THE DEHN FUNCTION OF BAUMSLAG’S METABELIAN GROUP

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ABSTRACT. Baumslag’s group is a finitely presented metabelian group with a \( \mathbb{Z} \wr \mathbb{Z} \) subgroup. There is an analogue with an additional torsion relation in which this subgroup becomes \( \mathbb{C}_m \wr \mathbb{Z} \). We prove that Baumslag’s group has an exponential Dehn function. This contrasts with the torsion analogues which have quadratic Dehn functions.

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1. Introduction

Baumslag’s group \( \Gamma \) is presented by

\[
\langle a, s, t \mid [a, a^t] = 1, [s, t] = 1, a^t = aa \rangle.
\]

[Our conventions are \([x, y] = x^{-1}y^{-1}xy\) and \(x^ny = y^{-1}x^ny\) for group elements \(x, y\) and integers \(n\).]

Baumslag gave \( \Gamma \) in [4] as the first example of a finitely presented group with an abelian normal subgroup of infinite rank — namely, the derived subgroup \( \Gamma'_\Gamma \). So \( \Gamma \) is metabelian but not polycyclic. The subgroup \( \langle a, t \rangle \) of \( \Gamma \) is

\[
\mathbb{Z} \wr \mathbb{Z} = \left( \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} \right) \rtimes \mathbb{Z} = \langle a, t \mid [a, a^t] = 1 (k \in \mathbb{Z}) \rangle.
\]

Introducing the relation \( a^m = 1 \), where \( m \geq 2 \), gives a family \( \Gamma_m = \langle \Gamma \mid a^m = 1 \rangle \), in which the subgroup \( \langle a, t \rangle \) is \( C_m \wr \mathbb{Z} \), where \( C_m \) denotes the cyclic group of order \( m \).

These groups appear in other guises: both \( \Gamma \) and \( \Gamma_m \) are groups of affine matrices and have Cayley graphs that are horocyclic products of trees [3]; for \( p \) prime, \( \Gamma_p \) is a cocompact lattice in \( \text{Sol}_3 (\mathbb{F}_p (t)) \) [12].

Dehn functions are invariants of finitely presentable groups that are both geometric and combinatorial in character.

The geometric perspective is that Dehn functions are analogous to classical isoperimetric functions for simply connected Riemannian manifolds — these record the infimal \( N \) such that any loop of length at most \( \ell \) can be spanned by a disc of area \( N \).

The combinatorial perspective, our predominant point–of–view in this article, is that Dehn functions measure the complexity of a head–on attack on the word problem. Suppose words \( w = w(A) \) and \( w' = w'(A) \) represent the same element of a finitely presented group \( \langle A \mid R \rangle \). The cost of converting \( w \) to \( w' \) is the minimal \( N \) such that there is a sequence \( w = w_0, \ldots, w_N = w' \) in which, for \( 0 \leq i < N \), there are words \( u_i v_i \) and \( u_i \alpha_i v_i \) freely equal to \( w_i \) and \( w_{i+1} \), respectively, such that \( \alpha_i \beta_i^{-1} v_i \in R^\pm \). When \( w = w(A) \) represents the identity we define \( \text{Area}(w) \) to be the cost of converting \( w \) to the empty word. (So, when \( w \)
and \( w' \) represent the same group element, the cost of converting \( w \) to \( w' \) is \( \text{Area}(w^{-1} w') \). Equivalently, \( \text{Area}(w) \) in the minimal \( N \) such that \( w \) freely equals \( \prod_{i=1}^{N} u_i^{-1} r_i u_i \) for some words \( u_i = u_i(A) \) and some \( r_i \in R^{e_i} \).

The Dehn function \( \text{Area} : \mathbb{N} \to \mathbb{N} \) of \( \langle A \mid R \rangle \) is defined by setting \( \text{Area}(n) \) to be the maximum of \( \text{Area}(w) \) over all words \( w = w(A) \) that have length at most \( n \) and represent the identity.

For \( f, g : \mathbb{N} \to \mathbb{N} \), we write \( f \preceq g \) when there exists \( C > 0 \) such that for all \( n \),

\[
    f(n) \leq C g(n) + Cn + C.
\]

This gives an equivalence relation capturing qualitative agreement of growth rates: \( f \preceq g \) if and only if \( f \preceq g \) and \( g \preceq f \). Any two finite presentations of the same group yield Dehn functions that are \( \preceq – \) equivalent; indeed, up to \( \preceq \), Dehn functions provide a quasi–isometry invariant for finitely presentable groups — see, for example, [8].

Our main results are the following.

**Theorem 1.1.** The Dehn function of \( \Gamma \) satisfies \( \text{Area}(n) \preceq 2^n \).

**Theorem 1.2.** For all \( m \), the Dehn functions of \( \Gamma_m \) satisfy \( \text{Area}(n) \preceq n^3 \).

Our proofs of the upper bounds are via direct analysis of manipulations of words by relations. The exponential lower bound in Theorem 1.1 stems from a calculation of subgroup distortion in an extension.

A polynomial bound on the Dehn functions of \( \Gamma_m \) in an earlier version of this article spurred de Cornulier and Tessera to prove in [12] that the Dehn function of \( \Gamma_p \) is quadratic for all prime \( p \). They view \( \Gamma_p \), for \( p \) prime, as a cocompact lattice in \( \text{Sol}_3(\mathbb{F}_p(\mathbb{F})) \), which they prove enjoys a quadratic isoperimetric function by adapting Gromov’s proof from [18] of the corresponding result for \( \text{Sol}_3(\mathbb{R}) \). They argue that their methods can be elaborated to cover \( \Gamma_m \) for all \( m \).

Alternatively, as we are grateful to an anonymous referee for explaining, this quadratic bound can be obtained by combining results in [3] and [14]. The description of a Cayley graph of \( \Gamma_m \) in [3] as a horocyclic product is essentially the same as a horosphere corresponding to a barycentric ray in the product of the three trees. (It is contained in the horosphere and it is at finite Hausdorff distance from it.) Theorem 1.1, (1) in [14] gives a quadratic Dehn function for the horosphere, and so for \( \Gamma_m \).

We will give a brief account of our proof of Theorem 1.2 — it is elementary and it is interesting to see how the exponential upper bound for the Dehn function of \( \Gamma \) improves in \( \Gamma_m \).

Another strategy for establishing upper bounds on the Dehn functions of \( \Gamma_m \) for all \( m \) has been suggested by N. Brady [7]. With respect to a suitable finite presentation, the Cayley 2–complex \( C \) of \( \Gamma_m \) is the horocyclic product of three \((m+1)\)–valent infinite trees [3] and so sits inside a CAT(0) space, namely the direct product of three such trees. A loop in \( C \) can be spanned by a disc in the ambient CAT(0) space whose area is at most quadratic in the length of the loop. Brady proposes pushing this disc into \( C \) in the manner of [1, 13] to give a filling disc with at most a quartic area.

**Background.** Much is already known about the isoperimetry of solvable groups. The Dehn functions of finitely generated nilpotent groups admit upper bounds of \( \preceq n^{c+1} \) where \( c \) is the nilpotency class [16, 19], and yet for all \( c \) there are class \( c \) examples with Dehn function \( \preceq n^{c+1} \) [6, 15, 22] and others with Dehn function \( n^3 \) [24]. There are also nilpotent examples with Dehn function \( \preceq n^2 \log n \) [23]. There are polycyclic groups such as lattices \( \text{Sol}_3(\mathbb{R}) \)
with exponential Dehn function, and there are non-nilpotent polycyclic groups such as higher-dimensional analogues of $\text{Sol}_{2m+1}(\mathbb{R})$ that have quadratic Dehn functions [14, 21] for $m \geq 2$. Venturing beyond polycyclic groups there are many metabelian groups with Dehn function $= 2^n$ — the best known example is $\langle x, y \mid x^2 = y^2 \rangle$. The first non-polycyclic solvable group with Dehn function bounded above by a polynomial were constructed in [2] — their Dehn functions grow at most cubically.

The groups $\Gamma$ and $\Gamma_m$ have received considerable attention in other contexts. The 3-dimensional integral homology group of $\Gamma$ is not finitely generated [5]. In [17] it is shown that $\Gamma_2$ is a counterexample to a strong version of the Atiyah Conjecture on $L^2$–Betti numbers. Random walks on Cayley graphs of $\Gamma_m$ (Diestel–Leider graphs) are studied in [3]. In [11], $\Gamma_2$ was given as the first known example of a finitely presented group with unbounded dead-end depth — a property of the shapes of balls in the Cayley graph. In [10] it is shown that $\langle a, t \rangle \cong \mathbb{Z} \wr \mathbb{Z}$ is exponentially distorted inside $\Gamma$.

The organisation of this article. Our exponential lower bound on the Dehn function of $\Gamma$ is established in Section 2 and is proved to be sharp in Section 3. In Section 4 we show how, for $\Gamma_m$, the upper bound can be improved to quartic.

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2. A Dehn function lower bound via extensions

We will use the following general result.

**Proposition 2.1.** Suppose $H$ is a normal subgroup of a group $G$ and $A$ is a finite generating set for $G$. Suppose $\langle A \mid R \rangle$ is a finite presentation for $G/H$. Then each $r \in R$ can be regarded as representing an element of $H$. Let $B = \langle r^k \mid r \in R, g \in G \rangle \subseteq H$.

If a word $w = w(A)$ represents the identity in $G/H$, then there exists a word $w' = w'(B)$ that equals $w$ in $G$ and has length equal to the area of $w$ in $\langle A \mid R \rangle$.

**Proof.** As $w$ represents the identity in $G/H$, it freely equals $\prod_{i=1}^{N} r_{i}^{e_{i}u_{i}}$ for some words $u_{i} = u_{i}(A)$, some $r_{i} \in R$ and some $e_{i} = \pm 1$, and $N = \text{Area}(w)$. But then $w = \prod_{i=1}^{N} (r_{i}^{e_{i}})^{u_{i}}$ in $G$ and as each $r_{i}^{e_{i}}$ represents an element of $B$, the result is proved. \qed

**Corollary 2.2.** If $B$ is finite, then the distortion of $H$ in $G$ is a lower bound for the Dehn function of $G$.

This was used in [6] in the case where $H$ is a central subgroup of $G$ (and so $B = R$), as in the following example.

**Example 2.3.** Here is an unusual proof that the Dehn function of $\mathbb{Z}^2$ is at least quadratic. The 3-dimensional integral Heisenberg group

$$\mathcal{H}_3 = \langle a, b, c \mid [a, b] = c, \ [a, c] = 1, \ [b, c] = 1 \rangle = \begin{pmatrix}
1 & \mathbb{Z} & \mathbb{Z} \\
0 & 1 & \mathbb{Z} \\
0 & 0 & 1
\end{pmatrix}$$

is a central extension of $\mathbb{Z}^2 = \langle a, b, c \mid [a, b] = c, \ c = 1 \rangle$ by $\mathbb{Z} = \langle c \rangle$. The word $[a^n, b^n]$ represents 1 in $\mathbb{Z}^2$ and $c^{n^2}$ in $\mathcal{H}_3$. As $c^{n^2}$ has word length $n^2$ with respect to the generating set $B = \langle c \rangle$, the area of $[a^n, b^n]$ in $\langle a, b, c \mid [a, b] = c, \ c = 1 \rangle$ is at least $n^2$. 

We are now ready to establish the exponential lower bound on the Dehn function of \( \Gamma \), claimed in Theorem 1.1. We will apply Proposition 2.1 and Corollary 2.2 to

\[
\Gamma = \left\langle a, p, q, s, t \mid [a, a^t] = p, \quad a^t = d^q, \quad s^1 ps = p^{-1}, \quad t^{-1}pt = p^{-1}, \quad [a, p] = 1 \right\rangle
\]

and its normal subgroup \( H := \langle p, q \rangle \). Baumslag’s group \( \Gamma \) is \( \Gamma / H \).

In this case, \( H \) is not central but Corollary 2.2 applies none---the---less because, if \( g \in \Gamma \) and \( \hat{h} \in H \) then either \( h^\hat{g} = h \) or \( h^\hat{g} = h^{-1} \), and so \( B \) is finite. The next two lemmas identify \( H \) and show it is exponentially distorted in \( \Gamma \).

**Lemma 2.4.** \( H \) is isomorphic to \( \mathbb{Z}^2 \).

**Proof.** Define \( R \) to be the ring \( \mathbb{Z}[x, x^{-1}, (x + 1)^{-1}] \) and

\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & -2x - 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & -x - 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},
\]

\[
P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ x & 0 & -x^2 - x \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -x^2 - x & 0 & 1 \end{pmatrix}.
\]

Consider the analogues of the defining relations of \( \Gamma \). In \( \text{GL}_3(R) \),

\[
[A, A^T] = P, \quad AA^T = A^2, \quad [S, T] = 1, \quad [P, Q] = 1, \quad [A, P] = 1, \quad [A, Q] = 1
\]

hold, but

\[
S^{-1}PS = P^{-1}, \quad S^{-1}QS = Q^{-1}, \quad T^{-1}PT = P^{-1}, \quad T^{-1}QT = Q^{-1}
\]

fail since, for \( f \in R \),

\[
S^{-1} \begin{pmatrix} 1 & 0 & f \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} S = T^{-1} \begin{pmatrix} 1 & 0 & f \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 & -(x^2 + x)f \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]

Let \( R^* \) denote the invertible elements of \( R \). The matrices \( A, P, Q, S, T \) are in

\[
G := \begin{pmatrix} 1 & R & R \\ R^* & R & R \\ R^* & R & R^* \end{pmatrix} \leq \text{GL}_3(R).
\]

Let \( \tau = (-1 + \sqrt{5})/2 \). Applying the ring homomorphism \( R \to \mathbb{Z}[\tau] \), defined by \( f(x) \mapsto f(\tau) \), to the top--right entry maps \( G \) to the group

\[
\hat{G} := \begin{pmatrix} 1 & R & \mathbb{Z}[\tau] \\ R^* & R \end{pmatrix}.
\]

[Lifting to \( \hat{G} \) and multiplying there leads to the group operation on \( \hat{G} \).

The calculation (2) shows that in \( \hat{G} \) the (images of the) relations (1) hold since \( \tau^2 + \tau = 1 \). So mapping \( a \mapsto A, \quad p \mapsto P, \quad q \mapsto Q, \quad s \mapsto S, \quad t \mapsto T \) induces a homomorphism \( \Gamma \to \hat{G} \). (Incidentally, it is possible to show that this is an injection, but that is not a result we need here.) The images of \( P \) and \( Q \) inside \( \hat{G} \) are the matrices that differ from the identity only in that they have \( -\sqrt{5} \) and \((-1 - \sqrt{5})/2 \), respectively, as their top--right entries, and so they generate

\[
\begin{pmatrix} 1 & 0 & \mathbb{Z}[\tau] \\ 1 & 0 & 1 \end{pmatrix}.
\]
That these commutators represent the identity is the key point in Baumslag’s proof in Lemma 2.5.

Let $F_n$ denote the $n$–th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

**Lemma 2.5.** $[a, a^n] = p^{(-1)^{n+1} F_3}$ in $\Gamma$ for all $n \geq 0$.

**Proof.** The commutator $[a, a^n]$ represents the identity in $\Gamma$ — a fact whose proof we postpone to Lemma 3.1 — and so, by the previous lemma, represents an element $p^{q^-}q^n$ of $H$ in $\Gamma$. We can find $\lambda$ and $\mu$ using the matrices from our proof of of Lemma 2.4. We calculate that in $G$,

$$[A, A^T] = \begin{pmatrix} 1 & 0 & (-x - 1)^n - x^n \\ 1 & 0 & 1 \end{pmatrix},$$

which has image

$$\begin{pmatrix} 1 & 0 & (-1)^n \sqrt{5}F_n \\ 1 & 0 & 1 \end{pmatrix}$$

in $\tilde{G}$. Now, the matrix in $\tilde{G}$ corresponding to $p^{q^-}q^n$ has $\lambda(-\sqrt{5}) + \mu(-1 - \sqrt{5})/2$ in the upper right corner, and otherwise agrees with the identity matrix. This entry equals $(-1)^n \sqrt{5}F_n$ precisely when $\lambda = (-1)^{n+1}F_n$ and $\mu = 0$.

We conclude that the Dehn function of $\Gamma$ grows $\geq 2^n$ by Corollary 2.2.

3. An exponential upper bound on the Dehn function of $\Gamma$

Throughout this section we calculate using the presentation

$$\left\{ a, s, t \mid [a, a] = 1, [s, t] = 1, a^t = a a^t \right\}$$

for $\Gamma$ introduced in Section 1.

We begin by focussing on a particular family of words. For $n \in \mathbb{Z}$, define

$$C(n) := \text{Area } [a, a^n],$$

the minimum cost to convert $[a, a^n]$ to the empty word, or equivalently $aa^n$ to $a^n a$, in $\Gamma$. That these commutators represent the identity is the key point in Baumslag’s proof in [4] that $\Gamma$ is metabelian.

**Lemma 3.1.** $[a, a^n] = 1$ in $\Gamma$ and $C(n) \leq 4^n$ for all $n \geq 1$.

**Proof.** We induct on $n$. The cases $n = 1, 2, 3$ can be checked directly. For the induction step, assume $n \geq 3$. We have $aa^n = a^n a$ at a cost of at most $C(n)$. So $(aa^n)^t = (a^n a)^t$ and therefore $aa^t(a^n a)^t = (a^n a)^t aa^n$ at an additional cost of $6 + 4n$. Thus we have $aa^t a a^n = a^{n+1} a^t a^n$. Then, using the commutator relations $[a, a^{n-1}] = 1$ and $[a, a^n] = 1$ once and twice, respectively, at a cost of $C(n-1) + 2C(n)$, gives $aa^{n+1} a^t a^n = a^n a a^n a^t a^n$ and therefore $aa^{n+1} = a^{n+1} a$. Thus $C(n + 1) \leq 3C(n) + C(n - 1) + 4n + 6 \leq 3 \cdot 4^n + 4^{n-1} + 4n + 6$ which is at most $4^n+1$ when $n \geq 3$. 

\[ \square \]
The notation and techniques used in our next lemma are similar to those in Section 4.3 of [20], which in turn draws on [9]. It concerns the impact of the relations $a^r = aa^r$ and $a^r = a^r a$ in $\Gamma$. For a polynomial $f(x) = \sum_{i=0}^n c_i x^i$ in $\mathbb{Z}[x]$ and letters $a$ and $r$, define the word

$$\|a\|_i^j := a^r r^1 a^r r^2 \ldots a^r r^i,$$

which freely equals $a^r a^r \ldots a^r a^r$.

**Lemma 3.2.** For all polynomials $f(x) = \sum_{i=0}^n c_i x^i$ satisfying $\max_i |c_i| \leq c$,

$$\|a\|_{f(0)}^{f(n)} = \|a\|_{f(x+1)}^{f(x+1)}$$

in $\Gamma$ and the cost of the equality is at most $D_i(n) := c^2 4^n$.

**Proof.** The lemma is trivial in the case $c = 0$, so we can assume that $c \geq 1$. We induct on $n$. When $n = 0$ the words are identical. For the induction step assume $n \geq 1$. Let $\tilde{f}(x) = \sum_{i=1}^n c_i x^{i-1}$, so that $f(x) = c_0 + x \tilde{f}(x)$.

\[
\|a\|_{\tilde{f}(0)}^{\tilde{f}(x+1)} = a^r \left( \|a\|_{\tilde{f}(x)}^{\tilde{f}(x+1)} \right)^x \equiv a^r \|a\|_{\tilde{f}(x+1)}^{\tilde{f}(x+1)},
\]

where (I) is a free equality, (II) costs at most $D_i(n-1)$ by the induction hypothesis, (III) costs at most $2n$ applications of $[s, t] = 1$, (IV) uses $a^r = aa^r$ and costs the sum of the absolute value of the coefficients of $\tilde{f}(x+1)$ and so at most $c 2^n$. (V) will be explained momentarily, and (VI) is a free equality. For each coefficient $m$ of $\tilde{f}(x+1)$, equality (V) uses $[a, a^r] = 1$ to transform a word $(aa^r)^m$ to $a^r a^m$ at a cost of no more than $m^2$. So a crude upper bound for the total cost of (V) is $c^2$ times the square of the sum of the coefficients of $\sum_{i=1}^n (x+1)^{i-1} = c^2 \left( \sum_{i=1}^n 2^{i-1} \right)^2 \leq c^2 4^n$. Thus $\|a\|_i^j$ can be transformed to $\|a\|_i^{f(x+1)}$ at a total cost of at most

$$D_i(n-1) + 2n + c 2^n + c^2 4^n \leq D_i(n).$$

The following proposition combines with Lemma 3.1 to prove that the Dehn function of $\Gamma$ admits an exponential upper bound, and so completes our proof of Theorem 1.1.

**Proposition 3.3.** There exists $K > 0$ such that for all $n \geq 1$, the Dehn function of $\Gamma$ satisfies

$$\text{Area}(n) \leq K^n \max \{ C(i) \mid 0 \leq i \leq 6n \}.$$  

**Proof.** Suppose a word $w = w(a, s, t)$ has length $n$ and represents 1 in $\Gamma$. Let

$$\rho : \Gamma \to \mathbb{Z}^2 = \langle s, t \rangle$$

be the retract arising from killing $a$. We may assume $w$ contains at least one letter $a^s$ for otherwise $w$ represents 1 in $\mathbb{Z}^2 = \langle s, t \rangle$ and so Area($w$) \leq $n^2$. We will convert $w$ to successive words $w_1, \ldots, w_5$ and then to the empty word, and will then sum the costs.

After each prefix $u$ of $w$ that ends in a letter $a^s$, insert the word $s^a \rho(t^b s^{-a})$ where $t^b s^{-a}$ equals $\rho(u)$ in $\mathbb{Z}^2$. The resulting word $w_1$ freely equals $w$. Note that $|a| + |b| \leq n$.

At a cost of at most $2n^3$, use the relation $[s, t] = 1$ to change $w_1$ to a word

$$w_2 = \prod_{i=1}^k a^{e_i s^{i_1} t^{i_2}}$$

in which each $e_i = \pm 1$, $|a| + |b| \leq n$ and $k \leq n$ — in front of each $a^s$ in $w_1$, there is a subword on $s^1$ and $t^1$ of length at most $2n$ to convert to $t^b s^{-a}$, for some $a_i$ and $b_i$; each of the at most $n$ such subwords costs at most $2n^2$. 

Next we would like to apply Lemma 3.2 with \( f(x) = x^n \) to each \( a^{x_i} \) to reach a word that is a product of conjugates of \( a^\pm 1 \) by powers of \( t \). But this presupposes that each \( a_i \) is non–negative, which may not be the case. We work around this issue as follows. Define \( \alpha := \min \{ a_i \} \). Then \( 0 \leq a_i - \alpha \leq 2n \) for all \( i \). Instead of continuing to transform \( w_2 \), we will work with \( w_2^{x_w} \). We may do this because in any finitely presented group the area of a word \( v \) representing 1 is the same as that of \( v^\rho \) for any word \( u \).

It costs at most \( 2n^2 \) to convert \( a^{x_i^{\rho_1}} a^{x_i^\rho_2} \) to \( a^{x_i^{\rho_1}+\rho_2} \) since \( |a_i|, |\beta_i| \leq n \). So, as \( k \leq n \), at most \( 2n^3 \) relations are required to convert \( w_2^{x_w} \) to

\[
w_3 := \prod_{i=1}^{k} a^{x_i^{\rho_1}+\rho_2}.
\]

We can now apply Lemma 3.2 since each \( a_i - \alpha \geq 0 \). It converts each \( a^{x_i^\rho} \) in \( w_3 \) to \( [a_i]_{1+x} \rho_i \), and so yields

\[
w_4 := \prod_{i=1}^{k} ([a_i]_{1+x} \rho_i)^{\epsilon_i^{\rho_i}}
\]

at a total cost of at most \( nD_1(\alpha_i - \alpha) \leq nD_1(2n) \). But then \( w_4 \) freely equals a word

\[
w_5 = \prod_{i=1}^{l} a^{\mu_i^{\gamma_i}}
\]

for some integers \( l, \mu_i, \gamma_i \) satisfying the following (crude) inequalities

\[
l \leq (2n + 1)n,
\]

\[
|\mu_i| \leq \max \left\{ \binom{\gamma_i}{i} \right\} 1 \leq j \leq 2n, 0 \leq i \leq j \leq 2^2n,
\]

\[
|\gamma_i| \leq \max(\alpha_i - \alpha + \beta_i) \leq 3n.
\]

Now, \( w_5 \) represents the identity in the subgroup

\[
\langle a, t \rangle \in \mathbb{Z} = \left\langle a, t \left| \begin{array}{c} [a, a^t] = 1 \text{ (in } \mathbb{Z} \text{)} \end{array} \right. \right\rangle
\]

and is freely equal to a product of at most \( 2^{2n}(2n + 1)n \) terms of the form \( a^{x_i^\rho} \) in which \( |i| \leq 5n \). So reordering these terms so as to collect all those in which the power of \( t \) agree will give a word which freely reduces to the empty word. This reordering can be achieved at a cost of no more than \( \left( \frac{2^{2n}(2n + 1)n}{2} \right)^2 \) commutators \( [a, a^t] \) in which \( 0 \leq i \leq 6n \).

Summing our cost estimates, we have

\[
\text{Area}(n) \leq 2n^3 + 2n^3 + nD_1(2n) + \left( 2^{2n}(2n + 1)n \right)^2 \max \{|C(i)| \mid 0 \leq i \leq 6n \},
\]

and the result follows.

\[
\square
\]

4. A quartic upper bound on the Dehn function of \( \Gamma_m \)

In this section we calculate using this presentation for \( \Gamma_m \):

\[
\left\langle a, s, t \left| [a, a^t] = 1, [s, t] = 1, a^s = aa^t, a^m = 1 \right. \right\rangle
\]

The disparity between \( \Gamma_m \) and \( \Gamma \) first appears in the following analogue of Lemma 3.2. Adapting the notation of Lemma 3.2, for a polynomial \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) in \( \mathbb{Z}[x] \) and letters
4.2 where the second equality is free, and the first and third stem from applications of Lemma Proof.

For \( n \in \mathbb{N} \), Lemmas 4.3.

We are now ready to prove Theorem 1.2. Suppose a word \( w = w(a, s, t) \) has length \( n \) and represents 1 in \( \Gamma_m \). At a cost of at most \( 2n^3 \), proceed to a word

\[ w_2 = \prod_{i=1}^k a^{t_i} s^{r_i} t_i \]
for which \( \epsilon_i = \pm 1 \), and \( |\alpha| + |\beta| \leq n \), and \( k \leq n \) exactly as in our proof of Proposition 3.3. Define \( \alpha := \min \alpha_i \) and \( \beta := \min \beta_i \). Then \( w_2 \) has the same area as \( w_2^{r+\gamma} \). It costs at most \( 2n^3 \) applications of \( [s, t] = 1 \) to convert \( w_2^{r+\gamma} \) to

\[
w_3' := \prod_{i=1}^{k} a_i^{r_i \alpha_i \gamma_i \beta_i}.
\]

Let \( f_i(x) := x^{\alpha_i} \). Then \( w_3' \) freely equals \( \prod_{i=1}^{k} (\|a\|_{t, i} f_i(x))^{r_i \alpha_i \gamma_i \beta_i} \) which, by Lemma 4.1, becomes

\[
w_4' := \prod_{i=1}^{k} (\|a\|_{t, i} f_i(x+1))^{r_i \alpha_i \gamma_i \beta_i}
\]

at a cost of at most \( n|K_m(n)| \). By applying Proposition 4.4 in the case \( g = -f \), we can change each \( (\|a\|_{t, i} f_i(x+1)) \) for which \( \epsilon_i = -1 \) to \( \|a\|_{t, i} f_i(x+1) \) at a total cost of at most \( 6n^2 |K_m(n)| + 4mn^2 \).

We then have a word freely equal to

\[
w_5' := \prod_{i=1}^{k} \|a\|_{t, i} g_i
\]

where \( g_i(x) := \epsilon_i x^{\alpha_i} f_i(x+1) \). For \( 1 \leq i < k \), Proposition 4.4 establishes the equality

\[
\|a\|_{t, i} \prod_{i=j+1}^{k} \|a\|_{t, i} = \|a\|_{t, i} \prod_{i=j+2}^{k} \|a\|_{t, i}
\]

at a cost of \( 6n|K_m(n)| + 4mn \). So \( w_5' \) can be transformed to \( w_6' := \|a\|_{t, i} \prod_{i=j+1}^{k} g_i \) at a cost of at most \( 6n^2 |K_m(n)| + 4mn^2 \). Since no letters \( s^{\pm 1} \) occur in \( w_6' \), it represents the identity in

\[
C_m \approx \langle a, t \mid a^n = 1, [a, t] = 1 (k \in \mathbb{Z}) \rangle
\]

and so \( \sum_{i=1}^{k} g_i = 0 \) and \( w_6' \) is in fact the empty word.

Theorem 1.2 then follows from summing the cost estimates:

\[
\text{Area}(n) \leq 4n^2 + n|K_m(n)| + 12n^2 |K_m(n)| + 8mn^2.
\]

Remark 4.5. The Dehn function of \( \Gamma_m \) is bounded below by \( n^2 \). (Use the argument in Example 2.3 or deduce this from the fact that \( \Gamma_m \) is not hyperbolic since it has a \( \mathbb{Z}^2 \) subgroup.) It seems that one might be able to do the estimates above more carefully and use essentially the method to obtain a cubic upper bound for the Dehn function. However, we would be surprised if these types of combinatorial arguments could lead to the quadratic upper bound obtained via geometric methods.

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