Nonnegative Ricci curvature, small linear diameter growth and finite generation of fundamental groups

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1 Introduction

In 1968, Milnor conjectured that a complete noncompact manifold, $M^n$, with nonnegative Ricci curvature has a finitely generated fundamental group [Mi]. This was proven for a manifold with nonnegative sectional curvature by Cheeger and Gromoll [ChGl1]. However, it remains an open problem even for manifolds with strictly positive Ricci curvature.

The conjecture is of particular interest because, if it is true, then by work of Cheeger-Gromoll, Milnor and Gromov, the fundamental group is almost nilpotent [ChGl2] [Mi] [Gr]. On the other hand, given any finitely generated torsion free nilpotent group, Wei has constructed an example of a manifold with positive Ricci curvature that has the given group as a fundamental group [Wei].

Schoen and Yau have proven the conjecture in dimension 3 for manifolds with strictly positive Ricci curvature [SchYau]. In fact they have proven that such a manifold is diffeomorphic to Euclidean space.

Anderson and Li have each proven that if a manifold with nonnegative Ricci curvature has Euclidean volume growth, then the fundamental group is actually finite [And] [Li]. Anderson uses volume comparison arguments while Li uses the heat equation to prove this theorem.

Abresch and Gromoll have proven that manifolds with small diameter growth, $(o(r^{1/n}))$, nonnegative Ricci curvature and sectional curvature bounded away from negative infinity have finite topological type [AbrGrl, Thm A]. Thus the fundamental group is finitely generated in such manifolds. Their theorem is proven using an inequality referred to as the Excess Theorem, [AbrGrl, Prop. 2.3], which is a crucial ingredient in this paper as well.

Abresch and Gromoll have also proven that a manifold with nonnegative Ricci curvature has linear almost intrinsic diameter growth [AbrGrl, Prop 1.1]. That is, given any $\epsilon > 0$, there exists an explicit constant, $C(\epsilon, n)$, such that given any $p$ and $q$ in $\partial B_{x_0}(r)$ joined by a curve in the annulus, $Ann_{x_0}(r-\epsilon r, r+\epsilon r)$, if $\sigma$ is the shortest such curve then $L(\sigma) \leq C(\epsilon, n)r$. This proposition is proven using volume comparison arguments.

In this paper we prove that a manifold with small linear diameter growth has a finitely generated fundamental group.

Theorem 1 There exists a universal constant,

$$S_n = \frac{n}{n-1} \frac{1}{4} \frac{1}{3^n} \left(\frac{n-2}{n-1}\right)^{n-1},$$

(1)
such that if \( M^n \) is complete and noncompact with nonnegative Ricci curvature and has small linear diameter growth,

\[
\limsup_{r \to \infty} \frac{\text{diam}(\partial B_p(r))}{r} < 4S_n,
\]

then it has a finitely generated fundamental group.

Note that unlike the theorem of Abresch and Gromoll we only control the fundamental group. However, we do not require a lower sectional curvature bound and the requisite diameter control is much weaker.

We also prove a more general theorem, Theorem 10. If \( M^n \) has nonnegative Ricci curvature and an infinitely generated fundamental group then it has a tangent cone at infinity which is not polar. Currently it is not known whether a manifold with nonnegative Ricci curvature can have a tangent cone at infinity which is not polar [ChCo].

The definitions of tangent cone and polar length spaces are reviewed in Section 5 [Defn 8, Defn 9] along with the precise statement of Theorem 10. The precise statement involves the universal constant, \( S_n \), defined in (1).

Note that if \( M^n \) has Euclidean volume growth then by Cheeger and Colding, [ChCo1], its tangent cones at infinity have poles, and thus \( M^n \) has a finitely generated fundamental group. However, in this case, Anderson and Li have each proven that the fundamental group is actually finite. [And, Cor 1.5], [Li].

On the other hand, if \( M^n \) has linear volume growth then by [So], its diameter growth is sublinear and its tangent cone at infinity is \([0, \infty)\) or \((-\infty, \infty)\), thus we have the following corollary of either theorem.

**Corollary 2** If \( M^n \) is complete with nonnegative Ricci curvature and linear volume growth then it has a finitely generated fundamental group.

The proofs of both Theorem 1 and Theorem 10 are based on two main lemmas.

The Halfway Lemma, [Lemma 5], concerns complete Riemannian manifolds with infinitely generated fundamental groups. It does not require the Ricci curvature condition and makes a special selection of generators and representative loops, such that the loops are minimal halfway around. It is stated and proven in Section 2.

The Uniform Cut Lemma, [Lemma 7], gives a uniform estimate on special cut points which are the halfway points of noncontractible geodesic loops in
a manifold with nonnegative Ricci curvature. The proof applies the Excess Theorem of Abresch and Gromoll. It appears in Section 3.

Section 4 contains the proof of Theorem \ref{thm:main} and Section 5 contains the proof and background material for Theorem \ref{thm:thm10}. Both of these sections require the results of Sections 1 and 2 but are independent of each other.

The author would like to thank Professor Cheeger for suggesting that Theorem \ref{thm:main} be strengthened to its current form and for his assistance during the revision process. Background material can be found in [Ch] and [GrLaPa].

2 The Halfway Lemma

In this section our manifold, \(N^n\), is a complete Riemannian manifold but does not have a bound on its Ricci curvature. It may or may not be noncompact. Recall that a group is infinitely generated if it does not have a presentation with a finite set of generators. If \(\pi_1(N)\) is not finitely generated, \(N^n\) must be noncompact.

Definition 3 If \(G\) is a group, we say that \(\{g_1, g_2, \ldots\}\) is a ordered set of independant generators of \(G\) if each \(g_i\) can not be expressed as a word in the previous generators and their inverses, \(g_1, g_1^{-1}, \ldots, g_i^{-1}\).

Definition 4 Given \(g \in \pi_1(N)\), we say \(\gamma\) is a minimal representative geodesic loop of \(g\) if \(\gamma = \pi \circ \tilde{\gamma}\), where \(\tilde{\gamma}\) is minimal from \(\tilde{x}_0\) to \(g\tilde{x}_0\) in \(\tilde{N}\). Note that \(L(\gamma) = d_{\tilde{M}}(\tilde{x}_0, g\tilde{x}_0)\).

We now state the Halfway Lemma.

Lemma 5 [Halfway Lemma] Let \(x_0 \in N^n\) where \(N^n\) is a complete Riemannian manifold with a fundamental group \(\pi_1(N, x_0)\). Then there exists an ordered set of independant generators \(\{g_1, g_2, g_3, \ldots\}\) of \(\pi_1(N, x_0)\) with minimal representative geodesic loops, \(\gamma_k\), of length \(d_k\) such that

\[
d_N(\gamma_k(0), \gamma_k(d_k/2)) = d_k/2.
\] (2)

If \(\pi_1(N)\) is infinitely generated we have a sequence of such generators.
Definition 6: Given $x_0 \in N$, we will call an ordered set of generators of $\pi_1(N, x_0)$ as constructed in Lemma 5, a set of halfway generators based at $x_0$ of $\pi_1(N)$.

Proof: In order to prove the Halfway Lemma, we need to choose a sequence of generators whose representative curves don’t have redundant extra looping. Let $G = \pi_1(N^n, x_0)$.

We first define the sequence of generators $\{g_1, g_2, g_3, \ldots\}$. Define $g_1 \in G$ such that it minimizes distance,

$$d_N(\tilde{x}_0, g_1 \tilde{x}_0) \leq d_N(\tilde{x}_0, g \tilde{x}_0) \quad \forall g \in G.$$ 

Let $G_i \in G$ be the elements of $G$ generated by $g_1, g_2, \ldots, g_i$ and their inverses. So $G_1 = \{e, g_1, g_1^{-1}, g_2, \ldots\}$. Define each $g_k \in G$ iteratively such that each minimizes distance among all elements in $G \setminus G_{k-1}$,

$$d_N(\tilde{x}_0, g_k \tilde{x}_0) \leq d_N(\tilde{x}_0, g \tilde{x}_0) \quad \forall g \in G \setminus G_{k-1}.$$ 

Note that $G \setminus G_{k-1}$ is nonempty for all $k$ when we have an infinitely generated fundamental group.

Thus if there exists $h \in G$ with

$$d_N(\tilde{x}_0, h \tilde{x}_0) < d_N(\tilde{x}_0, g_k \tilde{x}_0)$$  \hspace{1cm} (3)$$

then $h \in G_{k-1}$.

Let $\gamma_k$ be a minimal geodesic in $\tilde{N}$ from $\tilde{x}_0$ to $g_k \tilde{x}_0$. Let $\gamma_k(t) = \pi(\gamma_k(t))$.

Then $\gamma_k$ has no conjugate points on $[0, \frac{d_k}{2}]$.

Given a curve $C : [0, d] \to N$, let $C(t_1 \to t_2)$ represent the segment of $C$ running from $t_1$ to $t_2$. We allow $t_2 < t_1$.

Suppose that there is a $k \in \mathbb{N}$ such that

$$d_N(\gamma_k(0), \gamma_k(\frac{d_k}{2})) < \frac{d_k}{2}.$$  \hspace{1cm} (4)$$

Then there exists $T < \frac{d_k}{2}$ such that $\gamma_k(T)$ is a cut point of $x_0 = \gamma(0)$. Thus there exists a geodesic $\sigma$ from $\gamma_k(T) = \sigma(0)$ to $x_0 = \sigma(T)$. This geodesic segment cannot overlap $\gamma_k(T \to d_k)$ because it has a shorter length, $T < \frac{d_k}{2}$.

Then there exists elements $h_1 = [\sigma(0 \to T) \circ \gamma_k(0 \to T)] \in \pi_1(N)$ and $h_2 = [\gamma_k(T \to d_k) \circ \sigma(T \to 0)] \in \pi_1(N)$. Furthermore, since $\sigma$ meets $\gamma$ at an angle, the lift is not a minimal geodesic, so

$$d_N(\tilde{x}_0, h_1 \tilde{x}_0) < L(\sigma(0 \to T) \circ \gamma_k(0 \to T)) < 2T < d_k$$
and
\[ d_N(x_0, h_2 x_0) < L(\gamma_k(T \to d_k) \circ \sigma(T \to 0)) < T + (d_k - T) = d_k. \]

Thus, by (3), \( h_1 \) and \( h_2 \) are in \( G_{k-1} \). So \( g_k = h_2 \circ h_1 \in G_k \) contradicting our choice of \( g_k \in G \setminus G_{k-1} \). So our assumption in (4) is false.

\[ \square \]

3 The Uniform Cut Lemma

In this section, \( M^n \) is a complete manifold with nonnegative Ricci curvature of dimension, \( n \geq 3 \). It may or may not be compact. Recall that when \( n = 2 \), Ricci curvature is just sectional curvature and Theorems 1 and 10 follow from [ChrGr1].

The Uniform Cut Lemma describes special cut points which are the halfway points, \( \gamma(D/2) \), of geodesic loops, \( \gamma \), from a base point, \( x_0 \). Recall that no geodesic from \( x_0 \) through a cut point of \( x_0 \) is minimal after passing through that cut point. So if \( x \in \partial B_{x_0}(RD) \) where \( R > 1/2 \), then any minimal geodesic from \( x \) to \( x_0 \) does not hit the cut point \( \gamma(D/2) \). Thus
\[ d_M(x, \gamma(D/2)) > (R - 1/2)D. \]

The Uniform Cut Lemma gives a uniform and scale invariant estimate for this inequality.

Lemma 7 (Uniform Cut Lemma) Let \( M^n \) be a complete with nonnegative Ricci curvature and dimension \( n \geq 3 \). Let \( \gamma \) be a noncontractible geodesic loop based at a point, \( x_0 \in M^n \), of length, \( L(\gamma) = D \), such that the following conditions hold:

i) If \( \sigma \) based at \( x_0 \) is a loop homotopic to \( \gamma \) then \( L(\sigma) \geq D \)

ii) The loop \( \gamma \) is minimal on \([0, D/2]\) and is also minimal on \([D/2, D]\).

Then there is a universal constant \( S_n \), defined in (3), such that if \( x \in \partial B_{x_0}(RD) \) where \( R \geq (1/2 + S_n) \) then
\[ d_M(x, \gamma(D/2)) \geq (R - 1/2)D + 2S_nD. \]

Note that the minimal representative geodesic loops of the halfway generators constructed in the Halfway Lemma satisfy the hypothesis of the Uniform Cut Lemma.

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Proof We first prove the lemma for $R = (1/2 + S_n)$.

Assume on the contrary, that there is a point $x \in M^n$ such that $d_M(x, x_0) = (1/2 + S_n)D$ and $d_M(x, \gamma(D/2)) = H < 3S_nD$. Let $C : [0, H] \mapsto M^n$ be a minimal geodesic from $\gamma(D/2)$ to $x$.

Let $\hat{M}$ be the universal cover of $M$, let $\hat{x}_0 \in \hat{M}$ be a lift of $x_0$, and let $g \in \pi_1(M, x_0)$ be the element represented by the given loop, $\gamma$. By the first condition on $\gamma$, its lift, $\hat{\gamma}$, is a minimal geodesic running from $\hat{x}_0$ to $g\hat{x}_0$. Thus

\[
d_{\hat{M}}(\hat{x}_0, g\hat{x}_0) = D.
\]

We can lift the joined curves, $C(0 \mapsto H) \circ \gamma(0 \mapsto D/2)$, to a curve in the universal cover, $\hat{C} \circ \hat{\gamma}$, which runs from $\hat{x}_0$ through $\hat{\gamma}(D/2)$ to a point $\hat{x} \in \hat{M}$. Note that $L(\hat{C}) = L(C) = H$.

We can examine the triangle formed by $\hat{x}_0$, $g\hat{x}_0$ and $\hat{x}$ using the Excess Theorem of Abresch and Gromoll [AbrGrl]. For easy reference we introduce their variables $r_0$ and $r_1$.

By our assumption on $x$,

\[
r_0 = d_{\hat{M}}(\hat{x}, \hat{x}_0) \geq d_M(x, x_0) = (1/2 + S_n)D. \tag{5}
\]

Furthermore,

\[
r_1 = d_{\hat{M}}(g\hat{x}, \hat{x}_0) \geq d_M(x, x_0) = (1/2 + S_n)D.
\]

The excess of $\hat{x}$ relative to $\hat{x}_0$ and $g\hat{x}_0$ satisfies

\[
e(\hat{x}) := r_0 + r_1 - d(\hat{x}_0, g\hat{x}_0) \geq 2(1/2 + S_n)D - D = 2S_nD. \tag{6}
\]

On the other hand, by the Excess Theorem [AbrGrl, Prop 2.3], we can estimate the excess from above in terms of the distance, $l$, from $\hat{x}$ to the minimal geodesic, $\hat{\gamma}$. In particular for $n \geq 3$, $\text{Ricci} \geq 0$, they have proven that

\[
e(\hat{x}) \leq 2 \left( \frac{n-1}{n-2} \right) \left( \frac{1}{2} C_3 l^n \right)^{1/(n-1)} \tag{7}
\]

where

\[
C_3 = \frac{n-1}{n} \left( \frac{1}{r_0 - l} + \frac{1}{r_1 - l} \right) \tag{8}
\]

if $l < \min\{r_0, r_1\}$. 

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We now need to estimate $l$ from above. Suppose that the closest point on $\tilde{\gamma}$ to $\tilde{x}$ occurs at a point $\tilde{\gamma}(t_0)$. Then

$$l = d_{\tilde{M}}(\tilde{\gamma}(t_0), \tilde{x}) \leq d_{\tilde{M}}(\tilde{\gamma}(D/2), \tilde{x}) \leq L(C) = HD < 3S_nD.$$  

Since $S_n < 1/20$ for $n \geq 3$,

$$r_0 - l \geq (1/2 + S_n)D - 3S_nD > D/4.$$  

and, similarly,

$$r_1 - l > D/4.$$  

In particular, $l < \min\{r_0, r_1\}$. Substituting this into the Abresch and Gromoll’s estimate (7, 8), we have

$$e(\tilde{x}_k) < 2\left(\frac{n-1}{n-2}\right)\left(\frac{1}{2}\right)\left(\frac{n-1}{n}\right)\left(\frac{2}{D/4}\right)(3S_nD)^{n/(n-1)}$$

$$\leq 2D\left(\frac{n-1}{n-2}\right)\left(4\left(\frac{n-1}{n}\right)(3S_n)^n\right)^{1/(n-1)}.$$  

Combining this with (3) and cancelling $D$, we get

$$2S_n < 2\left(\frac{n-1}{n-2}\right)\left(4\left(\frac{n-1}{n}\right)(3S_n)^n\right)^{1/(n-1)}. \quad (9)$$

Cancelling 2 and exponentiating, we have

$$S_n^{n-1} = 4\left(\frac{n-1}{n-2}\right)^{n-1}\frac{3^n(n-1)}{n}S_n^n,$$

and

$$S_n > \frac{n-1}{n-1}4^{3^n}\left(\frac{n-2}{n-1}\right)^{n-1}.$$  

This contradicts the definition of $S_n$ in (4) and we’ve proven the theorem for $R = (1/2 + S_n)$. If $R \geq (1/2 + S_n)$, let $x \in \partial B_{x_0}(RD)$ and let $y \in \partial B_{x_0}((1/2 + S_n)D)$ be a point on a minimal geodesic from $x$ to $\gamma(D/2)$. Then, by the above case,

$$d_{\tilde{M}}(x, \gamma(D/2)) = d_{\tilde{M}}(x, y) + d_{\tilde{M}}(y, \gamma(D/2)) \geq (RD - (1/2 + S_n)D) + 3S_nD = (R - 1/2)D + 2S_nD.$$  

$\Box$
4 The Small Linear Diameter Growth Theorem

In this section we prove Theorem 1.

Proof of Theorem 1: Assume that $M^n$ has an infinitely generated fundamental group $\pi_1(M^n, x_0)$. Then by the Halfway Lemma, [Lemma 5], there is a sequence of halfway generators, $g_k$, whose minimal representative geodesic loops based at $x_0$, $\gamma_k$, satisfy the hypothesis of the Uniform Cut Lemma [Lemma 7]. Let $d_k = L(\gamma_k)$. Note that $d_k$ diverges to infinity.

By the Uniform Cut Lemma, if $x_k \in \partial B_{x_0}((1/2 + S_n)d_k)$ then $d_M(x_k, \gamma(d_k/2)) \geq 3S_n d_k$. Thus the point $y_k \in \partial B_{x_0}((1/2)d_k)$ on the minimal geodesic from $x_k$ to $x_0$, satisfies,

$$d_M(y_k, \gamma_k(d_k/2)) \geq d_M(x_k, \gamma_k(d_k/2)) - d(x_k, y_k) \geq (3S_n d_k) - (S_n d_k) = 2S_n d_k.$$

This allows us to estimate the diameter growth,

$$\limsup_{r \to \infty} \frac{diam(\partial B_{x_0}(r))}{r} \geq \limsup_{k \to \infty} \frac{d(y_k, \gamma_k(d_k/2))}{(d_k/2)} \geq \limsup_{k \to \infty} \frac{2S_n d_k}{d_k/2} = 4S_n.$$

This contradicts the small linear diameter growth of $M^n$. \(\square\)

5 The Pole Group Theorem

In this section we state and prove Theorem 10. We begin with some background material.

Given a complete noncompact manifold, $M^n$, with nonnegative Ricci curvature, we can define tangent cones at infinity by taking any pointed sequence of rescalings of the manifold. A subsequence of such a sequence must converge in the Gromov Hausdorff topology to a pointed length space, $(X, x_0)$, by Gromov’s Compactness Theorem. [GrLaPa]
**Definition 8** A pointed length space, \((X, x_0)\), is called a tangent cone at infinity of \(M\) if there exists a point \(p \in M\) and a sequence \(r_k\) of positive real numbers diverging to infinity such that for all \(R > 0\),

\[
d_{GH}( (B_R(x_0) \in X, x_0, d_X), (B_R(p) \in M, p, d_M/r_k) ) \to 0,
\]
as \(r_k \to \infty\). Here \(d_M\) is the length space distance function on \(M\) induced by the Riemannian metric \(g_M\) and \(d_{GH}\) is the Gromov-Hausdorff distance.

Note that \((X, x_0)\) need not be unique. See [Per] and [ChCo, 8.37]. Furthermore, \((X, x_0)\) need not be a metric cone unless the manifold has Euclidean volume growth [ChCo].

**Definition 9** [ChCo, sect 4] A length space, \(X\), has a pole at a point \(x \in X\) if for all \(y\) not equal to \(x\) there exists a curve \(\gamma : [0, \infty) \to X\) such that \(\gamma(0) = x\), \(d_X(\gamma(t), \gamma(s)) = |s - t|\) for all \(s, t \geq 0\), and \(\gamma(d(x, y)) = y\).

There is no known example of a manifold with nonnegative Ricci curvature with a tangent cone at infinity which does not have a pole at its base point [ChCo]. In order to find an example of such a manifold, intuitively one would need to construct a sequence of cut points on the manifold which remain uniformly cut even after rescaling. By Lemmas 5 and 7, such cut points exist if the manifold has an infinitely generated fundamental group. This is the intuition behind Theorem 10 and Theorem 1.

**Theorem 10** [Pole Group Theorem] If a complete noncompact manifold, \(M^n\), with nonnegative Ricci curvature has a fundamental group which is not finitely generated, then it has a tangent cone at infinity, \((Y, y_0)\), which does not have a pole at its base point.

In fact, if \((Z, z_0)\) is a length space with

\[
d_{GH}( (B_{z_0}(1), z_0, d_Z), (B_{y_0}(1), y_0, d_Y) ) < S_n/4.
\]

where \(S_n\) was defined in (1), then \(Z\) does not have a pole at \(z_0\).

**Proof:**

First we choose a special sequence of rescalings of \(M^n\) and a corresponding tangent cone at infinity. Let \(x_0\) be any base point in \(M\). By the Halfway
Lemma, there is a sequence of halfway generators, $g_k$, of lengths, $d_k$, corresponding to a base point $x_0$. Take a subsequence of this sequence such that $M_k = (M^n, x_0, d_M/d_k)$ converges to a tangent cone $Y = (Y, y_0, d_Y)$. [GrLaPa]

By the Halfway Lemma and the Uniform Cut Lemma, we know that for all $k \in \mathbb{N}$, for all $r \geq 1/2 + S_n$ and for all $x \in \partial B_{x_0}(rd_k)$ then

$$d_M(x, \gamma_k(d_k/2)) \geq (r - 1/2 + 2S_n)d_k. \quad (11)$$

We want to find a “cut point” in $Y$ and in $Z$. By Definition 8, taking $R = 1$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $d_{GH}(B_{y_0}(1) \subset Y, y_0, d_Y, B_{x_0}(1) \subset M_k, x_0, d_{M_k}) < S_n/12$. Thus, by (11), for all $k \geq N$, the length space $Z$ satisfies

$$d_{GH}(B_{z_0}(1) \subset Z, z_0, d_Z, B_{x_0}(1) \subset M_k, x_0, d_{M_k}) < \varepsilon_n = S_n/3. \quad (12)$$

So there is a map $F_k : B_{z_0}(1) \subset M_k \mapsto B_{z_0}(1) \subset Z$, with $F_k(x_0) = z_0$ that is $\varepsilon_n$ almost distance preserving,

$$|d_{M_k}(x_1, x_2) - d_Z(F_k(x_1), F_k(x_2))| < \varepsilon_n \quad \forall x_1, x_2 \in X, \quad (13)$$

and $\varepsilon_n$ almost onto.

$$\forall z \in B_{z_0}(1) \subset Z, \ \exists x_z \in B_{z_0}(1) \subset M_k \ s.t. \ d_Z(F_k(x_z), x) < \varepsilon_n. \quad (14)$$

The map $F_k$ can also be thought of as a map defined on $B_{z_0}(d_k) \subset M^n$.

Note that $F_k$ maps the halfway points, $\gamma_k(d_k/2)$, into an annulus,

$$F_k(\gamma_k(d_k/2)) \subset \text{Ann}_{z_0}(1/2 - \varepsilon_n, 1/2 + \varepsilon_n).$$

Thus there is subsequence of the $F_k(\gamma_k(d_k/2))$ that converges to a point

$$z_1 \in \text{Cl}(\text{Ann}_{z_0}(1/2 - \varepsilon_n, 1/2 + \varepsilon_n)). \quad (15)$$

In particular, we can choose a $k \geq N$ such that

$$d_Z(F_k(\gamma_k(d_k/2)), z_1) < \varepsilon_n. \quad (16)$$

We will show that this $z_1$ is our “cut point”. To find the cut point in $Y$, just set $Z = Y$. 

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We claim that \( z_1 \) has no ray based at \( z_0 \) passing through it. Suppose, on the contrary, that it does. Then there is a curve, \( C : [0,1) \rightarrow Z \) such that \( c(0) = 0, c(t_1) = z_1 \) and \( d_Y(c(t), c(s)) = |s - t| \quad \forall t, s \in [0,1). \)

Let \( z_2 = C(1/2 + h) \) where \( h \in [3S_n, 1/2). \) By (15), \( t_1 \geq 1/2 - \varepsilon_n \), so

\[
d_Z(z_2, z_1) = (1/2 + h) - t_1 \leq (1/2 + h) - (1/2 - \varepsilon_n) = h + \varepsilon_n. \tag{17}
\]

By (14), there exist \( x_2 \in B_{x_0}(1) \subset M_k \) which is mapped almost onto \( z_2 \),

\[
d_Z(F_k(x_2), z_2) < \varepsilon_n. \tag{18}
\]

By (13), the triangle inequality and (18), we know that

\[
d_M(x_0, x_2) = d_{M_k}(x_0, x_2) d_k > (d_Z(F_k(x_0), F_k(x_2)) - \varepsilon_n) d_k
\geq (d_Z(z_0, z_2) - d_Z(z_2, F_k(x_2)) - \varepsilon_n) d_k > ((1/2 + h) - 2\varepsilon_n) d_k.
\]

Recall that \( h \geq 3S_n \) and \( \varepsilon_n = S_n/3 \), so \( x_2 \in \partial B_{x_0}(r d_k) \) with \( r \geq 1/2 + S_n. \)

Thus by the uniform cut property, (11),

\[
d_M(x_2, \gamma_k(d_k/2)) \geq (r - 1/2 + 2S_n) d_k \geq (h - 2\varepsilon_n + 2S_n) d_k. \tag{19}
\]

However, \( F_k \) is \( \varepsilon_n \) almost distance preserving, (13). So applying the triangle inequality, (18), (17), and (13), we have

\[
d_M(x_2, \gamma_k(d_k/2)) = d_{M_k}(x_2, \gamma_k(d_k/2)) d_k
< (d_Z(F_k(x_2), F_k(\gamma_k(d_k/2))) + \varepsilon_n) d_k
\leq (d_Z(F_k(x_2), z_2) + d_Z(z_2, z_1) + d_Z(z_1, F_k(\gamma_k(d_k/2))) + \varepsilon_n) d_k
< (\varepsilon_n + (h + \varepsilon_n) + \varepsilon_n + \varepsilon_n) d_k.
\]

Combining this equation with (19), we get

\[
(h + \varepsilon_n + 3\varepsilon_n) d_k > (h - 2\varepsilon_n + 2S_n) d_k.
\]

So \( \varepsilon_n > S_n/3 \), contradicting (12). \( \square \)

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