Holographic Thermodynamic on the Brane in Topological Reissner-Nordström de Sitter Space

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Abstract

We consider the brane universe in the bulk background of the topological Reissner-Nordström de Sitter black holes. We show that the thermodynamic quantities (including entropy) of the dual CFT take usual special forms expressed in terms of Hubble parameter and its time derivative at the moment, when the brane crosses the black hole horizon or the cosmological horizon. We obtain the generalized Cardy-Verlinde formula for the CFT with an charge and cosmological constant, for any values of the curvature parameter $k$ in the Friedmann equations.

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1 Introduction

The holographic duality which connects $n+1$-dimensional gravity in Anti-de Sitter (AdS) background with $n$-dimensional conformal field theory (CFT) has been discussed vigorously for some years\cite{1}. But it seems that we live in a universe with a positive cosmological constant which will look like de Sitter space–time in the far future. Therefore, we should try to understand quantum gravity or string theory in de Sitter space preferably in a holographic way. Of course, physics in de Sitter space is interesting even without its connection to the real world; de Sitter entropy and temperature have always been mysterious aspects of quantum gravity\cite{2}.

While string theory successfully has addressed the problem of entropy for black holes, dS entropy remains a mystery. One reason is that the finite entropy seems to suggest that the Hilbert space of quantum gravity for asymptotically de Sitter space is finite dimensional, \cite{3, 4}. Another, related, reason is that the horizon and entropy in de Sitter space have an obvious observer dependence. For a black hole in flat space (or even in AdS) we can take the point of view of an outside observer who can assign a unique entropy to the black hole. The problem of what an observer venturing inside the black hole experiences, is much more tricky and has not been given a satisfactory answer within string theory. While the idea of black hole complementarity provides useful clues, \cite{5}, rigorous calculations are still limited to the perspective of the outside observer. In de Sitter space there is no way to escape the problem of the observer dependent entropy. This contributes to the difficulty of de Sitter space.

Recently much attention has been paid for the duality between de Sitter (dS) gravity and CFT by the analogy of the AdS/CFT correspondence\cite{6, 7, 8, 9} (for a very good review see also \cite{10}), because the isometry of $n+1$-dimensional de Sitter space, $SO(n+1,1)$, exactly agrees with the conformal symmetry of $n$-dimensional Euclidean space. Thus it might be natural to expect the correspondence between $n+1$-dimensional gravity in de Sitter space and $n$-dimensional Euclidean CFT (the dS/CFT correspondence). Moreover the holographic principle between the radiation dominated Friedmann-Robertson-Walker (FRW) universe in $n$-dimensions and same dimensional CFT with a dual $n+1$-dimensional AdS description was studied in \cite{11}. Especially, we can see the correspondence between black hole entropy and the entropy of the CFT which is derived by making the appropriate identifications for FRW equation with the Cardy-Verlinde formula.

In this paper we consider the brane universe in the bulk background of the topological Reissner-Nordström de Sitter (TRNdS) black holes. We show that the thermodynamic quantities (including entropy) of the dual CFT take usual special forms expressed in terms of Hubble parameter and its time derivative at the moment, when the brane crosses the black hole horizon or the cosmological horizon. We obtain the generalized Cardy-Verlinde formula for the CFT with an charge and cosmological constant, for any values of the curvature parameter $k$ in the Friedmann equations.
2 FRW equations in the background of TRNdS Black Holes

The topological Reissner-Nordström dS black hole solution in \((n + 2)\)-dimensions has the following form

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\gamma_{ij}dx^i dx^j,
\]

\[
f(r) = k - \frac{\omega_n M}{r^{n-1}} + \frac{n\omega_n^2 Q^2}{8(n-1)r^{2n-2}} - \frac{r^2}{l^2}, \tag{1}
\]

where

\[
\omega_n = \frac{16\pi G}{n \text{Vol}(\Sigma)}, \quad \phi = -\frac{n}{4(n-1)} \frac{\omega_n Q}{r^{n-1}}, \tag{2}
\]

where \(Q\) is the electric/magnetic charge of Maxwell field, \(M\) is assumed to be a positive constant, \(l\) is the curvature radius of de Sitter space, \(\gamma_{ij}dx^i dx^j\) denotes the line element of an \(n\)-dimensional hypersurface \(\Sigma_k\) with the constant curvature \(n(n-1)k\) and its volume \(V(\Sigma_k)\). \(\Sigma_k\) is given by spherical \((k = 1)\), flat \((k = 0)\), hyperbolic \((k = -1)\), \(\phi\) is the electrostatic potential related to the charge \(Q\). When \(k = 1\), the metric Eq.(1) is just the Reissner-Nordström-de Sitter solution. For general \(M\) and \(Q\), the equation \(f(r) = 0\) may have four real roots. Three of them are real, the largest one is the cosmological horizon \(r_c\), the smallest is the inner (Cauchy) horizon of black hole, the middle one is the outer horizon \(r_+\) of the black hole. And the fourth is negative and has no physical meaning.

When \(k = 0\) or \(k = -1\), there is only one positive real root of \(f(r)\), and this locates the position of cosmological horizon \(r_c\).

In the case of \(k = 0\), \(\gamma_{ij}dx^i dx^j\) is an \(n\)-dimensional Ricci flat hypersurface, when \(M = Q = 0\) the solution Eq.(1) goes to pure de Sitter space

\[
ds^2 = \frac{r^2}{l^2}dt^2 - \frac{l^2}{r^2}dr^2 + r^2 dx^2_n, \tag{3}
\]

in which \(r\) becomes a timelike coordinate.

When \(Q = 0\), and \(M \to -M\) the metric Eq.(1) is the TdS (Topological de Sitter) solution [12, 13], which have a cosmological horizon and a naked singularity.

For the purpose of getting the Friedmann-Robertson-Walker (FRW) metric, we impose the following condition[11],

\[
\frac{1}{f(r)} \left( \frac{dr}{d\tau} \right)^2 - f(r) \left( \frac{dt}{d\tau} \right)^2 = -1, \tag{4}
\]

which leads to a timelike brane. Substituting Eq.(4) into the TRNdS solution Eq.(1), one has the induced brane metric which takes FRW form

\[
ds^2 = -d\tau^2 + r^2(\tau)\gamma_{ij}dx^i dx^j, \tag{5}
\]

the equation of motion of the brane is given by[14]

\[
K_{ij} = \frac{\sigma}{n} h_{ij}, \tag{6}
\]
where $K_{ij}$ is the extrinsic curvature, and $h_{ij}$ is the induced metric on the brane, $\sigma$ is the brane tension. The extrinsic curvature, $K_{ij}$, of the brane can be calculated and expressed in term of function $r(\tau)$ and $t(\tau)$. Thus one rewrites the equations of motion (6) as

$$\frac{dt}{d\tau} = \frac{\sigma r}{f(r)}.$$  (7)

Using Eqs.(4,7), we can drive FRW equation with $H = \dot{r}$,

$$H^2 = -\frac{f(r)}{r^2} + \sigma^2 = \frac{\omega_n M}{r^{n+1}} - \frac{n \omega_n^2 Q^2}{8(n-1)r^{2n}} - \frac{k}{r^2} + \frac{1}{l^2} + \sigma^2,$$  (8)

where, $H$ is the Hubble parameter. Here we cannot make any fine-tuning to obtain a flat brane. We have thus an effectively de Sitter brane. Making use of the fact that the metric for the boundary CFT can be determined only up to a conformal factor, we rescale the boundary metric for the CFT to be of the following form

$$ds_{CFT}^2 = \lim_{r \to \infty} \left[ \frac{l^2}{r^2} ds_{n+2}^2 \right] = -dt^2 + l^2 \gamma_{ij} dx^i dx^j.$$  (9)

Then the thermodynamic relations between the boundary CFT and the bulk TRNdS are given by

$$E_{CFT} = M \frac{l}{r}, \quad \Phi_{CFT} = \phi \frac{l}{r},$$
$$T_{CFT} = T_{TRNdS} \frac{l}{r}, \quad S_{CFT} = S_{TRNdS}$$  (10)

where black hole horizon Hawking temperature $T_{TRNdS}^b$ and entropy $S_{TRNdS}^b$ are given by

$$T_{TRNdS}^b = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left( (n-1) - (n+1) \frac{r_+^2}{l^2} - \frac{n \omega_n^2 Q^2}{8r_+^{2n-2}} \right),$$
$$S_{TRNdS}^b = \frac{r_+^n Vol(\Sigma)}{4G},$$  (11)

where $r = r_+$ is black hole horizon and $v_+ = r_+^n Vol(\Sigma)$ is area of it in $(n+2)$–dimensional asymptotically dS space. The Hawking temperature $T_{TRNdS}^c$ and entropy $S_{TRNdS}^c$ associated with the cosmological horizon are

$$T_{TRNdS}^c = -\frac{f'(r_c)}{4\pi} = \frac{1}{4\pi r_c} \left( -(n-1)k + (n+1) \frac{r_c^2}{l^2} + \frac{n \omega_n^2 Q^2}{8r_c^{2n-2}} \right),$$
$$S_{TRNdS}^c = \frac{r_c^n Vol(\Sigma)}{4G},$$  (12)

where $V_c = r_c^n Vol(\Sigma)$ is area of the cosmological horizon.In terms of the energy density $\rho_{CFT} = E_{CFT}/V$, the pressure $p_{CFT} = \rho_{CFT}/n$, the charge density $\rho_{Q\text{CFT}} = Q/V$ and the electrostatic potential $\Phi_{CFT} = \phi \frac{l}{r}$ of the CFT within the volume $V = r^n Vol(\Sigma)$, the first Friedmann equation take the following form

$$H^2 = \frac{16\pi G}{n(n-1)} \left( \rho_{CFT} - \frac{1}{2} \Phi \rho_{Q\text{CFT}} \right) - \frac{k}{r^2} + \frac{2\Lambda_{n+1}}{n(n+1)},$$  (13)
with the positive cosmological constant \( \Lambda_{n+1} = \frac{n(n+1)}{2}(1/\ell^2 + \sigma^2) \), and \( G = \frac{(n-1)G_{n+2}}{f} \).

Taking the \( \tau \)-derivative of Eq. (8), we obtain the second Friedmann equation

\[
\dot{H} = -\frac{n+1}{2} \frac{\omega_n M}{r^{n+1}} + \frac{n^2 w_n^2 Q^2}{8(n-1)r^{2n}} + \frac{k}{r^2}.
\] (14)

Similar to the first Friedmann equation we can rewrite the second equation as following

\[
\dot{H} = -\frac{8\pi G}{n-1} (\rho_{CFT} + p_{CFT} - \Phi_{CFT}\rho_{QCF}) + \frac{k}{r^2},
\] (15)

So, the cosmological evolution is determined by the energy density \( \rho_{CFT} \) and the pressure \( p_{CFT} \), electric potential energy and cosmological constant. The Friedmann equations (13,15) can be respectively put into the following forms resembling thermodynamic formulas of the CFT,

\[
S_H = \frac{2\pi}{n} r^{n-1} \sqrt{E_{BH}[2(E_{CFT} + E_\Lambda - \frac{1}{2}\Phi_{CFT}Q) - kE_{BH}]},
\] (16)

\[
kE_{BH} = n(E_{CFT} + pV - \Phi_{CFT}Q - T_H S_H),
\] (17)

in terms of the Hubble entropy \( S_H \) and the Bekenstein-Hawking energy \( E_{BH} \) the cosmological energy \( E_\Lambda \) and the Hubble temperature \( T_H \), where

\[
S_H \equiv (n-1)\frac{HV}{4G}, \quad E_\Lambda \equiv \frac{\Lambda(n-1)V}{8\pi G(n+1)}, \quad E_{BH} \equiv n(n-1)\frac{V}{8\pi Gr^2}, \quad T_H \equiv -\frac{\dot{H}}{2\pi H}.
\] (18)

Now we study thermodynamics of the CFT at the moment when the brane crosses the black hole horizon \( r = r_+ \) in the case \( k = 1 \), and crosses the cosmological horizon \( r = r_c \) for the cases \( k = 0 \) and \( k = -1 \) defined as the largest root of \( f(r) = 0 \),

\[
k - \frac{\omega_n M}{r^{n-1}} + \frac{n\omega_n^2 Q^2}{8(n-1)r^{2n-2}} - \frac{r^2}{\ell^2} = 0.
\] (19)

From Eqs.(8,19), after setting \( \sigma = \frac{1}{\ell} \), we have

\[
H^2 = \frac{1}{\ell^2} \quad \text{at} \quad r = r_{+,c}.
\] (20)

The total entropy \( S_{CFT} \) of the CFT remains constant, but the entropy density,

\[
s \equiv \frac{S_{CFT}}{V} = \frac{r_+^n Vol(\Sigma)}{4G_{n+2} r^n Vol(\Sigma)} = (n-1)\frac{r_+^n}{4Gr^m},
\] (21)

varies with time. Using Eq. (20), we see that \( s \) at \( r = r_{+,c} \) can be expressed in terms of \( H \) as following

\[
s = (n-1)\frac{H}{4G} \quad \text{at} \quad r = r_{+,c},
\] (22)

then we have

\[
S = S_H \quad \text{at} \quad r = r_{+,c}
\] (23)
Using the formula $H^2 = \sigma^2 - f(r)/r^2$ that follows from Eq. (8), we see that the CFT temperature $T_{CFT} = f'(r_+)/l/(4\pi r_+)$, at the moment when the brane crosses the black hole horizon, can be expressed in terms of $H$ and $\dot{H}$ in the following way

$$T_{CFT} = -\frac{\dot{H}}{2\pi H} \quad \text{at} \quad r = r_+.$$ (24)

In the cases $k = 0$ and $k = -1$, at the moment when the brane crosses the cosmological horizon, the CFT temperature $T_{CFT} = -f'(r_c)/l/(4\pi r_c)$ can be expressed as the following

$$T_{CFT} = \frac{\dot{H}}{2\pi H} \quad \text{at} \quad r = r_c.$$ (25)

Using Eqs.(17,23,24) in the case $k = 1$ we can write

$$E_C = E_{BH} \quad \text{at} \quad r = r_+$$ (26)

where $E_C$ is the Casimir energy defined as

$$E_C \equiv n(E_{CFT} + p_{CFT}V - \Phi_{CFT}Q - TS).$$ (27)

If we redefine the Hubble temperature as $T_H \equiv \frac{\dot{H}}{2\pi H}$ for the cases $k = 0$ and $k = -1$, then we can write

$$E_C = kE_{BH} \quad \text{at} \quad r = r_{+,c}.$$ (28)

Therefore, for the case of topological Reissner-Nordstr"om de Sitter black holes, thermodynamic quantities of the CFT can be expressed in terms of the Hubble parameter and its time derivative, when the brane crosses the black hole horizon in the case $k = 1$ and crosses the cosmological horizon in the cases $k = 0, k = -1$.

We can rewrite Eq.(16) in term of the Casimir energy $E_C$, by using Eqs.(23,28)

$$S = \frac{2\pi r}{n} \sqrt{\frac{E_C}{k}} \left| (2(E_{CFT} + E_A - 1/2\phi_{CFT}Q) - E_C) \right|,$$ (29)

this is the generalized Cardy-Verlinde formula for the CFT with cosmological constant $\Lambda$ and charge $Q$. Therefore the generalized Cardy-Verlinde Eq.(29) for boundary CFT, coincides with the cosmological Cardy formula Eq.(16) when the brane crosses the black hole horizon $r = r_+$ or cosmological horizon $r = r_c$.

### 3 Conclusion

One of the striking results for the dynamic dS/CFT correspondence is that the Cardy-Verlinde’s formula on the CFT-side coincides with the Friedmann equation in cosmology when the brane crosses the horizon $r = r_+$ of the topological Reissner-Nordstr"om black hole. This means that the Friedmann equation knows the thermodynamics of the CFT. In this paper we have considered the brane universe in the bulk background of the topological Reissner-Nordstr"om de Sitter black holes. We have shown that in TRNdS space in contrast with TRNAdS space we cannot make any fine-tuning to obtain a flat brane. We have thus an effectively de Sitter brane. We have shown that thermodynamic quantities of the CFT can be expressed in terms of the Hubble parameter and its time derivative, when
the brane crosses the black hole horizon or cosmological horizon respectively for the case \( k = 1 \) and the cases \( k = 0, k = -1 \). We obtain the generalized Cardy-Verlinde formula for the CFT with an charge and cosmological constant, for any values of the curvature parameter \( k \) in the Friedmann equations.

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