Passing Tests without Memorizing: Two Models for Fooling Discriminators

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Abstract
We introduce two mathematical frameworks for foolability in the context of generative distribution learning. In a nutshell, fooling is an algorithmic task in which the input sample is drawn from some target distribution and the goal is to output a synthetic distribution that is indistinguishable from the target with respect to some fixed class of tests. This framework received considerable attention in the context of Generative Adversarial Networks (GANs) – a recently proposed algorithmic approach which achieves impressive empirical results.

From a theoretical viewpoint this problem seems difficult to model. This is due to the fact that in its basic form, the notion of foolability is susceptible to a type of overfitting called memorizing. This raises a challenge of devising notions and definitions that separate between fooling algorithms that generate new synthetic data versus algorithms that merely memorize or copy the training set.

The first model we consider is called GAM–Foolability and is inspired by GANs. Here the learning algorithm (called the generator) has only an indirect access to the target distribution via a discriminator. The second model, called DP–Foolability, exploits the notion of differential privacy as a candidate criterion for non-memorization.

We proceed and characterize foolability within these two models as well as study their interrelations. We show that DP–Foolability implies GAM–Foolability and prove partial results with respect to the converse. It remains, though, an open question whether GAM–Foolability implies DP–Foolability.

We also present an application in the context of differentially private PAC learning: we show that from a statistical perspective, for any class $H$, learnability by a private proper learner is equivalent to the existence of a private sanitizer for $H$. This can be seen as an analogue of the equivalence between uniform convergence and learnability in classical PAC learning.

1 Introduction

With the recent development of Generative Adversarial Networks (GANs) [24], there has been a growing interest to study algorithms that receive as input a sample of examples and are able to generate new and original synthetic data. For example, consider an algorithm that receives as input some tunes from a specific music genre (e.g. jazz, rock, pop) and then outputs a new tune. Providing a mathematical model for this setting appears to be a challenging task, and at least a priori, it seems to not fall into the classical framework of classification.

GANs is a recent framework for the construction of such algorithms that received considerable attention due to empirically impressive results. The training process within GANs proceeds by letting two neural networks, called generator and discriminator, compete with each other in a repeated game. The discriminator holds a sample drawn from the target distribution, and the generator’s goal is to output a synthetic distribution which is close to it. At each step of the game the generator proposes a synthetic distribution, and the discriminator tries to distinguish it from the target distribution; then, both networks receive a score that reflects to which extent the discriminator was able to distinguish between the two distributions, and they perform an update before proceeding to the next step of the game.

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This naturally suggests to investigate GANs as an optimization algorithm, where the generator’s objective is to find a distribution that minimizes a certain pseudo-distance from the target distribution that is induced by a class of discriminating functions [4]. Specifically, let $\mathcal{D}$ be the class of all possible neural networks that may be used by the discriminator then we interpret the generator’s objective as minimizing the Integral Probability Metric (IPM)\(^{[37]}\) distance between the two distributions:

$$\text{IPM}_D(p, q) = \sup_{d \in \mathcal{D}} \left| \mathbb{E}_{x \sim p}[d(x)] - \mathbb{E}_{x \sim q}[d(x)] \right|$$  \hspace{1cm} (1)

However, as observed by [4, 5], there is a critical gap between the task of generating original synthetic data and the IPM minimization problem: if the discriminator class is “too small” then the IPM framework allows certain “bad” solutions that memorize. Specifically, let $S$ be a sufficiently large independent sample from the target distribution and consider the uniform distribution over $S$ (also known as the empirical distribution) as a candidate solution to the IPM minimization problem. Standard arguments from statistical learning theory show that with high probability, the IPM distance between the empirical and the target distribution vanishes as $|S|$ grows (the rate in which it vanishes depends on the statistical capacity of $\mathcal{D}$ – e.g. its size or its VC dimension, etcetera). Allowing classes with large capacity does not remedy the problem: if the discriminator class is too large then in general it is information theoretically impossible to output a distribution whose IPM distance from the target is small.

To illustrate the problem, imagine that our goal is to generate new jazz tunes. Let us consider the discriminating class of all human music experts. The solution suggested above uses the empirical distribution and simply “generates” a tune from the training set\(^2\). This clearly misses the goal of generating new and original tunes but the IPM distance minimization framework does not discard this solution.

The starting point of this paper is the above simple observation: memorizing the data leads to successful IPM minimization. A quantitative version of this phenomenon is then given in Observation 1. It follows that in order to capture algorithms that generate original data one has to consider stronger notions, and this work proposes and explores two such notions which we discuss next.

1.1 Two Models of Foolability

GAM-Foolability. A remarkable property of GANs that potentially reduces the likeliness of memorization is that the generator’s access to the sample is masked by the discriminator. In more detail, only the discriminator has access to the target distribution (or rather to the training set). The generator only has restricted access to the training set via the feedback from the discriminator. Thus, potentially, the generator may avoid degenerate solutions that memorize as it never directly observed the sample.

This naturally suggests to investigate GANs as a sequential process [26, 32, 4], and we introduce the setting of Generative Adversarial Machines (GAMs): we model a sequential game between a generator (player G) and a discriminator (player D). At each iteration player G proposes a distribution and player D outputs a discriminating function from a prespecified binary class $\mathcal{D}$. The game stops when player G proposes a distribution that is close in IPM$_D$ distance to the true target distribution. As we are only concerned with the theoretical limits of the model, we ignore the optimization and computational complexity aspects and we assume that both players are ominous in terms of their computational power.

Regarding the question of memorization, even though the generator is not given a direct access to the training data, it could still be that information about this data could "leak" through the feedback it receives from the discriminator. This raises the question of whether GAM–Foolability can provide guarantees against memorization, and perhaps more importantly, in what sense? Our second model will thus take a more formal approach for preventing memorization. This will enable us to develop more concrete and quantifiable guarantees that certify non-memorization.

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\(^1\)This is also sometimes called the MMD distance [25], or in the context of neural networks and GANs, this is called the neural net distance [4].

\(^2\)There are at most $7 \cdot 10^9$ music experts in the world. Hence, by standard concentration inequalities a sample of size roughly $\frac{2}{\epsilon^2} \log 10$ suffices to achieve IPM distance at most $\epsilon$ with high probability.
**DP–Foolability.** At the heart of the DP-Fooling model lies a new interpretation of the notion of differential privacy [20, 19, 17] and we invoke it as a criterion for originality. Formally then, this model considers differentially private fooling algorithms.

To illustrate our interpretation of differential privacy as a criterion for originality consider the following situation: imagine that Lisa is a learning painter. She has learned to paint by observing samples of painting, some of them produced by a mentor painter Mona. After a learning process, she draws a new painting \( L \). Mona agrees that this new painting is a valid work of art, but Mona claims the result is not an original painting but a mere copy of a painting, say \( M \), produced by Mona.

How can Lisa argue that paint \( L \) is not a plagiary? The easiest argument would be that she had never observed \( M \). However, this line of defence is not always realistic as she must observe some paintings. Instead, we will argue using the following thought experiment: What if Lisa never observed \( M \)? Might she still create \( L \)? If this is the case, then one could argue similarly that \( L \) is not a plagiary.

The last argument is captured by the notion of differential privacy. In a nutshell, a randomized algorithm that receives a sequence of data points \( \bar{x} \) as input is differentially private if removing/replacing a single data point in its input, does not affect its output \( y \) by much; more accurately, for any event \( E \) over the output \( y \) that has non-negligible probability on input \( \bar{x} \), then the probability remains non-negligible even after modifying one data point in \( \bar{x} \).

### 1.2 Our contribution

- We first give characterization of foolable classes in each of these models.
  - We show that a class is GAM–Foolable if and only if it has finite Littlestone dimension. Littlestone classes have been extensively studied in the context of binary online prediction [10, 34]. Therefore, given the sequential nature of the GAM setting, it is natural that they arise in this context. Nevertheless, the characterization of GAM–Foolability by Littlestone classes turns out to be technically challenging, and to the best of the authors knowledge, does not follow via immediate reduction to online prediction.
  - We show that a class is DP–Foolable if and only if it is Privately (proper) PAC learnable [30]. Interestingly, to prove that Private learning implies DP–Foolability we exploit the GAM framework. In a nutshell, we construct a DP–fooling algorithm using a GAM with a private discriminator which is derived from the assumed private PAC learner.

- We next turn to explore the relationship between DP–Foolability and GAM–Foolability.
  - In one direction we show that DP–Foolability implies GAM–Foolability. Moreover, the proof reveals that if a class is DP–foolable by some algorithm then it is DP–foolable by a GANs-like algorithm in the sequential GAMs–foolability setting.
  - The other direction – whether GAM–Foolability implies DP–Foolability – seems more elusive and we leave it as an open problem. Nevertheless we shed some light on this problem by relating it to an existing open problem in differential privacy [1].

- Finally, as an application we prove a result in differentially private learning. This result can be seen as analogue of the equivalence between proper PAC learning and uniform convergence (see, e.g. Theorem 6 in the book [44]). We show that private proper PAC learning is equivalent to private uniform convergence. Our proof of this fact heavily exploits the GAM–foolability setting as well as the combinatorial structure of Littlestone classes. This is perhaps surprising because privacy, learnability, and uniform convergence are notions in statistics and are seemingly unrelated to the combinatorial/sequential nature of GAM or Littlestone dimension.

The main results are depicted in Fig. 1 (see the later sections for the formal definitions).

### 2 Related Work

As discussed, the work in this paper is highly inspired by the recent framework of GANs [24]. Following [4, 5] we relax the Wasserstein distance [3] and consider the IPM distance [37, 25] as the objective to be minimized by the generator.
The framework of GANs is a strip-mine of new theoretical problems, and has presented numerous challenges, such as mode collapse [36, 3, 47], questions on stability [33, 2, 39], as well as problems on the diversity of the output [4, 5].

The work presented here focuses on the sole problem of memorization. The question of memorization can be thought of as the problem of generalization within the framework of generative tasks such as GANs.

**The problem of memorization.** For qualitative assessment of memorization, it is a common practice to show synthetic samples next to their nearest neighbor within the training set. [48] demonstrated how the use of Euclidean distance may be misleading. Alternative metrics are often used [29] such as feature-space distance. [52] tries to estimate the log likelihood of the output via Annealed Importance Sampling. [39] suggests “walking on the manifold that is learnt”, and checks for large transition to indicate if memorization has happened.

While it seems there is some inclination to accept that memorization is not an issue [42, 6, 52], currently most work focuses on empirical studies, experiments and evidence. In fact the notion of memorization is often not even defined. Here we try to tackle the problem of memorization in a more formal manner, and aim for a rigorous model that would help study the problem.

In all its characteristic, memorization captures notions such as plagiarism and creativity. Hence it would be extremely challenging to model it perfectly. However, here we propose a formal criterion DP, which we believe to be strong enough to ensure lack of memorization. We then proceed to study which classes are foolable in a DP manner.

**Distribution Learning.** The model we consider in this work restricts the discriminating function class and we obtain a game between a discriminator and an ominous generator that is allowed to return an arbitrary distribution. When training GANs, the class of distributions is also parametrized by neural networks hence have finite capacity. It is possible, then, to consider a dual approach in which one restricts the generating class; perhaps more akin with traditional generative learning models such as GMM, HMM etc. (e.g. [38]).

Within the framework of GANs [22] shows how the class of Gaussian can be learned when the discriminators are quadratic. [6] extends and shows how one can learn, in Wasserstein distance, certain generator classes such as Mixture of Gaussian, exponential families, and classes generated by neural networks through choosing a right class of discriminating functions.

**DP GANs.** We are not the first to suggest differentially private GANs. In fact, one of the motivation of gans is to provide synthetic data that preserves privacy, and several papers propose optimization methods for GANs that are differentially private, [7, 53, 49, 13]. Our work differs in several respects. First, this paper is not concerned, directly, with the privacy of the data sets on which we train our algorithms. Instead we propose a new interpretation of privacy as a criterion for
the originality of the outcome, or lack of overfitting. Second, we focus less on providing algorithmic solutions for optimizing DP-GANs, and we aim to characterize the classes of discriminators that can be learnt within the framework, efficiency put aside.

**Private Synthetic Data.** The problem of generating, privately, synthetic data has also seen a focus of attention within several DP papers. Private synthetic data is a special case of the problem of sanitization introduced in [12] and further studied in [8]. Theorem 2 provides a reduction from private (proper and agnostic) PAC learning to the task of generating private synthetic data. Several other papers suggest private synthetic data generation in the presence of a private learner, [28, 23]. Most recently [43] provides an efficient algorithm for synthetic data generation that scales logarithmically with the size of the query class (given a polynomially sized universal identification sets). However, to the best of our knowledge all previous bounds depend on the size of the query set or domain.

The results here do not scale with the size of the class. Instead we provide both upper and lower bounds that scale with the Littlestone dimension of the class.

Like previous work our method relies on a zero sum game between a data player and a query player (which in GANs terminology are named generator and discriminator), and we employ a non-regret algorithm over a dual class of functions resulted from swapping the roles of query points and data points.

The main challenge, when dealing with finite Littlestone classes and not finite classes, is that existing no-regret algorithms on which we build upon ([10, 34]), do not immediately yield synthetic distributions over the domain (as for example expert advice or FTPL that can be employed to learn finite classes). We thus need to develop techniques for producing a distribution when the no-regret algorithm does not provide one immediately.

### 3 Prelimineries

In this section we review some of the basic notations we will use as well as recall some standard definitions and notions in differential privacy and online learning (a more extensive background is also given in Section 7.1). Throughout the paper we will study classes \( \mathcal{D} \) of boolean functions defined on a domain \( \mathcal{X} \). However, we will often use a dual point of view where we think of \( \mathcal{X} \) as the class of functions and on \( \mathcal{D} \) as the domain. Therefore, in order to avoid confusion, in this section we let \( \mathcal{W} \) denote the domain and \( \mathcal{H} \subseteq \{0,1\}^\mathcal{W} \) to denote the functions class.

#### 3.1 Notations

For a finite\(^3\) set \( \mathcal{W} \), let \( \Delta(\mathcal{W}) \) denote the space of probability measures over \( \mathcal{W} \). Note that \( \mathcal{W} \) naturally embeds in \( \Delta(\mathcal{W}) \) by identifying \( w \in \mathcal{W} \) with the Dirac measure \( \delta_w \) supported on \( w \). Therefore, every \( f : \Delta(\mathcal{W}) \to \mathbb{R} \) induces a \( \mathcal{W} \to \mathbb{R} \) function via this identification. In the other direction, every \( f : \mathcal{W} \to \mathbb{R} \) naturally extends to a linear\(^4\) map \( \hat{f} : \Delta(\mathcal{W}) \to \mathbb{R} \) which is defined by \( \hat{f}(p) = \mathbb{E}_p[f] \) for every \( p \in \Delta(\mathcal{W}) \).

We will often deal with boolean functions \( f : \mathcal{W} \to \{0,1\} \), and in some cases we will treat \( f \) as the subset of \( \mathcal{W} \) that it indicates. For example, given a distribution \( p \in \Delta(\mathcal{W}) \) we will use \( p(f) \) to denote the measure of the subset that \( f \) indicates (i.e. \( p(f) = \mathbb{P}_{p \sim \mathcal{W}}[f(w) = 1] \)). Given a class of functions \( F \subseteq \{0,1\}^\mathcal{W} \), its dual class is a class of \( F \to \{0,1\} \) functions, where each function in it is associated with \( w \in \mathcal{W} \) and acts on \( F \) according to the rule \( f \mapsto f(w) \). By a slight abuse of notation we will denote the dual class with \( \mathcal{W} \) and use \( w(f) \) to denoted the function associated with \( w \) (i.e. \( w(f) := f(w) \) for every \( f \in F \)).

Given a sample \( S = (w_1, \ldots, w_m) \in \mathcal{W}^m \), the empirical distribution induced by \( S \) is the discrete distribution \( p_S \) defined by

\[
p_S(w) = \frac{1}{m} \sum_{i=1}^m 1[w = w_i].
\]

\(^3\)The same notation will be used for infinite classes also. However we will properly define the the measure space and \( \sigma \)-algebra at later sections when we extend the results to the infinite regime.

\(^4\)A function \( g : \Delta(\mathcal{W}) \to \mathbb{R} \) is linear if \( g(\alpha p_1 + (1-\alpha)p_2) = \alpha g(p_1) + (1-\alpha)g(p_2) \), for all \( \alpha \in [0,1] \)
3.2 Differential Privacy and Private Learning

Differential Privacy [20, 17] is a statistical formalism which aims at capturing algorithmic privacy. It concerns problems whose input contains databases with private records and it enables to design algorithms for these problems that are formally guaranteed to protect the private information. For more background see the surveys [21, 50].

The formal definition is as follows: let \( W^m \) denote the input space. An input instance \( \Omega \in W^m \) is called a database, and two databases \( \Omega', \Omega'' \in W^m \) are called neighbours if there exists a single \( i \leq m \) such that \( \Omega'_i \neq \Omega''_i \). Let \( \alpha, \beta > 0 \) be the privacy parameters, a randomized algorithm \( M : W^m \to \Sigma \) is called \((\alpha, \beta)\)-differentially private if for every two neighbouring \( \Omega', \Omega'' \in W^m \) and for every event \( E \subseteq \Sigma \):

\[
\Pr[M(\Omega') \in E] \leq e^\alpha \Pr[M(\Omega'') \in E] + \beta.
\]

An algorithm \( M : \cup_{m=1}^\infty W^m \to Y \) is called differentially private if for every \( m \) its restriction to \( W^m \) is \((\alpha(m), \beta(m))\)-differentially private, where \( \alpha(m) = O(1) \) and \( \beta(m) \) is negligible\(^5\). Concretely, we will think of \( \alpha(m) \) as a small constant (say, 0.1) and \( \beta(m) = O(m^{-\log m}) \).

**Private Learning.** Differentially private learning algorithms [30] have been studied extensively in recent years. In this context the input database is the training set of the algorithm.

Within the PAC model, given a hypothesis class \( \mathcal{H} \) over a domain \( W \), we say that \( \mathcal{H} \subseteq \{0,1\}^W \) is privately PAC learnable if it can be learned by a differentially private algorithm. That is, if there is a differentially private algorithm \( M \) and a sample complexity bound \( m(\epsilon, \delta) = \poly(1/\epsilon, 1/\delta) \) such that for every \( \epsilon, \delta > 0 \) and every distribution \( \mathbb{P} \) over \( W \times \{0,1\} \), if \( M \) receives an independent sample \( S \sim \mathbb{P}^m \) then it outputs an hypothesis \( h_S \) such that with probability at least \( 1 - \delta \):

\[
L_H(h_S) \leq \min_{h \in H} L_H(h) + \epsilon,
\]

where \( L_H(h) = \mathbb{E}_{(w, y) \sim \mathbb{P}} [1[h(w) \neq y]] \). If the algorithm \( M \) is proper, namely \( h_S \in \mathcal{H} \) for every input sample \( S \), then \( \mathcal{H} \) is said to be Privately Agnostically and Properly PAC learnable (PAP-PAC-learnable).

In some of our proofs it will be convenient to consider private learning algorithms whose privacy parameter \( \alpha \) satisfies \( \alpha \leq 1 \) (rather than \( \alpha = O(1) \) as in the definition of private algorithms). This can be done without loss of generality due to privacy amplification theorems similar to Lemma 4. We refer the reader to [50] (Definition 8.2) and references within for further details (see also discussion after Lemma 4).

**Sanitization.** The notion of sanitization has been introduced in [12] and further studied in [8]. Let \( \mathcal{H} \subseteq \{0,1\}^W \) be a class of functions. An \((\epsilon, \delta, \alpha, \beta, m)\)-sanitizer for \( \mathcal{H} \) is an \((\alpha, \beta)\)-private algorithm \( M \) that receives as an input a sample \( S \in W^m \) and outputs a function \( \text{Est} : \mathcal{H} \to [0,1] \) such that with probability at least \( 1 - \delta \),

\[
(\forall h \in \mathcal{H}) : \left| \frac{\text{Est}(h) - \{|w \in S : h(w) = 1\}|}{|S|} \right| \leq \epsilon.
\]

A common type of sanitizers acts as follows: given an input sample \( S \), they publish a sample \( T \) such that with probability at least \( 1 - \delta \),

\[
(\forall h \in \mathcal{H}) : \left| \frac{|\{w \in T : h(w) = 1\}|}{|T|} - \frac{|\{w \in S : h(w) = 1\}|}{|S|} \right| \leq \epsilon.
\]

In other words, they use the estimate \( \text{Est}(h) = \frac{|\{w \in T : h(w) = 1\}|}{|T|} \), which is encoded by \( T \).

We say that \( \mathcal{H} \) is sanitizable if there exists an algorithm \( M \) and a bound \( m(\epsilon, \delta) = \poly(1/\epsilon, 1/\delta) \) such that for every \( \epsilon, \delta > 0 \), the restriction of \( M \) to samples of size \( m = m(\epsilon, \delta) \) is an \((\epsilon, \delta, \alpha, \beta, m)\)-sanitizer for \( \mathcal{H} \) with \( \alpha = o(m) = O(1) \) and \( \beta = \beta(m) \) negligible.

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\(^5\)I.e. \( \beta(m) = o(m^{-k}) \) for every \( k > 0 \).
**Private Uniform Convergence.** A basic concept in Statistical Learning Theory is the notion of *uniform convergence*. In a nutshell, a class of hypotheses \( \mathcal{H} \) satisfies the uniform convergence property if for any unknown distribution \( P \) over examples, one can uniformly estimate the expected losses of all hypotheses in \( \mathcal{H} \) given a large enough sample from \( P \). Uniform convergence and statistical learning are closely related. For example, the Fundamental Theorem of PAC Learning asserts that they are equivalent for binary-classification [44].

This notion extends to the setting of private learning: a class \( \mathcal{H} \) satisfies the *Private Uniform Convergence* property if there exists a differentially private algorithm \( M \) and a sample complexity bound \( m(\epsilon, \delta) = \text{poly}(1/\epsilon, 1/\delta) \) such that for every distribution \( P \) over \( \mathcal{W} \times \{0, 1\} \) the following holds: if \( M \) is given an input sample \( S \) of size at least \( m(\epsilon, \delta) \) which is drawn independently from \( P \), then it outputs an estimator \( \hat{L} : \mathcal{H} \to [0, 1] \) such that with probability at least \( (1 - \delta) \) it holds that

\[
(\forall h \in \mathcal{H}) : |\hat{L}(h) - L_P(h)| \leq \epsilon.
\]

Note that without the privacy restriction, the estimator

\[
\hat{L}(h) = L_S(h) := \frac{|\{(w_i, y_i) \in S : h(w_i) \neq y_i\}|}{|S|}
\]

satisfies the requirement for \( m = \tilde{O}(d/\epsilon^2) \), where \( d \) is the VC-dimension of \( \mathcal{H} \); indeed, this follows by the celebrated VC-Theorem [51, 44].

### 3.3 Littlestone Dimension

The Littlestone dimension is a combinatorial parameter that characterizes regret bounds in online learning⁶ [34, 10], but also have recently been related to other concepts in machine learning such as differentially private learning [1]. Perhaps surprisingly, the notion also plays a central role in Model Theory ([46, 16], and see [1] for further discussion).

The definition of this parameter uses the notion of *mistake-trees*: these are binary decision trees whose internal nodes are labelled by elements of \( \mathcal{W} \). Any root-to-leaf path in a mistake tree can be described as a sequence of examples \((w_1, y_1), ..., (w_d, y_d)\), where \( w_i \) is the label of the \( i \)'th internal node in the path, and \( y_i = +1 \) if the \( (i + 1) \)'th node in the path is the right child of the \( i \)'th node, and otherwise \( y_i = 0 \). We say that a tree \( T \) is *shattered* by \( \mathcal{H} \) if for any root-to-leaf path \((w_1, y_1), ..., (w_d, y_d)\) in \( T \) there is \( h \in \mathcal{H} \) such that \( h(w_i) = y_i \), for all \( i \leq d \).

The Littlestone dimension of \( \mathcal{H} \), denoted by \( \text{Ldim}(\mathcal{H}) \), is the maximum depth of a complete tree that is shattered by \( \mathcal{H} \).

The dual Littlestone Dimension which we will denote by \( \text{Ldim}^*(\mathcal{H}) \) is the Littlestone dimension of the dual class (i.e. we consider \( \mathcal{W} \) as the hypothesis class and \( \mathcal{H} \) is the domain). We will use the following fact:

**Lemma 1.** [Corollary 3.6 in [11]] Every class \( \mathcal{H} \) has a finite Littlestone dimension if and only if it has a finite dual Littlestone dimension. Moreover we have the following bound:

\[
\text{Ldim}^*(\mathcal{H}) \leq 2^{\text{Ldim}(\mathcal{H}) + 2} - 2
\]

### 4 Foolability models

In this section we define the three models of foolability that are discussed in this paper.

Let \( \mathcal{X} \) be a domain⁷ and let \( \mathcal{D} \subseteq \{0, 1\}^\mathcal{X} \) be a class of functions. The class \( \mathcal{D} \) is referred to as the *discriminating functions class* and its members \( d \in \mathcal{D} \) are called *discriminating functions* or *distinguishers*.

Given two distributions \( p, q \in \Delta(\mathcal{X}) \), let \( \text{IPM}_D(p, q) \) denote the IPM distance between \( p \) and \( q \):

\[
\text{IPM}_D(p, q) = \sup_{d \in \mathcal{D}} |p(d) - q(d)|.
\]

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⁶See Section 7.1.2 for further discussion on online learning.

⁷Our arguments exploit standard tools from probability (e.g. Uniform Convergence for VC classes) and analysis (i.e. Vonneuman’s Minmax Theorem). Applying these results when \( \mathcal{X} \) is infinite requires measurability/topological assumptions. We omit discussing these assumptions in the main text and refer the reader to Appendix B for more details.
It will be convenient to assume that $\mathcal{D}$ is symmetric, i.e. that whenever $d \in \mathcal{D}$ then also its complement, $1 - d \in \mathcal{D}$. Assuming that $\mathcal{D}$ is symmetric will not lose generality and will help simplify notations. Under this assumption we can remove the absolute value from the definition of IPM:

$$\text{IPM}_\mathcal{D}(p, q) = \sup_{d \in \mathcal{D}} (p(d) - q(d)).$$

(2)

4.1 Foolability

We begin with the first and most basic model, called foolability, the goal of the fooling algorithm is to find any distribution which is indistinguishable from the target distribution. As we will see, this form of foolability has the disadvantage of accommodating memorizing algorithms. We then proceed to define the refinements of GAM–Foolability and DP–Foolability which, to a certain extent, prevent memorization. As argued in the introduction, DP–Foolability prevents memorization in a stronger (and better defined) way.

**Definition 1 (Fooling Algorithm).** A fooling algorithm for $\mathcal{D}$ with sample complexity $m(\epsilon, \delta)$ is an algorithm $M$ that receives as input a sample $S$ of points from $X$ and parameters $\epsilon, \delta$ such that the following holds: for every $\epsilon, \delta > 0$ and every target distribution $p_{\text{real}}$, if $S$ is an independent sample of size at least $m(\epsilon, \delta)$ from $p_{\text{real}}$ then

$$\Pr \left[ \text{IPM}_\mathcal{D}(p_{\text{syn}}, p_{\text{real}}) < \epsilon \right] \geq 1 - \delta,$$

where $p_{\text{syn}} := M(S)$ is the distribution outputted by $M$, and the probability is taken over $S \sim (p_{\text{real}})^m$ as well as over the randomness of $M$.

We will say that a class is foolable if it can be fooled by an algorithm whose sample complexity is $\text{poly}(\frac{1}{\epsilon}, \frac{1}{\delta})$. Our starting point is the next characterization of foolability. This characterization is an immediate corollary (or rather a reformulation) of the celebrated VC Theorem ([51]).

**Observation 1 ([51]).** The following statements are equivalent for class $\mathcal{D} \subseteq \{0, 1\}^X$:

1. $\mathcal{D}$ is PAC-learnable.
2. $\mathcal{D}$ is foolable.
3. $\mathcal{D}$ satisfies the uniform convergence property.
4. $\mathcal{D}$ has a finite VC-dimension.
5. $M_{\text{emp}}$ is a fooling algorithm for $\mathcal{D}$ with sample complexity $m = O\left(\frac{\log \frac{1}{\delta}}{\epsilon^2}\right)$.

Observation 1 shows that foolability is equivalent to PAC-learnability (i.e. finite VC dimension). We will later see analogous results for DP–Foolability (which is equivalent to differentially private PAC learnability) and GAM–Foolability (which is equivalent to online learnability). Observation 1 highlights the fact that any foolable class can be fooled by an algorithm which memorizes its input. Thus, we would like to refine definition of foolability in a way that discards memorizing algorithms. This brings us to the stronger models of DP–Foolability and GAM–Foolability which are defined next.

4.2 GAM–Foolability

We now describe the second model of foolability which is inspired by GANs and provides an abstract setting of sequential adversarial training.
**Generative Adversarial Machines.** A Generative Adversarial Machine (GAM) can be thought of as a sequential game between two players called the generator (denoted by $G$) and the discriminator (denoted by $D$). At the beginning of the game, the discriminator $D$ receives the target distribution which is denoted by $p_{\text{real}}$. The goal of the generator $G$ is to find a distribution $p$ such that $p$ and $p_{\text{real}}$ are $\epsilon$-indistinguishable with respect to some prespecified discriminating class $\mathcal{D}$ and an error parameter $\epsilon > 0$ (both players know $\mathcal{D}, \epsilon$), i.e.

$$\text{IPM}_D(p, p_{\text{real}}) \leq \epsilon.$$  

The game proceeds in rounds, where in each round $t$ the generator $G$ submits to the discriminator a candidate distribution $p_t$ and the discriminator replies according to the following rule: if $\text{IPM}_D(p_t, p_{\text{real}}) \leq \epsilon$ then the discriminator replies “WIN” and the game terminates. Else, the discriminator picks $d_t \in \mathcal{D}$ such that $|p_t(d_t) - p_{\text{real}}(d_t)| \geq \epsilon$, and sends $d_t$ to the generator along with a bit which indicates whether $p_t(d_t) > p_{\text{real}}(d_t)$ or $p_t(d_t) < p_{\text{real}}(d_t)$. Equivalently, instead of transmitting an extra bit, we assume that the discriminator always sends $d_t \in \mathcal{D} \cup (1 - \mathcal{D})$ s.t.

$$p_{\text{real}}(d_t) - p_t(d_t) > \epsilon. \quad (3)$$

**Definition 2** (GAM–Foolability). Let $\epsilon > 0$ and let $\mathcal{D}$ be a discriminating class.

1. $\mathcal{D}$ is called $\epsilon$-GAM–Foolable if there exists a generator $G$ and a bound $T = T(\epsilon)$ such that $G$ wins any discriminator $D$ with any target distribution $p_{\text{real}}$ after at most $T$ rounds.

2. The round complexity of GAM–Fooling $D$ is defined as the minimal upper bound $T(\epsilon)$ on the number of rounds that suffice to $\epsilon$–Fool $D$.

3. $\mathcal{D}$ is called GAM–Foolable if it is $\epsilon$-GAM foolable for every $\epsilon > 0$ with $T(\epsilon) = \text{poly}(1/\epsilon)$.

In the next section we will see that if $\mathcal{D}$ is $\epsilon$-GAM–Foolable for some fixed $\epsilon < 1/2$ then it is GAM–Foolable with round complexity $T(\epsilon) = O(1/\epsilon^2)$.

**Randomness.** We stress out that we assume that both the generator and discriminator are deterministic. This assumption is made to simplify the presentation but does not restrict the validity of our results, as we explain next. Consider a setting where the players may use randomness and the definition of round complexity is modified by taking expectation. Assuming a deterministic discriminator does not lose generality because in each round the discriminator plays only after seeing the generator’s submitted distribution and so it may always respond deterministically. Restricting the generator to be deterministic is more subtle. However, in terms of upper bounds it only strengthen our result (because we only use deterministic generators, whereas randomized ones could potentially be stronger), and in terms of our lower bound (Theorem 1, Item 2), its proof applies verbatim to the expected round complexity of randomized generators.

### 4.3 DP–Foolability

We next introduce the notion of DP–Fooling algorithms and DP–Foolability which is central to this work. As discussed in the introduction, these are algorithms that fool a discriminating class $\mathcal{D}$, while avoiding memorization or overfitting of their training set.

**Definition 3** (DP-Fooling Algorithm). A DP-Fooling algorithm $M$ for a class $\mathcal{D}$ is an algorithm that receives as an input a finite sample $S$ and two parameters $(\epsilon, \delta)$ and satisfies:

- **Differential Privacy.** For every $m$, the restriction of $M$ to input samples $S$ of size $m$ is $(\alpha(m), \beta(m))$-differentially private, where $\alpha(m) = O(1)$ and $\beta(m)$ is negligible.

- ** Fooling.** $M$ fools $\mathcal{D}$: there exists a sample complexity bound $m = m(\epsilon, \delta)$ such that for every target distribution $p_{\text{real}}$ if $S$ is a sample of at least $m$ examples from $p_{\text{real}}$ then $\text{IPM}_D(p_{\text{syn}}, p_{\text{real}}) \leq \epsilon$ with probability at least $1 - \delta$, where $p_{\text{syn}}$ is the output of $M$ on the input sample $S$.

We will say in short that a class $\mathcal{D}$ is DP– Foolable if there exists a DP-Fooling algorithm for the class $\mathcal{D}$ with sample complexity $m = \text{poly}(1/\epsilon, 1/\delta)$.
5 Results

5.1 GAM–Foolability

Our first main result characterizes the GAM–Foolable classes and provides quantitative upper and lower bounds on their round complexity in terms of the Littlestone dimension.

**Theorem 1** (Quantitative round-complexity bounds). Let $D$ be class with dual Littlestone dimension $\ell^*$ and let $T(\epsilon)$ denote the round complexity of GAM–Fooling $D$. Then,

1. $T(\epsilon) = O\left(\frac{\ell^*}{\epsilon^2 \log \frac{\ell^*}{\epsilon}}\right)$ for every $\epsilon$.
2. $T(\epsilon) \geq \frac{\ell^*}{2}$ for every $\epsilon < \frac{1}{2}$.

To prove Item 1 we construct a generator with winning strategy which we outline in Section 6. A complete proof of Theorem 1 appears in Section 7.2.1.

As a corollary we get the following characterization of GAM–Foolability:

**Corollary 1** (Characterization of GAM–Foolability). The following are equivalent for a class $D \subseteq \{0, 1\}^X$:

1. $D$ is GAM–Foolable.
2. $D$ is $\epsilon$-GAM–Foolable for some $\epsilon < 1/2$.
3. $D$ has a finite dual Littlestone dimension.
4. $D$ has a finite Littlestone dimension.

Corollary 1 follows directly from Theorem 1 (which gives the equivalences $1 \iff 2 \iff 3$) and from Lemma 1 (which gives the equivalence $3 \iff 4$). We point out that Lemma 1 was known prior to this work.

Tightness of $\epsilon = \frac{1}{2}$. The implication Item 2 $\Rightarrow$ Item 1 can be seen as a boosting result: i.e. “weak” foolability for some fixed $\epsilon < 1/2$ implies “strong” foolability for every $\epsilon$. The following example demonstrates that the dependence on $\epsilon$ in Item 2 cannot be improved beyond $\frac{1}{2}$: let $X$ be the unit circle in $\mathbb{R}^2$, and let $D$ consist of all arcs whose length is exactly half of the circumference. It is easy to verify that the uniform distribution $\mu$ over $X$ satisfies $\text{IPM}_D(\mu, \text{real}) \leq \frac{1}{2}$ for any target distribution $\text{real}$ (since $\mu(d) = \frac{1}{2}$ for all $d \in D$). Therefore $D$ is $(\epsilon = \frac{1}{2})$-GAM–Foolable with round complexity $T(\frac{1}{2}) = 1$. On the other hand, $D$ has an infinite Littlestone dimension and therefore is not GAM–Foolable.

5.2 DP–Foolability

Our second main result characterizes DP–Foolability in terms of basic notions from differential privacy and PAC learning.

**Theorem 2** (Characterization of DP–Fooling). The following statements are equivalent for a class $D \subseteq \{0, 1\}^X$:

1. $D$ is privately and properly learnable in the agnostic PAC setting.
2. $D$ is DP–Foolable.
3. $D$ has a finite dual Littlestone dimension.
4. $D$ has a finite Littlestone dimension.

The implication Item 3 $\Rightarrow$ Item 1 was known prior to this work and was proven in [8]. The equivalence among Items 2 to 4 is natural and expected. Indeed, each of them expresses the existence of a private algorithm that publishes certain estimates of all functions in $D$ (e.g. in uniform convergence losses are estimated and in sanitization averages are estimated). The fact that Item 1 implies the other three items is perhaps more surprising. Interestingly, our proof of that exploits the GAMs framework from the previous section. In a nutshell, we show that Item 1 implies Item 2 by constructing a DP–fooling algorithm which is based on a GAM with a private discriminator. This private discriminator is derived from the private PAC learner whose existence is assumed by Item 1. See Section 7.3 for a complete proof.
Private learnability versus private uniform convergence. The equivalence Item 1 \iff Item 4 is between private learning and private uniform convergence. The non-private analogue of this equivalence is a cornerstone in statistical learning; it reduces the statistical challenge of minimizing an unknown population loss to an optimization problem of minimizing a known empirical estimate. In particular, it yields the celebrated Empirical Risk Minimization (ERM) principle: “Output \( h \in \mathcal{H} \) that minimizes the empirical loss”. We therefore highlight this equivalence in the following corollary:

**Corollary 2** (Private proper learning = private uniform convergence). Let \( \mathcal{H} \subseteq \{0, 1\}^X \). Then \( \mathcal{H} \) is privately and properly PAC learnable if and only if \( \mathcal{H} \) satisfies the private uniform convergence property.

5.3 GAMs versus DP-Fooling

So far we have introduced and characterized two formal approaches towards handling memorization (or ensure originality). The first approach uses the setting of GAMs which attempts to prevent memorization by masking the algorithm’s access to the training set via a discriminator. The second approach suggests the definition of differential privacy to restrict the algorithm from memorizing its input sample and enjoy formal guarantees that support non-memorization/originality. It is therefore natural to compare and seek connections between these two approaches. We first note that the DP setting may only be more restrictive than the GAMs setting:

**Corollary 3** (DP–Foolability implies GAM–Foolability). Let \( \mathcal{D} \) be a finite class that is DP–Foolable. Then \( \mathcal{D} \) has finite Littlestone dimension and in particular is GAM–Foolable.

Corollary 3 follows from Theorem 2: indeed, the latter yields that DP–Foolability is equivalent to PAP-PAC learnability, and by [1, 14] PAP-PAC learnability implies a finite Littlestone dimension which by Corollary 1 implies GAM–Foolability.

The universality of GAMs-based DP algorithms. Our proofs reveal a stronger phenomenon than Corollary 3. In more detail, Theorem 2 shows that DP-Foolability and proper agnostic private learning (PAP-PAC) are equivalent. Proving the direction that PAP-PAC implies DP-Foolability obtains a DP algorithm by invoking a GAM with a differentially private discriminator. This establishes universality of such GAMs-based algorithms in the following sense: for any \( \mathcal{D} \), if it can be fooled by some DP algorithm then it can also be fooled by a DP algorithm which is based on a GAM with a differentially private discriminator.

**Towards a converse of Corollary 3.** By the above it follows that the family of classes \( \mathcal{D} \) that can be fooled by a DP algorithm is contained in the family of all GAM–Foolable classes; specifically, those which admit a GAM with a differentially private discriminator.

We do not know whether the converse holds; i.e. whether “GAM–Foolability \( \implies \) DP–Foolability”. Nevertheless, the implication “PAP-PAC learnability \( \implies \) DP–Foolability” (Theorem 2) can be regarded as an intermediate step towards this converse. Indeed, as discussed above, PAP-PAC learnability implies GAM–Foolability. It is therefore natural to consider the following question, which is equivalent\(^8\) to the converse of Corollary 3:

**Question 1.** Let \( \mathcal{D} \) be a class that has finite Littlestone dimension. Is \( \mathcal{D} \) properly and privately learnable in the agnostic PAC setting?

A weaker form of this question – Whether every Littlestone class is privately PAC Learnable? – was posed by [1] as an open question.

6 A strategy for GAM–Fooling

In this section we present the generator’s strategy which is used in the proof of Theorem 1, Item 1 to fool a class \( \mathcal{D} \) with dual Littlestone dimension \( \ell^* \). We will assume that \( \mathcal{D} \) is symmetric (i.e. that \( \mathcal{D} = 1 - \mathcal{D} \)). This assumption does not affect generality since one can symmetrize \( \mathcal{D} \) by adding to

\(^8\)I.e. an affirmative answer to Question 1 is equivalent to the converse of Corollary 3.
it all functions in $1 - D$. This modification does not change the dual Littlestone dimension nor the associated GAM game.

The generator uses an online learner $\mathcal{A}$ for the dual class of $\Delta(D) \subseteq [0, 1]^X$ whose existence is proved in Corollary 4, and we refer the reader to Section 7.1 for further background in online learning as well as the exact statements. In a nutshell, $\mathcal{A}$ receives, sequentially, labelled examples, $(d_t, y_t)$, from the domain $\Delta(D) \times \{0, 1\}$ and returns at each step $t$ a predictor $\hat{f}_t$ of the type $f_t(d) = \mathbb{E}_{d \sim \bar{d}}[f_t(d)]$ for some function $f_t$ over the domain $D$. Moreover $\mathcal{A}$ has the following guarantee over what we define as its regret:

$$\text{REGRET}_T(\mathcal{A}) := \sum_{t=1}^{T} |\hat{f}_t(d_t) - y_t| - \min_{x \in X} \sum_{t=1}^{T} |\mathbb{E}_{d \sim \bar{d}}[x(d)] - y_t| \leq \sqrt{\frac{1}{2} T \log T},$$

Recall from Section 3.1 that each hypothesis in the dual class is associated with an $x \in X$ that acts on $D$ by “$d \mapsto d(x)$”, and that we use $\mathcal{X}$ to denote the dual class and $x$ to denote the function associated with $x$ (i.e. $x(d) := d(x)$). We further extend each $x$ to a linear function over $\Delta(D)$ by $x(q) := \mathbb{E}_{d \sim q}[x(d)]$ (as in Section 3.1)).

- Let $D$ be a symmetric class with $\text{Ldim}^*(D) = \ell^*$, and let $\epsilon > 0$ be the error parameter.
  - Pick $\mathcal{A}$ to be an online learner for the dual class $\mathcal{X}$ like in Corollary 4, and set
    $$T = \left\lceil \frac{4\ell^*}{\epsilon^2} \log \frac{4\ell^*}{\epsilon^2} \right\rceil = O\left(\frac{\ell^*}{\epsilon^2} \log \frac{\ell^*}{\epsilon}\right),$$
  - Set $\hat{f}_1(d) = \mathbb{E}_{d \sim \bar{d}}[f_1(d)]$ as the predictor of $\mathcal{A}$ at its initial state.
  - For $t = 1, \ldots, T$
    1. If there exists $p_t \in \Delta(\mathcal{X})$ such that
        $$\langle \forall d \in \mathcal{D} : \mathbb{E}_{x \sim p_t}[f_t(d) - x(d)] \leq \frac{\epsilon}{2},$$
        then
        - pick such a $p_t$ and submit it to the discriminator.
          * If the discriminator replies with “Win” then output $p_t$.
          * Else, receive from the discriminator $d_t \in D$ such that $p_{\text{real}}(d_t) - p_t(d_t) \geq \epsilon$
          * Set $\hat{d}_t = \delta_{d_t}$, and $y_t = 1$.
    2. Else
        - Find $\bar{d} \in \Delta(D)$ such that
          $$\langle \forall x \in \mathcal{X} : \mathbb{E}_{d \sim \bar{d}}[f_t(d) - x(d)] > \frac{\epsilon}{2},$$
          (if no such $\bar{d}$ exists then output “error”).
          - Set $y_t = 0$.
          - Submit $p_t = p_{t-1}$ to the discriminator and proceed to item 3 below (i.e. here the generator sends a dummy distribution to the discriminator and ignores the answer).
    3. Update $\mathcal{A}$ with the observation $(\hat{d}_t, y_t)$, receive $\hat{f}_{t+1}$, set $f_{t+1}$ such that $\hat{f}_{t+1}(d) = \mathbb{E}_{d \sim \bar{d}}[f_{t+1}(d)]$ (such $f_{t+1}$ exists by the assumed properties of $\mathcal{A}$ – see Corollary 4), and proceed to the next iteration.
- Output “Lost” (we will prove that this point is never reached).

Figure 2: A fooling strategy for the generator with respect to a symmetric discriminating class $D$. 

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Proof overview of Theorem 1, Item 1. We begin by considering a simpler setting where it is assumed that the learner $A$ is $\Delta(\mathcal{X})$-proper in the sense that at each iteration $t$ it uses a predictor $f_t = \mathbb{E}_p[f_t]$, where $f_t$ is a weighted average of hypotheses in $\mathcal{X}$; namely,
\[
(\forall d \in \mathcal{D}) : f_t(d) = p_t(d) = \mathbb{E}_{x \sim p_t} [x(d)], \tag{4}
\]
for some $p_t \in \Delta(\mathcal{X})$. Let us denote this learner by $A_\Delta$. We note in passing that Eq. (4) holds, for example, if one considers finite domains $\mathcal{X}$ and allows the regret to scale with the size of the domain instead of the Littlestone dimension. In this case we can concretely choose our online learner to be a Weighted majority algorithm [35], which satisfies Eq. (4). Indeed, similar uses of the Weighted majority have been applied to generate synthetic data over finite domains [28, 27].

Coming back to the proof overview, the crucial point is that if $f_t$ satisfies Eq. (4) then the generator can submit $p_t \in \Delta(\mathcal{X})$ to the discriminator. Specifically, the generator can use $A_\Delta$ as follows: at each iteration $t$, submit $p_t$ to the discriminator; then, unless $p_t$ fools $D$ and the generator wins, receive a discriminator $d_t$ and obtain $f_{t+1}$ by feeding the labelled example $(\delta_{d_t}, 1)$ to $A_\Delta$.

We claim that after at most $O(T^2)$ iterations the generator outputs a distribution that fools $D$: indeed, if the algorithm continues for more than $T$ iterations then for each $t \leq T$,
\[ p_{\text{real}}(d_t) - f_t(\delta_{d_t}) = p_{\text{real}}(d_t) - p_t(d_t) \geq \epsilon. \]

Therefore, such a $T$ must satisfy:
\[
\epsilon \cdot T \leq \sum_{t=1}^{T} p_{\text{real}}(d_t) - p_t(d_t) \leq \sum_{t=1}^{T} p_{\text{real}}(d_t) - f_t(d_t) = \sum_{t=1}^{T} |y_t - f_t(d_t)| - |y_t - p_{\text{real}}(d_t)|
\]
\[
= \sum_{t=1}^{T} |y_t - f_t(d_t)| - \sum_{t=1}^{T} |y_t - \mathbb{E}_{x \sim p_t} [x(d_t)]| \leq \sum_{t=1}^{T} |y_t - f_t(d_t)| - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} |y_t - x(d_t)|
\]
\[
= \text{REGRET}_T(A_\Delta) = \tilde{O}(\sqrt{T^2 T}),
\]

where the first equality is true since $y_t = 1$. This implies that $T = O(T^2)$ as required.

We next proceed to the general case. The main challenge is that existing online classification algorithms (including the algorithm implied by Corollary 4) are not necessarily $\Delta(\mathcal{X})$-proper. To bypass it, we first observe that one can relax the requirement that $A$ is $\Delta(\mathcal{X})$-proper to the requirement that $A$ is $\Delta(\mathcal{X})$-dominated in the sense that
\[
(\forall d \in \mathcal{D}) : f_t(d) \leq p_t(d), \tag{5}
\]
for some $p_t \in \Delta(\mathcal{X})$. Indeed, the above calculation remains valid under this weaker assumption. With this definition in hand we employ the minimax theorem to identify the following win-win situation: we check whether the predictor $f_t$ which is provided by $A$ is sufficiently close to satisfying Eq. (5) (see the condition in the “If” statement in Item 1 of Fig. 2) and proceed as follows:

- If $f_t$ is sufficiently close to satisfying Eq. (5) then continue like before: in this case $f_t(d) \leq \mathbb{E}_{x \sim p_t} [x(d)] + O(\epsilon)$ for every $d \in \mathcal{D}$. The generator then submits $p_t$ to the discriminator and uses the discriminator $d_t$ provided by the discriminator as before to feed $A$ with the example $(d_t, 1)$. By a similar calculation like above, this yields an increase of $\Omega(\epsilon)$ to the regret of $A$. This case is depicted in Item 1 in Fig. 2.

- In the complementing case, a minimax argument implies that there exists $\hat{d}_t \in \Delta(\mathcal{D})$ that separates $f_t$ from all dual hypotheses $x \in \mathcal{X}$ (see Lemma 5 below):
\[
(\forall x \in \mathcal{X}) : \mathbb{E}_{d \sim \hat{d}_t} [f_t(d)] > \mathbb{E}_{d \sim \hat{d}_t} [x(d)] + \frac{\epsilon}{2}
\]

By linearity, a corollary of the above equation is that $\mathbb{E}_{d \sim \hat{d}_t} [f_t(d)] > \mathbb{E}_{d \sim \hat{d}_t} [p_{\text{real}}(d)] + \frac{\epsilon}{2}$.

We, Thus, interpret $\hat{d}_t$ as a distinguishing function, and provide it to the learner $A$ with a label $y_t = 0$ and yield an increase of $\Omega(\epsilon)$ to its regret. Note that here the discriminator is not used to find $d_t$. This case is depicted in Item 2 in Fig. 2.

To summarize, in each of the two cases, the regret of $A$ is increased by $\Omega(\epsilon)$. Therefore, by the bound on $A$’s regret, it follows that after at most $O(\ell^t/\epsilon^2)$ rounds, the generator finds a fooling distribution.
7 Proofs

7.1 Background

7.1.1 Basic properties of Differential Privacy

We will use the following three basic properties of algorithmic privacy.

**Lemma 2** (Post-Processing (Lemma 2.1 in [50])). If $M : W^m \rightarrow \Sigma$ is $(\alpha, \beta)$-differentially private and $F : \Sigma \rightarrow Z$ is any (possibly randomized) function, then $F \circ M : W^m \rightarrow Z$ is $(\alpha, \beta)$-differentially private.

**Lemma 3** (Composition (Lemma 2.3 in [50])). Let $M_1, ..., M_k : W^m \rightarrow \Sigma$ be $(\alpha, \beta)$-differentially private algorithms, and define $M : W^k \rightarrow \Sigma$ by

$$M(\Omega) = (M_1(\Omega), M_2(\Omega), ..., M_k(\Omega)).$$

Then, $M$ is $(k\alpha, k\beta)$-differentially private.

**Lemma 4** (Privacy Amplification (Lemma 4.12 in [14])). Let $\alpha \leq 1$ and let $M$ be a $(\alpha, \beta)$-differentially private algorithm operating on databases of size $u$. For $v > 2u$, construct an algorithm $M'$ that on input database $\Omega \in W^v$ subsamples (with replacement) $u$ points from $\Omega$ and runs $M$ on the result. Then $M'$ is $(\tilde{\alpha}, \tilde{\beta})$-differentially private for

$$\tilde{\alpha} = 6\alpha u / v \quad \tilde{\beta} = \exp(6\alpha u / v)4u / v \beta.$$

We remark that the requirement $\alpha \leq 1$ can be replaced by $\alpha \leq c$ for any constant $c$ at the expanse of increasing the constant factors in the definitions of $\tilde{\alpha}$ and $\tilde{\beta}$. This follows by the same argument that is used to prove Lemma 4 in [14].

7.1.2 Online Learning

The Online learnability of Littlestone classes has been established by [34] in the realizable case and by [10] in the agnostic case. Ben-David et al’s [10] agnostic Standard Online Algorithm (SOA) will serve as a workhorse for our main results and we thus recall the online learning setting and state the relevant results. For a more exhaustive survey on online learning we refer the reader to [15, 45].

In the a binary online setting we assume a domain $W$ and a space of hypotheses $H \subseteq \{0, 1\}^W$. We consider the following oblivious setting which can be described as a repeated game between a learner $L$ and an adversary continuing for $T$ rounds; the horizon $T$ is fixed and known in advanced to both players. At the beginning of the game, the adversary picks a sequence of labelled examples $(w_t, y_t)_{t=1}^T \subseteq W \times \{0, 1\}$. Then, at each round $t \leq T$, the learner chooses (perhaps randomly) a mapping $f_t : W \rightarrow [0, 1]$ and then gets to observe the labelled example $(w_t, y_t)$. The performance of the learner $L$ is measured by her regret, which is the difference between her loss and the loss of the best hypothesis in $H$:

$$\text{REGRET}_T(L; \{w_t, y_t\}_{t=1}^T) = \sum_{t=1}^T \mathbb{E}[|f_t(w_t) - y_t|] - \min_{h \in H} \sum_{t=1}^T |h(w_t) - y_t|.$$

(6)

where the expectation is taken over the randomness of the learner. Define

$$\text{REGRET}_T(L) = \sup_{\{w_t, y_t\}_{t=1}^T} \text{REGRET}_T(L; \{w_t, y_t\}_{t=1}^T).$$

The following result establishes that Littlestone classes are learnable in this setting:

**Theorem 3.** [10] Let $H$ be a class with Littlestone dimension $\ell$ and let $T$ be the horizon. Then, there exists an online learning algorithm $L$ such that

$$\text{REGRET}_T(L) \leq \sqrt{\frac{1}{2} \ell \cdot T \log T}.$$
We will need the following corollary of Theorem 3. Recall that \( \Delta(W) \) denotes the class of distributions over \( W \), and that every \( f : W \to [0, 1] \) extends linearly to \( \Delta(W) \) by \( \hat{f}(p) = \mathbb{E}_{w \sim p}[f(w)] \).

The next statement concerns an online setting where the labelled example are of the form \( (p_t, y_t) \in \Delta(W) \times \{0, 1\} \), and the regret of a learner \( L \) with respect to \( \mathcal{H} \subseteq \{0, 1\}^W \) is defined by replacing each \( h \) by its linear extension \( \hat{h} \):

\[
\text{REGRET}_T(L; (p_t, y_t)_{t=1}^T) = \sum_{t=1}^T \mathbb{E}[f_t(p_t) - y_t] - \min_{h \in \mathcal{H}} \sum_{t} |\hat{h}(p_t) - y_t|
\]

\[
= \sum_{t=1}^T \mathbb{E}[f_t(p_t) - y_t] - \min_{h \in \mathcal{H}} \sum_{x \sim p_t} |\mathbb{E}[\hat{h}(w)] - y_t|.
\]

**Corollary 4.** Let \( \mathcal{H} \) be a finite class with Littlestone dimension \( \ell \) and let \( T \) be the horizon. Then, there exists a deterministic online learner \( L \) that receives labelled examples from the domain \( \Delta(W) \) such that

\[
\text{REGRET}_T(L) \leq \sqrt{\frac{1}{2} T \log T}
\]

Moreover, at each iteration \( t \) the predictor used by \( L \) is of the form \( \hat{f}_t(p) = \mathbb{E}_{w \sim p}[f_t(w)] \), where \( f_t \) is some \( W \to [0, 1] \) function.

Corollary 4 follows from Theorem 3; see Appendix A for a proof.

### 7.2 Proof of Theorem 1

#### 7.2.1 Upper Bound: Proof of Item 1

In this section we prove the upper bound presented in Theorem 1 in the case where \( \mathcal{X} \) is finite (and in turn, \( \mathcal{D} \subset \{0, 1\}^\mathcal{X} \) is also finite). The general case is proven in a similar fashion but is somewhat more delicate. The general proof is then given in Appendix B.

First note that we may assume without loss of generality that \( \mathcal{D} \) is symmetric. Indeed, if \( \mathcal{D} \) is not symmetric then we may replace \( \mathcal{D} \) with \( \mathcal{D} \cup (1 - \mathcal{D}) \), noting that this does not affect the GAM game, namely (i) IPM\(_D\) = IPM\(_{D \cup (1 - D)}\) (and so the goal of the generator remains the same), and (ii) the set of distinguishers the discriminator may use remains the same (recall that the discriminator is allowed to use distinguishers from \( 1 - \mathcal{D} \)). Also, one can verify that this modification does not change the dual Littlestone dimension (i.e. Ldim\(_D\) = Ldim\(_{D \cup (1 - D)}\)).

Therefore, we assume \( \mathcal{D} \) is a finite symmetric class with dual Littlestone dimension \( \ell^* \). The generator used in the proof is depicted in Fig. 2. The generator uses an online learner \( \mathcal{A} \) for the dual class \( \mathcal{X} \) with domain \( \Delta(\mathcal{D}) \) as in Corollary 4, where the horizon is set to be \( T = \lceil \frac{\ell^*}{2} \log \frac{1}{\epsilon} \rceil \).

Let \( D \) be an arbitrary discriminator, let \( p_{real} \in \Delta(\mathcal{X}) \) be the target distribution, and let \( \epsilon > 0 \) be the error parameter. The proof follows from the next lemma:

**Lemma 5.** Let \( \mathcal{D} \) be a finite set of discriminators, let \( f : \mathcal{D} \to [0, 1] \), Assume that,

\[
(\forall p \in \Delta(\mathcal{X}))(\exists d \in \mathcal{D}) : \mathbb{E}_{x \sim p}[f(d) - x(d)] > \epsilon/2.
\]

Then:

\[
(\exists d \in \Delta(\mathcal{D}))(\forall x \in \mathcal{X}) : \mathbb{E}_{d \sim d}[f(d) - x(d)] > \epsilon/2.
\]

Before proving this lemma, we show how it implies the desired upper bound on the round complexity. We first argue that the algorithm never outputs “error”: indeed, since \( \mathcal{A} \) only uses predictors of the form \( \hat{f}_t(d) = \mathbb{E}_{d}[f_t] \), Lemma 5 implies that whenever Item 2 in the “For” loop is reached then an appropriate \( d_t \in \Delta(\mathcal{D}) \) exists and therefore the algorithm never outputs “error”.

Next, we bound the number of rounds; let \( T' \leq T \) be the number of iterations performed when the generator \( G \) runs against the discriminator \( D \). The only way for the generator to lose is if the “For” loop ends without its winning and \( T' = T \). Thus, It suffices to show that \( T' < T \). The argument proceeds by showing that the regret of \( \mathcal{A} \) in each iteration \( t \leq T' \) increases by at least \( \epsilon/2 \). This, combined with the bound on \( \mathcal{A} \)'s regret (from Corollary 4) will yield the desired bound.
We begin by analyzing the increase in A’s regret. Let \((\hat{d}_1, y_1), \ldots, (\hat{d}_T, y_T)\) and \(\hat{f}_1, \ldots, \hat{f}_T\) be the sequences obtained during the execution of the algorithm as defined in Fig. 2. Recall from Corollary 4 that \(\hat{f}_t(d) = \mathbb{E}_{d \sim \hat{d}[t]}[f_t(d)]\), where \(f_t : \mathcal{D} \rightarrow [0,1]\). We claim that the following holds:

\[
(\forall t \leq T') : \begin{cases} 
\mathbb{E}_{d \sim \hat{d}[t]}[\text{preal}(d) - \hat{f}_t(d)] \geq \frac{\epsilon}{2} & \text{if } y_t = 1, \\
\mathbb{E}_{d \sim \hat{d}[t]}[\hat{f}_t(d) - \text{preal}(d)] \geq \frac{\epsilon}{2} & \text{if } y_t = 0.
\end{cases}
\] (7)

Indeed, if \(y_t = 1\) then by Fig. 2, the chosen \(p_t\) satisfies

\[
(\forall d \in \mathcal{D}) : \hat{f}_t(d) - \mathbb{E}_{x \sim p_t} [x(d)] \leq \frac{\epsilon}{2}.
\]

Since the discriminator replies with \(d_t\) such that \(\text{preal}(d_t) - p_t(d_t) \geq \epsilon\), and \(d_t = \delta_{\hat{d}_t}\), it follows that

\[
\mathbb{E}_{d \sim \hat{d}_t} [\text{preal}(d) - \hat{f}_t(d)] = \mathbb{E}_{d \sim \hat{d}_t} [\text{preal}(d_t)] - \mathbb{E}_{d \sim \hat{d}_t} [f_t(d_t)] \geq \mathbb{E}_{x \sim \text{preal}} [x(d_t)] - \mathbb{E}_{x \sim p_t} [x(d_t)] + \epsilon/2
\]

\[
\geq \mathbb{E}_{x \sim \text{preal}} [x(d_t)] - (p_t(d_t) + \epsilon/2)
\]

\[
\geq \frac{\epsilon}{2},
\]

which is the first case in Eq. (7). Next consider the case when \(y_t = 0\). Since the algorithm never outputs “error”, Fig. 2 implies that:

\[
(\forall x \in \mathcal{X}) : \hat{f}_t(d_t) - \mathbb{E}_{d \sim d_t} [x(d)] > \frac{\epsilon}{2}.
\]

Therefore, by linearity of expectation, \(\mathbb{E}_{d \sim \hat{d}_t} [\hat{f}_t(d) - \text{preal}(d)] = \hat{f}_t(d_t) - \mathbb{E}_{d \sim \hat{d}_t} [\text{preal}(d)] \geq \frac{\epsilon}{2}\), which amounts to the second case in Eq. (7).

We are now ready to conclude the proof by showing that \(T' < T\). Assume towards contradiction that \(T' = T\). Therefore, by Eq. (7):

\[
T \epsilon \frac{T}{2} \leq \sum_{t=1}^{T} \mathbb{E}_{d \sim \hat{d}_t} [\text{preal}(d) - \hat{f}_t(d)]
\]

\[
= \sum_{t=1}^{T} \left| y_t - \mathbb{E}_{d \sim \hat{d}_t} [\hat{f}_t(d)] \right| - \left| y_t - \mathbb{E}_{d \sim \hat{d}_t} [\text{preal}(d_t)] \right| \quad (y_t = 1 \iff \mathbb{E}_{d \sim \hat{d}_t} [\text{preal}(d_t)] \geq \mathbb{E}_{d \sim \hat{d}_t} [f_t(d)])
\]

\[
= \sum_{t=1}^{T} \left| y_t - \hat{f}_t(d_t) \right| - \mathbb{E}_{x \sim \text{preal}} \left[ y_t - \mathbb{E}_{d \sim \hat{d}_t} [x(d_t)] \right]
\]

\[
\leq \sum_{t=1}^{T} \left| y_t - \hat{f}_t(d_t) \right| - \min_{x \in \mathcal{X}} \left| y_t - \mathbb{E}_{d \sim \hat{d}_t} [x(d)] \right|
\]

\[
\leq \text{REGRET}_T(A).
\]

\[
\leq \sqrt{\frac{1}{2} \ell^* T \log T}
\]

Thus, we obtain that \(\frac{T}{\log T} \leq \frac{2 \ell^*}{\epsilon^2}\), however our choice of \(T = \left[ \frac{4 \ell^*}{\epsilon^2} \log \frac{4 \ell^*}{\epsilon^2} \right]\) ensures that this is impossible. Indeed:

\[
\frac{T}{\log T} \geq \frac{4 \ell^*}{\log \frac{4 \ell^*}{\epsilon^2} + \log \log \frac{4 \ell^*}{\epsilon^2}}
\]

\[
= \frac{4 \ell^*}{1 + \frac{\log \log \frac{4 \ell^*}{\epsilon^2}}{\log \frac{4 \ell^*}{\epsilon^2}}}
\]

\[
> \frac{4 \ell^*}{2}
\]

\[
= \frac{2 \ell^*}{\epsilon^2}.
\]
This finishes the proof of Item 1.

We end this section by proving Lemma 5.

**Proof of Lemma 5.** The proof hinges on Von Neuman’s Minimax Theorem. Let $D, f$ as in the formulation of the theorem, and consider the following zero-sum game: the pure strategies of the maximizer are indexed by $d \in D$, the pure strategies of the minimizer are indexed by $x \in X$, and the payoff (for pure strategies) is defined by $m(d, x) = f(d) - x(d)$. Note that the payoff function for mixed strategies $d \in \Delta(D), p \in \Delta(X)$ satisfies

$$m(d, p) = \mathbb{E}_{z \sim p}[f(d) - \mathbb{E}_{d \sim d}[x(d)] = \mathbb{E}_{d \sim d}[f(d) - \mathbb{E}_{x \sim p}[x(d)]]$$

We next apply Von Neuman’s Minimax Theorem on this game (Here we use the assumption that $X$ and, in turn, $D$ are finite). The premise of the lemma amounts to

$$\min_{p \in \Delta(X)} \max_{d \in D} m(d, p) > \epsilon/2.$$  

Therefore, by the Minimax Theorem also

$$\max_{d \in \Delta(D)} \min_{x \in X} m(d, x) > \epsilon/2,$$

which amounts to the conclusion of the lemma. \hfill \Box

**A remark.** A natural variant of the GAM setting follows by letting the discriminator $D$ to adaptively change the target distribution $p_{\text{real}}$ as the game proceeds ($D$ would still be required to maintain the existence of a distribution $p_{\text{real}}$ which is consistent with all of its answers). This modification allows for stronger discriminators and therefore, potentially, for a more restrictive notion of GAM–Foolability. However, the above proof extends to this setting verbatim.

### 7.2.2 Lower Bound: Proof of Item 2

Let $D$ be a class as in the theorem statement, let $G$ be a generator for $D$, and let $\epsilon < \frac{1}{2}$. We will construct a discriminator $D$ and a target distribution $p_{\text{real}}$ such that $G$ requires at least $\frac{\ell^*}{T}$ rounds in order to find $p$ such that $\text{IPM}_D(p, p_{\text{real}}) \leq \epsilon$.

To this end, pick a shattered mistake-tree $T$ of depth $\ell^*$ whose internal nodes are labelled by elements of $D$ and whose leaves are labelled by elements of $X$.

**The discriminator.** The target distribution will be a Dirac distribution $\delta_x$ where $x$ is one of the labels of $T$’s leaves. We will use the following discriminator $D$ which is defined whenever $p_{\text{real}}$ is one of these distributions: assume that $p_{\text{real}} = \delta_x$, and consider all functions in $D$ that label the path from the root towards the leaf whose label is $x$,

$$d_1, d_2, \ldots, d_{\ell^*}.$$ 

Let $p_1$ be the distribution the generator submitted in the first round. Then the discriminator picks the first $i$ such that $|p_i(d_1) - p_{\text{real}}(d_1)| > \epsilon$, and sends the generator either $d_i$ or $1 - d_i$ according to the convention in Eq. (3). If no such $d_i$ exists, the discriminator outputs WIN. Similarly, at round $t$ let $i_{t-1}$ denote the index of the distinguisher sent in the previous round; then, the discriminator acts the same with the modification that it picks the first $i_{t-1} + 1 \leq i \leq \ell^*$ such that $|p_i(d_i) - p_{\text{real}}(d_i)| > \epsilon$.

**Analysis.** The following claim implies that for every generator $G$, there exists a distribution $\delta_x$ such that if $p_{\text{real}} = \delta_x$ then the above discriminator $D$ forces $G$ to play at least $\ell^*/2$ rounds.

**Claim 1.** Let $G$ be a generator for $D$. Pick $p_{\text{real}}$ uniformly at random from the set $\{\delta_x : x \text{ labels a leaf in } T\}$. Then the expected number of rounds in the GAM game when $G$ is the generator and $D = D(T)$ is the discriminator is at least $\frac{\ell^*}{T}$.
Proof. For every \( i \leq \ell^* \), let \( X_i \) denote the indicator of the event that the \( i \)’th function on the path towards the leaf corresponding to \( p_{\text{real}} \) was used by \( D \) as a distinguisher. Note that the number of rounds \( X \) satisfies \( X = \sum_{i=1}^{\ell^*} X_i \). Thus, by linearity of expectation it suffices to argue that

\[
\mathbb{E}[X_i] = \Pr[X_i = 1] \geq \frac{1}{2}
\]

Consider \( X_1 \): let \( p_1 \) denote the first distribution submitted by \( G \). Note that \( X_1 = 1 \) if

(i) \( p_1(d_1) \geq \frac{1}{2} \) and the leaf labelled \( x \) belongs to the left subtree from the root, or

(ii) \( p_1(d_1) < \frac{1}{2} \) and the leaf labelled \( x \) belongs to the right subtree from the root.

In either way \( \Pr[X_1 = 1] \geq \frac{1}{2} \), since this leaf is drawn uniformly. Similarly, for every conditioning on the values of \( X_1, \ldots, X_{i-1} \) we have \( \Pr[X_i = 1|X_1 \ldots X_{i-1}] \geq \frac{1}{2} \) (follows from the same argument applied on subtrees corresponding to the conditioning). This yields that \( \mathbb{E}[X_i] = \Pr[X_i = 1] \geq \frac{1}{2} \) for every \( i \) as required.

\[ \square \]

7.3 Proof of Theorem 2

Proof Roadmap. We will show the following entailments: \( 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1 \). Then, given the equivalence between Items 1 to 3 we will show that \( 1 \iff 4 \). This will conclude the proof.

Overview of \( 1 \Rightarrow 2 \). We next overview the derivation of \( 1 \Rightarrow 2 \) which is the most involved derivation. Let \( p_{\text{real}} \) denote the target distribution we wish to fool. The argument relies on the following simple observation: let \( S \) be a sufficiently large independent sample from \( p_{\text{real}} \). Then, it suffices to privately output a distribution \( p_{\text{syn}} \) such that IPM \( D(p_{\text{syn}}, p_S) \leq \frac{\epsilon}{2} \), where \( p_S \) is the empirical distribution. Indeed, if \( S \) is sufficiently large then by standard uniform convergence bounds: IPM \( D(p_S, p_{\text{real}}) \leq \frac{\epsilon}{2} \), which implies that IPM \( D(p_{\text{syn}}, p_{\text{real}}) \leq \epsilon \) as required.

The output distribution \( p_{\text{syn}} \) is constructed using a carefully tailored Generative Adversarial Machine with a private discriminator \( D \). That is, \( D \)'s input distribution is the empirical distribution \( p_S \), and for every submitted distribution \( p_t \), it either replies with a discriminating function \( d_t \) or with “WIN” if no discriminating function exists. The crucial point is that it does so in a differentially private manner with respect to the input sample \( S \). The existence of such a discriminator \( D \) follows via the assumed PAP-PAC learner.

Once the private discriminator \( D \) is constructed, we turn to find a generator \( G \) with a bounded round complexity. This follows from Theorem 1 and a result by [1, 14]: by [1, 14] PAP-PAC learnability implies a finite Littlestone dimension, and therefore by Theorem 1 there is a generator \( G \) with a bounded round complexity. The desired DP fooling algorithm then follows by letting \( G \) and \( D \) play against each other and outputting the final distribution that \( G \) obtains. The privacy guarantee follows by the composition lemma (Lemma 3) which bounds the privacy leakage in terms of the number of rounds (which is bounded by the choice of \( G \)) and the privacy leakage per round (which is bounded by the choice of \( D \)).

One difficulty that is handled in the proof arises because the discriminator is differentially private and because the PAP-PAC algorithm may err with some probability. Indeed, these prevent \( D \) from satisfying the requirements of a discriminator as defined in the GAM setting. In particular, \( D \) cannot reply deterministically whether IPM \( D(p_S, p_t) < \epsilon \) as this could compromise privacy. Also, whenever the assumed PAP-PAC algorithm errs, \( D \) may reply with an illegal distinguisher that does not satisfy Eq. (3).

To overcome this difficulty we ensure that \( D \) satisfies the following with high probability: if IPM \( D(p_S, p_t) > \epsilon \) then \( D \) outputs a legal \( d_t \), and if IPM \( D(p_S, p_t) < \frac{\epsilon}{2} \) then it outputs WIN as required. When \( \frac{\epsilon}{2} \leq \text{IPM}(p_S, p_t) \leq \epsilon \) it may either output WIN or a legal discriminator \( d_t \). As we show in the proof, this behaviour of \( D \) will not affect the correctness of the overall argument.

Proof of Theorem 2. The equivalence is proven by showing: \( 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1 \) and \( 1 \iff 4 \).
Let $p_{\text{real}}$ denote the unknown target distribution and let $\epsilon_0, \delta_0$ be the error and confidence parameters. Draw independently from $p_{\text{real}}$ a sufficiently large input sample $S$ of size $|S|$ to be specified later. At this point we require $|S|$ to be large enough so that $\text{IPM}_\mathcal{D}(p_{\text{real}}, p_S) \leq \frac{\delta}{2}$ with probability at least $1 - \frac{1}{2}$. By standard uniform convergence bounds ([51]) it suffices to require

$$|S| \geq \Omega \left( \frac{d + \log(1/\delta_0)}{\epsilon_0^2} \right),$$

where $d$ is the VC-dimension of $\mathcal{D}$ (observe that $\mathcal{D}$ must have a finite VC dimension as it is PAC learnable). By the triangle inequality, this reduces our goal to privately output a distribution $p_{\text{syn}}$ so that $\text{IPM}_\mathcal{D}(p_S, p_{\text{syn}}) \leq \frac{\delta}{2}$ with probability $1 - \frac{\delta}{2}$ (this will imply that $\text{IPM}_\mathcal{D}(p_{\text{real}}, p_{\text{syn}}) \leq \epsilon_0$ with probability $1 - \delta_0$).

As explained in the proof outline, the latter task is achieved by a carefully tailored Generative Adversarial Machine which we will next describe. In order to construct the desired GAM, we first observe that $\mathcal{D}$ is GAM-Forbidable. Indeed, by Corollary 1 it suffices to argue that $\mathcal{D}$ has a finite Littlestone dimension, which follows by [1] since $\mathcal{D}$ is privately learnable.

Now, pick a generator $G$ that fools $\mathcal{D}$ with round complexity $T(\epsilon)$ as in Theorem 1, and pick a discriminator $D$ as in Fig. 3. Note that $D$ uses a PAP-PAC learner for the class $\mathcal{D} \cup (1 - \mathcal{D})$ whose existence follows by from the PAP-PAC learnability of $\mathcal{D}$ via standard arguments (which we omit). The next lemma summarizes the properties of $D$ that are needed for the proof.

**Lemma 6.** Let $D$ be the discriminator defined in Fig. 3 with input parameters $(\epsilon, \delta, \tau)$ and input sample $S$, and let $M$ be the assumed PAP-PAC learner for $\mathcal{D} \cup (1 - \mathcal{D})$ with sample complexity $m(\epsilon, \delta)$ and privacy parameters $(\alpha, \beta)$. Then, $D$ is $(6\tau\alpha(\tau|S|) + \tau, 4e^{6\tau\alpha(\tau|S|)\tau}\beta(\tau|S|))$-private, and if $S$ satisfies

$$|S| \geq \max \left( \frac{m(\epsilon/8, \tau\delta/2)}{\tau}, \frac{64\log(\tau\delta/2)}{\epsilon\tau} \right)$$

then following holds with probability at least $(1 - \tau\delta)$

(i) If $D$ outputs $d_t$, then $p_S(d_t) - p_d(d_t) \geq \frac{\epsilon}{2}$.

(ii) If $D$ outputs “WIN” then $\text{IPM}_\mathcal{D}(p_S, p_d) \leq \epsilon$.

We first use Lemma 6 to conclude the proof of 1$\Rightarrow$2 and then prove Lemma 6.

**The fooling algorithm we consider proceeds as follows.**

- Set the number of rounds $T_0 = |S|^{0.99}$.
- Set $G$ to be a generator with round complexity $T(\epsilon)$ and set its error parameter to be $\frac{\epsilon}{2}$.
- Set $D$ be the discriminator depicted in Fig. 3 and set its parameters to be $(\epsilon, \delta, \tau) = (\frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{1}{10})$ and its input sample to be $S$.
- Let $G$ and $D$ to play against each other for (at most) $T_0$ rounds.
- Output the final distribution which is held by $G$.

We next prove the privacy and fooling properties as required by a DP algorithm.

**Privacy.** We argue that the algorithm is $(\alpha', \beta')$-private, with $\alpha'(|S|) = O(1)$ and $\beta'(|S|)$ negligible. Note that since $G$ is deterministic then the output distribution $p_{\text{out}}$ is completely determined by the sequence of discriminating functions $d_1, \ldots, d_{T'}$ outputted by the discriminator.

For simplicity and without loss of generality we assume that $T' = T_0$: indeed, if $T' < T_0$ then extend it by repeating the last discriminating function; this does not change the fact that $p_{\text{out}}$ is determined by the sequence $d_1, \ldots, d_{T'}, \ldots, d_{T_0}$.

Recall that by Lemma 6 $D$ is $((6\tau_0\alpha(\tau_0|S|) + \tau_0), (4e^{6\tau_0\alpha(\tau_0|S|)\tau_0}\beta(\tau_0|S|)))$-private. Therefore, since the number of rounds in which $D$ is applied is $T_0$, by composition (Lemma 3) and post-processing (Lemma 2) it follows that the entire algorithm is

$$\left( T_0(6\tau_0\alpha(\tau_0|S|) + \tau_0), T_0(4e^{6\tau_0\alpha(\tau_0|S|)\tau_0}\beta(\tau_0|S|)) \right)$$

- private.

\[^{9}\text{In fact, since } \mathcal{D} \text{ is properly privately learnable the result is already a corollary of the lower bound for proper private learning by [14], combined with Theorem 3 in [1].}\]
Our choices of \( \tau_0 = \frac{1}{e^0} \) and \( T_0 \) guarantee that \( 1/\tau_0 < m^{0.99} \), and plugging it in yields privacy guarantee of \((6\alpha(|S|^{0.001}) + 1, 4e^{O(1)}\beta(|S|^{0.001}) \). As \( \alpha(|S|^{0.001}) = O(1) \) and \( \beta(|S|^{0.001}) \) is negligible, the desired privacy guarantee follows.

**Fooling.** First note that if \( S \) satisfies Eq. (9) with \((\epsilon, \delta, \tau) := (\epsilon_0, \delta_0, \tau_0) \) then with probability at least \( 1 - \frac{1}{e^0} \) the following holds: in every iteration \( t \leq T_0 \), either \( p_{S}(d_i) - p_{c}(d_i) \geq \frac{\epsilon}{e^0} \), or the discriminator yields WIN and \( \text{IPM}_{\mathcal{D}}(p_{S}, p_{c}) \leq \frac{\epsilon}{e^0} \). This follows by a union bound via the utility guarantee in Lemma 6. Assuming this event holds, we claim that if \(|S|\) is set to satisfy \(|S|^{0.99} \geq T(\frac{\epsilon}{e^0})\) then the output distribution \( p_{\text{syn}} \) satisfies \( \text{IPM}_{\mathcal{D}}(p_{\text{real}}, p_{\text{syn}}) \leq \frac{\epsilon}{e^0} \). This follows since as long as the GAM game proceeds the generator suffers a loss of at least \( \frac{\epsilon}{e^0} \) in every round, and the number of rounds is set as \(|S|^{0.99}\). Therefore we require

\[
|S|^{0.99} \geq T\left(\frac{\epsilon_0}{e^0}\right) = \Omega\left(\frac{e^\ell}{\epsilon_0^\ell} \log \frac{e}{\epsilon_0}\right).
\]

(10)

To conclude, if \(|S|\) is set to satisfy Eqs. (8) to (10) then with probability at least \( 1 - \frac{1}{\delta_0} \) both \( \text{IPM}_{\mathcal{D}}(p_{\text{real}}, p_{S}) \leq \frac{\epsilon}{e^0} \) and \( \text{IPM}_{\mathcal{D}}(p_{S}, p_{\text{syn}}) \leq \frac{\epsilon}{e^0} \), which implies that \( \text{IPM}_{\mathcal{D}}(p_{\text{real}}, p_{\text{syn}}) \leq \epsilon_0 \) as required. This concludes the proof of 1 \( \Rightarrow \) 2.

**Proof of Lemma 6.** Let \( S \) be the input sample, let \( p_{S} \) denote the uniform distribution over \( S \), and let \( p_{c} \) denote the distribution submitted by the generator. The discriminator operates as follows (see Fig. 3): it feeds the assumed PAP-PAC learner a labeled sample \( S_{t} = \{(x_i, y_i)\} \) that is drawn from the following distribution \( q_{t} \): first the label \( y_i \) is drawn uniformly from \( \{0, 1\} \); if \( y_i = 0 \) then draw \( x_i \sim p_{S} \) and if \( y_i = 1 \) then draw \( x_i \sim p_{c} \). Let \( d_{t} \) denote the output of the PAP-PAC learner on the input sample \( S \). Observe that the loss \( L_{q_{t}}(\cdot) \) satisfies

\[
L_{q_{t}}(d) = \frac{p_{S}(d) + (1 - p_{c}(d))}{2} = \frac{1 + p_{S}(d) - p_{c}(d)}{2}.
\]

(11)

Next, the discriminator checks whether \( p_{S}(d_{t}) - p_{c}(d_{t}) \geq \frac{\epsilon}{e^0} \) (equivalently, if \( L_{q_{t}}(d_{t}) < \frac{1 - \frac{1}{e^0}}{2} \)), and sends \( d_{t} \) the generator if so, and reply with “WIN” otherwise. The issue is that checking this "If" condition naively may violate privacy, and in order to avoid it we add noise to this check by a mechanism from [20] (see Fig. 4): roughly, this mechanism receives a data set of scalars \( \Sigma = \{\sigma_{i}\}_{i=1}^{m} \), a threshold parameter \( c \) and a margin parameters \( N \), and outputs \( \pm \) if \( \sum_{i=1}^{m} \sigma_{i} > c + O(1/N) \) or \( - \) if \( \sum_{i=1}^{m} \sigma_{i} < c - O(1/N) \). The distinguisher applies this mechanism over the sequence of scalars \( \{d_{t}(x_{1}), \ldots, d_{t}(x_{m})\} \).

We next formally establish the privacy and utility guarantees of \( D \). In what follows, assume that the input sample \( S \) satisfies Eq. (9).

**Privacy.** The discriminator \( D \) is a composition of two procedures, \( M_{1} \) and \( M_{2} \), where \( M_{1} \) applies the PAP-PAC learner \( M \) on the random subsample \( S_{t} \), and \( M_{2} \) runs the procedure THRESH. Thus, the privacy guarantee will follow from the composition lemma (Lemma 3) if we show that \( M_{1} \) is \((6\alpha(\tau m), 4e^{6\alpha(\tau m)}\tau\beta(\tau m))\)-private and \( M_{2} \) is \((\tau, 0)\)-private. The privacy guarantee of \( M_{1} \) follows by applying\textsuperscript{10} Lemma 4 with \( v := |S| \) and \( n := |S_{t}| = \tau|S| \), and the privacy guarantee of \( M_{2} \) follows from the statement in Fig. 4 since \( \frac{N}{|S|} = \frac{|S_{t}|}{|S|} = \tau \).

**Utility.** Let \( q_{t} \) denote the distribution from which the subsample \( S_{t} \) is drawn. Note that by Eq. (9), \( S_{t} = \tau \cdot |S| \geq m(\epsilon/8, \tau \delta/2) \). Therefore, since \( M \) PAC learns \( D \), its output \( d_{t} \) satisfies:

\[
L_{q_{t}}(d_{t}) \leq \min_{d \in D_{t}(1-D)} L_{q_{t}}(d) + \frac{\epsilon}{8},
\]

with probability at least \( 1 - \tau \delta/2 \). By Eq. (11) this is equivalent to

\[
p_{S}(d_{t}) - p_{c}(d_{t}) \geq \max_{d \in D_{t}(\tau-D)} (p_{S}(d) - p_{c}(d)) - \epsilon/4.
\]

(12)

Now, by plugging in the statement in Fig. 4: \((\Sigma, c, N) := \{d_{t}(x)\}_{x \in S}, p_{c}(d_{t}) + \frac{\epsilon}{e^0}, |S_{t}|\), and \( \gamma := \tau \delta/2 \) and conditioning on the event that both \( M \) and THRESH succeed (which occurs with probability at least \( 1 - \tau \delta \)) it follows that

\textsuperscript{10}Note that in order to apply Lemma 4 on \( M_{1} \), we need to assume that \( M \) satisfies \((\alpha, \beta)\) privacy with \( \alpha \leq 1 \). This assumption does not lose generality -- see the paragraph following the definition of Private PAC Learning.
Let $M$ be a PAP-PAC learner for the class $\mathcal{D} \cup (1 - \mathcal{D})$ with sample complexity $m(\epsilon, \delta)$.

Let $\epsilon, \delta, \tau$ be the input parameters.

Let $S$ be the input sample, let $p_S$ be the uniform distribution over $S$, and let $p_t$ be the distribution submitted by the generator.

Draw a labelled sample $S_\ell = \{(x_i, y_i)\}$ of size $\tau \cdot |S|$ independently as follows: draw the label $y_i$ uniformly from $\{0, 1\}$

(i) if $y_i = 0$ then draw $x_i \sim p_S$,

(ii) if $y_i = 1$ then draw $x_i \sim p_t$.

Apply the learner $M$ on the sample $S_\ell$ and set $d_\ell \in \mathcal{D}$ as its output.

Compute $Z := \text{THRESH} (\{d_\ell(x)\}_{x \in S}, p_t(d_\ell) + \frac{\epsilon}{\tau}, |S_\ell|)$.

(i) If $Z = \top$ then send the generator with $d_t$,

(ii) else, $Z = \bot$ and reply the generator with “Win”.

This concludes the proof of Lemma 6.

\[ 2 \Rightarrow 3 \]

This follows directly from the definition of a DP-Fooling algorithm. Indeed, given a DP-Fooling algorithm with sample complexity $m(\epsilon, \delta)$ and a sample $S$ outputs a distribution $p_{\text{sym}}$ such that $\text{IPM}(p_{\text{sym}}, p_S) \leq \epsilon$, with probability at least $(1 - \delta)$ and satisfies $(\alpha, \beta)$-privacy, with $\alpha = O(1)$ and $\beta$ negligible. To obtain a sanitizer, output the estimate $\text{EST} : \mathcal{D} \rightarrow [0, 1]$, where $\text{Est}(d) = \mathbb{E}_{x \sim p_{\text{sym}}} [d(x)]$.
3⇒1. This follows from Theorem 5.5 in [8].

4⇒1. This is an immediate corollary of post-processing for differential privacy (Lemma 2). Indeed, by the private uniform convergence property we can privately estimate the losses of all hypotheses in $\mathcal{D}$, and then output any hypothesis in $\mathcal{D}$ that minimizes the estimated loss.

1⇒4. Suppose $\mathcal{D}$ is PAP-PAC learnable by an algorithm $A$. For every function $d \in \mathcal{D}$, let $d'$ denote the $(X \times \{0, 1\}) \to \{0, 1\}$ function defined by $d'((x,y)) = 1[d(x) \neq y]$, and let $\mathcal{D}' = \{d' : d \in \mathcal{D}\}$. Observe that for every sample $S \subseteq (X \times \{0, 1\})^m$:

$$L_S(d) = p_S(d'),$$

(13)

where $L_S(d)$ denotes the empirical loss of $d$ and $p_S$ denotes the empirical measure of $d'$.

We claim that $\mathcal{D}'$ is also PAC-PAC learnable: for a $\mathcal{D}'$-example $z' = ((x, y), y')$ let $z$ denote the $\mathcal{D}$-example $(x, |y' - y|)$, and note that $d'$ errs on $z'$ if and only if $d$ errs on $z$. Therefore, a PAC-PAC learner for $\mathcal{D}'$ follows by using this transformation to convert the $\mathcal{D}'$-input sample $S' = \{z'_i\}_{i=1}^m$ to a $\mathcal{D}$ input sample $S = \{z_i\}_{i=1}^m$, applying $A$ on $S$ and outputting $d'$, where $d = A(S)$.

Therefore, by 1 ⇒ 3 it follows that $\mathcal{D}'$ is sanitizable by a sanitizer $M$ with sample complexity $m_1(\epsilon, \delta)$. We next use $M$ to show that $\mathcal{D}$ satisfies private uniform convergence: let $\mathbb{P}$ be a distribution over $\mathcal{X} \times \{0, 1\}$ and $\epsilon, \delta$ be the error and confidence parameters. Consider the following algorithm:

- Draw a sample $S$ from $\mathbb{P}$ of size $m(\epsilon, \delta) = \max\{m_1(\frac{\epsilon}{2}, \frac{\delta}{2}), m_2(\frac{\epsilon}{2}, \frac{\delta}{2})\}$, where

$$m_2 = O\left(\frac{\text{VC}(\mathcal{D}) + \log(1/\delta)}{\epsilon^2}\right)$$

is the uniform convergence rate of $\mathcal{D}$ (note that by PAC learnability, $\text{VC}(\mathcal{D}) < \infty$).

- Apply $M$ on $S$ to obtain an estimator $\text{EST}' : \mathcal{D}' \to [0, 1]$ and output the estimator $\text{EST} : \mathcal{D} \to [0, 1]$ defined by $\text{EST}(d) = \text{EST}'(d')$.

We want to show that

$$\forall d \in \mathcal{D} : \text{EST}(d) - L_P(d) \leq \epsilon,$$

with probability $1 - \delta$. Indeed, since $m \geq m_2(\frac{\epsilon}{2}, \frac{\delta}{2})$ it follows that

$$\forall d \in \mathcal{D} : |L_S(d) - L_P(d)| \leq \epsilon \frac{\delta}{2},$$

with probability at least $1 - \frac{\epsilon}{2}$, and since $m \geq m_1(\frac{\epsilon}{2}, \frac{\delta}{2})$,

$$\forall d \in \mathcal{D} : |\text{EST}(d) - L_S(d)| = |\text{EST}'(d') - p_S(d')|$$

(by Eq. (13))

$$\leq \epsilon / 2,$$

with probability $1 - \frac{\delta}{2}$. The desired bound thus follows by a union bound and the triangle inequality.

\[\square\]

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A Proof of Corollary 4

We begin by defining the predictors \( \hat{f}_t \)'s that \( L \) uses: let \( L_0 \) be the learner implied by Theorem 3. We first turn \( L_0 \) into a deterministic learner whose input is \((p_1, y_1), \ldots, (p_T, y_T)\) and that outputs at each iteration \( f_t : \mathcal{W} \to [0, 1] \). Then, we extend \( f_t \) linearly to \( \hat{f}_t \) as discussed in Section 3.1. Let \((p_1, y_1), \ldots, (p_T, y_T) \in \Delta(\mathcal{W}) \times \{0, 1\}\), given \( w \in \mathcal{W} \), the value \( f_t(w) \) is the expected output of the following random process:

- sample \( w_t \sim p_t \) for \( i \leq t - 1 \),
- apply \( L_0 \) on the sequence \((w_1, y_1), \ldots, (w_{t-1}, y_{t-1})\) to obtain the predictor \( \hat{f}_t \), and
- output \( \hat{f}_t \).

That is,
\[
f_t(x) = \mathbb{E}_{w_{1:t-1}} \left[ \mathbb{E}_{f_t \sim L_0} \left[ \hat{f}_t(w) \mid w_1 \ldots w_{t-1} \right] \right].
\]

where \( \mathbb{E}_{w \sim [i]} \) denotes the expectation over sampling each \( w_i \) from \( p_i \) independently, and \( \mathbb{E}_{\hat{f} \sim L_0} \) denotes the expectation over the internal randomness of the algorithm \( L_0 \) at iteration \( t \). Finally, \( \hat{f}_t(p) = \mathbb{E}_{w \sim [i]}[f_t(w)] \) is the predictor that \( L \) uses at the \( t \) th round. Note that indeed \( \hat{f}_t \) is determined (deterministically) from \((p_1, y_1), \ldots, (p_{t-1}, y_{t-1})\).

We next bound the regret: for every \( h \in \mathcal{H} \):

\[
\sum_{t=1}^T \left| \hat{f}_t(p_t) - y_t \right| - \left| \hat{h}(p_t) - y_t \right| = \sum_{t:y_t=0} \hat{f}_t(p_t) - \hat{h}(p_t) + \sum_{t:y_t=1} \hat{h}(p_t) - \hat{f}_t(p_t)
\]

\[
= \sum_{t:y_t=0} \mathbb{E}_{p_{1:t-1}} \left[ \mathbb{E}_{L_0 \sim p_t} \left[ f_t(w_t) \right] \mid \{w_i\}_{i=1}^{t-1} \right] - \mathbb{E}_{p_{1:T}} \left[ h(x_t) \right]
\]

\[
+ \sum_{t:y_t=1} \mathbb{E}_{p_{1:T}} \left[ h(w_t) \right] - \mathbb{E}_{p_{1:T}} \left[ \mathbb{E}_{L_0 \sim p_t} \left[ f_t(w_t) \right] \mid \{w_i\}_{i=1}^{t-1} \right]
\]

\[
= \mathbb{E}_{p_{1:T}} \left[ \mathbb{E}_{L_0 \sim p_t} \left[ \sum_{y_t=0} f_t(w_t) - h(w_t) + \sum_{y_t=1} h(w_t) - f_t(w_t) \right] \mid \{w_i\}_{i=1}^{t-1} \right]
\]

\[
= \mathbb{E}_{p_{1:T}} \left[ \mathbb{E}_{L_0 \sim p_t} \left[ \sum_{t=1}^T \left| f_t(w_t) - y_t \right| - \left| h(w_t) - y_t \right| \right] \mid \{w_i\}_{i=1}^{T} \right]
\]

\[
\leq \mathbb{E}_{p_{1:T}} \left[ \text{REGRET}_T(L_0, \{w_t, y_t\}_{t=1}^T) \right]
\]

\[
\leq \sqrt{\frac{1}{2} T \log T}.
\]

\[
\square
\]

B Extending Theorem 1, Item 1 to infinite classes

Here we extend the proof of the upper bound in Theorem 1 to the case where \( \mathcal{X} \) may be infinite.

**Technical assumptions.** The first technical milestone we need to consider in order to make the GAM setting well-defined is to pick a \( \sigma \)-algebra on \( \mathcal{X} \), which specifies the domain \( \Delta(\mathcal{X}) \) from which the generator’s distributions are taken. Clearly, every \( d \in \mathcal{D} \) need to be measurable and we therefore pick any\(^\dagger\) \( \sigma \)-algebra that contains \( \mathcal{D} \).

\(^\dagger\)Since the intersections of \( \sigma \)-algebras is a \( \sigma \)-algebra, one can simply take the intersection of all \( \sigma \)-algebras that contain \( \mathcal{D} \).
Next, the proposed protocol in Fig. 2 chooses at each round a distribution \( \bar{d} \in \Delta(D) \). Thus we also need to define a \( \sigma \)-algebra over the class \( D \) and specify the space \( \Delta(D) \). For this, we equip \( \{0,1\}^X \) with the product topology (see Appendix B.1 for a definition) and consider \( D \subseteq \{0,1\}^X \) with the induced subspace topology. This allows us to identify \( \Delta(D) \) as the corresponding space of Borel-probability measures.

Finally, we assume that \( X \) is well-behaved as a class of \( D \to \{0,1\} \) functions in the sense that the standard uniform convergence property of VC classes applies to it\(^{12}\): for any measure \( d \in \Delta(D) \) and \( \epsilon > 0 \), a sufficiently large independent sample \( d_1 \ldots d_m \sim d \) satisfies with high probability that

\[
(\forall x \in X) : \left| \mathbb{E}_{d \sim d} [d(x)] - \frac{1}{m} \sum_{i=1}^{m} d_i(x) \right| \leq \epsilon.
\]

This uniform convergence property is implied by the finiteness of VC(\( X \)) under certain measurability assumptions [51]. For the sake of brevity we simply assume this uniform convergence statement and refer the reader to [9, 18] for standard measure theoretic assumptions that imply it.

**A modification to the generator in Fig. 2.** We will make some technical modifications in the generator depicted in Fig. 2. The modification is depicted in Fig. 5. The first modification

Consider Fig. 2 with the following modification, at the Else Step:

- Find \( \bar{d}_t \in \Delta(D) \), with finite support such that

\[
(\forall x \in X) : \mathbb{E}_{d \sim \bar{d}_t} [f_t(d) - x(d)] > \epsilon \frac{4}{2}
\]

(if no such \( \bar{d}_t \) exists then output “error”).

![Figure 5: Modifying the algorithm in Fig. 2](image)

we make is that in the Else step, the generator picks \( \bar{d}_t \) with finite support. For the finite case, the requirement that \( \bar{d}_t \) has finite support is met trivially. The second modification is to allow a further slack for the distinguisher (require advantage of \( \geq \frac{\epsilon}{2} \) instead of \( \geq \frac{\epsilon}{2} \)).

**Proof outline.** To extend the proof to the infinite case it suffices to ensure that the generator in Fig. 2 (with the modification in Fig. 5) never outputs “error” in the 2nd item of the “For” loop. Towards this end, let us add the following notation that is consistent with the algorithm in Fig. 2.

Let \( f : D \to [0,1] \) be measurable.

1. If there exists \( p \in \Delta(X) \) such that

\[
(\forall d \in D) : \mathbb{E}_{x \sim p} [f(d) - x(d)] \leq \frac{\epsilon}{2},
\]

we say that \( f \) satisfies Item 1.

2. If there exists \( \bar{d} \in \Delta(D) \) such that

\[
(\forall x \in X) : \mathbb{E}_{d \sim \bar{d}} [f(d) - x(d)] > \frac{\epsilon}{2},
\]

we say that \( f \) satisfies Item 2.

3. \( f \) is amenable if it satisfies either Item 1 or Item 2.

When \( X \) and \( D \) are finite, every \( f \) satisfies one of Items 1 or 2 (and hence amenable). This is the content of Lemma 5 which is proved using LP duality (in the form of the Minmax Theorem). However, the case when \( X \) and \( D \) are infinite is more subtle. Specifically, the Minmax Theorem does not necessarily hold in this generality. Nevertheless, the next lemma guarantees the existence of a learner \( A \) which only outputs amenable functions. Recall that \( f : \Delta(D) \to [0,1] \) denotes the linear extension of \( f \) and is defined by \( f(d) = \mathbb{E}_{d \sim D} [f(d)] \).

\(^{12}\)note that VC(\( X \)) \( < \infty \) since it is the dual class of \( D \), and VC(\( D \)) \( \leq \text{Ldim}(D) < \infty \).
Lemma 7. Let $\mathcal{D}$ be a discriminating class with dual Littlestone dimension $\ell^*$, and let $T$ be the horizon. Then, there exists a deterministic online learning algorithm $A$ for the dual class $\mathcal{X}$ that receives labelled examples from the domain $\Delta(\mathcal{D})$ and uses predictors of the form $\hat{f}_t$ for some $f_t: \mathcal{D} \rightarrow [0,1]$, such that:

1. $A$’s regret is $O(\sqrt{\ell^* T \log T})$, and
2. For all $t \leq T$, if in the sequence of observed examples $(\bar{d}_1, y_1), \ldots, (\bar{d}_{t-1}, y_{t-1})$ up to iteration $t$, each $\bar{d}_t$ has a finite support then $f_t$ is amenable (in particular $f_1$ is amenable).

The next Lemma says that the algorithm in Fig. 2 with the modification depicted in Fig. 5 indeed never outputs error:

Lemma 8. Consider the generator in Fig. 2 with the modification depicted in Fig. 5. Assume $A$ is the online learner whose existence is implied by Lemma 7. Then for all $t \leq T$ the generator never outputs error.

Proof. We want to show that at every step $t$, if the algorithm reaches the else step in Fig. 5 then there exists a distribution $\bar{d}_t$ with finite supports which satisfies

$$\left(\forall x \in \mathcal{X}\right): \mathbb{E}_{d \sim \bar{d}_t} [f_t(d) - x(d)] > \frac{\epsilon}{4}.$$

Indeed, by the modification in Fig. 5 up to iteration $t$ the generator fed the learner $A$ with $(\bar{d}_1, y_1), \ldots, (\bar{d}_{t-1}, y_{t-1})$ such that each $\bar{d}_i$’s has a finite support. Therefore, by Lemma 7, Item 2 the function $f_t$ is amenable and thus there exists $\bar{d}$ such that

$$\left(\forall x \in \mathcal{X}\right): \mathbb{E}_{d \sim \bar{d}} [f_t(d) - x(d)] > \frac{\epsilon}{2}.$$

To get the desired finitely supported $\bar{d}_t$ we use uniform convergence: consider $\mathcal{X}$ as an hypothesis class over $\mathcal{D}$ and note that since $\mathcal{X}$ has a finite Littlestone then it also has a finite VC dimension. Therefore, by uniform convergence for VC classes there exists a finite sequence $d_1, \ldots, d_m \sim \bar{d}$ such that:

$$\left(\forall x \in \mathcal{X}\right): \left| \mathbb{E}_{d \sim \bar{d}} [f_t(d) - x(d)] - \frac{1}{m} \sum_{i=1}^{m} [f_t(d_i) - x(d_i)] \right| \leq \frac{\epsilon}{4}.$$

(by uniform convergence, a sufficiently large independent sample from $\bar{d}$ satisfies it with high probability). In particular, $\bar{d}_t$ can be chosen to be the empirical distribution induced by $d_1, \ldots, d_m$.

Lemma 7, together with Lemma 8, implies the upper bound in Theorem 1, Item 1 via the same argument as in the finite case. This follows by picking the online learner used by the generator in Fig. 2 as in Lemma 7; the amenability of the $f_t$’s (and Lemma 8) implies that the protocol never outputs “error”, and the rest of the argument is exactly the same like in the finite case (with slight deterioration in the constants).

Corollary 5. Let $A$ be an algorithm like in the above Lemma. Then, if one uses $A$ as the online learner in the algorithm in Fig. 2, together with the modification in Fig. 5, then the round complexity of it is at most $O(\frac{\ell^*}{\epsilon^2} \log \frac{T}{\epsilon})$, as in Theorem 1, Item 1.

In the remainder of this section we prove Lemma 7.

B.1 Preliminaries

We first present standard notions and facts from topology and functional analysis that will be used. We refer the reader to [41, 40] for further reading.

---

\[13\] I.e. with probability mass function $\bar{d}_t(d) = \frac{1}{m} \sum_{i=1}^{m} 1[d = d_i]$
**Weak* topology.** Given a compact Hausdorff space $K$, let $\Delta(K)$ denote the space of Borel measures over $K$, and let $C(K)$ denote the space of continuous real functions over $K$. The weak* topology over $\Delta(K)$ is defined as the weakest\textsuperscript{14} topology so that for any continuous function $f \in C(K)$ the following $\"\Delta(K) \to \mathbb{R}\"$ mapping is continuous

$$T_f(\mu) = \int f(k) d\mu(k).$$

We will rely on the following fact, which is a corollary of Banach–Alaoglu Theorem (see e.g. Theorem 3.15 in [40]) and the duality between $C(K)$ and $\mathcal{B}(K)$, the class of Borel measures over $K$:

**Claim 2.** Let $K$ be a compact Hausdorff space. Then $\Delta(K)$ is compact in the weak* topology.

**Upper and lower semicontinuity.** Recall that a real function $f$ is called upper semicontinuous (u.s.c) if for every $\alpha \in \mathbb{R}$ the set $\{x : f(x) \geq \alpha\}$ is closed. Note that $\limsup_{x \to x_0} f(x) \leq f(x_0)$ for any $x_0$ in the domain of $f$. Similarly, $f$ is called lower semicontinuous (l.s.c) if $-f$ is u.s.c. We will use the following fact:

**Claim 3.** Let $K$ be a compact Hausdorff space and assume $E \subseteq K$ is a closed set. Consider the “$\Delta(K) \to [0,1]$” mapping $T_E(\mu) = \mu(E)$. Then $T_E$ is u.s.c with respect to the weak* topology on $\Delta(X)$.

**Proof.** This fact can be seen as a corollary of Urysohn’s Lemma (Lemma 2.12 in [41]). Indeed, Borel measures are regular (see definition 2.15 in [41]). Thus, for every closed set $E$ we have

$$\mu(E) = \inf_{\{U : E \subseteq U, \ U \text{ is open}\}} \mu(U).$$

Fix a closed set $E$. Urysohn’s Lemma implies that for every open set $U \supseteq E$, there exists a continuous function $f_U \in C(K)$ such that $\chi_E \leq f_U \leq \chi_U$, where $\chi_A$ is the indicator function over the set $A$ (i.e. $\chi_A(x) = 1$ if and only if $x \in A$).

Thus, we can write $\mu(E) = \inf_{\{U : E \subseteq U, \ U \text{ is open}\}} \mu(f_U)$, where $\mu(f_U) = \mathbb{E}_{x \sim \mu}[f_U]$. Now, by continuity of $f_U$, it follows that the mapping $\mu \mapsto \mu(f_U)$ is continuous with respect to the weak* topology on $\Delta(X)$. Finally, the claim follows since the infimum of continuous functions is u.s.c. $\square$

**Sion’s Theorem.** We next state the following generalization of Von-Neumann’s Theorem for u.s.c/l.s.c payoff functions.

**Theorem 4** (Sion’s Theorem). Let $W$ be a compact convex subset of a linear topological space and $U$ a convex subset of a linear topological space. If $F$ is a real valued function on $W \times U$ with

- $F(w, \cdot)$ is l.s.c and convex on $U$ and
- $F(\cdot, u)$ is u.s.c and concave on $W$

then,

$$\max_{w \in W} \min_{u \in U} F(w, u) = \min_{u \in U} \max_{w \in W} F(w, u)$$

**Tychonof’s space.** The last notion we introduce is the topology we will use on $\{0,1\}^X$. Given an arbitrary set $W$, the space $\mathcal{F} = \{0,1\}^W$ is the space of all functions $f : W \to \{0,1\}$. The product topology on $\mathcal{F}$ is the weakest topology such that for every $w \in W$ the mapping $\Pi_w : \mathcal{F} \to \{0,1\}$, defined by $\Pi_w(f) = f(w)$ is continuous.

A basis of open sets in the product topology is provided by the sets $U_{w_1,\ldots,w_m}(g)$ of the form:

$$U_{w_1,\ldots,w_m}(g) = \{f : g(w_i) = f(w_i) \ i = 1,\ldots,m\},$$

where $w_1,\ldots,w_m$ are arbitrary elements in $W$ and $g \in \mathcal{F}$.

A remarkable fact about the product topology is that the space $\mathcal{F}$ is compact for any domain $W$ (see for example [31]). We summarize the above discussion in the following claim

**Claim 4.** Let $W$ be an arbitrary set and consider $\mathcal{F} = \{0,1\}^W$ equipped with the product topology. Then $\mathcal{F}$ is compact and $\Pi_w \in C(\mathcal{F})$ for every $w \in W$, where $\Pi_w$ is defined as $\Pi_w(f) = f(w)$.

\textsuperscript{14}In the sense that every other topology with this property contains all open sets in the weak* topology.
B.2 Two Technical Lemmas

The proof of Lemma 7 follows from the following two lemmas. Throughout the proofs we will treat \( \mathcal{D} \) as a topological subspace in \( \{0,1\}^X \) with the product topology, and \( \Delta(\mathcal{D}) \) as a topological space equipped with the weak* topology which is induced by topology on \( \mathcal{D} \).

**Lemma 9** (Analog of Lemma 5). Assume \( \mathcal{D} \subseteq \{0,1\}^X \) is closed and let \( f : \mathcal{D} \to [0,1] \). Assume that \( \hat{f} \) is u.s.c (with respect to the weak* topology on \( \Delta(\mathcal{D}) \)) then \( f \) is amenable.

**Lemma 10** (Analog of Corollary 4). Let \( \mathcal{D} \subseteq \{0,1\}^X \) be closed and let \( \ell^* \) denote its dual Littlestone dimension. Then, there exists a deterministic online learner that receives labelled examples from the domain \( \Delta(\mathcal{D}) \) such that for every sequence \((d_t,y_t)_{t=1}^T\) we have that:

\[
\text{REGRET}_T(L) \leq \sqrt{\frac{1}{2} \ell^* \log T}
\]

Moreover, at each iteration \( t \) the predictor, \( \hat{f}_t \), used by \( L \) is of the form \( \hat{f}_t[d] = \mathbb{E}_{d \sim \hat{d}}(f_t(d)) \) for some \( f_t : \mathcal{D} \to [0,1] \). Finally, for every \( t \leq T \) if each \( \hat{d}_t \) for \( i < t \) has a finite support then \( \hat{f}_t \) is u.s.c.

We first show how to conclude the proof of Lemma 7 using these lemmas and later prove the two lemmas.

**Concluding the proof of Lemma 7.** The proof follows directly from the two preceding Lemmas. Given a discriminating class \( \mathcal{D} \subseteq \{0,1\}^X \) there is no loss of generality in assuming \( \mathcal{D} \) is closed, since closing the class with respect to the product topology does not increase its dual Littlestone dimension.

Now, take the learner \( \mathcal{A} \) whose existence follows from Lemma 10. Since each \( \hat{f}_t \) is u.s.c we obtain via Lemma 9 that each \( f_t \) is also amenable. \( \square \)

**Proof of Lemma 9.** Lemma 9 extends Lemma 5 to the infinite case. Similar to the proof of Lemma 5 which hinges on Von-Neumann’s Minmax Theorem, the proof here hinges on Sion’s Theorem which is valid in this setting.

Before proceeding with the proof we add the following notation: let \( \mathbb{R}_+^X \) denote the space of real-valued functions \( v : X \to \mathbb{R} \) with finite support, i.e. \( v(x) = 0 \) except for maybe a finite many \( x \in X \). We equip \( \mathbb{R}_+^X \) with the topology induced by the \( \ell_1 \) norm, namely a basis of open sets is given by the open balls \( U_{\epsilon,x} = \{ u : \sum_{x \in X} |v(x) - u(x)| < \epsilon \} \). \( \mathbb{R}_{fin}(X) \) is indeed a linear topological space (i.e. the vector addition and scalar multiplication mappings are continuous). Finally, define

\[
\Delta_{fin}(X) := \{ p \in \mathbb{R}_+^X : p(x) \geq 0, \sum_{x \in X} p(x) = 1 \}.
\]

Next, let \( f : \mathcal{D} \to [0,1] \) be such that \( \hat{f} \) is u.s.c. Our goal is to show that \( f \) is amenable. Set \( F \) to be the following real-valued function over \( \Delta(\mathcal{D}) \times \Delta_{fin}(X) \):

\[
F(\hat{d}, p) = \mathbb{E}_{d \sim \hat{d}} \left[ f(d) - \sum_{x \in X} p(x)x(d) \right]
\]

It suffices to show that

\[
\max_{d \in \Delta(\mathcal{D})} \inf_{p \in \Delta_{fin}(X)} F(\hat{d}, p) = \inf_{p \in \Delta_{fin}(X)} \max_{d \in \Delta(\mathcal{D})} F(\hat{d}, p) \geq \frac{\epsilon}{2}.
\]

Indeed, the assumption that Item 1 does not hold implies in particular that

\[
\inf_{p \in \Delta_{fin}(X)} \max_{d \in \Delta(\mathcal{D})} F(\hat{d}, p) \geq \frac{\epsilon}{2}.
\]

Eq. (14) then states that

\[
\max_{d \in \Delta(\mathcal{D})} \inf_{x \in X} \mathbb{E}_{d \sim \hat{d}} |f(d) - x(d)| \geq \frac{\epsilon}{2}.
\]
which proves that Item 2 holds.

Eq. (14) follows by an application of Theorem 4 on the function $F$. Thus, we next show the premise of Theorem 4 is satisfied by $F$. Indeed, $W = \Delta(D)$ is compact and convex, and $U = \Delta_{f_{\infty}}(X)$ is convex. We show that $F(\cdot, p)$ is concave and u.s.c for every fixed $p \in \Delta_{f_{\infty}}(X)$: indeed, $F(\cdot, p)$ is in fact linear and therefore concave. We show that $F(\cdot, p)$ is u.s.c by showing that it is the sum of (i) a u.s.c function (i.e. $E_{d \sim 0}[f(d)]$) and (ii) finitely many continuous functions (i.e. $\sum_{x \in X} p(x) E_{d \sim 0}[x(d)]$). Indeed, (i) by assumption $\hat{f}(d) = E_{d \sim 0}[f(d)]$ is u.s.c, and (ii) by Claim 4, the mapping $\Pi_x(d)$ is continuous for every $x \in X$ which, by the definition of the weak* topology, implies that $d \to E_{d \sim 0} \Pi_x(d) = E_{d \sim 0} [x(d)]$ is continuous.

Finally, because $E_{d \sim 0} [x(d)] \leq 1$ is bounded, it follows that $F(d, \cdot)$ is linear and continuous in $p$ for every fixed $d$: indeed treating $\hat{f}(d)$ and $\{E_{d \sim 0} [x(d)]\}_{x \in X}$ as bounded constants, we have that:

$$F(d, p) = \hat{f}(d) - \sum_{x \in X} p(x) E_{d \sim 0} [x(d)]$$

**Proof of Lemma 10.** We will show that the learner which is derived in Corollary 4 satisfies the conclusion of Lemma 10. The regret bound and the fact that the learner outputs a predictor of the form $f_t = E_{d \sim 0} [f_t(d)]$ follows from Corollary 4.

Therefore, it suffices to show that the $f_t$’s can be chosen to be u.s.c. This follows from an examination of the proof provided in [10] for Theorem 3 and our extension of it to Corollary 4 as detailed next. The authors in [10] use the following type of functions. Call a function $s : D \to \{0, 1\}$ an SOA-type function if there exists a hypothesis class $H \subseteq X$ such that

$$s(d) = \begin{cases} 0 & \text{Ldim}(H_{(d, 0)}) = \text{Ldim}(H) \\ 1 & \text{else} \end{cases}$$

where $H_{(d, 0)} = \{h \in H : h(d) = 0\}$. In the proof by [10] of Theorem 3 the authors construct an online learner which in each iteration $t$ uses a randomized predictor (i.e. a distribution over predictors) which only uses SOA-type functions. Namely, the algorithm maintains a distribution $q_t$ over a finite set of SOA type functions $\{s_k\}$, and predicts according to $s_k$ with probability $q_t(s_k)$.

The extension in Corollary 4 of this predictor to the domain $\Delta(D)$ is done by setting:

$$f_t(d) = \mathbb{E}_{d_1, \tau} \left[ \mathbb{E}_{s \sim \text{L}_0} [s(d)d_1, \ldots, d_{t-1}] \right] = \mathbb{E}_{d_1, \tau} \left[ \sum_{k} q_t(s_k) s_k(d)d_1, \ldots, d_{t-1} \right].$$

Namely, $f_t(d)$ is defined by taking expectation both over the choice of the algorithm and over the sequence $d_1, \ldots, d_{t-1}$, which is drawn from $d_1, \ldots, d_{t-1}$. Since $d_1, \ldots, d_{t-1}$ have finite support we can summarize these expectations and write:

$$f_t = \sum \lambda_k s_k,$$

for some choice of SOA-type functions and weights $\lambda_k \geq 0$ where $\sum \lambda_k = 1$. Now, since the sum of u.s.c functions is u.s.c and since the multiplication of a u.s.c function with positive scalar is u.s.c, it is enough to prove that every SOA-type function $s$ induces a u.s.c function over $\Delta(D)$ via the rule $\mu \mapsto \mu \left( \{d : s(d) = 1\} \right)$. By Claim 3 it is enough to show that the set $s^{-1}(0)$ is open. To this end we show that for every $d \in s^{-1}(0)$ there is an open neighborhood of $d$ which is contained in $s^{-1}(0)$. Indeed, if $d \in s^{-1}(0)$, then there exist $x_1, \ldots, x_{2^\ell}$ that $d(x_i) = 0$ for all $i$, and they shatter a tree. Consider the open neighborhood of $d$ defined by $U = \cap_i \{d : d(x_i) = 0\}$. $U \subseteq s^{-1}(0)$ since if there were $d' \in U$ such that $s(d') = 1$ then $\text{Ldim}(H_{(d', 0)}) < \text{Ldim}(H) = \ell$. However, since $d' \in U$ then $x_1, \ldots, x_{2^\ell} \in H_{(d', 0)}$ and they shatter a tree of depth $\ell$ which is a contradiction. 

\footnote{SOA stands for Standard Optimal Algorithm; this definition was used by Littlestone [34].}