H-log spaces of continuous functions, potentials, and elliptic boundary value problems.

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Abstract

In these notes we study a family of Banach spaces, denoted $D^{0, \alpha}(\Omega)$, $\alpha \in \mathbb{R}^+$, and called H-log spaces. For $0 < \lambda \leq 1$, one has $C^{0, \lambda}(\Omega) \subset D^{0, \alpha}(\Omega) \subset C(\Omega)$, with compact embedding. These spaces present the following "intermediate" regularity behavior. Solutions $u$ of second order linear elliptic boundary value problems, under "external forces" $f \in D^{0, \alpha}(\Omega)$ for some $\alpha > 1$, satisfy $\nabla^2 u \in D^{0, \alpha-1}(\Omega)$. This result is optimal, since $\nabla^2 u \in D^{0, \beta}(\Omega)$, for some $\beta > \alpha - 1$, is false in general. We present a preliminary study on this subject.

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1 Motivation.

Assume that one looks for subspaces $X(\Omega)$ of $C(\Omega)$, as large as possible, such that solutions $u$ of second order linear elliptic boundary value problems, under "external forces" $f \in X(\Omega)$, have second order continuous derivatives $\nabla^2 u$ up to the boundary. It is well known that $X(\Omega) = C(\Omega)$ is too large. On the other hand, the classical Hölder spaces $C^{0, \lambda}(\Omega)$ are too narrow. In reference [2], dedicated to the Euler two dimensional evolution equations, we have appealed to a Banach space $C_*(\Omega)$, suitable for the above purposes. If $f \in C_*(\Omega)$ then $\nabla^2 u \in C(\Omega)$. However, full regularity $\nabla^2 u \in C_*(\Omega)$ is unreachable. This leads to the following question. For generical data $f \in C_*(\Omega)$, is there some additional, significant regularity for $\nabla^2 u$, besides continuity? Recall that, in the framework of Hölder spaces, full regularity is restored, since $\nabla^2 u$ and $f$ have the same regularity. This different behavior leads us to look for intermediate situations. In these notes we define and study a family of Banach spaces, denoted here $D^{0, \alpha}(\Omega)$, $\alpha > 0$, and called H-log spaces. For $\alpha > 1 > \beta > 0$, and $0 < \lambda \leq 1$, one has $C^{0, \lambda}(\Omega) \subset D^{0, \alpha}(\Omega) \subset C_*(\Omega) \subset D^{0, \beta}(\Omega) \subset C(\Omega)$, with compact embeddings. These spaces present the following "intermediate" regularity behavior. If $f \in D^{0, \alpha}(\Omega)$ for some $\alpha > 1$, then $\nabla^2 u \in D^{0, \alpha-1}(\Omega)$. This result is optimal. We present here a preliminary study on this subject. For some information on related references see the acknowledgment, at the end of section [4].
2 Results.

In the following Ω is an open, bounded, connected set in \( \mathbb{R}^n \), locally situated on one side of its boundary Γ. The boundary Γ is of class \( C^{2,\lambda} \), for some \( \lambda \), \( 0 < \lambda \leq 1 \).

By \( C(\Omega) \) we denote the Banach space of all real continuous functions \( f \) defined in \( \Omega \). The classical "sup" norm is denoted by \( \| f \| \). We also appeal to the classical spaces \( C^k(\Omega) \) endowed with their usual norms \( \| u \|_k \), and to the Hölder spaces \( C^{0,\lambda}(\Omega) \), endowed with the standard semi-norms and norms, denoted here by the non-standard symbols

\[ [f]_{H(\lambda)}, \quad \text{and} \quad \| f \|_{H(\lambda)}. \]

The meaning of \( \| f \|_{H(k,\lambda)} \) is obvious. Further, \( C^\infty(\Omega) \) denotes the set of all restrictions to \( \Omega \) of indefinitely differentiable functions in \( \mathbb{R}^n \). Norms in function spaces, whose elements are vector fields, are defined in the usual way by means of the corresponding norms of the components.

The symbol \( c \) denotes general positive constants depending at most on \( n \) and \( \Omega \). The symbol \( C \) denotes positive constants depending at most on \( n \), \( \Omega \), and \( \alpha \), see below. We may use the same symbol to denote different constants. \( X \preceq Y \) means that the immersion is compact.

For an arbitrary, stationary or evolution, boundary value problem we say that solutions are classical if all derivatives appearing in the equations and boundary conditions are continuous up to the boundary on their domain of definition. For brevity, the problem of looking for minimal conditions on the data, which still lead to classical solutions, is called by us "the minimal assumptions problem". In reference [2] we considered this problem for 2-DEuler equations in bounded domains. Since the study of this classical problem leads to considering the auxiliary problem

\[
\begin{align*}
-\Delta u &= f \quad \text{in} \; \Omega, \\
u &= 0 \quad \text{on} \; \Gamma,
\end{align*}
\]

(2.1)

it follows that the "minimal assumptions problem" for the Euler equations led to the corresponding problem for (2.1). As already remarked in section 1 a Hölder continuity assumption on \( f \) is unnecessarily restrictive. On the other hand, continuity of \( f \) is not sufficient. This situation led us to introduce in reference [2] an "intermediate" Banach space \( C^*(\overline{\Omega}) \), see section 3 below,

\[ C^{0,\lambda}(\overline{\Omega}) \subset C^*(\overline{\Omega}) \subset C(\overline{\Omega}), \]

for which, in particular, the following result holds (Theorem 4.5, in [2]).

**Theorem 2.1.** Let \( f \in C^*(\overline{\Omega}) \) and let \( u \) be the solution of problem (2.1). Then \( u \in C^{2}(\overline{\Omega}) \), moreover,

\[
\| \nabla^2 u \| \leq c \| f \|_*. \tag{2.2}
\]

Contrary to Hölder’s continuity case, where full regularity is restored in the sense described in section 1 no significant additional regularity is obtained.
above, besides continuity of $\nabla^2 u$. This set-up led us to look for intermediate, significant, situations. We define a new family of Banach spaces, denoted $D^{0,\alpha}(\Omega)$, $\alpha > 0$, and called H-log spaces, for which, the following holds. For $\alpha > 1 > \beta > 0$ and $0 < \lambda \leq 1$, one has

$$ C^{0,\lambda}(\Omega) \subset D^{0,\alpha}(\Omega) \subset C_*(\Omega) \subset D^{0,\beta}(\Omega) \subset C(\Omega), $$

with compact embedding. These new spaces present the following "intermediate" regularity behavior. Consider the simplest case of constant coefficients, second order, elliptic operators

$$ L = \sum_{1}^{n} a_{i,j} \partial_i \partial_j. $$

Without loss of generality, we assume that the matrix is symmetric. Below, we sketch a proof of the following result. A real proof is shown merely for interior regularity, see section 6. Optimality of the result is shown at the end of section 4.

**Theorem 2.2.** Let $f \in D^{0,\alpha}(\Omega)$ for some $\alpha > 1$, and let $u$ be the solution of problem

$$ \begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} $$

Then $u \in D^{2,\alpha-1}(\Omega)$, moreover

$$ \| \nabla^2 u \|_{\alpha-1} \leq C' \| f \|_{\alpha}. $$

The above result is optimal. If $\beta > \alpha - 1$, then $\nabla^2 u \in D^{0,\beta}(\Omega)$, is false in general.

The constant $C'$ depends on $n$, $\Omega$, $\alpha$, on the ellipticity constants of $L$, and on $\| \sigma \|$ (defined in the sequel).

Since $D^{0,\alpha}(\Omega) \subset C_*(\Omega)$, for $\alpha > 1$, it follows that (independently of the result claimed in the above theorem) the solution $u$ of problem (2.4) belongs to $C^{2}(\Omega)$.

We adapt here the argument developed in the classical treatise [6] to prove the classical Schauder estimates, in the framework of Hölder spaces, by appealing to the Hölder-Korn-Lichtenstein-Giraud inequality (see [6], part II, section 5.3). Following [6], we consider singular kernels $K(x)$ of the form

$$ K(x) = \frac{\sigma(x)}{|x|^n}, $$

where $\sigma(x)$ is infinitely differentiable for $x \neq 0$, and satisfies the properties $\sigma(t x) = \sigma(x)$, for $t > 0$, and

$$ \int_{S} \sigma(x) dS = 0, $$

where $S = \{ x : |x| = 1 \}$. It follows easily that, for $0 < R_1 < R_2$,

$$ \int_{R_1 < |x| < R_2} K(x) dx = \int_{R_1 < |x|} K(x) dx = \int K(x) dx = 0, $$

and

$$ \int_{|x| < R_1} K(x) dx = \int_{|x| < R_1} \frac{\sigma(x)}{|x|^n} dx = \int \frac{\sigma(x)}{|x|^n} dx = 0. $$

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where the last integral is in the Cauchy principal value sense.

For continuous functions \( \phi \) with compact support, the convolution integral, extended to the whole space \( \mathbb{R}^n \),

\[
(K \ast \phi)(x) = \int K(x - y) \phi(y) \, dy
\]

exists finite, as a Cauchy principal value.

Set \( I(R) = \{ x : |x| \leq R \} \), \( D^{0,\alpha}(R) = D^{0,\alpha}(I(R)) \), and similar for norms and semi-norms labeled by \( R \). The following theorem is the H-log counterpart of the classical result concerning Hölder spaces, sometimes called Hölder-Korn-Lichtenstein-Giraud inequality, see [6]. We show \( D^{0,\alpha-1}(\Omega) \) regularity for second order derivatives of potentials generated by a \( D^{0,\alpha}(\Omega) \) charge. The following result will be proved in section 5 below.

**Theorem 2.3.** Let \( K(x) \) be a singular kernel, enjoying the properties described above. Further, let \( \phi \in D^{0,\alpha}(R) \), for some \( \alpha > 1 \), vanish for \( |x| \geq R \). Then \( K \ast \phi \in D^{0,\alpha-1}(R) \), moreover

\[
\|(K \ast \phi)\|_{(\alpha-1),R} \leq C \|\phi\|_{\alpha,R},
\]

\[
C = c(\alpha)(\alpha - 1)^{-1} (\|\sigma\| + \|\nabla \sigma\|).
\]

One could also apply to ideas developed in references [1], [10], [15], and [16].

3 On the \( C_*(\Omega) \) space.

In the context of [2], the Theorem 2.1 was marginal. So, we did not published the proof, even though we were not able to find it in the current literature, for boundary value problems. At that time, we have proved the result for elliptic boundary value problems, like (2.4). Our proof depend only on the behavior of the related Green’s functions, hence it applies to larger classes of elliptic boundary value problems. Recently, by following the same ideas, we have proved the result for the Stokes system (see the Theorem 1.1 in [3]):

**Theorem 3.1.** For every \( f \in C_*(\Omega) \) the solution \((u, p)\) to the Stokes system

\[
\begin{cases}
- \Delta u + \nabla p = f & \text{in } \Omega, \\
\nabla \cdot u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma
\end{cases}
\]

belongs to \( C^2(\Omega) \times C^1(\Omega) \). Moreover, there is a constant \( c_0 \), depending only on \( \Omega \), such that the estimate

\[
\|u\|_2 + \|\nabla p\| \leq c_0 \|f\|_*, \quad \forall f \in C_*(\Omega),
\]

holds.

It is worth noting that similar results hold for elliptic equations with variable, sufficiently smooth, coefficients. In fact proofs depend only on suitable properties of the Green’s functions related to the particular problem.
For the readers convenience we recall definition and main properties of $C^\ast(\Omega)$. For $f \in C(\Omega)$, and for each $r > 0$, define $\omega_f(r)$ as in equation (4.1) below. In [2] we have introduced the semi-norm

$$[f]_* = [f]_\ast \equiv \int_0^\delta \frac{\omega_f(r)}{r} \, dr.$$  

The finiteness of the above integral is known as Dini’s continuity condition, see [3], equation (4.47). The space $C^\ast(\Omega)$ is defined as follows.

$$C^\ast(\Omega) \equiv \{ f \in C(\Omega) : [f]_* < \infty \}.$$  

A norm is introduced in $C^\ast(\Omega)$ by setting

$$\|f\|_* \equiv [f]_* + \|f\|.$$  

The following are some of the main properties of $C^\ast(\Omega)$ (for complete proofs see [3]):

- $C^\ast(\Omega)$ is a Banach space.
- The embedding $C^\ast(\Omega) \subset C(\Omega)$ is compact.
- The set $C^\infty(\Omega)$ is dense in $C^\ast(\Omega)$.

Remark 3.1. The results obtained in the framework of $C^\ast(\Omega)$ spaces led us to also consider the problem of their possible extension to larger functional spaces of continuous functions. This was done, at least partially, in references [4] and [5], to which the interested reader is referred.

4 $D^{0, \alpha}(\Omega)$ spaces: Definition and properties.

The above setup led us to define and study a family of Banach spaces $D^{0, \alpha}(\Omega)$, $\alpha > 0$, significant in our context. We call these spaces H-log spaces. They enjoy typical properties of functional spaces, suitable in PDEs theory.

We set

$$\omega_f(r) \equiv \sup_{x, y \in \Omega : 0 < |x - y| \leq r} |f(x) - f(y)|.$$  

In the sequel $0 < r < 1$. For $\alpha > 0$, $\alpha \neq 1$, one has

$$\int r^{-\alpha} (-\log r)^{1-\alpha} \, dr = \frac{1}{\alpha - 1} (-\log r)^{1-\alpha}.$$  

For $\alpha = 1$ the right hand side of the above equation should be replaced by $-\log (-\log r)$. It follows, in particular, that the $C^\ast(\Omega)$ semi-norm is finite if

$$\omega_f(r) \leq \text{Const.} (-\log r)^{-\alpha},$$  

for some $\alpha > 1$. This led us to define, for each fixed $\alpha > 0$, the semi-norm

$$[f]_\alpha \equiv \sup_{x, y \in \Omega : 0 < |x - y| < 1} \frac{|f(x) - f(y)|}{(-\log |x - y|)^{1-\alpha}} = \sup_{r \in (0, 1)} \frac{\omega_f(r)}{(-\log r)^{-\alpha}}.$$  

The space $D^{0, \alpha}(\Omega)$ is defined as follows.
Definition 4.1. For each real positive $\alpha$, we define

\begin{equation}
D^{0,\alpha}(\Omega) \equiv \{ f \in C(\Omega) : [f]_\alpha < \infty \}.
\end{equation}

A norm is introduced in $D^{0,\alpha}(\Omega)$ by setting

\[ \| f \|_\alpha \equiv [f]_\alpha + \| f \| . \]

The reason for the assumption $|x - y| < 1$ in (4.4) is due to the behavior of the function $\log r$ for $r \geq 1$. This is clearly not restrictive from the conceptual point of view.

Note that if we replace $0 < |x - y| < 1$ by $0 < |x - y| < \delta_0$, where $0 < \delta_0 < 1$, and set

\begin{equation}
[f]_{\alpha;\delta_0} \equiv \sup_{x, y \in \Omega, 0 < |x - y| < \delta_0} \frac{|f(x) - f(y)|}{(-\log |x - y|)^{-\alpha}},
\end{equation}

then,

\begin{equation}
[f]_{\alpha;\delta_0} \leq [f]_{\alpha} \leq [f]_{\alpha;\delta_0} + \frac{2}{(-\log \delta_0)^{-\alpha}} \| f \|.
\end{equation}

Hence, the norms $\| f \|_\alpha$ and $\| f \|_{\alpha;\delta_0}$ are equivalent. We may also define, in the canonical way, spaces $D^{k,\alpha}(\Omega)$, for integers $k \geq 1$. By writing (4.4) in the form

\[ [f]_\alpha = \sup_{r \in (0, 1)} \omega_f(r) (\log \frac{1}{r})^\alpha \]

we realize that we have merely replaced in the definition of Hölder spaces the quantity

\[ \frac{1}{r} \text{ by } \log \frac{1}{r}, \]

and allow $\alpha$ to be arbitrarily large. Finally, we may define Banach spaces $D^{k,\alpha}(\Omega)$ endowed with the norm

\[ \| f \|_{k,\alpha} \equiv \| f \|_k + [\nabla^k f]_{\alpha}. \]

Next we prove some properties of $D^{0,\alpha}$ spaces. Proofs are similar to those followed for Hölder spaces (much simpler than the corresponding proofs for $C^*_w(\Omega)$). For the reader’s convenience, without any claim of particular originality, we adapt to our case some proofs, well known in a Hölder’s framework.

It immediately follows from definitions that, for $\alpha > 1$, the embedding $D^{0,\alpha}(\Omega) \subset C^*(\Omega)$ is continuous (actually compact, see below). Furthermore,

\[ [f]_{*;\delta_0} \leq \frac{1}{(\alpha - 1) (-\log \delta_0)^{\alpha - 1}} [f]_{\alpha;\delta_0}, \]

for fixed $0 < \delta_0 < 1$.

Theorem 4.1. For each $\alpha > 0$, $D^{0,\alpha}(\Omega)$ is a Banach space.
Proof. Let \( f_n \) be a Cauchy sequence in \( D^{0, \alpha}(\overline{\Omega}) \). Since \( C(\overline{\Omega}) \) is complete, \( f_n \) is uniformly convergent to some \( f \in C(\overline{\Omega}) \). Let now \( x, y \in \overline{\Omega} \), with \(|x - y| < 1\). One has

\[
\frac{|f_n(x) - f_n(y)|}{(- \log |x - y|)^{-\alpha}} \leq |f_n|_\alpha \leq \text{Const}.
\]

By passing to the limit as \( n \to \infty \) one shows that \( f \in D_{0, \alpha} \). On the other hand, for each arbitrary couple \( x, y \in \overline{\Omega} \) satisfying \(|x - y| < 1\), one has

\[
|\frac{(f(x) - f_n(x)) - (f(y) - f_n(y))}{(- \log |x - y|)^{-\alpha}}| = \lim_{m \to \infty} \frac{|(f_m(x) - f_m(x)) - (f_m(y) - f_m(y))|}{(- \log |x - y|)^{-\alpha}} \leq \limsup_{m \to \infty} |f_m - f_n|_\alpha.
\]

Hence,

\[
|f - f_n|_\alpha \leq \limsup_{m \to \infty} |f_m - f_n|_\alpha.
\]

By appealing to the Cauchy sequence hypothesis, one shows that the right hand side of the above equation goes to zero as \( n \to \infty \).

**Theorem 4.2.** For \( \alpha > \beta > 0 \), the embedding \( D^{0, \alpha}(\overline{\Omega}) \subset D^{0, \beta}(\overline{\Omega}) \) is compact.

**Proof.** Let \( f_n \) be a bounded sequence in \( D_{0, \alpha}(\overline{\Omega}) \). Then, by Ascoli-Arzela’s theorem, there is a subsequence (denoted again by the same symbol \( f_n \)), and a function \( f \in C(\overline{\Omega}) \), such that \( f_n \) converges uniformly to \( f \). As in the proof of Theorem 4.1 one shows that \( f \in D_{0, \alpha} \). Hence, without loss of generality, by replacing \( f_n \) by \( f_n - f \), we may assume that \( f = 0 \). Furthermore, for \( x, y \in \overline{\Omega} \), with \(|x - y| < 1\),

\[
\frac{|f_n(x) - f_n(y)|}{(- \log |x - y|)^{-\alpha}} \leq \left( \frac{|f_n(x) - f_n(y)|}{(- \log |x - y|)^{-\alpha}} \right)^\beta |f_n(x) - f_n(y)|^{1 - \frac{\beta}{\alpha}} \leq |f_n|_{\alpha} \beta |f_n(x) - f_n(y)|^{1 - \frac{\beta}{\alpha}} \leq |f_n|_{\alpha} \left( 2 \| f_n \| \right)^{1 - \frac{\beta}{\alpha}},
\]

which tends to zero as \( n \) goes to infinity.

**Corollary 4.1.** The embedding \( D^{0, \alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \) is compact, if \( \alpha > 1 \).

The result follows by appealing to the continuity of the embedding \( D^{0, \alpha}(\overline{\Omega}) \subset C_*(\overline{\Omega}) \), for some \( \beta \) satisfying \( \alpha > \beta > 1 \).

**Theorem 4.3.** Set

\[
\Omega_\rho \equiv \{ x : \text{dist}(x, \Omega) < \rho \}.
\]

There is a \( \rho > 0 \) such that the following holds. There is a linear continuous map \( T \) from \( C(\overline{\Omega}) \) to \( C(\overline{\Omega}_\rho) \), and from \( D^{0, \alpha}(\overline{\Omega}) \) to \( D^{0, \alpha}(\overline{\Omega}_\rho) \), such that \( T f(x) = f(x) \), for each \( x \in \overline{\Omega} \).

**Proof.** For a sufficiently small positive \( \rho \), which depends only on \( \Omega \), we can construct a system of parallel surfaces \( \Gamma_r \), \( 0 \leq r \leq \rho \), such that the surface \( \Gamma_r \) lies outside \( \Omega \), at a distance \( r \) from \( \Gamma_0 = \Gamma \). For each \( x \in \overline{\Omega}_\rho - \Omega \), denote by \( \overline{x} \) the orthogonal projection of \( x \) upon the boundary \( \Gamma \). We define the extension \( T f = f \) by setting \( f(x) = f(x) \) if \( x \in \overline{\Omega} \), and \( f(x) = f(\overline{x}) \) if \( x \in \overline{\Omega}_\rho - \overline{\Omega} \). For convenience, assume that \( \rho < 1 \).

Since \( \Gamma \) is smooth and compact, the map \( x \to \overline{x} \) is Lipschitz continuous. This guarantees the existence of the constant \( k \), which depends only on \( \Omega \),
considered below. The map $Tf = \tilde{f}$ is clearly linear continuous from $C(\overline{\Omega})$ to $C(\overline{\Omega}_\rho)$. Define
\[
\omega_{f, \Omega_\rho}(r) = \sup_{x, y \in \Omega_\rho, |x - y| < r} |f(x) - f(y)|,
\]
Next, we show that
\[
(4.10) \quad \omega_{f, \Omega_\rho}(r) \leq \omega_f(kr),
\]
for $r \in (0, \rho)$, and some $k = k(\Omega) \geq 1$. Assume $|x - y| \leq \rho$. If $x \in \Omega_\rho$ and $y \in \Omega_\rho$, then
\[
|f(x) - f(y)| = |f(x) - f(y)|, \quad \text{and} \quad |x - y| \leq k|x - y|.
\]
If $x, y \in \Omega_\rho$, then
\[
|f(x) - f(y)| = |f(x) - f(y)|, \quad \text{and} \quad |x - y| \leq k|x - y|.
\]
Note that, in a neighborhood of a flat portion of $\Gamma$, one has $k = 1$. Equation (4.10) follows easily. Hence, for $\delta_0 \leq \rho$,
\[
[f]_{\alpha, \delta_0; \Omega_\rho} = \sup_{0 < r < \delta_0} \frac{\omega_{f, \Omega_\rho}(r)}{(-\log r)^{-\alpha}} \leq \sup_{0 < r < k\delta_0} \frac{\omega_f(r)}{(-\log r)^{-\alpha}}.
\]
Further, note that for $k > 0$, $0 < r < \frac{1}{k^{1+b}}$, and $0 < b < 1$, one has
\[
(4.11) \quad \log \frac{r}{k} \leq b \log r.
\]
By setting $b = \frac{1}{2}$, one easily shows that
\[
[f]_{\alpha, \delta_0; \Omega_\rho} \leq 2^\alpha [f]_{\alpha, k\delta_0}.
\]
Clearly, we impose to $\delta_0$ the constraint $k\delta_0 < 1$, to give sense to the right hand side of the above inequality. Equivalence of full norms is guaranteed by (4.7).

The next result shows that regular functions are dense in $D^{0, \alpha}(\overline{\Omega})$.

**Theorem 4.4.** The set $C^\infty(\overline{\Omega})$ is dense in $D^{0, \alpha}(\overline{\Omega})$.

The proof of this result, left to the reader, follows a well known argument, namely, a suitable combination of Theorem 4.3 with the classical Friedrichs’ mollification technique (left to the reader).

We end this section by showing that the regularity result claimed in Theorem 2.2 is optimal. We assume $L = \Delta$. The “singular point” is here the origin. Consider the function
\[
(4.12) \quad u(x) = (\log |x|)^{-\alpha} \sum_{i \neq j} x_i x_j,
\]
where $\alpha > 0$, and $|x| < 1$. Direct calculations show that each second order derivative of $u(x)$ is a combination of negative powers of $-\log |x|$, with $x$ dependent coefficients, homogeneous of degree zero. However, the larger negative
power is \(- (\alpha + 1)\) for double derivatives, and \(- \alpha\) for mixed derivatives. It follows that 
\[ \partial^2_i u \in D^{0, \alpha + 1}(\Omega) , \]
for each \(i = 1, \ldots, n\), in particular,
\[ \Delta u \in D^{0, \alpha + 1}(\Omega) . \]
However, for \(i \neq j\),
\[ \partial_i \partial_j u \not\in D^{0, \beta}(\Omega) , \]
if \(\beta > \alpha\), due to the presence of a \((- \log |x|)^{- \alpha}\) term.

By multiplication of \(u(x)\) by an infinitely differentiable function with compact support inside the unit sphere, and equal to 1 in a small neighborhood of the origin, we extend the above counterexample to homogeneous boundary value problems.

The above counter-example appears formally similar to that given in reference [11], at the end of section 2.6, where the author shows that the potential of a continuous function has not necessarily continuous second order derivatives.

Note that \(u\) is here the solution, not the "external force" \(f = - \Delta u\).

Taking into account the formal relation between the above, and the Hölder spaces, it would be interesting to define the spaces \(D^{0, \alpha}(\Omega)\) by means of an integral formulation. For instance, set
\[
[f]_p, \lambda \equiv \sup_{x_0 \in \Omega, 0 < \rho < 1} (- \log \rho)^\lambda \int_{\Omega(x_0, \rho)} |u(x) - u_{x_0, \rho}|^p |x - x_0|^{- \alpha} dx ,
\]
where \(u_{x_0, \rho}\) denotes the mean value of \(u(x)\) in \(\Omega(x_0, \rho)\), and \(\lambda > 0\). One easily shows that
\[
[f]_p, \lambda \leq c [f]_\alpha , \quad \text{where} \quad \alpha = \frac{1 + \lambda}{p} .
\]
It should be not difficult to show that inequality (4.14) is false for small values of the parameter \(\alpha\).

**Acknowledgement 4.1.** Concerning references, the author is particularly grateful to Francesca Crispo for calling our attention to the treatise [8], to which the reader is referred, and also to some related papers. In particular, as claimed in the introduction of the above volume, spaces \(D^{0, \alpha}(\Omega)\), in the particular case \(\alpha = 1\), were introduced in reference [14]. See also definition 2.2 in reference [8]. Other main references are [7], [13], [18], [19], and [20].

**5 Proof of Theorem 2.3.**

In this section we prove the Theorem 2.3, which is the counterpart of the classical Hölder-Korn-Lichtenstein-Giraud inequality in Hölder spaces.

**Proof.** Let \(x_0, x_1 \in I(R)\), \(0 < |x_0 - x_1| = \delta < \frac{1}{16}\). From (2.7) it follows that
\[
(K \ast \phi)(x) = \int (\phi(y) - \phi(x)) K(x - y) dy .
\]
Hence, with abbreviate notation,
Moreover, 

\[ (5.1) \]

\[ (K + \phi)(x_0) - (K + \phi)(x_1) = \]

\[ \int \{ (\phi(y) - \phi(x_0)) K(x_0 - y) - (\phi(y) - \phi(x_1)) K(x_1 - y) \} \, dy = \]

\[ \int_{|y - x_0| < 2\delta} \{ \ldots \} \, dr + \int_{2\delta < |y - x_0| < \frac{1}{2}} \{ \ldots \} \, dr + \int_{\frac{1}{2} < |y - x_0|} \{ \ldots \} \, dr \equiv I_1 + I_2 + I_4 . \]

Note that

\[ \{ y : |y - x_1| < 2\delta \} \subset \{ |y - x_0| < 3\delta \} . \]

Moreover, \( 3\delta < 1 \). So

\[ \int_{|y - x_0| < 2\delta} |\phi(y) - \phi(x_1)| |K(x_1 - y)| \, dy \leq \]

\[ (5.2) \]

\[ \int_{|y - x_1| < 3\delta} |\phi(y) - \phi(x_1)| |K(x_1 - y)| \, dy \leq \]

\[ [\phi]_a \int_{|y - x_1| < 3\delta} \frac{|\sigma(x_1 - y)|}{|x_1 - y|} \left( -\log |x_1 - y| \right)^{-\alpha} \, dy . \]

Next, by appealing to polar-spherical coordinates with \( r = |x_1 - y| \), by appealing to the fact that \( \sigma \) is positive homogeneous of order zero, and by appealing to (5.2) it readily follows that

\[ \int_{|y - x_0| < 2\delta} |\phi(y) - \phi(x_1)| |K(x_1 - y)| \, dy \leq \]

\[ (5.3) \]

\[ \frac{c}{\alpha - 1} \left( -\log 3\delta \right)^{-\alpha} \left\| \sigma \right\| \left[ \phi \right]_a . \]

A similar, simplified, argument shows that equation (5.3) holds by replacing \( x_1 \) by \( x_0 \) and \( 3\delta \) by \( 2\delta \). So,

\[ |I_1| \leq \frac{2c}{\alpha - 1} \left( -\log 3\delta \right)^{-\alpha} \left\| \sigma \right\| \left[ \phi \right]_a . \]

Since, by (4.11), \( \log (k \delta) \leq \frac{1}{\alpha} \log \delta \) for \( 0 < \delta \leq \frac{1}{k} \), it follows that

\[ |I_1| \leq \frac{c}{\alpha - 1} \left( -\log 3\delta \right)^{-\alpha} \left\| \sigma \right\| \left[ \phi \right]_a . \]

On the other hand

\[ I_2 = \int_{2\delta < |y - x_0| < \frac{1}{4}} (\phi(x_1) - \phi(x_0)) K(x_0 - y) \, dy + \]

\[ \int_{2\delta < |y - x_0| < \frac{1}{2}} (\phi(y) - \phi(x_1)) \left( K(x_0 - y) - K(x_1 - y) \right) \, dy . \]

The first integral vanishes, due to (2.7). Hence,

\[ |I_2| \leq \int_{2\delta < |y - x_0| < \frac{1}{2}} |\phi(y) - \phi(x_1)| \left| K(x_0 - y) - K(x_1 - y) \right| \, dy . \]
Further, by the mean-value theorem, there is a point $x_2$, between $x_0$ and $x_1$, such that
$$|K(x_0 - y) - K(x_1 - y)| \leq |\nabla K(x_2 - y)\delta|.$$

Since
$$\partial_i K(x) = \frac{1}{|x|^{n+1}} \left[ (\partial_i \sigma) \left( \frac{x}{|x|} \right) - n \frac{x_i}{|x|} \sigma(x) \right],$$
it readily follows that
(5.5) $$|K(x_0 - y) - K(x_1 - y)| \leq c\|\sigma\| \|\phi\| \left( -\log \left( \frac{3}{2} |x_0 - y| \right) \right)^{-\alpha}.$$  

The above estimates show that
(5.6) $$|I_2| \leq C \|\sigma\| \|\phi\| \int_{2\delta}^{t} (-\log \left( \frac{3}{2} r \right))^{-\alpha} r^{-2} dr.$$  

Next, by appealing to L’Hôpital’s rule, one gets
$$\lim_{\delta \to 0} \frac{\int_{2\delta}^{t} (-\log \left( \frac{3}{2} r \right))^{-\alpha} r^{-2} dr}{\delta^{\alpha-1} (-\log \delta)^{-\alpha}} = \frac{1}{2}.$$  

It follows that there is a positive constant $\delta_0 = \delta_0(\alpha)$ such that
(5.7) $$\delta \int_{2\delta}^{t} (-\log \left( \frac{3}{2} r \right))^{-\alpha} r^{-2} dr \leq (-\log \delta)^{-\alpha}, \quad \forall \delta \in (0, \delta_0).$$  

We fix $\delta_0$ satisfying $\delta_0 \leq \frac{1}{9}$. One has
(5.8) $$|I_2| \leq C \|\sigma\| \|\phi\| \left( -\log \delta \right)^{-\alpha}, \quad \forall \delta \in (0, \delta_0).$$  

Finally we consider $I_3$. By arguing as for $I_2$, in particular by appealing to (2.7) and (5.5), one shows that
$$|I_3| \leq C \delta \|\sigma\| \int_{|y-x_0| > \frac{1}{9}} \left| \frac{\phi(y) - \phi(x_1)}{|y - x_0|^{n+1}} \right| dy \leq C \delta \|\sigma\| \|\phi\|.$$  

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This last inequality, together with (5.6) and (5.4), shows that
\begin{equation}
(5.9) \quad |(\mathcal{K} \ast \phi)(x_0) - (\mathcal{K} \ast \phi)(x_1)| \leq C (-\log \delta)^{-(\alpha - 1)} \| \phi \|_\alpha ,
\end{equation}
for each couple of points \( x_0, x_1 \) such that \( 0 < |x_0 - x_1| \leq \delta_0(\alpha) \).

In agreement to definition (4.4), we must estimate \(|(\mathcal{K} \ast \phi)(x_0) - (\mathcal{K} \ast \phi)(x_1)|\) for all couple of points for which \( 0 < |x_0 - x_1| < 1 \). For brevity, we appeal here to a rough argumentation. Assume that \( \delta_0 < |x_0 - x_1| = \delta < 1 \). Let \( N > 1 \) be an integer such that \((N - 1) \delta_0 < \delta \leq N \delta_0 \). For simplicity, we assume that \( \delta = N \delta_0 \). Consider the sequence of points \( z_m = x_0 + m \frac{\delta_0}{N} (x_1 - x_0) \), for \( m = 0, \ldots, N \). One has \( z_0 = x_0, z_N = x_1 \), and \( |z_m + 1 - z_m| = \delta_0 \).

By appealing to equation (5.9), applied to each couple of points \( z_m, z_{m+1} \), a standard argument shows that
\begin{equation}
(5.10) \quad |(\mathcal{K} \ast \phi)(x_0) - (\mathcal{K} \ast \phi)(x_1)| \leq C N (-\log \delta_0)^{-(\alpha - 1)} \| \phi \|_\alpha
\end{equation}
\[ \leq \frac{C}{\delta_0} (-\log \delta)^{-(\alpha - 1)} \| \phi \|_\alpha . \]

\[ \square \]

6 On elliptic regularity.

In this chapter we apply the theorem 2.3 to prove regularity, in the framework of H-log spaces, for solutions of the elliptic equation (2.4). We follow the proof developed in Hölder spaces in [6], part II, section 5. So, in the sequel, the reader will be often referred to the above reference. We assume that \( n \geq 3 \).

By a fundamental solution of the differential operator \( L \) one means a distribution \( J(x) \) in \( \mathbb{R}^n \) such that
\begin{equation}
L \ J(x) = \delta(x) .
\end{equation}

The celebrated Malgrange-Ehrenpreis theorem states that every non-zero linear differential operator with constant coefficients has a fundamental solution (see, for instance, [17], Chap. VI, sec. 10). We recall that the analogue for differential operators whose coefficients are polynomials (rather than constants) is false, as shown by a famous Hans Lewy’s counter-example.

In particular, for a second order elliptic operator with constant coefficients and only higher order terms, one can construct explicitly a fundamental solution \( J(x) \) which satisfies the properties (i), (ii), and (iii), claimed in [6], namely,
(i) \( J(x) \) is a real analytic function for \( |x| \neq 0 \).
(ii) Since \( n \geq 3 \),
\begin{equation}
J(x) = \frac{\sigma(x)}{|x|^{n-2}} ,
\end{equation}
where \( \sigma(x) \) is positive homogeneous of degree 0.
(iii) Equation (6.1) holds. In particular, for every sufficiently regular, compact supported, function \( \phi \), one has
\[ \phi(x) = \int J(x - y) (L\phi)(y) \, dy = L \int J(x - y) \phi(y) \, dy . \]
For a second order elliptic operator as above, one has

\[ J(x) = c \left( \sum A_{ij} x_i x_j \right)^{\frac{2-n}{2}}, \]

where \( A_{ij} \) denotes the cofactor of \( a_{ij} \) in the determinant \( |a_{ij}| \).

Following [6], we denote by \( S \) the operator

\[ (S \phi)(x) = \int J(x - y) \phi(y) \, dy = (J * \phi)(x). \]

Note that in the constant coefficients case considered here, the operator \( T \) introduced in reference [6] vanishes.

Due to the structure of the function \( \sigma(x) \) appearing in equation (6.2), it readily follows that second order derivatives of \( (S \phi)(x) \) have the form

\[ \partial_i \partial_j S \phi = K_{ij} \phi, \]

where the \( K_{ij} \) enjoy the properties described for \( K \) in section 5.

We write, in abbreviate form,

\[ \nabla^2 S \phi(x) = \int K(x - y) \phi(y) \, dy, \]

where \( K(x) \) enjoys the properties described at the beginning of section 5.

As remarked in [6] “Lemma” A, if \( \phi \) is compact supported and sufficiently regular (for instance of class \( C^2 \)), then

\[ \phi = SL \phi, \]

Furthermore, if \( \phi = S f \), then \( L \phi = f \). Formally, one has \( SL = LS = I_d \).

Lets prove Theorem 2.2. Fix a no-negative \( C^\infty \) function \( \theta \), defined for \( 0 \leq t \leq 1 \) such that \( \theta(t) = 1 \) for \( 0 \leq t \leq \frac{1}{2} \), and \( \theta(t) = 0 \) for \( \frac{1}{2} \leq t \leq 1 \). Further fix a positive real \( R \), for convenience \( 0 < R < \frac{1}{2} R \), and define

\[ \zeta(x) = \begin{cases} 1 & \text{for } |x| \leq R, \\ \theta\left(\frac{|x| - R}{R}\right) & \text{for } R \leq |x| \leq 2R. \end{cases} \]

Let \( u \in D^{2,\alpha-1}(2R) \) be such that \( L u \in D^{2,\alpha}(2R) \), and set

\[ \phi = \zeta u. \]

Note that the support of \( \phi \) is contained in \( |x| < 2R \). By appealing to (6.6), we may write

\[ \| \nabla^2 \phi \|_{\alpha-1; 2R} = \| \nabla^2 SL \phi \|_{\alpha-1; 2R}. \]

From (6.5) and Theorem 2.2 one gets

\[ \| \nabla^2 SL \phi \|_{\alpha-1; 2R} \leq C \| L \phi \|_{\alpha; 2R}. \]

On the other hand,

\[ L \phi = \zeta L u + N, \]

where \( \| N \|_{\alpha; 2R} \leq C \| \zeta \|_{2; \alpha; 2R} \| u \|_{1, \alpha; 2R}. \) From the above estimates it readily follows that

\[ \| \nabla^2 \phi \|_{\alpha-1; 2R} \leq \| \zeta \|_{\alpha; 2R} \| L u \|_{\alpha; 2R} + C \| \zeta \|_{2, \alpha; 2R} \| u \|_{1, \alpha; 2R}. \]
So,

$$\| \nabla^2 u \| _{\alpha-1; R} \leq \| \zeta \| _{2, \alpha; 2R} \left( \| Lu \| _{\alpha; 2R} + C \| u \| _{1, \alpha; 2R} \right).$$

(6.8)

Next, we estimate $$\| \zeta \| _{2, \alpha; 2R}.$$ Recall that that $$2R < 1.$$ We consider points $$|x|$$ such that $$R \leq |x| \leq 2R,$$ and left to the reader different situations. Moreover, due to symmetry, it is sufficient to consider the one dimensional case

$$\zeta(t) = \theta \left( \frac{t - R}{R} \right) \quad \text{for} \quad R \leq t \leq 2R.$$

Hence

$$\zeta'(t) = \theta' \left( \frac{t - R}{R} \right) \frac{1}{R},$$

and

$$\zeta''(t) = \theta'' \left( \frac{t - R}{R} \right) \frac{1}{R^2}.$$ 

Further,

$$R^2 |\zeta''(t_2) - \zeta''(t_1)| \leq \left| \theta'' \left( \frac{t_2 - R}{R} \right) - \theta'' \left( \frac{t_1 - R}{R} \right) \right|,$$

where

$$\left| \frac{t_2 - R}{R} - \frac{t_1 - R}{R} \right| = \left| \frac{t_2 - t_1}{R} \right| \leq \frac{1}{3} < 1.$$

So

$$\frac{|\zeta''(t_2) - \zeta''(t_1)|}{(- \log |t_2 - t_1|)^{-\alpha}} \leq \left[ \theta'' \right]_{H(\alpha)} \frac{1}{R^{2+\alpha}} \frac{|t_2 - t_1|^\alpha}{(- \log |t_2 - t_1|)^{-\alpha}}.$$ 

Since $$0 \leq -r \log r \leq 1,$$ for $$0 < r \leq 1,$$ one shows that

$$[\zeta'']_{\alpha; 2R} \leq \left[ \theta'' \right]_{H(\alpha)} R^{-(2+\alpha)}.$$

It readily follows the estimate

$$\| \zeta \| _{2, \alpha; 2R} \leq C \| \theta \| _{H(2, \alpha)} \left( 1 + R^{-1} + R^{-2} + R^{-(2+\alpha)} \right) \equiv C_\theta(R).$$

(6.9) 

By appealing to (6.8), we show that

$$\| \nabla^2 u \| _{\alpha-1; R} \leq C_\theta(R) \left( \| Lu \| _{\alpha; 2R} + C \| u \| _{1, \alpha; 2R} \right),$$

(6.10) 

for $$0 < 2R < 1.$$ 

The above interior regularity result can be extended up to the boundary by following the argument sketched in part II, section 5.6, reference [6], which is essentially independent of the particular functional space. One starts by showing that the estimate (6.10) also holds on half-spheres, under the zero boundary condition on the flat boundary. This is achieved by means of a "reflection" to the corresponding whole sphere, through the flat boundary, as an odd function on the orthogonal direction. Then, sufficiently small neighborhoods of boundary points may be regularly mapped, one to one, onto half-spheres. This procedure allows the desired extension of the estimate (6.10) to functions $$u$$ defined on sufficiently small neighborhoods of boundary points, vanishing on the boundary. A well known finite covering argument lead to the thesis.
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