Induced structures on submanifolds in almost product Riemannian manifolds

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Abstract

We give some fundamental properties of the induced structures on submanifolds immersed in almost product or locally product Riemannian manifolds. We study the induced structure by the composition of two isometric immersions on submanifolds in an almost product Riemannian manifold. We give an effective construction for some induced structures on submanifolds of codimension 1 or 2 in Euclidean space.

Introduction

The geometry of submanifolds with induced structures in Riemannian manifolds was widely studied by many geometers, such as K. Yano and M. Kon ([36], [37], [38], [38]). An investigation of the properties of the almost product or locally product Riemannian manifolds has been made by M. Okumura ([29]), T. Adati and T. Miyazawa ([1], [2]), M. Anastasiei ([4]), G. Pitiş ([30]), X. Senlin and N. Yilong ([33]), A. G. Walker ([35]), M. Atçeken, S. Keleş and B. Şahin ([5], [32]), etc. Also, the properties of the almost r-paracontact structures were studied by A. Bucki and A. Miernovski ([9], [10]), T. Adati and T. Miyazawa ([3]), S. Ianuş and I. Mihai ([20]), J. Nikic ([28]), etc.

The purpose of this paper is to give some properties of the submanifolds with a $(P, g, \varepsilon \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ structure induced by a $\tilde{P}$ structure defined on a Riemannian manifold $(\tilde{M}, \tilde{g})$, with $\tilde{P}^2 = \varepsilon I$ and the compatibility equality (1.2) between $\tilde{g}$ and $\tilde{P}$ (where $I$ is the identity on $\tilde{M}$ and $\varepsilon = \pm 1$). The $(P, g, \varepsilon \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ structure is determined by an $(1,1)$-tensor field $P$ on $M$, tangent vector fields $\xi_\alpha$ on $M$, 1-forms $u_\alpha$ on $M$ and a $r \times r$ matrix $(a_{\alpha\beta})_r$, where its entries $a_{\alpha\beta}$ are real functions on $M$ ($\alpha, \beta \in \{1, \ldots, r\}$). Particularly, for $\varepsilon = 1$, the $\tilde{P}$ structure on $(\tilde{M}, \tilde{g})$ becomes an almost product structure.

This paper is organized as follows: in section 1 we construct the structure $(P, g, \varepsilon \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$, induced on a submanifold in the Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ with the conditions (1.1) and (1.2), in the same manner like in [2].

In section 2, we give the fundamental formulae for $(P, g, \varepsilon \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ induced structure on a submanifold in the Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with $\nabla \tilde{P} = 0$. 

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In sections 3 and 4, we shall investigate the necessary and sufficient conditions for a submanifold with \((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) induced structure, immersed in a locally Riemannian product manifold to be normal (relative to the commutativity of the endomorphism \(P\) and the Weingarten operators \(A_\alpha\) on \(M\)) and show further properties of this kind of submanifold.

In section 5, we prove that the composition of two isometric immersions \(M \hookrightarrow \mathcal{M} \hookrightarrow \tilde{M}\) induces on a submanifold \(M\) of codimension 2 in an almost product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) a \((P, g, u_1, u_2, \xi_1, \xi_2, (a_{\alpha\beta}))\) structure determined by a \((P, g, u_1, \xi_1, a_{11})\) structure on \(M\) (induced by \((P, g)\)) and a \((P, g, u_2, \xi_2, a_{22})\) structure on \(M\) (induced by \((P, g, u_2, \xi_2)\)), where \(a_{12} = a_{21} = g(\xi_2^\perp, N_1)\) and \(N_1\) is a unit normal vector field on \(M\).

In section 6, we show some properties of \((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) induced structures on the submanifold of codimension 1 or 2 in an almost product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\).

In section 7, we give an effective construction for some \((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) induced structures on hyperspheres or submanifolds of codimension 2 in Euclidean space.

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1 \((P, g, \varepsilon \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) induced structure on submanifolds in Riemannian manifold

Let \((\tilde{M}, \tilde{g})\) be a Riemannian manifold, equipped with a Riemannian metric tensor \(\tilde{g}\) and a (1,1) tensor field \(\tilde{P}\) such that

\[(1.1) \quad \tilde{P}^2 = \varepsilon I, \quad \varepsilon = \pm 1\]

where \(I\) is the identity on \(\tilde{M}\). We suppose that \(\tilde{g}\) and \(\tilde{P}\) are compatible in the sense that for each \(U, V \in \chi(\tilde{M})\) we have that

\[(1.2) \quad \tilde{g}(\tilde{P}U, \tilde{P}V) = \tilde{g}(U, V),\]

which is equivalent with

\[(1.3) \quad \tilde{g}(\tilde{P}U, V) = \varepsilon \tilde{g}(U, \tilde{P}V), \quad (\forall) U, V \in \chi(\tilde{M})\]

for each \(U, V \in \chi(\tilde{M})\), where \(\chi(M)\) is the Lie algebra of the vector fields on \(\tilde{M}\).

For \(\varepsilon = 1\), \(\tilde{P}\) is an almost product structure and the Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with the compatibility relation (1.2), becomes an almost product Riemannian manifold.

Let \(M\) be an \(n\)-dimensional submanifold of codimension \(r\) in a Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) which satisfied the relations (1.1) and (1.2).
We make the following notations throughout all of this paper: \( X, Y, Z, \ldots \) are tangential vector fields on \( M \). We denote the tangent space of \( M \) at \( x \in M \) by \( T_x(M) \) and the normal space of \( M \) in \( x \) by \( T^\bot_x(M) \). Let \((N_1, \ldots, N_r) := (N_\alpha)\) be an orthonormal basis in \( T^\bot_x(M) \), for every \( x \in M \). In the following statements, the indices range is fixed in this way: \( \alpha, \beta, \gamma \ldots \in \{1, \ldots, r\} \). We shall use the Einstein convention for summation.

The decomposition of the vector fields \( \tilde{P}X \) and \( \tilde{P}N_\alpha \) respectively, in the tangential and normal components of \( M \) has the form:

\[
(1.4) \quad \tilde{P} X = P X + \sum_{\alpha} u_\alpha(X)N_\alpha,
\]

for any \( X \in \chi(M) \) and

\[
(1.5) \quad \tilde{P} N_\alpha = \varepsilon \xi_\alpha + \sum_{\beta} a_{\alpha\beta} N_\beta, \quad (\varepsilon = \pm 1)
\]

where \( P \) is a \((1,1)\) tensor field on \( M \), \( \xi_\alpha \) are tangent vector fields on \( M \), \( u_\alpha \) are 1-forms on \( M \) and \((a_{\alpha\beta})_r\) is an \( r \times r \)-matrix and its entries \( a_{\alpha\beta} \) are real functions on \( M \). If \( \varepsilon = 1 \), the formulae in the next theorem were demonstrated by T. Adati (in [2]).

**Theorem 1.1.** Let \( M \) be an \( n \)-dimensional submanifold of codimension \( r \) in a Riemannian manifold \((\tilde{M}, \tilde{g})\), equipped by an \((1,1)\)-tensor field \( \tilde{P} \), such that \( \tilde{g} \) and \( \tilde{P} \) verify the conditions (1.1) and (1.2). The \((\tilde{M}, \tilde{g}, \tilde{P})\) structure induces on the submanifold \( M \) a \((P, g, u_\alpha, \varepsilon \xi_\alpha, (a_{\alpha\beta})_r)\) Riemannian structure which verifies the following properties:

\[
(1.6) \quad \begin{cases} 
(i) & P^2 X = \varepsilon (X - \sum_\alpha u_\alpha(X)\xi_\alpha), \\
(ii) & u_\alpha(P X) = -\sum_\beta a_{\alpha\beta} u_\beta(X), \\
(iii) & a_{\alpha\beta} = \varepsilon a_{\beta\alpha}, \\
(iv) & u_\alpha(\xi_\beta) = g(\xi_\alpha, \xi_\beta) = \delta_{\alpha\beta} - \varepsilon \sum_\gamma a_{\alpha\gamma} a_{\gamma\beta}, \\
(v) & P \xi_\alpha = -\sum_\beta a_{\alpha\beta} \xi_\beta
\end{cases}
\]

and

\[
(1.7) \quad \begin{cases} 
(i) & u_\alpha(X) = g(X, \xi_\alpha), \\
(ii) & g(P X, Y) = \varepsilon g(X, P Y), \\
(iii) & g(P X, P Y) = g(X, Y) - \sum_\alpha u_\alpha(X)u_\alpha(Y),
\end{cases}
\]

for any \( X, Y \in \chi(M) \).

**Proof:** Applying \( \tilde{P} \) in the equality (1.4) we obtain that

\[
\tilde{P}^2 X = \tilde{P}(P X) + \sum_{\alpha=1}^r u_\alpha(X)\tilde{P}(N_\alpha),
\]

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for every $X \in \chi(M)$.

From (1.1), (1.4) and (1.5) we have

$$\varepsilon X = P^2 X + \varepsilon \sum_{\alpha} u_{\alpha}(X) \xi_{\alpha} + \sum_{\beta} (u_{\beta}(PX) + \sum_{\alpha} u_{\alpha}(X)a_{\alpha\beta})N_{\beta},$$

for every $X \in \chi(M)$, and from this, it results (i) and (ii) from the relations (1.6). Furthermore, the equality (i) can be written in the following form

\[(i)' \quad P^2 = \varepsilon (I - \sum_{\alpha} u_{\alpha} \otimes \xi_{\alpha})\]

Applying the equality (1.3) to the normal vector fields $N_{\alpha}$ and $N_{\beta}$ respectively and using the equality (1.5) follows that

$$\tilde{g}(\varepsilon \xi_{\alpha} + \sum_{\gamma=1} a_{\alpha\gamma}N_{\gamma}, N_{\beta}) = \varepsilon \tilde{g}(N_{\alpha}, \varepsilon \xi_{\beta} + \sum_{\gamma=1} a_{\beta\gamma}N_{\gamma})$$

and from this we obtain the equality (iii) from (1.6).

From $\tilde{P}^2 N_{\alpha} = \varepsilon N_{\alpha}$, using the relations (1.4) and (1.5) we obtain

$$\varepsilon N_{\alpha} = \tilde{P}^2 N_{\alpha} = \tilde{P}(\varepsilon \xi_{\alpha} + \sum_{\beta} a_{\alpha\beta}N_{\beta}) = \varepsilon \tilde{P} \xi_{\alpha} + \sum_{\beta} a_{\alpha\beta}\tilde{P} N_{\beta} =$$

$$= \varepsilon (P \xi_{\alpha} + \sum_{\beta} a_{\alpha\beta}\xi_{\beta}) + \sum_{\beta} [\varepsilon u_{\beta}(\xi_{\alpha}) + \sum_{\gamma} a_{\alpha\gamma}a_{\gamma\beta}]N_{\beta}$$

so

$$\varepsilon N_{\alpha} = \varepsilon (P \xi_{\alpha} + \sum_{\beta} a_{\alpha\beta}\xi_{\beta}) + \sum_{\beta} [\varepsilon u_{\beta}(\xi_{\alpha}) + \sum_{\gamma} a_{\alpha\gamma}a_{\gamma\beta}]N_{\beta}$$

Identifying the tangential components from the last equality we obtain (v) from (1.6) and identifying the normal components from the last equality we obtain (iv) from (1.6). Applying the equality (1.3) to the vector fields $X$ and $N_{\alpha}$ respectively, we obtain

$$\tilde{g}($$$$PX, N_{\alpha}) = \varepsilon \tilde{g}(X, \tilde{P} N_{\alpha})$$

and from this it follows that

$$\tilde{g}(P X + \sum_{\beta} u_{\beta}(X)N_{\beta}, N_{\alpha}) = \varepsilon \tilde{g}(X, \varepsilon \xi_{\alpha} + \sum_{\beta} a_{\alpha\beta}N_{\beta})$$

for any tangent vector fields $X$ on $M$, so we obtain the equality (i)(1.7).

Applying the relations (1.3) and (1.4) to the tangential vector fields $X$ and $Y$ on $M$ we obtain

$$g(PX,Y) = g(\tilde{P}X,Y) = g(\tilde{P}^2 X, \tilde{P}Y) = \varepsilon g(X, \tilde{P}Y) = \varepsilon g(X, PY)$$
and from this we have (ii) from (1.7).

Replacing $Y$ by $PY$ in the equality (ii) from (1.7) and using the equality (i) from (1.6) we obtain

$$g(PX, PY) = \varepsilon g(X, P^2Y) = \varepsilon^2 (g(X, Y) - \sum_{\alpha=1} u_\alpha(Y)g(X, \xi_\alpha)),$$

for any $X, Y \in \chi(M)$. From $\varepsilon^2 = 1$ and using (i) from (1.7), we obtain the equality (iii) from (7). □

**Remark 1.1.** Particularly, for $\varepsilon = -1$ and $a = 0$ (if we omit the metric $g$), we obtain a $(P, u_{\alpha}, -\xi_{\alpha})$ induced structure, which has the following properties:

$$
\begin{align*}
(i) & \quad P^2X = -X + \sum_{\alpha} u_\alpha(X)\xi_\alpha, \quad (\forall) X \in \chi(M) \\
(ii) & \quad u_\alpha \circ P = 0, \\
(iii) & \quad u_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \\
(iv) & \quad P\xi_\alpha = 0.
\end{align*}
$$

Applying $P$ in the equality (i) from (1.8) we obtain

$$P^3 + P = 0$$

Therefore, $P$ is an $f(3, 1)$-structure and the manifold $M$ endowed with a $(P, u_{\alpha}, -\xi_{\alpha})$ structure is a framed manifold. Besides, we can call this structure an almost $r$-contact structure. If $r=1$, then we call this structure an almost contact structure.

**Definition 1.1.** For $\varepsilon = -1$ and $a = 0$, the $(P, g, u_{\alpha}, -\xi_{\alpha})$ induced structure by $\tilde{P}$ from $(\tilde{M}, \tilde{g})$, with the properties (1.7) and (1.8) is called an $f(3, 1)$ Riemannian structure.

**Remark 1.2.** Particularly, for $\varepsilon = 1$ and $a = 0$ the $(P, u_{\alpha}, \xi_{\alpha})$ induced structure on $M$ has the following properties:

$$
\begin{align*}
(i) & \quad P^2X = X - \sum_{\alpha} u_\alpha(X)\xi_\alpha, \quad (\forall) X \in \chi(M) \\
(ii) & \quad u_\alpha \circ P = 0, \\
(iii) & \quad u_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \\
(iv) & \quad P\xi_\alpha = 0.
\end{align*}
$$

Applying $P$ in the equality (i) from (1.10) we obtain

$$P^3 - P = 0$$

Therefore, $P$ is an $f(3, -1)$ structure on $M$, and $(P, u_\alpha, \xi_\alpha)$ induced structure on $M$, with the properties (1.10) is called an $f(3, -1)$ framed structure. Besides, this kind of structure is called an almost $r$-paracontact structure. For $r=1$ we obtain an almost paracontact structure.
**Definition 1.2.** For \( \varepsilon = 1 \) and \( a = 0 \), the \((P, g, u_\alpha, \xi_\alpha)\) induced structure on \( M \), with the properties (1.10) and (1.7) is an \( f(3, -1) \) Riemannian structure.

**Remark 1.3.** Therefore, the \((P, g, u_\alpha, \varepsilon \xi_\alpha, (a_{\alpha \beta})_r)\) induced structure on \( M \) by the \( \widetilde{P} \) structure on \((\widetilde{M}, \widetilde{g})\) (which verifies the relations (1.1) and (1.2)) is a generalization of an almost r-contact Riemannian structure and an almost r-paracontact Riemannian structure, respectively.

Thus, we have the following situations:
(I): For \( \varepsilon = -1 \) and \( a = 0 \), we obtain an \( f(3, 1) \) structure and the structure \((P, g, u_\alpha, -\xi_\alpha)\) becomes an almost r-contact Riemannian structure;
(II): For \( \varepsilon = 1 \) and \( a = 0 \), we obtain an \( f(3, -1) \) structure and the structure \((P, g, u_\alpha, \xi_\alpha)\) becomes an almost r-paracontact Riemannian structure.

**Definition 1.3.** A \((P, g, u_\alpha, \varepsilon \xi_\alpha, (a_{\alpha \beta})_r)\) structure on a submanifold \( M \) of codimension \( r \) in a \((\widetilde{M}, \widetilde{g}, \widetilde{P})\) Riemannian manifold with the proprieties (1.1) and (1.2), which verifies the properties (1.6) and (1.7) is called an \((a, \varepsilon)\) \( f \) Riemannian structure.

**Remark 1.4.** The case of the \((a, -1)\) \( f \) Riemannian structure was studied by K. Yano and M. Okumura (in [37] and [38]).

In the following issue, we suppose that \( \varepsilon = 1 \). Therefore, the \( \widetilde{P} \) structure on the Riemannian manifold \((\widetilde{M}, \widetilde{g})\), which satisfies the relations (1.1) and (1.2), is an almost product structure.

**Remark 1.5.** If we suppose that \( \xi_1, \ldots, \xi_r \) are linearly independent tangent vector fields on \( M \), it follows that the 1-forms \( u_1, \ldots, u_r \) are linearly independent, too. The equality
\[
\sum_{\alpha=1}^{r} \lambda^\alpha u_\alpha(X) = 0
\]
is equivalent with
\[
0 = \sum_{\alpha} \lambda^\alpha g(X, \xi_\alpha) = g(X, \sum_{\alpha} \lambda^\alpha \xi_\alpha), \ (\forall) X \in \chi(M)
\]
thus
\[
\sum_{\alpha=1}^{r} \lambda^\alpha \xi_\alpha = 0 \Rightarrow \lambda^\alpha = 0
\]
and from this we obtain that \( u_1, \ldots, u_r \) are linearly independent on \( M \).

**Remark 1.6.** We denote by
\[
(1.12) \quad D_x = \{X_x \in T_x M : u_\alpha(X_x) = 0\},
\]
for any \( \alpha \in \{1, \ldots, r\} \). We remark that \( D_x \) is an \((n - r)\)-dimensional subspace in \( T_x M \) and the function
\[
(1.13) \quad D : x \mapsto D_x, \ (\forall) x \in M
\]
is a distribution locally defined on \( M \). If \( X \in D \), from (1.6)(ii) we have
\[
(1.14) \quad u_\alpha(PX) = -\sum_\beta a_{\beta\alpha}u_\beta(X) = 0,
\]
for any \( X \in D \), then \( PX \in D \). Therefore \( D \) is an invariant distribution with respect to \( P \).

If \( D^\perp_x \) is an orthogonal supplement of \( D_x \) in \( T_x M \), then we obtain the distribution \( D^\perp : x \mapsto D^\perp_x \). Furthermore, we have the decomposition
\[
(1.15) \quad T_x M = D_x \oplus D^\perp_x,
\]
in any point \( x \in M \). From (1.7)(i) it follows that the vector fields \( \xi_\alpha \neq 0 \) are orthogonal on \( D_x \) and \( \xi_\alpha \in D^\perp_x \). Thus, if \( \xi_\alpha \neq 0 \) for any \( \alpha \in \{1, \ldots, r\} \), then \( D^\perp_x \) is generated by \( \xi_1, \ldots, \xi_r \) and \( D^\perp_x \) is \( r \)-dimensional in \( T_x M \).

From (1.6)(v) we remark that the space \( D_x \) is \( P \)-invariant and \( P \) satisfies
\[
(1.16) \quad P^2 X = X,
\]
and
\[
(1.17) \quad g(PX, PY) = g(X, Y),
\]
for all \( X, Y \in D \). Thus \( P \) is an almost product Riemannian structure on \( D \). Furthermore, \( \text{rank}(P) = n - r \) on \( D \) and its eigenvalues are 1 and -1.

**Remark 1.7.** Let \( \{N_1, \ldots, N_r\} \) and \( \{N'_1, \ldots, N'_r\} \) be two orthonormal basis on a normal space \( T^\perp_x M \). The decomposition of \( N'_\alpha \) in the basis \( \{N_1, \ldots, N_r\} \) is the following
\[
(1.18) \quad N'_\alpha = \sum_{\gamma=1}^r k^\gamma_{\alpha} N_\gamma,
\]
for any \( \alpha \in \{1, \ldots, r\} \), where \( (k^\gamma_{\alpha}) \) is an \( r \times r \) orthogonal matrix and we have (from [2]):
\[
(1.19) \quad u'_\alpha = \sum_{\gamma} k^\gamma_{\alpha} u_\gamma
\]
\[
(1.20) \quad \xi'_\alpha = \sum_{\gamma} k^\gamma_{\alpha} \xi_\gamma
\]
and
\[
(1.21) \quad a'_{\alpha\beta} = \sum_{\gamma} k^\gamma_{\alpha} a_{\gamma\delta} k^\delta_{\beta}
\]
From (1.20) we have that if \( \xi_1, \ldots, \xi_r \) are linearly independent vector fields, then \( \xi'_1, \ldots, \xi_r \) are also linearly independent.
2 The fundamental equations of submanifolds with 
\((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) structures

In this section, we suppose that the Riemannian manifolds \((\tilde{M}, \tilde{g})\) are endowed with a \((1,1)\) tensor field \(\tilde{P}\) on \(\tilde{M}\), which verifies the equalities (1.1), (1.2), and the structure \(\tilde{P}\) is parallel with respect to the Levi-Civita connection \(\tilde{\nabla}\) of \(\tilde{g}\). Let \(M\) be an \(n\)-dimensional Riemannian submanifold of codimension \(r\), isometric immersed in \(\tilde{M}\) and \((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\) is the induced structure by the structure \(\tilde{P}\) on \((\tilde{M}, \tilde{g})\). We denoted by \(\nabla\) the induced Levi-Civita connection on \(M\). We assume that \((N_1, ..., N_r) := (N_\alpha)\) is an orthonormal basis in the normal space \(T_xM^\perp\) at \(M\) in every point \(x \in M\). In the following, we shall identify the vector fields in \(M\) and their images under the differential mapping, that is, if we denote the immersion of \(M\) in \(N\) by \(i\) and \(X\) is a vector field in \(M\), we identify \(X\) and \(i^*X\), for all \(X \in \chi(M)\).

The Gauss and Weingarten formulae are:

\[(2.1)\quad \tilde{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=1}^{r} h_\alpha(X, Y)N_\alpha,\]

and

\[(2.2)\quad \tilde{\nabla}_X N_\alpha = -A_\alpha X + \nabla^\perp_X N_\alpha,\]

respectively, where

\[(2.3)\quad h_\alpha(X, Y) = g(A_\alpha X, Y),\]

for every \(X, Y \in \chi(M)\).

For the normal connection \(\nabla^\perp_X N_\alpha\), we have the decomposition

\[(2.4)\quad \nabla^\perp_X N_\alpha = \sum_{\beta=1}^{r} l_{\alpha\beta}(X)N_\beta,\]

for every \(X \in \chi(M)\). Therefore, we obtain an \(r \times r\) matrix \((l_{\alpha\beta}(X))_r\) of 1-forms on \(M\). From \(\bar{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}\) we get

\[\bar{g}(\nabla^\perp_X N_\alpha, N_\beta) = 0\]

which is equivalent with

\[\bar{g}(\sum_{\gamma} l_{\alpha\gamma}(X)N_\gamma, N_\beta) + \bar{g}(N_\alpha; \sum_{\gamma} l_{\beta\gamma}(X)N_\gamma) = 0,\]

for any \(X \in \chi(M)\). Thus, we have

\[(2.5)\quad l_{\alpha\beta} = -l_{\beta\alpha},\]

for any \(\alpha, \beta \in \{1, ..., r\}\).

For \(\varepsilon = 1\), the following formulae were demonstrated by T.Adati (in [2]).
**Theorem 2.1.** Let \( M \) be an \( n \)-dimensional submanifold of codimension \( r \) in a Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) (with the properties (1.1) and (1.2)). If the structure \( \tilde{P} \) is parallel with respect to the Levi-Civita connection \( \tilde{\nabla} \) of \( \tilde{g} \), then the \((\tilde{P}, g, u_\alpha, \epsilon \xi_\alpha, (a_\alpha)_r)\) induced structure on the submanifold \( M \) has the following properties:

\[
\begin{align*}
(i) \quad & (\nabla_X \tilde{P})(Y) = \varepsilon \sum_\alpha h_\alpha(X, Y) \xi_\alpha + \sum_\alpha u_\alpha(Y) A_\alpha X, \\
(ii) \quad & (\nabla_X u_\alpha)(Y) = -h_\alpha(X, PY) + \sum_\beta u_\beta(Y) l_{\alpha\beta}(X) + \sum_\beta h_\beta(X, Y) a_{\beta\alpha}, \\
(iii) \quad & \nabla_X \xi_\alpha = -\varepsilon P(A_\alpha X) + \varepsilon \sum_\beta a_{\alpha\beta} \xi_\beta + \sum_\alpha l_{\alpha\beta}(X) \xi_\beta, \\
(iv) \quad & X(a_{\alpha\beta}) = -\varepsilon u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X) a_{\gamma\beta} + l_{\beta\gamma}(X) a_{\alpha\gamma}].
\end{align*}
\]

**Proof:** From the assumption that \( \tilde{\nabla} \tilde{\nabla} \tilde{P} = 0 \) we have

\[
(\nabla U)(\tilde{P} V) = \tilde{P}(\nabla U V),
\]

for any tangential vector fields \( U \) and \( V \) on \( \tilde{M} \). Using the Gauss and Weingarten formulae, we obtain from (2.4) that

\[
\tilde{\nabla}_X (\tilde{P} Y) = \tilde{\nabla}_X P Y + \sum_\alpha X(u_\alpha(Y)) \nabla_X N_\alpha + \sum_\alpha u_\alpha(Y) \nabla_X N_\alpha =
\]

\[
= \nabla_X PY - \sum_\alpha u_\alpha(Y) A_\alpha X + \sum_\alpha [h_\alpha(X, PY) + X(u_\alpha(Y)) + \sum_\beta u_\beta(Y) l_{\beta\alpha}(X)] N_\alpha.
\]

On the other hand, we have

\[
\tilde{P}(\tilde{\nabla}_X Y) = \tilde{P}(\nabla_X Y) + \sum_\alpha h_\alpha(X, Y) \tilde{P} N_\alpha
\]

and from this we obtain

\[
\tilde{P}(\tilde{\nabla}_X Y) = P(\nabla_X Y) + \varepsilon \sum_\alpha h_\alpha(X, Y) \xi_\alpha +
\]

\[
+ \sum_\alpha [u_\alpha(\nabla_X Y) + \sum_\beta h_\beta(X, Y) a_{\beta\alpha}] N_\alpha.
\]

From [11], we know that:

\[
(\nabla_X P)(Y) = \nabla_X (PY) - P(\nabla_X Y)
\]

and

\[
(\nabla_X u_\alpha)(Y) = X(u_\alpha(Y)) - u_\alpha(\nabla_X Y)
\]

Using the relations (2.8), (2.9) in (2.7), we obtain (i) and (ii) from (2.6), from the equality of the tangential components of \( M \) (and the normal components of \( M \), respectively) from the both parts of the equality (2.7).
In the next, we apply $\tilde{\nabla}_X$ in (1.5) (with $X \in \chi(M)$) and using the equality (2.7), (with $Y = N_\alpha$), we get

\[(2.12) \quad \tilde{\nabla}_X (\tilde{P} N_\alpha) = \tilde{P} (\tilde{\nabla}_X N_\alpha)\]

From (2.12) we obtain

\[(2.13) \quad \tilde{\nabla}_X (\tilde{P} N_\alpha) = \tilde{\nabla}_X (\varepsilon \xi_\alpha + \sum \beta a_{\alpha\beta} N_\beta) =
\]

\[= \varepsilon \nabla_X \xi_\alpha - \sum \beta a_{\alpha\beta} A_{\beta} X + \sum \beta [X (a_{\alpha\beta}) + \varepsilon h_{\beta} (X, \xi_\alpha) + \sum \gamma a_{\alpha\gamma} \cdot l_{\gamma\beta} (X)] N_\beta\]

and

\[(2.14) \quad \tilde{P} (\tilde{\nabla}_X N_\alpha) = \tilde{P} (-A_\alpha X + \sum \beta l_{\alpha\beta} N_\beta) =
\]

\[= -P (A_\alpha X) + \varepsilon \sum \beta l_{\alpha\beta} (X) \xi_\beta - \sum \beta [u_\beta (A_\alpha X) - \sum \gamma a_{\alpha\beta} l_{\gamma\beta} (X)] N_\beta\]

Using the relations (2.13) and (2.14) in the equality (2.12) and identifying the tangential and the normal components at $M$, respectively, we obtain the relations (iii) and (iv) from (2.6). □

**Remark 2.1.** The compatibility condition $\tilde{\nabla} \tilde{P} = 0$, where $\tilde{\nabla}$ is Levi-Civita connection with respect of the metric $\tilde{g}$ implies the integrability of the structure $\tilde{P}$ which is equivalent with the vanishing of the Nijenhuis torsion tensor field of $\tilde{P}$:

\[(2.15) \quad N_{\tilde{P}} (X, Y) = [\tilde{P} X, \tilde{P} Y] + \tilde{P}^2 [X, Y] - \tilde{P} [\tilde{P} X, Y] - \tilde{P} [X, \tilde{P} Y].\]

For this assumption, we have the next general lemma:

**Lemma 2.1.** We suppose that we have an almost product structure $Q$ on a manifold $M$ and a linear connection $D$ with the torsion $T$. If $N_Q$ is Nijenhuis torsion tensor field of $Q$, then we obtain:

\[(2.16) \quad N_Q (X, Y) = (D_{QX} Q) (Y) - (D_{QY} Q) (X) + (D_X Q) (QY) -
\]

\[-(D_Y Q) (QX) - T (QX, QY) - T (X, Y) + QT (QX, Y) + QT (X, QY),\]

for any $X, Y \in \chi(M)$.

**Proof:** From the definition of the torsion $T$ follows that:

\[(2.17) \quad \quad [X, Y] = D_X Y - D_Y X - T (X, Y),\]

and from this we get

\[(2.18) \quad [QX, QY] = D_{QX} QY - D_{QY} QX - T (QX, QY),\]
\begin{align*}
(2.19) 
[QX,Y] &= D_{QX}Y - D_YQX - T(QX,Y), \\
(2.20) 
[X,QY] &= D_XQY - D_{QY}X - T(X,QY).
\end{align*}

Using the relations 
\( (D_XQ)Y = D_XQY - Q(D_XY) \), 
\( Q^2 = I \) and 
\( D_XY = D_XQ^2Y = D_XQ(QY) \) and replacing the relations (2.17), (2.18), (2.19) and (2.20) in the formula of Nijenhuis torsion tensor field of \( Q \), we obtain:

\[
N_Q (X,Y) = (D_{QX}Q)(Y) - (D_{QY}Q)(X) + D_XY - D_YX + Q(D_YQX) - Q(D_XQY) - T(QX,QY) - T(X,Y) + Q(T(QX,Y) + Q(T(X,QY) = \\
= (D_{QX}Q)(Y) - (D_{QY}Q)(X) + (D_XQ(QY) - Q(D_XQY)) - (D_YQ(QX) - Q(D_YQX)) - T(QX,QY) - T(X,Y) + Q(T(QX,Y) + Q(T(X,QY)
\]

so, we obtain the equality (2.16). □

**Remark 2.2.** From the last mentioned lemma, we remark that if we have 
\( T = 0 \) and \( DQ = 0 \) then the structure \( Q \) is integrable. An integrable almost product structure is also called locally product structure.

**Corollary 2.1.** If \( M \) is a totally geodesic submanifold in a locally product manifold \((\tilde{M}, \tilde{P}, \tilde{g})\) and the normal connection \( \nabla^\perp \) vanishes identically (that is \( t_{\alpha\beta} = 0 \)), then the \((P,g,u_\alpha, \xi_\alpha,(a_{\alpha\beta}),\tau)\) induced structure on \( M \) has the following properties:

\[
\nabla P = 0, \quad \nabla u = 0, \quad \nabla \xi = 0, \quad a = \text{constant}
\]

In the following issue, we suppose that \((\tilde{M}, \tilde{g}, \tilde{P})\) is an almost product Riemannian manifold, endowed by a linear connection \( \tilde{\nabla} \) such that \( \tilde{\nabla} P = 0 \) and with the torsion \( \tilde{T} \neq 0 \). Let \((M,g)\) be a submanifold of the \((\tilde{M}, \tilde{g}, \tilde{P})\) Riemannian manifold, endowed with the linear metric \( g \) induced on \( M \) by the metric \( \tilde{g} \) and let \( \nabla \) be the induced connection on \( M \) by the connection \( \tilde{\nabla} \) of \( \tilde{M} \).

The Gauss formula has the usual form:

\[
\tilde{\nabla}_XY = \nabla_XY + h(X,Y) (\forall) X, Y \in \chi(M)
\]

where

\[
h(X,Y) = \sum_\alpha h^\alpha(X,Y)N_\alpha
\]

and \( \nabla \) is a metric connection on \( M \) (i.e \( \nabla g = 0 \)) but it is not the Levi-Civita connection of \( g \) and his torsion has the form

\[
(2.21) 
T(X,Y) = \tilde{T}(X,Y) - h(X,Y) + h(Y,X),
\]

11
for any $X, Y \in \chi(M)$. We remark that the second fundamental form $h$ is bilinear in $X$ and $Y$, but it is not symmetric. If the torsion $\tilde{T} = 0$, then $T = 0$ if and only if the second fundamental form $h$ is symmetric.

Let $c : [a, b] \rightarrow M$, $t \mapsto c(t)$ a smooth curve on $M$. We denote by

$$
\dot{c} : t \mapsto \dot{c}(t) = \frac{dx^i}{dt} \frac{\partial}{\partial x^i}(c(t))
$$

the tangent vector field to $c$.

**Definition 2.1.** If $\nabla_{\dot{c}} \dot{c} = 0$ then we say that the curve $c$ is $M$-autoparallel.

**Proposition 2.1.** If $c$ is $\tilde{M}$-autoparallel curve, then $c$ is also, $M$-autoparallel curve and $h(\dot{c}, \dot{c}) = 0$.

**Proof:** From the Gauss formula for $X = Y = \dot{c}$ we get

$$
\tilde{\nabla}_{\dot{c}} \dot{c} = \nabla_{\dot{c}} \dot{c} + h(\dot{c}, \dot{c})
$$

But $c$ is $\tilde{M}$-autoparallel, so $\tilde{\nabla}_{\dot{c}} \dot{c} = 0$ and using the equality (2.23) we obtain $\nabla_{\dot{c}} \dot{c} = 0$ and $h(\dot{c}, \dot{c}) = 0$. □

**Definition 2.2.** A submanifold $M$ is said to be autoparallel in $\tilde{M}$ if any $M$-autoparallel curve of submanifold $M$ in $\tilde{M}$ is also $\tilde{M}$-autoparallel.

We denote by

$$
s_h(X, Y) = \frac{1}{2}(h(X, Y) + h(Y, X))
$$

the symmetric part and by

$$
a_h(X, Y) = \frac{1}{2}(h(X, Y) - h(Y, X))
$$

the skew-symmetric part, respectively, of the bilinear form $h$. We remark that

$$
h(X, Y) = s_h(X, Y) + a_h(X, Y),
$$

for any $X, Y \in \chi(M)$.

**Proposition 2.2.** A submanifold $M \subset \tilde{M}$ is autoparallel in $\tilde{M}$ if and only if $s_h = 0$.

**Proof:** From the Gauss formula for $X = Y = \dot{c}$ we obtain

$$
\tilde{\nabla}_{\dot{c}} \dot{c} = \nabla_{\dot{c}} \dot{c} + s_h(\dot{c}, \dot{c})
$$

If $s_h = 0$ follows that $\tilde{\nabla}_{\dot{c}} \dot{c} = \nabla_{\dot{c}} \dot{c}$, so any $M$-autoparallel curve is also $\tilde{M}$-autoparallel.

Conversely, if any $M$-autoparallel curve is also $\tilde{M}$-autoparallel, then $s_h(X, X) = 0$ for any $X \in \chi(M)$. Particularly, we have $s_h(X + Y, X + Y) = 0$ and from this we obtain $s_h(X, Y) = 0$ for any $X, Y \in \chi(M)$, so $s_h = 0$. □
Remark 2.3. The Weingarten formula is not affected by the non-vanishing of the torsion $\tilde{T}$ on $\tilde{M}$ and of the torsion $T$ on $M$, thus

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X \xi,$$

for any $X \in \chi(M)$ and $N \in \Gamma(TM^\perp)$. If $Y \in \chi(M)$ we have $\tilde{g}(Y, N) = 0$ so, $X(\tilde{g}(Y, N)) = 0$ for any $X, Y \in \chi(M)$ and this equality is equivalent with

\[(2.26) \quad \tilde{g}(\tilde{\nabla}_X Y, N) + \tilde{g}(Y, \tilde{\nabla}_X N) = 0\]

Using the Gauss and Weingarten formulae in (2.26), we obtain

\[(2.27) \quad g(A_N X, Y) = \tilde{g}(h(X, Y), N),\]

and

\[(2.27)' \quad g(A_N Y, X) = \tilde{g}(h(Y, X), N),\]

for any $X, Y \in \chi(M), N \in \Gamma(TM^\perp)$. Thus, from (2.27) and (2.27)' we get

\[(2.28) \quad g(A_N Y, X) + g(A_N X, Y) = 2\tilde{g}(s_h(X, Y), N),\]

for any $X, Y \in \chi(M), N \in \Gamma(TM^\perp)$.

If $s_h = 0$ then we obtain $g(A_N Y, X) = -g(A_N X, Y)$.

**Proposition 2.3.** A submanifold $M \subset \tilde{M}$ is autoparallel in $\tilde{M}$ if and only if

\[(2.29) \quad g(A_N Y, X) + g(A_N X, Y) = 0,\]

for any $X, Y \in \chi(M)$ and $N \in \Gamma(TM^\perp)$.

In the following statements, we suppose that $\tilde{\nabla}_X \tilde{P} Y \neq \tilde{P}(\tilde{\nabla}_X Y)$.

We denoted by $\tilde{P}$ the (1,2)-tensor field on $\tilde{M}$, such that

\[(2.30) \quad \tilde{P}(X, Y) = \tilde{\nabla}_X \tilde{P} Y - \tilde{P}(\tilde{\nabla}_X Y),\]

and

\[(2.31) \quad \tilde{P}(X, N) = \tilde{\nabla}_X \tilde{P} N - \tilde{P}(\tilde{\nabla}_X N),\]

for any $X \in \chi(M)$ and $N \in \Gamma(TM^\perp)$.

We denote the tangential and normal components on $M$ of $\tilde{P}(X, Y)$ by $\tilde{P}(X, Y)^\top$ and $\tilde{P}(X, Y)^\perp$, respectively, and the tangential and normal components on $M$ of $\tilde{P}(X, N_\alpha)$ by $\tilde{P}(X, N_\alpha)^\top$ and $\tilde{P}(X, N_\alpha)^\perp$, respectively.

If we omit to put the condition $\tilde{\nabla}_X \tilde{P} = 0$ in the Theorem 2.1, then we obtain a generalization of this:
Theorem 2.2. Let $M$ be an $n$-dimensional submanifold of codimension $r$ in a Riemannian manifold $(\overline{M}, \overline{g}, \overline{P})$ (which satisfied the conditions (1.1) and (1.2)). Then the structure $(P, g, u, \varepsilon, \alpha, (a_{\alpha\beta})_{r})$ induced on $M$ by the structure $\overline{P}$ has the following properties:

\[
\begin{align*}
(i) (\nabla_X P)(Y) &= \nabla_X PY - \sum_{\alpha} u_{\alpha}(Y) A_{\alpha}X, \\
(ii) (\nabla_X u_{\alpha})(Y) &= \overline{g}(P(X, Y), N_{\alpha}) - h_{\alpha}(X, PY) + \sum_{\beta} (u_{\beta}(Y) a_{\beta\alpha} + h_{\alpha}(X, Y) a_{\beta\alpha}), \\
(iii) (\nabla_X \xi_{\alpha}) &= \overline{g}(P(X, N_{\alpha}), N_{\alpha}) - \varepsilon P(A_{\alpha}X) + \varepsilon \sum_{\beta} a_{\alpha\beta}A_{\beta}X + \sum_{\beta} l_{\alpha\beta}(X)\xi_{\beta}, \\
(iv) (X a_{\alpha\beta}) &= \overline{g}(P(X, N_{\alpha}), N_{\beta}) - \varepsilon u_{\alpha}(A_{\beta}X) - u_{\beta}(A_{\alpha}X) + \sum_{\gamma} l_{\alpha\gamma}(X) a_{\beta\gamma} + l_{\beta\gamma}(X) a_{\alpha\gamma}
\end{align*}
\]

for any $X, Y \in \chi(M)$.

Proof: From (2.8) we have

\[
\begin{align*}
\tilde{\nabla}_X (\overline{P}Y) &= \nabla_X PY - \sum_{\alpha} u_{\alpha}(Y) A_{\alpha}X + \sum_{\alpha} [h_{\alpha}(X, PY) + X(u_{\alpha}(Y)) + \sum_{\beta} u_{\beta}(Y) l_{\beta\alpha}(X)]N_{\alpha}
\end{align*}
\]

and from (2.9) we have

\[
\begin{align*}
\overline{P}(\tilde{\nabla}_X Y) &= P(\nabla_X Y) + \varepsilon \sum_{\alpha} h_{\alpha}(X, Y)\xi_{\alpha} + \sum_{\alpha} [u_{\alpha}(\nabla_X Y) + \sum_{\beta} h_{\beta}(X, Y) a_{\beta\alpha}] N_{\alpha}
\end{align*}
\]

From the last two equalities we obtain

\[
\begin{align*}
P(X, Y) &= (\nabla_X P)(Y) - \sum_{\alpha} u_{\alpha}(Y) A_{\alpha}X - \varepsilon \sum_{\alpha} h_{\alpha}(X, Y)\xi_{\alpha} + \sum_{\alpha} [h_{\alpha}(X, PY) + (\nabla_X u_{\alpha})(Y) + \sum_{\beta} u_{\beta}(Y) l_{\beta\alpha}(X) + \sum_{\beta} h_{\beta}(X, Y) a_{\beta\alpha}] N_{\alpha}
\end{align*}
\]

Thus, identified the tangent and the normal parts respectively, from (2.33) we obtain (i) and (ii) from (2.32).

From (2.13) we have

\[
\begin{align*}
\tilde{\nabla}_X (\overline{P}N_{\alpha}) &= \varepsilon \nabla_X \xi_{\alpha} - \sum_{\beta} a_{\alpha\beta} A_{\beta}X + \sum_{\beta} [X(a_{\alpha\beta}) + \varepsilon h_{\beta}(X, \xi_{\alpha}) + \sum_{\gamma} a_{\alpha\gamma} l_{\gamma\beta}(X)] N_{\beta}
\end{align*}
\]

and from (2.14) we have

\[
\begin{align*}
\overline{P}(\tilde{\nabla}_X N_{\alpha}) &= -P(A_{\alpha}X) + \varepsilon \sum_{\beta} l_{\alpha\beta}(X)\xi_{\beta} - \sum_{\beta} [u_{\beta}(A_{\alpha}X) - \sum_{\gamma} a_{\alpha\gamma} l_{\gamma\beta}(X)] N_{\beta}
\end{align*}
\]

Replacing the last two equalities in (2.31) we obtain

\[
\begin{align*}
P(X, N) &= \varepsilon \nabla_X \xi_{\alpha} + P(A_{\alpha}X) - \varepsilon \sum_{\beta} l_{\alpha\beta}(X)\xi_{\beta} - \sum_{\beta} a_{\alpha\beta} A_{\beta}X + \sum_{\beta} \sum_{\gamma} l_{\alpha\gamma}(X) a_{\gamma\beta} + l_{\beta\gamma}(X) a_{\alpha\gamma}
\end{align*}
\]
Identifying the tangential and normal components, respectively, of $P(X, N_\alpha)$ from (2.34), we obtain the relations (iii) and (iv) from (2.32). □

3 The normality conditions of the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ structure

Let $M$ be an n-dimensional submanifold of codimension $r$ in a Riemannian almost product manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. We suppose that the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$, is an $(a, 1)$ Riemannian structure, where the elements $P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r$ were defined in section 1. Let $\tilde{\nabla}$ and $\nabla$ be the Levi-Civita connections defined on $M$ and $\tilde{M}$ respectively, with respect to $\tilde{g}$ and $g$ respectively.

The Nijenhuis torsion tensor field of $P$ has the form

\[
N_P(X, Y) = [PX, PY] + P^2[X, Y] - P[PY, X] - P[X, PY],
\]

for any $X, Y \in \chi(M)$.

As in the case of an almost paracontact structure ([25]), one can defined the normal $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ structure on $M$.

**Definition 3.1.** If we have the equality

\[
N_P(X, Y) - 2\sum_\alpha du_\alpha(X, Y)\xi_\alpha = 0,
\]

for any $X, Y \in \chi(M)$, then the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on submanifold $M$ in a Riemannian almost product manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ is said to be normal.

First of all, we mention a general proposition:

**Proposition 3.1.** If $(M, g, P)$ is an almost product manifold, then the Nijenhuis tensor of $P$ verifies that

\[
N_P(X, Y) = (\nabla_{PX}P)(Y) - (\nabla_{PY}P)(X) - P[(\nabla_XP)(Y) - (\nabla_YP)(X)],
\]

for any $X, Y \in \chi(M)$, where $\nabla$ is the Levi-Civita connection on $M$.

**Proof:** From the equality

\[
(\nabla_XP)(Y) = \nabla_X(PY) - P(\nabla_XY),
\]

for any $X, Y \in \chi(M)$, we obtain

\[
\nabla_X(PY) = (\nabla_XP)(Y) + P(\nabla_XY),
\]
for any $X, Y \in \chi(M)$. Inverting $X$ with $Y$ in (3.5) we obtain
\begin{equation}
(3.5)'
\nabla_Y(PX) = (\nabla_Y P)(X) + P(\nabla_Y X),
\end{equation}
for any $X, Y \in \chi(M)$. So, from the relations (3.5) and (3.5)', it follows that
\begin{equation}
(3.6)
\nabla_X(PY) - \nabla_Y(PX) = (\nabla_X P)(Y) - (\nabla_Y P)(X) + P([X,Y]),
\end{equation}
for any $X, Y \in \chi(M)$. Then, inverting $X$ with $P_X$ in the equality (3.4) we obtain
\begin{equation}
(3.7)
\nabla_X(PY) - \nabla_Y(PX) = (\nabla_X P)(Y) - (\nabla_Y P)(X) + P([X,Y]),
\end{equation}
for any $X, Y \in \chi(M)$. Inverting $X$ with $Y$ in the last relation, we obtain
\begin{equation}
(3.7)'
(\nabla_Y P)(X) = \nabla_Y P - P\nabla_Y X, \quad (\forall) X, Y \in \chi(M)
\end{equation}
Using (3.7) and (3.7)' in
\begin{equation}
\nabla_X(PY) - \nabla_Y(PX) = (\nabla_X P)(Y) - (\nabla_Y P)(X) + P([X,Y]),
\end{equation}
it follows that
\begin{equation}
(3.8)
\nabla_X(PY) - \nabla_Y(PX) = (\nabla_X P)(Y) - (\nabla_Y P)(X) + P([X,Y]),
\end{equation}
for any $X, Y \in \chi(M)$. On the other hand, we have
\begin{equation}
(3.9)
P[X,Y] = P\nabla_X Y - P\nabla_Y X, \quad (\forall) X, Y \in \chi(M)
\end{equation}
and
\begin{equation}
(3.10)
P[X,Y] = P\nabla_X Y - P\nabla_Y X, \quad (\forall) X, Y \in \chi(M)
\end{equation}
From (3.8), (3.9), (3.10) we obtain
\begin{equation}
N_P(X,Y) = (\nabla_X P)(Y) - (\nabla_Y P)(X) + P\nabla_X Y - P\nabla_Y X + P^2([X,Y]) - P\nabla_X Y + P\nabla_Y X - P([\nabla_X P)(Y) - \nabla_Y (PX)]
\end{equation}
and using (3.6) in the last equality we have
\begin{equation}
N_P(X,Y) = (\nabla_X P)(Y) - (\nabla_Y P)(X) + P^2([X,Y]) - P([\nabla_X P)(Y) - \nabla_Y (P[X,Y])
\end{equation}
and from this we obtain the equality (3.3). □

From [21] we have:

**Definition 3.2.** If $(\widetilde{M}, \tilde{g}, \tilde{P})$ is an almost product Riemannian manifold such that $\bigtriangledown \tilde{P} = 0$, then we say that $\tilde{M}$ is a locally product Riemannian manifold.
**Corollary 3.1.** If $M$ is a totally geodesic submanifold of a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$, then we have $N_P(X, Y) = 0$, for any $X, Y \in \chi(M)$.

**Theorem 3.1.** If $M$ is a submanifold of a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure and $\nabla$ is the Levi-Civita connection defined on $M$ with respect to $g$ then, the Nijenhuis torsion tensor field of $P$ has the form:

$$(3.11) \quad N_P(X, Y) = -\sum_\alpha g((PA_\alpha - A_\alpha P)(X), Y)\xi_\alpha - \sum_\alpha g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X) + \sum_\alpha g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y)$$

for any $X, Y \in \chi(M)$.

**Proof:** From (2.6)(i) (Theorem 2.1) for $\varepsilon = 1$, we obtain

$$(3.12) \quad (\nabla_X P)(Y) = \sum_\alpha g(A_\alpha X, Y)\xi_\alpha + \sum_\alpha g(Y, \xi_\alpha)A_\alpha X$$

and if we invert $X$ by $Y$, we obtain

$$(3.12)' \quad (\nabla_Y P)(X) = \sum_\alpha g(A_\alpha Y, X)\xi_\alpha + \sum_\alpha g(X, \xi_\alpha)A_\alpha Y$$

Replacing $X$ with $PX$ in the equality (3.12), we obtain

$$(3.13) \quad (\nabla_{PX} P)(Y) = \sum_\alpha g(A_\alpha PX, Y)\xi_\alpha + \sum_\alpha g(Y, \xi_\alpha)A_\alpha (PX)$$

If we invert $X$ by $Y$ then we obtain

$$(3.13)' \quad (\nabla_{PY} P)(X) = \sum_\alpha g(A_\alpha PY, X)\xi_\alpha + \sum_\alpha g(X, \xi_\alpha)A_\alpha PY$$

Replacing the relations (3.12), (3.12)', (3.13), (3.13)' in the equality (3.3) it follows that

$$(3.14) \quad N_P(X, Y) = \sum_\alpha [g(A_\alpha PX, Y) - g(A_\alpha PY, X)]\xi_\alpha + \sum_\alpha [g(Y, \xi_\alpha)A_\alpha (PX) - g(X, \xi_\alpha)A_\alpha (PY) - (g(A_\alpha X, Y) - g(A_\alpha Y, X)) P\xi_\alpha] - \sum_\alpha [g(Y, \xi_\alpha)P(A_\alpha X) - g(X, \xi_\alpha)P(A_\alpha Y)]$$

But we have

$$(3.15) \quad g(A_\alpha PY, X) = g(PY, A_\alpha X) = g(Y, PA_\alpha X) = g(PA_\alpha X, Y)$$

and using the equality (3.15) in (3.14) we obtain (3.11). □
Corollary 3.2. Let $M$ be a submanifold of codimension $r$ in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$ and let $\nabla$ be the Levi-Civita connection defined on $M$ with respect to $g$. If $(1,1)$ tensor field $P$ on $M$ commutes with the Weingarten operators $A_\alpha$ (that is $PA_\alpha = A_\alpha P$, for any $\alpha \in \{1, ..., r\}$) then, the Nijenhuis torsion tensor field of $P$ vanishes on $M$ (that is $N_P(X,Y) = 0$, for any $X, Y \in \chi(M)$).

Proposition 3.2. Let $M$ be a submanifold of codimension $r$ in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. If $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is induced structure on $M$ and $\nabla$ is the Levi-Civita induced connection on $M$ by $\tilde{\nabla}$ from $\tilde{M}$ then, the 1-forms $u_\alpha$ verify the equality:

\begin{equation}
2du_\alpha(X,Y) = -g((PA_\alpha - A_\alpha P)(X), Y) + \sum_\beta [l_{\alpha\beta}(X)g(Y, \xi_\beta) - l_{\alpha\beta}(Y)g(X, \xi_\beta)]
\end{equation}

for any $X, Y \in \chi(M)$, where $l_{\alpha\beta}$ are the coefficients of the normal connection in the normal bundle $T^\perp(M)$.

Proof: We know that

\begin{equation}
2du_\alpha(X,Y) = X(u_\alpha(Y)) - Y(u_\alpha(X)) - u_\alpha([X,Y]),
\end{equation}

for any $X, Y \in \chi(M)$ and $(\nabla_X u_\alpha)(Y) = X(u_\alpha(Y)) - u_\alpha(\nabla_X Y)$. Thus, we obtain

\begin{equation}
X(u_\alpha(Y)) = (\nabla_X u_\alpha)(Y) + u_\alpha(\nabla_X Y), \ (\forall)X, Y \in \chi(M)
\end{equation}

Inverting $X$ by $Y$ in the last relation, we obtain

\begin{equation}
y(u_\alpha(X)) = (\nabla_Y u_\alpha)(X) + u_\alpha(\nabla_Y X), \ (\forall)X, Y \in \chi(M)
\end{equation}

From the relations (3.17), (3.18) and (3.18)' we have

\begin{equation}
2du_\alpha(X,Y) = (\nabla_X u_\alpha)(Y) - (\nabla_Y u_\alpha)(X) + u_\alpha(\nabla_X Y - \nabla_Y X - [X,Y]), = 0
\end{equation}

for any $X, Y \in \chi(M)$ so

\begin{equation}
2du_\alpha(X,Y) = (\nabla_X u_\alpha)(Y) - (\nabla_Y u_\alpha)(X),
\end{equation}

for any $X, Y \in \chi(M)$. From (2.6)(ii) we have

\begin{equation}
(\nabla_X u_\alpha)(Y) = -g(A_\alpha X, PY) + \sum_\beta g(A_\beta X, Y)a_{\alpha\beta} + \sum_\beta g(Y, \xi_\beta)l_{\alpha\beta}(X)
\end{equation}

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and inverting $X$ by $Y$ in (3.20), it follows that

$$(\nabla_Y u_\alpha)(X) = -g(A_\alpha Y, PX) + \sum_\beta g(A_\beta Y, X)a_{\alpha\beta} + \sum_\beta g(X, \xi_\beta)l_{\alpha\beta}(Y)$$

Replacing (3.20) and (3.21) in the equality (3.19) we have

$$(3.21) \quad 2du_\alpha(X, Y) = -[g(A_\alpha X, PY) - g(A_\alpha Y, PX)] +$$

$$+ \sum_\beta a_{\alpha\beta}[g(A_\beta X, Y) - g(A_\beta Y, X)] + \sum_\beta [l_{\alpha\beta}(X)g(Y, \xi_\beta) - l_{\alpha\beta}(Y)g(X, \xi_\beta)].$$

Furthermore, we have $g(A_\beta X, Y) = g(A_\beta Y, X)$ and

$$g(A_\alpha X, PY) - g(A_\alpha Y, PX) = g((PA_\alpha - A_\alpha P)(X), Y)$$

and replacing the last two relations in (3.21), we obtain (3.16). □

**Corollary 3.3.** Under the assumptions of the last proposition, if the normal connection of $M$ vanishes identically (i.e. $l_{\alpha\beta} = 0$) then a necessary and sufficient condition for $du_\alpha = 0$ is the commutativity between $P$ and $A_\alpha$ (that is $PA_\alpha = A_\alpha P$, for any $\alpha \in \{1, \ldots, r\}$).

**Proposition 3.3.** If $M$ is a submanifold of codimension $r$ in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$ and $\nabla$ is the Levi-Civita connection defined on $M$ with respect to $g$, then we have

$$(3.22) \quad N_P(X, Y) - 2\sum_\alpha du_\alpha(X, Y)\xi_\alpha = \sum_\alpha g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) -$$

$$- \sum_\alpha g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X) + \sum_{\alpha, \beta}[g(X, \xi_\beta)l_{\alpha\beta}(X) - g(Y, \xi_\beta)l_{\alpha\beta}(Y)]\xi_\alpha,$$

for any $X, Y \in \chi(M)$.

**Proof:** The equality (3.22) is obtained from (3.11) and (3.16). □

**Corollary 3.4.** Let $M$ be a submanifold of codimension $r$ in a Riemannian locally product manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with a $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$. If the normal connection of $M$ vanishes identically (i.e. $l_{\alpha\beta} = 0$) and the $(1,1)$ tensor field $P$ on $M$ commutes with the Weingarten operators $A_\alpha$ (that is $PA_\alpha = A_\alpha P$, for any $\alpha \in \{1, \ldots, r\}$) then, $M$ has a normal $(a,1)$f Riemannian structure.

**Corollary 3.5.** Let $M$ be a submanifold of codimension $r$ in a Riemannian locally product manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. If the induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ on $M$ is normal then we obtain

$$(3.23) \quad \sum_\alpha [g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) - g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X)] +$$
\[ + \sum_{\alpha, \beta} [g(X, \xi_\beta) l_{\alpha \beta}(X) - g(Y, \xi_\beta) l_{\alpha \beta}(Y)] \xi_\alpha = 0, \]

for any \( X, Y \in \chi(M) \).

**Corollary 3.6.** Let \( M \) be a submanifold of codimension \( r \) in a Riemannian locally product manifold \((\tilde{M}, \tilde{g}, \tilde{P})\). If the induced structure \((P, g, u_\alpha, \xi_\alpha, (a_{\alpha \beta})_r)\) on \( M \) is normal and the normal connection \( \nabla^\perp \) of \( M \) vanishes identically (that is \( l_{\alpha \beta} = 0 \)), then we obtain the equality

\[ (3.24) \quad \sum_{\alpha} g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) = \sum_{\alpha} g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X), \]

for any \( X, Y \in \chi(M) \).

**Proposition 3.4.** Under the assumptions of the last corollary, the equality \((3.24)\) does not depend on the choice of a basis in the normal space \( T^\perp_x(M) \), for any \( x \in M \).

**Proof:** If \( \{N_\alpha'\} \) is another basis in \( T^\perp_x(M) \), then we have

\[ (3.25) \quad N'_\alpha = \sum_{\beta} s_{\alpha \beta} N_\beta \]

where \((s_{\alpha \beta})_r\) is an orthogonal matrix.

From the condition \( \tilde{\nabla}_X N'_\alpha = 0 \) we obtain \( \sum_{\beta} X(s_{\alpha \beta})N_\beta = 0 \) for any \( X \in M \), thus \( s_{\alpha \beta} \) are a constant functions on \( M \). On the other hand,

\[ (3.26) \quad \tilde{\nabla}_X N'_\alpha = -A'_\alpha X \]

and

\[ (3.27) \quad \tilde{\nabla}_X N'_\alpha = \sum_{\beta} X(s_{\alpha \beta})N_\beta - s_{\alpha \beta} A_\beta X \]

Thus, from the relations \((3.25)\), \((3.26)\) and \((3.27)\) we obtain

\[ (3.28) \quad A'_\alpha X = \sum_{\beta} s_{\alpha \beta} A_\beta X \]

Therefore, we have

\[ (3.29) \quad \tilde{P} N'_\alpha = \varepsilon \xi_\alpha' + \sum_{\beta} a'_{\alpha \beta} N_\beta = \varepsilon \xi'_\alpha + \sum_{\beta, \gamma} a'_{\alpha \beta} s_{\beta \gamma} N_\gamma \]

and

\[ (3.30) \quad \tilde{P} N'_\alpha = \sum_{\beta} s_{\alpha \beta} \tilde{P} N_\beta = \varepsilon \sum_{\beta} s_{\alpha \beta} \xi_\beta + \sum_{\beta, \gamma} s_{\alpha \beta} a_{\beta \gamma} N_\gamma \]

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So, from (3.29) and (3.30) we obtain

\[(3.31) \quad \xi'_\alpha = \sum_\beta s_{\alpha\beta} \xi_\beta \]

and

\[(3.32) \quad \sum_{\beta, \alpha} a'_{\alpha\beta} s_{\beta\gamma} = \sum_{\beta, \gamma} s_{\alpha\beta} a_{\beta\gamma} \]

On the basis \(\{N'_1, ..., N'_r\}\), the condition (3.24) becomes

\[(3.33) \quad \sum_{\alpha} g(X, \xi'_\alpha) \cdot (PA'_\alpha - A'_\alpha P)(Y) = \sum_{\alpha} g(Y, \xi'_\alpha)(PA'_\alpha - A'_\alpha P)(X) \]

From (3.28) and (3.31), we obtain

\[(3.34) \quad \sum_{\alpha} g(X, s_{\alpha\beta} \xi_\beta)(PS_{\alpha\gamma}A_\gamma - s_{\alpha\gamma}A_\gamma P)(Y) - \sum_{\alpha} g(Y, s_{\alpha\beta} \xi_\beta)(PS_{\alpha\gamma}A_\gamma - s_{\alpha\gamma}A_\gamma P)(X) = \]

\[= \sum_{\alpha} s_{\alpha\beta} s_{\alpha\gamma} [g(X, \xi_\beta) \cdot (PA_\gamma - A_\gamma P)(Y) - g(Y, \xi_\beta)(PA_\gamma - A_\gamma P)(X)] = 0. \]

From the orthogonality of the matrix \((s_{\alpha\beta})_r\) (that is \(\sum_\beta s_{\alpha\beta} s_{\gamma\beta} = \delta_{\alpha\gamma}\)) it follows that

\[(3.35) \quad \sum_{\alpha} [g(X, \xi_\alpha) \cdot (PA_\alpha - A_\alpha P)(Y) - g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X)] = 0 \]

Therefore, the condition (3.24) does not depend on the choice of a basis in the normal space \(T^\perp_x(M)\) (for any \(x \in M\)). □

In the following we denoted by

\[(3.36) \quad \begin{cases} (i) & B_\alpha = PA_\alpha - A_\alpha P \\ (ii) & C_\alpha(X, Y) = g(B_\alpha X, Y), \end{cases} \]

for any \(X, Y \in \chi(M)\)

**Lemma 3.1.** Let \(M\) be a submanifold in a Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\) which verifies the conditions (1.1) and (1.2). Let \((P, g, u_\alpha, \varepsilon_\xi_\alpha, (a_{\alpha\beta})_r)\) be the induced structure on \(M\). Then, the tensor field \(C_\alpha\) on \(M\) is skew-symmetric, thus we have

\[(3.37) \quad C_\alpha(X, Y) = -C_\alpha(Y, X), \quad (\forall) X, Y \in \chi(M) \]
Proof: From (3.36)(ii) we have:

\[ C_\alpha(X, Y) = g(B_\alpha X, Y) = g(PA_\alpha X - A_\alpha PX, Y) = \]

\[ = g(PA_\alpha X, Y) - g(A_\alpha PX, Y) = \]

\[ = g(X, A_\alpha PY) - g(X, PA_\alpha Y) = \]

\[ = -g(PA_\alpha Y - A_\alpha PY, X) = -C_\alpha(Y, X) \]

so \( C_\alpha \) is skew-symmetric. □

Remark 3.1. Under the assumptions of theorem 3.1, if we use the notations (3.36) then the identities (3.11) and (3.16) have the forms:

\[ N_P(X, Y) = \sum_\alpha [g(X, \xi_\alpha)B_\alpha(Y) - g(Y, \xi_\alpha)B_\alpha(X) - C_\alpha(X, Y)\xi_\alpha] \quad (3.38) \]

and

\[ 2du_\alpha(X, Y) = -C_\alpha(X, Y) + \sum_\beta [l_{\alpha\beta}(X)g(Y, \xi_\beta) - l_{\alpha\beta}(Y)g(X, \xi_\beta)] \quad (3.39) \]

for any \( X, Y \in \chi(M) \).

More of them, if the induced structure on \( M \) is normal, the equality (3.23) becomes

\[ \sum_\alpha [g(X, \xi_\alpha)B_\alpha(X) - g(X, \xi_\alpha)B_\alpha(Y)] = \]

\[ = \sum_{\alpha, \beta} [g(X, \xi_\beta)l_{\alpha\beta}(X) - g(Y, \xi_\beta)l_{\alpha\beta}(Y)]\xi_\alpha \quad (3.40) \]

for any \( X, Y \in \chi(M) \).

Remark 3.2. Using the model for an almost paracontact structure (25), we can compute the components \( N^{(1)}, N^{(2)}, N^{(3)} \) and \( N^{(4)} \) of the Nijenhuis torsion tensor field of \( P \) for the \( (P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r) \) induced structure on a submanifold \( M \) in an almost product Riemannian manifold \( (\tilde{M}, \tilde{g}, \tilde{P}) \):

\[
\begin{align*}
(i) \quad N^{(1)}_{\alpha}(X, Y) &= N_P(X, Y) - 2\sum_{\alpha=1}^r du_\alpha(X, Y)\xi_\alpha, \\
(ii) \quad N^{(2)}_{\alpha}(X, Y) &= (L_{P_X}u_\alpha)Y - (L_{P_Y}u_\alpha)X, \\
(iii) \quad N^{(3)}_{\alpha}(X) &= (L_{\xi_\alpha}P)X, \\
(iv) \quad N^{(4)}_{\alpha\beta}(X) &= (L_{\xi_\alpha}u_\beta)X,
\end{align*}
\]

for any \( X, Y \in \chi(M) \) and \( \alpha, \beta \in \{1, ..., r\} \), where \( N_P \) is the Nijenhuis torsion tensor field of \( P \) and \( L_X \) means the Lie derivative with respect to \( X \).
Remark 3.3. Let M be a submanifold in a Riemannian almost product manifold \((\bar{M}, \bar{g}, \bar{P})\). From (3.41)(i) we remark that the \((\bar{P}, \bar{g}, \xi_\alpha, u_\alpha, (a_\alpha)_r)\) induced structure on M is normal if and only if \(N^{(1)} = 0\).

Proposition 3.5. Let M be a submanifold in an almost product Riemannian manifold \((\bar{M}, \bar{g}, \bar{P})\), with \((\bar{P}, \bar{g}, u_\alpha, (a_\alpha)_r)\) induced structure on M. If the normal connection \(\bar{\nabla}^\perp = 0\) on the normal bundle \(T^\perp M\) vanishes identically (that is \(l_{\alpha\beta} = 0\)), then the components \(N^{(1)}, N^{(2)}, N^{(3)}\) and \(N^{(4)}\) of the Nijenhuis torsion tensor field of \(\bar{P}\) for the structure \((\bar{P}, \bar{g}, \xi_\alpha, u_\alpha, (a_\alpha)_r)\) induced on M have the forms:

\[
\begin{align*}
(i) \; N^{(1)}(X, Y) &= \sum_\alpha g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) - \\
&(\sum_\alpha g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X), \alpha) \\
(ii) \; N^{(2)}_\alpha(X, Y) &= -\sum_\beta a_{\alpha\beta}g((PA_\beta - A_\beta P)(X) + \)
+ \sum_\beta a_{\alpha\beta}u_\beta([X, Y]) + \sum_\beta [u_\beta(X)u_\alpha(A_\beta Y) - u_\beta(Y)u_\alpha(A_\beta X)], \\
(iii) \; N^{(3)}_\alpha(X) &= \sum_\beta a_{\alpha\beta}((PA_\beta - A_\beta P)(X) - \\
-P(\bar{P}A_\alpha - A_\alpha \bar{P})(X) + \sum_\beta [u_\alpha(A_\beta X)\xi_\beta + u_\beta(X)A_\beta \xi_\alpha], \\
(iv) \; N^{(4)}_{\alpha\beta}(X) &= -u_\alpha(A_\beta PX) - u_\beta(PA_\alpha X) + \\
+ \sum_\gamma [a_{\alpha\gamma}u_\beta(A_\gamma X) + a_{\gamma\beta}u_\alpha(A_\gamma X)]
\end{align*}
\]

for any \(X, Y \in \chi(M)\).

Proof: From the equality (3.22), with the condition \(l_{\alpha\beta} = 0\), we obtain (i). Using the equality (3.41)(ii) we have

\[
n^{(2)}_\alpha(X, Y) = PX(u_\alpha(Y)) - u_\alpha([PX, Y]) - PY(u_\alpha(X)) + u_\alpha([PY, X]) = \\
= [\underbrace{PX(u_\alpha(Y)) - u_\alpha(\nabla_P X Y)}_{(\nabla_P X u_\alpha)(Y)}] - [\underbrace{PY(u_\alpha(X)) - u_\alpha(\nabla_P Y X)}_{(\nabla_P Y u_\alpha)(X)}] + \\
+ u_\alpha(\nabla_Y PX) - u_\alpha(\nabla_X PY) = \\
= (\nabla_P X u_\alpha)(Y) - (\nabla_P Y u_\alpha)(X) + [X(u_\alpha(PY)) - u_\alpha(\nabla_X PY)] - \\
- [Y(u_\alpha(PX)) - X(u_\alpha(PY)) + Y(u_\alpha(PX)) = \\
= [\underbrace{(\nabla_P X u_\alpha)(Y)}_{2du_\alpha(X, Y)}] - [\underbrace{(\nabla_P Y u_\alpha)(X)}_{2du_\alpha(PY, X)}] + \\
+ \sum_\beta a_{\alpha\beta}u_\beta(Y) - Y(\sum_\beta a_{\alpha\beta}u_\beta(X)) = \\
= 2du_\alpha(PX, Y) + 2du_\alpha(X, PY) + \sum_\beta X(a_{\alpha\beta})u_\beta(Y) - \sum_\beta Y(a_{\alpha\beta})u_\beta(X) +
\]
so we obtain

\[(3.43) \quad N_\alpha^{(2)}(X, Y) = 2du_\alpha(PX, Y) + 2du_\alpha(X, PY) + 2\sum_\beta a_{\alpha\beta}du_\beta(X, Y) + \]

\[+ \sum_\beta a_{\alpha\beta}u_\beta([X, Y]) + \sum_\beta u_\beta(X)\left(\frac{u_\alpha(A_\beta Y) + u_\beta(A_\alpha Y)}{-Y(A_{\alpha\beta})}\right) - \]

\[\sum_\beta u_\beta(Y)\left(\frac{u_\alpha(A_\beta X) + u_\beta(A_\alpha X)}{-X(A_{\alpha\beta})}\right)\]

Using the relations (3.16) (with \(l_{\alpha\beta} = 0\), (1.6)(i) and (1.7)(iii) we obtain

\[2du_\alpha(PX, Y) + 2du_\alpha(X, PY) = \]

\[= -g((PA_\alpha - A_\alpha P)(PX), Y) - g((PA_\alpha - A_\alpha P)(X), PY) = \]

\[= -g(PA_\alpha PX, Y) + g(A_\alpha P^2 X, Y) - g(PA_\alpha X, PY) + g(A_\alpha PX, PY)\]

and from this we have

\[(3.44) 2du_\alpha(PX, Y) + 2du_\alpha(X, PY) = -\sum_\beta u_\beta(X)u_\beta(A_\alpha Y) + \sum_\beta u_\beta(Y)u_\beta(A_\alpha X)\]

Replacing the equality (3.44) in (3.43) we obtain

\[N_\alpha^{(2)}(X, Y) = -\sum_\beta (u_\beta(X)u_\beta(A_\alpha Y) + u_\beta(Y)u_\beta(A_\alpha X)) - \sum_\beta a_{\alpha\beta}g((PA_\beta - A_\beta P)(X), Y) + \]

\[+ \sum_\beta (a_{\alpha\beta}u_\beta([X, Y]) + u_\beta(X)u_\alpha(A_\beta Y) + u_\beta(X)u_\beta(A_\alpha Y) - u_\beta(Y)u_\alpha(A_\beta X) - u_\beta(Y)u_\beta(A_\alpha X))\]

and from this we have the equality (3.42)(ii).

Estimating \(N_\alpha^{(3)}(X)\) from the equality (3.41)(iii) we obtain

\[N_\alpha^{(3)}(X) = \nabla_{\xi_\alpha} PX - \nabla_{PX} \xi_\alpha - P(\nabla_{\xi_\alpha} X) + P(\nabla_X \xi_\alpha) = \]

\[= (\nabla_{\xi_\alpha} P)(X) - \nabla_{PX} \xi_\alpha + P(\nabla_X \xi_\alpha)\]

and using the relations (2.6) with the conditions \(\varepsilon = 1\) and \(l_{\alpha\beta} = 0\) we have

\[N_\alpha^{(3)}(X) = \sum_\beta (u_\alpha(A_\beta X)\xi_\beta + u_\beta(X)A_\beta \xi_\alpha + a_{\alpha\beta}(PA_\beta - A_\beta P)(X)) - P(PA_\alpha - A_\alpha P)(X)\]

and from this we obtain the equality (iii)(3.42).
Estimating $N^{(4)}_{\alpha\beta}(X)$ from the equality (3.41)(iv) we have
\[
N^{(4)}_{\alpha\beta}(X) = \xi_\alpha(u_\beta(X)) - u_\beta([\xi_\alpha, X]) = \\
= \xi_\alpha(u_\beta(X)) - u_\beta(\nabla_{\xi_\alpha}X) + u_\beta(\nabla_X\xi_\alpha) = (\nabla_{\xi_\alpha}u_\beta)(X) + u_\beta(\nabla_X\xi_\alpha)
\]
and using the relations (2.6) with the conditions $\varepsilon = 1$ and $l_{\alpha\beta} = 0$ we obtain
\[
N^{(4)}_{\alpha\beta}(X) = -h_\beta(\xi_\alpha, PX) + \sum_\gamma [a_{\gamma\beta} h_\gamma(\xi_\alpha, X) + a_{\alpha\gamma} u_\beta(A_\gamma X)] - u_\beta(\nabla_{\xi_\alpha}A_\beta X)
\]
so
\[
N^{(4)}_{\alpha\beta}(X) = -g(A_\beta PX, \xi_\alpha) - g(\nabla_{\xi_\alpha}A_\beta X, \xi_\beta) + \sum_\gamma [a_{\gamma\beta} h_\gamma(\xi_\alpha, X) + a_{\alpha\gamma} u_\beta(A_\gamma X)]
\]
and from this we have (3.42)(iv).

**Corollary 3.7.** Under the assumptions of the last proposition, if $P$ and the Weingarten operators $A_\alpha$ commute (that is $PA_\alpha = A_\alpha P$, for every $\alpha \in \{1, \ldots, r\}$) then we obtain

\[
\begin{align*}
(i) & \ N^{(4)}_{\alpha\beta}(X, Y) = 0, \\
(ii) & \ N^{(4)}_{\alpha\beta}(X, Y) = \sum_\beta (a_{\alpha\beta} u_\beta([X, Y]) + u_\beta(X) u_\alpha(A_\beta Y) - u_\beta(Y) u_\alpha(A_\beta X)), \\
(iii) & \ N^{(3)}_{\alpha}(X) = \sum_\beta [u_\alpha(A_\beta X) \xi_\beta + u_\beta(X) A_\beta \xi_\alpha], \\
(iv) & \ N^{(4)}_{\alpha\alpha}(X) = 2 \sum_\gamma a_{\alpha\gamma} u_\alpha(A_\gamma X) - 2u_\alpha(\nabla_{\xi_\alpha}A_\alpha X)
\end{align*}
\]

### 4 The commutativity of $P$ and $A_\alpha$ on submanifolds with $(P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$ normal structure

Let $M$ be an $n$-dimensional submanifold of codimension $r$ in an almost product Riemannian manifold $(M, \tilde{g}, \tilde{P})$. We suppose that $M$ is endowed with a $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$ by the $\tilde{P}$ and the normal connection $\nabla^\perp$ on the normal bundle $T^\perp(M)$ vanishes identically.

First of all, we search for necessary conditions for the linearly independent of the tangent vector fields $\xi_1, \ldots, \xi_r$ (with $r \geq 2$). In this situation, we will show that the condition of the normality of induced structure on $M$ is equivalent with the commutativity between the tensor field $P$ and the Weingarten operators $A_\alpha$ (for $\alpha \in \{1, \ldots, r\}$). We denoted by

\[
(4.1) \quad A := (a_{\alpha\beta})_r
\]
the matrix from the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on $M$. 

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Proposition 4.1. Let $M$ be a submanifold of codimension $r$ (with $r \geq 2$) in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with $(P, g, u_{\alpha}, \xi_{\alpha}, (a_{\alpha \beta}), r)$ induced structure on $M$ by $\tilde{P}$. If the normal connection $\nabla^\perp$ vanishes identically on the normal bundle $T^\perp(M)$ (i.e. $l_{\alpha \beta} = 0$) then, the tangent vector fields $\{\xi_1, ..., \xi_r\}$ are linearly independent if and only if the determinant of the matrix $(I_r - A^2)$ does not vanish in any point $x \in M$, (where $I_r$ is the $r \times r$ identity matrix).

Proof: Let $k_1, ..., k_r$ the real number with the properties that

\[ k_1 \xi_1 + ... + k_r \xi_r = 0 \]

in any point $x \in M$. From the equality (1.6)(iv), for $\varepsilon = 1$, we obtain

\[ g(\xi_\alpha, \xi_\beta) = \delta_{\alpha \beta} - \sum_\gamma a_{\alpha \gamma} a_{\gamma \beta} \]

Multiplying the equality (4.2) by $\xi_\alpha$ (for any $\alpha \in \{1, ..., r\}$) and using the equality (4.3) we obtain:

\[ \begin{cases} 
 k_1(1 - \sum_\gamma a_{1 \gamma} a_{1 \gamma}) + k_2(1 - \sum_\gamma a_{1 \gamma} a_{2 \gamma}) + ... + k_r(1 - \sum_\gamma a_{1 \gamma} a_{r \gamma}) = 0 \\
 k_1(1 - \sum_\gamma a_{2 \gamma} a_{2 \gamma}) + k_2(1 - \sum_\gamma a_{2 \gamma} a_{3 \gamma}) + ... + k_r(1 - \sum_\gamma a_{2 \gamma} a_{r \gamma}) = 0 \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad...
for any $X \in \chi(M)$ and $\alpha \in \{1, \ldots, r\}$

**Proof:** If the normal connection vanishes identically (thus $l_{\alpha\beta} = 0$), the equality (3.40) can be written in the form

$$
(4.6) \quad \sum_{\alpha} g(X, \xi_{\alpha}) B_{\alpha}(Y) = \sum_{\alpha} g(Y, \xi_{\alpha}) B_{\alpha}(X),
$$

for any $X, Y \in \chi(M)$ and $\alpha \in \{1, \ldots, r\}$.

Multiplying with $Z \in \chi(M)$ the equality (4.6) and using (3.36)(ii) we obtain:

$$
(4.7) \quad \sum_{\alpha} g(X, \xi_{\alpha}) C_{\alpha}(Y, Z) = \sum_{\alpha} g(Y, \xi_{\alpha}) C_{\alpha}(X, Z),
$$

for any $X, Y \in \chi(M)$. Inverting $Y$ by $Z$ in the equality (4.7), we have

$$
(4.7)' \quad \sum_{\alpha} g(X, \xi_{\alpha}) C_{\alpha}(Z, Y) = \sum_{\alpha} g(Z, \xi_{\alpha}) C_{\alpha}(X, Y),
$$

Summating the relations (4.7) and (4.7)', we obtain

$$
(4.8) \quad \sum_{\alpha} g(Y, \xi_{\alpha}) C_{\alpha}(X, Z) + \sum_{\alpha} g(Z, \xi_{\alpha}) C_{\alpha}(X, Y) = 0,
$$

because $C_{\alpha}$ is skew symmetric (i.e. $C_{\alpha}(Y, Z) + C_{\alpha}(Z, Y) = 0$). The equality (4.8) is equivalent with

$$
\sum_{\alpha} g(g(Y, \xi_{\alpha}) B_{\alpha}(X) + g(B_{\alpha}(X), Y) \xi_{\alpha}, Z) = 0, \ (\forall)X, Y, Z \in \chi(M)
$$

so

$$
(4.9) \quad \sum_{\alpha} g(Y, \xi_{\alpha}) B_{\alpha}(X) + \sum_{\alpha} g(B_{\alpha}(X), Y) \xi_{\alpha} = 0, \ (\forall)X, Y \in \chi(M)
$$

Using (3.36)(ii), the equality (4.9) can be written in the form

$$
(4.10) \quad \sum_{\alpha} g(Y, \xi_{\alpha}) B_{\alpha}(X) + \sum_{\alpha} C_{\alpha}(X, Y) \xi_{\alpha} = 0, \ (\forall)X, Y \in \chi(M)
$$

Inverting $X$ by $Y$ in the equality (4.10) we obtain

$$
(4.10)' \quad \sum_{\alpha} g(X, \xi_{\alpha}) B_{\alpha}(Y) + \sum_{\alpha} C_{\alpha}(Y, X) \xi_{\alpha} = 0, \ (\forall)X, Y \in \chi(M)
$$

Summating the equalities (4.10) and (4.10)' we obtain

$$
(4.11) \quad \sum_{\alpha} g(Y, \xi_{\alpha}) B_{\alpha}(X) + \sum_{\alpha} g(X, \xi_{\alpha}) B_{\alpha}(Y) = 0, \ (\forall)X, Y \in \chi(M)
$$
because \( C_\alpha(Y, Z) + C_\alpha(Z, Y) = 0 \). Therefore, from the relations (4.6) and (4.11) it follows that:

\[
(4.12) \quad \sum_\alpha g(Y, \xi_\alpha)B_\alpha(X) = 0, \ (\forall) X, Y \in \chi(M).
\]

From \( \det(A^2 - I_r) \neq 0 \) we obtain that the \( \xi_1, ..., \xi_r \) tangential vector fields are linearly independent in any point \( x \in M \), for \( r \geq 2 \). Because we have \( r \) linearly independent tangent vector fields on \( M \), then we have \( r \leq n \) and from this it follows that there exist a tangent vector field \( Y \in \chi(M) \) which is orthogonal on the space spanned by \( \{\xi_1, ..., \xi_r\} - \{\xi_\alpha\} \) and \( g(Y, \xi_\alpha) \neq 0 \).

From the equality (4.12) we have \( B_\alpha(X) = 0 \), for any \( X \in \chi(M) \) and \( \alpha \in \{1, ..., r\} \ (r > 1) \) and from this we have (4.5).

For \( r = 1 \), from the equality (4.12) we have \( g(Y, \xi)B(X) = 0 \) (where \( B = PA - AP \)). For \( Y = \xi \) we obtain \( g(\xi, \xi)B(X) = 0 \). But \( g(\xi, \xi) = 1 - a^2 \) and from \( 1 - a^2 \neq 0 \) we have \( B(X) = 0 \), so \( PA = AP \). □

From the Corollary 3.5 and Theorem 4.1 we obtain:

**Theorem 4.2.** Let \( M \) be an \( n \)-dimensional submanifold of codimension \( r \) in a locally product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with the normal connection \( \nabla_\perp \) vanishes identically \( (i.e. I_{\alpha\beta} = 0) \). If the determinant of the matrix \( (I_r - A^2) \) does not vanish in any point \( x \in M \), then the \( (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r) \) induced structure on \( M \) is normal if and only if the \((1,1)\) tensor field \( P \) commutes with the Weingarten operators \( A_\alpha \) \( (\text{for any } \alpha \in \{1, ..., r\}) \).

**Remark 4.1.** Under the assumptions of the last theorem, if the submanifold \( M \) in \( \tilde{M} \) is totally umbilical (or totally geodesic), then the commutativity between the \((1,1)\) tensor field \( P \) and the Weingarten operators \( A_\alpha \) \( (\text{for any } \alpha \in \{1, ..., r\}) \) has done.

5 On the composition of the immersions on manifolds with \((P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)\)-structure

Let \((\tilde{M}, \tilde{g}, \tilde{P})\) be an almost product Riemannian manifold and let \((\overline{M}, \overline{g})\) be a submanifold of codimension 1, isometric immersed in \( \tilde{M} \) (with the induced metric \( \overline{g} \) on \( \overline{M} \) by \( \tilde{g} \) and \( N_2 \) an unit vector field, normal on \( \overline{M} \) in \( \tilde{M} \)). On the other hand, we suppose that \((M, g)\) is an isometric immersed submanifold of codimension 1 in \( \overline{M} \) and let \( N_1 \) be an unit vector field, normal on \( M \) in \( \overline{M} \). Therefore, \((M, g)\) is an isometric immersed Riemannian submanifold in \((\tilde{M}, \tilde{g})\). We may assume that \( M \) is imbedded in \( \overline{M} \) and \( \overline{M} \) is imbedded in \( \tilde{M} \).

From the decomposed of the vector fields \( \tilde{P}X \) \((X \in \chi(\overline{M}))\) and \( \tilde{P}N_2 \) respectively, in tangential and normal components at \( \overline{M} \) in \( M \), we obtain

\[
(5.1) \quad \tilde{P}X = \overline{P}X + u_2(X)N_2,
\]
for any \( X \in \chi(M) \) and

\[
\lbrack \tilde{P} \rbrack N_2 = \xi_2 + a_{22}N_2,
\]

where \( \tilde{P} \) is an \((1,1)\) tensor field on \( \tilde{M} \), \( u_2 \) is an 1-form on \( \tilde{M} \), \( \xi_2 \) is a tangent vector field on \( \tilde{M} \) and \( a_{22} \) is a real function on \( \tilde{M} \).

For \( r=1 \), the relations (1.6) and (1.7) for the submanifold \( \tilde{M} \) in \( \tilde{M} \) are written in the next proposition:

**Proposition 5.1.** The almost product Riemannian structure \((\tilde{P}, \tilde{g})\) on a manifold \( \tilde{M} \) induces, on any submanifold \( \tilde{M} \) of codimension 1 in \( \tilde{M} \), a \((P, g, u_2, \xi_2, a_{22})\) Riemannian structure (where \( P \) is an \((1,1)\) tensor field on \( \tilde{M} \), \( u_2 \) is an 1-form on \( \tilde{M} \), \( \xi_2 \) is a tangent vector field on \( \tilde{M} \) and \( a_{22} \) is a real function on \( \tilde{M} \)) with the following properties:

\[
\begin{align*}
(i) \quad & \tilde{P}^2 X = X - u_2(\tilde{X})\xi_2, \quad (\forall) \tilde{X} \in \chi(\tilde{M}), \\
(ii) \quad & u_2(\tilde{P} \tilde{X}) = -a_{22}u_2(\tilde{X}), \quad (\forall) \tilde{X} \in \chi(\tilde{M}), \\
(iii) \quad & u_2(\xi_2) = 1 - a_{22}^2, \\
(iv) \quad & \tilde{P} \xi_2 = -a_{22}\xi_2,
\end{align*}
\]

and

\[
\begin{align*}
(i) \quad & u_2(\tilde{X}) = \tilde{g}(\tilde{X}, \xi_2), \quad (\forall) \tilde{X} \in \chi(\tilde{M}), \\
(ii) \quad & \tilde{g}(\tilde{P} \tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{P} \tilde{Y}), \quad (\forall) \tilde{X}, \tilde{Y} \in \chi(\tilde{M}), \\
(iii) \quad & \tilde{g}(\tilde{P} \tilde{X}, \tilde{P} \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \quad (\forall) \tilde{X}, \tilde{Y} \in \chi(\tilde{M}).
\end{align*}
\]

From the decomposed of the vector fields \( \tilde{P} \tilde{X} \ (X \in \chi(M)) \) and \( \tilde{P}N_1 \) respectively, in the tangential and normal components on \( M \) in \( \tilde{M} \), we obtain

\[
\tilde{P} \tilde{X} = P \tilde{X} + u_1(X)N_1, \quad (\forall) \tilde{X} \in \chi(\tilde{M})
\]

and

\[
\tilde{P} \tilde{N}_1 = \xi_1 + a_{11}N_1,
\]

where \( P \) is an \((1,1)\) tensor field on \( M \), \( u_1 \) is an 1-form on \( M \), \( \xi_1 \) is a tangential vector field on \( M \) and \( a_{11} \) is a real function on \( M \).

On the other hand, the vector field \( \xi_2 \in \chi(\tilde{M}) \) can be decomposed in the tangential and normal components on \( M \) in \( \tilde{M} \):

\[
\xi_2 = \xi_2^\top + \xi_2^\perp,
\]

and we remark that \( \xi_2^\perp \) and \( N_1 \) are collinear.
Proposition 5.2. The vector fields $\tilde{P}X$ ($X \in \chi(M)$), $\tilde{P}N_1$ and $\tilde{P}N_2$ have the next decomposes in the tangential and normal parts on $M$ in $\tilde{M}$:

$$
\begin{align*}
(i) & \quad \tilde{P}X = PX + u_1(X)N_1 + u_2(X)N_2, \quad (\forall)X \in \chi(M) \\
(ii) & \quad \tilde{P}N_1 = \xi_1 + a_{11}N_1 + a_{12}N_2, \\
(iii) & \quad \tilde{P}N_2 = \xi_2^\top + a_{12}N_1 + a_{22}N_2
\end{align*}
$$

(5.8)

where $P$ is an $(1,1)$ tensor field on $M$, $u_1$ is an 1-form on $M$, $\xi_1$ is a tangent vector field on $M$, $(a_{\alpha\beta})$ (with $\alpha, \beta \in \{1, 2\}$) is an $r \times r$ matrix where its entries $a_{11}$, $a_{22}$ and $a_{12} = a_{21} = \tilde{g}(\xi_2^\top, N_1)$ are real functions on $M$ and $u_2, \xi_2^\top, a_{22}$ were defined in the last proposition.

**Proof:** From the relations (5.1) and (5.5) we obtain

$$
\tilde{P}X = P X + u_2(X)N_2 = PX + u_1(X)N_1 + u_2(X)N_2
$$

so, (i) has done.

From the relations (5.1) and (5.6) we obtain

$$
\tilde{P}N_1 = PN_1 + u_2(N_1)N_1 = \xi_1 + a_{11}N_1 + u_2(N_1)N_2.
$$

(5.9)

We denote by $u_2(N_1) := a_{12}$. Thus,

$$
a_{12} = \tilde{g}(\xi_2, N_1) = \tilde{g}(\xi_2^\top, N_1)
$$

(5.10)

and from this it follows that

$$
\xi_2^\top = \tilde{g}(\xi_2, N_1)N_1 = u_2(N_1)N_1 = a_{12}N_1.
$$

(5.11)

So, from the equality (5.9) we have (ii) from (5.8).

From the relations (5.2) and (5.7) we have

$$
\tilde{P}N_2 = \xi_2^\top + \xi_2^\top + a_{22}N_2
$$

and using (5.11) we obtain (iii) from (5.8) (where $a_{21} = a_{12}$). □

**Theorem 5.1.** The structure $(\overline{P}, \overline{g}, u_2, \xi_2, a_{22})$ (induced on a submanifold $(\overline{M}, \overline{g})$ of codimension 1 in an almost product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$) also induces, on a submanifold $(M, g)$ of codimension 1 in $\overline{M}$, a Riemannian structure $(P, g, u_1, u_2, \xi_1, \xi_2^\top, (a_{\alpha\beta}))$ (where $P, u_1, u_2, \xi_1, \xi_2^\top, (a_{\alpha\beta})$ were defined in the last two propositions) which has the following properties:
\[
\begin{align*}
(i) & \quad P^2X = X - u_1(X)\xi_1 - u_2(X)\xi_2^\top, \ (\forall)X \in \chi(M) \\
(ii) & \quad u_1(PX) = -a_{11}u_1(X) - a_{12}u_2(X), \ (\forall)X \in \chi(M) \\
(iii) & \quad u_2(PX) = -a_{21}u_1(X) - a_{22}u_2(X), \ (\forall)X \in \chi(M) \\
(iv) & \quad u_1(\xi_1) = 1 - a_{11}^2 - a_{12}^2, \\
(v) & \quad u_2(\xi_1) = -a_{11}a_{12} - a_{12}a_{22}, \\
(vi) & \quad u_1(\xi_2^\top) = -a_{11}a_{12} - a_{12}a_{22}, \\
(vii) & \quad u_2(\xi_2^\top) = 1 - a_{11}^2 - a_{22}^2, \\
(viii) & \quad P(\xi_1) = -a_{11}\xi_1 - a_{12}\xi_2^\top, \\
(ix) & \quad P(\xi_2^\top) = -a_{12}\xi_1 - a_{22}\xi_2^\top,
\end{align*}
\]

(5.12)

and the properties which depends on the metric \( g \) are:

\[
\begin{align*}
(i) & \quad u_1(X) = g(X, \xi_1), \\
(ii) & \quad u_2(X) = g(X, \xi_2^\top), \\
(iii) & \quad g(PX, Y) = g(X, PY), \\
(iv) & \quad g(PX, PY) = g(X, Y) - u_1(X)u_1(Y) - u_2(X)u_2(Y),
\end{align*}
\]

(5.13)

for any \( X, Y \in \chi(M) \).

**Proof:** From \( \tilde{P}(PX) = X \) and (5.8) it follows that

\[
\tilde{P}(PX + u_1(X)N_1 + u_2(X)N_2) = X
\]

thus we have

\[
P^2X + u_1(PX)N_1 + u_2(PX)N_2 + u_1(X)(\xi_1 + a_{11}N_1 + a_{12}N_2) +
\]

\[
+ u_2(X)(\xi_2^\top + a_{12}N_1 + a_{22}N_2) = X
\]

Identifying the tangential components on \( M \) from the last equality, we obtain (i) from (5.12). Then, multiplying the last equality by \( N_1 \) and \( N_2 \) respectively, and using the equality (5.11) we obtain the relations (ii) and (iii) from (5.12).

On the other hand, from \( \tilde{P}(\tilde{P}N_1) = N_1 \) we obtain

\[
N_1 = \tilde{P}(\tilde{P}N_1) = \tilde{P}(\xi_1 + a_{11}N_1 + a_{12}N_2)
\]

and using the relations (5.8) it follows that

\[
N_1 = P\xi_1 + u_1(\xi_1)N_1 + u_2(\xi_1)N_2 +
\]

\[
+ a_{11}(\xi_1 + a_{11}N_1 + a_{12}N_2) + a_{12}(\xi_2^\top + a_{21}N_1 + a_{22}N_2)
\]

Identifying the tangential components on \( M \) from the last equality, and multiplying this relation by \( N_1 \) and \( N_2 \) respectively, we obtain, from the equality (5.11), the relations (iv), (v) and (viii) respectively, from (5.12).
From $\tilde{P}(\tilde{P}N_2) = N_2$ we obtain
\[ N_2 = \tilde{P}(\tilde{P}N_2) = \tilde{P}(\xi_2^T + a_{12}N_1 + a_{22}N_2) \]
and using the relations (5.8) it follows that
\[ N_2 = P(\xi_2^T) + u_1(\xi_2^T)N_1 + u_2(\xi_2^T)N_2 + a_{12}\xi_1 + a_{11}N_1 + a_{12}N_2 + a_{22}(\xi_2^T + a_{21}N_1 + a_{22}N_2) \]
Identifying the tangential components on $M$ from the last equality and multiplying this relation by $N_1$ and $N_2$ respectively, we obtain, from (5.11) the relations (vi), (vii) and (ix) from (5.12).

From $g(PX, Y) = \tilde{g}(\tilde{P}X - u_1N_1 - u_2N_2, Y) = \tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, PY + u_1Y + u_2(N_2)) = g(X, PY)$
we obtain the equality (iii) from (5.13).

From $\tilde{g}(\tilde{P}X, N_1) = \tilde{g}(X, \tilde{P}N_1)$
and using the relations (5.8), we have
\[ \tilde{g}(PX + u_1(X)N_1 + u_2(X)N_2, N_1) = \tilde{g}(X, \xi_1 + a_{11}N_1 + a_{12}N_2) \]
Thus, $u_1(X) = \tilde{g}(X, \xi_1) = g(X, \xi_1)$ and from this we obtain the equality (i) from (5.13).

From $\tilde{g}(\tilde{P}X, N_2) = \tilde{g}(X, \tilde{P}N_2)$
and using the relations (5.8) we have
\[ \tilde{g}(PX + u_1(X)N_1 + u_2(X)N_2, N_2) = \tilde{g}(X, \xi_2^T + a_{12}N_1 + a_{22}N_2) \]
Thus, $u_2(X) = \tilde{g}(X, \xi_2^T) = g(X, \xi_2^T)$ and from this we have the equality (ii) from (5.13).

From $g(PX, Y) = g(X, PY)$, replacing $Y$ with $PY$ and using the equality (i) from (5.12) we obtain
\[ g(PX, PY) = g(X, P^2Y) = g(X, Y - u_1(Y)\xi_1 - u_2(Y)\xi_2^T) \]
Thus, from the relations (i) and (ii) from (5.13), it follows that
\[ g(PX, PY) = g(X, Y) - u_1(X)u_1(Y) - u_2(X)u_2(Y) \]
and from this, we obtain the equality (iv) from (5.13). □
Definition 5.1. We say that a Riemannian immersion is an isometric immersion between two Riemannian manifolds.

Corollary 5.1. Let $M$ be a submanifold of codimension 1, isometric immersed in $\tilde{M}$, which is also of codimension 1 and isometric immersed in an almost product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, such that we have the Riemannian immersions:

$$(M, g) \hookrightarrow (\tilde{M}, \tilde{g}) \hookrightarrow (\tilde{M}, \tilde{g})$$

Then, the induced structure on $M$ by the structure $(\tilde{P}, \tilde{g})$ from $\tilde{M}$ is a $(P, g, u_1, u_2, \xi_1, \xi_2, (a_{\alpha\beta}))$- Riemannian structure (where $P$, $u_1$, $u_2$, $\xi_1$, $\xi_2$, $(a_{\alpha\beta})$ were defined in the last theorem). This structure is determined by the structure $(\tilde{P}, \tilde{g}, u_2, \xi_2, a_{22})$ (induced on $\overline{M}$ by the structure from $\tilde{M}$) and by the structure $(P, g, u_1, \xi_1, a_{11})$ (induced on a $M$ by the structure from $\tilde{M}$).

Corollary 5.2. Let $M := M_1$ be a submanifold of codimension $r$ (with $r \geq 2$) in an almost product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. We make the following notations: $\tilde{M} := M_{r+1}$, $\tilde{g} := g^{r+1}$, $\tilde{P} := P_{r+1}$, such that we have the sequence of Riemannian immersions

$$(M_1, g^1) \hookrightarrow (M_2, g^2) \hookrightarrow \ldots \hookrightarrow (M_r, g^r) \hookrightarrow (\tilde{M}, \tilde{g}, \tilde{P})$$

where $g^i$ is an induced metric on $M^i$ by the metric $g^{i+1}$ from $M_{i+1}$, $(i \in \{1, \ldots, r\})$ and each one of $(M_i, g^i)$ is a submanifold of codimension 1, isometric immersed in the manifold $(M_{i+1}, g^{i+1})$ $(i \in \{1, \ldots, r\})$. Then, the structure $(\tilde{P}_1, g^1, \xi^1_{\alpha_1}, u^1_{\alpha_1}, (a^1_{\alpha_1\beta_1}))$, which is successive induced by the structures $(\tilde{P}_i, g^i, \xi^i_{\alpha_i}, u^i_{\alpha_i}, (a^i_{\alpha_i\beta_i}))$ on the manifolds $M_i$ $(i \in \{2, \ldots, r\})$, $\alpha_i \in \{i, \ldots, r\}$ is the same as the induced structure on $(M_1, g^1)$ by the almost product structure $\tilde{P}$ on $\tilde{M}$, where the vectorial fields $\xi^1_{\alpha_1}$ on $M_1$ are the tangential components at $M$ of the tangent vectorial fields $\xi^i_{\alpha_i}$ from $M_i$, the 1-forms $u^i_{\alpha_i}$ are the restrictions on $M$ of the 1-forms $u^i_{\alpha_i}$ from $M_i$ (for any $i \in \{2, \ldots, r\}$ and $\alpha_i \in \{i, \ldots, r\}$), and the entries of the $r \times r$ matrix $(a^1_{\alpha_1\beta_1})$ are defined by

$$(5.14) \quad \begin{cases} a^1_{\alpha_1, \alpha_1} = g^1(P_1(N_{\alpha_1}), N_{\alpha_1}), \\ a^1_{\alpha_1, \beta_1} = a^1_{\beta_1, \alpha_1} = g^1(\xi^1_{\alpha_1}, N_{\beta_1}), \quad \alpha_1 > \beta_1 \end{cases}$$

for any $\alpha_1, \beta_1 \in \{1, \ldots, r\}$.

6 $(P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta}))$ induced structure on submanifolds of codimension 1 or 2 in almost product Riemannian manifolds

The hypersurfaces immersed in an almost product Riemannian manifolds were studied by T. Adati ([1], [2]), T. Miyazawa ([26]) and M. Okumura ([29]).
In the following, we assume that $M$ is a submanifold of codimension 1 in an almost product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. We denoted by $A_N := A$ (where $N$ is an unit normal vector field at submanifold $M$ in $\tilde{M}$), $u_1 := u$, $\xi_1 := \xi$ and $a_{\alpha\beta} := a$.

From (1.4), (1.5) we obtain

$$(6.1) \quad \tilde{P}X = PX + u(X)N,$$

for any $X \in \chi(M)$, and

$$(6.2) \quad \tilde{P}N = \xi + aN$$

where $P$ is a tensor field on $M$, $u$ is an 1-form on $M$, $a$ is a real function on $M$ and $\xi$ is a tangent vector field on $M$. The induced structure on $M$ is a $(P, g, u, \xi, a)$ Riemannian structure. From (1.6) we have:

$$(6.3) \begin{cases} (i) & P^2X = X - u(X)\xi, \\ (ii) & u(PX) = -au(X), \\ (iii) & u(\xi) = 1 - a^2, \\ (iv) & P(\xi) = -a\xi \end{cases}$$

for any $X \in \chi(M)$.

From the relations (1.7) we obtain

$$(6.4) \begin{cases} (i) & u(X) = g(X, \xi), \\ (ii) & g(PX, PY) = g(X, Y) - u(X)u(Y) \end{cases}$$

for any $X, Y \in \chi(M)$.

**Remark 6.1.** For $a=0$, from the relations (1.3), we remark that $(P, u, \xi)$ is an almost paracontact structure on $M$ and $(P, g, u, \xi)$ is a Riemannian almost paracontact structure.

The Gauss and Weingarten formulae on the hypersurface $M$, isometric immersed in $\tilde{M}$, have the forms:

$$(6.5) \quad \tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$$

and

$$(6.6) \quad \tilde{\nabla}_X N = -AX$$

respectively, for any $X, Y \in \chi(M)$.

If $M$ is a locally product Riemannian manifold, the formulae (2.6) become:

$$(6.7) \begin{cases} (i) & (\nabla_X P)(Y) = u(Y)AX + g(AX, Y)\xi, \\ (ii) & (\nabla_X u)(Y) = -g(AX, PY) + a \cdot g(AX, Y), \\ (iii) & \nabla_X \xi = -P(AX) + a \cdot AX, \\ (iv) & X(a) = -2u(AX) = -2g(AX, \xi) = -2g(X, A\xi) \end{cases}$$

for any $X, Y \in \chi(M)$. 

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Remark 6.2. From the relations (6.3)(iii) and (6.4)(i), we remark that \( a \in (-1, 1) \) for \( \xi \neq 0 \).

Remark 6.3. If \( a = 0 \), then the relations (6.3) have the forms:

\[
\begin{align*}
(i) & \quad P^2X = X - u(X)\xi, \\
(ii) & \quad u(PX) = 0, \\
(iii) & \quad u(\xi) = 1, \\
(iv) & \quad P(\xi) = 0,
\end{align*}
\]

(6.8)

for any \( X \in \chi(M) \).

Remark 6.4. Let \( M \) be a hypersurface in an almost product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\). From the relations (1.3)(iii) and (1.4)(i) we remark that \( u(\xi) = 0 \) so, \( g(\xi, \xi) = 0 \), for \( a^2 = 1 \) and from this we have \( \xi = 0 \). Therefore, from the equality (1.2) we obtain \( \tilde{P}N = a \cdot N \) and \( P^2X = X \). Thus, the induced structure \( P \) on \( M \) becomes an almost product structure and an eigenvalue of \( \tilde{P} \) is \( a \).

Remark 6.5. Let \( M \) be a hypersurface in an almost product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\). From (6.3)(iv) we remark that \( \xi \) is an eigenvector of \( a \) tensor field \( P \) with the eigenvalue \(-a\).

In the following we denote by \( \xi^\perp = \{X \in \chi(M)/X \perp \xi\} \).

Remark 6.6. From \( u(PX) = g(PX, \xi) \) and \( u(PX) = -au(X) = -ag(X, \xi) \) we obtain \( g(PX + aX, \xi) = 0 \) and \( (PX + aX) \perp \xi, (\forall)X \in \chi(M) \). We remark that there is a vector \( V \in \xi^\perp \) such that \( PX = -aX + V \).

Proposition 6.1. Let \( M \) be a hypersurface in an almost product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with the structure \((P, g, \xi, u, a)\) induced on \( M \) by the structure \( \tilde{P} \) from \( \tilde{M} \). We suppose that \( \xi \neq 0 \). A necessary and sufficient condition for \( M \) to be totally geodesic in \( \tilde{M} \) is that \( \nabla_X P = 0 \), for any \( X \in \chi(M) \).

Proof: If \( M \) is totally geodesic, then \( \nabla_X P = 0 \) \((\forall X \in \chi(M))\) from the equality (6.7)(i).

Conversely, we suppose that \( \nabla_X P = 0 \) and from the equality (6.7)(i) we obtain

\[
g(AX, Y)\xi + g(Y, \xi)AX = 0.
\]

We may have one of the following situations:

(i) If \( AX \) and \( \xi \) are linearly dependent vector fields, then there exist a real number \( \alpha \) such that \( AX = \alpha \xi \) and from this we obtain \( g(Y, \xi) = 0 \) for any \( Y \in \chi(M) \). Thus, for \( Y = \xi \) we obtain \( g(\xi, \xi) = 0 \) which is equivalent with \( \xi = 0 \) and this is an impossible case.

(ii) If \( AX \) and \( \xi \) are linearly independent vector fields, then \( g(AX, Y) = 0 \) (for any \( X, Y \in \chi(M) \)). Thus \( A = 0 \) and from this we have that \( M \) is a totally geodesic submanifold in \( \tilde{M} \). \( \square \)
Proposition 6.2. If $M$ is a hypersurface in an almost product Riemannian manifold $(\tilde{M}, \tilde{\mathcal{P}}, \tilde{g})$, with structure $(P, g, \xi, u, a)$ induced on $M$ by $\tilde{P}$, then the following equalities are equivalent:

$$\nabla_X u = 0 \iff \nabla_X \xi = 0.$$  

Proof: If $\nabla_X u = 0$ then we obtain $g(AX, PY) = ag(AX, Y)$, from the equality (6.7)(ii).

From $g(AX, PY) = g(P(AX), Y)$ we have $g(P(AX) - aAX, Y) = 0$, for any $X, Y \in \chi(M)$, and using the equality (6.7)(iii) we have $\nabla_X \xi = 0$.

Conversely, we suppose that $\nabla_X \xi = 0$ (for any $X \in \chi(M)$) and we have $g(\nabla_X \xi, Y) = 0$ from (6.7)(iii). Thus

$$g(P(AX) - aAX, Y) = 0 \iff g(AX, PY) - ag(AX, Y) = 0$$

for any $X, Y \in \chi(M)$. Therefore, we obtain $\nabla_X u = 0$, from (ii)(6.7). \Box

From the theorem 4.2 we have the following property:

Proposition 6.3. Let $M$ be a hypersurface of a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with $(P, g, \xi, u, a)$ induced structure on $M$ by $\tilde{P}$. We suppose that $a \neq \pm 1$. A necessary and sufficient condition for the normality of structure $(P, g, \xi, u, a)$ is that the tensor field $P$ commutes by the Weingarten operator $A$ (that is $PA = AP$).

From the proposition (3.2) it follows that :

Proposition 6.4. Let $M$ be a hypersurface in a locally product manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the induced structure $(P, g, \xi, u, a)$. The necessary and sufficient condition for the 1-form $u$ on $M$ to be closed (i.e. $du = 0$) is that the tensor field $P$ commutes by the Weingarten operator $A$.

Proposition 6.5. Let $M$ be a hypersurface in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$ with $(P, g, \xi, u, a)$ induced structure on $M$. Then $\xi$ is a Killing vector field with respect to $g$ on $M$ if and only if we have

$$2aA = PA + AP$$

where $A$ is the Weingarten operator on $M$.

Proof: We have that $\xi$ is a Killing vector field on $M$ if and only if

$$\begin{align*}
(L_\xi g)(Y, Z) &= 0, \; (\forall) Y, Z \in \chi(M) \\
&\iff \xi g(Y, Z) - g([\xi, Y], Z) - g(Y, [\xi, Z]) = 0 \\
&\iff \xi g(Y, Z) - g(\nabla_\xi Y - \nabla_Y \xi, Z) - g(Y, \nabla_\xi Z - \nabla_Z \xi) = 0 \\
&\iff \xi g(Y, Z) - g(\nabla_\xi Y, Z) - g(Y, \nabla_\xi Z) + g(Y, \nabla_Z \xi) = 0 \\
&= 0
\end{align*}$$

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\[ (6.10) \quad g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = 0, \quad (\forall) Y, Z \in \chi(M) \]

Using the equality (6.7)(iii) we obtain that (6.10) is equivalent with

\[ g(-PA_N Y + aA_N Y, Z) + g(-PA_N Z + aA_N Y, Z) = 0 \]

\[ \iff g(2aA_N Y - PA_N Y - A_N(\mathbf{P}Y), Z) = 0, \quad (\forall) Y, Z \in \chi(M) \]

which is equivalent with the equality (6.9).

**Corollary 6.1.** Let \( M \) be a hypersurface of a locally product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with the normal induced structure \((P, g, \xi, u, a)\). Then \( \xi \) is a Killing vector field with respect to \( g \) on \( M \) if and only if we have

\[ (6.11) \quad aA = PA = AP \]

where \( A \) is the Weingarten operator on \( M \).

**Proposition 6.6.** Let \( M \) be a hypersurface of a locally product Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with a normal induced structure \((P, g, \xi, u, a)\) and \( \xi \) is a Killing vector field. If \( a^2 \neq 1 \), then \( \text{rank } A = 1 \) and \( \xi \) is an eigenvector of the Weingarten operator \( A \), with the eigenvalue \( \frac{x(a)}{2(a^2 - 1)} \).

**Proof:** From the corollary 6.1 we have \( PA = aA \). Thus, \( P^2 A = aPA = a^2 A \) and we have

\[ P^2(AX) = a^2 AX, \quad (\forall) X \in \chi(M) \]

Using the equality (6.3)(i) we obtain \( AX - u(AX)\xi = a^2 AX \), so

\[ (a^2 - 1)AX = -u(AX)\xi, \quad (\forall) X \in \chi(M) \]

From (6.7)(iv) and (6.12) (with \( a^2 \neq 1 \)) we have

\[ AX = \frac{X(a)}{2(a^2 - 1)}\xi \quad (\forall) X \in \chi(M) \]

and from this we remark that \( \text{rank } A = 1 \). More of this, if we put \( X = \xi \) in the equality (6.13) we obtain

\[ (6.14) \quad A(\xi) = \frac{x(a)}{2(a^2 - 1)}\xi \quad (\forall) \xi \in \chi(M) \]

thus \( \xi \) is an eigenvector of Weingarten operator \( A \), and its eigenvalue is \( \frac{x(a)}{2(a^2 - 1)} \).
Remark 6.7. Under the assumption of the last proposition, if

\[ e_1 = \frac{\xi}{\sqrt{1 - a^2}}, \]

then

\[ A(e_1) = A\left(\frac{\xi}{\sqrt{1 - a^2}}\right) = \frac{1}{\sqrt{1 - a^2}} A(\xi) = \frac{\xi(a)}{2(a^2 - 1)} \sqrt{1 - a^2} = \frac{\xi(a)}{2(a^2 - 1)} e_1 \]

Therefore, \( e_1 \) is eigenvector of the Weingarten operator \( A \) and its eigenvalue is \( \frac{\xi(a)}{2(a^2 - 1)} \).

Let \( (e_1, \ldots, e_n) \) be an orthonormal basis of a tangent space \( T_x M \), for any \( x \in M \). From (6.13) we have \( \ker A = n - 1 \) and we obtain \( A(e_i) = 0 \) for any \( i \in \{2, \ldots, n\} \). Thus, the first eigenvalue of the Weingarten operator is

\[ k_1 = \frac{\xi(a)}{2(a^2 - 1)} \]

and the others are

\[ k_2 = \ldots = k_n = 0 \]

More of them, the mean vector field \( H \) of hipersurface \( M \) in \( \tilde{M} \) is

\[ H = \frac{1}{n} \text{trace} A \cdot N = \frac{1}{n} k_1 N \]

and we have

\[ H = \frac{\xi(a)}{2n(a^2 - 1)} N. \]

Proposition 6.7. Let \( M \) be a hypersurface in a locally product Riemannian manifold \( (\tilde{M}, \tilde{g}, \tilde{P}) \), with the normal induced structure \( (P, g, \xi, u, a) \). If we have \( a^2 \neq 1 \) and \( \xi(a) = 0 \), then \( M \) is totally geodesic in \( \tilde{M} \).

Proof: From (6.16) (with \( \xi(a) = 0 \)) and (6.17) we obtain

\[ k_1 = k_2 = \ldots = k_n = 0, \]

so, the Weingarten operator \( A = 0 \) and we have that \( M \) is totally geodesic in \( \tilde{M} \). \( \Box \)

Proposition 6.8. Let \( M \) be a totally umbilical hypersurface (i.e. \( A = \lambda I \)) in a locally product manifold \( (\tilde{M}, \tilde{g}, \tilde{P}) \), with the induced structure \( (P, g, \xi, u, a) \). Then we have:

\[ \begin{align*}
(i) & \quad (\nabla_X P)(Y) = \lambda g(Y, \xi) X + \lambda g(X, Y) \xi, \\
(ii) & \quad (\nabla_X u)(Y) = -\lambda g(X, PY) + a \lambda g(X, Y), \\
(iii) & \quad \nabla_X (\xi) = -\lambda PX + a \lambda X, \quad \nabla_\xi \xi = 2a \lambda \xi, \\
(iv) & \quad X(a) = -2\lambda g(X, \xi)
\end{align*} \]

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for any $X, Y \in \chi(M)$.

**Proof:** We have
\[ g(A_N X, Y) = \lambda g(X, Y) \]
because $M$ is totally umbilical in $\tilde{M}$ and using (6.7) we obtain (6.19). □

**Proposition 6.9.** If $M$ is a totally umbilical hypersurface in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the induced structure $(P, g, \xi, u, a)$, then the 1-form $u$ is closed.

**Proof:** Using the equalities (6.19)(ii) and (3.19), we obtain
\[ du(X, Y) = (\nabla_X u)(Y) - (\nabla_Y u)(X) = \lambda (g(PX, Y) - g(X, PY)) = 0 \]
Thus, the 1-form $u$ is closed. □

**Corollary 6.2.** Let $M$ be a totally umbilical submanifold in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the induced structure $(P, g, \xi, u, a)$. Then, it follows that

\[
\begin{align*}
(i) \quad (\nabla_X P)(\xi) &= \lambda (1 - a^2) X, \\
(ii) \quad (\nabla_\xi P)(X) &= 2 \lambda g(X, \xi) \xi, \\
(iii) \quad (\nabla_X u)(\xi) &= 2 a \lambda g(X, \xi),
\end{align*}
\]
for any $X \in \chi(M)$.

**Proof:** For $Y = \xi$ in the relations (6.19)(i),(ii) and using the equality (6.3)(iii), we obtain (i) and (iii) from (6.20). If $X = \xi$ in the equality (6.19)(i) we obtain (ii) from (6.20). □

**Corollary 6.3.** Let $M$ be a totally umbilical submanifold in a locally product manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the induced structure $(P, g, \xi, u, a)$ and $X \in \xi^\perp$ (where $\xi^\perp = \{X \in \chi(M)/X \perp \xi\}$). Then we have

\[
\begin{align*}
(i) \quad (\nabla_\xi P)(X) &= 0, \\
(ii) \quad (\nabla_X u)(\xi) &= 0, \\
(iii) \quad X(a) = 0 \implies a = \text{constant}
\end{align*}
\]
for any $X \in \chi(M)$.

**Remark 6.8.** Let $M$ be a hypersurface in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the induced structure $(P, g, \xi, u, a)$. We suppose that $(e_1, ..., e_n)$ is an orthonormal basis of the tangent space $T_x M$, (for any $x \in M$). Then $\text{div} \xi = \text{trace}(e_i \rightarrow \nabla_{e_i} \xi)$ and using (6.19)(iii) we obtain $\nabla_{e_i} \xi = \lambda (a I - P)(e_i)$. So,

\[ \text{div} \xi = \lambda (na - \text{trace} P) \]
Therefore, if \( \text{div}\xi = 0 \) it follows that \( \text{trace} P = na \).

In the following we assume that \( M \) is an \( n \)-dimensional submanifold of codimension 2 in an almost product Riemannian manifold \((\widetilde{M}, \tilde{g}, \tilde{P})\), with induced structure \((P, g, u, \xi, (a_{\alpha\beta})_{2})\) on \( M \) \((\alpha, \beta \in \{1, 2\})\). We suppose that the normal connection vanishes identically (thus \( l_{\alpha\beta} = 0 \)). In these conditions, the relations (1.6) and (1.7) from Theorem 1.1 have the following forms:

\[(6.23)\]
\[
\begin{align*}
(i) & \quad P^2 X = X - u_1(X)\xi_1 - u_2(X)\xi_2, \\
(ii) & \quad u_1(PX) = -a_{11}u_1(X) - a_{12}u_2(X), \\
(iii) & \quad u_2(PX) = -a_{21}u_1(X) - a_{22}u_2(X), \\
(iv) & \quad u_1(\xi_1) = 1 - a^2_1 - a^2_2, \\
(v) & \quad u_2(\xi_2) = 1 - a^2_1 - a^2_2, \\
(vi) & \quad u_1(\xi_2) = u_2(\xi_1) = -a_{12}(a_{11} + a_{22}), \\
(vii) & \quad P(\xi_1) = -a_{11}\xi_1 - a_{12}\xi_2, \\
(viii) & \quad P(\xi_2) = -a_{21}\xi_1 - a_{22}\xi_2, \\
(ix) & \quad g(PX, PY) = g(X, Y) - u_1(X)u_1(Y) - u_2(X)u_2(Y)
\end{align*}
\]

for any \( X, Y \in \chi(M) \).

We denote by \( A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \).

**Remark 6.9.** A simplifier assumption for these relations is \( a_{11} + a_{22} = 0 \) thus, \( \text{trace} A = 0 \), which is equivalent with \( \xi_1 \perp \xi_2 \). Under this assumption, if we denote \( a_{11} = -a_{22} = a \), \( a_{12} = a_{21} = b \) and \( 1 - a^2 - b^2 = \sigma \), from the relations (ii)-(vii) (2.1), we easily see that

\[(6.24)\]
\[
\begin{align*}
(i) & \quad u_1(\xi_1) = u_2(\xi_2) = \sigma \iff g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = \sigma, \\
(ii) & \quad u_2(\xi_1) = u_1(\xi_2) = 0 \iff g(\xi_1, \xi_2) = 0, \\
(iii) & \quad u_1(PX) = -au_1(X) - bu_2(X), \\
(iv) & \quad u_2(PX) = -bu_1(X) + au_2(X), \\
(v) & \quad P(\xi_1) = -a\xi_1 - b\xi_2, \\
(vi) & \quad P(\xi_2) = -b\xi_1 + a\xi_2.
\end{align*}
\]

Furthermore, from (2.6), under the assumption that the normal connection \( \nabla^\perp \) vanishes identically (i.e. \( l_{\alpha\beta} = 0 \)), we obtain

\[(6.25)\]
\[
(\nabla_X P)(Y) = g(A_1X, Y)\xi_1 + g(A_2X, Y)\xi_2 + \\
+ g(Y, \xi_1)A_1X + g(Y, \xi_2)A_2X,
\]

\[(6.26)\]
\[
\begin{align*}
(\nabla_X u_1)(Y) & = -g(A_1X, PY) + ag(A_1X, Y) + bg(A_2X, Y), \\
(\nabla_X u_2)(Y) & = -g(A_2X, PY) + bg(A_1X, Y) - ag(A_2X, Y).
\end{align*}
\]

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and

\[
\begin{align*}
\nabla_X \xi_1 &= -P(A_1 X) + aA_1 X + bA_2 X, \\
\nabla_X \xi_2 &= -P(A_2 X) + bA_1 X - aA_2 X.
\end{align*}
\]

and

\[
\begin{align*}
(i) & \quad X(a) = -2g(A_1 X, \xi_1), \\
(ii) & \quad X(b) = -g(A_1 X, \xi_2) - g(A_2 X, \xi_1)
\end{align*}
\]

for any \(X, Y \in \chi(M)\).

**Lemma 6.1.** Let \(M\) be an \(n\)-dimensional submanifold of codimension 2 in an locally product Riemann manifold \((\tilde{M}, \tilde{g}, \tilde{P})\), with the normal induced structure \((P, g, u_\alpha, \xi_\alpha, (a_\alpha)_{\alpha})\). If the normal connection \(\nabla^\perp\) vanishes identically (i.e. \(l_{\alpha} = 0\)), then the following equation is good

\[
g(Y, \xi_1)B_1(X) + g(Y, \xi_2)B_2(X) + C_1(X, Y)\xi_1 + C_2(X, Y)\xi_2 = 0,
\]

for any \(X, Y \in \chi(M)\).

**Proof:** By virtue of (3.40) we obtain:

\[
g(X, \xi_1)B_1(Y) + g(X, \xi_2)B_2(Y) = g(Y, \xi_1)B_1(X) + g(Y, \xi_2)B_2(X),
\]

for any \(X, Y \in \chi(M)\). Multiplying by \(Z \in \chi(M)\) the equality (6.30) we have

\[
g(X, \xi_1)g(B_1(Y), Z) + g(X, \xi_2)g(B_2(Y), Z) = g(Y, \xi_1)g(B_1(X), Z) + g(Y, \xi_2)g(B_2(X), Z),
\]

and using the notation (ii) from (3.36) it follows that

\[
g(X, \xi_1)C_1(Y, Z) + g(X, \xi_2)C_2(Y, Z) = g(Y, \xi_1)C_1(X, Z) + g(Y, \xi_2)C_2(X, Z),
\]

for any \(X, Y, Z \in \chi(M)\). Inverting \(Y\) by \(Z\) in the last equality we obtain

\[
g(X, \xi_1)C_1(Z, Y) + g(X, \xi_2)C_2(Z, Y) = g(Z, \xi_1)C_1(X, Y) + g(Z, \xi_2)C_2(X, Y).
\]

Summatting the relations (6.30) and (6.31)', and using the skew-symmetry of \(C_1\) and \(C_2\) (from (3.36)(ii)), we obtain

\[
g(Y, \xi_1)C_1(X, Z) + g(Y, \xi_2)C_2(X, Z) + g(Z, \xi_1)C_1(X, Y) + g(Z, \xi_2)C_2(X, Y) = 0
\]

which is equivalent with

\[
g(Y, \xi_1)g(B_1(X), Z) + g(Y, \xi_2)g(B_2(X), Z) +
\]

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+g(Z,ξ_1)g(B_1(X),Y)+g(Z,ξ_2)g(B_2(X),Y) = 0

Hence,
\[ g(g(Y,ξ_1)B_1(X),Z) + g(g(Y,ξ_2)B_2(X),Z)+ 
+g(Z,g(B_1(X),Y)ξ_1) + g(Z,g(B_2(X),Y)ξ_2) = 0 \]
and it follows that
\[ g([g(Y,ξ_1)B_1(X)+g(Y,ξ_2)B_2(X)+g(B_1(X),Y)ξ_1+g(B_2(X),Y)ξ_2]), Z) = 0 \]
for any \( Z \in \chi(M) \) and from this we obtain (6.29). □

**Lemma 6.2.** Let \( M \) be an \( n \)-dimensional submanifold of codimension 2 in an locally product Riemannian manifold \((M, \tilde{g}, P)\), with the normal induced structure \((P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta}), i)\). If the normal connection \( \nabla \perp \) vanishes identically \( (i.e. l_{\alpha\beta} = 0) \) and \( \sigma \neq 0 \) then, the following equalities are good

\[
\begin{align*}
(i) & \quad B_1(ξ_1) = 0, \\
(ii) & \quad B_2(ξ_2) = 0, \\
(iii) & \quad B_1(ξ_2) = 0, \\
(iv) & \quad B_2(ξ_1) = 0.
\end{align*}
\]

**Proof:** By virtue of (6.29) with \( X = Y = ξ_1 \), we obtain

\[
\begin{align*}
g(ξ_1,ξ_1)B_1(ξ_1) + g(ξ_1,ξ_2)B_2(ξ_1) + \\
+g(B_1(ξ_1),ξ_1)ξ_1 + g(B_2(ξ_1),ξ_1)ξ_2 = 0.
\end{align*}
\]

Using \( g(ξ_1,ξ_1) = \sigma \neq 0, g(ξ_1,ξ_2) = 0, g(B_1(ξ_1),ξ_1) = C_1(ξ_1,ξ_1) = 0 \) and \( g(B_2(ξ_1),ξ_1) = C_2(ξ_1,ξ_1) = 0 \) we obtain \( B_1(ξ_1) = 0 \), from the skew-symmetry of \( C_1 \) and \( C_2 \).

From the equality (6.29) with \( X = Y = ξ_2 \), we obtain

\[
\begin{align*}
g(ξ_2,ξ_1)B_1(ξ_2) + g(ξ_2,ξ_2)B_2(ξ_2) + \\
+g(B_1(ξ_2),ξ_2)ξ_1 + g(B_2(ξ_2),ξ_2)ξ_2 = 0.
\end{align*}
\]

Using that \( g(ξ_2,ξ_2) = \sigma \neq 0, g(ξ_1,ξ_2) = 0, g(B_1(ξ_2),ξ_2) = C_1(ξ_2,ξ_2) = 0 \) and \( g(B_2(ξ_2),ξ_2) = C_2(ξ_2,ξ_2) = 0 \) we obtain \( B_2(ξ_2) = 0 \), from the skew-symmetry of \( C_1 \) and \( C_2 \).

If we put \( X = ξ_1 \) and \( Y = ξ_2 \) in (6.29), we obtain

\[
\begin{align*}
g(ξ_2,ξ_1)B_1(ξ_1) + g(ξ_2,ξ_2)B_2(ξ_1) + \\
+g(B_1(ξ_1),ξ_2)ξ_1 + g(B_2(ξ_1),ξ_2)ξ_2 = 0,
\end{align*}
\]

Using \( g(ξ_2,ξ_2) = \sigma \neq 0, g(ξ_1,ξ_2) = 0, B_1(ξ_1) = 0 \) and

\[
g(B_2(ξ_1),ξ_2) = C_2(ξ_1,ξ_2) = -C_2(ξ_2,ξ_1) = -g(B_2(ξ_2),ξ_1) = 0
\]

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and replacing these in (6.35) we obtain $B_2(\xi_1) = 0$.

Using the equality (6.29) with $X = \xi_2$ and $Y = \xi_1$ we obtain

\begin{equation}
(6.36)
g(\xi_1, \xi_1)B_1(\xi_2) + g(\xi_1, \xi_2)B_2(\xi_2) +
+ g(B_1(\xi_2), \xi_1) + g(B_2(\xi_2), \xi_1)\xi_2 = 0,
\end{equation}

and from $g(\xi_1, \xi_1) = \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, $B_2(\xi_2) = 0$, $B_1(\xi_1) = 0$ and

\begin{equation}
g(B_1(\xi_2), \xi_1) = C_1(\xi_2, \xi_1) = -C_1(\xi_1, \xi_2) = -g(B_1(\xi_1), \xi_2) = 0
\end{equation}

we have $B_1(\xi_2) = 0$. □

Under the assumption for the codimension $r=2$, the following proposition can be considered as a particular case of the theorem 4.1. Using the last two lemmas, we could make another proof of this, which will be done below:

**Proposition 6.10.** We suppose that $M$ is a submanifold of codimension 2 in a locally product Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$, with the normal induced structure $(P, g, u, \xi_\alpha, \xi_\beta)$. If the normal connection vanishes identically (i.e. $\tilde{\omega}_{\alpha\beta} = 0$), trace$\mathcal{A} = 0$ and $\sigma \neq 0$, then $P$ commutes with the Weingarten operators $A_\alpha$ ($\alpha \in \{1, 2\}$), thus the following relations take place:

\begin{equation}
(6.37)
\begin{cases}
(i) & (PA_1 - A_1P)(X) = 0, \quad (\forall)X \in \chi(M) \\
(ii) & (PA_2 - A_2P)(X) = 0, \quad (\forall)X \in \chi(M)
\end{cases}
\end{equation}

**Proof:** With $Y = \xi_1$ in the equality (6.29) we obtain

\begin{equation}
(6.38)
g(\xi_1, \xi_1)B_1(X) + g(\xi_1, \xi_2)B_2(X) +
+ g(B_1(X), \xi_1)\xi_1 + g(B_2(X), \xi_1)\xi_2 = 0,
\end{equation}

and from $g(\xi_1, \xi_1) = \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$ we have

\begin{equation}
(6.39)
g(B_\alpha(X), \xi_\beta) = C_\alpha(X, \xi_\beta) = -C_\alpha(\xi_\beta, X) = -g(B_\alpha(\xi_\beta), X) = 0
\end{equation}

where $\alpha, \beta \in \{1, 2\}$. From the last lemma we have $B_\alpha(\xi_\beta) = 0$, for any $\alpha, \beta \in \{1, 2\}$. Therefore we obtain $B_1X = 0$, for any $X \in \chi(M)$, so we have (i) from (6.37).

With $Y = \xi_2$ in (6.29), we obtain

\begin{equation}
(6.40)
g(\xi_2, \xi_1)B_1(X) + g(\xi_2, \xi_2)B_2(X) +
+ g(B_1(X), \xi_2)\xi_1 + g(B_2(X), \xi_2)\xi_2 = 0.
\end{equation}

From $g(\xi_2, \xi_2) = \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, using the equality (6.40) and the relations (ii) and (iii) from the last lemma then we obtain $B_2X = 0$, for any $X \in \chi(M)$, so we have (ii) from (6.37). □
Proposition 6.11. Let $M$ be a submanifold of codimension 2 in a locally product Riemannian manifold $(\bar{M}, \bar{g}, P)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_2)$. If the normal connection $\nabla^\perp$ vanishes identically (i.e. $l_{\alpha\beta} = 0$), trace $A = 0$ and $\sigma \neq 0$, then the relations occur:

\[
\begin{align*}
(i) & \quad A_1 \xi_1 = \frac{1}{\sigma} h_1(\xi_1, \xi_1) \xi_1 + \frac{1}{\sigma} h_1(\xi_1, \xi_2) \xi_2, \\
(ii) & \quad A_1 \xi_2 = \frac{1}{\sigma} h_1(\xi_1, \xi_2) \xi_1 + \frac{1}{\sigma} h_1(\xi_2, \xi_2) \xi_2, \\
(iii) & \quad A_2 \xi_1 = \frac{1}{\sigma} h_2(\xi_1, \xi_1) \xi_1 + \frac{1}{\sigma} h_2(\xi_1, \xi_2) \xi_2, \\
(iv) & \quad A_2 \xi_2 = \frac{1}{\sigma} h_2(\xi_1, \xi_2) \xi_1 + \frac{1}{\sigma} h_2(\xi_2, \xi_2) \xi_2.
\end{align*}
\]

Proof: Applying $P$ in (6.37)(i) it follows that

\[
P^2 A_1 X = PA_1 PX,
\]

for any $X \in \chi(M)$. Using the equality (6.23)(i) we obtain

\[
A_1 X - u_1(A_1 X) \xi_1 - u_2(A_1 X) \xi_2 = PA_1 PX,
\]

for any $X \in \chi(M)$.

If we put in (6.43) $X = \xi_1$ and $X = \xi_2$, respectively, we obtain

\[
A_1 \xi_1 + a PA_1 \xi_1 + b PA_1 \xi_2 = h_1(\xi_1, \xi_1) \xi_1 + h_1(\xi_1, \xi_2) \xi_2
\]

and

\[
A_1 \xi_2 + b PA_1 \xi_1 - a PA_1 \xi_2 = h_1(\xi_1, \xi_2) \xi_1 + h_1(\xi_2, \xi_2) \xi_2.
\]

from the equalities (6.24)(v) and (vi).

We replace $X \to PX$ in the equality (6.37)(i) so, $PA_1 PX = A_1 P^2 X$ and using the equality (6.23)(i), we obtain

\[
PA_1 PX = A_1 X - u_1(X) A_1 \xi_1 - u_2(X) A_1 \xi_2,
\]

for any $X \in \chi(M)$.

If we put $X = \xi_1$ and $X = \xi_2$ respectively in (6.46) we obtain

\[
(\sigma - 1) A_1 \xi_1 - a PA_1 \xi_1 - b PA_1 \xi_2 = 0,
\]

and

\[
(\sigma - 1) A_1 \xi_2 - b PA_1 \xi_1 + a PA_1 \xi_2 = 0.
\]

from the equalities (6.24)(v) and (vi).

Summating the relations (6.44) and (6.47), for $\sigma \neq 0$, we obtain (i) from (6.41). Summating the relations (6.45) and (6.48) we obtain, for $\sigma \neq 0$, the equality (ii) from (6.41).
Applying $P$ in the equality (6.37)(ii), it follows that
\begin{equation}
(6.49) \quad P^2 A_2 X = PA_2 PX,
\end{equation}
for any $X \in \chi(M)$ and using (6.23)(i), we obtain
\begin{equation}
(6.50) \quad A_2 X - u_1(A_2 X)\xi_1 - u_2(A_2 X)\xi_2 = PA_2 PX,
\end{equation}
for any $X \in \chi(M)$.

For $X = \xi_1$ and $X = \xi_2$, respectively in (6.50), and using the equalities
(6.24)(v) and (vi), we obtain
\begin{equation}
(6.51) \quad A_2 \xi_1 + aPA_2 \xi_1 + bPA_2 \xi_2 = h_2(\xi_1, \xi_1)\xi_1 + h_2(\xi_1, \xi_2)\xi_2
\end{equation}
and
\begin{equation}
(6.52) \quad A_2 \xi_2 + bPA_2 \xi_1 - aPA_2 \xi_2 = h_2(\xi_1, \xi_2)\xi_1 + h_2(\xi_2, \xi_2)\xi_2.
\end{equation}

We replace $X \to PX$ in the equality (6.37)(ii) so, $PA_2 PX = A_2 P^2 X$
and using the equality (6.23)(i), we obtain
\begin{equation}
(6.53) \quad PA_2 PX = A_2 X - u_1(X)A_2 \xi_1 - u_2(X)A_2 \xi_2,
\end{equation}
for any $X \in \chi(M)$.

For $X = \xi_1$ and $X = \xi_2$, respectively in (6.53), and using the equalities
(6.24)(v) and (vi), we obtain
\begin{equation}
(6.54) \quad (\sigma - 1)A_2 \xi_1 - aPA_2 \xi_1 - bPA_2 \xi_2 = 0,
\end{equation}
and
\begin{equation}
(6.55) \quad (\sigma - 1)A_2 \xi_2 - bPA_2 \xi_1 + aPA_2 \xi_2 = 0.
\end{equation}

Summating the relations (6.51) and (6.55) we obtain, for $\sigma \neq 0$, the equality
(iii) from (6.41). Summating the relations (6.52) and (6.55) we obtain, for
$\sigma \neq 0$, the equality (iv) from (6.41). □

7 Some examples of structures $(P, g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r)$
induced on submanifolds of Euclidean space

Exemple 1. We suppose that the ambient space is $\tilde{M} = E^{2p}$ and for
any $x \in E^{2p}$ we have $x = (x^1, ..., x_p, y^1, ..., y_p) := (x^i, y^j)$. The tangent space
$T_xE^{2p}$ is isomorphic to $E^{2p}$. Let $\tilde{P} : E^{2p} \to E^{2p}$ an almost product structure
on $E^{2p}$ such that
\begin{equation}
(7.2) \quad \tilde{P}(x^1, ..., x^p, y^1, ..., y^p) = (y^1, ..., y^p, x^1, ..., x^p)
\end{equation}
Thus, \((\bar{P}, \langle \cdot, \cdot \rangle)\) is an almost product Riemannian structure on \(E^{2p}\). We show that any hypersphere \(S^{2p-1} \hookrightarrow E^{2p}\) has an \((a, 1)\)f Riemannian structure, by constructing it in an effective way.

The equation of sphere \(S^{2p-1}(R)\) is

\[(7.3) \quad (x^1)^2 + \ldots + (x^p)^2 + (y^1)^2 + \ldots + (y^p)^2 = R^2\]

where \(R\) is its radius and \((x^1, \ldots, x^p, y^1, \ldots, y^p)\) are the coordinates of any point \(x \in S^{2p-1}(R)\). We have that

\[(7.4) \quad N = \frac{1}{R}(x^1, \ldots, x^p, y^1, \ldots, y^p)\]

is a unit normal vector in \(x\) on sphere \(S^{2p-1}(R)\) and

\[(7.5) \quad \bar{P}N = \frac{1}{R}(y^1, \ldots, y^p, x^1, \ldots, x^p)\]

We denote by \((X^1, \ldots, X^p, Y^1, \ldots, Y^p)\) a tangent vector in \(x\) at \(S^{2p-1}(R)\). Hence we have

\[(7.6) \quad \sum_{i=1}^{p} x^i X^i + \sum_{i=1}^{p} y^i Y^i = 0\]

If we decompose \(\bar{P}N\) in the tangential and normal components, we obtain

\[(7.7) \quad \bar{P}N = \frac{1}{R}(\xi^1, \ldots, \xi^p, \eta^1, \ldots, \eta^p) + a \cdot \frac{1}{R}(x^1, \ldots, x^p, y^1, \ldots, y^p)\]

From the relations (7.5) and (7.7) we have

\[(7.8) \quad \xi^i = y^i - ax^i, \quad \eta^i = x^i - ay^i\]

But \((\xi^1, \ldots, \xi^p, \eta^1, \ldots, \eta^p)\) is tangent at the sphere \(S^{2p-1}(R)\) and from this we obtain

\[(7.9) \quad \sum_{i=1}^{p} x^i \xi^i + \sum_{i=1}^{p} y^i \eta^i = 0\]

Using (7.8), it follows that

\[\sum_{i=1}^{p} x^i y^i - a \sum_{i=1}^{p} (x^i)^2 + \sum_{i=1}^{p} x^i y^i - a \sum_{i=1}^{p} (y^i)^2 = 0\]

So, from the equation (7.3) we have

\[(7.10) \quad a = \frac{2}{R^2} \sum_{i=1}^{p} x^i y^i\]
Therefore, the matrix \( A \) becomes a real function \( a \) on \( S^{2p-1}(R) \). Moreover, from equality (7.8) we obtain the tangential component of \( PN \) at sphere \( S^{2p-1}(R) \)

\[
(7.11) \quad \xi = \frac{1}{R} \left( y^1 - ax^1, ..., y^p - ax^p, x^1 - ay^1, ..., x^p - ay^p \right)
\]

where \( a \) was defined in (7.10).

Using the equality (7.11) in \( u(X) = \langle X, \xi \rangle \), with \( X = (X^1, ..., X^p, Y^1, ..., Y^p) \) a tangent vector in \( x \) at sphere \( S^{2p-1}(R) \), then we have

\[
\begin{align*}
    u(X) &= \frac{1}{R} \left( \sum_{i=1}^{p} y^i X^i - a \sum_{i=1}^{p} (x^i Y^i + y^i Y^i) + \sum_{i=1}^{p} x^i Y^i \right) \\
\end{align*}
\]

and from (7.6) we obtain

\[
(7.12) \quad u(X) = \frac{1}{R} \left( \sum_{i=1}^{p} x^i Y^i + \sum_{i=1}^{p} y^i X^i \right)
\]

From \( PX = \vec{P}X - u(X)N \) we obtain

\[
PX = (Y^1, ..., Y^p, X^1, ..., X^p) - \frac{u(X)}{R} (x^1, ..., x^p, y^1, ..., y^p)
\]

and from this we have

\[
(7.13) \quad PX = (Y^1 - \frac{u(X)}{R} x^1, ..., Y^p - \frac{u(X)}{R} x^p, X^1 - \frac{u(X)}{R} y^1, ..., X^p - \frac{u(X)}{R} y^p)
\]

where \( X = (X^1, ..., X^p, Y^1, ..., Y^p) \) is a tangent vector in \( x = (x^1, ..., x^p, y^1, ..., y^p) \) at sphere and \( u(X) \) was defined in (7.12). Moreover, \( PX \) is tangent at \( S^{2p-1}(R) \) because

\[
\langle PX, N \rangle = \sum_{i=1}^{p} x^i Y^i - \frac{u(X)}{R} \sum_{i=1}^{p} (x^i)^2 + \sum_{i=1}^{p} y^i X^i - \frac{u(X)}{R} \sum_{i=1}^{p} (y^i)^2 = \]

\[
= \sum_{i=1}^{p} (x^i Y^i + y^i X^i) - \frac{u(X)}{R} \sum_{i=1}^{p} [(x^i)^2 + (y^i)^2] = \sum_{i=1}^{p} (x^i Y^i + y^i X^i) - u(X) \cdot R = 0
\]

so we have \( \langle PX, N \rangle = 0 \).

Furthermore, if \( X \) and \( Y \) are tangent vectors of the sphere \( S^{2p-1}(R) \), then we have

\[
\langle PX, Y \rangle = \langle \vec{P}X, Y \rangle = \langle \vec{P}^2 X, \vec{P}Y \rangle = \langle X, \vec{P}Y \rangle = \langle X, PY \rangle.
\]

For \( Y = (Y^1, ..., Y^p, Y^1, ..., Y^p) \) we have \( PY = (Y^i - \frac{u(Y)}{R} x^i, X^i - \frac{u(Y)}{R} y^i) \)

Thus,

\[
\langle PX, PY \rangle = \sum_{i} (Y^i - \frac{u(X)}{R} x^i) (Y^i - \frac{u(Y)}{R} x^i) + \sum_{i} (X^i - \frac{u(X)}{R} y^i) (X^i - \frac{u(Y)}{R} y^i) =
\]

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\[
\begin{align*}
&= \sum_i (X^i X^i + Y^i Y^i) + \frac{u(X) u(Y)}{R^2} \sum_i ((x^i)^2 + (y^i)^2) - \\
&- \frac{u(X)}{R} \sum_i (x^i Y^i + y^i X^i) - \frac{u(Y)}{R} \sum_i (x^i Y^i + y^i X^i)
\end{align*}
\]

and from this we have \(< P X, P Y > = < X, Y > - u(X) u(Y)\), for any tangent vectors \(X\) and \(Y\) on the sphere \(S^{2p-1}(R)\), in any point \(x \in M\).

Therefore, from the relations (7.10), (7.11), (7.12), (7.13) we have a \((P, \xi, u, a)\) induced structure by \(\overline{P}\) from \(E^{2p}\) on the sphere \(S^{2p-1}(R)\). This structure is a normal \((a, 1)\) Riemannian structure, because \(S^{2p-1}(R)\) is the totally umbilical hypersurface in \(E^{2p}\) and from this we have that the tensor field \(P\) commutes with the Weingarten operator \(A\).

**Example 2.** Let \(S^{2p-1}(1)\) a hypersphere of the Euclidean space \(E^{2p}\) \((p \geq 2)\), endowed with an almost product Riemannian structure \((\overline{P}, < >)\) from the previous example. We have seen, from above, that on any hypersphere \(S^{2p-1}(1)\) we have an induced structure \((P, \xi, u, a)\). It is obvious that \(E^{2p} = E^{p} \times E^{p}\) and in each of \(E^{p}\) we can get a hypersphere

\begin{equation}
S^{p-1}(r_1) = \{(x^1, \ldots, x^p), \sum_{i=1}^{p} (x^i)^2 = r_1^2\},
\end{equation}

and

\begin{equation}
S^{p-1}(r_2) = \{(y^1, \ldots, y^p), \sum_{i=1}^{p} (y^i)^2 = r_2^2\},
\end{equation}

respectively, with the assumption that \(r_1^2 + r_2^2 = 1\).

We construct the product manifold \(S^{p-1}(r_1) \times S^{p-1}(r_2)\) (such in \([17]\), pg.116-117).

Any point \(x \in S^{p-1}(r_1) \times S^{p-1}(r_2)\) has the coordinates \((x^1, \ldots, x^p, y^1, \ldots, y^p)\) with the properties \(\sum_{i=1}^{p} (x^i)^2 = r_1^2\) and \(\sum_{i=1}^{p} (y^i)^2 = r_2^2\). This manifold is a submanifold of codimension 2 in \(E^{2p}\). Furthermore, \(S^{p-1}(r_1) \times S^{p-1}(r_2)\) (with \(r_1^2 + r_2^2 = 1\)) is a submanifold of codimension 1 in \(S^{2p-1}(1)\). Therefore we have the successive imbedded

\begin{equation}
S^{p-1}(r_1) \times S^{p-1}(r_2) \hookrightarrow S^{2p-1}(1) \hookrightarrow E^{2p}
\end{equation}

The tangent space in a point \(x = (x^1, \ldots, x^p, y^1, \ldots, y^p)\) at the product of spheres \(S^{p-1}(r_1) \times S^{p-1}(r_2)\) is

\[T_{(x^1, \ldots, x^p, y^1, \ldots, y^p)} S^{p-1}(r_1) \oplus T_{(y^1, \ldots, y^p)} S^{p-1}(r_2)\]

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A vector \((U^1, ..., U^p)\) from \(T_x E^p\) (with \(x = (x^1, ..., x^p)\)) is tangent to \(S^{p-1}(r_1)\) if

\[
\sum_{i=1}^{p} U^i x^i = 0
\]

and it can be identified by \((U^1, ..., U^p, 0, ..., 0)\) from \(E^{2p}\).

A vector \((V^1, ..., V^p)\) from \(T_y E^p\) (with \(y = (y^1, ..., y^p)\)) is tangent to \(S^{p-1}(r_2)\) if

\[
\sum_{i=1}^{p} V^i y^i = 0
\]

and it can be identified by \((0, ..., 0, V^1, ..., V^p)\) from \(E^{2p}\).

Consequently, for any \((x, y) := (x^1, ..., x^p, y^1, ..., y^p) \in S^{p-1}(r_1) \times S^{p-1}(r_2)\) we have

\[
(U^1, ..., U^p, V^1, ..., V^p) := (U^i, V^i) \in T_{(x,y)}(S^{p-1}(r_1) \times S^{p-1}(r_2))
\]

if and only if

\[
\sum_{i=1}^{p} U^i x^i = 0, \quad \sum_{i=1}^{p} V^i y^i = 0.
\]

Furthermore, from (7.19) we remark that \(\sum_{i=1}^{p} (U^i x^i + V^i y^i) = 0\) so \((U^i, V^i)\) is a tangent vector at \(S^{2p-1}(1)\) in any \((x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)\). From this it follows that

\[
T_{(x,y)}(S^{p-1}(r_1) \times S^{p-1}(r_2)) \subset T_{(x,y)}S^{2p-1},
\]

for any \((x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)\).

We denote by \(N_2\) the normal unit vector at \(S^{2p-1}(1)\) in a point \((x, y)\). Thus, we have

\[
N_2 = (x^1, ..., x^p, y^1, ..., y^p),
\]

from \(\sum_{i=1}^{p} (x^i)^2 + \sum_{i=1}^{p} (y^i)^2 = r_1^2 + r_2^2 = 1\). On the other hand, \(N_2\) is a normal vector field at \((S^{p-1}(r_1) \times S^{p-1}(r_2))\), when it is considered in its points. Let

\[
N_1 = (\frac{r_2}{r_1} x^1, ..., \frac{r_2}{r_1} x^p, -\frac{r_1}{r_2} y^1, ..., -\frac{r_1}{r_2} y^p) := (\frac{r_2}{r_1} x^i, -\frac{r_1}{r_2} y_i),
\]

be a vector in any \((x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)\). We remark that

\[
< N_1, N_2 > = \frac{r_2}{r_1} \sum_i (x^i)^2 - \frac{r_1}{r_2} \sum_i (y^i)^2 = \frac{r_2}{r_1} r_1^2 - \frac{r_1}{r_2} r_2^2 = 0,
\]

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Therefore $N_2$ is orthogonal to $N_1$ in any $(x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)$ and from this we have that $N_1$ is a tangent vector at $S^{2p-1}(1)$. Moreover, $N_1$ is a unit vector because

$$< N_1, N_1 > = \frac{r_2}{r_1} \sum_i (x^i)^2 + \frac{r_1}{r_2} \sum_i (y^i)^2 = r_2^2 + r_1^2 = 1$$

in any $(x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)$.

Let $U = (U^1, ..., U^p, V^1, ..., V^p) := (U^i, V^i)$ be a tangent vector at $S^{p-1}(r_1) \times S^{p-1}(r_2)$ in any its point $(x^i, y^i)$. From (7.19) we have

$$< U, N_1 > = \frac{r_2}{r_1} \sum_i x^i U^{i} - \frac{r_1}{r_2} \sum_i y^i V^{i} = 0$$

so $N_1$ is a normal vector field at $S^{p-1}(r_1) \times S^{p-1}(r_2)$ in any $(x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)$, and $(N_1, N_2)$ is an orthonormal basis in $T_{(x, y)}^\perp S^{p-1}(r_1) \times S^{p-1}(r_2)$ in any point $(x, y) \in S^{p-1}(r_1) \times S^{p-1}(r_2)$.

We have the induced structure $(P, \xi, u, a)$ on $S^{2p-1}(1)$ which was constructed in (7.10), (7.11), (7.12), (7.13). In the following we find the induced structure on $S^{p-1}(r_1) \times S^{p-1}(r_2)$ by the structure $(P, \xi, u, a)$ on $S^{2p-1}(1)$, using the propositions 5.1 and 5.2. Thus, we shall have a $(P_0, \xi_0, \xi^\top, u_0, u, (a_{\alpha\beta}))$ induced structure on the submanifold $S^{p-1}(r_1) \times S^{p-1}(r_2)$ by the $(P, \xi, u, a)$ structure on $S^{2p-1}(1)$.

Using the relations (7.12) and (7.22) we have

$$u(N_1) = \sum_i (-x^i y^i \frac{r_1}{r_2} + \frac{r_2}{r_1} x^i y^i) = \left( \frac{r_2}{r_1} - \frac{r_1}{r_2} \right) \sum_i x^i y^i$$

We denote by $\lambda := \frac{r_1}{r_2} - \frac{r_2}{r_1}$ and by $\sigma := \sum_i x^i y^i$. Then, it follows that

$$(7.23) \quad u(N_1) = \lambda \sigma$$

From the equality (7.10), with $R = 1$, we have

$$(7.24) \quad a = 2 \sum_{i=1}^{p} x^i y^i = 2\sigma.$$

If we decomposed $P(N_1)$ in normal and tangential components at $S^{p-1}(r_1) \times S^{p-1}(r_2)$ in $S^{2p-1}(1)$ we obtain

$$(7.25) \quad P(N_1) = (\xi^1, ..., \xi^p, \eta^1, ..., \eta^p) + bN_1$$

where $(\xi^1, ..., \xi^p, \eta^1, ..., \eta^p)$ is a tangent field at $S^{p-1}(r_1) \times S^{p-1}(r_2)$ and $b$ is a real function on this submanifold. Using the equality (7.13) we obtain

$$(7.26) \quad P(N_1) = \left( -\frac{r_1}{r_2} y^1 - \lambda \sigma x^1, ..., -\frac{r_1}{r_2} y^p - \lambda \sigma x^p, \frac{r_2}{r_1} x^1 - \lambda \sigma y^1, ..., \frac{r_2}{r_1} x^p - \lambda \sigma y^p \right)$$

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Thus, from (7.22), (7.25) and (7.26), we obtain

\[
(7.27) \begin{align*}
(i) & \xi^i = -\frac{br_1}{r_2}x^i - \frac{r_2}{r_1}y^i - \lambda \sigma x^i, \\
(ii) & \eta^i = \frac{br_1}{r_2}y^i + \frac{r_2}{r_1}x^i - \lambda \sigma y^i.
\end{align*}
\]

Hence, from (7.27)(i) and using that \(\sum_i \xi^i x^i = 0\) (because \((\xi^1, \ldots, \xi^p, \eta^1, \ldots, \eta^p)\) is tangent to \(S^{p-1}(r_1) \times S^{p-1}(r_2)\)), we obtain

\[br_1 r_2 = -\frac{r_1}{r_2} \sigma - \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)r_1^2 \sigma.\]

Furthermore, from \(r_1^2 + r_2^2 = 1\) we have

\[
(7.28) \quad b = -2\sigma.
\]

From (7.27) and (7.28), the tangential component at \(P(N_1)\) is

\[
(7.29) \quad \xi_0 = (\xi^1, \ldots, \xi^p, \eta^1, \ldots, \eta^p) := (\xi^i, \eta^i) = \left(\frac{\sigma}{r_1 r_2} y^i - \frac{r_1}{r_2} x^i, \frac{r_2}{r_1} x^i - \frac{\sigma}{r_1 r_2} y^i\right).
\]

From \(u_0(X) = \langle X, \xi_0 \rangle\) (where \(X = (X^i, Y^i)\) is a tangent vector field of \(S^{p-1}(r_1) \times S^{p-1}(r_2)\)), we can find the 1-form \(u_0\). So, using \(\sum_i X^i x^i = 0\) and \(\sum_i Y^i y^i = 0\), we obtain

\[
(7.30) \quad u_0(X) = \frac{r_2}{r_1} \sum_i x^i Y^i - \frac{r_1}{r_2} \sum_i y^i X^i.
\]

We denoted by \(P_0\) the tangent component of the \((1,1)\) tensor field \(P\) (defined in (7.13)) at \(S^{p-1}(r_1) \times S^{p-1}(r_2)\). For the tangent vector field \((X^1, \ldots, X^p, Y^1, \ldots, Y^p) := (X^i, Y^i)\) at \(S^{p-1}(r_1) \times S^{p-1}(r_2)\), we have

\[
(7.31) \quad P_0(X^i, Y^i) = P(X^i, Y^i) - u_0(X^i, Y^i) N_1
\]

From (7.13), (7.22) and (7.31) it follows that

\[
(7.32) \quad P_0(X^i, Y^i) = (Y^i - \frac{1}{r_1^2} \sum_{j=1}^p x^j Y^j) x^i, X^i - \frac{1}{r_2^2} \sum_{j=1}^p X^j y^j) y^i.
\]

On the other hand, from (5.11) we have

\[
\xi^\perp = \langle \xi, N_1 \rangle = N_1 = u(N_1) N_1
\]

and from (7.23) it follows that

\[
(7.33) \quad \xi^\perp = \lambda \sigma N_1 = \sigma\left(\frac{r_2^2}{r_1^2} - 1\right)x^i, \left(\frac{r_2^2}{r_1^2} - 1\right)y^i.
\]

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From $\xi^\top = \xi - \xi^\perp$ and using the relations (7.11) and (7.33) we obtain

$$\xi^\top = (y^i - \frac{\sigma}{r_1^2} x^i, x^i - \frac{\sigma}{r_2^2} y^i)$$

(7.34)

From (5.10) we can find the entries $a_{12} = a_{21}$ of the $2 \times 2$ matrix $A$. Hence,

$$a_{12} = <\xi, N_1> = u(N_1) = \lambda \sigma$$

(7.35)

Therefore, from the relations (7.24), (7.28) and (7.35) we obtain the matrix $A = (a_{\alpha\beta})$ with its entries

$$a_{11} = a = 2\sigma, \quad a_{22} = b = -2\sigma, \quad a_{12} = a_{21} = \lambda \sigma$$

which is

$$A = \begin{pmatrix} 2\sigma & \lambda \sigma \\ \lambda \sigma & -2\sigma \end{pmatrix}$$

(7.36)

where $\lambda = \frac{r_1}{r_1} - \frac{r_2}{r_2}$ and $\sigma = \sum_i x^i y^i$.

Consequently, from the corollary 5.1 we obtain the $(P_0, \xi_0, \xi^\top, u_0, u, (a_{\alpha\beta}))$ induced structure on $S^{p-1}(r_1) \times S^{p-1}(r_1)$ by the almost product Riemannian structure $(\tilde{P}, <>)$ on $E^{2p}$, which is effectively determined by the relations (7.11), (7.12), (7.30), (7.32), (7.33), (7.36) and it is an $(a, 1)$f Riemannian structure.

**Example 3.** We suppose that the ambient space is the Euclidean space $E^{2p+1}$ and let $\tilde{P}$ be an almost product Riemannian structure defined by

$$\tilde{P} : E^{2p+1} \to E^{2p+1}$$

(7.37)

$$\tilde{P}(x^1, ..., x^p, t, y^1, ..., y^p) = (y^1, ..., y^p, t, x^1, ..., x^p)$$

We show that any hypersphere $S^{2p}(R)$ in $E^{2p+1}$ has a normal $(a, 1)$f Riemannian structure. The equation of sphere $S^{2p}(R)$ in $x = (x^1, ..., x^p, t, y^1, ..., y^p) := (x^i, y^i)$ is

$$\sum (x^i)^2 + \sum (y^i)^2 + t^2 = R^2$$

(7.38)

where $R$ is its radius. Then, we remark that

$$N = \frac{1}{R}(x^1, ..., x^p, t, y^1, ..., y^p)$$

(7.39)

is a unit normal vector on $S^{2p}(R)$ in any point $x \in S^{2p}(R)$. From this we have

$$\tilde{P}N = \frac{1}{R}(y^1, ..., y^p, t, x^1, ..., x^p)$$

(7.40)
Let \((X^1, ..., X^p, T, Y^1, ..., Y^p)\) be a tangent vector field on \(S^{2p}(R)\). Thus, from (7.39) it follows that

\[
(7.41) \quad \sum_{i=1}^{p} x^i X^i + tT + \sum_{i=1}^{p} y^i Y^i = 0.
\]

We decomposed \(\tilde{P}N\) in the tangential and normal components:

\[
(7.42) \quad \tilde{P}N = \frac{1}{R}(\xi^1, ..., \xi^p, \tau, \eta^1, ..., \eta^p) + a \cdot \frac{1}{R}(x^1, ..., x^p, t, y^1, ..., y^p)
\]

From the relations (7.40) and (7.42) we obtain \(y^i = \xi^i + ax^i, t = \tau + a \cdot t\) and \(x^i = \eta^i + ay^i\) (for \(i \in \{1, ..., p\}\)), so

\[
(7.43) \quad \xi^i = y^i - ax^i, \quad \tau = t(1 - a), \quad \eta^i = x^i - ay^i
\]

But \((\xi^1, ..., \xi^p, \tau, \eta^1, ..., \eta^p)\) must to be tangent on \(S^{2p}\), thus it follows that

\[
(7.44) \quad \sum_{i=1}^{p} x^i \xi^i + \tau \cdot t + \sum_{i=1}^{p} y^i \eta^i = 0
\]

and from this, we obtain

\[
\sum_{i=1}^{p} x^i y^i - a \sum_{i=1}^{p} (x^i)^2 + t^2(1 - a) + \sum_{i=1}^{p} x^i y^i - a \sum_{i=1}^{p} (y^i)^2 = 0
\]

and from this we have

\[
(7.45) \quad a = \frac{1}{R^2}(2 \sum_{i=1}^{p} x^i y^i + t^2)
\]

From (7.43), we have

\[
(7.46) \quad \xi = \frac{1}{R}(y^1 - ax^1, ..., y^p - ax^p, t(1 - a), x^1 - ay^1, ..., x^p - ay^p)
\]

where \(a\) was defined in (7.45).

Using (7.46) in \(u(X) = <X, \xi>\), where \(X = (X^1, ..., X^p, T, Y^1, ..., Y^p)\) is a tangent vector field on sphere, we have

\[
u(X) = \frac{1}{R}\left[\left(\sum_{i=1}^{p} y^i X^i + \sum_{i=1}^{p} x^i Y^i + Tt\right) - a \left(\sum_{i=1}^{p} (x^i X^i + y^i Y^i) + tT\right)\right]\]

and from (7.41), we obtain

\[
(7.47) \quad u(X) = \frac{1}{R}\left(\sum_{i=1}^{p} x^i Y^i + \sum_{i=1}^{p} y^i X^i + tT\right)
\]
From \( PX = \tilde{P}X - u(X)N \) we obtain
\[
PX = (Y^1, ..., Y^p, T, X^1, ..., X^p) - \frac{u(X)}{R} (x^1, ..., x^p, t, y^1, ..., y^p)
\]
Therefore, it follows that
\[
(7.48) \quad PX = (Y^i - \frac{u(X)}{R}x^i, T - \frac{u(X)}{R}t, X^i - \frac{u(X)}{R}y^i),
\]
for \( i \in \{1, ..., p\} \).

We verify that \( PX \) is tangent at \( S^{2p-1} \). Using the relations (7.48), we obtain
\[
< PX, N > = \sum_{i=1}^{p} x^i Y^i - \frac{u(X)}{R} \sum_{i=1}^{p} (x^i)^2 + (T - \frac{u(X)}{R}t) \sum_{i=1}^{p} y^i X^i - \frac{u(X)}{R} \sum_{i=1}^{p} (y^i)^2 = \sum_{i=1}^{p} (x^i Y^i + tT + y^i X^i) - u(X) \cdot R = 0
\]
thus \( < PX, N > = 0 \) so \( PX \) is tangent sphere \( S^{2p} \).

Furthermore, if \( X \) and \( Y \) are tangent vector fields on sphere \( S^{2p} \), then
\[
< PX, Y > = < \tilde{P}X, Y > = < \tilde{P}^2X, \tilde{P}Y > = < X, \tilde{P}Y > = < X, PY >
\]
If \( Y := (X'^1, ..., X'^p, T', Y'^1, ..., y'^p) \) is a tangent vector in any point \( x \) at sphere \( S^{2p} \) then, from (7.50), we have
\[
(7.49) \quad PY = (Y'^i - \frac{u(Y)}{R} x^i, T' - \frac{u(Y)}{R} t, X'^i - \frac{u(Y)}{R} y^i), \quad i \in \{1, ..., p\}
\]
and from (7.48) and (7.49) we have
\[
< PX, PY > = \sum_{i} (Y^i - \frac{u(X)}{R} x^i)(Y'^i - \frac{u(Y)}{R} x^i) +
+ (T - \frac{u(X)}{R}t)(T' - \frac{u(Y)}{R}t) + \sum_{i} (X^i - \frac{u(X)}{R} y^i)(X'^i - \frac{u(Y)}{R} y^i) =
= \sum_{i} (X^i X'^i + TT' + Y'^i Y'^i) + \frac{u(X)u(Y)}{R^2} \sum_{i} ((x^i)^2 + t^2 + (y^i)^2) -
- \frac{u(X)}{R} \sum_{i} (x^i Y'^i + tT' + y^i X'^i) - \frac{u(Y)}{R} \sum_{i} (x^i Y^i + tT + y^i X^i)
\]

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Hence, we have
\[
<PX, PY> = <X, Y> - u(X)u(Y)
\]
for any tangent vector fields \(X\) and \(Y\) on \(S^{2p-1}\).

Consequently, from the relations (7.45), (7.46), (7.47), (7.48) we have that \((\tilde{P}, \xi, u, a)\) is an induced structure on \(S^{2p}\) by the almost product Riemannian structure \((\tilde{\mathbb{P}}, <\cdot, \cdot>)\) from \(E^{2p+1}\). This structure induced on \(S^{2p}\) is a normal \((a, 1)f\)-Riemannian structure, because the sphere \(S^{2p}\) is a totally umbilical hypersurface in \(E^{2p+1}\) thus, \(P\) commutes by the Weingarten operator \(A\).

**Example 4.** Let \(\tilde{M} = E^{p+q}\) be the ambient space and we define an almost product Riemannian structure \((\tilde{\mathbb{P}}, <\cdot, \cdot>)\) on the Euclidean space \(E^{p+q}\) such that \(\tilde{\mathbb{P}} : E^{p+q} \rightarrow E^{p+q}\) and

\[
(7.50) \quad \tilde{\mathbb{P}}(x^1, ..., x^p, y^1, ..., y^q) = (x^1, ..., x^p, -y^1, ..., -y^q)
\]

We show that any hypersphere \(S^{p+q-1}(R)\) in \(E^{p+q}\) has an \((a, 1)f\) Riemannian induced structure by \(\tilde{\mathbb{P}}\).

We denote by \((x^1, ..., x^p, y^1, ..., y^q) := (x^i, y^j)\), with \(i \in \{1, ..., p\}\) and \(j \in \{1, ..., q\}\).

The equation of the sphere \(S^{p+q-1}(R)\) in a point \(x = (x^i, y^j)\) is

\[
(7.51) \quad (x^1)^2 + \cdots + (x^p)^2 + (y^1)^2 + \cdots + (y^q)^2 = R^2
\]

The unit normal vector at sphere \(S^{p+q-1}(R)\) is

\[
(7.52) \quad N = \frac{1}{R}(x^1, ..., x^p, y^1, ..., y^q)
\]

for any point \(x = (x^i, y^j) \in S^{p+q-1}\). From (7.50) and (7.52) we have

\[
(7.53) \quad \tilde{\mathbb{P}}N = \frac{1}{R}(x^1, ..., x^p, -y^1, ..., -y^q)
\]

We denoted by \((X^1, ..., X^p, Y^1, ..., Y^q)\) a tangent vector field at \(S^{p+q-1}\). This is orthogonal on \(N\), so we obtain

\[
(7.54) \quad \sum_{i=1}^{p} x^i X^i + \sum_{j=1}^{q} y^j Y^j = 0
\]

We decompose \(\tilde{\mathbb{P}}N\) in the tangential and normal components:

\[
(7.55) \quad \tilde{\mathbb{P}}N = \frac{1}{R}(\xi^1, ..., \xi^p, \eta^1, ..., \eta^q) + a \cdot \frac{1}{R}(x^1, ..., x^p, y^1, ..., y^q)
\]

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From (7.52) and (7.54) we obtain

\begin{align}
(7.56) \quad & \xi^i = (1 - a)x^i, \quad \eta^j = -(1 + a)y^j
\end{align}

for any \( i \in \{1, \ldots, p\} \) and \( j \in \{1, \ldots, q\} \).

Because \((\xi^1, \ldots, \xi^p, \eta^1, \ldots, \eta^q)\) is tangent at \( S^{p+q-1} \), it follows that

\begin{align}
(7.57) \quad & \sum_{i=1}^{p} x^i \xi^i + \sum_{j=1}^{q} y^j \eta^j = 0
\end{align}

From (7.56), we have

\begin{align}
(1 - a) \sum_{i=1}^{p} (x^i)^2 - (1 + a) \sum_{j=1}^{q} (y^j)^2 = 0
\end{align}

and using the equality (7.51) we obtain

\begin{align}
(7.58) \quad & a = \frac{1}{R^2} [\sum_{i=1}^{p} (x^i)^2 - \sum_{j=1}^{q} (y^j)^2]
\end{align}

Therefore, the matrix \( \mathcal{A} \), is reduces to a real function on \( S^{p+q-1} \).

From (7.55) we have

\begin{align}
(7.59) \quad & \xi = \frac{1}{R} ((1 - a)x^i, -(1 + a)y^j)
\end{align}

for \( i \in \{1, \ldots, p\} \) and \( j \in \{1, \ldots, q\} \).

Using (7.59) in \( u(X) = \left< X, \xi \right> \), with \( X = (X^1, \ldots, X^p, Y^1, \ldots, Y^q) \) a tangent vector field of sphere, we have

\begin{align}
\left< X, \xi \right> = \frac{1}{R} \left( (1 - a) \sum_{i=1}^{p} x^i X^i - (1 + a) \sum_{j=1}^{q} y^j Y^j \right)
\end{align}

and from (7.54) we obtain

\begin{align}
(7.60) \quad & u(X) = \frac{1}{R} \left( \sum_{i=1}^{p} x^i X^i - \sum_{j=1}^{q} y^j Y^j \right)
\end{align}

Let \( X = (X^1, \ldots, X^p, Y^1, \ldots, Y^q) \) a tangent vector field on sphere \( S^{p+q-1}(R) \). From \( PX = PX - u(X)N \) we have

\begin{align}
PX = (X^1, \ldots, X^p, -Y^1, \ldots, -Y^q) - \frac{u(X)}{R} (x^1, \ldots, x^p, y^1, \ldots, y^p)
\end{align}

and from this we obtain

\begin{align}
(7.61) \quad & PX = (X^i - \frac{u(X)}{R} x^i, -Y^j - \frac{u(X)}{R} y^j)
\end{align}
for $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Furthermore, $PX$ is tangent at $S^{p+q-1}$, because we have

$$< PX, N > = \left( \sum_{i=1}^{p} x^i X^i - \sum_{j=1}^{q} y^j Y^j \right) - \frac{u(X)}{R} \left[ \sum_{i=1}^{p} (x^i)^2 + \sum_{j=1}^{q} (y^j)^2 \right] = 0$$

On the other hand, if $X$ and $Y$ are tangent vector fields on sphere $S^{p+q-1}$, then

$$< PX, Y > = < \tilde{P} X, Y > = < \tilde{P}^2 X, \tilde{P} Y > = < X, \tilde{P} Y > = < X, PY >$$

Let $Y = (X^1, \ldots, X^p, Y^1, \ldots, Y^q)$ be a tangent vector in a point $x = (x^i, y^j)$ at sphere. From (7.61) we have

$$PY = (X^n - \frac{u(Y)}{R} x^i, -Y^j - \frac{u(Y)}{R} y^j)$$

for $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Thus, we have

$$< PX, PY > = \sum_{i=1}^{p} (X^i - \frac{u(X)}{R} x^i)(X^i - \frac{u(Y)}{R} x^i) + \sum_{j=1}^{q} (-Y^j - \frac{u(Y)}{R} y^j)(-Y^j - \frac{u(Y)}{R} y^j) =$$

$$= \left( \sum_{i=1}^{p} (x^i)^2 + \sum_{j=1}^{q} (y^j)^2 \right) - \frac{u(X)}{R^2} \left( \sum_{i=1}^{p} x^i X^i - \sum_{j=1}^{q} y^j Y^j \right) - \frac{u(Y)}{R^2} \left( \sum_{i=1}^{p} x^i X^i - \sum_{j=1}^{q} y^j Y^j \right)$$

so

$$< PX, PY > = < X, Y > - u(X)u(Y),$$

for any tangent vector fields $X$ and $Y$ on sphere $S^{p+q-1}$.

Consequently, the relations (7.58), (7.59), (7.60), (7.61) give the structure $(P, \xi, u, a)$ induced on the sphere $S^{p+q-1}(R)$ by $\tilde{P}$ from $E^{p+q}$. This structure is a normal $(a, 1)$ f Riemannian structure because, the sphere $S^{p+q-1}$ is a hypersurface totally umbilical in $E^{p+q}$ thus, $P$ commutes by the Weingarten operator $A$.

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