Isometric embeddings of a class of separable metric spaces into Banach spaces

S.K. Mercourakis and G. Vassiliadis

Abstract

Let \((M, d)\) be a bounded countable metric space and \(c > 0\) a constant, such that
\[d(x, y) + d(y, z) - d(x, z) \geq c,\]
for any pairwise distinct points \(x, y, z\) of \(M\). For such metric spaces we prove that they can be isometrically embedded into any Banach space containing an isomorphic copy of \(\ell_\infty\).

Introduction

Let \((M, d)\) be a metric space; following [4] we will call it concave, when the triangle inequality is strict, i.e. when \(d(x, y) + d(y, z) > d(x, z)\) for any pairwise distinct points \(x, y, z\) of \(M\).

In this note we are interested in (concave) metric spaces satisfying the stronger property: there is a constant \(c > 0\), such that \(d(x, y) + d(y, z) - d(x, z) \geq c\), for any pairwise distinct points \(x, y, z\). Let us call these spaces strongly concave metric spaces.

The main result we prove is an infinite dimensional version of Theorem 4.3 of [4], that is, if a Banach space \(X\) contains an isomorphic copy of \(\ell_\infty\), then \(X\) contains isometrically any bounded countable strongly concave metric space (Th.2). An immediate consequence of this result is that any Banach space containing an isomorphic copy of \(c_0\), admits an infinite equilateral set (Th.3). This result was first proved (by similar methods) in [5] (Th.2).

A subset \(S\) of a metric space \((M, d)\) is said to be equilateral, if there is a \(\lambda > 0\) such that for \(x \neq y \in S\) we have \(d(x, y) = \lambda\); we also call \(S\) a \(\lambda\)-equilateral set (see [8]).

If \(X\) is any (real) Banach space, then \(B_X\) and \(S_X\) denote its closed unit ball and unit sphere respectively. \(X\) is said to be strictly convex, if for any \(x \neq y \in S_X\) we have \(\|x + y\| < 2\). The Banach-Mazur distance between two isomorphic Banach spaces \(X\) and \(Y\) is \(d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T\text{ is an isomorphism}\}\).

Strongly concave metric spaces

We start by presenting some examples of concave metric spaces

\[\text{2010 Mathematics Subject Classification:}\quad \text{Primary 46B20, 46E15; Secondary 46B26, 54D30.}\]

\text{Key words and phrases: concave metric space, isometric embedding, separated set.}
Examples 1
(1) a) Let \((M, d)\) be a discrete metric space (i.e. \(d(x, y) = 1\) when \(x \neq y\)). Clearly 
\[1 = d(x, z) < d(x, y) + d(y, z) = 2\text{ for any pairwise distinct triplet } x, y, z \in M.\] 
Therefore \((M, d)\) is a concave metric space. In particular, every \(\lambda\)-equilateral subset of any metric space is a concave metric space.
b) More generally, every \textit{ultrametric} space is concave. This holds since for any \(x, y, z\) pairwise distinct points we have 
\[d(x, z) \leq \max\{d(x, y), d(y, z)\} < d(x, y) + d(y, z).\]
(2) Let \((X, \|\cdot\|)\) be a strictly convex Banach space. As is well known, if \(x, y, z\) are non collinear points of \(X\) then 
\[\|x - z\| < \|x - y\| + \|y - z\|.
It then follows that the unit sphere \(S_X\) and every affinely independent subset \(A\) of \(X\) 
with the norm metric are concave metric spaces (in any case no three pairwise distinct 
points are collinear).
(3) Let \((X, \|\cdot\|)\) be a Banach space and \(A \subseteq B_X\) such that \(x \neq y \in A \Rightarrow \|x - y\| > 1\) (see \[2\]). Then for any \(x, y, z\) pairwise distinct points of \(A\) we have 
\[\|x - y\| + \|y - z\| - \|x - z\| > 1 + 1 - \|x - z\| \geq 1 + 1 - 2 = 0.\] Hence \(A\) with the norm metric is concave.
(4) Let \((M, d)\) be any metric space and \(p \in (0, 1)\). Then it is rather easy to show that 
\(d^p\) is a concave metric on \(M\). This follows from the fact that, given \(a, b, c > 0\) with 
\(a \leq b + c\) then \(a^p < b^p + c^p\). The metric \(d^p\) is then called the snowflaked version of \(d\) 
(see \[6]\).
We are interested in concave metric spaces \((M, d)\) satisfying the stronger property: 
there is a constant \(c > 0\) such that for any pairwise distinct points \(x, y, z\) of \(M\) we have 
\[d(x, y) + d(y, z) - d(x, z) \geq c,\] equivalently \(d(x, z) + c \leq d(x, y) + d(y, z)\). Let us call 
these spaces \textit{strongly concave} spaces.

Lemma 1. Every strongly concave metric space is separated (or uniformly discrete).

\textit{Proof.} Assume that \((M, d)\) is a \(c\)-strongly concave metric space. We claim that \(x \neq y \in M \Rightarrow d(x, y) \geq \frac{c}{2}\). Assume for the purpose of contradiction that there is a pair 
\(\{x, y\} \subseteq M\) with \(d(x, y) < \frac{c}{2}\). Let also \(z \in M \setminus \{x, y\}\). We then have 
\[d(x, y) + d(y, z) - d(x, z) = 2d(x, y) + d(x, z) \Rightarrow d(x, y) + d(y, z) - d(x, z) \leq 2d(x, y) < 2\frac{c}{2} = c.\] 
The last inequality clearly contradicts the fact that \(M\) is \(c\)-strongly concave.

The following are examples of strongly concave metric spaces.

Examples 2
(1) Every finite concave metric space is clearly strongly concave.
(2) Let \(A\) be a \(\lambda\)-equilateral subset of any metric space \((M, d)\). For any pairwise distinct points \(x, y, z\) of \(A\) we have 
\[d(x, y) + d(y, z) - d(x, z) = \lambda + \lambda - \lambda = \lambda,\] so \(A\) is a \(\lambda\)-strongly concave metric subspace of \((M, d)\).
(3) Let \((X, \|\cdot\|)\) be a Banach space. Also let \(A \subseteq B_X\) with the property that \(x \neq y \in A \Rightarrow \|x - y\| \geq 1 + \varepsilon\), where \(\varepsilon > 0\) is a constant. Then we have 
\[\|x - y\| + \|y - z\| - \|x - z\| > (1 + \varepsilon) + (1 + \varepsilon) - 2 = 2\varepsilon\] (cf. Examples 1(3)). Therefore \(A\) with the norm metric is a 
\(2\varepsilon\)-strongly concave metric space.

Note that if \(\dim X = \infty\), then by a result of Elton and Odell (\[2\]) there is \(A \subseteq S_X\) 
infinite and \(\varepsilon > 0\) such that \(x \neq y \in A \Rightarrow \|x - y\| \geq 1 + \varepsilon.\)
Remarks 1

(1) Clearly every separable strongly concave metric space $M$ is at most countable (this is so because $M$ is separated, hence it has the discrete topology).

(2) Every subspace of a concave (resp. strongly concave) space has the same property.

The following result is classical (see [6]).

Theorem 1. (Frechét) Every separable metric space $(M, d)$ embeds isometrically into $\ell_\infty$.

Proof. Let $(x_n) \subseteq M$ be a dense sequence in $M$. Then the map
$$\varphi : x \in M \mapsto (d(x, x_n) - d(x_1, x_n))_{n \geq 1} \in \ell_\infty$$
satisfies our claim.

Remark 2 Let $(M, d)$ be a separable metric space. We define a map
$$\sigma : M \to \mathbb{R}^N \text{ with } \sigma(x) = (d(x, x_n))_{n \geq 1}$$
where $(x_n)$ is any dense sequence in $M$. Then the Frechét embedding of $M$ into $\ell_\infty$ is the map
$$\varphi(x) = \sigma(x) - \sigma(x_1), \ x \in X$$
Note that if the space $(M, d)$ is bounded (that is, there is $K > 0$ such that $d(x, y) \leq K$ for all $x, y \in M$), then the map $\sigma$ is already an isometric embedding of $M$ into $\ell_\infty$, which we will still call the Frechét embedding of $M$ into $\ell_\infty$.

Proposition 1. Let $(M, d)$ be a bounded countable infinite metric space. Then there is an infinite subset $N$ of $M$ such that the Frechét embedding of $N$ into $\ell_\infty$ takes values into the space $c$.

Proof. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a one-to-one enumeration of $M$. Then $\sigma(x_k) = (d(x_k, x_n))_{n \geq 1} \in \ell_\infty$, for $k \in \mathbb{N}$, since $d$ is a bounded metric. We construct by induction a subsequence $\{x'_{n_1}, x'_{n_2}, \ldots, x'_{n_k}, \ldots\}$ of $(x_n)$ satisfying our claim.

Since $(d(x_1, x_n))_{n \geq 1}$ is a bounded sequence of real numbers, there is $A_1 \subseteq \mathbb{N}$ infinite, such that $d(x_1, x_n) \xrightarrow{n \in A_1} \alpha_1$. Set $n_1 = 1$.

Let $n_2 = \min A_1$, for which we may assume that $n_2 > n_1$. Then for the sequence $(d(x_{n_2}, x_n))_{n \in A_1}$, there is $A_2 \subseteq A_1$ infinite with $n_3 = \min A_2 > n_2$ such that $d(x_{n_2}, x_n) \xrightarrow{n \in A_2} \alpha_2$.

Then for the sequence $(d(x_{n_3}, x_n))_{n \in A_2}$, there is $A_3 \subseteq A_2$ infinite with $n_4 = \min A_3 > n_3$ such that $d(x_{n_3}, x_n) \xrightarrow{n \in A_3} \alpha_3$.

The inductive process should be clear. Now set $A = \{n_1 < n_2 < \cdots < n_k < \ldots\}$. Clearly $\{n_k, n_{k+1}, \ldots\} \subseteq A_k$ for $k \geq 1$ and hence $d(x_{n_k}, x_n) \xrightarrow{n \in A} \alpha_k$ for all $k \geq 1$. It is clear that the set $N = \{x'_{n_k} = x_{n_k} : k \geq 1\}$ satisfies our requirements. 

3
The following theorem is the main result of this note; its proof resembles the proof of Theorem 4.3 of [4] and the proof of Theorem 2 of [5] (we use Schauder’s fixed point theorem the same way we did in [5]). The origins of these ideas can be traced in Brass (see [1] and [8]) and Swanepoel and Villa (see [9] and [10]).

**Theorem 2.** Let $X$ be any Banach space containing an isomorphic copy of $\ell_\infty$. Then $X$ contains isometrically any bounded separable strongly concave metric space.

**Proof.** We shall use a kind of non distortion property of $\ell_\infty$ proved independently by Talagrand ([11]) and Partington ([7]). Let us denote by $\| \cdot \|$ the usual norm of $\ell_\infty$.

**Claim.** Let $(M, d)$ be any bounded separable strongly concave metric space. There is $\delta > 0$, such that if $\| \cdot \|$ is any equivalent norm on $\ell_\infty$ with Banach Mazur distance

$$d((\ell_\infty, \| \cdot \|), (\ell_\infty, \| \cdot \|)) \leq 1 + \delta$$

then the space $(M, d)$ embeds isometrically into $(\ell_\infty, \| \cdot \|)$.

**Proof of the Claim:** Since $(M, d)$ is strongly concave, there is $\eta > 0$ such that $d(x, y) + d(y, z) - d(x, z) \geq \eta$, for each triplet $x, y, z$ of pairwise distinct points of $M$. We may assume that $\|x\| \leq \|x\|_\infty \leq (1 + \delta)\|x\|$ for $x \in \ell_\infty$, where $\delta > 0$ is to be determined.

Let $I = \{(m, n) : n < m, \ n, m \in \mathbb{N}\}$; denote by $K$ the compact cube $[0, \eta]^I$. Since $M$ is (strongly concave and) separable, it is at most countable, so let $M = \{x_1, x_2, \ldots, x_n, \ldots\}$. For $\varepsilon = (\varepsilon_{(m, n)}) \in K$ set

$$p_1(\varepsilon) = (d(x_1, x_1) - d(x_1, x_1), d(x_1, x_1) - d(x_1, x_2), \ldots, d(x_1, x_n) - d(x_1, x_n), \ldots) = (0, 0, \ldots)$$

$$p_2(\varepsilon) = (d(x_2, x_1) - d(x_1, x_1) + \varepsilon_{(2, 1)}, d(x_2, x_2) - d(x_1, x_2), \ldots, d(x_2, x_n) - d(x_1, x_n), \ldots)$$

$$\vdots$$

$$p_n(\varepsilon) = (d(x_n, x_1) - d(x_1, x_1) + \varepsilon_{(n, 1)}, \ldots, d(x_n, x_{n-1}) - d(x_1, x_{n-1}) + \varepsilon_{(n, n-1)}, d(x_n, x_n) - d(x_1, x_n), \ldots)$$

$$\vdots$$

(Note that $x_n \mapsto p_n(0)$ is the Frechét embedding of $M$ into $(\ell_\infty, \| \cdot \|)$.)

For $n < m$ we have

$$\|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty = \sup_k |d(x_n, x_k) + \varepsilon_{(n, k)} - (d(x_m, x_k) + \varepsilon_{(m, k)})|$$

where we set $\varepsilon_{(k, l)} = 0$, for $l \geq k$. This supremum is equal to $d(x_n, x_m) + \varepsilon_{(n, m)}$, as for $k \neq n, m$ we have

$$d(x_n, x_k) - d(x_m, x_k) + \varepsilon_{(n, k)} - \varepsilon_{(m, k)} \leq d(x_n, x_m) - \eta + \varepsilon_{(n, k)} - \varepsilon_{(m, k)} \leq d(x_n, x_m).$$

We define a function

$$\varepsilon = (\varepsilon_{(m, n)}) \in K \xrightarrow{\phi} \phi(\varepsilon) = (\varphi_{(m, n)}(\varepsilon)) \in K,$$
by the rule \( \varphi_{(m,n)}(\varepsilon) = d(x_n, x_m) + \varepsilon_{(m,n)} - \|p_n(\varepsilon) - p_m(\varepsilon)\| \). Note that \( \varphi_{(m,n)}(\varepsilon) \geq d(x_n, x_m) + \varepsilon_{(m,n)} - \|p_n(\varepsilon) - p_m(\varepsilon)\|_{\infty} = 0 \) (using the computation above and the fact that the norm \( \| \cdot \|_{\infty} \) dominates \( \| \cdot \| \)). We also have
\[
\frac{1}{1+\delta} (d(x_n, x_m) + \varepsilon_{(m,n)}) \leq \|p_n(\varepsilon) - p_m(\varepsilon)\|.
\]
Therefore
\[
\varphi_{(m,n)}(\varepsilon) = d(x_n, x_m) + \varepsilon_{(m,n)} - \|p_n(\varepsilon) - p_m(\varepsilon)\| \\
\leq d(x_n, x_m) + \varepsilon_{(m,n)} - \frac{1}{1+\delta} (d(x_n, x_m) + \varepsilon_{(m,n)}) \\
= \frac{\delta}{1+\delta} (d(x_n, x_m) + \varepsilon_{(m,n)}).
\]
It then follows from (this inequality and) the fact that \( M \) is bounded that if \( \delta \) is quite small, then \( \varphi_{(m,n)}(\varepsilon) \leq \eta \), for \( \varepsilon \in K \).

Since each coordinate function \( \varphi_{(m,n)} \) is continuous (as dependent on finite coordinates, i.e. from the set \( \{(k, l) : 1 \leq l < k \leq m\} \)) it follows that \( \varphi \) is also continuous. By a classical result of Schauder, \( \varphi \) has a fixed point \( \varepsilon' = (\varepsilon'_{(m,n)}) \in K \), that is \( \varphi(\varepsilon') = \varepsilon' \), which implies \( \|p_n(\varepsilon') - p_m(\varepsilon')\| = d(x_n, x_m) \), for all \( n, m \in \mathbb{N} \). The proof of the Claim is complete.

Denote by \( \| \cdot \| \) the norm of \( X \) and let \( Y \) be a subspace of \( X \) isomorphic to \( \ell_{\infty} \). By the non distortion property of \( (\ell_{\infty}, \|\cdot\|_{\infty}) \) there is a subspace \( Z \subseteq Y \) (isomorphic to \( \ell_{\infty} \)) such that
\[
d((Z, \|\cdot\|), (\ell_{\infty}, \|\cdot\|_{\infty})) \leq 1 + \delta
\]
(this is the \( \delta > 0 \) postulated in the Claim). It follows immediately from the Claim that the space \((Z, \|\cdot\|)\) contains an isometric copy of \((M, d)\).

In the special case when \((M, d)\) is the countable infinite discrete metric space we get the following result first proved in [5] (Th.2), essentially with the same method

**Theorem 3.** Every Banach space \( X \) containing an isomorphic copy of \( c_0 \) admits an infinite equilateral set.

**Proof.** Take in the proof of the previous theorem \((M, d)\) to be the countable infinite discrete space. Then \( \eta = 1 \) and the resulting family \((p_n(\varepsilon))_{n \geq 1}, \varepsilon \in K = [0,1]^I \) takes values in \( c_0 \) (remember that \( x_n \mapsto p_n(0) \) is the Frechet embedding of \((M, d)\) into \( c_0 \)). Since \((c_0, \|\cdot\|_{\infty})\) is non distortable, we get the conclusion.

Theorem 2 can be improved in the following way

**Theorem 4.** Let \((M, d)\) be an infinite bounded separable strongly concave metric space. Then there is \( N \subseteq M \) infinite such that the metric space \((N, d)\) can be isometrically embedded into any Banach space containing an isomorphic copy of the space \( c_0 \).
Proof. By Proposition 1, there is $N \subseteq M$ infinite such that the Frechét embedding $\sigma : N \to \ell_\infty$ takes values into $c$. Then the proof of Theorem 2 gives us a family of embeddings $(p_n(\varepsilon))_{n \geq 1}$, $\varepsilon \in K = [0, \eta]^I$ taking values into $c$. Since $c$ is isomorphic to $c_0$, we are done. 

References

[1] P. Brass, On equilateral simplices in normed spaces, Beiträge Algebra Geom. 40 (1999), 303–307.

[2] J. Elton and E. Odell, The unit ball of every infinite-dimensional normed linear space contains a $(1 + \varepsilon)$-separated sequence, Colloq. Math. 44 (1981), no.1,105–109.

[3] E. Glakousakis and S. K. Mercourakis, On the existence of 1-separated sequences on the unit ball of a finite-dimensional Banach space, Mathematika, 61 (2015), 547–558.

[4] J. Kilbane, On embeddings of finite subsets of $\ell_2$, arXiv:1609.08971v2 [math.FA] (2016), 12 pages.

[5] S.K. Mercourakis and G. Vassiliadis, Equilateral sets in infinite dimensional Banach spaces, Proc. Amer. Math. Soc. 142 (2014), 205–212.

[6] M.J. Ostrovskii, Metric embeddings, Bilipschitz and coarse embeddings into Banach spaces, De Gruyter Studies in Mathematics, 49, De Gruyter, Berlin, 2013.

[7] J. R. Partington, Subspaces of certain Banach sequence spaces, Bull. London Math. Soc. 13 (1981), 162–166.

[8] K.J. Swanepoel, Equilateral sets in finite-dimensional normed spaces, Seminar of Mathematical Analysis, vol. 71, Univ. Sevilla Secr. Publ. (2004), 195–237.

[9] K.J Swanepoel and R. Villa, A lower bound for the equilateral number of normed spaces, Proc. Amer. Math. Soc. 136 (2008) no.1,127–131.

[10] K.J. Swanepoel and R. Villa, Maximal equilateral sets, Discrete and Computational Geometry 50 (2013) no.2, 354–373.

[11] M. Talagrand, Sur les espaces de Banach contenant $\ell_1(\tau)$, Israel J. Math. 40 (1981), 324–330 (French. English summary).

S.K.Mercourakis, G.Vassiliadis
University of Athens
Department of Mathematics
15784 Athens, Greece
e-mail: smercour@math.uoa.gr
gorgevassil@hotmail.com