Cohomologies of a Lie algebra with a derivation and applications

Rong Tang, Yael Frégier, Yunhe Sheng

Abstract

The main object of study of this paper is the notion of a LieDer pair, i.e. a Lie algebra with a derivation. We introduce the concept of a representation of a LieDer pair and study the corresponding cohomologies. We show that a LieDer pair is rigid if the second cohomology group is trivial, and a deformation of order n is extensible if its obstruction class, which is defined to be an element in the third cohomology group, is trivial. We classify central extensions of LieDer pairs using the second cohomology group with the coefficient in the trivial representation. For a pair of derivations, we define its obstruction class and show that it is extensible if and only if the obstruction class is trivial. Finally, we classify Lie2Der pairs using the third cohomology group of a LieDer pair.

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Keyword: Lie algebras, derivations, cohomologies, Lie 2-algebras, deformations, central extensions
MSC: 17B40, 17B70, 18D35.

*Research supported by NSFC (11471139) and NSF of Jilin Province (20170101050JC).
1 Introduction

A classical approach to study a mathematical structure is to associate to it invariants. Among these, cohomology theories occupy a central position as they enable for example to control deformations or extension problems. In particular, cohomology theories of various kinds of algebras have been developed with a great success ([8, 15, 19, 20]). The deformation of algebraic structures began with the seminal work of Gerstenhaber ([16]) for associative algebras and followed by its extension to Lie algebras by Nijenhuis and Richardson ([25]). After that, deformation of algebra morphisms and simultaneous deformations are widely studied ([11, 12, 13, 14, 17, 23, 24, 29]).

On the other hand, algebras are also useful via their derivations. For example one can mention their use in control theory ([1, 2]). One can also mention the higher derived bracket construction of Voronov in [28] that produces out of some special derivation of a graded Lie algebra a homotopy Lie algebra. Moreover, algebras and their derivations have proven to be a very efficient tool to encode, via Koszul duality of operads of [18], many different types of structures as homological vector fields (formal geometry and Q-manifolds) ([20, 27]). They turn out to also play a fundamental role in the study of gauge theories in quantum field theory via the BV-formalism of [5]. Indeed, this formalism relies on a Q-vector field (S, ·) which is a derivation of the 1-shifted bracket (·, ·). In [22], Loday studied the operad of associative algebras with derivation.

These facts motivate us to construct in this paper a cohomology theory that controls, among other things, simultaneous deformations of a Lie algebra with a derivation.

We begin by introducing in Section 2 a convenient categorical framework by defining the category LieDer whose objects are Lie algebras with a derivation. The point is that this enables us to introduce modules and semi-direct products in this category as “Beck-modules”, i.e. abelian group objects in the slice category. This is important since modules are a key ingredient in a cohomology theory.

We then define in Section 3 a complex associated to a LieDer pair and a module over it, and prove that it is indeed a complex. The rest of the paper is devoted to show that the usual interpretations of different cohomology groups are still valid in this framework. We first show in Section 4 that the second cohomology group governs infinitesimal deformations modulo equivalences, and that obstructions to extension to higher order deformations are given by 3-cocycles. We then consider in Section 5 the classical problem of central extensions and its characterization in terms of second cohomology groups. In Section 6 we study the extension problem of a pair of derivations. Finally we show in Section 7 that third cohomology groups classify certain categorifications (skeletal 2-objects).

There are many questions left open. The first one is to equip this complex with a compatible graded Lie algebra structure enabling to write the deformation equation as a Maurer-Cartan equation. It is probable that the dgLa can be obtained by a twist of a graded Lie algebra whose Maurer-Cartan elements are LieDer pairs. A by product of this approach would be to obtain at once the analogous theory for $L_\infty$ algebras with a (homotopy)-derivation. We expect to be able to obtain this more fundamental graded Lie algebra via the theory of operads, by showing that LieDer pairs are algebras over a suitable Koszul operad. Another line of research, suggested by the Beck-modules approach that we used to define our modules, could be to compare our complex with a complex that could be obtained by the Barr-Beck triple cohomology ([2]). We intend to answer these questions in forthcoming papers.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$.
2 LieDer pairs and their modules

We need a suitable definition of module in the category of LieDer pairs. We follow Quillen’s approach who characterized modules as monoid objects in slice categories ([10]). This is why we start by introducing in Subsection 2.1 the category of interest for us. We then recall in Subsection 2.2 the notion of slice category and monoid objects. We show that, given a Lie algebra \( g \), monoid objects in the slice category of Lie algebras over \( g \) are equivalent to \( g \)-modules. This result is apparently well known to experts ([4]), but we were not able to find its details in print. We then build on this result to obtain the LieDer version of module.

2.1 The category of LieDer pairs

Let \( g \) be a Lie algebra. A derivation of \( g \) is a linear map \( \varphi : g \to g \) which satisfies the Leibniz relation:

\[
\varphi[x,y] = [\varphi(x),y] + [x,\varphi(y)].
\]

One denotes by \( \text{Der}(g) \) the set of derivations of the Lie algebra \( g \).

**Definition 2.1.** A LieDer pair is a Lie algebra \( g \) with a derivation \( \varphi \in \text{Der}(g) \). One denotes it by \((g,\varphi)\). It is called abelian if the Lie bracket of \( g \) is trivial, that is \([x,y] = 0\) for all \( x,y \in g \).

**Notation 2.2.** We will use \([\cdot,\cdot]_g\) instead of \([\cdot,\cdot]\) if precision is needed. The same goes for \( \varphi_g \) instead of \( \varphi \).

**Definition 2.3.** Let \((g,\varphi_g)\) and \((h,\varphi_h)\) be LieDer pairs. A morphism \( f \) from \((g,\varphi_g)\) to \((h,\varphi_h)\) is a Lie algebra morphism \( f : g \to h \) such that

\[
f \circ \varphi_g = \varphi_h \circ f.
\]

We denote by LieDer the category of LieDer pairs and their morphisms.

2.2 Monoid objects in slice categories

**Definition 2.4.** For a category \( C \) and an object \( A \) in \( C \). The slice category \( C/A \) is the category whose

- objects \((B,\pi)\) are \( C \)-morphisms \( \pi : B \to A \), \( B \in C \), and
- morphisms \((B',\pi') \to (B'',\pi'')\) are commutative diagrams of \( C \)-morphisms:

\[
\begin{array}{ccc}
B' & \xrightarrow{f} & B'' \\
\downarrow \pi' & & \downarrow \pi'' \\
A & = & A
\end{array}
\]

**Definition 2.5.** Let \( C \) be a category with finite products and a terminal object \( T \). A monoid object in \( C \) is an object \( X \in \text{Ob}(C) \) together with two morphisms \( \mu : X \times X \to X \) and \( \eta : T \to X \) such that following diagrams commute:
• the associativity of $\mu$:

\[ X \times X \times X \xrightarrow{\mu \times \text{Id}_X} X \times X \]

\[ \xrightarrow{\text{Id}_X \times \mu} \]

\[ X \times X \xrightarrow{\mu} X, \]

• the neutrality of $\eta$:

\[ X \times X \xrightarrow{\mu} X \]

\[ \xleftarrow{\eta \times \text{Id}_X} \]

\[ X \times X \xrightarrow{\mu} \]

\[ \xrightarrow{\eta} \]

\[ X \times X \]

\[ \xrightarrow{\text{Id}_X \times \eta} \]

\[ X \times T \leftarrow (\text{Id}_X, t_X) \]

\[ X \xrightarrow{(t_X, \text{Id}_X)} T \times X, \]

where $t_X : X \rightarrow T$ is the unique map.

Let $C_m$ be the category whose objects are monoid objects $(X, \mu, \eta)$ in $C$ as above and the hom-set $\text{Hom}_{C_m}((X, \mu, \eta), (X', \mu', \eta'))$ is the set of all $f \in \text{Hom}_C(X, X')$ for which $\mu' \circ (f \times f) = f \circ \mu$ and $\eta' = f \circ \eta$.

2.3 The Lie case

Denote by $\text{Lie}$ the category of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. We show in Subsection 2.3.1 how monoid objects in $\text{Lie}/\mathfrak{g}$ give rise to $\mathfrak{g}$-modules. We then show in the following subsection that they form equivalent categories.

2.3.1 Deciphering the definition of a monoid object in the Lie case

It is obvious that the terminal object of the slice category $\text{Lie}/\mathfrak{g}$ is $T = \mathfrak{g} \xrightarrow{\text{Id}} \mathfrak{g}$. Let $X = \mathfrak{g} \xrightarrow{\kappa} \mathfrak{g}$ be a monoid object in $\text{Lie}/\mathfrak{g}$ with $\mu$ and $\eta$ as above. The information contained in $\eta$ gives this first result:

**Lemma 2.6.** There exists a vector space $V$ such that

\[ \mathfrak{g} = s(\mathfrak{g}) \oplus V. \]

**Proof.** It suffices to prove that $\kappa$ is a splitting epimorphism and take $V := \ker(\kappa)$. But remark that $\eta : T \rightarrow X$ in the slice category actually means that there exists a commutative diagram

\[ \begin{array}{ccc}
\mathfrak{g} & \xrightarrow{s} & \mathfrak{g} \\
\downarrow{\text{Id}_\mathfrak{g}} & & \downarrow{\kappa} \\
\mathfrak{g} & \xrightarrow{\kappa} & \mathfrak{g}
\end{array} \]

i.e. $\kappa \circ s = \text{Id}$, so we have $\mathfrak{g} = s(\mathfrak{g}) \oplus V$. \hfill \qed

The rest of this section aims to show that $V$ is an abelian Lie algebra and that it comes with an action of $\mathfrak{g}$. But for that we need an expression of $\mu$ in terms of $s$, the section of $\kappa$ that appeared in the previous proof.
First, notice that \( X \times X \) is given by \( k \times g \kappa \to g \), where \( k \times g k = \{ (t, t') \in k \oplus k \mid \kappa(t) = \kappa(t') \} \), with bracket defined by \( [(t, t'), (r, r')] := ([t, r], [t', r']) \) and \( \bar{\kappa}(t, t') := \kappa(t) \). In the slice category, \( X \times X \) amounts to the commutative diagram

\[
\begin{array}{ccc}
\times & \rightarrow & g \\
\downarrow \kappa & \downarrow \kappa \\
\rightarrow & \rightarrow & g \\
\end{array}
\]

**Lemma 2.7.** With the above notations, we have

\[
M(t, t') = t + t' - s(\kappa(t)).
\]  (3)

**Proof.** In terms of \( M \), the neutrality of \( \eta \) amounts to the set of equations

\[
\begin{align*}
M \circ (s \times \text{Id}) \circ (\kappa, \text{Id}) &= \text{Id} \\
M \circ (\text{Id} \times s) \circ (\text{Id}, \kappa) &= \text{Id}.
\end{align*}
\]

We used here the fact that in the slice category, \( t_X : X \to T \) translates in the commutative diagram

\[
\begin{array}{ccc}
\times & \rightarrow & g \\
\downarrow \kappa \downarrow \kappa & \downarrow \kappa \\
\rightarrow & \rightarrow & g \\
\end{array}
\]

Thus, for any \( t \in \mathfrak{k} \) we have

\[
M(s(\kappa(t)), t) = M(t, s(\kappa(t))) = t.
\]  (4)

Let \( (t, t') \in \mathfrak{k} \times \mathfrak{k} \), since \( \mathfrak{k} \times \mathfrak{k} \) is a vector space, we have

\[
(t, t') = \left( t - s(\kappa(t)) + s(\kappa(t)), t' - s(\kappa(t')) + s(\kappa(t')) \right)
= \left( s(\kappa(t)), s(\kappa(t')) \right) + (t - s(\kappa(t)), 0) + (0, t' - s(\kappa(t'))).
\]

Since \( M \) is a morphism in \( \text{Lie} \), it is in particular linear, hence

\[
M(t, t') = M(s(\kappa(t)), s(\kappa(t'))) + M(t - s(\kappa(t)), 0) + M(0, t' - s(\kappa(t'))).
\]

By (4), we have

\[
M(t, t') = s(\kappa(t)) + t - s(\kappa(t)) + t' - s(\kappa(t'))
= t + t' - s(\kappa(t)).
\]

The proof is finished. \( \blacksquare \)

**Remark 2.8.** It can be seen from (3) that a \( \mu \) satisfying this equation is automatically associative. Therefore, a monoid object in \( \text{Lie}/g \) is uniquely determined by the neutral map \( \eta \).

**Lemma 2.9.** \( V := \ker(\kappa) \) is an abelian sub Lie algebra of \( \mathfrak{k} \).
Proof. Since $\mathfrak{t} \times \mathfrak{t} \xrightarrow{M} \mathfrak{t}$ is a map in Lie,
\[ M(\left( (k, 0), (0, k') \right)) = [M(k, 0), M(0, k')] = [k, k'] \]
by (3) if moreover $k, k' \in \ker(\kappa)$. On the left side, we have
\[ M(\left( (k, 0), (0, k') \right)) = M(\left( k, 0 \right), \left( 0, k' \right)) = M(0, 0) = 0. \]
Thus, we have $[k, k'] = 0$ for all $k, k' \in \ker(\kappa)$. \[ \blacksquare \]

We recall that a representation of a Lie algebra $\mathfrak{g}$ on a vector space $V$ is a Lie algebra morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Lemma 2.10. The expression
\[ \rho(x)(k) = [s(x), k], \quad \forall x \in \mathfrak{g}, k \in V \quad (5) \]
defines a representation of $\mathfrak{g}$ on $V$.

Proof. Let us first check that $\rho$ is well defined, i.e. that for all $x \in \mathfrak{g}, k \in V$, the right hand side of (5) is in $V$. It follows from
\[ \kappa([s(x), k]) = [x, \kappa(k)]_\mathfrak{g} = [x, 0]_\mathfrak{g} = 0, \]
and $V = \ker(\kappa)$. We now show that $\rho$ is a morphism. Recall that, by the Jacobi identity, $\text{ad} : \mathfrak{t} \to \mathfrak{gl}(\mathfrak{t})$ is a Lie algebra morphism. One concludes by the remarks that $\rho = \text{ad} \circ s$ and that $s$ is also a Lie algebra morphism. \[ \blacksquare \]

Remark 2.11. Hadn’t we known the definition of a representation, studying the properties of $\rho$ of Lemma 2.10 would have led us to the correct definition. This is what we will do in Section 2.4.1 to obtain the definition of a representation of a LieDer pair.

2.3.2 Monoids in $\text{Lie/}\mathfrak{g}$ and $\mathfrak{g}$-representations form equivalent categories

We build two functors $\xrightarrow{\ker} \text{(Lie/}\mathfrak{g})_m \xrightarrow{\rho} \mathfrak{g} - \text{Rep}$, and show that they induce equivalences of categories.

We start with the functor $\ker : (\text{Lie/}\mathfrak{g})_m \to \mathfrak{g} - \text{Rep}$. Given $X = \mathfrak{t} \xrightarrow{\kappa} \mathfrak{g}$ in $(\text{Lie/}\mathfrak{g})_m$, one defines $\ker(X)$ to be $V := \ker(\kappa)$. The previous section insures that $\ker(X)$ is in $\mathfrak{g} - \text{Rep}$. We leave it as an exercise for the reader to verify that the restrictions of morphisms to the kernels of the maps to $\mathfrak{g}$ induce maps of representations.

We now construct the functor $\kappa : \mathfrak{g} - \text{Rep} \to (\text{Lie/}\mathfrak{g})_m$. Recall [?] that one can associate to a representation $V$ of $\mathfrak{g}$ the semi-direct product $\mathfrak{g} \ltimes V$:

Proposition-definition 2.12. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. The space $\mathfrak{g} \oplus V$ becomes a Lie algebra with the bracket
\[ [x + u, y + v]_\rho = [x, y]_\mathfrak{g} + \rho(x)(v) - \rho(y)(u), \quad \forall x, y \in \mathfrak{g}, u, v \in V. \]
We denote this Lie algebra by $\mathfrak{g} \ltimes V$. 

6
A first remark is that since the canonical projection \( p : g \ltimes V \to g \) is a morphism of Lie algebras, \( X := g \ltimes V \xrightarrow{p} g \) is an object in \( \text{Lie}/g \). It therefore suffices to equip \( X \) with a suitable product \( \mu \) and unit \( \eta \) to complete the definition of the functor \( \kappa \).

As already seen in the proof of Lemma 2.6, a map \( \eta : T \to X \) amounts to a splitting of the map \( X = t \to g \). In the case \( X = g \ltimes V \), such a splitting, and hence our map \( \eta \), is given by the canonical inclusion \( i : g \to g \ltimes V \).

By Lemma 2.7, \( \mu \) is determined by \( M : (g \ltimes V) \times (g \ltimes V) \to g \ltimes V \), where \( (g \ltimes V) \times (g \ltimes V) = \{(x + u, x + v) | x \in g, u, v \in V \} \) and \( M(x + u, x + v) = x + u + v \).

Therefore, \( g \ltimes V \) is a monoid object in \( \text{Lie}/g \), the image \( \ltimes (V) \) of \( V \) by the functor \( \kappa \), which achieves the construction of this functor.

**Theorem 2.13.** The functors \( \text{(Lie}/g)_m \xrightarrow{\kappa} g - \text{Rep} \) induce an equivalence of categories. 

**Proof.** It is straightforward to see that \( \ker \circ \kappa = \text{Id} \). On the other hand, for \( t \xrightarrow{\kappa} g \in \text{(Lie}/g)_m \), we have \( (\kappa \circ \ker)(t \xrightarrow{\kappa} g) = (g \ltimes \ker(\kappa)) \xrightarrow{p} g \).

We define \( \Psi : t \to g \oplus \ker(\kappa) \) by \( \Psi(t) = \kappa(t) + t - (s \circ \kappa)(t) \). Moreover, \( \Psi \) is a Lie algebra isomorphism from \( t \) to \( g \oplus \ker(\kappa) \) and

\[
\begin{align*}
p(\Psi(t)) &= p(\kappa(t) + t - (s \circ \kappa)(t)) = \kappa(t), \\
\Psi(s(x)) &= \kappa(s(x)) + s(x) - (s \circ \kappa)(s(x)) = x = i(x).
\end{align*}
\]

Thus, \( \Psi \) is an isomorphism from the monoid object \( t \xrightarrow{\kappa} g \) to the monoid object \( g \ltimes \ker(\kappa) \xrightarrow{p} g \). We have defined a natural equivalence from the identify functor to \( \kappa \circ \ker \). ■

### 2.4 The LieDer case

There is a forgetful functor \( \text{LieDer} \xrightarrow{F} \text{Lie} \) which consists in forgetting the derivations and that the maps intertwine the derivations. Fix an object \( (g, \varphi_g) \) in \( \text{LieDer} \). The functor \( F \) induces a functor

\[
\text{(LieDer}/(g, \varphi_g))_m \xrightarrow{F} \text{(Lie}/g)_m
\]

between monoid objects in the slice categories.

The aim of this section is to lift the equivalence of the previous section at the level of \( \text{LieDer} \) by completing the following diagram

\[
\begin{array}{ccc}
\text{(LieDer}/(g, \varphi_g))_m & \xrightarrow{F} & \text{(Lie}/g)_m \\
\downarrow & & \downarrow \kappa \\
\text{g - Rep.} & & \\
\end{array}
\]

That is, by introducing the correct definition of module in the category \( \text{LieDer} \) and understanding its relationship with monoid objects in \( \text{LieDer}/(g, \varphi_g) \).
2.4.1 Deciphering the definition of a monoid object in the LieDer case

According to Remark 2.11, an analysis of monoid objects in \( \text{LieDer}/(g, \varphi_g) \) should lead us to the correct definition of module. We therefore consider an element \( X = (t, \varphi_t) \rightarrow (g, \varphi_g) \) in \( (\text{LieDer}/(g, \varphi_g))_m \) with \( \mu \) and \( \eta \) as above.

The information contained in \( \eta \) gives this first result:

**Lemma 2.14.** One can decompose \( \varphi_t \) as

\[
\varphi_t = \varphi_t|_{s(g)} \oplus \varphi_t|_V
\]

where \( s(g) \) and \( V \) are given by the decomposition \( t = s(g) \oplus V \), see Lemma 2.6.

**Proof.** The morphism \( \eta : T \rightarrow X \) is given by the commutative diagram

![Diagram](https://via.placeholder.com/150)

Since \( (t, \varphi_t) \rightarrow (g, \varphi_g) \) is in \( \text{LieDer} \), that is, \( \kappa \circ \varphi_t = \varphi_g \circ \kappa \). For all \( v \in \text{ker}(\kappa) = V \), we have

\[
\kappa(\varphi_t(v)) = \varphi_g(\kappa(v)) = 0.
\]

Thus, we deduce that \( \varphi_t(V) \subset V \). Similarly, \( (g, \varphi_g) \rightarrow (t, \varphi_t) \) is in \( \text{LieDer} \), for all \( x \in g \), we have

\[
\varphi_t(s(x)) = s(\varphi_g(x)).
\]

Thus, we deduce that \( \varphi_t(s(g)) \subset s(g) \). The proof is finished. 

We now interpret in terms of \( \varphi_g \) and \( \varphi_t|_V \) the fact that \( \varphi_t \) is a derivation.

**Lemma 2.15.** The following is satisfied

\[
\varphi_t|_V[s(x), v] = [s(\varphi_g(x)), v] + [s(x), \varphi_t|_V(v)].
\]

**Proof.** Apply the Leibniz rule for \( \varphi_t = \varphi_t|_{s(g)} \oplus \varphi_t|_V \) to the element \([s(x), v]\) of \( t = s(g) \oplus V \) and \( \varphi_t \circ s = s \circ \varphi_g \). 

We can now, according to Remark 2.11 and the previous two lemmas, give the following :

**Definition 2.16.** A representation of a LieDer pair \((g, \varphi_g)\) on a vector space \( V \) with respect to \( \varphi_V \in \mathfrak{gl}(V) \) is a Lie algebra morphism \( \rho : g \rightarrow \mathfrak{gl}(V) \) such that for all \( x \in g \), the following equality is satisfied:

\[
\varphi_V \circ \rho(x) = \rho(\varphi_g(x)) + \rho(x) \circ \varphi_V.
\]

(6)

We denote a representation by \((\rho, V, \varphi_V)\). For all \( x \in g \), we define \( \text{ad}_x : g \rightarrow g \) by

\[
\text{ad}_x(y) = [x, y]_g, \quad \forall y \in g.
\]

(7)

Then \((\text{ad}, g, \varphi_g)\) is a representation of the LieDer pair \((g, \varphi_g)\) on \( g \) with respect to \( \varphi_g \), which is called the adjoint representation.

A representation \((\rho, V, \varphi_V)\) of a LieDer pair \((g, \varphi_g)\) is said to be trivial if \( \rho = 0 \).

A similar study of morphisms of monoid objects leads to the following
Definition 2.17. Let \((\rho, V, \varphi_V)\) and \((\rho', V', \varphi_{V'})\) be two representations of the \textsc{LieDer} pair \((\mathfrak{g}, \varphi_{\mathfrak{g}})\). A morphism from \((\rho, V, \varphi_V)\) to \((\rho', V', \varphi_{V'})\) is a morphism of Lie algebra representations \(f : V \to V'\) such that

\[
f \circ \varphi_V = \varphi_{V'} \circ f.
\]

\[\text{(8)}\]

Notation 2.18. Let \((\mathfrak{g}, \varphi_{\mathfrak{g}})\) be a \textsc{LieDer} pair. We denote by \((\mathfrak{g}, \varphi_{\mathfrak{g}})\text{-Rep}\) the category of the representations of the \textsc{LieDer} pair \((\mathfrak{g}, \varphi_{\mathfrak{g}})\) and their morphisms.

To sum-up the discussion and as an immediate corollary of Definition 2.10 and Lemmas 2.10, 2.14 and 2.15 one has

Corollary 2.19. Given a monoid object \(X = (\mathfrak{f}, \varphi_{\mathfrak{f}}) \xrightarrow{\kappa} (\mathfrak{g}, \varphi_{\mathfrak{g}})\) in \textsc{LieDer}/\((\mathfrak{g}, \varphi_{\mathfrak{g}})\), the expression

\[
\rho(x)(k) = [s(x), k]_{\mathfrak{f}}
\]

defines a representation of \((\mathfrak{g}, \varphi_{\mathfrak{g}})\) on \(V = \ker(\kappa)\) with respect to \(\varphi_{\mathfrak{f}}|_V \in \mathfrak{gl}(V)\).

2.4.2 Monoids in \textsc{LieDer}/\((\mathfrak{g}, \varphi_{\mathfrak{g}})\) and \((\mathfrak{g}, \varphi_{\mathfrak{g}})\)-representations form equivalent categories

In the previous section we have partially completed the diagram, lifting in Corollary 2.19 the functor \(\times\) to the \textsc{LieDer} level

![Diagram]

We now focus on the task of showing that the functor \(\times\) can also be lifted, i.e. we want to show that it is compatible with derivations.

Proposition 2.20. Given a representation \((\rho, V, \varphi_V)\) of a \textsc{LieDer} pair \((\mathfrak{g}, \varphi_{\mathfrak{g}})\), define \(\varphi_{\mathfrak{g}} + \varphi_V : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V\) by

\[
(\varphi_{\mathfrak{g}} + \varphi_V)(x + u) = \varphi_{\mathfrak{g}}(x) + \varphi_V(u).
\]

Then \((\mathfrak{g} \oplus V, \varphi_{\mathfrak{g}} + \varphi_V)\), with the Lie structure of Proposition 2.12, is a \textsc{LieDer} pair which we call the semi-direct product of the \textsc{LieDer} pair \((\mathfrak{g}, \varphi_{\mathfrak{g}})\) by the representation \((\rho, V, \varphi_V)\) and denote it by \(\mathfrak{g} \ltimes_{\text{LieDer}} V\).

Proof. It suffices to show that \(\varphi_{\mathfrak{g}} + \varphi_V\) is a derivation. On one hand, we have

\[
(\varphi_{\mathfrak{g}} + \varphi_V)[x + u, y + v] = (\varphi_{\mathfrak{g}} + \varphi_V)([x, y]_\mathfrak{g} + \rho(x)(v) - \rho(y)(u))
\]

\[
= \varphi_{\mathfrak{g}}([x, y]_\mathfrak{g}) + (\varphi_V \circ \rho)(x)(v) - (\varphi_V \circ \rho)(y)(u).
\]

On the other hand, we have

\[
[(\varphi_{\mathfrak{g}} + \varphi_V)(x + u), y + v]_{\mathfrak{g}} + [x + u, (\varphi_{\mathfrak{g}} + \varphi_V)(y + v)]_{\mathfrak{g}}
\]

\[
= [\varphi_{\mathfrak{g}}(x) + \varphi_V(u), y + v]_{\mathfrak{g}} + [x + u, \varphi_{\mathfrak{g}}(y) + \varphi_V(v)]_{\mathfrak{g}}
\]

\[
= [\varphi_{\mathfrak{g}}(x), y]_{\mathfrak{g}} + \rho(\varphi_{\mathfrak{g}}(x))(v) - (\rho(y) \circ \varphi_V)(u) + [x, \varphi_{\mathfrak{g}}(y)]_{\mathfrak{g}} + (\rho(x) \circ \varphi_V)(v) - \rho(\varphi_{\mathfrak{g}}(y))(u).
\]

One concludes by \(\text{(8)}\) and the fact that \(\varphi_{\mathfrak{g}}\) is a derivation. \(\blacksquare\)
Theorem 2.21. The functors \((\text{LieDer}/(g, \varphi_g))_m\) and \((g, \varphi_g) - \text{Rep}\) induce an equivalence of categories.

Proof. By Theorem 2.13, we only need to prove that
\[
\Psi \circ \varphi_t = (\varphi_g \oplus \varphi_t|_{V}) \circ \Psi.
\]
Since \(s\) and \(\kappa\) are LieDer pairs morphisms. For all \(t \in \mathfrak{t}\), we have
\[
(\varphi_g \oplus \varphi_t|_{V})(\Psi(t)) = (\varphi_g \oplus \varphi_t|_{V})(\kappa(t) + t - (s \circ \kappa)(t))
\]
\[
= \varphi_g(\kappa(t)) + \varphi_t(t - (s \circ \kappa)(t))
\]
\[
= \kappa(\varphi_t(t)) + \varphi_t(t) - s(\varphi_t(\kappa(t)))
\]
\[
= (\Psi \circ \varphi_t)(t).
\]
The proof is finished. \(\blacksquare\)

3 Cohomologies of LieDer pairs

Let \((\rho, V)\) be a representation of a Lie algebra \(g\). The Chevalley-Eilenberg cohomology of the Lie algebra \(g\) with the coefficient in \(V\) is the cohomology of the cochain complex \(C^n(g; V) = \text{Hom}(\wedge^n g, V)\) with the coboundary operator \(d : C^n(g; V) \rightarrow C^{n+1}(g; V)\) defined by
\[
(df_n)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^i f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})
\]
\[
+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).
\]
Denoted the set of closed \(n\)-cochains by \(Z^n(g; V)\) and the set of exact \(n\)-cochains by \(B^n(g; V)\). We denote by \(H^n(g; V) = Z^n(g; V)/B^n(g; V)\) the corresponding cohomology group.

Let \((\rho, V, \varphi_V)\) be a representation of a LieDer pair \((g, \varphi_g)\). We define the set of LieDer pair 0-cochains to be 0, and define the set of LieDer pair 1-cochains to be \(C^1_{\text{LieDer}}(g; V) = \text{Hom}(g, V)\). For \(n \geq 2\), we define the set of LieDer pair \(n\)-cochains by
\[
C^n_{\text{LieDer}}(g; V) \triangleq C^n(g; V) \times C^{n-1}(g; V).
\]
For \(n \geq 1\), we define an operator \(\delta : C^n(g; V) \rightarrow C^n(g; V)\) by
\[
\delta f_n = \sum_{i=1}^{n} f_n \circ (\text{Id} \otimes \cdots \otimes \varphi_g \otimes \cdots \otimes \text{Id}) - \varphi_V \circ f_n.
\]
Define \(\partial : C^1_{\text{LieDer}}(g; V) \rightarrow C^2_{\text{LieDer}}(g; V)\) by
\[
\partial f_1 = (df_1, (-1)^1 \delta f_1), \quad \forall f_1 \in \text{Hom}(g, V).
\]
Then for $n \geq 2$, we define $\partial : C^n_{\text{LieDer}}(g; V) \to C^{n+1}_{\text{LieDer}}(g; V)$ by

$$
\partial(f_n, g_{n-1}) = \left( df_n, dg_{n-1} + (-1)^n \delta f_n \right), \quad \forall f_n \in C^n(g; V), \ g_{n-1} \in C^{n-1}(g; V).
$$

(10)

The following lemma gives the relation between the operator $d$ and the operator $\delta$, which plays important role in the proof of that $\partial$ is a coboundary operator. We omit the proof which is straightforward tedious computations.

**Lemma 3.1.** The map $d$ and $\delta$ are commutative with each other, i.e. $d \circ \delta = \delta \circ d$.

**Theorem 3.2.** The map $\partial$ is a coboundary operator, i.e. $\partial \circ \partial = 0$.

**Proof.** For $n \geq 1$, since $d \circ \delta = \delta \circ d$, we have

$$
(\partial \circ \partial)(f_n, g_{n-1}) = \partial(df_n, dg_{n-1} + (-1)^n \delta f_n)
$$

$$
= \left( d(df_n), d(dg_{n-1} + (-1)^n \delta f_n) + (-1)^{n+1} \delta(df_n) \right)
$$

$$
= \left( 0, (-1)^n(d \circ \delta - \delta \circ d)(f_n) \right)
$$

$$
= 0.
$$

Thus, the map $\partial$ is a coboundary operator. ■

Associated to the representation $(\rho, V, \varphi_V)$, we obtain a complex $(C^*_\text{LieDer}(g; V), \partial)$. Denoted the set of closed $n$-cochains by $Z^n_{\text{LieDer}}(g; V)$ and the set of exact $n$-cochains by $B^n_{\text{LieDer}}(g; V)$. We define the corresponding cohomology group by

$$
H^n_{\text{LieDer}}(g; V) = Z^n_{\text{LieDer}}(g; V)/B^n_{\text{LieDer}}(g; V).
$$

**Proposition 3.3.** Let $(\rho, V, \varphi_V)$ be a representation of a LieDer pair $(g, \varphi_g)$. Then we have

$$
H^1_{\text{LieDer}}(g; V) = \{ f | f \in Z^1(g; V), \ f \circ \varphi_g = \varphi_V \circ f \}.
$$

**Proof.** For any $f \in C^1_{\text{LieDer}}(g; V)$, we have

$$
\partial f = (df, -\delta f).
$$

Therefore, $f$ is closed if and only if $f \in Z^1(g; V)$ and $f \circ \varphi_g = \varphi_V \circ f$. The conclusion follows from the fact that there is no exact 1-cochain. ■

### 4 Deformations of a LieDer pair

In this section, we study formal deformations and deformations of order $n$ of a LieDer pair.

#### 4.1 Formal deformations of a LieDer pair

In this subsection, we study 1-parameter formal deformations of a LieDer pair. We show that if the second cohomology group of a LieDer pair with the coefficient in the adjoint representation is trivial, then the LieDer pair is rigid.
Let $(g, \varphi)$ be a LieDer pair. In the sequel, we will also denote the Lie bracket $[\cdot, \cdot]$ by $\omega$. Consider a $t$-parametrized family of linear operations
\[ \omega_t = \sum_{i \geq 0} \omega_i t^i, \quad \omega_i \in C^2(g; g), \]
\[ \varphi_t = \sum_{i \geq 0} \varphi_i t^i, \quad \varphi_i \in C^1(g; g). \]

**Definition 4.1.** If all $(\omega_t, \varphi_t)$ endow the $\mathbb{K}[\![t]\!]$-module $\mathfrak{g}[\![t]\!]$ the LieDer-pair structure with $(\omega_0, \varphi_0) = (\omega, \varphi)$, we say that $(\omega_t, \varphi_t)$ is a 1-parameter formal deformation of the LieDer pair $(g, \varphi)$.

A pair $(\omega_t, \varphi_t)$, as given above, is a 1-parameter formal deformation of the LieDer pair $(g, \varphi)$ if and only if for all $x, y, z \in g$, the following equalities hold:
\[ \omega_t(\omega_t(x, y), z) + \omega_t(\omega_t(y, z), x) + \omega_t(\omega_t(z, x), y) = 0, \]  \hspace{1cm} (11)
\[ \varphi_t(\omega_t(x, y)) - \omega_t(\varphi_t(x), y) - \omega_t(x, \varphi_t(y)) = 0. \]  \hspace{1cm} (12)

Expanding the equations in (11), (12) and collecting coefficients of $t^n$, we see that (11) and (12) are equivalent to the system of equations
\[ \sum_{i+j \geq 0} \left( \omega_i(\omega_j(x, y), z) + \omega_i(\omega_j(y, z), x) + \omega_i(\omega_j(z, x), y) \right) = 0, \]  \hspace{1cm} (13)
\[ \sum_{i+j \geq 0} \left( \varphi_i(\omega_j(x, y)) - \omega_j(\varphi_i(x), y) - \omega_j(x, \varphi_i(y)) \right) = 0. \]  \hspace{1cm} (14)

**Remark 4.2.** For $n = 0$, condition (13) is equivalent to the usual Jacobi identity of $\omega$, and (14) is equivalent to the fact that $\varphi$ is a derivation.

**Proposition 4.3.** Let $(\omega_t, \varphi_t)$ be a 1-parameter formal deformation of the LieDer pair $(g, \varphi)$. Then $(\omega_t, \varphi_t) \in C^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{g})$ is a 2-cocycle of the LieDer pair $(g, \varphi)$ with the coefficient in the adjoint representation.

**Proof.** For $n = 1$, (13) is equivalent to $d\omega_1 = 0$, and (14) is equivalent to $d\varphi_1 + \delta \omega_1 = 0$. Thus for $n = 1$, (13) and (14) are equivalent to that $(\omega_t, \varphi_t)$ is a 2-cocycle. \( \blacksquare \)

**Definition 4.4.** The 2-cocycle $(\omega_t, \varphi_t)$ is called the infinitesimal of the 1-parameter formal deformation $(\omega_t, \varphi_t)$ of the LieDer pair $(g, \varphi)$.

**Definition 4.5.** Let $(\tilde{\omega}_t, \tilde{\varphi}_t)$ and $(\omega_t, \varphi_t)$ be 1-parameter formal deformations of a LieDer pair $(g, \varphi)$. A formal isomorphism from $(\tilde{\omega}_t, \tilde{\varphi}_t)$ to $(\omega_t, \varphi_t)$ is a power series $\Phi_t = \sum_{i \geq 0} \phi_i t^i : \mathfrak{g}[\![t]\!] \to \mathfrak{g}[\![t]\!]$, where $\phi_i \in C^1(\mathfrak{g}; \mathfrak{g})$ with $\phi_0 = \text{Id}$, such that
\[ \Phi_t \circ \tilde{\omega}_t = \omega_t \circ (\Phi_t \times \Phi_t), \]  \hspace{1cm} (15)
\[ \Phi_t \circ \tilde{\varphi}_t = \varphi_t \circ \Phi_t. \]  \hspace{1cm} (16)

Two 1-parameter formal deformations $(\tilde{\omega}_t, \tilde{\varphi}_t)$ and $(\omega_t, \varphi_t)$ are said to be equivalent if there exists a formal isomorphism $\Phi_t : (\tilde{\omega}_t, \tilde{\varphi}_t) \to (\omega_t, \varphi_t)$.

\(^{1}\) The notation $\mathfrak{g}[\![t]\!]$ means the vector space of formal power series in $t$ with coefficients in $\mathfrak{g}$, that is, for all $x_t \in \mathfrak{g}[\![t]\!]$, we have $x_t = x_0 + x_1 t + x_2 t^2 + \cdots$, for $x_0, x_1, x_2, \cdots \in \mathfrak{g}$.

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Thus, we have

\[ \Phi_t \circ \omega_t(x, y) = \omega_t \circ (\Phi_t \times \Phi_t)(x, y), \]
\[ \Phi_t \circ \varphi_t(x) = \varphi_t \circ \Phi_t(x). \]

Expanding the above identities and comparing coefficients of \( t \), we have

\[ \omega_t(x, y) = \omega_1(x, y) + \omega(\phi_1(x), y) + \omega(x, \phi_1(y)) - \phi_1(\omega(x, y)) \]
\[ \varphi_t(x) = \varphi_1(x) + \varphi(\phi_1(x)) - \phi_1(\varphi(x)). \]

Thus, we have \((\omega_1, \varphi_1) = (\omega_1, \varphi_1) + \partial(\phi_1)\), which implies that \([\omega_1, \varphi_1] = [\omega_1, \varphi_1] \in \mathcal{H}^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{g})\). The proof is finished. \( \blacksquare \)

**Definition 4.7.** A 1-parameter formal deformation \((\omega_t, \varphi_t)\) of a LieDer pair \((\mathfrak{g}, \Phi)\) is said to be trivial if it is equivalent to \((\omega, \varphi)\), i.e. there exists \( \Phi_t = \sum_{i \geq 0} \phi_i t^i : \mathfrak{g}[t] \to \mathfrak{g}[t] \), where \( \phi_i \in C^i(\mathfrak{g}; \mathfrak{g}) \) with \( \phi_0 = \text{Id} \), such that

\[ \Phi_t \circ \omega_t = \omega \circ (\Phi_t \times \Phi_t), \]
\[ \Phi_t \circ \varphi_t = \varphi \circ \Phi_t. \]

**Definition 4.8.** A LieDer pair \((\mathfrak{g}, \Phi)\) is said to be rigid if every 1-parameter formal deformation of \((\mathfrak{g}, \Phi)\) is trivial.

**Theorem 4.9.** If \( \mathcal{H}^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{g}) = 0 \), then the LieDer pair \((\mathfrak{g}, \Phi)\) is rigid.

**Proof.** Let \((\omega_t, \varphi_t)\) be a 1-parameter formal deformation of the LieDer pair \((\mathfrak{g}, \Phi)\). By Proposition 4.3, we have \((\omega_1, \varphi_1) \in \mathcal{Z}^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{g})\). By \( \mathcal{H}^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{g}) = 0 \), there exists a 1-cochain \( \phi_1 \in C^1(\mathfrak{g}; \mathfrak{g}) \) such that

\[ [\omega_1, \varphi_1] = -\partial(\phi_1). \]

Then setting \( \Phi_t = \text{Id} + \phi_1 t \), we have a deformation \((\bar{\omega}_t, \bar{\varphi}_t)\), where

\[ \bar{\omega}_t(x, y) = (\Phi_t^{-1} \circ \omega_t \circ (\Phi_t \times \Phi_t))(x, y), \]
\[ \bar{\varphi}_t(x) = (\Phi_t^{-1} \circ \varphi_t \circ \Phi_t)(x). \]

Thus, \((\bar{\omega}_t, \bar{\varphi}_t)\) is equivalent to \((\omega_t, \varphi_t)\). Moreover, we have

\[ \bar{\omega}_t(x, y) = (\text{Id} + \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i)(\omega_1(x + \phi_1(x)t, y + \phi_1(y)t)), \]
\[ \bar{\varphi}_t(x) = (\text{Id} + \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i)(\varphi_1(x + \phi_1(x)t)). \]

Thus, we have

\[ \bar{\omega}_t(x, y) = \omega(x, y) + (\omega_1(x, y) + \omega(\phi_1(x), y) + \omega(x, \phi_1(y)) - \phi_1(\omega(x, y)))t + \bar{\omega}_2(x, y)t^2 + \cdots, \]
\[ \bar{\varphi}_t(x) = \varphi(x) + (\varphi(\phi_1(x)) + \varphi_1(x - \phi_1(\varphi(x)))t + \bar{\varphi}_2(x)t^2 + \cdots. \]

By \((19)\), we have

\[ \bar{\omega}_t(x, y) = \omega(x, y) + \omega_2(x, y)t^2 + \cdots, \]
\[ \bar{\varphi}_t(x) = \varphi(x) + \bar{\varphi}_2(x)t^2 + \cdots. \]

Then by repeating the argument, we can show that \((\omega_t, \varphi_t)\) is equivalent to \((\omega, \varphi)\). \( \blacksquare \)

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4.2 Deformations of order $n$ of a LieDer pair

In this subsection, we introduce a cohomology class associated to any deformation of order $n$ of a LieDer pair. We show that a deformation of order $n$ of a LieDer pair is extensible if and only if this cohomology class is trivial. Thus we call this cohomology class the obstruction class of a deformation of order $n$ being extensible.

**Definition 4.10.** A deformation of order $n$ of a LieDer pair $(g, \varphi)$ is a pair $(\omega_t, \varphi_t)$ such that $\omega_t = \sum_{i=0}^{n} \omega_i t^i$ and $\varphi_t = \sum_{i=0}^{n} \varphi_i t^i$ endow the $\mathbb{K}[t]/(t^{n+1})$-module $g[[t]]/(t^{n+1})$ the LieDer pair structure with $(\omega_0, \varphi_0) = (\omega, \varphi)$.

**Definition 4.11.** Let $(\omega_t, \varphi_t)$ be a deformation of order $n$ of a LieDer pair $(g, \varphi)$. If there exists a 2-cocycle $(\omega_{n+1}, \varphi_{n+1}) \in C^2_{\text{LieDer}}(g; g)$, such that the pair $((\omega_t, \varphi_t), (\omega_{n+1}, \varphi_{n+1}))$ is a deformation of order $n+1$ of the LieDer pair $(g, \varphi)$, then we say that $(\omega_t, \varphi_t)$ is extensible.

Let $(\omega_t, \varphi_t)$ be a deformation of order $n$ of a LieDer pair $(g, \varphi)$. Define $(\text{Ob}^3_{(\omega_t, \varphi_t)}, \text{Ob}^2_{(\omega_t, \varphi_t)}) \in C^3_{\text{LieDer}}(g; g)$ by

\[
\text{Ob}^3_{(\omega_t, \varphi_t)}(x, y, z) = \sum_{i+j=n+1, i, j \geq 0} \left( \omega_i(\omega_j(x, y, z)) + \omega_i(\omega_j(y, z, x)) + \omega_i(\omega_j(z, x, y)) \right),
\]

\[
\text{Ob}^2_{(\omega_t, \varphi_t)}(x, y) = \sum_{i+j=n+1, i, j \geq 0} \left( \varphi_i(\omega_j(x, y)) - \omega_j(\varphi_i(y, x)) - \omega_j(\varphi_i(x, y)) \right).
\]

In the sequel, we show that $(\text{Ob}^3_{(\omega_t, \varphi_t)}, \text{Ob}^2_{(\omega_t, \varphi_t)})$ is a 3-cocycle. We need some preparations. Let $g$ be a Lie algebra. The Nijenhuis-Richardson bracket $[\cdot, \cdot]$ on the graded vector space $C^*(g; g) = \oplus_k \mathbb{K} C^{k+1}(g; g)$ is given by [25]:

\[
[P, Q] = P \circ Q - (-1)^{|P|} Q \circ P, \quad \forall P \in C^{p+1}(g; g), Q \in C^{q+1}(g; g),
\]

where $P \circ Q \in C^{p+q+1}(g; g)$ is defined by

\[
P \circ Q(x_1, \ldots, x_{p+q+1}) = \sum_{\sigma \in \text{unsh}(q+1, p)} (-1)^{|\sigma|} P(Q(x_{\sigma(1)}, \ldots, x_{\sigma(q+1)}), x_{\sigma(q+2)}, \ldots, x_{\sigma(p+q+1)}).
\]

Furthermore, $(C^*(g; g), [\cdot, \cdot], \partial = [\omega, -])$ is a differential graded Lie algebra. The Chevalley-Eilenberg coboundary operator $d$ of the Lie algebra $g$ with the coefficient in the adjoint representation can be reformulated as follows:

\[
df = (-1)^{|k|-1} \partial f = (-1)^{|k|-1} [\omega, f], \quad \forall f \in C^k(g; g).
\]

**Proposition 4.12.** Let $(\omega_t, \varphi_t)$ be a deformation of order $n$ of a LieDer pair $(g, \varphi)$. The 3-cocohain $(\text{Ob}^3_{(\omega_t, \varphi_t)}, \text{Ob}^2_{(\omega_t, \varphi_t)})$ defined by (20) and (21) is a 3-cocycle of the LieDer pair $(g, \varphi)$ with the coefficient in the adjoint representation.

**Proof.** We use the Nijenhuis-Richardson bracket to write $\text{Ob}^3_{(\omega_t, \varphi_t)}$ and $\text{Ob}^2_{(\omega_t, \varphi_t)}$ as follows:

\[
\text{Ob}^3_{(\omega_t, \varphi_t)} = \frac{1}{2} \sum_{i+j=n+1, i, j \geq 0} [\omega_i, \omega_j], \quad \text{Ob}^2_{(\omega_t, \varphi_t)} = \sum_{i+j=n+1, i, j \geq 0} [\varphi_i, \omega_j].
\]
Since \((\omega, \varphi)\) is a deformation of order \(n\) of the \textbf{LieDer} pair \((g, \varphi)\), for all \(0 \leq i \leq n\), we have

\[
\sum_{k+l \geq i, k, l \geq 0} \left( \omega_k(\omega_l(x, y), z) + \omega_k(\varphi_l(x, z), x) + \omega_k(\omega_l(z, x), y) \right) = 0, \tag{26}
\]

\[
\sum_{k+l \geq i, k, l \geq 0} \left( \varphi_k(\omega_l(x, y)) - \omega_l(\varphi_k(x), y) - \omega_l(x, \varphi_k(y)) \right) = 0. \tag{27}
\]

Thus, the equations (26) and (27) are equivalent to

\[
\frac{1}{2} \sum_{k+l \geq i, k, l \geq 0} [\omega_k, \omega_l] = -[\omega, \omega_i], \tag{28}
\]

\[
\sum_{k+l \geq i, k, l \geq 0} [\varphi_k, \omega_l] = -[\varphi, \omega_i] + [\omega, \varphi_i]. \tag{29}
\]

Then we have

\[
d\text{Ob}^3_{(\omega, \varphi)} = (-1)^2 [\omega, \text{Ob}^3_{(\omega, \varphi)}] = \frac{1}{2} \sum_{i+j=n+1} [\omega_i, [\omega_j, \omega]]
\]

\[
= \frac{1}{2} \sum_{i+j=n+1} ([\omega_i, [\omega_j, \omega]] - [\omega_i, [\omega, \omega_j]])
\]

\[
= \frac{1}{2} \sum_{i+j=n+1} ([\omega_i, [\omega_j, \omega]] - [\omega_i, [\omega, \omega_j]])
\]

\[
= \frac{1}{4} \sum_{i+j=n+1} [\omega_i, [\omega_j, \omega]] + \frac{1}{4} \sum_{i+j=n+1} [\omega_i, [\omega, \omega_j]]
\]

\[
= \frac{1}{4} \sum_{i+j=n+1} [\omega_i, [\omega_j, \omega]] + \frac{1}{4} \sum_{i+j=n+1} [\omega_i, [\omega, \omega_j]]
\]

\[
= \frac{1}{2} \sum_{i+j=n+1} [\omega_i, [\omega_j, \omega]]
\]

\[
= 0.
\]

Moreover, for all \(f \in C^k(g; g)\) we have

\[
\delta f = -[\varphi, f]. \tag{30}
\]
Thus, we have

\[
\begin{align*}
\text{dOb}_2^3(\omega_1, \varphi_i) &+ (-1)^3 \delta \text{Ob}_2^3(\omega_1, \varphi_i) \\
&= -[\omega_1, \text{Ob}_2^3(\omega_1, \varphi_i)] + [\varphi_i, \text{Ob}_2^3(\omega_1, \varphi_i)] \\
&= - \sum_{i,j>0} [\omega_1, [\varphi_i, \omega_j]] + \frac{1}{2} \sum_{i,j>0} [\varphi_i, [\omega_i, \omega_j]] \\
&= - \sum_{i,j>0} \left( [[\omega_1, \varphi_i], \omega_j] + [\varphi_i, [\omega_i, \omega_j]] \right) + \frac{1}{2} \sum_{i,j>0} \left( [[\varphi_i, \omega_j], \omega_j] + [\omega_j, [\varphi_i, \omega_j]] \right) \\
&= - \sum_{i,j>0} \left( [[\omega_1, \varphi_i], \omega_j] + [\varphi_i, [\omega_1, \omega_j]] \right) + \sum_{i,j>0} \left( [[\varphi_i, \omega_j], \omega_j] + [\omega_j, [\varphi_i, \omega_j]] \right) \\
&= 0.
\end{align*}
\]

Therefore, we have

\[
\delta(\text{Ob}_2^3(\omega_1, \varphi_i), \text{Ob}_2^3(\omega_1, \varphi_i)) = (\text{dOb}_2^3(\omega_1, \varphi_i), \text{dOb}_2^3(\omega_1, \varphi_i)) + (-1)^3 \delta \text{Ob}_2^3(\omega_1, \varphi_i) = 0.
\]

The proof is finished. 

\[\blacksquare\]

**Definition 4.13.** Let \((\omega_1, \varphi)\) be a deformation of order \(n\) of a LieDer pair \((g, \varphi)\). The cohomology class \([\text{Ob}_2^3(\omega_1), \text{Ob}_2^3(\omega_1, \varphi_i)]\) \(\in H^3_{\text{LieDer}}(g; g)\) is called the obstruction class of \((\omega_1, \varphi)\) being extensible.

**Theorem 4.14.** Let \((\omega_1, \varphi)\) be a deformation of order \(n\) of a LieDer pair \((g, \varphi)\). Then \((\omega_1, \varphi_i)\) is extensible if and only if the obstruction class \([\text{Ob}_2^3(\omega_1, \varphi_i), \text{Ob}_2^3(\omega_1, \varphi_i)]\) is trivial.

**Proof.** Suppose that a deformation \((\omega_1, \varphi)\) of order \(n\) of the LieDer pair \((g, \varphi)\) extends to a deformation of order \(n + 1\). Then \(26\) and \(27\) hold for \(i = n + 1\). Thus, we have

\[
\text{Ob}_2^3(\omega_1, \varphi_i) = \omega_{n+1}, \quad \text{Ob}_2^3(\omega_1, \varphi_i) = \varphi_{n+1} + \delta \omega_{n+1},
\]

which implies that

\[
(\text{Ob}_2^3(\omega_1, \varphi_i), \text{Ob}_2^3(\omega_1, \varphi_i)) = \partial(\omega_{n+1}, \varphi_{n+1}).
\]

Thus, the obstruction class \([\text{Ob}_2^3(\omega_1, \varphi_i), \text{Ob}_2^3(\omega_1, \varphi_i)]\) is trivial. 

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Conversely, if the obstruction class $[(\text{Ob}_3^\ell, \text{Ob}_2^\ell), \text{Ob}_3^\ell, \text{Ob}_2^\ell)]$ is trivial, suppose that
\[
(\text{Ob}_3^\ell, \text{Ob}_2^\ell) = \partial(\omega_{n+1}, \varphi_{n+1}),
\]
for some 2-cochain $(\omega_{n+1}, \varphi_{n+1}) \in C^2_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g})$. Set
\[
(\tilde{\omega}_t, \tilde{\varphi}_t) = (\omega_t + \omega_{n+1} t^{n+1}, \varphi_t + \varphi_{n+1} t^{n+1}).
\]
Then $(\tilde{\omega}_t, \tilde{\varphi}_t)$ satisfies (26)-(27) for $0 \leq i \leq n+1$, so $(\tilde{\omega}_t, \tilde{\varphi}_t)$ is a deformation of order $n+1$, which implies that $(\omega_t, \varphi_t)$ is extensible. □

**Corollary 4.15.** If $\mathcal{H}_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g}) = 0$, then every 2-cocycle in $C^2_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g})$ is the infinitesimal of some 1-parameter formal deformation of the LieDer pair $(\mathfrak{g}, \varphi)$.

### 5 Central Extensions of a LieDer Pair

In this section, we study central extensions of a LieDer pair and show that central extensions of a LieDer pair $(\mathfrak{g}, \varphi)$ are controlled by the second cohomology of $(\mathfrak{g}, \varphi)$ with the coefficient in the trivial representation.

**Definition 5.1.** Let $(\mathfrak{h}, \varphi_\mathfrak{h})$ be an abelian LieDer pair and $(\mathfrak{g}, \varphi_\mathfrak{g})$ a LieDer pair. An exact sequence of LieDer pair morphisms
\[
0 \longrightarrow \mathfrak{h} \overset{\iota}{\longrightarrow} \hat{\mathfrak{g}} \overset{p}{\longrightarrow} \mathfrak{g} \longrightarrow 0
\]
is called a central extension of $(\mathfrak{g}, \varphi_\mathfrak{g})$ by $(\mathfrak{h}, \varphi_\mathfrak{h})$, if $[h, g]_\mathfrak{g} = 0$, that is $[h, x]_\mathfrak{g} = 0$ for all $h \in \mathfrak{h}$ and $x \in \hat{\mathfrak{g}}$. Here we identify $\mathfrak{h}$ with the corresponding subalgebra of $\hat{\mathfrak{g}}$. Therefore, we have $\varphi_\mathfrak{g}|_\mathfrak{h} = \varphi_\mathfrak{h}$.

**Definition 5.2.** Let $(\mathfrak{g}_1, \varphi_{\mathfrak{g}_1})$ and $(\mathfrak{g}_2, \varphi_{\mathfrak{g}_2})$ be two central extensions of $(\mathfrak{g}, \varphi_\mathfrak{g})$ by $(\mathfrak{h}, \varphi_\mathfrak{h})$. They are said to be isomorphic if there exists a LieDer pair morphism $\zeta : (\mathfrak{g}_1, \varphi_{\mathfrak{g}_1}) \rightarrow (\mathfrak{g}_2, \varphi_{\mathfrak{g}_2})$ such that we have the following commutative diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & (\mathfrak{h}, \varphi_\mathfrak{h}) \\
\downarrow & & \downarrow \zeta \\
0 & \longrightarrow & (\mathfrak{g}_1, \varphi_{\mathfrak{g}_1}) \\
\downarrow & & \downarrow p_1 \\
0 & \longrightarrow & (\mathfrak{g}_2, \varphi_{\mathfrak{g}_2}) \\
\downarrow & & \downarrow p_2 \\
& & \longrightarrow 0
\end{array}
\]

A section of a central extension $(\hat{\mathfrak{g}}, \varphi_\mathfrak{g})$ of $(\mathfrak{g}, \varphi_\mathfrak{g})$ by $(\mathfrak{h}, \varphi_\mathfrak{h})$ is a linear map $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ such that $p \circ s = \text{Id}$.

Let $(\hat{\mathfrak{g}}, \varphi_\mathfrak{g})$ be a central extension of a LieDer pair $(\mathfrak{g}, \varphi_\mathfrak{g})$ by an abelian LieDer pair $(\mathfrak{h}, \varphi_\mathfrak{h})$ and $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ a section. Define linear maps $\psi : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{h}$ and $\chi : \mathfrak{g} \rightarrow \mathfrak{h}$ respectively by
\[
\psi(x, y) = [s(x), s(y)]_\mathfrak{g} - s(x, y)_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g},
\]
\[
\chi(x) = \varphi_\mathfrak{g}(s(x)) - s(\varphi_\mathfrak{g}(x)), \quad \forall x \in \mathfrak{g}.
\]

Observe that $\hat{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Transfer the LieDer pair structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus \mathfrak{h}$, we obtain a LieDer pair $(\mathfrak{g} \oplus \mathfrak{h}, \varphi_\mathfrak{g})$, where the Lie bracket $[\cdot, \cdot]$ is given by
\[
[x + h, y + l]_\mathfrak{g} = [x, y]_\mathfrak{g} + \psi(x, y), \quad \forall x, y \in \mathfrak{g}, h, l \in \mathfrak{h},
\]
\[
\varphi_\mathfrak{g}(x + h) = \varphi_\mathfrak{g}(x) + \chi(x) + \varphi_\mathfrak{h}(h), \quad \forall x \in \mathfrak{g}, h \in \mathfrak{h}.
\]
The following proposition gives the conditions on $\psi$ and $\chi$ such that $(\mathfrak{g} \oplus \mathfrak{h}, \varphi_\chi) = \text{LieDer}$ pair.

**Proposition 5.3.** With the above notations, $(\mathfrak{g} \oplus \mathfrak{h}, \varphi_\chi) = \text{LieDer}$ pair if and only if $(\psi, \chi)$ is a 2-cocycle of the LieDer pair $(\mathfrak{g}, \varphi_\psi)$ with the coefficient in the trivial representation $(\rho = 0, \mathfrak{h}, \varphi_\chi)$, i.e. $(\psi, \chi)$ satisfy the following equalities:

\[
\begin{align*}
\psi([x, y]_\mathfrak{g}, z) + \psi([y, z]_\mathfrak{g}, x) + \psi([z, x]_\mathfrak{g}, y) &= 0, \\
\chi([x, y]_\mathfrak{g}) + \varphi_\psi(\psi(x, y)) - \psi(\varphi_\psi(x), y) - \psi(x, \varphi_\psi(y)) &= 0.
\end{align*}
\]

(39) (40)

**Proof.** If $(\mathfrak{g} \oplus \mathfrak{h}, \varphi_\chi) = \text{LieDer}$ pair, by

\[
[[x + h, y + l]_\psi, z + t]_\psi + c.p. = 0,
\]

we deduce that (39) holds. By

\[
\varphi_\chi([x + h, y + l]_\psi) = [\varphi_\chi(x + h), y + l]_\psi + [x + h, \varphi_\chi(y + l)]_\psi,
\]

we deduce that (40) holds.

Conversely, if (39) and (40) hold, it is straightforward to see that $(\mathfrak{g} \oplus \mathfrak{h}, \varphi_\chi) = \text{LieDer}$ pair. The proof is finished. $\blacksquare$

**Theorem 5.4.** Let $(\mathfrak{h}, \varphi_\mathfrak{h})$ be an abelian LieDer pair and $(\mathfrak{g}, \varphi_\psi)$ a LieDer pair. Then central extensions of $(\mathfrak{g}, \varphi_\psi)$ by $(\mathfrak{h}, \varphi_\mathfrak{h})$ are classified by the second cohomology group $H^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{h})$ of the LieDer pair $(\mathfrak{g}, \varphi_\psi)$ with the coefficient in the trivial representation $(\rho = 0, \mathfrak{h}, \varphi_\mathfrak{h})$.

**Proof.** Let $(\hat{\mathfrak{g}}, \varphi_{\hat{\psi}})$ be a central extension of $(\mathfrak{g}, \varphi_\psi)$ by $(\mathfrak{h}, \varphi_\mathfrak{h})$. By choosing a section $s : \mathfrak{g} \to \hat{\mathfrak{g}}$, we obtain a 2-cocycle $(\psi, \chi)$. Now we show that the cohomological class of $(\psi, \chi)$ does not depend on the choice of sections. In fact, let $s_1$ and $s_2$ be two different sections. Define $\phi : \mathfrak{g} \to \mathfrak{h}$ by $\phi(x) = s_1(x) - s_2(x)$. Then we have

\[
\begin{align*}
\psi_1(x, y) &= [s_1(x), s_1(y)]_{\hat{\mathfrak{g}}} - s_1[x, y]_{\mathfrak{g}} \\
&= [s_2(x) + \phi(x), s_2(y) + \phi(y)]_{\hat{\mathfrak{g}}} - s_2[x, y]_{\mathfrak{g}} - \phi([x, y]_{\mathfrak{g}}) \\
&= [s_2(x), s_2(y)]_{\mathfrak{g}} - s_2[x, y]_{\mathfrak{g}} - \phi([x, y]_{\mathfrak{g}}) \\
&= \psi_2(x, y) - \phi([x, y]_{\mathfrak{g}}),
\end{align*}
\]

and

\[
\begin{align*}
\chi_1(x) &= \varphi_{\hat{\psi}}(s_1(x) - s_1(\varphi_\psi(x)) \\
&= \varphi_{\hat{\psi}}(s_2(x) + \phi(x)) - \varphi_{\hat{\psi}}(s_2 + \phi)(\varphi_\psi(x)) \\
&= \varphi_{\hat{\psi}}(s_2(x)) - \varphi_{\hat{\psi}}(\varphi_\psi(x)) + \varphi_{\hat{\psi}}(\phi(x)) - \phi(\varphi_\psi(x)) \\
&= \chi_2(x) + \phi_\psi(\phi(x)) - \phi(\varphi_\psi(x)).
\end{align*}
\]

Thus, we obtain $(\psi_1, \chi_1) = (\psi_2, \chi_2) + \partial(\phi)$. Therefore, $(\psi_1, \chi_1)$ and $(\psi_2, \chi_2)$ are in the same cohomological class.

Now we go on to prove that isomorphic central extensions give rise to the same element in $H^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{h})$. Assume that $(\hat{\mathfrak{g}}_1, \varphi_{\hat{\psi}_1})$ and $(\hat{\mathfrak{g}}_2, \varphi_{\hat{\psi}_2})$ are two isomorphic central extensions of $(\mathfrak{g}, \varphi_\psi)$ by $(\mathfrak{h}, \varphi_\mathfrak{h})$, and $\zeta : (\hat{\mathfrak{g}}_1, \varphi_{\hat{\psi}_1}) \to (\hat{\mathfrak{g}}_2, \varphi_{\hat{\psi}_2})$ is a LieDer pair morphism such that we have the commutative diagram in Definition 5.2. Assume that $s_1 : \mathfrak{g} \to \hat{\mathfrak{g}}_1$ is a section of $\hat{\mathfrak{g}}_1$. By $p_2 \circ \zeta = p_1$, we have

\[
p_2 \circ (\zeta \circ s_1) = p_1 \circ s_1 = \text{Id}.
\]
Thus, we obtain that \( \zeta \circ s_1 \) is a section of \( \hat{\mathfrak{g}}_2 \). Define \( s_2 = \zeta \circ s_1 \). Since \( \zeta \) is a morphism of \( \text{LieDer} \) pair and \( \zeta |_h = 1d \), we have

\[
\psi_2(x, y) = [s_2(x), s_2(y)]_{\hat{\mathfrak{g}}_2} - s_2[x, y]_\mathfrak{g} = [(\zeta \circ s_1)(x), (\zeta \circ s_1)(y)]_{\hat{\mathfrak{g}}_2} - (\zeta \circ s_1)[x, y]_\mathfrak{g} \\
= \zeta([s_1(x), s_1(y)]_{\hat{\mathfrak{g}}_1} - s_1[x, y]_\mathfrak{g}) \\
= [s_1(x), s_1(y)]_{\hat{\mathfrak{g}}_1} - s_1[x, y]_\mathfrak{g} \\
= \psi_1(x, y),
\]

and

\[
\chi_2(x) = \varphi_{\hat{\mathfrak{g}}_2}(s_2x) - s_2(\varphi_\mathfrak{g}(x)) = \varphi_{\hat{\mathfrak{g}}_2}((\zeta \circ s_1)(x)) - (\zeta \circ s_1)(\varphi_\mathfrak{g}(x)) \\
= \zeta(\varphi_{\hat{\mathfrak{g}}_1}(s_1x) - s_1(\varphi_\mathfrak{g}(x))) \\
= \varphi_{\hat{\mathfrak{g}}_1}(s_1x) - s_1(\varphi_\mathfrak{g}(x)) \\
= \chi_1(x).
\]

Thus, the isomorphic central extensions give rise to the same element in \( H^2_{\text{LieDer}}(\mathfrak{g}; \mathfrak{h}) \).

Conversely, given two 2-cocycles \((\psi_1, \chi_1)\) and \((\psi_2, \chi_2)\), we can construct two central extensions \((\mathfrak{g} \oplus \mathfrak{h}, \varphi_{\chi_1})\) and \((\mathfrak{g} \oplus \mathfrak{h,} \varphi_{\chi_2})\), as in \((37) \) and \((38) \). If they represent the same cohomological class, i.e. there exists \( \phi : \mathfrak{g} \to \mathfrak{h} \), such that \((\psi_1, \chi_1) = (\psi_2, \chi_2) + \partial(\phi)\), we define \( \zeta : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{h} \) by

\[
\zeta(x + h) = x + \phi(x) + h.
\]

Then we can deduce that \( \zeta \) is an isomorphism between central extensions. We omit details. This finishes the proof.■

6 Extensions of a pair of derivations

In \([7]\), the authors study extensions of a pair of automorphisms of Lie algebras. Since derivations are infinitesimals of automorphisms, we are interested in extensions of a pair of derivations. In this section, associated to a central extension \( \hat{\mathfrak{g}} \) of a Lie algebra \( \mathfrak{g} \) by an abelian Lie algebra \( \mathfrak{h} \) and a pair of derivations \((\varphi_\mathfrak{h}, \varphi_\mathfrak{g})\) \( \in \text{Der}(\mathfrak{h}) \times \text{Der}(\mathfrak{g}) \), we define a cohomology class \([\text{Ob}_{(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})}^h] \in H^2(\mathfrak{g}; \mathfrak{h})\). We show that \((\varphi_\mathfrak{h}, \varphi_\mathfrak{g})\) is extensible if and only if the cohomology class \([\text{Ob}_{(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})}^h]\) is trivial. Thus we call \([\text{Ob}_{(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})}^h]\) the obstruction class of \((\varphi_\mathfrak{h}, \varphi_\mathfrak{g})\) being extensible.

**Definition 6.1.** Let \( 0 \to \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0 \) be a central extension of Lie algebras. A pair of derivations \((\varphi_\mathfrak{h}, \varphi_\mathfrak{g})\) \( \in \text{Der}(\mathfrak{h}) \times \text{Der}(\mathfrak{g}) \) is said to be extensible if there exists a derivation \( \varphi_\mathfrak{g} \in \text{Der}(\hat{\mathfrak{g}}) \) such that we have the following exact sequence of \( \text{LieDer} \) pair morphisms

\[
0 \to \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0
\]

Equivalentely, \((\hat{\mathfrak{g}}, \varphi_\mathfrak{g})\) is a central extension of \((\mathfrak{g}, \varphi_\mathfrak{g})\) by \((\mathfrak{h}, \varphi_\mathfrak{h})\).

Let \( s : \mathfrak{g} \to \hat{\mathfrak{g}} \) be an arbitrary section of the central extension \( 0 \to \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0 \). Then any element of \( \hat{\mathfrak{g}} \) can be written uniquely as \( s(x) + h \) for some \( x \in \mathfrak{g} \) and \( h \in \mathfrak{h} \). Define \( \psi : \lambda^2 \hat{\mathfrak{g}} \to \mathfrak{h} \) by

\[
\psi(x, y) = [s(x), s(y)]_{\hat{\mathfrak{g}}} - s[x, y]_\mathfrak{g}.
\]
Thus, we have
\[ \rho \] which implies that
\[ \text{Ob} \]
Then we have
\[ \text{representation} \]
\[ \text{Proof.} \]
\[ \text{Proof.} \]
\[ \psi : \land^2 g \to h \] defined by (11) is a 2-cocycle of the Lie algebra \( g \) with the coefficient in the trivial representation \( (\rho = 0, h) \). Moreover, the cohomology class \[ \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)} \in H^2(\mathfrak{g}; h) \] does not depend on the choice of sections.

The cohomology class \[ \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)} \in H^2(\mathfrak{g}; h) \] is called the obstruction class of \( (\varphi_\delta, \varphi_\rho) \) being extensible.

\[ \psi([x, y]_g, z) + \psi([y, z]_g, x) + \psi([z, x]_g, y) = 0. \]

Then we have
\[ (d\text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)})(x, y, z) = -\text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}([x, y]_g, z) - \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}([y, z]_g, x) - \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}([z, x]_g, y) \]
\[ = -\varphi_\delta(\psi([x, y]_g, z)) + \psi(\varphi_\delta([x, y]_g, z)) + \psi([x, y]_g, \varphi_\delta(z)) \]
\[ = -\varphi_\delta(\psi([x, y]_g, z)) + \psi(\varphi_\delta([x, y]_g, z)) + \psi([x, y]_g, \varphi_\delta(z)) \]
\[ = 0, \]
which implies that \[ \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)} \] is a 2-cocycle of the Lie algebra \( g \) with the coefficient in the trivial representation \( (\rho = 0, h) \).

Let \( s_1 \) and \( s_2 \) be two different sections. Define \( \phi : g \to h \) by \( \phi(x) = s_1(x) - s_2(x) \). Then we have
\[ \psi_1(x, y) = \psi_2(x, y) + \phi([x, y]_g). \]

Moreover, we have
\[ \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)} = \varphi_\delta(\psi_1(x, y)) - \psi_1(\varphi_\delta(x), y) - \psi_1(x, \varphi_\delta(y)) \]
\[ = \varphi_\delta(\psi_2(x, y)) - \varphi_\delta(\phi([x, y]_g)) - \psi_2(\varphi_\delta(x), y) + \phi([\varphi_\delta(x), y]_g) \]
\[ = \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}^\delta \]
\[ = \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}^\delta + \text{d}(\varphi_\delta \circ \phi - \phi \circ \varphi_\delta)([x, y]_g) \]
Thus, we have \[ \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}^1 = \text{Ob}^\delta_{(\varphi_\delta, \varphi_\rho)}^2 \in H^2(\mathfrak{g}; h) \]. The proof is finished. 

Now we are ready to give the main result in this section.
Theorem 6.3. Let $0 \to \frakh \xrightarrow{\iota} \hat{\frakh} \xrightarrow{\rho} \frakg \to 0$ be a central extension of Lie algebras. Then a pair $(\varphi_{\frakh}, \varphi_{\frakg}) \in \text{Der}(\frakh) \times \text{Der}(\frakg)$ is extensible if and only if the obstruction class $[\text{Ob}^\frakh_{(\varphi_{\frakh}, \varphi_{\frakg})}] \in H^2(\frakg; \frakh)$ is trivial.

**Proof.** Let $s : \frakg \to \hat{\frakg}$ be a section of the central extension of Lie algebras $0 \to \frakh \xrightarrow{\iota} \hat{\frakg} \xrightarrow{\rho} \frakg \to 0$. Suppose that $(\varphi_{\frakh}, \varphi_{\frakg})$ is extensible. Then there exists a derivation $\hat{\varphi} \in \text{Der}(\hat{\frakg})$ such that we have the exact sequence of $\text{LieDer}$ pair morphisms in Definition 6.1. By $\varphi_{\frakh} \circ p = p \circ \hat{\varphi}$, we obtain $\varphi_{\frakh}(s(x)) = \varphi_{\frakh}(s(x)) \in \frakh$. Define $\lambda : \frakg \to \frakh$ by

$$\lambda(x) = \varphi_{\frakh}(s(x)) - s(\varphi_{\frakg}(x)).$$

Then we have

$$\varphi_{\frakh}(s(x) + h) = \varphi_{\frakh}(s(x)) + \varphi_{\frakh}(h) = \varphi_{\frakh}(s(x)) - s(\varphi_{\frakh}(x)) + s(\varphi_{\frakh}(x)) + \varphi_{\frakh}(h) = s(\varphi_{\frakh}(x)) + \lambda(x) + \varphi_{\frakh}(h).$$

Let $s(x) + h$ and $s(y) + l$ be any two elements of $\hat{\frakg}$. Since $\varphi_{\frakh}$ is a derivation of $\hat{\frakg}$, on one hand, we have

$$\varphi_{\frakh}([s(x) + h, s(y) + l])_\hat{\frakg} = \varphi_{\frakh}([s(x), s(y)]_\frakg) = \varphi_{\frakh}(s[x, y]_\frakg + [s(x), s(y)]_\frakg - s[x, y]_\frakg) = s(\varphi_{\frakh}(x, y) + \psi(x, y)) = s(\varphi_{\frakh}(x, y) + \lambda(x, y) + \varphi_{\frakh}(\psi(x, y))).$$

On the other hand, we have

$$\begin{align*}
[\varphi_{\frakh}(s(x) + h), s(y) + l]_\hat{\frakg} + [s(x) + h, \varphi_{\frakh}(s(y) + l)]_\hat{\frakg} &= [s(\varphi_{\frakh}(x)) + \lambda(x) + \varphi_{\frakh}(h), s(y) + l]_\frakg + [s(x) + h, s(\varphi_{\frakh}(y)) + \lambda(y) + \varphi_{\frakh}(l)]_\frakg \\
&= [s(\varphi_{\frakh}(x)), s(y)]_\frakg + [s(x), s(\varphi_{\frakh}(y))]_\frakg + s(\varphi_{\frakh}(x), y)_\frakg + [s(x, \varphi_{\frakh}(y))]_\frakg - s[x, \varphi_{\frakh}(y)]_\frakg \\
&= s(\varphi_{\frakh}(x), y)_\frakg + \psi(\varphi_{\frakh}(x), y) + s[x, \varphi_{\frakh}(y)]_\frakg + \psi(x, \varphi_{\frakh}(y)).
\end{align*}$$

Thus, we have

$$\varphi_{\frakh}(\psi(x, y)) = -\psi(\varphi_{\frakh}(x), y) = -\lambda(x, y)_\frakg,$$

which implies that

$$\text{Ob}^\frakh_{(\varphi_{\frakh}, \varphi_{\frakh})} = d\lambda.$$ 

Therefore, the obstruction class is trivial.

Conversely, if the obstruction class is trivial, then there exists a $\lambda : \frakg \to \frakh$ such that $\text{Ob}^\frakh_{(\varphi_{\frakh}, \varphi_{\frakh})} = d\lambda$. For any element $s(x) + h \in \hat{\frakg}$, define $\varphi_{\frakh}$ by

$$\varphi_{\frakh}(s(x) + h) = s(\varphi_{\frakh}(x)) + \lambda(x) + \varphi_{\frakh}(h).$$

By (43), we obtain the exact sequence of $\text{LieDer}$ pair morphisms in Definition 6.1. Thus, $(\varphi_{\frakh}, \varphi_{\frakh})$ is extensible. The proof is finished. $lacksquare$
Corollary 6.4. Let $0 \to \mathfrak{h} \overset{\iota}{\to} \mathfrak{g} \overset{\rho}{\to} \mathfrak{g} \to 0$ be a central extension of Lie algebras. If $H^2(\mathfrak{g}; \mathfrak{h}) = 0$, then any pair $(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})$ in $\text{Der}(\mathfrak{h}) \times \text{Der}(\mathfrak{g})$ is extensible.

At the end of this section, we give the condition on a pair of derivations $(\varphi_\mathfrak{h}, \varphi_\mathfrak{g}) \in \text{Der}(\mathfrak{h}) \times \text{Der}(\mathfrak{g})$ such that it is extensible in every central extension of $\mathfrak{g}$ by $\mathfrak{h}$. By Proposition [6.2], we can define a linear map $\Theta: \text{Der}(\mathfrak{h}) \times \text{Der}(\mathfrak{g}) \to \mathfrak{gl}(H^2(\mathfrak{g}; \mathfrak{h}))$ by

$$\Theta(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})([\psi]) = [\varphi_\mathfrak{h} \circ \psi - \psi(\varphi_\mathfrak{g} \odot \text{Id}) - \psi(\text{Id} \odot \varphi_\mathfrak{g})].$$

Theorem 6.5. Let $\mathfrak{h}$ be an abelian Lie algebra and $\mathfrak{g}$ a Lie algebra. A pair of derivations $(\varphi_\mathfrak{h}, \varphi_\mathfrak{g}) \in \text{Der}(\mathfrak{h}) \times \text{Der}(\mathfrak{g})$ is extensible in every central extension of $\mathfrak{g}$ by $\mathfrak{h}$ if and only if $\Theta(\varphi_\mathfrak{h}, \varphi_\mathfrak{g}) = 0$.

Proof. We suppose $\Theta(\varphi_\mathfrak{h}, \varphi_\mathfrak{g}) = 0$. For any central extension $0 \to \mathfrak{h} \overset{\iota}{\to} \mathfrak{g} \overset{\rho}{\to} \mathfrak{g} \to 0$, we choose a section $s: \mathfrak{g} \to \mathfrak{g}$. Then $\psi : \wedge^2 \mathfrak{g} \to \mathfrak{h}$ defined by $\psi(x, y) = [s(x), s(y)] - s[x, y]$ is a 2-cocycle. Moreover, we obtain that

$$[\text{Ob}_{\mathfrak{g}, \mathfrak{h}}^\mathfrak{h}(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})] = [\varphi_\mathfrak{h} \circ \psi - \psi(\varphi_\mathfrak{g} \odot \text{Id}) - \psi(\text{Id} \odot \varphi_\mathfrak{g})] = \Theta(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})([\psi]) = 0.$$

By Theorem [6.3], $(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})$ is extensible in the above central extension.

Conversely, for any element $[\psi] \in H^2(\mathfrak{g}; \mathfrak{h})$, there exists a central extension $0 \to \mathfrak{h} \overset{\iota}{\to} \mathfrak{g} \oplus \mathfrak{h} \overset{\rho}{\to} \mathfrak{g} \to 0$, where the bracket on $\mathfrak{g} \oplus \mathfrak{h}$ is defined by

$$[x + g, y + h] = [x, y]_\mathfrak{g} + \psi(x, y).$$

Since $(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})$ is extensible in every central extension of $\mathfrak{g}$ by $\mathfrak{h}$, by Theorem [6.3] we have

$$\Theta(\varphi_\mathfrak{h}, \varphi_\mathfrak{g})([\psi]) = [\varphi_\mathfrak{h} \circ \psi - \psi(\varphi_\mathfrak{g} \odot \text{Id}) - \psi(\text{Id} \odot \varphi_\mathfrak{g})] = 0.$$

Therefore, we have $\Theta(\varphi_\mathfrak{h}, \varphi_\mathfrak{g}) = 0$. The proof is finished. \qed

7 Classification of skeletal Lie2Der pairs

In this section, we call a Lie 2-algebra with a derivation of degree 0 a Lie2Der pair and show that the third cohomology group $H^3_{\text{Lie Der}}(\mathfrak{g}; V)$ classifies skeletal Lie2Der pairs. See [6] for more details about Lie 2-algebras.

Definition 7.1. A Lie 2-algebra $\mathcal{V}$ consists of the following data:

- a complex of vector spaces $V_i \overset{l_i}{\longrightarrow} V_{i+j}$,
- bilinear maps $l_2 : V_i \times V_j \longrightarrow V_{i+j}$, where $0 \leq i + j \leq 1$,
- a skew-symmetric trilinear map $l_3 : V_0 \times V_0 \times V_0 \longrightarrow V_1$, such that for all $w, x, y, z \in V_0$ and $m, n \in V_1$, the following equalities are satisfied:
  (a) $l_2(x, y) = -l_2(y, x)$, $l_2(x, m) = -l_2(m, x)$,
  (b) $l_1l_2(x, m) = l_2(x, l_1m)$, $l_2(l_1m, n) = l_2(m, l_1n)$,
  (c) $l_1l_3(x, y, z) = l_2(x, l_2(y, z)) + l_2(y, l_2(z, x)) + l_2(z, l_2(x, y))$. 

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the following equalities hold:

\[ l_3(x, y, l_1 m) = l_2(x, l_2(y, m)) + l_2(y, l_2(m, x)) + l_2(m, l_2(x, y)), \]

\[ l_3(l_2(w, x), y, z) + l_2(l_2(w, x, z), y) + l_3(w, l_2(x, z), y) \]
\[ + l_3(l_2(w, x), y) = l_2(l_2(w, x, y), z) + l_3(l_2(w, y), x, z) \]
\[ + l_3(w, l_2(x, y), z) + l_2(l_3(w, y, z), x) + l_3(w, l_2(y, z), x). \]

We usually denote a Lie 2-algebra by \((V_1, V_0, l_1, l_2, l_3)\) or simply by \(V\). A Lie 2-algebra is called **skeletal** if \(l_1 = 0\). There is a one-to-one correspondence between skeletal Lie 2-algebras and triples \((g, (\rho, V), l_3)\), where \(g\) is a Lie algebra, \((\rho, V)\) is a representation of \(g\), and \(l_3\) is a 3-cocycle on \(g\) with the coefficient in \(V\). More precisely, for a skeletal Lie 2-algebra \(V = (V_1, V_0, l_1 = 0, l_2, l_3)\), it is straightforward to see that \((V_0, l_2)\) is a Lie algebra. Define \(\rho\) from \(V_0\) to \(\mathfrak{gl}(V_1)\) by

\[ \rho(x)(u) = l_2(x, u). \]

Then, \((\rho, V_1)\) is a representation of the Lie algebra \((V_0, l_2)\) and \(l_3\) is a 3-cocycle on \(V_0\) with the coefficient in \(V_1\).

**Definition 7.2.** Let \(V\) and \(V'\) be Lie 2-algebras. A **morphism** \(f\) from \(V\) to \(V'\) consists of:

- a chain map \(f : V \rightarrow V'\), which consists of linear maps \(f_0 : V_0 \rightarrow V'_0\) and \(f_1 : V_1 \rightarrow V'_1\) satisfying \(f_0 \circ l_1 = l'_1 \circ f_1\),

- a skew-symmetric bilinear map \(f_2 : V_0 \times V_0 \rightarrow V'_1\) such that for all \(x, y, z \in V_0\) and \(m \in V_1\), the following equalities hold:

  \[ l'_2(f_2(x, y)) = l'_2(f_0(x), f_0(y)), \]

  \[ f_2(x, l_1 m) = f_1(l_2(x, m)) - l'_2(f_0(x), f_1(m)), \]

  \[ l'_2(f_0(x, f_2(y, z)) + l'_2(f_0(y, f_2(z, x)) + l'_2(f_0(z), f_2(x, y)) + l'_2(f_0(x), f_0(y)), f_0(z)) \]

- \( = f_2(l_2(x, y), z) + f_2(l_2(y, z), x) + f_2(l_2(z, x), y) + f_1(l_3(x, y, z)). \)

A morphism is called an isomorphism if \(f_0\) and \(f_1\) are invertible.

**Definition 7.3.** A **derivation** of degree 0 of a Lie 2-algebra \(V\) is a triple \((X_0, X_1, l_X)\), in which \(X = (X_0, X_1) \in \text{End}(V_0) \oplus \text{End}(V_1)\) and \(l_X : V_0 \wedge V_0 \rightarrow V_1\) is a linear map, such that for all \(x, y, z \in V_0, m \in V_1\), the following equalities hold:

- \(X_0 \circ l_1 = l_1 \circ X_1,\)
- \(l_1 l_X(x, y) = X_0(l_2(x, y)) - l_2(X_0x, y) - l_2(x, X_0y),\)
- \(l_X(x, l_1 m) = X_1(l_2(x, m)) - l_2(X_0x, m) - l_2(x, X_1m),\)
- \(l_X(x, l_2(y, z)) = l_2(x, l_2(y, z)) + l_2(x, l_X(y, z)) + l_3(X_0x, y, z) + c.p.(x, y, z).\)

See [9] [21] for more details about derivations of Lie 2-algebras and \(L_\infty\)-algebras. We denote a Lie 2-algebra with a derivation of degree 0 by \((V; (X_0, X_1, l_X))\) and call it a **Lie2Der** pair. In particular, a skeletal Lie 2-algebra with a derivation of degree 0 will be called a skeletal **Lie2Der** pair.
**Definition 7.4.** Let \((V; (X_0, X_1, l_X))\) and \((V'; (X'_0, X'_1, l'_X))\) be Lie2Der pairs. An isomorphism \(\tau\) from \((V; (X_0, X_1, l_X))\) to \((V'; (X'_0, X'_1, l'_X))\) consists of linear maps \(f_0 : V_0 \to V'_0, f_1 : V_1 \to V'_1, f_2 : V_0 \land V_0 \to V'_1\) and \(B : V_0 \to V'_1\) such that \((f_0, f_1, f_2)\) is a Lie 2-algebra isomorphism from \(V\) to \(V'\) and the following equalities hold for all \(x, y \in V_0, m \in V_1:\)

(a) \(X'_0(f_0(x)) - f_0(X_0(x)) = l'_1(B(x)),\)

(b) \(X'_1(f_1(m)) - f_1(X_1(m)) = B(l_1(m)),\)

(c) \(f_2(x,y) + f_2(x, y) - X'_1(f_2(x, y)) - l'_X(f_0(x), f_0(y)) = l'_2(B(x), f_0(y)) + l'_2(f_0(x), B(y)) - B(l_2(x, y)).\)

Let \(V = (V_0, V_1, l_0, l_2, l_3)\) be a skeletal Lie 2-algebra and \((X_0, X_1, l_X)\) be a derivation of degree 0 of \(V\). By condition (b) in Definition 7.4, we deduce that \(X_0\) is a derivation of the Lie algebra \((V_0, l_2).\)

Condition (c) in Definition 7.4 implies that \((\rho, V_1, X_1)\) is a representation of the LieDer pair \((V_0, X_0)\). Condition (d) in Definition 7.4 implies that \((l_3, -l_X) \in \mathbb{Z}^3_{\text{LieDer}}(V_0; V_1).\)

Conversely, if \((\rho, V, \varphi_V)\) is a representation of the LieDer pair \((g, \varphi_g)\) and \((\theta_3, \theta_2) \in \mathbb{Z}^3_{\text{LieDer}}(g; V),\) then we can deduce that \((\varphi_g, \varphi_V, -\theta_2)\) is a derivation of the skeletal Lie 2-algebra \((V \overset{\rho}{\to} g, l_2, l_3 = \theta_2),\) where \(l_2\) is defined by

\[
l_2(x, y) = [x, y]_g, \quad l_2(x, u) = -l_2(u, x) = \rho(x)(u), \quad \forall x, y, u \in g.\]

Summarize the above discussion, we have

**Proposition 7.5.** There is a one to one correspondence between skeletal Lie2Der pairs and triples \(((g, \varphi_g), (\rho, V, \varphi_V), (\theta_3, \theta_2))\), where \((g, \varphi_g)\) is a LieDer pair, \((\rho, V, \varphi_V)\) is a representation of \((g, \varphi_g),\) and \((\theta_3, \theta_2)\) is a 3-cocycle on \((g, \varphi_g)\) with the coefficient in \((\rho, V, \varphi_V).\)

In the sequel, we give the equivalence relation between triples \(((g, \varphi_g), (\rho, V, \varphi_V), (\theta_3, \theta_2))\) and show that there is a one-to-one correspondence between equivalence classes of such triples and isomorphism classes of skeletal Lie2Der pairs.

**Definition 7.6.** Let \(((g, \varphi_g), (\rho, V, \varphi_V), (\theta_3, \theta_2))\) and \(((g', \varphi_{g'}), (\rho', V', \varphi_{V'}), (\theta'_3, \theta'_2))\) be triples as described in Proposition 7.5. They are said to be equivalent if there exist Lie algebra isomorphism \(\alpha : g \to g',\) linear isomorphism \(\beta : V \to V'\) and two linear maps \(\gamma : g \land g \to V', \eta : g \to V'\) such that the following equalities hold for all \(x, y, z \in g, u \in V:\)

(a) \(\varphi_{g'}(\alpha(x)) = \alpha(\varphi_g(x)),\)

(b) \(\varphi_{V'}(\beta(u)) = \beta(\varphi_V(u)),\)

(c) \(\beta(\rho(\alpha(x))(u)) = \rho'(\alpha(x))(\beta(\alpha(x))),\)

(d) \(\rho'(\alpha(x))(\gamma(y, z)) + \rho'(\alpha(y))(\gamma(x, z)) + \rho'(\alpha(z))(\gamma(x, y)) + \theta'_2(\alpha(x), \alpha(y), \alpha(z))\]
\[
= \gamma([x, y]_g, z) + \gamma([y, z]_g, x) + \gamma([z, x]_g, y) + \beta(\theta_3(x, y, z)),\]

(e) \(-\beta(\theta_2(x, y)) + \gamma(x, \varphi_g(y)) + \gamma(x, \varphi_V(y)) - \varphi_{V'}(\gamma(x, y)) + \theta'_2(\alpha(x), \alpha(y))\]
\[
= -\rho'(\alpha(y))(\eta(x)) + \rho'(\alpha(x))(\eta(y)) - \eta([x, y]_g).\]

**Theorem 7.7.** There is a one-to-one correspondence between isomorphism classes of skeletal Lie2Der pairs and equivalence classes of triples \(((g, \varphi_g), (\rho, V, \varphi_V), (\theta_3, \theta_2))\), where \((g, \varphi_g)\) is a LieDer pair, \((\rho, V, \varphi_V)\) is a representation of \((g, \varphi_g),\) and \((\theta_3, \theta_2)\) is a 3-cocycle on \((g, \varphi_g)\) with the coefficient in \((\rho, V, \varphi_V).\)
Proof. Let \( ((g, \varphi_g), (\rho, V, \varphi_V, (\theta_3, \theta_2)) \) and \( ((g', \varphi_{g'}), (\rho', V', \varphi_{V'}, (\theta'_3, \theta'_2)) \) be equivalent triples. By Proposition 7.5, we have two skeletal Lie 2-algebras, given by

\[ V = \left( V \overset{\alpha}{\to} g, l_2, l_3 = \theta_3 \right), \quad V' = \left( V' \overset{\alpha}{\to} g', l'_2, l'_3 = \theta'_3 \right). \]

Moreover, \( (X_0 = \varphi_g, X_1 = \varphi_V, l_X = -\theta_2) \) and \( (X'_0 = \varphi_{g'}, X'_1 = \varphi_{V'}, l'_{X} = -\theta'_2) \) are degree 0 derivations of \( V \) and \( V' \) respectively.

We define \( f = (f_0 = \alpha, f_1 = \beta, f_2 = \gamma) \), and \( B = \eta \). By condition (c) and condition (d) in Definition 7.6 and the fact that \( \alpha \) is a Lie algebra isomorphism, \( f \) is a Lie 2-algebra isomorphism from \( V \) to \( V' \).

Moreover, by conditions (a), (b) and (e) in Definition 7.6, we deduce that \( (V; (X_0, X_1, l_X)) \) is isomorphic to \( (V'; (X'_0, X'_1, l'_{X})) \).

The converse part can be proved similarly and we omit details.

References

[1] V. Ayala, E. Kizil and I. de Azevedo Tribuzy, On an algorithm for finding derivations of Lie algebras. Proyecciones 31 (2012), 81-90.
[2] V. Ayala and J. Tirao, Linear control systems on Lie groups and controllability. In Differential geometry and control (Boulder, CO, 1997), volume 64 of Proc. Sympos. Pure Math. pages 47-64. Amer. Math. Soc. Providence, RI, 1999.
[3] M. Barr and J. Beck, Homology and standard constructions, 1969 Sem. on Triples and Categorical Homology Theory (ETH, Zurich, 1966/67) pp. 245-335 Springer, Berlin
[4] M. Barr, Cartan-Eilenberg cohomology and triples, J. Pure Appl. Algebra 112 (1996), no. 3, 219-238.
[5] I. A. Batalin and G. A. Vilkovisky, Gauge algebra and quantization. Phys. Lett. B 102 (1981), 27-31.
[6] J. C. Baez and A. S. Crans, Higher-dimensional algebra. VI. Lie 2-algebras, Theory Appl. Categ. 12 (2004), 492-538.
[7] V. G. Bardakov and M. Singh, Extensions and automorphisms of Lie algebras, J. Algebra Appl. 16 (2017), 15 pp.
[8] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras. Trans. Amer. Math. Soc. 63 (1948), 85-124.
[9] M. Doubek and T. Lada, Homotopy derivations, J. Homotopy Relat. Struct. 11 (2016), no. 3, 599-630.
[10] M. Doubek, M. Markl and P. Zima, Deformation theory (lecture notes), Arch. Math. (Brno) 43 (2007), no. 5, 333-371.
[11] Y. Frégier, M. Markl and D. Yau, The \( L_\infty \)-deformation complex of diagrams of algebras, New York J. Math. 15 (2009), 353-392.
[12] Y. Frégier, A new cohomology theory associated to deformations of Lie algebra morphisms, Lett. Math. Phys. 70 (2004), no. 2, 97-107.
[13] Y. Fréguier and M. Zambon, Simultaneous deformations and Poisson geometry, *Compos. Math.* 151 (2015), no. 9, 1763-1790.

[14] Y. Fréguier and M. Zambon, Simultaneous deformations of algebras and morphisms via derived brackets, *J. Pure Appl. Algebra* 219 (2015), no. 12, 5344-5362.

[15] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math. (2)* 78 (1963), 267-288.

[16] M. Gerstenhaber, On the deformation of rings and algebras, *Ann. of Math. (2)* 79 (1964), 59-103.

[17] M. Gerstenhaber and S. D. Schack, On the deformation of algebra morphisms and diagrams, *Trans. Amer. Math. Soc.* 279 (1983), no. 1, 1-50.

[18] V. Ginzburg and M. Kapranov, Koszul duality for operads. *Duke Math. J.* 76 (1994), 203-272.

[19] D. K. Harrison, Commutative algebras and cohomology, *Trans. Amer. Math. Soc.*, 104 (1962), 191-204.

[20] G. Hochschild, On the cohomology groups of an associative algebra. *Ann. of Math. (2)*, 46 (1945), 58-67.

[21] H. Lang, Z. Liu and Y. Sheng, Integration of derivations for Lie 2-algebras, *Transform. Groups* 21 (2016), no. 1, 129-152.

[22] L. J. Loday, On the operad of associative algebras with derivation, *Georgian Math. J.* 17 (2010), no. 2, 347-372.

[23] M. Markl, Intrinsic brackets and the $L_\infty$-deformation theory of bialgebras, *J. Homotopy Relat. Struct.* 5 (2010), no. 1, 177-212.

[24] A. Mandal, Deformation of Leibniz algebra morphisms, *Homology Homotopy Appl.* 9 (2007), no. 1, 439-450.

[25] A. Nijenhuis and R. Richardson, Cohomology and deformations in graded Lie algebras, *Bull. Amer. Math. Soc.* 72 (1966), 1-29.

[26] J. Stasheff, Homotopy associativity of $H$-spaces. I, II. *Trans. Amer. Math. Soc.* 108 (1963), 275-292.

[27] A. Yu. Vaintrob, Lie algebroids and homological vector fields, *Uspekhi Mat. Nauk*, 52 (1997), 161-162.

[28] Th. Th. Voronov, Higher derived brackets for arbitrary derivations, In *Travaux mathématiques. Fasc. XVI*, volume 16 of *Trav. Math.*, pages 163-186. Univ. Luxemb., Luxembourg, 2005.

[29] D. Yau, Deformations of coalgebra morphisms, *J. Algebra* 307 (2007), no. 1, 106-115.

Tang Rong and Yunhe Sheng
Department of Mathematics, Jilin University, Changchun 130012, China
Email: shengyh@jlu.edu.cn, tangrong16@mails.jlu.edu.cn
Yael Fréguier
LML, Artois University, Lens 62307, France
Email: yael.fregier@gmail.com