A short note on Casimir force and radius stabilization in QFT with non-commutative target space

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(Dated: November 12, 2020)

Stable radius of cylindrical space due to additional repulsion caused by noncommutativity of two-component field values is found.

PACS numbers: 11.10.-, 11.10.Kk.

I. INTRODUCTION

Since the famous paper of Snyder [1] the idea to build up quantum theory on noncommutative spaces has been widely studied by many authors. Quantum physics on noncommutative space became one of the main trends in modern physics as possible tool to study how space-time structure itself acts upon matter at very small scales and how space-time itself can be understood as quantum object, or quantized. Appearance of Casimir force in quantum fields under some nontrivial geometrical or topological situations has been studied widely in both commutative and non-commutative cases, as for example in [3]. Special attention is often paid to the effect of radius stabilization of suitable manifolds. In this short note we transfer the notion of non-commutativity into the abstract space of values of quantized field in very clear explained treatment of the work [4], where the reader can find references to other related papers.

II. THE COMMUTATIVE MODEL

Let us start with two (in fact, there is no way how to proceed with single-valued fields, on the other way, one may try to use the quaternions, see [2]) component real scalar \((\varphi^1, \varphi^2)\), \(\varphi^a = \varphi^a(x,t)\) defined over two dimensional cylinder, with spatial period equal to \(2\pi R\). This means we assume the periodic conditions: \(\varphi(x,t) = \varphi(x + 2\pi R,t), \varphi_x(x,t) = \varphi_x(x + 2\pi R,t)\) hold. Such fields can be decomposed into its Fourier spatial components as follows

\[
\varphi^a(x,t) = \sum_{n \in \mathbb{Z}} e^{\frac{i n x}{R}} \varphi^a_n(t), \quad \varphi^a_n = \varphi^a_{-n}, a = 1, 2,
\]

where the last condition guarantees reality of \(\varphi^a\). Let us consider \(\varphi^a\) be a pair of massless free scalars, this means the dynamics is given by the lagrangian

\[
L = \frac{1}{2} \int dx \left[ (\partial_t \varphi^1)^2 + (\partial_t \varphi^2)^2 - (\partial_x \varphi^1)^2 - (\partial_x \varphi^2)^2 \right] = \frac{R}{2} \sum_{a,n} \left[ \phi_n^a \dot{\phi}_n^a - \left( \frac{2\pi n}{R} \right)^2 \phi_n^a \phi_n^a \right]. \tag{1}
\]

Defining momenta:

\[
\pi_n^a = \frac{\partial L}{\partial \dot{\phi}_n^a} = R \phi_n^a \left( \pi_n^a \right)^* \]

one easily constructs the Hamiltonian of our system:

\[
H = \sum_a \frac{(\pi_n^a)^2}{2R} + \frac{1}{2R} \sum_{a,n \neq 0} \left[ \pi_n^a \pi_n^a - (2\pi n)^2 \phi_n^a \phi_n^a \right], \tag{2}
\]

where

\[
[\phi^a_m, \phi^b_n] = [\pi^a_m, \pi^b_n] = 0, \quad [\phi^a_m, \pi^b_n] = i \delta_{mn} \delta^{ab}.
\]

III. THE MODEL WITH NONCOMMUTATIVE PLANE AS TARGET SPACE

Motivated by the work [3] we shall consider the following modification of the canonical commutation relations

\[
\begin{align*}
\left[ \phi^a_m, \phi^b_n \right] = & \frac{i}{R} e^{ab} \theta(n) \delta_{m+n,0}, \\
\left[ \pi^a_m, \pi^b_n \right] = & 0, \\
\left[ \phi^a_m, \pi^b_n \right] = & i \delta^{ab} \delta_{mn},
\end{align*} \tag{3}
\]

where the smearing function \(\theta\) depends upon index of the mode:

\[
\theta(n) = \vartheta e^{-\frac{2\pi^2 n^2}{R^2}}. \tag{4}
\]

In this expression \(\vartheta\) is the parameter of non-commutativity of dimension \(L\). The commutation relations (3) (the first one considering coordinates) can be rewritten for spatially dependent field coordinates as the...
following equal-time commutation relation with Gaussian smearing on the right-hand side:

\[ [\hat{\varphi}^a(x, t), \hat{\varphi}^b(y, t)] = \frac{ie^{ab} \theta}{\sqrt{2\pi\sigma}} e^{-\frac{(x-y)^2}{2\sigma^2}}. \]

This reduces, in the limit \( \sigma \to 0 \), to the local commutation rule:

\[ [\hat{\varphi}^a(x, t), \hat{\varphi}^b(y, t)] = ie^{ab} \theta \delta(x - y), \]

by means of the standard gaussian approximation of Dirac mapping. We shall consider, accordingly with \([3]\), the system which dynamics is given by the following hamiltonian:

\[ H = \frac{1}{2gR} \sum_{a,n} \left[ \hat{\pi}^a_n \hat{\phi}^a_{n-} + (2\pi n)^2 \hat{\phi}^a_n \hat{\phi}^a_{-n} \right]. \quad (5) \]

Now, the idea is, that the commutation relations \([3]\) can be transformed to the canonical ones by the linear transformation \([3]\):

\[
\begin{align*}
\hat{\phi}^a_n &= \phi^a_n - \frac{1}{2R} e^{ab} \theta(n) \pi^b_{-n}, \\
\hat{\pi}^a_n &= \pi^a_n.
\end{align*}
\]

This transforms our hamiltonian \([5]\) into the form:

\[ H = \sum_{a,n} \left\{ 1 + \left( \frac{\pi n \theta(n)}{R} \right)^2 \pi^a_n \pi^a_{-n} \right\} + (2\pi n)^2 \hat{\phi}^a_n \hat{\phi}^a_{-n} \]

\[ - \sum_{a,b,n} \frac{2\pi n}{R} e^{ab} \theta(n) \phi^a_n \pi^b_{-n}. \quad (6) \]

The second term in this hamiltonian is proportional to the \((1, 2)\) component of the "angular momentum" operator corresponding to the rotations in field plane. This term can be interpreted as an effective interaction in field plane with auxiliary magnetic-like field. The hamiltonian can be simply diagonalised with help of properly chosen annihilation and creation operators. In order to do this, let us define some useful notations:

\[ \omega_n = \frac{2\pi |n|}{R}, \quad \Omega^2_n = 1 + \left( \frac{\pi n \theta(n)}{R} \right)^2. \]

We introduce the operators:

\[ a^a_n = \sqrt{\frac{\Delta_n}{2}} \left( \phi^a_n + i \frac{\pi^a_n}{\Delta_n} \right), \quad a^{a\dagger}_n = \sqrt{\frac{\Delta_n}{2}} \left( \phi^a_n - i \frac{\pi^a_n}{\Delta_n} \right), \]

where

\[ \Delta_n = \frac{R \omega_n}{\Omega^2_n} = \frac{2\pi |n|}{\Omega^2_n}. \]

This pair of operators obeys the canonical commutation relations:

\[ [a^a_n, a^b_m] = [a^{a\dagger}_n, a^{b\dagger}_m] = 0, \quad [a^a_n, a^{b\dagger}_m] = \delta^{ab} \delta_{mn}, \]

and our hamiltonian \([6]\) is given by

\[ H = \sum_a \left( \frac{\pi^a_0}{2R} \right)^2 + \sum_{a,n \neq 0} \omega_n \Omega^2_n a^a_n a^{a\dagger}_{-n} + \frac{i}{2} \sum_{a,b,n \neq 0} \theta(n) \omega_n e^{ab} a^a_n a^{b\dagger}_{-n}. \quad (7) \]

The following canonical transformation

\[ A^1_n = \frac{1}{\sqrt{2}} (a^a_n + ia^b_n), \quad A^2_n = \frac{1}{\sqrt{2}} (a^a_n - ia^b_n) \]

transforms our hamiltonian into the final diagonal form:

\[ H = \sum_n \omega_n \left[ \Lambda_n A^1_n A^{1\dagger}_n + \Lambda^2_n A^2_n A^{2\dagger}_n \right], \quad (8) \]

where we have introduced the notation:

\[ \Lambda^1_n = \Omega^2_n - \frac{\pi |n| \theta(n)}{R}, \quad \Lambda^2_n = \Omega^2_n + \frac{\pi |n| \theta(n)}{R}. \]

This result tells us that the magnetic like term causes splitting of the energy levels of the free-field hamiltonian.

### IV. CASIMIR ENERGY

We shall rewrite the hamiltonian \([8]\) into a more suitable form:

\[ H = \sum_{n \neq 0} \left\{ \omega_n \Omega^2_n \left[ \Lambda^1_n A^1_n A^{1\dagger}_n + \Lambda^2_n A^2_n A^{2\dagger}_n \right] + \frac{\pi |n| \theta(n)}{R} \left[ A^2_n A^{2\dagger}_n - A^1_n A^{1\dagger}_n \right] \right\}. \quad (9) \]

We are interested in vacuum expectation value of this operator. It is evident that the \(1 \leftrightarrow 2\)-antisymmetric term does not contributes to the vacuum expectation value. We can write

\[ E_C = \langle 0 | H | 0 \rangle = \]

\[ \sum_{n \neq 0} \omega_n \Omega^2_n \langle 0 | A^1_n A^{1\dagger}_n + A^2_n A^{2\dagger}_n | 0 \rangle = \]

\[ \sum_{n \neq 0} \omega_n \langle 0 | A^1_n A^{1\dagger}_n + A^2_n A^{2\dagger}_n | 0 \rangle + \]

\[ \frac{\pi^2 g^2}{R^2} \sum_{n \neq 0} \omega_n n^2 \theta^2(n) \langle 0 | A^1_n A^{1\dagger}_n + A^2_n A^{2\dagger}_n | 0 \rangle \]

\[ \equiv E^0_C + E^1_C. \]

The term \( E^0_C \) that does not contain noncommutativity parameter is the standard one and its value (after necessary regularization) can be found in e.g. \([4]\), or in an exhaustive explanation of methods of computation is given in the paper \([7]\). \( E^0_C \) reads

\[ E^{brun}_C = -\frac{1}{6R}. \]
The $\vartheta$-dependent contribution to the Casimir energy is already regularized by $\sigma$ in (4) and reads
\[ E^1_C(\sigma) = \frac{4\pi^3\vartheta^2}{R^3} \sum_{n=1}^{\infty} \frac{n^3}{n^2} e^{-\frac{2\pi^2}{\vartheta^2} n^2}. \]

Renormalizing it we obtain
\[ E^{1\text{ren}}_C = \frac{4\pi^3\vartheta^2}{R^3} \zeta(-3) = \frac{\pi^3\vartheta^2}{30R^3}. \]

Finally, we have the Casimir energy given by
\[ E_C = -\frac{1}{6R} + \frac{\pi^3\vartheta^2}{30R^3}. \quad (10) \]

V. DISCUSSION

If $\vartheta \neq 0$ then the Casimir energy (10) is not a monotonous function of the radius $R$. This fact makes the formula for the Casimir energy in our situation different from the standard result that is obtained letting $\vartheta = 0$. There is a radius $R_0$ at which the Casimir energy attains its minimal value, namely
\[ R_0 = \left( \frac{3\pi^3}{5} \right)^{1/2} \vartheta. \quad (11) \]

We see that the non-locality of the field field commutators generates in (10) the repulsive force at small distances $R < R_0$, and effectively this force can stabilize the radius of the space.

Acknowledgments

This work was supported by the grant scheme VEGA 1/0785/19 of the Ministry of Education, Science, and Sport of the Slovak Republic.

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