FLAT-TOPPED PROBABILITY DENSITY FUNCTIONS FOR MIXTURE MODELS

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Abstract. This paper investigates probability density functions (PDFs) that are continuous everywhere, nearly uniform around the mode of distribution, and adaptable to a variety of distribution shapes ranging from bell-shaped to rectangular. From the viewpoint of computational tractability, the PDF based on the Fermi-Dirac or logistic function is advantageous in estimating its shape parameters. The most appropriate PDF for \( n \)-variate distribution is of the form:

\[
p(x) \propto \left[ \cosh \left( \left[ (x - m)^T \Sigma^{-1} (x - m) \right]^{n/2} \right) + \cosh \left( r^n \right) \right]^{-1}
\]

where \( x, m \in \mathbb{R}^n \), \( \Sigma \) is an \( n \times n \) positive definite matrix, and \( r > 0 \) is a shape parameter. The flat-topped PDFs can be used as a component of mixture models in machine learning to improve goodness of fit and make a model as simple as possible.

1. Introduction

In machine learning, mixture models [1, 2, 3] are valuable for modeling and analyzing complex real-world data. The Gaussian mixture model (GMM) is widely used due to its simplicity and fundamentality. It is easy to estimate the parameters of a single Gaussian (normal) distribution using maximum likelihood (ML) estimation, which simplifies the M-step of the expectation–maximization (EM) algorithm [4]. Besides, exponential families have conjugate priors that help derive analytical expressions in maximum a posteriori (MAP) estimation and Bayesian inference such as Variational Bayes (VB) [5].

In practice, however, bell-shaped distributions are not always appropriate for modeling real data sets. There can be a variety of data coming from non-Gaussian distributions. A uniform (rectangular) distribution \( U(a, b) \) will be proper for data points uniformly distributed in line, area, or volume elements, which often appear in spatial analysis. It attaches importance to the distribution boundaries, \( a \) and \( b \), contrasted with the normal distribution characterized by the central tendency and deviation from the mean. Both concepts are essential to develop various methods for data clustering and classification. The uniform distribution is also fundamental but has drawbacks for the components of mixture models. Its probability density function (PDF) is zero outside the interval \([a, b]\) and discontinuous at \( a \) and \( b \), which are disadvantageous for ML estimation and numerical optimization algorithms. To avoid these inconveniences, it should be modified to have smooth steps at its boundaries.

Key words and phrases. flat-topped distribution, generalized Fermi-Dirac distribution, hyperbolic function, compound distribution, mixture model, generalized EM algorithm.
Some univariate PDF families have desirable properties. For example, the generalized normal distribution \cite{6} (or the exponential power distribution \cite{7, 8}), the generalized Cauchy distribution \cite{9} (or the generalized Pearson VII distribution \cite{7, 10}), and the Ferreri distribution \cite{11} are supported on the whole real line, continuous, unimodal, and can be flat-topped about the mode. They are used not only for data clustering via mixture models \cite{12} but also for the studies of laser beam shapes \cite{13}, uncertainty in measurements \cite{14}, and noise distributions \cite{15}. Unfortunately, their shape parameters are difficult to estimate. In practice, alternative PDFs have been proposed for the mixtures of rectangles \cite{16, 17}, though there remains a need for more detailed research.

In this paper, we study a variety of flat-topped PDFs useful for finite mixture models. The condition of flatness is described in the following section. The flat-topped PDFs based on various combinations of sigmoid functions or generalization of the Cauchy distribution are categorized into four general types in Section 3 and illustrated with some specific forms in Section 4. From the viewpoint of computational tractability, the combination of the logistic functions is most appropriate for building mixture models. Furthermore, a generalized Fermi-Dirac distribution and its variant using hyperbolic functions are advantageous for modeling multivariate elliptical distributions. We also discuss the ML estimation of model parameters using an iterative method in Section 5, a practical procedure using the generalized EM algorithm \cite{4} to build mixture models in Section 6, and the usefulness of the flat-topped PDFs with some simulation examples in Section 7.

2. Preliminaries

To deal with the vague concept of “flat-topped” PDF, we determine quantitative criteria for describing its property. We verify that the generalized normal distribution satisfies this property under certain conditions and see how difficult it is to estimate its shape parameter.

2.1. Condition of flat-topped PDF. Let \( p(x) \) be a PDF that is continuous for all \( x \in \mathbb{R} \). If it is twice differentiable, let \( p'(x) \) and \( p''(x) \) be its first and second derivatives, respectively. Suppose that \( p(x) \) is unimodal and \( x_m \) denotes the mode defined by

\[
x_m \in \arg \max \; p(x),
\]

so that \( p'(x_m) = 0 \) and \( (x - x_m)p'(x) \leq 0 \). The concept of the flat-topped \( p(x) \) is illustrated in Figure 2.1, where \( a, b \in \mathbb{R} \) are location parameters indicating the boundaries of the main part and \( a < x_m < b \). If \( a \) and \( b \) satisfy

\[
\int_{-\infty}^{a} p(x) \, dx = \int_{a}^{x_m} (p(x_m) - p(x)) \, dx,
\]

\[
\int_{x_m}^{b} (p(x_m) - p(x)) \, dx = \int_{b}^{\infty} p(x) \, dx,
\]

then we have

\[
a = x_m - \frac{1}{p'(x_m)} \int_{-\infty}^{x_m} p(x) \, dx \quad \text{and} \quad b = x_m + \frac{1}{p'(x_m)} \int_{x_m}^{\infty} p(x) \, dx,
\]

which implies \( p(x_m) = (b - a)^{-1} \). The interval \( [a, b] \) is expected, but not required, to be close to the full width at half maximum (FWHM), i.e., \( p(a) \approx p(x_m)/2 \approx \)}
p(b). Within its middle part $[x_1, x_2]$ such that $x_m \in [x_1, x_2] \subset (a, b)$, we assume $p(x)$ is nearly constant. Then let us say that $p$ is $(\Delta, \varepsilon)$-flat-topped if for a given $\varepsilon > 0$ there exists $\Delta = x_2 - x_1 > 0$ that satisfies

$$1 - \frac{1}{p(x_m)} \int_{x_1}^{x_1 + \Delta} p(x) \, dx < \varepsilon.$$  \hfill (2.1)

If $p(x)$ is concave within $[x_1, x_2]$, this condition may be substituted by

$$1 - \frac{p(x_1) + p(x_2)}{2p(x_m)} < \varepsilon.$$  \hfill (2.2)

For example, the PDF of $U(a, b)$ defined by

$$p_U(x \mid a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$  \hfill (2.3)

is $(\Delta, \varepsilon)$-flat-topped for any $\varepsilon > 0$ and $0 < \Delta < b - a$. Therefore, any PDF that approaches $p_U(x \mid a, b)$ can be $(\Delta, \varepsilon)$-flat-topped if close enough. Of course, $p(x)$ is required neither to be smooth (class $C^\infty$) nor flat (all derivatives vanish at $x \in [x_1, x_2]$) in a calculus sense.

In practice, however, a rigorous evaluation of $(\Delta, \varepsilon)$ is not necessary. Alternatively, without using $\Delta$, we simply say that $p$ is $\varepsilon$-flat-topped if for a given $\varepsilon > 0$ there exist $a$ and $b$ such that

$$|p''(x_m)| \frac{a-b}{p'(a) - p'(b)} < \varepsilon.$$  \hfill (2.4)

In the following sections, this inequality is mainly used for estimating parameters that determine the flat-topped shape, though there is no clear boundary between flat-topped and bell-shaped, even if $\varepsilon \ll 1$.

![Figure 2.1. The PDF of a typical flat-topped distribution and its derivatives](image)

Figure 2.1. The PDF of a typical flat-topped distribution and its derivatives

2.2. Generalized normal distribution. The generalized normal (or exponential power) distribution [6, 7, 8] is defined by the following PDF for all $x \in \mathbb{R}$:

$$p_{GN}(x \mid \mu, s, \beta) = \frac{\beta}{2s \Gamma(1/\beta)} \exp \left(- \frac{|x - \mu|^\beta}{s} \right),$$  \hfill (2.5)
where \( \mu \in \mathbb{R} \) is a location parameter, \( s > 0 \) is a scale parameter, \( \beta > 0 \) is a shape parameter, and \( \Gamma \) denotes the gamma function. This includes the normal distribution \( \mathcal{N}(\mu, \sigma^2) \), which is given by

\[
p_N(x \mid \mu, \sigma^2) = p_{GN}(x \mid \mu, \sqrt{2\sigma}, 2).
\]  \( \text{(2.6)} \)

The cumulative distribution function (CDF) is expressed as

\[
P_{GN}(x \mid \mu, s, \beta) = \int_{-\infty}^{x} p_{GN}(y \mid \mu, s, \beta) \, dy
\]

\[
= \frac{1}{2} + \frac{\text{sgn}(x - \mu)}{2\Gamma(1/\beta)} \cdot \gamma\left(\frac{1}{\beta}, \frac{|x - \mu|}{s}\right)
\]

where \( \text{sgn}(\cdot) \) denotes the sign function and \( \gamma(\cdot, \cdot) \) denotes the lower incomplete gamma function.

Let \( \mu_f(n) \) denote the \( n \)-th central moment of a function \( f \). If \( n \) is even, then

\[
\mu_{p_{GN}}(n) = \frac{s^n \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{s^n \Gamma\left(\frac{n+1}{2} + 1\right)}{(n+1) \Gamma\left(\frac{1}{2} + 1\right)}
\]

where we have used \( \Gamma(x+1) = x\Gamma(x) \), and so the kurtosis \( \kappa \) is given by

\[
\kappa_{p_{GN}} = \frac{\mu_{p_{GN}}(4)}{\mu_{p_{GN}}(2)^2} = \frac{9 \Gamma(5/\beta + 1) \Gamma(1/\beta + 1)}{5 \Gamma(3/\beta + 1)^2}.
\]  \( \text{(2.7)} \)

Hence, \( \lim_{\beta \to \infty} \kappa_{p_{GN}} = 9/5 \), which is equal to the kurtosis of \( p_U \).

The condition \( (2.4) \) for flat-topped shape is checked as follows. The first and second derivatives of \( p_{GN}(x) \) for \( x \neq \mu \) are

\[
p'_{GN}(x \mid \mu, s, \beta) = -\frac{\text{sgn}(x - \mu) \beta^2}{2s^2 \Gamma(1/\beta)} \cdot \frac{|x - \mu|^{\beta-1}}{s} \exp\left(-\frac{|x - \mu|^\beta}{s}\right)
\]

\[
p''_{GN}(x \mid \mu, s, \beta) = -\frac{\beta^2}{2s^2 \Gamma(1/\beta)} \cdot \frac{\beta - 2 - \frac{\beta - 1}{s} |x - \mu|^\beta}{s} \exp\left(-\frac{|x - \mu|^\beta}{s}\right).
\]

If \( a = \mu - s (\ln 2)^{1/\beta} \) and \( b = \mu + s (\ln 2)^{1/\beta} \), then

\[
p_{GN}(x \mid \mu, s, \beta) = \frac{\beta (\ln 2)^{1/\beta}}{(b - a) \Gamma(1/\beta)} 2^{-\frac{2b - a - b}{s^2}} |\cdot|^\beta
\]

\[
p'_{GN}(a) - p'_{GN}(b) = -\frac{\beta^2}{4s^3 \Gamma(1/\beta)} (\ln 2)^{1-2/\beta}.
\]

The mode is at \( x_m = \mu \) so that we have

\[
p''_{GN}(\mu) = \begin{cases} \frac{-2}{\sqrt{\pi} s^3} & \text{if } \beta = 2, \\ 0 & \text{if } \beta > 2. \end{cases}
\]

Therefore, \( p_{GN} \) is \( \varepsilon \)-flat-topped for any \( \varepsilon > 0 \) if \( \beta > 2 \). The interval \([x_1, x_2]\) of \( x \) that satisfies the condition \( 1 - p_{GN}(x) / p_{GN}(\mu) \leq \varepsilon \) increases with increasing \( \beta \) according to the equation

\[
\frac{|x_1 - x_2|}{a - b} = |\log_2 (1 - \varepsilon)|^{1/\beta}
\]
as shown in Fig. 2.2. This is consistent with the characteristics of \( \kappa_{p_{GN}} \) in (2.7) and \( \lim_{\beta \to \infty} p_{GN}(x \mid \mu, s, \beta) = p_U(x \mid \mu - s, \mu + s) \) pointwise. Thus \( \beta \) is the dominant parameter controlling the flat-topped shape.

\[ \lim_{\beta \to \infty} p_{GN}(x \mid \mu, s, \beta) = p_U(x \mid \mu - s, \mu + s) \]

\[ \beta \]

**Figure 2.2.** The ratio of the flat-topped interval \( |x_2 - x_1| \) to \( |a - b| \)

Unfortunately, it is not easy to estimate \( \beta \) using ML estimation. Let \( X = \{X_1, \ldots, X_N\} \) be an independent and identically distributed (i.i.d.) sample with density \( p_{GN} \). The log-likelihood is given by

\[
l_{GN}(\mu, s, \beta \mid X) = \sum_{i=1}^{N} \ln p_{GN}(X_i \mid \mu, s, \beta)
= -N \ln \left( 2s \Gamma \left( \frac{1}{\beta} + 1 \right) \right) - \sum_{i=1}^{N} \left| X_i - \mu \right| \frac{1}{s}.
\]

The ML estimator \( \left( \hat{\mu}, \hat{s}, \hat{\beta} \right) = \arg \max_{\mu, s, \beta} \{ l_{GN} \} \) has no closed-form solution. It is necessary to use iterative numerical optimization algorithms to obtain an approximate solution, as discussed in [6, 8]. In addition, another disadvantage of \( p_{GN} \) is that it is not suitable for modeling asymmetrically distributed data.

3. **General Forms of Flat-topped PDFs**

There can be four types of flat-topped PDF. Type A is a compound distribution, and Type B is its modified version to fit asymmetric distributions. Type C is a generalization of the Cauchy distribution and provides a new multivariate elliptical PDF. Type D is a function obtained by flattening the peak of arbitrary unimodal functions. These types are not disjoint, and some PDFs may belong to more than one type.

3.1. **Type A: Compounding with uniform distribution.** The PDF of Type A is defined by

\[
p_A(x \mid a, b, s) = \int_{-\infty}^{\infty} f(x; u, s) p_U(u \mid a, b) \, du
= \frac{1}{b-a} (F(x; a, s) - F(x; b, s))
\] (3.1)
where \( a \) and \( b \) are location parameters, \( s \) is a scale parameter, \( f(x; u, s) \) is a continuous unimodal PDF with location parameter \( u \), and \( F(x; u, s) \) is its CDF such that
\[
F(x; u, s) = \int_{-\infty}^{x} f(y; u, s) \, dy = \int_{-\infty}^{x} \frac{y - u}{s} \, dy. \tag{3.2}
\]
The shape of \( p_A \) varies with parameters, from bell-shaped to rectangular. The change of parameters
\[
m = \frac{a + b}{2} \quad \text{and} \quad r = \frac{b - a}{2}
\tag{3.3}
\]
yields another integral form
\[
p_A(x \mid a, b, s) = p_A(x \mid m - r, m + r, s) = \frac{1}{2r} \int_{-\infty}^{\frac{x-m}{r}} f(y; 0, 1) \, dy
\]
so that we have
\[
\lim_{r/s \to 0} p_A(x \mid a, b, s) = \frac{1}{s} f \left( \frac{x-m}{s}; 0, 1 \right) = f(x; m, s).
\]
On the other hand, if \( F \) approaches a step function, then \( p_A \) approaches \( p_U \). To be more precise, let \( P_A \) and \( P_U \) be the CDFs of \( p_A \) and \( p_U \), respectively. If \( F \) has the property that \( \lim_{s \to 0} F(x; u, s) = H(x-u) \) where \( H(\cdot) \) denotes the Heaviside step function, then it follows that \( \lim_{s \to 0} P_A(x \mid a, b, s) = P_U(x \mid a, b) \). This implies \( p_A(x \mid a, b, s) \) approaches \( p_U(x \mid a, b) \) (hereafter abbreviated as \( p_A \to p_U \)) as \( s \to 0 \).

The distributional derivative of \( H(x) \) is the Dirac delta function \( \delta(x) \) so that this property can be expressed in the form
\[
\lim_{s \to 0} p_A(x \mid a, b, s) = \int_{-\infty}^{\infty} p_U(y \mid a, b) \delta(x-y) \, dy = p_U(x \mid a, b).
\]

In general, if a distribution obtained by compounding \( f(x) \) with \( g(x) \) for \( x \in \mathbb{R}^n \) has the form of the convolution \( p(x) = \int_{y} f(x-y) g(y) \, dy \), then it can be expected that \( p(x) \to f(x) \) as \( g(y) \to \delta(y) \) and \( p(x) \to g(x) \) as \( f(y) \to \delta(y) \).

If \( m \) is the mean of \( p_A \), then the kurtosis \( \kappa \) of \( p_A(x \mid m-r, m+r, s) \) is given by
\[
\kappa_{p_A} = \frac{\mu_f(4) + 2\mu_f(2) \left( \frac{x}{s} \right)^2 + \frac{1}{3} \left( \frac{x}{s} \right)^4}{\left( \mu_f(2) + \frac{1}{3} \left( \frac{x}{s} \right)^2 \right)^2} \tag{3.4}
\]
where \( \mu_f(n) \) denotes the \( n \)-th central moment of \( f(x; 0, 1) \) (see Appendix A). If \( \mu_f(4) \) and \( \mu_f(2) \) are finite, then \( \lim_{r/s \to 0} \kappa_{p_A} = \kappa_f \) and \( \lim_{r/s \to \infty} \kappa_{p_A} = \kappa_{p_U} = 9/5 \). There are a variety of PDFs available for \( f \), such as normal (\( \kappa = 3 \)), logistic (\( \kappa = 4.2 \)), Laplace (\( \kappa = 6 \)), Student’s t (\( \kappa = 3+6/(\nu - 4) \)) and Cauchy distributions (\( \kappa \) is undefined). If \( f \) is leptokurtic, then \( p_A \) includes a variety of PDFs ranging from leptokurtic to platykurtic.

If \( f \) is symmetric about the mode \( m \) and satisfies the condition (2.4), i.e.,
\[
2r \left( \frac{f'(r/s; 0, 1)}{f(0; 0, 1) - f(-2r/s; 0, 1)} \right) < \varepsilon,
\]
then \( p_A \) is \( \varepsilon \)-flat-topped. Considering that \( p_A \) approaches \( p_U \) as \( s \to 0 \), this condition can be simplified in the form \( s/r < \varepsilon_s \) or \( 1 - F(m; a, s) + F(m; b, s) < \varepsilon_F \).
3.2. **Type B: Product of sigmoid functions.** The PDF of Type B is defined by

\[
p_B(x | a, b, s, t) = c F(x; a, s) (1 - G(x; b, t))
\]

(3.5)

where \( F \) and \( G \) are sigmoid functions given by (3.2) in Type A and \( c > 0 \) is a normalizing constant. In much the same way as Type A, \( p_B \) approaches \( p_U \) as \( s, t \to 0 \), so that it must be flat-topped when \( s \) and \( t \) are small enough. The advantage of Type B is that \( s \) and \( t \) are independently modifiable to fit the lower and upper tails to asymmetrically distributed data. A disadvantage is that the dependence of \( c \) on the parameters cannot generally be expressed in closed form. However, if \( F'(x; 0, 1) \) and \( G'(x; 0, 1) \) are symmetric about zero and \( p_B \) is flat-topped, i.e., \( 1 - F(x_m; a, s) + G(x_m; b, t) < \varepsilon_F \ll 1 \), then \( c \) can be approximated as \( c \approx (b - a)^{-1} \) as shown later. In that case, the ML estimation becomes simpler.

3.3. **Type C: Generalization of Cauchy distribution.** The third type can be expressed as

\[
p_C(x) = \frac{c}{h + g(x)}
\]

(3.6)

where \( h > 0 \) is a positive constant and \( g(x) \) is a non-negative continuous \( U \)-shaped function that satisfies \( \lim_{x \to \pm\infty} g(x) = \infty \). Furthermore, \( g(x) \) is supposed to satisfy \( (x - x_m) g'(x) \geq 0, 0 \leq g(x_m) \ll h \) for the mode \( x_m \in \arg \min_{x} g(x) \), and \( g(a) = h = g(b) \) for the location parameters \( a < x_m < b \). The condition (2.4) is rewritten as

\[
\frac{|g''(x_m)|}{(h + g(x_m))^2} \left| \frac{4h^2(a - b)}{g'(a) - g'(b)} \right| < \varepsilon.
\]

This type is so fundamental that it includes the Cauchy (Lorentz) distribution and can be easily extended to multivariate distributions.

3.4. **Type D: Peak flattening.** The PDF of this type is obtained by flattening the peak of bell-shaped functions, which can be expressed as

\[
p_D(x) = c \Psi(\alpha f(x))
\]

(3.7)

where \( c > 0 \) is a normalizing constant, \( \alpha > 0 \) is a shape parameter, \( f(x) > 0 \) is a unimodal function of \( x \), and \( \Psi(x) \) is a concave function satisfying \( \Psi(0) = 0, \Psi'(x) \geq 0, \Psi''(x) < 0 \) for \( x \geq 0 \). If \( \Psi \) is a “saturation” function such that \( \lim_{x \to \infty} \Psi'(x) = 0 \), it is easy to flatten the peak of \( f(x) \). The advantage of this type is that the flat-topped shape can be applied to various unimodal function \( f \), even if it has heavy tails. If \( \Psi \) is invertible, any flat-topped PDF can be expressed in this form, though it might be more complicated.

4. **Specific Examples of Flat-topped PDFs**

This section presents some computationally tractable examples of the flat-topped PDFs. It seems that the Gaussian is typical for \( f \); However, the logistic function is advantageous for \( F \), as shown in 4.2 and 4.3. Its successful extensions for multivariate elliptical distributions are proposed in 4.5.
4.1. Uniform Gaussian Mixture. The most likely component of the compound distribution of Type A is the normal distribution. Let \( f_N \) and \( F_N \) be the PDF and CDF of \( N(m, \sigma^2) \), respectively, i.e., \( f_N(x | u, s) = p_N(x | u, s^2) \) and
\[
F_N(x | u, s) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{x - u}{\sqrt{2} s} \right) \right)
\]
where \( \text{erf}(\cdot) \) is the error function. It follows from (3.1) that we have
\[
p_{AN}(x | a, b, s) = \frac{1}{2(b - a)} \left( \text{erf} \left( \frac{x - a}{\sqrt{2} s} \right) - \text{erf} \left( \frac{x - b}{\sqrt{2} s} \right) \right).
\]
According to (2.4), \( p_{AN}(x | a, b, s) \) is \( \varepsilon \)-flat-topped if
\[
\left| \frac{(a - b)}{2} \frac{p''_{AN}(x_m)}{p'_{AN}(a) - p'_{AN}(b)} \right| < \frac{2}{\pi} \frac{(\frac{a - b}{2})^2}{\exp \left( \frac{1}{2} \left( \frac{a - b}{2} \right)^2 \right) - 1} < \varepsilon.
\]
The CDF of \( p_{AN} \) is expressed as
\[
P_{AN}(x) = \frac{1}{2} + \frac{s}{2(b - a)} \left[ \frac{x - a}{s} \text{erf} \left( \frac{x - a}{\sqrt{2} s} \right) + \sqrt{\frac{2}{\pi}} \exp \left( - \frac{(x - a)^2}{2 \sigma^2} \right) \right.
\]
\[
- \left. \frac{x - b}{s} \text{erf} \left( \frac{x - b}{\sqrt{2} s} \right) - \sqrt{\frac{2}{\pi}} \exp \left( - \frac{(x - b)^2}{2 \sigma^2} \right) \right].
\]
A drawback is that the non-elementary function \( \text{erf}(\cdot) \) makes calculations somewhat intractable.

4.2. Symmetric Type A using Logistic Function. The most useful function for \( F \) must be a logistic function such that
\[
F_L(x | a, s) = \frac{1}{1 + \exp \left( \frac{a - x}{s} \right)} = \frac{1}{2} \left( 1 + \tanh \left( \frac{x - a}{2 s} \right) \right). \tag{4.2}
\]
The PDF \( p_{AL}(x) \) for \( x \in \mathbb{R} \) is expressed as
\[
p_{AL}(x | a, b, s) = \frac{1}{b - a} \left( \frac{1}{1 + \exp \left( \frac{a - x}{s} \right)} - \frac{1}{1 + \exp \left( \frac{b - x}{s} \right)} \right)
\]
\[
= \frac{1 - \exp \left( \frac{a - b}{s} \right)}{(b - a) \left( 1 + \exp \left( \frac{a - x}{s} \right) \right) \left( 1 + \exp \left( \frac{b - x}{s} \right) \right)}
\]
\[
= \frac{1}{2r} \left( \frac{\sinh \left( \frac{r}{2} \right)}{\cosh \left( \frac{x - m}{s} \right) + \cosh \left( \frac{r}{s} \right)} \right). \tag{4.3}
\]
where \( m = (a + b)/2 \) and \( r = (b - a)/2 \). The first equation analogous to a simple neural network model has been applied to a mixture model \cite{17}. The second equation shows that \( p_{AL}(x) \) is a special case of the Perks distribution \cite{18} and also belongs to types B and C. The CDF is expressed as
\[
P_{AL}(x) = \frac{s}{b - a} \ln \left( \frac{1 + \exp \left( \frac{x - m}{2 s} \right)}{1 + \exp \left( \frac{x - b}{2 s} \right)} \right)
\]
\[
= \frac{1}{2} + \frac{s}{r} \tanh \left( \frac{r}{2 s} \right) \tan \left( \frac{r}{2 s} \right).
\]
The inverse cumulative distribution function (quantile function) is

\[ P_{AL}^{-1}(v) = a + s \ln \left( \frac{1 - \exp \left( \frac{a - b}{s} v \right)}{\exp \left( \frac{a - b}{s} v \right) - \exp \left( \frac{a - b}{s} \right)} \right) \]

\[ = m + 2s \text{artanh} \left( \tanh \left( \frac{r}{s} \left( v - \frac{1}{2} \right) \right) \coth \left( \frac{r}{2s} \right) \right) \]

for a probability \( v \in (0, 1) \).

**Figure 4.1.** Plots of \( p_{AL} \) and \( P_{AL} \) for different parameter values.

The shapes of \( p_{AL} \) and \( P_{AL} \) depend on \( r \) and \( s \), as shown in Figure 4.1. Since \( p_{AL}(x) \) is symmetric about \( x_m = m \), the mean is \( m \) and the skewness is zero. The kurtosis is given by

\[ \kappa_{p_{AL}} = \frac{9}{5} + \frac{12}{5} \left( 1 + \left( \frac{r}{\pi s} \right)^2 \right) \]

which follows from (3.4), so that \( 1.8 < \kappa_{p_{AL}} \leq 4.2 \) (see Appendix B). If \( r/s = \pi \), then \( \kappa = 3 \), i.e., the excess kurtosis \( \kappa - 3 = 0 \) (mesokurtic), which is similar to a normal distribution. In practice, however, a better approximation to \( N(0, 1) \) is given by \( p_{AL}(x \mid -r_N, r_N, s_N) \) where \( r_N = \sqrt{\ln 4 - 0.2} \approx 0.97741 \) and \( s_N = r_N/\pi + 0.166 \approx 0.47712 \), for which the error is estimated as

\[ |p_N(x \mid 0, 1) - p_{AL}(x \mid -r_N, r_N, s_N)| < 0.0043. \]

Hence, \( N(\mu, \sigma^2) \) can be approximated as

\[ p_N(x \mid \mu, \sigma^2) \approx p_{AL}(x \mid \mu - \sigma r_N, \mu + \sigma r_N, \sigma s_N), \quad (4.4) \]
though it is slightly leptokurtic ($\kappa \approx 3.48$).

According to (2.4), $p_{AL}(x \mid m-r, m+r, s)$ is $\varepsilon$-flat-topped if

$$(4r/s) \csch (r/s) < \varepsilon$$

(see Appendix C). Under this condition, $p_{AL}$ can be substituted for the above-mentioned $p_{AN}$ given by (4.1), and vice versa, by using the following approximation:

$$p_{AN}(x \mid a, b, s) \approx p_{AL}(x \mid a, b, 0.5877 s).$$

This is because $F_N(x;0,1)$ can be approximated by $F_L(x;0,0.5877)$ [19], where $|F_N(x;0,1) - F_L(x;0,0.5877)| < 0.01$. If $p_{AL}$ is flat-topped, the effect of the difference is limited to near the boundaries $a$ and $b$ so that it may be less influential in ML estimation.

4.3. Asymmetric Type B using Logistic Function. Type B is useful for adjusting asymmetric tails to skewed data distributions. In the following PDF, for example, its lower and upper tails mainly depend on $s$ and $t$, respectively:

$$p_{BL}(x \mid a, b, s, t) = cF_L(x,a,s) (1 - F_L(x,b,t))$$

$$= \frac{c}{(1 + \exp \left( \frac{a-x}{s} \right)) \left( 1 + \exp \left( \frac{x-b}{t} \right) \right)}$$

(4.6)

where we have used $F_L(-x;-b,t) = 1 - F_L(x,b,t)$. This $p_{BL}(x \mid a, b, s, t)$ is $\varepsilon$-flat-topped if

$$6 \left( \frac{b-a}{s} \coth \left( \frac{b-a}{2s} \right) + \frac{b-a}{t} \coth \left( \frac{b-a}{2t} \right) \right) \exp \left( \frac{a-b}{s+t} \right) < \varepsilon$$

(see Appendix D). Unfortunately, $p_{BL}$ cannot be integrated in closed form for $s \neq t$. In practice, however, if $p_{BL}$ is flat-topped, then $c \approx 1/(b-a)$ and $p_{BL}$ can be approximated simply as

$$p_{BL}(x \mid a, b, s, t) \approx \begin{cases} \frac{1}{b-a} F_L(x,a,s) & x < x_m, \\ \frac{1}{b-a} (1 - F_L(x,b,t)) & x \geq x_m, \end{cases}$$

(4.7)

which is convenient to estimate the parameters (see Appendix E).

The second-best $F$ may be the CDF of the double exponential (Laplace) distribution defined by

$$F_D(x;a,s) = \begin{cases} \frac{1}{2} \exp \left( \frac{x-a}{s} \right) & x < a, \\ 1 - \frac{1}{2} \exp \left( - \frac{x-a}{s} \right) & x \geq a, \end{cases}$$

and then we have

$$p_{BD}(x \mid a, b, s, t) = cF_D(x,a,s) (1 - F_D(x,b,t))$$

(4.8)

where

$$c = \left( b-a + \left( \frac{s^2}{s^2-t^2} \right) \frac{s}{2} \exp \left( \frac{a-b}{s} \right) + \left( \frac{t^2}{t^2-s^2} \right) \frac{t}{2} \exp \left( \frac{a-b}{t} \right) \right)^{-1}.$$
4.4. **Asymmetric Type A.** The PDF of Type A can be asymmetric depending on the asymmetry of $F$ such as a CDF of the skew normal distribution [20]. From the viewpoint of tractability, it is better to use elementary functions, for example:

$$F_{LS}(x; a, s, \lambda) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{x - a}{2s} + \lambda \left[ \sqrt{\left( \frac{x - a}{2s} \right)^2 + 1} - 1 \right] \right) \right]$$

where $\lambda \in (-1, 1)$ is a skew parameter. This parameterization is designed to satisfy $F_{LS}(x; a, s, 0) = F_L(x; a, s)$, $F_{LS}(a) = 1/2$, and $F_{LS}^\prime(a) = 1/(4s)$. As $\lambda$ increases, the lower tail becomes heavier and the upper tail becomes lighter. However, the lower and upper tails cannot be adjusted independently, which is less convenient than the above-mentioned Type B.

4.5. **Symmetric Type C.** This section shows three examples of $g(x)$ in (3.6): $y^\beta$, $\exp(y^\beta)$, and $\cosh(y^\beta)$ where $y = |x - m|/s$, $m \in \mathbb{R}$, and $s, \beta > 0$.

First, a special case of the generalized Cauchy (generalized Pearson VII) distribution [7, 9, 10] is defined by

$$p_{CC}(x \mid m, s, \beta) = \frac{\beta}{2s B \left( 1 - \frac{1}{\beta}, \frac{1}{\beta} \right) \left[ 1 + \frac{|x - m|^{\beta}}{s^{\beta}} \right]}$$

(4.10)

where $B \left( 1 - 1/\beta, 1/\beta \right) = \pi/ \sin (\pi/\beta)$ is the beta function. If $\beta > 2$, then $p_{CC}$ is $\varepsilon$-flat-topped for any $\varepsilon > 0$. If $\beta = 4$, it is called the Laha distribution [21]. For $\beta = 6$, the kurtosis of $p_{CC}(x \mid 0, 1, 6)$ is 4 so that it is leptokurtic. A disadvantage is that the shape parameter $\beta$ is difficult to estimate, as in $p_{GN}$.

Second, a new PDF is defined by

$$p_{CF}(x \mid m, r, s, \beta) = \frac{\beta}{2s \Gamma \left( \frac{1}{\beta} \right) F_{\frac{1}{\beta} - 1} \left( \frac{s^\beta}{x^\beta} \right) \left[ 1 + \exp \left( \frac{|x - m|^{\beta} - r^{\beta}}{s^{\beta}} \right) \right]}$$

(4.11)

where $r > 0$ and $F_j(x)$ is the complete Fermi-Dirac integral [22] (see Appendix F). Note that $F_j(x)$ has a numerical subscript (not to be confused with $F$ having a capital letter subscript used for types A and B). We call $p_{CF}$ a generalized Fermi-Dirac distribution, though the Fermi function of the Fermi-Dirac statistics in physics is not a PDF. The kurtosis of $p_{CF}$ is given by

$$\kappa_{p_{CF}} = \frac{\Gamma(1/\beta) F_{1/\beta - 1} \left( r^{\beta}/s^{\beta} \right) \Gamma(5/\beta) F_{5/\beta - 1} \left( r^{\beta}/s^{\beta} \right)}{\left[ \Gamma(3/\beta) F_{3/\beta - 1} \left( r^{\beta}/s^{\beta} \right) \right]^{2}}.$$  

The special case of $\beta = 1$ is a variant of the Fermi function with normalizing constant:

$$p_{CF}(x \mid m, r, s, 1) = \frac{1}{2s \ln \left( 1 + \exp \left( \frac{r}{s} \right) \right) \left[ 1 + \exp \left( \frac{|x - m| - r}{s} \right) \right]}.$$

The shape of this PDF varies with $r$ and $s$, not with $\beta$, which is much better than the generalized Gaussian $p_{GN}$. According to the condition (2.2) for $x_1 = m - r/2$ and $x_2 = m + r/2$, if $\exp (-r/2s) < \varepsilon$, then $p_{CF}(x \mid m, r, s, 1)$ is $(r, \varepsilon)$-flat-topped and similar but not superior to $p_{AL}$. The special case of $\beta = 2$ is a Ferreri distribution [11] and rewritten using parameters $a$ and $b$ instead of $m$ and $r$ as

$$p_{CE}(x \mid a, b, s) = \frac{1}{\sqrt{\pi s} F_{-1/2} \left( r^2/s^2 \right) \left[ 1 + \exp \left( \frac{a-x-b}{s} \right) \right]}.$$  

(4.12)
This seems to be simple with respect to \( x \), but unfortunately it cannot be integrated in closed form. This \( p_{CE} \) is \( \varepsilon \)-flat-topped if \( \text{sech}^2 \left( \frac{h-a}{2\varepsilon} \right) < \varepsilon \). If \( \beta > 2 \), then \( p_{CE} \) is \( \varepsilon \)-flat-topped for any \( \varepsilon > 0 \).

For multivariate elliptical distributions of \( n \) dimensional vectors, using the Mahalanobis distance
\[
d_M(\mathbf{x}, \mathbf{m}, \Sigma) = \left( (\mathbf{x} - \mathbf{m})^T \Sigma^{-1} (\mathbf{x} - \mathbf{m}) \right)^{1/2}
\] (4.13)
where \( \mathbf{x}, \mathbf{m} \in \mathbb{R}^n \) and \( \Sigma \) is an \( n \times n \) positive-definite matrix like a covariance matrix, \( p_{CE} \) can be extended to be
\[
p_{CM}(\mathbf{x} | \mathbf{m}, \Sigma, r, t) = \frac{c_M}{1 + \exp \left( \left[ d_M(\mathbf{x}, \mathbf{m}, \Sigma) - r^n \right] / t \right)}
\] (4.14)
where \( r \) is a dispersion parameter, \( t = 1/s^n \) is a shape parameter used for adjusting the slope of boundaries, and \( c_M \) is a normalizing constant given by
\[
c_M = \frac{t \Gamma (n/2 + 1)}{\pi^{n/2} \ln (1 + \exp (r^nt)) |\Sigma|^{1/2}}
\]
(see Appendix G). A similar PDF has been proposed by Gasparini and Ma [23]: the multivariate Fermi-Dirac distribution defined as the form of
\[
f(\mathbf{x}) = \frac{c_F}{1 + \exp \left( \alpha + \lambda^2 (\mathbf{x} - \mathbf{m})^T \Sigma^{-1} (\mathbf{x} - \mathbf{m}) \right) / \beta}
\]
where \( \lambda^2 (\alpha) = F_{1/2} (\alpha) / F_{-1/2} (\alpha) \) and the normalizing constant \( c_F \) is
\[
c_F = \frac{\Gamma (n/2) \lambda^n (\alpha)}{\pi^{n/2} F_{n/2-1} (\alpha) |\Sigma|^{1/2}}.
\]
This is disadvantageous because the dependence of \( c \) on \( \alpha \) cannot be expressed in closed form except for \( n = 2 \).

Third, one more new PDF is defined by
\[
p_{CH}(x | m, r, s, \beta) = \frac{c_H \sinh (r^\beta / s^\beta)}{\cosh \left( |x - m|^{\beta} / s^\beta \right) + \cosh (r^\beta / s^\beta)}
\] (4.15)
where \( r, s, \beta > 0 \) and
\[
c_H = \frac{\beta}{2s \Gamma (1/\beta) \left[ F_{1/\beta-1} (r^\beta / s^\beta) - F_{1/\beta-1} (-r^\beta / s^\beta) \right]}
\]
The kurtosis of \( p_{CH} \) is given by
\[
\kappa_{p_{CH}} = \frac{\Gamma \left( \frac{1}{\beta} \right) \left[ F_{1/\beta-1} \left( \frac{r^\beta}{s^\beta} \right) - F_{1/\beta-1} \left( -\frac{r^\beta}{s^\beta} \right) \right] \Gamma \left( \frac{3}{\beta} \right) \left[ F_{1/\beta-1} \left( \frac{r^\beta}{s^\beta} \right) - F_{1/\beta-1} \left( -\frac{r^\beta}{s^\beta} \right) \right]}{\Gamma \left( \frac{3}{\beta} \right) \left[ F_{1/\beta-1} \left( \frac{r^\beta}{s^\beta} \right) - F_{1/\beta-1} \left( -\frac{r^\beta}{s^\beta} \right) \right]^2}
\]
The special case of \( \beta = 1 \) is identical to \( p_{AL} \) defined by (4.3). If \( \beta \geq 2 \), then \( p_{CH} \) is always flat-topped. For multivariate elliptical distributions, in much the same way as \( p_{CM} \), we have
\[
p_{CL}(x | m, \Sigma, r, t) = \frac{c_L \sinh (r^nt)}{\cosh \left( d_M(\mathbf{x}, \mathbf{m}, \Sigma)^n / t \right) + \cosh (r^nt)}
\] (4.16)
where
\[ c_L = \frac{\Gamma(n/2 + 1)}{\pi^{n/2} r^n |\Sigma|^{1/2}}. \]
This is very natural because \( \pi^{n/2} r^n / \Gamma(n/2 + 1) \) corresponds to the volume of an n-dimensional ball of radius \( r \). The examples of 2-dimensional \( p_{CL} \) for various values of \( \{\Sigma, r, t\} \) are shown in Fig. 4.2. The parameters \( \Sigma \) and \( t = 1/s^n \) are redundant so that we can impose a constraint on them such as \( |\Sigma| = 1 \), and so the number of necessary parameters is \( (n + 1)(n + 2)/2 \). However, redundant parameters are helpful for quickly finding solutions to parameter optimization problems. The log-likelihood equations for \( p_{CM} \) and \( p_{CL} \) are expressed by elementary functions of their parameters, which seems relatively simple in this type.

**Figure 4.2.** Multivariate flat-topped distributions: \( p_{CL} \) \((n = 2)\) for different parameter values.

### 4.6. **Symmetric Type D.**

There can be a variety of PDFs given by (3.7). The typical examples of \( \Psi(x) \) are “saturation” functions such as \( 1 - \exp(-x) \), \( \tanh(x) \), and \( \arctan(x) \). The unimodal functions for \( f(x; m, s) \) can be not only the PDFs of bell-shaped distributions but also simpler functions like \( y - \beta \) and \( \exp(-y^\beta) \) where \( y = |x - m|/s \) and \( \beta > 0 \). For example, a simple one is

\[ p_{DE}(x \mid m, s) = \frac{1}{2\sqrt{\pi}s} \left( 1 - \exp\left\{ -\left( \frac{x - m}{s} \right)^2 \right\} \right) \]  

which is \( \varepsilon \)-flat-topped for any \( \varepsilon > 0 \) with heavy tails. The CDF is expressed as

\[ P_{DE}(x \mid m, s) = \frac{1}{2} \left[ 1 + \left( \frac{x - m}{\sqrt{\pi}s} \right) \left( 1 - \exp\left\{ -\left( \frac{s}{x - m} \right)^2 \right\} \right) + \text{sgn}(x - m) - \text{erf}\left( \frac{s}{x - m} \right) \right]. \]

Unfortunately, there are few other simple flat-topped PDF of this type, mainly because \( c \) is rarely expressed in closed form.
5. Maximum Likelihood Estimation

This section discusses ML estimation for \( p_{\text{AL}} \), \( p_{\text{BL}} \), and \( p_{\text{CL}} \). The likelihood equations of the flat-topped PDFs have no closed-form solution, so it is necessary to use iterative methods to obtain approximate solutions.

5.1. Simplified gradient ascent. Let \( X = \{x_1, \ldots, x_N\} \) be a given data set. Assuming \( X \) is i.i.d. with density \( p_{\text{AL}}(x \mid a, b, s) \) given by (4.3), the model parameters \( a, b, \) and \( s \) are estimated by maximizing the log-likelihood function:

\[
\ell_{\text{AL}}(a, b, s; X) = \sum_{i=1}^{N} \ln p_{\text{AL}}(x_i \mid a, b, s). \tag{5.1}
\]

Its partial derivatives are

\[
\begin{align*}
\partial_a \ell_{\text{AL}} &= \frac{1}{s} \sum_{i=1}^{N} \left( \frac{s}{b-a} - \frac{1}{\exp \left( \frac{b-a}{s} \right) - 1} - 1 + F_L(x_i; a, s) \right), \\
\partial_b \ell_{\text{AL}} &= -\frac{1}{s} \sum_{i=1}^{N} \left( \frac{s}{b-a} - \frac{1}{\exp \left( \frac{b-a}{s} \right) - 1} - F_L(x_i; b, s) \right), \\
\partial_s \ell_{\text{AL}} &= -\frac{1}{s^2} \sum_{i=1}^{N} \left( \frac{b-a}{\exp \left( \frac{b-a}{s} \right) - 1} + (x_i - a) \left( 1 - F_L(x_i; a, s) \right) \\
&\quad - (x_i - b) F_L(x_i; b, s) \right)
\end{align*}
\]

(see Appendix H). The likelihood equations obtained by setting these derivatives equal to zero have no closed-form solution. However, we can impose a constraint that

\[
\min \{x_i\} < a < b < \max \{x_i\} \quad \text{and} \quad \frac{b-a}{4N} < s < \sigma,
\]

where \( \sigma \) is the standard deviation of \( X \). It seems to be easy to find approximately optimal parameters by using iterative methods. Although Newton’s method is sometimes inappropriate for the case of ill-conditioned Hessians, a simplified procedure that modifies each parameter one by one as in a coordinate descent algorithm [24] works well in practice. An example of iteration is as follows:

\[
\begin{align*}
a^{\text{new}} &= a^{\text{old}} + \eta_a \partial_a \ell_{\text{AL}} \left( a^{\text{old}}, b^{\text{old}}, s^{\text{old}}, X \right), \\
b^{\text{new}} &= b^{\text{old}} + \eta_b \partial_b \ell_{\text{AL}} \left( a^{\text{new}}, b^{\text{old}}, s^{\text{old}}, X \right), \\
s^{\text{new}} &= s^{\text{old}} + \eta_s \partial_s \ell_{\text{AL}} \left( a^{\text{new}}, b^{\text{new}}, s^{\text{old}}, X \right)
\end{align*}
\]

where \( \eta_a, \eta_b, \) and \( \eta_s \) are coefficients for step-size control such as \( \eta_a \propto |\partial_a \ell_{\text{AL}}|^{-1} \). Metaheuristic optimization techniques using adaptive step-size control are also applicable.

For \( p_{\text{BL}} \) given by (4.6), if it is flat-topped, then the partial derivatives of its log-likelihood \( \ell_{\text{BL}} \) can be approximated as:

\[
\partial_a \ell_{\text{BL}} \approx \frac{N}{b-a} - \frac{1}{s} \sum_{i=1}^{N} (1 - F_L(x_i; a, s)), \quad \partial_b \ell_{\text{BL}} \approx -\frac{N}{b-a} + \frac{1}{t} \sum_{i=1}^{N} F_L(x_i; b, t),
\]
\[ \partial_s l_{BL} \approx \frac{1}{s^2} \sum_{i=1}^{N} (a - x_i) (1 - F_L(x_i; a, s)), \quad \partial_t l_{BL} \approx \frac{1}{t^2} \sum_{i=1}^{N} (x_i - b) F_L(x_i; b, t). \]

These approximations may be relatively easier than those of the other flat-topped PDFs.

For \( p_{CL} \) given by (4.16), the log-likelihood is

\[ l_{CL} = N \ln \left( \frac{\Gamma(n/2 + 1) \sinh (r^n t)}{\pi^{n/2} r^n |\Sigma|^{1/2}} \right) - \sum_{i=1}^{N} \ln (\cosh (\rho_i^n t) + \cosh (r^n t)) \]

where \( \rho_i = d_M(x_i, m, \Sigma) \) is given by (4.13). Thus, the partial derivatives of \( l_{CL} \) with respect to parameters \( \{m, \Sigma^{-1}, r^n, t\} \) can be evaluated using elementary functions. For instance,

\[ \partial_{\Sigma^{-1}} l_{CL} = \frac{N}{2} \Sigma - \frac{1}{2} \sum_{i=1}^{N} \frac{\sinh (\rho_i^n t) n \rho_i^{n-2} t}{\cosh (\rho_i^n t) + \cosh (r^n t)} (x_i - m) (x_i - m)^T. \]

Unfortunately, the computational cost of matrix operations is expensive for large \( n \). It is important to simplify the model by decomposing it into factorized PDFs of fewer variables to reduce the cost.

5.2. Advantage of the flat-topped PDF. The flat-topped PDF can be adapted to fit a variety of distribution shapes ranging from bell-shaped to rectangular, and it brings about the increase of the log-likelihood. For example, suppose \( X \) is an i.i.d. sample from the uniform distribution \( \mathcal{U}(a, b) \). If this data set is modeled by a normal distribution, using ML estimation, the best fit \( p_N^* \) is estimated to be \( \mathcal{N}(m, r^2/3) \) where \( m = (a + b)/2 \) and \( r = (b - a)/2 \). Hence, the expected log-likelihood is

\[ \mathbb{E} \left[ \ln p_N^* (x \mid m, r^2/3) \right] = \int_{-\infty}^{\infty} p_U(x \mid a, b) \ln p_N^* (x \mid m, r^2/3) \, dx \]

\[ = -\ln 2r - \frac{1}{2} \ln \frac{\pi e}{6}. \]

If it is exactly modeled by \( \mathcal{U}(a, b) \), the expected log-likelihood increases by the Kullback-Leibler (KL) divergence of \( \mathcal{U}(a, b) \) with respect to \( \mathcal{N}(m, r^2/3) \), i.e.,

\[ D_{KL} (p_U \parallel p_N^*) = \frac{1}{2} \ln \frac{\pi e}{6} \approx 0.176. \]

Although this value appears small, it should not be neglected, because the values of \( \ln (p_U(x)/p_N^*(x)) \) can be positive or negative and cancel each other out in averaging. In fact, the \( L_1 \) distance between them is

\[ D_{L_1} (p_U, p_N^*) = \int_{-\infty}^{\infty} |p_U(x \mid a, b) - p_N^*(x \mid m, r^2/3)| \, dx \]

\[ = 2 \left( 1 - \sqrt{\frac{1}{3} \ln \left( \frac{6}{\pi} \right)} \right) \text{erf} \left( \sqrt{\frac{1}{2} \ln \left( \frac{6}{\pi} \right)} - \text{erf} \left( \frac{\sqrt{3}}{2} \right) \right) \approx 0.395. \]

This is not trivial considering that \( \sup \{D_{L_1} (p_U, p_N^*)\} = 2 \).
In an $n$-dimensional space, let $p_{MU}(x \mid m, r)$ denote the PDF of a multivariate uniform distribution such that

$$p_{MU}(x \mid m, r) = \lim_{t \to \infty} p_{CL}(x \mid m, I, r, t) = \begin{cases} \frac{\Gamma(n/2+1)}{\pi^{n/2}r^n} & \text{if } \|x - m\|_2 \leq r, \\ 0 & \text{otherwise} \end{cases} \quad (5.2)$$

where $x, m \in \mathbb{R}^n$ are $n$-dimensional vectors and $I$ is the $n \times n$ identity matrix substituted for $\Sigma$ in (4.16), then its best-fit model using a multivariate normal distribution is given by $p_{MN}^*(x \mid m, \hat{\Sigma})$ where the elements of $\hat{\Sigma}$ are $\sigma_{ii} = r^2/(n+2)$ for $i = 1, 2, \ldots, n$ and $\sigma_{ij} = 0$ for $i \neq j$. The KL divergence of $p_{MU}(x \mid m, r)$ with respect to $p_{MN}^*(x \mid m, \hat{\Sigma})$ is calculated as

$$D_{KL}(p_{MU} \parallel p_{MN}^*) = \ln \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2}\right)} - \frac{n}{2} \ln \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2}\right)} + \frac{n}{2}$$

and the $L_1$ distance between them is

$$D_{L_1}(p_{MU}, p_{MN}^*) = 2 \left(1 - \chi_n^n - \frac{\Gamma\left(n/2, (n/2 + 1) \chi_n^2\right) - \Gamma\left(n/2, n/2 + 1\right)}{\Gamma\left(n/2\right)}\right)$$

where

$$\chi_n = \sqrt{\frac{2}{n+2} \ln \frac{(n/2 + 1)^{n/2}}{\Gamma\left(n/2 + 1\right)}}$$

and $\Gamma(\cdot, \cdot)$ is the incomplete gamma function (see Appendix I). If $n = 2$, then $\chi_2^2 = \ln \sqrt{2}, D_{KL}(p_{MU} \parallel p_{MN}^*) = 1 - \ln 2 \approx 0.307$, and $D_{L_1}(p_{MU}, p_{MN}^*) = 1 - \ln 2 + 2/e^2 = 0.578$ where we have used $\Gamma(1, x) = e^{-x}$. Thus, both $D_{KL}(p_{MU} \parallel p_{MN}^*)$ and $D_{L_1}(p_{MU}, p_{MN}^*)$ monotonically increase with $n$ so that the flat-topped PDF is more effective in high dimensional spaces.

6. **Mixture Models**

The mixture of flat-topped PDFs can be useful for improving the goodness of fit of the GMM.

6.1. **Outline of model fitting.** A practical estimation procedure consists of three steps as follows:

1. Create a finite GMM using the EM (or VB) algorithm.
2. Improve the model by replacing each Gaussian component with a symmetric $p_{AL}(x \mid a, b, s)$ and using a generalized EM algorithm [4].
3. If the optimized $p_{AL}(x \mid a, b, s)$ is flat-topped, replace it with an asymmetric $p_{BL}(x \mid a, b, s, t)$ and optimize in the same way.

It is possible to build a mixture model using only flat-topped PDFs from scratch. However, the GMM is easier to build first and becomes a standard for comparison. The GMM can be smoothly transformed into a mixture of flat-topped PDFs and further optimized, which increases the log-likelihood. Even though the log-likelihood improvement may be small, it is important to understand the characteristics of the boundary regions of subpopulations. Moreover, $p_{AL}$ can be replaced with $p_{AN}$ that is a uniform Gaussian mixture given by (4.1), or we can restore the previously optimized GMM if it is reasonable. A similar approach can also be applied to modeling multivariate elliptical distributions using $p_{CM}$ or $p_{CL}$.
6.2. Mixture of flat-topped distributions. We consider a mixture model of the form

\[ p_{FM}(x \mid \theta) = \sum_{k=1}^{K} \pi_k p_{AL}(x \mid a_k, b_k, s_k) \]

where \( K \) is the number of mixture components, \( \theta = \{\pi_k, a_k, b_k, s_k \mid k = 1, \ldots, K\} \) denotes model parameters, and \( \pi_k \in [0, 1] \) is the mixing coefficients. Since \( \sum_{k=1}^{K} \pi_k = 1 \), the number of free parameters is \( 4K - 1 \). There is no closed-form solution for maximizing the likelihood \( \prod_{i=1}^{N} p_{FM}(x_i \mid \theta) \) for an i.i.d data set \( X = \{x_1, \ldots, x_N\} \). However, by introducing a latent variable \( Z = \{z_{i,k} \in \{0, 1\} \mid \sum_{k=1}^{K} z_{i,k} = 1\} \) and considering the problem of maximizing the likelihood for the complete data set \( \{X, Z\} \) such that

\[ L(\theta; X, Z) = p(X, Z \mid \theta) = \prod_{i=1}^{N} \prod_{k=1}^{K} [\pi_k p_{AL}(x_i \mid a_k, b_k, s_k)]^{z_{i,k}} \]

we can find an approximate solution using a generalized EM algorithm. For example,

1. Choose an initial setting for the parameters. If a GMM has been already obtained, each Gaussian component can be replaced with \( p_{AL}(x \mid a_k, b_k, s_k) \) using (4.4).
2. E-step: Evaluate the expected complete data log-likelihood given by

\[ Q = \sum_{i=1}^{N} \sum_{k=1}^{K} w_{i,k} [\ln \pi_k + \ln p_{AL}(x_i \mid a_k, b_k, s_k)] \]

where \( w_{i,k} \) denotes the probability that component \( k \) is responsible for generating \( x_i \). This probability can be estimated as a posterior probability with respect to the latent variables using Bayes’ theorem as follows:

\[ w_{i,k} = \mathbb{E}[z_{i,k}] = \frac{\pi_k p_{AL}(x_i \mid a_k, b_k, s_k)}{\sum_{j=1}^{K} \pi_j p_{AL}(x_i \mid a_j, b_j, s_j)} \]

3. M-step: Update the parameters to increase \( Q \) using \( w_{i,k} \) as follows:

\[ \pi_k^{\text{new}} = \frac{1}{N} \sum_{i=1}^{N} w_{i,k} \]

\[ a_k^{\text{new}} = a_k^{\text{old}} + \eta_\alpha (a_k^{\text{old}}) \sum_{i=1}^{N} w_{i,k} \left[ \frac{\partial}{\partial a_k} \ln p_{AL}(x_i \mid a_k^{\text{old}}, b_k^{\text{old}}, s_k^{\text{old}}) \right] \]

\[ b_k^{\text{new}} = b_k^{\text{old}} + \eta_\beta (b_k^{\text{old}}) \sum_{i=1}^{N} w_{i,k} \left[ \frac{\partial}{\partial b_k} \ln p_{AL}(x_i \mid a_k^{\text{new}}, b_k^{\text{old}}, s_k^{\text{old}}) \right] \]

\[ s_k^{\text{new}} = s_k^{\text{old}} + \eta_\sigma (s_k^{\text{old}}) \sum_{i=1}^{N} w_{i,k} \left[ \frac{\partial}{\partial s_k} \ln p_{AL}(x_i \mid a_k^{\text{new}}, b_k^{\text{new}}, s_k^{\text{old}}) \right] \]

where \( \eta_\theta (\theta_k) \) is a coefficient for step-size control of \( \theta_k \) such that

\[ \eta_\theta (\theta_k) \propto \left| \frac{\partial^2}{\partial \theta_k^2} \sum_{i=1}^{N} w_{i,k} \ln p_{AL}(x_i \mid \theta_k) \right|^{-1} \]
(4) Repeat E- and M-steps until the estimates converge.

The M-step is almost the same as a single iteration of the iterative method described in Section 5.1, except the log-likelihood has the coefficient $w_{i,k}$. If the optimized $p_{AL}$ is flat-topped, it can be replaced with $p_{BL}$ to further improve the log-likelihood in much the same way.

7. Experiments

In this section, the usefulness of the flat-topped PDF is demonstrated with simulation examples.

7.1. ML estimation for univariate distribution. The iterative method for the ML estimation of $p_{AL}$ described in Section 5 generally works well. An example is illustrated in Fig. 7.1, which shows the three PDFs $p_N$, $p_{AL}$, and $p_{BL}$ fitted for a test sample of $N = 55$ data points: 40 are from $U(0, 100)$, and 15 are from $N(60, 35^2)$. The parameters of $p_{AL}$ are initially set to approximate $p_N$ using (4.4). The iterations almost converge rapidly and bring about a reasonable increase in log-likelihood. If the sample $X$ is likely to be uniformly distributed, it is advisable to start from $s = 4s_{min} = (\max(X) - \min(X))/N$, $a = \min(X) + s$, and $b = \max(X) - s$, and keep the constraint $s \geq s_{min}$ to avoid overflow and underflow.

![Figure 7.1. ML estimation for $p_{AL}$ and $p_{BL}$. Data points are marked on each of the curves.](image)

7.2. Bivariate mixture modeling. The advantage of the mixture model using the flat-topped PDFs (hereafter abbreviated FTM) over GMM is demonstrated in modeling the following two-dimensional synthetic data. The focus is not only on goodness of fit, as discussed in Section 5.2, but also on parsimonious modeling based on AIC [25] and BIC [26].
Figure 7.2. Contours of the PDFs of GMM (left) and FTM (right) fitted for the sample of 406 data points (orange dots). Both models consist of four components ($K = 4$).

Figure 7.3. The 3D plots of the PDFs of GMM (left: $K = 9$) and FTM (right: $K = 4$) that minimize AIC.

The goodness of fit of the estimated models in the following simulations can be qualitatively assessed by simply looking at PDF plots. The data points are generated as $u + \epsilon$ where $u$ is a two-dimensional vector representing a random point from a uniform distribution on line segments in a plane and $\epsilon$ is a small isotropic Gaussian random vector with mean zero. For the sake of convenience, the line segments are aligned with the coordinate axes, and so the mixture components of FTM can be modeled by $p_{FM}(x,y) = p_{AL}(x)p_{AL}(y)$ where $(x,y)$ denotes Cartesian coordinates. Figure 7.2 shows the contours of two PDFs fitted for the data points ($N = 427$); the left plot shows a GMM ($K = 4$) estimated using the EM algorithm implemented in Scikit-learn [27] and the right plot a FTM ($K = 4$) fitted using the generalized EM algorithm presented in Section 6.3. Both models have the lowest BIC values. Figure 7.3 shows surface plots of the PDFs of a GMM ($K = 9$) and the
same FTM \((K = 4)\) that have the lowest AIC values. The obvious disadvantages of the GMMs are excess peaks, unreal tails, and unclear boundaries in this case.

![Figure 7.4. AIC and BIC](image)

**Table 1.** The minimum values of AIC and BIC

| X | N  | K | GMM it | \(L/N\) | AIC  | BIC  | FTM it | \(L/N\) | AIC  | BIC  |
|---|----|---|--------|--------|------|------|--------|--------|------|------|
| A | 427| 4 | 31     | -4.895 | 4227 | 4320 | 49     | -4.755 | 4115 | 4224 |
|  |    | 9 | 9      | -4.468 | 4169 | 4384 | 100    | -4.711 | 4147 | 4398 |
| B | 1281| 4 | 20     | -4.874 | 12534| 12652| 61     | -4.772 | 12280| 12419|
|  |    | 5 | 11     | -4.854 | 12494| 12643| 62     | -4.771 | 12292| 12467|
|  |    | 15| 25     | -4.745 | 12834| 12793| 300    | -4.726 | 12317| 12853|

(it: the number of iterations, \(L/N\): average log-likelihood)

The difference between AIC and BIC is significant for model selection. Figure 7.4 shows the plot of AIC and BIC values, where the numbers of free parameters in the two-dimensional GMM and FTM are \(6K - 1\) and \(7K - 1\), respectively. Naturally, both the AIC and BIC values of FTM are minimum at the number of the given line segments \((K = 4)\). On the other hand, for the optimal number of GMM components, BIC indicates just the same \(K = 4\), but AIC suggests \(K = 9\). The AIC values reflect subtle situation of model fitting. Table 1 shows the lowest values (indicated by italics) of AIC evaluated for the above models. It signifies that the number of \(K\) minimizing AIC for GMM increases with \(N\). In other words, the AIC values imply that the GMM \((K = 4)\) is insufficient for improving goodness of fit and needs more Gaussian components, even though they considerably overlap each other. That is quite reasonable, considering that the optimal model is almost equivalent to an infinite uniform mixture of Gaussians. The flat-topped PDF can approximate such a model using minimal parameters.

In the basis function decomposition of an arbitrary PDF, it is essential to choose appropriate basis functions. The FTM \((K = 4)\) seems much better than the GMM \((K = 9)\) in goodness of fit in the case of Figure 7.3. However, the log-likelihood
values in Table 1 indicate not much difference between them. That implies the likelihood is not the best measure of goodness of fit, and neither is KL divergence. It is desirable to develop another criterion for model selection to compare a wider variety of models.

8. Concluding Remarks

The most tractable univariate flat-topped PDF is $p_{AL}$ defined by (4.3). It is obtained by compounding a logistic distribution with a uniform distribution, and its shape varies with its parameters, from bell-shaped to rectangular. For asymmetric flat-topped distributions, $p_{BL}$ defined by (4.6) is available. Furthermore, a generalized Fermi-Dirac distribution $p_{CM}$ defined by (4.14) and its variant $p_{CL}$ defined by (4.16) are advantageous for modeling multivariate elliptical distributions. Although there is no closed-form solution for the ML estimates of model parameters, we can obtain approximate solutions using iterative methods. Thus, they are useful as a component of a mixture model that can be optimized using the generalized EM algorithm. Even in GMM, if it contains some data points distributed uniformly, it will be worthwhile to replace the Gaussians with flat-topped PDFs to improve goodness of fit and make the model as parsimonious as possible. In such a situation, AIC values may suggest that the Gaussian components are needed more than that indicated by BIC values.

Appendix A. Kurtosis of $p_A$

The $n$-th central moment of $p_A(x | -r, r, s)$ given by (3.1), for a positive even integer $n$, is evaluated as

$$
\mu_{p_A}(n) = \int_{-\infty}^{\infty} x^n \left\{ \int_{-\infty}^{\infty} f(x; u, s) p_U(u | -r, r) du \right\} dx
$$

$$
= \int_{-\infty}^{\infty} p_U(u | -r, r) \left\{ \int_{-\infty}^{\infty} x^n f\left(\frac{x-u}{s}; 0, 1\right) \frac{1}{s} dx \right\} du
$$

$$
= \frac{1}{2r} \int_{-r}^{r} \left\{ \int_{-\infty}^{\infty} (sy+u)^n f(y; 0, 1) dy \right\} du
$$

$$
= \frac{1}{2r} \int_{-r}^{r} \left\{ \int_{-\infty}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (sy)^{n-i} u^i f(y; 0, 1) dy \right\} du
$$

$$
= \frac{1}{2r} \sum_{i=0}^{n} \binom{n}{i} s^{n-i} \mu_f(n-i) \frac{r^{i+1} + (-r)^{i+1}}{i+1}
$$

where $y = (x-u)/s$ and $\mu_f(n)$ denotes the $n$-th central moment of $f(x; 0, 1)$. It follows that

$$
\mu_{p_A}(2) = s^2 \mu_f(2) + \frac{r^2}{3};
$$

$$
\mu_{p_A}(4) = s^4 \mu_f(4) + 6s^2 \mu_f(2) \frac{r^2}{3} + \frac{r^4}{5}.
$$

Thus, we have

$$
\kappa_{p_A} = \frac{\mu_{p_A}(4)}{\mu_{p_A}(2)^2} = \frac{\mu_f(4) + 2\mu_f(2) \left(\frac{r}{s}\right)^2 + \frac{1}{5} \left(\frac{r}{s}\right)^4}{\left(\mu_f(2) + \frac{1}{3} \left(\frac{r}{s}\right)^2\right)^2}.
$$

(8.1)
Appendix B. Central moments of $p_{AL}$

The second and fourth central moments of $f_L(x; 0, 1) = F'_L(x; 0, 1) = \text{sech}^2(x/2)/4$ are evaluated as $\mu_{f_L}(2) = \pi^2/3$ and $\mu_{f_L}(4) = 7\pi^4/15$, respectively (see reference [28]). It follows from (8.1) that we have

$$\kappa_{p_{AL}} = \frac{9}{5} + \frac{12}{5 (1 + (\frac{x}{\pi})^2)^2}.$$ 

Alternatively, the central moments of $p_{AL}$ given by (4.3) can be evaluated directly using the complete Fermi–Dirac integral:

$$F_j(x) = \frac{1}{\Gamma(j + 1)} \int_0^\infty \frac{t^j}{e^{t-x} + 1} dt = -\text{Li}_{j+1}(-e^x)$$ (8.2)

where $\text{Li}_n$ is the polylogarithm function defined by

$$\text{Li}_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}.$$ 

Furthermore, $\text{Li}_n$ satisfies the following relation [29]

$$\text{Li}_n(-z) + (-1)^n \text{Li}_n(-1/z) = -\frac{1}{n!} (\ln z)^n + 2 \sum_{k=1}^{[n/2]} (\ln z)^{n-2k} (n-2k)! \text{Li}_{2k}(-1)$$

where $[x]$ denotes the greatest integer less than or equal to $x$. Since $\text{Li}_2(-1) = -\pi^2/12$ and $\text{Li}_4(-1) = -7\pi^4/720$, we have

$$\text{Li}_3(-z) - \text{Li}_3(-1/z) = \frac{\pi^2}{6} \ln z,$$

$$\text{Li}_5(-z) - \text{Li}_5(-1/z) = \frac{1}{120} (\ln z)^5 - \frac{\pi^2}{36} (\ln z)^3 - \frac{7\pi^4}{360} \ln z.$$ 

By using these relations, the $n$-th central moment $\mu_{p_{AL}}(n)$ of $p_{AL}$, for a positive even integer $n$, is expressed as

$$\mu_{p_{AL}}(n) = \int_{-\infty}^\infty \frac{(x-m)^n}{b-a} \left( \frac{1}{1 + \exp \left( \frac{a-x}{s} \right)} - \frac{1}{1 + \exp \left( \frac{b-x}{s} \right)} \right) dx$$

$$= \frac{s^{n+1}}{r} \int_0^\infty \left( \frac{y^n}{1 + \exp \left( \frac{r-y}{s} \right)} - \frac{y^n}{1 + \exp \left( \frac{r+y}{s} \right)} \right) dy$$

$$= \frac{s^{n+1}}{r} \Gamma(n+1) \left\{ -\text{Li}_{n+1}\left( -\exp \left( \frac{r}{s} \right) \right) + \text{Li}_{n+1}\left( -\exp \left( \frac{-r}{s} \right) \right) \right\}$$

where $b-a = 2r > 0$, $m = (a+b)/2$, and $y = (m-x)/s$. It follows that

$$\mu_{p_{AL}}(2) = \frac{2s^3}{r} \left( \frac{1}{6} \left( \frac{r}{s} \right)^3 + \frac{\pi^2}{6} \left( \frac{r}{s} \right) \right) = \frac{s^2\pi^2}{3} \left( \frac{r}{\pi s} \right)^2 + 1$$

$$\mu_{p_{AL}}(4) = \frac{s^5}{r} 4! \left( \frac{1}{120} \left( \frac{r}{s} \right)^5 + \frac{\pi^2}{36} \left( \frac{r}{s} \right)^3 + \frac{7\pi^4}{360} \left( \frac{r}{s} \right) \right)$$

$$= \frac{s^4\pi^4}{15} \left( 3 \left( \frac{r}{\pi s} \right)^4 + 10 \left( \frac{r}{\pi s} \right)^2 + 7 \right).$$

Thus, we obtain the same result from $\kappa_{p_{AL}} = \mu_{p_{AL}}(4)/\mu_{p_{AL}}(2)^2$.
Appendix C. Condition for flat-topped $p_{AL}$

The PDF $p_{AL}(x)$ is given by
\[
p_{AL}(x) = \frac{1}{2r} \left( \frac{\sinh \left( \frac{x}{r} \right)}{\cosh \left( \frac{x-a}{s} \right) + \cosh \left( \frac{x}{r} \right)} \right)
\]
in (4.3). Its first and second derivatives are
\[
p'_{AL}(x) = \frac{1}{2rs} \left( \frac{\sinh \left( \frac{x}{r} \right) \sinh \left( \frac{x-a}{s} \right)}{\cosh \left( \frac{x-a}{s} \right) + \cosh \left( \frac{x}{r} \right)^2} \right),
\]
\[
p''_{AL}(x) = \frac{1}{2rs^2} \left( 2 \sinh \left( \frac{x}{r} \right) \sinh^2 \left( \frac{x-a}{s} \right) - \sinh \left( \frac{x}{r} \right) \cosh \left( \frac{x-a}{s} \right) \right) \frac{\sinh \left( \frac{x-a}{s} \right)}{\left( \cosh \left( \frac{x-a}{s} \right) + \cosh \left( \frac{x}{r} \right)^2 \right)}.
\]
Hence,
\[
p'_{AL}(a) = -p'_{AL}(b) = \frac{1}{8rs} \tanh^2 \left( \frac{r}{s} \right),
\]
\[
p''_{AL}(m) = -\frac{1}{2rs^2} \frac{\sinh (r/s)}{(1 + \cosh (r/s))^2}.
\]
Therefore, we have
\[
|p''_{AL}(m)| \left|\frac{a-b}{p'_{AL}(a) - p'_{AL}(b)}\right| = \frac{4r}{s} \frac{\cosh^2 (r/s)}{\sinh (r/s) (1 + \cosh (r/s)))} < 4 (r/s) \text{csch} (r/s).
\]

Appendix D. Condition for flat-topped $p_{BL}$

The PDF $p_{BL}(x)$ given by (4.6) can be rewritten as
\[
p_{BL}(x | a, b, s, t) = c F_L \left( \frac{x-a}{s} \right) F_L \left( \frac{b-x}{t} \right)
\]
where $F_L(x)$ is the abbreviation of $F_L(x; 0, 1)$. Since $F_L(-x) = 1 - F_L(x)$ and $F_L'(x) = F_L(x) F_L(-x)$, the first and second derivatives of $p_{BL}(x)$ are expressed as
\[
p'_{BL}(x) = c \left[ F_L \left( \frac{x-a}{s} \right) F_L \left( \frac{b-x}{t} \right) \left[ \frac{1}{s} F_L \left( \frac{a-x}{s} \right) - \frac{1}{t} F_L \left( \frac{x-b}{t} \right) \right] \right]
\]
\[
= p_{BL}(x) \left[ \frac{1}{s} F_L \left( \frac{a-x}{s} \right) - \frac{1}{t} F_L \left( \frac{x-b}{t} \right) \right]
\]
\[
p''_{BL}(x) = p'_{BL}(x) \left[ \frac{1}{s} F_L \left( \frac{a-x}{s} \right) - \frac{1}{t} F_L \left( \frac{x-b}{t} \right) \right]
\]
\[
- p_{BL}(x) \left[ \frac{1}{s^2} F_L \left( \frac{x-a}{s} \right) F_L \left( \frac{a-x}{s} \right) + \frac{1}{t^2} F_L \left( \frac{x-b}{t} \right) F_L \left( \frac{b-x}{t} \right) \right].
\]
Hence, we have
\[
p'_{BL}(a) = \frac{c}{8} \left[ \frac{1}{s} \left( 1 + \tanh \left( \frac{b-a}{2t} \right) \right) - \frac{1}{t} \left( 1 - \tanh^2 \left( \frac{b-a}{2t} \right) \right) \right]
\]
\[
p'_{BL}(b) = \frac{c}{8} \left[ \frac{1}{s} \left( 1 - \tanh^2 \left( \frac{b-a}{2s} \right) \right) - \frac{1}{t} \left( 1 + \tanh \left( \frac{b-a}{2s} \right) \right) \right].
\]
and the difference of $p'_{BL} (a) - p'_{BL} (b)$ satisfies the following inequality:

$$p'_{BL} (a) - p'_{BL} (b) \geq \frac{c}{4} \sqrt{\frac{1}{s t}} \left[ \frac{1}{s} \tanh \left( \frac{b-a}{2s} \right) + \frac{1}{t} \tanh \left( \frac{b-a}{2t} \right) \right]$$

$$\geq \frac{c}{4} \sqrt{\frac{1}{s t}} \tanh \left( \frac{b-a}{2s} \right) \tanh \left( \frac{b-a}{2t} \right)$$

$$\geq \frac{c}{2} \left[ t \coth \left( \frac{b-a}{2t} \right) + s \coth \left( \frac{b-a}{2s} \right) \right]$$

(8.3)

where we have used the AM-GM inequality twice. As concerns $p''_{BL} (x_m)$, it follows from $p'_{BL} (x_m) = 0$ that

$$\frac{1}{s} F_L \left( \frac{a-x_m}{s} \right) = \frac{1}{t} F_L \left( \frac{x_m-b}{t} \right)$$

and hence

$$p''_{BL} (x_m) = -p_{BL} (x_m) \left( F_L \left( \frac{x_m-a}{s} \right) F_L \left( \frac{x_m-b}{t} \right) + F_L \left( \frac{a-x_m}{s} \right) F_L \left( \frac{b-x_m}{t} \right) \right)$$

$$= -p_{BL} (x_m) \frac{F_L \left( \frac{x_m-a}{s} \right) F_L \left( \frac{b-x_m}{t} \right)}{st} \left( F_L \left( \frac{x_m-b}{t} \right) + F_L \left( \frac{a-x_m}{s} \right) \right)$$

$$= -\frac{c}{st} F_L \left( \frac{x_m-a}{s} \right)^2 F_L \left( \frac{b-x_m}{t} \right)^2 \left( \exp \left( \frac{x_m-b}{t} \right) + \exp \left( \frac{a-x_m}{s} \right) \right).$$

Let $y = (at + bs) / (s + t)$. Since $p_{BL} (x) \leq p_{BL} (x_m)$ holds for every $x$, we have

$$p_{BL} (y) = \frac{c}{(1 + \exp \left( \frac{a-b}{s+t} \right))^2} \leq \frac{c}{(1 + \exp \left( \frac{a-x_m}{s} \right)) (1 + \exp \left( \frac{x_m-b}{t} \right))}.$$ 

It follows that

$$\exp \left( \frac{a-x_m}{s} \right) + \exp \left( \frac{x_m-b}{t} \right) < \left( 2 + \exp \left( \frac{a-b}{s+t} \right) \right) \exp \left( \frac{a-b}{s+t} \right)$$

and therefore

$$|p''_{BL} (x_m)| < \frac{3c}{st} \exp \left( \frac{a-b}{s+t} \right).$$

From this inequality and (8.3), we have

$$|p''_{BL} (x_m)| < \frac{6c}{s t} \exp \left( \frac{a-b}{s+t} \right) \left[ \frac{b-a}{2s} \coth \left( \frac{b-a}{2s} \right) + \frac{b-a}{2t} \coth \left( \frac{b-a}{2t} \right) \right].$$

Appendix E. Approximation of the normalizing constant of $p_{BL}$

The normalization condition of $p_B$ given by (3.5) can be expressed as

$$1/c = \int_{-\infty}^{\infty} F (x; a, s) (1 - G (x; b, t)) \, dx$$

$$= \int_{-\infty}^{\infty} \{ F (x; a, s) - F (x; b, s) + (1 - F (x; a, s)) G (x; b, t) \} \, dx$$

$$= b - a + \int_{-\infty}^{\infty} (1 - F (x; a, s)) G (x; b, t) \, dx.$$
if \( \int_{-\infty}^{\infty} (F (x; b, s) - G (x; b, t)) \, dx = 0 \). Let \( \delta \) be the last term of the integral such that
\[
\delta = \int_{-\infty}^{\infty} (1 - F (x; a, s)) \, G (x; b, t) \, dx
\]
\[< \min_y \left\{ \int_{-\infty}^{y} G (x; b, t) \, dx + \int_{y}^{\infty} (1 - F (x; a, s)) \, dx \right\} .
\]

For \( p_{BL} (x) = F_L (x; a, s) (1 - F_L (x; b, t)) \), letting \( y \) be a point such that \( F_L (y; a, s) = 1 - F_L (y; b, t) \) gives
\[
\delta < t \ln \left( 1 + \exp \left( \frac{y - b}{t} \right) \right) + s \ln \left( 1 + \exp \left( \frac{a - y}{s} \right) \right).
\]

If \( p_{BL} \) is flat-topped under the condition \( \exp \left( \left( a - y \right)/s \right), \exp \left( \left( y - b \right)/t \right) < \varepsilon \ll 1 \), that is \( 1 - F_L (y; a, s) = F_L (y; b, t) < \varepsilon \), then it can be approximated by (4.7) and \( \delta \) must be very small so that the error of the approximation can be less influential.

Appendix F. Integration of \( p_{CF} \) and \( p_{CH} \)

Let \( p_{CF} \) be a PDF defined by
\[
p_{CF} (x) = c_F \left\{ 1 + \exp \left( \frac{|x|^\beta - r \beta}{s^\beta} \right) \right\}^{-1}
\]
where \( c_F, r, s, \beta > 0 \) are constants. Let \( k \) be a non-negative even integer. The \( k \)-th central moment of \( p_{CF} \) is given by
\[
\int_{-\infty}^{\infty} x^k p_{CF} (x) \, dx = 2c_F \int_{0}^{\infty} x^k \left\{ 1 + \exp \left( \frac{x^\beta - r \beta}{s^\beta} \right) \right\}^{-1} \, dx
\]
\[
= c_F \frac{2s}{\beta} \int_{0}^{\infty} \frac{s^k u^{k/\beta} u^{1/\beta - 1}}{1 + \exp \left( u - r \beta / s^\beta \right)} \, du
\]
\[
= c_F \frac{2s^{k+1}}{\beta} \Gamma \left( \frac{k + 1}{\beta} \right) F_{\frac{k + 1}{\beta} - 1} \left( \frac{r \beta}{s^\beta} \right).
\]
where \( u = x^\beta / s^\beta \), \( \Gamma \) is the gamma function, and \( F_j (\cdot) \) is the complete Fermi-Dirac integral given by (8.2). It follows from the normalization condition for \( k = 0 \) and \( \Gamma (x + 1) = x \Gamma (x) \) that
\[
c_F = \left( 2s \Gamma \left( \frac{1}{\beta} + 1 \right) F_{\frac{1}{\beta} - 1} \left( \frac{r \beta}{s^\beta} \right) \right)^{-1}.
\]
The CDF of \( p_{CF} \) is expressed as
\[
P_{CF} (x) = \frac{1}{2} \left( 1 + \text{sgn} (x) \left\{ 1 - \frac{F_{1/\beta - 1} \left( r \beta / s^\beta, |x|^\beta / s^\beta \right)}{F_{1/\beta - 1} \left( r \beta / s^\beta \right)} \right\} \right)
\]
where
\[
F_j (x, u) = \frac{1}{\Gamma (j + 1)} \int_{u}^{\infty} \frac{t^j}{e^{t-x} + 1} \, dt
\]
is the incomplete Fermi–Dirac integral for an index \( j \). If \( \beta = 1 \), then \( F_0 (x, u) = \ln (1 + \exp (x - u)) \) and we have

\[
P_{CF} (x) = \frac{1}{2} \left( 1 + \text{sgn} (x) \left\{ 1 - \frac{\ln (1 + \exp (r - |x|))}{\ln (1 + \exp (\frac{x}{s}))} \right\} \right).
\]

Let \( p_{CH} \) be a PDF defined by

\[
p_{CH} (x) = \frac{c_H \sinh (r^\beta / s^\beta)}{\cosh (|x|^{\beta} / s^\beta) + \cosh (r^\beta / s^\beta)}.
\]

As in \( p_{CF} \), the \( k \)-th central moment of \( p_{CH} \) is given by

\[
\int_{-\infty}^{\infty} x^k p_{CH} (x) \, dx = 2 c_H \int_{0}^{\infty} \left\{ \frac{x^k}{1 + \exp \left( \frac{x^\beta - r^\beta}{s^\beta} \right)} - \frac{x^k}{1 + \exp \left( \frac{x^\beta + r^\beta}{s^\beta} \right)} \right\} \, dx
\]

\[
= c_H \frac{2 s^{k+1}}{\beta} \Gamma \left( \frac{k + 1}{\beta} \right) \left\{ F_{k+1} \left( \frac{r^\beta}{s^\beta} \right) - F_{k+1} \left( -\frac{r^\beta}{s^\beta} \right) \right\}.
\]

where

\[
c_H = \left[ 2 s \Gamma \left( \frac{1}{\beta} + 1 \right) \left\{ F_{k+1} \left( \frac{r^\beta}{s^\beta} \right) - F_{k+1} \left( -\frac{r^\beta}{s^\beta} \right) \right\} \right]^{-1}.
\]

The CDF of \( p_{CH} \) is expressed as

\[
P_{CH} (x) = \frac{1}{2} \left( 1 + \text{sgn} (x) \left\{ 1 - \frac{F_{k+1} \left( \frac{r^\beta}{s^\beta}, \frac{|x|^{\beta}}{s^\beta} \right) - F_{k+1} \left( -\frac{r^\beta}{s^\beta}, \frac{|x|^{\beta}}{s^\beta} \right)}{F_{k+1} \left( \frac{r^\beta}{s^\beta} \right) - F_{k+1} \left( -\frac{r^\beta}{s^\beta} \right)} \right\} \right).
\]

If \( \beta = 1 \), then we have

\[
P_{CH} (x) = \frac{s}{2 r} \ln \left( \frac{1 + \exp \left( \frac{x}{s} \right)}{1 + \exp \left( \frac{-x}{s} \right)} \right).
\]

Appendix G. Integration of \( p_{CM} \) and \( p_{CL} \)

Let \( p_C \) be a PDF for \( n \)-multivariate distribution defined by

\[
p_C (x \mid m, \Sigma) = \frac{c}{h + g \left\{ (x - m)^T \Sigma^{-1} (x - m) \right\}^{n/2}}
\]

where \( x \) and \( m \) are \( n \) dimensional vectors and \( \Sigma \) is an \( n \times n \) positive-definite matrix. Based on the eigendecomposition of \( \Sigma \) with an orthogonal matrix \( Q \) such that \( \Sigma^{-1} = Q \Lambda^{-1} Q^{-1} \) where \( \Lambda \) is a diagonal matrix and \( |\Sigma| = |\Lambda| = \prod_{i=1}^{n} \lambda_i \), we make the changes of variables \( y = Q^{-1} (x - m) \) and \( z_i = y_i / \sqrt{\lambda_i} \) that satisfy

\[
(x - m)^T \Sigma^{-1} (x - m) = y^T \Lambda^{-1} y = \sum_{i=1}^{n} z_i^2.
\]

The Jacobian determinant of the transformation from \( x \) to \( y \) is 1 and that from \( y \) to \( z \) is \( \prod_{i=1}^{n} \sqrt{\lambda_i} = \sqrt{|\Sigma|} \). The integral of \( p_C \) for \( n \geq 2 \) can be evaluated by using
the further change of variables from Cartesian to polar coordinates as follows:

\[
\int_{\mathbb{R}^n} p_C (\mathbf{x} \mid \mathbf{m}, \boldsymbol{\Sigma}) \, d\mathbf{x} \\
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{c}{h + g \left( \sum_{i=1}^{n} z_i^2 \right)^{n/2}} \sqrt{\lambda_1 \cdots \lambda_n} \, dz_1 \cdots dz_n \\
= |\boldsymbol{\Sigma}|^{1/2} \int_0^{2\pi} \cdots \int_0^{\pi} \left( \int_0^{\pi} \frac{c}{h + g (\rho^n)} \, J_n \, d\rho \right) \, d\varphi_1 \cdots d\varphi_{n-1} \\
= |\boldsymbol{\Sigma}|^{1/2} \left( 2\pi \prod_{i=1}^{n-2} \int_0^{\frac{\pi}{2}} \sin^{n-1-i} \varphi_i \, d\varphi_i \right) \left( \int_0^{\infty} \frac{c \rho^{n-1}}{h + g (\rho^n)} \, d\rho \right) \\
= \frac{\pi^{n/2} |\boldsymbol{\Sigma}|^{1/2}}{\Gamma(n/2)} \left( \frac{1}{n} \int_0^{\infty} \frac{c}{h + g (u)} \, du \right) \\
= \int_0^{\infty} \frac{c}{h + g (u)} \, du \tag{8.4}
\]

where \( z_1 = \rho \cos (\varphi_1), \ z_2 = \rho \sin (\varphi_1) \cos (\varphi_2), \ z_3 = \rho \sin (\varphi_1) \sin (\varphi_2) \cos (\varphi_3), \ldots, \)
\( z_{n-1} = \rho \sin (\varphi_1) \cdots \sin (\varphi_{n-2}) \cos (\varphi_{n-1}), \ z_n = \rho \sin (\varphi_1) \cdots \sin (\varphi_{n-2}) \sin (\varphi_{n-1}), \)
\( u = \rho^n, \ J_n \) is the Jacobian determinant such that \( J_n = \rho^{n-1} \prod_{i=1}^{n-2} \sin^{n-1-i} \varphi_i \) and the integral with respect to \( \varphi_1, \ldots, \varphi_{n-1} \) is represented by the following special functions:

\[
\prod_{i=1}^{n-2} \int_0^{\pi} \sin^{n-1-i} \varphi_i \, d\varphi_i = \prod_{i=1}^{n-2} B \left( \frac{n-i}{2}, \frac{1}{2} \right) = \prod_{i=1}^{n-2} \frac{\Gamma \left( \frac{n-i}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{n+i-1}{2} \right) \Gamma \left( \frac{n}{2} \right)} = \pi^{n/2-1} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)}.
\]

The integral of \( p_{CM} \) can be obtained by substituting \( g (u) = \exp (ut - r^n t) \) and \( h = 1 \) into (8.4) as follows:

\[
\int_0^{\infty} \frac{c}{h + g (u)} \, du = \int_0^{\infty} \frac{c}{1 + \exp (ut - r^n t)} \, du = \frac{c}{t} F_0 (r^n t) = \frac{c}{t} \ln (1 + \exp (r^n t)).
\]

In much the same way, the integral of \( p_{CM} \) can be obtained by substituting \( g (u) = \cosh (ut) / \sinh (r^n t) \) and \( h = \coth (r^n t) \) into (8.4) as follows:

\[
\int_0^{\infty} \frac{c}{h + g (u)} \, du = \int_0^{\infty} \frac{c \sinh (r^n t)}{\cosh (r^n t) + \cosh (ut)} \, du \\
= \int_0^{\infty} \left( \frac{c}{1 + \exp (ut - r^n t)} - \frac{c}{1 + \exp (ut + r^n t)} \right) \, du \\
= \frac{c}{t} \left( F_0 (r^n t) - F_0 (-r^n t) \right) \\
= \frac{c}{t} \left( \ln (1 + \exp (r^n t)) - \ln (1 + \exp (-r^n t)) \right) \\
= cr^n.
\]
Appendix H. Partial derivatives of $l_{AL}$

The PDF $p_{AL}$ given by (4.3) is rewritten as

$$p_{AL}(x \mid a, b, s) = \frac{1}{2(b-a)} \left( \tanh \left( \frac{x-a}{2s} \right) - \tanh \left( \frac{x-b}{2s} \right) \right)$$

$$= \frac{1}{2(b-a)} \left( \frac{\sinh \left( \frac{b-a}{2s} \right)}{\cosh \left( \frac{x-a}{2s} \right) \cosh \left( \frac{x-b}{2s} \right)} \right).$$

The log-likelihood $l_{AL}$ for an i.i.d. sample $\{x_1, \ldots, x_N\}$ is

$$l_{AL} = \sum_{i=1}^{N} \ln p_{AL}(x_i \mid a, b, s)$$

$$= N \ln \left( \frac{\sinh \left( \frac{b-a}{2s} \right)}{2(b-a)} \right) - \sum_{i=1}^{N} \ln \left\{ \cosh \left( \frac{x_i-a}{2s} \right) \cosh \left( \frac{x_i-b}{2s} \right) \right\}.$$

Thus the partial derivatives of the log-likelihood are as follows:

$$\partial_a l_{AL} = \frac{N}{b-a} - \frac{N}{2s} \coth \left( \frac{b-a}{2s} \right) + \frac{1}{2s} \sum_{i=1}^{N} \tanh \left( \frac{x_i-a}{2s} \right),$$

$$\partial_b l_{AL} = -\frac{N}{b-a} + \frac{N}{2s} \coth \left( \frac{b-a}{2s} \right) + \frac{1}{2s} \sum_{i=1}^{N} \tanh \left( \frac{x_i-b}{2s} \right),$$

$$\partial_s l_{AL} = -\frac{N}{s} \frac{(b-a)}{2s} \coth \left( \frac{b-a}{2s} \right)$$

$$+ \frac{1}{s} \sum_{i=1}^{N} \left\{ \left( \frac{x_i-a}{2s} \right) \tanh \left( \frac{x_i-a}{2s} \right) + \left( \frac{x_i-b}{2s} \right) \tanh \left( \frac{x_i-b}{2s} \right) \right\},$$

$$\partial_{aa} l_{AL} = \frac{N}{(b-a)^2} - \frac{N}{4s^2} \text{csch}^2 \left( \frac{b-a}{2s} \right) - \frac{1}{4s^2} \sum_{i=1}^{N} \text{sech}^2 \left( \frac{x_i-a}{2s} \right),$$

$$\partial_{bb} l_{AL} = \frac{N}{(b-a)^2} - \frac{N}{4s^2} \text{csch}^2 \left( \frac{b-a}{2s} \right) - \frac{1}{4s^2} \sum_{i=1}^{N} \text{sech}^2 \left( \frac{x_i-b}{2s} \right),$$

$$\partial_{ss} l_{AL} = \frac{N}{s^2} \left\{ \left( \frac{b-a}{2s} \right) \coth \left( \frac{b-a}{2s} \right) - \left( \frac{b-a}{2s} \right)^2 \text{csch}^2 \left( \frac{b-a}{2s} \right) \right\}$$

$$- \frac{1}{s^2} \sum_{i=1}^{N} \left\{ \left( \frac{x_i-a}{2s} \right) \tanh \left( \frac{x_i-a}{2s} \right) + \left( \frac{x_i-a}{2s} \right)^2 \text{sech}^2 \left( \frac{x_i-a}{2s} \right) \right\}$$

$$- \frac{1}{s^2} \sum_{i=1}^{N} \left\{ \left( \frac{x_i-b}{2s} \right) \tanh \left( \frac{x_i-b}{2s} \right) + \left( \frac{x_i-b}{2s} \right)^2 \text{sech}^2 \left( \frac{x_i-b}{2s} \right) \right\},$$

$$\partial_{ab} l_{AL} = -\frac{N}{(b-a)^2} + \frac{N}{4s^2} \text{csch}^2 \left( \frac{b-a}{2s} \right),$$
\[ \partial_{s} l_{\text{AL}} = \frac{N}{2s^2} \left\{ \coth \left( \frac{b-a}{2s} \right) - \left( \frac{b-a}{2s} \right) \text{csch}^2 \left( \frac{b-a}{2s} \right) \right\} - \frac{1}{2s^2} \sum_{i=1}^{N} \left\{ \tanh \left( \frac{x_i-a}{2s} \right) + \left( \frac{x_i-a}{2s} \right) \text{sech}^2 \left( \frac{x_i-a}{2s} \right) \right\} , \]

\[ \partial_{s} l_{\text{AL}} = -\frac{N}{2s^2} \left\{ \coth \left( \frac{b-a}{2s} \right) - \left( \frac{b-a}{2s} \right) \text{csch}^2 \left( \frac{b-a}{2s} \right) \right\} - \frac{1}{2s^2} \sum_{i=1}^{N} \left\{ \tanh \left( \frac{x_i-b}{2s} \right) + \left( \frac{x_i-b}{2s} \right) \text{sech}^2 \left( \frac{x_i-b}{2s} \right) \right\} . \]

Appendix I. KL divergence between \( p_{\text{MU}} \) and \( p_{\text{MN}} \)

The variance \( \sigma_{11} \) of \( p_{\text{MU}} (x \mid 0, r) \) given by (5.2) is evaluated in much the same way as in (8.4):

\[ \sigma_{11} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^2 p_{\text{MU}} (x \mid 0, r) \, dx_1 \cdots dx_n \]

\[ = \int_{0}^{\pi} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \rho^2 \cos \varphi_1 \, q_{\text{MU}} (\rho) \, J_n \, d\rho \, d\varphi_1 \cdots d\varphi_{n-1} \]

\[ = \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2} \rho^{n}} \frac{1}{2\pi B \left( \frac{n-1}{2}, \frac{3}{2} \right)} \left( \prod_{i=2}^{n-2} B \left( \frac{n-i}{2}, \frac{1}{2} \right) \right) \int_{0}^{r} \rho^{n+1} d\rho \]

\[ = \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2} \rho^{n}} \frac{2\pi}{n \sqrt{\pi}} \frac{1}{\Gamma \left( \frac{n-1}{2} \right)} \frac{\Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2} \rho^{n}} \frac{r^{n+2}}{n + 2} \]

\[ = \frac{r^{2}}{n + 2} \]

where \( p_{\text{MU}} (x \mid 0, r) \) is transformed into the radial function

\[ q_{\text{MU}} (\rho) = \begin{cases} \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2} \rho^{n}} & \rho \leq r, \\ 0 & \text{otherwise.} \end{cases} \]

Hence, the fitted normal distribution \( p_{\text{MN}}^* (x \mid 0, \Sigma) \) is expressed by the following radial function with respect to \( \rho = (x^T x)^{1/2} \):

\[ q_{\text{MN}} (\rho) = \left( \frac{n+2}{2 \pi \rho^2} \right)^{n/2} \exp \left( -\frac{n+2}{2 \rho^2} \rho^2 \right) . \]

The expected log-likelihoods of \( p_{\text{MU}} \) and \( p_{\text{MN}}^* \) are

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\text{MU}} (x \mid 0, r) \ln p_{\text{MU}} (x \mid 0, r) \, dx_1 \cdots dx_n \]

\[ = \int_{0}^{\pi} \int_{0}^{\pi} \cdots \int_{0}^{\pi} q_{\text{MU}} (\rho) \ln q_{\text{MU}} (\rho) \, J_n \, d\rho \, d\varphi_1 \cdots d\varphi_{n-1} \]

\[ = \ln \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2} \rho^{n}} , \]
The partial integrals are evaluated as follows:

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{MU}(x | 0, r) \ln p_{MN}^* (x | 0, \Sigma) \, dx_1 \cdots dx_n
\]

\[
= 2\pi \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} q_{MU}(\rho) \ln q_{MN}(\rho) J_n \, d\rho \, d\varphi_1 \cdots d\varphi_{n-1}
\]

\[
= 2\pi \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \int_0^r \frac{\Gamma(n/2 + 1)}{\pi^{n/2} r^n} \left( n + 2 \ln \frac{n + 2}{2\pi r^2} \rho^2 \right) J_n \, d\rho \, d\varphi_1 \cdots d\varphi_{n-1}
\]

\[
= 2 \ln \frac{n + 2}{2\pi r^2} - \frac{n}{2}.
\]

Therefore the KL divergence is

\[
D_{KL}(p_{MU} \parallel p_{MN}^*) = \ln \frac{\Gamma(n/2 + 1)}{\pi^{n/2} r^n} - \left( \frac{n}{2} \ln \frac{n + 2}{2\pi r^2} - \frac{n}{2} \right)
\]

\[
= \ln \left( \frac{n}{2} + 1 \right) - \frac{n}{2} \ln \left( \frac{n}{2} + 1 \right) + \frac{n}{2}
\]

and the L1 distance is

\[
D_{L1}(p_{MU}, p_{MN}^*)
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |p_{MU}(x | 0, r) - p_{MN}^*(x | 0, \Sigma)| \, dx_1 \cdots dx_n
\]

\[
= 2\pi \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \int_0^r |q_{MU}(\rho) - q_{MN}(\rho)| J_n \, d\rho \, d\varphi_1 \cdots d\varphi_{n-1}
\]

\[
= I_{\varphi} \left\{ \int_0^u (q_{MN} - q_{MU}) \rho^{n-1} d\rho + \int_u^r (q_{MU} - q_{MN}) \rho^{n-1} d\rho + \int_r^\infty q_{MN} \rho^{n-1} d\rho \right\}
\]

\[
= \left( 1 - 2u^n r^n \right) + \frac{1}{\Gamma(n/2)} \left\{ \Gamma(n/2) - 2\Gamma \left( \frac{n}{2}, \frac{n + 2}{2\pi r^2} u^2 \right) + 2\Gamma \left( \frac{n}{2}, \frac{n + 2}{2\pi r^2} r^2 \right) \right\}
\]

\[
= 2 \left\{ 1 - \frac{\chi_n}{\Gamma(n/2)} - \frac{\Gamma(n/2, (n/2 + 1) \chi_n^2)}{\Gamma(n/2)} - \Gamma(n/2, (n/2 + 1)) \right\}.
\]

where \( \Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} \, dt \) is the upper incomplete gamma function and \( u = \chi_n r \) satisfies the equation \( q_{MU}(u) = q_{MN}(u) \), i.e.,

\[
\chi_n = \frac{u}{r} = \sqrt{\frac{1}{(n/2 + 1)} \ln \frac{(n/2 + 1)^{n/2}}{\Gamma(n/2 + 1)}},
\]

The partial integrals are evaluated as follows:

\[
I_{\varphi} = \int_0^{2\pi} \left( \int_0^{\pi} \cdots \int_0^{\pi} \left( \prod_{i=1}^{n-2} \sin^{n-2-i} \varphi_i \right) d\varphi_1 \cdots d\varphi_{n-2} \right) d\varphi_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)},
\]

\[
\int_u^r q_{MU}(\rho) \rho^{n-1} d\rho = \int_u^r \frac{\Gamma(n/2 + 1)}{\pi^{n/2} r^n} \rho^{n-1} d\rho = \frac{\Gamma(n/2)}{2\pi^{n/2} r^n} (r^n - u^n),
\]
\[ \int_{u}^{r} q_{MN}(\rho) \rho^{n-1} d\rho = \int_{u}^{r} \left( \frac{n+2}{2\pi^{n/2}} \right)^{n/2} \exp \left( -\frac{n+2}{2\pi^{2}} r^{2} \right) \rho^{n-1} d\rho \]
\[ = \frac{1}{2\pi^{n/2}} \int_{(n/2+1)u^{2}/r^{2}}^{n/2+1} t^{n/2-1} e^{-t} dt \]
\[ = \frac{1}{2\pi^{n/2}} \left\{ \Gamma \left( \frac{n}{2}, \frac{n}{2} + 1 \right) \left( \frac{u^{2}}{r^{2}} \right) - \Gamma \left( \frac{n}{2}, n \right) \right\} . \]

The $L_1$ distance $D_{L_1}(p_{MU}, p_{MN})$ increases with $n$, depending on the characteristics of $\chi_n$ such that $0 < \chi_n < 1$, $\lim_{n \to \infty} \chi_n^2 = 1$, and $\lim_{n \to \infty} \chi_n^n = 0$.

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