1/$N_C$ Corrections to $g_A$ in the Light of PCAC†

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Abstract

We comment on recently discovered 1/$N_C$ corrections to the nucleon axial current in the framework of the Nambu–Jona-Lasinio soliton. This kind of corrections arises only for a special treatment of ordering ambiguities of collective coordinates and operators in the semiclassical quantization. In addition to the missing derivation of this special quantization scheme from first principles its naïve application violates partial conservation of the axial current (PCAC). We show how within this scheme PCAC can be restored and determine the corresponding 1/$N_C$ corrections to the equation of motion for the chiral soliton. The resulting self-consistent solution allows to evaluate the nucleon axial coupling constant $g_A$ directly as a matrix element of the axial current as well as indirectly from the pion profile function. Enhancement is found for those baryon properties which are sensitive to the long range behavior of the pion profile.

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Chiral symmetry and its spontaneous breaking are generally accepted as guiding principles for the construction of models describing hadron physics. A prominent feature of chiral symmetry is the partial conservation of the axial vector current (PCAC). PCAC relates the divergence of the axial vector current \( A_\mu^a \) to the pseudoscalar-isovector pion field \( \pi^a \):

\[
\partial_\mu A_\mu^a = m_\pi^2 f_\pi \pi^a. \tag{1}
\]

Here \( m_\pi = 138\text{MeV} \) and \( f_\pi = 93\text{MeV} \) denote the pion mass and decay constant, respectively. Since the axial current is of the nature of a Noether current its (partial) conservation heavily relies on the pion fields satisfying their equation of motion; i.e. eqn. (1) is equivalent to the equation of motion for the pion field. Therefore variations of the axial current imply corresponding changes in the equation of motion for the pion field.

In baryon physics the axial current plays an important role since its matrix element between nucleon states, the nucleon axial coupling constant \( g_A \), directly relates to the neutron \( \beta \) decay amplitude. It has been a long standing problem for chiral soliton models of baryons that the predicted value for \( g_A \) is only of the order 0.6 to 0.9 underestimating the experimental value 1.26\(^{[1]}\). Recently it has been claimed \(^{[3]}\) that \( 1/N_C \) corrections might provide the missing amount to match the empirical value. However, these corrections are only present if a certain ordering prescription for the generators appearing in the semiclassical quantization scheme is adopted. We will resolve this point below. Additionally, the way these corrections are treated in ref. \(^{[3]}\) violates PCAC. In that calculation only the corrections of the axial current were taken into account disregarding the corresponding changes of the soliton profile. In this letter we will demonstrate that these corrections are mitigated but nevertheless sizable when PCAC is properly accounted for.

Let us briefly review the connection between the equation of motion for the meson profile function and PCAC in chiral soliton models. The static solution to the Euler-Lagrange equations acquires the hedgehog shape for the non-linear representation of the meson fields:

\[
U_0(\mathbf{r}) = \exp(i\tau \cdot \hat{\mathbf{r}} \Theta(\mathbf{r})), \tag{2}
\]

\( \Theta(\mathbf{r}) \) being the chiral angle. The equation of motion for \( \Theta(\mathbf{r}) \) may generally be expressed in the form \( F(\Theta) = 0 \). The hedgehog configuration (2) is quantized by introducing collective coordinates for the (iso-)rotational zero modes:

\[
U(\mathbf{r}, t) = R(t)U_0(\mathbf{r})R^\dagger(t), \quad R(t) \in SU(2). \tag{3}
\]

Due to the vanishing time component of the axial current for the configuration (3) PCAC reduces to (4)

\[
\partial_i A_i^a = D_{ab} \hat{r}_b F(\Theta) - f_\pi^2 m_\pi^2 D_{ab} \hat{r}_b \sin \Theta. \tag{4}
\]

\( D_{ab} = (1/2)\text{tr}(\tau_a R \tau_b R^\dagger) \) denotes the adjoint representation of the collective rotation. The PCAC relation is obviously recovered if the chiral angle satisfies the appropriate equation of motion which is obtained upon inversion of eqn. (4):

\[
0 = F(\Theta) = \hat{r}_b D_{ab} \partial_i A_i^a + f_\pi^2 m_\pi^2 \sin \Theta. \tag{5}
\]

\(^a\text{Note, however, that in the chiral quark model } g_A \text{ is somewhat overestimated}^{[2]}.

\(^b\text{For our sign conventions the pion field of the hedgehog configuration is given by } \pi^a = -f_\pi D_{ab} \hat{r}_b \sin \Theta.\)
The axial coupling constant \( g_A \) is defined as the matrix element of the axial current between proton states at zero momentum transfer:

\[
g_A = 2\langle p \uparrow | A^3_3 | p \uparrow \rangle. \tag{6}
\]

The defining equation for the pion decay constant provides the asymptotic form of the axial current:

\[
\lim_{r \to \infty} A^a_i = f_\pi \lim_{r \to \infty} \partial_i \pi^a = -f_\pi^2 \lim_{r \to \infty} \partial_i (D_{ab} \hat{r}_b \Theta). \tag{7}
\]

The identity \( \partial_j (r_i A^a_j) = A^a_i + r_j \partial_i A^a_j \) then allows to determine \( g_A \) from the pion profile via PCAC (4):

\[
g_A = \frac{4\pi}{3} f_\pi^2 \lim_{R \to \infty} R^3 \frac{\partial \Theta}{\partial r} \bigg|_{r=R} - \frac{8\pi}{9} f_\pi^2 m_\pi^2 \int dr r^3 \sin \Theta \tag{8}
\]

wherein extensive use has been made of the matrix element \( \langle D_{33} \rangle_{\text{proton}} = -1/3 \) in the two flavor case. Note the appearance of the factor 3/2 in the surface term which arises at zero momentum transfer in the chiral limit \( (m_\pi = 0) \) [4]. We would like to emphasize that the equivalence of eqns. (6) and (8) manifests a direct consequence of PCAC for chiral soliton models. Phrased otherwise: PCAC allows to read off \( g_A \) directly and uniquely from the pion profile.

So far these considerations have been fairly general and apply to various kinds of chiral soliton models. Let us now turn to the specific case of a two flavor Nambu–Jona-Lasinio (NJL) model of scalar and pseudoscalar interactions in the isospin limit [5]. After bosonization [6] the action \( \mathcal{A}_{NJL} \) may be expressed as a sum \( \mathcal{A}_{NJL} = \mathcal{A}_F + \mathcal{A}_m \) of a fermion determinant

\[
\mathcal{A}_F = \text{Tr} \log_A (i \hat{D}) = \text{Tr} \log_A (i \hat{\phi} - MU^{\gamma_5}) \tag{9}
\]

and a purely mesonic part

\[
\mathcal{A}_m = \frac{mM}{4G} \int d^4x \text{Tr} (U + U^{-1} - 2) \tag{10}
\]

where \( G \) is the NJL coupling constant. For a finite pion mass it may be eliminated via the relation \( G = mM/m_\pi^2 f_\pi^2 \). For a given constituent quark mass \( M \) the cut-off \( \Lambda \) is determined such as to reproduce the empirical value of the pion decay constant while the current quark mass \( m \) is obtained by solving the gap equation [6]. Throughout this letter we will apply Schwinger’s proper time regularization prescription [7] to the diverging fermion determinant. The NJL model supports static solitons [8] which are obtained as the solutions of the Euler-Lagrange equations to the static energy functional [8, 9, 10].

Only the fermion determinant (9) contributes to the axial current. Since \( \mathcal{A}_F = \mathcal{A}_{\text{val}} + \mathcal{A}_{\text{sea}} \) splits into valence and sea quark parts we have a corresponding decomposition of the axial current:

\[
A^a_i = A^a_i^{(\text{val})} + A^a_i^{(\text{sea})}. \tag{11}
\]

\(^c\)For simplicity we treat only the chiral field \( U \) as space–dependent quantity and keep the scalar field equal to its vacuum expectation value \( \langle \phi \rangle = M \).
The valence quark part is given by

$$A^{(\text{val})}_i = N_C \Psi^{(\text{val})}_i \gamma^a \frac{\tau^a}{2} \Psi^{(\text{val})}$$

(12)

with $N_C$ being the number of color degrees of freedom. In order to determine the valence quark wave function $\Psi^{(\text{val})}_i$ we need to construct the eigenfunctions $\psi_\mu$ of the static Hamiltonian $h_0$:

$$h_0 \psi_\mu = (\mathbf{\alpha} \cdot \mathbf{p} + \beta \mathbf{M} \gamma^5) \psi_\mu = \epsilon_\mu \psi_\mu.$$

(13)

Denoting the corresponding eigenstate $|\mu\rangle$ and the state with the lowest positive energy eigenvalue $|0\rangle$, the valence quark wave function $\Psi^{(\text{val})}_i$ is given as a perturbation expansion in the angular velocities $(i/2) \tau \cdot \mathbf{\Omega} = R^\dagger(t) R(t)$:

$$\Psi^{(\text{val})}_i = R \left( \psi_0 + \frac{1}{2} \sum_{\mu \neq 0} \psi_\mu \frac{\langle \mu | \tau \cdot \mathbf{\Omega} | 0 \rangle}{\epsilon_0 - \epsilon_\mu} \right) + \mathcal{O}(\mathbf{\Omega}^2).$$

(14)

Substitution of eqn. (14) into eqn. (12) yields

$$A^{(\text{val})}_i = N_C D_{ab} \psi_0^\dagger \alpha_i \gamma^5 \frac{\tau^b}{2} \psi_0 + \frac{1}{2} (\Omega_j D_{ab}) \sum_{\mu \neq 0} \frac{\langle \mu | \tau_j | 0 \rangle}{\epsilon_0 - \epsilon_\mu} \psi_\mu \alpha_i \gamma^5 \frac{\tau^b}{2} \psi_0$$

$$+ \frac{1}{2} (D_{ab} \Omega_j) \sum_{\mu \neq 0} \frac{\langle 0 | \tau_j | \mu \rangle}{\epsilon_0 - \epsilon_\mu} \psi_\mu \alpha_i \gamma^5 \frac{\tau^b}{2} \psi_\mu + \mathcal{O}(\mathbf{\Omega}^2).$$

(15)

The terms linear in $\mathbf{\Omega}$ on the RHS of eqn. (13) give vanishing contribution to $g_A$ as long as the angular velocities are considered to be pure $c$-numbers; i.e. there is no $1/N_C$ correction to $g_A$ if $\mathbf{\Omega}$ is treated as a classical quantity. However, this is no longer the case when the semiclassical quantization prescription

$$\Omega_i \rightarrow \frac{J_i}{\alpha^2}$$

(16)

is imposed. Here $J_i$ are the spin operators in the space of the collective coordinates and $\alpha^2$ denotes the (iso-) rotational moment of inertia. The analytic expression for $\alpha^2$ in the NJL model may be found in ref. [11]. Using furthermore the commutation relation

$$[J_j, D_{ab}] = i \epsilon_{jbm} D_{am}$$

(17)

the collective operator involved in the correction to $g_A$ is then found to be of the same structure as in the leading term in agreement with more general considerations [12]. Up to linear order in $1/\alpha^2$ the valence quark contribution to the nucleon axial coupling constant is obtained to be

$$g_A^{(\text{val})} = -\frac{N_C}{3} \left( \langle 0 | \Sigma_3 \tau_3 | 0 \rangle + \frac{i}{\alpha^2} \sum_{\mu \neq 0} \frac{\langle 0 | \tau_1 | \mu \rangle \langle \mu | \Sigma_3 \tau_2 | 0 \rangle}{\epsilon_0 - \epsilon_\mu} \right).$$

(18)

As $\alpha^2$ is of the order $N_C$ the correction in (18) is of order $1/N_C$. The result (18) is, of course, ambiguous. This is due to ordering ambiguities of the semiclassical quantization prescription [14] arising when the classical collective coordinates are substituted by the collective
generators. Since there is no derivation of this prescription from first principles uncertainties of this kind are unavoidable. Ignoring, for the time being, this kind of ambiguities we will explain the relevance of PCAC once this special quantization prescription is adopted. In order to derive the corresponding $1/N_C$ corrections to the equation of motion we need to evaluate the divergence of (13). Using the eigenequation (13) we find:

$$\frac{1}{N_C} \hat{r}_b D_{ab} \partial_i A_i^{(val)\alpha} = \not{M} \bar{\psi}_0 \beta (\sin\Theta + i \not{\tau} \cdot \hat{\gamma}_5 \cos\Theta) \psi_0$$

$$+ \frac{i M}{4\alpha^2} \sum_{\mu \neq 0} \langle 0| \tau_j | \mu \rangle \psi_0 \beta \gamma_5 [\not{T} \hat{r}, \tau_j] \psi_0 \cos\Theta. \tag{19}$$

The first term on the RHS of eqn. (19) is just the valence quarks’ contribution to the classical equation of motion [8] for chiral angle while the second term represents the $1/N_C$ correction we are looking for. According to ref. [3] no $1/N_C$ corrections to $g_A$ arise from the sea-part of the action $A_{sea}$:

$$g_A^{sea} = \frac{N_C}{6} \sum_{\mu} \text{erfc}(\frac{\epsilon_\mu}{\Lambda}) \langle \mu | \Sigma_3 \tau_3 | \mu \rangle \tag{20}$$

and thus there is also no further change to equation of motion. Note that in the chiral limit ($m_\pi = 0$) an overall factor $3/2$ appears in eqns. (19,20) [4]. The absence of $1/N_C$ corrections to $g_A^{sea}$ has been doubted and small corrections may indeed be present [13]. We will not go into this point further but rather concentrate on the $1/N_C$ corrections to the equation of motion stemming from the valence quark wave function since this suffices to illuminate the relevance of PCAC. The complete equation of motion we would like to solve then reads

$$\cos\Theta(r) \left[ \int d\Omega \left( \bar{\psi}_0 \not{T} \cdot \hat{\gamma}_5 \psi_0 + \frac{i}{4\alpha^2} \sum_{\mu \neq 0} \langle 0| \tau_j | \mu \rangle \bar{\psi}_0 \gamma_5 [\not{T} \hat{r}, \tau_j] \psi_0 + \text{tr}(i \not{T} \cdot \hat{\gamma}_5 \rho_S^{sea}) \right) \right]$$

$$= \sin\Theta(r) \left[ \int d\Omega \left( \bar{\psi}_0 \psi_0 + \text{tr}(\rho_S^{sea}) \right) - \frac{4\pi m_\pi^2 f_\pi^2}{N_C M} \right]. \tag{21}$$

wherein $\rho_S^{sea}$ denotes the sea contribution to the scalar density defined in ref. [8].

As the moment of inertia $\alpha^2$ is a functional of the meson profile the appearance of $\alpha^2$ in (21) transforms the original algebraic equation into an integral equation. Its numerical solution is obtained by iteration. For a given $\alpha^2$ the self-consistent solution to eqns. (13) and (21) is constructed. This solution then yields a new $\alpha^2$ to be substituted in eqn. (21). Since the resulting $\alpha^2$ deviates only moderately from the one obtained when the $1/N_C$ correction is omitted a converging solution is already obtained after a few iterations.

As can be seen from fig. (1) the pion profile at large radii is increased if the $1/N_C$ correction is taken into account. This increases the moment of inertia $\alpha^2$ which in turn decreases the correction term in the equation of motion (21) and in the expression for $g_A$ (13). This behavior guarantees the stability of the $1/N_C$ corrected soliton and limits the change of most quantities, see table (1). E.g. the soliton energy $E_{total}$ (11) is increased by a few percent only. As the slope of the soliton profile close to the origin becomes moderately larger, the valence quark energy $\epsilon_0$ decreases accordingly. Also the moment of inertia increases by a small amount yielding an even smaller nucleon–Δ mass splitting $M_\Delta - M_n = 3/2 \alpha^2$. A similar behavior is found for the isoscalar root mean square radius $\langle r^2 \rangle^{1/2}$. In the chiral limit it is even overestimated when the $1/N_C$ correction is taken into account. In contrast to
Fig. 1. The chiral angle $\Theta$ as the self-consistent solution of eqns. (13,21): solid lines. The dashed lines refer to the omission of the $1/\alpha^2$ term in (21). Displayed are the cases $m_\pi = 0$ (a) and $m_\pi = 138$MeV (b).

Table 1. Static properties for the soliton solution of eqns. (13,21): ‘$\Theta$ corr’. ‘$\Theta$ not corr’ refers to the omission of the $1/\alpha^2$ term in the equation of motion (21). The superscript for $g_A$ denotes the order of $1/\alpha^2$ in eqns. (18) and (20). $g_A^{(\text{tot})} = g_A^{(0)} + g_A^{(1)}$ is the final result for the nucleon axial coupling constant. $\epsilon_0$, $E_{\text{total}}$, $\alpha^2$ and $\langle r^2 \rangle_{I=0}^{1/2}$ are the valence quark eigenenergy, total energy of the soliton, moment of inertia and the isoscalar root mean square radius, respectively. We have used the constituent quark mass $M = 400$MeV.

| $m_\pi$ = 0 | $m_\pi$ = 138MeV |
|----------------|------------------|
| $\Theta$ not corr. | $\Theta$ corr. | $\Theta$ not corr. | $\Theta$ corr. |
| $g_A^{(0)}$ | 0.83 | 0.82 | 0.75 | 0.81 |
| $g_A^{(1)}$ | 0.47 | 0.35 | 0.39 | 0.30 |
| $g_A^{\text{tot}}$ | 1.30 | 1.17 | 1.14 | 1.12 |
| $g_A$ from eqn. (8) | 0.82 | 1.13 | 0.76 | 1.14 |
| $\epsilon_0$/MeV | 190 | 151 | 215 | 167 |
| $E_{\text{total}}$/MeV | 1203 | 1233 | 1253 | 1266 |
| $\alpha^2/(1/\text{GeV})$ | 6.49 | 7.50 | 5.86 | 6.09 |
| $\langle r^2 \rangle_{I=0}^{1/2}$/fm | 0.90 | 0.98 | 0.75 | 0.80 |
the moderate change of other quantities the first order correction for \( g_A \) drops significantly. Whereas in the chiral limit this change is reflected also in the total sum for \( g_A \) there is almost no net effect in the case of the physical pion mass because the zeroth order value for \( g_A \) increases.

Table (1) reveals also that without correction for the meson profile the value of \( g_A \) differs significantly from the value as obtained using eq. (8), i.e. PCAC is strongly violated. As expected the PCAC relation (8) provides values which are equal to the zeroth order contribution. On the other hand, with the corrected meson profile PCAC is fulfilled within numerical uncertainties. Note that the extraction of \( g_A \) from (8) becomes the more difficult the larger the soliton is, as all numerical calculations are done in finite boxes. We would also like to mention that we find some discrepancies from the numerical results of ref. [3]. Some of these discrepancies can be resolved by multiplying their result for \( g_A^{(1)} \) for \( m_\pi = 0 \) by 3/2. The necessity of this factor has been pointed out in ref. [4].

In conclusion, we have calculated the NJL soliton including \( 1/N_C \) corrections as they result from the ambiguous quantization “recipe” (16). As claimed already in ref. [3] these corrections solve the problem of a too small value of the axial coupling of the nucleon \( g_A \) in hedgehog soliton models. Taking into account these corrections in a way that PCAC is still fulfilled decreases the value of the correction for \( g_A \). Nevertheless, the total result is less than ten percent off the experimental value. Here we argued on the basis of PCAC that the soliton profile has to change also if we adopt the quantization (16,17). It would be interesting to construct the energy functional which corresponds to that operator ordering and therefore provides the meson profile calculated in this letter from a variational principle. Since the soliton mass is of order \( N_C \) this correction has to be of order \( N_C^0 \). On the other hand, as long as the existence of such a formalism cannot be shown, the application of “recipe” (16,17) to different orderings of collective coordinates and operators will at least be doubtful. After all, it would be somewhat astonishing if the problem of a too small \( g_A \) for hedgehog soliton models turned out to be only a result of a peculiar quantization.

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Table 1: Static properties for the soliton solution of eqns. 13, 21: ‘Θ corr’, ‘Θ not corr’ refers to the omission of the $1/\alpha^2$ term in the equation of motion 21. The superscript for $g_A$ denotes the order of $1/\alpha^2$ in eqns. 18 and 20. $g_A^{(tot)} = g_A^{(0)} + g_A^{(1)}$ is the final result for the nucleon axial coupling constant. $\epsilon_0$, $E_{total}$, $\alpha^2$ and $\langle r^2 \rangle_{I=0}$ are the valence quark eigenenergy, total energy of the soliton, moment of inertia and the isoscalar root mean square radius, respectively. We have used the constituent quark mass $M = 400$MeV.

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| $g_A$ from eqn. 8 | 0.82        | 0.76             | 1.14             |
| $\epsilon_0$/MeV | 190         | 215              | 167              |
| $E_{total}$/MeV | 1203        | 1253             | 1266             |
| $\alpha^2/(1/GeV)$ | 6.49        | 5.86             | 6.09             |
| $\langle r^2 \rangle_{I=0}$/a/fm | 0.90        | 0.75             | 0.80             |

Figure 1: The chiral angle $\Theta$ as the self-consistent solution of eqns. 13, 21: solid lines. The dashed lines refer to the omission of the $1/\alpha^2$ term in 21. Displayed are the cases $m_\pi = 0$ (a) and $m_\pi = 138$MeV (b).
