MEKLER’S CONSTRUCTION AND TREE PROPERTIES

JINHOO AHN

Abstract. Mekler constructed a way to produce a pure group from any given structure where the construction preserves $\kappa$-stability for any cardinal $\kappa$. Not only the stability, it is known that his construction preserves various model-theoretic properties such as simplicity, NIP, and NTP$_2$. Inspired by the last result, we show that the construction also preserves NTP$_1$ (NSOP$_2$) and NSOP$_1$. As a corollary, we obtain that if there is a theory of finite language which is non-simple NSOP$_1$, or which is NSOP$_2$ but has SOP$_1$, then there is a pure group theory with the same properties, respectively.

1. Introduction

Suppose a structure $M$ of finite language has more than one element, then there is a graph $N$ which is bi-interpretable with $M$ [6, Thm 5.5.1]. This implies if $M$ has a model-theoretic property like stability and simplicity, then one can find a graph which has the same properties of $M$. Unlike the graph, it is not easy to see whether there is a group which preserves a model-theoretic property of $M$. A partial answer to this problem was found by Mekler. In [10], he constructed a group $G$ where Th($G$) has the same stability spectrum as Th($M$). This group is not bi-interpretable with $M$, however, so it does not preserve all the properties of $M$. For example, even though $M$ is $\aleph_0$-categorical, the group $G$ may not be.

Later, it is proved that many other properties related with Shelah’s classification program are preserved by Mekler’s construction. Baudisch and Pentzel proved that simplicity is preserved by the construction, and assuming stability, Baudisch proved that CM-triviality is also preserved [1]. Recently, Chernikov and Hempel proved that the construction preserves NIP, $k$-dependence, and NTP$_2$ [3]. Thus, it is natural to expect that the construction preserves NSOP$_1$ and other non-simple theories [3, Conjecture 1].

In this paper we show that the conjecture is true for the following tree properties; NTP$_1$ (NSOP$_2$) and NSOP$_1$. To prove them, we analogously follow the argument used in the proof of preservation of NTP$_2$ in [9]. The difference is that parameters witnessing TP$_2$ formula is an array, not a tree. Hence, we need an appropriate generalized indiscernibility for each properties substituting the role of mutual indiscernibility.

We use one of the tree indiscernibility, called strong indiscernibility (see Definition 2.4) [9, 13]. SOP$_1$ formula also have parameters of a tree, but the strong indiscernibility is not much helpful. We recall the result of [7] to obtain parameters of array $\omega \times 2$. From the equivalent conditions of NSOP$_1$ [7, Proposition 2.4], we define another indiscernibility, called comb indiscernibility (see Definition 3.10).

We preview the corollaries of the main results.

Corollary 1.1. (1) There is a non-simple NSOP$_1$ pure group theory.

(2) If there is an NSOP$_2$ theory which has SOP$_1$, then there is a pure group theory with the same properties.

The first one is obtain by the preservation of simplicity and NSOP$_1$. Any example of non-simple NSOP$_1$ theory on finite language can be transformed into a pure group by the construction. Similarly, any NSOP$_2$ theory with SOP$_1$ on finite language can be transformed into a pure group. Thus, NSOP$_1$ and NSOP$_2$ are equivalent if and only if they are equivalent on the pure group theories.
In section 2, we introduce the notions about strong indiscernibility on trees from [9] and [13]. In section 3, using strong indiscernibility, we find equivalent conditions of NTP₂. And then, we define a comb indiscernibility and find equivalent conditions of NSOP₁. In section 4, we describe and summarize definitions and facts of Mekler’s construction following by [6] and [3]. In section 5, we first observe some combinatorial remarks on trees in [6], then show our main results that Mekler’s construction preserves NTP₁ and NSOP₁.

2. TREE INDISCERNIBILITY

Consider a tree <λ of height λ which has κ many branches. Each element in the tree can be considered as a string. We denote () as an empty string, 0 as a string of α many zeros, and α as a string (α) of length one.

Definition 2.1. Fix a tree <λ, and let η, ν, κ ∈ <λ.
(1) (Ordering) η < ν if ν’(α) = η for some ordinal α ∈ dom(ν).
(2) (Meet) η = η ∧ ν if η is the meet of η and ν, i.e., η = η[β], when β = \( \bigcup \{ \alpha \leq \text{dom}(\eta) \cap \text{dom}(\nu) \mid \eta[\alpha] = \nu[\alpha] \} \). For \( \eta \in <\lambda \), \( \nu \) is the meet closure of \( \eta \) if \( \nu = \{ \eta_1 \land \eta_2, \eta_1, \eta_2 \in \eta \} \).
(3) (Incomparability) η ⊥ ν if they are ≤-incomparable, i.e., ¬(η ≤ ν) and ¬(ν ≤ η).
(4) (Lexicographic order) η < lex ν if
(a) η < ν, or
(b) η ⊥ ν and \( 2\alpha[\eta] = \nu[\alpha] \) and \( \eta(\alpha) < \nu(\alpha) \).

Definition 2.2. A strong language \( L_0 \) is defined by the collection \{<, ∧, < lex\}.

We may view the tree <λ as an \( L_0 \)-structure.

Definition 2.3. A tree \( B = <\lambda \) of a set of parameters \( A = <\lambda \) if \( B \subseteq A \) and the inclusion map is an embedding in the language \{≤, < lex\}.

Fix a complete first order theory \( T \) (with language \( L \)). Let \( M \models T \) be a monster model. From now on, we will work in this \( M \).

Definition 2.4. Fix a structure \( I \) with language \( L_I \). For a set \( \{b_i \mid i \in I\} \), we say it is \( I \)-indexed indiscernible if for any finite \( i \) and \( j \) from \( I \),
\[
\text{qftp}(i)_x = \text{qftp}(j)_x \Rightarrow (b_i)_{i \in I} \equiv (b_j)_{j \in I}.
\]
\( I \) is called the index structure. In particular, we say a set \( \{b_i \mid i \in <\lambda \} \) is strongly indiscernible if it is \( I \)-indexed indiscernible for \( I \) the \( L_0 \)-structure on <λ.

Remark 2.5. Let \( \{a_0 \mid \eta \in <\lambda \} \) be a strongly indiscernible tree.
(1) For all \( \nu, \nu_2 < \lambda \), \( (a_{\nu_1}(\eta))_{\eta < \lambda} \equiv (a_{\nu_2}(\eta))_{\eta < \lambda} \).
(2) For all \( \eta_1, \eta_2 < \lambda \), if \( \nu_1 \) is the meet-closure of \( \eta_1 \), then \( \text{qftp}(\nu_1) = \text{qftp}(\eta_1) \).
(3) For all \( \eta \in <\lambda \), \( \eta < lex \nu \Rightarrow a_0 \nu_\eta \equiv a_0 \nu \).
(4) For any \( \eta \in <\lambda \), the tree \( (a_0 < \eta \eta)_{\eta < <\lambda} \) is strongly indiscernible over \( (a_{\nu_1}(\eta)_{\eta \in \text{dom}(\eta)} \).

Proof. See [4] and [6]. □

Definition 2.6. Let \( I \) be an index structure.
(1) The EM-type of a set of parameters \( A = \{a_i \mid i \in I\} \), EM₂(A), is the collection of formulas \( \varphi(x_1, \ldots, x_n) \) in \( L \) with variables \( \{x_i \mid i \in I\} \) such that for all \( j_1, \ldots, j_n \in I \), if \( j_1 \ldots j_n \equiv \varphi 1 \ldots i_1 \ldots i_n \), then \( \varphi(a_{j_1}, \ldots, a_{j_n}) \).
(2) A set \( B = \{b_\eta \mid \eta \in I \} \) is based on a set \( A = \{a_\nu \mid \nu \in I \} \) if for all \( \varphi(x_1, \ldots, x_n) \) in \( L \) and for all \( \eta_1, \ldots, \eta_n \in I \), there exists some \( \nu_1, \ldots, \nu_n \in I \) such that
(a) \( \nu_1 \ldots \nu_n \equiv \varphi \eta_1 \ldots \eta_n \), and
(b) \( b_{\eta_1} \ldots b_{\eta_n} \equiv a_{\nu_1} \ldots a_{\nu_n} \).

In particular, when \( I \) is \( L_0 \)-structure \( <\lambda \), we say \( B \) is strongly based on \( A \) whenever \( B \) is based on \( A \).

Remark 2.7. Let \( B = \{b_\eta \mid \eta \in I \} \) and \( A = \{a_\nu \mid \nu \in I \} \). Then \( B \) is based on \( A \) if and only if \( B \models \text{EM}_2(A) \).
Definition 3.1. We say a subset \( \{ \eta_i \mid i < k \} \subseteq \kappa^\omega \) is a collection of \( k \) distant siblings if given \( i_1 \neq i_2 \) and \( j_1 \neq j_2 \), all of which are less than \( k \), \( \eta_{i_1} \land \eta_{i_2} = \eta_{j_1} \land \eta_{j_2} \).

We may extend the result in Proposition 2.5(3).

Remark 3.2. Let \( \{ a_\eta \mid \eta \in \kappa^\omega \} \) be a strongly indiscernible tree. For any collection of distant siblings \( \{ \eta_i \mid i < k \} \subseteq \kappa^\omega \) and for any \( \xi \in \kappa^\omega \), if \( \eta_i <_{\text{lex}} \eta_j \) for each \( i < j < k \) then \( a_{\eta_0} \cdots a_{\eta_{k-1}} \equiv a_{\xi_0}^{\kappa^\omega} \cdots a_{\xi_{k-1}}^{\kappa^\omega} \).

Definition 3.3. [5, 11] Fix \( k \geq 2 \).

1. \( \varphi(x; y) \) has SOP\(_1\) if there is a \( (a_\eta \mid \eta \in \kappa^\omega) \) such that
   a. For all \( \eta \in \kappa^\omega \), \( \{ \varphi(x; a_{\eta_0}) \mid \alpha < \omega \} \) is consistent,
   b. For all \( \xi, \nu \in \kappa^\omega \), if \( \xi \cap \nu \neq \emptyset \), then \( \{ \varphi(x; a_{\xi_i}), \varphi(x; a_{\nu_j}) \} \) is inconsistent.

2. \( \varphi(x; y) \) has SOP\(_2\) if there is a \( (a_\eta \mid \eta \in \kappa^\omega) \) such that
   a. For all \( \eta \in \kappa^\omega \), \( \{ \varphi(x; a_{\eta_0}) \mid \alpha < \omega \} \) is consistent,
   b. For all \( \xi, \nu \in \kappa^\omega \), if \( \xi \cap \nu \neq \emptyset \), then \( \{ \varphi(x; a_\xi), \varphi(x; a_\nu) \} \) is inconsistent.

3. \( \varphi(x; y) \) has the tree property of the first kind (TP\(_1\)) if there is \( (a_\eta \mid \eta \in \kappa^\omega) \) such that
   a. For all \( \eta \in \kappa^\omega \), \( \{ \varphi(x; a_{\eta_0}) \mid \alpha < \omega \} \) is consistent,
   b. For all \( \xi, \nu \in \kappa^\omega \), \( \{ \varphi(x; a_{\xi_i}), \varphi(x; a_{\nu_j}) \} \) is inconsistent.

4. \( \varphi(x; y) \) has weak-TP\(_1\) if there is \( (a_\eta \mid \eta \in \kappa^\omega) \) such that
   a. For all \( \eta \in \kappa^\omega \), \( \{ \varphi(x; a_{\eta_0}) \mid \alpha < \omega \} \) is consistent,
   b. For all \( \eta, \nu \in \kappa^\omega \), \( \{ \varphi(x; a_{\eta_i}), \varphi(x; a_{\nu_j}) \} \) is inconsistent.

5. We say \( T \) has TP\(_1\) (resp. SOP\(_1\), SOP\(_2\)) if there is a formula having TP\(_1\) (resp. SOP\(_1\), SOP\(_2\)) if not, we say \( T \) is NTP\(_1\) (resp. NSOP\(_1\), NSOP\(_2\)). We say \( T \) has weak-TP\(_1\) if there is a formula having k-TP\(_1\) for some \( k \).

Fact 3.4. [4]

1. \( \varphi(x; y) \) has TP\(_1\) if and only if there is a strongly indiscernible tree \( (a_\eta \mid \eta \in \kappa^\omega) \) such that
   a. \( \{ \varphi(x; a_{\eta_0}) \mid \alpha < \omega \} \) is consistent for some \( \eta \in \kappa^\omega \).
   b. \( \{ \varphi(x; a_{\eta_{i<j}}) \mid i < \omega \} \) is pairwise inconsistent for some \( \nu \in \kappa^\omega \).

2. \( T \) has weak-TP\(_1\) if and only if \( T \) has TP\(_1\).

Remark 3.5. \( \varphi(x; y) \) has weak-TP\(_1\) if and only if there is a strongly indiscernible tree \( (a_\eta \mid \eta \in \kappa^\omega) \) such that

a. \( \{ \varphi(x; a_{\eta_i}) \mid i < \omega \} \) is consistent.

b. \( \{ \varphi(x; a_{\eta_{i<j}}) \mid i < \omega \} \) is \( k \)-inconsistent for some \( \nu \in \kappa^\omega \) and \( k \geq 2 \).

To prove the main theorem, we first establish the characterization of the given model theoretic property. For example, in [4], Chernikov and Hempel stated the following proposition cited from [2];

Fact 3.6. Let \( T \) be a theory and \( M \models T \) a monster model. Let \( \kappa := |T|^+ \). The following are equivalent:

1. \( T \) is NTP\(_2\).

2. For any array \( (a_{i,j} : i \in \kappa, j \in \omega) \) of finite tuples with mutually indiscernible rows and a finite tuple \( b \), there is some \( \alpha \in \kappa \) satisfying the following:
   a. For all \( i > \alpha \), there is some \( b' \) such that
      a. \( (a_{i,j} \mid j < \omega) \) is indiscernible over \( b' \), and
Proposition 3.7. Let $\kappa > 2^{\|T\|}$ be some sufficiently large regular cardinal. Then TFAE.

(1) $T$ is NTP$_1$.

(2) For any strongly indiscernible tree $(a_\alpha \mid \eta \in \kappa)$ of finite tuples and a finite tuple $b$, there is some $\beta < \kappa$ and $b'$ such that

- $(a_{\alpha \cup \beta})_i \mid i < \omega$ is indiscernible over $b'$, and
- $tp(b/a_{\alpha \cup \beta} - \eta) = tp(b'/a_{\alpha \cup \beta} - \eta)$.

(3) For any strongly indiscernible tree $(a_\alpha \mid \eta \in \kappa)$ of finite tuples and a finite tuple $b$, there is some $\gamma < \kappa$ satisfying the following:

- for all $\beta > \gamma$, there is some $b'$ such that
  - $tp(b/a_{\alpha \cup \beta} - \eta) = tp(b'/a_{\alpha \cup \beta} - \eta)$.

Proof. (1) $\Rightarrow$ (2). Assume $T$ has NTP$_1$, and let $A = (a_\alpha \mid \eta \in \kappa)$ and $b$ be given.

By pigeonhole principle, there is a subsequence $(a_{\alpha_i} \mid i < \omega)$ in the set of successor ordinals smaller than $\kappa$ such that for all $i < j < \omega$, $a_i < a_j < \kappa$ and $tp(a_{\alpha_i}/b) = tp(a_{\alpha_j}/b)$. We inductively define a subtree $(a_\alpha \mid \eta \in \kappa)$ in $A$ as follows:

- $a_0 = a_{\alpha_0}$,
- for any $\eta \in \kappa$, if $a_\eta = a_\nu$ for some $\rho < \kappa$ with $Dom(p) = \alpha_i$, then for each $j < \omega$, $a_{\eta' \cup \beta_j} = a_{\nu' \cup \beta_j}$ where $\beta_j = \rho^{j+1}0^f$ for some $\delta$ so that $Dom(p^{j+1}) = \alpha_{i+j}$.

Note that $(a_{\alpha_\eta} \mid i < \omega)$ is a subsequence of $(a_{\alpha_\eta} \mid \alpha < \kappa)$, and for any $\eta, \nu \in \kappa$, if $a_\eta = a_\nu$, $a_\eta' = a_\nu'$, $a_{\eta \cup \nu} = a_\rho$ for $\eta', \nu'$, and $\xi < \kappa$, then $\xi \in \eta' \wedge \nu'$. Moreover, the subtree is strongly indiscernible, too.

Let $p(x, a_\eta') := tp(b/a_{\eta'}b)$ and $q(x) := \bigcup_{\eta < \kappa} p(x, a_\eta')$. We claim that $q$ is consistent.

Suppose not. Then by compactness and strong indiscernibility, there is a formula $\varphi(x, y)$ such that $\varphi(x, a_\eta') \in p(x, a_\eta')$ and $\{\varphi(x, a_\eta') \mid i < \omega\}$ is $k$-inconsistent for some $k < \omega$. On the other hand, $b$ realizes $\bigcup_{\eta < \kappa} p(x, a_{\eta'})$, so it realizes $\{\varphi(x, a_{\eta'}) \mid i < \omega\}$. As a result, $\varphi$ has weak $k$-TP$_1$, and fact [3.4] further says that $T$ has TP$_1$.

By claim, we can find a realization $b' \models q(x)$.

We may assume $(a_{\alpha_\eta} \mid i < \omega)$ is indiscernible over $b'$ by Ramsey and compactness. Note that $tp(b/a_{\eta'}b') = tp(b'/a_{\eta'}b')$ since $b' \models p(x, a_\eta')$. Thus, if $a_\eta = a_{\alpha_\eta}$ for some $\beta$, then this $\beta$ and $b'$ is the desired one.

(2) $\Rightarrow$ (3). Assume (2). For a strongly indiscernible tree $(a_\alpha \mid \eta \in \kappa)$ and a finite tuple $b$, we will say $Q(\beta)$ holds on $(a_{\alpha} \mid \eta \in \kappa)$ and $b$ when there is a $b'$ such that $(a_{\alpha \cup \beta})_i \mid i < \omega$ is indiscernible over $b'$, and $tp(b/a_{\alpha \cup \beta} - \eta) = tp(b'/a_{\alpha \cup \beta} - \eta)$. Suppose there is a strongly indiscernible tree $(a_{\eta} \mid \eta \in \kappa)$ and a finite tuple $b$ such that for any $\gamma < \kappa$, there is a $\beta > \gamma$ which does not satisfy $Q$. Then, since $cf(\kappa) = \kappa$, we can find a cofinal map $f : \kappa \to \kappa$ such that for any $i, j < \kappa$, $1 < f(i)^+ < f(j)$, and $f(i)$ does not satisfy $Q$.

Now, construct the following map $g : \kappa \to \kappa$. First, if $\eta = \langle \rangle$, then $g(\eta) = \langle \rangle$. For other non-empty string $\eta \in \kappa$, $Dom(g(\eta)) = \Sup\{f(l)^+ \mid l \in Dom(\eta)\}$, and for each $i < Dom(g(\eta))$,

$$(g(\eta))(i) = \begin{cases} 
\eta(j) & \text{if } i = f(j)^+ \text{ for some } j < \kappa, \\
0 & \text{otherwise}.
\end{cases}$$

Then we define $a_\eta' = a_{g(\eta)}$. The subtree $(a_{\eta} \mid \eta \in \kappa)$ is strongly indiscernible, and for each $\beta < \kappa$, $Q(\beta)$ does not hold on $(a_{\eta} \mid \eta \in \kappa)$ and $b$. This contradicts to (2).

(3) $\Rightarrow$ (1). Assume (3). Suppose $T$ has TP$_1$ witnessed by $\varphi(x; y)$ and $(a_{\alpha} \mid \eta \in \kappa)$. We may assume the tree $(a_{\alpha} \mid \eta \in \kappa)$ is strongly indiscernible by the modeling property. Note the inconsistency and path consistency conditions still hold. Take $b \models \{\varphi(x, a_{\alpha}) \mid \alpha < \kappa\}$.

By the assumption, we have some ordinals $\gamma < \beta < \kappa$ and $b'$ such that

(a) $(a_{\alpha \cup \beta} - \eta) \mid i < \omega$ is indiscernible over $b'$, and
(b) $\text{tp}(b/a_{b}\beta\alpha\gamma\eta) = \text{tp}(b'/a_{b}\beta\alpha\gamma\eta)$.

Choose an automorphism $\sigma$ where it fixes $a_{b}\beta\alpha\gamma\eta$ and sends $b'$ to $b$. Denote $(a'_{i} | i < \omega)$ to $(\sigma(a_{b}\beta\alpha\gamma\eta) | i < \omega)$, then it is indiscernible over $b$. Since $a'_{0}$ is as same as $a_{b}\beta\alpha\gamma\eta$, $b \models \varphi(x,a'_{0})$, and then the indiscernibility implies that for all $i < \omega$, $b \models \varphi(x,a'_{i})$. But this is a contradiction because $\{\varphi(x,a'_{i}) | i < \omega\} \text{ is 2-inconsistent}$, as well as $\{\varphi(x, a_{b}\beta\alpha\gamma\eta) | i < \omega\}$ is.

□

Analogously, We find a lemma for SOP$_{1}$. Unlike the case of TP$_{1}$, we cannot use the strong indiscernibility. For instance, let $A$ be a tree which has the inconsistency condition of SOP$_{2}$ and let $B$ be a strong indiscernible tree based on $A$. There is no guarantee that $B$ has the inconsistency condition. Hence, we need another indiscernibility which matches up to SOP$_{1}$.

To find this, we recall results in [4] and [7].

Fact 3.8. [H] Suppose $\varphi(x;y)$ with the tree $(c_{\eta} | \eta \in <\omega 2)$ have SOP$_{1}$ where $\kappa \geq 2^{[T]}$. Then there is a sequence $(\eta_{i}, \nu_{i})_{i < \omega}$ of elements of $<\kappa 2$ such that

1. $c_{0} \equiv c_{0', \sigma \nu_{i}, c_{1}} c_{1}$ for all $i < \omega$.
2. $\{\varphi(x; c_{0}) | i < \omega\}$ is consistent, and
3. $\{\varphi(x, c_{1}) | i < \omega\}$ is 2-inconsistent.

Kaplan and Ramsey [7] proved more general result about SOP$_{1}$.

Fact 3.9. The following are equivalent;

1. $\varphi$ has SOP$_{1}$.
2. There is an array $(c_{i,j})_{i < \omega, j < 2}$ so that
   (a) $c_{0,0} \equiv c_{0,0 \sigma \nu_{i}, c_{1}} c_{1}$ for all $i < \omega$
   (b) $\{\varphi(x; c_{0}) | i < \omega\}$ is consistent
   (c) $\{\varphi(x, c_{1}) | i < \omega\}$ is 2-inconsistent.
3. There is an array $(c'_{i,j})_{i < \omega, j < 2}$ so that
   (a) $c'_{0,0} \equiv c_{0,0 \sigma \nu_{i}, c_{1}} c_{1}$ for all $i < \omega$
   (b) $\{\varphi(x; c'_{0}) | i < \omega\}$ is consistent
   (c) $\{\varphi(x, c'_{1}) | i < \omega\}$ is $k$-inconsistent for some $k \geq 2$.

Inspired from these facts, we define a new kind of indiscernibility.

Definition 3.10. We say an array $(c_{i,j})_{i < \omega, j < 2}$ is comb indiscernible over $A$ if

1. $(c_{0,0} c_{1})_{i < \omega}$ is an (order) indiscernible sequence over $A$, and
2. $c_{0,0} \equiv c_{0,0 \sigma \nu_{i}, c_{1}} c_{1}$ for all $i < \omega$.

Note that we may replace the array in [34] to a comb indiscernible one.

Remark 3.11. Let a comb indiscernible array $(c_{i,j})_{i < \omega, j < 2}$ be given.

1. For any $\kappa > \omega$, there is a comb indiscernible array $(c'_{i,j})_{i < \kappa, j < 2}$ such that $c_{i,j} = c'_{i,j}$ for all $i < \omega$ and $j < 2$.
2. For any $n < \omega$, $(c_{i,j})_{n \leq i < \omega, j < 2}$ is comb indiscernible over $\{c_{i,j} | i < n, j < 2\}$.

Proposition 3.12. Let $\kappa > 2^{[T]}$ be some sufficiently large regular cardinal. Then TFAE.

1. $T$ is NSOP$_{1}$
2. For any comb indiscernible array $(a_{i,j})_{i < \kappa, j < 2}$ of finite tuples and a finite tuple $b$, there is some $\beta < \kappa$ and some $b'$ such that
   (a) $\text{tp}(b/a_{0}) = \text{tp}(b'/a_{1})$,
   (b) $(a_{1,1} | i < \omega)$ is indiscernible over $b'$.
3. For any comb indiscernible array $(a_{i,j})_{i < \kappa, j < 2}$ of finite tuples and a finite tuple $b$, there is some $\gamma < \kappa$ satisfying the following:
   for any number $\beta > \gamma$, there is some $b'$ such that
   (a) $\text{tp}(b/a_{0}) = \text{tp}(b'/a_{1})$,
   (b) $(a_{1,1} | i < \omega)$ is indiscernible over $b'$.
Proof. (1) ⇒ (2). Assume $T$ is NSOP$_1$, and let $(a_{i,j})_{i< \kappa, j< 2}$ and $b$ in (2) be given. Using the pigeonhole principle, take a comb indiscernible subarray $(a'_{i,j}, a''_{i,j})_{i< \kappa}$ in $(a_{i,j}, b_{i,j})_{i< \kappa}$ where $tp(a'_{i,j}/b) = tp(a''_{i,j}/b)$ for all $i, j < \omega$.

Now let $p(x, y)$ in $p(x, y)$ such that $\{\varphi(x, a'_{i,j}) \mid i < \omega\}$ is $k$-inconsistent for some natural number $k$. On the other hand, since $\varphi(x, a'_{i,j})$ is in $tp(b/a_{i,j})$ for each $i < \omega$, $\{\varphi(x, a'_{i,j}) \mid i < \omega\}$ is consistent. Hence $\varphi$ has SOP$_1$ by Fact 3.5 which is a contradiction.

Let $\bar{y} = (y_i)_{i< \kappa}$ where for each $i < \omega$, $|y_i| = |a'_{i,j}|$, and let $\Pi(x, \bar{y})$ be the union of $\bigcup_{i< \kappa} p(x, y_i) \cup \bigcup_{i< \kappa} \Phi(y_i)$, and $\Psi(x, y)$ where $\Phi(y_i) = tp((a'_{i,j}, a''_{i,j})_{i< \omega})$ and $\Psi(x, y)$ means that $(y_i)_{i< \omega}$ is indiscernible over $x$. By Ramsey and compactness, $\Pi(x, \bar{y})$ is consistent. Let $(b'', (a''_{i,j})_{i< \kappa})$ be a realization of $\Pi(x, \bar{y})$. Since $\Phi(y_i) = tp((a'_{i,j}, a''_{i,j})_{i< \kappa})$, we have $(a_{i,j} \mid i < \omega)$ is indiscernible over some $b'$ where $tp(b'', a_{i,j}) = tp(b'', a_{i,j}) = p(x, y_0) = tp(b, a_{i,j})$. Note $a_{0,0} = a_{1,0}$ for some $\beta < \kappa$.

(2) ⇒ (3). For a comb indiscernible array $(a_{i,j})_{i< \kappa, j< 2}$ and a finite tuple $b$, we will say $Q(\beta)$ holds on $(a_{i,j})_{i< \kappa, j< 2}$ and $b$ when there is a $b'$ such that $tp(b/a_{i,j}) = tp(b'/a_{i,j})$, and $(a_{i,j} \mid i < \omega)$ is indiscernible over $b'$. Assume $T$ is NSOP$_1$, but (3) does not hold, that is, there is a comb indiscernible array $(a_{i,j})_{i< \kappa, j< 2}$ and a tuple $b$ such that for any $\gamma < \kappa$, there is some $\beta > \gamma$ which does not satisfy $Q$. From this and $cf(\kappa) = \kappa$, we choose an increasing sequence $(\beta_i)_{i< \kappa}$ of ordinal numbers such that for each $\beta_i$, $Q$ does not hold.

Now take a subarray $(a_{i,j})_{i< \kappa, j< 2}$ in $(a_{i,j})_{i< \kappa, j< 2}$. This array is still comb indiscernible, so (by 2), there is some $j < \kappa$ and some $b'$ such that $tp(b/a_{i,j}) = tp(b'/a_{i,j})$, and $(a_{i,j} \mid i < \omega)$ is indiscernible over $b'$. Since $tp((a_{i,j})_{i< \omega}) = tp((a_{i,j})_{i< \omega})$ by indiscernibility, we may assume that there is some $j < \kappa$ and some $b'$ such that $tp(b/a_{i,j}) = tp(b'/a_{i,j})$, and $(a_{i,j} \mid i < \omega)$ is indiscernible over $b'$. This contradicts that $Q(\beta_i)$ does not hold.

(3) ⇒ (1). Assume (3). Suppose $T$ has SOP$_1$. By fact 3.5(2) and compactness, we have a formula $\varphi(x, y)$ which witnesses SOP$_1$ with a comb indiscernible array $(a_{i,j})_{i< \kappa, j< 2}$. Let $b$ be a realization of $\bigcup_{i< \kappa} \varphi(x, a_{i,j})$. By assumption, there is some ordinals $\gamma < \beta < \kappa$ and some $b'$ such that

(a) $tp(b/a_{i,j}) = tp(b'/a_{i,j})$,
(b) $(a_{i,j} \mid i < \omega)$ is indiscernible over $b'$.

From (a), we have $|= \varphi(b', a_{i,j})$, and then from (b), we have $|= \varphi(b', a_{i,j})$. This contradicts that $\{\varphi(x, a_{i,j}) \mid i < \kappa\}$ is 2-inconsistent. \qed

4. Mekler’s Construction

We first recall the definitions and facts from [3].

For a graph $A$ and its vertices $a$ and $b$, we say $R(a, b)$ if $a$ and $b$ are connected by a single edge in $A$.

Definition 4.1. A graph $A$ which has at least two vertices is called nice if

(a) For any two distinct vertices $a$ and $b$, there is some vertex $c \neq a, b$ such that $R(a, c)$ but $\neg R(b, c)$;
(b) There are no triangles or squares.

Definition 4.2. [11] [10] Fix an odd prime $p$. For a nice graph $A$, let $F(A)$ be the free nilpotent group of class 2 and exponent $p$ generated freely by the vertices of $A$. Assume that $A$ is enumerated with some relation $<$ not in the original language. Then the Mekler group of $A$, denoted by $G(A)$, is defined as follows;

$$G(A) = F(A)/\langle\langle a, b \mid a, b \in A, a < b, \text{ and } A \models R(a, b)\rangle\rangle.$$ 

In other words, $G(A)$ is a group defined in the variety of nilpotent groups of class 2 and exponent $p$ such that the generators are the vertices of $A$ and that for any $a$ and $b$ in $G(A)$, $[a, b] = 1$ if and only if $a < b$ and $A \models R(a, b)$. 


We can see the definition in the point of view of vector spaces. Let \( Z(F(A)) \) be the center of \( F(A) \). Then both \( Z(F(A)) \) and \( F(A)/Z(F(A)) \) are all elementary abelian \( p \)-groups so they can be considered as a \( \mathbb{F}_p \)-vector space with basis \( \{a,b| a,b \in A, a < b \} \) and \( \{a/Z(F(A))| a \in A \} \) respectively. The same is true for the Mekler group \( G(A) \). If \( Z(G(A)) \) is the center of \( G(A) \), then both \( Z(G(A)) \) and \( G(A)/Z(G(A)) \) can be considered as a \( \mathbb{F}_p \)-vector space. The basis of \( Z(G(A)) \) is \( \{a,b| a,b \in A, a < b, \text{ and } A \models \neg R(a,b) \} \), and the basis of \( G(A)/Z(G(A)) \) is \( \{a/Z(F(A))| a \in A \} \).

**Definition 4.3.** For any element \( g,h \) of \( G(A) \), we say

1. \( g \sim h \) if \( C(g) = C(h) \), where \( C(g) \) is the centraliser of \( g \) in \( G(A) \),
2. \( g \equiv h \) if for some \( c \) in the center \( Z(G) \) and some \( r \) (\( 0 \leq r < p \)), \( h = g^r \cdot c \),
3. \( g \equiv_Z h \) if \( g \cdot Z(G) = h \cdot Z(G) \).

**Remark 4.4.** For any element \( g,h \) of \( G(A) \), \( g \equiv_Z h \Rightarrow g \equiv h \Rightarrow g \sim h \).

**Definition 4.5.** Let \( g \) be an element in \( G(A) \).

1. \( g \) is isolated if every non-central element of \( G(A) \) which commutes with \( g \) is \( \sim \)-equivalent to \( g \).
2. We say an element \( g \) is of type \( q \) if \( q \) is the number of \( \sim \)-classes in the \( \sim \)-class of \( g \).
3. We say \( g \) is of type \( q^+ \) (resp. \( q^- \)) if \( g \) is of type \( q \) and isolated (resp. of type \( q \) and not isolated).

**Definition 4.6.** For every element \( g \) of type \( p \), we say an element \( b \) is a handle of \( g \) if it is of type \( 1^p \) and commutes with \( g \).

As a remark, we note that for any \( g \) of type \( p \), the handle of \( g \) exists and is unique up to \( \sim \)-equivalency.

**Fact 4.7.** Every non-central element of \( G(A) \) is of exactly one of the four type \( 1^+, 1^-, p−1, p \); the classes of elements of each type are \( 0 \)-definable.

Now, let \( G \) be a model of \( \text{Th}(G(A)) \). Let us say an element of \( G \) is proper if it is not a product of any elements of type \( 1^p \) in \( G \).

**Definition 4.8.**

1. A \( 1^- \)-transversal of \( G \), denoted by \( X^r \), is a set consisting of one representative for each \( \sim \)-class of elements of type \( 1^r \) in \( G \).
2. A \( p \)-transversal of \( G \), denoted by \( X^p \), is a set of pairwise \( \sim \)-inequivalent proper elements of type \( p \) in \( G \) which is maximal with the property that if \( Y \) is a finite subset of \( X^p \) and all elements of \( Y \) have the same handle, then \( Y \) is a independent modulo the subgroup generated by all elements of type \( 1^p \) in \( G \) and \( Z(G) \).
3. A \( 1^+ \)-transversal of \( G \), denoted by \( X^+ \), is a set of representatives of \( \sim \)-classes of proper elements of type \( 1^+ \) in \( G \) which is maximal independent modulo the subgroup generated by all elements of types \( 1^- \) and \( p \) in \( G \), together with \( Z(G) \).
4. A subset \( X \) of \( G \) is called a transversal of \( G \) if it is the union of some \( 1^- \)-transversal \( X^- \), a \( p \)-transversal \( X^p \), and a \( 1^+ \)-transversal \( X^+ \) of \( G \).

Note that all the sets in the above definition are definable.

**Fact 4.9.** Let \( A \) be a nice graph. For a model \( G \models \text{Th}(G(A)) \), define an interpretation \( \Gamma \) such that \( \Gamma(G) \) is a graph where the set of vertices is \( \{g \in G| g \text{ is a noncentral element of type } 1^r\} / \sim \) and the edge relation is \( \{([g]_\sim, [h]_\sim)| [g,h] = 1 \text{ in } G \} \). Then \( \Gamma(G) \models \text{Th}(A) \).

From [47] we see that if \( X^r \) is a \( 1^- \)-transversal, then the set can be regarded as a graph which models \( \text{Th}(A) \).

**Fact 4.10.** Let \( C \) be an infinite nice graph, and \( G \models \text{Th}(G(C)) \). If \( X = X^v \cup X^p \cup X^r \) is a transversal of \( G \), then there is a subgroup \( H_X \leq Z(G) \) such that \( G = \langle X \rangle \times H_X \) for some \( H_X \leq Z(G) \). Moreover, if \( G \) is saturated and uncountable, then both \( \Gamma(G) \) and \( H_X \) are also saturated.

Since \( H_X \) is an elementary abelian \( p \)-group, we sometimes say \( G \) is isomorphic to \( \langle X \rangle \times \langle H_X \rangle \).
Fact 4.11. Let $G$ be a saturated model of $\text{Th}(G(C))$ and let $\kappa = |G|$. If $X = X^\nu \cup X^p \cup X^* \cup X^i$ is a transversal of $G$, then

(a) for any $x^\nu \in X^\nu$, the cardinality of $\{x^p \in X^p \mid x^\nu \text{ is the handle of } x^p\}$ is either zero or $\kappa$, and

(b) $|X^*| = \kappa$

As we mentioned in [13] $X^\nu$ can be regarded as a graph where two vertices are joined (connected by a single edge) if they commute in $G$. In this point of view, we can find a supergraph by extending the set of vertices to $X$ and then giving the edge relation with the same rule. Then each $x^p \in X^p$ is joined to a unique vertex in $X^\nu$, which is the handle of $x^p$, while each $x^i \in X^i$ is joined to no vertex.

This kind of supergraph is called a cover. See [3] for more precise proof.

We give more facts from [3].

Fact 4.12. (1) Let $G$ be a saturated model of $\text{Th}(G(C))$, and let $X$ and $H_X$ be the sets in [13, 14] so that $G = (X) \times H_X$. If $f$ is a bijection between two small sets $Y = Y^\nu \cup Y^p \cup Y^*$ and $Z = Z^\nu \cup Z^p \cup Z^*$ of $X$ with the following properties;

(a) $f(Y^\nu) = Z^\nu$, $f(Y^p) = Z^p$, and $f(Y^*) = Z^*$,

(b) for any $y^p \in Y^p$, the handle of $y^p$ is same as the handle of $f(y^p)$,

(c) $\text{tp}(Y^\nu) = \text{tp}(Y^p)$ in $G$;

then $f$ can be extended to an automorphism $\sigma$ of $G$.

Moreover, if $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$ in $H_X$, then we may assume $\sigma$ sends $\bar{a}$ to $\bar{b}$.

(2) Let $G$ be a model of $\text{Th}(G(C))$, and $\bar{x} = x^\nu - x^p - x^* - x^i$ and $\bar{y}$ be two small tuples of variables. Then there is a partial type $\pi(\bar{x}, \bar{y})$ such that $G \models \pi(\bar{a}, \bar{b})$ if and only if we can extend $\bar{a}$ to a transversal $X$ of $G$ and find $H$ containing $\bar{b}$ so that $H$ is an independent set in $\text{Z}(G)$ and $G = (X) \times (H)$.

5. Preservation

In [3], Dzamonja and Shelah introduced a way to choose a monochromatic subtree when the given tree $\kappa^+ 2$ is colored by $\theta$ many colors where $|\theta| < \kappa$ and $\kappa$ is a regular cardinal. The following is the key observation.

Fact 5.1. Let $\kappa$ be a regular cardinal and let $\theta$ be a set of colors such that $|\theta| < \kappa$. For any coloring $f : \kappa^+ 2 \rightarrow \theta$, there is a color $c \in \theta$ and an element $\nu^* \in \kappa^+ 2$ such that for any $\nu \in \kappa^+ 2$ satisfying $\nu^* \leq \nu$, there is $\rho \in \kappa^+ 2$ with $\nu \leq \rho$, $f(\rho) = c$.

Let us say a subtree $B \subseteq A$ is $f$-monochromatic if there is a color $c$ such that for all $b \in B$, $f(b) = c$.

Lemma 5.2. Let $\kappa$ be an uncountable regular cardinal and $f : \kappa^+ 2 \rightarrow \omega$ be a coloring.

(1) If $\varphi(x; y)$ and $(a_\eta | \eta \in \kappa^+ 2)$ witness SOP$_2$, then there is a $f$-monochromatic subtree $(a'_\eta | \eta \in \kappa^+ 2)$ of $(a_\eta | \eta \in \kappa^+ 2)$ such that $\varphi$ and $(a'_\eta | \eta \in \kappa^+ 2)$ witness SOP$_2$.

(2) If $\varphi(x; y)$ and $(a_\eta | \eta \in \kappa^+ 2)$ witness SOP$_1$, then there is a $f$-monochromatic subtree $(a'_\eta | \eta \in \kappa^+ 2)$ of $(a_\eta | \eta \in \kappa^+ 2)$ such that $\varphi$ and $(a'_\eta | \eta \in \kappa^+ 2)$ witness SOP$_1$.

(3) If $\varphi(x; y)$ and $(a_\eta | \eta \in \kappa^+ 2)$ witness TP$_1$, then there is a $f$-monochromatic subtree $(a'_\eta | \eta \in \kappa^+ \omega)$ of $(a_\eta | \eta \in \kappa^+ \omega)$ such that $\varphi$ and $(a'_\eta | \eta \in \kappa^+ \omega)$ witness TP$_1$.

Proof. (1) Let $c$ and $\nu^*$ be the element in Fact 5.1. We construct a subtree $(a'_\eta | \eta \in \kappa^+ 2)$ in $(a_\eta | \eta \in \kappa^+ 2)$ inductively as follows;

- $a'_0 = a_0$, where $\rho$ is any element satisfying $\nu^* \leq \rho$ and $f(\rho) = c$,

- for any $\eta \in \kappa^+ 2$, if $a'_0 = a_\rho$ for some $\rho \in \kappa^+ 2$, then $a''_0 = a_\rho$ where $\rho^0 \leq \rho'$ and $f(\rho') = c$, and $a''_{\eta-1} = a_{\rho'}$ where $\rho^1 \leq \rho'$ and $f(\rho') = c$.

$\varphi$ and the subtree witness SOP$_2$.

(2) We use a different version of the Fact 5.1 which has one more condition on $\rho$: $\rho$ is of the form $\xi^1$ for some $\xi$. This can be proved by the same way used in [5, 1] so we omit the proof.
Define $a'_0$ as same as the previous one, but give a small modification in the induction step. For any $\eta \in <\kappa$, let $a'_\eta = a_\rho$ for some $\rho \in <\kappa$. Take $a''_{\eta\rho} = a_\rho'$ where $\rho \leq \rho'$, $f(\rho') = c$ and $\rho'$ is of the form $\xi + 1$ for some $\xi$. Then take $a''_{\eta\rho} = a_{\rho'}$ where $\xi = 0 \leq \rho''$ and $\eta \in f(\rho'') = c$. $\varphi$ and the subtree witness SOP$_1$, too.

(3) In $(a_n | \eta \in <\kappa)$, consider the subtree $(a_n | \eta \in <\kappa)$. Find $(a'_n | \eta \in <\kappa)$ as in the proof of (1), and then construct $(a''_n | \eta \in <\kappa)$ inductively as follows:

- $a''_n = a'_n$,
- for any $\eta \in <\omega$, if $a''_n = a_\rho'$ for some $\rho \in <\kappa$, then $a''_{\eta\rho} = a'_{\rho''-1\rho} - a_0$ for each $i < \omega$.

\[ \varphi \] and the last subtree witness TP$_1$, too.

\[ \square \]

**Theorem 5.3.** For any infinite nice graph $C$, $Th(G(C))$ is NTP$_1$ if and only if $Th(C)$ is NTP$_1$.

**Proof.** Since $C$ is interpretable in $G(C)$, if $Th(C)$ has TP$_1$, then $Th(G(C))$ also has TP$_1$.

Suppose $Th(C)$ is NTP$_1$ and $Th(G(C))$ has TP$_1$. Let $G$ be a monster model of $Th(G(C))$, $X$ be a transversal of $G$ so that $G = \langle X \rangle \times \langle H \rangle$ for some $H$ as in Fact 4.11. We have a formula $\varphi(x, y)$ and a tree $(a_n | \eta \in <\kappa)$ of finite tuples in $G$ for some sufficiently large regular cardinals $\kappa$ so that they witness TP$_1$. Note that for each $\eta \in <\kappa$, $a_\eta$ is of the form $t_\eta(y, h_\eta)$ for some terms $t_\eta \in L_G$, and for some small tuples $x_\eta = x_\eta \langle \bar{y}_\eta \rangle X$, and $h_\eta \in H$.

By Lemma 4.3, we may assume $t_\eta = t_\in L_G$, and $|\bar{x}_{\eta0}|, |\bar{x}_{\eta1}|, |\bar{x}_{\eta2}|, |\bar{h}_\eta|$ are constant for all $\eta \in <\omega$. To obtain handle correspondence, add handles of elements in the tuple $\bar{x}_\eta$ to the beginning of $\bar{x}_{\eta0}$ for all $\eta \in <\omega$.

Taking $\varphi'(x, y') := \varphi(x, t(y'))$ with $|y'| = |\bar{y}_\eta \bar{h}_\eta|$ and $\bar{h}_\eta := \bar{x}_\eta \bar{h}_\eta$, we have $\varphi' \in L_G$ and the tree $(\bar{h}_\eta | \eta \in <\omega)$ still satisfy TP$_1$.

By modeling property of strong indiscernibility and compactness, we can find $(\bar{c}_\eta | \eta \in <\kappa)$ with $\bar{c}_\eta = \bar{y}_\eta \bar{m}_\eta = \bar{y}_\eta \bar{y}_\eta \bar{m}_\eta$ to be a strongly indiscernible tree where $(\bar{c}_\eta | \eta \in <\kappa)$ based on $(\bar{h}_\eta | \eta \in <\omega)$. Note $\varphi'$ and the tree still witness TP$_1$. Also, by 4.12, we can assume each $\bar{y}_\eta$ and $\bar{m}_\eta$ is some $y_\eta$ and $m_\eta$ where $<\kappa$ is an independent set in $Z(G)$ and $G = \langle Y \rangle \times \langle M \rangle$. Let $c$ be a realization of $\bigwedge_{\eta < \kappa} \varphi'(x, c_\eta)$. Write $c = s(y, m)$ for some terms $s \in L_G$, and for some tuples $y = y_\eta \bar{y}_\eta \bar{y}_\eta y' \in Y$, and $m \in M$. Again, To obtain handle correspondence, add handles of elements in the tuple $y_\eta$ to the beginning of $y'$. Take $\psi(x', y') = \varphi'(s(x'), y')$, then for all $\eta \fr Y \in <\kappa$, $\psi(x', c_\eta, y'(x', c_\eta))$ is inconsistent and $y_\eta m$ realizes $\bigwedge_{\eta < \kappa} \psi(x', c_\eta)$. Since $y_\eta m \cap \bigcup \bar{c}_\eta \alpha < \kappa$ is finite, we may assume the tree is strongly indiscernible over $y_\eta m \cap \bigcup \bar{c}_\eta \alpha < \kappa$.

Now, consider $y_\eta$ and the tree $(\bar{y}_\eta | \eta \in <\kappa)$ in $Y_\eta$. Applying 4.9 we can regard the elements as vertices of a graph. This graph satisfies NTP$_1$ theory $Th(C)$, so there is some $\gamma$ satisfying 3.7(2). Then for each $\beta^+ > \gamma$, we have a tuple $y_\rho$ such that $\text{tp}_C(y_\rho/\bar{y}_\eta \bar{m}_\eta \bar{y}_\eta \bar{m}_\eta) = \text{tp}_C(y_\rho/\bar{y}_\eta \bar{m}_\eta \bar{y}_\eta \bar{m}_\eta)$. And $\beta^+ > \gamma$, let $y_\rho$ be the tuples given by Corollary 3.7. Recall that the tree $(\bar{y}_\eta \bar{m}_\eta | \eta < \omega)$ is strongly indiscernible over $y_\eta m \cap \bigcup \bar{c}_\eta \alpha < \omega$. So, as in 3. Theorem 5.6, we can find a handle preserving bijection which can be extended by Fact 4.13(1) to have

1. $\text{tp}_{C}(y_\rho/\bar{y}_\eta \bar{m}_\eta \bar{y}_\eta \bar{m}_\eta) = \text{tp}_{C}(y_\rho/\bar{y}_\eta \bar{m}_\eta \bar{y}_\eta \bar{m}_\eta)$,
2. $\text{tp}_{C}(\bar{y}_\eta \bar{y}_\eta \bar{y}_\eta \bar{y}_\eta) = \text{tp}_{C}(\bar{y}_\eta \bar{y}_\eta \bar{y}_\eta \bar{y}_\eta)$.

From these conditions, we have $G = \psi(y_\rho m', \bar{y}_\eta \bar{y}_\eta \bar{y}_\eta \bar{y}_\eta) \wedge \psi(y_\rho m', \bar{y}_\eta \bar{y}_\eta \bar{y}_\eta \bar{y}_\eta)$, but this contradicts that for any $\eta \fr Y \in <\kappa$, $\psi(x', \bar{y}_\eta \bar{m}_\eta)$, $\psi(x', \bar{y}_\eta \bar{m}_\eta)$ is inconsistent.

\[ \square \]

**Theorem 5.4.** For any infinite nice graph $C$, $Th(G(C))$ is NSOP$_1$ if and only if $Th(C)$ is NSOP$_1$. 

Proof. Since $C$ is interpretable in $G(C)$, if $\text{Th}(C)$ has SOP$_1$, then $\text{Th}(G(C))$ also has SOP$_1$.

Suppose $\text{Th}(C)$ is NSOP$_1$ and $\text{Th}(G(C))$ has SOP$_1$. Again, we take $G$ to be a monster model of $\text{Th}(G(C))$, $X$ to be a transversal of $G$ so that $G = \langle X \rangle \times \langle H \rangle$ for some $H$ as in Fact 4.10. We have a formula $\varphi(x, y)$ and a tree $(a_\eta | \eta < \omega)_{\eta < \omega}$ of finite tuples in $G$ for some sufficiently large regular cardinals $\kappa$. Note that for each $\eta \in \omega$, $a_\eta$ is of the form $t_\eta(\vec{x}_\eta, \vec{h}_\eta)$ for some terms $t_\eta \in L_G$, and for some small tuples $\vec{x}_\eta = \vec{x}_\eta^\nu \vec{x}_\eta^\mu \vec{x}_\eta^\nu$ in $X$, and $\vec{h}_\eta \in H$.

By Lemma 7.2 we may assume $t_\eta = t \in L_G$, and $|\vec{x}_\eta^\nu|, |\vec{x}_\eta^\mu|, |\vec{h}_\eta|$ are constant for all $\eta \in \omega$. To obtain handle correspondence, add handles of elements in the tuple $\vec{x}_\eta^\nu$ to the beginning of $\vec{x}_\eta^\mu$ for all $\eta \in \omega$.

Taking $\varphi'(x, y') := \varphi(x, t(y'))$ with $|y'| = |\vec{x}_\eta^\nu \vec{h}_\eta|$ and $a_\eta' := \vec{x}_\eta^\mu \vec{h}_\eta$, we have a formula $\varphi' \in L_G$ and the tree $(a_\eta' | \eta < \omega)_{\eta < \omega}$ also witness SOP$_1$.

By Fact 7.2 we can choose an array $(\vec{b}, i)_{i<\omega,j<2}$ in $(a_\eta' | \eta < \omega)$ such that

1. $\vec{b}_{i,0} = (\vec{b}, i)_{0,0,0,0}$ for all $i < \omega$,
2. $\{\vec{b}(x, \vec{b}_{i,0}) | i < \omega\}$ is consistent, and
3. $(\vec{b}(x, \vec{b}_{i,1}) | i < \omega)$ is 2-inconsistent.

Consider an indiscernible sequence $(\vec{c}, 0, i, 1, 0)_{i<\kappa}$ realizing $EM((\vec{b}, 0, \vec{b}_{i,1})_{i<\omega})$. The new array $(\vec{c}, i, 0, i, 1)_{i<\kappa}$ is comb indiscernible and satisfies aforementioned conditions. Also, $\vec{c}, j, i$ is of the form $\vec{y}, \vec{m}, _i \vec{m}$ and there are some $Y$ and $M$ such that $Y$ is a transversal of $G$ extended from $\{\vec{y}, j | i < \kappa, j < 2\}$, $M$ is an independent set in $Z(G)$ containing $\{\vec{m}, j | i < \kappa, j < 2\}$, and $G = \langle Y \rangle \times \langle M \rangle$.

Let be a realization of $\bigwedge_{i<\kappa} \varphi(x, \vec{c}, i, 0)$. Write $c = s(y, m)$ for some terms $s \in L_G$, and for some tuples $y = y^\nu y^\mu y^\nu$ in $Y$, and $m \in M$. Again, To obtain handle correspondence, add handles of elements in the tuple $y^\nu$ to the beginning of $y^\mu$. Take $\psi(x', y') = \varphi(s(x', y'))$, then for all $i < j < \kappa$, $\{\psi(x', \vec{c}, i, 1), \psi(x', \vec{c}, i, 1)\}$ is inconsistent and $y^\mu m$ realizes $\bigwedge_{i<\kappa} \varphi(x', \vec{c}, i, 0)$. Since $y^\mu m \cap \bigcup \{y_{c,0} | i < \kappa\}$ is finite, we may assume the array is comb indiscernible over $y^\mu m \cap \bigcup \{y_{c,0} | i < \kappa\}$.

Now, consider $y^\nu$ and the array $(\vec{y}, i | i < \kappa, j < 2)$ in $Y^\nu$. Applying 4.12 we can regard the elements as vertices of a graph. This graph is a model of $\text{NSOP}_1$ theory $\text{Th}(C)$, so there is some $\gamma$ satisfying 4.12(2). Then for each $\beta > \gamma$, we have a tuple $y''$ such that $\text{tp}_\gamma(y''/\vec{y}, i) = \text{tp}_\gamma(y''/\vec{y}, i)$ and $\text{tp}_\gamma(y''/\vec{y}, i) = \text{tp}_\gamma(y''/\vec{y}, i)$.

On the other hand, observe that $\text{Th}(\langle M \rangle)$ is a theory of vector spaces, so that the theory is stable and has quantifier elimination. Then for the tuple $m$ and the array $(\vec{m}, i | i < \kappa, j < 2)$ in $M$, we can apply Corollary 8.3.2 to have some $\gamma'$ satisfying 8.3.2.

Fix some ordinal $\beta$ larger then $\gamma$ and $\gamma'$, and let $y''$ and $m'$ be the tuples given by 8.3.2. We will find a handle preserving bijection which can be extended by Fact 4.12(1) to have

1. $\text{tp}_\gamma(y''/\vec{y}, i) = \text{tp}_\gamma(y''/\vec{y}, i)$, and
2. $\text{tp}_\gamma(y''/\vec{y}, i) = \text{tp}_\gamma(y''/\vec{y}, i)$,

for some tuple $y' = y'' y'' y''$.

From the above two type equivalence, we have $G \models \psi(y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y'' y' y'' y'' y'$.

We finish our proof by illustrating the way to tuple $y' = y'' y'' y''$. For a finite tuple $\vec{z}$ and a natural number $k$, denote $(\vec{z}, k)$ to be the $k$-th element of $\vec{z}$.

Let $y'$ be the tuple of length $l$. For each $i < l$, if $(y')_i$ is $(\vec{y}, i, 0)_i k$ for some $k$, then choose $y''_i$ to be $(\vec{y}, i, 1, 0)_i k$, and if $(y')_i$ is not in $\vec{y}, i, 0$, then choose $(y'')_i$ to be any element in $Y \setminus \vec{y}, i, 0$. Let $y''$ be the tuple of length $l'$. For each $i < l'$, either $(y'')_i$ is in $\vec{y}, i, 0$ or not. Assume first that $(y'')_i$ is $(\vec{y}, i, 0)_i k$ for some $k$. Then choose $(y'''')_i$ to be $(\vec{y}, i, 1, 0)_i k$. Second, suppose $(y'')_i$ is not in $\vec{y}, i, 0$. Recall that $(y'')_i$ is the handle of $(y'')_i$. We have either $(y'')_i$ is in $\vec{y}, i, 0$ or not. If $(y'')_i$ is in $\vec{y}, i, 0$, then choose $(y'''')_i$ to be any element in $Y \setminus \vec{y}, i, 0$ whose handle is $(\vec{y}, i, 0)_i k$. Otherwise, choose $(y'''')_i$ to be any element in $Y' \setminus \vec{y}, i, 0$ whose handle is $(\vec{y}, i, 0)_i k$.

Mapping each element in $\vec{y}, i, 0$ to $y''$ by their natural order, we have a bijection which satisfies hypotheses in 8.3.2(1). As a result, \text{tp}_\gamma(y''/\vec{y}, i) = \text{tp}_\gamma(y''/\vec{y}, i)$.

It remains to check the bijection from $\vec{y}, i, 0$ to $\vec{y}, i, 0$ is well-defined. Suppose the $(\vec{y}, i, 1, 0)_i k$ is equal to the $(y'')_{i'}$ for some $k$ and $i'$. Since $\text{tp}_\gamma(y''/\vec{y}, i) = \text{tp}_\gamma(y''/\vec{y}, i)$.
This element is in \(y^{-m} \cap \bigcup \{c_i, 0 \mid i < \kappa\}\), hence for every \(i < \kappa\), \((\bar{y}_i, 0)_{\kappa'}\) is \((y)_{\kappa'}\). By comb indiscernibility, the \((\bar{y}_1)_{\kappa'}\) is \((y)_{\kappa'}\), too. Thus \((y')_{\kappa'}\), \((\bar{y}_{0,1})_{\kappa}, \bar{y}_{1,1,1})_{\kappa}\), and \((y)_{\kappa'}\) are all the same elements. This comes out again if we assume \(k\)-th element of \(\bar{y}_1\) is equal to the \(k'\)-th element of \(y'\). Therefore, we have a well-defined bijection from \(\bar{y}_{0,1}y'\) to \(\bar{y}_{1,1}y'\).

\[\Box\]

**Corollary 5.5.**

(1) There is a non-simple NSOP\(_1\) pure group theory.

(2) If there is an NSOP\(_2\) theory which has SOP\(_1\), then there is a pure group theory with the same properties.

**Proof.** (1) Fix a structure \(M\) of finite language such that \(\text{Th}(M)\) is non-simple and NSOP\(_1\). Then by \[6\] Theorem 5.5.1, Exercise 5.5.9, there is a nice graph \(C\) bi-interpretable with \(M\). Since Mekler’s construction preserves simplicity and NSOP\(_1\), the theory of Mekler group of \(C\) is non-simple and NSOP\(_1\), too.

(2) Follow the same argument above. \[\Box\]

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