THE EQUITABLE BASIS FOR \( \mathfrak{sl}_2 \)

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ABSTRACT. This article contains an investigation of the equitable basis for the Lie algebra \( \mathfrak{sl}_2 \). Denoting this basis by \( \{x, y, z\} \), we have

\[
[x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x.
\]

We determine the group of automorphisms \( G \) generated by \( \exp(\text{ad} x^*), \exp(\text{ad} y^*), \exp(\text{ad} z^*) \), where \( \{x^*, y^*, z^*\} \) is the basis for \( \mathfrak{sl}_2 \) dual to \( \{x, y, z\} \) with respect to the trace form \( (u, v) = \text{tr}(uv) \) and study the relationship of \( G \) to the isometries of the lattices \( L = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \) and \( L^* = \mathbb{Z}x^* \oplus \mathbb{Z}y^* \oplus \mathbb{Z}z^* \). The matrix of the trace form is a Cartan matrix of hyperbolic type, and we identify the equitable basis with a set of simple roots of the corresponding Kac-Moody Lie algebra \( g \), so that \( L \) is the root lattice and \( 1/2 L^* \) is the weight lattice of \( g \). The orbit \( G(x) \) of \( x \) coincides with the set of real roots of \( g \) and show that each isotropic root has multiplicity 1. We describe the finite-dimensional \( \mathfrak{sl}_2 \)-modules from the point of view of the equitable basis.

In the final section, we establish a connection between the Weyl group orbit of the fundamental weights of \( g \) and Pythagorean triples.

1. Introduction

The purpose of this article is to investigate systematically a certain basis, called the equitable basis, for the Lie algebra \( \mathfrak{sl}_2 \) of \( 2 \times 2 \) trace zero matrices over a field \( F \) of characteristic zero. This basis has already appeared in the theory of tridiagonal pairs [HI], [H2] and of the three-point loop algebra \( \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t - 1)^{-1}] \) ([IT], [BT], see also [ITW]). As we will show, it exhibits many striking features and has connections with the theory of Kac-Moody Lie algebras.

In Section 2, we introduce the equitable basis \( \{x, y, z\} \) and its dual basis \( \{x^*, y^*, z^*\} \) with respect to the trace form \( (u, v) = \text{tr}(uv) \) on \( \mathfrak{sl}_2 \). In Section 3, we study the group \( G \) generated by \( \exp(\text{ad} x^*), \exp(\text{ad} y^*), \exp(\text{ad} z^*) \) and show in Theorem 3.10 that \( G \) is isomorphic to the modular group \( \text{PSL}_2(\mathbb{Z}) \). We then turn our attention to the lattices \( L = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \) and \( L^* = \mathbb{Z}x^* \oplus \mathbb{Z}y^* \oplus \mathbb{Z}z^* \), and in Theorem 4.11 give a characterization of the orbit \( G(x) \) as the elements \( u \in L \) with \( (u, u) = 2 \). Since there is an automorphism of \( \mathfrak{sl}_2 \) in \( G \) which cyclically permutes \( x, y, z \), this is the same as the orbit of

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2. The Equitable Basis

Throughout, \( \{e, f, h\} \) will denote the basis for \( \mathfrak{sl}_2 \) given by

\[
e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

and having products \([e, f] = h, [h, e] = 2e, [h, f] = -2f\).

The equitable basis \( \{x, y, z\} \) for \( \mathfrak{sl}_2 \) consists of the matrices

\[
x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h,
\]
\[
y = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = 2e - h,
\]
\[
z = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = -2f - h,
\]

whose products satisfy
\[ [x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x. \quad (2.2) \]

From this it follows that there is a Lie algebra automorphism \( \varrho \) of \( \mathfrak{sl}_2 \) of order 3 such that
\[
\varrho(x) = y, \quad \varrho(y) = z, \quad \varrho(z) = x. \quad (2.3)
\]

Note that
\[
y = \exp(\text{ad}_e)(-h), \quad z = \exp(\text{ad}_f)(-h),
\]
where \( \text{ad} u(v) = [u, v] \) and \( \exp(w) = \sum_{n=0}^{\infty} w^n/n! \). We will relate the automorphisms \( \exp(\text{ad}_e) \) and \( \exp(\text{ad}_f) \) to \( \varrho \) in Section 3.

In our work we will use the trace form \( (u, v) := \text{tr}(uv) \) for \( u, v \in \mathfrak{sl}_2 \). We could use instead the Killing form \( \kappa(u, v) := \text{tr}(\text{ad}_u \text{ad}_v) = 4(u, v) \), but the trace form has some aesthetic advantages. Relative to the equitable basis, the matrix of the trace form is given by
\[
A = \begin{bmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2 \\
\end{bmatrix}. \quad (2.4)
\]

This is a (generalized) Cartan matrix as defined in ([K] §1.1, [MP] §3.4); the corresponding Kac-Moody Lie algebra will be related to the equitable basis in Section 6.

Let \( \{x^*, y^*, z^*\} \) denote the basis for \( \mathfrak{sl}_2 \) that is dual to the equitable basis in the sense that \( (u, v^*) = 2\delta_{u,v} \) for all \( u, v \in \{x, y, z\} \) (the factor of 2 is inessential but convenient). Then
\[
x + y = -2z^*, \quad y + z = -2x^*, \quad z + x = -2y^* \quad (2.5)
\]
and
\[
\varrho(x^*) = y^*, \quad \varrho(y^*) = z^*, \quad \varrho(z^*) = x^*. \quad (2.6)
\]

Relative to the basis \( \{x^*, y^*, z^*\} \) the matrix of the trace form is
\[
4A^{-1} = \begin{bmatrix}
0 & -1 & -1 \\
-1 & 0 & -1 \\
-1 & -1 & 0 \\
\end{bmatrix}. \quad (2.7)
\]

The equitable basis and its dual are related by the following multiplication tables:
\[
\begin{array}{c|ccc}
\cdot & x^* & y^* & z^* \\
\hline
x^* & 0 & z & -y \\
y^* & -z & 0 & x \\
z^* & y & -x & 0 \\
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{c|ccc}
\cdot & x & y & z \\
\hline
x & 0 & -4z^* & 4y^* \\
y & 4z^* & 0 & -4x^* \\
z & -4y^* & 4x^* & 0 \\
\end{array}
\quad (2.8)
We also have

\[
\begin{array}{|c|c|c|c|}
\hline
 & x & y & z \\
\hline
x^* & y - z & y + z & -y - z \\
y^* & -z - x & z - x & z + x \\
z^* & x + y & -x - y & x - y \\
\hline
\end{array}
\]

(2.9)

and

\[
x - y = 2(x^* - y^*), \quad y - z = 2(y^* - z^*), \quad z - x = 2(z^* - x^*),
\]

\[
x^* + y^* = z^* - z, \quad y^* + z^* = x^* - x, \quad z^* + x^* = y^* - y,
\]

(2.10)

\[
x + y + z = -x^* - y^* - z^*.
\]

By (2.11), each of the matrices \(x, y, z\) is semisimple (diagonalizable) with eigenvalues 1 and \(-1\). Since

\[
x^* = h - e + f, \quad y^* = f, \quad z^* = -e,
\]

(2.11)

each of the dual basis elements \(u \in \{x^*, y^*, z^*\}\) is nilpotent with \(u^2 = 0\) and \((ad u)^3 = 0\).

3. Connections with the modular group

Let \(G\) denote the subgroup of the automorphism group \(\text{Aut}_F(\mathfrak{sl}_2)\) generated by \(\exp(\text{ad} x^*), \exp(\text{ad} y^*), \exp(\text{ad} z^*)\). In this section we will prove that \(G\) is isomorphic to the modular group \(\text{PSL}_2(\mathbb{Z})\). Recall that \(\text{PSL}_2(\mathbb{Z})\) is obtained from the group \(\text{SL}_2(\mathbb{Z})\) of \(2 \times 2\) integral matrices of determinant 1 by factoring out the subgroup consisting of the matrices \(\pm I\). It is a free product of a cyclic group of order 2 and a cyclic group of order 3 (see for example, [A1]). To establish the isomorphism with \(G\), we first locate generators for \(G\) of order 2 and 3.

**Definition 3.1.** Let \(\sigma_x, \sigma_y, \sigma_z\) be the automorphisms of \(\mathfrak{sl}_2\) defined by

\[
\sigma_x = \exp(\text{ad} x^*), \quad \sigma_y = \exp(\text{ad} y^*), \quad \sigma_z = \exp(\text{ad} z^*).
\]

Using the table in (2.9), we obtain

**Lemma 3.2.** The matrices of \(\sigma_x, \sigma_y, \sigma_z\) relative to the equitable basis are given by

\[
\sigma_x \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma_y \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix}, \quad \sigma_z \rightarrow \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

**Lemma 3.3.** (i) \(\rho \sigma_x \rho^{-1} = \sigma_y\), \(\rho \sigma_y \rho^{-1} = \sigma_z\), \(\rho \sigma_z \rho^{-1} = \sigma_x\);
(ii) $\varrho$ is equal to each of the products $\sigma_x\sigma_y$, $\sigma_y\sigma_z$, $\sigma_z\sigma_x$. In particular $\varrho \in G$.

Proof: Part (i) follows from the well-known identity

$$\varphi \exp(ad u) \varphi^{-1} = \exp(ad \varphi(u))$$

which holds for all $\varphi \in \text{Aut}_F(\mathfrak{sl}_2)$ and all nilpotent $u \in \mathfrak{sl}_2$. To see that $\varrho = \sigma_x\sigma_y$, use (2.3) and Lemma 3.2 to verify that $\varrho$ and $\sigma_x\sigma_y$ agree on the elements of the equitable basis. To obtain the other two expressions for $\varrho$, apply part (i) above. □

Lemma 3.4. For $\sigma_x$, $\sigma_y$, $\sigma_z$ as in Definition 3.1, we have the following:

(a) $\sigma_x = \sigma_y\sigma_z\sigma_y^{-1}$;
(b) $\sigma_x = \sigma_z^{-1}\sigma_y\sigma_z$;
(c) $(\sigma_y\sigma_z)^3 = 1$;
(d) $\sigma_y\sigma_z\sigma_y = \sigma_z\sigma_y\sigma_z$;
(e) $(\sigma_y\sigma_z\sigma_y)^2 = 1$;
(f) $G$ is generated by $\sigma_y$ and $\sigma_z$.

Proof: These properties can be deduced from Lemma 3.3. □

Definition 3.5. Let $\tau_x$, $\tau_y$, $\tau_z$ denote the elements of $G$ defined by

$$\tau_x = \sigma_y\sigma_z\sigma_y^{-1};$$
$$\tau_y = \sigma_z\sigma_x\sigma_z^{-1};$$
$$\tau_z = \sigma_z\sigma_x\sigma_z^{-1};$$

Lemma 3.6. For $\tau_x$, $\tau_y$, $\tau_z$ as in Definition 3.1, the following relations hold:

(a) $\varrho \tau_x \varrho^{-1} = \tau_y$, $\varrho \tau_y \varrho^{-1} = \tau_z$, $\varrho \tau_z \varrho^{-1} = \tau_x$;
(b) $\tau_x^2 = \tau_y^2 = \tau_z^2 = 1$;
(c) $\sigma_z = \tau_x \varrho^{-1}$ and $\sigma_y = \varrho^{-1} \tau_x$;
(d) $\tau_x(x) = -x$, $\tau_x(y) = 2x + z$, $\tau_x(z) = 2x + y$.

Proof: Part (a) follows from Lemma 3.3(i), while (b) comes from (a) and Lemma 3.2(e). Concerning (c), the first (resp. second) equation follows from $\varrho = \sigma_y\sigma_z$ and $\tau_x = \sigma_z\sigma_y\sigma_z$ (resp. $\tau_x = \sigma_y\sigma_z\sigma_y$). To get (d), use $\tau_x = \sigma_z \varrho$ together with (2.3) and Lemma 3.2. □

Combining Lemma 3.6 with Lemma 3.4(f), we have

Corollary 3.7. Each of the following is a generating set for the group $G$.

(i) $\varrho$, $\tau_x$;
(ii) $\varrho$, $\tau_y$;
(iii) $\varrho$, $\tau_z$. 

Proof: These properties can be deduced from Lemma 3.3. □
For $\theta \in \text{SL}_2(\mathbb{Z})$, conjugation by $\theta$ determines an automorphism $\hat{\theta}$ of $\mathfrak{sl}_2$:

$\hat{\theta}: u \mapsto \theta u \theta^{-1}$.

The map

$$
\begin{align*}
\text{SL}_2(\mathbb{Z}) & \rightarrow \text{Aut}_F(\mathfrak{sl}_2) \\
\theta & \mapsto \hat{\theta}
\end{align*}
$$

(3.8)

is a group homomorphism with kernel $\{\pm I\}$. This map induces an embedding

$$
i: \text{PSL}_2(\mathbb{Z}) \rightarrow \text{Aut}_F(\mathfrak{sl}_2).
$$

(3.9)

**Theorem 3.10.** The image of the embedding $i: \text{PSL}_2(\mathbb{Z}) \rightarrow \text{Aut}_F(\mathfrak{sl}_2)$ coincides with $G$. Therefore $G$ is isomorphic to $\text{PSL}_2(\mathbb{Z})$.

**Proof:** The matrices

$$
A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}
$$

(3.11)

are in $\text{SL}_2(\mathbb{Z})$ and satisfy $C = BAB^{-1}$. Let $a, b, c$ denote the images of $A, B, C$ respectively under the canonical homomorphism $\text{SL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z})$, and note that $c = bab^{-1}$. By [A2], the elements $a, b$ generate $\text{PSL}_2(\mathbb{Z})$, so $b, c$ generate $\text{PSL}_2(\mathbb{Z})$. One checks that $\hat{B} = \tau_x$ and $\hat{C} = g$ so $i(b) = \tau_x$ and $i(c) = g$. The result then follows in view of Corollary 3.7 (i). \qed

4. The $G$-orbit of $x$

In this section we describe the orbit $G(x)$ of $x$ under the group $G$ generated by $\exp(\text{ad} x^*), \exp(\text{ad} y^*), \exp(\text{ad} z^*)$. Since $g$ belongs to $G$ and cyclically permutes the elements of the equitable basis, $G(x)$ coincides with the $G$-orbit of $y$ and the $G$-orbit of $z$. Later in the paper we relate $G(x)$ to the set of real roots for the Kac-Moody Lie algebra associated with the Cartan matrix $A$ from (2.4). We begin by determining the stabilizer of $x$ in $G$.

**Lemma 4.1.** Suppose $g \in G$ and $g(x) = x$. Then $g = 1$.

**Proof:** By Theorem 3.10 and the paragraph preceding it, there exists $\theta \in \text{SL}_2(\mathbb{Z})$ such that $\hat{\theta} = g$. Therefore $\theta x \theta^{-1} = g(x) = x$ gives $\theta x = x \theta$, and this along with the fact that $x = \text{diag}(1, -1)$ implies $\theta$ is diagonal. The diagonal entries of $\theta$ are integers whose product is 1, so they are both 1 or both $-1$; thus $\theta = \pm I$ so $g = 1$. \qed
Corollary 4.2. The map 
\[ G \to G(x) \]
\[ g \mapsto g(x) \]

is a bijection.

We turn our attention now to the lattice 
\[ L := \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z. \quad (4.3) \]

Our goal is to prove that 
\[ G(x) = \{ u \in L \mid (u, u) = 2 \}, \]
but this necessitates a few comments about \( L \). Note that \( L \) is closed under the Lie bracket and invariant under the group \( G \). Further observe that

\[ L = \left\{ \begin{bmatrix} p & q \\ r & -p \end{bmatrix} \right| p, q, r \in \mathbb{Z}, \quad q, r \text{ even} \right\}. \quad (4.4) \]

This realization of \( L \) shows that it is the Lie algebra analogue of the congruence subgroup of \( \text{PSL}_2(\mathbb{Z}) \).

Recall that an isometry of \( \mathfrak{sl}_2 \) is an \( \mathbb{F} \)-linear bijection \( \varphi : \mathfrak{sl}_2 \to \mathfrak{sl}_2 \) such that \((\varphi(u), \varphi(v)) = (u, v)\) for all \( u, v \in \mathfrak{sl}_2 \).

Lemma 4.5. Each automorphism of \( \mathfrak{sl}_2 \) is an isometry of \( \mathfrak{sl}_2 \).

Proof: Each automorphism of a finite-dimensional Lie algebra is an isometry relative to the Killing form. Since the trace map is a multiple of the Killing form, the result is apparent. □

The next lemma provides some useful formulae for the square norm \((u, u)\) of an element \( u \in \mathfrak{sl}_2 \).

Lemma 4.6. For \( u = \alpha x + \beta y + \gamma z \in \mathfrak{sl}_2 \), the expression \((u, u)/2\) is equal to each of the following:

\[ \alpha^2 + \beta^2 + \gamma^2 - 2(\alpha \beta + \beta \gamma + \gamma \alpha), \]
\[ 2(\alpha^2 + \beta^2 + \gamma^2) - (\alpha + \beta + \gamma)^2, \]
\[ (\alpha + \beta - \gamma)^2 - 4\alpha \beta, \]
\[ (\beta + \gamma - \alpha)^2 - 4\beta \gamma, \]
\[ (\gamma + \alpha - \beta)^2 - 4\gamma \alpha. \]

Proof: The above five scalars are mutually equal; this can be checked by algebraic manipulation. Observe that \((u, u) = (\alpha, \beta, \gamma)A(\alpha, \beta, \gamma)^t\) where \( A \) is from (2.4). Evaluating this triple product by matrix multiplication, we find that \((u, u)/2\) is equal to the first expression above, so the result follows. □
**Definition 4.7.** Define \( R = \{ u \in L \mid (u, u) = 2 \} \). We note that \( R \) is \( G \)-invariant by Lemma 4.5.

**Lemma 4.8.** For \( u = \alpha x + \beta y + \gamma z \) in the set \( R \) in Definition 4.7, the coefficients \( \alpha, \beta, \gamma \) are either all nonnegative or all are nonpositive.

**Proof:** We assume the result is false and reach a contradiction. There exists a pair of coefficients having opposite signs; without loss in generality we may assume they are \( \alpha \) and \( \beta \). Thus \( \alpha \beta \leq -1 \). Using \( (u, u) = 2 \) and the third expression in Lemma 4.6, we find that

\[-4 \geq 4\alpha\beta = (\alpha + \beta - \gamma)^2 - 1 \geq -1.\]

This is a contradiction, so the result must be true. \( \Box \)

**Definition 4.9.** For the set \( R \) in Definition 4.7 and for \( u = \alpha x + \beta y + \gamma z \in R \), we define the **height** of \( u \) to be the sum \( \text{ht}(u) = \alpha + \beta + \gamma \). Let \( R^+ = \{ u \in R \mid \text{ht}(u) > 0 \} \) and \( R^- = \{ u \in R \mid \text{ht}(u) < 0 \} \).

By Definition 4.7 and Lemma 4.8 we have \( R = R^+ \cup R^- \), \( R^- = -R^+ \), and \( g(R^\pm) = R^\pm \). Next we describe how the automorphisms \( \tau_x, \tau_y, \tau_z \) act on the set \( R^+ \).

**Lemma 4.10.** For \( u \in \{ x, y, z \} \), the map \( \tau_u \) sends \( u \) to \( -u \) and permutes the elements of \( R^+ \setminus \{ u \} \).

**Proof:** There is no loss in generality in assuming \( u = x \). Recall that \( \tau_x(x) = -x \) by Lemma 3.6(d). Now suppose we are given \( v = \alpha x + \beta y + \gamma z \in R^+ \) such that \( \tau_x(v) \in R^- \). It suffices to argue that \( v = x \). Using Lemma 3.6(d), we have

\[
\tau_x(v) = \alpha(-x) + \beta(2x + z) + \gamma(2x + y)
= (2\beta + 2\gamma - \alpha)x + \gamma y + \beta z.
\]

Observe \( \beta \geq 0 \) since \( v \in R^+ \) and \( \beta \leq 0 \) since \( \tau_x(v) \in R^- \) so \( \beta = 0 \). Similarly \( \gamma = 0 \). Now \( \alpha = 1 \) since \( (v, v) = 2 \). Therefore \( v = x \) and the result follows. \( \Box \)

**Theorem 4.11.** \( G(x) = R = \{ u \in L \mid (u, u) = 2 \} \).

**Proof:** The set \( R \) contains \( x \) and is \( G \)-invariant so \( G(x) \subseteq R \). To show equality holds, we assume there exists \( u \in R \setminus G(x) \) and arrive at a contradiction. Without loss we may further assume that \( u \in R^+ \) and that \( u \) has minimal height with this property. Note that \( u \) is not one of \( x, y, z \) as they are in \( G(x) \). Write \( u = \alpha x + \beta y + \gamma z \). By Lemma 4.10, each of \( \tau_x(u), \tau_y(u), \tau_z(u) \) belongs to \( R^+ \setminus G(x) \). By our minimality assumption all these elements have height at least \( \text{ht}(u) \). Evaluating these inequalities we determine that

\[
\alpha \leq \beta + \gamma, \quad \beta \leq \gamma + \alpha, \quad \gamma \leq \alpha + \beta.
\]
Since the situation is cyclically symmetric, we may assume that \( \alpha \geq \beta \). Using \( (u, u) = 2 \) and the third expression in Lemma 4.6 we see that

\[
(\alpha + \beta - \gamma)^2 = 1 + 4\alpha \beta > 4\beta^2,
\]

so that \( \alpha + \beta - \gamma > 2\beta \) and then \( \alpha > \beta + \gamma \). This is a contradiction, so it must be that \( G(x) = R \). \( \square \)

**Note 4.12.** Combining Theorem 3.10, Corollary 4.2, and Theorem 4.11 and using Theorem 4.6, we get a bijection between \( \text{PSL}_2(\mathbb{Z}) \) and the set of integral solutions \((\alpha, \beta, \gamma)\) to the quadratic equation

\[
2(\alpha^2 + \beta^2 + \gamma^2) - (\alpha + \beta + \gamma)^2 = 1.
\]

We close this section with a result about the coefficients of elements of \( G(x) \).

**Proposition 4.13.** For \( \alpha x + \beta y + \gamma z \in R \), exactly one of the coefficients \( \alpha, \beta, \gamma \) is odd.

**Proof:** Let \( S \) denote the set of elements in \( G(x) \) with exactly one odd coefficient. Note that \( S \) contains \( x \). For \( u = \alpha x + \beta y + \gamma z \in S \) we have \( \tau_x(u) = (2\beta + 2\gamma - \alpha)x + \gamma y + \beta z \), and modulo 2, this element has the same coefficients as \( u \). Therefore \( \tau_x(u) \in S \). Since \( \varrho(u) \in S \) also, we see that \( S \) is \( G \)-invariant. Consequently \( G(x) \subseteq S \) so \( G(x) = S \). \( \square \)

5. Automorphisms, Antiautomorphisms, and Isometries

In this section we continue our study of the lattice \( L \) from (4.3). We use the equitable basis to determine the precise relationship between the following four groups: (i) the group \( G \) from Section 3, (ii) the group of automorphisms of \( \mathfrak{s}\mathfrak{l}_2 \) that preserve \( L \), (iii) the group of automorphisms and antiautomorphisms of \( \mathfrak{s}\mathfrak{l}_2 \) that preserve \( L \), and (iv) the group of isometries of \( \mathfrak{s}\mathfrak{l}_2 \) that preserve \( L \).

By an **antiautomorphism** of \( \mathfrak{s}\mathfrak{l}_2 \) we mean an \( \mathbb{F} \)-linear bijection \( \phi : \mathfrak{s}\mathfrak{l}_2 \to \mathfrak{s}\mathfrak{l}_2 \) such that \( \phi([u, v]) = [\phi(v), \phi(u)] \) for \( u, v \in \mathfrak{s}\mathfrak{l}_2 \). Here are some examples. The map \( -1 : u \to -u \) is an antiautomorphism of \( \mathfrak{s}\mathfrak{l}_2 \) (in fact of any Lie algebra). For distinct \( u, v \in \{x, y, z\} \), the \( \mathbb{F} \)-linear map \( (u v) : \mathfrak{s}\mathfrak{l}_2 \to \mathfrak{s}\mathfrak{l}_2 \) that interchanges \( u, v \) and fixes the remaining element in \( \{x, y, z\} \) is an antiautomorphism of \( \mathfrak{s}\mathfrak{l}_2 \). Let \( \text{AAut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2) \) denote the group consisting of the automorphisms and antiautomorphisms of \( \mathfrak{s}\mathfrak{l}_2 \). Then \( \text{Aut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2) \) is a normal subgroup of \( \text{AAut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2) \) of index 2, and

\[
\text{AAut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2) = \{\pm 1\} \rtimes \text{Aut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2).
\]

Define

\[
\text{AAut}_{\mathbb{Z}}(L) = \{ \varphi \in \text{AAut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2) \mid \varphi(L) = L \},
\]

\[
\text{Aut}_{\mathbb{Z}}(L) = \{ \varphi \in \text{Aut}_{\mathbb{F}}(\mathfrak{s}\mathfrak{l}_2) \mid \varphi(L) = L \}
\]
and note that
\[ \text{AAut}_Z(L) = \{ \pm 1 \} \ltimes \text{Aut}_Z(L). \]  
(5.2)

We remark that AAut$_Z$(L) is the group of automorphisms and anti-automorphisms of L, viewed as a Lie algebra over $\mathbb{Z}$, since every such map over $\mathbb{Z}$ can be extended linearly to an automorphism or anti-automorphism in AAut$_F$(sl$_2$). By the construction $G \subseteq \text{Aut}_Z(L)$. Let Isom$_F$(sl$_2$) denote the group of all isometries of sl$_2$ and define
\[ \text{Isom}_Z(L) = \{ \varphi \in \text{Isom}_F(sl_2) \mid \varphi(L) = L \}. \]

Since $-1$ is an isometry of sl$_2$, by Lemma 5.3 we have AAut$_Z$(L) $\subseteq$ Isom$_Z$(L). So far we know that
\[ G \subseteq \text{Aut}_Z(L) \subseteq \text{AAut}_Z(L) \subseteq \text{Isom}_Z(L). \]

Before describing this chain in more detail we compute the stabilizer of $x$ in Isom$_Z$(L). In what follows $\langle S \rangle$ means the group generated by the set $S$.

**Lemma 5.3.** The stabilizer of $x$ in Isom$_Z$(L) is $\langle (y z), -\tau_x \rangle$ where $\tau_x$ is from Definition 5.4. Hence, this stabilizer is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof:** Concerning the first assertion, one inclusion is clear since each of the maps $(y z), -\tau_x$ fixes $x$ and is contained in Isom$_Z$(L). To obtain the other inclusion, we pick $\varphi \in \text{Isom}_Z(L)$ such that $\varphi(x) = x$ and show $\varphi \in \langle (y z), -\tau_x \rangle$. The subspace $\text{Span}_\mathbb{R}\{y^*, z^*\}$ is the orthogonal complement of $x$ relative to the trace form, so this subspace is $\varphi$-invariant. Write $\varphi(y^*) = ay^* + bz^*$ and $\varphi(z^*) = cy^* + dz^*$. Using $\varphi(x) = x$ and $x + z = -2y^*$, $x + y = -2z^*$, we have that $x + \varphi(z) = a(x + z) + b(x + y)$; this shows $a, b \in \mathbb{Z}$ since $\varphi(z) \in L$. Similarly $c, d \in \mathbb{Z}$. By (2.7) the matrix representing the trace form relative to $\{y^*, z^*\}$ is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Since $\varphi$ is an isometry,
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  0 & -1 \\
  -1 & 0
\end{bmatrix}
\begin{bmatrix}
  a & c \\
  b & d
\end{bmatrix}
= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}
\]
and this yields $ab = 0$, $cd = 0$, $ad + bc = 1$ after a brief calculation. By these equations $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is one of
\[
\begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
  0 & -1 \\
  -1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
  -1 & 0 \\
  0 & -1
\end{bmatrix}
\]
and these solutions correspond to $\varphi = 1$, $\varphi = (y z)$, $\varphi = -\tau_x$, $\varphi = -(y z)\tau_x$ respectively. In any event, $\varphi \in \langle (y z), -\tau_x \rangle$ and the first assertion follows. The second assertion is a direct consequence of the first. \quad \square

**Theorem 5.4.** Isom$_Z$(L) is equal to each of the groups
\[ \langle (x y), -1 \rangle \ltimes G, \quad \langle (y z), -1 \rangle \ltimes G, \quad \langle (z x), -1 \rangle \ltimes G. \]

In particular Isom$_Z$(L) is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes G$. 

Proof: We first show that $\text{Isom}_\mathbb{Z}(L) = \langle (y z), -1 \rangle \ltimes G$. Certainly $-1$ normalizes $G$ since $-1$ commutes with everything in $G$. The element $(y z)$ also normalizes $G$ since $(y z)\tau_x = \tau_x(y z)$, $(y z)\varphi = \varphi^2(y z)$, and $\tau_x \varphi$ together generate $G$. The group $G$ has trivial intersection with $\langle (y z), -1 \rangle$ because $G$ has trivial intersection with $\langle (y z), -\tau_x \rangle$ by Lemma 4.1 and Lemma 5.3 and because $\tau_x \in G$. To see that $\text{Isom}_\mathbb{Z}(L)$ is generated by $G$, $(y z)$, $-1$, choose $\varphi \in \text{Isom}_\mathbb{Z}(L)$. Since $\varphi$ is an isometry of $\mathfrak{sl}_2$, the set $R$ from Definition 4.7 is $\varphi$-invariant. Recall that $x \in R$, so $\varphi(x) \in R$. But $R = G(x)$ so there exists $g \in G$ such that $\varphi(x) = g(x)$. Now $g^{-1} \varphi(x) = x$, so $g^{-1} \varphi \in \langle (y z), -\tau_x \rangle$ in view of Lemma 5.3. Thus, $\varphi$ is in the subgroup of $\text{Isom}_\mathbb{Z}(L)$ generated by $G$, $(y z)$, $-1$. Therefore $\text{Isom}_\mathbb{Z}(L)$ is generated by $G$, $(y z)$, $-1$. By the above comments, $\text{Isom}_\mathbb{Z}(L) = \langle (y z), -1 \rangle \ltimes G$. The other assertions follow by symmetry or a routine argument. \hfill \square

Theorem 5.5. $\text{AAut}_\mathbb{Z}(L) = \text{Isom}_\mathbb{Z}(L)$.

Proof: We know already that $\text{AAut}_\mathbb{Z}(L) \subseteq \text{Isom}_\mathbb{Z}(L)$. By Theorem 5.4 we have $\text{Isom}_\mathbb{Z}(L) = \langle (y z), -1 \rangle \ltimes G$. But $(y z)$, $-1$, $G$ are all contained in $\text{AAut}_\mathbb{Z}(L)$, so $\text{Isom}_\mathbb{Z}(L) \subseteq \text{AAut}_\mathbb{Z}(L)$ holds as well. \hfill \square

Theorem 5.6. $\text{Aut}_\mathbb{Z}(L)$ is equal to each of the groups
\[ \langle -(x y) \rangle \ltimes G, \quad \langle -(y z) \rangle \ltimes G, \quad \langle -(z x) \rangle \ltimes G. \]
In particular $\text{Aut}_\mathbb{Z}(L)$ is isomorphic to $\mathbb{Z}_2 \ltimes G$.

Proof: Combine (5.2), Theorem 5.4 and Theorem 5.5 \hfill \square

6. THE LATTICE $L^*$

In previous sections we have discussed the lattice $L = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z$. Here we consider the lattice
\[ L^* := \mathbb{Z}x^* \oplus \mathbb{Z}y^* \oplus \mathbb{Z}z^* \] (6.1)
from a similar point of view. By (2.11), $L^*$ is equal to $\mathfrak{sl}_2(\mathbb{Z})$ and contains $L$. Regarding $L$ and $L^*$ as free abelian groups, we see from (4.4) that $L$ has index 4 in $L^*$ and $L^*/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The duality between $\{x, y, z\}$ and $\{x^*, y^*, z^*\}$ gives
\[ L^* = \left\{ u \in \mathfrak{sl}_2 \mid (u, v) \in 2\mathbb{Z} \quad \forall v \in L \right\}, \quad (6.2) \]
\[ L = \left\{ u \in \mathfrak{sl}_2 \mid (u, v) \in 2\mathbb{Z} \quad \forall v \in L^* \right\}. \quad (6.3) \]
Define
\[ \text{Aut}_\mathbb{Z}(L^*) = \left\{ \varphi \in \text{Aut}_\mathbb{Z}(\mathfrak{sl}_2) \mid \varphi(L^*) = L^* \right\}, \]
\[ \text{AAut}_\mathbb{Z}(L^*) = \left\{ \varphi \in \text{AAut}_\mathbb{Z}(\mathfrak{sl}_2) \mid \varphi(L^*) = L^* \right\}, \]
\[ \text{Isom}_\mathbb{Z}(L^*) = \left\{ \varphi \in \text{Isom}_\mathbb{Z}(\mathfrak{sl}_2) \mid \varphi(L^*) = L^* \right\}. \]
Theorem 6.4. We have

(i) \( \text{Aut}_\mathbb{Z}(L^*) = \text{Aut}_\mathbb{Z}(L) \),
(ii) \( \text{AAut}_\mathbb{Z}(L^*) = \text{AAut}_\mathbb{Z}(L) \),
(iii) \( \text{Isom}_\mathbb{Z}(L^*) = \text{Isom}_\mathbb{Z}(L) \).

Proof: By (6.2) and (6.3), for each isometry \( \sigma \) of \( \mathfrak{sl}_2 \) we have

\[
\sigma(L) = L \iff \sigma(L^*) = L^*.
\] (6.5)

Part (iii) is immediate from this. Parts (i) and (ii) also follow, since by Lemma 4.5 and (5.1) each of the groups \( \text{Aut}_F(\mathfrak{sl}_2) \), \( \text{AAut}_F(\mathfrak{sl}_2) \) is contained in \( \text{Isom}_F(\mathfrak{sl}_2) \). \( \square \)

Since \( \bar{\rho} \in G \) cyclically permutes \( x^*, y^*, z^* \), it follows that \( G(x^*) = G(y^*) = G(z^*) \). We will describe the orbit \( G(z^*) \) after first determining the stabilizer of \( z^* \) in \( G \).

Theorem 6.6. The stabilizer of \( z^* \) in \( G \) is the subgroup generated by \( \exp(\text{ad} z^*) \).

Proof: The subgroup of \( G \) generated by \( \exp(\text{ad} z^*) \) is clearly contained in the stabilizer of \( z^* \).

For the reverse containment, we observe by (3.8) that to any \( g \in G \) there corresponds \( \theta \in \text{SL}_2(\mathbb{Z}) \) such that \( \hat{\theta} = g \). Writing \( \theta = \begin{bmatrix} m & p \\ n & q \end{bmatrix} \) and using \( z^* = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \), we see that

\[
g(z^*) = \theta z^* \theta^{-1} = \begin{bmatrix} mn & -m^2 \\ n^2 & -mn \end{bmatrix}.
\] (6.7)

When \( g(z^*) = z^* \), we must have \( n = 0 \), which implies that \( mq = \det(\theta) = 1 \). Hence \( m = q \in \{1, -1\} \). Then either \( \theta \) or \( -\theta \) is an integral power of \( \exp(z^*) \), putting \( g \) in the subgroup generated by \( \exp(\text{ad} z^*) \). \( \square \)

Theorem 6.8.

(i) The orbit \( G(z^*) \) consists of \( y^*, z^* \) together with all matrices of the form \( \begin{bmatrix} mn \\ n^2 \end{bmatrix} -m^2 \\ -mn \) where \( m, n \in \mathbb{Z} \) are nonzero and relatively prime.

(ii) \( G(z^*) \) consists of \( x^*, y^*, z^* \) together with all vectors of the form

\[
-\frac{1}{2}(a^2 x + b^2 y + c^2 z),
\]

where \( a, b, c \) are relatively prime positive integers such that one of the relations \( a = b + c \), \( b = c + a \), \( c = a + b \) holds.

Proof: For statement (i), we assume \( g \in G \) and \( g(z^*) \) is as in (6.7) above. Since \( \det(\theta) = mq - np = 1 \), either \( m, n \) are nonzero and relatively prime, or one of integers \( m, n \) is zero. If \( m = 0 \) then \( n = -p \in \{1, -1\} \) and \( g(z^*) = y^* \). We have seen in the proof of Theorem 6.6 that when \( n = 0 \) then \( g(z^*) = z^* \). Hence (i) holds.
Part (ii) expresses a matrix from part (i) in terms of the equitable basis. This conversion to a combination of \(x, y, z\) just amounts to the identity
\[
\begin{pmatrix}
mn & -m^2 \\
n^2 & -mn
\end{pmatrix} = -\frac{1}{2} \left((m-n)^2 x + m^2 y + n^2 z \right).
\tag{6.9}
\]

\[\square\]

**Theorem 6.10.** The sets \(G(z^*)\) and \(-G(z^*)\) are disjoint and the following sets coincide:

(i) \(G(z^*) \cup (-G(z^*))\)
(ii) the orbit of \(z^*\) under \(\text{Aut}_\mathbb{Z}(L^*)\)
(iii) the orbit of \(z^*\) under \(\text{AAut}_\mathbb{Z}(L^*) = \text{Isom}_\mathbb{Z}(L^*)\).

**Proof:** That \(G(z^*)\) and \(-G(z^*)\) are disjoint can be seen from Theorem 6.8. The fact that the sets in (i)-(iii) are all equal is a direct consequence of Theorems 5.4, 5.5, 5.6, and 6.4. \[\square\]

7. Connections with a hyperbolic Kac-Moody Lie algebra

In this section we make explicit the relationship between the equitable basis for \(\mathfrak{sl}_2\) and the Kac-Moody Lie algebra \(\mathfrak{g} = \mathfrak{g}(\mathcal{A})\) over \(\mathbb{F}\) associated with the Cartan matrix \(\mathcal{A}\) from (2.4). (All the terminology and necessary background material used here can be found in [K, Ch. 1].) The Coxeter-Dynkin diagram corresponding to \(\mathcal{A}\) is

Since each subdiagram is the diagram of a Cartan matrix of finite type \(A_1\) or of affine type \(A_1^{(1)}\), the matrix \(\mathcal{A}\) is of hyperbolic type (see [K, §4.10]). We adopt the point of view that \(x, y, z\) are the simple roots for \(\mathfrak{g}\) and that there is a symmetric bilinear form \(\langle , \rangle\) on \(\text{Span}_\mathbb{F}\{x, y, z\}\) whose values on these simple roots are specified by \(\mathcal{A}\). Since \(\mathcal{A}\) is nonsingular, the simple roots are linearly independent and the bilinear form is nondegenerate. The root lattice may be identified with \(L = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z\) in our earlier notation. Note that \(L\) comes equipped with the Lie product \([ , ]\). Note also that the group \(\text{Isom}_\mathbb{Z}(L)\) makes sense in the present context. Three elements belonging to this group are the simple reflections \(r_x, r_y, r_z\). For \(u \in \{x, y, z\}\) we have \((u, u) = 2\), so the reflection \(r_u\) is given by
\[
r_u(v) = v - \frac{2(u, v)}{(u, u)} u = v - (u, v)u
\tag{7.1}
\]
for all \(v \in L\). For example,
\[
 r_x(x) = -x, \quad r_x(y) = y + 2x, \quad r_x(z) = z + 2x.
\]
Comparing this with Lemma 3.6 (d) and using symmetry, we have

\[ r_x = (y z) \tau_x, \quad r_y = (z x) \tau_y, \quad r_z = (x y) \tau_z. \]  \hspace{1cm} (7.2)

The subgroup \( W \) of \( \text{Isom}_Z(L) \) generated by the reflections \( r_x, r_y, r_z \) is the Weyl group. There is another subgroup of \( \text{Isom}_Z(L) \) that comes up naturally here. Observe that \((x y), (y z), (z x)\) and 1, \( \varphi \), \( \varphi^2 \) together form a subgroup of \( \text{Isom}_Z(L) \) that is isomorphic to the symmetric group \( S_3 \); we identify this group with \( S_3 \) for the rest of the paper. In what follows, we adopt \( \pm S_3 \) as a shorthand for \( \langle \pm 1 \rangle \times S_3 \). We note that \( S_3 \) is the group of diagram automorphisms associated with \( A \) in the sense of [K, p. 68]. The following theorem is implied by [K, Cor. 5.10 (b)].

**Theorem 7.3.** For the Cartan matrix \( A \) in (2.4),

\[ \text{Isom}_Z(L) = \pm S_3 \ltimes W. \]

We now describe how \( W \) is related to \( G \). Instead of doing this directly, we will relate them both to a certain normal subgroup of \( W \) denoted \( W^+ \). Recall that for \( w \in W \) the length of \( w \) is the number of factors \( r_x, r_y, r_z \) in a reduced expression for \( w \). Let \( W^+ \) denote the subgroup of \( W \) consisting of the elements of even length. Then \( W^+ \) is a normal subgroup of \( W \) with index 2.

**Proposition 7.4.** \( W^+ = W \cap \text{Aut}_Z(L) \). Moreover \( W^+ \) is a normal subgroup of \( \text{Isom}_Z(L) \) with index 24.

**Proof:** To get the first assertion, note that each of \( r_x, r_y, r_z \) is an antiautomorphism of \( \mathfrak{sl}_2 \) preserving \( L \). The second assertion follows from the first, using Theorem 7.3 and the fact that \( \text{Aut}_Z(L) \) is normal in \( \text{Isom}_Z(L) \) with index 2. \( \square \)

The following result is immediate from the definition of \( W^+ \).

**Proposition 7.5.** \( W \) is equal to each of the groups

\[ \langle r_x \rangle \ltimes W^+, \quad \langle r_y \rangle \ltimes W^+, \quad \langle r_z \rangle \ltimes W^+. \]

In particular \( W \) is isomorphic to \( \mathbb{Z}_2 \ltimes W^+ \).

**Proposition 7.6.** \( W^+ \) is a normal subgroup of \( G \), and the cosets of \( W^+ \) in \( G \) are

\[ W^+, \quad \tau_x W^+, \quad \tau_y W^+, \quad \tau_z W^+, \quad \varphi W^+, \quad \varphi^2 W^+. \] \hspace{1cm} (7.7)

The quotient group \( G/W^+ \) is isomorphic to \( S_3 \).

**Proof:** The group \( W^+ \) is contained in \( G \) since

\[ r_x r_y = \varphi \tau_z \tau_y, \quad r_y r_z = \varphi \tau_x \tau_z, \quad r_z r_x = \varphi \tau_y \tau_x. \] \hspace{1cm} (7.8)

Moreover, \( W^+ \) is normal in \( G \) because \( W^+ \) is normal in \( \text{Isom}_Z(L) \). The index of \( W^+ \) in \( G \) is 6, since the index of \( W^+ \) in \( \text{Isom}_Z(L) \) is 24 and the index of
Proposition 7.9. $\text{Isom}_Z(L)/W \cong \langle \pm 1 \rangle \times D$, where $D$ is the dihedral group of order 12.

Proof: Let $E = \text{Isom}_Z(L)/W^+$. To argue that $E \cong \langle \pm 1 \rangle \times D$, we will produce elements $\eta, \vartheta \in E$ such that $\eta$ has order 2, $\vartheta$ has order 6, and

$$\eta \vartheta \eta = \vartheta^{-1}. \quad (7.10)$$

Set $\eta = (yz)W^+$ and $\vartheta = (yz)\tau_y W^+ = \tau_z(yz)W^+$. Note that $\eta$ has order 2, and since

$$\begin{align*}
(yz)((yz)\tau_y)(yz) &= \tau_y(yz) = \tau_y^{-1},
\end{align*}$$

equation $(7.10)$ holds. Moreover,

$$\left((yz)\tau_y\right)^2 = (yz)\tau_y(yz)\tau_y = \tau_z\tau_y = (xy)r_z(zx)r_y = (xy)(zx)rxr_y = \vartheta^2rxr_y \equiv \vartheta^2 \mod W^+$$

so that $\vartheta^6 = (\vartheta^2W^+)^3 = W^+$. Thus, $\vartheta$ has order 6. Together $\eta, \vartheta$ generate a subgroup of $E$ isomorphic to $D$. Since $\langle \pm 1W^+ \rangle$ is a central subgroup of $E$ and $|E| = 24$ by Proposition 7.4, we have that $E \cong \langle \pm 1 \rangle \times D$, as claimed. 

Let $\Delta$ denote the set of roots attached to the Cartan matrix $\mathcal{A}$. Then $\Delta = \Delta_+ \cup \Delta_-$ where $\Delta_+$ (resp. $\Delta_-$) is the set of roots that are nonnegative (resp. nonpositive) integral linear combinations of the simple roots $x, y, z$. We have $\Delta_- = -\Delta_+$. A root is real if it lies in the $W$-orbit of a simple root; otherwise it is imaginary. Thus $\Delta$ decomposes into the disjoint union of the sets of real and imaginary roots: $\Delta = \Delta^\text{re} \cup \Delta^\text{im}$. The following theorem is implied by [K, Prop. 5.10 (a)].

Theorem 7.11. For the Cartan matrix $\mathcal{A}$ in $(2.4)$ the set of real roots $\Delta^\text{re}$ coincides with the set $R$ from Definition 4.7.

With the above theorem in mind, we have the following result which gives an interpretation of Proposition 4.13.
Proposition 7.12. For the Cartan matrix $A$ from (2.4), the corresponding Weyl group $W$ and simple roots satisfy

(i) $W(x) = \{\alpha x + \beta y + \gamma z \in \Delta^{\text{re}} \mid \alpha \equiv 1, \beta \equiv 0, \gamma \equiv 0 \mod 2\}$;
(ii) $W(y) = \{\alpha x + \beta y + \gamma z \in \Delta^{\text{re}} \mid \beta \equiv 1, \alpha \equiv 0, \gamma \equiv 0 \mod 2\}$;
(iii) $W(z) = \{\alpha x + \beta y + \gamma z \in \Delta^{\text{re}} \mid \gamma \equiv 1, \alpha \equiv 0, \beta \equiv 0 \mod 2\}$.

Proof: Each element in $W(x)$ is obtained from $x$ by applying a product of reflections in the simple roots. The root $x$ belongs to the set on the right-hand side of (i), and whenever $u = \alpha x + \beta y + \gamma z$ belongs to that set, then so do

$$r_x(u) = (2\beta + 2\gamma - \alpha)x + \beta y + \gamma z,$$
$$r_y(u) = \alpha x + (2\alpha + 2\gamma - \beta)y + \gamma z,$$
$$r_z(u) = \alpha x + \beta y + (2\alpha + 2\beta - \gamma)z.$$

Consequently $W(x)$ is contained in the right side of (i). Similar results apply in parts (ii) and (iii). Thus $\Delta^{\text{re}} = W(x) \cup W(y) \cup W(z)$ is contained in the (disjoint) union of the three sets on the right, which forces equality to hold in each case. □

Consider the set of imaginary roots $\Delta^{\text{im}}$ associated with the Cartan matrix $A$ from (2.4). It follows from [M] or [K, Prop. 5.2] that

$$\Delta^{\text{im}} = \{u \in \Delta \mid (u, u) \leq 0\}. \quad (7.13)$$

An important special case is the set of isotropic roots

$$\Delta^0 = \{u \in \Delta \mid (u, u) = 0\}. \quad (7.14)$$

This set will be our focus for the rest of this section. Each of the following four propositions contains a characterization $\Delta^0$.

Proposition 7.15. [K Prop. 5.10 (c)] The set of isotropic roots corresponding to the Cartan matrix $A$ from (2.4) is given by

$$\Delta^0 = \{u \in L \setminus \{0\} \mid (u, u) = 0\}.$$

The following result is implied by [K Prop. 5.7].

Proposition 7.16. For the Cartan matrix $A$ in (2.4), a vector is an isotropic root if and only if it is $W$-equivalent to a nonzero integer multiple of at least one of

$$2x^* = -(y + z), \quad 2y^* = -(z + x), \quad 2z^* = -(x + y).$$
Proposition 7.17. For the Cartan matrix \( A \) in (2.4), a vector is an isotropic root if and only if it is \( \text{Isom}_\mathbb{Z}(L) \)-equivalent to a positive even integer multiple of \( z^* \) if and only if it is \( G \)-equivalent to a nonzero even integer multiple of \( z^* \). Thus, the set of isotropic roots is given by

\[
\Delta^0 = \bigcup_{n \in \mathbb{Z}, n \neq 0} 2nG(z^*).
\]

Proof: This follows from Proposition 7.16, from the fact that \( \text{Isom}_\mathbb{Z}(L) = \pm S_3 \ltimes W = \langle (y, z), -1 \rangle \ltimes G \), and from the fact that \( \Delta^0 \) is \( G \)-invariant by Proposition 7.15. \( \square \)

Proposition 7.18. The isotropic roots corresponding to the Cartan matrix \( A \) in (2.4) are precisely the vectors of the form \( n(a^2x + b^2y + c^2z) \) for some \( n \in \mathbb{Z} \setminus \{0\} \) and \( a, b, c \in \mathbb{Z}_{\geq 0} \) (not all 0) such that at least one of

\[
a = b + c, \quad b = c + a, \quad c = a + b.
\]

Proof: Combine Theorem 6.8 (iii) and Proposition 7.17. \( \square \)

Suppose now that \( e_i, f_i, h_i \) (1 ≤ \( i \) ≤ 3) are the Chevalley generators for the Kac-Moody Lie algebra \( \mathfrak{g} \) over \( \mathbb{F} \) corresponding to the Cartan matrix \( A \). They satisfy the Serre relations (see [K, (0.3.1)]). The Lie algebra has a decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \), where \( \mathfrak{h} \) is the span of the \( h_i \). For \( u = \alpha x + \beta y + \gamma z \in \Delta_+ \), \( \mathfrak{g}_u \) is the subspace of \( \mathfrak{g} \) consisting of all products of \( e_i \) such that the number of \( e_1 \)'s appearing is \( \alpha \), \( e_2 \)'s is \( \beta \) and \( e_3 \)'s is \( \gamma \). A similar statement applies for \( u \in \Delta_- \) with \( f_i \)'s replacing the \( e_i \)'s. Each subspace \( \mathfrak{g}_u \) is finite-dimensional, and its dimension is said to be the multiplicity of the root \( u \). There is an automorphism of \( \mathfrak{g} \) interchanging \( e_i \) and \( f_i \) and sending \( h_i \) to \(-h_i \). Thus, the multiplicities of \( u \) and \(-u \) are the same. Since

\[
\dim \mathfrak{g}_{\sigma u} = \dim \mathfrak{g}_u
\]

holds for all roots \( u \) and all \( \sigma \in W \), it follows that the multiplicity of each real root is 1. For the isotropic roots, we conclude the following.

Corollary 7.20. For the Cartan matrix \( A \) in (2.4), each isotropic root has multiplicity 1.

Proof: Let \( u \in \Delta^0 \). By Proposition 7.16 we may assume that \( u \) is a nonzero integer multiple of \( 2x^* \), \( 2y^* \), or \( 2z^* \). Since the situation is cyclically symmetric, and since \( u \) and \(-u \) have the same multiplicity, there is no loss in generality in assuming \( u = n(x + y) \) for \( n \in \mathbb{Z}_{>0} \). Thus, elements in \( \mathfrak{g}_u \) are commutators with \( n \) factors equal to \( e_1 \) and \( n \) factors equal to \( e_2 \). Since no \( e_3, f_3, h_3 \) are involved and since the relations governing \( e_i, f_i, h_i \) for \( i = 1, 2 \) are the same as for the affine Lie algebra \( \mathfrak{g}(A_1^{(1)}) \) corresponding to the matrix
Proof: Using (8.1), we have
\[
\begin{bmatrix}
2 & -2 \\
-2 & 2
\end{bmatrix},
\]
the multiplicity of \( u \) is the same as the multiplicity of \( n(x+y) \)
in \( g(A_1) \), which is known to be 1 (see [K] Cor. 7.4, for example).
\[ \square \]

8. \( \mathfrak{sl}_2 \)-modules and the equitable basis

Let \( \mathcal{V} \) denote a finite-dimensional \( \mathfrak{sl}_2 \)-module. Let \( \varphi : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathcal{V}) \) be the representation afforded by \( \mathcal{V} \), so that \( \varphi(u)(v) = u.v \) for all \( u \in \mathfrak{sl}_2 \) and \( v \in \mathcal{V} \). For all \( u \in \mathfrak{sl}_2 \), the map \( \varphi(u) : \mathcal{V} \to \mathcal{V} \) is nilpotent; therefore the map \( \exp (\varphi(u)) = \sum_{n=0}^{\infty} \varphi(u)^n/n! \) is well-defined. We note that \( \exp (\varphi(u)) \) is invertible with inverse \( \exp (-\varphi(u)) \), and that
\[
\exp (\varphi(u))\varphi(t)\exp (\varphi(u))^{-1} = \exp (\text{ad} \varphi(u))(\varphi(t))
\]
(8.1)
for all \( t \in \mathfrak{sl}_2 \). Using the dual basis \( \{x^*, y^*, z^*\} \) for \( \mathfrak{sl}_2 \), we define the maps
\[
P = \exp (\varphi(x^*))\exp (\varphi(y^*)), \quad T_x = \exp (\varphi(y^*))\exp (\varphi(z^*))\exp (\varphi(y^*)), \quad T_y = \exp (\varphi(z^*))\exp (\varphi(x^*))\exp (\varphi(z^*)), \quad T_z = \exp (\varphi(x^*))\exp (\varphi(y^*))\exp (\varphi(x^*)).
\]
If \( \varphi \) is the adjoint representation, then these are just the maps \( \varrho, \tau_x, \tau_y, \tau_z \) from Section 3. The next results say that analogues of the relations in Lemmas 3.3 and 3.4 hold for arbitrary representations \( \varphi \).

Lemma 8.2. Let \( \varphi : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathcal{V}) \) be a finite-dimensional \( \mathfrak{sl}_2 \)-representation. Then the corresponding maps \( P, T_x, T_y, T_z \) satisfy (i)–(v) below.

(i) \( P\varphi(x)P^{-1} = \varphi(y) \), \( P\varphi(y)P^{-1} = \varphi(z) \), \( P\varphi(z)P^{-1} = \varphi(x) \);
(ii) \( P\varphi(x^*)P^{-1} = \varphi(y^*) \), \( P\varphi(y^*)P^{-1} = \varphi(z^*) \), \( P\varphi(z^*)P^{-1} = \varphi(x^*) \);
(iii) \( P \exp (\varphi(x^*))P^{-1} = \exp (\varphi(y^*)) \), \( P \exp (\varphi(y^*))P^{-1} = \exp (\varphi(z^*)) \), \( P \exp (\varphi(z^*))P^{-1} = \exp (\varphi(x^*)) \);
(iv) \( PT_xP^{-1} = T_y \), \( PT_yP^{-1} = T_z \), \( PT_zP^{-1} = T_x \);
(v) \( T_x\varphi(x)T_x^{-1} = -\varphi(x) \), \( T_x\varphi(y)T_x^{-1} = 2\varphi(x) + \varphi(z) \), \( T_x\varphi(z)T_x^{-1} = 2\varphi(x) + \varphi(y) \).

Proof: Using (8.1), we have
\[
P\varphi(x)P^{-1} = \exp (\text{ad} \varphi(x^*))\exp (\text{ad} \varphi(y^*))(\varphi(x)) = \varphi(\exp(\text{ad} x^*)\exp(\text{ad} y^*)(x)).
\]
Recall \( \exp(\text{ad} x^*)\exp(\text{ad} y^*) = \varrho \) by Lemma 3.3(ii) and \( \varrho(x) = y \) by (2.3) so \( P\varphi(x)P^{-1} = \varphi(y) \).

The remaining assertions can be deduced from similar arguments. \( \square \)

Lemma 8.3. Let \( \varphi : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathcal{V}) \) be a finite-dimensional \( \mathfrak{sl}_2 \)-representation. Then the corresponding maps \( P, T_x, T_y, T_z \) satisfy (i)–(vi) below.


To prove that parts (i)–(iv) follow easily from Lemma 8.2 (iii), so consider part (v).

Proof: Parts (i)–(iv) follow easily from Lemma 8.2 (iii), so consider part (v). To prove that \( P^3 = T_x^2 \), in the left-hand side of \( T_x T_x = T_x^2 \) evaluate the first factor (resp. second factor) using the first (resp. second) equation in equations 8.3. Then simplify the result using \( P = \exp (\varphi(x^*)) \exp (\varphi(y^*)) \exp (\varphi(z^*)) \).

The equations \( P^3 = T_y^2 \) and \( P^3 = T_z^2 \) are obtained similarly. Concerning (vi), note by Lemma 8.2(i) that \( P^3 \) commutes with each of \( \varphi(x), \varphi(y), \varphi(z) \) and recall \( x, y, z \) is a basis for \( \mathfrak{sl}_2 \).

□

Remark 8.4. Let \( B_3 \) be Artin’s braid group given by generators \( s_1, s_2 \) and the relation \( s_1 s_2 s_1 = s_2 s_1 s_2 \). It can be seen from Lemma 8.3 that each finite-dimensional \( \mathfrak{sl}_2 \)-representation \( \varphi : \mathfrak{sl}_2 \to \gl(V) \) determines a representation of \( B_3 \) given by

\[
\begin{align*}
    s_1 & \mapsto \exp (\varphi(x^*)), \\
    s_2 & \mapsto \exp (\varphi(y^*)).
\end{align*}
\]

(Of course we could replace the pair \( x, y \) by \( y, z \) or \( z, x \) in the above line.) The center of \( B_3 \) is generated by \( (s_1 s_2)^3 \) and this maps to \( P^3 = T_x^2 \). In Corollary 8.8 below, we will show that \( P^3 \) must act as a scalar multiple of the identity when \( V \) is an irreducible \( \mathfrak{sl}_2 \)-module, and we will determine the exact value of that scalar. More general information on irreducible \( B_3 \)-modules, particularly those of dimension \( \leq 5 \), can be found in [TW].

For each integer \( d \geq 0 \), there exists a unique irreducible \( \mathfrak{sl}_2 \)-module of dimension \( d + 1 \) up to isomorphism. This module, which we denote \( V(d) \), has a basis \( \{v_i\}_{i=0}^d \) such that

\[
\begin{align*}
    h.v_i &= (d - 2i)v_i, \\
    f.v_i &= (i + 1)v_{i+1}, \\
    e.v_i &= (d - i + 1)v_{i-1}
\end{align*}
\]
for $0 \leq i \leq d$, where $v_{-1} = 0$ and $v_{d+1} = 0$. We call $\{v_i\}_{i=0}^d$ a standard basis of $V(d)$. By (2.11) we have
\[
\begin{align*}
  x.v_i &= (d - 2i)v_i, \\
  y.v_i &= 2(d - i + 1)v_{i-1} + (2i - d)v_i, \\
  z.v_i &= (2i - d)v_i - 2(i + 1)v_{i+1}
\end{align*}
\]
for $0 \leq i \leq d$, and by (2.11) we have
\[
\begin{align*}
  x^*.v_i &= (i - d - 1)v_{i-1} + (d - 2i)v_i + (i + 1)v_{i+1}, \\
  y^*.v_i &= (i + 1)v_{i+1}, \\
  z^*.v_i &= (i - d - 1)v_{i-1}
\end{align*}
\]
for $0 \leq i \leq d$. Let $\varphi_d$ denote the $\mathfrak{sl}_2$ representation afforded by $V(d)$.

**Lemma 8.5.** With respect to a standard basis for $V(d)$,
\[
\begin{align*}
  (i) & \text{ the matrix representing } \exp(\varphi_d(y^*)) \text{ is lower triangular with (i, j) entry } \binom{i}{j} \text{ for } 0 \leq j \leq i \leq d; \\
  (ii) & \text{ the matrix representing } \exp(-\varphi_d(y^*)) \text{ is lower triangular with (i, j) entry } (-1)^{i-j}\binom{i}{j} \text{ for } 0 \leq j \leq i \leq d; \\
  (iii) & \text{ the matrix representing } \exp(\varphi_d(z^*)) \text{ is upper triangular with (i, j) entry } (-1)^{j-i}\binom{d-i}{j-i} \text{ for } 0 \leq i \leq j \leq d; \\
  (iv) & \text{ the matrix representing } \exp(-\varphi_d(z^*)) \text{ is upper triangular with (i, j) entry } \binom{d-i}{j-i} \text{ for } 0 \leq i \leq j \leq d.
\end{align*}
\]

**Proof:** This is a routine calculation using the definition of the exponential and the actions of $y^*$, $z^*$ on the standard basis. □

**Lemma 8.6.** For a standard basis $\{v_i\}_{i=0}^d$ of $V(d)$,
\[
T_xv_i = (-1)^iv_{d-i} \quad (0 \leq i \leq d). \quad (8.7)
\]

**Proof:** By construction $v_i$ is an eigenvector for $x$ with eigenvalue $d-2i$. This combined with the first equation of Lemma 8.2(v) implies that $T_xv_i$ is an eigenvector for $x$ with eigenvalue $2i - d$. Therefore there exists $\alpha_i \in \mathbb{F}$ such that $T_xv_i = \alpha_i v_{d-i}$. We show $\alpha_i = (-1)^i$. One readily checks that $\alpha_0 = 1$ using the definition of $T_x$ and the data in Lemma 8.5. For $1 \leq i \leq d$, we apply the second equation of Lemma 8.2(v) to the vector $T_xv_i$; this yields $\alpha_i = -\alpha_{i-1}$ after a brief calculation. By the above comments $\alpha_i = (-1)^i$ for $0 \leq i \leq d$ and the result follows. □

**Corollary 8.8.** For the maps $P$, $T_x$, $T_y$, $T_z$ corresponding to $\varphi_d$,
\[
P^3 = T_x^2 = T_y^2 = T_z^2 = (-1)^dI. \quad (8.9)
\]
Proof: Let \( \{v_i\}_{i=0}^{d} \) denote a standard basis for \( \mathcal{V}(d) \). By Lemma \ref{lem:8.6} for \( 0 \leq i \leq d \) we see that \( T_x v_i = (-1)^i v_{d-i} \) and \( T_x v_{d-i} = (-1)^{d-i} v_i \). Therefore \( T_x^2 = (-1)^d I \). The result follows in view of Lemma \ref{lem:8.3}(v). \( \square \)

Starting with a standard basis \( \{v_i\}_{i=0}^{d} \) of \( \mathcal{V}(d) \) and using the map \( P \) corresponding to \( \varphi \), we obtain three different bases for \( \mathcal{V}(d) \):

\[
\{v_i\}_{i=0}^{d}, \quad \{P v_i\}_{i=0}^{d}, \quad \{P^2 v_i\}_{i=0}^{d}.
\]  
(8.10)

One significance of these bases is that for \( 0 \leq i \leq d \) the vector \( v_i \) (resp. \( P v_i \), resp. \( P^2 v_i \)) is an eigenvector for \( x \) (resp. \( y \), resp. \( z \)) with eigenvalue \( d - 2i \); this can be checked using Lemma \ref{lem:8.2}(i). Our next goal is to describe how the three bases \( \text{(8.10)} \) are related. To this end the following lemma will be useful.

**Lemma 8.11.** For a standard basis \( \{v_i\}_{i=0}^{d} \) of \( \mathcal{V}(d) \) and for the map \( P \) associated with \( \varphi \),

\[
\exp (\varphi_d(y^*)) v_i = (-1)^{d-i} P^2 v_{d-i},
\]  
(8.12)

\[
\exp (-\varphi_d(z^*)) v_i = (-1)^{d-i} P v_{d-i}
\]  
(8.13)

for \( 0 \leq i \leq d \).

**Proof:** Equation \( \text{(8.12)} \) follows from

\[
P^{-2} \exp (\varphi_d(y^*)) v_i = (-1)^d P \exp (\varphi_d(y^*)) v_i
\]

\[
= (-1)^d \exp (\varphi_d(y^*)) \exp (\varphi_d(y^*)) v_i
\]

\[
= (-1)^d T_x v_i
\]

\[
= (-1)^{d-i} v_{d-i}
\]

and line \( \text{(8.13)} \) can be derived similarly. \( \square \)

**Corollary 8.14.** For a standard basis \( \{v_i\}_{i=0}^{d} \) of \( \mathcal{V}(d) \) and for the map \( P \) associated with \( \varphi \),

\[
P v_0 = \sum_{i=0}^{d} v_i, \quad P^2 v_0 = \sum_{i=0}^{d} P v_i, \quad (-1)^d v_0 = \sum_{i=0}^{d} P^2 v_i.
\]  
(8.15)

**Proof:** To derive the equation on the left in \( \text{(8.15)} \), set \( i = d \) in \( \text{(8.13)} \) and evaluate the result using Lemma \ref{lem:8.5}(iv). The other two relations in \( \text{(8.15)} \) can be obtained similarly using \( P^3 = (-1)^d I \). \( \square \)

**Proposition 8.16.** For a standard basis \( \{v_i\}_{i=0}^{d} \) of \( \mathcal{V}(d) \) and for the map \( P \) corresponding to \( \varphi \) the following (i)-(iii) hold for \( 0 \leq i \leq d \).

(i) The image of \( (\varphi_d(z^*))^{d-i} \) on \( \mathcal{V}(d) \) is equal to each of the subspaces \( \text{Span}_P\{v_0, \ldots, v_i\} \) and \( \text{Span}_P\{P v_0, \ldots, P v_{d-i}\} \).
(ii) The image of \((\varphi_d(x^*))^{d-i}\) on \(\mathcal{V}(d)\) is equal to each of the subspaces
\[
\text{Span}_F\{Pv_0, \ldots, Pv_i\}, \quad \text{Span}_F\{P^2v_d, \ldots, P^2v_{d-i}\}.
\]

(iii) The image of \((\varphi_d(y^*))^{d-i}\) on \(\mathcal{V}(d)\) is equal to each of the subspaces
\[
\text{Span}_F\{P^2v_0, \ldots, P^2v_i\}, \quad \text{Span}_F\{v_d, \ldots, v_{d-i}\}.
\]

Proof: (i) Recall \(z^*v_i = (i - d - 1)v_{i-1}\) so the image of \((\varphi_d(z^*))^{d-i}\) on \(\mathcal{V}(d)\) is \(\text{Span}_F\{v_0, \ldots, v_i\}\). Also \(\text{Span}_F\{v_0, \ldots, v_i\} = \text{Span}_F\{Pv_d, \ldots, Pv_{d-i}\}\) in view of Lemma 8.5(iv) and (8.13).

(ii), (iii): Apply \(P\) and \(P^2\) to the equations in part (i). \(\square\)

We have described how the three bases (8.10) are related. To visualize this description it is helpful to draw some diagrams. In these pictures, the following convention will be adopted. Given bases \(\{w_i\}_{i=0}^d\) and \(\{w'_i\}_{i=0}^d\) for \(\mathcal{V}(d)\), the display below will mean that \(\text{Span}_F\{w_0, \ldots, w_i\} = \text{Span}_F\{w'_d, \ldots, w'_{d-i}\}\) for \(0 \leq i \leq d\):

With this convention in mind, Proposition 8.16 tells us that

By Corollary 8.14, the sum of the vectors on each edge of the triangle is a scalar multiple of the vector at the opposite vertex.

For the special case of the adjoint module \(\mathcal{V}(2)\), the elements
\[
v_0 = z^* = -e, \quad v_1 = x = h, \quad v_2 = y^* = f
\]
form a standard basis. In this case the picture is

![Diagram](image)

By (2.10) we have

\[ z^* + x + y^* = x^*, \quad x^* + y + z^* = y^*, \quad y^* + z + x^* = z^*, \]

which is just (8.15) in this special case.

It is well known that each irreducible \( \mathfrak{sl}_2 \)-module \( V(d) \) can be realized explicitly as the space of homogeneous polynomials over \( F \) of total degree \( d \) in two variables. The symmetry in the equitable basis for \( \mathfrak{sl}_2 \) is reflected in how it acts on these polynomials, as we now discuss.

The action of \( \mathfrak{sl}_2 \) on \( V(1) \) extends to an action of \( \mathfrak{sl}_2 \) on the symmetric algebra \( S := S(V(1)) \) by derivations, so that \( d.(uv) = (d.u)v + u(d.v) \) holds for all \( d \in \mathfrak{sl}_2 \) and all \( u, v \in S \). We regard \( S \) as the polynomial algebra \( F[s, t] \) in two commuting indeterminates \( s, t \) and set \( r = -s - t \). Then

\[ S = F[r, s] = F[s, t] = F[t, r] \quad \text{and} \quad r + s + t = 0. \]

We identify the variables \( t, s \) with the standard basis elements \( v_0, v_1 \) respectively, and get the actions

- \( x: r \mapsto s - t \), \( y: s \mapsto t - r \), \( z: t \mapsto r - s \)
- \( s \mapsto -s \), \( t \mapsto -t \), \( r \mapsto -r \)
- \( t \mapsto t \), \( r \mapsto r \), \( s \mapsto s \),

\[ x^*: r \mapsto 0 \quad y^*: s \mapsto 0 \quad z^*: t \mapsto 0 \]
- \( s \mapsto r \), \( t \mapsto s \), \( r \mapsto t \)
- \( t \mapsto -r \), \( r \mapsto -s \), \( s \mapsto -t \).

**Proposition 8.17.** For an integer \( d \geq 0 \) the following hold.

(i) The homogeneous polynomials of degree \( d \) form an \( \mathfrak{sl}_2 \)-submodule of \( S \) that is isomorphic to \( V(d) \);
(ii) each of the sets \( \{s^it^d-i\}_{i=0}^d \), \( \{t^ir^d-i\}_{i=0}^d \), \( \{r^is^d-i\}_{i=0}^d \) is a basis for this submodule;
(iii) for \( 0 \leq i \leq d \) the vector \( s^it^d-i \) (resp. \( t^ir^d-i \), resp. \( r^is^d-i \)) is an eigenvector for \( x \) (resp. \( y \), resp. \( z \)) with eigenvalue \( d - 2i \).
For \( d = 3 \) these three bases appear on the perimeter of the triangle below.

9. Connections with the Poincaré Disk and Pythagorean Triples

Recall that \((v, w^*) = 2\delta_{v,w}\) for \(v, w \in \{x, y, z\}\). Thus, the elements \(\frac{1}{2}x^*, \frac{1}{2}y^*, \frac{1}{2}z^*\) are the fundamental weights of the Kac-Moody algebra \(g\), and the lattice \(\frac{1}{2}L^* = \mathbb{Z}(\frac{1}{2}x^*) \oplus \mathbb{Z}(\frac{1}{2}y^*) \oplus \mathbb{Z}(\frac{1}{2}z^*)\) is the weight lattice. The elements in the set \(\Omega := W(x^*) \cup W(y^*) \cup W(z^*)\) are the Weyl group images of twice the fundamental weights. Here we consider the set \(\Omega\), and relate it to Pythagorean triples. By Proposition 7.16, Proposition 7.18, and Theorem 6.8, the elements \(u \in \Omega\) have the form

\[ u = -\frac{1}{2}(a^2 x + b^2 y + c^2 z) \]

where \(a, b, c \in \mathbb{Z}_{\geq 0}\) (not all 0) and at least one of the following holds:

\[ a = b + c, \quad b = c + a, \quad c = a + b. \]

Each element in \(\Omega\) corresponds to a Pythagorean triple \((\alpha, \beta, \gamma)\) in the following way. When \(c = a + b\), the Pythagorean triple \((\alpha, \beta, \gamma)\) can be obtained from the matrix equation \(M(a^2, b^2, c^2)^t = (\alpha, \beta, \gamma)^t\) where

\[
M = \begin{bmatrix}
-1 & 1 & 1 \\
0 & -1 & 1 \\
0 & 1 & 1
\end{bmatrix},
\]

and \(\gamma^2 = \alpha^2 + \beta^2\). Similarly, when \(a = b + c\), then for \((\beta, \gamma, \alpha)^t := M(b^2, c^2, a^2)^t\) we have \(\alpha^2 = \beta^2 + \gamma^2\); and when \(b = c + a\) then for \((\gamma, \alpha, \beta)^t := M(c^2, a^2, b^2)^t\) we have \(\beta^2 = \gamma^2 + \alpha^2\).

There is a beautiful way to visualize the set \(\Omega\) that brings together many of the ideas in this paper. Starting with the picture for \(V(2)\) in Section 8, and applying reflections in \(W\) we obtain the Poincaré disk \(\mathcal{P}\). We have displayed a portion of the disk in Figure 1 below and have given the corresponding triple \(a^2, b^2, c^2\) for each displayed point on the circumference between \(x^*\) and \(y^*\). For each of these points, \(c = a + b\). The other labels can be obtained
by permuting $x^*, y^*, z^*$ and $x, y, z$. The vertices on the perimeter are the elements of $\Omega$; that is, they are the Weyl group reflections of the fundamental weights of the hyperbolic Kac-Moody algebra, after each one is multiplied by a factor of 2.

The group $\text{Isom}_\mathbb{Z}(L) = \langle (yz), -1 \rangle \rtimes G = \pm S_3 \rtimes W$ is the group of automorphisms $\text{Aut}(\mathcal{P})$ of the disk. Indeed, the Weyl group $W$ permutes the triangles. By multiplying any $\xi \in \text{Aut}(\mathcal{P})$ by an appropriate element of $W$, we can assume that $\xi$ maps the central triangle to itself. Such an automorphism can be seen to belong to $\pm S_3$. For the other realization of $\text{Aut}(\mathcal{P})$, observe that by multiplying an automorphism $\xi \in \text{Aut}(\mathcal{P})$ by a suitable element of $G$, we can assume that $\xi$ fixes the edge labeled by $x$ and $-x$. Since the stabilizer of $x$ in $G$ is trivial according to Lemma 4.1, such an automorphism must belong to $\langle (yz), -1 \rangle$. 
Figure 1
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