Chiral corrections to the isovector double scattering term for the pion-deuteron scattering length

N. Kaiser

Physik Department T39, Technische Universität München, D-85747 Garching, Germany

Abstract

The empirical value of the real part of the pion-deuteron scattering length can be well understood in terms of the dominant isovector $\pi N$-double scattering contribution. We calculate in chiral perturbation theory all one-pion loop corrections to this double scattering term which in the case of $\pi N$-scattering close the gap between the current-algebra prediction and the empirical value of the isovector threshold T-matrix $T_{\pi N}^-$. In addition to closing this gap there is in the $\pi d$-system a loop-induced off-shell correction for the exchanged virtual pion. Its coordinate space representation reveals that it is equivalent to $2\pi$-exchange in the deuteron. We evaluate the chirally corrected double scattering term and the off-shell contribution with various realistic deuteron wave functions. We find that the off-shell correction contributes at most -8% and that the isovector double scattering term explains at least 90% of the empirical value of the real part of the $\pi d$-scattering length.

To be published in *The Physical Review C* (2002), Brief Reports

The tool to systematically investigate the consequences of spontaneous and explicit chiral symmetry breaking in QCD is chiral perturbation theory. Recently, methodology has been developed [1] which allows one to relate scattering processes involving a single nucleon to nuclear scattering processes. For instance, one can relate pion-nucleon scattering to pion-deuteron scattering. The non-perturbative effects responsible for deuteron binding are accounted for by (phenomenological) deuteron wave functions. Although this introduces some model dependence one can evaluate matrix elements of chiral operators in a variety of wave functions in order to estimate the theoretical error induced by the off-shell behavior of different NN-potentials.

In a recent precision experiment the PSI-group has extracted the isoscalar and isovector pion-nucleon scattering lengths $a^{\pm}_{\pi N}$ from the shift and the width of the 1s-level in pionic hydrogen, with the result [2]:

\[ T_{\pi N}^+ = 4\pi \left( 1 + \frac{m_\pi}{M_p} \right) a_{\pi N}^+ = (-0.045 \pm 0.088) \text{ fm}, \]
\[ T_{\pi N}^- = 4\pi \left( 1 + \frac{m_\pi}{M_p} \right) a_{\pi N}^- = (1.847 \pm 0.086) \text{ fm}. \]
One concludes from these results that the isoscalar \( \pi N \)-scattering length \( a_{\pi N}^\pm \) is compatible with zero and that the (present) errors of \( a_{\pi N}^\pm \) are anti-correlated. The aim of future PSI-experiments is to measure the level width \( \Gamma_{1s} \sim a_{\pi N}^\pm \) with the same precision as the level shift \( \epsilon_{1s} \sim a_{\pi N}^\pm + a_{\pi N} \). We have converted in eqs.(1,2) scattering lengths into equivalent threshold T-matrices by multiplying with the appropriate kinematical factor. \( M_p = 938.27 \text{ MeV} \) and \( m_\pi = 139.57 \text{ MeV} \) denote the proton and charged pion mass.

In an analogous PSI-experiment the complex-valued pion-deuteron scattering length \( a_{\pi d} \) has been extracted from the shift and the width of the 1s-level in pionic deuterium \([3, 4]\). Again converted into a threshold T-matrix the (averaged) experimental result of refs.\([3, 4]\) reads for the real part,

\[
\text{Re} \, T_{\pi d} = 4\pi \left(1 + \frac{m_\pi}{M_d}\right) \text{Re} \, a_{\pi d} = (-0.496 \pm 0.016) \text{ fm}, \tag{3}
\]

where \( M_d = 1875.6 \text{ MeV} \) denotes the deuteron mass.

Because of the large size (and weak binding) of the deuteron it is possible to establish an accurate relationship \([5]\) between the real part \( \text{Re} \, T_{\pi d} \) and \( \pi N \) threshold T-matrices \( T_{\pi N}^\pm \). The single scattering contribution \( T_{\pi d}^{(s)} = 2T_{\pi N}^+ \) is small and within present error bars compatible with zero. The empirical value of the real part \( \text{Re} \, T_{\pi d} \) can be well understood in terms of the dominant isovector double scattering term \([5]\) which reads,

\[
T_{\pi d}^{(d)} = -\left(T_{\pi N}^\right)^2 \int_0^\infty dr \frac{\rho(r)}{\pi r}, \quad \rho(r) = u^2(r) + w^2(r). \tag{4}
\]

Here, \( u(r) \) and \( w(r) \) denote the conventional s- and d-state deuteron wavefunctions, normalized to \( \int_0^\infty dr \rho(r) = 1 \). The relation eq.(4) can be easily derived from the diagrams shown in Fig.1. The four-momentum of the in- and out-going pion (marked by arrows) is \((m_\pi, \vec{q})\) and that of the exchanged virtual pion is \((m_\pi, \vec{q})\). After Fourier-transforming the product of the corresponding pion-propagator \( 1/\vec{q}^2 \) and the deuteron momentum-distributions into coordinate space one gets \( 1/(4\pi r) \) times the deuteron density \( \rho(r) \). The remaining factor \(-4\) originates from the isospin factors of the diagrams. The advantage of working with threshold T-matrices (which are directly related to Feynman diagrams) is that there appear no additional kinematical factors in eq.(4). Other three-body interaction terms induced by \( \pi\pi \)-interaction (the in- and out-going pion couple to a virtual pion in flight) which are formally of the same chiral order have been evaluated in refs.\([1, 3]\) and they were found to be negligibly small. For extensive earlier works on the pion-deuteron scattering length and related issues, see refs.\([2, 5, 7, 8, 9, 10, 11, 12]\).

The shortcoming of the present chiral perturbation theory calculations of \( \text{Re} \, T_{\pi d} \) is that the isovector double scattering contribution eq.(4) is (and for reasons of consistency has to be) evaluated with the leading order expression \( T_{\pi N}^{-\text{(tree)}}(\text{tree}) = m_\pi/2f_\pi^2 = 1.61 \text{ fm} \) for the isovector \( \pi N \) threshold T-matrix. As usual \( f_\pi = 92.4 \text{ MeV} \) denotes the weak pion decay constant. Since this Tomozawa-Weinberg prediction \( T_{\pi N}^{-\text{(tree)}}(\text{tree}) = m_\pi/2f_\pi^2 \) is about 13% smaller than the empirical (central) value of \( T_{\pi N}^{-\text{(see eq.(2))}} \) the (leading order) chiral isovector double scattering contribution to \( \text{Re} \, T_{\pi d} \) comes out about 25% too small, as can be seen from the numbers given in Table 1 of ref.\([3]\). In fact, Weinberg argued in ref.\([4]\) that because of its dominance the isovector double scattering contribution eq.(4) should be calculated including vertex corrections and he simply took the measured \( \pi N \)-scattering length \( a_{\pi N}^- \). The purpose of the present paper is to study in the systematic framework of chiral perturbation theory the corrections to this simple substitution rule.
Fig. 1: Isovector double scattering contributions to pion-deuteron scattering at threshold. The grey disk symbolizes all one-loop graphs of isovector $\pi N$-scattering. Diagrams for which the role of the in- and out-going pion is interchanged are not shown.

It has been shown in ref. [13] that the about 13% gap between the current algebra prediction $T^\pi_{\pi N}(\text{tree}) = m_\pi/2f_\pi^2$ and the empirical value of $T^\pi_{\pi N}$ is closed by chiral one-pion loop corrections at order $O(m_\pi^3)$. All other possible corrections from resonances etc. are much too small to close this gap. Keeping only the important pion-loop correction the chiral expansion of the on-shell isovector $\pi N$ threshold T-matrix reads [13],

$$T^\pi_{\pi N} = \frac{m_\pi}{2f_\pi^2} + \frac{m_\pi^3}{16\pi^2 f_\pi^4} \left(1 - 2\ln \frac{m_\pi}{\lambda}\right),$$

(5)

with $\lambda$ a scale parameter introduced in dimensional regularization and minimal subtraction. With $\lambda = 1.04$ GeV the central empirical value of $T^\pi_{\pi N}$ is reproduced. The scale parameter $\lambda$ can also be interpreted as a momentum space cut-off $\Lambda$. By comparing with the expression of relevant (logarithmically divergent) loop integral obtained in cut-off regularization one finds the relation $\Lambda = \sqrt{\pi} \lambda/2 = 857$ MeV. This value of the cut-off $\Lambda$ is comparable to the usual estimate of the chiral symmetry breaking scale $\Lambda_{\chi} = 2\sqrt{2}\pi f_\pi \approx 820$ MeV.

In order to improve on the isovector double scattering contribution $T^{(d)}_{\pi d}$ one should therefore include pion-loop corrections at the $\pi N \rightarrow \pi N$ transition vertices. The grey disk in the diagrams in Fig. 1 symbolizes all one-loop graphs of (isovector) $\pi N$-scattering. Now, since the exchanged pion with four-momentum $(m_\pi, \vec{q})$ is off its mass-shell the loop diagrams included in the grey disk will lead for such a half off-shell kinematics to more than the order $O(m_\pi^3)$ loop correction to $T^\pi_{\pi N}$ given in eq.(5).

Fig. 1: Those one-loop graphs of isovector $\pi N$-scattering which generate the off-shell correction $\delta T^\pi_{\pi N}($off$)$ for the exchanged virtual pion in the $\pi d$-system.
In addition there is an off-shell contribution $\delta T_{\pi N}^-(\text{off})$ which turns out to be generated entirely by the two graphs shown in Fig. 2 involving the chiral $4\pi$-vertex. This off-shell contribution reads explicitly,

$$
\delta T_{\pi N}^-(\text{off}) = \frac{m_\pi}{96\pi^2 f_\pi^4} \left[ q^2 (1 + 5g_\Lambda^2) \ln \frac{m_\pi}{\Lambda} + \frac{q^2}{6} (1 + 17g_\Lambda^2) + \left[ 4m_\pi^2 (1 + 2g_\Lambda^2) + q^2 (1 + 5g_\Lambda^2) \right] \left[ s \ln \frac{s + q}{2m_\pi} - 1 \right] \right]
= \frac{m_\pi}{2f_\pi} G^V_{E}(-q^2)_{\text{loop}},
$$

(6)

with the abbreviation $s = \sqrt{4m_\pi^2 + q^2}$. As indicated it is proportional to the one-pion loop contribution to the isovector electric form factor of the nucleon $G_E^V(-q^2)$ (normalized to unity at $q^2 = 0$). We also note that the off-shell correction $\delta T_{\pi N}^-(\text{off})$ in eq.(6) is independent of the choice of the interpolating pion-field even though the (off-shell) $4\pi$-vertex is not.

With inclusion of chiral pion-loop corrections to the threshold T-matrix $T_{\pi N}$ the isovector double scattering term takes the form,

$$
T^{(d)}_{\pi d} = \frac{m_\pi^2}{4f_\pi^4} \left[ -1 + \frac{m_\pi^2}{2\pi^2 f_\pi^2} \ln \frac{m_\pi}{2\Lambda} \right] \int_{1/\Lambda}^{\infty} dr \frac{\rho(r)}{\pi r},
$$

(7)

which is practically equivalent to inserting the empirical value of $T_{\pi N}^-$ into eq.(4). The effect of the off-shell correction eq.(6) to Re$T^{(d)}_{\pi d}$ is obtained by multiplying $\delta T_{\pi N}^-(\text{off})$ with the pion-propagator $1/q^2$ and Fourier-transforming this product into coordinate space. The terms in the first line of eq.(6) give this way a delta-function $\delta^3(\vec{r})$ which exclusively probes the deuteron wavefunction at the origin. The Fourier-transform of the other non-polynomial terms in eq.(6) is most conveniently obtained with the help of a dispersion relation and one finds

$$
\delta T^{(o)}_{\pi d} = \frac{m_\pi^2}{96\pi^3 f_\pi^6} \int_{1/\Lambda}^{\infty} \frac{dr}{r} \int_{2m_\pi}^{\infty} d\mu \frac{e^{-\mu r}}{\sqrt{\mu^2 - 4m_\pi^2}} \left[ 1 + 5g_\Lambda^2 - \frac{4m_\pi^2}{\mu^2} (1 + 2g_\Lambda^2) \right].
$$

(8)

One observes that the weight function multiplying the deuteron density $\rho(r)$ goes like $r^{-3}$ for $r \to 0$. On the other hand its asymptotic behavior for $r \to \infty$ is $e^{-2m_\pi r} r^{-5/2}$, the typical form of a $2\pi$-exchange potential. In contrast to the double scattering term eq.(7) the off-shell correction eq.(8) is mainly sensitive to the short distance behavior of the deuteron wavefunction. The $r^{-3}$-singularity in eq.(8) originates from the large-$q$ behavior of the (one-loop) form factor in eq.(6), which clearly is not realistic for large momentum transfers. We regularize the $r^{-3}$ short distance singularity (in a minimal way) by starting the radial integration at a nonzero $r_{\text{min}} = 1/\Lambda = 0.23$ fm, where $\Lambda = 857$ MeV is the (equivalent) momentum cut-off entering the chiral logarithm in eq.(5).

| Potential | Paris | Bonn-CD | Idaho | NL5 | NL6 | NNL5 | NNL6 |
|-----------|-------|---------|-------|-----|-----|------|------|
| $T^{(d)}_{\pi d} [\text{fm}]$ | -0.479 | -0.495 | -0.466 | -0.478 | -0.459 | -0.500 | -0.499 |
| $\delta T^{(o)}_{\pi d} [\text{fm}]$ | 0.015 | 0.025 | 0.017 | 0.023 | 0.008 | 0.036 | 0.036 |

Tab.1: Numerical values of the chirally corrected double scattering contribution $T^{(d)}_{\pi d}$ and the off-shell correction $\delta T^{(o)}_{\pi d}$ for various deuteron wavefunctions.
In Table 1, we present numerical values of the chirally corrected double scattering contribution $T^{(d)}_{\pi d}$ and the off-shell correction $\delta T^{(o)}_{\pi d}$ for various realistic deuteron wavefunctions taken from refs.\[14, 15, 16, 17\]. One observes that the chirally corrected isovector double scattering contribution $T^{(d)}_{\pi d}$ is rather stable and close to the empirical value of $\text{Re} T_{\pi d}$ in eq.(3). When starting the radial integration at $r = 0$ instead of $r_{\text{min}} = 0.23 \text{ fm}$ these numbers are affected at most in the last digit. The off-shell correction $\delta T^{(o)}_{\pi d}$ which is mainly sensitive to the short distance behavior of $\rho(r)$ varies more with the deuteron wavefunction. It is however always a small attractive effect whose relative magnitude does not exceed 8%.

We conclude that the (chirally corrected) isovector double scattering term explains at least 90% of the empirical value of the real part of the $\pi d$ threshold T-matrix. This is in agreement with the findings of ref.\[18\] were a host of other small corrections to $\text{Re} T_{\pi d}$ has been investigated. For example, in ref.\[18\] an off-shell correction of the form eq.(6) proportional to a phenomenological dipole form factor (minus unity) has been considered. The off-shell correction eq.(6) obtained in the present work has in fact a well-defined conceptual and physical origin. It is the unique byproduct of the pion-loop graphs which close the gap between the current-algebra prediction and the empirical value of the isovector $\pi N$ threshold T-matrix $T_{\pi N}$. Its effect on $\text{Re} T_{\pi d}$ is comparable to the inherent theoretical uncertainty of the isovector double scattering contribution $T^{(d)}_{\pi d}$.

**Acknowledgements**

I thank E. Epelbaum and R. Machleidt for providing me tabulated deuteron wavefunctions.

**References**

[1] S. Weinberg, *Phys. Lett.* B295, 114 (1992).
[2] H.Ch. Schröder et al., *Phys. Lett.* B469, 25 (1999).
[3] D. Chatellard et al., *Nucl. Phys.* A625, 855 (1997).
[4] P. Hauser et al., *Phys. Rev.* C58, R1869 (1998).
[5] T. Ericson and W. Weise, *Pions and Nuclei*, Clarendon Press, Oxford, 1988; chapt. 4.
[6] S.R. Beane, V. Bernard, T.-S.H. Lee, and Ulf-G. Meißner, *Phys. Rev.* C57, 424 (1998).
[7] V.M. Kolybasov and A.E. Kudryatsev, *Nucl. Phys.* B41, 510 (1972).
[8] F. Myhrer and R.R. Silbar, *Phys. Lett.* B50, 299 (1974).
[9] I.R. Afnan and A.W. Thomas, *Phys. Rev.* C10, 109 (1974).
[10] G. Fäldt, *Phys. Scr.* 16, 81 (1977).
[11] A.W. Thomas and R.H. Landau, *Phys. Reports* 58, 121 (1980); and refs. therein.
[12] V.V. Baru and A.E. Kudryatsev, *Phys. Atom. Nucl.* 60, 1475 (1997).
[13] V. Bernard, N. Kaiser, and Ulf-G. Meißner, *Phys. Lett.* B309, 421 (1993); *Phys. Rev.* C52, 2185 (1995).
[14] M. Lacombe et al., *Phys. Rev.* C21, 861 (1980).
[15] R. Machleidt, *Phys. Rev.* C63, 024001 (2001).
[16] D.R. Entem and R. Machleidt, *Phys. Lett.* B524, 93 (2002).
[17] E. Epelbaum, W. Glöckle, and Ulf-G. Meißner, *Nucl. Phys.* A671, 295 (2000).
[18] T. Ericson, B. Loiseau, and A.W. Thomas, hep-ph/0009312, and refs. therein.