Powerlaw spectra from stochastic acceleration

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ABSTRACT

Numerical simulations of particle acceleration in magnetized turbulence have recently observed powerlaw spectra where pile-up distributions are rather expected. We interpret this as evidence for particle segregation based on acceleration rate, which is likely related to a non-trivial dependence of this acceleration efficiency on phase space variables other than the momentum. We describe the corresponding transport in momentum space using continuous-time random walks, in which the time between two consecutive momentum jumps becomes a random variable. We show that powerlaws indeed emerge when the experimental (simulation) timescale does not encompass the full extent of the distribution of waiting times. We provide analytical solutions, which reproduce dedicated numerical Monte Carlo realizations of the stochastic process, as well as analytical approximations. Our results can be readily extrapolated for applications to astrophysical phenomenology.

Key words: particle acceleration – turbulence

1 INTRODUCTION

In Fermi’s original model of particle acceleration (Fermi 1949; Fermi 1954), charged particles can increase their energy in a stochastic fashion through repeated interactions with moving magnetized structures. Magnetized turbulence has thus been recognized as a probable source of suprathermal particles in space plasmas, and indeed, it may well be responsible for the non-thermal radiation emitted by a wide variety of astrophysical objects, from the Sun (e.g., Miller et al. 1996; Petsosian & Liu 2004; Seltzowitz & Blackman 2004; Bian et al. 2012, and references therein) to the remote Universe (e.g., Lacome 1977; Schlickeiser 1984; Schlickeiser & Dermer 2000; Katarzyński et al. 2006; Brunetti & Lazarian 2007; Petsosian & East 2008; Tramacere et al. 2011; Asano & Hayashida 2018, and references therein).

This broad interest has brought about a vast literature on the theoretical aspects of stochastic acceleration. Analytically, it has been addressed through calculations based on resonant wave-particle interactions (e.g., Schlickeiser 2002 and references therein) to the remote Universe (e.g., Lacome 1977; Schlickeiser 1984; Schlickeiser & Dermer 2000; Katarzyński et al. 2006; Brunetti & Lazarian 2007; Petsosian & East 2008; Tramacere et al. 2011; Asano & Hayashida 2018, and references therein).

Astrophysical applications borrow the predicted or observed transport (diffusion) coefficients and rely on Fokker-Planck-type equations to determine the particle distribution function as a function of time, see e.g., Schlickeiser (1984), Becker et al. (2006), Stawarz & Petsosian (2008), Mertsch (2011). Pile-up distributions then emerge as a generic prediction of stochastic acceleration, in the absence of energy losses or particle escape. Yet, the recent numerical PIC simulations, which by construction work in a closed box and neglect energy losses, have produced distribution functions with extended soft powerlaw tails, in sharp contrast with those expectations. These results are of prime importance, because they cast into question a wealth of phenomenological applications to astrophysics.

The generation of a powerlaw as the result of competition between energy gain and energy loss or escape can be regarded as the gist of Fermi-type acceleration. Hence, the simplest explanation for the emergence of a powerlaw in...
those simulations is the existence of some trapping mechanism that inhibits acceleration for a fraction of the particles – at least, on the timescale of the simulations – and thereby acts as an effective escape mechanism. In this paper, we develop this line of thought in order to interpret the results of those numerical experiments, having in mind their extrapolation to astrophysical cases of interest.

On a formal level, random walks with trapping times belongs to the class of continuous-time random walks (Montroll & Weiss 1965), see Bouchaud & Georges (1990) for a general review, and Balescu (1995) for an application to anomalous transport in plasmas. These stochastic processes are characterized by a random, continuous time step, which is itself characterized by a probability density. If the distribution of jump time intervals has a finite mean (τ), then diffusion is normal, meaning that the probability of undergoing n jumps on timescale t converges at large n to the normal distribution, with σ(t) = t/τ. In the absence of energy losses or particle escape, the momentum distribution can be described by the Green function of a standard random walk with fixed time step (τ) in the large-time limit. More generally speaking, the Fokker-Planck formalism can be used to describe the transport in momentum space. By contrast, heavy-tailed distributions, who do not possess a finite mean, characterize Lévy flights and are more properly described by fractional transport equations, see for instance Zimbardo & Perri (2013), Zimbardo et al. (2017) and Isliker et al. (2017a).

In the following, we study these two classes of continuous-time random walks, and provide two toy models, which represent in our view the simplest models that can account for the emergence of a powerlaw on the finite timescale of numerical simulations. The first model assumes that the mean waiting time between two jumps in momenta is finite, but that the distribution of waiting times is such that for some particles, it allows acceleration on the simulation timescale, while for others, it does not. The second model assumes that the waiting time is distributed according to a one-sided stable (Lévy) distribution with infinite mean waiting time.

We do not aim at elucidating the origin of this segregation here but suggest that it arises from the hidden dependency of the acceleration rate on phase space variables other than the particle momentum. Consider for instance the pitch-angle cosine μ of the particle, as defined with respect to the direction of the magnetic field line. It can be regarded as an internal degree of freedom that is averaged out when one treats the bulk of the suprathermal particle population, which one does implicitly when considering a Fokker-Planck equation in momentum space. If scattering is slow (on the simulation timescale) for some range of μ, then particles in that range of μ effectively remain trapped in momentum space, given that scattering is a requisite of stochastic acceleration. As a possible realization, consider the interaction of particles with magnetic mirrors moving along the magnetic field lines: particles outside the loss cone bounce on the mirror and thus gain (or lose) energy, while particles inside the loss cone ignore the mirror and therefore undergo little energy gain/loss. This picture appears in qualitative agreement with the observation that high-energy particles are strongly peaked near μ = 0 in the simulations of Comisso & Sironi (2019), while low-energy particles rather show |μ| ≲ 1.

Alternatively, one may consider a situation in which the efficiency of acceleration is inhomogeneous in space, as suggested by some other simulations (Trotta et al. 2020). At each time step, a fraction of the particles happens to be in a region in which scattering, hence acceleration, is efficient, while the rest of particles mainly drift along the magnetic field lines. In this case, the internal degree of freedom that has been integrated out in deriving the Fokker-Planck equation is the position, but the general statistical description of the acceleration process as a continuous-time random walk remains legitimate.

Ultimately, one would like to study the full Fokker-Planck equation, including the dependence on the variables μ or x, but this introduces by definition an infinite number of degrees of freedom associated to the functional form of the transport coefficients. In this sense, our toy models provide the simplest approach to this problem.

Our paper is laid out as follows. We first discuss models with finite waiting time in Sec. 2 and address the second case of Lévy α-stable distributions in Sec. 3. In Sec. 4.2, we consider how these models are modified when one accounts for escape losses. We provide conclusions in Sec. 5. The diffusion coefficient in momentum space, (ΔpΔp) /2Δt is written $D_{pp}$ and throughout, unless otherwise noted, it is assumed that $D_{pp} \propto p^2$, in accord with the results of the above PIC simulations, and with theoretical expectations.

## 2 A BINARY MODEL FOR STOCHASTIC ACCELERATION

### 2.1 Analytical solution

Continuous-time random walks possess the following formal solution, known as the Montroll-Weiss formula: if ψ(Δt) denotes the distribution of waiting time Δt, and ϕ(Δt,Δp) the distribution of (log) jump increments Δt log Δp in momentum space, then the probability density for observing a shift of log-momentum Δp at time t, $P(\ln p; t)$, can be obtained from the inverse Fourier-Laplace transform

$$\mathcal{P}(\ln p; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \, e^{i k \ln p} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \, e^{i \lambda t} \hat{\phi}(k; \lambda),$$

with (Montroll & Weiss 1965):

$$\hat{\phi}(k; \lambda) = \frac{1 - \hat{\psi}(\lambda)}{\lambda} \frac{1}{1 - \hat{\psi}(\lambda)\hat{\phi}(k)}.$$

Here the hat symbol represents a Fourier transform from Δlnp to k, and the tilde symbol a Laplace transform from Δt to λ.

In practice, however, the calculations become prohibitive for non-trivial distribution functions, and one must rely on approximations. Consider for instance the following distributions, which characterize acceleration on two possible timescales $\tau_-$ and $\tau_+$ with (fixed) energy gain g:

$$\psi(\Delta t) = p - \delta(\Delta t - \tau_-) + p_+ \delta(\Delta t - \tau_+),$$

$$\phi(\Delta \ln p) = \delta(\Delta \ln p - \delta),$$

$$\mu$$
with $p_\ast = 1 - p_-$, corresponding to
\[
\hat{\psi}(\lambda) = p_\ast e^{-4\pi\lambda} + p_+ e^{-4\pi\lambda},
\]
\[
\hat{\psi}(\kappa) = e^{-i\kappa g}.
\] (4)

To simplify Eq. (2), we take the limit $\tau_\ast \to +\infty$, which describes a situation in which $\tau_\ast$ is effectively much larger than the times $t$ on which we probe the distribution function, e.g., the simulation timescale. Then $\hat{\psi}(\kappa; \lambda)$ presents poles in $\lambda$ at $\lambda = 0$ and $\lambda = \tau^{-1}(-i\kappa g + 2i\pi n)$, for all $n \in \mathbb{Z}$. The former provides the late-time (stationary regime) behavior and we concentrate on it. We thus obtain the stationary distribution $\mathcal{P}_s(p)$ as
\[
\mathcal{P}_s(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa e^{i\kappa p} \Re \hat{\psi}(\kappa; \lambda) \bigl|_{\lambda=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa e^{i\kappa p} \bigl( 1 - p - e^{-4\pi\kappa g} \bigr) = \sum_{n=0}^{\infty} p^n \delta(\ln p - n\gamma) - e^{\ln p \ln p./g}.
\] (5)

Therefore, the spectrum $p^2 f(p) \propto \mathcal{P}_s dp$ is a powerlaw with exponent $-1 + \ln p./g$. We recover here the formula of Bell (1978) for particle acceleration in a process with energy gain $g$, and escape probability $1 - p_-$, as indeed, the limit $\tau_\ast \to +\infty$ turns $1 - p_-$ into a probability of escape from the acceleration process.

To generalize the above solution to a time-dependent regime, including a more realistic description of the diffusion in momentum, we resort to an alternative description of the problem, borrowing on previous work by Malkov & Diamond (2006). These authors studied the transport (in configuration and momentum space) of particles subject to interactions with nonlinear fronts (weak shocks) in the precursor of a strong shock. The population of particles was split into an ensemble of particles that were trapped and convected with the moving fronts, because of their small pitch-angle cosines, and another population of particles that could explore the train of moving fronts, because of their larger pitch-angle cosines. This general framework nicely applies to the situation at hand, and we thus break our distribution function into two populations at each time step: $f_t(p, t)$ characterizes the subset of particles for which acceleration is inhibited, while $f_f(p, t)$ represents the group that undergoes acceleration at some rate $\nu_{acc} = D_{pp}/p^2 \propto p^0$.

To complete the model, we also define the transition rate of population 1 towards 0 as $\nu_1$, and the transition probability from population 0 to population 1 as $\nu_0$. These frequencies characterize the rates at which particles can respectively scatter out of, or into, the acceleration region in phase space. We thus write the following transport equations in momentum space for both populations, as in Malkov & Diamond (2006), replacing however the regular energy gain with a diffusion operator:
\[
\frac{\partial}{\partial t} f_0(p, t) = -\nu_0 f_0(p, t) + \nu_1 f_1(p, t),
\]
\[
\frac{\partial}{\partial t} f_1(p, t) = +\nu_0 f_0(p, t) - \nu_1 f_1(p, t)
\] + \frac{1}{p^2} \frac{\partial}{\partial p} \left( D_{pp} p^2 \frac{\partial}{\partial p} f_1(p, t) \right).
\] (6)

From the point of view of acceleration, a trap is created if $\nu_0 \ll \nu_1$ and the timescales $t$ on which we probe the distribution verifies $t \lesssim \nu_0^{-1}$. For PIC simulations, $t \sim O(10L_{\text{max}}/c)$ in terms of the maximum scale of the turbulence $L_{\text{max}}$, hence the present interpretation suggests that $\nu_0 L_{\text{max}}/c \lesssim 0.1$. The following assumes, for simplicity, but also in line with the scaling of $\nu_{acc}$: $\nu_0$ and $\nu_1$ are independent of $p$.

We provide in App. A a full solution of the above system of equations in integral form. For the sake of commodity, we make it explicit here for generic initial conditions:
\[
f_0(p, t) = e^{-\nu_0 t} f_0(p, 0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_0 \left( \frac{p}{p_0} \right)^{-3/2} \left\{ \int_{-\lambda_{acc}/\nu_{acc}}^{\lambda_{acc}/\nu_{acc}} + \int_{-\lambda_{acc}/\nu_{acc}}^{\infty} ds \right\} e^{-s \nu_{acc}} \cos \left( \frac{\pi}{2} \ln \left| \frac{p}{p_0} \right| / \nu_{acc} \right) \Gamma_0(p_0; -s \nu_{acc}) \bigg/ \left( \frac{\nu_{acc} p_0}{\nu_{acc} p_0} + \frac{9}{4} \right)^{1/2},
\]
\[
\nu_1(p, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_0 \left( \frac{p}{p_0} \right)^{-3/2} \left\{ \int_{-\lambda_{acc}/\nu_{acc}}^{\lambda_{acc}/\nu_{acc}} + \int_{-\lambda_{acc}/\nu_{acc}}^{\infty} ds \right\} e^{-s \nu_{acc}} \cos \left( \frac{\pi}{2} \ln \left| \frac{p}{p_0} \right| / \nu_{acc} \right) \Gamma_1(p_0; -s \nu_{acc}) \bigg/ \left( \frac{\nu_{acc} p_0}{\nu_{acc} p_0} + \frac{9}{4} \right)^{1/2},
\] (7)

where
\[
\frac{\nu_{acc} p_0}{\nu_{acc} p_0} + \frac{9}{4} = \left( \frac{\nu_{acc} p_0}{\nu_{acc} p_0} + \frac{9}{4} \right)^{1/2},
\]
\[
\lambda_{acc} = \frac{\nu_{acc}}{2} \left( \frac{\nu_{acc} p_0}{\nu_{acc} p_0} + \frac{9}{4} \right)^{1/2},
\] (8)
\[
\Gamma_0(p_0; -s \nu_{acc}) = \frac{\nu_0 \nu_{10}}{\nu_{acc} + \nu_{10}} f_0(p_0, 0) + \frac{\nu_0 \nu_{10}}{\nu_{acc} + \nu_{10}} f_0(p_0, 0) + f_1(p_0, 0).
\] (9)

As expected, the distribution function is controlled by two parameters only, the ratios of $\nu_0$ and $\nu_1$ to $\nu_{acc}$.

For $\nu_0 \to 0$, $f_0(p_0, 0) \to 0$ and $f_1(p_0, 0) \to f_0(p_0, 0)$, we obtain in particular,
\[
f_0(p, t) = \frac{f_0^0}{4 \sqrt{1 + \frac{8 \nu_0}{\nu_{acc}}}} \left( \frac{p}{p_0} \right)^{-3/2} e^{-\nu_0 t \sqrt{1 + \frac{8 \nu_0}{\nu_{acc}}}} \left[ \text{erfc} \left( \frac{\log \left( \frac{p}{p_0} \right)}{\sqrt{4 \nu_{acc} t}} / \sqrt{4 \nu_{acc} t} \right) / \left( \frac{p}{p_0} \right)^{3/2} \right] + \frac{f_0^0}{2 \sqrt{\pi \nu_{acc}}} \left( \frac{p}{p_0} \right)^{-3/2} e^{-\nu_0 t \sqrt{2 \nu_{acc}} \left[ -\log \left( \frac{p}{p_0} \right)^2 \nu_{acc} \right]}. \] (10)
In these equations, \( \epsilon_p = +1 \) if \( p > p_0 \), and \( \epsilon_p = -1 \) if \( p \leq p_0 \). The stationary distribution function can be obtained as \( f_s(p) \rightarrow \lim f_0(p; t) \). It can be read off the equation for \( f_0(p, t) \) above, noting that the brackets containing erfc functions tend to 2 as \( t \rightarrow +\infty \). This distribution function is thus characterized by a powerlaw at high momenta \( p > p_0 \), with

\[
p^2 f_s(p) \propto p^2\epsilon_p \sqrt{\frac{\nu_0}{\nu_0 - \nu}}.
\]  

(11)

In Figs. 1 and 2, we plot examples of such spectra, at different times, as measured in units of \( \nu_{\text{acc}} \). When \( \nu_{\text{acc}}t \ll 1 \), as is the case in Fig. 1, the solution takes the form of a powerlaw at high momenta. As \( \nu_{\text{acc}}t \ll 1 \) and \( \nu_{\text{acc}}t \ll \nu_{\text{acc}} \) in the present case, this index lies close to the theoretical value for \( \nu_{\text{acc}}t \rightarrow 0 \), given in Eq. (11). The maximum momentum \( p_{\text{max}} \) of the accelerated population, beyond which the powerlaw turns over into a decreasing exponential, increases with time. Its scaling can be derived as the evolution of \( \langle p \rangle \) (with the average taken over \( p^2f \)) from the solution of \( p^2f_t \) in Eq. (10). This gives \( p_{\text{max}} \sim p_0 \exp(4\nu_{\text{acc}}t) \).

At late times, viz. when \( \nu_{\text{acc}}t \gg 1 \), as in Fig. 2, the spectra slowly move from a powerlaw to a pile-up type population. However, its mean momentum increases much faster than at early times, as a consequence of trapping. The discretized model that we provide in the following will make this explicit.

2.2 Random walk with trapping

We now provide an alternative discretized description, which is more easily amenable to a numerical implementation, and which provides simple analytical estimates. Assume that, at each time step, of fixed extent \( \Delta t \), a particle of the untrapped (accelerating) population can become trapped with probability \( p_{\text{tr}} \), but otherwise shifts in momentum, from \( p \) to \( p' = p + \Delta p \), according to the probability law

\[
G_v(p'|p; \Delta t) = \frac{(p'/p)^{\nu}}{\nu!\Delta t} \exp \left[ -\frac{\nu}{\Delta t} \ln(\nu')^{2} \right]\bigg/ 4\nu \Delta t .
\]  

(12)

with \( \nu = \nu_{\text{acc}} \). This corresponds to the propagator of the particle density \( p^2f(p) \) for the Fokker-Planck equation, if \( D_{pp} \propto p^2 \), see Eq. (10) in the limit \( \nu_{\text{acc}}t \rightarrow 0 \), or Becker et al. (2006).

Conversely, at each time step, a particle of the trapped (non-accelerating) population can become untrapped with probability \( p_{\text{de}} \), without however gaining energy during \( \Delta t \). The trapping probability is given by \( p_{\text{tr}} = 1 - \exp(-\nu_{\text{acc}}t) \approx \nu_{\text{acc}}t \), while the untrapping probability (for trapped particles) is \( p_{\text{de}} = 1 - \exp(-\nu_{\text{acc}}t) \approx \nu_{\text{acc}}t \) for small \( \Delta t \).

The distribution function at time \( t \) can then be expressed as

\[
p^2f(p; t) = \sum_{n=1}^{\infty} \frac{dP_{n}}{dp}(p; n) P_{n}(n; t),
\]  

(13)

with \( dP_{n}/dp \) the differential probability of reaching momentum \( p \) after \( n \) jumps of size \( \Delta t \), and \( P_{n}(n; t) \) the probability of obtaining \( n \) jumps in a time interval \( t \). The former is simply

\[
G_{\nu_{\text{acc}}}(p|p_0, n\Delta t).
\]

The probability of achieving at least \( n \) jumps during \( \Delta t \), i.e. \( P_{n}(\geq n; t) \) corresponds to the probability that the total amount of time taken by these \( n \) jumps does not exceed \( t' \), or equivalently, that the time the particle spends in traps does not exceed \( t' = t - n\Delta t \), where \( \Delta t \) is the length of a step notwithstanding trapping. If the particle encounters \( m \) traps during these \( n \) jumps, and if the waiting time for each jump is exponentially distributed with mean \( \nu_{\text{acc}}^{-1} \), the probability
density $d\rho_m/d\tau$ of the sum $\tau$ of the $m$ waiting times reads
\[ \frac{d\rho_m}{d\tau} = \frac{\tau^{m-1}e^{-\tau}}{\Gamma(m)}, \quad (14) \]
whose cumulative distribution up to time $t' \geq 0$ is
\[ \int_0^{t'} d\tau \frac{d\rho_m}{d\tau} = 1 - \Gamma[m, \nu_0 t']/\Gamma(m). \quad (15) \]
Consequently, the cumulative probability $P_n(\geq n; t)$ can be written
\[ P_n(\geq n; t) = \sum_{m=0}^{n} \binom{n}{m} p_{01}^m (1 - p_{10})^{n-m} \left(1 - \Gamma[m, \nu_0 t']/\Gamma(m)\right) \Theta(t'). \quad (16) \]
Finally, $P_n(n; t) = P_n(\geq n; t) - P_n(\geq n + 1; t)$. This provides a formal solution to Eq. (13).

To derive approximate expressions, we consider both limits $\nu_0 t' \gg 1$ and $\nu_0 t' \ll 1$. The former describes a situation where particles can hop in and out of the traps frequently enough to homogenize the acceleration process among the particle population, in which case we can expect to recover the standard propagator $G_r(p|p_0, t \sim t_0)$ with some modified acceleration rate $\nu'$, to be determined. In the latter limit, however, a powerlaw should emerge and we seek to characterize its spectral index.

### 2.2.1 Small traps, $\nu_0 t' \gg 1$

For $\nu_0 t' \gg 1$,
\[ 1 - \frac{\Gamma[m, \nu_0 t']}{\Gamma(m)} \sim \Theta(\nu_0 t' - m). \quad (17) \]

For large $n$, one can approximate the binomial distribution in Eq. (16) with a normal distribution:
\[ P_n(\geq n; t) \approx \int_0^n \frac{dm}{2\pi p_{10}} \left( \Theta[\nu_0 t' - m] - \Theta[\nu_0 t' - np_{10}] \right) \exp \left[ -\frac{(m - np_{10})^2}{2np_{10}(1 - p_{10})} \right], \quad (18) \]
the last equality following from the large $n$ limit, in which the gaussian for the variable $m/(np_{10})$ tends to a Dirac distribution centered on unity.

Therefore,
\[ P_n(\geq n; t) \approx \Theta \left[ \frac{t}{p_{10}^{-1} + \Delta \tau} - n \right], \quad (19) \]

\text{hence}
\[ P_n(n; t) \approx \delta \left[ n - \frac{t}{\langle \tau \rangle} \right], \quad (20) \]

where $\langle \tau \rangle = p_{10}^{-1} + \Delta \tau$ gives the mean waiting time between two jumps, including the effect of trapping. The distribution is then given by
\[ p^2 f(p, t) = G_{\nu_0 t'}(p|p_{10}, \Delta \tau/\langle \tau \rangle) \approx G_{\nu_0 t'}(p|p_{10}, t). \quad (21) \]

Most of the particles are thus pushed to large Lorentz factors with $(p) = p_{10} \exp(4\nu_0 t'/\langle \tau \rangle)$, as in standard stochastic acceleration, except that the effective rate of acceleration has
\[ 1 - \frac{\Gamma[m, \nu_0 t']}{\Gamma(m)} \approx \frac{(\nu_0 t')^{n}}{\Gamma(m + 1)}. \quad (22) \]
approximate the sum in Eq. (13) with an integral, giving
\[ p^2 f(p, t) \approx -\ln(1 - p_{10}) \int_1^{\infty} d\nu \exp(\nu(1 - p_{10}) / (p_{10} n\Delta t)) \]
\[ = \left( \frac{p}{p_{10}} \right)^{\frac{1}{2}} \exp\left( \frac{\nu(1 - p_{10})}{2p_{10} \nu^2} \right) \ldots, \tag{24} \]
noting that \( \ln(1 - p_{10}) = -\nu_{10}\Delta t \). The terms in brackets that have not been explicitly match exactly the remainder of the solution for \( f_0(p, t) \), as written in Eq. (10). We thus obtain the same powerlaw behavior as before, to leading order in \( \nu_{10}^2 \).

In Figs. 3, 4, 5 and 6, we provide illustrations of this discretized process, as obtained from numerical Monte Carlo calculations, for various choices of parameters. Figure 3 shows the typical trajectories of particles in the presence of traps and illustrates how those particles that escape the traps can reach very high energies, while those that get trapped at some given time effectively escape the acceleration process, at least on timescale \( \sim \nu_{10}^{-1} \). Note that the exponential time behavior of \( p \) is typical of the dependence \( D_{pp} \propto p^2 \), which we have assumed here.

Figure 4 plots spectra at a given time for different values of \( \nu_{10}/\nu_{acc} \), and compares them to the analytical approximation Eq. (24), which applies in this case since \( \nu_{10}^2 \ll 1 \). The correspondence is quite satisfactory. Figure 5 shows spectra at different times, for given values of \( \nu_{10}/\nu_{acc} \) and \( \nu_{10}/\nu_{acc} \), with \( \nu_{10}^2 \ll 1 \). This figure confirms that the spectrum takes the form of a powerlaw, whose spectral index does not change in time on those timescales. Finally, Fig. 6 shows how such spectra evolve on longer timescales, in particular \( \nu_{10}^2 \gtrsim 1 \), slowly departing from a powerlaw to converge towards the (pile-up form) solution without traps, given in Eq. (21).

### 3 STOCHASTIC ACCELERATION WITH LÉVY TRAPS

Lévy flights are characterized by heavy-tailed distributions whose mean waiting time is infinite. Among this class of distributions, the family of one-sided stable distribution functions behaves as an attractor for the sums of (identically distributed) random variables, in analogy with the normal distribution for finite variance. For a positive real variable \( \nu \) (time), these distributions \( L_\nu(\nu) \) are defined through their characteristic function \( L_\nu(\nu) \), which is itself characterized by a real parameter \( \alpha \in [0, 1] \),
\[ L_\alpha(\omega) \equiv \int \hat{f} e^{i\nu\hat{\nu}} L_\nu(\nu) = \exp\left[ -i\omega|\hat{\nu}|^\alpha \right], \tag{25} \]
with \( \zeta = \exp(\pm i\pi/2) \), and \( \hat{\nu} = \nu / \Delta \); \( \Delta \) represents a reference timescale for the timestep duration. In the large time limit, \( L_\alpha(\nu) \propto \nu^{1-\alpha} \), hence the average waiting time is indeed infinite for \( \alpha < 1 \). In the following, we restrict our study to this family of distribution functions.

Consider therefore the following setup: at each step \( n \), the momentum of the particle jumps by a random quantity \( \Delta \nu \), which is distributed according to Eq. (12) as before. However, before entering a new cycle of energy gain, the particle lingers over a time interval \( \Delta t \), with \( \Delta t \) distributed according to \( L_\alpha(\nu) \). We should stress that Eq. (12) computes...
\( \Delta p \) assuming that the particle undergoes acceleration over a time interval \( \Delta t \), while the actual time spent between two cycles, in the present description, is \( \hat{t} \Delta t \), and \( \hat{t} \) can in principle be smaller than unity. However, \( L_\alpha(\hat{t}) \) goes rapidly to zero for values \( \hat{t} \) smaller than unity, so that this choice does not impact our conclusions. Some typical trajectories in momentum space are shown in Fig. 7 for \( \alpha = 0.5 \) and \( \nu_{\rm acc} \Delta t = 0.1 \).

Formula (13) remains valid in the present case, and the probability \( dP(p; n)/dp \) is still given by the propagator \( G(p|\rho_0, n\Delta t) \). We compute the probability of the particles executing at least \( n \) energy gaining jumps within an interval \( t \) as the probability of having the sum of the \( n \) waiting times less than \( t \). The sum of \( n \) variables, each distributed according to \( L_\alpha(\hat{t}) \), is distributed as \( n^{-1/\alpha}L_\alpha(\hat{t}/n^{1/\alpha}) \). Consequently, the probability of achieving at least \( n \) jumps within \( t \) can be written as:

\[
P_n(n; t) = \int_0^t \frac{d\tau}{n^{1/\alpha}} L_\alpha \left( \frac{\tau}{n^{1/\alpha}} \right)
\]

In the large-\( n \) limit, we obtain the probability \( P_n(n; t) \) from (minus) the derivative with respect to \( n \), which gives

\[
P_n(n; t) \simeq \frac{1}{\alpha} \hat{t} n^{-1-\frac{1}{\alpha}} L_\alpha \left( \hat{t} n^{-\frac{1}{\alpha}} \right).
\]

Approximating the discrete sum in Eq. (13) with an integral, we obtain

\[
p^2 f(p) \simeq \int_0^\infty dn \left( \hat{t} n^{-1-\frac{1}{\alpha}} L_\alpha \left( \hat{t} n^{-\frac{1}{\alpha}} \right) G_{\nu_{\rm acc}}(p|\rho_0, n\Delta t) \right).
\]

We can obtain useful approximations to this expression by changing variables for \( u = n^{-1/\alpha} \), and breaking the integrals into two parts, one over the integral \( u \in [0, 1] \), the other over \( u \in [1, \hat{t}] \) and using both the small and large argument approximations for \( L_\alpha(x) \) (Mikusiński 1959; Penson & Górska 2010; Saa & Venereoles 2011):

\[
\begin{align*}
L_\alpha(x) &\simeq \left( \frac{\alpha}{2\pi(1-\alpha)} \right)^{\frac{1}{2}} x^{\frac{2-\alpha}{2(1-\alpha)}} \exp \left( -(1-\alpha)\frac{\alpha}{\pi} x^{\frac{\alpha}{\pi}} \right), \\
L_\alpha(x) &\simeq \frac{1}{\pi} \sum_{k=1}^{k_{\rm max}} \frac{(-1)^{k+1}}{k!} x^{-1-\alpha} \Gamma(1+\alpha k) \sin(\pi \alpha k).
\end{align*}
\]

With \( k_{\rm max} \rightarrow +\infty \), the sum converges to \( L_\alpha(x) \) for all \( x \), but convergence is slow at \( x \ll 1 \) and the former expression is more useful. For the calculations that follow, it suffices to choose \( k_{\rm max} \sim 3 \) in practice. Define the integral

\[
I_{\nu_{\rm acc}}(\kappa, \mu, \nu, \rho) \equiv \int_0^\mu du u^{\kappa} \exp \left[ -\mu u - \nu u^\alpha - \rho u^{-\frac{\alpha}{\pi}} \right].
\]

as well as \( \hat{v} = \nu_{\rm acc} \Delta t \) and \( q_p = \log(p/p_0)^2/(4\hat{v}) \). We obtain

\[
p^2 f(p, t) \simeq \frac{(p/p_0)^{1/2}}{\sqrt{4\pi \hat{v}}} \int_0^\infty \frac{dq_p}{q_p^{1/2}} \\
\times \left( 1 + \hat{v} \int_0^\mu \frac{du}{u} u^{\kappa} \exp \left[ -\mu u - \nu u^\alpha - \rho u^{-\frac{\alpha}{\pi}} \right] \right).
\]

Although the expression appears cumbersome, it boils down to the evaluation of a few integrals and offers a convenient expression for the resulting spectrum, as shown in Figs. 8 and 9. Figure 8 shows some spectra of accelerated particles for \( \alpha \in [0.3, 0.5, 0.7, 0.9, 2] \) at a given time \( \nu_{\rm acc} = 2.5 \). The overall shape is close to a powerlaw for small values of \( \alpha \), as expected at early times. The value \( \alpha = 2 \) implies a finite mean waiting time, and the solution therefore converges to the standard propagator of the Fokker-Planck equation without traps. As \( \hat{t} \) increases, the functional shape of the distribution function evolves in a non-trivial way for all values of \( \alpha \), as illustrated in Fig. 9.

We note that the integral over the interval \([0, 1]\) provides the scaling at large \( p \), while that over \([1, \hat{t}]\) determines the low \( p \) behavior. In principle, one can further approximate these integrals, e.g., through steepest descent, but no strict powerlaw emerges from the resulting expression. The general scaling is that of an exponential of some power of \( \ln(p/p_0) \), which reproduces the rough powerlaw behavior seen in Figs. 8 and 9.

4 DISCUSSION

4.1 Other approaches

In principle, the Fokker-Planck equation can contain both diffusion and advection terms, e.g.

\[
\frac{\partial}{\partial t} f(p, t) = -\frac{1}{p^2} \frac{\partial}{\partial p} \left( A_p f(p, t) \right) + \frac{1}{p^2} \frac{\partial}{\partial p} \left( D_{pp} p^2 \frac{\partial}{\partial p} f(p, t) \right),
\]

with \( A_p \) the advection coefficient, here carrying the same dimensions as \( D_{pp} \). \( A_p \) may of course depend on momentum;
the analytical approximation Eq. (31) for walks with Lévy distributed waiting times, as characterized by the Figure 8.

random walk with a gaussian distribution of waiting time. For simplicity, we assume here $A_p \propto p^2$, unless the advection is strongly negative, $A_p = \nu_{\text{adv}} p^2$. This Fokker-Planck equation has a solution, see Becker et al. (2006), which generally retains a pile-up form. In particular, the mean momentum evolves as

$$\langle p \rangle = p_0 \exp \left( \nu_{\text{adv}} + 4 \nu_{\text{acc}} \right) t. \quad (33)$$

Its evolution is thus exponential in time for our choice unless the advection is strongly negative, $A_p = -4D_{pp}$. For such a choice of transport coefficients, one can show that the stationary distribution function scales according to $p^2 f_s(p) \propto p^{-1}$ at $p \ll p_0$ and $p^2 f_s(p) \propto p^{-2}$ at $p \gg p_0$. Therefore, a powerlaw shape is preserved, but most particles accumulate at low momenta.

In order to interpret the results of their PIC simulations of stochastic acceleration, Wong et al. (2020) have measured the advection and diffusion coefficients and shown that the numerical solution of the Fokker-Planck equation determined with those coefficients reproduce satisfactorily the observed spectra. In their case, $D_{pp} \propto p^2$ but the advection coefficient has a non-trivial sign (positive at low momenta, negative at large momenta), a non-trivial energy dependence and its physical origin is not obvious.

In our model of Sec. 2, this advection coefficient was set to zero, but the presence of trapping allowed to recover the general powerlaw behavior seen in similar numerical simulations. In our view, the present description is more physically motivated than an ad-hoc choice of an advection coefficient, and it also offers a simpler way of extracting physical solutions. For reference, we remark that $A_p = 0$ matches the prediction of quasilinear theory in the diffusion approximation (Schlickeiser 1989).

Regarding the Lévy random walk model, most studies consider Lévy jumps for the momentum (or for the position, when spatial transport is considered) with fixed jumps in time, see for instance Zimmer & Perri (2013), Isliker et al. (2017b) and Isliker et al. (2017a). In Sec. 3, we have rather considered momentum jumps characterized by a diffusive propagator, with waiting times distributed according to a Lévy distribution. Both choices are possible, in principle, and they have different physical meaning. In stochastic acceleration, the typical momentum gain is $(\Delta p) \sim u^\nu p$ per scattering event, $u$ denoting the velocity of the scattering center. Hence, a Lévy walk in momentum space at fixed time intervals might represent a situation in which the velocities are distributed according to some heavy-tailed distribution.

Such models typically produce hard powerlaw. To see this, consider Eq. (13) with jumps of fixed size in time, for instance $P_n(n, t) = \delta (n - \nu_{\text{acc}} t)$. We assume that at each jump, the log-momentum changes by an amount $\Delta \ln p = \Delta \ln p_0 \hat{a}$, where $\hat{a}$ is distributed as $L_{\beta}(\hat{a})$ and $\Delta \ln p_0$ is a reference scale for jumps in momentum. Consequently, $dP_n(p, n)/dp = p^{-1} \Delta \ln p_0 \beta^{-1} n^{-1/\beta} \Gamma\left(n^{-1/\beta}\right) \hat{a}$, which is strongly suppressed at momenta such that $\hat{a} \lesssim n^{1/\beta}$ and which scales as $p^{-1} n (p/p_0)^{1/\beta}$ at larger momenta. Such distributions cannot therefore reproduce the soft powerlaws seen in numerical simulations.

4.2 Consequences for phenomenology

So far, our discussion has concerned the time evolution of the distribution function for particles subject to stochastic acceleration only, without considering the possible impact of escape, or even energy losses. Such loss terms nevertheless play an important in shaping the spectra in phenomenological applications, see e.g., Schlickeiser (1984) or Stawarz & Petsiosian (2008). Without entering into the details, we wish to discuss here how the above distribution functions evolve on long time scales, once possible escape losses are considered. We assume, for simplicity, that the scattering timescale of the particles is independent of momentum, a choice consistent with our scaling $D_{pp} \propto p^2$. This implies that escape can be characterized by a momentum-independent scattering frequency $v_{\text{esc}}$. 

![Figure 8](image_url)

Figure 8. Distributions $p^2 f(p, t)$ as a function of $p/p_0$ for random walks with Lévy distributed waiting times, as characterized by the parameters $\alpha = 0.3, 0.5, 0.7, 0.9, 2$ (from soft to hard, or blue to red) at time $\nu_{\text{acc}} t = 2.5$, with $\Delta \nu_{\text{acc}} = 0.1$. The dotted lines show the analytical approximation Eq. (31) for $k_{\max} = 3$. For $\alpha = 2$, we recover the standard pile-up distribution, as expected for a random walk with a gaussian distribution of waiting time.

![Figure 9](image_url)

Figure 9. Spectra for the Lévy random walk model, plotted for a given value $\alpha = 0.5$ at different times $\nu_{\text{acc}} t = 0.2, 0.4, 1., 2.5, 6.3$, with $\Delta \nu_{\text{acc}} = 0.1$. As expected, the spectra become harder as the ratio $t/\Delta t$ increases.
4.2.1 Stochastic acceleration in the presence of traps and escape

The model that we have developed in Sec. 2 can be directly generalized to this case. Consider for instance the analytical solution Eq. (7). To account for escape losses, we include a distribution $f_{\text{esc}}$ that characterizes the population of particles that have escaped the system and rewrite Eq. (6) as follows,

$$
\frac{\partial}{\partial t} f_{\text{esc}}(p, t) = +\nu_{\text{esc}, 0} f_0(p, t) + \nu_{\text{esc}, 1} f_1(p, t),
$$

$$
\frac{\partial}{\partial t} f_0(p, t) = -(\nu_{01} + \nu_{\text{esc}, 0}) f_0(p, t) + \nu_{10} f_1(p, t),
$$

$$
\frac{\partial}{\partial t} f_1(p, t) = +\nu_{01} f_0(p, t) - (\nu_{10} + \nu_{\text{esc}, 1}) f_1(p, t)
$$

$$
+ \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D_{pp} \frac{\partial}{\partial p} f_1(p, t) \right).
$$

(34)

Then, redefining $f_{0/1}(p, t) = \exp\left(-\nu_{\text{esc}, 0/1} t\right) g_{0/1}(p, t)$, the functions $g_0(p, t)$ and $g_1(p, t)$ obey the original system (6), so that their solution is given by Eq. (7), while

$$
f_{\text{esc}}(p, t) = \int_0^t \frac{d \tau}{\nu_{\text{esc}, 0}} e^{-\nu_{\text{esc}, 0} \tau} g_0(p, \tau) + \nu_{\text{esc}, 1} e^{-\nu_{\text{esc}, 1} t} g_1(p, t),
$$

(35)

and the integral can be actually carried out explicitly in Eq. (7) to obtain an expression for $f_{\text{esc}}$ that is similar to that for $f_0$ and $f_1$. In the following, we assume $\nu_{\text{esc}, 0} = \nu_{\text{esc}, 1} = \nu_{\text{esc}}$ for simplicity.

We can expect the following behavior. Consider first the limit $\nu_{\text{esc}} \gg \nu_{01}$. Then $f_0(p, t) \ll f_1(p, t)$ and $f_{\text{esc}}(p, t)$, because particles have a larger probability of escaping the system than being trapped for some time in the $f_0$ population. This situation is thus similar to that discussed in Sec. 2 with $\nu_{01} \rightarrow 0$, since particles that escape the system never reenter it. We introduce the superscript $\text{(0)}$ to index parameters and distribution functions in the absence of escape losses (Sec. 2). We can obtain an approximate solution from Eq. (10), provided we make the substitutions $\nu_{t0}^\text{(0)} \rightarrow \nu_{\text{esc}}$ and $f_0^\text{(0)} \rightarrow f_{\text{esc}}$. Its powerlaw index becomes a function of the ratio $\nu_{\text{esc}}/\nu_{\text{acc}}$ and the powerlaw shape holds at all times, even when $\nu_{\text{esc}} \gg 1$.

Consider now the limit $\nu_{\text{esc}} \ll \nu_{01}$. We assume $\nu_{01} < \nu_{t0} \ll \nu_{\text{esc}}$ as before. In this case, we need to consider three regimes, $t \ll \nu_{01}^{-1}$, $\nu_{01}^{-1} < t < \nu_{\text{esc}}^{-1}$ and $t \gg \nu_{\text{esc}}^{-1}$. In the first two, $f_{\text{esc}} \ll f_0 + f_1$, since $\nu_{\text{esc}} \ll 1$. The behavior of $f_0$ and $f_1$ is adequately described by our earlier solution, Eq. (7), given that escape losses can be neglected on those early timescales. Correspondingly, the solution is a powerlaw at early times ($\nu_{01} t \ll 1$) and it evolves toward an evolving pile up distribution at late times ($\nu_{01} t \gg 1$).

At late times, $f_1 \ll f_0$ and $f_0$ becomes a pile up distribution, as we have seen (Fig. 2). Therefore, the present system including $f_{\text{esc}}$ is similar to that studied previously, if we make the substitutions $\nu_{t0}^\text{(0)} \rightarrow 0$, $\nu_{01}^\text{(0)} \rightarrow \nu_{\text{esc}}$, $\nu_{t0}^\text{(esc)} \rightarrow \nu_{\text{esc}}/(1 + \nu_{01}/\nu_{t0})$, $\nu_{01}^\text{(esc)} \rightarrow \nu_{\text{esc}}$ and $\nu_{10}^\text{(esc)} \rightarrow f_{\text{esc}}$. The replacement $\nu_{\text{esc}} \rightarrow \nu_{\text{esc}}/(1 + \nu_{01}/\nu_{t0})$ must be introduced, as when $\nu_{\text{esc}} \gg 1$, acceleration proceeds with a rate that is effectively $\nu_{\text{esc}}/(1 + \nu_{01}/\nu_{t0})$, not $\nu_{\text{esc}}$, see Eq. (21). Consequently, for $\nu_{\text{esc}} \ll 1 \ll \nu_{01}$, we expect to recover a pile up distribution for $f_0$ and a powerlaw for $f_1$, in accord with Eq. (10).

With the above substitutions, this implies a powerlaw for $p^2 f_{\text{esc}}(p, t)$ of exponent $1/2 - [9/4 + \nu_{\text{esc}}/\nu_{\text{acc}} (1 + \nu_{01}/\nu_{t0})]^{1/2}$. At late times, $\nu_{\text{esc}} \gg 1$, $f_0$ disappears and only the powerlaw for $f_{\text{esc}}$ remains.

We illustrate this behavior with Fig. 10, which shows the time evolution of $f_0 + f_1$ and $f_{\text{esc}}$ for the choice $\nu_{\text{esc}} = 0.4, 10, 80$, for $\nu_{01}/\nu_{\text{acc}} = 0.2$, $\nu_{01}/\nu_{\text{esc}} = 6$, and $\nu_{\text{esc}}/\nu_{\text{acc}} = 0.05$. At early times, both $f_{\text{esc}}$ and $f_0 + f_1$ scale as powerlaws, whose index is given by Eq. (10), as expected. At late times, $f_0 + f_1$ evolve toward pile up distribution, while $f_{\text{esc}}$ retains a powerlaw shape, but with a different index, close to that predicted above in terms of $\nu_{\text{esc}}/\nu_{\text{acc}}$ and $\nu_{01}/\nu_{t0}$.

4.2.2 Lévy random walks including escape terms

Consider now a Lévy random walker, including the possibility of escape at frequency $\nu_{\text{esc}}$. This means that, at each jump in momentum, $p \rightarrow p + dp$, the particle has a probability $p_{\text{esc}} = 1 - \exp(\nu_{\text{esc}} |\Delta t|)$ of escaping the system, where $\Delta t$ is distributed according to the stable distribution $L_{\alpha}(t)$. This effect can be easily included in the discretized random walk, and its impact on the spectrum can be estimated as follows.

At early times, $\nu_{\text{esc}} \ll 1$, escape plays little role and it can be neglected. The shape of the spectrum is therefore not modified with respect to that obtained in Eq. (31). At late times, $\nu_{\text{esc}} \gg 1$, escape is bound to shape the spectrum and to turn it into an approximate powerlaw. A first approximation for its exponent can be obtained from Eq. (5), namely $-1 + \ln(1 - \nu_{\text{esc}} p)/g$, where $g \approx 4\nu_{\text{esc}} \Delta t$ represents $\Delta \ln p$. The escape probability is given by

$$
p_{\text{esc}} \approx \int_0^{\nu_{\text{esc}} \Delta t} \frac{d \Delta t}{\nu_{\text{esc}} \Delta t} \nu_{\text{esc}} \Delta t \sim \frac{1}{\pi} (\nu_{\text{esc}} \Delta t)^{\frac{\alpha}{2}} \Gamma(\alpha) \sin(\alpha \pi).
$$

(36)
Figure 11. Evolution in time (ordered from red to blue, or left to right) of the distribution function for a random walk in momentum space with Lévy waiting times, for $\alpha = 0.5$, $\nu_{\text{acc}} \Delta t = 0.1$ and $\nu_{\text{esc}} t = 1, 2.5, 6.3, 13, 28$, including escape at frequency $\nu_{\text{esc}} = 0.3 \nu_{\text{acc}}$. The dotted lines represent the analytical solutions without escape, while the dashed line shows the powerlaw with the index discussed in the text, function of $\alpha$ and $\nu_{\text{esc}}$. At late times, the numerical solution converges to a powerlaw, with an index slightly steeper (exponent $= -1.5$ for $p^2f$) than our simple estimate ($-1.3$).

It thus depends on $\alpha$, giving a spectrum that is harder with increasing $\alpha$.

In Fig. 11, we plot the evolution of the distribution function in the presence of escape, for the case $\alpha = 0.5$. As before, we assume $\nu_{\text{acc}} \Delta t = 0.1$, and we choose here $\nu_{\text{esc}} = 0.3 \nu_{\text{acc}}$. At early times $\nu_{\text{esc}} t = 1, 2.5$ corresponding to $\nu_{\text{esc}} t < 1$, the solution agrees with the analytical solution without escape, as anticipated. At later times, $\nu_{\text{esc}} t > 1$, while the distribution for $\nu_{\text{esc}} = 0$ [Eq. (31)] departs from a powerlaw form and becomes harder and harder, the numerical solution that considers finite escape losses converges to a powerlaw with an index that is not very different from our prediction ($-1.5$ measured for $\alpha = 0.5$ vs $-1.3$ predicted by the above estimate).

5 CONCLUSIONS

This paper addresses the physics of stochastic particle acceleration for continuous-time random walks, in which the time span that separates two energy-jump events is distributed as a continuous random variable. This study is motivated by the result of recent numerical simulations of particle acceleration in magnetized turbulence, which have produced powerlaw spectra where pile-up distributions were theoretically expected. As we have argued in Sec. 1, such an observation is an indication for the existence of some “trapping”, which inhibits acceleration for some of the particles, and as such, acts as a form of escape on the finite timescale of those simulations. The powerlaw then results from the competition between energy gain and escape/trapping, a common trait of Fermi-type acceleration.

This segregation of particles is likely related to a non-trivial dependence on the acceleration rate on phase space variables other than the momentum, e.g. the pitch-angle of the particle, or its spatial position. Both dependencies are indeed averaged out when one considers momentum diffusion only. Our description of stochastic acceleration in terms of continuous-time random walks provides a simple way to describe the consequences of such hidden dependencies.

In Sec. 2, we have discussed random walks with finite mean waiting time, considering in particular distributions characterized by two timescales of acceleration, one slow and one fast. We have shown that a powerlaw indeed emerges as a natural consequence of stochastic acceleration if the timescale on which one probes the distribution function, e.g. the simulation timescale, is shorter than the slow timescale. The slope of the powerlaw can then be expressed in terms of the fast acceleration timescale and of the typical time over which a given particle transits into the region of phase where acceleration takes place on the slow timescale. On longer timescales, the distribution of accelerated particles converges to a pile-up distribution, as expected, albeit with an effective acceleration timescale which is significantly enlarged by the trapping. We have provided a general analytical solution for the distribution function as well as simplified analytical estimates in both limits.

In Sec. 3, we have discussed the other general class of continuous-time random walks, that of heavy-tailed distributions of waiting time, with infinite mean. We have considered in particular one-sided Lévy-stable distributions, which behave as attractors for that class of distribution functions. Here as well, we have provided analytical estimates which match dedicated numerical Monte Carlo simulations of the stochastic process. By construction, one cannot define here a slow and a fast timescale. The distribution cannot therefore be fully described by a powerlaw at high energies, although the running of the powerlaw exponent with momentum is rather mild. As one waits longer and longer, the distribution becomes harder and harder, until the mean momentum itself starts to increase, the distribution then turning into a pile-up form.

Our study thus provides a simple interpretation of the observation of powerlaws in recent numerical simulations and it clearly highlights the need for an improved understanding of the possible hidden dependencies of the acceleration rate. If confirmed by future numerical experiments, the shape and time dependence of the accelerated distribution could be used to characterize the distribution of waiting times. Our results can be generalized and applied to concrete astrophysical scenarios, by adding in the possible influence of energy losses, escape losses etc. As an illustration, we have discussed the influence of escape losses assuming a momentum-independent scattering timescale, and shown that such losses lead to a softened powerlaw distribution.

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DATA AVAILABILITY

The data underlying this article are available in the article.

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APPENDIX A: ANALYTICAL SOLUTION FOR MODEL 1

To solve the system of equations (6), we perform a Laplace transform in time to write (λ denotes the Laplace conjugate variable for t)

\[
\begin{align*}
\frac{(λ + ν_{01}) f_{0} - ν_{10} f_{1}}{λ + ν_{01}} &= \frac{d f_{0}}{d (p ν)} \\
\left(\frac{λ + ν_{01}}{λ + ν_{01}} \right) f_{0} &= \frac{1}{p ν} \left( ν_{acc} \frac{∂}{∂ p ν} f_{1} \right) + f_{0}^{10},
\end{align*}
\]

where \( f_{0}^{10} = f_{0}(p, t = 0) \) and \( f_{1}^{10} = f_{1}(p, t = 0) \). It proves convenient to switch variables from \( p \) to \( q = p^{-3} \), which leads to

\[
\begin{align*}
\int_{f_{0}}^{f_{1}} \frac{∂^{2}}{∂ q^{2}} \tilde{f}_{1} - \frac{λ}{ν_{acc}} \left( 1 + \frac{ν_{10}}{λ + ν_{01}} \right) \tilde{f}_{1} &= \frac{1}{ν_{acc}} \frac{ ν_{01}}{λ + ν_{01}} \tilde{f}_{1}^{0} - \frac{1}{ν_{acc}} f_{0},
\end{align*}
\]

The Green’s function \( F(q, q') \) such that

\[
q \frac{∂^{2}}{∂ q^{2}} F - T F = (q - q'), \quad (A1)
\]

where

\[
\Gamma_{1}(q, λ) = \frac{λ}{ν_{acc}} \left( 1 + \frac{ν_{10}}{λ + ν_{01}} \right), \quad (A2)
\]

can be expressed as

\[
F(q, q') = -\frac{v_{10}}{2q_{01}^{3} \sqrt{v_{10}} + \frac{1}{2} \sqrt{v_{10}} \sqrt{v_{acc}}} e^{-\frac{[ln(q) + ln(q')]}{2v_{10}}} \quad (A3)
\]

Hence, the Laplace transform of \( f_{1}(p, t) \) can be written as

\[
\tilde{f}_{1}(q, λ) = \frac{1}{ν_{acc}} \int dq_{0} F(q, q') \Gamma_{1}(q', λ), \quad (A4)
\]

with

\[
\Gamma_{1}(q', λ) = \left( \frac{ν_{01}}{λ + ν_{01}} f_{0}^{10} + f_{1}^{10} \right). \quad (A5)
\]

The initial distributions are evaluated at \( p' = q'^{-1/3} \) in the above expression. In the following, we consider initial data.
of the form $\delta(p - p_0)$ and thus operate the substitution $f_0^0 \to f_0^0 p_0 \delta(p - p_0)$. $f_1^0 \to f_1^0 p_0 \delta(p - p_0)$ to obtain

$$f_1(q, \lambda) = \frac{\sqrt{q/q_0 \cdot \Gamma(Y) + \frac{1}{4}}}{6v_{\text{acc}} \Gamma(q_0; \lambda)}.$$  

(A6)

To simplify further the notations, we define $x = |\ln(q/q_0)|$. The distribution function $f_1$ is then obtained through the inverse Laplace transform

$$f_1(p; t) = \frac{\sqrt{q/q_0}}{12\pi v_{\text{acc}}} \int_L \frac{e^{i \sqrt{T(Y)} + \frac{1}{4} x}}{\sqrt{T(Y) + \frac{1}{4}}} \Gamma(q_0, \lambda),$$  

(A7)

hence the solution for $f_0$:

$$f_0(p; t) = e^{-\nu_0 t} f_0^0 + \frac{\sqrt{q/q_0}}{12\pi v_{\text{acc}}} \int_L \frac{e^{i \sqrt{T(Y)} + \frac{1}{4} x}}{\sqrt{T(Y) + \frac{1}{4}}} \Gamma(q_0, \lambda).$$  

(A8)

with

$$\Gamma_0(q, \lambda) = \frac{\nu_0}{\lambda + \nu_0} \Gamma(q, \lambda) = \frac{\nu_0 \nu_0}{(\lambda + \nu_0)^2} f_0^0 + \frac{\nu_0}{\lambda + \nu_0} f_1^0.$$  

(A9)

The Bromwich integrals are of the form

$$I = \frac{1}{2\pi i} \int_L \frac{e^{i \sqrt{T(Y)} + \frac{1}{4} x}}{\sqrt{T(Y) + \frac{1}{4}}} \Gamma(Y).$$  

(A10)

and contain branch cuts on the negative real axis where the argument of the square root $\sqrt{T(Y) + \frac{1}{4}}$ becomes negative. In detail,

$$\sqrt{T(Y) + \frac{1}{4}} = \frac{1}{3\sqrt{\text{acc}}} \sqrt{(\lambda - \lambda_+)(\lambda - \lambda_-)} \frac{1}{\lambda + \nu_0},$$  

(A11)

with

$$\lambda_{\pm} = \frac{1}{2} \left( \sqrt{\nu_0 + \frac{9}{4} \nu_{\text{acc}} + \nu_0} \right) \pm \sqrt{\left( \nu_0 + \frac{9}{4} \nu_{\text{acc}} - \nu_0 \right)^2 + 4\nu_0 \nu_0} \right)^{1/2}. $$  

(A12)

Both roots are negative and ordered according to $\lambda_- < -\nu_0 < \lambda_+$. The branch cuts are at $\Re \lambda < \lambda_-$ and $-\nu_0 < \Re \lambda < \lambda_+$, which gives the two contours of integration $C_0$ and $C_1$

picture on Fig. A1. We thus obtain

$$I = \frac{1}{\pi} \left[ \int_{-\lambda_-}^{0} + \int_{-\lambda_-}^{\infty} \right] \frac{\cos \sqrt{T(Y) + \frac{1}{4}}} {\sqrt{T(Y) + \frac{1}{4}}} \Gamma(Y).$$  

(A13)

The poles of $\Gamma(-\lambda)$ at $\nu_1$ (which appear up to second order in $f_0$ and $f_1$) do not provide any additional contribution to the contour.

Finally, changing variables $\lambda = s\nu_{\text{acc}}$, and defining, for the sake of clarity

$$\Sigma(s) = \sqrt{s + \frac{1}{\nu_{\text{acc}}}} \left(s + \frac{1}{\nu_{\text{acc}}} \right),$$  

(A14)

we obtain:

$$f_0(p; t) = e^{-\nu_0 t} f_0^0$$

$$+ \frac{1}{2\pi} \left( \frac{p}{\nu_0} \right)^{-3/2} \int_{-\lambda_-}^{0} \int_{-\lambda_-}^{\infty} \frac{\left[ \nu_0 \nu_{\text{acc}} \right]} {\nu_0 - \nu_0} \left[ \nu_0 \nu_{\text{acc}} \right] \Gamma(Y)$$

$$\frac{\cos \left[ \frac{\nu_0 \nu_{\text{acc}} \left( \nu_0 \nu_{\text{acc}} \right)} {\nu_0 - \nu_0} \right]} {\nu_0 - \nu_0} \Gamma(-\lambda).$$  

(A15)

In the limit $\nu_0 \to 0$, $\lambda_+ \to 0$ hence the integral over the contour $C_1$ vanishes, and $\lambda_- \to -\left( \nu_0 + \frac{9}{4} \nu_{\text{acc}} \right)$. Assuming for simplicity $f_0^0 = 0$, changing variables $y = \Sigma(s)$, we obtain

$$f_1(p; t) = \frac{1}{\pi} \left( \frac{p}{\nu_0} \right)^{-3/2} \int_0^{\infty} dy e^{-\left( \nu_0 + \nu_{\text{acc}} \right)} \nu_{\text{acc}} y^2$$

$$\frac{\cos \left[ \ln(p/\nu_0) \right]} {\nu_0 - \nu_0} \Gamma(Y)$$

$$\frac{1}{2\nu \nu_{\text{acc}}} \left( \frac{p}{\nu_0} \right)^{-3/2} e^{-\left( \nu_0 + \nu_{\text{acc}} \right)} \nu_{\text{acc}} y^2.$$

(A16)

The distribution for $f_0$ can be obtained in a similar way, although it proves more convenient to directly integrate Eq. (6) in this case. The resulting expression is given in the main text, see Eq. (10).

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