The Dynamical Systems Approach to the Equations of a Linearly Viscous Compressible Barotropic Fluid

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Abstract

We develop a dynamical systems theory for the compressible Navier-Stokes equations based on global in time weak solutions. The following questions will be addressed:

- Global existence and critical values of the adiabatic constant;
- dissipativity in the sense of Levinson - bounded absorbing sets;
- asymptotic compactness;
- the long-time behaviour and attractors.

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1. Introduction

The long-time behaviour of solutions to the evolutionary equations arising in the mathematical fluid mechanics has been the subject of many theoretical studies. This type of problems is apparently related to the phenomena of turbulence, and there is still a significant gap between many formal “scenarios” and mathematically rigorous results.

The dynamical systems in question are usually related to a system of partial differential equations and, consequently, they are defined in an infinite dimensional phase space. On the other hand, the presence of dissipative terms in the equations due to viscosity results in the existence of global attractors-compact invariant sets attracting uniformly in time all trajectories emanating from a given bounded set. Such a theory is well developed for the incompressible linearly viscous fluids, and

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the reader may consult the monographs of BABIN and VISHIK [1], TEMAM [21] or CONSTANTIN et al. [2] for the recent state of art.

On the other hand, much less seems to be known for the compressible fluids. While there is an existence theory of the weak solutions for the incompressible Navier-Stokes equations due to LERAY [18], its “compressible” counterpart appeared only recently in the work of LIONS [19]. Even in the incompressible case, there is a qualitative difference between the two-dimensional case solved by LADYZHENSKAYA [17], and the three-dimensional case representing one of the most challenging unsolved problems of modern mathematics. It is worth-noting that a similar gap divides the one and more-dimensional problems for the compressible fluids.

The time evolution of the fluid density $\rho = \rho(t, x)$ and the velocity $\mathbf{u} = \mathbf{u}(t, x)$ is governed by the Navier-Stokes system of equations:

$$\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= \text{div} \mathbf{S} + \rho \mathbf{f} ,
\end{align*}$$

where $p$ is the pressure, $\mathbf{S}$ the viscous stress tensor, and $\mathbf{f}$ a given external force.

In what follows, we consider linearly viscous (Newtonian) fluids where the viscous stress is related to the velocity by the constitutive law

$$\mathbf{S} = \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T \right) + \lambda \text{div} \mathbf{u} \, I ,$$

where the viscosity coefficients satisfy

$$\mu > 0, \quad \lambda + \mu \geq 0 .$$

Generally speaking, the pressure $p$ depends on the density and the internal energy (temperature) of the fluid. If it is the case, the system (1.1), (1.2) is not closed and should be complemented by the energy equation. Unfortunately, however, the available global existence results for this full system allow for only for small initial data (cf. MATSUMURA and NISHIDA [20]).

There are physically relevant situations when one can assume the flow is barotropic, i.e., the pressure depends solely on the density. This is the case when either the temperature (the isothermal case) or the entropy (the isentropic case) are supposed to be constant. The typical constitutive relation between the pressure and the density then reads

$$p = p(\rho) = a \rho^\gamma, \quad a > 0 ,$$

where $\gamma = 1$ in the isothermal case, and $\gamma > 1$ represents the adiabatic constant in the isentropic regime. More general and even non-monotone pressure-density constitutive laws arise in nuclear plasma physics (see [4]). In the barotropic regime, the equations (1.1), (1.2) form a closed system and complemented by suitable initial and boundary conditions represent a (at least formally) well-posed problem.
If the problem is posed on a spatial domain \( \Omega \subset \mathbb{R}^N \), one usually assumes that the fluid adheres completely to the boundary which is mathematically expressed by the no-slip boundary conditions for the velocity:

\[
\vec{u}|_{\partial \Omega} = 0. \tag{1.6}
\]

Note that for viscous fluids such a condition is in a very good agreement with physical experiment.

In accordance with the deterministic principle, the state of the system at any time \( t > t_0 \) should be given by the initial conditions

\[
\rho(t_0) = \rho_I, \quad (\rho \vec{u})(t_0) = \vec{q}_I. \tag{1.7}
\]

The function \( \rho_I \) is non-negative and the momentum \( \vec{q}_I \) satisfies the compatibility condition

\[
\vec{q}_I = 0 \text{ a.a. on the set } \{ \rho_I = 0 \}. \tag{1.8}
\]

The reason why we impose the initial conditions for the momentum \( \rho \vec{u} \) rather than for the velocity \( \vec{u} \) is that the former quantity is weakly continuous with respect to time while the instantaneous values of the velocity are determined only almost anywhere with respect to time. Clearly such a problem does not arise provided the initial density \( \rho_I \) is strictly positive. However, it is an interesting open problem whether or not this property is preserved at any positive time for any distributional solution of the problem satisfying the natural energy estimates (cf. HOFF and SMOLLER [16]).

2. Finite energy weak solutions and well-posedness

In order to study the long-time behaviour, one should first make sure that the class of objects one deals with is not void. More precisely, one should be able to prove the existence of global-in-time solution for any initial data \( \rho_I, \vec{q}_I \) satisfying some physically relevant hypothesis.

Multiplying the equations of motion by \( \vec{u} \) and integrating by parts one deduces the energy inequality

\[
\frac{d}{dt} E[\rho, \vec{u}] + \int_{\Omega} \mu |\nabla \vec{u}|^2 + (\lambda + \mu) |\text{div} \ \vec{u}|^2 \ dx \leq \int_{\Omega} \rho \vec{f} \cdot \vec{u} \ dx, \tag{2.1}
\]

where the total energy \( E \) is given by the formula

\[
E[\rho, \vec{u}] = \int_{\Omega} \rho |\vec{u}|^2 + P(\rho) \ dx
\]

with

\[
P'(\rho) \rho - P(\rho) = p(\rho).
\]

Note that in the most common isentropic case, the function \( P \) can be taken in the form

\[
P(\rho) = \frac{a}{\gamma - 1} \rho^\gamma.
\]
The energy inequality is the main (and almost the only one) source of a priori estimates. Accordingly, “reasonable” solutions of the problem \((1.1, 1.2)\) defined on a bounded time interval \(I \subset \mathbb{R}\) should belong to the class
\[
\rho \geq 0, \quad \rho \in L^\infty(I; L^\gamma(\Omega)), \quad \vec{u} \in L^2(I; W^{1,2}_0(\Omega, R^N)). \tag{2.2}
\]

The energy inequality (2.1) represents an additional constraint imposed on any solution \(\rho, \vec{u}\) of the problem. Similarly as in the theory of the variational (weak) solutions developed for the incompressible case by Leray, it is not clear if it is satisfied for any weak solution of the problem.

Following DiPERNA and LIONS \([3]\) we shall say that \(\rho, \vec{u}\) is a renormalized solution of the continuity equation (1.1) if the identity
\[
\partial_t b(\rho) + \text{div}(b(\rho)\vec{u}) + \left(b'(\rho)\rho - b(\rho)\right)\text{div} \vec{u} = 0 \tag{2.3}
\]
holds in the sense of distributions for any function \(b \in C^1(\mathbb{R})\) such that
\[
b'(\rho) = 0 \quad \text{for all } \rho \geq \text{const}(b). \tag{2.4}
\]

Similarly as for the energy inequality, it is not known if any weak solution \(\rho, \vec{u}\) of (1.1) satisfies (2.3).

**Definition 2.1** We shall say that \(\rho, \vec{u}\) is a finite energy weak solution of the problem (1.1 - 1.6) on a set \(I \times \Omega\) if the following conditions are satisfied:

- The functions \(\rho, \vec{u}\) belong to the function spaces determined in (2.2);
- the energy inequality (2.1) is satisfied in \(D'(I)\);
- the continuity equation (1.1) as well as its renormalized version (2.3) hold in \(D'(I \times R^N)\) provided \(\rho, \vec{u}\) were extended to be zero outside \(\Omega\);
- the momentum equation (1.2) is satisfied in \(D'(I \times \Omega)\).

The most general available existence result reads as follows:

**Theorem 2.1** Let \(\Omega \subset R^N, N = 2, 3\) be a bounded Lipschitz domain. Let \(I = (0, T)\), and let the initial data \(\rho_I, \vec{q}_I\) satisfy (1.8) together with
\[
\rho_I \in L^\gamma(\Omega), \quad \frac{|\vec{q}_I|^2}{\rho_I} \in L^1(\Omega).
\]
Let \(\vec{f}\) be a bounded measurable function of \(t \in I, x \in \Omega\). Finally, let the pressure \(p \in C[0, \infty) \cap C^1(0, \infty)\) be given by a constitutive law
\[
p = p(\rho), \quad \frac{1}{a} \rho^{\gamma-1} - b \leq p'(\rho) \leq a\rho^{\gamma-1} + b, \quad \text{for all } \rho > 0,
\]
where \(a > 0, b \geq 0,\) and
\[
\gamma > \frac{N}{2}.
\]

Then the problem (1.1 - 1.6) admits a finite energy weak solution \(\rho, \vec{u}\) on \(I \times \Omega\) satisfying the initial conditions (1.7).

In his pioneering work, LIONS \([19]\) proved Theorem 2.1 for \(\Omega\) regular, \(p\) monotone, and \(\gamma \geq \frac{3}{2}\) if \(N = 2\), and \(\gamma \geq \frac{4}{3}\) for \(N = 3\). The hypotheses concerning \(\gamma\) were relaxed in \([9, 12]\), the case of a general bounded domain \(\Omega\) treated in \([11]\), and the hypothesis of monotonicity of the pressure removed in \([5, 4]\).
3. Ultimate boundedness

The first issue to be discussed when describing the long-time asymptotics of a given dynamical system is ultimate boundedness or dissipativity. This means there exists an absorbing set bounded in a suitable topology. Here “suitable topology” is of course that one induced by the total energy $E$. One of possible results in this direction is contained in the following theorem.

**Theorem 3.1** Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded Lipschitz domain. Let $\mathbf{f}$ be a bounded measurable function such that

$$\text{ess sup}_{t \in R, x \in \Omega} |\mathbf{f}(t, x)| \leq F.$$ 

Let the pressure $p$ be given by the isentropic constitutive law

$$p = a \rho^\gamma$$

with $\gamma > 1$ if $N = 2$, $\gamma > \frac{5}{3}$ for $N = 3$.

Finally, set

$$\int_{\Omega} \rho \, dx = m > 0.$$ 

Then there exists a constant $E_\infty$ depending solely on $m$ and $F$ having the following property:

Given $E_I$, there exists a time $T = T(E_I)$ such that

$$E[\rho, \mathbf{u}](t) \leq E_\infty$$

for a.a. $t > T$ provided

$$\text{ess lim sup}_{t \to 0^+} E[\rho, \mathbf{u}](t) \leq E_I$$

and $\rho, \mathbf{u}$ is a finite energy weak solution of the problem (1.1 - 1.6) on $(0, \infty) \times \Omega$.

The proof of Theorem 3.1 can be found in [14] and [7]. The reader will have noticed the “critical” exponent $\gamma > \frac{5}{3}$ which is larger than in the existence Theorem 2.1 and, as a matter of fact, does not include any physically relevant case. Indeed the value $\frac{5}{3}$ happens to be the largest adiabatic constant corresponding to a monoatomic gas.

Under the hypotheses of Theorem 3.1, the dissipative mechanism induced by viscosity is strong enough to prevent any “resonance” phenomena, i.e., the existence of unbounded solutions driven by a bounded external force. Another interesting feature is the existence of periodic solutions provided $\mathbf{f}$ is periodic in time.

**Theorem 3.2** In addition to the hypotheses of Theorem 3.1, assume that $\mathbf{f}$ is time periodic, i.e.,

$$\mathbf{f}(t + \omega, x) = \mathbf{f}(t, x)$$

for a.a. $t \in R, x \in \Omega$

with a certain period $\omega > 0$.

Then there exists at least one finite energy weak solution of the problem (1.1 - 1.6) on $R \times \Omega$ which is $\omega$-periodic in time, i.e.,

$$\rho(t + \omega) = \rho(t), \ (\rho \mathbf{u})(t + \omega) = (\rho \mathbf{u})(t)$$

for all $t \in R$.

See [10] for the proof.
4. Asymptotic compactness

The property of asymptotic compactness of a given dynamical system plays a crucial role in the proof of existence of a global (compact) attractor. Here, the main problem is the density component which is bounded only in $L^\gamma(\Omega)$, and, consequently, compact only with respect to the weak topology on this space. Moreover, given the hyperbolic character of the continuity equation, one cannot hope any possible oscillations of the density to be killed at a finite time. However, the amplitude of possible oscillations is decreasing in time uniformly on trajectories emanating from a given bounded set.

**Theorem 4.1** Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ be a bounded Lipschitz domain. Let the pressure $p$ be given by the isentropic constitutive relation

$$p = a \varphi^\gamma, \ a > 0, \ \gamma > 1 \text{ if } N = 2, \ \gamma > \frac{5}{3} \text{ for } N = 3.$$  

Let $\vec{f}_n$ be a sequence of functions such that

$$\text{ess sup}_{t \in \mathbb{R}, \ x \in \Omega} |\vec{f}_n(t, x)| \leq F \ \text{independently of } n = 1, 2, ... .$$

Finally, let $\varrho_n, \vec{u}_n$ be a sequence of finite energy weak solutions to the problem (1.1 - 1.6) on $(0, \infty) \times \Omega$ such that

$$\int_\Omega \varrho_n \ dx = m, \ \text{ess lim sup}_{t \to 0^+} E[\varrho_n, \vec{u}_n](t) \leq E_I$$

independently of $n = 1, 2, ... .$

Then any sequence of times $t_n \to \infty$ contains a subsequence such that

$$\varrho_n(t_n + t) \to \varrho(t) \ \text{strongly in } L^1(\Omega) \ \text{and weakly in } L^\gamma(\Omega),$$

$$(\varrho_n \vec{u}_n)(t_n + t) \to (\varrho \vec{u})(t) \ \text{weakly in } L^1(\Omega, \mathbb{R}^N)$$

for any $t \in \mathbb{R}$.

Moreover,

$$\int_J \int_\Omega |\varrho_n(t_n + t, x) - \varrho(t, x)|^\gamma \ dx \ dt \to 0,$$

$$\int_J \int_\Omega |(\varrho_n \vec{u}_n)(t_n + t, x) - (\varrho \vec{u})(t, x)| \ dx \ dt \to 0$$

for any bounded interval $J \subset \mathbb{R}$. The limit functions $\varrho, \vec{u}$ represent a globally defined (for $t \in \mathbb{R}$) finite energy weak solution of the problem (1.1 - 1.6) driven by a force

$$\vec{f} = \lim_{n \to \infty} \vec{f}_n(t_n + \cdot) \ \text{in the weak star topology of the space } L^\infty(\mathbb{R} \times \Omega, \mathbb{R}^N),$$

and such that

$$\text{ess sup}_{t \in \mathbb{R}} E[\varrho, \vec{u}](t) < \infty.$$ 

The proof of Theorem 4.1 is given in \cite{13} (see also \cite{8}).
5. The long-time behaviour, attractors

The results presented in Sections 3., 4. allow us to develop a theory of attractors analogous to that one for the incompressible flows (see e.g. TEMAM [21]). Assume, for the sake of simplicity, that the driving force \( \vec{f} \) is independent of \( t \). We introduce

\[
\mathcal{A} = \{ [\varrho_I, \vec{q}_I] \mid \varrho_I = \varrho(0), \quad \vec{q}_I = (\varrho \vec{u})(0) \}
\]

where \( \varrho, \vec{u} \) is a finite energy weak solution of the problem (1.1 - 1.6) on \( \mathbb{R} \times \Omega \) with \( E[\varrho, \vec{u}] \in L^\infty(\mathbb{R}) \).

In other words, the set \( \mathcal{A} \) is formed by all globally defined (for \( t \in \mathbb{R} \)) trajectories whose energy is uniformly bounded.

The next result shows that \( \mathcal{A} \) is a global attractor in the sense of FOIAS and TEMAM [15].

**Theorem 5.1** Let \( \Omega \subset \mathbb{R}^N, \; N = 2, 3 \) be a bounded Lipschitz domain. Let \( \vec{f} = \vec{f}(x) \) be a bounded measurable function independent of time, and let the pressure \( p \) be given by the isentropic constitutive law

\[
p = a \varrho^\gamma, \quad a > 0, \quad \gamma > 1 \quad \text{for} \quad N = 2, \quad \gamma > \frac{5}{3} \quad \text{if} \quad N = 3.
\]

Then the set \( \mathcal{A} \) defined by (5.1) is compact in the space

\[
L^\alpha(\Omega) \times L^{\frac{2\alpha}{\gamma+1}}(\Omega),
\]

and

\[
\sup_{[\varrho, \vec{u}] \in \mathcal{A}} \left[ \inf_{[\varrho_I, \vec{q}_I] \in \mathcal{B}(E_I)} \left( \| \varrho(t) - \varrho_I \|_{L^\alpha(\Omega)} + \| \int_\Omega ((\varrho \vec{u})(t) - \vec{q}_I) \cdot \phi \, dx \| \right) \right] \to 0
\]

for \( t \to \infty \)

for any \( 1 \leq \alpha < \gamma \) and any \( \phi \in L^{\frac{2\alpha}{\gamma+1}}(\Omega, \mathbb{R}^N) \), where the symbol \( \mathcal{B}(E_I) \) stands for the set of all finite energy weak solutions of the problem (1.1 - 1.6) on \( (0, \infty) \times \Omega \) such that

\[
\text{ess lim sup}_{t \to 0} E[\varrho, \vec{u}](t) \leq E_I.
\]

The proof can be found in [6].

To conclude, we give another result concerning the long-time behaviour of solutions on condition that the driving force is a gradient of a scalar potential.

**Theorem 5.2** Let \( \Omega \subset \mathbb{R}^N, \; N = 2, 3 \) be a bounded Lipschitz domain. Let the pressure \( p \) be given by the isentropic constitutive relation

\[
p = a \varrho^\gamma, \quad a > 0, \quad \gamma > \frac{N}{2}.
\]

Let the driving force \( \vec{f} \) be of the form

\[
\vec{f} = \nabla F,
\]
where $F = F(x)$ be a scalar potential which is globally Lipschitz on $\Omega$ and such that the upper level sets

$$[F > k] = \{ x \in \Omega \mid F(x) > k \}$$

are connected for any $k \in \mathbb{R}$.

Then any finite energy weak solution $\varrho$, $\vec{u}$ of the problem (1.1 - 1.6) satisfies

$$\varrho(t) \to \varrho_s \text{ in } L^\gamma(\Omega), \quad (\varrho \vec{u})(t) \to 0 \text{ in } L^1(\Omega) \text{ as } t \to \infty,$$

where $\varrho_s$ is a solution of the static problem

$$a \nabla \varrho_s^\gamma = \varrho_s \vec{f} \text{ on } \Omega.$$

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