Analysis of the Fisher-KPP equation with a time-dependent Allee effect

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Keywords: Reaction-diffusion model, Allee threshold, traveling wave, wave speed

Abstract
In this short note, we study the Fisher-KPP population model with a time-dependent Allee threshold. We consider the time dependence as sinusoidal functions and rational functions as they relate to varying environmental situations of the model. Employing the generalized Riccati equation mapping method, we obtain exact traveling wave solutions. Also, when the time-dependent Allee threshold decays to a constant value, we recover the traveling wave solution of the degenerate Fitzhugh-Nagumo equation from our general solution.

1. Introduction

One-dimensional reaction-diffusion equations of the form

\[ \partial_t u = \partial_{xx} u + f(u), \; \; t > 0, \; \; x \in (-\infty, \infty), \]  

(1.1)

where \( f(u) \) is a sufficiently smooth function of \( u \), are used to model phenomena that occur in many areas of science and engineering [1–5]. Equation (1.1) reduces to the well known Fisher-KPP equation when \( f(u) = u(1 - u) \) and is given by [6, 7]

\[ \partial_t u = \partial_{xx} u + u(1 - u), \]

where \( u(x, t) \) is the density of the population of genes at time \( t \) and space position \( x \). Here, \( u(1 - u) \) is known as the logistic growth function.

However, a growth function is said to be of the Allee type if it satisfies, for some \( \rho \in (0, 1/2) \),

\[ f(u) = u(1 - u)(u - \rho), \; \; \text{for all } u \in (0, 1). \]  

(1.2)

The parameter \( \rho \) corresponds to the so-called ‘Allee threshold’ [8]. Equation (1.1) with (1.2) is the Fisher-KPP equation with Allee effect and is given by

\[ \partial_t u = \partial_{xx} u + u(1 - u)(u - \rho). \]  

(1.3)

This equation can also be thought of as the degenerate Fitzhugh-Nagumo equation [3] which is used to model impulse propagation in nerve axons.

The Allee effect has to do with the fact that the fitness of small populations is sometimes negative, i.e., if the population density is too small, the species or group of individuals will not survive. To model this effect an extra factor \( (u - \rho) \) is added to the logistic growth of the Fisher-KPP equation. In realistic situations, there is no need for the Allee threshold to be a constant. With a changing population habitat due to seasonal variations and/or other external impacts on environmental conditions, the Allee threshold itself could depend on time and possibly, space. The Allee effect is covered in detail in [8–11].

If one considers the Allee threshold to be just time-dependent, that is \( \rho = \rho(t) \), then equation (1.3) can be written as,

\[ \partial_t u = \partial_{xx} u + u(1 - u)(u - \rho(t)). \]  

(1.4)
In a recent work, a generalized Fisher equation with time-dependent coefficients given by

$$\partial_t u = f(t)\partial_{xx}u + g(t)u - h(t)u^2,$$

where, $f(t)$ represents a time-dependent diffusion coefficient, $g(t)$, a growth coefficient and $h(t)$, a competition parameter was studied by Hammond [12]. The time-dependent coefficients in (1.5) can be thought of as representing long term changes in climate or short-term seasonality [12, 13]. Assuming $f(t)$ to be a constant, (1.5) was analyzed in [14] and [15] using the Painlevé property for partial differential equations and the generalized Riccati equation expansion method respectively. A somewhat similar equation to (1.5) that is given by

$$\partial_t u = f(t)\partial_{xx}u + g(t)u - h(t)u^3,$$

where, the only difference is a $u^3$ term instead of a $u^2$ term, was also studied recently by Trikia and Wazwaz [16] and Öğün and Kart [17] in order to find exact solutions.

In this short note, our objective is to find exact traveling wave solutions for the Fisher-KPP equation with time-dependent Allee threshold (1.4). Our analysis considers a variety of different practically relevant functions, as they relate to seasonal changes and/or externally impacted environment, for the Allee threshold. As far as we can gather from the literature, such a study has not been carried out yet.

2. Existence of traveling wave solutions

In order to look for a traveling wave solution of equation (1.4), we define

$$u(x, t) = U(\xi), \quad \xi = \kappa x - \omega(t),$$

where $\omega(t)$ is a time-dependent differentiable function and $\kappa$ is a non-zero real number. Then equation (1.4) can be rewritten as

$$\kappa^2 U_{\xi\xi} + \omega'(t)U_\xi + U(1 - U)(U - \rho(t)) = 0.$$  

(2.1)

By letting $U_\xi = W$, we formulate equation (2.2) into a system of first order differential equations

$$U_\xi = W, \quad \kappa^2 W_\xi = -\omega'(t)W - U(1 - U)(U - \rho(t)).$$

(2.2)

To determine whether the system (2.3) has bounded nontrivial traveling wave solutions with boundary values 0 and 1 as $\xi \rightarrow \pm\infty$, we assume that there is a heteroclinic orbit (in the $U$-$W$ phase plane) in the shape of a parabola, and therefore we make the ansatz $W = bU(1 - U) = b(U - U^2)$. Then, $\kappa^2 W_\xi = b\kappa^2(1 - 2U)U_\xi = b\kappa^2(1 - 2U)W = b\kappa^2(1 - 2U)(U - U^2) = b^2\kappa^2U - 3b^2\kappa^2U^2 + 2b^2\kappa^2U^3.$

(2.3)

From system (2.3), we also have

$$\kappa^2 W_\xi = -\omega'(t)W - U(1 - U)(U - \rho(t)) = -b\omega'(t)(U - U^2) - U(1 - U)(U - \rho(t)) = [-b\omega'(t) + \rho(t)]U + [b\omega'(t) - 1 - \rho(t)]U^2 + U^3.$$  

(2.4)

Equating equations (2.4) and (2.5) gives

$$[2b^2\kappa^2 - 1]U^3 - [3b^2\kappa^2 + b\omega'(t) - 1 - \rho(t)]U^2 + [b^2\kappa^2 + b\omega'(t) - \rho(t)]U = 0.$$  

Since this last equation is valid for all $U$, equating the coefficients of $U^3, U^2,$ and $U$ to zero, we obtain $b = \pm\frac{1}{\sqrt{2}\kappa}$ and $b\omega'(t) = \rho(t) - \frac{1}{2}$. So, the time-dependent wave speed $c'(t)$ is governed by

$$c'(t) = \frac{\omega'(t)}{\kappa} = \pm\frac{1}{\sqrt{2}}(2\rho(t) - 1).$$

(2.5)

Note that for $\rho(t) = \frac{1}{2}$, we find $c'(t) = 0$ and thus we have a standing wave solution for the Fisher-KPP equation when the Allee threshold takes the maximum possible value. Also, for any $\rho(t)$, there are two traveling wave solutions with different speeds.

On the other hand, assuming $\rho(t) = \rho > 0$ and $\omega(t) = \omega t$, where $\omega \neq 0$, and by making the ansatz $W = bU(1 - \rho)$, the system (2.3) has bounded nontrivial traveling wave solutions with boundary values 0 and $\rho$ as $\xi \rightarrow \pm\infty$ and wave speed $c = \frac{\omega}{\kappa} = \frac{\omega}{\frac{\rho}{\sqrt{2}}}(\rho - 2)$. Also, by making the ansatz $W = b(U - \rho)(1 - U)$, the system (2.3) has bounded nontrivial traveling wave solutions with boundary values $\rho$ and 1 as $\xi \rightarrow \pm\infty$ and wave speed $c = \frac{\omega}{\kappa} = \frac{\omega}{\frac{\rho}{\sqrt{2}}}(\rho + 1)$. Figure 1 shows the 2-D graphs of the three possible kink wave solutions to the Fisher-KPP equation (1.4).
3. Exact traveling wave solutions

In the past, the exact solutions for nonlinear evolution equations have been investigated by many authors who were interested in nonlinear phenomena. Many effective and powerful methods have been presented such as Hirota’s bilinear method [18], inverse scattering transform [19], homogeneous balance method [20], auxiliary equation method [21], hyperbolic function method [22], Exp-function method [23], G'/G-expansion method [24], and many more. In [25], Zhu presents a so-called generalized Riccati equation mapping method to construct exact traveling wave solutions of nonlinear evolution equations by assuming that the traveling wave solutions can be expressed by a polynomial in \( f(\xi) \), where \( f(\xi) \) is a solution of the generalized Riccati equation of the form

\[
\phi'(\xi) = \alpha \phi^2(\xi) + \beta \phi(\xi) + \gamma, \quad (3.1)
\]

where \( \alpha, \beta, \) and \( \gamma \) are real constants. The degree of the polynomial can be determined by the homogeneous balance method, and the coefficients can be obtained by solving a set of algebraic equations.

For the present work, we will focus on only the three solutions, given by,

\[
\phi_1(\xi) = -\frac{1}{2\alpha}\beta + \frac{\sqrt{\Delta}}{2}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right), \quad (3.2)
\]

\[
\phi_2(\xi) = -\frac{1}{2\alpha}\beta + \frac{\sqrt{\Delta}}{2}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right), \quad (3.3)
\]

and

\[
\phi_3(\xi) = -\frac{1}{2\alpha}\beta + \frac{\sqrt{\Delta}}{2}\left(\tanh(\sqrt{\Delta} \xi) \pm i \text{sech}(\sqrt{\Delta} \xi)\right), \quad (3.4)
\]

where \( \Delta = \beta^2 - 4\alpha\gamma > 0 \) and \( \alpha \neq 0 \). The reader interested in other forms of solutions for equation (3.1) is referred to [25].

In order to look for the traveling wave solution of equation (1.4), as in section 2, we make the transformation \( u(x, t) = U(\xi) \), where \( \xi = \kappa x - \omega(t) \), and change equation (1.4) into the following ordinary differential equation

\[
\kappa^2 U_{\xi\xi} + \omega(t) U_{\xi} + U(1 - U)(U - \rho(t)) = 0. \quad (3.5)
\]

Considering the homogeneous balance between \( U_{\xi\xi} \) and \( U^3 \) in equation (3.5), \( K + 2 = 3K \Leftrightarrow K = 1 \), we simply seek the solutions of equation (3.5) of the form

\[
U(\xi) = \gamma_0 + \gamma_1\phi, \quad \gamma_1 = 0, \quad (3.6)
\]

where \( \phi = \phi(\xi) \) satisfies the generalized Riccati equation (3.1) and \( \gamma_0 \) and \( \gamma_1 \) are constants to be determined later.

Substituting equation (3.1) and equation (3.6) into equation (3.5) and collecting coefficients of polynomials of \( \phi^k (k = 0, 1, \ldots, 6) \), then setting each coefficient to zero, we obtain a system of algebraic equations for \( \alpha, \beta, \gamma, \gamma_0, \gamma_1, \) and \( \kappa \). Solving the resulting system with the aid of Maple, we obtain

\[
\gamma_0 = \frac{\alpha + \beta\gamma_1}{2\alpha}, \quad \gamma = \frac{\beta^2\gamma_1^2 - \alpha^2}{4\alpha\gamma_1^2}, \quad \kappa = \frac{\pm\gamma_1}{\sqrt{2}\alpha}, \quad \text{and} \quad \omega'(t) = \frac{\gamma_1(1 - 2\rho(t))}{2\alpha}, \quad (3.7)
\]

\[
\gamma_0 = \frac{\alpha \rho + \beta\gamma_1}{2\alpha}, \quad \gamma = \frac{\beta^2\gamma_1^2 - \alpha^2\rho^2}{4\alpha\gamma_1^2}, \quad \kappa = \frac{\pm\gamma_1}{\sqrt{2}\alpha}, \quad \text{and} \quad \omega = \frac{\gamma_1(\rho - 2)}{2\alpha}, \quad (3.8)
\]
and
\[ \gamma = -\frac{\alpha + 1 + \beta \gamma_1}{2\alpha}, \]
\[ \gamma = -\frac{\beta^2 \gamma_1 + 2\alpha^2 \rho - \alpha^2 (\rho^2 + 1)}{4\alpha \gamma_1^2}, \]
\[ \kappa = \pm \frac{\gamma_1}{\sqrt{2} \alpha}, \quad \text{and} \quad \omega = \frac{\gamma_1 (\rho + 1)}{2 \alpha}, \]
(3.9)
where \( \alpha, \beta, \) and \( \gamma_1 \) are arbitrary non-zero real numbers.

Substituting (3.7), noting \( \Delta = \frac{\alpha^2 \rho^2}{\gamma_1^2} > 0, \) with the solutions (3.2), (3.3), and (3.4) of equation (3.1) into equation (3.6), we obtain the exact traveling wave solutions of (1.4) as follows:
\[ u(x, t) = -\frac{1}{2} \left( \frac{\sqrt{2}}{4} x - \frac{1}{4} t + \frac{\rho}{2} t \right), \]
(3.10)
\[ u(x, t) = \frac{1}{2} \left( \frac{\sqrt{2}}{4} x - \frac{1}{4} t + \frac{\rho}{2} t \right), \]
(3.11)
and
\[ u(x, t) = \frac{1}{2} \left( \frac{\sqrt{2}}{4} x - \frac{1}{4} t + \frac{\rho}{2} t \right), \]
(3.12)
where
\[ \omega(t) = \int \frac{\gamma_1 (1 - 2 \rho(t))}{2\alpha} dt, \]
(3.13)
\( \varepsilon = \pm 1, \) \( \eta = \pm 1, \) and \( \alpha \) and \( \gamma_1 \) are arbitrary non-zero real numbers.

Equations (3.10), (3.11), and (3.12) are the exact traveling wave solutions of (1.4) with time-dependent Allee effect \( \rho(t) \), traveling at time-dependent speed \( c(t) = \frac{\alpha \rho}{t} \). Equation (3.10) represents a kink/anti-kink wave solution, while equation (3.12) represents a complex-valued wave solution with real kink/anti-kink and imaginary bell/anti-bell wave shapes. Equation (3.11) gives a solution in the form of a hyperbolic function. Clearly, the exact traveling wave solutions of (1.3) can be obtained from equations (3.10), (3.11), and (3.12) by assuming that \( \rho(t) = \rho_1 \) a constant. For instance, the exact anti-kink wave solution with asymptotic values 0 and 1, given by,
\[ u(x, t) = \frac{1}{2} \left( \frac{\sqrt{2}}{4} x - \frac{1}{4} t + \frac{\rho}{2} t \right), \]
(3.14)
is obtained from equation (3.10) when \( \varepsilon = 1, \) \( \alpha = 1, \) and \( \gamma_1 = 1. \) Figure 2 shows the exact solutions obtained from (3.10) with different \( \rho(t) \) functions, when \( \rho(t) = \sin(t) \) and \( \rho(t) = \frac{1}{4} + \frac{1}{t + 1} \), respectively. Here, one can relate the sinusoidal function \( \sin(t) \) to natural seasonal variations and the function \( \frac{1}{4} + \frac{1}{t + 1} \) to any external impacts on the environment.

Now, substituting (3.8), noting \( \Delta = \frac{\alpha^2 \rho^2}{\gamma_1^2} > 0, \) with the solution (3.2) of equation (3.1) into equation (3.6), we obtain the exact traveling wave solution of (1.3) as follows:
\[ u(x, t) = \frac{1}{2} \left( \frac{\sqrt{2}}{4} x - \frac{1}{4} t + \frac{\rho}{2} t \right), \]
(3.15)
where \( \varepsilon = \pm 1 \) and \( \alpha \) and \( \gamma_1 \) are arbitrary non-zero real numbers.

Equation (3.15) represents a kink/anti-kink wave solution with asymptotic values 0 and \( \rho \). Exact solution in the form of a hyperbolic function or complex-valued wave solution with real kink/anti-kink and imaginary bell/anti-bell wave shapes can be obtained in the same manner as before using (3.3), and (3.4), respectively.

Moreover, substituting (3.9), noting \( \Delta = \frac{\alpha^2 \rho^2 - \rho^2}{\gamma_1^2} > 0, \) with the solution (3.2) of equation (3.1) into equation (3.6), we obtain the exact traveling wave solution of (1.3) as follows:
\[ u(x, t) = \frac{1}{2} \left( \frac{\sqrt{2}}{4} x - \frac{1}{4} t + \frac{\rho}{2} t \right), \]
(3.16)
where \( \varepsilon = \pm 1 \) and \( \alpha \) and \( \gamma_1 \) are arbitrary non-zero real numbers.

Equation (3.16) represents a kink/anti-kink wave solution with asymptotic values \( \rho \) and 1. Also, other types of exact solutions can be obtained in the same manner as before using (3.3), and (3.4).

In the case, where the effect of seasonal variation is combined to any externally impacted environment, one could choose the Allee threshold as
Using equation (3.13), we have
\[ \omega(t) = \frac{19}{100}(t + 1) - \frac{1}{4}\ln(t + 1) + \text{Ci}(t + 1), \]
and hence,
\[ \lim_{t \to \infty} \rho(t) = 0.31 \quad \text{and} \quad \lim_{t \to \infty} \frac{\omega(t)}{t} = 0.19, \]
where \( \text{Ci} \) is the Cosine Integral function.

Therefore, equation (3.10), with the time-dependent Allee effect (3.17), approaches the wave solution (3.14) of equation (1.3) as \( t \to \infty \), as shown in figure 3. The solution in figure 3 is obtained when \( \varepsilon = 1, \alpha = 1, \) and \( \gamma_1 = 1 \). This behavior is also valid for the other solutions (3.11) and (3.12).
It should be noted that these new exact traveling wave solutions with interesting characteristics were obtained assuming that the traveling wave can be expressed as a first order polynomial, see (3.6), in terms of a solution of the generalized Riccati equation. If one is to express the traveling wave as a higher order polynomial, it may be possible to find other exciting new solutions for the Fisher-KPP equation with a time-dependent Allee threshold. However, such an analysis can be tedious and it is beyond the scope of this short note.

4. Conclusions

In this work, we obtained exact traveling wave solutions of kink/anti-kink types for the Fisher-KPP equation with a time-dependent Allee effect. We considered the time-dependency in the form of sinusoidal and exponentially decaying functions as they relate to seasonal variations and/or external impacts on the environment. In the case of the decaying Allee effect, when the limit of decay is a constant, we were able to recover the traveling wave solution of the degenerate Fitzhugh-Nagumo equation from our general solution. The results presented here are new and they will add to the body of knowledge on reaction-diffusion equations with Allee effect. Any future work in this area could focus on an Allee threshold that is both time and space dependent. No new data were created or analysed in this study.

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