Long Solutions of Sequence A348480 of the On-Line Encyclopedia of Integer Sequences

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December 2021

Abstract

For numbers $x$ coprime to 10 there exist infinitely many binary numbers $b$ such that the greatest common divisor of $b$ and $\text{rev}(b) = x$ and the sum of digits of $b = x$ ($\text{rev}(b)$ is the digit reversal of $b$). In most cases, the smallest $b$ that fulfill these two constraints contain just a few zeros. But in some cases like for $x = 7, 11, 13$ and 37, $b$ must contain more zeros than ones and these $b$ are called long solutions. For 11 and 37 it follows directly from the fact that these are porous numbers. For 7 and 13, the proofs that they have long solutions are presented in this paper.

1 Introduction

For numbers $x$ coprime to 10, i.e. numbers ending with 1, 3, 7 or 9, there exist infinitely many binary numbers $b$ such that the greatest common divisor of $b$ and $\text{rev}(b) = x$ and the sum of digits of $b = x$. $\text{rev}(b)$ is the digit reversal of $b$, e.g. $\text{rev}(123) = 321$. The solutions $b$ converted to decimal numbers form the sequence A348480 of the On-Line Encyclopedia of Integer Sequences [1].

For example

$$\gcd(1011, 1101) = 3$$

and since there is no smaller binary number that satisfies this constraint, $b = 1011$ is the solution for $x = 3$. $b$ contains 3 ones and 1 zero and is called a short solution because there are more ones than zeros. If there are just a few zeros in the solution a computer search program will find the solution very quickly. However, in the cases $x = 7, 11, 13$ and 37, there are many more zeros in the solution than ones and only for $x = 7$ and 11, the computer program is able to find the solution in a reasonable amount of time. The solution for $x = 13$ has 73 digits, which is beyond the capabilities of a systematic search.

Nevertheless, requirements for the long solutions can be established and the search space will collapse to a single degree of freedom. In the case of the porous
numbers $x = 11$ and $37$, it was demonstrated in [2] that every number $m$ with sum of digits $= x$ and $x$ is a divisor of both $m$ and digit reversal of $m$ must have repeating zeros in their digits. And since a palindrome $m$ where $m =$ rev$(m)$ like 10101 cannot have 11 or 37 as greatest common divisor of $m$ and rev$(m)$, more zeros need to be added and the solutions for $x = 11$ and 37 have more zeros than ones.

For the sequence A348480, the requirements are even stricter: not only must $x$ be a divisor of both $m$ and digit reversal of $m$, $x$ must also be the greatest common denominator of the two numbers. In the case of 7 and 13, this complicates the search for the smallest solution, because for short numbers $m$ for which 7 is a divisor of both $m$ and digit reversal of $m$, both $m$ and rev$(m)$ are also divisible by 13 and vice versa. And therefore no short solution exist for $x = 7$ and 13, as will be shown in this paper.

In the last chapter, the special case of $x = 39$ will be treated shortly. 39 does not have a long solution, but it also has no really short solution like all other cases up to and including $x = 71$. The explanation is that the multiplicative order of 10 modulo 39 is 6, the same as the multiplicative order of 10 modulo 7 or modulo 13 and therefore possible solutions are often also divisible by 7 or 13.

2 Required mathematical tools

For our proof, only a very few simple mathematical tools or fundamentals are required. The first fundamental deals with the parity of numbers. The following statement can be made:

If, $x, y, p$ and $q$ are integer numbers and

$$x + y = p + q$$

(1)

then $x + p - y - q$ is an even number.

Proof:

$a + b$ has the same parity as $a - b$. Then $x + p - y - q = (x - y) + (p - q)$ is the sum of two numbers with the same parity which gives an even number.

A second tool that will be applied are divisibility rules. A number is divisible by 7 if the sum of blocks with length 6 is divisible by 7: You cut the number $b$ in blocks of length 6 starting from the right and sum up the blocks and check if the sum is divisible by 7. The same rule applies for all numbers $x$ where the multiplicative order 10 modulo $x$ is 6, like for 13 and for 39.

The last required prowess deals with Diophantine equations. For example,
if $r$ and $s$ are integers and

$$5r + 8s$$

is a multiple of 13

then all solutions can be parameterized by:

$$r = 13n + s$$

with $n$ being any integer.

With these tools in hand we can start the proof.

3 Requirements for a solution for $x = 7$

The first requirement is that the sum of the digits of the solution $b$ is 7.

The second requirement is derived from the fact that 7 divides $b$ since 7 is the greatest common divisor of $x$ and $\text{rev}(x)$. The multiplicative order 10 modulo 7 is 6. Hence a number $b$ is divisible by 7 if the sum of blocks with length 6 is divisible by 7.

Let "$b_{s-1}...b_3b_2b_1b_0$" be a number $b$ with $s$ digits. We define:

$$A = b_0 + b_6 + b_{12} + ...$$
$$B = b_1 + b_7 + b_{13} + ...$$
$$C = b_2 + b_8 + b_{14} + ...$$
$$D = b_3 + b_9 + b_{15} + ...$$
$$E = b_4 + b_{10} + b_{16} + ...$$ and
$$F = b_5 + b_{11} + b_{17} + ...$$

When $10^i$ ($i > 1$) is divided by 7, the first six remainders are 1, 3, 2, 6, 4 and 5 and then this sequence is repeating. We define:

$$\delta_i = \begin{cases} 
1 & \text{if } \text{mod}(i, 6) = 0 \\
3 & \text{if } \text{mod}(i, 6) = 1 \\
2 & \text{if } \text{mod}(i, 6) = 2 \\
6 & \text{if } \text{mod}(i, 6) = 3 \\
4 & \text{if } \text{mod}(i, 6) = 4 \\
5 & \text{if } \text{mod}(i, 6) = 5 
\end{cases}$$

With these definitions the powers of 10 can be written as:

$$10^i = j_i \cdot 7 + \delta_i$$

with some integer $j_i$. Then $b$ can be written as:

$$b = \sum_{i=0}^{s-1} b_i 10^i = \sum_{i=0}^{s-1} b_i (j_i \cdot 7 + \delta_i)$$
Since 7 must divide this sum, the multiples of 7 can be dropped and it follows
\[ \sum_{i=0}^{s-1} b_i \delta_i = l_1 \cdot 7 \]  
(2)

Hence we get this requirement for the distribution of the 7 ones:
\[ A + 3B + 2C + 6D + 4E + 5F = l_1 \cdot 7 \]  
(3)

The third requirement is derived from the fact that 7 also divides \( \text{rev}(x) \) since 7 is the greatest common divisor of \( x \) and \( \text{rev}(x) \). As above \( \text{rev}(b) \) can be written as:
\[ b = \sum_{i=0}^{s-1} b_{s-i-1} 10^i = \sum_{i=0}^{s-1} b_{s-i-1} (j_i \cdot 7 + \delta_i) \]

Again, the multiples of 7 can be dropped and it follows
\[ \sum_{i=0}^{s-1} b_{s-i-1} \delta_i = m_1 \cdot 7 \]  
(4)

If 7 divides \( b \) then 7 also divides 10\( b \) and therefore we can add as many zeros to \( b \) as needed such that \( s \), without loss of generality, can be assumed to be a multiple of 6\(^1\). In this case, eq. 4 translates into our third requirement for the distribution of the ones in \( b \):
\[ 5A + 4B + 6C + 2D + 3E + F = m_1 \cdot 7 \]  
(5)

Adding eq. 3 and eq. 5 yields:
\[ 6(A + F) + C + D = l_2 \cdot 7 \]  
(6)

Note the term 8\( C \) reduces to \( C \) because 7\( C \) can be subtracted on both sides and is absorbed in a new \( l_2 \) on the right-hand side of the equation. The same holds for 7\( E \) and 7\( B \) which disappear on the left-hand side.

Subtracting eq. 3 from eq. 5 yields:
\[ 4(A + C - D - F) + B - E = m_2 \cdot 7 \]  
(7)

In summary, these are the 3 requirements for the positive integers \( A, B, C, D, E \) and \( F \) such that the greatest common divisor of \( b \) and \( \text{rev}(b) \) equals 7 and the sum of the digits of \( b \) equals 7:

\(^1\)To be on the safe side, the author repeated the proof individually for the other 5 cases of \( s \) where \( 1 \leq \text{mod}(s, 6) \leq 5 \).
\[
\begin{align*}
A + B + C + D + E + F &= 7 \\
6(A + F) + C + D &= l_2 \cdot 7 \\
4(A + C - D - F) + B - E &= m_2 \cdot 7
\end{align*}
\]

4 Requirements for a solution for \(x = 13\)

The first requirement is that the sum of the digits of the solution \(c\) is 13.

The second requirement is derived from the fact that 13 divides \(c\) since 13 is the greatest common divisor of \(x\) and \(\text{rev}(x)\). The multiplicative order 10 modulo 13 is 6. Hence a number \(c\) is divisible by 13 if the sum of blocks with length 6 is divisible by 13.

Let "\(c_{t-1}c_3c_2c_1c_0\)" be a number \(c\) with \(t\) digits. We define:

- \(A' = c_0 + c_6 + c_{12} + \ldots\)
- \(B' = c_1 + c_7 + c_{13} + \ldots\)
- \(C' = c_2 + c_8 + c_{14} + \ldots\)
- \(D' = c_3 + c_9 + c_{15} + \ldots\)
- \(E' = c_4 + c_{10} + c_{16} + \ldots\)
- \(F' = c_5 + c_{11} + c_{17} + \ldots\)

When \(10^i (i > 1)\) is divided by 13, the first six remainders are 1, 10, 9, 12, 3 and 4 and then this sequence is repeating. We define:

\[
\theta_i = \begin{cases} 
1 & \text{if } \text{mod}(i, 6) = 0 \\
10 & \text{if } \text{mod}(i, 6) = 1 \\
9 & \text{if } \text{mod}(i, 6) = 2 \\
12 & \text{if } \text{mod}(i, 6) = 3 \\
3 & \text{if } \text{mod}(i, 6) = 4 \\
4 & \text{if } \text{mod}(i, 6) = 5
\end{cases}
\]

With these definitions the powers of 10 can be written as:

\[10^i = k_i \cdot 13 + \theta_i\]

with some integer \(k_i\). Then \(c\) can be written as:

\[
c = \sum_{i=0}^{t-1} c_i 10^i = \sum_{i=0}^{t-1} c_i (k_i \cdot 13 + \theta_i)
\]

Since 13 must divide this sum, the multiples of 13 can be dropped and it follows

\[
\sum_{i=0}^{t-1} c_i \theta_i = l_3 \cdot 13 
\]
Hence we get this requirement for the distribution of the 13 ones:

\[ A' + 10B' + 9C' + 12D' + 3E' + 4F' = l_3 \cdot 13 \]  \hspace{1cm} (9)

The third requirement is derived from the fact that 13 also divides \( \text{rev}(x) \) since 13 is the greatest common divisor of \( x \) and \( \text{rev}(x) \). As above \( \text{rev}(c) \) can be written as:

\[ c = \sum_{i=0}^{t-1} c_{t-i-1}10^i = \sum_{i=0}^{t-1} c_{t-i-1}(k_i \cdot 13 + \theta_i) \]

Again, the multiples of 13 can be dropped and it follows

\[ \sum_{i=0}^{t-1} c_{t-i-1}\theta_i = m_3 \cdot 13 \]  \hspace{1cm} (10)

In this case, again without loss of generality assuming that \( t \) is a multiple of 6, eq. 10 translates into our third requirement for \( c \):

\[ 4A' + 3B' + 12C' + 9D' + 10E' + F' = m_3 \cdot 13 \]  \hspace{1cm} (11)

Adding eq. 9 and eq. 11 yields:

\[ 5(A' + F') + 8(C' + D') = l_4 \cdot 13 \]  \hspace{1cm} (12)

Subtracting eq. 9 from eq. 11 yields:

\[ 3(A' + C' - D' - F') + 6(B' - E') = m_4 \cdot 13 \]  \hspace{1cm} (13)

In summary, these are the 3 requirements for the positive integers \( A', B', C', D', E' \) and \( F' \):

\[
\begin{align*}
A' + B' + C' + D' + E' + F' &= 13 \\
5(A' + F') + 8(C' + D') &= l_4 \cdot 13 \\
3(A' + C' - D' - F') + 6(B' - E') &= m_4 \cdot 13 
\end{align*}
\]

5 Proof that 7 has a long solution

We will prove that 5 out of the 6 positions \( A, B, C, D, E \) and \( F \) will be zero and all 7 ones must be located in one position. If we define \( R = A + F \) and \( S = C + D \) then eq. 6 translates into:

\[ 6R + S = l_2 \cdot 7 \]

This Diophantine equation has the solution space

\[ R = 7n_1 + S \]
with $n_1$ being any integer.

Since $0 \leq R, S \leq 7$, $n_1$ can only be -1, 0 or +1.

1. $n_1 = -1 \Rightarrow R = 0, S = 7$. This means all ones are in positions $C$ and $D$ and $A = B = E = F = 0$. Then it follows from eq. 7 that $4(C - D) = m_2 \cdot 7$ or $C - D = m_2 \cdot 7$. Since $C + D = 7$ this is only possible if $C$ or $D$ equals 7. Hence all ones are located in one position, either $C$ or $D$, qed.

2. $n_1 = +1 \Rightarrow R = 7, S = 0$. This means all ones are in positions $A$ and $F$ and $B = C = D = E = 0$. Then it follows from eq. 7 that $4(A - F) = m_2 \cdot 7$ or $A - F = m_2 \cdot 7$. But this is only possible if $A$ or $F$ equals 7. Hence all ones are located in one position, either $A$ or $F$, qed.

3. $n_1 = 0 \Rightarrow R = S$. In this case we can conclude two facts: first, if we define $U = A + C - D - F$ then $U$ is even (see Chapter 2) and second, eq. 12 holds because $5R + 8S$ is a multiple of 13.

This means either we have proved that 7 has a long solution or $U$ must be even and eq. 12 holds.

In the latter case, we define $V = B - E$. Then eq. 7 translates into:

$$4U + V = m_2 \cdot 7$$

This Diophantine equation has the solution space

$$U = 7n_2 - 2V$$

with $n_2$ being any integer.

Since $U$ is even, also $n_2$ must be even and since $-7 \leq U, V \leq 7$, $n_2$ can only be -2, 0 or +2.

1. $n_2 = -2 \Rightarrow U = 0, V = -7$. This means all ones are in position $E$, qed.

2. $n_2 = +2 \Rightarrow U = 0, V = 7$. This means all ones are in position $B$, qed.

3. $n_2 = 0 \Rightarrow U = -2V$. In this case eq. 13 holds because $3U + 6V = -6V + 6V = 0 = m_4 \cdot 13$.

So either we have a long solution or both equations 12 and 13 hold. But if both equations are true, then $b$ and rev($b$) are divisible by 13 and the greatest common divisor cannot be 7. Hence 7 has a long solution, qed.

The shortest "long solution" for $x = 7$ is

$$100000000001000001000001000001000001000001,$$

with all ones in position $A$. Digit $b_{36}$ must also be a zero, because the solution cannot be a palindrome, otherwise the greatest common divisor will not be 7.
6 Proof that 13 has a long solution

We will prove that 5 out of the 6 positions $A', B', C', D', E'$ and $F'$ will be zero and all 13 ones must be located in one position. If we define $R' = A' + F'$ and $S' = C' + D'$ then eq. 6 translates into:

$$5R' + 8S' = l_4 \cdot 13$$

This Diophantine equation has the solution space $R = 13n_3 + S$ with $n_3$ being any integer.

Since $0 \leq R, S \leq 13$, $n_3$ can only be -1, 0 or +1.

1. $n_3 = -1 \Rightarrow R' = 0, S' = 13$. This means all ones are in positions $C'$ and $D'$ and $A' = B' = E' = F' = 0$. Then it follows from eq. 13 that $3(C' - D') = m_4 \cdot 13$ or $C' - D' = m_4 \cdot 13$. But this is only possible if $C'$ or $D'$ equals 13. Hence all ones are located in one position, either in $C'$ or $D'$, qed.

2. $n_3 = +1 \Rightarrow R' = 13, S' = 0$. This means all ones are in positions $A'$ and $F'$ and $B' = C' = D' = E' = 0$. Then it follows from eq. 13 that $4(A' - F') = m_4 \cdot 13$ or $A' - F' = m_4 \cdot 13$. But this is only possible if $A'$ or $F'$ equals 13. Hence all ones are located in one position, either in $A'$ or $F'$, qed.

3. $n_3 = 0 \Rightarrow R' = S'$. In this case we can conclude two facts: first, if we define $U' = A' + C' - D' - F'$ then $U'$ is even (see Chapter 2) and second, eq. 6 holds because $6R' + S'$ is a multiple of 7.

This means either we have proved that 13 has a long solution or $U'$ must be even and eq. 6 holds.

In the latter case, we define $V' = B' - E'$. Then eq. 13 translates into:

$$3U' + 6V' = m_4 \cdot 13$$

This Diophantine equation has the solution space $U' = 13n_4 - 2V'$ with $n_4$ being any integer.

Since $U'$ is even, also $n_4$ must be even and since $-13 \leq U', V' \leq 13$, $n_4$ can only be -2, 0 or +2.

1. $n_4 = -2 \Rightarrow U' = 0, V' = -13$. This means all ones are in position $E'$, qed.
2. \( n_4 = +2 \Rightarrow U' = 0, V' = 13 \). This means all ones are in position \( B' \), qed.

3. \( n_4 = 0 \Rightarrow U' = -2V' \). In this case eq. 7 holds because \( 4U' + V' = -8V' + V' = -7V' \) is a multiple of 7.

So either we have a long solution or both equations 6 and 7 hold. But if both equations are true, then \( b \) and \( \text{rev}(b) \) are divisible by 7 and the greatest common divisor cannot be 13. Hence 13 has a long solution, qed.

## 7 39 has quite a long ”short solution”

Since the multiplicative order 10 modulo 39 is also 6, it turns out that for most numbers \( b \) where the sum of digits of \( b \) is 39 and 39 divides both \( b \) and \( \text{rev}(b) \), also 7 divides both \( b \) and \( \text{rev}(b) \) and therefore 39 is not the greatest common divisor. In fact the smallest \( b \) where the greatest common divisor is 39 has 16 zeros, i.e. \( b \) has 55 digits in total. With a computer program, an iterative search was performed given these three constraints:

\[
\begin{align*}
A + B + C + D + E + F &= 39 \\
A + 10B + 22C + 25D + 16E + 4F &= l_5 \cdot 39 \\
4A + 16B + 25C + 22D + 10E + F &= m_5 \cdot 39
\end{align*}
\]

The result was \( A = 10, B = C = 9, D = 6, E = 0 \) and \( F = 5 \). In binary terms, this translates into \( b = 10001100011000110001111111111011111111011111111111111111 \) or in decimal form 20016007615544303 which is term 16 of A348480.

## References

[1] R. Jehn, *Sequence A348480 of the On-Line Encyclopedia of Integer Sequences*. [https://oeis.org/A348480](https://oeis.org/A348480), 2021.

[2] R. Jehn, *Porous numbers*. [https://arxiv.org/abs/2104.02482](https://arxiv.org/abs/2104.02482), 2021.