POINCARÉ TYPE AND SPECTRAL GAP INEQUALITIES WITH FRACTIONAL LAPLACIANS ON HAMMING CUBE

DONG LI, ALEXANDER VOLBERG

Abstract. We prove here some dimension free Poincaré-type inequalities on Hamming cube for functions with different spectral properties and for fractional Laplacians. In this note the main attention is paid to estimates in $L^1$ norm on Hamming cube. We build the examples showing that our assumptions on spectral properties of functions cannot be dropped in general.

1. Introduction

Let $C_n := \{-1,1\}^n$ denote the Hamming cube, and let $x_i, i = 1, \ldots, n$, be its coordinate functions assuming the values $\pm 1$. If $S$ denotes a subset of $\{1, \ldots, n\}$, then a monomial $x^S$ is just $x_{i_1} \cdots x_{i_k}$, where $S = \{i_1, \ldots, i_k\}$. If $S$ is the empty set, then we set $x^\emptyset = 1$. There is a standard measure $\mu$ on $C_n$ (all points are charged by $2^{-n}$) and standard expectation with respect to this measure, it will be called $E$. For any $f: C_n \to \mathbb{R}$, one can develop the expansion

$$f(x) = \sum_S a_S x^S = a_\emptyset + \sum_{S \neq \emptyset} a_S x^S,$$

where $a_S = E(f(x)x^S) = \hat{f}(S)$ is usually called the Fourier coefficient of $f$. The $L^2$ isometry takes the form

$$E|f|^2 = \sum_S |a_S|^2.$$ 

For $j \in \{1, \ldots, n\}$, define $\nabla_j$ as

$$(\nabla_j f)(x) = \frac{f(x_1, \ldots, x_j = 1, \ldots, x_n) - f(x_1, \ldots, x_j = -1, \ldots, x_n)}{2}.$$ 

Then the adjoint operator $\nabla_j^*$ has the form

$$\nabla_j^* f = x_j E f = x_j \frac{f(x_1, \ldots, x_j = 1, \ldots, x_n) + f(x_1, \ldots, x_j = -1, \ldots, x_n)}{2}.$$ 

One can then introduce the Laplacian $\Delta = -\sum_{j=1}^n \nabla_j^* \nabla_j = -\sum_{j=1}^n x_j \nabla_j$. Clearly

$$-E f \Delta g = E(\sum_{j=1}^n \nabla_j f \nabla_j g).$$ 

On monomials the Laplacian acts by the rule

$$\Delta(x^S) = -|S|x^S,$$

where $|S|$ denotes the cardinality of $S$, and thus semigroup $e^{t\Delta}$ acts by the rule

$$e^{t\Delta} f = a_\emptyset + \sum_{S \neq \emptyset} a_S e^{-|S| t} x^S$$

for $f = \sum_S a_S x^S$. The length of the gradient of $f$, $|\nabla f|(x)$ is defined as

$$|\nabla f|^2(x) = \sum_{y \sim x} \left( \frac{f(x) - f(y)}{2} \right)^2,$$
where $y \sim x$ denotes all neighbours of $x$. A point $y \in C_n$ is called a neighbour of $x$, denoted as $y \sim x$, if for some $i_0 \in \{1, \cdots , n\}$, we have $y_i = x_i$ for all $i \neq i_0$, and $y_{i_0} = -x_{i_0}$. It is easy to see that

$$|\nabla f|^2(x) = \sum_{j=1}^{n} |\nabla_j f(x)|^2,$$

and consequently

$$-E(f \cdot \Delta f) = E|\nabla f|^2 = \sum_{S} |S||a_S|^2.$$

Then clearly, via the $L^2$ isometry mentioned earlier,

$$E|f - Ef|^2 \leq E|\nabla f|^2 = -E(f \cdot \Delta f), \quad (1.1)$$

and

$$\|e^{t\Delta}(f - Ef)\|_2 \leq e^{-\gamma t}\|f - Ef\|_2. \quad (1.2)$$

The first one is the Poincaré inequality in $L^2(\mu)$, the second one can be called the spectral gap inequality in $L^2(\mu)$.

Below we are interested in such dimension free inequalities, where $L^2$ is replaced by $L^p$, especially for $p = 1$ and when $\Delta$ is replaced by $\Delta_{\gamma}$, $0 < \gamma < 1$, where we define fractional Laplacian by

$$\Delta_{\gamma} := -(-\Delta)^\gamma.$$

Such inequalities were studied in many situations, for us the starting point was [1]. The analogs on Hamming cube have some interesting properties and raise questions—especially about the sharp constants. But we do not address here the problem of sharp constants. We wish to mention that certain estimates for fractional Laplacian on Hamming cube were considered in [1]. Our estimates are different, but in conjunction with the estimates of [1], they naturally generate another set of questions which we plan to address elsewhere.

In dealing with spectral gap estimates for $e^{t\Delta_{\gamma}}, 1 < p < \infty$, we are led to the same estimate as (1.2) with the only difference that $e^{-t}$ gets substituted by $e^{-c_{p,\gamma} t}, c_{p,\gamma} > 0$. We do not calculate $c_{p,\gamma}$ very precisely, but it is readily seen that it blows down to zero if $p \to 1$. Moreover, we show that the inequality

$$\|e^{t\Delta_{\gamma}}(f - Ef)\|_1 \leq e^{-c_{1,\gamma} t}\|f - Ef\|_1 \quad (1.3)$$

cannot generally hold with $c_{1,\gamma} > 0$.

For $\gamma = 1$ this effect was carefully researched in [3], where for $p > 1$ the constant $c_{p,1}$ is considered in the following heat smoothing (or spectral gap) inequality:

$$\|e^{t\Delta}(f - Ef)\|_p \leq e^{-c_{p,1} t}\|f - Ef\|_p \quad (1.4)$$

and it is shown that it blows down to zero when $p \to 1$. Moreover, this constant in calculated.

To have (1.3) one needs something like extra assumption on the spectral properties of $f$. In the spirit of [1] we call $f$ band limited (or with band spectrum) if in the Fourier decomposition $f = \sum_{S} a_S x^S$ of $f$ one has all $a_S$ zero unless the length $|S|$ belongs to a finite set (say, set $\{1, 2, 3\}$).

**Remark 1.1.** For band limited $f$ we prove estimate (1.3), but only if $\gamma < 1$! For $\gamma = 1$ there is a counterexample (see Section 2) to (1.3) even for $f$ with band spectrum.

It goes without saying that we need all constants met below to be independent of the dimension $n$ of cube $C_n$.

Our spectral gap estimates are the combination of Poincaré type estimates in various $L^p(\mu)$ (especially for $p = 1$), hypercontractivity, and some standard convexity arguments. The Poincaré estimate at $p = 1, 0 < \gamma < 1$, obtained below seems to be unusual. And even Poincaré inequalities for $p > 1$ seem to be different from the standard ones. The next section is devoted to them.
2. Poincaré-type inequalities with Laplacian

**Lemma 2.1.** Let $0 < \beta \leq 2$. Let $(\Omega, d\mu)$ be a probability space. Then for any random variable $g : \Omega \to \mathbb{R}$ with $E|g|^2 < \infty$, we have

$$E|g - Eg|^2 \geq c_1 E|g|^2 - 2^{\frac{1}{\beta}} \cdot |E[|g|^\beta \text{ sgn}(g)]|^\frac{2}{\beta},$$

where $c_1 > 0$ is an absolute constant.

**Proof.** Without the loss of generality we assume $E|g|^2 = 1$. Let $c_1 > 0$ be a sufficiently small absolute constant. If $E|g - Eg|^2 \geq c_1$ we are done. Now assume $E|g - Eg|^2 < c_1 \ll 1$. Together with the condition $E|g|^2 = 1$ we infer that $0 \leq 1 - |Eg| \ll 1$. Replacing $g$ by $-g$ if necessary we may assume $|1 - Eg| \ll 1$. Let $\eta = \frac{1}{c_1^{\frac{1}{\beta}}}$. Then for $c_1$ sufficiently small (below the smallness of $c_1$ is independent of $\beta$ since $0 < \beta \leq 2$), we have

$$\int |g|^\beta \text{ sgn}(g) d\mu \geq \int_{|g-1| \leq \eta} |g|^\beta \text{ sgn}(g) d\mu - \int_{|g-1| > \eta} |g|^\beta d\mu \geq \frac{3}{4} - 4 \int_{|g-1| > \eta} |g-1|^2 d\mu \geq \frac{1}{\sqrt{2}}. \quad \Box$$

The desired inequality then obviously follows.

**Proposition 2.2.** Let $0 < \beta \leq 2$. Then for any $g : \{-1, 1\}^n \to \mathbb{R}$, we have

$$E|\nabla g|^2 \geq c_1 E|g|^2 - 2^{\frac{1}{\beta}} |E(|g|^\beta \text{ sgn}(g))|^\frac{2}{\beta},$$

where $c_1 > 0$ is an absolute constant.

**Proof.** This follows from the Poincaré inequality with $p = 2$ on Hamming cube:

$$E|\nabla g|^2 \geq E|g - Eg|^2$$

and the previous lemma. \qed

Next is an elementary lemma.

**Lemma 2.3.** Let $a, b \in \mathbb{R}$, $p > 1$. Then there exists $\tilde{c}_p > 0$ such that

$$(a - b)(|a|^{p-1} \text{ sgn } a - |b|^{p-1} \text{ sgn } b) \geq \tilde{c}_p (|a|^\frac{p}{2} \text{ sgn } a - |b|^\frac{p}{2} \text{ sgn } b)^2.$$

Moreover,

$$(2.1) \quad \tilde{c}_p = \min_{0 \leq t \leq 1} \frac{1 - t^p}{1 - t} \cdot \frac{1 - t^2}{1 - t} \geq 2 \min \left( \frac{1}{p}, \frac{1}{p'} \right).$$

**Proof.** Notice that by symmetry we can think that either $a, b$ are both positive or that $a > 0 > b$. Then by homogeneity the case $a > 0 > b$ is reduced to the estimate

$$(1 + x)(1 + x^{p-1}) \geq (1 + x^p)^2, x \geq 0,$$

which is the same as $2x^p \leq x + x^{p-1}$. The latter inequality is just $2AB \leq A^2 + B^2$.

The case when both $a, b$ are positive becomes

$$(1 - x)(1 - x^{p-1}) \geq \tilde{c}_p (1 - x^p)^2, 0 \leq x \leq 1.$$

Notice that this inequality is false for $p = 1$, but it holds for $p > 1$. This is just because after the change of variable $x = t^{\frac{1}{p}}$ one can observe that

$$\lim_{t \to 1^-} \frac{1 - t^p}{1 - t} > 0, \quad \lim_{t \to 1^-} \frac{1 - t^2}{1 - t} > 0.$$
From this one sees immediately that
\[ \tilde{c}_p := \inf_{0 \leq x \leq 1} \frac{(1 - x)(1 - x^{p-1})}{(1 - x^{p})^2} > 0. \]

**Theorem 2.4.** Let $1 < p < \infty$. Then for any $f : \{-1, 1\}^n \to \mathbb{R}$, we have

\[ -\mathbb{E}(\Delta f|f|^{p-1}\text{sgn}(f)) \geq C_1 \cdot c_p \cdot \mathbb{E}|f|^p - 2\tilde{c}_p \cdot \mathbb{E}|f|^p, \]

where $C_1 > 0$ is an absolute constant, and $c_p = 2\min\left(\frac{1}{p}, \frac{1}{p'}\right)$.

**Proof.** By Lemma [2.3] (note that we need $p > 1$), we have

\[ -\mathbb{E}\Delta f|f|^{p-1}\text{sgn}(f) = \mathbb{E}\left(\sum_{i=1}^{n} \nabla_i f \nabla_i ([|f|^{p-1}\text{sgn}(f)]) \right) \]
\[ \geq \mathbb{E}\left(\sum_{i=1}^{n} \tilde{c}_p |\nabla_i ([|f|^{\frac{p}{2}}\text{sgn}(f))]^2 \right) \geq c_p \|
abla ([|f|^{\frac{p}{2}}\text{sgn}(f)])\|^2. \]

Now we make a change of variable and denote $g(x) = |f(x)|^{\frac{p}{2}}\text{sgn}(f(x))$. Note that $g$ and $f$ have the same sign. Clearly

\[ \mathbb{E}f = \mathbb{E}[|g|^\beta \text{sgn}(g)], \]

where $\beta = \frac{2}{p} \in (0, 2)$ since $1 < p < \infty$. The desired inequality then clearly follows from Proposition [2.2].

---

3. **Fractional Laplacian on Hamming cube and its spectral gap estimates**

For $0 < \gamma \leq 1$, we introduce

\[ \Delta_\gamma = -(-\Delta)^\gamma, \]

the fractional Laplacian operator on Hamming cube via Fourier transform as

\[ (\Delta_\gamma f)(x) = -\sum_S |S|^\gamma a_S x^S, \]

for any $f = \sum_S a_S x^S$. In yet other words $\Delta_\gamma$ is simply the Fourier multiplier $-|S|^\gamma$.

The first claim of the next theorem is very well known for $p = 2$. It is the claim that Laplacian on Hamming cube has a spectral gap. It is interesting that this “spectral gap” estimate can be extrapolated to $1 < p < \infty$, and even, as we will see later, for $p = 1$ sometimes.

In Section 4 we will see that with extra spectral assumptions on $f$ it holds even for $p = 1$.

**Theorem 3.1.** Let $1 < p < \infty$. Then for any $f : \{-1, 1\}^n \to \mathbb{R}$, we have

\[ \|e^{t\Delta}(f - \mathbb{E}f)\|_p \leq e^{-k_1 t}\|f - \mathbb{E}f\|_p, \quad \forall t > 0, \]

where $k_1 = C_1 \cdot c_p, C_1 > 0$ is an absolute constant and $c_p = 2\min\left(\frac{1}{p}, \frac{1}{p'}\right)$. Similarly for $\Delta_\gamma$,

\[ \|e^{t\Delta_\gamma}(f - \mathbb{E}f)\|_p \leq e^{-k_\gamma t}\|f - \mathbb{E}f\|_p, \quad \forall t > 0, \]

where the constant $k_\gamma = k_1^\gamma$.

**Proof.** Without loss of generality we can assume $\mathbb{E}f = 0$. Denote $I(t) = \mathbb{E}(|e^{t\Delta}f|^p)$. Since $\mu$ is uniform counting measure, we can directly differentiate this and gives

\[ \frac{d}{dt}I(t) = p\mathbb{E}(\Delta g|g|^{p-1}\text{sgn}(g)), \]

where $g = e^{\Delta}f$. Note that $\mathbb{E}g = 0$. Thus by Theorem [2.4] we have

\[ \frac{d}{dt}I(t) \leq -p \cdot C_1 \cdot c_p I(t). \]
Integrating in time then yields the desired inequality with $k_1 = C_1 \cdot c_p$. For the fractional Laplacian case, we can use the subordination identity (see Lemma 5.1)

$$e^{-\lambda \gamma} = \int_0^\infty e^{-r \lambda} d\rho(\tau), \quad \lambda \geq 0,$$

where $d\rho(\tau)$ is a probability measure on $[0, \infty)$. Clearly then

$$e^{-\lambda \gamma} = \int_0^\infty e^{-r \lambda} d\rho(\tau).$$

It follows that

$$\|e^{t \Delta f}\|_p \leq \int_0^\infty e^{-k_1 \frac{1}{\gamma} \lambda} d\rho(\tau) \|f\|_p = e^{-k_2 \|f\|_p}, \quad k_2 = k_1^\gamma.$$

**4. Band spectrum and $p = 1$. The first proof**

We first prove a certain Poincaré-type inequality involving $\Delta f, 0 < \gamma < 1$ in $L^1(\{-1,1\}^n)$. It will work for functions with band spectrum. Then we derive from it the inequality of “spectrum gap type” for functions in $L^1(\{-1,1\}^n)$ having band spectrum. Namely, we get

**Theorem 4.1.** For every $\gamma \in (0, 1)$ there exists $c_\gamma > 0$ independent of $n$ such that for every $n$ and every $f \in L^1(\{-1,1\}^n)$ with band spectrum (meaning that it has only, say, 1-mode and 2-mode only), or, more generally, finite number of modes and $\mathbb{E}f = 0$, we have

$$\|e^{t \Delta f}\|_1 \leq e^{-c_\gamma t \|f\|_1}.$$  

**Remark 4.2.** This result will be proved, in fact, by two different methods. The second method shows, in particular, that the $L^1$-norm can be changed to any shift invariant norm (as $\{-1,1\}^n$ is isomorphic to $\mathbb{F}_2^n$ and shift can be understood on this group). In particular, one gets the spectral gap theorem on any Lorentz or Orlicz space on cube $C_n$.

**Remark 4.3.** This result is false if $\gamma = 1$ even for band limited $f$. The counterexample is in Section 6.

However, the Poincaré inequality in $L^1(\{-1,1\}^n)$ from the subsection 4.2 below seems to have an independent interest and it looks slightly unusual.

But first we need a known result on hypercontractivity.

**4.1. Hypercontractivity helps.**

**Theorem 4.4.** Let $f$ be Fourier localized to finite number of (say $k$) modes with $\mathbb{E}f = 0$. Then

$$\|e^{t \Delta f}\|_1 \leq e^{-\frac{1}{2} t \|f\|_1}, \quad t \geq 3k \log 3.$$  

**Proof.** This follows easily from Theorem 9.22 of [9]. We will repeat the reasoning for the sake of convenience of the reader. By using the Bonami lemma (see pp. 247 of [9]), we have

$$\|f\|_4 \leq 3^{\frac{1}{2}} \|f\|_2 \leq 3^{\frac{1}{2}} \|f\|_1^{\frac{1}{2}} \|f\|_2^{\frac{1}{2}}.$$  

This implies $\|f\|_4 \leq 3^{\frac{3}{4}} \|f\|_1$. Thus

$$\|e^{t \Delta f}\|_1 \leq \|e^{t \Delta f}\|_2 \leq e^{-t \|f\|_2} \leq e^{-t \|f\|_4} \leq e^{-t \|f\|_4} \leq e^{-t \|f\|_4}.$$  

**Remark 4.5.** The inequality

$$\|e^{t \Delta f}\|_1 \leq e^{-ct \|f\|_1}$$

is not true for small $t$ even for band limited $f$. The counterexample in subsection 6.3 shows that.
4.2. Poincaré inequality with $\Delta \gamma$ in $L^1$. The first proof. Recall that $\Delta \gamma = -(-\Delta)^7$.

**Theorem 4.6.** For every $\gamma \in (0, 1)$ there exists $b_\gamma > 0$ independent of $n$ such that for every $n$ and every $f \in L^1(\{-1, 1\}^n)$ with finite number of Fourier $k$ modes (i.e. localized to 1-mode, $\cdots$, $k$-mode) and $E f = 0$, we have

\[
\tag{4.3}
\| \cdot \|_1 \leq \| E[(\Delta \gamma f) \cdot \text{sgn} f \cdot 1_{f \neq 0}] - E[|\Delta \gamma f| \cdot 1_{f = 0}] \|,
\]

where $\alpha_k = k^{-\gamma} \cdot 3^{-3k}$.

**Proof.** Let $\gamma \in (0, 1)$, put

\[
C_\gamma := \int_0^\infty (1 - e^{-u}) \frac{du}{u^{1+\gamma}} < \infty.
\]

It is then obvious that for any function $f$ on the cube such that $E f = 0$ one has

\[
-\Delta \gamma f = C_\gamma^{-1} \int_0^\infty (f - e^{t\Delta} f) \frac{dt}{t^{1+\gamma}}.
\]

Note that here and below convergence is not an issue since we are on the Hamming cube.

Now clearly

\[
C_\gamma (E[(\Delta \gamma f) \cdot \text{sgn} f \cdot 1_{f \neq 0}] - E[|\Delta \gamma f| \cdot 1_{f = 0}])
\]

\[
= E\int_0^\infty (|f| - \text{sgn}(f) \cdot 1_{f \neq 0} e^{t\Delta f}) \frac{dt}{t^{1+\gamma}} - E(\int_0^\infty (e^{t\Delta} f) \frac{dt}{t^{1+\gamma}} \cdot 1_{f = 0})
\]

\[
\geq \int_0^\infty (\| f \|_1 - \| e^{t\Delta} f \|_1) \cdot \frac{dt}{t^{1+\gamma}}.
\]

Since $\| e^{t\Delta} f \|_1 \leq \| f \|_1$ for each $t \geq 0$. We can restrict the integral to $3k \log 3 \leq t \leq 6k \log 3$ and then apply Thorem 4.1. \qed

4.3. The first proof of Theorem 4.1 via Poincaré inequality in $L^1$. Denote

\[
I(t) = E|e^{t\Delta \gamma} f|.
\]

We want to estimate $\frac{d}{dt} I(t)$ for a test function $f$. Let $F := F_\varepsilon := e^{\varepsilon \Delta \gamma} f$. Then for $\varepsilon > 0$, we have

\[
|e^{-\varepsilon (\Delta \gamma)} F| - |F| = \begin{cases} 
\varepsilon \text{sgn}(F) \cdot (\Delta \gamma F) + O(\varepsilon^2), & \text{if } F(x) \neq 0; \\
|\Delta \gamma F| + O(\varepsilon^2), & \text{if } F(x) = 0.
\end{cases}
\]

One should keep in mind that we are on the discrete Hamming cube and as such interchanging integrals with differentiation should not be an issue. Now if we look at $\frac{d}{dt} I(t)$ as the expression

\[
\frac{d}{dt} I(t) := \lim_{\varepsilon \to 0} \frac{I(t + \varepsilon) - I(t)}{\varepsilon},
\]

we notice that the limit exists and that we can go to the limit under the sign of $E$. So we get from (4.4) that

\[
\frac{d}{dt} I(t) = E\left( \text{sgn} F_\varepsilon \cdot (\Delta \gamma F_\varepsilon) \cdot 1_{F_\varepsilon \neq 0} \right) - E(|\Delta \gamma F_\varepsilon| \cdot 1_{F_\varepsilon = 0}) \leq -\tilde{b}_\gamma E|F_\varepsilon|.
\]

The last inequality follows from Theorem 4.6. Hence,

\[
\frac{d}{dt} I(t) \leq -\tilde{b}_\gamma I(t), \quad I(0) = \| f \|_1.
\]

Therefore, (4.1) is proved for test functions $f$ with universal constant, and so Theorem 4.1 is proved.
5. **The second proof of Theorem 4.1 via the modification of the kernel of** \( e^{t\Delta_\gamma} \)

We begin with a well-known fact connected with the subordination of fractional heat operators. We need some sharp asymptotics which will play some role in the perturbation argument later. For the sake of completeness we include the proof (even for some well-known facts).

**Lemma 5.1.** Let \( 0 < \gamma < 1 \). Then
\[
e^{-\lambda \gamma} = \int_0^\infty e^{-\lambda \tau} p_\gamma(\tau) d\tau, \quad \lambda \geq 0,
\]
where \( p_\gamma \) is a probability density function on \( \mathbb{R} \) satisfying:

- \( p_\gamma \) is infinitely differentiable with bounded derivatives of all orders, and \( p_\gamma(\tau) = 0 \) for any \( \tau \leq 0 \).
- \( \lim_{\tau \to \infty} \tau^{1+\gamma} p_\gamma(\tau) = C_\gamma \), where
\[
\frac{1}{C_\gamma} = \int_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+\gamma}} d\tau.
\]

**Remark 5.2.** For \( \gamma = 1/2 \) it is well-known that \( p_{1/2} \) admits an explicit representation. One can just observe
\[
e^{-|x|} = \frac{1}{\pi} \int_{x+i\infty}^{x+i\infty} e^{-\gamma^2} e^{iz} dz.
\]

*Proof.* For simplicity we shall write \( p_\gamma \) as \( p \). We first show its existence. For any \( z = re^{i\theta} \) with \( \theta \in [-\pi/2, \pi/2] \), we fix the branch of \( z^\gamma \) such that \( z^\gamma = r^\gamma e^{i\gamma \theta} \). Define for \( x > 0 \):
\[
p(\tau) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{-z^\gamma} e^{z\tau} dz.
\]
By deforming the contour it is easy to check that the integral is independent of \( x \). By using a large semi-circle to the right one can verify that \( p(\tau) \) vanishes for \( \tau < 0 \). Furthermore one can take the limit \( x \to 0^+ \) to obtain
\[
(5.1) \quad p(\tau) = \frac{1}{\pi} \int_0^\infty e^{-y^\gamma} \cos(\frac{\gamma \pi}{2}) dy.
\]
From this it is evident that \( p \) has bounded derivatives of all orders. For \( z \in \{x+iy : x > 0, y \in \mathbb{R} \} \), we have
\[
e^{-z^\gamma} = \int_0^\infty e^{-\tau z} p(\tau) d\tau.
\]
In particular this identity holds for any \( z = \lambda > 0 \). Now to show \( p \geq 0 \) one can just appeal to the Bernstein theory. More directly one can just use the fact that
\[
\lim_{\lambda \to \infty} e^{-\lambda \tau} \sum_{k \leq \lambda x} \frac{(\lambda \tau)^k}{k!} = \begin{cases} 1, & \text{if } 0 \leq \tau \leq x; \\ 0, & \text{otherwise}. \end{cases}
\]
Since \( e^{-\lambda \gamma} \) is completely monotone, one can then deduce
\[
\int_{x_1}^{x_2} p(\tau) d\tau \geq 0, \quad \text{for any } 0 \leq x_1 < x_2 < \infty.
\]
This then yields \( p \geq 0 \). Finally to show the asymptotic of \( p \), we use (5.1) and partial integration to write
\[
\pi \tau p(\tau) = \gamma \text{Re} \left[ \frac{2\gamma}{i} \int_0^\infty e^{-y^\gamma} e^{iy} y^{\gamma-1} dy \right].
\]
where \( z_0 = e^{i\pi/2} \). By a further change of variable, we get
\[
\pi \tau^{1+\gamma} p(\tau) = \gamma \Re \left[ \frac{z_0}{i} \int_0^\infty e^{-\frac{2\pi}{\tau} y} e^{iy(1-\gamma)} dy \right].
\]
Now one can deform the contour integral inside the square bracket slightly to a straight line making a very small angle with the positive real axis. This then easily yields the existence of the limit as \( \tau \to \infty \). To calibrate this constant, one can use the simple relation
\[
1 - e^{-R - \gamma} = \int_0^\infty (1 - e^{-\tau^{1+\gamma}}) p(\tau) d\tau = \int_0^\infty (1 - e^{-\tau}) R p(\tau R) d\tau.
\]
This gives
\[
R^\gamma (1 - e^{-R - \gamma}) = \int_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+\gamma}} p(\tau R) \cdot (\tau R)^{1+\gamma} d\tau.
\]
Sending \( R \to \infty \) then yields the constant. \( \square \)

Our next lemma is the heart of the matter. It shows that a carefully chosen perturbation of the fractional heat kernel can leave invariant the “band-limited” portion and decrease the \( L^1 \) operator norm. Compared with the continuous setting in \cite{LiVolberg3} the discrete Hamming cube case requires a new and nontrivial twist.

**Lemma 5.3.** Let \( 0 < \gamma < 1 \). There exists \( t_0 = t_0(\gamma) \in (0, 1) \) such that the following hold. Consider the kernel \( K_t^\gamma \) corresponding to \( e^{t\Delta^\gamma} \):
\[
K_t^\gamma (x) = \sum_{S \subset [n]} e^{-t |S|^{\gamma}} x^S.
\]
there exists a modification of \( K_t^\gamma \) which we denoted as \( \hat{K}_t^\gamma \), such that:
\begin{itemize}
  \item \( \hat{K}_t^\gamma \) is still non-negative, and \( \| \hat{K}_t^\gamma (\cdot) \|_{L^1} \leq e^{-c_0 t} \), for all \( 0 < t \leq t_0 \), where \( c_0 > 0 \) depends only on \( \gamma \);
  \item \( \hat{K}_t^\gamma (S) = \hat{K}_t^\gamma (S) \), for any \( S \subset [n] \) with \( 1 \leq |S| \leq 2 \).
\end{itemize}

**Proof.** By using Lemma 5.1 we can write
\[
K_t^\gamma (x) = \int_0^\infty \prod_{j=1}^n (1 + e^{-\tau x_j}) p_\gamma (\tau) d\tau.
\]
By Lemma 5.4 we may choose \( R_0 = R_0(\gamma) > 10 \) sufficiently large such that
\begin{equation}
(5.2)
p_\gamma (\tau) \geq \frac{1}{2^{\gamma}} 2^{-(-1+\gamma)C_\gamma}, \quad \forall \tau \geq R_0.
\end{equation}
Now define \( t_0 = R_0^{-\gamma} \). Let us assume \( S = \{1, 2\} \). Any other \( S \) with \( |S| = 2 \) will be treated in the same way. Moreover, as the reader will see any finite \( |S| \) can be treated in exactly same way. For \( 0 < t \leq t_0 \), we construct the modified kernel function as
\[
\hat{K}_t^\gamma (x) = \int_0^\infty \prod_{j=1}^n (1 + e^{-\tau x_j}) (p_\gamma (\tau) - \kappa t^{\frac{1+\gamma}{\gamma}} \varphi(t^{\frac{1}{\gamma}} \tau)) d\tau,
\]
where \( \kappa > 0 \) is a sufficiently small constant, and \( \varphi \) is a bump function supported in \([1, 2]\) satisfying:
\[
\int_0^\infty e^{-\tau} \varphi(\tau) d\tau = 0,
\]
\[
\int_0^\infty e^{-2\tau} \varphi(\tau) d\tau = 0, \quad \int_0^\infty \varphi(\tau) d\tau > 0.
\]
Note that the first two equalities easy imply that \( \hat{K}_t^\gamma (S) = \hat{K}_t^\gamma (S) \), for our \( S = \{1, 2\} \).

On the other hand, on the support of \( \varphi \), we have \( \tau \sim t^{-\frac{1}{\gamma}} \), and one can easily choose (by using \( \kappa \)) \( \kappa \) sufficiently small such that
\[
p_\gamma (\tau) - \kappa t^{\frac{1+\gamma}{\gamma}} \varphi(t^{\frac{1}{\gamma}} \tau) \geq 0.
\]
Since $\tilde{K}_t^\gamma$ is non-negative, we clearly have

$$\|\tilde{K}_t^\gamma\|_{L^1_x} = 0\text{-mode of }\tilde{K}_t^\gamma = 1 - \kappa t \int_0^\infty \varphi(\tau) d\tau.$$  

Notice that any $S$ with fixed finite $|S|$ can be treated by the same approach. For example, if $S$ were $\{2, 3, 7\}$ we would replace the previous requirements by the following ones:

$$\int_0^\infty e^{-2\tau} \varphi(\tau) d\tau = 0,$$

$$\int_0^\infty e^{-3\tau} \varphi(\tau) d\tau = 0,$$

$$\int_0^\infty e^{-7\tau} \varphi(\tau) d\tau = 0, \quad \int_0^\infty \varphi(\tau) d\tau > 0.$$

\[\Box\]

**Remark 5.4.** Of course $\kappa$ depends on $|S|$, and even on $S$ itself. But if one fixes the “band” $S$ and starts to increase the dimension $n$, this constant $\kappa$ will not be depending on $n$. We choose function $\varphi$ with orthogonality conditions as above that depend on $S$ but have nothing to do with $n$. It would be interesting to measure the dependence on $S$.

We have the following general inequality for band localized functions $f$ for some universal $c_\gamma > 0$ (if $0 < \gamma < 1$).

**Theorem 5.5.** Let $0 < \gamma < 1$. Let the function $f : C_n \to \mathbb{R}$ is band localized to, say, the first and the second mode only, then independent of $n$ and for all such $f$ we have

$$\|e^{t\Delta^\gamma} f\|_p \leq e^{-c_\gamma t} \|f\|_p, \quad \forall t \geq 0, \ 1 \leq p \leq \infty,$$

where $c_\gamma > 0$ depends only on $\gamma$. Moreover, the norm $\| \cdot \|_p, 1 \leq p \leq \infty$ can be replaced here by the norm of any shift invariant Banach space on Hamming cube.

**Proof.** Since $e^{t\Delta^\gamma} f$ is still band localized, it suffices to prove the result for $0 < t < t_0$ with $t_0 = t_0(\gamma)$ small. This follows directly from Lemma 5.3 and Young’s inequality. \[\Box\]

**Remark 5.6.** For $p > 1$ and $\gamma = 1$ we have even stronger Theorem [5.7]. It is stronger because it can be formulated as

$$\|e^{t\Delta} f\|_p \leq e^{-c_1 t} \|f\|_p, \quad t \geq 0, \ 1 < p < \infty,$$

independently of $n$ for all functions $f$ that are very weakly spectral localized, namely, for $f$ such that only $0$-mode vanishes: $\mathbb{E} f = 0$.

**Remark 5.7.** Also for $p = 1$, $\gamma = 1$ one has the estimate [5.4]—but only for large $t$, see Theorem [4.4]. As to the case $p = 1$, $\gamma = 1$, $t$ is small, and $f$ is band localized, subsection 6.3 shows that such drop of norm can be false. So this is the case when even for band localized functions we do not have “spectral gap” type inequality. But as soon as either 1) $p > 1$ and any $\gamma \leq 1$ or 2) $\gamma < 1, p = 1$ we have “spectral gap” inequality

$$\|e^{t\Delta^\gamma} f\|_p \leq e^{-ct} \|f\|_p, \quad c > 0.$$

In case 1) we just need very weak spectral localization, namely, just $\mathbb{E} f = 0$. In case 2) we used that $f$ is band localized. This condition cannot be dropped as counterexample is subsection 6.2 shows.
6. COUNTEREXAMPLES

6.1. Counterexample to \( \|e^{t\Delta}f\|_1 \leq e^{-ct}\|f\|_1 \), \( \mathbb{E}f = 0 \). One cannot get independent of \( n \) estimate of Theorem 3.1 for \( p = 1 \). In fact, let \( f(1,\ldots,1) = 2^{n-1}, f(-1,\ldots,-1) = -2^{n-1} \), and \( f(x) = 0 \) for all other points \( x \in \{-1,1\}^n \). Then \( \mathbb{E}f = 0, \|f\|_1 = 1, \) and

\[
e^{t\Delta}f(x) = 2^{-1}(\sum_{i=1}^{n}(1+e^{-t}x_i) - \sum_{i=1}^{n}(1-e^{-t}x_i)).
\]

Hence,

\[
\|e^{t\Delta}f\|_1 = \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} |(1+e^{-t})^{n-k}(1-e^{-t})^k - (1-e^{-t})^{n-k}(1+e^{-t})^k| = \frac{1}{2^n} \sum_{0 \leq k \leq n/2} \binom{n}{k} ((1+e^{-t})^{n-k}(1-e^{-t})^k - (1-e^{-t})^{n-k}(1+e^{-t})^k).
\]

Now let us assume that there exists a universal constant \( \kappa < 1 \) such that for all \( n \) and all functions \( f \in L^1(\{-1,1\}^n), \mathbb{E}f = 0 \), there exists \( t_0 \) such that for all \( t \geq t_0 \)

\[
\|e^{t\Delta}f\|_1 \leq \kappa \|f\|_1, \text{ if } \mathbb{E}f = 0.
\]

Then by semi-group property \( \|e^{t_1\Delta}f\|_1 < \frac{1}{2}\|f\|_1, \) if \( \mathbb{E}f = 0 \).

**Proposition 6.1.** Let \( 0 < \epsilon \leq 1/2 \). Then for \( n \) sufficiently large, we have

\[
\frac{1}{2^n} \sum_{0 \leq k \leq n/2} \binom{n}{k} \cdot ((1+\epsilon)^{n-k}(1-\epsilon)^k - (1+\epsilon)^k(1-\epsilon)^{n-k}) \geq \frac{1}{2}(1 - (1 - \epsilon^2)^{\frac{n}{2}}).
\]

**Proof.** We have

\[
2 \cdot \text{LHS} \geq \sum_{0 \leq k \leq n/2} \frac{1}{2^n} \binom{n}{k} \cdot (1+\epsilon)^{n-k}(1-\epsilon)^k + \sum_{k > n/2} \frac{1}{2^n} \binom{n}{k} \cdot (1+\epsilon)^{n-k}(1-\epsilon)^k
\]

\[
- \sum_{0 \leq k \leq n/2} \frac{1}{2^{n-1}} \binom{n}{k} \cdot (1+\epsilon)^{n-k}(1-\epsilon)^k \geq 1 - (1 - \epsilon^2)^{\frac{n}{2}},
\]

where in the last inequality we may assume \( n \) is an odd integer so that \( k = n/2 \) cannot be obtained. If \( n \) is even, one can get a similar bound. \( \square \)

Now we use (6.1) and the Proposition to come to contradiction with (6.3). Hence (6.2) is false too.

6.2. Counterexample to \( \|e^{t\Delta}f\|_{L^1} \leq e^{-ct}\|f\|_{L^1} \) for \( f \) with \( \mathbb{E}f = 0 \). Fix \( 0 < \gamma < 1 \). Again we shall argue by contradiction. Assume the desired estimate is true. Similar to the Laplacian case, this would imply that there exists universal \( t_1 > 0 \) independent of \( n \), such that for all \( f \) with \( \mathbb{E}f = 0 \), we have

\[
\|e^{t_1\Delta}f\|_1 \leq \frac{1}{4}\|f\|_1.
\]

Now take the same \( f \) as in the Laplacian case. By using the subordination formula

\[
e^{-t\lambda^2} = \int_0^\infty e^{-t\tau^2} \lambda^2 d\rho(\tau),
\]
we get
\[(e^{t\Delta} f)(x) = \frac{1}{2} \int_0^\infty \left( \prod_{j=1}^n (1 + e^{-\tau t^+} x_j) - \prod_{j=1}^n (1 - e^{-\tau t^+} x_j) \right) d\rho(\tau).\]

Hence
\[\|e^{t\Delta} f\|_1 = \frac{1}{2^n} \sum_{0 \leq k \leq \frac{n}{2}} \left( \begin{array}{c} n \\ k \end{array} \right) \int_0^\infty \left( (1 + e^{-\tau t^+})^{n-k}(1 - e^{-\tau t^+})^k - (1 - e^{-\tau t^+})^{n-k}(1 + e^{-\tau t^+})^k \right) d\rho(\tau) \geq \frac{1}{2} \int_0^\infty (1 - (1 - e^{-2\tau t^+})^\frac{n}{2}) d\rho(\tau).
\]

Now take \(t = t_1\) and send \(n\) to infinity. We clearly arrive at a contradiction!

6.3. **Counterexample to \(\|e^{t\Delta} f\|_1 \leq e^{-ct}\|f\|_1\) for band-limited \(f\) with small \(t\).** Consider the Gaussian space case. Let \(\rho(x) = e^{-x^2/2}\) and consider \(f(x) = x^3 = \text{He}_3(x) + 3\text{He}_1(x)\). Denote \(\Delta_{ou} f = f'' - xf'\). Then one can verify that
\[
\int_{f \neq 0} (-\Delta_{ou} f) \text{sgn}(f)\rho(x)dx = 0.
\]

This in turn implies that
\[\|e^{t\Delta} f\|_1 \geq \|f\|_1 - O(t^2),\]

for small \(t\), which of course contradicts \(\|e^{t\Delta} f\|_1 \leq e^{-ct}\|f\|_1 \leq (1 - c_0 t + O(t^2))\|f\|_1,\ c_0 > 0\).

**References**

[1] L. Ben Efraim, F. Lust-Piquard, Poincaré type inequalities on the discrete cube and in the CAR algebra, Probab. Theory Related Fields 141 (2008), no. 3-4, 569–602.

[2] R.M. Blumenthal and R.K. Getoor, *Some theorems on stable processes*, Trans. Amer. Math. Soc. 95 (1960), 263–273.

[3] Diego Chamorro and Pierre Gilles Lemarié-Rieusset, *Quasi-geostrophic equations, nonlinear Bernstein inequalities and α-stable processes*, Rev. Mat. Iberoam. 28 (2012), no. 4, 1109–1122.

[4] Dong Li, *On a frequency localized Bernstein inequality and some generalized Poincaré-type inequalities*, Math. Res. Lett. 20 (2013), no. 5, 933–945.

[5] Leonard Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975), no. 4, 1061–1083. 521 citations.

[6] S. Heilmann, E. Mossel, AND K. Oleszkiewicz, Strong contraction and influences in tail spaces, arXiv:1406.7855v3, pp. 1–20.

[7] Paata Ivanisvili, Fedor Nazarov and Alexander Volberg, *Square function and the Hamming cube: Duality*, Discrete Analysis, 2018.

[8] Paata Ivanisvili, Alexander Volberg, *Isoperimetric functional inequalities via the maximum principle: the exterior differential systems approach*, arXiv: 1511.06895, Operator Theory: Advances and Applications, Vol. 261, 279–303, Birkhäuser volume dedicated to V. P. Khavin.

[9] R. O’Donnell, Analysis of Boolean functions, Cambridge University Press, 2014.

[10] G. Samorodnitsky, M. Taqqu, *Stable non-Gaussian Random Processes*. Chapman and Hall, New York, London, 1994, 632 pp.

Department of Mathematics, Hong Kong University of Science and Technology
E-mail address: madli@ust.hk (Dong Li)

Department of Mathematics, Michigan State University, East Lansing, MI 48823, USA
E-mail address: volberg@math.msu.edu (A. Volberg)