A Proof of the Strong Converse Theorem for Gaussian Multiple Access Channels

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Abstract

We prove that $N$-user Gaussian multiple access channels (MACs) admit the strong converse, i.e., every sequence of codes with asymptotic average error probabilities smaller than one has rate tuples that lie in the capacity (pentagonal) region. Our proof consists of four key ingredients: First, we perform an expurgation step to convert the code defined in terms of the average probability of error to one defined in terms of the maximum error without too much loss in rate. Second, we use a scalar quantizer of increasing precision with the blocklength to discretize the input spaces so that the wringing procedure to be performed can yield a useful bound on the correlation among users’ codewords. Third, we use a wringing technique to further expurgate appropriate codewords so that the resultant quantized code distribution can be approximated by a product distribution over users’ inputs. Finally, we obtain upper bounds on achievable sum-rates in terms of the type-II error of a binary hypothesis test through a judicious choice of output distributions. Our strong converse result carries over to the two sender two-receiver Gaussian interference channel under strong interference as long as the sum of the asymptotic average error probabilities is smaller than one.

Index Terms

Gaussian multiple access channel, Strong converse, Binary hypothesis testing, Expurgation, Wringing technique

I. INTRODUCTION

The multiple access channel (MAC) is one of the most well-studied problems in network information theory [1]. The capacity region of the discrete memoryless MAC was independently derived by Ahlswede [2] and Liao [3] in the early 1970s. In this paper, we are interested in the Gaussian version of this problem for which the channel output $Y$ corresponding to the inputs $(X_1, X_2, \ldots, X_N)$ is

$$Y = \sum_{i=1}^{N} X_i + Z,$$

(1)

where $Z$ is standard Gaussian noise. We assume an average transmission power constraint of $P_i$ corresponding to each transmitter $i \in \{1, 2, \ldots, N\}$. The capacity region was derived by Cover [4] and Wyner [5] and is the set of all rate tuples $(R_1, R_2, \ldots, R_N) \in \mathbb{R}_+^N$ that satisfy

$$\sum_{i \in T} R_i \leq \frac{1}{2} \log \left( 1 + \sum_{i \in T} P_i \right)$$

(2)

for all subsets $T \subseteq \{1, 2, \ldots, N\}$. The pentagonal region of rate tuples in (2) is known as the Cover-Wyner region and is illustrated for the $N = 2$ case in Figure 1.

Despite our seemingly complete understanding of fundamental limits of the Gaussian MAC, it is worth highlighting that in the above-mentioned seminal works [2]–[5], it is assumed that the average error probability tends to zero as the length of the code grows without bound. This implies that those established converses are, in fact, weak converses. Fano’s inequality [1, Section 2.1] is typically used as a key tool to establish such weak converses. In this work, we strengthen the results of Cover [4] and Wyner [5] and show that all sequences (in the blocklength) of Gaussian multiple access codes with asymptotic average error probabilities strictly less than one (and not necessarily

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tending to zero) must have rate tuples that lie in the Cover-Wyner region. This is a \textit{strong converse} statement, akin to the works of Wolfowitz \cite{Wolfowitz64}. It indicates that the boundary of the Cover-Wyner region designates a sharp phase transition between achievable rate tuples whose asymptotic error probabilities tend to zero and unachievable rate tuples whose asymptotic error probabilities necessarily tend to one (and are not simply bounded away from zero). Thus, this work augments our understanding of the first-order fundamental limits of Gaussian MACs. Additionally, it may also serve as a stepping stone for studying the second-order asymptotics \cite{Shamai09, Hassani10, Etkin12} or upper bounds (e.g., the sphere-packing bound) on the reliability function of Gaussian MACs (cf. \cite{ElGamal16Theorem4}).

\section*{A. Related Work}

The study of MACs has a long history and we refer the reader to the excellent exposition in El Gamal and Kim \cite[Chapter 4]{ElGamal16} for a thorough discussion and accompanying references. Dueck \cite{Dueck84} proved that the strong converse for the (two-user) discrete memoryless MAC holds by using the technique of blowing up decoding sets originally due to Ahlswede, Gács and Körner \cite{Ahlswede83}, combined with a novel strategy known as the \textit{wringing technique}. The technique of blowing up decoding sets is known as the \textit{blowing-up lemma} \cite{Ahlswede83, Dueck85} (see also \cite[Chapter 5]{ElGamal16} or \cite[Section 3.6]{Shamai09}), which is useful for establishing strong converse results for memoryless channels with finite output alphabet. The wringing technique was used to wring out any residual dependence between the codewords transmitted by the $N$ encoders. Wringing is necessary for establishing tight sum-rate bounds, because the sum-rate capacity of the MAC is expressed as the supremum of mutual information terms over all independent input distributions (the independence is due to the fact that the $N$ encoders do not cooperate). Ahlswede \cite{Ahlswede85} provided an elementary proof of the strong converse for the (two-user) discrete memoryless MAC by using Augustin’s non-asymptotic converse bound \cite{Augustin99} and a version of the wringing technique without recourse to the blowing-up lemma. However, the proofs of Dueck and Ahlswede are specific to the discrete (finite alphabet) setting and it is not clear by examining the proofs that the same strong converse statement follows in a straightforward way for the Gaussian MAC with power constraints.

Another approach to proving the strong converse for a general MAC is due to Han \cite{Han83}, who used the information spectrum technique \cite{Cover83} to provide a general formula for MACs and stated a condition \cite[Theorem 6]{Han83} for the strong converse to hold. However, unlike for the point-to-point setting \cite[Section 3.6–3.7]{Cover83}, the property is difficult to verify for various classes of memoryless MACs.

In view of the above works, we are motivated in this work to provide a self-contained proof for the strong converse of the Gaussian MAC.

\section*{B. Challenges in Establishing Strong Converse for Gaussian MAC}

Before discussing our proof techniques, we state some challenges concerning the extension of the strong converse techniques for the discrete memoryless case (by Dueck \cite{Dueck84} and Ahlswede \cite{Ahlswede85}) to the Gaussian case.

First, the use of the blowing-up lemma in Dueck’s proof \cite{Dueck84} for proving a strong converse theorem is applicable to only channels with finite output alphabet (the input alphabet can be uncountable). A crucial step in Dueck’s proof is obtain a useful bound on the list size of highly likely messages for every output sequence. If this step
fails, the blowing-up lemma cannot lead to a strong converse theorem. Since the crucial step relies on the finiteness of the output alphabet and the output alphabet of the Gaussian MAC is uncountably infinite, it is not immediately apparent how to extend this step to the Gaussian case.

Second, although Ahlswede’s wringing technique for the two-user case in [17] does not rely on the blowing-up lemma, it is not apparent how to adapt his techniques to obtain a strong converse bound on the sum-rate. More specifically, Ahlswede’s wringing technique leads to the following sum-rate bound for any blocklength $n$, any average decoding error probability $\varepsilon \in [0, 1)$ and any $(n, \varepsilon)$-achievable $(R_1, R_2)$:

$$R_1 + R_2 \leq I(X_1, X_2; Y) + O\left(\frac{\log n}{\sqrt{n}}\right) |X_1||X_2||Y|$$

(3)

where $X_1$ and $X_2$ are independent (the last term can be deduced directly from Equation (5.3) in [17]). However, the bound in (3) is sensitive to the sizes of the input and output alphabets, which prevents one from extending Ahlswede’s proof to the Gaussian case with uncountable alphabets. A naive attempt to extend Ahlswede’s proof to the Gaussian case is to quantize the input and output spaces of the Gaussian MAC so that the term $O\left(\frac{\log n}{\sqrt{n}}\right) |X_1||X_2||Y|$ in (3) vanishes as $n$ tends to infinity. However, quantization arguments that are used to prove information-theoretic statements for continuous-valued alphabets are usually applied to the achievable parts of coding theorems. For example, a quantization argument is used in [1, Section 3.4.1] for leveraging the achievability proof for the discrete memoryless channel (DMC) with cost constraints to demonstrate the achievability part of the capacity of the AWGN channel. To the best of our knowledge, standard quantization arguments for achievability parts do not work for strong converse proofs due to the following reason: In strong converse proofs, one is provided with a sequence of codes with asymptotic error probability $\varepsilon$. Upon quantization, one has to ensure that this error probability is approximately maintained after quantization, which is the main difficulty.

We believe that the aforementioned challenges are the main reasons why the strong converse for the Gaussian MAC has been open for more than 30 years since the strong converse for the discrete case was first proved. This is further evidenced by numerous attempts to study the second-order coding rates of Gaussian MACs (including those by Scarlett, Martinez, and Guillén i Fàbregas [8] and MolavianJazi and Laneman [9]) but all the initial efforts have all been aimed towards finding achievable second-order rates and difficulties have been encountered in the converse parts [21, Appendix D]. Our work is a necessary first step in making progress towards characterizing the full second-order asymptotics of the Gaussian MAC. In this work, we use a novel combination of a quantization strategy and the wringing technique that helps to extend Ahlswede’s strong converse proof to the Gaussian case. Unlike quantization strategies in existing achievability proofs, our quantization scheme facilitates the choosing of a suitable subcode such that the resultant codewords are approximately independent among the senders.

C. Discussion of the Proof Technique

Our proof technique contains the following four main ingredients.

First, we perform an expurgation step, similar to that done by Dueck [12], to convert the code defined in terms of the average probability of decoding error to one defined in terms of the maximum decoding error. We show that restricted to this new code, the resultant rates are essentially unaffected.

Second, we use a scalar quantizer of increasing precision with the blocklength to discretize input spaces of the channel so that the wringing technique to be performed can yield a useful bound on the correlation among different users’ codewords. Note that if the quantizer’s precision is too small or too large, then the resultant converse bound obtained from the remaining procedures will become loose and hence not useful in proving the strong converse. In addition, the idea of quantizing inputs for proving a converse bound is novel because, to the best of our knowledge, existing capacity proofs involving quantization arguments are only for the achievability rather than the converse parts. For example, see [1, Section 3.4.1] for the achievability proof of the channel coding theorem for the AWGN channel, and [22, Section X-B] for the achievability proof of the second-order coding rate theorem for the AWGN channel.

Third, similar to Ahlswede’s proof of the strong converse for the discrete memoryless MAC [17], we use a wringing technique to further expurgate appropriate codewords so that the resultant quantized code distribution can be approximated by a product distribution over users’ inputs. Our wringing technique combined with the quantization strategy in the previous step enables us to obtain a vanishing residual term that no longer depends on
which is in stark contrast to the residual term in (3) obtained from Ahlswede’s proof. See also [21, Appendix D.6] for the explanation of why Ahlswede’s original wringing technique works for only MACs with finite input alphabets but not Gaussian MACs.

Finally, we leverage a non-asymptotic converse bound by Wang, Colbeck and Renner [23, Lemma 1] (see also the work of Polyanskiy, Poor and Verdú [24, Section III-E]) relating binary hypothesis testing to channel coding. We relax this non-asymptotic converse bound using Chebyshev’s inequality, essentially obtaining a bound analogous to that used by Wolfowitz in the strong converse proof for discrete memoryless channels [6]. A careful choice of the auxiliary conditional output distributions for the non-asymptotic bound, together with a product distribution that approximates the quantized code distribution, enables us to bound relevant moments and to establish the strong converse. Note that in order to obtain a good asymptotic bound from the non-asymptotic bound, the choice of the auxiliary output distributions (which is afforded to us due to our use of ideas from binary hypothesis testing and the information spectrum method) not only depends on the peak power values as usual, but also the quantized code distribution (and hence the quantizer). The judicious choice of the auxiliary output distributions in our proof is in stark contrast to that in Ahlswede’s proof where the auxiliary output distributions are simply chosen to be the induced code distributions. The flexibility of the choice of output distributions is essential in the Gaussian problem, and without a good choice in terms of the peak powers and the precision of the quantizers, we opine that the strong converse may be difficult to establish.

The combination of the above proof techniques enables us to prove the following sum-rate bound for the two-sender Gaussian MAC: For any blocklength $n$, any average error probability $\epsilon \in [0, 1)$ and any $(n, \epsilon)$-achievable $(R_1, R_2)$,

$$R_1 + R_2 \leq \frac{1}{2} \log(1 + P_1 + P_2) + O\left(\sqrt{\frac{\log n}{n}}\right).$$

Our bound on the second-order term in (4) for the Gaussian case is slightly better than Ahlswede’s bound on the second-order term in (3), the best existing upper bound for the discrete case. Of course, it would be desirable to reduce the residual term to $O\left(\frac{\log n}{\sqrt{n}}\right)$ to match existing second-order achievability bounds [8], [9] but this seems out of reach due to the use of the wringing lemma.

An auxiliary contribution is a strong converse proof for the two-sender two-receiver Gaussian interference channel (IC) under strong interference [25], [26] as long as the sum of asymptotic error probabilities is strictly smaller than one.

### D. Paper Outline

In the next subsection, we state the notation used in this paper. In Section II, we describe the system model and define the $\epsilon$-capacity region of the Gaussian MAC. In Section III, we present the main result of the paper. We present a few preliminaries for the proof in Section IV. The complete proof is then presented in Section V. Section VI extends our strong converse result to the two-sender two-receiver Gaussian IC under strong interference.

### E. Notation

We use the upper case letter $X$ to denote an arbitrary (discrete or continuous) random variable with alphabet $\mathcal{X}$, and use a lower case letter $x$ to denote a realization of $X$. We use $X^n$ to denote the random tuple $(X_1, X_2, \ldots, X_n)$.

The following notations are used for any arbitrary random variables $X$ and $Y$ and any mapping $g$ whose domain includes $\mathcal{X}$. We let $p_{X,Y}$ and $p_{Y|X}$ denote the probability distribution of $(X, Y)$ (can be both discrete, both continuous or one discrete and one continuous) and the conditional probability distribution of $Y$ given $X$ respectively. We let $p_{X,Y}(x, y)$ and $p_{Y|X}(y|x)$ be the evaluations of $p_{X,Y}$ and $p_{Y|X}$ respectively at $(X, Y) = (x, y)$. To avoid confusion, we do not write $\Pr\{X = x, Y = y\}$ to represent $p_{X,Y}(x, y)$ unless $X$ and $Y$ are both discrete. To make the dependence on the distribution explicit, we let $\Pr_{p_X}\{g(X) \in A\}$ denote $\int_{x \in \mathcal{X}} p_{X}(x) 1\{g(x) \in A\} \, dx$ for any real-valued function $g$ and any set $A$. The expectation and the variance of $g(X)$ are denoted as $\mathbb{E}_{p_X}[g(X)]$ and $\text{Var}_{p_X}[g(X)] = \mathbb{E}_{p_X}[(g(X) - \mathbb{E}_{p_X}[g(X)])^2]$ respectively, where we again make the dependence on the underlying distribution $p_X$ explicit. We let $\mathcal{N}(\cdot; \mu, \sigma^2) : \mathbb{R} \rightarrow [0, \infty)$ denote the probability density function of a Gaussian random variable whose mean and variance are $\mu$ and $\sigma^2$ respectively. This means that

$$\mathcal{N}(z; \mu, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z - \mu)^2}{2\sigma^2}\right).$$
We will take all logarithms to base 2 throughout this paper. The Euclidean norm of a vector $x^n \in \mathbb{R}^n$ is denoted by $\|x^n\| = \sqrt{\sum_{k=1}^{n} x_k^2}$.

II. GAUSSIAN MULTIPLE ACCESS CHANNEL

We consider a Gaussian MAC that consists of $N$ sources and one destination. Let

$$\mathcal{I} \triangleq \{1, 2, \ldots, N\}$$

be the index set of the sources (or encoders), and let $d$ denote the destination (or decoder). The $N$ message sources transmit information to the destination in $n$ time slots (channel uses) as follows. For each $i \in \mathcal{I}$, node $i$ chooses message

$$W_i \in \{1, 2, \ldots, M_i^{(n)}\}$$

and sends $W_i$ to node $d$ where $M_i^{(n)} = |\mathcal{W}_i|$. Based on $W_i$, each node $i$ prepares a codeword $X_i^n \in \mathbb{R}^n$ to be transmitted and $X_i^n$ should satisfy

$$\sum_{k=1}^{n} X_{i,k}^2 \leq n P_i,$$

where $P_i$ denotes the power constraint for the codeword transmitted by node $i$. Then for each $k \in \{1, 2, \ldots, n\}$, each node $i$ transmits $X_{i,k}$ in time slot $k$ and node $d$ receives the real-valued symbol

$$Y_k = \sum_{i \in \mathcal{I}} X_{i,k} + Z_k,$$

where $Z_1, Z_2, \ldots, Z_n$ are i.i.d. and $Z_1$ is a standard Gaussian random variable. After $n$ time slots, node $d$ declares $\{\hat{W}_i\}_{i \in \mathcal{I}}$ to be the transmitted $\{W_i\}_{i \in \mathcal{I}}$ based on $Y^n$.

To simplify notation, we use the following convention for any $T \subseteq \mathcal{I}$. For any random tuple $(X_1, X_2, \ldots, X_N)$, we let

$$X_T \triangleq (X_i | i \in T)$$

be its subtuple, whose generic realization and alphabet are denoted by $x_T$ and

$$X_T = \prod_{i \in T} X_i$$

respectively. Similarly, for any $k \in \{1, 2, \ldots, n\}$ and any random tuple $(X_{1,k}, X_{2,k}, \ldots, X_{N,k}) \in \mathcal{X}_T$, we let

$$X_{T,k} \triangleq (X_{i,k} | i \in T)$$

be its subtuple, whose realization is denoted by $x_{T,k}$. The following five definitions formally define a Gaussian MAC and its capacity region.

**Definition 1:** Let $T$ be a non-empty subset in $\mathcal{I}$. An $(n, M_T^{(n)}, P_T, A, T)$-code for the Gaussian MAC, where $M_T^{(n)} \triangleq (M_1^{(n)}, M_2^{(n)}, \ldots, M_N^{(n)})$ and $P_T \triangleq (P_1, P_2, \ldots, P_N)$, consists of the following:

1) A message set $\mathcal{W}_i \triangleq \{1, 2, \ldots, M_i^{(n)}\}$ at node $i$ for each $i \in \mathcal{I}$.
2) A support set of the message tuple $W_T$ denoted by $A \subseteq W_T$ where $W_T$ is uniform on $A$. In addition, all the $w_T$’s in $A$ have the same $w_T^*$, i.e., there exists a $w_T^* \in W_T$ such that for all $w_T \in A$, we have $w_T^* = w_T^*$. Define

$$A_T \triangleq \{w_T \in W_T | \text{There exists a } \tilde{w}_T \in A \text{ such that } w_T = \tilde{w}_T\}$$

3) An encoding function $f_i : \mathcal{W}_i \rightarrow \mathbb{R}^n$ for each $i \in \mathcal{I}$, where $f_i$ is the encoding function at node $i$ such that $X_i^n = f_i(W_i)$ and

$$||f_i(w_i)||^2 \leq n P_i$$

for all $w_i \in \mathcal{W}_i$. The set of codewords $\{f_i(1), f_i(2), \ldots, f_i(M_i^{(n)})\}$ is called the **codebook** for $W_i$. For each
\[ i \in \mathcal{I}, \text{ the finite alphabet} \]
\[ X_i = \{ x \in \mathbb{R} \mid x \text{ is a component of } f_i(w_i) \text{ for some } w_i \in \mathcal{W}_i \} \]  \hspace{1cm} (14)

is called the support of symbols transmitted by \( i \) because \( f_i(\mathcal{W}_i) \subseteq X_i^n \). Note that
\[ |X_i| \leq nM_i^{(n)} \]  \hspace{1cm} (15)

for each \( i \in \mathcal{I} \) by (14).

4) A decoding function \( \varphi : \mathbb{R}^n \to \mathcal{A} \), where \( \varphi \) is the decoding function for \( W_T \) at node \( d \) such that \( \hat{W}_T = \varphi(Y^n) \).

We allow \( \varphi \) to be stochastic.

If \( \mathcal{A} = \mathcal{W}_T \) and \( T = \mathcal{I} \), then \( W_T \) is uniformly distributed on \( \mathcal{W}_T \), which implies that the \( N \) messages are mutually independent. Since \((n, M_T^{(n)}, P_T, \mathcal{W}_T, \mathcal{I})\)-codes are of our main interest, they are also called \((n, M_T^{(n)}, P_T)\)-codes for notational convenience. However, in the present work, it is necessary to allow \( \mathcal{A} \) and \( T \) to be strict subsets of \( \mathcal{W}_T \) and \( \mathcal{I} \) respectively so the generality afforded in the above definition is necessary. In this case, the \( N \) messages need not be independent. In the rest of this paper, if we fix a code with encoding function \( f_i \), then \( X_i \) as defined in (14) denotes the support of symbols transmitted by each \( i \in \mathcal{I} \).

**Definition 2:** A Gaussian MAC is characterized by the probability density function \( q_{Y_i|X_T}(y|x_T) \) satisfying
\[ q_{Y_i|X_T}(y|x_T) = N \left( y; \sum_{i \in \mathcal{I}} x_i, 1 \right) \]  \hspace{1cm} (16)

for all \( x_T \in \mathbb{R}^N \) and all \( y \in \mathbb{R} \) such that the following holds for any \((n, M_T^{(n)}, P_T, \mathcal{A}, T)\)-code: Let \( p_{W_T,X_T^n,Y^n} \) be the probability distribution induced by the \((n, M_T^{(n)}, P_T, \mathcal{A}, T)\)-code. Then,
\[ p_{W_T,X_T^n,Y^n}(w_T, x_T^n, y^n) = p_{W_T}(w_T) \left( \prod_{i \in \mathcal{I}} 1 \{ x_i^n = f_i(w_i) \} \right) \left( \prod_{k=1}^{n} p_{Y_k|X_T,k}(y_k|x_{T,k}) \right) \]  \hspace{1cm} (17)

for all \( (w_T, x_T^n, y^n) \in \mathcal{A} \times X_T^n \times \mathbb{R}^n \) where
\[ p_{Y_k|X_T,k}(y_k|x_{T,k}) \triangleq q_{Y_i|X_T}(y_k|x_{T,k}). \]  \hspace{1cm} (18)

Since \( p_{Y_i|X_T,k} \) does not depend on \( k \) by (18) and (16), the channel is stationary.

For any \((n, M_T^{(n)}, P_T, \mathcal{A}, T)\)-code defined on the Gaussian MAC, let \( p_{W_T,X_T^n,Y^n,W_T} \) be the joint distribution induced by the code. Since \( \hat{W}_T \) is a function of \( Y^n \) by Definition 1, it follows that
\[ p_{W_T,X_T^n,Y^n,\hat{W}_T} = p_{W_T,X_T^n,Y^n}\hat{W}_T|Y^n, \]  \hspace{1cm} (19)

which implies from (17) that
\[ p_{W_T,X_T^n,Y^n,\hat{W}_T} = p_{W_T,X_T^n} \left( \prod_{k=1}^{n} p_{Y_k|X_T,k} \right) p_{\hat{W}_T|Y^n}. \]  \hspace{1cm} (20)

**Definition 3:** For an \((n, M_T^{(n)}, P_T)\)-code defined on the Gaussian MAC, we can calculate according to (20) the average probability of decoding error which is defined as
\[ \Pr \{ \hat{W}_T \neq W_T \}. \]  \hspace{1cm} (21)

An \((n, M_T^{(n)}, P_T)\)-code with average probability of decoding error no larger than \( \varepsilon \) is called an \((n, M_T^{(n)}, P_T, \varepsilon)_{\text{avg}}\)-code. Similarly for an \((n, M_T^{(n)}, P_T, \mathcal{A}, T)\)-code, we can calculate the maximal probability of decoding error defined as
\[ \max_{W_T \in \mathcal{A}} \Pr \{ \hat{W}_T \neq W_T \mid W_T = w_T \}. \]  \hspace{1cm} (22)

An \((n, M_T^{(n)}, P_T, \mathcal{A}, T)\)-code with maximal probability of decoding error no larger than \( \varepsilon \) is called an \((n, M_T^{(n)}, P_T, \varepsilon)_{\text{max}}\)-code.
Then for each sequence of $A$. Remarks Concerning Theorem for each of all $T$, $\varepsilon$.

**Definition 4:** A rate tuple $R_T \triangleq (R_1, R_2, \ldots, R_N)$ is $\varepsilon$-achievable for the Gaussian MAC if there exists a sequence of $(n, M^{(n)}_T, P_T, \varepsilon_n)_{\text{avg}}$-codes on the Gaussian MAC such that

$$\liminf_{n \to \infty} \frac{1}{n} \log M^{(n)}_i \geq R_i$$

for each $i \in \mathcal{I}$ and

$$\limsup_{n \to \infty} \varepsilon_n \leq \varepsilon.$$  \hspace{1cm} (24)

**Definition 5:** For each $\varepsilon \in [0, 1)$, the $\varepsilon$-capacity region of the Gaussian MAC, denoted by $C_\varepsilon$, is the set consisting of all $\varepsilon$-achievable rate tuples $R_T$. The capacity region is defined to be the 0-capacity region $C_0$.

### III. MAIN RESULT

The following theorem is the main result in this paper.

**Theorem 1:** Define

$$\mathcal{R}_{\text{cw}} \triangleq \bigcap_{T \subseteq \mathcal{I}} \left\{ R_T \in \mathbb{R}^N_+ \, | \sum_{i \in T} R_i \leq \frac{1}{2} \log \left( 1 + \sum_{i \in T} P_i \right) \right\}.$$  \hspace{1cm} (25)

Then for each $\varepsilon \in [0, 1)$,

$$C_\varepsilon \subseteq \mathcal{R}_{\text{cw}}.$$  \hspace{1cm} (26)

**A. Remarks Concerning Theorem 1**

We now present three remarks concerning Theorem 1.

1) Note that $\mathcal{R}_{\text{cw}}$ is the Cover-Wyner [4], [5] region for an $N$-user Gaussian MAC. The theorem says that regardless of the admissible average error probability (as long as it is strictly smaller than 1), all achievable rate tuples must lie in $\mathcal{R}_{\text{cw}}$. Since all rate tuples in $\mathcal{R}_{\text{cw}}$ are 0-achievable [1, Section 4.7], we have for every $\varepsilon \in [0, 1)$

$$C_\varepsilon = \mathcal{R}_{\text{cw}}.$$  \hspace{1cm} (27)

2) In fact, the proof allows us to additionally assert the following: For any non-vanishing average error probability $\varepsilon \in [0, 1)$ and any subset $T \subseteq \mathcal{I}$, it can be shown that the sum-rate of the messages indexed by $T$ of any sequence of $(n, M^{(n)}_T, P_T, \varepsilon_n)_{\text{avg}}$-codes satisfying the constraint in (24) also satisfies

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n \log n}} \left[ \sum_{i \in T} \log M^{(n)}_i - \frac{n}{2} \log \left( 1 + \sum_{i \in T} P_i \right) \right] \leq \Upsilon(\varepsilon, T, P_T) < \infty$$

for some finite constant $\Upsilon(\varepsilon, T, P_T)$. See (178) in the proof of Theorem 1. Even though the normalizing speed of $\sqrt{n \log n}$ is not the desired $\sqrt{n}$ (as usually defined in second-order asymptotic analyses [7]), the techniques in this work may serve as a stepping stone to establish an outer bound for the second-order coding rate region [7] for the Gaussian MAC. The best inner bound for the second-order coding rates for the Gaussian MAC was established independently by Scarlett, Martínez, and Guillén i Fábregas [8] and MolavianJazi and Laneman [9]. According to the inner bounds in [8], [9] and the relation between second-order coding rates and second-order asymptotics of sum-rates in [10],

$$\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \left[ \sum_{i \in T} \log M^{(n)}_i - \frac{n}{2} \log \left( 1 + \sum_{i \in T} P_i \right) \right] \geq \Upsilon(\varepsilon, T, P_T) > -\infty$$

for some finite constant $\Upsilon(\varepsilon, T, P_T)$. Our normalizing speed of $\sqrt{n \log n}$ in (28) is slightly better than in Ahlswede’s work on the discrete memoryless MAC [17], which is $\sqrt{n \log n}$. We have attempted to optimize (reduce) the exponent of the logarithm $\zeta > 0$ in the normalizing speed $\sqrt{n \log n} \zeta$. However, using our wringing technique, it does not appear that $\zeta$ can be further reduced from 1/2. For both the discrete and Gaussian MACs, the challenge of proving the exact normalizing speed of the second-order term being equal
to $\sqrt{n}$ is due to the use of wringing technique, which prevents one from obtaining a matching converse for the normalizing speed. Unless new techniques are invented to replace the wringing argument in the strong converse proof for the MAC (such techniques have remained elusive for over 30 years), the exact normalizing speed of the second-order term for the discrete and Gaussian MACs will remain an open problem.

In the next section, we will present a few preliminaries for the proof of Theorem 1, which will be detailed in Section V.

IV. PRELIMINARIES FOR THE PROOF OF THEOREM 1

A. Expurgation of Message Tuples

The following lemma is based on the technique of expurgating message tuples introduced by Dueck [12, Section II], and the proof is provided in the Appendix for completeness.

**Lemma 1:** Let $\epsilon \in [0, 1)$. Suppose an $(n, M_T^{(n)}, P_T, \epsilon)_{\text{avg}}$-code for the Gaussian MAC is given. Then for each nonempty $T \subseteq I$, such that

$$
\left( \frac{1 - \epsilon}{1 + \epsilon} \right) \prod_{i \in T} M_i^{(n)} \geq \left( \frac{1 - \epsilon}{2(1 + \epsilon)} \right) \prod_{i \in T} M_i^{(n)},
$$

(30)

there exist a set $A \subseteq \mathcal{W}_T$ and an $(n, M_T^{(n)}, P_T, A, T, \frac{1 + \epsilon}{2})_{\text{max}}$-code such that

$$
|A_T| = |A| \geq \left( \frac{1 - \epsilon}{2(1 + \epsilon)} \right) \prod_{i \in T} M_i^{(n)},
$$

(31)

where $A_T$ is defined in (12). As a consequence, if we let $p_{W_T, X_T^{(n)}, Y_T^{(n)}, \hat{W}_T}$ denote the probability distribution induced on the Gaussian MAC by the $(n, M_T^{(n)}, P_T, A, T, \frac{1 + \epsilon}{2})_{\text{max}}$-code, then we have for each $w_T \in A_T$

$$
p_{W_T}(w_T) \leq \frac{1}{\prod_{i \in T} M_i^{(n)}} \left( \frac{2(1 + \epsilon)}{1 - \epsilon} \right).
$$

(32)

**Remark 1:** Lemma 1 says that restricted to the set $A_T$, the $i$th (for $i \in T$) codebooks have almost the same sizes as the original codebooks. In addition, the conditional probability of decoding error for each message tuple in this restricted codebook is upper bounded by $\frac{1 + \epsilon}{2}$, which is still smaller than one because $\epsilon \in [0, 1)$. According to (32), the probability of each message tuple cannot be greater than its original value by a factor of $\frac{2(1 + \epsilon)}{1 - \epsilon}$.

B. Wringing Technique

The following lemma forms part of the wringing technique proposed by Ahlswede and its proof can be found in [17, Lemma 4].

**Lemma 2:** Let $\mathcal{X}$ be a finite alphabet, let $p_{X^n}$ and $u_{X^n}$ be two probability mass functions defined on $\mathcal{X}^n$ and let $c > 0$ be a real number such that

$$
p_{X^n}(x^n) \leq (1 + c)u_{X^n}(x^n)
$$

(33)

for all $x^n \in \mathcal{X}^n$. Fix any $0 < \lambda < 1$. Then for any $0 < \delta < c$, there exist $\ell$ natural numbers in $\{1, 2, \ldots, n\}$, denoted by $t_1, t_2, \ldots, t_\ell$, and $\ell$ elements of $\mathcal{X}$ denoted by $\bar{x}_{t_1}, \bar{x}_{t_2}, \ldots, \bar{x}_{t_\ell}$, such that the following three statements hold:

(I) $\ell \leq \frac{c}{\delta}$.

(II) $\Pr_{X^n} \{ (X_{t_1}, X_{t_2}, \ldots, X_{t_\ell}) = (\bar{x}_{t_1}, \bar{x}_{t_2}, \ldots, \bar{x}_{t_\ell}) \} \geq \lambda^\ell$.

(III) For all $k \in \{1, 2, \ldots, n\} \setminus \{t_1, t_2, \ldots, t_\ell\}$, we have

$$
p_{X^n|X_{t_1}, X_{t_2}, \ldots, X_{t_\ell}}(x_k|\bar{x}_{t_1}, \bar{x}_{t_2}, \ldots, \bar{x}_{t_\ell}) \leq \max \{(1 + \delta)u_{X^n|X_{t_1}, X_{t_2}, \ldots, X_{t_\ell}}(x_k|\bar{x}_{t_1}, \bar{x}_{t_2}, \ldots, \bar{x}_{t_\ell}), \lambda\}
$$

(34)

for all $x_k \in \mathcal{X}$.
In order to use Lemma 2 for proving Theorem 1, an important step involves controlling the size of $\mathcal{X}$ in Lemma 2. To this end, we use the following scalar quantizer to quantize the alphabet $\mathcal{X}$ in (14) which is originally exponential in the blocklength $n$ (cf. (15)) to another alphabet whose size is now polynomial in the blocklength.

**Definition 6:** Let $L$ be a natural number and $\Delta$ be a positive real number, and let

$$\mathbb{Z}_{L,\Delta} \triangleq \{-L\Delta, (-L + 1)\Delta, \ldots, L\Delta\}$$

be a set of $2L + 1$ quantization points where $\Delta$ specifies the quantization precision. A scalar quantizer with domain $[-L\Delta, L\Delta]$ and precision $\Delta$ is the mapping

$$\Omega_{L,\Delta} : [-L\Delta, L\Delta] \to \mathbb{Z}_{L,\Delta}$$

such that

$$\Omega_{L,\Delta}(x) = \begin{cases} [x/\Delta] \Delta & \text{if } x \geq 0, \\ [-x/\Delta] \Delta & \text{otherwise.} \end{cases}$$

In other words, $\Omega_{L,\Delta}(x)$ maps $x$ to the closest quantized point whose value is smaller than or equal to $x$ if $x \geq 0$, and to the closest quantized point whose value is larger than or equal to $x$ if $x < 0$. In addition, define the scalar quantizer for a real-valued tuple as

$$\Omega_{L,\Delta}^{(n)} : [-L\Delta, L\Delta]^n \to \mathbb{Z}_{L,\Delta}^n$$

such that

$$\Omega_{L,\Delta}^{(n)}(x^n) \triangleq (\Omega_{L,\Delta}(x_1), \Omega_{L,\Delta}(x_2), \ldots, \Omega_{L,\Delta}(x_n)).$$

By our careful choice of the quantizer in Definition 6, we have the following property for all $x \in \mathbb{R}$:

$$|\Omega_{L,\Delta}(x)| \overset{\text{(37)}}{=} \begin{cases} [x/\Delta] \Delta & \text{if } x \geq 0, \\ [-x/\Delta] \Delta & \text{otherwise} \end{cases}$$

$$= \begin{cases} [x/\Delta] \Delta & \text{if } x \geq 0, \\ [-x/\Delta] \Delta & \text{otherwise} \end{cases}$$

$$\leq |x|.$$

Although the following lemma looks similar to [17, Corollary 2] and they both rely on Lemma 2, the proof of the following lemma is more involved due to the additional consideration of the quantizer’s precision and the quantized input symbols. If the quantizer’s precision is too small or too large, then the resultant bound obtained from the following lemma will not be useful in proving the strong converse. See Section V-H for a detailed discussion on the appropriate choice for the quantizer’s precision.

**Lemma 3:** Suppose we are given an $(n, M_x^{(n)}(n), P_x, \mathcal{A}', T, \frac{1+\epsilon}{2})_{\max}$-code such that

$$|\mathcal{A}'_T| = |\mathcal{A}'| \geq \left( \frac{1 - \epsilon}{2(1 + \epsilon)} \right) \prod_{i \in T} M_i^{(n)}$$

and

$$p_{W_T}(w_T) \leq \frac{1}{\prod_{i \in T} M_i^{(n)}} \cdot \left( \frac{2(1 + \frac{\epsilon}{2})}{1 - \epsilon} \right)$$

for each $w_T \in \mathcal{A}'_T$ where $p_{W_T}(w_T)$ denotes the probability distribution induced on the Gaussian MAC by the $(n, M_x^{(n)}(n), P_x, \mathcal{A}', T, \frac{1+\epsilon}{2})_{\max}$-code. Then, there exists an $(n, M_x^{(n)}(n), P_x, \mathcal{A}, T, \frac{1+\epsilon}{2})_{\max}$-code with

$$|\mathcal{A}_T| = |\mathcal{A}| \geq n^{ \frac{-4T(1+\epsilon)}{(1-\epsilon)} \sqrt{\log n} \left( \frac{1 - \epsilon}{2(1 + \epsilon)} \right) \prod_{i \in T} M_i^{(n)}$$

such that the following holds: Let $p_{W_T, X_T, Y^n, \bar{W}_T}$ denote the probability distribution induced on the Gaussian MAC.
by the \((n, M^{(n)}_I, P, \mathcal{A}, T, \frac{1+\varepsilon}{2})\text{max}\)-code. In addition, let

\[
\hat{X}^n_i = \Omega^{(n)}_{\lfloor n\sqrt{MT} \rfloor, n-i} (X_i^n),
\]

(47)
define the alphabet

\[
\hat{X}_i \triangleq \mathbb{Z}_{\lfloor n\sqrt{MT} \rfloor, n-i}
\]

(48)for each \(i \in T\) \((\hat{X}_i^n)\) is always in the domain of \(\mathbb{Z}_{\lfloor n\sqrt{MT} \rfloor, n-i}\) because of (47), (43) and (13), and hence \(\hat{X}_i^n \in \hat{X}_i^n\), define

\[
\hat{X}_T \triangleq \prod_{i \in T} \hat{X}_i
\]

(49)and define

\[
p_{W,T,X_2,Y^n,\hat{W}}(w_T, x_T^n, \hat{x}_T^n, y^n, \hat{w}_T)
\]

\[
\triangleq p_{W,T,X_2,Y^n,\hat{W}}(w_T, x_T^n, y^n, \hat{w}_T) \prod_{i \in T} 1 \left\{ \hat{x}_i^n = \Omega^{(n)}_{\lfloor n\sqrt{MT} \rfloor, n-i} (x_i^n) \right\}.
\]

(50)for all \((w_T, x_T^n, \hat{x}_T^n, y^n, \hat{w}_T) \in \mathcal{A} \times \hat{X}_T^n \times \hat{X}_T^n \times \mathbb{R}^n \times \mathcal{A}\). Then there exists a distribution \(u_{\hat{X}_T^n}\) such that for all \(k \in \{1, 2, \ldots, n\}\), we have

\[
p_{X_T,k}(\hat{x}_{T,k}) \leq \max \left\{ 1 + \sqrt{\frac{\log n}{n}} \prod_{i \in T} u_{\hat{X}_{i,k}}(\hat{x}_{i,k}), \frac{1}{n^4|T|} \right\}
\]

(51)for all \(\hat{x}_{T,k} \in \hat{X}_T^n\) and

\[
\sum_{i \in T} \sum_{k=1}^n E_{u_{\hat{X}_{i,k}}} \left[ \hat{X}_{i,k}^2 \right] \leq \sum_{i \in T} n P_i.
\]

(52)

The importance of Lemma 3 is that for each time slot \(k\), we can find a subcode such that the resultant probability distribution of the quantized transmitted symbol \(\hat{X}_{T,k}\) can be approximated by a product distribution \(\prod_{i \in T} u_{\hat{X}_{i,k}}\) through (51), which is the main purpose of the wringing technique which wrings out the independence among the random variables corresponding to the different encoders \(\{X_{i,k} \mid i \in T\}\).

Proof: Let \(p'_{W,T,X_2,Y^n,\hat{W}}(w_T, x_T^n, \hat{x}_T^n, y^n, \hat{w}_T)\) be the probability distribution induced on the Gaussian MAC by the \((n, M^{(n)}_I, P, \mathcal{A}, T, \frac{1+\varepsilon}{2})\text{max}\)-code that satisfies (44) and (45), and let

\[
p'_{W,T,X_2,Y^n,\hat{W}}(w_T, x_T^n, \hat{x}_T^n, y^n, \hat{w}_T)
\]

\[
\triangleq p'_{W,T,X_2,Y^n,\hat{W}}(w_T, x_T^n, y^n, \hat{w}_T) \prod_{i \in T} 1 \left\{ \hat{x}_i^n = \Omega^{(n)}_{\lfloor n\sqrt{MT} \rfloor, n-i} (x_i^n) \right\}.
\]

(53)Define a probability mass function \(u'_{W,T,X_2,Y^n,\hat{X}_T^n}\) as

\[
u'_{W,T,X_2,Y^n,\hat{X}_T^n}(w_T, x_T^n, \hat{x}_T^n) \triangleq \prod_{i \in T} \frac{1 \{ f_i(w_i) \} \cdot 1 \left\{ \hat{x}_i^n = \Omega^{(n)}_{\lfloor n\sqrt{MT} \rfloor, n-i} (x_i^n) \right\}}{M_i^{(n)}}
\]

(54)for all \((w_T, x_T^n, \hat{x}_T^n) \in \mathcal{W}_T \times \hat{X}_T^n \times \hat{X}_T^n\) (cf. (14) and (48)), where \(f_i\) represents the encoding function for \(W_i\) of the \((n, M^{(n)}_I, P, \mathcal{A}, T, \frac{1+\varepsilon}{2})\text{max}\)-code (cf. Definition 1). The distribution \(u'_{W,T,X_2,Y^n,\hat{X}_T^n}\) is well-defined (the probability
masses sum to one) through (54) because
\[
\sum_{(w_T, x^n_T, \hat{x}^n_T) \in \mathcal{W}_T \times \hat{X}^n_T} u'_W, x^n_T, \hat{x}^n_T (w_T, x^n_T, \hat{x}^n_T) \tag{55}
\]
\[
\overset{(54)}{=} \sum_{w_T \in \mathcal{W}_T} \prod_{i \in T} \frac{1}{M_i(n)} \sum_{x^n_T \in \hat{X}^n_T} \prod_{i \in T} 1 \{ x^n_i = f_i(w_i) \} \sum_{\hat{x}^n_T \in \hat{X}^n_T} \prod_{i \in T} 1 \left\{ \hat{x}^n_i = \frac{\Omega(n)}{n^{\sqrt{n}P_i}}, n^{-1}(x^n_i) \right\} \tag{56}
\]
\[
= 1. \tag{57}
\]
Using (54), we obtain
\[
u'_W, x^n_T, \hat{x}^n_T \left( \sum_{i \in T} \sum_{k=1}^{n} X^2_{i,k} \right) \leq \sum_{i \in T} n P_i \right) = 1. \tag{60}
\]

Since \( \hat{X}^2_{i,k} \leq X^2_{i,k} \) for all \( i \in T \) and all \( k \in \{1, 2, \ldots, n\} \) by (53) and (43), it follows from (60) that
\[
\Pr_{\nu'_W, x^n_T, \hat{x}^n_T} \left\{ \sum_{i \in T} \sum_{k=1}^{n} \hat{X}^2_{i,k} \leq \sum_{i \in T} n P_i \right) = 1. \tag{61}
\]

We will use Lemma 2 to prove the existence of a subcode of the \((n, M_L(n), P_T, A', T, \frac{1+\varepsilon}{2})_{\text{max}}\)-code such that the subcode satisfies (46), (51) and (52). To this end, we first consider the following chain of inequalities for each \( \hat{x}^n_T \in \hat{X}^n_T \) such that \( p'_{X^n_T}(\hat{x}^n_T) > 0 \):
\[
p'_{X^n_T}(\hat{x}^n_T) = \sum_{w_T, x^n_T, \hat{x}^n_T \in \hat{X}^n_T} p'_W, x^n_T, \hat{x}^n_T (w_T, x^n_T, \hat{x}^n_T) \tag{62}
\]
\[
= \sum_{w_T, x^n_T, \hat{x}^n_T \in \hat{X}^n_T} p'_W (w_T) p'_X(x^n_T, \hat{x}^n_T | w_T) \tag{63}
\]
\[
\overset{(a)}{=} \sum_{w_T, x^n_T, \hat{x}^n_T \in \hat{X}^n_T} p'_W (w_T) \prod_{i \in T} \left( 1 \{ x^n_i = f_i(w_i) \} \cdot 1 \left\{ \hat{x}^n_i = \frac{\Omega(n)}{n^{\sqrt{n}P_i}}, n^{-1}(x^n_i) \right\} \right) \tag{64}
\]
\[
\overset{(45)}{\leq} \sum_{w_T, x^n_T, \hat{x}^n_T \in \hat{X}^n_T} \prod_{i \in T} \frac{1}{M_i(n)} \cdot \left( \frac{2(1+\varepsilon)}{1-\varepsilon} \right) \prod_{i \in T} \left( 1 \{ x^n_i = f_i(w_i) \} \cdot 1 \left\{ \hat{x}^n_i = \frac{\Omega(n)}{n^{\sqrt{n}P_i}}, n^{-1}(x^n_i) \right\} \right) \tag{65}
\]
\[
\overset{(54)}{=} \frac{2(1+\varepsilon)}{1-\varepsilon} \sum_{w_T, x^n_T, \hat{x}^n_T \in \hat{X}^n_T} u'_W, x^n_T, \hat{x}^n_T (w_T, x^n_T, \hat{x}^n_T) \tag{66}
\]
\[
\leq \frac{2(1+\varepsilon)}{1-\varepsilon} \sum_{w_T, x^n_T, \hat{x}^n_T \in \hat{X}^n_T} u'_W, x^n_T, \hat{x}^n_T (w_T, x^n_T, \hat{x}^n_T) \tag{67}
\]
\[
= \frac{2(1+\varepsilon)}{1-\varepsilon} \cdot u'_{X^n_T}(\hat{x}^n_T) \tag{68}
\]
where (a) follows from (17) and (53). It follows from (68) and Lemma 2 with the identifications
\[
\mathcal{X} \triangleq \hat{X}_T, \quad c \triangleq \frac{1+3\varepsilon}{1-\varepsilon}, \quad \lambda \triangleq \frac{1}{n^{4|T|}}, \quad \delta \triangleq \sqrt{\log n} \tag{69}
\]
that there exist \( \ell \) natural numbers in \( \{1, 2, \ldots, n\} \), denoted by \( t_1, t_2, \ldots, t_\ell \), and \( \ell \) real-valued \( |T| \)-dimensional tuples in \( \hat{X}_T \), denoted by \( \hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell} \), such that the following three statements hold:

(I) \( \ell \leq \left( \frac{1+3\varepsilon}{1-\varepsilon} \right) \sqrt{\frac{n}{\log n}}. \)

(II) \( \Pr_{x^n} \left\{ \left( \hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell} \right) = (\hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell}) \right\} \geq \frac{1}{n^{4|T|}}. \)

(III) For all \( k \in \{1, 2, \ldots, n\} \setminus \{t_1, t_2, \ldots, t_\ell\} \), we have

\[
\begin{align*}
\Pr_{x^n} \left\{ \left( \hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell} \right) = (\hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell}) \right\} & \leq \max \left\{ \left( 1 + \sqrt{\frac{n}{\log n}} \right) \prod_{i \in T} u'_{\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}} (\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}), \frac{1}{n^{4|T|}} \right\} \\
& \leq \max \left\{ \left( 1 + \sqrt{\frac{n}{\log n}} \right) \prod_{i \in T} u'_{\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}} (\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}), \frac{1}{n^{4|T|}} \right\}
\end{align*}
\]

for all \( \hat{x}_{T,k} \in \hat{X}_T \).

Using Statement (II), Statement (III) and (44), we can construct an \( (n, M_{\hat{n}}(n), p_{\hat{z}}, A, T, \frac{1+\varepsilon}{2})_{\max} \)-code by collecting all the codewords \( x^\hat{n}_k \) for the \( (n, M_{\hat{n}}(n), p_{\hat{z}}, A', T, \frac{1+\varepsilon}{2})_{\max} \)-code which satisfy

\[
\begin{align*}
(\hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell}) = (\hat{x}_{T,t_1}, \hat{x}_{T,t_2}, \ldots, \hat{x}_{T,t_\ell})
\end{align*}
\]

such that the following two statements hold:

(i) \( |A_T| = |A| \geq n^{-4|T|\ell} \left( \frac{1-\varepsilon}{2(1+\varepsilon)} \right) \prod_{i \in T} M_{i}^{(n)} \).

(ii) Let \( p_{w^n,x^n_2,y^n,\hat{w}_2} \) denote the probability distribution induced on the Gaussian MAC by the \( (n, M_{\hat{n}}(n), p_{\hat{z}}, A, T, \frac{1+\varepsilon}{2})_{\max} \)-code, and let

\[
\begin{align*}
p_{w^n,x^n_2,y^n,\hat{w}_2} (w^n, x^n_2, \hat{x}_n^n, y^n, \hat{w}_n) \equiv p_{w^n,x^n_2,y^n,\hat{w}_2} (w^n, x^n_2, y^n, \hat{w}_n) \prod_{i \in T} 1 \left\{ x_i^n = \Omega^{(n)}_{|nM_{\hat{n}}|, n^{-2}}(x_i^n) \right\}.
\end{align*}
\]

Then,

\[
\begin{align*}
\Pr_{p_{X^n}} \left\{ \bigcap_{m=1}^\ell \{ \hat{X}_{T,t_m} = \hat{x}_{T,t_m} \} \right\} = 1,
\end{align*}
\]

and we have for all \( k \in \{1, 2, \ldots, n\} \setminus \{t_1, t_2, \ldots, t_\ell\} \)

\[
\begin{align*}
\Pr_{p_{X^n}} \left\{ \hat{x}_{T,k} \leq \max \left\{ \left( 1 + \sqrt{\frac{n}{\log n}} \right) \prod_{i \in T} u'_{\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}} (\hat{x}_{i,k}), \frac{1}{n^{4|T|}} \right\} \\
\end{align*}
\]

for all \( \hat{x}_{T,k} \in \hat{X}_T \).

Since for each \( k \in \{t_1, t_2, \ldots, t_\ell\} \)

\[
\begin{align*}
p_{X^n_k} (\hat{x}_{T,k}) \equiv 1 \left\{ \hat{x}_{T,k} = \hat{x}_{T,k} \right\} = \prod_{i \in T} u'_{\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}} (\hat{x}_{i,k})
\end{align*}
\]

for all \( \hat{x}_{T,k} \in \hat{X}_T \), it follows from (75) that the following statement holds:

(iii) For all \( k \in \{1, 2, \ldots, n\} \), we have

\[
\begin{align*}
\Pr_{p_{X^n}} \left\{ \hat{x}_{T,k} \leq \max \left\{ \left( 1 + \sqrt{\frac{n}{\log n}} \right) \prod_{i \in T} u'_{\hat{x}_{i,k}|\hat{x}_{i,t_1}, \hat{x}_{i,t_2}, \ldots, \hat{x}_{i,t_\ell}} (\hat{x}_{i,k}), \frac{1}{n^{4|T|}} \right\} \\
\end{align*}
\]

for all \( \hat{x}_{T,k} \in \hat{X}_T \).
To conclude the proof, we obtain from Statement (i) and Statement (I) that (46) holds, and (51) follows from Statement (iii) by letting
\[ u_{X_k} \triangleq \prod_{k=1}^{n} \prod_{i \in T} u_{X_i, k} | X_i, t_1 = x_i, t_2 = x_i, t_3 = x_i, \ldots, t_T = x_i, t_T. \] (78)

In addition, (52) follows from the following chain of inequalities:
\[ \sum_{i \in T} \sum_{k=1}^{n} \mathbb{E}_{u_{X_i, k}} \left[ \hat{X}_{i, k}^2 \right] \]
\[ \hspace{1cm} \overset{(78)}{=} \sum_{i \in T} \sum_{k=1}^{n} \mathbb{E}_{u_{X_i, k}} \left[ \hat{X}_{i, k}^2 | X_i, t_1 = x_i, t_2 = x_i, \ldots, t_T = x_i, t_T \right] \]
\[ \hspace{1cm} \overset{(58)}{=} \sum_{i \in T} \sum_{k=1}^{n} \mathbb{E}_{u_{X_i, k}} \left[ \hat{X}_{X_i, t_1 = x_i, t_2 = x_i, \ldots, t_T = x_i, t_T}^2 \right] \]
\[ \hspace{1cm} \overset{(58)}{=} \sum_{i \in T} \sum_{k=1}^{n} \mathbb{E}_{u_{X_i, k}} \left[ \hat{X}_{X_i, t_1 = x_i, t_2 = x_i, \ldots, t_T = x_i, t_T}^2 \right] \left[ \sum_{j \in T} \sum_{k=1}^{n} \hat{X}_{i, k}^2 \right] \]
\[ \hspace{1cm} \overset{(61)}{\leq} \sum_{i \in T} n P_i. \] (83)

C. Binary Hypothesis Testing

The following definition concerning the non-asymptotic fundamental limits of a simple binary hypothesis test is standard. See for example [24, Section III-E].

**Definition 7:** Let \( p_X \) and \( q_X \) be two probability distributions on some common alphabet \( \mathcal{X} \). Let
\[ \mathcal{Q}(\{0, 1\} | \mathcal{X}) \triangleq \{ r_Z | X \mid Z \text{ and } X \text{ assume values in } \{0, 1\} \text{ and } \mathcal{X} \text{ respectively} \} \]
be the set of randomized binary hypothesis tests between \( p_X \) and \( q_X \) where \( \{ Z = 0 \} \) indicates the test chooses \( q_X \), and let \( \delta \in [0, 1] \) be a real number. The minimum type-II error in a simple binary hypothesis test between \( p_X \) and \( q_X \) with type-I error no larger than \( 1 - \delta \) is defined as
\[ \beta_\delta(p_X \| q_X) \triangleq \inf_{r_Z : Z \in \mathcal{Q}(\{0, 1\} | \mathcal{X})} \int_{x \in X} r_Z | X (1|x) q_X (x) \, dx. \] (84)

The existence of a minimizing test \( r_Z | X \) is guaranteed by the Neyman-Pearson lemma.

We state in the following lemma and proposition some important properties of \( \beta_\delta(p_X \| q_X) \), which are crucial for the proof of Theorem 1. The proof of the following lemma can be found in, for example, the paper by Wang, Colbeck, and Renner [23, Lemma 1].

**Lemma 4:** Let \( p_X \) and \( q_X \) be two probability distributions on some alphabet \( \mathcal{X} \), and let \( g \) be a function whose domain contains \( \mathcal{X} \). Then, the following two statements hold:
1. Data processing inequality (DPI):
\[ \beta_\delta(p_X \| q_X) \leq \beta_\delta(p_{g(X)} \| q_{g(X)}). \] (85)
2. For all \( \xi > 0 \),
\[ \beta_\delta(p_X \| q_X) \geq \frac{1}{\xi} \left( \delta - \int_{x \in X} p_X (x) \mathbf{1} \left\{ \frac{p_X (x)}{q_X (x)} \geq \xi \right\} \, dx \right). \] (86)

The proof of the following proposition is similar to Lemma 3 in [23] and therefore omitted.
Proposition 5: Let $p_{U,V}$ be a probability distribution defined on $\mathcal{W} \times \mathcal{W}$ for some finite alphabet $\mathcal{W}$. In addition, let $q_V$ be a distribution defined on $\mathcal{W}$, and let
\[ \alpha = \max_{u \in \mathcal{W}} \Pr\{ V \neq u | U = u \} \]
be a real number in $[0, 1)$ where $(U, V)$ is distributed according to $p_{U,V}$. Then for each $u \in \mathcal{W}$,
\[ \beta_{1-\alpha}(p_{V | U = u} || q_V) \leq q_V(u). \]

V. Proof of Theorem 1

A. Expurgation to Obtain a Maximum Error Code

Let $\varepsilon \in [0, 1)$ and suppose $R_T$ is an $\varepsilon$-achievable rate tuple. By Definition 4, there exists a $\gamma \in [0, 1)$ and a sequence of $(n, M_T^{(n)}, P_T, \varepsilon_n)_\text{avg}$-codes such that
\[ \varepsilon_n \leq \gamma \]
for all sufficiently large $n$ and
\[ \liminf_{n \to \infty} \frac{1}{n} \log M_i^{(n)} \geq R_i \]
for each $i \in I$. Fix a non-empty set $T \subseteq I$. Our goal is to prove that
\[ \sum_{i \in T} R_i \leq \frac{1}{2} \log \left( 1 + \sum_{i \in T} P_i \right). \]
Since (91) holds trivially if $\sum_{i \in T} R_i = 0$, we assume without loss of generality that
\[ \sum_{i \in T} R_i > 0. \]
It follows from (90) and (92) that
\[ \left( \frac{1-\gamma}{1+\gamma} \right) \prod_{i \in T} M_i^{(n)} \geq \frac{1}{2} \left( \frac{1-\gamma}{1+\gamma} \right) \prod_{i \in T} M_i^{(n)} \]
for all sufficiently large $n$. Fix a sufficiently large $n$ and the corresponding $(n, M_T^{(n)}, P_T, \varepsilon_n)_\text{avg}$-code for the Gaussian MAC such that (89) and (93) hold. Using Lemma 1, Lemma 3 and Definition 1, there exists an $(n, M_T^{(n)}, P_T, A, T, \frac{1 + \gamma}{2})_\text{max}$-code, which induces a probability distribution on the Gaussian MAC denoted by $p_{W_T, X_T, Y^n, \hat{W}_T}$, such that the following four statements hold:

(i) For all $w_T \in A$ and all $w_T \in A_T$,
\[ p_{W_T}(w_T) = \frac{1}{|A|} \quad \text{and} \quad p_{W_T}(w_T) = \frac{1}{|A_T|}. \]

(ii) There exists a $w_{T_v}^* \in W_{T_v}$ such that for all $w_T \in A$, we have $w_{T_v} = w_{T_v}^*$.

(iii) The support of $W_T$ satisfies
\[ |A_T| = |A| \geq n^{-\frac{4T/(1+\gamma)}{(1-\gamma)} \sqrt{\log n}} \left( \frac{1-\gamma}{2(1+\gamma)} \right) \prod_{i \in T} M_i^{(n)}. \]

(iv) Define
\[ p_{W_T, X_T, Y^n, \hat{W}_T}(w_T, x_T^n, \hat{x}_T^n, y^n, \hat{w}_T) \]
\[ \overset{\Delta}{=} p_{W_T, X_T, Y^n, \hat{W}_T}(w_T, x_T^n, y^n, \hat{w}_T) \prod_{i \in T} 1 \left\{ \hat{x}_i^n = \Omega_{n \sqrt{M_i} }^{(n)}(x_i^n) \right\} \]
for all $(w_T, x_T^n, \hat{x}_T^n, y^n, \hat{w}_T) \in A \times X_T^n \times \hat{X}_T \times \mathbb{R}^n \times A$, where
\[ \hat{X}_T \overset{\Delta}{=} \prod_{i \in T} \mathbb{Z}_{n \sqrt{M_i} }^{(n)}(x_i^n). \]
Then there exists a distribution \( u_{\hat{x}_T^n} \) such that for all \( k \in \{1, 2, \ldots, n\} \), we have
\[
p_{X_T,k}(\hat{x}_{T,k}) \leq \max \left\{ 1 + \sqrt{\frac{\log n}{n}} \left( \prod_{i \in T} u_{X_i,k}(\hat{x}_{i,k}) \right), \frac{1}{n^{|T|}} \right\}
\]
for all \( \hat{x}_{T,k} \in \hat{X}_T \) and
\[
\sum_{i \in T} \sum_{k=1}^n E_{u_{\hat{x}_i}}[\hat{X}_{i,k}^2] \leq \sum_{i \in T} n P_i.
\]
Note that \( p_{W_T|X_T^n,Y^n;\hat{W}_T} \) is not the distribution induced by the original \((n, M_T^{(n)}, P_T, \varepsilon_n)_{\text{avg}}\)-code but rather it is induced by the expurgated \((n, M_T^{(n)}, P_T, A, T, \frac{1+\gamma}{2})_{\text{max}}\)-code.

B. Lower Bounding the Error Probability using Binary Hypothesis Testing

Now, let
\[
s_{W_T|X_T^n,Y^n;\hat{W}_T} \triangleq p_{W_T|X_T^n} \left( \prod_{k=1}^n s_{Y_k|X_T,k} \right) p_{W_T|Y^n}
\]
be a distribution such that for each \( k \in \{1, 2, \ldots, n\} \), the auxiliary conditional output distribution is chosen to be
\[
s_{Y_k|X_T,k}(y_k|x_{T,k}) = \mathcal{N} \left( y_k; \sum_{i \in T} E_{u_{X_i,k}}[\hat{X}_{i,k}] + \sum_{j \in T \setminus T_{c}} x_{j,k}, 1 + \sum_{i \in T} P_i \right)
\]
for all \( x_{T,k} \in X_{T,k} \) and \( y_k \in \mathbb{R} \). It can be seen from (100) and (101) that \( s_{W_T|X_T^n,Y^n;\hat{W}_T} \) depends on the choice of \( T \) we fixed at the start of the proof and the distribution \( u_{\hat{x}_T^n} \) in Statement (iv). We shall see later that this choice of \( s_{W_T|X_T^n,Y^n;\hat{W}_T} \), in particular the mean of the distribution in (101) namely \( \sum_{i \in T} E_{u_{\hat{x}_i}}[\hat{X}_{i,k}] + \sum_{j \in T \setminus T_{c}} x_{j,k} \), combined with Proposition 5 and Lemma 4 enables us to prove (91). We do not index \( s_{W_T|X_T^n,Y^n;\hat{W}_T} \) by \( T \) nor \( u_{\hat{x}_T^n} \) for notational brevity. To simplify notation, let \( \hat{\gamma} \triangleq (1 + \gamma)/2 \) be the maximal probability of decoding error of the \((n, M_T^{(n)}, P_T, A, T, \frac{1+\gamma}{2})_{\text{max}}\)-code, where \( \hat{\gamma} < 1 \) because \( \gamma < 1 \). Then for each \( w_T \in \mathcal{A} \), since
\[
s_{W_T}(w_T) \overset{(100)}{=} p_{W_T}(w_T) \overset{(94)}{>} 0,
\]
it follows from Proposition 5 and Definition 1 with the identifications \( U \equiv W_T, V \equiv \hat{W}_T, p_{U,V} \equiv p_{W_T,\hat{W}_T|W_T \neq \hat{W}_T}, q_{V} \equiv s_{\hat{W}_T|W_T = \hat{W}_T} \) and \( \alpha \equiv \max_{w_T \in \mathcal{A}} \Pr\{\hat{W}_T \neq W_T|W_T = w_T\} \leq \hat{\gamma} \) that
\[
\begin{align*}
\beta_{1-\hat{\gamma}}(p_{W_T|W_T = w_T} \| s_{W_T|W_T = w_T}) &\leq \beta_{1-\alpha}(p_{W_T|W_T = w_T} \| s_{\hat{W}_T|W_T = \hat{W}_T}) \\
&\leq s_{\hat{W}_T|W_T}(w_T) \| w_T
\end{align*}
\]
(C. Using the DPI to Introduce the Channel Inputs and Output

Consider the following chain of inequalities for each \( w_T \in \mathcal{A} \):
\[
\begin{align*}
\beta_{1-\hat{\gamma}}(p_{\hat{W}_T|W_T = w_T} \| s_{\hat{W}_T|W_T = \hat{W}_T}) &\overset{(a)}{\geq} \beta_{1-\hat{\gamma}}(p_{Y^n,W_T|W_T = w_T} \| s_{Y^n,\hat{W}_T|W_T = \hat{W}_T}) \\
&= \beta_{1-\hat{\gamma}}(p_{Y^n|W_T = w_T, \hat{W}_T} p_{\hat{W}_T|Y^n,W_T = w_T} \| s_{Y^n,\hat{W}_T|W_T = \hat{W}_T}) \\
&\overset{(b)}{=} \beta_{1-\hat{\gamma}}(p_{Y^n|W_T = w_T, \hat{W}_T} \| s_{Y^n,W_T} \| s_{\hat{W}_T|W_T}) \\
&\overset{(c)}{\geq} \beta_{1-\hat{\gamma}}(p_{W_T|Y^n,P_X^n,Y^n|W_T = w_T} \| p_{X_T^n|X_T^n,W_T = w_T} s_{X_T^n,Y^n,W_T} \| s_{\hat{W}_T|W_T})
\end{align*}
\]
\[ (100) \quad \beta_{1-\gamma} \left( p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \middle| p_{\hat{W}_Z}[Y^n|X^n_{T^c}, W_Z = w_Z] \right) \equiv p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \prod_{k=1}^{n} s_y_k[X_{T^c,k}] \]  
\[ (109) \quad \beta_{1-\gamma} \left( p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \middle| p_{\hat{W}_Z}[Y^n|X^n_{T^c}, W_Z = w_Z] \right) \equiv p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \prod_{k=1}^{n} s_y_k[X_{T^c,k}] \]  
\[ (110) \quad \beta_{1-\gamma} \left( p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \middle| p_{\hat{W}_Z}[Y^n|X^n_{T^c}, W_Z = w_Z] \right) \equiv p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \prod_{k=1}^{n} s_y_k[X_{T^c,k}] \]  
\[ (111) \quad \beta_{1-\gamma} \left( p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \middle| p_{\hat{W}_Z}[Y^n|X^n_{T^c}, W_Z = w_Z] \right) \equiv p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \prod_{k=1}^{n} s_y_k[X_{T^c,k}] \]  
\[ (112) \quad \beta_{1-\gamma} \left( p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \middle| p_{\hat{W}_Z}[Y^n|X^n_{T^c}, W_Z = w_Z] \right) \equiv p_{\hat{W}_Z}[Y^n|X^n_{T^c}, Y^n | W_Z = w_Z] \prod_{k=1}^{n} s_y_k[X_{T^c,k}] \]  
where
(a) follows from the DPI of $\beta_{1-\gamma}$ by introducing the channel output $Y^n$.
(b) follows from the fact that $W_T \rightarrow Y^n \rightarrow \hat{W}_T$ forms a Markov chain under the distribution $p_{W_Z,Y^n,W_Z}$.
(c) follows from the DPI of $\beta_{1-\gamma}$ by introducing the channel input $X^n_{T}$.
(d) follows from Definition 1, which says $X^n_{T^c}$ is a function of $W_T$.

D. Relaxation via Chebyshev’s Inequality

Following (112), we consider
\[ P_{X^n_{T^c},Y^n,W_Z[W_Z = w_Z]} = P_{X^n_{T^c},Y^n,W_Z[W_Z = w_Z]} \prod_{k=1}^{n} p_{Y_k}[X_{T^c,k}] \]
and we obtain from Lemma 4 and (114) that for each $w_T \in A$ and each $\xi_{w_T} > 0$,
\[ \beta_{1-\gamma} \left( P_{X^n_{T^c},Y^n,W_Z[W_Z = w_Z]} \prod_{k=1}^{n} p_{Y_k}[X_{T^c,k}] \right) \geq \frac{1}{\xi_{w_T}} \left( 1 - \gamma - \text{Pr}_{p_{X^n_{T^c},Y^n,W_Z[W_Z = w_Z]} = 1} \left( \prod_{k=1}^{n} \frac{p_{Y_k}[X_{T^c,k}] (Y_k[X_{T^c,k}])}{s_y_k[X_{T^c,k}] (Y_k[X_{T^c,k}])} \geq \xi_{w_T} \right) \right) . \]  
\[ \text{(115)} \]
Combining (104), (112) and (115), we obtain for each $w_T \in A$ and each $\xi_{w_T} > 0$
\[ s_{\hat{W}_T}[W_{T^c}(w_T|w_{T^c})] \geq \frac{1}{\xi_{w_T}} \left( 1 - \gamma - \text{Pr}_{p_{X^n_{T^c},Y^n,W_Z[W_Z = w_Z]} = 1} \left( \prod_{k=1}^{n} \frac{p_{Y_k}[X_{T^c,k}] (Y_k[X_{T^c,k}])}{s_y_k[X_{T^c,k}] (Y_k[X_{T^c,k}])} \geq \xi_{w_T} \right) \right) , \]  
\[ \text{(116)} \]
which implies that
\[ \log \left( \frac{1}{s_{\hat{W}_T}[W_{T^c}(w_T|w_{T^c})]} \right) \leq \log \xi_{w_T} - \log \left( 1 - \gamma - \text{Pr}_{p_{X^n_{T^c},Y^n,W_Z[W_Z = w_Z]} = 1} \left( \sum_{k=1}^{n} \log \left( \frac{p_{Y_k}[X_{T^c,k}] (Y_k[X_{T^c,k}])}{s_y_k[X_{T^c,k}] (Y_k[X_{T^c,k}])} \right) \geq \log \xi_{w_T} \right) \right) . \]  
\[ \text{(117)} \]
For each $w_T \in A$, let
\[ \log \xi_{w_T} \triangleq \mathbb{E}_{p_{X^n_{T^c},Y^n,W_Z[W_Z = w_T]}} \left[ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k}[X_{T^c,k}] (Y_k[X_{T^c,k}])}{s_y_k[X_{T^c,k}] (Y_k[X_{T^c,k}])} \right) \right] \]
\[ + \sqrt{\frac{2}{1 - \gamma}} \text{Var}_{p_{X^n_{T^c},Y^n,W_Z[W_Z = w_T]}} \left[ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k}[X_{T^c,k}] (Y_k[X_{T^c,k}])}{s_y_k[X_{T^c,k}] (Y_k[X_{T^c,k}])} \right) \right] . \]  
\[ \text{(118)} \]
Using Chebyshev’s inequality, it follows from (118) that for each $w_I \in \mathcal{A}$

$$\Pr_{p_{x_T,Y^n|w_T=w_I}} \left\{ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k|X_{I,k}}(Y_k|X_{I,k})}{s_{Y_k|X_{I,k}}(Y_k|X_{I,k})} \right) \geq \log \xi_{w_I} \right\} \leq \frac{1 - \tilde{\gamma}}{2},$$

(119)

which implies from (117) that

$$\log \left( \frac{1}{s_{W_T|W_{T^c}|w_T}(w_T|w_{T^c})} \right) \leq \log \xi_{w_I} + \log \left( \frac{2}{1 - \gamma} \right).$$

(120)

Since $t \mapsto \log \frac{1}{t}$ is convex for $t > 0$, by Jensen’s inequality

$$\sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \log \left( \frac{1}{s_{W_T|W_{T^c}}(w_T|w_{T^c})} \right) \geq \log \left( \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) s_{W_T|W_{T^c}}(w_T|w_{T^c}) \right).$$

(121)

We have

$$\sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) s_{W_T|W_{T^c}}(w_T|w_{T^c}) \overset{(94)}{=} \frac{1}{|\mathcal{A}|} \sum_{w_T \in \mathcal{A}} s_{W_T|W_{T^c}}(w_T|w_{T^c})$$

$$\overset{(a)}{=} \frac{1}{|\mathcal{A}|} \sum_{w_T \in \mathcal{A}} s_{W_T|W_{T^c}}(w_T|w_{T^c})$$

$$\overset{(120)}{=} \log |\mathcal{A}|.$$  

(122)

Taking expectation with respect to $p_{W_T}$ on both sides of (120) and applying (126), we obtain

$$\log |\mathcal{A}| \leq \left( \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \log \xi_{w_I} \right).$$

(127)

### E. Simplification of Log-Likelihood Terms

In order to simplify (127), we will simplify the log-likelihood term in $\log \xi_{w_I}$ defined in (118). To this end, we first let $x^n_i(w_i) \triangleq f_i(w_i)$ ($f_i$ is the encoding function at node $i$ defined in Definition 1) and we also let $x_{i,k}(w_i)$ denote the $k^{th}$ component of $x^n_i(w_i)$ for each $i \in \mathcal{I}$ and each $k \in \{1, 2, \ldots, n\}$ such that

$$x^n_i(w_i) = (x_{i,1}(w_i), x_{i,2}(w_i), \ldots, x_{i,n}(w_i)).$$

(128)

In addition, we let

$$x_{I,k}(w_I) \triangleq (x_{1,k}(w_1), x_{2,k}(w_2), \ldots, x_{N,k}(w_N)),$$

(129)

and we let

$$x_{T^c,k}(w_{T^c}) \triangleq (x_{j,k}(w_j) \mid j \in T^c)$$

(130)

be a subtuple of $x_{I,k}(w_I)$. Similarly, let

$$x^n_I(w_I) \triangleq (x^n_1(w_1), x^n_2(w_2), \ldots, x^n_N(w_N)),$$

(131)

and let

$$x^n_{T^c}(w_{T^c}) \triangleq (x^n_j(w_j) \mid j \in T^c)$$

(132)
be a subtuple of $x^n(w_I)$. Using the fact that $X^n_i$ is a function of $W_i$ for all $i \in I$ and the notations defined above, we obtain from (118) that

$$
\log \xi_{w_I} = \mathbb{E}_{p_{Y^n|W_I = w_I, X^n_I = x^n_I}} \left[ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right] + \sqrt{\frac{2}{1 - \gamma}} \text{Var}_{p_{Y^n|W_I = w_I, X^n_I = x^n_I}} \left[ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right],
$$

which implies from (17) that

$$
\log \xi_{w_I} = \mathbb{E}_{\prod_{k=1}^{n} p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \left[ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right] + \sqrt{\frac{2}{1 - \gamma}} \text{Var}_{\prod_{k=1}^{n} p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \left[ \sum_{k=1}^{n} \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right],
$$

which then implies that

$$
\log \xi_{w_I} = \sum_{k=1}^{n} \mathbb{E}_{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \left[ \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right] + \sqrt{\frac{2}{1 - \gamma}} \sum_{k=1}^{n} \text{Var}_{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \left[ \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right].
$$

Following (135), we use (18), (16) and (101) to obtain

$$
\log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) = \frac{1}{2} \log \left( 1 + \sum_{i \in T} P_i \right) + \frac{\log e}{2(1 + \sum_{i \in T} P_i)} \left( - \left( \sum_{i \in T} P_i \right) \left( Y_k - \sum_{i \in T} x_i,k(w_i) \right)^2 \right)
+ \left( \sum_{i \in T} (x_i,k(w_i) - E_{u_{X_i,k}}[\hat{X}_i,k]) \right) \left( Y_k - \sum_{i \in T} x_i,k(w_i) \right) + \left( \sum_{i \in T} (x_i,k(w_i) - E_{u_{X_i,k}}[\hat{X}_i,k]) \right)^2.
$$

For each $w_I \in A$ and each $k \in \{1, 2, \ldots, n\}$, it follows from Definition 2 that $Y_k - \sum_{i \in T} x_i,k(w_i)$ is a standard normal random variable if $Y_k$ is distributed according to $p_{Y_k|X_{T^c,k}=x_{T^c,k}(w_I)}$, which then implies that

$$
\mathbb{E}_{p_{Y_k|X_{T^c,k}=x_{T^c,k}(w_I)}} \left[ \log \left( \frac{p_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))}{s_{Y_k|X_{T^c,k}}(Y_k|X_{T^c,k}(w_I))} \right) \right] \equiv \frac{1}{2} \log \left( 1 + \sum_{i \in T} P_i \right) + \frac{\log e}{2(1 + \sum_{i \in T} P_i)} \left( - \left( \sum_{i \in T} P_i \right) + \left( \sum_{i \in T} (x_i,k(w_i) - E_{u_{X_i,k}}[\hat{X}_i,k]) \right)^2 \right).
$$
and
\[
\Var_{p_{X_t} | X_{t-1} = x_{t-1}(w_T)} \left[ \log \frac{p_{Y_t} | X_{t-1} = y_{t-1}(w_T)}{p_{Y_t} | X_{t-1} = y_{t-1}(w_T)} \right] \\
\overset{(136)}{=} \left( \frac{\log e}{2(1 + \sum_{i \in T} P_i)} \right)^2 \Var_{p_{X_t} | X_{t-1} = x_{t-1}(w_T)} \left[ - \left( \sum_{i \in T} P_i \right) \left( Y_k - \sum_{i \in T} x_{i,k}(w_i) \right)^2 \\
+ 2 \sum_{i \in T} (x_{i,k}(w_i) - \mathbb{E}_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \left( Y_k - \sum_{i \in T} x_{i,k}(w_i) \right) \right] \\
= \frac{\left( \sum_{i \in T} P_i \right)^2 + 2 \left( \sum_{i \in T} (x_{i,k}(w_i) - \mathbb{E}_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right)^2 (\log e)^2}{2(1 + \sum_{i \in T} P_i)^2}. \tag{139}
\]

Define
\[
|P_T| \doteq \sum_{i \in T} P_i \tag{140}
\]
and
\[
\bar{x}_{i,k}(w_i) \doteq x_{i,k}(w_i) - \mathbb{E}_{u_{X_{i,k}}} [\hat{X}_{i,k}] \tag{141}
\]
Combining (127), (135), (137), (139), (140) and (141), we obtain for each \( w_T \in \mathcal{A} \)
\[
\log |\mathcal{A}| \leq \frac{n}{2} \log (1 + |P_T|) + \frac{\sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( -n|P_T| + \sum_{k=1}^n \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \right) \log e}{2(1 + |P_T|)} \\
+ \frac{\sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \sqrt{\left( \sum_{k=1}^n \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \right) \log e}}{(1 + |P_T|) \sqrt{1 - \gamma}} + \log \left( \frac{2}{1 - \gamma} \right), \tag{142}
\]
which implies from Jensen’s inequality (\( t \mapsto \sqrt{t} \) is concave for \( t \geq 0 \)) that
\[
\log |\mathcal{A}| \leq \frac{n}{2} \log (1 + |P_T|) + \frac{\left( -n|P_T| + \sum_{k=1}^n \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \right) \log e}{2(1 + |P_T|)} \\
+ \frac{\sqrt{n|P_T|^2 + 2 \sum_{k=1}^n \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \log e}}{(1 + |P_T|) \sqrt{1 - \gamma}} + \log \left( \frac{2}{1 - \gamma} \right). \tag{143}
\]
In the following, we will obtain an upper bound on the crucial term \( \sum_{k=1}^n \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \) which appears in the second and third terms on the right-hand-side of (143).

F. Introducing the Quantized Input Distribution to Simplify the Upper Bound

Following (143), we consider for each \( k \in \{1, 2, \ldots, n\} \)
\[
\sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \\
= \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right) \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right) \tag{144}
\]
\[
= \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right) \tag{145}
\]
\[
\leq \sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2. \tag{146}
\]
Since $X_i^n$ is a function of $W_i$ for each $i \in T$, it follows from (141) that for each $k \in \{1, 2, \ldots, n\}$

$$
\sum_{w_T \in W_T} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 = \sum_{x_{T,k} \in X_T} p_{X_{T,k}}(x_{T,k}) \left( \sum_{i \in T} \left( x_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2,
$$

which implies from (146) that

$$
\sum_{w_T \in \mathcal{A}} p_{W_T}(w_T) \left( \sum_{i \in T} \bar{x}_{i,k}(w_i) \right)^2 \leq \sum_{x_{T,k} \in X_T} p_{X_{T,k}}(x_{T,k}) \left( \sum_{i \in T} \left( x_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2.
$$

Recalling the definition of $\hat{X}_T^n$ and $\hat{X}_{T,k}$ in (96) and (97) respectively, we write for each $k \in \{1, 2, \ldots, n\}$

$$
\sum_{x_{T,k} \in X_T} p_{X_{T,k}}(x_{T,k}) \left( \sum_{i \in T} \left( x_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2
$$

$$
= \sum_{x_{T,k}, \hat{x}_{T,k} \in X_T} p_{X_{T,k},\hat{X}_{T,k}}(x_{T,k}, \hat{x}_{T,k}) \left( \sum_{i \in T} \left( x_{i,k} - \hat{x}_{i,k} + \hat{x}_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2
$$

$$
= \sum_{x_{T,k}, \hat{x}_{T,k} \in X_T} p_{X_{T,k},\hat{X}_{T,k}}(x_{T,k}, \hat{x}_{T,k}) \left( \sum_{i \in T} \left( x_{i,k} - \hat{x}_{i,k} \right) \right)^2
$$

$$
+ 2 \sum_{x_{T,k}, \hat{x}_{T,k} \in X_T} p_{X_{T,k},\hat{X}_{T,k}}(x_{T,k}, \hat{x}_{T,k}) \left( \sum_{i \in T} \left( x_{i,k} - \hat{x}_{i,k} \right) \right) \left( \sum_{i \in T} \left( \hat{x}_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)
$$

$$
+ \sum_{\hat{x}_{T,k} \in \hat{X}_T} p_{\hat{X}_{T,k}}(\hat{x}_{T,k}) \left( \sum_{i \in T} \left( \hat{x}_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2.
$$

$$
\leq \sum_{x_{T,k}, \hat{x}_{T,k} \in X_T} p_{X_{T,k},\hat{X}_{T,k}}(x_{T,k}, \hat{x}_{T,k}) \sum_{i \in T} \left| x_{i,k} - \hat{x}_{i,k} \right|^2
$$

$$
+ 2 \sum_{x_{T,k}, \hat{x}_{T,k} \in X_T} p_{X_{T,k},\hat{X}_{T,k}}(x_{T,k}, \hat{x}_{T,k}) \sum_{i \in T} \left( x_{i,k} - \hat{x}_{i,k} \right) \sum_{i \in T} \left( \hat{x}_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right)
$$

$$
+ \sum_{\hat{x}_{T,k} \in \hat{X}_T} p_{\hat{X}_{T,k}}(\hat{x}_{T,k}) \left( \sum_{i \in T} \left( \hat{x}_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2
$$

$$
\leq \frac{|T|^2}{n^2} + \frac{4|T|}{\sqrt{n}} \left( \sum_{i \in T} \sqrt{P_i} \right) + \sum_{\hat{x}_{T,k} \in \hat{X}_T} p_{\hat{X}_{T,k}}(\hat{x}_{T,k}) \left( \sum_{i \in T} \left( \hat{x}_{i,k} - E_{u_{X_{i,k}}}[\hat{X}_{i,k}] \right) \right)^2.
$$
where (a) follows from the facts below for each \( i \in T \), each \( k \in \{1, 2, \ldots, n\} \) and each \( x_{i,k} \in X_i \) (recall the definition of \( \hat{x}_{i,k} \) in (96)):

\[
|x_{i,k} - \hat{x}_{i,k}| \leq \frac{1}{n} \tag{154}
\]

and

\[
|\hat{x}_{i,k}| \leq |x_{i,k}| \leq \sqrt{nP_i}. \tag{155}
\]

**G. Approximating the Quantized Input Distribution by a Product Distribution**

In order to bound the last term in (153), we use the bound in (98) for bounding \( p_{\hat{X}_{T,k}}(\hat{x}_{T,k}) \) in terms of \( u_{\hat{X}_{T,k}}(\hat{x}_{T,k}) \) to obtain

\[
\sum_{\hat{x}_{T,k} \in \hat{X}_T} p_{\hat{X}_{T,k}}(\hat{x}_{T,k}) \left( \sum_{i \in T} \left( \hat{x}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}] \right) \right)^2 \\
\leq \sum_{\hat{x}_{T,k} \in \hat{X}_T} \left( \left( 1 + \frac{\log n}{n} \right) \prod_{i \in T} u_{X_{i,k}}(\hat{x}_{i,k}) + \frac{1}{n^4|T|} \right) \left( \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \tag{156}
\]

\[
= \sum_{\hat{x}_{T,k} \in \hat{X}_T} \left[ \left( 1 + \frac{\log n}{n} \right) \prod_{i \in T} u_{X_{i,k}}(\hat{x}_{i,k}) \left( \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \\
+ \frac{1}{n^4|T|} \left( \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \right] \tag{157}
\]

for each \( k \in \{1, 2, \ldots, n\} \). Note that the size of the quantized alphabet \( \hat{X}_T \) satisfies

\[
|\hat{X}_T| \leq \prod_{i \in T} \left( 2 \left[ n\sqrt{nP_i} \right] + 1 \right), \tag{158}
\]

by (97) and the construction of \( Z_{L,\Delta} \) in (35) in Definition 6.

The bound in (157) consists of two distinct terms which we now bound separately. Consider the following two chains of inequalities for each \( k \in \{1, 2, \ldots, n\} \):

\[
\sum_{\hat{x}_{T,k} \in \hat{X}_T} \left( \prod_{i \in T} u_{X_{i,k}}(\hat{x}_{i,k}) \right) \left( \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \\
= E_{\prod_{i \in T} u_{X_{i,k}}} \left[ \left( \sum_{i \in T} (\hat{X}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \right] \tag{159}
\]

\[
= \sum_{i \in T} \left[ E_{u_{X_{i,k}}} \left( (\hat{X}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}])^2 \right) + 2 \sum_{j \in T, j < i} E_{u_{X_{j,k}}} E_{u_{X_{i,k}}} \left[ (\hat{X}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) (\hat{X}_{j,k} - E_{u_{X_{j,k}}} [\hat{X}_{j,k}]) \right] \right] \tag{160}
\]

\[
= \sum_{i \in T} \left[ E_{u_{X_{i,k}}} \left( (\hat{X}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}])^2 \right) \\
+ 2 \sum_{j \in T, j < i} E_{u_{X_{j,k}}} \left( (\hat{X}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}]) \right) E_{u_{X_{j,k}}} \left( (\hat{X}_{j,k} - E_{u_{X_{j,k}}} [\hat{X}_{j,k}]) \right) \right] \tag{161}
\]

\[
= \sum_{i \in T} E_{u_{X_{i,k}}} \left( (\hat{X}_{i,k} - E_{u_{X_{i,k}}} [\hat{X}_{i,k}])^2 \right) \tag{162}
\]

\[
\leq \sum_{i \in T} E_{u_{X_{i,k}}} \left( \hat{X}_{i,k}^2 \right) \tag{163}
\]
and

\[ \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} \left( \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{x_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \]

\leq \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} \left( |T| \max_{i \in T} \left\{ |\hat{x}_{i,k} - E_{u_{x_{i,k}}} [\hat{X}_{i,k}]| \right\} \right)^2 \tag{164}

= |T|^2 \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} \max_{i \in T} \left\{ (\hat{x}_{i,k} - E_{u_{x_{i,k}}} [\hat{X}_{i,k}])^2 \right\} \tag{165}

\leq |T|^2 \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{x_{i,k}}} [\hat{X}_{i,k}])^2 \tag{166}

\leq 2|T|^2 \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} \sum_{i \in T} (\hat{x}_{i,k}^2 + (E_{u_{x_{i,k}}} [\hat{X}_{i,k}])^2) \tag{167}

\leq 2|T|^2 \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} \sum_{i \in T} 2nP_i \tag{168}

\leq 4n|T|^2 |P_T| \left( \prod_{i \in T} \left( 2 \left\lceil n \sqrt{nP_i} \right\rceil + 1 \right) \right) \tag{169}

< 4n|T|^2 |P_T| \left( \prod_{i \in T} \left( 2n \sqrt{nP_i} + 3 \sqrt{n} \right) \right) \tag{170}

= 4n^{3|T|^2 - 1} |T|^2 |P_T| \prod_{i \in T} \left( 2\sqrt{P_i} + 3 \right) \tag{171}

< 4n^{3|T|^2} |T|^2 |P_T| \prod_{i \in T} \left( 2\sqrt{P_i} + 3 \right), \tag{172}

where

(a) follows from the fact that \((a - b)^2 \leq 2a^2 + 2b^2\) for all real numbers \(a\) and \(b\).

(b) follows from the definition of \(|P_T|\) in (140) and the upper bound on the size of the quantized alphabet in (158).

Combining (157), (163) and (172), we obtain for each \(k \in \{1, 2, \ldots, n\}\)

\[ \sum_{\hat{x}_{T,k} \in \mathcal{X}_T} p_{\hat{x}_{T,k}} (\hat{x}_{T,k}) \left( \sum_{i \in T} (\hat{x}_{i,k} - E_{u_{x_{i,k}}} [\hat{X}_{i,k}]) \right)^2 \]

\leq \left( 1 + \sqrt{\frac{\log n}{n}} \right) \sum_{i \in T} E_{u_{x_i}} \left\lceil \hat{X}_{i,k}^2 \right\rceil + 4n^{-|T|} |T|^2 |P_T| \prod_{i \in T} \left( 2\sqrt{P_i} + 3 \right), \tag{173}

which implies from (148) and (153) that

\[ \sum_{w_i \in A} p_{W \gamma} (w_i) \left( \sum_{i \in T} \hat{x}_{i,k} (w_i) \right)^2 \]

\leq \frac{|T|^2}{n^2} + \frac{4|T|}{\sqrt{n}} \left( \sum_{i \in T} \sqrt{P_i} \right) + \left( 1 + \sqrt{\frac{\log n}{n}} \right) \sum_{i \in T} E_{u_{x_i}} \left\lceil \hat{X}_{i,k}^2 \right\rceil + 4n^{-|T|} |T|^2 |P_T| \prod_{i \in T} \left( 2\sqrt{P_i} + 3 \right). \tag{174}
Using (174) and (52) and recalling that \(|T| \geq 1\) (because \(T\) is non-empty), we obtain
\[
\sum_{k=1}^{n} \sum_{w \in \mathcal{A}} R_{k}(w) \left( \sum_{i \in T} \bar{x}_{i,k}(w) \right)^2 
\leq n|P_T| + \sqrt{n \log n} |P_T| + 4 \sqrt{n} |T| \left( \sum_{i \in T} \sqrt{P_i} \right) + 4 |P_T|^{2} \prod_{i \in T} (2 \sqrt{P_i} + 3) + \frac{|T|^2}{n}. \tag{175}
\]
To simplify notation, let

\[
\kappa_1 \triangleq 4 |T| \left( \sum_{i \in T} \sqrt{P_i} \right) \quad \text{and} \quad \kappa_2 \triangleq 4 |P_T|^{2} \prod_{i \in T} (2 \sqrt{P_i} + 3)
\]

be two constants that are independent of \(n\). Then, we combine (143) and (175) to yield
\[
\log |\mathcal{A}| \leq \frac{n}{2} \log (1 + |P_T|) + \frac{\sqrt{n \log n} |P_T| + \sqrt{n} \kappa_1 + \kappa_2 + n^{-1} |T|^2 \log e}{2(1 + |P_T|)}
+ \frac{\sqrt{n} |P_T|^{2} \prod_{i \in T} (2 \sqrt{P_i} + 3)}{(1 + |P_T|) \sqrt{1 - \gamma}} + \log \left( \frac{2}{1 - \gamma} \right). \tag{177}
\]
Combining (95) and (177), we obtain
\[
\left( -4 |T| (1 + 3 \bar{\gamma}) \right) \sqrt{n \log n} + \log \left( \frac{1 - \bar{\gamma}}{2 (1 + \bar{\gamma})} \right) \quad \text{and} \quad \sum_{i \in T} \log M_i^{(n)}
\leq \frac{n}{2} \log (1 + |P_T|) + \frac{\sqrt{n \log n} |P_T| + \sqrt{n} \kappa_1 + \kappa_2 + n^{-1} |T|^2 \log e}{2(1 + |P_T|)}
+ \frac{\sqrt{n} |P_T|^{2} \prod_{i \in T} (2 \sqrt{P_i} + 3)}{(1 + |P_T|) \sqrt{1 - \gamma}} + \log \left( \frac{2}{1 - \gamma} \right). \tag{178}
\]
Dividing both sides of (178) by \(n\) and taking limit inferior as \(n\) goes to infinity, we obtain from (90) that (91) holds as desired. This completes the proof of Theorem 1.

**H. Discussion on the Appropriate Choice of the Quantizer’s Precision**

We would now like to emphasize the importance of the appropriate choice of the quantizer’s precision. Our choice for the quantizer’s precision is \(1/n\) as shown in (97). If the precision is made too small (which decays much faster than \(1/n\)), then the resultant upper bound on \(|\mathcal{V}_T|\) in (158) as well as the upper bound obtained at step (b) in the chain of inequalities leading to (172) would become too loose. On the other hand, if the precision is chosen to be too large (which decays much slower than \(1/n\)), the resultant upper bound on \(|x_{i,k} - \hat{x}_{i,k}|\) in (154) as well as the upper bound obtained at step (a) in the chain of inequalities leading to (153) would become too loose.

**VI. INTERFERENCE CHANNEL UNDER STRONG INTERFERENCE REGIME**

The capacity region of a two-sender two-receiver Gaussian interference channel (IC) under strong interference was derived by Han and Kobayashi [25] and Sato [26]. Let \(P_1, P_2\) be the received signal-to-noise ratios and let \(I_1, I_2\) be the received interference-to-noise ratios [1, Section 6.4]. Under the formulation of the Gaussian IC under strong interference, it is assumed that \(I_2 \geq P_1\) and \(I_1 \geq P_2\). Under this condition, the capacity region was shown in [25, Theorem 5.2] to be the set of all rate pairs \((R_1, R_2)\) belonging to
\[
\mathcal{R}_{\text{IKS}} \triangleq \left\{ (R_1, R_2) \in \mathbb{R}_+^2 \mid \begin{array}{c}
R_1 \leq \frac{1}{2} \log (1 + P_1), \\
R_2 \leq \frac{1}{2} \log (1 + P_2), \\
R_1 + R_2 \leq \min \left\{ \frac{1}{2} \log (1 + P_1 + I_1), \frac{1}{2} \log (1 + P_2 + I_2) \right\} \end{array} \right\}. \tag{179}
\]
By applying Theorem 1 to each of the decoders of a two-sender two-receiver Gaussian IC, we can derive that the corresponding \((\varepsilon_1, \varepsilon_2)\)-capacity region \(\mathcal{C}_{\varepsilon_1, \varepsilon_2}\) is outer bounded as
\[
\mathcal{C}_{\varepsilon_1, \varepsilon_2} \subseteq \mathcal{R}_{\text{IKS}} \tag{180}
\]
as long as $\varepsilon_1 + \varepsilon_2 < 1$, where $\varepsilon_i$ characterizes the asymptotic average probability of decoding error of message $i$ at decoder $i$. Since the rate pairs in $R_{\text{HK,s}}$ are $(0,0)$-achievable via simultaneous non-unique decoding [1, Section 6.4], we have

$$C_{\varepsilon_1, \varepsilon_2} = R_{\text{HK,s}}$$

(181)
as long as $\varepsilon_1 + \varepsilon_2 < 1$. The strong converse (in fact, the complete second-order asymptotics) for the Gaussian IC under the more restrictive condition of strictly very strong interference was shown by Le, Tan, and Motani [27]. In the rest of this section, we will provide the formulation of the Gaussian IC under strong interference and present the corresponding strong converse result. The proof of the strong converse result will be presented in Section VI-B.

A. Problem Formulation and Main Result

We follow the standard setting of the Gaussian IC under strong interference as given in [25, Section V]. The Gaussian IC under strong interference consists of two senders, denoted by $s_1$ and $s_2$ respectively, and two receivers, denoted by $d_1$ and $d_2$ respectively. For each $i \in \{1, 2\}$, $s_i$ chooses a message $W_i$ and transmits $X_{i}^n$ in $n$ time slots, and $d_i$ receives $Y_{i}^n$ in $n$ time slots and declares $\hat{W}_i$ to be the transmitted $W_i$. The channel law in each time slot $k$ is

$$
\begin{bmatrix}
Y_{1,k} \\
Y_{2,k}
\end{bmatrix} =
\begin{bmatrix}
1 & g_{12} \\
g_{21} & 1
\end{bmatrix}
\begin{bmatrix}
X_{1,k} \\
X_{2,k}
\end{bmatrix} +
\begin{bmatrix}
Z_{1,k} \\
Z_{2,k}
\end{bmatrix},
$$

(182)
where $g_{12}$ and $g_{21}$ are two real constants characterizing the channel gains of the interference links, and \(\{(Z_{1,k}, Z_{2,k})\}_{k=1}^n\) are $n$ copies of some joint Gaussian random variable denoted by $(Z_1, Z_2)$ (need not be independent) such that

$$E[Z_1] = E[Z_2] = 0$$

(183)
and

$$E[Z_1^2] = E[Z_2^2] = 1.$$ (184)

For each $i \in \{1, 2\}$, the codewords transmitted by $s_i$ should satisfy the peak power constraint

$$\Pr\{\|X_i^n\|^2 \leq nP_i\} = 1$$

(185)
for some $P_i > 0$. We assume that the IC is under strong interference, i.e., $g_{12}^2 \geq 1$ and $g_{21}^2 \geq 1$, which implies that

$$I_1 \triangleq g_{12}^2 P_2 \geq P_2$$

(186)
and

$$I_2 \triangleq g_{21}^2 P_1 \geq P_1,$$ (187)
where $I_1$ and $I_2$ characterize the interference power received at $d_1$ and $d_2$ respectively (cf. (182)). The Gaussian IC can be characterized by some $q_{Y_1,Y_2|X_1,X_2}$ and we define the Gaussian IC as in Definition 2 such that (182), (183) and (184) hold. In addition, we define a length-$n$ code for the Gaussian IC as follows.

**Definition 8:** An $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2)$-code for the Gaussian IC consists of the following:

1) A message set $W_i \triangleq \{1, 2, \ldots, M_i^{(n)}\}$ at node $i$ for each $i \in \{1, 2\}$, where $W_i$ is uniform on $W_i$.
2) An encoding function $f_i : W_i \rightarrow \mathbb{R}^n$ for each $i \in \{1, 2\}$, where $f_i$ is the encoding function at node $i$ such that $X_i^n = f_i(W_i)$ and $\|f_i(w_i)\|^2 \leq nP_i$ for all $w_i \in W_i$.
3) A decoding function $\varphi_i : \mathbb{R}^n \rightarrow W_i$ for each $i \in \{1, 2\}$, where $\varphi_i$ is the decoding function for $W_i$ at node $d_i$ such that $\hat{W}_i = \varphi_i(Y_i^n)$. We allow $\varphi_1$ and $\varphi_2$ to be stochastic.

We define an $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}$ code as follows.

**Definition 9:** For an $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2)$-code defined on the Gaussian IC, the average probability of decoding error for $W_i$ for each $i \in \{1, 2\}$ is defined as

$$\Pr\{\hat{W}_i \neq W_i\}.$$ (188)
An \((n, M_1(n), M_2(n), P_1, P_2)\)-code with average probabilities of decoding error for \(W_1\) and \(W_2\) no larger than \(\varepsilon_1\) and \(\varepsilon_2\) respectively is called an \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}\)-code.

For each \(\varepsilon_1 \in [0, 1)\) and each \(\varepsilon_2 \in [0, 1)\), we define an \((\varepsilon_1, \varepsilon_2)\)-achievable rate pair as in Definition 4, and we define the \((\varepsilon_1, \varepsilon_2)\)-capacity region, denoted by \(C_{\varepsilon_1, \varepsilon_2}\), to be the set of \((\varepsilon_1, \varepsilon_2)\)-achievable rate pairs. The following theorem is the main result in this section.

**Theorem 2:** For each \(\varepsilon_1 \in [0, 1)\) and each \(\varepsilon_2 \in [0, 1)\) such that \(\varepsilon_1 + \varepsilon_2 < 1\),

\[
C_{\varepsilon_1, \varepsilon_2} = R_{\text{HK,S}}.
\]

### B. Proof of Theorem 2

We need the following definitions and lemma before presenting the proof of Theorem 2. The definition below differs from Definition 8 in the decoding functions only, but we state the whole definition for clarity.

**Definition 10:** An \((n, M_1(n), M_2(n), P_1, P_2)\)-multicast code for the Gaussian IC consists of the following:

1) A message set \(W_i \triangleq \{1, 2, \ldots, M_i(n)\}\) at node \(i\) for each \(i \in \{1, 2\}\), where \(W_i\) is uniform on \(W_i\).
2) An encoding function \(f_i : W_i \rightarrow \mathbb{R}^n\) for each \(i \in \{1, 2\}\), where \(f_i\) is the encoding function at node \(i\) such that \(X_i^n = f_i(W_i)\) and \(||f_i(w_i)||^2 \leq nP_i\) for all \(w_i \in W_i\).
3) A decoding function \(\varphi_i : \mathbb{R}^n \rightarrow W_1 \times W_2\) for each \(i \in \{1, 2\}\), where \(\varphi_i\) is the decoding function for both \(W_1\) and \(W_2\) at node \(i\); such that \((\hat{W}_1, \hat{W}_2, d_i) = \varphi_i(Y_i^n)\) is the pair of message estimates at \(d_i\). We allow \(\varphi_1\) and \(\varphi_2\) to be stochastic.

We define an \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}\)-multicast code as follows.

**Definition 11:** For an \((n, M_1(n), M_2(n), P_1, P_2)\)-multicast code defined on the Gaussian IC, the **average probability of decoding error at** \(d_i\) for each \(i \in \{1, 2\}\) is defined as

\[
\text{Pr}\left\{ \hat{W}_1, d_i \neq W_1 \right\} \cup \left\{ \hat{W}_2, d_i \neq W_2 \right\}.
\]

An \((n, M_1(n), M_2(n), P_1, P_2)\)-multicast code with average probabilities of decoding error no larger than \(\varepsilon_1\) and \(\varepsilon_2\) respectively is called an \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}\)-code.

The following lemma plays a crucial role in extending our strong converse result for the Gaussian MAC to the Gaussian IC under strong interference, because it relates the error probabilities for codes defined on the Gaussian IC in Definition 9 to the error probabilities for multicast-codes defined in Definition 11 where the latter error probabilities can be viewed as error probabilities for codes defined on the Gaussian MAC in Definition 3.

**Lemma 6:** For each \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}\)-code, there exists an \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2)_{\text{avg}}\)-multicast code.

**Proof:** Suppose we are given an \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}\)-code whose encoding and stochastic decoding functions are denoted by \(\{f_1, f_2\}\) and \(\{\varphi_1, \varphi_2\}\) respectively (cf. Definition 8). Let \(p_{W_1, W_2, X_1^n, X_2^n, Y_1^n, Y_2^n, Z_1^n, Z_2^n}\) be the probability distribution induced by the \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}\)-code. By Definition 9, we have for each \(i \in \{1, 2\}\)

\[
\text{Pr}_{p_{W_1, Y_i^n}} \{ \varphi_i(Y_i^n) \neq W_i \} \leq \varepsilon_i,
\]

which implies from (182) that

\[
\text{Pr}_{p_{W_1, W_2, Z_1^n}} \{ \varphi_1(f_1(W_1) + g_{12}f_2(W_2) + Z_1^n) \neq W_1 \} \leq \varepsilon_1
\]

and

\[
\text{Pr}_{p_{W_1, W_2, Z_2^n}} \{ \varphi_2(g_{21}f_1(W_1) + f_2(W_2) + Z_2^n) \neq W_2 \} \leq \varepsilon_2.
\]

In the rest of the proof, we would like to construct additional stochastic decoding functions at \(d_1\) and \(d_2\), denoted by \(\varphi_1'\) and \(\varphi_2'\) respectively, such that \((\varphi_1, \varphi_1')\) and \((\varphi_2, \varphi_2')\) can be viewed as the stochastic decoding functions of an \((n, M_1(n), M_2(n), P_1, P_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2)_{\text{avg}}\)-multicast code. To this end, we first define \(\tilde{Z}_1^n\) and \(\tilde{Z}_2^n\) to be \(n\)
independent copies of the standard normal random variable such that $\tilde{Z}_1^n, \tilde{Z}_2^n$ and $(X_1^n, X_2^n, Y_1^n, Y_2^n, Z_1^n, \tilde{Z}_2^n)$ are independent. In addition, there exist $w_1^* \in \mathcal{W}_1$ and $w_2^* \in \mathcal{W}_2$ such that

$$\Pr_{pw_1, w_2, Z^n} \{|\varphi_1(Y_2^n) \neq W_2|W_1 = w_1^*\} = \arg \min_{w_1 \in \mathcal{W}_1} \Pr_{pw_1, w_2, Z^n} \{|\varphi_1(Y_2^n) \neq W_2|W_1 = w_1\}$$

(194)

and

$$\Pr_{pw_1, w_2, Z^n} \{|\varphi_1(Y_1^n) \neq W_1|W_2 = w_2^*\} = \arg \min_{w_2 \in \mathcal{W}_2} \Pr_{pw_1, w_2, Z^n} \{|\varphi_1(Y_1^n) \neq W_1|W_2 = w_2\},$$

(195)

which implies from (192) and (193) that

$$\Pr_{pw_1, Z^n} \{|\varphi_1(g_1 f_2(w_1^*) + f_2(W_2) + Z^n_2) \neq W_2\} \leq \varepsilon_2$$

(196)

and

$$\Pr_{pw_1, Z^n} \{|\varphi_1(f_1(W_1) + g_1 f_2(w_2^*) + Z^n_1) \neq W_1\} \leq \varepsilon_1.$$ 

(197)

Then, we define the stochastic decoders

$$\varphi'_1(Y_1^n) \triangleq \varphi_2 \left( g_2 f_1(w_1^*) + \frac{Y_1^n - f_1(\varphi_1(Y_1^n))}{g_{12}} + \sqrt{1 - \frac{1}{g_{12}^2}} \tilde{Z}_2^n \right)$$

(198)

and

$$\varphi'_2(Y_2^n) \triangleq \varphi_1 \left( \frac{Y_2^n - f_2(\varphi_2(Y_2^n))}{g_{21}} + g_{12} f_2(w_2^*) + \sqrt{1 - \frac{1}{g_{21}^2}} \tilde{Z}_1^n \right),$$

(199)

where the randomness properties of the stochastic functions originate from not only $\varphi_1$ and $\varphi_2$ but also $\tilde{Z}_1^n$ and $\tilde{Z}_2^n$. Since

$$g_{21} f_1(w_1^*) + f_2(W_2) + Z^n_2$$

(200)

and

$$g_{21} f_1(w_1^*) + \frac{Y_1^n - f_1(W_1)}{g_{12}} + \sqrt{1 - \frac{1}{g_{12}^2}} \tilde{Z}_2^n$$

(201)

have the same distribution by (182), it follows from (196) and (198) that

$$\Pr_{pw_1, w_2, Z^n, \tilde{Z}_2^n} \left\{ \{|\varphi'_1(Y_1^n) \neq W_2\} \cap \{|\varphi_1(Y_1^n) = W_1\} \right\}$$

$$\leq \Pr_{pw_1, w_2, Z^n, \tilde{Z}_2^n} \left\{ \varphi_2 \left( g_2 f_1(w_1^*) + \frac{Y_1^n - f_1(W_1)}{g_{12}} + \sqrt{1 - \frac{1}{g_{12}^2}} \tilde{Z}_2^n \right) \neq W_2 \right\}$$

(202)

$$\leq \varepsilon_2.$$ 

(203)

Combining (198) and (203), we obtain

$$\Pr_{pw_1, w_2, Z^n, \tilde{Z}_2^n} \{ |\varphi_1(Y_1^n) \neq W_1 \text{ or } \varphi'_1(Y_1^n) \neq W_2 \}$$

$$= \Pr_{pw_1, w_2, Z^n} \{ |\varphi_1(Y_1^n) \neq W_1 \} + \Pr_{pw_1, w_2, Z^n, \tilde{Z}_2^n} \{ |\varphi'_1(Y_1^n) \neq W_2 \} \cap \{|\varphi_1(Y_1^n) = W_1\}$$

(204)

$$\leq \varepsilon_1 + \varepsilon_2.$$ 

(205)

Following similar procedures for deriving (205), we obtain the following inequality by using (182), (197) and (199):

$$\Pr_{pw_1, w_2, Z^n, \tilde{Z}_2^n} \{ |\varphi_2(Y_2^n) \neq W_2 \text{ or } \varphi'_2(Y_2^n) \neq W_1 \} \leq \varepsilon_1 + \varepsilon_2.$$ 

(206)

Replacing the decoding functions of the $(n, M_1^n, M_2^n, P_1, P_2, \varepsilon_1, \varepsilon_2)_{\text{avg}}$-code with $(\varphi_1, \varphi'_1)$ and $(\varphi_2, \varphi'_2)$ and keeping the encoding functions unchanged, we conclude from (205) and (206) that the resultant code is an $(n, M_1^n, M_2^n, P_1, P_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2)_{\text{avg}}$-multicast code.

We are now ready to prove the strong converse theorem for the Gaussian IC under strong interference.
Proof of Theorem 2: Fix $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that
\begin{equation}
\varepsilon_1 + \varepsilon_2 < 1.
\end{equation}
As discussed at the beginning of Section VI, it follows from Theorem 5.2 in [25] that $C_{0,0} = R_{\text{HK-S}}$ where the quantities $I_1$ and $I_2$ in $R_{\text{HK-S}}$ are defined in (186) and (187) respectively. Since $C_{0,0} \subseteq C_{\varepsilon_1, \varepsilon_2}$ for all non-negative real numbers $\varepsilon_1$ and $\varepsilon_2$ by definition,
\begin{equation}
R_{\text{HK-S}} \subseteq C_{\varepsilon_1, \varepsilon_2}.
\end{equation}
Therefore, it suffices to prove
\begin{equation}
C_{\varepsilon_1, \varepsilon_2} \subseteq R_{\text{HK-S}}.
\end{equation}
To this end, fix a rate pair $(R_1, R_2) \in C_{\varepsilon_1, \varepsilon_2}$. By definition, there exists a sequence of $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2, \varepsilon_1^{(n)}, \varepsilon_2^{(n)})_\text{avg}$-codes such that
\begin{equation}
\liminf_{n \to \infty} \frac{1}{n} \log M_i^{(n)} \geq R_i
\end{equation}
and
\begin{equation}
\limsup_{n \to \infty} \varepsilon_i^{(n)} \leq \varepsilon_i
\end{equation}
for each $i \in \{1, 2\}$. It then following from Lemma 6 and (211) that there exists a sequence of $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2, \varepsilon_1^{(n)}, \varepsilon_2^{(n)})_\text{avg}$-multicast codes such that
\begin{equation}
\limsup_{n \to \infty} \varepsilon_i^{(n)} \leq \varepsilon_1 + \varepsilon_2
\end{equation}
for each $i \in \{1, 2\}$.

Construct a subnetwork of the Gaussian IC formed by deleting $d_2$ as well as the links connecting to it. By inspection, the resultant subnetwork is a two-user Gaussian MAC and the sequence of $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2, \varepsilon_1^{(n)}, \varepsilon_2^{(n)})_\text{avg}$-multicast codes for the Gaussian IC induces a sequence of $(n, M_1^{(n)}, M_2^{(n)}, P_1, P_2, \varepsilon_1^{(n)}, \varepsilon_2^{(n)})_\text{avg}$-codes for the two-user Gaussian MAC. It then follows from (210) and (212) that $(R_1, R_2)$ is $(\varepsilon_1 + \varepsilon_2)$-achievable for the two-user Gaussian MAC, which implies from Theorem 1, (207) and (182) that
\begin{equation}
R_1 \leq \frac{1}{2} \log (1 + P_1),
\end{equation}
\begin{equation}
R_2 \leq \frac{1}{2} \log (1 + g_{12}^2 P_2)
\end{equation}
and
\begin{equation}
R_1 + R_2 \leq \frac{1}{2} \log (1 + P_1 + g_{12}^2 P_2).
\end{equation}
Similarly, if we repeat the above procedures for the other two-user Gaussian MAC resulting from deleting $d_1$ from the Gaussian IC, we obtain
\begin{equation}
R_1 \leq \frac{1}{2} \log (1 + g_{21}^2 P_1),
\end{equation}
\begin{equation}
R_2 \leq \frac{1}{2} \log (1 + P_2)
\end{equation}
and
\begin{equation}
R_1 + R_2 \leq \frac{1}{2} \log (1 + g_{21}^2 P_1 + P_2).
\end{equation}
Combining the bounds in (213), (215), (217), (218), the capacity region in (179), and the strong interference conditions in (186) and (187), we have $(R_1, R_2) \in R_{\text{HK-S}}$. Consequently, (209) holds, and the theorem follows from (208) and (209).
Proof of Lemma 1: Suppose an \((n, M_I^{(n)}, P_I, \varepsilon)_{\text{avg}}\)-code is given for some \(\varepsilon \in [0, 1)\), and let
\[
e_{w_T} \triangleq \Pr\{\tilde{W}_I \neq w_I \mid W_I = w_I\} \tag{219}
\]
be the probability of decoding error given that \(w_I\) is the message tuple transmitted by the sources. Then by choosing \(w_I\) one by one in an increasing order of \(e_{w_T}\), we can construct a set \(D \subseteq W_I\) such that
\[
\Pr\{\tilde{W}_I \neq w_I \mid W_I = w_I\} \leq \frac{1 + \varepsilon}{2} \tag{220}
\]
for all \(w_I \in D\) and
\[
|D| \geq \left[\frac{1 - \varepsilon}{1 + \varepsilon} \prod_{i \in I} M_i^{(n)}\right]. \tag{221}
\]
This is essentially an expurgation argument. The bound in (220) means that there exists an \((n, M_I^{(n)}, P_I, D, I, \frac{1+\varepsilon}{2})_{\text{max}}\)-code such that (221) holds. Fix a nonempty \(T \subseteq I\). Define
\[
D_{w_T} \triangleq \{\tilde{w}_I \in D \mid \tilde{w}_{T^c} = w_{T^c}\} \tag{222}
\]
for each \(w_{T^c} \in W_{T^c}\) such that
\[
\sum_{w_{T^c} \in W_{T^c}} |D_{w_{T^c}}| = |D|. \tag{223}
\]
Since \(|W_{T^c}| = \prod_{i \in T^c} M_i^{(n)}\), it follows from (221) and (223) that there exists an \(w^*_T \in W_{T^c}\) such that
\[
|D_{w^*_T}| \geq \left[\frac{1 - \varepsilon}{1 + \varepsilon} \prod_{i \in T^c} M_i^{(n)}\right], \tag{224}
\]
or otherwise we would obtain the following chain of inequalities which would eventually contradict (221):
\[
|D| \overset{(223)}{=} \sum_{w_{T^c} \in W_{T^c}} |D_{w_{T^c}}| \overset{(225)}{=} < |W_{T^c}| \left[\frac{1 - \varepsilon}{1 + \varepsilon} \prod_{i \in T^c} M_i^{(n)}\right] \overset{(226)}{=} \prod_{i \in T^c} M_i^{(n)} \left[\frac{1 - \varepsilon}{1 + \varepsilon} \prod_{i \in T^c} M_i^{(n)}\right] \overset{(227)}{=} \left[\frac{1 - \varepsilon}{1 + \varepsilon} \prod_{i \in I} M_i^{(n)}\right], \tag{228}
\]
which contradicts (221). Due to (224), we can construct an \((n, M_I^{(n)}, P_I, D_{w^*_T}, T, \frac{1+\varepsilon}{2})_{\text{max}}\)-code based on the \((n, M_I^{(n)}, P_I, D, I, \frac{1+\varepsilon}{2})_{\text{max}}\)-code such that they have the same message sets, encoding functions and decoding function and differ in only the support set of the message tuple \(W_I\) (cf. Definition 1). In particular, the second statement in Definition 1 is satisfied because of the following reasons:

1) By construction, \(W_I\) is uniform on \(D_{w^*_T}\).

2) For all \(w_I \in D_{w^*_T}\), we have \(w_{T^c} = w^*_T\) by (222).

Let \(A \triangleq D_{w^*_T}\). It remains to show that (31) and (32) hold for the \((n, M_I^{(n)}, P_I, A, T, \frac{1+\varepsilon}{2})_{\text{max}}\)-code. Recalling the definition of \(A_T\) in (12), we obtain from (222) that
\[
|A| = |A_T| = |D_{w^*_T}|, \tag{229}
\]
which implies from (224) that
\[
|\mathcal{A}| = |\mathcal{A}_T| \geq \left\lfloor \frac{1 - \varepsilon}{1 + \varepsilon} \prod_{i \in T} M_i^{(n)} \right\rfloor. 
\] (230)
Consequently, (31) follows from (229), (230) and (30). It remains to prove (32). To this end, let \( p_{W_T, X_T^n, Y_T^n, W_T} \) denote the probability distribution induced on the Gaussian MAC by the \( (n, M_T^{(n)}, P_T, A, T, \frac{1 + \varepsilon}{2})_{\max} \)-code, where
\[
p_{W_T}(w_T) = \frac{1}{|\mathcal{A}_T|} 
\] (231)
for all \( w_T \in \mathcal{A}_T \) by Definition 1. Using (231) and (31), we obtain
\[
p_{W_T}(w_T) \leq \frac{1}{\prod_{i \in T} M_i^{(n)} (\frac{2(1 + \varepsilon)}{1 - \varepsilon})} 
\] (232)
for each \( w_T \in \mathcal{A}_T \).

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