ON $\gamma$-CLEAN RINGS

Shaimaa S. Esa * & Hewa S. Faris

Dept. of Mathematics, College of Basic Education, University of Duhok, Kurdistan Region – Iraq. (shaimaa.essa@uod.ac)

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Abstract:
In this paper we introduce the concept of $\gamma$-clean ring and we discuss some relations between $\gamma$- clean ring and other rings with explaining by some examples. Also, we give some basic properties of it.

1. INTRODUCTION

The concept of Von Neumann regular rings was first introduced by Von Neumann (1936), called R (briefly) regular (resp. strongly regular) if for every $r \in R$ there exists $b \in R$ such that $r = rb = br$ (resp. $r = r^2 b$). Mohammad and Salih (2006) called R is $\gamma$-regular (resp. strongly $\gamma$-regular) ring if $a = ax^n a$ (resp. $a = a^2 x^n$) for some $x \in R$ and $n \neq 1$ positive integer. Clearly every $\gamma$-regular rings is regular rings. Esa (2010), investigated some concept of $\gamma$-regular called an element $a$ has a $\gamma$-reflexive inverse if there exists $0 \neq x \in R$ and $n \neq 1$ positive integer such that $x^n = x^n ax^n$, and she proved that every $\gamma$-regular element has a $\gamma$-reflexive inverse. An element $x$ of a ring $R$ is called clean if $x = u + e$, where $u$ is a unit in $R$ and $e$ is an idempotent element in $R$. Esa (2015), had extended the definition of clean rings but not $\gamma$-regular rings which are called as clean rings. In this paper we view some basic definitions and relations of $\gamma$-regular rings.

Theorem 2.1. A commutative regular ring is a clean ring (Anderson and Camillo, 2002).

Lemma 2.2. Let $R$ be an abelian ring. Let $a \in R$ be a clean element in $R$ and let $e \in Id(R)$ (Asharafi and Nasibi, 2013). Then:

(i) The element $ae$ is clean.

(ii) If $-a$ is clean, then $a + e$ is also clean.

Definition 2.3. A ring $R$ is said to be a quasi-commutative if for every $a, b \in R$, when $1 \neq a$, there exists positive integer $m$ such that $ab = b^m a$. (Mohammad and Salih, 2006)

Remark 2.4. Let $R$ be a ring, then:

(i) For every $a, b \in R$ when $1 \neq a$ there exists $m > 1$ positive integer such that $ab = b^m a$. For $a = 1$ the above condition does not satisfy.

(ii) For $1 \cdot b = b^m \cdot 1$ then $b = b^m$ and this is a trivial case where $m = 1$.

Lemma 2.5. If $R$ be a reduced ring. Then $R$ is an abelian ring (Esa, 2015).

3. $\gamma$-CLEAN RINGS

In this section we introduce the definition of $\gamma$-clean ring and we discuss some relations of $\gamma$-clean rings with other rings such as $\gamma$-regular rings, r-clean rings, and $\gamma$-Von Neumann regular rings. Also, we illustrate these relations by examples.

Definition 3.1. An element $x$ in a ring $R$ is called $\gamma$-clean if $x$ can be written as $x = a + e$, where $a$ is a $\gamma$-regular element and $e$ is an idempotent.

Some Examples:

1. $M_2(\mathbb{Z}_2)$ is $\gamma$-clean ring, because, for all $X \in M_2(\mathbb{Z}_2)$, we can write $X$ as $A + 1$, where the entries of $A \times \mathbb{Z}_2$ are $\gamma$-regular element and $1 \in Id(M_2(\mathbb{Z}_2))$.

2. Boolean rings are $\gamma$-clean rings.

3. Every field is a $\gamma$-clean ring as well as $\mathbb{Z}_p$ where $p$ is a prime.

4. It is clear that every $\gamma$-regular rings which are clean rings are $\gamma$-clean rings.

Remark 3.2. Clearly every $\gamma$-regular ring is $\gamma$-clean ring, but the converse is not true, for example the ring $M_2(\mathbb{Z}_2)$ is $\gamma$-clean ring but not $\gamma$-regular ring because at least $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$, where $\gamma = 2$ is not $\gamma$-regular element.

2. PRELIMINARIES

In this section we view some basic definitions and relations on $\gamma$-regular rings.

* Corresponding author

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Theorem 3.3. Let R be an abelian ring. If R is γ-clean, then R is γ-regular ring.

Proof: Let R be a γ-clean ring, then for every \( x \in R \), we can write \( x = a + e \), where a is a γ-regular element and e is an idempotent. Assume \( e = ab^n = ba^n \), where \( n \geq 1 \), \( b \in R \) an idempotent, then \( e = ab^n = ba^n \). Since \( a \) has a γ-reflexive inverse then \( ab^n b = ba^n b \). Assume \( y = b^2 \), so \( y = ay \gamma \), hence a is γ-regular. Then if \( a \neq 0 \) then \( x = a + e = ab^n + ba^n + a = a + e \). Since \( a = ae \) then \( a = ae = 0 \), so \( (1 - e) = 0 \). But if a = 0 then \( 1 - e = 1 \), hence \( a = a + (1 - e) = a + 1 \). By Theorem 4.7, \( 1 - x \) is also γ-clean then \( x = -a = -(ab^n a) = (a - b)^n \). Therefore \( x \) is γ-regular, so R is γ-regular.

Remark 3.4. Every γ-clean ring is a r-clean ring, since every γ-regular ring is regular ring; But the converse is not true, for example \( Z_4 \) is r-clean (Ashrafi and Nasibi, 2013), but not γ-clean since \( Z \in Z_4 \) is not γ-regular so \( Z_4 \) is not γ-clean.

Theorem 3.5. Let R be a quasi-commutative ring as defined in Remark 2.4.(i). If R is r-clean, then R is γ-clean.

Proof: Let \( R \) be r-clean ring, then for every \( x \in R \), we can write \( x = r + e \), where \( r \) is a regular element and e is an idempotent. Now, \( x = r + e = rbr + e \), for some \( b \in R \). Since \( R \) is a quasi-commutative ring, then for every pair \( a, b \in R \), \( ab^n = ba^n \), where \( n \neq 1 \) a positive integer. Then \( x = r^2a + e \) so \( x = br + r + e \). Thus \( x \) is γ-clean, hence R is γ-clean.

Remark 3.6. Every γVNL-ring is γ-clean ring, but the converse is not true, for example we can use the trivial and nontrivial idempotent in γ-clean ring, such as \( M_2(Z_2) \), but in the definition of γVNL-ring we use only the trivial idempotent to obtain the definition of γVNL-ring (Esa, 2015).

Theorem 3.7. Let R be a ring such that 0 and 1 are the only idempotents in R. Then R is γ-clean ring if and only if it is γVNL-ring.

Proof: Let R be a γ-clean ring and assume that 0 and 1 are the only idempotents in R. Then for any \( a \in R \), we have \( x = a + e \), where a is γ-regular element and e ∈ I(R). Now, if \( x = a = 0 + 1a \), then x is γ-regular. And if \( x = a + 1 \), then \( 1 - x = -a \), since (a) is γ-regular. So, \( 1 - x \) is γ-regular. Hence R is γVNL-ring.

Conversely, let R be a γVNL-ring. Then, for any \( a \in R \), either \( a + 1 = 0 \) or \( a - 1 = 0 \). Now, if a is γ-regular, by Remark 3.2., then a is γ-clean. And if \( a - 1 = 0 \) is γ-regular, also by Remark 3.2., then \( a - 1 = 0 \) is γ-clean. Hence, a is γ-clean.

Now, we discuss the relation between γ-clean rings and clean rings by giving some conditions on γ-clean rings to be clean rings.

Theorem 3.8. Let R be a quasi-commutative ring under the Remark 2.4.(i). If R is directly finite, then R is clean ring γ-clean ring.

Proof: Let R γ-clean, then every \( x \in R \), \( x = a + e \), where \( a \) is γ-regular element and e ∈ I(R). Now, if \( a = 0 \) then \( x = 0 + 1 = 0 + (2e - 1) + (1 - e) \). Since \( (2e - 1) \in U(R) \) and \( (1 - e) \in I(R) \), hence x is clean. And if \( a \neq 0 \), there exists b \( \in R \) and \( n \) \( \in \mathbb{N} \) a positive integer such that \( a = ab^n \), but R is quasi-commutative then \( ab^n b = ba^n b \). Assume \( e = ab^n b \). By hypothesis either \( ab^n = 0 \) or \( ab = 1 \). So, if \( ab = 0 \), then \( a = aba = 0 \), which is contradiction.

Therefore, \( ab = 1 \), so R is directly finite. Thus, \( ab = ba = 1 \) then \( a \in U(R) \). So, x is clean and hence R is clean.

Theorem 3.9. Let R be a commutative γ-clean ring and each pair of idempotent in R is orthogonal. Then R is clean.

Proof: If R is a commutative ring then every γ-regular ring is regular and so by Theorem 2.1. R is clean ring. Now, for any \( x \in R \) we can write x = \( e_1 + e_2 + u \) where \( e_1, e_2 \in I(R) \) and \( u \in U(R) \). Since \( e_1 + e_2 \) are orthogonal then \( e_1 + e_2 = x \in I(R) \). Hence, \( x = e + u \) is clean, so R is clean.

Theorem 3.10. Let R be an abelian ring. If R is γ-clean, then R is clean.

Proof: Let R be a γ-clean ring, then for every \( x \in R \), we have \( x = a + e \), where a is γ-regular and e ∈ I(R). So, \( x = ab^n a = c \), for some \( b \in R \) and \( n \neq 1 \), positive integer. Assume \( e = ab^n b \), then \( (1 - e) = (b^n + (1 - e)) = 1 \). Since R is abelian so for every \( x \in R \), \( x = e(a) \) and \( eb^n b = b^n b \). Then \( (a + (1 - e))(b^n e + (1 - e)) = ab^n b + ae(1 - e) + (1 - e)eb^n b = e^2 + ae - ae^2 + b^n e - eb^n e + (1 - e)^2 \). Hence, \( x = e = (1 - e) \). Usually, \( (a + (1 - e)) \) is a unit, moreover \( u = e(a) \) is an idempotent. Then \( u + f \) is a unit, also \((-a + f) \) is a unit. Since f is an idempotent, so \( -a = f + (-ue + f) \), since a is regular.

Then by Lemma 2.2. (ii), we get x is clean.

Corollary 3.11. Let R be a reduced ring. If R is γ-clean, then R is clean.

Proof: By Lemma 2.5., the proof is complete.

Theorem 3.12. Let R be a ring without zero divisor. If R is γ-clean, then R is clean.

Proof: Let R be a γ-clean ring, then for every \( x \in R \), we can write x as \( x = a + e \), where a is γ-regular and e ∈ I(R). If a = 0, then \( x = 0 + e = e(2e - 1) + (1 - e) \). So, \( (2e - 1) \in U(R) \) and \( (1 - e) \in I(R) \), then x is clean. Now, if a ≠ 0, then \( x = ab^n a + e \). Since R has no zero divisor, so inverse must exist. So, \( ab^n = ab^n a + 1 = a \) \( U(R) \). Hence x is clean, so R is clean.

4. OTHER RESULTS ON γ-CLEAN RINGS

In this section, some other results of γ-clean rings will be given.

Theorem 4.1. Let R be a γ-clean ring and I an ideal of a ring R. Then \( R/I \) is a γ-clean ring.

Proof: Let R be a γ-clean ring and I an ideal of a ring R. Then \( R/I \) is a γ-clean ring. So, \( (a + I) = ab^n a + I \). Therefore, \( (ab^n a) = a \). So a is γ-regular and \( x = x \in I(R) \). It follows that \( R/I \) is γ-clean.

Remark 4.2. In general, the converse of Theorem 4.1 is not true, for example; if \( p \) is a prime number then \( Z/pZ \) and its clearly that \( Z/pZ \cong Z \) is a field, so \( Z/pZ \) is γ-clean, but \( Z \) is not γ-clean.

Now, we discuss the center of γ-clean ring by using the center γ-ring as follows:

Lemma 4.3.[1, Theorem 2.4]. The center of γ-ring is also γ-regular.

Theorem 4.4. Let R be a γ-clean ring with only idempotents 0 and 1. Then the center of R is also γ-clean.

Proof: Let R be a γ-clean ring and let \( x \in Z(R) \), then \( x = a + e \), where a is γ-regular element and e ∈ I(R). By hypothesis, either \( x = a \) or \( x = a + e \). Now, if \( x = a \) then \( x = ab^n a \), for some \( b \in R \) and \( n \neq 1 \) positive integer. By Lemma 4.3., a is γ-regular in Z(R), hence \( x = a + 0 \) is γ-clean in Z(R).

But if \( x = a + 1 \), then \( x = x - 1 \). Since a is γ-regular then \( (x - 1) = (x - 1)ab^n (x - 1) \) for some \( b \in R \) and \( n \neq 1 \) positive integer. By Lemma 4.3., \( (x - 1) \) is γ-regular in Z(R).
and so $(x - 1)$ is $γ$-clean in $Z(R)$. Therefore $Z(R)$ is $γ$-clean. 

**Theorem 4.5.** If $e$ is an idempotent element of $R$ and $eRe$ and $(1 - e)R(1 - e)$ are both $γ$-clean, then $R$ is also $γ$-clean.

**Proof:** Let $R$ be a ring and let $e \in lId(R)$. By using Pierce decomposition for $R$:

$$R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e).$$

Since $e$ is central in $R$, so

$$R = eRe \oplus eR(1 - e)R^1,$$

For each $X \in R$, give $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x \in eRe$ and $y \in (1 - e)R(1 - e)$. By hypothesis $x, y$ are $γ$-clean element. Thus $x = a_1 + e_1$ and $y = a_2 + e_2$, where $a_1, a_2$ are $γ$-regular element and $e_1, e_2 \in lId(R)$. So $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = (a_1 + e_1 \ 0 \ a_2 + e_2) = (a_1 \ 0 \ a_2) + (e_1 \ 0 \ e_2)$

But there exists $b_1, b_2 \in R$ and $n \neq 1$ positive integer such that $a_1 = a_1b_1a_1, a_2 = a_2b_2a_2$.

Therefore,

$$e_1 (a_1 \ 0 \ a_2) + e_2 (0 \ a_2) = (a_1 \ 0 \ a_2) + e_1 (0 \ a_2) + e_2 (0 \ a_2)$$

Hence $X = (a_1 \ 0 \ a_2)$ is $γ$-regular element and $(e_1 \ 0 \ e_2) \in lId(R)$. Hence $R$ is $γ$-clean ring.

**Theorem 4.6.** Let $e_1, e_2, \ldots, e_n$ be orthogonal central idempotent with $e_1 + e_2 + \cdots + e_n = 1$. Then $e_iRe_1$ is $γ$-clean for each $i$ if and only if $R$ is $γ$-clean.

**Proof:** By using Theorem 4.5, and by induction. Then $e_1Re_1$ is a $γ$-clean ring.

On the other hand, Let $R$ be a $γ$-clean ring and $e_1, e_2, \ldots, e_n$ be orthogonal central idempotent with $e_1 + e_2 + \cdots + e_n = 1$. Since $R = e_1Re_1 \oplus \ldots \oplus e_nRe_n$, then by Theorem 4.3, $e_1Re_1$ is $γ$-clean for each $i$.

**Theorem 4.7.** Let $R$ be a ring. Then $R$ is $γ$-clean if and only if for every $x \in R$ can be written as $x = a - e$, where $a$ is $γ$-regular element and $e \in lId(R)$.

**Proof:** Let $R$ be a $γ$-clean ring, then for every $x \in R$, then $x = a + e$, where $a$ is $γ$-regular element and $e \in lId(R)$. So $-x \in R$, we can write $-x = a + e$. Hence $x = (-a) - e$, where $-a$ is $γ$-regular and $e \in lId(R)$.

Conversely, suppose that for every $x \in R$ we can write $x = a - e$ where $a$ is $γ$-regular and $e \in lId(R)$. So, for $-x \in R$, we can write $-x = a + e$. Hence $x = (-a) + e$, where $-a$ is $γ$-regular and $e \in lId(R)$.

**Theorem 4.8.** Let $R$ be a ring. Then $x \in R$ is $γ$-clean element if and only if $1 - x$ is $γ$-clean.

**Proof:** Let $x \in R$ be a $γ$-clean then $x = e + a$ where $a$ is $γ$-regular element and $e \in lId(R)$.

Thus $1 - x = 1 - (e + a) = 1 - e - a = 1 - (1 - e) + (-a)$

But there exists $b \in R$ and $n \neq 1$ positive integer such that $a = ab^n a$

Hence $(-a)(-b)^n(-a) = -(ab^n a) = -a$

So, $-a$ is $γ$-regular and $(1 - e) \in lId(R)$.

Therefore, $1 - x$ is $γ$-clean.

**Corollary 4.9.** Let $R$ be a ring and $x \in J(R)$. Then $x$ is $γ$-clean.

**Proof:** Let $x \in J(R)$ then $1 - x \in U(R)$ so $1 - x$ is regular. Hence $1 - x$ is $γ$-clean.

Therefore, by Theorem 4.8, $x$ is $γ$-clean.

**5. CONCLUSION**

After all these relations and properties, we conclude that:

i. Every $γ$-regular ring is $γ$-clean and the converse is not true, see Theorem 3.3.

ii. Every $γ$-clean ring is $r$-clean and the converse is not true, see Theorem 3.5.

iii. Every $γVNL$-ring is $γ$-clean and the converse is not true, see Theorem 3.7.

iv. There is no relation between $γ$-clean ring and clean, we can get this diagram:

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