FULL AMALGAMATION CLASSES WITH INTRINSIC TRANSCENDENTALS

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ABSTRACT. We develop some basic results about full amalgamation classes with intrinsic transcendentals. These classes have generics whose models may have finite subsets whose intrinsic closure is not contained in its algebraic closure. We will show that under fairly natural conditions the generic will have an essentially undecidable theory, but we will also exhibit strictly superstable and strictly simple examples. Separating types over a model into those that are intrinsic and those that are extrinsic, we will demonstrate that the complexity exceeding that of a simple theory in the classes with essentially undecidable theories of [4] comes from the intrinsic types by deriving a class from them which has a strictly simple theory with few intrinsic types.

1. Introduction

This paper is concerned with the complexity of generic structures arising from full amalgamation classes with what we term intrinsic transcendentals. The latter are elements which will be contained in the closure of a finite subset of a model of the generic but will not be algebraic over that set. Specifically, let \((K, \leq)\) be a full amalgamation class with intrinsic transcendentals and let \(T\) be the theory of the \((K, \leq)\)-generic. We will give a combinatorial condition under which \(T\) is essentially undecidable; this condition will be met by many classes derived from a pre-dimension function in the usual way. We will introduce the notion of a support system, which is used to measure the complexity of the ways certain intrinsic extensions can attach to a model. We will show that by limiting that complexity we can tame the model-theoretic complexity of \(T\) to a certain degree. Specifically, for \(M \models T\) we divide the types in \(S(M)\) into those that are intrinsic and those that are extrinsic. If the class \((K, \leq)\) has limited supports, then the number of intrinsic types over \(M\) will be limited. While the resulting generic may still be unstable (due to the number of extrinsic types), we will show that in a particular family of amalgamation classes the number of and structure of the extrinsic types will not exceed the conditions of simplicity. Thus if a class has a generic with a non-simple theory, this will be due to the number of intrinsic types over models of that theory.

The history of the Hrushovski construction is well-detailed in many sources (for example, see [11, 3, 15]) to which the reader is referred for a full accounting. The basic idea, which we will reprise in Section 2, is that a class of finite structures \(K\) which are partially ordered by a notion of strong substructure \(\leq\) can be amalgamated to produce a canonical generic of the class. The model theory of the generic is determined by the partial order \(\leq\), and by choosing appropriate notions of strong substructure Hrushovski was able to refute a number of then-current conjectures in geometric stability theory [9, 7, 8]. The construction continues to play a prominent role in many ongoing research programmes.
Every instance of the construction is associated with a closure operator which is determined by $K$ and $\leq$. For many of the studied examples of the construction [9, 3, 2, 13] the relation $\leq$ is chosen so that the closure of finite subset of a model of the theory the generic would be contained in the algebraic closure of that set. Pourmahdian [14] called this the algebraic closure property (AC) and made a study of certain classes which did not possess the property. He showed that performing a kind of partial Morleyization on the descriptions of the closures of sets yields a kind of quantifier elimination. He also showed the simplicity of the resulting theory. This paper contains somewhat analogous results; our introduction of closure types in Section 3 provides a similar kind of quantifier elimination that does not require an expansion of the language. Our result that reducing the number of intrinsic types results in a simple structure (Theorem 6.15) is similar in flavor to Pourmahdian’s simplicity result. On the other hand, we note that a previous paper [4] and more generally the results of Section 4 answer Pourmahdian’s question of whether the first-order theory of the generic need be simple (Question 4.10 of [14]) in the negative.

In part, the significance of our results is that the methods described here can potentially produce well-behaved generics with rich geometries that can shed light on the nature of amalgamation constructions. For example, Baldwin has conjectured that no generic with a small theory will be strictly superstable. We will exhibit a non-small strictly superstable generic in section 5, and the techniques developed here may ultimately shed some light on the main conjecture.

The organization of the paper is as follows. In Section 2 we outline Hrushovski amalgamation construction, define intrinsic transcendentals, and give examples of generics which have them. Section 3 introduces a syntactic device for counting types when automorphism arguments are not available (this will be the usual case in classes with intrinsic transcendentals, since the closure of a model will in general have the same power as the universal domain in which it lives). Section 4 will show that classes with intrinsic transcendentals often have essentially undecidable theories. The following section, 5, gives a condition under which the number of types represented in the closure of a model is limited. In Section 6, we will employ this condition to derive a generic with a simple theory from a class whose generic has an essentially undecidable theory. The main technique in this section will be to adapt ideas from Hrushovski’s collapse to limit the structure of the intrinsic closure.

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2. Background

2.1. Hrushovski’s Amalgamation Construction. Hrushovski’s amalgamation construction generalizes the Fraïssé construction to allow for the production of a more general class of structures. In particular, rather than amalgamating a class of finite structures partially ordered by the substructure relation, the Hrushovski construction amalgamates a class of finite structures partially ordered by a strong...
(or closed) substructure relation. This allows for a finer level of control on the resulting structure. The choice of the strong substructure relation forces various model-theoretic properties on the final (generic) structure. In particular, the relation determines a closure operator on the generic which in turn determines much of its model-theory.

We present the construction in an arbitrary finite relational language even though our main examples in this paper will either be graphs or unary-predicate expansions of graphs. For any finite relational language $L$, we want to work with a class of finite $L$-structures $K$ ordered by a strong substructure relation $\leq$ so that the pair $(K, \leq)$ satisfies the axioms in Definition 2.2.2 below. It is worth noting that our requirements (especially the assumption of full amalgamation) are significantly stronger than the minimal set of assumptions needed to carry out the construction. The assumption of full amalgamation will allow us to simplify our analyses, and specifically allow the tools developed in Section 3 to work.

**Convention 2.1.** For $A, B$ subsets of a common superstructure, $AB$ denotes the union of $A$ and $B$ and $A \subseteq \omega B$ denotes that $A$ is a finite substructure of $B$.

**Definition 2.2.** For any finite relational language $L$, $K$ a class of finite $L$-structures and $\leq$ a binary relation on $K$, we say that $(K, \leq)$ is a full amalgamation class if the following properties are satisfied for any $A, B, C \in K$:

1. The class $K$ is closed under substructures and isomorphisms.
2. The relation $\leq$ is isomorphism-invariant: if $A \leq B$ and $f : B \approx B'$, then $f(A) \leq B'$.
3. $A \leq A$.
4. If $A \leq B$, then $A \subseteq B$.
5. $\emptyset \leq A$.
6. If $A \leq B$ and $B \leq C$, then $A \leq C$.
7. If $A \leq B$ and $A \subseteq C \subseteq B$, then $A \leq C$.
8. For $A \leq B$ from $K$, we have $A \cap C \leq B \cap C$ for every $C \in K$.
9. The pair $(K, \leq)$ has the full amalgamation property, discussed below.

For $A, B \in K$, if $A \leq B$ then we will say that $A$ is strong (or closed) in $B$.

The amalgamation property says that we can coherently join two strong extensions of a structure $A$ into a new structure $D \in K$. Some form of amalgamation is central to any form of this construction; it allows us to canonically join elements of $K$ to produce a unique countable structure, the $(K, \leq)$-generic.

Specifically, let us say that a map $f : A \to B$ is strong if $f(A) \leq B$ for $A, B \in K$. Then, given $A, B, C \in K$ and $f$ as in the diagram below (where $\leq$ above an arrow indicates that the corresponding map should be strong, and arrows labeled $i$ are inclusions), the amalgamation property states that we can find $D \in K$ and a strong map $g$ for which the diagram commutes.

![Amalgamation Diagram](image)

We will call $D$ an amalgam of $B$ and $C$ over $A$. 
We will want to work with classes which have a particularly strong form of the amalgamation property. These are the full amalgamation classes defined by Baldwin and Shi (see [3]).

**Definition 2.3.** Suppose \( A, B, C \) are elements of \( K \) with \( A = B \cap C \), and let \( D \) be the structure whose universe is \( BC \) and whose relations are precisely those of \( B \) and those of \( C \). Then we will denote \( D \) by \( B \oplus_A C \) if:

- If \((K, \leq)\) is an amalgamation class in which \( B \oplus_A C \) is an amalgam of \( B \) and \( C \) over \( A \), then we will call \( B \oplus_A C \) the free amalgam of \( B \) and \( C \) over \( A \) and say that \((K, \leq)\) is a free amalgamation class.
  
  As notation, if \( \{B_i : i < n\} \) is a family of structures with \( B_i \cap B_j = A \) for \( i \neq j \), then we write \( \oplus_{i<n}(B_i/A) \) to denote \( (\oplus_{i<n-1}(B_i/A)) \oplus_A B_{n-1} \), when \( n > 2 \)
- A free amalgamation classes is full if for \( A, B, C \in K \) and \( A \leq B, A \subseteq C \), then \( C \leq D \), where \( D = B \oplus_A C \).

If \( N \) is any \( L \)-structure whose finite subsets are all in \( K \), then the closed substructure relation can be extended to arbitrary subsets of \( N \), this in turn determines a closure operator on \( N \). The definition depends on the notion of a minimal pair \((A, B)\): for \( A \subseteq \omega \) \( B \in K \), \((A, B)\) is a minimal pair if \( A \not\subseteq B \) but for every \( A \subseteq B_0 \subseteq B \), \( A \leq B_0 \). These represent minimal instances of extensions which are not strong; the intuition is that a (possibly infinite) set will be closed if it is closed under the operation of extending finite subsets by minimal pairs.

**Definition 2.4.** Let \((K, \leq)\) be a full amalgamation class and let \( N \) be an \( L \)-structure whose finite subsets are elements of \( K \).

- If \( A \subseteq_\omega B \subseteq_\omega N \), then \( A \leq B \) is determined by the \( \leq \)-relation on \( K \).
- If \( A \subseteq_\omega M \) (with \( M \) possibly infinite), then \( A \leq M \) exactly when \( A \leq B \) for every \( A \subseteq B \subseteq_\omega M \).
- If \( M \subseteq N \), then \( M \leq N \) exactly when for \( A \subseteq_\omega M \), if \( (A, B) \) is a minimal pair for some \( B \subseteq_\omega N \), then \( B \subseteq M \).

This leads to the critical notion of a closure: for \( M \subseteq N \) the \( N \)-closure of \( M \), denoted \( cl_N(M) \) is the smallest \( M' \) such that \( M \subseteq M' \leq N \). Our axioms (particularly 2.2.8) will guarantee that this is well-defined.

The closure operator determines a pre-geometry (without exchange) on the structures \( N \) whose finite substructures are in \( K \); it is largely through determining the properties of the closure that the Hrushovski construction forces various model-theoretic properties in the generic, which we now define.

If \((K, \leq)\) is a full amalgamation class, then we can imitate the construction of the Fraïssé limit (amalgamating over strong substructures rather than over all substructures) and produce the \((K, \leq)\)-generic. This is a countable \( L \)-structure \( M \) (unique up to isomorphism) that satisfies the following constitutive properties.

1. For \( A \subseteq_\omega M \), \( A \in K \)
2. If \( A \leq M \) and \( A \leq B \), then there is a strong embedding of \( B \) into \( M \) over \( A \).
3. For any finite \( A \subseteq_\omega M \), \( cl_M(A) \) is finite.

We adopt the following conventions.

**Convention 2.5.**
• For \((K, \leq)\) a full amalgamation class, we will write \(T_{(K, \leq)}\) to denote the theory of the \((K, \leq)\)-generic and let \(C_{(K, \leq)}\) denote a universal domain for \(T_{(K, \leq)}\). That is, \(C_{(K, \leq)}\) is chosen to be a \(\kappa\)-saturated, strongly \(\kappa\)-homogeneous model of \(T_{(K, \leq)}\) for \(\kappa\) larger than the size of any model we’re working with. If context makes the amalgamation class clear, we may simply write \(C\).

• We will use \((K, \leq)\) to denote an arbitrary amalgamation class and introduce extra notation (e.g. \((K_r, \leq_r)\)) to indicate a specific class.

2.2. Intrinsic Transcendentals. Our main interest in this paper will be in amalgamation classes which do not have what Pourmahdian [14] calls the algebraic closure property. Specifically, we will be interested in classes which satisfy the following condition.

**Definition 2.6.** The class \((K, \leq)\) has intrinsic transcendentals if there is a minimal pair \((A, B)\) from \((K, \leq)\) such that for any \(n \in \omega\) there is some \(D_n \in K\) which contains a copy of the free amalgamation \( \bigoplus_{i<n} B/A \).

The primary examples of such classes arise from predimension functions. These are functions \(\delta : K \to \mathbb{R}\) which are used to define a strong substructure relation \(\leq\) on \(K\). When \(K\) is a class of graphs, then for any real \(\alpha \geq 0\) we will be particularly interested in the following.

**Definition 2.7.** For any graph \(A\) and real \(\alpha \geq 0\), the predimension of \(A\), \(\delta_\alpha(A)\) is given by \(\delta_\alpha(A) = |A| - \alpha e(A)\) where \(e(A)\) denotes the number of edges in \(A\). For \(A \subseteq \omega B\), the relative predimension of \(B\) over \(A\), is given by \(\delta_\alpha(B/A) = \delta_\alpha(B) - \delta_\alpha(A)\).

For any such \(\alpha\), this gives us two distinct ways to define an amalgamation class.

**Definition 2.8.** Fix a real \(\alpha \geq 0\).

• \(K_\alpha\) is the class of all finite graphs which have hereditarily non-negative predimension. That is \(K_\alpha = \{ A : \delta_\alpha(A') \geq 0 \text{ for } A' \subseteq A \}\). For \(A \subseteq \omega B \in K_\alpha\), we say that \(A \leq_\alpha B\) if for \(A \subseteq B_0 \subseteq B\), \(\delta_\alpha(B_0/A) \geq 0\).

• \(K_{\alpha+}\) is the class of all finite graphs which have hereditarily positive predimension. That is \(K_{\alpha+} = \{ A : \delta_\alpha(A') > 0 \text{ for } A' \subseteq A, A' \neq \emptyset \}\). For \(A \subseteq \omega B \in K_{\alpha+}\), we say that \(A <_\alpha B\) if for \(A \subseteq B_0 \subseteq B\), \(\delta_\alpha(B_0/A) > 0\).

Note that for irrational \(\alpha\) the two classes coincide since \(\delta_\alpha\) is never 0. For rational \(r\), it is shown in [3] that \(T_{(K_r, \leq_r)}\) is \(\omega\)-stable while [4] shows that \(T_{(K^r_{\langle r, \leq_r\rangle})}\) (for \(0 < r < 1\)) is essentially undecidable. The primary difference between these two classes is that in the former class extensions of relative predimension 0 are strong and must occur over any finite set, while in the latter class such extensions can occur or not in arbitrarily complex configurations. Such classes form our primary example of classes with intrinsic transcendentals; in Section 6 we will modify these classes to produce tamer theories.

3. The Closure Type

For a given class \((K, \leq)\), our intuition is that the model theory of \(T_{(K, \leq)}\) is determined by the complexity of the associated closure operation. This is made precise in the \((K, \leq)\)-generic by noting that isomorphisms of closed subsets extend to automorphisms of the generic. This will not extend to the universal domain \(C_{(K, \leq)}\)
since closed sets will in general have the same cardinality as \( \mathcal{C} \) and we can thus not appeal to strong homogeneity. We can, however, get a similar result by working with back-and-forth equivalence rather than automorphisms. In particular, we will show that if \( X \) and \( Y \) are closed subsets of \( \mathcal{C} \) which are elementarily equivalent as \( L \)-structures (i.e. \( X \equiv Y \)) then they are equivalent as substructures of \( \mathcal{C} \) (i.e. \( (\mathcal{C}, X) \equiv (\mathcal{C}, Y) \)).

Our technical device for showing this is the closure type, which determines the elementary structure of the closure of a set. Throughout this section, fix a full amalgamation class \( (\mathcal{K}, \preceq) \) and let \( \mathcal{C} = \mathcal{C}_{(\mathcal{K}, \preceq)} \). Our main result will be that if finite sets have the same closure type, then they realize the same complete type in \( \mathcal{C} \).

The closure type is modeled on the game-normal formulae used in the proof of the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3]. The latter paper noticed the Fraïssé-Hintikka theorem (see [6]) and makes fundamental use of a way of decomposing the closure of a set noted by Baldwin and Shi in [3].

Imagine a modified Ehrenfeucht-Fraïssé game in which two players are working on the closures of tuples \( \bar{a} \) and \( \bar{b} \), and at each round both players extend the vertices already played by a minimal pair. Thus in a 1-round game the “spoiler” might pick an extension \( A \) where \((\bar{a}, A)\) is a minimal pair, and it will suffice for the “duplicator” to find a copy of \( A \) over \( \bar{b} \). If playing a 2-round game the duplicator will need to be more careful about her copy of \( A \) – it will have to be a copy for which she knows that she can respond appropriately to the next move of the spoiler. In particular, she will have to choose \( B \) a copy of \( A \) for which the minimal pairs over \( \bar{b}B \) correspond precisely to those over \( \bar{a}A \). If the game will go for \((k+1)\)-rounds, then the duplicator will need to choose a copy of \( A \) which can support all the possible sequences of \( k \) moves by the spoiler.

The idea of an intrinsic formula is to code this information syntactically. In particular, a \( k \)-intrinsic formula over \( \bar{a} \) will describe a minimal extension along with the possible combinations of \( k - 1 \) moves that could be supported by that extension.

Formally, we define a 0-intrinsic formula over \( \bar{a} \) to be of the form

\[
\phi(\bar{x}; \bar{a}) := \Delta_B(\bar{a}\bar{x})
\]

where \((\bar{a}_0, B)\) is a minimal pair for \( \bar{a}_0 \subseteq \bar{a} \) and \( \Delta_B(\bar{a}\bar{x}) \) asserts that \( \bar{a}_0\bar{x} \) is isomorphic to \( B \) (note that the notation is ambiguous – \( \Delta_B(\bar{a}\bar{x}) \) could refer to several different formulae depending on which subset of \( \bar{a} \) is picked out. This should not cause any problems in what follows.)

Having defined \( k \)-intrinsic formulae, we define a \( k + 1 \)-intrinsic formula to be of the form

\[
\phi(\bar{x}; \bar{a}) := \Delta_B(\bar{a}\bar{x}) \land \bigwedge_{i<n} \exists \bar{w}_i. \phi_i(\bar{w}_i, \bar{a}\bar{x}) \land \bigwedge_{j<n} \neg \exists \bar{z}_j. \psi_j(\bar{z}_j; \bar{a}\bar{x})
\]

where again \((\bar{a}_0, B)\) is a minimal pair for some \( a_0 \subseteq A \) and the \( \phi_i, \psi_j \) are \( k \)-intrinsic formulae.

We will call a formula \( \phi(\bar{x}; \bar{a}) \) intrinsic over \( \bar{a} \) when it is \( k \)-intrinsic over \( \bar{a} \) for some \( k \in \omega \).
Lemma 3.2. Let \( p \) be a \( \kappa \)-saturated formula such that any finite fragment of \( p \) is finitely satisfiable. Thus \( p \) will be partially isomorphic, and \( p \) will allow us to establish back-and-forth equivalence. The proof will divide into two cases, based on whether or not \( a \) is realized by some \( A \) in a minimal chain \( A \). We will do this by showing that for tuples \( \bar{a} \) with cltp(\( \bar{a}/M \)) a minimal pair and \( M \) any set, the closure type of \( \bar{a} \) over \( M \) is defined by cltp(\( \bar{a}/M \)) = \( \bigcup_{m \subseteq \omega} \text{cltp}(\bar{a} \bar{m}) \).

We want to show that for tuples \( \bar{a} \) and \( \bar{b} \), if cltp(\( \bar{a}/M \)) = cltp(\( \bar{b}/M \)), then \( \text{tp}_\mathcal{C}(\bar{a}/M) = \text{tp}_\mathcal{C}(\bar{b}/M) \). We will do this by showing that for \( a_0 \) with cltp(\( \bar{a}_0 \bar{a}_0 \)) = cltp(\( \bar{b}_0 \bar{b}_0 \)). Since this implies that \( \bar{a}_0 \) and \( \bar{b}_0 \) are partially isomorphic, this will allow us to establish back-and-forth equivalence. The proof will divide into two cases, based on whether or not \( a_0 \in \text{cl}(\bar{a}) \).

Lemma 3.2. Let \( \bar{a}, \bar{b} \) be tuples with cltp(\( \bar{a} \)) = cltp(\( \bar{b} \)). Then for any \( a_0 \in \text{cl}(\bar{a}) \) there is a \( b_0 \) with cltp(\( \bar{a}_0 \bar{a}_0 \)) = cltp(\( \bar{b}_0 \bar{b}_0 \)).

Proof. Suppose first that there is some \( A \) with (\( \bar{a}, A \)) a minimal pair and \( a_0 \in A \). Let \( p \) be the type

\[
\{ \Delta_A(\bar{x}) \} \cup \bigcup \{ \exists \bar{y} \phi(\bar{y}; \bar{b}x_0) : \exists \bar{y} \phi(\bar{y}; \bar{a}a_0) \in \text{cltp}(\bar{a}a_0) \}
\cup \bigcup \{ \neg \exists \bar{z} \phi(\bar{z}; \bar{b}x_0) : \neg \exists \bar{z} \phi(\bar{z}; \bar{a}a_0) \in \text{cltp}(\bar{a}a_0) \}
\]

where we assume without loss that \( A \) is enumerated as \( a_0 a_1 \ldots a_n \) and \( \bar{x} \) is \( x_0 x_1 \ldots x_n \). Note that any finite fragment of \( p \) will be implied by a single intrinsic formula of the form

\[
\Delta_A(\bar{x}) \wedge \bigwedge_i \exists \bar{y} \phi_i(\bar{y}; \bar{b}x_0) \wedge \bigwedge_j \neg \exists \bar{z} \phi(\bar{z}; \bar{b}x_0)
\]

Thus \( p \) is finitely satisfiable since cltp(\( \bar{b} \)) = cltp(\( \bar{a} \)). Also, since \( \mathcal{C} \) is \( \omega \)-saturated, \( p \) is realized by some \( b_0 \ldots b_n \). Then by definition cltp(\( \bar{b}_0 \bar{b}_0 \)) = cltp(\( \bar{a}_0 \bar{a}_0 \)).

If \( a_0 \) is not contained in a minimal extension as above, then it will be contained in a minimal chain \( A_0 \subseteq A_1 \ldots \subseteq A_l \) where \( (A_i, A_{i+1}) \) is a minimal pair. Thus iterating the above argument will suffice for the general case.

To handle the case where \( a_0 \notin \text{cl}(\bar{a}) \), we will want to employ the following relative notion of closure. This was noted in [3] and [2] where it was used to define semi-genericity.

Definition 3.3. For \( A \subseteq B \subseteq M \), we say that \( B \) is closed over \( A \) (in \( M \)) if cl\(_M\)(\( B \)) = cl\(_M\)(\( A \)) \( \cup \) \( B \).

The idea is that the any minimal pairs which originate in \( B \) actually originate in \( A \), so that extending to \( B \) adds no new minimal pairs.

The following lemma was noted by Baldwin and Shi in [3]
Lemma 3.4. Let $A \subseteq B \in K$. Let $C_1, \ldots, C_m$ be extensions of $B$ for which $B \not\subseteq C_i$ but $A \subseteq (C_i \setminus B)$. Then for $M \models T(K, \leq)$, any embedding of $A$ into $M$ extends to an embedding of $B$ into $M$ which does not extend to an embedding of any of the $C_i$.

Together with the $\omega$-saturation of $C$, we have

Corollary 3.5. For any $A \subseteq \omega C$, if $A \subseteq B$ then $A$ extends to an embedding of $B$ which is closed over $A$.

In order to apply this to the case in which $a_0 \notin \text{cl}(\bar{a})$, we show that any finite fragment of $\text{cltp}(\bar{a}_0a)$ can be realized by embedding a finite closed set over $\text{cl}(\bar{b})$.

Lemma 3.6. Suppose $\bar{a}, \bar{b}$ are finite tuples with $\text{cltp}(\bar{a}) = \text{cltp}(\bar{b})$. Fix $a_0 \in C \setminus \text{cl}(\bar{a})$, and let $\Sigma$ be a finite fragment of $\text{cltp}(\bar{a}_0a)$. Then there is a finite $D \subseteq \text{cl}(\bar{a}_0a)$ such that if $f$ is an embedding of $D$ with $f : \bar{a} \mapsto \bar{b}$ and $f(D)$ closed over $\text{cl}(\bar{b})$, then there some $b_0 \in f(D)$ so that $C \models \Sigma(bb_0)$.

Proof. We may assume without loss that $\Sigma$ consists of $k$-intrinsic formulae for some $k$. Starting with $\Sigma_0 = \Sigma$ and $E_{-1} = \varnothing$, we inductively construct sets of formulae $\Sigma_t$ and sets $E_t$ which are realizations of the positive part of $\Sigma_t$. For each positive $\exists \bar{x} \phi(\bar{x})$ in $\Sigma_t$, we add a tuple $\bar{e}_\phi$ to $E_t$ such that $C \models \phi(\bar{e}_\phi)$. We then create $\Sigma_{t+1}$ to ensure that $E_{t+1} = \Sigma_{t+1}$. We do this by adding every positive and negative formula in $\Sigma_{t+1}$ to $\Sigma_{t+1}$. We also need to ensure that $E_k = \neg \exists \bar{x} \psi(x)$ when the latter is in $\Sigma_t$. We note that $\neg \exists \bar{x} \psi(x)$ will be equivalent to

$$\forall \bar{x} \left[ \Delta_B(\bar{x}\bar{a}_0\bar{e}) \rightarrow \bigvee \neg \exists \bar{y} \phi_\gamma(\bar{y}; \bar{x}\bar{a}_0\bar{e}) \land \bigvee \exists \bar{w} \psi_\delta(\bar{w}; \bar{x}\bar{a}_0\bar{e}) \right]$$

for some tuple of parameters $\bar{e}$. Let $\bar{b}_0, \ldots, \bar{b}_t$ enumerate all realizations of $B$ in $E_t$ (if any). For each $\bar{b}_t$ we can choose a formula $\theta^t_\gamma$ which witnesses the disjunction (thus, $\theta^t_\gamma$ will be either $\neg \exists \bar{y} \phi_\gamma$ or $\exists \bar{w} \psi_\delta$). We add all such $\theta^t_\gamma$ to $\Sigma_t$ as well.

Let $D = E_k$ and suppose $f$ is a closed embedding of $D$ which maps $\bar{a} \mapsto \bar{b}$. Letting $b_0$ denote the image of $a_0$ under this embedding, it is straightforward to show that $C \models \Sigma(bb_0)$ as desired.

Lemma 3.7. Suppose $\text{cltp}(\bar{m}\bar{a}) = \text{cltp}(\bar{m}\bar{b})$ for tuples $\bar{m}, \bar{a}, \bar{b}$. Then for any $a_0 \in C \setminus \text{cl}(\bar{m}\bar{a})$ there is a $b_0$ such that $\text{cltp}(\bar{m}\bar{a}_0a) = \text{cltp}(\bar{m}\bar{b}_0b)$

Proof. Let $\Sigma(\bar{m}\bar{a}_0a)$ be a finite fragment of $\text{cltp}(\bar{m}\bar{a}_0b)$. By the previous lemma, there is a finite extension $E$ of $\bar{m}\bar{a}_0a$ so that for any $b_0$, if $F$ is a closed embedding of $E$ over $\bar{m}\bar{b}_0b$, then $\Sigma(\bar{m}\bar{b}_0b)$ will hold. Since $(K, \leq)$ is a full amalgamation class, we can find a closed embedding of $E$ over $\bar{m}\bar{b}_0b$ by Corollary 3.5; we let $b_0$ correspond to $a_0$ under such an embedding. Thus we can realize every finite fragment of $\text{cltp}(\bar{m}\bar{a}_0a)$ while fixing $\bar{m}\bar{b}_0b$, so that by compactness we can find a $b_0$ as required.

Corollary 3.8. Suppose $\bar{a}, \bar{b}$ are tuples with $\text{cltp}(\bar{a}) = \text{cltp}(\bar{b})$. Then for any $c_0 \in C$, there are tuples $\bar{c}, \bar{d}$ with $c_0 \in \bar{c}$ and $\text{cltp}(\bar{a}\bar{c}) = \text{cltp}(\bar{b}\bar{d})$

Proof. This is a combination of Lemmas 3.2 and 3.7.

Finally, we prove that elements with the same closure type over a set have the same type over that set.

Proposition 3.9. Let $M \subseteq C$ with $|M| < |C|$. Let $a, b \in \text{cl}(M)$ and suppose $\text{cltp}(a/M) = \text{cltp}(b/M)$. Then $\text{tp}(a/M) = \text{tp}(b/M)$
Proof. We show that for $\bar{m} \subseteq \omega M$, $(\mathcal{C}, \bar{m}a) \approx_k (\mathcal{C}, \bar{m}b)$ for any finite $k$, where $\approx_k$ denotes $k$-move back-and-forth equivalence. By 3.8, for any tuples $\bar{c}, \bar{d}$ with $\text{cltp}(\bar{ma}) = \text{cltp}(\bar{mb})$ and $\bar{c}'$, we can find a $\bar{d}'$ so that $\text{cltp}(\bar{ma}\bar{c}') = \text{cltp}(\bar{mb}\bar{d}')$. This implies that $(\mathcal{C}, \bar{ma}\bar{c}') \approx_{k-1} (\mathcal{C}, \bar{mb}\bar{d}')$, and establishes the back-and-forth equivalence. By the Fraïssé-Hintikka theorem and compactness, this establishes that $\text{tp}(a/M) = \text{tp}(b/M)$. □

4. Classes which Interpret Arithmetic

In Section 2 we introduced the classes $(\mathbf{K}_r^+, \preccurlyeq_r)$ for rational $r$. These classes have intrinsic transcendentals and essentially undecidable theories [4]. We note in this section that it is the presence of intrinsic transcendentals in conjunction with an ability to combine such extensions in arbitrary configurations that leads to the essential undecidability of the resulting theory. In Section 6 we derive a class from $(\mathbf{K}_r^+, \preccurlyeq_r)$ which will limit the existence of such configurations and produce a simple theory.

Our result here is simply a generalized restatement of the main theorem of [4], which asserts that under certain conditions the theory of the generic will interpret Robinson’s $R$ and hence be essentially undecidable.

**Theorem 4.1.** Let $(\mathbf{K}, \leq)$ be a full amalgamation class, and suppose there is an intrinsically transcendental pair $(A, B)$ for which there are $V \in \mathbf{K}, U \subseteq B \setminus A$, and $X \in \mathbf{K}$ such that:

1. $U \oplus_V U' \subseteq X$, where $U' \approx U$.
2. $(U \oplus_V U', X)$ is an intrinsically transcendental biminimal pair (that is, it is a minimal pair and there is no $A \subset U \oplus_V U'$ for which $(A, X)$ is minimal pair).
3. For $n \in \omega$, $\bigoplus_{i<n}(U_i \oplus U'_i) \leq \bigoplus_{i<n}(X_i/V)$ where $X_i \approx X$.

Then the theory of the $(\mathbf{K}, \leq)$-generic interprets Robinson’s $R$ and is hence essentially undecidable.

The details of the proof are omitted here; it is a straightforward generalization of the special case proved in [4]. The main ideas are that each copy of $A$ will code a natural number via the number of copies of $B$ that appear over it. Copies of $V$ will be codes for bijections between copies of $B$ over different copies of $A$, and the structures $\bigoplus_{i<n}(X_i/V)$ will witness the presence or absence of such a bijection. The structures $U$ are simply appropriate substructures of $B$ used to define the bijection; $V$ and $U$ were simply points in [4]. Using this coding, one can interpret the graphs of addition and multiplication on the natural numbers. It is worth noting that the presence of the appropriate configurations in [4] relied on the presence of sufficiently large intrinsically transcendentals extensions and the neutrality of the minimal pair relation $(A, B)$ to the internal structure of $A$.

5. Limiting Intrinsic Types

In this section we give a condition on classes with intrinsic transcendentals that will limit the number of intrinsic types, and in the next section we will give a non-trivial example of such a class.

Our approach will be to associate intrinsic types with *supports*; these will be subsets of a model which determine the type. The definition of a support will
depend on that of an external closure; this will consist of unions of chains of minimal pairs with a base in $M$ but all other extensions external to $M$.

**Definition 5.1.** Let $(K, \leq)$ be a full amalgamation class, let $M \models T_{(K, \leq)}$ and let $\mathcal{C} = \mathcal{C}_{(K, \leq)}$. For $X \subseteq M$, we define the external closure $\text{ecl}_M(X)$ as follows.

- Let $J_0$ be the union of all $B \subseteq \mathcal{C}$ with $(X', B)$ a minimal pair for some $X' \subseteq X$, and $B \cap M \subseteq X$.
- Given $J_i$, let $J_{i+1}$ be the union of all $B \subseteq \mathcal{C}$ with $(X', B)$ a minimal pair for $X' \subseteq J_i$ and $B \cap M \subseteq X$.

We then define $\text{ecl}_M(X)$ as $X \cup \bigcup_{i \in \omega} J_i$.

**Remark.** Note that for $a \in \text{ecl}_M(X) \setminus M$, there is a minimal chain $X = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k$ with $a \in X_k$ and $X_i \cap M = X$ for all $i$.

**Definition 5.2.** Let $(K, \leq)$ be a full amalgamation class with $\mathcal{C} = \mathcal{C}_{(K, \leq)}$. Given a model $M \models T_{(K, \leq)}$, a support system for $\text{cl}(M)$ over $M$ is a set $S$ of subsets of $M$ such that for $a \in \text{cl}(M) \setminus M$, there is some $S \in S$ so that $a \in \text{ecl}(S)$. Such an $S$ is called a support for $a$ over $M$.

Moreover,

- $S$ is bounded by $\mu$ if every $S \in S$ has cardinality less than $\mu$;
- $S$ has unique supports if for $a \in \text{cl}(M) \setminus M$ there is exactly one $S \in S$ with $a \in \text{ecl}(S)$. In such cases $S$ may be denoted by $\text{supp}_M(a)$;
- $S$ has free closures if for $S, T \in S$,
  $$\text{ecl}_M(ST) = \text{ecl}_M(S) \oplus_{ST} \text{ecl}_M(T)$$
- $S$ is edge-closed if $S$ has unique supports and for $a \in \text{cl}(M) \setminus M$, $\bar{m} \subseteq \text{ecl}(M)$, if $R(a\bar{m})$ holds for some relation $R$ of the language, then $\bar{m} \subseteq \text{supp}_M(a)$.

We note that a support system with free closures will also have unique supports. We say that the class $(K, \leq)$ has limiting supports if there is a cardinal $\lambda$ such that any sufficiently large model has a support system which is bounded by $\lambda$, has free closures and is edge-closed.

The thrust of the notion of limiting supports is that for any model $M$ and $a \in \text{cl}(M) \setminus M$, any interaction between $a$ and $M$ will be mediated by $\text{supp}_M(a)$. The support of $a$ will thus determine $\text{tp}(a/M)$. Since there will be a uniform bound on the number of such supports, for sufficiently large models $M$ there will be at most $|M|$ intrinsic types.

The notion of a limiting support system is motivated by the following example.

**Example 5.3.** The class $(K^1_+, \leq_1)$ consists of finite acyclic graphs with $A \leq_1 B$ when the connected components of $B \setminus A$ are disjoint from the components of $A$ (i.e. $B = (B \setminus A) \oplus A$). For $M$ a model, any vertex $a \in \text{cl}(M) \setminus M$ must be contained in a tree which has its root vertex in $M$ and all other vertices in $\text{ecl}(M)$. The root vertices can be taken as supports, so that the set of all single vertices of $M$ forms a support system for $M$. In fact this defines a limiting support system. It is not hard to show that there are at most $2^{\aleph_0}$ extrinsic types over any model of $T_{(K^1_+, \leq_1)}$ so by Corollary 5.9 below, the theory is strictly superstable.\(^1\)

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\(^1\)A published form (see [1]) of a conjecture of Baldwin’s stated that no generic had a strictly superstable theory. While this example provides a counterexample (see [10] for another); it is clear from the literature (e.g. Example 2.35 of [3]) that Baldwin meant to deny the existence of a generic with a small strictly superstable theory.
The above example generalizes slightly to any class of graphs in which for $M \models T_{(\mathbf{K}, \leq)}$ and $a, b \in M$, there are only finitely many pairwise disjoint paths from $a$ to $b$ in $M$. We also present two examples which do not have limiting supports.

**Example 5.4.** Consider the class $(\mathbf{K}_f^+, \preceq_r)$ for rational $r$ with $0 < r < 1$ as discussed above. For $M$ a model of $T_{(\mathbf{K}_f^+, \preceq_r)}$, an obvious choice for a support system would be the set of finite subsets of $M$. Such a support system would not have free closures or even unique supports. In fact, it is not hard to see that no support system for this class with bounded supports could have unique supports. Choose $X, Y \in \mathbf{K}_f^+$ so that there are $W, Z \in \mathbf{K}_f^+$ with $X \subseteq W, Y \subseteq Z, X \cap Y = \emptyset, W \cap Z \neq \emptyset$, and $\delta_r(W/X) = 0 = \delta_r(Z/Y)$. Then (by compactness) we can choose a model $M$ with copies of $X, Z$ in different supports with intersecting external closures.

**Example 5.5.** Consider the class $(\mathbf{K}_f, \preceq)$ where $\mathbf{K}_f$ is the class of all finite graphs, and $A \preceq B$ is defined as $A \preceq_1 B$ for $A \neq \emptyset$ (that is $\delta_1(B_0/A) > 0$ for $A \subseteq B_0 \subseteq B$) and $\emptyset \preceq B$ for every $B \in \mathbf{K}_f$. Then it is straightforward to show that $(\mathbf{K}_f, \preceq)$ satisfies the properties 2.2.1-2.2.8. It is not a full amalgamation class, since for arbitrary $B, C$ we do not have $C \preceq_1 B \oplus \emptyset C$. It does however satisfy Corollary 3.5, hence Lemma 3.9 holds in this context.

Note that for any finite $A$, if $B$ is the one point extension of $A$ that connects the only vertex of $B \setminus A$ to every vertex of $A$ then $(A, B)$ will be a minimal pair. A simple compactness argument shows that for any $\kappa < |\mathcal{C}|$, there is a model $M_\kappa$ and an $A_\kappa \subseteq M_\kappa$ with $|A_\kappa| = \kappa = |M_\kappa|$ and $a, b$ such that for $a \in A_\kappa$, $(a, b)$ is an edge. Since $\operatorname{tp}(b/A)$ is non-algebraic, it has $|\mathcal{C}|$ realizations and hence some realizations in $\mathcal{C} \setminus M_\kappa$. Let $b_\kappa$ be such a realization; then for any $A_0 \subseteq A_\kappa$ we have $b_\kappa \in \text{ecl}(A_0)$. Thus $b_\kappa$ has $2^\kappa$ different supports.

We note that no support system could be limiting. If a support system $S$ is bounded by $\lambda$, then for $\kappa > \lambda$ there will be $\kappa$ supports for $b_\kappa$ in $A_\kappa$ so that the supports would not be unique. Further the system would not be edge-closed.

In what follows, when a class has limiting supports we will implicitly choose a support system that witnesses this for whatever model $M$ is under discussion. Thus when the next lemma chooses a support for an element over a model, it should be understood that the support is chosen from a system with the properties witnessing limiting supports.

To show that classes with limiting supports have few intrinsic types, we will argue that the type of an element over a model is determined by its support. The main part of this is contained in the following lemma.

**Lemma 5.6.** Assume that $(\mathbf{K}, \leq)$ is a full amalgamation class with limiting supports. Let $M \models T_{(\mathbf{K}, \leq)}$, let $X \subseteq M$ be a support, and fix tuples $\bar{a}, \bar{b} \subseteq_\omega \mathcal{C}$. Suppose

- $\operatorname{tp}(\bar{a}/X) = \operatorname{tp}(\bar{b}/X)$
- For every $a \in \bar{a} \setminus M$ and every $b \in \bar{b} \setminus M$, $\text{supp}_M(a) = X = \text{supp}_M(b)$.
- $\bar{a} \cap M = \bar{b} \cap M$

Then for $\bar{m} \subseteq_\omega M$ and $\bar{c}$ with $(\bar{a}\bar{m}, \bar{c})$ a minimal pair, there is a tuple $\bar{d}$ with

1. $\bar{a}\bar{m}\bar{c} \equiv \bar{b}\bar{d}\bar{a}$ as witnessed by some isomorphism $\alpha$
2. $\operatorname{tp}(\bar{a}\bar{c}/X) = \operatorname{tp}(\bar{b}\bar{d}/X)$
3. For $c \in \bar{c} \setminus M$, $\text{supp}_M(c) = X$ if and only if $\alpha(c) \in \bar{d} \setminus M$ and $\text{supp}_M(\alpha(c)) = X$
4. $\bar{c} \cap M = \bar{d} \cap M$
Proof. Let \(\bar{c}_0 := \{ c \in \bar{c} : c \not\in M, \text{supp}_M(c) = X \} \) and let \(\bar{c}_1 = \bar{c} \setminus (\bar{c}_0 X)\). Note that for \(c \in \bar{c}_1\), either \(\text{supp}(c) \neq X\) or \(c \in M\). In the first case, free closures ensure that there is no relation which holds between \(c\) and any element of \(\bar{c}_0\), while in the latter the same is guaranteed by the definition of a support. Thus, letting \(\bar{x} = \bar{c} \cap X\), we have \(\bar{c} = \bar{a}m\bar{x}\bar{c}_0 \oplus_{\bar{a}m\bar{x}} \bar{a}m\bar{x}\bar{c}_1\). Since \((\bar{a}m, \bar{a}m\bar{c})\) is a minimal pair, we have \(\bar{a}m \leq \bar{a}m\bar{x} \leq \bar{c}_0 \oplus_{\bar{a}m\bar{x}} \bar{c}_1\) if \(\bar{c}_0 \neq \emptyset \neq \bar{c}_1\), contradicting the minimality of \((\bar{a}m, \bar{a}m\bar{c})\). Thus \(\bar{c} = \bar{c}_0\) or \(\bar{c} = \bar{c}_1\).

If \(\bar{c} = \bar{c}_0\), let \(q\) be obtained from \(\text{tp}(\bar{c}/\bar{a}X)\) by mapping \(\bar{a}\) to \(\bar{b}\). Then \(q\) is consistent since \(\text{tp}(\bar{a}/X) = \text{tp}(\bar{b}/X)\). By saturation, it has more than \(|M|\) realizations (since \(M\) is algebraically closed). Choose \(d\) to be a realization contained in \(E \setminus M\).

If \(\bar{c} = \bar{c}_1\bar{x}\), then we let \(\bar{d} = \bar{c}\). We want to show that \((\bar{a}m, \bar{c}) \approx (\bar{b}m, \bar{c})\). If \((a, m)\) is an edge with \(a \in \bar{a} \setminus M, m \in \bar{m}\), we have that \(m \in X\) (since \(\mathcal{E}\) is edge-closed) and similarly for \(b \in \bar{b} \setminus M\). Thus we have \(\bar{a}m \approx \bar{b}m\). Since no element of \(\bar{c}_1 \setminus M\) has support \(X\), by free closures there is no relation which holds between \(\bar{a} \setminus M\) and \(\bar{c} \setminus M\). Thus \(\bar{a}m\bar{c} \approx \bar{b}m\bar{c}\) since \(\bar{a} \cap M = \bar{b} \cap M\).

This establishes the first statement. In both cases, the remaining statements are clear. \(\square\)

Corollary 5.7. Assume that \((K, \leq), M\) and \(X\) are as above. Fix tuples \(\bar{a}, \bar{b} \subseteq_{\omega} \mathcal{E}\). Suppose

1. \(\text{tp}(\bar{a}/X) = \text{tp}(\bar{b}/X)\)
2. For \(a \in \bar{a} \setminus M\) and \(b \in \bar{b} \setminus M\), \(\text{supp}(a) = X = \text{supp}(b)\).
3. \(\bar{a} \cap M = \bar{b} \cap M\)

Then

1. For \(\bar{m} \subseteq \omega\) and \(\exists \phi(\bar{z}; \bar{a}m) \in \text{cltp}(\bar{a}m)\), we have \(\exists \phi(\bar{z}; \bar{b}m) \in \text{cltp}(\bar{b}m)\).
2. \(\text{cltp}(\bar{a}/M) = \text{cltp}(\bar{b}/M)\).

Proof. The second statement is an immediate consequence of the first; we prove (1) by induction on \(k\), the least number such that \(\phi\) is \(k\)-intrinsic. The case when \(k = 0\) is immediate from the previous lemma. For the inductive step, fix \(\bar{c}\) with \(\phi(\bar{c}; \bar{a}m)\). By the previous lemma, there is some \(d\) with \(\text{cl}(\bar{b}m, d) \approx (\bar{a}m, \bar{c})\), \(\text{tp}(\bar{a}c/X) = \text{tp}(bd/X)\), \(\bar{c} \cap M = d \cap M\) and every element of \(d \setminus M\) has support \(X\). Thus the hypotheses of the corollary are satisfied, so by the inductive hypothesis the \((k - 1)\)-intrinsic formulae realized over subsets \(bdm\) are precisely those realized over subsets of \(a\bar{c}m\). This establishes that \(\phi(\bar{d}; \bar{b}m)\) holds. \(\square\)

Proposition 5.8. If \((K, \leq)\) is a full amalgamation class with limiting supports, then any sufficiently large model \(M\) of the theory of the \((K, \leq)\)-generic has \(|M|\) intrinsic types.

Proof. Let \(\mu\) be a cardinal so that all supports can be chosen to be of cardinality less than \(\mu\); fix \(M\) a model with cardinality \(\kappa \geq 2^\mu\). For \(a, b \in c(M) \setminus M\) with \(\text{supp}_M(a) = \text{supp}_M(b)\) and \(\text{tp}(a/\text{supp}_M(a)) = \text{tp}(b/\text{supp}_M(b))\), the preceding corollary implies that \(\text{cltp}(a/M) = \text{cltp}(b/M)\) and hence \(\text{tp}(a/M) = \text{tp}(b/M)\). Since there are most \(\kappa\) choices for \(\text{supp}(a)\) and \(2^\mu\) choices for \(\text{tp}(a/X)\), we have at most \(\kappa \cdot 2^\mu = \kappa\) intrinsic types. \(\square\)

Corollary 5.9. If \((K, \leq)\) is a full amalgamation class with limiting supports, then:
If for any \( M \models T_{(\mathbb{K}, \leq)} \), \( |M| = \kappa \geq 2^{\aleph_0} \) implies that \( S(M) \) contains has at most \( \kappa \) extrinsic types, then \( T_{(\mathbb{K}, \leq)} \) is superstable.

If for any \( M \models T_{(\mathbb{K}, \leq)} \), \( |M| = \kappa \geq 2^{\aleph_0} \) implies that \( S(M) \) contains has at most \( \kappa \) extrinsic types, then the theory of the generic is stable.

### 6. The Anti-Collapse

In this section we apply the results of Section 5 to derive a class with few intrinsic types from an arithmetic class. The generic of this class will have a strictly simple theory.

Fixing \( r \) rational from \((0, 1)\), we work with the class \((\mathbb{K}^+_r, \leq_r)\) discussed in the introduction. Recall that this class has intrinsic transcendentals and the generic has an essentially undecidable theory. It is straightforward to show that for any \( M \models T_{(\mathbb{K}^+_r, \leq_r)} \), there are \( 2^{|M|} \) intrinsic types over \( M \).

We will limit the structure of intrinsically transcendental extensions to meet the conditions of Proposition 5.8. The procedure is analogous to Hrushovski's collapse (see [15]), but reversed in the following sense. For \( B, M \) any graphs embedded in a common superstructure, the basis of \( B \) in \( M \) is the set of vertices of \( M \) which have an edge to some vertex in \( B \). We denote it by \( B \leq M := \{ m \in M : (m, b) \text{ is an edge for some } b \in B \} \). Then while the original collapse limited the number of extensions over a fixed basis with relative predimension 0, we instead limit the structure of the possible bases of such extensions. We thus term our construction an anti-collapse.

#### Notation 6.1.

We adopt the following notations:

- \( \delta \) refers to \( \delta_r \).
- We extend \( \delta \) to all pairs \( A, B \) embedded in a common superstructure by setting \( \delta(B/A) = \delta(AB/A) \).

The following facts are easily established:

#### Lemma 6.2.

For \((\mathbb{K}^+_r, \leq_r)\) and \( \delta \) as above:

1. \((\mathbb{K}^+_r, \leq_r)\) is a full amalgamation class
2. \( \delta(\oplus_{i<n}(B_i/A)) = \sum_{i<n} \delta(B_i/A) \) for \( A, B_i \in \mathbb{K}^+_r \)
3. If \( A, B, C \in \mathbb{K}^+_r \) are embedded in a common superstructure, then \( \delta(B/AC) \leq \delta(B/A) \)
4. For \( A \subseteq B \in \mathbb{K}^+_r \), \( \delta(B/A) = |B \setminus A| - re(B, A) \), where \( e(B, A) \) denotes the number of edges from \( B \setminus A \) to \( A \)

Note that intrinsically transcendental extensions will correspond to intrinsic extensions \( B \supseteq A \) with \( \delta(B/A) = 0 \). For any class, a biminimal pair \( (A, B) \) is a minimal pair for which there is no \( A' \subsetneq A \) with \( (A', B) \) a minimal pair. Our approach will be to limit the copies of \( A \) which can give rise to copies of \( B \) – this will maintain the presence of intrinsic transcendentals but force a tree-like structure onto the closure and limit the number of intrinsic types.

Our basic procedure is to fix a parameter \( N \in \omega \) and partition vertices into equivalence classes of size at most \( N \). Intuitively, a set of vertices \( A \) whose elements all belong to the same class can serve as the basis for an arbitrary number of copies of an intrinsically transcendental extension \( B \), while sets which contain elements of different classes can only serve as the basis for finitely many such extensions.
To proceed formally, let $L^* = L_G \cup \{ S(x, y) \}$ where $L_G$ is the language of graphs; we will amalgamate classes of $L^*$-structures. $S(x, y)$ will be used to define an equivalence relation on the vertices; a set of vertices whose elements are all in the same class will be referred to as $S$-homogeneous.

**Notation 6.3.** Our derived amalgamation class will be based on the following parameters.

- Choose $N \in \omega$ with $N > 1$. This will determine the size of the equivalence classes under $S$.
- Let $\gamma$ denote the smallest rational number for which $\delta(B/A) \ll -\gamma$ whenever $\delta(B/A) < 0$ (this is a simple case of the granularity in [13]; its existence is immediate from the rationality of $r$).
- We choose a function $\mu(X, Y) : K^+_r \times K^+_r \to \mathbb{R}$ to be any function with
  \[
  \mu(X, Y) \geq \frac{2(\delta(X) + \delta(Y))}{\gamma}
  \]
  where $X_0 \sqcup X_1$ denotes the disjoint union of $X_0$ and $X_1$.

We will need to work with a generalization of biminimal pairs which we call proper 0-extensions. These represent minimal extensions of relative predimension 0 which cannot be decomposed into independent extensions. We need this generalization because simply limiting the occurrence of biminimal pairs over non-homogeneous bases would still allow dependencies between the closures of different homogeneous sets; this would result in a failure to have free closures on the resulting support system.

**Definition 6.4.** For $X \subseteq Z \in K^+_r$, we say that $(X, Z)$ is a 0-extension if there is a minimal chain $X = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k = Z$ with $\delta(X_{i+1}/X_i) = 0$. We will call the extension proper if there is no $X_0 \subseteq X$ with $(X_0, Z)$ a 0-extension and there are no non-empty $Z_0, Z_1$ with $Z = XZ_0 \oplus_X XZ_1$.

We note that if $(X, Z)$ is a 0-extension, then there is no $Y$ with $X \subseteq Y \subseteq Z$ and $\delta(Y/X) < 0$. Also note that biminimal 0-extensions are examples of proper 0-extensions.

We seek to limit the appearance of proper 0-extensions while preserving the notion of a closed substructure. Because biminimal extensions can be created during amalgamation, we cannot require that every biminimal extension occur over a base which is $S$-homogeneous. Our obstruction to doing so, however, is only finite and will not affect the external closures since models are algebraically closed.

**Definition 6.5.** We say that an $L^*$-structure $A$ is admissible if it satisfies the following conditions.

1. $A \restriction L_G \in K^+_r$
2. $S$ defines an equivalence relation on $A$, with classes of cardinality less than $N$.
3. For $X \subseteq A$, if $X$ is not $S$-homogeneous, then for $(X, Y)$ a proper 0-extension, there are at most $\nu(X)$ pairwise disjoint copies of $Y$ over $X$ in $A$.

We copy of $Y$ over $X$ is a $Y'$ so that there is some isomorphism $f : Y \to Y'$ which fixes $X$. This condition will restrict the number of intrinsic types.
Let \( K^*_\ell \) denote the class of all finite admissible \( L^\ast \)-structures. We extend \( \delta \) to \( K^*_\ell \) by defining \( \delta(A) = \delta(A|L_G) \).

**Definition 6.6.** For \( A, B \in K^*_\ell \), we say that \( A \preccurlyeq^\ast B \) if

1. \( A \subseteq B \)
2. If \( X \subseteq B \) is \( S \)-homogeneous, then \( X \subseteq A \) or \( X \subseteq B \setminus A \).
3. \( (A|L_G) \preccurlyeq (B|L_G) \)

We want to show that \((K^*_\ell, \preccurlyeq^\ast)\) is a full amalgamation class. The proof relies on Hrushovski’s insight in the collapse that the number of new biminimal pairs which can be created during free amalgamation is bounded (see [15], pp. 174-5). The proposition below and its proof are modifications of his argument (as presented by Wagner).

**Notation 6.7.** If \( D = B \oplus_A C \), then for \( X \subseteq D \):

- \( X_B := X \cap (B \setminus A) \)
- \( X_A := X \cap A \)
- \( X_{BC} := X \cap (BC \setminus A) \)

**Proposition 6.8.** Fix \( A, B, C \in K^*_\ell \) with \( A = B \cap C, A \preccurlyeq^\ast B, A \preccurlyeq^\ast C \) and let \( D = B \oplus_A C \). For \( U \subseteq D \) with \( U_B \neq \emptyset \neq U_C \), there are at most \( \nu(U) \) pairwise disjoint proper \( 0 \)-extensions of \( U \) in \( D \)

**Proof.** It suffices to show that for \( U = X \uplus X' \) with \( X_B \neq \emptyset \neq X_C \), if \( \{(XX', Z_i) : i < k\} \) is a sequence of proper \( 0 \)-extensions which are pairwise disjoint over \( XX' \), then \( k < \mu(X, X') \). For each \( i < k \), we want to replace \( X, X' \) with \( Y_i, Y_i' \) so that \((Y_i Y_i', Z_i)\) is a proper \( 0 \)-extension which is minimal in the sense that there are no \( Y_i, Y_i' \) extensions of \( Y_i, Y_i' \) for which \((Y_i Y_i', Z_i)\) is a proper \( 0 \)-extension. We choose \( Y_i, Y_i' \subseteq Z_i \) to be maximal extensions of \( X \) and \( X' \) so that \( \delta(Y_i/X) = 0 = \delta(Y_i'/X') \) and \( Y_i, Y_i' \) are disjoint over \( X, X' \).

Letting \( W_i = Z_i \setminus Y_i Y_i' \), the following are straightforward

- \( W_i \neq \emptyset \)
- \( \delta(W_i/Y_i Y_i') = 0 \)
- \( W_i Y_i' \neq \emptyset \neq W_i Y_i' \)
- \( Y_i X X' \subseteq X, Y_i' X X' \subseteq X' \)

**Case 1.** Remebering as needed, let \( Z_1 \ldots Z_m \) be such that there is \( V_i \subseteq Z_i \) with \((Y_i Y_i', V_i)\) a minimal \( 0 \)-extension and \((V_i)_B \neq \emptyset \neq (V_i)_C \). We will show that \( \delta((V_i)_{BC}/A(Y_i Y_i')_{BC}) < 0 \); since \( A \preccurlyeq^\ast BC \) this will limit the number of possible such \( V_i \) and hence limit \( m \).

It is easily established that \((V_i)_A \neq \emptyset\) and \( V_i Y_i' \neq \emptyset \neq V_i Y_i' \) (the latter fact comes from our choice of \( Y_i, Y_i' \)). By minimality, \( \delta((V_i)_{BC}/A(Y_i Y_i')_{BC}) < 0 \) so \( \delta((V_i)_{BC}/A(Y_i Y_i')_{BC}) < 0 \) as well.

Let \( YY' := \bigcup_{i=1}^m Y_i Y_i' \) and let \( Q = \bigcup_{i=1}^m (V_i)_{BC} \). Then note that

\[
\delta(Q/A(YY')_{BC}) \leq \sum_i \delta((V_i)_{BC}/A(YY')_{BC}) \\
\leq \sum_i \delta((V_i)_{BC}/A(Y_i Y_i')_{BC}) \\
\leq \gamma m
\]
Since \( A \preceq^* D \) we have
\[
0 < \delta((YY')_{\bar{BC}Q/A}) = \delta(Q/A(YY')_{\bar{BC}}) + \delta((YY')_{\bar{BC}/A}) \\
\leq -\gamma m + \delta((YY')_{\bar{BC}/A}) \\
\leq -\gamma m + \delta((YY')_{\bar{BC}}/(XX')_A(YY')_A) \\
= -\gamma m + \delta(XX') - \delta(XAX_A'YAY_A') \\
\leq -\gamma m + \delta(X) + \delta(X')
\]
where the final equality holds since \( \delta(YY'/XX') = 0 \). Thus \( m < \frac{\delta(X) + \delta(X')}{\gamma} \).

**Case 2.** We show that \( k - m \leq \frac{\delta(X) + \delta(X')}{\gamma} \). For \( i > m \), every minimal 0-extension \( (Y_iY_i', V_i) \) satisfies \( (V_i)_B \) or \( (V_i)_C \) empty. Let \( Z_{m+1}, \ldots, Z_l \) denote those copies with \( (V_i)_B = \varnothing \). Since \( W_{Y_i}^{Y_i} \neq \varnothing \), we can choose \( V_i \) so that \( (V_i)_Y \neq \varnothing \). Thus we have
\[
\delta(Y_i/Y_iX_BX_A) < \delta(Y_i/X_BX_A) = \delta(Y_i/X) = 0.
\]
Letting \( Y^* = \bigcup_{i=m+1}^l Y_i \) and letting \( V^* = \bigcup_{i=m+1}^l V_i \) we have the following inequalities:
\[
\delta(XBY^*/AC) - \delta(XBY^*/X_A) \leq \delta(XBY^*/V^*X) - \delta(XBY^*/X_A) \\
= \delta(Y^*/V^*X) + \delta(X_B/V^*X_A) - \delta(X_B/X_A) \\
\leq \delta(Y^*/V^*X) + \delta(X_B/X_A) - \delta(X_B/X_A) \\
\leq -\gamma (l - m)
\]
Since \( AC \preceq^* ABC \), we have \( \delta(XBY^*/AC) > 0 \). Thus
\[
l - m \leq \frac{\delta(XBY^*/X_A) - \delta(XBY^*/AC)}{\gamma} \\
\leq \frac{\delta(XBY^*/X_A)}{\gamma} \\
\leq \frac{\delta(X)}{\gamma}
\]

The same reasoning shows that \( k - l \leq \frac{\delta(X')}{\gamma} \).

We thus have \( k \leq \mu(X, X') \) as desired.

**Proposition 6.9.** \( (K_r^*, \preceq^*) \) is a full amalgamation class which satisfies 2.2.1 through 2.2.8.

**Proof.** We need to show that \( (K_r^*, \preceq^*) \) is closed under free amalgamation and has full amalgamation in addition to satisfying the axioms of 2.2.

We first show that \( (K_r^*, \preceq^*) \) is closed under free amalgamation. Fix \( A, B, C \in K_r^* \) with \( A = B \cap C, A \preceq^* B, A \preceq^* C \). Letting \( D = B \oplus_A C \), we first have to show that \( D \in K_r^* \) and that \( B \preceq^* D, C \preceq^* D \).

Of the properties defining admissibility, only 6.6.3 needs comment (6.5.2 holds by 6.6.2). Fix \( X' \subseteq D \) with \( X \) not \( S \)-homogeneous, and let \( (X, Y) \) be a proper 0-extension. Since \( B, C \in K_r^* \), we may assume without loss that \( X_B \) and \( X_C \) are
both non-empty. Then Proposition 6.8 implies that there are most \( \nu(X) \) copies of
\( Y \) over \( X \) in \( D \). That \( B \preceq^* D \) and \( C \preceq^* D \) are straightforward.

For full amalgamation, we have to show that if \( A, B, C, D \) are as above with the
exception that we only require \( A \subseteq C \), then \( C \preceq^* D \). We know that \( C|L_G \preceq_r D|L_G \)
since \((K_1^+, \preceq_r)\) is a full amalgamation class. Thus it suffices to show that for \( X \subseteq C \),
if \( X \) is not \( S \)-homogeneous and \((X, Y)\) is a proper 0-extension, then there are fewer
than \( \nu(X) \) copies of \( Y \) over \( X \) in \( D \). Since \( A \preceq^* B \), any proper 0-extension of \( X \)
must be contained in \( C \). Since \( C \in K_1^+ \), we have \( C \preceq^* D \) as required.

The verifications of 2.2.1 through 2.2.8 are routine. \( \Box \)

We now show that the generic has few intrinsic types by applying the results of
Section 5. Again, let \( \mathcal{E} \) be a universal domain for the theory of the \((K_1^+, \preceq^*)\)-
generic.

Note that for \( X \subseteq M \) with \( M \models T(K_1^+, \preceq^*) \), if \((X, Y)\) is a minimal pair where
\( Y \) enlarges an \( S \)-class of \( X \), then by the second criterion for admissibility, \( Y \) is
algebraic over \( X \) and hence contained in \( M \). Thus the only intrinsic extensions of \( X \)
which can be external to a model will be intrinsic in the sense of \((K_1^+, \preceq_r)\).

**Lemma 6.10.** Let \( M \models T(K_1^+, \preceq^*) \) and fix \( X \subseteq \omega M \). For \( X = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k \)
a minimal chain over \( X \) with \( X_i \not\subseteq M \) for \( i > 0 \), \( \delta(X_k/X) = 0 = \delta(X_k/M) \)

**Proof.** Note that if \( \delta(X_k/X) < 0 \) then \( X_k \) is an algebraic extension and we have
\( X_k \subseteq M \). Since \( 0 \leq \delta(X_k/M) \leq \delta(X_k/X) \), \( \delta(X_k/M) = 0 \) as well. \( \Box \)

**Lemma 6.11.** Let \( M \models T(K_1^+, \preceq^*) \) and let \( X \subseteq M \). Then
\[ \text{ecl}_M(X) = \bigcup \{ \text{ecl}_M(X') : X' \subseteq X, X' \text{\ S-homogeneous} \} \]

**Proof.** One direction of the equality is obvious, so it suffices to show that if \( a \in \text{ecl}_M(X) \), then for some \( X' \subseteq X \) with \( X' \text{\ S-homogeneous} \), \( a \in \text{ecl}_M(X') \). Let
\( X = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k \) be a minimal chain with \( a \in X_k \). We have to show that
\( X_i \cap M = X_0 \).

By Lemma 6.10, \((X_i, X_{i+1})\) is a 0-extension for \( i \geq 0 \). Letting \( X'' = X_i^M \), we
have that \((X'', X_1)\) is a biminimal 0-extension. By property 6.6.3, if \( X'' \) is not \( S \)-
homogeneous then \( X_1 \) is an algebraic extension of \( X'' \), contradicting that \( X_1 \not\subseteq M \). Letting
\( X' \) be the \( S \)-closure of \( X'' \), I claim that for \( i > 1 \) and \( Y' = X_i^M \), \( Y' \subseteq X' \).
If not, then \( X', Y' \) are in different \( S \)-classes, and it is easy to see that \( X_1 \) is a proper 0
extension of a non \( S \)-homogeneous subset of \( X'Y' \). Thus by 6.5.3, there can only be
finitely many copies of \( X_1 \) over \( X'Y' \), contradicting the algebraic closure of \( M \). \( \Box \)

**Proposition 6.12.** For \( a \in \text{cl}(M) \setminus M \), there is \( X \subseteq \omega M \) with \( X \text\ S\text-homogeneous \)
and \( a \in \text{ecl}_M(X) \).

**Proof.** Let \( a \in \text{cl}(M) \setminus M \). Fix \( X \subseteq \omega M \) with \( a \in \text{cl}(X) \). We show that \( a \in \text{ecl}_M(X) \).
Let \( X = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k \) be a minimal 0-chain with \( a \in X_k \).
Without loss, we may assume that \( X_i \not\subseteq M \) for \( i > 0 \) (otherwise we can replace \( X \)
with \( \bigcup X_j \) for \( X_j \subseteq M \)).

Suppose, by way of contradiction, that \( a \not\in \text{ecl}_M(X) \). Then choose the least value
\( j \) so that \( X_{j+1} \cap M \neq X \). Let \( W = (X_{j+1} \setminus X_j) \cap M, V = (X_{j+1} \setminus X_j) \setminus M \). Then
\( \delta(WV/X_j) = 0, \delta(V/X_jW) < 0 \) (by minimality). Note that \( \delta(X_j/VXW) =
\delta(V/X_jW) + \delta(X_j/X) \leq \delta(X_j/X) < 0 \). Thus \( X_j \subseteq M \), a
contradiction which establishes that \( a \in \text{ecl}_M(X) \). We can choose \( X \text\ S\text-homogeneous \)
and \( S\text-closed \) by Lemma 6.11. \( \Box \)
Suppose that $X \subseteq M$ : $X$ is $S$-homogeneous. Proposition 6.14. if $(X \subseteq M)$ such.

Proof. the theory of the $\mathbb{K}^*_r$, $\prec^*$) has limiting supports.

Proof. For a fixed model $M$, let $T$ be $\{ X \subseteq M : X$ is $S$-homogeneous $\}$. Then $T$ is clearly a support system bounded by $\omega$ which is edge-closed. We show that $T$ has free closures. We need to show that for $X, Y$ supports,

$$\text{ecl}_M(X) \text{ecl}_M(Y) = \text{ecl}_M(X) \bigoplus \text{ecl}_M(Y)_{XY}$$

Suppose that $X \neq Y$ are $S$-homogeneous subsets of $M$, $X = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k$ and $Y = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_l$ are minimal 0-chains with $X_i, Y_i \not\subseteq M$ for $i > 0$. Then if $(X_k \setminus Y_{k-1}) \cap (Y_l \setminus Y_{l-1}) \neq \emptyset$, $X_kY_l$ would contain a proper 0-extension over some non-homogeneous subset of $XY$, contradicting that there can be only finitely many such.

We now show that the theory of the generic is strictly simple.

Proposition 6.14. For any rational $r \in (0, 1)$, there is an $N_r$ so that for $N > N_r$, the theory of the $(\mathbb{K}^*_r, \prec^*)$ generic has the independence property.

Proof. Lemma 3.2 of [4] shows that for sufficiently large $N$ we can find a biminimal 0-extension $(A, C)$ from $\mathbb{K}_r$ with $|A| = N$. By the definition of admissibility, we can expand $A$ to an $L^*$-structure so that for some $a \in A$, $A \setminus \{a\}$ is $S$-homogeneous. Let $\phi(x, \bar{b})$ be the $L^*$ formula stating that $x\bar{b}$ has the quantifier-free type of $A$ and $x\bar{b}$ extends to a copy of $C$. Fix $M \models T^{(\mathbb{K}^*_r, \prec^*)}$, and let $\bar{b}_1, \ldots, \bar{b}_n$ be $L^*$-isomorphic to $\bar{b}$. By biminimality, $\bar{b} \prec aC_\bar{b}$ for $C_\bar{b}$ a copy of $C$ over $ab$. For $\eta \in 2^\omega$, let $D_\eta = \bigoplus_{\eta(i) = 1} (C_\bar{b}/a)$. Then by Lemma 2.5, there is a strong embedding of $D_\eta$ over $M$, so that $\models \phi(a, \bar{b}_i)$ exactly when $\eta(i) = 1$.

This highlights that simply limiting the number of intrinsic types over a model is not sufficient to guarantee the stability of the theory. We do have a simple theory, however.

Theorem 6.15. The theory of the $(\mathbb{K}^*_r, \prec^*)$-generic is simple.

Proof. Employing Theorem 2.8 of [5] we show that for $\kappa, \lambda$ infinite cardinals, if $W$ is any set of pairwise incompatible types of size at most $\lambda$ over a parameter set $A$ of size $\kappa$, then $|W| \leq \kappa^\omega + 2^\lambda$. Since we have already shown that there are at most $\kappa$ intrinsic types over $A$, we may assume that $W$ consists of extrinsic types.

Let us call two types $p, q$ over $A$ closure-type incompatible if there is some finite $\bar{a} \subseteq \omega A$ and an intrinsic formula $\phi(x; \bar{a})$ such that $p \vdash \phi$ and $q \vdash \neg \phi$. Imitating 3.3 in [5] we will show that if $X$ is a set of pairwise mutually closure-type incompatible types over $A$, with each $p \in X$ of power at most $\lambda$, then $|X| \leq 2^\lambda$. Indeed, we can uniquely associate each $p \in X$ with a function $f_p$ from $A^{<\omega}$ to the set of intrinsic types over $\emptyset$ by $f_p(\bar{a}) = \{ \phi(\bar{x}; \bar{y}) : p(x) \vdash \phi(x; \bar{a}) \}$. Thus we can map $X$ to an anti-chain from $\text{Fn}_{\lambda^+}(\kappa, 2^{\aleph_0})$, the set of all partial maps from a cardinality $\lambda$ subset of $\kappa$ to a subset of $2^{\aleph_0}$. Lemma IV.7.5 of [12] then implies $|X| \leq (2^{\aleph_0})^\lambda = 2^\lambda$.

We note that if $p, q$ are incompatible types, then one of the following must hold ( $a, b$ are chosen so that $a \models p$ and $b \models q$)

1. $p$ and $q$ are closure-type incompatible over $A$
(2) The relative predimension of $\text{cl}(Aa)$ over $Aa$ is negative, as is the relative predimension of $\text{cl}(Ab)$ over $Ab$, so that any realization of $p \cup q$ would have to be intrinsic over $A$. Specifically, $\delta(\text{cl}(Aa)/Aa) < 0$, $\delta(\text{cl}(Ab)/Ab) < 0$ and $\delta(\text{cl}(Aa)/Aa) + \delta(\text{cl}(Ab)/Ab) \leq -1$.

(3) The number of edges in from $a$ to $A$ and $b$ to $A$ would make any realization of $p \cup q$ intrinsic over $A$. Specifically, this occurs when $1 - r|a^A \cup b^A| \leq 0$.

For $p \in W$, let $A_p \subseteq_\omega A$ be a minimal subset of $A$ such that for $a \models p$, $a^A \subseteq A_p$ and every minimal chain over $Aa$ with negative relative dimension over $Aa$ will be over $A_p$; let $F$ denote the map $p \mapsto A_p$. Note that for $p, q \in W$, if $A_p = A_q$ then $p$ and $q$ must be incompatible for reason (1) (in this case, reason (2) reduces to (1)); thus for a fixed $A_p$ there are at most $2^\lambda$ types $q \in W$ with $F(q) = A_p$. Since there are at most $\kappa$ possible $A_p$, there are at most $\kappa 2^\lambda = \kappa + 2^\lambda$ types in $W$.

\[ \square \]

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