HOCHSCHILD TWO-COCYCLES AND THE GOOD TRIPEL
\((As, Hoch, Mag^\infty)\)

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Abstract. The aim of this paper is to introduce the category of Hoch-algebras whose objects are associative algebras equipped with an extra magmatic operation \(\succ\) verifying the following relation motivated by the Hochschild two-cocycle identity:

\[ R_2 : (x \succ y) \ast z + (x \ast y) \succ z = x \succ (y \ast z) + x \ast (y \succ z). \]

Such algebras appear in mathematical physics with \(\succ\) associative under the name of compatible products. Here, we relax the associativity condition. The free Hoch-algebra over a \(K\)-vector space is then given in terms of planar rooted trees and the triple of operads \((As, Hoch, Mag^\infty)\) endowed with the infinitesimal relations is shown to be good. Hence, according to Loday’s theory, we then obtain an equivalence of categories between connected infinitesimal Hoch-bialgebras and \(Mag^\infty\)-algebras.

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Notation: In the sequel \(K\) is a field. We adopt Sweedler notation for the binary cooperation \(\Delta\) on a \(K\)-vector space \(V\) and set \(\Delta(x) = x_{(1)} \otimes x_{(2)}\). For a \(K\)-vector space \(V\), we set \(\bar{T}(V) := \bigoplus_{n>0} V^\otimes n\).

1. Introduction

The well-known Poincaré-Birkhoff-Witt and the Cartier-Milnor-Moore theorems together can be rephrased as follows:

**Theorem 1.1.** (CMM-PBW) [6] For any cocommutative (associative) bialgebra \(\mathcal{H}\), \(Com^c - As\)-bialgebra for short, the following are equivalent.

1. \(\mathcal{H}\) is connected;
2. \(\mathcal{H}\) is isomorphic to \(U(\text{Prim } \mathcal{H})\) as a bialgebra;
3. \(\mathcal{H}\) is isomorphic to \(Com^c(\text{Prim } \mathcal{H})\) as a coalgebra,
where \( U \) is the usual enveloping functor and \( \text{Prim} \mathcal{H} \) the usual Lie algebra of the primitive elements of \( \mathcal{H} \).

In the theory developed by J.-L. Loday [6], this result is rephrased by saying that the triple of operads \((\text{Com}, \text{As}, \text{Lie})\), endowed with the usual Hopf relation, is good, where \( \text{Com}, \text{As}, \text{and Lie} \) stand respectively for the operads of commutative, associative and Lie algebras. Other good triples of operads equipped with other relations than the usual Hopf one, have been found since. A summary can be found in [6], see also [3,4] for other examples.

It has been shown in [3] that the triple of operads \((\text{As}, \text{Dipt}, \text{Mag}^\infty)\) endowed with the semi-infinitesimal relations is good. The operad \( \text{Dipt} \) is governed by dipterous algebras which are associative algebras equipped with an extra left module on themselves, see also [7], and \( \text{Mag}^\infty \) is governed by \( \text{Mag}^\infty \)-algebras, i.e., \( K \)-vector spaces having one \( n \)-ary (magmatic) generating operation for each integer \( n > 1 \). We then obtained that the category of connected infinitesimal dipterous bialgebras, \( \text{As}^c - \text{Dipt} \)-bialgebras for short, was equivalent to the category of \( \text{Mag}^\infty \)-algebras.

In this paper, we propose another equivalence of categories involving \( \text{Mag}^\infty \): the category of connected infinitesimal \( \text{Hoch} \)-bialgebras is equivalent to the category of \( \text{Mag}^\infty \)-algebras.

In Section 2, we introduce \( \text{Hoch} \)-algebras, give examples and an explicit construction of the free \( \text{Hoch} \)-algebra over a \( K \)-vector space. In Section 3, we introduce the notion of (connected) infinitesimal \( \text{Hoch} \)-bialgebras. In Section 4 we prove the announced equivalence of categories. In Section 5, we deal with unital \( \text{Hoch} \)-algebras and close by Section 6 with two other good triples involving the operad \( \text{Hoch} \).

2. The free \( \text{Hoch} \)-algebra

A \( \text{Hoch} \)-algebra \( G \) is a \( K \)-vector space equipped with an associative operation * and a magmatic operation \( \succ \) verifying:

\[
\mathcal{R}_2 : (x \succ y) \ast z + (x \ast y) \succ z = x \succ (y \ast z) + x \ast (y \succ z),
\]

for all \( x, y, z \in G \).

Remark 2.1. 1) Recall that a formal deformation of an associative algebra \((A, \ast)\) is a \( K[[t]] \)-bilinear multiplication law \( m_t : A[[t]] \otimes_{K[[t]]} A[[t]] \rightarrow A[[t]] \) on the space \( A[[t]] \) of formal power series in a variable \( t \) with coefficients in \( A \), satisfying the following properties:

\[
m_t(a,b) = a \ast b + m_1(a,b)t + m_2(a,b)t^2 + \ldots,
\]
for \(a, b \in A\) where \(m_t\) is associative, that is: the equation \(m_t(m_t(a, b), c) = m_t(a, m_t(b, c))\) for \(a, b, c \in A\) holds. It is well known that \(m_1\) satisfies \(R_2\) if and only if \(m_t\) is associative modulo \(t^2\). In this case, \(A[[t]]\) equipped with the initial associative operation \(*\) and \(m_1\) is a \(Hoch\)-algebra.

2) when \(\succ\) turns out to be associative, such algebras appear in mathematical physics in the works of A. Odesskii and V. Sokolov [9,8], A.B. Goncharov [2] and V. Dotsenko [1]. See also [5] for others examples of such algebras on matrices. In this paper, we relax the associativity condition on \(\succ\).

Let \(V\) be a \(K\)-vector space. The free \(Hoch\)-algebra over \(V\) is defined as follows. It is equipped with a linear map \(i : V \rightarrow Hoch(V)\) and for any \(Hoch\)-algebra \(G\) and any linear map \(f : V \rightarrow G\), there exists a unique \(Hoch\)-algebra morphism \(\phi : Hoch(V) \rightarrow G\) such that \(\phi \circ i = f\). We now give an explicit construction of the free \(Hoch\)-algebra over a \(K\)-vector space.

Denote by \(T_n\) the set of rooted planar trees (degrees at least 2) with \(n\) leaves. The cardinalities of \(T_n\) are registered under the name \(A001003\) little Schroeder numbers of the Online Encyclopedia of Integer Sequences. For \(n = 1, 2, 3\), we get:

\[T_1 = \{ \mid \}, \quad T_2 = \{ \bigtriangledown \}, \quad T_3 = \{ \bigtriangledown , \bigtriangledown , \bigtriangledown \}.\]

Define grafting operations by:

\[\cdot, \ldots, \cdot : T_{n_1} \times \ldots \times T_{n_p} \rightarrow T_{n_1 + \ldots + n_p}, \quad (t_1, \ldots, t_p) \mapsto \left[ t_1, \ldots, t_p \right] := t_1 \lor \ldots \lor t_p,\]

where the tree \(t_1 \lor \ldots \lor t_p\) is the tree whose roots of the \(t_i\) have been glued together and a new root has been added. Observe that any rooted planar tree \(t\) can be decomposed in a unique way via the grafting operation as \(t_1 \lor \ldots \lor t_p\). Set \(T_\infty := \bigoplus_{n>0} KT_n\). Define over \(\check{T}(T_\infty)\), the following binary operations, first on trees, then by bilinearity:

\[\text{Concatenation} : \quad (t_1 \ldots t_p) * (s_1 \ldots s_q) := t_1 \ldots t_p s_1 \ldots s_q,\]

\[(t_1 \ldots t_p) \succ (s_1 \ldots s_q) := \sum_{k=1}^q \sum_{i=1}^p t_1 \ldots t_{p-i} | t_{p-(i-1)}, \ldots, t_p, s_1, \ldots, s_k | s_{k+1} \ldots s_q.\]

For instance we get:

\[\mid | \mid \succ | := | | \bigtriangledown + | \bigtriangledown + \bigtriangledown,\]

\[| \succ | \bigtriangledown := \bigtriangledown \bigtriangledown + \bigtriangledown\]

**Theorem 2.2.** The \(K\)-vector space \(\check{T}(T_\infty)\) endowed with the operations \(*\) and \(\succ\) is the free \(Hoch\)-algebra over \(K\).
Proof. Let $x := x_1 \ldots x_m$, $y := y_1 \ldots y_n$ and $z := z_1 \ldots z_p$. We get:

$$(x \triangleright y) \ast z + (x \ast y) \triangleright z = \sum_{k=1}^{n} \sum_{i=0}^{m-1} x_1 \ldots x_{m-(i+1)}[x_{m-i}, \ldots, x_m, y_1, \ldots, y_k]y_{k+1} \ldots y_n z_1 \ldots z_p$$

$$+ \sum_{k=1}^{p} \sum_{i=0}^{n-1} x_1 \ldots x_m y_1 \ldots y_{n-(i+1)} \ast [y_{n-i}, \ldots, y_n, z_1, \ldots, z_k] z_{k+1} \ldots z_p$$

$$+ \sum_{k=1}^{p} \sum_{i=0}^{m-1} x_1 \ldots x_{m-(i+1)}[x_{m-i}, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_k] z_{k+1} \ldots z_p$$

showing that

$$(x \triangleright y) \ast z + (x \ast y) \triangleright z = x \triangleright (y \ast z) + x \ast (y \triangleright z),$$

holds for all forests of planar rooted trees $x, y, z$. Observe that any rooted planar tree $t := [t_1, \ldots, t_n]$ can be rewritten as:

$$t = (t_1 \ast (t_2 \ldots t_{n-1})) \triangleright t_n - t_1 \ast ((t_2 \ldots t_{n-1}) \triangleright t_n)$$

Let $G$ be a Hoch-algebra and $g \in G$ and $f : K \to G$ be a linear map. Consider the embedding $i : K \hookrightarrow \tilde{T}(T_\infty)$ defined by $i(1_K) := |$ and define by induction the map $\phi : \tilde{T}(T_\infty) \to G$ as follows:

$$\phi(|) = g,$$

$$\phi(t_1 \ldots t_n) = \phi(t_1) \ast_G \phi(t_2) \ast_G \ldots \ast_G \phi(t_n),$$

$$\phi(t) = (\phi(t_1) \ast_G (\phi(t_2) \ldots \phi(t_{n-1}))) \triangleright_G \phi(t_n) - \phi(t_1) \ast_G ((\phi(t_2) \ldots \phi(t_{n-1})) \triangleright_G \phi(t_n))$$

for any $t := [t_1, \ldots, t_n]$ and extend $\phi$ by linearity. By construction, $\phi$ is a morphism of associative algebras. Using the fact that:

$$(x \ast y) \triangleright z - x \ast (y \triangleright z) = x \triangleright (y \ast z) - (x \triangleright y) \ast z,$$

and changes of indices in the involving sums, one shows that $\phi$ is also a morphism for the magmatic operations. It is then the only Hoch-morphism such that $\phi \circ i = f$.  \[\square\]
As the operad $Hoch$ is nonsymmetric, the following holds. Let $V$ be a $K$-vector space. The free $Hoch$-algebra over $V$ is the $K$-vector space:

$$Hoch(V) := \bigoplus_{n>0} Hoch_n \otimes V^\otimes n,$$

with $Hoch(K) := \bigoplus_{n>0} Hoch_n \simeq \tilde{T}(T_\infty)$ (hence $Hoch_n$ is explicitly described in terms of forests of rooted planar trees) equipped with the operations $\ast$ and $\succ$ defined as follows:

$$((t_1 \ldots t_n) \otimes \omega) \ast ((s_1 \ldots s_p) \otimes \omega') = (t_1 \ldots t_n s_1 \ldots s_p) \otimes \omega \omega',$$

$$((t_1 \ldots t_n) \otimes \omega) \succ ((s_1 \ldots s_p) \otimes \omega') = ((t_1 \ldots t_n) \succ (s_1 \ldots s_p)) \otimes \omega \omega',$$

for any $\omega \in V^\otimes n, \omega' \in V^\otimes p$. The embedding map $i : V \hookrightarrow Hoch(V)$ is defined by:

$$v \mapsto v \otimes 1.$$

Since the generating function associated with the Schur functor $\tilde{T}$ is $f_{\tilde{T}}(x) := \frac{x}{1-x}$ and with the Schur functor $T_\infty$ is $f_{T_\infty}(x) := \frac{1+x-\sqrt{1-6x+x^2}}{4} = x + x^2 + 3x^3 + 11x^4 + 45x^5 + \ldots$, the generating function of the operad $Hoch$ is $f_{\tilde{T}} \circ f_{T_\infty}$, that is:

$$f_{Hoch}(x) := \frac{1+x-\sqrt{1-6x+x^2}}{3-x+\sqrt{1-6x+x^2}} = x + 2x^2 + 6x^3 + 22x^4 + \ldots.$$

The sequence $(1, 2, 6, 22, 90, \ldots)$ is registered as A006318 under the name Large Schroeder numbers on the Online Encyclopedia of Integer Sequences.

**Remark 2.3.** When $\succ$ is associative, the corresponding free algebra has been constructed in [1].

### 3. Infinitesimal $Hoch$-bialgebras

By definition, an infinitesimal $Hoch$-bialgebra (or an $As^c - Hoch$-bialgebra for short) $(\mathcal{H}, \ast, \succ, \Delta)$ is a $Hoch$-algebra equipped with a coassociative coproduct $\Delta$ verifying the following so-called nonunital infinitesimal relations:

$$\Delta(x \succ y) := x(1) \otimes (x(2) \succ y) + (x \succ y(1)) \otimes y(2) + x \otimes y,$$

$$\Delta(x \ast y) := x(1) \otimes (x(2) \ast y) + (x \ast y(1)) \otimes y(2) + x \otimes y.$$

It is said to be connected when $\mathcal{H} = \bigcup_{r>0} F_r \mathcal{H}$ with the filtration $(F_r \mathcal{H})_{r>0}$ defined as follows:

(The primitive elements) $F_1 \mathcal{H} := Prim \mathcal{H} = \ker \Delta$,

Set $\Delta^{(1)} := \Delta$ and $\Delta^{(n)} := (\Delta \otimes id_{n-1})\Delta^{(n-1)}$ with $id_{n-1} = id \otimes \ldots \otimes id$. Then,

$$F_r \mathcal{H} := \ker \Delta^{(r)}.$$
**Theorem 3.1.** Let $V$ be a $K$-vector space. Define on $Hoch(V)$, the free Hoch-algebra over $V$, the cooperation $\Delta : Hoch(V) \to Hoch(V) \otimes Hoch(V)$ recursively as follows:

$$\Delta(i(v)) := 0, \text{ for all } v \in V,$$

$$\Delta(x \triangleright y) := x_{(1)} \otimes (x_{(2)} \triangleright y) + (x \triangleright y_{(1)}) \otimes y_{(2)} + x \otimes y,$$

$$\Delta(x \star y) := x_{(1)} \otimes (x_{(2)} \star y) + (x \star y_{(1)}) \otimes y_{(2)} + x \otimes y,$$

for all $x, y \in Hoch(V)$. Then $(Hoch(V), \Delta)$ is a connected infinitesimal Hochbialgebra.

**Proof.** This result can be proved by hand or can be seen as a corollary of the Theorem 4.2 in the next section. \qed

**4. A good triple of operads**

It can be useful to have the following result when searching for good triples.

**Lemma 4.1.** Let $C, A, Z, Q$ and $Prim$ be operads. Suppose the triples of operads $(C, A, Prim)$ and $(C, Z, Vect)$ equipped with the same compatibility relations, between products and coproducts, to be good. Suppose $A = Z \circ Q$, then $Prim = Q$.

**Proof.** Since $(C, Z, Vect)$ is good, the notion of $C^c - Z$-bialgebra has a meaning and the following are equivalent:

1. The $C^c - Z$-bialgebra $\mathcal{H}$ is connected.
2. As $Z$-algebra, $\mathcal{H}$ is isomorphic to the free $Z$-algebra over its primitive elements.
3. As $C^c$-coalgebra, $\mathcal{H}$ is isomorphic to the cofree $C^c$-coalgebra over its primitive elements.

As $(C, A, Prim)$ is good, the isomorphism of Schur functors $A \simeq C^c \circ Prim$ holds. Therefore, if $V$ is a $K$-vector space, then $A(V) = C^c(Prim(V))$. As $A(V) = C^c(Prim(V))$, it obeys the third item, consequently the second item. But by hypothesis $A(V) = Z(Q(V))$. As $A(V)$ is isomorphic to the free $Z$-algebra over its primitive elements, we get $Q(V) = Prim(V)$. Consequently, $Prim = Q$. \qed

**Theorem 4.2.** The triple of operads $(As, Hoch, Mag^\infty)$ endowed with the infinitesimal relation is good.

**Proof.** Fix an integer $n > 0$. By $[n] - Mag$ we mean the nonsymmetric binary operad generated by $n$ magmatic (binary) operations. In [4, Theorems 4.4 (and 4.5)], it has been shown that for each integer $n > 0$ the triples of operads $(As, [n] -$
Mag, Prim [n] − Mag) endowed with the infinitesimal relations were good. For
n = 2, the operadic ideal J generated by the primitive operations:
\[ *(> \otimes id) + > (* \otimes id) - > (id \otimes *) - *(id \otimes >), \]
\[ *(> \otimes id) - *(id \otimes *), \]
yields another good triple of operads \((As, [2] − Mag/J, Prim ([2] − Mag/J))\) (cf. [6, Proposition 3.1.1] on quotient triples), which turns out to be the triple
\((As, Hoch, Prim Hoch)\). As \((As, As, Vect)\) endowed with the infinitesimal relation
is good (cf [7]) and \(Hoch = As \circ Mag^\infty\) using Section 2, we get \(Prim Hoch = Mag^\infty\)
by using Lemma 4.1.

**Remark 4.3.** We give here another proof. The triple \((As, Hoch, Prim_Hoch)\) can
be shown to be good via [6, Theorem 2.5.1] checking hypotheses \(H0, H1\) and
\(H2\text{epi}\) of this theorem. The two first hypotheses are straightforward. Let \(V\) be a
\(K\)-vector space. For the last one, recall that the projection map \(Hoch(V) \to V\)
determines a unique coalgebra map \(\phi(V) : Hoch(V) \to As^c(V)\) mapping any tree
of \(Hoch(n)\), with \(n \in \mathbb{N}^*\) to \(1_n \in As^c(n)\). Consider \(s(V) : As^c(V) \to Hoch(V)\)
mapping \(1_n\) to the tree \(|\ldots| (n\text{ times})\). It is also a coalgebraic morphism
and \(\phi(V) \circ s(V) = id_{As^c(V)}\). Hence \(H2\text{epi}\) holds. Hence, using [6, Theorem 2.5.1],
\((As, Hoch, Prim_Hoch)\) is good. By construction \(Hoch(K) = As(T_\infty)\) and \(Hoch(V)\)
is isomorphic to \(As(Mag^\infty(V))\) using Section 2. As \((As, As, Vect)\), endowed with
the same compatibility relations between products and coproducts, that is the in-
finitesimal one, is good, we get \(Prim_Hoch = Mag^\infty\).

We then obtain another equivalence of categories involving the operad \(Mag^\infty\).

**Corollary 4.4.** The category of connected infinitesimal As\(^c\)−Hoch-bialgebras and
the category of Mag\(^\infty\)-algebras are equivalent.
\[
\{\text{conn. As}^c − \text{Hoch} − \text{bialg.}\} \xrightarrow{U} \{\text{Mag}^\infty − \text{alg.}\},
\]
where \(U\) and Primitive are respectively the universal enveloping functor and the
primitive functor.

**Proof.** Apply [6, Theorem 2.6.3].

**Remark 4.5.** The functor Primitive is obviously given as follows. If \((\mathcal{H}, *, >)\)
is a connected infinitesimal Hoch-bialgebra, then for all integer \(n > 1\) and for all
primitive elements \(x_1, \ldots, x_n \in \mathcal{H}\), the element:
\[
[x_1, \ldots, x_n]_n := (x_1 * \ldots * x_{n-1}) > x_n - x_1 * ((x_2 * \ldots * x_{n-1}) > x_n),
\]
will be primitive. The functor $U$ acts as follows. Let $(M, ([\ldots, [], n]_{n > 1})$ be a $Mag^\infty$-algebra with the $[\ldots, [], n]$ being its generating $n$-ary operations. Then $U(M)$ is given by $Hoch(M)/\sim$, where the equivalence relation $\sim$ consists in identifying,

$$(x_1 \ast \ldots \ast x_{n-1}) \sim x_n - x_1 \ast ((x_2 \ast \ldots \ast x_{n-1}) \sim x_n),$$

with $[x_1, \ldots, x_n]_n$, for all $x_1, \ldots, x_n \in M$.

5. Extension to a unit

Unital $Hoch$-algebras are $Hoch$-algebras equipped with a unit 1 whose compatibility with operations are defined as follows:

$$x \succ 1 = x = 1 \succ x, \quad x \ast 1 = x = 1 \ast x.$$ 

For instance, $Hoch_+(V) := K.1_K \oplus Hoch(V)$, where $Hoch(V)$ is the free $Hoch$-algebra over a $K$-vector space $V$ is a unital $Hoch$-algebra with unit $1_K$. This gives birth to unital $Mag^\infty$-algebras which are $Mag^\infty$-algebras such that the generating operations are related with the unit as follows:

$$[1, \ldots, [], n] = 0,$$

$$[, \ldots, 1, \ldots, ] \equiv [\ldots, \cdot, \cdot]_{n-1},$$

$$[, \ldots, , 1] = 0.$$

Over $Hoch_+(V)$, one has a unital infinitesimal coproduct $\delta$ defined via the former coproduct $\Delta$ as follows:

$$\delta(x) = 1_K \otimes x + x \otimes 1_K + \Delta(x),$$

for any $x \in Hoch(V)$. The compatibility relations are the so-called unital infinitesimal relations defined as follows:

$$\Delta(x \succ y) := x(1) \otimes (x(2) \succ y) + (x \succ y(1)) \otimes y(2) - x \otimes y,$$

$$\Delta(x \ast y) := x(1) \otimes (x(2) \ast y) + (x \ast y(1)) \otimes y(2) - x \otimes y.$$

We then obtain the good triple of operads $(As, Hoch, Mag^\infty)$ equipped with the unital infinitesimal relations.
6. Other triples of operads

The triple of operads \((As, Hoch, Mag^\infty)\) endowed with the infinitesimal relations are not the only one involving the operad \(Hoch\). By changing the compatibility relations, two other good triples of operads \((Com, Hoch, Prim_{Com} Hoch)\) and \((As, Hoch, Prim_{As} Hoch)\) endowed respectively with the Hopf relations and the semi-Hopf relations can be proposed. But contrarily to the case of the triple \((As, Hoch, Mag^\infty)\) the explicit descriptions of operads of the primitive elements of these two other triples are open problems.

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