CHARACTERIZATION OF AFFINE $\mathbb{G}_m$-SURFACES OF HYPERBOLIC TYPE

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ABSTRACT. In this note we extend the result from [LRU20] and prove that if $S$ is an affine non-toric $\mathbb{G}_m$-surface of hyperbolic type that admits a $\mathbb{G}_a$-action and $X$ is an affine irreducible variety such that Aut($X$) is isomorphic to Aut($S$) as an abstract group, then $X$ is a $\mathbb{G}_m$-surface of hyperbolic type. Further, we show that a smooth Danielewski surface $D_p = \{xy = p(z)\} \subset \mathbb{A}^3$, where $p$ has no multiple roots, is determined by its automorphism group seen as an ind-group in the category of affine irreducible varieties.

1. Introduction and main results

In this note we work over the field of complex numbers $\mathbb{C}$ and assume all varieties to be irreducible. We denote the multiplicative and additive group of $\mathbb{C}$ respectively by $\mathbb{G}_m$ and $\mathbb{G}_a$, and call a variety that admits a regular $\mathbb{G}_m$-action/$\mathbb{G}_a$-action to be a $\mathbb{G}_m$-variety/$\mathbb{G}_a$-variety.

Affine $\mathbb{G}_m$-surfaces appear in three types with respect to the dynamical behavior of the $\mathbb{G}_m$-action: parabolic, corresponding to the case with infinitely many fixed points, elliptic, where an attractive fixed point exists, and hyperbolic, with finitely many non-attractive fixed points. In addition, in the case of a non-toric affine $\mathbb{G}_m$-surface $S$, the dynamical type of $S$ does not depend on the choice of the $\mathbb{G}_m$-action ([FZ05a, Corollary 4.3]). [LRU22, Theorem 1.2] shows that if there exists a group isomorphism of automorphism groups of normal affine $\mathbb{G}_m$-surfaces $S$ and $S'$ with $S$ being also a non-toric $\mathbb{G}_a$-surface, then $S$ and $S'$ have the same dynamical type. We generalize this statement for the hyperbolic dynamical type, which can be viewed as a rigidity result, and observe that it does not extend to parabolic and elliptic dynamical types in Remark 1.

Theorem 1. Let $S$ and $X$ be affine $\mathbb{G}_m$-varieties with $S$ being also a normal non-toric $\mathbb{G}_a$-surface. If there exists a group isomorphism $\varphi: \text{Aut}(S) \to \text{Aut}(X)$, then $\dim X = 2$. Moreover, if $X$ is normal, then $X$ is also a hyperbolic non-toric $\mathbb{G}_m$-surface.

Remark 1. The only elliptic $\mathbb{G}_m$-surfaces $S$ that admit a $\mathbb{G}_a$-action are toric, thus Theorem 1 does not extend to this case. Indeed, by [Lie10, Lemma 1.10] $S$ admits a root subgroup (see Section 2.4) in its automorphism group with respect to a subtorus of Aut($S$) induced by the $\mathbb{G}_m$-action. Hence, by [FZ05b, Theorem 3.3] $S$ is toric. Moreover, at the end of this note, Example 1 demonstrates that Theorem 1 does not extend to the parabolic case.

One of the most important class of hyperbolic $\mathbb{G}_m$-surfaces are Danielewski surfaces which were originally studied in [Dan89] with the aim to find a counterexample for the generalized version of Zariski Cancellation Problem. A smooth Danielewski
surface $D_p$ is $\{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\}$, where $p$ is a polynomial without multiple roots.

**Remark 2.** [Da04, Lemma 2.10] shows that $D_p$ and $D_q$ are isomorphic if and only if there exists an automorphism $F$ of $\mathbb{C}[z]$ such that $\frac{F(p)}{q} \in \mathbb{C}^*$. In particular if $\deg p = \deg q = 2$, then $D_p$ and $D_q$ are isomorphic.

We call a surface $D_p$ generic if there is no affine automorphism of the affine line $\mathbb{C}$ that permutes the roots of $p$. For two generic surfaces $D_p$ and $D_q$ with $\deg p \geq 3$ and $\deg q \geq 3$ there is an isomorphism $\text{Aut}(D_p) \cong \text{Aut}(D_q)$ of abstract groups. Indeed, in [ML90, Theorem and Remark (3) on p. 256] and more precisely in [KL16, Theorem 2.7] it is shown that for a generic Danielewski surface $D_p$, we have $\text{Aut}(D_p) \cong (\mathbb{C}[x] \ast \mathbb{C}[y]) \rtimes (\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z})$ and the semidirect product structure does not depend on $p(z)$ (see also [LR20, Remark 7]). On the other hand, by [LR20, Theorem 3] $\text{Aut}(D_p)$ and $\text{Aut}(D_q)$ are isomorphic as ind-groups (see Section 2.1 for details) if and only if $D_p$ is isomorphic to $D_q$ as a variety. In this paper we prove the following result.

**Theorem 2.** Let $X$ be an affine irreducible variety. Assume $\text{Aut}(X)$ is isomorphic to $\text{Aut}(D_p)$ as an ind-group, then $X$ is isomorphic to $D_p$ as a variety.

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2. Preliminaries

2.1. Ind-groups. The notion of an ind-group goes back to Shafarevich, who called them infinite dimensional groups, see [Sh66]. We refer to [FK18] and [KPZ15] for basic notions.

**Definition 1.** An ind-variety is a set $V$ together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V$ such that the following is satisfied:

(1) $V = \bigcup_{k \in \mathbb{N}} V_k$;

(2) each $V_k$ has the structure of an algebraic variety;

(3) for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski topology.

A morphism between two ind-varieties $V = \bigcup_{k \in \mathbb{N}} V_k$ and $W = \bigcup_{l \in \mathbb{N}} W_l$ is a map $\phi: V \to W$ such that for each $k \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $\phi(V_k) \subset W_l$ and such that the induced map $V_k \to W_l$ is a morphism of algebraic varieties. Isomorphisms of ind-varieties are defined in the usual way.

An ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ has a natural topology: a subset $S \subset V$ is called closed, respectively open, if $S_k := S \cap V_k \subset V_k$ is closed, respectively open, for all $k$. A closed subset $S \subset V$ has the natural structure of an ind-variety. It is called an ind-subvariety if each variety $V_k$ is affine. In the sequel we consider only affine ind-varieties and for simplicity we call them just ind-varieties.

A set theoretical product of ind-varieties admits a natural structure of an ind-variety. This allows us to introduce the following definition.

**Definition 2.** An ind-variety $G$ is called an ind-group if the underlying set $G$ is a group such that the map $G \times G \to G$, defined by $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties.
Note that any closed subgroup $H$ of $G$, i.e., $H$ is a subgroup of $G$ and is a closed subset, is again an ind-group under the closed ind-subvariety structure on $G$. A closed subgroup $H$ of an ind-group $G$ is an algebraic subgroup if $H$ is an algebraic subset of $G$ i.e., $H$ is a closed subset of some $G_i$, where $G_1 \subset G_2 \subset \ldots$ is a filtration of $G$.

The next result can be found in [FK18, Section 5].

**Proposition 1.** Let $X$ be an affine variety. Then $\text{Aut}(X)$ has the structure of an ind-group such that for any algebraic group $G$, a regular $G$-action on $X$ induces an ind-group homomorphism $G \to \text{Aut}(X)$.

For example, if $X = \mathbb{A}^n$, the ind-group filtration of $\text{Aut}(\mathbb{A}^n)$ is given by

$\text{Aut}(\mathbb{A}^n)_d = \{f = (f_1, \ldots, f_n) \in \text{Aut}(\mathbb{A}^n) \mid \deg f = \max_i f_i \leq d, \deg f^{-1} \leq d\}$.

The following observation will turn out to be useful in the proof of Theorem 1:

**Lemma 1** (Lemma 2.4, [LRU22]). Let $X$ be an affine variety and let $U \subset \text{Aut}(X)$ be a commutative subgroup that coincides with its centraliser. Then $U$ is a closed subgroup of $\text{Aut}(X)$.

The next statement follows from [CRX19, Theorem B] (see also [RvS21, Corollary 3.2]). We will need it in the proof of Theorem 1.

**Proposition 2.** Let $X$ be an affine variety and let $G$ be a commutative connected closed subgroup of $\text{Aut}(X)$. Then $G$ is the countable union of an increasing filtration by commutative connected algebraic subgroups of $\text{Aut}(X)$.

### 2.2. Algebraic and divisible elements.

We call an element $f$ in a group $G$ divisible by $n$ if there exists an element $g \in G$ such that $g^n = f$. An element is called divisible if it is divisible by all $n \in \mathbb{Z}^+$. An element $f$ in the automorphism group of an affine variety $X$ is algebraic if it is contained in an algebraic subgroup $G$ of $\text{Aut}(X)$ with respect to its ind-group structure.

In the proof of Theorem 1 we will need the following result that follows from [LRU22, Theorem 3.1] and [LRU22, Corollary 2.6] and that connects the notions of divisibility and algebraicity in the automorphism group of an affine surface.

**Proposition 3.** Let $S$ be an affine irreducible algebraic surface. Then the following two conditions are equivalent:

1. there exists a $k > 0$ such that $f^k$ is divisible;
2. $f$ is algebraic.

### 2.3. Lie algebras of vector fields.

We denote by $\text{Vec}(X)$ the Lie algebra of vector fields on an affine variety $X$. A vector field $\nu \in \text{Vec}(X)$ is called locally nilpotent if for any $f \in \mathcal{O}(X)$ there exists $s \in \mathbb{N}$ such that $\nu^s(f) = 0$. By $\langle \text{LNV}(X) \rangle$ we denote the Lie subalgebra of $\text{Vec}(X)$ generated by all locally nilpotent vector fields. We have the following lemma.

**Lemma 2** (Lemma 1, [LR20]). Let $X$ and $Y$ be affine algebraic varieties and let $\varphi: \text{Aut}(X) \to \text{Aut}(Y)$ be a homomorphism of ind-groups. Then $\varphi$ induces the homomorphisms of Lie algebras

$d\varphi: \langle \text{LNV}(X) \rangle \to \langle \text{LNV}(Y) \rangle$.

Moreover, if $\varphi$ is an isomorphism, then the homomorphism $d\varphi$ is also the isomorphism.
2.4. **Root subgroups.** In this section we describe root subgroups of the automorphism group $\text{Aut}(X)$ of an affine variety $X$ with respect to a subtorus.

**Definition 3.** Let $T \subset \text{Aut}(X)$ be a subtorus in $\text{Aut}(X)$, i.e. a closed algebraic subgroup isomorphic to a torus. A closed algebraic subgroup $U \subset \text{Aut}(X)$ isomorphic to $\mathbb{G}_a$ is called a root subgroup with respect to $T$ if the normalizer of $U$ in $\text{Aut}(X)$ contains $T$. Such an algebraic subgroup $U$ corresponds to a non-trivial $\mathbb{G}_a$-action on $X$, whose image in $\text{Aut}(X)$ is normalized by $T$.

Assume $U \subset \text{Aut}(X)$ is a root subgroup with respect to $T$. Since $T$ normalizes $U$, we can define an action $\varphi: T \to \text{Aut}(U)$ of $T$ on $U$ given by $t.u = t \circ u \circ t^{-1}$ for all $t \in T$ and $u \in U$. Moreover, since $\text{Aut}(U) \simeq \mathbb{G}_m$, such an action corresponds to a character of the torus $\mu: T \to \mathbb{G}_m$, which does not depend on the choice of an isomorphism between $\mathbb{G}_m$ and $\text{Aut}(U)$. Such a character is called the weight of $U$. The subgroup of $\text{Aut}(X)$ generated by algebraic subgroups $T$ and $U$ is isomorphic to $\mathbb{G}_a \rtimes_{\mu} T$.

Assume that the algebraic torus $T$ acts linearly and rationally on a vector space $A$ of countable dimension. We say that $A$ is multiplicity-free if the weight spaces $A_{\mu}$ are all of dimension less or equal than one for every character $\mu: T \to \mathbb{G}_m$ of the torus $T$. In the proof of Theorem 1 we will use the following lemma that is due to Kraft:

**Lemma 3 (Lemma 5.2, [Kr17]).** Let $X$ be a normal affine variety and let $T \subset \text{Aut}(X)$ be a torus. If there exists a root subgroup $U \subset \text{Aut}(X)$ with respect to $T$ such that $O(X)^U$ is multiplicity-free, then $\dim T \leq \dim X \leq \dim T + 1$.

Let $X$ be an affine variety and consider a nontrivial algebraic action of $\mathbb{G}_a$ on $X$, given by $\lambda: \mathbb{G}_a \to \text{Aut}(X)$. If $f \in O(X)$ is a $\mathbb{G}_a$-invariant regular function, then the modification $f \cdot \lambda$ of $\lambda$ is defined in the following way (see [FK18, Section 8.3]):

$$(f \cdot \lambda)(s)x = \lambda(f(x)s)x$$

for $s \in \mathbb{C}$ and $x \in X$. This is again a $\mathbb{G}_a$-action. If $X$ is irreducible and $f \neq 0$, then $f \cdot \lambda$ and $\lambda$ have the same invariants. If $U \subset \text{Aut}(X)$ is a closed algebraic subgroup isomorphic to $\mathbb{G}_a$ and if $f \in O(X)^U$ is a $U$-invariant, then similarly as above we define the modification $f \cdot U$ of $U$. Pick an isomorphism $\lambda: \mathbb{G}_a \to U$ and set

$$f \cdot U = \{(f \cdot \lambda)(s) \mid s \in \mathbb{G}_a\}.$$

If $U \subset \text{Aut}(X)$ is a root subgroup with respect to $T$ and $f \in O(X)^U$ is a $T$-semi-invariant, then $f \cdot U$ is again a root subgroup with respect to $T$.

2.5. **Hyperbolic surfaces.** We will use the next two lemmas proved in [LRU22] in the proof of Theorem 1.

**Lemma 4 (Lemma 4.16, [LRU22]).** A non-toric $\mathbb{G}_m$-surface $S$ admits root subgroups of different weights if and only if $S$ is hyperbolic. Furthermore, in this case all root subgroups have different weights.

**Lemma 5 (Lemma 4.15, [LRU22]).** Let $S$ be a non-toric normal affine $\mathbb{G}_m$-surface and denote by $T \subset \text{Aut}(S)$ the subgroup isomorphic to $\mathbb{G}_m$ induced by the $\mathbb{G}_m$-action. Let $H \subset \text{Aut}(S)$ be an abelian subgroup containing only algebraic elements such that $T \subset H$. Then there exists a finite group $F$ such that $H \simeq T \times F$. 
3. Proof of Theorem 1

Assume \( \varphi: \text{Aut}(S) \to \text{Aut}(X) \) is an isomorphism of groups. Denote by \( T \) a one-dimensional subtorus of \( \text{Aut}(S) \), i.e. \( T \subset \text{Aut}(S) \) is an algebraic subgroup isomorphic to \( \mathbb{G}_m \). Further, denote by \( Z \) a maximal abelian subgroup of the centralizer of \( T \) in \( \text{Aut}(S) \) that contains \( T \). The group \( Z \) coincides with its centralizer in \( \text{Aut}(S) \) which in turn implies that \( Z \subset \text{Aut}(S) \) is closed (see Lemma 1). We claim that \( Z \) is a countable extension of \( T \). Indeed, otherwise if \( Z \) is an uncountable extension of \( T \), then the connected component \( Z^0 \) is again an uncountable extension of \( T \) (as \( Z^0 \subset Z \) is a countable index subgroup) and by Proposition 2 \( Z \) contains a two-dimensional commutative non-unipotent algebraic subgroup which is not possible as \( S \) is non-toric (see [LRU22, Lemma 4.17]). Further, since \( \varphi(Z) \) coincides with its centralizer in \( \text{Aut}(X) \), \( \varphi(Z) \subset \text{Aut}(X) \) is closed and \( \varphi(Z)^0 \) is a union of commutative algebraic groups (see Proposition 2).

By assumption \( S \) admits a regular \( \mathbb{G}_a \)-action. Hence, \( \text{Aut}(S) \) contains a root subgroup with respect to \( T \) (see for example [Lie10, Lemma 1.10]). Choose a root subgroup \( U \subset \text{Aut}(S) \) with respect to \( T \). Note that the weight of the root subgroup \( U \) is non-zero since \( S \) is non-toric (see [LRU22, Lemma 4.17]).

Claim 1. The subgroup \( \varphi(U) \subset \text{Aut}(X) \) is closed.

The subgroup \( U \subset \text{Aut}(S) \) is normalized by \( T \) and in particular, there is \( t_0 \in T \) such that

\[
t_0 \circ u \circ t_0^{-1} = u^2,
\]

where \( u \in U \). We claim that

\[
U = \{ u \in \text{Aut}(S) \mid t_0 \circ u \circ t_0^{-1} = u^2 \}.
\]

Indeed, if there is some \( h \in \text{Aut}(S) \setminus U \) such that \( t_0 \circ h \circ t_0^{-1} = h^2 \), then the group generated by \( t_0 \) normalizes the group generated by \( h \). Hence, \( T = \langle t_0 \rangle \) normalizes \( \langle h \rangle \). Observe that \( \langle h \rangle \) is one-dimensional algebraic subgroup since otherwise \( \langle h \rangle \) would contain a two-dimensional commutative non-unipotent algebraic subgroup which contradicts the assumption that \( S \) is non-toric (see [LRU22, Lemma 4.17]). Hence, \( \langle h \rangle^0 \simeq \mathbb{G}_a \) (see [LRU22, Lemma 4.10]). In another words, \( \langle h \rangle^0 \subset \text{Aut}(S) \) is the root subgroup with respect to \( T \). Since all root subgroups of \( \text{Aut}(S) \) with respect to \( T \) have different weights (see Lemma 4) we have that \( \langle h \rangle^0 \subset U \) which proves (1). The Claim 1 follows from (1).

Claim 2. \( \varphi(U) = \varphi(U)^0 \).

As \( T \) acts transitively on \( U \setminus \{ \text{id}_S \} \) by conjugations, it follows that \( \varphi(T) \) acts transitively on \( \varphi(U) \setminus \{ \text{id}_X \} \) by conjugations. Note that \( \varphi(U)^0 \setminus \{ \text{id}_X \} \) is a subset of \( \varphi(U) \setminus \{ \text{id}_X \} \) that is left invariant under the \( \varphi(T) \)-action. As \( \varphi(U) \) is uncountable and \( \varphi(U)^0 \) has countable index in \( \varphi(U) \), it follows that \( \varphi(U)^0 \setminus \{ \text{id}_X \} \) is non-empty. Hence, \( \varphi(U)^0 \setminus \{ \text{id}_X \} = \varphi(U) \setminus \{ \text{id}_X \} \) and thus \( \varphi(U)^0 = \varphi(U) \). This proves Claim 2.

Claim 3. \( \overline{\varphi(T)}^0 \) is isomorphic to \( \mathbb{G}_m \).

The subgroup \( \overline{\varphi(T)}^0 \subset \text{Aut}(X) \) is closed. Moreover, \( \varphi(T) \) acts on \( \varphi(U) \) by conjugations and hence, \( \overline{\varphi(T)}^0 \) acts on \( \varphi(U)^0 = \varphi(U) \) by conjugations. By Proposition 2 the subgroups \( \overline{\varphi(T)}^0, \varphi(U) \subset \text{Aut}(X) \) are the unions of commutative connected algebraic groups \( \bigcup_{i \geq 1} G_i \) and \( \bigcup_{i \geq 1} H_i \) respectively. Moreover, since \( \varphi(U) \) does not
contain elements of finite order, $H_i \simeq \mathbb{G}_a$ for each $i \geq 1$. Assume there is $k \in \mathbb{N}$ such that $G_k$ contains an algebraic subgroup $K \simeq \mathbb{G}_a$. Then it follows from the Lie-Kolchin Theorem (see [Hum75, §17.6]) that there is $l \in \mathbb{N}$ and $v \in H_l$ such that $K$ fixes $v$. Equivalently, $\varphi^{-1}(K)$ fixes $\varphi^{-1}(v) = u \in U$.

All elements of $\varphi^{-1}(K)$ are divisible which in turn implies that all elements of $\varphi^{-1}(K)$ are algebraic (see Proposition 3). Hence, by Lemma 5 the commutative subgroup $\varphi^{-1}(K) \subset \text{Aut}(S)$ is a subgroup of $T \times F$ for some finite group $F$. Since all divisble elements of $T \times F$ are contained in $T$ we conclude that $\varphi^{-1}(K) \subset T$. Further, because $\varphi^{-1}(K)$ fixes $\varphi^{-1}(v) = u \in U$ it follows that $T = \varphi^{-1}(K)$ acts trivially on $(u) = U$ which is not the case. Therefore, $G_i \simeq \mathbb{G}_a^r$ for some $r_i \in \mathbb{N}$. Moreover, since all elements of $\mathbb{G}_a^r$ are divisible, similarly as above, $\varphi^{-1}(G_i) \subset T$. Hence, there is no copy of $(\mathbb{Z}/p\mathbb{Z})^2$ in $\varphi^{-1}(G_i)$, where $p > 1$ and $i \in \mathbb{N}$. This implies that $G_i \simeq \mathbb{G}_m$ for any $i$ and we conclude that $\varphi(T) \simeq \mathbb{G}_m$.

Claim 4. $\varphi(T)$ is isomorphic to $\mathbb{G}_m$ and $\varphi(U)$ is isomorphic to $\mathbb{G}_a$.

We first claim that $\varphi(T) = \varphi(T)$. Indeed, since all elements of $\varphi^{-1}(\varphi(T))$ are divisible, by Proposition 3 they are all algebraic. By Lemma 5 the group $\varphi^{-1}(\varphi(T))$ is a subgroup of $T \times F$ for some finite group $F$. Moreover, since all divisble elements of $T \times F$ are contained in $T$ we conclude that $\varphi^{-1}(\varphi(T)) \subset T$ or equivalently $\varphi(T) \subset \varphi(T)$. Because both $\varphi(T)$ and $\varphi(T)$ act on $\varphi(U) \setminus \{\text{id}_X\}$ transitively with finite kernels, it follows that the subgroup $\varphi(T) \subset \varphi(T)$ has a finite index. Finally, because $\varphi(T)$ and $\varphi(T)$, seen as abstract groups, are isomorphic to $\mathbb{G}_m$, the torsion subgroups of $\varphi(T)$ and $\varphi(T)$ coincide and we conclude that $\varphi(T) = \varphi(T) \simeq \mathbb{G}_m$. Since $\varphi(T)$ acts on $\varphi(U) \setminus \{\text{id}_X\}$ transitively with a finite kernel, $\varphi(U)$ is a one-dimensional algebraic group which implies that $\varphi(U) \simeq \mathbb{G}_a$ (see [LRU22, Lemma 4.10]).

Claim 5. $\varphi(T)$ acts on the invariant ring $\mathcal{O}(X)^{\varphi(U)}$ multiplicity freely.

Assume towards a contradiction that $f, g \in \mathcal{O}(X)^{\varphi(U)}$ are linearly independent $\varphi(T)$-semi-invariants of the same $\varphi(T)$-weight. Hence, the subgroups $f \cdot \varphi(U)$ and $g \cdot \varphi(U)$ of $\text{Aut}(X)$ are root subgroups with respect to $\varphi(T)$ with the same weight. Then $\varphi^{-1}(f \cdot \varphi(U))$ and $\varphi^{-1}(g \cdot \varphi(U))$ are root subgroups with respect to $\varphi^{-1}(\varphi(T)) = T$. Indeed, $\varphi^{-1}(f \cdot \varphi(U)) \setminus \{\text{id}_S\}$ is a $T$-orbit and hence is a quasi-affine curve in $\text{Aut}(S)$. Therefore, $\varphi^{-1}(f \cdot \varphi(U)) = (\varphi^{-1}(f \cdot \varphi(U)) \setminus \{\text{id}_S\}) \circ (\varphi^{-1}(f \cdot \varphi(U)))$ is an algebraic subgroup of $\text{Aut}(S)$ that is normalized by algebraic torus $T$. Finally, by [LRU22, Lemma 4.10] $\varphi^{-1}(f \cdot \varphi(U)) \simeq \mathbb{G}_a$ and analogously $\varphi^{-1}(g \cdot \varphi(U)) \simeq \mathbb{G}_a$.

By Lemma 4 root subgroups $\varphi^{-1}(f \cdot \varphi(U))$ and $\varphi^{-1}(g \cdot \varphi(U))$ with respect to $T$ have different weights which means that $T$ acts on $\varphi^{-1}(f \cdot \varphi(U))$ and $\varphi^{-1}(g \cdot \varphi(U))$ with different kernels. This contradicts the assumption that $f \cdot \varphi(U)$ and $g \cdot \varphi(U)$ are $\varphi(T)$-root subgroups with the same weight. The claim follows.

Since $\varphi(T)$ acts on the invariant ring $\mathcal{O}(X)^{\varphi(U)}$ multiplicity freely, by Lemma 3 dim $X \leq 2$. Moreover, we claim that dim $X \neq 1$. Indeed, if dim $X = 1$, then since $X$ admits an action of $\varphi(U) \simeq \mathbb{G}_a$, $X$ is isomorphic to the affine line $\mathbb{A}^1$. Hence, $\text{Aut}(X = \mathbb{A}^1)$ is a two-dimensional algebraic group. But this is not possible since there is a non-trivial semi-invariant $f \in \mathcal{O}(S)^U$ and then $f \cdot U$ is a root subgroup of $\text{Aut}(S)$ with respect to $T$ that is different from $U$. Therefore, by Claim 5 $\varphi(U)$ and
$\varphi(f \cdot U)$ are root subgroups of $\text{Aut}(X)$ with respect to $\varphi(T)$, i.e. $\text{Aut}(X)$ contains a three-dimensional algebraic subgroup $\varphi(T) \times (\varphi(U) \times \varphi(f \cdot U))$. This proves that $\dim X = 2$. Finally, by [LRU22, Theorem 1.3] $X$ is non-toric and by [LRU22, Theorem 1.2] $X$ is a hyperbolic $\varphi(T) \simeq \mathbb{G}_m$-surface.

4. PROOF OF THEOREM 2

In the first paragraph of the proof of [Sie96, Proposition 1] it is shown that the Lie algebra of vector fields of a singular affine variety $X$ is non-simple. The same proof is suitable for the next statement. For the convenience of the reader we provide the detailed proof.

**Proposition 4.** Assume $X$ is a singular affine variety and assume that $\langle \text{LNV}(X) \rangle$ is non-trivial. Then the Lie algebra $\langle \text{LNV}(X) \rangle$ is not simple.

**Proof.** Denote by $I \subset \mathcal{O}(X)$ the ideal corresponding to the singular locus of $X$. By [Sei67, Theorem 5] any vector field $\mu \in \text{Vec}(X)$ preserves $I$, i.e. $\mu(I) \subset I$. Hence, $\mu(I^k) \subset I^k$ for any $k \in \mathbb{N}$. Denote by

$$\langle \text{LNV}(X, I^k) \rangle = \{ \nu \in \langle \text{LNV}(X) \rangle \mid \nu(\mathcal{O}(X)) \subset I^k \}.$$ 

This is the ideal of the Lie algebra $\langle \text{LNV}(X) \rangle$ since

$$[\nu, \mu](f) = \nu(\mu(f)) - \mu(\nu(f)) \in I^k,$$

where $f \in \mathcal{O}(X)$, $\nu \in \langle \text{LNV}(X, I^k) \rangle$ and $\mu \in \langle \text{LNV}(X) \rangle$. It is clear that for a big enough $k$, $\langle \text{LNV}(X, I^k) \rangle \subset \langle \text{LNV}(X) \rangle$ is proper. Finally, $\langle \text{LNV}(X, I^k) \rangle$ is non-zero. Indeed, assume $\mu$ a locally nilpotent vector field on $X$ and $f \in I$. Hence, there exists $l \in \mathbb{N}$ such that $\mu^l(f) = 0$ and $g = \mu^{-1}(f) \neq 0$. This means that $\mu(g) = 0$. Note that $g \in I$ by [Sei67, Theorem 5]. Therefore, $g^k \mu$ is a locally nilpotent vector field and $g^k \mu(\mathcal{O}(X)) \subset I^k$. □

**Proof of Theorem 2.** Let $\varphi: \text{Aut}(D_p) \to \text{Aut}(X)$ be an isomorphism of ind-groups. By Lemma 2 $\varphi$ induces the isomorphism of Lie algebras

$$d\varphi: \langle \text{LNV}(D_p) \rangle \to \langle \text{LNV}(X) \rangle.$$ 

By [LR20, Theorem 1] $\langle \text{LNV}(D_p) \rangle$ is simple. Hence, from Proposition 4 it follows that $X$ is smooth. Moreover, by Theorem 1 $\dim X = 2$. Now by [LR20, Theorem 1] $X$ is isomorphic to some $D_q$ for some polynomial $q$ and by [LR20, Theorem 3] $X$ is isomorphic to $D_p$. □

**Example 1.** Consider the product $\mathbb{A}^1 \times C$, where $C$ is a non-rational affine curve with the trivial automorphism group and no non-constant invertible regular functions. The surface $\mathbb{A}^1 \times C$ is a $\mathbb{G}_m$-surface of parabolic type as each point of curve $C$ is the fixed $\mathbb{G}_m$-point. Assume $Z$ is an affine variety with the trivial automorphism group that contains no rational curves and admits no non-constant invertible regular functions. We claim that $\text{Aut}(\mathbb{A}^1 \times C)$ and $\text{Aut}(\mathbb{A}^1 \times Z)$ are isomorphic as ind-groups.

Let $\varphi: \mathbb{A}^1 \times Z \to \mathbb{A}^1 \times Z$ be an automorphism of $\mathbb{A}^1 \times Z$. Assume $z \in Z$. The image $\varphi(\mathbb{A}^1 \times \{ z \})$ is a subvariety of $\mathbb{A}^1 \times Z$ isomorphic to $\mathbb{A}^1$. Since $Z$ does not contain rational curves, $\varphi(\mathbb{A}^1 \times \{ z \}) = \mathbb{A}^1 \times \{ z' \}$ for some $z' \in Z$. Therefore, $\varphi$ induces an automorphism of $Z$ which is trivial by assumption. So, $\varphi(x, z) = (\psi(x, z), z)$, where $\psi: \mathbb{A}^1 \times Z \to \mathbb{A}^1$ is a regular function. For each $z \in Z$, $\psi$ induces an isomorphism of $\mathbb{A}^1$. Hence, $\psi(x) = f(z)x + g(z)$, where $f, g \in \mathcal{O}(Z)$. Moreover,
since there are no non-constant invertible regular functions on $\mathbb{Z}$, $f \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

We conclude that

$$\text{Aut}(\mathbb{A}^1 \times \mathbb{Z}) = \{(x, z) \mapsto (fx + g(z), z) \mid f \in \mathbb{C}^*, g \in \mathcal{O}(\mathbb{Z})\} \simeq \mathbb{G}_m \ltimes \mathcal{O}(\mathbb{Z})$$

and analogously

$$\text{Aut}(\mathbb{A}^1 \times \mathbb{C}) = \{(x, c) \mapsto (fx + g(c), c) \mid f \in \mathbb{C}^*, g \in \mathcal{O}(\mathbb{C})\} \simeq \mathbb{G}_m \ltimes \mathcal{O}(\mathbb{C})$$

These two groups are isomorphic as ind-groups since there is an isomorphism of ind-groups $\phi: \mathcal{O}(\mathbb{Z}) \rightarrow \mathcal{O}(\mathbb{C})$. This proves the claim.

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