The Blume-Emery-Griffiths neural network: dynamics for arbitrary temperature

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Abstract

The parallel dynamics of the fully connected Blume-Emery-Griffiths neural network model is studied for arbitrary temperature. By employing a probabilistic signal-to-noise approach, a recursive scheme is found determining the time evolution of the distribution of the local fields and, hence, the evolution of the order parameters. A comparison of this approach is made with the generating functional method, allowing to calculate any physical relevant quantity as a function of time. Explicit analytic formulae are given in both methods for the first few time steps of the dynamics. Up to the third time step the results are identical. Some arguments are presented why beyond the third time step the results differ for certain values of the model parameters. Furthermore, fixed-point equations are derived in the stationary limit. Numerical simulations confirm our theoretical findings.

1 Introduction

Recently, we have studied the Blume-Emery-Griffiths (BEG) neural network dynamics at zero temperature using the probabilistic signal-to-noise analysis (SNA). The interest of such a study arises from the fact that the BEG-model optimises the mutual information content for three-state networks\textsuperscript{2,3}. In this way the best retrieval properties including, e.g., the largest retrieval overlap, loading capacity, basin of attraction and convergence time, are guaranteed in comparison with other three-state networks studied in the literature\textsuperscript{4}. As has been pointed out in\textsuperscript{4} the system allows for new stable stationary states, i.e. quadrupolar states, when the temperature becomes high enough. Therefore, it is interesting to study the dynamics of the model at arbitrary temperatures. This can be done in the framework of the SNA by introducing auxiliary thermal fields\textsuperscript{5} to express the stochastic dynamics within the gain function formulation of the deterministic dynamics. The calculations are rather involved because the feedback in these systems\textsuperscript{6} causes the
appearance of discrete noise, besides the Gaussian one, involving the neurons at all previous
time steps. This prevents a closed-form solution of the dynamics although it allows for a
recursive scheme to find the order parameters, starting from the time evolution of the
distribution of the local fields. Hence, it seems worthwhile to also apply the generating
functional approach (GFA) \cite{7}, \cite{8} to solve this feedback dynamics. This method enables
us to find all relevant physical order parameters at any time step via the derivatives of the
generating functional.

Both methods are then compared for the first few time steps and the results are analyt-
ically shown to be identical up to the third time step. Numerical simulations are performed
for systems up to 7000 neurons and they confirm the analytical results we have obtained.
Beyond the third time step it is found numerically that the SNA approach certainly leads
to different results than those obtained through simulations for values of the model pa-
rameters corresponding to spin-glass behaviour of the system. This is traced back to the
treatment of the residual overlap representing the influence of the non-condensed patterns
in the time evolution of the system. Finally, as in the \( T = 0 \) case, we derive fixed-point
equations in the stationary limit and compare them with existing results within a replica
symmetric thermodynamic approach \cite{4}.

The paper is organized as follows. In Section 2 we introduce the BEG model and
the order parameters of interest. Section 3 is devoted to the solution of the dynamics of
the model with the SNA. The first few time steps are obtained for the main overlap, the
neural activity, the activity overlap and the variance of the residual overlaps. Moreover,
the stationary limit of the dynamics is discussed. Furthermore, numerical simulations of
this dynamics are presented for the first three time steps showing agreement with the
theoretical findings. Section 4 discusses the GFA approach to the BEG dynamics. In
Section 5 we compare both methods analytically up to the third time step and numerically
beyond that. Some concluding remarks are given in Section 6. All technical details are
referred to Appendices.

2 The BEG model

Consider a neural network consisting of \( N \) neurons which can take values \( \{\sigma_i| i = 1, \ldots, N\} \)
from the discrete set \( S = \{s_k| k = 1, 2, 3\} \equiv \{-1, 0, +1\} \). The \( p = \alpha N \) patterns to be stored
in this network are supposed to be a collection of independent and identically distributed
random variables (i.i.d.r.v.), \( \{\xi^\mu_i| \mu = 1, \ldots, p; i = 1, \ldots, N\} \) with a probability distribution

\[
p(\xi^\mu_i) = \frac{a}{2} \delta(\xi^\mu_i - 1) + \frac{a}{2} \delta(\xi^\mu_i + 1) + (1 - a) \delta(\xi^\mu_i)
\]

with \( a \) the activity of the patterns so that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i}^N (\xi^\mu_i)^2 = a.
\]
Given the network configuration at time \( t \), \( \sigma_N(t) \equiv \{ \sigma_j(t) | j = 1, \ldots, N \} \), the neurons are updated according to the stochastic parallel spin-flip dynamics defined by the transition probabilities

\[
\text{Prob}(\sigma_i(t+1) = s_k \in S | \sigma_N(t)) = \frac{\exp\{ -\beta \epsilon_i[s_k | \sigma_N(t)] \}}{\sum_{s \in S} \exp\{ -\beta \epsilon_i[s | \sigma_N(t)] \}}.
\]

(3)

The configuration \( \sigma_N(0) \) is chosen as input. The energy potential \( \epsilon_i[s | \sigma_N(t)] \) is defined by

\[
\epsilon_i[s | \sigma_N(t)] = -sh_i(\sigma_N(t)) - s^2 \theta_i(\sigma_N(t)),
\]

(4)

where the following local fields in neuron \( i \) carry all the information

\[
h_i(\sigma_N(t)) = \sum_{j \neq i} J_{ij} \sigma_j(t), \quad \theta_i(\sigma_N(t)) = \sum_{j \neq i} K_{ij} \sigma_j^2(t).
\]

(5)

The synaptic couplings \( J_{ij} \) and \( K_{ij} \) are of the Hebb-type

\[
J_{ij} = \frac{1}{a^2 N} \sum_{\mu=1}^p \xi_i^\mu \xi_j^\mu, \quad K_{ij} = \frac{1}{N} \sum_{\mu=1}^p \eta_i^\mu \eta_j^\mu
\]

(6)

with

\[
\eta_i^\mu = \frac{1}{\tilde{a}}((\xi_i^\mu)^2 - a), \quad \tilde{a} = a(1 - a).
\]

(7)

The order parameters relevant for the present discussion of this model are

\[
m_N^\mu(t) = \frac{1}{aN} \sum_i \xi_i^\mu \sigma_i(t), \quad l_N^\mu(t) = \frac{1}{N} \sum_i \eta_i^\mu \sigma_i^2(t), \quad q_N(t) = \frac{1}{N} \sum_i \sigma_i^2(t)
\]

(8)

where \( m_N^\mu(t) \) is the retrieval overlap, \( l_N^\mu(t) \) related to the activity overlap, and \( q_N(t) \) the neural activity [2], [3].

3 The SNA for arbitrary temperatures

3.1 Recursive dynamical scheme

In [4] we discussed the SNA for the BEG model at temperature \( T = 0 \). In order to generalize this approach to finite temperatures we introduce auxiliary thermal fields [5] in order to express the stochastic dynamics within the gain function formulation of the deterministic dynamics. Consider for each \( i \) and \( t \) the auxiliary thermal fields, \( \Psi_i(t) = \{ \psi_l^i(t) | l = 1, 2, 3 \} \), which are i.i.d.r.v. with respect to \( i \) and \( t \), such that [2]

\[
\text{Prob}[\sigma_i(t+1) = s_k \in S | \sigma_N(t)] =
\]

\[
\left[ \prod_{l \neq k} \Theta(\epsilon_i[s_l | \sigma_N(t)] - \psi_l^i(t+1) - \epsilon_i[s_k | \sigma_N(t)] + \psi_k^i(t+1)) \right] \Psi
\]

(9)
with \( \langle \cdot \rangle \) the expectation value with respect to \( \Psi_i(t) \). The joint probability density of the \( \Psi_i(t) \) with respect to \( l \) follows from comparing (3)-(11) with (9) and the usual normalisation to 1. The result is

\[
f_{\beta}(\Psi_i(t)) = \beta^2 3! \frac{\exp \left[ -\beta \sum_{i=1}^3 \psi_i^3(t) \right]}{\left( \sum_{i=1}^3 \exp \left[ -\beta \psi_i^3(t) \right] \right)^3} \right]^{\delta} \left( \sum_{i=1}^3 \psi_i^3(t) \right).
\] (10)

To make the link between the stochastic dynamics and the deterministic dynamics in a convenient way we observe that for each realization of the \( \Psi_i(t+1) \) with density \( f_{\beta}(\cdot) \) the network state evolves according to the deterministic rule

\[
\sigma_i(t+1) = \sum_{k=1}^3 s_k \prod_{l \neq k} \Theta(\epsilon_i[s_l|\sigma_N(t)]) - \psi_i(t+1) + \epsilon_i[s_k|\sigma_N(t)] + \psi_i^k(t+1)
\]

\[
\equiv g_{\psi}[h_i(\sigma_N(t)), \theta_i(\sigma_N(t))].
\] (11)

In this way the problem of deriving recursion relations for \( T \neq 0 \) becomes tractable. We start from the following initial conditions

\[
E[\sigma_i(0)] = 0, \quad Var[\sigma_i(0)] = q_0,
\]

\[
E[\xi^\mu_i \sigma_j(0)] = \delta_i, j \delta_\mu, 1 m_0^1 a, \quad m_1^0 > 0, \quad E[\eta_i^l \sigma_j^2(0)] = \delta_i, j \delta_\mu, 1 l_0^1.
\] (13)

So we assume that the initial configuration is correlated with only one (i.e. the condensed) stored pattern. By the law of large numbers (LLN) Eqs. (8) and (12)-(13) determine the order parameters \( m_1^1(0), q_1^1(0) \) and \( l_0^1(0) \) at \( t = 0 \) in the limit \( N \to \infty \). In the following we leave out the pattern index 1.

For a general time step we find from Eq. (11) and the LLN in the limit \( N \to \infty \) for the order parameters (eqs. 8)

\[
m_1^1(t+1) \equiv \frac{1}{a} \left[ \langle \xi, g_{\psi}(h_i(t), \theta_i(t)) \rangle \right],
\]

\[
q(t+1) \equiv \left[ \langle g_{\psi}(h_i(t), \theta_i(t)) \rangle \right],
\]

\[
l_1^1(t+1) \equiv \left[ \langle \eta, g_{\psi}(h_i(t), \theta_i(t)) \rangle \right],
\] (14)

where \( h_i(t) = \lim_{N \to \infty} h_i(\sigma_N(t)) \) with an analogous formula for \( \theta_i(t) \), and where the convergence is in probability. In the above average, denoted by \( \langle \cdot \rangle \), has to be taken over both the distribution of the \( \{\xi_i^\mu\} \) and the \( \{\sigma_i(0)\} \). Note that the \( \{\sigma_i(0)\} \) is hidden in the local fields through the updating rule (11). For the Eqs. (11)-(16) the average over the thermal fields can be done explicitly by using the relations (9)-(10) leading to

\[
m_1^1(t+1) \equiv \frac{1}{a} \left[ \langle \xi V_\beta(h_i(t), \theta_i(t)) \rangle \right]
\]

\[
q(t+1) \equiv \left[ \langle W_\beta(h_i(t), \theta_i(t)) \rangle \right]
\]

\[
l_1^1(t+1) \equiv \left[ \langle \eta_\beta W_\beta(h_i(t), \theta_i(t)) \rangle \right],
\] (17)

(18)

(19)
with
\[
V_\beta(h_i(t), \theta_i(t)) = \frac{1}{2} \exp (-\beta \theta_i(t)) + \cosh(\beta h_i(t))
\]
(20)
\[
W_\beta(h_i(t), \theta_i(t)) = \frac{1}{2} \exp (-\beta \theta_i(t)) + \cosh(\beta h_i(t)).
\]
(21)

In order to obtain the recursion relations for the local fields we can then follow the derivation of the zero temperature case \[1\] (see Section 3) step by step. We do not repeat this calculation here but write down the final results
\[
h_i(t+1) = \frac{\xi_i}{a} m(t+1) + \chi_h(t) \left\{ h_i(t) - \frac{1}{a} \xi_i m(t) + \frac{\alpha}{a} \sigma_i(t) \right\} + \mathcal{N} \left( 0, \frac{\alpha^2}{a^2} q(t+1) \right)
\]
(22)
\[
\theta_i(t+1) = \eta_l(t+1) + \chi_\theta(t) \left\{ \theta_i(t) - \eta_l(t) + \frac{\alpha}{a} \sigma_i^2(t) \right\} + \mathcal{N} \left( 0, \frac{\alpha^2}{a^2} q(t+1) \right).
\]
(23)

From this it is clear that the local fields at time \( t + 1 \) consist out of a discrete part and a normally distributed part
\[
h_i(t) = M_i(t) + \mathcal{N}(0, \alpha a D(t))
\]
(24)
\[
\theta_i(t) = L_i(t) + \mathcal{N}(0, \alpha E(t))
\]
(25)

where
\[
M_i(t+1) = \chi_h(t) \left[ M_i(t) - \frac{\xi_i}{a} m(t) + \frac{\alpha}{a} \sigma_i(t) \right] + \frac{\xi_i}{a} m(t+1)
\]
(26)
\[
L_i(t+1) = \chi_\theta(t) \left[ L_i(t) - \eta_l(t) + \frac{\alpha}{a} \sigma_i^2(t) \right] + \eta_l(t+1)
\]
(27)

with \( D(t) \) and \( E(t) \) the variances of the residual overlaps defined by
\[
r^\mu(t) = \lim_{N \to \infty} \frac{1}{a^2 \sqrt{N}} \sum_j \xi_j^\mu \sigma_j(t), \quad s^\mu(t) = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_j \eta_j^\mu \sigma_j^2(t), \quad \mu > 1,
\]
(28)

and taking into account the noise produced by the non-condensed patterns in the local fields. They satisfy the recursion relations
\[
D(t+1) = \frac{q(t+1)}{a^2} + \chi_h^2(t) D(t) + 2 \chi_h(t) \text{Cov}[\tilde{r}^\mu(t), r^\mu(t)]
\]
(29)
\[
E(t+1) = \frac{q(t+1)}{a} + \chi_\theta^2(t) E(t) + 2 \chi_\theta(t) \text{Cov}[\tilde{s}^\mu(t), s^\mu(t)].
\]
(30)

In these expressions, the modified residual overlaps \( \tilde{r}^\mu, \tilde{s}^\mu, \mu > 1 \) are introduced because in the residual overlaps the local fields strongly depend upon \( \xi_i^\mu \) and \( \eta_i^\mu \) respectively. Therefore, the local fields for finite \( N \) have been modified as follows
\[
\tilde{h}_{N,i}^\mu(t) = h_{N,i}(t) - \frac{1}{\sqrt{N}} \xi_i^\mu r_{N}^\mu(t), \quad \tilde{\theta}_{N,i}^\mu(t) = \theta_{N,i}(t) - \frac{1}{\sqrt{N}} \eta_i^\mu s_{N}^\mu(t),
\]
(31)
such that, employing the central limit theorem (CLT)

\[
\bar{r}^\mu(t) = \lim_{N \to \infty} \frac{1}{a^2 \sqrt{N}} \sum_j \xi_j^\mu g(\bar{h}_{N,j}(t), \bar{\theta}_{N,j}(t)) \xrightarrow{P_r} N\left(0, \frac{q(t + 1)}{a^3}\right) \tag{32}
\]

\[
\bar{s}^\mu(t) = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_j \eta_j^\mu g(\bar{h}_{N,j}(t), \bar{\theta}_{N,j}(t)) \xrightarrow{P_r} N\left(0, \frac{q(t + 1)}{a}\right), \tag{33}
\]

where we have now assumed that the \(\bar{r}^\mu, \bar{s}^\mu, \mu > 1\) depend only weakly on \(\xi_i^\mu\) and \(\eta_i^\mu\) respectively. We remark that in the thermodynamic limit the density distributions of the modified local fields \(\bar{h}_i(t)\) and \(\bar{\theta}_i(t)\) equal the density distributions of the local fields \(h_i(t)\) and \(\theta_i(t)\) itself. The \(\chi_h(t)\) and \(\chi_\theta(t)\) are the susceptibilities corresponding to the fields \(h_i(t)\) and \(\theta_i(t)\)

\[
\chi_h(t) = \frac{1}{a} \left\langle \int d\bar{h} \int d\bar{\theta} \rho_{h_i(t)}(\bar{h}) \rho_{\theta_i(t)}(\bar{\theta}) \left( \frac{\partial V_\beta(h(t), \theta(t))}{\partial h(t)} \right) \right|_{(\bar{h}, \bar{\theta)}} \tag{34}
\]

\[
\chi_\theta(t) = \frac{1}{a} \left\langle \int d\bar{h} \int d\bar{\theta} \rho_{h_i(t)}(\bar{h}) \rho_{\theta_i(t)}(\bar{\theta}) \left( \frac{\partial W_\beta(h(t), \theta(t))}{\partial \theta(t)} \right) \right|_{(\bar{h}, \bar{\theta)}} \tag{35}
\]

In these expressions, \(\rho_{h_i(t)}(x)\) and \(\rho_{\theta_i(t)}(x)\) are the probability densities of the local fields

\[
\rho_{h_i(t)}(z) = \int \left( \prod_{s=0}^{t-2} dx_s dy_s \right) dx_t dy_t \frac{1}{\sqrt{\det(2\pi C_h)}} \frac{1}{\sqrt{\det(2\pi C_\theta)}}
\]

\[
\times \exp \left( -\frac{1}{2} x C_h^{-1} x - \frac{1}{2} y C_\theta^{-1} y \right) \delta(z - M_i(t) - \sqrt{\alpha a E(t)} x_t) \tag{36}
\]

\[
\rho_{\theta_i(t)}(z) = \int \left( \prod_{s=0}^{t-2} dx_s dy_s \right) dx_t dy_t \frac{1}{\sqrt{\det(2\pi C_h)}} \frac{1}{\sqrt{\det(2\pi C_\theta)}}
\]

\[
\times \exp \left( -\frac{1}{2} x C_h^{-1} x - \frac{1}{2} y C_\theta^{-1} y \right) \delta(z - L_i(t) - \frac{\alpha}{a} E(t) y_t) \tag{37}
\]

with \(x = (x_0, \ldots, x_{t-2}, x_t)\) and \(y = (y_0, \ldots, y_{t-2}, y_t)\) two sets of correlated normally distributed variables which we choose to normalize. The correlations matrices between the different time steps are then given by the elements \(C_h(t, t') = E[x_t x_t']\) and \(C_\theta(t, t') = E[y_t y_t']\).

This concludes the explanation of the recursive scheme from which the order parameters of the system can be obtained. A practical difficulty that remains is the calculation of the explicit correlations at different time steps. As an illustration we present the first few time steps in Appendix A.
3.2 Stationary equations

Stationary equations for non-zero temperature can, in principle, be obtained from the zero temperature expressions obtained in [1] by introducing auxiliary thermal fields and averaging over them. However, extreme care has to be taken concerning these thermal fluctuations. The point is that in the case of the stationary limit at zero temperature it is well known that there is no difference between the neural activity and the Edwards-Anderson order parameter, but for non-zero temperatures they become distinct. Therefore, in order to obtain the stationary limit, we assume that for \( t \to \infty \) the local fields can be replaced by their thermal averages

\[
\lim_{t \to \infty} \left[ \sigma_i^n(t + 1) \right]_\Psi = \lim_{t \to \infty} \left[ g^n_h \left( h_i(t), \theta_i(t) \right) \right]_\Psi \approx \lim_{t \to \infty} \left[ g^n_h \left( \left[ h_i(t) \right]_\Psi, \left[ \theta_i(t) \right]_\Psi \right) \right]_\Psi \quad n = 1, 2. \tag{38}
\]

The first step is to find the thermally averaged fields. Starting from the residual overlaps (28) we write (we focus on the \( h \)-quantities)

\[
[r^\mu(t + 1)]_\Psi = [\tilde{r}^\mu(t)]_\Psi + \chi h(t) r^\mu(t)_\Psi \tag{39}
\]

where now

\[
[\tilde{r}^\mu(t)]_\Psi = \lim_{N \to \infty} \frac{1}{a^2 N} \sum_i \xi_i^{\mu*} \left[ g_\Psi \left( \tilde{h}_N^\mu(t), \tilde{\sigma}_N^\mu(t) \right) \right]_\Psi \mathcal{P}_r \equiv N \left( 0, \frac{1}{a^2} q_1(t + 1) \right) \tag{40}
\]

with

\[
q_1(t) = \left\langle \left[ \sigma_i(t) \right]^2_\Psi \right\rangle. \tag{41}
\]

The recursion relation for the field then becomes

\[
[h_i(t + 1)]_\Psi = \frac{\xi_i}{a} m(t + 1) + \chi h(t) \left[ h_i(t) - \frac{\xi_i}{a} m(t) + \frac{\alpha}{a} \sigma_i(t) \right]_\Psi + \mathcal{N} \left( 0, \frac{q_1(t + 1)}{a} \right) \tag{42}
\]

where we have used the LLN for \( m(t) \) and \( \chi h(t) \), viz.

\[
m(t) = \frac{1}{a N} \sum_i \xi_i \sigma_i(t) \mathcal{P}_r \equiv \frac{1}{a} \left\langle \xi_i \left[ \sigma_i(t) \right]_\Psi \right\rangle
\]

\[
\chi h(t) = \frac{1}{a^2 N} \sum_i (\xi_i^{\mu*})^2 \frac{\partial g_\Psi(t)}{\partial h_i(t)} \mathcal{P}_r \equiv \frac{1}{a} \left\langle \left[ \frac{\partial g_\Psi(\ldots)}{\partial h} \right]_\Psi \right\rangle. \tag{43}
\]

Completely analogous equations can be written down for \( \langle \theta_i(t + 1) \rangle_\Psi \) and \( \langle \tilde{s}^\mu(t + 1) \rangle_\Psi \) with

\[
p_1(t) = \left\langle \left[ \sigma_i^2(t) \right]^2_\Psi \right\rangle. \tag{44}
\]
Next, in order to obtain the stationary limit we assume that the thermal averages of the local fields and of the order parameters do not change in time. This leads to

\[
[h]_\psi = \frac{\xi}{a}m + N\left(0, \frac{\alpha q_1}{a^2(1 - \chi_h)^2}\right) + \frac{\alpha}{a}\eta_h[\sigma]_\psi \equiv \bar{h} + \frac{\alpha}{a}\eta_h[\sigma]_\psi \\
[\theta]_\psi = \eta l + N\left(0, \frac{\alpha p_1}{a^2(1 - \chi_\theta)^2}\right) + \frac{\alpha}{a}\eta_\theta[\sigma^2]_\psi \equiv \bar{\theta} + \frac{\alpha}{a}\eta_\theta[\sigma^2]_\psi
\]  

with \(\eta_x = \chi_x/(1 - \chi_x)\), \(x = h, \theta\). The solution for \([\sigma]_\psi\) is a self-consistent equation of the form

\[
[\sigma]_\psi = V_\beta \left(\bar{h} + \frac{\alpha}{a}\eta_h[\sigma]_\psi, \bar{\theta} + \frac{\alpha}{a}\eta_\theta[\sigma^2]_\psi\right)
\]  

In order to solve this equation we consider the case \(T = 0\). There the averages over the thermal noise disappear, yielding

\[
\lim_{\beta \to \infty} V_\beta (h, \theta) = \text{sign}(h)\Theta(|h| + \theta)
\]  

and a Maxwell construction [10] leads to

\[
\sigma = \text{sign}(h)\Theta(|h| + \theta + \Delta)
\]

where

\[
\Delta = \frac{1}{2}\left(\frac{\alpha}{a}\eta_h + \frac{\alpha}{a}\eta_\theta\right).
\]  

We introduce then a set of auxiliary thermal fields and average over them to arrive at the fixed point equations in the limit \(N \to \infty\) (dropping the index \(i\))

\[
m = \left\langle \left\langle \frac{\xi}{a} \int Dz \int Dy V_\beta(h(z), \theta(y) + \Delta) \right\rangle \right\rangle
\]

\[
l = \left\langle \left\langle \eta \int Dz \int Dy W_\beta(h(z), \theta(y) + \Delta) \right\rangle \right\rangle
\]

\[
q = \left\langle \left\langle \int Dz \int Dy W_\beta(h(z), \theta(y) + \Delta) \right\rangle \right\rangle
\]

\[
\chi_h = \frac{1 - \chi_h}{\sqrt{\alpha q_1}} \left\langle \int Dz \int Dy V_\beta(h(z), \theta(y) + \Delta) \right\rangle
\]

\[
\chi_\theta = \frac{1 - \chi_\theta}{\sqrt{\alpha p_1}} \left\langle \int Dz \int Dy W_\beta(h(z), \theta(y) + \Delta) \right\rangle
\]

with

\[
h(z) = \frac{\xi}{a}m + \frac{\sqrt{\alpha q_1}}{a(1 - \chi_h)}z, \quad \theta(y) = \eta l + \frac{\sqrt{\alpha p_1}}{a(1 - \chi_\theta)}y.
\]
and \( q_1 \) and \( p_1 \) given by

\[
q_1 = \left\langle \left\langle \int Dz \int Dy V_\beta^2(h(z), \theta(y) + \Delta) \right\rangle \right\rangle
\]

\[
p_1 = \left\langle \left\langle \int Dz \int Dy W_\beta^2(h(z), \theta(y) + \Delta) \right\rangle \right\rangle.
\]

Furthermore, in agreement with the fluctuation-dissipation theorem \[11\]

\[
\chi_h = \beta \frac{a}{q - q_1}, \quad \chi_\theta = \beta \frac{\tilde{a}}{q - p_1}
\]

with \( \chi_h \) and \( \chi_\theta \) the susceptibilities defined before, in the stationary limit. It is interesting to remark that the equations (50)-(54) are the same as the fixed-point equations derived from a replica symmetric thermodynamic approach with a hamiltonian for sequential dynamics \[4\].

### 3.3 Numerical results and simulations

The results derived in Sections 2-3 and appendix A have been studied numerically and compared with simulations for systems up to \( N = 7000 \) neurons averaged over 500 runs.

As in the \( T = 0 \) case the first few time steps, given by the explicit formula in appendix A, and the simulations agree very well over the whole range of \( \alpha \). As an illustration we refer to Fig. 1 presenting the order parameters as a function of \( \alpha \) for uniform patterns, \( m_0 = l_0 = 0.6, q_0 = 0.5 \) and \( T = 0.5 \). We remark that the critical capacity for this system is \( \alpha_c \approx 0.06 \) (11). We then learn that the first time steps do give us a reasonable estimate for the critical capacity, especially through the order parameter \( l \).

In Fig. 2 we examine the order parameters \( m \) and \( l \) in the quadrupolar phase \((m = 0, l > 0)\) versus the paramagnetic phase \((m = 0, l = 0)\), for several values of \( \alpha \). We see that a few time steps do give us already the characteristic behaviour. When time increases \( m \) decreases while \( l \) differentiates between the phases, as is seen in the theoretical results as well as in the simulations. For the quadrupolar phase \((\alpha = 0.001)\) \( l \) increases, deep inside the paramagnetic phase \((\alpha = 0.1)\) \( l \) decreases, while in the intermediate region \((\alpha = 0.01)\) the rate of increase of \( l \) quickly diminishes and \( l \) itself goes to zero.

At this point, we remark that there is a small but visible discrepancy between the theory and simulations especially in \( l(3) \). It is of the order \( O(10^{-3}) \) and attributed to finite-size effects. In this respect we report that earlier calculations \[12\] of further time steps of the overlap order parameter in the Hopfield model also showed a discrepancy between simulations and numerical results, e.g., for \( \alpha = 0.10 \) the difference \( m_{th}(3) - m_{sim}(3) \) is again of the order \( O(10^{-3}) \). However, for \( 0.1 < m_0 < 0.4 \), the difference \( m_{th}(4) - m_{sim}(4) \) is about 0.018 and always positive both for \( N = 6000 \) and \( N = 7000 \). This, and the fact that the SNA does not give a closed form solution of the dynamics, has motivated us to look in Section 4 at a second, alternative approach for solving the dynamics.
Figure 1: Order parameters $m(t)$, $l(t)$ and $q(t)$ as a function of the capacity $\alpha$ for the first three time steps for uniform patterns ($a = 2/3$), $m_0 = l_0 = 0.6$, $q_0 = 0.5$ and $T = 0.5$. Theoretical results (solid lines) versus simulations (time 1, 2 and 3 represented by a circle, a plus respectively a times symbol) are shown.

4 The generating functional approach

The second approach we use to discuss the dynamics of the BEG-model is an extension of the generating functional approach (GFA) [7] [8] (for a recent review see, e.g., [13] and references therein). It is an exact procedure based on the idea of looking at the probability for finding a microscopic path in time

$$\text{Prob}[\sigma_N(0), \ldots, \sigma_N(t)] = \text{Prob}(\sigma(0)) \prod_{t' = 1}^{t} \prod_{i = 1}^{N} \text{Prob} (\sigma_i(t') = s_k \in S|\sigma_N(t' - 1))$$

(58)
where the transition probabilities are given by (3)-(4) with the local fields \( h_i(\sigma_N(t)) \) and \( \theta_i(\sigma_N(t)) \) replaced by

\[
\begin{align*}
    h_i(\sigma_N(t)) &= \sum_{j \neq i} J_{ij} \sigma_j(t) + \gamma_{h,i}(t) \\
    \theta_i(t) &= \sum_{j \neq i} K_{ij} \sigma_j^2(t) + \gamma_{\theta,i}(t).
\end{align*}
\] (59)

Here the \( \gamma_{h,i}(t) \) and \( \gamma_{\theta,i}(t) \) are time-dependent external fields introduced in order to define a response function.

The basic tool is then the generating functional

\[
Z_N[\Psi, \Phi] = \left\langle \exp \left\{ -i \sum_{s=1}^{t} \sum_{i=1}^{N} \left( \psi_i(s) \sigma_i(s) + \phi_i(s) \sigma_i^2(s) \right) \right\} \right\rangle_{\text{paths}}
\]

\[
= \sum_{\sigma_N(0)} \ldots \sum_{\sigma_N(t)} \text{Prob}[\sigma_N(0), \ldots, \sigma_N(t)] 
\times \exp \left\{ -i \sum_{s=1}^{t} \sum_{i=1}^{N} \left( \psi_i(s) \sigma_i(s) + \phi_i(s) \sigma_i^2(s) \right) \right\}
\] (60)

from which we can find all the relevant observables by calculating derivatives with respect to \( \{\psi_i(t)\} \) and \( \{\phi_i(t)\} \) and taking these zero afterwards. In the sequel we omit the subscript indicating the path average. This path average of the spin, the correlation functions and

Figure 2: Order parameters \( m(t) \) (bottom three lines) and \( l(t) \) (top three lines) as a function of time for uniform patterns \( (a = 2/3), m_0 = l_0 = 0.1, q_0 = 0.5, T = 1.1 \) and several values of \( \alpha \). Theoretical results (open symbols) are shown versus simulations (full lines for \( \alpha = 0.001 \), dashed lines for \( \alpha = 0.01 \) and dotted lines for \( \alpha = 0.1 \)).
the response functions are given by

\[ \langle \sigma_i(t) \rangle = \lim_{\psi,\Phi \to 0} \frac{\partial Z_N[\Psi, \Phi]}{\partial \psi_i(t)}, \quad \langle \sigma_i^2(t) \rangle = \lim_{\psi,\Phi \to 0} \frac{\partial Z_N[\Psi, \Phi]}{\partial \phi_i(t)}, \]

\[ C_{h,ij}(t,t') = -\lim_{\psi,\Phi \to 0} \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \psi_i(t) \partial \psi_j(t')}, \quad C_{\theta,ij}(t,t') = -\lim_{\psi,\Phi \to 0} \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \phi_i(t) \partial \phi_j(t')}, \]

\[ G_{h,ij}(t,t') = \lim_{\psi,\Phi \to 0} \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \psi_i(t) \partial \gamma_{h,j}(t')}, \quad G_{\theta,ij}(t,t') = \lim_{\psi,\Phi \to 0} \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \phi_i(t) \partial \gamma_{\theta,j}(t')} \]  

(61)

The generating function is averaged over all pattern realizations, i.e., over the disorder, before the path average is taken. In the limit \( N \to \infty \), this results in a theory with effective single spin local fields which, for the model at hand, are given by

\[ h(t) = \xi m(t) + \gamma h(t) + \frac{1}{a} \sum_{s'=0}^{t} R_h(t, s') \sigma(s') + \sqrt{\alpha} z_h(t) \]  

(62)

\[ \theta(t) = \eta l(t) + \gamma \theta(t) + \frac{1}{a} \sum_{s'=0}^{t} R_{\theta}(t, s') \sigma^2(s') + \sqrt{\alpha} z_{\theta}(t) \]  

(63)

where the noises \( z_h(t), z_{\theta}(t) \) are a set of correlated normally distributed variables with the measure

\[ w(z_h, z_{\theta}) = \exp \left\{ -\frac{1}{2} \sum_{s,s'=0}^{t-1} z_h(s)[w_h^{-1}](s, s') z_h(s') - \frac{1}{2} \sum_{s,s'=0}^{t-1} z_{\theta}(s)[w_{\theta}^{-1}](s, s') z_{\theta}(s') \right\} \]

\[ \sqrt{(2\pi)^t \det(w_h) \sqrt{(2\pi)^t \det(w_{\theta})}} \]  

(64)

\[ w_h = (aI - G_h)^{-1} C_h \left( aI - G_h^\dagger \right)^{-1} \]  

(65)

\[ w_{\theta} = (a\tilde{I} - G_{\theta})^{-1} C_{\theta} \left( a\tilde{I} - G_{\theta}^\dagger \right)^{-1} \]  

(66)

The retarded self-interactions appearing in (62)-(63) are then given by

\[ R_h = (aI - G_h)^{-1} G_h, \quad R_{\theta} = (a\tilde{I} - G_{\theta})^{-1} G_{\theta}. \]  

(67)
Furthermore, the order parameters can be written as

\[ m(t) = \lim_{N \to \infty} \lim_{\Psi, \Phi \to 0} \frac{1}{aN} \sum_i \xi_i \frac{\partial Z_N[\Psi, \Phi]}{\partial \psi_i(t)} = \frac{1}{a} \langle \xi \sigma(t) \rangle \]  
(68)

\[ l(t) = \lim_{N \to \infty} \lim_{\Psi, \Phi \to 0} \frac{1}{N} \sum_i \eta_i \frac{\partial Z_N[\Psi, \Phi]}{\partial \phi_i(t)} = \langle \eta \sigma^2(t) \rangle \]  
(69)

\[ C_h(t, t') = -\lim_{N \to \infty} \lim_{\Psi, \Phi \to 0} \frac{1}{N} \sum_i \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \psi_i(t) \partial \psi_i(t')} = \langle \sigma(t) \sigma(t') \rangle \]  
(70)

\[ C_\theta(t, t') = -\lim_{N \to \infty} \lim_{\Psi, \Phi \to 0} \frac{1}{N} \sum_i \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \phi_i(t) \partial \phi_i(t')} = \langle \sigma^2(t) \sigma(t') \rangle \]  
(71)

\[ G_h(t, t') = \lim_{N \to \infty} \lim_{\Psi, \Phi \to 0} \frac{1}{N} \sum_i \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \psi_i(t) \partial \gamma_{h,i}(t')} = \langle \frac{\partial \sigma(t)}{\partial \gamma_h} \rangle \]  
(72)

\[ G_\theta(t, t') = \lim_{N \to \infty} \lim_{\Psi, \Phi \to 0} \frac{1}{N} \sum_i \frac{\partial^2 Z_N[\Psi, \Phi]}{\partial \phi_i(t) \partial \gamma_{\theta,i}(t')} = \langle \frac{\partial \sigma^2(t)}{\partial \gamma_\theta} \rangle \]  
(73)

where the overline denotes the disorder average and \( \langle \rangle \) denotes the averages over the noises \( z_h, z_\theta \), and the initial conditions and the condensed pattern, according to the measure

\[ \langle f[\sigma(0), \ldots, \sigma(t)] \rangle = \left\langle \int dz_h dz_\theta w[z_h, z_\theta] \text{Prob}[\sigma(0), \ldots, \sigma(t)|z_h, z_\theta] f[\sigma(0), \ldots, \sigma(t)] \right\rangle_{\xi} \]  
(74)

Here

\[ \text{Prob}[\sigma(0), \ldots, \sigma(t)|z_h, z_\theta] = \sum_{\sigma(0), \ldots, \sigma(t)} \text{Prob}(\sigma(0)) \prod_{s=0}^{t-1} \frac{\exp(\beta \sigma(s+1)h(s) + \beta \sigma^2(s+1)\theta(s))}{2 \exp(\beta \theta(s)) \cosh(\beta h(s)) + 1} \]  
(75)

with \( h(s) \) and \( \theta(s) \) given by \( [02] \) and \( [03] \); \( \gamma_h, \gamma_\theta = 0 \). For more details of this calculation we refer to Appendix B.

For later convenience we remark that \( G_h(t, t') = 0 \) for \( t \leq t' \) due to causality (analogously for \( G_\theta(t, t') \)), and that for \( t > t' \)

\[ G_h(t, t') = \beta \langle \sigma(t) [\sigma(t' + 1) - V_\beta(h(t'), \theta(t'))] \rangle \]  
(76)

\[ G_\theta(t, t') = \beta \langle \sigma^2(t) [\sigma^2(t' + 1) - W_\beta(h(t'), \theta(t'))] \rangle . \]  
(77)

This set of equations completely solves the dynamics for the BEG-model. We calculate the first few time steps in Appendix C, in order to compare the results with those of the SNA method.
5 Comparison between SNA and GFA

By comparing the explicit calculations described in the Appendices A and C for the first two time steps in the dynamics of the BEG model we find that they are identical and that in the GFA response functions, restricting ourselves for the moment to the $h$-field,

\[ G_h(1, 0) = a\chi_h(0) \quad G_h(2, 1) = a\chi_h(1). \]  

(78)

The third time step requires a bit more work. Looking at the response function in general (recall (76)) and considering $t' = t - 2$, the sums over $\sigma(t)$ and $\sigma(t - 1)$ can be done explicitly because of the specific time behaviour of the local fields, leading to

\[ G_h(t, t - 2) = 0. \]  

(79)

Using this, together with the causality requirement on $G_h$ and noting that a similar treatment is valid for $G_\theta$, we find that the result for the third time step in the GFA method coincides with the corresponding SNA result. Comparing the next time steps explicitly is more cumbersome. Also from the local fields as expressed by Eqs. (22)-(23) in the SNA approach and Eqs. (62)-(63) in the GFA method no immediate obvious analytic connection is apparent. Therefore, a separate detailed analytic study of these relationships for the simpler Hopfield model is appropriate [14].

In the sequel we present some strong numerical evidence that the SNA and the GFA approach will lead to slightly different results for certain values of the model parameters, specifying spin-glass behaviour, from the fourth time step onwards. This is consistent with our remark at the end of Section 3 that a difference, although a very small one, has been noticed between the SNA approach and simulations at zero temperature in the fourth time step of the parallel Hopfield dynamics [12]. This result was not entirely conclusive because of the possible entanglement with finite-size effects.

Reviewing the SNA approach one knows that one of the basic ingredients is the introduction of a modified residual overlap $\tilde{r}(t)$ which converges to a normal distribution in the thermodynamic limit (recall eqs. (32)-(33)).

We have done some numerical experiments for different values of the model parameters comparing this limiting normal distribution in the SNA method with simulations for different time steps. Some results are shown in Fig. 3. We compare the normal distribution with simulations for time steps $t = 2$ and $t = 9$ for systems with $N = 2000$ neurons averaged over 250 runs for the initial conditions $m_0 = l_0 = 0.6, q_0 = 0.5, a = 2/3$ and temperature $T = 0.2$ as a function of $\alpha$. We note that the critical $\alpha$ for these parameter values is $\alpha_c = 0.086$ [4].

We conclude that in the retrieval region ($\alpha_c < 0.086$) the simulations results coincide quite well with the limiting distribution, while in the spin-glass region, certainly from $\alpha \sim 0.11$ onwards, the results for $t = 9$ start diverting systematically. We remark that a set of analogous results can be obtained for $\tilde{s}(t)$. This may indicate that for $t > 3$
Figure 3: Simulations of the distribution of the residual overlap $\tilde{r}(t)$ for time steps 2 and 9, compared with the limiting normal distribution (full and dotted lines) as a function of the capacity $\alpha$. The simulations are done for $N = 2000$ neurons averaged over 250 samples. The initial conditions for the dynamics are $m_0 = l_0 = 0.6, q_0 = 0.5, a = 2/3$ and $T = 0.2$. 
the SNA works quite well in a large part of the retrieval phase since most of the time
the system reaches the attractor already after a few time steps. However, in the spin-glass
phase the SNA method does not estimate correctly the noise of the non-condensed patterns
represented by $\tilde{r}(t)$, suggesting that the assumption on the residual noise discussed above
is no longer fulfilled. Resolving these issues analytically is not immediately straightforward
and beyond the scope of the present work. They will be worked out first for the Hopfield
model [14].

6 Concluding remarks

The parallel dynamics of the fully connected BEG neural network model for finite temper-
atures has been solved using both the signal to noise analysis and the generating functional
approach. It is shown that both methods give identical results up to the first three time
steps.

Furthermore, by studying the distribution of the residual noise in the signal to noise
analysis numerically and comparing it with further time steps up to $t = 9$ it is found that
the signal to noise method as it is used in the literature does not estimate correctly the
modified residual overlaps in the spin-glass region. However, in a large part of the retrieval
region it can be considered as accurate. Numerical simulations confirm these findings. The
signal to noise approach can be made exact by carefully studying the long time correlations
[14].

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A The first three time steps in the SNA

Employing the general recursive scheme of Section 3 evolution equations are derived for the
first three time steps of the dynamics in the SNA approach. The following initial conditions
are taken

\begin{align}
m^1(0) &= m_0, \quad l^1(0) = l_0, \quad q(0) = q_0, \quad (80) \\
h_i(0) &= \frac{\xi^1_i}{a} m_0 + \mathcal{N}\left(0, \frac{\alpha q_0}{a^2}\right), \quad \theta_i(0) = \eta^1_i l_0 + \mathcal{N}\left(0, \frac{\alpha q_0}{a^2}\right), \quad (81) \\
D(0) &= \frac{q_0}{a^3}, \quad E(0) = \frac{q_0}{a} \quad (82)
\end{align}

From now we forget about the superscript 1 to indicate the condensed pattern. Further-
more, we do not write down all details but we refer to [1] for the analogous $T = 0$ case.
A.1 First time step

From [17]-[19] we immediately find

\[ m(1) = \frac{1}{a} \left\langle \xi \int Dz \int Dy V_\beta(h'_0(z), \theta'_0(y)) \right\rangle , \]  
\[ q(1) = \left\langle \int Dz \int Dy W_\beta(h'_0(z), \theta'_0(y)) \right\rangle , \]  
\[ l(1) = \left\langle \eta \int Dz \int Dy W_\beta(h'_0(z), \theta'_0(y)) \right\rangle , \]

where

\[ h'_0(z) = \frac{1}{a} \xi m_0 + \sqrt{\alpha a D(0)} z, \quad \theta'_0(y) = \eta l_0 + \sqrt{\frac{\alpha}{\tilde{a}}} E(0) y \]

and where \( Dx = dx \exp(-x^2/2) / \sqrt{2\pi} \) is the Gaussian measure. Because of the modification of the local fields we know that

\[ \text{Cov}[\tilde{r}_\mu(0), r_\mu(0)] = E[\tilde{r}_\mu(0)r_\mu(0)] = E\left[\frac{1}{a}\sigma_\gamma(0)g_{\Psi}(\tilde{h}_i^\mu(0), \tilde{\theta}_i^\mu(0))\right] \]  
\[ \text{Cov}[\tilde{s}_\mu(0), s_\mu(0)] = E[\tilde{s}_\mu(0)s_\mu(0)] = E\left[\frac{1}{a}\sigma_\gamma^2(0)g_{\Psi}^2(\tilde{h}_i^\mu(0), \tilde{\theta}_i^\mu(0))\right] \]

Using the recursion relations [24]-[26] we obtain

\[ D(1) = \frac{q(1)}{a^3} + \chi^2 h(0) D(0) + 2\chi h(0) R(1,0) \]  
\[ E(1) = \frac{q(1)}{a^3} + \chi^2(0) E(0) + 2\chi(0) S(1,0) , \]

where the correlation parameters \( R(t, t') \) and \( S(t, t') \) are defined by

\[ R(t, t') = \frac{1}{a^3} E\left[g_{\Psi}(\tilde{h}(t-1), \tilde{\theta}(t-1))g_{\Psi}(\tilde{h}(t'-1), \tilde{\theta}(t'-1))\right] \]  
\[ R(t, 0) = \frac{1}{a^3} E\left[g_{\Psi}(\tilde{h}(t-1), \tilde{\theta}(t-1))\sigma(0)\right] \]  
\[ S(t, t') = \frac{1}{a} E\left[g_{\Psi}^2(\tilde{h}(t-1), \tilde{\theta}(t-1))g_{\Psi}^2(\tilde{h}(t'-1), \tilde{\theta}(t'-1))\right] \]  
\[ S(t, 0) = \frac{1}{a} E\left[g_{\Psi}^2(\tilde{h}(t-1), \tilde{\theta}(t-1))\sigma^2(0)\right] \]

Since the density distribution of \( h \) and \( \tilde{h} \) are the same in the limit \( N \to \infty \) we have for the first time step

\[ R(1,0) = \frac{1}{a^3} \left\langle \sigma(0) \int Dz \int Dy V_\beta(h'_0(z), \theta'_0(y)) \right\rangle \]
\[ S(1, 0) = \frac{1}{a} \left\langle \left\langle \sigma^2(0) \int Dz \int Dy W_\beta(h'_0(z), \theta'_0(y)) \right\rangle \right\rangle \]  \quad (96)

Furthermore, since there are no feedback correlations at time zero
\[ \chi_h(0) = \frac{1}{a} \left\langle \left\langle \int Dz \int Dy \left\langle \frac{\partial V_\beta(h(t), \theta(t))}{\partial h(t)} \right\rangle \right\rangle \right\rangle \]  \quad (97)
\[ \chi_\theta(0) = \frac{1}{a} \left\langle \left\langle \int Dz \int Dy \left\langle \frac{\partial W_\beta(h(t), \theta(t))}{\partial \theta(t)} \right\rangle \right\rangle \right\rangle \]  \quad (98)

### A.2 Second time step

First, we need the distribution of the local fields at time step 1 using (24)-(27)
\[ h_i(1) = \frac{\xi_i}{a} m(1) + \frac{\alpha}{a} \chi_h(0) \sigma_i(0) + \mathcal{N}(0, \alpha a D(1)) \]  \quad (99)
\[ \theta_i(1) = \eta_i l(1) + \frac{\alpha}{a} \chi_\theta(0) \sigma_i^2(0) + \mathcal{N}(0, \alpha \tilde{a} E(1)) \]  \quad (100)

These results enable us to write down the important order parameters at time 2:
\[ m(2) = \frac{1}{a} \left\langle \left\langle \xi \int Dz \int Dy V_\beta(h'_1(z), \theta'_1(y)) \right\rangle \right\rangle \]  \quad (101)
\[ q(2) = \left\langle \left\langle \int Dz \int Dy W_\beta(h'_1(z), \theta'_1(y)) \right\rangle \right\rangle \]  \quad (102)
\[ l(2) = \left\langle \left\langle \eta \int Dz \int Dy W_\beta(h'_1(z), \theta'_1(y)) \right\rangle \right\rangle \]  \quad (103)

defining
\[ h'_1(z) = \frac{1}{a} \xi m(1) + \frac{\alpha}{a} \chi_h(0) \sigma(0) + \sqrt{\alpha a D(1)} z \]  \quad (104)
\[ \theta'_1(y) = \eta l(1) + \frac{\alpha}{a} \chi_\theta(0) \sigma^2(0) + \sqrt{\alpha \tilde{a} E(1)} y \]  \quad (105)

The calculation of the variance of the residual overlaps proceeds as follows. From the recursion relations (29)-(30) we get
\[ D(2) = \frac{q(2)}{a^3} + \chi_h^2(1) D(1) + 2 \chi_h(1)(R(2, 1) + \chi_h(0) R(2, 0)) \]  \quad (106)
\[ E(2) = \frac{q(2)}{a} + \chi_\theta^2(1) E(1) + 2 \chi_\theta(1)(S(2, 1) + \chi_\theta(0) S(2, 0)) \]  \quad (107)
where

\[ R(2,0) = \frac{1}{a^3} \left\langle \left\langle \sigma(0) \int Dz \int Dy V_\beta(h'_1(z), \theta'_1(y)) \right\rangle \right\rangle \]  

(108)

\[ S(2,0) = \frac{1}{a} \left\langle \left\langle \sigma^2(0) \int Dz \int Dy W_\beta(h'_1(z), \theta'_1(y)) \right\rangle \right\rangle \]  

(109)

To obtain the further correlations introduced we remark that because \( \sigma(1) \) appears in the expressions, we have to introduce the correlation between time steps 0 and 1. The relevant correlation coefficients are given by

\[ \rho_h(t, t') = \frac{E[(h(t) - M(t))(h(t') - M(t'))]}{\sqrt{aaD(t)}\sqrt{aaD(t')}} \]  

(110)

\[ \rho_\theta(t, t') = \frac{E[(\theta(t) - L(t))(\theta(t') - L(t'))]}{\sqrt{(\alpha/\tilde{a})E(t)\sqrt{(\alpha/\tilde{a})E(t')}}} \]  

(111)

and the joint distribution

\[ D\omega^a_b(z, y) = \frac{dz dy}{2\pi \sqrt{1 - \rho^2(a, b)}} \exp \left( -\frac{z^2 - 2zy\rho_x(a, b) + y^2}{2(1 - \rho^2(a, b))} \right) \]  

(112)

Then we arrive at

\[ R(2,1) = \frac{1}{a^3} \left\langle \left\langle \int D\omega^1_0(z, s) D\omega^1_0(y, t) V_\beta(h'_1(z), \theta'_1(y)) V_\beta(h'_0(s), \theta'_0(t)) \right\rangle \right\rangle \]  

(113)

\[ S(2,1) = \frac{1}{a} \left\langle \left\langle \int D\omega^1_0(z, s) D\omega^1_0(y, t) W_\beta(h'_1(z), \theta'_1(y)) W_\beta(h'_0(s), \theta'_0(t)) \right\rangle \right\rangle \]  

(114)

Finally, the susceptibilities at time step 1 are given by

\[ \chi_h(1) = \frac{1}{a} \left\langle \left\langle \int Dz \int Dy \left( \frac{\partial V_\beta(h(t), \theta(t))}{\partial h(t)} \right) \bigg|_{h'_1(z), \theta'_1(y)} \right\rangle \right\rangle \]  

(115)

\[ \chi_\theta(1) = \frac{1}{a} \left\langle \left\langle \int Dz \int Dy \left( \frac{\partial W_\beta(h(t), \theta(t))}{\partial \theta(t)} \right) \bigg|_{h'_1(z), \theta'_1(y)} \right\rangle \right\rangle \]  

(116)

### A.3 Third time step

We have all the quantities needed to write down the local fields at time \( t = 2 \)

\[ h_i(2) = \frac{\xi}{a} m(2) + \frac{\alpha}{a} \chi_h(1)(\sigma_i(1) + \chi_h(0)\sigma(0)) + \mathcal{N}(0, \alpha a D(2)) \]  

(117)

\[ \theta_i(2) = \eta l(2) + \frac{\alpha}{a} \chi_\theta(1)(\sigma_i^2(1) + \chi_\theta(0)\sigma^2(0)) + \mathcal{N}(0, \frac{\alpha}{a} E(2)) \]  

(118)
This gives for the order parameters

\[ m(3) = \frac{1}{a} \left\langle \xi \int D\omega^2_0(z,s) D\omega^2_0(y,t) \sum_{\sigma(1)} V_\beta(h'_2(z),\theta'_2(y)) U_\beta(h'_0(s),\theta'_0(t),\sigma(1)) \right\rangle \]  

(119)

\[ q(3) = \left\langle \int D\omega^2_0(z,s) D\omega^2_0(y,t) \sum_{\sigma(1)} W_\beta(h'_2(z),\theta'_2(y)) U_\beta(h'_0(s),\theta'_0(t),\sigma(1)) \right\rangle \]  

(120)

\[ l(3) = \left\langle \eta \int D\omega^2_0(z,s) D\omega^2_0(y,t) \sum_{\sigma(1)} W_\beta(h'_2(z),\theta'_2(y)) U_\beta(h'_0(s),\theta'_0(t),\sigma(1)) \right\rangle \]  

(121)

with

\[ U_\beta(h'_0(s),\theta'_0(t),\sigma(1)) = \frac{\exp[\beta\sigma(1)h'_0(s) + \beta\sigma^2(1)\theta'_0(t)]}{2 \exp(\beta\theta'_0(t)) \cosh(\beta h'_0(s)) + 1} \]  

(122)

and

\[ h'_2(z) = \frac{\xi}{a} m(2) + \frac{\alpha}{a} \chi(1)(\sigma(1) + \chi(0)\sigma(0)) + \sqrt{\alpha a D(2)} z \]  

(123)

\[ \theta'_2(y) = \eta l(2) + \frac{\alpha}{a} \chi(1)(\sigma^2(1) + \chi(0)\sigma^2(0)) + \sqrt{\frac{\alpha}{a} E(2)} y \]  

(124)

In an analogous way further time steps could be calculated.

**B The complete GFA calculation**

Starting from the path with the transition probabilities given by the generating function can be written as

\[ Z[\Psi,\Phi] = \sum_{\sigma(0)} \sum_{\sigma(t)} \text{Prob}(\sigma(0)) \prod_{i=1}^{N} \prod_{s=1}^{t} \exp[-i\sigma_i(s) (\psi_i(s) + i\beta h_i(\sigma(s-1)))] \]

\[ \times \exp[-i\sigma_i^2(s) (\phi_i(s) + i\beta \theta_i(\sigma(s-1)))] \]

\[ \times \exp[-\ln(2 \exp(\beta \theta_i(\sigma(s-1))) \cosh(\beta h_i(\sigma(s-1))))) + 1)] \]  

(125)

20
where we have left out the subscript $N$. We then isolate the local fields by inserting the appropriate delta distributions

$$Z[Ψ, Φ] = \sum_{σ(0)} ... \sum_{σ(t)} \text{Prob}(σ(0)) \int \frac{dh \hat{h} dθ d\hat{θ}}{(2π)^{2N(t+1)}}$$

$$\times \prod_{s=1}^{t} \left\{ \exp \left[ i \sum_{i=1}^{N} \hat{h}_i(s)(h_i(s) - γ_{h,i}(s)) + i \sum_{i=1}^{N} \hat{θ}_i(s)(θ_i(s) - γ_{θ,i}(s)) \right] \right.$$ 

$$\times \exp \left[ -i \sum_{i=1}^{N} σ_i(s)(ψ_i(s) + iβh_i(s - 1)) - i \sum_{i=1}^{N} σ_i^2(s)(ϕ_i(s) + iβθ_i(s - 1)) \right]$$

$$\times \exp \left[ -\ln\{2 \exp(βθ_i(s - 1)) \cosh(βh_i(s - 1)) + 1\} \right] \right\} \right.$$ 

$$\times \exp \left[ i \sum_{i=1}^{N} \hat{h}_i(0)h_i(0) + i \sum_{i=1}^{N} \hat{θ}_i(0)θ_i(0) \right] \right.$$ 

$$\times \exp [F]$$

where $\exp [F]$ includes all $J$ and $K$ dependent terms

$$\exp [F] \equiv \exp \left[ -iN \sum_{s=0}^{t} \left( \frac{1}{N} \sum_{i=1}^{N} \hat{h}_i(s) \right) \left( \sum_{j=1}^{N} J_{ij}σ_j(s) \right) \right]$$

$$\times \exp \left[ -iN \sum_{s=0}^{t} \left( \frac{1}{N} \sum_{i=1}^{N} \hat{θ}_i(s) \right) \left( \sum_{j=1}^{N} K_{ij}σ_j^2(s) \right) \right]$$

$$\times \exp \left[ i \sum_{s=0}^{t} \sum_{i=1}^{N} \left( \hat{h}_i(s)J_{ii}σ_i(s) + \hat{θ}_i(s)K_{ii}σ_i^2(s) \right) \right]$$

(127)

We remark that in the expression above, the diagonal terms are excluded by the introduction of the last factor, which we denote by $M$.

We first concentrate on the terms containing the disorder, i.e., the terms containing $J_{ij}$ and $K_{ij}$, of which we need to know the limit $N \to \infty$. Using the learning rule $\text{(6)}$ and taking for simplicity one condensed pattern again, say $μ = 1$, we can study the condensed and the non-condensed part of the noise (127) by splitting

$$\exp [F] = \exp [F_c + F_{nc}] .$$

(128)

With respect to the condensed pattern it is seen that the following single time variables...
are relevant

\[ m(s) \equiv \frac{1}{aN} \sum_{i=1}^{N} \xi_{i}^1 \sigma_{i}(s) \quad k_{h}(s) \equiv \frac{1}{aN} \sum_{i=1}^{N} \xi_{i}^1 \hat{h}_{i}(s) \]  \hspace{1cm} (129)\\

\[ l(s) \equiv \frac{1}{N} \sum_{i=1}^{N} \eta_{i}^1 \sigma_{i}^2(s) \quad k_{\theta}(s) \equiv \frac{1}{N} \sum_{i=1}^{N} \eta_{i}^1 \hat{\theta}_{i}(s) \]  \hspace{1cm} (130)

To achieve site factorisation we then introduce appropriate delta distributions for (129)-(130) and their conjugated variables denoted with a hat. Consequently,

\[
\exp \left[ \mathcal{F}_{c} \right] = \int dm \hat{m}dk_{h} \hat{k}_{h} dk_{\theta} \hat{k}_{\theta} \frac{N^{4(t+1)}}{(2\pi)^{4(t+1)}} \\
\times \exp \left\{ -iN \left( k_{h} \cdot m + k_{\theta} \cdot l \right) \right\} \\
\times \exp \left\{ -iN \left( \hat{m} \cdot m + \hat{k}_{h} \cdot k_{h} + \hat{l} \cdot 1 + \hat{k}_{\theta} \cdot k_{\theta} \right) \right\} \\
\times \exp \left\{ -i \sum_{s=0}^{t} \sum_{i=1}^{N} \left( \hat{m}(s) \frac{\xi_{i}^1}{a} \sigma_{i}(s) + \hat{k}_{h}(s) \frac{\xi_{i}^1}{a} \hat{h}_{i}(s) + \hat{l}(s) \eta_{i}^1 \sigma_{i}^2(s) + \hat{k}_{\theta}(s) \eta_{i}^1 \hat{\theta}_{i}(s) \right) \right\}  \hspace{1cm} (131)
\]

With respect to the non-condensed patterns, we first introduce gaussian integration

\[
Dx \equiv \prod_{s=0}^{t} ((2\pi)^{-1/2} dx(s) \exp \left\{ -\frac{1}{2} x(s)^{2} \right\})
\]

to arrive at (we forget about superindex 1 for the condensed pattern)

\[
\exp \left[ \mathcal{F}_{nc} \right] = \left\langle \int Dx_{1} Dx_{2} Dy_{1} Dy_{2} \\
\times \exp \left\{ \frac{1}{\sqrt{2iN}} \sum_{i=1}^{N} \xi_{i} \sum_{s=0}^{t} \left( x_{1}(\hat{h}_{i}(s) + \sigma_{i}(s)) + iy_{1}(s)(\hat{h}_{i}(s) - \sigma_{i}(s)) \right) \right\} \\
\times \exp \left\{ \frac{1}{\sqrt{2iN}} \sum_{i=1}^{N} \eta_{i} \sum_{s=0}^{t} \left( x_{2}(\hat{\theta}_{i}(s) + \sigma_{i}^{2}(s)) + iy_{2}(s)(\hat{\theta}_{i}(s) - \sigma_{i}^{2}(s)) \right) \right\} \right\rangle^{p-1} \\
\times \exp \left\{ i\alpha \sum_{s,s'=0}^{N} \sum_{i=1}^{N} \left( \frac{1}{a} \hat{h}_{i}(s') \sigma_{i}(s) + \frac{1}{a} \hat{\theta}_{i}(s') \sigma_{i}^{2}(s) \right) \delta_{ss'} \right\}  \hspace{1cm} (132)
\]

with clear notation. We have also rewritten \( M \) in a more suitable way. Performing the average over the patterns and expanding the exponentials up to order \( O(N^{-1}) \) it is seen
that the following two time variables turn up

\[
q(s, s') = \frac{1}{Na} \sum_{i=1}^{N} \sigma_i(s)\sigma_i(s') \quad Q_h(s, s') = \frac{1}{Na} \sum_{i=1}^{N} \hat{h}_i(s)\hat{h}_i(s')
\] (133)

\[
p(s, s') = \frac{1}{Na} \sum_{i=1}^{N} \sigma_i^2(s)\sigma_i^2(s') \quad Q_{\theta}(s, s') = \frac{1}{Na} \sum_{i=1}^{N} \hat{\theta}_i(s)\hat{\theta}_i(s')
\] (134)

\[
K_h(s, s') = \frac{1}{Na} \sum_{i=1}^{N} \sigma_i(s)\hat{h}_i(s') \quad K_{\theta}(s, s') = \frac{1}{Na} \sum_{i=1}^{N} \sigma_i^2(s)\hat{\theta}_i(s')
\] (135)

Introducing again appropriate delta distributions we obtain after some matrix algebra

\[
\exp[\mathcal{F}_{nc} = \left(\frac{N}{2\pi}\right)^{12(t+1)} \int dqd\hat{q}dQ_hd\hat{Q}_hdK_hd\hat{K}_hdpd\hat{p}dQ_{\theta}d\hat{Q}_{\theta}dK_{\theta}d\hat{K}_{\theta}
\]

\[
\times \exp \left[iN\text{Tr} \left(\hat{q}Q_h + \hat{Q}_hK_h^\dagger + \hat{K}_hQ_h^\dagger + \hat{p}\hat{p}^\dagger + \hat{Q}_{\theta}Q_{\theta}^\dagger + \hat{K}_{\theta}K_{\theta}^\dagger\right)\right]
\]

\[
\times \exp \left[i\alpha N\text{Tr} \left(K_h + K_{\theta}\right)\right]
\]

\[
\times \exp \left[-i \sum_{s, s'} \sum_{i=1}^{N} \left(\hat{q}(s, s')\sigma_i(s)\sigma_i(s') + \hat{Q}_h(s, s')\hat{h}_i(s)\hat{h}_i(s') + \hat{K}_h(s, s')\sigma_i(s)\hat{h}_i(s')\right)\right]
\]

\[
\times \exp \left[-i \sum_{s, s'} \sum_{i=1}^{N} \left(\hat{p}(s, s')\sigma_i^2(s)\sigma_i^2(s') + \hat{Q}_{\theta}(s, s')\hat{\theta}_i(s)\hat{\theta}_i(s') + \hat{K}_{\theta}(s, s')\sigma_i^2(s)\hat{\theta}_i(s')\right)\right]
\]

\[
\times \exp \left[-\frac{p}{2} \left[\ln \left(Q_hq - (iI - K_h^\dagger)(iI - K_h)\right) + \ln \left(Q_{\theta}p - (iI - K_{\theta}^\dagger)(iI - K_{\theta})\right)\right]\right]
\]

(136)

with \(I\) the unit matrix of dimension \((t+1)\).

This concludes the discussion of the averaging over the disorder with as result a generating functional of the form \(Z \sim \exp N...\) which can be evaluated in the limit \(N \to \infty\) via the saddle-point method.

\[
\overline{Z[\Psi, \Phi]} \propto \int \{\ldots\} \exp \left[N \left(\alpha \ln \Omega + \Xi + \Lambda\right)\right]
\] (137)

with

\[
\Omega = \left(|Q_hq - (iI - K_h^\dagger)(iI - K_h)||Q_{\theta}p - (iI - K_{\theta}^\dagger)(iI - K_{\theta})|\right)^{-\frac{1}{2}}
\] (138)

\[
\Xi = i \left(\hat{m} \cdot m + \hat{K}_h \cdot k_h + k_h \cdot m + \hat{I} \cdot 1 + \hat{K}_{\theta} \cdot k_{\theta} + k_{\theta} \cdot 1\right)
\]

\[
+ i\text{Tr} \left(\hat{q}Q_h^\dagger + \hat{Q}_hK_h^\dagger + \hat{p}\hat{p} + \hat{Q}_{\theta}Q_{\theta}^\dagger + \hat{K}_{\theta}K_{\theta}^\dagger + \alpha(K_h + K_{\theta})\right)
\] (139)
\[ \Lambda = \frac{1}{N} \sum_{i=1}^{N} \ln \left\{ \int d\mathbf{h} d\mathbf{\theta} d\hat{\mathbf{\theta}} \sum_{\sigma(0)\ldots\sigma(t)} \text{Prob}(\sigma(0)) \right\} \]
\[
\times \exp \left[ i \sum_{s=0}^{t} \hat{h}_i(s)(h_i(s) - \gamma_{h,i}(s) - \hat{k}_h(s) \xi_i) \right] \]
\[
\times \exp \left[ i \sum_{s=0}^{t} \hat{\theta}_i(s)(\theta_i(s) - \gamma_{\theta,i}(s) - \hat{k}_\theta(s) \eta_i) - \sum_{s=1}^{t} \ln \left( 2 \exp \left( \beta \theta(s-1) \cosh(\beta h(s-1)) + 1 \right) \right) \right] \]
\[
\times \exp \left[ -i \sum_{s=0}^{t} \left( \sigma(s)(\psi_i(s) + m(s) \xi_i + i\beta h(s - 1)) + \sigma^2(s)(\phi_i(s) + l(s) \eta_i + i\beta \theta(s - 1)) \right) \right] \]
\[
\times \exp \left[ -\frac{i}{a} \sum_{s,s'=0}^{t} \hat{q}(s,s')\sigma(s)\sigma(s') + \hat{K}_h(s,s')\sigma(s)\hat{h}(s') + \hat{Q}_h(s,s')\hat{h}(s)\hat{h}(s') \right] \]
\[
\times \exp \left[ -\frac{i}{a} \sum_{s,s'=0}^{t} \hat{p}(s,s')\sigma^2(s)\sigma^2(s') + \hat{K}_\theta(s,s')\sigma^2(s)\hat{\theta}(s') + \hat{Q}_\theta(s,s')\hat{\theta}(s)\hat{\theta}(s') \right] \right\} \tag{140} \]

At this point, some remarks are in order. First, we have been able to factorize the generating functional with respect to the sites. Further, the external fields are only present in \( \Lambda \). Therefore it is useful to introduce the effective single spin measure

\[
\langle f \rangle_{s,i} = \frac{\int d\mathbf{h} d\mathbf{\theta} d\hat{\mathbf{\theta}} \sum_{\sigma(0)\ldots\sigma(t)} \text{Prob}(\sigma(0)) M_i \left[ \sigma, \xi_i, h, \hat{h}, \theta, \hat{\theta} \right] f}{\int d\mathbf{h} d\mathbf{\theta} d\hat{\mathbf{\theta}} \sum_{\sigma(0)\ldots\sigma(t)} \text{Prob}(\sigma(0)) M_i \left[ \sigma, \xi_i, h, \hat{h}, \theta, \hat{\theta} \right]} \tag{141} \]
where the auxiliary fields $\psi_i$ and $\phi_i$ have been taken to be zero and $\mathcal{M}_i$ is given by

$$
\mathcal{M}_i[...] = \exp \left[ i \sum_{s=0}^{t} \hat{h}(s)(h(s) - \gamma_h(s) - m(s)\frac{\xi_s}{a}) + i \sum_{i=1}^{N} \hat{\theta}(s)(\theta(s) - \gamma_\theta(s) - l(s)\eta_i) \right]
\times \exp \left[ -i \sum_{s=1}^{t} \sigma(s) \left( i\beta h(s - 1) + k_h(s)\frac{\xi_s}{a} \right) + \sigma^2(s) \left( i\beta \theta(s - 1) + k_\theta(s)\eta_i \right) \right]
\times \exp \left[ -i \sum_{s=1}^{t} \sum_{s',s''} \hat{q}(s,s')\sigma(s)\sigma(s') + \hat{Q}_h(s,s')\hat{h}(s') + \hat{K}_h(s,s')\sigma(s)\hat{h}(s') \right]
\times \exp \left[ -\frac{i}{a} \sum_{s,s'} \left( \hat{p}(s,s')\sigma^2(s') + \hat{Q}_\theta(s,s')\hat{\theta}(s') + \hat{K}_\theta(s,s')\sigma^2(s')\hat{\theta}(s') \right) \right]
$$

(142)

The parameters $m, k_h, l, k_\theta, \hat{Q}_h, \hat{K}_h, \hat{p}, \hat{Q}_\theta, \hat{K}_\theta$ in $\mathcal{M}_i[...]$ are defined as the solutions of the saddle-point equation $d(\Xi + \Lambda) = 0$. Furthermore, we have dropped the subindices $i$ from the external fields since they are taken to be site-independent.

Before determining the saddle-point solutions we write down the following identities

$$
\lim_{\psi, \phi \to 0} \frac{\partial Z[\Psi, \Phi]}{\partial \psi_i(s)} = \lim_{\psi, \phi \to 0} \frac{\partial Z[\Psi, \Phi]}{\partial \phi_i(s)} = \int d\mathcal{M}...d\mathcal{K}_\theta e^{N(\Xi + \Lambda) + \alpha N \ln(\Omega) + O(\ln(N))} \left[ N \frac{\partial \Lambda}{\partial \psi_i(s)} \right] = -i\langle \sigma(s) \rangle_{s,i}
$$

(143)

$$
\lim_{\psi, \phi \to 0} \frac{\partial^2 Z[\Psi, \Phi]}{\partial \psi_i(s) \partial \psi_j(s')} = -\delta_{ij} \left[ \langle \sigma(s)\sigma(s') \rangle_{s,i} - \langle \sigma(s) \rangle_{s,i} \langle \sigma(s') \rangle_{s,j} \right] - \langle \sigma(s) \rangle_{s,i} \langle \sigma(s') \rangle_{s,j}
$$

(145)

$$
\lim_{\psi, \phi \to 0} \frac{\partial^2 Z[\Psi, \Phi]}{\partial \phi_i(s) \partial \phi_j(s')} = -\delta_{ij} \left[ \langle \sigma^2(s)\sigma^2(s') \rangle_{s,i} - \langle \sigma^2(s) \rangle_{s,i} \langle \sigma^2(s') \rangle_{s,j} \right] - \langle \sigma^2(s) \rangle_{s,i} \langle \sigma^2(s') \rangle_{s,j}
$$

(146)
where we recall that the overline denotes the disorder average and where we used the LLN, which leads to
\[
\langle h^\hat{\Psi}, s \rangle = \langle h \hat{\Psi}, s \rangle + \langle h^\Phi, s \rangle.
\]
Recalling (61) we then have obtained the following identities
\[
\langle \sigma_i(s) \rangle = \langle \sigma(s) \rangle_{s,i}
\]
\[
\langle \sigma_i^2(s) \rangle = \langle \sigma^2(s) \rangle_{s,i}
\]
\[
C_{h,ij}(s, s') = \delta_{ij} \langle \sigma(s)\sigma(s') \rangle_{s,i} + (1 - \delta_{ij}) \langle \sigma(s) \rangle_{s,i} \langle \sigma(s') \rangle_{s,j}
\]
\[
C_{\theta,ij}(s, s') = \delta_{ij} \langle \sigma^2(s)\sigma^2(s') \rangle_{s,i} + (1 - \delta_{ij}) \langle \sigma^2(s) \rangle_{s,i} \langle \sigma^2(s') \rangle_{s,j}
\]
\[
G_{h,ij}(s, s') = \delta_{ij} \langle \sigma(s)\hat{h}(s') \rangle_{s,i}
\]
\[
G_{\theta,ij}(s, s') = \delta_{ij} \langle \sigma^2(s)\hat{\theta}(s') \rangle_{s,i}
\]
from which we obtain by using the LLN
\[
C_h(s, s') \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} C_{h,ii}(s, s') = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \langle \sigma(s)\sigma(s') \rangle_{s,i} = \langle \langle \sigma(s)\sigma(s') \rangle_{s,i} \rangle_{\xi}
\]
\[
C_\theta(s, s') \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} C_{\theta,ii}(s, s') = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \langle \sigma^2(s)\sigma^2(s') \rangle_{s,i} = \langle \langle \sigma^2(s)\sigma^2(s') \rangle_{s,i} \rangle_{\xi}
\]
\[
G_h(s, s') \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} G_{h,ii}(s, s') = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \gamma_{h,i}(s')} \langle \sigma(s) \rangle_{s,i} = -i \langle \langle \sigma(s)\hat{h}(s') \rangle_{s,i} \rangle_{\xi}
\]
\[
G_\theta(s, s') \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} G_{\theta,ii}(s, s') = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \gamma_{\theta,i}(s')} \langle \sigma^2(s) \rangle_{s,i} = -i \langle \langle \sigma^2(s)\hat{\theta}(s') \rangle_{s,i} \rangle_{\xi}
\]
where $\langle \cdot \rangle_\xi$ denotes the average over the condensed pattern. Finally, we write down the saddle-point equations. Variation with respect to $m, k_h, l$ and $k_\theta$ gives

$$\hat{m}(s) = k_h(s) \quad \hat{k}_h(s) = m(s) \quad \hat{l}(s) = k_\theta(s) \quad \hat{k}_\theta(s) = l(s) \quad (162)$$

Next, we calculate the variation with respect to $\hat{m}, \hat{k}_h, \hat{l}$ and $\hat{k}_\theta$, leading to

$$m(s) = \langle \xi (\sigma(s))_s \rangle_\xi \quad k_h(s) = \langle \xi (\hat{h}(s))_s \rangle_\xi = 0 \quad (163)$$
$$l(s) = \langle \eta (\sigma^2(s))_s \rangle_\xi \quad k_\theta(s) = \langle \eta (\hat{\theta}(s))_s \rangle_\xi = 0 \quad (164)$$

Variation with respect to $\hat{q}, \hat{Q}_h$ and $\hat{K}_h$ gives

$$q(s, s') = \frac{1}{a} C_h(s, s') \quad Q_h(s, s') = 0 \quad K_h(s, s') = \frac{i}{a} G_h(s, s') \quad (165)$$

and with respect to $\hat{p}, \hat{Q}_\theta$ and $\hat{K}_\theta$

$$p(s, s') = \frac{1}{a} C_\theta(s, s') \quad Q_\theta(s, s') = 0 \quad K_\theta(s, s') = \frac{i}{a} G_\theta(s, s') \quad (166)$$

Finally, variation with respect to of $q, Q_h, K_h, p, Q_\theta$ and $K_\theta$ leads to, after some algebra

$$\tilde{q}(s, s') = 0 \quad \tilde{p}(s, s') = 0 \quad (167)$$
$$\tilde{Q}_h(s, s') = -\frac{ia}{2} \left[ (aI - G_h)^{-1} C_h \left( aI - G_h^\dagger \right)^{-1} \right] (s, s') \quad (168)$$
$$\tilde{K}_h(s, s') = \frac{ia}{2} \left[ (aI - G_h)^{-1} \right] (s, s') \quad (169)$$
$$\tilde{Q}_\theta(s, s') = -\frac{ia}{2} \left[ (aI - G_\theta)^{-1} C_\theta \left( aI - G_\theta^\dagger \right)^{-1} \right] (s, s') \quad (170)$$
$$\tilde{K}_\theta(s, s') = \alpha \left[ (aI - G_\theta)^{-1} \right] (s, s') \quad (171)$$

After the introduction of the saddle-point solutions, the single spin measure is expressed in terms of the physical order parameters, $\{m, l, C_h, C_\theta, G_h, G_\theta\}$. Rewriting the average in the following way

$$\langle f[\sigma(0), \ldots, \sigma(t)] \rangle_\xi \equiv \left\langle \int dh(0) \ldots dh(t-1) d\theta(0) \ldots d\theta(t-1) \sum_{\sigma(0) \ldots \sigma(t)} \text{Prob} \left[ h(0), \ldots, h(t-1), \theta(0), \ldots, \theta(t-1); \sigma(0), \ldots, \sigma(t) \right] f[\sigma(0), \ldots, \sigma(t)] \right\rangle_\xi$$

27
with

\[
\text{Prob}[...] \sim \text{Prob}(\sigma(0)) \int \prod_{s=0}^{t-1} dz_h(s) dz_\theta(s)
\]

\[
\times \delta \left[ h(s) - \gamma_h(s) - m(s) \xi - \alpha \sum_{s'=0}^{t} \left( (aI - G_h^{-1} \sigma(s') + \frac{\alpha}{a} \sigma(s) - \sqrt{\alpha} z_h(s) \right) \right] \cdot \delta \left[ \theta(s) - \gamma_\theta(s) - l(s) \eta - \alpha \sum_{s'=0}^{t} \left( (\tilde{a}I - G_\theta^{-1} \sigma^2(s') + \frac{\alpha}{a} \sigma^2(s) - \sqrt{\alpha} z_\theta(s) \right) \right] \cdot \exp \left( \frac{\beta}{2} \sum_{s, s'=0}^{t} \left( (aI - G_h^{-1} \sigma(s') - 1 \right) \cdot (aI - G_h^+ \sigma(s') \cdot (s, s') h(s) h(s') \right) \cdot \exp \left( \frac{\beta}{2} \sum_{s, s'=0}^{t} \left( (\tilde{a}I - G_\theta^{-1} \sigma^2(s') - 1 \right) \cdot (\tilde{a}I - G_\theta^+ \sigma^2(s') \cdot (s, s') \hat{\theta}(s) \hat{\theta}(s') \right) \cdot \exp \left( i \sqrt{\alpha} \sum_{s=0}^{t} \left( \hat{h}(s) z_h(s) + \hat{\theta}(s) z_\theta(s) \right) \right) \right]
\]

(172)

the integration over \( \hat{h} \) and \( \hat{\theta} \) is easy since the exponents multiplying the terms \( \hat{h}(s) \hat{h}(s') \) and \( \hat{\theta}(s) \hat{\theta}(s') \) are symmetric matrices. Doing the integration and including the normalization factors, we find the results of Section 4.

C First two time steps in the GFA

We use the GFA results to calculate the first two time steps of the dynamics explicitly. The initial conditions are chosen as follows: \( \gamma(0) = \beta(0) = 0 \), i.e., no external field, \( m(0) = m_0 \), \( l(0) = l_0 \), \( C_h(0, 0) = C_\theta(0, 0) = q_0 \), \( G_h(0, 0) = G_\theta(0, 0) = G_h(0, 1) = G_\theta(0, 1) = 0 \).

C.1 First time step

For time 1, the relevant probabilities are

\[
w[z_h(0), z_\theta(0)] = \frac{a \tilde{a}}{2 \pi q_0} \exp \left[ -\frac{a^2}{2q_0} (z_h(0))^2 - \frac{\tilde{a}^2}{2q_0} (z_\theta(0))^2 \right] \quad (173)
\]

\[
\text{Prob}[\sigma(0), \sigma(1)|z_h(0), z_\theta(0)] = \text{Prob}(\sigma(0)) \frac{\exp \left( \beta \sigma(1) h(0) + \beta \sigma^2(1) \theta(0) \right)}{2 \exp (\beta \theta(0)) \cosh(\beta h(0)) + 1} \quad (174)
\]
with \( h(0) = \xi m_0/a + \sqrt{\alpha} z_h(0) \) and \( \theta(0) = l_0/\eta + \sqrt{\alpha} z_\theta(0) \). Then, using (74) and (75),

\[
m(1) = \frac{1}{a} \langle \langle \xi \sigma(1) \rangle \rangle = \int \int Dz_h(0) Dz_\theta(0) \left\{ \frac{\exp(\beta \theta^+(0)) \sinh(\beta h^+(0))}{2 \exp(\beta \theta^+(0)) \cosh(\beta h^+(0)) + 1} \right. \\
+ \left. \frac{\exp(\beta \theta^+(0)) \sinh(\beta h^-(0))}{2 \exp(\beta \theta^+(0)) \cosh(\beta h^-(0)) + 1} \right\}
\]

(175)

with, as usual, \( Dz \) a gaussian measure defined in (173) and where \( h^\pm(0) = \pm m_0/a + \sqrt{\alpha} z_h(0) \) and \( \theta^+(0) = l_0/\eta + \sqrt{\alpha} z_\theta(0) \). In the same way we find the values for \( l(1) \) and \( q(1) \), i.e. \( l(1) = \langle \langle \eta \sigma^2(1) \rangle \rangle \) and \( q(1) = \langle \langle \sigma^2(1) \rangle \rangle \).

The correlation and response functions are

\[
C_{h}(1,0) = C_{h}(0,1) = \langle \langle \sigma(0) \sigma(1) \rangle \rangle \\
C_{\theta}(1,0) = C_{\theta}(0,1) = \langle \langle \sigma(0) \sigma(1) \rangle \rangle \\
G_{h}(1,0) = \beta \langle \langle \sigma(1) [\sigma(1) - V_\beta(h(0), \theta(0))] \rangle \rangle \\
G_{\theta}(t,t') = \beta \langle \langle \sigma^2(1) [\sigma^2(1) - W_\beta(h(0), \theta(0))] \rangle \rangle
\]

(176) \hspace{1cm} (177) \hspace{1cm} (178) \hspace{1cm} (179)

C.2 Second time step

In this time step correlations start to appear.

\[
(aI - G_{h}^{t=2})^{-1} = \frac{1}{a^2} \left( \begin{array}{cc} a & 0 \\ G_{h}(1,0) & a \end{array} \right) \\
(\tilde{a}I - G_{\theta}^{t=2})^{-1} = \frac{1}{\tilde{a}^2} \left( \begin{array}{cc} \tilde{a} & 0 \\ G_{\theta}(1,0) & \tilde{a} \end{array} \right)
\]

The explicit single-site average for time 2 is determined by

\[
w[z_h(0), z_h(1), z_\theta(0), z_\theta(1)] = \frac{1}{4\pi^2 \sqrt{(q_0 A_h - B_h^2)(q_0 A_\theta - B_\theta^2)}} \\
\times \exp \left[ -\frac{1}{2 C_h} (z_h(0))^2 A_h + z_h(1)^2 q_0 - 2 z_h(0) z_h(1) B_h \right] \\
\times \exp \left[ -\frac{1}{2 C_\theta} (z_\theta(0))^2 A_\theta + z_\theta(1)^2 q_0 - 2 z_\theta(0) z_\theta(1) B_\theta \right]
\]

(180)

\[
\text{Prob}\{\sigma(0), \sigma(1) | z_h(0), z_\theta(0)\} = \text{Prob}\{\sigma(0)\} \frac{\exp(\beta \sigma(1) h(0) + \beta \sigma^2(1) \theta(0))}{2 \exp(\beta \theta(0)) \cosh(\beta h(0)) + 1} \\
\times \frac{\exp(\beta \sigma(2) h(1) + \beta \sigma^2(2) \theta(1))}{2 \exp(\beta \theta(1)) \cosh(\beta h(1)) + 1}
\]

(181)
with the local fields at time $t = 1$ given by

\begin{align}
    h(1) &= \frac{\xi}{a} m(1) + \frac{\alpha}{a^2} G_h(1,0) \sigma(0) + \sqrt{\alpha} z_h(1) \\
    \theta(1) &= l(1) \eta + \frac{\alpha}{a^2} G_h(1,0) \sigma^2(0) + \sqrt{\alpha} z_\theta(1)
\end{align}

and the coefficients defined by

\begin{align}
    A_h &\equiv a^2 q(1) + 2a C_h(1,0) G_h(1,0) + q_0 (G_h(1,0))^2 \\
    B_h &\equiv a^2 C_h(1,0) + a q_0 G_h(1,0) \\
    C_h &\equiv q_0 q(1) - C_h^2(1,0) \\
    A_\theta &\equiv \tilde{a}^2 q(1) + 2\tilde{a} C_\theta(1,0) G_\theta(1,0) + q_0 (G_\theta(1,0))^2 \\
    B_\theta &\equiv \tilde{a}^2 C_\theta(1,0) + \tilde{a} q_0 G_\theta(1,0) \\
    C_\theta &\equiv q_0 q(1) - C_\theta^2(1,0)
\end{align}

From these relations any quantity can be computed. We present the final expressions for $m(2)$, $l(2)$ and $q(2)$.

\begin{equation}
    m(2) = \frac{1}{a} \langle \xi \sigma(2) \rangle
\end{equation}

\begin{equation}
    = \frac{\tilde{a}}{2\pi \sqrt{A_h A_\theta}} \int d z_h(1) d z_\theta(1) \exp \left[ -\frac{1}{2} \left( \frac{(z_h(1))^2}{A_h} + \frac{(z_\theta(1))^2}{A_\theta} \right) \right] \\
    \times \left\{ A \beta(h^{++}(1), \theta^{++}(1)) - C \beta(h^{-+}(1), \theta^{++}(1)) + D \beta(h^{+0}(1), \theta^{+0}(1)) \\
    - D \beta(h^{-0}(1), \theta^{+0}(1)) + C \beta(h^{+-(1), \theta^{++}(1)} - A \beta(h^{--}(1), \theta^{++}(1)) \right\}
\end{equation}

where the coefficients $A$, $C$ and $D$ coming from the distribution

\begin{equation}
    \text{Prob}(\sigma(0), \xi) = A \delta(\sigma(0) - 1) \delta(\xi - 1) + B \delta(\sigma(0) - 1) \delta(\xi) \\
    + C \delta(\sigma(0) - 1) \delta(\xi + 1) + D \delta(\sigma(0)) \delta(\xi - 1) + E \delta(\sigma(0)) \delta(\xi) \\
    + D \delta(\sigma(0)) \delta(\xi + 1) + C \delta(\sigma(0) + 1) \delta(\xi - 1) \\
    + B \delta(\sigma(0) + 1) \delta(\xi) + A \delta(\sigma(0) + 1) \delta(\xi + 1)
\end{equation}

defined by

\begin{align}
    A &= \frac{a}{4} (m_0 + (1 - a) l_0 + q_0) \\
    B &= \frac{1 - a}{2} (q_0 - a l_0) \\
    C &= \frac{a}{4} (-m_0 + (1 - a) l_0 + q_0) \\
    D &= \frac{a}{2} (1 - (1 - a) l_0 + q_0) \\
    E &= (1 - a) (1 + a l_0 - q_0)
\end{align}
The local field contributions appearing in these final equations have the following form, due to the average over the initial conditions

\[ h^{u,v}(1) = u \frac{m(1)}{a} + v \frac{\alpha}{a^2} G_h(1,0) + \sqrt{\alpha} z_h(1) \] (198)

\[ \theta^{x,y}(1) = \frac{x-a}{a} l(1) + y \frac{\alpha}{a^2} G_\theta(1,0) + \sqrt{\alpha} z_\theta(1) \] (199)

where \( u, v \in \{-1, 0, 1\} \) and \( x, y \in \{0, 1\} \).

In the same way we find \( q(2) \) and \( l(2) \).

References

[1] D. Bollé, J. Busquets Blanco and G.M. Shim, Phys. A 318, 613 (2003)
[2] D.R.C. Dominguez and E. Korutcheva, Phys. Rev. E, 62, 2620 (2000)
[3] D. Bollé and T. Verbeiren, Phys. Lett. A, 297, 156 (2002)
[4] D. Bollé and T. Verbeiren, J. Phys. A: Math. and Gen. 35, 295 (2003).
[5] A.E. Patrick and V.A. Zagrebnov, J. Phys. France, 51, 1129 (1990)
[6] E. Barkai, I. Kanter and H. Sompolinsky, Phys. Rev. A, 41, 590 (1990).
[7] E.D. Siggia, P.C. Martin and H.A. Rose, Phys. Rev. A 8, 423 (1973).
[8] C. De Dominicis, Phys. Rev. B 18, 4913 (1978)
[9] D. Bollé, G.M. Shim and B. Vinck, J. Stat. Phys. 74, 583 (1994).
[10] M. Shiino and T. Fukai, Phys. Rev. 48, 867 (1993)
[11] K.H. Fischer and J.A. Hertz, Spin glasses, (Cambridge University Press, 1991).
[12] D. Bollé, G. Jongen and G.M. Shim, unpublished results.
[13] A.C.C. Coolen, in Handbook of Biological Physics Vol 4, ed. by F. Moss and S. Gielen (Elsevier Science B.V., 2001), p. 597
[14] D. Bollé, J. Busquets Blanco and T. Verbeiren, in preparation.