Witt vectors, commutative and non-commutative

D. B. Kaledin

Abstract. A review of the classical construction of Witt vectors is presented, and some recent generalizations of it to the non-commutative case are described.

Bibliography: 37 titles.

Keywords: Witt vectors, Hochschild homology, de Rham–Witt complex.

Contents

Introduction 1
1. Classical theory 4
2. Non-commutative theory 7
3. Trace functors 11
   3.1. Trace maps 11
   3.2. Small categories 12
   3.3. Fibrations 13
   3.4. Edgewise subdivision 15
4. Hochschild homology 16
   4.1. Generalities about cyclic homology 16
   4.2. Twisting 18
   4.3. Comparison 19
5. Further developments and open questions 21
   5.1. Frobenius maps 21
   5.2. Multiplication 23
   5.3. Extending the definition 25
   5.4. Big Witt vectors 27
Bibliography 28

Introduction

Witt vectors occupy a curious place in the mathematical grand scheme of things: above the radar, but only just. Most mathematicians are vaguely aware that such a thing exists, and no more than that. Those who do encounter Witt vectors in the course of their own work often develop quite an attachment towards the subject, and
a bit of a personal viewpoint. Such an encounter can happen quite unexpectedly, and from many different directions, so that these viewpoints vary greatly. Somehow, the beautiful discovery of Witt refuses to take its richly deserved place in history and insists on staying alive; moreover, it even refuses to be pigeon-holed, and reveals a new side to itself every time one takes a new look.

The goal of this paper is to give an overview of some old results on Witt vectors and present some new ones. However, given the shifting nature of the subject, it is perhaps prudent to drop all pretence of objectivity right away and state clearly that what is presented here is the viewpoint of the author, as idiosyncratic and personal as anyone else’s. To get some context, we start with a bit of a historical overview, but again, this is not history in the proper sense: rather, a story.

The story starts in Germany in about 1936, with Oswald Teichmüller and Ernst Witt. Teichmüller discovered a new way to think about $p$-adic integers: instead of treating $\mathbb{Z}_p$ as a completion of $\mathbb{Z}$ whose elements are convergent sequences modulo an equivalence relation, he noted that there is a completely canonical identification between the set $\mathbb{Z}_p$ and the set $W(\mathbb{F}_p)$ of infinite sequences $(a_0, a_1, \ldots)$ of elements in the prime field $\mathbb{F}_p$. Witt then showed [37] how to turn the set $W(\mathbb{F}_p)$ into an abelian group and then into a commutative ring, by constructing explicit polynomials that provide addition and multiplication.

The key word here is ‘canonical’: Teichmüller’s representation of $\mathbb{Z}_p$ was independent of any choices, thus sufficiently natural, and it is this naturality that allows Witt universal polynomials to exist.

However, as a byproduct of the construction, Teichmüller and Witt actually obtained much more. Namely, Witt polynomials have integer coefficients, and thus make sense not only in $\mathbb{F}_p$ but in any commutative ring. So, although they did not think in these terms—which did not even exist at the time—what Teichmüller and Witt discovered was a functor from commutative rings to commutative rings.

For the next step, fast-forward twenty years and move to France, where modern algebraic geometry was being created. One of the driving forces behind this creation was the quest for a ‘Weil cohomology theory’ of algebraic varieties in characteristic $p$ that could be used to prove the Weil Conjectures. As a minimum requirement, such a theory should have coefficients of characteristic 0. An early attempt to construct it was due to Serre [35], who used exactly the functoriality of Witt vectors. By functoriality, one can do the construction locally with respect to the Zariski topology and endow an algebraic variety $X$ over a field of characteristic $p$ with the sheaf of rings $W(\mathcal{O}_X)$ that is in good cases has no $p$-torsion. Taking its cohomology groups and inverting $p$, one obtains a cohomology theory with coefficients in characteristic 0, and one can check whether it has the other required properties.

Unfortunately, it does not. However, in an alternative universe it is pretty clear where the story would go. The reason Serre’s cohomology theory did not work was that it was not topological: it was a characteristic-0 lifting of the cohomology groups $H^*(X, \mathcal{O}_X)$ and not of, say, the de Rham cohomology groups $H^*_{\text{DR}}(X)$. Thus, a natural thing to do was to find a functorial lifting $W\Omega^*_X$ of the whole de Rham complex $\Omega^*_X$, or in other words, to add higher-degree terms to $W(\mathcal{O}_X) = W\Omega^0_X$.

As we know, this is not what happened in real life. In the short run, a much more successful suggestion for Weil cohomology appeared in the form of étale cohomology of Grothendieck — this has coefficients in $\mathbb{Q}_l$, not $\mathbb{Q}_p$, but for the Weil Conjectures
this works just as well. Slightly later, Grothendieck turned his attention to de Rham cohomology and developed his theory of crystalline cohomology. For a variety in characteristic $p$, this has coefficients in $\mathbb{Z}_p$, works for the Weil Conjectures, and in fact gives a functorial characteristic-0 lifting of de Rham cohomology in a very precise sense. The price to pay is a very high-tech definition of crystalline cohomology: it requires the full force of topos theory, and does not provide cohomology groups in any explicit form.

However, our story does not end here—and for the next step, we need to fast-forward twenty more years, to the late 1970s. We stay in France, and surprisingly, we rejoin the alternative universe: using earlier work by Bloch [2] and input from Deligne, Illusie discovered that crystalline cohomology of a smooth algebraic variety $X$ can in fact be computed by a functorial lifting $W\Omega^*_X$ of the de Rham complex $\Omega^*_X$ to characteristic 0. This lifting is known as the *de Rham–Witt complex*.

What is remarkable here is that Bloch was in fact working with algebraic $K$-theory, not cohomology, and $K$-theory exists in much greater generality—it is defined for any associative ring $A$, with no commutativity assumptions. Moreover, it was known since 1962 that differential forms also exist in the same generality—by a theorem of Hochschild, Kostant, and Rosenberg, differential forms of degree $i$ on an affine smooth algebraic variety $X = \text{Spec } A$ are canonically identified with elements in the $i$th Hochschild homology group $HH_i(A)$, and Hochschild homology makes sense in the non-commutative case. But in the 1970s these ideas were pretty far outside the mainstream—in fact, Bloch was using Milnor $K$-theory that requires commutativity, since good technology for working with the full Quillen $K$-theory did not yet exist.

To get to the next step of the story, we have to skip twenty more years that saw, among other things, the discovery of cyclic homology by Connes and Tsygan, and the appearance of Non-Commutative Geometry as a separate subject. We also have to move to Denmark, and change the area from algebraic geometry to algebraic topology. A topological version of Hochschild homology was introduced by Bökstedt in 1985, and in 1992, in a groundbreaking paper [3], Bökstedt, Hsiang, and Madsen expanded it greatly in their theory of Topological Cyclic Homology and cyclotomic trace. Among a wealth of other results, the paper associates a certain spectrum $TR(A, p)$ to any ring spectrum $A$ and prime $p$. In particular, a ring $A$ is tautologically a ring spectrum, so $TR(A, p)$ is well-defined. Moreover, if $A$ is annihilated by $p$, then $TR(A, p)$ is an Eilenberg–Mac Lane spectrum, thus effectively a complex of abelian groups. The homotopy groups $\pi_*TR(A, p)$ are the homology groups of this complex, and it has been shown by Hesselholt [13], [15] that when $A$ is a commutative algebra over a perfect ring $k$ of characteristic $p$ with smooth spectrum $X = \text{Spec } A$ — that is, in the situation of the Hochschild–Kostant–Rosenberg Theorem — the groups $\pi_*TR(A, p)$ are naturally identified with the terms $H^0(X, W\Omega^*_X)$ of the de Rham–Witt complex. In particular, in degree 0 we recover the $p$-typical Witt vectors $W(A)$.

For a general associative $A$, $\pi_0TR(A, p)$ was constructed in a purely algebraic manner, again by Hesselholt [14], [16]; he called it the *group of non-commutative Witt vectors* $W(A)$. For higher homotopy groups, a purely algebraic construction is not known.
We are now running out of time to skip: fast-forwarding another twenty years brings us up more-or-less to today. What we want to present in this paper, then, is a sort of a postscript to the long story sketched above. For any associative unital algebra $A$ over a perfect field $k$ of characteristic $p$, the Hochschild–Witt homology groups $W\text{HH}_q(A)$ have been constructed recently in [26] and [27]. These ought to coincide with the homotopy groups of the spectrum $TR(A,p)$, but at present this has not been checked. What has been proved is two comparison theorems: one says that in the Hochschild–Kostant–Rosenberg situation, $W\text{HH}_q(A)$ is identified with de Rham–Witt forms, and the other says that in the general case, $W\text{HH}_0(A)$ is identified with the non-commutative Witt vectors $W(A)$ of Hesselholt.

The paper is organized as follows. We start with a brief overview of the classical story, presented in a certain way. This is §1. Then §2 shows that with very few strategically placed modifications, the classical theory gives much more. However, what it gives is not a theory for non-commutative rings: rather, we obtain a notion of ‘Witt vectors’ of a vector space. Here we actually follow the approach sketched in [22]—in order to generalize Hochschild homology, one first has to treat it as a theory of two variables, an algebra $A$ over a field $k$ and an $A$-bimodule $M$, and work out what happens when $A$ is just $k$ (but $M$ is arbitrary). Then a general machine of [22] shows that as soon as the functor on vector spaces is equipped with an additional structure, it automatically extends to a functor on pairs $(A, M)$. The structure in question is that of a trace functor, and this is the subject of §3. In §4, we turn to homology: we construct our generalization of $\text{HH}_*(A)$ that we call Hochschild–Witt homology, and we explain how it is related to the non-commutative Witt vectors of Hesselholt and to the de Rham–Witt complex. Finally, in §5, we discuss some parts of the theory that we skipped in §1 (multiplication, Frobenius map) and discuss possible extensions of the existing theory.

I am very grateful to S. Gorchinskiy for many useful remarks.

1. Classical theory

Fix a prime $p$. Denote by

$$R: \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}, \quad n \geq 1,$$

the projection maps, and let

$$\mathbb{Z}_p = \lim_{\substack{\longrightarrow \cr n}} \mathbb{Z}/p^n\mathbb{Z}$$

be the ring of $p$-adic integers. Alternatively, elements $a \in \mathbb{Z}_p$ can be thought of as formal power series

$$a = \sum_{i \geq 0} a_i p^i,$$

where $a_i$ lies in any fixed set $S \subset \mathbb{Z}$ of representatives of residues $\overline{a} \in F_p = \mathbb{Z}/p\mathbb{Z}$. For example, one can take $S = \{0, 1, \ldots, p-1\}$ or $\{1, 2, \ldots, p\}$. There is no canonical choice.

The key observation of Teichmüller is that we do have a canonical choice if we allow $S$ to lie in $\mathbb{Z}_p$ itself, not in $\mathbb{Z}$. Namely, one has the following.
Lemma 1.1. For any \( x \in \mathbb{Z}/p\mathbb{Z} \) there exists a unique element \( [x] \in \mathbb{Z}_p \) such that \( [x]^p = [x] \) and \( [x] = x \mod p \).

Proof. Take \( a \in \mathbb{Z}_p \) such that \( a^p = a \). If \( a \) is divisible by \( p \), then by induction, it is divisible by \( p^n \) for any \( n \), so that \( a = 0 \). If \( a \) is not divisible by \( p \), then for any \( n \geq 1 \) it is invertible with respect to multiplication in \( \mathbb{Z}/p^n\mathbb{Z} \), and thus a root of unity of order \( p-1 \). All such roots form a subgroup \( \mu_{p-1} \) in the group \( (\mathbb{Z}/p^n\mathbb{Z})^* \) of invertible elements in \( \mathbb{Z}/p^n\mathbb{Z} \), and we need to prove that the projection \( R_n^{-1}: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{F}_p \) induces an isomorphism \( \mu_{p-1} \cong \mathbb{F}_p^* \). This is clear, since \( (\mathbb{Z}/p^n\mathbb{Z})^* \) has order \( (p-1)p^{n-1} \), and thus splits canonically into a product of \( \mathbb{F}_p^* \) and an abelian group of order \( p^{n-1} \). \( \square \)

The element \( [x] \) of Lemma 1.1 is known as the Teichmüller representative of the residue class \( x \). By virtue of uniqueness, the correspondence \( x \mapsto [x] \) is multiplicative, and thus defines a multiplicative map \( T: \mathbb{F}_p \to \mathbb{Z}_p \) that splits the projection \( R: \mathbb{Z}_p \to \mathbb{F}_p \) (that is, we have \( R \circ T = \text{id} \)). More generally, we obtain a functorial isomorphism

\[
T_* : \prod_{i \geq 0} \mathbb{F}_p \cong \mathbb{Z}_p, \quad T_*(⟨a_0, a_1, \ldots⟩) = \sum_i p^i[a_i]. \tag{1.1}
\]

This is only an isomorphism of sets. The question is, how do we write down the ring operations in \( \mathbb{Z}_p \) in terms of the components \( a_0, a_1, \ldots \)?

In fact, there are two questions here: how do we write down addition, and then how do we write down multiplication? Since the Teichmüller map is multiplicative, the second question is easier, and we will concentrate on the first one (and postpone the second until §5.2).

To get a good answer to the question, we need to generalize it. Namely, for any commutative ring \( A \) and integer \( n \geq 1 \), denote by \( W_n(A) \) the \( n \)-fold product of copies of \( A \) — that is, the set of collections \( ⟨a_0, \ldots, a_{n-1}⟩ \) of elements \( a_*, \in A \) (these are the ‘Witt vectors’). What we want to construct is a collection of abelian group structures on \( W_n(A) \) that is functorial with respect to \( A \), and such that (1.1) is a group isomorphism when \( A = \mathbb{F}_p \).

Moreover, this last condition can also be generalized so that it makes sense for an arbitrary \( A \). Namely, while the Teichmüller map \( T \) cannot be written by an explicit formula, one can write down explicitly the composition map

\[
\mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{R^n} \mathbb{F}_p \xrightarrow{T} \mathbb{Z}_p \xrightarrow{\text{id}} \mathbb{Z}/p^{n+1}\mathbb{Z}.
\]

To do this, one proves the following effective version of Lemma 1.1.

Lemma 1.2. An element \( x \in \mathbb{Z}/p^{n+1}\mathbb{Z} \) is a Teichmüller representative of some residue class if and only if \( x^{p^n} = x \). For any \( x \in \mathbb{Z}/p^{n+1}\mathbb{Z} \) with residue \( \bar{x} = R^n(x) \in \mathbb{F}_p \),

\[
x^{p^n} = [\bar{x}] = T(R^n(x)) \mod p^{n+1}.
\]

Proof. As in Lemma 1.1, use the canonical abelian group decomposition \( (\mathbb{Z}/p^{n+1}\mathbb{Z})^* \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{F}_p^* \). \( \square \)
Now, for any commutative ring $A$ and any integer $n \geq 1$, define the ghost map $w_n: W_{n+1}(A) \to A$ by

$$w_n(\langle a_0, \ldots, a_n \rangle) = \sum_{i=0}^{n} p^i a_0^{n-i} = a_0^n + pw_{n-1}(\langle a_1, \ldots, a_n \rangle). \quad (1.2)$$

Since the restriction map $W_n(R^n): W_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to W_n(\mathbb{F}_p)$ is surjective, the map $T_n: W_n(\mathbb{F}_p) \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ induced by (1.1) is compatible with the group structure if and only if so is the composition

$$T_n \circ W_n(R^n): W_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

By Lemma 1.2 and (1.1), this composition is exactly the ghost map (1.2) for $A = \mathbb{Z}/p^{n+1}\mathbb{Z}$. Thus if we require that our functorial group structures on $W_*(A)$ make (1.2) additive for any $A$, then the map (1.1) will automatically be an isomorphism of abelian groups.

Now, let $R: W_{n+1}(A) \to W_n(A)$ and $V: W_n(A) \to W_{n+1}(A)$ be the maps given by

$$R(\langle a_0, \ldots, a_n \rangle) = \langle a_0, \ldots, a_{n-1} \rangle, \quad V(\langle a_0, \ldots, a_{n-1} \rangle) = \langle 0, a_0, \ldots, a_{n-1} \rangle,$$

where $V$ traditionally stands for *Verschiebung*, German for ‘shift’. With this notation, Witt’s theorem — or rather, its additive part — can be formulated as follows.

**Theorem 1.3.** There exists a unique set of functorial abelian group structures on $W_n(A)$, $n \geq 1$, such that $R$, $V$, and $w_n$, $n \geq 1$, are additive.

While this theorem can be proved directly, it is more instructive to deduce it from the following elementary statement.

**Lemma 1.4.** There exists a unique collection of polynomials $c_i(-,-)$, $i \geq 1$, with integer coefficients such that for any $n$ and commuting variables $x_0$ and $x_1$,

$$(x_0 + x_1)^{p^n} = x_0^{p^n} + x_1^{p^n} + \sum_{i=1}^{n} p^i c_i(x_0,x_1)^{p^{n-i}}. \quad (1.3)$$

**Proof.** Uniqueness is obvious. Existence can be deduced from the standard divisibility properties of binomial coefficients; we skip this since we will prove a more general statement later in Proposition 2.2. □

**Proof of Theorem 1.3.** Since $w_0 = \text{id}$, we must have $W_1(A) = A$. Assume by induction that we have proved the existence and uniqueness of the functorial group structures on $W_i(A)$, $i \leq n$, with additive $R$, $V$, and $w_i$, $i \leq n$. We then have to construct a functorial extension

$$0 \longrightarrow W_n(A) \xrightarrow{V} W_{n+1}(A) \xrightarrow{R^n} W_1(A) = A \longrightarrow 0$$

of abelian groups, where $W_{n+1}(A) \cong A \times W_n(A)$ as sets. With this decomposition, the operation in $W_{n+1}(A)$ must be given by

$$\langle a_0, b_0 \rangle + \langle a_0, b_0 \rangle = \langle a_0 + a_1, b_0 + b_1 + c_1(a_0,a_1) \rangle \quad (1.4)$$
for some cocycle $c_\bullet (\cdot,\cdot)$ that is functorial in $A$ and symmetric with respect to the transposition of variables. Moreover, since $c_\bullet (\cdot,\cdot)$ is functorial in $A$, it is a map between affine schemes, and thus it must be polynomial in $a_0$ and $a_1$. Then $w_n$ is additive with respect to the operation (1.4) if and only if $c_\bullet (\cdot,\cdot)$ satisfies (1.3), so that Lemma 1.4 proves uniqueness. To prove existence, it suffices to define the group operation by (1.4) with the polynomials $c_\bullet (\cdot,\cdot)$ of Lemma 1.4, and check that it is indeed associative and commutative. By functoriality, it suffices to check that it is indeed associative and commutative. By functoriality, it suffices to check this for $A = \mathbb{Z}$: if two polynomials are equal at all integer points, they coincide. But then the map

$$w_\bullet = \langle w_0, \ldots, w_n \rangle : W_{n+1}(\mathbb{Z}) \to \mathbb{Z}^{n+1}$$

is injective, so that the required identities can be checked after composing with $w_\bullet$, and they follow from the additivity of $w_\bullet$. □

We note that by construction, for any integers $m, n \geq 1$ and any ring $A$, we have a functorial short exact sequence

$$0 \longrightarrow W_m(A) \xrightarrow{v^n} W_{m+n}(A) \xrightarrow{R^m} W_n(A) \longrightarrow 0 \quad (1.5)$$

of abelian groups. In particular, we can take $m = 1$; in this case we see that $W_{n+1}(A)$ is a functorial extension of $W_n(A)$ by $A$.

## 2. Non-commutative theory

To adapt the construction of the Witt vectors $W(A)$ to a non-commutative ring $A$, we start with Lemma 1.4. Does the recursive formula (1.4) admit a non-commutative refinement? Naively, one would like to have non-commutative polynomials $c_i(\cdot,\cdot)$, $i \geq 1$, of degrees $p^i$ such that for any $n$

$$(x_0 + x_1)^{\otimes p^n} = x_0^{\otimes p^n} + x_1^{\otimes p^n} + \sum_{i=1}^{n} p^i c_i(x_0, x_1)^{\otimes p^{n-i}} \in T^{p^n}(x_0, x_1), \quad (2.1)$$

where $T^{p^n}(x_0, x_1)$ is the component of degree $p^n$ of the free associative $\mathbb{Z}$-algebra $T^i(x_0, x_1)$ on two variables $x_0$ and $x_1$. A moment’s reflection shows that this is not possible already for $n = 1$, the non-commutative polynomial $(x_0 + x_1)^{\otimes p} - x_0^{\otimes p} - x_1^{\otimes p}$ is not divisible by $p$. Something has to be modified in (2.1). This something turns out to be the coefficient $p^i$.

**Notation 2.1.** For any free $\mathbb{Z}$-module $N$ and integer $i \geq 1$, we denote by $\sigma : N^{\otimes i} \to N^{\otimes i}$ the permutation consisting of one cycle of length $i$.

In particular, for any $i$ we have $T^i(x_0, x_1) = (\mathbb{Z} x_0 \oplus \mathbb{Z} x_1)^{\otimes i}$, so that we have the permutation $\sigma : T^i(x_0, x_1) \to T^i(x_0, x_1)$.

**Proposition 2.2.** There exists a collection of non-commutative polynomials $c_i(\cdot,\cdot)$, $i \geq 1$, of degree $p^i$ such that for any $n \geq 1$

$$(x_0 + x_1)^{\otimes p^n} = x_0^{\otimes p^n} + x_1^{\otimes p^n} + \sum_{i=1}^{n} (\text{id} + \sigma + \cdots + \sigma^{p^{i-1}}) c_i(x_0, x_1)^{\otimes p^{n-i}}. \quad (2.2)$$
We note that Proposition 2.2 is a refinement of the existence part of Lemma 1.4 (to obtain commutative polynomials, just project onto the symmetric algebra and note that then $\sigma$ intertwines the identity map). Like Lemma 1.4, Proposition 2.2 admits a direct combinatorial proof, although it is rather non-trivial (Proposition 1.2.3 in [14] claims slightly less but actually proves exactly (2.2)). Let us present an alternative proof. It is somewhat roundabout but requires many fewer computations and gives useful by-products.

We start by observing that if for some integer $m > 1$ we have already constructed polynomials $c_i(-, -), i < m$, such that (2.2) holds for $n < m$, then to construct $c_m(-, -)$ we need to show that the difference

$$(x_0 + x_1)\otimes p^m - x_0\otimes p^m - x_1\otimes p^m - \sum_{i=1}^{m-1} (\text{id} + \sigma + \cdots + \sigma^{p-1})c_i(x_0, x_1)\otimes p^{m-i} \quad (2.3)$$

lies in the image of the map $\text{id} + \sigma + \cdots + \sigma^{p-1}$. We note that this difference is obviously invariant under $\sigma$.

**Definition 2.3.** For a free $\mathbb{Z}$-module $N$ and an integer $n > 1$, $Q_n(N)$ is the cokernel of the map

$$\left( N\otimes p^n \right)_\sigma \xrightarrow{\text{id} + \sigma + \cdots + \sigma^{p-1}} \left( N\otimes p^n \right)\sigma. \quad (2.4)$$

Equivalently,

$$Q_n(N) = \tilde{H}^0(\mathbb{Z}/p^n\mathbb{Z}, N\otimes p^n)$$

is the 0th Tate cohomology group of the cyclic group $\mathbb{Z}/p^n\mathbb{Z}$ acting on $N\otimes p^n$ via the permutation $\sigma$ (for an interested reader we remark that $\tilde{H}^1(\mathbb{Z}/p^n\mathbb{Z}, N\otimes p^n) = 0$).

**Lemma 2.4.** The map $N \rightarrow Q_n(N), x \mapsto x\otimes p^n$, factors as

$$N \longrightarrow N/pN \xrightarrow{T_n} Q_n(N),$$

for some functorial map $T_n : N/pN \rightarrow Q_n(N)$.

**Proof.** We need to show that for any $x_0, x_1 \in N$, $(x_0 + px_1)\otimes p^n - x_0\otimes p^n$ lies in the image of the map $\text{id} + \sigma + \cdots + \sigma^{p-1}$. This is an elementary direct computation (see [26], Lemma 2.2, for details). $\square$

**Remark 2.5.** We note that $Q_n(N)$ is a torsion group, but not a $p$-torsion group if $n \geq 2$: while $p^nQ_n(N) = 0$, the group $p^{n-1}Q_n(N)$ is non-trivial.

**Corollary 2.6.** There exists a unique functor $W_n$ from $\mathbb{F}_p$-vector spaces to abelian groups such that for any free $\mathbb{Z}$-module $N$,

$$Q_n(N) \cong W_n(N/pN),$$

and this isomorphism is functorial in $N$.

**Proof.** Since every $\mathbb{F}_p$-vector space $M$ is of the form $M = N/pN$ for some free $\mathbb{Z}$-module $N$, defining $W_n$ on objects is trivial, and the issue is morphisms: we need to show that $Q_n$ factors through the full essentially surjective functor $N \mapsto N/pN$. Explicitly, this means that for any two maps $a, a' : N_0 \rightarrow N_1$ between free $\mathbb{Z}$-modules
we have \( Q_n(a) = Q_n(a + pa') \). But by definition, the map \( Q_n(a) \) is induced by 
\( a\otimes p^n : N_0^\otimes p^n \to N_1^\otimes p^n \), and similarly for \( Q_n(a + pa') \). Since \( Q_n(-) \) is obviously
a pseudotensor functor, the natural map
\[
(\text{Hom}(N_0, N_1)^\otimes p)^\sigma \to \text{Hom}(N_0^\otimes p, N_1^\otimes p)^\sigma \to \text{Hom}(Q_n(N_0), Q_n(N_1))
\]
factors through \( Q_n(\text{Hom}(N_0, N_1)) \), so we are done, by Lemma 2.4. □

**Definition 2.7.** For any \( \mathbb{F}_p \)-vector space \( M \) the abelian group \( W_n(M) \) is the group of
\( n \)-truncated polynomial Witt vectors of the vector space \( M \).

Later, in §4, we will discuss the exact relationship between the usual Witt vectors of
commutative rings and our polynomial Witt vectors of vector spaces. For now,
let us just note that the polynomial Witt vectors of Definition 2.7 behave very
much like the usual Witt vectors. Indeed, Lemma 2.4 provides a canonical map
\( T_n: M \to W_n(M) \), an analogue of the Teichmüller representative map. Moreover,
for any free \( \mathbb{Z} \)-module \( N \) and any integer \( n \geq 1 \) we have
\[
N^\otimes p^n \cong (N^\otimes p)^{p^{n-1}},
\]
with the \( \mathbb{Z}/p^n \mathbb{Z} \)-action generated by \( \sigma^p \), and the order-\( p \) permutation \( \sigma \) on \( N^\otimes p^n \)
induces an order-\( p \) automorphism \( \sigma \) on \( Q_{n-1}(N^\otimes p) \). This descends to a functorial
automorphism
\[
\sigma: W_{n-1}(M^\otimes p) \to W_{n-1}(M^\otimes p)
\]
of order \( p \), and we have a functorial additive map
\[
V = \text{id} + \sigma + \cdots + \sigma^{p-1}: W_{n-1}(M^\otimes p) \to W_n(M),
\]
an analogue of the Verschiebung map (note that \( \bar{\sigma} \) is different from \( W_{n-1}(\sigma) \)). To
complete the analogy, we need to define restriction maps \( R \). To do this, we need
the following version of Definition 2.3.

**Definition 2.8.** For a free \( \mathbb{Z} \)-module \( N \) and an integer \( n \geq 1 \), \( Q'_n(N) \) is the cokernel
of the map
\[
(N^\otimes p^n)^\sigma \xrightarrow{\text{id} + \sigma + \cdots + \sigma^{p+1}} (N^\otimes p^n)^\sigma.
\]

**Lemma 2.9.** The map \( N \to Q'_n(N) \), \( x \mapsto xp^n \), factors through a functorial map
\( T'_n: N/pN \to Q'_n(N) \). Moreover, there exists a unique functor \( W'_n \) from \( \mathbb{F}_p \)-vector
spaces to abelian groups such that \( Q'_n(N) \cong W'_n(N/pN) \) functorially in \( N \), and \( T'_n \)
descends to a functorial map \( T'_n: M \to W'_n(M) \) for any \( \mathbb{F}_p \)-vector space \( M \).

**Proof.** Same as Lemma 2.4 and Corollary 2.6 (for details, see [26], Lemma 2.2). □

**Remark 2.10.** If \( N = \mathbb{Z} \) is the free module of rank 1, then \( Q_n(N) = \mathbb{Z}/p^n\mathbb{Z} \), and
Lemma 2.4 asserts that for any integer \( a \) the residue \( a^p \mod p^n \) depends only on \( a \mod p \). It is well known (and proved in Lemma 1.2) that more is true: even
the residue \( a^p \mod p^{n+1} \) depends only on \( a \mod p \). This is exactly the claim of
Lemma 2.9.
On the other hand, the permutation $\sigma: N^{\otimes p^n} \to N^{\otimes p^n}$ has order $p^n$, so that
\[
id + \sigma + \cdots + \sigma^{p^n+1} = p(id + \sigma + \cdots + \sigma^{p^n}),
\]
and we have a natural functorial projection $W'_n(M) \to W_n(M)$.

**Definition 2.11.** For any free $\mathbb{Z}$-module $N$, an additive map $c: N \to (N^{\otimes p})^\sigma$ is **standard** if the induced map $\overline{c}: N/pN \to Q_1(N)$ is the Teichmüller map $T_1$ of Lemma 2.4.

Standard maps exist (for example, choose a basis in $N$, and let $c$ send any basis element $e$ to $e^{\otimes p}$) and enjoy the following property.

**Proposition 2.12.** For any standard map $c: N \to N^{\otimes p}$ and any integer $n \geq 1$, the map $Q_n'(N) \to Q_{n+1}(N)$ induced by $c^{\otimes p^n}$ is an isomorphism that does not depend on the choice of $c$, and $c^{\otimes p^n} \circ T'_n = T_{n+1}$.

**Proof.** See [26], Proposition 2.6. □

**Remark 2.13.** In the case $n = 1$ we have $Q_1'(N) = N/p$, and by definition the map in question is $T_1$. In this case Proposition 2.12 simply claims that the Teichmüller map $T$ is an isomorphism.

As a corollary of Proposition 2.12, we have a functorial isomorphism $W'_n(M) \cong W_{n+1}(M)$ for any $\mathbb{F}_p$-vector space $M$. Composing the inverse isomorphism with the projection $W'_n(M) \to W_n(M)$, we obtain a functorial map
\[R: W_{n+1}(M) \to W_n(M),\]

an analogue of the restriction map. The interconnection between the maps $R$ and $V$ is also analogous to what one has for classical Witt vectors—in particular, it was proved in [26], Lemma 3.7, that for any $n, m \geq 0$ and any $\mathbb{F}_p$-vector space $M$ we have an exact sequence
\[
\begin{array}{ccccccccc}
W_m(M^{\otimes p^n}) & \xrightarrow{V^n} & W_{m+n}(M) & \xrightarrow{R^m} & W_n(M) & \longrightarrow & 0
\end{array}
\]
of abelian groups, a vector-space counterpart of the sequence (1.5).

**Proof of Proposition 2.2.** Assume by induction that we have defined the polynomials $c_i(\cdot, \cdot)$ for $i \leq n - 1$. Then we have the equality (2.2) in $T_{p^n-1}(x_0, x_1)$, and both sides are obviously $\sigma$-invariant, so we can project it onto $Q_{n-1}'((\mathbb{Z}x_0 \oplus \mathbb{Z}x_1))$. We see that
\[
T_{n-1}'(x_0 + x_1) = T_{n-1}'(x_0) + T_{n-1}'(x_1) + \sum_{i=1}^{n-1} V^i(T_{n-1-i}'(c_i(x_0, x_1))).
\]
However, $Q_{n-1}'((\mathbb{Z}x_0 \oplus \mathbb{Z}x_1)) \cong Q_n((\mathbb{Z}x_0 \oplus \mathbb{Z}x_1))$ by Proposition 2.12, and in terms of the group $Q_n((\mathbb{Z}x_0 \oplus \mathbb{Z}x_1))$ this equality can be rewritten as
\[
(x_0 + x_1)^{\otimes p^n} = x_0^{\otimes p^n} + x_1^{\otimes p^n} + \sum_{i=1}^{n-1}(\id + \sigma + \cdots + \sigma^{p-1})c_i(x_0, x_1)^{\otimes p^{n-i}}.
\]
In other words, (2.3) vanishes after projecting to $Q_n((\mathbb{Z}x_0 \oplus \mathbb{Z}x_1))$. As we have noted, this implies the existence of $c_n(\cdot, \cdot)$ and proves the assertion. □
3. Trace functors

3.1. Trace maps. One obvious difference between Lemma 1.4 and Proposition 2.2 is that the latter does not claim uniqueness. Uniqueness is in fact wrong; formally, this is reflected in the fact that the sequence (2.7) is not exact on the left. If $m = 1$, then one can show that the image of the map

$$V^n: W_1(M^{\otimes p^n}) \to W_{n+1}(M)$$

is the space $M^{\otimes p^n}$ of coinvariants with respect to the permutation $\sigma$. At the opposite extreme, if $n = 1$, then the image of the map

$$V: W_m(M^{\otimes p}) \to W_{m+1}(M)$$

turns out to be isomorphic to the group of coinvariants $W_m(M^{\otimes p})_\sigma$ with respect to the permutation $\sigma$ in (2.6). To understand the behaviour in general, we observe that $W_q(-)$ carry an additional structure: that of a trace functor in the sense of [22], §2.

**Definition 3.1.** A trace functor from a unital monoidal category $\mathcal{C}$ to a category $\mathcal{E}$ is a collection of a functor $F: \mathcal{C} \to \mathcal{E}$ and isomorphisms

$$\tau_{M,N}: F(M \otimes N) \cong F(N \otimes M)$$

for any $M, N \in \mathcal{C}$ such that $\tau_{M,N}$ are functorial in $M$ and $N$, $\tau_{1,M} = \text{id} = \tau_{M,1}$ for any object $M \in \mathcal{C}$, and for any three objects $M, N, L \in \mathcal{C}$ we have

$$\tau_{L,M,N} \circ \tau_{N,L,M} \circ \tau_{M,N,L} = \text{id},$$

where for any $A, B, C \in \mathcal{C}$, $\tau_{A,B,C}$ is the composition of the map $\tau_{A,B \otimes C}$ and the associativity isomorphism $(B \otimes C) \otimes A \cong B \otimes (C \otimes A)$.

If a monoidal category $\mathcal{C}$ is symmetric, such as for example the category of vector spaces over a field, then any functor $\mathcal{C} \to \mathcal{E}$ is tautologically a trace functor, with $\tau_{-, -, -}$ induced by the commutativity isomorphism in $\mathcal{C}$. However, even in this case there are also non-trivial trace functor structures. The basic example considered in [22] is the following.

**Example 3.2.** Let $\mathcal{C}$ be the category of free $\mathbb{Z}$-modules, and fix an integer $l \geq 2$. For any $M \in \mathcal{C}$, let $\sigma_M = \sigma: M^{\otimes l} \to M^{\otimes l}$ be as in Notation 2.1, and for any $M, N \in \mathcal{C}$ let

$$\tilde{\tau}_{M,N}: (M \otimes N)^{\otimes l} \to (N \otimes M)^{\otimes l}$$

be the composition of the $l$th tensor power of the commutativity isomorphism and the automorphism $\sigma_N \otimes \text{id}_{M^{\otimes l}}$ of $(N \otimes M)^{\otimes l} \cong N^{\otimes l} \otimes M^{\otimes l}$. Then we obviously have

$$\sigma_N^{\otimes M} \circ \tilde{\tau}_{M,N} = \tilde{\tau}_{M,N} \circ \sigma_{M^{\otimes N}},$$

so that $\tilde{\tau}_{M,N}$ induces a map

$$\tau_{M,N}: (M \otimes N)^{\otimes l}_\sigma \to (N \otimes M)^{\otimes l}_\sigma$$

of the modules of coinvariants with respect to $\sigma$. Setting $C_l(M) = M^{\otimes l}_\sigma$ with these structure maps $\tau_{M,N}$ gives a trace functor from $\mathcal{C}$ to itself.
Remark 3.3. The maps (3.3) themselves do not define a trace functor structure since they do not satisfy (3.2).

Alternatively, one can consider the module $C^l(M) = (M^\otimes l)^\sigma$ of invariants with respect to $\sigma$; then (3.3) turns $C^l(M)$ into a trace functor as well. Moreover, if $l = p^n$, then (2.4) is a map of trace functors, so that the functor $Q_n$ of Definition 2.3 is naturally a trace functor from free $\mathbb{Z}$-modules to abelian groups. This trace functor structure then descends to a trace functor structure on the Witt vectors functor $W_n(-)$.

For any trace functor $F: \mathcal{C} \to \mathcal{E}$ and for any $l$ objects $M_1, \ldots, M_l \in \mathcal{C}$ we have a natural map

$$
\tau: F(M_1 \otimes \cdots \otimes M_l) \to F(M_2 \otimes \cdots \otimes M_l \otimes M_1),
$$

(3.4)

obtained by composing $\tau_{M_1, M_2 \otimes \cdots \otimes M_l}$ with the associativity isomorphisms, and it is not difficult to show that these maps satisfy an $l$-variable version of (3.2). In particular, for any $M \in \mathcal{C}$ we have an automorphism $\tau: F(M^\otimes l) \to F(M^\otimes l)$ of order $l$. Then it was proved in [26], Lemma 3.7, that for any $n, m \geq 1$ the sequence (2.7) can be refined to a functorial short exact sequence

$$
0 \longrightarrow W_m(M^\otimes p^n)_{\tau} \longrightarrow W_{m+n}(M) \longrightarrow W_n(M) \longrightarrow 0,
$$

(3.5)

where $\tau: W_m(M^\otimes p^n) \to W_m(M^\otimes p^n)$ is the automorphism of order $p^n$ induced by the trace functor structure on $W_m$. If $m = 1$, then $\tau = \sigma$ is the permutation, and if $n = 1$, then $\tau = \sigma$ is as in (2.6). We caution the reader that for $m \geq 2$ the trace functor structure on $W_m$ is non-trivial, so that $\tau$ is distinct from the map induced by the permutation $\sigma: M^\otimes p^n \to M^\otimes p^n$.

3.2. Small categories. While constructing the map (3.4) directly is not difficult, this was not done in [22]. Instead, the maps are deduced from a convenient packaging of a trace functor structure using the category $\Lambda$ introduced by Connes in [4]. This is a small category whose objects $[n]$ are indexed by positive integers $n$. Maps between $[n]$ and $[m]$ can be defined in various equivalent ways (for example, see [30] or [8]); for the convenience of the reader, we recall the most geometric of the descriptions.

- The object $[n] \in \Lambda$ is thought of as a ‘wheel’ — a cellular decomposition of the unit circle $S^1 \subset \mathbb{C}$ with $n$ vertices corresponding to the $n$th roots of unity, and $n$ edges. A continuous map $f: [n'] \to [n]$ is good if it is cellular (that is, sends vertices to vertices) and has degree 1, and the induced map $\tilde{f}: \mathbb{R} \to \mathbb{R}$ between the universal covers is non-decreasing. Morphisms from $[n']$ to $[n]$ in the category $\Lambda$ are homotopy classes of good maps $f: [n'] \to [n]$.

We will denote the set of vertices of $[n] \in \Lambda$ by $V([n])$. Every map $f: [n'] \to [n]$ in $\Lambda$ induces a map

$$
V(f): V([n']) \to V([n]),
$$

and for any $v \in V([n])$ the pre-image $V(f)^{-1}(v) \subset V([n'])$ has a natural total order induced by the clockwise orientation of the circle (note that $f^{-1}(v) \subset S^1$ is an interval).

Now, for any unital monoidal category $\mathcal{C}$, define a category $\mathcal{C}^2$ as follows:
objects are pairs \(\langle [n], \{c_v\} \rangle\) consisting of an object \([n] \in \Lambda\) and a collection of objects \(c_v \in C\) indexed by vertices \(v \in V([n])\);

- morphisms from \(\langle [n], c_v \rangle\) to \(\langle [n'], c'_v \rangle\) are given by pairs \(\langle f, \{f_v\} \rangle\) consisting of a morphism \(f: [n] \to [n']\) and a collection of morphisms

\[
f_v: \bigotimes_{v' \in V(f)^{-1}(v)} c_{v'} \to c_v, \quad v \in V([n]),
\]

where the product is taken in the natural order on \(V(f)^{-1}(v)\).

A morphism \(\langle f, \{f_v\} \rangle\) in \(C^2\) is said to be cartesian if all the maps \(f_v\) are invertible. With these definitions, one has the following result.

**Lemma 3.4** ([22], Lemma 2.3). Giving a trace functor from \(C\) to some category \(\mathcal{E}\) is equivalent to giving a functor \(C^2 \to \mathcal{E}\) that sends any cartesian map to an invertible map.

Here is a sketch of the correspondence of Lemma 3.4. The category \(C\) is naturally embedded into \(C^2\) by sending \(c \in C\) to \(\langle [1], c \rangle\). Thus, any functor \(F^2: C^2 \to \mathcal{E}\) gives by restriction a functor \(F: C \rightarrow \mathcal{E}\). To see the map (3.1), consider the object \(\langle [2], \{M, N\} \rangle \in C^2\) and note that there are two maps \(s, t: [2] \to [1]\) in \(\Lambda\) that impose opposite orders on the set \(V([2])\). These maps lift to cartesian maps

\[
\tilde{s}: \langle [2], \{M, N\} \rangle \to \langle [1], M \otimes N \rangle \quad \text{and} \quad \tilde{t}: \langle [2], \{M, N\} \rangle \to \langle [1], N \otimes M \rangle,
\]

and we then have \(\tau_{M,N} = F^2(\tilde{t}) \circ F^2(\tilde{s})^{-1}\). The constraint (3.2) is encoded in the structure of the category \(C^2\), and so is the map (3.4) for \(l \geq 3\). Indeed, a choice of a map \(s: [l] \to [1]\) induces an order on the set of vertices \(V([l])\), so we can number them by the integers \(1, \ldots, l\), and a choice of a cartesian lifting \(\tilde{s}\) then induces an identification

\[
F^2(\langle [l], \{M_1, \ldots, M_l\} \rangle) \cong F(M_1 \otimes \cdots \otimes M_l).
\]

But the group \(\text{Aut}(l)\) of automorphisms of the object \(l \in \Lambda\) is the cyclic group \(\mathbb{Z}/l\mathbb{Z}\) whose generator \(\tau\) is the clockwise rotation by \(2\pi/l\), and it lifts to a map \(\tilde{\tau}\) in \(C^2\) that then gives (3.4) after applying \(F^2\).

### 3.3. Fibrations.

To describe the functors \(W^2_n\) corresponding to the trace functor structures on the polynomial Witt vectors functors \(W_n\), we need to make a digression about the homology of small categories. Recall that for any small category \(I\) and any ring \(k\) the category \(\text{Fun}(I, k)\) of functors from \(I\) to \(k\)-modules is abelian. For any functor \(\gamma: I \to I'\) between small categories we have the natural pullback functor

\[
\gamma^*: \text{Fun}(I', k) \to \text{Fun}(I, k), \quad E \mapsto E \circ \gamma,
\]

and it has a left-adjoint and a right-adjoint *Kan extension functor*

\[
\gamma_!, \gamma_*: \text{Fun}(I, k) \to \text{Fun}(I', k).
\]

For example, if \(I' = \text{pt}\) is the one-point category, then

\[
\gamma_! = \text{colim}_I \quad \text{and} \quad \gamma_* = \lim_I
\]
are the colimit and limit functors; furthermore, if \( I = \text{pt}_G \) is the groupoid with one object with automorphism group \( G \), then \( \text{Fun}(I, k) \) is the category of representations of \( G \) in \( k \)-modules, and \( \gamma_! \) and \( \gamma^* \) send a representation \( E \) to its module of \( G \)-coinvariants and its module of \( G \)-invariants, respectively:

\[
\gamma_!(E) = E_G \quad \text{and} \quad \gamma^*(E) = E^G.
\]

For a more general target category \( I' \), computing Kan extensions can be cumbersome, but there is one situation where it is still easy.

**Definition 3.5.** A functor \( \pi: I_0 \to I_1 \) is a **fibration in groupoids** if

(i) for any morphism \( f_1: i'_1 \to i_1 \) in \( I_1 \) and any object \( i_0 \in I_0 \) with \( \pi(i_0) = i_1 \) there exists a morphism \( f_0: i'_0 \to i_0 \) in \( I_0 \) such that \( \pi(f_0) = f_1 \), and

(ii) for any two such morphisms \( f'_0: i'_0 \to i_0 \) and \( f''_0: i''_0 \to i_0 \) there exists a unique morphism \( g: i'_0 \to i''_0 \) such that

\[
\pi(g) = \text{id}_{i'} \quad \text{and} \quad f'_0 = f''_0 \circ g.
\]

A functor \( \pi \) is a **bifibration in groupoids** if both \( \pi \) and the opposite functor \( \pi^o: I_0 \to I_1^o \) are fibrations in groupoids.

Definition 3.5 is a special case of the general formalism of [12], but we will not need the full generality—it suffices to know that for any bifibration \( \pi: I' \to I \) and any morphism \( f: i' \to i \) in \( I \) there exists a pair of adjoint functors

\[
f^*: I'_i \to I'_i \quad \text{and} \quad f_1: I'_i \to I'_i
\]

between the fibres \( I'_i = \pi^{-1}(i) \), \( I'_i = \pi^{-1}(i') \) of the bifibration \( \pi \). Note that by virtue of uniqueness, the map \( g \) in Definition 3.5, (ii) is automatically invertible, so if \( \pi \) is a bifibration in groupoids in the sense of Definition 3.5, then its fibres are indeed groupoids. Therefore, \( f^* \) and \( f_1 \) are mutually inverse equivalences of categories. We also need the following special case of the base change lemma in [19], Lemma 1.7.

**Lemma 3.6.** Assume given a bifibration in groupoids \( \pi: I' \to I \) with small categories \( I' \) and \( I \) and a functor \( \gamma: J \to I \) from some small category \( J \). Consider the pullback square

\[
\begin{array}{ccc}
J' & \stackrel{\gamma'}{\longrightarrow} & I' \\
\varphi \downarrow & & \downarrow \pi \\
J & \stackrel{\gamma}{\longrightarrow} & I
\end{array}
\]

Then \( \varphi \) is a bifibration in groupoids, and the natural maps

\[
\varphi_! \circ \gamma'^* \to \gamma^* \circ \pi_! \quad \text{and} \quad \gamma^* \circ \pi_* \to \varphi_* \circ \gamma'^*,
\]

adjoint to the isomorphism \( \gamma'^* \circ \pi^* \cong \varphi^* \circ \gamma^* \) are themselves invertible.

As a corollary of Lemma 3.6, we see that if we are given a category \( I' \), a functor \( E: I' \to k\text{-mod} \) to the category of modules over a ring \( k \), and a bifibration in groupoids \( \pi: I' \to I \) with small fibres, then \( \pi_!E \) and \( \pi_*E \) are well defined even if
$I'$ and $I$ are large—indeed, for any small category $I_0 \subset I$ with pre-image $I'_0 = \pi^{-1}(I_0) \subset I'$, the induced functor $\pi: I'_0 \to I_0$ is a bifibration in groupoids, so that $\pi_1 E$ and $\pi_1^* E$ are well defined on $I_0$, and both do not depend on the choice of $I_0 \subset I$. Moreover, applying Lemma 3.6 to the embedding $\gamma: pt \to I$ onto an object $i \in I$, we obtain canonical identifications

$$\pi_1 E(i) \cong \operatorname{colim}_{I'} E \quad \text{and} \quad \pi_1^* E(i) \cong \operatorname{lim}_{I'} E,$$

where as before, $I'_i = \pi^{-1}(i) \subset I'$ is the fiber of the bifibration $\pi$.

### 3.4. Edgewise subdivision.

A typical example of the situation in Definition 3.5 occurs in the study of Connes’ category $\Lambda$. Namely, fix an integer $l \geq 1$, and note that for any $n \geq 1$ the object $[nl] \in \Lambda$ has an automorphism $\sigma = \tau^n: [nl] \to [nl]$ of order $l$ given by a clockwise rotation by $2\pi/l$. Let $\Lambda_l$ be the category whose objects $[n]$ correspond to integers $n \geq 1$, and with maps from $[n]$ to $[n']$ given by $\sigma$-equivariant maps from $[nl]$ to $[n'l]$ in $\Lambda$. We then have a tautological embedding

$$i_l: \Lambda_l \to \Lambda, \quad [n] \mapsto [nl],$$

known as the *edgewise subdivision functor*. On the other hand, we can identify the quotient $S^1/\sigma$ with $S^1$ by the $l$th-power map $z \mapsto z^l$, and this identification is compatible with our cellular decompositions. Then sending a morphism in $\Lambda_l$ to the induced map of the quotient circles defines a functor $\pi_l: \Lambda_l \to \Lambda, [n] \mapsto [n]$. This functor is a bifibration in groupoids in the sense of Definition 3.5. Its fibres are naturally identified with the groupoid $pt_l = pt_{Z/lZ}$ with one object with automorphism group $Z/lZ$.

More generally, assume given a unital monoidal category $\mathcal{C}$, consider the category $\mathcal{C}^\sharp$ in Lemma 3.4, and define a category $\mathcal{C}^\sharp_l$ by the fibred product square

$$\begin{array}{ccc}
\mathcal{C}^\sharp_l & \xrightarrow{\pi_l} & \mathcal{C}^\sharp \\
\downarrow & & \downarrow \\
\Lambda_l & \xrightarrow{\pi_l} & \Lambda
\end{array}$$

Then $\pi_l: \mathcal{C}^\sharp_l \to \mathcal{C}^\sharp$ is also a bifibration in groupoids with fiber $pt_l$. Moreover, the edgewise subdivision functor

$$i_l: \Lambda_l \to \Lambda$$

extends to a functor $i_l: \mathcal{C}^\sharp_l \to \mathcal{C}^\sharp$ sending $([n], \{e_i\})$ to $[nl] = i_l([n])$ with the collection of objects $\{e_{q(i)}\}$, where $q: V([nl]) \to V([n])$ is induced by the quotient map $q: S^1 \to S^1/\sigma$.

Now if we take $\mathcal{C}$ to be the category of free $\mathbb{Z}$-modules, then the tautological embedding from $\mathcal{C}$ to the category $\operatorname{Ab}$ of abelian groups has the trivial structure of a trace functor, and thus defines a functor $I^\sharp: \mathcal{C}^\sharp \to \operatorname{Ab}$. Therefore, Lemma 3.6 shows that for any $l \geq 1$ both functors

$$\pi_{l*} i_l^* I^\sharp, \pi_{l*} i_l^* I^\sharp: \mathcal{C}^\sharp \to \operatorname{Ab}$$

send cartesian maps to invertible maps, and hence correspond to trace functors by Lemma 3.4. Moreover, the map $\operatorname{tr}_l = \text{id} + \sigma + \cdots + \sigma^l$ defines a map of functors

$$\operatorname{tr}_l: \pi_{l*} i_l^* I^\sharp \to \pi_{l*} i_l^* I^\sharp, \quad (3.7)$$
and if we take \( l = p^n \), then the cokernel \( Q_n^p \) of the map (3.7) corresponds to the trace functor structure on the functor \( Q_n \) of Definition 2.3. By virtue of Corollary 2.6, \( Q_n^p \) actually factors through a functor

\[
W_n^\sharp: \mathcal{C}^\sharp \to \text{Ab},
\]

where \( \mathcal{C} \) is the category of \( \mathbb{F}_p \)-vector spaces; the functor \( W_n^\sharp \) then corresponds to the trace functor structure on the Witt vectors functor \( W_n \). One then shows (see [26], Proposition 4.3) that the restriction maps \( R \) extend to maps \( R: W_{n+1}^\sharp \to W_n^\sharp \), and the Verschiebung maps \( V \) induce maps

\[
\pi_p^! \circ \pi_p^* W_n^\sharp \to W_{n+1}^\sharp.
\]

More generally, for any \( n, m \geq 0 \) we have a short exact sequence

\[
0 \to \pi_p^! \circ \pi_p^* W_n^\sharp \to W_{n+m}^\sharp \to W_m^\sharp \to 0
\]

of functors from \( \mathcal{C}^\sharp \) to \( \text{Ab} \), a further refinement of the sequence (3.5).

4. Hochschild homology

4.1. Generalities about cyclic homology. While the category \( \Lambda \) does nicely describe the trace functors of Definition 3.1, this was not why it was originally introduced in [4]—the intended application was to cyclic homology.

To understand this, recall that for any small category \( I \) and ring \( k \) the category \( \text{Fun}(I, k) \) of functors from \( I \) to \( k \)-mod is abelian, with enough injectives and projectives. For any \( E \in \text{Fun}(I, k) \) the homology and cohomology modules of \( I \) with coefficients in \( E \) are given by

\[
H_q(I, E) = L_q \text{colim}_I E \quad \text{and} \quad H^q(I, E) = R_q \text{lim}_I E.
\]

If \( E = k \) is the constant functor with value \( k \), and \( k \) is commutative, then \( H^*(I, k) \) is an algebra, and for any \( E \in \text{Fun}(I, k) \) both \( H_*(I, E) \) and \( H^*(I, E) \) are modules over \( H^*(I, k) \).

**Example 4.1.** Let \( \Delta \) be the category of finite non-empty totally ordered sets \( [n] = \{0, \ldots, n\} \) and order-preserving maps, and let \( \Delta^o \) be the opposite category. Then \( \text{Fun}(\Delta^o, k) \) is the category of simplicial \( k \)-modules, and for any \( E \in \text{Fun}(\Delta^o, k) \), the homology modules \( H_*(\Delta^o, E) \) can be computed by the standard chain complex of the simplicial \( k \)-module.

Consider now the category \( \Lambda \), and let \( \Lambda/[1] \) be the category of objects \( [n] \in \Lambda \) equipped with a morphism \( f: [n] \to [1] \). Then \( \text{Fun}([1]) \) consists of a single vertex \( o \), and the pre-image \( V(f)^{-1}(o) \) carries a natural total order, so that we can define a functor \( \Lambda/[1] \to \Delta \) by sending \( f: [n] \to [1] \) to \( V(f)^{-1}(o) \). It turns out that this is an equivalence of categories. Composing the inverse equivalence with the forgetful functor \( \Lambda/[1] \to \Lambda \) sending \( f: [n]\to [1] \) to \( [n] \), we obtain a natural embedding \( j: \Delta \to \Lambda \). Moreover, \( \Lambda \) is self-dual: we have an equivalence \( \Lambda \cong \Lambda^o \) sending a cellular decomposition of \( S^1 \) to the dual cellular decomposition. Therefore, we also have an embedding \( j^o: \Delta^o \to \Lambda \).
Definition 4.2. For any ring $k$ and functor $E \in \text{Fun}(\Lambda, k)$ the Hochschild and cyclic homology groups of $E$ are given by

$$HH_* (E) = H_*(\Delta^o, j^o E) \quad \text{and} \quad HC_* (E) = H_*(\Lambda, E).$$

Remark 4.3. Explicitly, $HH_* (E)$ can be computed by the standard complex

$$b \rightarrow E([n]) \rightarrow \cdots \rightarrow b \rightarrow E([1]) \quad (4.1)$$

of the simplicial $k$-module $j^o E$. Its terms are the groups $E([n])$ placed in homological degree $n$, and the differential $b$ is the alternating sum of the face maps. In particular, $b : E([2]) \rightarrow E([1])$ is given by $b = E(s) - E(t)$, where $s, t : [2] \rightarrow [1]$ are the two maps as in (3.6).

Now assume that the ring $k$ is commutative. It turns out (for example, see [30], Chap. 6) that the cohomology $H^* (\Lambda, k)$ with coefficients in $k$ is naturally identified with the polynomial algebra $k[u]$ in one generator $u$ of degree 2, so that for any $E \in \text{Fun}(\Lambda, k)$ the cyclic homology $HC_* (E)$ is a $k[u]$-algebra. One has a $k[u]$-equivariant spectral sequence

$$HH_* (E)[u^{-1}] \Rightarrow HC_* (E), \quad (4.2)$$

known as a Hodge-to-de Rham spectral sequence. One can also invert the generator $u$ and define the periodic cyclic homology $HP_* (E)$ by

$$HP_* (E) = \varprojlim u HP_* (E),$$

where $\varprojlim u$ is the derived functor of the inverse limit functor. Then the spectral sequence (4.2) gives rise to a spectral sequence

$$HH_* (E)((u)) \Rightarrow HP_* (E), \quad (4.3)$$

where on the left-hand side we have formal Laurent power series in the generator $u$.

Now assume given an associative unital $k$-algebra $A$, and consider the functor $A^\natural \in \text{Fun}(\Lambda, k)$ that sends an object $[n] \in \Lambda$ to $A^\otimes n$, with copies numbered by vertices $v \in V([n])$, and sends a morphism $f : [n'] \rightarrow [n]$ to the map

$$A^\natural (f) = \bigotimes_{v \in V([n])} m_{V(f)^{-1} (v)}, \quad (4.4)$$

where for any finite totally ordered set $S$

$$m_S : A^\otimes S \rightarrow A$$

is the product map. The Hochschild and cyclic homology of the algebra $A$ are then given by

$$HH_* (A) = HH_* (A^\natural), \quad HC_* (A) = HC_* (A^\natural), \quad HP_* (A) = HP_* (A^\natural), \quad (4.5)$$

and one has the following comparison result (see [30], Chap. 3, and [8]).
Theorem 4.4. Assume that a $k$-algebra $A$ is flat, commutative, and finitely generated, and $X = \text{Spec} A$ is smooth over $k$. Then there are canonical identifications

$$HH_i(A) \cong H^0(X, \Omega^i_X)$$

for all $i \geq 0$, and under these identifications the first differential

$$B: HH_i(A) \to HH_{i+1}(A)$$

in the spectral sequences (4.2) and (4.3) is identified with the de Rham differential $d$.

The identification (4.6) is the famous theorem of Hochschild, Kostant, and Rosenberg [17] (they used the original definition of $HH_*(A)$ via the complex (4.1)). The differential $B$ was discovered by Rinehart [33], forgotten, and then rediscovered as a part of the cyclic homology package, independently by Connes [5] and Tsygan [36], in about 1982. The packaging using the category $\Lambda$ was discovered by Connes one year later. The name ‘Hodge-to-de Rham spectral sequence’ for the sequences (4.2), (4.3) is motivated by Theorem 4.4. If $k$ contains $\mathbb{Q}$, then one can prove more: the Hodge-to-de Rham spectral sequences degenerate at the second term, and in particular,

$$HP_*(A) \cong H^*_{\text{DR}}(X)((u)),$$

the de Rham cohomology of the variety $X = \text{Spec} A$. The theorem itself works over any $k$, but even if $k$ is a field not containing $\mathbb{Q}$, it is currently unknown whether (4.3) degenerates (unless $A$ is a polynomial algebra; see [30], Chap. 3.2).

4.2. Twisting. We now observe that Lemma 3.4 allows us to generalize (4.5) in the following way. Assume given a unital monoidal category $\mathcal{C}$, and consider the category $\mathcal{C}^\natural$ with the forgetful functor $\rho: \mathcal{C}^\natural \to \Lambda$. Then an associative unital algebra object $A$ in $\mathcal{C}$ gives rise to a section $\alpha: \Lambda \to \mathcal{C}^\natural$ of $\rho$ sending $[n]$ to $\langle [n], \{A\} \rangle$, the collection of copies of the object $A$ numbered by vertices $v \in V([n])$. If we also have a trace functor $F$ from $\mathcal{C}$ to $k$-mod for some ring $k$, then we can consider the corresponding functor $F^\natural: \mathcal{C}^\natural \to k$-mod. Composing $\alpha$ and $F^\natural$, we obtain a functor

$$FA^\natural = F^\natural \circ \alpha \in \text{Fun}(\Lambda, k).$$

**Definition 4.5.** The twisted Hochschild homology and periodic cyclic homology of the algebra object $A$ with respect to the trace functor $F$ are given by

$$FHH_*(A) = HH_*(FA^\natural) \quad \text{and} \quad FHP_*(A) = HP_*(FA^\natural).$$

**Example 4.6.** If $\mathcal{C}$ is the category $k$-mod, and $F: \mathcal{C} \to k$-mod is the identity functor, then an associative unital algebra object $A$ in $\mathcal{C}$ is simply an associative unital $k$-algebra, and $FA^\natural$ is simply the functor $A^\natural$ given by (4.4).

**Remark 4.7.** For general $\mathcal{C}$, $F$, and $A$, $FHH_*(A)$ can still be computed by the complex (4.1); its terms are $F(A^{\otimes n})$, $n \geq 0$, and the differential $b$ is an alternating sum of terms that involve the trace functor structure on $F$. In particular,

$$b: F(A^{\otimes 2}) \to F(A)$$

is given by $b = F(m) - F(m) \circ \tau_{A,A}$, where $m: A^{\otimes 2} \to A$ is the product map, and $\tau_{A,A}$ is the map (3.1).
And in particular, we can take \( k = \mathbb{Z} \), let \( \mathcal{C} \) be the category of \( \mathbb{F}_p \)-vector spaces, and take the polynomial Witt vectors functor \( W_n \) for some integer \( n \). Then for any associative unital \( \mathbb{F}_p \)-algebra \( A \) we obtain the Hochschild–Witt homology groups \( W_nH_* (A) \) and the corresponding periodic cyclic homology groups \( W_nHP_* (A) \). The restriction maps \( R \) induce functorial maps

\[
R : W_{n+1}H_* (A) \to W_nH_* (A),
\]

so that the groups \( W_nH_* (A) \), \( n \geq 1 \), form a projective system. For any \( n, m \geq 0 \), the exact sequence (3.9) induces an exact sequence

\[
0 \to \pi_p ^* W_n A \to V_m W_{n+m} A \to R_n W_m A \to 0
\]

in \( \text{Fun}(\Lambda, \mathbb{Z}) \). Moreover, for any integer \( l \geq 1 \) the embedding \( j^o : \Delta^o \to \Lambda \) lifts canonically to an embedding

\[
j^o_l : \Delta \to \Lambda^l
\]

such that \( \pi_l \circ j^o_l \cong j^o \) and \( i_l \circ j^o_l = j^o \circ i_l \) for a certain functor \( i_l : \Delta^o \to \Delta^o \). The classical Edgewise Subdivision Lemma [34] then asserts, among other things, that for any ring \( k \) and object \( E \in \text{Fun}(\Delta^o, k) \) the natural map

\[
H_* (\Delta^o, i^*_l E) \to H_* (\Delta^o, E)
\]

is an isomorphism. Then for any \( E \in \text{Fun}(\Lambda, k) \) the map \( j^o_* \circ j^o \circ \pi_l ! \) adjoint to the isomorphism \( j^o_* \circ \pi_l ^* \cong j^o_* \) induces a natural map

\[
HH_* (E) \cong H_* (\Delta^o, j^o_* i^*_l E) \to H_* (\Delta^o, j^o_* \pi_l ! i^*_l E) = HH_* (\pi_l ! i^*_l E).
\]

Taking \( E = W_n A \) and \( l = p \) and composing this map with the Verschiebung map (3.8), we obtain a functorial map

\[
V : W_nH_* (A) \to HH_* (\pi_p ^* i^*_p W_n A) \to W_{n+1}H_* (A),
\]

the Hochschild–Witt homology version of the Verschiebung map.

### 4.3. Comparison

The first of the comparison theorems for Hochschild–Witt homology proved in [27] works for any associative unital \( \mathbb{F}_p \)-algebra \( A \), but only in degree 0. Here is the statement.

**Theorem 4.8** ([27], Theorem 5.4). For any \( n \geq 1 \) and any unital associative \( \mathbb{F}_p \)-algebra \( A \), there exists a functorial isomorphism

\[
W_nH_0 (A) \cong W_n^H (A)
\]

of abelian groups, where the \( W_n^H (A) \) are the non-commutative Witt vector groups introduced by Hesselholt in [14]. These isomorphisms are compatible with the restriction maps \( R \) of (4.8) and the Verschiebung maps (4.10).

When \( A \) is commutative, Hesselholt’s Witt vectors coincide with the classical ones, which we will from now on denote by \( W_n^{\text{cl}} (A) \) to avoid confusion. For any \( A \),
as a part of the construction, Hesselholt’s Witt vector groups fit into the short exact sequences

\[ A/[A, A] \xrightarrow{V^n} W^H_{n+1}(A) \xrightarrow{R} W^H_n(A) \longrightarrow 0, \quad (4.11) \]

where \( A/[A, A] = HH_0(A) = W^I_1(A) \) is the quotient of \( A \) by the subspace spanned by commutators. It was further noted in [16] that in general the sequence is not exact on the left. In terms of Theorem 4.8, the sequence (4.11) is the degree-0 part of the homology long exact sequence induced by (4.9), and the fact that it is not exact on the left corresponds to the fact that the connecting differential does not have to be trivial (with an explicit example given in [16]).

The second comparison theorem works in all degrees but only for commutative algebras. To state it, recall that for any smooth algebraic variety \( X \) over \( \mathbb{F}_p \) the de Rham–Witt complex \( W_\ast\Omega_X \) of sheaves on \( X \) was constructed in [18]. This is actually a projective system of complexes \( W_n\Omega_X^n \), \( n \geq 1 \), of sheaves on \( X \) in the Zariski topology that are functorial with respect to \( X \), and equipped with restriction maps \( R: W_{n+1}\Omega_X^n \rightarrow W_n\Omega_X^n \). The first term \( W_1\Omega_X^1 \) is the usual de Rham complex, and for any \( n \) we have \( W_n\Omega_X^n \cong W_n(\mathcal{O}_X) \), the ring of Witt vectors of the structure sheaf \( \mathcal{O}_X \). The complexes \( W_n\Omega_X^n \) are also equipped with functorial Verschiebung maps \( V: W_n\Omega_X^n \rightarrow W_{n+1}\Omega_X^n \) that extend the usual Verschiebung maps on \( W_n\Omega_X^n \). We note that \( V \) does not commute with the differential — instead, one has \( pdV = Vd \).

**Theorem 4.9** ([27], Theorem 6.14). Let \( k = \mathbb{F}_p \), and let \( A \) be an \( \mathbb{F}_p \)-algebra satisfying the assumptions of Theorem 4.4. Then there exist functorial isomorphisms

\[ W_nHH_i(A) \cong H^0(X, W_n\Omega_X^i), \quad n \geq 1, \quad i \geq 0, \quad (4.12) \]

that are compatible with the maps \( R \) and \( V \) and send the Connes–Tsygan differential \( B \) to the de Rham–Witt differential \( d \).

For \( i = 0 \) the identification (4.12) is the same as in Theorem 4.8 (and this is in fact used in the proof of Theorem 4.9). Theorem 4.9 is similar to Theorem 4.4 in that it establishes the Hochschild–Witt homology groups as the correct non-commutative generalization of the de Rham–Witt forms. However, it also clarifies somewhat the structure of the original de Rham–Witt complex of Bloch, Deligne, and Illusie. For example, the limit \( W\Omega_X^n \) of the projective system \( W_n\Omega_X^n \) has a natural decreasing filtration, so that for any \( n \geq 2 \) we have a short exact sequence of complexes

\[ 0 \longrightarrow \text{gr}^n W\Omega_X^n \longrightarrow W_n\Omega_X^n \longrightarrow W_{n-1}\Omega_X^n \longrightarrow 0. \]

Note that \( \text{gr}^n W\Omega_X^n \cong \mathcal{O}_X \) does not depend on \( n \), but in higher degrees the sheaves \( \text{gr}^n \Omega_X^n \) are new functorial sheaves on \( X \) that cannot be expressed in terms of the cotangent bundle and its tensor powers. This is true already for \( n = 2 \). Namely, denote by \( \mathcal{B}_X \subset \mathcal{B}_X \subset \Omega_X^n \) the subsheaves of locally exact forms and locally closed forms, respectively, and recall that we have the functorial Cartier isomorphism

\[ C: \mathcal{B}_X \cong \Omega_X^n. \quad (4.13) \]

Then Illusie proves in [18] that one has a functorial short exact sequence

\[ 0 \longrightarrow \Omega_X^n/\mathcal{B}_X \longrightarrow \text{gr}^2 W\Omega_X^n \longrightarrow \mathcal{B}_X \longrightarrow 0. \quad (4.14) \]
Thus, both $\Omega^*_X = \text{gr}^1 W\Omega^*_X$ and $\text{gr}^2 W\Omega^*_X$ have a two-step filtration with the same associated graded quotients, but what is a subobject in one is the quotient object in the other and vice versa. The extension class represented by the sequence (4.14) is a new characteristic class of smooth algebraic varieties over $\mathbb{F}_p$; there is no way to construct it, short of doing the whole rather opaque construction of the de Rham–Witt complex.

However, if one uses Theorem 4.9, then the quotients $\text{gr}^n W\Omega^*_X$ become much more transparent. Namely, a non-commutative generalization of the Cartier isomorphism (4.13) was given in [24] for odd $p$ and then in [25], §4, for $p = 2$. Using these results, one easily shows that for $A$ as in Theorem 4.9 the connecting differential in the Hochschild homology long exact sequence induced by (4.9) vanishes, so that we have a natural isomorphism

$$H^0(X, \text{gr}^n W\Omega^*_X) \cong HH_i(\pi_{p^n!\pi^*_n A^\natural}).$$

For $n = 2$ the tilting-type relation between $\Omega^*_X$ and $\text{gr}^2 \Omega^*_X$ and the exact sequence (4.14) can be straightforwardly deduced from this if one uses the non-commutative Cartier map from [24]. For the other $n$ the behavior is similar; it is elucidated in [27], Remark 6.13.

5. Further developments and open questions

5.1. Frobenius maps. Let us now describe some additional structures carried by Witt vectors, both commutative and non-commutative. We start with the Frobenius map. In the classical situation, the statement is as follows.

**Proposition 5.1.** There exists a unique collection of maps

$$F: W^\text{cl}_{n+1}(A) \to W^\text{cl}_n(A), \quad n \geq 1,$$

that are additive, functorial with respect to the commutative ring $A$, and satisfy

$$F \circ R = R \circ F \quad \text{and} \quad w_n \circ F = w_{n+1}, \quad n \geq 1.$$

Moreover, these maps also satisfy

$$F \circ V = p \text{id} \quad \text{and} \quad F(T(a)) = a^p, \quad a \in A,$$

where $T: A \to W_*(A)$ are the Teichmüller representative maps.

**Proof.** To find the maps $F$ such that

$$w_n \circ F = w_{n+1} \quad \text{and} \quad F \circ R = R \circ F,$$

it suffices to find universal polynomials $f_i(a_0, \ldots, a_i), i \geq 1$, such that for any $n \geq 1$,

$$f_1^{p^n-1} + pf_2^{p^n-2} + \cdots + p^{n-1}f_n = a_0^{p^n} + pa_1^{p^n-1} + \cdots + p^n a_n. \quad (5.1)$$

To do this, assume by induction that we have found the $f_i$ for $i < n$. Then to show that (5.1) provides a well-defined polynomial $f_n$ with integer coefficients, it suffices to observe that by Lemma 1.2,

$$b^{p^n-1} = b^{p^n} \mod p^n$$

for any integer $n \geq 1$ and any integer $b$. 

To prove that the resulting maps $F$ are unique, additive, and satisfy all the equalities claimed, it suffices to consider the universal situation $A = \mathbb{Z}$. But then the ghost maps

$$w_\ast = \langle w_1, \ldots, w_n \rangle : W_n^{cl}(A) \to A^n$$

are injective, so all the claims follow from the compatibility between $w_\ast$ and $F$. \(\square\)

We note that it also follows from (5.1) that $f_n = a_{n-1}^p \mod p$, $n \geq 1$. Therefore, if $A$ consists entirely of $p$-torsion, and thus has a Frobenius endomorphism $\varphi : A \to A$, $a \mapsto a^p$, then $F : W_{n+1}(A) \to W_n(A)$ is given by $F = W_n^{cl}(\varphi) \circ R$ and provides a canonical additive lifting of the Frobenius map $\varphi$ to Witt vectors. By abuse of terminology, $F$ is also called the Frobenius endomorphism for a general ring $A$.

A special feature of the case $pA = 0$ is the additional identity $VF = p\text{id}$. Indeed, this is an identity on universal polynomials with coefficients in $\mathbb{F}_p$, so to check it, it suffices to check that it holds after evaluation at elements in the algebraic closure $\overline{\mathbb{F}}_p$. In other words, we have to prove that $VF = p\text{id}$ for $A = \mathbb{F}_p$. But then $\varphi : \mathbb{F}_p \to \mathbb{F}_p$ is a bijection, so that $F = W(\varphi)$ is a bijection as well, and $FVF = pF$ implies that $VF = p\text{id}$.

Defining a Frobenius map is even simpler for polynomial Witt vectors $W_n(M)$ with $M$ an $\mathbb{F}_p$-vector space. Namely, for any free $\mathbb{Z}$-module $N$ and integer $n \geq 1$, the tautological identification $(N \otimes p^n) \cong (N \otimes p)^{\otimes p^{n-1}}$ induces a map of functors $Q_n(N) \to Q_{n-1}(N \otimes p)$. This map is compatible with the trace functor structure, and descends to a functorial map

$$F : W_n(M) \to W_{n-1}(M \otimes p)^{\overline{\sigma}}$$

for any $\mathbb{F}_p$-vector space $M$, where $\overline{\sigma}$ is induced by the trace functor structure. It is also immediately obvious from the construction that

$$F \circ V = \text{id} + \overline{\sigma} + \cdots + \overline{\sigma}^{p-1},$$

where $V$ is the Verschiebung map (2.6). For any associative unital $\mathbb{F}_p$-algebra $A$ the map (5.2) induces a map

$$F : W_nA^\natural \to \pi_{p!}i_p^\ast W_{n-1}A^\natural$$

(5.3)

of functors from $\Lambda$ to abelian groups, and

$$F \circ V : \pi_p!i_p^\ast W_nA^\natural \to \pi_p!i_p^\ast W_nA^\natural$$

is the trace map $\text{tr}_p$ of (3.7). On the level of Hochschild–Witt homology we have maps

$$F : W_{n+1}HH_\ast(A) \to W_nHH_\ast(A),$$

and it was proved in [27], Corollary 1.10, that $\overline{\sigma}$ acts trivially on individual Hochschild homology groups, so that

$$F \circ V = p\text{id} : W_nHH_\ast(A) \to W_nHH_\ast(A).$$
Moreover, it was proved in [27], Lemma 2.7, that

\[ FBV = B, \]  

(5.4)

where \( B : W_n \mathcal{H} \mathcal{H}_\ast(A) \to W_n \mathcal{H} \mathcal{H}_{\ast+1}(A) \) is the Connes–Tsygan differential.

As far as the comparison theorems in §4.3 are concerned, we note that Hesselholt does not directly construct a Frobenius endomorphism on his Witt vector groups, but they inherit it from the Topological Hochschild Homology spectrum, since he does prove that the limit group \( W^H(A) \) coincides with \( \pi_0(TR(A, p)) \) in [3]. The identification in Theorem 4.8 is probably compatible with the Frobenius maps but this has not been checked. The de Rham–Witt complex \( W \Omega^q_X \) of a smooth algebraic variety \( X \) over \( \mathbb{F}_p \) also has a Frobenius endomorphism \( F \); in fact, this is an integral part of the construction in [18]. The endomorphisms \( F \) and \( \nu \) of the de Rham–Witt complex commute: \( F \nu = \nu F = \nu_1 \), and moreover, \( F d \nu = d \), where \( d \) is the differential in the complex. Our identification in Theorem 4.9 is compatible with \( F \), and the equality \( F d \nu = d \) corresponds to the general non-commutative equality (5.4).

### 5.2. Multiplication

Next, let us describe the structure that we omitted in Theorem 1.3—namely, the structure of a commutative ring.

**Theorem 5.2.** There exists a unique collection of functorial unital commutative ring structures on the abelian groups \( W^{\mathrm{cl}}_n(A), \ n \geq 1 \), such that the restriction maps \( R \) and the ghost maps \( w_n \) are ring maps. Moreover, with respect to these ring structures,

\[ F(a \cdot b) = F(a) \cdot F(b), \quad V(a) \cdot b = V(a \cdot F(b)), \quad a, b \in W^{\mathrm{cl}}_n(A), \]

and

\[ T(ab) = T(a) \cdot T(b), \quad a, b \in A. \]

**Proof.** By induction, we may assume that we have constructed the ring structures on the \( W^{\mathrm{cl}}_l(A) \) for \( l \leq n \). We decompose

\[ W^{\mathrm{cl}}_{n+1}(A) = A^{n+1} = A \times A^n \cong A \times W^{\mathrm{cl}}_n(A), \]

so that \( R^n : W^{\mathrm{cl}}_{n+1}(A) \to A \) is the projection onto the first factor, and we define the product by

\[ \langle a_0, b_0 \rangle \cdot \langle a_1, b_1 \rangle = \langle a_0 a_1, b_0 \cdot F(\langle a_1, b_1 \rangle) + a_0^p b_1 \rangle, \]  

(5.5)

where \( F : W^{\mathrm{cl}}_{n+1}(A) \to W^{\mathrm{cl}}_n(A) \) is the Frobenius map of Proposition 5.1. Then it follows immediately from the inductive assumption that \( w_n \) is multiplicative with respect to the product (5.5). Now, as in Proposition 5.1, to check that (5.5) defines a commutative associative unital product compatible with the additive structure and satisfying all the identities, it suffices to consider the case \( A = \mathbb{Z} \), where everything follows from compatibility with the ghost map. The same applies to the uniqueness assertion. \( \square \)

We note that as an immediate corollary of Theorem 5.2, we see that the image \( VW^{\mathrm{cl}}(A) \subset W^{\mathrm{cl}}(A) \) of the Verschiebung map is an ideal in the ring \( W^{\mathrm{cl}}(A) \), and

\[ A = W^{\mathrm{cl}}(A) / VW^{\mathrm{cl}}(A). \]
If $pA = 0$, then the equality $VF = p\text{id}$ implies that $pW^{cl}(A) \subset W^{cl}(A)$ is contained in $VW^{cl}(A)$. If moreover $A$ is perfect—that is, the Frobenius map $\varphi: A \to A$ is a bijection—then $VW^{cl}(A) = pW^{cl}(A)$, so that $W^{cl}(A)$ is a $p$-adic lifting of the ring $A$ in the most naive sense.

For polynomial Witt vectors, the product survives in the form of external multiplication. Namely, for any two free $\mathbb{Z}$-modules $N_0$, $N_1$ and any integer $l$, we have a canonical isomorphism

$$(N_1 \otimes N_2)^{\otimes l} \cong N_1^{\otimes l} \otimes N_2^{\otimes l},$$

and this induces a functorial map

$$\mu: W_n(M_0) \otimes W_n(M_1) \to W_n(M_0 \otimes M_1) \quad (5.6)$$

for any $\mathbb{F}_p$-vector spaces $M_0$, $M_1$ and any integer $n \geq 1$.

**Proposition 5.3.** The map (5.6) is commutative, associative, and unital in the obvious sense. Moreover,

$$R \circ \mu = \mu \circ (R \otimes R), \quad F \circ \mu = \mu \circ (F \otimes F), \quad \text{and} \quad \mu \circ (V \otimes \text{id}) = V \circ \mu \circ (\text{id} \circ F). \quad (5.7)$$

**Proof.** See [26], Lemma 1.5 and Proposition 3.10. \qed

**Corollary 5.4.** For any $\mathbb{F}_p$-vector space $M$ and integer $n$,

$$V \circ F = p\text{id}: W_n(M) \to W_n(M).$$

**Proof.** Take $M_0 = M$, $M_1 = \mathbb{F}_p$, and apply (5.7) and the identity $VF = p\text{id}$ in $W(\mathbb{F}_p) = \mathbb{Z}_p$. \qed

Here is another application of the product (5.6). Assume given a finite-dimensional $\mathbb{F}_p$-vector space $M$, and let $M^*$ be the dual vector space. Then for any $n \geq 1$ the natural pairing $M \otimes M^* \to \mathbb{F}_p$ combined with the product map (5.6) induces a functorial map

$$W_n(M) \otimes W_n(M^*) \to W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}.$$  

One shows (see [26], Lemma 3.12) that this is in fact a perfect pairing, and $W_n(M)$ and $W_n(M^*)$ are dual $\mathbb{Z}/p^n\mathbb{Z}$-modules (note that typically they are not flat, but since $\mathbb{Z}/p^n\mathbb{Z}$ is Gorenstein, the duality is well behaved). The duality interchanges the Verschiebung map $V$ and the Frobenius map $F$. In fact, it has been shown in [26], Lemma 3.7, that the short exact sequence (3.5) has a functorial dual sequence

$$0 \to W_n(M) \to W_{n+1}(M) \xrightarrow{F} W_n(M \otimes p^n)^\tau \to 0,$$

where $C: W_n(M) \to W_{n+1}(M)$ is a certain functorial corestriction map dual to the restriction map $R$ and satisfying $RC = CR = p\text{id}$.

In the simplest non-trivial case $n = 2$ the picture can be explained as follows. For any $\mathbb{F}_p$-vector space $M$ one has a natural functorial exact sequence

$$0 \to M \xrightarrow{\psi} (M \otimes p)_\sigma \xrightarrow{\text{tr}_p} (M \otimes p)_\sigma \xrightarrow{\hat{\psi}} M \to 0, \quad (5.8)$$

$$\text{tr}_p: (M \otimes p)_\sigma \to (M \otimes p)_\sigma$$
where \( \psi \) sends \( m \in M \) to \( m \otimes p \) and \( \hat{\psi} \) is dual. This sequence represents a certain \( \text{Ext}^2 \)-class in the category of functors from \( \mathbb{F}_p \)-vector spaces to \( \mathbb{F}_p \)-vector spaces, and this class is non-trivial, so that (5.8) does not admit a functorial splitting (for more details on this, see [23], §6). However, the exact sequence splits if we consider it in the category of functors from \( \mathbb{F}_p \)-vector spaces to abelian groups, and \( W_2(M) \) is exactly the splitting object. It has a functorial three-step filtration \( F^* \) with associated graded terms \( M, \Phi(M), \) and \( M, \) where \( \Phi(M) \) is the image of the map \( \text{tr}_p \) in (5.8), and with terms

\[
F^2W_1(M) \cong M, \quad F^1W_2(M) \cong (M \otimes p)_\sigma, \quad \text{and} \quad F^0W_2(M) = W_2(M),
\]

while the quotients are given by

\[
W_2(M)/F^1W_2(M) \cong M \quad \text{and} \quad W_2(M)/F^2W_2(M) \cong (M \otimes p)^\sigma.
\]

In other words, \( W_2(M) \) can be represented as either an extension of \( M \) by \( (M \otimes p)_\sigma \) or as an extension of \( (M \otimes p)^\sigma \) by \( M \). As an abelian group, \( W_2(M) \) is annihilated by \( p^2 \) but not by \( p \); multiplication by \( p \) acts by the composition

\[
W_2(M) \xrightarrow{R} W_1(M) \cong M \xrightarrow{C} W_2(M)
\]

of the projection \( R \) onto the top quotient \( F^0/F^1 \) of the filtration and the embedding \( C \) of the bottom piece \( F^2 \). For higher \( n \) the picture is similar, although the filtration now has \( n(n+1)/2 \) associated graded pieces; for details see [26], §3.1.

The map (5.6) is also compatible with the trace functor structure. The precise meaning of this compatibility can be found in [26], §4.3, and the practical implication is that there are natural maps

\[
\mu: W_nA^\natural \otimes W_nB^\natural \to W_n(A \otimes B)^\natural
\]

for any two associative unital \( \mathbb{F}_p \)-algebras \( A \) and \( B \). Composing these maps with the Küneth isomorphism, we obtain maps

\[
\mu: W_nHH_*(A) \otimes W_nHH_*(B) \xrightarrow{L} W_nHH_*(A \otimes B), \quad (5.9)
\]

where \( W_nHH_*(A) \) and \( W_nHH_*(B) \) are tacitly understood as objects in the derived category of abelian groups, and \( \otimes \) is the derived tensor product. These maps also satisfy (5.7). If \( A \) is commutative, then the multiplication map \( A \otimes A \to A \) is an algebra map, so that \( W_nHH_*(A) \) becomes a commutative associative algebra. In the situation of Theorem 4.9 this algebra structure is identified with the standard algebra structure on the de Rham–Witt complex under the isomorphisms (4.12).

### 5.3. Extending the definition

So far, for simplicity we have only defined polynomial Witt vectors for vector spaces over \( \mathbb{F}_p \). However, when defining \( W_n(M) \), it is not necessary to represent \( M \) as a quotient \( N/pN \) of a free \( \mathbb{Z} \)-module \( N \)—it was shown in [26], Lemma 2.1, that already for a free \( \mathbb{Z}/p^n\mathbb{Z} \)-module \( N \) with \( M \cong N/pN \) we have \( W_n(M) \cong Q_n(N) \). Therefore, if we fix a perfect field \( k \) of characteristic \( p \), then we can consider the category of free modules over \( W_n(k) \) and the ring of
n-truncated Witt vectors of the field $k$, and repeat Definition 2.3 verbatim for such free modules. Then Corollary 2.6 still holds, and we obtain a functor $W_n$ from $k$-vector spaces to abelian groups. All the other results about polynomial Witt vectors also hold—in fact, [26] and [27] work in this greater generality from the very beginning. In particular, we have the Verschiebung maps $V$, the Frobenius maps $F$, the restriction maps $R$, and the corestriction maps $C$ (although one has to keep in mind that these maps are not necessarily $W(k)$-linear—for example, $F$ is semilinear with respect to the Frobenius endomorphism of $W(k)$). We also have the Hochschild–Witt homology groups and the product (5.6). Theorem 4.9 also holds in this greater generality, with $W_n^*\Omega^n_X$ taken to be the relative de Rham–Witt complex over $k$.

Another and more substantial extension concerns algebraic varieties that are not necessarily affine. One could try to incorporate this case by considering sheaves of non-commutative algebras, but this is not what arises in practical applications. In fact, it is a well-established principle of non-commutative geometry that 'every non-commutative variety is affine in the derived sense’, or in other words, that the correct basic object to consider is an associative unital DG-algebra, or a small DG-category (for example, see [29] or [32], §2, [31], §1, for a detailed discussion of this point). What one would like to have then is the theory of Hochschild–Witt homology groups $W_n^q HH_q(A_q)$ of a DG-algebra $A_q$ over $k$.

A machine providing just such a theory has in fact been constructed in [22]. As an input, it needs a trace functor $F$ from $k$-vector spaces to abelian groups that has two additional properties: it should be balanced in the sense of [22], Definition 3.9, and localizing in the sense of [22], Definition 5.5. Both properties are closed under extensions (see [22], Lemmas 3.11 and 5.6) and hold for the cyclic power functor $M \mapsto M^\otimes_n$, $n \geq 1$ (see [22], Lemma 5.7 and Proposition 5.10). Therefore, by (3.5) with $m = 1$ they also hold for $W_n$, $n \geq 1$. What the machine produces is a collection of twisted Hochschild homology functors $F HH_q(A_q)$ for small DG-categories $A_q$ over $k$. These functors are derived-Morita invariant in the sense of [29], §4.6, and moreover, are additive invariants in the sense of [29], §5.1. Plugging in the polynomial Witt vectors functors $W_n$, we obtain Hochschild–Witt homology functors $W_n^q HH_q(A_q)$, $n \geq 1$, that are additive invariants, just as they should be. They are also related by the maps $F$, $V$, $R$, and $C$, and carry the external product (5.9).

Unfortunately, the machine of [22] is rather indirect and cumbersome (it works by replacing a chain complex by a simplicial-cosimplicial abelian group). There might be a more straightforward approach to the problem that starts by extending Definition 2.3 directly to chain complexes. This has not been done yet, although a way to do it for $W_2$ was sketched in [23], §6.

One further result that is not at all automatic is a Künneth-type isomorphism for Hochschild–Witt homology. The difficulty here is that it certainly does not exist on the level of Hochschild homology groups—nor of de Rham–Witt forms, in the commutative case. For de Rham–Witt forms, we do have a Künneth isomorphism once we turn on the differential (this gives crystalline cohomology). This suggests that for general DG-algebras $A_q$ and $B_q$ the map (5.9) should induce an isomorphism on periodic cyclic homology groups. So far, this has not been checked.
Another interesting question that comes up in applications to DG-algebras is the degeneration of the Hodge-to-de Rham spectral sequence (4.3). For DG-algebras $A_q$ over a ring $k$ that contains $\mathbb{Q}$, it was conjectured by Kontsevich and Soibelman that (4.3) degenerates at the first term if $A_q$ is homologically smooth and homologically proper (these properties correspond to the usual smoothness and properness of algebraic varieties; see [29] and [32], §2 for more details). The conjecture was recently proved in [25], and an even more general statement was established in [7]. The method of the proof is that of Deligne and Illusie [6], and it works essentially by reducing the question to positive characteristic. However, the argument in [6] appeared in fact as a simplification of a degeneration established already in [18] for the de Rham–Witt complex. Thus, it is natural to expect that the sequence (4.3) for Hochschild–Witt homology also degenerates when the DG-category $A_q$ is smooth and proper—either literally, or at least when one takes the limit with respect to $n$ and inverts $p$. However, at present this has not been studied at all.

5.4. Big Witt vectors. Yet another generalization of the Witt vectors definition is actually quite old, and exists already on the classical level—this is the theory of ‘big’ or ‘universal’ Witt vectors that combines together the theories for all primes $p$.

Classically, for any unital commutative ring $A$ one considers the product $W(A) = A^N$ of copies of $A$ numbered by positive integers $m \geq 1$ and defines the universal ghost map $\hat{w}_m : W(A) \to A^N$ by

$$\hat{w}_m(a_1, \ldots, a_m) = \sum_{d|m} d a_d^{m/d}, \quad (5.10)$$

where the sum is over all the integers $d \geq 1$ that divide $m$. We note that for any prime $p$ and any integer $n \geq 1$ the map $\hat{w}_{p^n}$ depends only on the components $a_{p^i}$, $0 \leq i \leq n$, and up to a renumbering of these components it coincides with the ghost map of (1.2), said to be $p$-typical in this context. One then equips $W(A)$ with the product topology and proves that there exists a unique functorial continuous commutative ring structure on $W(A)$ such that (5.10) is a ring map for any $m \geq 1$.

The proof can be done along the same line as Theorem 1.3, but there is an attractive alternative: the additive group structure on $W(A)$ is visible right away. Namely, one shows that there is a functorial isomorphism

$$W(A) \cong 1 + tA[[t]] \subset A[[t]], \quad (5.11)$$

where $A[[t]]^*$ is the group of invertible formal power series in one formal variable $t$ with coefficients in $A$, considered with respect to multiplication. Explicitly, the isomorphism (5.11) sends an element $\langle a_1, a_2, \ldots \rangle \in W(A)$ to the series

$$\sum_{i \geq 1} (1 - a_i t^i) \in 1 + tA[[t]].$$

The fact that this is an isomorphism is obvious by induction, and the fact that (5.10) is additive is a simple computation.

The product in the ring $W(A)$ is not immediately obvious from the isomorphism (5.11). To see it, one can use an interpretation due to Almkvist [1] that
exhibits a dense subgroup in $\mathbb{W}(A)$ spanned by characteristic polynomials of matrices over $A$, with addition corresponding to the direct sum of matrices—and then one shows that the product corresponds to the tensor product. Alternatively, it has been shown in [20] how to write down the product on the whole of $\mathbb{W}(A)$ in terms of the Tate residue (or more precisely, the Contou-Carrère residue; for example, see [9]–[11] and references there). To obtain this interpretation, one starts by observing that $\mathbb{W}(A)$ is the kernel of the split surjective map 

$$K_1(A[[t]]) \to K_1(A)$$

of the first algebraic $K$-groups of $A[[t]]$ and $A$. This also suggests a way to obtain a polynomial version of the functor $\mathbb{W}$. Namely, for any perfect field $k$ and a $k$-vector space $M$, let $\widehat{T}^*(M/k)$ be the completion of the tensor algebra $T^*(M/k)$ generated by $M$ over $k$ with respect to the augmentation ideal $T^\geq 1(M/k) \subset T^*(M/k)$. Then we also have a split surjective map 

$$K_1(\widehat{T}^*(M/k)) \to K_1(k)$$

and can define $\mathbb{W}(M/k)$ as the kernel of this map. It seems that this gives a good object equipped with all the additional structures one would like, such as Frobenius and Verschiebung maps for each prime $p$, a product similar to (5.6), and a trace functor structure that allows one to define the universal Hochschild–Witt homology groups. However, at present there is no written proof of all this, and we will return to the topic in the forthcoming paper [28].

To connect the universal and the $p$-typical theory, one can compute the ring $\mathbb{W}(\mathbb{Z})$ of universal Witt vectors. As a group, we have

$$\mathbb{W}(\mathbb{Z}) = \mathbb{Z}\langle \varepsilon_1, \varepsilon_2, \ldots \rangle,$$

the group of (possibly infinite) linear combinations of the elements $\varepsilon_n$, $n \geq 1$. The product is given by

$$\varepsilon_i \varepsilon_j = \frac{ij}{\{i,j\}} \varepsilon_{\{i,j\}},$$

where $\{i,j\}$ is the least common multiple of the integers $i, j \geq 1$. Then by functoriality $\varepsilon_i$ acts on $\mathbb{W}(A)$ for any commutative ring $A$, and if $A$ is $p$-local—that is, any integer $n$ prime to $p$ acts on $A$ by an invertible map—then $(1/n)\varepsilon_n$ is an idempotent endomorphism of the ring $\mathbb{W}(A)$. Taken together, these idempotents generate a decomposition

$$\mathbb{W}(A) = \prod_n W(A)$$

into the product of copies of $W(A)$ numbered by integers $n \geq 1$ prime to $p$ (this is known as the $p$-typical decomposition). For polynomial Witt vectors the same decomposition seems to exist; to construct it one needs to use the technology of $\mathbb{Z}$-Mackey profunctors developed in [21], §9.2.

Bibliography

[1] G. Almkvist, “The Grothendieck ring of the category of endomorphisms”,

$J. \text{ Algebra}$ $28$:3 (1974), 375–388.
[2] S. Bloch, “Algebraic K-theory and crystalline cohomology”, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 187–268.

[3] M. Bökstedt, W.C. Hsiang, and I. Madsen, “The cyclotomic trace and algebraic K-theory of spaces”, *Invent. Math.* **111**:3 (1993), 465–539.

[4] A. Connes, “Cohomologie cyclique et foncteur Ext“”, *C. R. Acad. Sci. Paris Sér. I Math.* **296**:23 (1983), 953–958.

[5] A. Connes, “Non-commutative differential geometry”, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360.

[6] P. Deligne and L. Illusie, “Relèvements modulo $p^2$ et décomposition du complexe de de Rham”, *Invent. Math.* **89**:2 (1987), 247–270.

[7] А.И. Ефимов, “Обобщенная гипотеза о некоммутативном вырождении”, *Современные проблемы математики, механики и математической физики*, Сборник статей, Тр. МИАН, **290**, МАИК, М. 2015, с. 7–17; English transl., А.И. Ефимов, “Generalized non-commutative degeneration conjecture”, *Proc. Steklov Inst. Math.* **290**:1 (2015), 1–10.

[8] B.L. Feigin and B.L. Tsygan, “Additive K-theory”, *K-theory, arithmetic and geometry* (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin 1987, pp. 67–209.

[9] С.О. Горчинский, Д.В. Осипов, “Многомерный символ Конту-Каррера: локальная теория”, *Матем. сб.* **206**:9 (2015), 21–98; English transl., S.O. Gorchinskiy and D.V. Osipov, “A higher-dimensional Contou-Carrè re symbol: local theory”, *Sb. Math.* **206**:9 (2015), 1191–1259.

[10] С.О. Горчинский, Д.В. Осипов, “Явная формула для многомерного символа Конту-Каррера”, *УМН* **70**:1(421) (2015), 183–184; English transl., S.O. Gorchinskiy and D.V. Osipov, “Explicit formula for the higher-dimensional Contou-Carrè re symbol”, *Russian Math. Surveys* **70**:1 (2015), 171–173.

[11] С.О. Горчинский, Д.В. Осипов, “Многомерный символ Конту-Каррера и непрерывные автоморфизмы”, *Функц. анализ и его прил.* **50**:4 (2016), 26–42; English transl., S.O. Gorchinskiy and D.V. Osipov, “Higher-dimensional Contou-Carrè re symbol and continuous automorphisms”, *Funct. Anal. Appl.* **50**:4 (2016), 268–280.

[12] M. Raynaud, “Catégories fibrée et descente”, *Revêtements étales et groupe fondamental (SGA 1)*, Séminaire de géométrie algébrique du Bois Marie 1960–61, 2nd ed., Doc. Math. (Paris), vol. 3 (A. Grothendieck, ed.), Soc. Math. France, Paris 2003, Exp. VI.

[13] L. Hesselholt, “On the $p$-typical curves in Quillen’s K-theory”, *Acta Math.* **177**:1 (1996), 1–53.

[14] L. Hesselholt, “Witt vectors of non-commutative rings and topological cyclic homology”, *Acta Math.* **178**:1 (1997), 109–141.

[15] L. Hesselholt, “Algebraic K-theory and trace invariants”, *Proceedings of the international congress of mathematicians*, vol. II: *Invited lectures* (Beijing 2002), Higher Education Press, Beijing 2002, pp. 415–425.

[16] L. Hesselholt, Correction to: ‘Witt vectors of non-commutative rings and topological cyclic homology’ [Erratum to [14]], *Acta Math.* **195** (2005), 55–60.

[17] G. Hochschild, B. Kostant, and A. Rosenberg, “Differential forms on regular affine algebras”, *Trans. Amer. Math. Soc.* **102**:3 (1962), 383–408.

[18] L. Illusie, “Complexe de de Rham–Witt et cohomologie cristalline”, *Ann. Sci. École Norm. Sup.* (4) **12**:4 (1979), 501–661.

[19] D. Kaledin, “Non-commutative Hodge-to-de Rham degeneration via the method of Deligne–Illusie”, *Pure Appl. Math. Q.* **4**:3 (2008), 785–875.
D. Kaledin, “Universal Witt vectors and the ‘Japanese cocycle’”, *Mosc. Math. J.* **12**:3 (2012), 593–604.

D. Kaledin, *Mackey profunctors*, 2014, 104 pp., arXiv:1412.3248.

D. Kaledin, “Trace theories and localization”, *Stacks and categories in geometry, topology, and algebra*, Contemp. Math., vol. 643, Amer. Math. Soc., Providence, RI 2015, pp. 227–262.

D. Kaledin, “Bockstein homomorphism as a universal object”, *Adv. in Math.* **324** (2018), 267–325; 2015, 63 pp., arXiv:1510.06258.

D. Kaledin, “Cartier isomorphism for unital associative algebras”, *Proc. Steklov Inst. Math.* **290**:1 (2015), 35–51.

D. Kaledin, “Spectral sequences for cyclic homology”, *Algebra, geometry, and physics in the 21st century*, Progr. Math., vol. 324 (D. Auroux, L. Katzarkov, T. Pantev, Y. Soibelman, Y. Tschinkel, eds.), Birkhäuser, Cham 2017, pp. 99–129, arXiv:1601.00637.

D. Kaledin, “Witt vectors as a polynomial functor”, *Selecta Math. (N. S.)* **24**:1 (2018), 359–402; vers. 3, 2017 (v1 – 2016), 49 pp., arXiv:1602.04254v3.

D. Orlov, “Smooth and proper noncommutative schemes and gluing of DG categories”, *Adv. Math.* **302** (2016), 59–105.

G. Segal, “Configuration-spaces and iterated loop-spaces”, *Invent. Math.* **21**:3 (1973), 213–221.

J.-P. Serre, “Sur la topologie des variétés algébriques en caractéristique $p$”, *International symposium on algebraic topology*, Universidad Nacional Autónoma de México and UNESCO, Mexico City 1958, pp. 24–53.

E. Witt, “Zyklische Körper und Algebren der Charakteristik $p$ vom Grad $p^n$. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik $p^n$”, *J. Reine Angew. Math.* **1937**:176 (1937), 126–140.