Equivalence and/or quantum duality of the non/supersymmetric noncommutative field theories related by the $\theta$-exact Seiberg-Witten map

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In this article we expound a discovery of the quantum equivalence/duality of U(N) noncommutative quantum field theories (NC QFT) related by the $\theta$-exact Seiberg-Witten (SW) maps and at all orders in the perturbation theory with respect to the coupling constant. We show that this proof holds for Super Yang-Mills (SYM) theories with $\mathcal{N} = 0, 1, 2, 4$ supersymmetry. In short, Seiberg-Witten map does commute with the quantization of the U(N) NCQFT independently, with or without supersymmetry.

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I. INTRODUCTION

In accord with the very essence of the coupling constant perturbative description of the quantum field theory, our approach to the Seiberg-Witten map issue is to build the SW map by using the expansion in terms of the coupling constant $\theta$ [2,3]. Thus, the $\theta$ dependence of the the coupling constant perturbative definition of the theory is treated in an exact way and, then, the UV/IR mixing effect pops up [2], inducing the noncommutative quadratic IR divergence and signaling an IR instability [5]. This IR instability can be cured in unmapped theory by making the theory supersymmetric, since supersymmetry removes the corresponding quadratic noncommutative IR divergences [3,8]. However, it was shown in [10] that if the noncommutative fields carry a linear realization of supersymmetry their ordinary duals under the Seiberg-Witten map carry a nonlinear realization of supersymmetry. Hence, it is far from trivial that the supersymmetry cancelation mechanism between the one-loop noncommutative quadratic IR divergences coming from bosonic and fermionic degrees of freedom works when the classical noncommutative theory is formulated, first, in terms of the ordinary fields and then quantized. And yet, it has been shown in [11,12] that SUSY cancelation mechanism just mentioned works for all the two-point functions when we have $\mathcal{N} = 1, 2$ and 4 supersymmetry.

The occurrence of the UV/IR mixing phenomenon in both these quantum field theories (without and with SW map) give strong support to the idea that they are dual descriptions of the same underlying quantum field theory, at least in the perturbative regime defined by the coupling constant. However, in the U(1) YM theory, the UV divergent part of the two-point function of the noncommutative gauge field is local, whereas the UV divergent bit of the two-point function of the ordinary theory obtained by using the $\theta$-exact SW map contains unusual $\theta$-dependent nonlocal contributions, at least in the Feynman gauge [13,10]. Their existence cast doubts on the truth of the quantum duality conjecture at hand. Of course, UV divergent contributions to the two-point functions are gauge dependent.

We present here our answer(s), given in [11,12,17], to the following 16 years old but important question:

– Does the Seiberg-Witten map between different NCYMs persist after quantization?

Our answer will also solve the following closed related problems

– Is the UV/IR mixing effect–signaling a vacuum instability–a gauge-fixing independent characteristic of both versions of theories?

– Whether or not the UV divergent nonlocal terms in ordinary theory are really physically relevant?

– Does SUSY help to remove the UV and IR divergences in the above NCQFT in general?

II. MOTIVATIONS

Among other, like plasmon and Z forbidden and invisible decays [4,6], or holography [8] and reheating phase after inflation [9], in the framework of the noncommutativity of spacetime, our first main phenomenological motivation for using/applying the $\theta$-exact SW-map roots in physics of "the ultra-high energy neutrino experiments", where relevant scatterings of extreme energetic neutrinos (originating from cosmic rays, so called cosmogenic neutrinos) on nucleons, as presented in our Fig. [1] were analyzed in both NCQFT models, in $\theta$-expanded and $\theta$-
at $E_\nu = 10^{11}$ GeV ($4 \times 10^{-3}$ mb for the FKRT neutrino flux \cite{24}), in $\theta$-expanded model one can infer that the scale of noncommutativity $\Lambda_{NC}$ to be greater than 455 TeV, a really strong bound \cite{3}. One should however be careful and suspect this result as it has been obtained from the conjecture that the $\theta$-expansion stays well-defined in the kinematical region of interest, and the more reliable limits on $\Lambda_{NC}$ are expected to be placed precisely by examining low-energy processes \cite{22}. Although a heuristic criterion for the validity of the perturbative $\theta$-expansion, $\sqrt{s}/\Lambda_{NC} \lesssim 1$, with $s = 2E_\nu M_N$, would underpin our result on $\Lambda_{NC}$, a more thorough inspection on the kinematics of the process does reveal a more stronger energy dependence $E_\nu^{1/2} s^{1/4}/\Lambda_{NC} \lesssim 1$. In spite of an additional phase-space suppression for small $x$’s in the $\theta^2$-contribution \cite{23} of the cross section relative to the $\theta$-contribution, we find an unacceptably large ratio $\sigma(\theta^2)/\sigma(\theta) \approx 10^4$, at $\Lambda_{NC} = 455$ TeV. Hence, the bound on $\Lambda_{NC}$ obtained this way is incorrect, and our last resort was to modify the model adequately to include the full-$\theta$ resummation leading to the $\theta$-exact Feynman rules \cite{4}, thereby allowing us to compute nonperturbatively in $\theta$. Result, presented in Fig. 2 was unexpectedly successful showing nice convergent behavior of the total cross section as a function of the scale of noncommutativity $\Lambda_{NC}$ at two different extreme neutrino energies.

Finally we give in Fig. 3 as an example of the convergent form of the physically relevant result, a plot of the scale of noncommutativity $\Lambda_{NC}$ versus the plasmon frequency $x_{pl}$ obtained from the computation of the plasmon decay into neutrino pair rate in the neutrino mass extended NCSM \cite{4}. In addition in Fig. 4 there is a convergent plot of the scale $\Lambda_{NC}$ versus BBN decoupling temperature $T_{dec}$, obtained from the assumption that the plasmon decay into sterile neutrino pair rate $\Gamma(\gamma_{pl} \rightarrow \bar{\nu}_R \nu_R)$ is mostly due to the noncommutativity of spacetime, see details in \cite{4}.

III. NONCOMMUTATIVE GAUGE THEORY

A. Definitions/properties of the Moyal $\star$-product

The Moyal space means $[\hat{x}^i, \hat{x}^j] = \theta^{ij}$ is constant, with $\hat{x}^i$ being operators. This can be realized by the Moyal-Weyl star-$(\star)$ product

$$f(x) \star g(x) = \frac{1}{(2\pi)^{2n}} \int d^{2n}k d^{2n}p \tilde{f}(k) \tilde{g}(p) \cdot \exp[i(k_i + p_i)x^i - i\theta^{ij}k_ip_j] \tag{2}$$

$$\rightarrow [x^i, x^j] = x^i \star x^j - x^j \star x^i = i\theta^{ij},$$

which is associative and invariant under the cyclicity of integration.
B. The noncommutative U(1) gauge action

The pure gauge action $S_g$

\[ S_g = -\frac{1}{4} \int F_{\mu\nu} \star F^{\mu\nu}, \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \gamma^\alpha A_\nu], \]

\[ \delta A_\mu = \partial_\mu \lambda + i[\lambda \gamma^\alpha A_\mu], \]

\[ \delta F_{\mu\nu} = i[\lambda \gamma^\alpha F_{\mu\nu}], \]

is interacting, yet perturbatively workable:

\[ S_g = -\frac{1}{4} \int \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 \]

\[ -2i(\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu \gamma^\alpha A^\nu - [A_\mu \gamma^\alpha A_\nu]^2. \]

Introducing Moyal product turns the commutative U(1) theory, which is a free theory, into the NC U(1) gauge theory which is a nonlocally interacting theory. Also, due to the cyclicity, field theories on Moyal space admits relatively simple pertubative quantization. Nonlocal factor regularize part of the Schwinger parameterized loop integral, turning it into integral over modified bessel function $K_\alpha$ which is IR divergent. $K_\alpha$’s contain logarithmic function in its expansion, which could cause unitarity problem when performing Wick rotation. This study is restricted to the Euclidean space only.

C. Supersymmetric U(1) NCQFT

Classical SUSY is easy to realize on Moyal space:

\[ S_{N=1} = S_g + i\bar{\Lambda}_\alpha A^\mu \partial_\mu [A]\Lambda_\alpha + \frac{1}{2} D^{(nc)} D^{(nc)}, \]

since the SUSY transformations are linear:

\[ \delta_\xi A_\mu = i\xi \partial_\mu A_\mu - A_\mu \partial_\mu \xi, \]

\[ \delta_\xi D^{(nc)} = (\xi \partial^\mu D_\mu \bar{\Lambda} - D_\mu \Lambda \partial_\mu \xi), \]

and there exists superfield formalism.

Extended SUSY actions can also be constructed, as given below:

\[ S_{N=2} = S_g + (D_\mu [A]\phi)^\dagger D^{(nc)} [A]\phi \]

\[ - \frac{\epsilon^2}{2} [\phi^\dagger \phi]^2 + i\bar{\Lambda}\tilde{\sigma}^\mu D_\mu [A]\Lambda \]

\[ + i\Psi \tilde{\sigma}^\mu D_\mu [A]\Psi + i\epsilon \sqrt{2}[\Psi^\dagger [\Lambda^\ast : \Phi^\dagger] + i\epsilon \sqrt{2}[\bar{\Psi}^\dagger [\Lambda^\ast : \Phi^\dagger]], \]

\[ S_{N=4} = S_g + i\bar{\Lambda}^i \tilde{\sigma}^{ij} D_\mu [A]\Lambda^j \]

\[ + \frac{1}{2} D_\mu [A]\Phi_\mu \tilde{D}^{(nc)} [A]\Phi_\mu + \left( \frac{\epsilon}{2} [\Phi_\mu : \Phi_\mu] + \frac{\epsilon}{2} [\Phi_\mu : \Phi_\mu] \right)^2 \]

\[ + i\epsilon \sqrt{2}(\tilde{\sigma}^{-1})^{ij} \Lambda^i [\Lambda^j : \Phi^j] - i\epsilon \sqrt{2}(\tilde{\sigma})^{ij} \Lambda^i [\Lambda^j : \Phi^j]. \]

(6)
D. Photon polarization tensor computed in the Feynman gauge

The one-loop quantum corrections in the NC U(1) gauge theory on Moyal space has the structure where the UV, quadratic and logarithmic IR divergences coexist. Photon polarization tensor computed in the pure U(1) with Feynman gauge shows UV and IR divergences:

$$\Pi_{\text{photon}}^{\mu\nu}(p) \sim \frac{g^2}{(4\pi)^2} \left( \frac{10}{3} g^{\mu\nu} p^2 - p^\mu p^\nu \right) \cdot \left( \frac{2}{\epsilon} - \ln \frac{p^2}{\mu^2} + \ln(p^2(\theta p)^2) \right) + 32 \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^4} \cdot \ln(\mu^2(\theta p)^2) \right) \right),$$

(7)

The $\beta$-function obtained from the planar UV divergences in the photon two and three point functions suggests asymptotic freedom. Yet the quadratic IR divergence is an even bigger concern, while logarithmic term is actually the one that really mixes with the UV term.

Question is whether supersymmetry could help? Answer is yes since SUSY controls IR divergences \[11\]. For the $N = 1$ SUSY suppresses the quadratic IR divergence,

$$\Pi_{N=1}^{\mu\nu}(p) = \Pi_{\text{photon}}^{\mu\nu}(p) + \Pi_{\text{photino}}^{\mu\nu}(p) \sim \frac{g^2}{(4\pi)^2} \left( \frac{10}{3} g^{\mu\nu} p^2 - p^\mu p^\nu \right) \cdot \left( \frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right) + 32 \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^4} \cdot \ln(\mu^2(\theta p)^2) \right) \right).$$

(8)

while the $N = 4$ SUSY renders the theory finite. This we shall explicitly at the end of this article.

IV. QUANTUM DUALITY OF QFT’S RELATED BY THE $\theta$-EXACT SW MAP

Classical noncommutative field theories admit an equivalent representation in terms of ordinary fields formulated by employing the Seiberg-Witten (SW) map \[1\]. However we still do not know whether this equivalence holds at the quantum level, i.e., whether the quantum theory defined in terms of the NC fields is the same as the quantum theory defined in terms of commutative fields, and obtained from the NC action by using the $\theta$-exact SW map \[1\]. In this article we prove that the $\theta$-exact Seiberg-Witten map establishes an equivalence relation between perturbative—in the coupling constant—quantum field theories defined with respect to the noncommutative and commutative fields, by showing that the corresponding on-shell DeWitt effective actions \[24, 27\] can be SW-mapped one to each other \[12, 17\]. We also give an explicit check of our verdict in the (supersymmetric) NC U(1) gauge theory.

This result gives further robustness to the quantum duality conjecture between the formulation in terms of ordinary fields and the description in terms of noncommutative fields. However, the nonlocal UV divergent structure still persists after introducing supersymmetry into the game. But, by using two different gauge-fixing terms, it was shown in \[11\] that the nonlocal UV divergent contributions are gauge dependent and, therefore, it could be possible to remove them. This is unlike the noncommutative quadratic IR divergences which do not exist. Photon polarization tensor computed in the pure U(1) with Feynman gauge shows UV and IR divergences:

$$\Pi_{\text{photon}}^{\mu\nu}(p) \sim \frac{g^2}{(4\pi)^2} \left( \frac{10}{3} g^{\mu\nu} p^2 - p^\mu p^\nu \right) \cdot \left( \frac{2}{\epsilon} - \ln \frac{p^2}{\mu^2} + \ln(p^2(\theta p)^2) \right) + 32 \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^4} \cdot \ln(\mu^2(\theta p)^2) \right) \right),$$

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$$\Pi_{N=1}^{\mu\nu}(p) = \Pi_{\text{photon}}^{\mu\nu}(p) + \Pi_{\text{photino}}^{\mu\nu}(p) \sim \frac{g^2}{(4\pi)^2} \left( \frac{10}{3} g^{\mu\nu} p^2 - p^\mu p^\nu \right) \cdot \left( \frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right) + 32 \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^4} \cdot \ln(\mu^2(\theta p)^2) \right) \right).$$

(8)

while the $N = 4$ SUSY renders the theory finite. This we shall explicitly at the end of this article.

A. DeWitt effective action in the path integral formulation

The on-shell DeWitt effective action \[27\] with respect to the noncommutative/hatted fields, $\hat{\Gamma}_{\text{DeW}}[\hat{B}_\mu]$, is given by the following path integral formulation

$$e^{i\hat{\Gamma}_{\text{DeW}}[\hat{B}_\mu]} = \int d\hat{Q}_\mu d\hat{C}^a d\hat{\epsilon}^a d\hat{F}^a,$$

(9)

$$e^{i\hat{S}_{\text{NCYM}}[\hat{B}_\mu, \hat{\chi}^a_\mu, \hat{\epsilon}, \hat{C}, \hat{C}]} = \hat{\delta}_{\text{BRS}} X_{\text{gf}}[\hat{B}_\mu, \hat{\chi}^a_\mu, \hat{\epsilon}, \hat{C}, \hat{C}],$$

(10)

Above $X_{\text{gf}}[\hat{B}_\mu, \hat{\chi}^a_\mu, \hat{\epsilon}, \hat{C}, \hat{C}]$ is an arbitrary gauge-fixing functional. The noncommutative U(N) BRST transformations $\hat{\delta}_{\text{BRS}} \hat{A}_\mu = \hat{D}_\mu \hat{C}$ and $\hat{\delta}_{\text{BRS}} \hat{C} = -i \hat{\epsilon} \ast \hat{C}$ induce the following BRST transformations after introducing the noncommutative background-field splitting $A_\mu \rightarrow \hat{B}_\mu + h \hat{\chi}^a_\mu$.

$$\hat{\delta}_{\text{BRS}} \hat{B}_\mu = 0, \quad \hat{\delta}_{\text{BRS}} \hat{\chi}^a_\mu = h \hat{D}_\mu \hat{B}_\mu + h^2 \hat{\chi}^a_\mu \hat{C},$$

(11)

$$\hat{\delta}_{\text{BRS}} \hat{\epsilon} = -i \hat{\epsilon} \ast \hat{C}, \quad \hat{\delta}_{\text{BRS}} \hat{C} = h^{\frac{1}{2}} \hat{F}, \quad \hat{\delta}_{\text{BRS}} \hat{F} = 0.$$
of the field operators\textsuperscript{28, 29}:

\begin{align}
\hat{A}_\mu [A_\mu, \theta] (x) &= A_\mu (x) + \sum_{n=2}^{\infty} \mathcal{A}^{(n)}_\mu (x), \\
\hat{C} [A_\mu, C, \theta] (x) &= C(x) + \sum_{n=1}^{\infty} \mathcal{C}^{(n)} (x),
\end{align}

where

\begin{align}
\mathcal{A}^{(n)}_\mu (x) &= \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} \sum_{i \in \mathbb{Z}} (p_i) \times \\
&\quad \cdot \mathbb{Q}^{(n)}_\mu [(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); \theta] \\
&\quad \cdot \hat{A}^{a_1}_{\mu_1} (p_1) \ldots \hat{A}^{a_n}_{\mu_n} (p_n), \\
\mathcal{C}^{(n)} (x) &= \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} \sum_{i \in \mathbb{Z}} (p_i) \times \\
&\quad \cdot \mathbb{C}^{(n)} [(a_1, \mu_1, p_1), \ldots, (a_n, \mu_n, p_n); (a, p); \theta] \\
&\quad \cdot \hat{A}^{a_1}_{\mu_1} (p_1) \ldots \hat{A}^{a_n}_{\mu_n} (p_n) C^n (p).
\end{align}

The quantities \(\mathbb{Q}^{(n)}_\mu\) and \(\mathbb{C}^{(n)}\) are totally symmetric under the permutations with respect to the set of the parameter-triples \{\((a_i, \mu_i, p_i)\)\} \(i = 1, \ldots, n\), which have the property –of key importance– that only the momenta which are not contracted with \(\theta^{\mu \nu}\) build up polynomials which never occur in the denominator\textsuperscript{22, 25}.

Using the ordinary background-field splitting

\begin{equation}
A_\mu = B_\mu + \hbar \hat{A}^{\mu} Q_\mu,
\end{equation}

with the corresponding BRS transformations,

\begin{equation}
\delta_{\text{BRS}} B_\mu = 0, \quad \delta_{\text{BRS}} Q_\mu = h^{\hat{\mu}} D_\mu [B_\mu + h^{\hat{\mu}} Q_\mu] C,
\end{equation}

where \(B_\mu\) is the commutative background field and \(Q_\mu\) the commutative quantum fluctuation, for the SW map\textsuperscript{12} we find the background-field splitting

\begin{align}
\hat{A}_\mu [B_\mu + h^{\hat{\mu}} Q_\mu, \theta] &= \hat{A}_\mu [B_\mu, \theta] + h^{\hat{\mu}} \hat{Q}_\mu [B_\mu, \theta] + \hbar^{\hat{\mu}} \hat{Q}_\mu [B_\mu, Q_\mu, h, \theta] \\
&= B_\mu [B_\mu, \theta] + h^{\hat{\mu}} \hat{Q}_\mu [B_\mu, Q_\mu, h, \theta] \\
\hat{C} [B_\mu + h^{\hat{\mu}} Q_\mu, C, \theta] &= \hat{C} [B_\mu, C, \theta] + h^{\hat{\mu}} \hat{C} [B_\mu, Q_\mu, C, h, \theta],
\end{align}

which ensures that the ordinary BRS transformations\textsuperscript{19} induces the NC BRS transformations\textsuperscript{11}.

Now the on-shell DeWitt action with respect to the ordinary fields, \(\Gamma_{\text{DeW}} [B_\mu]\), is given by the path integral

\begin{equation}
\hat{e}^{\Gamma_{\text{DeW}} [B_\mu]} = \int d^4 Q_\mu dC^a d\hat{C}^a d\hat{F}^a e^{\frac{i}{\hbar} \hat{S}_{\text{NCYM}} [B_\mu, \hbar^{\hat{\mu}} Q_\mu] + i S_{\text{ef}} [B_\mu, Q_\mu, F, \hat{\mathcal{C}}, C]},
\end{equation}

in which we change variables: \(C^a \to \hat{C}^a\) and \(Q^a \to \hat{Q}^a\), so that it transforms into the new path integral

\begin{equation}
\hat{e}^{\Gamma_{\text{DeW}} [B_\mu]} = \int d\hat{Q}^a_\mu d\hat{C}^a d\hat{C}^a d\hat{F}^a J_{-1}^a [B, Q] J_2 [B, Q] e^{\hat{S}_{\text{NCYM}} [B_\mu, \hbar^{\hat{\mu}} Q_\mu] + i S_{\text{ef}} [B_\mu, Q_\mu, F, \hat{\mathcal{C}}, C]},
\end{equation}

containing the Jacobian determinants \(J_1 [B^a, Q^a]\) and \(J_2 [B^a, Q^a]\) who are defined as follows

\begin{equation}
J_1 [B^a, Q^a] = \text{det} \frac{\delta \hat{Q}^a_\mu (x)}{\delta Q^a (y)} = \text{exp} \text{ Tr} \ln \frac{\delta C^n (x)}{\delta C^n (y)},
\end{equation}

\begin{equation}
J_2 [B^a, Q^a] = \text{det} \frac{\delta \hat{C}^a (x)}{\delta C^n (y)} = \text{exp} \text{ Tr} \ln \frac{\delta C^a (x)}{\delta C^n (y)}.
\end{equation}

B. Triviality of the Jacobian determinants

Under the assumption that both above Jacobians are equal to one, we can prove that the right hand side of\textsuperscript{22} equals to the right hand side of\textsuperscript{23}, so that

\begin{equation}
\Gamma_{\text{DeW}} [B_\mu] = \hat{\Gamma}_{\text{DeW}} [\hat{B}_\mu [B_\mu]].
\end{equation}

Note that above result is valid on-shell, i.e. when \(\hat{B}_\mu [B_\mu]\) satisfies the NC YM equations of motions

\begin{equation}
\hat{D}_\mu [\hat{B}_\mu [B_\mu]] \hat{F}^{\mu \nu} [\hat{B}_\mu [B_\mu]] = 0,
\end{equation}

and the reason is the on-shell uniqueness of DeWitt effective action\textsuperscript{25, 30}.

Using the SW map expansion\textsuperscript{10} and the background-field splitting\textsuperscript{20} one can show that

\begin{equation}
\frac{\delta \hat{Q}^a_\mu (x)}{\delta Q^a (y)} = \frac{1}{h^{\hat{\mu}}} \delta \hat{Q}^a_\mu (x-y) + \sum_{n=2}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} \text{ e}^{i \sum_{i=1}^{n-1} p_i (x-y) M_{(n)}^{(a \nu)} b^{\mu} (p_1, p_2, \ldots, p_{n-1}; p_n; \theta),}
\end{equation}

where

\begin{equation}
M_{(n)}^{(a \nu)} b^{\mu} (p_1, p_2, \ldots, p_{n-1}; p_n; \theta) = n \text{ tr} \left[ \hat{R}^a [Q^{(n)}] (a_1, \mu_1, p_1), \ldots, (a_{n-1}, \mu_{n-1}, p_{n-1}) \right]
\end{equation}

\begin{equation}
(b, \nu, p_n; \theta) \left[ \hat{A}^{a_1}_{\mu_1} (p_1) \ldots \hat{A}^{a_{n-1}}_{\mu_{n-1}} (p_{n-1}) \right]
\end{equation}

Note that \(\hat{A}^{a_i}_{\mu_i} (p_i) = \hat{B}^{a_i}_{\mu_i} (p_i) + h^{\hat{\mu}} \hat{Q}^{a_i}_{\mu_i} (p_i)\) for all \(i\). Let \(l_i, i = 1, \ldots, m+1\) be given by

\begin{equation}
l_1 = \sum_{i=1}^{n_{m+1}} p_{1,i_1}, \ldots, l_{m+1} = \sum_{i_{m+1}}^{n_{m+1}} p_{m,i_{m+1}},
\end{equation}

then, by taking into account\textsuperscript{20} and carrying out a
in each circle gives the assignment of 
Fig. 5: The one-loop diagram interpretation/illustration of 
(28): Each circle corresponds one \( M^{(n_i)} \), wavy lines denote the gauge field operators, either background or quantum, 
within the \( M^{(n_i)} \). The \( l_i \)'s are then just the total momentum brought in by these field operators. The solid line flows 
in each circle gives the assignment of \( q - \sum_{i=2}^{m} l_i \) into the corresponding \( M^{(n_i)} \) in (29).

lengthy straightforward computation one gets

\[
\ln J_1[B,Q] = \text{Tr} \ln \left( \frac{\delta Q^n}{\delta \theta^q(x)} \right) = \sum_{n=2}^{\infty} \int \prod_{i=1}^{n-1} \frac{d^3p_i}{(2\pi)^3} \delta \left( \sum_{i=1}^{n-1} p_i \right) 
\cdot \sum_{m=1}^{\infty} \sum_{n=2}^{m} \int \prod_{i=1}^{n-1} \frac{d^3p_{i+1}}{(2\pi)^3} \int \prod_{i=m+1}^{n} \frac{d^3p_{i+1,m+1}}{(2\pi)^3} 
\cdot \frac{1}{m+1} 
\cdot \delta \left( \sum_{i=1}^{n-1} l_i \right) \int \frac{d^3p_{n,1}}{(2\pi)^3} \left[ M^{(n_1)}_{a_1 \mu_1, a_2 \mu_2} (p_{1,1}, p_{1,2}, \ldots, p_{n,1-1}; q; \theta) \right. 
\cdot \left. M^{(n_2)}_{a_2 \mu_2, a_3 \mu_3} (p_{2,1}, p_{2,2}, \ldots, p_{n,2-1}; q - l_2; \theta) \right. 
\cdot \left. \ldots \right. 
\cdot \left. M^{(n_{m+1})}_{a_{m+1} \mu_{m+1}, a_{m+1} \mu_{m+1}} (p_{m+1,1}, p_{m+1,2}, \ldots, p_{m+1,n_{m+1}-1}, q - \sum_{i=2}^{m+1} l_i; \theta) \right].
\]

(28)

The general structure of the master integral (28) above can be visualized as a kind of one-loop diagram, given in Fig. 5.

Hence, in view of the above equations (27) and (28), to compute \( J_1[B,Q] \) one has to work out the following dimensionally regularized type of integrals over the internal momenta \( q^\nu \):

\[
\Omega = \int \frac{d^D q}{(2\pi)^D} \left[ \text{tr} T^{a_1 \mu_1, a_2 \mu_2} (b_{1,1}, \nu_{1,1}, p_{1,1}), \ldots (b_{1,n_1-1,1}, \nu_{1,n_1-1,1}, p_{1,n_1-1,1}; a_1, \mu_1, q; \theta) \right] 
\cdot \text{tr} T^{a_m \mu_m} (b_{m+1,1}, \nu_{m+1,1,1}, p_{m+1,1,1}), \ldots (b_{m+1,n_{m+1}-1,1}, \nu_{m+1,n_{m+1}-1,1}, p_{m+1,n_{m+1}-1,1}; a, \mu, q - \sum_{i=2}^{m+1} l_i; \theta) \right].
\]

(29)

However, the previous integral in (29) is a linear combination of integrals of the type

\[
\mathcal{I} = \int \frac{d^D q}{(2\pi)^D} \Omega(q) I(q \theta k_i, k_i \theta k_j),
\]

(30)

under dimensional regularization (31). By substituting \( \Omega = 0 \) in (28), we obtain that in dimensional regularization the following result holds

\[
\ln J_1[B,Q] = 0;
\]

(32)

proving that indeed \( J_1[B,Q] = 1 \).

It is straightforward to see that identical arguments apply to \( J_2[B,Q] \) as well, thus the Seiberg-Witten map equivalence between quantum theories defined in terms of noncommutative fields and in terms of ordinary fields indeed holds up to all orders in the perturbation theory.

Now, since the \( \theta \)-exact Seiberg-Witten map for matter fields –see (29)– have expressions analogous to that of the ghost field, it is clear that the Jacobian of the transformation from ordinary matter fields to noncommutative matter fields is also trivial in dimensional regularization. Hence, the conclusion that we have reached above when no matter fields are included remains valid when the latter are included: the on-shell DeWitt action of the theory defined in terms of noncommutative fields is the same as the on-shell DeWitt action of the ordinary theory obtained by using the \( \theta \)-exact Seiberg-Witten map.

C. Two-point functions in the background field gauge

We have checked the equivalence established above by computing the one-loop quantum correction to the
quadratic part of the effective action of the U(1) NCGFT in the NC background-field gauge prior to and after the Seiberg-Witten map. In this specific case the general equivalence reduces to a simple relation

\[ \tilde{\Pi}^{\mu\nu}(p) = \Pi^{\mu\nu}(p) \Big|_{on-shell}. \]  

The standard procedure for computing DeWitt effective action of the NC U(1) gauge theory perturbatively in the background-field formalism\(^{25, 26}\) evaluates 1-PI diagrams with all background-field external legs and all integrand field (\(Q_\mu, \hat{C}, \hat{\bar{C}}, \hat{F}\)) internal lines using the following action:

\[
\hat{S}_{loop} = S_{BFG} + S_{NCYM}[\hat{B}_\mu + \hat{Q}_\mu] - S_{NCYM}[\hat{B}_\mu] - \int \left( \frac{\delta}{\delta \hat{B}_\mu} S_{NCYM}[\hat{B}_\mu] \right) \cdot \hat{Q}_\mu.
\]

(34)

We choose \(\theta\)-exact SW map from \(\hat{S}_{loop} \rightarrow S_{loop}\), and then use the resulted action

\[
S_{loop} = S_{BFG}[B_{\mu}, Q_{\mu}, \hat{C}, \hat{\bar{C}}, \hat{F}] + S_{NCYM}[\hat{B}_\mu[B_{\mu}] + \hat{Q}_\mu[Q_{\mu}, B_{\mu}]] - S_{NCYM}[\hat{B}_\mu[B_{\mu}]] - \int \left( \frac{\delta}{\delta \hat{B}_\mu} S_{NCYM}[\hat{B}_\mu] \right) [B_{\mu}] \cdot \hat{Q}_\mu[B_{\mu}, Q_{\mu}],
\]

(35)

for one-loop computation of the effective action with respect to the ordinary fields. This choice can be shown to be equivalent to the subtraction of commutative equations of motions of \(\frac{\delta}{\delta B_{\mu}} S_{NCYM}[\hat{B}_\mu[B_{\mu}]] = 0\) on-shell as long as the Seiberg-Witten map is invertible.

In the follow-on computation, by using the extended version of dimensional regularization scheme\(^{11}\), we find that photon one loop 1-PI two point functions from (34) and (35) turns out to be actually exactly the same:

\[ \tilde{\Pi}^{\mu\nu}(p) = \Pi^{\mu\nu}(p). \]  

(36)

Owing to the fact that \(\theta\)-exact Seiberg-Witten map for the matter fields is analogous to that of the ghost field \(\pi = \pi\), the same conclusion as above (36) holds for the super partners, i.e.

\[
\hat{\Gamma}^{\alpha\dot{\alpha}}(p) = \hat{\Gamma}^{\alpha\dot{\alpha}}(p), \quad \hat{\Gamma}^{(\phi)}(p) = \hat{\Gamma}^{(\phi)}(p),
\]

(37)

which altogether verifies the equivalence relation (24).  

As a consequence of (37), of importance is that once we turn on supersymmetry\(^{11}\) both, the photon polarization tensor IR and UV cancellation results

\[
\Pi^{\mu\nu}_{BFG,\text{total}} \Big|_{UV} = \frac{g^2}{(4\pi)^2} \left( \frac{22}{3} - \frac{4}{3}N_f - \frac{1}{3}N_s \right) \cdot \left( g^{\mu\nu} p^\rho p^\nu \right) \left( \frac{2}{\epsilon} + \ln(\mu^2 (\theta p)^2) \right),
\]

\[
\Pi^{\mu\nu}_{\text{total}} \Big|_{IR} = \frac{g^2}{(4\pi)^2} \left( 32 - 32N_f + 16N_s \right) (\theta p)^\mu (\theta p)^\nu, \quad \left( \theta p \right)^4
\]

found prior to the Seiberg-Witten map now hold precisely after the Seiberg-Witten map.

V. DISCUSSION

In this presentation we state a result concerning the equivalence of two formulations of noncommutative quantum field theories. On the one hand there is the intrinsic formulation of noncommutative U(N) gauge theory, and on the other hand there is a re-formulation using only commutative fields via the SW map, which is a change of field variables. We claim that these two formulations, using their respective path integral quantization, lead to perturbatively equivalent quantum field theories, or in other words the SW map commutes with quantization. The non-trivial part of this claim is that certain Jacobians in this change of field variables are trivial. We extend the SW map valid for classical noncommutative U(N) gauge fields \(A_{\mu} \rightarrow \hat{A}_{\mu}\) and corresponding ghost fields \(C \rightarrow \hat{C}\). Then we compare on-shell DeWitt actions \(\Gamma_{\text{DeW}}[B_{\mu}]\) and \(\Gamma_{\text{DeW}}[\hat{B}_{\mu}]\) given by functional integrals. Both functional integrals are related by Seiberg-Witten map. We show that both expressions are equal to all orders in coupling constant perturbation theory. This duality holds also for Super Yang-Mills theory with \(N = 4\).

We proved remarkable main result that the perturbative gauge theory in NC space derived from classical fields \(\hat{A}_{\mu}\) and \(\hat{A}_{\mu}\) related by SW map are equivalent and related to each other again by SW map. We have also explicitly computed, by using the Feynmann rules derived from the classical action, the one-loop two-point contribution to the on-shell DeWitt action for U(1) SYM with \(N = 0, 1, 2\) and 4 supersymmetry and found complete agreement with general result obtained by carrying out changes of variables in the path integral. These results should be useful to guide towards a proper use of the Seiberg-Witten map.

As shown by our explicit 1-loop result, the same quadratic noncommutative IR divergences that occur in nonsupersymmetric noncommutative U(N) gauge theories formulated in terms of noncommutative fields occur in the ordinary theory obtained from the former by using the \(\theta\)-exact Seiberg-Witten map and that this UV/IR mixing effect –signaling a vacuum instability– is a gauge-fixing independent characteristic of the ordinary gauge theory, in keeping with the duality statement. On the

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1 Explicit computations of the photon polarization tensor \(\Pi^{\mu\nu}\), as well as the super partners two point functions, with full technical details are presented in\(^{11, 17}\).
other hand, all nasty non-local noncommutative UV divergences which occur in the one-loop 1PI functional in the Feynman gauge, computed in \cite{11,13,14,12} are mere gauge artifacts since they do not occur in the one-loop two-point contribution to the on-shell DeWitt action – which is a gauge-fixing independent object– and therefore they do not contribute to any physical quantity. Finally, the quadratic noncommutative IR diverges can be removed by considering supersymmetric versions of the theory, even though supersymmetry is not linearly realized in terms of the ordinary fields \cite{10}.

One final comment regarding the validity of our all-order result: The preexistence of a self-consistent NCGFT which closes on the U(N) Lie algebra without Seiberg-Witten map, admits a sound perturbative quantization by itself and, more importantly, an invertible Seiberg-Witten map. The invertibility is achievable here since the the U(N) Lie algebra generators (in the fundamental representation) form the basis of NxN complex matrices when considered as the generators of a complex linear space. The SU(N) Lie algebra, as a subspace of the U(N) Lie algebra loses this property, therefore one must require U(N) gauge symmetry for an analysis involving inversion of the SW map. There exists, however, deformed SU(N) gauge theories \cite{32} possessing a noncommutativity of spacetime coordinates and having an algebra which closes on the enveloping algebra of the SU(N) algebra only. In that case the equivalence relation cannot be applied since we lack an intrinsic formulation of the quantum theory in terms of noncommutative fields. However, the results presented here indicate that their current definition in terms of ordinary fields \cite{10}.

VI. CONCLUSION

We have proven that at the quantum level the $\theta$-exact SW map provides –at least in perturbative theory with respect to the coupling constant– a dual description, in terms of ordinary fields, of the noncommutative U(N) YM theory with or without supersymmetry. We achieve that by performing appropriate changes of variables in the path integral defining the on-shell DeWitt effective action in dimensional regularization.

There remain to be seen how the results presented here carry over to the nonperturbative regime in the coupling constant. In this regard the analysis of the nonperturbative features of $N = 2$ and 4 supersymmetric gauge theories looks particularly interesting.

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