Global Symmetries of Quaternion-Kähler
$\mathcal{N} = 4$ Supersymmetric Mechanics

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Abstract
We analyze the global symmetries of $\mathcal{N} = 4$ supersymmetric mechanics involving
4n-dimensional Quaternion-Kähler (QK) 1D sigma models on projective spaces $\mathbb{H}^n$ and $\mathbb{H}P^n$ as the bosonic core. All Noether charges associated with global
worldline symmetries are shown to vanish as a result of equations of motion, which implies that we deal with a severely constrained hamiltonian system. The
complete hamiltonian analysis of the bosonic sector is performed.

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1 Introduction

After appearance of the first version of supersymmetric quantum mechanics (SQM) in the seminal paper [1], this class of theories was intensively and extensively studied in numerous articles and reviews (see, e.g., [2] - [5]). The SQM models reveal interesting quantum and geometric properties, some of which cannot be reproduced in the framework of the standard dimensional reduction from the higher-dimensional supersymmetric field theories. Of special interest are SQM models with extended $\mathcal{N} = 4$ and $\mathcal{N} = 8$ worldline supersymmetries. Some of these 1D sigma models admit, e.g., hyper-Kähler manifolds as their bosonic target spaces and are capable to provide a good laboratory for analyzing various properties of the higher-dimensional theories associated with such kind of targets.

Until recently, the majority of SQM models (including those with $\mathcal{N} \geq 4$) were constructed under assumption that the worldline (or super worldline in the case of superfield models) are “flat”, i.e. 1D supersymmetry is rigid. In the paper [6], using $\mathcal{N} = 4, 1D$ harmonic superspace approach [7], we constructed a new type of $\mathcal{N} = 4$ supersymmetric mechanics involving $4n$-dimensional Quaternion-Kähler (QK) 1D sigma models as the bosonic core. The basic distinguishing features of the new SQM models constructed are local worldline $\mathcal{N} = 4$ supersymmetry and the presence of the appropriate $\mathcal{N} = 4, 1D$ supergravity multiplet for ensuring this local invariance.

When restricted to the $H^n$ or $HP^n$ target manifolds, the bosonic sector of these models is identical to the dimensionally reduced homogeneous QK sigma models of refs. [8], [9], [10]. In ref. [6] the component action was also obtained for the fermionic sector, therefore the whole component action in an arbitrary gauge is available. With so much detailed information accessible it is worth trying some further insights into these models. These models have rather large gauge symmetries, so that the quantization in a covariant manner seems challenging. In this paper we will restrict ourselves to the study of the global symmetries which these models possess, restricting the gauge transformations to constant parameters. We will start by presenting a short review of the QK component model followed by a review of their transformation properties. Then we will concentrate on the global properties of these models, restricting the gauge transformations to constant parameters. After a somehow involved algebra we will show that the corresponding Noether charges are vanishing as a consequence of the equations of motion, and therefore give rise to the gauge constraints, in a similar fashion, e.g., to the appearance of Virasoro constraints in the string theory. The hamiltonian analysis of the bosonic sectors of these homogeneous models is performed as a prerequisite to their quantization.
2 Superfield $\mathbb{H}H^n$ and $\mathbb{H}P^n$ actions

The particular $\mathbb{H}H^n$ or $\mathbb{H}P^n$ case of the general 1D superfield action describing $\mathcal{N} = 4$ supersymmetric QK mechanics looks very simple \[^6\]

$$S_{HP} = \frac{1}{8} \int \mu_H \left[ E \left( \gamma q^a - \dot{Q}^r \right) + \beta \sqrt{E} \right], \quad (2.1)$$

where $\mu_H$ is the measure of integration over the whole $\mathcal{N} = 4, 1D$ superspace, $a$ and $r$ are indices of the fundamental representations of the internal groups $Sp(1)$ and $Sp(n)$, $a = 1, 2; r = 1, \ldots, 2n$, $q^a := D^- q^a$, $\dot{Q}^r := D^- \dot{Q}^r$, and $q^a$ and $\dot{Q}^r$ are analytic $\mathcal{N} = 4$ superfields subjected to the additional harmonic constraints

$$D^{++} q^a = D^{++} \dot{Q}^r = 0.$$ 

The details of 1D harmonic superspace formalism including the explicit form of the harmonic derivatives $D^{\pm\pm}$ can be found in \cite{7} and \cite{6}. An important object appearing in (2.1) is the harmonic-independent (in the central basis) supervielbein $E$ incorporating fields of some non-minimal version of $\mathcal{N} = 4, 1D$ “supergravity”. Its presence secures invariance of (2.1) under the appropriate local extension of $\mathcal{N} = 4, 1D$ supersymmetry. The real parameter $\beta$ is arbitrary, while $\gamma = \pm 1$, and with the signs $\pm$ the action (2.1) is invariant, respectively, under the extended global $Sp(n, 1)$ or $Sp(n + 1)$ groups realized as

$$\delta q^a = -\gamma \Lambda^{a r} \dot{Q}^r, \quad \delta \dot{Q}^r = \Lambda^{a r} q^a,$$ \quad (2.2)$$

where $\Lambda^{a r}$ are the constant $Sp(n, 1)/[Sp(1) \times Sp(n)]$ or $Sp(n + 1)/[Sp(1) \times Sp(n)]$ parameters. The $Sp(1)$ and $Sp(n)$ subgroups in both cases are realized as symplectic rotations with respect to the indices $a$ and $r$, respectively. For $\gamma = \pm 1$ the action (2.1) in the bosonic sector, after fixing some gauges with respect to local symmetries, describes 1D sigma models on the $\mathbb{H}H^n$ and $\mathbb{H}P^n$ target spaces.

3 Component QK Lagrangians

Three classes of the superfields which appear in (2.1) have the following field contents:

- The vielbein $E$ encompasses the multiplet $8 + 8$ of the “non-minimal” $\mathcal{N} = 4, 1D$ “supergravity”. Its bosonic sector consists of the dynamical field $h(t)$ (“graviton”) and the auxiliary fields $M(t), \bar{M}(t), \mu(t), D(t), L^{ik}(t)$ where $i = 1, 2$ are the doublet indices of the $SU(2)$ automorphism group of the supersymmetry algebra. The fermionic sector involves the dynamical fermionic fields $\phi_i(t), \bar{\phi}_i(t)$, (“gravitino”) and the auxiliary fields $\sigma^i(t), \bar{\sigma}^i(t)$. The conjugation rules for the bosonic fields are evident, while for the fermionic fields they are:

$$\overline{\phi_i} = \bar{\phi}^i, \quad \overline{\sigma_i} = \bar{\sigma}^i.$$ \quad (3.1)
• Two other superfields, \( q^{+a} \) and \( \hat{Q}^{+r} \), describe 1D “matter” and both encompass the \( \mathcal{N} = 4, 1D \) multiplets \((4,4,0)\). First of them contains the dynamical fields \( f^{ia}, \chi^a, \bar{\chi}^a \). It is 1D analog of the “conformal compensator” \( q^{+} \) superfield of the harmonic superspace formulation of \( \mathcal{N} = 2, 4D \) supergravity [11], [12]. The set of \((4,4,0)\) fields entering \( \hat{Q}^{+r} \) contains the dynamical fields \( \hat{F}^{ia}, \chi^r, \bar{\chi}^r \). The conjugation properties of the relevant fields are:

\[
(f_{ia}) = f^{ia}, \quad (\dot{f}^{ia}) = \dot{f}_{ia}; \quad (\chi^a) = \bar{\chi}^a, \quad (\dot{\chi}^a) = -\bar{\chi}^a, \quad (\chi^r) = \bar{\chi}^r.
\]

(3.2)

and similar ones for \( \hat{F}^{ir}, \chi^r \). It is assumed that the indices \( i, a, \) and \( r, \) are raised and lowered in the standard way by the skew -symmetric tensors \( \varepsilon_{ik}, \varepsilon^{ik}, \varepsilon_{ab}, \varepsilon^{ab} \) and \( \Omega_{rs}, \Omega^{rs} \).

The precise way how all these fields enter the respective superfields can be found in [6].

We will be interested in the component Lagrangian \( \mathcal{L}_{HP} \) which corresponds to the superfield one (2.1) and is obtained from the latter after integrating there over Grassmann and harmonic variables. It is a sum of the three Lagrangians:

• The gauge-covariantized kinetic terms of the bosonic compensator and matter fields:

\[
\mathcal{L}_{HP}^{b} = \frac{1}{2} h \left( \dot{\hat{F}}^{ir} \hat{F}_{ir} - \gamma \dot{f}^{ia} \dot{f}_{ia} \right) + L_{ik} \left( \ddot{\hat{F}}^{(ir} \hat{F}^{k)}_{r} - \gamma f^{(ia} \dot{f}^{k)a} \right)
+ \frac{1}{4} D \left( \gamma f^{ia} f_{ia} - \hat{F}^{ir} \hat{F}_{ir} + \frac{\beta}{\sqrt{h}} \right)
+ \frac{\beta}{4} \frac{1}{\sqrt{hh}} \left[ L^{ik} L_{ik} - \frac{1}{8} (M \bar{M} + \mu^2 + \dot{h}^2) \right].
\]

(3.3)

• The gauge covariantization of the kinetic terms of the fermionic compensator and matter fields:

\[
\mathcal{L}_{HP}^{f(1)} = \frac{i}{4} h \left[ \gamma \left( \chi^a \dot{\bar{X}}_a - \bar{\chi}^a \dot{X}_a \right) - \chi^r \dot{\bar{X}}_r + \bar{\chi}^r \dot{X}_r \right]
+ \frac{i}{2} \bar{\sigma}_i \left( \gamma \dot{\bar{f}}^{ia} \bar{X}_a - \hat{F}^{ir} \bar{X}_r \right) - \frac{i}{2} \bar{\sigma}_i \left( \gamma \dot{\bar{f}}^{ia} \chi_a - \hat{F}^{ir} \chi_r \right)
+ \frac{M}{8} \left( \gamma \bar{X}_a \dot{\bar{X}}_a - \bar{\chi}^r \dot{X}_r \right) - \frac{M}{8} \left( \gamma \bar{X}_a \chi_a - \bar{\chi}^r \chi_r \right)
+ \frac{\mu}{4} \left( \gamma \bar{X}_a \chi_a - \bar{\chi}^r \chi_r \right).
\]

(3.4)
• The remaining Lagrangian that involves fermionic fields of the 1D “supergravity” multiplet and comes solely from the last term in (2.1):

\[
\mathbb{L}_{f(2)}^{\hat{H}P} = \beta \frac{i}{32 \hbar^{3/2}} \left( \phi^i \dot{\phi}_i - \tilde{\phi}^i \dot{\tilde{\phi}}_i + 4 \sigma^i \tilde{\phi}_i - 4 \tilde{\sigma}^i \phi_i \right) \\
+ \beta \frac{3}{64 \hbar^{5/2}} \left( 4iL^{ik} \phi_i \phi_k + \frac{M}{2} \phi^i \phi_i - \frac{M}{2} \tilde{\phi}^i \phi_i + \mu \phi^i \phi_i \right) \\
+ \beta \frac{15}{64 \cdot 8 h^{7/2}} (\phi^k \phi_k)(\tilde{\phi}^i \tilde{\phi}_i). \tag{3.5}
\]

The total off-shell Lagrangian is the sum of these three ones:

\[
\mathbb{L}_{HP} = \mathbb{L}_{f(1)}^b + \mathbb{L}_{f(2)}^b + \mathbb{L}_{f(2)}^{\hat{H}P}. \tag{3.6}
\]

4 Transformation properties of QK Lagrangian

The above Lagrangian is invariant under the local transformations with the parameters \(b(t), \lambda^i(t), \tau^i_k(t)\) associated, respectively, with the time reparametrizations, local \(\mathcal{N} = 4, 1D\) supersymmetry and local \(SU(2)\) R-symmetry. Various sets of fields have the following transformation laws:

1. The fields \(f^{ia}, \chi^a, \bar{\chi}^a\):

\[
\delta_b f^{ia} = -2b \dot{f}^{ia} - \dot{b} f^{ia}, \quad \delta_b \chi^a = -2b \dot{\chi}^a - 2b \chi^a, \quad \delta_b \bar{\chi}^a = -2b \dot{\bar{\chi}}^a - 2b \bar{\chi}^a, \tag{4.1}
\]

\[
\delta_{\lambda} f^{ia} = -\lambda^i \chi^a - \lambda^i \bar{\chi}^a, \quad \delta_{\lambda} \chi^a = 2i \partial_t (\lambda^i f_i^a), \quad \delta_{\lambda} \bar{\chi}^a = 2i \partial_t (\lambda^i \bar{f}_i^a), \tag{4.2}
\]

\[
\delta_{\tau} f^{ia} = \tau^i_k f^{ka}, \quad \delta_{\tau} \chi^a = \delta_{\tau} \bar{\chi}^a = 0. \tag{4.3}
\]

2. The fields \(\hat{F}^{ir}, \chi^r, \bar{\chi}^r\):

\[
\delta_b \hat{F}^{ir} = -2b \dot{\hat{F}}^{ir} - \dot{b} \hat{F}^{ir}, \quad \delta_b \chi^r = -2b \dot{\chi}^r - 2b \chi^r, \quad \delta_b \bar{\chi}^r = -2b \dot{\bar{\chi}}^r - 2b \bar{\chi}^r, \tag{4.4}
\]

\[
\delta_{\lambda} \hat{F}^{ir} = -\lambda^i \chi^r + \lambda^i \bar{\chi}^r, \quad \delta_{\lambda} \chi^r = 2i \partial_t (\lambda^i \hat{F}_i^r), \quad \delta_{\lambda} \bar{\chi}^r = 2i \partial_t (\lambda^i \bar{F}_i^r), \tag{4.5}
\]

\[
\delta_{\tau} \hat{F}^{ir} = \tau^i_k \hat{F}^{kr}, \quad \delta_{\tau} \chi^r = \delta_{\tau} \bar{\chi}^r = 0. \tag{4.6}
\]

3. \(\mathcal{N} = 4, 1D\) “supergravity” fields:

\[
\delta_b h = -2b \dot{h} + 4b \dot{b} h, \quad \delta_b M = -2b \dot{M} + 2b \dot{b} M, \quad \delta_b \mu = -2b \dot{\mu} + 2b \dot{b} \mu, \tag{4.7}
\]

\[
\delta_b L^{(ik)} = -2b \dot{L}^{(ik)} + 2b L^{(ik)}, \quad \delta_b D = -2b \dot{D} + 2b \partial_t (b h), \\
\delta_b \phi^i = -2b \dot{\phi}^i + 3b \phi^i, \quad \delta_b \sigma^i = -2b \dot{\sigma}^i + \dot{b} \sigma^i + \ddot{b} \phi^i.
\]

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\[ \delta \lambda h = \lambda^i \phi_i - \dot{\lambda}^i \dot{\phi}_i, \quad \delta \lambda M = 2i \dot{\lambda}^i \phi_i + i \ddot{\lambda}^i (4\sigma_i - 2\dot{\phi}_i), \]
\[ \delta \lambda \dot{M} = -2i \dddot{\lambda}^i \phi_i - i \lambda^i (4\dot{\sigma}_i - 2\ddot{\phi}_i), \]
\[ \delta \lambda \mu = -i (\dot{\lambda}^i \phi_i + \ddot{\lambda}^i \phi_i) - i [\lambda^i (2\sigma_i - \dot{\phi}_i) + \dddot{\lambda}^i (2\ddot{\sigma}_i - \dddot{\phi}_i)], \]
\[ \delta \lambda L^{(ik)} = \lambda^i \sigma^k - \lambda^i \sigma^k - [\dot{\lambda}^i \phi^k - \dot{\lambda}^i \phi^k], \]
\[ \delta \lambda \dot{D} = \lambda^i \dot{\sigma}_i - \dddot{\lambda}^i \dot{\phi}_i + \dot{\lambda}^i \sigma_i + \partial_t (\dot{\lambda}^i \phi_i - \ddot{\lambda}^i \phi_i), \]
\[ \delta \lambda \phi^i = \lambda^i M + \dddot{\lambda}^i (\mu + \dot{h}) + 4i \dot{\lambda}^k L^i_k - 4i \dot{\lambda}^k h, \]
\[ \delta \lambda \dddot{\phi}^i = \dddot{\lambda}^i M - \dddot{\lambda}^i (\mu - \dot{h}) + 4i \lambda^k L^i_k - 4i \dot{\lambda}^k h, \]
\[ \delta \lambda \sigma^i = \lambda^i \dot{M} - \dot{\lambda}^i (\mu + \dot{h}) + 2i \lambda^k \dot{L}^i_k + i \dot{\lambda}^i D - 2i \ddot{\lambda}^i h, \]
\[ \delta \lambda \dddot{\sigma}^i = \dddot{\lambda}^i \dot{M} - \dddot{\lambda}^i (\mu + \dot{h}) + 2i \lambda^k \dot{L}^i_k + i \dot{\lambda}^i D - 2i \dddot{\lambda}^i h, \] (4.8)
\[ \delta \tau h = \delta \tau M = \delta \tau \mu = 0, \quad \delta \tau L^{(ik)} = \hbar \dddot{\tau}^{(ik)} - 2\tau^{(im)} L^k_m, \quad \delta \tau D = -2 \ddot{\tau}^{(ik)} L^{(ik)}, \]
\[ \delta \tau \phi^i = \tau^i \phi^k, \quad \delta \sigma^i = \tau^i \sigma^k - \dddot{\tau}^{(ik)} \phi_k. \] (4.9)

The standard rigid \( \mathcal{N} = 4,1D \) supersymmetry and \( R \)-symmetry \( SU(2) \) transformations of the component fields are recovered upon choosing the constant parameters in (4.1) - (4.9), \( \dot{\lambda}^i = \dot{\tau}^{ik} \) = 0. The \( \dot{\tau} = 0 \) transformations are just the constant time shifts.

The internal symmetry transformations (2.2) uniformly act in the evident way on the indices \( a \) and \( r \) of matter bosonic and fermionic fields, properly mixing \( f^{ia} \) with \( \bar{F}^{ir} \) and \( \chi^a \) with \( \chi^r \).

It is not so easy to check that the Lagrangian \( L_{HP} \) given by eq. (3.6) is indeed invariant, up to a total time derivative, under all these sets of the field transformations. Nevertheless, this can be done, in complete agreement with the corresponding invariances of the superfield action (2.1) proved in [6].

## 5 Equations of motion

It is straightforward to derive the classical equations of motion following from the action with the Lagrangian \( L_{HP} \), eq. (3.6). They are divided into sets of non-dynamical algebraic equations, as well as the dynamical equations, of the second order in \( \partial_t \) for bosonic fields and of the first order for fermionic fields.

1. Non-dynamical equations:

\[ D : \quad \gamma f^2 - \bar{F}^2 + \frac{\beta}{\hbar^{1/2}} = 0, \] (5.1)
2. Dynamical equations:

\[ L^{ik} : \quad L^{ik} = -2\frac{h^{3/2}}{\beta} \left[ \dot{F}^{(ir}\dot{F}^{k)r} - \gamma f^{(ia}\dot{f}^{k)a} \right] + \frac{3i}{8h} \bar{\phi}^{(i} \phi^{k)} , \]  
\[ M : \quad M = \frac{4h^{3/2}}{\beta} \left( \chi^r \chi_r - \gamma \chi^a \chi_a \right) - \frac{3}{4h} \phi^i \dot{\phi}_i , \]  
\[ \bar{M} : \quad \bar{M} = \frac{4h^{3/2}}{\beta} \left( \gamma \bar{\chi}^a \bar{\chi}_a - \bar{\chi}^r \bar{\chi}_r \right) + \frac{3}{4h} \bar{\phi}^i \dot{\bar{\phi}}_i , \]  
\[ \mu : \quad \mu = \frac{4h^{3/2}}{\beta} \left( \gamma \bar{\chi}^a \chi_a - \bar{\chi}^r \chi_r \right) + \frac{3}{4h} \phi^i \dot{\bar{\phi}}_i , \]  
\[ \bar{\sigma}_i : \quad \phi^i = \frac{4h^{3/2}}{\beta} \left( \gamma f^{ia} \chi_a - \dot{F}^{ir} \chi_r \right) , \quad \sigma_i : \quad \bar{\phi}^i = \frac{4h^{3/2}}{\beta} \left( \gamma f^{ia} \bar{\chi}_a - \dot{\bar{F}}^{ir} \bar{\chi}_r \right). \]  

2. Dynamical equations:

\[ \dot{F}^{ir} : \quad \partial_i (\hbar \dot{F}^{ir}) + \frac{1}{2} D \dot{F}^{ir} - 2L^{ir} \dot{F}^{k)^r} - \bar{L}_{ik} \dot{F}_r^{k} - \frac{i}{2} \left( \phi_i \dot{\bar{\chi}}_r - \bar{\phi}_i \chi_r \right) + \frac{i}{2} \left[ (\sigma_i - \dot{\phi}_i) \chi_r - (\bar{\sigma}_i - \dot{\bar{\phi}}_i) \bar{\chi}_r \right] = 0 \]  
(equation for \( f^{ia} \) has the same form, with the evident substitution of indices \( r \rightarrow a \)).

\[ h : \quad \partial_i^2 (h^{-\frac{1}{2}}) = \frac{4}{\beta} (\dot{F}^2 - \gamma f^2) + V , \]  
\[ V = -\frac{D}{h^{3/2}} - \frac{3}{h^{3/2}} [L^{ik} L_{ik} - \frac{1}{8} (M \bar{M} + \mu^2 + \dot{h}^2) + \frac{i}{8} (\phi^i \dot{\phi}_i - \bar{\phi}^i \dot{\bar{\phi}}_i + 4\sigma^i \dot{\phi}_i - 4\bar{\sigma}^i \bar{\phi}_i)] + \frac{2i}{\beta} \left[ \gamma (\chi^a \dot{\bar{\chi}}_a - \dot{\chi}^a \bar{\chi}_a) - \chi^r \dot{\bar{\chi}}_r + \dot{\chi}^r \bar{\chi}_r \right] - \frac{15}{16h^{7/2}} (4i L^{ik} \phi_k \bar{\phi}_i + M \bar{\phi}^j \dot{\phi}_i - \frac{M}{2} \phi^j \dot{\bar{\phi}}_i + \mu \phi^j \dot{\bar{\phi}}_i) - \frac{7 \times 15}{128h^{9/2}} \phi^k \phi_k \bar{\phi}^j \dot{\phi}_i . \]  

Then follow the equations for matter fermions:

\[ \check{\chi}^r : \quad \check{\chi}_r + \frac{1}{h} [ - \phi_i \dot{F}_i^r - \sigma_i \dot{\bar{F}}_i^r + \frac{i}{2} M \check{\chi}_r + \frac{i}{2} \mu \chi_r + \frac{1}{2} \dot{h} \chi_r ] = 0 , \]  
\[ \chi^r : \quad \check{\chi}_r + \frac{1}{h} [ - \bar{\phi}_i \dot{\bar{F}}_i^r - \bar{\sigma}_i \dot{\bar{F}}_i^r + \frac{i}{2} M \bar{\chi}_r - \frac{i}{2} \mu \bar{\chi}_r + \frac{1}{2} \dot{h} \bar{\chi}_r ] = 0 \]  
(and similar equations for \( \chi_a \) and \( \bar{\chi}_a \)).
Finally, we obtain the equations for the “gravitino” $\phi^i$ and $\bar{\phi}^i$:

$$\bar{\phi}^i : \frac{1}{16} h^{-2} \bar{\phi}^i (\phi^k \phi_k) = 0,$$

$$\phi^i : \frac{15i}{16} h^{-2} \phi_i (\phi^k \phi_k) = 0.$$

(5.9)

### 6 Noether charges

Noether charges are calculated in the standard way for the transformations with constant parameters.

1. **R-symmetry.** We start with the $SU(2)$ current associated with the parameters $\tau_{ik}$. We define

$$\delta_{\tau} \Phi_A \frac{\partial L}{\partial \dot{\Phi}_A} = \tau_{ik} J_{ik},$$

whence

$$J_{ik} = h \left[ \hat{F}_{i(r} \hat{F}_{k)}^r - \gamma f_{(i} f_{k)}^a \right] + \frac{i}{2} \left[ \phi_i (\hat{F}_{k}) \bar{\chi}_r - \gamma f_{k(a} \bar{\chi}_r) \right]$$

$$- \frac{1}{2} (\hat{F}^2 - \gamma f^2) L_{ik} + i \beta \frac{1}{16 h^{3/2}} \phi_i (\phi_k).$$

(6.1)

Using the bosonic constraint (5.1) and the on-shell expression for $L_{ik}$ and $\phi_i, \bar{\phi}_i$ (from the non-dynamical equations (5.2) and (5.4)), it is easy to show that on shell

$$J_{ik} = 0.$$

(6.2)

2. **Supersymmetry.** Next we construct supercharges associated with the constant parameters $\lambda^i$. We define

$$\delta_{\lambda} L_{HP} = \partial_i U, \quad \delta_{\lambda} \Phi_A \frac{\partial L_{HP}}{\partial \dot{\Phi}_A} - U = \lambda^k Q_k + \bar{\lambda}^k \bar{Q}_k.$$

(6.3)
with
\[ U = \chi^k \left\{ \frac{\hbar}{2} (\dot{F}_k \chi_r - \gamma \dot{f}_k^a \chi_a) + L_k^i (\dot{F}_i \chi_r - \gamma f_i^a \chi_a) - \frac{1}{4} \sigma_k (\dot{F}^2 - \gamma f^2) \right\} \\
- i \frac{1}{4} \dot{\phi}_k (\chi^r \dot{x}_r - \gamma \chi^a \chi_a) + i \frac{1}{4} \Bar{\phi}_r (\chi^r \dot{x}_r - \gamma \chi^a \chi_a) \\
+ \beta \frac{4h^{1/2}}{4h^{1/2}} [ - \frac{i}{8 \hbar} (M \dot{\phi}_k + \mu \dot{\phi}_k - i \hbar \phi_k + 4i L_k \phi_i) - \frac{3i}{32 h^2} \dot{\phi}_k \phi^i \phi^i ] \} \\
\]
whence
\[ Q_k = h \left\{ \dot{F}_k \chi_r - \gamma \dot{f}_k^a \chi_a \right\} + \frac{1}{4} \sigma_k (\dot{F}^{jr} \dot{F}_j r - \gamma f^{ak} f_{ak}) \\
- i \frac{1}{4} \dot{\phi}_k (\chi^r \dot{x}_r - \gamma \chi^a \chi_a) + i \frac{1}{4} \Bar{\phi}_r (\chi^r \dot{x}_r - \gamma \chi^a \chi_a) \\
+ \frac{\beta}{16 h^{3/2}} \left[ - 4 \hbar \sigma_k - \hbar \phi_k - 4L_k^i \phi_i + \frac{3i}{8 \hbar} \phi_k (\phi^i \phi^i) \right], \tag{6.5} \]
and
\[ \Bar{Q}_k = h \left\{ \dot{F}_k \chi_r - \gamma \dot{f}_k^a \chi_a \right\} + \frac{1}{4} \sigma_k (\dot{F}^{jr} \dot{F}_j r - \gamma f^{ak} f_{ak}) \\
+ i \frac{1}{4} \dot{\phi}_k (\chi^r \dot{x}_r - \gamma \chi^a \chi_a) - i \frac{1}{4} \phi_k (\chi^a \dot{x}_r - \gamma \chi^a \chi_a) \\
- \frac{\beta}{16 h^{3/2}} \left[ 4 \hbar \sigma_k + \hbar \phi_k + 4L_k^i \phi_i + \frac{3i}{8 \hbar} \phi_k (\phi^i \phi^i) \right]. \tag{6.6} \]

Let us show that $Q_i = 0$ on shell. We identically rewrite the expression in the first term in (6.5) as
\[ \dot{F}_k \chi_r - \gamma \dot{f}_k^a \chi_a = \partial_i (\dot{F}_k \chi_r - \gamma f_i^a \chi_a) - (\dot{F}_k \dot{x}_r - \gamma f_k^a \dot{x}_a) \]
\[ = -\beta \left( \frac{1}{4h^{3/2}} \dot{\phi}_k - \frac{3}{8h^{5/2}} \hbar \phi_i \right) - \frac{1}{\hbar} \phi_j (\dot{F}^r \dot{F}_j - \gamma f^a f_j^a) \]
\[ + \frac{\beta}{2h^{3/2}} \sigma_k - i \frac{\beta}{8h^{5/2}} \left( M \dot{\phi}_k + \mu \dot{\phi}_k + i \hbar \phi_k \right), \tag{6.7} \]
where we made use of the bosonic constraint (5.1), eq. (5.8), together with the analogous one for $\chi_a$ and (a few times) of the fermionic constraint (5.4). As the next steps we represent
\[ - \frac{1}{\hbar} \phi_j (\dot{F}_k \dot{F}_j^r - \gamma f_k^a f_j^a) = - \frac{\beta}{8h^{5/2}} \hbar \phi_k + \frac{1}{\hbar} \phi^i (\dot{F}_k \dot{F}_j^r - \gamma f_k^a \dot{f}_j^a) \]
substitute it in (6.7) and then use the equation of motion for $\phi_i$, eq. (5.9). After some
algebra, we obtain
\[
\dot{\hat{F}}_{kr} - \gamma \dot{f}_{kr} = \frac{\beta}{16h^{5/2}} \dot{\phi}_k - \frac{1}{h} \phi^j \left[ \hat{F}_{kj} - \gamma f_{kj} a \right] - i \frac{\beta}{16h^{5/2}} \left[ M \bar{\phi}_k + \mu \phi_k + 12iL_k \phi_l + \frac{15}{4h} \bar{\phi}_k (\phi^l \phi_l) \right]. \tag{6.8}
\]

Finally, using (5.2), we obtain
\[
\dot{\hat{F}}_{kr} - \gamma \dot{f}_{kr} = \frac{\beta}{16h^{5/2}} \dot{\bar{\phi}}_k - i \frac{\beta}{16h^{5/2}} \left[ M \bar{\phi}_k + \mu \phi_k + 4iL_k \phi_l + \frac{3}{2h} \bar{\phi}_k (\phi^l \phi_l) \right], \tag{6.9}
\]
and this expression exactly cancels the remaining terms in (6.5), taking into account that the total coefficient of \(\sigma_i\) in (6.5) is vanishing as a consequence of the constraint (5.1).

In a similar fashion or just by conjugation we obtain:
\[
\dot{\hat{F}}_{kr} - \gamma \dot{f}_{kr} = \frac{\beta}{16h^{5/2}} \dot{\bar{\phi}}_k - i \frac{\beta}{16h^{5/2}} \left[ M \bar{\phi}_k - \mu \bar{\phi}_k + 4iL_k \bar{\phi}_l - \frac{3}{2h} \bar{\phi}_k (\phi^l \phi_l) \right], \tag{6.10}
\]
Thus on shell
\[
Q_k = \bar{Q}_k = 0. \tag{6.11}
\]

It is worth mentioning that, using the identities (6.9), (6.10) deduced above, the equations for the auxiliary fields \(\sigma_i, \bar{\sigma}_i\) (5.9) can be simplified:
\[
\dot{\phi}_i - 2\sigma_i + \frac{i}{4h} \left[ 4iL_i \phi_k + M \bar{\phi}_i + (\mu + i\hbar) \phi_i \right] + \frac{3i}{16} h^{-2} \bar{\phi}_i (\phi^k \phi_k) = 0, \tag{6.12}
\]
\[
\dot{\bar{\phi}}_i - 2\bar{\sigma}_i - \frac{i}{4h} \left[ - 4iL_i \bar{\phi}_k - M \phi_i + (\mu - i\hbar) \bar{\phi}_i \right] - \frac{3i}{16} h^{-2} \bar{\phi}_i (\bar{\phi}^k \bar{\phi}_k) = 0.
\]

3. Time translations. We define the conserved charge associated with \(b\) transformations (Hamiltonian) as
\[
\delta_b \mathbb{L}_{HP} = -2b \mathbb{L}_{HP}, \quad \dot{\Phi}^a A \frac{\partial \mathbb{L}_{HP}}{\partial \Phi^a} - \mathbb{L}_{HP} = \mathbb{H}, \tag{6.13}
\]
whence
\[
\mathbb{H} = \mathbb{H}^b + \mathbb{H}^{f(1)} + \mathbb{H}^{f(2)},
\]
where
\[
\mathbb{H}^b = \frac{1}{2} h \left( \dot{\hat{F}}_{iv} \hat{F}_{ir} - \gamma \hat{f}_{iv} \hat{f}_{ia} \right) - \frac{1}{4} D \left( \gamma \hat{f}_{iv} \hat{f}_{ia} - \hat{F}_{iv} \hat{F}_{ir} + \frac{\beta}{\sqrt{h}} \right) - \frac{\beta}{4} \frac{1}{h^{3/2}} \left[ L_{ik} L_{ik} - \frac{1}{8} \left( M \bar{\phi}_k + \mu^2 - \hbar^2 \right) \right], \tag{6.14}
\]

\[ \mathbb{H}^{(1)} = -\frac{i}{2} \sigma_i \left( \gamma f^{ia} \bar{\chi}_a - \hat{F}^{ir} \bar{\chi}_r \right) + i \frac{\bar{\gamma}}{2} \left( \gamma f^{ia} \chi_a - \hat{F}^{ir} \chi_r \right) - \frac{M}{8} \left( \gamma \tilde{\chi}_a \bar{\chi}_a - \bar{\chi}_r \tilde{\chi}_r \right) + \frac{M}{8} \left( \gamma \chi_a \bar{\chi}_a - \bar{\chi}_r \chi_r \right) - \frac{i}{4} \left( \gamma \tilde{\chi}_a \bar{\chi}_a - \bar{\chi}_r \chi_r \right), \] (6.15)

\[ \mathbb{H}^{(2)} = -\beta \frac{3}{64h^{5/2}} \left( 4i L^{ik} \phi_i (\tilde{\phi}_k) + \frac{M}{2} \tilde{\phi}_i \phi_i - \frac{M}{2} \phi_i \phi_i + \mu \phi_i \tilde{\phi}_i \right) - i \frac{\bar{\beta}}{8h^{3/2}} \left( \bar{\sigma}^i \tilde{\phi}_i - \bar{\phi}^i \phi_i \right) - \beta \frac{15}{64} \frac{1}{8h^{7/2}} (\phi^k \phi_k)(\tilde{\phi}^i \tilde{\phi}_i). \] (6.16)

Putting these formulas together and using some equations for the auxiliary fields we get:

\[ \mathbb{H} = \frac{1}{2} h \left( \hat{F}^{ir} \hat{F}_{ir} - \gamma \dot{f}^{ia} \dot{f}_{ia} \right) - \frac{\beta}{4h^{3/2}} \left[ L^{ik} L_{ik} - \frac{1}{8} (M \ddot{M} + \mu^2 - h^2) \right] - \frac{3i \beta}{16h^{5/2}} L^{ik} \phi_i \tilde{\phi}_k - \frac{i}{\beta M} \frac{15 \bar{\beta}}{64} \frac{1}{8h^{7/2}} (\phi^k \phi_k)(\tilde{\phi}^i \tilde{\phi}_i). \] (6.17)

It still remains to show that

\[ \mathbb{H} = 0 \] (6.18)
on shell.

The proof of (6.18) is more involved compared to (6.11). The basic step is to represent the first term in the first line of (6.14) as

\[ \frac{h}{2} \left( \hat{F}^{ir} \hat{F}_{ir} - \gamma \dot{f}^{ia} \dot{f}_{ia} \right) = \frac{1}{4} \partial_i \left[ h \partial_i (\ddot{F}^2 - \gamma \dot{f}^2) \right] - \frac{1}{2} \left[ \hat{F}^{ir} \partial_i (h \hat{F}_{ir}) - \gamma f^{ia} \partial_i (h \dot{f}_{ia}) \right]. \] (6.19)

Then, making use of the equations of motion, we obtain:

\[ \frac{1}{2} h \left( \hat{F}^{ir} \hat{F}_{ir} - \gamma \dot{f}^{ia} \dot{f}_{ia} \right) = -\beta \frac{1}{4} \partial^2 (h^{1/2}) + W, \] (6.20)

where

\[ W = \frac{1}{2} \hat{F}^{ir} \left\{ \frac{1}{2} D \hat{F}_{ir} - 2L_{ik} \hat{F}_{ik} - L_{ik} \hat{F}_{ik} - \frac{i}{2} (\phi \tilde{\chi}_r - \bar{\phi} \bar{\chi}_r) \right\} + \frac{i}{2} \left[ (\sigma - \bar{\sigma} \tilde{\chi}_r - (\bar{\sigma} - \bar{\phi} \chi_r) \right\] - \frac{1}{2} (\sigma \tilde{\chi}_a - \tilde{\phi} \chi_a) + \frac{i}{2} \left[ (\bar{\sigma} - \bar{\phi} \chi_a) - (\bar{\sigma} - \bar{\phi} \tilde{\chi}_a) \right\]. \] (6.21)

As the next step, we cast (5.6) in the form:

\[ -\frac{1}{h} \partial^2 (h^{1/2}) + \frac{\dot{h}^2}{2h^{5/2}} = 4 \frac{\beta}{3} (\dot{\hat{F}}^2 - \gamma \dot{f}^2) + V. \] (6.22)
Using this equation, we rewrite (6.20) as the following identity:

\[
\frac{h}{2} (\dot{F}_{ir} \dot{F}_{ir} - \gamma \dot{f}_{ia} \dot{f}_{ia}) = -\frac{\beta h}{4} V + \frac{\beta}{8 h^{3/2}} \hat{h}^2 - W . \tag{6.23}
\]

Now, the strategy will be following: the terms which contain \( \dot{h} \) are going to eventually cancel, the other terms with time derivatives, or matter fermions, can be replaced by using the equations of motion. In this way we end up with an expression for \( \mathbb{H} \) which contains only terms related to the supergravity multiplet and its auxiliary fields ordered by odd integer powers of \( h^{-\frac{1}{2}} \), and the coefficients of each individual power of this sort must vanish. This can be rather easily checked for the terms with \( h^{-\frac{1}{2}} \) and \( h^{-\frac{3}{2}} \). More involved calculation shows that all terms with the higher inverse degrees of \( h \) also vanish.

4. \( Sp(n, 1) \) and \( Sp(n + 1) \) symmetries. The general definition of the relevant conserved current is

\[
\delta_{\Lambda} \Phi_{A} \frac{\partial L}{\partial \dot{\Phi}_{A}} = \Lambda_{ar} J_{ar} .
\]

Then

\[
J_{ar} = f_{a}^{i} \left[ h \dot{F}_{ir} - \frac{i}{2} (\phi_{i} \bar{\chi}_{r} - \bar{\phi}_{i} \chi_{r}) \right] + \dot{F}_{r}^{i} \left[ h \dot{f}_{ia} - \frac{i}{2} (\phi_{i} \bar{\chi}_{a} - \bar{\phi}_{i} \chi_{a}) \right] - 2 L_{ik} f_{a}^{i} \dot{F}_{r}^{k} + \frac{i}{2} h \left( \chi_{r} \bar{\chi}_{a} - \bar{\chi}_{r} \chi_{a} \right) . \tag{6.24}
\]

The \( Sp(1) \) and \( Sp(n) \) currents are calculated analogously. All these currents are conserved, but non-vanishing on shell, because they correspond to the internal global symmetries as opposed to the previously presented currents associated with the worldline symmetries.

7 Canonical momenta

The canonical momenta are calculated straightforwardly:

\[
P_{ir}^{(F)} = \frac{\partial L}{\partial \dot{F}_{ir}} = h \dot{F}_{ir} - L_{ik} \dot{F}_{r}^{k} - \frac{i}{2} (\phi_{i} \bar{\chi}_{r} - \bar{\phi}_{i} \chi_{r}) ,
\]

\[
P_{ia}^{(f)} = \frac{\partial L}{\partial \dot{f}_{ia}} = -\gamma h \dot{f}_{ia} + \gamma L_{ik} f_{a}^{k} + \frac{i}{2} \gamma (\phi_{i} \bar{\chi}_{a} - \bar{\phi}_{i} \chi_{a}) ,
\]

\[
P^{(h)} = \frac{\partial L}{\partial \dot{h}} = -\frac{\beta}{16 h^{3/2}} \hat{h} , \tag{7.1}
\]

\[
P_{r}^{(\chi)} = \frac{\partial L}{\partial \dot{\chi}_{r}} = -\frac{i}{4} h \bar{\chi}_{r} , \quad P_{r}^{(\bar{\chi})} = \frac{\partial L}{\partial \dot{\bar{\chi}}_{r}} = -\frac{i}{4} h \chi_{r} ,
\]

\[
P_{a}^{(\chi)} = \frac{\partial L}{\partial \dot{\chi}_{a}} = \frac{i}{4} h \bar{\chi}_{a} , \quad P_{a}^{(\bar{\chi})} = \frac{\partial L}{\partial \dot{\bar{\chi}}_{a}} = \frac{i}{4} h \chi_{a} ,
\]

\[
P_{i}^{(\phi)} = \frac{\partial L}{\partial \dot{\phi}_{i}} = i \beta \frac{1}{32 h^{3/2}} \bar{\phi}_{i} , \quad P_{i}^{(\bar{\phi})} = \frac{\partial L}{\partial \dot{\bar{\phi}}_{i}} = -i \beta \frac{1}{32 h^{3/2}} \phi_{i} , \tag{7.2}
\]
where the derivatives with respect to $\dot{\chi}^r, \ddot{\chi}^r, \dot{\chi}^a, \ddot{\chi}^a, \dot{\phi}^i$ and $\ddot{\phi}^i$ are understood as the right ones.

The supercharges can be expressed in terms of the bosonic canonical momenta as

$$Q_k = \mathcal{P}^{(F)r}_k \chi_r + \mathcal{P}^{(f)a}_k \chi_a + \mathcal{P}^{(h)} \phi_k + \frac{3i\beta}{16 \cdot 8h^{3/2}} \bar{\phi}_k (\phi^l \phi_l)$$

$$+ \frac{i}{4} [\phi_k (\bar{\chi}^r \chi_r - \gamma \bar{\chi}^a \chi_a) - \bar{\phi}_k (\chi^r \chi_r - \gamma \chi^a \chi_a)],$$

(7.3)

$$\bar{Q}_k = \mathcal{P}^{(F)r}_k \bar{\chi}_r + \mathcal{P}^{(f)a}_k \bar{\chi}_a + \mathcal{P}^{(h)} \bar{\phi}_k - \frac{3i\beta}{16 \cdot 8h^{3/2}} \phi_k (\bar{\phi}^l \bar{\phi}_l)$$

$$+ \frac{i}{4} [\phi_k (\bar{\chi}^r \bar{\chi}_r - \gamma \bar{\chi}^a \bar{\chi}_a) - \bar{\phi}_k (\bar{\chi}^r \bar{\chi}_r - \gamma \bar{\chi}^a \bar{\chi}_a)].$$

(7.4)

We used here the algebraic equations (5.1) and (5.4). Analogously, one can reexpress the $SU(2)$ current and the Hamiltonian

$$J_{kl} = \mathcal{P}^{(F)r}_{(k)l} + \mathcal{P}^{(f)a}_{(k)l} + \frac{i\beta}{16h^{3/2}} \phi_k (\phi^l \phi_l),$$

(7.5)

$$\mathbb{H} = \frac{1}{2h} \left[ (\mathcal{P}^{(F)})^2 - \gamma (\mathcal{P}^{(f)})^2 - \frac{16h^{5/2}}{\beta} (\mathcal{P}^{(h)})^2 \right]$$

$$- \frac{i\beta}{4h^{3/2}} L^{(ik)} \phi_k (\bar{\phi}_k) - \frac{\beta}{32h^{3/2}} (M \bar{M} + \mu^2) - \frac{3\beta}{384h^{7/2}} (\phi^i \phi_i) (\bar{\phi}^j \bar{\phi}_j)$$

$$+ \frac{1}{h} L^{ik} J_{ik} + \frac{i}{2h} (\phi_i \bar{Q}^i - \bar{\phi}_i Q^i).$$

(7.6)

Since $J_{ik} = Q^i = \bar{Q}^i = 0$ on shell, the last line in the expression (7.6) can be suppressed. The auxiliary fields $M, \bar{M}$ and $\mu$ can be replaced by their on-shell expressions.

Finally, the internal symmetry current $J_{ar}, \,(6.24),$ is expressed as

$$J_{ar} = f^i_a \mathcal{P}^{(F)}_{ir} - \gamma \bar{\mathcal{F}}^i_r \mathcal{P}^{(f)}_{ia} + \frac{i}{2h} (\chi_r \bar{\chi}_a - \bar{\chi}_r \chi_a).$$

(7.7)

Now one can define the Poisson brackets, quantize them and find quantum expressions for the (super)charges. The wave function $|\Phi>$ should satisfy the conditions

$$J_{ik} |\Phi> = Q^i |\Phi> = \mathbb{H} |\Phi> = 0$$

(7.8)

and, after solving these equations, be expressed in terms of irreps of $Sp(n + 1)$ (for $\gamma = -1$) or $Sp(1, n)$ (for $\gamma = 1$).

Due to the first-class constraints $J_{kl} \simeq 0 \quad Q^i \simeq 0 \quad \bar{Q}_i \simeq 0 \quad \mathbb{H} \simeq 0$ the considered system, prior to quantization, should be exposed to an accurate Hamiltonian analysis, which for the bosonic sector will be performed in the next section. We will finish this section by presenting the set of Poisson brackets for the dynamical variables.

**Bosonic brackets.**

$$\{\mathcal{F}^{ir}, \mathcal{P}^{(F)}_{ks}\} = \delta^i_k \delta_r^s, \quad \{f^{ia}, \mathcal{P}^{(f)}_{kb}\} = \delta^i_a \delta^a_b, \quad \{h, \mathcal{P}^{(h)}\} = 1.$$  

(7.9)
Fermionic brackets.

To unambiguously define the brackets involving fermionic fields we need to apply to the Dirac method. As is seen from the expressions for the fermionic momenta \((7.2)\), there is a set of second-class constraints

\[
\varphi_r^{(x)} := \mathcal{P}_r^{(x)} + \frac{i}{4} \hbar \tilde{X}_r \simeq 0, \quad \varphi_r^{(x)} := \mathcal{P}_r^{(x)} - \frac{i}{4} \hbar \chi_r \simeq 0,
\]

\[
\varphi_a^{(x)} := \mathcal{P}_a^{(x)} - \frac{i}{4} \gamma \hbar \tilde{\chi}_a \simeq 0, \quad \varphi_a^{(x)} := \mathcal{P}_a^{(x)} + \frac{i}{4} \gamma \hbar \chi_a \simeq 0,
\]

\[
\varphi_i^{(\phi)} := \mathcal{P}_i^{(\phi)} - i \beta \frac{1}{32 \hbar^{3/2}} \tilde{\phi}_i \simeq 0, \quad \varphi_i^{(\phi)} := \mathcal{P}_i^{(\phi)} + i \beta \frac{1}{32 \hbar^{3/2}} \phi_i \simeq 0 \quad (7.10)
\]

with the following non-zero canonical brackets:

\[
\{ \varphi_r^{(x)}, \varphi_s^{(x)} \} = \frac{i}{2} \hbar \Omega_{rs}, \quad \{ \varphi_a^{(x)}, \varphi_b^{(x)} \} = -\frac{i}{2} \gamma \hbar \varepsilon_{ab}, \quad \{ \varphi_i^{(\phi)}, \varphi_k^{(\phi)} \} = -i \beta \frac{1}{16 \hbar^{3/2}} \varepsilon_{ik}. \quad (7.11)
\]

Then the standard Dirac procedure yields the following non-zero brackets involving the fermionic variables:

\[
\{ \chi_r, \chi_s \}_D = 2i \frac{1}{\hbar} \Omega_{rs}, \quad \{ \chi_a, \tilde{\chi}_b \}_D = -2i \gamma \frac{1}{\hbar} \varepsilon_{ab}, \quad \{ \tilde{\phi}_i, \tilde{\phi}_k \}_D = -16i \frac{h^{3/2}}{\beta} \varepsilon_{ik}. \quad (7.12)
\]

\[
\{ \mathcal{P}^{(h)}, \chi_r \}_D = \frac{1}{2 \hbar} \chi_r, \quad \{ \mathcal{P}^{(h)}, \tilde{\chi}_r \}_D = \frac{1}{2 \hbar} \tilde{\chi}_r,
\]

\[
\{ \mathcal{P}^{(h)}, \chi_a \}_D = \frac{1}{2 \hbar} \chi_a, \quad \{ \mathcal{P}^{(h)}, \tilde{\chi}_a \}_D = \frac{1}{2 \hbar} \tilde{\chi}_a,
\]

\[
\{ \mathcal{P}^{(h)}, \tilde{\phi}_i \}_D = -\frac{3}{4 \hbar} \tilde{\phi}_i, \quad \{ \mathcal{P}^{(h)}, \phi_i \}_D = -\frac{3}{4 \hbar} \phi_i. \quad (7.13)
\]

In what follows, we will omit the index \(^{\text{"D"}}\) on these brackets. Using them, one can calculate the brackets between \(J_{kl}, Q^i, \tilde{Q}_i\), and \(\mathbb{H}\) and convince oneself that they form a closed superalgebra. Here we present the brackets between the \(Sp(n,1)(Sp(n+1))\) currents \(J_{ar}\):

\[
\{ J_{ar}, J_{bs} \} = -2 \gamma \left[ \Omega_{rs} J_{(ab)} + \varepsilon_{ab} J_{(rs)} \right], \quad (7.14)
\]

where

\[
J_{(ab)} = f^i \mathcal{P}_{ib}^{(f)} - \frac{i}{2} \gamma \hbar \chi(a \tilde{X}_b), \quad J_{(rs)} = \tilde{F}^i \mathcal{P}_{ia}^{(F)} + \frac{i}{2} \hbar \chi(r \tilde{X}_s) \quad (7.15)
\]

are \(Sp(1)\) and \(Sp(n)\) currents. Note that the brackets of \(\mathcal{P}^{(h)}\) with the internal symmetry currents \(J_{ar}, J_{(ab)}\) and \(J_{(rs)}\) are vanishing, as it should be. The same concerns the brackets with the Hamiltonian \(\mathbb{H}\). It is rather easy to check that \(J_{ar}, J_{(ab)}\) and \(J_{(rs)}\) have zero bracket with the quadratic combination of the currents

\[
C^{(2)} = \gamma J_{bs} J_{bs} - 2 \left[ J_{(ab)} J_{(ab)} + J_{(rs)} J_{(rs)} \right], \quad (7.16)
\]
which can thus be identified with the second-order Casimir of $Sp(n, 1)$ (for $\gamma = 1$) or $Sp(n + 1)$ (for $\gamma = -1$).

Actually, besides the dynamical variables, the full ungauged (super)charges involve the auxiliary fields, bosonic and fermionic, which have no kinetic terms in the Lagrangian and so have vanishing conjugate momenta. This produces new Hamiltonian constraints, which, for the time being, are difficult to analyze in a full generality. For this reason, in this paper we limit our consideration to the bosonic sector of the whole system.

8 Dirac analysis of the bosonic model

In what follows we will restrict our attention to the bosonic part (3.3):

$$L_{HP} = \frac{1}{2} h \left( \dot{F}_{ir} F_{ir} - \gamma \dot{f}_{ia} f_{ia} \right) + L_{ik} \left( \dot{F}^{(ir)} F_{ik} - \gamma f^{(ia)} f_{ik} \right) + \frac{1}{4} D \left( \gamma f^{ia} f_{ia} - \dot{F}_{ir} F_{ir} + \frac{\beta}{\sqrt{h}} \right) + \frac{\beta}{4} \frac{1}{\sqrt{h}} \left( L_{ik} L_{ik} - \frac{1}{8} \dot{h}^2 \right),$$

(8.1)

where we ignored the fields $M, \bar{M}$ and $\mu$ which fully decouple in the absence of fermionic variables. We will perform the Dirac analysis of the above Lagrangian.

Digression: relativistic particle.

As a warmup, we replay the Dirac formalism in the application to the well known case of the massive relativistic particle:

$$L = -m \left( -\dot{X}^2 \right)^{1/2}, \quad (\dot{X})^2 = -\dot{X}^0^2 + \dot{X}^2.$$

(8.2)

We have:

$$\mathbb{P} = \frac{\partial L}{\partial \dot{X}} = \frac{m \dot{X}}{\sqrt{-X^2}}, \Rightarrow \mathbb{P}^2 = -(P_0)^2 + \mathbb{P}^2 = \frac{m^2 (\dot{X})^2}{-X^2} = -m^2,$$

(8.3)

that is, we arrive at the primary constraint:

$$\Phi(t) = \mathbb{P}^2 + m^2 \simeq 0.$$

(8.4)

The corresponding canonical Hamiltonian vanishes and so (8.4) is the only first-class constraint. The Hamiltonian becomes:

$$\mathbb{H} = \frac{1}{2} \ell \left( \mathbb{P}^2 + m^2 \right).$$

(8.5)

\footnote{The Hamiltonian analysis of the simplest $\mathcal{N} = 1, 1D$ “supergravity” system, with the worldline multiplet consisting of a real “graviton” and one real “gravitino”, was accomplished in [13].}
We consider now the Hamiltonian form of the Lagrangian for the relativistic massive particle:

\[ L = \dot{X} P - \frac{\ell}{2}(P^2 + m^2). \]  

(8.6)

This Lagrangian has a local symmetry:

\[ \delta X = \alpha(t) P, \quad \delta P = 0, \quad \delta \ell = \dot{\alpha}(t), \]  

(8.7)

for which reason we naturally expect first-class constraints. Indeed, \( X \) and \( P \) are conjugate variables, while the momentum corresponding to the variable \( \ell \) vanishes:

\[ \Pi_\ell = \frac{\partial L}{\partial \dot{\ell}} = 0 \Rightarrow \Phi_\ell = \Pi_\ell \simeq 0, \]  

(8.8)

so that \( \Phi_\ell \simeq 0 \) is a primary constraint. Introducing the Hamiltonian:

\[ H = \dot{X} P - L = \frac{\ell}{2}(P^2 + m^2), \]  

(8.9)

together with the relevant non-vanishing equal-time Poisson brackets,

\[ \{ X, P \} = 1, \quad \{ \ell, \Pi_\ell \} = 1, \]  

(8.10)

and then imposing the condition of conservation of the constraint \( \Phi_\ell \),

\[ \dot{\Phi}_\ell = \{ \Phi_\ell, H \} = -\frac{1}{2}(P^2 + m^2) \simeq 0, \]  

(8.11)

we arrive at the mass-shell condition \( (P^2 + m^2) \simeq 0 \) as a secondary first-class constraint. This is the complete set of constraints, and the quantum theory is obtained by replacing the non-vanishing Poisson brackets by the commutators:

\[ [X, P] = i, \quad [\ell, \Pi_\ell] = i. \]  

(8.12)

Thus \( X, \ell \) can be viewed as standard multiplication operators, while the corresponding momenta are \( P = \partial/(i\partial X) \), \( \Pi_\ell = \partial/(i\partial \ell) \). These operators act on the space of wave functions \( \Psi(X, \ell) \), and the corresponding first-class constraints tell us that the wave function does not depend on \( \ell \) and obeys the Klein Gordon equation:

\[ (P^2 + m^2) \Psi(X) = 0. \]  

(8.13)

Getting back to (8.6), we can eliminate \( P \):

\[ \dot{X} = \ell P, \quad \Rightarrow \quad L = \frac{\dot{X}^2}{2\ell} - \frac{\ell m^2}{2} \]  

(8.14)

and, furthermore, eliminate \( \ell \):

\[ -\frac{\dot{X}^2}{2\ell^2} - \frac{m^2}{2} = 0, \quad \Rightarrow \quad L = -m \left( -\frac{\dot{X}^2}{2} \right)^{1/2}. \]  

(8.15)
Then it becomes clear that the Lagrangian in the Hamiltonian form (8.6) is more general than (8.2), as it also permits the case $m = 0$. For the massless particle the configuration space action involves an additional field $\ell$. The auxiliary field $\ell$ is necessary to guarantee the gauge invariance:

$$\delta X = \frac{\alpha(t)}{\ell} \dot{X}, \quad \delta \ell = \dot{\alpha}(t).$$

(8.16)

This auxiliary field cannot be eliminated as this would lead to a vanishing action. Its role is to impose the gauge invariant mass shell constraint (this is nicely explained in [14]). One can then fix the gauge to obtain a simpler action. In what follows we will proceed for (8.1) in a similar fashion.

*Back to (8.1).*

A first glance at (8.1) tells us that it is not in the Hamiltonian form analogous to (8.6). Exploiting the Dirac method, we will arrive at the corresponding Hamiltonian form.

We start by defining the canonical momenta:

$$\mathcal{P}^{(F)}_{ir} = \frac{\partial \mathcal{L}_{HP}}{\partial \dot{F}^{ir}} = \hbar \dot{F}^{ir} - L_{ik} \dot{F}^{kr}, \quad \mathcal{P}^{(f)}_{ia} = \frac{\partial \mathcal{L}_{HP}}{\partial \dot{f}^{ia}} = -\gamma \hbar \dot{f}^{ia} + \gamma L_{ik} f^{k}_{a},$$

$$\mathcal{P}^{(h)} = \frac{\partial \mathcal{L}_{HP}}{\partial \dot{h}} = -\frac{\beta}{16 \hbar \gamma^2} \dot{\hbar}, \quad \mathcal{P}^{(L)}_{ik} = \frac{\partial \mathcal{L}_{HP}}{\partial \dot{L}^{ik}} = 0, \quad \mathcal{P}^{(D)} = \frac{\partial \mathcal{L}}{\partial D} = 0.$$  

(8.17)

An inspection of the above formulas tells us that we have now two primary sets of commuting constraints:

$$\Phi^{(L)}_{ik} := \mathcal{P}^{(L)}_{ik} \simeq 0, \quad \Phi^{(D)} := \mathcal{P}^{(D)} \simeq 0.$$  

(8.18)

Indeed, using the non-vanishing Poisson brackets:

$$\{ \dot{F}^{ir}, \mathcal{P}^{(F)}_{js} \} = \delta^{i}_{j} \delta^{r}_{s}, \quad \{ \dot{f}^{ia}, \mathcal{P}^{(f)}_{jb} \} = \delta^{i}_{j} \delta^{a}_{b}, \quad \{ D, \mathcal{P}^{(D)} \} = 1,$$

$$\left\{ L^{ik}, \mathcal{P}^{(L)}_{jl} \right\} = \frac{1}{2} \left( \delta^{j}_{l} \delta^{k}_{i} + \delta^{k}_{l} \delta^{i}_{j} \right), \quad \text{and} \quad \{ h, \mathcal{P}^{(h)} \} = 1,$$

(8.19)

we can calculate:

$$\left\{ \Phi^{(L)}_{ik}, \Phi^{(D)} \right\} = 0.$$  

(8.20)

Introducing the Hamiltonian of our system as

$$\mathcal{H}' = \dot{\mathcal{H}}^{(F)}_{ir} \mathcal{P}^{(F)}_{ir} + \dot{f}^{ia} \mathcal{P}^{(f)}_{ia} + \dot{h} \mathcal{P}^{(h)} + \left[ \dot{L}^{ik} \mathcal{P}^{(L)}_{ik} + \dot{D} \mathcal{P}^{(D)} \right] = 0 | \mathcal{L}_{HP},$$

we obtain:

$$\mathcal{H}' = \frac{1}{2 \hbar} \left( \mathcal{P}^{(F)}_{ir} + L_{ik} \dot{F}^{kr} \right) \left( \mathcal{P}^{(F)}_{ir} - L^{ik'} \dot{F}^{kr} \right) - \frac{\gamma}{2 \hbar} \left( \mathcal{P}^{(f)}_{ia} - \gamma L_{ik} f^{k}_{a} \right) \left( \mathcal{P}^{(f)}_{ia} + \gamma L^{ik'} f^{k'}_{a} \right).$$

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- \frac{8 \hbar^{5/2}}{\beta} (\mathcal{P}(\hbar))^2 - \frac{D}{4} \left( \gamma f^2 - \hat{F}^2 + \frac{\beta}{h^{3/2}} \right) - \frac{\beta}{4h^{3/2}} L^{ij} L_{ij}. \quad (8.21)

Imposing the requirement of weak vanishing of the Poisson bracket of the primary constraints \(8.18\) with the Hamiltonian \(8.21\), we obtain two new sets of first-class constraints:

\[
\Phi^{F-f} := \gamma f^2 - \hat{F}^2 + \frac{\beta}{h^{3/2}} \simeq 0, \quad \Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})} := \hat{F}_{l'(l)l} + f_{l'(l)}^{(F)} \simeq 0.
\]

(8.22)

Using the definitions \(8.17\) in \(8.21\) and taking into account the constraints themselves, we get:

\[
\mathbb{H}' \simeq \mathbb{H}' \bigg|_{(M, M, \hat{\phi}, \hat{\phi}) = 0},
\]

\[
(8.23)
\]

where \(\mathbb{H}\) is given by \(6.17\). Using eq. \(8.17\) in \(\Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})}\) we get:

\[
\Phi_{ik}^{F,\mathcal{P}(\mathcal{F})} = - J_{ik} \bigg|_{(\phi, \bar{\phi}) = 0} = h \hat{F}_{k(i)l} + \frac{1}{2} \hat{F}^2 L_{ik},
\]

where \(J_{ik}\) is given by eq. \(6.1\). This means that the constraint \(\Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})}\) is just the \(SU(2) - R\) charge discussed earlier.

Further, one calculates:

\[
\begin{aligned}
\left\{ \Phi^{F-f}, \Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})} \right\} &= 0, \\
\left\{ \Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})}, \Phi_{k'F}^{F,\mathcal{P}(\mathcal{F})} \right\} &= \frac{1}{2} \left( \epsilon_{lk} \Phi_{l'k'}^{F,\mathcal{P}(\mathcal{F})} + \epsilon_{l'k} \Phi_{lk'}^{F,\mathcal{P}(\mathcal{F})} + \epsilon_{lk'} \Phi_{l'k}^{F,\mathcal{P}(\mathcal{F})} + \epsilon_{l'k'} \Phi_{lk}^{F,\mathcal{P}(\mathcal{F})} \right).
\end{aligned}
\]

At this stage, we can use these constraints to simplify the Hamiltonian \(8.21\):

\[
\mathbb{H}' = \frac{1}{2h} \left[ (\mathcal{P}(\mathcal{F}))^2 - \frac{1}{\gamma} (\mathcal{P}(\mathcal{F})^2 - 16h^{5/2} \mathcal{P}(\mathcal{h}))^2 \right] - \frac{1}{4} \left( \frac{L^{ij} L_{ij}}{h} + D \right) \Phi^{F-f} - 2L^{ij} \Phi_{ij}^{F,\mathcal{P}(\mathcal{F})} \simeq H, \quad (8.24)
\]

\[
H = \frac{1}{2h} \left[ (\mathcal{P}(\mathcal{F}))^2 - \frac{1}{\gamma} (\mathcal{P}(\mathcal{F})^2 - 16h^{5/2} \mathcal{P}(\mathcal{h}))^2 \right]. \quad (8.25)
\]

Then, \(\Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})}\) commutes with \(H\), while commuting \(\Phi^{F-f}\) with \(H\) gives rise to the new constraint:

\[
\Phi^{\mathcal{P}(\mathcal{F})}_F := \frac{1}{h} \left( \mathcal{P}(\mathcal{F}) i(a f_{ia} + \mathcal{P}(\mathcal{F}) i r \hat{F}_{ir}) - 4 \mathcal{P}h \right) \simeq 0, \quad (8.26)
\]

Poisson bracket of which with \(H\) defined in \(8.25\) is equal to:

\[
\left\{ \Phi^{\mathcal{P}(\mathcal{F})}_F , H \right\} \simeq - \frac{2}{h} H.
\]

We also have:

\[
\left\{ \Phi^{\mathcal{P}(\mathcal{F})}_F , \Phi_{l'F}^{F,\mathcal{P}(\mathcal{F})} \right\} = 0, \quad \left\{ \Phi^{\mathcal{P}(\mathcal{F})}_F , \Phi^{F-f} \right\} = - \frac{2}{h} \Phi^{F-f}.
\]
Taking into account that the new constraint $H$ is weakly equivalent to $H'$, which in its turn is weakly equivalent to $H_{\bar{M}, M, \mu, \phi, \delta_k} = 0$, the properly restricted (6.17) is weakly equivalent to the new constraint $H$. It is thereby established that, in the present approach, the generators of global symmetries of our model correspond to some first-class constraints. The algebra of the constraints $(\Phi^F - f, \Phi^{F \varphi^{(x)}}, \Phi^{(f)\bar{f}}, H)$, closes in the weak sense and the Hamiltonian finally becomes:

$$\mathcal{H} = H_0(t)H - A''(t)\Phi^{F \varphi^{(x)}} + B(t)\Phi^{(f)\bar{f}} - D(t)\Phi^F - f.$$

Now we are able to write the action in the Hamiltonian form:

$$S = \int_0^{t_1} dt \left[ \dot{F}^{ir}(t) \mathcal{P}^{(F)}_{ir}(t) + f^{ia}(t) \mathcal{P}^{(f)}_{ia}(t) + h(t) \mathcal{P}^{(h)}(t) - \mathcal{H}(t) \right]. \tag{8.27}$$

Using the definitions (8.17) in (8.26), one can show that

$$\Phi^{(f)\bar{f}} = -\frac{1}{2} \frac{d}{dt} \Phi^F - f. \tag{8.28}$$

Therefore, integrating by parts in the action, one can absorb the constraint $\Phi^{(f)\bar{f}}$ into a redefinition of the Lagrange multiplier $D$. We will not further pursue this approach, because, when we deduce the corresponding gauge transformations, some of them may turn out singular on the surface of constraints. The action (8.27) should have the corresponding gauge invariances specified by the transformations which are generated by the first-class constraints incorporated in this action. In what follows we will spell the local transformations which should not be singular on the surface of constraints.

A. Transformations corresponding to the constraint $H$ (8.25):

$$\delta \hat{F}^{ir} = \frac{b(t)}{h} \mathcal{P}^{(F)}_{ir}, \quad \delta f^{ia} = -\frac{b(t)}{\gamma h} \mathcal{P}^{(f)}_{ia}, \quad \delta h = -\frac{16h^{3/2}}{\beta} \mathcal{P}^{(h)}, \quad \delta A'' = -\frac{2}{h} \mathcal{P}^{(h)} B. \tag{8.29}$$

B. Transformations corresponding to the constraint $\Phi^{F \varphi^{(x)}}$ in (8.22):

$$\delta \hat{F}^{ir} = \tau^{il} \hat{F}^l_i, \quad \delta \mathcal{P}^{F}_{ir} = \tau^{il} \mathcal{P}^{(F)}_{il}, \quad \delta f^{ia} = \tau^{il} f^{ia}, \quad \delta \mathcal{P}^{(f)ia}_{ia} = \tau^{il} \mathcal{P}^{(f)ia}_{al}, \quad \delta A'' = -\tau^{il} - \tau^{l} A^{kl} - \tau^{k} A^{il}. \tag{8.30}$$

C. Next, transformations corresponding to the constraint $\Phi^F - f$ in (8.22):

$$\delta \mathcal{P}^{F}_{ir} = 2\alpha \hat{F}^{ir}, \quad \delta \mathcal{P}^{f}_{ia} = -2\gamma \alpha f^{ia}, \quad \delta \mathcal{P}^{(h)} = \frac{\beta \alpha}{2h^{3/2}}. \tag{8.31}$$
\[ \delta D = -\dot{\alpha} - \frac{2\alpha}{\hbar} B, \quad \delta B = 2\alpha H_0. \]  

(8.32)

D. Finally, transformations corresponding to the constraint \( \Phi^{\Phi(F)\Phi} \) in (8.26):

\[ \delta \hat{F}^{ir} = \frac{c}{h} \hat{F}^{ir}, \quad \delta \mathcal{P}^{(F)}_{ir} = -\frac{c}{h} \mathcal{P}^{(F)}_{ir}, \quad \delta f^{ia} = \frac{c}{h} f^{ia}, \quad \delta \mathcal{P}^{(f)}_{ia} = -\frac{c}{h} \mathcal{P}^{(f)}_{ia}, \quad \delta h = -4c, \]

\[ \delta \mathcal{P}^{(h)} = \frac{c}{h^2} \left( f^{ia} \mathcal{P}^{(f)} + \hat{F}^{ir} \mathcal{P}^{(F)}_{ir} \right), \quad \delta H_0 = -\frac{2c}{h} H_0, \quad \delta B = \dot{c} + \frac{16h^{1/2}c}{\beta} \mathcal{P}^{(h)} H_0, \quad \delta D = -\frac{2c}{h} D. \]

We thus have established that the action (8.27) is invariant under the local transformations listed above. We have shown that this action has six gauge invariances, while for the original action (8.1) we exhibited only four explicit gauge invariances, \( \text{viz.} \), the local \( SU(2) \), and time reparametrizations. An interesting question is as to whether the action (8.1) also exhibits the remaining two gauge invariances. However, it may happen that such additional gauge invariances are specific just to the Hamiltonian form of the action. As in the case of the Hamiltonian form of the action for the massive particle, we expect the presence of the relevant first-class constraints. Indeed, \( \mathcal{P}^{(F)}_{ir}(t), \mathcal{P}^{(f)}_{ia}(t), \mathcal{P}^{(h)}(t) \) are variables canonically conjugated to \( \hat{F}^{ir}(t), f^{ia}(t), h(t) \), while the momenta corresponding to \( H_0, A^{ii}, B, D, \) vanish, so that the standard Poisson brackets restore the previous four constraints as secondary first-class ones. The quantization of the above system is problematic as at the moment we are not aware of the general solution to the first-class constraints on the space of wave functions \( \Psi(\hat{F}^{ir}, f^{ia}, h) \). Gauge fixing might simplify the above constraints, but we are interested to glimpse the covariant quantization.

It remains to prove the equivalence with the original action. To this end, like for the massive particle in (8.14), we substitute the corresponding momenta in (8.27) by their explicit expressions:

\[ \mathcal{P}^{(F)}_{ir} = \frac{h}{H_0} \left( \hat{F}^{ir} + A^{il} \hat{F}_l - B \hat{F}^{ri} \right), \quad \mathcal{P}^{(f)}_{ia} = -\frac{\gamma h}{H_0} \left( f^{ia} + A^{il} f^{ia}_l - B f^{ia} \right), \]

\[ \mathcal{P}^{(h)} = -\frac{\beta}{16H_0 h^{3/2}} \left( \dot{h} + 4B \right). \]

After the redefinition \( L^{ij} = h A^{ij} \), some rearrangements and integrations by parts we represent the Lagrangian in (8.27) as:

\[ L = \frac{1}{H_0} \left[ \frac{2h}{4} \Phi^{F-f} - \frac{D}{4} \Phi^{F-f} \right] + \left[ \frac{4H_0 + 1}{4H_0} D - \frac{1}{2} \frac{d}{dt} \left( \frac{B}{H_0} \right) - \frac{B^2}{2} \frac{1}{2H_0} \frac{d}{dt} \left( \frac{B}{H_0} \right) - \frac{L^{ij} L^{ij}}{4H_0} \right] \Phi^{F-f}. \]

If we eliminate \( H_0 \) from the above expression we obtain that the Lagrangian is weakly vanishing. This can be easily confirmed using the available constraints. Like in the case of the massless particle, it is a step which can be done but it does not appear to be useful. To obtain our original action it is in fact enough to fix the gauge with respect to the transformations (8.29) by the condition \( H_0 = 1 \). In order to be convinced that this
gauge choice is permissible, we assume that $H_0 = 1 + \epsilon$ and, using (8.29), determine the infinitesimal transformation parameter to be:

$$b = \left( \int^t dt'' \epsilon(t'') \exp \left\{ 2 \int^t du \frac{B_h(u)}{h} \right\} \right) \exp \left\{ -2 \int^t dt' \frac{B_h(t')}{h} \right\}.$$

Finally, in the gauge $H_0 = 1$ we redefine the auxiliary field $D$ as

$$\frac{D'}{4} = \frac{5D}{4} - \frac{1}{2} \frac{dB}{dt} - \frac{B^2}{2h} - \frac{L_{ij}L_{ij}}{4h},$$

and come back to the initial Lagrangian (8.1). We therefore conclude that the action (8.27) and the action corresponding to (8.1) are equivalent.

9 Concluding remarks

In this paper we continued the study of the new class of $\mathcal{N} = 4$ supersymmetric mechanics models introduced in [6], the Quaternion-Kähler (QK) ones. We limited our attention to their simplest representatives, with the 1D sigma models on the homogeneous projective manifolds $\mathbb{H}^n$ or $\mathbb{H}P^n$ as the bosonic core. We started from the total off-shell component actions of these supersymmetric models, wrote down the local gauge transformations leaving these actions invariant, and explicitly presented the corresponding global invariances, together with the Noether currents associated with the latter. The full set of the equations of motion for different fields, involving both dynamical and algebraic equations, was accurately written down. The currents corresponding to global symmetries the gauging of which yields the total local symmetries of the action, were found to be vanishing on the shell of the equations of motion, while those related to the global isometries $Sp(n + 1)$ (or $Sp(n, 1)$) do not vanish. The vanishing of the first type of currents is quite analogous of the on-shell vanishing of Virasoro currents in bosonic string theory and/or the vanishing of $\mathcal{N} = 4$ supercurrents in the spinning particle coupled to a non-propagating $\mathcal{N} = 4, 1D$ supergravity (see, e.g., [15]). The vanishing of these currents is associated with the local worldline $\mathcal{N} = 4, 1D$ supersymmetry of the models considered and so this property should be equally valid for the most general $\mathcal{N} = 4$ QK mechanics model which also respects this local supersymmetry and the superfield action of which (as well as the bosonic component action) were given in [6]. For the same reason, the hamiltonian analysis of Sect. 8 should also be directly applicable to the general case. We hope to address, from this point of view, examples of more general QK mechanics models (with the

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2An important difference from the spinning particle is that the models of $\mathcal{N} = 4$ QK mechanics are coupled to an extended (“non-minimal”) worldline $\mathcal{N} = 4, 1D$ supergravity multiplet involving, besides the gauge fields for local 1D reparametrizations, local $\mathcal{N} = 4$ supersymmetry and local $SU(2)$ R-symmetry, also some auxiliary fields. One of them (the field $D$) is a Lagrange multiplier for the important bosonic constraint (see (3.3)) ensuring the correct number of physical bosonic fields in the theory.
reduced isometry groups) elsewhere. An important property of $\mathcal{N} = 4$ QK mechanics models is the possibility to add, to the sigma-model type action, the locally $\mathcal{N} = 4$ supersymmetric Wess-Zumino term. It would be interesting to see how the inclusion of such terms (even in the simplest $\mathbb{H}H^n$ or $\mathbb{H}P^n$ cases) will affect the analysis carried out in the present paper.

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**References**

[1] E. Witten, *Dynamical Breaking of Supersymmetry*, Nucl. Phys. B **188** (1981) 513.

[2] R.A. Coles, G. Papadopoulos, *The Geometry of the one-dimensional supersymmetric nonlinear sigma models*, Class. Quant. Grav. **7** (1990) 427-438, doi:10.1088/0264-9381/7/3/016.

[3] C.M. Hull, *The Geometry of supersymmetric quantum mechanics*, [arXiv:hep-th/9910028](https://arxiv.org/abs/hep-th/9910028).

[4] E.A. Ivanov, A.V. Smilga, *Dirac Operator on Complex Manifolds and Supersymmetric Quantum Mechanics*, Int. J. Mod. Phys. A **27** (2012) 1230024, [arXiv:1012.2069](https://arxiv.org/abs/1012.2069) [hep-th].

[5] S. Fedoruk, E. Ivanov, O. Lechtenfeld, *Superconformal mechanics*, J. Phys. A **45** (2012) 173001, [arXiv:1112.1947](https://arxiv.org/abs/1112.1947) [hep-th].

[6] E. Ivanov, L. Mezincescu, *Quaternion-Kähler $\mathcal{N} = 4$ supersymmetric mechanics*, JHEP **1712** (2017) 016, [arXiv:1709.02286](https://arxiv.org/abs/1709.02286) [hep-th].

[7] E. Ivanov, O. Lechtenfeld, *$\mathcal{N} = 4$ supersymmetric mechanics in harmonic superspace*, JHEP **0309** (2003) 073, [arXiv:hep-th/0307111](https://arxiv.org/abs/hep-th/0307111).

[8] J. Bagger, E. Witten, *Matter couplings in $N=2$ supergravity*, Nucl. Phys. B **222** (1983) 1-10.

[9] K. Galicki, *Quaternionic Kähler and Hyperkähler Nonlinear $\sigma$ Models*, Nucl. Phys. B **271** (1986) 402-416.
[10] E. Ivanov, G. Valent, Quaternionic metrics from harmonic superspace: Lagrangian approach and quotient construction, Nucl. Phys. B 576 (2000) 543-577, arXiv:hep-th/0001165.

[11] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, N=2 supergravity in superspace: Different versions and matter couplings, Class. Quant. Grav. 4 (1987) 1255-1265.

[12] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, and E.S. Sokatchev, Harmonic superspace, Cambridge, UK: Univ. Pr. (2001) 306 p.

[13] J.W. van Holten, D = 1 supergravity as a constrained system, J. Phys. Conf. Ser. 1194 (2019) no.1, 012107, arXiv:1901.08816 [hep-th].

[14] M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory. Vol. 1: Introduction,” Cambridge, Uk: Univ. Pr. (1987) page 18. (Cambridge Monographs On Mathematical Physics)

[15] A.I. Pashnev, D.P. Sorokin, n = 4 superfield description of relativistic spinning particle mechanics, Phys. Lett. B 253 (1991) 301-305.