Approximation of a supports stiffness influence on support coefficient values for a spring-hinged beam by quadratic functions

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Abstract. The paper proposes the approximation for the dependence of the beam first frequency on the torsional stiffness of its supports using quadratic analytical functions. For this purpose, a numerical solution of a beam's free vibration equation for a large range of torsional stiffness of supports is made and support coefficients are calculated. The obtained values are divided into three subbands, each of which is approximated by a quadratic function. This approach reduced the approximation error and allowed to use a quadratic polynomial. This allows solving the direct problem of analytical approximation, as well as solving the inverse problem of finding the required stiffness of the supports to achieve the required coefficient of supports and, accordingly, the frequency of vibration of the beam. Normalizing the obtained support coefficients is proposed for a more convenient estimation of the effect of their values on the first frequency of beam vibration.

1. Introduction
Free vibrations of a hinged beam are well known in the theory of vibrations [1-20]. The analytical solutions obtained for it are simple and widely used in analytical calculations and reference books [21]. However, in practice, a truly hinged joint is impossible and elastic forces arise in the hinges, which are taken into account by the stiffness values \(k_1\) and \(k_2\). This stiffness will affect the dynamic behavior of the beam (figure 1).

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Free vibration of a beam supported by spring-hinges.
The free vibration equation of the beam has the form [1-10]:

\[ E J_{\text{min}} \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = 0. \]  

(1)

where \( y = y(x) \) is the beam deflection; \( E \) is the elastic modulus; \( J_{\text{min}} \) is the minimum moment of inertia of the beam cross-section; \( m \) is the mass per beam length.

The first frequency of a hinged beam’s free vibration is defined as [1-10]:

\[ f_i = \frac{\alpha^2}{2\pi l^2} \sqrt{\frac{E J_{\text{min}}}{m}}. \]  

(2)

where \( \alpha \) is the natural frequency parameter or support coefficient. It is the dimensionless natural frequency parameter, depends on the boundary conditions.

The well-known literature gives the values of support coefficients for common types of restraints: hinge support, rigid, etc. [21,22]. If the beam is fixed in hinges with some stiffness (figure 1), the support coefficient will be a function:

\[ \alpha = \alpha (k_1, k_2). \]  

(3)

To find out the values of the function (3), the stiffness of the supports is taken into account through the boundary conditions of the beam restraint, which have the form:

\[ y(0,t) = y(l,t) = 0; \quad E J_{\text{min}} \frac{\partial^2 y(0,t)}{\partial x^2} = k_1 \frac{\partial y(0,t)}{\partial x}; \quad E J_{\text{min}} \frac{\partial^2 y(l,t)}{\partial x^2} = -k_2 \frac{\partial y(l,t)}{\partial x}. \]  

(4)

The general solution can be represented as

\[ y(x,t) = \sum_{i=1}^{\infty} Y_i(x) \sin \omega_i t. \]  

(5)

where \( Y_i(x) \) is the \( i \)-th mode of vibration, given by

\[ Y_i(x) = A_i \sin p_i x + B_i \cos p_i x + C_i \sinh p_i x + D_i \cosh p_i x. \]  

(6)

Using equation (5) in conjunction with the boundary conditions yields the following frequency equation, similar to the equation in [23]:

\[ 2\alpha_i^4 \tan \alpha_i^2 \cdot \tanh \alpha_i^2 + \alpha_i^2 l \frac{k_1 + k_2}{E J_{\text{min}}} \left( \tan \alpha_i^2 - \tanh \alpha_i^2 \right) - \frac{k_1 k_2 l^2}{E^2 J_{\text{min}}} \left( 1 - \frac{1}{\cos \alpha_i^2 \cdot \cosh \alpha_i^2} \right) = 0. \]  

(7)

where \( \alpha_i^2 = p_i l \) is the square of support coefficient for \( i \)-th natural mode of vibration.

The analytical solution of equation (7) using simple functions is impossible and requires the use of numerical methods. In this work, solutions to equation (7) are given for some combinations of stiffness values of supports, which are then approximated by quadratic functions. This makes it possible to obtain simple analytical solutions for determining support coefficients and, accordingly, beam vibration frequencies at known support stiffness. Quadratic functions allow analytically solving the inverse problems of finding the necessary stiffness of supports in order to achieve the required value of support coefficients and beam vibration frequency.

2. Approximation of support coefficient values

In this paper, a numerical solution of equation (7) was made for more than 500 combinations of stiffness values of beam supports. The limited size of the article does not allow you to give all the results in full.
Table 1 shows some of the results of the calculation of the support coefficients $\alpha$ with respect to the relative stiffness $C_i$, which is calculated as:

$$C_1 = k_1 \frac{l}{EJ_{\text{min}}}; \quad C_2 = k_2 \frac{l}{EJ_{\text{min}}}.$$

(8)

Table 1. Values of support coefficients $\alpha(C_1, C_2)$.

| $C_2$ | 0   | 0.01 | 0.1  | 1    | 5    | 10   | 50   | 100  | 500  | 1000 | $\infty$ |
|-------|-----|------|------|------|------|------|------|------|------|------|---------|
|       |     |      |      |      |      |      |      |      |      |      |         |
| 0     | 3.142 | 3.143 | 3.157 | 3.273 | 3.534 | 3.665 | 3.855 | 3.889 | 3.919 | 3.923 | 3.926 |
| 0.01  | 3.143 | 3.145 | 3.159 | 3.275 | 3.534 | 3.666 | 3.856 | 3.890 | 3.920 | 3.924 | 3.927 |
| 0.1   | 3.157 | 3.159 | 3.173 | 3.288 | 3.548 | 3.678 | 3.868 | 3.902 | 3.932 | 3.936 | 3.940 |
| 1     | 3.273 | 3.275 | 3.288 | 3.399 | 3.652 | 3.781 | 3.970 | 4.004 | 4.034 | 4.038 | 4.041 |
| 5     | 3.534 | 3.534 | 3.548 | 3.652 | 3.897 | 4.026 | 4.156 | 4.354 | 4.390 | 4.422 | 4.426 |
| 10    | 3.665 | 3.666 | 3.678 | 3.781 | 4.026 | 4.156 | 4.354 | 4.390 | 4.422 | 4.426 | 4.430 |
| 50    | 3.855 | 3.856 | 3.868 | 3.970 | 4.219 | 4.354 | 4.563 | 4.602 | 4.636 | 4.640 | 4.645 |
| 100   | 3.889 | 3.890 | 3.902 | 4.004 | 4.254 | 4.390 | 4.602 | 4.641 | 4.676 | 4.681 | 4.685 |
| 500   | 3.919 | 3.920 | 3.932 | 4.034 | 4.241 | 4.422 | 4.636 | 4.676 | 4.711 | 4.716 | 4.720 |
| 1000  | 3.923 | 3.924 | 3.936 | 4.038 | 4.245 | 4.426 | 4.640 | 4.681 | 4.716 | 4.721 | 4.725 |
| $\infty$ | 3.926 | 3.927 | 3.940 | 4.041 | 4.249 | 4.430 | 4.645 | 4.685 | 4.720 | 4.725 | 4.730 |

Figure 2 shows the dependence of support factor $\alpha$ on support stiffness $C_2$ at some constant support stiffness $C_i$.

The graph is highly nonlinear, which does not allow you to approximate it with a simple analytical function. Therefore, we limit the considered range of stiffness change $C_i=0$-1000 and divide it into three zones, I: 0-10, II: 10-100, III: 100-1000. Stiffness values $C_i>1000$ actually correspond to fixed support.
Inside each range, we approximate the obtained numerical values of the support coefficients by the quadratic functions using the least squares method [24-26]. As a result, we obtain a solution for the first zone $C_i = 0-10$ in the form:

$$\alpha_i(C_i) = 3.142 + 0.107 \cdot (C_i + C_2) - 5.63 \cdot 10^{-3} \cdot (C_i^2 + C_2^2).$$  \hspace{1cm} (9)

The solution for the second zone must be divided into two parts depending on the combination of stiffness values $C_1$ and $C_2$:

$$\alpha_{II}(10 \leq C_{12} \leq 10^3) = 4.03 + 7.5 \cdot 10^{-3} \cdot (C_i + C_2) - 4.35 \cdot 10^{-5} \cdot (C_i^2 + C_2^2)$$  \hspace{1cm} (10)

$$\alpha_{II}(C_i \leq 10, 10 \leq C_2 \leq 10^3) = 3.142 + 0.107 \cdot (C_i + 10) - 5.63 \cdot 10^{-3} \cdot (C_i^2 + 100) + 7.5 \cdot 10^{-3} \cdot (C_2 - 10) - 4.35 \cdot 10^{-5} (C_2^2 - 100)$$  \hspace{1cm} (11)

At last, the solution in the third zone will also depend on a combination of values of stiffness $C_i$, $C_2$ and will be divided into three equations:

$$\alpha_{III}(10^2 \leq C_{12} \leq 10^3) = 4.62 + 1.25 \cdot 10^{-4} \cdot (C_i + C_2) - 7.4 \cdot 10^{-8} \cdot (C_i^2 + C_2^2);$$  \hspace{1cm} (12)

$$\alpha_{III}(10 \leq C_i \leq 100, 10^2 \leq C_2 \leq 10^3) = 4.03 + 7.5 \cdot 10^{-3} (C_i + 100) - 4.35 \cdot 10^{-5} (C_i^2 + 10^4) + 1.25 \cdot 10^{-4} (C_2 - 10^4) - 7.4 \cdot 10^{-8} (C_2^2 - 10^4);$$  \hspace{1cm} (13)

$$\alpha_{III}(C_i \leq 10, 10^2 \leq C_2 \leq 10^3) = 3.386 + 0.107 \cdot (C_i + 10) - 5.63 \cdot 10^{-3} \cdot (C_i^2 + 100) + 1.25 \cdot 10^{-4} \cdot (C_2 - 100) - 7.4 \cdot 10^{-8} \cdot (C_2^2 - 10^4).$$  \hspace{1cm} (14)

The obtained approximating functions (9-14) allow calculating the values of the support coefficient $\alpha$ for equation (2) with an error of not more than 1.87% in all considered range of support stiffness.

3. Squares of support coefficient values

In the free vibration frequency equation (2), the support coefficient is squared, so it makes sense to approximate the squares of their values. Table 2 shows the squares of the support coefficient values.

**Table 2. Values of support coefficients $\alpha(C_i, C_2)$.**

| $C_2$ | $0$ | $0.01$ | $0.1$ | $1$ | $5$ | $10$ | $50$ | $100$ | $500$ | $1000$ | $\infty$ |
|-------|-----|-------|------|----|----|-----|-----|-----|------|-------|--------|
|       | 9.870 | 9.880 | 9.968 | 10.71 | 12.49 | 13.43 | 14.86 | 15.13 | 15.36 | 15.39 | 15.41 |
| 0.01  | 9.880 | 9.890 | 9.977 | 10.72 | 12.49 | 13.44 | 14.87 | 15.14 | 15.37 | 15.40 | 15.43 |
| 0.1   | 9.968 | 9.977 | 10.07 | 10.81 | 11.55 | 12.59 | 13.53 | 14.96 | 15.23 | 15.46 | 15.49 |
| 1     | 10.71 | 10.72 | 10.81 | 11.55 | 12.59 | 13.34 | 14.29 | 15.76 | 16.03 | 16.27 | 16.30 |
| 5     | 12.49 | 12.49 | 12.59 | 12.59 | 13.34 | 15.19 | 16.21 | 17.80 | 18.10 | 18.36 | 18.43 |
| 10    | 13.43 | 13.44 | 13.53 | 14.29 | 16.21 | 17.27 | 18.95 | 19.27 | 19.55 | 19.59 | 19.62 |
| 50    | 14.86 | 14.87 | 14.96 | 15.76 | 17.80 | 18.95 | 20.82 | 21.18 | 21.49 | 21.53 | 21.57 |
| 100   | 15.13 | 15.14 | 15.23 | 16.03 | 18.10 | 19.27 | 21.18 | 21.54 | 21.86 | 21.91 | 21.95 |
| 500   | 15.36 | 15.37 | 15.46 | 16.27 | 18.36 | 19.55 | 21.49 | 21.86 | 22.20 | 22.24 | 22.28 |
| 1000  | 15.39 | 15.40 | 15.49 | 16.30 | 18.39 | 19.59 | 21.53 | 21.91 | 22.24 | 22.28 | 22.32 |
| $\infty$ | 15.41 | 15.43 | 15.52 | 16.33 | 18.43 | 19.62 | 21.57 | 21.95 | 22.28 | 22.32 | 22.36 |

Using the same approach, we divide the entire stiffness range into three zones, and by the least squares method we get the coefficient values for approximating quadratic functions for each zone. The first zone $C_i=0-10$:
\[ \alpha_i^2 (C_i) = 9.87 + 0.736 \cdot (C_1 + C_2) - 3.66 \cdot 10^{-2} \cdot (C_i^2 + C_i^2). \] 

(15)

The solution for the second zone must be divided into two parts depending on the combination of stiffness values \(C_1\) and \(C_2\):

\[ \alpha_2^2 \left( 10 \leq C_{1, 2} \leq 10^2 \right) = 16.06 + 6.38 \cdot 10^{-2} \cdot (C_1 + C_2) - 4 \cdot 10^{-4} \cdot (C_1^2 + C_2^2) \]

(16)

\[ \alpha_2^2 \left( C_1 \leq 10, 10 \leq C_2 \leq 10^3 \right) = 9.87 + 0.736 \cdot (C_1 + 10) - 3.66 \cdot 10^{-2} \cdot (C_1^2 + 100) + 6.38 \cdot 10^{-2} \cdot (C_2 - 10) - 4 \cdot 10^{-4} \cdot (C_2^2 - 100) \]

(17)

At last, the solution in the third zone will also depend on a combination of values of stiffness \(C_1, C_2\) and will be divided into three equations

\[ \alpha_{iii}^2 \left( 10^2 \leq C_{1, 2} \leq 10^3 \right) = 21.3 + 1.15 \cdot 10^{-3} \cdot (C_1 + C_2) - 6.5 \cdot 10^{-7} \cdot (C_1^2 + C_2^2); \]

(18)

\[ \alpha_{iii}^2 \left( 10 \leq C_1 \leq 100, 10^2 \leq C_2 \leq 10^3 \right) = 16.06 + 6.38 \cdot 10^{-2} \cdot (C_1 + 100) - 4 \cdot 10^{-4} \cdot (C_1^2 + 10^4) + 1.15 \cdot 10^{-3} \cdot (C_2 - 100) - 6.5 \cdot 10^{-7} \cdot (C_2^2 - 10^4); \]

(19)

\[ \alpha_{iii}^2 \left( C_1 \leq 10, 10 \leq C_2 \leq 10^3 \right) = 11.652 + 0.736 \cdot (C_1 + 10) - 3.66 \cdot 10^{-2} \cdot (C_1^2 + 100) + 1.15 \cdot 10^{-3} \cdot (C_2 - 10^2) - 6.5 \cdot 10^{-7} \cdot (C_2^2 - 10^4). \]

(20)

The obtained approximating functions allow calculating the squares of the support factor \(\alpha^2\) for equation (2) with an error of not more than 3.12% in the entire considered range of support stiffness.

4. Discussion

The values of the support coefficients calculated from the approximating functions (9-20) are in good agreement with the known values, for example, in [21,23]. It should be noted that the results of calculations for the proposed approximating functions (9-20) significantly depend on the coefficients included in these equations and it is undesirable to round them to simpler numbers in calculations.

Comparison of calculation results on approximating functions (9-20) with data in the well-known reference book [21] made it possible to detect errors there: on page 169 in table 4.10 the matrix of support coefficients is asymmetric and the maximum value corresponding to the fixed support is 4.712 instead of 4.73.

A comparative review of the obtained values of the support coefficients confirmed the operability and reliability of the proposed approximation of the support coefficients. The proposed solution is valid not only for a single-span hinge beam, but also for a multi-span beam, provided that the spans and stiffness of each hinge support are equal.

5. Conclusion

Quadratic approximation of dependence of values of support coefficients on stiffness of beam hinge supports is made in this work. The obtained solution can be used to determine the beam support coefficient at any known stiffness of supports with an error of not more than 2%. Also, the obtained analytical quadratic approximations make it possible to solve the inverse problem of finding the necessary stiffness of supports at the known required value of the support coefficient or vibration frequency of a given beam. As a result, an analytical solution was obtained to control the natural frequency of beam vibration by setting the required stiffness of the beam supports.
Acknowledgements
The research was funded by RFBR, Krasnoyarsk Territory and Krasnoyarsk Regional Fund of Science, project number 20-48-242922.

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