Spin and Statistics in Galilean Covariant Field Theory

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The existence of a possible connection between spin and statistics is explored within the framework of Galilean covariant field theory. To this end fields of arbitrary spin are constructed and admissible interaction terms introduced. By explicitly solving such a model in the two particle sector it is shown that no spin and statistics connection can be established.

I. Introduction

The possibility of a connection between spin and statistics was first explored by Pauli who considered the problem within the framework of relativistic quantum field theory. He showed that integer spin fields must satisfy Bose-Einstein statistics and that half-integral fields must satisfy Fermi-Dirac statistics provided that no other (i.e., parastatistics) possibilities are considered. Crucial to his proof are the requirements that the energy be positive and that observables commute for space-like separations. Since both of these concepts have no counterparts in the nonrelativistic theory, it has reasonably been concluded that there is no spin and statistics connection for the nonrelativistic case. This has been confirmed by the analysis of Messiah and Greenberg who showed that in nonrelativistic quantum mechanics both symmetric and antisymmetric wave functions are allowable for any spin.

However, this conclusion has been challenged by Peshkin who claims to have established the need for symmetric wave functions in the spin zero case. This in turn has been rebutted by Allen and Mondragon and counter-rebutted by Peshkin. More recently Shaji and Sudarshan have claimed to establish the usual spin and statistics connection in the nonrelativistic theory provided that the underlying Lagrangian has a certain symmetry property.

The present paper is motivated by this continuing debate which seemingly indicates a certain reluctance to be limited by formal analyses such as those of ref. It is based on two simple observations. i) The most restrictive group of space-time transformations which can be viewed as a limit of special relativity is that of the Galilei group. This leads one to the conclusion that if a connection between spin and statistics is to exist anywhere outside the realm of relativistic quantum field theory, it is to be found in the corresponding Galilean limit. ii) If nontrivial counterexamples to the usual spin and statistics connection can be found, it immediately becomes clear that no claim to a general connection can be tenable.

In the following section the spin one-half Galilean wave equation is reviewed together with its extension to the arbitrary spin case by means of the multispinor formalism. This is used in section III to construct possible interaction terms. The resulting system is solved in the two particle sector to obtain a bound state solution as well as the two particle scattering phase shift. It is found that the solution is consistent independent of the type of statistics satisfied by the particle (or field).

II. Arbitrary Spin Galilean Fields

In principle the wave equation for a particle of arbitrary spin can be found by using the spin one-half equation as the basic ingredient in a multispinor approach. In particular if one takes a rank 2s multispinor which is symmetric in all its indices one can expect to obtain an equation for a particle of spin s. This approach is totally successful and has in fact been used by Hurley and the author to derive the g-factor of an arbitrary spin particle.

To carry out the indicated program one begins with the spin one-half Galilean wave equation. This has been derived by Lévy-Leblond some years ago and a brief review of his result is essential as a preliminary to its application to the multispinor approach. In units in which \( \hbar = 1 \) the relevant wave equation for a particle of mass \( m \) is derived from the Lagrangian

\[
\mathcal{L} = \overline{\psi} G \psi
\]

where

\[
G = (1/2)(1 + \rho_3) \left[ i \frac{\partial}{\partial t} - U_0 \right] + i \rho_1 \sigma \cdot \nabla + m(1 - \rho_3)
\]  

(1)

where two commuting sets of Pauli matrices \( \rho_i \) and \( \sigma_i \) have been used to span the \( 4 \times 4 \) dimensional spinor space. The quantity \( U_0 \) which appears in conjunction with the time derivative term is the so-called internal energy term.
Under the Galilean transformation

\[
\begin{align*}
x' &= Rx + v + a \\
t' &= t + b
\end{align*}
\]

the wave function \( \psi \) transforms as

\[
\psi'(x', t') = \exp[i f(x, t)] \Delta^{\frac{1}{2}}(v, R) \psi(x, t)
\]

where

\[
f(x, t) = \frac{1}{2} m v^2 t + m v \cdot Rx
\]

and

\[
\Delta^{\frac{1}{2}}(v, R) = \left( \frac{1}{2} \sigma \cdot v \ 0 \right) D^{\frac{1}{2}}(R).
\]

Note that \( D^{\frac{1}{2}}(R) \) is the usual two-dimensional (i.e., spin one-half) representation of rotations which acts in the space of the \( \sigma \) matrices. It is clear from the form of \( \Delta^{\frac{1}{2}}(v, R) \) that the upper components of \( \psi \) do not mix with the lower components under a Galilean transformation, an important consequence of which is the fact that the matrix

\[
\Gamma = \frac{1}{2}(1 + \rho_3)
\]

is invariant under all Galilean transformations.

Upon writing \( \psi \) in terms of the two-component spinors \( \phi \) and \( \chi \)

\[
\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}
\]

one can write the Lagrangian equation \( G \psi = 0 \) as

\[
E \phi + \sigma \cdot p \chi = U_0 \phi
\]

\[
\sigma \cdot p \phi + 2m \chi = 0
\]

where \( E = i \frac{\partial}{\partial t} \) and \( p = -i \nabla \). These equations are combined to yield the free Schrödinger equation

\[
\left( E - \frac{p^2}{2m} \phi \right) = U_0 \phi.
\]

One now proceeds to construct the spin \( s \) wave function as a completely symmetrized \( 2s \)-rank spinor \( \psi_{a_1...a_{2s}} \), where each \( a_i \) ranges from 1 to 4. In the absence of additional restrictions such an object has \( \frac{1}{6}(2s + 3)(2s + 2)(2s + 1) \) components. Using the fact that \( \Gamma \) is an invariant matrix one readily sees that an appropriate multispinor extension of the spin one-half Lagrangian is given by

\[
\mathcal{L} = \frac{1}{2s} \psi^\dagger_{a_1...a_{2s}} \left[ \sum_{i=1}^{2s} \Gamma_{a_1 a_i'...a_{i-1} a_{i-1}'...a_{2s}} G_{a_1 a_i'...a_{i-1} a_{i-1}'...a_{2s}} \psi_{a_1...a_{2s}} \right] \psi_{a_1...a_{2s}}^\prime.
\]

The resulting wave equation is

\[
\sum_{i=1}^{2s} \Gamma_{a_1 a_i'...a_{i-1} a_{i-1}'...a_{2s}} G_{a_1 a_i'...a_{i-1} a_{i-1}'...a_{2s}} \psi_{a_1...a_{2s}} = 0.
\]

In the case of special relativity it is not possible to remove the summation in Eq.(3) to obtain standard Bargmann-Wigner equations. However, in Galilean relativity it can be shown \( [10] \) that the summation can be eliminated to obtain

\[
\Gamma_{a_1 a_i'...a_{i-1} a_{i-1}'...a_{2s}} G_{a_1 a_i'...a_{i-1} a_{i-1}'...a_{2s}} \psi_{a_1...a_{2s}} = 0.
\]
Thus a Bargmann-Wigner set of equations has been obtained. Because the matrix $\Gamma$ projects out only upper components, the presence of $(2s - 1)$ $\Gamma$ matrices in (4) means that those components of $\psi$ in which more than one index is a 3 or 4 drop out of the equations of motion. One thus defines the $2s + 1$ components

$$\psi_{a_1 \ldots a_{2s}} = \phi_{a_1 \ldots a_{2s}}$$

for $a_i = 1, 2$ and the $4s$ components

$$\psi_{a_1 \ldots a_{2s-1}r} = \chi_{a_1 \ldots a_{2s-1}}^{r-2}$$

for $a_i = 1, 2; r = 3, 4$ to obtain

$$(E - U_0)\phi_{a_1 \ldots a_{2s}} + \frac{1}{2s} \left[ \sum_{i=1}^{2s} \sigma_{a_i r} \cdot p \right] \chi_{a_1 \ldots a_{i-1}a_{i+1} \ldots a_{2s}}^{r} = 0$$

(5)

and

$$\sigma_{ra_2s} \cdot p \phi_{a_1 \ldots a_{2s}} + 2m\chi_{a_1 \ldots a_{2s-1}}^{r} = 0$$

(6)

Upon solving (6) for $\chi_{a_1 \ldots a_{2s-1}}^{r}$ and inserting the result into (5) one obtains the free Schrödinger equation for the independent components $\phi_{a_1 \ldots a_{2s}}$

$$(E - U_0 - \frac{p^2}{2m})\phi_{a_1 \ldots a_{2s}} = 0.$$ 

III. Interacting Arbitrary Spin Fields

As is well known Galilean field theory is characterized by the existence of a central extension of the algebra of the Galilei group. It is designated as the mass operator and (in the context of the preceding section) is given by \[11\]

$$M = \int d^3x \phi_{a_1 \ldots a_{2s}}^\dagger \phi_{a_1 \ldots a_{2s}}.$$ 

The operator $M$ has the property that it commutes with all the operators of the Galilei group and implies the so-called Bargmann superselection rule. The latter has the consequence that a given Galilean field theory factors into sectors labelled by the eigenvalue of $M$. Thus the vacuum state is the state which is the null eigenvalue of $M$ and satisfies

$$\phi_{a_1 \ldots a_{2s}} |0\rangle = 0.$$ 

This has the consequence that the usual two-point function is no longer a time ordered function but has the degenerate form

$$G_{a_1 \ldots a_{2s},a'_1 \ldots a'_{2s}}(x - x', t - t') = -i\theta(t - t')\langle 0|\phi_{a_1 \ldots a_{2s}}(x, t)\phi_{a'_1 \ldots a'_{2s}}^\dagger (x', t')|0\rangle.$$ 

This clearly implies that the two-point function is insensitive to the statistics of the field in question. Alternatively stated, the one-particle state (like the vacuum) cannot be used to test the consistency of a particular type of statistics. Accordingly, attention is now focused on the two particle sector of the spin $s$ theory formulated in section II.

The goal here is to determine whether the choice of statistics for the $\phi$ particle is constrained in any way by the value of the spin. While this issue could be pursued in the free field limit, it is clearly of greater interest to examine it for the case of a nontrivial interaction. One possible Galilean invariant interaction is given by the Lagrangian \[12\]

$$\mathcal{L}' = \frac{\lambda}{2} \mathbf{J}^\dagger \cdot \mathbf{J}$$

(7)

where

$$\mathbf{J} = \int d\xi \phi_{a_1 \ldots a_{2s}} \left( x + \frac{1}{2} \xi, t \right) \nabla f^\ast(\xi) \Sigma_{a_1 \ldots a_{2s},a'_1 \ldots a'_{2s}} \phi_{a'_1 \ldots a'_{2s}} \left( x - \frac{1}{2} \xi, t \right)$$
where
\[\Sigma_{a_1 \ldots a_{2s};a'_1 \ldots a'_{2s}} = \sigma^2 a^2_{a_1 \ldots a_{2s}} a'^2_{a'_1 \ldots a'_{2s}}\]

and \(\sigma^2\) is the usual Pauli matrix. Such an interaction is Galilean invariant for all \(f(\xi)\) which depend only on the scalar \(|\xi|\). It is important to note that one seeks here to impose the “wrong” statistics relation
\[
\phi_{a_1 \ldots a_{2s}}(x, t)\phi_{a'_1 \ldots a'_{2s}}(x', t) + (\lambda^2 - 1)\phi_{a_1 \ldots a_{2s}}(x, t)\phi_{a_1 \ldots a_{2s}}(x, t) = 0.
\]

Had one chosen to apply the usual statistics relation a scalar-scalar coupling (i.e., no derivative of \(f(\xi)\)) rather than the vector-vector coupling (7) would be the simplest interaction choice. This is readily seen to be a consequence of the transpose relation
\[(\Sigma^2)^T = (\lambda^2)^2\Sigma^2.\]

Since the internal energy is irrelevant for purposes of this discussion, it can be discarded, thereby leading to the Hamiltonian
\[
H = -\frac{1}{2m} \int d\mathbf{x} \phi^\dagger_{a_1 \ldots a_{2s}} \nabla^2 \phi_{a_1 \ldots a_{2s}} - \frac{\lambda}{2} \int d\mathbf{x} J^\dagger \cdot \mathbf{J}.
\]

Thus the equation to be solved is
\[
H|N\rangle = E_N|N\rangle
\]
where \(E_N\) is the energy of the \(N\)-particle state. Note that the \(N\)-particle state also satisfies the eigenvalue equation
\[
M|N\rangle = Nm|N\rangle.
\]

For the cases \(N = 0\) and \(N = 1\) Eq. (8) is trivially solved since the interaction term is inoperative. Thus subsequent attention is devoted entirely to the case \(N = 2\).

One looks for a solution of the form
\[
|2\rangle = \int \int d\mathbf{x} d\mathbf{x}' \phi^\dagger_{a_1 \ldots a_{2s}}(\mathbf{x}) \Sigma^2_{a_1 \ldots a_{2s};a'_1 \ldots a'_{2s}} \phi^\dagger_{a'_1 \ldots a'_{2s}}(\mathbf{x}') \exp[i\mathbf{P} \cdot (\mathbf{x} + \mathbf{x}')]|\psi(\mathbf{x} - \mathbf{x}')\rangle 0\]

corresponding to a center-of-mass momentum \(\mathbf{P}\). This leads to an equation of the form
\[
\left[ E - \frac{\mathbf{P}^2}{4m} + \frac{1}{m} \nabla^2 \right] \psi(x) = -\lambda 2^2 \nabla^2 \psi(x) \int d\mathbf{x}' \nabla^2 f(x) f(x') \psi(\mathbf{x'}). \]

This is a well known example of an integral equation with a separable kernel which by standard techniques leads to the eigenvalue condition
\[
1 = -\lambda 2^2 \int \frac{dp}{(2\pi)^3} \frac{p^2 |f(p)|^2}{\Omega - p^2/m}
\]

where
\[
\Omega \equiv E - \frac{\mathbf{P}^2}{4m}
\]

is the renormalized internal energy of an assumed bound state solution. Such a \(\Omega < 0\) solution will exist when the condition
\[
m\lambda 2^2 \lambda 2^2 \int \frac{dp}{(2\pi)^3} |f(p)|^2 > 1
\]

is satisfied. Although the local limit \(f(p) \rightarrow 1\) is not allowable in a finite theory, this has no bearing on the overall consistency of the theory since any rotationally invariant \(f(p)\) is in fact consistent with Galilean invariance with “wrong” statistics.
Scattering solutions can also be obtained for Eq.(11). In this case one infers for a particle of incoming momentum \( k \) and internal energy \( U = \frac{k^2}{m} \) a solution of the form

\[
\psi(p) = (2\pi)^3 \frac{1}{2} \left[ \delta(p - k) - \delta(p + k) \right] - \lambda \frac{2^{2s}}{U - p^2/m} \int \frac{dp}{(2\pi)^3} e^{ip \cdot x} \frac{p_0 \cdot p f(p) f^*(p_0)}{U - p^2/m}
\]

which leads to the explicit result for \( \psi(x) \)

\[
\psi(x) = \frac{1}{2} \left( e^{ik \cdot x} - e^{-ik \cdot x} \right) - \lambda \frac{2^{2s}}{3} \int \frac{dq}{(2\pi)^3} \frac{q^2 |f(q)|^2}{U - q^2/m} \left[ 1 + \lambda \frac{2^{2s}}{3} \int \frac{dq}{(2\pi)^3} \frac{q^2 |f(q)|^2}{U - q^2/m} \right]^{-1}.
\]

Upon performing the integration over \( p \) this is found to yield

\[
\psi(x) = \frac{1}{2} \left( e^{ik \cdot x} - e^{-ik \cdot x} \right) + m \lambda \frac{2^{2s}}{3} \int \frac{dq}{(2\pi)^3} \frac{q^2 |f(q)|^2}{U - q^2/m} \left[ 1 + \lambda \frac{2^{2s}}{3} \int \frac{dq}{(2\pi)^3} \frac{q^2 |f(q)|^2}{U - q^2/m} \right]^{-1},
\]

thereby displaying the fact that there is scattering only of the \( P \) wave.

This allows identification of the \( P \) wave phase shift \( \delta_1 \) as

\[
e^{i\delta_1} \sin \delta_1 = m \lambda \frac{2^{2s}}{3} k^3 \left[ 1 + \lambda \frac{2^{2s}}{3} \int \frac{dq}{(2\pi)^3} \frac{q^2 |f(q)|^2}{U - q^2/m} \right]^{-1}.
\]

This in turn allows one to establish exact equivalence to the effective range formula appropriate to \( P \) wave scattering

\[
k^3 \cot \delta_1 = -\frac{1}{a} + \frac{1}{2} r_s k^2
\]

where the scattering length (or scattering volume) is given by

\[
-\frac{1}{a} = \frac{4\pi \zeta \Omega}{m} - \frac{1}{2} (-m\Omega)^{\frac{3}{2}}
\]

where

\[
\zeta \equiv \int \frac{dq}{(2\pi)^3} \frac{q^2 |f(q)|^2}{(\Omega - q^2/m)^2}.
\]

Similarly one finds for \( r_0 \)

\[
r_0 = \frac{-8\pi \zeta}{m^2} - 3 (-m\Omega)^{\frac{3}{2}}.
\]

In sum the Galilean spin \( s \) theory is totally consistent despite its having been quantized with “wrong” statistics. It has two divergences in the local limit \( f(p) = 1 \), the linearly divergent \( \zeta \) and the cubically divergent internal energy \( \Omega \). Although it is not feasible to extend the solution to the general sectors of the model, it seems clear that no complications are likely to arise in such cases. Were such to occur, they would lead to the conclusion that the statistics of particle pairs are unavoidably linked to the presence of other particles. Such possibilities are not to be found in the arguments normally raised against theories with “wrong” statistics, however.

**IV. Conclusion**

The question as to whether “wrong” statistics field theories can be excluded on the basis of very general arguments has been examined here within the framework of a Galilean multispinor formalism. It has in fact been demonstrated that there is no particular difficulty in constructing such theories. While the imposition of conventional statistics is somewhat more natural in the sense that they allow scalar-scalar interactions to be accommodated, it has been demonstrated here that vector-vector interactions pose no special difficulty.

As a final remark it should be noted that the usual symmetric multispinor method applies only to nonzero spin—namely, one cannot construct zero spin from a symmetric multispinor. In that special case it is, however, sufficient to observe that an antisymmetric spinor suffices to derive the spinless counterpart of Eq.(8). The details of such an approach are to be found in [8].

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11. It should be remarked here that although the considerations of section II apply equally to quantum field theory and ordinary quantum mechanics, henceforth it is to be understood that one is dealing exclusively with the case of a quantum field theory.
12. This is a straightforward extension to arbitrary spin of a special case of the Galilean invariant four field interaction given in J. M. Lévy-Leblond, Commun. Math. Phys. 4, 157 (1967).