Manifestations of geometric phases in a proton electric-dipole-moment experiment in an all-electric storage ring

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Abstract. We consider a proton electric-dipole-moment experiment in an all-electric storage ring when the spin is frozen and local longitudinal and vertical electric fields alternate. In this experiment, the geometric (Berry) phases are very important. Due to these phases, the spin rotates about the radial axis. The corresponding systematic error is rather important while it can be canceled with clockwise and counterclockwise beams. The geometric phases also lead to the spin rotation about the radial axis. This effect can be canceled with clockwise and counterclockwise beams as well. The sign of the azimuthal component of the angular velocity of the spin precession depends on the starting point where the spin orientation is perfect. The radial component of this quantity keeps its value and sign for each starting point. When the longitudinal and vertical electric fields are joined in the same sections without any alternation, the systematic error due to the geometric phases does not appear but another systematic effect of the spin rotation about the azimuthal axis takes place. It has opposite signs for clockwise and counterclockwise beams.

1. Introduction
The geometric (Berry) phase [1, 2] is an important effect of classical and quantum mechanics. It is a phase difference acquired over the course of a cycle, when a system is subjected to cyclic adiabatic processes, which results from the geometrical properties of the parameter space of the Hamiltonian. The assumption that the action of the perturbing fields is much weaker than that of the main field means the adiabatic approximation, wherein the time scale over which a time-dependent Hamiltonian varies is long compared to typical quantum-mechanical oscillation periods. In this case, a quantum-mechanical system remains in the same eigenstate but develops a dynamical phase factor being the geometric (Berry) phase.

The geometric phases are rather important in spin physics. In particular, they manifest themselves in the muon $g-2$ experiment [3]. In this experiment, the motional magnetic field in the muon rest frame caused by electric focusing leads to a change of the $g-2$ frequency (so-called pitch correction [4, 5, 6, 7]). An importance of these phases for electric-dipole-moment (EDM) experiments is caused by their potential to mimic the EDM. They condition the most important systematic error in the neutron EDM experiment [8] and play an important role in storage-ring EDM experiments. The Berry phases resulting in the false EDM effect can originate from the
noncommutativity of spin rotations about different axes [9]. In three dimensions, when the spin rotates first about one axis and then about a second one, and then the rotations repeat with opposite sign, the total spin rotation is equal to the product of the two rotation angles with respect to the third axis [9, 10]. For an EDM experiment in an all-electric storage ring, this situation takes place for the false EDM effect conditioned by a vertical and a longitudinal component of a magnetic field. However, the corresponding systematic error [9, 10, 11, 12] is small due to magnetic shielding.

We consider another effect conditioned by the geometric phase. It takes place in the proton EDM experiment in an all-electric storage ring and appears owing to the noncommutativity of impacts of the vertical and longitudinal electric fields.

2. Spin behavior in an all-electric storage ring

A spin behavior in an all-electric storage ring is defined by the Thomas-Bargmann-Michel-Telegdi equation [13] generalized to the case of a particle with the EDM [14, 15, 16, 17, 18]. The main distinguishing feature of the all-electric storage ring for the proton EDM experiment is the use of radial electric field electrodes for bending, quadrupoles for focusing, and sextupoles for a chromaticity and a long spin coherence time [9]. When the magnetic field is absent, the spin motion equation in the Frenet-Serret (FS) coordinate system (see Ref. [19] for details) reduces to

\[ \frac{d\mathbf{K}}{dt} = \mathbf{\Omega} \times \mathbf{\zeta}, \quad \mathbf{\Omega} = \frac{e}{mc} \left\{ \left( G - \frac{1}{\sqrt{1-\gamma^2}} \right) \left( \mathbf{\beta} \times \mathbf{E} \right) - \frac{\eta}{2} \left[ \mathbf{E} - \frac{7}{8} \mathbf{\beta} (\mathbf{\beta} \cdot \mathbf{E}) \right] \right\}, \tag{1} \]

where \( G = (g - 2)/2 \), \( \mathbf{\beta} = \mathbf{v}/c \), \( \eta = 4dm/e \), \( d \) is the EDM. When the FS coordinate system is used, one describes the momentum evolution in the cylindrical coordinate system and the spin motion relative to the momentum direction. Except for field misalignments and imperfections, the electric field is orthogonal to \( \mathbf{\beta} \). Thus, we can neglect the term proportional to \( \mathbf{\beta} \cdot \mathbf{E} \). When the particle momentum in the all-electric EDM storage ring satisfies the condition

\[ p = mc\sqrt{\gamma^2 - 1} = \frac{mc}{\sqrt{G}}, \tag{2} \]

the spin in frozen. In this case, the spin follows the momentum even if the beam is deviated from the ideal orbit. When the condition (2) is exactly satisfied, systematic errors are vanished. We should mention that the cylindrical coordinates are usually more convenient for a description of the spin dynamics than the FS ones [19]. However, the latter coordinates define the spin motion relative to the momentum direction. Since this motion is the only reason for an appearance of systematic errors in the considered case, the use of the FS coordinates is preferable. In the cylindrical coordinate system, the beam with the initial azimuthal polarization can acquire a vertical polarization component even if Eq. (2) is exact. As a result, the description of a spin evolution in this system may lead to a consideration of some fictitious sources of systematic errors.

While the effect of the geometric phase owing to the vertical and longitudinal magnetic fields is rather small, the residual radial magnetic field can lead to a significant false EDM signal. This field produces a vertical spin rotation signal which cannot be canceled with clockwise (CW) and counterclockwise (CCW) beams. In addition, the radial magnetic field causes a vertical separation of two counter-rotating beams [10].

It has been mentioned in Ref. [10] that systematic errors in the proton EDM experiment can be conditioned by the vertical electric field and deviations of the beam momentum from the value (2). In the present work, we show that these errors can be rather significant and evaluate them. We also determine main properties of spin dynamics caused by these systematic errors. We consider the spin rotation about the radial and longitudinal axes. Unlike precedent investigations, we give a detailed analysis for some special conditions.
In the all-electric storage ring, the longitudinal electric field does not rotate the spin and the main radial electric field rotating the spin about the vertical axis cannot cause a false EDM signal. Therefore, the electric fields cannot create the Berry phase from the noncommutativity of spin rotations. The false EDM signal appears owing to an alternation of the vertical and longitudinal electric fields. When the condition (2) is violated, the first field causes the spin rotation about the radial axis imitating the EDM signal. The second field changing the beam momentum brings a violation of this condition. The resulting effect is caused by misalignments and imperfections of the both fields and these fields do not have a strong impact on the spin. It can be easily checked that the effect considered meets conditions of an appearance of the Berry phase.

If the averaged beam momentum does not satisfy Eq. (2), the spin rotation about the vertical axis appears. Since this rotation can be easily recognized, we suppose that the beam momentum satisfies Eq. (2) on the average. To simplify the analysis, we specify starting points of the beam trajectory where the beam momentum complies with Eq. (2) (∆γ = 0) and neglect very small deviations of Ωz from zero. When Eq. (2) is satisfied even locally, the spin rotates about the radial axis only due to the EDM.

Any spin effect conditioned by the magnetic dipole moment (MDM) and mimicking the EDM signal may take place only when the condition (2) is violated. This situation takes place when the particle momentum is changed by a longitudinal electric field. We consider second-order systematic errors (bilinear in small effects caused by the MDM). A violation of the above-mentioned condition by a vertical electric field and by imperfections of the main radial electric field can be neglected because p2r and p2z are small quantities of the second order. Certainly, the average longitudinal electric field is equal to zero because perpetuum mobile cannot exist. However, this field can be nonzero in some sections of the ring.

3. Spin rotation mimicking the EDM signal

It can be easily shown that the angular velocity of the spin motion in the all-electric storage ring with small perturbations of the azimuthal particle momentum is given by

\[ \Omega = \frac{2e}{mc} G^2 \gamma (\beta \times E) \Delta \gamma - \frac{en}{2mc} E, \]

where \( \Delta \gamma \) is the deviation from (2). The main systematic error consists in local spin rotations about the radial axis and is caused by the vertical electric field \( E^{(v)} \)

\[ E^{(v)} = \pm |E^{(v)}| e_z = \pm E^{(v)} e_z. \]

Let us consider the simplest case leading to this systematic error. This case is shown in Figs. 1, 2. The main electric field is always antiparallel to \( e_r \). We denote \( \beta = |\beta| \). Regions with nonzero \( \Delta \gamma \) and \( E^{(v)} \) bounded by the points A, B, C, and D alternate. We will analyze four cases when both the momentum and the spin are collinear to the azimuthal direction, \( e_\phi \), in these starting points.

3.1. Clockwise beam motion (Fig. 1)

a) The starting point is A. At the points B and C, \( \Delta \gamma > 0 \). The angular velocity of the spin motion on the path BC (without the EDM effect) is

\[ \Omega_{BC} = - \frac{2e}{mc} G^2 \beta \gamma E^{(v)} |\Delta \gamma| e_z. \]

At the points D and A, \( \Delta \gamma = 0 \). On the path CA, \( \Omega_{CA} = 0 \). As a result, the average angular velocity is given by [20]

\[ \Omega = \frac{1}{4} \Omega_{BC} = - \alpha e_z, \quad \alpha = \frac{e}{2mc} G^2 \beta \gamma E^{(v)} |\Delta \gamma|. \]
This is a systematic error imitating the EDM effect. We should note that the longitudinal (azimuthal) electric field brings also the radial velocity component, $\beta^{(r)}$. However, the vector product $\beta^{(r)} \times E$ with allowance for the factor $\Delta \gamma$ leads to a third-order correction to $\Omega$. This correction can be neglected.

b) The starting point is B. At the points C and D, $\Delta \gamma = 0$ and $\Delta \gamma < 0$, respectively. The false effect is only on the path DA: $\Omega_{DA} = -4ae_z$ and $\Omega_{AB} = 0$. As a result, $\Omega = -ae_z$.

The analysis is similar for the starting points C and D. The false effect does not depend on the starting point. This effect can be calculated [20] and is given by

$$|\Omega| = \alpha = \frac{e}{2mc} G^2 \beta \gamma E^{(v)} \Delta \gamma_{\text{max}}.$$  

(6)

3.2. Counterclockwise beam motion (Fig. 2)

a) The starting point is A. At the points D and C, $\Delta \gamma = 0$ and $\Delta \gamma < 0$, respectively. The false effect is only on the path CB: $\Omega_{CB} = 4ae_z$. On the path BA, $\Omega_{BA} = 0$. As a result, $\Omega = \alpha e_z$.

b) The starting point is B. At the points A and D, $\Delta \gamma < 0$. The false effect is only on the path AD: $\Omega_{AD} = 4ae_z$ and $\Omega_{DB} = 0$. As a result, $\Omega = \alpha e_z$.

For the starting points C and D, the analysis is similar [20]. The systematic error due to the geometric phases has different signs for the two directions of the beam rotation. The EDM effect remains the same in this case. Thus, the considered systematic error can be canceled with CW and CCW beams.

4. Spin rotation about the longitudinal direction

There exists also an effect which does not influence spin dynamics of longitudinally polarized beams in all-electric storage rings. This is the spin rotation about the azimuthal axis which is also conditioned by the longitudinal and vertical electric fields. We suppose that the spin precession about the vertical axis is vanished.

Let us return to Figs. 1, 2. The vertical electric field in the sections BC and DA changes the vertical component of the particle momentum. The maximum value of this component is equal to

$$p_{\text{max}}^{(v)} = \frac{eR}{c\beta} \int_{\phi_1}^{\phi_2} E^{(v)} d\phi.$$  

(7)
In the considered case,
\[ p_{\text{max}}^{(v)} = \frac{\pi e R}{2 \epsilon_0} E^{(v)}. \]  
(8)

The vertical component of the particle velocity is given by
\[ \beta_{\text{max}}^{(v)} = \frac{\pi e R}{2 mc^2 \beta_\gamma} E^{(v)}. \]  
(9)

Evidently,
\[ E = \frac{mc^2 \beta^2 \gamma}{e R}, \quad \beta_{\text{max}}^{(v)} E = \frac{\pi}{2} \beta E^{(v)}, \]  
(10)

where \( E \) is the main electric field.

We analyze the same cases which have been considered in the precedent section. In the current section, we calculate only the angular velocity of the spin motion acting on the radial spin component.

4.1. **Clockwise beam motion** (Fig. 1)

The momentum distribution shown at this figure is valid when the starting points are A and B.

a) The starting point is A. At the points B and C, \( \Delta \gamma > 0 \). The average value of \( \beta_\gamma \) on the path BC is equal to \( \beta_{\text{max}}^{(v)} / 2 \). The angular velocity of the spin motion on this path reads
\[ \Omega_{BC} = -\frac{e}{mc} G^{(v)} \beta_{\text{max}}^{(v)} \gamma E \Delta \gamma_{\text{max}} e_\phi = -\pi \alpha e_\phi. \]  
(11)

At the points D and A, \( \Delta \gamma = 0 \). On the path CD, \( \beta_\gamma = \beta_{\text{max}}^{(v)} = \text{const} \) and the average value of \( \Delta \gamma \) is equal to \( \Delta \gamma_{\text{max}} / 2 \). As a result, the angular velocity of the spin motion on this path is given by
\[ \Omega_{CD} = -\pi \alpha e_\phi. \]  
(12)

b) The starting point is B. At the points B and C, \( \Delta \gamma = 0 \). On the path BC, \( \Omega_{BC} = 0 \). The average value of \( \Delta \gamma \) on the path CD is equal to \( -\Delta \gamma_{\text{max}} / 2 \). The angular velocity of the spin motion on this path reads
\[ \Omega_{CD} = \pi \alpha e_\phi. \]  
(13)

On the path DA, the average value of \( \beta_\gamma \) is equal to \( \beta_{\text{max}}^{(v)} / 2 \) and \( \Omega_{DA} = \pi \alpha e_\phi \). As a result, \( \Omega = \frac{\pi}{2} \alpha e_\phi \).

For the starting points C and D, the analysis is similar [20].

4.2. **Counterclockwise beam motion** (Fig. 2)

The momentum distribution shown at Fig. 2 is valid when the starting points are A and B.

a) The starting point is A. At the points C and B, \( \Delta \gamma = \Delta \gamma_{\text{max}} \). The average value of \( \Delta \gamma \) on the path DC is equal to \( \Delta \gamma_{\text{max}} / 2 \). The angular velocity of the spin motion on this path reads
\[ \Omega_{DC} = \pi \alpha e_\phi. \]  
(14)

On the path CB, \( \Delta \gamma = \Delta \gamma_{\text{max}} = \text{const} \) and the average value of \( \beta_\gamma \) is equal to \( -\beta_{\text{max}}^{(v)} / 2 \). As a result, the angular velocity of the spin motion on this path is given by
\[ \Omega_{CB} = \pi \alpha e_\phi. \]  
(15)

b) The starting point is B. At the points A and D, \( \Delta \gamma = -\Delta \gamma_{\text{max}} \). On the path AD, the average value of \( \beta_\gamma \) is equal to \( -\beta_{\text{max}}^{(v)} / 2 \) and \( \Omega_{AD} = -\pi \alpha e_\phi \). On the path DC, the average value of \( \Delta \gamma \) is equal to \( -\Delta \gamma_{\text{max}} / 2 \) and \( \Omega_{DC} = -\pi \alpha e_\phi \). As a result, \( \Omega = -\frac{\pi}{2} \alpha e_\phi \).

The analysis is similar for the starting points C and D [20]. The longitudinal component of the angular velocity of the spin precession has different signs for the CW and CCW directions of the beam rotation. Therefore, this systematic error can be canceled with CW and CCW beams. Even if the spin precession about the vertical axis is not vanished, this property allows one to eliminate the systematic errors caused by the Berry phases. The azimuthal component of \( \Omega \), contrary to the radial one, depends on the starting point where the spin orientation is perfect.
5. Spin evolution in joined longitudinal and vertical electric fields

In this section, we consider the spin evolution on condition that the longitudinal and vertical electric fields are joined in the same sections without any alternation. This consideration shows that only an alternation of these fields leads to a systematic error mimicking the EDM signal.

![Figure 3. Spin motion in joined longitudinal and vertical electric fields.](image)

Let us consider the simple case shown in Fig. 3. In this case, the longitudinal and vertical electric fields are constant into the paths AB and CD and the fields averaged over the ring are equal to zero. The starting point is F. Since $\Delta \gamma = 0$ at the points A and D and the beam rotates CW, the average value of $\Delta \gamma$ on the path AB is given by Eq. (6) and is positive. The average value of $\Delta \gamma$ on the path CD is the same. However, the values of $E^{(v)}$ are opposite. As a result, the total spin rotation is equal to zero. The same situation takes place when the beam rotates CCW. Thus, the systematic error due to the Berry phases is absent when the longitudinal and vertical electric fields are joined in the same sections without any alteration.

However, another systematic effect of the spin rotation about the azimuthal axis takes place. Figure 3 shows that the signs of the vertical components of the momentum and velocity are the same in the sections AB and CD. Specifically, we obtain the relations

$$
\beta_z = -\beta_{max}^{(v)} x, \quad \Delta \gamma = \Delta \gamma_{max} x, \quad x = \frac{\phi(X) - \phi(A)}{\phi(B) - \phi(A)},
$$

(13)

for any point $X \in [A, B]$. As a result, the average angular velocity of the spin motion is given by

$$
\Omega_{AB} = \frac{2e}{3mc} G^2 \beta_{max}^{(v)} E \Delta \gamma_{max} x e_\phi = \frac{2}{3} \pi \alpha e_\phi.
$$

The average angular velocity of the spin motion on the path CD is the same: $\Omega_{CD} = \Omega_{AB}$. As a result [20],

$$
\Omega = \frac{2[\phi(B) - \phi(A)]}{3} \alpha e_\phi.
$$

(14)

When the beam rotates CCW, $\gamma$ remains the same but the vertical components of the momentum and velocity change the signs. In this case,

$$
\Omega = -\frac{2[\phi(B) - \phi(A)]}{3} \alpha e_\phi.
$$

Thus, the angular velocity of the spin rotation has opposite directions for the CW and CCW beams.
6. Discussion and conclusions
The present study shows that local longitudinal and vertical electric fields in the all-electric storage ring may lead to the systematic effects caused by the Berry phases. This takes place when sections with the longitudinal and vertical electric fields alternate. If these fields are perfectly joined, such systematic effects do not appear. Estimates show that the systematic error caused by the Berry phases is not small. Nevertheless, it can be canceled with CW and CCW beams. This possibility is conditioned by different signs of the radial component of the angular velocity of the spin precession $\Omega$ for the CW and CCW directions of the beam rotation. The azimuthal component of $\Omega$ also possesses this property. However, the sign of this component depends on the starting point (where the spin orientation is perfect) while the radial component of $\Omega$ keeps its value for each starting point. When the longitudinal and vertical electric fields are joined in the same sections without any alternation, the systematic error due to the Berry phases does not appear. However, another systematic effect of the spin rotation about the azimuthal axis takes place. The angular velocity of the spin rotation has opposite signs for CW and CCW beams.

It is important that both components of $\Omega$ are caused by the same local longitudinal and vertical electric fields. Therefore, the azimuthal component of $\Omega$ can be even helpful because it allows one to evaluate an order of magnitude of the radial component even without the use of the beam with the opposite direction of rotation.

Another important systematic error in the proton EDM experiment in the all-electric storage ring is the residual radial magnetic field. However, this field can be measured by high-precision magnetometers and the map of magnetic field over the ring can be obtained. In this case, the systematic error due to the radial magnetic field can be transformed into the systematic correction.

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