PSEUDO-RIEMANNIAN $G_{2(2)}$-MANIFOLDS WITH DIMENSION AT MOST 21

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Abstract. Let $G_{2(2)}$ be the non-compact connected simple Lie group of type $G_2$ over $\mathbb{R}$, and let $M$ be a connected analytic complete pseudo-Riemannian manifold that admits an isometric $G_{2(2)}$-action with a dense orbit. For the case $\dim(M) \leq 21$, we provide a full description of the manifold $M$, its geometry and its $G_{2(2)}$-action. The latter are always given in terms of a Lie group geometry related to $G_{2(2)}$, and in one case $M$ is essentially the quotient of $SO_0(3,4)$ by a lattice.

1. Introduction

A fundamental problem in both geometry and dynamics is to understand the actions of a connected simple Lie group $G$ on manifolds. This is particularly interesting when one of such $G$-actions preserves a geometric structure on a manifold $M$. A basic example to consider is the left $G$-action on the manifold $H/\Gamma$ where $H$ is a semisimple Lie group, $\Gamma$ is a lattice of $H$ and the action is given by a non-trivial homomorphism $G \to H$. There are two well known properties for such an example. In the first place, the $G$-action on $H/\Gamma$ is ergodic and so has a dense orbit. And secondly, the Killing form of the Lie algebra of $H$ induces a bi-invariant pseudo-Riemannian metric on $H$ that descends to a metric on $H/\Gamma$ preserved by the $G$-action.

It has been conjectured that every finite volume preserving ergodic $G$-action on a manifold is essentially one of the examples $H/\Gamma$ just described (see [16]). In this direction, Zimmer’s program proposes to study ergodic $G$-actions to understand their rigid properties. Many efforts on this line of research have shown very useful to consider actions that preserve a geometric structure. In particular, this has lead to the development of a set of tools widely known as Gromov-Zimmer’s machinery (see for example [2, 15, 11]).

In this work we consider the case $G = G_{2(2)}$, the connected non-compact exceptional Lie group of type $G_2$ over $\mathbb{R}$, and an isometric $G_{2(2)}$-action on a finite volume pseudo-Riemannian manifold $M$. Following Zimmer’s program, the final goal is to prove that $M$ is closely related to the group $G_{2(2)}$ itself, from the viewpoint of all the structures involved. We achieve this objective for the case $\dim(M) \leq 21$. We note that 21 is the dimension of the Lie group $SO(3,4)$ and that there is a homomorphism $G_{2(2)} \to SO(3,4)$ that realizes the irreducible non-trivial representation of $G_{2(2)}$ with lowest dimension (see Section 4 for details). The following result proves that $M$ is always related to $G_{2(2)}$ and in one case that it is essentially given

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by a non-trivial homomorphism $G_{2(2)} \to \text{SO}(3, 4)$. In a sense, we thus provide a geometric/dynamic characterization of the homomorphism $G_{2(2)} \to \text{SO}(3, 4)$ that defines the irreducible 7-dimensional representation of $G_{2(2)}$.

**Theorem 1.1** (Manifold and action type). Let $(M, h)$ be a connected analytic complete pseudo-Riemannian manifold with finite volume that admits an isometric $G_{2(2)}$-action with a dense orbit. If $\dim M \leq 21$, then there exists a finite covering map $\tilde{\pi} : \tilde{M} \to M$ so that $\tilde{M}$ satisfies one of the following properties.

1. There exist a connected pseudo-Riemannian manifold $N$ and a discrete subgroup $\Gamma \subset G_{2(2)} \times \text{Iso}(N)$ such that $\tilde{M} = (G_{2(2)} \times N)/\Gamma$.

Furthermore, the $G_{2(2)}$-action on $M$ lifted to $\tilde{M}$ is precisely the left $G_{2(2)}$-action induced from the action on the first factor of $G_{2(2)} \times N$.

2. There exists a lattice $\Gamma \subset \text{SO}_0(3, 4)$ so that $\tilde{M} = \text{SO}_0(3, 4)/\Gamma$.

Furthermore, the $G_{2(2)}$-action on $M$ lifted to $\tilde{M}$ is precisely the left $G_{2(2)}$-action induced from a non-trivial homomorphism $G_{2(2)} \to \text{SO}_0(3, 4)$ and the left translation action on $\text{SO}_0(3, 4)$.

The next result proves that the pseudo-Riemannian metric on $M$ can also be related to natural metrics.

**Theorem 1.2** (Metric type). With the hypotheses and notation of Theorem 1.1, one of the following holds according to the cases of such theorem.

1. For $\tilde{M} = (G_{2(2)} \times N)/\Gamma$, the covering map $\tilde{\pi} : (\tilde{M}, \tilde{h}) \to (M, h)$ is locally isometric for the metric $\tilde{h}$ on $\tilde{M}$ induced from the product metric on $G_{2(2)} \times N$ where $G_{2(2)}$ carries a bi-invariant metric.

2. For $\tilde{M} = \text{SO}_0(3, 4)/\Gamma$, there is new $G_{2(2)}$-invariant metric $\tilde{h}$ on $M$ so that the covering map $\tilde{\pi} : (\tilde{M}, \tilde{h}) \to (M, h)$ is locally isometric for the metric $\tilde{h}$ on $\tilde{M}$ induced from the bi-invariant metric on $\text{SO}_0(3, 4)$ given by the Killing form of $\mathfrak{so}(3, 4)$.

It is well known that the group $\text{SO}_0(3, 4)$ is weakly irreducible for the bi-invariant metric defined by the Killing form of its Lie algebra. We recall that a pseudo-Riemannian manifold is weakly irreducible if its not locally isomorphic to a product of pseudo-Riemannian manifolds. This property can be used to distinguish between the two cases of the theorems above.

As for the organization of the work, in Section 2 we present some basic facts on $G_{2(2)}$, its Lie algebra $\mathfrak{g}_{2(2)}$ and their representations. Section 3 applies the Gromov-Zimmer’s machinery to describe the centralizer $\mathcal{H}$ of the $G_{2(2)}$-action on the universal covering space $\tilde{M}$ in the Lie algebra of Killing fields. Finally, Section 4 provide the proofs of the results stated in this Introduction.
2. Preliminaries on $G_2(2)$

We introduce the exceptional Lie group $G_2(2)$ and recall some properties that we will use in this work. This includes some properties of $g_2(2)$, the Lie algebra of $G_2(2)$. We refer to [5, 12] for further details.

We define $G_2(2)$ as the connected group of automorphisms of the split Cayley algebra $C$ over $\mathbb{R}$. We recall that $C$ is a composition algebra whose norm is a split quadratic form. In other words, the norm of $C$ is a quadratic form whose associated bilinear form has signature $(4, 4)$. The group $G_2(2)$ preserves the unit $e$ of $C$ and so it preserves the orthogonal complement $e^\perp$ which is precisely the space of pure imaginary elements of $C$: the set of $a \in C$ such that $\overline{a} = -a$. The bilinear form of $C$ restricted to $e^\perp$ has signature $(3, 4)$ and so we will denote $e^\perp = \mathbb{R}^{3, 4}$. This yields a faithful representation $G_2(2) \to \text{SO}(3, 4)$, that we will call the linear realization of $G_2(2)$. Correspondingly, there is a Lie algebra representation $g_2(2) \to \mathfrak{so}(3, 4)$, that exhibits $g_2(2)$ as the Lie algebra of derivations of $C$ restricted to $e^\perp$. We will call this representation the linear realization of $g_2(2)$.

**Proposition 2.1.** The Lie algebra $g_2(2)$ is the split exceptional Lie algebra of type $G_2$ over $\mathbb{R}$. The linear realization $g_2(2) \to \mathfrak{so}(3, 4)$ turns the space $\mathbb{R}^{3, 4}$ into an irreducible $g_2(2)$-module. Furthermore, $\mathbb{R}^{3, 4}$ is the $g_2(2)$-module corresponding to the first fundamental weight of $g_2(2)$ and every other irreducible $g_2(2)$-module has dimension at least 14.

**Proof.** The first claim is well known (see [5, 12]).

Next, we recall that the irreducible representations of the split form $g$ of a simple complex Lie algebra $g^C$ are all real forms of the corresponding irreducible representations of $g^C$ (see [9]). On the other hand, the irreducible representation of $g_2^C$ corresponding to the first fundamental weight has dimension 7, and all other irreducible non-trivial representations have dimension at least 14 (see [4]). Hence, the irreducible representation of $g_2(2)$ associated to the first fundamental weight has (real) dimension 7. Since the linear realization homomorphism $g_2(2) \to \mathfrak{so}(3, 4)$ is non-trivial, it defines such irreducible representation.

Finally, we recall that Weyl’s dimension formula implies that the irreducible representations of $g_2^C$ corresponding to the first and second fundamental weights have dimensions 7 and 14, and that every other irreducible representations has dimension strictly larger (see [4]). Hence, the last claim follows from the above remarks on split forms. \qed

In the rest of this work, $\mathbb{R}^{3, 4}$ will be considered as a $g_2(2)$-module with the structure given by Proposition 2.1.

The following result establishes the uniqueness of the $g_2(2)$-invariant scalar product on $\mathbb{R}^{3, 4}$.

**Proposition 2.2.** The $g_2(2)$-module $\mathbb{R}^{3, 4}$ carries a unique, up to a constant multiple, scalar product invariant under $g_2(2)$. In particular, any such scalar product has signature either $(3, 4)$ or $(4, 3)$.

**Proof.** The existence of the scalar product is clear from the construction of the $g_2(2)$-module $\mathbb{R}^{3, 4}$ in terms the composition algebra $C$. 

Recall that there is a natural isomorphism of vector spaces between the space of $g_{2(2)}$-invariant bilinear forms on $\mathbb{R}^{3,4}$ and the real algebra $\text{End}_{g_{2(2)}}(\mathbb{R}^{3,4})$ of homomorphisms of $g_{2(2)}$-modules of $\mathbb{R}^{3,4}$. Hence, it is enough to prove that $\text{End}_{g_{2(2)}}(\mathbb{R}^{3,4})$ is 1-dimensional.

By Schur’s Lemma and the irreducibility of $\mathbb{R}^{3,4}$, the algebra $\text{End}_{g_{2(2)}}(\mathbb{R}^{3,4})$ is a division algebra over $\mathbb{R}$, and so it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or the quaternion numbers. If $\text{End}_{g_{2(2)}}(\mathbb{R}^{3,4})$ is not 1-dimensional, then there is a $g_{2(2)}$-invariant complex structure on $\mathbb{R}^{3,4}$, which is absurd since this space is odd-dimensional. □

We recall the following elementary property.

**Lemma 2.3.** Let $E$ be a finite dimensional vector space with scalar product $\langle \cdot, \cdot \rangle$. Then, the assignment
\[ u \wedge v \mapsto \langle \cdot, u \rangle v - \langle \cdot, v \rangle u, \]
defines an isomorphism $\varphi : \wedge^2 E \to \mathfrak{so}(E)$ of $\mathfrak{so}(E)$-modules. In particular, $\varphi$ yields an isomorphism of $\mathfrak{g}$-modules for every Lie subalgebra $\mathfrak{g}$ of $\mathfrak{so}(E)$.

As a consequence $\wedge^2 \mathbb{R}^{3,4} \cong \mathfrak{so}(3,4)$ as $g_{2(2)}$-modules for the structures defined by the linear realization of $g_{2(2)}$. The next proposition describes some useful properties of these $g_{2(2)}$-modules.

**Remark 2.4.** If $\mathfrak{h}$ is a Lie algebra, then the Jacobi identity implies that the linear map
\[ \wedge^2 \mathfrak{h} \to \mathfrak{h} \]
\[ X \wedge Y \mapsto [X, Y], \]
is a homomorphism of $\mathfrak{h}$-modules. In particular, if $\mathfrak{h}_1$ is a Lie subalgebra of $\mathfrak{h}$ and $V_1, V_2$ are $\mathfrak{h}_1$-submodules of $\mathfrak{h}$ (for the $\mathfrak{h}_1$-module structure defined by the Lie brackets), then $[V_1, V_2]$ is an $\mathfrak{h}_1$-module (again, by the Jacobi identity) whose irreducible $\mathfrak{h}_1$-submodules must be among those that appear in $V_1 \otimes V_2$. Similarly, there is a corresponding remark for $[V_1, V_1]$ and $\wedge^2 V_1$. We will use these facts in the rest of this work.

**Proposition 2.5.** The following isomorphism of $g_{2(2)}$-modules holds
\[ \wedge^2 \mathbb{R}^{3,4} \cong \mathfrak{so}(3,4) \cong \mathbb{R}^{3,4} \oplus g_{2(2)}, \]
where $g_{2(2)}$ is the $g_{2(2)}$-module given by the adjoint representation. If we let $V$ denote the $g_{2(2)}$-submodule of $\mathfrak{so}(3,4)$ isomorphic to $\mathbb{R}^{3,4}$, then
\[ [V, V] = \mathfrak{so}(3,4), \]
with respect to the Lie brackets of $\mathfrak{so}(3,4)$.

**Proof.** Since $g_{2(2)}$ is $g_{2(2)}$-submodule of $\mathfrak{so}(3,4)$ (for the structure mentioned above) and since $g_{2(2)}$ is simple, there is a $g_{2(2)}$-submodule $V$ of $\mathfrak{so}(3,4)$ such that
\[ \mathfrak{so}(3,4) = g_{2(2)} \oplus V. \]
In particular, $V$ has dimension 7. By Proposition 2.1 either $V$ is a direct sum of trivial 1-dimensional modules or $V \cong \mathbb{R}^{3,4}$ as $g_{2(2)}$-modules. If the former occurs, then Remark 2.4 implies that $[V, V]$ is a sum of 1-dimensional $g_{2(2)}$-modules as well and so $[V, V] \subset V$. This implies that $V$ is a proper ideal of $\mathfrak{so}(3,4)$, which is absurd. This proves the first claim.
On the other hand, by Remark 2.3 the space $[V, V]$ is either 0 or a sum of the irreducible $\mathfrak{g}_{2(2)}$-modules that appear in $\mathfrak{so}(3, 4)$. We have already ruled out that $[V, V] \subset V$. If $[V, V] \subset \mathfrak{g}_{2(2)}$, then $(\mathfrak{so}(3, 4), \mathfrak{g}_{2(2)})$ is a symmetric pair. But an inspection of Table II from [1] shows that no such symmetric pair exists. Therefore, the only possibility left is to have $[V, V] = \mathfrak{so}(3, 4)$. □

Remark 2.6. Note that $\mathfrak{g}_{2(2)}$, as a module over itself, is the irreducible representation of $\mathfrak{g}_{2(2)}$ corresponding to the second fundamental weight. Hence, Proposition 2.5 says that the $\mathfrak{g}_{2(2)}$-module $\mathbb{R}^2 \mathbb{R}^{3, 4} \simeq \mathfrak{so}(3, 4)$ is the sum of the irreducible representations corresponding to the fundamental weights.

3. CENTRALIZER OF THE ISOMETRIC $G_{2(2)}$-ACTION

In this section we specialize some known results for actions of non-compact simple Lie groups to the our case of $G_{2(2)}$-actions. Our main references are [6] and [17].

We will assume the hypotheses of Theorem 1.1 through out this section. Under such conditions, it is well known that the $G_{2(2)}$-action on $M$ is everywhere locally free (see [13]). Hence, the set of orbits defines a foliation $O$ on $M$, whose tangent bundle will be denoted by $TO$. In particular, the map $M \times \mathfrak{g}_{2(2)} \to TO$ given by the assignment $(x, X) \mapsto X_x^*$ is an isomorphism of bundles. We recall that for $X \in \mathfrak{g}_{2(2)}$ we denote by $X^*$ the vector field on $M$ whose local flow is $\exp(tX)$. Also, we will denote by $TO^\perp$ the bundle whose fibers are the subspaces orthogonal to the fibers of $TO$. Then, the condition $\dim(M) \leq 21$ ensures that both $TO$ and $TO^\perp$ are non-degenerate and so that $TM = TO \oplus TO^\perp$ (see Lemma 1.4 from [6]). In what follows, we will use the same symbols $O$, $TO$ and $TO^\perp$ for the corresponding objects on $M$.

The following result is fundamental for our work. See Proposition 2.3 from [11] for a proof for arbitrary non-compact simple Lie groups (see also [2],[15]). For a pseudo-Riemannian manifold $N$ we denote by $\text{Kill}(N)$ the Lie algebra of globally defined Killing vector fields on $N$. Also, we will denote by $\text{Kill}_0(N, x)$ the Lie subalgebra of $\text{Kill}(N)$ consisting of those vector fields that vanish at $x$.

**Proposition 3.1.** For $M$ as above, there is a dense conull subset $A \subset \tilde{M}$ such that for every $x \in A$ the following properties are satisfied.

1. There is a homomorphism $\rho_x : \mathfrak{g}_{2(2)} \to \text{Kill}(\tilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}_{2(2)})$.

2. Every element of $\rho_x(\mathfrak{g}_{2(2)})$ vanishes at $x$: $\rho_x(\mathfrak{g}_{2(2)}) \subset \text{Kill}_0(\tilde{M}, x)$.

3. For every $X, Y \in \mathfrak{g}_{2(2)}$ we have: $[\rho_x(X), Y^\ast] = [X, Y]^\ast = -[X^\ast, Y^\ast]$.

In particular, the elements in $\rho_x(\mathfrak{g}_{2(2)})$ and their corresponding local flows preserve both $O$ and $TO^\perp$.

The following local homogeneity result is well known and it is a particular case of Gromov’s open dense orbit theorem. For its proof for general actions of non-compact simple Lie groups we refer to [2] and [15].

**Proposition 3.2.** For $M$ satisfying the above conditions, there is an open dense conull subset $U \subset \tilde{M}$ such that for every $x \in U$ the evaluation map $ev_x : \mathcal{H} \to T_x \tilde{M}$ given by $Z \mapsto Z_x$ is surjective.
For $A$ as in Proposition 3.1 let $x \in A$ be given and consider the map
\[
\hat{\rho}_x : \mathfrak{g}_{2(2)} \to \text{Kill}(\tilde{M})
\]
\[
\hat{\rho}_x(x) = \rho_x(x) + X^*.
\]
Then, Proposition 3.1(3) implies that $\hat{\rho}_x$ is an injective homomorphism of Lie algebras. We will denote its image by $\mathcal{G}(x)$, which is thus a Lie subalgebra of $\mathcal{H}$ isomorphic to $\mathfrak{g}_{2(2)}$. In particular, the Lie brackets induce a $\mathfrak{g}_{2(2)}$-module structure on $\mathcal{H}$. Furthermore, through the isomorphism $\hat{\rho}_x$ between $\mathcal{G}(x)$ and $\mathfrak{g}_{2(2)}$ every $\mathcal{G}(x)$-module can be considered as a $\mathfrak{g}_{2(2)}$-module.

Proposition 3.2 allows us to define a $\mathcal{G}(x)$-module structure on $T_x\tilde{M}$ through the following construction.

Let $A$ and $U$ be as in Propositions 3.1 and 3.2 respectively. Fix some point $x \in A \cap U$. We consider the map $\lambda_0 : \mathcal{G}(x) \to \mathfrak{so}(T_x\tilde{M})$ given by
\[
\lambda_0(Z)(v) = [Z, V]|_x,
\]
where $V \in \mathcal{H}$ is such that $V_x = v$. It is easy to see that this is a well defined homomorphism of Lie algebras. Furthermore, it is also known that the evaluation map $ev_0 : \mathcal{H} \to T_x\tilde{M}$ is a homomorphism of $\mathcal{G}(x)$-modules that satisfies $ev_0(\mathcal{G}(x)) = T_x\mathcal{O}$. In particular, $T_x\mathcal{O}$ is a $\mathcal{G}(x)$-module isomorphic to the $\mathfrak{g}_{2(2)}$-module $\mathfrak{g}_{2(2)}$. As a consequence the subspace $T_x\mathcal{O}^\perp$ is a $\mathcal{G}(x)$-submodule of $T_x\tilde{M}$.

For $x \in A \cap U$, in the rest of this work we consider $\mathcal{H}$ and $T_x\tilde{M}$ endowed with the $\mathcal{G}(x)$-module structures defined above.

On the other hand, we denote by $\mathcal{H}_0(x) = \ker(ev_0)$ which, by the previous remarks, is a $\mathcal{G}(x)$-submodule of $\mathcal{H}$. Also, it is clear that $\mathcal{H}_0(x)$ is a Lie subalgebra of $\mathcal{H}$ as well. In particular, $\mathcal{G}(x) + \mathcal{H}_0(x)$ is a Lie subalgebra of $\mathcal{H}$ that contains $\mathcal{H}_0(x)$ as an ideal. Hence, $\mathcal{H}$ can be considered as a module over $\mathcal{G}(x) + \mathcal{H}_0(x)$ through the Lie brackets.

By Proposition 3.5 from [7] we can extend $\lambda_x$ from $\mathcal{G}(x)$ to the map
\[
\lambda_x : \mathcal{G}(x) + \mathcal{H}_0(x) \to \mathfrak{so}(T_x\tilde{M})
\]
\[
\lambda_x(Z)(v) = [Z, V]|_x,
\]
where for a given $v \in T_x\tilde{M}$ we choose $V \in \mathcal{H}$ such that $V_x = v$. As before, it is proved that $\lambda_x$ is a well defined homomorphism of Lie algebras, thus defining a $\mathcal{G}(x) + \mathcal{H}_0(x)$-module structure on $T_x\tilde{M}$ for which both $T_x\mathcal{O}$ and $T_x\mathcal{O}^\perp$ are submodules. Furthermore, the evaluation map $ev_0 : \mathcal{H} \to T_x\tilde{M}$ is a homomorphism of $\mathcal{G}(x) + \mathcal{H}_0(x)$-modules. In particular, we have a representation
\[
\lambda^{\perp}_0 : \mathcal{G}(x) + \mathcal{H}_0(x) \to \mathfrak{so}(T_x\mathcal{O}^\perp)
\]
\[
\lambda^{\perp}_0(Z) = \lambda_x(Z)|_{T_x\mathcal{O}^\perp}.
\]
Furthermore, by Proposition 3.6 from [7], the restriction $\lambda^{\perp}_0 : \mathcal{H}_0(x) \to \mathfrak{so}(T_x\mathcal{O}^\perp)$ is injective and its image is both a Lie subalgebra and a $\mathcal{G}(x)$-submodule of $\mathfrak{so}(T_x\mathcal{O}^\perp)$.

The fact that $\mathcal{G}(x) \simeq \mathfrak{g}_{2(2)}$ as a Lie algebra allows us to obtain the following decomposition of the centralizer $\mathcal{H}$.

**Proposition 3.3.** Let $A$ and $U$ be as in Propositions 3.1 and 3.2 respectively. For a fixed point $x \in A \cap U$ there exists a $\mathcal{G}(x)$-submodule $\mathcal{V}(x)$ of $\mathcal{H}$ such that
\[
\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x), \quad T_x\mathcal{O}^\perp = ev_\perp(\mathcal{V}(x)).
\]
Next we consider the analytic map

\[ \omega : T\tilde{M} \to \mathfrak{g}_{2(2)} \]

given by the orthogonal projection onto \( T\tilde{O} \) followed by the fiberwise isomorphism \( T\tilde{O} \to \mathfrak{g}_{2(2)} \) described at the beginning of this section. Let us also consider the analytic \( \mathfrak{g}_{2(2)} \)-valued 2-form \( \Omega \) defined by

\[ \Omega_x = d\omega_x |_{\bigwedge^2 T_x\tilde{O}^\perp}, \]

for every \( x \in \tilde{M} \). If \( X, Y \) are smooth sections of \( T\tilde{O}^\perp \), then \( \omega(X) = \omega(Y) = 0 \) and so we have

\[ \Omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = -\omega([X,Y]), \]

which implies the following result (see [2, 11]).

**Lemma 3.4.** For \( G \) and \( M \) as above, assume that \( T\tilde{M} = T\tilde{O} \oplus T\tilde{O}^\perp \). Then, \( T\tilde{O}^\perp \) is integrable if and only if \( \Omega \equiv 0 \).

By Lemma 2.3 we obtain from the map \( \Omega_x : \bigwedge^2 T_x\tilde{O}^\perp \to \mathfrak{g}_{2(2)} \) a corresponding map \( \mathfrak{so}(T_x\tilde{O}^\perp) \to \mathfrak{g}_{2(2)} \) given by \( \Omega_x \circ \varphi_x^{-1} \), where \( \varphi_x : \bigwedge^2 T_x\tilde{O}^\perp \to \mathfrak{so}(T_x\tilde{O}^\perp) \) is the isomorphism defined by Lemma 2.3. This does not change the \( \mathfrak{so}(T_x\tilde{O}^\perp) \)-module structure on the domain. Hence, we will denote with the same symbol \( \Omega_x \) the linear map given by the 2-form \( \Omega \) when considered as a map \( \mathfrak{so}(T_x\tilde{O}^\perp) \to \mathfrak{g} \).

It turns out that the forms \( \omega_x \) and \( \Omega_x \) satisfy special intertwining properties with respect to the module structure over \( \mathcal{G}(x) + \mathcal{H}_0(x) \). These are stated and proved in Proposition 3.10 from [7] for general non-compact simple Lie group actions. For our given setup, the following hold for every \( x \in A \cap U \).

1. The linear map \( \Omega_x : \bigwedge^2 T_x\tilde{O}^\perp \to \mathfrak{g}_{2(2)} \) intertwines the homomorphism of Lie algebras \( \tilde{\rho}_x : \mathfrak{g}_{2(2)} \to \mathcal{G}(x) \) for the actions of \( \mathfrak{g}_{2(2)} \) on \( \mathfrak{g}_{2(2)} \) and of \( \mathcal{G}(x) \) on \( T_x\tilde{O}^\perp \) via \( \lambda_2^\perp \).
2. The linear map \( \Omega_x : \mathfrak{so}(T_x\tilde{O}^\perp) \to \mathfrak{g}_{2(2)} \) is \( \mathcal{H}_0(x) \)-invariant via \( \lambda_2^\perp \). In particular, we have

\[ [\lambda_2^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x\tilde{O}^\perp)] \subset \ker(\Omega_x). \]

Given the previous discussion there are two natural cases to consider: either \( \Omega \equiv 0 \), and \( T\tilde{O}^\perp \) is integrable, or for some \( x \in A \cap U \) the linear map \( \Omega_x \) is non-zero, and the above properties for \( \Omega_x \) impose strong restrictions on the centralizer \( \mathcal{H} \). The following result provides the description of \( \mathcal{H} \) in the latter case.

**Proposition 3.5.** For a \( G_{2(2)} \)-action on \( M \) as above, for \( A \) and \( U \) as in Propositions 3.4 and 3.3, respectively, let \( x \in A \cap U \) be such that \( \Omega_x \neq 0 \). If \( \mathcal{V}(x) \) is a \( \mathcal{G}(x) \)-submodule of \( \mathcal{H} \) given as in Proposition 3.3 then \( \mathcal{V}(x) \simeq \mathbb{R}^{3,4} \) as \( \mathfrak{g}_{2(2)} \)-modules and \( \dim(M) \leq 21 \). Furthermore, \( \mathcal{H}_0(x) = 0 \) and \( \mathcal{H} \simeq \mathfrak{so}(3,4) \) both as Lie algebras and as modules over \( \mathcal{G}(x) \simeq \mathfrak{g}_{2(2)} \).

**Proof.** For our given \( x \), property (3.1) implies that \( T_x\tilde{O}^\perp \) is a non-trivial \( \mathfrak{g}_{2(2)} \)-module. Hence, Proposition 2.1 shows that \( \mathcal{V}(x) \simeq T_x\tilde{O}^\perp \simeq \mathbb{R}^{3,4} \) as \( \mathfrak{g}_{2(2)} \)-modules. Furthermore, by Proposition 2.2 this isomorphism is an isometry up to a constant. In particular, the representation \( \lambda_2^\perp : \mathcal{G}(x) \to \mathfrak{so}(T_x\tilde{O}^\perp) \) discussed above is naturally equivalent to the linear realization \( \mathfrak{g}_{2(2)} \to \mathfrak{so}(3,4) \).
In particular, \( f \) shows that straightforward computation (compare with the proof of Proposition 4.4 from [7])
\[
\mathfrak{so}(T_xO^\perp) = \lambda_x^+(\mathcal{G}(x)) \oplus V,
\]
for the brackets of \( \mathfrak{so}(T_xO^\perp) \). Thus, the map \( \Omega_x : \mathfrak{so}(T_xO^\perp) \to \mathfrak{g}_2(2) \) is naturally identified with the projection onto the summand \( \lambda_x^+(\mathcal{G}(x)) \), which implies that \( \ker(\Omega_x) = V \). If we apply property [3.2] and the fact that \( [V, V] = \mathfrak{so}(T_xO^\perp) \), we conclude that \( \lambda_x^+(\mathcal{H}_0(x)) \) is an integral submanifold of \( T_x\mathcal{H}_0 = 0 \). By the previous remarks in this section this yields \( H(x) = 0 \).

Hence, we have \( \mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x) \) where \( \mathcal{V}(x) \) is a \( \mathcal{G}(x) \)-submodule isomorphic through \( ev_x \) to \( T_xO^\perp \simeq \mathbb{R}^{3,4} \).

If we consider a Levi decomposition \( \mathcal{H} = \mathcal{L} \oplus \text{rad}(\mathcal{H}) \) such that \( \mathcal{G}(x) \subset \mathcal{L} \), then such sum is a decomposition into \( \mathcal{G}(x) \)-submodules as well. In particular, either \( \text{rad}(\mathcal{H}) = 0 \) or \( \text{rad}(\mathcal{H}) = \mathcal{V}(x) \). In the latter case, we obtain a semi-direct product \( \mathcal{H} = \mathcal{G}(x) \ltimes \mathcal{V}(x) \).

Suppose that \( \text{rad}(\mathcal{H}) = \mathcal{V}(x) \) and choose \( R \) a simply connected Lie group whose Lie algebra is \( \mathcal{V}(x) \). Hence, the Lie group \( G_{2(2)} \ltimes R \), with the semi-direct product structure, has Lie algebra \( \mathcal{H} \). Let \( \psi : \mathfrak{g}_{2(2)} \ltimes \mathcal{V}(x) \to \mathcal{H} \) be the isomorphism whose restriction to \( \mathfrak{g}_{2(2)} \) is \( \hat{\omega}_x \) and that maps \( \mathcal{V}(x) \) to itself by the identity. By Lemma 1.11 from [6] (or by the results from [8]), the completeness of \( \mathcal{M} \) and the fact that \( \mathcal{H} \subset \text{Kill}(\mathcal{M}) \) imply the existence of a right \( G_{2(2)} \ltimes R \)-action on \( \mathcal{M} \) such that
\[
\psi(X) = X^*,
\]
for every \( X \in \mathfrak{g}_{2(2)} \ltimes \mathcal{V}(x) \), where \( X^* \) denotes the Killing field generated by the (right) action of the 1-parameter subgroup \( (\exp(tX))_t \) of \( G_{2(2)} \ltimes R \). Consider the analytic map
\[
f : G_{2(2)} \ltimes R \to \mathcal{M}
\]
\[
f(g, r) = x(g, r),
\]
given by the \( G_{2(2)} \ltimes R \)-orbit at \( x \) and which is clearly \( G_{2(2)} \ltimes R \)-equivariant. A straightforward computation (compare with the proof of Proposition 4.4 from [7]) shows that \( df_{(e, r)} \) is an isomorphism that maps
\[
df_{(e, r)}(\mathfrak{g}_{2(2)}) = T_xO, \quad df_{(e, r)}(\mathcal{V}(x)) = T_xO^\perp.
\]
In particular, \( f \) is a local diffeomorphism from a neighborhood of the identity onto a neighborhood of \( x \). If we choose \( N = f(\{e\} \times R) \), then \( N \) is a submanifold of \( \mathcal{M} \) in a neighborhood of \( x \) such that
\[
T_xN = T_xO^\perp.
\]
Furthermore, the equivariance of \( f \) is easily seen to imply that
\[
T_{f(e, r)}N = T_{f(e, r)}O^\perp,
\]
for every \( r \) in a neighborhood of \( e \) in \( R \). In other words, \( N \) is an integral submanifold of \( TO^\perp \) passing through \( x \). Next, the equivariance with respect to \( G_{2(2)} \) implies that there is an integral submanifold of \( TO^\perp \) passing through every point in a neighborhood of \( x \). By the analyticity of \( TO^\perp \), we conclude that this vector bundle is integrable. This yields a contradiction since we assumed that \( \Omega \neq 0 \).
From the previous discussion we conclude that \( \text{rad}(H) = 0 \) and so that \( H \) is semisimple. We observe that every decomposition of \( H \) into simple ideals is also a decomposition into \( G(x) \)-submodules. Since \( H \) is the sum of the two inequivalent irreducible \( G(x) \)-submodules \( G(x) \) and \( V(x) \), if \( H \) is not simple, then both submodules are ideals. But this is impossible because \( [G(x), V(x)] \neq 0 \). We conclude that \( H \) is a simple Lie algebra of dimension 21. In particular, \( H \) is a noncompact real form of the 21-dimensional simple complex Lie algebra \( H^C \). An inspection of the list of simple complex Lie algebras (see [3]) shows that the only possibilities are either \( H^C \cong \mathfrak{so}(7, \mathbb{C}) \) or \( H^C \cong \mathfrak{sp}(6, \mathbb{C}) \). The latter and the fact that \( \mathfrak{g}_2(2) \cong G(x) \subset H \) would imply the existence of a non-trivial 6-dimensional representation of \( \mathfrak{g}_2(2) \), which is absurd. We conclude that \( H \cong \mathfrak{so}(3, 4) \), and so that \( H \cong \mathfrak{so}(p, q) \) for some \( p, q \geq 1 \) such that \( p + q = 7 \). Considering again the inclusion \( \mathfrak{g}_2(2) \cong G(x) \subset H \) we obtain a non-trivial representation \( \mathfrak{g}_2(2) \rightarrow \mathfrak{so}(3, 4) \). Then, Propositions 2.1 and 2.2 imply that we must have \( \{p, q\} = \{3, 4\} \) and so we can in fact assume that \( p = 3, q = 4 \). In other words, we conclude that \( H \cong \mathfrak{so}(3, 4) \).

4. PROOF OF THE MAIN RESULTS.

In what follows we will assume that the hypotheses of Theorem 1.1 hold. We will consider two cases according to whether \( TO^\perp \) is integrable or not. In the first case, the results from [11] imply that the first conclusion from both Theorems 1.1 and 1.2 hold. So we can assume that the conclusions from Proposition 3.5 hold at some point \( x_0 \).

Hence, the isomorphism \( \hat{\rho}_{x_0} : \mathfrak{g}_2(2) \rightarrow G(x_0) \) can be extended to an isomorphism \( \psi : \mathfrak{so}(3, 4) \rightarrow H \).

As in the proof of Proposition 3.5 and by the geodesic completeness of \( \tilde{M} \), we can apply Lemma 1.11 from [6] or the results from [8] to obtain an isometric right \( \tilde{SO}_0(3, 4) \)-action on \( \tilde{M} \) such that \( \psi(X) = X^* \), for every \( X \in \mathfrak{so}(3, 4) \). Recall that \( X^* \) is the Killing field obtained from the (right) action of the 1-parameter subgroup \( (\exp(tX))_t \) of \( \tilde{SO}_0(3, 4) \).

Let us denote by
\[
\varphi : \tilde{SO}_0(3, 4) \rightarrow \tilde{M} \\
g \mapsto x_0g,
\]
the \( \tilde{SO}_0(3, 4) \)-orbit map at \( x_0 \). From the previous remarks it follows that
\[
d\varphi_e(X) = ev_{x_0}(\psi(X))
\]
for every \( X \in \tilde{SO}_0(3, 4) \), and so defines an isomorphism. Since \( \varphi \) is \( \tilde{SO}_0(3, 4) \)-equivariant, we conclude that \( \varphi \) is a local diffeomorphism.

Let us denote with \( K \) the Killing form of \( \mathfrak{so}(3, 4) \) and let \( h_K \) be the bi-invariant pseudo-Riemannian metric on \( \tilde{SO}_0(3, 4) \) induced by \( K \). It is well known that \( \tilde{SO}_0(3, 4) \) is complete with the pseudo-Riemannian metric \( h_K \).
Let $V$ be the $\mathfrak{g}_{2(2)}$-submodule of $\mathfrak{so}(3,4)$ complementary to $\mathfrak{g}_{2(2)}$, as given by Proposition 2.5. We have proved that $d\varphi_c = ev_{x_o} \circ \psi$ and so it defines an isomorphism of modules from $\mathfrak{so}(3,4)$ onto $T_{x_o}\widetilde{M}$ for the module structures over $\mathfrak{g}_{2(2)}$ and $\mathcal{G}(x_o)$, respectively, and with respect to the isomorphism $\hat{\rho}_{x_o} : \mathfrak{g}_{2(2)} \rightarrow \mathcal{G}(x_o)$. Furthermore, we also have

$$d\varphi_c(\mathfrak{g}_{2(2)}) = T_{x_o}\mathcal{O}, \quad d\varphi_c(V) = T_{x_o}\mathcal{O}^\perp.$$ 

On the other hand, the restrictions of the metric $h_{x_o}$ to both $T_{x_o}\mathcal{O}$ and $T_{x_o}\mathcal{O}^\perp$ are non-degenerate and $\mathcal{G}(x_o)$-invariant. It follows that the bilinear forms

$$\varphi_c^*(h|_{T_{x_o}\mathcal{O}}), \quad \varphi_c^*(h|_{T_{x_o}\mathcal{O}^\perp})$$

on $\mathfrak{g}_{2(2)}$ and $V$, respectively, are non-degenerate and $\mathfrak{g}_{2(2)}$-invariant. By Proposition 2.2 there exists non-zero constants $c_1, c_2$ such that

$$K|_{\mathfrak{g}_{2(2)}} = c_1\varphi_c^*(h|_{T_{x_o}\mathcal{O}}), \quad K|_V = c_2\varphi_c^*(h|_{T_{x_o}\mathcal{O}^\perp}).$$

If we consider the pseudo-Riemannian metric on $M$ given by

$$\overline{h} = c_1h|_{T\mathcal{O}} \oplus c_2h|_{T\mathcal{O}^\perp},$$

then the above discussion shows that the map

$$d\varphi_c : (\mathfrak{so}(3,4), K) \rightarrow (T_{x_o}\widetilde{M}, \overline{T}_{x_o})$$

is an isometry. Furthermore, the equivariance of $\varphi$ implies that it defines a local isometry $(\widetilde{SO}(3,4), h_K) \rightarrow (\widetilde{M}, \overline{h})$. The completeness of $h_K$ and the results from [10] prove that $\varphi$ is in fact an isometry.

Let us consider the (left) $G_{2(2)}$-action on $\widetilde{M}$ lifted from the corresponding action on $M$. This yields from the isometry $\varphi$ a homomorphism

$$\rho : G_{2(2)} \rightarrow \text{ISO}(\widetilde{SO}(3,4), h_K).$$

The latter group of isometries is given by $L(\widetilde{SO}(3,4))R(\widetilde{SO}(3,4))$, the group of left and right translations of $\widetilde{SO}(3,4)$, and so we obtain a pair of homomorphisms

$$\rho_1, \rho_2 : G_{2(2)} \rightarrow \widetilde{SO}(3,4),$$

such that

$$\rho(g) = L_{\rho_1(g)}R_{\rho_2(g)}^{-1},$$

for every $g \in G_{2(2)}$. We note that this $G_{2(2)}$-action commutes with the right $\widetilde{SO}(3,4)$-action on $\widetilde{M}$ and so both actions commute when acting on $\widetilde{SO}(3,4)$. This implies that

$$\rho_2(G_{2(2)}) \subset Z(\widetilde{SO}(3,4)), $$

thus showing that $\rho_2 = e$. In particular, we have $\rho = L_{\rho_1}$, i.e. the $G_{2(2)}$-action defined by $\rho$ is given by left translations by $\rho_1$. Hence, $\varphi$ is $G_{2(2)}$-equivariant for the $G_{2(2)}$-action on the domain given by the non-trivial homomorphism $\rho_1 : G_{2(2)} \rightarrow \widetilde{SO}(3,4)$ and left translations.

Let us identify $\widetilde{M}$ with $\widetilde{SO}(3,4)$ through the isometry $\varphi$. By the previous discussion we have

$$\pi_1(M) \subset \text{ISO}(\widetilde{SO}(3,4), h_K).$$

Since $\text{ISO}(\widetilde{SO}(3,4))$ has finite index in $\text{ISO}(\widetilde{SO}(3,4))$ (see for example [10]) we conclude that the discrete subgroup

$$\Gamma_1 = \pi_1(M) \cap \text{ISO}(\widetilde{SO}(3,4), h_K) = \pi_1(M) \cap L(\widetilde{SO}(3,4))R(\widetilde{SO}(3,4))$$
is a finite index subgroup of $\pi_1(M)$. Every element $\gamma \in \Gamma_1$ corresponds to an isometry

$$\gamma = L_{g_1}R_{g_2},$$

where $g_1, g_2 \in \tilde{SO}_0(3, 4)$. Since the $\Gamma_1$-action and the lifted $G_2(2)$-action on $\tilde{M}$ commute with each other, it follows that $g_1 \in Z = Z_{\tilde{SO}_0(3, 4)}(\rho_1(G_2(2)))$, the centralizer in $\tilde{SO}_0(3, 4)$ of the image of $\rho_1 : G_2(2) \to \tilde{SO}_0(3, 4)$. Hence, we conclude that

$$\Gamma_1 \subset L(Z)R(\tilde{SO}_0(3, 4)).$$

We now prove the following.

**Lemma 4.1.** For every non-trivial homomorphism $\rho_1 : G_2(2) \to \tilde{SO}_0(3, 4)$, the centralizer $Z = Z_{\tilde{SO}_0(3, 4)}(\rho_1(G_2(2)))$ of the image of $\rho_1$ in $\tilde{SO}_0(3, 4)$ is a finite subgroup.

**Proof.** Consider the corresponding non-trivial homomorphism $d\rho_1 : g_2(2) \to so(3, 4)$, and let $V$ a $g_2(2)$-submodule of $so(3, 4)$ complementary to $g_2(2)$. By Proposition 2.1 we have $V \cong \mathbb{R}^{3,4}$ as $g_2(2)$-modules. It follows that $d\rho_1(g_2(2))$ is a maximal subalgebra of $so(3, 4)$. Since $d\rho_1(g_2(2)) + 3$ is a Lie subalgebra of $so(3, 4)$, where $3$ is the Lie algebra of $Z$, we conclude that $3 = 0$. Hence, $Z$ is a discrete subgroup. Finally, Lemma 1.1.3.7 from [14] implies that $Z$ is contained in any maximal compact subgroup of $\tilde{SO}_0(3, 4)$. Hence, $Z$ is a finite subgroup. \[\square\]

By Lemma 4.1 we conclude that

$$\Gamma = \Gamma_1 \cap R(\tilde{SO}_0(3, 4)),$$

is a finite index subgroup of $\Gamma_1$ and so of $\pi_1(M)$. The group $\Gamma$ is clearly identified with a discrete subgroup of $\tilde{SO}_0(3, 4)$ such that

$$\pi : \tilde{M} = \tilde{SO}_0(3, 4)/\Gamma \to \tilde{M}/\pi_1(M) = M$$

defines a finite cover of $M$.

On the other hand, a proof similar to that of Lemma 3.4 from [6] shows that $M$ has finite volume on the metric $\tilde{h}$. Hence, $\Gamma$ is a lattice of $\tilde{SO}_0(3, 4)$. This proves that the cases (2) of Theorems 1.1 and 1.2 are satisfied, thus completing the proof of these theorems.

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