Abstract

We consider non-bijective piecewise rotations of the plane. These maps belong to a family introduced in previous papers by Boshernitzan and Goetz. We derive in this paper some upper bounds to the size of the limit set. This improves results of [5].

1 Introduction

Consider a line $D$ in $\mathbb{R}^2$, it splits the plane in two half-planes. Now define a dynamical system $T$ on $\mathbb{R}^2$ such that the restriction to each half-plane of $T$ is given by a euclidean rotation. The two rotations are of the same angle but with different centers, which may well lie outside of the corresponding half-plane.

This type of map is called a piecewise isometry. A lot of examples have been studied in the last years, see [10], [2] or [6]. The maps studied here have the advantage of belonging to a family with a lot of interesting properties, see [5], [11] and [7] for references.

Throughout this paper we follow the notation and conventions of Boshernitzan-Goetz [5]. They studied non-bijective maps. They prove that either every orbit is bounded uniformly by some constant $M$, or else every point outside a ball of radius $M$, has a divergent orbit.

The bijective case was not treated in this paper; it has been done by Goetz and Quas for a rational angle in the symmetric case (i.e. the middle of the segment between the two centers of rotations belongs to the line $D$), see [11]. In [4] two authors of this paper give a complete description of the bijective symmetric map if the angle belongs to a specified finite set. In [3] they describe the non-symmetric bijective maps, assuming the angle belongs to a particular finite set.

The aim of this paper is to find, for any non-bijective map in the family, an explicit bound for $M$, for every given angle. If the angle is rational, the strategy is to use the proof of [5] and improve it, where this is possible. For the irrational case, a complete different approach is presented.

1.1 Outline of the paper

In Section 2 we recall the definition of our maps, give some notations and introduce some tools used later. In Section 3 we introduce the notion of limit sets which allows us to state correctly a previously known result, see Theorem [15] Moreover we give some results on limit sets and periodic islands, see Proposition [19] and Proposition [20].

In Section 4 we consider the case where the angle is an irrational multiple of $\pi$ and we prove Theorem [23]. Finally in Section 5 we consider the rational case $p/q$ where $q > 2$ is an even number, see Theorem [31].
2 Definitions

2.1 Definitions of a piecewise rotation

Consider the euclidean space $\mathbb{R}^2$ equipped with an orientation. In the following we will use the classical identification between $\mathbb{C}$ and the euclidean plane. Let us fix a line $D$. The line cuts the plane in two half planes, denoted by $P_0$ and $P_1$. Moreover we assume that $P_1$ is closed, and that $P_0$ is open, in order to have a partition of $\mathbb{C}$.

Definition 1. Now we fix a real number $\alpha$ and two points $C_0, C_1$. We define $\mathcal{T}$ as the set of maps $T$:

$$
\mathbb{C} \to \mathbb{C}
$$

$$
z \mapsto T(z) = e^{i\alpha}(z - C_j) + C_j \quad \text{if} \quad z \in P_j, j \in \{0, 1\}.
$$

Remark 2. The cases with $C_0 = C_1$ or $\alpha = 0$ will not be treated in the sequel. But the dynamic is obvious in these cases.

Notations 1. Let us fix some notation, see Figure 1:

- We parametrize the line $D$ by a point $z_0$ and an angle $\gamma$. The angle $\gamma$ is chosen so that $z_0 + ie^{i\gamma}$ belongs to $P_0$.
- We call $r_0, r_1$ the isometries of the plane which coincide with the restriction of $T$ to the two half planes: for all $z \in \mathbb{C}$, $r_j(z) = e^{i\alpha}(z - C_j) + C_j$ with $j \in \{0, 1\}$.
- The two centers have complex coordinates given by $C_j = |C_j|e^{i\beta_j}$, and we denote $\beta = \arg(C_1 - C_0)$.
- In the following we will need the quantity $\Delta = -2|C_1 - C_0| \times \sin(\alpha/2) \cos(\gamma + \alpha/2 - \beta)$.

These notations shows that the map $T$ is given by some parameters. The following remark will explain how to reduce the number of such parameters.

Remark 3 (Conjugacy). Let $T \in \mathcal{T}$, and consider the maps $f_0 : z \mapsto \rho e^{i\theta}z + b$ and $f_1 : z \mapsto \rho e^{i\theta}z + b \quad \text{with} \quad \rho \in \mathbb{R}_+, \theta \in \mathbb{R} \quad \text{and} \quad b \in \mathbb{C}$. Then we denote $T_j = f_j \circ T \circ f_j^{-1}$ for $j \in \{0, 1\}$. This map is in $\mathcal{T}$, and we have the following parameters:

| $T$  | $\alpha$ | $C_0$  | $C_1$  | $\beta$ | $\gamma$ | $\Delta$ |
|-----|----------|--------|--------|--------|----------|--------|
| $T_0$ | $\alpha$ | $f_0(C_0)$ | $f_0(C_1)$ | $\theta + \beta$ | $\theta + \gamma$ | $\rho \Delta$ |
| $T_1$ | $-\alpha$ | $f_1(C_0)$ | $f_1(C_1)$ | $\theta - \beta$ | $\theta - \gamma$ | $-\rho \Delta$ |

Thus we can always assume $0 \in D$, and the given data for $\gamma, \alpha, C_0, C_1$ determine the map $T$. We call these data the parameters of $T$. 

Figure 1: Example of an application $T \in \mathcal{T}$ with $C_0 = 1 + 1.5i, C_1 = 4.5 + 0.25i, \gamma = \pi - \arctan(2/3)$. 

Thus we can always assume $0 \in D$, and the given data for $\gamma, \alpha, C_0, C_1$ determine the map $T$. We call these data the parameters of $T$. 

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Figure 2: Injective example: same parameters as in Figure 1 with an angle \( \alpha = -\pi/10 \).

Figure 3: Surjective example: same parameters as in Figure 1 with an angle \( \alpha = \pi/10 \).

Now we recall a known result for the maps from \( \mathcal{T} \), see [5]. For completeness we include a proof.

**Lemma 4.**

- A map \( T \in \mathcal{T} \) is surjective if and only if \( \Delta \geq 0 \). In this case the strip \( T(P_0) \cap T(P_1) \) has width \( 2\Delta \).
- A map \( T \in \mathcal{T} \) is injective if and only if \( \Delta \leq 0 \). In this case the set \( \mathbb{R}^2 \setminus (T(P_0) \cup T(P_1)) \) is a strip of width \( -2\Delta \).

**Proof.** The image of \( D \) by \( r_0, r_1 \) are two parallel lines. Thus the images of the two half planes intersect or not. It is enough to compute the following scalar product to conclude, see Figure 2. We have for all \( z \in D \):

\[
\langle r_1(z - ie^{iy}) - r_1(z), r_0(z) - r_1(z) \rangle = \langle -e^{ia}e^{iy}, e^{ia}(C_1 - C_0) - (C_1 - C_0) \rangle \\
= \langle -e^{ia}e^{iy}, (e^{ia} - 1)(C_1 - C_0) \rangle \\
= \langle e^{ia}e^{iy}, 2i \sin(a/2)e^{ia/2}|C_1 - C_0|e^{i\beta} \rangle \\
= -2|C_1 - C_0| \sin(a/2) \left\langle e^{ia}e^{iy}, ie^{ia/2}e^{i\beta} \right\rangle = \Delta \quad \blacksquare
\]

### 2.2 Tools

**Definition 5.** Consider a map \( T : \mathbb{C} \rightarrow \mathbb{C} \). We define:

\[
\|T\| = \sup \{ |T(z)| - |z|, z \in \mathbb{C} \} \in \mathbb{R}_+ \cup \{+\infty\}
\]

**Lemma 6.** For all \( k \geq 0 \) and \( z \in \mathbb{C} \) and for every map \( T \in \mathcal{T} \) one has:

\[
|T^k(z) - e^{ik\alpha}z| \leq k\|T\| \\
\|T\| \leq 2 \times |\sin(a/2)| \times \max(|C_0|, |C_1|).
\]
Proof. For \( z \in \mathbb{C} \) we have \( T(z) = e^{i\alpha}(z - C_j) + C_j \), and thus:

\[
|T(z)| - |z| = |T(z)| - |e^{i\alpha}z| \leq |T(z) - e^{i\alpha}z| = |C_j(1 - e^{i\alpha})| \leq 2|\sin(\alpha/2)|\max(|C_0|, |C_1|).
\]

Hence we have proved the formula for \( k = 1 \) and \( ||T|| \leq 2|\sin(\alpha/2)|\max(|C_0|, |C_1|) \).

We obtain by triangular inequality

\[
|T^k(z) - e^{ik\alpha}z| \leq \sum_{l=0}^{k-1} |e^{i(k-l)\alpha}T^l(z) - e^{i(k-l)\alpha}T^l(z)|
\]

\[
\leq \sum_{l=0}^{k-1} |T(T^l(z)) - e^{i\alpha}T^l(z)| \leq \sum_{l=0}^{k-1} ||T|| \leq k||T||.
\]

\( \blacksquare \)

Lemma 7. Let \( y \in \mathbb{C} \) with \( R > |y| \). Then we have:

\[
||R + y| - R - Re(y)|| \leq \frac{|y|^2}{R - |y|}.
\]

Proof. Remark that \( |R + y|^2 = R^2 + 2RRe(y) + |y|^2 \):

\[
|R + y| - R - Re(y) = \frac{|R + y|^2 - R^2}{|R + y| + R} - Re(y) = \frac{|y|^2 + 2RRe(y)}{|R + y| + R} - Re(y) = \frac{|y|^2 + (R - |R + y|)Re(y)}{|R + y| + R}.
\]

If we denote \( |R + y| - R - Re(y) = h \), then we have \( h = \frac{|y|^2 - (h + Re(y))Re(y)}{|R + y| + R} = \frac{\Im^2(y - Re(y)h)}{|R + y| + R} \). We deduce \( |h| \leq \frac{|y|^2 + |h||y|}{R} \) and \( 0 \leq |y|^2 + |h||(y) - R) \), which gives the result.

\( \blacksquare \)

Definition 8. Let us define the map \( g : \mathbb{R} \rightarrow \mathbb{R} \):

\[
x \mapsto g(x) = \begin{cases} 
-2|C_0| \sin(\alpha/2) \sin(x + \alpha/2 - c_0) & \text{if } \gamma < x < \gamma + \pi \mod 2\pi. \\
-2|C_1| \sin(\alpha/2) \sin(x + \alpha/2 - c_1) & \text{otherwise}.
\end{cases}
\]

Proposition 9. Consider a map \( T \in \mathbb{C} \). Let \( z = Re^{i\theta} \in \mathbb{C} \) such that \( |z| > ||T|| \), see Definition 5. Then we obtain the following inequalities, where \( \theta_1 \) is the argument of \( T(z) \):

\[
||T(z)| - |z| - g(\theta)|| \leq \frac{||T||^2}{|z| - ||T||}.
\]

\[
||T|| = 2|\sin(\alpha/2)|\max(|C_0|, |C_1|).
\]

\[
|\theta_1 - (\theta + \alpha) \mod 2\pi| \leq \frac{||T||}{|z| - ||T||}.
\]

Proof. Proof of (1) and (2). Let \( z = Re^{i\theta} \in \mathbb{C} \) such that \( R > 2|\sin(\alpha/2)|\max(|C_0|, |C_1|) \). Since \( 0 \in D = e^{i\pi/2} \mathbb{R} \), we have

\[
z \in P_0 \iff \gamma < \theta < \gamma + \pi \mod 2\pi.
\]

Let us denote \( j \in \{0, 1\} \) such that \( z \in P_j \). We obtain:

\[
|T(z)| = |e^{i\alpha}(z - C_j) + C_j| = |Re^{i\theta} - C_j + e^{-i\alpha}C_j| = |R + C_j e^{-i\theta}(e^{-i\alpha} - 1)|
\]

Now we define

\[
y = C_j e^{-i\theta}(e^{-i\alpha} - 1) = C_j e^{-i(\theta + \alpha/2)}(e^{-i\alpha/2} - e^{i\alpha/2}) = 2\sin(\alpha/2)C_j e^{-i(\theta + \alpha/2 + \pi/2)}
\]
Note that Re(\(y\)) = g(\(\theta\)), with \(g\) defined in Definition 8. We apply Lemma 7 with \(R = |z|\) and obtain:

\[ |T(z)| - |z| = |R + y| - R - Re(y) \leq \frac{|y|^2}{|z| - |y|} \]

For \(|z| > |y|\) we deduce

\[ |T(z)| - |z| \geq |g(\theta)| - \frac{|y|^2}{|z| - |y|}. \]

We conclude \(||T|| \geq |y|\) using the fact that \(|y| = \sup |g(\theta)|\) and \(\lim_{|z| \to +\infty} \frac{|y|^2}{|z| - |y|} = 0\). We conclude by means of Lemma 6 to obtain Equation (2). Then we deduce (1).

**Proof of (3).**

\[
\left| \frac{T(z)e^{-ia}}{z} - 1 \right| = \left| \frac{T(z)}{z} - e^{ia} \right| = \left| \frac{e^{ia}(z - C_j) + C_j}{z} - e^{ia} \right| = \left| \frac{C_j(1 - e^{ia})}{z} \right| \leq \frac{||T||}{|z|}.
\]

If \(\frac{||T||}{|z|} < 1\) we deduce:

\[
\left| \frac{\text{Im}\left( \frac{T(z)e^{-ia}}{z} \right)}{z} \right| = \left| \text{Im}\left( \frac{T(z)e^{-ia}}{z} - 1 \right) \right| \leq \left| \frac{T(z)e^{-ia}}{z} - 1 \right| \leq \frac{||T||}{|z|},
\]

\[
\left| \frac{\text{Re}\left( \frac{T(z)e^{-ia}}{z} \right)}{z} \right| = 1 + \text{Re}\left( \frac{T(z)e^{-ia}}{z} - 1 \right) \geq 1 - \frac{||T||}{|z|}.
\]

Thus we obtain

\[
|\tan(\theta_1 - a - \theta)| = \left| \frac{\text{Im}\left( \frac{T(z)e^{-ia}}{z} \right)}{\text{Re}\left( \frac{T(z)e^{-ia}}{z} \right)} \right| \leq \frac{||T||}{1 - \frac{||T||}{|z|}} = \frac{||T||}{|z| - ||T||}.
\]

Thus we conclude

\[ 0 \leq |\theta_1 - (\theta + \alpha)| \leq \arctan\left( \frac{||T||}{|z| - ||T||} \right) \leq \frac{||T||}{|z| - ||T||}. \quad \square \]

### 3 Limit sets

#### 3.1 Definitions

**Definition 10.** Let \(T : \mathbb{C} \to \mathbb{C}\) be an application. For \(M > 0\), let \(B(0, M)\) be the open ball of radius \(M\) centered at 0. We define the following sets for any \(n \in \mathbb{N}:\)

\[
A_{M,n}(T) = T^n B(0, M) \quad \text{and} \quad B_{M,n}(T) = T^{-n} B(0, M).
\]

\[
A(T) = \bigcup_{M>0} \bigcap_{n \geq 0} A_{M,n}(T) \quad \text{and} \quad B(T) = \bigcup_{M>0} \bigcap_{n \geq 0} B_{M,n}(T).
\]

These sets will be called limit sets in the rest of the paper.

**Remark 11.** Let \(z \in \mathbb{C}\). The previous sets can be characterized as follows:

- \(z \in A(T) \iff \exists (y_n) \in \mathbb{C}^\mathbb{N}\) a bounded sequence such that \(\forall n \in \mathbb{N}, z = T^n(y_n)\).
- \(z \in B(T)\) if and only if \((T^n(z))_n\) is a bounded sequence.

**Remark 12.** By construction we obtain the following inclusions:

- \(\forall n \in \mathbb{N}, A(T) \subset A(T^n)\) and \(B(T) \subset B(T^n)\).
- \(A(T) \subset \cap_{n \geq 0} \text{Im}(T^n)\).
3.2 Global attraction and global repulsion

**Definition 13.** Let $T$ be an application from $C$ to $C$.

- The application $T$ is globally attractive if there exists $M \geq 0$ such that $\forall L \geq 0$, $\limsup_{n \to +\infty} \sup_{|z| \leq L} |T^n(z)| \leq M$.
- The map $T$ is globally repulsive if there exists $M \geq 0$ such that $\forall L \geq M$, $\liminf_{n \to +\infty} \inf_{|z| \geq L} |T^n(z)| = +\infty$.

We will denote $M(T)$ the infimum of such real numbers $M$.

**Remark 14.** Let $T$ be an application from $C$ to $C$.

- If $T$ is globally attractive, then $A(T)$ is bounded and included in $B(0, M(T))$.
- If $T$ is globally repulsive, then $B(T)$ is bounded and included in $B(0, M(T))$.

We are interested in the size of the limit sets. Boshernitzan and Goetz have proved the following formula, reformulated here using our terminology.

**Theorem 15.**

- Let $T \in \mathcal{T}$ be a surjective and non-injective map, then $T$ is globally attractive and $A(T)$ is a bounded set.
- Let $T \in \mathcal{T}$ be an injective and non-surjective map, then $T$ is globally repulsive and $B(T)$ is a bounded set.

We obtain an upper bound for $M(T)$, in the irrational and the rational case, by distinct methods presented in Sections 4 and 5.

3.3 Periodic islands and limit sets

In this part we want to describe the limit sets $B(T)$ and $A(T)$. Note that these sets can be quite complicated, as explained in [9] and [8]. In Proposition 19 we describe some explicit subsets obtained by periodic islands (see Definition 18). Moreover, in Proposition 20 and Corollary 22 we explain how these sets change when the parameters change, see Figure 4.

Let us fix $\alpha, C_0, C_1$. Now let $u = u_0, \ldots, u_{n-1} \in \{0, 1\}^n$ be a finite word and consider the maps $r_k(z) = e^{i\alpha}(z - C_k) + C_k$. We observe that

$$r_{u_{n-1}} \circ \cdots \circ r_{u_0}(z) = e^{i\alpha} z + \sum_{k=0}^{n-1} e^{-(n-k-1)i\alpha} C_{u_k} (1 - e^{i\alpha}) .$$

(4)

**Definition 16.** Consider the word $u = u_0 \ldots u_{n-1} \in \{0, 1\}^n$ with $n \geq 1$ and assume $e^{i\alpha} \neq 1$. Then we define the almost periodic point associated to $u$ as the point

$$z_u = \frac{1 - e^{i\alpha}}{1 - e^{i\alpha}} \sum_{k=0}^{n-1} e^{-(n-k-1)i\alpha} C_{u_k} .$$

The following lemma appears already in [11] and [7].

**Lemma 17.** Let $T \in \mathcal{T}$ and $u = u_0 \ldots u_{n-1} \in \{0, 1\}^n$ such that $e^{i\alpha} \neq 1$. Then the almost periodic point $z_u$ is a periodic point of period $n$ for $T$ if and only if for all $k \in \{0, n-1\}$, $T^k(z_u) \in P_{u_k}$.
A coding of the map $T$ is a map from $\mathbb{C}$ to $\{0, 1\}^\mathbb{N}$ which sends the point $z$ to the sequence $(u_n)_{n \in \mathbb{N}}$ where $T^n z \in P_{u_n}$.

**Definition 18.** Let $n$ be such that $e^{i n \alpha} \neq 1$ and consider a word $u = u_0 \ldots u_{n-1} \in \{0, 1\}^n$. Assume that $z_u$ is a periodic word for $T$ with coding $u^\alpha$ (a periodic sequence of period $u$). Then we define the periodic island as the set of complex numbers $z$ such that $T^k(z) \in P_{u_k \mod n}$ for all integers $k \geq 0$.

Moreover we define the weight of $z_u$ as $w(z_u) = \min \{d(T^k(z), D); k \in [0, n-1]\}$.

Note that a periodic island is a convex set obtained as intersection of half-planes. If $z_u$ is a periodic point, then $B(z_u, w(z_u))$ is included inside the periodic island associated to $z_u$, see [11] (this inclusion becomes an equality if $\alpha$ is irrational, see Section 2 of [11]).

**Proposition 19.** For any $T \in \mathcal{T}$ we have

$$\bigcup_{z_u \in \text{Per}} B(z_u, w(z_u)) \subset B(T) \quad \text{and} \quad \bigcup_{z_u \in \text{Per}} B(z_u, w(z_u)) \subset A(T),$$

where $\text{Per} = \{z_u \text{ periodic point of some period } n \text{ such that } e^{i n \alpha} \neq 1\}$.

**Proof.** Let $z_u$ be a periodic point of period $n$ with $e^{i n \alpha} \neq 1$, and let $z \in B(z_u, w(z_u))$. Notice that the restriction of $T^n$ to $B(z, w(z_u))$ is a rotation of angle $n \alpha$ with center $z_u$.

- Let $k \in \mathbb{N}$ and consider the euclidean division: $k = qn + r$ with $0 \leq r < n$. The point $T^{qn}(z)$ is in $B(z_u, w(z_u))$. By definition of $\|T\|$ and Lemma 6, we deduce

$$|T^k(z)| = |T^r \circ T^{qn}(z)| \leq r \|T\| + |T^{qn}(z)| < n \|T\| + |z_u| + w(z_u).$$

Thus we obtain $\sup \{|T^k z| \mid k \in \mathbb{N}\} < n \|T\| + |z_u| + w(z_u) < +\infty$ and $z \in B(T)$.

- The restriction of $T$ to $\bigcup_{i=0}^{n-1} T^i B(z_u, w(z_u))$ is a bijection onto its image, which is the same set. Let us denote this map by $\tilde{T}$. Then for all $k \in \mathbb{N}$, if we denote $y_k = \tilde{T}^{-k}(z)$, we have $z = T^k(y_k)$ and we conclude $z \in A(T)$.

\[\square\]

**Proposition 20.** Let us consider an element $T$ of $\mathcal{T}$ with parameters $\alpha$, $\gamma$, $D$ and $C_j$. For all $\varepsilon > 0$, let us denote by $T_\varepsilon$ the element of $\mathcal{T}$ with the same parameters, except for the discontinuity line $D_\varepsilon = e^{i \varepsilon} D$.

Consider a periodic point $z_u$ of $T$ of weight $w$ associated to the word $u$ of length $n$. Then for all $\varepsilon > 0$ such that

$$w_\varepsilon = w - 2(|z_u| + n \|T\|) \times |\sin(\varepsilon/2)| > 0,$$

the point $z_u$ is a periodic point for $T_\varepsilon$ with weight bigger or equal to $w_\varepsilon$.

**Proof.** Let us consider $u = u_0 \ldots u_{n-1} \in \{0, 1\}^n$ and a periodic point $z_u$ for $T$ of weight $w$, associated to $u$. By definition we have:

$$z_u = \frac{1 - e^{i \alpha}}{1 - e^{i \alpha}} \sum_{k=0}^{n-1} e^{-(n-k) i \alpha} C_{u_k}$$

Now we fix $\varepsilon$ small enough to have

$$w_\varepsilon = w - 2(|z_u| + n \|T\|) \times |\sin(\varepsilon/2)| > 0.$$

Let us prove that $z_u$ is a periodic point for $T_\varepsilon$. We denote by $P_0^\varepsilon$ and $P_1^\varepsilon$ the half-planes defined by $T_\varepsilon$.  

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• We need to check that \( k \in [0, n-1] \), \( T^k(z_u) \in P^k_u \) and \( d(T^k(z_u), e^{i\epsilon}D) \geq w_\epsilon \).

We'll give a proof by induction over \( k \).

• For \( k = 0 \) we have by hypothesis

\[
d(e^{i\epsilon}z_u, z_u) = |z_u e^{-i\epsilon} - z_u| = 2|z_u| \times |\sin(\epsilon/2)|.
\]

We deduce \( z_u \in P^0_u \) and

\[
d(z_u, e^{i\epsilon}D) = d(z_u, e^{i\epsilon}D) - d(z_u, e^{i\epsilon}z) = d(z_u, D) - d(z_u, e^{i\epsilon}z)
\geq w - 2|z_u| \times |\sin(\epsilon/2)| \geq w_\epsilon.
\]

• Assume that the result is true for \( k < n - 1 \). Then \( T^k(z_u) \) and \( T^k(z_u) \) are in the same half planes \( P^k_u \cap P^k_u \) and then we deduce

\[
d(T^{k+1}(z_u), e^{i\epsilon}T^{k+1}(z_u)) = |T^{k+1}(z_u) - e^{i\epsilon}T^{k+1}(z_u)| = |1 - e^{i\epsilon} \times |T^{k+1}(z_u)|
\leq 2|\sin(\epsilon/2)| \times (|z_u| + (k + 1)||T||).
\]

Thus we obtain

\[
d(T^{k+1}(z_u), e^{i\epsilon}D) \geq d(e^{i\epsilon}T^{k+1}(z_u), e^{i\epsilon}D) - d(T^{k+1}(z_u), e^{i\epsilon}T^{k+1}(z_u))
\geq d(T^{k+1}(z_u), D) - d(T^{k+1}(z_u), e^{i\epsilon}T^{k+1}(z_u))
\geq w - 2|\sin(\epsilon/2)| \times (|z_u| + (k + 1)||T||) \geq w_\epsilon.
\]

We have proved our claim by induction.

By using the conjugacy by \( z \mapsto e^{i\epsilon}z \) and Remark 3 we can rephrase the previous proposition as follows:

**Proposition 21.** Let us consider the element \( T \) of \( T \) with parameters \( \alpha, \gamma, D \) and \( C_k \). For all \( \epsilon \), let us denote by \( T_\epsilon \) the element of \( T \) with the same parameters but with centers \( e^{i\epsilon}C_k \).

Consider a periodic point \( z_u \) of \( T \) of weight \( w \) associated to the word \( u \) of length \( n \). Then for all \( \epsilon \) such that

\[
w_\epsilon = w - 2(|z_u| + n||T||) \times |\sin(\epsilon/2)| > 0
\]

we have that \( e^{i\epsilon}z_u \) is a periodic point for \( T_\epsilon \) with weight bigger or equal to \( w_\epsilon \).
We deduce:

**Corollary 22.** For all $M > 0$, there exists a non-bijective piecewise rotation $T \in \mathcal{T}$ such that $B$ or $A$ are non-empty and contain a ball of radius $M$.

**Proof.** We apply Proposition 21 with $\beta = a/2 + y - \pi/2$. In this case we have a bijective map, thus it has some periodic islands. Hence for some $\varepsilon > 0$ we can find some non-bijective maps $T_\varepsilon$ such that $B$ or $A$ are non-empty and we apply Proposition 19 to conclude. \qed

## 4 Size of the limit set for irrational angle

### 4.1 Statement of the result

Let $T = T(\alpha, C_0, C_1, D, y) \in \mathcal{T}$ be a non-bijective map. Let us denote $\alpha = 2\pi a$ and assume that $a$ is an irrational number. Recall that $M(T)$ is defined in Definition 13.

Let $(p_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$ be the sequence of convergents of $a$ in the continued fraction expansion. Let $\ell_0$ be such that

$$\frac{1}{q_{\ell_0}} \left( 8 + \frac{\pi}{2q_{\ell_0}} \right) \|T\| + 4|\Delta| < \frac{|\Delta|}{\pi}.$$ 

Then we will prove the following:

**Theorem 23.** Let $T \in \mathcal{T}$ be a non-bijective map with an irrational angle. We obtain the following bound:

$$M(T) \leq q_{\ell_0} \|T\| \left( \frac{2}{\pi} q_{\ell_0} + 1 \right).$$

To prove this theorem we first need some background on the Denjoy-Kosma inequality and the Three gaps theorem.

### 4.2 Bounded variation function and three gaps theorem

**Definition 24.** Consider a periodic function $f : \mathbb{R} \to \mathbb{R}$ of period $T$. If the following quantity exists we say that $f$ has bounded variation with variation $\text{Var}(f)$:

$$\text{Var}(f) = \sup \left\{ \sum_{k=0}^{P-1} \left| f(x_{k+1}) - f(x_k) \right| ; \{x_0, \ldots, x_P\} \in \mathcal{P} \right\},$$

where $\mathcal{P}$ is the set of subdivisions $P = \{x_0, \ldots, x_{P-1}\}$ of $[0, T]$ such that

$$0 = x_0 \leq x_1 \leq \cdots \leq x_{P-1} \leq x_P = T.$$

**Proposition 25.** Let $f : \mathbb{R} \to \mathbb{R}$ be a $2\pi$-periodic function which is piecewise $C^1$. Assume that the discontinuity points of $f$ on $[0, 2\pi]$ are denoted by $d_1, \ldots, d_m$, and set $d_0 = 0$. Then $f$ has bounded variation and

$$\text{Var}(f) = \int_0^{2\pi} |f'(t)| \, dt + \sum_{k=0}^{m} \left| f(d_k^+) - f(d_k^-) \right|.$$

We associate to any irrational positive real number $0 < a < 1$ its continued fraction expansion:

$$a = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

that we denote by $a = [0, a_1, \ldots, a_r, \ldots]$. We also consider two sequences of integers $(p_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$, such that for any integer $\ell \geq 0$ one has $[0, a_1, \ldots, a_\ell] = \frac{p_\ell}{q_\ell}$. They are defined by induction as follows,
\begin{align*}
\begin{cases}
p_{-2} = 0 & p_{-1} = 1 \\
q_{-2} = 1 & q_{-1} = 0
\end{cases}
\quad \text{and for any integer } \ell \geq 0 : \begin{cases}
p_\ell = a_\ell p_{\ell-1} + p_{\ell-2} \\
q_\ell = a_\ell q_{\ell-1} + q_{\ell-2}
\end{cases}
\end{align*}

We recall Denjoy–Koksma inequality:

**Theorem 26.** [12] Let \( a \in [0, 1] \) be an irrational number, and let us denote by \( \frac{p_\ell}{q_\ell} \) the sequence of partial quotients in the continued fraction expansion of \( a \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( 2\pi \)-periodic function, piecewise \( C^1 \). Then for all \( x \in \mathbb{R} \) and for all \( \ell \in \mathbb{N}^* \) one has:

\[
\left| \frac{1}{q_\ell} \sum_{k=0}^{q_\ell-1} f(x + 2\pi k a) - \frac{1}{2\pi} \int_0^{2\pi} f(t) dt \right| \leq \frac{1}{q_\ell} \text{Var}(f).
\]

The following theorem has been proved by V. Sos:

**Theorem 27** (Three gaps theorem). [11] Let \( 0 < a < 1 \) be an irrational number and \( \ell \) a positive integer. Let us denote by \( p_\ell/q_\ell \) a partial quotient. The points \( \{ka\}, 0 \leq k \leq q_\ell - 1 \) partition the unit circle into \( q_\ell \) intervals. The lengths of the latter take on three values, and the minimal value is equal to \( |q_\ell a - p_{\ell-1}| \geq \frac{1}{2q_\ell} \).

### 4.3 Auxiliary function

**Lemma 28.** Consider the map \( g \) from Definition 8. We have the following properties:

- The map \( g \) is of bounded variation and we have \( \text{Var}(g) \leq 2|\Delta| + 4||T|| \).
- \( \int_{-\pi}^{\pi} g(x) \, dx = -2\Delta \).

**Proof.**

- The map \( g \) has two discontinuities, in \( \gamma \) and \( \gamma + \pi \), and we have

\[
\begin{align*}
|g(y^+) - g(y^-)| + |g(y + \pi^+) - g(y + \pi^-)| \\
&= 2|\sin(\alpha/2)| \times |C_\ell| \sin(\alpha/2 - c_\ell) \\
&\quad + 2|\sin(\alpha/2)| \times |C_\ell| \sin(\alpha/2 + c_\ell) \\
&= 2|\sin(\alpha/2)| \times \left| \text{Im}(\bar{C}_\ell e^{i(\gamma+\alpha/2)}) - \text{Im}(\bar{C}_\ell e^{i(\gamma+\alpha/2)}) \right| \\
&\quad + 2|\sin(\alpha/2)| \times \left| \text{Im}(C_\ell e^{i(\gamma+\alpha/2)}) \right| \\
&= 4|\sin(\alpha/2)| \times |C_\ell - C_\ell| \times |\sin(\alpha/2 - \beta)| \\
&= 2|\Delta|.
\end{align*}
\]

We obtain for the computation of the integral

\[
\int_{-\pi}^{\pi} |g'(x)| \, dx + \int_{-\pi}^{\pi+\pi} |g'(x)| \, dx
\]

\[
= 2|C_\ell| |\sin(\alpha/2)| \int_{-\pi}^{\pi} |\cos(\alpha/2 - c_\ell)| \, dx + 2|C_\ell| |\sin(\alpha/2)| \int_{-\pi}^{\pi+\pi} |\cos(\alpha/2 - c_\ell)| \, dx
\]

\[
= 4|\sin(\alpha/2)||(C_\ell + |C_\ell|)| \leq 8|\sin(\alpha/2)| \max(|C_\ell|, |C_\ell|).
\]

We use (1) in Proposition 9 and Proposition 25 to conclude at the first point.
• We obtain:

\[
\int_{-\pi}^{\pi} g(x) \, dx = \int_{-\pi}^{\theta} g(x) \, dx + \int_{\theta}^{\pi} g(x) \, dx
\]

\[
= -2|C_1| \sin(\alpha/2) \int_{-\pi}^{\theta} \sin(x + \alpha/2 - c_1) \, dx - 2|C_0| \sin(\alpha/2) \int_{\theta}^{\pi} \sin(x + \alpha/2 - c_0) \, dx
\]

\[
= 2|C_1| \sin(\alpha/2) \int_{-\pi}^{\theta} \cos(x + \alpha/2 - c_1) \, dx + 2|C_0| \sin(\alpha/2) \int_{\theta}^{\pi} \cos(x + \alpha/2 - c_0) \, dx
\]

\[
= 4|C_1| \sin(\alpha/2) \cos(\gamma + \alpha/2 - c_1) - 4|C_0| \sin(\alpha/2) \cos(\gamma + \alpha/2 - c_0)
\]

where we use the fact that \( \Re(\tilde{C}_1 e^{i(\alpha/2+\gamma)}) - \Re(\tilde{C}_0 e^{i(\alpha/2+\gamma)}) = \Re\left(\left(\overline{C_1 - C_0}\right) e^{i(\alpha/2+\gamma)}\right) \).

\[
\]

Now we state the key part in the proof of Theorem 23

**Proposition 29.** Consider \( \ell \in \mathbb{N}, z \in \mathbb{C} \) and assume \(|z| > q_\ell \|T\| (\frac{3}{2} q_\ell + 1)\). Then we obtain

\[
\left| \frac{|T^{q_\ell} z| - |z|}{q_\ell} + \frac{\Delta}{\pi} \right| \leq \frac{1}{q_\ell} \left( 8 + \frac{\pi}{2q_\ell} \right) \|T\| + 4|\Delta| \right).
\]

\[
\]

**Proof.** We will need the following notations. If \( z \in \mathbb{C} \), then we denote by \( \theta \) its argument, and for all integer \( k \geq 0 \) we set \( z_k := T^k(z) \) and \( \theta_k := \arg(z_k) \). Moreover we will denote, by abuse of notations, \( d(\theta, \theta') = |\theta - \theta' \mod 2\pi| \). We obtain:

\[
\left| \frac{|T^{q_\ell} z| - |z|}{q_\ell} + \frac{\Delta}{\pi} \right| \leq \frac{1}{q_\ell} \sum_{k=0}^{q_\ell-1} \left( |z_{k+1}| - |z_k| \right) + \frac{\Delta}{\pi} \right|.
\]

By Lemma 6 and equation (2) we obtain, if \(|z| > ||T||q_\ell|, |z_k| > ||T|| \) for \( 0 \leq k \leq q_\ell - 1 \).

Then by equation (1) we have:

\[
\left| \frac{|T^{q_\ell} z| - |z|}{q_\ell} + \frac{\Delta}{\pi} \right| \leq \frac{1}{q_\ell} \sum_{k=0}^{q_\ell-1} \frac{||T||^2}{|z_k| - ||T||} + \frac{1}{q_\ell} \sum_{k=0}^{q_\ell-1} g(\theta_k) + \frac{\Delta}{\pi} \right|.
\]

We will produce upper bound for each term.

(A) First term:

\[
\frac{1}{q_\ell} \sum_{k=0}^{q_\ell-1} \frac{||T||^2}{|z_k| - ||T||} \leq \frac{1}{q_\ell} \sum_{k=0}^{q_\ell-1} \frac{||T||^2}{|z| - (k + 1)||T||} \leq \frac{||T||^2}{|z| - q_\ell ||T||}.
\]

(B) Second term:
- By equation (3) in Proposition 3, if \( |z| > \|T\| \), we obtain \( d(\theta_1, \theta_0 + \alpha) \leq \frac{\|T\|}{|z| - \|T\|} \).
- If \( |z| > 2\|T\| \), then \( |z_1| = |T(z)| \geq |z| - \|T\| > \|T\| \) and we deduce
  \[
  d(\theta_2, \theta_0 + 2\alpha) \leq d(\theta_2, \theta_1 + \alpha) + d(\theta_1 + \alpha, \theta_0 + 2\alpha) \\
  \leq d(\theta_2, \theta_1 + \alpha) + d(\theta_1, \theta_0 + \alpha) \\
  \leq \frac{\|T\|}{|z_1| - \|T\|} + |z| - \|T\| \\
  \leq \frac{\|T\|}{|z| - 2\|T\|} + |z| - \|T\| \\
  
  \]
- Now if \( |z| > n\|T\| \), we have \( \forall k \in \mathbb{I}0, n - 1 \mathbb{I}, |z_k| > \|T\| \) and
  \[
  d(\theta_n, \theta_0 + n\alpha) \leq \sum_{k=0}^{n-1} d(\theta_{n-k} + k\alpha, \theta_{n-k-1} + (k + 1)\alpha) \\
  \leq \sum_{k=0}^{n-1} d(\theta_{n-k}, \theta_{n-k-1} + \alpha) \\
  \leq \sum_{k=0}^{n-1} \frac{\|T\|}{|z_k| - \|T\|} \\
  \leq \sum_{k=1}^{n} \frac{\|T\|}{|z| - k\|T\|} \\
  \leq \frac{n\|T\|}{|z| - n\|T\|} \\
  
  \]
- By Theorem 27, the smallest interval in the subdivision \( \{k\alpha, 0 \leq k \leq q_t - 1\} \) has length at least \( \frac{2\pi}{2q_t} = \frac{\pi}{q_t} \).
- we observe the following fact:
  \[
  (q_t - 1) \frac{\|T\|}{|z| - (q_t - 1)\|T\|} \leq \frac{\pi}{2q_t} \iff \frac{2\pi}{q_t} (q_t - 1)\|T\| < |z| - (q_t - 1)\|T\| \\
  \iff |z| > (q_t - 1)\|T\| \left( \frac{2\pi}{q_t} + 1 \right). 
  \]
Thus if \( |z| > (q_t - 1)\|T\| \left( \frac{2\pi}{q_t} + 1 \right) \) and \( |z| > q_t\|T\| \) then for all \( 0 \leq k \leq q_t - 1 \), we obtain
\[
  d(\theta_k, \theta_0 + k\alpha) < \frac{\pi}{2q_t}.
  \]
- Thus if \( |z| > q_t\|T\| \left( \frac{2\pi}{q_t} + 1 \right) \), we obtain for the second term:
  \[
  \left| \frac{1}{q_t} \sum_{k=0}^{q_t-1} g(\theta_k) - g(\theta_0 + k\alpha) \right| \leq \frac{1}{q_t} \text{Var}(g).
  \]
(C) Third term: we use the Denjoy–Koksma inequality (Theorem 26 and Lemma 28) to deduce:
\[
  \left| \frac{1}{q_t} \sum_{k=0}^{q_t-1} g(\theta_0 + k\alpha) + \frac{A}{\pi} \right| = \left| \frac{1}{q_t} \sum_{k=0}^{q_t-1} g(\theta_0 + k\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) dx \right| \leq \frac{1}{q_t} \text{Var}(g).
  \]
Thus if we have $|z| > q_\ell \|T\| \left( \frac{2}{\pi} q_\ell + 1 \right)$, we deduce from Lemma 28 that

$$\left| \frac{|T^{q_\ell} z| - |z|}{q_\ell} + \frac{\Delta}{\pi} \right| \leq \frac{\|T\|^2}{|z| - q_\ell \|T\|} + \frac{2}{q_\ell} \text{Var}(g)$$

$$\leq \frac{\|T\|^2}{q_\ell \|T\| (\frac{2}{\pi} q_\ell + 1) - q_\ell \|T\|} + \frac{2}{q_\ell} (2|\Delta| + 4\|T\|)$$

$$\leq \frac{1}{q_\ell} \left( \frac{\|T\|^2}{\frac{2}{\pi} q_\ell} + 4|\Delta| + 8\|T\| \right) = \frac{1}{q_\ell} \left( 8 \frac{\pi}{2q_\ell} \|T\| + 4|\Delta| \right)$$

which proves the result. \qed

4.4 Conclusion of the proof

Let $\ell_0$ such that $\frac{1}{q_{\ell_0}} \left( 8 \frac{\pi}{2q_{\ell_0}} \|T\| + 4|\Delta| \right) < \frac{|\Delta|}{\pi}$ and let us denote $\epsilon = \frac{|\Delta|}{\pi} - \frac{1}{q_{\ell_0}} \left( 8 \frac{\pi}{2q_{\ell_0}} \|T\| + 4|\Delta| \right)$.

- If $\Delta > 0$, then by Proposition 29 for $|z| > q_{\ell_0} \|T\| \left( \frac{2}{\pi} q_{\ell_0} + 1 \right)$ one has:

$$|T^{q_{\ell_0}} z| \leq |z| - q_{\ell_0} \epsilon$$

Thus we obtain

$$A(T^{q_{\ell_0}}) \subset B \left( 0, q_{\ell_0} \|T\| \left( \frac{2}{\pi} q_{\ell_0} + 1 \right) - q_{\ell_0} \epsilon \right) \subset B \left( 0, q_{\ell_0} \|T\| \left( \frac{2}{\pi} q_{\ell_0} + 1 \right) \right).$$

By definition of $A(T)$, we obtain (see Remark 12):

$$A(T) \subset A(T^{q_{\ell_0}}) \subset B \left( 0, q_{\ell_0} \|T\| \left( \frac{2}{\pi} q_{\ell_0} + 1 \right) \right).$$
• If \( \Delta < 0 \), then by Proposition \[ \ref{prop:29} \) we obtain for \( |z| > q_6 ||T|| \left( \frac{2}{\pi} q_6 + 1 \right) \):

\[
|T^{q_6} z| \geq |z| + q_6 \varepsilon
\]

Thus for every integer \( n \) we deduce

\[
|T^{nq_6} z| \geq |z| + nq_6 \varepsilon
\]

and \( z \notin B \left( 0, q_6 ||T|| \left( \frac{2}{\pi} q_6 + 1 \right) \right) \). Then by Remark \[ \ref{rem:12} \) we conclude:

\[
B(T) \subset B \left( 0, q_6 ||T|| \left( \frac{2}{\pi} q_6 + 1 \right) \right).
\]

### 4.5 Example

Consider \( T \) with the following parameters:

| \( \alpha \) | \( \varphi \) | \( C_0 \) | \( C_1 \) | \( y \) | \( \beta \) | \( \Delta \) | \( ||T|| \) |
|---|---|---|---|---|---|---|---|
| \( \sqrt{2} \pi \) | \( \frac{\sqrt{2} \pi}{4} \) | \( -e^{1.4i} \) | \( 1.8 e^{1.4i} \) | \( \frac{5}{4} \) | \( \sqrt{2} \) | \( i \mathbb{R} \) | \( 1.14 \) | \( -0.13 > \Delta > -0.14 \) | \( 2.68 < |3 \sin(\alpha/2)| \leq 2.69 \)

Since \( \Delta < 0 \) the map \( T \) is injective and is not bijective.

Notice that the continued fraction of \( a \) is equal to \( a = [0, 2, 1, 4, 1, 4, 1, 4, \ldots] \)

| \( \ell \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( p_\ell \) | 0 | 1 | 5 | 6 | 29 | 169 | 204 | 985 |
| \( q_\ell \) | 1 | 2 | 14 | 17 | 82 | 99 | 478 | 577 | 2786 |

We obtain

\[
\frac{1}{q_8} \left( 8 + \frac{\pi}{2q_8} \right) ||T|| + 4|\Delta| < 0.0383 < \frac{|\Delta|}{\pi} < 0.0460 < \frac{1}{q_7} \left( 8 + \frac{\pi}{2q_7} \right) ||T|| + 4|\Delta|.
\]

and, see Figure \[ \ref{fig:6} \) we deduce the bound \( M = q_8 ||T|| \left( \frac{2}{\pi} q_6 + 1 \right) < 571283 \). Notice that the real size of the set seems to be much smaller.

In fact we can obtain a better bound due to the following remark:

**Remark 30.** The bound from \( M \) can be improved. In order to avoid technicalities we only explain this for the above example. Consider some conjugation in order to change the origin: let us define \( \rho \) as the point of \( D \) which minimizes both distances to \( C_0 \) and \( C_1 \). This point exists by a compactness argument. If \( \rho \) is the new origin, then we obtain new coordinates for the centers of rotations, and \( ||T'||\| \leq ||T|| \). In this example we obtain:

The point \( \rho \) is at the intersection of \( D \) with the bisection line of \( [C_0, C_1] \). This point is given by \( \rho = \frac{0.25}{\sin(1.14)} i \approx 0.275 i \). The new centers of rotations are \( C'_0 = C_0 - \rho \) and \( C'_1 = C_1 - \rho \). Then the new map fulfills \( ||T'||\| = 2 \max(|C'_0|, |C'_1|) \sin(\alpha/2) \approx 2.249 \). We obtain \( \frac{1}{q_8} \left( 8 + \frac{\pi}{2q_8} \right) ||T'|| + 4|\Delta| \approx 2.249 \). We deduce the new bound

\[
M' = q_8 ||T'|| \left( \frac{2}{\pi} q_8 + 1 \right) < 536048 < M.
\]

## 5 Size of the limit set for the rational angle

Now we prove the same type of result for a piecewise rotation of rational angle.
5.1 Notations and statement

Consider a non-bijective map \( T = T(\alpha, C_0, C_1, D, \gamma) \in \mathcal{T} \). In all of this section we will use the following notation:

\[
\alpha = \frac{p}{q} 2\pi \quad \text{with} \quad q > 2 \quad \text{an even number and} \quad \gcd(p, q) = 1.
\]

We will prove:

**Theorem 31.** Let \( T \in \mathcal{T} \) a non-bijective map with a rational angle \( \alpha \) such that \( \alpha = \frac{p}{q} 2\pi \) for some even number \( q > 2 \) and \( \gcd(p, q) = 1 \). Then we obtain:

\[
M(T) \leq q \|T\| \left( \frac{1}{2 \tan(|\beta - \alpha/2|)} + \frac{1}{2 \tan(\pi/q)} + 1 \right).
\]

**Remark 32.** Up to a conjugacy by \( z \mapsto e^{-i(\gamma + \pi/2)} \) we can always assume that \( \gamma = \pi/2 \) and that the discontinuity line is \( D = i\mathbb{R} \). We will assume this in the proof and thus we will prove:

\[
M(T) \leq q \|T\| \left( \frac{1}{2 \tan(|\beta - \alpha/2|)} + \frac{1}{2 \tan(\pi/q)} + 1 \right).
\]

We finish this paragraph with another remark.

**Remark 33.** If we conjugue by \( z \mapsto \bar{z} \) (see Remark 3), then we can assume \( \alpha \in ]0, \pi[ \). This allows us to assume \( \sin(\alpha/2) > 0 \).

5.2 Vectors of translations

First of all, we consider the following partition of the complement of \( B(0, q \|T\|) \), see Figure 7:

- For \( k = 0, \ldots, q-1 \), we define the cone \( C_k = \left\{ z \mid \frac{\pi}{2} + 2\pi \frac{k}{q} < \arg(z) < \frac{\pi}{2} + 2\pi \frac{(k+1)}{q} \right\} \).
- For \( k = 0, \ldots, q-1 \), let us consider \( E_k \) which is the translate of \( C_k \) such that its vertex is the point

\[
O_k = \frac{q \|T\|}{\sin(\pi/q)} \exp \left( i \left( \frac{\pi}{q} + 2\pi \frac{k}{q} + \frac{\pi}{2} \right) \right).
\]
- For \( k = 0, \ldots, q-1 \), let us denote \( G_k \) the strip which is the intersection of the set of points of modulus bigger than \( q \|T\| \) and of the closed \( q \|T\| \)-neighborhood of \( C_{k-1} \cap C_k \) where \( C_{-1} = C_{q-1} \).
The map $T^q$ is clearly a piecewise translation. In the next lemma we compute the vectors of translation which define this map. We identify complex numbers with coordinates of points, in order to simplify.

**Lemma 34.** Let us denote: $v = \frac{2\sin(\alpha/2)}{\sin(\pi/q)} |C_1 - C_0|$ and $w = \frac{2\sin(\alpha/2)}{\tan(\pi/q)} |C_1 - C_0|$. For $k = 0, \ldots, q - 1$, let us consider the vectors:

$$v_k = ve^{i(\beta - \frac{\pi}{q} + k \frac{\pi}{q})}, \quad w_k = we^{i(\beta - \frac{\alpha}{q} + k \frac{\pi}{q})}.$$

Then the map $T^q$ is a piecewise translation with vectors $t_i$ with $i \in I$ for some finite set $I$. We have for each $k \in \{0, \ldots, q - 1\}$.

- The set $E'_k$, where the vector of translation of $T^q$ is $v_k$, is a subcone of $C_k$ which contains $E_k$, see Figure 7.

- The restriction of $T^q$ to $G_k$ is a translation by some vector $t_k$ which is equal to $v_{k-1}$, $w_k$ or $v_k$.

More precisely,

- consider the subset $G'_k$ of $G_k$ where $T^q$ is a translation by $w_k$. Then $G'_k$ is a strip of $G_k$.

- There exists a subset of $G_k$ inside $C_{k-1}$ where $t_k = v_{k-1}$.

Recall that we have assumed that $\alpha$ satisfies $\sin(\alpha/2) > 0$ (cf Remark 33). Thus we have $v = |v_k|$ and $w = |w_k|$.

**Remark 35.** A statement similar to Lemma 34 above appears already in [Bosh.Goet.03]. Their statement, however, was not quite correct: they claimed that the vectors of translation for $T^q$ restricted to $G_k$ all have the same norm. Moreover the description on the sets $G'_k$, $C'_k$ was not in their paper.

**Proof of Lemma 34** Let us fix $k = 0, \ldots, q - 1$. We consider several different cases.

- Let $z \in E_k$. We have $e^{i\ell \alpha} O_k = O_{k+pf}$ by definition of $O_k$ for all $\ell \in \mathbb{N}$. This gives us $e^{i\ell \alpha} z \in E_{k+pf}$.

Then by Lemma 6 we obtain for $\ell \leq q - 1$, $d(T^\ell(z), E_{k+pf}) < q ||T||$. We deduce

$$T^\ell(z) \in C_{k+pf} \quad 0 \leq \ell \leq q - 1.$$
Thus the first $q - 1$ iterates of $z$ under the action of $T$ meet all the cones $C_i$ and in particular no two subsequent iterates lie in the same cone.

Thus the restriction of $T^q$ to $E_k$ is an honest translation and not a piecewise translation. Now we will compute the vector of translation of $T^q$. We use the notation

$$\eta(i) = \begin{cases} 0 & i \leq q/2, \\ 1 & i > q/2. \end{cases}$$

We compute the vector $T^q(z)$ for $z \in E_k$ by the following formula, see Equation (4)

$$T^q(z) = z + (1 - e^{i\alpha}) \sum_{n=0}^{q-1} C_{n} e^{i\alpha(q-n-1)} = z + (1 - e^{i\alpha}) \sum_{n=0}^{q-1} e^{i\alpha} C_{n} e^{i\alpha(-n-1)}.$$ 

Modulo $q$ the map $n \mapsto k + np$ is a bijection since $\gcd(p, q) = 1$

$$T^q(z) = z + (1 - e^{i\alpha}) \sum_{m=0}^{q/2-1} e^{2i\pi \frac{p-mk}{q}} + (1 - e^{i\alpha}) C_k \sum_{m=q/2}^{q-1} e^{2i\pi \frac{p-mk}{q}} \tag{5}$$

By means of well known trigonometric formula we obtain

$$T^q(z) = z + (1 - e^{i\alpha}) e^{2i\pi \frac{p-k\beta}{q}} \left( C_0 \sum_{m=0}^{q/2-1} e^{-2i\pi \frac{m}{q}} + C_1 \sum_{m=q/2}^{q-1} e^{-2i\pi \frac{m}{q}} \right)$$

$$= z + (1 - e^{i\alpha}) e^{2i\pi \frac{k-p\alpha}{q}} (C_0 - C_1) \frac{2}{1 - e^{-2i\pi/q}}$$

$$= z + 2 \frac{\sin(\alpha/2)}{\sin(\pi/q)} |C_1 - C_0| \exp \left( i \left( \beta + 2\pi \frac{k-p}{q} + \frac{\alpha}{2} + \frac{\pi}{q} \right) \right)$$

$$= z + 2 \frac{\sin(\alpha/2)}{\sin(\pi/q)} |C_1 - C_0| e^{i (\beta - \frac{\pi}{q} + \frac{2\pi k}{q})}$$

$$= z + v_k.$$ 

- Now consider $z \in G_k$. We will denote by $R$ the rotation with center 0 and angle $\alpha$. If there exists some $n \geq 0$ such that $e^{i\alpha z}$ and $T^n(z)$ are in the two different half planes defined by $\Re(z) \geq 0$ and $\Re(z) \leq 0$, then this implies that the two points are both in $G_0$ or $G_1$. Indeed by Equation (6) they are at distance at most $q||T||$ of each other. Thus there are several choices for the different codings: either the codings of the orbits of $T^n z, R^n z$ for $z \in G_k$ differ in one position $j_1$, or else they differ at two positions $j_1, j_2$. Note that $j_1, j_2$ are defined by $j_1, j_2 = q/2 \mod q$ or $k + j_2 p = 0 \mod q$, according to the points in $G_0$ or $G_1$. In other words: $R^{j_1}(z)$ and $T^{j_1}(z)$ are contained in different half spaces.

Now we want to do the same computation as in the case $z \in E_k$. We can resume the situation as indicated in the lines below: the first one concerns the coding of a point in $E_k$, and the next three lines correspond to one change in the coding if the change is at position $j_1$ or $j_2$, or else to two changes in the coding.

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We consider the two possibilities, starting from Equation (5).

1. Assume that there is only one change. For example if one 0 becomes a 1, we need to change in Equation (5) the $C_0 \sum_{0}^{q/2-1}$ into $C_0 \sum_{1}^{q/2-1}$ and $C_1 \sum_{q/2}^{q-1}$ into $C_1 \sum_{q/2}^{q}$. We obtain the following:

$$T^q(z) = z + (1 - e^{i\alpha})e^{2i\pi \frac{k-p}{q}} \left( C_0 \sum_{m=1}^{q/2-1} e^{2i\pi \frac{2m}{q}} + C_1 \sum_{m=q/2}^{q} e^{2i\pi \frac{2m}{q}} \right)$$

$$= z + (1 - e^{i\alpha})e^{2i\pi \frac{k-p}{q}} \left( C_0 \sum_{m=0}^{q/2-1} e^{2i\pi \frac{2m}{q}} - C_0 + C_1 \sum_{m=q/2}^{q} e^{2i\pi \frac{2m}{q}} + C_1 \right)$$

$$= z + (1 - e^{i\alpha})e^{2i\pi \frac{k-p}{q}} (C_0 - C_1) \left( \sum_{m=0}^{q/2-1} e^{2i\pi \frac{2m}{q}} - (C_0 - C_1) \right)$$

$$= z + (1 - e^{i\alpha})e^{2i\pi \frac{k-p}{q}} (C_0 - C_1) \frac{2}{1 - e^{-2i\pi/q}} - (C_0 - C_1)$$

$$= z + (1 - e^{i\alpha})e^{2i\pi \frac{k-p}{q}} (C_0 - C_1) \frac{1 + e^{-2i\pi/q}}{1 - e^{-2i\pi/q}}.$$

$$= z + \cos \left( \frac{\pi}{q} \right) e^{-i\pi/4} v_k$$

$$= z + w_k.$$

2. The case where one 1 becomes a 0 is similar: we replace $C_0 \sum_{0}^{q/2-1}$ by $C_0 \sum_{1}^{q/2}$ and $C_1 \sum_{q/2}^{q-1}$ by $C_1 \sum_{q/2+1}^{q}$. We obtain the following:

$$T^q(z) = z + v_k + (1 - e^{i\alpha})C_0 e^{2i\pi \frac{p-q/2k}{q}} - (1 - e^{i\alpha})C_1 e^{2i\pi \frac{p-q/2k}{q}}$$

$$= z + v_k - (1 - e^{i\alpha})C_0 e^{2i\pi \frac{p}{q}} - (1 - e^{i\alpha})C_1 e^{2i\pi \frac{p+k}{q}}$$

$$z + \cos \left( \frac{\pi}{q} \right) e^{-i\pi/4} v_k$$

$$= z + w_k.$$

3. In the last case (where we exchange one 0 and one 1), we need to change the $C_0 \sum_{0}^{q/2-1}$ into
Notations 2. Let us denote $\Delta$ polygons will have the following properties:

In this part we assume the disjoint union of all $P$, big enough, some polygon $P_x$. These polygons will have the following properties:

- the disjoint union of all $P_x$ cover the plane outside the ball $B(0, q\|T\|)$.

5.3 Construction of polygons

5.3.1 The case $\Delta < 0$

In this part we assume $\Delta < 0$. By Definition of $\Delta$ and Remark 32 we deduce that $\beta - \alpha/2$ belongs to $(0, \pi/2)$. The goal is to construct for each positive number $x$, big enough, some polygon $P_x$. These polygons will have the following properties:

- the disjoint union of all $P_x$ cover the plane outside the ball $B(0, q\|T\|)$.
Lemma 37. Proof. We refer to Figure 9. There is a polygon with vertices \(a_k\), \(b_k\), \(B_{k+1}\) such that:

- the map \(x \mapsto P_x\) is increasing, with respect to the partial order on subsets of the plane given by the inclusion,
- if \(z\) lies outside of outside \(P_x\), then the piecewise translation \(T^y\) maps \(z\) to a point outside \(P_{x'}\) for some \(x' > x\), see Proposition 40.

This will be the key point to finish the proof of Theorem 31.

Definition 36. Consider \(x > q\|T\|\), then we define a polygon \(P_x\) with vertices \(A_0, B_0, ..., A_{q-1}, B_{q-1}\) where (see Figure 8):

- the point \(A_k\) is defined in \(E_k' \cap G'_k\) as the point such that the orthogonal projection \(A'_k\) of \(A_k\) on \([0, i e^{2\pi k/q}]\) has modulus \(x\).
- Similarly we define \(B_k \in E'_k \cap G'_{k+1}\) such that the orthogonal projection \(B'_k\) of \(B_k\) on \([0, i e^{2\pi (k+1)/q}]\) has modulus \(x\).

Moreover we define \(\delta_k(x)\) as the angle (taken in \([0, \pi/2]\)) between the lines \((A_kB_k)\) and \((A'_kB'_k)\).

Lemma 37. For all \(x > q\|T\|\) and for all \(z\) on the boundary of \(P_x\) we obtain

\[
x \cos(\pi/q) \leq |z| \leq x^2 + q^2\|T\|^2.
\]

The proof of this lemma is left to the reader with the help of Figure 8.

Lemma 38. Let us consider \(\tilde{x} = q\|T\|\left(\frac{1}{2 \tan(\beta - \alpha/2)} + \frac{1}{2 \tan(\pi/q)}\right)\). Then one has:

- the map \(x \mapsto \delta_k(x)\) is decreasing
- \(\lim_{x \to +\infty} \delta_k(x) = 0\)
- For \(x > \tilde{x}\), we have \(0 \leq \delta_k(x) < \beta - \alpha/2\).

Proof. We refer to Figure 9. There is a polygon with vertices \(A'_k, A_k, B_k, B'_k\) and we define points \(C_k, L_k\) such that:

- If \(a_k \geq b_{k+1}\), then \(C_k\) belongs to \([A'_k, A_k]\) and \(|A'_kC_k| = b_{k+1}\). The point \(L_k\) is the orthogonal projection of \(A_k\) on \([B_k, C_k]\).
- If \(b_{k+1} > a_k\), \(C_k\) belongs to \([B'_k, B_k]\) and \(|B'_kC_k| = a_k\). The point \(L_k\) is defined as the orthogonal projection of \(A_k\) on \([A_k, C_k]\).

Let us denote \(r_k = \max(a_k, b_{k+1}) - \min(a_k, b_{k+1})\) and \(s_k = \min(a_k, b_{k+1})\). Note that \(0 \leq r_k \leq q\|T\|\).
We present the proof in the case $a_k \geq b_{k+1}$; the other case is similar. We have $\overline{B_k C_k A_k} = \overline{B_k A'_k A_k} = \pi/q$ and
\[
\tan(\delta_k(x)) = \tan(\overline{C_k B_k A_k}) = \frac{|A_k L|}{|B_k L|} = \frac{|A_k L_k|}{|B_k C_k| - |C_k L_k|} = \frac{r_k \sin(\pi/q)}{\frac{r_k}{2}(x - s_k) \sin(\pi/q) - r_k \cos(\pi/q)}.
\]
We obtain from a classical geometry argument that $|B_k C_k| = \frac{x - s_k}{x}|B'_k A'_k| = 2(x - s_k) \sin(\pi/q)$. We deduce
\[
\tan(\delta_k(x)) = \frac{r_k \sin(\pi/q)}{2(x - s_k) \sin(\pi/q) - r_k \cos(\pi/q)} \implies \delta_k(x) = \arctan\left(\frac{r_k \sin(\pi/q)}{2(x - s_k) \sin(\pi/q) - r_k \cos(\pi/q)}\right).
\]
Thus we obtain that the map $x \mapsto \delta_k(x)$ is decreasing and tends to 0 when $x$ approaches $+\infty$.

The angle $\delta_k(x)$ is maximal if $r_k = q\|T\|$ and $s_k = 0$:
\[
\tan(\delta_k(x)) \leq \frac{q\|T\| \sin(\pi/q)}{2x \sin(\pi/q) - q\|T\| \cos(\pi/q)}.
\]
Thus we have:
\[
\begin{align*}
\delta_k(x) < \beta/2 & \iff \tan(\delta_k(x)) < \tan(\beta/2) \\
& \iff \frac{q\|T\| \sin(\pi/q)}{2x \sin(\pi/q) - q\|T\| \cos(\pi/q)} < \tan(\beta/2) \\
& \iff \frac{q\|T\| \sin(\pi/q)}{\tan(\beta/2)} < \frac{2x \sin(\pi/q) - q\|T\| \cos(\pi/q)}{\tan(\beta/2)} \\
& \iff \frac{q\|T\| \sin(\pi/q)}{\tan(\beta/2)} < \frac{1}{2} \tan(\pi/q) \\
& \iff \frac{q\|T\|}{2 \tan(\beta/2)} < \frac{1}{2} \tan(\pi/q) \\
& \iff \frac{x}{q\|T\|} < \frac{1}{2} \tan(\pi/q) \\
& \iff \hat{x} < x.
\end{align*}
\]

**Remark 39.** We can also show $x \leq |A_k| = \sqrt{x^2 + a_k^2} \leq \sqrt{x^2 + q\|T\|}$ and $x \leq |B_k| = \sqrt{x^2 + b_k^2} \leq \sqrt{x^2 + q\|T\|}$. Using Proposition 38, we deduce that $\frac{1}{x}P_x$ converges in the Hausdorff topology to the regular polygon centered at 0 with q sides, and with i as a vertex.

**Proposition 40.** For any $x > \hat{x}$, let us define $K_x = \min(v \sin(\beta - \alpha/2 - \delta_k(x)), w \sin(\beta - \alpha/2))$, see Lemma 34 for notations. Then, if $z \in P_x$, we have $T^q(z) \in P_{x'}$ with $x' \geq x + K_x$. Moreover, the map $x \mapsto K_x$ is increasing.

**Proof.** First of all, note that by Lemma 38 we have $\beta - \alpha/2 - |\delta_k| > 0$. From now on we will use the notation from Figure 9.

We split the proof into four cases, due to Lemma 34. In each case we consider $z \in P_x$ and compute $x'$ such that $T^qz \in P_{x'}$.

- **Assume $z \in E'_k$ and $T^q(z) \in E'_k$.** The angle between $v_k$ and the blue segment is equal to $\beta - \alpha/2 - \delta_k(x)$. Indeed, $\beta - \alpha/2$ is the angle between $A'_k B'_k$ and $v_k$, see Figure 8. Thus we deduce $x' = x + v_k \sin(\beta - \alpha/2 - \delta_k(x))$.

- **If $z \in E'_k$ and $T^q(z) \in G'_k$, then $x' - x$ is bigger than previous value (see green arrow in Figure 8).**
\[ s_k = b_{k+1} \]

Figure 9: The quantity \( \delta_k(x) \) used in Lemma 38.

- Assume \( z \in G_k' \) and \( T^q(z) \in G_k' \)
  Then we obtain \( x' = x + w \sin(\beta - \alpha/2) \).

- Assume \( z \in G_k' \) and \( T^q(z) \in E_k' \). In this case we can fall back onto the previous case \( x' \geq x + w \sin(\beta - \alpha/2) \).

Since \( x \mapsto \delta_k(x) \) is decreasing, we deduce that \( x \mapsto K_x \) is increasing.

5.3.2 The case \( \Delta > 0 \)

The same construction as in the case \( \Delta < 0 \) can be done with a family of polygons \( Q_x \):

- As in Lemma 37 we can prove:
  \[
  \forall z \in \mathbb{C}, \quad z \in Q_x \implies x \cos(\pi/q) \leq |z| \leq \sqrt{x^2 + q^2 \|T\|^2}
  \]

- As in Proposition 40 for all \( x > \bar{x} \) (with \( \bar{x} \) defined as in Proposition 38), there exists \( L_x > 0 \) such that for \( z \in Q_x \) we have \( T^q(z) \in P_{x'} \) with \( x' \leq x - L_x \). Moreover, the map \( x \mapsto L_x \) is increasing and tends to 0 if \( x \mapsto \bar{x} \).

5.4 Proof of Theorem 31

We consider two cases:

- First case: \( \Delta < 0 \).

Consider \( z \) such that \( |z| > \sqrt{x^2 + q^2 \|T\|^2} \). By Lemma 37 we deduce \( z \in P_{x_0} \), with
  \[
  x_0 = \sqrt{|z|^2 - q^2 \|T\|^2} > \sqrt{x^2} = \bar{x}.
  \]
  Thus we have \( x_0 > \bar{x} \) and by Proposition 40 for all \( n \in \mathbb{N} \), we obtain \( T^n z \in P_{x_n} \) with
  \[
  x_n \geq x_0 + K_{x_0} + \cdots + K_{x_{n-1}} \geq x_0 + n K_{x_0} \text{ since } x \mapsto K_x \text{ is increasing}.
  \]
  Now by Lemma 37 we have
  \[
  |T^n z| \geq x_n \cos(\pi/q) \geq (x_0 + K_{x_0} n) \cos(\pi/q) \to +\infty.
  \]
Thus we deduce \( M \leq \sqrt{x^2 + q^2 \|T\|^2} \). We obtain finally

\[
\sqrt{x^2 + q^2 \|T\|^2} = q \|T\| \left( \frac{1}{2 \tan(\frac{\beta - \alpha}{2})} + \frac{1}{2 \tan(\pi/q)} \right)^2 + 1
\]

\[
\leq q \|T\| \left( \frac{1}{2 \tan(\frac{\beta - \alpha}{2})} + \frac{1}{2 \tan(\pi/q)} + 1 \right).
\]

- Second case: \( \Delta > 0 \). Let us denote by \( \text{conv}(Q_x) \) the convex hull of \( Q_x \).

We fix \( M > 0 \), and we deduce \( B(0, M) \subset \text{conv}(Q_{x_0}) \) with \( x_0 = M/\cos(\pi/q) \).

If \( x_0 \leq \bar{x} \) then the proof is finished.

Otherwise, consider a sequence \( (x_n) \) of real numbers, defined by induction according to the following instructions: If \( x_{n-1} > \bar{x} \), then there exists \( x_n > 0 \) such that \( T^{q_n}B(0, M) \subset \text{conv}(Q_{x_n}) \) and \( x_n < x_{n-1} < \cdots < x_0 \).

If there exists \( n_1 \) such that \( x_{n_1} < \bar{x} \), then the proof is finished. Otherwise the sequence \( (x_n) \) is a decreasing bounded sequence, thus it converges to some value \( x_{\infty} \). If \( x_{\infty} > \bar{x} \), then \( L_x \) has limit value 0 as \( x \) tends to \( x_{\infty} \), and by a fact directly analogous to what has been shown in Proposition 40 we deduce \( L_{x_{\infty}} = 0 \), which is a contradiction. We conclude \( x_{\infty} = \bar{x} \).

The convex polygons \( (\text{conv}(Q_x))_x \) form a decreasing family (again with respect to the inclusion) and from the definition of \( A(T) \) we deduce

\[
A(T) \subset \bigcap_{n \geq 0} T^{q_n}B(0, M) \subset \bigcap_{n \geq 0} \text{conv}(x_n) = \text{conv}(Q_{\bar{x}}).
\]

We conclude the proof with the observation that \( \text{conv}(Q_{\bar{x}}) \subset B(0, \sqrt{x^2 + q^2 \|T\|^2}) \).

**Remark 41.** The above derived result can be improved slightly: indeed, it is possible to show \( B(T) \subset \text{conv}(P_{\bar{x}}) \) and \( A(T) \subset \text{conv}(Q_{\bar{x}}) \).

### 5.5 Rational example similar to Section 4.5

We consider the irrational angle of Section 4.5 and truncate it with the convergents of the continued fraction. In other words, we consider the following:

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( \Delta \) |
|---|---|---|---|
| \( \frac{169}{478} \) | \( \frac{7}{169} \) | \( 1.5e^{1.14i} \) | \( 1.14 \) |
| \( \frac{181}{478} \) | \( \frac{7}{169} \) | \( 1.5e^{1.14i} \) | \( 1.14 \) |

Here we obtain \( M \approx 120,968 \). For the irrational angle, we have obtained \( M \approx 571283 \).

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