STABILITY RESULTS FOR SECTIONS OF CONVEX BODIES

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Abstract. It is shown by Makai, Martini, and Ódor that a convex body $K$, all of whose maximal sections pass through the origin, must be origin-symmetric. We prove a stability version of this result. We also discuss a theorem of Koldobsky and Shane about determination of convex bodies by fractional derivatives of the parallel section function and establish the corresponding stability result.

1. Introduction

Let $K$ be a convex body in $\mathbb{R}^n$, i.e. a compact convex set with non-empty interior. More generally, a body is a compact subset of $\mathbb{R}^n$ which is equal to the closure of its interior. Throughout the paper, we assume all bodies include the origin as an interior point. Now, we say $K$ is origin-symmetric if $K = -K$. The parallel section function of $K$ in the direction $\xi \in S^{n-1}$ is defined by

$$A_K,\xi(t) = \text{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}.$$ 

Here, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$ is the hyperplane passing through the origin and orthogonal to the vector $\xi$.

For the study of central sections it is often more natural to consider a larger class of bodies than the class of convex bodies. Recall that if $K$ is a body containing the origin in its interior and star-shaped with respect to the origin, its radial function is defined by

$$\rho_K(\xi) = \max\{a \geq 0 : a\xi \in K\}, \quad \xi \in S^{n-1}.$$ 

Geometrically, $\rho_K(\xi)$ is the distance from the origin to the point on the boundary in the direction of $\xi$. If $\rho_K$ is continuous, then $K$ is called a star body. Every convex body (with the origin in its interior) is a star body. The intersection body of a star body $K$ is the star body $IK$ with radial function

$$\rho_{IK}(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1}.$$ 

Intersection bodies were introduced by Lutwak in [10] and have been actively studied since then. For example, they played a crucial role in the solution of the Busemann-Petty problem (see [8] for details).

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The cross-section body of a convex body $K$ is the star body $CK$ with radial function
$$\rho_{CK}(\xi) = \max_{t \in \mathbb{R}} A_{K,\xi}(t), \quad \xi \in S^{n-1}.$$ Cross-section bodies were introduced by Martini [12]. For properties of these bodies and related questions see [2], [4], [5], [11], [13], [14].

Brunn’s theorem asserts that the origin-symmetry of a convex body $K$ implies
$$A_{K,\xi}(0) = \max_{t \in \mathbb{R}} A_{K,\xi}(t)$$ for all $\xi \in S^{n-1}$. In other words, $CK = IK$. The converse statement was proved by Makai, Martini and Ódor [11].

**Theorem 1** (Makai, Martini and Ódor). If $K$ is a convex body in $\mathbb{R}^n$ such that $CK = IK$, then $K$ is origin-symmetric.

The goal of the present paper is to provide a stability version of Theorem 1. For star bodies $K$ and $L$ in $\mathbb{R}^n$, the radial metric is defined as
$$\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$ We prove the following result.

**Theorem 2.** Let $K$ be a convex body in $\mathbb{R}^n$ contained in a ball of radius $R$, and containing a ball of radius $r$, where both balls are centred at the origin. If there exists $0 < \varepsilon < \min \left\{ \left( \frac{\sqrt{2}r}{6\sqrt{3}\pi r+32\pi} \right)^2, \frac{r^2}{16} \right\}$ so that
$$\rho(CK, IK) \leq \varepsilon,$$ then
$$\rho(K, -K) \leq C(n, r, R) \varepsilon^q \quad \text{where} \quad q = \begin{cases} \frac{1}{2} & \text{if } n = 2, \\ \frac{1}{2(n+1)} & \text{if } n = 3, 4, \\ \frac{1}{(n-2)(n+1)} & \text{if } n \geq 5. \end{cases}$$ Here, $C(n, r, R) > 0$ are constants depending on the dimension, $r$, and $R$.

**Remark.** In the proof of Theorem 2 we give the explicit dependency of $C(n, r, R)$ on $r$ and $R$.

The following corollary is a straightforward consequence of the Lipschitz property of the parallel section function (Lemma 9) and Theorem 2. Roughly speaking: if, for every direction $\xi \in S^{n-1}$, the convex body $K$ has a maximal section perpendicular to $\xi$ that is close to the origin, then $K$ is close to being origin-symmetric.
Corollary 3. Let $K$ be a convex body in $\mathbb{R}^n$ contained in a ball of radius $R$, and containing a ball of radius $r$, where both balls are centred at the origin. Let $L = L(n)$ be the constant given in Lemma 4. If there exists

$$0 < \varepsilon < \min \left\{ \frac{r}{2}, \frac{3r^3}{LR^{n-1}(6\sqrt{3\pi r} + 32\pi)^2}, \frac{r^3}{16LR^{n-1}} \right\}$$

so that, for each direction $\xi \in S^{n-1}$, $A_{K,\xi}$ attains its maximum at some $t = t(\xi)$ with $|t(\xi)| \leq \varepsilon$, then

$$\rho(K, -K) \leq \tilde{C}(n, r, R) \varepsilon^q.$$ 

Here, $\tilde{C}(n, r, R) > 0$ are constants depending on the dimension, $r$, and $R$, and $q = q(n)$ is the same as in Theorem 2.

The proof of Theorem 2 is given in Section 4 and consists of a sequence of lemmas from Section 3. The main idea is the following. If $K$ is of class $C^\infty$, then we use Brunn’s theorem and an integral formula from [3] to show that $\rho(CK, IK)$ being small implies that $\int_{S^{n-1}} |A_{K,\xi}(0)|^2 d\xi$ is also small. (Recall that $K$ is called $m$-smooth or $C^m$, if $\rho_K \in C^m(S^{n-1}).$) If $K$ is not smooth, we approximate it by smooth bodies, for which the above integral is small. Then we use the Fourier transform techniques from [15] and the tools of spherical harmonics similar to those from [6] to finish the proof.

As we will see below, the same methods can be used to obtain a stability version of a result of Koldobsky and Shane [9]. It is well known that the knowledge of $A_{K,\xi}(0)$ for all $\xi \in S^{n-1}$ is not sufficient for determining the body $K$ uniquely, unless $K$ is origin-symmetric. However, Koldobsky and Shane have shown that if $A_{K,\xi}(0)$ is replaced by a fractional derivative of non-integer order of the function $A_{K,\xi}(t)$ at $t = 0$, then this information does determine the body uniquely.

**Theorem 4** (Koldobsky and Shane). Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$. Let $-1 < p < n - 1$ be a non-integer, and $m$ be an integer greater than $p$. If $K$ and $L$ are $m$-smooth and

$$A^{(p)}_{K,\xi}(0) = A^{(p)}_{K,\xi}(0),$$

for all $\xi \in S^{n-1}$, then

$$K = L.$$ 

The following is our stability result.

**Theorem 5.** Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ contained in a ball of radius $R$, and containing a ball of radius $r$, where both balls are centred at the origin. Let $-1 < p < n - 1$ be a non-integer, and $m$ be an integer greater than $p$. If $K$ and $L$ are $m$-smooth and

$$\sup_{\xi \in S^{n-1}} \left| A^{(p)}_{K,\xi}(0) - A^{(p)}_{L,\xi}(0) \right| \leq \varepsilon$$

for all $\xi \in S^{n-1}$, then

$$K = L.$$
for some $0 < \varepsilon < 1$, then

$$\rho(K, L) \leq C(n, p, r, R) \varepsilon^q \quad \text{where} \quad q = \begin{cases} \frac{2}{n+1} & \text{if } n \leq 2p + 2, \\ \frac{4}{(n-2p)(n+1)} & \text{if } n > 2p + 2. \end{cases}$$

Here, $C(n, p, r, R) > 0$ are constants depending on the dimension, $p, r,$ and $R$.

**Remark.** In the proof of Theorem 5, we give the explicit dependency of $C(n, p, r, R)$ on $r$ and $R$. Furthermore, our second result remains true when $p$ is a non-integer greater than $n - 1$. However, considering such values for $p$ would make our arguments less clear.

2. **Preliminaries**

Throughout our paper, the constants

$$\kappa_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} \quad \text{and} \quad \omega_n := n \cdot \kappa_n$$

give the volume and surface area of the unit Euclidean ball in $\mathbb{R}^n$, where $\Gamma$ denotes the Gamma function. Whenever we integrate over Borel subsets of the sphere $S^{n-1}$, we are using non-normalized spherical measure; that is, the $(n - 1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, scaled so that the measure of $S^{n-1}$ is $\omega_n$.

Let $K$ be a convex body in $\mathbb{R}^n$ containing the origin in its interior. The maximal section function of $K$ is defined by

$$m_K(\xi) = \max_{t \in \mathbb{R}} \text{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}) = \max_{t \in \mathbb{R}} A_{K,\xi}(t), \quad \xi \in S^{n-1}.$$

Note that $m_K$ is simply the radial function for the cross-section body $CK$. For each $\xi \in S^{n-1}$, we let $t_K(\xi) \in \mathbb{R}$ be the closest to zero number such that

$$A_{K,\xi}(t_K(\xi)) = m_K(\xi).$$

Towards the proof of our first stability result, we use the formula

$$f_K(t) := \frac{1}{\omega_n} \int_{S^{n-1}} A_{K,\xi}(t) d\xi = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{n-1}{2}\right)} \int_{\mathbb{R}^n \setminus \{0\}} \frac{1}{|x|} \left(1 - \frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} dx; \quad (1)$$

refer to Lemma 1.2 in [3] or Lemma 1 in [1] for the proof.

The Minkowski functional of $K$ is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}, \quad x \in \mathbb{R}^n.$$

It easy to see that $\rho_K(\xi) = \|\xi\|_K^{-1}$ for $\xi \in S^{n-1}$. The latter also allows us to consider $\rho_K$ as a homogeneous degree $-1$ function on $\mathbb{R}^n \setminus \{0\}$. The support function of $K$ is defined by

$$h_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n.$$
The function $h_K$ is the Minkowski functional for the polar body $K^\circ$ associated with $K$. Given another convex body $L$ in $\mathbb{R}^n$, define

$$\delta_2(K, L) = \left( \int_{S^{n-1}} |h_K(\xi) - h_L(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

and

$$\delta_\infty(K, L) = \sup_{\xi \in S^{n-1}} |h_K(\xi) - h_L(\xi)|.$$ 

These functions are, respectively, the $L^2$ and Hausdorff metrics for convex bodies in $\mathbb{R}^n$. The following theorem, due to Vitale [17], relates these metrics; refer to Proposition 2.3.1 in [7] for the proof.

**Theorem 6.** Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$, and let $D$ denote the diameter of $K \cup L$. Then

$$\frac{2\kappa_n}{n(n+1)} \delta_\infty(K, L)^{n+1} \leq \delta_2(K, L)^2 \leq \omega_n \delta_\infty(K, L)^2.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be any $n$-tuple of non-negative integers. We will use the notation

$$[\alpha] := \sum_{j=1}^n \alpha_j$$

to define the differential operator

$$\partial^{[\alpha]} := \partial^{[\alpha]}_{x^1} \cdots \partial^{[\alpha]}_{x^n}.$$

We let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz test functions; that is, functions in $C^\infty(\mathbb{R}^n)$ for which all derivatives decay faster than any rational function. The Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a test function $\mathcal{F}\phi$ defined by

$$\mathcal{F}\phi(x) = \hat{\phi}(x) = \int_{\mathbb{R}^n} \phi(y)e^{-i(x,y)} \, dy, \quad x \in \mathbb{R}^n.$$ 

The continuous dual of $\mathcal{S}(\mathbb{R}^n)$ is denoted as $\mathcal{S}'(\mathbb{R}^n)$, and elements of $\mathcal{S}'(\mathbb{R}^n)$ are referred to as distributions. The action of $f \in \mathcal{S}'(\mathbb{R}^n)$ on a test function $\phi$ is denoted as $\langle f, \phi \rangle$. The Fourier transform of $f$ is a distribution $\hat{f}$ defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n);$$

$\hat{f}$ is well-defined as a distribution because $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is a continuous and linear bijection.

For any $f \in C(S^{n-1})$ and $p \in \mathbb{C}$, the $-n + p$ homogeneous extension of $f$ is given by

$$f_p(x) = |x|^{n-p} f \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.$$
When $R^p > 0$, $f^p$ is locally integrable on $\mathbb{R}^n$ with at most polynomial growth at infinity. In this case, $f^p$ is a distribution on $S(\mathbb{R}^n)$ acting by integration, and we may consider its Fourier transform. Goodey, Yaskin, and Yaskina show in [6] that, for $f \in C^\infty(S^{n-1})$, the additional restriction $R^p < n$ ensures the action of $\hat{f}^p$ is also by integration, with $\hat{f}^p \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

We make extensive use of the mapping $I_p : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ defined in [6], which sends a function $f$ to the restriction of $\hat{f}^p$ to $S^{n-1}$. For $0 < R^p < n$ and $m \in \mathbb{Z}_{\geq 0}$, Goodey, Yaskin and Yaskina show $I_p$ has an eigenvalue $\lambda_m(n, p)$ whose eigenspace includes all spherical harmonics of degree $m$ and dimension $n$. These eigenvalues are given explicitly in the following lemma; refer to [6] for the proof.

**Lemma 7.** If $0 < R^p < n$, then the eigenvalues $\lambda_m(n, p)$ are given by

$$\lambda_m(n, p) = \frac{2^p \pi^{\frac{n}{2}} (-1)^\frac{m}{2} \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m+n-p}{2}\right)} \text{ if } m \text{ is even,}$$

and

$$\lambda_m(n, p) = i \frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{m-1}{2}} \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m+n-p}{2}\right)} \text{ if } m \text{ is odd.}$$

The spherical gradient of $f \in C(S^{n-1})$ is the restriction of $\nabla f \left(\frac{x}{|x|}\right)$ to $S^{n-1}$. It is denoted by $\nabla_0 f$.

An extensive discussion on spherical harmonics is given in [7]. A spherical harmonic $Q$ of dimension $n$ is a harmonic and homogeneous polynomial in $n$ variables whose domain is restricted to $S^{n-1}$. We say $Q$ is of degree $m$ if the corresponding polynomial has degree $m$. The collection $H_n^m$ of all spherical harmonics with dimension $n$ and degree $m$ is a finite dimensional Hilbert space with respect to the inner product for $L^2(S^{n-1})$. If, for each $m \in \mathbb{Z}_{\geq 0}$, $B_m$ is an orthonormal basis for $H_n^m$, then the union of all $B_m$ is an orthonormal basis for $L^2(S^{n-1})$. Given $f \in L^2(S^{n-1})$, and defining

$$\sum_{Q \in B_m} \langle f, Q \rangle Q =: Q_m \in H_n^m,$$

we call $\sum_{m=0}^\infty Q_m$ the condensed harmonic expansion for $f$. The condensed harmonic expansion does not depend on the particular orthonormal bases chosen for each $H_n^m$.

Let $m \in \mathbb{N} \cup \{0\}$, and let $h : \mathbb{R} \to \mathbb{C}$ be an integrable function which is $m$-smooth in a neighbourhood of the origin. For $p \in \mathbb{C} \setminus \mathbb{Z}$ such that $-1 < R^p < m$, we define the fractional derivative of the order $p$ of $h$ at zero
as

\[ h^{(p)}(0) = \frac{1}{\Gamma(-p)} \int_0^1 t^{-1-p} \left( h(-t) - \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k!} t^k \right) dt \]

\[ + \frac{1}{\Gamma(-p)} \int_1^\infty t^{-1-p} h(-t) dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k! (k-p)}. \]

Given the simple poles of the Gamma function, the fractional derivatives of \( h \) at zero may be analytically extended to the integer values 0, \ldots, \( m - 1 \), and they will agree with the classical derivatives.

Let \( K \) be an infinitely smooth convex body. By Lemma 2.4 in [8], \( A_{K,\xi} \) is infinitely smooth in a neighbourhood of \( t = 0 \) which is uniform with respect to \( \xi \in S^{n-1} \). With the exception of a sign difference, the equality

\[ A^{(p)}_{K,\xi}(0) = \frac{\cos \left( \frac{np}{2} \right)}{2\pi(n-1-p)} \left( \|x\|^{-n+1+p} + \| -x\|^{-n+1+p} \right) \wedge (\xi) \quad (2) \]

\[ + i \frac{\sin \left( \frac{np}{2} \right)}{2\pi(n-1-p)} \left( \|x\|^{-n+1+p} - \| -x\|^{-n+1+p} \right) \wedge (\xi), \]

was proven by Ryabogin and Yaskin in [15] for all \( \xi \in S^{n-1} \) and \( p \in \mathbb{C} \) such that \(-1 < \text{Re}(p) < n-1\). The sign difference results from their use of \( h(x) \) rather than \( h(-x) \) in the definition of fractional derivatives.

3. Auxiliary Results

We first prove some auxiliary lemmas.

**Lemma 8.** Let \( m \) be a non-negative integer. Let \( K \) be an \( m \)-smooth convex body in \( \mathbb{R}^n \) contained in a ball of radius \( R \), and containing a ball of radius \( r \), where both balls are centred at the origin. There exists a family \( \{K_\delta\}_{0<\delta<1} \) of infinitely smooth convex bodies in \( \mathbb{R}^n \) which approximate \( K \) in the radial metric as \( \delta \) approaches zero, with

\[ B_{(1+\delta)^{-1}R}(0) \subset K_\delta \subset B_{(1-\delta)^{-1}R}(0). \]

Furthermore,

\[ \lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \sup_{|t| \leq \frac{r}{\delta}} |A_{K,\xi}(t) - A_{K_\delta,\xi}(t)| = 0, \]

and

\[ \lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{K,\xi}(0) \right| = 0 \]

for every \( p \in \mathbb{R}, -1 < p \leq m \).

**Proof.** For each \( 0 < \delta < 1 \), let \( \phi_\delta : [0, \infty) \to [0, \infty) \) be a \( C^\infty \) function with support contained in \([\delta/2, \delta]\), and

\[ \int_{\mathbb{R}^n} \phi_\delta(|z|) dz = 1. \]
It follows from Theorem 3.3.1 in [16] that there is a family \( \{ K_\delta \}_{0 < \delta < 1} \) of \( C^\infty \) convex bodies in \( \mathbb{R}^n \) such that
\[
\| x \|_{K_\delta} = \int_{\mathbb{R}^n} \| x + |z| \| K_\phi_\delta(|z|) \, dz,
\]
and
\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left( \| \xi \|_{K_\delta} - \| \xi \|_K \right) = 0.
\]
For each \( \xi \in S^{n-1} \) and \( z \in \mathbb{R}^n \) with \( |z| \leq \delta \), we have
\[
\| \xi + |z| \|_K = \| \xi + z \|_K = \| \lambda \eta \|_K = \lambda \| \eta \|_K
\]
for some \( \eta \in S^{n-1} \) and \( 0 < 1 - \delta \leq \lambda \leq 1 + \delta \). It then follows from the support of \( \phi_\delta \) and the inequality \( R^{-1} \leq \| \eta \|_K \leq r^{-1} \) that
\[
\| \xi \|_{K_\delta} = \int_{\mathbb{R}^n} \| \xi + z \| K_\phi_\delta(|z|) \, dz \leq (1 + \delta) r^{-1},
\]
and
\[
\| \xi \|_K = \int_{\mathbb{R}^n} \| \xi + z \| K_\phi_\delta(|z|) \, dz \geq (1 - \delta) R^{-1},
\]
which gives
\[
B_{(1+\delta)^{-1} r}(0) \subset K_\delta \subset B_{(1-\delta)^{-1} R}(0).
\]
This containment, with the limit of the difference of Minkowski functionals above, implies
\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \rho_{K_\delta}(\xi) - \rho_K(\xi) \right| = 0. \tag{3}
\]
Therefore, \( \{ K_\delta \}_{0 < \delta < 1} \) approximate \( K \) with respect to the radial metric.

Furthermore, the radial functions \( \{ \rho_{K_\delta} \}_{0 < \delta < 1} \) approximate \( \rho_K \) in \( C^m(S^{n-1}) \).

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be any \( n \)-tuple of non-negative integers such that
\( 1 \leq [\alpha] \leq m \), and consider the function
\[
f(y, z) := \left. \frac{\partial^{[\alpha]} x^\alpha}{\partial x^\alpha} \| x + |z| \|_K \right|_{x=y}.
\]
Observe that \( f \) is uniformly continuous on
\[
\{ y \in \mathbb{R}^n, 2^{-1} \leq |y| \leq 2 \} \times \{ z \in \mathbb{R}^n, |z| \leq 2^{-1} \}
\]
since \( K \) is \( m \)-smooth. Therefore, we have
\[
\frac{\partial^{[\alpha]} x^\alpha}{\partial x^\alpha} \left( \| x \|_{K_\delta} - \| x \|_K \right)_{x=\xi} = \int_{\mathbb{R}^n} \frac{\partial^{[\alpha]} x^\alpha}{\partial x^\alpha} \left( \| x + |z| \| K_\phi_\delta(|z|) \| x \|_K \right)_{x=\xi} \, dz
\]
for all \( \xi \in S^{n-1} \) and \( \delta < 1/2 \), which implies
\[
\sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]} x^\alpha}{\partial x^\alpha} \left( \| x \|_{K_\delta} - \| x \|_K \right)_{x=\xi} \right| \leq \sup_{\xi \in S^{n-1}, |z| < \delta} \sup_{\xi \in S^{n-1}} \left| f(\xi, z) - f(\xi, 0) \right|.
\]
Noting that \( |(\xi, z) - (\xi, 0)| = |z| < \delta \), the uniform continuity of \( f \) then implies

\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \left( \|x\|_{K_\delta} - \|x\|_K \right) \right|_{x=\xi} = 0. \tag{4}
\]

It follows from the relation \( \rho_K(x) = \|x\|_K^{-1} \) that \( \frac{\partial^{|\alpha|}}{\partial x^\alpha} \rho_K \big|_{x=\xi} \) may be expressed as a finite linear combination of terms of the form

\[
\rho_K^{d+1}(\xi) \prod_{j=0}^{d} \frac{\partial^{|\beta_j|}}{\partial x^{|\beta_j|}} \|x\|_K \big|_{x=\xi},
\]

where \( d \in \mathbb{Z}^>0 \), and each \( \beta_j \) is an \( n \)-tuple of non-negative integers such that \( |\beta_j| \geq 1 \) and \( |\alpha| = \sum_j |\beta_j| \). Of course, \( \frac{\partial^{|\alpha|}}{\partial x^\alpha} \rho_K \big|_{x=\xi} \) may be expressed similarly. Equations (3) and (4) then imply

\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \left( \rho_{K_\delta} - \rho_K \right) \right|_{x=\xi} = 0, \tag{5}
\]

once we note that \( \rho_K \) and the partial derivatives of \( \|x\|_K \), up to order \( m \), are bounded on \( S^{n-1} \).

Our next step is to uniformly approximate the parallel section function \( A_{K,\xi} \). Fix \( \xi \in S^{n-1} \), and define the hyperplane

\[
H_t = \xi^\perp + t\xi
\]

for any \( t \in \mathbb{R} \) such that \( |t| < r \). Let \( S^{n-2} \) denote the Euclidean sphere in \( H_t \) centred at \( t\xi \), and let \( \rho_{K \cap H_t} \) denote the radial function for \( K \cap H_t \) with respect to \( t\xi \) on \( S^{n-2} \). Then, for \( |t| < r \),

\[
A_{K,\xi}(t) = \frac{1}{n-1} \int_{S^{n-2}} \rho_{K \cap H_t}^{n-1}(\theta) d\theta. \tag{6}
\]

For \( |t| < r/2 \) and \( 0 < \delta < 1 \), \( A_{K_\delta,\xi}(t) \) may be expressed similarly. Fixing \( \theta \in S^{n-2} \), and with angles \( \alpha \) and \( \beta \) as in Figure 1, we have

\[
|\rho_{K \cap H_t}(\theta) - \rho_{K_\delta \cap H_t}(\theta)| \leq \frac{\sin \beta}{\sin \alpha} |\rho_K(\eta) - \rho_{K_\delta}(\eta)|.
\]

By restricting to \( |t| \leq r/4 \), \( \alpha \) may be bounded away from zero and \( \pi \). Indeed, if \( \alpha < \pi/2 \), then

\[
\tan \alpha \geq \frac{r/2 - |t|}{R} \geq \frac{r}{4R},
\]

and if \( \alpha > \pi/2 \), then

\[
\tan(\pi - \alpha) \geq \frac{r/2 + |t|}{R} \geq \frac{r}{2R}.
\]

Therefore

\[
0 < \arctan \left( \frac{r}{4R} \right) \leq \alpha \leq \pi - \arctan \left( \frac{r}{4R} \right) < \pi.
\]
Figure 1. The diagrams represent two extremes: when the angle $\alpha$ is small ($\alpha < \pi/2$), and when it is large ($\alpha > \pi/2$). The point $O$ represents the origin in $\mathbb{R}^n$, and $|OT| = t$ where $0 \leq t \leq r/4$. The points $A$ and $C$ are the boundary points for $K$ and $K_\delta$ in the direction $\theta$, with two obvious possibilities: either $|TA| = \rho_{K \cap H_t}(\theta)$ and $|TC| = \rho_{K_\delta \cap H_t}(\theta)$, or the opposite. The point $B$ is a boundary point for the same convex body as $A$, but in the direction $\eta_1$. The point $D$ lies outside of the convex body for which $A$ and $B$ are boundary points.

We now have

$$|\rho_{K \cap H_t}(\theta) - \rho_{K_\delta \cap H_t}(\theta)| \leq \frac{1}{\sin \left(\arctan \left(\frac{t}{r}\right)\right)} \sup_{\eta \in S^{n-1}} |\rho_K(\eta) - \rho_{K_\delta}(\eta)|,$$

where the upper bound is independent of $\xi \in S^{n-1}$, $t$ with $|t| \leq r/4$, and $\theta \in S^{n-2}$. This inequality, the integral expression (6), and equation (3) imply

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \sup_{|t| \leq \frac{r}{4}} |A_{K_\xi}(t) - A_{K_\delta \xi}(t)| = 0.$$

Lemma (2.4) in [8] establishes the existence of a small neighbourhood of $t = 0$, independent of $\xi \in S^{n-1}$, on which $A_{K_\xi}$ is $m$-smooth. The following is an elaboration of Koldobsky’s proof, so that we may uniformly approximate the derivatives of $A_{K_\xi}$. Again fix $\xi \in S^{n-1}$, and fix $\theta \in S^{n-2} \subset H_t$. Let
\( \rho_{K,\theta} \) denote the \( m \)-smooth restriction of \( \rho_K \) to the two dimensional plane spanned by \( \xi \) and \( \theta \), and consider \( \rho_{K,\theta} \) as a function on \([0, 2\pi]\), where the angle is measured from the positive \( \theta \)-axis. A right triangle then gives the equation

\[
\rho_{K\cap H_t}(\theta) + t^2 = \rho_{K,\theta}^2 \left( \frac{t}{\rho_{K\cap H_t}(\theta)} \right),
\]

which we can use to implicitly differentiate \( y(t) := \rho_{K\cap H_t}(\theta) \) as a function of \( t \). Indeed,

\[
F(t, y) := y^2 + t^2 - \rho_{K,\theta}^2 \left( \arctan \left( \frac{t}{y} \right) \right)
\]
is differentiable away from \( y = 0 \), with

\[
F_y(t, y) = 2y + \frac{2t}{y^2 + t^2} \rho_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) \rho'_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right).
\]
The containment \( B_r(0) \subset K \subset B_R(0) \) implies \( \rho_{K,\theta} \) is bounded above on \( S^{n-1} \) by \( R \), and

\[
\rho_{K\cap H_t}(\theta) \geq \frac{\sqrt{15}r}{4}
\]
for \( |t| \leq r/4 \). If

\[
M = 1 + \sup_{\xi \in S^{n-1}} |\nabla \rho_K(\xi)| < \infty,
\]
and \( \lambda \in \mathbb{R} \) is a constant such that

\[
0 < \lambda < \min \left\{ \frac{15\sqrt{15}r^3}{128RM} \right\},
\]
then

\[
|F_y(t, \rho_{K\cap H_t}(\theta))| > \frac{\sqrt{15}r}{4}
\]
for \( |t| \leq \lambda \). Therefore, by the Implicit Function Theorem, \( y(t) = \rho_{K\cap H_t}(\theta) \) is differentiable on \((-\lambda, \lambda)\), with

\[
y'(t) = \frac{\rho_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) \rho'_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) (y^2 + t^2)^{-1}y - t}{y + t \rho_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) \rho'_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) (y^2 + t^2)^{-1}}.
\]

Recursion shows that \( \rho_{K\cap H_t}(\theta) \) is \( m \)-smooth on \((-\lambda, \lambda)\), independent of \( \xi \in S^{n-1} \) and \( \theta \in S^{n-2} \). It follows from the integral expression \( \text{6} \) that \( A_{K,\xi} \) is \( m \)-smooth on \((-\lambda, \lambda)\) for every \( \xi \in S^{n-1} \). This argument also shows that \( A_{K,\delta,\xi} \) is \( m \)-smooth on the same interval, for \( \delta > 0 \) small enough. Using the resulting expressions for the derivatives of \( A_{K,\xi} \) and \( A_{K,\delta,\xi} \), and applying equations \( \text{3}, \text{5}, \text{6} \), and the inequality \( \text{7} \), we have

\[
limit_{\delta \to 0} \sup_{\xi \in S^{n-1}} \sup_{|t| \leq \lambda} \left| A_{K,\xi}^{(k)}(t) - A_{K,\delta,\xi}^{(k)}(t) \right| = 0
\]
for $k = 1, \ldots, m$.

Finally, for any $p \in \mathbb{R}$ such that $-1 < p < m$ and $p \neq 0, 1, \ldots, m - 1$, we will uniformly approximate $A_{K,\xi}^{(p)}(0)$. With $\lambda > 0$ as chosen above, we have

$$A_{K,\xi}^{(p)}(0) = \frac{1}{\Gamma(-p)} \int_0^\lambda t^{-1-p} \left( A_{K,\xi}(-t) - \sum_{k=0}^{m-1} \frac{(-1)^k A_{K,\xi}^{(k)}(0)}{k!} t^k \right) dt$$

$$+ \frac{1}{\Gamma(-p)} \int_\lambda^\infty t^{-1-p} A_{K,\xi}(-t) dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k \lambda^{k-p} A_{K,\xi}^{(k)}(0)}{k!(k-p)}.$$

The first integral in this equation can be rewritten as

$$\int_0^\lambda t^{-1-p} \int_0^t \frac{A_{K,\xi}^{(m)}(-z)}{(m-1)!} (t-z)^{m-1} dz dt,$$

using the integral form of the remainder in Taylor’s Theorem. We also have

$$\int_\lambda^\infty t^{-1-p} A_{K,\xi}(-t) dt$$

$$= \int_{K \cap \{x, -\xi\} \geq \lambda} \langle x, -\xi \rangle^{-1-p} dx$$

$$= \int_{B_K(\xi)} \langle \eta, -\xi \rangle^{-1-p} \rho_K(\eta) \int_{\lambda\langle\eta, -\xi\rangle^{-1}} r^{n-2-p} dr d\eta$$

$$= \frac{1}{n-1-p} \int_{B_K(\xi)} \left( \langle \eta, -\xi \rangle^{-1-p} \rho_K^{n-1-p}(\eta) - \lambda^{n-1-p} \langle \eta, -\xi \rangle^{-n} \right) d\eta,$$

where

$$B_K(\xi) = \left\{ \eta \in S^{n-1} \mid \langle \eta, \xi \rangle < 0 \text{ and } \rho_K(\eta) \geq \lambda \langle \eta, -\xi \rangle^{-1} \right\}.$$
Therefore, with the set $B_{K\delta}(\xi)$ defined similarly, we have

$$
\left| A_{K,\xi}^{(p)}(0) - A_{K,\xi}^{(p)}(0) \right| \cdot |\Gamma(-p)|
\leq \frac{1}{(m-1)!} \left( \sup_{|z| \leq \lambda} \left| A_{K,\xi}^{(m)}(z) - A_{K,\xi}^{(m)}(z) \right| \right) \int_0^t \int_0^t t^{-1-p}(t-z)^{m-1} \, dz \, dt \tag{8}
$$

$$
+ \left( \sup_{\eta \in S_n} \left| p_K^{n-1-p}(\eta) - p_{K\delta}^{n-1-p}(\eta) \right| \right) \int_{B_K(\xi) \cap B_{K\delta}(\xi)} \frac{\langle \eta, -\xi \rangle^{-1-p}}{n-1-p} \, d\eta \tag{9}
$$

$$
+ \int_{B_K(\xi) \setminus B_{K\delta}(\xi)} \left| \frac{\langle \eta, -\xi \rangle^{-1-p} p_K^{n-1-p}(\eta) - \lambda^{n-1-p}(\eta, -\xi)^{n}}{n-1-p} \right| \, d\eta \tag{10}
$$

$$
+ \int_{B_{K\delta}(\xi) \setminus B_K(\xi)} \left| \frac{\langle \eta, -\xi \rangle^{-1-p} p_{K\delta}^{n-1-p}(\eta) - \lambda^{n-1-p}(\eta, -\xi)^{n}}{n-1-p} \right| \, d\eta \tag{11}
$$

$$
+ \sum_{k=0}^{m-1} \frac{\lambda^{k-p}}{k! (k-p)} \left| A_{K,\xi}^{(k)}(0) - A_{K,\xi}^{(k)}(0) \right|
$$

for $\delta > 0$ small enough. The integrals in expressions (8) and (9) are finite, with

$$
\int_0^t \int_0^t t^{-1-p}(t-z)^{m-1} \, dz \, dt = \frac{\lambda^{m-p}}{m(m-p)},
$$

since $p$ is a non-integer less than $m$, and

$$
\int_{B_K(\xi) \cap B_{K\delta}(\xi)} \langle \eta, -\xi \rangle^{-1-p} \, d\eta \leq \left( \frac{R}{\lambda} \right)^{1+p} \omega_n.
$$

Furthermore, the integrands in expression (10) and (11) are bounded above by

$$
\left( \frac{2R}{\lambda} \right)^{1+p} (2R)^{n-1-p} + \lambda^{n-1-p} \left( \frac{2R}{\lambda} \right)^n \quad \text{if } p < n-1,
$$

and

$$
\left( \frac{2R}{\lambda} \right)^{1+p} \left( \frac{R}{2} \right)^{n-1-p} + \lambda^{n-1-p} \left( \frac{2R}{\lambda} \right)^n \quad \text{if } p > n-1,
$$

noting that $B_{r/2}(0) \subset K_\delta \subset B_{2R}(0)$ for $\delta < 1/2$.

It is now sufficient to prove

$$
\lim_{\delta \to 0} \sup_{\xi \in S_n} \int_{B_K(\xi) \cap B_{K\delta}(\xi)} \chi_{B(\xi, \delta)} \, d\eta = 0,
$$

where

$$
B(\xi, \delta) = B_K(\xi) \Delta B_{K\delta}(\xi)
$$

$$
= \left\{ \eta \in S_n \left| \rho_K(\eta) \geq \frac{\lambda}{\langle \eta, -\xi \rangle} > \rho_{K\delta}(\eta) \text{ or } \rho_{K\delta}(\eta) \geq \frac{\lambda}{\langle \eta, -\xi \rangle} > \rho_K(\eta) \right\}.
$$
We will prove the equivalent statement
\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} \chi_{B(-\xi, \delta)} \, d\eta = 0,
\]
where the sign of \( \xi \) has changed, so that we may use Figure 1.

Towards this end, fix any \( \theta \in S^{n-2} \), and consider Figure 1 specifically when \( t = \lambda \). In this case,
\[
|\overline{OA}| = \rho_K(\eta_2) = \lambda(\eta_2, \xi)^{-1} \quad \text{and} \quad |\overline{OC}| = \rho_{K_\delta}(\eta_1) = \lambda(\eta_1, \xi)^{-1}
\]
or
\[
|\overline{OC}| = \rho_K(\eta_2) = \lambda(\eta_2, \xi)^{-1} \quad \text{and} \quad |\overline{OA}| = \rho_{K_\delta}(\eta_1) = \lambda(\eta_1, \xi)^{-1}.
\]
Any \( \eta \in B(-\xi, \delta) \) lying in the right half-plane spanned by \( \xi \) and \( \theta \) will lie between \( \eta_1 \) and \( \eta_2 \). Furthermore, the angle \( \omega \) converges to zero as \( \delta \) approaches zero, uniformly with respect to \( \xi \in S^{n-1} \) and \( \theta \in S^{n-2} \). Indeed, we have
\[
0 \leq \sin \omega \leq \frac{2 \sin \beta \sin \gamma}{r \sin \alpha} |\rho_K(\eta_1) - \rho_{K_\delta}(\eta_1)|,
\]
using the fact that both \( K \) and \( K_\delta \) contain a ball of radius \( r/2 \), and with \( \sin \alpha \) uniformly bounded away from zero as before. It follows that the spherical measure of \( B(-\xi, \delta) \) converges to zero as \( \delta \) approaches zero, uniformly with respect to \( \xi \in S^{n-1} \).

\[\square\]

**Lemma 9.** Let \( K \subset \mathbb{R}^n \) be a convex body contained in a ball of radius \( R \), and containing a ball of radius \( r \), where both balls are centred at the origin. If
\[
L(n) = 8(n - 1)\pi^{\frac{n-1}{2}} \left[ \Gamma \left( \frac{n + 1}{2} \right) \right]^{-1},
\]
then
\[
|A_{K, \xi}(t) - A_{K, \xi}(s)| \leq L(n) R^{n-1} r^{-1} |t - s|
\]
for all \( s, t \in [-r/2, r/2] \) and \( \xi \in S^{n-1} \).

**Proof.** For \( \xi \in S^{n-1} \), Brunn’s Theorem implies \( f := A_{K, \xi}^{-1} \) is concave on its support, which includes the interval \([-r, r]\). Let
\[
L_0 = \max \left\{ \left| \frac{f \left( -\frac{3r}{4} \right) - f(-r)}{-\frac{3r}{4} - (-r)} \right|, \left| \frac{f(r) - f \left( \frac{3r}{4} \right)}{r - \frac{3r}{4}} \right| \right\},
\]
and suppose \( s, t \in [-r/2, r/2] \) are such that \( s < t \). If
\[
\frac{f(t) - f(s)}{t - s} > 0,
\]
then
\[
\frac{f \left( -\frac{3r}{4} \right) - f(-r)}{-\frac{3r}{4} - (-r)} \geq \frac{f(s) - f \left( -\frac{3r}{4} \right)}{s - \left( -\frac{3r}{4} \right)} \geq \frac{f(t) - f(s)}{t - s} > 0;
\]
and
otherwise, we will obtain a contradiction of the concavity of \( f \). Similarly, if
\[
\frac{f(t) - f(s)}{t - s} < 0,
\]
then
\[
\frac{f(r) - f(\frac{3r}{4})}{r - \frac{3r}{4}} \leq \frac{f(\frac{3r}{4}) - f(t)}{\frac{3r}{4} - t} \leq \frac{f(t) - f(s)}{t - s} < 0.
\]
Therefore,
\[
|A_{\frac{1}{n-1}K,\xi}(t) - A_{\frac{1}{n-1}K,\xi}(s)| \leq L_0 |t - s|
\]
for all \( s, t \in [-r/2, r/2] \). Now, we have
\[
|A_{K,\xi}(t) - A_{K,\xi}(s)| \leq (n - 1) \left( \max_{\xi(t) \in \mathbb{R}} A_{K,\xi}(t_0) \right) \frac{n-2}{n-1} \left| A_{\frac{1}{n-1}K,\xi}(t) - A_{\frac{1}{n-1}K,\xi}(s) \right|
\]
by the Mean Value Theorem, and
\[
L_0 \leq \frac{4}{r} \cdot 2 \left( \max_{\xi(t) \in \mathbb{R}} A_{K,\xi}(t_0) \right) ^{\frac{1}{n-1}} = \frac{8}{r} A_{\frac{1}{n-1}K,\xi}(t_0) (t_0(\xi)).
\]
Finally, since \( K \) is contained in a ball of radius \( R \), we have
\[
A_{K,\xi}(t_0(\xi)) \leq \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} R^{n-1}.
\]
Combining these inequalities gives
\[
|A_{K,\xi}(t) - A_{K,\xi}(s)| \leq L(n) R^{n-1} r^{-1} |t - s|
\]
for all \( s, t \in [-r/2, r/2] \) and \( \xi \in S^{n-1} \). \( \square \)

We now prove two lemmas that will be the core of the proof of Theorem 2.

**Lemma 10.** Let \( K \) be a convex body in \( \mathbb{R}^n \) contained in a ball of radius \( R \), and containing a ball of radius \( r \), where both balls are centred at the origin. Let \( \{K_\delta\}_{0<\delta<1} \) be as in Lemma 8. If there exists \( 0 < \varepsilon < \frac{r^2}{16} \) so that
\[
\rho(CK, IK) \leq \varepsilon,
\]
then, for \( \delta > 0 \) small enough,
\[
\int_{S^1} |A'_{K_\delta,\xi}(0)| \, d\xi \leq \left( 6\pi + \frac{32\pi}{\sqrt{3}} \right) \sqrt{\varepsilon} \quad \text{when } n = 2,
\]
\[
\int_{S^{n-1}} |A'_{K_\delta,\xi}(0)|^2 \, d\xi \leq C(n) \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \sqrt{\varepsilon} \quad \text{when } n \geq 3.
\]
Here, \( C(n) > 0 \) are constants depending only on the dimension.
Proof. By Lemma 8 we may choose $0 < \alpha < 1/2$ small enough so that for every $0 < \delta < \alpha$,

$$
\sup_{\xi \in S^{n-1}} \sup_{|t| \leq r/4} |A_{K,\xi}(t) - A_{K,\xi}(t)| \leq \varepsilon.
$$

We first show that for each $0 < \delta < \alpha$ and $\xi \in S^{n-1}$, there exists a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \leq \sqrt{\varepsilon}$ for which

$$
|A'_{K,\xi}(c_\delta(\xi))| \leq 3\sqrt{\varepsilon}.
$$

Indeed, if $\xi \in S^{n-1}$ is such that $|t_{K,\xi}(\xi)| \leq \sqrt{\varepsilon}$, then

$$
A'_{K,\xi}(t_{K,\xi}(\xi)) = 0,
$$

and we may take $c_\delta(\xi) = t_{K,\xi}(\xi)$.

Assume $\xi \in S^{n-1}$ is such that $|t_{K,\xi}(\xi)| > \sqrt{\varepsilon}$. Letting $s$ denote the sign of $t_{K,\xi}(\xi)$, we have

$$
|A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0)| = A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0)
= \left( A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0) \right) + \left( A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(s\sqrt{\varepsilon}) \right)
+ \left( A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0) \right)
\leq \sup_{\xi \in S^{n-1}} \max_{t \in \mathbb{R}} |A_{K,\xi}(t) - A_{K,\xi}(0)| + 2 \sup_{\xi \in S^{n-1}} \sup_{|t| \leq r/4} |A_{K,\xi}(t) - A_{K,\xi}(t)|
\leq 3\varepsilon.
$$

It then follows from the Mean Value Theorem that there is a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \leq \sqrt{\varepsilon}$ for which

$$
|A'_{K,\xi}(c_\delta(\xi))| = \frac{A_{K,\xi}(s\sqrt{\varepsilon}) - A_{K,\xi}(0)}{\sqrt{\varepsilon} - 0} \leq 3\sqrt{\varepsilon}.
$$

With the numbers $c_\delta(\xi)$ as above, for the case $n = 2$ we have

$$
\int_{S^1} |A'_{K,\xi}(0)| d\xi
\leq \int_{S^1} \left( |A'_{K,\xi}(c_\delta(\xi))| + \int_{c_\delta(\xi)}^0 A''_{K,\xi}(t) dt \right) d\xi
\leq 6\pi \sqrt{\varepsilon} + \int_{S^1} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |A''_{K,\xi}(t)| dt d\xi. \tag{12}
$$

When $0 < \delta < 1/2$, $K_\delta$ is contained in a ball of radius $2R$, and contains a ball of radius $r/2$. Lemma 9 then implies

$$
\sup_{\xi \in S^{n-1}} \sup_{t \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})} |A'_{K,\xi}(t)| \leq \frac{2L(n)(2R)^{n-1}}{r}.
$$
So, when \( n \geq 3 \),
\[
\int_{S^{n-1}} |A_{K_{\delta},\xi}(0)|^2 \, d\xi
\]
\[
\leq \int_{S^{n-1}} \left( |A_{K_{\delta},\xi}(c_\delta(\xi))|^2 + \left| \int_{c_\delta(\xi)}^0 2A''_{K_{\delta},\xi}(t)A_{K_{\delta},\xi}(t) \, dt \right| \right) \, d\xi
\]
\[
\leq 9 \omega_n \varepsilon + \frac{4L(n)(2R)^{n-1}}{r} \int_{S^{n-1}} \int_{-\sqrt{r}}^{\sqrt{r}} |A''_{K_{\delta},\xi}(t)| \, dt \, d\xi
\]  
(13)

Considering inequalities (12) and (13), we still need to bound
\[
\int_{S^{n-1}} \int_{-\sqrt{r}}^{\sqrt{r}} |A''_{K_{\delta},\xi}(t)| \, dt \, d\xi
\]
for arbitrary \( n \). Rearranging the equation
\[
\frac{d^2}{dt^2}A_{n-1}^{-1}_{K_{\delta},\xi}(t) = \frac{d}{dt} \left( \frac{1}{n-1} A_{n-1}^{-1}_{K_{\delta},\xi}(t) A'_{K_{\delta},\xi}(t) \right)
\]
\[
= \frac{2 - n}{(n-1)^2} A_{n-1}^{-1}_{K_{\delta},\xi}(t) \left( A'_{K_{\delta},\xi}(t) \right)^2 + \frac{1}{n-1} A_{n-1}^{-1}_{K_{\delta},\xi}(t) A''_{K_{\delta},\xi}(t)
\]
gives
\[
A''_{K_{\delta},\xi}(t) = (n-1)A_{n-1}^{-1}_{K_{\delta},\xi}(t) \frac{d^2}{dt^2} A_{n-1}^{-1}_{K_{\delta},\xi}(t) + \frac{n - 2}{n - 1} \left( A'_{K_{\delta},\xi}(t) \right)^2.
\]

Brunn’s Theorem implies that the second derivative of \( A_{n-1}^{-1} \) is non-positive for \( |t| < r \), so
\[
|A''_{K_{\delta},\xi}(t)| \leq (1-n)A_{n-1}^{-1}_{K_{\delta},\xi}(t) \frac{d^2}{dt^2} A_{n-1}^{-1}_{K_{\delta},\xi}(t) + \frac{n - 2}{n - 1} \left( A'_{K_{\delta},\xi}(t) \right)^2
\]
\[
= -A''_{K_{\delta},\xi}(t) + 2 \left( \frac{n - 2}{n - 1} \right) \left( A'_{K_{\delta},\xi}(t) \right)^2.
\]

Because \( K_{\delta} \) contains a ball of radius \( r/2 \) centred at the origin, we have
\[
A_{K_{\delta},\xi}(t) \geq \frac{1}{\Gamma \left( \frac{n+1}{2} \right)} \left( \frac{3\pi r^2}{16} \right)^{\frac{n-1}{2}}
\]
for \( |t| \leq r/4 \), and so
\[
\frac{n - 2}{n - 1} \left( A'_{K_{\delta},\xi}(t) \right)^2 \leq \frac{n - 2}{n - 1} \Gamma \left( \frac{n + 1}{2} \right) \left( \frac{2L(n)(2R)^{n-1}}{r} \right)^2 \left( \frac{16}{3\pi r^2} \right)^{\frac{n-1}{2}}
\]
\[
= \frac{\tilde{L}(n) R^{2n-2}}{r^{n+1}}
\]
for all $|t| \leq \sqrt{\varepsilon}$, where $\tilde{L}(n)$ is a constant depending only on $n$. Therefore,

$$
\int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |A''_{K_{\delta}}(t)| \, dt \, d\xi \\
\leq \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left( - A''_{K_{\delta}}(t) \right) \, dt \, d\xi + \frac{4 \omega_n \tilde{L}(n) R^{2n-2}}{r^{n+1}} \sqrt{\varepsilon}.
$$

(14)

We will bound the first term on the final line above using formula (1). Letting

$$
\tilde{C}(n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)},
$$

formula (1) becomes

$$
f_{K_{\delta}}(t) = \tilde{C}(n) \int_{S^{n-1}} \int_{|t|}^{\rho_{K_{\delta}}(\xi)} \frac{1}{r} \left( 1 - \frac{t^2}{r^2} \right)^{\frac{n-3}{2}} \, r^{n-1} \, dr \, d\xi
$$

$$=
\tilde{C}(n) \int_{S^{n-1}} \int_{|t|}^{\rho_{K_{\delta}}(\xi)} r \left( r^2 - t^2 \right)^{\frac{n-3}{2}} \, dr \, d\xi
$$

$$=
\frac{\tilde{C}(n)}{(n-1)} \int_{S^{n-1}} \left( \rho_{K_{\delta}}^2(\xi) - t^2 \right)^{\frac{n-3}{2}} \, d\xi.
$$

The derivatives of $A_{K_{\delta},\xi}$ and $\left( \rho_{K_{\delta}}^2(\xi) - t^2 \right)^{\frac{n-3}{2}}$ are bounded on $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ uniformly with respect to $\xi \in S^{n-1}$, so

$$f'_{K_{\delta}}(t) = \frac{1}{\omega_n} \int_{S^{n-1}} A'_{K_{\delta},\xi}(t) \, d\xi = -\tilde{C}(n) t \int_{S^{n-1}} \left( \rho_{K_{\delta}}^2(\xi) - t^2 \right)^{\frac{n-3}{2}} \, d\xi.
$$

Observing $\tilde{C}(2) = \pi^{-1}$, and using that $0 < \varepsilon < r^2/16$ and $r/2 \leq \rho_{K_{\delta}} \leq 2R$ for $\delta < 1/2$, we have

$$
\left| \int_{S^{n-1}} A'_{K_{\delta},\xi}(\pm \sqrt{\varepsilon}) \, d\xi \right|
= \omega_n \left| f'_{K_{\delta}}(\pm \sqrt{\varepsilon}) \right| = \tilde{C}(n) \omega_n \sqrt{\varepsilon} \int_{S^{n-1}} \left( \rho_{K_{\delta}}^2(\xi) - \varepsilon \right)^{\frac{n-3}{2}} \, d\xi
\leq \begin{cases} 
16 \pi \left( \sqrt{3} r \right)^{-1} \sqrt{\varepsilon} & \text{if } n = 2, \\
\tilde{C}(n) \omega_n^2 (2R)^{n-3} \sqrt{\varepsilon} & \text{if } n \geq 3.
\end{cases}
$$

This implies

$$
\left| \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} A''_{K_{\delta}}(\xi) \, dt \, d\xi \right|
= \left| \int_{S^{n-1}} (A'_{K_{\delta},\xi}(- \sqrt{\varepsilon}) - A'_{K_{\delta},\xi}(\sqrt{\varepsilon})) \, d\xi \right|
\leq \begin{cases} 
32 \pi \left( \sqrt{3} r \right)^{-1} \sqrt{\varepsilon} & \text{if } n = 2, \\
2 \tilde{C}(n) \omega_n^2 (2R)^{n-3} \sqrt{\varepsilon} & \text{if } n \geq 3.
\end{cases}
$$

(15)
Noting that $\tilde{L}(2) = 0$, inequalities (12), (14), and (15) give
\[
\int_{S^1} |A'_{K,\xi}(0)| \, d\xi \leq \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon}
\]
when $n = 2$. For $n \geq 3$, inequalities (13), (14), and (15) give
\[
\int_{S^{n-1}} |A'_{K,\xi}(0)|^2 \, d\xi \leq C(n) \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \sqrt{\varepsilon},
\]
where $C(n)$ is a constant depending on $n$. \hfill \Box

**Lemma 11.** Let $K$ and $L$ be infinitely smooth convex bodies in $\mathbb{R}^n$ which are contained in a ball of radius $R$, and contain a ball of radius $r$, where both balls are centred at the origin. Let $p \in (0, n)$. If $\varepsilon > 0$ is such that
\[
\left\| I_p \left( \|\xi\|_{K}^{-n+p} - \|\xi\|_{L}^{-n+p} \right) \right\|_2 \leq \varepsilon,
\]
then when $n \leq 2p$,
\[
\rho(K, L) \leq C(n, p) R^2 r^{-\frac{3n-1+2p}{n+1}} \frac{2}{\varepsilon^{n+1}},
\]
and when $n > 2p$,
\[
\rho(K, L) \leq C(n, p) R^2 r^{-\frac{3n-1+2p}{n+1}} \left( \varepsilon^2 + \frac{R^{2(n+1-p)}}{r^2} \right)^{\frac{n-2p}{(n+2-2p)(n+1)}} \varepsilon^{\frac{n+4}{(n+2-2p)(n+1)}}.
\]
Here, $\| \cdot \|_2$ denotes the norm on $L^2(S^{n-1})$, and $C(n, p) > 0$ are constants depending on the dimension and $p$.

**Proof.** Define the function
\[
f(\xi) := \|\xi\|_{K}^{-n+p} - \|\xi\|_{L}^{-n+p}
\]
on $S^{n-1}$. Towards bounding the radial distance between $K$ and $L$ by $\|f\|_2$, the $L^2(S^{n-1})$ norm of $f$, note that the identity
\[
\rho_K(\xi) - \rho_L(\xi) = \rho_K(\xi)\rho_L(\xi) \left( \|\xi\|_{L} - \|\xi\|_{K} \right)
\]
implies
\[
\left| \rho_K(\xi) - \rho_L(\xi) \right| \leq R^2 \|\xi\|_{K} - \|\xi\|_{L}.
\]
By Theorem 6 we have
\[
\delta_{\infty}(K^\circ, L^\circ) \leq C(n) D^{\frac{2n}{n+1}} \left( \delta_2(K^\circ, L^\circ) \right)^{\frac{n}{n+1}},
\]
where $C(n) > 0$ is a constant depending on $n$, and $D$ is the diameter of $K^\circ \cup L^\circ$. Both $K^\circ$ and $L^\circ$ are contained in a ball of radius $r^{-1}$ centred at the origin. We then have $D \leq 2r^{-1}$, and
\[
\sup_{\xi \in S^{n-1}} \left| \|\xi\|_{K} - \|\xi\|_{L} \right| \leq C(n) r^{\frac{1-n}{n+1}} \left( \int_{S^{n-1}} \left( \|\xi\|_{K} - \|\xi\|_{L} \right)^2 \, d\xi \right)^{\frac{1}{n+1}}
\]
for some new constant \(C(n)\). There exists a function \(g : S^{n-1} \to \mathbb{R}\) such that

\[
(||\xi||K - ||\xi||L)g(\xi) = ||\xi||_K^{-n+p} - ||\xi||_L^{-n+p}.
\]

If \(\xi \in S^{n-1}\) is such that \(||\xi||K \neq ||\xi||L\), then an application of the Mean Value Theorem to the function \(t^{-n+p}\) on the interval bounded by \(||\xi||K\) and \(||\xi||L\) gives

\[
|g(\xi)| \geq (n-p)\left(\max\{||\xi||K, ||\xi||L\}\right)^{-n+1+p} \geq (n-p)p^{n+1-p}.
\]

Therefore,

\[
||\xi||_K - ||\xi||_L \leq (n-p)^{-1}p^{-n+1+p}|f(\xi)|.
\]

Combining the above inequalities, we get

\[
\sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)| \leq C(n,p)R^{2(p-n+1+p)} \left\| f \right\|_p^{2(p-n+1+p)}, \quad (16)
\]

for some constant \(C(n,p)\).

We now compare the \(L^2\) norm of \(f\) to that of \(I_p(f)\) by considering two separate cases based on the dimension \(n\), as in the proof of Theorem 3.6 in [6]. In both cases, we let \(\sum_{m=0}^\infty Q_m\) be the condensed harmonic expansion for \(f\), and let \(\lambda_m(n,p)\) be the eigenvalues from Lemma 7. As in [6], the condensed harmonic expansion for \(I_pf\) is then given by \(\sum_{m=0}^\infty \lambda_m(n,p)Q_m\).

Assume \(n \leq 2p\). An application of Stirling’s formula to the equations given in Lemma 7 shows that \(\lambda_m(n,p)\) diverges to infinity as \(m\) approaches infinity. The eigenvalues are also non-zero, so there is a constant \(C(n,p)\) such that \(C(n,p)\left|\lambda_m(n,p)\right|^2\) is greater than one for all \(m\). Therefore,

\[
\left\| f \right\|_2^2 = \sum_{m=0}^\infty \left\| Q_m \right\|_2^2 \\
\leq C(n,p) \sum_{m=0}^\infty \left|\lambda_m(n,p)\right|^2 \left\| Q_m \right\|_2^2 = C(n,p)\left\| I_p(f) \right\|_2^2 \leq C(n,p)\varepsilon^2.
\]

Combining this inequality with (16) gives the first estimate in the theorem.

Assume \(n > 2p\). Hölder’s inequality gives

\[
\left\| f \right\|_2^2 = \sum_{m=0}^\infty \left\| Q_m \right\|_2^2 \\
= \sum_{m=0}^\infty \left(\left|\lambda_m(n,p)\right|^{n+2-2p} \left\| Q_m \right\|_2^{\frac{n+2-2p}{2}}\right) \cdot \left(\left|\lambda_m(n,p)\right|^{-n+2-2p} \left\| Q_m \right\|_2^{-\frac{n+2-2p}{2}}\right) \\
\leq \left(\sum_{m=0}^\infty \left|\lambda_m(n,p)\right|^2 \left\| Q_m \right\|_2^2\right)^{\frac{n-2p}{n+2-2p}} \cdot \left(\sum_{m=0}^\infty \left|\lambda_m(n,p)\right|^{-n+2-2p} \left\| Q_m \right\|_2^{-2}\right)^{-\frac{n+2-2p}{n-2p}}.
\]
where we again note that the eigenvalues are all non-zero. It follows from Lemma [7] and Stirling’s formula that there is a constant $C(n, p)$ such that

$$|\lambda_m(n, p)|^{\frac{-4}{n-2p}} \leq C(n, p)$$

for all $m \geq 1$, and

$$|\lambda_0(n, p)|^{\frac{-4}{n-2p}} \leq C(n, p).$$

Using the identity

$$\|\nabla o f\|_2^2 = \sum_{m=1}^{\infty} m(m+n-2)\|Q_m\|_2^2$$

(17)
given by Corollary 3.2.12 in [7], we then have

$$\|f\|_2^2 \leq C(n, p) \left( \|I_{p}(f)\|_2^{\frac{n+2-2p}{n+2-2p}} (\|Q_0\|_2^2 + \|\nabla o f\|_2^{\frac{n-2p}{n+2-2p}}) \right).$$

The Minkowski functional of a convex body is the support function of the corresponding polar body, so

$$\nabla o \|\xi\|_K^{-n+p} = (-n + p)\|\xi\|_K^{-n-1+p} \nabla o h_{K^0}(\xi).$$

Because $K^0$ is contained in a ball of radius $r^{-1}$, it follows from Lemma 2.2.1 in [7] that

$$|\nabla o h_{K^0}(\xi)| \leq 2r^{-1}$$

for all $\xi \in S^{n-1}$. We now have

$$\left\| \nabla o \|\xi\|_K^{-n+p} \right\|_2^2 \leq 4(n-p)^2 R^{2(n+1-p)} r^{-2} \omega_n.$$

This constant bounds the squared $L^2$ norm of $\nabla o \|\xi\|_K^{-n+p}$ as well, so

$$\|\nabla o f\|_2^2 \leq 16(n-p)^2 R^{2(n+1-p)} r^{-2} \omega_n.$$

Therefore,

$$\|f\|_2^2 \leq C(n, p) \varepsilon^{\frac{-4}{n+2-2p}} \left( \varepsilon^2 + R^{2(n+1-p)} r^{-2} \right)^{\frac{n-2p}{n+2-2p}},$$

where the constant $C(n, p) > 0$ is different from before. This inequality with (16) gives the second estimate in the theorem. □

4. PROOFS OF STABILITY RESULTS

We are now ready to prove our stability results.

**Proof of Theorem 2** Let $\{K_\delta\}_{0 < \delta < \alpha}$ be the family of smooth convex bodies from Lemma [8]. We will show that $\rho(K_\delta, -K_\delta)$ is small for $0 < \delta < \alpha$, where $\alpha$ is the constant from the proof of Lemma [10]. The bounds in the theorem will then follow from

$$\rho(K, -K) \leq \lim_{\delta \to 0} \left( 2\rho(K, K_\delta) + \rho(K_\delta, -K_\delta) \right) = \lim_{\delta \to 0} \rho(K_\delta, -K_\delta).$$
Figure 2. $K_\delta$ is a convex body in $\mathbb{R}^2$, and $\xi \in S^1$.

We begin by separately considering the case $n = 2$. Let the radial function $\rho_{K_\delta}$ be a function of the angle measured counter-clockwise from the positive horizontal axis. For any $\xi \in S^1$, let the angles $\phi_1$ and $\phi_2$ be functions of $t \in (-r, r)$ as indicated in Figure 3. If $\xi$ corresponds to the angle $\theta$, then the parallel section function for $K_\delta$ may be written as

$$A_{K_\delta, \theta}(t) = \rho_{K_\delta}(\theta + \phi_1) \sin \phi_1 + \rho_{K_\delta}(\theta - \phi_2) \sin \phi_2.$$  

Implicit differentiation of

$$\cos \phi_j = \frac{t}{\rho_{K_\delta}(\theta - (-1)^j \phi_j)} \quad (j = 1, 2)$$

gives

$$\left. \frac{d\phi_j}{dt} \right|_{t=0} = \frac{(-1)}{\rho_{K_\delta}(\theta - (-1)^j \frac{\pi}{2})},$$

so

$$A'_{K_\delta, \theta}(0) = -\frac{\rho'_{K_\delta}(\theta + \frac{\pi}{2})}{\rho_{K_\delta}(\theta + \frac{\pi}{2})} + \frac{\rho'_{K_\delta}(\theta - \frac{\pi}{2})}{\rho_{K_\delta}(\theta - \frac{\pi}{2})}.$$

Since $f(\phi) := \rho_{K_\delta}(\phi + \pi/2) - \rho_{K_\delta}(\phi - \pi/2)$ is a continuous function on $[0, \pi]$ with

$$f(0) = \rho_{K_\delta}(\pi/2) - \rho_{K_\delta}(-\pi/2) = -(\rho_{K_\delta}(-\pi/2) - \rho_{K_\delta}(\pi/2)) = -f(\pi),$$

there exists an angle $\theta_0 \in [0, \pi]$ such that $\rho_{K_\delta}(\theta_0 + \pi/2) = \rho_{K_\delta}(\theta_0 - \pi/2)$. With this $\theta_0$, we get the inequality

$$\left| \int_{\theta_0}^{\theta} \left( -\frac{\rho'_{K_\delta}(\phi + \frac{\pi}{2})}{\rho_{K_\delta}(\phi + \frac{\pi}{2})} + \frac{\rho'_{K_\delta}(\phi - \frac{\pi}{2})}{\rho_{K_\delta}(\phi - \frac{\pi}{2})} \right) d\phi \right| \leq \int_{\theta_0}^{2\pi} |A'_{K_\delta, \phi}(0)| d\phi.$$
Integrating the left side of this inequality, and applying Lemma 10 to the right side, gives

\[ \left| \log \left( \frac{\rho_{K_{\delta}}(\theta - \frac{\pi}{2})}{\rho_{K_{\delta}}(\theta + \frac{\pi}{2})} \right) \right| \leq \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon}. \]

This implies

\[ 1 - \exp \left[ \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon} \right] \leq \exp \left[ - \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon} \right] - 1 \leq \frac{\rho_{K_{\delta}}(\theta - \frac{\pi}{2})}{\rho_{K_{\delta}}(\theta + \frac{\pi}{2})} - 1 \]

\[ \leq \exp \left[ \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon} \right] - 1. \]

It follows that

\[ -2 \left( \exp \left[ \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon} \right] - 1 \right) R \leq \rho_{K_{\delta}}(\theta - \frac{\pi}{2}) - \rho_{K_{\delta}}(\theta + \frac{\pi}{2}) \]

\[ \leq 2 \left( \exp \left[ \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon} \right] - 1 \right) R, \]

since \( K_{\delta} \) is contained in a ball of radius \( 2R \). Viewing \( \rho_{K_{\delta}} \) again as a function of vectors, we have

\[ \sup_{\xi \in S^1} |\rho_{K_{\delta}}(\xi) - \rho_{K_{\delta}}(-\xi)| \leq 2 \left( \exp \left[ \left( 6\pi + \frac{32\pi}{\sqrt{3} r} \right) \sqrt{\varepsilon} \right] - 1 \right) R. \]

The inequality \( e^t - 1 \leq 2t \) is valid when \( 0 < t < 1 \); therefore, if

\[ \varepsilon < \left( \frac{\sqrt{3} r}{6\sqrt{3} \pi r + 32\pi} \right)^2, \]

then

\[ \sup_{\xi \in S^1} |\rho_{K_{\delta}}(\xi) - \rho_{K_{\delta}}(-\xi)| \leq \left( 24\pi + \frac{128\pi}{\sqrt{3} r} \right) R \sqrt{\varepsilon}. \]

Consider the case when \( n > 2 \). For \( K_{\delta} \) with \( p = 1 \), Equation (2) becomes

\[ I_2 \left( \|x\|_{K_{\delta}}^{-n+2} - \| - x\|_{K_{\delta}}^{-n+2} \right)(\xi) = -2\pi i \left( n - 2 \right) A'_{K_{\delta},\xi}(0), \]

so

\[ \left\| I_2 \left( \|x\|_{K_{\delta}}^{-n+2} - \| - x\|_{K_{\delta}}^{-n+2} \right) \right\|_2 = 2\pi (n - 2) \left( \int_{S^{n-1}} |A'_{K_{\delta},\xi}(0)|^2 d\xi \right)^{\frac{1}{2}} \]

\[ \leq \tilde{C}(n) \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \]
by Lemma 10. Finally, by Lemma 11
\[ \rho(K_\delta, -K_\delta) \leq C(n) \frac{R^2}{r^{n+1}} \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \frac{1}{\varepsilon^{2(n+1)}} \]
when \( n = 3 \) or 4, and
\[ \rho(K_\delta, -K_\delta) \leq C(n) \left[ \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \frac{1}{\varepsilon^{(n-2)(n+1)}} \right] \]
\[ \cdot \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \frac{1}{\varepsilon^{(n-2)(n+1)}} \]
when \( n \geq 5 \), where \( C(n) > 0 \) are constants depending on the dimension. \( \square \)

We now present the proof of our second stability result.

**Proof of Theorem 2.** Apply Lemma 8 to \( K \) and \( L \); let \( \{K_\delta\}_{0<\delta<1} \) and \( \{L_\delta\}_{0<\delta<1} \) be the resulting families of smooth convex bodies. For each \( \delta \), define the constant
\[ \varepsilon_\delta := \sup_{\xi \in S^{n-1}} \left| \mathcal{A}_{K,\delta,\xi}^{(p)}(0) - \mathcal{A}_{K,\xi}^{(p)}(0) \right| + \sup_{\xi \in S^{n-1}} \left| \mathcal{A}_{L,\delta,\xi}^{(p)}(0) - \mathcal{A}_{L,\xi}^{(p)}(0) \right| + \varepsilon. \]
Defining the auxiliary function
\[ f_\delta(\xi) := \|\xi\|_{K_\delta}^{-n+1+p} - \|\xi\|_{L_\delta}^{-n+1+p}, \]
we have
\[ \cos \left( \frac{\pi}{2} \right) I_{1+p}(f_\delta(x) + f_\delta(-x))(\xi) + i \sin \left( \frac{\pi}{2} \right) I_{1+p}(f_\delta(x) - f_\delta(-x))(\xi) \]
\[ = 2\pi(n-1-p) \left( \mathcal{A}_{K,\delta,\xi}^{(p)}(0) - \mathcal{A}_{L,\delta,\xi}^{(p)}(0) \right) \]
from Equation 2. The function of \( \xi \) on the left side of this equality is split into its even and odd parts, because \( I_{1+p} \) preserves even and odd symmetry. Therefore,
\[ \frac{\cos \left( \frac{\pi}{2} \right)}{\pi(n-1-p)} I_{1+p}(f_\delta(x) + f_\delta(-x))(\xi) \]
\[ = \left( \mathcal{A}_{K,\delta,\xi}^{(p)}(0) - \mathcal{A}_{L,\delta,\xi}^{(p)}(0) \right) + \left( \mathcal{A}_{K,\delta,-\xi}^{(p)}(0) - \mathcal{A}_{L,\delta,-\xi}^{(p)}(0) \right) \]
and
\[ \frac{i \sin \left( \frac{\pi}{2} \right)}{\pi(n-1-p)} I_{1+p}(f_\delta(x) - f_\delta(-x))(\xi) \]
\[ = \left( \mathcal{A}_{K,\delta,\xi}^{(p)}(0) - \mathcal{A}_{L,\delta,\xi}^{(p)}(0) \right) - \left( \mathcal{A}_{K,\delta,-\xi}^{(p)}(0) - \mathcal{A}_{L,\delta,-\xi}^{(p)}(0) \right) \]
By the definition of $\varepsilon_\delta$, 
\[
|I_{1+p}(2f_\delta)(\xi)| \leq |I_{1+p}(f_\delta(x) + f_\delta(-x))(\xi)| + |I_{1+p}(f_\delta(x) - f_\delta(-x))(\xi)| 
\leq \frac{2\pi(n-1-p)}{\cos\left(\frac{p\pi}{2}\right)} \varepsilon_\delta + \frac{2\pi(n-1-p)}{\sin\left(\frac{p\pi}{2}\right)} \varepsilon_\delta,
\]
which implies 
\[
\|I_{1+p}(f_\delta)\|_2 \leq \pi \sqrt{\omega_n} \frac{1}{n-1-p} \left( \left| \sec\left(\frac{p\pi}{2}\right) \right| + \left| \csc\left(\frac{p\pi}{2}\right) \right| \right) \varepsilon_\delta.
\]
Both $K_\delta$ and $L_\delta$ are contained in a ball of radius $2R$ when $0 < \delta < 1/2$, and contain a ball of radius $r/2$. It now follows from Lemma 11 that 
\[
\rho(K_\delta, L_\delta) \leq C(n, p) R^2 \left( \frac{-3n+1}{n+1} \right) \frac{(n-2-2p)}{(n-2p)(n+1)} \varepsilon_\delta
\]
when $n \leq 2p + 2$, and 
\[
\rho(K_\delta, L_\delta) \leq C(n, p) R^2 \left( \frac{-3n+1}{n+1} \right) \left( \varepsilon_\delta^2 + \frac{R^2(n-p)}{n-2p} \right) \frac{(n-2-2p)}{(n-2p)(n+1)} \varepsilon_\delta
\]
when $n > 2p + 2$, where $C(n, p) > 0$ are constants depending on the dimension and $p$. Finally, the bounds in the theorem statement follow from the observations 
\[
\rho(K, L) \leq \lim_{\delta \to 0} \left( \rho(K, K_\delta) + \rho(L, L_\delta) + \rho(K_\delta, L_\delta) \right) = \lim_{\delta \to 0} \rho(K_\delta, L_\delta),
\]
and $\lim_{\delta \to 0} \varepsilon_\delta = \varepsilon$. 

\[\square\]

References

[1] S. Bobkov and A. Koldobsky, On the central limit property of convex bodies, Geometric aspects of functional analysis, 44–52, Lecture Notes in Math. 1807, Springer, Berlin, 2003.
[2] U. Brehm, Convex bodies with non-convex cross-section bodies, Mathematika 46 (1999), 127–129.
[3] U. Brehm and J. Voigt, Asymptotics of cross-sections for convex bodies, Beiträge Algebra Geom. 41 (2000), 437–454.
[4] M. Fradelizi, Hyperplane sections of convex bodies in isotropic position, Beiträge Algebra Geom. 40 (1999), 163-183.
[5] R. J. Gardner, D. Ryabogin, V. Yaskin, and A. Zvavitch, A problem of Klee on inner section functions of convex bodies, J. Differential Geom. 91 (2012), 261–279.
[6] P. Goodey, V. Yaskin, and M. Yaskina, Fourier transforms and the Funk-Hecke theorem in convex geometry, J. London Math. Soc. (2) 80 (2009), 388–404.
[7] H. Groemer, Geometric Applications of Fourier Series and Spherical Harmonics, Cambridge University Press, New York, 1996.
[8] A. Koldobsky, Fourier Analysis in Convex Geometry, American Mathematical Society, Providence RI, 2005.
[9] A. Koldobsky and C. Shane, The determination of convex bodies from derivatives of section functions, Arch. Math. 88 (2007), 279–288.
[10] E. Lutwak, Intersection bodies and dual mixed volumes, Adv. Math. 71 (1988), 232-261.
[11] E. Makai, H. Martini, T. Ődor, Maximal sections and centrally symmetric bodies, Mathematika 47 (2000), 19–30.
[12] H. Martini, Cross-sectional measures, in: Intuitive geometry (Szeged, 1991), 269-310, Colloq. Math. Soc. János Bolyai, 63, North-Holland, Amsterdam, 1994.
[13] M. Meyer, Maximal hyperplane sections of convex bodies, Mathematika 46 (1999), 131-136.
[14] F. Nazarov, D. Ryabogin, and A. Zvavitch, An asymmetric convex body with maximal sections of constant volume, Journal of Amer. Math. Soc. 27 (2014), 43–68.
[15] D. Ryabogin and V. Yaskin, Detecting symmetry in star bodies, J. Math. Anal. Appl. 395 (2012), 509–514.
[16] R. Schneider, Convex bodies: the Brunn-Minkowski theory, Cambridge University Press, Cambridge, 1993.
[17] R. Vitale, $L_p$ metrics for compact, convex sets, J. Approx. Theory 45 (1985), 280–287.

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