On a stability problem of a preliminary compressed console plate

M V Belubekyan and V M Belubekyan*

Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia
E-mail: *vbelub@gmail.com

Abstract. A rectangular plate with two opposite free edges is considered. One of the remaining edges is clumped and the other is free. It is assumed that the plate is uniformly compressed in the direction to the two opposite free edges. The commonly adopted solution under the approximation of elongated plate is analyzed. It is established that along this solution, additional solutions exist within same approximation.

1. Overview

Let the plate to be defined by the following region in the Cartesian coordinate system \((x, y, z)\):
\[0 \leq x \leq a, -0.5b \leq y \leq 0.5b, -h \leq z \leq h,\]
as shown in figure 1.

The plate is preliminary compressed with a uniform load \(P\) along the \(x\) coordinate. Equation of static stability of plate, also known as Euler’s equation of stability following to [1] can be written in the following form:
\[D \Delta^2 w + P \frac{\partial^2 w}{\partial x^2} = 0,\]
where \(w(x, y)\) is the deflection of the plate, \(D\) is its flexural stiffness, \(E\) is the Young module and \(\nu\) is the Poisson’s ratio.

On the edges of the plate the following boundary conditions are assumed:
\[w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{for} \quad x = 0,\]
\[M_x = 0, \quad \bar{\bar{N}}_x + P \frac{\partial w}{\partial x} = 0 \quad \text{for} \quad x = a.\]

Taking into account expansions of the bending moment \(M_x\) and transverse force [1], the conditions (3) can be rewritten as follows:
\[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{for} \quad x = a.\]

The edges \(y = \pm 0.5b\) of the plate are free, i.e:
\[M_y = 0, \quad \bar{\bar{N}}_y = 0 \quad \text{for} \quad y = \pm 0.5b,\]
Figure 1. Uniformly compressed console plate

or in other terms:

\[
\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial y \partial x^2} = 0 \quad \text{for} \quad y = \pm 0.5b. \tag{6}
\]

A non-trivial solution of equation is sought, satisfying to boundary conditions (2), (4), and (6).

2. Approximation approach

Known solutions of the considered problem were obtained within an approximation of elongated plate, where it was assumed that the shape of buckling of the plate does not depend on the y coordinate [1, 2]. The specific boundary conditions of the considered problem make the method of separation of variables inapplicable. Application of the Bubnov-Galerkin’s method, due to the fact that finding appropriate coordinate-functions, satisfying to momentum and load conditions (6) at the free edge is problematic, also leads to significant difficulties, as described in monograph by N. Alfutov [1]. So the above discussion yields to the conclusion cited below from the monograph [1]: “Going on to other cases of exact integration of the basic linearized equation, it can be noted that the results obtained for an elongated plate can be applied to a plate of finite dimensions with two free edges, as well. In this case, we can assume, to sufficient accuracy, that \( w = w(x) \). The conditions at the free edges will not be exactly satisfied but for \( h/b \ll 1 \) this will not affect the critical-force values.”

Assuming that \( w = w(x) \), the eigenfunctions of the problem are obtained as:

\[
w_n = A_n \left[ 1 - \cos(\alpha_n x) \right], \quad \alpha_n = \frac{(2n - 1)\pi}{2a}, \quad n = 1, 2, \ldots \tag{7}
\]

The eigenvalues, representing the critical loads are obtained as well:

\[
P_n = D\alpha_n^2, \quad \min P_n = P_1 = \frac{\pi^2D}{4a^2}. \tag{8}
\]

The solution (7) satisfies to equation (1) and the boundary conditions (2) and (4), but not to the boundary conditions of free edge (6) at \( y = \pm 0.5b \). However this solution (7) satisfies to following boundary conditions:

\[
\frac{\partial w}{\partial y} = 0, \quad \frac{\partial^3 w}{\partial y^3} = 0 \quad \text{for} \quad y = \pm 0.5b. \tag{9}
\]
i.e. to the conditions of sliding contact [3], as well to following conditions:

\[
\frac{\partial^2 w}{\partial y^2} = 0, \quad \frac{\partial^3 w}{\partial y^3} = 0 \quad \text{for} \quad y = \pm 0.5b, \tag{10}
\]

which are the free end boundary conditions for a beam.

It is obvious that the following expansion for the deflection

\[
w_n(x, y) = A_n [1 - \cos(\alpha_n x)]y, \tag{11}
\]

also satisfies to equation (1) and the boundary conditions (2) and (4), but not to the boundary conditions (6). Instead of conditions (6) the function (11) satisfies to conditions of free end of a beam (10), and thus this function can also serve as an approximation to solution of the original problem.

3. Alternative approximation

An alternative approximation to the solution of the original problem is proposed as follows:

\[
w_n(x, y) = A[1 - \cos(\alpha_1 x)] \left( 1 - \gamma \frac{y}{b} \right). \tag{12}
\]

A method to adopt linear functions to solution of flexural problems for plate with two opposite free edges earlier was introduced in [4]. Based on the expression (12) we apply the energy method as follows: write down the potential energy of deformation, which yields:

\[
U = \frac{1}{2} D \int_{-b}^{b} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx \, dy = A^2 \frac{\pi^2 b}{64a^2} D \left( 1 + \gamma + \frac{\gamma^2}{3} + 2 \frac{1-\nu}{\pi^2 b^2} \gamma^2 \right). \tag{13}
\]

And the mechanical work of external force, which yields:

\[
W = \frac{h}{2} \int_{-b}^{b} \sigma_x \left[ \int_{0}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx \right] dy = P A^2 \frac{\pi^2 b}{16a} \left( 1 + \gamma + \frac{\gamma^2}{3} \right). \tag{14}
\]

The critical load is determined, according to energy method by a requirement: \( U=W \). Thus for the critical load we ultimately obtain:

\[
P = \frac{\pi^2 D}{4a^2} \left[ 1 + \frac{1-\nu}{\pi^2 b^2} \gamma^2 \left( 1 + \gamma + \frac{\gamma^2}{3} \right) \right]. \tag{15}
\]

4. Determining the buckling shape

It is generally known that solving stability problems for plates, when opposite edges are either free or clamped, requires overcoming significant complexity. On other hand, if the conditions of clamped edge would be replaced by somewhat “softened” conditions of hinged edge, then obtaining exact closed form analytical solutions turns to be straightforward, as the problem then would permit separation of variables. Analogously, replacing the conditions of free edge by somewhat more limiting conditions of sliding contact lead to analytical solutions. From the above we conclude, that the buckling force of console plate could be evaluated by means of two additional easy solvable problems: buckling of a beam and as buckling of a plate with opposite edges subjected to sliding contacts.

In the following, for the problem of pt. 1 (1)–(3) the following boundary conditions are assumed:

\[
\frac{\partial w}{\partial y} = 0, \quad \frac{\partial^3 w}{\partial y^3} = 0 \quad \text{for} \quad y = \pm 0.5b. \tag{16}
\]
General solution of the equation (1) with boundary conditions (15) can be expanded as follows:

\[ w = \sum_{n=0}^{\infty} f_n(x) \cos(\lambda_n y), \quad \lambda_n = \frac{n\pi}{b}. \]  

(17)

Substituting expressions (17) into equation (1) yields to an infinite sequence of ordinary differential equations

\[
\begin{align*}
  f_0^{IV} + \alpha^2 f_0'' &= 0, \\
  f_n^{IV} - 2\lambda_n^2 (1 - \alpha_n^2) f_n'' + \lambda_n^4 f_n &= 0,
\end{align*}
\]

where

\[
\alpha^2 = \frac{P}{2D}, \quad \alpha_n^2 = \frac{P}{2\lambda_n^2 D}.
\]

(19)

General solutions of the equations (18) have the form

\[
\begin{align*}
  f_0 &= A_0 + B_0 x + C_0 \sin(\alpha x) + D_0 \cos(\alpha x), \\
  f_n &= [A_n \sin(\alpha_n s_2 x) + B_n \cos(\alpha_n s_2 x)] \sinh(\lambda_n s_1 x) \\
  &\quad + [C_n \sin(\alpha_n s_2 x) + D_n \cos(\alpha_n s_2 x)] \cosh(\lambda_n s_1 x),
\end{align*}
\]

where

\[
s_1 = \sqrt{1 - \alpha_n^2 / 2}, \quad s_2 = \alpha_n / \sqrt{2}
\]

(21)

and \(A_n, B_n, C_n,\) and \(D_n\) are arbitrary constants.

The requirement for solution (17) to satisfy to clamped edge boundary conditions (2) yields to determination of unknown functions \(f_0, f_n\) through two arbitrary constants:

\[
\begin{align*}
  f_0 &= A_0 [1 - \cos(\alpha x)] + C_0 [\sin(\alpha x) - \alpha x], \\
  f_n &= [A_n \sin(\lambda_n s_2 x) + \frac{s_2}{s_1} C_n \cos(\lambda_n s_2 x)] \sinh(\lambda_n s_1 x) + C_n \sin(\lambda_n s_2 x) \cosh(\lambda_n s_1 x)
\end{align*}
\]

(22)

5. Localized instability

Let’s consider the case of freely supported edge at \(x = a.\) Taking also the conditions (16) we obtain the following conditions for the functions \(f_0, f_n:\)

\[
\begin{align*}
  f_0'' &= 0, \quad f_0''' + \alpha_0^2 f_0' = 0, \quad f_n'' - \alpha_n^2 f_n = 0, \quad f_n''' + (2 - \nu) \lambda_n^2 f_n' + \alpha_n^2 f_n = 0 \quad \text{for} \ x = a.
\end{align*}
\]

(23)

Substituting \(f_0\) from (20) confirms the results obtained under pt. 1, (7), (8) along other results also yielding to determination of minimal load of buckling. Substituting \(f_n\) from (20) into boundary conditions (23) yield to a system homogeneous linear algebraic equations with respect to the constants \(A_n, C_n:\)

\[
\begin{align*}
  [s_1^2 - s_2^2 - \nu] \sin \xi_2 \sinh \xi_1 + 2 s_1 s_2 \cos \xi_2 \cosh \xi_1] A_n \\
  + \left[ \frac{s_2}{s_1} (1 + \nu) \cos \xi_2 \sinh \xi_1 + (1 - \nu) \sin \xi_2 \cosh \xi_1 \right] C_n &= 0,
\end{align*}
\]

\[
\begin{align*}
  [s_2 (3 s_1^2 - s_2^2 - 2 + \nu + 2 \alpha_n^2) \cos \xi_2 \sinh \xi_1 + s_1 (s_1^2 - 3 s_2^2 - 2 + \nu + 2 \alpha_n^2) \sin \xi_2 \cosh \xi_1] A_n \\
  + \frac{1}{s_1} [(s_1^2 - s_2^2 - 2 + \nu + 2 \alpha_n^2) \sin \xi_2 \sinh \xi_1 + 2 s_1 s_2 \cos \xi_2 \cosh \xi_1] C_n &= 0,
\end{align*}
\]

where

\[
\xi_k = \lambda_n s_k a.
\]

(24)

(25)
The requirement for the determinant of this system to be zero, can be reduced to the form:

\[
[(1 - \nu - \alpha_n^2)^2 \tan^2 \xi_2 + s_1^2(1 + \nu)^2 \tanh^2 \xi_1 - s_1^2[4s_2^2 + (1 + \nu)^2 \tan^2 \xi_2] = 0. \tag{26}
\]

In the limit case of semi-infinite plate \( a \to \infty \), or under approximation \( \tanh \xi_n \approx 1 \), for the parameter characterizing the load of buckling we obtain:

\[
\alpha_n^2 = 2 - 0.5(1 + \nu)^2. \tag{27}
\]

From the equation (24), it follows that the equation (26) may have a solution satisfying to condition of existence of localized buckling, as described in [6]:

\[
0 < \alpha_n^2 < 2. \tag{28}
\]

The equation (26), has the root of \( s_1 = 0 \) \( (\alpha_n^2 = 2) \), which divides the parameter domain corresponding to localized instability from the parameter domain corresponding to solution of overall instability. Eliminating the root \( s_1 = 0 \) from equation (26) and applying limit \( \alpha_n \to \infty \), we obtain the equations determining the parameters \( \lambda_n a \) and \( \nu \):

\[
[(1 + \nu)^2 \tan^2(\lambda_n a) + (1 + \nu)^2(\lambda_n a)^2 - 4 - (1 - \nu)^2 \tan^2(\lambda_n a) = 0. \tag{29}
\]

The root \( \lambda_n a \) of equation (28) depending on the Poisson ratio \( \nu \) defines the relative dimensions of the sides of the plates, at which localized instability first appears.

Below let’s consider a question whether the critical load of localized buckling can ever be less than the critical load of overall buckling. Consider the following example. The equation (28) has the root \( \lambda_n a = \pi \) and \( \nu \approx 0.274 \), then the minimal critical load of localized buckling would be

\[
\alpha_1^2 < 2 \quad \Rightarrow \quad \min P_{loc} < \frac{2\pi^2 D}{b^2}. \tag{30}
\]

From (7), (8) it follows that \( \min P < \frac{\pi^2 D}{(4a^2)} \). Comparing \( P \) and \( P_{loc} \) we see that in this example the critical force of localized buckling is less than the critical force of overall buckling if the following condition holds:

\[
\frac{b}{a} > 2\sqrt{2}.
\]

From equation (29) also a simple condition of appearance of localized buckling can be obtained under the approximation of \( \tanh(\lambda_n a) \approx 1 \):

\[
\lambda_n a < \frac{2 + 0.5(1 - \nu)^2}{(1 + \nu)^2}.
\]

References

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