Abstract

We investigate the finite-temperature behavior of the Yukawa model in which $N_f$ fermions are coupled with a scalar field $\phi$ in the limit $N_f \to \infty$. Close to the chiral transition the model shows a crossover between mean-field behavior (observed for $N_f = \infty$) and Ising behavior (observed for any finite $N_f$). We show that this crossover is universal and related to that observed in the weakly-coupled $\phi^4$ theory. It corresponds to the renormalization-group flow from the unstable Gaussian fixed point to the stable Ising fixed point. This equivalence allows us to use results obtained in field theory and in medium-range spin models to compute Yukawa correlation functions in the crossover regime.
1 Introduction

The finite-temperature transition in QCD has been extensively studied in the last twenty years and is becoming increasingly important because of the recent experimental progress in the physics of ultrarelativistic heavy-ion collisions. Some general features of the transition, which is associated with the restoration of chiral symmetry, can be studied in dimensionally-reduced three-dimensional models [1, 2]. However, a detailed understanding requires a direct analysis in QCD. Being the phenomenon intrinsically nonperturbative, our present knowledge comes mainly from numerical simulations [3, 4]. Due to the many technical difficulties—finite-size effects, proper inclusion of fermions, etc.—results are not yet conclusive and thus it is worthwhile to study simplified models that show the same basic features but are significantly simpler. In this paper we shall consider a Yukawa model in which $N_f$ fermions are coupled with a scalar field through a Yukawa interaction. The action of the model in $d+1$ dimensions is

$$S = DN_f \int d^{d+1}x \left( \frac{1}{2} (\partial \phi)^2 + \frac{\mu}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right) + \sum_{f=1}^{N_f} \int d^{d+1}x \bar{\psi}_f \left( \slashed{\partial} + g\phi + M \right) \psi_f, \quad (1)$$

where $\text{tr} \gamma_\mu^2 = D \ (D = 2^{d/2}$ if $d$ is even, $D = 2^{(d+1)/2}$ if $d$ is odd), the integration is over $\mathbb{R}^d \times [0, T^{-1}]$, and $\lambda \geq 0$ to ensure the stability of the quartic potential. Along the thermal direction we take periodic boundary conditions for the bosonic field $\phi$ and antiperiodic ones for the fermionic fields $\psi_f$. The theory must be properly regularized. We shall consider a sharp-cutoff regularization, restricting the momentum integrations in the spatial directions to $p < \Lambda$. However, the discussion presented here can be extended without difficulty to any other regularization that maintains at least a remnant of chiral symmetry.

In the limit $N_f \to \infty$ this model can be solved analytically and one finds that there is a range of parameters in which it shows a transition analogous to that observed in QCD [5, 6]. It separates a low-temperature phase in which chiral symmetry is broken from a high-temperature phase in which chiral symmetry is restored. For $N_f = \infty$ this transition shows mean-field behavior, in contrast with general arguments that predict the transition to belong to the Ising universality class. This apparent contradiction was explained in Ref. [7] where, by means of scaling arguments, it was shown that the width of the Ising critical region scales as a power of $1/N_f$, so that only mean-field behavior can be observed in the limit $N_f = \infty$. An analogous behavior was observed in a generalized $O(N) \sigma$ model in Ref. [8]: for finite values of $N$ the transition was expected to be in the Ising universality class, while the $N = \infty$ solution predicted mean-field behavior. In Ref. [9] we performed a detailed calculation of the $1/N$ corrections, explaining the observed behavior in terms of a critical-region suppression. The analytic technique discussed in Ref. [9] can be applied to model (1). It allows us to obtain an analytic description of the crossover from mean-field to Ising behavior that occurs when $N_f$ is large and to extend the discussion of Ref. [7] to the case $M \neq 0$. More importantly, we are able to show that the phenomenon is universal. In field-theoretical terms, it can be characterized as a crossover between two fixed points: the Gaussian fixed point and the Ising fixed point. This implies that quantitative predictions for model (1) can be obtained in completely different settings. One can use field theory and compute the crossover functions by resumming the perturbative series [10, 11, 12, 13]. Alternatively, one can use the fact that the field-theoretical crossover is equivalent to the critical crossover that occurs in models
with medium-range interactions \[14, 15, 13, 16, 17\]. This allows one to use the wealth of results available for these spin systems \[14, 18, 19, 15, 13, 17\]. In this case the interaction range \(R\) is essentially equivalent to a power of \(N\), \(N \sim R^d\). Finally, we should note that the phenomenon is quite general and occurs in any situation in which there is a crossover from the Gaussian fixed point to a nonclassical stable fixed point. For instance, similar considerations have been recently presented for finite-temperature QCD in some very specific limit \[20\].

The paper is organized as follows. In Sec. 2 we review the behavior in the limit \(N_f = \infty\). In Sec. 3 we consider the \(1/N_f\) fluctuations and determine the effective theory of the excitations that are responsible for the Ising behavior at the critical point. These modes are described by an effective weakly-coupled \(\phi^4\) Hamiltonian. In Sec. 4.1 we present a general discussion of the critical crossover limit. These considerations are applied to the Yukawa model in Sec. 4.2 and in Sec. 4.3. We determine the relevant scaling variables and show how to compute the crossover behavior of the correlation functions. Finally, in Sec. 5 we present our conclusions. In the appendix we discuss the relations among medium-ranged spin models, field theory, and the Yukawa model. A short summary of this work was presented in Ref. \[21\].

2 Behavior for \(N_f = \infty\)

The solution of the model for \(N_f = \infty\) is quite standard. We briefly summarize here the main steps, following the presentation of Ref. \[6\]. As a first step we integrate the fermionic fields obtaining an effective action \(S_{\text{eff}}[\phi]\) given by

\[
e^{-D N_f S_{\text{eff}}[\phi]} = \int \prod_{f=1}^{N_f} d\psi_f d\bar{\psi}_f e^{-S[\phi, \bar{\psi}, \psi]},
\]

where

\[
S_{\text{eff}}[\phi] = \int d^{d+1}x \left( \frac{1}{2} (\partial \phi)^2 + \frac{\mu}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right) - \frac{1}{D} \int d^{d+1}x \ \text{tr} \log \left( \partial + g \phi + M \right).
\]

For \(N_f \to \infty\) one can expand around the saddle point \(\phi = \overline{\phi}\), that is determined by the gap equation

\[
\overline{\mu} m + \frac{\lambda}{6} m^3 = (m + M) T \sum_{n \in \mathbb{Z}} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + \omega_n^2 + (m + M)^2},
\]

where we define the frequencies \(\omega_n \equiv (2n + 1)\pi T\), and

\[
m \equiv g \overline{\phi} \quad \overline{\mu} \equiv \mu g^{-2} \quad \overline{\lambda} \equiv \lambda g^{-4}.
\]

The action corresponding to a saddle-point solution \(m\) is:

\[
S_{\text{eff}}(m, M, T) = \frac{\overline{\mu}}{2} m^2 + \frac{\overline{\lambda}}{4!} m^4 - \frac{T}{2} \sum_{n \in \mathbb{Z}} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} \log \left[ \frac{p^2 + \omega_n^2 + (m + M)^2}{p^2 + \omega_n^2} \right],
\]
where we have added a mass-independent counterterm to regularize the sum \[6\]. Such a quantity has been chosen so that the action for \(M = m = 0\) vanishes. Summations can be done analytically \[6\]. The gap equation can then be written as

\[
p(m) = (m + M) \mathcal{G}(m + M, T),
\]

(6)

where the functions \(p(m)\) and \(\mathcal{G}(x, T)\) are defined by

\[
p(m) \equiv \mu m + \frac{\lambda}{6} m^3,
\]

(7)

\[
\mathcal{G}(x, T) \equiv \int_{p<\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{p^2 + x^2}} \left( \frac{1}{2} - \frac{1}{e^{\sqrt{p^2 + x^2}/T} + 1} \right).
\]

(8)

Analogously we can rewrite Eq. (5) as

\[
S_{\text{eff}}(m, M, T) = \frac{\mu}{2} m^2 + \frac{\lambda}{4!} m^4
\]

\[\int_{p<\Lambda} \frac{d^d p}{(2\pi)^d} \log \left( \cosh \frac{\sqrt{p^2 + (m + M)^2}}{2T} \right) - \log \cosh \frac{p}{2T} \].

(9)

Using Eqs. (6) and (9) we can determine the phase diagram of the model. Given \(\mu\) and \(\lambda\), for each value of \(T\) and \(M\) we determine the solutions \(m\) of the gap equations. In general, we find either one solution or three different solutions \(m_0, m_+, \) and \(m_-\) with \(m_- \leq m_0 \leq m_+\) (for some specific values of the parameters two of them may coincide). When the solutions are more than one, the physical solution is the one with the lowest action \(S_{\text{eff}}(m, M, T)\).

Note that Eqs. (6) and (9) are invariant under the transformations \(m \rightarrow -m\) and \(M \rightarrow -M\). Thus, we can limit our study to the case \(M \geq 0\). There are four different possibilities:

(a) If \(\mu \geq \mathcal{G}(0, 0) = C_0 \Lambda^{d-1}\) with

\[
C_0 \equiv \left[ 2^d \pi^{d/2} (d-1) \Gamma \left( \frac{d}{2} \right) \right]^{-1}.
\]

(10)

then, for every \(M \geq 0\), there is only one solution \(m_+ \geq 0\); for \(M = 0\) we have \(m_+ = 0\).

(b) If \(0 < \mu < C_0 \Lambda^{d-1}\), there exists a critical temperature \(T_c(\mu)\). For \(T > T_c(\mu)\) and any \(M\) there is only one solution \(m_+ \geq 0\) (for \(M = 0\) we have \(m_+ = 0\)). For \(T < T_c(\mu)\) and \(M < \bar{M}\) there are three solutions \(m_0, m_+\), and \(m_-\) with \(m_- \leq m_0 \leq m_+\) and \(m_0 \geq 0\) and \(m_- \leq 0\). For \(T < T_c(\mu)\) and \(M > \bar{M}\) there is only one solution corresponding to \(m_+\). The physical solution is always \(m_+\) so that \(\bar{M}\) has no physical meaning. Moreover, for \(T < T_c(\mu)\) and \(M = 0, m_+ > 0\). The critical temperature can be computed from the following relation:

\[
\bar{\mu} = \mathcal{G}(0, T_c) = T_c \sum_{n \in \mathbb{Z}} \int_{p<\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + \omega_{c,n}^2}
\]

\[= \Lambda^{d-1} \sum_{n \in \mathbb{Z}} \int_{p<\Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^{d-1}} \left[ \frac{\cos \omega_{c,n} t_c}{\sin \omega_{c,n} t_c} + 1 \right],
\]

(11)

where \(\omega_{c,n} \equiv (2n + 1) \pi T_c\) and \(T_c(\mu) = t_c(\mu) \Lambda\). For \(\bar{\mu} \rightarrow 0\), we have \(T_c(\mu) \rightarrow \infty\).
(c) If $-C_1 < \overline{\mu} \leq 0$, with
\[ C_1^{3/2} \equiv \lambda^{1/2} \frac{3\Lambda^d}{2^{d+2}\pi^{d/2}\sqrt{2}} \Gamma\left(\frac{d+2}{2}\right)^{-1}, \]  
(12)

there is a critical mass $\tilde{M}$ such that there are three solutions for $M < \tilde{M}$, two of them coincide for $M = \tilde{M}$, while for $M > \tilde{M}$ the only solution is $m_+$. The physical solution—the one with the lowest action—is always $m_+$ so that $\tilde{M}$ has no physical meaning. Note that $m_+ > 0$ for $M = 0$.

(d) For $\overline{\mu} \leq -C_1$ there are three solutions for all values of $T$ and $M$. The relevant solution is always $m_+ > 0$.

In case (a) chiral symmetry is never broken, while in cases (c) and (d) chiral symmetry is never restored. Thus, the only case of interest—and the only one we shall consider in the following—is case (b), in which there is a chirally-symmetric high-temperature phase and a low-temperature phase in which chiral symmetry is broken.

The nature of the transition is easily determined. We expand
\[ G(x, T) = \sum_{m, n} g_{mn} x^{2m}(T - T_c)^n, \]  
(13)

where
\[ g_{00} = T_c \sum_{n \in \mathbb{Z}} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + \omega_{c,n}^2}, \]
\[ g_{01} = \sum_{n \in \mathbb{Z}} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} \frac{p^2 - \omega_{c,n}^2}{(p^2 + \omega_{c,n}^2)^2} = -\frac{1}{T_c^2} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{(e^{p/T_c} + 1)^2}, \]
\[ g_{10} = -T_c \sum_{n \in \mathbb{Z}} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + \omega_{c,n}^2)^2}, \]  
(14)

and $\omega_{c,n} \equiv (2n + 1)\pi T_c$. Since $g_{00} = G(0, T_c) = \overline{\mu}$ [Eq. (11)], the gap equation (6) becomes:
\[ \frac{\lambda}{6} M^3 = \overline{\mu} M + g_{01}(T - T_c) (m + M) + g_{10}(m + M)^3 + \cdots \]  
(15)

where we have neglected subleading terms in $m + M$ and $T - T_c$. Defining
\[ u_h \equiv \frac{6\overline{\mu}}{6g_{10} - \lambda} M, \quad u_t \equiv \frac{6g_{01}}{6g_{10} - \lambda} (T - T_c), \]  
(16)

$C_1$ is the solution of the equation $p(-x) = \lim_{M \to \infty} MG(M, 0)$ with $x = (2C_1/\lambda)^{1/2}$. The value $x$ corresponds to the position of a maximum of $p(m)$ for $\overline{\mu} = -C_1$. 



and taking the limit \( u_h, u_t \to 0 \) at fixed \( x \equiv u_t/u_h^{2/3} \) we obtain the equation of state

\[
\begin{align*}
m &= u_h^{1/3} f(x) \\
0 &= f(x)^3 + x f(x) + 1.
\end{align*}
\]

Note that the prefactor of \( T - T_c \) in \( u_t \) is always positive to ensure the existence of only one solution for \( T > T_c \). Such an equation is exactly the mean-field equation of state that relates magnetization \( \varphi \), magnetic field \( h \), and reduced temperature \( t \). Indeed, if we consider the mean-field Hamiltonian

\[
\mathcal{H} = h \varphi + \frac{t}{2} \varphi^2 + \frac{u}{24} \varphi^4,
\]

the stationarity condition gives

\[
h + t \varphi + \frac{u}{6} \varphi^3 = 0,
\]

which is solved by \( \varphi = Ah^{1/3} f(Bt|h|^{-2/3}) \), where \( f(x) \) satisfies Eq. (18), and \( A \) and \( B \) are constants depending on \( u \). This identification also shows that \( M \) plays the role of an external field, while \( m \sim \varphi \) is the magnetization.

### 3 Effective theory for the zero mode

In order to perform the \( 1/N_f \) calculation, we expand the field \( \phi \) around the saddle-point solution,

\[
\phi(x) = \bar{\varphi} + \frac{1}{g\sqrt{N}} \hat{\varphi}(x),
\]

where \( N \equiv DN_f \), and \( \hat{\varphi}(x) \) in Fourier modes:

\[
\hat{\phi}(x_d, x_{d+1}) = T \sum_{n \in \mathbb{Z}} e^{2 \pi n T x_{d+1}} \int \frac{d^d p}{(2\pi)^d} \hat{\phi}_n(p) e^{i p \cdot x_d}.
\]

In the following we will refer to the integers \( n \)—or more precisely to \( 2\pi n T \)—as frequencies. In this way we obtain the following expansion for the effective action:

\[
\mathcal{S}_{\text{eff}}[\hat{\phi}_n] = N \{ \mathcal{S}_{\text{eff}}[\phi] - \mathcal{S}_{\text{eff}}[\bar{\varphi}] \} = \frac{T}{2} \sum_n \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} P(p, n) \hat{\phi}_n(p) \hat{\phi}_{-n}(-p)
\]

\[
+ \sum_{l \geq 3} \frac{T^{l-1}}{l!N^{l/2-1}} \sum_{\{n_i\}} \int_{p_i < \Lambda} \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_l}{(2\pi)^d} (2\pi)^d \delta \left( \sum_{j=1}^l p_j \right) \delta_{\sum_i n_i, 0} \hat{\phi}_{n_1}(p_1) \cdots \hat{\phi}_{n_l}(p_l) V^{(l)}(p_1, n_1; \ldots; p_l, n_l).
\]

Note the Kronecker \( \delta \) on the frequencies that ensures that vertices are nonvanishing only for \( \sum_i n_i = 0 \). In particular—this property will be important below—if only one frequency is nonvanishing, we have \( V^{(l)}(p_1, n; p_2, 0; \cdots; p_l, 0) = 0 \).
The vertices appearing in expansion (23) are easily computed. The fermion contribution is obtained by considering the one-loop fermionic graphs in theory (1). If we define the free fermion propagator

$$\Delta_F(p, n; m) \equiv \frac{-i \sum_{j=1}^{d} \gamma_j p_j - i \gamma_{d+1} \omega_n + m}{p^2 + \omega_n^2 + m^2}$$

(24)

with $\omega_n \equiv (2n + 1)\pi T$, the fermion contribution is

$$V_f^{(l)}(p_1, n_1; \cdots; p_l, n_l) =$$

$$(-1)^l \frac{T}{D} \sum_{a \in \mathbb{Z}} \int_{q < \Lambda} \frac{d^d q}{(2\pi)^d} \text{tr} \left[ \prod_{i=1}^{l} \Delta_F\left( q + \sum_{j=1}^{i} p_j; a + \sum_{j=1}^{i} n_j; m + M \right) \right]$$

+ permutations,

(25)

where the permutations make the vertex completely symmetric [there are $(l - 1)!$ terms]. Note that the frequencies $\omega_n$ never vanish and thus the vertices have a regular expansion in powers of $m + M$. Vertices $V_f^{(l)}$ satisfy an important symmetry relation. First, note that

$$\gamma_{d+1} \Delta_F(p, n; m) \gamma_{d+1} = -\Delta_F(p, -n - 1; -m).$$

(26)

It follows

$$\sum_{a \in \mathbb{Z}} \text{tr} \left[ \prod_{i=1}^{l} \Delta_F\left( q_i + a + b_i; m \right) \right] = \sum_{a \in \mathbb{Z}} \text{tr} \left[ \prod_{i=1}^{l} \gamma_{d+1} \Delta_F\left( q_i + a + b_i; m \right) \gamma_{d+1} \right] =$$

$$(-1)^l \sum_{a \in \mathbb{Z}} \text{tr} \left[ \prod_{i=1}^{l} \Delta_F\left( q_i - a - b_i - 1; -m \right) \right] = (-1)^l \sum_{a \in \mathbb{Z}} \text{tr} \left[ \prod_{i=1}^{l} \Delta_F\left( q_i - a - b_i; -m \right) \right].$$

(27)

In the second step we used $\gamma_{d+1}^2 = 1$, while in the last one we redefined $a \to -a + 1$. This relation implies (we write here explicitly the dependence of the vertices on $m$ and $M$)$^2$

$$V_f^{(l)}(p_1, n_1; \cdots; p_l, n_l; m + M) = (-1)^l V_f^{(l)}(p_1, -n_1; \cdots; p_l, -n_l; -m - M).$$

(28)

For every $l > 4$, the vertex is due only to the fermion loops, so that $V^{(l)} = V_f^{(l)}$. For $l \leq 4$ we must also take into account the contribution of the bosonic part of the action, so that

$$V^{(3)}(p, n_1; q, n_2; r, n_3) = \mathcal{L} m + V_f^{(3)}(p, n_1; q, n_2; r, n_3),$$

$$V^{(4)}(p, n_1; q, n_2; r, n_3; s, n_4) = \mathcal{L} + V_f^{(4)}(p, n_1; q, n_2; r, n_3; s, n_4).$$

(29)

Finally, for the inverse propagator we have

$$P(p, n) = \frac{p^2}{g^2} + \frac{(2\pi T)n^2}{g^2} + \mathcal{L} + \frac{\lambda}{2} n^2 + V_f^{(2)}(-p, -n; p, n).$$

(30)

$^2$If $d$ is odd, one can repeat the same argument using $\gamma_{d+2} = \prod_{i=1}^{d+1} \gamma_i$. It shows that vertices with $l$ legs are multiplied by $(-1)^l$ if one changes the sign of $m + M$ at fixed momenta and frequencies.
Vertices $V(0)$ also satisfy the symmetry relation $\hat{\phi}_n = 0$, while the inverse propagator $P(p, n)$ satisfies $P(p, n; m, M) = P(p, -n; -m, -M)$. Finally, note that $V(4)(0, 0; 0, 0; 0, 0; 0, 0)$ is positive at the transition. Indeed, one obtains explicitly (note that $\lambda \geq 0$ to ensure the stability of the quartic potential)

$$V(4)(0, 0; 0, 0; 0, 0; 0, 0) = \bar{\chi} + 6 T_c \sum_a \int_{q<\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \omega^2_{\varepsilon, n}} > 0. \quad (31)$$

It is easy to verify that $P(0, 0)$ vanishes at the transition. Indeed, for $m = M = 0$ we have

$$P(0, 0) = \bar{\mu} - T \sum_{n \in \mathbb{Z}} \int_{q<\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \omega^2_{n}}$$

$$= -g_{01}(T - T_c) + O(T - T_c)^2, \quad (32)$$

where we used Eqs. (13) and (11). Thus, the mode with $n = 0$ is singular. It is exactly this singularity that forbids a standard $1/N$ expansion at $T = T_c$ and gives rise to the Ising behavior. This type of singular behavior is completely analogous to that observed in Ref. [9].

The strategy proposed there consists in integrating all the nonsingular modes $\hat{\phi}_n$, $n \neq 0$, and study the effective theory for the zero mode $\hat{\phi}_0$.

Integrating all fields $\hat{\phi}_n$ with $n \neq 0$ we obtain the effective action

$$e^{-\mathcal{S}_{\text{eff}}[\hat{\phi}_0]} = \int \prod_{n \neq 0} d\hat{\phi}_n e^{-\mathcal{S}_{\text{int}}[\hat{\phi}_n]}, \quad (33)$$

with

$$\mathcal{S}_{\text{eff}} = \sqrt{N} H \hat{\phi}_0(0) + \frac{T}{2} \int_{q<\Lambda} \frac{d^d p}{(2\pi)^d} \hat{\phi}_0(p) \tilde{P}(p) \hat{\phi}_0(-p)$$

$$+ \sum_{l \geq 3} \frac{T^{l-1}}{l! N^{l/2 - 1}} \int_{p_i<\Lambda} \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_l}{(2\pi)^d} d\delta \left( \sum_{i=1}^l p_i \right)$$

$$\times \tilde{V}^{(l)}(p_1, \ldots, p_l) \hat{\phi}_0(p_1) \cdots \hat{\phi}_0(p_l)$$

where $H$, $\tilde{P}$, and $\tilde{V}^{(l)}$ have an expansion in powers of $1/N$. The computation of these quantities is quite simple. The contribution of order $1/N^k$ to $\tilde{V}^{(l)}$ is obtained by considering all $k$-loop diagrams contributing to the $l$-point connected correlation function of $\hat{\phi}_0$ and considering only the nonsingular fields (i.e. propagators with $n \neq 0$) on the internal lines. Frequency conservation implies that all tree-level diagrams with more than one vertex vanish.\(^3\) Therefore, $H = O(N^{-1})$, $\tilde{P}(p) = P(p, 0) + O(N^{-1})$, and $\tilde{V}^{(l)} = V^{(l)}_{n_i=0} + O(N^{-1})$ ($V^{(l)}_{n_i=0}$ is the vertex $V^{(l)}$ with all frequencies set to zero). For the inverse propagator $\tilde{P}$ and

\(^3\)For a tree-level graph, the usual topological arguments give the relation $\sum_n (n - 2) N_n = -2$, where $N_n$ is the number of vertices belonging to the graph such that $n$ legs belong to internal lines. Since $n \geq 1$ if there is more than one vertex, the previous equality requires $N_1 \geq 2$. But frequency conservation implies that $V_l$ vanishes if all frequencies but one vanish. Therefore, each nontrivial tree-level diagram vanishes.
for the magnetic field $\tilde{H}$ we shall also need the $1/N$ corrections. We obtain

$$\tilde{H} = \frac{H_1}{N} + O(N^{-2})$$  

$$\tilde{P}(\mathbf{p}) = P(\mathbf{p}, 0) + \frac{P^{(1)}(\mathbf{p})}{N} + O(N^{-2}),$$

with

$$H_1 = \frac{T}{2} \sum_{n \neq 0} \int_{p < \Lambda} \frac{d^d\mathbf{p}}{(2\pi)^d} V_3(\mathbf{p}, n) P(\mathbf{p}, n)^{-1}$$

$$P^{(1)}(0) = \frac{T}{2} \sum_{n \neq 0} \int_{p < \Lambda} \frac{d^d\mathbf{p}}{(2\pi)^d} \left[ V_4(\mathbf{p}, n) P(\mathbf{p}, n)^{-1} - V_3(\mathbf{p}, n)^2 P(\mathbf{p}, n)^{-2} \right],$$

where $V_1(\mathbf{p}, n) \equiv V^{(l)}(\mathbf{p}, n; -\mathbf{p}, -n; 0, 0; \ldots)$. Note that relation (28) implies an analogous symmetry relation for $\tilde{V}^{(l)}$:

$$\tilde{V}^{(l)}(\mathbf{p}_1, \ldots, \mathbf{p}_l; m, M) = (-1)^l \tilde{V}^{(l)}(\mathbf{p}_1, \ldots, \mathbf{p}_l; -m, -M),$$

where we have written explicitly the dependence on $m$ and $M$. Analogously $\tilde{P}(\mathbf{p})$ and $\tilde{H}$ are respectively symmetric and antisymmetric under $m, M \rightarrow -m, -M$.

In order to obtain the final effective theory we introduce a new field $\chi(\mathbf{p})$ such that the corresponding zero-momentum three-leg vertex vanishes for any value of the parameters. For this purpose we write

$$\alpha \chi(\mathbf{p}) = T \hat{\phi}_0(\mathbf{p}) + \sqrt{N} k \delta(\mathbf{p}),$$

where $\alpha$ and $k$ are functions to be determined. If we write $a_l \equiv \tilde{V}^{(l)}(0, 0; \ldots; 0, 0)$, $k$ is determined by the equation

$$\sum_{l=0} \frac{(-1)^l k(m, M)^l}{l!} a_{l+3}(m, M) = 0,$$

where we have written explicitly the dependence on $m$ and $M$. Now, symmetry (39) implies also

$$\sum_{l=0} \frac{(-1)^l [-k(-m, -M)^l]{l!}}{l!} a_{l+3}(m, M) = 0,$$

so that $k(m, M) = -k(-m, -M)$. Therefore, $k$ has an expansion of the form

$$k = \sum_{ab, a+b \text{ odd}} k_{ab} M^a M^b,$$

where the coefficients $k_{ab}$ have a regular expansion in powers of $1/N$. The leading behavior close to the transition is easily computed:

$$k = \frac{a_3}{a_4} + O(m^a M^b, a + b = 3).$$
In terms of \( \chi \) the effective action can be written as
\[
\mathcal{S}_{\text{eff}} = N^{1/2} \mathcal{H} \chi(0) + \frac{1}{2} \int_{q < \Lambda} \frac{d^d p}{(2\pi)^d} \chi(p) \mathcal{F}(p) \chi(-p)
\]
(45)
\[
+ \sum_{l \geq 3} \frac{1}{l! N^{l/2 - 1}} \int_{p < \Lambda} \frac{d^d p_1}{(2\pi)^d} \ldots \frac{d^d p_l}{(2\pi)^d} (2\pi)^d \delta \left( \sum_{i=1}^{l} p_i \right) \tilde{V}^{(l)}(p_1, \ldots, p_l) \chi(p_1) \cdot \cdots \chi(p_l).
\]

The quantities \( \mathcal{H}, \mathcal{F}, \) and \( \tilde{V}^{(l)} \) have an expansion in terms of \( m, M, \) and \( 1/N \). Explicitly we have:
\[
\mathcal{H} = \alpha T^{-1}[\tilde{H} - k \tilde{P}(0) + \frac{k^2}{2} \tilde{V}_3(0) - \frac{k^3}{6} \tilde{V}_4(0) + O(m^a M^b, a + b = 5)],
\]
(46)
\[
\mathcal{F}(p) = \alpha^2 T^{-1}[\tilde{P}(p) - k \tilde{V}_3(p) + \frac{k^2}{2} \tilde{V}_4(p) + O(m^a M^b, a + b = 4)],
\]
(47)
\[
\tilde{V}^{(2i+1)}(p_1, \ldots, p_{2i+1}) = \alpha^{2i+1} T^{-1}[\tilde{V}^{(2i+1)}(p_1, \ldots, p_{2i+1}) - k \tilde{V}^{(2i+2)}(p_1, \ldots, p_{2i+1}, 0)
+ O(m^a M^b, a + b = 3)],
\]
(48)
\[
\tilde{V}^{(2i)}(p_1, \ldots, p_{2i}) = \alpha^{2i} T^{-1} \tilde{V}^{(2i)}(p_1, \ldots, p_{2i}) + O(m^a M^b, a + b = 2).
\]
(49)

Up to now we have not defined the parameter \( \alpha \). We will fix it by requiring
\[
\left. \frac{d\mathcal{F}(p)}{dp^2} \right|_{p=0} = 1,
\]
(50)
for all values of the parameters. The parameter \( \alpha \) is a function of \( m, M, \) and \( 1/N \). The symmetry properties of \( k \) and of the vertices imply that \( \alpha \) is invariant under \( m, M \rightarrow -m, -M \). As a consequence, under \( m, M \rightarrow -m, -M \), the quantities \( \mathcal{H}, \mathcal{F}, \) and \( \tilde{V}^{(l)} \) have the same symmetry properties as \( \tilde{H}, \tilde{P}, \) and \( \tilde{V}^{(l)} \).

In the following we shall need the expansions of \( \mathcal{H}, \mathcal{F}(0), \) and \( \tilde{V}^{(3)}(p, -p, 0) \) close to the critical point. For this purpose we will use the expansions
\[
P(0, 0) \approx \tilde{\chi} m^2 - (T - T_c)g_{01} - 3(M + m)^2 g_{10},
\]
\[
V_3(0, 0) \approx \lambda m - 6(M + m) g_{10},
\]
\[
V_4(0, 0) \approx \lambda - 6 g_{10},
\]
(51)
\[
[V_n(p, 0) \equiv V^{(n)}(p, 0; -p, 0; 0, 0; 0, 0, \ldots)] \text{ and the relation}
\]
\[
V_f^{(3)}(p, n; -p, -n; 0, 0; m) = m V_f^{(4)}(p, n; -p, -n; 0, 0; 0, 0; m) + O(m^3),
\]
(52)
where we have explicitly written the mass dependence of the vertices. We expand \( \mathcal{H} \) and \( \mathcal{F}(0) \) is powers of \( 1/N \) as
\[
\left. \frac{T \mathcal{H}}{\alpha} \right|_{\alpha} = h_0 + \frac{h_1}{N} + O(N^{-2}),
\]
(53)
\[
\left. \frac{T \mathcal{F}(0)}{\alpha^2} \right|_{\alpha} = p_0 + \frac{p_1}{N} + O(N^{-2}).
\]
(54)
By using expansions \([51]\) we obtain

\[
\begin{align*}
    h_0 & \approx -\frac{V_3(0,0)}{V_4(0,0)} P(0,0) + \frac{1}{3} \frac{V_3(0,0)^3}{V_4(0,0)^2} \\
    & \approx -\mu M + \frac{g_{01}}{6 g_{10} - \lambda} M(T - T_c) - \frac{g_{10}}{6 g_{10} - \lambda} \lambda T^3 + O(m^a M^b, a + b = 5), \\
    h_1 & \approx H_1 - \frac{V_3(0,0)}{V_4(0,0)} P^{(1)}(0) \\
    & \approx -\frac{\lambda M T}{2(6 g_{10} - \lambda)} \sum_{n \neq 0} \int_{p < \Lambda} \frac{d^d p}{(2\pi)^d} [6 g_{10} + V_f^{(4)}(p, n; -p, -n; 0, 0; 0, 0)] P(p, n)^{-1} \\
    & + O(m^a M^b, a + b = 3), \\
    p_0 & \approx P(0, 0) - \frac{V_3(0,0)^2}{2 V_4(0,0)} \\
    & \approx -g_{01}(T - T_c) + \frac{3 g_{10}}{6 g_{10} - \lambda} M^2 + O(m^a M^b, a + b = 4), \\
    p_1 & \approx P^{(1)}(0, 0) \approx e + O(m^a M^b, a + b = 2),
\end{align*}
\]

where \(e\) is the value of \(P^{(1)}(0,0)\) for \(M = m = 0\). Note that several terms that are allowed by the symmetry \(m, M \rightarrow -m, -M\) are missing in these expansions. In the case of \(h_0\) we used the gap equation to eliminate the term proportional to \(m^3\). This substitution is responsible for the appearance of the term linear in \(M\) and cancels the terms proportional to \(m(T - T_c)\), \(m^2 M\), and \(m M^2\). In the case of \(h_1\) and \(p_0\) note that the terms proportional to \(m\), and \(m^2\), \(m M\) cancel out. Finally, we compute the three-leg vertex. At leading order in \(1/N\) we obtain

\[
\begin{align*}
    T \frac{\alpha^3}{\lambda} \nabla^{(3)}(p, -p, 0) = V_3(p, 0) - \frac{V_3(0,0)}{V_4(0,0)} V_4(p, 0) \\
    & \approx -\frac{\lambda M}{6 g_{10} - \lambda} [6 g_{10} + V_f^{(4)}(p, 0; -p, 0; 0, 0; 0, 0)] + O(m^a M^b, a + b = 3). \quad (56)
\end{align*}
\]

Note that the term proportional to \(m\) is missing as a consequence of relation \([52]\).

4 The critical crossover limit

The manipulations presented in the previous section allowed us to compute the effective action for the zero mode \(\chi(p)\). Far from the critical point \(\overline{P}(p) \neq 0\) for all momenta and thus one can perform a standard \(1/N_f\) expansion. At the critical point instead this expansion fails because \(\overline{P}(0) = 0\). At the critical point, for \(N \rightarrow \infty\) the long-distance behavior is controlled by the action

\[
\begin{align*}
    \tilde{S}_{\text{eff}} & \approx \int d^d x \left[ \frac{1}{2} (\partial \chi)^2 + \frac{u}{4!} \chi^4 \right] + O(N^{-2}), \quad (57)
\end{align*}
\]

where

\[
\begin{align*}
    u & = \frac{1}{N} \nabla^{(4)}(0, 0, 0, 0). \quad (58)
\end{align*}
\]
Here we have used the fact that vertices with an odd number of fields vanish at the critical point and the normalization condition (50). Moreover, since the critical mode corresponds to \( p = 0 \), we have performed an expansion in powers of the momenta, keeping only the leading term. Since for \( N \to \infty \), \( V^{(4)}(0, 0, 0, 0) = \alpha^4 T^{-1} V^{(4)}(0, 0, 0, 0) \), inequality (51) implies \( u > 0 \) at the critical point. Eq. (57) is the action of the critical \( \phi^4 \) theory which should be studied in the weak-coupling limit \( u \to 0 \). In this regime the model shows an interesting scaling behavior—we name it critical crossover—that describes the crossover between mean-field and Ising behavior. This allows us to obtain quantitative predictions for the critical-region suppression observed for \( N_f \to \infty \).

### 4.1 The general theory

In this section we wish to review some basic results on the critical crossover limit. An extensive discussion can be found in Refs. [10, 13, 9]. Let us first consider the standard \( \phi^4 \) theory in \( d \) dimensions with \( d < 4 \).

\[
S_{\text{cont}}[\varphi] = \int d^d r \left[ H \varphi + \frac{1}{2} (\partial \varphi)^2 + \frac{r}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right]. \tag{59}
\]

We assume the theory to be regularized with the introduction of a momentum cutoff. The results are however independent of the chosen regularization and one could equally well use a lattice regularization. Then, we define a new bosonic field \( \psi(s) \) as

\[
\psi(s) = u^{(d-2)/[2(4-d)]} \varphi(s u^{-1/(4-d)}). \tag{60}
\]

Formally, the action can be rewritten as

\[
S_{\text{cont}}[\psi] = \int d^d s \left[ \tilde{H} \psi + \frac{1}{2} (\partial \psi)^2 + \frac{\tilde{r}}{2} \psi^2 + \frac{1}{4!} \psi^4 \right], \tag{61}
\]

where

\[
\tilde{H} \equiv H u^{-(d+2)/[2(4-d)]}, \quad \tilde{r} \equiv r u^{-2/(4-d)}. \tag{62}
\]

Thus, formally, once the action is expressed in terms of \( \psi \), the bare parameters appear only in the combinations \( \tilde{H} \) and \( \tilde{r} \). Then, consider the zero-momentum connected correlation function \( \chi_n \). We have

\[
\chi_n \equiv \int d^d \mathbf{r}_2 \ldots d^d \mathbf{r}_n \langle \varphi(0) \varphi(\mathbf{r}_2) \ldots \varphi(\mathbf{r}_n) \rangle_{\text{conn}} = u^{[2d-n(2+d)]/[2(4-d)]} \int d^d s_2 \ldots d^d s_n \langle \psi(0) \psi(s_2) \ldots \psi(s_n) \rangle_{\text{conn}} = u^{[2d-n(2+d)]/[2(4-d)]} f_n(\tilde{H}, \tilde{r}), \tag{63}
\]

i.e. \( u^{-[2d-n(2+d)]/[2(4-d)]} \chi_n \) is a scaling function of \( \tilde{H} \) and \( \tilde{r} \). Analogously, one can determine the scaling behavior of the correlation length \( \xi \):

\[
\xi^2 = \frac{1}{2d \chi_2} \int d^d \mathbf{r} r^2 \langle \varphi(0) \varphi(\mathbf{r}) \rangle = u^{-2/(4-d)} f_\xi(\tilde{H}, \tilde{r}). \tag{64}
\]
The above-reported discussion is valid only at the formal level since we have not taken into account the presence of the cutoff that breaks scale invariance. For $d < 4$ only a mass renormalization (a redefinition of the parameter $r$) is needed in order to take care of divergencies. By a proper treatment one can show that there is a function $r_c(u)$ such that the correlation function $\chi_n$ and $\xi$ satisfy the scaling relations (63) and (64) with $\tilde{t} = tu^{-2/(4-d)}$, $t \equiv r - r_c(u)$, replacing $r$:

$$\chi_n = u^{2d-n(2+d)}/2^{4(d-d)} f_n(\tilde{H}, \tilde{t}) \quad \xi^2 = u^{-2/(4-d)} f(\tilde{H}, \tilde{t}).$$

(65)

The function $r_c(u)$ takes care of the ultraviolet divergent diagrams. In two dimensions, only the tadpole is primitively divergent and we have

$$r_c(u) = \frac{u}{8\pi} \ln u + Ku.$$  

(66)

In $d = 3$ divergences appear at one and two loops, so that

$$r_c(u) = -\frac{\Lambda}{4\pi^2}u + \frac{u^2}{96\pi^2} \ln u + Ku^2.$$  

(67)

In both cases the arbitrary constant $K$ can be chosen so that $t = 0$ corresponds to the critical point. The function $r_c(u)$ depends on the chosen regularization (the expressions we report above correspond to a sharp-cutoff regularization). On the other hand, the scaling functions $f(\tilde{H}, \tilde{t})$ are regularization-independent (universal) once a specific normalization for the fields, the coupling constant, and the scaling variables is chosen. They are the crossover functions that relate mean-field and Ising behavior. Consider, for instance, the case $H = 0$. For $t$ fixed and $u \to 0$ we obtain the standard perturbative expansion; thus, $\tilde{t} \to \infty$ corresponds to the mean-field limit. On the other hand, for $t \to 0$ at $u$ fixed, Ising behavior is obtained; $\tilde{t} = 0$ is the nonclassical limit. By varying $t$ between 0 and $\infty$ one obtains the full universal crossover behavior.

In Ref. [9] we extended these considerations to the general two-dimensional Hamiltonian

$$S_{\text{eff}}[\varphi] = H\varphi(0) + \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} [K(p) + r] \varphi(p)\varphi(-p)$$

$$+ \sum_{l \geq 3} \frac{u^{1/2-1}}{l!} \int \frac{d^2p_1}{(2\pi)^2} \cdots \frac{d^2p_l}{(2\pi)^2} 2^d \delta \left( \sum_i p_i \right) \mathcal{V}^{(l)}(p_1, \ldots, p_l) \varphi(p_1) \cdots \varphi(p_l),$$

where $K(p) = p^2 + O(p^4)$, $\mathcal{V}^{(3)}(0, 0, 0) = 0$, and $\mathcal{V}^{(4)}(0, 0, 0, 0) = 1$. The presence of vertices with an odd number of legs requires an additional counterterm for the magnetic field. Indeed, we showed that it was possible to find functions $r_c(u)$ and $H_c(u)$ such that for $t \equiv r - r_c(u)$ (infrared limit), $h \equiv H - H_c(u)$, $u \to 0$ (weak-coupling limit), at fixed $t/u$, $h/u$ one has

$$\chi_n = u^{1-n} f_n(\tilde{h}, \tilde{t}),$$

(69)

where the scaling function $f_n(x, y)$ is the same as that computed in the continuum theory. In particular, $\chi_n u^{n-1}$ vanishes in the crossover limit if $n$ is odd. The counterterms are regularization-dependent. In the continuum theory with a cutoff we have

$$h_c = -\frac{\sqrt{u}}{2} \int_{p < \Lambda} \frac{d^2p}{(2\pi)^2} \frac{\mathcal{V}_3(p)}{K(p)},$$

(70)

$$r_c = \frac{u}{8\pi} \ln u + \frac{u}{2} \int_{p < \Lambda} \frac{d^2p}{(2\pi)^2} \frac{\mathcal{V}_3(p)^2}{K(p)^2} + A_0u,$$
with \( \mathcal{V}_3(p) \equiv \mathcal{V}^{(3)}(p, -p, 0) \) and

\[
A_0 = -D_2 - \frac{3}{8\pi} + \frac{1}{\pi} \log \frac{3}{8\pi} - \frac{1}{2} \int_{p<\Lambda} \frac{d^2p}{(2\pi)^2} \left[ \frac{\mathcal{V}^{(4)}(p, -p, 0, 0)}{K(p)} - \frac{1}{p^2} \right].
\]

(72)

The nonperturbative constant \( D_2 \) was estimated in Ref. [13]: \( D_2 = -0.0524(2) \).

### 4.2 Scaling behavior

In this section we wish to use the previous results to compute the crossover behavior of model (1) in 2+1 dimensions. Since \( u \sim 1/N \) the relevant scaling variables are

\[
x_h = \frac{NT_c}{\alpha}(N^{1/2}H - H_c) \\
x_t = \frac{NT_c}{\alpha^2}(\mathcal{P}(0) - r_c),
\]

(73)

where the factors \( \alpha/T_c \) and \( \alpha^2/T_c \) are introduced for convenience. The critical crossover limit is obtained by tuning \( T, M \), and \( N \) close to the critical point so that \( x_h \) and \( x_t \) are kept constant. The expansions of \( \mathcal{H} \) and \( \mathcal{P}(0) \) are reported in Eq. (55). The expansions of \( H_c \) and \( r_c \) are easily derived. For \( H_c \) we have

\[
H_c = -\frac{1}{2\sqrt{N}} \int_{p<\Lambda} \frac{d^2p}{(2\pi)^2} \frac{\mathcal{V}^{(3)}(p, -p, 0)}{\mathcal{P}(p)},
\]

(74)

where all quantities are computed for \( M = m = 0 \). Using Eq. (56) we have

\[
H_c = \frac{\alpha h_{c0}}{T_c} + O(m^a M^b, a + b = 3),
\]

(75)

where \( h_{c0} \) is a constant. Using Eq. (71) we obtain for \( r_c \) the expansion

\[
r_c = \frac{\alpha^2}{T_c N} (r_0 \ln N + r_1) + O(m^a M^b, a + b = 2),
\]

(76)

where

\[
r_0 = -\frac{\alpha^2 V_4(0, 0)}{8\pi},
\]

(77)

\[
r_1 = \frac{\alpha^2 V_4(0, 0)}{8\pi} \left[ \frac{1}{8\pi} \ln \frac{3\alpha^4 V_4(0, 0)}{8\pi T_c A^2} - D_2 - \frac{3}{8\pi} \right] \\
- \frac{1}{2 \int_{p<\Lambda} \frac{d^2p}{(2\pi)^2} \left[ \frac{T_c V_4(p, 0)}{\mathcal{P}(p, 0)} - \frac{\alpha^2 V_4(0, 0)}{p^2} \right]}. \]

(78)

Note that the three-leg vertex that appears in Eq. (71) does not contribute to this order, since it vanishes for \( m = M = 0 \). Thus, Eqs. (73) can be written as

\[
x_h \approx N^{3/2} \left[ -\bar{\mu} M + a_0 M (T - T_c) + a_1 M^3 + \frac{a_2 M}{N} + \cdots \right] - h_{c0} M \sqrt{N},
\]

(79)

\[
x_t \approx N \left[ -g_{01} (T - T_c) + a_3 M^3 + \frac{c}{N} + \cdots \right] - r_0 \ln N - r_1,
\]

(80)
where \(a_0, a_1, a_2, a_3\), and \(e\) are coefficients that can be read from Eq. (55). These expansions show that

\[
\bar{\mu}M = -x_h N^{-3/2},
\]

\[
T - T_c = \frac{e - r_0 \ln N - r_1 - x_t}{N g_{01}}.
\]

(81)

The critical point is specified by the condition \(x_t = x_h = 0\). The symmetry under \(m, M \rightarrow -m, -M\) guarantees that the critical point corresponds to \(M = 0\). On the other hand, \(1/N\) fluctuations give rise to a shift of the critical temperature. If \(T_c(N)\) is the finite-\(N\) critical temperature, we obtain

\[
T_c(N) \approx T_c + \frac{1}{N} \frac{e - r_0 \ln N - r_1}{g_{01}}.
\]

(82)

Note that, beside the expected \(1/N\) correction there is also a \(\ln N/N\) term that is related to the nontrivial renormalization. It follows

\[
T - T_c(N) = -\frac{x_t}{N g_{01}}.
\]

(83)

Note that \(g_{01}\) is negative [see Eq. (14)] and thus we have \(x_t > 0\) for \(T > T_c(N)\), as expected. Using the gap equation we can also derive the behavior of \(m\) in the critical crossover limit. We obtain

\[
m \equiv \frac{m_0}{N^{1/2}},
\]

(84)

where \(m_0\) is a function of \(\ln N\) that satisfies the equation

\[
\frac{1}{6} (\lambda - 6 g_{10}) m_0^3 + (r_0 \ln N + r_1 - e + x_t) m_0 + x_h = 0.
\]

(85)

For \(N \rightarrow \infty\), \(m_0\) has an expansion in inverse powers of \(\ln N\), the leading term being

\[
m_0 \approx -\frac{x_h}{r_0 \ln N} + O(\ln^{-2} N).
\]

(86)

Note that \(m_0 \rightarrow 0\) as \(x_h \rightarrow 0\).

These results confirm the scaling predictions of Ref. [7]. For the massless theory with \(M = 0\), there are two regimes: for \(N(T - T_c(N)) \ll 1\) one observes Ising behavior, while for \(N(T - T_c(N)) \gg 1\) mean-field behavior occurs. If \(M \neq 0\) the same considerations apply, the relevant variable being \(MN^{3/2}\). It is important to note the role played in the derivation by the symmetry \(m, M \rightarrow -m, -M\), that is present because the regularization preserves chiral invariance. Even though vertices with an odd number of legs are present, the symmetry makes them irrelevant in the crossover limit. Thus, the additional renormalizations computed in Ref. [9] do not play any role here.

The results reported above can be extended to \(d\) dimensions for \(d < 4\). Eq. (62) implies that the relevant scaling variables are

\[
x_h \sim MN^{(d+2)/[2(4-d)]}, \quad x_t \sim [T - T_c(N)] N^{2/(4-d)}.
\]

(87)

In \(d = 3\), on the basis of Eq. (67), we also predict for \(T_c(N)\) an expansion of the form

\[
T_c(N) \approx T_c + \frac{a}{N} + \frac{b \ln N + c}{N^2},
\]

(88)

where \(a, b,\) and \(c\) are constants that can be computed as in the two-dimensional case.
4.3 Correlation functions

The results reported in Sec. 4.2 allow us to compute the scaling behavior of the correlation functions. For instance, we have

$$\langle \phi(x_d, x_{d+1}) \rangle = \bar{\phi} - \frac{k}{g} + \frac{\alpha}{g \sqrt{N}} \langle \chi(x_d) \rangle$$  \hspace{1cm} (90)

Using Eq. (65) with \( n = 1 \) and \( d = 2 \), we have \( \langle \chi(x_d) \rangle = f_1(x_h, x_t) \) in the critical crossover limit. The background term can be neglected in the crossover limit since

$$\bar{\phi} - \frac{k}{g} \approx \frac{1}{g} \left[ m - \frac{V_2(0,0)}{V_4(0,0)} \right] \approx \frac{M}{g} \frac{6g_{10}}{\lambda - 6g_{10}} \sim N^{-3/2}.$$  \hspace{1cm} (91)

Thus, we obtain

$$\langle \phi(x_d, x_{d+1}) \rangle \approx \frac{\alpha}{g \sqrt{N}} f_1(x_h, x_t)$$  \hspace{1cm} (92)

The factor \( 1/\sqrt{N} \) is related to the particular normalization of \( \phi \) used in [11] and disappears if we redefine \( \phi = \sqrt{N} \phi \) in order to have a canonical kinetic term for \( \phi \). The function \( f_1(x_h, x_t) \) is the scaling function for the magnetization in the Ising model. For instance, for \( x_h = 0 \) and \( x_t < 0 \) (low-temperature phase), we have \( f_1(0, x_t) \sim (-x_t)^{\beta_I} \) and \( f_1(0, x_t) \sim (-x_t)^{\beta_{MF}} \) respectively for \( |x_t| \ll 1 \) and \( |x_t| \gg 1 \), where \( \beta_I = 1/8 \) and \( \beta_{MF} = 1/2 \) are the magnetization exponents in the Ising and in the Gaussian model. The universality of the crossover allows us to compute the scaling functions in any other model in which there exists a crossover between the Gaussian and the Ising fixed point. In particular, we can use the results for systems with medium-range interactions [14, 18] (see also the appendix). In Ref. [18] (LBB) the authors report \( \langle |m| R \rangle \) versus \( t R^2 \) (see their Fig. 9), where \( m \) is the magnetization, \( t \) the reduced temperature, and \( R \) the effective interaction range. These results give us \( \langle \phi(x_d, x_{d+1}) \rangle \) for \( x_h = 0 \). One only needs to take into account the different normalizations of the fields, of the coupling constant, and of the scaling variable. In the crossover limit \( N \to \infty, T \to T_c(N) \) at fixed \( N(T - T_c(N)) \) we have

$$g \sqrt{N} \langle \phi(x_d, x_{d+1}) \rangle = K_{1,LBB} \langle |m|R \rangle_{LBB}$$  \hspace{1cm} (93)

$$\langle t R^2 \rangle_{LBB} = K_{LBB} N[T - T_c(N)].$$  \hspace{1cm} (94)

The nonuniversal constants \( K_{LBB} \) and \( K_{1,LBB} \) are computed in the Appendix.

It is customary to define an effective exponent \( \beta_{eff}(T) \) as

$$\beta_{eff}(T) = \frac{d}{dT} \ln \langle \phi(x_d, x_{d+1}) \rangle,$$  \hspace{1cm} (95)

for \( M = 0 \) and \( T < T_c(N) \). In the crossover limit \( T \to T_c(N) \), \( N \to \infty \) at fixed \( N[T - T_c(N)] \), the exponent \( \beta_{eff}(T) \) interpolates between the Ising value \( \beta_I = 1/8 \) and the mean-field \( \beta_{MF} = 1/2 \). Again, this effective exponent can be derived from the results of Ref. [18]. The curve reported in Fig. 15 of Ref. [18] gives \( \beta_{eff} \) in the Yukawa model once \( t R^2 \) is replaced by \( K_{LBB}[T - T_c(N)]N \).
The same considerations apply to the connected zero-momentum \( n \)-point function \( \chi_n \):

\[
\chi_n = \int \frac{\alpha^n}{T^{n-1}g^nN_n/2} \int d^d x_2 \ldots d^d x_n \langle \phi(0)\phi(x_2)\ldots\phi(x_n)\rangle^{\text{conn}}
\]

\[
= \frac{\alpha^n}{g^n N^{n/2} f_n(x_h, x_t)},
\]

For \( n = 2 \) the crossover function for \( x_h = 0 \) can be obtained from the results of Ref. \[18\], since \( g^2\chi_2 = K_{2,\text{LBB}}(\tilde{\chi}R^2)_{\text{LBB}} \). The constant \( K_{2,\text{LBB}} \) is given in the Appendix.

One can also use field theory to compute the crossover curves and thus use the results of Ref. \[13\]. For instance, in the high-temperature phase, for \( M = 0 \) we have in the crossover limit

\[
ge^2\chi_2 = K_{2,\text{FT}}^\phi F_\chi(\tilde{t}), \quad \tilde{t} = K_{\text{FT}}^\phi N[T - T_c(N)],
\]

where \( F_\chi(\tilde{t}) \) is reported in Ref. \[13\] and \( K_{\text{FT}}, K_{2,\text{FT}} \) are nonuniversal constants computed in the appendix.

In the discussion presented above we have focused on the case \( d = 2 \), but it is immediate to generalize all these considerations to the three-dimensional case. For \( d = 3 \) the universal crossover curves have been computed in Refs. \[10, 11, 13, 12\] (field theory) and in Ref. \[19\] (medium-range models). These results apply directly to the Yukawa model.

### 5 Conclusions

In this paper we have considered the Yukawa model in the limit \( N_f \to \infty \), focusing on the crossover between mean-field and Ising behavior. For this purpose we have determined the action of the mode that becomes critical at the transition. In the long-distance limit, it becomes equivalent to that of a weakly coupled \( \phi^4 \) theory. This identification allows us to use the results available for this model \[10, 9\] and, in particular, to identify a universal critical crossover occurring for \( N_f \to \infty, M \to 0, \) and \( T - T_c(N) \to 0 \) at fixed \( x_t \) and \( x_h \), see Eqs. \[81\], \[84\], \[88\]. In field-theoretical terms, this behavior represents the crossover induced by the flow from the unstable Gaussian fixed point to the stable Ising fixed point. Quantitative results for the Yukawa model can be obtained by using the field-theoretical results of Refs. \[10, 13, 12\], or the Monte Carlo results available for medium-ranged models \[18, 19\]. The necessary nonuniversal renormalization constants can be computed in perturbation theory. Results for \( d = 2 \) are reported in the appendix.

We should stress that our results are not specific of the chosen regularization, but can be extended to other regularizations as well. In particular, the extension to Kogut-Susskind fermions, the model considered in Ref. \[7\], is essentially straightforward. The Wilson case is more involved. Indeed, the absence of chiral symmetry implies that the symmetry relations satisfied here by the effective vertices [see Eq. \[23\]] are no longer valid. In turn, this may imply additional mixings as it happens in the generalized Heisenberg model \[9\].

Let us note that all calculations presented here refer to the model in infinite spatial volume. However, the crossover behavior can also be observed in the finite-size scaling limit. The discussion in Sec. \[4.1\] can be easily extended to this case too. It is trivial to
verify that the correct scaling variable is \( \tilde{L} = Lu^{1/(4-d)} \), i.e. \( \tilde{L} = LN^{-1/(4-d)} \) in the Yukawa model. Again, one can use universality and obtain predictions for the Yukawa model from the results obtained in other contexts. In particular, one can use the finite-size scaling results of Refs. [14, 18, 19] that refer to medium-range models at the critical point.

Finally, we should mention that one could also generalize the model and consider fermion fields \( \bar{\psi} \alpha_f \) transforming according to a representation of a group \( G \) and a coupling of the form \( \bar{\psi}_f T^a \phi^a \psi_f \), where \( T^a \) are the generators of the algebra of \( G \). The discussion is essentially unchanged, though in this case one would obtain the vector \( \phi^4 \) theory. Field-theory results relevant for this case are given in Refs. [10, 13, 12].

### A Relations among the Yukawa model, medium-range models, and field theory

In this appendix we relate the weakly coupled \( \phi^4 \) theory, medium-range models, and the Yukawa model for \( d = 2 \). For simplicity, we only consider the case \( H = 0 \), corresponding to \( M = 0 \) in the Yukawa model. The field-theory model has been discussed in Sec. 4.1, where it was shown that the \( n \)-point zero momentum connected correlation function \( \chi_{FT,n} \) shows a scaling behavior of the form

\[
\chi_{FT,n} = f_{FT,n}(\tilde{t}_{FT}) \quad \tilde{t}_{FT} \equiv (t - r_c(u))/u.
\]  

(98)

Next, we consider systems with medium-range interactions. Consider a square lattice, Ising spins \( \sigma_x \) at the sites of the lattice, and the Hamiltonian

\[
\mathcal{H} = -\frac{1}{2} \sum_{xy} J(x - y) \sigma_x \sigma_y.
\]  

(99)

We assume\(^4\) that \( J(x) = 1 \) for \( |x| \leq R_m \), \( J(x) = 0 \) for \( |x| > R_m \). The behavior of these models is very similar to that observed in the Yukawa model, \( R_m \) playing the role of \( N \). For any finite \( R_m \), the system belongs to the Ising universality class, while for \( R_m = \infty \) all spins are coupled together and one obtains mean-field behavior. In Ref. [14] it was shown that this model shows a crossover that interpolates between mean-field and Ising behavior. If one defines an effective interaction range \( R \) by

\[
R^2 = \frac{\sum_{x,y} (x - y)^2 J(x - y)}{\sum_{x,y} J(x - y)},
\]  

(100)

then for \( R, R_m \to \infty \), \( t \equiv (T - T_c(R))/T_c(R) \to 0 \) at fixed \( \tilde{t}_{MR} \equiv tR^2 \) one has

\[
R^{4-3n} \chi_{MR,n} = f_{MR,n}(\tilde{t}_{MR}),
\]  

(101)

where \( \chi_{MR,n} \) is the connected zero-momentum \( n \)-point correlation function of the fields \( \sigma \). In Ref. [13] it was shown that \( f_{MR,n}(x) \) and \( f_{FT,n}(x) \) are closely related. Indeed, we have

\[
f_{MR,n}(x) = \mu_{1,MR} \mu_{2,MR}^n f_{FT,n}(\lambda_{MR} x),
\]  

(102)

\(^4\)One can also consider a much more general class of medium-ranged models, see Ref. [13].
where \( \mu_{1,\text{MR}} \) and \( \lambda_{\text{MR}} \) are model-dependent constants that reflect the arbitrariness in the definitions of the fields, of the range \( R \), and of the scaling variable \( \tilde{t} \). The constants can be computed using the results of Ref. [13], Sec. 4.2.\(^5\) Explicitly, we have for the two-point and four-point zero-momentum connected correlation functions:

\[
\chi_{\text{MR},2} R^{-2} = \frac{1}{t_{\text{MR}}} + \frac{1}{4 \pi^2 t_{\text{MR}}^2} \left[ \ln \left( \frac{4 \pi \tilde{t}_{\text{MR}}}{3} \right) + 8 \pi D_2 + 3 \right] + O(\tilde{t}_{\text{MR}}^{-3}),
\]

\[
\chi_{\text{MR},4} R^{-8} = -\frac{2}{t_{\text{MR}}^2} + O(\tilde{t}_{\text{MR}}^{-5}).
\]

(103)

In the field-theory model we have instead

\[
u \chi_2 = \frac{1}{\tilde{t}} + \frac{1}{8 \pi \tilde{t}^2} \left[ \ln \frac{8 \pi \tilde{t}}{3} + 8 \pi D_2 + 3 \right] + O(\tilde{t}^{-3}).
\]

(104)

\[
u^2 \chi_4 = -\frac{1}{\tilde{t}^4} + O(\tilde{t}^{-5}).
\]

(105)

Comparing we obtain

\[
\mu_{1,\text{MR}} = 2, \quad \mu_{\text{MR}} = \lambda_{\text{MR}} = \frac{1}{2}.
\]

(106)

In Sec. 4 we have shown a similar relation for the Yukawa model. If \( x_t \equiv -g_{01} N[T - T_c(N)] \), we find for the zero-momentum correlation functions of the field \( \phi \)

\[
\tilde{\chi}_n = g^n \chi_n = N^{n/2-1} f_{Y,n}(x_t),
\]

(107)

and

\[
f_{Y,n}(x) = \mu_{1,Y} \mu_{2,Y} f_{\text{FT},n}(\lambda_Y x).
\]

(108)

In order to compute these constants we compare the one-loop expansions of the two-point function in field theory and in the Yukawa model. In the Yukawa model we find

\[
\int d^d x \langle \chi(0) \chi(x) \rangle = \frac{NT}{\alpha^2 x_t} - \left( \frac{NT}{\alpha^2 x_t} \right)^2 r_c
\]

\[-\frac{1}{2N} \left( \frac{NT}{\alpha^2 x_t} \right)^2 \int_{p<\Lambda} \frac{d^2 p}{(2\pi)^2} \frac{V^{(4)}(p, -p, 0, 0)}{P(p)} + O(x_t^{-3}).
\]

(109)

Using the explicit expression for \( r_c \) we obtain

\[
\tilde{\chi}_2 = \frac{1}{x_t} + \frac{\alpha^2 V_4(0, 0)}{8 \pi x_t^2} \left[ \ln \frac{8 \pi x_t}{3 \alpha^2 V_4(0, 0)} + 8 \pi D_2 + 3 \right] + O(x_t^{-3}).
\]

(110)

For the four-point function we have instead

\[
\tilde{\chi}_4 N^{-1} = -\frac{V_4(0, 0)}{x_t^4} + O(x_t^{-5}).
\]

(111)

\(^5\)Note that the function \( f_{\Phi}(\tilde{t}) \) defined in Ref. [13] refers to correlations of the fields \( \phi \) and not of the original fields \( \varphi \). However, relation (4.12) of Ref. [13] shows that in the critical crossover limit \( \sum_{x} \langle \varphi_0 \varphi_x \rangle \approx \sum_{x} \langle \phi_0 \phi_x \rangle \).

The same holds for \( \chi_4 \). The expression reported here are obtained from those reported in Ref. [13] by setting \( \pi_2 = 1, \pi_4 = -2, N = 1, c_0 = \hat{c}_0 = \tau, \) and \( \tilde{t}_{\text{MR}} = \tilde{t} + \hat{c}_0. \)
Comparing we obtain

\[ \mu_{1,Y} = \alpha^4 V_4(0,0), \quad \mu_{2,Y} = \frac{1}{\alpha^3 V_4(0,0)}, \quad \lambda_Y = \frac{1}{\alpha^2 V_4(0,0)}. \]  (112)

The constants reported in Sec. 4.3 are easily derived:

\[ K_{\text{FT}} = -\lambda_Y g_{01}, \quad K_{2,\text{FT}} = \mu_{1,Y} \mu_{2,Y}^2, \]  (113)

\[ K_{\text{LBB}} = -\frac{\lambda_Y g_{01}}{\lambda_{\text{MR}}}, \quad K_{n,\text{LBB}} = \frac{\mu_{1,Y} \mu_{2,Y}^n}{\mu_{1,\text{MR}} \mu_{2,\text{MR}}^n}, \]  (114)

where \( n = 1, 2 \). Note that \( g_{01} \) is negative, so that \( K_{\text{FT}} \) and \( K_{\text{LBB}} \) are positive as expected.

References

[1] R.D. Pisarski and F. Wilczek, Phys. Rev. D 29 (1984) 338.

[2] A. Butti, A. Pelissetto, and E. Vicari, J. High Energy Phys. 08 (2003) 029 [hep-ph/0307036]; F. Basile, A. Pelissetto, and E. Vicari, Proceedings of the XXIII International Symposium on Lattice Field Theory, Dublin, July 2005, Proc. of Science (LAT2005) 199 [hep-lat/0509018].

[3] F. Karsch, Lectures on Quark Matter, Proceedings of the 40 Internationale Universitätswochen für Kern und Teilchenphysik, Lecture Notes in Physics 583, edited by W. Plessas and L. Mathelisch (Springer, Berlin, 2002) hep-lat/0106019.

[4] F. Karsch and E. Laermann, Thermodynamics and in-medium hadron properties from lattice QCD, in Quark-Gluon Plasma 3, edited by R.C. Hwa (World Scientific, Singapore, 2003) hep-lat/0305025.

[5] B. Rosenstein, A.D. Speliotopoulos and H.L. Yu, Phys. Rev. D 49 (1994) 6822.

[6] M. Moshe and J. Zinn-Justin, Phys. Rept. 385 (2003) 69 hep-th/0306133.

[7] J.B. Kogut, M.A. Stephanov, and C.G. Strouthos, Phys. Rev. D 58 (1998) 096001 hep-lat/9805023.

[8] S. Caracciolo and A. Pelissetto, Phys. Rev. E 66 (2002) 016120 cond-mat/0202506.

[9] S. Caracciolo, B.M. Mognetti, and A. Pelissetto, Nucl. Phys. B 707 (2005) 458 cond-mat/0409536.

[10] C. Bagnuls and C. Berviller, Phys. Rev. B 32 (1985) 7209.

[11] C. Bagnuls, C. Berviller, D.I. Meiron, and B.G. Nickel, Phys. Rev. B 35 (1987) 3585; (Erratum) B 65 (2002) 149901 hep-th/0006187.

[12] C. Bagnuls and C. Berviller, Phys. Rev. E 65 (2002) 066132 [hep-th/011220].
[13] A. Pelissetto, P. Rossi, and E. Vicari, Nucl. Phys. B 554 (1999) 552 [cond-mat/9903410].

[14] E. Luijten, H.W.J. Blöte, and K. Binder, Phys. Rev. E 54 (1996) 4626 [cond-mat/9607019].

[15] A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. E 58 (1998) 7146 [cond-mat/9804264].

[16] S. Caracciolo, M.S. Causo, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. E 64 (2001) 046130 [cond-mat/0105160].

[17] A. Pelissetto and E. Vicari, Phys. Rep. 368 (2002) 549 [cond-mat/0012164].

[18] E. Luijten, H.W.J. Blöte, and K. Binder, Phys. Rev. E 56 (1997) 6540 [cond-mat/9706257].

[19] E. Luijten and K. Binder, Phys. Rev. E 58 (1998) 4060(R) [cond-mat/9807415]; 59 (1999) 7254.

[20] B. Bringoltz, hep-lat/0511058

[21] S. Caracciolo, B.M. Mognetti, and A. Pelissetto, Proceedings of the XXIII International Symposium on Lattice Field Theory, Dublin, July 2005, Proc. of Science (LAT2005) 187 hep-lat/0509063.