Extended slow-roll conditions and rapid-roll conditions

Takeshi Chiba\textsuperscript{1} and Masahide Yamaguchi\textsuperscript{2,3}

\textsuperscript{1} Department of Physics, College of Humanities and Sciences, Nihon University, Tokyo 156-8550, Japan
\textsuperscript{2} Department of Physics and Mathematics, Aoyama Gakuin University, Sagamihara 229-8558, Japan
\textsuperscript{3} Department of Physics, Stanford University, Stanford, CA 94305, USA
E-mail: chiba@phys.chs.nihon-u.ac.jp and gucci@phys.aoyama.ac.jp

Received 11 August 2008
Accepted 29 September 2008
Published 14 October 2008

Online at stacks.iop.org/JCAP/2008/i=10/a=021
doi:10.1088/1475-7516/2008/10/021

Abstract. We derive slow-roll conditions for a scalar field which is non-minimally coupled with gravity in a consistent manner and express spectral indices of scalar/tensor perturbations in terms of the slow-roll parameters. The conformal invariance of the curvature perturbation is proved without linear approximations. Rapid-roll conditions are also derived, and the relation with the slow-roll conditions is discussed.

Keywords: cosmological perturbation theory, inflation, physics of the early universe
Extended slow-roll conditions and rapid-roll conditions

1. Introduction

In curved spacetime, a scalar field may generally couple to the scalar curvature so that the potential of $\phi$ has an additional term $\xi R \phi^2 / 2$ [1, 2]. $\xi = 1/6$ corresponds to the conformal coupling: a massless scalar field is conformally invariant. Even if the scalar field is minimally coupled with the curvature at tree level, a non-minimal coupling is induced naturally by radiative corrections [3]. Moreover, in supergravity theories scalar fields typically have a curvature coupling from the Kähler potential. Furthermore, a warped brane inflation in string theory involves a non-minimally coupled scalar field [4]. Thus, a non-minimal coupling of the scalar field is well motivated from the theoretical point of view and has applications in cosmology (for example [5, 6]). Such a non-minimally coupled scalar field may also be interesting from the observational view point. Recent observations like the Wilkinson Microwave Anisotropy Probe (WMAP) 5-year results are sufficiently precise to constrain inflation models severely. For example, chaotic inflation with the quartic type of potential is excluded at a confidence level of more than 95% [7].

One method to circumvent such a constraint is to add another source of fluctuations [9]. Another interesting method is to introduce the non-minimal coupling to the curvature, which can reduce the tensor to scalar ratio to negligible levels [10, 11]. Then, a non-minimally coupled chaotic inflation with a quartic type of potential is still viable. Thus, it is very useful to derive generalized slow-roll conditions for such a non-minimally coupled inflaton field and provide the formulae of the scalar/tensor spectral indices in terms of the slow-roll parameters, as in the case of a minimally coupled inflaton. However, as far

such a chaotic inflation is predicted as a simple realization of chaotic inflation in supergravity [8].
as we know, such slow-roll conditions have not been derived in a fully consistent manner. Non-minimal coupling also realizes non-slow-roll inflation. Coupling to the curvature like \( \frac{\xi R \phi^2}{2} \) leads to the additional mass squared \( m^2 \simeq 12 \xi H^2 \), so that the inflaton cannot slow-roll with \( \xi = \mathcal{O}(1) \). However, in the case of conformal coupling, it has recently been shown that the inflation takes place with the rapidly rolling inflaton \(^{12}\).

In this paper we first define the slow-rolling of the scalar field in the Jordan frame and derive consistency conditions of it, which we call extended slow-roll conditions. We then apply these conditions to several examples. We also compute observational quantities (spectral indices of scalar/tensor perturbations and the ratio of tensor to scalar perturbation) and rewrite them in terms of these slow-roll parameters. Our formulae make it possible to calculate the observational quantities in either frame (Jordan or Einstein) using the functions appearing in the Lagrangian only. Next we define the rapid-rolling of the scalar field, derive rapid-roll conditions, discuss its relation with the slow-rolling of the scalar field and then provide several examples. In an appendix we give several formulae in the Einstein frame which are useful for calculations in the text and also prove the conformal invariance of the curvature perturbation.

2. Slow-roll inflation with a non-minimally coupled scalar field

In this section we derive the extended slow-roll conditions of the scalar field which couples non-minimally to gravity and express scalar/tensor spectral indices in terms of these slow-roll parameters.

The action in the Jordan frame metric \( g_{\mu \nu} \) is

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - F(\phi)R - \frac{1}{2}(\nabla \phi)^2 - V(\phi) \right].
\]

Here \( \kappa^2 \equiv 8\pi G \) is the bare gravitational constant and the \( F(\phi)R \) term corresponds to non-minimal coupling of the scalar field to gravity.

We assume that the universe is described by the flat, homogeneous and isotropic universe model with the scale factor \( a \). The field equations are then given by

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \rho_{\phi},
\]

\[
\rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V + 6H(\dot{F} + HF),
\]

\[
\dot{H} = -\frac{\kappa^2}{2} (\rho_{\phi} + p_{\phi}),
\]

\[
p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V - 2\dot{F} - 4H \dot{F} - 2F(2\dot{H} + 3H^2),
\]

\[
\ddot{\phi} + 3H \dot{\phi} + V' + 6F'(\dot{H} + 2H^2) = 0,
\]

where the dot denotes the derivative with respect to the cosmic time and \( V' = dV/d\phi \). The equation of motion of the scalar field, equation (6), is also derived from the energy–momentum conservation: \( \dot{\rho_{\phi}} + 3H(\rho_{\phi} + p_{\phi}) = 0 \).
2.1. Slow-roll conditions

Introducing $\Omega = 1 - 2\kappa^2 F$, which corresponds to a conformal factor between the Jordan frame and the Einstein frame, the equations of motions are rewritten as

$$H^2\Omega + H\dot{\Omega} = \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\phi}^2 + V \right),$$

(7)

$$\ddot{\Omega} - H\dot{\Omega} + 2H\dot{\Omega} = -\kappa^2 \dot{\phi}^2,$$

(8)

$$\ddot{\phi} + 3H\dot{\phi} + V' - \frac{3\Omega'}{\kappa^2} (\dot{H} + 2H^2) = 0.$$  

(9)

Since under the slow-roll approximations [13], the timescale of the motion of the scalar field is assumed to be much larger than the cosmic timescale $H^{-1}$, as an extended slow-rolling of the scalar field, we assume that $\dot{\phi}^2 \ll V, |\dot{\Omega}| \ll H\Omega, |\dot{\phi}| \ll H|\phi|$ and $|\dot{\phi}| \ll |V'|$. Then we obtain

$$H^2\Omega \simeq \frac{\kappa^2}{3} V,$$

(10)

$$3H\dot{\phi} \simeq -V' + 6\frac{\Omega'}{\kappa^2} H^2 \simeq -\Omega^2 \left( \frac{V}{\Omega^2} \right)' =: -V_{\text{eff}}',$$

(11)

where in the second equation we have assumed $|\dot{H}/H^2| \ll 1$, which should be checked later. Note that if $V \propto \Omega^2$, then $V_{\text{eff}}$ is flat and $V_{\text{eff}}' = 0$ identically.

In the following, we derive the consistency conditions for the extended slow-rolling of the scalar field (the extended slow-roll conditions). By computing $\ddot{\phi}$ from equation (11), we obtain

$$\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \frac{\dot{H}}{H^2} - \frac{V'_{\text{eff}}}{3H^2}.$$  

(12)

Moreover, from equations (10) and (11), using $\dot{\Omega} = \Omega' \dot{\phi}$

$$\frac{\dot{\phi}^2}{V} \simeq \frac{\Omega V_{\text{eff}}^2}{3\kappa^2 V^2},$$

(13)

$$\frac{\dot{\Omega}}{H\Omega} \simeq \frac{\Omega' V'_{\text{eff}}}{\kappa^2 V}.$$  

(14)

Hence, we finally introduce the following slow-roll parameters and obtain the extended slow-roll conditions:

$$\epsilon := \frac{\Omega V_{\text{eff}}^2}{2\kappa^2 V^2}; \quad \epsilon \ll 1,$$

(15)

$$\eta := \frac{\Omega V''_{\text{eff}}}{\kappa^2 V}; \quad |\eta| \ll 1,$$

(16)

$$\delta := \frac{\Omega' V'_{\text{eff}}}{\kappa^2 V}; \quad |\delta| \ll 1,$$

(17)

where we have introduced a factor of 2 in the definition of $\epsilon$ so that it accords with the standard notation of the slow-roll parameters [14, 15]. A useful bookkeeping rule is that
a term involving derivatives divided by $\kappa^2$ is to be treated as a small quantity. The importance of the last condition, equation (17), in the background dynamics of the scalar field has not been fully appreciated\textsuperscript{5}. However, it is necessary for slow-roll inflation in both the Jordan frame and the Einstein frame and is essential to relate these slow-roll parameters to the slow-roll parameters in the Einstein frame, which are discussed in the appendix.

We also need to make sure that $|\ddot{\phi}| \ll |V'|$ is satisfied in deriving equation (11). From (11) we have

$$\frac{\ddot{\phi}}{V'} \simeq -\frac{\dot{H}}{H} \frac{\phi}{V'} = \frac{V''_{\text{eff}} \phi}{3H V'} \simeq \frac{\dot{H}}{3H^2} \frac{V'}{V'} + \frac{V''_{\text{eff}} V'}{9H^2}.$$  

Therefore, comparing equations (16) and (12), $|\ddot{\phi}| \ll |V'|$ is satisfied as long as

$$\left| \frac{V''_{\text{eff}}}{V'} \right| = O(1).$$  

Note that since from equation (8) $|\dot{H}/H^2|$ is approximated as

$$|\dot{H}/H^2| \approx \left| \frac{\dot{\Omega}}{2H\Omega} - \frac{3\dot{\phi}^2}{2V} \right| \simeq \left| \Omega V'_{\text{eff}} V''_{\text{eff}} + \Omega V^2_{\text{eff}} \right|,$$

$|\dot{H}/H^2| \ll 1$ is guaranteed by these conditions.

To sum up, the extended slow-roll conditions consist of three main conditions—equations (15), (16) and (17)—and one subsidiary condition, (19).\textsuperscript{6}

### 2.2. Example: chaotic inflation with non-minimal coupling

As an example of extended slow-roll inflation, we consider a non-minimally coupled scalar field for chaotic inflation with

$$V(\phi) = \frac{\lambda}{n} \phi^n, \quad F(\phi) = \frac{1}{2} \xi \phi^2,$$

where $\xi$ is a dimensionless coupling parameter and $\xi = 1/6$ corresponds to the conformal coupling.

For $|\xi| \kappa^2 \phi^2 \gg 1$, from equation (10) $\xi < 0$ is required. Then $\Omega \simeq -\xi \kappa^2 \phi^2, \Omega' \simeq -2\xi \kappa^2 \phi, f \simeq 1 - 6\xi$ and $V_{\text{eff}} \simeq (n - 4)V/((1 - 6\xi)\phi)$ for $n \neq 4$. Hence slow-roll parameters become

$$\epsilon = -\frac{(n - 4)^2 \xi}{2(1 - 6\xi)^2}, \quad \eta = -\frac{(n - 4)(n - 1)\xi}{1 - 6\xi}, \quad \delta = -\frac{2(n - 4)\xi}{1 - 6\xi},$$

while a subsidiary condition becomes $|V'_{\text{eff}}/V'| = 1/|1 - 6\xi||(n - 4)/n|$. Therefore, for general $n$, $|\xi| \ll 1$ are required for slow-roll, which coincides with the conditions derived in [1]. An exception is the case of the quartic potential $n = 4$. In this case, from equation (11) for $|\xi| \kappa^2 \phi^2 \gg 1$, $V'_{\text{eff}} \simeq -\lambda \phi/((1 - 6\xi)\xi \kappa^2) = -4V/((1 - 6\xi)\xi \kappa^2 \phi^3)$ and $V''_{\text{eff}}$

\textsuperscript{5} In [11,17], a similar slow-roll parameter is introduced in the context of a scalar perturbation equation.

\textsuperscript{6} It is to be noted that the subsidiary condition is a sufficient condition for slow-roll and a necessary condition is that equation (19) multiplied by $\epsilon, \eta$ or $\delta$ is sufficiently small.
becomes vanishingly small, which corresponds to the flat plateau in the Einstein frame found by Futamase and Maeda [1]. The slow-roll parameters are

\[ \epsilon = -\frac{8}{(1 - 6\xi)^2 \kappa^4 \phi^4}, \quad \eta = \frac{4}{(1 - 6\xi) \kappa^2 \phi^2}, \quad \delta = \frac{8}{(1 - 6\xi) \kappa^2 \phi^2}. \]  

(23)

Hence, for \( n = 4 \) the slow-roll conditions are automatically satisfied irrespective of \( \xi \) as long as \( \xi < 0 \), which again coincides with the results in [1].

On the other hand for \( |\xi| \kappa^2 \phi^2 \ll 1, \Omega \simeq 1, \Omega' \simeq -2\xi \kappa^2 \phi, f \simeq 1 \) and \( V_{\text{eff}} \simeq nV/\phi \), so slow-roll parameters become

\[ \epsilon = \frac{n^2 \kappa^2 \phi^2}{2\kappa^2 \phi^2}, \quad \eta = \frac{n(n - 1) \kappa^2 \phi^2}{\kappa^2 \phi^2}, \quad \delta = -2\xi n, \]  

(24)

and \( V_{\text{eff}}/\Omega' \simeq 1 \). Hence \( |\xi| \ll 1 \) and \( \kappa^2 \phi^2 \gg 1 \) are required for slow-roll for \( |\xi| \kappa^2 \phi^2 \ll 1 \), which again coincides with the conditions given in [1].

### 2.3. Perturbations

In the appendix it is shown that the gauge invariant curvature perturbation \( \mathcal{R} \) is invariant under the conformal transformation into the Einstein frame. Then we can calculate the power spectrum \( P_\mathcal{R}(k) \) [11, 16, 17]:

\[ P_\mathcal{R}^{1/2}(k) = \frac{\hat{H}^2}{2\pi|d\tilde{\phi}/d\tilde{t}|} = \frac{H^2}{2\pi f^{1/2}|\phi|} = \frac{3H^3}{2\pi f^{1/2}|V_{\text{eff}}|} \simeq \frac{\kappa^3 V^{3/2}}{2\sqrt{3\pi} \Omega^{3/2} |V_{\text{eff}}|}, \]  

(25)

where \( k \) is a co-moving wavenumber at the horizon exit \( (k = aH) \), the hatted variables are those in the Einstein frame and we have used equations (A.5) and (A.14) and equation (A.15). In the last equality we have assumed the slow-roll approximation. Using

\[ \frac{d}{dk} = \frac{d}{d \ln k} = H d\phi/\dot{\phi} \simeq -\frac{\kappa^2 V}{\Omega V_{\text{eff}}^\prime} d\phi, \]  

(26)

to the first order in the slow-roll parameters\(^7\), the spectral index of scalar perturbation is then given by

\[ n_S - 1 \equiv \frac{d \ln P_\mathcal{R}}{d \ln k} = -6\epsilon + 2\eta - 3\delta. \]  

(27)

Moreover, using the relation equations (A.11) and (A.12), we finally obtain the simple formula

\[ n_S - 1 = -6\epsilon + 2\eta - 3\delta = -6\epsilon + 2\tilde{\eta} = \tilde{n}_S - 1, \]  

(28)

which proves the invariance of the spectral index under the conformal transformation.

Tensor perturbations are also invariant under the conformal transformation into the Einstein frame. Then we can calculate the tensor power spectrum \( P_h(k) \)

\[ P_h^{1/2}(k) = \frac{2\kappa H}{\sqrt{2\pi}} \simeq \frac{2\kappa \hat{H}}{\sqrt{2\Omega \pi}} \simeq \frac{2\kappa^2 V^{1/2}}{\sqrt{6\pi} \Omega}, \]  

(29)

\(^7\) Note that \( d \ln k \simeq -\kappa^2 \hat{V} \tilde{d}\phi/(d\tilde{V}/d\tilde{\phi}) \) under the same approximations.
Table 1. Slow-roll parameters and inflationary observables in the Jordan/Einstein frame.

| 参数 | Jordan frame $g_{\mu\nu}$ | Einstein frame $\hat{g}_{\mu\nu} = \Omega g_{\mu\nu}$ |
|------|-----------------------------|--------------------------------------------------|
| 慢roll 参数 | $\epsilon, \eta, \delta$ | $\hat{\epsilon} = \epsilon, \hat{\eta} = \eta - \frac{1}{2}\delta$ |
| 不平坦谱指数 $n_S$ | $1 - 6\epsilon + 2\eta - 3\delta$ | $1 - 6\hat{\epsilon} + 2\hat{\eta}$ |
| 张量谱指数 $n_T$ | $-2\epsilon$ | $-2\hat{\epsilon}$ |
| 张量/标度比 $r$ | $16\epsilon = -8n_T$ | $16\hat{\epsilon} = -8\hat{n}_T$ |

where in the last equality the slow-roll approximation is assumed. Then the tensor spectral index is given by

$$n_T \equiv \frac{d\ln P_h}{d\ln k} = -2\epsilon = -2\hat{\epsilon} = \hat{n}_T,$$

(30)

which is also conformally invariant. The tensor to scalar ratio $r$ is also calculated as

$$r \equiv \frac{P_h}{P_R} = 16\epsilon = 16\hat{\epsilon} = \hat{r}.$$

(31)

Again this is also conformally invariant. Therefore, the consistency relation for a single scalar field inflation

$$r = -8n_T$$

(32)

is conformally invariant [11]. These results are summarized in table 1.

It should be noted that the invariance refers to the equality between the quantities calculated using $V(\phi)$ with $g_{\mu\nu}$ and those using $\hat{V}(\hat{\phi})$ with $\hat{g}_{\mu\nu}$, not $\hat{V}(\phi)$ with $\hat{g}_{\mu\nu}$. Therefore, the observational quantities calculated with $V(\phi)$ with non-minimal coupling are different from those calculated by $\hat{V}(\phi)$ with minimal coupling.

To demonstrate this, as an example we calculate $n_S, n_T$ and $r$ with equation (21). For $|\xi|\kappa^2\phi^2 \gg 1$, from equation (22), they are calculated for $n \neq 4$ (note $\xi < 0$ and $|\xi| \ll 1$)

$$n_S - 1 = n_T \simeq (n - 4)^2\xi, \quad r \simeq -8(n - 4)^2\xi.$$

(33)

Here we note that these are independent of the e-folding number. This feature can be easily understood by calculating them in the Einstein frame. From equations (A.5) and (A.6), the canonical scalar field $\hat{\phi}$ and the potential $\hat{V}(\hat{\phi})$ in the Einstein frame are given by

$$\hat{\phi} \simeq \sqrt{\frac{1 - 6\xi}{|\xi|}} \frac{1}{\kappa} \log \frac{\phi}{\phi_0},$$

(34)

$$\hat{V}(\hat{\phi}) \simeq \hat{V}_0 \exp \left[ \sqrt{\frac{|\xi|}{1 - 6\xi}} (n - 4)\kappa \hat{\phi} \right],$$

(35)

where $\phi_0$ is a constant field value which yields the origin of $\hat{\phi}$ and $\hat{V}_0 = \lambda \phi_0^{n - 4}/(n\xi^2\kappa^4)$. Thus, the inflation becomes of power-law type with the exponential potential in the Einstein frame. In fact, the slow-roll parameters in the Einstein frame are given by $\hat{\epsilon} \simeq -(n - 4)^2\xi/2$ and $\hat{\eta} \simeq -(n - 4)^2\xi$, and inserting them into the formulae (28), (30) and (31) yields the same values as those calculated in the Jordan frame.
For $n = 4$ from equation (23), $\epsilon = -1/(8\xi N^2)$, $\eta = 1/2N$ and $\delta = 1/N$, where $N$ is the e-folding number until the end of inflation and is written for $n = 4$ as
$$N = \int_{t}^{t_e} H dt \simeq \frac{(1 - 6\xi)}{8} \kappa^2 \phi^2,$$
so that
$$n_s - 1 = \frac{3}{4\xi N^2} - \frac{2}{N}, \quad n_T = \frac{1}{4\xi N^2}, \quad r = -\frac{2}{\xi N^2}. \quad (36)$$

On the other hand, for a minimally coupled ($F = 0$) inflaton with the same potential, $n_s, n_T$ and $r$ are calculated in the standard manner [15]:
$$n_s - 1 = -\frac{n + 2}{2N}, \quad n_T = -\frac{n}{2N}, \quad r = \frac{4n}{N}, \quad (37)$$
which clearly shows that both (equation (33) or equation (37) versus equation (38)) are different. In particular, although the tensor–scalar ratio $r$ for a minimally coupled inflaton is generally large $r \simeq 0.13(n/2)(60/N)$ and it is so for a non-minimal coupling with $n \neq 4$, $r \simeq 0.32((1 - n_s)/0.04)$, for a non-minimally coupled inflaton with $n = 4$ it can be small, $r \simeq 0.056(0.01/|\xi|)(60/N)^2$, due to the extreme flatness of the effective potential $V_{\text{eff}}$ for $n = 4$.

3. Rapid-roll inflation

Rapid-roll inflation is a novel type of inflation with a non-minimal coupling in which inflation occurs even without slow-roll for the conformal coupling. In this section we derive the conditions for rapid-roll inflation for a more general non-minimal coupling and discuss their relation to the slow-roll conditions.

Firstly, we must define rapid-roll inflation. Rapid-roll inflation is a type of inflation without slow-roll of the scalar field. The motion of the scalar field is determined primarily by the curvature coupling rather than by the potential term, so that the timescale of the motion is determined by the Hubble parameter rather than by the effective mass of the potential, so that $|\dot{\phi}| \sim H\phi$. Nevertheless, as shown in [12], for conformal coupling, such a rapid motion of the scalar field does not affect the expansion rate of the universe and the universe inflates. In the following, we first derive necessary conditions of the coupling $F(\phi)$ for de Sitter expansion for constant $V(\phi) = V_0$, and then derive rapid-roll conditions for general $V(\phi)$. The equations of motion are given by
$$H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\phi}^2 + V + 6H(\dot{F} + HF) \right), \quad (39)$$
$$\ddot{\phi} + 3H\dot{\phi} + V' + 6F'(\dot{H} + 2H^2) = 0. \quad (40)$$
Then, by defining $\pi \equiv \dot{\phi} + 6HF'$, this can be written as
$$H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \pi^2 + V + 6H^2 F - 3F' \right), \quad (41)$$
$$\ddot{\pi} + 2H\dot{\pi} + V' + (1 - 6F'')H\dot{\phi} = 0. \quad (42)$$
Therefore, for constant $V = V_0$, if the following conditions are satisfied,
\begin{equation}
F'' = \frac{1}{6}, \quad F = 3F'^2, \quad (43)
\end{equation}
then from equation (42) $\pi$ decays as $\pi \propto a^{-2}$ and thus from equation (41) $H$ becomes constant: i.e. de Sitter expansion. The conditions equation (43) can be integrated to give
\begin{equation}
F(\phi) = \frac{1}{12}(\phi - v)^2, \quad (44)
\end{equation}
where $v$ is a constant, which may be called ‘shifted conformal coupling’. Thus if the conditions of equation (43) are satisfied, even if $\phi$ itself does not move slowly, inflation occurs and is called rapid-roll inflation.

### 3.1. Rapid-roll conditions

Next, we consider a general $V(\phi)$ and $F(\phi)$ and derive rapid-roll conditions in terms of $V$ and $F$. We assume that the timescale of $\dot{\pi}$ is also determined by the Hubble scale, so that $\dot{\pi} \simeq cH\pi$ and that the equations of motions are approximated by
\begin{equation}
H^2 \simeq \frac{\kappa^2}{3}V, \quad (45)
\end{equation}
\begin{equation}
(c+2)H\pi \simeq -V', \quad (46)
\end{equation}
where $c$ is a proportionality constant at most of the order $O(1)$ and will be explicitly given later (it is not given in [12]). Note that the left-hand side of equation (45) does not contain the conformal factor. Furthermore, for successful inflation, the variation of the Hubble parameter should be slow
\begin{equation}
\left| \frac{\ddot{H}}{H^2} \right| \ll 1. \quad (47)
\end{equation}

In the following, we shall derive the consistency conditions for rapid-roll inflation following [12]. In equation (45), the first and the last terms of the right-hand side in equation (41) are neglected. Since
\begin{equation}
\frac{\pi^2}{2V} + \frac{6H^2}{V}(F - 3F'^2) \simeq \frac{3V'^2}{2(c+2)^2\kappa^2V^2} + 2\kappa^2(F - 3F'^2), \quad (48)
\end{equation}
this is consistent if
\begin{equation}
\epsilon_c := \frac{V'^2}{2\kappa^2V^2} + \frac{2(c+2)^2}{3}\kappa^2(F - 3F'^2); \quad |\epsilon_c| \ll 1, \quad (49)
\end{equation}
Moreover, from the time derivative of equation (46)
\begin{equation}
\frac{\dot{\pi}}{(c+2)H\pi} \simeq -\frac{\dot{H}}{(c+2)H^2} - \frac{3V''}{(c+2)^2\kappa^2V} - \frac{6F'V''}{(c+2)V'}, \quad (50)
\end{equation}
Therefore, neglecting $\dot{\pi} - cH\pi + H(1 - 6F')\pi - 6HF'$ in equation (46) is consistent if
\[
\eta_c := \frac{V''}{\kappa^2 V} + \frac{2(c + 2)F'V''}{V'} + \frac{c(c + 2)}{3} - \frac{c + 2}{3}(1 - 6F') - \frac{2(c + 2)^2\kappa^2(1 - 6F''F')V'}{V'},
\]
(51)

$|\eta_c| \ll 1$, (52)

where we have assumed $|\dot{H}|H^2 \ll 1$, which also should be checked. From the time derivative of equation (45)
\[
\left|\frac{\dot{H}}{H^2}\right| \simeq \left|\frac{3 V'^2}{2(c + 2)\kappa^2 V^2} + \frac{3 F'V'}{V}\right|.
\]
(53)

Therefore assuming $|\dot{H}|H^2 \ll 1$ is consistent if
\[
\delta_c := \frac{V'^2}{2(c + 2)\kappa^2 V^2} + \frac{F'V'}{V}; \quad |\delta_c| \ll 1.
\]
(54)

To sum up, rapid-roll conditions consist of three conditions—equations (49), (52) and (54).

The constant $c$ may be expressed in terms of the potential $V$ and the coupling function $F$. The time derivative of equation (45) and equation (46) yields a quadratic equation for $c$, and the solutions of it are given by
\[
c = -\left(1 + \frac{3 F'V''}{V'} - \frac{3 F'V'}{2 V}\right) \pm \sqrt{\left(1 - \frac{3 F'V''}{V'} + \frac{3 F'V'}{2 V}\right)^2 - \frac{3 V''}{\kappa^2 V} + \frac{3 V'^2}{2 \kappa^2 V^2}}.
\]
(55)

For $F$ satisfying equation (43), using the rapid-roll conditions equation (49) and equation (54), it can be approximated as (note that this can also be derived by setting $\eta_c \simeq 0$)
\[
c \simeq -\left(1 + \frac{3 F'V''}{V'}\right) \pm \sqrt{\left(1 - \frac{3 F'V''}{V'}\right)^2 - \frac{3 V''}{\kappa^2 V}},
\]
(56)

and for $|c| \ll 1$ it reduces to
\[
c \simeq -\frac{3 V''}{2 \kappa^2 V} \left(1 + \frac{3 F'V''}{V'}\right)^{-1}.
\]
(57)

Hence $|c| \ll 1$ requires $|V''/\kappa^2 V| \ll 1$. Since $\Omega = 1 - \kappa^2 \phi^2 / 6 = \mathcal{O}(1)$, together with conditions (49) and (54), this implies that rapid-roll conditions are reduced to slow-roll conditions, which is also seen from $|\dot{\pi}| \simeq cH|\pi| \ll H|\pi|$. Thus the relation between rapid-roll and slow-roll is clarified: $c = \mathcal{O}(1)$ for rapid-roll; $|c| \ll 1$ for slow-roll.
3.2. Example

As an example of rapid-roll inflation, we consider a scalar field with the conformal coupling

$$F(\phi) = \frac{\phi^3}{12},$$

and the potential

$$V(\phi) = v^4 \pm \frac{1}{2} m^2 \phi^2,$$ (58)

where $v$ is the typical energy scale of inflation and $m$ is the inflaton mass. This type of potential often appears in hybrid inflation (plus sign) or new inflation (minus sign).

First we derive the rapid-roll conditions for this potential and then confirm the relation $\dot{\pi} \simeq c H \pi$ by directly solving the equation of motion.

The rapid-roll parameters are estimated as

$$\epsilon_c = \frac{m^4 \phi^2}{2 \kappa^2 V^2}, \quad \eta_c = \pm \frac{m^2}{\kappa^2} + \frac{(c + 2) \pm \sqrt{c(c + 2) - 4m^2 H_0^2}}{3}, \quad \delta_c = \frac{m^4 \phi^2}{2(c + 2) \kappa^2 V^2} \pm \frac{m^2 \phi^2}{6V}. \quad (59)$$

Hence from $|\delta_c| \ll 1$ and $\epsilon_c \ll 1$, $m^2 \phi^2 \ll v^4$, so that $V \simeq v^4$. From $\epsilon_c \ll 1$, $m^4 \phi^2 \ll \kappa^2 v^8$, $c$ is determined by solving $\eta_c = 0$ (equation (56)):

$$c \simeq \frac{3}{2} \pm \frac{1}{2} \sqrt{1 \mp \frac{4m^2}{H_0^2}}, \quad (60)$$

where $H_0 \equiv \kappa v^2 / \sqrt{3}$. Then the scalar field moves according to $\dot{\pi} \simeq c H \pi$ with $\pi = \dot{\phi} + H \phi$. Since $c$ is always negative as long as the determinant is positive, $\pi \to 0$ and the inflation occurs. Note that for new inflation (plus sign in the determinant), we can consider the case of $m^2 \gg H_0^2$, so that one of the usual slow-roll conditions, $|\eta| = |V''/\kappa^2 V| \ll 1$, is violated, which is related to the situation in so-called fast-roll inflation [18].

We can now confirm the assumption $\dot{\pi} \simeq c H \pi$ by solving the equation of motion for the scalar field $\phi$ directly. For $m^2 \phi^2 \ll v^4$, the equation of motion of $\phi$ is given by

$$\ddot{\phi} + 3H_0 \dot{\phi} + (2H_0^2 \pm m^2) \phi \simeq 0. \quad (61)$$

Inserting $\phi = \phi_0 e^{\omega t}$ into the above equation yields

$$\omega^2 + 3H_0 \omega + 2H_0^2 \pm m^2 = 0. \quad (62)$$

Then, $\omega$ is given by

$$\omega = -\frac{3}{2} H_0 \pm \frac{1}{2} \sqrt{H_0^2 + 4m^2} = c H_0. \quad (63)$$

Thus, $\pi = \dot{\phi} + H \phi = (\omega + H_0) \phi = (c + 1) H_0 \phi$, and so indeed $\dot{\pi} = c(c + 1) H_0^2 \phi = c H_0 \pi$.

Finally, we would like to point out that for the polynomial potential $V(\phi) \propto \phi^n$, which often appears in chaotic inflation, one of the rapid-roll parameters $\delta_c \simeq n/6 = O(1)$ violates the rapid-roll condition. Therefore, rapid-roll inflation does not occur for this type of potential.
4. Summary

In this paper, we have derived the slow-roll conditions for the scalar field non-minimally coupled to gravity in a consistent manner, which was made possible by rewriting the equation of motion using the conformal factor $\Omega$ and by introducing the effective potential $V_{\text{eff}}$. The slow-roll conditions consist of three main conditions—equations (15), (16) and (17)—and one subsidiary condition, (19). The third condition $|\delta| \ll 1$ (equation (17)) appears not to have been derived before in the context of the background dynamics of the scalar field. However, it is necessary for slow-roll inflation both in the Jordan frame and the Einstein frame. These conditions enable us to relate the slow-roll parameters in the Jordan frame ($\epsilon, \eta, \delta$) to those in the Einstein frame ($\hat{\epsilon}, \hat{\eta}$), so that observational quantities can be calculated in either frame and the conformal invariance of them can be proved. We have also derived the rapid-roll conditions by slightly generalizing those in [12]. The rapid-roll conditions consist of three conditions—equations (49), (52) and (54). We also discussed the relation between rapid-roll and slow-roll.

Our formulae, equations (25), (28), (29) and (30), allow us to calculate the observational quantities in either frame using the functions appearing in the Lagrangian only. Although we have shown the conformal invariance of the curvature perturbation beyond linear perturbation, the conformal invariance of the spectral indices of scalar/tensor perturbations is proved only in the first order in the slow-roll parameters. It would be interesting to extend the invariance to higher orders.

Acknowledgments

The authors would like to thank the participants of Summer Institute 2008 (Fujiyoshida, Japan, 3–13 August 2008), especially Hideo Kodama, Shinji Mukohyama and Misao Sasaki, for useful comments. This work was supported in part by Grant-in-Aid for Scientific Research from JSPS (no. 17204018 and no. 20540280 (TC), no. 18740157 and no. 19340054 (MY)) and from MEXT (no. 20040006 (TC)) and in part by Nihon University.

Appendix. Slow-roll conditions and perturbations in the Einstein frame

In this appendix we perform the conformal transformation to the Einstein frame, introduce slow-roll parameters and give their relations with those in the Jordan frame. Perturbations in the FRW metric are also given and the conformal invariance of the curvature perturbation is proved.

Introducing $\Omega(\phi) = 1 - 2\kappa^2 F(\phi)$, the action equation (1) is

$$ S = \int d^4x \sqrt{-g} \left[ \frac{\Omega(\phi)}{2\kappa^2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right]. \tag{A.1} $$

Introducing the Einstein metric $\hat{g}_{\mu\nu}$ by the conformal transformation

$$ \hat{g}_{\mu\nu} = \Omega(\phi) g_{\mu\nu}, \tag{A.2} $$
the action becomes that of a scalar field minimally coupled to Einstein gravity

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \hat{R} - \frac{1}{2} \left( 1 + \frac{3\Omega^2}{2\kappa^2} \Omega \right) \left( \hat{\nabla} \hat{\phi} \right)^2 - V \Omega \right] \]

(A.3)

\[ = \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa^2} \hat{R} - \frac{1}{2} \left( \hat{\nabla} \hat{\phi} \right)^2 - \hat{V} \hat{\phi} \right], \]

(A.4)

where in the second line we have introduced a canonical scalar field \( \hat{\phi} \) with a potential \( \hat{V} \)

\[ d\hat{\phi}^2 = \frac{1}{\Omega(\phi)} \left( 1 + \frac{3\Omega'\phi^2}{2\kappa^2} \right) =: f(\phi) \, d\phi^2, \]

(A.5)

\[ \hat{V}(\hat{\phi}) = \frac{V(\phi)}{\Omega(\phi)^2}. \]

(A.6)

**A.1. Slow-roll conditions**

The slow-roll conditions in the Einstein frame are simply given by

\[ \hat{\epsilon} := \frac{1}{2\kappa^2 \hat{V}^2} \left( \frac{d\hat{V}}{d\hat{\phi}} \right)^2; \quad \hat{\epsilon} \ll 1, \]

(A.7)

\[ \hat{\eta} := \frac{1}{\kappa^2 \hat{V}} \frac{d^2\hat{V}}{d\hat{\phi}^2}; \quad |\hat{\eta}| \ll 1. \]

(A.8)

In terms of \( \phi \), using equations (A.5) and (A.6), these parameters are rewritten as

\[ \hat{\epsilon} = \frac{\Omega V_{\text{eff}}'^2}{2\kappa^2 f V^2}, \]

(A.9)

\[ \hat{\eta} = \frac{\Omega^{5/2}}{\kappa^2 f^{1/2} V} \left( \frac{V_{\text{eff}}'}{f^{1/2} V^{3/2}} \right)', \]

(A.10)

where \( V_{\text{eff}}' \) is defined in equation (11). If we assign \( O(\epsilon^2) \) to the slow-roll parameters \( (\epsilon, \eta, \delta) \),\(^8\) then \( f = 1 + O(\epsilon^2) \), and \( f \approx 1 \) is satisfied under the slow-roll conditions equations (15) and (17). Therefore, under the slow-roll approximations, the slow-roll parameters in the Einstein frame are related to those in the Jordan frame as

\[ \hat{\epsilon} \approx \epsilon, \]

(A.11)

\[ \hat{\eta} \approx \frac{\Omega V_{\text{eff}}''}{\kappa^2 V} - \frac{3\Omega' V_{\text{eff}}'}{2\kappa^2 V} = \eta - \frac{3}{2}\delta. \]

(A.12)

\(^8\) We adopt the standard mathematical notation, according to which \( \epsilon = O(\epsilon^2) \) means that \( \epsilon \) falls like \( \epsilon^2 \) or faster as \( \epsilon \to 0 \).
A.2. FRW metric and perturbations

From equation (A.2), the line element in the Einstein frame is
\[ ds^2 = \Omega \, ds^2. \]
(A.13)

So, the cosmic time \( \hat{t} \) and the scale factor \( \hat{a} \) in the Einstein frame become
\[ d\hat{t} = \sqrt{\Omega} \, dt, \quad \hat{a}(\hat{t}) = \sqrt{\Omega(t)} a(t). \]
(A.14)

Hence the Hubble parameter \( \hat{H} \) and the acceleration in the Einstein frame become [11,17,19]
\[ \hat{H} = \frac{d\hat{a}/d\hat{t}}{\hat{a}} = \frac{1}{\sqrt{\Omega}} \left( H + \frac{\dot{\Omega}}{2\Omega} \right) , \]
(A.15)
\[ \frac{d^2\hat{a}/d\hat{t}^2}{\hat{a}} = \frac{1}{\Omega} \left( \frac{\ddot{a}}{a} + \frac{H\dot{\Omega}}{2\Omega} + \frac{1}{2} \left( \frac{\dot{\Omega}}{\Omega} \right) \right) . \]
(A.16)

These show that acceleration in the Einstein frame does not immediately imply acceleration in the Jordan frame. For the latter, \( |\dot{\Omega}| \ll H\Omega \) is required, which is realized by \( |\delta| \ll 1 \). If \( |\dot{\Omega}| \ll H\Omega \), then we obtain
\[ \hat{H} \sim \Omega^{-1/2} H, \]
\[ \frac{d^2\hat{a}/d\hat{t}^2}{\hat{a}} \sim \Omega^{-1} \frac{\ddot{a}}{a}. \]
(A.17)
(A.18)

Next, we consider scalar perturbations. In the longitudinal gauge
\[ ds^2 = -(1 + 2\Psi(t,x^i)) \, dt^2 + a^2(t)(1 + 2\Phi(t,x^i)) \delta_{ij} \, dx^i \, dx^j, \]
(A.19)
the gauge invariant curvature perturbation, which is the curvature perturbation on the co-moving (or velocity-orthogonal) slices,
\[ \mathcal{R} = \Phi - H \frac{\dot{\phi}}{\phi} \]
(A.20)
is constructed. Defining \( \delta\Omega(t,x^i) = \Omega(t,x^i) - \Omega(t) \) and noting that that \( \hat{a}(\hat{t}) = \sqrt{\Omega(t)} a(t) \), we obtain
\[ \hat{\Phi} = \Phi + \frac{\delta\Omega}{2\Omega(t)}. \]
(A.21)

Then the invariance of \( \mathcal{R} \) under the conformal transformation is immediately proved using equations (A.21) and (A.15) [11,16,17]
\[ \hat{\mathcal{R}} = \hat{\Phi} - \frac{\hat{H}}{d\phi/d\hat{t}} \delta\hat{\phi} \]
\[ = \Phi + \frac{\delta\Omega}{2\Omega} - \left( H + \frac{\dot{\Omega}}{2\Omega} \right) \frac{\delta\phi}{\phi} \]
\[ = \Phi - H \frac{\dot{\phi}}{\phi} \delta\phi = \mathcal{R}. \]
(A.22)
Extended slow-roll conditions and rapid-roll conditions

From equation (A.2), tensor metric perturbations \( g_{ij} = a^2 (\delta_{ij} + h_{ij}) \) are invariant under the conformal transformation [20].

Finally, we note that the conformal invariance of the curvature perturbation is not limited to the linear perturbation and can be proved in fully non-linear theory along the line of [21]. In the \((3+1)\)-decomposition of the metric [22]
\[
ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i)(dx^j + \beta^j),
\]
(A.23)
we write the three-metric \( \gamma_{ij} \) as a product of the scale factor and a perturbation \( \Phi \)
\[
\gamma_{ij} = a^2(t)e^{2\Phi(t,x^i)}. \tag{A.24}
\]
In linear theory, this reduces to equation (A.19). Then we define a quantity
\[
-\zeta \equiv \Phi - \int_{\phi(t)}^{\phi(t,x^i)} \frac{H}{\dot{\phi}} d\phi, \tag{A.25}
\]
which reduces to \( \mathcal{R} \) (equation (A.20)) in linear theory and coincides with \( \zeta \) in [21] and is conserved on super-horizon scales in the Einstein gravity. Since \( \widehat{ae}^\Phi = \sqrt{\Omega(t,x^i)}a e^\Phi \) and \( \widehat{a}(\widehat{t}) = \sqrt{\Omega(\widehat{t})}a(\widehat{t}) \), the conformal invariance of \( \zeta \) is immediately seen
\[
-\hat{\zeta} = \hat{\Phi} - \int_{\phi(\hat{t})}^{\phi(\hat{t},x^i)} \frac{\hat{H}}{d\phi/d\hat{t}} d\phi = \hat{\Phi} - \int_{\phi(\hat{t})}^{\phi(\hat{t},x^i)} \frac{H}{\dot{\phi}} d\phi = -\zeta. \tag{A.26}
\]
\( \hat{\zeta} \) is a conserved quantity because it is the curvature perturbation in the Einstein frame, which implies the conservation of \( \zeta \).

References

[1] Futamase T and Maeda K I, 1989 Phys. Rev. D 39 399 [SPIRES]
[2] Yokoyama J, 1988 Phys. Lett. B 212 273 [SPIRES]
[3] Yokoyama J, 1989 Phys. Rev. Lett. 63 712 [SPIRES]
[4] Allen B, 1983 Nucl. Phys. B 226 228 [SPIRES]
[5] Ishikawa K, 1983 Phys. Rev. D 28 2445 [SPIRES]
[6] Mazur P and Mottola E, 1986 Nucl. Phys. B 276 694 [SPIRES]
[7] Chiba T, 2001 Phys. Rev. D 64 103503 [SPIRES] [arXiv:astro-ph/0105550]
[8] Chiba T and Yamaguchi M, 2000 Phys. Rev. D 61 027301 [SPIRES] [arXiv:hep-ph/9907402]
[9] Komatsu E et al (WMAP Collaboration), 2008 arXiv:0803.0545 [astro-ph]
[10] Komatsu E and Futamase T, 1999 Phys. Rev. D 59 064029 [SPIRES] [arXiv:astro-ph/9901127]
Extended slow-roll conditions and rapid-roll conditions

[12] Kofman L and Mukohyama S, 2008 Phys. Rev. D 77 043519 [SPIRES] arXiv:0709.1952 [hep-th]
[13] Steinhardt P J and Turner M S, 1984 Phys. Rev. D 29 2162 [SPIRES]
[14] Liddle A R and Lyth D H, 1992 Phys. Lett. B 291 391 [SPIRES] arXiv:astro-ph/9208007
[15] Lyth D H and Riotto A, 1999 Phys. Rep. 314 1 [SPIRES] arXiv:hep-ph/9807278
[16] Makino N and Sasaki M, 1991 Prog. Theor. Phys. 86 103 [SPIRES]
[17] Hwang J C, 1997 Class. Quantum Grav. 14 1981 [SPIRES] arXiv:gr-qc/9605024
[18] Linde A, 2001 J. High Energy Phys. JHEP11(2001)052 [SPIRES] arXiv:hep-th/0110195
[19] Esposito-Farese G and Polarski D, 2001 Phys. Rev. D 63 063504 [SPIRES] arXiv:gr-qc/0009034
[20] Mukhanov V F, Feldman H A and Brandenberger R H, 1992 Phys. Rep. 215 203 [SPIRES]
[21] Lyth D H, Malik K A and Sasaki M, 2005 J. Cosmol. Astropart. Phys. JCAP05(2005)004 [SPIRES] arXiv:astro-ph/0411220
[22] Arnowitt R, Deser S and Misner C W, 1962 Gravitation: and Introduction to Current Research ed L Witten (New York: Wiley) pp 227–65 [arXiv:gr-qc/0405109]