The Irradiation Instability of Protoplanetary Disks

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Abstract

The temperature in most parts of a protoplanetary disk is determined by irradiation from the central star. Numerical experiments of Watanabe and Lin suggested that such disks, also called “passive disks,” suffer from a thermal instability. Here we use analytical and numerical tools to elucidate the nature of this instability. We find that it is related to the flaring of the optical surface, the layer at which starlight is intercepted by the disk. Whenever a disk annulus is perturbed thermally and acquires a larger scale height, disk flaring becomes steeper in the inner part and flatter in the outer part. Starlight now shines more overhead for the inner part and so can penetrate into deeper layers; conversely, it is absorbed more shallowly in the outer part. These geometric changes allow the annulus to intercept more starlight, and the perturbation grows. We call this the irradiation instability. It requires only ingredients known to exist in realistic disks and operates best in parts that are both optically thick and geometrically thin (inside 30 au, but can extend to further reaches when, e.g., dust settling is considered). An unstable disk develops traveling thermal waves that reach order unity in amplitude. In thermal radiation, such a disk should appear as a series of bright rings interleaved with dark shadowed gaps, while in scattered light it resembles a moving staircase. Depending on the gas and dust responses, this instability could lead to a wide range of consequences, such as ALMA rings and gaps, dust traps, vertical circulation, vortices, and turbulence.

Unified Astronomy Thesaurus concepts: Protoplanetary disks (1300); Circumstellar dust (236); Hydrodynamics (1963); Astrophysical fluid dynamics (101)

1. Introduction

Currently, the main bottleneck for understanding planet formation lies in an incomplete knowledge of the protoplanetary disk. In this work, we study the dynamics of passive disks, i.e., disks where stellar irradiation dominates the energetics. These include almost all parts of the disk, except perhaps for the innermost region. We show that passive disks suffer from an instability, and this could have a broad range of observational and theoretical ramifications.

1.1. Motivation

I. Gaps and Rings: Recent observations made using the Atacama Large Millimeter/submillimeter Array (ALMA) have shown that typical disks are not the smooth power laws beloved by theorists. Rather, bright rings and dark gaps are ubiquitous, on scales from tens to hundreds of au (e.g., ALMA Partnership et al. 2015; Andrews 2020; Huang et al. 2018). At the moment, these rings and gaps are most commonly attributed to the effects of unseen planets (e.g., Baruteau et al. 2014; Dong et al. 2015; Dipierro et al. 2016; Bae et al. 2017; Dong et al. 2017; Zhang et al. 2018). And in a few cases, there is strong kinematic evidence for planets, such as in the gap of HD 163296 (Pinte et al. 2018). But the near ubiquity of gaps and rings (Long et al. 2018; Nielsen et al. 2019) is at tension with the paucity of high-mass planets at these distances, as suggested by direct imaging surveys (Bowler 2016) and microlensing surveys (Suzuki et al. 2016; Gaudi 2021). While the planet hypothesis is difficult to exclude, given its large number of free parameters (such as planet mass and orbit, disk viscosity, and history), it is worthwhile to consider whether planets are the causes for the gaps and rings, or instead the products of such features.

Many alternative scenarios have been proposed to explain these features, including dust-drift-driven viscous ring instability (Wünsch et al. 2005; Dullemond & Penzlin 2018), secular gravitational instabilities in the dust (Takahashi & Inutsuka 2014), dead zones (Flock et al. 2015), snow lines (Okuzumi et al. 2016), MHD wind-driven structures (Bai 2014; Riols et al. 2020), and an eccentric disk instability (Li et al. 2021). Lastly, Siebenmorgen & Heymann (2012) and Ueda et al. (2019) propose that they may be triggered by an instability found in Monte Carlo simulations of irradiated disks. This last proposal may be closely related to the irradiation instability considered in this paper.

These features motivate us to study the stability of passive disks. An inherent instability could naturally explain the multiple gaps and rings in a given disk, without invoking an arbitrary number of planets. It would also directly impact the formation and migration of planets.

II. Dust Wafting and Migration: Radiation in disks is controlled by dust. Micron-sized grains absorb and scatter starlight, while larger (~millimeter-sized) grains provide the bulk of the opacity for the thermalized radiation. Conversely, the distribution of dust grains is strongly influenced by gas dynamics.

Both the spectral energy distribution (e.g., Chiang et al. 2001; Furlan et al. 2006; Woitke et al. 2019) and scattered-light images (e.g., Avenhaus et al. 2018) of protoplanetary disks suggest that micron-sized grains must reside at least a couple gas scale heights above the midplane. Although they are tightly coupled to the gas, vertical settling high up in the atmosphere is rapid (Dullemond & Dominik 2004b), even more so when dust coagulation occurs (Dullemond & Dominik 2005). These small grains must be replenished, by fragmentation and/or updraft. Both can be provided by gas turbulence or circulation.
The millimeter-sized dust, on the other hand, is much more weakly coupled to the gas, and so it settles closer to the midplane. Without any local pressure maxima, these grains can drift for large distances over the disk’s lifetime (Weidenschilling 1977). Pressure maxima, on the other hand, can halt this migration and trap drifting particles (Whipple 1972). ALMA images tantalizingly suggest that the observed rings and gaps are the smoking gun for particle traps (e.g., Andrews & Bärmstiel 2018; Dullemond et al. 2018).

A disk instability could therefore have important implications for the dust behavior in protoplanetary disks. It could generate turbulence or meridional flows that waft up the micron-sized grains. And it may also produce pressure maxima, leading to vortices or axisymmetric rings, natural barriers for inwardly drifting dust and welcoming cradles for planet formation.

III. Turbulence and Accretion: A crucial open question in the study of protoplanetary disks is why disks accrete (Lynden-Bell & Pringle 1974). It is unclear whether these disks are turbulent, and whether such turbulence can transport enough angular momentum to disperse disks in a few million years (see review by Klahr et al. 2018). Until recently, the most promising mechanism for generating turbulence was the magnetorotational instability (Balbus & Hawley 1998). But such MHD proposals require a sufficient degree of ionization so that the disk gas can couple effectively to the magnetic field. That is hard to achieve, particularly for disks that are very dusty (e.g., Bai 2015; Simon et al. 2015). A plethora of other instabilities have been investigated (as reviewed in Klahr et al. 2018). One example is the vertical shear instability (Urrn & Brandenburg 1998; Lin & Youdin 2015), which may produce sufficiently strong turbulence for accretion ($\alpha \sim 10^{-3}$, where $\alpha$ is the Shakura–Sunyaev parameter), but at large distances (Flock et al. 2020). This $\alpha$-value appears consistent with some upper limits placed at various disk locales (e.g., $\alpha \lesssim 7 \times 10^{-3}$; Flaherty et al. 2018). Alternatively, disks may also disappear via a nonturbulent mechanism such as disk winds (Bai & Stone 2013).

As the situation remains murky, we are motivated to look for a robust instability that could generate turbulence or fluid circulation and ultimately drive accretion.  

1.2. Prior Work on Passive Disks and Their Stability

For a protoplanetary disk accreting at a typical rate (say, $M \sim 10^{-8} M_{\odot}$ yr$^{-1}$), disk heating is dominated by stellar irradiation except inward of $\sim 1$ au (see, e.g., D’Alessio et al. 1998). Hence, most of the disk should be regarded as being “passively heated.” Kenyon & Hartmann (1987) showed that passive disks can account for the far-IR excesses of IRAS disks, provided that they are flared. The predicted flaring morphology was confirmed by Hubble images (e.g., Burrows et al. 1996).

Chiang & Goldreich (1997, hereafter CG97) set forth a simple model for such disks. Optical light from the star is absorbed high up in the disk by small grains. And radiation from these grains illuminates the disk midplane. In thermal and hydrostatic equilibrium, flared passive disks take a simple analytic form: $\frac{\rho}{\rho_0} \propto r^{2/7}$, where $h$ is the vertical scale height and $r$ the cylindrical radius (Kusaka et al. 1970; Cunningham 1976; Chiang & Goldreich 1997). Passive disk models have been very successful in explaining the spectral energy distributions of protoplanetary disks, provided that one also accounts for some vertical settling of the dust (Chiang et al. 2001; Dullemond & Dominik 2004b; D’Alessio et al. 2006).

The stability of passive disks was first investigated by Cunningham (1976) for disks around black holes. That work was extended to protoplanetary disks by D’Alessio et al. (1999). Their simple analysis showed that such disks are stable: thermal perturbations propagate inward and damp along the way.

The equilibrium solution for a stable disk should be easily obtained by iteration. Yet, mysteriously, such attempts are often plagued by convergence issues (e.g., Dullemond & Dominik 2004a; Min et al. 2009; Siebenmorgen & Heymann 2012; Wang & Goodman 2017; Ueda et al. 2019). Using Monte Carlo radiative transfer codes to describe the radiation effects more accurately than CG97, these authors iteratively solved the equations of hydrostatic equilibrium and thermal equilibrium. They often find no convergence, particularly for disks with realistically large dust surface densities. With successive iterations, new waves appear at large radii and propagate inward with order-unity amplitudes (see, e.g., Figure 7 of Ueda et al. 2019). It is unclear if such behavior is generic in physical disks, or if it is an artificial instability introduced by the iteration procedure. In any case, this issue hampers further study of realistic disks.

The work by Watanabe & Lin (2008), though receiving little attention (see Ueda et al. 2021, for a modern reincarnation), raises an interesting possibility. Using 1D simplified radiative transfer, halfway in complexity between CG97 and a Monte Carlo code, they found that passive disks are unstable. They argued that the instability is likely related to changes in the optical surface. Their numerical experiments showed that the unstable disk develops inward-traveling thermal waves that reach order-unity amplitudes.

In this work, we further elucidate the origin of this behavior, which we call the “irradiation instability.” Using both analytical and numerical tools (including the radiative transfer code RADMC-3D), we demonstrate that the instability is genuine, not numerical—although using RADMC-3D (Dullemond et al. 2012, henceforth called RADMC) to iterate is risky unless guided by analytical insights. We derive the conditions for such an instability and argue that they should be prevalent in observed disks.

Paper Overview: In Section 2 we present a cartoon view of the instability. In Sections 3–6 we do the math. Because of the complexities involved with solving the radiative transfer problem, we develop three models that are successively more complex and more realistic. Readers who prefer to skip the technical details may proceed to Section 7, where we summarize the main results. We end with an extensive discussion of the assumptions (Section 8) and a brief introduction of things to come (Section 9).

2. The Irradiation Instability—Cartoon Version

2.1. Passive Disks in Equilibrium

We follow the concepts and notations of CG97 to study passive disks. Figure 1 (left panel) shows a cartoon of a passive disk in equilibrium, with the background shading representing the density of dust grains, relative to their midplane density. The star’s optical light is absorbed by dust grains at altitude $H$, the “optical surface,” which lies a few scale heights above the midplane. This layer reradiates half of the luminosity it receives
down into the disk, and the latter then reemits this energy at longer wavelengths (thermalized radiation).

The amount of heating a disk receives is determined by the flaring of its optical surface. In Figure 1, the relevant star rays received by a radial zone of concern are those bound by the two green arrows. When balancing the heating against blackbody cooling and insisting on vertical hydrostatic equilibrium, a flared solution is found for these disks (Kusaka et al. 1970; Chiang & Goldreich 1997; Dullemond 2000).

2.2. Passive Disk Perturbed

Figure 1 illustrates what happens to a disk when a localized thermal perturbation increases the scale height $h(r)$. There are two effects. First, the optical surface rises in proportion to the scale height (dashed curve). This is what is considered in D’Alessio et al. (1999). The disk intercepts a bit more stellar flux, but not enough to overcome the extra blackbody cooling from a now hotter disk. As a result, the perturbation damps away.

But there is another effect. Consider first the inner half of the scale height perturbation. The slope of the optical surface is increased there, so the star’s rays shine more directly overhead (i.e., closer to the surface normal) and can penetrate more deeply into the disk. This effect is analogous to stellar limb darkening, but now for absorption. Conversely, at the outer part of the perturbation, starlight penetrates more shallowly owing to the more tangent slant. As a result, the opening angle between the two green rays is increased. This means excess heating, and given the right perturbation, it can overcome the excess cooling and drive an instability.

Interestingly, an extreme manifestation of stronger flaring leading to enhanced heating is the inner rim of a protoplanetary disk (Dullemond et al. 2001). In this region, the star’s rays enter the disk almost head-on. The strong heating puffs up the inner rim into a wall.

In the following sections, we make this picture quantitative.

3. Setting the Stage

To study the stability of a passively irradiated disk, we make a number of simplifying assumptions, both to highlight what we believe are the most relevant effects and to facilitate a short treatment. These include the following:

1. opacity is provided only by dust;
2. dust traces gas with a constant density ratio;
3. vertical hydrostatic equilibrium;
4. the radial profile of surface density does not vary;
5. axisymmetry;
6. gas temperature tracks dust temperature;
7. star is point-like, so all star rays are radial;
8. no inner hot rim, which would otherwise cast a shadow.

In addition to these, we also introduce a string of other simplifications, which we will describe as we go along. In Section 8, we assess how some of these assumptions and how varying them may qualitatively affect our results.

3.1. The Thermal Equation

We adopt cylindrical coordinates ($r$ and $z$) and assume that the disk is sufficiently thin that we may work to leading order in $z/r$ (the small-angle approximation). The midplane temperature $T$ is governed by the thermal equation (obtained in Appendix A under a number of assumptions)

$$\frac{3}{8} \tau_{\text{mm}} c_p \Sigma_{\text{gas}} \frac{\partial T}{\partial t} = \frac{1}{2} F_{\text{irr}}[T] - \sigma_{\text{SB}} T^4. \tag{1}$$

This describes the rate of thermal energy increase per unit disk area as a result of the imbalance between starlight heating and blackbody cooling. Here, $c_p$ is the specific heat per unit mass, $\Sigma_{\text{gas}}$ is the gas surface density, and $\tau_{\text{mm}}$ is the vertical optical depth for thermal radiation (more details below), which is assumed to be larger than unity. On the right-hand side, $F_{\text{irr}}$ is the stellar flux incident on the disk’s optical surface. The square brackets denote that $F_{\text{irr}}$ depends on the local profile (not just value) of temperature. And the factor of 1/2 multiplying $F_{\text{irr}}$ arises because half of the incident flux is reradiated by grains in this surface to space, without heating the disk interior.

In writing Equation (1), we have simplified the physics of vertical transport under the assumption that the timescale of variation is much longer than the thermal time (see Appendix A). A more accurate treatment should allow for...
vertical thermal waves. We argue in Appendix A.3 that this likely does not impact our results.

As we show in Appendix A, the $T$ that appears throughout Equation (1) should be the midplane temperature, not the surface temperature, provided that the factor of $\tau_{mm}$ is included on the left-hand side.\footnote{The factor of $\tau_{mm}$ has been incorrectly neglected in the literature (D’Alessio et al. 1999; Dullemond 2000; Watanabe & Lin 2008). But its neglect leads only to a change in timescale.}

The heating flux is (Safronov 1962; Kusaka et al. 1970; Chiang & Goldreich 1997)

$$F_{\text{irr}} = \frac{L_\star}{4\pi r^2} \sin \alpha,$$

where $\alpha$ is the grazing angle of the star rays relative to the optical surface. We denote the height of this surface above the midplane by $H$. In the limit that the disk is thin ($H/r \ll 1$) and the star is point-like, we can simplify the expression for $\alpha$ to arrive at (CG97, Chiang et al. 2001)

$$F_{\text{irr}} \approx \frac{L_\star}{4\pi r^2} \left( \frac{d}{d \ln r} \frac{H}{r} \right).$$

If $H/r$ decreases with radius, the disk falls into shadow cast by interior annuli, and one should instead set $F_{\text{irr}} \to 0$. Hidden in this equation is the implicit assumption that the stellar heating is only determined by the local gradient of the optical surface. This is reasonable as long as the variation length scale is much greater than $H$, as is certainly satisfied by the equilibrium disk. However, this may not be true for a perturbed disk, an issue we call “horizontal averaging” and return to later.

In hydrostatic equilibrium, the midplane temperature determines the gas scale height as

$$h = \frac{c_s}{\Omega} = \left( \frac{k_B}{\mu m_p G M_*} \right)^{1/2} r^{1/2},$$

where $M_*$ is the stellar mass, $m_p$ the proton mass, and $\mu$ the mean molecular weight (henceforth set to $\mu = 2.3$). The scale height in turn sets the dust density field, which we take to be

$$\rho_{\text{dust}}(r, z) = \frac{\Sigma_{\text{dust}}(r)}{\sqrt{2\pi h(r)}} \exp \left[ -\frac{z^2}{2h(r)^2} \right],$$

where $\Sigma_{\text{dust}}$ is the vertical column density of dust (both sides of equator). The dust density controls how far the stellar flux penetrates into the disk.

In this study, we assume that vertical hydrostatic equilibrium is maintained instantaneously, or Equations (3)–(4) remain valid as the disk heats and cools. This is likely valid in the region where the thermal time is longer than the orbital time, or inward of $\sim 50$ au for our fiducial disk (see below).

### 3.2. The Optical Surface

The key ingredient in this problem is the geometry of the optical surface. For our analytical study (not the RADMC simulations), we consider two approximate forms for $H$:

1. Simplistic Surface: following CG97, we introduce an important quantity $\chi$, the ratio between the optical surface height and the local gas scale height,

$$\chi \equiv \frac{H}{h}. \quad (5)$$

The value of $\chi$, for passive disks in equilibrium, depends on dust density logarithmically (see more below). CG97 found that the value of $\chi$ ranged from 5 to 4 between 3 and 100 au, for a minimum-mass solar nebula (MMSN) type disk. So at first sight it seems reasonable to assume that $\chi$ is a constant everywhere and does not vary when the disk is perturbed. We call this a “simplistic surface.” It is adopted by D’Alessio et al. (1999) for their stability analysis.

2. Realistic Surface: alternatively, one could self-consistently determine the optical surface using the definition that the optical depth to the star is unity. This brings in some algebraic difficulties but, as we show below, is essential for describing perturbed disks. The name “realistic” is a euphemism—this approach remains an approximation to reality, which can be addressed only using radiative transfer codes.

In Sections 4–5, we study passive disks under each of these approximations. The key result is that under the first approximation the disk is stable, whereas under the second, more accurate one it is unstable. We then perform RADMC simulations (Section 6) to demonstrate that true disks also exhibit instability.

#### 3.3. Fiducial Disk

We assume that the star has solar mass and luminosity, $M_*=M_\odot$, $L_*=L_\odot$. For our fiducial disk, we choose a dust surface density

$$\Sigma_{\text{dust}} = 20\ r_{\text{au}}^{-1}\ \text{g cm}^{-2},$$

where $r_{\text{au}}=(r/1\ \text{au})$. This density is similar to the MMSN at 1 au (Hayashi 1981) but falls off more gradually with radius and so is more consistent with many observed systems (e.g., Clevees et al. 2016; van Boekel et al. 2017).

We adopt the dust opacity from Figure 2 of Woitke et al. (2016), calculated for a power-law dust mixture with sizes from 0.05 $\mu$m to 3 mm, which we approximate as

$$\kappa_\lambda = 10^5 \left( \frac{\lambda}{0.5\ \mu\text{m}} \right)^{-1/2} \text{cm}^2/\text{g(dust)},$$

for total extinction. In the optical, this opacity falls below that adopted by CG97 by a factor of 40, as they assume that all grains are small (0.1 $\mu$m).

In our analytical study, the radiation field is described by fluxes at only two frequencies: that of starlight and that of disk thermal radiation. So the disk radiative properties can be encapsulated by two vertical optical depths. One is $\tau_V$, the optical depth in the visual band,

$$\tau_V \equiv \frac{1}{2} \kappa_\lambda \Sigma_{\text{dust}} = 10^4 \ r_{\text{au}}^{-1},$$

where the factor of 1/2 indicates integration from the midplane upward. The other is the optical depth for dust thermal radiation. We name it $\tau_{mm}$, with the actual wavelength determined by the peak of the local blackbody. Figure 2 (bottom panel) shows these two optical depths. Our fiducial
Our fiducial disk. The top panel compares the thermal time (red curve) and the dynamical time (orbital period, green curve); inward of ~50 au, it may be reasonable to assume that vertical hydrostatic equilibrium is reached quickly. The bottom panel displays the vertical optical depths for visual (blue) and for thermal radiation (red). The disk is optically thick throughout. Here, the disk temperature profile is that of the equilibrium disk, and $\chi = \frac{h}{r}$, both determined in Section 5.1.

disk remains optically thick to thermal radiation out to ~100 au.

Equation (1) induces us to define an evolutionary timescale, the thermal time,

$$t_{th} = \frac{3}{8} \frac{c_p}{\sigma_{SB}} \frac{\sum}{\tau_{mm}}. \tag{9}$$

We evaluate this timescale using the equilibrium disk temperature and a gas-to-dust ratio $\sum_{gas}/\sum_{dust} = 100$, and we plot it in Figure 2. It decays outward as $t_{th} \propto r^{-0.93}$ and intersects the disk dynamical timescale at around 50 au. Outside this radius, the assumption of vertical hydrostatic equilibrium fails.

4. Stability for Simplistic Surface

We solve the thermal equation under the assumption that $\chi = H/h$ is a constant in both space and time. While allowing for a spatially varying $\chi$ does not much alter our conclusions, a time-varying $\chi$ will, as we discuss later. We will present the equilibrium and a simple stability analysis, neither of which is new to this work. We do so both to connect to the previous (mis)understanding that passive disks are stable and to highlight the differences between such a treatment and a more correct one (Section 5).

4.1. Equilibrium Disk

The equilibrium solution has been presented by Kusaka et al. (1970), CG97, and Dullemond (2000). The thermal equation reads

$$\frac{L_\ast}{8\pi\sigma_{SB}} \frac{d}{d\ln r} \frac{H}{r} = r^2 T^4. \tag{10}$$

It may be integrated after setting $H = \chi h$. With $\chi$ being a constant and $h$ determined by hydrostatic equilibrium (Equation (3)), this equation has a power-law solution:

$$\frac{h}{r} = 0.02 \frac{\chi^{1/7}}{r_{au}^{2/7}} \frac{L_\ast}{L_\odot}^{1/7} \left( \frac{M_\ast}{M_\odot} \right)^{1/7} \tag{11}$$

and

$$T = 90K \frac{\chi^{2/7}}{r_{au}^{3/7}} \frac{L_\ast}{L_\odot}^{2/7} \left( \frac{M_\ast}{M_\odot} \right)^{1/7}. \tag{12}$$

These power laws reproduce Equations (14) in CG97, albeit with slightly different normalizations.

As described below (Section 5.1), $\chi$ can be solved for self-consistently. It varies slowly for a disk in equilibrium, so Equations (11)–(12) remain approximately correct. In particular, for our fiducial disk, $\chi$ may be adequately fitted as $\chi \approx 4.5 r_{au}^{-0.75}$, and we estimate the optical surface to lie at

$$\frac{H}{r} = \frac{\chi}{r} \approx 0.11 \frac{r_{au}}{0.21}. \tag{13}$$

4.2. Linear Perturbations

We repeat the stability analysis by Cunningham (1976) and D’Alessio et al. (1998), but using a form that will make generalization to a more complicated form of $F_{irr}$ straightforward. We perturb by setting $T \rightarrow T + \delta T$, and similarly $F_{irr} \rightarrow F_{irr} + \delta F_{irr}$. Linearizing the thermal equation yields

$$\frac{\partial \delta T}{\partial T} = \frac{1}{t_{th}} \left( \delta F_{irr} - \frac{4 \delta T}{T} \right). \tag{14}$$

The power-law equilibrium profile, together with a constant $\chi$, yields

$$\frac{\delta F_{irr}}{F_{irr}} = \left( \frac{d}{d\ln r} \frac{H}{r} \right)^{-1} \frac{\partial}{\partial \ln r} \left( \delta H \frac{H}{r} \right)$$

$$= \frac{7}{4} \frac{\partial}{\partial \ln r} \left( \frac{7 \delta T}{2 T} + \frac{1}{2} \right), \tag{15}$$

where the first term on the right-hand side bestows the traveling wave nature for the perturbation, while the second term provides positive feedback. Now the thermal equation becomes

$$\frac{\partial \delta T}{\partial T} = \frac{1}{t_{th}} \left( \frac{7}{4} \frac{\partial}{\partial \ln r} \frac{\delta T}{T} - \frac{7 \delta T}{2 T} \right). \tag{16}$$

This equation admits a decaying traveling wave solution. We consider a complex $\delta T$ of the form

$$\frac{\delta T}{T} \propto e^{\eta t + ik \ln r}, \tag{17}$$

with wavenumber $k$ (a real number, not restricted to integers) and growth rate $\eta$ (complex valued). The physical perturbations are understood to be the real parts of these. Inserting this into
Equation (16) then yields the growth rate
\[ s = \frac{1}{\ln h} \left( \frac{7}{4} jk - \frac{7}{2} \right), \] (18)
or the thermal perturbation propagates inward with a phase speed \( \frac{d \ln r}{d t} \big|_{\text{phase}} = -\frac{1}{4} \text{Im}[s] = -\frac{7}{4} \ln h \) and decays at a rate
\[ \text{Re}[s] = -\frac{7}{2} \frac{1}{\ln h} \text{ (D’Alessio et al. 1998)}. \]

Why is the disk stable? Naively, one might imagine that a hotter region rises up in scale height and can therefore intercept more stellar flux. However, under the assumption that \( \chi \) is constant, stellar heating scales with temperature to the half-power \((F_{*\text{irr}} \propto H \propto T^{1/2})\), while cooling scales with a higher power \((\propto T^4)\). So the perturbation decays in time.

To understand why perturbations propagate inward, we consider the disk surface in the presence of a local positive temperature perturbation. The inner half of the affected region has a steeper grazing angle than in equilibrium and so is heated more. This raises the local surface under hydrostatic adjustment. Conversely, in the outer half, the surface drops. As a result, the perturbation moves inward.

5. Instability for Realistic Surface

A crucial effect not accounted for in the simplistic surface model is that the depth of starlight penetration changes when the slope of the optical surface varies (Figure 1). That strengthens the heating response and leads to an instability.

Here we locate the disk surface using the definition that the optical depth to the star is unity. The optical depth along a slanted ray from the star that has an inclination angle \( \theta_H \) is (Watanabe & Lin 2008)
\[ \tau_{\text{slant}} = (1 + \tan^2 \theta_H)^{1/2} \int_0^r \kappa_V \rho_{\text{dust}} (r', z = \tan \theta_H r') dr' \approx \int_0^r \kappa_V \rho_{\text{dust}} (r', z = \theta_H r') dr', \] (19)

where the approximate sign holds when \( \theta_H \ll 1 \). With the density profile given by Equation (4) and the vertical optical depth \( \tau_V = \kappa_V \Sigma_{\text{dust}} / 2 \), we have the following equation to determine the more realistic optical surface:
\[ 1 = \tau_{\text{slant}} \approx \int_0^r dr' \frac{2 \tau_V (r')}{\sqrt{2 \pi} h (r')} e^{\frac{1}{2} \frac{H(r')^2}{r'^2} - \frac{\kappa_V}{h(r')^2}}. \] (20)

This is our revised equation for the optical surface. Given a disk opacity and temperature profile, one can use it to solve for \( H(r) \).

5.1. An Approximate Form and Qualitative Discussion

To better understand the implications of Equation (20), we derive a simpler form that is valid for \( \chi - \frac{H}{r} \gtrsim 1 \). The optical depth along a given line of sight (\( \theta_H \)) is mostly produced by material close to the optical surface, so we can write \( \tau_{\text{slant}} \approx \rho_{\text{dust}} (r, H) \kappa_V \Delta r \), where \( \Delta r \) is the local density scale height as experienced by the slanted ray. Expressing density as \( \rho_{\text{dust}} (r, H) \propto e^{-H^2/(2\sigma^2)} \propto e^{-\theta_H^2/(2\sigma^2)} \) (Equation (4)), we have
\[ \Delta r = \left( \frac{d \ln \rho_{\text{dust}}}{dr} \right)_{\text{constant } \theta_H}^{-1} = \frac{r}{\chi^2 \gamma}, \] (21)

after defining a flaring index \( \gamma \),
\[ \gamma \equiv \frac{d \ln (h/r)}{d \ln r}. \] (22)

This index equals \( 2/\gamma \) for the power-law disk (Equation (11)). Setting \( \tau_{\text{slant}} = 1 \) then yields our desired approximate form for the optical surface,
\[ \frac{\chi^2}{2} e^{\chi^2} \approx \frac{\tau_V}{\sqrt{2\pi} \gamma} \frac{1}{h(r)}. \] (23)

Although much simpler than Equation (20), this form still must be solved numerically. However, it makes explicit the dependencies of \( \chi \) on disk properties such as the optical depth and the flaring angle, key physical elements for the instability.

In the following, we discuss the main features of this model of a realistic surface.

1. Equilibrium disk. One can obtain the equilibrium disk by iteratively solving the optical surface equation (Equation (20)) and the steady-state thermal equation. Appendix B.2 explains how we perform this procedure and achieve convergence. The resulting value of \( \chi \) for our fiducial disk is shown in Figure 2. It drops only modestly over a large stretch of the disk. This can be easily understood from Equation (23): for \( \chi \gg 1 \), we can neglect terms outside the exponential to find that \( \chi \) depends logarithmically on \( \tau_V \), \( \chi \sim (2 \ln \tau_V)^{1/2} \). Such a weak dependence on the surface density arises owing to the rapid fall-off of density with vertical height and allows CG97 to assume a constant \( \chi \) throughout the disk and to derive a power-law equilibrium solution (Equation (11)).

2. Instability. The most interesting implication of Equation (23), however, lies in the relation between \( \chi \) and the flaring index \( \gamma \). When the flaring index is larger, \( \chi \) is smaller because the starlight shines closer to the disk’s surface normal (larger grazing angle) and so can penetrate deeper toward the midplane. It is this dependence on the grazing angle that is crucial for the irradiation instability (Section 2).

To illustrate this point, we retain the key factors in Equation (23) to recast it as
\[ e^{-\chi^2 / 2} \propto \frac{\partial (h/r)}{\partial r}. \] (24)

A thermal perturbation of the spatial form \( e^{ik \ln r} \) (with \( k \gg 1 \)) then perturbs the surface as
\[ \delta \chi \approx -ik \ln h \frac{\delta h}{\chi \gamma}. \] (25)

\[ 5 \quad \text{Equation (23) may also be derived directly from Equation (20) as follows:} \]
\[ \text{assuming that } \chi \text{ is large, the integrand in Equation (20) is exponentially suppressed unless } r' \text{ is close to } r. \text{ Therefore, we may approximate } r' \text{ close to } r, \text{ and we set } \frac{\sqrt{2 \pi} h(r')}{\kappa_V} \approx \chi^2 (1 - 2 \gamma r') \text{, in which case the integral yields Equation (23)} \]
\[ \text{(Garraud & Lin 2007).} \]

\[ 6 \quad \text{For a more exact form, see Equation (B13).} \]
This in turn affects the heating rate as (using Equation (2))
\[
\delta F_{\text{irr}} \propto \frac{\partial}{\partial \ln r} \delta H = \frac{\partial}{\partial \ln r} (\chi \delta h + h \delta \chi) \\
= ik(\chi \delta h + h \delta \chi). \tag{26}
\]
While the first term in the brackets is present in the model of a simplistic surface (constant \(\chi\)), the second term is not. It describes the change in the penetration depth as the grazing angle varies. Inserting Equation (25) into this expression, we find a positive-definite contribution to the heating,\footnote{For a more exact form, see Equation (B14).}
\[
\delta F_{\text{irr}} \propto ik\chi \delta h + \frac{k^2}{\chi \gamma} \delta h. \tag{27}
\]
Compared with Equation (15), the second term is new. At sufficiently high wavenumbers, it can overcome damping by radiative cooling and lead to a new instability, the irradiation instability.

3. The Smearing Length. There is a unique scale length associated with irradiation. Thus far, we have assumed that light is absorbed at a well-defined surface (the optical surface). But in truth, starlight is deposited over a distance \(\Delta r\) (Equation (21)). We call the latter the “smearing length” and define an associated wavenumber
\[
k_{\text{smear}} \equiv \frac{r}{\Delta r} = \chi^2 \gamma. \tag{28}
\]
With this new quantity, Equation (27) now reads
\[
\delta F_{\text{irr}} \propto \left(1 + \frac{k}{k_{\text{smear}}}\right) k \chi \delta h. \tag{29}
\]
For \(k \ll k_{\text{smear}}\), the simplistic surface model prevails and thermal perturbations lose out to radiative cooling. Modes with \(k \sim k_{\text{smear}}\) or larger, on the other hand, can be destabilized by the changes in the penetration depth of the starlight.

Moreover, mode growth rates are affected by the smearing length. While the above simple expression indicates that mode growth rate rises with \(k\) as \(k^2\), a more careful analysis (Appendix B.4) that accounts for a finite smearing length yields a saturated growth rate for modes with \(k \gg k_{\text{smear}}\) (see text below).

5.2. Linear Perturbations
After the above qualitative discussions, we now present results from more rigorous derivations. We call the following model “analytical,” to distinguish it from the RADMC simulations.

We employ Equations (1), (2), (3), and (20) to study the stability of a thermal perturbation. We first derive an approximate dispersion relation and then compare it against exact numerical solutions. Our analysis shows that perturbations of certain wavelengths indeed grow in amplitude, on a timescale that is of order the local thermal time.

Assuming a space/time dependence of the form \(e^{i \omega t + ik \ln r}\) (Equation (17)), we derive in Appendix B.4 an approximate analytic expression for the growth rate,
\[
\delta s_{\text{grow}} \equiv \text{Re}[s] = \frac{1}{t_{\text{th}}} \frac{k^2(\chi^2 - 8) - 7k_{\text{smear}}^2}{2(k_{\text{smear}}^2 + k^2)}, \tag{30}
\]
where \(k_{\text{smear}} = \chi^2 \gamma\) (Equation (28)). In the limit of long wavelengths \((k \ll k_{\text{smear}})\), variations of \(\chi\) are relatively insignificant, and so one recovers that the wave damps at the rate \(s_{\text{grow}} = -7/(2t_{\text{th}})\) (Equation (18), and D’Alessio et al. 1998).

Equation (30) allows unstable modes whenever \(\chi > \sqrt{8}\), i.e., when the disk is sufficiently opaque that the optical surface lies well above the gas scale height. As Figure 2 shows, this condition is satisfied throughout our fiducial disk. The unstable waves have short wavelengths,
\[
k \gg k_{\text{min}} = \sqrt{\frac{7}{\chi^2 - 8}} k_{\text{smear}}, \tag{31}
\]
with the value of \(k_{\text{min}}\) ranging from 4.3 to 4.8 in our fiducial disk (see Figure 5). For reference, the wavelength of a \(k = 2.7\) wave spans one decade in radius. So these unstable waves have \(\sim 2\) or more wavelengths per radial decade.

The growth rates of unstable waves first rise with \(k\) as \(k^2\), before saturating to a constant value for \(k \gg k_{\text{smear}}\). This saturation is related to the smearing length, i.e., the finite spread of the optical surface. When the wavelength is much shorter than this length, changes in stellar heating are subdued relative to that for a razor-sharp optical surface. As a result, the growth rate saturates.

To confirm these analytical findings, we obtain mode growth rates by numerically integrating the relevant equations. The numerical details are in Appendix B.1. Figure 3 shows the results of two such integrations. Starting from an equilibrium disk (Appendix B.2), we impose small initial sinusoidal perturbations of the form \(e^{i \omega t + ik \ln r}\). We observe that the first perturbation (with \(k = 2\)) damps with time as the wave travels inward, while the second one (\(k = 6\)) grows.

We can extract growth rates in integrations like these, following the procedure in Appendix B.3. The results for some
low-\(k\) waves are plotted in Figure 4. We find that the analytical expression (Equation (30)) agrees with the numerical results qualitatively. Both suggest that \(k \geq 4\) waves should be unstable over much of our fiducial disk, with a growth time that is of order the local thermal time.

6. Instability Also Found by RADMC

We turn to the radiative transfer code RADMC. We do so for multiple purposes. One is to substantiate our analytical results. Another is to use RADMC to remedy a major shortcoming in the analytical approach, i.e., horizontal averaging (see below). An additional advantage of RADMC is that it allows us to go beyond the two-frequency approximation.

Given an assumed dust density field, RADMC uses the Monte Carlo method to follow photon absorption and emission and to determine the local equilibrium temperature by balancing energy gain and loss. In principle, it can be combined with the equation of hydrostatic equilibrium to obtain, iteratively, the equilibrium disk profile. This has been attempted by a number of studies (e.g., Dullemond & Dominik 2004a; Min et al. 2009; Siebenmorgen & Heymann 2012; Ueda et al. 2019), either with RADMC or with an analogous code. But the procedure does not converge for optically thick disks. We show in Appendix C that this nonconvergence is partly caused by a numerical (nonphysical) instability. So nonconvergence does not prove that the disk is truly unstable. To tease out the physical instability requires some finesse.

6.1. Simulation Setup

We adopt the same fiducial disk as before. The density profile is assumed to stay in hydrostatic equilibrium (Equation (4)) as the midplane temperature \(T = T(r, t)\) evolves. RADMC treats the radiation field at multiple wavelengths. The opacity law is set following Equation (7).

To trace the thermal evolution, we replace the stellar heating term in Equation (1) by

\[
\frac{1}{2} F_{\text{in}}[T] \to \sigma_{\text{SB}}(T_{\text{RADMC}}[T])^4,
\]

where \(T_{\text{RADMC}}[T]\) is the midplane temperature profile output by RADMC.\(^8\) We use this as a proxy for stellar heating. The thermal equation becomes

\[
\frac{3}{8} \tau_{\text{nm}} c_p \Sigma_{\text{gas}} \frac{\partial T}{\partial t} = \sigma_{\text{SB}}((T_{\text{RADMC}}[T])^4 - T^4).
\]

At equilibrium, \(T = T_{\text{RADMC}}[T]\), as is desired. Whereas previous works have ignored the left-hand term when iteratively searching for an equilibrium solution, we integrate the full equation forward in time. At each time step, we inject 20 million stellar photon packets from the origin. This provides a sufficiently accurate map of the illumination pattern. We choose a time step that is a small fraction of the thermal time (typically 4\% of the thermal time at 10 au).

Before we proceed to present results of these integrations, we comment on three major issues. The first two cast some doubts on the RADMC results, and the third relates to a major improvement of RADMC over our analytical study.

First, the inner boundary of our disk is set to be 1 au. Between 1 and 1.3 au, we freeze the disk profile to avoid a puffed-up inner rim, which would otherwise cast a shadow farther out. Although such a shadow may indeed be realistic, we do not wish its effect to pollute what happens in the simulation domain. Second, Equation (33) itself contains an important shortcoming: excessive thermal diffusion. In realistic situations, temperature gradients in disks should be communicated on diffusive timescales. However, the way we hijack RADMC for our purpose effectively assumes that both vertical heat transfer and horizontal heat transfer are instantaneous. The former shortcoming (neglecting vertical diffusion) produces a greater perturbation on the midplane temperature than reality, boosting the instability; the latter one (horizontal diffusion), on the other hand, leads to an excessive damping of the instability. Clearly, a more thorough study that considers time-dependent radiative transfer is required.

The third major issue is what we term “horizontal averaging.” This is different from the horizontal diffusion discussed in the last paragraph. As grains at the perturbed optical surface receive more (less) stellar irradiation, they light up as brighter (dimmser) bulbs for the disk down below. In our analytical study, we have assumed that the midplane disk can only see bulbs lying directly above it (heating is only affected by the local gradient, Equation (2)). This assumption fails if the distance between the bulbs and the midplane (more accurately, the vertical photosphere in thermal radiation) is greater than the radial wavelength, \(H \geq r/\kappa\). In this case, the midplane feels a reduced heating (cooling) due to horizontal averaging of the different light bulbs, and we expect the instability to be quenched. This effect was modeled in Watanabe & Lin (2008)

\(^8\) We use the midplane temperature, because in RADMC simulations the disk is vertically isothermal below the optical surface, and the midplane temperature reflects the amount of stellar heating for the disk. In contrast, the optical surface is hotter (Chiang & Goldreich 1997).
by manually averaging the perturbations over a radial distance; in RADMC simulations, it is automatically captured.

To order of magnitude, horizontal averaging limits unstable waves to those with

$$k \leq k_{\text{max}} = \frac{r}{H}. \quad (34)$$

Together with the lower limit on unstable $k$ (Equation (31)), this defines the region in disks where irradiation instability can occur. For our fiducial disk, we expect unstable waves only inward of $\sim 30$ au (Figure 5).

6.2. Linear Evolution

We impose a small sinusoidal perturbation of the form $e^{ik \ln r}$ on the equilibrium disk (see Section 5). The initial linear evolution is shown in Figure 6. Comparing with the analogous plot for our analytical model (Figure 3), we see that the $k = 2$ perturbation decays in both calculations. On the other hand, while our analytical model predicts growth for the $k = 6$ perturbation at all radii, RADMC runs show that it only experiences growth inside 30 au. This reflects the shortcoming of the analytical model: it does not account for horizontal averaging, a problem that is more severe in the outer disk.

We design a special numerical procedure to extract mode growth rates from RADMC simulations. This is detailed in Appendix B.3. The results are shown in Figure 7 for a few wavenumbers. In this exercise, in order to use RADMC to accurately track changes in heating associated with a small perturbation, we launch a very large number of photon packets (20 billion). Despite this, the results still appear somewhat jagged. We are able to confirm that RADMC yields qualitatively similar growth rates to those from the analytical model (thick lines in Figure 4, also reproduced here). But there are two notable differences. First, RADMC shows that the outer disk (beyond 30 au) is stable to perturbations. As we explain above, this is related to the issue of horizontal averaging in the outer region. Second, the growth rates for unstable modes are somewhat lower in RADMC, indicating enhanced damping. We believe that this may be related to the fact that the RADMC procedure assumes instantaneous horizontal diffusion. A fully time-dependent radiative transfer code, together with a hydrodynamic solver, will be necessary to provide more accurate answers.

6.3. Nonlinear Development

RADMC also allows us to follow the instability to its nonlinear stage. For the following runs, we use 20M photon packets.
packets, and we have checked to confirm that the behavior does not depend on numerical parameters such as time step, grid spacing, initial conditions, and inner/outer boundary conditions.

Starting from an equilibrium profile, small perturbations (seeded only by numerical noise) grow in time. After around five thermal times at 10 au, the system reaches a quasi-steady state, in which waves are generated around 30 au and grow in amplitude as they propagate inward. A few snapshots are presented in Figure 8. Globally, the most prominent waves have wavenumbers \( k \sim 4 - 6 \) (corresponding to about two wavelengths per radius decade), as expected from their growth rates and horizontal averaging (Figure 5). While the disk outside 30 au remains largely unperturbed, inside 30 au these waves cause order-unity perturbations in the midplane temperature. The bottom panel of Figure 8 shows the optical surface at 0.5 \( \mu \)m. This looks like a staircase: the front edge of each stair is illuminated by the star ("stair riser"), and behind that the disk falls into shadow and the optical surface appears flat in \( H/r \) ("stair tread").

In the nonlinear stage, the temperature waves are asymmetric with steeper Sun-facing edges, as is also found by Watanabe & Lin (2008). This may arise because at large amplitudes the local flaring index (\( \gamma \)) of the front edge is higher, allowing waves of higher wavenumber to be unstable. As such, perturbations with initially long wavelengths can acquire sharp spatial gradients, leading to large pressure variations across radial scales as small as the disk scale height. In fact, as our RADMC procedure introduces enhanced horizontal heat diffusion, real disks may harbor waves that are even larger in amplitudes and sharper in scale.

7. Recap of Technical Results

We summarize results obtained in previous sections. We found that passive disks are susceptible to an irradiation instability. A slight thermal perturbation in such a disk can lead to the disk receiving even more stellar heating, thereby initiating unstable growth. We examined this instability using three models of increasing complexity and realism.

In the simplest model, we assumed that the height of the optical surface (the disk layer at which stellar photons are absorbed) varies in proportion to the gas scale height when the disk is thermally perturbed. We found that thermal perturbations propagate inward and decay as they do so, in agreement with D’Alessio et al. (1999). The modes are damped because the increase in stellar insolation in this model is insufficient to counteract the damping due to blackbody cooling.

In contrast, in a model where the optical surface is self-consistently determined, we found that sinusoidal perturbations can grow as they propagate inward. The key physics is that as a thermal perturbation increases the scale height in a disk annulus, the inner half of the annulus acquires a steeper flaring, and the outer half a gentler one. Starlight now shines more overhead for the inner half and so can penetrate into deeper layers; conversely, it is absorbed more shallowly in the outer part. These geometric changes allow the annulus to intercept more starlight, more so than in the first model, and give rise to instability. Thermal perturbations grow and travel inward in order the local thermal time.

In terms of unstable wavelength, very long waves are stable because they cause little change in the surface slope; only waves with wavelengths comparable to or shorter than the so-called “smearing length,” the slant length over which the star deposits its energy, can grow. More optically thick disks have shorter smearing lengths and therefore harbor a broader spectrum of unstable waves. For our fiducial disk, the unstable waves have wavenumber \( k \geq 4 \), i.e., with \( \sim 2 \) or more wavelengths per decade in radius.

To confirm those analytical results, we retooled the radiative transfer code RADMC to track the stellar heating. This also allowed us to address the issue of horizontal averaging, the complication that arises when the disk heating is not solely determined by the slope of the local optical surface, but also by nearby hot and cold patches on the optical surface. Using both scaling arguments and RADMC simulations, we found that unstable modes exist only inside \( \sim 30 \) au, or where the disk is geometrically thin (\( H \leq r/k \)).

We also used RADMC to study the nonlinear evolution of the irradiation instability. Within a few thermal times, waves generated far from the star have propagated to the inner region and have grown to order-unity amplitudes. The front edges of these waves can be as sharp as a pressure scale height, suggesting that they may be effective in trapping dust grains.

7.1. Relation to Watanabe & Lin

The irradiation instability was first discussed in Watanabe & Lin (2008). Calculating the optical surface self-consistently (as in our “realistic surface” model), they numerically integrated the thermal equation and observed that thermal waves are driven to large amplitudes as they travel inward. But unlike our RADMC simulations here, they found instability throughout the whole disk. This difference may partly be explained by the different ways we account for horizontal averaging: while

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*The inner edge of a wave could also be thought of as a puffed-up inner rim, casting a long shadow on the disk behind.*
8. Examining Assumptions

There remain a wide range of uncertainties associated with the irradiation instability. In this preliminary exploration, we have had to make a string of assumptions. Here, we discuss some of these. We suggest that modifying them can lead to an array of interesting results. But the essence of the irradiation instability, we feel, should prevail.

8.1. Hydrodynamics

We simplify the hydrodynamical response of the disk by assuming that the scale height reacts instantaneously to the midplane temperature, the perturbation is axisymmetric, and the surface density does not evolve. These are questionable for multiple reasons.

First, they are predicated on the assumption that the orbital time is shorter than the thermal time, which is only true inside of ~50 au in our fiducial disk (Figure 2). What happens at larger radii is uncertain. Perturbation analyses by Dullemond (2000) and Chiang (2000) show that a different instability may operate in that limit. But a more careful analysis, including effects such as starlight penetration depth, smearing length, etc., is needed. In addition, perturbations are likely no longer axisymmetric when the orbital time is long.

Second, even in the limit of long thermal time, hydrodynamical effects can dramatically alter the story. A local heating not only will change the gas scale height but also can expel material to neighboring annuli. This may set up meridional flows that advect heat and dust. Vortices or turbulence may also ensue (via, e.g., the Rossby wave instability; Lovelace et al. 1999; Lithwick 2009). Investigating these effects, in terms of both how they impact the irradiation instability and how they impact the long-term disk evolution, requires numerical simulations that coevolve hydrodynamics and radiation. We are currently pursuing this course. Another motivation for a full hydro+radiation treatment is to improve on our treatment of thermal diffusion. Thermal perturbations in optically thick disks should propagate as diffusive waves. However, even in our most sophisticated account of the thermal physics (the RADMC approach), this is not accurately captured. Our retooling of RADMC accelerates horizontal diffusion artificially. We suspect that this leads to enhanced damping for the waves, as well as reduced amplitudes during the nonlinear evolution. We have also ignored vertical thermal waves, which may have the potential to weaken the instability (but see Appendix A.3).

8.2. Dust Movement

We assume that dust and gas are cospatial, with a constant mass opacity everywhere in the disk. This is perhaps our most problematic assumption. It is expected that dust grains evolve in at least three dimensions: they settle vertically, migrate radially, and their sizes can change owing to conglomeration and fragmentation. This can change the opacity relative to what we adopt here.

Dust opacity, especially that in the optical, is the deciding factor for the irradiation instability. The discussion following Equation (30) indicates that instability requires a fairly optically thick disk. Opacity in the optical is mostly contributed by micron-sized grains. These grains are known to settle quickly from these heights (Dullemond & Dominik 2004a), though even weak turbulence or circulation can disrupt this process. While the jury is still out on the vertical distribution of these small grains, we explore a scenario where settling has occurred. In Figure 9, we repeat our RADMC calculations but now with all dust above 2h removed. The optical surface now lies much closer to the midplane (H < 2h). This mitigates the negative impact of horizontal averaging (Equation (34)). And our RADMC simulations show that the instability now extends to almost the entire disk (out to ~100 au).

In summary, it appears that a complete picture of the irradiation instability can only emerge after we understand dust physics, an endeavor that is further complicated by the fact that the hydrodynamical response of the disk affects dust settling and fragmentation (Section 8.1).

8.3. Inner Region

We simplified the physics in the inner region substantially, by ignoring the presence of a hot rim and by assuming that the star is point-like. Moreover, we have ignored viscous heating, which may dominate over starlight heating in the inner region. Here we briefly discuss how these three effects may impact the irradiation instability.

A hot rim is expected to form where the inner disk sees the star unobstructed (Natta et al. 2001; Dullemond et al. 2001). Observed around T Tauri stars (Muzerolle et al. 2003; Akeson et al. 2005) and Herbig Ae/Be stars (Natta et al. 2001), such a rim casts a long shadow on the disk behind (Dullemond et al. 2001;
Vinković et al. 2006; Siebenmorgen & Heymann 2012; Flock et al. 2016). In this report, we have opted to simplify the picture by freezing the disk profile near the inner edge. But our (additional) explorations with RADMC often find that disks near the shadow terminator are unstable, in agreement with some previous work (Siebenmorgen & Heymann 2012; Flock et al. 2016; Ueda et al. 2019). We suspect that it is related to the irradiation instability but have yet to provide firm evidence.

We also simplified the star into a point-like light source. This is adequate as long as $0.4R_\ast \ll 2H$ (Kusaka et al. 1970; Chiang & Goldreich 1997), where $R_\ast$ is the stellar radius, and outside 0.4 au for a star with a radius of 2.5 $R_\odot$. Inside this region, we expect the irradiation instability to be suppressed by the finite-source effect, and this is indeed demonstrated by the numerical experiments of Watanabe & Lin (2008).

Viscous heating associated with mass accretion can compete against stellar heating only in the very inner disk (inward of an au). The irradiation instability should be suppressed there (Watanabe & Lin 2008).

9. Conclusions

We have developed a model to study the irradiation instability in passively heated disks. Our approach, combining analytical and numerical tools, reveals the origin of the instability, identifies the unstable wavelengths, and obtains relevant growth rates. Our fiducial disk is unstable inside of 30 au and produces large-amplitude, inwardly propagating thermal waves. A preliminary exploration shows that the instability can extend to much further reaches, if some degree of dust settling is included.

Future work should relax many of the approximations we have made here. Of particular urgency are the assumptions of instantaneous hydrostatic adjustment, vertical isothermal structure, and dust evolution.

If the instability proves robust, it has many important potential implications. For example, the large-amplitude waves could be the forebears for the gaps and rings observed by ALMA. The latter features are observed at radial distances of 10–100 au and have radial scales that are compatible with the unstable wavelengths we obtain here (~2 wavelengths per decade in radius). The narrowness of the bright rings (Dullemond et al. 2018) is also consistent with the nonlinear development of the irradiation instability. Moreover, these waves can affect dust migration and conglomeration, which in turn feeds back on the instability. Last but not least, the same waves could also be hydrodynamically unstable and produce vertical circulation, vortices, and turbulence. This could provide the much-needed viscosity for disk accretion. Clearly, much exciting work lies ahead.

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Appendix A

Derivation of the Thermal Equation

A.1. Flux Splitting

In a passively irradiated disk, there are three temperatures of concern. The first is that at the optical surface, $T_s$. This layer receives the stellar flux $F_{\text{irr}}$ and then emits half down, half up. It has negligible thermal inertia and simply acts as a beam splitter. We have

$$ F_{\text{in}} = 2\sigma_{\text{SB}} T_s^4. $$

(A1)

The downward flux, $\sigma_{\text{SB}} T_s^4$, is rapidly reprocessed at the disk thermal photosphere, which has a temperature $T_{\text{ph}}$. This layer emits a blackbody flux upward and transmits a flux $F_{\text{in}}$ downward toward the optically thick midplane, leading to

$$ \sigma_{\text{SB}} T_s^4 = \sigma_{\text{SB}} T_{\text{ph}}^4 + F_{\text{in}}. $$

(A2)

In steady state, $F_{\text{in}} = 0$ and the disk is vertically isothermal, with midplane temperature $T_c = T_{\text{ph}} = T_s = (F_{\text{irr}}/2\sigma_{\text{SB}})^{1/4}$.

A.2. Simple Thermal Equation

We adopt a simple model for describing the thermal evolution of the disk. We assume the downward diffusive flux determined by the equation of radiative diffusion,

$$ F_{\text{in}} = \frac{4\sigma_{\text{SB}} T_{\text{ph}}^4 - T_c^4}{3 \tau_{\text{mm}}}, $$

(A3)

where $T_c$ is the midplane temperature and $\tau_{\text{mm}}$ is the optical depth to the midplane for thermal radiation. We then assume that this inward flux heats up gas near the midplane at the rate

$$ \frac{c_p \Sigma_{\text{gas}}}{2} \frac{\partial T_c}{\partial t} = F_{\text{in}}, $$

(A4)

where $T_c$ is the midplane temperature and $\Sigma_{\text{gas}}/2$ is the surface density from the midplane outward.

Here we have adopted $c_p$ as the gas heat capacity. Watanabe et al. (1990) have argued for an extra factor of $(\Gamma + 1)/2\Gamma$, where $\Gamma$ is the adiabatic index, when one also accounts for the vertical expansion of the disk under stellar gravity. Moreover, when the disk’s thermal state is changed, its rotational speed also has to adapt in order to stay in hydrostatic equilibrium. In this exploration, we have ignored these corrections, as they only change the timescales by an order-unity factor.

Combining Equations (A1)–(A4), we land at our desired thermal equation

$$ \frac{1}{2} \left( 1 + \frac{3}{4} \tau_{\text{mm}} \right) c_p \Sigma_{\text{gas}} \frac{\partial T_c}{\partial t} = \frac{1}{2} F_{\text{irr}} - \sigma_{\text{SB}} T_c^4. $$

(A5)

In the main text, we drop the 1 in the brackets, as is valid for $\tau_{\text{mm}} \gtrsim 1$. This derivation ignores heat diffusion in the radial direction, which is reasonable since the radial wavelength is typically much longer than the scale height. It also does not properly account for the vertical temperature profile, which we discuss next.

A.3. Vertical Thermal Waves

Equations (A3)–(A4) approximate the vertical transport of heat when the timescale of variation is longer than the thermal time. Since unstable modes have these two timescales comparable to each other, we examine here what happens in the opposite limit. Our principal conclusions are that (i) vertical thermal waves appear and (ii) although those waves could potentially kill off instability, they probably do not, particularly if the millimeter-sized dust has settled.
We solve for the vertical temperature structure when the perturbation $\delta T_{\text{ph}}\exp(-i\omega t)$. Assume that the disk is optically thick and energy transport is by radiative diffusion. Energy conservation gives at any height

$$
\rho_{\text{gas}}c_p \frac{\partial T}{\partial t} = \nabla \cdot \left( \frac{16\sigma_{\text{SB}}T^3}{3\kappa_{\text{dust}}} \nabla T \right). 
$$

(A6)

Here gas provides the thermal capacity ($\rho_{\text{gas}}c_p$ on the left-hand side), while dust provides the opacity ($\kappa_{\text{dust}}$ on the right-hand side).

It is preferable to use optical depth in lieu of the spatial coordinate, $\tau(z) = \int_{-\infty}^{z} \kappa_{\text{dust}} \, \mathrm{d}z$. Equation (A6) now becomes

$$
\frac{3c_p}{16\sigma_{\text{SB}}\kappa_{\text{dust}}} \rho_{\text{gas}} \frac{\partial T}{\partial t} = \frac{\partial}{\partial \tau_z} \left( \tau_z^3 \frac{\partial T}{\partial \tau_z} \right). 
$$

(A7)

This is valid even if the dust has settled relative to the gas. But currently we assume no settling. Note that Equation (A4) is essentially the vertical integral of Equation (A7), together with the approximation that $T \propto \tau_z$ in the unperturbed state.

The solution for the temperature perturbation, subject to the boundary conditions at the photosphere and at depth (i.e., $\delta T \to 0$ at large $\tau_z$), is

$$
\delta T(\tau_z, t) = \delta T_{\text{ph}}\exp\left(-\frac{\tau_z}{\tau'}\right) \cos\left(\omega t - \frac{\tau_z}{\tau'}\right), 
$$

(A8)

where

$$
\tau' = \left( \frac{32\sigma_{\text{SB}}\kappa T^3 \rho_{\text{dust}}}{3c_p \omega \rho_{\text{gas}}} \right)^{1/2} = \sqrt[8]{8} \tau_{\text{eff}}(z = 0) = \sqrt[8]{\omega h_{\text{th}}}. 
$$

(A9)

Equation (A8) describes a wave train that propagates downward. The amplitude of the wave train decays exponentially. The key dimensionless parameter is

$$
\frac{\tau_z}{\tau'}_{\text{midplane}} = \frac{1}{\sqrt[8]{8}} \sqrt[8]{\omega h_{\text{th}}} \approx \sqrt[8]{\frac{7k}{32}} = \sqrt{k/4.6}, 
$$

(A10)

where in the approximate equality we used the dispersion relation for irradiated thermal waves (Equation (18)). If that parameter is $\gtrsim \frac{7}{2}$, then the vertical temperature profile will be oscillatory, which will likely suppress the irradiation instability. Our unstable modes have $k \sim 6$, which suggests that the instability is not suppressed. But clearly a more careful treatment is needed to obtain more accurate values for the order-unity coefficients.

One should also account for dust settling, because the millimeter-sized grains responsible for the opacity tend to settle toward the midplane. The waves in Equation (A8) are in $\tau_z$ rather than $z$. So if the grains settle into a thin layer, then even if the parameter in Equation (A10) exceeds unity, the temperature throughout most of the lowest gas scale height (by volume) will be similar to that at the photosphere. Thus, dust settling will likely help prevent thermal waves from suppressing instability.

Appendix B

Methods for the Model with Realistic Surface

B.1. Time Integration

The model equations of motion are listed in Section 5.2. We evolve the thermal equation forward in time with the Euler method. The spatial derivative in Equation (2) is, at the $i$th grid point, taken to be proportional to $H/r_{i+1} - H/r_i$—i.e., the first-order upwind scheme. The grid in $r$ is logarithmic, with typically 250 grid points per decade. At each time step, $H(r)$ on the spatial grid is obtained from $T(r)$ as follows: after converting from $T$ to $h$ with the hydrostatic equation, the integral in the equation for the optical surface (Equation (20)) is performed numerically at the $i$th grid point for an assumed value of $H(r_i)$, and that $H(r_i)$ is then adjusted via a root-finder until the integral is unity. For the boundary conditions, we typically freeze the temperature near the inner and outer boundary.

B.2. Solving for Equilibrium

As stated in Section 5.1, we obtain the equilibrium disk profile by iteratively solving the steady-state thermal equation (Equation (10)) and the photosphere equation (Equation (20)). A naive application of this procedure is typically numerically unstable. But that instability may be avoided by using the integral of the thermal equation in place of the thermal equation itself. In particular, for any assumed $\chi(r)$ profile one may integrate Equation (10) to yield

$$
\frac{H}{r} = 0.02 \chi_{\text{eff}} \frac{r^{27/7}}{\ln(r)}, 
$$

(B1)

after using hydrostatic equilibrium and defining $\chi_{\text{eff}}$ via

$$
\frac{1}{r^{19/7}} \equiv \int_{r}^{\infty} \frac{2d\ln r'}{r'^{27/7}}. 
$$

Therefore, the iteration proceeds as follows:

(i) given a profile for $\chi(r)$ (initially taken to be constant), calculate $H$ via Equation (B1), and thence $h = H/\chi$; (ii) use that $h$ in the equation for the optical surface and solve for $H(r)$, and thence $\chi = H/h$; (iii) insert that $\chi$ back into step (i) and repeat until convergence.

B.3. Measuring Linear Growth Rates Numerically

We explain how we measure the exact growth rates shown in Figures 4 and 7. It is nontrivial because the growth rate is in general complex valued. We start from the linearized thermal equation, the integral in the equation for the optical surface (Equation (20)) is performed numerically at the $i$th grid point for an assumed value of $H(r_i)$, and that $H(r_i)$ is then adjusted via a root-finder until the integral is unity. For the boundary conditions, we typically freeze the temperature near the inner and outer boundary.

$$
\frac{\delta F_{\text{irr}}}{F_{\text{irr}}} = \frac{\delta F_{\text{irr}}}{F_{\text{irr}}}[T(1 + \Re(\epsilon_T e^{i\phi_{\text{irr}}})] - 1. 
$$

(B3)

To evaluate $\sigma$ at any desired $r$ and $k$, we choose a fixed value of $\epsilon_T \ll 1$ and various values of $\phi_T$ and evaluate Equation (B3) at those values. The resulting function of $\delta F_{\text{irr}}/F_{\text{irr}}$ versus $\phi_T$ is fit to a sinusoid with the form of Equation (B2), which allows us to extract the values of $\sigma$ and $\phi_T$—and hence $\sigma$.

With the value of $\sigma$ in hand, we substitute into Equation (14) the relation between the complex amplitudes, $\frac{\delta F_{\text{irr}}}{F_{\text{irr}}} = \sigma \epsilon_T$. 

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Equation (14) then yields the solution \( \epsilon_T \propto e^{s t} \), where the (complex) growth rate is
\[
s = \frac{1}{l_{th}}(\sigma - 4). \tag{B4}
\]

### B.4. Analytical Growth Rate

We derive the growth rate analytically, under a number of simplifying assumptions. To begin, we first determine the relationship between the perturbed temperature and the perturbed heating rate. When the temperature is perturbed by \( \delta T \), hydrostatic equilibrium provides the perturbation in scale height (\( \delta h \)), and the equation for the optical surface (Equation (20)) then yields \( \delta H \), which in turn leads to the perturbed heating rate.

Under perturbations \( \delta h(r) \) and \( \delta H(r) \) (but no change in surface density), the perturbed equation for the optical surface (Equation (20)) is Taylor-expanded to linear order, which yields
\[
0 = \int_0^r \frac{dr'}{r'} \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r')}{h(r')}}} {h(r')^3} \left[ \frac{\delta h(r')}{H} + \frac{\delta h(r')}{h(r')} \right] - \frac{\delta h(r')}{h(r')} \tag{B5}
\]
where \( H \) and \( \delta H \) are understood to be functions of \( r \). In the limit that \( \chi = H/h \gg 1 \), the last term inside the curly brackets is small and we obtain
\[
\frac{\delta H}{H} = \frac{\text{NUM}}{\text{DEN}}, \tag{B6}
\]
where the numerator and the denominator are, respectively,
\[
\text{NUM} = \int_0^r \frac{dr'}{r'} \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r')}{h(r')}}}{h(r')}^3 \delta h(r'),
\]
\[
\text{DEN} = \int_0^r \frac{dr'}{r'} \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r')}{h(r')}}}{h(r')}^3.
\]

One may use a trick to perform the DEN integral: taking the derivative of the equation for the optical surface (Equation (20)), one finds
\[
0 = \frac{d}{dr} \int_0^r \frac{dr'}{r'} \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r')}{h(r')}}}{h(r')}^3 \delta h(r') = \frac{ \frac{d}{dr} \left[ \frac{H(r')}{h(r')} \right] }{r} \cdot \frac{e^{-\frac{H(r')}{h(r')}}}{h(r')}^3 \tag{B8}
\]
\[
= \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r)}{h(r)}}}{h(r)}^3 \left[ - \frac{H(r)}{r} \frac{d \left( \frac{H(r)}{h(r)} \right)}{dr} \right] \tag{B9}
\]
\[
= \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r)}{h(r)}}}{h(r)}^3 \left[ - \frac{H(r) dH(r)}{r} \right] \times \text{DEN}, \tag{B10}
\]
where \( \chi(r) = H(r)/h(r) \). The latter equations may be solved for DEN to yield
\[
\frac{\delta H}{H} = \frac{1}{\chi^2} \frac{d}{dr} \left( \frac{H(r)}{h(r)} \right) \tag{B11}
\]
after approximating \( \frac{d \ln H/ H}{d \ln r} \approx \gamma \), with \( \gamma \) being the logarithmic slope of the background \( h/r \) (Equation (22)), as is valid for large enough \( \chi \).

To perform the NUM integral, we write the temperature perturbation to be in the complex form of Equation (17). We also have \( \frac{\delta h}{h} = \frac{\delta H}{H} = \frac{1}{\chi} \) in hydrostatic equilibrium. We may now approximate the NUM integral as we did for the background approximate equation for the optical surface (see footnote 3) to obtain
\[
\text{NUM} \approx \frac{1}{\chi^2} \frac{1}{2 \pi} \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r)}{h(r)}}}{h(r)}^3 \frac{\delta T}{T}, \tag{B12}
\]
Combining the above two results, we arrive at
\[
\frac{\delta H}{H} = \frac{1}{\chi^2} \frac{1}{2 \pi} \frac{2 \gamma}{\sqrt{2\pi}} \frac{e^{-\frac{H(r)}{h(r)}}}{h(r)}^3 \frac{\delta T}{T}. \tag{B13}
\]

For comparison, the simplistic surface model (constant \( \chi \)) gives \( \delta H/H = \delta h/h = \frac{1}{\chi} \frac{\delta T}{T} \), which agrees at small \( k \).

We may now insert \( \delta H/H \) into the heating rate (Equation (2)), which yields
\[
\frac{\delta F_{irr}}{F_{irr}} \approx \frac{d}{d \ln r} \left( \frac{\delta H}{H} \right) \left( \frac{d}{d \ln r} \right) \left( \frac{\delta H}{H} \right) \tag{B14}
\]
\[
= \left( \frac{\chi^2}{2} \frac{k \chi + \gamma}{2 \pi} \right) \frac{\delta T}{T},
\]
where in the second equality we assume that the spatial variation of Equation (B13) is dominated by the sinusoidal dependence, as is reasonable for \( k \gtrsim 1 \). Inserting this into the linearized thermal equation (Equation (14)) yields the complex growth rate
\[
s \approx \frac{1}{\chi^2} \left[ \frac{k \chi^2 - 8}{2 \pi} - 7 \chi^2 \right]. \tag{B15}
\]

### Appendix C

#### Numerical Instability in RADMC due to Iterations

As described in Section 6, a number of papers attempt to find the equilibrium state of passive disks with an iterative scheme, based on a radiative transfer code such as RADMC. These often find that the iterations do not converge, but instead produce large-amplitude propagating waves (e.g., Dullemond & Dominik 2004a; Min et al. 2009; Siebenmorgen & Heymann 2012; Ueda et al. 2019). Here we show that the reason for the nonconvergence is an instability closely connected with the irradiation instability described in this paper. However, the two instabilities are not identical: the iteration instability is partly polluted by a numerical (i.e., nonphysical) instability, and so if the disk is unstable under iterations, it is not necessarily unstable in reality.
As described in Section B.3, the key quantity governing the stability of a disk is the complex number $\sigma$ that relates heating perturbations to temperature perturbations, defined via
\[
\frac{\delta F_{\text{irr}}}{F_{\text{irr}}} = \sigma \frac{\delta T}{T}.
\] (C1)

In the above, the bold $\delta T$ is a complex amplitude, i.e., the real temperature perturbation is $\delta T_{\text{real}}(\delta T_{\text{irr}})$, and similarly for $\delta F_{\text{irr}}$; $\sigma$ is a function of both $r$ and $k$, and its value may be determined numerically (Section B.3). Given the value for $\sigma$, the linearized thermal equation (Equation (14)) shows that the disk is unstable if $\text{Real}(\sigma) > 4$; otherwise, it is stable.

Now, let us compare this behavior with that of the iteration scheme, in which the temperature at the $k$th iterative step ($T_k$) is the equilibrium solution of the thermal equation, i.e., from Equation (33), $T_k = T_{\text{RADMC}}[T_{k-1}]$. Equivalently, writing this in terms of $F_{\text{irr}}$ (Equation (32)),
\[
T_k = \left( \frac{F_{\text{irr}}[T_k - 1/2]}{\sigma_{SB}} \right)^{1/4}.
\] (C2)

Setting $T_k = T + \delta T_k$, where $T$ is the equilibrium temperature, and linearizing Equation (C2) yields
\[
\frac{\delta T_k}{T} = \frac{1}{4} \frac{\delta F_{\text{irr}}[T_{k-1}]}{F_{\text{irr}}} = \frac{\sigma}{4} \frac{\delta T_{k-1}}{T}.
\] (C3)

The solution to this difference equation is $\delta T_k = \text{const} \times \left( \frac{\sigma}{4} \right)^k$. Therefore, the solution is unstable if $|\sigma| > 4$, which is less stringent than the true criterion for instability (Real$(\sigma) > 4$).

We note that typically the real and imaginary parts of $\sigma$ are comparable to each other, and so the criterion for iterative instability is incorrect by an order-unity factor. For example, in the model with a realistic surface, the marginal $k$ for stability (plotted in Figure 4) is $\sim 2$ for the iterative instability—rather than the true value of $\sim 4$ as is shown in the figure.

**Appendix D**

**Gas–Dust Thermal Coupling**

In our work, we have assumed that the gas and dust components in the disk share the same temperature. This requires efficient energy transfer. We examine the validity of this assumption here.

We first focus on the midplane gas. As gas is inert radiatively, it only receives or loses energy when colliding with the dust grains. Assuming that all dust particles have size $s$ and bulk density $\rho_{\text{bulk}}$, we can estimate the timescale for changing the gas’s thermal energy as
\[
\tau_{\text{gas–dust}} \sim \frac{S_{\text{bulk}}}{c_s \rho_{\text{dust}}}.
\] (D1)

Comparing this to the dynamical time and evaluating at the midplane of our adopted disk (Equation (6)), we find
\[
\Omega \tau_{\text{gas–dust}} \sim \frac{S_{\text{bulk}}}{\Sigma_{\text{dust}}} \sim 0.005 \left( \frac{s}{1 \text{mm}} \right) \left( \frac{\rho_{\text{bulk}}}{1 \text{g cm}^{-3}} \right) r_{\text{au}}.
\] (D2)

This process is therefore fast throughout the disk midplane. The strong coupling between gas and dust may fail at high altitudes. Fortunately, this should not affect the irradiation instability.
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