GRAVITY ALGEBRA STRUCTURE ON THE NEGATIVE CYCLIC
HOMOLOGY OF CALABI-YAU ALGEBRAS

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Abstract. In this paper, we study the gravity algebra structure on the negative cyclic
homology or the cyclic cohomology of several classes of algebras. These algebras include:
Calabi-Yau algebras, symmetric Frobenius algebras, unimodular Poisson algebras, and
unimodular Frobenius Poisson algebras. The relationships among these gravity algebras
are also discussed under some additional conditions.

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1. Introduction

In 1994, Getzler [15] showed that, \{H_\bullet(\mathcal{M}_{0,n+1})\}, the collection of the homology classes
of the moduli spaces of Riemann spheres with \( n + 1 \) marked points, forms an operad,
which he called the gravity operad, and that an algebra over it consists a sequence of skew-
symmetric brackets satisfying the so-called generalized Jacobi identity, which he called
gravity algebra. He later showed in [16] that \{H_\bullet(\mathcal{M}_{0,n+1})\} is Koszul dual, in the sense
of Ginzburg and Kapranov [18], to the operad \{H_\bullet(\mathcal{M}_{0,n+1})\}, the collection of the ho-
mology classes of the compactified moduli spaces of Riemann spheres with \( n + 1 \) marked
points, algebras over which are also called hypercommutative algebras or formal Frobenius
manifolds (see also Manin [30, Ch. III]).

Besides examples from topological field theories ([15, Theorem 4.6]), the first nontrivial
example of gravity algebra, to the authors’ best knowledge, arises from string topology.
In [5], Chas and Sullivan showed that the \( S^1 \)-equivariant homology of the free loop space
of a smooth compact manifold, after shifting some degree, forms a gravity algebra. More
examples, more or less inspired by string topology, can be found in Westerland [38] and
Ward [37] (see also Menichi [29] for some partial results). These works show that the cyclic
cohomology of a symmetric Frobenius (or more generally cyclic $A_\infty$) algebra, which is the algebraic model of the $S^1$-equivariant homology of the free loop spaces, has a nontrivial gravity algebra structure.

A symmetric Frobenius algebra is an associative algebra with a non-degenerate symmetric pairing. Let us recall two facts on this algebra:

(i) if a symmetric Frobenius algebra is Koszul, then its Koszul dual is a Calabi-Yau algebra, a notion introduced by Ginzburg in [17] (see Van den Bergh [36, Theorem 12.1] for a proof);

(ii) the cyclic cohomology of an algebra is isomorphic to the negative cyclic homology of its Koszul dual (see [7, Theorem 37] for a proof).

From these two facts one deduces that there is a gravity algebra structure on the negative cyclic homology of a Koszul Calabi-Yau algebra (see Ward [37, Example 6.5]). It is then naturally expected that for arbitrary Calabi-Yau algebras which are not necessarily Koszul, such gravity structure still exists on their negative cyclic homology. In loc cit Ward wrote “How to directly construct the higher brackets comprising this [gravity] structure on the Calabi-Yau side is an open question”.

The purpose of this paper is to answer this open question. To this end, let us first recall that for a mixed complex $(C_\bullet, b, B)$, if we denote by $\text{HH}_\bullet$ and $\text{HC}^-\bullet$ its Hochschild homology (that is, the homology of $(C_\bullet, b)$) and its negative cyclic homology, then we have the long exact sequence

$$
\cdots \longrightarrow \text{HC}^-_{\bullet+2} \longrightarrow \text{HC}^-_{\bullet} \xrightarrow{\pi_*} \text{HH}_\bullet \xrightarrow{\beta} \text{HC}^-_{\bullet+1} \longrightarrow \cdots
$$

The main result of the current paper is the following slightly technical result:

**Theorem 1.1.** (1) For a mixed $(C_\bullet, b, B)$, if $\text{HH}_\bullet$ has a Batalin-Vilkovisky algebra structure such that $B$ is the generator of the Gerstenhaber bracket, then the following sequence of maps

$$(\text{HC}^-_{\bullet})^{\otimes n} \longrightarrow \text{HC}^-_{\bullet}$$

$$(x_1, \ldots, x_n) \longmapsto (-1)^{(n-1)|x_1|+(n-2)|x_2|+\cdots+|x_{n-1}|} \beta(\pi_*(x_1) \bullet \pi_*(x_2) \bullet \cdots \bullet \pi_*(x_n)),$$

for $n = 2, 3, \ldots$, where $\bullet$ is the product on the Hochschild homology, give on $\text{HC}^-_{\bullet}$ a gravity algebra structure.

(2) Suppose

$$f : (C_\bullet, b, B) \longrightarrow (C'_\bullet, b', B')$$

is a quasi-isomorphism of mixed complexes and furthermore suppose that $(\text{HH}_\bullet(C), B)$ and $(\text{HH}_\bullet(C'), B')$ are extended to Batalin-Vilkovisky algebras such that $f_*$ is a Batalin-Vilkovisky algebra isomorphism. Then $\text{HC}^-_{\bullet}(C)$ and $\text{HC}^-_{\bullet}(C')$ are isomorphic as gravity algebras.

The following complexes satisfy the conditions of the above theorem (more details of these complexes will be given in later sections):

(1) the mixed cyclic complex of a Calabi-Yau algebra,

(2) the mixed cyclic cochain complex of a symmetric Frobenius algebra,

(3) the mixed Poisson complex of a unimodular Poisson algebra, and

(4) the mixed Poisson cochain complex of a unimodular Frobenius Poisson algebra.
As a corollary, the negative cyclic homology or cyclic cohomology of the above four types of complexes all have a gravity algebra structure. These structures (the brackets) are nontrivial in general; for example, it contains a graded Lie algebra which is related to the deformations of the corresponding algebras, which are in general obstructed (see also Remark 4.7 (2)). We make the following two remarks:

**Remark 1.2.** (i) The above case (1) gives an answer to Ward’s problem at the homology level. In loc cit he also asked for a homotopical, or say, chain level, construction of the Batalin-Vilkovisky and gravity structure for general Calabi-Yau algebras. Such construction seems to exist; see Remark 2.6 for some more discussions.

(ii) The above case (2) covers the example of symmetric Frobenius algebras, which is not new. In fact, in [37] Ward showed that the mixed cyclic cochain complex of symmetric Frobenius algebras is an algebra over the (chain) gravity operad, hence Theorem 1.1 is a direct corollary of his result in this case. In the subsequent paper [3] Campos and Ward proved a more general result saying that any mixed complex over the (chain) gravity operad induces the gravity algebra structure on its negative cyclic homology (see loc cit Corollary 1.33 and Example 1.34).

It will then be very interesting to find the relationships among these gravity algebras. In fact, we find that:

(a) if the algebras in the above cases (1) and (2) are Koszul dual to each other, then the corresponding two gravity algebras are isomorphic (partially given in [7, Theorem 37]);

(b) if the Poisson algebras in the above cases (3) and (4) are Koszul dual to each other (this means the Poisson algebra is the polynomials with a quadratic Poisson structure), then the corresponding two gravity algebras are also isomorphic;

(c) by a result of Dolgushev [10] saying that the deformation quantization of a unimodular Poisson algebra is a Calabi-Yau algebra, the negative cyclic Poisson homology of the algebra in the above case (3) is then isomorphic to the negative cyclic homology of its deformation quantization given in the case (1);

(d) by a result of Felder-Shoikhet [11] and Willwacher-Calaque [40] saying that the deformation quantization of a unimodular Frobenius Poisson algebra is a symmetric Frobenius algebra, the cyclic Poisson cohomology of the algebra in the case (4) is then isomorphic to the cyclic cohomology of its deformation quantization given in the case (2).

In particular, starting from quadratic, unimodular Poisson polynomials, we have the following correspondence:

\[
\begin{array}{ccc}
\text{Quadratic unimodular Poisson algebra} & \xrightarrow{\text{Koszul duality}} & \text{Quadratic unimodular Frobenius Poisson algebra} \\
\text{deformation quantization} & & \text{deformation quantization} \\
\text{Calabi-Yau algebra} & \xrightarrow{\text{Koszul duality}} & \text{Symmetric Frobenius algebra}
\end{array}
\]  

(1.1)
And therefore in this case, all the gravity algebras arising from the above (1)–(4) cases are isomorphic.

This work is a sequel to [6], where the same classes of algebras are studied, and the associated Batalin-Vilkovisky algebras are identified. We had planned to put the results of this paper as an appendix of that one, which, however, turns out to be too long. Moreover, we think Theorem 1.1 and its corollaries have their own virtue, and it is better to write a separate paper. Finally, we mention that when we were finishing the current paper, we learned of the preprint of D. Fiorenza and N. Kowalzig [12], where the Lie algebra structure on the negative cyclic homology are also studied. Nevertheless, our primary goals and main interests are quite different.

Convention. Throughout this paper, we work over a ground field $k$ of characteristic 0 containing $\mathbb{R}$. All vector spaces, their tensors and morphisms are graded over $k$ unless otherwise specified. Algebras are unital and augmented over $k$. All DG algebras/modules are graded such that the differential has degree $-1$, and for a chain complex, its cohomology is $H^\bullet(-) := H_{-\bullet}(-)$; for example, an element in the $q$-th Hochschild cochain group $CH^q(A)$ has grading $-q$.

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2. FROM BATALIN-VILKOVISKY ALGEBRA TO GRAVITY ALGEBRA

Gravity algebras are closely related to Batalin-Vilkovisky algebras. In this section we first recall the definition of Batalin-Vilkovisky and gravity algebras; some more backgrounds, and especially their relationships to topological conformal field theories, can be found in Getzler [14, 15, 16]. After that, we prove Theorem 1.1.

Definition 2.1 (Batalin-Vilkovisky algebra). Suppose $(V, \ast)$ is a graded commutative algebra. A Batalin-Vilkovisky algebra structure on $V$ is the triple $(V, \ast, \Delta)$ such that

\begin{align}
(1) \quad & \Delta : V^i \to V^{i-1} \text{ is a differential, that is, } \Delta^2 = 0; \text{ and} \\
(2) \quad & \Delta \text{ is a second order operator, that is,}
\end{align}

$$
\Delta(a \ast b \ast c) = \Delta(a \ast b) \ast c + (-1)^{|a|} a \ast \Delta(b \ast c) + (-1)^{|a|-1} b \ast \Delta(a \ast c) \\
- (\Delta a) \ast b \ast c - (-1)^{|a|} a \ast (\Delta b) \ast c - (-1)^{|a|+|b|} a \ast b \ast (\Delta c),
$$

(2.1)

for all homogeneous $a, b, c \in V$.

Equivalently, if we set the bracket

$$[a, b] := (-1)^{|a|+1}(\Delta(a \ast b) - \Delta(a) \ast b - (-1)^{|a|} a \ast \Delta(b)),$$

then $[-, -]$ is a derivation with respect to $\ast$ for each argument. In other words, a Batalin-Vilkovisky algebra is a Gerstenhaber algebra $(V, \ast, [-, -])$ with a differential $\Delta : V^i \to V^{i-1}$ such that

$$[a, b] = (-1)^{|a|+1}(\Delta(a \ast b) - \Delta(a) \ast b - (-1)^{|a|} a \ast \Delta(b)),$$

for $a, b \in V$. $\Delta$ is called the Batalin-Vilkovisky operator, or the generator of the Gerstenhaber bracket.
From (2.1) one also obtains that $\Delta$ being of second order implies that

$$\Delta(x_1 \cdot x_2 \cdot \ldots \cdot x_n) = \sum_{i<j} (-1)^{\epsilon_{ij}} \Delta(x_i \cdot x_j) \cdot x_1 \cdot \ldots \cdot \hat{x}_i \cdot \ldots \cdot \hat{x}_j \cdot \ldots \cdot x_n - \sum_{i=1}^n (-1)^{|x_1|+\cdots+|x_{i-1}|} x_1 \cdot \ldots \cdot \Delta(x_i) \cdot \ldots \cdot x_n,$$

(2.2)

where $\epsilon_{ij} = (|x_1| + \cdots + |x_{i-1}|)|x_i| + (|x_1| + \cdots + |x_{j-1}|)|x_j| - |x_i||x_j|.$

**Definition 2.2** (Gravity algebra). Suppose $V$ is a (graded) vector space over $k$. Then a gravity algebra structure on $V$ consists of a sequence of (graded) skew symmetric operators (called the *higher Lie brackets*)

$$\{x_1, \ldots, x_n\} : V^\otimes n \to V, \quad n = 2, 3, \ldots$$

such that

$$\sum_{1 \leq i < j \leq n} (-1)^{\epsilon_{ij}} \{\{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n, y_1, \ldots, y_m\}$$

$$= \begin{cases} \{\{x_1, \ldots, x_n\}, y_1, \ldots, y_m\}, & \text{if } m > 0, \\ 0, & \text{otherwise,} \end{cases}$$

(2.3)

where $\epsilon_{ij} = (|x_i| + 1)(|x_1| + \cdots + |x_{i-1}| + i - 1) + (|x_j| + 1)(|x_1| + \cdots + |x_{j-1}| + j - 1) - (|x_i| + 1)(|x_j| + 1)$.

From the definition, one also obtains that $(V, \{-,-\})$ forms a graded Lie algebra.

**Definition 2.3** (Cyclic homology; cf. Jones [19] and Kassel [20]). Suppose $(C_\bullet, b, B)$ is a mixed complex, with $|d| = -1$ and $|B| = 1$. Let $u$ be a free variable of degree $-2$ which commutes with $b$ and $B$. The *negative cyclic*, *periodic cyclic*, and *cyclic* chain complex of $C_\bullet$ are the following complexes

$$(C_\bullet[u], b + uB),$$

$$(C_\bullet[u, u^{-1}], b + uB),$$

$$(C_\bullet[u, u^{-1}]/uC_\bullet[u], b + uB),$$

and are denoted by $CC^{-}_{\bullet}(C_\bullet), CC^{per}_{\bullet}(C_\bullet)$ and $CC_{\bullet}(C_\bullet)$ respectively. The associated homology are called the *negative cyclic*, *periodic cyclic* and *cyclic homology* of $C_\bullet$, and are denoted by $HC^{-}_{\bullet}(C_\bullet), HC^{per}_{\bullet}(C_\bullet)$ and $HC_{\bullet}(C_\bullet)$ respectively.

In the following, if $C_\bullet$ is clear from the context, we sometimes simply write, for example, $HC^{-}_{\bullet}$ instead of $HC^{-}_{\bullet}(C_\bullet)$. As usual, $\text{HH}_{\bullet}$ denotes the Hochschild homology of $C_\bullet$, which is the $b$-homology of the mixed complex.

**Remark 2.4** (Cyclic cohomology). Suppose $(C^\bullet, b, B)$ is a mixed cochain complex, namely $|b| = 1$ and $|B| = -1$. By negating the degrees of $C^\bullet$, we obtain a mixed chain complex, denoted by $(C_{-\bullet}, b, B)$ with $|b| = -1$ and $|B| = 1$. By our convention, the *cyclic cohomology* of $(C^\bullet, b, B)$, denoted by $HC^\bullet(C^\bullet)$, is the *cohomology* of the negative cyclic complex of $(C_{-\bullet}, b, B)$.
In fact, commutative, by commutativity it is sufficient to show

\[ \pi : \bigoplus_{i} \frac{CC_0^\ast}{x_i \cdot u^i} \rightarrow C_0 \]

is the projection. It induces functorially a long exact sequence

\[ \cdots \rightarrow \text{HC}_n \rightarrow \text{HC}_n \rightarrow \beta \rightarrow \text{HC}_{n+1} \rightarrow \cdots, \]

where we have applied that \( \text{HC}_n \cong \text{HC}_{n+1} \). It is obvious that \( \beta \circ \pi = 0 \) and we claim that (cf. [5][7]):

**Lemma 2.5.** With the above notations, we have

\[ \pi \circ \beta = B : \text{HH}_n(C_\ast, b) \rightarrow \text{HH}_{n+1}(C_\ast, b). \]

**Proof.** In fact, for any \( x \in C_\ast \) which is \( b \)-closed, from the following diagram

\[ \begin{array}{ccccccc}
0 & \rightarrow & u \cdot CC_0^\ast & \overset{\iota}{\rightarrow} & CC_0^\ast & \overset{\pi}{\rightarrow} & C_0 & \rightarrow & 0 \\
\downarrow b+uB & & \downarrow b+uB & & \downarrow b & & \\
0 & \rightarrow & u \cdot CC_{-1}^\ast & \overset{\iota}{\rightarrow} & CC_{-1}^\ast & \overset{\pi}{\rightarrow} & C_{-1} & \rightarrow & 0
\end{array} \]

we have, up to a boundary,

\[ \iota^{-1} \circ (b+uB) \circ \pi^{-1}(x) = \iota^{-1} \circ (b+uB)(x) = \iota^{-1}(u \cdot B(x)) = u \cdot B(x) \in u \cdot CC_{-1}^\ast. \]

Via the isomorphism \( u \cdot CC_0^\ast \cong CC_{n+1}^\ast \), this element \( u \cdot B(x) \) is mapped to \( B(x) \in CC_{n+1}^\ast \), and under \( \pi \) it is mapped to \( B(x) \). Thus \( \pi \circ \beta = B \) as desired.

**Proof of Theorem 1.1.** (1) Recall that for homogeneous \( x_1, x_2, \cdots, x_n \in \text{HC}_n^\ast \),

\[ \{x_1, x_2, \cdots, x_n\} := (-1)^{(n-1)|x_1|+(n-2)|x_2|+\cdots+|x_{n-1}|} \beta(\pi_s(x_1) \cdot \pi_s(x_2) \cdots \pi_s(x_n)). \]

We first show the graded skew-symmetry. Since the multiplication \( \cdot \) on \( \text{HH}_n \) is graded commutative, by commutativity it is sufficient to show

\[ \{x_1, x_2, \cdots, x_n\} + (-1)^{|x_1|+1} \{x_2, x_1, \cdots, x_n\} = 0. \]

In fact,

\[ \begin{align*}
\{x_1, x_2, \cdots, x_n\} & + (-1)^{|x_1|+1} \{x_2, x_1, \cdots, x_n\} \\
& = (-1)^{(n-1)|x_1|+(n-2)|x_2|+\cdots+|x_{n-1}|} \beta(\pi_s(x_1) \cdot \pi_s(x_2) \cdots \pi_s(x_n)) \\
& + (-1)^{|x_1|+1} \{x_2, x_1, \cdots, x_n\} + (-1)^{(n-1)|x_2|+(n-2)|x_1|+\cdots+|x_{n-1}|} \beta(\pi_s(x_2) \cdot \pi_s(x_1) \cdots \pi_s(x_n)) \\
& = (-1)^{(n-1)|x_1|+(n-2)|x_2|+\cdots+|x_{n-1}|} \beta(\pi_s(x_1) \cdot \pi_s(x_2) \cdots \pi_s(x_n)) \\
& - (-1)^{|x_1||x_2|+(n-2)|x_2|+(n-1)|x_1|+\cdots+|x_{n-1}|} \beta(\pi_s(x_1) \cdot \pi_s(x_2) \cdots \pi_s(x_n)).
\end{align*} \]
Next, we show the graded Jacobi identity. First,
\[
\sum_{i<j} (-1)^{i,j} \left\{ \{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right\} = \sum_{i<j} (-1)^{i,j} \beta(\pi_*(\beta(\pi_*(x_i) \cdot \pi_*(x_j))) \cdot \pi_*(x_1) \cdot \pi_*(\pi_*(x_j) \cdots \pi_*(x_n)) \\
= \sum_{i<j} (-1)^{i,j} \beta(\pi_*(x_i) \cdot \pi_*(x_j)) \cdot \pi_*(x_1) \cdot \pi_*(\pi_*(x_j) \cdots \pi_*(x_n)) \\
= \beta(\pi_*(x_1) \cdots \pi_*(x_n)) + \sum_{i} (-1)^{|x_1| + \cdots + |x_{i-1}|} \pi_*(x_1) \cdots \beta(\pi_*(x_i)) \cdots \pi_*(x_n)) \\
= 0,
\]
where \( \epsilon_{ij} = |x_i| + \cdots + |x_{i-1}| + |x_j| + |x_1| + \cdots + |x_{j-1}| - |x_i||x_j| \), and in the last equality, we have used the fact that \( \beta \circ B = \beta \circ \pi_* \circ \beta = 0 \) and \( B \circ \pi_* = \pi_* \circ \beta \circ \pi_* = 0 \). Second, by the same argument,
\[
\sum_{i<j} (-1)^{i,j} \left\{ \{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n, y_1, \ldots, y_t \right\} = \beta \left( (-1)^{i,j} (B(\pi_*(x_1) \cdots \pi_*(x_n)) \\
+ \sum_i (-1)^{|x_1| + \cdots + |x_{i-1}|} \pi_*(x_1) \cdots \beta(\pi_*(x_i)) \cdots \pi_*(x_n)) \pi_*(y_1) \cdots \pi_*(y_t) \right),
\]
which is then equal to
\[
\begin{align*}
\beta \left( \left(\pi_*(x_1) \cdots \pi_*(x_n)\right) \pi_*(y_1) \cdots \pi_*(y_t) \right) & = \beta \left( \pi_* \circ \beta \left(\pi_*(x_1) \cdots \pi_*(x_n)\right) \right) \pi_*(y_1) \cdots \pi_*(y_t) \\
& = \left\{ \{x_1, \ldots, x_n\}, y_1, \ldots, y_t \right\}.
\end{align*}
\]
This proves the statement.

(2) Since \( f : (C_*, b, B) \to (C'_*, b', B') \) is a quasi-isomorphism of mixed complexes, we have the following commutative diagram of long exact sequences
\[
\begin{array}{cccccc}
\cdots \longrightarrow & HC^{-2}_*(C) & \longrightarrow & HC^{-1}_*(C) & \longrightarrow & HH_*(C) \\
& f_* \cong & f_* \cong & f_* \cong & f_* \cong & f_* \cong \\
\cdots \longrightarrow & HC^{-2}_*(C') & \longrightarrow & HC^{-1}_*(C') & \longrightarrow & HH_*(C')
\end{array}
\]
The statement now follows directly from the construction (2.5) of the higher brackets. \( \square \)

Remark 2.6. In the above proof, in particular in (2.5), we have used the Batalin-Vilkovisky algebra structure on the Hochschild homology. This assumption seems to be too strong; in fact, a chain level (up to homotopy) Batalin-Vilkovisky structure is enough. This gives a hint to completely solve Ward’s question for general Calabi-Yau algebras on the homotopy level. The technical difficulty for us is that at present, we are not able to show the chain level noncommutative Poincaré duality and hence the homotopy Batalin-Vilkovisky structure for general Calabi-Yau algebras. We hope to address this problem in the future.
3. Differential calculus with duality and Batalin-Vilkovisky algebras

In this section, we present how to obtain the Batalin-Vilkovisky algebra structure from a special class of mixed complexes. Such mixed complexes arise from the so-called differential calculus with duality, a notion introduced by Lambre in [23].

Lambre’s notion is based on Tamarkin-Tsygan’s notion of differential calculus (see [32]), which is to capture the algebraic structure that occurs on the Hochschild cohomology and homology of associative algebras. He observed that, for algebras such as Calabi-Yau algebras and symmetric Frobenius algebras, the volume form which gives the so-called noncommutative Poincaré duality (studied by Van den Bergh [35]), can be encapsulated into the differential calculus and forms what he called differential calculus with duality.

**Definition 3.1** (Tamarkin-Tsygan [32]). Let $H^\bullet$ and $H_\bullet$ be graded vector spaces. A differential calculus is the sextuple

$$(H^\bullet, H_\bullet, \cup, \cap, \{-,-\}, B),$$

satisfying the following conditions

1. $(H^\bullet, \cup, \{-,-\})$ is a Gerstenhaber algebra;
2. $H_\bullet$ is a graded (left) module over $(H^\bullet, \cup)$ by the map
   $$\cap : H^n \otimes H_m \to H_{m-n}, \ f \otimes \alpha \mapsto f \cap \alpha,$$
   for any $f \in H^n$ and $\alpha \in H_m$, i.e., if we define $\iota_f(\alpha) := f \cap \alpha$, then $\iota_{fg}(\alpha) := \iota_f(\iota_g(\alpha))$;
3. There is a map $B : H_\bullet \to H_{\bullet+1}$ satisfying $B^2 = 0$ and moreover, if we set $L_f := [B, \iota_f] = B\iota_f - (-1)^{|f|}\iota_f B$, then
   $$[L_f, L_g] = L_{\{f,g\}},$$
   and
   $$(-1)^{|f|+1}\iota_{[f,g]} = [L_f, \iota_g] = L_{[f,g]} - (-1)^{|g|(|f|+1)}\iota_g L_f. \quad (3.1)$$

**Remark 3.2** (Comparison with Poisson modules). Recall that if $(R, \bullet, \{-,-\})$ is a Poisson algebra. A Poisson $R$-module, is a $k$-vector space, say $M$, equipped with the structures of an algebra module (denoted by $\circ$) and a Lie module (denoted by $\{-,-\}_M$) over $R$, together with the following compatibility condition

$$\{r, s \circ m\}_M = \{r, s\} \circ m + s \circ \{r, m\}_M, \quad (3.2)$$

for all $r, s \in R$ and $m \in M$. We may similarly consider modules over graded Poisson algebras, or even Gerstenhaber algebras. Now for a differential calculus $(H^\bullet, H_\bullet, \cup, \cap, \{-,-\}, B)$, given in above definition, (2) says that $H_\bullet$ is an algebra module over $H^\bullet$, and (3) says that $H_\bullet$ is a Lie module over $H^\bullet$ and that these two module structures are compatible by (3.1). In other words, for a differential calculus as above, $H_\bullet$ is a Gerstenhaber module over $H^\bullet$.

**Definition 3.3** (Lambre [23]). A differential calculus $(H^\bullet, H_\bullet, \cup, \cap, \{-,-\}, B)$ is called a differential calculus with duality if there exists an integer $n$ and an element $\eta \in H_n$ such that $\eta \cap 1 = \eta$ and $B(\eta) = 0$, and for any $i \in \mathbb{Z}$,

$$\text{PD}(\cdot) := \eta \cap - : H^i \to H_{n-i}$$

is an isomorphism. Such isomorphism is called the noncommutative Poincaré duality, and $\eta$ is called the volume form.
Suppose \((H^\bullet, H_\bullet, \cup, \cap, \{-,-\}, B)\) is a differential calculus with duality. Let \(\Delta : H^\bullet \to H^{\bullet-1}\) be the pull-back of \(B\) via the map \(\text{PD}\):

\[
\begin{array}{c}
H^\bullet \\
\downarrow \Delta \\
H_n^{\bullet-1}
\end{array}
\]

\[
\begin{array}{c}
\text{PD} \\
\downarrow \\
H_n^{\bullet-1} \cup \text{PD}
\end{array}
\]

In other words, \(\Delta = \text{PD}^{-1} \circ B \circ \text{PD}\), which is the divergence of \(B\).

**Theorem 3.4** (Lambre [23]). Suppose that \((H^\bullet, H_\bullet, \cup, \cap, \{-,-\}, B, \eta)\) is a differential calculus with duality. Then the triple \((H^\bullet, \cup, \Delta)\) is a Batalin-Vilkovisky algebra.

**Proof.** See Lambre [23] (see also [6, Theorem 5.3] for some more details). □

Alternatively, we may push the cup product on \(H^\bullet\) via \(\text{PD}\) to \(H_\bullet\), which is again denoted by \(\cap\), then \((H^\bullet, \cap, B)\), after degree shifting down by \(n\), forms a Batalin-Vilkovisky algebra.

In practice, \(H^\bullet\) is usually the homology of a mixed complex. In this case, we have the following.

**Theorem 3.5.** Suppose \((H^\bullet, H_\bullet, \cup, \cap, \{-,-\}, B, \eta)\) is a differential calculus with duality, where \(H_\bullet = H_\bullet(C_\bullet, b)\) for some mixed complex \((C_\bullet, b, B)\), then the negative cyclic homology \(HC_{-\bullet}^\circ(C_\bullet)\), after degree shifting down by \(n-2\), is a gravity algebra.

**Proof.** This is a direct corollary of Theorems 1.1 and 3.4. □

### 4. Calabi-Yau and Symmetric Frobenius Algebras

In this section, we show that the Hochschild cohomology and homology of Calabi-Yau algebras (respectively symmetric Frobenius algebras) satisfy the condition of Theorem 3.5 and thus their negative cyclic homology (respectively cyclic cohomology) has a gravity algebra structure. This completes the cases (1) and (2) in §1.

We refer the reader to Loday [26] for the definition of Hochschild homology and cohomology. Suppose \(A\) is an associative algebra over \(k\). Let \(\tilde{A} := A/k\) be its augmentation. Let \(A \to \tilde{A} : a \mapsto \bar{a}\) be the projection. Denote by \((\tilde{C}^\bullet(A), \delta)\) the reduced Hochschild cochain complex of \(A\). Let us recall that:

1. The Gerstenhaber cup product on \(\tilde{C}^\bullet(A)\) is given as follows: for any \(f \in \tilde{C}^n(A)\) and \(g \in \tilde{C}^m(A)\),

\[
f \cup g(\bar{a}_1, \ldots, \bar{a}_{n+m}) := (-1)^{nm} f(\bar{a}_1, \ldots, \bar{a}_n)g(\bar{a}_{n+1}, \ldots, \bar{a}_{n+m}),
\]

where \((\bar{a}_{n+1}, \ldots, \bar{a}_{n+m}) \in \tilde{A}^\otimes(n+m)\).

2. For any \(f \in \tilde{C}^n(A)\) and \(g \in \tilde{C}^m(A)\), let

\[
f \circ g(\bar{a}_1, \ldots, \bar{a}_{n+m-1})
\]

\[
:= \sum_{i=0}^{n-1} (-1)^{|g|+1} f(\bar{a}_1, \ldots, \bar{a}_i, g(\bar{a}_{i+1}, \ldots, \bar{a}_{i+m}), \bar{a}_{i+m+1}, \ldots, \bar{a}_{n+m-1}),
\]

then the Gerstenhaber bracket on \(\tilde{C}^\bullet(A)\) is defined to be

\[
\{f, g\} := f \circ g - (-1)^{|f|+1)}|g|+1) g \circ f.
\]
Theorem 4.1 (Gerstenhaber [13]). Let $A$ be an associative algebra. Then the Hochschild cohomology $HH^* (A)$ of $A$ equipped with the Gerstenhaber cup product and the Gerstenhaber bracket forms a Gerstenhaber algebra.

Now let $(\bar{C}^\bullet (A), \partial)$ be the reduced Hochschild chain complex of $A$. Then $\bar{C}^\bullet (A)$ acts on $\bar{C}^\bullet (A)$, called the cap product, as follows: for homogeneous $f \in \bar{C}^n (A)$ and $\alpha = (a_0, \bar{a}_1, \ldots, \bar{a}_m) \in \bar{C}_m (A)$, the cap product $\cap: \bar{C}^n (A) \times \bar{C}_m (A) \to \bar{C}_{m-n} (A)$ is given by
$$f \cap \alpha := (a_0 f (\bar{a}_1, \ldots, \bar{a}_n), \bar{a}_{n+1}, \ldots, \bar{a}_m),$$
for $m \geq n$, and zero otherwise. We denote by $\iota_f (-) := f \cap -$ the contraction operator, then $\iota_f \iota_g = (-1)^{|f||g|} \iota_g \iota_f$.

Recall the Connes operator $B: \bar{C}^\bullet (A) \to \bar{C}^\bullet +1 (A)$ is given by
$$B (\alpha) := \sum_{i=0}^{m} (-1)^{mi} (1, \bar{a}_i, \cdots, \bar{a}_m, \bar{a}_0, \cdots, \bar{a}_{i-1}).$$

It is known that $(\bar{C}^\bullet (A), \partial, B)$ forms a mixed complex.

From $\iota$ and $B$, we may define the Lie derivative of $\bar{C}^\bullet (A)$ on $\bar{C}^\bullet (A)$ by $L := [B, \iota]$. It is then direct to see $\partial = L_\mu$, where $\mu$ is the product of $A$, viewed as a cochain in $\bar{C}^{-2} (A)$.

Theorem 4.2 (Daletskii-Gelfand-Tsygan [8]; Tamarkin-Tsygan [32]). Let $A$ be an associative algebra. Denote by $HH^* (A)$ and $HH_\bullet (A)$ the Hochschild cohomology and homology of $A$ respectively. Then the following sextuple
$$(HH^* (A), HH_\bullet (A), \cup, \cap, \{-,-\}, B)$$
is a differential calculus.

We remark that on the chain level, some identities in the above theorem only hold up to homotopy, which is highly nontrivial.

In the rest of this section, we give two examples of differential calculus with duality, one from Calabi-Yau algebras and the other from symmetric Frobenius algebras. They are obtained by Lambre in [23], which, in particular, put the Batalin-Vilkovisky algebra structure, first obtained by Ginzburg [17, Theorem 3.4.3] and Tradler [34, Theorem 1], in a general framework. A new observation here, if there is any, is that these two examples satisfy the condition of Theorem 3.5, and hence we obtain a gravity algebra structure on the associated negative cyclic homology.

4.1. Calabi-Yau algebras. The notion of Calabi-Yau algebra is introduced by Ginzburg in [17]. In this subsection, we briefly recall the differential calculus structure on Calabi-Yau algebras.

Definition 4.3 (Calabi-Yau algebra). An algebra $A$ is called a Calabi-Yau algebra of dimension $d$ (or $d$-Calabi-Yau algebra) if

1. $A$ is homologically smooth, that is, $A$, viewed as an $A^e$-module, has a bounded resolution by finitely generated projective $A^e$-modules, and
2. there is an isomorphism
$$\eta: R\text{Hom}_{A^e} (A, A \otimes A) \to \Sigma^{-d} A$$ (4.1)
in the derived category $D(A^e)$ of (left) $A^e$-modules, where $A^e$ is the enveloping algebra of $A$ and $\Sigma^{-d}(-)$ is the $d$-fold desuspension functor.

**Theorem 4.4** (de Thanhoffer de Völcsey-Van den Bergh [9], Lambre [23]). If $A$ is $n$-Calabi-Yau, then there exists a volume form $\Omega \in \HH_n(A)$ such that

$$\text{PD} : \HH^i(A) \rightarrow \HH_{n-i}(A)$$

$f \mapsto f \cap \Omega$

is an isomorphism. That is,

$$(\HH^*(A), \HH_*(A), \cup, \cap, \{-, -\}, B)$$

is in fact a differential calculus with duality.

For a proof of this statement, see [9, Proposition 5.5] and also [23]. Note that $\HH_*(A)$ is the $b$-homology of the mixed complex $(\tilde{C}^\bullet(A), b, B)$, which satisfies the condition of Theorem 3.5, and hence we obtain a gravity algebra structure on its negative cyclic homology, which may be viewed as induced from the Batalin-Vilkovisky structure on the Hochschild cohomology.

**4.2. Symmetric Frobenius algebras.** For an associative algebra $A$, denote $A^* := \text{Hom}(A, k)$, and then $A^*$ is an $A$-bimodule. Let $\tilde{C}^\bullet(A; A^*)$ be the reduced Hochschild cochain complex of $A$ with values in $A^*$. Under the identities

$$\tilde{C}^\bullet(A; A^*) = \bigoplus_{n \geq 0} \text{Hom}(A \otimes^\mathbb{L} A \otimes A \otimes k, A^*) = \text{Hom}(\tilde{C}^\bullet(A), k),$$

it is proved, for example, in [26, §2.4], that the Hochschild coboundary of the leftmost is identical with the dual of the Hochschild boundary of the rightmost. Moreover, via the above identity, one may equip on $\tilde{C}^\bullet(A; A^*)$ the dual Connes differential, which is denoted by $B^*$, i.e., $B^*(g) := (-1)^{|g|} g \circ B$ for homogeneous $g \in \tilde{C}^\bullet(A; A^*)$. It is then direct to see that $B^*$ has square zero and commutes with the Hochschild coboundary operator.

The Hochschild cochain complex $\tilde{C}^\bullet(A)$ acts on $\tilde{C}^\bullet(A; A^*)$ via the following “cap product”

$$\cap^* : \tilde{C}^\bullet(A) \times \tilde{C}^\bullet(A; A^*) \rightarrow \tilde{C}^\bullet(A; A^*)$$

$$(f, \alpha) \mapsto (-1)^{|f||\alpha|} \alpha \circ f,$$

which respects the Hochschild coboundaries. We thus obtain the following.

**Theorem 4.5.** Let $A$ be an associative algebra. Then

$$(\HH^*(A), \HH^*(A; A^*), \cup, \cap^*, \{-, -\}, B^*)$$

is a differential calculus.

**Proof.** We already know from Theorem 4.2 that

$$(\HH^*(A), \HH_*(A), \cup, \cap, \{-, -\}, B)$$

is a differential calculus. Note that by (4.2), $\HH^*(A; A^*)$ is the linear dual of $\HH_*(A)$, and therefore by Remark 3.2, $\HH^*(A; A^*)$ is a Gerstenhaber module over $\HH^*(A)$ given by the adjoint action. Since $L^* = [B, \cap]^* = [B^*, \cap]^*$, the differential calculus structure is obtained as desired.

□
The above theorem can be directly generalized to graded algebras without any difficulty. Now let us recall that a graded associative algebra $A$ is called symmetric Frobenius (or cyclic associative) of degree $n$ if there is a non-degenerate bilinear pairing

$$\langle -, - \rangle : A \times A \to k$$

which is cyclically invariant; that is, $\langle a \cdot b, c \rangle = (-1)^{|c|(|a|+|b|)}\langle c \cdot a, b \rangle$, for all homogeneous $a, b, c \in A$. It is direct to see that this is equivalent to a degree $n$ isomorphism of $A$-bimodules

$$\eta : A \to A^*,$$

which is completely determined by the image of 1. Viewing $\eta \in \overline{C}^n(A; A^*)$, then that $\eta$ is an $A$-bimodule map means $\delta(\eta) = 0$ and hence $\eta$ is a Hochschild cocycle. The isomorphism of $\eta$ in fact means $[\eta] \neq 0$. Let

$$\text{PD} : \text{HH}^\bullet(A) \to \text{HH}^{\bullet-n}(A; A^*)$$

be the composition

$$\text{Hom}(\overline{A} \otimes q, A) \xrightarrow{\eta} \text{Hom}(\overline{A} \otimes q, A^*),$$

which passes to the cohomology level, then one obtains the following.

**Theorem 4.6** (Lambre [23]). Suppose $A$ is a symmetric Frobenius algebra. Then

$$(\text{HH}^\bullet(A), \text{HH}^\bullet(A; A^*))$$

forms a differential calculus with duality.

Again, $\text{HH}^\bullet(A; A^*)$ is the $\delta$-cohomology of the mixed complex $(\overline{C}^\bullet(A; A^*), \delta, B^*)$, and by Theorem 3.5 we obtain a gravity algebra structure on its cyclic cohomology, induced from the Batalin-Vilkovisky structure on the Hochschild cohomology.

**Remark 4.7.** (1) The Batalin-Vilkovisky algebra structure on the Hochschild cohomology has been obtained by Tradler [34] (see also Menichi [29] and Ward [37]).

(2) The two graded Lie algebras that are contained in the gravity algebras after Theorems 4.4 and 4.6 are nontrivial in general. In fact, it is proved by de Thanhoffer de Völcsey and Van den Bergh [9] and Terilla and Tradler in [33] that the DG Lie algebras that control the deformations of the Calabi-Yau algebras and symmetric Frobenius algebras, are quasi-isomorphic to the negative cyclic complex and cyclic cochain complex of these two algebras respectively; the Lie brackets on the homology level are exactly identical to the Lie brackets contained in the gravity algebra structure.

5. **Unimodular Poisson and Frobenius Poisson algebras**

In this section, we show that the Poisson chain and cochain complexes of unimodular Poisson and unimodular Frobenius Poisson algebras satisfy the condition of Theorem 3.5. The differential calculus with duality structure that appears in these two cases are already implicit in Xu [41] and Zhu, Van Oystaeyen and Zhang [42], where the Batalin-Vilkovisky algebra are also shown. Therefore the associated negative cyclic Poisson homology (respectively cyclic Poisson cohomology) has a gravity algebra structure. This completes the cases (3) and (4) in [41].
Suppose $A$ is a (possibly graded) commutative algebra, and $M$ an $A$-bimodule. Let $\Omega^p(A)$ be the set of $p$-th Kähler differential forms of $A$, and $\mathcal{H}^p(A; M)$ be the set of skew-symmetric multilinear maps $A^{\otimes p} \to M$ which are derivations in each argument. There is an isomorphism of left $A$-modules

$$\mathcal{H}^p(A; M) \cong \text{Hom}_A(\Omega^p(A), M).$$

If $M = A$, then we simply write $\mathcal{H}^p(A; M)$ as $\mathcal{H}^p(A)$.

**Definition 5.1** (Poisson homology; Koszul [22]). Suppose $A$ is a Poisson algebra with the Poisson structure $\pi$. Denote by $\Omega^p(A)$ the set of $p$-th Kähler differential forms of $A$. Then the *Poisson chain complex* of $A$, denoted by $\mathcal{C}^p(A)$, is

$$\cdots \to \Omega^{p+1}(A) \xrightarrow{\partial} \Omega^p(A) \xrightarrow{\partial} \Omega^{p-1}(A) \xrightarrow{\partial} \cdots \to \Omega^0(A) = A,$$

(5.1)

where $\partial$ is given by

$$\partial(f_0 df_1 \wedge \cdots \wedge df_p) = \sum_{i=1}^p (-1)^{i-1} \{f_0, f_i\} df_1 \wedge \cdots \widehat{df_i} \cdots \wedge df_p + \sum_{1 \leq i < j \leq p} (-1)^{i-j} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \widehat{df_i} \cdots \widehat{df_j} \cdots \wedge df_p.
$$

The associated homology is called the *Poisson homology* of $A$, and is denoted by $H_p(A)$.

In the above definition, $\{f_i, f_j\}$ means $\pi(f_i, f_j)$. It is direct to check (see also Xu [41]) that $\partial$ commutes with the de Rham differential $d$, and therefore $(\Omega^\bullet(A), \partial, d)$ is a mixed complex.

**Definition 5.2** (Poisson cohomology; Lichnerowicz [25]). Suppose $A$ is a Poisson algebra and $M$ is a left Poisson $A$-module. Let $\mathcal{H}^p(A; M)$ be the space of skew-symmetric multilinear maps $A^{\otimes p} \to M$ that are derivations in each argument. The *Poisson cochain complex* of $A$ with values in $M$, denoted by $\mathcal{C}^p(A; M)$, is the cochain complex

$$M = \mathcal{H}^0(A; M) \xrightarrow{\delta} \cdots \to \mathcal{H}^{p+1}(A; M) \xrightarrow{\delta} \mathcal{H}^p(A; M) \xrightarrow{\delta} \cdots$$

where $\delta$ is given by

$$\delta(P)(f_0, f_1, \cdots, f_p) := \sum_{0 \leq i \leq p} (-1)^i \{f_i, P(f_0, \cdots, \widehat{f_i}, \cdots, f_p)\}$$

$$+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} P(\{f_i, f_j\}, f_1, \cdots, \widehat{f_i}, \cdots, \widehat{f_j}, \cdots, f_p).$$

The associated cohomology is called the *Poisson cohomology* of $A$ with values in $M$, and is denoted by $H_p^\bullet(A; M)$. In particular, if $M = A$, then the cohomology is just called the *Poisson cohomology* of $A$, and is simply denoted by $H^\bullet(A)$.

Note that in the above definition, the Poisson cochain complex, viewed as a chain complex, is negatively graded, and the coboundary $\delta$ has degree $-1$. However, by our convention, the Poisson cohomology is positively graded. In the following, we present two versions of differential calculus on the Poisson (co)homology.
5.1. **Unimodular Poisson algebras.** Given a commutative algebra $A$, we have the following operations on $\mathfrak{X}^\bullet(A)$ and $\Omega^\bullet(A)$:

1. **Wedge (cup) product:** suppose $P \in \mathfrak{X}^p(A)$ and $Q \in \mathfrak{X}^q(A)$, then the wedge product of $P$ and $Q$, denoted by $P \wedge Q$, is a polyvector in $\mathfrak{X}^{p+q}(A)$, defined by

$$
(P \wedge Q)(f_1, f_2, \cdots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) \cdot Q(f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q)}),
$$

where $\sigma$ runs over all $(p, q)$-shuffles of $(1, 2, \cdots, p + q)$.

2. **Schouten bracket:** suppose $P \in \mathfrak{X}^p(A)$ and $Q \in \mathfrak{X}^q(A)$, then their Schouten bracket is an element in $\mathfrak{X}^{p+q-1}(A)$, which is denoted by $[P, Q]$ and is given by

$$
[P, Q](f_1, f_2, \cdots, f_{p+q-1}) := \sum_{\sigma \in S_{p,q-1}} \text{sgn}(\sigma) P(Q(f_{\sigma(1)}, \cdots, f_{\sigma(q)}), f_{\sigma(q+1)}, \cdots, f_{\sigma(p+q-1)})
$$

$$
- (-1)^{p-1} \sum_{\sigma \in S_{p,q-1}} \text{sgn}(\sigma) Q(P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}), f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q-1)}).
$$

3. **Contraction (inner product):** suppose $P \in \mathfrak{X}^p(A)$ and $Q = df_1 \wedge \cdots \wedge df_n \in \Omega^n(A)$, then the contraction (also called inner product or internal product) of $P$ with $Q$, denoted by $\iota_P(Q)$, is an $A$-linear map with values in $\Omega^{n-p}(A)$ given by

$$
\iota_P(Q) = \sum_{\sigma \in S_{p,n-p}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) d f_{\sigma(p+1)} \wedge \cdots \wedge d f_{\sigma(n)}, \quad \text{if } n \geq p,
$$

$$
0, \quad \text{otherwise.}
$$

**Proposition 5.3.** Suppose $A$ is a Poisson algebra. Then

$$(\text{HP}^\bullet(A), \text{HP}_\bullet(A), \wedge, \iota, [-, -], d)$$

forms a differential calculus, where $d$ is the de Rham differential.

**Proof.** We only need to show the operations given in the proposition respect the Poisson boundary and coboundary, which can be found in, for example, [24, Chapter 3].

In the above proposition, let $L := [d, \iota]$. Then similar to the Hochschild case, $\partial = L_\pi$, where $\pi$ is the Poisson structure.

For a Poisson algebra $A$, suppose there exists a form $\eta \in \Omega^n(A)$ such that $\iota_{(-)}\eta : \mathfrak{X}^\bullet(A) \to \Omega^{n+\bullet}(A)$ is an isomorphism, then $\eta$ is called a *volume form*. In this case, we have the following diagram

$$
\begin{array}{ccc}
\mathfrak{X}^\bullet(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n+\bullet}(A) \\
\downarrow{\delta} & & \downarrow{\partial} \\
\mathfrak{X}^{\bullet-1}(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n+\bullet-1}(A),
\end{array}
$$

which may not be commutative, i.e., $\eta$ may not be a Poisson cycle. If there exists a volume form $\eta$ such that the above diagram commutes, then we say $A$ is *unimodular*.

**Theorem 5.4.** Suppose $A$ is a unimodular Poisson algebra. Then

$$(\text{HP}^\bullet(A), \text{HP}_\bullet(A), \wedge, \iota, [-, -], d)$$

forms a differential calculus with duality.
The proof of this theorem is given in Xu [41], and in particular, as a corollary, \( \text{HP}^*(A) \) is a Batalin-Vilkovisky algebra, where the Batalin-Vilkovisky operator generates the Schouten-Nijenhuis bracket. The negative cyclic homology of the mixed complex \((\Omega^* A, \partial, d)\), denoted by \( \text{PC}^{-*}(A) \) and called the negative cyclic Poisson homology of \( A \), has a gravity algebra structure, induced from the Batalin-Vilkovisky structure on the Poisson cohomology.

5.1.1. Unimodular Frobenius Poisson algebras. Suppose \( A \) is a Poisson algebra, and let \( A^* \) be its linear dual space. For any \( P \in \mathcal{X}^p(A) \) and \( \phi \in \mathcal{X}^q(A; A^*) \), let \( \iota P(\phi) \in \mathcal{X}^{p+q}(A; A^*) \) be given by

\[
(\iota P(\phi))(f_1, \cdots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) \cdot \phi(f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q)}).
\] (5.4)

Observe that

\[
\mathcal{X}^{*}(A; A^*) = \text{Hom}_A(\Omega^* A, A^*).
\]

By dualizing the de Rham differential \( d \) on \( \Omega^* (A) \), we obtain a differential \( d^* \) on \( \text{Hom}(\Omega^* A, k) \), i.e., on \( \mathcal{X}^*(A; A^*) \), which commutes with the Poisson coboundary (see [42, Theorem 4.10] for a proof).

**Proposition 5.5.** Suppose \( A \) is a Poisson algebra. Then \( (\text{HP}^*(A), \text{HP}^*(A; A^*)) \) has a differential calculus structure, where \( A^* \) is the dual space of \( A \).

**Proof.** Parallel to the proof of Theorem 4.6. \( \square \)

Now, we go to unimodular Frobenius Poisson algebras, a notion introduced by Zhu, Van Oystaeyen and Zhang in [42]. Suppose \( A^! \) is a finite dimensional graded Poisson algebra. If there is an \( A^! \)-module isomorphism

\[
\eta^! : (A^!)^* \longrightarrow (A^!|n\cdot=^*), \quad \text{for some } n \in \mathbb{N},
\]

where \( A^i := (A^!|i^* = \text{Hom}(A^!, k) \), then we may view \( \eta^! \) as an element in \( \text{Hom}_{A^!}(A^i; A^i) \subset \mathcal{X}^*(A^!; A^i) \), and have a diagram

\[
\begin{array}{ccc}
\mathcal{X}^*(A^i) & \xrightarrow{(\iota)^{-1}(\eta^!)} & \mathcal{X}^{n+i}(A^i; A^i) \\
\downarrow \delta & & \downarrow \delta \\
\mathcal{X}^{n-1}(A^i) & \xrightarrow{(\iota)^{-1}(\eta^!)} & \mathcal{X}^{n+i-1}(A^i; A^i).
\end{array}
\] (5.5)

**Definition 5.6** (Unimodular Frobenius Poisson algebra; [42]). Suppose \( A^i \) is a finite dimensional (graded) Poisson algebra. If there is an \( \eta^! \in \mathcal{X}^*(A^i; A^i) \) (also called the volume form) such that the diagram (5.5) commutes, then \( A^i \) is called a unimodular Frobenius Poisson algebra of degree \( n \).
Alternatively described, a volume form is a nonzero element $\eta$ in the top degree of $A^i$. It gives the unimodular Poisson structure on $A^i$ if and only if it is a Poisson cycle. From the definition, one immediately deduces that:

**Theorem 5.7.** Suppose $A$ is a unimodular Frobenius Poisson algebra. Then

$$(\text{HP}^*(A), \text{HP}^*(A; A^*))$$

forms a differential calculus with duality.

The proof of this theorem is given in [42], although the authors did not express the statement in the above form. As a corollary, $\text{HP}^*(A)$ is a Batalin-Vilkovisky algebra, where the Batalin-Vilkovisky operator again generates the Schouten-Nijenhuis bracket. The cyclic cohomology of the mixed cochain complex $(X^*(A; A^*), \delta, d^*)$, denoted by $\text{PC}^*(A)$ and called the *cyclic Poisson cohomology* of $A$, therefore has a gravity algebra structure again induced from the Batalin-Vilkovisky structure on the Poisson cohomology.

6. **Koszul duality**

The purpose of this section is to relate the gravity algebras obtained in previous sections by means of Koszul duality. This completes relationships (a) and (b) listed in §1.

6.1. **Quadratic and Koszul algebras.** Let $V$ be a finite-dimensional (possibly graded) vector space over $k$. Denote by $TV$ the free algebra generated by $V$ over $k$. Suppose $R$ is a subspace of $V \otimes V$, and let $(R)$ be the two-sided ideal generated by $R$ in $TV$, then the quotient algebra $A := TV/(R)$ is called a quadratic algebra. There are two concepts associated to a quadratic algebra, namely, its *Koszul dual coalgebra* and *Koszul dual algebra*, which are given as follows:

1. Consider the subspace $U = \bigoplus_{n=0}^{\infty} U_n := \bigoplus_{n=0}^{\infty} \bigcap_{i+j+2=nm} V^i \otimes R \otimes V^j$ of $TV$, then $U$ is not an algebra, but a coalgebra, whose coproduct is induced from the de-concatenation of the tensor products. The *Koszul dual coalgebra* of $A$, denoted by $A^\¡$, is

$$A^\¡ = \bigoplus_{n=0}^{\infty} \Sigma^\otimes (U_n),$$

where $\Sigma$ is the degree shifting-up (suspension) functor. $A^\¡$ naturally has a graded coalgebra structure induced from that of $U$; for example, if all elements of $V$ have degree zero, then

$$(A^\¡)_0 = k, \quad (A^\¡)_1 = V, \quad (A^\¡)_2 = R, \quad \cdots$$

2. The *Koszul dual algebra* of $A$, denoted by $A^!$, is just the linear dual space of $A^\¡$, which is then a graded algebra. More precisely, Let $V^* = \text{Hom}(V, k)$ be the linear dual space of $V$, and let $R^\perp$ denote the space of annihilators of $R$ in $V^* \otimes V^*$. Shift the grading of $V^*$ down by one, denoted by $\Sigma^{-1} V^*$, then

$$A^! = T(\Sigma^{-1} V^*)/((\Sigma^{-1} \otimes \Sigma^{-1}) \circ R^\perp).$$
Choose a set of basis \( \{e_i\} \) for \( V \), and let \( \{e_i^*\} \) be their duals in \( V^* \). There is a natural chain complex associated to \( A \), called the Koszul complex:

\[
\cdots \longrightarrow A \otimes A_i^{i+1} \overset{\delta}{\longrightarrow} A \otimes A_i^i \overset{\delta}{\longrightarrow} \cdots \longrightarrow A \otimes A_i^0 \overset{\delta}{\longrightarrow} k,
\]

where for any \( r \otimes f \in A \otimes A^i \), \( \delta(r \otimes f) = \sum_i e_i r \otimes \Sigma^{-1} e_i^* f \).

**Definition 6.1** (Koszul algebra). A quadratic algebra \( A = TV/(R) \) is called Koszul if the Koszul chain complex (6.1) is acyclic.

### 6.2. Koszul duality for Calabi-Yau algebras.

For Koszul Calabi-Yau algebras, we have the following result due to Van den Bergh [36]: Suppose \( A \) is a Koszul algebra, and denote by \( A^! \) its Koszul dual algebra; then \( A \) is \( d \)-Calabi-Yau if and only if \( A^! \) is cyclic of degree \( d \). This can be seen as follows: Since \( A \) is Koszul, the following complex

\[
\cdots \longrightarrow A \otimes A_m^i \otimes A \overset{b}{\longrightarrow} A \otimes A_{m-1}^i \otimes A \overset{b}{\longrightarrow} \cdots \overset{b}{\longrightarrow} A \otimes A_0^i \otimes A
\]

with

\[
b(a \otimes c \otimes a') = \sum_i (e_i a \otimes e_i^* c \otimes a' + (-1)^m a \otimes e_i^* a') \tag{6.2}
\]

gives a free, minimal resolution of \( A \) as a (left) \( A^e \) module. Now suppose \( A \) is Calabi-Yau, we have

\[
\text{RHom}_{A^e}(A, A \otimes A) = \text{Hom}_{A^e}(A \otimes A^! \otimes A, A \otimes A) = A \otimes A^! \otimes A
\]

in \( D(A^e) \), where the differential of \( A \otimes A^i \otimes A \) is similar to (6.2). It is isomorphic to \( \Sigma^{-n} A \) in \( D(A^e) \) means

\[
A \otimes A^! \otimes A \cong A \otimes \Sigma^{-n} A^! \otimes A,
\]

which then implies \( A^! \cong \Sigma^{-n} A^i \) as \( A^! \) bimodules by the uniqueness of minimal resolutions. That is, \( A^! \) is a symmetric Frobenius algebra of dimension \( n \).

**Proposition 6.2.** For a Koszul Calabi-Yau algebra \( A \), denote \( A^! \) to be its Koszul dual algebra. Then:

1. there exists a quasi-isomorphism of DG algebras \( \tilde{C}^*(A) \simeq \tilde{C}^*(A^!) \), which induces, on the Hochschild cohomology level, an isomorphism of graded commutative algebras;
2. \( (\tilde{C}^*(A), b, B) \simeq (\tilde{C}^*(A^!; A^i), \delta, B^*) \) as mixed complexes, which, on the Hochschild homology level, maps the volume class to the volume class;
3. the DG algebra action (the inner product) of \( \tilde{C}^*(A) \) on \( \tilde{C}^*(A) \) and that of \( \tilde{C}^*(A^!) \) on \( \tilde{C}^*(A^!; A^i) \) are compatible on the homology level.

Since the Gerstenhaber bracket on the Hochschild cohomology is generated by the Batalin-Vilkovisky operator, this proposition implies that the two pairs

\[
(\text{HH}^*(A), \text{HH}_*(A)) \quad \text{and} \quad (\text{HH}^*(A^!), \text{HH}_*(A^!; A^i))
\]

are isomorphic as differential calculus with duality. Thus as a corollary, the first author together with Yang and Zhou proved in [7] that for Koszul Calabi-Yau algebras, \( \text{HH}^*(A) \cong \text{HH}^*(A^!) \) as Batalin-Vilkovisky algebras. For reader’s convenience we sketch the proof of
the above proposition in the following while leaving the details for the interested readers to refer to [7].

Proof of Proposition 6.2. (1) First, since $A$ is Koszul, its bar construction $B(A) \simeq A^i$ as DG coalgebras. With this quasi-isomorphism we obtain a quasi-isomorphism of DG algebras

$$\bar{C}^\bullet(A) \cong \text{Hom}(B(A), A) \simeq \text{Hom}(A^i, A) \cong A \otimes A^i,$$

where the differential on $A \otimes A^i$ is given by

$$\delta(a \otimes x) = \sum_i \left( e_i a \otimes e_i^* x + (-1)^{|x|} a e_i \otimes x e_i^* \right).$$

Second, with the same argument and with the differentials appropriately assigned, we obtain a quasi-isomorphism of DG algebras

$$\bar{C}^\bullet(A^i) \cong \text{Hom}(B(A^i), A^i) \simeq A^i \otimes \Omega(A^i) \cong A^i \otimes A,$$

where $\Omega(A^i)$ is the cobar construction of $A^i$, which is quasi-isomorphic to $A$ as DG algebras. Now the right-most two DG algebras in (6.3) and (6.4) are quasi-isomorphic as DG algebras via $a \otimes x \mapsto x \otimes a$, from which follows the desired quasi-isomorphism $\bar{C}^\bullet(A) \simeq \bar{C}^\bullet(A^i)$.

(2) Equip $A \otimes A^i$ with differential

$$b(a \otimes c) = \sum_i \left( a e_i \otimes c \cdot e_i^* + (-1)^m a e_i \otimes c \right).$$

It is proved in [7] Lemma 16] that we have a commutative diagram

$$\begin{array}{ccc}
\bar{C}^\bullet(A) & & \bar{C}^\bullet(A^i; A^i) \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
A \otimes A^i & & \\
\end{array}$$

\begin{equation}
(6.5)
\end{equation}

where $p_1$ and $p_2$ are quasi-isomorphisms of mixed complexes, all other maps are quasi-isomorphisms of $b$-complexes, and $p_2$ and $q_2$ are homotopy inverse to each other. This means that even though $A \otimes A^i$ has no mixed complex structure, on the homology level, it gives an isomorphism

$$\text{HH}_\bullet(A) \cong \text{HH}_\bullet(A \otimes A^i, b) \cong \text{HH}^\bullet(A^i; A^i)$$

which identifies the $B$ operator on the left-most term with the $B^*$ operator on the right-most term.

It is also proved in [7] that the volume class in $\text{HH}_\bullet(A)$ and $\text{HH}^\bullet(A^i; A^i)$, via the above isomorphism, is represented by a nonzero element $\omega \in A^i_n \equiv k \otimes A^i_n \subset A \otimes A^i$. 


(3) Since $A_1$ is a graded coalgebra and hence admits an action of $A^1$, we have that $A \otimes A^1$ acts on $A \otimes A_1$. Denote this action by $\circ$, then we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}^\bullet(A) & \xrightarrow{\cap} & \mathcal{C}^\bullet(A) \\
\approx & & \approx \\
A \otimes A^1 & \xrightarrow{\circ} & A \otimes A^1,
\end{array}
\] (6.6)

where the curved arrows mean the module actions. By exactly the same argument, we have that the following diagram

\[
\begin{array}{ccc}
\mathcal{C}^\bullet(A^1) & \xrightarrow{\cap^*} & \mathcal{C}^\bullet(A^1, A^i) \\
\approx & & \approx \\
A \otimes A^1 & \xrightarrow{\circ} & A \otimes A^1,
\end{array}
\] (6.7)

is commutative. Combining the above two diagrams (6.6) and (6.7) with (6.3), (6.4) and (6.5), we obtain (3).

Example 6.3 (Polynomials). Let $A = k[x_1, x_2, \ldots, x_n]$ be the space of polynomials (the symmetric tensor algebra), with each $x_i$ having degree zero. It is well-known (cf. [27]) that $A$ is a Koszul algebra, and its Koszul dual algebra $A^!$ is the graded symmetric algebra $\Lambda(\xi_1, \xi_2, \cdots, \xi_n)$, with grading $|\xi_i| = -1$. There is a non-degenerate symmetric pairing on $A^!$ given by

\[(\alpha, \beta) \mapsto \alpha \wedge \beta / \xi_1 \cdots \xi_n\]

so that $A^!$ is symmetric Frobenius, and therefore $A$ is Calabi-Yau. Equivalently, the pairing on $A^1$ that gives $A^! \cong \Sigma^{-n} A^1$ is the same as capping with the following form

\[A^* \to \Sigma^{-n} A^1: \quad x \mapsto x \cap \xi_1^* \cdots \xi_n^*.
\]

By pulling $\xi_1^* \cdots \xi_n^*$ via the quasiisomorphism $A \otimes A^! \simeq \mathcal{C}^\bullet(A)$, we get the volume form on $A$. Its homology class is exactly $dx_1 \cdots dx_n$ under the Hochschild-Kostant-Rosenberg map.

Corollary 6.4. Suppose $A$ is a $d$-Calabi-Yau algebra and $A^!$ is its Koszul dual. Then

\[HC^\bullet(A) \cong HC^\bullet(A^!)
\]
as gravity algebras.

Proof. Proposition 6.2 implies that

\[(CH_\bullet(A), b, B) \simeq (CH^\bullet(A^1; A^i), \delta, B^*)\]
as mixed complexes. Combine it with the quasi-isomorphism of the Hochschild cochain complexes, we obtain that

\[(HH^\bullet(A), HH_\bullet(A)) \quad \text{and} \quad (HH^\bullet(A^1), HH_\bullet(A^1; A^i))\]
are isomorphic as differential calculus with duality, which, by Theorems 4.4 and 4.6, induces an isomorphism

\[HH_\bullet(A) \cong HH^\bullet(A^1; A^i)\]
as Batalin-Vilkovisky algebras. The statement now follows from Theorem 1.1 (2). \qed
6.3. Koszul duality for quadratic Poisson polynomial algebras. For \( A = k[x_1, x_2, \cdots, x_n] \) with a quadratic bivector
\[
\pi = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{i_1 i_2}^{j_1 j_2} \in k, \tag{6.8}
\]
there is a bivector \( \pi' \) on \( A^! = \Lambda(\xi_1, \xi_2, \cdots, \xi_n) \), which we would call the Koszul dual of \( \pi \) and is given by
\[
\pi' := \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} \xi_{i_1} \xi_{i_2} \frac{\partial}{\partial \xi_{j_1}} \wedge \frac{\partial}{\partial \xi_{j_2}}. \tag{6.9}
\]
Shoikhet showed in [31] that for \( A = k[x_1, \cdots, x_n] \) with a bivector \( \pi \) in the form \([6.8]\).

Then \((A, \pi)\) is Poisson if and only if \((A^!, \pi')\) is Poisson. We have the following result, obtained in [9]:

**Proposition 6.5.** Suppose \( A = k[x_1, \cdots, x_n] \) is a quadratic Poisson algebra, and let \( A^! \) be its Koszul dual. Then \( A \) is unimodular if and only if \( A^! \) is unimodular Frobenius, and in this case we have \( \text{“}\cong\text{” means isomorphism} \):

1. \( \text{CP}^*(A) \cong \text{CP}^*(A^!) \) as DG algebras;
2. \( \text{CP}_*(A) \cong \text{CP}^*(A^!; A^!) \) as mixed complexes, and moreover, under the isomorphism, the volume form of the former is mapped to the volume form of the latter;
3. The DG algebra action (the inner product) of \( \text{CP}^*(A) \) on \( \text{CP}_*(A) \) and that of \( \text{CP}^*(A^!) \) on \( \text{CP}^*(A^!; A^!) \) are compatible under the above two isomorphisms.

**Proof.** (1) Since \( A = k[x_1, \cdots, x_n] \), we have an explicit expression for \( \Omega^*(A) \), which is
\[
\Omega^*(A) = \Lambda(x_1, \cdots, x_n, dx_1, \cdots, dx_n), \tag{6.10}
\]
where \( \Lambda \) means the graded symmetric tensor product, and \(|x_i| = 0 \) and \(|dx_i| = 1\), for \( i = 1, \cdots, n \). Similarly,
\[
\Omega^*(A^!) = \Lambda(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n), \tag{6.11}
\]
where \(|\xi_i| = -1\) and \(|d\xi_i| = 0\) for \( i = 1, \cdots, n \). From \((6.10)\) and \((6.11)\) we have the following:
\[
\mathfrak{X}^*(A) = \text{Hom}_A(\Omega^*(A), A) = \text{Hom}_A(x_1, \cdots, x_n, dx_1, \cdots, dx_n, \Lambda(x_1, \cdots, x_n)),
\]
where
\[
\mathfrak{X}^*(A^!) = \text{Hom}_A(\Omega^*(A^!), A^!) = \text{Hom}_A(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n, \Lambda(\xi_1, \cdots, \xi_n)).
\]
Under the identification
\[ x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i, \quad i = 1, \ldots, n, \] (6.12)
we get \( \mathfrak{X}^\bullet(A) \cong \mathfrak{X}^\bullet(A^!) \). It is now straightforward to check that this isomorphism is in fact an isomorphism of DG Gerstenhaber algebras.

(2) Similarly,
\[
\mathfrak{X}^\bullet(A^!; A^!) = \text{Hom}_A(\Omega^\bullet(A^!), A^!)
\]
\[ = \text{Hom}_{A^{(\xi_1, \ldots, \xi_n)}}(\Lambda(\xi_1, \ldots, \xi_n, d\xi_1, \ldots, d\xi_n), \text{Hom}(\Lambda(\xi_1, \ldots, \xi_n), k))
\]
\[ = \text{Hom}_{A^{(\xi_1, \ldots, \xi_n)}}(\Lambda(\xi_1, \ldots, \xi_n) \otimes \Lambda(d\xi_1, \ldots, d\xi_n), \text{Hom}(\Lambda(\xi_1, \ldots, \xi_n), k))
\]
\[ = \text{Hom}(\Lambda(\xi_1, \ldots, d\xi_n), \text{Hom}(\Lambda(\xi_1, \ldots, \xi_n), k))
\]
\[ = \text{Hom}(\Lambda(\xi_1, \ldots, d\xi_n) \otimes \Lambda(\xi_1, \ldots, \xi_n), k)
\]
\[ = \text{Hom}(\Lambda(d\xi_1, \ldots, d\xi_n, \xi_1, \ldots, \xi_n), k)
\]
\[ = \Lambda(\xi_1^*, \ldots, \xi_n^*, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n}).
\]

Under the identifications
\[ x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad dx_i \mapsto \xi_i^*, \quad i = 1, \ldots, n, \] (6.13)
we get \( \Omega^\bullet(A) \cong \mathfrak{X}^\bullet(A^!; A^!) \). It is again straightforward to show that this is an isomorphism of mixed complexes.

Note that the volume form of \( A \) is \( dx_1 dx_2 \cdots dx_n \) while the volume form of \( A^! \) is \( \xi_1^* \xi_2^* \cdots \xi_n^* \); under (6.12) and (6.13), the former is a Poisson cycle if and only if so is the latter.

(3) With identifications (6.12) and (6.13), it is direct to see that the inner products given by (5.2) and by (5.4) are compatible. \( \square \)

**Corollary 6.6.** Suppose \( A = k[x_1, \ldots, x_n] \) is a unimodular quadratic Poisson algebra, and let \( A^! \) be its Koszul dual. Then
\[ \text{PC}^\bullet_\mathbb{C}(A) \cong \text{PC}^\bullet_\mathbb{C}(A^!)
\]
as gravity algebras.

**Proof.** (Compare with the proof of Proposition 6.3) Proposition 6.5 together with Theorems 5.4 and 5.7 shows that
\[ (\text{CP}^\bullet_\mathbb{C}(A), \partial, d) \quad \text{and} \quad (\text{CP}^\bullet_\mathbb{C}(A^!; A^!), \delta, d^*),
\]
satisfy the conditions of Theorem 1.1 (2), from which the conclusion follows. \( \square \)

## 7. Deformation Quantization

The purpose of this section is to relate the gravity algebras obtained in previous sections by means of deformation quantization. This completes relationships (c) and (d) listed in §1.
In the following, we work over \( k[[\hbar]] \), where \( \hbar \) is a formal parameter. Recall that for a Poisson algebra \( A \) with bracket \( \{-, -\} \), its deformation quantization, denoted by \( A_\hbar \), is a (completed) \( k[[\hbar]] \)-linear associative product (called the star-product) on \( A[\hbar] \):

\[
a \ast b = a \cdot b + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots,
\]

where \( \hbar \) is the formal parameter and \( \mu_i \) are bilinear operators, satisfying

\[
\lim_{\hbar \to 0} \frac{1}{\hbar} (a \ast b - b \ast a) = \{a, b\}, \quad \text{for all } a, b \in A.
\]

In other words, a deformation quantization of \( A \) is a formal quantization of \( A \) in the direction of the Poisson bracket.

### 7.1. Deformation quantization of Calabi-Yau Poisson algebras

From now on, \( A = k[x_1, \ldots, x_n] \). By Example 6.3, it is a Calabi-Yau algebra of dimension \( n \) with volume class given by \( dx_1dx_2\cdots dx_n \).

Let \( \mu \in \bar{C}^2(A[\hbar]) \) be the Hochschild coboundary. For any \( \tilde{\mu} \in h \cdot \bar{C}^2(A[\hbar]) \),

\[
\mu + \tilde{\mu} : A[\hbar] \otimes_{k[[\hbar]]} A[\hbar] \to A[\hbar]
\]

defines a new product if and only if \( \tilde{\mu} \) is a Maurer-Cartan element, namely

\[
\mu(\tilde{\mu}) + \frac{1}{2}[\tilde{\mu}, \tilde{\mu}] = 0.
\]

Kontsevich proved in [21] that there is an \( L_\infty \)-quasi-isomorphism

\[
\mathcal{X}^\bullet(A[\hbar]) \to \bar{C}^\bullet(A[\hbar])
\]

(7.1)

between these two DG Lie algebras, whose first term is the classical Hochschild-Kostant-Rosenberg map. Here the differential of the former is zero. Thus as a corollary, up to gauge equivalences, the set of Maurer-Cartan elements of \( \bar{C}^2(A[\hbar]) \), which is exactly the set of Poisson structures on \( A[\hbar] \), is in one-to-one correspondence to the set of Maurer-Cartan elements of \( \bar{C}^2(A[\hbar]) \), which is exactly the set of deformation quantizations of \( A[\hbar] \).

Now suppose \( A \) is Poisson with Poisson structure \( \pi \). We equip with \( A[\hbar] \) the Poisson structure \( h\pi \). Let \( \mu + \tilde{\mu} \in \bar{C}^2(A[\hbar]) \) be the corresponding deformed product, where \( \tilde{\mu} \in h \cdot \bar{C}^2(A[\hbar]) \). Then the Cyclic Formality Conjecture for chains, proved by Willwacher in [39] Theorem 1.3 and Corollary 1.4, says that

\[
(\Omega^\bullet(A[[\hbar]]), L_{h\pi}, d) \simeq (C^\bullet(A[[\hbar]]), L_{\mu + \tilde{\mu}}, B)
\]

(7.2)

is a quasi-isomorphism of mixed complexes. Moreover, Dolgushev proved in [10] the following:

**Theorem 7.1** (Dolgushev). For a Calabi-Yau Poisson algebra \( A \), its deformation quantization, denoted by \( A_\hbar \), is Calabi-Yau over \( k[[\hbar]] \) if and only if \( A \) is unimodular.

**Proof.** See Dolgushev [10] Theorem 3] (an alternative proof by using differential graded Lie algebras can be found in [39] (1.3)).

**Corollary 7.2.** Suppose \( A \) is a unimodular Poisson Calabi-Yau algebra. Denote by \( A_\hbar \) Kontsevich’s deformation quantization of \( A \). Then

\[
\text{PC}^-_{\bullet}(A[[\hbar]]) \cong HC^-_{\bullet}(A_\hbar)
\]

as gravity algebras.
Proof. First, from (7.1), we obtain an $L_\infty$-quasi-isomorphism
\[
(\text{CP}^\bullet(A[\hbar]),\delta_{h\pi}) \simeq (\text{C}^\bullet(A[\hbar]),\delta_{\mu+\tilde{\mu}}).
\]
Later it is proved by Manchon and Torossian in [28, Théorème 1.2] that the above is also a quasi-isomorphism of DG algebras.

Second, just to repeat (7.2) we have that
\[
(\Omega^\bullet(A[\hbar]),L_{h\pi},d) \simeq (\overline{\text{C}}^\bullet(A[\hbar]),L_{\mu\tilde{\mu}},B)
\]
is a quasi-isomorphism of mixed complexes.

Third, it is proved by Calaque and Rossi in [2, Theorem 6.1] that there is a commutative diagram
\[
X^\bullet(A[\hbar]) \xrightarrow{\simeq} \Omega^\bullet(A[\hbar]) \xrightarrow{\simeq} \overline{\text{C}}^\bullet(A[\hbar]),
\]
where the curved arrows mean the DG algebra action.

Thus combining the above three results, we have that on the homology level,
\[
(\text{HP}^\bullet(A[\hbar]),\text{HP}_\bullet(A[\hbar])) \quad \text{and} \quad (\text{HH}^\bullet(A[\hbar]),\text{HH}_\bullet(A[\hbar]))
\]
are isomorphic as differential calculus. Theorem 7.1 implies that both pairs are in fact differential calculus with duality. Thus to show they are isomorphic as differential calculus with duality, we need to show that the volume class is mapped to the volume class. However, this is guaranteed by the Hochschild-Kostant-Rosenberg theorem. The corollary now follows from Theorem 3.5. □

7.2. Deformation quantization of Frobenius Poisson algebras. Cattaneo and Felder showed in [4, Appendix] that Kontsevich’s $L_\infty$-quasi-isomorphism holds also for graded manifolds, with exactly the same formula. That is, we have $L_\infty$-quasi-isomorphism
\[
X^\bullet(A[\hbar]) \sim \text{C}^\bullet(A[\hbar]), \quad \Omega^\bullet(A[\hbar]) \sim \text{C}^\bullet(A[\hbar]),
\]
where $A^\bullet = \Lambda(\xi_1, \cdots, \xi_n)$.

Now suppose $A^\bullet$ is Poisson with Poisson structure $\pi^\bullet$. We equip with $A^\bullet[\hbar]$ the Poisson structure $h\pi^\bullet$. Denote by $\mu^\bullet$ the product of $A^\bullet[\hbar]$ and let $\mu^\bullet + \tilde{\mu}^\bullet \in \overline{\text{C}}^\bullet(A^\bullet[\hbar])$ be the corresponding deformed product. Then the Cyclic Formality Conjecture for cochains, proposed by Felder and Shoikhet in [11] and proved by Willwacher and Calaque in [40, Theorem 2], says that
\[
(X^\bullet(A^\bullet[\hbar];A^\bullet[\hbar]),L^*_{\mu\tilde{\mu}},d^*) \simeq (\text{C}^\bullet(A^\bullet[\hbar];A^\bullet[\hbar]),L^*_{\mu\tilde{\mu}},B^*)
\]
is a quasi-isomorphism of mixed complexes. Moreover, Willwacher-Calaque also showed in [40] that:

**Theorem 7.3** (Willwacher-Calaque). Suppose $A^\bullet = \Lambda(\xi_1, \cdots, \xi_n)$ is a Frobenius Poisson algebra, then Kontsevich’s deformation quantization of $A^\bullet$, say $A^\bullet_h$, is symmetric Frobenius if and only if $A^\bullet$ is unimodular.

**Proof.** See Willwacher-Calaque [40, Theorem 37]. □
Corollary 7.4. Suppose $A^! = \Lambda(\xi_1, \cdots, \xi_n)$ is unimodular Frobenius Poisson. Denote by $A_h^!$ Kontsevich’s deformation quantization of $A^!$. Then

$$PC^*(A^![\hbar]) \cong HC^*(A_h^!)$$

as gravity algebras.

Proof. (Compare with Corollary 7.2). Manchon and Torossian’s result says that the $L_\infty$-quasi-isomorphism (7.4) is also a quasi-isomorphism of DG algebras. To relate with (7.5), let us note that

$$\mathfrak{X}^*(A^![\hbar]; A^i[\hbar]) \quad \text{and} \quad \Omega^*(A^i[\hbar])$$

are linear dual to each other, and so are

$$\mathring{C}^*(A^i[\hbar]; A^i[\hbar]) \quad \text{and} \quad \mathring{C}^*(A^i[\hbar]).$$

Thus in (7.3), replacing $A[h]$ with $A^i[\hbar]$ and then considering the associated adjoint action, we obtain the following commutative diagram

$$\begin{array}{cccc}
\mathfrak{X}^*(A^i[\hbar]) & \longrightarrow & \mathfrak{X}^*(A^i[\hbar]; A^i[\hbar]) \\
\cong & & \cong \\
\mathring{C}^*(A^i[\hbar]) & \longrightarrow & \mathring{C}^*(A^i[\hbar]; A^i[\hbar]).
\end{array}$$

Thus on the homology level, we obtain that

$$(HP^*(A^i[\hbar]), HP^*(A^i[\hbar]; A^i[\hbar])) \quad \text{and} \quad (HH^*(A^i[\hbar]), HH^*(A^i[\hbar]; A^i[\hbar]))$$

are isomorphic as differential calculus. Now Theorem 7.3 further implies that they are quasi-isomorphic as differential calculus with duality. The corollary now follows from Theorem 3.5. $\square$

7.3. Deformation quantization of quadratic unimodular Poisson algebras. As explained by Shoikhet in [31, §1], the Koszul duality theory can be extended to algebras over $k[\hbar]$, which is still valid. Now, for $A = k[x_1, \cdots, x_n]$ and $A^! = \Lambda(\xi_1, \cdots, \xi_n)$, it is shown by Shoikhet in [31 Theorem 7.1] (see also [1 Theorem 8.6]) that if $A$ and $A^!$ are Koszul dual to each other as Poisson algebras, then their deformation quantization $A_h^!$ and $A_h^i$ are also Koszul dual to each other as associative algebras over $k[\hbar]$. Therefore, $A$ being unimodular implies that $A_h^!$ and $A_h^i$ are Koszul dual as Calabi-Yau and symmetric Frobenius algebras. We have shown in Corollaries 6.4 and 6.6 that

$$HC_\sim^*(A_h^!) \cong HC_\sim^*(A_h^i) \quad \text{and} \quad PC_\sim^*(A^i[\hbar]) \cong PC_\sim^*(A^![\hbar])$$

as gravity algebras by Koszul duality, and in Corollaries 7.2 and 7.4 that

$$PC_\sim^*(A^i[\hbar]) \cong HC_\sim^*(A_h^!) \quad \text{and} \quad PC_\sim^*(A^i[\hbar]) \cong HC_\sim^*(A_h^i)$$

as gravity algebras via deformation quantization. Combining these results we have the following statement, which also verifies the commutative diagram 1.1 in [1]...
Theorem 7.5. Let $A = k[x_1, \ldots, x_n]$ be a quadratic unimodular Poisson algebra. Denote by $A^!$ the Koszul dual Poisson algebra of $A$, and by $A_\hbar$ and $A^!_\hbar$ Kontsevich’s deformation quantization of $A$ and $A^!$ respectively. Then the following diagram of algebra homomorphisms is commutative:

$$
\begin{array}{ccc}
PC^{-\bullet}(A[\hbar]) & \longrightarrow & PC^{\bullet}(A^![\hbar]) \\
\downarrow & & \downarrow \\
HC^{-\bullet}(A_\hbar) & \longrightarrow & HC^{\bullet}(A^!_\hbar),
\end{array}
$$

where $\hbar$ be a formal parameter, is a commutative diagram of isomorphisms of gravity algebras.

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