The numerical calculation method based on equivalent frequency-independent time-domain damping model

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Abstract
In order to realize the time-domain analysis based on hysteretic damping model, the frequency-independent time-domain damping model of single degree of freedom (SDOF) system is constructed. Based on the assumed relationship of vibration responses, the equivalent frequency-independent time-domain damping model in complex domain and real domain are proposed. The characteristic that the dissipated energy in each cycle is not related to the vibration frequency of external excitation is retained for the two equivalent damping models. Combined with Newmark-\(\beta\) method, the corresponding numerical methods are obtained. The numerical examples show that the free vibration responses are stably convergent based on equivalent damping models. The numerical results of vibration responses of SDOF system due to earthquake wave have high calculation accuracy. Compared with equivalent frequency-independent time-domain damping model in real domain, the computational accuracy of equivalent frequency-independent time-domain damping model in complex domain is higher, and the computational efficiency is lower.

Keywords
damping model, frequency-independent, equivalent, time-domain, numerical calculation

Introduction
The hysteretic damping model\textsuperscript{1–3} (or called complex damping model\textsuperscript{4}) is a common internal damping model. The hysteretic damping model can used to calculate structural vibration responses and provide reference for the oscillation theory.\textsuperscript{5,6} The hysteretic damping model is different from the viscous damping model.\textsuperscript{7} The characteristic of the hysteretic damping model is that the energy dissipation in each cycle is independent of the external excitation frequency for the steady state response.\textsuperscript{8} The characteristic is consistent with the harmonic vibration test results of most engineering materials.\textsuperscript{9} However, the viscous damping model does not have the characteristic.

The time-domain motion equation of hysteretic damping model for the single degree of freedom (SDOF) system is expressed as\textsuperscript{8,10}

\[
m \frac{d^2y(t)}{dt^2} + i\eta k y(t) + k y(t) = P e^{i\omega t} \quad (1)
\]

\[
y(t) = x(t) + i x'(t) \quad (2)
\]

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where \( m \) is the mass of system; \( k \) is the stiffness of system; \( \eta \) is the loss factor; \( i \) is the imaginary unit, \( i = \sqrt{-1} \); \( P \) is the vibration amplitude of harmonic wave; \( \theta \) is the vibration frequency of harmonic wave. \( x(t) \) is the displacement; \( x'(t) \) is the dual item of \( x(t) \).

The corresponding motion equation of free vibration response is expressed as
\[
m \frac{d^2 y(t)}{dt^2} + \eta k y(t) + k y(t) = 0
\]  
(3)

Using the complex plane method to solve equation (3), which is obtained as
\[
x(t) = [A_1 \cos(\omega_c t) + A_2 \sin(\omega_c t)]e^{-\alpha t} + [A_3 \cos(\omega_c t) + A_4 \sin(\omega_c t)]e^{\alpha t}
\]  
(4)

\[
\begin{align*}
\alpha_c &= \omega \sqrt{\frac{1 + \eta^2}{2}} \\
\omega_c &= \omega \sqrt{\frac{1 + \eta^2}{2}}
\end{align*}
\]  
(5)

where \( \omega_c \) is damped natural frequency; \( \alpha_c \) is attenuation coefficient; \( A_1, A_2, A_3, \) and \( A_4 \) are undetermined coefficients.

Equation (4) includes exponential increment term. The divergent term of the displacement is
\[
x_d(t) = [A_3 \cos(\omega_c t) + A_4 \sin(\omega_c t)]e^{\alpha t}
\]  
(6)

\( \alpha_c > 0 \) and \( e^{\alpha t} \) is exponential increment term. With the increase of the time, \( x_d(t) \) is divergent, and the corresponding free vibration response is divergent. The hysteretic damping model cannot be applied in time-domain, which is only applied to calculate the frequency-domain response.\(^{11}\) Pan et al.\(^ {12} \) proposed a new time-domain numerical method by aid of virtual initial condition, which obtained convergent results of hysteretic damping model. The method eliminated subjectively the divergent term of the free vibration general solution in essence. Jiang and Yuan\(^ {13} \) proposed a new direct for the element matrix updating problem, which can be applied in the hysteretic damping model.

In order to overcome the shortcoming of hysteretic damping model, the equivalent time-domain damping model was proposed. The Maxwell–Wiechert model can be equivalent to complex damping model.\(^ {14} \) Then, Genta and Amati\(^ {15} \) proposed an equivalent model that the Maxwell–Wiechert model with a number meter of spring damper branches. Luo et al.\(^ {16} \) paralleled a Maxwell element and a negative stiffness element, which is approximately equivalent to hysteretic damping model. Yang et al.\(^ {17} \) obtained an equivalent viscoelastic damping model with five parameters based on the least square method. Wang\(^ {18} \) determined the equivalent Rayleigh damping coefficients by aid of hysteretic frequency-domain damping model. To further simplify the equivalent process, an equivalent viscous damping model is obtained for SDOF systems, however, the natural frequency is only considered and there are no vibration modes involved in this process.\(^ {19} \)

In this paper, the frequency-independent time-domain damping model can be derived, which is equivalent to hysteretic frequency-domain damping model. The proposed damping model has time-domain characteristic.\(^ {20} \) The equivalent frequency-dependent damping model in complex domain can be obtained based on the relationship between velocity and displacement. The equivalent frequency-dependent damping model in real domain can be obtained based on the relationship between acceleration and displacement. Combined with Newmark-\( \beta \) method, the corresponding time-domain numerical calculation method is proposed based on the equivalent frequency-dependent damping models in complex domain and real domain. The procedure of the proposed method is presented in Figure 1. The free vibration response, harmonic vibration response and seismic vibration response of numerical examples are calculated, and the time-history results of different methods are compared.

**Equivalent frequency-independent time-domain damping model in complex domain**

**Based on velocity and displacement in complex domain**

Due to the harmonic wave, the frequency-domain motion equation of hysteretic damping model for SDOF system is expressed as
\[-m\omega^2 Y(\omega) + i\eta \text{sgn}(\omega) Y(\omega) + k Y(\omega) = P\pi[\delta(\omega + \theta) + \delta(\omega - \theta)] + i P\pi[\delta(\omega + \theta) - \delta(\omega - \theta)]\]  

(7)

where \(\omega\) is the vibration frequency of the response; \(X(\omega)\) is the frequency function of displacement. \(\text{sgn}(\omega)\) is signum, which is expressed as

\[\text{sgn}(\omega) = \begin{cases} 
1 & \omega > 0 \\
0 & \omega = 0 \\
-1 & \omega < 0 
\end{cases}\]  

(8)

Based on equation (7), the linear frequency-independent damping time-domain motion equation of SDOF system is equivalent to the frequency-domain motion equation of hysteretic damping model for steady state harmonic vibration response, which is expressed as in complex domain

\[m\frac{d^2 y(t)}{dt^2} + \frac{\eta k}{|\omega|} \frac{dy(t)}{dt} + ky(t) = Pe^{i\theta t}\]  

(9)

\[y(t) = x(t) + ix'(t)\]  

(10)

where \(x(t)\) is the displacement; \(x'(t)\) is the dual item of \(x(t)\). \(x'(t)\) is Hilbert transform of \(x(t)\) \(^{21,22}\) which is rewritten as

\[x'(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau\]  

(11)

The steady state response is expressed as

\[x(t) = A \sin(\theta t) + B \cos(\theta t)\]  

(12)

where \(A\) is the coefficient of sine item for steady state response; \(B\) is the coefficient of cosine item for steady state response. The dual item is expressed as
\[ x'(t) = -A \cos(\theta t) + B \sin(\theta t) \] (13)

In the steady state response, the vibration frequency is
\[ \omega = \theta \] (14)

The dissipated energy in each cycle is
\[ W = \int_0^{2\pi} \eta k \frac{dx(t)}{dt} \, dx(t) = \pi \eta A^2 \] (15)

Equation (15) shows that the energy is not related to the vibration frequency of external excitation, which is consistent with the conclusion of reference.9 The corresponding frequency-independent time-domain damping model is difficult to analyze. The reason is that there is unknown item \( \omega \) in equation (9), and the denominator of damping coefficient contains the vibration frequency. The damping coefficient will be amplified in the low frequency range, and the instability needs further improvement based on the tuned passive control approach.24 Therefore, the equivalent time-domain model needs to be constructed.

Based on the relationship of velocity and displacement in complex domain, the vibration frequency is expressed as
\[ \omega = \frac{1}{|y(t)|} \frac{dy(t)}{dt} \] (16)

Equation (16) is substituted into equation (9), which is rewritten as
\[ m \frac{d^2 y(t)}{dt^2} + \eta k \frac{dy(t)}{dt} \left| \frac{dy(t)}{dt} \right| + ky(t) = Pe^{int} \] (17)

The dissipated energy in each cycle is
\[ W = \int_0^{2\pi} \eta k \left| \frac{dx(t)}{dt} \right| \, dx(t) = \pi \eta A^2 \] (18)

Equation (18) shows that the energy is not related to the vibration frequency of external excitation, which is consistent with the frequency-independent time-domain damping model.

The scope of application is extended from harmonic wave to random external excitation. Based on equation (17), the equivalent frequency-independent time-domain damping model in complex domain is expressed as
\[ m \frac{d^2 y(t)}{dt^2} + \eta k \frac{dy(t)}{dt} \left| \frac{dy(t)}{dt} \right| + ky(t) = f(t) + if^H(t) \] (19)

where \( f(t) \) the acceleration of external excitation. \( f^H(t) \) is Hilbert transform of \( f(t) \).
\( f^H(t) \) can be expressed as23
\[ f^H(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(\tau)}{t-\tau} \, d\tau \] (20)

**Numerical calculation method**

The common numerical calculation methods can be used to solve equation (19), such as central difference method,7 Newmark-\( \beta \) method,25 Wilson-\( \theta \) method,26 and variational iteration method.27 Compared with other methods, Newmark-\( \beta \) method \((\alpha = 0.5, \beta = 0.25)\) is an unconditionally convergent method. Combined with Newmark-\( \beta \) method \((\alpha = 0.5, \beta = 0.25)\), an improved time-domain calculation method is proposed, which is used to calculate the vibration...
responses for the equivalent frequency-independent time-domain damping model in complex domain. equation (19) is rewritten as

\[
\begin{align*}
\frac{m}{\Delta t^2} \frac{d^2x(t)}{dt^2} + \eta k \frac{dx(t)}{dt} \left[ \frac{dx(t) + i\dot{x}(t)}{dt} \right] + kx(t) &= f(t) \\
\frac{m}{\Delta t^2} \frac{d^2\dot{x}(t)}{dt^2} + \eta k \frac{d\dot{x}(t)}{dt} \left[ \frac{d\dot{x}(t) + i\ddot{x}(t)}{dt} \right] + k\dot{x}(t) &= f^{ii}(t)
\end{align*}
\]

(21)

By the Fourier transform method, \(f(t)\) is expressed as

\[
f(t) = a_0 + \sum_{j=1}^{N} a_j \cos(\theta t) + \sum_{j=1}^{N} b_j \sin(\theta t)
\]

(22)

By the Hilbert transform method, \(f^{ii}(t)\) is obtained as

\[
f^{ii}(t) = \eta a_0 + \sum_{j=1}^{N} a_j \sin(\theta t) - \sum_{j=1}^{N} b_j \cos(\theta t)
\]

(23)

Time-history can be discretized, namely

\[
t = k\Delta t \quad (k = 0, 1, 2, \ldots)
\]

(24)

where \(\Delta t\) is time step.

The vibration frequency at time \(t_{k+1}\) is difficult to obtained, which involves the calculation of modules in complex domain. Therefore, here is the vibration frequency at time \(t_k\). At time \(t_{k+1}\), equation (21) is rewritten as

\[
\begin{align*}
\frac{m}{\Delta t^2} \frac{d^2x(t_{k+1})}{dt^2} + \eta k \frac{dx(t_{k+1})}{dt} \left[ \frac{dx(t_{k+1}) + i\dot{x}(t_{k+1})}{dt} \right] + kx(t_{k+1}) &= f(t_{k+1}) \\
\frac{m}{\Delta t^2} \frac{d^2\dot{x}(t_{k+1})}{dt^2} + \eta k \frac{d\dot{x}(t_{k+1})}{dt} \left[ \frac{d\dot{x}(t_{k+1}) + i\ddot{x}(t_{k+1})}{dt} \right] + k\dot{x}(t_{k+1}) &= f^{ii}(t_{k+1})
\end{align*}
\]

(25)

The average acceleration method based on Newmark-\(\beta\) method is unconditionally stable. The average acceleration method is used to calculate the vibration responses, which are obtained as

\[
\frac{dx(t_{k+1})}{dt} = \frac{dx(t_k)}{dt} + \frac{\Delta t}{2} \left[ \frac{d^2x(t_k)}{dt^2} + \frac{d^2x(t_{k+1})}{dt^2} \right]
\]

(26)

\[
x(t_{k+1}) = x(t_k) + \frac{\Delta t}{2} \left[ \frac{d^2x(t_k)}{dt^2} + \frac{d^2x(t_{k+1})}{dt^2} \right] + \frac{\Delta t^2}{4} \left[ \frac{d^2x(t_k)}{dt^2} + \frac{d^2x(t_{k+1})}{dt^2} \right]
\]

(27)

\[
\frac{d\dot{x}(t_{k+1})}{dt} = \frac{d\dot{x}(t_k)}{dt} + \frac{\Delta t}{2} \left[ \frac{d^2\dot{x}(t_k)}{dt^2} + \frac{d^2\dot{x}(t_{k+1})}{dt^2} \right]
\]

(28)

\[
\dot{x}(t_{k+1}) = \dot{x}(t_k) + \frac{\Delta t}{2} \left[ \frac{d^2\dot{x}(t_k)}{dt^2} + \frac{d^2\dot{x}(t_{k+1})}{dt^2} \right] + \frac{\Delta t^2}{4} \left[ \frac{d^2\dot{x}(t_k)}{dt^2} + \frac{d^2\dot{x}(t_{k+1})}{dt^2} \right]
\]

(29)

Substituting equations (26)–(29) into equation (25), \(\frac{dx(t_{k+1})}{dt}\) and \(\frac{d\dot{x}(t_{k+1})}{dt}\) are obtained as
Based on equation (9), the linear frequency-independent damping time-domain motion equation of SDOF system is expressed as in real domain

\[
\begin{align*}
\frac{d^2x(t_k)}{dt^2} &= f(t_k) - \left(\frac{\eta k \Delta t + k(\Delta t)^2}{2\omega_k}\right) \frac{d^2x(t_k)}{dt^2} - \left(\frac{\eta k}{\omega_k} + k\Delta t\right) \frac{dx(t_k)}{dt} - kx(t_k) \\
\frac{d^2x'(t_{k+1})}{dt^2} &= f'(t_{k+1}) - \left(\frac{\eta k \Delta t + k(\Delta t)^2}{2\omega_k}\right) \frac{d^2x'(t_{k+1})}{dt^2} - \left(\frac{\eta k}{\omega_k} + k\Delta t\right) \frac{dx'(t_{k+1})}{dt} - kx'(t_{k+1}) \\
\omega_k &= \left|\frac{dx(t_k)}{dt} + i \frac{dx'(t_k)}{dt}\right| / \left|x(t_k) + ix'(t_k)\right|
\end{align*}
\]

(30)

Substituting equation (30) into equations (26) – (29), \(x(t_{k+1}), x'(t_{k+1}), \frac{dx(t_{k+1})}{dt}, \), and \(\frac{dx'(t_{k+1})}{dt}\) are obtained. Repeated equations (26) – (29) can complete the time-history iterative calculation of the dynamic response for SDOF system.

In the time-history calculation process, when \(x(t_k) = 0\) or \(\omega_k = 0\), equation (30) cannot be calculated. In order to realize the calculation process, when \(x(t_k) = 0\) or \(\omega_k = 0\), \(x(t_k) = x(t_{k-1})\) and \(\omega_k = \omega_{k-1}\).

Here is also a problem that initial conditions of SDOF system should be determined first. The vibration frequency at the initial time assumed to be the damped natural frequency, which is

\[
\omega_0 = \omega \sqrt{1 + \frac{1 - \eta^2}{2}}
\]

(31)

where \(\omega\) is undamped natural frequency. \(x(t_0)\) and \(\frac{dx(t_0)}{dt}\) can usually be obtained directly. \(x'(t_0)\) and \(\frac{dx'(t_0)}{dt}\) assumed to be as

\[
\begin{align*}
x'(t_0) &= \frac{1}{\omega_0} \frac{dx(t_0)}{dt} \\
\frac{dx'(t_0)}{dt} &= \omega_0 x(t_0)
\end{align*}
\]

(32)

**Equivalent frequency-independent time-domain damping model in real domain**

**Based on acceleration and displacement in real domain**

Based on equation (9), the linear frequency-independent damping time-domain motion equation of SDOF system is equivalent to the frequency-domain motion equation of hysteretic damping model for steady state harmonic vibration response, which is expressed as in real domain

\[
m \frac{d^2x(t)}{dt^2} + \frac{\eta k}{\omega} \frac{dx(t)}{dt} + kx(t) = P \cos \omega t
\]

(33)

Based on the relationship of velocity and displacement in complex domain, the vibration frequency is expressed as

\[
\omega = \sqrt{\left|\frac{1}{x(t)} \frac{d^2x(t)}{dt^2}\right|}
\]

(34)

Equation (34) is substituted into equation (33), which is rewritten as

\[
m \frac{d^2x(t)}{dt^2} + \frac{\eta k}{\omega} \frac{dx(t)}{dt} \sqrt{\left|x(t) \frac{d^2x(t)}{dt^2}\right|} + kx(t) = P \cos \omega t
\]

(35)
The dissipated energy in each cycle is

\[ W = \int_0^{2\pi} \sqrt{\left| x(t) \frac{d^2 x(t)}{dt^2} \right|} \eta k \frac{dx(t)}{dt} \, dt = \pi \eta k A^2 \]  

(36)

Equation (36) shows that the energy is not related to the vibration frequency of external excitation, which is consistent with the conclusion of reference. The scope of application is extended from harmonic wave to random external excitation. Based on equation (36), the equivalent frequency-independent time-domain damping model in real domain is expressed as

\[ m \frac{d^2 x(t)}{dt^2} + \eta k \frac{dx(t)}{dt} \sqrt{\left| x(t) \frac{d^2 x(t)}{dt^2} \right|} + kx(t) = f(t) \]  

(37)

**Numerical calculation method**

The numerical method is proposed for solving equation (37), which is similar to the one of the equivalent frequency-independent time-domain damping model in complex domain, and is also combined with Newmark-β method.

The vibration frequency at time \( t_{k+1} \) is difficult to obtained, which involves the calculation of absolute value and quadratic root. Therefore, here is the vibration frequency at time \( t_k \). At time \( t_{k+1} \), equation (35) is rewritten as

\[ m \frac{d^2 x(t_{k+1})}{dt^2} + \eta k \frac{dx(t_{k+1})}{dt} \sqrt{\left| x(t_k) \frac{d^2 x(t_k)}{dt^2} \right|} + kx_{t_{k+1}} = f_{t_{k+1}} \]  

(38)

Based on Newmark-β method, substituting equations (26)–(29) into equation (38), \( \frac{dx(t_{k+1})}{dt} \) is obtained as

\[
\begin{align*}
\frac{d^2 x(t_{k+1})}{dt^2} &= f(t_{k+1}) - \left( \frac{\eta k \Delta t}{2 \sigma_k} + \frac{k (\Delta t)^2}{4} \right) \frac{d^2 x(t_k)}{dt^2} - \left( \frac{\eta k}{\omega_k} + k \Delta t \right) \frac{dx(t_k)}{dt} - kx(t_k) \\
\sigma_k &= \sqrt{\frac{1}{x(t_k)} \frac{d^2 x(t_k)}{dt^2}}
\end{align*}
\]  

(39)

Repeated equations (26)–(29) and (39) can complete the time-history iterative calculation of the dynamic response for SDOF system. \( x(t_0) \) and \( \frac{dx(t_0)}{dt} \) can usually be obtained directly. The vibration frequency at the initial time assumed to be the damped natural frequency, which is shown in equation (31).

**Numerical study**

**Free vibration response**

The numerical examples of SDOF systems are taken as Model A and Model B, and the model properties are shown in Table 1. The initial displacement is 10 cm. The initial velocity is 20 cm/s. Based on equivalent frequency-independent

| Model information | m, kg | k | η |
|-------------------|-------|---|---|
| Model A           | 10    | 100 N/m | 0.1 |
| Model B           | 10    | 100 N/m | 0.2 |
time-domain damping model in complex domain (CET) and real domain (RET), the displacement time-history and velocity time-history of Model A and Model B are calculated, which are shown in Figure 2 and Figure 3. The calculated results of CET and RET are both stably convergent, which are also approximately equal. CET and RET are different from the hysteretic frequency-domain damping model for free vibration response, which can overcome the divergent shortcoming.

**Harmonic vibration response**

The external excitation is harmonic wave that only includes a single vibration frequency, and the vibration frequency is 2 rad/s. The initial conditions are static. Based on CET and RET, the displacement time-history and velocity time-history of Model A and Model B are calculated, which are shown in Figure 2 and Figure 3. Besides, hysteretic frequency-domain damping model (HF)\(^{28}\) can only calculate the steady state responses. The displacement time-history and velocity time-history of Model A and Model B are obtained based on HF. The calculated results of HF only include the steady state response of SDOF system, and do not include the transient response. During the initially short time, Model A and Model B do not reach steady state and the free vibration response has not completely attenuated. The calculated results of CET and RET are slightly different from the one of HF. When Model A and Model B are in steady state, the displacement time-histories of CET, RET, and HF are approximately equal. However, the velocity time-history of RET is obviously different. Compared with RET, the velocity time-history of CET is closer to the one of HF. The results show that the computational accuracy of CET is higher.

**Seismic vibration response**

Earthquake wave can be regarded as harmonic waves with many different frequencies. El Centro wave, Taft wave, and Chichi wave are used to analyze the numerical models, and the information is shown in Table 2. The initial conditions are static. Based on HF, CET, and RET, the displacement time-history and velocity time-history of
Model A and Model B due to earthquake wave are calculated, which are shown in Figure 6, Figure 7, Figure 8, Figure 9, Figure 10, and Figure 11. The relative errors of peak displacement, peak velocity, and root mean square are used as indexes. The comparison of calculated results based on HF, CET, and RET are shown in Table 3, Table 4, and Table 5. For the external excitation of various frequency components, the equivalent principles of CET and RET no longer holds strictly. However, the calculated results of CET and RET have high precision. The maximum relative error of peak displacement for RET is 9.30%. The maximum relative error of peak velocity for RET is 10.13%. The maximum relative error of the root mean squares for RET is 8.62%. Compared with RET, the relative errors of CET are relatively smaller in general. The maximum relative error of peak displacement for CET is 5.41%. The maximum relative error of peak velocity for CET is 4.68%. The maximum relative error of the root mean squares for CET is 4.73%. Figure 5, Figure 6, Figure 7, Figure 8, Figure 9, and Figure 10 show that the displacement time-history and velocity time-history of CET are closer to the ones of HF locally. Therefore, the computational accuracy of CET is higher.

Using the same computing equipment, the computing time of HF, CET, and RET is shown in Table 6. The computing time of HF is longest, and HF cannot calculate structural free vibration response. Based on CET, Fourier transform of

Table 2. The information of earthquake waves.

| Earthquake name   | Peak ground acceleration (cm/s²) | Time (s) |
|-------------------|----------------------------------|----------|
| El centro wave    | 210.10                           | 35.00    |
| Taft wave         | 152.70                           | 40.00    |
| Chichi wave       | 1438.45                          | 36.00    |
Figure 6. The vibration responses of Model A due to El Centro wave: (a) displacement time-history, (b) velocity time-history.

Figure 7. The vibration responses of Model B due to El Centro wave: (a) displacement time-history, (b) velocity time-history.
Figure 8. The vibration responses of Model A due to Taft wave: (a) displacement time-history, (b) velocity time-history.

Figure 9. The vibration responses of Model B due to Taft wave: (a) displacement time-history, (b) velocity time-history.
Figure 10. The vibration responses of Model A due to Chichi wave: (a) displacement time-history, (b) velocity time-history.

Figure 11. The vibration responses of Model B due to Chichi wave: (a) displacement time-history, (b) velocity time-history.
### Table 3. Comparisons of calculated results with different methods due to El Centro wave.

|               | Model A |       | Model B |       |       |
|---------------|---------|-------|---------|-------|-------|
|               | HF  | CET  | RET  | HF  | CET  | RET  |
| $d_p$ (cm)    | 4.8107 | 4.8346 | 5.2580 | 4.0185 | 3.9475 | 4.2995 |
| $\delta_d$ (%)| —   | 0.50  | 9.30  | —   | 1.77  | 7.00  |
| $d_r$ (cm)    | 1.2062 | 1.1792 | 1.2697 | 0.8418 | 0.8503 | 0.8692 |
| $\lambda_d$ (%)| —   | 2.24  | 5.26  | —   | 1.01  | 3.25  |
| $v_p$ (cm/s)  | 48.6851 | 49.0230 | 53.6168 | 41.2293 | 40.7770 | 44.4176 |
| $\delta_v$ (%)| —   | 0.50  | 9.30  | —   | 1.77  | 7.00  |
| $v_r$ (cm)    | 12.1849 | 11.8646 | 12.6480 | 8.5365 | 8.4940 | 8.6444 |
| $\lambda_v$ (%)| —   | 2.63  | 3.74  | —   | 0.50  | 1.26  |

Note: $d_p$ is the peak displacement; $\delta_d$ is the relative error of peak displacement; $d_r$ is root mean square of the displacement time-history; $\lambda_d$ is the relative error of root mean square of the displacement time-history; $v_p$ is the peak velocity; $\delta_v$ is the relative error of peak velocity; $v_r$ is root mean square of the velocity time-history; $\lambda_v$ is the relative error of root mean square of the velocity time-history. The follows are the same.

### Table 4. Comparisons of calculated results with different methods due to Taft wave.

|               | Model A |       | Model B |       |       |
|---------------|---------|-------|---------|-------|-------|
|               | HF  | CET  | RET  | HF  | CET  | RET  |
| $d_p$ (cm)    | 3.4851 | 3.4918 | 3.7621 | 2.4836 | 2.3492 | 2.2804 |
| $\delta_d$ (%)| —   | 0.19  | 7.95  | —   | 5.41  | 8.18  |
| $d_r$ (cm)    | 0.9013 | 0.9256 | 0.9790 | 0.5877 | 0.5619 | 0.5471 |
| $\lambda_d$ (%)| —   | 2.70  | 8.62  | —   | 4.39  | 6.91  |
| $v_p$ (cm/s)  | 34.7673 | 33.1408 | 36.1468 | 25.2127 | 24.8463 | 24.3776 |
| $\delta_v$ (%)| —   | 4.68  | 3.97  | —   | 1.45  | 3.31  |
| $v_r$ (cm)    | 8.8997 | 8.6362 | 9.0977 | 5.7472 | 5.4752 | 5.3063 |
| $\lambda_v$ (%)| —   | 2.96  | 2.22  | —   | 4.73  | 7.67  |

### Table 5. Comparisons of calculated results with different methods due to Chichi wave.

|               | Model A |       | Model B |       |       |
|---------------|---------|-------|---------|-------|-------|
|               | HF  | CET  | RET  | HF  | CET  | RET  |
| $d_p$ (cm)    | 35.7924 | 36.6582 | 37.2320 | 29.8541 | 29.9997 | 30.3946 |
| $\delta_d$ (%)| —   | 2.42  | 4.02  | —   | 0.49  | 1.81  |
| $d_r$ (cm)    | 7.4403 | 7.4297 | 7.2808 | 6.0094 | 5.9362 | 6.1574 |
| $\lambda_d$ (%)| —   | 0.14  | 2.14  | —   | 1.22  | 2.46  |
| $v_p$ (cm/s)  | 374.2741 | 384.3137 | 362.9667 | 328.7988 | 336.2462 | 302.0978 |
| $\delta_v$ (%)| —   | 2.68  | 3.02  | —   | 2.26  | 8.12  |
| $v_r$ (cm)    | 74.5954 | 74.0246 | 72.8622 | 59.9985 | 60.4510 | 58.8020 |
| $\lambda_v$ (%)| —   | 0.77  | 2.32  | —   | 0.75  | 1.99  |

### Table 6. Computing time of different methods (in: second).

| Methods | El Centro wave | Taft wave | Chichi wave |
|---------|----------------|-----------|-------------|
| HF      | 0.72           | 0.88      | 0.75        |
| CET     | 0.30           | 0.38      | 0.32        |
| RET     | 0.05           | 0.07      | 0.05        |
external excitation is involved in the calculation process. Therefore, compared with RET, the computing time of CET is longer.

Conclusions

In the present work, the frequency-independent time-domain damping model is equivalent to the hysteretic frequency-domain damping model for steady state harmonic vibration response. However, the time-domain motion equation includes unknown vibration frequency. Based on the relationship of velocity and displacement, the equivalent frequency-independent time-domain damping model is proposed in complex domain. Based on the relationship of acceleration and displacement, the equivalent frequency-independent time-domain damping model is proposed in real domain. Combined with Newmark-$\beta$ method, the corresponding numerical methods are obtained in time-domain, respectively. The main conclusions are as follows.

1. In order to overcome the shortcoming that hysteretic damping model cannot be directly applied in time-domain, the equivalent frequency-independent time-domain damping models are proposed. The proposed damping models can reserve the characteristic that the energy dissipation in each cycle is independent of the external excitation frequency for the steady state vibration response. Besides, the problem that the vibration responses are divergent in the time-domain is solved.

2. The numerical methods of the equivalent frequency-independent time-domain damping models are realized. Based on the two equivalent frequency-independent time-domain damping models, the corresponding free vibration responses are convergent. And the time-domain numerical results have high calculation accuracy for harmonic wave and earthquake wave.

3. The proposed two methods can be applied to different conditions. Compared with equivalent frequency-independent time-domain damping model in real domain, the computational accuracy of equivalent frequency-independent time-domain damping model in complex domain is higher. However, Fourier transform of external excitation is involved in the calculation process. The computational efficiency of equivalent frequency-independent time-domain damping model in complex domain is lower.

By aid of the mode superposition method, the numerical method for multi degree of freedom (MDOF) systems can be decomposed into the numerical method for SDOF systems. The proposed method can be easily extended to MDOF systems.

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