frac{1}{p})}, \quad t > a.
\]
(2) If \(p = \varpi^*(d,s)\) and \(d + 2 > 4s,\) then \(\nabla G_f(x,t) \in L^{\frac{d}{d+1-4s}}(\mathbb{R}^d;\mathbb{R}^d),\) and it holds that
\[
||\nabla G_f(x,t)||_{\frac{d}{d+1-4s}} \leq C \left( \log \frac{t}{a} \right)^{\alpha-1}, \quad t > a.
\]

**Remark 4.** If \(d + 2 < 4s,\) according to inequality (27) of Theorem 2.7, we find that \(\nabla G_f(\cdot,t) \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)\) and \(||\nabla G_f(x,t)||_\infty \leq C (\log \frac{t}{a})^{\alpha-1-\frac{d(d+1)}{2s}}\) hold for any \(t > a.\)

It is easy known that \(\nabla u(x,t) = (\nabla G_\varphi(x,t)) * \varphi_a(x) + (\nabla G_\psi(x,t)) * \psi_a(x)\) when the source term \(f \equiv 0.\) For this situation, the following theorem holds whose proof is similar as that of Theorem 2.12 so is omitted here.

**Theorem 3.4.** Let \(d \in \mathbb{N}, 1 < \alpha < 2, 0 < s < 1.\) Suppose that \(1 \leq p \leq \varpi^*(d,s) = \frac{d}{d+1-2s}\) and \(f \equiv 0.\) Then \(u(x,t) = G_\varphi(x,t) * \varphi_a(x) + G_\psi(x,t) * \psi_a(x)\) has the following asymptotic behaviors.

(1) When \(p < \varpi^*(d,s),\) if \(1 \leq q, r \leq \infty, 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r},\) and \(\varphi_a(x), \psi_a(x) \in L^q(\mathbb{R}^d),\) then one has
\[
||\nabla u(x,t)|| \leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{q} - \frac{\alpha d}{q}(1-\frac{1}{p})} \||\varphi_a(x)||_q
\]
\[
+ C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{q} - \frac{\alpha d}{q}(1-\frac{1}{p})+1} \||\psi_a(x)||_q, \quad t > a.
\]

(2) When \(p = \varpi^*(d,s):\)

(i) If \(1 < q, r < \infty, \frac{1}{p} + \frac{2s}{d} = \frac{1}{q} + \frac{1}{\alpha},\) and \(\varphi_a(x), \psi_a(x) \in L^q(\mathbb{R}^d),\) then one has
\[
||\nabla u(x,t)|| \leq C \left( \log \frac{t}{a} \right)^{-\alpha} \||\varphi_a(x)||_q + C \left( \log \frac{t}{a} \right)^{-\alpha+1} \||\psi_a(x)||_q, \quad t > a.
\]

(ii) If \(\varphi_a(x), \psi_a(x) \in L^1(\mathbb{R}^d),\) then one has
\[
||\nabla u(x,t)|| \leq C \left( \log \frac{t}{a} \right)^{-\alpha} \||\varphi_a(x)||_1 + C \left( \log \frac{t}{a} \right)^{-\alpha+1} \||\psi_a(x)||_1, \quad t > a.
\]

Note that \(\nabla u(x,t) = (\nabla G_f(x,t)) * f(x,t)\) with \(\varphi_a = \psi_a \equiv 0,\) we can obtain the following result.

**Theorem 3.5.** Let \(d \in \mathbb{N}, 1 < \alpha < 2, 0 < s < 1,\) and \(\varphi_a = \psi_a \equiv 0.\) Let
\[
1 \leq p < \varpi^*(d,s) = \frac{d}{d+1-2s}, 1 \leq q, r < \infty, 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{\alpha}.\] Suppose that \(f(\cdot,t) \in L^q(\mathbb{R}^d)\) for each \(t \geq a\) and for some \(1 \leq q < \infty.\) Suppose further that the condition
\[
||f(x,t)||_q \leq C \left(1 + \log \frac{t}{a} \right)^{-\gamma}, \quad t > a,
\]
holds for some $\gamma > 0$. Then $u(x,t) = G_f(x,t) \ast f(x,t)$ has the following estimates

$$||\nabla u(x,t)||_r \leq C \left( \log \frac{t}{a} \right)^{a-\min\{1,\gamma\} - \frac{1}{\alpha} - \frac{d}{p} (1 - \frac{1}{p})}, \quad t > a,$$

if $\gamma \neq 1$, and

$$||\nabla u(x,t)||_r \leq C \left( \log \frac{t}{a} \right)^{a-\frac{d}{p} (1 - \frac{1}{p})} \log \left( 1 + \log \frac{t}{a} \right), \quad t > a,$$

if $\gamma = 1$.

We next study the large time behavior of the solution $u(x,t)$ to Equ. (1). Let us start with the following decomposing lemma.

**Lemma 3.6.** ([21]) Suppose $g(x) \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x| \cdot |g(x)|dx < \infty$. Then there exists $\Phi(x) \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$g(x) = \left( \int_{\mathbb{R}^d} g(x)dx \right) \delta + \text{div}(\Phi(x))$$

in the distributional sense and

$$||\Phi(x)||_{L^1(\mathbb{R}^d; \mathbb{R}^d)} \leq C_d \int_{\mathbb{R}^d} |x| \cdot |g(x)|dx,$$

where $\delta$ denotes the Dirac delta function and $\text{div}(\Phi(x))$ is the divergence of the function $\Phi(x)$.

With the help of this lemma, we have the following result about the large time behavior of the solution $u(x,t) = G_{\varphi}(x,t) \ast \varphi_a(x) + G_{\psi}(x,t) \ast \psi_a(x)$ in the case $f \equiv 0$.

**Theorem 3.7.** Let $d \in \mathbb{N}$, $1 < \alpha < 2$, $0 < s < 1$. Assume $1 \leq p < \frac{d}{d+1-2s}$ and $f \equiv 0$. Assume further that $\varphi_a(x), \psi_a(x) \in L^1(\mathbb{R}^d)$, $M_{\varphi} = \int_{\mathbb{R}^d} \varphi_a(x)dx$, and $M_{\psi} = \int_{\mathbb{R}^d} \psi_a(x)dx$.

1. If $||x|\varphi_a(x)||_1 < \infty$ and $||x|\psi_a(x)||_1 < \infty$, then it holds that

$$\left( \log \frac{t}{a} \right)^{\frac{d}{2d} (1 - \frac{1}{p})} \|u(x,t) - M_{\varphi} G_{\varphi}(x,t) - M_{\psi} G_{\psi}(x,t)\|_p$$

$$\leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{2d}} + C \left( \log \frac{t}{a} \right)^{-\frac{d}{2d} + 1}, \quad t > a.$$  

And in the limit case with $p = \frac{d}{d+1-2s}$,

$$\left( \log \frac{t}{a} \right)^{\frac{d(2s-1)}{2s}} \|u(x,t) - M_{\varphi} G_{\varphi}(x,t) - M_{\psi} G_{\psi}(x,t)\|_{\frac{d}{d+1-2s}} \to \infty$$

$$\leq C \left( \log \frac{t}{a} \right)^{-\frac{d}{2s}} + C \left( \log \frac{t}{a} \right)^{-\frac{d}{2s} + 1}, \quad t > a.$$  

2. It holds that

$$\left( \log \frac{t}{a} \right)^{\frac{d}{2d} (1 - \frac{1}{p})} \|u(x,t) - M_{\varphi} G_{\varphi}(x,t) - M_{\psi} G_{\psi}(x,t)\|_p \to 0, \quad t \to \infty,$$

whether or not the conditions hold in (1).
Proof. We first give the proof of (1). Due to \( \varphi_\alpha(x) \in L^1(\mathbb{R}^d) \) and \( |||x|\varphi_\alpha(x)||_1 < \infty \), Lemma 3.6 indicates that there exists a function \( \Psi_0(x) \in L^1(\mathbb{R}^d; \mathbb{R}^d) \) such that
\[
\varphi_\alpha(x) = M_\varphi \delta + \text{div}\Psi_0(x)
\]
with \( ||\Psi_0(x)||_1 \leq C|||x|\varphi_\alpha(x)||_1 \). Similarly, there is also a function \( \Psi_1(x) \in L^1(\mathbb{R}^d; \mathbb{R}^d) \) such that
\[
\psi_\alpha(x) = M_\psi \delta + \text{div}\Psi_1(x)
\]
with \( ||\Psi_1(x)||_1 \leq C|||x|\psi_\alpha(x)||_1 \).

As a result, one gets
\[
u(x, t) = G_\varphi(x, t) \ast \varphi_\alpha(x) + G_\psi(x, t) \ast \psi_\alpha(x)
= G_\varphi(x, t) \ast (M_\varphi \delta + \text{div}\Psi_0(x)) + G_\psi(x, t) \ast (M_\psi \delta + \text{div}\Psi_1(x))
= M_\varphi G_\varphi(x, t) + \text{div}\Psi_0(x) + \nabla G_\varphi(x, t) \ast \Psi_0(x)
+ M_\psi G_\psi(x, t) + \text{div}\Psi_1(x) + \nabla G_\psi(x, t) \ast \Psi_1(x).
\]

where \( \ast \) denotes the convolution of vector functions, i.e.,
\[
P(x) \ast Q(x) = \sum_{k=1}^{d} (P_k(x) \ast Q_k(x)).
\]

Hence,
\[
u(x, t) - M_\varphi G_\varphi(x, t) - M_\psi G_\psi(x, t)
= \nabla G_\varphi(x, t) \ast \Psi_0(x) + \nabla G_\psi(x, t) \ast \Psi_1(x).
\]

Applying Young’s inequality (57), Theorems 3.1 and 3.2, one has for \( 1 \leq p < \frac{d}{d+1-2\alpha} \),
\[
||u(x, t) - M_\varphi G_\varphi(x, t) - M_\psi G_\psi(x, t)||_p
\leq ||\nabla G_\varphi(x, t)||_p ||\Psi_0(x)||_1 + ||\nabla G_\psi(x, t)||_p ||\Psi_1(x)||_1
\leq C||\nabla G_\varphi(x, t)||_p \cdot ||x|\varphi_\alpha(x)||_1 + C||\nabla G_\psi(x, t)||_p \cdot ||x|\psi_\alpha(x)||_1
\leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{d} - \frac{\alpha}{d+1-2\alpha} \frac{1}{2} (1-\frac{1}{p})} + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{d} - \frac{\alpha}{d+1-2\alpha} \frac{1}{2} (1-\frac{1}{p}) + 1}.
\]

That is,
\[
\left( \log \frac{t}{a} \right) \frac{d}{2}(1-\frac{1}{p}) ||u(x, t) - M_\varphi G_\varphi(x, t) - M_\psi G_\psi(x, t)||_p
\leq C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{d} + \frac{1}{2}} + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha}{d} + 1}, \quad t > a.
\]

In the case \( p = \frac{d}{d+1-2\alpha} \), the corresponding result holds by virtue of equality (55) and Theorems 3.1 and 3.2.

We now prove (2). Take a sequence \( \{\xi_k(x)\} \subseteq C_0^\infty(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} \xi_k(x)dx = M_\varphi \) \( (k = 1, 2, \cdots) \) and \( \xi_k(x) \to \varphi_\alpha(x) \) in \( L^1(\mathbb{R}^d) \). Similarly, choose a sequence \( \{\eta_l(x)\} \subseteq C_0^\infty(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} \eta_l(x)dx = M_\psi \) \( (l = 1, 2, \cdots) \) and \( \eta_l(x) \to \psi_\alpha(x) \) in
\[ L^1(\mathbb{R}^d). \] Applying Young’s inequality (57) and the result in (1), one gets for any \( k, l \) that
\[
\begin{align*}
\| u(x, t) - M_p \varphi(x, t) - M_q \psi(x, t) \|_p \\
= & \| G_\varphi(x, t) * \varphi(x) - M_p \varphi(x, t) \|_p + \| G_\psi(x, t) * \psi(x) - M_q \psi(x, t) \|_p \\
= & \| G_\varphi(x, t) * (\varphi(x) - \xi_k(x)) + G_\psi(x, t) * \xi_k(x) - M_p \varphi(x, t) \|_p \\
& + \| G_\psi(x, t) * (\psi(x) - \eta_l(x)) + G_\psi(x, t) * \eta_l(x) - M_q \psi(x, t) \|_p \\
\leq & \| G_\varphi(x, t) * (\varphi(x) - \xi_k(x)) \|_p + \| G_\psi(x, t) * \xi_k(x) - M_p \varphi(x, t) \|_p \\
& + \| G_\psi(x, t) * (\psi(x) - \eta_l(x)) \|_p + \| G_\psi(x, t) * \eta_l(x) - M_q \psi(x, t) \|_p \\
\leq & \| G_\varphi(x, t) \|_p \| \varphi - \xi_k \|_1 + C_k \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} - \frac{d}{2} (1 - \frac{1}{p})} \\
& + \| G_\psi(x, t) \|_p \cdot \| \psi - \eta_l \|_1 + C_l \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} - \frac{d}{2} (1 - \frac{1}{p}) - 1} \\
\leq & C \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} (1 - \frac{1}{p})} \| \varphi - \xi_k \|_1 + C_k \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} (1 - \frac{1}{p})} \\
& + C \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} (1 - \frac{1}{p}) - 1} \| \psi - \eta_l \|_1 + C_l \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} (1 - \frac{1}{p}) - 1},
\end{align*}
\]
in which Theorems 2.9 and 2.10 are utilized. So one gets
\[
\begin{align*}
\left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} (1 - \frac{1}{p})} \| u(x, t) - M_p \varphi(x, t) - M_q \psi(x, t) \|_p \\
\leq C \left( \log \frac{t}{a} \right)^{-1} \| \varphi - \xi_k \|_1 + C \| \psi - \eta_l \|_1 + C_k \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} - 1} + C_l \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2}},
\end{align*}
\]
which gives
\[
\lim_{t \to \infty} \left( \log \frac{t}{a} \right)^{-\frac{\alpha d}{2} (1 - \frac{1}{p})} \| u(x, t) - M_p \varphi(x, t) - M_q \psi(x, t) \|_p \leq C \| \psi - \eta_l \|_1.
\]
By letting \( l \to \infty \), we obtain the result in (2). The proof is thus ended. \( \square \)

Likewise, we have the large time behaviors of the solution \( u(x, t) = G_f(x, t) * f(x, t) \) when the initial values \( \varphi_a = \psi_a \equiv 0. \)

**Theorem 3.8.** Let \( d \in \mathbb{N}, 1 < \alpha < 2, 0 < s < 1 \), and \( \varphi_a = \psi_a \equiv 0. \) Assume \( f(x, t) \in L^1(\mathbb{R}^d \times (a, \infty)) \) and \( M_f = \int_a^\infty \int_{\mathbb{R}^d} f(x, t)dx \frac{dt}{t} \). Assume further that \( f(x, t) \) satisfies the condition
\[
\| f(x, t) \|_1 \leq C \left( 1 + \log \frac{t}{a} \right)^{-\gamma}, \quad t > a,
\]
with some \( \gamma > 1 \). Then for \( u(x, t) = G_f(x, t) * f(x, t) \) there holds
\[
\left( \log \frac{t}{a} \right)^{-1 - \alpha + \frac{\alpha d}{2} (1 - \frac{1}{p})} \| u(x, t) - M_f G_f(x, t) \|_p \to 0, \quad t \to \infty,
\]
provided that \( 1 \leq p \leq \infty \) if \( d < 4s \), or \( 1 \leq p < \varpi(d, s) \) if \( d \geq 4s \).

**Proof.** The proof of Theorem 3.16 (3) in [26] can be almost verbatim copied. So we omit it here. \( \square \)
4. Conclusion. In this paper, we discuss the asymptotic behavior of the solution for fractional diffusion equation with Caputo–Hadamard derivative and fractional Laplacian in the hyperbolic case, that is, the order $\alpha$ of Caputo–Hadamard derivative lies in $(1, 2)$. In view of the technique of integral transforms, the fundamental solutions of Eqn. (1) are expressed by special function such as Fox $H$-function and its exact solution can be written as a convolution form. Then we establish the estimates of the fundamental solutions and exact solution to Eqn. (1). Finally, the large time behaviors of this solution are also displayed by using previous derived results.

Appendix. In the Appendix, we introduce some important results which are needed in this study. We first recall several useful results from the classical Fourier analysis [12].

Let $X$ be a measure space and let $\varrho$ be a positive, not necessarily finite, measure on $X$. $L^p(X, \varrho)$ denotes the set of all complex-valued $\varrho$-measurable functions on $X$ whose modulus to the $p$-th power is integrable with $0 < p < \infty$. The set of all $\varrho$-measurable functions $f$ such that

$$||f||_{L^p, \infty} := \sup \{ \gamma (d_f(\gamma))^{\frac{1}{\gamma}} : \gamma > 0 \} < \infty,$$

is known as the space weak $L^p(X, \varrho)$, where $d_f(\gamma) = \varrho(\{x \in X : |f(x)| > \gamma \})$ represents the distribution function of $f$. The weak $L^p(X, \varrho)$ space is denoted by $L^{p, \infty}(X, \varrho)$.

The following is Young’s equalities for convolution in the sense of $L^p$ or $L^{p, \infty}$ norm.

Let $1 \leq p, q, r \leq \infty$ satisfy $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then for $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ we have

$$||f * g||_{L^r(\mathbb{R}^d)} \leq ||f||_{L^p(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}. \quad (57)$$

Let $1 \leq p < \infty$ and $1 < q, r \leq \infty$ satisfy $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then for $f \in L^{p, \infty}(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ there exists a constant $C_{p,q,r}$ such that

$$||f * g||_{L^r(\mathbb{R}^d)} \leq C_{p,q,r} ||f||_{L^{p, \infty}(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}. \quad (58)$$

Let $1 < p, q, r \leq \infty$ satisfy $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then for $f \in L^{p, \infty}(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ there exists a constant $C_{p,q,r}$ such that

$$||f * g||_{L^r(\mathbb{R}^d)} \leq C_{p,q,r} ||f||_{L^{p, \infty}(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}. \quad (59)$$

In order to give representation formula of the fundamental solution, we need some special functions in the setting of fractional calculus, for example, Fox $H$-function. Next, we briefly introduce the definition and properties of Fox $H$-function, more detail content can be referred to [3, 18, 19, 33]. Let us begin with the definition.

Let $0 \leq m \leq \nu$ and $0 \leq n \leq \mu$ with $m, n, \mu, \nu \in \mathbb{Z}$. For $a_i, b_j \in \mathbb{C}$, and $\alpha_l, \beta_j \in \mathbb{R}^+$ $(l = 1, \cdots, \mu; j = 1, \cdots, \nu)$, the Fox $H$-function $H_{\mu\nu}^{mn}(z)$ is defined by

$$H_{\mu\nu}^{mn}(z) \equiv H_{\mu\nu}^{mn}\left(z \left| \begin{array}{l} a_1, \alpha_1, \cdots, a_n, \alpha_n; \quad (a_{n+1}, \alpha_{n+1}), \cdots, (a_{m+1}, \alpha_{m+1}) \\ b_1, \beta_1, \cdots, b_m, \beta_m; \quad (b_{m+1}, \beta_{m+1}), \cdots, (b_{\nu}, \beta_{\nu}) \end{array} \right. \right)
:= \frac{1}{2\pi i} \int_{C} \mathcal{H}_{\mu\nu}^{mn}(\tau) z^{-\tau} d\tau, \quad (60)$$

where

$$\mathcal{H}_{\mu\nu}^{mn}(\tau) := \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j \tau) \prod_{l=1}^{n} \Gamma(1 - a_l - \alpha_l \tau)}{\prod_{j=m+1}^{\nu} \Gamma(a_l + \alpha_l \tau) \prod_{l=n+1}^{\mu} \Gamma(1 - b_j - \beta_j \tau)}, \quad (61)$$

and

$$\gamma := \sup \{ \gamma (d_f(\gamma))^{\frac{1}{\gamma}} : \gamma > 0 \} < \infty.$$
and the poles
\[ b_{j\sigma} = -\frac{b_j + \sigma}{\beta_j} \quad (j = 1, \ldots, m; \sigma = 0, 1, 2, \ldots) \tag{62} \]
and the poles
\[ a_{lk} = \frac{1 - a_l + k}{a_l} \quad (l = 1, \ldots, n; k = 0, 1, 2, \ldots) \tag{63} \]
do not coincide, i.e.,
\[ \alpha_l(b_j + \sigma) \neq \beta_j(a_l - k - 1) \quad (j = 1, \ldots, m; l = 1, \ldots, n; \sigma, k = 0, 1, 2, \ldots). \tag{64} \]
\( C \) in (60) is the Mellin-Barnes contour which separates all the poles at \( \tau = b_{j\sigma} \) to the left and all the poles at \( \tau = a_{lk} \) to the right of \( C \).

Fox \( H \)-function has the following simple properties.

If one of \((a_l, \alpha_l)\) \((l = n + 1, \ldots, \mu)\) is equal to one of \((b_j, \beta_j)\) \((j = 1, \ldots, m)\), then \( H \)-function \( \mathcal{H}^{mn}_{\nu \nu}(z) \) reduces to a lower order one. That is, with \( m \geq 1 \) and \( \mu > n \),
\[ \mathcal{H}^{mn}_{\nu \nu}(z) \mid_{(a_0, \alpha_0), \ldots, (a_m, \alpha_m), (b_0, \beta_0), \ldots, (b_m, \beta_m)} = \mathcal{H}^{m-1}_{\nu-1}(z) \mid_{(a_0, \alpha_0), \ldots, (a_{m-1}, \alpha_{m-1}), (b_0, \beta_0), \ldots, (b_{m-1}, \beta_{m-1})}. \tag{65} \]

Let \( \zeta, c \in \mathbb{C}, \theta > 0 \), and \( k \in \mathbb{N}_0 \), then the following differential relation holds
\[
\left( \frac{d}{dz} \right)^k \left[ z^\zeta \mathcal{H}^{mn}_{\nu \nu} \right] \mid_{(a_1, \alpha_1), \ldots, (a_m, \alpha_m), (b_1, \beta_1), \ldots, (b_m, \beta_m)} = (-1)^k z^{\zeta - k} \mathcal{H}^{m+1}_{\nu+1}(z) \mid_{(a_1, \alpha_1), \ldots, (a_m, \alpha_m), (b_1, \beta_1), \ldots, (b_m, \beta_m)}. \tag{66} \]

The asymptotic expansions of \( H \)-function are very important at infinity and zero, which is used to estimate the fundamental solution. We denote
\[ a^* = \sum_{l=1}^n a_l - \sum_{l=n+1}^\mu a_l + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^\nu \beta_j. \]

Assume that the condition in (64) be satisfied and the poles in (63) be simple. If \( a^* > 0 \) and \( z \neq 0 \) with \( |\arg z| < \frac{\pi}{2} a^* \), then \( H \)-function (60) has the following power series expansion
\[ \mathcal{H}^{mn}_{\nu \nu}(z) = \sum_{l=1}^n \sum_{k=0}^\infty h_{lk} z^{\frac{1 + k - a_l}{a_l}}, \tag{67} \]
where
\[ h_{lk} = \frac{(-1)^k}{l! a_l} \prod_{j=1}^m \Gamma(b_j + (1 - a_l + k)\frac{\alpha_l}{\alpha_l}) \prod_{\lambda=1}^n \lambda \neq l, \lambda \neq \mu \Gamma(1 - a_\lambda - (1 - a_l + k)\frac{\alpha_l}{\alpha_l}) \prod_{\lambda=n+1}^\mu \Gamma(a_\lambda + (1 - a_l + k)\frac{\alpha_l}{\alpha_l}) \prod_{j=m+1}^\nu \Gamma(1 - b_j - (1 - a_l + k)\frac{\alpha_l}{\alpha_l}). \tag{68} \]

Suppose that the condition in (64) be satisfied and the poles in (63) be simple. Then we have the asymptotic expansions of \( H \)-function at infinity,
\[ \mathcal{H}^{mn}_{\nu \nu}(z) = \sum_{l=1}^n \left[ h_{l} z^{\frac{a_l - 1}{a_l}} + o\left( z^{\frac{a_l - 1}{a_l}} \right) \right], \quad z \to \infty, \tag{69} \]
with the condition \( a^* > 0 \), \( z \neq 0 \), and \( |\arg z| < \frac{\pi}{2} a^* \). Here the coefficient \( h_l = h_{l0} \) as in (67).
Let the condition in (64) be satisfied, $a^* > 0$, and $z \neq 0$ with $|\arg z| < \frac{\pi}{2}a^*$. Then we have the asymptotic expansions of $H$-function at zero,

\[
H_{\mu\nu}^m(z) = \sum_{j} \left[ h_j z^{b_j} + o \left( z^{b_j} \right) \right] + \sum_{j}'' \left[ H_j z^{b_j} \left( \log z \right)^{N_j - 1} + o \left( z^{b_j} \left( \log z \right)^{N_j - 1} \right) \right], \quad z \to 0,
\]

with

\[
h_j = \frac{1}{\beta_j} \prod_{k=1, k \neq j}^\infty \Gamma(b_k - b_j \frac{\alpha_k}{\beta_j})\prod_{k=m+1}^n \Gamma(1 - a_k + b_j \frac{\alpha_k}{\beta_j}),
\]

and

\[
H_j = \frac{(-1)^{N_j - 1}}{(N_j - 1)!} \prod_{l=1}^{N_j} \frac{(-1)^b \prod_{k=1}^m \Gamma(b_k - b_j \frac{\alpha_k}{\beta_j})\prod_{k=m+1}^n \Gamma(1 - a_k + b_j \frac{\alpha_k}{\beta_j})}{\Gamma(a_k - b_j \frac{\alpha_k}{\beta_j})\prod_{k=m+1} \Gamma(1 - b_k + b_j \frac{\alpha_k}{\beta_j})},
\]

where $\sum_j$ and $\sum_j''$ are summations taken over $j$ ($j = 1, \ldots, m$) such that the Gamma functions $\Gamma(b_j + \beta_j \tau)$ have simple poles and poles of order $N_j$ at the points $b_{j0}$, respectively.

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