A normalization formula for the Jack polynomials in superspace.

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(joint work with Luc Lapointe and Philippe Nadeau)
A normalisation formula for the Jack polynomials \((J^{(\alpha)}_{\Lambda})_{\Lambda}\) in superspace involves:

\[
[\theta_1 \ldots \theta_m x_1^{\Lambda_1} \ldots x_N^{\Lambda_N}] \sum_{\text{certain tableaux } T \text{ of "shape" } \Lambda} \sum_{\sigma \in S_N} \epsilon(\sigma) W_1(T, \sigma).
\]

Computational observations lead Desrosiers, Lapointe and Mathieu (2007) to conjecture a more compact expression for the coefficient of this double sum:

\[
\frac{1}{(3\alpha+5)(2\alpha+3)(\alpha+2)(\alpha+1)(\alpha+3)}
\]

- What are superspace and Jack polynomials?
- Where does the double sum come from?
- Some combinatorics to prove the conjecture by a simplification of the double sum.
I. The context where the double sum appears.

\[ [\theta_1 \ldots \theta_m x_1^{\Lambda_1} \ldots x_N^{\Lambda_N}] \sum \text{certain tableaux } T \text{ of "shape" } \Lambda \sum_{\sigma \in S_N} \epsilon(\sigma) W_1(T, \sigma). \]
Partial references

- *Jack polynomials in Superspace*, Desrosiers, Lapointe, and Mathieu, Commun. Math. Phys. (2003) 242 331-360,
- *Orthogonality of Jack polynomials in superspace*, Desrosiers, Lapointe and Mathieu, Advances in Mathematics (2007) 212 361-388.
- *A recursion and a combinatorial formula for Jack polynomials*, Knop and Sahi, Inventiones mathematicae (1997) 128 9-22,
- *A normalization formula for the Jack polynomials in superspace and an identity on partition*, Lapointe, Le Borgne, Nadeau
  arXiv:math.CO.0803.4182.
Symmetric functions in superspace

The space is made up of polynomials in $2N$ variables $\{x_i\}_i \cup \{\theta_i\}_i$ satisfying

$$x_ix_j = x_jx_i, \quad x_i\theta_j = \theta_jx_i, \quad \theta_i\theta_j = -\theta_j\theta_i \ (\implies \theta_i^2 = 0).$$

Diagonal action of the symmetric group $S_N$, $K_\sigma x_i\theta_j x_k \theta_k \equiv x_{\sigma(i)}\theta_{\sigma(i)}x_{\sigma(j)}\theta_{\sigma(k)}$.

A polynomial $P$ is symmetric iff $\forall \sigma \in S_N, K_\sigma P = P$.

Description of the basis $(m_\Lambda)_\Lambda$ of monomial symmetric functions indexed by superpartitions:

- Monomials as weight of a single tableau $T$ of shape the superpartition $\Lambda$. (here $x^{ev(T)}\theta^{ev(T)} = x_1^4\theta_1x_2^3\theta_2x_3^3\theta_3x_4\theta_4\theta_5$)

- Symmetrization:

$$m_\Lambda = \sum_{\sigma \in S_7} K_\sigma x_1^4\theta_1x_2^3\theta_2x_3^3\theta_3x_4\theta_4\theta_5$$

| $i$ | $x_i$ |
|-----|-------|
| $i$ | $\theta_i$ |
Jack polynomials in superspace

\((J^{(\alpha)}_{\Lambda})_{\Lambda}\) is another basis of symmetric functions parametrized by \(\alpha \in \mathbb{R}\). There are many characterizations of these polynomials:

- \((C_{\text{operator}})\) as simultaneous eigenfunctions of operators related to the Calogero-Moser-Sutherland Model that may involve Cherednik’s operators,
- \((C_{\text{orthogonality}})\) by a triangular decomposition in the monomial basis and orthogonality with respect to different scalar products,
- \((C_{\text{tableau}})\) as the symmetrization

\[
\sum_{\sigma \in S_N} K_\sigma \theta_1 \ldots \theta_m E^{(\alpha)}_{\Lambda}
\]

of non-symmetric polynomials \(E^{(\alpha)}_{\Lambda}\) satisfying a recurrence that can be interpreted in terms of tableaux [Knop, Sahi] :

\[
E^{(\alpha)}_{\Lambda} = \sum_{T \text{ 0-admissible tableau of “shape” } \Lambda} d^{(\alpha)}_T \chi^{\text{ev}}(T).
\]

Generalized to superspace, see [Desrosiers, Lapointe, Mathieu, Vinet . . .]
A first explicit combinatorial formula for $\langle \langle J^{(\alpha)}_{\Lambda} \mid J^{(\alpha)}_{\Omega} \rangle \rangle_{\alpha}$

Let $\langle \langle . \mid . \rangle \rangle_{\alpha}$ be a scalar product used in a characterization of type ($C_{orthogonality}$). Combining ($C_{orthogonality}$) and ($C_{operator}$) [DLM] showed that

$$\langle \langle J^{(\alpha)}_{\Lambda} \mid J^{(\alpha)}_{\Omega} \rangle \rangle_{\alpha} = \alpha^{f(\Lambda)} \frac{c_{\Lambda}^{\min}(\alpha)}{c_{\Lambda'}^{\min}(1/\alpha)} \delta_{\Lambda,\Omega},$$

where

$$J_{\Lambda} = c_{\Lambda}^{\min}(\alpha)m_{\Lambda \min} + \sum_{\Omega \neq \Lambda \min} c_{\Lambda,\Omega}(\alpha)m_{\Omega}.$$ 

Using ($C_{tableau}$) and the symmetry of $m_{\Lambda \min}$ we obtain

$$c_{\Lambda}^{\min}(\alpha) = [\theta_1 \ldots \theta_m x_1^{m-1} \ldots x_0 x_{m+1} \ldots x_N] \sum_{\sigma \in S_N} K_{\sigma} \theta_1 \ldots \theta_m \left( \sum_T d_T^{(\alpha)} x^{ev(T)} \right).$$
II. Combinatorics to simplify the double sum
A more direct description of $c_{\Lambda}^{\min}(\alpha)$.

$$c_{\Lambda}^{\min}(\alpha) = [\theta_1 \ldots \theta_m x_1^{m-1} \ldots x_m \ldots x_N] \sum_{\sigma \in S_N} \mathcal{K}_\sigma \theta_1 \ldots \theta_m \left( \sum_T d_T^{(\alpha)} x^\text{ev}(T) \right)$$

- $\mathcal{K}_\sigma \theta_1 \ldots \theta_m \propto \theta_1 \ldots \theta_m$ requires $\sigma = \sigma_1 \circ \sigma_2$
  
  where $\sigma_1 \in S_{\{1 \ldots m\}}$
  
  and $\sigma_2 \in S_{\{m+1 \ldots N\}}$.

- $\mathcal{K}_\sigma x^\text{ev}(T) \propto x^\text{ev}(\Lambda^{\min})$
  
  constrains the distribution of labels in $T$.

$\Lambda$, $\Lambda^{\min}$, $\mathcal{K}_{\sigma^{-1}}\Lambda^{\min}$, $\tilde{\Lambda}$
0-admissible tableaux related to $c_{\Lambda}^{\min}(\alpha)$

A tableau $T$ is 0-admissible if all labels in cells of the same column are different and when a label occurs in two consecutive columns, the right occurrence is not below the left one.

Each label defines the state of its cell:
- **Free**: The label is not critical.
- **Common**: The label is in a common position.
- **Critical**: The label is in a critical position.

The weight $d_T^{(\alpha)}$ is a product of hooks (not yet defined) related to critical cells.
0-admissible tableaux related to \( c_{\Lambda \min}^\alpha \)

A tableau \( T \) is 0-admissible if all labels in cells of the same column are different and when a label occurs in two consecutive columns, the right occurrence is not below the left one.

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- free
- common
- critical

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0-admissible tableaux related to $c_{\lambda}^{\min}(\alpha)$

A tableau $T$ is 0-admissible if all labels in cells of the same column are different and when a label occurs in two consecutive columns, the right occurrence is not below the left one.

![Diagram showing 0-admissible tableau]

Each label defines the state of its cell:
- **Free**
- **Common**
- **Critical**

The weight $d_T^{(\alpha)}$ is a product of hooks (not yet defined) related to critical cells.

$\mathcal{K}_{\sigma^{-1}\Lambda^{\min}} \Lambda$
A tableau $T$ is 0-admissible if all labels in cells of the same column are different and when a label occurs in two consecutive columns, the right occurrence is not below the left one.

```
| 3 | 3 | 3 |
|---|---|---|
| 1 | 1 | 1 |
| 4 | 4 |
| 6 |
| 12 |
| 5 |
| 8 |
| 14 |
| 7 |
| 11 |
| 9 |
| 13 |
| 15 |
| 10 |
```

$\epsilon(\sigma_1)$

- Each label defines the state of its cell:
  - free
  - common
  - critical

- The weight $d_T^{(\alpha)}$ is a product of hooks (not yet defined) related to critical cells.
Killing configurations with free cells

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Killing configurations with free cells

A normalization formula for the Jack polynomials in superspace.
The sign of the two configurations are opposite because $\epsilon((4\ 7) \circ \sigma_1) = -\epsilon(\sigma_1)$. The (not yet defined) remaining parts $d^{(\alpha)}_T$ of their weights are the same since they only involve critical cells and $\tilde{\Lambda}$. 
A configuration without free cells

A normalization formula for the Jack polynomials in superspace.
A configuration without free cells
A configuration without free cells
A configuration without free cells
An application of LGV lemma

The LGV lemma is based on a combinatorial interpretation of

\[
\det(M_{i,j})_{1 \leq i,j \leq m} = \sum_{\sigma} \left( \epsilon(\sigma) \prod_{j=1}^{m} M_{j,\sigma(j)} \right).
\]

in terms of systems of paths.

The weights of systems of paths where at least one vertex is shared by two paths cancel in pairs that can be described by an involution. Thus it remains only the configurations without free cells.
Evaluation of the determinant (not detailed yet)

In a particular case,

$$\det \left( \sum_{\text{paths } j \to i} j \right) \quad \begin{array}{c} \text{paths } j \to i \\ 1 \leq i, j \leq m \end{array}$$
Evaluation of the determinant (not detailed yet)

In a particular case,

\[ \det \left( \sum_{\text{paths } j \rightarrow i} j \right) \]

where \( 1 \leq i, j \leq m \)
In a particular case,

$$\det \left( \sum_{\text{paths } j \rightarrow i} j \right)_{1 \leq i, j \leq m}$$
Evaluation of the determinant (not detailed yet)

In a particular case,

\[
\det \left( \sum_{\text{paths } j \rightarrow i} \right)_{1 \leq i, j \leq m}
\]

In the general case,
Sorting rows for the "hook" formula
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A normalization formula for the Jack polynomials in superspace
Sorting rows for the “hook” formula

\[ d_{\Lambda}((j, i)) = 6\alpha + 9 \]
A normalization formula for the Jack polynomials in superspace.

\[ d^\Lambda((j, i)) = 6\alpha + 9 \]
A normalization formula for the Jack polynomials in superspace.
A normalization formula for the Jack polynomials in superspace.
Sorting rows for the “hook” formula

$$d^\Lambda((j, i)) = 6\alpha + 9$$
The normalization formula

\[
\langle J^{(\alpha)}_{\Lambda} | J^{(\alpha)}_{\Omega} \rangle_{\alpha} = \delta_{\Lambda, \Omega} \alpha^{1/2} \left( \frac{\alpha}{2} \right) \]

\[= \delta_{\Lambda, \Omega} \alpha^{1/2} \left( \frac{\alpha}{2} \right) \]

\[
\frac{(5/\alpha+4)(3/\alpha+3)(1/\alpha+2)(1/\alpha+1)^3 (3/\alpha+2)}{(3\alpha+5)(2\alpha+3)(\alpha+2)(\alpha+1)(\alpha+3)}
\]
Back to the determinant: $d_{\tilde{\Lambda}}((j, i)) = a_j + b_i$

We use parameters $(a_j)$ on rows and $(b_i)$ on columns to decompose the hooks.

A circle on row $j$ and column $i$ implies $b_i = 1 - a_j$. 
Circles and relations $b_i = 1 - a_j$

In this example $m = 5$ \implies \det(a_1, \ldots, a_5, b_1, \ldots, b_4)$

We have two relevant relations:

$b_2 = 1 - a_1$ and $b_4 = 1 - a_2$ \implies \det(a_1, \ldots, a_5, b_1, b_2)$

More surprisingly $\det(a_1, \ldots, a_5, b_1, b_2) = \prod_{1 \leq i < j \leq m} (1 + a_j - a_i)$.

Moreover we will treat $(a_j)$ and $(b_i)$ not related by relevant relations as formal variables independant of the shaded part of the superpartition.
A proof in two steps

Even if there is no relation between \((a_i)\) and \((b_j)\),
\[
\det(a_1, \ldots, a_m, b_1, \ldots, b_{m-1}) = \prod_{1 \leq i < j \leq m}(1 + a_j - a_i).
\]

When we add a circle related to a relevant relation \(b_i = 1 - a_j\), we also restrict the graph for the system of paths. These two modifications leaves invariant the determinant if initially it does not depend on \((b_i)\).
Shape Invariance proof

We suppose that the first sum depends only on \((a_j)\) so that it is not modified by the introduced relation \(b_4 = 1 - a_3\).

\[
\sum_{\text{systems}} = \sum_{\text{systems}} + \sum_{\text{systems}}
\]

We then obtain the equality of the two first terms by a cancellation (described by an involution) of the third sum.

The eventual horizontal step is weighted by

\[
a_3 + b_4 = 1
\]

due to the introduced relation.
Recursive computation for the full square

\[ P_{j,i}^{[k]} = \sum \Delta_{j,i}^{[k]} \equiv P_{j,i}^{[k]} - P_{j-1,i}^{[k]} = (a_j + 1 - a_{j-k-1}) P_{j,i}^{[k+1]} \]

Path \( p \) from \( j \) to \( i \) intersecting all rows \( j-1, \ldots j-k \)

\[
\begin{array}{cccc|cccc}
P_{1,1}^{[0]} & P_{1,2}^{[0]} & P_{1,3}^{[0]} & P_{1,4}^{[0]} & P_{1,1}^{[0]} & P_{1,2}^{[0]} & P_{1,3}^{[0]} & P_{1,4}^{[0]} \\
P_{2,1}^{[0]} & P_{2,2}^{[0]} & P_{2,3}^{[0]} & P_{2,4}^{[0]} & P_{2,1}^{[0]} & P_{2,2}^{[0]} & P_{2,3}^{[0]} & P_{2,4}^{[0]} \\
P_{3,1}^{[0]} & P_{3,2}^{[0]} & P_{3,3}^{[0]} & P_{3,4}^{[0]} & P_{3,1}^{[0]} & P_{3,2}^{[0]} & P_{3,3}^{[0]} & P_{3,4}^{[0]} \\
P_{4,1}^{[0]} & P_{4,2}^{[0]} & P_{4,3}^{[0]} & P_{4,4}^{[0]} & P_{4,1}^{[0]} & P_{4,2}^{[0]} & P_{4,3}^{[0]} & P_{4,4}^{[0]} \\
\end{array}
\]
A double-counting for the factorisation’s recurrence

\[ \Delta_{j,i}^{[k]} \equiv P_{j,i}^{[k]} - P_{j-1,i}^{[k]} = (a_j + 1 - a_{j-k-1})P_{j,i}^{[k+1]} \]

is equivalent to

\[ (a_j - a_{j-k-1})P_{j,i}^{[k+1]} = \left( P_{j,i}^{[k]} - P_{j,i}^{[k+1]} \right) - P_{j-1,i}^{[k]} \]

where both expressions are the weighted sum of “extended” paths:
Conclusion

Using LGV lemma, we identified a determinant in the double sum

\[
[\theta_1 \ldots \theta_m x_1^{\Lambda_1} \ldots x_N^{\Lambda_N}] \sum_{\text{certain tableaux } T \text{ of "shape" } \Lambda} \sum_{\sigma \in S_N} \epsilon(\sigma) W_1(T, \sigma).
\]

We computed this determinant and then proved the following conjectured expression for the double sum:

\[
(3\alpha+5)(2\alpha+3)(\alpha+2)(\alpha+1)(\alpha+3)
\]

This lead to a compacted normalization formula for Jack polynomials in superspace involving product of certain hooks of the superpartition.
Appendix. Some skipped definitions

- The Hamiltonian of the supersymmetric form of the trigonometric Calogero-Moser-Sutherland model seems to be

\[ \mathcal{H} = \sum_i (x_i \partial_{x_i})^2 + \frac{1}{\alpha} \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_{x_i} - x_j \partial_{x_j}) - \frac{2}{\alpha} \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2} (1 - \mathcal{K}(ij)). \]

- The scalar product for our normalization formula is

\[ \langle \langle p_{\Lambda} | p_{\Omega} \rangle \rangle_\alpha = (-1)^{m(m-1)/2} \alpha^l(\Lambda) z_{\Lambda^s} \delta_{\Lambda,\Omega} \]

where \( \Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda^s) \) and \( p_{\Lambda} := \tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \cdots p_{\Lambda_N} \).

- We are not using the following compatible scalar product

\[ \langle A(x, \theta) | B(x, \theta) \rangle_{\alpha,N} = \prod_{1 \leq j \leq N} \frac{1}{2\pi i} \oint \frac{dx_j}{x_j} \int d\theta_j \theta_j \prod_{1 \leq k \neq l \leq N} \left( 1 - \frac{x_k}{x_l} \right)^{1/\alpha} \frac{A(x, \theta)B(x, \theta)}{A(x, \theta)B(x, \theta)} \]

where \( \overline{x_j} = 1/x_j \) and \( \overline{\theta_{i_1} \cdots \theta_{i_m} \theta_{i_1} \cdots \theta_{i_m}} = 1. \)