A POSTERIORI ERROR ESTIMATES FOR THE ELECTRIC FIELD INTEGRAL EQUATION ON POLYHEDRA

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ABSTRACT. We present a residual-based a posteriori error estimate for the Electric Field Integral Equation (EFIE) on a bounded polyhedron \( \Omega \) with boundary \( \Gamma \). The EFIE is a variational equation formulated in \( H_{\text{div}}^{-1/2}(\Gamma) \). We express the estimate in terms of \( L^2 \)-computable quantities and derive global lower and upper bounds (up to oscillation terms).

1. Introduction

The Electric Field Integral Equation (EFIE) describes the scattering of electromagnetic waves on a perfectly conducting obstacle \( \Omega \) with surface \( \Gamma \), in our case a polyhedron. Assuming a time-harmonic dependence, the Stratton-Chu representation formula expresses the electric field \( E \) in terms of a surface potential as

\[
E(x) = E^{inc}(x) + \int_{\Gamma} \left( G_k(x, y)u(y) + \frac{1}{k^2} \text{grad}_\Gamma G_k(x, y) \text{div}_\Gamma u(y) \right) d\sigma(y),
\]

where \( k \) denotes the wave-number and \( E^{inc}(x) \) is the given incident wave that is scattered on \( \Gamma \). Invoking the boundary condition that the tangential component of the total electric field \( E \) vanishes on the surface \( \Gamma \), as corresponds to \( \Omega \) being perfectly conducting, the EFIE consists of seeking the surface current \( u \in H_{\text{div}}^{-\frac{1}{2}}(\Gamma) \) such that for all \( x \in \Gamma \)

\[
\int_{\Gamma} \left( G_k(x, y)u(y) + \frac{1}{k^2} \text{grad}_\Gamma G_k(x, y) \text{div}_\Gamma u(y) \right) d\sigma(y) = -\gamma_\parallel (E^{inc}(x)),
\]

where \( H_{\text{div}}^{-\frac{1}{2}}(\Gamma) \) is the space of traces of \( H(\text{curl}, \Omega) \) functions that are rotated by a right angle on the surface and \( \gamma_\parallel \) denotes the tangential trace onto \( \Gamma \).

Computing approximations of the EFIE by means of the Boundary Element Method (BEM), namely using a Galerkin approach based on the variational formulation of the EFIE, is expensive due to the dense matrix structure of the ensuing linear system. Although fast techniques such as the Fast Multipole Method exist, c.f. [15] as an example of a first work in this field, it is still crucial to locate the degrees of freedom efficiently, namely in regions of low regularity of the solution \( u \).

Since \( u \in H_{\text{div}}^{-\frac{1}{2}}(\Gamma) \), \( u \) exhibits in general rather low regularity and, as a consequence, \textit{a priori} estimates show extremely low convergence rates for quasi-uniform mesh refinements; see [10, 16]. In contrast, adaptive refinement techniques, based
on a posteriori error estimates, exploit much weaker regularity of $u$ in a nonlinear Sobolev scale and allow for optimal error decay in terms of degrees of freedom in situations where quasi-uniform meshes are suboptimal. The design and analysis of a posteriori error estimators is, however, problem dependent; we refer to [9, 17] for an account of the theory of adaptive finite element methods in the energy norm for linear second order elliptic partial differential equations in polyhedra.

The a posteriori error analysis and corresponding theory of adaptive mesh refinements for BEM is much less developed and an overview of different approaches for the former is given in [7, 13, 18]. It seems that this is the first contribution with specific application to electromagnetic scattering problems on polyhedra.

For integral equations, additional difficulties arise since the residual typically lies in a Sobolev space with fractional index that is possibly also negative, as in the present case. Since such norms are not computable in practice, this imposes additional challenges to the residual based approach of a posteriori error estimates.

In this paper we develop nevertheless a residual based a posteriori error estimator for the EFIE on polyhedra, and prove upper and lower global bounds. Residual based estimators are especially attractive due to their simplicity of derivation and computation, but they involve interpolation constants which can at best be estimated. Alternative estimators have been proposed, mostly for elliptic problems defined in $\Omega$, at the expense of their simplicity; we believe that our approach can be extended to those estimators as well. We derive computable $L^2$–integrable quantities to estimate the error of the BEM measured in the $H^{\frac{-1}{2}}_{\text{div}}(\Gamma)$ norm, which is the natural norm for EFIE. We therefore avoid evaluating fractional Sobolev norms.

For proving well-posedness of the exact solution and developing a priori error estimates it is important to decompose both the exact solution and test function using a Helmholtz decomposition [6, 16]. In contrast, to derive a posteriori error estimates, it is crucial to decompose the test function according to a regular decomposition which extends the Helmholtz decomposition; see [8] for $H(\text{div}; \Omega)$.

This paper is organized as follows. In Section 2 we recall the necessary functional analysis in order to derive a posteriori error estimates for the EFIE [2, 3, 4, 5]. We also present and study a Clément type interpolation operator for the Raviart-Thomas space, based on ideas from [1]. We discuss the EFIE integral equation in Section 3, and derive global upper and lower a posteriori error estimates in Section 4. Section 5 is finally left for conclusions.

2. Functional Spaces and Differential Operators

The functional analysis framework developed in [2, 3] will be used in this work. In this section we give a short introduction to the functional spaces and differential operators used in the following sections. However, for a detailed and thorough overview we refer to [2, 3, 5, 6]. References [5, 6] deal with non-smooth Lipschitz surfaces, thus the theory is also valid for polyhedra, and covers therefore a more general framework. However, we restrict our theory to polyhedral surfaces.

2.1. Spaces, norms and trace operators. Let $\Omega$ be a bounded polyhedron in $\mathbb{R}^3$, and denote its boundary by $\Gamma$ and its different faces by $\Gamma_j$, $j = 1, \ldots, N_F$. The exterior part $\Omega^+$ is defined by $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}$. Let $n(x)$, $x \in \Gamma$, denote the outer unit normal to the surface $\Gamma$, which is piecewise constant on $\Gamma$. We also indicate by $e_{ij} = \partial \Gamma_i \cap \partial \Gamma_j$ the edges of $\Gamma$ and by $\tau_{ij}$ the unit vectors parallel to $e_{ij}$, with
its orientation fixed but arbitrary. If $n_i = n_i|_{\Gamma_i}$, we further define

$$\tau_i = \tau_{ij} \times n_i, \quad \tau_j = \tau_{ij} \times n_j$$

to be unit vectors lying on the supporting planes of $\Gamma_i$ and $\Gamma_j$; see Figure 1 for an illustration.

![Figure 1. Local coordinate systems around an edge $e_{ij} = \partial \Gamma_i \cap \partial \Gamma_j$.](image)

On $\Gamma$, we define the space of square integrable tangential fields

$$L^2_t(\Gamma) = \{ v \in [L^2(\Gamma)]^3 \mid v \cdot n = 0 \ \text{a.e.} \}.$$ 

Moreover, we let $H^s(\Gamma)$ and $H^s(\Gamma) = [H^s(\Gamma)]^3$, with $s \in [-1, 1]$, denote the standard Sobolev spaces of complex-valued scalar and vector-valued functions on $\Gamma$ and denote their norms by $\| \cdot \|_{H^s(\Gamma)}$ and $\| \cdot \|_{H^s(\Gamma)}$, respectively; for negative Sobolev indices the norms are defined by duality. Furthermore, for $s \in (0, 1)$, we denote by

$$\gamma : H^{s+\frac{1}{2}}(\Omega) \to H^s(\Gamma), \quad \gamma : [H^{s+\frac{1}{2}}(\Omega)]^3 \to H^s(\Gamma)$$

the standard continuous trace operators, and by $R_\gamma$ and $R_\gamma$ their continuous right inverses.

For complex-valued vector functions we introduce the facewise $H^{\frac{1}{2}}$-broken spaces

$$H^{\frac{1}{2}}_i(\Gamma) = \{ v \in L^2_t(\Gamma) \mid v|_{\Gamma_i} \in H^{\frac{1}{2}}(\Gamma_i), 1 \leq i \leq N_F \},$$

with corresponding norm

$$\| v \|^2_{H^{\frac{1}{2}}_i(\Gamma)} = \sum_{j=1}^{N_F} \| v \|^2_{H^{\frac{1}{2}}(\Gamma_j)}.$$ 

Moreover, we define the spaces

$$H^{\frac{1}{2}}(\Gamma) = \left\{ v \in H^{\frac{1}{2}}_i(\Gamma) \mid v|_{\Gamma_i} \cdot \tau_{ij} \equiv v|_{\Gamma_j} \cdot \tau_{ij}, \text{ for every edge } e_{ij} \right\},$$

$$(1) \quad H^{-\frac{1}{2}}(\Gamma) = \left\{ v \in H^{-\frac{1}{2}}_i(\Gamma) \mid v|_{\Gamma_i} \cdot \tau_i \equiv v|_{\Gamma_j} \cdot \tau_j, \text{ for every edge } e_{ij} \right\},$$

where the relation $\equiv$ is understood in the sense that

$$\| v \|^{1/2}_{\Gamma_i} \equiv v_j \iff \int_{\Gamma_i} \int_{\Gamma_j} \frac{|v_i(x) - v_j(y)|^2}{\|x - y\|^3} d\sigma(x) d\sigma(y) < \infty.$$
The norms \( \| \cdot \| \) in (4)

right inverse maps are linear, surjective and continuous operators. In addition, there exists continuous in the following Proposition.

space \( L^2 \) when endowed with the norms (3)

We further define

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a polyhedron, for any edge \( e \)

\[ \gamma \]

\[ \gamma \]

\[ \| \cdot \| \]

\[ \| \cdot \| \]

\[ \| \cdot \| \]

for each edge \( e_{ij} \) of the polyhedron and denote by \( I_j \) the set of indices \( i \) such that \( \Gamma_j \) and \( \Gamma_i \) have a common edge \( e_{ij} \).

**Proposition 2.1** ([2, Prop. 2.6]). The spaces \( H^{\frac{1}{2}}_\parallel (\Gamma) \) and \( H^{\frac{1}{2}}_\perp (\Gamma) \) are Hilbert spaces when endowed with the norms

\[ \| v \|_{H^{\frac{1}{2}}_\parallel (\Gamma)}^2 := \| v \|_{H^{\frac{1}{2}}_\perp (\Gamma)}^2 + \sum_{j=1}^{N_F} \sum_{i \in I_j} N^i_{ij}(v), \]

\[ \| v \|_{H^{\frac{1}{2}}_\perp (\Gamma)}^2 := \| v \|_{H^{\frac{1}{2}}_\perp (\Gamma)}^2 + \sum_{j=1}^{N_F} \sum_{i \in I_j} N^i_{ij}(v). \]

In other words, \( v \in H^{\frac{1}{2}}_\parallel (\Gamma), H^{\frac{1}{2}}_\perp (\Gamma) \) satisfies \( v \in H^{\frac{1}{2}}_\parallel (\Gamma_i) \) on the faces \( \Gamma_i \) of \( \Gamma \), and the parallel resp. orthogonal component of the function \( v \) to edges \( e_{ij} \) of \( \Gamma \) are “\( H^{\frac{1}{2}} \)-continuous” in the sense of (2); c.f. [2] for further details.

We denote by \( H^{\frac{1}{2}}_\parallel (\Gamma), H^{\frac{1}{2}}_\perp (\Gamma) \) the dual spaces of \( H^{\frac{1}{2}}_\parallel (\Gamma), H^{\frac{1}{2}}_\perp (\Gamma) \) with pivot space \( L^2(\Gamma) \). The corresponding duality pairing is denoted by \( \langle \cdot, \cdot \rangle_\parallel \) resp. \( \langle \cdot, \cdot \rangle_\perp \). The norms \( \| \cdot \|_{H^{\frac{1}{2}}_\parallel (\Gamma)} \) and \( \| \cdot \|_{H^{\frac{1}{2}}_\perp (\Gamma)} \) are defined by duality.

For complex-valued functions \( v \in C^\infty(\Omega) \) the tangential traces are defined by

\[ \gamma_\parallel(v) := n \times (v \times n)|_\Gamma, \quad \gamma_\perp(v) := (v \times n)|_\Gamma. \]

We point out that \( \gamma_\parallel(v) = v - (v \cdot n)n \) gives the component of \( v \) tangential to \( \Gamma \), whereas \( \gamma_\perp(v) \) provides a tangent vector field perpendicular to \( \gamma_\parallel(v) \). Since \( \Gamma \) is a polyhedron, for any edge \( e_{ij} \) of \( \Gamma \) the components of \( \gamma_\parallel(v) \) and \( \gamma_\perp(v) \) tangential and normal to \( e_{ij} \) are continuous, namely,

\[ \gamma_\parallel(v)|_{\Gamma \setminus \gamma} - \gamma_\parallel(v)|_{\Gamma \setminus \gamma} = 0, \quad \gamma_\perp(v)|_{\Gamma \setminus \gamma} - \gamma_\perp(v)|_{\Gamma \setminus \gamma} = 0. \]

This means that both operators \( \gamma_\parallel \) and \( \gamma_\perp \) can be viewed as face-by-face projections; see [2, p.36]. Combining this observation with definitions (1), we realize that

\[ H^{\frac{1}{2}}_\parallel (\Gamma) \] and \( H^{\frac{1}{2}}_\perp (\Gamma) \) are the trace spaces of \( \gamma_\parallel, \gamma_\perp \) acting on \( H^1(\Gamma) \). This is stated in the following Proposition.

**Proposition 2.2** ([2, Prop. 2.7]). The trace operators

\( \gamma_\parallel : H^1(\Omega) \to H^{\frac{1}{2}}_\parallel (\Gamma), \quad \gamma_\perp : H^1(\Omega) \to H^{\frac{1}{2}}_\perp (\Gamma) \)

are linear, surjective and continuous operators. In addition, there exists continuous right inverse maps \( R_\parallel : H^{\frac{1}{2}}_\parallel (\Gamma) \to H^1(\Omega) \) and \( R_\perp : H^{\frac{1}{2}}_\perp (\Gamma) \to H^1(\Omega) \).

We can now establish a critical result for the upcoming analysis. Note that \( H^{\frac{1}{2}}_\perp (\Gamma) \) consists of tangential vector fields whereas \( H^{\frac{1}{2}}_\parallel (\Gamma) \) does not.
Lemma 2.3. There exists a continuous map $t_\perp : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}_\perp(\Gamma)$ with right inverse $t_\perp^{-1} : H^{\frac{1}{2}}_\perp(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$.

Proof. We define $t_\perp : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}_\perp(\Gamma)$ and $t_\perp^{-1} : H^{\frac{1}{2}}_\perp(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ by

$$t_\perp(w) = \gamma_\perp(R_\gamma(w)), \quad \forall w \in H^{\frac{1}{2}}(\Gamma),$$

$$t_\perp^{-1}(v) = \gamma(R_\perp(v)), \quad \forall v \in H^{\frac{1}{2}}_\perp(\Gamma)$$

where $\gamma$ and $R_\gamma$ are the trace and its right inverse, whereas $\gamma_\perp$ and $R_\perp$ are the operators of Proposition 2.2. The continuity of these operators implies the continuity of $t_\perp$ and $t_\perp^{-1}$.

To prove that $t_\perp^{-1}$ is the right inverse of $t_\perp$, we observe that

$$t_\perp(t_\perp^{-1}(v)) = \gamma_\perp(R_\gamma(\gamma(R_\perp(v)))), \quad \forall v \in H^{\frac{1}{2}}_\perp(\Gamma),$$

and that $\gamma_\perp$ projects face by face on $\Gamma$ [2, page 36]. If $w = t_\perp^{-1}(v) \in H^{\frac{1}{2}}_\perp(\Gamma)$ and $g = R_\gamma w \in H^1(\Omega)$, then $w = \gamma(R_\perp(v))$ and $t_\perp(w) = \gamma_\perp(g)$. Since $\gamma_\perp(g)|_{\Gamma_i} = \gamma(g)|_{\Gamma_i} \times n$ for each face $\Gamma_i$ of $\Gamma$, we obtain on $\Gamma_i$

$$t_\perp(t_\perp^{-1}(v)) = t_\perp(w) = \gamma_\perp(g) = \gamma(g) \times n$$

$$= \gamma(R_\gamma w) \times n = w \times n = \gamma(R_\perp(v)) \times n = \gamma_\perp(R_\perp(v)) = v.$$

Thus, $t_\perp^{-1}$ is indeed the right inverse of $t_\perp$. \qed

2.2. Tangential differential operators. We set $H^{\frac{1}{2}}(\Gamma) := \gamma(H^2(\Omega))$, and define the tangential operators $\text{grad}_\Gamma : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}_\parallel(\Gamma)$ and $\text{curl}_\Gamma : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}_\perp(\Gamma)$ by

$$\text{grad}_\Gamma \phi := \gamma_{\parallel}(\text{grad} \phi) \quad \text{curl}_\Gamma \phi := \gamma_{\perp}(\text{grad} \phi) \quad \forall \phi \in H^2(\Omega),$$

where $\text{grad}$ denotes the standard gradient in $\mathbb{R}^3$. According to definitions (5), $\text{grad}_\Gamma \phi$ is the orthogonal projection of $\text{grad} \phi$ on each face $\Gamma_i$ of $\Gamma$, whereas $\text{curl}_\Gamma \phi$ is obtained from the former by a $\pi/2$ rotation. It can be shown that the maps $\text{grad}_\Gamma$ and $\text{curl}_\Gamma$ are linear and continuous.

The adjoint operators $\text{div}_\Gamma : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ and $\text{curl}_\Gamma : H^{-\frac{1}{2}}_\parallel(\Gamma) \to H^{-\frac{1}{2}}_\perp(\Gamma)$ can be defined as follows

$$\langle \text{div}_\Gamma v, \phi \rangle_{\frac{1}{2}, \Gamma} = -\langle v, \text{grad}_\Gamma \phi \rangle_{\parallel, \Gamma},$$

$$\langle \text{curl}_\Gamma w, \phi \rangle_{\frac{1}{2}, \Gamma} = \langle w, \text{curl}_\Gamma \phi \rangle_{\perp, \Gamma},$$

for all $\phi \in H^{\frac{1}{2}}(\Gamma)$, $v \in H^{-\frac{1}{2}}(\Gamma)$ and $w \in H^{-\frac{1}{2}}_\perp(\Gamma)$.

In view of these definitions we now introduce the spaces

$$H^{-\frac{1}{2}}_{\text{div}}(\Gamma) := \left\{ v \in H^{-\frac{1}{2}}(\Gamma) \left| \text{div}_\Gamma v \in H^{-\frac{1}{2}}(\Gamma) \right\} \right.,$$

$$H^{-\frac{1}{2}}_{\text{curl}}(\Gamma) := \left\{ v \in H^{-\frac{1}{2}}_\perp(\Gamma) \left| \text{curl}_\Gamma v \in H^{-\frac{1}{2}}_\perp(\Gamma) \right\} \right..$$
The scalar and vector single layer potential are then defined respectively by

\[
\|v\|^2_{H^{\frac{1}{2}}(\Gamma)} := \|v\|^2_{H_{\text{div}}^{\frac{1}{2}}(\Gamma)} + \|\text{div}_{\Gamma} v\|^2_{H^{-\frac{1}{2}}(\Gamma)},
\]

\[
\|v\|^2_{H_{\text{curl}}^{\frac{1}{2}}(\Gamma)} := \|v\|^2_{H_{\text{curl}}^{\frac{1}{2}}(\Gamma)} + \|\text{curl}_{\Gamma} v\|^2_{H^{-\frac{1}{2}}(\Gamma)}.
\]

Let the natural solution space of Maxwell’s equations be denoted by

\[
\mathbf{H}(\text{curl}, \Omega) = \{v \in L^2(\Omega) \mid \text{curl} v \in L^2(\Omega)\}.
\]

**Theorem 2.4** ([3, Theorem 4.6]). The mappings \(\gamma_{\|}\) and \(\gamma_{\perp}\) admit linear and continuous extensions

\[
\gamma_{\|} : \mathbf{H}(\text{curl}, \Omega) \to H_{\text{curl}}^{\frac{1}{2}}(\Gamma), \quad \gamma_{\perp} : \mathbf{H}(\text{curl}, \Omega) \to H_{\text{div}}^{\frac{1}{2}}(\Gamma).
\]

Moreover, the following integration by parts formula holds true:

\[
\int_{\Omega} \left(\text{curl} v \cdot u - \text{curl} u \cdot v\right) \, d\Omega = \langle \gamma_{\perp} u, \gamma_{\|} v \rangle_{\|, \Gamma}, \quad \forall u \in \mathbf{H}(\text{curl}, \Omega), v \in H^1(\Omega).
\]

Furthermore, a duality pairing \(\langle \cdot, \cdot \rangle_{\|}\) between \(H_{\text{div}}^{\frac{1}{2}}(\Gamma)\) and \(H_{\text{curl}}^{\frac{1}{2}}(\Gamma)\) can be established by using an orthogonal decomposition of those spaces so that the following integration by parts formula still holds

\[
\int_{\Omega} \left(\text{curl} v \cdot u - \text{curl} u \cdot v\right) \, d\Omega = \langle \gamma_{\perp} u, \gamma_{\|} v \rangle_{\Gamma}, \quad \forall u, v \in \mathbf{H}(\text{curl}, \Omega).
\]

For more details, we refer to [3].

The differential operators \(\text{grad}_{\Gamma}\) and \(\text{curl}_{\Gamma}\) can be further extended as follows.

**Proposition 2.5** ([3, p.39]). The tangential gradient and curl operators introduced in (7) can be extended to linear and continuous operators defined on \(H^{\frac{1}{2}}(\Gamma)\)

\[
\text{grad}_{\Gamma} : H^{\frac{1}{2}}(\Gamma) \to H_{\text{div}}^{\frac{1}{2}}(\Gamma), \quad \text{curl}_{\Gamma} : H^{\frac{1}{2}}(\Gamma) \to H_{\text{curl}}^{\frac{1}{2}}(\Gamma).
\]

Moreover their formal \(L^2(\Gamma)\)-adjoints

\[
\text{div}_{\Gamma} : H_{\text{div}}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \quad \text{and} \quad \text{curl}_{\Gamma} : H_{\text{curl}}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)
\]

can be defined by

\[
\langle \text{div}_{\Gamma} v, \phi \rangle_{\frac{1}{2}, \Gamma} = -\langle v, \text{grad}_{\Gamma} \phi \rangle_{\perp, \Gamma},
\]

\[
\langle \text{curl}_{\Gamma} w, \phi \rangle_{\frac{1}{2}, \Gamma} = \langle w, \text{curl}_{\Gamma} \phi \rangle_{\parallel, \Gamma},
\]

for all \(\phi \in H^{\frac{1}{2}}(\Gamma), v \in H_{\text{div}}^{\frac{1}{2}}(\Gamma)\) and \(w \in H_{\text{curl}}^{\frac{1}{2}}(\Gamma)\).

2.3. **Potentials.** Let \(G_k\) denote the fundamental solution of the Helmholtz operator \(\Delta + k^2\), which is given by

\[
G_k(x, y) := \frac{\exp(ik|x-y|)}{4\pi|x-y|}.
\]

The scalar and vector single layer potential are then defined respectively by

\[
\Psi_{k}^V : H^{-\frac{1}{2}}(\Gamma) \to H_{\text{loc}}^1(\mathbb{R}^3), \quad \Psi_{k}^V(v)(x) := \int_{\Gamma} G_k(x, y)v(y) \, d\sigma(y),
\]

\[
\Psi_{k}^A : H^{-\frac{1}{2}}(\Gamma) \to H_{\text{loc}}^1(\mathbb{R}^3), \quad \Psi_{k}^A(v)(x) := \int_{\Gamma} G_k(x, y)v(y) \, d\sigma(y).
\]
These potentials are known to be continuous \([6, \text{Theorem 3.8}]\). Finally the scalar and vector single layer boundary operators are defined by
\[
V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \quad V_k := \gamma \circ \Psi_k^V,
\]
\[
A_k : H_\parallel^{\frac{1}{2}}(\Gamma) \to H_\parallel^{\frac{1}{2}}(\Gamma), \quad A_k := \gamma_\parallel \circ \Psi_k^A.
\]
The simultaneous continuity of the trace operators \(\gamma, \gamma_\parallel\) and the single layer potentials yield then the continuity of the single layer boundary operators, namely,
\[
\|V_k v\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|v\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad \|A_k v\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|v\|_{H^{-\frac{1}{2}}(\Gamma)},
\]
for all \(v \in H^{-\frac{1}{2}}(\Gamma)\) and \(v \in H^{-\frac{1}{2}}(\Gamma)\). In particular, if restricted to \(L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)\), then the range of \(V_k\) lies in \(H^1(\Gamma)\) \([6, \text{Theorem 3.8}]\), i.e.,
\[
\text{Im}(V_k(L^2(\Gamma)) \subset H^1(\Gamma).
\]
Likewise, for the vector case the corresponding result reads \([4, \text{Prop. 2}]\)
\[
\text{Im}(A_k(L^2(\Gamma)) \subset H^1(\Gamma).
\]

### 2.4. Interpolation of weighted spaces

In the following section we will be confronted with interpolation of weighted \(L^2\)-spaces. We thus recall in this section some basic results taken from Tartar’s book \([20]\), which are valid without regularity on the weights.

Let \(T\) be a family of shape-regular triangulations decomposing \(\Gamma\) into flat triangles such that the surface covered by the triangles coincides with \(\Gamma\). Denote the set of edges of the mesh by \(\mathcal{E}_T\). For a fixed triangulation let \(h_T\) denote the diameter of any element \(T \in T\) and let \(h\) be the piecewise constant function such that \(h|_T = h_T\).

**Lemma 2.6** ([20, Lemma 22.3, p.110]). If \(A\) is linear from \(E_0 + E_1\) into \(F_0 + F_1\) and maps \(E_0\) into \(F_0\) with \(\|Ax\|_{F_0} \leq M_0\|x\|_{E_0}\) for all \(x \in E_0\), and maps \(E_1\) into \(F_1\) with \(\|Ax\|_{F_1} \leq M_1\|x\|_{E_1}\) for all \(x \in E_1\), then \(A\) is linear continuous from \((E_0, E_1)_{\theta, p}\) into \((F_0, F_1)_{\theta, p}\) for all \(\theta, p\), and for \(0 < \theta < 1\) one has
\[
\|Aa\|_{(F_0, F_1)_{\theta, p}} \leq M_0^{1-\theta} M_1^\theta \|a\|_{(E_0, E_1)_{\theta, p}} \quad \text{for all } a \in (E_0, E_1)_{\theta, p}.
\]
The space \((E_0, E_1)_{\theta, p}\) denotes the interpolation space between \(E_0\) and \(E_1\) based on the \(L^p\)-inner product.

**Lemma 2.7** ([20, Lemma 23.1, p.115]). For a measurable positive function \(w\) on \(\Gamma\), let
\[
E(w) = \left\{ u \left| \int_{\Gamma} |u(x)|^2 w(x) \, dx < \infty \right. \right\} \quad \text{with} \quad \|u\|_W = \left( \int_{\Gamma} |u(x)|^2 w(x) \, dx \right)^{\frac{1}{2}}.
\]
If \(w_0, w_1\) are two such functions, then for \(0 < \theta < 1\) one has
\[
(E(w_0), E(w_1))_{\theta, 2} = E(w_\theta) \quad \text{with equivalent norms, where } w_\theta = w_0^{1-\theta} w_1^\theta.
\]

**Corollary 2.8.** Let \(0 < s < 1\) be arbitrary. Let \(A\) be a linear continuous map from \(L^2(\Gamma)\) into \(L^2(\Gamma)\) and from \(H^1(\Gamma)\) into \(L^2(\Gamma)\) with
\[
\|Av\|_{L^2(\Gamma)} \leq M_0 \|v\|_{L^2(\Gamma)} \quad \text{for all } v \in L^2(\Gamma),
\]
\[
\|h^{-1} Av\|_{L^2(\Gamma)} \leq M_1 \|v\|_{H^1(\Gamma)} \quad \text{for all } v \in H^1(\Gamma).
\]
Then $A$ is a linear map from $H^s(\Gamma) = (H^1(\Gamma), L^2(\Gamma))_{s,2}$ into $L^2(\Gamma)$ with
\[
\|h^{-s}Av\|_{L^2(\Gamma)} \leq M_0^{1-s}M_2^s\|v\|_{H^s(\Gamma)} \quad \text{for all } v \in H^{\frac{s}{2}}(\Gamma).
\]

**Proof.** Combine Lemma 2.6 with Lemma 2.7. \hfill \Box

2.5. **Discrete spaces and interpolation.** Let $RT_0(T)$ denote the local Raviart-Thomas space of complex-valued functions on $T \in \mathcal{T}$ defined by (cf. [14, 19])
\[
RT_0(T) := \left\{ v(\mathbf{x}) = \alpha + \beta \mathbf{x} \mid \alpha \in \mathbb{C}^2, \beta \in \mathbb{C} \right\}.
\]
The global Raviart-Thomas space is defined by
\[
RT_0 := \left\{ v \in H^0_{\text{div}}(\Gamma) \mid v|_T \in RT_0(T) \quad \forall T \in \mathcal{T} \right\},
\]
where $H^0_{\text{div}}(\Gamma)$ is defined in a standard manner
\[
H^0_{\text{div}}(\Gamma) := \left\{ v \in L^2(\Gamma) \mid \text{div} v \in L^2(\Gamma) \right\}.
\]
Further denote by $V_T$ the space of scalar complex-valued continuous functions that are piecewise linear, namely
\[
V(T) = \left\{ v \in H^1(\Gamma) \mid v|_K \in \mathbb{P}_1(T) \right\},
\]
where $\mathbb{P}_1(T)$ denotes the space of affine polynomials on $T$. Let $N(T)$ denote the set of all nodes $\nu$ of $\mathcal{T}$ and $\{\varphi_\nu\}_{\nu \in N(T)}$ be the family of nodal bases of $V(T)$.

**Definition 2.9.** Let the Clément type interpolation operator $I_T : L^2(\Gamma) \to V(T)$ be
\[
I_T v := \sum_{\nu \in N(T)} \phi_\nu(v) \varphi_\nu \quad \forall v \in L^2(\Gamma)
\]
where $\Gamma_\nu = \text{supp}(\varphi_\nu)$ and the degrees of freedom are given by
\[
\phi_\nu(v) := \frac{3}{|\Gamma_\nu|} \int_{\Gamma_\nu} v(\mathbf{x}) \varphi_\nu(\mathbf{x}) d\mathbf{x}.
\]

**Proposition 2.10.** If $v \in H^s(\Gamma)$ with $0 < s < 1$, then the interpolation operator $I_T$ satisfies the following interpolation properties
\[
\|h^{-s}(v - I_T v)\|_{L^2(\Gamma)} \leq \|v\|_{H^s(\Gamma)} \quad \text{for all } v \in H^s(\Gamma).
\]

**Proof.** This interpolation operator $I_T$ is also used in [12] and the following result is proven
\[
\|h^{-1}(v - \phi_\nu(v))\|_{L^2(\Gamma_\nu)} \leq \|\text{grad}_T v\|_{L^2(\Gamma_\nu)}
\]
for $v \in H^1(\Gamma)$ under the assumption of shape-regularity [12, (2.2.29)] (Note that in our case the mesh matches the surface and therefore equation (2.2.29) can be simplified). Following the arguments of the original paper of Clément [11, Proof of Theorem 1], it is now straightforward to prove that
\[
\|h^{-1}(v - I_T v)\|_{L^2(T)} \leq \sum_{\nu \in N(T)} \|\text{grad}_T v\|_{L^2(\Gamma_\nu)},
\]
where $N(T)$ denotes the set of nodes of the element $T$. Now, summing over all elements of the mesh $\mathcal{T}$ and using that the number of elements sharing a node is bounded, as a consequence of shape regularity of $\mathcal{T}$, we get
\[
\|h^{-1}(v - I_T v)\|_{L^2(\Gamma)} \leq \|v\|_{H^s(\Gamma)}.
\]
Furthermore, the operator can also be shown to be $L^2$-stable [12, (2.2.33)].
Therefore, the linear continuous operator \( A_T = \text{Id} - I_T : L^2(\Gamma) \to L^2(\Gamma) \) satisfies
\[
\|h^{-1}(v - I_Tv)\|_{L^2(\Gamma)} \leq \|v\|_{H^1(\Gamma)},
\]
\[
\|v - I_Tv\|_{L^2(\Gamma)} \leq \|v\|_{L^2(\Gamma)}.
\]
The asserted estimate (13) follows from Corollary 2.8. \(\square\)

Besides this for \( s = 1/2 \), we will also need a Raviart-Thomas type interpolation operator for functions in \( v \in H^1_\perp(\Gamma) \). Since \( \text{div}_\Gamma v \notin L^2(\Gamma) \), the standard degrees of freedom are no longer well-defined. Therefore, we will utilize an interpolation operator similar to that introduced in [1] for the first type Nédelec elements.

For any edge \( e \in E_T \) of the mesh we associate an arbitrary but fixed element \( T_e \) such that \( e \subset \partial T_e \). On \( T_e \) we denote by \( \pi_e \) the \( L^2(T_e) \)-projection onto constant functions. We let \( \{\psi_e\}_{e \in E_T} \) be the standard Raviart-Thomas basis of lowest order, sometimes also referred to as the Rao-Wilton-Glisson (RWG) basis in this context of electromagnetic scattering, such that
\[
\int_e \psi_e \cdot \nu_e \, ds = 1 \quad \text{and} \quad \int_e \psi_{e'} \cdot \nu_e \, ds = 0
\]
for any \( e' \in E_T \) such that \( e \neq e' \) and where \( \nu_e \) denotes the outer unit normal of \( T_e \) at the edge \( e \) which is coplanar with \( T_e \); see Fig 2.5.

**Definition 2.11.** Let the Clément type interpolation operator \( I_T : L^2_1(\Gamma) \to RT_0 \) for the Raviart-Thomas element of lowest order be given by
\[
I_T v := \sum_{e \in E_T} \alpha_e(v) \psi_e
\]
where the degrees of freedom are defined by
\[
\alpha_e(v) := \int_e \pi_e(v) \cdot \nu_e \, ds.
\]

**Remark 2.12.** Note that \( I_T \) does not satisfy the usual commutative property
\[
\text{div}_\Gamma(I_T v) \neq P_0(\text{div}_\Gamma v),
\]
where \( P_0 \) denotes the element-wise \( L^2 \)-projection of degree 0. This is important in the a priori analysis but not in the upcoming a posteriori error analysis.

**Lemma 2.13.** The degrees of freedom of the interpolation operator \( I_T : L^2_1(\Gamma) \to RT_0 \) are well-defined and \( I_T \) satisfies the local \( L^2 \)-stability bound
\[
\|I_T v\|_{L^2(T)} \leq c \|v\|_{L^2(\Delta T)} \quad \text{for all } T \in T,
\]
where $\Delta_T$ denotes the set of elements that share at least one edge with $T$.

Proof. The argument is similar to [1]. If $T \in \mathcal{T}$ is an arbitrary but fixed element and $\mathcal{E}(T)$ denotes the three edges of $T$, then
\[
\|I_T v\|_{L^2(T)} = \| \sum_{e \in \mathcal{E}(T)} \alpha_e \psi_e \|_{L^2(T)} \leq \sum_{e \in \mathcal{E}(T)} \| \alpha_e \psi_e \|_{L^2(T)} = \sum_{e \in \mathcal{E}(T)} |\alpha_e| \| \psi_e \|_{L^2(T)}.
\]

Invoking the Piola transformation, it can be shown that
\[
\| \psi_e \|_{L^2(T)} \leq c \| \hat{\psi}_e \|_{L^2(\hat{T})} \leq c
\]
since the basis functions $\hat{\psi}_e$ on the reference element $\hat{T}$ are uniformly bounded independent on $T$. Moreover, applying the Cauchy-Schwarz inequality, we get
\[
|\alpha_e| = \left| \int_e \pi_e (v) \cdot \nu_e \, ds \right| \leq \frac{1}{2} \| \pi_e v \|_{L^2(e)} \leq c h_T \| \pi_e v \|_{L^2(e)}
\]
where $\hat{\pi}_e$ denotes the $L^2(\hat{T})$-projection onto constant functions on the reference element $\hat{T}$. Note that $\hat{\pi}_e \hat{v}$ is defined via the affine transformation from $T_e$ (and not $T$) to $\hat{T}$. Norm equivalence of polynomials (constant functions in this case), the $L^2$-stability of $\hat{\pi}_e$ and a scaling argument yield
\[
|\alpha_e| \leq c h_T \| \hat{\pi}_e \hat{v} \|_{L^2(\hat{T})} \leq c h_T \| \hat{\pi}_e \hat{v} \|_{L^2(T_e)} \leq c \| \hat{\pi}_e \hat{v} \|_{L^2(T_e)}.
\]
Combining the above estimates implies the asserted stability bound of $I_T$. $\square$

To explore the accuracy of the interpolant $I_T$, we need the following lemmas.

**Lemma 2.14** (Local approximability). For any $v \in \mathbb{R}^3$, we have the error estimate
\[
\| v - I_T v \|_{L^2(T)} \leq \| v - \gamma_\perp c \|_{L^2(\Delta_T)} \text{ for all } T \in \mathcal{T},
\]
where $\Delta_T$ denotes the set of elements that share at least one edge with $T$.

Proof. Let $\hat{v} = \gamma_\perp c$ with $c \in \mathbb{R}^3$ which, in view of (5), is piecewise constant in $\mathcal{T}$. According to (6) the normal component of $\hat{v}$ is continuous across all edges of the mesh, including those of the polyhedron $\Gamma$, whence $\hat{v} \in \mathbb{R} T_0$.

We first observe that $I_T \hat{v} = \hat{v}$ because $\pi_e \hat{v} |_{T_e} = \hat{v} |_{T_e}$ for any edge $e \subset \partial T$ and
\[
\alpha_e (\hat{v}) = \int_e \pi_e \hat{v} \cdot \nu_e \, ds = \int_e \hat{v} \cdot \nu_e \, ds.
\]
Since these three local degrees of freedom on $T \in \mathcal{T}$ are unisolvent and they coincide for both $I_T \hat{v} |_T$ and $\hat{v} |_T$, we deduce $I_T \hat{v} |_T = \hat{v} |_T$. Consequently
\[
\| v - I_T v \|_{L^2(T)} \leq \| v - \hat{v} \|_{L^2(T)} + \| I_T (v - \hat{v}) \|_{L^2(T)}.
\]
By the local $L^2$-stability of Lemma 2.13 we conclude that
\[
\| v - I_T v \|_{L^2(T)} \leq \| v - \hat{v} \|_{L^2(\Delta_T)},
\]
as asserted. $\square$

**Lemma 2.15.** If $v \in H^1_\perp (\Gamma)$, then there exists $w \in H^{1/2} (\Gamma)$ such that $t_\perp (w) = v$ and $\| w \|_{H^{1/2} (\Gamma)} \leq \| v \|_{H^{1/2} (\Gamma)}$.

Proof. Simply set $w = t_\perp^{-1} (v)$, where $t_\perp^{-1}$ is defined in Lemma 2.3, and use the facts that $t_\perp^{-1}$ is the right inverse of $t_\perp$ and the continuity of $t_\perp^{-1}$. $\square$
Lemma 2.16. If $w \in H^\frac{1}{2}(\Gamma)$, then $\|t_\perp(w)\|_{L^2(\Delta_T)} \leq \|w\|_{L^2(\Delta_T)}$.

Proof. As in the proof of Lemma 2.3, let $g \in H^1(\Omega)$ be the function such that $g = R_\gamma w$ and $t_\perp(w) = \gamma_\perp(g)$. Therefore

$$
\gamma_\perp(g)|_T = \gamma_g|_T \times n, \quad \text{for a.e. } x \in T, \text{ for all } T \in T,
$$

whence

$$
\|t_\perp(w)\|^2_{L^2(\Delta_T)} = \sum_{T \subset \Delta_T} \|t_\perp(w)\|_{L^2(T)}^2 = \sum_{T \subset \Delta_T} \|\gamma_\perp(g) \times n\|_{L^2(T)}^2
\leq \sum_{T \subset \Delta_T} \|\gamma_\perp(g)\|_{L^2(T)}^2 = \|\gamma_g\|_{L^2(\Delta_T)}^2 = \|w\|_{L^2(\Delta_T)}^2,
$$

because $R_\gamma$ is the right inverse of $\gamma$ and thus $\gamma(g) = w$. \qed

Proposition 2.17 (Global approximability). If $v \in H^\frac{1}{2}(\Gamma)$, then the interpolation operator $I_T$ satisfies the following global error estimate

$$
\|h^{-\frac{1}{2}}(v - I_T v)\|_{L^2(\Gamma)} \leq \|v\|_{H^\frac{1}{2}(\Gamma)} \quad \text{for all } v \in H^\frac{1}{2}(\Gamma).
$$

Proof. Given $v \in H^\frac{1}{2}(\Gamma)$, there exists $w \in H^\frac{1}{2}(\Gamma)$ so that $t_\perp(w) = v$ according to Lemma 2.15. For each $T \in T$, we define $w_T = \int_{\Delta_T} w(x) dx \in \mathbb{R}^3$ and $\nu_T = t_\perp(w_T) \in H^\frac{1}{2}(\Gamma)$. Since the estimate of Lemma 2.14 is local, we have

$$
\|h^{-\alpha}(v - I_T v)\|_{L^2(\Gamma)}^2 = \sum_{T \in T} \|h^{-\alpha}(v - I_T v)\|_{L^2(T)}^2 \leq \sum_{T \in T} \|h^{-\frac{1}{2}}(v - \nu_T)\|_{L^2(\Delta_T)}^2
$$

for $\alpha = -1, 0$. Using Lemma 2.16 yields

$$
\|v - \nu_T\|_{L^2(\Delta_T)} = \|t_\perp(w - \nu_T)\|_{L^2(\Delta_T)} \leq \|w - \nu_T\|_{L^2(\Delta_T)}
$$

and stability of the $L^2$-projection together with the definition of $\nu_T$ implies

$$
\|w - \nu_T\|_{L^2(\Delta_T)} \leq \|w\|_{L^2(\Delta_T)},
\|h^{-\frac{1}{2}}(w - \nu_T)\|_{L^2(\Delta_T)} \leq \|w\|_{H^\frac{1}{2}(\Delta_T)},
$$

whence

$$
\|v - I_T v\|_{L^2(\Gamma)}^2 \leq \sum_{T \in T} \|w\|^2_{L^2(\Delta_T)} \leq \|w\|^2_{L^2(\Gamma)},
\|h^{-\frac{1}{2}}(v - I_T v)\|_{L^2(\Gamma)}^2 \leq \sum_{T \in T} \|w\|^2_{H^\frac{1}{2}(\Delta_T)} \leq \|w\|^2_{H^\frac{1}{2}(\Gamma)}.
$$

Applying Corollary 2.8 to vector-valued functions, we obtain

$$
\|h^{-\frac{1}{2}}(v - I_T v)\|_{L^2(\Gamma)} \leq \|w\|_{H^\frac{1}{2}(\Gamma)} \leq \|v\|_{H^\frac{1}{2}(\Gamma)}',
$$

where the last inequality results from Lemma 2.15. This concludes the proof. \qed
3. Problem Setting

The variational formulation of the Electric Field Integral Equation (EFIE), also called Rumsey principle, consists of seeking $u \in H^{-\frac{1}{2}}_{\text{div}}(\Gamma)$ such that

$$a(u, v) = \langle f, v \rangle_\Gamma$$

for all $v \in H^{-\frac{1}{2}}_{\text{div}}(\Gamma)$

where $f \in H^{-\frac{1}{2}}_{\text{curl}}(\Gamma)$, the sesquilinear form $a(\cdot, \cdot)$ is given by

$$a(u, v) := \langle V_k \text{div} \Gamma u, \text{div} \Gamma v \rangle_{\frac{1}{2}, \Gamma} - k^2 \langle A_k u, v \rangle_\Gamma,$$

$\langle \cdot, \cdot \rangle_\Gamma$ is the duality pairing between $H^{-\frac{1}{2}}_{\text{curl}}(\Gamma)$ and $H^{-\frac{1}{2}}_{\text{div}}(\Gamma)$, $\langle \cdot, \cdot \rangle_{\frac{1}{2}, \Gamma}$ is the duality pairing $H^{\frac{1}{2}}(\Gamma) - H^{-\frac{1}{2}}(\Gamma)$, and the integral operators $V_k, A_k$ has been defined in §2.3.

The discrete formulation reads: find $U \in RT_0$ such that

$$a(U, V) = \langle f, V \rangle_\Gamma$$

for all $V \in RT_0$.

Equation (16) is well-posed under the assumption that the wave number $k$ does not correspond to an interior eigenmode of the Maxwell problem on $\Gamma$. As a consequence, the following continuous inf-sup condition holds (see also [16]):

$$\|u\|_{H^{-\frac{1}{2}}_{\text{div}}(\Gamma)} \leq \sup_{v \in H^{-\frac{1}{2}}_{\text{div}}(\Gamma)} \frac{a(u, v)}{\|v\|_{H^{-\frac{1}{2}}_{\text{div}}(\Gamma)}}$$

for all $u \in H^{-\frac{1}{2}}_{\text{div}}(\Gamma)$.

Since the boundary element discretization is conforming, i.e. $RT_0 \subset H^{-\frac{1}{2}}_{\text{div}}(\Gamma)$, the following Galerkin orthogonality holds: if $u \in H^{-\frac{1}{2}}_{\text{div}}(\Gamma)$ is the solution of (16) and $U \in RT_0$ is the solution of (17), then

$$a(u - U, V) = 0$$

for all $V \in RT_0$.

In addition, as a direct consequence of the Cauchy-Schwarz inequality and the continuity (9) of the single layer boundary operators, the form $a(\cdot, \cdot)$ is continuous:

$$a(v, w) \leq \|v\|_{H^{\frac{1}{2}}_{\text{div}}(\Gamma)} \|w\|_{H_{\text{curl}}^{-\frac{1}{2}}(\Gamma)}$$

for all $v, w \in H_{\text{div}}^{-\frac{1}{2}}(\Gamma)$.

4. A Posteriori Error Analysis

As is customary in the theory of a posteriori error estimation, one has to assume a higher regularity of the right-hand side than it is needed for well-posedness in order to derive computable error bounds. Therefore we assume in this section that

$$f \in H_{\text{curl}}^{\frac{1}{2}}(\Gamma) \cap H_{\text{curl}}^0(\Gamma)$$

with $H_{\text{curl}}^{\frac{1}{2}}(\Gamma)$ given in Proposition 2.1 and $H_{\text{curl}}^0(\Gamma) = \{ v \in L^2(\Gamma) \mid \text{curl}_\Gamma v \in L^2(\Gamma) \}$.

We proceed as in Cascón, Nochetto, and Siebert [8] for flat domains. To this end, we start with some auxiliary results that will be useful for our analysis later.

Lemma 4.1 (Regular decomposition [5, Theorem 5.5]). The decomposition

$$H_{\text{div}}^{-\frac{1}{2}}(\Gamma) = \text{curl}_\Gamma(H^{\frac{1}{2}}(\Gamma)/\mathbb{C}) + H_{\text{curl}}^{\frac{1}{2}}(\Gamma)$$
is valid and is stable, i.e. for \( v = \Psi + \text{curl}_T \alpha \) with \( \Psi \in H^{1/2}_T(\Gamma) \) and \( \alpha \in H^{3/2}(\Gamma) \setminus \mathbb{C}, \)
\[
\| \Psi \|_{H^{1/2}_T(\Gamma)} + \| \alpha \|_{H^{3/2}(\Gamma)} \leq \| v \|_{H^{-1/2}_\text{div}(\Gamma)} \quad \text{for all } v \in H^{-1/2}_\text{div}(\Gamma).
\]

**Lemma 4.2.** For \( \mathcal{V}(\mathcal{T}) \) given by (12) there holds \( \text{curl}_T(\mathcal{V}(\mathcal{T})) \subset \mathcal{R}T_0. \)

*Proof.* By [5, Corollary 5.3] we have
\[
\ker(\text{div}_T) \cap L^2(\Gamma) = \text{curl}_T(H^1(\Gamma)).
\]
Thus for all \( \alpha \in \mathcal{V}(\mathcal{T}) \subset H^1(\Gamma) \) we infer that \( \text{curl}_T \alpha \in L^2(\Gamma) \) is piecewise constant and that \( \text{div}_T \text{curl}_T \alpha \equiv 0 \in L^2(\Gamma). \) This implies that \( \text{curl}_T \alpha \in H^0_\text{div}(\Gamma). \)

### 4.1. Upper Bound

Let \( u \in H^{1/2}_\text{div}(\Gamma) \) be the exact solution of (16) and \( U \in \mathcal{R}T_0 \) be its approximation defined by (17). By the Galerkin orthogonality observe that
\[
a(u - U, v - V) = a(u - U, v - V) \quad \text{for all } v \in H^{1/2}_\text{div}(\Gamma), V \in \mathcal{R}T_0.
\]
Decompose \( v \) as \( v = \Psi + \text{curl}_T \alpha \), according to Lemma 4.1, and define
\[
\delta \Psi := \Psi - \Psi_T, \quad \delta \alpha := \alpha - \alpha_T
\]
where \( \Psi_T \in \mathcal{R}T_0 \) and \( \alpha_T \in \mathcal{V}(\mathcal{T}) \) can be arbitrarily chosen. Thus we can write
\[
v - V = \delta \Psi + \text{curl}_T \delta \alpha
\]
and
\[
a(u - U, v - V) = \Delta \langle f, \delta \Psi + \text{curl}_T \delta \alpha \rangle - a(U, \delta \Psi + \text{curl}_T \delta \alpha)
\]
\[
= \langle f, \delta \Psi \rangle + \langle k^2 A_k U, \delta \Psi \rangle_{\Gamma, \Gamma}
\]
\[
+ \langle f, \text{curl}_T \delta \alpha \rangle + \langle k^2 A_k U, \text{curl}_T \delta \alpha \rangle_{\Gamma, \Gamma} = \langle V \text{div}_T U, \text{div}_T \delta \Psi \rangle_{\Gamma, \Gamma}
\]
for any \( v \in H^{1/2}_\text{div}(\Gamma), \Psi_T \in \mathcal{R}T_0 \) and \( \alpha_T \in \mathcal{V}(\mathcal{T}) \). We proceed in four steps.

\[ \Box \] We note that \( f \in L^2(\Gamma), k^2 A_k U \in H^{1/2}_\parallel(\Gamma) \subset L^2(\Gamma) \) and that \( \Psi \in H^{1/2}(\Gamma) \subset L^2(\Gamma) \) due to enhanced regularity of \( \Psi \) asserted in Lemma 4.1. Since \( \Psi_T \in \mathcal{R}T_0 \subset L^2(\Gamma) \), we can replace the duality pairing in \( I_1 \) by an integral and thus write
\[
I_1 = \int_\Gamma (f + k^2 A_k U) \cdot \delta \Psi \, d\sigma.
\]

\[ \Box \] Since \( f \in H^{1/2}_\parallel(\Gamma) \) the duality pairing \( \langle \cdot, \cdot \rangle \) can be interpreted as
\[
\langle f, \text{curl}_T \delta \alpha \rangle = \langle f, \text{curl}_T \delta \alpha \rangle_{\Gamma, \Gamma},
\]
namely as a duality pairing in \( H^{1/2}_\parallel(\Gamma) \). The definition (8) of \( \text{curl}_T \) now yields
\[
I_2 = \langle f + k^2 A_k U, \text{curl}_T \delta \alpha \rangle_{\Gamma, \Gamma} = \langle \text{curl}_T (f + k^2 A_k U), \delta \alpha \rangle_{\Gamma, \Gamma}
\]
Since \( \delta \alpha \in H^{1/2}(\Gamma) \) and \( \text{curl}_T (f + k^2 A_k U) \in L^2(\Gamma) \) because of (11) and (21), we can also write \( I_2 \) as an integral
\[
I_2 = \int_\Gamma \text{curl}_T (f + k^2 A_k U) \, \delta \alpha \, d\sigma.
\]

\[ \Box \] For the last term \( I_3 \), we integrate by parts according to (8), whence
\[
I_3 = -\langle \text{grad}_T (V_k \text{div}_T U), \delta \Psi \rangle_{\Gamma, \Gamma}.
\]
Since $\text{div}_T(\mathbf{R} \mathbf{T}_0) \subset L^2(\Gamma)$ we infer that $\text{grad}_T(\mathbf{V}_k \text{div}_T \mathbf{U}) \in L^2(\Gamma)$ in light of (10). This implies that $\mathcal{I}_3$ is also an integral

\begin{equation}
\mathcal{I}_3 = - \int_{\Gamma} \text{grad}_T(\mathbf{V}_k \text{div}_T \mathbf{U}) \cdot \partial \Psi \, d\sigma.
\end{equation}

Inserting (23)-(25) back into the sesquilinear form $a$ yields

\begin{equation}
a(\mathbf{u} - \mathbf{U}, \mathbf{v}) = \int_{\Gamma} \mathbf{R} \cdot \partial \Psi \, d\sigma + \int_{\Gamma} r \partial_\alpha \, d\sigma \quad \text{for all} \ \mathbf{v} \in H^{-\frac{1}{2}}_{\text{div}}(\Gamma),
\end{equation}

where $\mathbf{R} \in L^2(\Gamma)$ and $r \in L^2(\Gamma)$ are given element-by-element by

\begin{align}
\mathbf{R}_T &:= f + k^2 \mathbf{A}_k \mathbf{U} + \text{grad}_T(\mathbf{V}_k \text{div}_T \mathbf{U}) \quad \text{for all} \ T \in \mathcal{T}, \\
R_T &:= \text{curl}_T(f + k^2 \mathbf{A}_k \mathbf{U}) \quad \text{for all} \ T \in \mathcal{T}.
\end{align}

We now choose $\alpha_T = I_T \alpha$ and $\Psi_T = I_T \Psi$ where $I_T$ and $I_T$ are the interpolation operators of Definitions 2.9 and 2.11. Applying the Cauchy-Schwarz inequality and the interpolation estimates (13) and (15) yields

\begin{align}
a(\mathbf{u} - \mathbf{U}, \mathbf{v}) &\leq \|h^2 \mathbf{R}\|_{L^2(\Gamma)} \|h^\frac{1}{2} \partial \Psi\|_{L^2(\Gamma)} + \|h^2 r\|_{L^2(\Gamma)} \|h^{-\frac{1}{2}} \partial_\alpha\|_{L^2(\Gamma)} \\
&\leq \|h^2 \mathbf{R}\|_{L^2(\Gamma)} \|\Psi\|_{H^\frac{1}{2}_0(\Gamma)} + \|h^2 r\|_{L^2(\Gamma)} \|\partial_\alpha\|_{H^\frac{1}{2}(\Gamma)},
\end{align}

which together with the stability (22) of the regular decomposition leads to

\begin{equation}
a(\mathbf{u} - \mathbf{U}, \mathbf{v}) \leq \left( \|h^2 \mathbf{R}\|_{L^2(\Gamma)} + \|h^2 r\|_{L^2(\Gamma)} \right) \|\mathbf{v}\|_{H^{-\frac{1}{2}}_{\text{div}}(\Gamma)}.
\end{equation}

Combining this with the inf-sup condition (18) finally implies

\begin{equation}
\|\mathbf{u} - \mathbf{U}\|_{H_{\text{div}}^{-\frac{1}{2}}(\Gamma)} \leq \sup_{\mathbf{v} \in H_{\text{div}}^{-\frac{1}{2}}(\Gamma)} \frac{a(\mathbf{u} - \mathbf{U}, \mathbf{v})}{\|\mathbf{v}\|_{H_{\text{div}}^{-\frac{1}{2}}(\Gamma)}} \leq \|h^2 \mathbf{R}\|_{L^2(\Gamma)} + \|h^2 r\|_{L^2(\Gamma)}.
\end{equation}

We summarize this derivation in the following theorem.

**Theorem 4.3** (Upper bound). Let $f \in H^\frac{1}{2}_0(\Gamma) \cap H^0_{\text{curl}}(\Gamma)$, $\mathbf{u} \in H_{\text{div}}^{-\frac{1}{2}}(\Gamma)$ be the exact solution of (16) and $\mathbf{U} \in \mathbf{RT}_0$ be its approximation defined by (17). Then, there exists a constant $C_1 > 0$ depending on shape regularity of $\mathcal{T}$ such that the following bound holds

\begin{equation}
\|\mathbf{u} - \mathbf{U}\|_{H_{\text{div}}^{-\frac{1}{2}}(\Gamma)}^2 \leq C_1 \sum_{T \in \mathcal{T}} \eta^2_T(T),
\end{equation}

where the element indicators $\eta_T(T)$ are defined as follows in terms of the residuals $\mathbf{R} \in L^2(\Gamma)$ and $r \in L^2(\Gamma)$ given in (27)

\begin{equation}
\eta^2_T(T) := h_T \|\mathbf{R}\|_{L^2(\Gamma)}^2 + h_T \|r\|_{L^2(\Gamma)}^2.
\end{equation}

**Remark 4.4** (Trace regularity of an incident plane wave). It the right hand side $f$ is the tangential trace of a plane wave $\mathbf{E}_{\text{inc}}$, then we conclude from the analyticity of the plane wave and of all its derivatives that

\begin{equation}
f = \gamma \|E_{\text{inc}}\|_{H^\frac{1}{2}(\Gamma)}, \quad \gamma \|\partial_{\nu_0} E_{\text{inc}}\|_{H^\frac{1}{2}(\Gamma)}
\end{equation}

for $i = 1, 2, 3$. Therefore, $f$ satisfies the stated regularity assumption (21).
4.2. Lower Bound. We next show a global lower bounds for the error indicators \( \eta_{T}^2(\cdot) \). Since \( R \in L^2_T(\Gamma) \) and \( r \in L^2(\Gamma) \) we define the local constants
\[
R_T = \int_T R(x) \, d\sigma(x) \quad r_T = \int_T r(x) \, d\sigma(x), \quad \text{for all } T \in \mathcal{T},
\]
and their global piecewise constant counterparts \( R_0|_T = R_T \) and \( r_0|_T = r_T \).

**Theorem 4.5** (Global lower bound for the residual). Let \( u \in H^{1/2}_\text{div}(\Gamma) \) be the exact solution of (16) and \( U \in RT_0 \) be its approximation defined by (17). Then, there exists a constant \( C_2 > 0 \), only depending on shape regularity of \( T \), such that the following bound holds
\[
C_2 \| h^2 R \|_{L^2(\Gamma)} \leq \| u - U \|_{H^{1/2}_\text{div}(\Gamma)} + \| h^2 (R - R_0) \|_{L^2(\Gamma)}.
\]

**Proof.** Let \( b_T: \Omega \to \mathbb{R} \) be a bubble function, namely a Lipschitz function so that
\[
supp b_T \subset T, \quad \int_T b_T \, dx = |T| \approx \int_T b_T^2 \, dx,
\]
for a given \( T \in \mathcal{T} \). Such a function can be given by a polynomial of degree three on \( T \) consisting of the product of all three barycentric coordinates times a real scaling factor. Let \( \Psi_T = \sigma_T b_T \) with \( \sigma_T \in \mathbb{C}^2 \) and note that as a direct consequence of the first point we have
\[
\int_T \text{div}_T \Psi_T \, d\sigma = \int_{\partial T} \Psi_T \cdot n_T \, ds = 0.
\]
For the particular choice \( \sigma_T = h_T R_T \), we see that
\[
\int_T R_T \cdot \Psi_T \, dx = h_T \| R_T \|_{L^2(T)}^2
\]
and
\[
\| \Psi_T \|_{L^2(T)} \leq h_T \| R_T \|_{L^2(T)} \leq \| \Psi_T \|_{L^2(T)}.
\]
We construct a global function \( \Psi \) so that its restriction to \( T \) coincides with \( \Psi_T \) for all \( T \in \mathcal{T} \). We claim that \( \Psi \in H^{1/2}_\text{div}(\Gamma) \) because it is made of piecewise polynomials with vanishing normal component on the interelement boundaries of \( \mathcal{T} \). In view of Lemma 4.1, such a \( \Psi \) is an admissible test function in (26) and, together with the choices \( \Psi_T = 0 \) and \( \alpha = \alpha_T = 0 \), yields
\[
a(u - U, \Psi) = \int_T R \cdot \Psi \, d\sigma = \int_T (R - R_0) \cdot \Psi \, d\sigma + \int_T R_0 \cdot \Psi \, d\sigma
\]
\[
= \int_T (R - R_0) \cdot \Psi \, d\sigma + \| h^2 R_0 \|_{L^2(\Gamma)}^2.
\]
By the continuity of the sesquilinear form \( a(\cdot, \cdot) \), we have
\[
\| h^2 R_0 \|_{L^2(\Gamma)}^2 = a(u - U, \Psi) - \int_T (R - R_0) \cdot \Psi \, dx
\]
\[
\leq \| u - U \|_{H^{1/2}_\text{div}(\Gamma)} \| \Psi \|_{H^{1/2}_\text{div}(\Gamma)}^2 + \| h^2 (R - R_0) \|_{L^2(\Gamma)} \| h^2 R_0 \|_{L^2(\Gamma)}.
\]
It remains to estimate \( \| \Psi \|_{H^{1/2}_\text{div}(\Gamma)} \). For \( \varphi \in H^{1/2}(\Gamma) \), let \( \varphi_0 \) denote the elementwise average of \( \varphi \). The Bramble-Hilbert Lemma yields
\[
\| h^{-1/2}(\varphi - \varphi_0) \|_{L^2(\Gamma)} \leq \| \varphi \|_{H^{1/2}(\Gamma)},
\]
which in conjunction with (28) implies
\[\langle \text{div}_T \Psi, \varphi \rangle_{\frac{1}{2}, \Gamma} = \int_{\Gamma} \text{div}_T \Psi (\varphi - \varphi_0) \, dx \leq \|h^{\frac{1}{2}} \text{div}_T \Psi\|_{L^2(\Gamma)} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma)} \]
\[\leq \|h^{\frac{1}{2}} \Psi\|_{L^2(\Gamma)} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|h^{\frac{1}{2}} R_0\|_{L^2(\Gamma)} \]

because of the norm equivalence for the discrete function \(\Psi\). Now, by definition
\[\|\text{div}_T \Psi\|_{H^{-\frac{1}{2}}(\Gamma)} = \sup_{\varphi \in H^{\frac{1}{2}}(\Gamma)} \frac{\langle \text{div}_T \Psi, \varphi \rangle_{\frac{1}{2}, \Gamma}}{|\varphi|_{H^{\frac{1}{2}}(\Gamma)}} \leq \|h^{\frac{1}{2}} R_0\|_{L^2(\Gamma)},\]

and
\[\|\Psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|\Psi\|_{L^2(\Gamma)} \leq \|\Psi\|_{L^2(\Gamma)} R_0\|_{L^2(\Gamma)}.
\]

Consequently
\[\|\Psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|h^{\frac{1}{2}} R_0\|_{L^2(\Gamma)}\]

which together with (29) implies that
\[\|h^{\frac{1}{2}} R_0\|_{L^2(\Gamma)} \leq \|u - U\|_{H^{-\frac{1}{2}}(\Gamma)} + \|h^{\frac{1}{2}} (R - R_0)\|_{L^2(\Gamma)}\]

Invoking the triangle inequality finally finishes the proof. \(\square\)

It is important to realize the global nature of the above lower bound. This is due to the presence of integral operators \(V_{\kappa}, A_{\kappa}\) in the sesquilinear form \(a(\cdot, \cdot)\) which lead to a global norm for the error in (29) regardless of the support of \(\Psi\).

In a very similar fashion, the following theorem can also be proven.

**Theorem 4.6** (Global lower bound for the curl residual). Let \(u \in H^{\frac{1}{2}}_{\text{div}}(\Gamma)\) be the exact solution of (16) and \(U \in RT_0\) be its approximation defined by (17). Then, there exists a constant \(C_3 > 0\), only depending on shape regularity of \(T\), such that the following bound holds
\[C_3\|h^{\frac{1}{2}} r\|_{L^2(\Gamma)} \leq \|u - U\|_{H^{-\frac{1}{2}}_{\text{div}}(\Gamma)} + \|h^{\frac{1}{2}} (r - r_0)\|_{L^2(\Gamma)}\]

5. Conclusions

In this paper we develop the first a posteriori error estimates for the electric field integral equation on polyhedra. We choose, for simplicity, to derive residual based error estimates but believe that our theory extends to other non-residual estimators. We also choose to develop the theory for polyhedra, the most interesting and useful case in practice, but we expect the results to extend to smooth surfaces. For scattering problems on polyhedra, the solution \(u\) of the integral equation, or surface current, is not smooth whereas the regularity of the right-hand side \(f\) is dictated by the surface \(\Gamma\) because the incident wave is always smooth. This justifies our additional regularity assumption (21) which, coupled with the properties \(\text{grad}_T(V_{\kappa} \text{div}_T U) \in L^2(\Gamma), \text{curl}_T(A_{\kappa} U) \in L^2(\Gamma)\), allows us to evaluate the residuals \(R, r\) of (27) in \(L^2(\Gamma)\) and thus avoid dealing with fractional Sobolev norms. We derive computable global upper and lower a posteriori bounds for the estimator (up to oscillation terms). In contrast to PDE, the estimator is global and due to the presence of the potentials \(V_{\kappa}, A_{\kappa}\) in the definition of the sesquilinear form. However, the residuals \(R, r\) being evaluated in \(L^2(\Gamma)\) can be split elementwise and used
to drive an adaptive boundary element method (ABEM). The actual implementation of ABEM for EFIE is rather delicate and is not part of the current discussion, which focusses on the derivation and properties of the estimators.

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