Global existence, uniqueness and estimates of the solution to the Navier-Stokes equations

Alexander G. Ramm
Department of Mathematics, Kansas State University,
Manhattan, KS 66506, USA
ramm@math.ksu.edu

Abstract

The Navier-Stokes (NS) problem consists of finding a vector-function $v$ from the Navier-Stokes equations. The solution $v$ to NS problem is defined in this paper as the solution to an integral equation. The kernel $G$ of this equation solves a linear problem which is obtained from the NS problem by dropping the nonlinear term $(v \cdot \nabla)v$. The kernel $G$ is found in closed form. Uniqueness of the solution to the integral equation is proved in a class of solutions $v$ with finite norm

$$N_1(v) = \sup_{\xi \in \mathbb{R}^3, t \in [0,T]} (1 + |\xi|)(|v| + |\nabla v|) \leq c(\ast),$$

where $T > 0$ and $C > 0$ are arbitrary large fixed constants. In the same class of solutions existence of the solution is proved under some assumption. Estimate of the energy of the solution is given.

1 Introduction

There is a large literature on the Navier-Stokes (NS) problem, (see [2], [8] and references therein, including the papers by Leray, Hopf, Lions, Prodi, Kato and others). The global existence and uniqueness of a solution was not proved. Various notions of the solution were used, see [2], [3], [6]. In this paper a new notion of the solution is given: the solution to NS problem is defined as a solution to some integral equation. It is proved that this solution is unique, it exists in a certain class of functions globally, i.e., for all $t \geq 0$, it has finite energy, and an estimate of this solution is given.

The NS problem consists of solving the equations

$$v' + (v, \nabla)v = -\nabla p + \nu \Delta v + f, \quad x \in \mathbb{R}^3, \quad t \geq 0, \quad \nabla \cdot v = 0, \quad v(x,0) = v_0(x). \quad (1)$$

Vector-functions $v = v(x,t)$, $f = f(x,t)$ and the scalar function $p = p(x,t)$ decay as $|x| \to \infty$ uniformly with respect to $t \in [0,T]$, where $T > 0$ is an arbitrary large fixed number, $v' = v_t$, $\nu = \text{const} > 0$, $v_0$ is given, $\nabla \cdot v_0 = 0$, the velocity $v$ and the pressure $p$ are unknowns, $v_0$ and $f$ are known. Equations (1) describe viscous incompressible fluid with density $\rho = 1$. We assume

MSC: 35Q30; 76D05.

Key words: Navier-Stokes equations; global existence, uniqueness and estimates.
that $|f| + |\nabla f|$ and $|v_0| + |\nabla v_0|$ decay as $O\left(\frac{1}{(1+|x|)^a}\right)$ as $|x| \to \infty$, where $a > 3$ and for $f$ the decay with respect to $x$ holds for any $t \geq 0$, uniformly with respect to $t \in [0, T]$, where $T \geq 0$ is an arbitrary large fixed number.

Our approach consists of three steps. First, we construct tensor $G_{jm}(x,t)$, equivalent to Oseen’s tensor in [3], p.62, solving the problem:

$$G' - \nu \Delta G = \delta(x)\delta(t)\delta_{jm} - \nabla p_m, \quad \nabla \cdot G = 0; \quad G = 0, \quad t < 0. \quad (2)$$

Here $\delta(x)$ is the delta-function, $\delta_{jm}$ is the Kronecker delta, $p_m = p(x,t)e_m$, $p$ is the scalar, $p(x,t) = 0, t < 0; \nabla p_m = \frac{\partial p}{\partial x_j}e_j e_m$, $\{e_j\}_{j=1}^3$ is the orthonormal basis of $\mathbb{R}^3$, $e_i e_m$ is tensor. Our construction method differs from the one in [3].

Secondly, we prove that solving (1) is equivalent to solving the following integral equation, cf [3], p.62:

$$v(x,t) = \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)[f(y,s) - (v, \nabla)v]dy + \int_{\mathbb{R}^3} g(x - y, t)v_0(y)dy, \quad (3)$$

where

$$g(x,t) = \frac{e^{-\frac{|x|^2}{4\nu t \pi}}}{(4\nu t \pi)^{3/2}}, \quad g(x,0) = \delta(x), \quad g' - \nu \Delta g = \delta(x)\delta(t); \quad \int_{\mathbb{R}^3} g(x,t)dx = 1. \quad (4)$$

Thirdly, we prove by a new method, using the new assumption (21), see below, that equation (3) has a solution in the space $X$ of functions with finite norm $N_1(v) := \sup_{x \in \mathbb{R}^3, t \in [0,T]} \{(|v(x,t)| + |\nabla v(x,t)|)(1 + |x|)\}$ and this solution is unique in $X$. In Lemma 1 a new a priori estimate is obtained.

**Theorem 1.** Problem (1) has a unique solution in $X$. The solution to (1) exists in $X$ provided that $\sup_{n \geq 1} N_1(v_n) < \infty$, where $v_n$ is defined in (20). This solution has finite energy $E(t) = \int_{\mathbb{R}^3} |v(x,t)|^2dy$ for every $t \geq 0$.

## 2 Construction of $G$

Let $G = \int_{\mathbb{R}^3} H(\xi,t)e^{i\xi \cdot x}d\xi$. Taking Fourier transform of (2) yields

$$H' + \nu \xi^2 H = \frac{\delta(t)\delta_{jm}}{(2\pi)^3} - i\xi P_m(\xi,t), \quad \xi H = 0; \quad H(\xi,t) = 0, \quad t < 0. \quad (5)$$

The $H = H_{jm}$ is a tensor, $1 \leq j, m \leq 3$, $P_m(\xi,t) := (2\pi)^{-3} \int_{\mathbb{R}^3} p_m(x,t)e^{-i\xi \cdot x}dx$, $\xi = \xi e_j$, summation is understood here and below over the repeated indices, $1 \leq j \leq 3, \xi H := \xi_j H_{jm}$. Multiply (5) by $\xi$ from the left, use the equation $\nabla \cdot G = 0$ which implies $\xi H = 0, \xi H' = 0$, and get

$$(2\pi)^{-3}\delta(t)\xi_m = i\xi^2 P_m, \quad P_m := P_m(\xi,t) = -i\frac{\delta(t)\xi_m}{(2\pi)^3 \xi^2}, \quad \xi^2 := \xi_j \xi_j. \quad (6)$$

From (5)-(6) one obtains:

$$H'_{jm} + \nu \xi^2 H_{jm} = (2\pi)^{-3}\delta(t)\left(\delta_{jm} - \frac{\xi_j \xi_m}{\xi^2}\right). \quad (7)$$
Thus,
\[ H = H_{jm}(\xi, t) = (2\pi)^{-3} \frac{\xi_j \xi_m}{\xi^2} e^{-\nu \xi^2 t}; \quad H = 0, \ t < 0. \]  \hfill (8)
\[ G(x, t) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \left( \delta_{jm} - \frac{\xi_j \xi_m}{\xi^2} \right) e^{-\nu \xi^2 t} d\xi := I_1 + I_2. \]  \hfill (9)

The integrals \( I_1, I_2 \) are calculated in the Appendix:
\[ I_1 = \delta_{jm} g(x, t); \quad g(x, t) := \frac{e^{-\xi^2 x^2}}{(4\nu t \pi)^{3/2}}, \quad t > 0; \quad g = 0, \ t < 0, \]  \hfill (10)
\[ I_2 = \frac{\partial}{\partial x_j} \frac{1}{|x|^2} \left( \int_{0}^{\xi |/(4\nu t)\pi^{1/2}} e^{-s^2} ds \right), \quad t > 0; \quad I_2 = 0, \ t < 0. \]  \hfill (11)

Therefore,
\[ G(x, t) = \delta_{jm} g(x, t) + \frac{1}{2\pi^3/2} \frac{\partial^2}{\partial x_j \partial x_m} \left( \frac{1}{|x|} \int_{0}^{\xi |/(4\nu t)\pi^{1/2}} e^{-s^2} ds \right), \quad t > 0. \]  \hfill (12)

3 \ Solution to integral equation for \( v \) satisfies NS equations

Apply the operator \( L := \partial / \partial t - \nu \Delta \) to the left side of (3) and use (2) to get
\[ L v = \int_{0}^{t} ds \int_{\mathbb{R}^3} \left[ (x - y) \delta(t - s) \delta_{jm} + \partial_j \left( \frac{1}{2\pi^3/2} \int_{0}^{\xi |/(4\nu t)\pi^{1/2}} e^{-s^2} ds \right) \right] \times \]
\[ \left( f(y, s) - (v(y, s), \nabla) v(y, s) \right) dy = f(x, t) - (v(x, t), \nabla) v(x, t) - \nabla p_m(x, t), \]  \hfill (13)
where
\[ p_m(x, t) = -\int_{0}^{t} ds \int_{\mathbb{R}^3} \partial_j \left( \frac{1}{2\pi^3/2} \int_{0}^{\xi |/(4\nu t)\pi^{1/2}} e^{-s^2} ds \right) \left[ f(y, s) - (v(y, s), \nabla) v(y, s) \right] dy, \]  \hfill (14)
and \( v(x, 0) = v_0(x) \) because \( g(x - y, 0) = \delta(x - y) \). Using the formula \( \nabla \cdot G = 0 \), the relation
\[ \int_{\mathbb{R}^3} g(x - y, t)v_0(y) dy = \int_{\mathbb{R}^3} g(z, t)v_0(z + x) dz \]  and the formula \( \nabla \cdot v_0 = 0 \), one checks that \( \nabla \cdot v = 0 \). Thus, a solution to (3) solves (1).

4 \ Proof of the uniqueness of the solution to (3) in the space of functions with finite norm \( N_1(v) \)

Let \( X \) be the space \( C^1(\mathbb{R}^3; 1 + |x|) \) of vector-functions with the norm
\[ N_1(v) := \sup_{x \in \mathbb{R}^3, t \in [0, T]} \left( |v(x, t)| + |\nabla v(x, t)| \right) (1 + |x|), \]
where $T > 0$ is an arbitrary large fixed number, $\nabla$ stands for any first order derivative, and let $N_0(v) = \sup_{x \in \mathbb{R}^3}(|v(x, t)| + |\nabla v(x, t)|)$, $N_0(v) \leq N_1(v)$, $|v| := (\sum_{j=1}^3 |v_j|^2)^{1/2}$. Assume that there are two solutions $v_1, v_2$ to equation (3) with finite norms $N_1(v)$ and let $w = v_1 - v_2$. One has

$$w(x, t) = \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)[(v_2, \nabla)v_2 - (v_1, \nabla)v_1]dy =$$

$$\int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)[(w, \nabla)v_2 + (v_1, \nabla)w]dy. \tag{15}$$

Therefore

$$N_0(w) \leq \int_0^t dsN_0\left( \int_{\mathbb{R}^3} |G(x - y, t - s)|(1 + |y|)^{-1}dy \right)N_0(w)(N_1(v_2) + N_1(v_1)) \leq c \int_0^t \frac{N_0(w)}{(t - s)^{1/2}}ds. \tag{16}$$

Since we are proving uniqueness in the set of functions with finite norm $N_1(v)$, one has $N_1(v_2) + N_1(v_1) \leq c$, where $c > 0$ stands for various estimation constants. It is checked in the Appendix that

$$N_0\left( \int_{\mathbb{R}^3} |G(x - y, t)|(1 + |y|)^{-1}dy \right) \leq ct^{-1/2}, \tag{17}$$

so inequality (16) holds. From (16) by the standard argument one derives that $N_0(w) = 0$. Thus, $v_1 = v_2$. Uniqueness of the solution to (3) is proved in $X$. \hfill \square

5 Proof of the existence of the solution to (3) in the set of functions with finite norm $N_1(v)$

Rewrite (3) as

$$v(x, t) = -\int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)(v, \nabla)vdy + F(x, t), \tag{18}$$

where $F$ is known:

$$F(x, t) := \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)f(y, s)dy + \int_{\mathbb{R}^3} g(x - y, t)v_0(y)dy. \tag{19}$$

Equation (18) is of Volterra type, nonlinear, solvable, as we wish to prove, by iterations:

$$v_{n+1}(x, t) = -\int_0^t B(v_n, \nabla)v_n ds + F, \quad v_1 = F; \quad Bv := \int_{\mathbb{R}^3} G(x - y, t - s)v(y, s)dy. \tag{20}$$

Using (17), the formula $\int_0^t \frac{ds}{s^{1/2}} = 2t^{1/2}$, and assuming that

$$\sup_{n \geq 1} N_1(v_n) \leq c, \tag{21}$$

one gets:

$$N_0(v_{n+1} - v_n) \leq N_0\left( \int_0^t ds \int_{\mathbb{R}^3} \frac{|G(x - y, t - s)|}{1 + |y|}dy \right)[N_0(v_n - v_{n-1})(N_1(v_n) + N_1(v_{n-1})] \leq$$

$$ct^{1/2}N_0(v_n - v_{n-1}). \tag{22}$$
Therefore,
\[ N_0(v_{n+1} - v_n) \leq ct^{1/2} N_0(v_n - v_{n-1}). \tag{23} \]

If \( \tau \) is chosen so that \( c \tau^{1/2} < 1 \), then \( B \) is a contraction map on the bounded set \( N_0(v) \leq R \) for sufficiently large \( R \) and \( t \in [0, \tau] \). Thus, \( v \) is uniquely determined by iterations for \( x \in \mathbb{R}^3 \) and \( t \in [0, \tau] \). If \( t > \tau \) rewrite (18) as
\[
v = F - \int_0^\tau ds \int_{\mathbb{R}^3} G(x - y, t - s)(v, \nabla)v dy - \int_\tau^t ds \int_{\mathbb{R}^3} G(x - y, t - s)(v, \nabla)v dy := F_1 - \int_\tau^t ds \int_{\mathbb{R}^3} G(x - y, t - s)(v, \nabla)v dy,
\]
where \( F_1 \) is a known function since \( v \) is known for \( t \in [0, \tau] \). To this equation one applies the contraction mapping principle and get \( v \) on the interval \([0, 2\tau]\). Continue in this fashion and construct \( v \) for any \( t \geq 0 \). Process (20) converges to a solution to equation (3).

Let us make a remark concerning the mapping done by the operator in (18). In [4], p. 234, sufficient conditions are given for a singular integral operator to map a class of functions with a known power rate of decay at infinity into a class of functions with a suitable rate of decay at infinity. It follows from [4], Theorem 5.1 on p. 234, that if \( |f(x)| + |\nabla f| \leq c(1 + |x|)^{-a}, a > 3, \) then the part of the operator \( G \) in (18), responsible for the lesser decay of the iterations \( v_n \) at infinity, acts as a weakly singular operator similar to the operator \( Q \), where
\[ Qf := \int_{\mathbb{R}^3} \frac{b(|x - y|)f(y)}{|x - y|^3} dy \leq c\frac{1}{(1 + |x|)^3} \]
for large \( |x| \). This part of \( G \) yields the decay \( O\left(\frac{1}{(1 + |x|)^3}\right) \) at infinity. This decay, in general, cannot be improved. The first iteration \( v_1 \) yields a function decaying with its first derivatives as \( O\left(\frac{1}{(1 + |x|)^3}\right) \). The second iteration contains a function whose behavior for large \( |x| \) is determined by the decay of the functions \( F \). Since \( (v_1 \cdot \nabla)v_1 = O\left((1 + |x|)^{-7}\right) \), the decay of \( v_2 \) is again \( O\left((1 + |x|)^{-3}\right) \) because \( v_2 = O\left((1 + |x|)^{-3}\right) \) and \( |\nabla v_1| = O\left((1 + |x|)^{-4}\right) \). Thus, for \( |v_n| + |\nabla v_n| \) one gets the decay of the order \( O\left((1 + |x|)^{-3}\right) \) as \( |x| \to \infty \) for every fixed \( t \geq 0 \), provided that \( f \) and \( v_0 \) together with their first derivatives decay not slower than \( O\left((1 + |x|)^{-a}\right), a > 3 \).

6 Energy of the solution

In this Section we prove that the solution has finite energy in a suitable sense and give an estimate of the solution as \( t \to \infty \). Let us define the energy by the integral \( E(t) = \int_{\mathbb{R}^3} |v(x, t)|^2 dy \). If one multiplies (11) by \( v \) and integrate over \( \mathbb{R}^3 \), then one gets a known conservation law (see [2]):
\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} |v(x, t)|^2 dx + v \int_{\mathbb{R}^3} \nabla v_j(x, t) \cdot \nabla v_j(x, t) dx = \int_{\mathbb{R}^3} f \cdot v dx. \tag{24}
\]

Integrating (24) with respect to \( t \) over any finite interval \([0, t], 0 \leq t \leq T\), and denoting \( N(v) := (\int_{\mathbb{R}^3} |v(x, t)|^2 dy)^{1/2} = \sqrt{E(t)} \), one gets
\[
2\nu \int_0^t \|\nabla v\|^2 dt + E(t) \leq E(0) + 2 \int_0^t ds \int_{\mathbb{R}^3} |f \cdot v| dx \leq E(0) + 2 \int_0^T N(f)V(v) ds. \tag{25}
\]

Denote \( E_T := \sup_{t \in [0, T]} E(t) \). Maximizing inequality (25) with respect to \( t \in [0, T] \), and using the elementary inequality \( 2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2, \) \( a, b, \epsilon > 0 \), one derives from (25) the following
Let $p = \frac{1}{2}$. Then (25)-(26) allow one to estimate $E_T$ and $\nu \int_0^t \|\nabla v\|^2 \, dt$ through $f$ and $v_0$. In particular, we have proved the following theorem.

**Theorem 2.** Assume that $\int_0^T \mathcal{N}(f) \, dt < \infty$ for all $T > 0$ and $E(0) < \infty$. Then $\sup_{t \leq T} E(t) < \infty$ and $\int_0^T \|\nabla v(x, t)\|^2 \, dt < \infty$ for all $T > 0$.

**Remark 1.** It is proved in [5] that in a bounded domain $D \subset \mathbb{R}^3$ the solution to a boundary problem for equations (1) in a bounded domain with the Dirichlet boundary condition for large $t$ decays exponentially provided that $\int_0^\infty e^{bt} \mathcal{N}(f) \, dt < \infty$ for some $b = \text{const} > 0$.

### 7 Appendix

1. **Integral $I_1$.** One has $I_1 = I_{11}I_{12}I_{13}$, where $I_{1p} := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixp}\hat{\xi}_p \, d\xi_p$. No summation in $x_p\xi_p$ is understood here. One has

\[
\frac{1}{\pi} \int_0^\infty \cos(x_p\xi_p) e^{-v\xi^2} \, d\xi_p = \frac{1}{\pi} \frac{\pi^{1/2}}{2(\nu\pi)^{1/2}} e^{-\frac{x^2}{4\nu t}} = \frac{e^{-\frac{x^2}{4\nu t}}}{(4\nu\pi)^{1/2}}.
\]

After a multiplication (with $p = 1, 2, 3$) this yields formula (11).

**Integral $I_2$.** One has

\[
I_2 = \frac{\partial^2}{\partial x_j \partial x_m} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix} \frac{e^{-\nu\xi^2}}{\xi^2} \, d\xi = \frac{\partial^2}{\partial x_j \partial x_m} \left( \frac{1}{(2\pi)^2} \int_0^\infty dr e^{-\nu r^2} \int_1^{-1} e^{ir|x|} \, du \right),
\]

so

\[
I_2 = \frac{\partial^2}{\partial x_j \partial x_m} \left( \frac{1}{|x|} \frac{1}{2\pi^3} \mathcal{J}_{\mathbb{R}^3} \int_0^{12/\nu \pi} e^{-s^2} \, ds \right).
\]

Here we have used formula (2.4.21) from [1]:

\[
\int_0^\infty \frac{\sin(x) y}{s} e^{-as^2} \, ds = \frac{\pi}{2} \text{Erf}(\frac{y}{2a^{1/2}}); \quad \text{Erf}(y) := \frac{2}{\pi^{1/2}} \int_0^y e^{-s^2} \, ds.
\]

2. One has $\int_{\mathbb{R}^3} |\nabla g(x, t)| \, dx \leq (4\nu\pi)^{-\frac{3}{4}} \int_{\mathbb{R}^3} e^{-\frac{|x|^2}{16\nu t}} \, dx = \frac{2}{(\nu t)^{3/2}}$. This proves the estimate

\[
\int_{\mathbb{R}^3} |G(x - y, t) (1 + |y|)^{-1} \, dy \leq \frac{ct^{-\frac{3}{2}}}{2\pi^{3/2}} \quad \text{for the first term of $G$ in (12), namely for $g$.}
\]

The second term of $G$ is $J := \frac{\partial^2}{\partial x_j \partial x_m} \left( \frac{|x|}{|x|^4} \int_0^{12/(4\nu t)^{1/2}} e^{-s^2} \, ds \right)$ up to a factor $\frac{1}{2\pi^{3/2}}$.

One checks by direct differentiation that

\[
J = e^{-\frac{|x|^2}{16\nu t}} \left( \frac{\delta_{jm}}{|x|^2} - \frac{3x_j x_m}{|x|^4} - \frac{2x_j x_m}{|x|^2 (4\nu t)^{3/2}} \right) + \int_0^{12/(4\nu t)^{1/2}} e^{-s^2} \, ds \left( \frac{3x_j x_m}{|x|^5} - \frac{\delta_{jm}}{|x|^3} \right).
\]

Let $b(x) := b(\frac{|x|}{(4\nu t)^{1/2}}) := \int_0^{12/(4\nu t)^{1/2}} e^{-s^2} \, ds$. Note that $b(s) = O(s)$ as $s \to 0$ and $b(s) \to \frac{\pi^{1/2}}{2}$ as $s \to \infty$. It is sufficient to estimate the term with the strongest singularity, namely,
where the known estimate

It follows from (24) and (25) that

Thus, provided that assumptions of Theorem 2 hold. It follows from (32) that

Remark 2. In [2] it is shown that the smoothness properties of \( f \) and \( v_0 \) are improved when the smoothness properties of \( f \) and \( v_0 \) are improved. This also follows from equation (3) by the properties of weakly singular integrals and of the kernel \( g(x, t) \).

Let us prove a new a priori estimate.

Lemma 1. If the assumptions of Theorem 2 hold, then \((1 + \xi^2)|\hat{v}(\xi, t)|^2 \leq c + ct\) for all \( t \geq 0 \), where \( c > 0 \) does not depend on \( \xi, t \).

Proof. The Fourier transform of (18) yields

\[
\hat{v}(\xi, t) = \hat{F}(\xi, t) - \int_0^t ds H(\xi, t-s)(\hat{v}, \nabla)\hat{v}, \quad v(x, t) = \int_{\mathbb{R}^3} e^{i\xi \cdot x} \hat{v}(\xi, t) d\xi,
\]

where \( H \) is defined in (8),

\[
|\hat{v} \ast \xi_j \hat{v}_m| \leq \mathcal{N}(\hat{v})\mathcal{N}(\xi|\hat{v}),
\]

over the repeated indices \( j \) one sums up, \( \mathcal{N}(\hat{v}) := \|\hat{v}\|_{L^2(\mathbb{R}^3)} \), and \( \hat{v} \ast \psi \) denotes the convolution of two functions, so

\[
|\hat{v} \ast \xi_j \hat{v}_m| \leq \mathcal{N}(\hat{v})\mathcal{N}(\xi|\hat{v}) \leq c\mathcal{N}(\xi|\hat{v}),
\]

where the known estimate \( \mathcal{N}(\hat{v}) \leq c \) was used. One has

\[
|H| \leq ce^{-\nu t \xi^2}.
\]

It follows from (21) and (25) that

\[
(2\pi)^3 \int_0^t ds \int_{\mathbb{R}^3} |\xi|^2 |\hat{v}|^2 d\xi = \int_0^t ds \int_{\mathbb{R}^3} |\nabla \hat{v}|^2 dx \leq c, \quad \forall t \geq 0,
\]

provided that assumptions of Theorem 2 hold. It follows from (32) that

\[
|\hat{v}| \leq |\hat{F}| + c \int_0^t ds e^{-\nu(t-s)\xi^2} \mathcal{N}(\xi|\hat{v}) \leq |\hat{F}| + c \left( \int_0^t e^{-2\nu(t-s)\xi^2} ds \right)^{1/2} \left( \int_0^t ds \mathcal{N}^2(\nabla \hat{v}) \right)^{1/2}.
\]

Thus,

\[
|\xi|^2 |\hat{v}|^2 \leq 2|\xi|^2 |\hat{F}|^2 + 2c^2 |\xi|^2 \frac{1 - e^{-2\nu t \xi^2}}{2\nu \xi^2} \leq c + ct,
\]

provided that \( |\xi|^2 |\hat{F}|^2 \leq c \). This inequality holds if \( f \) and \( v_0 \) are smooth and rapidly decaying. Lemma 1 is proved. \( \square \)
References

[1] H. Bateman, A. Erdelyi, Tables of integral transforms, New York, McGraw-Hill, 1954.

[2] O. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach, New York, 1969.

[3] P. Lemarie-Rieusset, The Navier-Stokes problem in 21-st century, Chapman and Hall/CRC, 2016.

[4] S. Mikhlin, S. Prössdorf, Singular integral operators, Springer Verlag, New York, 1986.

[5] A. G. Ramm, Large-time behavior of the weak solution to 3D Navier-Stokes equations, Appl. Math. Lett., 26 (2013), 252-257.

[6] A. G. Ramm, Existence and uniqueness of the global solution to the Navier-Stokes equations, Applied Math. Letters, 49, (2015), 7-11.

[7] A. G. Ramm, Large-time behavior of solutions to evolution equations, Handbook of Applications of Chaos Theory, Chapman and Hall/CRC, 2016, pp. 183-200 (ed. C. Skiadas).

[8] R. Temam, Navier-Stokes equations. Theory and numerical analysis, North Holland, Amsterdam, 1984.