NECESSARY AND SUFFICIENT CONDITIONS
FOR THE SOLVABILITY OF THE $L^p$ DIRICHLET
PROBLEM ON LIPSCHITZ DOMAINS.

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Abstract. We study the homogeneous elliptic systems of order $2\ell$ with real constant coefficients on Lipschitz domains in $\mathbb{R}^n$, $n \geq 4$. For any fixed $p > 2$, we show that a reverse Hölder condition with exponent $\varepsilon$ is necessary and sufficient for the solvability of the Dirichlet problem with boundary data in $L^p$. We also obtain a simple sufficient condition. As a consequence, we establish the solvability of the $L^p$ Dirichlet problem for $n \geq 4$ and $2 - \varepsilon < p < \frac{2(n-1)}{n-4} + \varepsilon$. The range of $p$ is known to be sharp if $\ell \geq 2$ and $4 \leq n \leq 2\ell + 1$. For the polyharmonic equation, the sharp range of $p$ is also found in the case $n = 6, 7$ if $\ell = 2$, and $n = 2\ell + 2$ if $\ell \geq 3$.

1. Introduction

In this paper we study the higher order homogeneous elliptic systems with real constant coefficients on bounded domains in $\mathbb{R}^n$ with Lipschitz boundaries. For any fixed $p > 2$, we obtain necessary and sufficient conditions for the solvability of the Dirichlet problem with boundary data in $L^p$. As a consequence, we are able to establish the solvability of the $L^p$ Dirichlet problem for $2 - \varepsilon < p < \frac{2(n-1)}{n-4} + \varepsilon$. The range of $p$ is known to be sharp if $\ell \geq 2$ and $4 \leq n \leq 2\ell + 1$, where $2\ell$ is the order of the system. We also obtain the $L^p$ solvability for the sharp range of $p$ in the case of the polyharmonic equation $\Delta^\ell u = 0$ for $n = 6, 7$, if $\ell = 2$ and for $n = 2\ell + 2$, if $\ell \geq 3$.

More precisely, let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Consider the homogeneous elliptic system of order $2\ell$, $\mathcal{L}(D)u = 0$ in $\Omega$, where $u = (u^1, \ldots, u^m)$,

$$
(\mathcal{L}(D)u)^j = \sum_{k=1}^m \mathcal{L}^{jk}(D)u^k, \quad j = 1, \ldots, m,
$$

$$(1.1)$$

$$
\mathcal{L}^{jk}(D) = \sum_{|\alpha|=|\beta|=\ell} a^{jk}_{\alpha\beta} D^\alpha D^\beta,
$$

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and $D = (D_1, D_2, \ldots, D_n)$, $D_i = \partial/\partial x_i$ for $i = 1, 2, \ldots, n$. Also $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index with length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. Let

$$L^{jk}(\xi) = \sum_{|\alpha| = |\beta| = \ell} a^{jk}_{\alpha\beta} \xi^\alpha \xi^\beta, \quad \text{for } \xi \in \mathbb{R}^n.$$ 

Throughout this paper, we will assume that $a^{jk}_{\alpha\beta}$ are real constants satisfying the symmetry condition

$$L^{jk}(\xi) = L^{kj}(\xi)$$

and the Legendre-Hadamard ellipticity condition,

$$\mu|\xi|^{2\ell} \leq \sum_{j,k=1}^m L^{jk}(\xi)\eta_j\eta_k \leq \frac{1}{\mu}|\xi|^{2\ell}||\eta||^2,$$

for some $\mu > 0$, and all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$. We are interested in the Dirichlet problem,

$$\begin{cases}
  L(D)u = 0 & \text{in } \Omega, \\
  D^\alpha u = f_\alpha & \text{on } \partial\Omega \quad \text{for } |\alpha| \leq \ell - 2, \\
  \frac{\partial^{\ell-1} u}{\partial N^{\ell-1}} = g & \text{on } \partial\Omega,
\end{cases}$$

where $\frac{\partial^{\ell-1} u}{\partial N^{\ell-1}} = \sum_{|\alpha| = \ell - 1} \frac{(\ell - 1)!}{\alpha!} N^\alpha D^\alpha u$, and $N$ denotes the outward unit normal to $\partial\Omega$.

To describe the $L^p$ Dirichlet problem, we let $\hat{f} = \{f_\alpha : |\alpha| \leq \ell - 2\}$ be an array of functions on $\partial\Omega$. Following Verchota and Pipher [V3,PV3,V4], we consider the Dirichlet problem (1.4) with boundary data $(\hat{f}, g)$ taken from the space $(WA^p_{\ell-1}(\partial\Omega), L^p(\partial\Omega))$, where $WA^p_{\ell-1}(\partial\Omega)$ denotes the completion of the set of arrays $\hat{\psi} = \{D^\alpha \psi : |\alpha| \leq \ell - 2\}$, $\psi \in C^\infty_0(R^n)$ under the norm

$$\sum_{|\alpha| \leq \ell - 2} \|D^\alpha \psi\|_{L^p(\partial\Omega)} + \sum_{|\alpha| = \ell - 2} \|\nabla_t D^\alpha \psi\|_{L^p(\partial\Omega)} \quad \text{on } \partial\Omega.$$ 

Here $\nabla_t h$ denotes the tangential derivatives of $h$. The boundary values in (1.4) are taken in the sense of non-tangential convergence a.e. with respect to the surface measure $d\sigma$ on $\partial\Omega$. As such, we will require that the non-tangential maximal function $(\nabla^{\ell-1} u)^*$ on $\partial\Omega$ is in $L^p(\partial\Omega)$, where $\nabla^{\ell-1} u$ denotes the tensor of all partial derivatives of order $\ell - 1$ in $\mathbb{R}^n$ of $u$. Thus the $L^p$ Dirichlet problem (1.4) on $\Omega$ is said to be uniquely solvable if given any $(\hat{f}, g) \in (WA^p_{\ell-1}(\partial\Omega), L^p(\partial\Omega))$, there exists a unique $u$ satisfying (1.4) and $(\nabla^{\ell-1} u)^* \in L^p(\partial\Omega)$, and the unique solution $u$ satisfies the scale-invariant estimate

$$\|\nabla^{\ell-1} u\|_{L^p(\partial\Omega)} \leq C \left\{ \|g\|_{L^p(\partial\Omega)} + \sum_{|\alpha| = \ell - 2} \|\nabla_t f_\alpha\|_{L^p(\partial\Omega)} \right\}$$

for some constant $C$. This completes the proof.
with constant $C$ independent of the boundary data $(f,g)$.

For Laplace’s equation $\Delta u = 0$ in $\Omega$, the $L^p$ Dirichlet problem was solved by Dahlberg [D1,D2] for the optimal range $2 - \varepsilon < p \leq \infty$, where $\varepsilon > 0$ depends on $n$ and the Lipschitz character of $\Omega$ (also see [JK,DK1,K1] for the Neumann problem and [K2] for other related problems). For the general elliptic equations and systems $L(D)u = 0$ considered in this paper, the solvability of the $L^p$ Dirichlet problem has been established for $2 - \varepsilon < p < 2 + \varepsilon$. See [FKV,DKV2,K1,F,G] for second order elliptic systems, [DKV1,V2,V3] for the biharmonic and polyharmonic equations, and [PV3,V4] for general higher order elliptic equations and systems. It is known that the restriction $p > 2 - \varepsilon$ is necessary for general Lipschitz domains (see e.g. [K1]). However, due to the lack of the maximum principle for elliptic systems and higher order elliptic equations, it has been a challenging problem to determine the optimal ranges of $p$ for which one may solve the $L^p$ Dirichlet problem. Nevertheless in the case of $n = 2$ or 3, the $L^p$ Dirichlet problem for elliptic systems and higher order equations was solved for the optimal range $2 - \varepsilon < p \leq \infty$ [PV1,DK2,PV2,PV4,V4,S1,S2,MM]. This was done by establishing certain decay estimates on the Green’s functions.

Recently in [S5] we developed a new approach to the $L^p$ estimates for boundary value problems, via $L^2$ estimates, reverse Hölder inequalities, and a real variable argument. For the second order elliptic systems as well as the polyharmonic equation, we were able to show that the $L^p$ Dirichlet problem is uniquely solvable for

\begin{equation}
(1.7)
\quad n \geq 4 \quad \text{and} \quad 2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon.
\end{equation}

We remark that in the case of the polyharmonic equation $\Delta^\ell u = 0$, the range in (1.7) is known to be sharp for $\ell \geq 2$ and $4 \leq n \leq 2\ell + 1$. This was pointed out by Pipher and Verchota [PV1,PV3,PV4], using examples in [MNP,KM].

In this paper we continue the work in [S5] and study the general higher order elliptic equations and systems. Let $\Delta(Q,r) = B(Q,r) \cap \partial\Omega$ and $T(Q,r) = B(Q,r) \cap \Omega$ where $Q \in \partial\Omega$ and $0 < r < r_0$. One of the key ingredients in the approach we developed in [S5] is the following (weak) reverse Hölder inequality

\begin{equation}
(1.8)
\quad \left( \frac{1}{r^{n-1}} \int_{\Delta(Q,r)} |(\nabla^{\ell-1} v)^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q,2r)} |(\nabla^{\ell-1} v)^*|^2 \, d\sigma \right)^{1/2},
\end{equation}

for solutions of $L(D)v = 0$ in $\Omega$ satisfying $(\nabla^{\ell-1} v)^* \in L^2(\partial\Omega)$ and $D^\alpha v = 0$ for $|\alpha| \leq \ell - 1$ on $\Delta(Q,3r)$. We will show in this paper that given a general system of elliptic operators $L(D)$, a bounded Lipschitz domain $\Omega$ and $p > 2$, the reverse Hölder condition (1.8) with exponent $p$ for $L^2$ solutions with zero Dirichlet data on $\Delta(Q,3r)$ is necessary and sufficient for the solvability of the $L^p$ Dirichlet problem on $\Omega$. 


Theorem 1.9. Let $L(D)$ be a system of elliptic operators of order $2\ell$ given by (1.1) and satisfying conditions (1.2) and (1.3). For any bounded Lipschitz domain $\Omega$ and $p > 2$, the following are equivalent.

(i) The $L^p$ Dirichlet problem for $L(D)u = 0$ on $\Omega$ is uniquely solvable.

(ii) There exist $C > 0$ and $r_0 > 0$ such that for any $Q \in \partial \Omega$ and $0 < r < r_0$, the reverse Hölder condition (1.8) holds for any solution $v$ of $L(D)v = 0$ in $\Omega$ with the properties $(\nabla^{\ell-1}v)^* \in L^2(\partial \Omega)$ and $D^\alpha v = 0$ for $|\alpha| \leq \ell - 1$ on $\Delta(Q; 3r)$.

Since the reverse Hölder condition (1.8) has the self-improving property, it follows that the set of all exponents $p$ in $(2, \infty)$ for which the $L^p$ Dirichlet problem for $L(D)u = 0$ on $\Omega$ is solvable is an open interval $(2, q]$ with $2 < q \leq \infty$.

Using square function estimates for $L(D)$ [DKPV] as well as the regularity estimate (1.12) [PV3,V4], we also obtain a much simpler condition which implies (1.8) (see condition (1.11) as well as (2.15)). This leads to the following result.

Theorem 1.10. Let $L(D)$ be a system of elliptic operators given by (1.1) and satisfying the symmetry condition (1.2) and ellipticity condition (1.3). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 4$. Suppose that there exist constants $C_0 > 0$, $R_0 > 0$, and $\lambda \in (0, n]$ such that for $0 < r < R < R_0$ and $Q \in \partial \Omega$,

\[
\int_{T(Q,r)} |\nabla^{\ell-1}v|^2 \, dx \leq C_0 \left( \frac{r}{R} \right)^\lambda \int_{T(Q,R)} |\nabla^{\ell-1}v|^2 \, dx,
\]

whenever $v$ is a solution of $L(D)v = 0$ in $\Omega$ with the properties, $(\nabla^{\ell-1}v)^* \in L^2(\partial \Omega)$ and $D^\alpha v = 0$ on $\Delta(Q; R)$ for $|\alpha| \leq \ell - 1$. Then, if

\[
2 < p < 2 + \frac{4}{n - \lambda},
\]

the $L^p$ Dirichlet problem (1.4) on $\Omega$ is uniquely solvable.

For solutions of the higher order elliptic equations and systems $L(D)u = 0$ in $\Omega$, the following regularity estimate,

\[
\| (\nabla^\ell u)^* \|_{L^p(\partial \Omega)} \leq C \| \nabla_x \nabla^{\ell-1}u \|_{L^p(\partial \Omega)},
\]

was established by Pipher and Verchota [PV3,V4] for $n \geq 2$ and $2 - \varepsilon_1 < p < 2 + \varepsilon_1$. Using (1.13), it is not hard to show that condition (1.11) holds for some $\lambda > 3$. As a consequence, we obtain the following extension of the main results in [S5]. It gives an affirmative answer to an open question raised in [PV3].

Corollary 1.14. For a general higher order homogeneous elliptic system with real constant coefficients satisfying the symmetry condition and the Legendre-Hadamard ellipticity condition, the $L^p$ Dirichlet problem (1.4) on $\Omega$ is uniquely solvable for $n \geq 4$ and...
\[ 2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon, \quad \text{where} \quad \varepsilon > 0 \quad \text{depends on} \quad n, \ m, \ \ell, \ \mu \ \text{and the Lipschitz character of} \ \Omega. \]

Whether condition (1.11) is necessary for the conclusion of Theorem 1.10 remains open for \( p > 2(n-1)/(n-3) \) (see Remark 5.21). As we mentioned earlier, for the polyharmonic equation \( \Delta^\ell u = 0 \) in \( \Omega \) where \( \ell \geq 2 \), the \( L^p \) Dirichlet problem is not solvable in general on Lipschitz domains if \( 4 \leq n < 2\ell + 1 \) and \( p > 2(n-1)/(n-3) \). Thus the range of \( p \) in Corollary 1.13 is sharp in the case \( 4 \leq n \leq 2\ell + 1 \). If \( n \geq 2\ell + 2 \), the \( L^p \) Dirichlet problem is known to be not solvable in general for \( p > 2\ell/(\ell-1) \) [PV3,MNP]. Note that if \( \mathcal{L}(D) \) on \( \Omega \) satisfies condition (1.11) for some \( \lambda > n - 2\ell + 2 \), which would imply that \( v \) is Hölder continuous up to the boundary \( \Delta(Q,R) \), then the \( L^p \) Dirichlet problem is indeed solvable for \( 2 - \varepsilon < p < 2\ell/(\ell-1) + \varepsilon \) by Theorem 1.10. In this regards, the problem seems to be closely related to the Wiener’s type characterization of regularity for higher order elliptic equations studied in [M1,MN,M2,M3]. It is not hard to see that for the subclass of the operators \( \mathcal{L}(D) \) called positive with the weight \( F \) studied by Maz’ya in [M2,M3] (\( F \) is the fundamental solution of \( \mathcal{L}(D) \)), estimate (1.11) holds for some \( \lambda > n - 2\ell + 2 \) on Lipschitz domains. See Theorem 2.14. In particular, estimate (1.11) for some \( \lambda > n - 2\ell + 2 \) holds if \( \ell = 2 \) (the biharmonic equation) and \( n = 5, 6, \) or \( 7 \) [M1]. In the case \( \ell \geq 3 \), estimate (1.11) holds for some \( \lambda > n - 2\ell + 2 \) if \( n = 2\ell + 1 \) or \( 2\ell + 2 \) [MN]. This, combined with Theorem 1.10 as well as results in [DKV1,V3,PV1,PV2,PV4,S5], yields the following.

**Theorem 1.15.** For the biharmonic equation \( \Delta^2 u = 0 \) in \( \Omega \), the \( L^p \) Dirichlet problem is uniquely solvable if

\[
2 - \varepsilon < p \leq \infty \quad \text{if} \quad n = 2, \ 3, \\
2 - \varepsilon < p < 6 + \varepsilon \quad \text{if} \quad n = 4, \\
2 - \varepsilon < p < 4 + \varepsilon \quad \text{if} \quad n = 5, \ 6, \ 7, \\
2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon \quad \text{if} \quad n \geq 8.
\]

The ranges of \( p \) are sharp for \( 2 \leq n \leq 7 \).

**Theorem 1.16.** For the polyharmonic equation \( \Delta^\ell u = 0 \) in \( \Omega \) with \( \ell \geq 3 \), the \( L^p \) Dirichlet problem is uniquely solvable if

\[
2 - \varepsilon < p \leq \infty \quad \text{if} \quad n = 2, \ 3, \\
2 - \varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon \quad \text{if} \quad 4 \leq n \leq 2\ell + 1 \text{ or } n \geq 2\ell + 3, \\
2 - \varepsilon < p < \frac{2\ell}{\ell-1} + \varepsilon \quad \text{if} \quad n = 2\ell + 2.
\]

The ranges of \( p \) are sharp for \( 2 \leq n \leq 2\ell + 2 \).

We should remark that only the case \( n = 6, 7 \) in Theorem 1.15 and the case \( n = 2\ell + 2 \) in Theorem 1.16 are new. Also the problem of sharp ranges of \( p \) remains open for \( n \geq 8 \) if \( \ell = 2 \), and for \( n \geq 2\ell + 3 \) if \( \ell \geq 3 \).
The paper is organized as follows. In Section 2 we collect some basic estimates for solutions of the elliptic system $L(D)u = 0$ as well as some inequalities that will be used in later sections. In Section 3 we show that the reverse Hölder condition (1.8) is sufficient for the $L^p$ solvability of the Dirichlet problem (1.4) (see Theorem 3.2). That this condition is also necessary is proved in Section 4 (see Theorem 4.1). While Theorem 1.9 follows by combining Theorem 3.2 with Theorem 4.1, the proof of Theorem 1.10 as well as Corollary 1.14 will be given in Section 5. Finally we remark that we will make no effort to distinguish vector-valued functions from real valued functions. It should be clear from the context.

2. Basic Estimates and Inequalities

Most of the materials in this section are known.

Proposition 2.1. (Interior estimates) Let $u$ be a solution of $L(D)u = 0$ in $\Omega$. Suppose $B(x, 2r) \subset \Omega$. Then

\begin{equation}
|D^\alpha u(x)| \leq C_\alpha \frac{r^{n+|\alpha|}}{r^{n+|\alpha|}} \int_{B(x, r)} |u(y)| dy,
\end{equation}

for any multi-index $\alpha$.

Estimate (2.2) is well known (see e.g. [DN]). In the case when $n$ is odd, it follows from the potential representation of $u \phi$ by the fundamental solution homogeneous of degree $2\ell m - n$ for the elliptic operator $\det(L^j_k(D))_{m \times m}$. If $n$ is even and $2\ell m \geq n$, the fundamental solution contains the logarithmic function $\ln |x|$ (see e.g. [H], p.169). However in this case, one may reduce the problem to the odd dimensional case by adding an independent variable (the method of descending).

Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function such that $\psi(0) = 0$. For $r > 0$, define

\begin{equation}
I(r) = \{ (x', \psi(x')) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_1| < r, \ldots, |x_{n-1}| < r \},
\end{equation}

\begin{equation}
Z(r) = \{ (x', x_n) : |x_1| < r, \ldots, |x_{n-1}| < r, \psi(x') < x_n < C_0 r \},
\end{equation}

where $C_0 = 1 + 10\sqrt{n}\|\nabla \psi\|_\infty > 0$ is chosen so that $Z(r)$ is a star-shaped Lipschitz domain with Lipschitz constant independent of $r$.

Lemma 2.5. (Poincaré inequality) Let $1 \leq p < \infty$. Suppose that $u \in W^{1,p}(Z(r))$ and $u = 0$ on $I(r)$. Then

\begin{equation}
\int_{Z(r)} |u|^p dx \leq C r^p \int_{Z(r)} |\nabla u|^p dx,
\end{equation}

where $C$ depends only on $\|\nabla \psi\|_\infty$, $p$ and $n$.

Proof. The case $p = 1$ follows easily from the fundamental theorem of calculus. For $p > 1$, since $|\nabla |u|^p| \leq p|u|^{p-1}|\nabla u|$, one applies inequality (2.6) with $p = 1$ to the function $|u|^p$ and then use the Hölder’s inequality.
Lemma 2.7. (Sobolev inequality) Let $1 \leq p < n$ and $q = pn/(n-p)$. Suppose $u \in W^{1,p}(Z(r))$ and $u = 0$ on $I(r)$. Then
\begin{equation}
(\int_{Z(r)} |u|^q \, dx)^{1/q} \leq C \left( \int_{Z(r)} |\nabla u|^p \, dx \right)^{1/p},
\end{equation}
where $C$ depends only on $\|\nabla \psi\|_{\infty}$, $p$ and $n$.

Proof. Estimate (2.8) follows from (2.6) and the well known Sobolev inequality
\[ \|u\|_{L^q(Z(r))} \leq C \left\{ \|\nabla u\|_{L^p(Z(r))} + \frac{1}{r} \|u\|_{L^p(Z(r))} \right\}. \]

Lemma 2.9. (Cacciopoli's inequality) Let $\Omega = Z(3r)$. Suppose $L(D)u = 0$ in $\Omega$ and $\nabla^\ell u \in L^2(\partial \Omega)$. Also assume that $D^\alpha u = 0$ on $I(3r)$ for $|\alpha| \leq \ell - 1$. Then
\begin{equation}
\int_{Z(r)} |\nabla^\ell u|^2 \, dx \leq \frac{C}{r^2} \int_{Z(2r)} |\nabla^{\ell-1} u|^2 \, dx,
\end{equation}
where $C$ depends only on $\|\nabla \psi\|_{\infty}$, the ellipticity constant $\mu$ as well as $n$, $m$ and $\ell$.

Proof. Let $\varphi$ be a smooth cut-off function on $\mathbb{R}^n$ such that $\varphi = 1$ on $Z(r)$, $\varphi = 0$ on $Z(3r) \setminus Z(2r)$ and $|D^\alpha \varphi| \leq C/r^{2|\alpha|}$ for $|\alpha| \leq 2\ell$. To prove (2.10), we use the test function $u\varphi^2$ and proceed as in the proof of the Cacciopoli's inequality for second order elliptic systems ([Gr], pp.76-77). This gives
\[ \int_{Z(r)} |\nabla^\ell u|^2 \, dx \leq C \sum_{|\alpha| \leq \ell - 1} \frac{1}{r^{2\ell - 2|\alpha|}} \int_{Z(2r)} |D^\alpha u|^2 \, dx. \]

We remark that with the assumption $\nabla^\ell u \in L^2(\partial \Omega)$, the necessary integration by parts may be justified by approximating $\Omega$ from inside by a sequence of smooth domains [V1]. The desired estimate (2.10) now follows from Poincaré inequality (2.6).

Lemma 2.11. (Higher integrability) Under the same assumption as in Lemma 2.9, we have
\begin{equation}
\left( \frac{1}{r^n} \int_{Z(r)} |\nabla^\ell u|^q \, dx \right)^{1/q} \leq C \left( \frac{1}{r^n} \int_{Z(2r)} |\nabla^\ell u|^2 \, dx \right)^{1/2},
\end{equation}
where $q > 2$ depends only on $\|\nabla \psi\|_{\infty}$, $\mu$, $n$, $m$ and $\ell$.

Proof. It follows from Cacciopoli's inequality (2.10) and Sobolev inequality (2.8) that
\begin{equation}
\left( \frac{1}{r^n} \int_{Z(r)} |\nabla^\ell u|^2 \, dx \right)^{1/2} \leq C \left( \frac{1}{r^n} \int_{Z(2r)} |\nabla^\ell u|^{p_n} \, dx \right)^{1/p},
\end{equation}
where $p_n = 2n/(n+2)$. It is well known that boundary reverse Hölder inequality (2.13), together with the interior estimate (2.2), implies the inequality (2.12) (see [Gr], pp.122-123).

We end this section with a theorem concerning the condition (1.11). Recall that for $Q \in \partial \Omega$ and $r > 0$, $\Delta(Q,r) = B(Q,r) \cap \partial \Omega$ and $T(Q,r) = B(Q,r) \cap \Omega$. 
Theorem 2.14. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Fix $Q \in \partial \Omega$ and $R_0 > 0$ sufficiently small. Let $u$ be a solution of $\mathcal{L}(D)u = 0$ in $\Omega$ with the properties $(\nabla^\ell u)^* \in L^2(\Delta(Q, R_0))$ and $D^\alpha u = 0$ on $\Delta(Q, R_0)$ for $|\alpha| \leq \ell - 1$. Suppose that for some $\lambda_0 > 0$ and all $0 < r < R < R_0$, 

$$
(2.15) \quad \int_{T(Q, r)} |u|^2 \, dx \leq C \left( \frac{r}{R} \right)^{\lambda_0 + 2\ell - 2} \int_{T(Q, R)} |u|^2 \, dx.
$$

Then if $0 < \lambda < \lambda_0$, we have 

$$
(2.16) \quad \int_{T(Q, r)} |\nabla^{\ell-1} u|^2 \, dx \leq C_\lambda \left( \frac{r}{R} \right)^{\lambda} \int_{T(Q, R)} |\nabla^{\ell-1} u|^2 \, dx,
$$

for all $0 < r < R < R_0$.

Proof. By translation and rotation, it suffices to prove the theorem with $\Delta(Q, R_0)$, $T(Q, r)$ and $T(Q, R)$ replaced by $I(R_0)$, $Z(r)$ and $Z(R)$ respectively. We may also assume that $0 < r < R/2 < R_0/4$.

By the interpolation inequality ([A], p.79) and Poincaré inequality (2.6), we have 

$$
(2.17) \quad \|\nabla^{\ell-1} u\|_{L^2(Z(r))} \leq C \|\nabla^\ell u\|_{L^2(Z(2r))} \|u\|_{L^2(Z(r))}^{\frac{1}{\ell}} \leq C \|\nabla^{\ell-1} u\|_{L^2(Z(2r))} \cdot \left( \frac{r}{R} \right)^{\frac{\lambda_0 + 2\ell - 2}{\ell} - \frac{1}{\ell}} \|u\|_{L^2(Z(R))}^{\frac{1}{\ell}},
$$

where we also use Cacciopoli’s inequality (2.10) and assumption (2.15). It then follows from Poincaré inequality (2.6) and Hölder’s inequality that 

$$
(2.18) \quad \|\nabla^{\ell-1} u\|_{L^2(Z(r))} \leq C \|\nabla^{\ell-1} u\|_{L^2(Z(2r))} \cdot \left( \frac{r}{R} \right)^{\frac{\lambda_0}{\ell}} \|\nabla^{\ell-1} u\|_{L^2(Z(R))}^{\frac{1}{\ell}} \leq \varepsilon \|\nabla^{\ell-1} u\|_{L^2(Z(2r))} + C_\varepsilon \left( \frac{r}{R} \right)^{\frac{\lambda_0}{\ell}} \|\nabla^{\ell-1} u\|_{L^2(Z(R))},
$$

for any $\varepsilon > 0$ and any $0 < r < R/2$. By Lemma 8.23 in [GT] (p.201), estimate (2.18) implies 

$$
(2.19) \quad \int_{Z(r)} |\nabla^{\ell-1} u|^2 \, dx \leq C_\lambda \left( \frac{r}{R} \right)^{\lambda} \int_{Z(R)} |\nabla^{\ell-1} u|^2 \, dx
$$

for any $\lambda \in (0, \lambda_0)$ and for all $0 < r < R < R_0$. The proof is finished.

3. The Sufficiency of the Reverse Hölder Condition

The goal of this section is to show that for a given operator $\mathcal{L}(D)$ on a fixed Lipschitz domain $\Omega$, the reverse Hölder condition (1.8) with exponent $p$ for solutions with zero Dirichlet data on $\Delta(Q, 3r)$ is sufficient for the solvability of the $L^p$ Dirichlet problem on $\Omega$.

Recall that for a function $u$ defined on $\Omega$, its non-tangential maximal function $(u)^*$ is defined by 

$$
(3.1) \quad (u)^*(Q) = \sup \left\{ |u(x)| : x \in \Gamma(Q) \right\}, \quad \text{for } Q \in \partial \Omega,
$$

where $\Gamma(Q) = \left\{ x \in \Omega : |x - Q| < 2 \text{dist}(x, \partial \Omega) \right\}$.
Theorem 3.2. Let $\mathcal{L}(D)$ be an elliptic operator given by (1.1) and satisfying the symmetry condition (1.2) and ellipticity condition (1.3). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Fix $p > 2$. Suppose that for any $\Delta(Q,r) \subset \partial \Omega$, inequality (1.8) holds for all solutions of $\mathcal{L}(D)v = 0$ in $\Omega$ with the properties $(\nabla^{\ell-1}v)^* \in L^2(\partial \Omega)$ and $D^\alpha v = 0$ for $|\alpha| \leq \ell - 1$ on $\Delta(Q,3r)$. Then the $L^p$ Dirichlet problem (1.4) is uniquely solvable.

Since $p > 2$, the uniqueness in Theorem 3.2 follows from the uniqueness for the case $p = 2$ [V4]. For the existence as well as estimate (1.6), it suffices to establish the following lemma.

Lemma 3.3. Let $(\hat{f}, g) \in (W A_{\ell-1}^p(\partial \Omega), L^p(\partial \Omega))$. Let $u$ be the unique $L^2$ solution of (1.4) with boundary data $(\hat{f}, g)$, i.e., $u$ satisfies (1.4) and $(\nabla^{\ell-1}u)^* \in L^2(\partial \Omega)$. Then, under the same assumption as in Theorem 3.2, we have

\begin{equation}
\left\{ \frac{1}{r^{n-1}} \int_{\Delta(Q,r)} |(\nabla^{\ell-1}u)^*|^p d\sigma \right\}^{1/p} \leq C \left\{ \frac{1}{r^{n-1}} \int_{\Delta(Q,2r)} |(\nabla^{\ell-1}u)^*|^2 d\sigma \right\}^{1/2} + C \left\{ \frac{1}{r^{n-1}} \int_{\Delta(Q,2r)} \left( |g|^p + \sum_{|\alpha| = \ell - 2} |\nabla \ell f_\alpha|^p \right) d\sigma \right\}^{1/p},
\end{equation}

for any $\Delta(Q,r) \subset \partial \Omega$ with $Q \in \partial \Omega$ and $0 < r < r_0$.

The desired estimate (1.6) follows from (3.4) by covering $\partial \Omega$ with a finite number of surface balls $\Delta(Q,r)$.

The proof of Lemma 3.3 is essentially contained in [S5], where the cases of second order elliptic systems and the polyharmonic equation were considered. For completeness as well as reader’s convenience we will provide a detailed proof here.

We will need the following Poincaré type inequality on $\partial \Omega$.

Lemma 3.5. Suppose $\ell \geq 2$. Let $\hat{f} = \{ f_\alpha : |\alpha| \leq \ell - 2 \} \in W A_{\ell-1}^2(\partial \Omega)$ and $\Delta(Q,r) \subset \partial \Omega$. Then there exists a polynomial $h$ of degree at most $\ell - 2$ such that

\begin{equation}
\| f_\beta - D^\beta h \|_{L^2(\Delta(Q,r))} \leq C r^{\ell - |\beta| - 1} \sum_{|\alpha| = \ell - 2} \| \nabla \ell f_\alpha \|_{L^2(\Delta(Q,r))},
\end{equation}

for any multi-index $\beta$ with $|\beta| \leq \ell - 2$.

Proof. Let $h(x) = \sum_{|\alpha| \leq \ell - 2} C_\alpha x^\alpha$ be a polynomial where $C_\alpha$ is defined inductively by

\begin{align*}
C_\alpha &= \frac{1}{|\Delta(Q,r)|} \int_{\Delta(Q,r)} f_\alpha d\sigma \quad \text{if } |\alpha| = \ell - 2, \\
C_\alpha &= \frac{1}{|\Delta(Q,r)|} \int_{\Delta(Q,r)} \left\{ f_\alpha(P) - \sum_{\beta > \alpha : |\beta| \leq \ell - 2} \frac{C_\beta}{(\beta - \alpha)!} P^{\beta - \alpha} \right\} d\sigma(P), \quad \text{if } |\alpha| < \ell - 2.
\end{align*}
It is easy to check that
\[ (3.7) \quad \int_{\Delta(Q, r)} \{ f_{\alpha} - D^\alpha h \} \, d\sigma = 0 \quad \text{for all } |\alpha| \leq \ell - 2. \]

With (3.7), by using Poincaré inequality on $\Delta(Q, r)$ repeatedly, we obtain
\[
\| f_{\beta} - D^\beta h \|_{L^2(\Delta(Q, r))} \leq C r \sum_{|\alpha| = |\beta| + 1} \| f_{\alpha} - D^\alpha h \|_{L^2(\Delta(Q, r))}
\leq C r^{\ell - |\beta| - 2} \sum_{|\alpha| = \ell - 2} \| f_{\alpha} - D^\alpha h \|_{L^2(\Delta(Q, r))}
\leq C r^{\ell - |\beta| - 1} \sum_{|\alpha| = \ell - 2} \| \nabla_t f_{\alpha} \|_{L^2(\Delta(Q, r))},
\]
for any $|\beta| \leq \ell - 2$. The proof is finished.

To prove (3.4), we fix $\Delta(Q, r)$ with $Q \in \partial \Omega$ and $0 < r < r_0$. By rotation and translation, we may assume that $Q = 0$ and
\[ B(0, C_1 r_0) \cap \Omega = B(0, C_1 r_0) \cap \{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \psi(x') \}, \]
\[ B(0, C_1 r_0) \cap \partial \Omega = B(0, C_1 r_0) \cap \{ (x', \psi(x')) : x' \in \mathbb{R}^{n-1} \}, \]
where $\psi$ is a Lipschitz function on $\mathbb{R}^{n-1}$. Let $S = \{ (x', \psi(x')) : x' \in \mathbb{R}^{n-1} \}$. We will perform a Calderón-Zygmund decomposition on $S$ in the proof of (3.4). To do this, we need to introduce surface cubes on the set $S$.

Let $\Phi : S \to \mathbb{R}^{n-1}$ be a map defined by $\Phi(x', \psi(x')) = x'$. A subset $I$ of $S$ is said to be a cube on $S$ if $\Phi(I)$ is a cube in $\mathbb{R}^{n-1}$ with sides parallel to the coordinate planes. A cube $I$ on $S$ is said to be a dyadic subcube of $I'$ if $\Phi(I)$ is a dyadic subcube of $\Phi(I')$. Also for $\rho > 0$ and a cube $I$ on $S$, we will use $\rho I$ to denote $\Phi^{-1}(\rho \Phi(I))$.

For cube $I$ on $S$ and a function $f$ defined on $I$, we define a localized Hardy-Littlewood maximal function $M_I$ by
\[ (3.9) \quad M_I(f)(P) = \sup_{\text{cube } I' \subset I} \frac{1}{|I'|} \int_{I'} |f| \, d\sigma \quad \text{for } P \in I. \]

**Proof of Lemma 3.3.** The proof relies on a real variable argument which is motivated by the method used in [CP]. Let $I$ be a cube on $S$ such that $3I \subset S \cap B(0, 2r_0)$. For $\lambda > 0$, let
\[ (3.10) \quad E(\lambda) = \{ Q \in I : M_{2I}(|(\nabla^{\ell - 1} u)|^2)(Q) > \lambda \}. \]
Fix $2 < q < p$, let $A = 1/(2\delta^2/q)$ where $\delta > 0$ is a small constant to be determined. Let $F = |g|^2 + \sum_{|\alpha| = \ell - 2} |\nabla_t f_{\alpha}|^2$. We claim that there exist positive constants $\delta$, $\gamma$ and $C_0$
depending only on $n, m, \mu, \ell, \Omega$ as well as the constant $C$ in the reverse Hölder condition (1.8) such that

$$|E(A\lambda)| \leq \delta |E(\lambda)| + \{|Q \in \mathcal{I} : M_{2I}(F)(Q) > \gamma \lambda\}|$$

for all $\lambda \geq \lambda_0$, where

$$\lambda_0 = \frac{C_0}{|2I|} \int_{2I} |(\nabla^{\ell-1}u)^*|^2 \, d\sigma. \tag{3.12}$$

Assume the claim is true for a moment. We multiply both sides of (3.11) by $\lambda^{\frac{q}{2}-1}$ and integrate the resulting inequality in $\lambda$ over the interval $(\lambda_0, \Lambda)$. We obtain

$$\frac{1}{A^q} \int_{A\lambda_0}^A \lambda^{\frac{q}{2}-1} |E(\lambda)| \, d\lambda \leq \delta \int_{\lambda_0}^\Lambda \lambda^{\frac{q}{2}-1} |E(\lambda)| \, d\lambda + C \int_{2I} |F|^\frac{q}{2} \, d\sigma, \tag{3.13}$$

where we have used the boundedness of $M_{2I}$ on $L^{q/2}(2I)$. This implies that

$$\left(\frac{1}{A^{q/2}} - \delta\right) \int_0^\Lambda \lambda^{\frac{q}{2}-1} |E(\lambda)| \, d\lambda \leq C \int_0^{A\lambda_0} \lambda^{\frac{q}{2}-1} |E(\lambda)| \, d\lambda + C \int_{2I} |F|^\frac{q}{2} \, d\sigma$$

$$\leq C \lambda_0^{\frac{q}{2}} |I| + C \int_{2I} \left\{|g|^q + \sum_{|\alpha|=\ell-2} |\nabla_t f_\alpha|^q \right\} \, d\sigma.$$  

Observe that $\delta A^{q/2} \leq 1$. Let $\Lambda \to \infty$ in the above inequality, we obtain $(\nabla^{\ell-1}u)^* \in L^q(I)$ and

$$\left\{ \frac{1}{|I|} \int_I |(\nabla^{\ell-1}u)^*|^q \, d\sigma \right\}^{1/q} \leq C \left\{ \frac{1}{|2I|} \int_{2I} |(\nabla^{\ell-1}u)^*|^2 \, d\sigma \right\}^{1/2}$$

$$+ C \left\{ \frac{1}{|2I|} \int_{2I} \left( |g|^q + \sum_{|\alpha|=\ell-2} |\nabla_t f_\alpha|^q \right) \, d\sigma \right\}^{1/q}, \tag{3.14}$$

for any $q \in (2, p)$. Note that the reverse Hölder condition (1.8) is a self-improving property. That is, if $(\nabla^{\ell-1}u)^*$ satisfies condition (1.8) for some $p > 2$, then it satisfies condition (1.8) for some $\bar{p} > p$ ([Gi], pp.122-123). Thus (3.14) holds for $q = p$. Estimate (3.4) now follows from (3.14) by covering $\Delta(Q, r)$ with a finite number of sufficiently small surface cubes $I$.

It remains to prove the claim (3.11). To this end, we fix $\lambda \geq \lambda_0$. Note that $E(\lambda)$ is open relative to $I$. This implies that there exists a sequence of disjoint dyadic subcubes $\{I_j\}$ of $I$ such that $E(\lambda) = \bigcup I_j$ (up to a set of surface measure zero). We may assume that each $I_j$ is maximal in the sense that if $I' \supset I_j$ is a dyadic subcube of $I$, then $I' \not\subset E(\lambda)$ unless $I' = I_j$. Also, since

$$|E(\lambda)| \leq \frac{C}{\lambda} \int_{2I} |(\nabla^{\ell-1}u)^*|^2 \, d\sigma \leq \frac{C\lambda_0|I|}{C_0\lambda} \leq \frac{C|I|}{C_0}, \tag{3.15}$$
we may assume that $|E(\lambda)| \leq \delta |I|$ by taking $C_0$ sufficiently large. It follows that $|I_j| \leq \delta |I|$. In particular we may assume that $32I_j \subset 2I$.

We claim that it is possible to choose positive constants $\delta$, $\gamma$ and $C_0$ such that if

$$
\{ Q \in I_j : M_{2I}(F)(Q) \leq \gamma \lambda \} \neq \emptyset ,
$$

then $|E(A\lambda) \cap I_j| \leq \delta |I_j|$. Clearly, this yields estimate (3.11) by summation.

To prove the last claim, we fix $I_j$ which satisfies (3.16). Note that for any $Q \in I_j$,

$$
M_{2I}(|(\nabla^{\ell-1}u)^*|^2)(Q) \leq \max (M_{2I_j}(|(\nabla^{\ell-1}u)^*|^2)(Q), C_1 \lambda)
$$

where $C_1$ depends only on $n$ and $\|\nabla \psi\|_\infty$. This is because $I_j$ is maximal. We may assume that $A \geq C_1$ by taking $\delta$ small. It follows that

$$
|E(A\lambda) \cap I_j| = |\{ Q \in I_j : M_{2I_j}(|(\nabla^{\ell-1}u)^*|^2)(Q) > A\lambda \}|.
$$

Let $\varphi$ be a smooth cut-off function on $\mathbb{R}^n$ such that $\varphi = 1$ on $16I_j$, $\varphi = 0$ on $\partial \Omega \setminus 17I_j$, and $|D^\alpha \varphi| \leq C/\rho^{|\alpha|}$ for $|\alpha| \leq \ell - 1$, where $\rho = \rho_j$ is the diameter of $I_j$. Let $h$ be a polynomial of degree at most $\ell - 2$ given by Lemma 3.5, but with $\Delta(Q, r)$ replaced by $17I_j$. Let $w = w_j$ be the solution of the $L^2$ Dirichlet problem (1.4) with boundary data

$$
D^\alpha w = D^\alpha((u - h)\varphi) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (f_\beta - D^\beta h)D^{\alpha - \beta} \varphi
$$

for $|\alpha| \leq \ell - 2$ and $\frac{\partial^{\ell-1} w}{\partial N^{\ell-1}} = g\varphi$ on $\partial \Omega$. Note that

$$
\sum_{|\alpha| = \ell - 2} \int_{\partial \Omega} |\nabla_t D^\alpha w|^2 d\sigma
$$

$$
\leq C \left\{ \sum_{|\alpha| = \ell - 2} \int_{17I_j} |\nabla_t f_\alpha|^2 d\sigma + \sum_{|\beta| \leq \ell - 2} \frac{1}{\ell^{1+|\beta|}} \int_{17I_j} |f_\beta - D^\beta h|^2 d\sigma \right\}
$$

$$
\leq C \sum_{|\alpha| = \ell - 2} \int_{17I_j} |\nabla_t f_\alpha|^2 d\sigma,
$$

where we have used Lemma 3.5 in the last inequality. It follows from the $L^2$ estimates \cite{PV3,V4} and (3.16) that

$$
\int_{\partial \Omega} |(\nabla^{\ell-1}w)^*|^2 d\sigma \leq C \int_{17I_j} \left\{ |g|^2 + \sum_{|\alpha| = \ell - 2} |\nabla_t f_\alpha|^2 \right\} d\sigma
$$

$$
\leq C \gamma \lambda |I_j|.
$$

Now let $v = u - w - h$ in $\Omega$. Note that $v$ is a solution to the $L^2$ Dirichlet problem (1.4) with boundary data vanishing on $16I_j$. Indeed, $D^\alpha v = D^\alpha((u - h)(1 - \varphi))$ for $|\alpha| \leq \ell - 2$
and \( \frac{\partial^\ell v}{\partial \nu^\ell} = g(1 - \varphi) \) on \( \partial \Omega \). Hence we may apply the reverse Hölder condition (1.8) to \( v \) on \( 16I_j \). This gives

\[
\left( \frac{1}{|2I_j|} \int_{2I_j} |(\nabla^{\ell-1}v)^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{|4I_j|} \int_{4I_j} |(\nabla^{\ell-1}v)^*|^2 \, d\sigma \right)^{1/2}
\]

(3.21)

\[
\leq C \left( \frac{1}{|4I_j|} \int_{4I_j} |(\nabla^{\ell-1}w)^*|^2 \, d\sigma \right)^{1/2} + C \left( \frac{1}{|4I_j|} \int_{4I_j} |(\nabla^{\ell-1}w)^*|^2 \, d\sigma \right)^{1/2}
\]

\[
\leq C \lambda^{1/2},
\]

where we have used the fact \( \nabla^{\ell-1}u = \nabla^{\ell-1}w + \nabla^{\ell-1}v \) in \( \Omega \). In (3.21) we also use (3.20) as well as the observation 3I \( j \notin E(\lambda) \) for the last inequality. We should point out that the reverse Hölder condition on surface balls \( \Delta(Q,r) = B(Q,r) \cap \partial \Omega \) is equivalent to the reverse Hölder condition over surface cubes. This is because one may cover any surface cube by sufficiently small surface cubes with a finite overlap and vice versa.

Finally in view of (3.17), (3.20) and (3.21) we have

\[
|E(A\lambda) \cap I_j| = |\{ Q \in I_j : M_{2I_j}((\nabla^{\ell-1}w)^*)|^2)(Q) > A\lambda \}| \\
\leq |\{ Q \in I_j : M_{2I_j}((\nabla^{\ell-1}w)^*|^2)(Q) > \frac{A\lambda}{4} \}| \\
+ |\{ Q \in I_j : M_{2I_j}((\nabla^{\ell-1}v)^*|^2)(Q) > \frac{A\lambda}{4} \}|
\]

\[
\leq \frac{C}{A\lambda} \int_{2I_j} |(\nabla^{\ell-1}w)^*|^2 \, d\sigma + \frac{C}{(A\lambda)^{p/2}} \int_{2I_j} |(\nabla^{\ell-1}v)^*|^p \, d\sigma
\]

\[
\leq |I_j| \left\{ \frac{C\gamma}{A} + \frac{C}{Ap^{2/4}} \right\}
\]

\[
\leq \delta |I_j| \{ C\delta^{\frac{p-1}{4}} + C\gamma \delta^{\frac{p-1}{2}} \},
\]

where we have used \( A = 1/(2\delta^{2/4}) \) in the last inequality. Since \( q < p \), we may choose \( \delta > 0 \) so small that \( C\delta^{\frac{p-1}{4}} \leq 1/2 \). With \( \delta \) chosen, we then choose \( \gamma > 0 \) so small that \( C\gamma \delta^{\frac{p-1}{2}} \leq 1/2 \). This gives \( |E(A\lambda) \cap I_j| \leq \delta |I_j| \). The proof is now complete.

4. The Necessity of the Reverse Hölder Condition

In this section we will show that for a given elliptic operator \( \mathcal{L}(D) \) on a fixed Lipschitz domain \( \Omega \), the reverse Hölder condition (1.8) with exponent \( p \) is also necessary for the solvability of the \( L^p \) Dirichlet problem (1.4).

**Theorem 4.1.** Let \( \mathcal{L}(D) \) be an elliptic operator given by (1.1). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n, n \geq 4 \). Fix \( p > 2 \). Suppose that the \( L^p \) Dirichlet problem (1.4) on \( \Omega \) is uniquely solvable. Then the reverse Hölder inequality (1.8) holds for solutions of \( \mathcal{L}(D)(v) = 0 \) in \( \Omega \) with the properties \( (\nabla^{\ell-1}v)^* \in L^2(\partial \Omega) \) and \( D^\alpha v = 0 \) on \( \Delta(Q,3r) \) for \( |\alpha| \leq \ell - 1 \).
To prove Theorem 4.1, we will need a lemma on the traces of Riesz potentials. Let

\[ I_1(f)(x) = \int_{\Omega} \frac{f(y) \, dy}{|x-y|^{n-1}}. \]  

**Lemma 4.3.** Let \( 1 < q < n \) and \( p = q(n-1)/(n-q) \). Then

\[ \|I_1(f)\|_{L^p(\partial \Omega)} \leq C \|f\|_{L^q(\Omega)}. \]  

**Proof.** Since \( \Omega \) is a Lipschitz domain, there exists a smooth vector field \( \mathbf{V}(x) \) on \( \mathbb{R}^n \) such that \( \mathbf{V} \cdot N \geq c_0 > 0 \) on \( \partial \Omega \) [V1]. It follows from the divergence theorem that

\[ c_0 \int_{\partial \Omega} |I_1(f)|^p \, d\sigma \leq \int_{\partial \Omega} \mathbf{V} \cdot N |I_1(f)|^p \, d\sigma \]
\[ \leq C \int_{\Omega} |I_1(f)|^p \, dx + C \int_{\Omega} |I_1(f)|^{p-1} |\nabla I_1(f)| \, dx \]
\[ \leq C \int_{\Omega} |I_1(f)|^p \, dx + C \left( \int_{\Omega} |\nabla I_1(f)|^q \, dx \right)^{1/q} \left( \int_{\Omega} |I_1(f)|^{(n-1)q'} \, dx \right)^{1/q'}. \]

Observe that \( \frac{1}{(p-1)q'} = \frac{1}{q} - \frac{1}{n} \). It follows that the last term on the right side of (4.5) is bounded by \( C \|f\|_{L^q(\Omega)}^p \). To see this, one uses the well known estimates for fractional integrals and singular integrals [St]. It is clear that the first term on the right side of (4.5) is also bounded by \( C \|f\|_{L^q(\Omega)}^p \). The proof is complete.

**Proof of Theorem 4.1.** Fix \( Q_0 \in \partial \Omega \) and \( 0 < r < r_0 \). Let \( v \) be a solution of \( \mathcal{L}(D)v = 0 \) in \( \Omega \) with the properties \( (\nabla^{\ell-1}v)^* \in L^2(\partial \Omega) \) and \( D^\alpha v = 0 \) on \( \Delta(Q_0,3r) \) for \( |\alpha| \leq \ell - 1 \). For a function \( u \) on \( \Omega \), define

\[ \mathcal{M}_1(u)(Q) = \sup \{ |u(x)| : x \in \Gamma(Q) \text{ and } |x-Q| < c_0 r \}, \]
\[ \mathcal{M}_2(u)(Q) = \sup \{ |u(x)| : x \in \Gamma(Q) \text{ and } |x-Q| \geq c_0 r \} \]

for \( Q \in \partial \Omega \). Clearly, \( (\nabla^{\ell-1}v)^* = \max \{ \mathcal{M}_1(\nabla^{\ell-1}v), \mathcal{M}_2(\nabla^{\ell-1}v) \} \). Note that if \( x \in \Gamma(Q) \) for some \( Q \in \Delta(Q_0,r) \) and \( |x-Q| \geq c_0 r \), by interior estimate (2.2), we have

\[ |\nabla^{\ell-1}v(x)| \leq \frac{C}{r^\ell} \int_{B(x,cr)} |\nabla^{\ell-1}v(y)| \, dy \leq \frac{C}{r^{n-1}} \int_{\Delta(Q_0,2r)} |(\nabla^{\ell-1}v)^*| \, d\sigma. \]

It follows that for any \( p > 2 \),

\[ \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,r)} |\mathcal{M}_2(\nabla^{\ell-1}v)|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,2r)} |(\nabla^{\ell-1}v)^*|^2 \, d\sigma \right)^{1/2}. \]
The estimate of $\mathcal{M}_1(\nabla^{\ell-1}v)$ on $\Delta(Q_0, r)$ is much more involved. First, we choose a smooth cut-off function $\varphi$ on $\mathbb{R}^n$ such that $\varphi = 1$ on $B(Q_0, 2r)$, supp$\varphi \subset B(Q_0, 3r)$, and $|D^\alpha \varphi| \leq C/r^{1|\alpha|}$ for $|\alpha| \leq 2\ell$. Note that

\[
[\mathcal{L}(D)(v\varphi)]^j = \sum_{k=1}^{m} \sum_{|\alpha| = |\beta| = \ell} a_{ij}^{jk} D^\alpha(v^k \varphi)
\]

\[
= \sum_{k=1}^{m} \sum_{|\alpha| = |\beta| = \ell} a_{ij}^{jk} D^\alpha \left\{ D^\beta v^k \cdot \varphi + \sum_{\gamma < \beta} \frac{\beta!}{\gamma!(\beta - \gamma)!} D^\gamma v^k \cdot D^{\beta - \gamma} \varphi \right\}
\]

\[
= \sum_{k=1}^{m} \sum_{|\alpha| = |\beta| = \ell} \sum_{\gamma < \alpha} a_{ij}^{jk} \frac{\alpha!}{\gamma!(\alpha - \gamma)!} D^\beta v^k \cdot D^{\gamma - \alpha} \varphi
\]

\[
+ \sum_{k=1}^{m} \sum_{|\alpha| = |\beta| = \ell} \sum_{\gamma < \beta} a_{ij}^{jk} \frac{\beta!}{\gamma!(\beta - \gamma)!} D^\alpha (D^\gamma v^k \cdot D^{\beta - \gamma} \varphi),
\]

where we have used $\mathcal{L}(D)(v) = 0$ in $\Omega$. Let $G(x) = (G_{ij}(x))_{m \times m}$ denote a matrix of fundamental solutions on $\mathbb{R}^n$ to the operator $\mathcal{L}(D)$ with pole at the origin. We remark that if $n$ is odd or $2\ell < n$, $G_{ij}(x)$ is homogeneous of degree $2\ell - n$ and smooth away from the origin. However, if $n$ is even and $2\ell \geq n$, the logarithmic function $\ln |x|$ appears in $G(x)$. Indeed in this case, we have $G_{ij}(x) = G_{ij}^{(1)}(x) + G_{ij}^{(2)}(x) \ln |x|$ where $G_{ij}^{(1)}(x)$ is homogeneous of degree $2\ell - n$ and $G_{ij}^{(2)}(x)$ is a polynomial of degree $2\ell - n$ (see [F], p.76 or [H], p.169). To deal with the factor $\ln |x|$, we need to replace $\ln |x|$ by $\ln(|x|/r)$. This can be done because $G_{ij}^{(2)}(x)$ is a polynomial of degree $2\ell - n$.

Note that in both cases, we have

\[
|D^\alpha G(x)| \leq \frac{C_{\alpha}}{|x|^{n-2\ell+|\alpha|}} \quad \text{for } |\alpha| \geq 2\ell - n + 1,
\]

as the derivatives $D^\alpha$ eliminate the (possible) logarithmic singularity if $|\alpha| > 2\ell - n$.

Next we fix $y_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ so that $|y_0 - Q_0| = r = \text{dist}(y_0, \partial \Omega)$. Let $\tilde{G}(x, y) = G(x - y)$ and

\[
F_{ij}(x, y) = \tilde{G}_{ij}(x, y) - \sum_{|\gamma| \leq 2\ell - 1} \frac{(y - y_0)^\gamma}{\gamma!} D^\gamma y \tilde{G}_{ij}(x, y_0).
\]

Note that the summation term in (4.11) is a solution to $\mathcal{L}(D)u = 0$ in $\Omega$ in both $x$ and $y$ variables. It is added to $\tilde{G}(x, y)$ in order to create the desired decay when $|x - Q_0| \geq 5r$ and $|y - Q_0| \leq 3r$. Indeed, by the Taylor remainder theorem and (4.10), if $x \in \Omega \setminus T(Q_0, 5r)$ and $y \in T(Q_0, 3r)$, we have

\[
|\nabla_x^{\ell-1} F_{ij}(x, y)| \leq \frac{C r^{2\ell-|\alpha|}}{|x - y|^{n+\ell-1}} \quad \text{for } |\alpha| \leq 2\ell.
\]
Also, if \( x \in T(Q_0, 5r) \) and \( y \in T(Q_0, 3r) \),

\[
(4.13) \quad |\nabla_x^{\ell-1} D_y^\alpha F_{ij}(x, y)| \leq \frac{C r^{\ell-|\alpha|}}{|x - y|^{n-1}} \quad \text{for} \ |\alpha| \leq \ell.
\]

To see (4.13), one considers two cases: \( |\alpha| > \ell - n + 1 \) and \( |\alpha| \leq \ell - n + 1 \). In the first case, one uses estimate (4.10). For the second case, the (possible) term involving the logarithmic function in \( \nabla_x^{\ell-1} D_y^\alpha \tilde{G}_{ij}(x, y) \) is bounded by \( C |x - y|^{\ell-n-|\alpha|+1} \ln \left| \frac{|x-y|}{r} \right| \). Since \( |x - y| \leq C r \), it is clearly bounded by the right side of (4.13).

In view of (4.9), we let \( w(x) = (w^1(x), \ldots, w^m(x)) \) where

\[
w^i(x) = \sum_{j,k=1}^{m} \sum_{|\alpha|=|\beta|} (-1)^{\ell} a_{\alpha \beta}^j \frac{\alpha!}{\gamma!(\alpha - \gamma)!} \int_{\Omega} D_y^\beta \{ F_{ij}(x, y) D^\alpha \gamma \phi(y) \} D^\gamma v^k(y) \, dy
\]

\[+ \sum_{j,k=1}^{m} \sum_{|\alpha|=|\beta|} (-1)^{\ell} a_{\alpha \beta}^j \frac{\beta!}{\gamma!(\beta - \gamma)!} \int_{\Omega} D_y^\alpha F_{ij}(x, y) D^\gamma v^k(y) \cdot D^{\beta-\gamma} \phi(y) \, dy.\]

Then \( \mathcal{L}(D)(w) = \mathcal{L}(D)(v \phi) \) in \( \Omega \). To see this, one may fix \( B(x_0, 3s) \subset \Omega \) and write \( w = w_1 + w_2 \), where \( w_1 \) and \( w_2 \) are defined in the same way as \( w \) but with domain \( \Omega \) of both integrals replaced by \( B(x_0, 2s) \) and \( \Omega \setminus B(x_0, 2s) \) respectively. Clearly \( \mathcal{L}(D)(w_2) = 0 \) in \( B(x_0, s) \). To show \( \mathcal{L}(D)(w_1) = \mathcal{L}(D)(v \phi) \) in \( B(x_0, s) \), one uses integration by parts and (4.9).

To continue, we observe that on \( \Delta(Q_0, r) \),

\[
(4.14) \quad \mathcal{M}_1(\nabla^{\ell-1} v) = \mathcal{M}_1(\nabla^{\ell-1}(v \phi)) \leq \mathcal{M}_1(\nabla^{\ell-1} w) + \mathcal{M}_1(\nabla^{\ell-1}(v \phi - w)).
\]

It follows from (4.13) that for \( x \in T(Q_0, 5r) \),

\[
(4.15) \quad |\nabla^{\ell-1} w(x)| \leq C \sum_{|\gamma| \leq \ell-1} \int_{T(Q_0, 3r) \setminus T(Q_0, 2r)} \frac{|D^\gamma v(y)|}{|x - y|^{n-1}} \, dy.
\]

This implies that if \( Q \in \Delta(Q_0, r) \) and the constant \( c_0 \) in (4.6) is sufficiently small,

\[
\mathcal{M}_1(\nabla^{\ell-1} w)(Q) \leq C \sum_{|\gamma| \leq \ell-1} r^{\gamma - \ell} \int_{T(Q_0, 3r) \setminus T(Q_0, 2r)} \, dy
\]

\[\leq C \sum_{\ell-1}^{\ell+1} \left( \frac{1}{r^{n-1}} \int_{T(Q_0, 3r)} |D^\gamma v(y)|^2 \, dy \right)^{1/2}
\]

\[
(4.16) \quad \leq C \left( \frac{1}{r^n} \int_{T(Q_0, 3r)} |\nabla^{\ell-1} v|^2 \, dy \right)^{1/2}.
\]
where we have used Poincaré inequality (2.6) in the last step (the proof for Poincaré inequality on \( T(Q, r) \) is the same). Clearly, this gives

\[
\left( \frac{1}{r^{n-1}} \int_{\Delta(Q, r)} |\mathcal{M}_1(\nabla^{\ell-1}w)|^p d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q, 3r)} |(\nabla^{\ell-1}v)^*|^2 d\sigma \right)^{1/2}.
\]

(4.17)

It remains to estimate \( \mathcal{M}_1(\nabla^{\ell-1}(v\varphi - w)) \). It is here that we need to use the assumption that the \( L^p \) Dirichlet problem (1.4) on \( \Omega \) is uniquely solvable.

Note that \( \mathcal{L}(D)(v\varphi - w) = 0 \) in \( \Omega \). We also have \( (\nabla^{\ell-1}(v\varphi - w))^* \in L^2(\partial\Omega) \). To see this, by the square function estimates (see (5.1)-(5.2)), it suffices to show \( \delta(x)^{1/2}\nabla^{\ell}(v\varphi - w) \in L^2(\Omega) \), where \( \delta(x) = \text{dist}(x, \partial\Omega) \). But this is clear since \( \delta(x)^{1/2}\nabla^{\ell}(v\varphi) \in L^2(\Omega) \) by the square function estimates as well as the assumption \( (\nabla^{\ell-1}v)^* \in L^2(\partial\Omega) \), and \( \nabla^{\ell}w \in L^2(\Omega) \) by singular integral estimates [St] and \( \nabla^{\ell-1}v \in L^2(\Omega) \). Thus, by the \( L^2 \) uniqueness and estimate (1.6),

\[
\int_{\Delta(Q, r)} |\mathcal{M}_1(\nabla^{\ell-1}(v\varphi - w))|^p d\sigma \leq \int_{\partial\Omega} |(\nabla^{\ell-1}(v\varphi - w))^*|^p d\sigma
\]

(4.18)

\[
\leq C \int_{\partial\Omega} |\nabla^{\ell-1}(v\varphi - w)|^p d\sigma = C \int_{\partial\Omega} |\nabla^{\ell-1}w|^p d\sigma
\]

where we also used the fact \( \nabla^{\ell-1}(v\varphi) = 0 \) on \( \partial\Omega \).

Let \( p = q(n-1)/(n-q) \). Note that \( \frac{n-1}{p} = \frac{n}{q} - 1 \). It follows from (4.15) and Lemma 4.3 that

\[
\left( \frac{1}{r^{n-1}} \int_{\Delta(Q, 5r)} |\nabla^{\ell-1}w|^p d\sigma \right)^{1/p} \leq C r \left( \frac{1}{r^n} \int_{T(Q, 3r)} \left| \sum_{|\gamma| \leq \ell-1} r^{|\gamma|-\ell} |D^\gamma v|^q \right| dx \right)^{1/q}
\]

(4.19)

Clearly in (4.19) we may replace \( q \) by \( \tilde{q} = \max(q, 2) \). By Poincaré inequality (2.6), this gives

\[
\left( \frac{1}{r^{n-1}} \int_{\Delta(Q, 5r)} |\nabla^{\ell-1}w|^p d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^n} \int_{T(Q, 3r)} |(\nabla^{\ell-1}v)^\tilde{q}| dx \right)^{1/\tilde{q}}
\]

(4.20)

\[
\leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q, 3r)} |(\nabla^{\ell-1}v)^*|^\tilde{q} d\sigma \right)^{1/\tilde{q}}.
\]

Finally, if \( Q \in \partial\Omega \setminus \Delta(Q, 5r) \), we use estimate (4.12) to obtain

\[
|\nabla^{\ell-1}w(Q)| \leq \frac{C}{|Q - Q_0|^{n+\ell-1}} \int_{T(Q_0, 3r)} \sum_{|\gamma| \leq \ell-1} r^{|\gamma|} |D^\gamma v(y)| dy
\]

(4.21)

\[
\leq \frac{C r^{n+\ell-1}}{|Q - Q_0|^{n+\ell-1}} \left( \frac{1}{r^{n-1}} \int_{\Delta(Q, 3r)} |(\nabla^{\ell-1}v)^*|^2 d\sigma \right)^{1/2},
\]
as in (4.16). It follows that

\[(4.22) \left( \frac{1}{r^{n-1}} \int_{\partial \Omega \setminus \Delta(Q_0,5r)} |\nabla^{\ell-1} w|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,3r)} |(\nabla^{\ell-1} v)^*|^2 \, d\sigma \right)^{1/2}. \]

In view of (4.8), (4.14), (4.17), (4.20) and (4.22), we have proved that

\[(4.23) \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,r)} |(\nabla^{\ell-1} v)^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,3r)} |(\nabla^{\ell-1} v)^*|^\bar{q} \, d\sigma \right)^{1/\bar{q}}, \]

where \(\bar{q} = \max(q,2)\) and \(\frac{n-1}{p} = \frac{n}{q} - 1\). Observe that for \(p \geq 2\),

\[\frac{1}{q} - \frac{1}{p} = \frac{1}{n} (1 - \frac{1}{p}) \geq \frac{1}{2n}.\]

With this, one may iterate estimate (4.23) to obtain

\[(4.24) \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,cr)} |(\nabla^{\ell-1} v)^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(Q_0,r)} |(\nabla^{\ell-1} v)^*|^2 \, d\sigma \right)^{1/2},\]

starting with \(q = 2\). This is possible since the \(L^p\) solvability of the Dirichlet problem (1.4) implies the \(L^s\) solvability for any \(2 < s < p\).

By covering \(\Delta(Q,r)\) with sufficiently small surface balls \(\{\Delta(Q_j,cr)\}\), it is easy to see that estimate (4.24) is equivalent to the Hölder condition (1.8). The proof is finished.

**Remark 4.25.** The proof of Theorem 4.1 would be much simpler if one assumes that for the given \(p\), the \(L^p\) Dirichlet problem is uniquely solvable for all Lipschitz domains. In this case, one may use the localization techniques in [DK1] and apply the \(L^p\) estimate (1.6) on the domain \(\{(x',x_n) : |x'| \leq \rho r \text{ and } \psi(x') < x_n < \psi(x') + \rho r\} \) for \(\rho \in (1,2)\).

From Theorems 3.2 and 4.1 as well as the self-improving property of the reverse Hölder condition (1.8), we may deduce the following.

**Corollary 4.26.** Let \(L(D)\) be an elliptic operator given by (1.1). Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^n\), \(n \geq 4\). Then the set of exponents \(p \in (2,\infty)\) for which the \(L^p\) Dirichlet problem (1.4) on \(\Omega\) is uniquely solvable is an open interval \((2,q)\) with \(2 < q \leq \infty\).

## 5. The Proof of Theorem 1.10

In view of Theorem 3.2, to prove Theorem 1.10, it suffices to show that condition (1.11) implies the reverse Hölder condition (1.8) for \(p\) in the range given by (1.12). To do this, we will use the regularity estimate (1.13). The proof also depends on the following square function estimates established in [DKPV] for solutions of \(L(D)u = 0\) in \(\Omega\),

\[(5.1) \|S(\nabla^{\ell-1} u)\|_{L^p(\partial \Omega)} \leq C \|(\nabla^{\ell-1} u)^*\|_{L^p(\partial \Omega)},\]

\[(5.2) \|(\nabla^{\ell-1} u)^*\|_{L^p(\partial \Omega)} \leq C \|S(\nabla^{\ell-1} u)\|_{L^p(\partial \Omega)} + C |\nabla^{\ell-1} u(P_0)| |\partial \Omega|^{1/p},\]
where $0 < p < \infty$, $P_0 \in \Omega$ and $C$ depends on $n, m, \ell, \mu, P_0$ and the Lipschitz character of $\Omega$.

We first recall that the square function $S(w)$ is defined by

\begin{equation}
S(w)(Q) = \left\{ \int_{\Gamma(Q)} \frac{|\nabla w(x)|^2}{|x - Q|^{n-2}} \, dx \right\}^{1/2} \quad \text{for } Q \in \partial \Omega.
\end{equation}

(5.3)

Let

\begin{equation}
\tilde{S}(w)(Q) = \left\{ \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^2}{|x - Q|^{n-4}} \, dx \right\}^{1/2} \quad \text{for } Q \in \partial \Omega.
\end{equation}

(5.4)

It follows from Lemma 2 on p.216 of [St] that $\tilde{S}(w)(Q)$ is bounded by $CS(w)(Q)$ plus an interior term. Thus, by (5.2),

\begin{equation}
\| (\nabla^{\ell-1} u)^* \|_{L^p(\partial \Omega)} \leq C \| \tilde{S}(\nabla^{\ell-1} u) \|_{L^p(\partial \Omega)} + C \| (\nabla^{\ell-1} u)^* \|_{L^2(\partial \Omega)} \frac{|\partial \Omega|}{p - \frac{2}{p}}.
\end{equation}

(5.5)

**Lemma 5.6.** Let $p > 2$. Then for any $\gamma \in (0, 1)$ and $w \in C^2(\Omega)$, we have

\begin{equation}
\int_{\partial \Omega} |\tilde{S}(w)|^p \, d\sigma \leq C \gamma \{ \text{diam}(\Omega) \}^\gamma \int_{\Omega} |\nabla^2 w(x)|^p \left[ \delta(x) \right]^{2p-1-\gamma} \, dx,
\end{equation}

where $\delta(x) = \text{dist}(x, \partial \Omega)$.

**Proof.** Write

\begin{equation}
\tilde{S}(w)(Q) = \left\{ \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^2}{|x - Q|^{2(n+\gamma) - 4}} \cdot \frac{dx}{|x - Q|^{(p-2)n-2\gamma}} \right\}^{1/2}.
\end{equation}

(5.8)

Using Hölder's inequality with exponents $p/2$ and $(p/2)' = \frac{p}{p-2}$ in (5.8), we obtain

\begin{equation}
\tilde{S}(w)(Q) \leq C \{ \text{diam}(\Omega) \}^{\frac{\gamma}{p}} \left\{ \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^p}{|x - Q|^{n+\gamma-2p}} \, dx \right\}^{1/p}.
\end{equation}

(5.9)

From this, inequality (5.7) follows easily by integration.

**Lemma 5.10.** Let $p > 2$. Suppose $\mathcal{L}(D)u = 0$ in $\Omega$. Then for any $\gamma \in (0, 1)$,

\begin{equation}
\int_{\partial \Omega} (|\nabla^{\ell-1} u|)^p \, d\sigma \leq C \left\{ \int_{\partial \Omega} (|\nabla^{\ell-1} u|^2)^{\frac{p}{2}} \, d\sigma \right\}^{p/2} |\partial \Omega|^{1 - \frac{p}{2}}
\end{equation}

\begin{equation}
+ C \gamma \{ \text{diam}(\Omega) \}^\gamma \sup_{x \in \Omega} |\nabla^{\ell+1} u(x)|^{p-2} \left[ \delta(x) \right]^{2p-2-\gamma} \int_{\partial \Omega} (|\nabla^{\ell} u|)^2 \, d\sigma.
\end{equation}
Proof. It follows from (5.7) that

\[
\int_{\partial \Omega} |S(\nabla^{\ell-1} u)|^p \, d\sigma \leq C_\gamma \{\text{diam}(\Omega)\}^\gamma \int_{\Omega} |\nabla^{\ell+1} u|^p [\delta(x)]^{2p-1-\gamma} \, dx
\]

\[
\leq C_\gamma \{\text{diam}(\Omega)\}^\gamma \sup_{x \in \Omega} |\nabla^{\ell+1} u(x)|^{p-2} [\delta(x)]^{2p-2-\gamma} \int_{\Omega} |\nabla^{\ell+1} u|^2 \, \delta(x) \, dx
\]

\[
\leq C_\gamma \{\text{diam}(\Omega)\}^\gamma \sup_{x \in \Omega} |\nabla^{\ell+1} u(x)|^{p-2} [\delta(x)]^{2p-2-\gamma} \int_{\partial \Omega} |(\nabla^{\ell} u)^*|^2 \, d\sigma,
\]

where we have used square function estimate (5.1) (with \(p = 2\)) in the last inequality. This, together with (5.5), gives the desired estimate in Lemma 5.10.

We are now ready to give the proof of Theorem 1.10.

**Proof of Theorem 1.10.** We begin by fixing \(\Delta(Q_0, r)\) with \(Q_0 \in \partial \Omega\) and \(0 < r < r_0\). Let \(v\) be a solution of \(L(D)v = 0\) in \(\Omega\) with the properties \((\nabla^{\ell-1} v)^* \in L^2(\partial \Omega)\) and \(D^\alpha v = 0\) on \(\Delta(Q_0, 3r)\) for \(|\alpha| \leq \ell - 1\). Note that by condition (1.11) and interior estimate (2.2), for any \(x \in T(Q_0, r)\),

\[
[\delta(x)]^2 |\nabla^{\ell+1} v(x)| \leq C \left( \frac{\delta(x)}{r} \right)^{\frac{\lambda-n}{2}} \left( \frac{1}{r} \int_{T(Q_0, 2r)} |\nabla^{\ell-1} v|^2 \, dx \right)^{1/2}.
\]

It then follows that for any \(x \in T(Q_0, r)\),

\[
(5.11) \quad |\nabla^{\ell+1} v(x)| \leq C \left( \frac{\delta(x)}{\delta(x)} \right)^{\frac{\lambda-n}{2}} \left( \frac{1}{r} \int_{\Delta(0, 2r)} |(\nabla^{\ell-1} v)^*|^2 \, d\sigma \right)^{1/2}.
\]

By rotation and translation, we may assume that \(Q_0 = 0\) and \(r_0 = r_0(n, \Omega) > 0\) is so small that

\[
B(0, C_0 r_0) \cap \Omega = B(0, C_0 r_0) \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \psi(x')\},
\]

\[
B(0, C_0 r_0) \cap \partial \Omega = B(0, C_0 r_0) \cap \{(x', \psi(x')) : x' \in \mathbb{R}^{n-1}\}
\]

where \(\psi\) is a Lipschitz function on \(\mathbb{R}^{n-1}\). For \(\rho \in (1, 4)\), with slightly abused notation, we let

\[
I_{\rho r} = \{(x', \psi(x')) : |x'| < \rho c_2 r\},
\]

\[
Z_{\rho r} = \{(x', x_n) : |x'| < \rho c_2 r \quad \text{and} \quad \psi(x') < x_n < \psi(x') + \rho c_2 r\},
\]

where \(c_2 = c_2(n, \|\nabla \psi\|_\infty) > 0\) is small so that \(I_{3r} \subset \Delta(0, r)\) and \(Z_{3r} \subset B(0, r) \cap \Omega\). Let \(\mathcal{M}_1\) and \(\mathcal{M}_2\) be the operators defined by (4.6). As in (4.7) and (4.8), it is easy to see that

\[
(5.13) \quad \left( \frac{1}{r^{n-1}} \int_{I_r} |\mathcal{M}_2(\nabla^{\ell-1} v)|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{n-1}} \int_{\Delta(0, 2r)} |(\nabla^{\ell-1} v)^*|^2 \, d\sigma \right)^{1/2},
\]
by interior estimates.

To estimate $M_1(\nabla^\ell v)$ on $I_r$, we apply Lemma 5.10 to solution $v$ on the Lipschitz domain $Z_{\rho r}$ for $\rho \in (3/2, 2)$. This gives

$$\frac{1}{r^{n-1}} \int_{I_r} |M_1(\nabla^\ell v)|^p \, d\sigma \leq \frac{1}{r^{n-1}} \int_{\partial Z_{\rho r}} |(\nabla^\ell v)_\rho|^p \, d\sigma$$

(5.14)

$$\leq C \left( \frac{1}{r^{n-1}} \int_{\partial Z_{\rho r}} |(\nabla^\ell v)_\rho|^2 \, d\sigma \right)^{p/2}$$

$$+ C_\gamma r^\gamma \sup_{x \in Z_{\rho r}} |\nabla^{\ell+1} v(x)|^{p-2} \left[ \delta_\rho(x) \right]^{2p-2-\gamma} \cdot \frac{1}{r^{n-1}} \int_{\partial Z_{\rho r}} |(\nabla^\ell v)_\rho|^2 \, d\sigma,$$

where $\delta_\rho(x) = \text{dist}(x, \partial Z_{\rho r})$ and $(\nabla^\ell v)_\rho$ denotes the non-tangential maximal function of $\nabla^\ell v$ with respect to the domain $Z_{\rho r}$. By regularity estimate (1.13),

(5.15)

$$\int_{\partial Z_{\rho r}} |(\nabla^\ell v)_\rho|^2 \, d\sigma \leq C \int_{\partial Z_{\rho r}} |\nabla^\ell v|^2 \, d\sigma \leq C \int_{\Omega \cap \partial Z_{\rho r}} |\nabla^\ell v|^2 \, d\sigma,$$

since $\nabla^\ell v = 0$ on $\Delta(0, 3r)$.

Note that $\delta_\rho(x) \leq \delta(x) \leq C r$ for $x \in Z_{\rho r}$. Also observe that the condition (1.12) for $p$ is equivalent to

$$\frac{\lambda - n}{2} \cdot (p - 2) + 2 > 0.$$  

This implies that

$$\frac{\lambda - n}{2} \cdot (p - 2) + 2 - \gamma > 0.$$  

By (5.11), this implies that

$$r^\gamma \sup_{x \in Z_{\rho r}} |\nabla^{\ell+1} v(x)|^{p-2} \left[ \delta_\rho(x) \right]^{2p-2-\gamma}$$

(5.16)

$$\leq C r^2 \left( \frac{1}{r^{n-1}} \int_{\Delta(0, 2r)} |(\nabla^\ell v)|^2 \, d\sigma \right)^{\frac{p-2}{2}}.$$  

In view of (5.14), (5.15) and (5.16), we have proved that

$$\left( \frac{1}{r^{n-1}} \int_{I_r} |M_1(\nabla^\ell v)|^p \, d\sigma \right)^{2/p} \leq \frac{C}{r^{n-1}} \int_{\Omega \cap \partial Z_{\rho r}} |\nabla^\ell v|^2 \, d\sigma$$

$$+ C r^{\frac{4}{p}} \left( \frac{1}{r^{n-1}} \int_{\Delta(0, 2r)} |(\nabla^\ell v)|^2 \, d\sigma \right)^{\frac{p-2}{2}} \left( \frac{1}{r^{n-1}} \int_{\Omega \cap \partial Z_{\rho r}} |\nabla^\ell v|^2 \, d\sigma \right)^{2/p}.$$  

By Hölder’s inequality, it follows that
\[
\left(\frac{1}{r^{n-1}} \int_{I_r} |M_1(\nabla^\ell v)|^p d\sigma\right)^{2/p} \leq \frac{C}{r^{n-1}} \int_{\Delta(0,2r)} |(\nabla^\ell v)^*|^2 d\sigma
+ \frac{C}{r^{n-1}} \int_{\Omega \cap \partial Z_{\rho r}} |\nabla \ell v|^2 d\sigma + \frac{C}{r^{n-3}} \int_{\Omega \cap \partial Z_{\rho r}} |\nabla^\ell v|^2 d\sigma.
\]
Integrating the above inequality in $\rho \in (3/2, 2)$, we obtain
\[
\left(\frac{1}{r^{n-1}} \int_{I_r} |M_1(\nabla^\ell v)|^p d\sigma\right)^{2/p} \leq \frac{C}{r^{n-1}} \int_{\Delta(0,2r)} |(\nabla^\ell v)^*|^2 d\sigma
+ \frac{C}{r^{n-2}} \int_{Z_{2r}} |\nabla \ell v|^2 d\sigma + \frac{C}{r^{n-2}} \int_{Z_{2r}} |\nabla^\ell v|^2 d\sigma.
\]
Using Cacciopoli’s inequality (2.10), it is easy to see that the last two terms in the right side of (5.17) is dominated by the first term. This, together with (5.13), gives
\[
\left(\frac{1}{I_r} \int_{I_r} |(\nabla^\ell v)^*|^p d\sigma\right)^{1/p} \leq C \left(\frac{1}{r^{n-1}} \int_{\Delta(0,2r)} |(\nabla^\ell v)^*|^2 d\sigma\right)^{1/2}.
\]
By a simple covering argument, inequality (5.18) implies the reverse Hölder condition (1.8). The proof is complete.

**Proof of Corollary 1.14.** We will show that for any elliptic operator $\mathcal{L}(D)$ given by (1.1), condition (1.11) holds for some $\lambda > 3$. To this end, let $v$ be a solution of $\mathcal{L}(D)v = 0$ in $\Omega$ with the properties $(\nabla^\ell v)^* \in L^2(\partial\Omega)$ and $D^\alpha v = 0$ on $\Delta(Q_0, 5R)$ for $|\alpha| \leq \ell - 1$. We will assume $Q_0 = 0$ and use the same notation as in the proof of Theorem 1.10.

Let $0 < r < R/2$. Note that by Hölder’s inequality,
\[
\int_{Z_r} |\nabla^\ell v|^2 dx \leq Cr^3 \int_{I_r} |M_1(\nabla^\ell v)|^2 d\sigma
\leq Cr^{3+(n-1)(1-\frac{q}{2})} \left(\int_{I_{\rho R}} |M_1(\nabla^\ell v)|^q d\sigma\right)^{2/q},
\]
where $\rho \in (1/2, 1)$ and $q > 2$. Choose $q > 2$ so that the regularity estimate (1.13) holds on the Lipschitz domain $Z_{\rho R}$ uniformly for $\rho \in (1/2, 1)$. It follows that
\[
\left(\int_{Z_r} |\nabla^\ell v|^2 dx\right)^{q/2} \leq Cr^{3+(n-1)(1-\frac{q}{2})} \int_{\Omega \cap \partial Z_{\rho R}} |\nabla^\ell v|^q d\sigma.
\]
Integrating both sides of (5.19) in $\rho \in (1/2, 1)$, we obtain
\[
\left(\int_{Z_r} |\nabla^\ell v|^2 dx\right)^{q/2} \leq Cr^{3+(n-1)(1-\frac{q}{2})} \cdot \frac{1}{R} \int_{Z_R} |\nabla^\ell v|^q dx.
\]
We may assume that inequality (2.12) holds for this $q$. Hence,

$$\int_{Z_r} |\nabla^\ell v|^2 \, dx \leq C r^{n+2} \left( \frac{r}{R} \right)^{(1-n)(\frac{2}{q})} \cdot \frac{1}{R^n} \int_{Z_{2R}} |\nabla^\ell v|^2 \, dx \leq C \left( \frac{r}{R} \right)^{3+(n-1)(1-\frac{2}{q})} \int_{Z_{3R}} |\nabla^\ell-1 v|^2 \, dx,$$

where we have used the Cacciopoli’s inequality (2.10) in the last inequality. By a covering argument, estimate (5.20) implies the condition (1.11) with $\lambda = 3 + (n - 1)(1 - \frac{2}{q}) > 3$.

**Remark 5.21.** It would be very interesting to know whether condition (1.11) in Theorem 1.10 is also necessary in the case $p > 2(n - 1)/(n - 3)$. That is, does the $L^p$ solvability of the Dirichlet problem on $\Omega$ implies condition (1.11) for all $\lambda < n - \frac{4}{p-2}$? We point out that it is not hard to see that the $L^p$ solvability implies condition (1.11) for $\lambda = n - \frac{2(n-1)}{p}$. However, this is a weaker statement if $p > 2(n - 1)/(n - 3)$.

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