Menon-type identities concerning Dirichlet characters

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Abstract
Let \( \chi \) be a Dirichlet character (mod \( n \)) with conductor \( d \). In a quite recent paper Zhao and Cao deduced the identity \( \sum_{k=1}^{n} (k - 1, n) \chi(k) = \varphi(n) \tau(n/d) \), which reduces to Menon’s identity if \( \chi \) is the principal character (mod \( n \)). We generalize the above identity by considering even functions (mod \( n \)), and offer an alternative approach to proof. We also obtain certain related formulas concerning Ramanujan sums.

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1 Introduction

In a quite recent paper Zhao and Cao [13] derived the following identity. Let \( \chi \) be a Dirichlet character (mod \( n \)) with conductor \( d \) (\( d \mid n \)). Then

\[
\sum_{k=1}^{n} (k - 1, n) \chi(k) = \varphi(n) \tau(n/d) \quad (n \in \mathbb{N}),
\]

where \( (k - 1, n) \) stands for the greatest common divisor of \( k - 1 \) and \( n \), \( \varphi(n) \) is Euler’s totient function and \( \tau(n) = \sum_{d \mid n} 1 \) is the divisor function. If \( \chi \) is the principal character (mod \( n \)), that is \( d = 1 \), then (1.1) reduces to Menon’s identity

\[
\sum_{(k,n)=1}^{n} (k - 1, n) = \varphi(n) \tau(n) \quad (n \in \mathbb{N}).
\]

On the other hand, if \( \chi \) is a primitive Dirichlet character (mod \( n \)), then (1.1) gives (the case \( d = n \))

\[
\sum_{k=1}^{n} (k - 1, n) \chi(k) = \varphi(n) \quad (n \in \mathbb{N}).
\]
In fact, Zhao and Cao [13] first proved formula (1.3) and then deduced identity (1.1) by using the fact that every Dirichlet character is induced by a primitive character (Lemma 3.1). They showed that the left hand sides of (1.1) and (1.3) are multiplicative in \(n\) and computed their values for prime powers.

It is the goal of the present paper to generalize these identities by considering even functions \((\text{mod } n)\), and to offer an alternative approach to proof, based on direct manipulations of the corresponding sums, valid for any integer \(n \in \mathbb{N}\).

A function \(f : \mathbb{Z} \to \mathbb{C}\) is called an even function \((\text{mod } n)\) if \(f((k, n)) = f(k)\) holds for any \(k \in \mathbb{Z}\), where \(n \in \mathbb{N}\) is fixed. The term \(n\)-even function is also used in the literature. Examples of even functions \((\text{mod } n)\) are \(f(k) = (k, n)\), more generally \(f(k) = F((k, n))\), where \(F\) is an arbitrary arithmetic function; \(f(k) = c_n(k)\), representing the Ramanujan sum; the function \(N(k)\), counting the solutions \((x_1, \ldots, x_q) \in \mathbb{Z}_n^q\) of the congruence \(x_1 + \cdots + x_q \equiv k \pmod{n}\) such that \((x_1, n) = \cdots = (x_q, n) = 1\), with \(q \in \mathbb{N}\) fixed. General accounts of even functions \((\text{mod } n)\) can be found, e.g., in the books by McCarthy [4], Schwarz and Spilker [7], and the paper by the author and Haukkanen [12].

Different Menon-type identities were established by several authors. See, e.g., the papers by Haukkanen [1], Li and Kim, [2, 3], Miguel [5], Sita Ramaiah [8], T˘ arn˘ auceanu [9], the author [10, 11].

## 2 Main results

We prove the following results. The first one is a direct generalization of Menon’s identity, not involving characters, which will be used later in the proof. Let \(\mu\) denote, as usual, the Möbius function and let \(*\) denote the Dirichlet convolution of arithmetic functions.

**Theorem 2.1.** Let \(n, d \in \mathbb{N}, r, s \in \mathbb{Z}\) such that \(d \mid n\). Let \(f\) be an even function \((\text{mod } n)\). Then

\[
\sum_{\substack{k=1 \\ (k,n)=1 \\ k \equiv r \pmod{d}}}^{n} f(k - s) = \begin{cases} 
\frac{\varphi(n)}{\varphi(d)} \sum_{e \mid n} \frac{(\mu * f)(e)}{\varphi(e)} \varphi(e, d), & \text{if } (r, d) = 1, \\
0, & \text{if } (r, d) > 1.
\end{cases} \tag{2.1}
\]

If \(f(k) = F((k, n))\), where \(F\) is an arbitrary arithmetic function, \(d = 1\) and \((s, n) = 1\), then from (2.1) we reobtain the identity due to Sita Ramaiah [8, Th. 9.1] in the more general setting of regular arithmetic convolutions.

**Corollary 2.2.** Let \(n, d \in \mathbb{N}, r, s \in \mathbb{Z}\) such that \(d \mid n\). Then

\[
\sum_{\substack{k=1 \\ (k,n)=1 \\ k \equiv r \pmod{d}}}^{n} (k - s, n) = \begin{cases} 
\frac{\varphi(n)}{\varphi(d)} \sum_{e \mid n} \varphi((e, d)), & \text{if } (r, d) = 1, \\
0, & \text{if } (r, d) > 1.
\end{cases} \tag{2.2}
\]

If \(d = 1\) and \(s = 1\), then (2.2) reduces to Menon’s identity (1.2).
Corollary 2.3. Let \( n, d \in \mathbb{N}, r, s \in \mathbb{Z} \) such that \( d | n \). Then

\[
\sum_{k=1}^{n} c_n(k - s) = \begin{cases} 
\frac{\varphi(n)}{\varphi(d)} \sum_{e|n \atop (e,s)=1} \frac{e\mu(n/e)}{\varphi((e,d))}, & \text{if } (r,d) = 1, \\
0, & \text{if } (r,d) > 1.
\end{cases}
\]  

(2.3)

If \( d = 1 \), then (2.3) gives the first identity of the known formulas

\[
\sum_{k=1}^{n} c_n(k - s) = \varphi(n) \sum_{e|n \atop (e,s)=1} \frac{e\mu(n/e)}{\varphi((e,d))} = \mu(n)c_n(s),
\]

the second one being the Brauer-Rademacher identity. See [4, Ch. 2].

Theorem 2.4. Let \( \chi \) be a Dirichlet character \( \pmod{n} \) with conductor \( d \) \( (n,d \in \mathbb{N}, d | n) \). Let \( f \) be an even function \( \pmod{n} \) and let \( s \in \mathbb{Z} \). Then

\[
\sum_{k=1}^{n} f(k - s)\chi(k) = \varphi(n)\chi^*(s) \sum_{\delta|n/d \atop (\delta,s)=1} \frac{\mu* f(\delta d)}{\varphi(\delta d)},
\]

where \( \chi^* \) is the primitive character \( \pmod{d} \) that induces \( \chi \).

Corollary 2.5. Let \( \chi \) be a Dirichlet character \( \pmod{n} \) with conductor \( d \) \( (n,d \in \mathbb{N}, d | n) \) and let \( s \in \mathbb{Z} \). Then

\[
\sum_{k=1}^{n} (k - s, n)\chi(k) = \varphi(n)\chi^*(s) \sum_{\delta|n/d \atop (\delta,s)=1} 1.
\]  

(2.4)

If \( s = 1 \), then (2.4) reduces to the identity (1.1) of Zhao and Cao [13]. If \( \chi \) is the principal character \( \pmod{n} \), that is \( d = 1 \), then (2.4) gives

\[
\sum_{k=1}^{n} (k - s, n) = \varphi(n) \sum_{\delta|n \atop (\delta,s)=1} 1,
\]  

valid for any \( s \in \mathbb{Z} \). If \( (s,n) = 1 \), then the right hand side of (2.5) is \( \varphi(n)\tau(n) \), like in Menon’s classical identity (1.2).

Corollary 2.6. Let \( \chi \) be a Dirichlet character \( \pmod{n} \) with conductor \( d \) \( (n,d \in \mathbb{N}, d | n) \) and let \( s \in \mathbb{Z} \). Then

\[
\sum_{k=1}^{n} c_n(k - s)\chi(k) = d\varphi(n)\chi^*(s) \sum_{\delta|n/d \atop (\delta,s)=1} \frac{\delta\mu(n/(\delta d))}{\varphi(\delta d)}.
\]
We remark that the sums in Theorem 2.4 and Corollaries 2.5 and 2.6 vanish provided that $(s, d) > 1$.

**Theorem 2.7.** Let $\chi$ be a primitive Dirichlet character (mod $n$), where $n \in \mathbb{N}$. Let $f$ be an even function (mod $n$) and let $s \in \mathbb{Z}$. Then
\[ \sum_{k=1}^{n} f(k - s) \chi(k) = (\mu \ast f)(n) \chi(s). \]

The above results can be applied to other special even functions (mod $n$), as well. For example, we have

**Corollary 2.8.** Let $\chi$ be a primitive Dirichlet character (mod $n$), where $n \in \mathbb{N}$. Let $F$ be an arbitrary arithmetic function and let $s \in \mathbb{Z}$. Then
\[ \sum_{k=1}^{n} F((k - s, n)) \chi(k) = (\mu \ast F)(n) \chi(s). \] (2.6)

In particular,
\[ \sum_{k=1}^{n} \sigma((k - s, n)) \chi(k) = n \chi(s), \]
where $\sigma(n)$ is the sum-of-divisors function, and
\[ \sum_{k=1}^{n} \tau((k - s, n)) \chi(k) = \chi(s). \]

It turns out that if $F$ is a multiplicative function and $s = 1$, then the sum (2.6) is also multiplicative in $n$.

The sums in Theorem 2.7 and Corollary 2.8 vanish provided that $(s, n) > 1$.

### 3 Proofs

We need the following known results. For the first one see, e.g., [6, Th. 9.2].

**Lemma 3.1.** Let $\chi$ be a Dirichlet character (mod $n$) with conductor $d$. Then there is a unique primitive character $\chi^*$ (mod $d$) that induces $\chi$. That is,
\[ \chi(k) = \begin{cases} 
\chi^*(k), & \text{if } (k, n) = 1, \\
0, & \text{if } (k, n) > 1.
\end{cases} \]

For the next result see, e.g., [6, Th. 9.4]. However, it is not included in most of other textbooks. For the sake of completeness we present its (short) proof.
Lemma 3.2. Let $\chi$ be a primitive character (mod $n$). Then for any $d \mid n$, $d < n$ and any $s \in \mathbb{Z}$,

$$\sum_{k=1 \atop k \equiv s \pmod{d}}^{n} \chi(k) = 0.$$ 

Proof of Lemma 3.2. Since $\chi$ is a primitive character, for a given $d \mid n$, $d < n$ there exists $c \in \mathbb{Z}$ such that $(c, n) = 1$, $c \equiv 1 \pmod{d}$ and $\chi(c) \neq 1$. We have

$$S := \sum_{k=1 \atop k \equiv s \pmod{d}}^{n} \chi(k) = \sum_{t \pmod{n/d}} \chi(s + td).$$

Here, since $(c, n) = 1$, as $t$ runs through a complete residue system (mod $n/d$), the numbers $j = cs + tcd$ run through a complete residue system (mod $n$), where $j \equiv cs \equiv s \pmod{d}$. Hence,

$$S = \sum_{t \pmod{n/d}} \chi(cs + tcd) = \chi(c) \sum_{t \pmod{n/d}} \chi(s + td) = \chi(c)S.$$ 

Since $\chi(c) \neq 1$, it follows that $S = 0$. \hfill \square

Proof of Theorem 2.1. If $(k, n) = 1$ and $k \equiv r \pmod{d}$, then $(r, d) = (k, d) = 1$. Therefore, the sum is empty in the case $(r, d) > 1$.

Now assume that $(r, d) = 1$. Since $f$ is an even function (mod $n$),

$$f(k) = f((k, n)) = \sum_{d \mid (k,n)} (\mu * f)(d). \quad (3.1)$$

We have

$$T := \sum_{k=1 \atop (k, n) = 1 \atop k \equiv r \pmod{d}}^{n} f(k-s) = \sum_{k=1 \atop k \equiv r \pmod{d}}^{n} f(k-s) \sum_{\delta \mid (k,n)} \mu(\delta)$$

$$= \sum_{\delta \mid n} \mu(\delta) \sum_{k=1 \atop \delta \mid k \atop k \equiv r \pmod{d}}^{n} f(k-s) = \sum_{\delta \mid n} \mu(\delta) \sum_{j=1 \atop \delta j \equiv r \pmod{n}}^{n/\delta} f(\delta j - s).$$

According to (3.1),

$$T = \sum_{\delta \mid n} \mu(\delta) \sum_{j=1 \atop e \mid (\delta j - s,n)}^{n/\delta} (\mu * f)(e)$$

$$= \sum_{\delta \mid n} \mu(\delta) \sum_{e \mid n} (\mu * f)(e) \sum_{j=1 \atop \delta j \equiv s \pmod{e}}^{n/\delta} 1. \quad (3.2)$$
Let $\delta, d, e$ be fixed. The linear congruence $\delta j \equiv r \pmod{d}$ has solutions in $j$ if and only if $(\delta, d) \mid r$, equivalent to $(\delta, d) = 1$, since $(r, d) = 1$. Similarly, the congruence $\delta j \equiv s \pmod{e}$ has solutions in $j$ if and only if $(\delta, e) \mid s$. The above two congruences have common solutions in $j$ if and only if $(d, e) \mid r - s$. Furthermore, if $j_1$ and $j_2$ are solutions of these simultaneous congruences, then $\delta j_1 \equiv \delta j_2 \pmod{d}$ and $\delta j_1 \equiv \delta j_2 \pmod{e}$. This gives $j_1 \equiv j_2 \pmod{(d, e)}$, since $(r, d) = 1$. That is, $j_1 \equiv j_2 \pmod{(d, e)}$, the least common multiple of $d$ and $e/(\delta, e)$. We conclude that there are

$$N = \frac{n}{\delta \gcd(d, e/(\delta, e))} = \frac{n}{\delta \gcd(d, e)} \cdot \frac{n}{\delta k}$$

solutions (mod $n/\delta$). Therefore, the value of the last sum in (3.2) is $N$.

We deduce that

$$T = \sum_{\delta \mid n \atop (\delta, d) = 1} \mu(\delta) \sum_{e \mid n \atop (e, \delta) \mid s \atop (e, d) \mid r - s} (\mu \ast f)(e) \frac{n}{\delta \gcd(d, e)}$$

$$= \frac{n}{d} \sum_{e \mid n \atop (e, d) \mid r - s} \frac{(\mu \ast f)(e)}{e} \frac{(d, e)}{\gcd(e)} \sum_{\delta \mid n \atop (\delta, d) = 1 \atop (\delta, e) \mid s} \frac{\mu(\delta)(\delta, e)}{\delta}.$$ 

Here the inner sum is

$$\prod_{\substack{p \mid n \\ p \nmid d \\ (p, e) \mid s}} \left(1 - \frac{(p, e)}{p}\right),$$

which equals $(\varphi(n)/n)(\varphi(de)/de)^{-1}$ in the case $(e, s) = 1$ and zero otherwise. We obtain that

$$T = \varphi(n) \sum_{e \mid n \atop (e, d) \mid r - s \atop (e, s) = 1} \frac{(\mu \ast f)(e)}{\varphi(de)} \frac{\varphi(n)}{\varphi(d)} \sum_{e \mid n \atop (e, d) \mid r - s \atop (e, s) = 1} \frac{(\mu \ast f)(e)}{\varphi(e)} \varphi((d, e)).$$

**Proof of Corollary 2.2.** Apply Theorem 2.1. Let $f(k) = (k, n)$. Then for every $e \mid n$ we have

$$(\mu \ast f)(e) = \sum_{ab = e} \mu(a)(b, n) = \sum_{ab = e} \mu(a)b = \varphi(e).$$

**Proof of Corollary 2.3.** Apply Theorem 2.1. Select $f(k) = c_n(k)$ and use the familiar formula

$$c_n(k) = \sum_{e \mid (k, n)} e \mu(n/e).$$

It follows that $(\mu \ast c_n)(e) = e \mu(n/e)$ for any $e \mid n$. Also see [12, Sect. 3].
Proof of Theorem 2.4. We have, according to Lemma 3.1,

\[ S_f := \sum_{k=1}^{n} f(k-s)\chi(k) = \sum_{k=1}^{n} f(k-s)\chi^*(k) \]

\[ = \sum_{r=1}^{d} \sum_{k=1 \atop (k,n)=1}^{n} f(k-s)\chi^*(k) = \sum_{r=1}^{d} \chi^*(r) \sum_{k=1 \atop (k,n)=1}^{n} f(k-s). \]

Now, by using Theorem 2.1,

\[ S_f = \frac{\varphi(n)}{\varphi(d)} \sum_{r=1}^{d} \chi^*(r) \sum_{e \mid n \atop (e,s)=1} \frac{(\mu * f)(e)}{\varphi(e)} \varphi((e,d)) \]

\[ = \frac{\varphi(n)}{\varphi(d)} \sum_{e \mid n \atop (e,s)=1} \frac{(\mu * f)(e)}{\varphi(e)} \varphi((e,d)) \sum_{r=1 \atop r \equiv s \pmod{(e,d)}}^{d} \chi^*(r). \]

Here, by Lemma 3.2 the inner sum is zero, unless \((e,d) = d\), that is \(d \mid e\), and in this case the inner sum is \(\chi^*(s)\). We deduce that

\[ S_f = \frac{\varphi(n)}{\varphi(d)} \chi^*(s) \sum_{e \mid n \atop d \mid e \atop (e,s)=1} \frac{(\mu * f)(e)}{\varphi(e)} \varphi(d) = \varphi(n) \chi^*(s) \sum_{\delta \mid n/d \atop (\delta,s)=1} \frac{(\mu * f)(\delta d)}{\varphi(\delta d)}, \]

which vanishes if \((s,d) > 1\).

Proof of Corollary 2.5. Apply Theorem 2.4 to \(f(k) = (k,n)\), where \((\mu * f)(e) = \varphi(e)\) for every \(e \mid n\).

Proof of Corollary 2.6. Apply Theorem 2.4 by selecting \(f(k) = c_k(n)\). See the proof of Corollary 2.3.

Proof of Theorem 2.7. This is a direct consequence of Theorem 2.4 by taking \(d = n\). A short direct proof is the following: by using (3.1) and Lemma 3.2 we have

\[ \sum_{k=1}^{n} f(k-s)\chi(k) = \sum_{k=1}^{n} \chi(k) \sum_{e \mid (k-s,n)} (\mu * f)(e) \]

\[ = \sum_{e \mid n} (\mu * f)(e) \sum_{k=1 \atop k \equiv s \pmod{e}}^{n} \chi(k) = (\mu * f)(n)\chi(s). \]

Proof of Corollary 2.8. Use Theorem 2.7. Select \(f(k) = F((k,n))\) and then \(F = \sigma\) and \(F = \tau\), respectively.
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