Complexity of Bose-Hubbard Model: Quantum Phase Transition

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Abstract

The operator approach is applied to investigate the complexity of Bose-Hubbard model. We present a systematic method to expand the quantum complexity in series of coupling constant. We first study 2-sites system. For the ground state we can find the exact value of complexity which is the summation of all order. The complexity is divergent at critical value of coupling constant that indicates the quantum phase transition at this point. We then generalize the method to the N-sites closed chain and any dimensional system. The found properties are similar to those in 2-sites system. We also study the excited state and present the general formulas of Bose-Hubbard model complexity, which shows a similar form as that in $\lambda\phi^4$ theory studied in our previous paper.

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1 Introduction

Complexity is a quantity expected to probe the non-local property of a quantum system besides the entanglement entropy \[1, 2\]. In the context of the holography the complexity=volume (CV) conjecture \[3\] and complexity=action (CA) conjecture \[4, 5\] were proposed to calculate the quantity. The complexity in there is the volume of an extremal codimension-one bulk surface and gravitational action evaluated on the Wheeler-DeWitt (WDW) patch anchored on a time slice of the boundary respectively.

Complexity have been studied in the quantum field theory \[6, 7, 8, 9, 10, 11\]. The complexity therein is defined as the number of operations \(\{O^I\}\) needed to transform a reference state \(|\psi_R\rangle\) to a target state \(|\psi_T\rangle\). These operators are also called as quantum gates: the more gates one needs, the more complexity the target state has. To calculate the complexity one defines an affine parameter “s” associated to an unitary operator \(U(s)\) and use a set of function, \(Y^I(s)\), to characterize the quantum circuit. The unitary operation connecting the reference state and target state is

\[
U(s) = \mathcal{P} e^{\int_0^s Y_I(s) O_I}, \quad |\psi_R\rangle = U(0)|\psi_R\rangle, \quad |\psi_T\rangle = U(1)|\psi_R\rangle, \quad (1.1)
\]
where $\vec{P}$ is a time ordering along $s$. The complexity $C$ and circuit depth $D[U]$ (cost function) are [6]

\[
C = \min_{\{Y^1\}} D[U], \quad D[U] = \int_0^1 ds \sum_I |Y^I(s)|^2 .
\]

(1.2)

Above definitions were shown to be consistent with a gravitational computation [6].

The initial studies in field theory considered the Gaussian ground state wavefunctions in reference state and target state [6, 8, 9]. The theories studied so far are the free field theory or exponential type wavefunction in interacting model [10]. In this (wavefunction) approach, since that the excited-state wavefunction of harmonic oscillation is not pure exponential form the wavefunction approach is hard to work.

In our first paper [11] we adopt the operator approach, in which the transformation matrix between the second quantization operators of reference state and target state is regarded as the quantum gate, to evaluate the complexity in free scalar field theory. We examined the system in which the reference state is two oscillators with same frequency $\omega_f$ while the target state is two oscillators with frequency $\omega_1$ and $\omega_2$. The complexity in excited states is calculated and find that that the square of geodesic length in the general state $|N_1, N_2\rangle$ is

\[
D^2_{(N_1, N_2)} = (N_1 + 1) \left( \ln \sqrt{\frac{\omega_1}{\omega_f}} \right)^2 + (N_2 + 1) \left( \ln \sqrt{\frac{\omega_2}{\omega_f}} \right)^2 .
\]

(1.3)

The results was furthermore extended to the $N$ couple harmonic oscillators which correspond to the lattice version of free scalar field.

In our next paper [12] we included interaction to further study the complexity using the operator approach. We present a systematic method to evaluate the complexity of the $\lambda \phi^4$ field theory by the perturbation of small coupling constant. We describes the lattice scalar field as coupled oscillators. In two coupled oscillators, to the $\lambda^n$ order, the square distance of excited state between target and reference state is

\[
D^{(n)2}_{(N_1, N_2)} = (N_1 + 1) \left( \ln \sqrt{R_1^{(n)}} \right)^2 + (N_2 + 1) \left( \ln \sqrt{R_2^{(n)}} \right)^2 .
\]

(1.4)

in which $R_1^{(n)}$ and $R_2^{(n)}$ are described by simple recurrent relations

\[
R_1^{(n)} = \frac{\omega_1 + \frac{3\lambda}{2} \left( \frac{1+N_1 R_1^{(n-1)}}{2 \omega_f^2} + \frac{1+N_2 R_2^{(n-1)}}{\omega_1 \omega_2} \right)}{\omega_f + \frac{3\lambda}{2} \frac{1+N_1}{2 \omega_f^2}} , \quad R_2^{(n)} = \frac{\omega_2 + \frac{3\lambda}{2} \left( \frac{1+N_1 R_1^{(n-1)}}{\omega_1 \omega_2} + \frac{1+N_2 R_2^{(n-1)}}{2 \omega_f^2} \right)}{\omega_f + \frac{3\lambda}{2} \frac{1+N_2}{2 \omega_f^2}}
\]

(1.5)

with initial values $R_i^{(0)} = \frac{\omega_i}{\omega_f}$. We had also generalize it to the case of $N$ coupled oscillators which correspond to the lattice version of $\lambda \phi^4$ theory.

Our method in [12] is very general and can be applied to many-body model in condense matter. In this paper, along the same method, we will to study the complexity of Bose Hubbard model [13, 14].

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3The operator approach had also been used in [7, 9] to study the complexity of fermion theory.
In section II we briefly review the Bose Hubbard model. In section III we describe the method and use it to study the complexity of two-sites Bose-Hubbard model. A general perturbative formula of the associated complexity is presented. We see that the formula of complexity of Bose-Hubbard model is very similar to that of $\lambda\phi^4$ field theory. Use it we evaluate the complexity of Bose-Hubbard model and find that the complexity is divergent at a critical value of parameter $g$, which can be identified as the quantum phase transition point. In section IV we extend the method to the N-sites closed chain Bose-Hubbard model and any dimensional system. We conclude in last section.

2 Bose Hubbard Model

Quantum Phase transitions is known to occur when a coupling constant $g$ in the Hamiltonian is varied across some critical value $g = g_c$ at which the ground state dependence on $g$ becomes non-analytic \[15, 16\]. In this paper we are interesting in the quantum phase transition happens in a system of Bosons in the background of a periodic potential and a two-particle repulsive interaction. The Hamiltonian is given by

$$ H = \int d^d x \left( -\frac{\hbar^2}{2m} \psi^\dagger(x) \nabla^2 \psi(x) \right) + \frac{U_0}{2} \int d^d x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) \quad (2.1) $$

The constant $U_0$ is the short-ranged repulsion between the Bosons. At low energies, we can keep only the lowest vibrational state at each minima of $V_0(x)$ and the dynamics is given by the Bose Hubbard model \[13\]:

$$ \mathcal{H} = \frac{J}{N} \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{N} \sum_i \epsilon_i \tilde{n}_i + \frac{U}{N} \sum_i \tilde{n}_i(\tilde{n}_i - 1), \quad \tilde{n}_i = \hat{a}_i^\dagger \hat{a}_i \quad (2.2) $$

Here $\langle i, j \rangle$ refers to all nearest neighbor pairs, $J$ is a hopping term while $U$ is an on-site repulsion proportional to $U_0$. The factor $\frac{1}{N}$ is used to render $\mathcal{H}$ to be energy per site.

We consider the homogeneous case and set $\epsilon_i = 0$ at each site. This system has a superfluid-Mott insulator transition first studied by \[17\]. The insulating phase is characterized by zero compressibility and has a gap in the excitation spectrum. The Bose-Hubbard model can describe optical lattices and its properties, such as the excitation spectrum, had been experimentally probed \[18, 19\].

3 Complexity of Two-Sites Bose Hubbard Model

Hamiltonian of 2-sites Bose Hubbard Model can be written as

$$ \mathcal{H}_{1,2} = \frac{J}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2} \left( (\hat{a}_1^\dagger \hat{a}_1)^2 + (\hat{a}_2^\dagger \hat{a}_2)^2 - (\hat{a}_1^\dagger \hat{a}_1) - (\hat{a}_2^\dagger \hat{a}_2) \right) \quad (3.1) $$

in which we consider the periodic boundary as shown in figure 1.
Now, we define new operators $a$ and $b$

$$a \equiv \frac{1}{\sqrt{2}}(\tilde{a}_1 + \tilde{a}_2), \quad b \equiv \frac{1}{\sqrt{2}}(\tilde{a}_1 - \tilde{a}_2) \quad (3.2)$$

then, by $[\tilde{a}_i, \tilde{a}^\dagger_j] = \delta_{i,j}$, we have commutation relations

$$[a, a^\dagger] = \frac{1}{2}([\tilde{a}_1 + \tilde{a}_2, \tilde{a}^\dagger_1 + \tilde{a}^\dagger_2] = \frac{1}{2} + \frac{1}{2} = 1 \quad (3.3)$$

$$[b, b^\dagger] = \frac{1}{2}([\tilde{a}_1 - \tilde{a}_2, \tilde{a}^\dagger_1 - \tilde{a}^\dagger_2] = \frac{1}{2} + \frac{1}{2} = 1 \quad (3.4)$$

$$[a, b^\dagger] = \frac{1}{2}([\tilde{a}_1 + \tilde{a}_2, \tilde{a}^\dagger_1 - \tilde{a}^\dagger_2] = \frac{1}{2} - \frac{1}{2} = 0 = [a^\dagger, b] \quad (3.5)$$

$$[a, b] = [a^\dagger, b^\dagger] = 0 \quad (3.6)$$

Thus $\{a, a^\dagger\}$ commute with $\{b, b^\dagger\}$ while each set has standard commutation relation. We can then define the number operators $n_a$ and $n_b$

$$n_a = a^\dagger a, \quad n_b = b^\dagger b, \quad [n_a, n_b] = 0 \quad (3.7)$$

which define the Fock space

$$n_a |N_a\rangle = N_a |N_a\rangle, \quad n_b |N_b\rangle = N_b |N_b\rangle \quad (3.8)$$

and kinematic term $\mathcal{H}^{(0)}$ becomes

$$\mathcal{H}^{(0)}_{1,2} = \frac{J}{2}(\tilde{a}^\dagger_1 \tilde{a}_2 + \tilde{a}^\dagger_2 \tilde{a}_1) = \frac{J}{2}(a^\dagger a - b^\dagger b) \quad (3.9)$$

To proceed we begin to calculated the interaction terms:

$$\tilde{a}^\dagger_1 \tilde{a}_2 + \tilde{a}^\dagger_2 \tilde{a}_1 = \frac{1}{2}(a^\dagger + b^\dagger)(a - b) + \frac{1}{2}(a^\dagger - b^\dagger)(a + b) = a^\dagger a - b^\dagger b = n_a - n_b \quad (3.10)$$

$$\tilde{a}^\dagger_1 \tilde{a}_1 + \tilde{a}^\dagger_2 \tilde{a}_2 = \frac{1}{2}(a^\dagger + b^\dagger)(a + b) + \frac{1}{2}(a^\dagger - b^\dagger)(a - b) = a^\dagger a + b^\dagger b = n_a + n_b \quad (3.11)$$
and

\[
(\tilde{a}_1^\dagger \tilde{a}_1)^2 = \left( \frac{1}{2} (a^\dagger + b^\dagger)(a + b) \right)^2 = \frac{1}{4} \left( (a^\dagger a + b^\dagger b) + (a^\dagger b + ab^\dagger) \right)^2
\]

\[
= \frac{1}{4} \left( n_a^2 + n_b^2 + 2n_a n_b + (a^\dagger bab^\dagger + ab^\dagger a^\dagger b) + \text{irrelevant terms} \right)
\]

\[
= \frac{1}{4} \left( n_a^2 + n_b^2 + 2n_a n_b + (n_a(1 + n_b) + (1 + n_a)n_b) + \text{irrelevant terms} \right)
\]

\[
= \frac{1}{4} \left( n_a^2 + n_b^2 + 4n_a n_b + n_a + n_b + \text{irrelevant terms} \right)
\]

(3.12)

\[
(\tilde{a}_2^\dagger \tilde{a}_2)^2 = \left( \frac{1}{2} (a^\dagger - b^\dagger)(a - b) \right)^2 = \frac{1}{4} \left( (a^\dagger a + b^\dagger b) - (a^\dagger b + ab^\dagger) \right)^2
\]

\[
= \frac{1}{4} \left( n_a^2 + n_b^2 + 2n_a n_b - (a^\dagger bab^\dagger + ab^\dagger a^\dagger b) + \text{irrelevant terms} \right)
\]

\[
= \frac{1}{4} \left( n_a^2 + n_b^2 + 2n_a n_b - (n_a(1 + n_b) + (1 + n_a)n_b) + \text{irrelevant terms} \right)
\]

\[
= \frac{1}{4} \left( n_a^2 + n_b^2 - n_a - n_b + \text{irrelevant terms} \right)
\]

(3.13)

in which the "irrelevant terms" are those with different power of \(a\) and \(a^\dagger\) and/or with different power of \(b\) and \(b^\dagger\).

In this paper we investigate the quantity \(\langle N_a, N_b|\mathcal{H}_{1,2}|N_a, N_b \rangle\) for the states \(|N_a, N_b\rangle\) with fixed \(N_a\) and \(N_b\), and calculate the complexity associated to the states. In this way, only the terms that have the same power of \(a\) and \(a^\dagger\) and that have the same power of \(b\) and \(b^\dagger\) are relevant since that \(\langle N_a, N_b|\text{irrelevant terms}|N_a, N_b \rangle = 0\). Therefore we neglect the irrelevant terms hereafter.

Collect all then Hamiltonian of 2-sites Bose-Hubbard model becomes

\[
\mathcal{H}_{1,2} = \frac{J}{2} (\tilde{a}_1^\dagger \tilde{a}_2 + \tilde{a}_2^\dagger \tilde{a}_1) + \frac{U}{2} \left( (\tilde{a}_1^\dagger \tilde{a}_1)^2 - (\tilde{a}_1^\dagger \tilde{a}_1) + (\tilde{a}_2^\dagger \tilde{a}_2)^2 - (\tilde{a}_2^\dagger \tilde{a}_2) \right)
\]

\[
= \frac{J}{2} (n_a - n_b) + \frac{U}{2} \left( n_a^2 + n_b^2 + 2n_a n_b - n_a - n_b \right) + \text{irrelevant terms}
\]

(3.14)

We will use above Hamiltonian to find the complexity of the Bose-Hubbard model. We first consider the free theory without the interaction and then the interacting theory.

### 3.1 Complexity of Free Theory

Hamiltonian (3.14) is regarded as the target system which has the kinetic term and potential term. The target kinetic term is

\[
K_{a,b}^{(\text{tar})} = \frac{J}{2} (\tilde{a}_1^\dagger \tilde{a}_2 + \tilde{a}_2^\dagger \tilde{a}_1) = \frac{J}{2} a^\dagger a - \frac{J}{2} b^\dagger b
\]

(3.15)

in which, for later convenience, we define

\[
J_a = J_b = J
\]

(3.16)

To calculate the complexity we choose the following kinetic term as that of the reference state

\[
K_{a^{(\text{ref})}, b^{(\text{ref})}} = \frac{J_f}{2} (a^{(\text{ref})})^\dagger a^{(\text{ref})} - \frac{J_f}{2} (b^{(\text{ref})})^\dagger b^{(\text{ref})}
\]

(3.17)
where $J_f$ is the coupling strength of the reference state.

Now we see that with the replacement

$$a^{(\text{ref})} \rightarrow \sqrt{\frac{J_a}{J_f}} a, \quad b^{(\text{ref})} \rightarrow \sqrt{\frac{J_b}{J_f}} b$$

(3.18)

one can obtain $K^{(\text{tar})}$ from $K^{(\text{ref})}$, i.e.

$$K^{(\text{ref})} \rightarrow K^{(\text{tar})}$$

(3.19)

In the operator approach the gate matrix defined in (1.1) is constructed by the transformation from target operator to reference operator in above relation.

### 3.1.1 Ground State

Consider first the ground state. Target ground state is annihilated by $a, b$, i.e. $a|0,0\rangle = b|0,0\rangle = 0$, and reference ground state is annihilated by $a^{(\text{ref})}, b^{(\text{ref})}$, i.e. $a^{(\text{ref})}|0,0\rangle_{\text{ref}} = b^{(\text{ref})}|0,0\rangle_{\text{ref}} = 0$. In the operator approach to complexity the gate matrix $U(s)$, defined in (1.1), connecting the target operator to reference operator can be read from (3.18)

$$
\begin{pmatrix}
  a & b
\end{pmatrix} = U(1)
\begin{pmatrix}
  a^{(\text{ref})} \\
  b^{(\text{ref})}
\end{pmatrix}, \quad U(1) = \begin{pmatrix}
\sqrt{\frac{J_a}{J_f}} & 0 \\
0 & \sqrt{\frac{J_b}{J_f}}
\end{pmatrix}
$$

(3.20)

in which the initial condition is $U(0) = \text{diag}(1,1)$. Since the transformation matrix $U(1)$ is diagonal we can choose $O_T = 1$ in (1.1) and have a simple relation

$$U(1) = \begin{pmatrix}
  e^{\int_{0}^{1} ds Y_a(s)} & 0 \\
0 & e^{\int_{0}^{1} ds Y_b(s)}
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{J_a}{J_f}} & 0 \\
0 & \sqrt{\frac{J_b}{J_f}}
\end{pmatrix}
$$

(3.21)

The associated solutions of $Y_{(1,2)}(s)$ are

$$Y_a(s) = \ln \left( \sqrt{\frac{J_a}{J_f}} \right), \quad Y_b(s) = \ln \left( \sqrt{\frac{J_b}{J_f}} \right)$$

(3.22)

which satisfied the initial condition. For the convention used in later we define new functions

$$R_a^{(0)} = \frac{J_a}{J_f}, \quad R_b^{(0)} = \frac{J_b}{J_f}, \quad R^{(0)} = \frac{J}{J_f}$$

(3.23)

in which $(0)$ means a quantity of the zero order of interaction. The squared distance for ground state, denoted as $(D_{(0,0)})^2$, between target and reference state can be calculated by formula (1.2)

$$(D_{(0,0)})^2 = Y_a(1)^2 + Y_b(1)^2 = \left( \ln \left( R_a^{(0)} \right) \right)^2 + \left( \ln \left( R_b^{(0)} \right) \right)^2$$

(3.24)

in which $(0,0)$ means a quantity of the ground state of two sites system. This formula has a similar form as that in scalar field theory [6] [11].
3.1.2 Excited States

Consider next the \( \{ N_a^{th}, N_b^{th} \} \) excited state which is defined by

\[
|N_a, N_b\rangle = \frac{(a^\dagger)^{N_a+1} (b^\dagger)^{N_b+1}}{\sqrt{(N_a + 1)! (N_b + 1)!}} |0, 0\rangle, \quad a^{N_a+1} b^{N_b+1} |N_a, N_b\rangle = 0 \quad (3.25)
\]

In the operator approach to complexity the gate matrices can be read from the transformations

\[
\begin{align*}
\begin{array}{c}
\cdots a^{(\text{ref})} \\
\cdots a^{(\text{ref})}
\end{array}
& \Rightarrow \begin{array}{c}
\sqrt{J_a / J_f} a \\
\sqrt{J_b / J_f} b
\end{array}
\begin{array}{c}
\cdots a^{(\text{ref})} \\
\cdots a^{(\text{ref})}
\end{array}
\begin{array}{c}
\sqrt{J_a / J_f} a \\
\sqrt{J_b / J_f} b
\end{array}
\end{align*}
\]

Then, the gate matrix connecting the target operator with reference operator in (1.1) becomes a \((N_a + 1) \times (N_b + 1)\) diagonal matrix

\[
U(1) = \text{Diag} \left( \begin{array}{cccc}
\sqrt{J_a / J_f} & \cdots & \sqrt{J_a / J_f} \\
\sqrt{J_b / J_f} & \cdots & \sqrt{J_b / J_f}
\end{array} \right) \quad (3.27)
\]

which reduces to (3.20) in the case of ground state \( N_a = N_b = 0 \).

Follow the discussions in before the gate matrix \( U_i \) defined in (1.1) now becomes

\[
U_i(1) = \sqrt{J / J_f}, \quad \text{with} \quad U_i(0) = 1, \quad 1 \leq i \leq N_2 + N_1 + 2 \quad (3.28)
\]

after use the relation \( J_a = J_b = J \). The associated functions of \( Y_i(s) \) in (1.1) are

\[
Y_i(s) = \ln \left( \sqrt{J / J_f} \right), \quad 1 \leq i \leq N_a + N_b + 2 \quad (3.29)
\]

The squared distance for excited state, denoted as \( D^{(0)}_{(N_a,N_b)} \), between target and reference state calculated from (1.2) is

\[
(D^{(0)}_{(N_a,N_b)})^2 = \sum_{i=1}^{N_a+N_b+2} Y_i(1)^2 = (N_a + N_b + 2) \left( \ln \left( \sqrt{R^{(0)}} \right) \right)^2, \quad R^{(0)} = J / J_f \quad (3.30)
\]

This has a similar form as the result of scalar theory obtained earlier in [11].

Since that state wavefunction is described by \( \Psi_n(x) = \frac{1}{\sqrt{n!}} \langle x | (a^\dagger)^n |0\rangle \) the gate matrix of wavefunction, \( \Psi_n(x) \), is thus related to the gate matrix of field operators, \( (a^\dagger)^n \).

3.2 Complexity of Interacting Theory

We next study the correction to the complexity due to the interaction term. We write the Hamiltonian as the summation of kinetic (free) term and potential (interaction) term

\[
\mathcal{H}_{1,2} = \frac{J}{2} \left( \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \right) + \frac{U}{2} \left( (\hat{a}_1^\dagger \hat{a}_1)^2 - (\hat{a}_1^\dagger \hat{a}_1) (\hat{a}_2^\dagger \hat{a}_2) - (\hat{a}_2^\dagger \hat{a}_2)^2 \right)
\]

\[
= \frac{J}{2} (n_a - n_b) + \frac{U}{2} \left( n_a^2 + n_b^2 + 2n_a n_b - n_a - n_b + \text{irrelevant terms} \right)
\]

\[
= \kappa_{a,b}^{(\text{tar})} + \nu_{a,b} \quad (3.31)
\]
The interacting term is

$$V_{a,b}^{(\text{tar})} = \frac{U}{2} \left( n_a^2 + n_b^2 + 2n_an_b - n_a - n_b \right)$$

(3.32)

To consider $\langle N_a, N_b | V_{a,b} | N_a, N_b \rangle$ for the excited state $|N_a, N_b\rangle$ we write above equation as

$$V_{a,b}^{(\text{tar})} = \frac{U}{2} \left( N_a a^\dagger a + N_b b^\dagger b + N_a b^\dagger b + N_b a^\dagger a - a^\dagger a - b^\dagger b \right)$$

(3.33)

Therefore

$$\mathcal{H}_{a,b}^{(\text{tar})} = J \frac{1}{2} \left( 1 - \frac{U}{J} (N_a + N_b - 1) \right) a^\dagger a + J \frac{1}{2} \left( -1 - \frac{U}{J} (N_a + N_b - 1) \right) b^\dagger b$$

(3.34)

The associated Hamiltonian of the reference state is

$$H_{a, b}^{(\text{ref}), (\text{ref})} = J_{f} \frac{1}{2} \left( 1 - \frac{U}{J_f} (N_a - 1) \right) (a^{(\text{ref})})^\dagger a^{(\text{ref})} + J_{f} \frac{1}{2} \left( -1 - \frac{U}{J_f} (N_b - 1) \right) (b^{(\text{ref})})^\dagger b^{(\text{ref})}$$

(3.35)

Above choice satisfies a desirable property of the reference state that it does not contain any entanglement between operators $\{N_a, a^{(\text{ref})}, (a^{(\text{ref})})^\dagger\}$ and $\{N_b, b^{(\text{ref})}, (b^{(\text{ref})})^\dagger\}$. The property in operator approach is like as that in coordinate approach in which a desirable property of the reference state is that it should not contain any entanglement between the original coordinates $x_1$ and $x_2$, as discussed in [6][12].

### 3.2.1 First Order

In the case of zero-order of $U$ the Hamiltonian remain only the kinetic term. This is the free case discussed in previous section. Now consider the perturbation to the complexity for the two-sites Bose-Hubbard Model. At the first order of $U$ we can use (3.34) and (3.35) to find transformations

$$\begin{align*}
J \left( 1 - \frac{U}{J} (N_a + N_b - 1) \right) a^\dagger a & \to J_f \left( 1 - \frac{U}{J_f} (N_a - 1) \right) (a^{(\text{ref})})^\dagger a^{(\text{ref})} \\
J \left( 1 + \frac{U}{J} (N_a + N_b - 1) \right) b^\dagger b & \to J_f \left( 1 + \frac{U}{J_f} (N_b - 1) \right) (b^{(\text{ref})})^\dagger b^{(\text{ref})}
\end{align*}$$

(3.36)

To proceed we notice that the factors $N_{(a,b)}$ are within the coupling term, i.e. $\frac{U}{2J}$ and we only need to consider their zero-order transform. Thus, by (3.23), we can put following replacement

$$\begin{align*}
N_a & \to R_a^{(0)} N_a \\
N_b & \to R_b^{(0)} N_b
\end{align*}$$

(3.37) (3.38)

into (3.36) and final formulas in first-order transformations are

$$\begin{align*}
R_a^{(1)}(N_a, N_b) & = J \frac{1}{J_f} \left( 1 - \frac{U}{J_f} (N_a - 1) \right) \\
R_b^{(1)}(N_a, N_b) & = J \frac{1}{J_f} \left( 1 + \frac{U}{J_f} (N_b - 1) \right)
\end{align*}$$

(3.39)
The first-order square distance is
\[ (D_{(N_a,N_b)}^{(1)})^2 = (N_a + 1) \left( \ln \left( \sqrt{R_a^{(1)}(N_a,N_b)} \right) \right)^2 + (N_b + 1) \left( \ln \left( \sqrt{R_b^{(1)}(N_a,N_b)} \right) \right)^2 \] (3.40)
which reduce to the square distance formula of free theory in (3.24) for ground (i.e. \( N_a = N_b = 0 \))

### 3.2.2 n’th-Order

Extending to higher-order interactions is straightforward. In terms of new variables
\[ g = \frac{U}{J}, \quad \gamma = \frac{J}{J_f} \] (3.41)
the recursion relations are
\[
\begin{align*}
R_a^{(n)}(N_a, N_b) &= \frac{\gamma(1-g(R_a^{(n-1)}(N_a)+R_b^{(n-1)}N_b - 1))}{(1-g\gamma(N_a-1))}, \\
R_b^{(n)}(N_a, N_b) &= \frac{\gamma(1+g(R_b^{(n-1)}(N_a)+R_a^{(n-1)}N_a - 1))}{(1+g\gamma(N_b-1))},
\end{align*}
\] (3.42)
with initial values \( R^{(0)}_{(a,b)} \) defined in (3.23) with \( J_a = J_b = J \). For excited states, the \( n \)-order square distance is
\[ (D_{(N_a,N_b)}^{(n)})^2 = (N_a + 1) \left( \ln \left( \sqrt{R_a^{(n)}(N_a,N_b)} \right) \right)^2 + (N_b + 1) \left( \ln \left( \sqrt{R_b^{(n)}(N_a,N_b)} \right) \right)^2 \] (3.43)
which is the general formula of \( n'th \)-order complexity of two-sites Bose-Hubbard model.

### 3.3 Complexity and Quantum Phase Transition

We now use above formula to investigate the quantum phase transition in two-sites Bose-Hubbard model. Consider first the ground in which \( N_a = N_b = 0 \). Then (3.42) become
\[ R_a^{(n)}(0,0) = \frac{\gamma(1+g)}{1+g\gamma}, \quad R_b^{(n)}(0,0) = \frac{\gamma(1-g)}{1-g\gamma} \] (3.44)
in which \( g = \frac{U}{J}, \gamma = \frac{J}{J_f} \), defined in (3.41), describes the ratio of one-site repulsion and hopping constant. We expect that complexity will be divergent at critical value \( g_c \), which indicates the quantum phase transition occurs at this point.

To proceed we make following comments about above relation :
1. \( R_{a,b}^{(n)}(0,0) \) are the qualities at \( n \)'th order of interaction strength \( g \). Thus (3.44) gives
\[ R^{(0)}_{a,b}(0,0) = \frac{J}{J_f} = \gamma \] (3.45)
which are those in (3.23). To first order (3.44) gives
\[ R_a^{(1)}(0,0) = \gamma + \gamma(1-\gamma)g, \quad R_b^{(1)}(0,0) = \gamma - \gamma(1-\gamma)g \] (3.46)
and using (3.44) we can find any order of $R_{a,b}^{(n)}(0,0)$ by series expansion of $g$.

2. In fact the formulas (3.44) are the exact values and we can let $n \to \infty$. Note that being able to obtain the exact formula for ground state complexity is a special property in our method. Now we see that $R_{b}^{(n\to\infty)}(0,0)$ in (3.44) becomes zero at critical value $g = g_c = 1$. This implies that the associated complexity, $(D_{(0,0)}^{(n\to\infty)})^2$, is divergent and thus it is the quantum phase transition point.

3. We plot the figure 2 to explicitly see this property.

![Figure 2](image_url)

Figure 2: Dependence of complexity $(D_{(0,0)}^{(n\to\infty)})^2$ on “$g$” in 2-sites Bose-Hubbard model. The divergence at $g = g_c = 1$ indicates the quantum phase transition at this point. The figure is plotted with $\gamma_L = 1/2$ while $\gamma_R = 3/2$.

4. The value of $g$ we considered is positive. Thus the factor $(1 - g\gamma)$ in formulas (3.44) could become zero at $g_0 = \gamma^{-1}$ which depends on the reference state parameter $\gamma$. To avoid $g_0$ become a critical point the following scheme is proposed. For range of $g > g_c$ we choose $\gamma = \gamma_R > 1$ while for range of $g < g_c$ we choose $\gamma = \gamma_L < 1$. Under this prescription the value of $(1 - g\gamma)$ is always positive and it remains only one critical point at $g = g_c$.

5. Notice that, in this way, the complexity calculated for $g > g_c$ is that with $\gamma = \gamma_R > 1$ while the complexity calculated with $g < g_c$ is that with $\gamma = \gamma_L < 1$. Therefore, one could compare the magnitude of complexity between the systems of $g > g_c$. One could also compare the magnitude of complexity between the systems of $g < g_c$ since the reference state parameter $\gamma$ is different between them. The figure 2 is plotted with $\gamma_L = 1/2$ while $\gamma_R = 3/2$.

6. Consider next the excited ground in which $N_a = N_b = 1$. For $n \geq 1$ (3.42) becomes

\[
R_a^{(n)}(1,1) = \gamma (1 + g(1 - 2\gamma)) \quad \text{and} \quad R_b^{(n)}(1,1) = \gamma (1 - g(1 - 2\gamma)), \quad n \geq 1 \quad (3.47)
\]

As that in ground case, the formulas (3.47) are the exact values and we can let $n \to \infty$. We also see that either $R_a^{(n\to\infty)}(1,1)$ or $R_a^{(n\to\infty)}(1,1)$ becomes zero at critical value $g = g_c^{(N=1)} = \frac{1}{1 - 2\gamma}$. This implies that the associated complexity, $(D_{(1,1)}^{(n\to\infty)})^2$, is divergent at the critical value and
thus it is the quantum phase transition point. Note that the critical point is a physical property
and it shall not depend on the reference state parameter $\gamma$. However, the calculated value $g_c^{(N=1)}$ depends on $\gamma$. This feels short of author expectation. The reason may be that the model
described in (2.2) is that keeps only the lowest vibrational state and could only describe the
ground state property. The analysis in above, however, have shown a possible way to calculate
excited-state complexity in other many-body models.

4 Complexity of N-Sites Bose-Hubbard Model

Hamiltonian of N-sites closed chain of Bose-Hubbard model defined in (2.2) can be written as

$$\mathcal{H}_N = \frac{J}{2N} \sum_{j=0}^{N} (\tilde{a}_j^\dagger \tilde{a}_{j+1} + \tilde{a}_{j-1}^\dagger \tilde{a}_j) + \frac{U}{N} \sum_{j=0}^{N} \tilde{n}_j (\tilde{n}_j - 1), \quad \tilde{n}_j = \tilde{a}_j^\dagger \tilde{a}_j$$

(4.1)
in which we consider the periodic boundary as shown in figure 3.

Figure 3: Six-sites Bose-Hubbard Model model. We choose the model with periodic boundary
and there has six links. The associated complexity is plotted in figure 4.

We can generalize method in previous section to define the new operators

$$a_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(\frac{2\pi ik}{N} j\right) \tilde{a}_j, \quad a_k^\dagger = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(-\frac{2\pi ik}{N} j\right) \tilde{a}_j^\dagger$$

(4.2)

$$\tilde{a}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(-\frac{2\pi ik}{N} j\right) a_j, \quad \tilde{a}_k^\dagger = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(\frac{2\pi ik}{N} j\right) a_j^\dagger$$

(4.3)
in which we impose a periodic boundary condition $\tilde{a}_{k+N+1} = \tilde{a}_k$ and $\tilde{a}_{k+N+1}^\dagger = \tilde{a}_k^\dagger$.

Note that the relative sign between the Fourier series of $a_k$ and $a_k^\dagger$ is important to have
standard commutation relation [6]. Then, above definitions leads to simple relations

$$[\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0, \quad [\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{i,j} \quad \Rightarrow \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{i,j}$$

(4.4)

and

$$[a_j, a_k^\dagger] = \frac{1}{N} \sum_{m,\ell} e^{2\pi i (m-j-k\ell)/N} [a_m, a_\ell^\dagger] = \frac{1}{N} \sum_{m,\ell} e^{2\pi i (m-j-k\ell)/N} \delta_{m,\ell} = \frac{1}{N} \sum_{\ell} e^{2\pi i (m-k\ell)/N} = \delta_{j,k}$$

(4.5)
The general states can then be constructed by creation operators $a_k^\dagger$

$$|N_0, N_1 \cdots N_{N-1}\rangle = a_0^{N_0} a_1^{N_1} \cdots a_{N-1}^{N_{N-1}} |0\rangle$$ (4.6)

due to the commutation relation between the new operators.

We now express N-sites Hamiltonian $H_N$ in terms of new operators $a_j$ and $a_j^\dagger$. Kinematic term is

$$\sum_j \tilde{a}_j^\dagger \tilde{a}_j + \sum_j \tilde{a}_{j-1}^\dagger \tilde{a}_j = \frac{1}{N} \sum_j \sum_{m,k} e^{\frac{2\pi i}{N} (-m(j+1) + kj)} a_m^\dagger a_k + \frac{1}{N} \sum_j \sum_{m,k} e^{\frac{2\pi i}{N} (-m(j-1) + kj)} a_m^\dagger a_k$$

$$= \sum_k e^{-2\pi i k/N} a_k^\dagger a_k + \sum_k e^{-2\pi i m/N} a_k^\dagger a_k$$ (4.7)

$$= 2 \sum_{k=0}^{N-1} \cos \left( \frac{2\pi k}{N} \right) a_k^\dagger a_k$$ (4.8)

The kinematic term in $H_N$ can thus be expressed as

$$\mathcal{K}_N = \frac{J}{2N} \left( \sum_j \tilde{a}_j^\dagger \tilde{a}_j + \sum_j \tilde{a}_{j-1}^\dagger \tilde{a}_j \right) = \frac{J}{N} \sum_{k=0}^{N-1} \cos \left( \frac{2\pi k}{N} \right) a_k^\dagger a_k$$ (4.9)

which reduces to the formula of N=2 case in previous section. The interaction terms has two parts. One is

$$\sum_j \tilde{n}_j = \sum_j \tilde{a}_j^\dagger \tilde{a}_j = \frac{1}{N} \sum_j \sum_{m,k} e^{\frac{2\pi i}{N} (-m+j)} a_m^\dagger a_k = \sum_{m,k} \delta_{m,k} a_m^\dagger a_k = \sum_k n_k$$ (4.10)

Another one is

$$\sum_j \tilde{n}_j^2 = \sum_j \tilde{a}_j^\dagger \tilde{a}_j = \frac{1}{N^2} \sum_j \sum_{\alpha,\beta,\gamma,\delta} e^{\frac{2\pi i}{N} (+\alpha - \beta - \gamma + \delta)} a_\alpha^\dagger a_\beta a_\gamma^\dagger a_\delta$$

$$= \frac{1}{N} \sum_{\alpha,\beta,\gamma,\eta} a_\alpha^\dagger a_\beta a_\gamma^\dagger a_\eta \delta_{\alpha-\beta+\gamma-\eta,0}$$ (4.11)

The delta function in here means that once $\alpha = \beta$ then $\gamma = \eta$ and, if $\alpha \neq \beta$ then $\gamma \neq \eta$. In the first case we have term

$$\frac{1}{N} \sum_{\alpha,\gamma} a_\alpha^\dagger a_\alpha a_\gamma^\dagger a_\gamma = \frac{1}{N} \sum_{\alpha,\gamma} n_\alpha a_\gamma^\dagger a_\gamma$$ (4.12)

In the second case, following the discuss in previous section we have to choose the relevant term which is

$$\frac{1}{N} \sum_{\alpha \neq \gamma} a_\alpha a_\gamma b_{\alpha,\gamma} = \frac{1}{N} \sum_{\alpha \neq \gamma} n_\alpha a_\gamma^\dagger a_\gamma = \frac{1}{N} \sum_{\alpha \neq \gamma} n_\alpha (1 + a_\gamma^\dagger a_\gamma)$$

$$= \left( \frac{N-1}{N} \sum_{\alpha} n_\alpha \right) + \frac{1}{N} \sum_{\alpha \neq \gamma} n_\alpha a_\gamma^\dagger a_\gamma$$ (4.14)
Collect all and the interaction term in $H_N$ becomes
\[ V_N = \frac{U}{N} \left( \sum_{\alpha \gamma} n_{\alpha} a_{\alpha}^\dagger a_{\gamma} + \sum_{\alpha \neq \gamma} n_{\alpha} a_{\alpha}^\dagger a_{\gamma} - \sum_{\alpha} n_{\alpha} \right) + \text{irrelevant terms} \]
\[ = \frac{U}{N} \left( \sum_{k=0}^{N-1} n_k a_k^\dagger a_k + 2 \sum_{(k,j)=0}^{N-1} n_j a_k^\dagger a_k - \sum_{k=0}^{N-1} n_k \right) + \text{irrelevant terms} \] (4.15)
where $(k,j)$ denotes any pair at site $k$ and site $j$. When $N=2$ above formula reduces to that of 2-sites formula in (3.32).

### 4.1 Complexity of Free Theory

Let us first discuss the case of free theory. Following the scheme detailed in previous section the target system kinematic term and reference system kinematic term are
\[ K_N^{(\text{tar})} = \frac{J}{N} \sum_{k=0}^{N-1} \cos \left( \frac{2\pi k}{N} \right) a_k^\dagger a_k \] (4.16)
\[ K_N^{(\text{ref})} = \frac{J_f}{N} \sum_{k=0}^{N-1} \cos \left( \frac{2\pi k}{N} \right) a_{(\text{ref})}^\dagger a_{(\text{ref})} \] (4.17)
where $J_f$ is the coupling strength of the reference state.

Now we see that with the replacement
\[ a_{(\text{ref})}^\dagger a_k \rightarrow \sqrt{\frac{J}{J_f}} a_k \] (4.18)
one can obtain $K^{(\text{tar})}$ from $K^{(\text{ref})}$, i.e.
\[ K^{(\text{ref})} \rightarrow K^{(\text{tar})} \] (4.19)
As mentioned before, in the operator approach the gate matrix defined in (1.1) is constructed by the transformation from target operator to reference operator in above relation. Thus the complexity, square distance, for the general excited state $\{N_0, N_1, \cdots, N_{N-1}\}$ is easy to be found by the method discussed in the 2-site case. The result is
\[ D^2_{\{N_0, N_1, \cdots, N_{N-1}\}} = \sum_{k=0}^{N-1} \left( N_k + 1 \right) \left( \ln \left( \sqrt{R_k^{(0)}} \right) \right)^2 \] (4.20)
\[ R_k^{(0)} = \frac{J}{J_f} \] (4.21)
which is a simple extension of 2-sites formula in (3.30).

### 4.2 Complexity of Interacting Theory

To include the interaction we first collect the total Hamiltonian
\[ H_N = \frac{1}{N} \sum_{k=0}^{N-1} J \cos \left( \frac{2\pi k}{N} \right) a_k^\dagger a_k + \frac{U}{N} \left( \sum_{k=0}^{N-1} n_k a_k^\dagger a_k + 2 \sum_{(k,j)=0}^{N-1} n_j a_k^\dagger a_k - \sum_{k=0}^{N-1} a_k^\dagger a_k \right) \] (4.21)
\[ = \frac{1}{N} \sum_{k=0}^{N-1} \left( J \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + n_k + \sum_{j \neq k}^{N-1} n_j \right) \right) a_k^\dagger a_k \] (4.22)
where \( n_k = a_k^\dagger a_k \) is the number operator. Then, following the scheme described in previous section we furthermore write the total Hamiltonian of target state and reference state as

\[
H_{N}^{(\text{tar})} = \frac{1}{N} \sum_{k=0}^{N-1} \left( J \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + N_k + \sum_{j \neq k} N_j \right) \right) a_k^\dagger a_k \tag{4.23}
\]

\[
H_{N}^{(\text{ref})} = \frac{1}{N} \sum_{k=0}^{N-1} \left( J_f \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + N_k \right) \right) a_{k}^{(\text{ref})\dagger} a_{k}^{(\text{ref})} \tag{4.24}
\]

where \( N_k \) is an integral number defined by \( n_k |0\rangle = |N_k \rangle \). The reference operator in above satisfies a desirable property of the reference state that it does not contain any entanglement between operators \( a_k^{(\text{ref})} \) and \( a_j^{(\text{ref})} \) if \( k \neq j \), as discussed in \[6, 12\].

### 4.2.1 First Order

In the case of zero-order of \( U \) the Hamiltonian remain only the kinetic term. This is the free case. Now consider the perturbation to the complexity for the \( N \) sites Bose Hubbard Model. At the first order of \( U \) the transformation from target operator to reference operator is chosen as following

\[
\left( J \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + N_k + \sum_{j \neq k} N_j \right) \right) a_k^\dagger a_k \rightarrow \left( J_f \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + N_k \right) \right) a_{k}^{(\text{ref})\dagger} a_{k}^{(\text{ref})} \tag{4.25}
\]

Now, we notice that the factors \( N_k, N_j \) are within the coupling term, i.e. \( U \), and we only need to consider their zero-order transform. Thus, by \[4.20\], we shall put following replacement

\[
N_k \rightarrow R_k^{(0)} N_k \tag{4.26}
\]

into \[4.25\] and final formulas in first-order transformation is

\[
R_k^{(1)} (\{N_i\}) = \frac{J \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + R_k^{(0)} N_k + \sum_{j \neq k} N_j \right)}{J_f \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + N_k \right)} \tag{4.27}
\]

Therefore the first-order complexity or square distance is

\[
\left( D_{\{N_i\}}^{(1)} \right)^2 = \sum_{k=0}^{N} (N_k + 1) \left( \ln \left( \sqrt{R_k^{(1)} (\{N_i\})} \right) \right)^2 \tag{4.28}
\]

The case of \( \{N_i\} = \{N_a, N_b\} \) above formula reduce to the square distance formula of 2-site theory in \[3.40\].
4.2.2 n’th-Order

Extending to higher-order interactions is straightforward. The recursion relation is

\[ R_k^{(n)}(\{N_i\}) = \frac{J \cos \left( \frac{2\pi k}{N} \right) + U \left( -1 + R_k^{(n-1)} N_k + \sum_{j \neq k}^{N-1} N_j \right)}{J_f \cos \left( \frac{2\pi k}{N} \right) + U \left( 1 + N_k \right)} (4.29) \]

or

\[ = \frac{\gamma \left( \cos \left( \frac{2\pi k}{N} \right) + g \left( -1 + R_k^{(n-1)} N_k + \sum_{j \neq k}^{N-1} N_j \right) \right)}{\cos \left( \frac{2\pi k}{N} \right) + \gamma g \left( 1 + N_k \right)} (4.30) \]

with initial values \( R_k^{(0)}(a, b) \) defined in (4.20) and the n-order square distance is

\[ \left( D_{\{N_i\}}^{(n)} \right)^2 = \sum_{k=0}^{N_1} (N_k + 1) \left( \ln \left( \sqrt{R_k^{(n)}(\{N_i\})} \right) \right)^2 (4.31) \]

These complete our derivations.

4.3 Complexity and Quantum Phase Transition

Consider first the ground in which \( N_i = 0 \). Then (4.30) become

\[ R_k^{(n)}(\{0\}) = \frac{\gamma \left( \cos \left( \frac{2\pi k}{N} \right) - g \right)}{\cos \left( \frac{2\pi k}{N} \right) - \gamma g}, \quad k = 0, \ldots, N - 1 (4.32) \]

which reduces to formulas (3.44) for \( N=2 \) case. Likes as that in two-sites model we expect that complexity will be divergent at critical value \( g_c \), which indicates the quantum phase transition occurs at this point. Let us discuss the property of (4.32).

1. As that in 2-sites system, above formula is the exact value and we can let \( n \rightarrow \infty \). Being able to obtain the exact formula for ground state complexity is a special property in our method.

2. We see that \( R_k^{(n \rightarrow \infty)}(\{0\}) \) in (4.32) becomes zero at critical value \( g = g_c = 1 \). This implies that the associated complexity, \( (D_{\{0,0\}}^{(n \rightarrow \infty)})^2 \), is divergent and thus it is the quantum phase transition point. The critical value \( g_c = 1 \) is same as that in two-sites system.

3. We plot the figure 4 to explicitly see this property.
Figure 4: Dependence of complexity \( D_{(0,0)}^{(n \to \infty)} \) on “g” in 6-sites Bose-Hubbard model. The divergence at \( g = g_c = 1 \) indicates the quantum phase transition at this point. The figure is plotted with \( \gamma_L = 3/4 \) while \( \gamma_R = 2 \).

4. In two-sites system, for range of \( g > g_c \) we choose \( \gamma = \gamma_R > 1 \) while for range of \( g < g_c \) we choose \( \gamma = \gamma_L < 1 \). In this algorithm, we see that the complexity becomes infinity only at \( g = g_c \). In the N-sites system, however, \( \gamma \) tells us that complexity could become infinity at the point \( g = g_k < 1 \) if \( \cos \left( \frac{2\pi k}{N} \right) - g_k = 0 \). While we know that these are not physical critical points we have not yet found a simple algorithm to rule it out. Figure 4 is plotted for 6-sites Bose-Hubbard model with \( \gamma = 2 \). The complexity for \( g < 1 \) has three divergence points which depend on \( \gamma \) and are un-physical.

5. When consider the excited state in which \( \{ N \} \neq \{ 0 \} \) we can see that the associated complexity becomes divergent at the critical value which depends on the reference state parameter \( \gamma \). As that in 2-sites system the un-physical property reason may be that the model described in \[ (2.2) \] is that keeps only the lowest vibrational state and could only describe the ground state property.

6. Generalize the method to N dimensional Bose-Hubbard model is straightforward, as that in quantum field theory \[ [6] \]. We only need to extend the N-sites closed chain Hamiltonian \[ (4.1) \] to

\[
\mathcal{H}_N = \frac{J}{2} \sum_{\vec{j}} \left( \hat{a}_{\vec{j}+1}^\dagger \hat{a}_{\vec{j}} + \hat{a}_{\vec{j}-1}^\dagger \hat{a}_{\vec{j}} \right) + U \sum_{\vec{j}} \hat{n}_{\vec{j}} (\hat{n}_{\vec{j}} - 1), \quad \hat{n}_{\vec{j}} = \hat{a}_{\vec{j}}^\dagger \hat{a}_{\vec{j}}
\]

in which \( \vec{j} = (j_1, j_2, \ldots, j_N) \) is the N-dimensional vector with integral number. In this way, to find the formula of complexity we only need to extend the single parameter Fourier transform to N parameters transform and can get a general formula of complexity. For example the relation \[ (4.31) \] becomes

\[
\left( D_{\{N_i\}}^{(n)} \right)^2 = \sum_{\vec{k}} (N_{\vec{k}} + 1) \left( \ln \left( \sqrt{R_{\vec{k}}^{(n)}(\{N_i\})} \right) \right)^2, \quad \vec{k} = (k_1, k_2, \ldots, k_N)
\]

which is the complexity of N dimensional Bose-Hubbard model.

5 Concluding Remarks

We adopt the operator approach to compute the complexity of the Bose-Hubbard Model. Our prescription is to consider the value of \( \langle N_1 \cdots N_n | \mathcal{H} | N_1 \cdots N_n \rangle \). In this way, after proper mode expansion of \( \mathcal{H} \), many irrelevant terms become zero and the contributions of complexity from the remained terms could be calculated exactly through a perturbation algorithm. We then obtain the general formula of complexity therein. It is interesting to see that the formula of complexity of Bose-Hubbard model is very similar to that of \( \lambda \phi^4 \) field theory. Our analysis leads to a result that the critical point \( g_c = 1 \) is found in 2-sites model and N-sites system. The special property
that the critical value being independent of lattice number tells that our method is mean-field like trick. The more general method to compute complexity in many-body models remained to be found.

Finally, our algorithm can be applied to many quantum field theories and several many-body models in condense matter. We will study the complexity of Sachdev-Ye-Kitaev (SYK) model in the next paper to see how the complexity probes the topological phase transition.

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