Anomalous enhancement of quasiparticle current near a potential barrier in a Bose-Einstein condensate

Shunji Tsuchiya\(^1,2\) and Yoji Ohashi\(^1,2\)

\(^1\)Department of Physics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
\(^2\)CREST(JST), 4-1-8 Honcho, Saitama 332-0012, Japan

(Dated: June 15, 2009)

We investigate tunneling properties of Bogoliubov phonons in a Bose-Einstein condensate. We find the anomalous enhancement of the quasiparticle current \(J_q\) carried by Bogoliubov phonons near a potential barrier, due to the supply of the excess current from the condensate. This effect leads to the increase of quasiparticle transmission probability in the low energy region found by Kovrizhin et al. We also show that the quasiparticle current twists the phase of the condensate wavefunction across the barrier, leading to a finite Josephson supercurrent \(J_s\) through the barrier. This induced supercurrent flows in the opposite direction to the quasiparticle current so as to cancel out the enhancement of \(J_q\) and conserve the total current \(J = J_q + J_s\).

PACS numbers: 03.75.Kk, 03.75.Lm, 67.85.De

Phonon in a superfluid system is an example of Goldstone mode which appears in various fields of physics associated with spontaneous broken symmetry \([1]\). It is a manifestation of broken (global) \(U(1)\) symmetry which underlies the macroscopic quantum nature of the system, and a key to understand the low energy properties of superfluids. In particular, in a Bose superfluid, it plays fundamental roles for the superfluidity \([2]\). Phonon has been observed in various systems such as superfluid \(^4\)He \([3]\), superconducting films \([4]\), atomic superfluid fermi gases \([5]\), as well as Bose-Einstein condensation (BEC) of cold atomic gases \([6]\). Due to the high degree of controllability, BEC of cold atomic gases offers an opportunity to study novel properties of phonons in the superfluid phase.

Bogoliubov phonon \([7]\) in a BEC has a long-standing issue of investigation in cold atomic gases \([8]\). In the last few years, quantum tunneling of Bogoliubov phonon has attracted much attention \([9,10,11,12,13,14]\). Since Bogoliubov phonon is a collective excitation of a BEC, its tunneling property has specific features which are quite different from that of free particles. In fact, an anomalous tunneling property of Bogoliubov phonon has been predicted in \([9,10]\). It has been shown that the transmission probability of Bogoliubov phonon through a potential barrier increases at low energies and always remains unity in the zero-energy limit, irrespective of the height of the barrier \([9,10]\). Although several mechanisms were proposed in \([10,11]\), the underlying physics of this anomalous tunneling has not been understood yet.

In this paper, we report alternative anomalous tunneling properties of Bogoliubov phonons. We find that the quasiparticle current is not conserved, but greatly enhanced near the potential barrier at low incident energies, due to the excess current supplied from the condensate. This anomalous enhancement of the quasiparticle current increases the transmission probability of Bogoliubov phonon in the low energy region, which is consistent with the tunneling property in \([9,10]\). In addition, we show that the quasiparticle current twists the phase of the condensate wavefunction, leading to a Josephson supercurrent through the barrier. The excess part of the quasiparticle current is canceled out by this induced supercurrent, so that the total current, given by the sum of the quasiparticle current and supercurrent, is conserved.

We consider a tunneling of a Bogoliubov phonon at \(T = 0\) through a potential barrier which only depends on \(x\). This one-dimensional potential barrier was used in a recent experiment \([15]\). Ignoring the motion of atoms in the \(y-\) and \(z-\)direction, we can treat this model as a one-dimensional problem. For simplicity, we ignore effects of a harmonic trap. This is allowed in a box-shaped trap \([10]\). To describe the Bose-condensed phase, we divide the boson field operator \(\hat{\psi}(x)\) into the sum of the BEC order parameter \(\Psi_0(x)\) and the non-condensate part \(\delta \hat{\psi}(x)\) \([17]\), as \(\hat{\psi}(x) = \Psi_0(x) + \delta \hat{\psi}(x)\). The condensate wavefunction \(\Psi_0(x) = \langle \hat{\psi}(x) \rangle\) obeys the static Gross-Pitaevskii (GP) equation, given by (setting \(\hbar = 1\))

\[
\left(-\frac{1}{2m} \frac{d^2}{dx^2} + U(x) + g|\Psi_0|^2\right) \Psi_0 = \mu \Psi_0, \tag{1}
\]

where \(m\), \(\mu\), and \(U(x)\) represent the mass of a boson, chemical potential, and a potential barrier, respectively. \(g \equiv 4\pi a/m\) is the interaction between bosons, where \(a(>0)\) is the s-wave scattering length.

In the Bogoliubov mean-field theory, the non-condensate part has the form, \(\delta \hat{\psi} = \sum_j [u_j(x) \hat{\alpha}_j^\dagger - v_j(x) \hat{\alpha}_j]\). Here, \(\hat{\alpha}_j^\dagger\) (\(\hat{\alpha}_j\)) is the creation (annihilation) operator of an excitation in the \(j\)-th state, satisfying the bosonic commutation relation \([\hat{\alpha}_i, \hat{\alpha}_j^\dagger] = \delta_{ij}\). \(u_j\) and \(v_j\) are determined by the following Bogoliubov equation:

\[
\begin{pmatrix}
\hat{\hbar} & -g\Psi_0^* \\
-g(\Psi_0^*)^2 & \hat{\hbar}
\end{pmatrix}
\begin{pmatrix}
-u_j \\
v_j
\end{pmatrix}
= E_j \tau_3 \begin{pmatrix}
u_j \\
v_j
\end{pmatrix}, \tag{2}
\]

where \(\hat{\hbar} \equiv -\frac{1}{2m} \frac{d^2}{dx^2} + U(x) + 2g|\Psi_0|^2 - \mu\). \(\tau_3\) is the Pauli matrix, and \(E_j\) is the Bogoliubov excitation spectrum.
We consider a simple rectangular potential barrier $U(x) = U_0 \theta(d/2 - |x|)$ ($U_0 > 0$). In the absence of supercurrent, we can safely take the condensate wavefunction $\Psi_0$ to be real. In this case, the analytic solution of Eq. (1) is given in Ref. [10], as $\Psi_0(x) = \sqrt{n_0} \tanh \left([|x| - d/2] / \sqrt{2} \xi + \text{arctan} \gamma \right)$ ($|x| > d/2$), and $\Psi_0(x) = \sqrt{n_0} \beta / \text{cn} \sqrt{K^2 + \beta^2 x} / \sqrt{2} \xi$, $q$ ($|x| < d/2$), where $\text{cn}(x, q)$ is the Jacobi elliptic function, $\beta \equiv \Psi_0(0)/\sqrt{n_0}$, $\gamma \equiv \Psi_0(d/2)/\sqrt{n_0}$, $K \equiv \sqrt{\beta^2 + 2(U_0/\mu - 1)}$, and $q \equiv K / \sqrt{K^2 + \beta^2}$. $n_0 \equiv \mu / g$ is the condensate density far away from the barrier ($x \to \pm \infty$), and $\xi \equiv 1 / \sqrt{2mg\Omega_0}$ is the healing length.

In the absence of the barrier, Eqs. (1) and (2) give the chemical potential $\mu = gn_0$ and excitation energy $E = \sqrt{\varepsilon_p (\varepsilon_p + 2gn_0)}$ (where $\varepsilon_p = p^2 / 2m$). The wavefunction has the form $(u(x), v(x)) = (u_p, v_p) e^{ipx}$, where $u_p$ and $v_p$ are given by

$$
\begin{pmatrix}
  u_p \\
  v_p
\end{pmatrix} = \begin{pmatrix}
  a \\
  b
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{\sqrt{2V}} \left( \sqrt{\frac{e_p + gn_0}{E}} + 1 \right) \\
  \frac{1}{\sqrt{2V}} \left( \sqrt{\frac{e_p + gn_0}{E}} - 1 \right)
\end{pmatrix},
\quad (3)
$$

where $V$ is the volume of the system. $p = \pm k = \pm \sqrt{2m \left( \sqrt{E^2 + (gn_0)^2} - gn_0 \right)}$ describe the ordinary propagating waves in the $\pm x$-direction. In considering an inhomogeneous system, however, we note that, besides the propagating solutions in Eq. (3), Eq. (2) also has other localized solutions, having the form $(u_p, v_p) = (-b, a)$, where $p = \pm ik \equiv \pm \sqrt{2m \left( \sqrt{E^2 + (gn_0)^2} + gn_0 \right)}$. The normalization of these localized states is given by $u_p^2 - v_p^2 = -1 / V$. When the incident Bogoliubov phonon comes from $x = -\infty$, the asymptotic solution is given by

$$
\begin{cases}
  \begin{pmatrix}
  u \\
  v
\end{pmatrix} = \begin{pmatrix}
  a \\
  b
\end{pmatrix} e^{ikx} + r \begin{pmatrix}
  a \\
  b
\end{pmatrix} e^{-ikx} + A \begin{pmatrix}
  b \\
  a
\end{pmatrix} e^{ikx}, \\
  (x \to -\infty),
\end{cases}
\quad (4)
\begin{cases}
  \begin{pmatrix}
  u \\
  v
\end{pmatrix} = t \begin{pmatrix}
  a \\
  b
\end{pmatrix} e^{ikx} + B \begin{pmatrix}
  b \\
  a
\end{pmatrix} e^{-kx}, \\
  (x \to \infty).
\end{cases}
$$

Here, $r$ and $t$ are, respectively, the reflection and transmission amplitudes, satisfying $|r|^2 + |t|^2 = 1$. As discussed later, this condition results from the conservation of energy flux. In Eq. (4), $A$ and $B$ are the amplitudes of the localized components near the potential barrier.

We numerically solve the Bogoliubov equation Eq. (2) so as to satisfy the asymptotic solution in Eq. (4). For this purpose, we employ the finite element method.

Figure 1 shows the calculated transmission probability $W \equiv |t|^2$, as well as the phase shift $\delta \equiv \arg(t)$ as functions of the incident energy $E$. The anomalous tunneling behavior discussed in Ref. [10] can be clearly seen in Fig. 1, i.e., $W \to 1$ and $\delta \to 0$ when $E \to 0$. Around $E = 0$, one can see the enhancement of $W$, the region of which is wider for weaker potential barrier. When the incoming energy $E$ is very large ($E \gg \mu$), since the Bogoliubov phonon loses its collective nature, the tunneling property is close to that of a single particle.

The upper panel in Fig. 1 shows the resonance tunneling behavior ($W = 1$) at finite energies for $(d, U_0) = (4\xi, 2\mu)$. At each resonance energy, $u$ has a large amplitude in the barrier while $v$ is suppressed. This is quite different from the case of anomalous tunneling at $E \approx 0$, where $u$ and $v$ monotonically decrease under the barrier approaching the zero-mode $\Psi_0$. It was proposed that the anomalous tunneling is due to the quasiresonance scattering by the potential wells formed near the barrier [10]. However, we do not find any signature of it in $u$ and $v$.

To see the low energy quasiparticle transmission in more detail, we directly calculate the quasiparticle current $J_q$. First, we discuss current conservation for Bogoliubov phonons. In the Bogoliubov theory, the total number density $n = \langle \hat{\psi}^\dagger \hat{\psi} \rangle$ and current $J = (1/m) \text{Im}(\hat{\psi}^\dagger \partial_x \hat{\psi})$ are respectively given by

$$
n = n_s + \sum_j (n_{u_j} + n_{v_j}) \langle \hat{a}_j^\dagger \hat{a}_j \rangle + \sum_j n_{v_j},
\quad (5)
$$

$$
J = J_s + \sum_j (J_{u_j} - J_{v_j}) \langle \hat{a}_j^\dagger \hat{a}_j \rangle - \sum_j J_{v_j}.
\quad (6)
$$

Here, $n_{u_j} = |u_j|^2$, $n_{v_j} = |v_j|^2$, $J_{u_j} = (1/m) \text{Im}(u_j^\dagger \partial_x u_j)$, and $J_{v_j} = (1/m) \text{Im}(v_j^\dagger \partial_x v_j)$, where $n_s = |\Psi_0|^2$ is the condensate density, and $J_s = (1/m) \text{Im}(\Psi_0^\dagger \partial_x \Psi_0)$ is the supercurrent density associated with the phase-twisted condensate wavefunction. The total number density $n$ and total current $J$ satisfy the continuity equation $\partial_t n + \partial_x J = 0$. Since the second terms in Eqs. (5) and (6) describe the quasiparticle contributions, the quasiparticle...
density and quasiparticle current are, respectively, given by \( n_q = n_u + n_v \) and \( J_q = J_u - J_v \). In a uniform system, the creation of a Bogoliubov phonon with momentum \( k \) induces the quasiparticle current \( J_q = k/(mV) \). (Note that \( J_u = (k/m) a^2 \), \( J_v = (k/m) b^2 \).) The last term in Eq. (4) is the so-called quantum depletion at \( T = 0 \), which is the non-condensate density originating from the repulsive interaction between bosons.

Using the time-dependent Bogoliubov equations,

\[
i \tau_0 \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{h} & -g\Psi_0^2 \\ -g\Psi_0^2 & \hat{h} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},
\]

one obtains the continuity equations for \( u \) and \( v \), as \( \partial_t n_u + \partial_x J_u = S/2 \) and \( \partial_t n_v + \partial_x J_v = S/2 \), where the source term has the form \( S = -4\mu \text{Im}(\Psi_0^2 u^* v) \). Thus, the continuity equation for quasiparticles is obtained as

\[
\partial_t n_q + \partial_x J_q = S.
\]  

In a uniform case, one finds \( S = 0 \), so that the number of quasiparticles is conserved. On the other hand, as will be shown later, since the source term \( S \) is finite near the barrier in our inhomogeneous problem, the number of quasiparticles is not conserved. In particular, in the stationary state \( (\partial_t n_q = 0) \), Eq. (8) indicates that there is a source supplying excess quasiparticle current in addition to the incoming current injected from \( x = -\infty \).

In contrast to the non-conserved quasiparticles, the continuity equation for energy density \( \rho_q = E(n_u - n_v) \) has no source term, as \( \partial_t \rho_q + \partial_x Q_q = 0 \) [10], where \( Q_q = E(J_u + J_v) \) is the energy flux carried by quasiparticles. Namely, \( Q_q \) is conserved in the stationary state. Using the asymptotic form in Eq. (4), we obtain \( Q_q = E(k/m)(a^2 + b^2)(1 - |r|^2) \) and \( J_q = (k/m V) (1 - |r|^2) \) for \( x \to -\infty \), and \( Q_q = E(k/m)(a^2 + b^2)|t|^2 \) and \( J_q = (k/m V)|t|^2 \) for \( x \to \infty \). In the stationary state, noting that \( Q_q \) is conserved, one obtains \( |r|^2 + |t|^2 = 1 \). From this result, we find that \( J_q(x = -\infty) = J_q(x = \infty) \).

The upper panel in Fig. 2 shows the excess quasiparticle current \( \Delta J_q(x) \equiv J_q(x) - J_q(-\infty) \) measured from the value at \( x = -\infty \). Lower panel: Spatial variation of the source term \( S \). We take the rectangular barrier with \( (d, U_0) = (\xi/10) \mu \). The dash-dotted line indicates the potential barrier \( U(x) \).

\[ \Delta J_q(x = 0) \] is proportional to \( k \). Thus, \( \Delta J_q(0)/(k/m V) \) in Fig. 2 approaches a constant height which becomes larger for larger potential barrier.

The transmission probability of Bogoliubov phonon increases when the quasiparticle current is enhanced near the barrier. Comparing Fig. 2 with the result for \( (d, U_0) = (\xi/10) \mu \) in Fig. 1, one finds that the energy \( (E \sim 0.1 \mu) \) where the high transmission probability associated with the anomalous tunneling becomes remarkable coincides with the energy where the excess current in the barrier (normalized by \( k/m V \)) becomes large.

The excess quasiparticle current is supplied by the condensate. To see this, we note that the divergence of Eq. (6) in the stationary state gives \( \partial_x J = \partial_x J_s + \sum_j (\hat{\alpha}_j^\dagger \hat{\alpha}_j + \frac{1}{2} \sum_j S_j \hat{\alpha}_j^\dagger \hat{\alpha}_j) \). Since the supercurrent is conserved \( (\partial_x J_s = 0) \), it reduces to \( \partial_x J = \frac{1}{2} \sum_j S_j \) in the ground state \( (\hat{\alpha}_j^\dagger \hat{\alpha}_j = 0) \). Since the total current \( J \) is conserved, the sum of the source term vanishes \( (\sum_j S_j = 0) \). When quasiparticles are excited \( (\hat{\alpha}_j^\dagger \hat{\alpha}_j \neq 0) \), one obtains \( \partial_x J \neq 0 \), which contradicts with the conservation of the total current. This inconsistency is actually eliminated when one includes effects of quasiparticles on the condensate, as

\[
- \frac{1}{2m} d^2 \frac{d^2}{dx^2} + U(x) + g |\Psi_0|^2 \right) \Psi_0 = 2g \sum_j u_j v_j^* \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle \Psi_0^* = \mu \Psi_0.
\]

In this GP equation, the last term on the left side is the quasiparticle contribution, which originates from the so-called anomalous average term \( g \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle \Psi_0^* \) (which is neglected in deriving the GP equation within the Bo-
Here, we assume that only one Bogoliubov phonon with produced supercurrent $\Delta J$ is subject to change by the tunneling quasiparticle. The induced supercurrent is given by $\Delta J = \theta J $ where $\theta = \langle \hat{\alpha}^\dagger \hat{\alpha} \rangle$ and integrating the new continuity equation for $J_s$ in terms of $x$ from $-\infty$ to $x$ we obtain $\Delta J_s = \sum \Delta J_{sj} \langle \hat{\alpha}_{s}^\dagger \hat{\alpha}_{s} \rangle$, where $\Delta J_{s}(x)$ is the deviation of supercurrent from the value based on Eq. $\Psi$. This equation shows that the sum of supercurrent and quasiparticle current is always conserved. Thus, the excess quasiparticle current shown in the upper panel in Fig. 2 is found to be supplied from the condensate. At the same time, the quasiparticle current also affects the supercurrent. Since the quasiparticle current is enhanced near the barrier, the induced supercurrent flows only near the barrier, in the opposite direction to the quasiparticle current to conserve the total current.

If we assume that the influence of quasiparticle changes the condensate wavefunction by a phase shift $\theta(x)$, the induced supercurrent is given by $\Delta J_s = (1/m) n_s \theta(x) \partial_x \theta(x)$. Thus, we find that $\theta(x)$ is given by

$$\theta(x) = -m \sum_j \left( \int_{-\infty}^{x} dy \frac{\Delta J_{sj}(y)}{n_{s}(y)} \langle \hat{\alpha}_{sj} \hat{\alpha}_{sj} \rangle \right).$$

The inset in Fig. 3 shows the spatial variation of $N_{0} \theta(x)$, where $N_{0} = n_{0} V$ is the number of condensate atoms. Here, we assume that only one Bogoliubov phonon with energy $E$ exists. The phase $\theta(x)$ sharply varies around the barrier. This clearly shows that the Bogoliubov phonon twists the relative phase of the condensates on the left and right sides of the barrier. Since their coupling is weak at the barrier region, the relative phase is subject to change by the tunneling quasiparticle. The induced supercurrent $\Delta J_s$ can be naturally regarded as the Josephson current originating from the phase difference $\phi \equiv \theta(x=\infty) - \theta(x=-\infty)$.

Figure 3 shows the normalized phase difference $N_0 \phi/(k \xi)$ as a function of $E$. It is enhanced at low energies similarly to the normalized excess quasiparticle current $\Delta J_s/(k/mV)$ in Fig. 2 and indeed, normalized supercurrent $\Delta J_s(x=0)$ is proportional to $\phi$ at low energies. This result justifies regarding $J_s$ as the Josephson current due to $\phi$. In addition, Fig. 3 shows that, at the same energy, $\phi$ is larger for higher barriers. This implies that, since the coupling of the condensates across the barrier becomes weaker as the barrier gets higher, the phase of the condensate wavefunction is easily twisted by the Bogoliubov phonon for higher barriers.

In summary, we have investigated tunneling effect of Bogoliubov phonon in a BEC at $T = 0$. We found anomalous tunneling properties of Bogoliubov phonons. The quasiparticle current of Bogoliubov phonon is enhanced near the potential barrier due to the supply from the condensate. This enhancement is remarkable at low incident energies, and explains the increase of the transmission probability of Bogoliubov phonon in [9, 10]. Furthermore, we have shown that the quasiparticle current twists the phase of the condensate wavefunction across the barrier, which induces the counterflow of Josephson supercurrent to conserve the total current. This twisted phase could be observed in the interference pattern when a BEC is released from the trap potential as in [21].

S.T. acknowledges I. Danshita, K. Kamide, S. Inouye, and F. Dalfovo for useful discussions.

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[19] The anomalous average term has a quantum contribution $-g \sum_j u_j v_j\Psi_0$, which yields the additional term $\frac{1}{2} \sum_j S_j$. 
in the continuity equation. In this paper, we treat the problem perturbatively in the sense that \( \Psi_0 \) and \((u_j, v_j)\) obtained from Eq. (1) and (2) are used in evaluating backreaction effect of quasiparticles on condensate, i.e. the correction to \( \Psi_0 \). Within this treatment, one finds \( \sum_j S_j = 0 \). As a result, the quantum contribution in the continuity equation vanishes. To improve this, one needs to solve the generalized GP and Bogoliubov equations including the anomalous average terms self-consistently (Hartree-Fock-Bogoliubov approximation). However, it yields an excitation spectrum with a finite energy gap which is unphysical as discussed in [18].

[20] This assumption is valid as long as \( \theta(x) \) is small. Since \( \theta(x) \) is inversely proportional to the number of condensate atoms \( N_0 \), it is negligibly small. In this case, \( \Psi_0 \) can be obtained without solving Eq. (9).

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