Almost Complete Intersection Binomial Edge Ideals and Their Rees Algebras

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Abstract. Let $G$ be a simple graph on $n$ vertices and $J_G$ denote the corresponding binomial edge ideal in the polynomial ring $S = \mathbb{K}[x_1, ..., x_n, y_1, ..., y_n]$. In this article, we compute the second Betti number and obtain a minimal presentation of trees and unicyclic graphs. We also classify all graphs whose binomial edge ideals are almost complete intersection and we prove that the Rees algebra of their binomial edge ideal is Cohen-Macaulay. We also obtain an explicit description of the defining ideal of the Rees algebra of those binomial edge ideals.

1. Introduction

Let $G$ be a simple graph with $V(G) = [n] := \{1, ..., n\}$. Villarreal in [21] defined the edge ideal of $G$ as $I(G) := (x_ix_j : \{i, j\} \text{ is an edge of } G) \subset \mathbb{K}[x_1, ..., x_n]$. Herzog et al. in [7] and independently Ohtani in [16] defined the binomial edge ideal of $G$ as $J_G = (x_iy_j - x_jy_i : i < j \text{ and } \{i, j\} \text{ is an edge of } G) \subset \mathbb{K}[x_1, ..., x_n, y_1, ..., y_n]$. In the recent past, researchers have been trying to understand the connection between combinatorial invariants of $G$ and algebraic invariants of $I(G)$ and $J_G$. While this relation between $G$ and $I(G)$ is well explored (see for example [1] and the references therein), the connection between the properties of $G$ and $J_G$ are not very well understood, see [7, 10, 11, 12, 13, 19] for a partial list. It is known that the Rees algebra of an ideal encodes a lot of asymptotic properties of the ideal. In the case of monomial edge ideals, properties of their Rees algebra have been explored by several researchers (see [22] and the citations to this paper). In [22], Villarreal described the generators of the defining ideal of the Rees algebra of a graph. As a consequence of this, he proved that $I(G)$ is of linear type, i.e., the Rees algebra is isomorphic to the Symmetric algebra, if and only if $G$ is either a tree or a unicyclic graph with having a cycle of odd length. However, nothing much is known about the Rees algebra of binomial edge ideals. In this article, we initiate such a study.

A homogeneous ideal $I \subset S$ is said to be a complete intersection if $\mu(I) = \text{ht}(I)$, where $\mu(I)$ denotes the number of a minimal homogeneous generating set of $I$. It is said to be an almost complete intersection if $\mu(I) = \text{ht}(I) + 1$ and $I_p$ is a complete intersection for all minimal primes $p$ of $I$. It is known that for a connected graph $G$, $J_G$ is a complete intersection if and only if $G$ is a path, [5]. Rinaldo studied the Cohen-Macaulayness of certain subclasses of almost complete intersection binomial edge ideals, [18]. In this article, we characterize graphs whose binomial edge ideal is an almost complete intersection ideal. We prove that these are either a subclass of trees or a subclass of unicyclic graphs (Theorems 4.1, 4.2).

1

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the Rees algebra are known to be Cohen-Macaulay. In general, computing the depth of these blowup algebras is a non-trivial problem. If the ideal is generated by an almost complete intersection, the Cohen-Macaulayness of the Rees algebra and the associated graded ring are closely related by a result of Herrmann, Ribbe and Zarzuela (see Theorem 4.3). We prove that the associated graded ring and hence the Rees algebra of almost complete intersection binomial edge ideals are Cohen-Macaulay, (Theorems 4.4, 4.6). The Cohen-Macaulayness of the associated graded ring is proved using the above mentioned result of Herrmann et al, by proving that the ring $S/J_G$ has near maximal depth. For the computation of depth of $S/J_G$, we use a recent result of Conca and Varbaro, which says that for a homogeneous ideal $I$ in a polynomial ring $S$, if its initial ideal with respect to a term order is square-free, then $\text{depth}(S/I) = \text{depth}(S/\text{in}(I))$, [3]. With respect to the degree lexicographical order, the initial ideal of $J_G$ is known to be square-free for any graph $G$, [7]. Therefore, instead of computing $\text{depth}(S/J_G)$, we compute $\text{depth}(S/\text{in}(J_G))$ (Lemma 4.5).

Another problem of interest for commutative algebraists is to compute the defining ideal of the Rees algebra. Describing the defining ideal not only gives more insight into the structure of the Rees algebra, but it also helps in understanding other homological properties and invariants associated with the Rees algebra. For example, the maximal degree of a minimal generator of the defining ideal also serves as a lower bound for one of the most important homological and computational invariant, the Castelnuovo-Mumford regularity. In general, it is quite a hard task to describe the defining ideals of Rees algebras. Huneke proved that the defining ideal of the Rees algebra of an ideal generated by a $d$-sequence has a linear generating set, [8]. Such ideals are said to be of linear type. We prove that if $G$ is either a unicyclic graph of girth at least 4 or a tree and $J_G$ is an almost complete intersection ideal, then $J_G$ is generated by a $d$-sequence, (Theorem 4.9). We also prove that being almost complete intersection is not a necessary condition for the binomial edge ideal to have a generating set which is a $d$-sequence by proving that $J_{K_1,n}$ is generated by a $d$-sequence (Proposition 4.8). We then completely describe the defining ideals of the Rees algebras of almost complete intersection binomial edge ideals of trees and unicyclic graphs of girth at least 4, (Corollary 4.10, Remark 4.11).

It is known that for an ideal $I$ of linear type, the generators of the defining ideal of the Rees algebra can be obtained from the matrix of a minimal presentation of $I$, [9]. For describing the generating set of the defining ideal of Rees algebras, we compute a minimal presentation of ideals. In this process, we compute the second graded Betti numbers and generators of the second syzygy of $S/J_G$ when $G$ is a tree or a unicyclic graph, (Theorems 3.1 - 3.7). Here we do not assume that the binomial edge ideal is almost complete intersection. The main idea in the computation of the syzygies is the observation that we can get a minimal free presentation of $J_G$ from the mapping cone of two minimal free resolutions of certain associated binomial edge ideals. We use the same technique to obtain a minimal free resolution of $J_{K_1,n}$.

The article is organized as follows. The second section contains all the necessary definitions and notation required in the rest of the article. In Section 3, we describe the Betti numbers and syzygies of the binomial edge ideal of trees and unicyclic graphs. We study the Rees algebra of almost complete intersection binomial edge ideals in Section 4.
2. Preliminaries

Let \( G \) be a simple graph with the vertex set \([n]\) and edge set \( E(G)\). A graph on \([n]\) is said to be a complete graph, if \( \{i, j\} \in E(G) \) for all \( 1 \leq i < j \leq n \). The complete graph on \([n]\) is denoted by \( K_n \). For \( A \subseteq V(G) \), \( G[A] \) denotes the induced subgraph of \( G \) on the vertex set \( A \), that is, for \( i, j \in A \), \( \{i, j\} \in E(G[A]) \) if and only if \( \{i, j\} \in E(G) \). For a vertex \( v \), \( G \setminus v \) denotes the induced subgraph of \( G \) on the vertex set \( V(G) \setminus \{v\} \). A subset \( U \) of \( V(G) \) is said to be a clique if \( G[U] \) is a complete graph. A vertex \( v \) of \( G \) is said to be a simplicial vertex if \( v \) is contained in only one maximal clique. For a vertex \( v \), \( N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\} \) denotes the neighborhood of \( v \) in \( G \) and \( G_v \) is the graph on the vertex set \( V(G) \) and edge set \( E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\} \). The degree of a vertex \( v \), denoted by \( \deg_G(v) \), is \( |N_G(v)| \). A vertex \( v \) is said to be a pendant vertex, if \( \deg_G(v) = 1 \). Let \( c(G) \) denote the number of components of \( G \). A vertex \( v \) is called a cut vertex of \( G \) if \( c(G) < c(G \setminus v) \). For an edge \( e \) in \( G \), \( G \setminus e \) is the graph on the vertex set \( V(G) \) and edge set \( E(G) \setminus \{e\} \). An edge \( e \) is called a cut edge if \( c(G) < c(G \setminus e) \). Let \( u, v \in V(G) \) be such that \( e = \{u, v\} \notin E(G) \), then we denote by \( G_e \), the graph on vertex set \( V(G) \) and edge set \( E(G_e) = E(G) \cup \{\{x, y\} : x, y \in N_G(u) \text{ or } x, y \in N_G(v)\} \). A cycle is a connected graph \( G \) with \( \deg_G(v) = 2 \) for all \( v \in V(G) \). A graph is said to be a unicyclic graph if it contains exactly one cycle as a subgraph. A graph is a tree if it does not have a cycle. The girth of a graph \( G \) is the length of a shortest cycle in \( G \). A complete bipartite graph on \( m + n \) vertices, denoted by \( K_{m,n} \), is the graph having a vertex set \( V(K_{m,n}) = \{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_n\} \) and \( E(K_{m,n}) = \{\{u_i, v_j\} : 1 \leq i \leq m, 1 \leq j \leq n\} \). A claw is the complete bipartite graph \( K_{1,3} \). A claw \( \{u, v, w, z\} \) with center \( u \) is the graph with vertices \( \{u, v, w, z\} \) and edges \( \{\{u, v\}, \{u, w\}, \{u, z\}\} \). For a graph \( G \), let \( G_c \) denote the set of all induced claws in \( G \).

Let \( G \) be a graph on \([n]\). For an edge \( e = \{i, j\} \in E(G) \) with \( i < j \), we define \( f_e = f_{ij} = f_{ji} := x_i y_j - x_j y_i \). Let \( R = \mathbb{K}[x_1, \ldots, x_n] \) be a polynomial ring over an arbitrary field \( \mathbb{K} \) and \( M \) be a finitely generated graded \( R \)-module. Then \( M \) is said to be a finitely presented \( R \)-module if there exists an exact sequence of the form \( R^p \xrightarrow{\phi} R^q \xrightarrow{\psi} M \rightarrow 0 \). Then this exact sequence is called an \( R \)-presentation of \( M \). If \( \varphi(R^p) \subseteq \mathfrak{m} R^p \) and \( \psi(R^q) \subseteq \mathfrak{m} M \), where \( \mathfrak{m} \) is the unique homogeneous maximal ideal in \( R \), then this presentation is called a minimal presentation. Let

\[
0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-p - j)^{\beta_{p, p+j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0, j}(M)} \longrightarrow M \longrightarrow 0,
\]

be the minimal graded resolution of \( M \), where \( R(-j) \) is the free \( R \)-module of rank 1 generated in degree \( j \). The number \( \beta_{i,j}(M) \) is called the \((i, j)\)-th graded Betti number of \( M \). For \( T \subset [n] \), let \( \bar{T} = [n] \setminus T \) and \( c_T \) denote the number of components of \( G[\bar{T}] \). Also, let \( G_1, \ldots, G_{c_T} \) be the components of \( G[\bar{T}] \) and for every \( i \), \( G_i \) denote the complete graph on \( V(G_i) \). Let \( P_T(G) := (\bigcup_{i \in T} \{x_i, y_i\}, J_{G_1}, \ldots, J_{G_{c_T}}) \). Herzog et al. proved that \( J_G = \bigcap_{T \subseteq [n]} P_T(G) \), [7 Theorem 3.2]. This, in particular, implies that \( J_G \) is a radical ideal. A set \( T \subseteq [n] \) is said to have cut point property if, for every \( i \in T \), \( i \) is a cut vertex of graph \( G[\bar{T} \cup \{i\}] \). They showed that \( P_T(G) \) is a minimal prime of \( J_G \) if and only if either \( T = \emptyset \) or \( T \subset [n] \) has cut point property, [7 Corollary 3.9].
Mapping Cone Construction: For an edge \( e = \{i, j\} \in E(G) \), We consider the following exact sequence:

\[
0 \rightarrow \frac{S}{J_{G \setminus e} \ast f_e} \rightarrow \frac{S}{J_{G \setminus e} \ast f_e} \rightarrow \frac{S}{J_G} \rightarrow 0.
\]

By [15, Theorem 3.7], we have

\[
J_{G \setminus e} \ast f_e = J_{(G \setminus e) \ast e} + (g_{P,t} : P \text{ is a path of length } s + 1 \text{ between } i, j \text{ and } 0 \leq t \leq s),
\]

where for a path \( P : i, i_1, \ldots, i_s, j \), \( g_{P,t} = x_{i_1} \cdots x_{i_s} \) and for each \( 1 \leq t \leq s \), \( g_{P,t} = y_{t_1} \cdots y_{t_s}x_{i_{t+1}} \cdots x_{i_s} \). Let \((F_*, d^F_*)\) and \((G_*, d^G_*)\) be minimal \( S \)-free resolutions of \( S/J_{G \setminus e} \) and \( S/(J_{G \setminus e} \ast f_e) \) respectively. Let \( \varphi : (F_*, d^F_*) \rightarrow (G_*, d^G_*)\) be the complex morphism induced by the multiplication by \( f_e \). The mapping cone \((M(\varphi), \delta)\) is an \( S \)-free resolution of \( S/J_G \) such that \((M(\varphi))_i = F_i \oplus G_{i-1}\) and the differential maps are \( \delta_i(x, y) = (d^F_i(x) + \varphi_{i-1}(y), -d^G_{i-1}(y)) \) for \( x \in F_i \) and \( y \in G_{i-1}\). The mapping cone need not necessarily be a minimal free resolution. We refer the reader to [4] for more details on the mapping cone.

3. BETTI NUMBERS AND SYZYGIES OF BINOMIAL EDGE IDEALS

In this section, we describe the first Betti number and the first syzygy of binomial edge ideals of trees and unicyclic graphs. We first compute the second Betti number of \( S/J_G \), where \( G \) is a tree.

**Theorem 3.1.** Let \( G \) be a tree on \([n]\). Then

\[
\beta_2(S/J_G) = \beta_{2,4}(S/J_G) = \binom{n-1}{2} + \sum_{v \in V(G)} \binom{\text{deg}_G(v)}{3}.
\]

**Proof.** We prove this by induction on \( n \). If \( n = 2 \), then \( G = P_2 \) and \( J_G \) is a complete intersection so that \( \beta_2(S/J_G) = 0 \). Hence the assertion follows. We now assume that \( n > 2 \). Let \( e = \{u, v\}\) be an edge such that \( u \) is a pendant vertex. The long exact sequence of \( \text{Tor} \) in degree \( j \) component corresponding to the short exact sequence (1) is:

\[
\cdots \rightarrow \text{Tor}_{2,j}^S \left( \frac{S}{J_{G \setminus e} \ast f_e}, \mathbb{K} \right) \rightarrow \text{Tor}_{2,j}^S \left( \frac{S}{J_G}, \mathbb{K} \right) \rightarrow \text{Tor}_{1,j}^S \left( \frac{S}{J_{G \setminus e} \ast f_e}(-2), \mathbb{K} \right) \rightarrow \cdots
\]

Since \( e \) is a cut edge and \( u \) is a pendant vertex of \( G, (G \setminus e)_u = (G \setminus u)_v \sqcup \{u\} \). Thus it follows from [15, Theorem 3.7] that \( J_{G \setminus e} \ast f_e = J_{(G \setminus e) \setminus e} \). One can observe that

\[
\text{Tor}_{1,j} \left( \frac{S}{J_{(G \setminus e) \setminus e}(-2)}, \mathbb{K} \right) \simeq \text{Tor}_{1,j-2} \left( \frac{S}{J_{(G \setminus e) \setminus e}}, \mathbb{K} \right).
\]

Since \( G \setminus e = (G \setminus u) \sqcup \{u\} \), \( J_{G \setminus e} = J_{G \setminus u} \). Therefore by using induction, we obtain

\[
\beta_{2,4}(S/J_{G \setminus e}) = \binom{n-2}{2} + \sum_{w \in V(G) \setminus \{v\}} \binom{\text{deg}_G(w)}{3} + \binom{\text{deg}_G(v) - 1}{3}
\]

and \( \beta_{2,j}(S/J_{G \setminus e}) = 0 \) for \( j \neq 4 \). If \( j = 4 \), then

\[
\text{Tor}_{1,j-2} \left( \frac{S}{J_{G \setminus u}}, \mathbb{K} \right) = 0.
\]
Hence, \( \beta_{2,j}(S/J_G) = 0 \) if \( j \neq 4 \). Since, \( \beta_{2,2}(S/J_{G(u)}) = 0 \) and \( \beta_{1,4}(S/J_{G(v)}) = 0 \), we have \( \beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G(v)}) + \beta_{1,2}(S/J_{G(u)}) \). Now, \( \beta_{1,2}(S/J_{G(u)}) = |E((G \setminus u)_v)| = n - 2 + \binom{\deg_G(v)}{2} \). Hence, \( \beta_2(S/J_G) = \beta_{2,4}(S/J_G) = \left( \frac{n-1}{2} \right) + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} \).

We now describe the first syzygy of binomial edge ideals of trees. We crucially use the knowledge of the Betti numbers of \( J_G \) to compute a generating set. A tree on \([n]\) vertices has \( n - 1 \) edges. For convenience in writing the list of generators, we need some notation. For \( A \subseteq [n] \) and \( i \in A \), we define \( p_A(i) = |\{j \in A \mid j \leq i\}| \). The function \( p_A \) indicates the position of an element in \( A \) when the elements are arranged in the ascending order.

**Theorem 3.2.** Let \( G \) be a tree on \([n]\) vertices. Then the first syzygy of \( J_G \) is minimally generated by elements of the form

\[
\begin{align*}
(a) & \quad f_{i,j} e_{\{k,l\}} - f_{k,l} e_{\{i,j\}} \quad \text{where} \quad \{i,j\}, \{k,l\} \in E(G) \quad \text{and} \quad e_{\{i,j\}} : \{i,j\} \in E(G) \quad \text{is the standard basis of} \quad S^{n-1}; \\
(b) & \quad (-1)^{p_A(i)+p_A(j)+1} f_{k,l} e_{\{i,j\}} + (-1)^{p_A(i)+p_A(k)+1} f_{j,i} e_{\{i,k\}} + (-1)^{p_A(i)+p_A(l)+1} f_{j,k} e_{\{i,l\}}, \\
\end{align*}
\]

where \( A = \{i,j,k,l\} \in \mathcal{C}_G \) with center at \( i \).

**Proof.** From Theorem 3.1, we have \( \beta_{2,4}(S/J_G) = \left( \frac{n-1}{2} \right) + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} \) and \( \beta_{2,j}(S/J_G) = 0 \), for \( j \neq 4 \). Therefore the minimal presentation of \( J_G \) is of the form

\[
S(-4)^{\beta_{2,4}(S/J_G)} \xrightarrow{\varphi} S(-2)^{n-1} \xrightarrow{\psi} J_G \xrightarrow{} 0.
\]

Note that \( |\mathcal{C}_G| = \sum_{v \in V(G)} \binom{\deg_G(v)}{3} \). Since \( \beta_2(S/J_G) = \left( \frac{n-1}{2} \right) + |\mathcal{C}_G| \), we index the standard basis of \( S^{\beta_2(S/J_G)} \) accordingly.

Let

\[
\mathcal{S}_1 = \{ E_{\{i,j\}}, \{i,j\} : \{i,j\}, \{k,l\} \in E(G), i < j, k < l \quad \text{and} \quad (i,j) > \text{lex} (k,l) \}
\]

\[
\mathcal{S}_2 = \{ E_{\{j,k,l\}}^{i} : \{i,j,k,l\} \in \mathcal{C}_G \quad \text{with center at} \quad i \}
\]

and \( \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \) denote the standard basis of \( S^{\beta_2(S/J_G)} \). For a pair of edges \( \{i,j\}, \{k,l\} \in E(G) \), \( f_{i,j} f_{k,l} - f_{k,l} f_{i,j} = 0 \) gives a relation among the generators of \( J_G \). Let \( \{i,j,k,l\} \in \mathcal{C}_G \) be a claw with center at \( i \). Then, it can be easily verified that for \( A = \{i,j,k,l\} \),

\[
(-1)^{p_A(i)+p_A(j)+1} f_{k,l} f_{i,j} + (-1)^{p_A(i)+p_A(k)+1} f_{j,i} f_{i,k} + (-1)^{p_A(i)+p_A(l)+1} f_{j,k} f_{i,l} = 0,
\]

which gives another relation among the generators of \( J_G \). Define the maps \( \varphi \) and \( \psi \) as follows:

\[
\varphi( E_{\{i,j\}}, \{k,l\} ) = f_{i,j} e_{\{k,l\}} - f_{k,l} e_{\{i,j\}}; \\
\varphi( E_{\{j,k,l\}}^{i} ) = (-1)^{p_A(i)+p_A(j)+1} f_{k,l} e_{\{i,j\}} + (-1)^{p_A(i)+p_A(k)+1} f_{j,i} e_{\{i,k\}} + (-1)^{p_A(i)+p_A(l)+1} f_{j,k} e_{\{i,l\}}; \\
\psi( e_{\{i,j\}} ) = f_{i,j},
\]

where \( A = \{i,j,k,l\} \). Observe that \( \varphi(\mathcal{S}_1) \) is the collection of all the elements of type (a) in the statement of the Theorem and \( \varphi(\mathcal{S}_2) \) is the collection of all elements of type (b). Also, for any pair of edges \( \{i,j\}, \{k,l\} \) and a claw \( \{u,v,w,z\} \) with \( u \) as a center, we have \( \psi( \varphi( E_{\{i,j\}}, \{k,l\} ) ) = 0 \) and \( \psi( \varphi( E_{\{u,v,w,z\}}^{i} ) ) = 0 \). Since \( \beta_{2,j} = 0 \) for all \( j \neq 4 \), it follows that the first syzygy is generated in degree 4. Moreover, as \( \beta_2(S/J_G) = \beta_{2,4}(S/J_G) = |\mathcal{S}| \), to prove the assertion, it is enough to prove that the elements of \( \varphi(\mathcal{S}) \) are \( \mathbb{K} \)-linearly independent, equivalently, the columns of the matrix of \( \varphi \) are \( \mathbb{K} \)-linearly independent. For this, note that for each \( \{i,j\} \in E(G) \), the entries of the corresponding row are the coefficients of \( e_{\{i,j\}} \) in the expression for the images of elements in \( \mathcal{S} \) under \( \varphi \). The coefficient of \( e_{\{i,j\}} \) in \( \varphi( E_{\{i,j\}}, \{k,l\} ) \) or \( \varphi( E_{\{k,l\}}, \{i,j\} ) \) is \( \pm f_{k,l} \). Moreover, the entry will be zero in the column corresponding to \( \varphi( E_{\{u,v\}}, \{w,z\} ) \) for \( \{u,v\} \neq \{i,j\} \) and \( \{w,z\} \neq \{i,j\} \). Therefore, among the first \( \binom{n-1}{2} \) column
entries in the row corresponding to $e_{i,j}$, there will be $(n - 2)$ non-zero entries, namely the binomials corresponding to all the edges other than \{i, j\}. In $\varphi(E_u)$, the coefficient of $e_{i,j}$ is non-zero if and only if either $i = u$ and $j \in \{v, w, z\}$ or $j = u$ and $i \in \{v, w, z\}$. If $i = u$ and $j = v$ (similarly any one of the other three), then the coefficient of $e_{i,j}$ is $\pm f_{w,z}$. It may be noted here that $f_{w,z}$ does not correspond to an edge in $G$. Moreover, for two distinct basis elements $E^{u_1}_{\{v_1, w_1, z_1\}}$ and $E^{u_2}_{\{v_2, w_2, z_2\}}$, with the edge \{i, j\} in both the claws, then $\{u_1, v_1, w_1, z_1\} \setminus \{i, j\} \neq \{u_2, v_2, w_2, z_2\} \setminus \{i, j\}$. Hence the corresponding coefficients of $e_{i,j}$ in $\varphi(E^{u_1}_{\{v_1, w_1, z_1\}})$ and $\varphi(E^{u_2}_{\{v_2, w_2, z_2\}})$ will not be equal. From the above discussion one concludes that in the row corresponding to $e_{i,j}$, each nonzero entry is of the form $\pm f_{k,l}$ for some $k, l \in [n]$, $\{k, l\} \neq \{i, j\}$ and no two are equal. Therefore, the entries of this row can be seen as the minimal generating set of binomial edge ideal of a graph on $[n]$, possibly different from $G$ and hence they are $\mathbb{K}$-linearly independent. Therefore, the assertion follows. 

We now study the Betti numbers and syzygies of binomial edge ideal of unicyclic graphs. Let $G$ be a unicyclic graph on the vertex set $[n]$ of girth $m$. First, we compute $\beta_2(S/J_G)$, where $G$ is a unicyclic graph of girth 3.

**Theorem 3.3.** Let $G$ be a unicyclic graph on $[n]$ of girth 3. Let $v_1, v_2, v_3$ be the vertices of the cycle in $G$. Then

$$\beta_2(S/J_G) = \beta_{2,3}(S/J_G) + \beta_{2,4}(S/J_G) = 2 + \beta_{2,4}(S/J_G),$$

$$\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v \in V(G)} \left( \frac{\deg_G(v)}{3} \right) - \sum_{i=1,2,3} \deg_G(v_i) + 3.$$

**Proof.** We prove this by induction on $n$. By [19] Theorem 2.2, for any graph $G$, $\beta_{2,3}(S/J_G) = 2k_3(G)$, where $k_3(G)$ denotes the number of $K_3$’s appearing in $G$. If $n = 3$, then $G = K_3$ and hence the assertion follows from [19] Theorem 2.1. We now assume that $n > 3$. Let $e = \{u, v\}$ be an edge such that $u$ is a pendant vertex. Since $e$ is a cut edge and $u$ is a pendant vertex of $G$, $(G \setminus e)_v = (G \setminus u)_v \cup \{u\}$. Thus, $J_{G \setminus e} : f_e = J_{(G \setminus u)_v}$. Now we can conclude, by induction that $\beta_{2,3}(S/J_{G \setminus e}) = 2$ and

$$\beta_{2,4}(S/J_{G \setminus e}) = \binom{n-1}{2} + \sum_{w \in V(G \setminus \{v\})} \left( \frac{\deg_G(w)}{3} \right) + \left( \frac{\deg_G(v) - 1}{3} \right) - \sum_{i=1,2,3} \deg_G(v_i) + 3$$

and $\beta_{2,3}(S/J_{G \setminus e}) = 0$ for $j > 4$. If $j \neq 4$, then $\text{Tor}_{1, j-2} \left( \frac{S}{J_{(G \setminus u)_v}}, \mathbb{K} \right) = 0$. Hence, the long exact sequence [2] gives that $\beta_{2,j}(S/J_G) = 0$, if $j > 4$. Since $\beta_{2,2}(S/J_{(G \setminus u)_v}) = 0$ and $\beta_{1,4}(S/J_{G \setminus e}) = 0$, it follows from the long exact sequence [2] that $\beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G \setminus e}) + \beta_{1,2}(S/J_{(G \setminus u)_v})$.

If $v = v_i$ for some $i$, then $\beta_{1,2}(S/J_{(G \setminus u)_v}) = |E((G \setminus u)_v)| = |E(G)| - 1 + \left( \frac{\deg_G(v) - 1}{2} \right) - 1 = n - 2 + \left( \frac{\deg_G(v) - 1}{2} \right)$. Moreover, for this $i$, $\deg_{G \setminus e}(v_i) = \deg_{G}(v_i) - 1$. Hence we get the required expression for $\beta_{2,4}(S/J_G)$. If $v \neq v_i$, for any $i = 1, 2, 3$, then $\beta_{1,2}(S/J_{(G \setminus u)_v}) = |E((G \setminus u)_v)| = n - 1 + \left( \frac{\deg_G(v) - 1}{2} \right)$. Hence, $\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v \in V(G \setminus \{v\})} \left( \frac{\deg_G(v)}{3} \right) - \sum_{i=1,2,3} \deg_G(v_i) + 3$. 

We now compute the second graded Betti numbers $S/J_G$, where $G$ is a unicyclic graph of girth at least 4.

**Theorem 3.4.** If $G$ is a unicyclic graph on $[n]$ of girth $m \geq 4$, then

$$\beta_2(S/J_G) = \begin{cases} \beta_{2,4}(S/J_G), & \text{if } m = 4, \\ \beta_{2,4}(S/J_G) + \beta_{2,m}(S/J_G), & \text{if } m > 4, \end{cases}$$
Thus, for $m > 1$ follows:

\[ \beta_{2,4}(S/J_G) = \begin{cases} \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} & \text{if } m = 4, \\ \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} & \text{if } m > 4, \end{cases} \]

and $\beta_{2,m}(S/J_G) = m - 1$, if $m > 4$.

**Proof.** Let $e = \{u, v\}$ be an edge of the cycle in $G$. Then after removing the edge $e$, $G \setminus e$ becomes a tree. Therefore from Theorem 3.1, we have

\[ \beta_2(S/J_{G \setminus e}) = \beta_{2,4}(S/J_{G \setminus e}) = \binom{n - 1}{2} + \sum_{w \in V(G)} \binom{\deg_G(w)}{3} + \sum_{e \in \{u, v\}} \binom{\deg_G(w) - 1}{3}. \]

Note that $(G \setminus e)_e = ((G \setminus e)_e)_u$.

It follows from [15, Theorem 3.7] that $J_{(G \setminus e)_e} : f_e = J_{((G \setminus e)_e)_u} + I$, where

\[ I = (g_{P,t} : P : u, i_1, \ldots, i_s, v \text{ is a path between } u \text{ and } v \text{ in } G \setminus e \text{ and } 0 \leq t \leq s). \]

In $G \setminus e$, there is only one path between $u$ and $v$ and the corresponding $g_{P,t}$ has degree $m - 2$ for all $t$. Since, $\beta_{2,2}(S/J_{((G \setminus e)_e)_u}) = 0$ and $\beta_{1,4}(S/J_{(G \setminus e)_e}) = 0$, we have $\beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G \setminus e}) + \beta_{1,2}(S/J_{((G \setminus e)_e)_u}) + I)).$ For $m = 4$, $I = (y_2y_3, x_2y_3, x_2x_3)$. Therefore,

\[ \beta_{1,2}(S/(J_{(G \setminus e)_e} + I)) = 3 + |E((G \setminus e)_e)| = 3 + (n - 1) + \binom{\deg_G(v) - 1}{2} + \binom{\deg_G(u) - 1}{2}. \]

Hence,

\[ \beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G \setminus e}) + \beta_{1,2}(S/(J_{(G \setminus e)_e} + I)) = \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(u)}{3} + 3. \]

Also, $\beta_{2,j}(S/J_{G \setminus e}) = 0$ and $\beta_{1,j-2}(S/(J_{(G \setminus e)_e} + I)) = 0$, if $j \neq 4$. Therefore, $\beta_{2,j}(S/J_G) = 0$, if $j \neq 4$, $j \neq m$.

\[ \Tor_{1,j-2} \left( \frac{S}{J_{((G \setminus e)_e)_u} + I}, \mathbb{K} \right) = 0 \text{ and } \dim_{\mathbb{K}} \left( \Tor_{1,m-2} \left( \frac{S}{J_{((G \setminus e)_e)_u} + I}, \mathbb{K} \right) \right) = m - 1. \]

Hence it follows from the long exact sequence (2) that $\beta_{2,j}(S/J_G) = 0$, if $j \notin \{4, m\}$. Since, $\beta_{1,j}(S/J_{G \setminus e}) = 0$ for $j \neq 2$, we have

\[ \Tor_{1,m-2} \left( \frac{S}{J_{((G \setminus e)_e)_u} + I}, \mathbb{K} \right) \simeq \Tor_{2,m} \left( \frac{S}{J_G}, \mathbb{K} \right). \]

Thus, for $m > 4$, $\beta_{2,m}(S/J_G) = m - 1$. Now, $\beta_{1,2}(S/(J_{((G \setminus e)_e)_u} + I)) = |E(((G \setminus e)_e)_u)| = n - 1 + \binom{\deg_G(v) - 1}{2} + \binom{\deg_G(u) - 1}{2}$. Hence, $\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3}$. \hfill \Box

Now we obtain a generating set for the first syzygy of $J_{C_n}$ for $n \geq 4$. Let $G = C_n$ be a cycle on $[n]$ with edge set $E(C_n) = \{\{i, i+1\}, \{1, n\} : 1 \leq i \leq n - 1\}$.

**Theorem 3.5.** Let $C_n$ be the cycle on $n$ vertices, $n \geq 4$. Let $\{e_{k+1}, e_{1, n} : 1 \leq k \leq n - 1\}$ denote the standard basis of $S^n$ and $Y = y_1 \cdots y_n$. For $i = 1, \ldots, n - 1$, define $b_i \in S^n$ as follows:

\[ (b_i)_k = \frac{Y}{y_k y_{k+1}} \quad \text{for } 1 \leq k \leq n - 1, \quad (b_i)_n = \frac{Y}{y_1 y_n}. \]
Now, $S/J$ is the Koszul complex $(F, \varphi)$ be the minimal resolution of $S/J$.

Then the first syzygy of $J_{C_n}$ is minimally generated by

$$
\{ f_{k,l}e_{\{i,j\}} - f_{i,j}e_{\{k,l\}} : \{i, j\}, \{k, l\} \in E(C_n) \} \cup \left\{ \sum_{k=1}^{n-1} (b_{i})_{k}e_{\{k,k+1\}} - (b_{i})_{n}e_{\{1,n\}} : 1 \leq i \leq n - 1 \right\}.
$$

Proof. By [24, Corollary 16], $\beta_{2,4}(S/J_{C_n}) = \{ \frac{9}{(2)} \}$ if $n = 4$; $\beta_{2,2}(S/J_{C_n})$ is complete for $n > 4$. Therefore, we get the minimal presentation of $J_{C_n}$ in the form

$$
S(-4)(2) + S(-n)^{n-1} \rightarrow S(-2)^n \rightarrow J_{C_n} \rightarrow 0. \quad (3)
$$

Now we consider the following exact sequence

$$
0 \rightarrow \frac{S}{(f_{k,k+1} : 1 \leq k \leq n - 1)} : f_{1,n} \rightarrow \frac{S}{(f_{k,k+1} : 1 \leq k \leq n - 1)} \rightarrow J_{C_n} \rightarrow 0
$$

and apply the mapping cone construction. Note that $(f_{1,2}, \ldots, f_{n-1,n}) = J_{P_n}$ is complete intersection. Thus, the Koszul complex $(F, \varphi)$ gives the minimal free resolution for $S/(f_{1,2}, \ldots, f_{n-1,n})$. Let $\{e_{\{i,j\}}, e_{\{k,l\}} \mid \{i, j\} \neq \{k, l\} \in E(P_n)\}$ denote the standard basis of $S^{(n-1)}$ and $\{e_{\{j,j+1\}} \mid 1 \leq j \leq n - 1\}$ denote the standard basis of $S^{n-1}$. Set $d_{1}F(e_{\{j,j+1\}}) = f_{j,j+1}$ for $1 \leq j \leq n - 1$ and $d_{2}F(e_{\{i,j\}}, e_{\{k,l\}}) = f_{k,l}e_{\{i,j\}} - f_{i,j}e_{\{k,l\}}$ for $\{i, j\} \neq \{k, l\} \in E(P_n)$. It follows from [13, Theorem 3.7] that

$$(f_{1,2}, \ldots, f_{n-1,n}) : f_{1,n} = (f_{1,2}, \ldots, f_{n-1,n}) + (y_2 \cdots y_{n-1}, x_2y_3 \cdots y_{n-1}, \ldots, x_2 \cdots x_{n-1}).$$

Let $(G, d_{G})$ be the minimal resolution of $S/(f_{1,2}, \ldots, f_{n-1,n})(-2)$ with the differential maps given by $d_{1}F(E_{i,i+1}) = f_{i,i+1}$ for $1 \leq i \leq n - 1$ and $d_{2}F(E_{m}) = x_2 \cdots x_{m}y_{m+1} \cdots y_{n-1}$ for $1 \leq m \leq n - 1$, where $\{E_{i,i+1}, E_{m} : 1 \leq i \leq n - 1, 1 \leq m \leq n - 1\}$ denotes the standard basis of $G_1$. Clearly the map from $G_0$ to $F_0$ in the mapping cone complex is the multiplication by $f_{1,n}$. Define the map $\varphi : G_1 \rightarrow F_1$ by

$$
\varphi_1(E_{i,i+1}) = f_{1,n}e_{\{i,i+1\}} \quad 1 \leq i \leq n - 1,
$$

$$
\varphi_1(E_{m}) = 1 \leq m \leq n - 1,
$$

where $(b_{m})_{k}$’s are as defined in the statement of the Theorem. We show that the map $\varphi$ satisfy the property that for all $x \in G_1$, $d_{1}F(\varphi_1(x)) = f_{1,n} \cdot d_{1}G(x)$. It is enough to prove the property for the basis elements. Clearly $d_{1}F(\varphi_1(E_{i,i+1})) = f_{1,n}f_{i,i+1} = f_{1,n} \cdot d_{1}G(E_{i,i+1})$. Now $d_{1}F(\varphi_1(E_{i})) = d_{1}F(\sum_{k=1}^{n-1}(b_{k})_{k}e_{\{k,k+1\}}) = \sum_{k=1}^{n-1} \frac{y_k}{y_k y_{k+1}} f_{k,k+1}$. Note that $\sum_{k=1}^{n-1} \frac{y_k}{y_k y_{k+1}} f_{k,k+1}$. Now taking the summation over $k$, we get $d_{1}F(\varphi_1(E_{i})) = f_{1,n}y_2 \cdots y_{n-1} = f_{1,n} \cdot d_{1}G(E_{i})$. Let
Summing up these three terms together, we get
\[
\sum_{k=1}^{m-1} (b_m)_{k} f_{k,k+1} = Y \left[ \frac{x_2}{y_2} \ldots \frac{x_{m-1}}{y_{m-1}} \frac{x_{m+1}}{y_{m+1}} \left( \frac{x_1}{y_1} - \frac{x_m}{y_m} \right) \right],
\]
\[
(b_m)_{m} f_{m,m+1} = Y \left[ \frac{x_1}{y_1} \ldots \frac{x_{m-1}}{y_{m-1}} \left( \frac{x_m}{y_m} - \frac{x_{m+1}}{y_{m+1}} \right) \right],
\]
\[
\sum_{k=m+1}^{n-1} (b_m)_{k} f_{k,k+1} = Y \left[ \frac{x_2}{y_2} \ldots \frac{x_m}{y_m} \left( \frac{x_{m+1}}{y_{m+1}} - \frac{x_n}{y_n} \right) \right].
\]

Summing up these three terms together, we get
\[
d_1^F(\varphi_1(E_m)) = Y \left[ \frac{x_2}{y_2} \ldots \frac{x_m}{y_m} \left( \frac{x_1}{y_1} - \frac{x_n}{y_n} \right) \right] = x_2 \cdots x_m y_{m+1} \cdots y_{n-1} f_{1,n} = f_{1,n} \cdot d_1^G(E_m).
\]

Therefore by the mapping cone construction, we get a presentation of $J_{C_n}$ as
\[
F_2 \oplus G_1 \longrightarrow F_1 \oplus G_0 \longrightarrow J_{C_n} \longrightarrow 0.
\]

Since $F_2 \oplus G_1 \simeq S^{(2n)^{+}}$ and $F_1 \oplus G_0 \simeq S^n$ whose ranks coincide with the corresponding Betti numbers of $J_{C_n}$, we can conclude that this is a minimal presentation. Hence the first syzygy of $J_{C_n}$ is minimally generated by the images of the standard basis elements under the map $\Phi : F_2 \oplus G_1 \rightarrow F_1 \oplus G_0$, where $\Phi = \begin{bmatrix} d_2^F & \varphi_1 \\ 0 & -d_1^G \end{bmatrix}$. Then we have
\[
\Phi(e_{\{i,j\},\{k,l\}}) = d_2^F(e_{\{i,j\},\{k,l\}}) - f_{i,j} e_{\{k,l\}} - f_{k,l} e_{\{i,j\}} \text{ for } \{i,j\} \neq \{k,l\} \in E(P_n),
\]
\[
\Phi(E_{i,i+1}) = (\varphi_1 - d_1^G)(E_{i,i+1}) = f_{1,n} e_{\{i,i+1\}} - f_{i,i+1} e_{\{1,n\}} \text{ for } i = 1, \ldots, n-1, \text{ and }
\]
\[
\Phi(E_m) = \varphi_1(E_m) - d_1^G(E_1) = \sum_{k=1}^{n-1} (b_m)_{k} e_{\{k,k+1\}} - (b_m)_{n} e_{\{1,n\}} \text{ for } i = 1, \ldots, n-1.
\]

Hence the assertion follows.

We now describe a minimal generating set for the first syzygy of binomial edge ideals of unicyclic graphs. The syzygy structure is slightly different for unicyclic graphs of girth 3. We deal with that case first.

**Theorem 3.6.** Let $G$ be a unicyclic graph on $[n]$ of girth 3. Denote the vertices and the edges of the unique cycle of $G$ by $v_1 < v_2 < v_3$ and $e_1, e_2, e_3$ respectively. Let the standard basis of $S^n$ be denoted by $\{ e_{\{i,j\}} : \{i,j\} \in E(G), i < j \}$. Then the first syzygy of $J_G$ is minimally generated by the elements of the form
\[
(a) \ x_{v_1} e_{\{v_2,v_3\}} - x_{v_2} e_{\{v_1,v_3\}} + x_{v_3} e_{\{v_1,v_2\}}, \ y_{v_1} e_{\{v_2,v_3\}} - y_{v_2} e_{\{v_1,v_3\}} + y_{v_3} e_{\{v_1,v_2\}},
\]
\[
(b) \ f_{i,j} e_{\{p,l\}} - f_{p,l} e_{\{i,j\}}, \text{ where } \{i,j\}, \{p,l\} \in E(G) \text{ and } \{\{i,j\}, \{p,l\}\} \not\subset \{e_1, e_2, e_3\},
\]
\[
(c) \ (-1)^{p_{A(i)}+p_{A(j)}+1} f_{k,l} e_{\{i,j\}} + (-1)^{p_{A(i)}+p_{A(k)}+1} f_{j,l} e_{\{i,k\}} + (-1)^{p_{A(i)}+p_{A(l)}+1} f_{j,k} e_{\{i,l\}}, \text{ where } A = \{i,j,k\} \in C_G \text{ with center at } i.
\]

**Proof.** We prove by induction on $n = |V(G)| = |E(G)|$. For $n = 3$, $G$ is a complete graph i.e., $J_G$ is the ideal generated by the set of all $2 \times 2$ minor of a $2 \times 3$ matrix. Then it follows from Eagon-Northcott complex that the first syzygy of $J_G$ is minimally generated by
\[
\{ x_{v_1} e_{\{v_2,v_3\}} - x_{v_2} e_{\{v_1,v_3\}} + x_{v_3} e_{\{v_1,v_2\}}, y_{v_1} e_{\{v_2,v_3\}} - y_{v_2} e_{\{v_1,v_3\}} + y_{v_3} e_{\{v_1,v_2\}} \}.
\]
Now we assume that \( n > 3 \). From Theorem \[ \text{3.3} \] we know that the minimal presentation of \( J_G \) is of the form
\[
S(-4)^{\beta_2,4(S/J_G)} \oplus S(-3)^{\beta_2,3(S/J_G)} \xrightarrow{\varphi} S(-2)^n \xrightarrow{\psi} J_G \to 0,
\]

where \( \beta_2,4(S/J_G) = \binom{n}{2} + \sum_{v \in V(G)} \left( \deg_G(v) \right) \) and \( \beta_2,3(S/J_G) = 2 \).

Let \( e = \{ u, v \} \) be an edge in \( G \) such that \( u \) is a pendant vertex of \( G \). Since \( e \) is a cut edge and \( u \) is a pendant vertex of \( G \), \( (G \setminus e)_e = (G \setminus u)_v \cup \{ u \} \). Thus, \( J_{G \setminus e} : f_e = J_{(G \setminus u)_v} \). Since \( G \setminus e \) is also a unicyclic graph having the unique cycle of girth 3 and \( J_{G \setminus e} = J_{G \setminus u} \), by induction we get that the first syzygy of \( J_{G \setminus e} \) is generated by elements of the form

1. \( x_{v_1}e_{v_2,v_3} - x_{v_2}e_{v_1,v_3} + x_{v_3}e_{v_1,v_2} + y_{v_1}e_{v_2,v_3} - y_{v_2}e_{v_1,v_3} + y_{v_3}e_{v_1,v_2} \),
2. \( f_{i,j}e_{\{p,l\}} - f_{i,j}e_{\{i,j\}} \) where \( \{ i, j \}, \{ p, l \} \in E(G) \) and \( \{ i, j \}, \{ p, l \} \not\subseteq \{ e_1, e_2, e_3 \} \),
3. \( (-1)^{p_A(i)+p_A(j)+1}f_{i,j}e_{\{i,j\}} \) and \( (-1)^{p_A(i)+p_A(k)+1}f_{j,i}e_{\{i,k\}} \) where \( A = \{ i, j, k, l \} \in C_{G \setminus e} \) with center at \( i \).

Case-1: We assume that \( v \neq v_i \) for all \( 1 \leq i \leq 3 \). Now we apply the mapping cone construction to the short exact sequence \([\pi]\). Let \( (G, d^G) \) be a minimal free resolution of \( S/(J_{G \setminus e} : f_e) \). Then \( G_1 \simeq S^{\deg_G(G \setminus e)}(\deg_G(G \setminus e)) \). Also let \( (F, d^F) \) be a minimal free resolution of \( S/J_{G \setminus e} \). Then \( F_1 \simeq S^{\deg_G(G \setminus e)}(\deg_G(G \setminus e)) \) and \( F_2 \simeq S^{\deg_G(G \setminus e)}(\deg_G(G \setminus e)) \). By Theorem \[ \text{3.3} \] \( \beta_2(S/J_{G \setminus e}) = 2 + \beta_2(A(S/J_{G \setminus e})) \), where \( \beta_2(A(S/J_{G \setminus e})) = \binom{n-1}{2} + \sum_{v \in V(G) \setminus u} \deg_G(v) \). Set \( S_1 = \{ E_{\{i,j\}} : \{ i, j \} \in E(G \setminus u) \} \) and \( S_2 = \{ E_{\{i,j\}} : i, j \in N_G(G \setminus e) \} \). Then \( |S_1| = |E(G \setminus e)| = n - 1 \) and \( |S_2| = |E(G \setminus e) \setminus E(G \setminus u)| = \deg_G(G \setminus u) - 1 \). Let \( S_1 \cup S_2 \) denote the standard basis of \( G_1 \) and set \( d_1^G(E_{\{i,j\}}) = f_{i,j} \) for \( E_{\{i,j\}} \in S_1 \cup S_2 \). Also, let \( \{ e_{\{i,j\}} : \{ i, j \} \in E(G \setminus u) \} \) be the standard basis of \( F_1 \). By the mapping cone construction, the map from \( G_0 \to F_0 \) is multiplicity \( f_{u,v} \). Define \( \varphi_1 : G_1 \to F_1 \) by \( \varphi_1(E_{\{i,j\}}) = \left\{ \begin{array}{ll} f_{i,j}e_{\{i,j\}} & \text{if } E_{\{i,j\}} \in S_1, \\ (-1)^{p_A(i)+p_A(j)+1}f_{i,j}e_{\{i,j\}} + (-1)^{p_A(i)+p_A(j)+1}f_{i,j}e_{\{i,j\}} & \text{if } E_{\{i,j\}} \in S_2. \end{array} \right. \)

Then to prove that \( \varphi_1 \) is a lifting map from \( G_1 \) to \( F_1 \) in the mapping cone construction, it is enough to show that the corresponding diagram commutes i.e., \( d_1^F(\varphi_1(x)) = f_{u,v} \cdot d_1^G(x) \) for all \( x \in G_1 \). If \( i, j \in N_G(G \setminus u) \), then \( \{ v, u, i, j \} \) is an induced claw with center \( v \) and it can be easily seen that
\[
(-1)^{p_A(i)+p_A(j)+1}f_{i,j}f_{i,j} - (-1)^{p_A(i)+p_A(j)+1}f_{i,j}f_{i,j} - f_{i,j}f_{u,v} = 0.
\]

Therefore it follows that for \( E_{\{i,j\}} \in S_1 \cup S_2 \), \( d_1^F(\varphi_1(E_{\{i,j\}})) = f_{u,v} \cdot d_1^G(E_{\{i,j\}}) \). Hence the mapping cone construction gives a \( S \)-free presentation of \( J_G \), which is
\[
F_2 \oplus G_1 \to F_1 \oplus G_0 \to J_G \to 0. \tag{4}
\]

Since \( F_2 \oplus G_1 \simeq S^{\deg_G(G \setminus e)}(\deg_G(G \setminus e)) \) and \( F_1 \oplus G_0 \simeq S^n \), the above presentation is a minimal one.

Case-2: Let \( v = v_i \) for some \( 1 \leq i \leq 3 \). Assume that \( v = v_1 \). Then \( \{ v_2, v_3 \} \in E((G \setminus e) \setminus E(G \setminus e)) \). Hence \( \beta_1,2(S/J_{G \setminus u}) = \text{rank } G_1 = (n - 1) + \binom{\deg_G(G \setminus u) - 1}{2} - 1 \). Also, it follows from Theorem \[ \text{3.3} \] that
\[
\beta_2(S/J_{G \setminus e}) = 2 + \binom{n-1}{2} + \sum_{x \in V(G) \setminus u} \deg_G(x) - \sum_{i=1}^3 \deg_G(v_i) + 3.
\]
Note that $\deg_G(v_1) = \deg_{G \setminus e}(v_1) + 1$ and $\deg_G(x) = \deg_{G \setminus e}(x)$ for all $x \neq u$ and $x \neq v$. Substituting these values in the above expression and taking summation with rank $G_1$, we see that $\operatorname{rank} F_2 + \operatorname{rank} G_1 = \beta_S(G/J_G)$. Let $\mathcal{S}_1 = \{E_{(i,j)} : \{i,j\} \in E(G \setminus u)\}$ and $\mathcal{S}_2 = \{E_{(i,j)} : i,j \in N_G(v) \setminus u, \{i,j\} \neq \{v_2,v_3\}\}$. Define $\varphi_1 : G_1 \to F_1$ as in Case-1 and proceeding as in there, it can be proved that the mapping cone construction gives a minimal $S$-free presentation of $J_G$ as in [3]. The first syzygy is minimally generated by the images of the standard basis under the map $\Phi : F_2 \oplus G_1 \to F_1 \oplus G_0$ which is given by the matrix
\[
\begin{pmatrix}
\varphi_1 & G_{1} \\
0 & -d_{1}^{G}
\end{pmatrix}
\]. Now as done in the proof of Theorem [3,5] one concludes that the images under $\Phi$ are precisely the elements given in the assertion of the theorem. 

\section*{Theorem 3.7.}
Let $G$ be a unicyclic graph on $[n]$ of girth $m \geq 4$. Also let the vertex set of the unique cycle in $G$ be $\{1, \ldots, m\}$. Let $\{e_{i,j} : \{i,j\} \in E(G)\}$ denote the standard basis of $S^n$. Then the first syzygy of $J_G$ is minimally generated by elements of the form
\[
(a) \ f_{i,j}e_{[k,l]} - f_{k,l}e_{[i,j]}, \text{ where } \{i,j\}, \{k,l\} \in E(G),
\]
\[
(b) \ (-1)^{p_1(u)+p_1(v)+1}f_{u,v}e_{[u,v]} + (-1)^{p_1(u)+p_1(w)+1}f_{v,w}e_{[u,z]} + (-1)^{p_1(u)+p_1(w)+1}f_{u,x}e_{[u,w]},
\]
where $\{u,v,w,z\}$ forms a claw in $G$ with center $u$,
\[
(c) \ \sum_{k=1}^{m-1} (b_i)_k e_{[k,k+1]} - (b_i)_m e_{[1,m]}, \text{ where } 1 \leq i \leq m - 1, \text{ and } b_i's \text{ are as defined in Theorem } 3.4.
\]

\textbf{Proof.} We prove the assertion by induction on $n - m$. If $n = m$, then $G$ is a cycle and the result follows from Theorem [3,5]. Now we assume that $n > m$. From Theorem 3.4 we know that the minimal presentation of $J_G$ is of the form
\[
S^{\beta_2(S/J_G)} \to S^n \to J_G \to 0,
\]
where
\[
\beta_2(S/J_G) = \begin{cases} 
\beta_{24}(S/J_G) & \text{if } m = 4 \\
\beta_{24}(S/J_G) + \beta_{2,m}(S/J_G) & \text{if } m > 4, and \end{cases}
\]
\[
\beta_{24}(S/J_G) = \begin{cases} 
\frac{n}{2} + \sum_{v \in V(G)} \left( \frac{\deg_G(v)}{3} \right)^3 + 3 & \text{if } m = 4 \\
\frac{n}{2} + \sum_{v \in V(G)} \left( \frac{\deg_G(v)}{3} \right)^3 & \text{if } m > 4 \text{ and } \beta_{2,m}(S/J_G) = m - 1.
\end{cases}
\]

Let $e = \{u,v\}$ be an edge in $G$ such that $u$ is a pendant vertex of $G$. Since $e$ is a cut edge and $u$ is a pendant vertex of $G$, $(G \setminus e)_e = (G \setminus u)_e \sqcup \{u\}$. Thus, $J_G:e = J_G \setminus e = J_G \setminus u$. Since $G \setminus e$ is also a unicyclic graph having the unique cycle $C_m$ and $J_G \setminus e = J_G \setminus u$, by induction we get a minimal generating set of the first syzygy of $J_{G \setminus e}$ as
\[
(a) \ f_{i,j}e_{[k,l]} - f_{k,l}e_{[i,j]}, \text{ where } \{i,j\}, \{k,l\} \in E(G \setminus e),
\]
\[
(b) \ (-1)^{p_1(i)+p_1(j)+1}f_{k,l}e_{[i,j]} + (-1)^{p_1(i)+p_1(k)+1}f_{j,k}e_{[i,l]},
\]
where $\{i,j,k,l\}$ forms a claw in $G \setminus e$ with center at $i$,
\[
(c) \ \sum_{k=1}^{m-1} (b_i)_k e_{[k,k+1]} - (b_i)_m e_{[1,m]}, \text{ where } 1 \leq i \leq m - 1.
\]

Now we apply the mapping cone construction to the short exact sequence [1]. Let $(G_e,d_{G_e})$ and $(F_e,d_{F_e})$ be minimal free resolutions of $S/(J_G \setminus e) : f_e(-2)$ and $S/J_G : e$ respectively. Then
\[
G_1 \simeq S^{n-1+\left(\deg_G(e)-1\right)}, F_1 \simeq S^{n-1} \text{ and } F_0 \simeq S^{\beta_2(S/J_G \setminus e)}.
\]

Denote the standard basis of $G_1$ by $\mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = \{E_{(i,j)} : \{i,j\} \in E(G \setminus e)\}$ and $\mathcal{S}_2 = \{E_{(k,l)} : k,l \in N_G(v) \setminus u\}$. Note that $|\mathcal{S}_1| = n - 1$ and $|\mathcal{S}_2| = \left(\deg_G(e)-1\right)$. Set $d_{1}^{G_e}(E_{(i,j)}) = f_{i,j}$ for a basis element $E_{(i,j)}$. Also let $\{e_{(i,j)} : \{i,j\} \in E(G \setminus e)\}$ be the standard basis of $F_1$. By the mapping cone construction, the map from $G_0$ to $F_0$ is given by the
multiplication by \( f_e \). Now we define \( \varphi_1 \) from \( G_1 \) to \( F_1 \) by \( \varphi_1(\varphi_0) = f_e \cdot e_{(i,j)} \) for \( E_{(i,j)} \in S_1 \) and \( \varphi_1(\varphi_0) = (1-p_a)^{1+p_a(u-1)}f_u,0e_{(i,j)}(v,w) + (1-p_a)^{1+p_a(u-1)}f_u,w e_{(i,j)}(v,w) \) for \( G_{(i,j)} \in S_2 \). We need to prove that \( d^F(\varphi_1) = f_e \cdot d^G(\varphi_1) \) for any element \( x \in G_1 \). For a claw \( \{v, u, k, l\} \) with center at \( v \), we have the relation \( (1-p_a)^{1+p_a(u-1)}f_u,0e_{(i,j)}(v,w) + (1-p_a)^{1+p_a(u-1)}f_u,w e_{(i,j)}(v,w) \). This yields us the equality \( d^F(\varphi_1) = f_u,0e_{(i,j)}(v,w) \) for a basis \( E_{(i,j)} \) of \( G_1 \). So the mapping cone construction gives us a \( S \)-free presentation of \( J_G \) as

\[
F_2 \oplus G_1 \rightarrow F_1 \oplus G_0 \rightarrow F_0 \rightarrow J_G \rightarrow 0.
\]

Since \( F_2 \oplus G_1 \approx S^{\beta_2(S/J_G)} \) and \( F_1 \oplus G_0 \approx S^n \), this is a minimal free presentation. Hence the first syzygy of \( J_G \) is minimally generated by the images of basis elements under the map \( \Phi : F_2 \oplus G_1 \rightarrow F_1 \oplus G_0 \). Now the assertion can be proved just as done in the proof of Theorem 3.5. \( \Box \)

If \( e = \{u, v\} \) is a cut-edge in \( G \) such that both \( u \) and \( v \) are simplicial vertices, then the mapping cone construction on the exact sequence \( (\text{I}) \) gives a minimal free resolution of \( S/J_G \). However, this is not a necessary condition as we see below.

**Proposition 3.8.** The minimal free resolution of \( S/J_{K_{1,n}} \) is given by the mapping cone of \( S/J_{K_{1,n-1}} \) and \( S/J_{K_n} \). \( \Box \)

**Proof.** Let \( V(K_{1,n}) = \{1, \ldots, n, n+1\} \) with \( E(K_{1,n}) = \{i, n+1 \mid 1 \leq i \leq n\} \). For \( G = K_{1,n} \) and \( e = \{n, n+1\} \), note that \( J_{G/e} = J_{K_{1,n-1}} \) and \( J_{K_{1,n-1}} : f_e = J_{K_n} \). Since, \( K_{1,n} \) is a tree, it follows from \([5\text{, Theorem 1.1}] \) that \( \text{pd}(S/J_{K_{1,n}}) = n \). Also, by \([19\text{, Corollary 2.3}] \), \( \beta_{i,j}(S/J_{K_{1,n}}) = 0 \) for \( 1 \leq i \leq n \). Since, \( \text{reg}(S/J_{K_{1,n}}) = 2 \), \([20] \), and \( \text{reg}(S/J_{K_n}) = 1 \), \([19] \), \( \beta_{i,j}(S/J_{K_{1,n}}) = 0 \) for \( j \neq 2 \) and \( \beta_{i,j}(S/J_{K_n}) = 0 \) for \( j \neq 1 \). Corresponding to \( (\text{I}) \), we have the long exact sequence for all \( j \geq 1 \),

\[
\cdots \rightarrow \text{Tor}^S_{i,j}(S_{J_{K_{1,n-1}}}, \mathbb{K}) \rightarrow \text{Tor}^S_{i,j}(S_{J_{K_{1,n}}}, \mathbb{K}) \rightarrow \text{Tor}^S_{i,j}(S_{J_{K_{n-1}}}, \mathbb{K}) \rightarrow \cdots.
\]

Hence, \( \beta_{i,j}(S/J_{K_{1,n}}) = \beta_{i,j}(S/J_{K_{1,n-1}}) + \beta_{i,j-2}(S/J_{K_{n-1}}) \). If \( G \) denotes a minimal free resolution of \( S/J_{K_{1,n-1}} \), then the above equality implies that \( \beta_{i,j}(S/J_{K_{1,n}}) = rank F_i + rank G_{i-1} \). Hence, the mapping cone gives a minimal free resolution of \( S/J_{K_{1,n}} \). \( \Box \)

### 4. Rees Algebra

Let \( G \) be a graph on \( [n] \) and \( J_G \) be its binomial edge ideal. Let \( R = S[T_{i,j}] : \{i, j\} \in E(G) \) with \( i < j \). Let \( \delta : R \rightarrow S[t] \) be given by \( \delta(T_{i,j}) = f_{i,j}t \). Then \( \text{Im}(\delta) = \mathcal{R}(J_G) \) and \( \ker(\delta) \) is called the defining ideal of \( \mathcal{R}(J_G) \). We first characterize graphs whose binomial edge ideals are almost complete intersection. We begin by studying trees.

**Theorem 4.1.** If \( G \) is a tree which is not a path, then \( J_G \) is an almost complete intersection ideal if and only if \( G \) is obtained by adding an edge between two vertices of two paths.

**Proof.** Suppose \( G \) is obtained by adding an edge \( e \) between paths \( P_{n_1} \) and \( P_{n_2} \). Then \( J_{G/e} \) is a complete intersection ideal and \( J_G = J_{G/e} + f_eS \). Now, let \( g \in J_{G/e} : f_e^2 \), then \( g(f_e^2) \in J_{G/e} \) which further implies that \( g^2f_e^2 \in J_{G/e} \). Since, \( J_{G/e} \) is a radical ideal, therefore, \( g(f_e^2) \in J_{G/e} \). Hence, \( J_{G/e} : f_e^2 = J_{G/e} : f_e \). Thus, the desired result follows from \([6\text{, Proposition 1.3}] \).

Now, assume that \( G \) is not a graph obtained by adding an edge between two paths. Therefore, either there exists a vertex \( v \) such that \( \text{deg}_G(v) \geq 4 \) or there exist \( z, w \in V(G) \)
such that $\deg_G(z) \geq 3$, $\deg_G(w) \geq 3$ and \{z, w\} \notin E(G)$. Let $T = \{v\}$ in the first case and $T = \{z, w\}$ in the second case. By [7, Lemma 3.1], $ht(P_T(G)) = n - c(T) + |T|$. Since $z$ and $w$ are of degrees at least 3, \{z, w\} \notin E(G) and $G$ is a tree, $c(T) \geq 5$. Hence, $ht(P_T(G)) \leq n - 3$. Now, if $T = \{v\}$, then $c(T) \geq 4$ so that $ht(P_T(G)) \leq n - 3$. Note that in both cases $T$ has the cut point property so that $P_T(G)$ is a minimal prime. Thus $ht(J_G) \leq n - 3$. Since, $\mu(J_G) = n - 1$, $\mu(J_G) > ht(J_G) + 1$. Hence $J_G$ is not an almost complete intersection ideal. \hfill \square

Now that we have characterized the almost complete intersection trees, we move on to graphs containing cycles.

**Theorem 4.2.** Let $G$ be a connected graph on $[n]$ which is not a tree. Then $J_G$ is an almost complete intersection ideal if and only if $G$ is obtained by adding an edge between two vertices of a path or by attaching a path to each vertex of a $C_3$.

**Proof.** First assume that $J_G$ is an almost complete intersection ideal. Therefore, $\mu(J_G) = ht(J_G) + 1$. Since, $ht(J_G) \leq n - 1$, it follows that $\mu(J_G) \leq n$. Since, $G$ is not a tree, we have $\mu(J_G) = n$. Therefore, $G$ is a unicyclic graph and $ht(J_G) = n - 1$. Let $u$ be a vertex which does not belong to the unique cycle in $G$. If $\deg_G(u) \geq 3$, then for $T = \{u\}, P_T(G)$ is a minimal prime of $J_G$ of height $\leq n - 2$ which contradicts the fact that $ht(J_G) = n - 1$. Hence, $\deg_G(u) \leq 2$. Now, we claim that $\deg_G(u) \leq 3$, for every $u$ belongs to vertex set of the unique cycle in $G$. If $\deg_G(u) \geq 4$ for such a vertex $u$, then $G \setminus \{u\}$ has at least three components so that for $T = \{u\}, P_T(G)$ is a minimal prime of $J_G$ of height $\leq n - 2$ which is a contradiction. Hence $\deg_G(u) \leq 3$. If the girth of $G$ is $3$, then clearly it belongs to one of the category described in the theorem. We now assume that girth of $G$ is $\geq 4$. Suppose $u, v \in V(G)$ be two vertices of the unique cycle in $G$ with $\deg_G(u) = 3$ and $\deg_G(v) = 3$. If $\{u, v\} \notin E(G)$, then for $T = \{u, v\}, P_T(G)$ is a minimal prime of $J_G$ of height $\leq n - 2$ which is again a contradiction. Therefore $\{u, v\} \in E(G)$. Thus, the number of vertices of the cycle having degree three is at most 2 and if two vertices of the cycle have degree three, then they are adjacent. Therefore $G$ is obtained by adding an edge between two vertices of a path.

Now assume that $G$ is a graph obtained by adding an edge between two vertices, say $u$ and $v$, of a path. Let $e = \{u, v\}$. Observe that $J_{G_{\setminus e}}$ is a complete intersection ideal. Now, if $g \in J_{G_{\setminus e}} : f_e^2$, then $gf_e \in J_{G_{\setminus e}}$ which further implies that $g^2f_e^2 \in J_{G_{\setminus e}}$. Since, $J_{G_{\setminus e}}$ is a radical ideal, $gf_e \in J_{G_{\setminus e}}$. Hence, $J_{G_{\setminus e}} : f_e^2 = J_{G_{\setminus e}} : f_e$. Thus, it follows from [6, Proposition 1.3] that $J_{G_{\setminus e}}$ is an almost complete intersection ideal.

Now, suppose $G$ is a graph obtained by adding a path to each of the vertices of a $C_3$. Then, by [3, Theorem 1.1], $S/J_G$ is Cohen-Macaulay of dimension $n + 1$. Therefore, $ht(J_G) = n - 1 = \mu(J_G) - 1$. Now we have to prove that if $p$ is a minimal prime of $J_G$, then $(J_G)_p$ is a complete intersection ideal of $S_p$, i.e. $\mu((J_G)_p) = ht((J_G)_p) = n - 1$. Let $p$ be a minimal prime of $J_G$. It follows from [4, Corollary 3.9] that there exists $T \subset [n]$ having cut point property such that $p = P_T$. By Theorem 3.6 the minimal presentation of $J_G$ is

$$S(-4)^{\beta_2(S/J_G)} \oplus S(-3)^{\beta_2(S/J_G)} \xrightarrow{\varphi} S(-2)^n \rightarrow J_G \rightarrow 0.$$  

Moreover, the linear relations given in Theorem 3.6(a) show that $(x_{v_1}, y_{v_1}, x_{v_2}, y_{v_2}, x_{v_3}, y_{v_3}) \subset I_1(\varphi)$, the ideal generated by the entries of the matrix of $\varphi$. Now, if $I_1(\varphi) \subset p$, then $(x_{v_1}, y_{v_1}, x_{v_2}, y_{v_2}, x_{v_3}, y_{v_3}) \subset p$. Thus $\{v_1, v_2, v_3\} \subset T$, which is a contradiction to the fact that $T$ has cut point property. Therefore, $I_1(\varphi) \notin p$, and hence by [2, Lemma 1.4.8], $\mu((J_G)_p) \leq n - 1$. If $\mu((J_G)_p) < n - 1$, then by [14, Theorem 13.5], $ht(p) < n - 1$, which
is a contradiction. Thus, \( \mu((J_G)_p) = n - 1 \). Hence, \( J_G \) is an almost complete intersection ideal. \( \square \)

Below, we give representatives of four different types of graphs whose binomial edge ideals are almost complete intersection ideals.

We now study the Rees algebra of almost complete intersection binomial edge ideals. We prove that they are Cohen-Macaulay and we also obtain the defining ideals of these Rees algebras. We first recall a result which characterizes the Cohen-Macaulayness of the Rees algebra and the associated graded ring.

**Theorem 4.3.** [6 Corollary 1.8] Let \( A \) be a Cohen-Macaulay local (graded) ring and \( I \subset A \) be a (homogeneous) almost complete intersection ideal in \( A \). Then

1.  \( \text{gr}_A(I) \) is Cohen-Macaulay if and only if \( \text{depth}(A/I) \geq \dim(A/I) - 1 \).
2.  \( \mathcal{R}(I) \) is Cohen-Macaulay if and only if \( \text{ht}(I) > 0 \) and \( \text{gr}_A(I) \) is Cohen-Macaulay.

Therefore, in our situation, to prove that \( \mathcal{R}(J_G) \) is Cohen-Macaulay, it is enough to prove that \( \text{depth}(S/J_G) \geq \dim(S/J_G) - 1 \).

We first show that the associated graded ring and the Rees algebra of trees with almost complete intersection binomial edge ideals and of cycles are Cohen-Macaulay.

**Theorem 4.4.** If \( G = C_n \) for \( n \geq 3 \) or \( G \) is a tree such that \( J_G \) is an almost complete intersection ideal, then \( \text{gr}_S(J_G) \) and \( \mathcal{R}(J_G) \) are Cohen-Macaulay.

**Proof.** If \( G = C_n \), then it follows from [23 Theorem 4.5] that \( \dim(S/J_{C_n}) = n + 1 \) and \( \text{depth}(S/J_{C_n}) = n \). Moreover, by Theorem 4.2.1 \( J_{C_n} \) is an almost complete intersection ideal. Therefore by Theorem 4.3 \( \text{gr}_S(J_{C_n}) \) is Cohen-Macaulay and hence \( \mathcal{R}(J_{C_n}) \) is Cohen-Macaulay. Now, assume that \( G \) is a tree and \( J_G \) is almost complete intersection. It follows from [5 Theorem 2.1] and Theorem 4.1 that \( \text{depth}(S/J_G) = n + 1 = \dim(S/J_G) - 1 \). Hence by Theorem 4.3 \( \text{gr}_S(J_G) \) and \( \mathcal{R}(J_G) \) are Cohen-Macaulay. \( \square \)

4.1. **Discussion.** Suppose \( G \) is a unicyclic graph such that \( J_G \) is almost complete intersection. We may assume that \( G \) is not a cycle. If girth of \( G \) is 3, then by Theorem 4.2 and [5 Theorem 1.1], \( S/J_G \) is Cohen-Macaulay. Thus, \( \text{gr}_S(J_G) \) is Cohen-Macaulay and hence so is \( \mathcal{R}(J_G) \). Now, we assume that girth of \( G \) is at least 4 and \( n \geq 5 \).

Let \( G_1 \) and \( G_2 \) denote graphs on the vertex set \([n]\) with edge sets given by \( E(G_1) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-2, n-1\}, \{n-1, n\}, \{2, n-1\}\} \) and \( E(G_2) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{2, n\}\} \).

If \( G \) is a unicyclic graph on \([k]\), \( k \geq 5 \), which is not a cycle and having an almost complete intersection binomial edge ideal, then by Theorem 4.2 \( G \) is obtained by attaching a path to each of the pendant vertices of \( G_1 \) or \( G_2 \).

Let \( G \) denote the graph obtained by identifying the vertex 1 of \( G_i \) and a pendant vertex of \( P_m \). Then by [17 Theorem 2.7] \( \text{depth}(S/J_G) = \text{depth}(S_i/J_{G_i}) + \text{depth}(S_P/J_{P_m}) - 2 \),
where $S_i$ denotes the polynomial ring corresponding to the graph $G_i$ and $S_P$ denotes the polynomial ring corresponding to the graph $P_n$. Since $J_{P_n}$ is generated by a regular sequence of length $m - 1$, $\text{depth}(S_P/J_{P_n}) = m + 1$. Also $\text{dim}(S/J_G) = n + m$. Therefore, to prove that $\text{depth}(S/J_G) \geq n + m - 1$, it is enough to prove that $\text{depth}(S_i/J_{G_i}) \geq n$. Similarly, if $G$ is obtained by attaching a path each to two pendant vertices of $G_1$, then to prove $\text{depth}(S/J_G) \geq \text{dim}(S/J_G) - 1$, it is enough to prove that $\text{depth}(S_i/J_{G_i}) \geq \text{dim}(S_i/J_{G_i}) - 1$.

We now proceed to prove this.

Let $G$ be a graph with binomial edge ideal on $[n]$ and $J_G \subset S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We consider $S$ with lexicographical order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. It follows from [7, Theorem 2.1] that in $\mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, we get

$$\text{depth}(S/J_G) = \text{depth}(S/I_{G_1}) = \text{depth}(S/I_{G_2})$$

Now consider the graphs $G_1$ and $G_2$ as defined above. It follows from the labeling of the vertices of $G_1$ that the admissible paths in $G_1$ are the edges and the paths of the form $i, i-1, \ldots, 3, 2, n-1, n-2, \ldots, j$ with $2 \leq j - i \leq n - 4$. Similarly the admissible paths in $G_2$ are the edges and the paths of the form $i, i-1, \ldots, 3, 2, n, n-1, \ldots, j$ with $2 \leq j - i \leq n - 3$.

Consequently, the corresponding initial ideals are given by

$$\text{in}_<(J_{G_1}) = \left\{x_1y_2, \ldots, x_{n-1}y_n, x_2y_{n-1}, x_i x_{j+1} \cdots x_{n-1}y_2 \cdots y_{i-1}y_j : 2 \leq j - i \leq n - 4 \right\}$$

and

$$\text{in}_<(J_{G_2}) = \left\{x_1y_2, \ldots, x_{n-1}y_n, x_2y_{n-1}, x_i x_{j+1} \cdots x_{n-1}y_2 \cdots y_{i-1}y_j : 2 \leq j - i \leq n - 3 \right\}$$

We denote the monomials of degree $\geq 3$ by $v_1, \ldots, v_p$. We order these monomials such that $i < j$ if either $\deg v_i < \deg v_j$ or $\deg v_i = \deg v_j$ and $v_i >_{\text{lex}} v_j$. Set $J = (x_1y_2, \ldots, x_{n-1}y_n)$, $I_0(G_1) = J + (x_2y_{n-1})$, $I_0(G_2) = J + (x_2y_n)$ and, for $1 \leq k \leq p$, $I_k(G_i) = I_{k-1}(G_i) + (v_k)$ for $i = 1, 2$. Then $I_p(G_i) = \text{in}_<(J_{G_i})$ for $i = 1, 2$. We now compute the projective dimension, equivalently depth, of these ideals.

**Lemma 4.5.** For $0 \leq k \leq p$ and $i = 1, 2$, $\text{pd}(S/I_k(G_i)) \leq n$.

**Proof.** We prove the assertion by induction on $k$. If $k = 0$, then consider the following exact sequences:

$$0 \longrightarrow \frac{S}{J} \longrightarrow \frac{S}{I_0(G_1)} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{S}{J} \longrightarrow \frac{S}{I_0(G_2)} \longrightarrow 0.$$
Let \( 1 \leq \) generated by \( d \) depth\( J \). Then, \( S/J \) Cohen-Macaulay, by Theorem \( 4.3(b) \) and this happens if depth\( S \) is enough to prove that depth\( S/J \) \( -1 \). It follows from Discussion \( 4.1 \), to obtain the above inequality, it is enough to prove that pd\( S/I \) \( = n \). In a similar manner, using the exact sequence and the colon ideal, one can prove that pd\( S/I \) \( \leq n \). \( \square \)

Now we prove the Cohen-Macaulayness of the Rees algebra of almost complete intersection unicyclic graphs.

**Theorem 4.6.** If \( G \) is a unicyclic graph such that \( J_G \) is an almost complete intersection ideal, then \( \mathcal{R}(J_G) \) is Cohen-Macaulay.

**Proof.** To prove that \( \mathcal{R}(J_G) \) is Cohen-Macaulay, it is enough to prove that gr\( S(J_G) \) is Cohen-Macaulay, by Theorem \( 4.3(b) \) and this happens if depth\( S/J \) \( \geq \) dim\( S/J \) \( -1 \), by Theorem \( 4.3(a) \). It follows from Discussion \( 4.1 \) to obtain the above inequality, it is enough to prove that depth\( (S_i / J_G) \) \( \geq n \) for \( i = 1, 2 \). From \( 3 \) Corollary \( 2.7 \), we get depth\( (S_i / J_G) = \) depth\( (S_i / in_j(J_G)) \). It follows from Lemma \( 4.5 \) that depth\( (S_i / in_j(J_G)) = \) depth\( (S_i / J_{1,n}^i) \) \( \geq n \). Hence \( \mathcal{R}(J_G) \) is Cohen-Macaulay. \( \square \)

First note that since \( J_{P_n} \) is a complete intersection, the defining ideal of the \( \mathcal{R}(J_{P_n}) \) is generated by all \( 2 \times 2 \) minor of the matrix

\[
\begin{bmatrix}
  f_{1,2} & f_{2,3} & \cdots & f_{n-1,n} \\
 T_{\{1,2\}} & T_{\{2,3\}} & \cdots & T_{\{n-1,n\}}
\end{bmatrix}
\]

We now study binomial edge ideals which are of linear type. We first show that the \( J_{K_{1,n}} \) is of linear type. For this purpose, recall the definition of \( d \)-sequence.

**Definition 4.7.** Let \( R \) be a commutative ring. Set \( d_0 = 0 \). A sequence of elements \( d_1, \ldots, d_n \) is said to be a \( d \)-sequence if \( (d_0, d_1, \ldots, d_i) : d_{i+1}d_j = (d_0, d_1, \ldots, d_i) : d_j \) for all \( 0 \leq i \leq n-1 \) and for all \( j \geq i+1 \).

We refer the reader to the book \( [9] \) by Swanson and Huneke for more properties of \( d \)-sequences.

**Proposition 4.8.** The binomial edge ideal of \( K_{1,n} \) is of linear type.

**Proof.** Let \( K_{1,n} \) denote the graph on \( [n+1] \) with the edge set \( \{ \{i, n+1\} : 1 \leq i \leq n \} \). We claim that \( J_{K_{1,n}} \) is generated by the \( d \)-sequence \( d_1, d_2, \ldots, d_n \), where \( d_i = x_iy_{n+1} - x_{n+1}y_i \). Let \( 1 \leq i \leq n-1 \) and \( j \geq i+1 \). Let \( K_{i+1} \) denote the complete graph of the vertex set \( \{1, \ldots, i, n+1\} \). Then

\[
(d_0, d_1, \ldots, d_i) : d_{i+1}d_j = ((d_0, d_1, \ldots, d_i) : d_{i+1}) : d_j = J_{K_{i+1}} : d_j = J_{K_{i+1}},
\]

also \( (d_0, d_1, \ldots, d_i) : d_j = J_{K_{i+1}} \), where the last two equalities follow from \( [15] \) Theorem \( 3.7 \). Therefore, \( J_{K_{1,n}} \) is generated by \( d \)-sequence. Hence, by \( [9] \) Corollary \( 5.5.5 \), \( J_{K_{1,n}} \) is of linear type. \( \square \)

We now show that most of the almost complete intersection binomial edge ideals are generated by \( d \)-sequences.
Theorem 4.9. Let $G$ be either a tree or a unicyclic graph on $[n]$ of girth $m \geq 4$. If $J_G$ is an almost complete intersection ideal, then $J_G$ is generated by a $d$-sequence. In particular, $J_G$ is of linear type.

Proof. Suppose $G$ is a tree such that $J_G$ is almost complete intersection. Then by Theorem 4.1, $G$ is obtained by adding an edge between two paths, say $P_{n_1}$ and $P_{n_2}$. Let $e$ denote the edge between $P_{n_1}$ and $P_{n_2}$ and $f_e$ denote the corresponding binomial. Note that $G \setminus e$ is the disjoint union of two paths. Thus, by [5, Corollary 1.2], $J_{G \setminus e}$ is complete intersection. To prove that $J_G$ is generated by a $d$-sequence, it is enough to show that $J_{G \setminus e} : f_e^2 = J_{G \setminus e} : f_e$. But this has already been proved in the first paragraph of the proof of Theorem 4.1. Hence $J_G$ is generated by a $d$-sequence.

Assume now that $G$ is a unicyclic graph with unique cycle $C_m, m \geq 4$, such that $J_G$ is almost complete intersection. Then by Theorem 4.2, $G$ is obtained by adding an edge $e$ between two vertices of a path. If $f_e$ denotes the binomial corresponding to the edge $e$, then $J_{G \setminus e}$ is a complete intersection ideal and it follows from the proof of Theorem 4.2 that $J_{G \setminus e} : f_e^2 = J_{G \setminus e} : f_e$. Hence $J_G$ is generated by a $d$-sequence.

The second assertion that $J_G$ is of linear type is a consequence of [8, Theorem 3.1]. □

As a consequence, we obtain the defining ideal of the Rees algebra of binomial edge ideals of cycles.

Corollary 4.10. Let $\varphi : S[T_{1,n}, T_{i,i+1}] : i = 1, \ldots, n-1 \rightarrow R(J_{C_n})$ be the map defined by $\varphi(T_{i,j}) = f_{i,j}$. The defining ideal of $R(J_{C_n})$, the kernel of $\varphi$, is minimally generated by

$$\{ f_{i,j}T_{k,l} - f_{k,l}T_{i,j} : \{i,j\} \neq \{k,l\} \in E(G) \} \cup \left\{ \sum_{k=1}^{n-1} (b_i)_k T_{k,k+1} - (b_i)_n T_{1,n} : 1 \leq i \leq n-1 \right\},$$

where $b_i$’s are as defined in Theorem 3.5.

Proof. Let

$$S(-4)^{(2)} \oplus S(-n)^{n-1} \xrightarrow{\phi} S(-2)^n \rightarrow J_{C_n} \rightarrow 0$$

be the minimal presentation of $J_{C_n}$ given in the proof of Theorem 3.5. Since By Theorem 4.9, $J_{C_n}$ is of linear type, it follows from [9, Exercise 5.23] that the defining ideal of $R(J_{C_n})$ is generated by $TA$, where $A$ is the matrix of $\phi$ and $T = [T_{1,2}, \ldots, T_{n-1,n}, T_{1,n}]$. Hence the assertion follows directly from Theorem 3.5. □

Remark 4.11. Suppose $G$ is a unicyclic graph of girth $m \geq 4$ or a tree. If $J_G$ is almost complete intersection, then by Theorem 4.9, $J_G$ is of linear type. Therefore, as in Corollary 4.10, we can conclude that the defining ideal of $R(J_G)$ is generated by $TA$, where $T$ is the matrix consisting of variables and $A$ is the matrix of the presentation of $J_G$. Hence we obtain a minimal set of generators for the defining ideal of $R(J_G)$ by replacing the $e_{i,j}$’s by $T_{i,j}$’s in the list of generators given in the statements in Theorems 3.2, 3.7. In a similar manner, using Proposition 4.8 and using a minimal presentation of $J_{K_{1,n}}$, one can obtain the minimal generators of the defining ideal of the Rees algebra, $R(J_{K_{1,n}})$.

Remark 4.12. We have shown that if $G$ is a tree with an almost complete intersection binomial edge ideal $J_G$, then $J_G$ is of linear type. It would be interesting to know whether binomial edge ideals of trees, or more generally all bipartite graphs, are of linear type. Here we give an example to show that $J_G$ need not be of linear type for all bipartite graphs.
Let $G$ be the graph as given on the right. Then, it can be seen (for example, using Macaulay 2) that the defining ideal of $J_G$ is not of linear type. If $\delta : S[T_{i,j}] : \{i,j\} \in E(G) \rightarrow R(J_G)$ is the map given by $\delta(T_{i,j}) = f_{i,j}$, then $x_8T_{\{1,8\}}T_{\{3,8\}} - x_8T_{\{1,6\}}T_{\{3,6\}} + x_4T_{\{1,4\}}T_{\{3,6\}} - x_4T_{\{1,8\}}T_{\{3,4\}} + x_6T_{\{1,4\}}T_{\{3,6\}} - x_6T_{\{1,8\}}T_{\{3,4\}}$ is a minimal generator of $\ker(\delta)$.

It will be interesting to obtain an answer to:

**Question 4.13.** Classify all bipartite graphs whose binomial edge ideals are of linear type.

Note that the above bipartite graph is not a tree. We have enough experimental evidence to pose the following conjecture:

**Conjecture 4.14.**

1. If $G$ is a tree or a unicyclic graph, then $J_G$ is of linear type.
2. $R_s(J_{C_n}) = R_s(J_{C_n})$, where $R_s(J_{C_n})$ denote the symbolic Rees algebra of $J_{C_n}$.

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