Formation and Evaporation of a Naked Singularity in 2 d Gravity

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ABSTRACT

We describe a classical configuration of conformal matter forming a naked singularity and discuss its subsequent Hawking evaporation within the context of two dimensional dilaton gravity. The one loop analysis is credible for a large mass naked singularity and suggests the existence of a weak cosmological censorship that would cause it to explode into radiation upon forming.
The discrepancy between its classical and quantum histories seems to require a fundamental revision of our understanding of black hole physics. Perhaps more importantly, it raises serious questions of consistency and interpretation in quantum gravity. Hawking’s pioneering discovery that a black hole is not in fact quite “black” but, via quantum effects actually radiates away its energy in the form of thermal radiation at a characteristic temperature, has inspired a great amount of work attempting both to gain some insight into the ultimate fate of these objects and to understand the dilemma introduced by the apparent loss of quantum coherence in the process. Most attempts have focused on toy models of two dimensional gravity, some inspired by string theory, in the hope that some of these issues can be resolved while the theory yet retains some of the important features of its four dimensional (Einstein) counterpart. While black holes have received a great deal of attention, the formation and fate of naked singularities remains comparatively neglected and obscure. It turns out that some models of two dimensional gravity provide simplified arenas in which to study naked singularities. In this paper we investigate a modification, admitting positive energy naked singularities, of a recently proposed model for a full quantum treatment of two dimensional black hole evaporation. Our aim is to describe qualitatively the history of a naked singularity within the context of this model. This represents therefore a preliminary step to a detailed quantum treatment of naked singularities, at least in two dimensions.

The model we consider is described by the action

\[
S = \int d^2x \sqrt{-g} \left[ e^{-2\phi} (-R + 4(\nabla \phi)^2 + \Lambda) - \frac{1}{2} \sum_i (\nabla f_i)^2 \right]
\]

where \(\Lambda\) is the cosmological constant, the \(f_i(x)\) are \(N\) conformally coupled matter fields and \(\phi\) is the dilaton. The model with positive \(\Lambda\) was considered in ref (7) and shown to be unstable against gravitational collapse, admitting black hole solutions produced by incoming \(f^-\) waves. However, quantum effects in the one loop approximation made the \(f^-\) wave radiate away all its energy even before the formation of a horizon, so that black holes do not appear in the quantum spectrum. As we show, the model with negative \(\Lambda\) admits naked singularities, and their quantum evolution is very different. They are formed, but do not survive for long, radiating away their energy in a burst. The action in (1), with \(\Lambda > 0\), was shown to arise as the effective action describing the radial modes of extremal dilatonic black holes in four dimensions and, without the \(f^-\) fields, from string
theory to the lowest order in world sheet perturbation theory. Here it will be considered simply as a model of two dimensional gravity, and we will take $\Lambda = -4\lambda^2$ henceforth. Our sign conventions throughout are those of Weinberg\textsuperscript{9}.

Varying (1) with respect to the metric $g^{\mu\nu}$ gives the gravitational equation of motion

$$0 = T_{\mu\nu} = e^{-2\phi} \left[ 2\nabla_\mu \nabla_\nu \phi - \frac{1}{2} e^{2\phi} \sum_i \nabla_\mu f_i \nabla_\nu f_i \right. \left. + g_{\mu\nu} \left( -2 \nabla^2 \phi + 2(\nabla \phi)^2 + 2\lambda^2 + \frac{1}{4} e^{2\phi} \sum_i (\nabla f_i)^2 \right) \right]$$

where $\nabla_\mu$ is the covariant derivative with respect to $g_{\mu\nu}$. The three equations above form a set of constraints on the allowable solutions of the field equations. Passing to the conformal gauge, in which the metric has the form

$$g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu},$$

(1) may be re-expressed as

$$S = \int d^2x \left[ e^{-2\phi} \left( -2 \nabla^2 \rho + 4(\nabla \phi)^2 - 4\lambda^2 e^{2\rho} \right) - \frac{1}{2} \sum_i (\nabla f_i)^2 \right],$$

(4)

where all derivatives are with respect to the flat metric $\eta_{\mu\nu}$ and it is understood that the constraints in (2) are satisfied fields above. One then obtains the following equations of motion

$$2\nabla^2 \phi - \nabla^2 \rho - 2(\nabla \phi)^2 - 2\lambda^2 e^{2\rho} = 0$$

$$\nabla^2 \phi - 2(\nabla \phi)^2 - 2\lambda^2 e^{2\rho} = 0$$

$$\nabla^2 f_i = 0.$$

The solutions of interest are most readily obtained in light-cone coordinates $x^\pm = x^0 \pm x^1$ which we use hereafter. In the conformal gauge the constraint equations reduce to

$$T_{++} = e^{-2\phi} \left[ 2\partial_+ \partial_+ \phi - 4\partial_+ \phi \partial_+ \rho \right] - \frac{1}{2} \sum_i \partial_+ f_i \partial_+ f_i = 0$$

$$T_{--} = e^{-2\phi} \left[ 2\partial_- \partial_- \phi - 4\partial_- \phi \partial_- \rho \right] - \frac{1}{2} \sum_i \partial_- f_i \partial_- f_i = 0$$

$$T_{+-} = e^{-2\phi} \left[ -2\partial_+ \partial_- \phi + 4\partial_+ \phi \partial_- \phi - \lambda^2 e^{2\rho} \right] = 0$$

(6)
and the equations of motion are

\[-4\partial_+\partial_-\phi + 4\partial_+\phi\partial_-\phi + 2\partial_+\partial_-\rho - \lambda^2 e^{2\rho} = 0\]
\[-2\partial_+\partial_-\phi + 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho} = 0\]
\[\partial_+\partial_-f_i = 0.\]  \hspace{1cm} (7)

The first two equations in (7) imply that the \(\rho(x)\) is the same as \(\phi(x)\) up to a harmonic function, \(h(x)\). A choice of \(h(x)\) amounts to a choice of coordinate system, because the choice of conformal gauge does not fix the conformal subgroup of diffeomorphisms. Fixing this gauge freedom by taking \(h(x) = 0\) the general solution satisfying the constraints has the form \((e^{-2\phi} = e^{-2\rho} = \sigma)\)

\[\sigma = \lambda^2 x^+ x^- + \frac{M}{\lambda}\]  \hspace{1cm} (8)

where \(M\) is a positive constant which, as we show below, is the Bondi energy of the singularity.

When \(M = 0\), the metric is flat and the dilaton is linear in the spatial coordinate. This is the linear dilaton vacuum and has appeared before in various studies. When \(M \neq 0\), the metric is \(g_{\mu\nu} = e^{2\phi}\eta_{\mu\nu}\), and the curvature scalar

\[R = + 2e^{-3\rho}\nabla^2 e^\rho - 2e^{-4\rho}(\nabla e^\rho)^2\]
\[= + 4 \left[ \partial_+\partial_-\sigma - \frac{\partial_+\sigma\partial_-\sigma}{\sigma} \right] \]
\[= + \frac{4\lambda M}{\sigma},\]  \hspace{1cm} (9)

is singular at \(\sigma(x) = 0\). The Penrose diagram of the spacetime described above is given in figure 1. In the section of spacetime covered by \(\lambda^2 x^+ x^- + M/\lambda > 0\), labeled by I and II, the Killing vector \(\xi\),

\[(\xi^+, \xi^-) = \lambda(x^+, -x^-),\]  \hspace{1cm} (10)

is spacelike in region I and timelike in region II.

Since the naked singularity appears at spatial infinity, the ADM mass is a meaningless concept. We want to relate \(M\) to its Bondi mass. There is a standard procedure to evaluate
the Bondi energy of a configuration admitting a Killing vector. The existence of a Killing vector implies that the current density defined by

\[ j_\mu = T_{\mu\nu} \xi^\nu \]  

is conserved. Let \( t_{\mu\nu} \) be a linearization of \( T_{\mu\nu} \) about the dilaton vacuum so that to first order \( j_\mu = t_{\mu\nu} \xi^\nu \), and consider a solution of the dilaton field that is asymptotic to the vacuum with \( \phi = \phi^{(0)} + \delta \phi \), where \( \phi^{(0)} = -\ln(\lambda^2 x^+ x^-)/2 \). The conserved current density, (11), will take the form

\[ j^+ = 2\lambda \partial_+ \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_+ \delta \phi + x^- \partial_- \delta \phi \right] \right), \]

\[ j^- = 2\lambda \partial_- \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_+ \delta \phi + x^- \partial_- \delta \phi \right] \right). \]  

The conservation of \( j_\mu \) implies the existence of two charges, \( Q^+ \) and \( Q^- \), which evolve the system in the direction of increasing \( x^+ \) and \( x^- \). Integrating, for example, the conservation equation, \( \nabla \cdot j = \partial_+ j_- + \partial_- j_+ = 0 \), over \( x^+ \) shows that \( Q^- = \int_{\mathcal{I}_R} dx^+ j_+ \) is constant in \( x^- \). The current density \( j_+ \) is a total derivative, so its integral can be measured as a surface term on \( \mathcal{I}_R^+ \). Evaluating \( \delta \phi \) from (8), one obtains this conserved charge,

\[ Q^- = 2\lambda \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_+ \delta \phi + x^- \partial_- \delta \phi \right] \right)_{\mathcal{I}_R^+} = + M > 0. \]  

This is the Bondi mass of the naked singularity, and it is positive.

The naked singularity described above can be dynamically created by an incoming pulse of the minimally coupled scalar fields. Consider a shock wave of one of the \( N \) conformally coupled matter fields traveling in the \( x^- \) direction (at constant \( x^+ = x^+_0 \)) and set all the other matter fields to zero. If the shock wave has magnitude \( a \), it is described by the stress tensor

\[ \frac{1}{2} \partial_+ f \partial_+ f = a \delta(x^+ - x^+_0) \]  

Once again in the gauge \( h(x) = 0 \), the solution becomes

\[ \sigma = \lambda^2 x^+ x^- - a(x^+ - x^+_0) \Theta(x^+ - x^+_0), \]

which is just the linear dilaton vacuum when \( x^+ < x^+_0 \), and the naked singularity with mass \( M = ax^+_0 \lambda \) when \( x^+ > x^+_0 \). The resulting metric and curvature are identical to
those in (8), with a shift in retarded time, \( x^- \to x^- - a/\lambda^2 \). The Penrose diagram for this spacetime is given in figure 2. Note that the naked singularity is formed by the incoming shock wave in the distant past. This contrasts with the formation of the black hole which occurs after the shockwave has traversed a considerable part of spacetime.

We now want to include quantum effects to the one loop order. As is well known, in two dimensions (and only in two dimensions), the quantum stress energy tensor of evaporation can always be calculated exactly from the conservation equations and the trace anomaly. The latter is given by the well known expression

\[
\langle T_{\mu\mu} \rangle = -\frac{4}{\sigma} \langle T_{++} \rangle = -\alpha R,
\]

where \( \alpha \) is a spin dependent constant which, for bosonic fields, is equal to \( +1/24\pi \). The conservation equations, \( \nabla_\mu \langle T_{\nu}^\mu \rangle = 0 \), determine the components of the stress tensor in terms of its trace

\[
\langle T_{++} \rangle = - \int \frac{dx^-}{\sigma} \partial_+ (\sigma \langle T_{++} \rangle) + A(x^+)
\]

\[
\langle T_{--} \rangle = - \int \frac{dx^+}{\sigma} \partial_- (\sigma \langle T_{--} \rangle) + B(x^-)
\]

where \( A(x^+) \) and \( B(x^-) \) are boundary condition dependent functions of \( x^+ \) and \( x^- \) respectively. For the collapsing \( f^- \) wave, the natural boundary condition is that the stress tensor vanishes in the linear dilaton vacuum. This fixes both \( A(x^+) \) and \( B(x^-) \) giving, for the components of \( \langle T_{\mu\nu} \rangle \) the final expressions

\[
\langle T_{++} \rangle = + \frac{\alpha \lambda^4 (x^- - a/\lambda^2)^2}{2\sigma^2} - \frac{\alpha}{2x^+^2}
\]

\[
\langle T_{--} \rangle = + \frac{\alpha \lambda^2 x^+_0}{\sigma^2} - \frac{\alpha}{2x^-^2}
\]

\[
\langle T_{+-} \rangle = + \frac{\alpha \lambda^2 ax_0^-}{\sigma^2}.
\]

It is most convenient to analyze the above expressions in the coordinate system in which the metric is asymptotically flat. In region I of figure 2, define the new coordinates \( \sigma^\pm = t \pm x \) by

\[
x^+ = \frac{1}{\lambda} e^{\lambda \sigma^+}
\]

\[
x^- = \frac{1}{\lambda} e^{\lambda \sigma^-} + \frac{a}{\lambda^2}.
\]

Thus, \( \sigma^- \to -\infty \) corresponds to the light-like surface at \( x^- = a/\lambda^2 \) shown in figure 2. This transformation preserves the conformal gauge, and gives for the new metric in region
\[ ds^2 = \frac{dt^2 - dx^2}{1 + ax_0^+ e^{-2\lambda t}} \]  

(19)

Transforming the expressions above for \( \langle T_{\mu\nu} \rangle \) to the new coordinate system, we find on \( \mathcal{I}_R^+ \) (as \( \sigma^+ \to \infty \))

\[ \langle T^{(\sigma)}_{++} \rangle \to 0, \quad \langle T^{(\sigma)}_{+-} \rangle \to 0 \]

\[ \langle T^{(\sigma)}_{--} \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{(1 + (a/\lambda)e^{-\lambda \sigma})^2} \right]. \]  

(20)

Now, \( \langle T^{(\sigma)}_{--} \rangle \) represents the outgoing flux at \( \mathcal{I}_R^+ \). It approaches a maximum of \( \alpha \lambda^2/2 \) in the far past of \( \mathcal{I}_R^+ \) at \( x^- = a/\lambda^2 \), and decreases smoothly to zero in the far future of \( \mathcal{I}_R^+ \) as \( x^- \to \infty \). All components of the tensor vanish on \( \mathcal{I}_L^+ \).

This is the Hawking radiation from the naked singularity. At \( x^- = a/\lambda^2 \) on \( \mathcal{I}_R^+ \), it is independent of the mass of the singularity just as the radiation from two dimensional black holes approaches a constant independent of \( M \) near the horizon. The total energy lost by the collapsing \( f^- \) wave at a fixed retarded time, \( \sigma^- \), is the integrated flux along \( \mathcal{I}_R^+ \) from \( \sigma^- \to -\infty \) to \( \sigma^- \). This integrated flux is infinite because the flux approaches a steady state at early retarded times. Of course, the singularity cannot radiate away more energy than it possesses. The result is nonsense, a consequence of having neglected the back reaction of the radiation on the spacetime geometry.

It is safe to believe, however, that the radiation process is cataclysmic, and that the \( f^- \) wave radiates away most of its energy at a point very close to \( (x_0^+, a/\lambda^2) \). To justify more quantitatively this statement, consider the value of the dilaton field at this point, keeping in mind that \( e^\phi \) is the loop expansion parameter for dilaton gravity. One finds

\[ e^\phi = \frac{1}{\sqrt{ax_0^+}} \]  

(21)

which is indeed small for very energetic incoming \( f^- \) shock waves (large \( a \)), making the one loop approximation credible in this case. The situation here contrasts starkly with the black hole. In the latter case, the flux approaches zero at early retarded times approaching a constant as the horizon is approached on \( \mathcal{I}_R^+ \). It was shown in ref(7) that the dilaton coupling is not small at the point at which the \( f^- \) wave is expected to radiate itself away,
implying that the one-loop calculation breaks down before the $f$ − wave fully disappears. Likewise, for small mass naked singularities, the loop expansion parameter is large and the results of a one-loop approximation are not credible. Moreover, given a fixed, small mass naked singularity, proliferating the number of matter fields cannot remedy the situation because the presence of $N$ matter fields serves only to multiply the Hawking radiation by a factor of $N$. The naked singularity therefore evaporates even more catastrophically.

One might attempt to include quantum effects in our earlier computation of the configuration’s Bondi energy. To do so, one must add the quantum stress tensors derived above to the expressions we had for $t_{\mu\nu}$. The quantum corrected conserved current density is thus the sum $j_{\mu} = j_{\mu}^{(old)} + j_{\mu}^{Q}$, where $j_{\mu}^{Q}$ is due to the Hawking radiation, and $j_{\mu}^{(old)}$ is given in (12) after the appropriate shift in retarded time. Linearizing as before, the charge density is once again found to be a total derivative, and the mass of the naked singularity appears as a surface term on $I_{R}^{+}$,

$$M(x^{-}) = 2\lambda \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^{+} \partial_{+} \delta \phi + (x^{-} - \frac{a}{\lambda^{2}}) \partial_{-} \delta \phi \right] + \alpha \left[ x^{+} \partial_{+} \delta \phi + (x^{-} - \frac{a}{\lambda^{2}}) \partial_{-} \delta \phi \right] \right)_{I_{R}^{+}} = ax_{0}^{+} \lambda. \tag{22}$$

It is, however, independent of $x^{-}$ because the term proportional to $\alpha$ vanishes as a consequence of the boundary conditions which imply that there are no geometric invariants on $I_{R}^{+}$. This would seem to lead to the absurd conclusion that the quantum corrected Bondi energy is constant despite the evaporation, but it is once again a consequence of having neglected the back reaction on the spacetime geometry.

The picture that seems to emerge from the above is the following. If the full quantum theory permits the formation of a naked singularity, it will evaporate catastrophically (“explode”) due to its Hawking radiation. For large mass naked singularities, the one loop anomaly calculation is believable, as the loop expansion parameter at the explosion point is small. In a full quantum treatment however, the $f$ − wave may not instantly give up all its energy to the Hawking radiation as one is led to believe from the above. A large mass naked singularity will rapidly, but smoothly, give up its energy, diminishing in size in a short time. Because of the radiation, we expect that the mass of the singularity will be a function, $M(\sigma^{-})$, of the retarded time $\sigma^{-}$ (or $x^{-}$), that is, $M(\sigma^{-})$ should approximate zero very shortly in the retarded future of $a/\lambda^{2}$ on $I_{R}^{+}$. As $M(\sigma^{-})$ becomes small, the theory becomes strongly coupled and the one loop approximation breaks down. Without a full quantum treatment, it is therefore premature to predict its final fate.
One result that is expected to prevail in four dimensions is that if a naked singularity forms it will evaporate catastrophically. As the evaporation starts in the far past, at the approach to the naked singularity, quantum mechanical effects may actually prevent the latter from forming. Thus one can imagine that quantum cosmic censorship occurs either by preventing the formation of a naked singularity or causing it to explode immediately after formation. In the second alternative the naked singularity would be an "event" in spacetime.\(^\text{13}\)

**Acknowledgement**

This work was supported in part by NATO under contract number CRG 920096. L.W. acknowledges the partial support of the U. S. Department of Energy under contract number DOE-FG02-84ER40153.

**Figure Captions:**

1. Figure 1. Penrose diagram of the naked singularity represented by (8)

2. Figure 2. Penrose diagram of the naked singularity formed by an incoming $f$– shock wave.

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