On the validity of the Euler product inside the Critical Strip

Guilherme França* and André LeClair†

Cornell University, Physics Department, Ithaca, NY 14850

Abstract

The Euler Product Formula relates Riemann’s zeta function $\zeta(s)$ to an infinite product over primes, and is known to be valid for $\Re(s) > 1$ where it converges absolutely. We provide arguments that the formula is actually valid for $\Re(s) > 1/2$ and $\Im(s) \neq 0$, due to the “statistical convergence” of the infinite product in this regime. Namely, though it is formally divergent, it is still meaningful because its logarithm is Cesàro summable, that is, the average of the partial products converges to $\zeta(s)$. Our argument relies on four ingredients: the prime number theorem, an Abel transform, a Central Limit Theorem for the Random Walk of the Primes series $\sum_p \cos(t \log p)$, where $p$ is a prime number, and the Cauchy criterion for convergence. The significance of $\Re(s) > 1/2$ arises from the universality of the $N^{1/2}$ growth of fluctuations in various central limit theorems for independent and weakly dependent random processes, which are common in statistical physics for systems of size $N$. Numerical evidence of this surprising result is presented, and some of its consequences are discussed.

* guifranca@gmail.com
† andre.leclair@gmail.com
I. INTRODUCTION AND SUMMARY

The Riemann $\zeta$-function was originally defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$  \hspace{1cm} (1)

where $s$ is a complex number. This series converges absolutely for $\Re(s) > 1$. It can be analytically continued to the entire complex plane by extending an integral representation valid for $\Re(s) > 1$, except for the simple pole at $s = 1$. Using only the unique prime factorization theorem, one can derive the Euler Product Formula, which is the equality

$$\zeta(s) = P(s)$$

and $P(s)$ is the equality

$$P(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}$$  \hspace{1cm} (2)

and $p_n$ is the $n$-th prime number. It is this formula which is the key to Riemann’s result that relates the distribution of primes, namely the prime number counting function $\pi(x)$, to a series involving an infinite sum over zeros $\rho$ of the $\zeta$-function inside the critical strip $0 < \Re(s) < 1$. Henceforth it is implicit that $\zeta(s)$ inside the strip is defined by analytic continuation.

The product (2) also converges absolutely only for $\Re(s) > 1$, and is known not to converge for $\Re(s) \leq 1$. However, in their own studies, Berry and Keating used the Euler product inside the strip [1]. Away from the real line in the right half of the critical strip $1/2 < \Re(s) \leq 1$, the phases could be such that one can make sense of the infinite product, and this is the main idea studied in this article. Just to illustrate, if one introduces alternating signs into the series (1), it converges for $\Re(s) > 0$ and is the Dirichlet $\eta$-function. In this case, it converges by the simple alternating series test. But if the signs were not strictly alternating, convergence would be difficult to disprove or prove.

In this article we provide strong arguments that (2) is valid in a concrete statistical sense for $\Re(s) > 1/2$, i.e. on the right half of the critical strip, so long as $\Im(s) \neq 0$. Our argument invokes four ingredients: the prime number theorem, an Abel transformation (summation by parts), the Central Limit Theorem (CLT), and the Cauchy criterion for convergence. The most important ingredient is the CLT, and the significance of $\sigma = 1/2$ comes from the $\sqrt{N}$ growth of fluctuations. We will also provide convincing numerical verification of this surprising result. The hybrid approach developed in [2, 3] involves both a truncated product and Riemann zeros, and is thus different than the work presented here.
We first prove the following in the next section. Consider the series

\[ B = \lim_{N \to \infty} B_N = \lim_{N \to \infty} \sum_{n=1}^{N} \cos(t \log p_n) \quad (3) \]

where \( t \) is a non-zero parameter, which we will refer to as the \textit{Random Walk of the Primes} (RWP). If \( B_N \) grows as \( \sqrt{N} \), i.e. \( B_N = O(\sqrt{N}) \), then \( \mathcal{P}(s) \) “converges” for \( \Re(s) > 1/2 \).

This \( \sqrt{N} \) growth is robust and universal in statistics and statistical physics. For instance, diffusion grows as the square-root of time. For a random variable \( x \), with standard deviation \( \sigma \), the relative uncertainty goes as \( \sigma/N \sim 1/\sqrt{N} \) and thus becomes small for large \( N \). This is a consequence of the CLT. We provide strong arguments that the CLT applies to \( (3) \).

The specialty of \( \Re(s) = 1/2 \) is due to this square root. The beauty of this argument is that it does not rely on any details of the primes, on the contrary, it depends on their global pseudo-random behaviour \[4\]. This is analogous to the fact that one does not need to know the exact positions and velocities of \( N \sim 10^{24} \) molecules in a gas to predict its pressure. The situation for the RWP is even better compared to this, since the list of primes is very long, especially towards the end.

The Euler product \[(2)\] does not converge inside the critical strip in the conventional sense, otherwise this would contradict fundamental theorems on the domain of convergence of Dirichlet series, which are always half-planes, and our argument excludes the segment \( 1/2 < s \leq 1 \) for \( s \) real. However, some divergent series are still meaningful \[5\], for instance if the divergence amounts to small oscillations around a meaningful central value. Since we invoke the pseudo-randomness of the primes, this introduces a statistical aspect to the Euler product, and it converges in the sense that its logarithm is Cesàro summable, which is to say that its \textit{average} converges. Henceforth, “statistical convergence”, or simply “convergence”, refers to Cesàro summability of a series, or the convergence of the average of a product.

By “EPF” let us refer to the Euler Product Formula \[(2)\] for \( \Re(s) > 1/2 \). If it is indeed valid, there are many consequences. There is one which is immediate. The Euler product formula \( \zeta(s) = \mathcal{P}(s) \) implies that \( \zeta(s) \) has no zeros with \( \Re(s) > 1 \), since each term \( (1-p_n^{-s})^{-1} \) never vanishes. The prime number theorem is equivalent to the fact that there are no zeros of \( \zeta(s) \) with \( \Re(s) = 1 \). Since there is a unique way to analytically continue \( \zeta \) into the strip, then if \( \mathcal{P}(s) \) statistically converges, the same argument proves there are no zeros with \( \Re(s) > 1/2 \), which is of course the \textit{Riemann Hypothesis} (RH). Incidentally, the EPF also
gives a new proof of the prime number theorem. Combined with the functional equation

$$
\chi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \chi(1-s),
$$

this shows there are also no zeros with $\Re(s) < 1/2$. Ultimately it is the pseudo-randomness of the primes that underlies the validity of the RH. Other consequences will be discussed in the last section of this article. One is a formula that relates Riemann zeros to an infinite sum over primes, which is a kind of inverse of Riemann’s result that relates primes to sums over zeros.

II. A CRITERION FOR CONVERGENCE OF THE EULER PRODUCT

In this section we obtain a criterion for convergence that is independent of any randomness of the primes. Henceforth it is implicit that $\Im(s) \neq 0$. In this region of the upper half plane $\zeta$ is analytic since it excludes the pole at $s = 1$. The product in (2) converges if the following sum converges

$$
\log \mathcal{P}(s) = -\sum_{n=1}^{\infty} \log \left(1 - \frac{1}{p_n^s}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{p_n^s} + \frac{1}{2p_n^{2s}} + \ldots\right).
$$

(5)

Using the prime number theorem, one can establish Rosser’s theorem $[6]$

$$
p_n > n \log n \quad (n \geq 1).
$$

(6)

Let $s = \sigma + it$. The second term in (5) satisfies $1/|p_n^{2s}| < 1/(n \log n)^{2\sigma} < 1/n^{2\sigma}$, thus by comparison with the hyper-harmonic series, it converges absolutely for $\sigma > 1/2$. Thus convergence of the Euler product depends on the first term in (5), i.e. on the series

$$
X = \lim_{N \to \infty} X_N = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{e^{-it \log p_n}}{(p_n)^\sigma}.
$$

(7)

Chernoff $[7]$ considered the above series with $p_n$ replaced by $n \log n$, and showed it could be analytically continued for $\Re(s) > 0$; therefore the hypothetical zeta function based on this product has no zeros in the whole critical strip. As we will see, it is important not to do this in the phase.

Already one can see the role of $\sigma = 1/2$, but for elementary reasons that are clearly not enough for our purposes. The above series (7) clearly does not converge on the real line,
\( t = 0 \), except for \( \sigma > 1 \). For complex \( s \), it also only converges absolutely for \( \sigma > 1 \). The series actually also fails the Dirichlet test of convergence since \( |\sum_n e^{-it \log p_n}| \) is unbounded; if it were bounded then the series would converge for all \( \sigma > 0 \), which is certainly not the case, otherwise this would rule out Riemann zeros on the critical strip. Thus \( X \) fails the simplest convergence tests.

Let us consider convergence of the real and imaginary parts of \( X \) separately. If the following series converges

\[
S = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{n=1}^{N} a_n b_n
\]

where

\[
a_n = \frac{1}{(p_n)^\sigma}, \quad b_n = \cos(t \log p_n),
\]

then the real part of \( X \) converges. Analogous arguments apply to the imaginary part of \( X \), with \( b_n = \sin(t \log p_n) \).

As stated above, when \( t = 0 \), \( S \) converges only if \( \sigma > 1 \). However, the oscillations of \( b_n \) can conspire to make the series converge for \( \sigma \leq 1 \). The simplest example to illustrate this is to replace \( b_n \) by \((-1)^n\). In this case the alternating sign test shows that the series converges for \( \sigma > 0 \). For our series \( S \), the signs of \( b_n \) can be both positive or negative, but they do not strictly alternate. Rather, the situation here is between the two extremes of strictly alternating signs versus all positive signs, which suggests that \( S \) may converge for \( \sigma \) larger than some number between 0 and 1.

Through an Abel transformation [8], the partial sum in (8) can be written as

\[
S_N = a_N b_N - \sum_{n=1}^{N-1} B_n (a_{n+1} - a_n)
\]

where

\[
B_n = \sum_{k=1}^{n} b_k.
\]

From (10) we have the following difference between the partial sums:

\[
|S_M - S_N| < |a_M||b_M| + |a_N||b_N| + \sum_{n=N}^{M-1} |B_n||a_{n+1} - a_n|.
\]

We now apply the Cauchy criterion [9]. We have to show that for every \( \epsilon > 0 \) there is an integer \( K \) such that if \( M > N \geq K \) then \( |S_M - S_N| < \epsilon \).
Using the well known bounds \[10, 11\]

\[n(\log n + \log \log n - 1) < p_n < n(\log n + \log \log n) \quad (n \geq 6), \quad (13)\]

we have that

\[|a_{n+1} - a_n| = \left|\frac{1}{(p_{n+1})^\sigma} - \frac{1}{(p_n)^\sigma}\right| < \frac{2\sigma}{n \log n \sqrt{n}}, \quad (14)\]

Now choose a large enough \(K\) such that for \(M > N \geq K\) the first two terms in (12) go to zero due to \(|a_N| \to 0\). For \(n \geq N \geq K\), (14) also goes to zero, thus convergence depends essentially on the growth of \(B_N\). For the moment, let us assume that \(B_N = O(\sqrt{N})\), i.e. there is a constant \(C\) such that \(|B_N| \leq C\sqrt{N}\). This is a non-trivial assumption, and is going to be justified in the next section. Then we have

\[|S_M - S_N| < \sum_{n=N}^{M-1} \frac{2\sigma C}{n^{\sigma+1/2} \log n} < 2\sigma C \sum_{n=N}^{M-1} \frac{1}{n^{\sigma+1/2}} < \epsilon. \quad (15)\]

The last conclusion in (15) is because the last term is just the hyper-harmonic series which converges for \(\sigma > 1/2\) and it is therefore a Cauchy sequence, and thus satisfies \(\sum_{n=N}^{M-1} 1/n^{\sigma+1/2} < \epsilon\). Therefore, the series (8) converges for \(\sigma > 1/2\), so long as \(\Im(s) \neq 0\). Analogous arguments apply for the imaginary part of (7).

Thus if \(B_N = O(\sqrt{N})\) the series (7) converges for \(\sigma > 1/2\), clearly except for \(t = 0\) since our argument relies on the phases. Again \(\sigma = 1/2\) is singled out, but for entirely different reasons than before for the next to leading term.

**III. \(\sqrt{N}\) FROM PSEUDO-RANDOMNESS OF THE PRIMES**

In the last section, we showed that if \(B_N = O(\sqrt{N})\) then the Euler product “converges” on the right half of the critical strip for \(t \neq 0\). In this section, we provide two different but intrinsically related arguments for why this is the case, once one refines what one means by convergence by giving it a statistical meaning. These will be strong arguments because of the universality of \(\sqrt{N}\) in various versions of the Central Limit Theorem (CLT). The partial sum \(B_N\) in and of itself has no probabilistic aspect in the usual sense of probability theory, since it is not a random variable. Rather, it is a fixed number for each \(N\), since in principle each prime \(p_n\) can be determined. Nevertheless, statistical arguments can be valid for understanding its growth as a function of \(N\), which is all we need. The first argument
invokes pseudo-randomness of the primes themselves. The second argument is more specific and based on the well known fact that some gapped trigonometric sequences behave like sequences of independent and identically distributed (iid) random variables, and therefore obeys the CLT. We will study a series $L_N(u)$ that is a random variable, where $B_N$ is one member of its ensemble. We will argue that it satisfies the CLT because the primes have increasing gaps.

A. Pseudo-randomness of the Primes

The primes are deterministic, but the correlations between them are very weak. For instance, for $n > 1$ they are all odd and asymptotically behave like $p_n \sim n \log n$. The twin-prime conjecture can also be viewed as a weak correlation between primes [12]. Nevertheless, it is generally accepted that they behave pseudo-randomly1 [4]. Globally they behave regularly, while locally they appear irregular. This should be compared with the behaviour of an ideal gas in thermodynamic equilibrium (see for instance [13]). Macroscopically the gas is uniform, with a regular behaviour, while microscopically each individual atom or molecule behaves irregularly. However, the macroscopic behaviour of the gas can be deduced through statistical averages assuming very little knowledge about its individual constituents. This is so because in the limit of a large number of particles, the macroscopic behaviour is universally described by the CLT, and does not depend on specific details of its microscopic structure. Moreover, due to the lack of information about primes—there is no formula to predict the $n$-th prime—it is reasonable to assume that they are pseudo-random. Since we are not interested in any detailed property of primes—there is no formula to predict the $n$-th prime—it is reasonable to assume that they are pseudo-random. Since we are not interested in any detailed property of primes, such as gaps $p_{n+1} - p_n$, but rather just in the behaviour of the RWP \footnote{“God may not play dice with the universe, but something strange is going on with the prime numbers”}. In the large $N \to \infty$ limit, to this purpose, we can assume that $\{p_n\}$ mimics a random, independent, though increasing sequence.

Let us motivate our subsequent arguments as follows. Consider the partial sum (11), with $b_n$ defined in (9). The $\{b_n\}$ is a sequence of positive or negative numbers with $|b_n| \leq 1$. For simplicity, let us first replace $b_n$ by its sign. Then $\{b_n\}$ is a sequence of $+1$ and $-1$ that

\footnote{This is a misattributed quotation to P. Erdős, one of the pioneers in applying probabilistic methods to number theory, but actually it seems to be a comment from Carl Pomerance in a talk about the Erdős-Kac theorem, in response to Einstein’s famous assertion about quantum mechanics.}
FIG. 1: The partial sum $B_N = \sum_{n=1}^{N} \cos(t \log p_n)$ versus $N$ for a fixed $t = 1000$ (blue line). The solid black curve is $\sqrt{N}$.

appears random. Suppose it was truly random, i.e. each $b_n$ had equal probability to be $+1$ or $-1$. Then this sequence is just the Random Walk in one dimension. The CLT applies and $B_N/\sqrt{N}$ approaches a gaussian (normal) distribution at large $N$, which implies $B_N$ grows as $\sqrt{N}$.

Let us return now to our original $B_N$. Since $-1 \leq \cos(t \log p_n) \leq 1$ and the primes behave pseudo-randomly, if we assume no correlations between them, then $\{b_n\}$ behave like a sequence of iid random variables on the real interval $(-1, 1)$. The CLT applies and $B_N/\sqrt{N}$ converges to the gaussian distribution $\varphi_{0,1/2}$, with zero mean and variance $1/2$, since the average of $\cos^2$ is $1/2$. This implies $B_N = O(\sqrt{N})$.

The $\sqrt{N}$ growth is universal in statistics for systems of size $N$. If the primes are not completely uncorrelated, it is known [14] [16] that the CLT still applies with weak correlations between $b_n$’s, namely if $b_m$ and $b_n$ are independent for large $|m - n|$, which is certainly true of the primes $p_n$. Moreover, in some cases the CLT is valid for even strong correlations among the $b_n$’s, as will be discussed below.

The statistical convergence of $B_N/\sqrt{N}$ then does not rely on any special detailed properties of the primes, but rather the opposite, their pseudo-random behaviour. Recall though that we did use the prime number theorem in establishing the convergence of (7). In Figure 1 we plot the partial sums $B_N$ for $N$ up to 600 000, and one clearly sees this $\sqrt{N}$ growth, as predicted.
B. The Random Walk of Primes as a trigonometric series with gaps

Let us now characterize the pseudo-randomness of the primes in a more concrete manner. Important for our purposes, it is known that some lacunary\(^2\) trigonometric series behave like sums of iid random variables. The CLT is known to apply for these series, as long as the gaps grow fast enough. Consider for instance the lacunary series inspired by Hadamard, 

\[ C_N(u) = \sum_{k=1}^{N} c_k(u) \quad \text{with} \quad c_k = \cos(u n_k), \]

where \( u \) is a random variable uniformly distributed on the interval \((0, 2\pi)\) and \( n_k \) are integers satisfying \( n_{k+1}/n_k > q > 1 \) for all \( k \) (Hadamard gaps). Salem and Zygmund \[17\] demonstrated that in the limit of large \( N \), \( C_N/\sqrt{N} \to \varphi_{0,1/2} \), where \( \varphi_{0,1/2} \) is a gaussian distribution with zero mean and variance \( 1/2 \). For instance, if \( n_k = 2^k \) the theorem holds. Further developments that weaken the gap condition are in \[13–21\]. It is important to note that \( u \) is not chosen randomly for each \( c_k \) in the sum; rather one chooses a fixed \( u \), randomly, then computes the sum \( C_N(u) \). In this sense a single sum \( C_N(u) \) is completely deterministic. The ensemble \( \{C_N(u)\} \), generated by choosing a random \( u \) for each element in the set, has a gaussian probability distribution and obey the CLT. Thus the \( c_k(u) \) are clearly not iid. If for instance \( n_k = 2^k \), the \( c_k \) are strongly dependent, i.e. strongly correlated, but the CLT still applies; what is important are the increasing gaps.

Now we consider a series that is much closer to the actual \( B_N \) \[8\]. As stated above, there is no probabilistic aspect to the RWP in the conventional sense of probability theory, however one can create one. The theorem for the lacunary series \( C_N(u) \) described above is stated with \( n_k \) an integer. However, the most important aspect is the lacunary property, namely that \( \{n_k\} \) is an increasing gapped sequence. Let us take \( n_k = t \log p_k \). The pseudo-randomness of the primes makes this a lacunary series and the CLT should apply. Note also that the bound \( (13) \) shows that gaps between primes tends to increase.

Let us therefore introduce a random variable \( u \) uniformly distributed on the interval \((0, 2\pi)\) and define the partial sum

\[ L_N(u) = \sum_{n=1}^{N} \cos(u t \log p_n). \quad (16) \]

It satisfies the property \( \log p_{n+1} > \log p_n \) with irregular gaps which tend to grow on average.

Now consider an ensemble \( \{L_N(u_i)/\sqrt{N}\}_{i=1}^{E} \), where for each element of the set we choose

\(^2\) The word lacunary comes from lacuna, gap or vacancy.
FIG. 2: The probability distribution for $L_N/\sqrt{N}$ with $t = 1000$, $N = 10000$ and $E = 50000$ ensembles (histogram). The solid black line is the standard normal distribution $\varphi_{0,1/2}$ with zero mean and variance 1/2.

A random $u_i$ in the interval $(0, 2\pi)$. The methods of probability theory apply and one can consider its probability distribution. This is done numerically in Figure 2 for a fixed $t$, and one can see, as expected, the validity of the CLT for (16). This result confirms that $L_N/\sqrt{N}$ approaches a gaussian distribution at large $N$, with zero mean and variance 1/2. This histogram is essentially indistinguishable from the analogous one for the conventional random walk, or with a continuous iid random variable on the interval $(-1, 1)$. Numerically we observe that the result gets better for high $t$, since the latter creates larger gaps. If one replaces $p_n$ with $n \log n$ one loses this gaussian distribution.

A CLT for $L_N(u)/\sqrt{N}$ guarantees that $B_N = O(\sqrt{N})$, since $B_N = L_N(1)$, and for any $u$, $L_N(u)/\sqrt{N}$ falls on the gaussian curve. To summarize, we have argued that the pseudo-randomness of the primes creates gaps in the series (16), and this is what is mainly responsible for the validity of the CLT.

IV. STATISTICAL CONVERGENCE AND CESÀRO SUMMABILITY

The series (7) is a generalized Dirichlet series, which by definition has the form $\sum_n a_n e^{-s\lambda_n}$ with $\{\lambda_n\}$ a strictly increasing sequence of positive real numbers. There is a fundamental
theorem that the region of convergence of such series is a half-plane \[\Re(s) > \sigma_a\]. One speaks about the abscissa of absolute convergence \(\sigma_a\), where the series absolutely converges for the whole half-plane \(\Re(s) > \sigma_a\). For (7) we have \(\sigma_a = 1\). There is also an abscissa for conditional convergence \(\sigma_c\). Absolute convergence is more stringent, \(\sigma_c \leq \sigma_a\). In the case of an ordinary Dirichlet series with \(\lambda_n = \log n\), it can be shown that \(0 \leq \sigma_a - \sigma_c \leq 1\). Interestingly, for Dirichlet series with random coefficients, \(0 \leq \sigma_a - \sigma_c \leq 1/2\) \[23\]. For the series (7) we also have \(\sigma_c = 1\) due to the pole at \(s = 1\) and the half-plane convergence theorem. Thus (7) does not converge for \(\sigma \leq 1\) in the strict sense.

Since the above argument that \(B_N = O(\sqrt{N})\) invoked pseudo-randomness of the primes, the convergence of (7) and thus the Euler product (2) has a statistical aspect. Namely, for increasing \(N\), there are small fluctuations around a central value, and the average of \(X_N\) converges for large \(N\). This property is referred to as Cesàro summability, which for the series \(X\) should be a consequence of the CLT for \(B_N\) discussed in the last section. Therefore, in terms of the Euler product itself, we then propose that for \(\Re(s) > 1/2\) and \(\Im(s) \neq 0\),

\[
\zeta(s) = \lim_{N \to \infty} \langle P_N(s) \rangle, \quad P_N(s) = \prod_{n=1}^{N} \left(1 - \frac{1}{p_n^s}\right)^{-1},
\]

(17)

where \(\langle P_N(s) \rangle\) is its arithmetic average over \(N\):

\[
\langle P_N(s) \rangle = \frac{1}{N} \sum_{n=1}^{N} P_n(s).
\]

(18)

We illustrate this numerically in Figure 3. One sees that the averages \(\langle P_N \rangle\) converge to the correct value of \(\zeta\), while the product itself \(P_N\) oscillates around its average. At large \(N\), the oscillations are more and more regular.

V. NUMERICAL STUDIES

We have provided arguments that the Euler product (2) statistically converges in the region \(\sigma > 1/2\) and \(t \neq 0\). We now provide numerical evidence of the validity of this result. Throughout this section we plot the Euler product itself, rather than its average, since the resolution of the plots is not high enough to see the fluctuations, so that these plots are indistinguishable from the plots of the average. In Figure 4 one can see how the partial product in (17) converges to the \(|\zeta(s)|\) function as we increase \(N\). Below, we study the
FIG. 3: The solid black line is $\mathcal{P}_N$ and the dots are $\langle \mathcal{P}_N \rangle$ for $s = 0.9 + i \, 100$ and $N \in [10, 2000]$. Here $|\zeta(s)| = 1.75582$.

convergence of arg $\zeta$; these two numerical studies ensure that both the real and imaginary parts of the Euler product converge. As we approach the critical line $\sigma \to 1/2^+$ higher $N$ is of course required.

One can clearly see how the Euler product diverges for $\sigma \leq 1/2$ from Figure 5. The curves only match for $\sigma > 1/2$, and the dramatic change in behavior is abrupt at $\sigma = 1/2$.

Let us also verify convergence for arg $\zeta$, which plays a central role for the zeros on the critical line (see the next section). Using the EPF we have

$$S_\delta(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + \delta + it\right) = -\frac{1}{\pi} \lim_{N \to \infty} \Im \left[ \sum_{n=1}^{N} \log \left(1 - \frac{1}{p_n^{-1/2 - \delta - it}}\right) \right].$$

(19)

The above equality is verified in Figure 6.

We argued above that the convergence is expected to be slower for low $t$ since it is closer to $t = 0$ where there is no convergence. Also, clearly, the convergence is slower closer to the critical line $\sigma = 1/2$. This is illustrated in Figure 7 where one clearly sees the divergence near $t = 0$. It converges very well in the vicinity of the first zero on the critical line with $t \sim 14$, and even at $\sigma = 0.8$ one can see evidence for this first zero. Notice that the fluctuations are larger at low $t$.

In Table I we show some values of the average $|\langle \mathcal{P}_N \rangle|$ and the product itself $|\mathcal{P}_N|$. The convergence is slow, and high $N$ is required to have more precise results, although one can
see that $\langle P_N \rangle \to \zeta$ as $N \to \infty$, whereas the unaveraged $P_N$ continues to oscillate around $\zeta$. With $N = 10^5$ we obtain nearly 6 digit accuracy.

VI. SOME CONSEQUENCES OF THE EULER PRODUCT FORMULA

Having provided analytical arguments and numerical evidence, in this section we assume the EPF is valid for $\Re(s) > 1/2$, and discuss some possible consequences, without attempting fully rigorous arguments. It is implicit that we are referring to the statistical average of the Euler product.
FIG. 5: Left: the solid black line corresponds to $|\zeta(\sigma + it)|$ against $0 < \sigma < 1$ for a fixed $t = 10000$. The dashed red line is the partial product $|P_N(\sigma + it)|$ with $N = 10^4$. One clearly sees that it is not valid for $\sigma \leq 1/2$, as predicted. Right: the solid black line is the exact $|\zeta|$, and the red line is the partial product $|P_N|$ (with $N = 10^4$), against $t$. We took $\sigma = 0.4$. The blue dots correspond to the average $|\langle P_N \rangle|$.

FIG. 6: Verification of (19). The solid black line is the actual $\frac{1}{\pi} \arg \zeta(1/2 + \delta + it)$ and the red line is the RHS of (19) as a function of $t$. Red dots are added to the line to aid in visualization. We used $\delta = 10^{-1}$ and $N = 10^5$.

**A. The function $S(t)$**

Let $N(T)$ denote the number of zeros *in the entire critical strip*, $0 < \sigma < 1$, up to height $T$ where $T$ is not the ordinate of a zero. There is a known exact formula for $N(T)$ due to
FIG. 7: The solid black line is $|\zeta(0.8 + it)|$, the red line is $|\mathcal{P}_N|$, with $N = 10^4$, and the blue dots are $|\langle \mathcal{P}_N \rangle|$.

| $N$  | $|\langle \mathcal{P}_N \rangle|$ | $|\mathcal{P}_N|$ |
|------|----------------|----------------|
| $10^3$ | 0.976752 | 0.97221 |
| $2 \times 10^3$ | 0.976690 | 0.981506 |
| $3 \times 10^3$ | 0.977653 | 0.976654 |
| $4 \times 10^4$ | 0.977865 | 0.975735 |
| $5 \times 10^3$ | 0.977926 | 0.984674 |
| $6 \times 10^3$ | 0.977463 | 0.977983 |
| $7 \times 10^3$ | 0.978208 | 0.97651 |
| $8 \times 10^3$ | 0.977593 | 0.978773 |
| $9 \times 10^3$ | 0.978290 | 0.981781 |
| $10^4$ | 0.9779 | 0.971017 |

TABLE I: Convergence of the average $\langle \mathcal{P}_N \rangle$ and the Euler product $\mathcal{P}_N$. In the left table we have $s = 0.95 + i 20$ while in the right table $s = 0.95 + i 100$.

Backlund [24],

$$N(T) = \frac{1}{\pi} \vartheta(T) + 1 + S(T),$$

where $\vartheta(T)$ is the Riemann-Siegel $\vartheta$ function

$$\vartheta(t) = \arg \Gamma\left(\frac{1}{4} + i \frac{t}{2}\right) - t \log \sqrt{\pi}$$

and

$$S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right).$$
This result is obtained by the argument principle. Here, \( S(t) \) is defined by piecewise integration of \( \zeta'/\zeta \) from \( s = 2 \) to \( 2 + it \), then to \( \frac{1}{2} + it \). \( N(T) \) is a monotonically increasing staircase function, however it is discontinuous at the zeros where it jumps by the multiplicity of the zero. Since \( \vartheta(t) \) is smooth, these jumps come from \( S(t) \).

Now, if the EPF is valid, then there are no zeros to the right of the critical line. Then \( S(t) \) defined by piecewise integration does not encounter any zeros as one approaches the critical line and must be the same as in (19). This is an explicit formula for \( S(t) \), and for \( \delta \) strictly not zero, \( S_\delta(t) \) is continuous. We can also express this as

\[
S(t) = \lim_{\delta \to 0^+} S_\delta(t) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\phi_n(t)}{n}, \quad \phi_n(t) = \lim_{\delta \to 0^+} \sum_p \frac{\sin(nt \log p)}{p^{(1/2+\delta)}}. \tag{23}
\]

The function \( S(t) \) “knows” about the Riemann zeros since it jumps at each zero. Thus (19) or (23) is a relation between the Riemann zeros and the primes that is completely different than Riemann’s result for the prime number counting function \( \pi(x) \) expressed as a sum of zeros. For the latter, one needs to sum over all zeros to identify the primes. Our result is the inverse: to find zeros, one must sum over all primes. A stronger version of this is presented below. If one replaces \( p_n \) by \( n \log n \), then \( S(t) \) no longer jumps at the zeros. This indicates that the zeros themselves and their GUE statistics \([25, 26]\) arises from fluctuations in the primes.

### B. A transcendental equation for the \( n \)-th zero

Let us characterize precisely the zeros on the line \( \rho_n = \frac{1}{2} + it_n \) for \( n = 1, 2, 3, \ldots \). In \([27, 29]\) a transcendental equation for each \( t_n \) was proposed which depends only on \( n \). A more lengthy discussion of this result can be found in our lectures \([30]\). This transcendental equation for \( t_n \) is easy to describe. Let \( \theta(s) = \arg \chi(s) \) where \( \chi \) is defined in \([4]\). It was argued that the zeros are in one-to-one correspondence to the zeros of \( \cos \theta \), namely

\[
\lim_{\delta \to 0^+} \theta(\frac{1}{2} + \delta + it_n) = (n - \frac{3}{2})\pi, \quad \theta(s) = \arg \chi(s). \tag{24}
\]

We obtained this equation through a delicate limit that was not fully rigorous, and furthermore we were unable to prove that this equation has a unique solution for every \( n \). As we now describe, the EPF resolves these issues.
Let us first provide a different derivation of (24) based on the EPF. Using $S_\delta(T)$ in (20), $N(T)$ is now a monotonically increasing staircase function that is smoothed out at the jumps, i.e. it is continuous everywhere. Since it jumps at the ordinate of a zero $t_n$, and there are no zeros off the line, one can use $N(T)$ to find an equation for $t_n$. Assume for the moment that all zeros are simple; below we will give an argument for why this is the case. Then one simply replaces $T \rightarrow t_n$ and $N \rightarrow n - \frac{1}{2}$ in $N(T)$:

$$\vartheta(t_n) + \pi \lim_{\delta \to 0^+} S_\delta(t_n) = (n - \frac{3}{2})\pi.$$ (25)

This equation is identical to (24). The small $\delta$ is required to be positive because the EPF is only valid to the right of the critical line. Since the left hand side of the above equation is monotonic and continuous, there is a unique solution to (25) for every $n$. The zeros on the line are now enumerated and one can determine the counting formula $N_0(T)$ for them from (25) and of course $N_0(T) = N(T)$. Thus, the existence of a unique solution to (25) for each $n$ is also equivalent to the validity of the RH.

Using the above definition (19) for $S_\delta$, the above equation (25) no longer makes any reference to the $\zeta$-function itself. This indicates that every single individual zero depends on all of the primes.

The $S_\delta(t)$ term in (25) fluctuates and is very small compared with the $\vartheta(t)$ term for large $t$. If one ignores it, and uses Stirling’s approximation for the $\Gamma$-function, then the solution to the resulting equation can be expressed in terms of the Lambert $W$-function [28]:

$$t_n \approx 2\pi \frac{n - \frac{11}{8}}{W[e^{-1}(n - \frac{11}{8})]}.$$ (26)

The equation (25) was used to numerically calculate many zeros to very high accuracy, thousands of digits, up to the billion-th zero [28, 30]. The approximation (26) is also quite accurate; generally the integer part is correct, but it does not capture the fluctuations that satisfy GUE statistics. This is clear since this approximation does not capture any sum over primes. This suggests that the GUE statistics of the zeros originates from the pseudo-randomness of the primes.

As previously stated, the equation (25) is identical to the equation (24) which comes from $\cos \theta = 1$. In [28] the argument which led to (25) was entirely different than the one presented here, i.e. it did not assume the RH, and did not rely on the EPF nor knowledge of $N(T)$. It was obtained directly on the critical line using the functional equation without
assuming the zeros were simple. If the argument that (24) is valid can be made rigorous, then this would show that all zeros are indeed simple, since the zeros of the cosine are simple.

The Dirichlet \(L\)-functions and \(L\)-functions based on modular forms also have a functional equation and an Euler product. Our results clearly extend to these infinite classes of functions due to the universality of central limit theorem. The analogs of the transcendental equation above were already presented in [29].

C. A Counterexample

There is a well-known counterexample to the RH based on the Davenport-Heilbronn function, which is a linear combination of two Dirichlet \(L\)-functions. It satisfies a functional equation like [1]. The Dirichlet \(L\)-functions each have an Euler product, however the sum does not. The analog of (25) was studied for this function in [30]. It was found that the analog of \(S_\delta(t)\) becomes ill-defined in the vicinity of zeros off of the line and there are no solutions to the analog of (25). This is now perfectly clear, since there is no Euler product formula to smooth out \(S_\delta\) here. Thus, the validity of the RH depends on both the functional equation [4] and the Euler Product Formula.

Acknowledgments

We thank Denis Bernard for helpful discussions on probability theory and especially Keith Conrad for discussions on convergence of generalized Dirichlet series.

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