Exact density of states of a two-dimensional electron gas in a strong magnetic field and a long-range correlated random potential.

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We derive an exact result for the averaged Feynman propagator and the corresponding density of states of an electron in two dimensions in a perpendicular homogeneous magnetic field and a Gaussian random potential with long-range spatial correlations described by a quadratic correlation function.

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The problem of an electron moving in two dimensions (2D) in a strong homogeneous magnetic field \( B \) and a random potential has been studied particularly intensively since the discovery of the integer quantum Hall effect (IQHE) [1]. Especially the density of states (DOS) as the simplest quantity that characterizes the quantum mechanical spectrum has been the subject of continuous interest [2][4]. This was partly induced by the fact that it remains notoriously difficult to explain the unexpectedly large density of states between the Landau bands (LBs) detected experimentally [1][4]. The theoretical calculations, mostly based on models with short-range correlated randomness as, for instance, the Gaussian white noise potential, yielded in the high magnetic field limit well separated, disorder broadened LBs with a width proportional to \( \sqrt{B} \) and an exponentially small DOS in between the bands [15][16]. On the other hand, the experimental data indicate that there is a roughly constant background DOS which contains as many as 10% to 20% of the levels [1].

It is very tempting to suspect that spatially long-range correlations of the disorder, due to inhomogeneities, for instance, are responsible for this background DOS. Such a model was studied by using path integrals in connection with a cumulant expansion [3]. This reproduced the experimental result. However, the influence of the cumulant approximation which is uncontrollable in the presence of a magnetic field was unclear. Subsequent approximate treatments of the DOS in the presence of long-range correlated disorder, by using functional integrals and without the cumulant approximation, did not improve the situation [4]. Exact results for long-range correlated disorder exist only for the band tails [3].

In this paper, we consider a model that has already been treated before without magnetic field [20]. Also in the region of the quantum Hall effect, this model allows for the treatment of the influence of long-range correlations in the randomness. The essential assumption is a quadratic approximation of the correlation function. The range of validity of this model assumption is discussed below. The Feynman propagator can be calculated exactly, and the DOS evaluated numerically, but without employing further approximations.

The comparison with recent experimental results [21] shows that long-range correlations can qualitatively explain some aspects of the experimental data. However, the quantitative behavior, as a function of the magnetic field, cannot be understood. We conclude, that the large DOS between the LBs seen in experiment, is very likely related to interaction between the electrons.

We consider a spinless electron moving in 2D under the influence of a homogeneous magnetic field and a random impurity potential \( V(r) \). The Hamiltonian is

\[
H = \frac{1}{2} [\vec{p} \cdot \vec{A}(r)]^2 + V(r). \tag{1}
\]

For the vector potential, we assume the symmetric gauge \( \vec{A}(r) = \hat{e} \cdot r/2 \), where \( \hat{e} \) denotes the antisymmetric \((2 \times 2)\) matrix, so that \( \hat{e}^2 = -1 \). The magnetic field \( B \) defines the intrinsic energy, length, and time scales. Energies and lengths are measured here in units of the cyclotron energy \( \hbar \omega_c \equiv \hbar eB/m \) and the magnetic length \( \ell_c \equiv \sqrt{\hbar/eB} \), respectively (\( e \) elementary charge and \( m \) effective mass). The unit of time is then the inverse of \( \omega_c \). The probability distribution of the impurity potential \( V(r) \) is assumed to be Gaussian with zero mean \( \langle V(r) \rangle = 0 \) and correlation function

\[
C(r, r') \equiv \langle V(r)V(r') \rangle = \eta^2 \left[ 1 - \frac{1}{L^2} (r - r')^2 \right]. \tag{2}
\]

Angular brackets denote the ensemble average with respect to the probability distribution of the random variable \( V \). The parameters \( \eta \) and \( L \) determine the strength and the range of the impurity potential, respectively.

The above polynomial correlation function Eq. (2) is the first order expansion of a Gaussian

\[
C_G(r, r') = \eta^2 e^{-(r-r')^2/L^2}. \tag{3}
\]

The function \( C(r, r') \), Eq. (2), has been used previously [20] to describe approximately the DOS of an electron in 2D in a random impurity potential with the Gaussian correlation function, Eq. (3).
Here, we derive the exact result for the impurity averaged Feynman propagator of the model Eq. (2),
\[
\langle K(r; t; r_0) \rangle = \langle \int \mathcal{D}[r] \exp \left[ i \int_0^t dt L(r, \dot{r}; \tau) \right] \rangle,
\]
(4)
where \( \mathcal{D}[r] \) is the Feynman measure on the set of trajectories with boundary conditions \( r(0) = r_0 \) and \( r(t) = r \). The time evolution begins at \( t = 0 \), which is omitted in \( K(r, t; r_0) \) for simplicity. Equation (4) is a Feynman path integral and the Langrangian
\[
L(r, \dot{r}; \tau) = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^T \dot{r} - V(r)
\]
represents the usual expression for a charged particle in 2D in a magnetic field and a potential.

Since \( V(r) \) enters only linearly in the exponent of Eq. (4), the impurity average is easily performed,
\[
\langle K(r; t; r_0) \rangle = \int \mathcal{D}[r] \exp \left\{ \int_0^t dt \left[ \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^T \dot{r} \right] \right\} \times \exp \left\{ -\frac{1}{2} \int_0^t \dot{r}^2 C(r, \tau)(r') \right\}.
\]
(6)
The last factor in Eq. (6) results from the impurity average. It is purely real and non-local in time. This consequence of integrating out degrees of freedom is well known \[22\]. It is usually the end of an exact calculation. In our model, Eq. (2), however, we have the simplifying feature that \( C(r, r') \) is quadratic in \( r - r' \). Therefore, the path integral remains quadratic and can be evaluated exactly by using standard techniques \[22\] — in spite of the non-locality in time. Details are given in \[24\].

Denoting \( \langle K(r; t; r_0) \rangle = K_B(r; t; r_0) \), we obtain
\[
K_B(r, t; r_0) =
\frac{1}{2\pi i t} \sin \left[ \frac{(t^2/4)(\Omega^2 - 1/4)}{\sin \left[ (\Omega + 1/2) (t/2) \right] \cdot \sin \left[ (\Omega - 1/2) (t/2) \right]} \right]
\times \exp \left\{ -\frac{1}{2} \dot{r}^2 t^2 + if(r, r_0; t) \right\}
\]
(7)
and
\[
f(r, r_0; t) = \frac{\Omega}{2 \sin \Omega t} \left[ \frac{(r - r_0)^2}{2} \cos \Omega t \right.
- 2r_0^2 e^{\frac{i}{2}t} r + \frac{(r + r_0)^2}{2} \cos \frac{1}{2} t
- \frac{(r - r_0)^2}{2\Omega^2} \left( \Omega \sin \frac{1}{2} \frac{\dot{r}}{\Omega} - \frac{1}{2} \sin \Omega t \right)^2
+ \frac{2r_0^2 \dot{r}}{\Omega} \left( \Omega \sin \frac{1}{2} \frac{\dot{r}}{\Omega} - \frac{1}{2} \sin \Omega t \right) \bigg].
\]
(8)

The complex frequency \( \Omega = \frac{1}{2} \sqrt{1 + 8it(\eta/L)^2} \) depends on the time \( t \) of propagation. Configurational averaging leads to a path integral with a non-local action. Therefore, the above is not a conventional Feynman propagator. It does not have the group property,
\[
K_B(r, t; r_0) \neq \int d^2r' K_B(r, t - t'; r') K_B(r', t'; r_0),
\]
(9)
which would be valid for a unitary time evolution operator without the configurational average.

This has serious consequences for the analytical structure of the diagonal part
\[
K_B(t) \equiv K_B(r, t; r) = \frac{e^{-\eta t^2/2}}{2\pi i t} \times \sin \left[ (\Omega^2 - 1/4) (\Omega^2 - 1/4) \right]
\sin \left[ (\Omega + 1/2) (t/2) \right] \sin \left[ (\Omega - 1/2) (t/2) \right]
\]
(10)
of our propagator that determines the DOS. By investigating the analytical continuation of \( K_B(t) \) into the complex \( t \)-plane, we find not only a single pole for \( t = 0 \) on the real time axis, but also poles in the lower and upper complex plane. In the limit \( L \to \infty \), the poles in the lower plane approach the real time axis whereas the poles in the upper plane disappear towards infinity.

Evaluating the DOS for such a propagator, we have to take two conditions into account: first, causality should be fulfilled and second, the definition of the DOS for this model should reproduce the limit \( L = \infty \) of a model with constant spatial correlations. The DOS can be calculated exactly in this limit \[25,26\]. We choose the integration contour in the complex \( t \)-plane in such a way that coming from minus infinity all poles, \( t_n \), in the lower half plane are passed anti-clockwise. This choice is unique, and the limit \( L \to \infty \) is continuous. Deforming this contour back to the real axis gives the DOS
\[
D(E) = \frac{1}{4\pi} + \frac{1}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} dt K_B(t) e^{iEt}
\times \sum_{\text{Res}_{t=t_n}} [K_B(t) e^{iEt}] .
\]
(11)
Here, \( \mathcal{P} \int_{-\infty}^{\infty} dt \ldots \) denotes Cauchy’s principal value of the integral and \( \text{Res}_{t=t_n} \) the residue at \( t = t_n \).

Equation (11) is our central result. Figure 1 shows the result of the integration for \( L = 6 \) and \( \eta = 0.2, 0.3 \) and 0.4 (in the units defined above), normalized to the DOS at \( B = 0, D_0 \). Due to the impurity potential, the Landau peaks are broadened as expected. The broadening decreases for increasing energy which indicates that disorder becomes less effective. For large \( L \), Eq. (11) can be shown to yield the same as obtained for the model with a constant correlation function, a superposition of Gaussians centered at the Landau levels (LLs). However, Eq. (11) becomes negative if \( E \) exceeds a critical
value. This is a consequence of the unphysical shape of the polynomial correlation function Eq. (3), as we will now see.

While the correlation function Eq. (3) is positive definite, its truncated expansion Eq. (4) is not. Nevertheless, using Eq. (4) in calculating physical quantities may give the same result as using Eq. (3) and expanding afterwards in $L^{-1}$. It is instructive to take the limit in Eq. (4) in which one keeps only the lowest LL in Eq. (4). Formally, this is achieved by measuring the energy from the center of the lowest LL, scaling $\bar{E} = \hbar\omega_c(E - 1/2)$ and $\tilde{\eta} = \hbar\omega_c\eta$, and taking $\omega_c \to \infty$. We find for the DOS of the lowest LL with the random potential described by Eq. (2),

$$D_{LLL}(\bar{E}) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-\tilde{\eta}t^2/2} \frac{2(\tilde{\eta}t/L)^2 \cos \bar{E}t}{1 - \exp(-2(\tilde{\eta}t/L)^2)}.$$

(12)

The limit $L \to \infty$ is correctly reproduced. For $L > \sqrt{2}$, $D_{LLL}(\bar{E})$ is positive for all energies. Now, the following argument suggests that including more and more LLs yields an increasingly higher bound for $L$ until it diverges in the limit of all LLs taken into account. The classical cyclotron radius is proportional to $\sqrt{\bar{E}}$. Thus, the correlation function Eq. (2) is positive and the DOS well defined only for energies smaller than $E_{\text{max}} \approx (L/2)^2$, independent of the strength of the disorder $\eta$. Beyond this critical energy, the classical cyclotron radius, which is a characteristic length for the range of the electron, is of the order of $L$ and larger. Then, the expansion leading from Eq. (3) to Eq. (2) breaks down and, for such distances, the correlation function Eq. (2) becomes negative. Such a bound $E_{\text{max}}$, and its independence of $\eta$, can indeed be seen in Fig. 1.

Our model is the first order approximation of an impurity potential with Gaussian correlations, Eq. (3), which is believed to be typical for such materials as GaAs/AlGaAs heterostructures. In order to obtain information on the validity of our model, Eq. (4), we compare the DOS with previous results for the Gaussian model.

By neglecting the mixing of states between different LLs and assuming a Gaussian shape of each level, Broderix et al. [27] found for the width of the $n$th LB

$$\sigma_n^2 = \eta^2 \frac{L^2}{L^2 + 2} \left[ \frac{L^2 - 2}{L^2 + 2} \right]^n P_n \left( \frac{L^4 + 4}{L^4 - 4} \right),$$

(13)

with $P_n(x)$ the $n$th Legendre polynomial. Figure 2 shows both results, ours and that of [27], for $L = 10$ and $\eta = 0.4$. The agreement is quantitative. For increasing $L$, the agreement becomes even better. This demonstrates indeed that for large values, the length parameter $L$ has the meaning of the correlation length of a Gaussian correlation function. Also, our results support the conjecture that in the quantum Hall regime, LL mixing is negligible and the DOS is a superposition of Landau bands with Gaussian shapes. This is valid for even a broader energy range when $L$ is increased.

Armed with the exact result for the DOS of a model with spatially correlated randomness, we now compare with experiments, cf. Fig 3. By plotting the DOS at the cyclotron energy – between the lowest and the second lowest LL – against the magnetic field strength, we find that the various experiments on GaAs/AlGaAs heterostructures show the same qualitative behavior. However, this appears to be quite different from the prediction based on the model of non-interacting electrons with spatially correlated disorder. We find a strong decrease of the DOS when increasing the strength of the magnetic field, much stronger than is observed experimentally. Since a spatially correlated random potential energy represents the most general model of disorder, this leads to the question, whether or not the interaction between the electrons has to be taken into account, even for a theory of the DOS in the region of the IQHE which is commonly believed to be explainable without electron-electron interactions. More quantitative comparisons between experiment and theory are necessary, in order to clarify this point.

If our above suspicion proves to be true, the proper understanding of the IQHE would very probably need, in addition to disorder, electron-electron interaction as an important ingredient, as is already the case for the fractional quantum Hall effect.

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