Dynamical effects of a cosmological constant

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Abstract

The observational evidence for the existence of a non-zero cosmological constant is getting stronger. It is therefore timely to address the question of its eventual effect on the dynamics of galaxies, clusters and larger structures in the Universe. We find, contrary to a recent claim, that the influence of the cosmological constant has to be negligible for, e.g., the rotation curves of galaxies. On larger scales, starting with large galaxy clusters, there are potentially measurable effects from the repulsive addition to the Newtonian gravitational force caused by the cosmological constant.
During the past few years, remarkable progress has been made in cosmology, both observational and theoretical. One of the outcomes of these rapid developments is the increased confidence that most of the energy density of the observable universe is of an unusual form, i.e., not made up of the ordinary matter (baryons and electrons) that we see around us in our everyday world.

There are convincing arguments for the existence of a large amount of non-luminous, i.e., dark, matter. The matter content of the universe is at least a factor of 5 higher than the maximum amount of baryonic matter implied by big bang nucleosynthesis. This dark matter is thus highly likely to be “exotic”, i.e, non-baryonic.

There are also indications, although still not entirely conclusive, of the existence of vacuum energy, corresponding to the famous “cosmological constant” that Einstein introduced but later rejected (although without very good reasons) in his theory of general relativity. This possibility has recently been given increased attention due to results from Type Ia supernova surveys [1, 2]. (For a recent review of the observational status of dark matter and dark energy, see [3].)

It may be interesting to investigate possible consequences of the cosmological constant besides its influence on the geometry of the universe, and the redshift-dependence of the luminosity-distance relation for standard candles [4]. This is the subject of the present paper. We find disagreement with a recent paper [5] where the cosmological constant was claimed to influence the rotation curves of galaxies strongly. However, some small effects on the dynamics of galaxy clusters do not seem excluded.

Let us first set our conventions. Einstein’s equations read

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \Lambda g^{\rho\sigma} = 8\pi G_N T^{\mu\nu}. \]  

(1)

The energy density in the form of a cosmological constant \( \Lambda \) can be conveniently written in units of the density scaled to the critical density,

\[ \Omega_{\Lambda} \equiv \frac{\rho_{\Lambda}}{\rho_{\text{crit}}}, \]  

(2)

where

\[ \rho_{\Lambda} = \frac{\Lambda}{8\pi G_N} \]  

(3)
with $G_N = 1/m_{Pl}^2$ (the numerical value of the Planck mass is $m_{Pl} = 1.2 \cdot 10^{19}$ GeV) and the present value of the critical density

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G_N}. \quad (4)$$

Thus,

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \quad (5)$$

Writing $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$ (with $h \sim 0.6 \pm 0.1$ from observations), the present numerical value of the critical density is

$$\rho_{\text{crit}} \simeq 8 \cdot 10^{-47} h^2 \text{GeV}^4. \quad (6)$$

This was derived using particle physics units ($c = \hbar = 1$). Expressed in cgs units, the presently observationally favoured value $\Omega_\Lambda \simeq 0.7 \pm 0.2$ then translates to

$$\Lambda_{\text{obs}} \simeq 10^{-56} \text{cm}^{-2}. \quad (7)$$

In a recent preprint [5], an attempt was made to explain the flat rotation curves of galaxies as being due to the effect of a cosmological constant instead of the “traditional” explanation in terms of dark matter. However, values some three to four orders of magnitude larger than that in (7) were needed, something which is clearly in extreme disagreement with observations. In fact, there seems to be a further mistake, a sign error, in [5]. A positive cosmological constant, as favored by the observations, will tend to accelerate the expansion of the universe and, if anything, make matter in the outer regions of galaxies less rather than more bound.

While the effects of a cosmological constant thus are negligible on the length scale of galaxies, one might expect observable consequences for galaxy clusters. As a first attempt to see such an effect, we will consider the fate of circular orbits in a flat, expanding universe with a cosmological constant. To do this we start with the equation of motion for a particle in an expanding universe with an additional gravitational potential,

$$\ddot{\chi} + \frac{2\dot{a}}{a}\dot{\chi} = -\frac{\mathcal{F}}{a},$$

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where $\chi$ is the comoving coordinate and $a$ is the scale factor. In terms of physical distances, $R = a\chi$, one finds

$$\ddot{R} - \frac{\dot{a}}{a} \dot{R} = -\gamma.$$

As an example we consider the de Sitter case with $\Omega_\Lambda = 1$, i.e. a universe totally dominated by the cosmological constant. We then use

$$a(t) = e^{Ht}$$

where $H = \sqrt{\frac{\Lambda}{3}}$ is constant, to obtain

$$\frac{v^2}{R} = \frac{G_N m}{R^2} - \frac{\Lambda}{3} R^2.$$

for an object in orbit around a central mass $m$. One might note that the same result may be obtained by starting with the static form of the de Sitter metric, i.e.

$$ds^2 = \left(1 - \frac{2G_N m}{r} - \frac{\Lambda}{3} r^2\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2G_N m}{r} - \frac{\Lambda}{3} r^2\right)} - r^2 d\Omega^2,$$

and using a Newtonian analysis. This form of the metric is related to the cosmological form through a coordinate transformation. Equation (8) shows that for large enough $R$, i.e.

$$R > \left(\frac{3G_N m}{\Lambda}\right)^{1/3},$$

there are no longer bound orbits. Does this have observable consequences? Unfortunately it is easy to see that the effect becomes important only for orbits that are such that they have periods of the order of the age of the universe. Furthermore, the effect of the cosmological constant decreases rapidly for smaller orbits. Hence the concept of a rotation curve loses its meaning and we had better look elsewhere for a better approach to the possible effect of a cosmological constant. One possibility is the way clusters and superclusters of galaxies form.

As a first step, one may consider the infall of matter onto a galaxy cluster in the regime where linear perturbation theory is valid. This has in fact
been treated by Peebles [3], in the case $\Omega_M + \Omega_\Lambda = 1$ (i.e. zero curvature) which is the natural case in view of inflation, and which, incidentally, is now also indicated by the recent balloon measurements of the cosmic microwave background [7]. Peebles showed that, unfortunately, the dependence of the infall peculiar velocity $v$ for a cluster of proper radius $R$ and overdensity $\delta$ on $\Omega_\Lambda$ is quite weak, being well parametrized by

$$v = 0.3H_0 R \delta \Omega_M^{0.6}$$

(9)

for $0.03 < \Omega_M < 0.3$ and $1 < \delta < 3$, essentially independent of $\Omega_\Lambda$. This formula was generalized to arbitrary $\Omega_M + \Omega_\Lambda$ in [3].

In the non-linear, collapsing phase, we may obtain an estimate of the influence of $\Lambda$ by adapting the simple constant density, spherical collapse model [9] to the situation when the cosmological constant is present. We thus look at the situation when an overdense region expands to a maximal radius $R_{\text{max}}$, and then contracts to a virial radius $R_{\text{vir}}$.

To analyse this situation, we use the equation for the energy per unit mass (cf. [3], Eq. (20) or our equation (8)) of a mass shell of proper radius $R(t)$ containing a fixed mass $m$:

$$E = \frac{\dot{R}^2}{2} - \frac{G_N m}{R} - \frac{\Lambda R^2}{6},$$

(10)

where the three terms correspond to the kinetic energy, Newtonian gravitational energy, and vacuum energy, respectively. As in the standard analysis [9], we may employ the virial theorem relating the average value of the kinetic energy $T$ to the potential energy $V$

$$\langle 2T \rangle = \langle \vec{r} \cdot \frac{\partial V}{\partial \vec{r}} \rangle.$$  

(11)

Taking the average over a sphere of constant density of the energy equation (at the turn-around radius $R_{\text{max}}$ the kinetic energy is zero)

$$E = T_{\text{vir}} + V_{\text{vir}}^G + V_{\text{vir}}^\Lambda = V_{\text{max}}^G + V_{\text{max}}^\Lambda,$$

(12)

utilizing

$$\langle \Lambda r^2 \rangle = \frac{3\Lambda}{R^3} \int_0^R r^4 dr = \frac{3R^2}{5}$$

(13)

and

$$\langle G_N M(r) r^{-1} \rangle = \frac{3G_N M}{R^6} \int_0^R r^4 dr = \frac{3G_N M}{5R},$$

(14)
we recover in the case $\Lambda = 0$ the well-known result \[9\]

$$\frac{3G_N M}{5R_{\text{max}}} = \frac{3G_N M}{10R_{\text{vir}}},$$

(15)
i.e., $R_{\text{vir}} = R_{\text{max}}/2$. For a non-vanishing $\Lambda$, the corresponding equation is

$$\frac{3G_N M}{5R_{\text{max}}} + \frac{\Lambda}{10} R_{\text{max}}^2 = \frac{3G_N M}{10R_{\text{vir}}} + \frac{\Lambda}{5} R_{\text{vir}}^2,$$

(16)

Suppose that the mass overdensity contrast compared to the cosmological average mass density at the maximal (turn-around) radius is $\omega$ (\(= 5.6\) in the standard case), and assume $\Omega_M + \Omega_\Lambda = 1$. Then we can write $M = 4\pi R_{\text{max}}^3 \omega \rho_M/3$, and this inserted into (15) gives, by use of (13)

$$1 + \kappa = \frac{1}{2\mu} + 2\kappa \mu^2$$

(17)

where we have introduced $\mu = R_{\text{vir}}/R_{\text{max}}$ (\(= 0.5\) in the standard case) and

$$\kappa = \frac{1}{\omega} \frac{\Omega_\Lambda}{(1 - \Omega_\Lambda)(1 + z)^3}.$$

This result is written in a somewhat different form than, but agrees with, Eq. (26) of \[8\]. If we assume as a first approximation $\omega = 5.6$ as in the $\Lambda = 0$ case, and $\Omega_\Lambda = 0.7$, we find by solving (17) numerically for $z = 0$ that $\mu$ has decreased from 0.5 to around 0.39.

This means that the virialized radius is smaller, which is not unreasonable, since for a given mass, only more compact clusters can “survive” the repulsive force from a positive cosmological constant.

We can improve this analysis somewhat by taking into account the fact that $\omega$ also depends on $\Lambda$. It can be seen that this has the effect of increasing $\omega$ at intermediate redshifts. This increase causes an increase in $\mu$, meaning that we have overestimated the effect of $\Lambda$ above. The effect is, however, small. One can also estimate what the final density contrast of the cluster will be. To do this we have obtained an expression comparing the density at maximum expansion with the density of the universe today. This is given by

$$\tilde{\omega} = \frac{9\pi^2}{16} \frac{1}{f(a)^2} \frac{1}{(1 - \Omega_\Lambda)}$$

(18)
with
\[
f(a) = \frac{3}{2} \int_0^a \frac{da}{\sqrt{(1 - \Omega_\Lambda)/a + \Omega_\Lambda a^2}}.
\] (19)

The result is a further net relative compression of the clusters due to the cosmological constant above the one given by \(\mu\).

Since we have been dealing with an imagined situation where one can compare between a “standard” scenario without a cosmological constant, and one where \(\Lambda \neq 0\), and found only small effects on cluster scales, it seems difficult, but maybe not excluded, to draw conclusions about the value of \(\Omega_\Lambda\) in the real universe where observational uncertainties have to be taken into account. The effect to search for, is a tendency for virialized clusters to get smaller and more overdense for a positive \(\Lambda\).

It seems clear, however, that the effects on galactic scales are extremely tiny and negligible for rotation curves.

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