Elliptic recurrence representation
of the $\mathcal{N} = 1$ Neveu-Schwarz blocks

Leszek Hadasz†1, Zbigniew Jaskólski‡2 and Paulina Suchanek†3

† M. Smoluchowski Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland,
‡ Institute of Theoretical Physics, University of Wrocław, pl. M. Borna, 50-204 Wrocław, Poland.

Abstract

We apply a suitably generalized method of Al. Zamolodchikov to derive an elliptic recurrence representation of the Neveu-Schwarz superconformal blocks

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† e-mail: hadasz@th.if.uj.edu.pl
‡ e-mail: jask@ift.uni.wroc.pl
§ e-mail: suchanek@th.if.uj.edu.pl
1 Introduction

Conformal field theory proved to be very efficient tool in describing second order phase transitions in two-dimensional system and is a commonly used language of string theory. Correlation functions in CFT can be expressed as sums (or integrals) of three-point coupling constants and the conformal blocks, fully determined by the symmetry alone [1]. An explicit calculation of conformal blocks is still one of the most difficult problems in CFT. Not only the form of a general conformal block is unknown, but also its analytic properties are still conjectures rather than theorems.

On the other hand, there exist very efficient recursive methods of an approximate, analytic determination of a general 4-point conformal block [2–4]. They were used for instance in checking the conformal bootstrap in the Liouville theory with the DOZZ coupling constants [5], in study of the $c \to 1$ limit of minimal models [6] or in obtaining new results in the classical geometry of hyperbolic surfaces [7]. In a more general context of an arbitrary CFT model these methods allow for efficient numerical calculations of any 4-point function once the structure constants are known.

In the last year the recursion representations have been worked out for the super-conformal blocks related to the Neveu-Schwarz algebra [8–11]. The so called elliptic recursion was conjectured in [10] for one type of NS blocks and applied in the numerical verification of the consistency of $N = 1$ super-Liouville theory. The extension of this method to another type of NS blocks was proposed in [11] where also further numerical support for the consistency of the $N = 1$ super-Liouville theory was given.

The aim of the present paper is to provide a comprehensive derivation of the elliptic recursion for all types of NS blocks. This is done by an appropriate extension of the method originally developed in [3, 4] for the Virasoro case. In our derivation we also use an exact analytic expressions for certain $N = 1$ NS superconformal blocks of $c = \frac{3}{2}$ theory obtained in [12].

The organization of the paper is as follows. In Section 2 we present our notation and basic properties of NS blocks derived in [8]. Section 3 is devoted to the analysis of the classical limit of $N = 1$ NS supersymmetric Liouville theory. Using the path integral representation we show that in the classical limit of the supersymmetric Liouville correlators the leading terms are described by the classical bosonic Liouville action. This implies in particular that the exponential part of the classical limit of all NS blocks is given by the classical conformal block of the Virasoro theory.

One of the main points of the method proposed in [3, 4] is that the dependence of the first two terms of the $\frac{1}{\Delta}$ expansion of quantum conformal block on the external weights and the central charge can be read off from the $\frac{1}{\Delta}$ expansion of the classical block. Since the extension of this reasoning to the NS N=1 case is rather straightforward we present it in Section 4 mainly for completeness.

At this point one could use the results of [3, 4] to derive the large $\Delta$ asymptotics of NS
blocks. On the other hand one can follow the general strategy of [3, 4] within the NS theory. This is done in Section 5. The fact that the null vector decoupling equations of NS theory imply exactly the same equation for the classical conformal block that one gets in the Virasoro case can be seen as a consistency check of the path integral arguments used in Section 3.

Finally in Section 6, using the explicit analytic expressions for \( c = \frac{3}{2} \) superconformal blocks [12] we derive the recurrence relations for all type of NS superconformal blocks.

Some technical details of the elliptic Ansatz used in [3, 4] are given in Appendix A. In Appendix B it is shown that the recurrence formulae are in perfect agreement with an exact analytical form of the conformal blocks obtained in [12].

## 2 NS N = 1 superconformal blocks

The 4-point NS superconformal blocks are conveniently defined in terms of the 3-point block

\[
\rho_{\Delta_1 \Delta_2 \Delta_3}^{\Delta_4 \Delta_5 \Delta_6} : \mathcal{V}_{\Delta_3} \times \mathcal{V}_{\Delta_2} \times \mathcal{V}_{\Delta_1} \mapsto \mathbb{C},
\]

normalized by the condition

\[
\rho_{\Delta_1 \Delta_2 \Delta_3}^{\Delta_4 \Delta_5 \Delta_6} (\nu_3, \nu_2, \nu_1) = \rho_{\Delta_1 \Delta_2 \Delta_3}^{\Delta_4 \Delta_5 \Delta_6} (\nu_3, \ast \nu_2, \nu_1) = 1,
\]

where \( \nu_3, \nu_2, \nu_1 \) are super-primary states in NS superconformal Verma modules \( \mathcal{V}_{\Delta_3}, \mathcal{V}_{\Delta_2}, \mathcal{V}_{\Delta_1} \) and \( \ast \nu_2 = S_{-\frac{1}{2}} \nu_2 \).

Then the even,

\[
F^{\Delta_1 \Delta_2 \Delta_3}_{\Delta_4 \Delta_5 \Delta_6} (x) = x^{\Delta_1 - \Delta_2 - \Delta_3} \left( 1 + \sum_{m \in \mathbb{N}} x^m F^{m}_{\Delta_1 \Delta_2 \Delta_3} \right),
\]

and the odd,

\[
F^{\frac{1}{2} \Delta_1 \Delta_2 \Delta_3}_{\Delta_4 \Delta_5 \Delta_6} (x) = \sum_{k \in \mathbb{N} - \frac{1}{2}} x^k F^{k}_{\Delta_1 \Delta_2 \Delta_3},
\]

superconformal blocks are determined by their coefficients:

\[
F^{\Delta_1 \Delta_2 \Delta_3}_{\Delta_4 \Delta_5 \Delta_6} (x) = \sum_{|K| + |M| = |L| + |N| = f} \rho_{\Delta_1 \Delta_2 \Delta_3}^{\Delta_4 \Delta_5 \Delta_6} (\nu_4, \nu_3, \nu_2, \nu_1) \left[ B^{f}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4 \Delta_5 \Delta_6} \right]^{K, L, N}_{M},
\]

where \( [B^{f}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4 \Delta_5 \Delta_6}]^{K, L, N}_{M} \) is the matrix inverse to the Gram matrix of superconformal Verma module at level \( f \) with respect to the basis \( \{ \nu_{L, N} \} \), \( \Delta_i \) and \( \nu_i \) stand for \( \Delta_i \) or \( \ast \Delta_i \), and \( \nu_i \) or \( \ast \nu_i \), respectively, and \( x^{\Delta_1 - \ast \Delta_2 - \Delta_3} = x^\Delta - \Delta_2 - \Delta_1 - \frac{1}{2} \).

\( ^4 \text{We shall follow the notation of [8]} \)
It follows from (2.2) that the blocks’ coefficients are polynomials in the external weights \( \Delta \) and rational functions of the intermediate weight \( \Delta \) and the central charge \( c \). They can be expressed as a sum over the poles in \( \Delta \):

\[
F^f_{c,\Delta}[\Delta_1, \Delta_2] = h^f_{c,\Delta}[\Delta_3, \Delta_2] + \sum_{1 < rs \leq 2f \atop r+s \in 2N} \mathcal{R}^f_{c,rs}[\Delta_1, \Delta_2] \Delta - \Delta_{rs}(c),
\]

(2.3)

with \( \Delta_{rs}(c) \) given by Kac determinant formula for NS Verma modules:

\[
\Delta_{rs}(c) = -\frac{rs - 1}{4} + \frac{1 - r^2}{8} b^2 + \frac{1 - s^2}{8} b^2, \quad c = \frac{3}{2} + 3 \left( b + \frac{1}{b} \right)^2.\]

(2.4)

It was shown in [8] that the residue at \( \Delta = \Delta_{rs} \) takes the form

\[
\mathcal{R}^m_{c,rs}[\Delta_3, \Delta_2] = S_{rs}(\Delta_3) A^c_{rs} \frac{[\Delta_3, \Delta_2]}{[\Delta_1, \Delta_1]} F^m_{c,\Delta_{rs}}[\Delta_1, \Delta_2]
\]

(2.5)

for \( m \in \mathbb{N} \cup \{0\} \) and

\[
\mathcal{R}^k_{c,rs}[\Delta_3, \Delta_2] = S_{rs}(\Delta_3) A^c_{rs} \frac{[\Delta_3, \Delta_2]}{[\Delta_1, \Delta_1]} F^k_{c,\Delta_{rs}}[\Delta_1, \Delta_2]
\]

(2.6)

for \( k \in \mathbb{N} - \frac{1}{2} \). Here \( \tilde{\Delta} = \Delta, \Delta = \Delta, S_{rs}(\Delta) = 1, S_{rs}(\Delta) = (-1)^{rs} \) and

\[
A^c_{rs}[\Delta_3, \Delta_2] = A_{rs}(c) P^c_{rs}[\Delta_3, \Delta_1] P^c_{rs}[\Delta_2, \Delta_1],
\]

where

\[
P^c_{rs}[\Delta_2, \Delta_1] = \prod_{p=1-r}^{r-1} \prod_{q=1-s}^{s-1} \left( \frac{2a_1 - 2a_2 - pb - qb^{-1}}{2\sqrt{2}} \right) \left( \frac{2a_1 + 2a_2 + pb + qb^{-1}}{2\sqrt{2}} \right)
\]

(2.7)

with \( p + q - (r + s) \in 4\mathbb{Z} + 2 \),

\[
P^c_{rs}[\Delta_2, \Delta_1] = \prod_{p=1-r}^{r} \prod_{q=1-s}^{s-1} \left( \frac{2a_1 - 2a_2 - pb - qb^{-1}}{2\sqrt{2}} \right) \left( \frac{2a_1 + 2a_2 + pb + qb^{-1}}{2\sqrt{2}} \right)
\]

(2.8)

with \( p + q - (r + s) \in 4\mathbb{Z} \) and

\[
A_{rs}(c) = \frac{1}{2} \prod_{p=1-r}^{r} \prod_{q=1-s}^{s} \left( \frac{1}{\sqrt{2}} \left( pb + \frac{q}{b} \right) \right)^{-1}, \quad p + q \in 2\mathbb{Z}, \quad (p,q) \neq (0,0),(r,s).
\]

(2.9)

3 Supersymmetric Liouville theory and classical limit of superconformal blocks

Within a path integral approach the \( N = 1 \) super-Liouville theory is defined by the action:

\[
S_{\text{SLFT}} = \int d^2 z \left( \frac{1}{2\pi} |\partial \phi|^2 + \frac{1}{2\pi} (\bar{\psi} \partial \bar{\psi} + \bar{\psi} \partial \psi) + 2i\mu b^2 \bar{\psi} \psi e^{i\phi} + 2\pi b^2 e^{2i\phi} \right).
\]

(3.1)
Each super-primary field $V_a$ with conformal dimension $\Delta_a = \bar{\Delta}_a = \frac{a(Q-a)}{2}$ and all its descendant Virasoro primaries are represented by exponentials:

$$V_a = e^{a\phi},$$

$$A_a = [S_{-1/2}, V_a] = -ia\bar{\psi}e^{a\phi},$$

$$\bar{A}_a = [\bar{S}_{-1/2}, V_a] = -ia\bar{\psi}e^{a\phi},$$

$$\bar{V}_a = \{ S_{-1/2}, [\bar{S}_{-1/2}, V_a] \} = a^2\bar{\psi}\bar{\psi}e^{a\phi} - 2i\pi\mu bae^{(a+b)\phi}.$$ 

One has for instance

$$\langle V_4 V_3 \bar{V}_2 V_1 \rangle = \int D\phi D\bar{\psi} D\bar{\psi} e^{-S_{\text{SLFT}}[\phi, \bar{\psi}]} e^{a_4 \phi} e^{a_3 \phi} \left( a_2^2 \bar{\psi}\bar{\psi} e^{a_2 \phi} - 2i\pi\mu bae^{(a_2+b)\phi} \right) e^{a_1 \phi}. \quad (3.2)$$

In order to analyze the classical limit ($b \to 0$, $2\pi\mu b^2 \to m = \text{const}$) of this correlator one may integrate fermions out. Since the integration is gaussian and the operator $e^{b\phi}$ is light, one can expect in the case of heavy weights

$$a = \frac{\rho}{2} (1 - \lambda), \quad ba \to \frac{1-\lambda}{2}, \quad 2b^2 \Delta \to \delta = \frac{1-\lambda}{4},$$

the following asymptotic behavior

$$\langle V_4 V_3 \bar{V}_2 V_1 \rangle \sim \frac{1}{\pi} e^{-\frac{1}{2\pi} S \left[ \delta_1, \delta_2, \delta_3, \delta_4 \right]},$$

where $S \left[ \delta_1, \delta_2, \delta_3, \delta_4 \right]$ is the bosonic Liouville action

$$S[\phi] = \frac{1}{2\pi} \int d^2z \left( |\partial \phi|^2 + m^2 e^{2\phi} \right)$$

calculated on the classical configuration $\varphi$ satisfying the Liouville equation

$$\partial \bar{\partial} \varphi - m^2 e^{2\phi} = \sum_{i=1}^4 \frac{1 - \lambda_i}{4} \delta(z - z_i).$$

On the other hand in $N = 1$ supersymmetric Liouville theory the correlator $\langle V_4 V_3 \bar{V}_2 V_1 \rangle$ can be expressed as an integral over the spectrum. In the case of standard locations $z_4 = \infty, z_3 = 1, z_2 = x, z_1 = 0$ one has

$$\langle V_4 V_3 \bar{V}_2 V_1 \rangle = \int \frac{da}{2\pi i} \left( C_{43a} \bar{C}_{s21} \left| F_{\frac{\Delta_4}{2}} \left[ \frac{\Delta_1^* \Delta_2}{\Delta_4} \right] (x) \right|^2 + \bar{C}_{43a} C_{a21} \left| F_{\frac{\Delta_4}{2}} \left[ \frac{\Delta_1^* \Delta_2}{\Delta_4} \right] (x) \right|^2 \right). \quad (3.3)$$

where $C$ and $\bar{C}$ are the the two independent supersymmetric Liouville structure constants

$$C_{a21} = \langle V_a(\infty, \infty) V_{a_2}(1, 1) V_{a_1}(0, 0) \rangle, \quad \bar{C}_{a21} = \langle V_a(\infty, \infty) \bar{V}_{a_2}(1, 1) V_{a_1}(0, 0) \rangle.$$ 

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5The factor $\frac{1}{\pi}$ in front of the Liouville classical action comes from different parameterizations of the central charge in the NS and in the bosonic Liouville theory.
As in the case of 4-point functions the path integral representation yields the asymptotic behavior
\[
\begin{align*}
C_{\alpha_2 \alpha_1} & \sim e^{-\frac{1}{2\sigma^2} S_{\alpha_1}[\delta, \delta_2, \delta_1]}, \\
\tilde{C}_{\alpha_2 \alpha_1} & \sim \frac{1}{b^2} e^{-\frac{1}{2\sigma^2} S_{\alpha_1}[\delta, \delta_2, \delta_1]},
\end{align*}
\tag{3.4}
\]

where \( S_{\alpha_1}[\delta, \delta_2, \delta_1] \) is the 3-point classical bosonic Liouville action.

Following the reasoning of \([4, 5]\) one can apply the path integral arguments to the correlator (3.2) projected on the even and on the odd subspaces of the \( \Delta \) superconformal family of intermediate states. This leads to the following \( b \to 0 \) asymptotic behavior:
\[
\begin{align*}
\langle V_{\lambda_3}^{\text{(even)}} \tilde{V}_{\lambda_1} \rangle & \sim \frac{1}{b^2} e^{-\frac{1}{2\sigma^2} S_{\alpha_1}[\delta, \delta_3, \delta_2, \delta_1]}, \\
\langle V_{\lambda_3}^{\text{(odd)}} \tilde{V}_{\lambda_1} \rangle & \sim \frac{1}{b^2} e^{-\frac{1}{2\sigma^2} S_{\alpha_1}[\delta, \delta_3, \delta_2, \delta_1]},
\end{align*}
\tag{3.5}
\]

where the “\( \Delta \)-projected” classical action is given by
\[
S_{\alpha_1}[\delta_4, \delta_3, \delta_2, \delta_1|\delta] = S_{\alpha_1}[\delta_4, \delta_3, \delta] + S_{\alpha_1}[\delta, \delta_2, \delta_1] - f_b^{\delta_1}((x)) - f_b^{\delta_1}(\bar{x})
\]

and \( f_b^{\delta_1}(x) \) is the classical conformal block, defined in terms of the \( \hat{Q} \to \infty \) limit of the quantum conformal block in the Virasoro \( c = 1 + 6\hat{Q}^2 \) CFT:
\[
\mathcal{F}_{0+6\hat{Q}^2, \Delta}^{\hat{Q}\to \infty} \left( \frac{\Delta_3 \Delta_2}{\Delta_4 \Delta_1} \right) (x) \sim \exp \left\{ \hat{Q}^2 f_b^{\delta_1}(x) \right\}.
\tag{3.6}
\]

Equations (3.3) – (3.5) then imply:
\[
\mathcal{F}_{\Delta}^{\hat{Q}\to \infty} \left[ \frac{\Delta_3 \Delta_2}{\Delta_4 \Delta_1} \right] (x) \sim e^{\frac{1}{2\sigma^2} f_b^{\delta_3 \delta_2 \delta_1}(x)}, \quad \mathcal{F}_{\Delta}^{\hat{Q}\to \infty} \left[ \frac{\Delta_3 \Delta_2}{\Delta_4 \Delta_1} \right] (x) \sim e^{\frac{1}{2\sigma^2} f_b^{\delta_3 \delta_2 \delta_1}(x)};
\tag{3.7}
\]

Using representations analogous to (3.3) and the same reasoning for the other 4-point correlators of primary fields \( V_{\alpha}, \lambda_{\alpha}, \bar{\lambda}_{\alpha}, \tilde{V}_{\alpha} \) one gets
\[
\mathcal{F}_{\Delta} \left[ \frac{\Delta_3 \Delta_2}{\Delta_4 \Delta_1} \right] (x) \sim e^{\frac{1}{2\sigma^2} f_b^{\delta_3 \delta_2 \delta_1}(x)}, \quad \mathcal{F}_{\Delta} \left[ \frac{\Delta_3 \Delta_2}{\Delta_4 \Delta_1} \right] (x) \sim e^{\frac{1}{2\sigma^2} f_b^{\delta_3 \delta_2 \delta_1}(x)},
\tag{3.8}
\]

The properties of classical conformal block relevant for the the elliptic recurrence relations where already derived by Al. B. Zamolodchikov in the Virasoro CFT \([4]\). In the next two sections we shall nevertheless present a step by step derivation of these properties in NS SCFT. This can be seen as a nontrivial consistency check of heuristic path integral arguments of this section.

### 4 Large \( \Delta \) vs. classical asymptotic of superconformal blocks

As in the bosonic case the first step in the derivation of the elliptic recurrence is to find the large \( \Delta \) asymptotic of the conformal block. The method of calculations proposed in \([4]\) is based on the observation that the full dependence of the first two terms in the large \( \Delta \)
expansion on the variables $\Delta_i, c$ can be read off from the first two terms of the $\frac{1}{\Delta}$ expansion of the classical block. While in the case of even NS blocks the reasoning is essentially the same as in [4], the odd case is slightly more complicated.

Let us first note that on each level of the NS Verma module the determinant of the Gram matrix is proportional to $\Delta$ and each matrix element of its inverse contains a factor $\Delta^{-1}$. On the other hand it follows from the properties of the 3-point superconformal blocks [8] that in a generic case $\rho_{\infty}^{(1)} \Delta_1, \Delta_2 (\nu_4, \nu_3, \nu_{K, M})$ does not contain the factor $\Delta$, while for all odd levels $|K| \in \mathbb{N} - \frac{1}{2}$, $\rho_{\infty}^{(1)} \Delta_1, \Delta_2 (\nu_4, *\nu_3, \nu_{K, M})$ is proportional to $\Delta$. Since the power series defining the odd blocks (2.1) do not contain zeroth order terms it follows from the definition (2.2) that the functions

$$G_{\Delta}^{\frac{1}{2}} \left[ -\Delta_1, \Delta_2 \right] (x) = \ln F_{\Delta}^{\frac{1}{2}} \left[ -\Delta_1, \Delta_2 \right] (x)$$

admit power series expansions of the form

$$G_{\Delta}^{\frac{1}{2}} \left[ \Delta_1, \Delta_2 \right] (x) = (\Delta - \Delta_2 - \Delta_1) \ln x - \ln \Delta + \sum_{i=0}^{\infty} G_n x^n,$$

$$G_{\Delta}^{\frac{1}{2}} \left[ \Delta_1 + \Delta_2 \right] (x) = (\Delta - \Delta_2 - \Delta_1) \ln x + \sum_{i=0}^{\infty} G^*_n x^n,$$

$$G_{\Delta}^{\frac{1}{2}} \left[ \Delta_1 - \Delta_2 \right] (x) = (\Delta - \Delta_2 - \Delta_1) \ln x + \ln \Delta + \sum_{i=0}^{\infty} G^{**}_n x^n,$$

where the coefficients $G_n, G^*_n, G^{**}_n$ are rational functions of $\Delta, \Delta_i, c$.

We shall consider the first case in more detail. One has:

$$G_n = \frac{P_n (\Delta, \Delta_i, c)}{Q_n (\Delta, c)},$$

where $P_n (\Delta, \Delta_i, c)$ and $Q_n (\Delta, c)$ are polynomials in all their arguments. The existence of semiclassical limit (3.8) implies that the maximal homogeneous degree of $P_n (\Delta, \Delta_i, c)$,

$$P^{N_n+1}_n (\Delta, \Delta_i, c) = \Delta^{N_n} \left( A_n \Delta + \sum_{i=1}^{4} B^i_n \Delta_i + C_n c \right) + \Delta^{N_n-1} (\ldots),$$

is greater by 1 than the maximal homogeneous degree of $Q_n (\Delta, c)$,

$$Q^{N_n}_n (\Delta, c) = a_n \Delta^{N_n} + b_n \Delta^{N_n-1} c + c_n \Delta^{N_n-2} c^2 + \ldots$$

The coefficients $A_n, \ldots, c_n$ are by definition independent of $c, \Delta$ and $\Delta_i$.

It follows from (2.2) and the Kac determinant formula for NS supermodules that $a_n \neq 0$, hence

$$f^{\delta}_n \left[ \frac{\delta_1 \delta_2}{\delta_1 \delta_1} \right] = \frac{\delta^{N_n} \left( A_n \delta + \sum_{i=1}^{4} B^i_n \delta_i + 6 C_n \right) + \delta^{N_n-1} (\ldots)}{a_n \delta^{N_n} + 6 b_n \delta^{N_n-1} + 36 c_n \delta^{N_n-2} + \ldots}.$$
On the other hand the $\frac{1}{a}$ expansion of $G_n$ takes the form

$$G_n = \frac{A_n}{a_n} \Delta + \sum_{i=1}^{4} \frac{B_n^i}{a_n^j} \Delta_i + \frac{C_n}{a_n} + \frac{A_nb_n}{a_n} + \frac{D_n}{a_n} + O\left(\frac{1}{\Delta}\right), \quad (4.1)$$

where $D_n$ is the coefficient in front of $\Delta^{N_n}$ in the polynomial $P_n$.

5 Null vector decoupling equation

We consider 5-point correlators of primary fields $V_i = V_a(z_i, \bar{z}_i)$ or $\Lambda_i = \Lambda_a(z_i, \bar{z}_i)$ in the limit $a_5 \to -b$. The field $V_{-b}(z_5, \bar{z}_5)$ is degenerate and satisfies the null vector decoupling equation:

$$V_0(z_5, \bar{z}_5) \equiv \left(L_{-1} S_{-\frac{1}{2}} + b^2 S_{-\frac{1}{2}}\right) V_{-b}(z_5, \bar{z}_5) = 0.$$

Applying to correlators $\langle V_4 A_3 V_0 V_2 V_1 \rangle$, $\langle V_4 V_3 V_0 A_2 V_1 \rangle$, $\langle V_4 V_3 V_0 V_2 A_1 \rangle$, $\langle V_4 A_3 V_0 A_2 A_1 \rangle$ the conformal Ward identities one can obtain differential equations ($z_4 \to \infty$):

$$\left[ \partial_{z_5}^2 + b^2 \left( \frac{1}{z_{53}} \partial_3 + \frac{1}{z_{52}} \partial_2 + \frac{1}{z_{51}} \partial_1 + \frac{2\Delta_1}{z_{51}} \right) \right] \langle V_4 V_3 V_5 V_2 V_1 \rangle$$

$$= \left( \partial_{z_5} - \frac{b^2}{z_{52}} \right) \langle V_4 V_3 A_5 A_2 V_1 \rangle - \left( \partial_{z_5} - \frac{b^2}{z_{53}} \right) \langle V_4 A_3 A_5 V_2 V_1 \rangle$$

$$+ b^2 \left( \frac{1}{z_{53}} - \frac{1}{z_{52}} \right) \langle V_4 A_3 V_5 A_2 V_1 \rangle,$$

$$b^2 \left[ \left( \frac{1}{z_{15}} + \frac{1}{z_{25}} \right) \partial_2 + \frac{2\Delta_2}{z_{25}} \right] \langle V_4 V_3 V_5 V_2 V_1 \rangle$$

$$= b^2 \left( \frac{1}{z_{15}} + \frac{1}{z_{25}} \right) \langle V_4 A_3 V_5 A_2 V_1 \rangle - \left( \partial_{z_5} + \frac{b^2}{z_{15}} \right) \langle V_4 V_3 A_5 A_2 V_1 \rangle.$$  \quad (5.1)

$$b^2 \left[ \left( \frac{1}{z_{15}} + \frac{1}{z_{25}} \right) \partial_3 + \frac{2\Delta_3}{z_{25}} \right] \langle V_4 V_3 V_5 V_2 V_1 \rangle$$

$$= -b^2 \left( \frac{1}{z_{25}} + \frac{1}{z_{51}} \right) \langle V_4 A_3 V_5 A_2 V_1 \rangle + \left( \partial_{z_5} + \frac{b^2}{z_{15}} \right) \langle V_4 A_3 A_5 V_2 V_1 \rangle.$$  \quad (5.2)

Adding the second equation to the first one and subtracting from the result the third equation we obtain:

$$\left[ \partial_{z_5}^2 + b^2 \left( \frac{1}{z_{51}} \partial_1 + \left( \frac{1}{z_{15}} + \frac{2}{z_{25}} \right) \partial_2 + \left( \frac{1}{z_{15}} + \frac{2}{z_{2}} \right) \partial_3 + \frac{2\Delta_1}{z_{51}} + \frac{2\Delta_2}{z_{25}} + \frac{2\Delta_3}{z_{25}} \right) \right] \langle V_4 V_3 V_5 V_2 V_1 \rangle$$

$$= b^2 \left( \frac{1}{z_{51}} - \frac{1}{z_{25}} \right) \langle V_4 V_3 A_5 A_2 V_1 \rangle - b^2 \left( \frac{1}{z_{51}} - \frac{1}{z_{53}} \right) \langle V_4 A_3 A_5 V_2 V_1 \rangle.$$  \quad (5.2)

Since $V_{-b}$ and $\Lambda_{-b}$ are “light” fields, in the classical limit all the correlators in (5.2) have the form

$$\chi(z_5) e^{-\frac{2\pi}{a_5} S_c[\delta_4, \delta_3, \delta_2, \delta_1]}.$$
Therefore, for \( b \to 0 \) and \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) of order \( b^{-2} \), we have:

\[
\partial_1, \partial_2, \partial_3 = \mathcal{O} (b^{-2}), \quad \Delta_5, \partial_{z_5} = \mathcal{O} (1) .
\]

Keeping only the leading terms in (5.2) we thus get the closed equation for the classical limit of \( \langle V_4 V_3 V_2 V_1 \rangle \). In the standard locations \( z_1 = 0, z_3 = 1, z_5 = z, z_2 = x \), it takes the form:

\[
\left\{ \partial^2_z + 2b^2 \left[ \frac{\Delta_4 - \Delta_3 - \Delta_2 - \Delta_1}{z(z-1)} + \frac{\Delta_3}{(z-1)^2} + \frac{\Delta_2}{(z-x)^2} + \frac{\Delta_1}{z^2} \right] \right\} \langle V_4 V_3 V_2 V_1 \rangle \\
+ 2b^2 \frac{x(x-1)}{z(z-1)(z-x)} \frac{\partial}{\partial x} \langle V_4 V_3 V_2 V_1 \rangle = 0 .
\]

(5.3)

Let us consider the contribution to this correlation function from an even subspace of a single NS Verma module \( \mathcal{V}_\Delta \otimes \overline{\mathcal{V}_\Delta} \). In the classical limit one gets:

\[
\langle V_4 (\infty) V_3 (1, 1) V_{-b} (z, \bar{z}) |_{\text{even}} V_2 (x, \bar{x}) V_1 (0, 0) \rangle \sim \chi_\Delta (z) e^{\frac{i}{16} f^b_3 [\delta_3 \delta_2]} (x) ,
\]

where \( f^b_3 [\delta_3 \delta_2] (x) \) is the classical conformal block (3.6). Substituting into (5.3) one obtains a Fuchsian equation:

\[
\frac{d^2 \chi_\Delta (z)}{dz^2} + \left( \frac{\delta_4 - \delta_3 - \delta_2 - \delta_1}{z(z-1)} + \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \frac{\delta_3}{(z-1)^2} \right) \chi_\Delta (z)
\]

\[
+ \frac{x(x-1)C(x)}{z(z-x)(z-1)} \chi_\Delta (z) = 0 ,
\]

with the accessory parameter \( C(x) \) given by:

\[
C(x) = \frac{\partial}{\partial x} f^b_3 [\delta_3 \delta_2] (x) .
\]

(5.5)

We shall now calculate the monodromy properties of \( \chi_\Delta (z) \) along the contour encircling the points 0 and \( x \). There are only three conformal families in the OPE of the degenerate field \( V_{-b} \) with a super-primary field \( V_a \):

\[
V_{-b}(z, \bar{z}) V_a (0, 0) = C_{a,+,b,a} (z \bar{z}) \frac{\lambda_0 (1+\lambda)}{2\lambda} V_{a+} (0, 0) + C_{a,-,b,a} (z \bar{z}) \frac{\lambda_0 (1-\lambda)}{2\lambda} V_{a-} (0, 0)
\]

\[
+ \hat{C}_{a,+,b,a} (z \bar{z}) \frac{1}{(2\Delta_a)^2} \tilde{V}_a (0, 0) + \text{descendants},
\]

(5.6)

where \( a_\pm = a \pm b \).

In the classical limit the third term in (5.6) is sub-leading with respect to the first two. Hence in the space of solutions of (5.4) there is a basis \( \chi_\Delta^\pm (z) \) such that:

\[
\chi_\Delta^\pm (e^{2\pi i z}) = -e^{\pm i \pi \lambda} \chi_\Delta^\pm (z) .
\]

(5.7)

The problem of calculating \( C \) can then be formulated as follows: adjust \( C \) in such a way that the equation admits solutions with the monodromy around 0 and \( x \) given by (5.7). This is
exactly the monodromy problem obtained and solved in the context of Virasoro theory in [3, 4]. Details of these calculations are presented in Appendix A.

Taking into account the $\frac{1}{q}$ expansion of the classical block (A.14) and (4.1) one gets:

$$\begin{align*}
g_{\Delta}^{1/2} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x) &= - \ln \Delta + \pi \tau \left( \Delta - \frac{c}{24} \right) + \left( \frac{c}{8} \Delta_3 \Delta_4 - \Delta_2 - \Delta_3 - \Delta_4 \right) \ln K^2 (x) \\
&+ \left( \frac{c}{24} - \Delta_2 - \Delta_3 \right) \ln (1 - x) + \left( \frac{c}{24} - \Delta_1 - \Delta_2 \right) \ln (x) + f^\frac{1}{2} (x) + O \left( \frac{1}{\Delta} \right),
\end{align*}$$

(5.8)

where $K(x)$ is the complete elliptic integral of the first kind, $\tau \equiv \tau(x) = \frac{iK(1-x)}{K(x)}$ is the half-period ratio and $f^\frac{1}{2} (x)$ is a function of $x$ specific for each type of block and independent of $\Delta_i$ and $c$. One can obtain corresponding formulae for other types of blocks in a similar way. The exact form of the functions $f^{1/2}(x)$ can be derived from analytic expressions for $c = \frac{3}{2}$ NS superconformal blocks with external weights $\Delta_i = \frac{1}{8} [12]$.

### 6 Elliptic recursion relations

The large $\Delta$ asymptotic suggests the following form of superconformal blocks:

$$\begin{align*}
F^{1/2}_{\Delta} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x) &= (16q) \Delta - \frac{c-3}{24} x - \frac{c-3}{24} \Delta_1 - \Delta_2 \left( 1 - x \right) - \frac{c-3}{24} \Delta_2 - \Delta_3 \\
&\times \theta^\frac{c-3}{24} - 4(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \mathcal{H}^{1/2}_{\Delta} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x),
\end{align*}$$

(6.1)

where $q \equiv q(x) = e^{i\pi \tau}$ is the elliptic nome. The elliptic blocks $\mathcal{H}^{1/2}_{\Delta} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x)$ have the same analytic structure as superconformal ones:

$$\mathcal{H}^{1/2}_{\Delta} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x) = g^{1/2}_{-x} (x) + \sum_{m,n} h^{1/2}_{mn} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x).$$

The functions $g^{1/2}_{-x} (x)$ depend on the type of block and are independent of the external weights $\Delta_i$ and the central charge $c$. They have no singularities in $\Delta$ and are directly related to the functions $f^{1/2}(x)$ in (5.8). For instance, in the case of the odd block $F^{1/2}_{\Delta} \left[ \frac{-\Delta_3 + \Delta_2}{\Delta_4 - \Delta_1} \right] (x):

$$\exp f^{\frac{1}{2}} (x) = (16q)^{\frac{1}{24}} [x(1-x)]^{-\frac{1}{24}} \theta_3 (q)^{\frac{1}{2}} g^{\frac{1}{2}} (x).$$

The analytic form of these functions can be read off from the form of the elliptic blocks related to the $c = \frac{3}{2}$ conformal ones with $\Delta_i = \Delta_0 = \frac{1}{8} [12]:$

$$\begin{align*}
\mathcal{H}^{1}_{\Delta} \left[ \frac{\Delta_0 \Delta_0}{\Delta_0 \Delta_0} \right] (q) &= \theta_3 (q^2), \\
\mathcal{H}^{1}_{\Delta} \left[ \frac{\Delta_0 \Delta_0}{\Delta_0 \Delta_0} \right] (q) &= \frac{1}{\Delta} \theta_2 (q^2),
\end{align*}$$

(6.2)

$$\begin{align*}
\mathcal{H}^{1}_{\Delta} \left[ \frac{\Delta_0 \Delta_0}{\Delta_0 \Delta_0} \right] (q) &= \theta_3 (q^2), \\
\mathcal{H}^{1}_{\Delta} \left[ \frac{\Delta_0 \Delta_0}{\Delta_0 \Delta_0} \right] (q) &= \theta_2 (q^2),
\end{align*}$$

(6.3)

$$\begin{align*}
\mathcal{H}^{1}_{\Delta} \left[ \frac{\Delta_0 \Delta_0}{\Delta_0 \Delta_0} \right] (q) &= \theta_3 (q^2), \\
\mathcal{H}^{1}_{\Delta} \left[ \frac{\Delta_0 \Delta_0}{\Delta_0 \Delta_0} \right] (q) &= \theta_2 (q^2),
\end{align*}$$

(6.2)
Indeed the functions $g^{1\pm\frac{1}{2}}(x)$ are just given by terms regular for $\Delta \to 0$:

\[
\begin{align*}
g^1(x) &= \theta_3(q^2), \\
g^1_\Delta(x) &= 0, \\
g^1_\Lambda(x) &= \theta_3(q^2), \\
g^1_\frac{1}{2}(x) &= \theta_2(q^2), \\
g^1_\frac{1}{2\pm}(x) &= -\theta_2(q^2) \left( \Delta - q \theta_3^{-1}(q) \frac{\partial}{\partial q} \theta_3(q^2) + \frac{\theta_2(q^2)}{4} \right) \quad \text{(6.4)}
\end{align*}
\]

Taking into account the form of the residue at $\Delta_{mn}$ (equations (2.5), (2.6)) one gets the general elliptic recursion relations:

\[
\mathcal{H}_{\Delta}^{1\pm\frac{1}{2}} \left[ \frac{\Delta_3 - \Delta_2}{\Delta_4 - \Delta_1} \right] (x) = g^{1\pm\frac{1}{2}}(x)
+ \sum_{m,n>0 \atop m,n \in 2N} (16q)^{\frac{m+n}{2}} A_{rs}^c \mathcal{H}_{\Delta_{mn} + \frac{m+n}{2}}^{1\pm\frac{1}{2}} \left[ \frac{\Delta_3 - \Delta_2}{\Delta_4 - \Delta_1} \right] (x) \quad \text{(6.5)}
\]

\[
+ \sum_{m,n>0 \atop m,n \in 2N+1} (16q)^{\frac{m+n}{2}} S_{rs} \mathcal{H}_{\Delta_{mn} + \frac{m+n}{2}}^{1\pm\frac{1}{2}} \left[ \frac{\Delta_3 - \Delta_2}{\Delta_4 - \Delta_1} \right] (x).
\]

Formula (6.5) is the main result of the present paper.

As a nontrivial consistency check of (6.5) one can verify that each pair of elliptic blocks in (6.2), (6.3), satisfy recursion relations (6.5) with the corresponding functions (6.4). This is done in Appendix B.

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**Appendix A**

Consider the equation

\[
\frac{d^2 \chi(z)}{dz^2} + U(z) \chi(z) + \frac{x(x-1)C(x)}{z(z-x)(z-1)} \chi(z) = 0, \quad \text{(A.1)}
\]

with

\[
U(z) = \frac{1}{4} \left( \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_4^2 - 2}{z(z-1)} + \frac{1 - \lambda_1^2}{2^2} + \frac{1 - \lambda_2^2}{(z-x)^2} + \frac{1 - \lambda_3^2}{(z-1)^2} \right). \quad \text{(A.2)}
\]

We want to choose $C(x)$ such that (A.1) admits a pair of solutions $\chi^\pm(z)$ satisfying the monodromy condition

\[
\chi^\pm(e^{\pm i\pi}z) = -e^{\pm i\pi \lambda} \chi^\pm(z), \quad \text{(A.3)}
\]
where \( \chi(e^{2\pi i z}) \) denotes a function analytically continued in \( z \) along a closed path encircling points \( z = 0 \) and \( z = x \).

Following [4] we perform an elliptic change of variables:

\[
\xi(z) = \frac{1}{2} \int \frac{dt}{\sqrt{t(1-t)(1-xt)}}, \quad \tilde{\chi}(\xi) = \left( \frac{dz(\xi)}{d\xi} \right)^{-\frac{1}{2}} \chi(z(\xi)). \tag{A.4}
\]

This gives

\[
\frac{d^2}{dz^2} \chi(z) = -\frac{1}{2} (\xi')^{-\frac{1}{2}} \{\xi(z), z\} \tilde{\chi}(\xi) + (\xi')^{\frac{3}{2}} \frac{d^2 \tilde{\chi}(\xi)}{d\xi^2} \bigg|_{\xi = \xi(z)}, \tag{A.5}
\]

where \( \{\xi(z), z\} \) is the Schwarzian derivative of the map (A.4):

\[
\{\xi(z), z\} = \frac{3}{8} \left[ \frac{1}{z^2} + \frac{1}{(z-x)^2} + \frac{1}{(z-1)^2} \right] - \frac{1}{4} \left[ \frac{1}{z(z-x)} + \frac{1}{z(z-1)} + \frac{1}{(z-x)(z-1)} \right]. \tag{A.6}
\]

Using (A.4) and (A.5) we can rewrite equation (A.1) in the form of a Schrödinger equation

\[
\left[ -\frac{d^2}{d\xi^2} - \tilde{U}(\xi) \right] \tilde{\chi}(\xi) = 4x(x-1)C(x)\tilde{\chi}(\xi) \tag{A.7}
\]

with the (double periodic in \( \xi \)) potential

\[
\tilde{U}(\xi) = \left( \xi'(z) \right)^{-2} \left[ U(z) - \frac{1}{2} \{\xi(z), z\} \right] \bigg|_{z = z(\xi)}
= \left( \frac{1}{4} - \lambda_1^2 \right) \left( \frac{x}{z(\xi)} - 1 \right) + \left( \frac{1}{4} - \lambda_2^2 \right) \left[ \frac{x(x-1)}{z(\xi) - x} + 2x - 1 \right]
+ \left( \frac{1}{4} - \lambda_3^2 \right) \left( 1 - \frac{1-x}{1-z(\xi)} \right) + \left( \frac{1}{4} - \lambda_4^2 \right) (z(\xi) - x) + x - \frac{1}{2}. \tag{A.8}
\]

Continuing analytically the function \( \xi(z) \) along the closed path encircling the points 0 and \( x \) one gets:

\[
\xi(e^{2\pi i z}) = \xi(z) + \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}} = \xi(z) + 2K(x)
\]

where \( K(x) \) is the complete elliptic integral of the first kind:

\[
K(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}
\]

The monodromy condition (A.3) thus takes the form

\[
\tilde{\chi}^\pm \left( \xi + 2K(x) \right) = e^{\pm i\pi \lambda} \tilde{\chi}^\pm(\xi). \tag{A.9}
\]

We shall solve (A.7) in the large \( \lambda \) limit using a standard perturbative method. First, assume that

\[
\tilde{U}(\xi) = o(C(x)) \tag{A.10}
\]
so that in the leading order we can neglect in (A.7) the potential term and the solutions are just plane waves:

\[ \tilde{\chi}_0^\pm(\xi) = e^{\pm ip\xi}, \quad p^2 = 4x(x - 1)C(x). \] (A.11)

On the other hand the monodromy condition (A.9) implies

\[ e^{\pm 2ipK(x)} = e^{\pm i\pi \lambda} \Rightarrow p = -\frac{\pi \lambda}{2K(x)}, \]

what also proves the consistency of (A.10). Hence, in the leading order one obtains:

\[ C^{(0)}(x) = \frac{\pi^2 \lambda^2}{16x(x - 1)K^2(x)}. \] (A.12)

The first correction \( C^{(1)}(x) \) is given by:

\[ C^{(1)}(x) = -\frac{1}{8x(x - 1)K(x)} \int_{\xi_0}^{\xi_0 + 2K(x)} d\xi \tilde{\chi}_0^-(\xi) \tilde{U}(\xi) \tilde{\chi}_0^+(\xi) \] (A.13)

\[ = -\frac{1}{16x(x - 1)K(x)} \int_{[0,x]} \frac{\tilde{U}(\xi(z))dz}{\sqrt{z(1-z)(x-z)}}, \]

where in the first line \( \Im \xi_0 > 0 \) while in the second line \([0,x]\) denotes a positively oriented, closed contour in the complex \( z \) plane, surrounding the points 0 and \( x \).

Integrating one gets:

\[ \int_{[0,x]} \frac{\tilde{U}(\xi(z))dz}{\sqrt{z(1-z)(x-z)}} = \left\{ \begin{array}{l} (1 - 4\lambda_1^2) (I_1 - K(x)) + (1 - 4\lambda_2^2) (I_2 + (2x - 1)K(x)) \\
+ (1 - 4\lambda_3^2) (K(x) - I_3) + (1 - 4\lambda_4^2) (I_4 - xK(x)) + 4 \left( x - \frac{1}{2} \right) K(x) \end{array} \right\}, \]

where

\[ I_1 = \frac{1}{4} \int_{[0,x]} \frac{xdz}{z\sqrt{z(1-z)(x-z)}} = K(x) - E(x), \]

\[ I_2 = \frac{1}{4} \int_{[0,x]} \frac{x(1-x)dz}{(z-x)\sqrt{z(1-z)(x-z)}} = (1 - x)K(x) - E(x), \]

\[ I_3 = \frac{1}{4} \int_{[0,x]} \frac{(1-x)dz}{(1-z)\sqrt{z(1-z)(x-z)}} = E(x), \]

\[ I_4 = \frac{1}{4} \int_{[0,x]} \frac{zdz}{\sqrt{z(1-z)(x-z)}} = K(x) - E(x). \]

Here \( E(x) \) is the complete elliptic integral of the second kind:

\[ E(x) = \int_0^1 \frac{(1 - xt^2)dt}{\sqrt{(1 - t^2)(1 - xt^2)}}. \]
The correction to the accessory parameter takes the form:

\[
C^{(1)}(x) = \frac{-1}{4x(x-1)} \left\{ \frac{E(x)}{K(x)} \left( -1 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \right) + x \left( 1 - \lambda_2^2 + \lambda_4^2 \right) - (\lambda_3^2 + \lambda_4^2) \right\}
\]

Since \(C(x) = \partial_x f_\delta \left[ \delta_1 \delta_2 \right] (x)\) one can calculate the classical block:

\[
f_\delta \left[ \delta_1 \delta_2 \right] (x) = \int \frac{dx}{x(x-1)} \left\{ \frac{(\pi \lambda)^2}{4K^2(x)} \right\} + \frac{E(x)}{K(x)} \left( 1 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) - x \left( 1 - \lambda_2^2 + \lambda_4^2 \right) + \lambda_3^2 + \lambda_4^2 \right\} + O\left( \frac{1}{\lambda^2} \right).
\]

Using

\[
\int \frac{dx}{x(x-1)} \frac{1}{4K^2(x)} = \frac{1}{\pi} \frac{K(1-x)}{K(x)} = \frac{\tau}{i\pi}
\]

\[
\int \frac{dx}{x(x-1)} \frac{E(x)}{K(x)} = -\frac{1}{2} \ln K^4(x) - \ln x,
\]

one gets:

\[
f_\delta \left[ \delta_1 \delta_2 \right] (x) = \frac{1}{4} \left\{ -i\pi \lambda^2 - \frac{1}{2} \left( 1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) \ln K^4(x) \right\}
\]

\[
-(1 - \lambda_2^2 - \lambda_3^2) \ln(1 - x) - (1 - \lambda_2^2 - \lambda_3^2) \ln(x) \right\} + O\left( \frac{1}{\lambda^2} \right)
\]

or, in terms of \(\delta = \frac{1-\lambda_2^2}{4}, \delta_i = \frac{1-\lambda_i^2}{4}\),

\[
f_\delta \left[ \delta_1 \delta_2 \right] (x) = i\pi \left( \delta - \frac{1}{4} \right) + \frac{1}{2} \left( \frac{3}{4} - \delta_1 - \delta_2 - \delta_3 - \delta_4 \right) \ln K^4(x)
\]

\[
+ \left( \frac{1}{4} - \delta_2 - \delta_3 \right) \ln(1 - x) + \left( \frac{1}{4} - \delta_1 - \delta_2 \right) \ln(x) + O\left( \frac{1}{\delta} \right).
\]

The absence in the last two formulae of the \(x\)-independent integration constants follows from the normalization condition of the block \(F_{c,\Delta} \left[ \Delta_1 \Delta_2 \right] \Delta_0 \right\).

### Appendix B

Consider \(c = \frac{3}{2}\) theory with external weights \(\Delta_0 = \frac{1}{5}\). For \(r \neq s\) all coefficients \(A_{rs}^{c} \left[ \Delta_0 \Delta_0 \Delta_0 \right] \Delta_0 \Delta_0 \) are zero. There are however some non zero terms if \(r = s\):

\[
(16)^2 A_{rr}^c \left[ \Delta_0 \Delta_0 \Delta_0 \right] = \begin{cases} 
    -r^2 & \text{if } r \in 2\mathbb{N}, \\
    2 & \text{if } r \in 2\mathbb{N} + 1.
\end{cases}
\]
Moreover, $\Delta_{rr} = 0$. One can show that all elliptic blocks (6.2), (6.3) satisfy recursion relations (6.5) with corresponding $g^{1/2}$ functions (6.4). Indeed, for the blocks:

$$
H^1_{\Delta_0} \left[ \Delta \right] (x) = g^1(x), \\
H^1_{\Delta_0} \left[ \Delta \right] (x) = g^1_s(x), \\
H^1_{\Delta_0} \left[ \Delta \right] (x) = g^1_x(x), \\
H^1_{\Delta_0} \left[ \Delta \right] (x) = g^1_x^s(x),
$$

the relation (6.5) holds because all the residues at $\Delta = \Delta_{rs}$ are zero. In the other cases the formula (B.1) becomes helpful:

$$
H^\frac{1}{2}_{\Delta_0} \left[ \Delta \right] (x) = \sum_{r \in 2N} (16q)^{-2} \frac{A_{rr}[\Delta_0, \Delta_0]}{x} H^\frac{1}{2}_{\Delta_0}(x) + \sum_{r \in 2N} (16q)^{-2} \frac{A_{rr}[\Delta_0, \Delta_0]}{x} H^1_{\Delta_0}(x) + \frac{1}{\Delta} \sum_{r \in 2N+1} q^2 \theta_3(q^2).
$$

Substituting $H^\frac{1}{2}_{\Delta_0} \left[ \Delta \right] (x) = \frac{2}{r} \theta_2(q^2)$ and using the definitions of the theta functions:

$$
\theta_2(q^2) = \sum_{n=-\infty}^{\infty} q^{n^2} = 2 \sum_{n=0}^{\infty} q^{n^2}, \quad \theta_3(q^2) = \sum_{n=-\infty}^{\infty} q^{2n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{2n^2},
$$

one gets:

$$
H^\frac{1}{2}_{\Delta_0} \left[ \Delta \right] (x) = -\frac{2}{\Delta} \sum_{r \in 2N} q^2 \theta_2(q^2) + \frac{2}{\Delta} \sum_{r \in 2N+1} q^2 \theta_3(q^2) = -\frac{1}{\Delta} (\theta_3(q^2) - 1) \theta_2(q^2) + \frac{1}{\Delta} \theta_3(q^2) \theta_3(q^2) = \frac{1}{\Delta} \theta_2(q^2).
$$

The last block in (6.3), $H^1_{\Delta_0} \left[ \Delta \right] (x)$, also satisfies the recursion relation:

$$
H^1_{\Delta_0} \left[ \Delta \right] (x) = \theta_3(q^2) + \sum_{r \in 2N} \left( \frac{-r^2}{\Delta} \right) q^2 \theta_3(q^2) + \sum_{r \in 2N+1} \left( \frac{-r^2}{\Delta} \right) q^2 \theta_3(q^2)
$$

$$
= \theta_3(q^2) - \frac{2}{\Delta} \left( \sum_{r \in 2N} q^2 \theta_3(q^2) - \sum_{r \in 2N+1} q^2 \theta_3(q^2) \right) \left( q \theta_3^{-1} \frac{\partial}{\partial q} \theta_3(q) + \frac{\theta_3^2(q)}{4} \right)
$$

$$
= \theta_3(q^2) \left( 1 - \frac{q}{\Delta} \theta_3^{-1} \frac{\partial}{\partial q} \theta_3(q) + \frac{\theta_3^2(q)}{4} \right) + \frac{1}{\Delta} \left( q \theta_3^{-1} \frac{\partial}{\partial q} \theta_3(q) - \frac{\theta_3^2(q)}{4} - \frac{q}{2} \frac{\partial}{\partial q} \left( \theta_3^2(q^2) - \theta_2^2(q^2) \right) \right).
$$
From the identities
\[ \theta_3(q^2) = \theta_3(q) \left( \frac{1 + \sqrt{1 - x}}{2} \right)^{\frac{1}{2}}, \quad \theta_2(q^2) = \theta_3(q) \left( \frac{1 - \sqrt{1 - x}}{2} \right)^{\frac{1}{2}}, \]
it follows that
\[ \theta_3^2(q^2) - \theta_2^2(q^2) = \sqrt{1 - x} \theta_3^2(q). \]

Since
\[ \frac{dq(x)}{dx} = \frac{x^2q(x)}{4x(1-x)K^2(x)} = \frac{q(x)}{x(1-x)\theta_3^4(q)} \]
we have
\[ q \frac{\partial}{\partial q} = x(1-x) \theta_3^4(q) \frac{\partial}{\partial x} \]
and with the help of the relation \( x = \frac{\theta_3(q)}{\theta_2(q)} \) we finally get
\[ \frac{q}{2} \frac{\partial}{\partial q} \left( \sqrt{1-x} \theta_3^2(q) \right) = \left( q \theta_3^{-1}(q) \frac{\partial \theta_3(q)}{\partial q} - \frac{\theta_2^4(q)}{4} \right) \sqrt{1-x} \theta_3^2(q), \]
what demonstrates that the last line in (B.2) indeed vanishes.

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