Two coupled Jaynes-Cummings cells

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ABSTRACT

We develop a theoretical framework to evaluate the energy spectrum, stationary states, and dielectric susceptibility of two Jaynes-Cummings systems coupled together by the overlap of their respective longitudinal field modes, and we solve and characterize the combined system for the case that the two atoms and two cavities share a single quantum of energy.

Keywords: Jaynes-Cummings system, entanglement, susceptibility

1. INTRODUCTION

The Jaynes-Cummings (JC) system\textsuperscript{[1,2]} comprises a two-level atom (2LA) coupled to a single optical resonator mode. Although Jaynes and Cummings emphasized the robustness of the semiclassical description as compared to the fully quantum model, manifestations of quantum field effects are now ubiquitous\textsuperscript{[3–7]} leading to optimism that JC systems will soon be coupled together in one- or more-dimensional lattices\textsuperscript{[8]} thereby yielding novel condensed-matter phenomena\textsuperscript{[9–12]}. The first step to achieving coupled JC systems is to create and study a double-JC (DJC) system.

We develop a theoretical framework for the DJC system by calculating its stationary states, energy spectrum and dielectric susceptibility, and we show that this quadripartite system comprising two atoms and two field modes has fascinating features. The system is effectively characterized by two independent parameters: \( g \) for the coupling rate between the resonator and the single atom and \( \kappa \) for the coherent photon hopping rate between the two resonators and proportional to the overlap of the two resonator field modes. By varying \( g \) and \( \kappa \), quite different features emerge from the DJC system. Although we are interested in general properties of this system, our focus here is specifically on the case that the DJC system shares precisely one quantum of energy.

2. DOUBLE JAYNES-CUMMINGS SYSTEM

The JC system has an atomic dipole with frequency \( \omega_a \) coupled to a cavity with frequency \( \omega_c \) via the atom’s electric dipole between ground state \(|g\rangle\) and \(|e\rangle\). The dipole coupling frequency is \( g \), and the atom-cavity decoupling frequency is \( \Delta = \omega_a - \omega_c \). In the rotating-wave approximation, the JC Hamiltonian is (\( \hbar \equiv 1 \)):

\[
\hat{H}_{JC} = \omega_c \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \frac{1}{2} \omega_a \hat{\sigma}_z + g (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) ,
\]

where \( \hat{a}^\dagger \) and \( \hat{a} \) are creation and annihilation operators for the cavity field and \( \hat{\sigma}_\pm \) and \( \hat{\sigma}_z \) with \( \text{spec}(\hat{\sigma}_z) = \pm 1 \) are spin operators for the atoms. The energy spectrum of the JC system is

\[
\omega_{JC(0)} = -\Delta/2, \quad \omega_{JC(\pm)} = n \omega_c \pm \sqrt{n^2 g^2 + \Delta^2}/4
\]

for the ground state and the \( n^{th} \) JC doublet, respectively. The stationary states are the ground state singlet \(|0g\rangle\) and the excited state doublets \(|\pm\rangle_n\) given by

\[
|\pm\rangle_n = e^{\pm i \theta_n} \left( |n,g\rangle + e^{\pm i \theta_n} |n-1,e\rangle \right), \quad \theta_n = \tan^{-1} \left( 2g \sqrt{n}/\Delta \right)/2.
\]

Two neighboring cells (JC cavities) have overlapping evanescent mode functions resulting in an intercavity hopping rate \( \kappa \) depicted schematically in Fig. 2. For \( \hat{a}_i \) and \( \sigma_i \) the field annihilation operator and atomic electron energy lowering

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operator for the $i$th JC system ($i = 1, 2$), the DJC Hamiltonian is
\[
\hat{H} = \hat{H}_1^{JC} + \hat{H}_2^{JC} - \kappa(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger) = \bigoplus_{\nu} \hat{H}^{(\nu)}
\]
with $\nu$ the total number of quanta shared between the two atoms and the two cavity modes. Thus, the pure-state Hilbert space is a union of subspaces $\mathcal{H}^{(\nu)}$ with definite overall particle number $\nu$.

Although we are interested in characterizing this system for all $\nu$, our focus in this paper is solely on studying the $\nu = 1$, which is intriguing in its own right. The $\nu = 1$ case provides an enticing simplification: the field modes can be treated as two-level systems (known as ‘qubits’ in quantum information parlance). As the system has atoms coupled only to the field modes and the field modes coupled to each other, for $\nu = 1$, the system corresponds to a chain of four qubits as shown in Fig. 2 with at most one qubit in the upper state.

The spectrum and stationary states for the cases $\nu = 0, 1$ can be solved in closed form. Let $\mathcal{B}^{(\nu)} = \{ |n_1, c_1, n_2, c_2\rangle \}$ be a basis for $\mathcal{H}^{(\nu)}$ with $n_i$ denoting the number of photons in the $i$th mode and $c_i = (e, g)$ the state of the $i$th atom. For the trivial case of no excitation in the system $\nu = 0$, we have $\mathcal{B}^{(0)} = \{ |0g0g\rangle \}$. For one excitation ($\nu = 1$) there are four basis states
\[
\mathcal{B}^{(1)} = \{ |0e0g\rangle, |1g0g\rangle, |0g1g\rangle, |0g0e\rangle \},
\]
In this basis
\[
\hat{H}^{(1)} = \begin{pmatrix}
0 & g & 0 & 0 \\
g & -\Delta & -\kappa & 0 \\
0 & -\kappa & -\Delta & g \\
0 & 0 & g & 0
\end{pmatrix} + \omega_c I,
\]
with $I$ the $4 \times 4$ identity matrix. The Hamiltonian matrix is not diagonal due to the presence of the coupling $\kappa$. For independent cells, $\kappa = 0$, and then the $\hat{H}^{(1)}$ matrix becomes a block diagonal with two $2 \times 2$ matrices each corresponding to a single cell.

By diagonalizing the Hamiltonians $\hat{H}^{(\nu)}$ for $\nu = 0, 1$, the corresponding energy spectra are determined to be
\[
\omega^{(0)} = -\Delta = 2\omega^{JC(0)}, \omega^{(1)} \pm = \omega_c - \frac{1}{2}(\Delta + \epsilon\kappa) \pm \sqrt{g^2 + \frac{1}{4}(\Delta + \epsilon\kappa)^2}
\]
with $\epsilon = \pm$. The spectrum $\omega^{(0)}$ is composed of a single nondegenerate state that is independent of $g$ and $\kappa$, whereas the spectrum $\omega^{(1)}$ is composed of doubly degenerate pairs for $\kappa = 0$ and is non-degenerate for $\kappa \neq 0$. The coherent photon hopping effect ($\kappa \neq 0$) does not lead to avoided crossings of the energies. Rather coherent hopping shifts the crossing point from $\Delta = 0$ to $\Delta = \pm \kappa$. Only the atom-cavity coupling rate $g$ results in avoided crossings. This avoided-crossing phenomenon for $g$ and not for $\kappa$ is shown in Fig. 2 which shows the eigenvalues $\omega^{(1)}_{\pm}$ as a function of $\Delta$ for $\kappa \neq 0, g \neq 0$ (left frame) and $\kappa \neq 0, g \neq 0$ (right frame). Evidently the energies cross at $\Delta = \pm \kappa$ for $g = 0$, and $g \neq 0$ leads to avoided crossing.

For zero coupling $\kappa = 0$, the first-level spectral term reduces to two degenerate eigenvalues
\[
\omega^{(1)}_{\pm} |_{\kappa = 0} = \omega^{JC(0)} + \omega^{JC(1)}_{\pm}.
\]
with $\epsilon$ irrelevant. Thus, the spectral values for $\kappa = 0$ can be understood as sums of spectral values for each of the two isolated JC systems.

Now consider the $\nu = 1$ spectrum for the limited case of $g = 0$ and let $\Delta = 0$. In this case, $\omega^{(0)} = 0$, and $\omega^{(1)}_\pm |g=0=\Delta$ values are $\omega_c$ (doubly-degenerate) and $\omega_c \pm \kappa$. These spectra correspond to the spectra for coupled harmonic oscillators as expected. Thus, strong inter-cavity mode coupling (large $\kappa$) compared to atom-cavity coupling ($g$) is expected to make the DJC system behave nearly like coupled harmonic oscillators with a perturbation due to atom-cavity coupling.

In the other limit we consider small $\kappa$ with strong coupling $g$. For $\nu = 1$ we have the spectral values
\begin{equation}
\omega^{(1)}_\pm \approx \omega_c \pm g - \frac{\epsilon \kappa}{2} \pm \frac{\kappa^2}{8g}. \tag{9}
\end{equation}

The shift $\pm g$ is due to vacuum Rabi splitting, and $\epsilon \kappa/2$ is normal-mode splitting due to inter-cavity coupling. The next-order shift $\pm \kappa^2/8g$ is analogous to the ac Stark shift: the field of one cavity induces frequency pulling on the other cavity, which is detuned by $g$ due to vacuum Rabi splitting. Thus, the strong-$g$, weak-$\kappa$ limit is equivalent to a weakly driven strong-coupling JC model.

We now focus on the eigenstates of the system for $\nu = 0, 1$. The $\nu = 0$ case is trivial, composed of a singlet $|0g0g\rangle$ whose energy is independent of $g$ and $\kappa$. The eigenstates of the $\nu = 1$ Hamiltonian are
\begin{equation}
|\pm, r_e\rangle = u^{\pm}_e (|1g00\rangle - \epsilon |0g1g\rangle) + w^{\pm}_e (|0e0g\rangle - \epsilon |0g0e\rangle), \tag{10}
\end{equation}
in which
\begin{equation}
u^{\pm}_e = \frac{-r_e \pm \sqrt{1 + r_e^2}}{\sqrt{2 + 2 \left(r_e \mp \sqrt{1 + r_e^2}\right)^2}}, \quad w^{\pm}_e = \frac{1}{\sqrt{2 + 2 \left(r_e \mp \sqrt{1 + r_e^2}\right)^2}}, \tag{11}
\end{equation}
with
\begin{equation}
r_e = \frac{\Delta + \epsilon \kappa}{2g}. \tag{12}
\end{equation}
The states $|\pm, r_e\rangle$ in Eq. (10) are like W states, i.e., a superposition of one excitation in each of the four degrees of freedom.

In general, the states (10) are non-maximally entangled states with unequal weighting of the field and atomic states in the superposition. The states are maximally entangled for $u^\pm_e = w^\pm_e = 1/2$. A close look at Eq. (11) reveals that this could happen only for $r_e = 0$, which, according to (12), only takes place for $\Delta = \pm \kappa$. Thus, maximally entangled four-qubits states can be created at the thresholds $\Delta = -\kappa$ and $\Delta = \kappa$. 

Figure 2. The $\nu = 1$ energy spectrum vs detuning $\Delta$ illustrating avoided crossing effect to occur only for $g \neq 0$ and at $\Delta = \pm \kappa$: (a) $g = 0, \kappa = 2$ and (b) $g = 1, \kappa = 2$.
3. LINEAR SUSCEPTIBILITY

The spectral properties and stationary states reveal the nature of the DJC system, but ultimately these features need to be observed experimentally. One way to observe these properties is to measure the dielectric susceptibility for a probe field directed through both cavities sequentially with measurement of the output field. The susceptibility is especially important for characterizing the DJC system for two reasons: susceptibility is experimentally meaningful on a macroscopic scale (i.e., without needing to manipulate individual elements of the system such as a single atom or cavity), and the susceptibility would provide a direct signature of a quantum phase transitions in the JC lattice case.

The connection between susceptibility and spectrum is as follows. Given that the system is prepared on the ground state, the transitions from the higher energy states to the ground state are governed by the spontaneous emission, cavity damping, and whether the states are maximally entangled or not. This property contrasts the symmetry between the atoms and/or the cavity modes, i.e. when either the atoms or the field modes are not degenerate in frequency. The symmetry could also be broken by allowing the atoms and/or the cavity modes to be damped with different rates. If the atoms are damped with different rates, say \( \gamma_\alpha \), the transition rates from states with different frequencies differ from each other. Of course, the spontaneous emission, cavity damping, and whether the states are maximally entangled or not. This property contrasts the symmetry between the atoms and/or the cavity modes, i.e. when either the atoms or the field modes are not degenerate in frequency.

In this case, transitions from the two states corresponding to \( |\pm, r_\epsilon \rangle \) to the ground state \( |0 \rangle \) occur with the same probability \( \gamma_\epsilon |u_\epsilon| \). The total probability of the transitions \( \Gamma_{i,\epsilon} \) is a sum of the squares of the absolute values of the amplitudes of transitions from the energy states \( |\pm, r_\epsilon \rangle \) to the ground state \( |0 \rangle \) caused by spontaneous emission from the atoms, occurring with the rate \( \gamma_\epsilon \), and by damping of the modes of the JC cavities with the rate \( \gamma_c \).

The atomic dipole operators and the field operators have matrix elements

\[
\langle \pm, r_\epsilon | (\hat{\sigma}_1^\pm + \hat{\sigma}_2^\pm) |0 \rangle = (1 - \epsilon)u_\epsilon^\pm, \langle \pm, r_\epsilon | (\hat{a}_1^\dagger + \hat{a}_2^\dagger) |0 \rangle = (1 - \epsilon)u_c^\pm,
\]

which cause transitions from the \( \nu = 1 \) states \( |\pm, r_\epsilon \rangle \) to the ground state \( |0 \rangle \) to occur with probabilities

\[
\Gamma_{\pm,\epsilon} = (1 - \epsilon)^2 \left( \gamma |u_\epsilon^\pm|^2 + \gamma_c |u_c^\pm|^2 \right).
\]

Clearly, the transition rates from states with \( \epsilon = +1 \) are zero so that the states are dark (non-radiative) states irrespective of the spontaneous emission, cavity damping, and whether the states are maximally entangled or not. This property contrasts with the two-qubit case for which a dark state can be created between the qubits only if the qubits are identical.

In our case of four qubits, the modes of the JC cavities with the rate \( \gamma_\epsilon \) would provide a direct signature of a quantum phase transitions in the JC lattice case.
absorption spectrum of a probe field is expected to be symmetric regardless of whether the states $|\pm, r_+\rangle$ are maximally entangled or not. Thus, if the atom and the cavity mode of cell 1 are damped with the same rates and also the atom and the cavity mode of the cell 2 are damped with the same rates, which may or may not equal the damping rates of cell 1, the transition probabilities are independent of whether the states $|\pm, r_+\rangle$ are maximally entangled or not.

However, in the case of $\gamma \neq \gamma_c$, the spectrum could be symmetric only if the states $|\pm, r_+\rangle$ are maximally entangled. Otherwise, the spectrum is asymmetric. From Eqs. (13) and (16), the imaginary part of the susceptibility (absorption spectrum) is a sum of two Lorentzians:

$$\text{Im} \left[ \chi_0^{(1)} (\omega_p) \right] = \frac{\gamma_a \Gamma_{+,-}}{(\omega_{-1}^{(1)} - \omega_p)^2 + \gamma_a^2} + \frac{\gamma_a \Gamma_{-,-}}{(\omega_{-1}^{(1)} - \omega_p)^2 + \gamma_a^2}. \quad (19)$$

The symmetry of the spectrum depends on the ratio between $\Gamma_{+,-}$ and $\Gamma_{-,-}$, and the spectrum could be symmetric only for $\Gamma_{+,-} = \Gamma_{-,-}$. From Eq. (16) we see that $\Gamma_{+,-} = \Gamma_{-,-}$ holds only if $|u_{\pm}|^2 = |w_{\pm}|^2 = 1/4$. According to Eq. (16), it happens only when the states $|\pm, r_-\rangle$ are maximally entangled. Hence, an observation of the symmetric absorption spectrum when $\gamma \neq \gamma_c$ could be regarded as an indication of the presence of maximally entangled states in the system.

Figure 3 shows the absorption spectrum for $\gamma \neq \gamma_c$, $\kappa = 2g$, and different $\Delta$. We see that as long as $\Delta \neq \kappa$, the spectrum is composed of two peaks of unequal amplitudes. The spectrum becomes symmetric at $\Delta = \kappa$. In this case the states $|\pm, r_-\rangle$ are maximally entangled states. Thus, the symmetry of the spectrum can be regarded as an indication of the presence of maximally entangled states. Figure 4 shows the absorption spectra for $\gamma_1 \neq \gamma_2$ and $\gamma_c_1 \neq \gamma_c_2$ with $(\gamma_1, \gamma_2) \neq (\gamma_c_1, \gamma_c_2)$. In this case the spectrum is composed of four peaks and is always asymmetric. However, one can notice from the figure that, at $\Delta = -\kappa$, two peaks corresponding to transitions from the states $|\pm, r_+\rangle$ have equal amplitudes and are symmetrically located about $(\omega_c - \omega_p)/g = 0$. Similarly, at $\Delta = \kappa$, two other peaks that correspond to transitions from the states $|\pm, r_+\rangle$ now have equal amplitudes and are symmetrically located about $(\omega_c - \omega_p)/g = 0$. Again, equal amplitudes of the peaks indicate that the states which the transitions correspond to are maximally entangled states.

The presence of the threshold values for $\Delta$ at which maximally entangled four-qubit states are created could be predicted from the structure of the Hamiltonian of the system. Instead, working in terms of the two coupled JC systems, we can

![Figure 3. Absorption spectra](image-url)
Figure 4. Absorption spectra $\text{Im} \left[ \chi_0^{(1)}(\omega) \right]$ of two coupled JC cells plotted as a function of $(\omega_c - \omega_p)/g$ for $\gamma_a / g = 0.05$, $(\gamma_1 / g, \gamma_2 / g) = (0.01, 0.2)$, $(\gamma_{c1} / g, \gamma_{c2} / g) = (0.2, 0.01)$, $\kappa / g = 2$ and different $\Delta$: (a) $\Delta / g = 0$, (b) $\Delta / g = -2$, (c) $\Delta / g = 2$ and (d) $\Delta / g = 5$.

introduce symmetric and antisymmetric combinations of the atomic and field operators (collective modes)

$$
\hat{A}_1 = \frac{1}{\sqrt{2}} (\hat{a}_1 + \hat{a}_2), \quad \hat{A}_2 = \frac{1}{\sqrt{2}} (\hat{a}_1 - \hat{a}_2),
\hat{S}_1 = \frac{1}{\sqrt{2}} (\hat{\sigma}_-^1 + \hat{\sigma}_-^2), \quad \hat{S}_2 = \frac{1}{\sqrt{2}} (\hat{\sigma}_-^1 - \hat{\sigma}_-^2),
$$

and find that, for any $\nu$, the Hamiltonian of the system can be written as

$$
\hat{H} = (\omega_a + \kappa) \hat{A}_1^\dagger \hat{A}_1 + (\omega_a - \kappa) \hat{A}_2^\dagger \hat{A}_2 + \frac{1}{2} \omega_a \hat{S}_z + g \left( \hat{A}_1^\dagger \hat{S}_1 + \hat{A}_2^\dagger \hat{S}_2 + \text{H.c.} \right).
$$

Thus, the DJC system is equivalent to two independent and non-degenerate collective systems corresponding to symmetric and antisymmetric combinations of the modes. The collective bosonic modes are coupled to the collective atomic system with the same coupling strength $g$.

Hamiltonian (21) can be written as

$$
\hat{H} = \omega_a \left( \hat{A}_1^\dagger \hat{A}_1 + \hat{A}_2^\dagger \hat{A}_2 + \frac{1}{2} \hat{S}_z \right) - (\Delta - \kappa) \hat{A}_1^\dagger \hat{A}_1 - (\Delta + \kappa) \hat{A}_2^\dagger \hat{A}_2 + g \left( \hat{A}_1^\dagger \hat{S}_1 + \hat{A}_2^\dagger \hat{S}_2 + \text{H.c.} \right).
$$

with the first part representing the average free energy of the field modes and the atoms, the second part representing the shift of the energies of the superposition modes from the average energy, and the last part representing the interaction of the field modes with the collective atomic systems.

Notice that, at $\Delta = \kappa$, the systems $A_1$ and $S_1$ are resonant, so that the coupled two-qubit system can then be maximally entangled. Similarly, at $\Delta = -\kappa$, the systems $A_2$ and $S_2$ are resonant and therefore can be maximally entangled at that frequency. Clearly Eq. (10) implies that, at $\Delta = -\kappa$, the avoided crossing occurs between states $|\pm, r_+\rangle$ that involve antisymmetric combinations of the atomic and field states, whereas, at $\Delta = \kappa$, the avoided crossing occurs between states $|\pm, r_-\rangle$. Thus maximally entangled states can be created in the system for $\Delta = \kappa$ and $\Delta = -\kappa$. 

4. CONCLUSIONS

We have constructed a framework for calculating the energy spectrum, stationary states, and dielectric susceptibility of two Jaynes-Cummings systems coupled together by the overlap of their respective longitudinal field modes and solved it for $\nu = 0$ and $\nu = 1$ excitations of the system, which can be understood in terms of four coupled qubits. For weak coupling, the pair of systems is similar to a single Jaynes-Cummings system undergoing an AC Stark effect, and for strong coupling the behavior is similar to two coupled harmonic oscillators. For moderate coupling strengths, the pair of atoms and the pair of field modes can be highly entangled states, and, where the spectrum exhibits avoided crossings as a function of the detuning, the atoms and fields are found in maximally entangled four-qubit W-like states. We also show the susceptibility and the absorption of the system that explore the entangled features of the system.

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