Paul, Sean Timothy

Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. (English)

Ann. Math. (2) 175, No. 1, 255-296 (2012).

Let us consider a projective complex manifold $X$ and an ample line bundle $L$ on $X$. The Mabuchi energy $\nu$ is a crucial object to detect the existence of a constant scalar curvature Kähler metric (cscK for short) in $c_1(L)$. It is an energy functional defined on the Kähler potentials that enjoys some natural geometric properties when there exists a cscK metric in $c_1(L)$. Given the polarization $L$, one can also consider the space of holomorphic sections $H^0(X, L^k)$ and the associated symmetric space of Bergman metrics $H_k = GL(N_k, \mathbb{C})/U(N_k)$ where $N_k = \dim H^0(X, L^k)$. It is well known from the work of G. Tian than the space of Bergman metrics $H_k$ is dense in the set of Kähler metrics in the class $c_1(L)$. Therefore it is natural to understand the behavior of the Mabuchi energy over the space of Bergman metrics and to relate this behavior to the geometry of $(X, L)$.

This paper provides important results in that direction by giving a simple formula for the restriction of the Mabuchi energy $\nu$ to the space of Bergman metrics. This formula depends on the log norms of $\nu$ and the hyperdiscriminant $\Delta_X$ of $(X, L^k)$. Let us recall that the Chow form of $X$ is given by the equation of the divisor $[R_X] = \{ L \in \text{Gr}(N - 1, n) | L \cap X \neq \emptyset \}$ where $n = \dim X$. The hyperdiscriminant $\Delta_X$ of $X$ is the equation of the dual variety of $X \times \mathbb{P}^n$ in the corresponding Segre embedding. Namely, if $\phi_\sigma$ is the potential of the Bergman metric induced by $\sigma \in \text{SL}(N, \mathbb{C})$, then the author proves that

$$\nu(\phi_\sigma) = \deg(R_X) \log \frac{\|\sigma \cdot \Delta_X\|^2}{\|\Delta_X\|^2} - \deg(\Delta_X) \log \frac{\|\sigma \cdot R_X\|^2}{\|R_X\|^2}.$$

There are several nice consequences of this result. For instance, one can see the asymptotic behavior of the Mabuchi energy along any algebraic one parameter subgroup of a maximal algebraic torus of $\text{SL}(N, \mathbb{C})$. It is completely determined by the Chow polytope and the hyperdiscriminant polytope associated to $R_X$ and $\Delta_X$. Furthermore, the boundedness (or properness) of the Mabuchi energy along degenerations in $\text{SL}(N, \mathbb{C})$ can be rewritten in a geometric way in terms of inclusions of those polytopes (but unfortunately one needs to check the inclusion for all maximal algebraic tori). This leads the author to give a new definition of the notion of K-stability (which is a priori different from the one introduced by G. Tian and S.K. Donaldson) in terms of inclusion of polytopes. Contrarily to the classical notion of K-stability, the deformation of $X$ does not play the role and so does not use the delicate notion of test-configuration, but requires to check a property on all maximal tori of $\text{SL}(N, \mathbb{C})$. Then the stability of $(X, L^k)$ as introduced by the author is equivalent to a condition on the positions of the relative polytopes of $R_X$ and $\Delta_X$ and hence involves only classical projective complex geometry.

Reviewer: Julien Keller (Marseille)

MSC:

14L24 Geometric invariant theory
32Q20 Kähler-Einstein manifolds
51M20 Polyhedra and polytopes; regular figures, division of spaces

Keywords:

K-stability; polytope; resultant; discriminant; hyperdiscriminant; chow form; Mabuchi energy; properness; Kähler-Einstein; Bergman metrics; projective manifold; G.I.T

Full Text: DOI arXiv

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