Some thoughts on dynamic effective properties
– a working document

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The main purpose of this work is to address the question of the utility of “effective constitutive relations” for problems in dynamics. This is done in the context of longitudinal shear waves in an elastic medium that is periodically laminated, with attention restricted to plane waves propagating in the direction normal to the interfaces. The properties of such waves can be found by employing Floquet theory, implemented via a “transfer matrix” formulation. Problems occur at frequencies beyond those that define the first pass band, associated in part with the difficulty of assigning a unique wavenumber to the wave. This problem is examined, paying careful attention to the requirements of causality and passivity. The transmission of waves into a half-space is discussed by studying the impedance of the half-space, both directly and in the “effective medium” approximation, and an alternative way of looking at this problem, based on construction of the Green’s function, is developed.

1 Introduction

The construction of effective relations for the dynamics of composites displays some difficulties but these have largely been resolved. The general form of the effective relations is by now well-established. There is a complication associated with lack of uniqueness but this is understood and under control. In particular, it is known that one definition of the effective properties, which defines them uniquely, is to insist that they should apply in the presence of any prescribed “inelastic” deformation, such as induced by temperature change or plastic deformation. Furthermore, it is these properties that are delivered naturally and automatically from a formulation involving a comparison medium.

There remains, however, an important question relating to the utility of an effective medium formulation. It is (in principle) exact if it is expressed in terms of a Green’s function for the exact body, and type of boundary conditions, to which is is to be applied, but finding the Green’s function is tantamount to solving the original problem which is not helpful. The usual procedure is to formulate the effective medium problem for an infinite body in the hope that replacement of a finite composite medium by one with the “infinite medium”
effective properties will provide a useful approximation. This is a valid approach in the “homogenization limit” in which the scale of the microstructure is much smaller than any wavelength contained in the mean field that is generated: a rigorous proof in the context of electromagnetic waves has been given by Kohn and Shipman [1]. But what if this restriction is not met? This question has recently been addressed, through explicit study of a particular case, by Srivastava and Nemat-Nasser [2]. The present note offers further discussion along similar lines.

The problem to be addressed is for a half-space occupying $x > 0$, whose elastic constants and density are piecewise-constant periodic functions of $x$. Srivastava and Nemat-Nasser [2] adjoined a medium occupying $x < 0$, which was taken to be “effective medium”. They noted that, if the half-space $x > 0$ could reasonably be modelled as effective medium, there would be little reflection of a wave incident on the interface from the medium occupying $x < 0$. In the sections that follow, the problem is considered more generally, by comparing the impedance of the actual half-space with its “effective medium” approximation.

## 2 Waves in a periodic medium

Consider an $n$-phase infinite periodic medium. Phase $r$ occupies $x_{r-1} < x < x_r$ and has elastic constant $E_r$ and density $\rho_r$, for $1 \leq r \leq n$. The medium is repeated periodically with period $h = x_n - x_0$. Define also $h_r = x_r - x_{r-1} = p_r h$, so that $p_r$ is the volume fraction of phase $r$.

The equation of motion, Laplace transformed with respect to time, is

$$d\sigma/dx = sp,$$  \hspace{1cm} (2.1)

where stress $\sigma$ and momentum density $p$ satisfy the constitutive relations

$$\sigma = E \frac{du}{dx}, \quad p = \rho v \equiv \rho (su)$$  \hspace{1cm} (2.2)

in the transform domain, where $u$ represents displacement. Thus, $u$ satisfies

$$(d/dx)\{E(x)du/dx\} - s^2 \rho(x)u(x) = 0.$$  \hspace{1cm} (2.3)

The coefficients in this equation are periodic functions with period $h$. Hence it admits two linearly independent solutions of the form

$$u(x) = e^{i\mu x} \phi(x),$$  \hspace{1cm} (2.4)
where $\phi$ is periodic with period $h$.

In the case of the $n$-phase medium, $u(x)$ has the form

$$u(x) = A_r \cosh[k_r(x - x_{r-1})] + B_2 \sinh[k_r(x - x_{r-1})] \quad \text{for} \quad x_r < x < x_{r+1},$$

where

$$k_r = s/c_r, \quad c_r = \sqrt{E_r/\rho_r}. \quad (2.6)$$

The displacement $u$ and stress $E du/dx$ are continuous at all interfaces, and

$$u(x_n) = e^{\mu h}u(x_0) \quad \text{and} \quad \sigma(x_n) = e^{\mu h}\sigma(x_0). \quad (2.7)$$

The continuity conditions at $x_r$ for $1 \leq r \leq n - 1$ are satisfied by taking

$$\begin{pmatrix} A_{r+1} \\ B_{r+1} \end{pmatrix} = M_r \begin{pmatrix} A_r \\ B_r \end{pmatrix}, \quad (2.8)$$

where

$$M_r = \begin{pmatrix} \cosh(k_r h_r) & \sinh(k_r h_r) \\ (Z_r/Z_{r+1}) \sinh(k_r h_r) & (Z_r/Z_{r+1}) \cosh(k_r h_r) \end{pmatrix} \quad (2.9)$$

with the impedance $Z_r$ of phase $r$ given by

$$Z_r = E_r k_r/s = \sqrt{E_r \rho_r}. \quad (2.10)$$

Conditions (2.7) then require that

$$(M - e^{\mu h}I) \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0, \quad (2.11)$$

where

$$M = M_1 M_2 \cdots M_n. \quad (2.12)$$

The determinant of $M_r$ is $Z_r/Z_{r+1}$, with $Z_{n+1} = Z_1$. Hence, the determinant of $M$ is 1, and the equation for the eigenvalues $e^{\mu h}$ reduces to

$$\cosh(\mu h) = \frac{1}{2} \text{trace}(M). \quad (2.13)$$

In the case $n = 2$ this formula reproduces the dispersion relation of Rylov (1956).

Since inversion of the Laplace transform involves the time factor $e^{st}$, the wave travelling in the positive $x$-direction is characterized by the eigenvalue for which $\text{Re}(\mu) < 0$ when $s$ is
real. More generally, it is a requirement of causality that $\text{Re}(\mu) < 0$ when $\text{Re}(s) > 0$. This restriction on $\mu$ ensures that the wave amplitude remains bounded as $x \to +\infty$.

While there is no ambiguity in the definition of the eigenvalue $e^{\mu h}$, $\mu$ itself is not uniquely determined because if $\mu$ is a solution then so is $\mu + 2m\pi i/h$, for any integer $m$. This has no effect on the actual solution but it does raise a problem in relation to an effective “dispersion relation”. Figure 1 shows a plot of values of $\mu h$ against frequency $f$, when $s = -(2\pi f)i + \epsilon$ (with $\epsilon$ small and positive), as calculated directly using Matlab for an example two-phase medium which is discussed in more detail later. The imaginary part of $\mu$ is denoted $q$.

Figure 2 shows two alternatives. In the first, Fig. 2a, $qh$ is restricted to the interval $(-\pi, \pi]$. In the second, Fig. 2b, $qh$ is continuous. Since the corresponding time-dependence
is $e^{-i(2\pi ft)}$, the group velocity is positive in the pass bands and the same for all forms of $\mu$. Phase velocity depends on the choice of $\mu$ however. Perhaps the “correct” choice will depend on the psychology of perception and the importance assigned to “aliasing”.

### 3 Effective properties

The general form of the dynamic effective relations of a composite is

$$
\sigma^{\text{eff}} = E^{\text{eff}}e^{\text{eff}} + S_1v^{\text{eff}}, \quad p^{\text{eff}} = S_2e^{\text{eff}} + s\rho^{\text{eff}}v^{\text{eff}},
$$

(3.1)

where $e^{\text{eff}} = du^{\text{eff}}/dx$ and $v^{\text{eff}} = su^{\text{eff}}$. $E^{\text{eff}}$, $S_1$, $S_2$ and $\rho^{\text{eff}}$ depend on $s$ and are in general non-local operators in space, of convolution form if the medium is statistically uniform. A periodic medium fits this description if the exact position of any one interface is regarded as a random variable, uniformly over one period, $(0, h]$ say. These relations are supposed to apply independently of the equation of motion: any displacement $u$ can be maintained by application of appropriate body-force. If, however it is required only to identify some effective relations that reproduce just a free wave travelling in the direction of positive $x$, there is only one such wave in the present context. It has the spatial dependence $e^{\mu x}$ (times a periodic function) in any realization and it is natural and consistent to define the effective displacement to be $e^{\mu x}$ times the mean value of the periodic function. Since $\mu$ is known as a function of $s$, it is possible to replace the relations (3.1) by the local relations

$$
\sigma^{\text{eff}} = \hat{E}^{\text{eff}}e^{\text{eff}}, \quad p^{\text{eff}} = \hat{\rho}^{\text{eff}}v^{\text{eff}},
$$

(3.2)

where

$$
\hat{E}^{\text{eff}}(s) = \tilde{E}^{\text{eff}}(s, \mu(s)) + (s/\mu(s))\tilde{S}_1(s, \mu(s)),
$$

$$
\hat{\rho}^{\text{eff}}(s) = (\mu(s)/s)\tilde{S}_2(s, \mu(s)) + \tilde{\rho}^{\text{eff}}(s, \mu(s)),
$$

(3.3)

the tilde denoting Laplace transformation with respect to $x$. The fields $\sigma^{\text{eff}}$ and $p^{\text{eff}}$ are defined in the same way as $u^{\text{eff}}$ while $e^{\text{eff}} = du^{\text{eff}}/dx$.

Note, however, that, unless $S_1 = S_2 = 0$, a wave travelling from left to right would be described by different $\hat{E}^{\text{eff}}$ and $\hat{\rho}^{\text{eff}}$ because $\tilde{S}_1$ and $\tilde{S}_2$ will not be even functions of $\mu$.

Henceforth, the effective relations applicable to waves travelling in the positive direction will be described by equations (3.2) except that the hat symbols will be dropped. With this convention, the impedance at the surface of effective medium occupying the half-space $x > 0$
is $Z_{\text{eff}} \sqrt{E_{\text{eff}} \rho_{\text{eff}}}$. However, to avoid any possible confusion relating to the sign of the square root, the alternative definition $Z_{\text{eff}} = -\sigma_{\text{eff}}/v_{\text{eff}} \equiv -\sigma_{\text{eff}}/(su_{\text{eff}})$ is preferred.

Before concluding this section, one additional refinement has to be mentioned. While it is important that $\sigma_{\text{eff}}$ and $p_{\text{eff}}$ must be defined by averaging their periodic parts over an entire period (to ensure satisfaction of the equation of motion), it is possible to define $u_{\text{eff}}$ as a weighted average. In the most general case, this would mean averaging the periodic part of $u$, multiplied by any periodic function $w(x)$ with period $h$ and mean value 1. This includes, in particular, averaging the periodic part of $u$ over any one phase, or subset of phases. The resulting effective parameters of course depend on the weighting that is selected.

4 Comparisons between effective and actual impedance

The impedance at the surface of a half-space $x > X$, composed of the layered $n$-phase composite considered in Section 2, is $Z(X) = -\sigma(X)/su(X)$ and so, if $x_{r-1} \leq X < x_r$,

$$Z(X) = -\frac{Z_r(A_r \sinh[k_r(X - x_{r-1})] + B_r \cosh[k_r(X - x_{r-1})])}{(A_r \cosh[k_r(X - x_{r-1})] + B_r \sinh[k_r(X - x_{r-1})])},$$

having employed the relation $k_r E_r = sZ_r$ which follows from (2.6) and (2.10). An alternative view of the same result is that $Z(X)$ is the impedance at the surface of a half-space $x > 0$, composed of material in which interfaces in the above material have been shifted a distance $X$ to the left. In our “random material” interpretation, $X$ is a random variable, uniformly distributed on $[0, h)$.

Some comparisons between actual and effective impedance are presented below, with $s = -i(2\pi f) + \epsilon$, where $\epsilon$ is positive (to facilitate selection of the correct eigenvalue) but small enough to be disregarded. Thus, impedances are plotted as functions of frequency $f$.

4.1 A two-phase laminate

Following [2], the properties of the laminate are taken as

$E_1 = 8 \times 10^9 \text{ Pa}, \quad \rho_1 = 1180 \text{ kg/m}^3, \quad h_1 = 3 \text{ mm},$

$E_2 = 300 \times 10^9 \text{ Pa}, \quad \rho_2 = 8000 \text{ kg/m}^3, \quad h_2 = 1.3 \text{ mm}.$

This is the material whose dispersion relation is illustrated in Figs. 1 and 2. Figure 3 and 4 show plots of impedance at front and at the centre of each phase. The differences in the
Figure 3: Impedance versus frequency (a) at front of phase 1, (b) mid-phase 1.

Figure 4: Impedance versus frequency (a) at front of phase 2, (b) mid-phase 2.
impedances at different points are pronounced. Note, however, that all are pure imaginary in the stop bands, and in the pass bands all have positive real part. This corresponds to positive mean rate of working by the pressure at the surface.

Since any “effective” impedance will reflect some kind of averaged response, it is evident that it cannot approximate the actual response, except at low frequencies. To facilitate study of this, actual impedances are first plotted over a more restricted frequency range, in Figs. 5 and 6. Then, Fig. 7 shows the effective impedance, calculated employing the “unweighted” average for effective displacement.

Figure 8 shows the corresponding “unweighted” effective modulus and effective density. These values confirm that $Z_{	ext{eff}}$ in Fig. 7 is equal to $\sqrt{(E_{\text{eff}} \rho_{\text{eff}})}$. It is inevitable that
Figure 7: “Unweighted” effective impedance.

Figure 8: (a) effective modulus, (b) effective density, both unweighted.
the unweighted \( Z_{\text{eff}} \) cannot reproduce the actual impedance, because this varies from point to point. It is also not to be expected that \( Z_{\text{eff}} \) gives the exact mean value of \( Z \), because the mean value of a fraction is now equal to the fraction of the corresponding mean values.

Figure 9 shows plots of effective impedance, weighted on phase 1 (Fig. 9a), and on phase 2 (Fig. 9b). Figure 9a should be compared with Fig. 5b, and Fig. 9b should be compared with Fig. 6b. Reasonable agreement can be recognised for frequencies up to about 100 kHz – an appreciable range, but still within the first pass band.

It should be noted that the real parts of effective impedance, weighted or not, are qualitatively wrong (i.e. they are not zero) in the first stop band. This is in part associated with the ambiguity in the definition of \( \mu h \). In the figures plotted so far, its phase was taken as \(+\pi\), consistent with Fig. 2. If, however, the phase is taken instead as \(-\pi\), the resulting values for the weighted effective impedances are shown in Fig. 10. The corresponding plots in Figs. 9 and 10 display exactly the same values in the pass band but differ in the stop band, where the phases of \( \mu h \) are different, and hence the effective fields are calculated as the mean values of different periodic functions.

### 4.2 A three-material laminate

Some results are now presented for a laminate composed of three different materials, which was also studied in [2]. It is convenient to describe it, however, as a “five-phase” periodic laminate, with properties as follows.

\[
E_1 = 8 \times 10^9 \text{ Pa}, \quad \rho_1 = 1180 \text{ kg/mm}^3, \quad h_1 = 1.45 \text{ mm},
\]
\[
E_2 = 0.02 \times 10^9 \text{ Pa}, \quad \rho_2 = 1100 \text{ kg/mm}^3, \quad h_2 = 0.5 \text{ mm},
\]
Figure 10: Effective impedance (a) phase 1 average, (b) phase 2 average, with “−π” phase for $\mu h$ in the stop band.

$E_3 = 300 \times 10^9 \text{ Pa}, \quad \rho_3 = 8000 \text{ kg/m}^3, \quad h_3 = 0.4 \text{ mm},$

$E_4 = 0.02 \times 10^9 \text{ Pa}, \quad \rho_4 = 1100 \text{ kg/m}^3, \quad h_4 = 0.5 \text{ mm},$

$E_5 = 8 \times 10^9 \text{ Pa}, \quad \rho_5 = 1180 \text{ kg/m}^3, \quad h_5 = 1.45 \text{ mm}.$

It could equally well be described as a “four-phase” material but the specification given above shows the symmetry and was in fact used in the computations. The interesting feature of this composite is that the dense “middle” phase 3 is surrounded by soft phase 2 material and produces a resonance at a relatively low frequency. A plot of the normalized dispersion relation, $\mu h$ versus frequency $f$, is shown in Fig. 11a. There is, in fact, a small stop band around $f = 23 \text{ kHz}$. This is shown in the more detailed plot of Fig. 11b.

Effective response is subject to the limitations already exposed in the two-phase case, so the point will not be laboured again. Figure 12 shows the actual impedance at the leading edge of “phase 1”, which corresponds to the middle of “material 1”, for the extended frequency range. Figure 13a shows the detail, for frequencies around the first stop band, while Fig. 13b shows the effective impedance in the same frequency range, calculated using the corresponding weighted average of the displacement. Note, however, that the first stop band is narrow (from about 23 to 23.7 kHz) and that the effective impedance in the second pass band remains reasonably representative of, though somewhat larger than, the actual response.

5 An alternative view

What has been described so far is based on what could be regarded as the “conventional” view of effective relations, for composites with periodic microstructure. Regardless of the
Figure 11: Normalized dispersion relation for three-phase material (a) extended frequency range, (b) detail showing first stop band.

Figure 12: Actual impedance mid-material 1.
utility or otherwise of effective properties, it is disturbing – at least to me – that taking averages of fields, each one of which displays exactly the same stop band structure, should result in an “effective impedance” which depends on which branch of the (exact) dispersion relation should be selected, and which fails altogether to display even the first stop band. It is worth recalling that the exact definition of effective constitutive relations, involving non-local operators, is completely independent of the selection of any particular branch of the dispersion relation: these relations are obtained from taking ensemble averages of expressions involving the exact Green’s function. The formulae given in [4], for example, show that these ensemble averages contain summations over all equivalent wavenumbers and so require no judgement as to which branch is “relevant”.

Following on from this, it seems worth contemplating the ensemble average of the Green’s function itself, \( \langle G \rangle \) (or, more generally, the weighted average \( \langle wGw \rangle \) introduced in [4]). In the one-dimensional setting considered here, the point force at \( x = 0 \) generates a single wave in the half-space \( x < 0 \) which propagates from right to left (and, if \( s \) has positive real part, decays exponentially as \( x \to -\infty \)), and a wave in the half-space \( x > 0 \) which propagates from left to right and decays as \( x \to +\infty \). A reasonable definition of effective impedance of the half-space \( x > X \), with \( X > 0 \), is therefore to take

\[
Z^{\text{eff}}(X) = -\Sigma^{\text{eff}}(X)/sG^{\text{eff}}(X)
\]

where, for a general weight \( w \), \( G^{\text{eff}}(X) = \langle w(X)G(X)w(0) \rangle \) while \( \Sigma^{\text{eff}}(X) = \langle \Sigma(X) \rangle \), \( \Sigma(X) \) being the stress associated with the displacement \( G(X)w(0) \). Explicit expressions for both of these are given in [4].

The impedance \( Z^{\text{eff}}(X) \) so defined is, by its construction, independent of any choice of branch for the dispersion relation. The penalty, however, is that it does depend on \( X \): it
the results of some sample calculations follow.

5.1 Two-phase laminate

Figure 14 shows the effective impedance $Z^{\text{eff}}(X)$, evaluated (a) as $X \to 0$ and (b) at $X = h/2$. These plots should be compared with the exact impedances plotted in Figs. 5 and 6. It is already evident from Figs. 5 and 6 that no single effective impedance can be applicable over the whole of the frequency range that is shown. Figure 15 gives a plot $Z^{\text{eff}}(X)$ versus $X/h$ (a) when frequency $f = 100$ kHz and (b) when $f = 150$ kHz. The relatively small variation of $Z^{\text{eff}}$ at 100 kHz is consistent with the applicability of effective medium theory at that frequency, whereas Fig. 15(b) provides evidence that effective medium theory is not useful at 150 kHz. However, at least the stop band is correctly identified, with the present definition of $Z^{\text{eff}}$.

5.2 Three-phase laminate

Figure 16 shows plots (a) of the exact impedance of the three-material laminate introduced in Section 4.2, with boundary $x = 0$ coinciding with the front of the cell described as
Figure 15: Effective impedance at $X = 0$ (a) at frequency $f = 100$ kHz, (b) at $f = 150$ kHz.

Figure 16: (a) Exact impedance, mid-“material 1” for the three-phase material, (b) effective impedance at $X = 0$. 
“phase 1”, and (b) the effective impedance \(Z_{eff}(0)\). The close similarity of the two plots is noteworthy. They would differ at frequencies somewhat higher than shown but nevertheless the effective impedance captures the first stop band exactly and provides a better estimate of the impedance in the second pass band than the estimate reported in Section 4.2 – compare Figs. 13 and 16.

Figure 17 shows plots of \(Z_{eff}(X)\) against \(X/h\), (a) for frequency \(f = 17.5\) kHz and (b) \(f = 325\) kHz. These frequencies lie on either side of the first stop band. They demonstrate, even though the plots of Fig. 16 display reasonable agreement, that use of effective medium theory involves some loss of precision.

6 Discussion

The broad conclusion is that an “effective medium” description of a composite medium provides a reasonable approximation for its response, so long as the predicted “effective wavelength” is larger than \(2h\) – say at least \(2.5h\). This is consistent with the findings of Srivastava and Nemat-Nasser [2] though they expressed it somewhat differently. Here, \(h\) represents the period of the microstructure. For a more general random medium, a similar limitation is to be expected; an order of magnitude can be given for \(h\) but it is not defined precisely. Of course, the conclusion, even for a periodic medium, is a tentative one, being
based on only two simple examples.

A periodic medium throws up a very specific difficulty as soon as the effective wavelength is double the period. The imaginary part of the parameter called $\mu h$ in this work is mathematically determined only up to a multiple of $2\pi$. While it is entirely natural that $\mu h$ should tend to zero as frequency tends to zero, the choice of phase into and beyond the first stop band becomes problematic. Perhaps the most “natural” choice is that $\mu h$ should be a continuous function of frequency. The calculated phase speed then remains positive, even though the visually perceived phase speed may be negative, due to aliasing. The problem is a real one for effective property calculation because the choice of phase affects the definition of the periodic parts of the fields and their associated averaged values. The “alternative view” presented in Section 5 circumvents this difficulty: the formulation of that section reproduces stop bands exactly, and remains useful through and beyond the first stop band. The ambiguity in definition of the dispersion relation appears precisely at the first stop band. Note from Fig. 11 that $\text{Im}(\mu h) = \pi$ in the first stop band, and was chosen to be greater than $\pi$ thereafter.

When time permits, I hope to do further investigations, for a material with a stronger resonance at a low frequency for which “effective medium” theory should be more accurate, and try to throw some additional light on phenomena associated with apparently negative effective modulus and density.

References

[1] R.V. Kohn and S.P. Shipman. Magnetism and homogenization of microresonators. Multiscale Model. Simul. 7 (2008) 62–92.

[2] A. Srivastava and S. Nemat-Nasser. On the limit and applicability of dynamic homogenization. Manuscript dated October 1, 2012.

[3] S.M. Rytov. Acoustical properties of a thinly laminated medium. Sov. Phys. Acoustics 2 (1956) 6880.

[4] J.R. Willis. Exact effective relations for dynamics of a laminated body. Mechanics of Materials 41 (2009) 385–393.