Symmetric Reduction and Hamilton-Jacobi Equation of Underwater Vehicle with Internal Rotors

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Abstract. In this paper, we first give the regular point reduction by stages and Hamilton-Jacobi theorem of regular controlled Hamiltonian (RCH) system with symmetry on the generalization of a semidirect product Lie group. Next, as an application of the theoretical result, we consider the underwater vehicle with two internal rotors as a regular point reducible by stages RCH system, by using semidirect product Lie group and Hamiltonian reduction by stages. In the cases of coincident and non-coincident centers of buoyancy and gravity, we give explicitly the motion equation and Hamilton-Jacobi equation of reduced underwater vehicle-rotors system on a symplectic leaf by calculation in detail, respectively, which show the effect on controls in regular symplectic reduction by stages and Hamilton-Jacobi theory.

Keywords: underwater vehicle with internal rotors, regular controlled Hamiltonian system, non-coincident center, regular point reduction by stages, Hamilton-Jacobi equation.

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1 Introduction

Symmetry is a general phenomenon in the natural world, but it is widely used in the study of mathematics and mechanics. The reduction theory for mechanical system with symmetry in modern geometric formulation of symplectic manifolds and equivariant momentum maps is developed by Meyer, Marsden and Weinstein; see Abraham and Marsden [1] or Marsden and Weinstein [19] and Meyer [21]. The main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. On the other hand, we note that the theory of controlled mechanical systems has formed an important subject in recent years. Its research gathers together some separate areas of research such as analytical mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system in a way that helps both analysis and design. In particular, in Marsden et al. [18], the authors noted first that the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle $T^*Q$ can be not a Hamiltonian system on a cotangent bundle, that is, the set of Hamiltonian systems on the cotangent bundle is not complete under the regular reduction. Thus, a kind of regular controlled Hamiltonian (RCH) system on symplectic fiber bundle is introduced and the set of such systems is complete under the regular reduction. Next, a good expression of the dynamical vector field of RCH system is given in Marsden et al. [18], such that one can describe the feedback control law to modify the structure of RCH system and show an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. Moreover, Wang in [24] generalized the work to study the singular reduction theory of regular controlled Hamiltonian systems, and Wang and Zhang in [28] generalized the work to study optimal reduction theory of controlled Hamiltonian systems with Poisson structure and symmetry by using optimal momentum map and reduced Poisson tensor (or reduced symplectic form). In addition, Ratiu and Wang in [23] studied the Poisson reduction of controlled Hamiltonian system by controllability distribution, when there is no momentum map for our considered systems. These research work not only gave a variety of reduction methods for controlled Hamiltonian systems, but also showed a variety of relationships of controlled Hamiltonian equivalence of these systems.

At the same time, we note also that Hamilton-Jacobi theory is an important part of classical mechanics. On the one hand, it provides a characterization of the generating functions of certain time-dependent canonical transformations, such that a given Hamiltonian system in such a form that its solutions are extremely easy to find by reduction to the equilibrium, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [15]. On the other hand, it is possible in many cases that Hamilton-Jacobi theory provides an immediate way to integrate the equation of motion of system, even when the problem of Hamiltonian system itself has not been or cannot be solved completely. In addition, the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the development of numerical integrators that preserve the symplectic structure and in the study of stochastic dynamical systems, see Woodhouse [29], Ge and Marsden [5], Marsden and West [20] and Lázaro-Camí and Ortega [7]. Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is very important to describe it from the geometrical point
of view. In particular, Wang in [25] used the dynamical vector fields of Hamiltonian system and the regular reduced Hamiltonian system to describe the Hamilton-Jacobi theory for these systems. Moreover, Wang in [26] extended the Hamilton-Jacobi theory to the regular controlled Hamiltonian system and its regular reduced systems, and described the relationship between the RCH-equivalence for RCH systems and the solutions of corresponding Hamilton-Jacobi equations.

Now, it is a natural problem if there is a practical controlled Hamiltonian system and how to show the effect on controls in regular symplectic reduction and Hamilton-Jacobi theory of the system. In this paper, as an application of the regular point symplectic reduction by stages and Hamilton-Jacobi theory of RCH system with symmetry, we consider that underwater vehicle with internal rotors is a regular point reducible by stages RCH system, where the underwater vehicle with internal rotors is modeled as a Hamiltonian system with control, which is given in Leonard and Marsden [10], Bloch and Leonard [3], Bloch, Leonard and Marsden [4]. In the cases of coincident and non-coincident centers of buoyancy and gravity, we give explicitly the motion equation and Hamilton-Jacobi equation of reduced underwater vehicle-rotors system on a symplectic leaf by calculation in detail, respectively. These equations are more complex than that of Hamiltonian system without control and describe explicitly the effect on controls in symplectic reduction by stages and Hamilton-Jacobi theory.

A brief of outline of this paper is as follows. In the second section, we first review some relevant definitions and basic facts about underwater vehicle with two internal rotors, which will be used in subsequent sections. As an application of the theoretical result of symplectic reduction of RCH system given by Marsden et al. [18], in the third section, we first give the regular point reduction by stages of RCH system with symmetry on the generalization of a semidirect product Lie group. Then we regard the underwater vehicle with two internal rotors as a regular point reducible by stages RCH system on the generalization of semidirect product Lie group $SE(3) \times S^1 \times S^1$ and $(SE(3) \circledS \mathbb{R}^3) \times S^1 \times S^1$, respectively, and in the cases of coincident and non-coincident centers of buoyancy and gravity, we give explicitly the motion equations of their reduced RCH systems on the symplectic leaves by calculation in detail. Moreover, as an application of the theoretical result of Hamilton-Jacobi theory of regular reduced RCH system given by Wang [26], in the fourth section, we first give the Hamilton-Jacobi theorem of RCH system with symmetry on the generalization of a semidirect product Lie group. Then we give the Hamilton-Jacobi equations of the reduced underwater vehicle-rotors systems on the symplectic leaves by calculation in detail, respectively, in the cases of coincident and non-coincident centers of buoyancy and gravity. These research work develop the applications of symplectic reduction by stages and Hamilton-Jacobi theory of RCH systems with symmetry and make us have much deeper understanding and recognition for the structure of Hamiltonian systems and RCH systems.

2 The Underwater Vehicle with Internal Rotors

In this paper, our goal is to give the regular point reduction by stages and Hamilton-Jacobi theorem of underwater vehicle with two symmetric internal rotors. In order to do these, in this section, we first review some relevant definitions and basic facts about underwater vehicle with two symmetric internal rotors, which will be used in subsequent sections. We shall follow the notations and conventions introduced in Leonard [8,9], Leonard and Marsden [10], Marsden [12], Marsden and Ratiu [15], and Marsden et al. [18]. In this paper, we assume that all manifolds
are real, smooth and finite dimensional and all actions are smooth left actions. For convenience, we also assume that all controls appearing in this paper are the admissible controls.

2.1 The Underwater Vehicle-Rotors System with Coincident Centers

In the following we consider that the underwater vehicle is a neutrally buoyant, rigid body (often ellipsoidal) submerged in an infinitely large volume of incompressible, inviscid, irrotational fluid which is at rest at infinity. The dynamics of the body-fluid system are described by Kirchhoff’s equations, where we assume that the only external forces and torques acting on the system are due to buoyancy and gravity. In general, it is possible that the underwater vehicle’s center of buoyancy may not be coincident with its center of gravity. But, in this subsection we assume the system with coincident centers of buoyancy and gravity, and we fix an orthogonal coordinate frame to the body with origin located at the center of buoyancy and axes aligned with the principal axes of the displaced fluid, see Leonard and Marsden [10]. When these axes are also the principal axes of the body and the vehicle is an ellipsoid and it is symmetric about these axes, we assume that the inertia matrix of the body-fluid system is denoted by \( I = \text{diag}(I_1, I_2, I_3) \) and the mass matrix by \( M = \text{diag}(m_1, m_2, m_3) \), where these matrices include the "added" inertias and masses due to the fluid, see Leonard [9]. When the body is oriented so that the body-fixed frame is aligned with the inertial frame, the third principal axis aligns with the direction of gravity. From Leonard and Marsden [10] we know that the dynamics of the underwater vehicle with internal rotors can be viewed as Lie-Poisson dynamics. In fact, we put two rotors within the vehicle so that each rotor’s rotation axis is parallel to the first and the second principal axes of the body, and the rotors spins under the influence of a torque acting on the rotor. The configuration space is \( Q = W \times V \), where \( W = \text{SE}(3) = \text{SO}(3) \oplus \mathbb{R}^3 \) and \( V = S^1 \times S^1 \), with the first factor being the attitude and position of the underwater vehicle and the second factor being the angles of rotors. The corresponding phase space is the cotangent bundle \( T^*Q = T^*\text{SE}(3) \times T^*V \), where \( T^*V = T^*(S^1 \times S^1) \cong T^*\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \), with the canonical symplectic form.

Let \( I = \text{diag}(I_1, I_2, I_3) \) be the moment of inertia of the underwater vehicle in the body-fixed frame. Let \( J_i, i = 1, 2 \) be the moments of inertia of rotors around their rotation axes. Let \( J_{ik}, i = 1, 2, k = 1, 2, 3 \), be the moments of inertia of the \( i \)-th rotor with \( i = 1, 2 \) around the \( k \)-th principal axis, respectively, and denote by \( \bar{I}_i = I_i + J_{i1} + J_{i2} - J_{ii}, i = 1, 2, \) and \( \bar{I}_3 = I_3 + J_{31} + J_{32} \). Let \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \) and \( v = (v_1, v_2, v_3) \) be the angular and linear velocity vectors of the underwater vehicle computed with respect to the axes fixed in the body and \( (\Omega_1, \Omega_2, \Omega_3) \in \text{so}(3) \) and \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \). Let \( \theta_i, i = 1, 2, \) be the relative angles of rotors and \( \tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2) \) the relative angular velocity vector of rotor about the principal axes with respect to the body fixed frame of underwater vehicle, and \( M = \text{diag}(m_1, m_2, m_3) \) the total mass of the system. Consider the Lagrangian \( L(A, c, \Omega, v, \theta, \tilde{\theta}) : \text{SE}(3) \times \text{se}(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \), which is the total kinetic energy of the underwater vehicle plus the total kinetic energy of rotors of the system, given by

\[
L(A, c, \Omega, v, \theta, \tilde{\theta}) = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 + m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2 + J_1 (\Omega_1 + \tilde{\theta}_1)^2 + J_2 (\Omega_2 + \tilde{\theta}_2)^2),
\]

where \( (A, c) \in \text{SE}(3), \) \( (\Omega, v) \in \text{se}(3) \) and \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \in \text{so}(3) \), \( v \in \mathbb{R}^3 \), \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \), \( \tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2) \in \mathbb{R}^2 \). If we introduce the conjugate angular momentum and linear momentum, which is given by \( \Pi_i = \frac{\partial L}{\partial \Omega_i} = \bar{I}_i \Omega_i + J_i (\Omega_i + \tilde{\theta}_i), \) \( i = 1, 2, \) \( \Pi_3 = \frac{\partial L}{\partial \Omega_3} = \bar{I}_3 \Omega_3, \) \( P_k = \frac{\partial L}{\partial v_k} = \)
where \( A, \Omega, v, \theta, \dot{\theta} \) → \( (A, c, \Pi, P, \theta, l) \),
where \( \Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3) \), \( P = (P_1, P_2, P_3) \in \mathbb{R}^3 \), \( l = (l_1, l_2) \in \mathbb{R}^2 \), we have the Hamiltonian 

\[
H(A, c, \Pi, P, \theta, l) = \Theta \cdot \Pi + v \cdot P + \dot{\Theta} \cdot l - L(A, c, \Omega, v, \theta, \dot{\theta})
\]

\[
= \frac{1}{2} \left( \frac{(\Pi_1 - l_1)^2}{I_1} + \frac{(\Pi_2 - l_2)^2}{I_2} + \frac{\Pi_3^2}{I_3} + \frac{P_1^2}{m_1} + \frac{P_2^2}{m_2} + \frac{P_3^2}{m_3} + \frac{\dot{\Theta}_1^2}{J_1} + \frac{\dot{\Theta}_2^2}{J_2} \right].
\] (2.1)

In order to give the motion equation of underwater vehicle-rotors system, in the following we need to consider the symmetry and reduced symplectic structure of the configuration space 

\( Q = SE(3) \times S^1 \times S^1 \).

### 2.2 The Underwater Vehicle-Rotors System with Non-coincident Centers

Since it is possible that the underwater vehicle’s center of buoyancy may not be coincident with its center of gravity, in this subsection then we consider the system with non-coincident centers of buoyancy and gravity. We fix an orthogonal coordinate frame to the body with origin located at the center of buoyancy and axes aligned with the principal axes of the displaced fluid, and these axes are also the principal axes of the body, since the vehicle is an ellipsoidal and it is symmetric about these axes. The vector from the center of buoyancy to the center of gravity with respect to the body-fixed frame is \( h \chi \), where \( \chi \) is an unit vector on the line connecting the two centers which is assumed to be aligned along the third principal axis, and \( h \) is the length of this segment, see Leonard and Marsden [10]. When the body is oriented so that the body-fixed frame is aligned with the inertia frame, the third principal axis aligns with the direction of gravity. We assume that the inertia matrix of the body-fluid system is denoted by \( I = \text{diag}(I_1, I_2, I_3) \) and the mass matrix by \( M = \text{diag}(m_1, m_2, m_3) \), where these matrices include the ”added” inertias and masses due to the fluid, see Leonard [9]. The mass of the body alone is denoted \( m \) and the magnitude of gravitational acceleration is \( g \), and let \( \Gamma \) be the unit vector viewed by an observer moving with the body. In this case, the configuration space is \( Q = W \times V \), where \( W = SE(3) \otimes \mathbb{R}^3 = (SO(3) \otimes \mathbb{R}^3) \otimes \mathbb{R}^3 \) is a double semidirect product and \( V = S^1 \times S^1 \), with the first factor being the attitude and position of underwater vehicle as well as the drift of underwater vehicle in the rotational and translational process and the second factor being the angles of rotors, see Leonard and Marsden [10]. The corresponding phase space is the cotangent bundle \( T^*Q = T^*(SE(3) \otimes \mathbb{R}^3) \times T^*V \), where \( T^*V = T^*(S^1 \times S^1) \cong T^*\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \), with the canonical symplectic form.

Consider the Lagrangian 

\[
L(A, c, b, \Omega, v, \Gamma, \theta, \dot{\theta}) : SE(3) \times \mathbb{R}^3 \times \mathfrak{so}(3) \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},
\]

which is the total kinetic energy of the underwater vehicle plus the total kinetic energy of rotors minus potential energy of the system, given by

\[
L(A, c, b, \Omega, v, \Gamma, \theta, \dot{\theta}) = \frac{1}{2} \left[ I_1 \dot{\Omega}_1^2 + I_2 \dot{\Omega}_2^2 + I_3 \dot{\Omega}_3^2 + m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2 + J_1 (\dot{\Theta}_1)^2 + J_2 (\dot{\Theta}_2)^2 \right] - mgh \Gamma \cdot \chi,
\]

where \((A, c) \in SE(3), (\Omega, v) \in \mathfrak{so}(3) \) and \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3), b, v, \Gamma \in \mathbb{R}^3, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2, \dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2) \in \mathbb{R}^2 \). If we introduce the conjugate angular momentum and linear momentum,
which is given by

\[ \Pi_i = \frac{\partial L}{\partial \dot{\theta}_i} = \tilde{I}_i \Omega_i + J_i (\Omega_i + \dot{\theta}_i), \quad i = 1, 2, \Pi_3 = \frac{\partial L}{\partial \dot{\theta}_3} = \tilde{I}_3 \Omega_3, \quad P_k = \frac{\partial L}{\partial v_k} = m_k v_k, \quad k = 1, 2, 3, \quad l_i = \frac{\partial L}{\partial \dot{\theta}_i} = J_i (\Omega_i + \dot{\theta}_i), \quad i = 1, 2, \] and by the Legendre transformation

\[
FL : SE(3) \times \mathbb{R}^3 \times \mathfrak{se}(3) \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow SE(3) \times \mathbb{R}^3 \times \mathfrak{se}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2,

(A, c, b, \Omega, v, \Gamma, \theta, \dot{\theta}) \rightarrow (A, c, b, P, \Gamma, \theta, l),
\]

where \( \Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{se}^*(3) \), \( P = (P_1, P_2, P_3) \in \mathbb{R}^3 \), \( (\Pi, P) \in \mathfrak{se}^*(3) \), \( l = (l_1, l_2) \in \mathbb{R}^2 \), we have the Hamiltonian

\[
H(A, c, b, \Pi, P, \Gamma, \theta, l) = \Omega \cdot \Pi + v \cdot P + \theta \cdot l - L(A, c, b, \Omega, v, \Gamma, \theta, \dot{\theta})
\]

\[
= \frac{1}{2} \left( \frac{(\Pi_1 - l_1)^2}{I_1} + \frac{(\Pi_2 - l_2)^2}{I_2} + \frac{\Omega_3^2}{I_3} + \frac{P_1^2}{m_1} + \frac{P_2^2}{m_2} + \frac{P_3^2}{m_3} + \frac{l_1^2}{J_1} + \frac{l_2^2}{J_2} \right) + mgh \cdot \chi . \tag{2.2}
\]

In order to give the motion equation of underwater vehicle-rotors system, in the following we need to consider the symmetry and reduced symplectic structure of the configuration space

\[ Q = (SE(3) \ltimes \mathbb{R}^3) \times S^1 \times S^1. \]

### 3 Symmetric Reduction of Underwater Vehicle with Internal Rotors

In this section, as an application of the theoretical result of symplectic reduction of RCH system given by Marsden et al. [18], we first give the regular point reduction by stages of RCH system with symmetry on the generalization of a semidirect product Lie group. Then we regard the underwater vehicle with two internal rotors as a regular point reducible by stages RCH system on the generalizations of semidirect product Lie groups \((SO(3) \ltimes \mathbb{R}^3) \times S^1 \times S^1\) and \((SE(3) \ltimes \mathbb{R}^3) \times S^1 \times S^1\), respectively, and give explicitly the motion equations of their reduced RCH systems on the symplectic leaves by calculation in detail. It is worthy note that they are different from the symmetric reductions of Hamiltonian systems in Bloch and Leonard [3], Bloch, Leonard and Marsden [4], Leonard and Marsden [10] and Marsden [12], the reductions in this paper are all the controlled Hamiltonian reductions, that is, the symmetric reductions of regular controlled Hamiltonian systems, see Marsden et al. [18]. We also follow the notations and conventions introduced in Marsden et al. [13, 14, 16, 17], Marsden and Ratiu [15], Libermann and Marle [11], Ortega and Ratiu [22].

#### 3.1 Symmetric Reduction on the Generalization of Semidirect Product Lie Group

In order to describe the symplectic reduction of underwater vehicle with two internal rotors, we need to first give the regular point reduction by stages of RCH system with symmetry on the generalization of a semidirect product Lie group \(Q = W \times V\) as follows, where \(W = G \ltimes E\) is a semidirect product Lie group with Lie algebra \(\mathfrak{w} = \mathfrak{g} \ltimes E\), \(G\) is a Lie group with Lie algebra \(\mathfrak{g}\), \(E\) is an \(r\)-dimensional vector space and \(V\) is a \(k\)-dimensional vector space. See Marsden et al. [13, 14, 16, 17]. Assume that \(G\) acts on the left by linear maps on \(E\), and \(G\) also acts on the left on the dual space \(E^* \) of \(E\), and the action by an element \(g\) on \(E^*\) is the transpose of the action of \(g^{-1}\) on \(E\). As a set, the underlying manifold of \(W\) is \(G \times E\) and the multiplication on \(W\) is given by

\[
(g_1, x_1)(g_2, x_2) := (g_1 g_2, x_1 + \sigma(g_1) x_2), \quad g_1, g_2 \in G, \quad x_1, x_2 \in E \tag{3.1}
\]
where $\sigma : G \to \text{Aut}(E)$ is a representation of the Lie group $G$ on $E$, $\text{Aut}(E)$ denotes the Lie group of linear isomorphisms of $E$ onto itself, whose Lie algebra is $\text{End}(E)$, the space of all linear maps of $E$ to itself.

The Lie algebra of $W$ is the semidirect product of Lie algebras $\mathfrak{w} = \mathfrak{g} \otimes E$, $\mathfrak{w}^*$ is the dual of $\mathfrak{w}$, that is, $\mathfrak{w}^* = (\mathfrak{g} \otimes E)^*$. The underlying vector space of $\mathfrak{w}$ is $\mathfrak{g} \times E$ and the Lie bracket on $\mathfrak{w}$ is given by

\[
[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \sigma'([\xi_1, \xi_2])v_2 - \sigma'([\xi_2, \xi_1])v_1), \quad \forall \xi_1, \xi_2 \in \mathfrak{g}, \quad v_1, v_2 \in E
\]  

(3.2)

where $\sigma' : \mathfrak{g} \to \text{End}(E)$ is the induced Lie algebra representation given by

\[
\sigma'(\xi)v := \left. \frac{d}{dt} \right|_{t=0} \sigma(\exp t\xi)v, \quad \xi \in \mathfrak{g}, \quad v \in E
\]  

(3.3)

Identify the underlying vector space of $\mathfrak{w}^*$ with $\mathfrak{g}^* \times E^* \cong \mathfrak{g}^* \times E$, by using the duality pairing on each factor. One can give the formula for the Lie-Poisson bracket on the dual of semidirect product $\mathfrak{w}^* = (\mathfrak{g} \otimes E)^*$ as follows, that is, for $F, K : \mathfrak{w}^* \to \mathbb{R}$, their $(-)$-semidirect product Lie bracket is given by

\[
\{F, K\}_{\mathfrak{w}^*}(\mu, a) = -\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \rangle - \langle a, \frac{\delta F}{\delta a} \rangle \cdot \frac{\delta K}{\delta a} - \langle a, \frac{\delta K}{\delta a} \rangle \cdot \frac{\delta F}{\delta a}
\]  

(3.4)

where $(\mu, a) \in \mathfrak{w}^*$ and $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$, $\frac{\delta F}{\delta a} \in E$ are the functional derivatives. Moreover, the Hamiltonian vector field of a smooth function $H : \mathfrak{w}^* \to \mathbb{R}$ is given by

\[
X_H(\mu, a) = (\text{ad}^*_{\frac{\delta H}{\delta \mu}} \mu - \rho_{\frac{\delta H}{\delta a}} a, \frac{\delta H}{\delta \mu} \cdot a),
\]  

(3.5)

where the infinitesimal action of $\mathfrak{g}$ on $E$ can be denoted by $\xi \cdot v = \rho_v(\xi)$, for any $\xi \in \mathfrak{g}$, $v \in E$ and the map $\rho_v : \mathfrak{g} \to E$ is the derivative of the map $g \mapsto gv$ at the identity and $\rho_v^* : E^* \to \mathfrak{g}^*$ is its dual, see Marsden et al. [13].

At first, we consider a symplectic action of $W$ on a symplectic manifold $P$ and assume that this action has an $\text{Ad}^*$-equivariant momentum map $J_W : P \to \mathfrak{w}^*$. On the one hand, we can regard $E$ as a normal subgroup of $W$, which acts on $P$ and has a momentum map $J_E : P \to E^*$ given by $J_E = i_E^* \cdot J_W$, where $i_E : E \to \mathfrak{w}$; $x \mapsto (0, x)$ is the inclusion, and $i_E^* : \mathfrak{w}^* \to E^*$ is its dual. $J_E$ is called the second component of $J_W$. On the other hand, we can also regard $G$ as a subgroup of $W$, and have the inclusion $i_g : \mathfrak{g} \to \mathfrak{w}$, given by $\xi \mapsto (\xi, 0)$. Thus, $G$-action also has a momentum map $J_G : P \to \mathfrak{g}^*$ given by $J_G = i_g^* \cdot J_W$, where $i_g^* : \mathfrak{w}^* \to \mathfrak{g}^*$ is its dual of $i_g$, which is called the first component of $J_W$. Moreover, from the $\text{Ad}^*$-equivariance of $J_W$ under $W$-action, we know that $J_E$ is also $\text{Ad}^*$-equivariant under $G$-action. Thus, we can carry out reduction of $P$ by $W$ at a regular value $\tau = (\mu, a) \in \mathfrak{w}^*$ of the momentum map $J_W$ in two stages using the following procedure. (i)First reduce $P$ by $E$ at the value $a \in E^*$, and get the reduced space $P_a = J_E^{-1}(a)/E$. Since the reduction is by the Abelian group $E$, so the quotient is done using the whole of $E$. (ii)The isotropy subgroup $G_a \subset G$, consists of elements of $G$ that leave the point $a \in E^*$ fixed using the action of $G$ on $E^*$. One can prove that the group $G_a$ leaves the set $J_E^{-1}(a) \subset P$ invariant, and acts symplectically on the reduced space $P_a$ and has a naturally induced momentum map $J_a : P_a \to \mathfrak{g}^*_a$, where $\mathfrak{g}_a$ is the Lie algebra of the isotropy subgroup $G_a$ and $\mathfrak{g}^*_a$ is its dual. (iii)Reduce the first reduced space $P_a$ at the point $\mu_a = \mu|_{\mathfrak{g}_a} \in \mathfrak{g}_a^*$, one
can get the second reduced space \((P_a)_{\mu_a} = J^{-1}_a(\mu_a)/(G_a)_{\mu_a}\). Thus, we can give the following proposition on the reduction by stages for semidirect product Lie group, see Marsden et al. [13].

**Proposition 3.1** The reduced space \((P_a)_{\mu_a}\) is symplectically diffeomorphic to the reduced space \(P_{\tau}\) obtained by reducing \(P\) by \(W\) at the regular point \(\tau = (\mu, a) \in \mathfrak{w}^*\).

In particular, we can choose that \(P = T^*W\), where \(W = G\otimes E\) is a semidirect product Lie group, with the cotangent lift action of \(W\) on \(T^*W\) induced by left translation of \(W\) on itself. Since the reduction of \(T^*W\) by the action of \(E\) can give a space which is isomorphic to \(T^*G\), from the above reduction by stages proposition for semidirect product Lie group we can get the following semidirect product reduction proposition.

**Proposition 3.2** The reduction of \(T^*G\) by \(G_a\) at the regular values \(\mu_a = \mu|_{\mathfrak{g}_a^*}\) gives a space which is isomorphic to the coadjoint orbit \(O_{(\mu, a)} \subset \mathfrak{w}^*\) through the point \((\mu, a) \in \mathfrak{w}^*\), where \(\mathfrak{w}^*\) is the dual of the Lie algebra \(\mathfrak{w}\) of \(W\).

Next, we consider the action of \(W\) on the generalization of a semidirect product Lie group \(Q = W \times V\) and define the left \(W\)-action \(\Phi : W \times Q \to Q\), \(\Phi((g_1, x_1), (g_2, x_2), \theta) := ((g_1, x_1)(g_2, x_2), \theta)\), for any \((g_1, x_1), (g_2, x_2) \in W, \theta \in V\), that is, the \(W\)-action on \(Q\) is the left translation on the first factor \(W\), and \(W\) acts trivially on the second factor \(V\). Because \(T^*Q = T^*W \times T^*V\), and \(T^*V = V \times \mathfrak{w}^*\), by using the left trivialization of \(T^*W\), that is, \(T^*W = W \times \mathfrak{w}^*\), where \(\mathfrak{w}^*\) is the dual of \(\mathfrak{w} = \mathfrak{g} \otimes E\), and hence we have that \(T^*Q = W \times \mathfrak{w}^* \times V \times V^*\). If the left \(W\)-action \(\Phi : W \times Q \to Q\) is free and proper, then the cotangent lift of the action to its cotangent bundle \(T^*Q\), given by \(\Phi^* : W \times T^*Q \to T^*Q\), \(\Phi^*((g_1, x_1), (g_2, x_2), (\mu, a), \theta, \lambda) := ((g_1, x_1)(g_2, x_2), (\mu, a), \theta, \lambda)\), for any \((g_1, x_1), (g_2, x_2) \in W, (\mu, a) \in \mathfrak{w}^*, \theta \in V, \lambda \in V^*\), is also a free and proper action, and the orbit space \((T^*Q)/W\) is a smooth manifold and \(\pi : T^*Q \to (T^*Q)/W\) is a smooth submersion. Since \(W\) acts trivially on \(\mathfrak{w}^*, V\) and \(V^*\), it follows that \((T^*Q)/W\) is diffeomorphic to \(\mathfrak{w}^* \times V \times V^*\).

We know that \(\mathfrak{w}^* = (\mathfrak{g} \otimes E)^*\) is a Poisson manifold with respect to the \((-)\)-semidirect product Lie-Poisson bracket \(\{\cdot, \cdot\}_{\mathfrak{w}^*}\) defined by (3.4). For \((\mu, a) \in \mathfrak{w}^*\), the coadjoint orbit \(O_{(\mu, a)} \subset \mathfrak{w}^*\) has the induced orbit symplectic forms \(\omega_{O_{(\mu, a)}}\) given by

\[
\omega_{O_{(\mu, a)}}(X_{F_{(\mu, a)}}, X_{K_{(\mu, a)}}) = \{F_{(\mu, a)}, K_{(\mu, a)}\}_{\mathfrak{w}^*}|_{O_{(\mu, a)}},
\]

for \(F_{(\mu, a)}, K_{(\mu, a)} : O_{(\mu, a)} \to \mathbb{R}\), which are the restriction of the \((-)\)-semidirect product Lie-Poisson brackets on \(\mathfrak{w}^*\) to the coadjoint orbit \(O_{(\mu, a)}\). From the Symplectic Stratification theorem we know that the coadjoint orbits \((O_{(\mu, a)}, \omega_{O_{(\mu, a)}}), (\mu, a) \in \mathfrak{w}^*\), form the symplectic leaves of the Poisson manifolds \((\mathfrak{w}^*, \{\cdot, \cdot\}_{\mathfrak{w}^*})\). Let \(\omega_V\) be the canonical symplectic form on \(T^*V \cong V \times V^*\) given by

\[
\omega_V((\theta_1, \lambda_1), (\theta_2, \lambda_2)) = \langle \lambda_2, \theta_1 \rangle - \langle \lambda_1, \theta_2 \rangle,
\]

where \((\theta_i, \lambda_i) \in V \times V^*, i = 1, 2, < \cdot, \cdot >\) is the natural pairing between \(V^*\) and \(V\). Thus, we can induce a symplectic forms \(\omega_{O_{(\mu, a)} \times V \times V^*} = \pi^*_{O_{(\mu, a)}} \omega_{O_{(\mu, a)}} + \pi^*_V \omega_V\) on the smooth manifold \(O_{(\mu, a)} \times V \times V^*\), where the maps \(\pi_{O_{(\mu, a)}} : O_{(\mu, a)} \times V \times V^* \to O_{(\mu, a)}\) and \(\pi_V : O_{(\mu, a)} \times V \times V^* \to V \times V^*\) are canonical projections. On the other hand, note that for \(F, K : T^*V \cong V \times V^* \to \mathbb{R}\), by using the canonical symplectic form \(\omega_V\) on \(T^*V \cong V \times V^*\), we can define the Poisson bracket \(\{\cdot, \cdot\}_V\) on \(T^*V\) as follows

\[
\{F, K\}_V(\theta, \lambda) = \langle \frac{\delta F}{\delta \theta} \cdot \frac{\delta K}{\delta \lambda}, - \frac{\delta K}{\delta \theta} \cdot \frac{\delta F}{\delta \lambda} \rangle.
\]
If \( \theta_i, i = 1, \cdots, k \), is a base of \( V \), and \( \lambda_i, i = 1, \cdots, k \), a base of \( V^* \), then we have that
\[
\{F, K\}_V(\theta, \lambda) = \sum_{i=1}^k \left( \frac{\partial F}{\partial \theta_i} \frac{\partial K}{\partial \lambda_i} - \frac{\partial K}{\partial \theta_i} \frac{\partial F}{\partial \lambda_i} \right).
\]

(3.7)

Thus, by using the \((-\cdot, -\cdot\))-semidirect product Lie-Poisson brackets on \( \mathfrak{w}^* \) and the Poisson bracket \( \{\cdot, \cdot\}_V \) on \( T^*V \), for \( F, K : \mathfrak{w}^* \times V \times V^* \to \mathbb{R} \), we can define the Poisson bracket on \( \mathfrak{w}^* \times V \times V^* \) as follows
\[
\{F, K\}_- (\mu, a, \theta, \lambda) = \{F, K\}_{\mathfrak{w}^*}(\mu, a) + \{F, K\}_V(\theta, \lambda)
\]
\[
=- \langle \mu, \frac{\delta F}{\delta \mu} \frac{\delta K}{\delta a} \rangle = -\langle a, \frac{\delta F}{\delta \mu} \frac{\delta K}{\delta a} \rangle + \sum_{i=1}^k \left( \frac{\partial F}{\partial \theta_i} \frac{\partial K}{\partial \lambda_i} - \frac{\partial K}{\partial \theta_i} \frac{\partial F}{\partial \lambda_i} \right).
\]

(3.8)

See Krishnaprasad and Marsden [6]. In particular, for \( F_{(\mu, a)}, K_{(\mu, a)} : \mathcal{O}_{(\mu, a)} \times V \times V^* \to \mathbb{R} \), we have that \( \tilde{\omega}_{\mathcal{O}_{(\mu, a)} \times V \times V^*}(X_{F_{(\mu, a)}}, X_{K_{(\mu, a)}}) = \{F_{(\mu, a)}, K_{(\mu, a)}\}_{-}[\mathcal{O}_{(\mu, a)} \times V \times V^*] \).

On the other hand, from \( T^*Q = T^*W \times T^*V \) we know that there is a canonical symplectic form \( \omega_Q = \pi_1^* \omega_0 + \pi_2^* \omega_V \) on \( T^*Q \), where \( \omega_0 \) is the canonical symplectic form on \( T^*W \) and the maps \( \pi_1 : Q = W \times V \to W \) and \( \pi_2 : Q = W \times V \to V \) are canonical projections. Then the cotangent lift of the left \( W \)-action \( \Phi^+: W \times T^*Q \to T^*Q \) is also symplectic, and admits an associated \( \text{Ad}^* \)-equivariant momentum map \( J_Q : T^*Q \to \mathfrak{w}^* \) such that \( J_Q \cdot \pi_1^* = J_W \), where \( J_W : T^*W \to \mathfrak{w}^* \) is a momentum map of left \( W \)-action on \( T^*W \), and \( \pi_*^1 : T^*W \to T^*Q \). If \( (\mu, a) \in \mathfrak{w}^* \) is a regular value of \( J_Q \), then \( (\mu, a) \in \mathfrak{w}^* \) is also a regular value of \( J_W \) and \( J_Q^{-1}(\mu, a) \cong J_W^{-1}(\mu, a) \times V \times V^* \). Denote by \( W_{(\mu, a)} = \{(g, x) \in W | \text{Ad}^*_{(g, x)}^{-1}(\mu, a) = (\mu, a)\} \), the isotropy subgroup of coadjoint \( W \)-action at the point \( (\mu, a) \in \mathfrak{w}^* \). It follows that \( W_{(\mu, a)} \) acts freely and properly on \( J_W^{-1}(\mu, a) \), the regular point reduced space \( (T^*W)_{(\mu, a)} = J_W^{-1}(\mu, a) / W_{(\mu, a)} \cong (T^*W)_{(\mu, a)} \times V \times V^* \) of \( (T^*Q, \omega_Q) \) at \( (\mu, a) \), is a symplectic manifold with symplectic form \( \omega_{(\mu, a)} \) uniquely characterized by the relation \( \pi^*_{(\mu, a)} \omega_Q = i^*_{(\mu, a)} \omega, \pi^*_{(\mu, a)} \omega = i^*_{(\mu, a)} \pi^*_{(\mu, a)} \omega_Q + i^*_W \pi^*_{(\mu, a)} \omega_V \), where the map \( i_{(\mu, a)} : J_W^{-1}(\mu, a) \to T^*Q \) is the inclusion and \( \pi_{(\mu, a)} : J_W^{-1}(\mu, a) \to (T^*Q)_{(\mu, a)} \) is the projection. From the reduction by stages propositions for semidirect product Lie group, see Proposition 3.1 and Proposition 3.2, we know that \( (T^*W)_{(\mu, a)}, \omega_{(\mu, a)} \) is symplectically diffeomorphic to \( (\mathcal{O}_{(\mu, a)}, \tilde{\omega}_{\mathcal{O}_{(\mu, a)} \times V \times V^*}) \), and hence we have that \( (T^*Q)_{(\mu, a)}, \omega_{(\mu, a)} \) is symplectically diffeomorphic to \( (\mathcal{O}_{(\mu, a)} \times V \times V^*, \tilde{\omega}_{\mathcal{O}_{(\mu, a)} \times V \times V^*}) \), which is a symplectic leaf of the Poisson manifold \( \{\mathfrak{w}^* \times V \times V^*, \{\cdot, \cdot\}_-\} \).

We now consider Lagrangian \( L(g, x, \xi, v, \theta, \dot{\theta}) : TQ \cong W \times \mathfrak{w} \times TV \to \mathbb{R} \), which is usual the total kinetic minus potential energy of the system, where \( (g, x) \in W, (\xi, v) \in \mathfrak{w}, \) and \( \theta \in V, (\xi^i, v^j) \) and \( \dot{\theta}^j = \frac{\partial L}{\partial v^j} \) (\( i = 1, \cdots, n, j = 1, \cdots, k, n = \dim W, k = \dim V \)), regarded as the velocity of system. If we introduce the conjugate momentum \( \zeta_i = \frac{\partial L}{\partial v^i} = p_i, \) \( \pi_i = \frac{\partial L}{\partial \xi^i} = l_i = \frac{\partial L}{\partial \theta^j} \), \( i = 1, \cdots, n, j = 1, \cdots, k \), and by the Legendre transformation \( FL : TQ \cong W \times \mathfrak{w} \times V \times V \to T^*Q \cong W \times \mathfrak{w}^* \times V \times V^* \), \((g^i, x^j, \xi^i, v^j, \theta^j, \dot{\theta}^j) \to (g^i, x^j, \zeta_i, p_i, \theta^j, l_j) \), we have the Hamiltonian \( H(\xi, v, \theta, l) : T^*Q \cong W \times \mathfrak{w}^* \times V \times V^* \to \mathbb{R} \) given by
\[
H(g^i, x^j, \zeta_i, p_i, \theta^j, l_j) = \sum_{i=1}^n (\zeta_i \dot{\xi}^i + p_i \dot{v}^i) + \sum_{j=1}^k l_j \dot{\theta}^j - L(g^i, x^j, \xi^i, v^j, \theta^j, \dot{\theta}^j).
\]

(3.9)

If Hamiltonian \( H(\xi, v, \theta, l) : T^*Q \cong W \times \mathfrak{w}^* \times V \times V^* \to \mathbb{R} \) is left cotangent lifted \( W \)-action \( \Phi^+: \mathbb{R} \) invariant, for regular value \( (\mu, a) \in \mathfrak{w}^* \) we have the associated reduced Hamiltonian
The 6-tuple semidirect product Lie group \( \mu, a \) is reducible by stages RCH system. For the point \((\mu, a) \in \w^*\), the regular value of the momentum map \( J_Q : T^*Q \to \w^* \), and the given feedback control \( u : T^*Q \to C(\bigcup J_Q^{-1}(\mu, a)) \), the regular point reduced by stages system is the 5-tuple \((\O(\mu, a) \times V \times V^*, \o_{\O(\mu, a) \times V \times V^*}(\mu, a), f(\mu, a), u(\mu, a))\), where \( \O(\mu, a) \subset \w^* \) is the coadjoint orbit, \( \o_{\O(\mu, a) \times V \times V^*} \) is orbit symplectic form on \( \O(\mu, a) \times V \times V^* \), \( h(\mu, a) \cdot \o_{\O(\mu, a) \times V \times V^*} = H \cdot i(\mu, a) \), \( f(\mu, a) \cdot \o_{\O(\mu, a) \times V \times V^*} = \pi(\mu, a) \cdot F \cdot i(\mu, a) \in C(\bigcup J_Q^{-1}(\mu, a)) \), and \( u(\mu, a) \in C(\mu, a) = \pi(\mu, a)(C) \subset \O(\mu, a) \times V \times V^* \). Moreover, the dynamical vector field of the reduced RCH system can be expressed by

\[
X(\O(\mu, a) \times V \times V^*, \o_{\O(\mu, a) \times V \times V^*}(\mu, a), f(\mu, a), u(\mu, a)) = X_{h(\mu, a)} + vlift(f(\mu, a)) + vlift(u(\mu, a)),
\]

where \( X_{h(\mu, a)} \in T(\O(\mu, a) \times V \times V^*) \) is Hamiltonian vector field of the reduced Hamiltonian \( \mu, a \) reduction \( h(\mu, a) : \O(\mu, a) \times V \times V^* \to \mathbb{R} \), and \( vlift(f(\mu, a)) = vlift(f(\mu, a))X_{h(\mu, a)} \in T(\O(\mu, a) \times V \times V^*) \), \( vlift(u(\mu, a)) = vlift(u(\mu, a))X_{h(\mu, a)} \in T(\O(\mu, a) \times V \times V^*) \), and satisfies the condition

\[
X(\O(\mu, a) \times V \times V^*, \o_{\O(\mu, a) \times V \times V^*}(\mu, a), f(\mu, a), u(\mu, a)) \cdot \pi(\mu, a) = T\pi(\mu, a) \cdot X(\Theta_Q \cdot W, \o_{\Theta_Q \cdot W, \w^*}, \w) \cdot i(\mu, a).
\]

Note that \( vlift(u(\mu, a))X_{h(\mu, a)} \) is the vertical lift of vector field \( X_{h(\mu, a)} \) under the action of \( u(\mu, a) \) along fibers, that is,

\[
vlift(u(\mu, a))X_{h(\mu, a)}(\zeta, p, \theta, l) = vlift((Tu(\mu, a))X_{h(\mu, a)}(\zeta, p, \theta, l), \zeta, p, \theta, l)) = (Tu(\mu, a))X_{h(\mu, a)}(\zeta, p, \theta, l),
\]

where \( \sigma \) is a geodesic in \( \O(\mu, a) \times V \times V^* \) connecting \( u(\mu, a)(\zeta, p, \theta, l) \) and \( (\zeta, p, \theta, l) \), and \( (Tu(\mu, a))X_{h(\mu, a)}(\zeta, p, \theta, l) \) is the parallel displacement of vertical vector \( (Tu(\mu, a))X_{h(\mu, a)}(\zeta, p, \theta, l) \) along the geodesic \( \sigma \) from \( u(\mu, a)(\zeta, p, \theta, l) \) to \( (\zeta, p, \theta, l) \), and \( vlift(f(\mu, a))X_{h(\mu, a)} \) is defined in the similar manner, see Marsden et al. [18] and Wang [26]. Thus, we can get the following theorem.

**Theorem 3.3** The 6-tuple \((T^*Q, W, \o_{\Theta_Q \cdot W, \w^*}, H, F, C)\) is a regular point reducible by stages RCH system on the generalization of semidirect product Lie group \( Q = W \times V \), where \( W = G \times E \) is a semidirect product Lie group with Lie algebra \( \w = g \oplus E \), \( G \) is a Lie group with Lie algebra \( g \), \( E \) is a \( r \)-dimensional vector space and \( V \) is a \( k \)-dimensional vector space, and the Hamiltonian \( H : T^*Q \to \w^* \), the fiber-preserving map \( F : T^*Q \to T^*Q \) and the fiber submanifold \( C \) of \( T^*Q \) are all left cotangent lifted \( W \)-action \( \Phi^{T^*Q} \) invariant. For the point \((\mu, a) \in \w^*\), the regular value of the momentum map \( J_Q : T^*Q \to \w^* \), and the given feedback control \( u : T^*Q \to C(\bigcup J_Q^{-1}(\mu, a)) \), the regular point reduced by stages system is the 5-tuple \((\O(\mu, a) \times V \times V^*, \o_{\O(\mu, a) \times V \times V^*}(\mu, a), f(\mu, a), u(\mu, a))\), where \( \O(\mu, a) \subset \w^* \) is the coadjoint orbit, \( \o_{\O(\mu, a) \times V \times V^*} \) is orbit symplectic form on \( \O(\mu, a) \times V \times V^* \), \( h(\mu, a) \cdot \o_{\O(\mu, a) \times V \times V^*} = H \cdot i(\mu, a) \), \( f(\mu, a) \cdot \o_{\O(\mu, a) \times V \times V^*} = \pi(\mu, a) \cdot F \cdot i(\mu, a) \in C(\bigcup J_Q^{-1}(\mu, a)) \), and \( u(\mu, a) \in C(\mu, a) = \pi(\mu, a)(C) \subset \O(\mu, a) \times V \times V^* \). Moreover, the dynamical vector field of the reduced RCH system is given by (3.10).

In the following we regard the underwater vehicle with two symmetric internal rotors as a regular point reducible by stages RCH system on the generalization of semidirect product Lie group \((SO(3) \oplus \mathbb{R}^3) \times S^1 \times S^1\) and \((SE(3) \oplus \mathbb{R}^3) \times S^1 \times S^1\), respectively, and give explicitly the motion equations of their reduced RCH systems on the symplectic leaves by calculation in detail.
3.2 Symmetric Reduction of Vehicle-Rotors System with Coincident Centers

We first give the regular point reduction by stages of underwater vehicle-rotors system with coincident centers of buoyancy and gravity. Assume that semidirect product Lie group $W = \text{SO}(3) \circledast \mathbb{R}^3 = \text{SE}(3)$ acts freely and properly on $Q = \text{SE}(3) \times V$ by the left translation on $\text{SE}(3)$, then the action of $\text{SE}(3)$ on the phase space $T^*Q$ is by cotangent lift of left translation on $\text{SE}(3)$ at the identity, that is, $\Phi : \text{SE}(3) \times T^*\text{SE}(3) \times T^*V \cong \text{SE}(3) \times \text{SE}(3) \times \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \text{SE}(3) \times \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2$, given by $\Phi((B, s)|(A, c), (\Pi, w), (\theta, l)) = ((BA, c), (\Pi, w), \theta, l)$, for any $A, B \in \text{SO}(3)$, $\Pi \in \mathfrak{se}^* (3)$, $s, c, w \in \mathbb{R}^3$, $\theta, l \in \mathbb{R}^2$, which is also free and proper, and admits an associated $\text{Ad}^*$-equivariant momentum map $J_Q : T^*Q \cong \text{SE}(3) \times \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{se}^* (3)$ for the left $\text{SE}(3)$ action. If $(\Pi, w) \in \mathfrak{se}^* (3)$ is a regular value of $J_Q$, then the regular point reduced space $(T^*Q)_{(\Pi, w)} = J_Q^{-1}(\Pi, w)/\text{SE}(3)$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\Pi, w)} \times \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2$, where $\text{SE}(3)_{(\Pi, w)}$ is the isotropy subgroup of coadjoint $\text{SE}(3)$-action at the point $(\Pi, w) \in \mathfrak{se}^* (3)$.

We know that $\mathfrak{se}^* (3) = \mathfrak{so}^* (3) \circledast \mathbb{R}^3$ is a Poisson manifold with respect to its semidirect product Lie-Poisson bracket, that is, heavy top Lie-Poisson bracket defined by

$$\{F, K\}_{\mathfrak{se}^* (3)}(\Pi, P) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) - P \cdot (\nabla_\Pi F \times \nabla_P K - \nabla_\Pi K \times \nabla_P F),$$

(3.12)

where $F, K \in C^\infty(\mathfrak{se}^* (3))$, $(\Pi, P) \in \mathfrak{se}^* (3)$. For $(\mu, a) \in \mathfrak{se}^* (3)$, the coadjoint orbit $\mathcal{O}_{(\mu, a)} \subset \mathfrak{se}^* (3)$ has the induced orbit symplectic form $\omega^-_{(\mu, a)}$, which is coincide with the restriction of the Lie-Poisson bracket on $\mathfrak{se}^* (3)$ to the coadjoint orbit $\mathcal{O}_{(\mu, a)}$, and the coadjoint orbits $(\mathcal{O}_{(\mu, a)}, \omega^-_{(\mu, a)}), (\mu, a) \in \mathfrak{se}^* (3)$, form the symplectic leaves of the Poisson manifold $(\mathfrak{se}^* (3), \{\cdot, \cdot\}_{\mathfrak{se}^* (3)})$. Let $\omega_{\mathbb{R}^2}$ be the canonical symplectic form on $T^*\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$, and it induces a canonical Poisson bracket $\{\cdot, \cdot\}_{\mathbb{R}^2}$ on $T^*\mathbb{R}^2$ given by (3.7). Thus, we can induce a symplectic form $\tilde{\omega}_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2} = \pi_{\mathcal{O}_{(\mu, a)}}^* \omega^-_{\mathcal{O}_{(\mu, a)}} + \pi_{\mathbb{R}^2}^* \omega_{\mathbb{R}^2}$ on the smooth manifold $\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2$, where the maps $\pi_{\mathcal{O}_{(\mu, a)}} : \mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{O}_{(\mu, a)}$ and $\pi_{\mathbb{R}^2} : \mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ are canonical projections, and induce a Poisson bracket $\{\cdot, \cdot\} = \pi_{\mathfrak{se}^* (3)}^* \{\cdot, \cdot\}_{\mathfrak{se}^* (3)} + \pi_{\mathbb{R}^2}^* \{\cdot, \cdot\}_{\mathbb{R}^2}$ on the smooth manifold $\mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2$, where the maps $\pi_{\mathfrak{se}^* (3)} : \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{se}^* (3)$ and $\pi_{\mathbb{R}^2} : \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ are canonical projections, and such that $(\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega}_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2})$ is a symplectic leaf of the Poisson manifold $(\mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2, \{\cdot, \cdot\}_{\mathbb{R}^2})$.

From the above expression (2.1) of the Hamiltonian, we know that $H(A, c, \Pi, P, \theta, l)$ is invariant under the left $\text{SE}(3)$-action. For the case that $(\mu, a) \in \mathfrak{se}^* (3)$ is the regular value of $J_Q$, we have the reduced Hamiltonian $h_{(\mu, a)}(\Pi, P, \theta, l) : \mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $h_{(\mu, a)}(\Pi, P, \theta, l) = \mu_{(\mu, a)}(H(A, c, \Pi, P, \theta, l)) = H(A, c, \Pi, P, \theta, l)|_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}$. From the heavy top bracket on $\mathfrak{se}^* (3)$ and the Poisson bracket on $T^*\mathbb{R}^2$, we can get the Poisson bracket on $\mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2$, that is, for $F, K : \mathfrak{se}^* (3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, we have that

$$\{F, K\}_{\mathbb{R}^2}(\Pi, P, \theta, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) - P \cdot (\nabla_\Pi F \times \nabla_P K - \nabla_\Pi K \times \nabla_P F) + \{F, K\}_{\mathbb{R}^2}(\theta, l).$$

In particular, for $F_{(\mu, a)}, K_{(\mu, a)} : \mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, we have that

$$\tilde{\omega}_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}(X_{F_{(\mu, a)}}, X_{K_{(\mu, a)}}) = \{F_{(\mu, a)}, K_{(\mu, a)}\}_{\mathbb{R}^2}.$$
vector field $X_{h(\mu,a)}(K(\mu,a)) = \{K(\mu,a), h(\mu,a)\} - |\sigma(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2|$, and hence we have that

$$
X_{h(\mu,a)}(\Pi)(\Pi, P, \theta, l) = \{\Pi, h(\mu,a)\} - (\Pi, P, \theta, l) = -\Pi \cdot (\nabla_\Pi \times \nabla_\Pi h(\mu,a))
$$

$$
- P \cdot (\nabla_\Pi \times \nabla_P h(\mu,a) - \nabla_\Pi h(\mu,a) \times \nabla_P \Pi) + \sum_{i=1}^{2} \left( \frac{\partial P}{\partial \theta_i} \frac{\partial h(\mu,a)}{\partial l_i} - \frac{\partial h(\mu,a)}{\partial \theta_i} \frac{\partial P}{\partial l_i} \right)
$$

$$
= -\nabla_\Pi \Pi \cdot (\nabla_\Pi h(\mu,a) \times \Pi) - \nabla_\Pi \Pi (\nabla_P h(\mu,a) \times P) = \Pi \times \Omega + P \times v,
$$
since $\nabla_\Pi \Pi = 1$, $\nabla_P P = 0$, $\nabla_\Pi h(\mu,a) = \Omega$, $\nabla_P h(\mu,a) = v$, and $\frac{\partial \Pi}{\partial \theta_i} = \frac{\partial h(\mu,a)}{\partial l_i} = 0$, $i = 1, 2$.

$$
X_{h(\mu,a)}(P)(\Pi, P, \theta, l) = \{P, h(\mu,a)\} - (\Pi, P, \theta, l) = -\Pi \cdot (\nabla_\Pi \times \nabla_P h(\mu,a))
$$

$$
- P \cdot (\nabla_\Pi \times \nabla_P h(\mu,a) - \nabla_\Pi h(\mu,a) \times \nabla_P \Pi) + \sum_{i=1}^{2} \left( \frac{\partial P}{\partial \theta_i} \frac{\partial h(\mu,a)}{\partial l_i} - \frac{\partial h(\mu,a)}{\partial \theta_i} \frac{\partial P}{\partial l_i} \right)
$$

$$
= \nabla_P P \cdot (P \times \nabla_P h(\mu,a)) = P \times \Omega,
$$
since $\nabla_P P = 1$, $\nabla_P P = 0$, $\nabla_\Pi h(\mu,a) = \Omega$, and $\frac{\partial P}{\partial \theta_i} = \frac{\partial h(\mu,a)}{\partial l_i} = 0$, $i = 1, 2$.

$$
X_{h(\mu,a)}(\theta)(\Pi, P, \theta, l) = \{\theta, h(\mu,a)\} - (\Pi, P, \theta, l) = -\Pi \cdot (\nabla_\Pi \theta \times \nabla_P h(\mu,a))
$$

$$
- P \cdot (\nabla_\Pi \theta \times \nabla_P h(\mu,a) - \nabla_\Pi h(\mu,a) \times \nabla_P \theta) + \sum_{i=1}^{2} \left( \frac{\partial \theta}{\partial \theta_i} \frac{\partial h(\mu,a)}{\partial l_i} - \frac{\partial h(\mu,a)}{\partial \theta_i} \frac{\partial \theta}{\partial l_i} \right)
$$

$$
= (-\frac{(\Pi_1 - l_1)}{I_1} + \frac{l_1}{J_1} - \frac{(\Pi_2 - l_2)}{I_2} + \frac{l_2}{J_2})
$$
since $\nabla_\Pi \theta = \nabla_P \theta = 0$, $\frac{\partial \theta}{\partial \theta_i} = 1$, $j = i$, $\frac{\partial \theta}{\partial \theta_i} = 0$, $j \neq i$, $\frac{\partial \theta}{\partial \theta_i} = 0$, $i = 1, 2$, and $\frac{\partial h(\mu,a)}{\partial \theta_i} = -\frac{(\Pi_i - l_i)}{l_i} + \frac{l_i}{J_i}$, $i = 1, 2$.

$$
X_{h(\mu,a)}(l)(\Pi, P, \theta, l) = \{l, h(\mu,a)\} - (\Pi, P, \theta, l) = -\Pi \cdot (\nabla_\Pi l \times \nabla_\Pi h(\mu,a))
$$

$$
- P \cdot (\nabla_\Pi l \times \nabla_P h(\mu,a) - \nabla_\Pi h(\mu,a) \times \nabla_P l) + \sum_{i=1}^{2} \left( \frac{\partial l}{\partial \theta_i} \frac{\partial h(\mu,a)}{\partial l_i} - \frac{\partial h(\mu,a)}{\partial \theta_i} \frac{\partial l}{\partial l_i} \right) = (0, 0),
$$
since $\nabla_\Pi l = \nabla_P l = 0$, and $\frac{\partial l}{\partial \theta_i} = \frac{\partial h(\mu,a)}{\partial l_i} = 0$, $i = 1, 2$. If we consider the underwater vehicle-rotors system with a control torque $u : T^*Q \rightarrow C$ acting on the rotors, and $u \in C \subset J^{-1}_Q((\mu,a))$ is invariant under the left SE(3)-action, and its reduced control torque $u_{(\mu,a)} : \mathcal{O}(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathcal{C}(\mu,a)$ is given by $u_{(\mu,a)}(\Pi, P, \theta, l) = \pi_{(\mu,a)}(u(A, c, \Pi, P, \theta, l)) = u(A, c, \Pi, P, \theta, l)|_{\mathcal{O}(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2}$, where $\pi_{(\mu,a)} : J^{-1}_Q((\mu,a)) \rightarrow \mathcal{O}(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2$, $\mathcal{C}(\mu,a) = \pi_{(\mu,a)}(\mathcal{C})$. Moreover, assume that the vertical lift of vector field $X_{h(\mu,a)}$ under the action of $u_{(\mu,a)}$ along fibers vlift$(u_{(\mu,a)})X_{h(\mu,a)} \in T(C(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2)$ is given by

$$
\text{vlift}(u_{(\mu,a)})X_{h(\mu,a)}(\Pi, P, \theta, l) = (\Pi_1, \Pi_P, \Pi_0, \Pi_t) \in T_x(\mathcal{O}(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2),
$$
where $x = (\Pi, P, \theta, l) \in \mathcal{O}(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2$, and $\Pi_1, \Pi_P \in \mathbb{R}^3$, and $\Pi_0, \Pi_t \in \mathbb{R}^2$. Thus, in the case of coincident centers of buoyancy and gravity, from (3.10) we can obtain the equations of
motion for reduced underwater vehicle-rotors system with the control torque \( u \) acting on the rotors, which are given by

\[
\begin{align*}
\frac{d\Pi}{dt} &= \Pi \times \Omega + P \times v + U\Pi, \\
\frac{dP}{dt} &= P \times \Omega + U_P, \\
\frac{d\theta}{dt} &= (\frac{-\Pi_1 - l_1}{I_1} + \frac{l_1}{J_1} - (\frac{\Pi_2 - l_2}{I_2} + \frac{l_2}{J_2}) + U_\theta, \\
\frac{dl_1}{dt} &= U_l.
\end{align*}
\]

(3.13)

To sum up the above discussion, we have the following theorem.

**Theorem 3.4** In the case of coincident centers of buoyancy and gravity, the underwater vehicle-rotors system with the control torque \( u \) acting on the rotors, that is, the 5-tuple \((T^*Q, SE(3), \omega_Q, H, u)\), where \( Q = SE(3) \times S^1 \times S^1 \), is a regular point reducible RCH system. For a point \((\mu, a) \in se^*(3)\), the regular value of the momentum map \( J_Q: SE(3) \times se^*(3) \times \mathbb{R}^3 \times \mathbb{R}^2 \rightarrow se^*(3)\), and \( u \in C \subset J_Q^{-1}(\mu, a) \) is invariant under the left \( SE(3) \)-action, the regular point reduced system is the 4-tuple \((O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2, \omega_{O_{(\mu, a)}}, h_{(\mu, a)}, u_{(\mu, a)})\), where \( O_{(\mu, a)} \subset se^*(3) \) is the coadjoint orbit, \( \omega_{O_{(\mu, a)}}, \mathbb{R}^2 \times \mathbb{R}^2 \) is orbit symplectic form on \( O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \), \( h_{(\mu, a)}(\Pi, P, \theta, l) = \pi_{(\mu, a)}(H(A, c, \Pi, P, \theta, l)) = H(A, c, \Pi, P, \theta, l)|_{O_{(\mu, a)}, \mathbb{R}^2 \times \mathbb{R}^2}, \) and \( u_{(\mu, a)}(\Pi, P, \theta, l) = \pi_{(\mu, a)}(u(A, c, \Pi, P, \theta, l)) = u(A, c, \Pi, P, \theta, l)|_{O_{(\mu, a)}, \mathbb{R}^2 \times \mathbb{R}^2} \), and its equations of motion are given by (3.13).

**Remark 3.5** When the underwater vehicle does not carry any internal rotor, in this case the configuration space is \( Q = W = SE(3) \), the motion of underwater vehicle is just the rotation and translation motion of a rigid body, the above symmetric reduction of underwater vehicle-rotors system is just the Marsden-Weinstein reduction of a heavy top at a regular value of momentum map, and the motion equation (3.13) of reduced underwater vehicle-rotors system becomes the motion equation of a reduced heavy top on a coadjoint orbit of Lie group \( SE(3) \). See Marsden and Ratiu [15].

### 3.3 Symmetric Reduction of Vehicle-Rotors System with Non-coincident Centers

In the following we shall give the regular point reduction of underwater vehicle-rotors system with non-coincident centers of buoyancy and gravity. Because the drift in the direction of gravity breaks the symmetry and the underwater vehicle-rotors system is no longer \( SE(3) \) invariant. In this case, its physical phase space is \( T^*SE(3) \times T^*V \) and the symmetry group is \( S^1 \otimes \mathbb{R}^3 \cong SE(2) \times \mathbb{R} \), regarded as rotations about the third principal axis, that is, the axis of gravity, and translation. By the semidirect product reduction theorem, see Marsden et al. [13] or Leonard and Marsden [10], we know that the reduction of \( T^*SE(3) \) by \( S^1 \otimes \mathbb{R}^3 \) gives a space which is symplectically diffeomorphic to the reduced space obtained by the reduction of \( T^*W = T^*(SE(3) \otimes \mathbb{R}^3) \) by left action of \( W = SE(3) \otimes \mathbb{R}^3 \), that is the coadjoint orbit \( O_{(\mu, a_1, a_2)} \subset se^*(3) \times \mathbb{R}^3 \cong T^*W/W \). In fact, in this case, we can identify the phase space \( T^*SE(3) \) with the reduction of the cotangent bundle of the double semidirect product Lie group \( W = SE(3) \otimes \mathbb{R}^3 = (SO(3) \otimes \mathbb{R}^3) \otimes \mathbb{R}^3 \) by the Euclidean translation subgroup \( \mathbb{R}^3 \) and identifies the symmetry group \( S^1 \otimes \mathbb{R}^3 \) with isotropy group \((G_{a_1})(\mu, a_1, a_2) \otimes \mathbb{R}^3 \), where \( G_{a_1} = \{ A \in SO(3) \mid Aa_1 = a_1 \} = S^1 \), which is Abelian and
In particular, for manifold $\tilde{O}$ of $\mathfrak{so}(3)$, an associated $\mathrm{Ad}^*$-equivariant momentum map $J_Q : T^*Q \cong (\mathfrak{se}^*(3)\mathfrak{\otimes R}^3) \times \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \rightarrow \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ for the left $\mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ action. If $(\Pi, w_1, w_2) \in \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ is a regular value of $J_Q$, then the regular point reduced space $(T^*Q)_{(\Pi, w_1, w_2)} = J_Q^{-1}(\Pi, w_1, w_2)/(\mathfrak{se}^*(3)\mathfrak{\otimes R}^3)_{(\Pi, w_1, w_2)}$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\Pi, w_1, w_2)} \times \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2$, where $(\mathfrak{se}^*(3)\mathfrak{\otimes R}^3)_{(\Pi, w_1, w_2)}$ is the isotropy subgroup of coadjoint $\mathfrak{se}^*(3)\mathfrak{\otimes R}^3$-action at the point $(\Pi, w_1, w_2) \in \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$.

We know that $\mathfrak{w}^* = \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ is a Poisson manifold with respect to its semidirect product Lie-Poisson bracket defined by

$$ \{F, K\}_{\mathfrak{w}^*}(\Pi, P, \Gamma) = -\Pi \cdot (\nabla F \times \nabla K) - P \cdot (\nabla F \times \nabla P K - \nabla K \times \nabla P F) - \Gamma \cdot (\nabla F \times \nabla K - \nabla K \times \nabla F),$$

(3.14) where $F, K \in C^\infty(\mathfrak{w}^*)$, $(\Pi, P, \Gamma) \in \mathfrak{w}^* = \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$. For $(\mu, a_1, a_2) \in \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$, the coadjoint orbit $\mathcal{O}_{(\mu, a_1, a_2)} \subset \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ has the induced orbit symplectic form $\omega_{\mathcal{O}_{(\mu, a_1, a_2)}}$, which is coincide with the restriction of the semidirect product Lie-Poisson bracket on $\mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ to the coadjoint orbit $\mathcal{O}_{(\mu, a_1, a_2)}$, and the coadjoint orbits $(\mathcal{O}_{(\mu, a_1, a_2)}), (\mathfrak{se}^*(3)\mathfrak{\otimes R}^3), (\mu, a_1, a_2) \in \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$, form the symplectic leaves of the Poisson manifold $(\mathfrak{se}^*(3)\mathfrak{\otimes R}^3), (\cdot, \cdot)_{\mathfrak{w}^*}$. Let $\omega_{\mathfrak{w}^*}$ be the canonical symplectic form on $T^*\mathfrak{w}^* \cong \mathfrak{w}^* \times \mathfrak{w}^*$, and it induces a canonical Poisson bracket $(\cdot, \cdot)_{\mathfrak{w}^*}$ on $T^*\mathbb{R}^2$ given by (3.7). Thus, we can induce a symplectic form $\tilde{\omega}_{\mathcal{O}} = \pi_{\mathcal{O}_{(\mu, a_1, a_2)}} \omega_{\mathcal{O}_{(\mu, a_1, a_2)}} + \pi_{\mathbb{R}^2} \omega_{\mathbb{R}^2}$ on the smooth manifold $\mathcal{O} = \mathcal{O}_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^2$, where the maps $\pi_{\mathcal{O}_{(\mu, a_1, a_2)}} : \mathcal{O} \rightarrow \mathcal{O}_{(\mu, a_1, a_2)}$ and $\pi_{\mathbb{R}^2} : \mathcal{O} \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$ are canonical projections, and induce a Poisson bracket $(\cdot, \cdot)_{\mathfrak{w}^*} = (\cdot, \cdot)_{\mathbb{R}^2}$ on the smooth manifold $\mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$, where the maps $\pi_{\mathfrak{w}^*} : \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2 \rightarrow \mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ and $\pi_{\mathbb{R}^2} : \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2 \rightarrow \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2$ are canonical projections, and such that $(\mathcal{O}, \tilde{\omega}_{\mathcal{O}})$ is a symplectic leaf of the Poisson manifold $(\mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2, (\cdot, \cdot)_{\mathfrak{w}^*})$.

From the above expression (2.2) of the Hamiltonian, we know that $H(A, c, b, \Pi, P, \Gamma, \theta, l)$ is invariant under the left $W$-action. For the case that $(\mu, a_1, a_2) \in \mathfrak{w}^*$ is the regular value of $J_Q$, we have the reduced Hamiltonian $h_{(\mu, a_1, a_2)}(\Pi, P, \Gamma, \theta, l) = \tilde{\mathcal{O}} = \mathcal{O}_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^2(\subset \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2) \rightarrow \mathbb{R}$ given by $h_{(\mu, a_1, a_2)}(\Pi, P, \Gamma, \theta, l) = \pi_{(\mu, a_1, a_2)}(H(A, c, b, \Pi, P, \Gamma, \theta, l)) = H(A, c, b, \Pi, \Omega, \Gamma, \theta), \Omega)$. From the semidirect product Lie-Poisson bracket on $\mathfrak{se}^*(3)\mathfrak{\otimes R}^3$ and the Poisson bracket on $T^*\mathbb{R}^2$, we can get the Poisson bracket on $\mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$, that is, for $F, K : \mathfrak{se}^*(3)\mathfrak{\otimes R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, we have that

$$ \{F, K\}_{\mathfrak{se}^*(3)\mathfrak{\otimes R}^3}(\Pi, P, \Gamma, \theta, l) = -\Pi \cdot (\nabla F \times \nabla K) - P \cdot (\nabla F \times \nabla P K - \nabla K \times \nabla P F) - \Gamma \cdot (\nabla F \times \nabla K - \nabla K \times \nabla F) + \{F, K\}_{\mathbb{R}^2}(\theta, l).$$

In particular, for $F_{(\mu, a_1, a_2)}, K_{(\mu, a_1, a_2)} : \tilde{\mathcal{O}} \rightarrow \mathbb{R}$, we have that

$$ \tilde{\omega}_{\tilde{\mathcal{O}}}(X_{F_{(\mu, a_1, a_2)}}, X_{K_{(\mu, a_1, a_2)}}) = \{F_{(\mu, a_1, a_2)}, K_{(\mu, a_1, a_2)}\}_{\tilde{\mathcal{O}}}. $$
Moreover, for reduced Hamiltonian $h_{(\mu, a, 1, 2)}(\Pi, P, \Gamma, \theta, l) : \mathcal{O} \to \mathbb{R}$, we have the Hamiltonian vector field $X_{h_{(\mu, a, 1, 2)}}(K_{(\mu, a, 1, 2)}) = \{K_{(\mu, a, 1, 2)}, h_{(\mu, a, 1, 2)}\} \big|_{\mathcal{O}}$, and hence we have that

$$X_{h_{(\mu, a, 1, 2)}}(\Pi)(\Pi, P, \Gamma, \theta, l) = \{\Pi, h_{(\mu, a, 1, 2)}\} \big|_{\mathcal{O}}$$

$$= -\Pi \cdot (\nabla_{\Pi} \times \nabla_{\Pi} h_{(\mu, a, 1, 2)}) - P \cdot (\nabla_{\Pi} \times \nabla_{P} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} h_{(\mu, a, 1, 2)} \times \nabla_{P} \Pi$$

$$- \Gamma \cdot (\nabla_{\Pi} \times \nabla_{\Gamma} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} \Gamma h_{(\mu, a, 1, 2)} \times \nabla_{\Gamma} \Pi$$

$$= -\nabla_{\Pi} \Pi \cdot (\nabla_{\Pi} h_{(\mu, a, 1, 2)} \times \Pi) - \nabla_{\Pi} \Pi(\nabla_{P} h_{(\mu, a, 1, 2)} \times P) - \nabla_{\Pi} \Pi(\nabla_{\Gamma} h_{(\mu, a, 1, 2)} \times \Gamma)$$

$$= \Pi \times \Omega + P \times v + mgh\chi \times \chi,$$

since $\nabla_{\Pi} \Pi = 1$, $\nabla_{P} \Pi = \nabla_{\Gamma} \Pi = 0$, $\nabla_{\Pi} h_{(\mu, a, 1, 2)} = \Omega$, $\nabla_{P} h_{(\mu, a, 1, 2)} = v$, $\nabla_{\Gamma} h_{(\mu, a, 1, 2)} = mgh\chi$, and $\frac{\partial h}{\partial \theta} = 0$, $i = 1, 2$.

$$X_{h_{(\mu, a, 1, 2)}}(P)(\Pi, P, \Gamma, \theta, l) = \{P, h_{(\mu, a, 1, 2)}\} \big|_{\mathcal{O}}$$

$$= -\Pi \cdot (\nabla_{\Pi} P \times \nabla_{\Pi} h_{(\mu, a, 1, 2)}) - P \cdot (\nabla_{\Pi} P \times \nabla_{P} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} h_{(\mu, a, 1, 2)} \times \nabla_{P} \Pi$$

$$- \Gamma \cdot (\nabla_{\Pi} P \times \nabla_{\Gamma} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} \Gamma h_{(\mu, a, 1, 2)} \times \nabla_{\Gamma} \Pi$$

$$= \nabla_{P} \Pi \cdot (P \times \nabla_{\Pi} h_{(\mu, a, 1, 2)}) = P \times \Omega,$$

since $\nabla_{P} \Pi = 1$, $\nabla_{\Pi} \Pi = \nabla_{\Gamma} \Pi = 0$, $\nabla_{\Pi} h_{(\mu, a, 1, 2)} = \Omega$, and $\frac{\partial P}{\partial \theta} = 0$, $i = 1, 2$.

$$X_{h_{(\mu, a, 1, 2)}}(\Gamma)(\Pi, P, \Gamma, \theta, l) = \{\Gamma, h_{(\mu, a, 1, 2)}\} \big|_{\mathcal{O}}$$

$$= -\Pi \cdot (\nabla_{\Pi} \Gamma \times \nabla_{\Pi} h_{(\mu, a, 1, 2)}) - P \cdot (\nabla_{\Pi} \Gamma \times \nabla_{P} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} h_{(\mu, a, 1, 2)} \times \nabla_{P} \Pi$$

$$- \Gamma \cdot (\nabla_{\Pi} \Gamma \times \nabla_{\Gamma} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} \Gamma h_{(\mu, a, 1, 2)} \times \nabla_{\Gamma} \Pi$$

$$= \nabla_{\Gamma} \Pi \cdot (\Gamma \times \nabla_{\Pi} h_{(\mu, a, 1, 2)}) = \Gamma \times \Omega,$$

since $\nabla_{\Gamma} \Pi = 1$, $\nabla_{P} \Pi = \nabla_{\Pi} \Pi = 0$, $\nabla_{\Pi} h_{(\mu, a, 1, 2)} = \Omega$, and $\frac{\partial \Gamma}{\partial \theta} = 0$, $i = 1, 2$.

$$X_{h_{(\mu, a, 1, 2)}}(\theta)(\Pi, P, \Gamma, \theta, l) = \{\theta, h_{(\mu, a, 1, 2)}\} \big|_{\mathcal{O}}$$

$$= -\Pi \cdot (\nabla_{\Pi} \theta \times \nabla_{\Pi} h_{(\mu, a, 1, 2)}) - P \cdot (\nabla_{\Pi} \theta \times \nabla_{P} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} h_{(\mu, a, 1, 2)} \times \nabla_{P} \theta$$

$$- \Gamma \cdot (\nabla_{\Pi} \theta \times \nabla_{\Gamma} h_{(\mu, a, 1, 2)}) - \nabla_{\Pi} \Gamma h_{(\mu, a, 1, 2)} \times \nabla_{\Gamma} \theta$$

$$= -\left(\frac{\Pi_{1} - l_{1}}{l_{1}}\right) + \frac{l_{1}}{J_{1}} - \left(\frac{\Pi_{2} - l_{2}}{l_{2}}\right) + \frac{l_{2}}{J_{2}},$$

since $\nabla_{\Pi} \theta = \nabla_{P} \theta = \nabla_{\Gamma} \theta = 0$, $\frac{\partial \theta}{\partial \theta} = 0$, $j = i$, $\frac{\partial \theta}{\partial \theta} = 0$, $j \neq i$, $\frac{\partial \theta}{\partial \theta} = 0$, $i = 1, 2$, and $\frac{\partial h_{(\mu, a, 1, 2)}}{\partial \theta} = -\left(\frac{\Pi_{i} - l_{i}}{l_{i}}\right) + \frac{l_{i}}{J_{i}}$, $i = 1, 2$. 

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\[
X_{h(u(a_1,a_2))}(l)(\Pi, P, \Gamma, \theta, l) = \{l, h(u(a_1,a_2))\} - (\Pi, P, \Gamma, \theta, l) \\
= -\Pi \cdot (\nabla_P l \times \nabla h(u(a_1,a_2))) - P \cdot (\nabla_P l \times \nabla h(u(a_1,a_2))) - \nabla h(u(a_1,a_2)) \times \nabla P l \\
- \Gamma \cdot (\nabla_P l \times \nabla h(u(a_1,a_2)) - \nabla h(u(a_1,a_2)) \times \nabla l) + \sum_{i=1}^{2} \left( \frac{\partial h(u(a_1,a_2))}{\partial \theta_i} \frac{\partial l}{\partial l_i} - \frac{\partial h(u(a_1,a_2))}{\partial \theta_i} \frac{\partial l}{\partial l_i} \right) \\
= (0, 0),
\]

since \( \nabla_P l = \nabla_P l = \nabla l = 0 \) and \( \frac{\partial l}{\partial \theta_i} = \frac{\partial h(u(a_1,a_2))}{\partial \theta_i} = 0, \quad i = 1, 2 \). If we consider the underwater vehicle-rotors system with a control torque \( u : T^*Q \rightarrow C \) acting on the rotors, and \( u \in C \subset J_Q^{-1}((\mu, a_1, a_2)) \) is invariant under the left \( SE(3) \otimes \mathbb{R}^3 \)-action, and its reduced control torque \( u(\mu, a_1, a_2) : \tilde{\Omega} \rightarrow \mathcal{C}(\mu, a_1, a_2) \) is given by \( u(\mu, a_1, a_2)(\Pi, P, \Gamma, \theta, l) = \pi_{(\mu, a_1, a_2)}(u(A, c, b, \Pi, P, \Gamma, \theta, l)) = u(A, c, b, \Pi, P, \Gamma, \theta, l) |_{\Omega} \), where \( \pi_{(\mu, a_1, a_2)} : J_Q^{-1}((\mu, a_1, a_2)) \rightarrow \tilde{\Omega}, \quad \mathcal{C}(\mu, a_1, a_2) = \pi_{(\mu, a_1, a_2)}(C) \). Moreover, assume that the vertical lift of vector field \( X_{h(u(a_1,a_2))} \), under the action of \( u(\mu, a_1, a_2) \) along fibers \( vlift(u(\mu, a_1, a_2))X_{h(u(a_1,a_2))} \in T(\mathcal{O}(\mu, a_1, a_2) \times \mathbb{R}^2 \times \mathbb{R}^2) \) is given by

\[
vlift(u(\mu, a_1, a_2))X_{h(u(a_1,a_2))}(\Pi, P, \Gamma, \theta, l) = (\mathcal{U}_1, \mathcal{U}_P, \mathcal{U}_\Gamma, \mathcal{U}_\theta, \mathcal{U}_l) \in T_x(\mathcal{O}(\mu, a_1, a_2) \times \mathbb{R}^2 \times \mathbb{R}^2),
\]

where \( x = (\Pi, P, \Gamma, \theta, l) \in \mathcal{O}(\mu, a_1, a_2) \times \mathbb{R}^2 \times \mathbb{R}^2 \), and \( \mathcal{U}_1, \mathcal{U}_P, \mathcal{U}_\Gamma \in \mathbb{R}^2 \), and \( \mathcal{U}_\theta, \mathcal{U}_l \in \mathbb{R}^2 \). Thus, in the case of non-coincident centers of buoyancy and gravity, from (3.10) we can obtain the equations of motion for reduced underwater vehicle-rotors system with the control torque \( u \) acting on the rotors, which are given by

\[
\begin{align*}
\frac{d\Pi}{dt} &= \Pi \times \Omega + P \times v + \frac{mgh}{\Gamma} \times \mathcal{U}_1, \\
\frac{dP}{dt} &= P \times \Omega + \mathcal{U}_P, \\
\frac{d\Gamma}{dt} &= \Gamma \times \Omega + \mathcal{U}_\Gamma, \\
\frac{d\theta}{dt} &= \left( -\frac{(\Pi_1 - l_1)}{I_1} + \frac{l_1}{J_1} \right) - \left( \frac{(\Pi_2 - l_2)}{I_2} + \frac{l_2}{J_2} \right) + \mathcal{U}_\theta, \\
\frac{dl}{dt} &= \mathcal{U}_l, 
\end{align*}
\]

(3.15)

To sum up the above discussion, we have the following theorem.

**Theorem 3.6** In the case of non-coincident centers of buoyancy and gravity, the underwater vehicle-rotors system with the control torque \( u \) acting on the rotors, that is, the 5-tuple \( (T^*Q, SE(3) \otimes \mathbb{R}^3, \omega_Q, H, u) \), where \( Q = SE(3) \otimes \mathbb{R}^3 \times S^1 \times S^1 \), is a regular point reducible by stages RCH system. For a point \( (\mu, a_1, a_2) \in se^*(3) \otimes \mathbb{R}^3 \), the regular value of the momentum map \( J_Q : SE(3) \otimes \mathbb{R}^3 \rightarrow se^*(3) \otimes \mathbb{R}^3 \), and \( u \in C \subset J_Q^{-1}((\mu, a_1, a_2)) \) is invariant under the left \( SE(3) \otimes \mathbb{R}^3 \)-action, the regular point reduced by stages system is the 4-tuple \( (\tilde{\Omega} = \mathcal{O}(\mu, a_1, a_2) \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega} = h(\mu, a_1, a_2), h(\mu, a_1, a_2), u(\mu, a_1, a_2)), \) where \( \mathcal{O}(\mu, a_1, a_2) \subset se^*(3) \otimes \mathbb{R}^3 \) is the coadjoint orbit, \( \tilde{\omega} \) is orbit symplectic form on \( \mathcal{O}(\mu, a_1, a_2) \times \mathbb{R}^2 \times \mathbb{R}^2, h(\mu, a_1, a_2)(\Pi, P, \Gamma, \theta, l) = \pi_{(\mu, a_1, a_2)}(H(A, c, b, \Pi, P, \Gamma, \theta, l)) = H(A, c, b, \Pi, P, \Gamma, \theta, l) |_{\Omega}, \quad u(\mu, a_1, a_2)(\Pi, P, \Gamma, \theta, l) = \pi_{(\mu, a_1, a_2)}(u(A, c, b, \Pi, P, \Gamma, \theta, l)) = u(A, c, b, \Pi, P, \Gamma, \theta, l) |_{\Omega}, \) and its equations of motion are given by (3.15).
Remark 3.7 When the underwater vehicle does not carry any internal rotor, in this case the configuration space is $Q = W = SE(3) \otimes \mathbb{R}^3$, the motion of underwater vehicle is just the rotation and translation motion with drift of a rigid body, the above symmetric reduction of underwater vehicle-rotors system is just the Marsden-Weinstein reduction by stages at a regular value of momentum map, and the motion equation (3.15) of reduced underwater vehicle-rotors system becomes the equation on a coadjoint orbit of the semidirect product Lie group $SE(3) \otimes \mathbb{R}^3$. See Leonard and Marsden [10], Marsden et al. [13], Marsden and Ratiu [15].

4 Hamilton-Jacobi Equation of Underwater Vehicle with Internal Rotors

It is well-known that Hamilton-Jacobi theory provides a characterization of the generating functions of certain time-dependent canonical transformations, such that a given Hamiltonian system in such a form that its solutions are extremely easy to find by reduction to the equilibrium, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [15]. In general, we know that it is not easy to find the solutions of Hamilton’s equation. But, if we can get a solution of Hamilton-Jacobi equation of the Hamiltonian system, by using the relationship between Hamilton’s equation and Hamilton-Jacobi equation, it is easy to give a special solution of Hamilton’s equation. Thus, it is very important to give explicitly the Hamilton-Jacobi equation of a Hamiltonian system. In this section, as an application of the theoretical result of Hamilton-Jacobi theory of regular reduced RCH system given by Wang [26], we first give the Hamilton-Jacobi theorem of RCH system with symmetry on the generalization of a semidirect product Lie group. Then we give explicitly the Hamilton-Jacobi equations of the reduced underwater vehicle-rotors systems on the symplectic leaves by calculation in detail, respectively, in the cases of coincident and non-coincident centers of buoyancy and gravity.

4.1 H-J Theorem on the Generalization of Semidirect Product Lie Group

In order to describe the Hamilton-Jacobi Equation of underwater vehicle with two internal rotors, we need to first give the Hamilton-Jacobi theorem of regular point reducible by stages RCH system with symmetry on the generalization of a semidirect product Lie group $Q = W \times V$, where $W = G \otimes E$ is a semidirect product Lie group with Lie algebra $w = g \otimes E$. From §3.1 we know that if the Hamiltonian $H(g, x, \zeta, p, \theta, l) : T^*Q \cong W \times \mathfrak{w}^* \times V \times V^* \to \mathbb{R}$ is left cotangent lifted $W$-action $\Phi^{T^*}$ invariant, for regular value $(\mu, a) \in \mathfrak{w}^*$ we have the associated regular point reduced by stages Hamiltonian $h_{(\mu, a)}(\zeta, p, \theta, l) : (T^*Q)_{(\mu, a)} \cong \mathcal{O}_{(\mu, a)} \times V \times V^* \to \mathbb{R}$, defined by $h_{(\mu, a)} \cdot \pi_{(\mu, a)} = H \cdot i_{(\mu, a)}$, and the associated reduced Hamiltonian vector field $X_{h_{(\mu, a)}}$ given by $X_{h_{(\mu, a)}}(K_{(\mu, a)}) = \{K_{(\mu, a)}, h_{(\mu, a)}\}$. Thus, if the fiber-preserving map $\bar{F} : T^*Q \to T^*Q$ and the fiber submanifold $\mathcal{C}$ of $T^*Q$ are all left cotangent lifted $W$-action $\Phi^{T^*}$ invariant, then the 6-tuple $(T^*Q, W, \omega_Q, H, F, \mathcal{C})$ is a regular point reducible by stages RCH system. For a point $(\mu, a) \in \mathfrak{w}^*$, the regular value of the momentum map $J_Q : T^*Q \to \mathfrak{w}^*$, the regular point reduced by stages system is the 5-tuple $(\mathcal{O}_{(\mu, a)} \times V \times V^*, \mathcal{C}_{(\mu, a)} \subset \mathcal{O}_{(\mu, a)} \times V \times V^*, h_{(\mu, a)} \cdot \pi_{(\mu, a)} = u_{(\mu, a)} \cdot \pi_{(\mu, a)} = u \cdot i_{(\mu, a)}$.

Moreover, assume that the map $\pi_{\mathcal{C}} : T^*Q \to \mathcal{C}$ induces the map $\tilde{T} \pi_{\mathcal{C}} : TT^*Q \to \mathcal{T}Q$, and $\gamma : Q \to T^*Q$ is an one-form on $Q$, and it is closed with respect to $T \pi_{\mathcal{C}} : TT^*Q \to \mathcal{T}Q$, and $\lambda = \gamma \cdot \pi_{\mathcal{C}} : T^*Q \to T^*Q$ is symplectic. Moreover, assume that $\text{Im}(\gamma) \subset J_Q^{-1}(\mu, a)$, and it is $W_{(\mu, a)}$-invariant. Denote by $\tilde{\gamma} = \pi_{(\mu, a)}(\gamma) : Q \to \mathcal{O}_{(\mu, a)} \times V \times V^*$, and $\tilde{\lambda} = \pi_{(\mu, a)}(\lambda) : T^*Q \to \mathcal{O}_{(\mu, a)} \times V \times V^*$. From (3.10) we know that the dynamical vector field of the reduced by stages
RCH system can be expressed by

\[ X_{(\mathcal{O}_{(\mu,a)} \times V \times V^*, \tilde{\omega}_{(\mu,a)} \times V \times V^*, h_{(\mu,a)} \times f_{(\mu,a)} \times u_{(\mu,a)})} = X_{h_{(\mu,a)}} + \text{vlift}(f_{(\mu,a)}) + \text{vlift}(u_{(\mu,a)}). \]

By using the same way in the proof of Hamilton-Jacobi theorem for the regular point reduced RCH system, see Wang [26], we can get the following theorem.

**Theorem 4.1** For the regular point reducible by stages RCH system \( (T^*Q, W, \omega, H, F, C) \) on the generalization of a semidirect product Lie group \( Q = W \times V, \) where \( W = G \mathbb{S} E \) is a semidirect product Lie group with Lie algebra \( \mathfrak{w} = \mathfrak{g} \mathbb{S} \mathfrak{e}, \) \( G \) is a Lie group with Lie algebra \( \mathfrak{g}, \) \( E \) is a \( r \)-dimensional vector space and \( V \) is a \( k \)-dimensional vector space. Assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q, \) and it is closed with respect to \( T \pi_Q : T^*Q \to TQ, \) and \( \lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q \) is symplectic, and \( \tilde{X}^\lambda = T \pi_Q \cdot \tilde{X} \cdot \lambda, \) where \( \tilde{X} = X_{(T^*Q, W, \omega', H, F, a)} \) is the dynamical vector field of the regular point reducible by stages RCH system \( (T^*Q, W, \omega, H, F, C) \) with a control law \( u \in C. \) Moreover, assume that \( (\mu, a) \in \mathfrak{w}^* \) is the regular value of the momentum map \( J_Q : T^*Q \to \mathfrak{w}^*, \) and \( u \in C \subset J_Q^{-1}(\mu, a)) \) is invariant under the left \( W \)-action, and \( \text{Im}(\gamma) \subset J_Q^{-1}(\mu, a), \) and it is \( W_{(\mu,a)} \)-invariant. Denote by \( \bar{\gamma} = \pi_{(\mu,a)}(\gamma) : Q \to \mathcal{O}_{(\mu,a)} \times V \times V^*, \) and \( \bar{\lambda} = \pi_{(\mu,a)}(\lambda) : T^*Q \to \mathcal{O}_{(\mu,a)} \times V \times V^*. \) Then the following two assertions are equivalent: (i) \( T \bar{\gamma} \cdot \tilde{X}^\lambda = \tilde{X}_{(\mu,a)} \cdot \bar{\lambda}, \) where \( \tilde{X}_{(\mu,a)} = X_{(\mathcal{O}_{(\mu,a)} \times V \times V^*, \tilde{\omega}_{(\mu,a)} \times V \times V^*, h_{(\mu,a)} \times f_{(\mu,a)} \times u_{(\mu,a)})} \) is the dynamical vector field of regular point reduced by stages RCH system \( (\mathcal{O}_{(\mu,a)} \times V \times V^*, \tilde{\omega}_{(\mu,a)} \times V \times V^*, h_{(\mu,a)} \times f_{(\mu,a)} \times u_{(\mu,a)}) \); (ii) \( X_{h_{(\mu,a)}}, \bar{\lambda} + \text{vlift}(f_{(\mu,a)} \cdot \bar{\lambda}) + \text{vlift}(u_{(\mu,a)} \cdot \bar{\lambda}) = 0. \) Moreover, \( \bar{\lambda} \) is a solution of the Hamilton-Jacobi equation \( X_{h_{(\mu,a)}}, \bar{\lambda} + \text{vlift}(f_{(\mu,a)} \cdot \bar{\lambda}) + \text{vlift}(u_{(\mu,a)} \cdot \bar{\lambda}) = 0, \) if and only if \( \bar{\lambda} \) is a solution of the Hamilton-Jacobi equation \( X_{h_{(\mu,a)}}, \bar{\lambda} + \text{vlift}(f_{(\mu,a)} \cdot \bar{\lambda}) + \text{vlift}(u_{(\mu,a)} \cdot \bar{\lambda}) = 0. \)

In particular, when \( Q = W, \) we can obtain the Hamilton-Jacobi theorem for the regular point reduced by stages RCH system on semidirect product Lie group \( W. \) In this case, note that the symplectic structure on the coadjoint orbit \( \mathcal{O}_{(\mu,a)} \) is induced by the (-)-semidirect product Lie-Poisson brackets on \( \mathfrak{w}^*, \) then the Hamilton-Jacobi equation \( X_{h_{(\mu,a)}}, \bar{\lambda} + \text{vlift}(f_{(\mu,a)} \cdot \bar{\lambda}) + \text{vlift}(u_{(\mu,a)} \cdot \bar{\lambda}) = 0 \) for regular point reduced by stages RCH system \( (\mathcal{O}_{(\mu,a)} \times \tilde{\omega}_{(\mu,a)} \times h_{(\mu,a)} \times f_{(\mu,a)} \times u_{(\mu,a)}) \) is also called Lie-Poisson Hamilton-Jacobi equation. See Wang [25], Marsden and Ratiu [15], and Ge and Marsden [5].

As an application of the theoretical result, in the following we consider the underwater vehicle with two internal rotors as a regular point reducible by stages RCH system on the generalizations of semidirect product Lie groups \( (\text{SO}(3) \mathbb{S} \mathbb{R}^3) \times S^1 \times S^1, \) and \( (\text{SE}(3) \mathbb{S} \mathbb{R}^3) \times S^1 \times S^1, \) respectively, and give explicitly the Hamilton-Jacobi equations of their reduced RCH systems on the symplectic leaves by calculation in detail, which show the effect on controls in Hamilton-Jacobi theory. We shall follow the notations and conventions introduced in Leonard and Marsden [10], Marsden et al. [13], Marsden and Ratiu [15], Marsden [12], Marsden et al. [18], and Wang [26].

### 4.2 H-J Equation of Vehicle-Rotors System with Coincident Centers

In the following we first give the Hamilton-Jacobi equation for regular point reduced underwater vehicle-rotors system with coincident centers of buoyancy and gravity. From the expression (2.1) of the Hamiltonian, we know that \( H(A, c, \Pi, P, \theta, l) \) is invariant under the left \( \text{SE}(3)\)-action. For the case \( (\mu, a) \in \text{se}^*(3) \) is the regular value of \( J_Q, \) we have the reduced
Hamiltonian $h_{(\mu,a)}(P,\theta,l) : \mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, which is given by $h_{(\mu,a)}(P,\theta,l) = \pi_{(\mu,a)}(H(A,c,\Pi,\theta,l)) = H(A,c,\Pi,\theta,l)|_{\mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2}$, and we have the reduced Hamiltonian vector field $X_{h_{(\mu,a)}}(K_{(\mu,a)}) = \{K_{(\mu,a)}, h_{(\mu,a)}\}|_{\mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2}$.

Assume that $\gamma : SE(3) \times S^1 \times S^1 \to T^*(SE(3) \times S^1 \times S^1)$ is an one-form on $SE(3) \times S^1 \times S^1$, and it is closed with respect to $T\pi_Q : T^*(SE(3) \times S^1 \times S^1) \to T(SE(3) \times S^1 \times S^1)$, and $\lambda = \gamma \cdot \pi_Q : T^*(SE(3) \times S^1 \times S^1) \to T^*(SE(3) \times S^1 \times S^1)$ is symplectic, and $\text{Im}(\gamma) \subset T^{-1}_Q((\mu,a))$, and it is $SE(3)_{(\mu,a)}$-invariant. Denote by $\gamma = \pi_{(\mu,a)}(\gamma) : SE(3) \times S^1 \times S^1 \to \mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2$, and $\lambda = \pi_{(\mu,a)}(\lambda) : T^*(SE(3) \times S^1 \times S^1) \to \mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2$. Moreover, assume that $\lambda(A,c,\Pi,\theta,l) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10})(A,c,\Pi,\theta,l) \in \mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2(\subset \text{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2)$, that is, $\Pi = (\Pi_1, \Pi_2, \Pi_3) = (\lambda_1, \lambda_2, \lambda_3)$, $P = (P_1, P_2, P_3) = (\lambda_4, \lambda_5, \lambda_6)$, $\theta = (\theta_1, \theta_2) = (\lambda_7, \lambda_8)$ and $l = (l_1, l_2) = (\lambda_9, \lambda_{10})$. Then $h_{(\mu,a)} \cdot \lambda : T^*(SE(3) \times S^1 \times S^1) \to \mathbb{R}$ is given by

\[
\lambda(A,c,\Pi,\theta,l) = \frac{1}{2} \left( \frac{\lambda_1 - \lambda_9}{I_1} + \frac{\lambda_2 - \lambda_{10}}{I_2} + \frac{\lambda_3}{I_3} + \frac{\lambda_4}{m_1} + \frac{\lambda_5}{m_2} + \frac{\lambda_6}{m_3} + \frac{\lambda_7}{J_1} + \frac{\lambda_8}{J_2} \right),
\]

and the vector field

\[
T\lambda X_{h_{(\mu,a)}} \cdot \lambda(\Pi) = X_{h_{(\mu,a)}}(\Pi) \cdot \lambda = \{\Pi, h_{(\mu,a)}\}|_{\mathcal{O}_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2} \cdot \lambda(A,c,\Pi,\theta,l)
\]

\[
= -\Pi \cdot \left( \nabla_{\Pi} \nabla_{\Pi}(h_{(\mu,a)}) \cdot \lambda - P \cdot \left( \nabla_{\Pi} \nabla_{P}(h_{(\mu,a)}) - \nabla_{\Pi}(h_{(\mu,a)}) \right) \right) \cdot \lambda
\]

\[
+ \frac{2}{\partial_i} \left( \frac{\partial h_{(\mu,a)}}{\partial \Pi_i} - \frac{\partial h_{(\mu,a)}}{\partial \Pi_j} \right) \cdot \lambda
\]

\[
= \left( \Pi_1, \Pi_2, \Pi_3 \right) \times \left( \frac{\Pi_1 - l_1}{I_1}, \frac{\Pi_2 - l_2}{I_2}, \frac{\Pi_3}{I_3} \right) \cdot \lambda + \left( P_1, P_2, P_3 \right) \times \left( \frac{P_1}{m_1}, \frac{P_2}{m_2}, \frac{P_3}{m_3} \right) \cdot \lambda
\]

\[
= \left( \frac{I_2 - I_3}{I_2 I_3} I_3 \lambda_2 \lambda_3 + \frac{I_3}{I_2 I_3} I_3 \lambda_3 \lambda_1 + \frac{(m_2 - m_3) I_3 \lambda_5 \lambda_6}{m_2 m_3} \right)
\]

\[
+ \left( \frac{I_3 - I_1}{I_3 I_1} I_1 \lambda_3 \lambda_1 + \frac{I_1}{I_3 I_1} I_1 \lambda_1 \lambda_3 + \frac{(m_3 - m_1) I_3 \lambda_6 \lambda_4}{m_3 m_1} \right)
\]

\[
+ \left( \frac{I_1 - I_2}{I_1 I_2} I_2 \lambda_2 \lambda_2 + \frac{I_2}{I_1 I_2} I_2 \lambda_2 \lambda_4 + \frac{(m_1 - m_2) I_1 \lambda_4 \lambda_5}{m_1 m_2} \right),
\]

since $\nabla_{\Pi} \Pi = 1$, $\nabla_{P} \Pi = 0$, $\frac{\partial \Pi_i}{\partial \theta_j} = \frac{\partial \Pi_i}{\partial \Pi_j} = 0$, $i = 1, 2$, $P = (P_1, P_2, P_3)$, $\nabla_{P_j}(h_{(\mu,a)}) = \frac{\Pi_j}{m_j}$, $j = 1, 2, 3$, and $\nabla_{\Pi_i}(h_{(\mu,a)}) = \left( \Pi_i - l_i \right)/I_i$, $i = 1, 2$, $\nabla_{\Pi_3}(h_{(\mu,a)}) = \Pi_3/I_3$.
\[ T \lambda X_{h(\mu,a)}(\lambda(P)) = X_{h(\mu,a)}(P) \cdot \lambda = \{ P, h(\mu,a) \} \cdot \lambda = \{ \lambda, h(\mu,a) \} \cdot \lambda(A, c, \Pi, P, \theta, l) \\
= -\Pi \cdot (\nabla P \cdot \nabla h(\mu,a)) \cdot \lambda - \lambda \cdot (\nabla P \cdot \nabla h(\mu,a)) - \nabla h(\mu,a) \cdot \lambda \]
\[ + \{ P, h(\mu,a) \} \cdot \lambda = \frac{1}{I_1} \left( \frac{\nabla}{\partial l_i} \lambda_i \right) \]

since \( \nabla P = 1, \nabla P = 0, \frac{\partial P}{\partial \theta_i} = \frac{\partial P}{\partial \theta_i} = 0, i = 1, 2. \]

\[ T \lambda X_{h(\mu,a)}(\lambda(\theta)) = X_{h(\mu,a)}(\theta) \cdot \lambda \]
\[ = \{ \theta, h(\mu,a) \} \cdot \lambda = \{ \theta, \lambda \} \cdot \lambda(A, c, \Pi, P, \theta, l) \\
= -\Pi \cdot (\nabla \theta \cdot \nabla h(\mu,a)) \cdot \lambda - \lambda \cdot (\nabla \theta \cdot \nabla h(\mu,a)) - \nabla h(\mu,a) \cdot \lambda \]
\[ + \{ \theta, h(\mu,a) \} \cdot \lambda = \frac{1}{I_1} \left( \frac{\nabla}{\partial l_i} \lambda_i \right) \]

since \( \nabla \theta = \nabla \theta = 0, \frac{\partial \theta_i}{\partial \theta_i} = 1, j = i, \frac{\partial \theta_i}{\partial \theta_i} = 0, j \neq i, \frac{\partial \theta_i}{\partial \theta_i} = 0, i = 1, 2. \]

\[ T \lambda X_{h(\mu,a)}(\lambda(l)) = X_{h(\mu,a)}(l) \cdot \lambda = \{ l, h(\mu,a) \} \cdot \lambda = \{ l, \lambda \} \cdot \lambda(A, c, \Pi, P, \theta, l) \\
= -\Pi \cdot (\nabla l \cdot \nabla h(\mu,a)) \cdot \lambda - \lambda \cdot (\nabla l \cdot \nabla h(\mu,a)) - \nabla h(\mu,a) \cdot \lambda \]
\[ + \{ l, h(\mu,a) \} \cdot \lambda = \frac{1}{I_1} \left( \frac{\nabla}{\partial l_i} \lambda_i \right) \]

since \( \nabla l = \nabla l = 0, \frac{\partial l_i}{\partial l_i} = 0, \frac{\partial h(\mu,a)}{\partial l_i} = 0, i = 1, 2. \]
\( \mathbb{R}^2 \times \mathbb{R}^2 \), where \( x = (\Pi, P, \theta, l) \in \mathcal{O}_{(\mu, a)} \) \( \times \mathbb{R}^2 \times \mathbb{R}^2 \), and \( U_i(A, c, \Pi, P, \theta, l) : T^* (SE(3) \times S^1 \times S^1) \to \mathbb{R}, \ i = 1, 2, \cdots , 10 \), then from the Hamilton-Jacobi equations \( X_{h_{(\mu, a)}} \cdot \lambda + \text{vlift}(u_{(\mu, a)} \cdot \lambda) = 0 \), we have that

\[
0 = T \lambda (X_{h_{(\mu, a)}} \cdot \lambda + \text{vlift}(u_{(\mu, a)} \cdot \lambda)) = X_{h_{(\mu, a)}} \cdot \lambda + \text{vlift}(u_{(\mu, a)} \cdot \lambda) = X_{(\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2, \omega_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}, h_{(\mu, a), \mu, a})} \cdot \lambda,
\]

that is, the dynamical vector field along \( \lambda \) of regular point reduced underwater vehicle-rotors system \( (\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2, \omega_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}, h_{(\mu, a), \mu, a}) \) is its equilibrium. Thus, in the case of coincident centers of buoyancy and gravity, the Hamilton-Jacobi equations for underwater vehicle-rotors system with the control torque \( u \) acting on the rotors are given by

\[
\begin{align*}
(I_2 - I_3) & \bar{\lambda}_2 \bar{\lambda}_3 + I_3 \bar{\lambda}_3 \bar{\lambda}_{10} + \frac{(m_2 - m_3) \bar{\lambda}_5 \bar{\lambda}_6}{m_2 m_3} + U_1 = 0, \\
(I_3 - I_1) & \bar{\lambda}_3 \bar{\lambda}_1 - I_3 \bar{\lambda}_3 \bar{\lambda}_9 + \frac{(m_3 - m_1) \bar{\lambda}_6 \bar{\lambda}_4}{m_3 m_1} + U_2 = 0, \\
(I_1 - I_2) & \bar{\lambda}_1 \bar{\lambda}_2 - I_2 \bar{\lambda}_2 \bar{\lambda}_9 + \frac{(m_1 - m_2) \bar{\lambda}_4 \bar{\lambda}_5}{m_1 m_2} + U_3 = 0, \\
I_2 & \bar{\lambda}_5 \bar{\lambda}_3 - I_3 \bar{\lambda}_6 (\bar{\lambda}_2 - \bar{\lambda}_{10}) + U_4 = 0, \\
I_3 \bar{\lambda}_6 (\bar{\lambda}_1 - \bar{\lambda}_9) - I_1 \bar{\lambda}_4 \bar{\lambda}_2 + U_5 = 0, \\
I_1 \bar{\lambda}_4 (\bar{\lambda}_2 - \bar{\lambda}_{10}) - I_2 \bar{\lambda}_5 (\bar{\lambda}_1 - \bar{\lambda}_9) + U_6 = 0, \\
\frac{\bar{\lambda}_1 - \bar{\lambda}_9}{I_1} & + \frac{\bar{\lambda}_9}{J_1} + U_7 = 0, \\
\frac{\bar{\lambda}_2 - \bar{\lambda}_{10}}{I_2} & + \frac{\bar{\lambda}_{10}}{J_2} + U_8 = 0, \\
U_9 & = U_{10} = 0.
\end{align*}
\]

To sum up the above discussion, we have the following theorem.

**Theorem 4.2** In the case of coincident centers of buoyancy and gravity, for a point \( (\mu, a) \in \mathfrak{se}^*(3) \), the regular value of the momentum map \( J_Q : SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{se}^*(3) \), and \( u \in C \subset J_Q^{-1}((\mu, a)) \) is invariant under the left \( SE(3) \)-action, the regular point reduced system of underwater vehicle-rotors system with the control torque \( u \) acting on the rotors \( (T^* Q, SE(3), \omega_Q, H, u) \), where \( Q = SE(3) \times S^1 \times S^1 \), is the 4-tuple \( (\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2, \omega_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}, h_{(\mu, a), \mu, a}) \), where \( \mathcal{O}_{(\mu, a)} \subset \mathfrak{se}^*(3) \) is the coadjoint orbit, \( \omega_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2} \) is orbit symplectic form on \( \mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \), \( h_{(\mu, a)}(\Pi, P, \theta, l) = \pi_{(\mu, a)}(H(A, c, \Pi, P, \theta, l)) = H(A, c, \Pi, P, \theta, l)|_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}, \) \( \text{and} \ u_{(\mu, a)}(\Pi, P, \theta, l) = \pi_{(\mu, a)}(u(A, c, \Pi, P, \theta, l)) = u(A, c, \Pi, P, \theta, l)|_{\mathcal{O}_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2} \). Assume that \( \gamma : SE(3) \times S^1 \times S^1 \to T^* (SE(3) \times S^1 \times S^1) \) is an one-form on \( SE(3) \times S^1 \times S^1 \), and it is closed with respect to \( T \pi_Q : T T^* (SE(3) \times S^1 \times S^1) \to T(SE(3) \times S^1 \times S^1) \), and \( \lambda = \gamma \cdot \pi_Q : T^* (SE(3) \times S^1 \times S^1) \to T^* (SE(3) \times S^1 \times S^1) \) is symplectic, and \( \text{Im}(\gamma) \subset J_Q^{-1}((\mu, a)) \), and it is \( SE(3)_{(\mu, a)} \)-invariant, where \( SE(3)_{(\mu, a)} \) is the isotropy subgroup of coadjoint \( SE(3) \)-action at the point \( (\mu, a) \in \mathfrak{se}^*(3) \).
Denote by \( \tilde{\gamma} = \pi_{(\mu,a)}(\gamma) : SE(3) \times S^1 \times S^1 \to O_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \), and \( \tilde{\lambda} = \pi_{(\mu,a)}(\lambda) : T^*(SE(3) \times S^1 \times S^1) \to O_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \). Then \( \tilde{\lambda} \) is a solution of the Hamilton-Jacobi equation of reduced underwater vehicle-rotors system given by (4.1), if and only if \( T\tilde{\gamma} \cdot \tilde{X}^\lambda = \tilde{X}_{(\mu,a)} \cdot \tilde{\lambda} \), where \( \tilde{X}^\lambda = T\pi_Q \cdot \tilde{X} \cdot \lambda \), \( \tilde{X} = X_{(T\pi_Q,SE(3),\omega_Q,H,a)} \), and \( \tilde{X}_{(\mu,a)} = X_{(O_{(\mu,a)} \times \mathbb{R}^2 \times \mathbb{R}^2,\omega_{O_{(\mu,a)}},h_{(\mu,a)} \circ \mu_{(\mu,a)})} \).

**Remark 4.3** When the underwater vehicle does not carry any internal rotor, in this case the configuration space is \( Q = W = SE(3) \), the above Hamilton-Jacobi equation (4.1) of reduced underwater vehicle-rotors system is just the Lie-Poisson Hamilton-Jacobi equation of Marsden-Weinstein reduced Hamiltonian system \( (O_{(\mu,a)},\omega_{O_{(\mu,a)}} h_{(\mu,a)}) \) on the coadjoint orbit of semidirect product Lie group \( SE(3) = SO(3) \circledast \mathbb{R}^3 \), since the symplectic structure on the coadjoint orbit \( O_{(\mu,a)} \) is induced by the \((-)\)-semidirect product Lie-Poisson brackets on \( \text{se}^*(3) \). See Marsden and Ratiu [15], Ge and Marsden [5], and Wang [25].

### 4.3 H-J Equation of Vehicle-Rotors System with Non-coincident Centers

In the following we shall give the Hamilton-Jacobi equation for regular point reduced underwater vehicle-rotors system with non-coincident centers of buoyancy and gravity. From the expression (2.2) of the Hamiltonian, we know that \( H(A,c,b,P,\Gamma,\theta,l) \) is invariant under the left \( SE(3) \circledast \mathbb{R}^3 \)-action. For the case \( (\mu,a_1,a_2) \in \text{se}^*(3) \circledast \mathbb{R}^3 \) is the regular value of \( J_Q \), we have the regular point reduced by stages Hamiltonian \( h_{(\mu,a_1,a_2)}(P,\Gamma,\theta,l) : \tilde{O} \to \mathbb{R} \), which is given by \( h_{(\mu,a_1,a_2)}(P,\Gamma,\theta,l) = \pi_{(\mu,a_1,a_2)}(H(A,c,b,P,\Gamma,\theta,l)) = H(A,c,b,P,\Gamma,\theta,l) \mid_{\tilde{O}} \), and we have the associated reduced Hamiltonian vector field \( X_{h_{(\mu,a_1,a_2)}}(K_{(\mu,a_1,a_2)}) \equiv \{ K_{(\mu,a_1,a_2)}, h_{(\mu,a_1,a_2)} \} \mid_{\tilde{O}} \).

Assume that \( \gamma : SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1 \to T^*(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \) is an one-form on \( SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1 \), and it is closed with respect to \( T\pi_Q : TT^*(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \to T(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \), and \( \lambda = \gamma \cdot \pi_Q : T^*(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \to T^*(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \) is symplectic, and \( \text{Im}(\gamma) \subset J_Q^{-1}(\mu,a_1,a_2)) \), and it is \( (SE(3) \circledast \mathbb{R}^3)(\mu,a_1,a_2)) \)-invariant. Denote by \( \tilde{\gamma} = \pi_{(\mu,a_1,a_2)}(\gamma) : SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1 \to \tilde{O} \), and \( \tilde{\lambda} = \pi_{(\mu,a_1,a_2)}(\lambda) : T^*(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \to \tilde{O} \). Moreover, denote by \( \lambda(A,c,b,P,\Gamma,\theta,l) = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13}) \) \( (A,c,b,\Pi,P,\Gamma,\theta,l) \in \tilde{O} \), that is, \( \Pi = (\Pi_1,\Pi_2,\Pi_3) = (\lambda_1,\lambda_2,\lambda_3), P = (P_1,P_2,P_3) = (\lambda_4,\lambda_5,\lambda_6), \Gamma = (\Gamma_1,\Gamma_2,\Gamma_3) = (\lambda_7,\lambda_8,\lambda_9), \theta = (\theta_1,\theta_2) = (\lambda_{10},\lambda_{11}) \) and \( l = (l_1,l_2) = (\lambda_{12},\lambda_{13}) \). Then \( h_{(\mu,a_1,a_2)} \cdot \lambda : T^*(SE(3) \circledast \mathbb{R}^3 \times S^1 \times S^1) \to \mathbb{R} \) is given by

\[
\begin{align*}
&h_{(\mu,a_1,a_2)} \cdot \tilde{\lambda}(A,c,b,P,\Gamma,\theta,l) = H|_{\tilde{O}} \cdot \tilde{\lambda}(A,c,b,P,\Gamma,\theta,l) \\
&= \frac{1}{2} \left[ \frac{\tilde{\lambda}_1^2}{I_1} + \frac{\tilde{\lambda}_2^2}{I_2} + \frac{\tilde{\lambda}_3^2}{I_3} + \frac{\tilde{\lambda}_4^2}{m_1} + \frac{\tilde{\lambda}_5^2}{m_2} + \frac{\tilde{\lambda}_6^2}{m_3} + \frac{\tilde{\lambda}_7^2}{J_1} + \frac{\tilde{\lambda}_8^2}{J_2} \right] \\
&\quad + mg \tilde{\lambda}_2 \cdot \chi_1 + \tilde{\lambda}_8 \cdot \chi_2 + \tilde{\lambda}_9 \cdot \chi_3,
\end{align*}
\]
and the vector field

\[ T\bar{\lambda}X_{h_{(\mu, a_1, a_2)}}(\Pi) = X_{h_{(\mu, a_1, a_2)}}(\Pi) \cdot \bar{\lambda} = \{\Pi, h_{(\mu, a_1, a_2)}\} - |\bar{\lambda}(A, c, b, \Pi, P, \Gamma, \theta, l)| \]

\[ \nabla_{\Pi} P \times \nabla P(h_{(\mu, a_1, a_2)}) \cdot \bar{\lambda} - P \cdot (\nabla_{\Pi} P \times \nabla P(h_{(\mu, a_1, a_2)}) - \nabla_{\Pi}(h_{(\mu, a_1, a_2)}) \times \nabla P \Pi) \cdot \bar{\lambda} \]

\[ = \frac{\Pi}{\Pi_1 - l_1} \frac{l_2}{l_2 - l_2} \frac{l_3}{l_3 - l_3} \cdot \bar{\lambda} + mgh(\Pi_1, \Pi_2, \Pi_3) \times (\chi_1, \chi_2, \chi_3) \cdot \bar{\lambda} \]

\[ = \frac{(\tilde{I}_2 - \tilde{I}_3)\tilde{\lambda}_2\tilde{\lambda}_3 + \tilde{I}_3\tilde{\lambda}_3\tilde{\lambda}_{13}}{\tilde{I}_2 \tilde{I}_3} + \frac{(m_2 - m_3)\tilde{I}_5\tilde{\lambda}_6 + mgh(\tilde{\lambda}_8\chi_3 - \tilde{\lambda}_9\chi_2)}{m_2m_3} \]

since \( \nabla_{\Pi} P = 1, \ \nabla P \Pi = \nabla_{\Gamma} P = 0, \ \frac{\partial P}{\partial \theta_i} = \frac{\partial P}{\partial l_i} = 0, \ i = 1, 2, 3, \ P = (P_1, P_2, P_3, \chi = (\chi_1, \chi_2, \chi_3), \ \nabla P_j(h_{(\mu, a_1, a_2)}) = \frac{p_j}{m_j}, \ j = 1, 2, 3, \ \text{and} \ \nabla_{\Pi_j}(h_{(\mu, a_1, a_2)}) = (\Pi_j - l_j)/\Pi_j, \ i = 1, 2, \ \nabla_{\Pi_j}(h_{(\mu, a_1, a_2)}) = \Pi_j/\Pi_j, \ \nabla_{\Gamma_j} h_{(\mu, a)} = mgh\chi_j, \ j = 1, 2, 3. \]
\[ T \lambda X_{h(\mu, a_1, a_2)} : \lambda(\Gamma) = X_{h(\mu, a_1, a_2)}(\Gamma) \cdot \lambda = \{ \Gamma, h(\mu, a_1, a_2) \} - \lambda\cdot \tilde{\lambda}(A, c, b, \Pi, P, \Gamma, \theta, l) \]

\[ = -\Pi \cdot (\nabla_{\Pi} \nabla \Pi(h(\mu, a_1, a_2))) \cdot \lambda - P \cdot (\nabla_{\Pi} \nabla P(h(\mu, a_1, a_2))) - \nabla_{\Pi}(h(\mu, a_1, a_2)) \times \nabla_{\Pi} \lambda \]

\[ \lambda - \Gamma \cdot (\nabla_{\Pi} \nabla \Pi(h(\mu, a_1, a_2))) - \nabla_{\Pi}(h(\mu, a_1, a_2)) \times \nabla_{\Pi} \lambda) \cdot \lambda + \{ \Gamma, h(\mu, a_1, a_2) \} \in \mathbb{R}^2 | \tilde{\lambda} = \lambda \]

\[ \lambda = \nabla_{\Pi} \lambda = \nabla_{\Pi} \Gamma = 0, \quad \partial_{\Pi} = 0, \quad i = 1, 2. \]

\[ T \lambda X_{h(\mu, a_1, a_2)} : \lambda(l) = X_{h(\mu, a_1, a_2)}(l) \cdot \lambda = \{ l, h(\mu, a_1, a_2) \} - \lambda\cdot \tilde{\lambda}(A, c, b, \Pi, P, \Gamma, \theta, l) \]

\[ = -\Pi \cdot (\nabla_{\Pi} l \times \nabla_{\Pi}(h(\mu, a_1, a_2))) \cdot \lambda - P \cdot (\nabla_{\Pi} l \times \nabla_{\Pi} P(h(\mu, a_1, a_2))) - \nabla_{\Pi}(h(\mu, a_1, a_2)) \times \nabla_{\Pi} l) \cdot \lambda \]

\[ \lambda - \Gamma \cdot (\nabla_{\Pi} l \times \nabla_{\Pi}(h(\mu, a_1, a_2))) - \nabla_{\Pi}(h(\mu, a_1, a_2)) \times \nabla_{\Pi} l) \cdot \lambda + \{ l, h(\mu, a_1, a_2) \} \in \mathbb{R}^2 | \tilde{\lambda} = \lambda \]

\[ \lambda = (0, 0), \]

since \( \nabla_{\Pi} l \times \nabla_{\Pi} P \times \nabla_{\Pi} \Gamma \times 0 \), and \( \partial_{\Pi} \partial_{\Pi} = 0, \quad i = 1, 2. \] If we consider the underwater vehicle-rotors system with a control torque \( u : T^*Q \rightarrow C \) acting on the rotors, and \( u \in C \subset J^{-1}_Q((\mu, a_1, a_2)) \) is invariant under the left \( \text{SE}(3)S^3 \)-action, and its reduced control torque \( u(\mu, a_1, a_2) : \hat{O} \rightarrow C(\mu, a_1, a_2) \) is given by \( u(\mu, a_1, a_2)(\Pi, P, \Gamma, \theta, l) = \pi(\mu, a_1, a_2)(u(A, c, b, \Pi, P, \Gamma, \theta, l)) = u(A, c, b, \Pi, P, \Gamma, \theta, l)|_{\hat{O}}, \) where \( \pi(\mu, a_1, a_2) : J^{-1}_Q((\mu, a_1, a_2)) \rightarrow \hat{O}. \) The dynamical vector field along \( \lambda \) of regular point reduced by stages underwater vehicle-rotors system \( (\hat{O}, \omega_{\hat{O}}, h(\mu, a_1, a_2), u(\mu, a_1, a_2)) \) is given by

\[ X(\lambda, h(\mu, a_1, a_2), u(\mu, a_1, a_2)) \cdot \lambda = X_{h(\mu, a_1, a_2)} \cdot \lambda + \text{vlift}(u(\mu, a_1, a_2)) \cdot \lambda, \]

where \( \text{vlift}(u(\mu, a_1, a_2)) \cdot \lambda = T \lambda \cdot \text{vlift}(u(\mu, a_1, a_2)) \cdot \lambda = T \lambda \cdot \text{vlift}(u(\mu, a_1, a_2)) \cdot \lambda)X_{h(\mu, a_1, a_2)} \lambda \in T(\hat{O}). \) Assume that \( \text{vlift}(u(\mu, a_1, a_2)) \cdot \lambda(A, c, b, \Pi, P, \Gamma, \theta, l) = (U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9, U_{10}, U_{11}, U_{12}, U_{13}) \in T_x(\hat{O}), \)
where \( x = (\Pi, P, \Gamma, \theta, l) \in \tilde{O} = \mathcal{O}_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^2 \), and \( U_i(A, c, b, \Pi, P, \Gamma, \theta, l) : T^* (SE(3) \otimes \mathbb{R}^3 \times S^1 \times S^1) \to \mathbb{R}, \ i = 1, 2, \cdots, 13 \), then from the Hamilton-Jacobi equations \( X_{h(\mu, a_1, a_2)} \bar{\lambda} + \text{vlift}(u(\mu, a_1, a_2) \cdot \bar{\lambda}) = 0 \), we have that

\[
0 = T\bar{\lambda}(X_{h(\mu, a_1, a_2)} \cdot \bar{\lambda} + \text{vlift}(u(\mu, a_1, a_2) \cdot \bar{\lambda})) = X_{h(\mu, a_1, a_2)} \cdot \bar{\lambda} + \text{vlift}(u(\mu, a_1, a_2) \cdot \bar{\lambda}) = X_{(\tilde{O}, \tilde{\omega}, h(\mu, a_1, a_2), u(\mu, a_1, a_2))} \cdot \bar{\lambda},
\]

that is, the dynamical vector field along \( \bar{\lambda} \) of regular point reduced by stages underwater vehicle-rotors system \((\tilde{O}, \tilde{\omega}, h(\mu, a_1, a_2), u(\mu, a_1, a_2))\) is its equilibrium. Thus, in the case of non-coincident centers of buoyancy and gravity, the Hamilton-Jacobi equations for underwater vehicle-rotors system with the control torque \( u \) acting on the rotors are given by

\[
\begin{align*}
(I_2 - I_3)\bar{\lambda}_2\bar{\lambda}_3 + I_3\bar{\lambda}_3\bar{\lambda}_{13} + (m_2 - m_3)\bar{\lambda}_5\bar{\lambda}_6 - m_2 m_3 \bar{\lambda}_6\bar{\lambda}_8 + mgh(\bar{\lambda}_s\bar{\lambda}_3 - \bar{\lambda}_9\bar{\lambda}_2) + U_1 &= 0, \\
(I_3 - I_1)\bar{\lambda}_3\bar{\lambda}_1 - I_3\bar{\lambda}_3\bar{\lambda}_{12} + (m_3 - m_1)\bar{\lambda}_6\bar{\lambda}_4 - m_3 m_1 \bar{\lambda}_6\bar{\lambda}_8 + mgh(\bar{\lambda}_9\bar{\lambda}_1 - \bar{\lambda}_7\bar{\lambda}_3) + U_2 &= 0, \\
(I_1 - I_2)\bar{\lambda}_1\bar{\lambda}_2 - I_1\bar{\lambda}_1\bar{\lambda}_{13} + I_2\bar{\lambda}_2\bar{\lambda}_{12} + (m_1 - m_2)\bar{\lambda}_4\bar{\lambda}_5 - m_1 m_2 \bar{\lambda}_4\bar{\lambda}_8 + mgh(\bar{\lambda}_7\bar{\lambda}_2 - \bar{\lambda}_8\bar{\lambda}_1) + U_3 &= 0, \\
I_2\bar{\lambda}_3\bar{\lambda}_3 - I_3\bar{\lambda}_6(\bar{\lambda}_2 - \bar{\lambda}_{13}) + U_4 &= 0, \\
I_2\bar{\lambda}_6(\bar{\lambda}_1 - \bar{\lambda}_{12}) - I_1\bar{\lambda}_4\bar{\lambda}_3 + U_5 &= 0, \\
I_1\bar{\lambda}_4(\bar{\lambda}_2 - \bar{\lambda}_{13}) - I_2\bar{\lambda}_5(\bar{\lambda}_1 - \bar{\lambda}_{12}) + U_6 &= 0, \\
I_1 I_2 + U_7 &= 0, \\
I_2\bar{\lambda}_3\bar{\lambda}_3 - I_3\bar{\lambda}_9(\bar{\lambda}_2 - \bar{\lambda}_{13}) + U_8 &= 0, \\
I_3\bar{\lambda}_9(\bar{\lambda}_1 - \bar{\lambda}_{12}) - I_1\bar{\lambda}_7\bar{\lambda}_3 + U_9 &= 0, \\
I_1\bar{\lambda}_7(\bar{\lambda}_2 - \bar{\lambda}_{13}) - I_2\bar{\lambda}_8(\bar{\lambda}_1 - \bar{\lambda}_{12}) + U_{10} &= 0, \\
- \frac{\bar{\lambda}_1 - \bar{\lambda}_{12}}{I_1} + \frac{\bar{\lambda}_{12}}{J_1} + U_{11} &= 0, \\
- \frac{\bar{\lambda}_2 - \bar{\lambda}_{13}}{I_2} + \frac{\bar{\lambda}_{13}}{J_2} + U_{12} &= 0, \\
U_{12} &= U_{13} = 0.
\end{align*}
\]

To sum up the above discussion, we have the following theorem.

**Theorem 4.4** In the case of non-coincident centers of buoyancy and gravity, for a point \((\mu, a_1, a_2) \in \text{sc}^*(3) \otimes \mathbb{R}^3\), the regular value of the momentum map \(J_Q : SE(3) \otimes \mathbb{R}^3 \times \text{sc}^*(3) \otimes \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \text{sc}^*(3) \otimes \mathbb{R}^3\), and \( u \in \mathcal{C} \subset J_Q^{-1}(\mathbb{R}^2) \) is invariant under the left \(SE(3) \otimes \mathbb{R}^3\)-action, the regular point reduced by stages system of underwater vehicle-rotors system with the control torque \( u \) acting on the rotors \((T^* Q, SE(3) \otimes \mathbb{R}^3, \omega, H, u)\), where \( Q = SE(3) \otimes \mathbb{R}^3 \times S^1 \times S^1\), is the 4-tuple \((\tilde{O}, \tilde{\omega}, h(\mu, a_1, a_2), u(\mu, a_1, a_2))\), where \( \tilde{O} = \mathcal{O}_{(\mu, a_1, a_2)} \times \mathbb{R}^2 \times \mathbb{R}^2\), \( \mathcal{O}_{(\mu, a_1, a_2)} \subset \text{sc}^*(3) \otimes \mathbb{R}^3\) is the coadjoint orbit, \( \tilde{\omega}\) is orbit symplectic form on \( \tilde{O}, h(\mu, a_1, a_2)(\Pi, P, \Gamma, \theta, l) = \).
\[ \pi_{(\mu, a_1, a_2)}(H(A, c, b, \Pi, P, \Gamma, \theta, l)) = H(A, c, b, \Pi, P, \Gamma, \theta, l) \big|_{\mathcal{O}} \], u_{(\mu, a_1, a_2)}(\Pi, P, \Gamma, \theta, l) = \pi_{(\mu, a_1, a_2)}(u(A, c, b, \Pi, P, \Gamma, \theta, l)) = u(A, c, b, \Pi, P, \Gamma, \theta, l) \big|_{\mathcal{O}}. \] Assume that \( \gamma : SE(3)\times \mathbb{R}^3 \times S^1 \times S^1 \rightarrow T^*(SE(3)\times \mathbb{R}^3 \times S^1 \times S^1) \) is an one-form on \( SE(3)\times \mathbb{R}^3 \times S^1 \times S^1 \), and it is closed with respect to \( T_{\pi_Q} : TT^*(SE(3)\times \mathbb{R}^3 \times S^1 \times S^1) \rightarrow T(SE(3)\times \mathbb{R}^3 \times S^1 \times S^1) \), and \( \lambda = \gamma \cdot \pi_Q : T^*(SE(3)\times \mathbb{R}^3 \times S^1 \times S^1) \rightarrow T^*(SE(3)\times \mathbb{R}^3 \times S^1 \times S^1) \) is sympletic, and \( \text{Im}(\gamma) \subset J^{-1}(\{\mu, a_1, a_2\}) \), and it is \( (SE(3)\times \mathbb{R}^3)_{\pi_{(\mu, a_1, a_2)}} \)-invariant, where \( (SE(3)\times \mathbb{R}^3)_{\pi_{(\mu, a_1, a_2)}} \) is the isotropy subgroup of coadjoint \( SE(3)\times \mathbb{R}^3 \)-action at the point \( (\mu, a_1, a_2) \in se^*(3)\times \mathbb{R}^3 \). Denote by \( \tilde{\gamma} = \pi_{(\mu, a_1, a_2)}(\gamma) : SE(3)\times \mathbb{R}^3 \times S^1 \times S^1 \rightarrow \mathcal{O} \), and \( \tilde{\lambda} = \pi_{\{\mu, a_1, a_2\}}(\lambda) : SE(3)\times \mathbb{R}^3 \times S^1 \times S^1 \rightarrow \mathcal{O} \). Then \( \tilde{\lambda} \) is a solution of the Hamilton-Jacobi equation of reduced underwater vehicle-rotors system given by (4.2), if and only if \( T\tilde{\gamma} \cdot \tilde{X}^\lambda = \tilde{X}_{(\mu, a_1, a_2)} \cdot \tilde{\lambda} \), where \( \tilde{X}^\lambda = T\pi_Q \cdot \tilde{X} \cdot \lambda \), \( \tilde{X} = X(T\pi_Q, SE(3)\times \mathbb{R}^3, \omega_Q, H, a) \), and \( \tilde{X}_{(\mu, a_1, a_2)} = X_{(\tilde{\mathcal{O}}(\tilde{\omega}_{\gamma}, h(\mu, a_1, a_2)), \tilde{u}(\mu, a_1, a_2))} \).

**Remark 4.5** When the underwater vehicle does not carry any internal rotor, in this case the configuration space is \( Q = W = SE(3)\times \mathbb{R}^3 \), the above Hamilton-Jacobi equation (4.2) of reduced underwater vehicle-rotors system is just the Lie-Poisson Hamilton-Jacobi equation of semidirect product Marsden-Weinstein reduced Hamiltonian system \( (\mathcal{O}_{(\mu, a_1, a_2)}, \tilde{\omega}_{\gamma}, h(\mu, a_1, a_2)) \) on the coadjoint orbit of semidirect product Lie group \( SE(3)\times \mathbb{R}^3 \), since the symplectic structure on the coadjoint orbit \( \mathcal{O}_{(\mu, a_1, a_2)} \) is induced by the (-)semidirect product Lie-Poisson brackets on \( se^*(3)\times \mathbb{R}^3 \). See Marsden and Ratiu [15], Ge and Marsden [5], and Wang [25].

The theory of controlled mechanical systems is a very important subject, following the theoretical and applied development of geometric mechanics, a lot of important problems about this subject are being explored and studied. In this paper, as an application of the symplectic reduction by stages and Hamilton-Jacobi theory of regular controlled Hamiltonian systems with symmetry on the generalization of a semidirect product Lie group, in the cases of coincident and non-coincident centers of buoyancy and gravity, we give explicitly the motion equation and Hamilton-Jacobi equation of reduced underwater vehicle-rotors system on a symplectic leaf by calculation in detail, respectively, which show the effect on controls in regular symplectic reduction by stages and Hamilton-Jacobi theory. By using the same way, Wang [27] also studies the symmetric reduction and Hamilton-Jacobi theory of rigid spacecraft with a rotor. But if we define a controlled Hamiltonian system on the cotangent bundle \( T^*Q \) by using a Poisson structure, see Wang and Zhang in [28] and Ratiu and Wang in [23], and the way given in this paper cannot be used, what and how we could do? This is a problem worthy to be considered in detail. In addition, we also note that there have been a lot of beautiful results of reduction theory of Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. Thus, it is an important topic to study the application of reduction theory of controlled Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. These are our goals in future research.

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