Universal lattices and property $\tau$

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Abstract
We prove that the universal lattices – the groups $G = \text{SL}_d(R)$ where $R = \mathbb{Z}[x_1, \ldots, x_k]$, have property $\tau$ for $d \geq 3$. This provides the first example of linear groups with $\tau$ which do not come from arithmetic groups. We also give a lower bound for the $\tau$-constant with respect to the natural generating set of $G$. Our methods are based on bounded elementary generation of the finite congruence images of $G$, a generalization of a result by Dennis and Stein on $K_2$ of some finite commutative rings and a relative property $T$ of $(\text{SL}_2(R) \ltimes R^2, R^2)$.

Introduction
The groups $\text{SL}_d(O)$, where $O$ is a ring of integers is in a number field $K$, have many common properties – they all have Kazhdan property $T$, a positive solution of the congruence subgroup problem and super rigidity. In [16], Y. Shalom conjectured that many of these properties are inherited from the group $\text{SL}_d(\mathbb{Z}[x])$. He called the groups $\text{SL}_d(\mathbb{Z}[x_1, \ldots, x_k])$ universal lattices, because they can be mapped onto many lattices in groups $\text{SL}_d(K)$ for different fields $K$. Almost nothing is known about the representation theory of these groups.

The main result in this paper is Theorem 5, which says that the universal lattices have property $\tau$, provided that $d \geq 3$. However unlike the classical lattices $\text{SL}_d(O)$ these groups do not have congruence subgroup property and have infinite (even infinitely generated) congruence kernel.

Let $G$ be a topological group and consider the space $\tilde{G}$ of all equivalence classes of unitary representations of $G$ on some Hilbert space $\mathcal{H}$. This space has a naturally defined topology, called the Fell topology, as explained in [12] §1.1 or [11] Chapter 3 for example. Let $1_G$ denote the trivial 1-dimensional representation of $G$ and let $\tilde{G}_0$ be the set of representations in $\tilde{G}$ which do not contain $1_G$ as a subrepresentation (i.e., do not have invariant vectors).

Definition 1 A group is $G$ is said to have Kazhdan property $T$ if $1_G$ is isolated from $\tilde{G}_0$ in the Fell topology of $\tilde{G}$.

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A discrete group with property $T$ is finitely generated: see [10]. Since in this paper we shall be concerned only with discrete groups we give the following equivalent reformulation of property $T$:

**Equivalent definition:** Let $G$ be a discrete group generated by a finite set $S$. Let $\rho : G \to U(\mathcal{H})$ be a unitary representation of the group $G$. A vector $v \in \mathcal{H}$ is called $\varepsilon$-invariant vector iff $\|\rho(s)v - v\| < \varepsilon\|v\|$ for all $s \in S$. Then $G$ has the Kazhdan property $T$ if there is $\varepsilon > 0$ such that every irreducible unitary representation $\rho : G \to U(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, which contains a $\varepsilon$-invariant vector is isomorphic to the trivial representation. The largest $\varepsilon$ with this property is called the Kazhdan Constant for $S$ and is denoted by $\mathcal{K}(G; S)$.

The property $T$ depends only on the group $G$ and does not depend on the choice of the generating set $S$, however the Kazhdan constant depends also on the generating set.

Property $T$ implies certain group theoretic conditions on $G$ (finite generation, FP, FAB etc) and can be used for construction of expanders from the finite images of $G$. For this last application the following weaker property $\tau$ (introduced by Lubotzky in [11]) is sufficient:

Let $\tilde{G}^f$ and $\tilde{G}_0^f$ denote the finite representations of $\tilde{G}$, resp. $\tilde{G}_0$ (i.e., the representations which factor through a finite index subgroup).

**Definition 2** A group is $G$ is said to have property $\tau$ if $1_G$ is isolated from $\tilde{G}^f$ in the induced Fell topology of $\tilde{G}^f$. Equivalently: the group $G$ with the profinite topology has property $T$.

The two definitions are equivalent because any continuous irreducible representation of $G$ in $U(\mathcal{H})$ is then finite.

Again, for a discrete finitely generated group $G$ there is an equivalent definition of property $\tau$:

Let $G$ be an discrete group generated by a finite\footnote{There are examples of groups with property $\tau$ which are not finitely generated. This definition can be modified to cover also this case. We are not going to need this because by a result of Suslin [18] the universal lattices are finitely generated.} set $S$. Then $G$ has the property $\tau$ if there exists $\varepsilon > 0$ such that for every finite irreducible unitary representation $\rho : G \to U(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (necessary finite dimensional), which contains an $\varepsilon$-invariant vector is trivial. The largest $\varepsilon$ with this property is called the $\tau$-constant and is denoted by $\tau(G; S)$.

Our approach to property $\tau$ is inspired by a paper by Shalom [16] which relates property $T$ to bounded generation. In this paper we will work only with the groups $\text{SL}_d(R)$, where $d > 2$ and $R$ is a finitely generated commutative ring. The arguments can be easily generalized to any high rank Chevalley group over $R$. It is also possible to extend some parts of the argument to Chevalley groups over noncommutative rings [9].

Let $R$ be a commutative (unital) ring, and for $i \neq j \in \{1, 2, \ldots, d\}$ let $E_{i,j}$ denote the set of elementary $d \times d$ matrices $\{\text{Id} + r \cdot e_{i,j} \mid r \in R\}$. Also set
\[ E = E(R) = \bigcup_{i \neq j} E_{i,j} \] and let \( EL(d; R) \) be the subgroup in \( GL_n(R) \) generated by \( E(R) \). By a result of Suslin we have that \( SL_d(R) = EL(d; R) \), i.e., that \( SK_1(d, R) = 1 \) in the case of \( d \geq 3 \) and \( R = \mathbb{Z}[x_1, \ldots, x_k] \); for \( d = 2 \) this is not true in general.

**Definition 3** The group \( G = EL(d; R) \) is said to have bounded elementary generation property if there is a number \( N = BE_d(R) \) such that every element of \( G \) can be written as a product of at most \( N \) elements from \( E(R) \).

Examples of \( R \) satisfying the above definition are rings of integers \( \mathcal{O} \) in number fields \( K \) (for \( d \geq 3 \)), see [3]. In this classical case this property is known as bounded generation because each group \( E_{i,j} \simeq (\mathcal{O}, +) \) is a product of finitely many cyclic groups.

The following theorem was proved in [10] and the method of its proof forms the basis of our results.

**Theorem 4** Suppose that \( d \geq 3 \), \( R \) is a \( k \)-generated commutative ring such that \( SL_d(R) = EL(d; R) \) has bounded elementary generation property. Then \( SL_d(R) \) has property \( T \) (as a discrete group). Moreover the Kazhdan constant \( K(G, S) \) is bounded from below by

\[ K(G, S) \geq \frac{1}{BE_d(R)22^k+1}, \]

for a specific generating set \( S \) (defined below).

The generating set \( S \) in Theorem [4] is defined as follows: Suppose that the ring \( R \) is generated by 1 and \( \alpha_1, \ldots, \alpha_k \in R \). Then \( S = S_{d,k} := F_1 \cup F_2 \), where \( F_1 = \{ \text{Id} \pm e_{i,j} \} \) is the set of \( 2(d^2 - d) \) unit elementary matrices, and

\[ F_2 = \{ \text{Id} \pm \alpha_l \cdot e_{i,j} \mid |i - j| = 1, 1 \leq l \leq k \}, \]

is the set of \( 4(d - 1)k \) elementary matrices with generators of the ring \( R \) next to the main diagonal.

A very interesting conjecture is whether the group \( G_{d,1} := SL_d(\mathbb{Z}[x]) \) and the other universal lattices have bounded elementary generation property. In view of Theorem [4] this would imply that \( G_{d,1} \) (and therefore all of its images which include \( SL_d(\mathcal{O}) \) for many rings of algebraic integers \( \mathcal{O} \)) has property \( T \). We are unable to say anything about property \( T \), but we shall prove that \( G_{d,1} \) at least has \( \tau \). More generally:

**Theorem 5** Let \( d \geq 3 \), \( k \geq 0 \) denote the ring \( \mathbb{Z}[x_1, \ldots, x_k] \) by \( R_k \). Then the universal lattice \( G_{d,k} := SL_d(R_k) \) has property \( \tau \).
Moreover we have the following explicit bound for the $\tau$-constant $\tau(G_{d,k}; S)$ with respect to the generating set $S_{d,k}$ as above:

$$\tau(G_{d,k}; S) > \frac{1}{(2d^2 + 18k + 30)22^{k+1}}$$

It is a nontrivial fact proved by Suslin [18] that the group $G_{d,k}$ is generated by the set $S_{d,k}$. However this is independent from our result because a group $G$ can have property $\tau$ and a positive $\tau$-constant with respect to some set $S$ which does not generate the group, it is sufficient that $S$ generates a dense subgroup (in the profinite topology) in $G$.

Let $\hat{G}_{d,k}$ be the profinite completion of the universal lattice $G_{d,k}$ and define the pro-elementary subgroup $\hat{E}_{i,j}$ to be the closure of $E_{i,j}$ in $\hat{G}_{d,k}$. Each $\hat{E}_{i,j}$ is isomorphic to the additive group of the profinite completion $\hat{R}$ of the ring $R$.

We shall prove

**Theorem 6** The profinite completion $\hat{G}_{d,k}$ is ‘boundedly pro-elementary’ generated: it is a finite product of the groups $\hat{E}_{i,j}$.

In fact $\hat{G}$ can be written as a product of at most $(3d^2 - d - 2)/2 + 18(k + 2)$ pro-elementary subgroups $\hat{E}_{i,j}$ in some fixed order.

In order to prove this Theorem we need Lemma 1.2 below on bounded elementary generation of $\text{SL}_d$ over a finite ring and the following result which may be of independent interest:

**Theorem 7** Let $\bar{R}$ be a finite commutative ring generated by $k$ elements. Then every element of $K_2(\bar{R})$ is a product of at most $k + 2$ Steinberg symbols $\{a, b\}$.

Once Theorem 6 has been proved, the general techniques from [16] are applied to prove that $\hat{G}$ has property $T$. As noted above this gives that $G$ has property $\tau$.

Most of the known examples of discrete finitely generated groups $G$ with property $\tau$ arise as lattices in higher rank semi-simple Lie groups. In particular they almost have property $T$ and ‘rigidity’ (even super-rigidity): their representation theory is controlled by the representations of the ambient Lie group. There are also many examples of ‘randomly presented’ hyperbolic groups with property $T$, e.g. [7, 20], but it is not known if their profinite completion is infinite.

The universal lattices seem to be the first residually finite ‘non-arithmetic’ groups with property $T$ discovered so far. They have infinitely (continuously) many irreducible (but not unitary) representations of fixed finite degree. In this light the question whether they have property $T$ is even more interesting.

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3The bound for the Kazhdan constant can be significantly improved, see [9].

4In some sense these groups are arithmetic subgroups of $\text{SL}_d(K)$ for some huge field $K$ like $\mathbb{R}(\langle x_1^{-1} \rangle \cdots \langle x_k^{-1} \rangle)$, however they are not cocompact subgroups and the standard theory of arithmetic groups can not be applied in this case.
Finally we remark that Theorem 5 gives a lower bound for the $\tau$-constant which is asymptotically $O(d^{-2}22^{-k})$ in $d$ and $k$. However it is possible to improve this estimate to $O \left( d^{-1/2}(1 + (k/d)^{3/2})^{-1} \right)$. The proof of this result involves some new ideas from \[8\], namely relative property T of of groups with big radical and use of generalized elementary matrices. If the reader is interested in the values of the Kazhdan constants and generalization of Theorem 5 to non commutative rings, he is encourage to go over the sequel of this paper \[9\]. It is interesting to note that using a different generating set it is possible to improve the the $\tau$-constant for $G_{d,k}$ to $O((1 + k/d)^{-1/2})$.

A few words about the structure of the rest of the paper:

Section 1 contains the proof of Theorem 5 modulo Lemma 1.2 (proved in Section 2), Theorem 7 (proved in Section 3) together with some technical results from \[10\]. We conclude with Section 4 where we investigate the implication Theorem 7 has to the structure of the congruence kernel of $G$ and prove Theorem 6.

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Notation: For the rest of this paper we will denote $R_k := \mathbb{Z}[x_1, \ldots, x_k]$ and $G_{d,k} := \text{SL}_d(R_k)$. We will fix a generating set $S_{d,k} = F_1 \cup F_2$ of the group $G_{g,k}$ where $F_1$ is the set of $2(d^2 - d)$ elementary matrices with $\pm 1$ off the diagonal and $F_2$ is the set $4k(d-1)$ elementary matrices $\text{Id} \pm x_n e_{ij}$ with $|i - j| = 1$. Unless we explicitly need to specify the number of generators of the ring $R_k$ or the size of matrices $G_{d,k}$ we will denote $R_k$ by $R$, $G_{d,k}$ by $G$ and $S_{d,k}$ by $S$. We will assume that all rings are commutative with 1.

1 Proof of Theorem 5

Let $\rho : G_{d,k} \to U(\mathcal{H})$ be a unitary representation of the universal lattice $G_{d,k}$ which factors through a finite index subgroup. Suppose that $v \in \mathcal{H}$ is an $\epsilon$-almost invariant unit vector for the set $S$, where $\epsilon \leq K_{d,k}$

The following result is Corollary 3.5 to Theorem 3.4 proved in \[10\].

Lemma 1.1 There is a constant $M(k) < 11.22^k$ with the following property: Let $(\rho, \mathcal{H})$ be a unitary representation of the group $G_{d,k} = \text{SL}_d(R_k)$. Let $v \in \mathcal{H}$ be a unit vector such that $||\rho(g)v - v|| < \epsilon$ for $g \in S_{d,k}$. Then for every elementary matrix $g = \text{Id} + re_{ij}$ we have $||\rho(g)v - v|| < 2M(k)\epsilon$.

We shall prove that the $\tau$-constant for the universal lattice $G_{d,k}$ is bounded from below by:

$$\tau(G_{d,k}; S_{d,k}) \geq K_{d,k} := [(3d^2 - d - 2) + 36(k + 2)]^{-1}M(k)^{-1}.$$
Applying Lemma \ref{lemma1} gives us that
\[ ||\rho(g)v - v|| < 2M(k)\epsilon \]
for every elementary matrix \( g \in E \).

Let \( H = \ker \rho < G \), this is a normal subgroup of finite index in \( G \). Define subgroups \( H_{i,j} \) of \( H \) and subsets \( U_{i,j} \) of the ring \( R \) for \( i \neq j \) as follows
\[ H_{i,j} := H \cap E_{i,j}; \quad U_{i,j} := \{ r \in R \mid 1 + r \cdot e_{i,j} \in H \}. \]

Using elements in the Weyl group we see that the subgroups \( E_{i,j} \) are pairwise conjugate in \( \text{SL}_d(R) \). This gives us that the sets \( U_{i,j} \) do not depend on the indices \( i,j \). Using commutation with a suitable elementary matrix we can see that \( U_{i,j} \) is also closed under multiplication by elements in \( R \) (this is valid only for \( d \geq 3 \), i.e., \( U = U_{i,j} \) is an ideal of the ring \( R \). This ideal is of finite index in \( R \) because \( H \) is a subgroup of finite index in \( G \).

Let \( \text{EL}(d;U) \) be the normal subgroup in \( G \) generated by \( H_{i,j} \):
\[ \text{EL}(d;U) := \langle H_{i,j} \rangle^G < H \]

Let \( SK_1(R,U;d) = G(U)/\text{EL}(d;U) \) where \( G(U) \) is the principal congruence subgroup of \( G \) modulo \( U \), i.e., the matrices in \( G \) congruent to \( \text{Id}_d \) modulo \( U \). We have the following diagram
\[
\begin{array}{cccc}
1 & \rightarrow & SK_1(R,U;d) & \rightarrow G/\text{EL}(d;U) & \rightarrow G(R/U) = \text{SL}_d(\bar{R}) & \rightarrow 1 \\
\downarrow & & \downarrow & & \downarrow \\
G/H & & & & \bar{1} \\
\end{array}
\]
where the row and the column are exact.

The next lemma is well known in the case of local fields, for general commutative rings is slightly stronger than Remark 10 in \cite{[6]}:

**Lemma 1.2** Let \( \bar{R} \) be a finite commutative ring. Then \( \text{SL}_d(\bar{R}) \) has uniform bounded elementary generation property. In fact every matrix in \( \text{SL}_d(\bar{R}) \) can be written as a product of \( (3d^2 - d - 2)/2 \) elementary matrices.

It remains to deal with the group \( SK_1(R,U;d) \). It is always a finite abelian group and it measures the departure of \( \text{EL}(d;U) \) from being a congruence subgroup. We note that while \( SK_1(R,U;d) \) is finite, it is not bounded in terms of \( d \) and \( R \) and can be arbitrarily large for some ideals \( U \) (unless \( R = \mathbb{Z} \), when \( SK_1(R,U;d) \) is always trivial), see Section \ref{section4}.

Nevertheless we have the following:
Lemma 1.3 Each element of the group $G(U)/EL(d; U) = SK_1(R, U; d)$ is a product of at most $18(k + 2)$ elementary matrices. In fact it is a product of at most $k + 2$ Steinberg symbols $\{a, b\}$, for $a, b$ invertible elements in $\bar{R}$, coming from the surjection $K_2(\bar{R}) \to SK_1(R, U; d)$ (to be defined in Section 3).

We obtain this Lemma as a corollary to Theorem 7 proved in Section 3.

To finish the proof of Theorem 5 we use an argument due to Y. Shalom from [16]: Let $g$ be any element in the group $G$. Using Lemmas 1.1 and 1.3 every $g \in G$ can be written as a product $g = h \prod_{s=1}^{N} u_s$ where $h \in EL(d; U) \subset H$ and $u_s \in E(R)$ are elementary matrices, here $N$ denotes the number $(3d^2 - d - 2)/2 + 18(k + 2)$. By Lemma 1.1

$$||\rho(g)v - v|| \leq ||\rho(h)v - v|| + \sum_{s=1}^{N} ||\rho(u_s)v - v|| < 0 + \sum_{s=1}^{N} 2M(k)\epsilon = \frac{(3d^2 - d - 2) + 36(k + 2)}{2} 2M(k)\epsilon.$$ 

Recall that $\epsilon \leq K_{d, k}$ i.e.,

$$||\rho(g)v - v|| < \frac{(3d^2 - d - 2) + 36(k + 2)}{2} 2M(k)K_{d, k} = 1.$$ 

This shows that every element in the group $G$ moves the vector $v$ by less than 1.

Let $V$ denote the the $G$-orbit of the vector $v$ in $H$. The center of mass $c_V$ of the set $V$ is invariant under the action of $G$ and is not zero because the whole orbit $V$ lies entirely in the half space $\{v' \in H \mid \Re(v, v') > 0\}$. Therefore $H$ contains a nontrivial $G$-invariant vector, which completes the proof that the group $G$ has property $\tau$.

Finally, the estimate for $M(k)$ show that the constant $K_{d, k}$ satisfies the inequality

$$K_{d, k} \geq \frac{1}{(2d^2 + 18k + 30) 2^{2k+1}}.$$ 

Theorem 5 is now proved, modulo Theorem 7 and Lemma 1.2.

2 Uniform bounded generation of $\text{SL}_d(\bar{R})$

Lemma 1.2 is a well-known result for fields and the same proof works for any local ring $\bar{R}$: Where the classical algorithm (described below) uses a nonzero ‘pivot’ field element in a row or column operation, the algorithm for $\bar{R}$ uses the corresponding element outside the maximal ideal $\bar{I}$ of $\bar{R}$, which is therefore a unit. This argument can be generalized to semi-local rings if one is careful
about the order of the elementary matrices which appear in the product. See also [6], where a more general result is proved for rings with Bass stable range at most 1.

It is well known that any finite commutative ring is semi-local. In fact:

**Theorem 2.1** Any commutative unital finite ring is a direct sum of local rings.

**Proof:** This is known as a generalized Chinese Reminder Theorem. The proof is by induction on the size of the finite ring $\bar{R}$. Let $\bar{J}$ be the Jacobson radical $\bar{R}$. Then $\bar{R}/\bar{J}$ is a finite direct product of finite fields. If it is a field there is nothing to prove because $\bar{J}$ is a maximal ideal and $\bar{R}$ is a local ring. Otherwise let $e_0 \in \bar{R}$ map to a nontrivial idempotent in $\bar{R}/\bar{J}$. Since $\bar{J}$ is nilpotent ideal we can find $e \in \bar{R}$ such that $e \equiv e_0 (\mod \bar{J})$ and $e$ is an (nontrivial) idempotent in $\bar{R}$. Then we have $\bar{R} = e\bar{R} \oplus (1 - e)\bar{R}$. Finally we can apply the induction hypothesis to the rings $e\bar{R}$ and $(1 - e)\bar{R}$. □

It is clear that if $\bar{R}$ is a direct sum $\bar{R} = \bar{R}_1 \oplus \bar{R}_2 \oplus \cdots \oplus \bar{R}_s$ then the linear group $\text{SL}_d(\bar{R})$ decomposes as a direct product

$$\text{SL}_d(\bar{R}) \cong \text{SL}_d(\bar{R}_1) \times \cdots \times \text{SL}_d(\bar{R}_s).$$

This observation together with Theorem 2.1 reduces the proof of Lemma 1.2 to the case of local rings $\bar{R}$, namely the following Lemma 2.2:

**Lemma 2.2** Let $\bar{R}$ be a local ring. Then $\bar{G} = \text{SL}_d(\bar{R})$ has bounded generation with respect to $\bigcup_{i,j} E_{i,j}(\bar{R})$. In fact $\bar{G}$ can be written as

$$\bar{G} = \prod_{s=1}^{(3d^2 - d - 2)/2} E(s),$$

where each of the $(3d^2 - d - 2)/2$ terms $E(s)$ in the product is one from the groups $E_{i,j}(\bar{R})$ (with entries in $\bar{R}$) in some fixed order.

**Proof:** Since $\bar{R}$ is a local ring Lemma 2.2 is a consequence of the familiar argument that a matrix $g \in \text{SL}_d(K)$ can be reduced by successive applications of row and column operation to the identity matrix: Each of these operations is in fact a multiplication by an elementary matrix from left or right.

The following well-known algorithm produces such decomposition:

Let $g \in \text{SL}_d(\bar{R})$. By multiplying with $d - 1$ elementary matrices to the right we can ensure that the last entry on the first row is an invertible element in $\bar{R}$. Using an extra multiplication to the right we can make the $d,d$ entry equal to 1, next with $d - 1$ left and $d - 1$ right multiplications by elementary matrices we can transform the matrix $g$ to an element in $\text{SL}_{d-1} \times 1$ (sitting in the top left corner of $\text{SL}_d$). Thus the reduction form $\text{SL}_d$ to $\text{SL}_{d-1}$ can be done using $3d - 2$ elementary matrices, and by induction using $(3d^2 - d - 2)/2$ we can transform any matrix to the identity.

This proves Lemma 2.2 and Lemma 1.2 follows. □
3 \( K_2 \) of finite commutative rings: Theorem \(^7\)

In this section we allow \( \bar{R} \) to be any commutative ring with 1. Let \( d > 2 \). In order to simplify the argument a bit we require that \( SL_d(\bar{R}) \) is generated by the elementary matrices \( E_{i,j}(r) \) with entries in \( \bar{R} \). This assumption is indeed true for any finite ring \( \bar{R} \) (Lemma \(^1\) is even stronger). Recall the definition of the Steinberg group \( St_d(\bar{R}) \):\(^5\)

\[\text{Definition 3.1} \quad St_d(\bar{R}) \text{ is the group generated by elements } x_{i,j}(r) \text{ for distinct } i,j \in \{1,2,\ldots,d\} \text{ and } r \in \bar{R} \text{ subject to the relations:} \]

1. \( x_{i,j}(u)x_{i,j}(v) = x_{i,j}(u + v) \),
2. \( [x_{i,j}(u), x_{j,k}(v)] = x_{i,k}(uv) \),
3. \( [x_{i,j}(u), x_{k,l}(v)] = 1 \)

for all distinct indices \( i,j,k,l \in \{1,2,\ldots,d\} \) and all \( u,v \in \bar{R} \).

We have an obvious surjection

\[\phi : St_d(\bar{R}) \to EL(d; \bar{R}) = SL_d(\bar{R})\]

given by \( x_{i,j}(u) \mapsto \text{Id} + ue_{i,j}(u) \). The kernel of \( \phi \) is of great interest in classical \( K \)-theory and related areas and is denoted by \( K_2(\bar{R};d) \).\(^6\) Unless we need to specify the size of the matrices we will denote \( K_2(\bar{R};d) \) by \( K_2(\bar{R}) \).

Some naturally defined elements of it are defined below:

For a unit \( u \) in \( \bar{R} \) and a pair of indices \( i,j \) consider the elements

\[w_{i,j}(u) := x_{i,j}(u)x_{j,i}(-u^{-1})x_{i,j}(u) \quad \text{and} \quad h_{i,j}(u) := w_{i,j}(u)w_{i,j}(-1).\]

For any units \( u,v \) we define the Steinberg symbol

\[\{u,v\}_{i,j} = h_{i,j}(uv)h_{i,j}(u)^{-1}h_{i,j}(v)^{-1} \in St_d(\bar{R}).\]

It can be shown that the element \( \{u,v\}_{i,j} \in St_d(\bar{R}) \) is independent on the choice of \( i,j \) and lies in the kernel of \( \phi \). This element in \( K_2(\bar{R}) \) is called a Steinberg symbol and is denoted by \( \{u,v\} \). It has the properties

1. bimultiplicative: \( \{uv,w\} = \{u,w\}\{v,w\} \),
2. skew-symmetric: \( \{u,v\} = \{-v,u\} \),
3. \( \{u,-u\} = 1 \)
4. \( \{u,1-u\} = 1 \)

\(^5\) Analogues of the Steinberg group can be defined for the other Chevalley groups, see \(^1\) and the references therein. The results from this section also hold in that case.

\(^6\) For a finite ring \( \bar{R} \) it is in fact independent of \( d \geq 3 \) by stability results in \(^6\), but we shall not need this.
for all units \( u, v, w \) (the last identity only if \( 1 - u \) is also a unit). See [12], Chapter 9 for details.

**Theorem 3.2 (Dennis and Stein, [4])** When \( \bar{R} \) is a semi-local commutative ring then \( \ker \phi = K_2(\bar{R}; d) \) is central in \( \text{St}_d(\bar{R}) \) and is generated by the Steinberg symbols \( \{a, b\} \) for all \( a, b \) invertible in \( R \).

Let \( \pi : \text{SL}_d(R)/\text{EL}(d; U) \to \text{SL}_d(\bar{R}) \) be the reduction modulo \( U \). Then \( \phi \) factors through \( \pi \).

Indeed, define a map \( \psi : \text{St}_d(\bar{R}) \to \text{SL}_d(R)/\text{EL}(d; U) \) by \( x_{i,j}(\bar{r}) \mapsto \text{Id} + r \cdot e_{i,j} \). This is well defined and extends to a homomorphism \( \psi \) with the property that \( \phi = \pi \circ \psi \). We thus see that \( \ker \pi = \text{SL}_d(U)/\text{EL}(d; U) = SK_1(R, U; d) \) is an image of \( K_2(\bar{R}; d) \) under \( \psi \) and is therefore central in \( \text{SL}_d(R)/\text{EL}(d; U) \) and generated by the images \( \psi(\{a, b\}) \) of symbols. Another way to see this is to use the exact sequence from classical \( K \)-theory cf. Theorem 6.2 in [13]:

\[
\cdots \to K_2(R; d) \to K_2(\bar{R}; d) \xrightarrow{\psi} SK_1(R, U; d) \to SK_1(R; d) \to SK_1(\bar{R}; d) \to \cdots
\]

and by the result of Suslin [18] \( SK_1(R) = SK_1(\bar{R}) = 1 \), hence our map \( \psi \) is surjective.

By definition each \( \psi(\{a, b\}) \) is a product of 18 images of elementary matrices in \( \text{SL}_d(R)/\text{EL}(d; U) \) (it is actually a product of only 13 images of elementary matrices because there are some cancelations) and therefore Lemma 1.3 follows from Theorem 7 in the Introduction.

The rest of this section is devoted to the proof of Theorem 7.

First, note that the functor \( K_2 \) respects direct sums of rings i.e.

\[
K_2(R_1 \oplus \cdots \oplus R_s) = K_2(R_1) \oplus \cdots \oplus K_2(R_s).
\]

Hence by Theorem 2.1 we may assume that \( \bar{R} \) is a finite local ring with a maximal ideal \( \bar{I} \).

**Lemma 3.3** If the finite local ring \( \bar{R} \) is an image of \( R_k = \mathbb{Z}[x_1, \ldots, x_k] \) then the maximal ideal \( \bar{I} \) in \( \bar{R} \) is generated by at most \( k + 1 \) elements.

**Proof:** Indeed \( \bar{I} \) is an image of a maximal ideal \( I \) of finite index in \( R_k = \mathbb{Z}[x_1, \ldots, x_k] \). If \( p \) is the characteristic of \( R/I \) then \( I/pR \) is a maximal ideal of \( \mathbb{F}_p[x_1, \ldots, x_k] \) and Theorem 24 of Chapter VII in [14] gives that \( I/pR \) is a \( k \)-generated ideal. This proves the claim. There is another, more conceptual, proof of this lemma based on the fact that any finite field has a presentation as a ring with 2 generators and 2 relations. \( \square \)

In their paper [4] Dennis and Stein proved several new identities for the Steinberg symbols besides the standard one we quoted:
Theorem 3.4 ([4] Proposition 1.1) Let \( v, p, q, r \) be elements of \( \bar{R} \) such that \( v, 1 - p, 1 - q, 1 - r, 1 - qv, 1 - pv, 1 - pqv, 1 - pq, 1 - pr, 1 - qr, 1 - pqr \) are units in \( \bar{R} \). Then the following identities hold in \( K_2(\bar{R}) \):

\[
\{v, 1 - pqv\} = \left\{ \frac{1 - qv}{1 - p}, \frac{1 - pqv}{1 - q} \right\} \left\{ \frac{1 - pv}{1 - q}, \frac{1 - pq}{1 - r} \right\} = 1
\]

and

\[
\left\{ \frac{1 - qr}{1 - p}, \frac{1 - pqr}{1 - q} \right\} \left\{ \frac{1 - pr}{1 - q}, \frac{1 - pqr}{1 - r} \right\} = 1
\]

The basic idea of the following construction is derived from [4], where the authors prove the \( K_2(\bar{R}) \) is a cyclic group if \( \bar{R} \) is a factor of a discrete valuation ring. We can not apply directly this result because we need a result which holds for any finite local ring and there are many finite local rings which do not arise from discrete valuation rings.

For a nonzero element \( a \in \bar{R} \) define the level \( l(a) \) to be the largest \( n \) such that \( a \in I^n \) and set \( l(0) = \infty \). In particular \( l(a) = 0 \) if and only \( a \) is a unit of \( \bar{R} \).

For \( n \in \mathbb{N} \), let \( K_2^{(n)} \) be the subgroup of \( K_2(\bar{R}; d) \) generated by all symbols \( \{a, b\} \) with \( a, b \) units such that \( l(a - 1) + l(b - 1) \geq n \).

Since \( I \) is nilpotent we have that \( K_2^{(n)} = \{1\} \) for all large \( n \). Clearly we have \( K_2^{(0)} = K_2(\bar{R}) \), then

\[
K_2(\bar{R}) = K_2^{(0)} \geq K_2^{(1)} \geq \cdots \geq \{1\}
\]

is a filtration of \( K_2(\bar{R}) \) terminating at \( \{1\} \).

Lemma 3.5 Let \( v \) be a unit of \( \bar{R} \), \( k \in \mathbb{N} \) and \( p, q, r \in I \).

(i1) If \( l(p) + l(q) \geq n \) then

\[
\{v, 1 - pqv\}\{1 - qv, 1 - p\}\{1 - pv, 1 - q\} \equiv 1 \mod K_2^{(n+1)}.
\]

(i2) If \( l(p) + l(q) + l(r) \geq n \) then

\[
\{1 - p, 1 - qr\}\{1 - q, 1 - pr\}\{1 - r, 1 - pq\} \equiv 1 \mod K_2^{(n+1)}.
\]

Proof: Using the basic properties of the Steinberg symbols we have

\[
\left\{ \frac{1 - qv}{1 - p}, \frac{1 - pqv}{1 - p} \right\} = \{1 - qv, 1 - pqv\} \left\{ \frac{1 - qv}{1 - p}, \frac{1}{1 - p} \right\} \times
\]

\[
\times \left\{ \frac{1}{1 - p}, \frac{1 - pqv}{1 - p} \right\} = \{1 - qv, 1 - pqv\} \{1 - p, 1 - qv\} \times
\]

\[
\times \{1 - p, 1 - pqv\}^{-1} \{1, 1 - pqv\}.
\]
The first and the third multiplier are in $K_2^{(n+1)}$. The forth multiplier is the identity if the characteristic of $\overline{R}/\overline{I}$ is 2 and in $K_2^{(n+1)}$ otherwise. This gives us
\[
\left\{ \frac{1 - pqv}{1 - p}, \frac{1 - puq}{1 - p} \right\} \equiv \{1 - p, 1 - qv\} \pmod{K_2^{(n+1)}}
\]
Applying this reduction in the identity (11) leads to the (i1). In the same way we can obtain (i2) from the identity (2). □

Identity (i2) from Lemma 3.5 has the following:

**Corollary 3.6** Let $r, q_1, q_2, ..., q_t \in \overline{I}$ and suppose that $l(r) + \sum_i l(q_i) \geq n$. Then we can write \( \{1 - q_1q_2\cdots q_t, 1 - r\} \) as
\[
\{1 - q_1q_2\cdots q_t, 1 - r\} = \prod_{i=1}^{t} \{1 - q_i, 1 - u_i\} \pmod{K_2^{(n+1)}}
\]
for some $u_i \in \overline{I}$ with $l(q_i) + l(u_i) \geq n$ for each $i = 1, 2, \ldots, t$.

Now, fix $\theta$ to be an element in $\overline{R}$ such that the image of $\theta$ generates the multiplicative group $\overline{R}/\overline{I}$, and let $a_j, j = 1, 2, \ldots, k + 1$ be some fixed set of generators of the ideal $\overline{I}$.

We claim that the identities above and the fact that the ideal $\overline{I}$ is nilpotent allow us to write (not uniquely) every Steinberg symbol $\{a, b\}$ as a product
\[
\{a, b\} = \{\theta, *\} \prod_{j=1}^{k+1} \{1 - a_j, *\}.
\]

Let $T$ denote the following subset of $K_2(\overline{R})$:
\[
T := \left\{ \{\theta, s_0\} \prod_{j=1}^{k+1} \{1 - a_j, s_j\} \mid s_j \in \overline{R}^*, s = 0, 1, \ldots, k \right\}.
\]

Bimultiplicativity of $\{ -, - \}$ implies that $T$ is a subgroup of $K_2(\overline{R})$.

We shall prove by induction on $n$ that any symbol $\{a, b\}$ can be written as $\{a, b\} = tu$ with $t \in T$ and $u \in K_2^{(n)}$. As $K_2^{(n)} = \{1\}$ for large enough $n$ this will prove the above claim and Theorem 7.

Now, every unit $a \in \overline{R}$ is congruent to $\theta^l$ for some integer $l$, and thus $a = \theta^l a'$ with $a' \in 1 + I$. Then
\[
\{a, b\} = \{\theta^l, b\} \{a', b\} = \{\theta, b'\} \{a', b\}
\]
and $\{a', b\} \in K_2^{(1)}$, giving the first step $n = 1$ of the induction.

Suppose our claim has been proved for $n - 1$. We need to express a symbol $\{a, b\}$ with $l(1 - a) + l(1 - b) = n$ in the required form modulo $K_2^{(n+1)}$. The
same argument from the base step \( n = 1 \) gives that it is sufficient to consider the case \( a = 1 - r, b = 1 - q \) with \( r, q \in I \) and \( l(r) + l(q) = n \). The ideal \( \overline{I} \) is generated by \( a_i \) modulo \( \overline{I}^2 \). Therefore \( r \) may be written as

\[ r \equiv \sum_{i=1}^{s} v_i a_{m_{1,i}} a_{m_{2,i}} \cdots a_{m_{t,i}} \mod \overline{I}^{t+1}, \]

where \( t = l(q) \) and the \( v_i \) are units in \( \overline{R} \). Now

\[ 1 - r \equiv \prod_{i=1}^{s} (1 - v_i a_{m_{1,i}} a_{m_{2,i}} \cdots a_{m_{t,i}}) \mod \overline{I}^{t+1} \]

giving that

\[ \{a, b\} \equiv \prod_{i=1}^{s} \{1 - v_i a_{m_{1,i}} a_{m_{2,i}} \cdots a_{m_{t,i}}, b\} \mod K_2^{(n+1)}. \]

By Corollary \( \Box \), each term \( \{1 - v_i a_{m_{1,i}} a_{m_{2,i}} \cdots a_{m_{t,i}}, b\} \) in the product on the right hand side is equal to

\[ \{1 - v_i a_{m_{1,i}}, 1 - u_{i,1}\} \prod_{j=2}^{t} \{1 - a_{m_{j,i}}, 1 - u_{i,j}\} \mod K_2^{(n+1)} \]

for the appropriate \( u_{i,j} \in \overline{I} \), \( (j = 1, 2, \ldots, t) \). These symbols are all in the required from, except maybe the first one: \( \{1 - v_i a_{m_{1,i}}, 1 - u_{i,1}\} \).

Recall that by Corollary \( \Box \) we have \( l(a_{m_{1,i}}) + l(u_{i,1}) \geq n \) and therefore (ii) of Lemma \( \Box \) (with \( v = v_i, p = u_{i,1}, q = a_{m_{1,i}} \)) gives

\[ \{1 - v_i a_{m_{1,i}}, 1 - u_{i,1}\} \equiv \{v_i, 1 - u_{i,1} a_{m_{1,i}}\}^{-1} \{1 - a_{m_{1,i}}, 1 - v_i u_{i,1}\} \mod K_2^{(n+1)}. \]

The unit \( v_i \) can be written as \( v_i \equiv \theta^{-w_i} \mod \overline{I} \) for some \( w_i \in \mathbb{Z} \). Now

\[ \{v_i, 1 - u_{i,1} a_{m_{1,i}}\}^{-1} \equiv \{\theta, 1 - u_{i,1} a_{m_{1,i}}\}^{w_i} \equiv \{\theta, (1 - u_{i,1} a_{m_{1,i}})^{w_i}\} \mod K_2^{(n+1)} \]

and we have expressed \( \{a, b\} \) as an element of \( T \) modulo \( K_2^{(n+1)} \).

This completes the induction and proves our claim and Theorem \( \Box \)

### 4 On the profinite completion of \( G \)

In this section we shall prove Theorem \( \Box \) and several basic properties of the congruence kernel of \( G \).

As noted in Section \( \Box \) Theorem \( \Box \) together with Lemma \( \Box \) imply that any finite image \( G/H \) of \( G \) is a product of at most \( N := (3d - d - 2)/2 + 18(k + 2) \) of
the images $E_{i,j}H$ of the elementary groups $E_{i,j}$ in some fixed order. We make this statement more precise:

There is a map $s \mapsto E(s) = E_{i,j} \in \{E_{i,j} \mid i \neq j\}$ for $s = 1, 2, \ldots, N$ such that

$G/H = \left( \prod_{s=1}^{N} E(s) \right) H$

for any normal subgroup $H$ of finite index in $G$.

Recall that $\hat{G}$ is the profinite completion of $G$ and $\hat{E}_{i,j} \simeq \hat{R}$ is the closure of $E_{i,j}$ in $\hat{G}$ (where $\hat{R}$ is the profinite completion of $R$).

Set $\hat{E}(s) = \hat{E}_{i,s,j} \subseteq \hat{G}$. It is a closed subgroup of $\hat{G}$ and the identity above shows that $\Pi := \prod_{s=1}^{N} \hat{E}(s)$ is dense in $\hat{G}$. However the set $\Pi$ is a finite product of closed sets in a compact Hausdorff group and is therefore closed. Hence $\Pi = \hat{G}$ and Theorem 6 is proved. □

Let $\hat{G}_c = \hat{G}_{d,k,c}$ be the congruence completion of $G_{d,k} = G_{d,k}$, i.e.,

$\hat{G}_c = \lim_{\leftarrow U \triangleleft R} G(G(U)) \simeq \lim_{U \triangleleft R} G(R/U) \simeq G(\hat{R})$.

We have natural surjection $p : \hat{G} \rightarrow \hat{G}_c$.

Define $C_{d,k} := \ker p$, this is called the congruence kernel of $G_{d,k}$. From the argument in Section 4 we see that

$C_{d,k} \simeq \lim_{U \triangleleft R} G(U)/EL(d, U) = \lim_{U \triangleleft R} SK_1(R, U; d)$.

The centrality of $K_2(R/U; d)$ in $St_d(R/U)$ (see [4]) gives that $C_{d,k}$ is central in $\hat{G}$. We thus have the first part of the following Theorem:

**Theorem 4.1** Suppose $d \geq 3$ and $k \geq 1$. The congruence kernel $C_{d,k}$ is a central closed subgroup of $\hat{G}$. Moreover for $d_1, d_2 > k + 2$ the abelian groups $C_{d_1,k} \simeq C_{d_2,k}$ are isomorphic and not finitely generated as profinite groups.

To prove the second claim note that the polynomial ring $R$ satisfies the Bass stable range condition $sr(R) \leq k + 2$ (see Definition 3.1 and Theorem 3.5 of Chapter V in [1]).

A result of Vaserstein [19], improving on [1] gives an isomorphism

$SK_1(R, U; d_1) \simeq SK_1(R, U; d_2) \simeq SK_1(R, U)$ (3)

for $d_i > k + 2$. Recall the exact sequence from Section 8

$K_2(R) \rightarrow K_2(R/U) \rightarrow SK_1(R, U) \rightarrow 1$

A classical result by Quillen [14] is that $K_2(R) \simeq K_2(\mathbb{Z}) \simeq \{\pm 1\}$. On the other hand there are various examples of finite images $R/U$ such that $K_2(R/U)$ has
large rank, e.g. if $R = \mathbb{Z}[x]$ and $U = \langle p^2, x^d \rangle$ then $K_2(R/U)$ is an elementary abelian $p$-group of rank $\geq l$ by [5], proof of Theorem 2.8. We conclude that $SK_1(R,U)$ can have arbitrary large rank and by [3] the same is true for the finite images $SK_1(R,U;d)$ of $C_{d,k}$. Therefore the congruence kernels $C_{d,k}$ are not finitely generated for $d > k + 2$ and $k \geq 1$, which proves the second claim.

We remark that Theorem [4] can be restated in the following form:

**Remark 4.2** The congruence kernel $C_{d,k}$ has finite width with respect to the Steinberg symbols $\{a,b\} \in \text{St}_d(\hat{R})$ with arguments $a, b \in \hat{R}^*$.

Theorem [4] implies the following corollary which gives a negative answer to Question 1.27 from [12].

**Corollary 4.3** Provided $d > k + 2$ and $k \geq 1$ the congruence completion $\hat{G}_{d,k}$ of the universal lattice $G_{d,k}$ is a profinite group with property $\tau$ which is not finitely presented.

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