Abstract

Since the 1990s, artificial intelligence (AI) systems have achieved ‘superhuman performance’ in major zero-sum games, where winning has an unambiguous definition. However, most economic and social interactions are non-zero-sum, where measuring ‘performance’ is a non-trivial task. In this paper, I introduce a novel benchmark, super-Nash performance, and a solution concept, optimin, whereby every player maximizes their minimal payoff under unilateral profitable deviations of the others. Optimin achieves super-Nash performance in that, for every Nash equilibrium, there exists an optimin where each player not only receives but also guarantees super-Nash payoffs, even if other players deviate unilaterally and profitably from the optimin. Further, optimin generalizes and unifies several key results across domains: it coincides with (i) the maximin strategies in zero-sum games, and (ii) the core in cooperative games when the core is nonempty, though it exists even if the core is empty; additionally, optimin generalizes (iii) Nash equilibrium in n-person constant-sum games. Finally, optimin is consistent with the direction of non-Nash deviations in games in which cooperation has been extensively studied, including the finitely repeated prisoner’s dilemma, the centipede game, the traveler’s dilemma, and the finitely repeated public goods game.

Keywords: maximin criterion, noncooperative games, cooperative games, Nash equilibrium, traveler’s dilemma, public goods game, repeated prisoner’s dilemma


1 Introduction and motivating examples

Since the early 1990s, artificial intelligence (AI) systems have gradually achieved ‘superhuman performance’ in major competitive games such as checkers, chess, Go, and poker (Campbell et al., 2002; Schaeffer et al., 2007; Silver et al., 2016; Brown and Sandholm, 2019). These games belong to a class of zero-sum games, where “winning” has a clear definition, and approximating a Nash equilibrium typically yields successful performance against human players. However, most economic and social interactions occur in non-zero-sum settings, and there is a gap in the literature that rigorously defines what constitutes successful performance in these type of games from a game-theoretical perspective.

In this paper, I define a novel benchmark for successful performance in $n$-person games. A strategy profile is said to achieve super-Nash performance if each player (i) receives a super-Nash payoff—defined as a payoff exceeding that of a Nash equilibrium—and (ii) guarantees a super-Nash payoff under any unilateral profitable deviation by other players. I justify this definition with both theoretical and empirical arguments. First, a “good” measure of performance should not rely solely on the direct payoffs received by the players but should also consider the counterfactual payoffs—i.e., how well these players would perform if they played against players that “exploited” their strategies. For example, two AIs that play cooperate in a prisoner’s dilemma would receive a payoff strictly greater than their Nash equilibrium payoff. However, they would perform poorly against an opportunistic AI that defects. Second, extensive experimental evidence suggests that humans already achieve super-Nash performance in a variety of non-zero-sum games in which there are gains from cooperation. These include the finitely repeated prisoner’s dilemma, the finitely repeated public goods game, the centipede game, and the traveler’s dilemma (Axelrod, 1980; McKelvey and Palfrey, 1992; Capra et al., 1999; Goeree and Holt, 2001; Rubinstein, 2007; Lugovskyy et al., 2017; Embrey et al., 2017). In these games, humans not only consistently achieve super-Nash payoffs but also guarantee these super-Nash payoffs even when other players anticipate non-Nash behavior and respond “opportunistically.”

I further introduce a novel solution concept, denoted as “optimin,” designed to achieve super-Nash performance in any $n$-person game. This concept extends the maximin strategy introduced by von Neumann (1928) from zero-sum to non-zero-sum contexts. Informally, an optimin is a strategy profile in which each player simultaneously maximizes their minimal payoff, under unilateral profitable deviations by other players. Optimin achieves super-Nash performance in the sense that in every game for each Nash equilibrium there exists an optimin where each player not only receives a super-Nash payoff but also guarantees a super-Nash payoff even if the others deviate unilaterally and profitably from the optimin. As is well-known, a profile of maximin strategies can never Pareto dominate a Nash equilibrium. Sim-
ilarly, a Nash equilibrium can never Pareto dominate an optimin point.

Moreover, optimin generalizes and unifies results across various subfields. Optimin coincides with (i) Wald’s statistical decision-making criterion when Nature is antagonistic (Proposition 2), and (ii) the core in cooperative games when the core is nonempty (Proposition 3), but it even exists when the core is empty (Theorem 2). Additionally, optimin generalizes (iii) Nash equilibrium in \( n \)-person constant-sum games (Remark 3) and (iv) stable matchings in matching models (Remark 5). Furthermore, a pure strategy optimin point exists in every finite \( n \)-person game when the game is restricted to the pure strategies (Remark 4). In contrast, it is well-known that a pure Nash equilibrium does not exist in general and that computing even approximate mixed strategy Nash equilibria is “hard” (Gilboa and Zemel, 1989; Daskalakis, Goldberg, and Papadimitriou, 2009). This is not to say that finding a mixed strategy optimin point is computationally more efficient. But the advantage of the pure strategy optimin is that it suffices to restrict attention to the set of pure strategy profiles, which is a finite set, as opposed to the set of mixed strategy profiles. This property can be especially useful for finding a solution in large games in which dealing with mixed strategies is computationally demanding.

While the theoretical advantages of optimin over Nash equilibrium appear promising, it is not immediately clear how optimin would perform in experimental games where players consistently exhibit non-Nash behavior yet obtain super-Nash payoffs. I analytically derive optimin predictions in four well-studied experimental games—the centipede game, the traveler’s dilemma, the finitely repeated \( n \)-person public goods game, and the finitely repeated prisoner’s dilemma—in which Nash equilibrium and optimin predictions are in stark contrast. I show that, unlike Nash equilibrium, optimin is consistent with the comparative statics of behavior in these games.

1.1 Short formal definition and informal discussion

Let \((\Delta X_i, u_i)_{i \in N}\) be an \( n \)-person noncooperative game in mixed extension. Define the following sets: \( B_i(p) = \{ p'_i \in \Delta X | u_i(p'_i, p_{-i}) > u_i(p_i, p_{-i}) \} \cup \{ p_i \} \) and \( B_{-i}(p) = \times_{j \in N \setminus \{ i \}} B_j(p) \). A strategy profile \( p^* \in \Delta X \) is said to satisfy the optimin criterion, or called an optimin point, if for every player \( i \), \( p^* \) solves the following multi-objective optimization problem.

\[
p^* \in \arg \max_{q \in \Delta X} \inf_{p_{-i} \in B_{-i}(q)} u_i(q, p_{-i}').
\]

In plain words, a strategy profile \( p^* \) is called an optimin if every player simultaneously maximizes their minimal payoff under the unilateral profitable deviations of other players. To elucidate the concept of optimin, consider a situation where \( p^* \) is a Nash equilibrium. By definition, no unilateral profitable deviation exists in
Figure 1: An illustrative game (left) and its minimal payoffs (right).

Note: The unique optimin point is (Top, Left), whereas every strategy is a maximin strategy in this game. The unique Nash equilibrium is (Bottom, Right).

1.2 Illustrative example

To illustrate the optimin criterion, consider the game depicted in Figure 1 (left), where, for simplicity, attention is restricted to pure strategies. Notice that the maximin strategy concept lacks predictive power in this game because every action is a maximin strategy, guaranteeing a payoff of 0. Now consider strategy profile (Top, Left). Even if player 2 (he) has a profitable deviation to ‘Center’, player 1 (she), who plays Top, would still secure 100 at profile (Top, Center). Admittedly, player 2 could also deviate to ‘Right’ —a viable choice under the maximin strategy concept— but doing so would be implausible as he would incur a substantial loss. Thus, the
minimal payoff associated with (Top, Left) is 100 for each player under unilateral profitable deviation of the opponent. For another example, consider strategy profile (Middle, Center), which also seems attractive. The minimal payoff associated with this profile is 0 for player 1 because player 2 could profitably deviate to ‘Right’, in which case player 1 would receive 0 (and vice versa). It emerges that (Top, Left) is the unique optimin point, maximizing the minimal payoffs as illustrated in Figure 1 (right).

It is natural to ask whether it is possible to modify some of the payoffs in this game and make the Nash equilibrium of the modified game somewhat “better” than the optimin point. Next, I show that this is not possible.

1.3 Optimin and super-Nash performance

Optimin achieves super-Nash performance in the following sense: for every Nash equilibrium $q^*$ in a game $G$ there exists an optimin point $p^*$ such that for every player $i$, $u_i(p^*) \geq u_i(q^*)$ and, more generally, $\inf_{p'_i \in B_{-i}(p^*)} u_i(p^*_i, p'_{-i}) \geq u_i(q^*)$. In simple words, for every Nash equilibrium in a game there is an optimin where every player not only receives but also guarantees a payoff that is at least as high as their Nash equilibrium payoff.

The sketch of the proof proceeds as follows. To reach a contradiction, suppose that there exists a Nash equilibrium $q^*$ for every optimin $p^*$ there exists a player $j$ such that

$$\inf_{p'_j \in B_{-j}(p^*)} u_j(p^*_j, p'_{-j}) < u_j(q^*).$$

(1.1)

However, this implies that $q^* \in \arg \max_{q \in \Delta X} \inf_{p'_i \in B_{-i}(q)} u_i(q_i, p'_{-i})$—i.e., $q^*$ is an optimin point, which leads to a contradiction because inequality 1.1 does not hold for $p^* = q^*$.

It is well-known that for every Nash equilibrium there is a Pareto efficient profile in which every player is (weakly) better off. However, if a player profitably deviates from a Pareto efficient profile—like deviating from $(C,C)$ in the prisoner’s dilemma—it can be disastrous for the non-deviators.

1.4 Economic applications

It is not immediately clear how different optimin and Nash equilibrium predictions can be in economic games. In section 5, I apply optimin to four well-studied experimental games, the traveler’s dilemma, the finitely repeated $n$-person public goods game, and the finitely repeated prisoner’s dilemma, in which Nash equilibrium and optimin predictions are in sharp contrast with each other. I show that the optimin

\[1\]For the calculation of minimal payoffs in each case, see section 3.

\[2\]See Proposition 1 for details. The existence of optimin is shown by Theorem 1.
criterion predictions are in line with the observed comparative statics of human behavior in these games, in stark contrast to the Nash equilibrium, which often fails to do so.

The traveler’s dilemma is a two-person game depicted in Figure 2, where each player selects a number between 2 and 100. The player choosing the smaller number, \( n \), receives \( n + r \), and the other receives \( n - r \). If both choose \( n \), each receives \( n \) (Basu, 1994). In the original game \( r = 2 \). Experimental results show that subjects’ choices depend crucially on the reward parameter \( r \). When \( r \) is “small,” as in the original game, the subjects’ behavior converges towards the maximum number, whereas when \( r \) is “large,” their behavior converges towards the minimum. Nash equilibrium predictions are invariant to \( r \), always selecting the smallest number. By contrast, the optimin point is responsive to \( r \) and is consistent with the direction of play: the unique optimin point coincides with the Nash equilibrium when \( r \) is large, but when \( r \) is small, only the highest pair of numbers satisfies the optimin criterion. The underlying reason is that, as the reward parameter increases, the minimal payoffs of cooperation decrease, and at some point, the minimal payoffs for the maximum number (100) become smaller than the minimal payoffs for the minimum number (2).

The finitely repeated \( n \)-person public goods game is a repeated game in which players simultaneously choose to contribute to a public pot in the stage game. Not contributing anything (i.e., free-riding) is a dominant strategy for every player. However, if everyone contributes (i.e., cooperates), then all participants are better off. Experimental research indicates four key observations about cooperation in this setting: (i) cooperation is significantly greater in finitely repeated public goods games with a high marginal per capita return (MPCR) compared to those with low MPCR; (ii) cooperation decreases as the game progresses; (iii) cooperation restarts if the finitely repeated game is replayed; and (iv) cooperation is amplified by pre-play communication. While the unique subgame perfect equilibrium predicts zero contribution in every round, regardless of parameters such as MPCR, the optimin criterion offers an explanation for these experimental findings. Comparative statics
Figure 3: The performance of SPNE vs the performance of TFT in a $k$-time repeated game ($k = 1, 10, 20, 30$).

Note: In a $k$-time repeated PD, the performance of SPNE is simply $k \times 1$. The performance of TFT is $(k - 1) \times 3$ because a player guarantees $(k - 1) \times 3$ even if the other profitably deviates from the TFT.

on the game’s exogenous parameters reveal two main regularities. First, for high values of MPCR, cooperative behavior satisfies the optimin criterion, whereas for low values, free-riding behavior does. Second, as the game progresses, the minimal payoffs for free-riding approach, and eventually exceed, those for cooperation. However, if the finitely repeated game is replayed, the minimal payoffs for cooperation at the start of the game once again exceed those for free-riding, accounting for the observed “restart” effect.

The finitely repeated prisoner’s dilemma is a well-known two-person repeated game in which defection is the dominant strategy in the stage game. As is well-known, the unique subgame perfect equilibrium prescribes defection in every round. Experimental findings suggest two main regularities: (i) initial cooperation increases as the number of rounds increases, and (ii) cooperation decays as the end of the game approaches. The optimin criterion provides an explanation for these regularities. In the one-shot game, the unique optimin point coincides with the unique Nash equilibrium. However, in the finitely repeated prisoner’s dilemma, strategy profiles such as the Tit-for-Tat (TFT) profile and the grim trigger profile often satisfy the optimin criterion. This is because, even if a player attempts to exploit cooperative behavior, the minimal payoff for the cooperator is generally greater than the subgame perfect equilibrium payoff. As the number of rounds increases, the minimal payoffs for cooperation rise. However, these payoffs gradually
diminish as the game progresses, eventually becoming less than the minimal payoffs for defection, which accounts for the decreasing levels of cooperation in the final rounds.

|       | Cooperate | Defect |
|-------|-----------|--------|
| Cooperate | 3,3      | 0,5   |
| Defect   | 5,0      | 1,1   |

Figure 3 illustrates how the performance of TFT, which is an optimin, grows compared to the performance of the subgame perfect Nash equilibrium (SPNE) as the number of repetitions of the PD with the above stage game increases. For instance, consider the PD repeated for 10 rounds. Because there is a profitable deviation in the last round, the TFT is evidently not an SPNE. Nonetheless, by the last round in which a player deviates from the TFT profile, each player is already guaranteed to receive a payoff of $27 = 9 \times 3$, which is nearly three times the payoff of a player in the unique subgame perfect equilibrium. For a more comprehensive analysis of TFT and other cooperative strategies, see section 5.4.

1.5 Cooperative games

The optimin criterion can also be applied to cooperative games in characteristic function form. When the core is nonempty, an allocation is in the core if and only if it satisfies the optimin criterion. But as I show in subsection 6.1, optimin points exist even when the core is empty. To illustrate, consider the following cooperative game in characteristic function form in which $N = \{1, 2, 3\}$, $u(\{1\}) = 35$, $u(\{2\}) = 30$, $u(\{3\}) = 25$, $u(\{1, 2\}) = 90$, $u(\{1, 3\}) = 80$, $u(\{2, 3\}) = 70$, and $u(N) = 110$, where $u(S)$ denotes the worth of coalition $S$. Although the core of this game is empty, points satisfying the optimin criterion can be characterized by the following set, as depicted in Figure 4:\(^3\)

$$\{x \in \mathbb{R}^3 \mid x_1 = 40, x_2 + x_3 = 70, x_2 \geq 30, x_3 \geq 25\}.$$  

The Shapley value is (44.16, 36.66, 29.16), and the nucleolus is (46.66, 36.66, 26.66). Observe that under each of these solutions, player 2 and player 3 can profitably break away from the grand coalition to receive a joint payoff of 70.\(^4\) As a result, the minimal payoff of player 1 would be equal to her individual payoff, $u(\{1\}) = 35$, under both the Shapley value and the nucleolus. Notably, under the optimin criterion, player 1 receives less than both the Shapley value and the nucleolus. However, this reduction is counterbalanced by the fact that coalition $\{2, 3\}$ does not

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\(^3\)For the formal definition and calculations, see subsection 6.1.

\(^4\)This is not surprising because the Shapley value is generally regarded as an a priori assessment of the game.
have any incentive to deviate from an optimin point, enabling player 1 to secure a (worst-case) payoff of 40, in contrast to the minimal payoff of 35 under the Shapley value or the nucleolus.

2 Relevant literature

The closest concept to the optimin criterion is that of the maximin criterion, which has been studied in different contexts by a number of researchers, including Borel (1921), von Neumann (1928), Wald (1939), and Rawls (1971). There are also axiomatizations of the maximin criterion proposed by including Milnor (1954) and Gilboa and Schmeidler (1989). In their seminal work, Gilboa and Schmeidler (1989) show that maximin expected utility can be characterized by a set of intuitive axioms for cautious decision makers with a set $C$ of (multi-prior) subjective beliefs. Just like one can vary $C$ in the Gilboa-Schmeidler maxmin model and rationalize different choices, one can vary constraint $B_i$ in the optimin model and rationalize different choices in an interactive setting. For another related axiomatization, see Puppe and Schlag (2009), who show that the axioms of Milnor (1954) that characterize the maximin decision rule are consistent with ignoring some states in which all payoffs are “small.”

The literature on solution concepts which incorporates various levels of cautiousness in games includes Selten (1975), Basu and Weibull (1991) and more recently Perea et al. (2006), Renou and Schlag (2010), and Iskakov, Iskakov, and d’Aspremont (2018). Prominent equilibrium concepts under ambiguity include Dow and Werlang (1994), Lo (1996), Klibanoff (1996), Marinacci (2000), and more recently Azrieli and
Teper (2011), Bade (2011), Riedel and Sass (2014), and Battigalli et al. (2015). For an overview of the ambiguity models in games, see Mukerji and Tallon (2004) and Beauchêne (2014).

This paper also contributes to the literature on theoretical explanations for experimental deviations from Nash equilibrium towards cooperation. The prominent models in the literature usually explain systematic deviations by focusing on social preferences (e.g., Klumpp, 2012), cognitive hierarchy of players (e.g., Stahl, 1993; Nagel, 1995; Camerer, Ho, and Chong, 2004), dynamic reasoning (e.g., Brams, 1994), reciprocity (e.g., Ambrus and Pathak, 2011), common knowledge (e.g., Aumann, 1992; Binmore, 1994), incomplete information, bounded rationality (e.g., Radner, 1980, 1986; McKelvey and Palfrey, 1995), preferences over strategies (e.g., Segal and Sobel, 2007), repeated game with random matching (e.g., Kandori, 1992; Heller and Mohlin, 2017), and learning (e.g., Mengel, 2014). Some of the early works include Kreps and Wilson (1982), Kreps et al. (1982), Sobel (1985), Fudenberg and Maskin (1986), and Neyman (1999), who show that incomplete information about various aspects of the game can explain cooperation in finitely repeated games. Neyman (1999) shows that cooperation can be induced in finitely repeated PD if players are restricted to using strategies with bounded complexity. Non-utility maximizing models include imitation-based decision-making in games (e.g., Eshel, Samuelson, and Shaked, 1998). For relevant literature on experiments and more details about theoretical explanations, see section 5 and the references therein.

The optimin criterion diverges from the aforementioned models and solution concepts along three main dimensions: (i) conceptual/cognitive background, (ii) scope of application, and (iii) predictive power. First, the optimin criterion operates as a non-equilibrium concept wherein players maximize their minimal payoffs subject to the constraints of unilateral and profitable deviations. In doing so, it offers a novel extension of maximin reasoning from two-person zero-sum games to non-zero-sum n-person games. Second, while the optimin criterion is principally designed for noncooperative games, it can also be applied to cooperative games, matching models, and statistical decision theory. Third, in contrast to other solution concepts, the optimin criterion aligns with the direction of non-Nash deviations in games that have been extensively analyzed for cooperation, including the finitely repeated prisoner’s dilemma, the centipede game, the traveler’s dilemma, and the finitely repeated public goods game.

All in all, it seems unlikely that complex human behavior can be captured by a single solution concept. For example, the ‘11-21’-type games introduced by Arad and Rubinstein (2012) seem to naturally invoke level-k reasoning (Stahl, 1993), whereas in complex games, such as Blotto games, players—who are unable to calculate optimal strategies—look at the characteristics of strategies rather than the strategies themselves (for a formalization of this type of reasoning, see Arad and Rubinstein, 2019). It is important to note, however, that a solution concept and a “reasoning
process” are generally two distinct notions. Despite over seventy years of research in game theory, it remains unclear whether any reasoning process can lead players to arrive at a Nash equilibrium. In the context of optimin points, a promising avenue for future research would be the investigation of reasoning processes that might converge to an optimin.

3 Optimin criterion in noncooperative games

3.1 Definition

Let \((\Delta X_i, u_i)\) be an \(n\)-person noncooperative game in mixed extension, where \(N = \{1, ..., n\}\) is the finite set of players, \(\Delta X_i\) the set of all probability distributions over the finite action set \(X_i\), \(u_i: \Delta X_i \rightarrow \mathbb{R}\) the von Neumann-Morgenstern expected utility function of player \(i \in N\), and \(p \in \Delta X\) a (mixed) strategy profile.\(^5\)

I first formally define the optimin criterion and then informally discuss the concept in the next subsection.

**Definition 1.** Sets \(B_i(p)\) and \(B_{-i}(p)\) are defined as follows

\[
B_i(p) = \{p'_i \in \Delta X_i | u_i(p'_i, p_{-i}) > u_i(p_i, p_{-i})\} \cup \{p_i\}, \text{ and } B_{-i}(p) = \bigtimes_{j \in N \setminus \{i\}} B_j(p).
\]

**Definition 2.** A profile \(p^* \in \Delta X\) is said to satisfy the optimin criterion or called an optimin point if \(p^*\) solves the following multi-objective optimization problem. For every \(i\) simultaneously

\[
p^* \in \arg\max_{q \in \Delta X} \inf_{p'_i \in B_{-i}(q)} u_i(q_i, p'_{-i}). \quad (3.1)
\]

The problem can be stated less compactly as follows:

\[
p^* \in \arg\max_{q \in \Delta X} \left( \inf_{p'_1 \in B_{-1}(q)} u_1(q_1, p'_{-1}), \inf_{p'_2 \in B_{-2}(q)} u_2(q_2, p'_{-2}), ..., \inf_{p'_n \in B_{-n}(q)} u_n(q_n, p'_{-n}) \right).
\]

3.2 The intuition behind the definitions and potential variations

3.2.1 Informal discussion

In plain words, strategy profile \(p^*\) is called an optimin point if each player simultaneously maximizes their minimal payoff, subject to the constraint of unilateral

\(^5\)For a detailed discussion of the mixed-strategy concept, see Luce and Raiffa (1957, p. 74). For a more recent discussion, see Rubinstein (1991).
profitable deviations (i.e., opportunistic behavior) by the other players. As mentioned in the introduction, when $p^*$ is a Nash equilibrium, no unilateral profitable deviation is possible. However, if $p^*$ is not a Nash equilibrium, then at least one player (say, $j$) has a unilateral profitable deviation from $p^*$, and the remaining players who conform to their part of the profile $p^*$ are assured of specific payoffs even if player $j$ deviates from $p^*$.

It is instructive to differentiate between optimin and maximin. In the case of the latter, a player selects an action to maximize their minimum utility, considering any potential choice—even if that choice would be disadvantageous for the deviator—made by the other players. By contrast, under the optimin, players are restricted to making only profitable deviations from $p^*$.

The optimin criterion assumes fully noncooperative play: players select their strategies independently, without collaborating with any other player. Furthermore, a player may make any unilateral and profitable deviation, without regard for its impact on the payoffs of other players. It is important to note that coalitional or correlated profitable deviations are not accounted for in the definition of optimin, although the definition can be amended to include such concepts (see section 1.5).

### 3.2.2 Illustrative example

To illustrate the optimin, I return to the game I discussed in the introduction (Figure 1, left). Figure 1 (right) shows that the minimal payoffs (i.e., the performance) of (Top, Left) is (100, 100). This is because even if, for example, player 2 unilaterally and profitably deviates to ‘Center’, player 1 would still receive a payoff of 100. Notice that this is the only profitable deviation from (Top, Left), because a unilateral deviation to ‘Right’ would be implausible. To give another example, the performance of (Bottom, Center) is (5, 0) because (i) player 1 has no profitable deviation from it, and (ii) player 2 may profitably deviate to ‘Right’, in which case player 1 would receive a payoff of 5. The minimal payoff of (Top, Center) is 100 for player 1 because player 2 has no profitable deviation from it. By contrast, for player 2 the minimal payoff of (Top, Center) is 0 because player 1 can profitably deviate to Bottom, which decreases player 2’s payoff to 0. All in all, as discussed in the introduction, (Top, Left) maximizes the minimal payoffs of the players under unilateral profitable deviations. Thus, it is the unique optimin point.

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6In Ismail (2014), I called $p^*$ a “maximin equilibrium” if either (i) it is an optimin point, or (ii) for every $i$, $p_i^* \in \arg \max_{q_i \in \Delta X_i} \inf_{p'_{-i} \in B_{-i}(q_i,p_i^*)} u_i(q_i,p'_{-i})$, which could be thought of as the equilibrium counterpart of the optimin.
3.2.3 Alternative formulation and deviations

In the definition of optimin, one might also consider types of deviations other than strict better-response deviations, such as weak better-response deviations and best-response deviations. I next reformulate the optimin in a slightly different but equivalent manner to demonstrate this.

First, define the minimal payoffs of a mixed strategy profile \( p \). Let \( \bar{B}_{-i}(p) \) denote the set of beliefs held by player \( i \) regarding the other players’ potential deviations from \( p \).

**Definition 3.** Given a profile \( p \in \Delta X \) and \( i \in N \), the \( i \)’th component of the (optimin) performance function \( \pi : \Delta X \rightarrow \mathbb{R}^n \) is defined as

\[
\pi_i(p) = \inf_{p', q \in \bar{B}_{-i}(p)} u_i(p_i, p'_{-i}).
\]

In other words, player \( i \)’s value from a profile \( p \) is defined as the minimum payoff the player receives (i) from \( p \), or (ii) under the deviations of the other players. The subsequent step involves comparing evaluations of different strategy profiles. In this context, the multi-variable performance function is maximized, a process known as multi-objective maximization or Pareto-optimization. A strategy profile is designated an optimin point if it maximizes this performance function.

Note that the performance function in Definition 3 assigns to each player a unique value or minimal payoff for every strategy profile \( p \). One may interpret this definition as follows. Suppose that before playing a noncooperative game players can make a tacit (non-binding) agreement to play a strategy profile \( p \). Then, each player chooses his or her strategy simultaneously and independently, leaving each with the option to either honor or break the non-binding agreement. The optimin value assumes that players evaluate such a profile cautiously, ruling out implausible deviations from the agreement—i.e., the deviations which do not strictly improve the payoff of the deviator, holding the others’ strategies fixed. Accordingly, a player’s value or minimal payoff of following a tacit agreement is defined as the minimum utility the player receives either (i) from the agreement or (ii) under the constraint of \( \bar{B}_{-i} \), which is the set of (possibly correlated) beliefs held by player \( i \) about potential deviations by the other players. Obviously, it might be that \( \bar{B}_{-i}(p) \) might be defined as the weak better-response correspondence or the best-response correspondence.

In any case, Definitions 1–3 remain well-defined even when correlated or best-response deviations are incorporated. The main theorem shows that under mild conditions optimin points exist. If we stipulate that \( B_j \) should be either best-response correspondence or better-response correspondence with weak inequality, as opposed to strict inequality, the main theorem remains applicable. Nevertheless, should we consider weak-better-response deviations as viable, a Nash equilibrium
would no longer exist in $n$-person games. It would cease to be a self-enforcing agreement generally, as weak-better-response deviations from a Nash equilibrium are possible. For example, there is always a weak-better-response deviation from a mixed strategy Nash equilibrium. All in all, using these definitions would not change the optimin's capacity to account for non-Nash behavior in experiments, which I examine in section 5.

Recall that a player may possess a profitable deviation from an optimin point, as it is not a Nash equilibrium. The performance function captures the minimal payoff of the non-deviator in case of any unilateral profitable deviation, which may be a strictly dominant strategy. While I have yet to find a game where all optimin points incorporate strictly dominated strategies, one can conceive of a somewhat peculiar game as described below. Consider a game in which there is a unique Nash equilibrium in possibly mixed strategies and a unique optimin point from which a player can deviate to a strictly dominating strategy. In such a scenario, every player's optimin profile payoff and the associated minimal payoff must exceed their Nash payoff, with at least one strict inequality. If players rank strategy profiles based on these minimal payoffs, the optimin point becomes attractive in the following sense: each player is assured a higher payoff than their Nash payoff even if others unilaterally and profitably deviate from the optimin. This attractiveness amplifies if the game is repeated finitely many times. In the repeated context, the optimin profile cannot be sustained as a subgame-perfect equilibrium but becomes even more appealing due to its higher stage-game payoffs. To illustrate, consider the finitely repeated prisoner's dilemma (PD). Generally, players each receive strictly greater payoffs than their Nash payoffs when employing the Tit-for-Tat (TFT) strategy. Additionally, their minimal payoffs remain strictly greater than their Nash payoffs even when an opponent profitably deviates from the TFT. In this light, the TFT, which is an optimin point, emerges as a compelling strategy. In section 5, I delve further into the experimental evidence in some finitely repeated games and discuss why the optimin criterion is compatible with non-Nash behavior, including the TFT, in such contexts.

### 3.2.4 Evaluation and comparison method

The optimin point is an application of the evaluation and comparison method I propose for evaluating performance of strategy profiles in games. As I have described

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7In addition, the performance function evaluates strategy profiles based on their minimal payoffs, but it is conceivable to use a different function for the evaluation of the profiles such as minimax regret or the Hurwicz criterion. Using the minimal payoffs is perhaps the most conservative approach, but it suffices to explain some well-documented non-Nash deviations in experimental games, as discussed in section 5.

8Note that PD is not a ‘peculiar’ game as described earlier; the unique optimin point and the Nash equilibrium coincide in the one-shot game.
above, for a more general application, we may interpret $\bar{B}_j(p)$ as being the belief of some player $i$ about player $j$’s (or a coalition’s) potential deviations, and apply the evaluation step accordingly. Analogous to varying $C$ in the Gilboa-Schmeidler maxmin model to rationalize different choices, one can also vary $\bar{B}_j$ and rationalize different choices in a game setting. Maximin strategy corresponds to the case in which a player’s belief about her opponent’s deviations is the whole strategy set of the opponent. That is, player $i$ does not take individual rationality of the opponent into account. The optimin principle can be incorporated with stronger or weaker individual rationality assumptions, even with different ones for different players, by following the same method I follow in this section.

3.2.5 Optimin and efficiency

Note that there is no logical relationship between Pareto optimality and the optimin point. In the battle of the sexes game, for example, the two optimin points are (Football, Football) and (Opera, Opera), which are Pareto optimal. However, an optimin point may be Pareto dominated. In the prisoner’s dilemma, the unique optimin point is (Defect, Defect), which is Pareto dominated.

4 Results in noncooperative games

4.1 Existence of optimin in $n$-person games

The following lemma presents a property of the performance function which will be used in the following existence result.

**Lemma 1.** For every $i$, $\pi_i$ is upper semi-continuous.

**Proof.** In several steps, I will show that the performance function $\pi_i$ of player $i$ in a game $\Gamma = (\Delta X_1, \Delta X_2, u_1, u_2)$ is upper semi-continuous. The extension of the arguments to $n$-person case is completely analogous as long as $n$ is finite, as is assumed.

First, we decompose the performance function as

$$\pi_i(p) = \min\{ \inf_{p'_j \in B'_j(p)} u_i(p_i, p'_j), u_i(p) \},$$

where $B'_j(p)$ is the (strict) better response correspondence of player $j$ with respect to $p$, representing the set of profitable deviations, which is defined as

$$B'_j(p) = \{ p'_j \in \Delta X_j | u_j(p_i, p'_j) > u_j(p) \}.$$
I next show that the correspondence $B'_j : \Delta X_i \times \Delta X_j \to \Delta X_j$ is lower semi-continuous. For this, it is enough to show the graph of $B'_j$ defined as follows is open.

$$Gr(B'_j) = \{(q, p_j) \in \Delta X_i \times \Delta X_j \times \Delta X_j \mid p_j \in B'_j(q)\}.$$ 

$Gr(B'_j)$ is open in $\Delta X_i \times \Delta X_j \times \Delta X_j$ if and only if its complement is closed. Let $\{(p_j, q_i, q_j^k)\}_{k=1}^\infty$ be a sequence in $[Gr(B'_j)]^c = (\Delta X_i \times \Delta X_j \times \Delta X_j) \setminus Gr(B'_j)$, converging to $(p_j, q_i, q_j)$ where $p_j^k \notin B'_j(q^k)$ for all $k$. That is, we have $u_j(p_j^k, q_i^k) \leq u_j(q^k)$ for all $k$. Continuity of $u_j$ implies that $u_j(p_j, q_i) \leq u_j(q)$, which means $p_j \notin B'_j(q)$. Hence $[Gr(B'_j)]^c$ is closed, implying that $B'_j$ is lower semi-continuous.

Next, we define $\hat{u}_i : \Delta X_i \times \Delta X_j \times \Delta X_j \to \mathbb{R}$ by $\hat{u}_i(q_i, q_j, p_j) = u_i(p_j, q_i)$ for all $(q_i, q_j, p_j) \in \Delta X_i \times \Delta X_j \times \Delta X_j$. Since $u_i$ is continuous, $\hat{u}_i$ is also continuous. In addition, we define $\tilde{u}_i : Gr(B'_j) \to \mathbb{R}$ as the restriction of $\hat{u}_i$ to $Gr(B'_j)$, that is $\tilde{u}_i = \hat{u}_i|_{Gr(B'_j)}$. The continuity of $\hat{u}_i$ implies the continuity of its restriction $\tilde{u}_i$, which in turn implies $\tilde{u}_i$ is upper semi-continuous.

By Theorem 1 of Berge (1959, p. 115), lower semi-continuity of $B'_j$ and lower semi-continuity of $-\tilde{u}_i : Gr(B'_j) \to \mathbb{R}$ imply that the function $-\bar{\pi}_i : \Delta X_i \times \Delta X_j \to \mathbb{R}$—defined as $-\bar{\pi}_i(q) = \sup_{p_j \in B'_j(q)} -\tilde{u}_i(p_j, q)$—is lower semi-continuous. This implies that the function $\bar{\pi}_i(q) = \inf_{p_j \in B'_j(q)} \tilde{u}_i(p_j, q)$ is upper semi-continuous.

As a result, the performance function $\pi_i(q) = \min\{\bar{\pi}_i(q), u_i(q)\}$ of player $i$ is upper semi-continuous because the minimum of two upper semi-continuous functions is also upper semi-continuous.

The following theorem shows that the optimin point exists in mixed strategies.

**Theorem 1.** Every mixed extension of a finite game has an optimin point.

**Proof.** Define $\pi_i^{max} = \arg \max_{q \in \Delta X_i \times \Delta X_j} \pi_i(q)$, which is a nonempty compact set because $\Delta X_i \times \Delta X_j$ is compact, and $\pi_i$ is upper semi-continuous by Lemma 1. Since $\pi_i^{max}$ is compact and $\pi_i$ is also upper semi-continuous, set $\pi_{ij}^{max} = \arg \max_{q \in \Delta X_i \times \Delta X_j} \pi_{ij}(q)$ is nonempty and compact. Clearly, the profiles in $\pi_{ij}^{max}$ are Pareto optimal with respect to the performance function, implying that $\pi_{ij}^{max}$ is a nonempty compact subset of the set of optimin points in the game. Analogously, the set $\pi_{ij}^{max}$ is also a nonempty compact subset of the set of optimin points. (Note that these arguments can be applied to games with any finite number of players.)

Notice that we have used neither the convexity of the strategy sets nor the concavity of the utility functions in the proof of Lemma 1 or Theorem 1. Thus, the latter result can be stated more generally as follows: any game with continuous utility functions and compact strategy spaces possesses an optimin point.

---

9I follow the terminology, especially the definition of upper hemi-continuity, presented in Aliprantis and Border (1994, p. 569).

10I use the fact that a function $f$ is lower semi-continuous if and only if $-f$ is upper semi-continuous.
4.2 Properties of optimin

First, I show that for every Nash equilibrium there is an optimin point in which not only is every player (weakly) better off but also guarantees to be better off even if the other players deviate unilaterally and profitably from the optimin.

**Proposition 1** (Super-Nash performance). For every Nash equilibrium \( q^* \) there exists an optimin point \( p^* \) such that for every player \( i \) \( \inf_{p'_i \in B_{-i}(p^*)} u_i(p_i^*, p'_{-i}) \geq u_i(q^*) \). Moreover, a Nash equilibrium \( q^* \) can never Pareto dominate an optimin point \( p^* \).

**Proof.** First, notice that if \( q^* \) is an optimin point, then we are done. So I assume that \( q^* \) is not an optimin point. To reach a contradiction, suppose that there exists a Nash equilibrium \( q^* \) for every optimin \( p^* \) there exists a player \( j \) such that \( \inf_{p'_j \in B_{-j}(p^*)} u_j(p_j^*, p'_{-j}) < u_j(q^*) \).

This implies that for every \( i \) \( q^* \in \arg\max_{q \in \Delta X} \inf_{p'_i \in B_{-i}(q)} u_i(q_i, p'_{-i}) \) because for every player \( i \) \( \inf_{p'_i \in B_{-i}(q^*)} u_i(q_i^*, p'_{-i}) = u_i(q^*) \). This means that \( q^* \) is an optimin point, which contradicts to our supposition that it is not.

Next, I show that a Nash equilibrium \( q^* \) cannot Pareto dominate an optimin point \( p^* \). To reach a contradiction suppose that \( q^* \) Pareto dominates \( p^* \). Note that for every player \( j \) \( \inf_{p'_j \in B_{-j}(q^*)} u_j(q_j^*, p'_{-j}) = u_j(q^*) \) and \( u_j(q^*) \geq u_j(p^*) \) (with at least one strict inequality) by our supposition. However, then \( p^* \) is not an optimin point, which contradicts our supposition. \( \square \)

Note that Proposition 1 would remain true if we interchange Nash equilibrium with a profile of maximin strategies in an \( n \)-person game. This is because it is a well-known fact that a player’s payoff from a Nash equilibrium cannot be strictly lower than their maximin strategy payoff.

The following remark illustrates when the optimin criterion coincides with the maximin criterion.

**Remark 1.** Suppose that in the definition of the performance function \( B_{-i}(p) \) is replaced with \( \Delta X_{-i} \). Then, the optimin criterion solution reduces to a profile of maximin strategies.

**Proof.** We take the infimum over \( \Delta X_{-i} \) instead of taking it over \( B_{-i}(q) \) in the definition of optimin. Hence, we have that for every player \( i \)

\[
\max_{q \in \Delta X} \inf_{p'_i \in B_{-i}(q)} u_i(q_i, p'_{-i}) = \max_{q \in \Delta X} \min_{p'_i \in \Delta X_{-i}} u_i(q_i, p'_{-i}).
\]

Thus, if \( p^* \in \arg\max_{q \in \Delta X} \min_{p'_i \in \Delta X_{-i}} u_i(q_i, p'_{-i}) \), then it is clear that \( p_i^* \) is a maximin strategy of player \( i \). \( \square \)
Harsanyi and Selten (1988, p. 70) argued that invariance with respect to positive linear transformations of the utilities is a fundamental requirement for a solution concept. This requirement is satisfied by the optimin point as the next proposition shows.

**Remark 2.** Optimin points are invariant to positive linear transformation of the utilities.

**Proof.** Let $\Gamma$ and $\hat{\Gamma}$ be two games such that for some player $j$ $\hat{u}_j = \alpha u_j + \beta$ for some $\alpha > 0$ and some constant $\beta$. Then, the set of optimin points of $\Gamma$ and $\hat{\Gamma}$ are the same because (i) strict better response correspondence does not change under positive linear transformations, and (ii) we can take $\alpha$ and $\beta$ out of the infimum in the maximization problem 3.1 without affecting the maximizers. \hfill $\square$

There is a large literature on contests, which dates back to the Blotto game first introduced by Borel (1921). Many contest are constant-sum games, and the next remark illustrates that the optimin criterion in constant-sum games generalizes Nash equilibrium.

**Remark 3.** Every Nash equilibrium in an $n$-person constant-sum game is an optimin point.

**Proof.** Let $q^*$ be a Nash equilibrium in an $n$-person constant-sum game. Then, for every player $i$ $\inf_{p'_{-i} \in B_{-i}(q^*)} u_i(q^*_i, p'_{-i}) = u_i(q^*)$. This implies that for every $i$

$$q^* \in \arg \max_{q \in \Delta X} \inf_{p'_{-i} \in B_{-i}(q)} u_i(q_i, p'_{-i})$$

because every strategy profile is Pareto optimal in $n$-person constant-sum games. \hfill $\square$

The following remark shows the existence of an optimin point in pure strategies when the game is restricted to pure strategies.

**Remark 4.** Every finite $n$-person game restricted to pure strategies has an optimin point in pure strategies.

The proof of this remark is straightforward because there are finitely many pure strategies in finite games, so there exists $x^* \in X$ that solves for every $i$ $\arg \max_{x \in X} \inf_{x'_{-i} \in B_{-i}(x)} u_i(x_i, x'_{-i})$. This result is useful in part because it guarantees the existence of pure optimin points when attention is restricted to pure strategies, as finding mixed strategy equilibria can be tedious in many economic applications.
4.3 Statistical games

Wald’s (1950) theory is based on the idea that a statistician should use a maximin strategy to minimize the maximum risk in a carefully constructed game against Nature. The statistician faces a decision problem under uncertainty and assumes that Nature wants to maximize the risk, which makes the game between the statistician and Nature a zero-sum game. This approach views statistical decision-making as a game against Nature. Formally, a statistical game is denoted by a tuple $S = (Y_1, Y_2, u_1, u_2)$ where $Y_1$ and $Y_2$ denote the set (which is not necessarily finite) of strategies of the statistician and the Nature, respectively.

To illustrate, consider the following simplified version of Bulmer’s (1979, p. 416) game. Suppose that a possibly unfair coin has a probability of either $1/4$ or $1/2$ of coming up heads. The task is to choose between (i) $p = 1/4$ or (ii) $p = 1/2$ upon tossing the coin once.

What is the “optimal” decision in this problem? Figure 5 illustrates four actions of the experimenter: (1) never choose $p = 1/4$; (2) always choose $p = 1/4$; (3) choose $p = 1/4$ if it comes up heads; (4) choose $p = 1/4$ if it comes up tails. Nature has two (pure) actions, (i) $p = 1/4$ or (ii) $p = 1/2$. Payoffs are simply the expected probability of guessing right in each case.

In this game against Nature, the experimenter’s optimal maximin strategy is to play $(1/5, 0, 0, 4/5)$ and Nature’s optimal strategy is to play $(2/5, 3/5)$. The experimenter’s probability of guessing it right is $3/5$, which is significantly higher than a random guess. There is no other strategy that can guarantee a higher probability of being correct. As Bulmer (1979, p. 416) shows, if one is allowed to toss the coin twice, then the probability of being correct increases to $6/11$.

In what follows, I show that the optimin criterion coincides with Wald’s maximin criterion (with a pessimist Nature) and von Neumann’s (1928) maximin strategies in zero-sum games. The following proposition shows that a strategy profile is an optimin point if and only if it is a pair of maximin strategies in zero-sum games.

**Proposition 2.** Let $S$ be a statistical game. A profile $(y_1^*, y_2^*)$ is an optimin point if and only if $y_1^* \in \arg \max_{y_1} \inf_{y_2} u_1(y_1, y_2)$ and $y_2^* \in \arg \max_{y_2} \inf_{y_1} u_2(y_1, y_2)$.

*Proof.* ‘⇒’ First, we show that $\pi_i(y_i, y_j) = \inf_{y'_j \in Y_j} u_i(y_i, y'_j)$ for each $i \neq j$. Suppose

| Say $p = 1/4$ | $p = 1/4$ | $p = 1/2$ |
|---------------|-----------|-----------|
| never         | 0         | 1         |
| always        | 1         | 0         |
| if it comes up heads | 1/4 | 1/2 |
| if it comes up tails | 3/4 | 1/4 |

Figure 5: A statistical game against Nature, which picks $p = 1/4$ or $p = 1/2$. 

that there exists $\bar{y}_j \in Y_j$ such that $\bar{y}_j \in \arg\min_{y'_j \in Y_j} u_i(y_i, y'_j)$. Then, we have that

$$\pi_i(y_i, y_j) = \min_{y'_j \in Y_j} u_i(y_i, y'_j) = u_i(y_i, \bar{y}_j).$$

Suppose, otherwise, that for all $y'_j \in Y_j$ there exists $y''_j \in Y_j$ such that $u_i(y_i, y''_j) < u_i(y_i, y'_j)$. This implies that

$$\pi_i(y_i, y_j) = \inf_{y'_j : u_i(y_i, y'_j) < u_i(y_i, y_j)} u_i(y_i, y'_j) = \inf_{y'_j \in Y_j} u_i(y_i, y'_j).$$

Next, we show that the performance of an optimin point $(y^*_1, y^*_2)$ must be Pareto dominant in a zero-sum game. By contraposition, suppose that its value is not Pareto dominant, that is, there is another optimin point $(\bar{y}_1, \bar{y}_2)$ such that $\pi_i(y^*_1, y^*_2) > \pi_i(\bar{y}_1, \bar{y}_2)$ and $\pi_j(y^*_1, y^*_2) < \pi_j(\bar{y}_1, \bar{y}_2)$ for $i \neq j$. Then, we have $v_1(y^*_1, y^*_2) = v_1(\bar{y}_1, \bar{y}_2)$ and $v_2(y^*_1, y^*_2) = v_2(\bar{y}_1, \bar{y}_2)$. This implies the performance of $(y^*_1, y^*_2)$ Pareto dominates the performance of $(y^*_1, y^*_2)$, which is a contradiction to our supposition that $(y^*_1, y^*_2)$ is an optimin point. Since the performance of $(y^*_1, y^*_2)$ is Pareto dominant, each strategy is a maximin strategy of the respective players.

\[=\] Suppose that for each $i$ we have $y^*_i \in \arg\max_{y_i} \inf_{y_j} u_i(y_i, y_j)$. This implies that for all $(y'_1, y'_2) \in Y_1 \times Y_2$ and for each $i$ we have $\pi_i(y^*_1, y^*_2) \geq \pi_i(y'_1, y'_2)$. Hence the performance of $(y^*_1, y^*_2)$ is Pareto dominant, which implies that it is an optimin point.

\[\Box\]

5 Economic Applications: Optimin and non-Nash Behavior

Proposition 1 illustrates that for every Nash equilibrium there is an optimin point in which every player guarantees a super-Nash payoff under the opportunistic behavior of the other players. While this is a promising theoretical guarantee, it is not immediately clear how different the predictions of the optimin can be. In this section, I apply optimin to four well-studied experimental games, the traveler’s dilemma, the finitely repeated prisoner’s dilemma, and the finitely repeated $n$-person public goods game, in which Nash equilibrium and optimin predictions are in sharp contrast with each other. I show that the optimin criterion predictions are consistent with the comparative statics of behavior in these games; whereas it is well-documented that Nash equilibrium is not.

5.1 The centipede game

First, consider the famous centipede game introduced by Rosenthal (1981). Let the tuple $(\{1, 2\}, \{C, S\}, u_1, u_2, m)$ denote a centipede game, which is two-person extensive-form game of perfect information where each player $i$ can choose either $C$ (continue) or $S$ (stop) at each node. There are $m$, finite and even, decision nodes, and players take turns moving at each node with player 1 moving at the first node.
Figure 6: Centipede game payoff function (up) and the performance function (down).

Note: The number of decision nodes is $m \geq 4$, which is an even number. The decision node at which player $i$ acts is denoted by $k_i = 1, 2, ..., \frac{m}{2}$.
The main characteristic of this game is that the unique subgame perfect equilibrium is to choose $S$ at every decision node. Let $k_i = 1, 2, ..., m$ denote the $(k_i)$th decision node at which player $i$ acts. Aumann’s (1998) variation of this game is characterized by the payoff function illustrated in Figure 6 (up).

Assume that $m \geq 4$. Figure 6 (down) illustrates the optimin performance function for this game. Consider the case in which player 1 plays $S$ at $k_1$ and player 2 at $k_2 \geq k_1$. Player 1 has either no profitable deviation or has no profitable deviation that decreases player 2’s payoff. Thus, player 2’s worst-case payoff (i.e., value) is equal to its payoff, i.e., $2k_1 - 1$. By contrast, player 2 has a (unique) profitable deviation to $k_1 - 1$, which undercuts player 1. As a result, player 1’s worst-case payoff would decrease to $2(k_1 - 1) - 1 = 2k_1 - 3$. Now, assume that player 2 stops $k_2 < k_1$. Player 2 has either no profitable deviation or has no profitable deviation that decreases player 1’s payoff. Thus, player 1’s value is equal to its payoff, i.e., $2k_2 - 1$. By contrast, player 1 has a (unique) profitable deviation to $k_2$. As a result, player 2’s worst-case payoff would decrease to $2k_2 - 1$. At the cooperative strategy profile, “play always $C$,” player 1 has no profitable deviation, and player 2 has a profitable deviation to $S$ at node $m$, which decreases player 1’s payoff to $2k_1 - 3 = m - 1$, so the values at this node are $(m - 1, m + 1)$.

Next, I show that the cooperative strategy profile is the unique profile that satisfies the optimin criterion whenever $m \geq 4$. First, note that the performance of the cooperation is $(m - 1, m + 1)$. If player 1 stops at $k_1 \leq k_2$, and either $k_1 < \frac{m}{2}$ or $k_2 < \frac{m}{2}$, then the performance would be $(2k_1 - 3, 2k_1 - 1)$, which is (Pareto) dominated. If player 2 stops at $k_2 < k_1$, then take $k_2 = \frac{m}{2} - 1$, and observe that the performance would be equal to $(m - 1, m - 1)$, which is dominated. For lower values of $k_1$ (i.e., $k_1 < \frac{m}{2}$) with $k_1 > k_2$, the associated value would be at most $(2(\frac{m}{2} - 1) - 1, 2k_1 - 1) = (m - 3, 2k_1 - 1)$, which is also Pareto dominated. Thus, we find that the cooperative strategy profile is the unique optimin point.\[11\]

Alternatively, consider constant-sum centipede games—i.e., class of centipede games denoted by $C = \{1, 2\}, \{C, S\}, u_1, u_2, m\}$ where $u_1 + u_2$ is a constant. By Proposition 3 and Proposition 2, the Nash equilibria coincide with the optimin points because the game $C$ is of constant-sum. Thus, in constant-sum centipedes the optimin criterion uniquely suggests noncooperative behavior—i.e., stopping in the first node.

Centipede games have been studied experimentally starting from McKelvey and Palfrey (1992) to including Fey et al. (1996), Nagel and Tang (1998), Rubinstein (2007), and Levitt et al. (2011). One of the most common and replicated finding is that, on average, subjects show the most cooperative behavior in increasing-sum

\[11\text{Aumann’s (1998) centipede game is a special case of increasing-sum centipede games. Analogous calculations would show that the cooperative strategy profile would be the unique optimin point in increasing-sum centipede games with } m' \text{ or more nodes, where } m' \text{ would depend on the specific payoff function.}\]
centipedes and the most noncooperative (equilibrium-like) behavior in constant-sum centipedes. (For a meta-study of almost all published centipede experiments, see Krockow, Colman and Pulford, 2016). The direction of these findings is consistent with the optimin criterion as the unique optimin criterion coincides with the equilibrium prediction in constant-sum centipedes, whereas the unique optimin criterion leads to cooperation in increasing-sum centipedes whenever the number of decision nodes is greater than or equal to four. Moreover, as the number of decision nodes increases the optimin value between cooperation and defection becomes larger in increasing-sum centipedes, but this gap decreases as the number of decision nodes decreases. Eventually, the optimin value for defection becomes greater than the optimin value for cooperation as the game progresses. This provides an explanation as to why cooperation may decrease as the game proceeds.

5.2 The traveler’s dilemma

Figure 8 illustrates the traveler’s dilemma, which was introduced by Basu (1994). It is a symmetric two-person game in which players can pick a number from 2 to 100 and the one who picks the lower number receives the dollar amount equal to her choice plus a $2 reward, and the other receives a $2 punishment. If both choose the same number they get what they choose. The payoff function of player \( i \in \{1, 2\} \) if she plays \( a \) and her opponent plays \( b \) is defined as

\[
u_i(a, b) = \min \{a, b\} + r \cdot \text{sgn}(b - a)
\]

for all \( a, b \) in \( X = \{2, 3, ..., 100\} \), where \( r > 1 \) determines the magnitude of reward and punishment, which is 2 in the original game. Regardless of the magnitude of the reward/punishment, the unique Nash equilibrium is \( (2, 2) \).

It has been shown in many experiments that players do not on average choose the Nash equilibrium strategy and that changing the reward/punishment parameter affects the behavior observed in experiments. Goeree and Holt (2001) found that when the reward is high, 80% of the subjects choose the Nash equilibrium strategy, but when the reward is small, about the same percent of the subjects choose the highest number. This finding is a confirmation of Capra et al. (1999). There, play converged towards the Nash equilibrium over time when the reward was high but converged towards the other extreme when the reward was small. On the other hand, Rubinstein (2007) found (in a web-based experiment without payments) that
Figure 8: Traveler’s dilemma with reward/punishment parameter $r$.

55\% of 2985 subjects choose the highest amount and only 13\% choose the Nash equilibrium when the reward was small.

To find the optimin points, we first need to compute the performance function of the traveler’s dilemma. The performance function of player $i$ if she plays $a$ and her opponent plays $b$ with $r > 1$ is given below.

$$
\pi_i(a, b) = \begin{cases} 
  b - r, & \text{if } a > b \text{ for } a \in X \\
  a + 1 - 2r, & \text{if } a = b, a \neq 2, \text{ and } (a + 1 - r) \geq 2 \\
  2 - r, & \text{if } a = b, a \neq 2, \text{ and } (a + 1 - r) < 2 \\
  2, & \text{if } a = 2 \\
  a + 1 - 3r, & \text{if } a < b, a \neq 2, \text{ and } (a + 1 - 2r) \geq 2 \\
  2 - r, & \text{if } a < b, a \neq 2, \text{ and } (a + 1 - 2r) < 2.
\end{cases}
$$

Observe that the performance function is maximized when $(a, b) = (100, 100)$ provided that $r < 50$. Hence, choosing the highest number is the unique optimin point whenever $r < 50$. Note that as the reward parameter $r$ increases, the performance of the optimin point decreases as Figure 9 illustrates. When $r$ is greater than or equal to 50, the unique optimin point becomes the profile $(2, 2)$, which is also the unique Nash equilibrium of the game regardless of parameter $r$.\textsuperscript{12} The optimin criterion explains both the convergence of play to the highest number when the reward gets smaller and the convergence of play to the lowest number when the reward gets larger.

5.3 The finitely repeated $n$-person public goods game

Another important class of games in economics are public goods games. The main characteristic of these games is that it is a dominant strategy for every player not

\textsuperscript{12}The unique rationalizable strategy profile (Bernheim, 1984; Pearce, 1984) and the correlated equilibrium (Aumann, 1974) coincide with the Nash equilibrium irrespective of $r$. 

---

| 100 | 100, 100 | 99 - $r$, 100 + $r$ | $\cdots$ | 3 - $r$, 3 + $r$ | 2 - $r$, 2 + $r$ |
|-----|------------|---------------------|--------------|-----------------|------------------|
| 99  | 100 + $r$, 99 - $r$ | 99, 99 | $\cdots$ | 3 - $r$, 3 + $r$ | 2 - $r$, 2 + $r$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\vdots$ |
| 3   | 3 + $r$, 3 - $r$ | 3 + $r$, 3 - $r$ | $\cdots$ | 3, 3 | 2 - $r$, 2 + $r$ |
| 2   | 2 + $r$, 2 - $r$ | 2 + $r$, 2 - $r$ | $\cdots$ | 2 + $r$, 2 - $r$ | 2, 2 |
to contribute anything into a public pot; but if everyone contributes, then everyone would be better off. I consider finitely \((k \text{ times})\) repeated public goods games without discounting. Let the following be the stage game utility function of player \(i\) in an \(n\)-person (linear voluntary contribution) public goods game:

\[
u_i(a_i, a_{-i}) = \bar{a} - a_i + \frac{m}{n} \sum_{j=1}^{n} a_j,
\]

where \(\bar{a}\) denotes the maximal amount each player can contribute, \(a_i \geq 0\) the contribution of player \(i\), and \(\frac{m}{n}\) marginal per capita return (MPCR) with \(n > m > 1\). Each stage game is called a round.

To give an example, assume that \(\bar{a} = 10, k = 10, n = 4,\) and \(m = 2\). Assume further that every player \(i\) contributes \(a_i = 10\) in the first round, i.e., the 10th round from the last. Then, each player \(i\) receives \(u_i(a_i, a_{-i}) = 10 - 10 + \frac{2}{4} \times 40 = 20\) from the first round. In this game, MPCR is \(\frac{m}{n} = \frac{2}{4}\).

Now, I define a specific conditional cooperation strategy \(\bar{\sigma}_i\) as follows: player \(i\) contributes \(a_i = \bar{a}\) unless there is a round \(k' < k\) in which another player \(j \neq i\) contributes \(a_j < \bar{a}\) in which case \(a_i = 0\) from the following round \(k' + 1\) onwards (including round \(k' + 1\)).

Let \(r\) denote the \(r\)'th round from the last. Note that the utility of player \(i\) from
\( \sigma \) starting from round \((k - r)\) onwards would be

\[ rm\bar{a}. \]  

(5.1)

By deviating to free-riding behavior (i.e., contributing 0) in round \(k - r\) onwards, player \(i\) would receive

\[ \left( \frac{m(n - 1)\bar{a}}{n} + \bar{a} \right) + (r - 1)\bar{a}, \]

(5.2)

where \( \frac{m(n - 1)\bar{a}}{n} + \bar{a} \) denotes the payoff player \(i\) receives in round \((k - r)\), and \((r - 1)\bar{a}\) denotes the payoff player \(i\) receives in the remaining \((r - 1)\) rounds in which everyone contributes 0. If (5.1) is greater than or equal to (5.2), then there is no unilateral profitable deviation from the conditional cooperation profile \(\sigma\) in \(r\)-round or longer public goods games.\(^{13}\) To simplify, \( rm\bar{a} \geq \frac{m(n - 1)\bar{a}}{n} + \bar{a} + (r - 1)\bar{a} \) if and only if \( r \geq \frac{m(n - 1)}{n(m - 1)} \). As a result, in every \(k\)-round repeated public goods game with \(k \geq \frac{m(n - 1)}{n(m - 1)}\), the performance of the conditional cooperation \(\sigma\) is weakly greater than the performance of the SPNE. This is in part because when there are \(r < \frac{m(n - 1)}{n(m - 1)}\) rounds left in the game, playing 0 in round \(k - r\) (i.e., the \(r\)’th from the last) is a profitable deviation from conditional cooperation, whose minimal payoffs are less than the free-riding profile in which everyone contributes 0.

In general, the performance or minimal payoffs of conditional cooperation profile \(\sigma\) can be calculated as follows. In a \(k\)-round game, let \(r' \leq k\) be such that \(r' \geq \frac{m(n - 1)}{n(m - 1)}\) and \(r' - 1 < \frac{m(n - 1)}{n(m - 1)}\). Then, the minimal payoffs of \(\sigma\) can be calculated based on a deviation in round \((k - r' + 1)\). Note that the minimal payoff of following \(\sigma\) in round \((k - r' + 1)\) is \(\frac{m\bar{a}}{n}\) and from round \((k - r' + 2)\) onwards players each receive \((r' - 2)\bar{a}\). Thus, the performance or minimal payoff of \(\sigma\) is given by

\[ (k - r')m\bar{a} + \frac{m\bar{a}}{n} + (r' - 2)\bar{a}. \]  

(5.3)

By contrast, the minimal payoff of free-riding behavior to player \(i\) is simply \(i\)’s payoff

\[ \pi_i(0, ..., 0) = k\bar{a}, \]  

(5.4)

because there is no unilateral profitable deviation from it. Conditional cooperation, \(\sigma\), is an optimin if (5.3) is weakly greater than (5.4), the performance of SPNE.

In finitely repeated public goods games, the unique subgame perfect equilibrium is to contribute 0 in every round, which is not sensitive to the parameters such as marginal per capita return or the number of rounds. However, this insensitivity sharply contrasts experimental findings: see, e.g., Isaac, Walker, and Thomas

\(^{13}\)Note that if (5.2) is strictly greater than (5.1), then the minimal payoffs of conditional cooperation would be less than the minimal payoffs of zero contribution.
Figure 10: The performance of SPNE vs the performance of conditional cooperation for different values of MPCR in a $k$-time repeated game ($k = 1, \ldots, 10$).

Note: The number of players, $n = 4$, and the maximal contribution, $\bar{a} = 10$.

Experimental research indicates that cooperation (i) is significantly greater in games with high MPCR compared to the ones with low MPCR, (ii) decreases as the end of the game approaches, (iii) restarts if the finitely repeated game is played again, and (iv) is magnified by pre-play communication.

The optimin criterion is consistent with these experimental findings. First notice that the greater the MPCR, the greater the gap between the optimin value (i.e., the minimal payoff) of conditional cooperation and free-riding, as Figure 10 illustrates. For low values of MPCR, free-riding behavior satisfies the optimin criterion, unless the game is repeated for a long time (so that cooperation may eventually pay off). For example, the performance of conditional cooperation for MPCR $= 0.33$ exceeds the performance of SPNE for $k \geq 14$. By contrast, for high values of MPCR, conditional cooperation satisfies the optimin criterion if the game is repeated a few times or more. Second, as the end of the repeated game approaches, the gap between the minimal payoffs of conditional cooperation and free-riding behavior closes. The optimin value of defection eventually becomes greater than the optimin value of cooperation (e.g., see Figure 10). But if the finitely repeated game is played again, the optimin value of cooperation at the beginning of the game is (generally) greater than the optimin value of defection, which can explain the “restart” effect. Finally, the finding that pre-play communication increases cooperation is consistent with tacit agreement interpretation of the optimin criterion: pre-play communication facilitates players in agreeing to cooperation, whose minimal payoffs are generally greater.
than the minimal payoffs of defection—though these are certainly tacit agreements because they are non-binding.

5.4 The finitely repeated prisoner’s dilemma

In another example, consider the finitely repeated prisoner’s dilemma (PD) with the stage game shown below. Assume that $T > R > P > S$, where $T$ stands for temptation payoff, $R$ the reward, $P$ the punishment payoff, and $S$ the sucker’s payoff. Recall that the unique optimin point in the one-shot game is (Defect, Defect). However, new solutions emerge when the game is repeated. Note that at every Nash equilibrium each player defects in each round in a finitely repeated prisoner’s dilemma.

|            | Cooperate | Defect |
|------------|-----------|--------|
| Cooperate  | $R, R$    | $S, T$ |
| Defect     | $T, S$    | $P, P$ |

As mentioned, a player who plays the TFT strategy starts by cooperating in the first round and then does whatever the opponent did in the previous round. TFT is perhaps the most famous form of conditional cooperation in the PD. Another conditional cooperation strategy is called the grim trigger strategy, which is more forgiving than the TFT, in which a player cooperates until a defection is observed and defects from then on.

I next give the conditions under which the conditional cooperation (i.e., TFT and grim trigger strategy profiles) satisfy the optimin criterion in the $k$-time repeated PD. Normalize $S = 0$ without loss of generality. By following steps similar to subsection 5.3, one can obtain that there is no unilateral profitable deviation from neither the TFT strategy profile nor the grim-trigger strategy profile in a $k \geq r$ round PD when (i) $3R \geq 2T$ and (ii) $r \geq \frac{T - P}{R - P}$ the grim-trigger is still an optimin point if (i) is interchanged with $2R \geq T$.

14 The performance or the minimal payoffs of conditional cooperation (TFT or grim trigger) can be calculated as follows. In a $k$-round game, let $r' \leq k$ be such that $r' \geq \frac{T - P}{R - P}$ and $r' - 1 < \frac{T - P}{R - P}$. It implies that there is a profitable deviation in round $(k - r' + 1)$. Then, the minimal payoff of conditional cooperation is given by

$$(k - r')R + S + (r' - 2)P.$$  \hspace{1cm} (5.5)

The minimal payoff of subgame perfect equilibrium to a player is simply $kP$. Conditional cooperation is an optimin in a $k \geq r$ round PD if, in addition to (i) and

14 Notice that a two-person public goods game in which each player has only two actions, “contribute” and “don’t contribute” is a type of prisoner’s dilemma—though PD is more general.
Figure 11: The performance of SPNE vs the performance of TFT for different values of \( T \) in a \( k \)-time repeated game (\( k = 1, \ldots, 10 \)).

Note: \( R = 3, P = 1, \) and \( S = 0. \)

(ii), (5.5) is weakly greater than \( kP \). In summary, if the minimal payoffs from cooperation are “large enough” then conditional cooperation satisfies the optimin criterion.\(^{15}\)

**Illustrations**

|        | Cooperate | Defect  |
|--------|-----------|---------|
| Cooperate | 3, 3     | 0, \( T \) |
| Defect  | \( T, 0 \) | 1, 1    |

Figure 11 illustrates the performances of SPNE and TFT in a \( k \)-time repeated PD for different values of temptation payoff \( T \) given the above payoff matrix. The performance of SPNE does not depend on \( T \), whereas the performance of TFT decreases as \( T \) increases. As described earlier, when the \( T \) is very large (22 in the game shown below), the performance of SPNE exceeds the performance of TFT.

\(^{15}\)In contrast, when \( 2R < T + S \), there emerge different “cooperative” strategy profiles whose minimal payoffs are greater than the TFT. For example, suppose that the payoffs are as follows: \( T = 10, R = 3, \) and \( P = 1. \) Consider the strategy profile in which players play (D, C), (C, D), (D, C), ..., unless there is a deviation in which case players play always D. Notice that this profile would have a greater minimal payoff than the usual TFT or the grim trigger.
Figure 12: Contour of $\frac{1-P}{R-P}$.

Note: If the PD is repeated $\frac{1-P}{R-P}$-times, then the performance of TFT is greater than the performance of the SPNE.

Next, normalize $S = 0$ and $T = 1$. Figure 12 illustrates the contour of $\frac{1-P}{R-P}$. Note that $k \geq \frac{1-P}{R-P}$ implies that that in every $k'$-time repeated PD such that $k' \geq k$, the performance of TFT is greater than the performance of the SPNE. Note also that this does not necessarily imply that TFT is an optimin because, depending on the payoffs, there might be another strategy profile whose performance is even greater than the performance of the TFT (see footnote 15).

The literature on finitely repeated prisoner’s dilemma games is huge: see, e.g., Axelrod (1980), Selten and Stoecker (1986), recent meta studies Mengel (2017), Embrey, Fréchette, and Yuksel (2017), and the references therein. It has been well established that players cooperate more often than the subgame perfect equilibrium predicts. More specifically, (i) initial cooperation becomes more likely as the number of rounds increases, (ii) cooperation decays as the end of the game approaches. The optimin criterion is consistent with these regularities. Cooperation generally satisfies optimin criterion in the repeated PD because, even if a player tries to take advantage of cooperative behavior, the minimal payoff of the cooperator (expression 5.5) is greater than the subgame perfect equilibrium payoff. As the number of rounds increases in a game, notice that the optimin performance or the minimal payoffs of cooperation increase. However, these minimal payoffs gradually decrease as the game progresses, and they eventually become less than the minimal payoffs of defection, which is consistent with the finding that cooperation levels decrease as the end of the game approaches.
6 Cooperative games

6.1 Games in characteristic function form

In their groundbreaking book, von Neumann and Morgenstern (1944) introduced cooperative games in which the characteristic function assigns a unique number to each coalition or subset of players. In this section, I assume transferable utility—i.e., the utility of a coalition can be redistributed among its members. The concepts introduced in this section can be analogously extended to games with nontransferable utility.

Let \( N = \{1, ..., n\} \) denote the finite set of players and \( S \subseteq N \) denote a nonempty coalition. An \( n \)-person cooperative game in characteristic function form is a tuple \((N, u)\), where \( u : 2^N \to \mathbb{R} \) is the characteristic function satisfying coherence, i.e., for every partition \( \{S_1, ..., S_K\} \) of \( N \), \( u(N) \geq \sum_{k=1}^{K} u(S_k) \). Coherence restricts the attention to games in which the grand coalition \( N \) forms.

Let \((x_i)_{i \in S}\) denote a payoff distribution (i.e., allocation) for coalition \( S \) where \( x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \), and \( x(S) = \sum_{i \in S} x_i \). A vector \( x \in \mathbb{R}^n \) is called feasible if \( x(N) \leq u(N) \). Let \( X \subseteq \mathbb{R}^n \) be the set of all feasible payoff distributions. A feasible \( x \) is called an imputation if \( x_i \geq u(i) \) for all \( i \in N \) and \( x(N) = u(N) \). The set of all imputations is denoted by \( I(u) \). A vector \( y \in \mathbb{R}^n \) is said to dominate \( z \in \mathbb{R}^n \) via coalition \( S \) if for all \( i \in S \) and \( z(S) < y(S) \leq u(S) \), in which case \( S \) has a profitable deviation from \( z \). An allocation \( z \) is dominated by another allocation \( y \) if \( y \) dominates \( z \). Let

\[
D_{-i}(x) = \{ S \subseteq N \setminus i \mid x \text{ is dominated via coalition } S \}
\]

be the set of all coalitions excluding player \( i \) that dominate the payoff distribution \( x \), and let \( D(x) = \{ S \subseteq N \mid x \text{ is dominated via coalition } S \} \). The set of all imputations that are not dominated by another imputation is called the core, which may be empty.

**Definition 4 (Performance function).** The performance function of a cooperative game \((N, u)\) is a mapping \( \Pi : X \to \mathbb{R}^n \) that satisfies the following conditions. Let \( \Pi'_i : 2^{N \setminus i} \to \mathbb{R} \) be a real-valued function. The \( i \)'th component of \( \Pi \), denoted by \( \Pi_i : X \to \mathbb{R} \), is defined as follows.

\[
\Pi_i(x) = \begin{cases} 
  x_i, & \text{if } D_{-i}(x) = \emptyset \\
  \min_{S \in D_{-i}(x)} \Pi'_i(N \setminus S), & \text{else}. 
\end{cases}
\]

Note that characteristic function is usually called value function denoted by \( v \) in the literature; for a reference textbook, see, e.g., Peters (2015). To avoid confusion with the performance function I define earlier in this paper, I call the characteristic function utility function denoted by \( u \).

Coalition \( S \) "blocks" allocation \( z \).
The intuition underpinning the performance function is as follows. Let \( \Pi_i(x) \) represent the performance of a payoff distribution \( x \) for player \( i \). This performance is defined as the minimum of two quantities: (i) the player’s actual payoff \( x_i \), and (ii) the minimum payoff that the player would receive under coalitional profitable deviations from \( x \). Specifically, if \( x \) is dominated by \( S \), then player \( i \) would consider her minimal payoff under such a deviation by \( S \). There are in general many ways to define the “minimal payoff” under (ii). For instance, the performance function can be defined as follows.

\[
\Pi_i(x) = \min\{x_i, \min_{S \in D_i(x)} \left( \frac{u(N \setminus S)}{|N \setminus S|} \right) \}.
\]

Here, the rationale is that the total utility of the non-deviating coalition \( N \setminus S \) is equally distributed among its members.

Of course, it is possible, and perhaps desirable, to consider “farsighted” profitable coalitional deviations when calculating the minimal payoffs of a payoff distribution. For example, Harsanyi (1974) criticized the core as it is based on “myopic” deviations because a deviating coalition does not consider the possibility of another coalition deviating further. Harsanyi’s (1974) observation has led to a large literature on solution concepts with farsighted individuals. First, it is possibility to define the performance of a player under profitable farsighted deviations by changing the definition from dominance to farsighted dominance. (For a comprehensive survey and references, see, e.g., Ray and Vohra, 2015). Second, there may be cases in which a payoff distribution, \( x \), is dominated via some \( S \). Then, the minimal payoff of player \( i \) in that situation would depend on the worth of nondeviating coalition \( N \setminus S \) which includes \( i \), and how this worth is distributed among players in \( N \setminus S \). Of course, distributing worth of a coalition among its members defines another game, which may be solved recursively. Third, there may be two different subsets of \( N \setminus i \) that have a profitable deviation from \( x \), in which case it would be sensible to consider only “maximal” or “best-response” deviations—i.e., deviations that give the largest payoff to a deviating coalition.

While these are all potential research directions, the purpose of this section is to illustrate how “evaluate and compare” method and its specific application, the optimin criterion, I introduced in this paper can be applied to cooperative games. As before, the evaluation step gives a performance value to each payoff distribution based on deviations that are deemed “reasonable,” and the comparison step makes comparison among these performance evaluations.

As mentioned, the performance function can take many forms. The following three assumptions impose some natural restrictions on the performance function. Let \( i \in N \), \( x \in X \), and \( y \in X \).

1. **Consistency**: for every \( i \) and every \( x \in X \), \( \Pi_i(x) \leq x_i \).
2. **Monotonicity**: (i) If \(D_{-i}(x) = D_{-i}(y)\) and \(x_i \geq y_i\), then \(\Pi_i(x) \geq \Pi_i(y)\). (ii) If \(D_{-i}(x) \subseteq D_{-i}(y)\), \(D_{-i}(x) = \emptyset\), and \(u(i) \leq x_i\), then \(\Pi_i(x) > \Pi_i(y)\).

3. **Individual rationality**: for every \(x\) and every \(i\), if \(N \setminus i \in D_{-i}(x)\) and \(u(i) \leq x_i\), then \(\Pi_i(x) = u(i)\).

The functional form of the performance function is based on the notion that the performance of a payoff distribution captures what a player “guarantees” under the profitable deviations by other players. Consistent with this idea, the performance of a payoff distribution for a player cannot exceed the payoff that the player obtains from that distribution; this is referred to as the Consistency assumption.

Monotonicity posits that, given two payoff distributions \(x\) and \(y\), the distribution with the larger set of profitable deviations will have a weakly lower value. The inequality becomes strict if one distribution has no profitable deviation while the other does. Specifically, if there exists a profitable deviation from \(y\) but not from \(x\), i.e., \(D_{-i}(x) = \emptyset\), then player \(i\) cannot guarantee as much payoff from \(y\) as from \(x\) due to the existence of deviations from \(y\).\(^{18}\)

Individual Rationality states that if player \(i\) receives at least \(u(i)\) from a payoff distribution \(x\), and if coalition \(N \setminus i\) has a profitable deviation from \(x\), then the value of \(x\) for player \(i\) must equal \(u(i)\). The rationale is straightforward: if player \(i\) was supposed to receive a payoff greater than \(u(i)\) but the rest of the coalition deviates, then player \(i\) will simply receive their individual payoff. Although this is a natural assumption, it is not essential for proving the existence of optimin points.

**Definition 5.** A feasible payoff distribution \(x \in \mathbb{R}^n\) is said to satisfy the optimin criterion or called an **optimin point** if for every player \(i \neq j\) and every feasible \(x' \in \mathbb{R}^n\), \(\Pi_i(x') > \Pi_i(x)\) implies that there is some \(j\) with \(\Pi_j(x') < \Pi_j(x)\).

As before, if the performance of a feasible payoff vector is Pareto optimal, then it is called an optimin point.\(^{19}\) I next give an algorithm for finding an optimin point before presenting two useful results and giving an illustrative example.

### 6.2 Results

**6.2.1 Existence of optimin via an algorithm**

**Theorem 2.** There exists an optimin point in every cooperative game in characteristic function form in which the performance function satisfies Consistency and Monotonicity.

\(^{18}\)To further narrow the set of functions satisfying Definition 4, one could introduce “strong monotonicity,” wherein \(D_{-i}(x) \subseteq D_{-i}(y)\) would imply \(\Pi_i(x) < \Pi_i(y)\). However, this stronger assumption is not necessary for the results presented in this section.

\(^{19}\)One could also define “optimin core” as the core of the cooperative game with respect to the minimal payoffs function.
Proof. I establish the existence of an optimin through the construction of an algorithm, which considers three cases: (i) the core is nonempty; (ii) the core is empty but there exists some player $i$ and allocation $x \in \mathbb{R}^n$ such that $D_{-i}(x)$ is empty; (iii) both (i) and (ii) are false. Let $(N, u)$ be a cooperative game.

Case (i): it is well-known that when the core is nonempty the set of core allocations are precisely the set of solutions to the following linear program. Fix $i$.

Consider the following linear program (LP).

\[
\begin{align*}
\text{minimize:} & \quad \sum_{i \in N} x_i \\
\text{subject to:} & \quad \sum_{i \in S} x_i \geq u(S) \quad \text{for all } S \subseteq N(y) \\
& \quad u(N) \geq \sum_{i \in N} x_i.
\end{align*}
\]

By the assumption of case (i), the LP is feasible and hence has a solution, denoted by $x^*$. If $x^*$ is in the core, then for every $i$, $D_{-i}(x^*)$ is empty, and hence $\Pi_i(x^*) = x^*_i$.

To reach a contradiction, suppose that there exists another feasible allocation $x$ such that for some $i$, $\Pi_i(x) > \Pi_i(x^*)$. Given that $x$ is feasible and for every $i$, $x_i \geq \Pi_i(x)$ by the consistency of $\Pi$, there exists $j$ such that $x^*_j > x_j \geq \Pi_j(x)$. Thus, we obtain a contradiction to the supposition that the performance of $x$ Pareto dominates the performance of $x^*$.

Case (ii): the core is empty but there exists some player $j$ and allocation $x$ such that $D_{-j}(x)$ is empty. Observe that for every $i$, $D_{-i}(\cdot) \subseteq 2^{N \setminus i}$, which implies that the correspondence $D_{-i}$ is finite-valued. I proceed to construct a feasible allocation $x^*$ using strong induction on the number of players satisfying the assumption of case (ii).

Base case ($LP_1$): fix $j_1$. Consider the following linear program (LP).

\[
\begin{align*}
\text{minimize:} & \quad \sum_{i \in N \setminus \{j_1\}} x_i \\
\text{subject to:} & \quad \sum_{i \in S} x_i \geq u(S) \quad \text{for all } S \subseteq N \setminus \{j_1\} \\
& \quad u(N) \geq \sum_{i \in N \setminus \{j_1\}} x_i.
\end{align*}
\]

Let $(\bar{x}_i)_{i \in N \setminus \{j_1\}}$ be a solution to the above LP and define $x^*_{j_1} = u(N) - \sum_{i \in N \setminus \{j_1\}} \bar{x}_i$.

Induction step: let $M = \{j_1, j_2, ..., j_m\} \subseteq N$ be a set of players satisfying the following condition: for every $j_i$ there exists an allocation $x$ such that $D_{-j_i}(x)$ is empty. Assume that for every $m$ satisfying $2 \leq m \leq n$, $LP_{m-1}$ is defined. Let $(x^*_{j_i})_{i=1}^{m-1}$ be the sequence of solutions where $x^*_{j_i}$ is a solution to of $LP_{j_i}$. I next define
Let \( LP_m \) as follows.

\[
\begin{align*}
\text{minimize:} & \quad \sum_{i \in N \setminus M} x_i \\
\text{subject to:} & \quad \sum_{i \in S \setminus M} x_i \geq u(S) - \sum_{i \in S \cap M} x_i^* \quad \text{for all } S \subseteq N \setminus \{j_m\} \\
& \quad u(N) - \sum_{i \in M \setminus \{j_m\}} x_i^* \geq \sum_{i \in N \setminus M} x_i.
\end{align*}
\]

Let \( (\bar{x}_i)_{i \in N \setminus \{j_m\}} \) be a solution to the above LP and define

\[
x_{j_m}^* = u(N) - \sum_{i \in N \setminus M} \bar{x}_i - \sum_{i \in M \setminus \{j_m\}} x_i^*.
\]

Note that the set of players is finite, implying that the induction process terminates after a finite number of iterations. Let \( LP_{\bar{m}} \) be the last LP as defined above.\(^{20}\) Consequently, allocations \( (x_{j_i}^*)_{i=1}^n \) have been defined. I next define the remaining allocations \( (x_{j_i}^*)_{i=m+1}^n \) as follows. For every \( k > \bar{m} \), choose the allocations \( x_{j_k}, x_{j_{k+1}}, x_{j_{k+2}}, \ldots, x_{j_n} \) such that the performance \( \Pi_{j_k} \) is maximum, given \( (x_{j_i}^*)_{i=1}^{k-1} \) and that

\[
x_{j_1}^*, x_{j_2}^*, \ldots, x_{j_k}^*, x_{j_{k+1}}, x_{j_{k+2}}, \ldots, x_{j_n}
\]

is a feasible allocation. The maximum exists because for every \( k > \bar{m} \), there is no feasible \( x \) such that \( D_{j_k}(x) \) is empty. This implies that, for every \( k > \bar{m} \) and every \( x, \Pi_{j_k} \) has finitely many values by definition of \( \Pi \).

Now, notice that \( x_{j_1}^* \) is the maximum value player \( j_1 \) can attain in the game due to the monotonicity assumption and \( LP_{j_1} \). Analogously, for every \( k < \bar{m} \), \( x_{j_k}^* \) is the maximum value player \( j_{k+1} \) can attain, given \( (x_{j_i}^*)_{i=k}^{k+1} \), due to the monotonicity assumption and \( LP_{j_{k+1}} \). Furthermore, for every \( k > \bar{m} \), \( x_{j_k}^* \) is the maximum value player \( j_{k+1} \) can attain, given \( (x_{j_i}^*)_{i=1}^{k-1} \), by construction.

Finally, to reach a contradiction, suppose that \( y \) is a feasible allocation and its value Pareto dominates \( x^* \). Let \( k \geq 1 \) be the smallest integer such that \( \Pi_{j_k}(y) > \Pi_{j_k}(x^*) \). It implies that for every \( k' < i \), \( \Pi_{j_k}(y) = \Pi_{j_{k'}}(x^*) \). Note that \( i > 1 \) because by construction of \( x^* \), \( x_{j_1}^* \) attains the maximum value for player \( j_1 \). So, \( \Pi_{j_1}(y) = \Pi_{j_1}(x^*) \). Given \( (x_{j_i}^*)_{i=1}^{k-1} \), \( x_{j_k}^* \) is constructed such that \( \Pi_{j_k} \) is as high as possible, which implies that \( \Pi_{j_k}(y) > \Pi_{j_k}(x^*) \) cannot hold. Therefore, we obtain a contradiction to the supposition that \( y \)'s value Pareto dominates \( x^* \)'s value, which implies that \( x^* \) is an optimin.

Case (iii): both (i) and (ii) are false, i.e., there is no feasible \( x \) and no player \( i \) such that \( D_i(x) \) is empty. This implies that, for every \( i \) and every \( x \), the performance...

\(^{20}\)Notice that \( n - 1 \geq \bar{m} \geq 1 \) because if it were \( n \), then the core would be nonempty, which is case (i).
\(\Pi_i\) attains finitely many values by definition of \(\Pi\). Therefore, performance function \(\Pi\) attains Pareto optima, i.e., there exists an optimin point.

The proof strategy can be explained in simpler terms in several steps.

1. **Case (i):** The core of the game is nonempty. In this case, it is shown that the core allocations are precisely the set of solutions to a specific linear program (LP). By assumption, the LP is feasible, so it has a solution, denoted as \(x^*\). The proof shows that if another feasible allocation, \(x\), Pareto dominates \(x^*\), it leads to a contradiction. Hence, \(x^*\) is an optimin.

2. **Case (ii):** The core is empty, but there exists a player \(i\) and allocation \(x\) such that \(D_{-i}(x)\) is empty.
   - The proof first notes that for every \(i\), \(D_{-i}(\cdot)\) is finite-valued. It then constructs a feasible allocation, \(x^*\), using strong induction on the number of players satisfying the assumption of case (ii).
   - **Base case:** The first part of the induction is the base case, where \(LP_1\) is defined, and a solution to \(LP_1\) is used to define \(x^*_1\).
   - **Induction step:** For the induction step, it is assumed that \(LP_{m-1}\) is defined for every \(m\) satisfying \(2 \leq m \leq n\), and \(LP_m\) is then defined. A solution to \(LP_m\) is used to define \(x^*_m\), and the induction continues until \(LP_{\bar{m}}\), the last LP, is defined. The remaining allocations are then defined to maximize the performance of \(\Pi_{jk}\) for \(k > \bar{m}\), given the previous allocations. The maximum exists because for every \(k > \bar{m}\), there is no feasible \(x\) such that \(D_{jk}(x)\) is empty, and hence \(\Pi_{jk}\) is finite-valued by definition of \(\Pi\).
   - It is then shown that if there exists another feasible allocation, \(y\), whose value Pareto dominates \(x^*\), it leads to a contradiction. Hence, \(x^*\) is an optimin.

3. **Case (iii):** Both (i) and (ii) are false. This means there is no feasible \(x\) and no player \(i\) such that \(D_i(x)\) is empty. In that case, the performance function is finite-valued by definition. Thus, the performance function \(\Pi\) attains Pareto optima, which implies that there exists an optimin point.

### 6.2.2 Optimin and the core

**Proposition 3.** Suppose that the core is nonempty. Then, a feasible payoff distribution \(x\) is in the core if and only if \(x\) is an optimin point.
Proof. Suppose that \( x \) is in the core, which is nonempty by assumption. By definition, there is no individual or coalitional profitable deviation from an element \( x \) in the core. Thus, the definition of the performance function implies that every \( i \in N, \Pi_i(x) = x_i \). To reach a contradiction, suppose that there exists a feasible payoff distribution \( y \) whose value Pareto dominates the performance of \( x \), i.e., for every \( i \in N, \Pi_i(y) \geq \Pi_i(x) \) with at least one strict inequality. By Consistency of \( \Pi \), for every \( i \in N, y_i \geq \Pi_i(y) \). However, this implies that \( \sum_{i \in N} y_i > \sum_{i \in N} x_i \), i.e., \( y \) is not a feasible allocation, a contradiction.

Conversely, to reach a contradiction suppose that \( x \) satisfies the optimin criterion but is not in the core. Let \( d_x = \{ i \in N | S \subseteq N, i \in S, x(S) \geq u(S) \} \), i.e., \( d_x \) is the set of all players who are not part of any deviating coalition. Let \( e_i \geq 0 \) be a constant (“excess”) satisfying \( y_i = x_i - e_i \) such that \( i \in d_{(x_i, y_i)} \) and for every \( \epsilon > 0 \), \( i \notin d_{(x_i, y_i - \epsilon)} \). In words, if \( i \) is a player who is not part of any deviating coalition given payoff distribution \( x \), then it is possible to weakly decrease \( x_i \) so that \( i \) is still not part of any deviating coalition. Since we assume that \( x \) is not in the core, there always exists a player \( i \in d_x \) such that we can strictly decrease \( x_i \) and that

\[
i \in d_{(x_i, x_i - e_i)} \quad \text{with} \quad e_i > 0.
\]

Let \( e = \sum_{i \in d_x} e_i \) be the sum of the excesses as defined above. Because the core exists, it is possible to redistribute \( e \) among players in \( N \setminus d_x \) such that no player \( j \in N \setminus d_x \) has a profitable deviation from the new payoff distribution, denoted by \( y^* \). By Monotonicity assumption part (ii), for every \( i \in d_x, \Pi_i(y^*) > \Pi_i(x) \) because \( D_{-i}(y^*) = \emptyset \) and \( D_{-i}(x) \neq \emptyset \). Analogously, for \( j \in N \setminus d_x \), if \( D_{-j}(x) = \emptyset \) and \( D_{-j}(y^*) = \emptyset \), then \( \Pi_j(y^*) > \Pi_j(x) \). For \( j \in N \setminus d_x \), if \( D_{-j}(x) = \emptyset \) and \( D_{-j}(y^*) = \emptyset \), then \( y_j^* \geq x_j \) and part (i) of the Monotonicity assumption imply that \( \Pi_j(y^*) \geq \Pi_j(x) \). As a result, for every \( i \in N, \Pi_i(y^*) \geq \Pi_i(x) \) with at least one strict inequality. Therefore, the performance of \( y^* \) Pareto dominates the performance of \( x \), which is a contradiction to the supposition that \( x \) satisfies the optimin criterion. We obtain that if \( x \) is an optimin point, which we know it exists by Theorem 2, and the core is nonempty, then the optimin point it is in the core.

**Corollary 1.** The nucleolus satisfies the optimin criterion whenever the core is nonempty.

Proof. When the core is nonempty the nucleolus is in the core. Thus, by Theorem 3 nucleolus satisfies the optimin criterion.

### 6.2.3 Optimin, the Shapley value, and the nucleolus

As Theorem 3 illustrates the set of optimin points is equivalent to the core whenever the core is nonempty. Corollary 1 shows that when the nucleolus is in the core it satisfies the optimin criterion. But when the core is empty, this result no longer holds as the following example illustrates. The game is adapted from Kahan and
Rapoport (1984, p. 61) to compare the core, the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969), and the optimin criterion.

Example: a game with an empty core. Suppose that $N = \{1, 2, 3\}$ and $u(\{1\}) = 35$, $u(\{2\}) = 30$, $u(\{3\}) = 25$, $u(\{1, 2\}) = 90$, $u(\{1, 3\}) = 80$, $u(\{2, 3\}) = 70$, and $u(N) = 110$.

Solution of the game: observe that the core of this game is empty because $x_1 \geq 50$, $x_2 \geq 40$, and $x_3 \geq 30$, which lead to a contradiction. The Shapley value of this game can be calculated by taking the average of marginal contributions, which is $44.166, 36.666, 29.166$. The nucleolus of the game is $(46.666, 36.666, 26.666)$.

Next, I show that the set of points that satisfy the optimin criterion (satisfying the assumptions 1–3) can be characterized by

$$O = \{x \in \mathbb{R}^3 \mid x_1 = 40, x_2 + x_3 = 70, x_2 \geq 30, x_3 \geq 25\}.$$  

Consider an optimin point $x$ such that any coalition of two players could profitably deviate. Then, the performance of player $i$ from $x$ must be $u(i)$ or less by Individual Rationality. More specifically, if $u(i) \leq x_i$, then $\Pi_i(x) = u(i)$, and if $u(i) > x_i$, then $\Pi_i(x) = x_i$. Suppose that $x_1 + x_3 = 80$. Under this case, coalition $\{1, 3\}$ has no incentive to deviate. However, player 2 would receive only 30, which is not greater than its individual payoff $u(\{2\}) = 30$. Similarly, if $x_1 + x_2 = 90$, then coalition $\{1, 2\}$ would not deviate, but player 3 would receive only 20, less than its individual payoff. Lastly, consider $x_2 + x_3 = 70$. In this case, coalition $\{2, 3\}$ would not deviate, and $\Pi_1(x) = 40$, which actually exceeds player 1’s individual payoff of 35. By Monotonicity, 40 is the greatest performance player 1 can obtain. The performances of 2 and 3 would be $\Pi_2(x) = 30$ and $\Pi_3(x) = 25$, respectively, because at distribution $x = (40, x_2, x_3)$ with $x_2 + x_3 = 70$, $x_2 \geq 30$, and $x_3 \geq 25$, player 1 can form a coalition with either 2 or 3 and deviate profitably. Thus, the payoff that $i \in \{2, 3\}$ can guarantee individually would be $u(\{i\})$. For these reasons, every $x \in O$ has the performance $(40, 30, 25)$, which is Pareto optimal.

To compare the optimin criterion with the Shapley value, notice that every two-player coalition would like to deviate from the payoff distribution suggested by the Shapley value. This is not surprising because, according to Shapley (1953),

“... the value is best regarded as an a priori assessment of the situation, based on either ignorance or disregard of the social organization of the players” (emphasis added).

For example, $\{2, 3\}$ would profitably deviate from the Shapley value distribution and get 70 together, as a result of which 1’s payoff would decrease to 35. Compared to the Shapley value (44) and the nucleolus (47), player 1’s payoff is lower under
the optimin criterion (40) in a way that gives 2 and 3 just enough payoff to prevent them from deviating (because they receive 70 in total). Thus, having a lower payoff gives player 1 the safety to enjoy the (worst-case) payoff of 40.

Let’s modify Example 7 so that $u(N) = 110 + c$ with $c > 0$, everything else being equal. As $c$ increases, the set of optimin points follows a pattern similar to before

$$\left\{ x \in \mathbb{R}^3 | x_1 = 40 + c, x_2 + x_3 = 70, x_2 \geq 30, x_3 \geq 25 \right\},$$

up to $c = 10$, in which case the optimin point becomes unique, which is $(50, 40, 30)$ as is illustrated in Figure 13. This is because if $x_1 > 50$, then $\{2, 3\}$ would jointly deviate from $N$ to receive 70, in which case player 1’s value would decrease to 35. By similar arguments, one can show that $(50, 40, 30)$ is indeed the unique optimin point. It turns out that $(50, 40, 30)$ is also the unique element in the core of the game when $u(N) = 120$, and the Shapley value is $(47.5, 40, 32.5)$. When $c > 10$, the core gets larger, hence the set of the optimin points.

6.3 Matching markets

Gale and Shapley (1962) published in the American Mathematical Monthly, a paper that is generally considered to have initiated matching theory. Another remarkable point about this paper is that it contained almost no explicit mathematics such as formulas. In this paper, the authors introduced a “two-sided” matching model in which there are two sets of invididuals (or objects) that need to be paired or matched.
Two-sided matching problems include the marriage market, school choice, medical labor markets; one-sided matching problems include housing markets and kidney exchange. The literature on matching markets has grown considerably since the publication of Gale and Shapley (1962) and another seminal paper by Shapley and Scarf (1974).

Let \((A, B, (>_i)_{i \in A \cup B})\) be a marriage problem where \(A\) and \(B\) are two disjoint finite sets, in which each individual \(i\) in a set \(A\) or \(B\) ranks those potential partners in the other set. For convenience, I assume that there are \(n\) individuals in each set and preferences are strict. Preference of \(i\) is captured by \(>_i\), which is over \(C \cup i\) where \(C\) is \(A\) or \(B\) and \(i \notin C\). Notation \(i >_i j\) means that individual \(i\) would not like to marry \(j\).

Matching in a marriage problem is a function, \(\mu : A \cup B \to A \cup B\) that satisfies the following properties:

1. For all \(a\) in \(A\), \(\mu(a) \notin B\) implies that \(\mu(a) = a\);
2. For all \(b\) in \(B\), \(\mu(b) \notin A\) implies that \(\mu(b) = b\);
3. For all \(a\) in \(A\) and \(b\) in \(B\), \(\mu(a) = b\) if and only if \(\mu(b) = a\).

A matching \(\mu\) is called individually rational if there is no individual \(i\) such that \(i >_i \mu(i)\). A matching is called stable if it is individually rational and there are no \(a \in A\) and \(b \in B\) who are not married to each other yet prefer each other to their current partners.

The optimin criterion’s application to a matching problem is similar to its application to cooperative games. Let \(\mu\) be a matching and \(I \subset A \cup B\) be a group of players in which each player in \(I\) is either single or matched with another partner in \(I\). Then, \(I\) is said to be a profitable deviation from a matching \(\mu\) if every player in \(I\) prefers their new partner to their current partner. As before, the first step is to evaluate a matching. The performance of a matching \(\mu\) to an individual \(i\) is the worst-case outcome under (i) the matching \(\mu\) or (ii) any profitable deviation by an individual or group, \(I\). Let \(\mathbf{v}_i\) denote the preferences of \(i\) based on the performance of a matching—i.e., if the performance of matching \(\mu\) is greater than or equal to the performance of matching \(\mu'\), then \(\mu \geq_{\mathbf{v}_i} \mu'\). The second step would be to make comparisons among the evaluations of various matchings. Accordingly, a matching is said to satisfy the optimin criterion if its value is Pareto optimal—no individual can improve his or her worst-case outcome without decreasing someone else’s worst-case outcome.

Definition 6. A matching \(\mu\) is said to satisfy the optimin criterion if for every player \(i \neq j\) and every \(\mu'\), \(\mu' >_{\mathbf{v}_i} \mu\) implies that there is some \(j\) with \(\mu >_{\mathbf{v}_j} \mu'\).

Remark 5 shows that every stable matching in the marriage problem is an optimin point.
Remark 5. Every stable matching satisfies the optimin criterion.

Proof. Because there is neither a unilateral nor a group profitable deviation from a stable matching (see, e.g., Roth and Sotomayor, 1992)), each individual’s value of a stable matching is equal to the matching’s “payoff” to the individual. (This is similar to the fact that the performance of a Nash equilibrium is exactly its payoff distribution.) It is left to show that the performance of a stable matching is Pareto optimal, which is true because every stable matching is Pareto optimal (see, e.g., Abdulkadiroğlu and Sönmez, 2013).

By similar reasoning, it can be shown that the result would extend to college admission problems (many-to-one matching). However, there are problems such as the roommate problem, in which the existence of stable matchings is not guaranteed. In such situations, a matching that satisfies the optimin criterion would always exist as long as there are finitely many individuals or objects to be matched. I omit this existence proof as it is essentially the same as the proof of the existence of pure optimin points in strategic games (Proposition 4).

Shapley and Scarf (1974) proposed a housing market (one-sided matching) model in which a set of houses is to be assigned to a set of individuals who have initial endowments. (For formal model see, e.g., Sönmez and Ünver, 2011). Gale’s Top Trading Cycles (TTC) algorithm gives a rather strong solution to this problem: it chooses unique matching in the core of the housing market, which is Pareto efficient and individually rational (Roth and Postlewaite, 1977). The definition of the optimin in one-sided matching contexts would be analogous to its definition in two-sided markets. The aforementioned properties of the TTC algorithm show that its outcome satisfies the optimin criterion.

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