INTERSECTION DENSITY OF TRANSITIVE GROUPS WITH CYCLIC POINT STABILIZERS

Ademir Hujdurović\textsuperscript{a,h,n,*}, István Kovács\textsuperscript{a,b} Klavdija Kutnar\textsuperscript{a,b} and Dragan Marušič\textsuperscript{a,b,c}

\textsuperscript{a}University of Primorska, UP IAM, Muzejski trg 2, 6000 Koper, Slovenia
\textsuperscript{b}University of Primorska, UP FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
\textsuperscript{c}IMFM, Jadranska 19, 1000 Ljubljana, Slovenia

Abstract

For a permutation group $G$ acting on a set $V$, a subset $F$ of $G$ is said to be an intersecting set if for every pair of elements $g, h \in F$ there exists $v \in V$ such that $g(v) = h(v)$. The intersection density $\rho(G)$ of a transitive permutation group $G$ is the maximum value of the quotient $|F|/|G_v|$ where $G_v$ is a stabilizer of a point $v \in V$ and $F$ runs over all intersecting sets in $G$. If $G_v$ is a largest intersecting set in $G$ then $G$ is said to have the Erdős-Ko-Rado (EKR)-property. This paper is devoted to the study of transitive permutation groups, with point stabilizers of prime order with a special emphasis given to orders 2 and 3, which do not have the EKR-property. Among other, constructions of infinite family of transitive permutation groups having point stabilizer of order 3 with intersection density $4/3$ and of infinite families of transitive permutation groups having point stabilizer of order 3 with arbitrarily large intersection density are given.

Keywords: intersection density, transitive permutation group, derangement graph.

Math. Subj. Class.: 05C25, 20B25.

1 Introductory remarks

The Erdős-Ko-Rado theorem \cite{3}, one of the central results in extremal combinatorics, which gives a bound on the size of a family of intersecting $k$-subsets of a set and classifies the families satisfying the bound, has been extended in various ways. This paper is concerned with an extension of this theorem to the ambient of transitive permutation groups.

For a finite set $V$ let $\text{Sym}(V)$ denote the corresponding symmetric group, and if $|V| = n$, the notation $S_n$ will be adopted. Given a permutation group $G \leq \text{Sym}(V)$, a subset $F$ of $G$ is called intersecting if, for any two $g, h \in F$, there exists $v \in V$ such that $g(v) = h(v)$. The intersection density $\rho(F)$ of the intersecting set $F$ is defined to be the quotient

$$\rho(F) = \frac{|F|}{\max\{|G_v| : v \in V\}}.$$
where \( G_v \) is the point stabilizer of \( v \in V \), and the intersection density \( \rho(G) \) (see [8]) of a group \( G \), is the maximum value of \( \rho(F) \) where \( F \) runs over all intersecting sets in \( G \). Clearly, every coset \( gG_v, v \in V \) and \( g \in G \), is an intersecting set, referred to as a canonical intersecting set. Consequently, \( \rho(G) \geq 1 \). We say that \( G \) has the Erdős-Ko-Rado property (in short EKR-property), if the size of a maximum intersecting set is equal to the order of the largest point stabilizer, and is said to have the strict Erdős-Ko-Rado property (in short strict-EKR-property) if every maximum intersecting set of \( G \) is a coset of a point stabilizer. It is clear that strict-EKR-property implies EKR-property, but the converse does not hold. In particular, for a transitive group \( G \) it follows that \( \rho(G) = 1 \) if and only if the maximum cardinality of an intersecting set is \(|G|/|V|\). Note that if \( F \) is an intersecting set of \( G \) and \( f \in F \) then \( f^{-1}F \) is an intersecting of \( G \) (see [4, Proposition 2.3]). Therefore without loss of generality one can consider those intersecting sets containing the identity. We will refer to such sets as basic intersecting sets.

The investigation of the EKR-property of transitive permutation groups and related concepts is an active topic of research (see [4, 5, 6, 8, 13, 14, 16, 15, 18, 19]). This paper initiates a program aimed at obtaining deeper understanding of transitive permutation groups, not having the EKR-property, with small point stabilizers. As a starting point we consider groups with cyclic stabilizers of prime order, in particular those isomorphic to \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \). We remark that in this context the question on whether their orbitals are connected and self-paired or not is essential, see Section 2.

The paper is organized as follows. In Section 2 we gather basic definitions and fix the notation. Section 3 deals with groups having point stabilizers of order 2 while Section 4 deals with solvable groups having point stabilizers of odd prime order. In Section 5 some special results on intersecting sets in groups with point stabilizer of order 3 are proved. In Section 6 a construction of an infinite family of groups with point stabilizer of order 3 and intersection density 4/3 is given. Finally, in Section 7 infinite families of groups with point stabilizer of order 3 and arbitrarily large intersection density are constructed.

## 2 Preliminaries

### 2.1 Permutation groups and orbital digraphs

Let \( G \leq \text{Sym}(V) \). The orbitals of \( G \) are the orbits of \( V \times V \) under the canonical action of \( G \) on \( V \times V \). The set \( \Delta^* := \{(u,v) : (v,u) \in \Delta \} \) is also an orbital, and in the case when \( \Delta = \Delta^* \), \( \Delta \) is called self-paired. Supppose that \( G \) is transitive. Then the set \( \{(v,v) : v \in V \} \) is an orbital, and any orbital distinct from this is called non-trivial. For a non-trivial orbital \( \Delta \), let \( \Gamma_\Delta \) be the digraph with vertex set \( V \) and arc set \( \Delta \), that is, the so-called orbital digraph \( \Gamma_\Delta = \Gamma(G, \Delta) \) of \( G \) relative to \( \Delta \). We say that \( \Delta \) is connected if so is the associated digraph \( \Gamma_\Delta \), that is, for any two vertices \( u \) and \( v \), there is a sequence of vertices \( u = w_1, w_2, \ldots, w_k = v \) such that for every \( 1 \leq i \leq k-1 \), \( (w_i, w_{i+1}) \in \Delta \) or \( (w_{i+1}, w_i) \in \Delta \). Using a standard terminology, for a prime \( p \) a \( p \)-element is an element of order a power of \( p \).

**Proposition 2.1** Let \( G \leq \text{Sym}(V) \) be a transitive permutation group and let \( \Delta \) be a non-trivial orbital containing the pair \( (u,v) \). Then

1. \( \Delta \) is self-paired if and only if there exists a 2-element \( g \in G \) such that \( g(u) = v \) and \( g^2 \in G_u \).
2. \( \Delta \) is connected if and only if \( G = \langle G_u, g \rangle \) for every \( g \in G \) with \( g(u) = v \).
Let $H$ be a point stabilizer and $\Delta$ an orbital of a transitive permutation group $G$. Then it is well-known that the orbital digraph $\Gamma_\Delta = \Gamma (G, \Delta)$ corresponds to the so-called double coset graph $\text{Cos}(G, H, HgH)$, where $g \in G$ such that $(u, u^g) \in \Delta$ and $H = G_u$. In general, for a subset $S \subseteq G$, the double coset graph is defined to have vertex set $G/H$ (the set of left cosets of $H$ in $G$) and arc set consisting of pairs $(xH, yH)$, $x, y \in G$ and $x^{-1}y \in HSH$.

We end this subsection with a result about quotient graphs of arc-transitive graphs which will be needed in Section 4. For terms not defined here we refer the reader to [11] (see also [9]).

**Proposition 2.2** [11] Lemma 2.5 Let $\Gamma$ be a connected $G$-arc-transitive graph of valency $p$ for an odd prime $p$, and let $N \triangleleft \text{Aut}(\Gamma)$ be a semiregular subgroup with at least 3 orbits. Then $\Gamma_N$ is a connected $\bar{G}$-arc-transitive graph of valency $p$, where $\bar{G}$ is the image of $G$ under its action on $V(\Gamma_N)$.

### 2.2 Derangement graphs

Let $G \leq \text{Sym}(V)$. The fixed-point-free elements of $G$ are also called *derangements*. Following [16], the *derangement graph* of $G$ is defined to be the Cayley graph $\Gamma_G = \text{Cay}(G, D)$ with vertex set $G$ and edge set consisting of all pairs $(g, h) \in G \times G$ such that $g^{-1}h \in D$, where $D$ is the set of all derangements of $G$. Clearly, $D$ is closed under conjugation by elements of $G$, which shows that $\text{Inn}(G) \leq \text{Aut}(\Gamma_G)$, where $\text{Inn}(G)$ denotes the group of all *inner automorphisms* of $G$. The lemma below gives an additional kind of automorphisms of a derangement graph.

**Lemma 2.3** For every $G \leq \text{Sym}(V)$, the inverse map $\iota : G \to G$ defined by $\iota(g) = g^{-1}$ ($g \in G$) is an automorphism of the derangement graph $\Gamma_G$.

**Proof.** Suppose that $x$ and $y$ are two non-adjacent vertices of $\Gamma_G$. Then there exists points $v, u \in V$ such that $x(v) = u = y(v)$. Clearly, $x^{-1}(u) = v = y^{-1}(u)$, implying that $x^{-1}$ and $y^{-1}$ are also non-adjacent in $\Gamma_G$. 

In the terminology of derangement graphs, an intersecting set of a transitive permutation group $G$ is an independent set of $\Gamma_G$, or equivalently, a clique of its complement $\overline{\Gamma_G}$. Therefore, $\rho(G) = \omega(\overline{\Gamma_G})/|G_v|$, where the *clique number* $\omega(\overline{\Gamma_G})$ is the size of a maximum clique of $\overline{\Gamma_G}$. In the special case with stabilizers of order 2 or 3, the complement of the derangement graph is arc-transitive.

**Lemma 2.4** Let $G$ be a transitive group with point stabilizers of order 2 or 3. Then the complement $\overline{\Gamma_G}$ is arc-transitive.

**Proof.** Observe that $\overline{\Gamma_G} = \text{Cay}(G, S)$ where $S$ is the set of all non-identity elements of $G$ that fix a point. As $G$ is transitive, its point stabilizers are conjugate. Thus $S = C \cup C^{-1}$, where $C$ is the conjugacy class of some $g \in G$ fixing a point. Using the fact that $\text{Inn}(G) \leq \text{Aut}(\overline{\Gamma_G})$ and that $\iota \in \text{Aut}(\overline{\Gamma_G})$, we obtain that the stabilizer of the identity element $1 = 1_G$ of $G$ in $\text{Aut}(\overline{\Gamma_G})$ acts transitively on $S$. Hence $\overline{\Gamma_G}$ is arc-transitive.

For a prime power $q$ with $q \equiv 1 \pmod{4}$, the *Paley graph* $P_q$ is defined to be the Cayley graph of $\mathbb{F}_q^\times$, the additive group of the finite field $\mathbb{F}_q$ with $q$ elements, whose connection set consists of all non-zero squares in $\mathbb{F}_q$.

**Proposition 2.5** [2] Theorem 1] Let $q = p^{2n}$ where $p$ is a prime such that $q \equiv 1 \pmod{4}$. Then $\omega(P_q) = \sqrt{q}$. 

3
2.3 Intersection density

In this subsection we list four results from [4].

Proposition 2.6 ([4, Proposition 2.6]) Let $G$ be a transitive permutation group containing a semiregular subgroup $H$ with $k$ orbits. Then $\rho(G) \leq k$. In particular, if $H$ is regular then $\rho(G) = 1$.

Proposition 2.7 ([4, Proposition 3.1]) Let $G$ be a transitive permutation group admitting a semiregular subgroup $H$ whose orbits form a $G$-invariant partition $\mathcal{B}$, and let $\bar{G}$ be the image of $G$ under its action on $\mathcal{B}$. Then $\rho(G) \leq \rho(\bar{G})$.

Proposition 2.8 ([4, Theorem 1.4]) For a transitive permutation group $G$ of prime power degree the intersection density $\rho(G)$ is equal to 1.

We end this subsection with a lemma that can be extracted from the proof of [4, Theorem 1.5]

Lemma 2.9 ([4]) Let $p$ be a prime and let $G$ be a transitive permutation group of degree $2p$. If $G$ admits $G$-invariant partition $\mathcal{B}$ consisting of two blocks then $\rho(G) = 1$.

2.4 Properties of $\text{PSL}(2,q)$

We will need two well-known results about the group $\text{PSL}(2,q)$. The first one can be easily derived from the structure of subgroups of $\text{PSL}(2,q)$ (see, for example, [20]), so we omit the proof. For the second one we refer the reader to [12].

Proposition 2.10 Let $q = 3^n$ and $n > 1$. Then $\text{PSL}(2,q)$ has one conjugacy class of subgroups of order 3 if $n$ is odd, and two otherwise.

Proposition 2.11 ([12]) Unless $q = 9$ the group $\text{PSL}(2,q)$ is generated by an element of order 2 and an element of order 3.

Finally, it is known that if $F$ is an algebraic closed field of characteristic $p$ and $A \in \text{SL}(2,F)$ such that $A$ is not in the center $Z = Z(\text{SL}(2,F))$, then $A^3 \in Z$ if and only if $\text{Tr}(A)$ is either 1 or $-1$ (see [20] (6.19) in Chapter 3)). This implies the following statement.

Proposition 2.12 Let $A \in \text{SL}(2,q)$ be a matrix such that $A \neq \pm I$. Then $A^3 = \pm I$ if and only if its trace $\text{Tr}(A)$ is either 1 or $-1$.

Proposition 2.13 Let $q = p^k$ for a prime $p$, $p \neq 3$. Further let $G = \text{PSL}(2,q)$ be considered in its transitive action on the cosets of a subgroup isomorphic to $\mathbb{Z}_3$. Then a non-canonical basic intersecting set in $G$ contains no point stabilizer as a proper subset.

Proof. Let us assume that there is an intersecting set $\mathcal{F}$ in $G$ containing $I$, $B$, $B^2$ and $C$, where $B$ and $C$ are two elements of order 3 in $G$ such that $C \notin \{B, B^2\}$.

Consider the above two elements $B$ and $C$ in $G$ as ‘matrices’, that is elements in $\text{SL}(2,q)$. Then the minimal polynomial $m_B(x)$ of $B$ is either $x^2 + x + 1$ or $x^3 - x + 1$, depending on whether $B^3 = I$ or $B^3 = -I$ in $\text{SL}(2,q)$. If $m_B(x) = x^2 + x + 1$ consider the product $C(I + B + B^2)$. We have

$$0 = C \cdot 0 = C(I + B + B^2) = CI + CB + CB^2.$$
Consequently, the sum of traces $\text{Tr}(CI)$, $\text{Tr}(CB)$ and $\text{Tr}(CB^2)$ equals zero. However, by our assumption each of $C$, $CB$ and $CB^2$ is an element of order 3, and therefore its trace (as an ‘element’ of $\text{SL}(2,q)$) is either 1 or $-1$. Therefore, since $p \neq 3$, the sum of these three traces cannot be equal to zero. The argument in case when $m_B(x) = x^2 - x + 1$ is analogous. We just have to consider the expression $C(I - B + B^2)$, completing the proof of Proposition 2.13.

### 3 Groups with point stabilizers of order 2

**Lemma 3.1** Let $G$ be a transitive group with point stabilizers of order 2, and let $\mathcal{F}$ be a basic intersecting set. Then $(\mathcal{F})$ is an elementary abelian 2-group.

**Proof.** Let $x, y \in \mathcal{F}$ be distinct and non-identity. Since $x$ and $y$ are intersecting with the identity, it follows that each fixes a point, hence has order 2. Moreover, since $x$ and $y$ are intersecting, it follows that $x^{-1}y = xy$ is also of order 2, which implies that $x$ and $y$ commute.

In the next proposition we provide some sufficient condition for the EKR property of a transitive group with point stabilizers of order 2.

**Proposition 3.2** A transitive group $G$ with point stabilizers of order 2 has the EKR property if

(i) $G$ has a cyclic Sylow 2-subgroup, or

(ii) $G$ is a nilpotent group.

**Proof.** Part (i) is a direct consequence of Lemma 3.1.

To show part (ii) we proceed by induction on the number of prime divisors of $|G|$. If $G$ is a 2-group, then the statement follows from Proposition 2.8. Assume that $|G|$ is divisible by an odd prime $p$, and let $P$ be a Sylow $p$-subgroup of $G$. Then $P \lhd G$ because $G$ is nilpotent. As $P$ is also semiregular, Proposition 2.7 can be applied with $B$ being the set of $P$-orbits. It follows that $\rho(G) \leq \rho(\bar{G})$, where $\bar{G}$ is the image of $G$ under its action on $B$. Moreover, $G$ is transitive with point stabilizers of order 2, it is also nilpotent with the smaller number of prime divisors than $|G|$. Hence the induction hypothesis yields $\rho(\bar{G}) = 1$, and (ii) follows.

It can be deduced from Lemma 3.1 that a maximal intersecting set of transitive group with point stabilizer of order 2 cannot have size 3. Namely, if $\mathcal{F} = \{1, x, y\}$ is an intersection set, then it is easy to see that so is $\{1, x, y, xy\}$. Nonetheless, the example below shows that maximum intersecting sets in transitive groups with stabilizers of order 2 can have any size $d \geq 2$ and $d \neq 3$.

**Example 3.3** Let $G = E \rtimes Q$ be the semidirect product of the group $E = \langle e_1, \ldots, e_n \rangle \cong \mathbb{Z}_2^n$ with the group $Q = \langle a, b \rangle$, where the action of $a$ and $b$ by conjugation on $E$ is defined by

$$e_i^a = \begin{cases} e_{i+1} & \text{if } 1 \leq i < n - 1, \\ e_1 & \text{if } i = n. \end{cases} \quad e_i^b = \begin{cases} e_{i+1} & \text{if } 1 \leq i < n - 1, \\ e_n & \text{if } i = n. \end{cases}$$

Consider the permutation group induced by the action of $G$ on the cosets of $\langle e_1 \rangle$.

It is easy to check that the conjugacy class of $G$ containing $e_1$ is equal to the set $C := \{e_i, e_ie_j : 1 \leq i, j \leq n\}$. This shows that the set $\{1, e_1, \ldots, e_n\}$ is an intersecting set. We prove that, if $\mathcal{F}$
is an arbitrary intersecting set, then $|\mathcal{F}| \leq n + 1$. We may assume without loss of generality that $1 \in \mathcal{F}$, hence $\mathcal{F} \setminus \{1\} \subset C$. Write

$$\mathcal{F}_1 = \mathcal{F} \cap \{e_i : 1 \leq i \leq n\} \text{ and } \mathcal{F}_2 = \mathcal{F} \cap \{e_i e_j : 1 \leq i, j \leq n\}.$$ 

Note that, from the set theoretic point of view the sets $\{i, j\}, e_i e_j \in \mathcal{F}$, form a classical intersecting family of $\{1, \ldots, n\}$, hence $|\mathcal{F}_2| \leq n - 1$. On the other hand, it is easy to see that $|\mathcal{F}_1| \leq 1$ if $|\mathcal{F}_2| \geq 2$. All these yield that $|\mathcal{F}| \leq n + 1$.

We remark that the group $G$ above admits no connected orbitals. If that was not the case then $G = \langle e_1, g \rangle$ would hold for some element $g \in G$ due to Proposition 2.1(ii). But then $G = \langle E, g \rangle = E \rtimes \langle g \rangle$, so $Q$ is a cyclic group, contradicting the fact that $ab \neq ba$.

We conclude this subsection with a connection to character theory. In fact, the EKR property of a given transitive group $G$ with stabilizers of order 2 can be read off the character table of $G$.

**Proposition 3.4** Let $G$ be a transitive group with point stabilizers of order 2. Then $G$ has the EKR property if and only if

$$\sum_{\chi} \frac{\chi(g)^3}{\chi(1)} = 0,$$

where $\chi$ runs though the set of all irreducible characters of $G$, and $g$ is any non-identity element of $G$ fixing a point.

**Proof.** Let $C_1, \ldots, C_r$ be the conjugacy classes of $G$ and let $a_{ijk}$ be the class algebra constants of $G$, that is, for $1 \leq i, j, k \leq r$,

$$C_i C_j = \sum_{k=1}^{r} a_{ijk} C_k$$

holds in the group algebra $\mathbb{C}G$ (see [10, Chapter 30]). For a subset $X \subseteq G$, $X$ stands for the element $\sum_{g \in G} a_g g$ with $a_g = 1$ if $g \in X$, and $a_g = 0$ otherwise. Then we have

$$a_{ijk} = \frac{|G|}{|C_G(g_i)||C_G(g_j)||C_G(g_k)|} \sum_{\chi} \frac{\chi(g_i) \chi(g_j) \chi(g_k)}{\chi(1)},$$

(1)

where $g_i \in C_i, g_j \in C_j$ and $g_k \in C_k$, and $\chi$ runs through the set of all irreducible characters of $G$ (see [10, Theorem 30.4]). Now, take $g_1$ to be $g$. It is clear that having the EKR property is the same as requiring $a_{111} = 0$, so the proposition follows directly from (1).

4 Solvable groups with point stabilizers of odd prime order

We start by observing that solvable transitive permutation groups with point stabilizers of prime order and admitting a connected self-paired orbital have the EKR property. This is a consequence of the following more general result formulated in a graph-theoretic language.

**Proposition 4.1** Let $p$ be a prime, $\Gamma$ be a connected $p$-valent arc-transitive graph, and let $G \leq Aut(\Gamma)$ be a solvable group acting transitively on arcs of $\Gamma$. Then $G$ has the EKR property.
Proof. Assume first that \( p = 2 \). Then \( \Gamma \) is a cycle, and so \( G \) contains a regular subgroup, and thus Proposition 2.6 implies that \( G \) has the EKR property.

Assume from now on that \( p \) is an odd prime. Assume on the contrary that \( \rho(G) > 1 \) with \( \Gamma \) being the smallest counterexample. In other words, if \( \bar{\Gamma} \) is a connected \( p \)-valent \( \bar{G} \)-arc-transitive graph for some solvable group \( \bar{G} \) and \( |V(\bar{\Gamma})| < |V(\Gamma)| \), then \( \rho(\bar{G}) = 1 \).

Let \( N \) be a minimal normal subgroup of \( G \). Then \( N \cong \mathbb{Z}_r^l \) for some prime \( r \) and \( l \geq 1 \). If \( N \) is transitive, then it is regular, but this contradicts the fact that \( \rho(G) > 1 \) in view of Proposition 2.6.

If \( N \) has more than 2 orbits, by Proposition 2.2, \( \Gamma_N \) is a connected \( p \)-valent \( G \)-arc-transitive graph with \( \bar{G} \cong G/N \), where \( \bar{G} \) is the image of \( G \) under its action on the orbits of \( N \). The group \( \bar{G} \) is solvable, hence by the minimality of \( \Gamma \), \( \rho(G) = 1 \). On the other hand, by Proposition 2.7, \( 1 < \rho(G) \leq \rho(G) \), a contradiction.

It remains to consider the case when \( N \) has two orbits. If \( N \) is semiregular, then the image of the action of \( G \) on the two \( N \)-orbits is \( S_2 \), hence by Proposition 2.7, \( 1 < \rho(G) \leq \rho(S_2) = 1 \), again a contradiction. If \( N \) is not semiregular, then it is easy to show that \( r = p, N \cong \mathbb{Z}_p^2 \) and \( \Gamma = K_{p,p} \).

Clearly the bipartition sets of \( \Gamma \) form a block system for \( G \). It follows then from Lemma 2.9 that \( \rho(G) = 1 \), a contradiction, completing the proof of Proposition 4.1.

In the example below we show that the above proposition does not extend to solvable transitive permutation groups with the point stabilizer of order \( p \) but without connected self-paired orbitals. These groups act on edge-transitive graphs, first studied by Praeger and Xu [17].

Example 4.2 Let \( G = \text{AGL}(1,q) \), where \( q = p^e \), \( p \) an odd prime and \( e > 1 \). Then \( G \) consists of the linear transformations \( x \mapsto ax + b \), where \( a \in \mathbb{F}_q^* \) and \( b \in \mathbb{F}_q \). Fix non-zero elements \( a, b \in \mathbb{F}_q \) such that \( a \) is a primitive element, and let \( \sigma : x \mapsto x + b \) and \( \tau : x \mapsto ax \). Note that \( \langle \sigma, \tau \rangle = G \).

Then let \( \Gamma \) be the double coset graph \( \text{Cos}(G,H,H\{\tau,\tau^{-1}\}H) \), where \( H = \langle \sigma \rangle \). It follows from basic properties of double coset graphs (see, for example, [7 Lemma 2.1]) that \( \Gamma \) is a connected \( G \)-edge-transitive graph. It is easy to see that \( H\tau H \neq H\tau^{-1}H \) (the corresponding orbitals are non-self-paired), hence, by [7 Lemma 2.4], \( \Gamma \) has valency \( 2|H|/|H \cap H^\tau| = 2p \). The conjugacy class of \( G \) containing \( \sigma \) consists of all transformations \( x \mapsto x + u, u \neq 0 \). Consequently, these transformations together with the identity form an intersecting set, and so \( \rho(G) > 1 \). Note also that a self-paired orbital of the action of \( G \) on \( H = \langle \sigma \rangle \) is necessarily disconnected.

5 Intersecting sets in groups with point stabilizers of order 3

In view of Proposition 4.1 in order to construct transitive permutation groups having point stabilizers of prime order \( p \) admitting a connected self-paired orbital and without EKR-property, nonsolvable groups need to be brought into consideration. Hereafter, we restrict ourselves to \( p = 3 \). The lemma below gives a lower bound on the cardinality of a maximal intersecting set containing a point stabilizer in a transitive group with point stabilizer of order 3.

Lemma 5.1 Let \( G \) be a transitive permutation group. Let \( H, K \leq G \) such that \( H \cup K \) is an intersecting set. Then \( HK \) is also an intersecting set.

Proof. Since \( H \cup K \) is an intersecting set, it follows that \( hk \) and \( kh \) fixes a point for each \( h \in H \) and \( k \in K \). Hence 1 is adjacent with every element of \( HK \) in \( \Gamma_G \). Let \( hk, h_1k_1 \) be arbitrary in
HK. By ~ we denote the adjacency relation in $\Gamma_G$. Then we have

$$hk \sim h_1k_1 \iff k \sim h^{-1}h_1k_1$$

(Using left multiplication by $h^{-1}$ which is an automorphism)

$$\iff k^{-1} \sim k_1^{-1}h_1^{-1}h$$

(Using the inversion automorphism)

$$\iff 1 \sim kk_1^{-1}h_1^{-1}h$$

(Using left multiplication by $k$)

$$\iff 1 \sim k'h'.$$

Observe that $k'h'$ is adjacent with 1 in $\Gamma_G$ for every $h' \in H$ and every $k' \in K$ since $H \cup K$ is an intersecting set in $G$. This shows that $HK$ is a clique in $\Gamma_G$.

Proposition 5.2 Let $G$ be a transitive group acting on a set $V$, with point stabilizer $H = \langle x \rangle \cong \mathbb{Z}_3$ of order 3, and admitting a maximal intersecting set $\mathcal{F}$ containing $H$ as a proper subset. Then

(i) $|\mathcal{F}| \geq 9$.

(ii) For every $y \in \mathcal{F} \setminus H$, the group $\langle x, y \rangle$ is isomorphic either to $\mathbb{Z}_3^2$ or to the unique non-abelian group of order 27 with exponent 3.

Proof. By assumption there exists $y \in \mathcal{F} \setminus H$. It follows that $\{1, x, x^2, y \}$ is a clique in $\Gamma_G$. Since, by Lemma 2.3 the mapping $v : g \mapsto g^{-1}$ ($g \in G$) is an automorphism of $\Gamma_G$, it follows that $\{1, x, x^2, y, y^2 = y^{-1}\}$ is an intersecting set of $G$, and so (i) follows by Lemma 5.1.

As for part (ii), let $P = \langle x, y \rangle$. If $z = [x, y] = 1$, then clearly $P \cong \mathbb{Z}_3^2$. Hence assume that $z \neq 1$. We show that $[x, z] = [y, z] = 1$. It follows from $(x^2y)^3 = 1$ that $y^2xy = x^2yx$ and from $(xy)^3 = 1$ that $x^2y^2x^2 = yxy$ and $(yx)^2 = x^2y^2$. Using these we compute

$$z^x = x[x,y]x^2 = y^2xyx^2 = (y^2xy)^2x^2 = x^2y(x^2y^2x^2) = x^2y^2xy = z,$$

$$z^y = y[x,y]y^2 = yx^2y^2x = xy(x^2y) = yxyx^2 = (yx)^2xy = x^2y^2xy = z.$$

Since $xz = y^2xy$ is a conjugate of $x$ it has order 3, and since $[x, z] = 1$, we conclude that so does $z$. It follows that $\langle z \rangle \cong \mathbb{Z}_3$, $P/\langle z \rangle \cong \mathbb{Z}_3^2$, and therefore, $P$ is isomorphic to the unique non-abelian group of order 27 with exponent 3.

Corollary 5.3 Let $G$ be a transitive group acting on a set $V$, with point stabilizer of order 3, and admitting a maximal basic intersecting set $\mathcal{F}$ of $G$ such that $|\mathcal{F}| = 4$. Then no point stabilizer $G_v$, $v \in V$, is contained in $\mathcal{F}$. In other words, $\mathcal{F} = \{1, x, y, z\}$ with $x$, $y$ and $z$ belonging to different point stabilizers.

Example 5.4 There exists a connected cubic symmetric graph of order 720 with automorphism group $A$ of order 2160. Using MAGMA [1], one can obtain that the group $A$ is isomorphic to $(3 \cdot A_6) \rtimes \mathbb{Z}_2$, where $3 \cdot A_6$ denotes the triple cover of $A_6$. Let $P$ be a Sylow 3-subgroup of $A$. Then $P \cong \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = 1, [b, c] = 1 \rangle$ is the unique non-abelian group of order 27 with exponent 3. Again, we established with MAGMA that maximum intersecting sets of $A$ are of size 9 and furthermore every maximum intersecting set of $A$ containing identity is contained in $P$ or one of its 10 conjugates in $A$. All non-central elements of order 3 from $P$ are conjugate in $A$, and the subgraph of the derangement graph of $A$ induced by these 24 non-central elements of $P$ is a disjoint union of 8 triangles of the form $\{x, xc, xc^2\}$, with $x$ a non-central element of $P$. Then the maximum intersecting sets containing 1 that lie inside $P$ are just the transversals of $\langle c \rangle = Z(P)$ in $P$. Observe that there are $3^8$ such transversals. Consequently, all possibilities between 0 and 4 full vertex-stabilizers, contained in such a maximum intersecting set of $A$, can occur.
Groups with point stabilizer of order 3 and intersection density 4/3

In this section we give an infinite family of transitive permutation groups with intersection density 4/3. These groups arise from the groups $\text{PSL}(2, q)$ acting on cosets of a subgroup of order 3. In view of Proposition 2.13, the corresponding non-canonical basic intersecting sets are of the form $\{1, x, y, z\}$ where $x, y$ and $z$ are from different point stabilizers.

**Theorem 6.1** Let $p$ be a prime, let $q = p^e \equiv 1 \pmod{3}$, and let $G$ be a transitive permutation group arising from $\text{PSL}(2, q)$ acting on cosets of a subgroup of order 3. Then

$$\rho(G) = \begin{cases} 
4/3 & \text{if } p \neq 5, \\
2 & \text{if } p = 5.
\end{cases} \quad (2)$$

**Proof.** Observe that all cyclic subgroups of order 3 are conjugate in $\text{PSL}(2, q)$, while the elements of order 3 form two conjugacy classes, with two elements $x$ and $x^{-1}$ of order 3 belonging to different conjugacy classes. In $\text{PGL}(2, q)$, however, these two conjugacy classes merge into a single class. Consequently, every element of order 3 in $\text{PSL}(2, q)$ belongs to a point stabilizer (with regards to the action considered in this theorem).

Let $\Gamma$ be the subgraph of the complement of the derangement graph $\Gamma_G$ of $G$ induced by the set of neighbours of the identity element of $G$. In other words, the vertices of $\Gamma$ are the elements of order 3, with $h, g$ adjacent if and only if $h^{-1}g$ is of order 3. Since a clique in $\Gamma$ together with the identity element gives rise to an intersecting set for $G$, the statement (2) will be proved if we show that

$$\omega(\Gamma) = \begin{cases} 
3 & \text{if } p \neq 5, \\
5 & \text{if } p = 5,
\end{cases} \quad (3)$$

where $\omega(\Gamma)$ denotes the clique number of $\Gamma$.

As before we will think of elements of $\text{PGL}(2, q)$ as ‘matrices’, that is, as elements of $\text{GL}(2, q)$. Note that, $A, B \in \text{SL}(2, q)$ represent the same element in $\text{PSL}(2, q)$ if and only if $B = -A$.

In view of the observation made in the first paragraph, the action of $\text{PGL}(2, q)$ on $V(\Gamma)$ by conjugation is transitive, and the corresponding image is therefore a transitive subgroup of $\text{Aut}(\Gamma)$. In particular, every vertex of $\Gamma$ is contained in a maximum clique. Let us fix the vertex

$$A_0 = \begin{bmatrix} r & 0 \\
0 & r^2 \end{bmatrix},$$

where $r^3 = 1$ and $r \neq 1$. The existence of such an $r$ is guaranteed by the condition $q \equiv 1 \pmod{3}$. Let $K$ be a maximum clique containing $A_0$. The vertex stabilizer of $A_0$ in $\text{PGL}(2, q)$ coincides with the centralizer $\Lambda = \{L_a : a \in \mathbb{F}_q^*\}$ of $A_0$ in $\text{PGL}(2, q)$, isomorphic to $\mathbb{Z}_{q-1}$, where

$$L_a := \begin{bmatrix} a & 0 \\
0 & 1 \end{bmatrix}, \quad a \in \mathbb{F}_q^*.$$ 

We now show that the neighborhood $\Gamma(A_0)$ splits into the following $\Lambda$-orbits: $\{A_0^{-1}\}, \ O_1, \ O'_1$.
without loss of generality, assume that $A_0 \in A$ and $O_1$ if $p = 2$, where

$$O_1 = \{ A_x = \begin{bmatrix} -r^2 & x \\ 0 & -r \end{bmatrix} \mid x \in \mathbb{F}_q^* \},$$

$$O_1' = \{ A^T_x : x \in \mathbb{F}_q^* \},$$

$$O_2 = \{ B_x = \begin{bmatrix} r & -2x \\ \frac{r}{x^2} - \frac{1}{r} \end{bmatrix} : x \in \mathbb{F}_q^* \}.$$

It is clear that $A_0$ and $A_0^{-1}$ are adjacent in $\Gamma$. In view of Proposition 2.12, we may choose $A \in \Gamma(A_0) \setminus \{A_0^{-1}\}$ with $\text{Tr}(A) = 1$. Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some $a, b, c, d \in \mathbb{F}_q$ with $ad - bc = 1$ and $a + d = 1$. Also, $\text{Tr}(A_0^{-1}A) = r^2a + rd = \pm 1$.

Suppose first that $p \neq 2$. Using the fact that $a + d = 1$, we have $(r^2 - r)a = \pm 1 - r$. If $(r^2 - r)a = 1 - r$, then we have $a = -r^2, d = -r$, and $bc = ad - 1 = 0$. If $c = 0$, then $A \in O_1$, and if $b = 0$, then $A \in O_1'$. Suppose now that $(r^2 - r)a = -1 - r = r^2$. Then we have that $a = r/(r - 1), d = -1/(r - 1)$, and $bc = ad - 1 = r/(r - 1) - 1 = -r/(-3r) - 1 = -2/3$. It follows that $A \in O_2$. We have therefore shown that $\Gamma(A_0) = \{A_0^{-1}\} \cup O_1 \cup O_1' \cup O_2$.

Suppose now that $p = 2$. Then $r^2a + rd = 1$, and since $d = 1 - a$ it follows that $a = (r + r^2)a = r + 1 = r^2, d = r$ and $bc = 0$. We obtain that $\Gamma(A_0) = \{A_0^{-1}\} \cup O_1 \cup O_1'$.

Straightforward computations show that, for every $x \in \mathbb{F}_q^*$,

$$A_x = L_xA_1L_x^{-1}, \ A^T_x = L_x^{-1}A_1^TL_x, \text{ and } B_x = L_xB_1L_x^{-1}.$$

Thus each of $O_1, O_1'$ and $O_2$ is a $\Lambda$-orbit, as claimed. It can be directly checked that for every $x \in \mathbb{F}_q^*$

$$A_x^{-1} = \begin{bmatrix} -r & -x \\ 0 & -r^2 \end{bmatrix} \text{ and } B_x^{-1} = \begin{bmatrix} \frac{r^2 + 3}{3x} & 2x \\ -\frac{1}{x} & 2r + 1 \end{bmatrix}.$$

Using these, the traces $\text{Tr}(XY)$ for $X \in \{A_x^{-1}, (A^T_x)^{-1}, B_x^{-1}\}$ and $Y \in \{A_y, A^T_y, B_y\}$ can be computed directly. The results are collected in Table 1. In particular, $A_1$ and $A_1^T$ are adjacent, and so $|\mathcal{K}| \geq 3$. Combining this with Proposition 2.13 we have that $A_0^{-1} \notin \mathcal{K}$.

Assume for the moment that $\mathcal{K} \setminus \{A_0\}$ is not contained in $O_2$. It is easy to show that the mapping $\tau : X \mapsto X^T, X \in V(\Gamma)$, is an automorphism of $\Gamma$. This automorphism fixes $A_0$ and $A_0^{-1}$, swaps $O_1$ with $O_1'$, and maps $O_2$ onto itself. Thus $\langle \Lambda, \tau \rangle$ is transitive on $O_1 \cup O_1'$, and so we may, without loss of generality, assume that $A_1 \in \mathcal{K}$. Now, if $A \in \mathcal{K} \setminus \{A_0, A_1\}$ then $\text{Tr}(A_1^{-1}A) = \pm 1$.

|       | $A_y$ | $A^T_y$ | $B_y$ |
|-------|-------|---------|-------|
| $A_x^{-1}$ | 2     | $2 - xy$ | $-\frac{x}{3y}$ |
| $(A^T_x)^{-1}$ | $2 - xy$ | 2 | $2xy$ |
| $B_x^{-1}$ | $-\frac{y}{3x}$ | $2xy$ | $\frac{2}{3}(1 + \frac{x}{y} + \frac{y}{x})$ |

Table 1: The traces $\text{Tr}(XY), X \in \{A_x^{-1}, (A^T_x)^{-1}, B_x^{-1}\}$ and $Y \in \{A_y, A^T_y, B_y\}$. 
Using Table 1, we find $A \in \{A_1^T, A_2^T, B_{1/3}, B_{-1/3}\}$ if $p \neq 2$, and $A = A_1^T$ if $p = 2$. This shows that $K = \{A_0, A_1, A_1^T\}$ if $p = 2$. Let $p \neq 2$. If $p \neq 5$, then using Table 1 again, we obtain that no two vertices in the set $\{A_1^T, A_2^T, B_{1/3}, B_{-1/3}\}$ are adjacent, and hence $|K| = 3$. If $p = 5$, then a direct computation yields $K = \{A_0, A_1, A_1^T, B_{1/3}, B_{-1/3}\}$, and so $|K| = 5$.

Observe that the above arguments show that $\omega(\Gamma) = 3$ for $p = 2$, as claimed in (3). It remains to consider the case $p \neq 2$ and $K \setminus \{A_0\} \subseteq O_2$. As $O_2$ is a $\Lambda$-orbit, we may assume that $B_1 \in K$. Also, assume that $B_z \in K$ for $z \neq 1$. Then $(\text{Tr}(B_1^{-1}B_z) - 1)(\text{Tr}(B_1^{-1}B_z) + 1) = 0$ holds. Using Table 1, we find that $z$ is a root of the polynomial

$$f(x) = \left(x^2 - \frac{1}{2}x + 1\right)\left(x + 2\right)(x + \frac{1}{2}).$$

(4)

If $p = 5$ then $x^2 - \frac{1}{2}x + 1 = (x + 1)^2$, $x + \frac{1}{2} = x + 3$, and so $z \in \{2, 3, 4\}$. Consequently, $K = \{A_0, B_1, B_2, B_3, B_4\}$, $|K| = 5$, and $\omega(\Gamma) = 5$, as claimed in (3). Suppose now that $p \neq 5$ and that there exists a root $w$ of $f(x)$, $w \notin \{1, z\}$, such that $B_w \in K$. Then, by Table 1, we have

$$\frac{2}{3}\left(1 + \frac{z}{w} + \frac{w}{z}\right) = \pm 1.$$  

(5)

We complete the proof of Theorem 6.1 by showing that (5) together with the fact that $z$ and $w$ are roots of the polynomial $f(x)$ given in (4) leads to a contradiction. We go through all possibilities for $z$ and $w$.

If $z = -2$ and $w = -1/2$ then (5) yields $15 = 0$ or $27 = 0$ in $\mathbb{F}_q$, a contradiction.

If $z = -2$ and $w^2 - 1/2w + 1 = 0$ then (5) implies $2 - 4/w - w = \pm 3$, and so $w = -2$ or $w = 2/3$. Hence, $6 = 0$ or $10 = 0$, a contradiction.

If $z = -1/2$ and $w^2 - 1/2w + 1 = 0$ then (5) implies $2 - 1/w - 4w = \pm 3$, and so $w = 1$ or $w = -1$. Hence $3 = 0$ or $5 = 0$, a contradiction.

Finally, if $z, w$ are both roots of $x^2 - 1/2x + 1$ then $zw = 1$ and $z + w = 1/2$, and so $1 + z^2 + w^2 = -3/4$. Dividing both sides with $zw = 1$, we have $1 + z/w + w/z = -3/4$. Combining this with (5), we conclude that $-3/4 = \pm 3/2$, a contradiction, completing the proof of Theorem 6.1.

7 Groups with point stabilizer of order 3 and large intersection density

In the example below, using an action of the symmetric group $S_n$, $n \geq 4$, we show that maximum intersecting sets in transitive groups with point stabilizers of order 3 can be arbitrarily large.

Example 7.1 Let $n \geq 4$. First observe that for two 3-cycles $x = (i j k), y \in S_3$ we have that $x^{-1}y$ is also a 3-cycle if and only if $y = (i k j)$ or $y \in \{(i j l), (j k l), (k i l) : 1 \leq l \leq n-1, l \notin \{i, j, k\}\}$.

Let $G$ be the permutation group induced by the action of $S_n$ on cosets of $\langle (1 2 3) \rangle$. We show that a maximum intersecting set of $G$ has size $n-1$. Let $\Gamma$ be the complement of the derangement graph of $G$. Observe that the set of neighbours of $id$ in $\Gamma$ is the set of all 3-cycles from $S_n$. Let $\Gamma_1$ be the subgraph of $\Gamma$ induced by the set of common neighbours of $id$ and $(1 2 3)$. In view of the first paragraph it follows that $V(\Gamma_1) = \{(1 3 2) \cup \{(1 2 i), (2 3 i), (3 1 i) : i \in \{4, 5, \ldots, n\}\}$. Furthermore, it follows that the set $\{(1 2 i) : i \in \{4, 5, \ldots, n\}\}$ induces a clique in $\Gamma_1$. Analogously,
that contains this edge is the one induced by $K_n$ into $3$ and $\text{PGL}(2, A)$ can assume that this neighbour is of the form $\langle a, b \rangle$. The points are the one-dimensional subspaces $\langle T \rangle$.

In what follows, we shall consider the action of $\text{PSL}(2, A \in F_3)$, each conjugate corresponding to a Sylow 3-subgroup of the stabilizer of some projective point. Observe also that $\text{PSL}(2,3^n)$ is a Sylow 3-subgroup of $\text{PSL}(2,3^n)$. Observe also that $K$ consists precisely of those elements of $\text{PSL}(2,3^n)$ fixing the subspace $\langle [1,0] \rangle$. There are $3^n+1$ conjugates of $K$, each conjugate corresponding to a Sylow 3-subgroup of the stabilizer of some projective point. In what follows, we shall consider the action of $\text{PSL}(2,3^n)$ on the projective line, whose points are the one dimensional subspaces $\langle [a,b] \rangle$, $a, b \in F_q$.

Suppose first that $n$ is odd. Then, by Proposition 2.11 any two subgroups of order 3 in $\text{PSL}(2,3^n)$ are conjugate. Let $\Gamma$ be the subgraph of the complement of the derangement graph of $G$ induced by the neighbours of the identity matrix $I \in G$. The vertex set $V(\Gamma)$ can be partitioned into $3^n+1$ cliques, each of size $3^n - 1$, corresponding to $K \setminus \{I\}$ and its conjugates.

Consider a neighbour of $M_x$ outside of $K$. (Note that, since Sylow 3-subgroups of $\text{PSL}(2,3^n)$ and $\text{PGL}(2,3^n)$ are of the same order, all elements of order 3 are contained in $\text{PSL}(2,3^n)$.) We can assume that this neighbour is of the form $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$. Using the assumption that $M_x^2 A$ is an elements of order 3, it follows that its trace must be equal to $\pm 1$. If $2cx + 1 = 1$, it follows that $c = 0$, but this would imply that $A$ is in $K$, a contradiction. Hence $2cx + 1 = -1$, and we obtain that $c = -1/x$. This implies that all neighbours of $M_x$ outside of $K$ have the entry in the first column and second row equal to $-1/x$. Consequently, no two different elements of $K$ have a common neighbour outside of $K$. This shows that given an edge of $K$, the maximum clique that contains this edge is the one induced by $K$. Hence, the maximum clique of $\Gamma$ can be the one

Theorem 7.2 Let $G$ be a transitive permutation group arising from $\text{PSL}(2,3^n)$, $n \geq 3$, acting on cosets of a subgroup of order 3. Then

$$\rho(G) = \begin{cases} 3^{n-1} & \text{if } n \text{ is odd}, \\ 3^{n/2-1} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Consider the subgroup $K$ of $\text{PSL}(2,3^n)$ consisting of matrices

$$M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in F_3,$$

(where a matrix is a ‘matrix’ as in the proof of Theorem 6.1).

Observe that $K \cong \mathbb{Z}_3^3$ is a Sylow 3-subgroup of $\text{PSL}(2,3^n)$. Theorem 7.2

for $n \geq 3$ the group $\text{PSL}(2,3^n)$ is generated by an involution and an element of order 3, and thus the theorem below indeed gives transitive permutation groups with point stabilizer of order 3 having connected self-paired orbital.

Theorem 7.2 Let $G$ be a transitive permutation group arising from $\text{PSL}(2,3^n)$, $n \geq 3$, acting on cosets of a subgroup of order 3. Then

$$\rho(G) = \begin{cases} 3^{n-1} & \text{if } n \text{ is odd}, \\ 3^{n/2-1} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Consider the subgroup $K$ of $\text{PSL}(2,3^n)$ consisting of matrices

$$M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in F_3,$$

(where a matrix is a ‘matrix’ as in the proof of Theorem 6.1).

Observe that $K \cong \mathbb{Z}_3^3$ is a Sylow 3-subgroup of $\text{PSL}(2,3^n)$. Theorem 7.2

for $n \geq 3$ the group $\text{PSL}(2,3^n)$ is generated by an involution and an element of order 3, and thus the theorem below indeed gives transitive permutation groups with point stabilizer of order 3 having connected self-paired orbital.

Theorem 7.2 Let $G$ be a transitive permutation group arising from $\text{PSL}(2,3^n)$, $n \geq 3$, acting on cosets of a subgroup of order 3. Then

$$\rho(G) = \begin{cases} 3^{n-1} & \text{if } n \text{ is odd}, \\ 3^{n/2-1} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Consider the subgroup $K$ of $\text{PSL}(2,3^n)$ consisting of matrices

$$M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in F_3,$$

(where a matrix is a ‘matrix’ as in the proof of Theorem 6.1).

Observe that $K \cong \mathbb{Z}_3^3$ is a Sylow 3-subgroup of $\text{PSL}(2,3^n)$. Theorem 7.2

for $n \geq 3$ the group $\text{PSL}(2,3^n)$ is generated by an involution and an element of order 3, and thus the theorem below indeed gives transitive permutation groups with point stabilizer of order 3 having connected self-paired orbital.

Theorem 7.2 Let $G$ be a transitive permutation group arising from $\text{PSL}(2,3^n)$, $n \geq 3$, acting on cosets of a subgroup of order 3. Then

$$\rho(G) = \begin{cases} 3^{n-1} & \text{if } n \text{ is odd}, \\ 3^{n/2-1} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Consider the subgroup $K$ of $\text{PSL}(2,3^n)$ consisting of matrices

$$M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in F_3,$$

(where a matrix is a ‘matrix’ as in the proof of Theorem 6.1).

Observe that $K \cong \mathbb{Z}_3^3$ is a Sylow 3-subgroup of $\text{PSL}(2,3^n)$. Theorem 7.2

for $n \geq 3$ the group $\text{PSL}(2,3^n)$ is generated by an involution and an element of order 3, and thus the theorem below indeed gives transitive permutation groups with point stabilizer of order 3 having connected self-paired orbital.

Theorem 7.2 Let $G$ be a transitive permutation group arising from $\text{PSL}(2,3^n)$, $n \geq 3$, acting on cosets of a subgroup of order 3. Then

$$\rho(G) = \begin{cases} 3^{n-1} & \text{if } n \text{ is odd}, \\ 3^{n/2-1} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Consider the subgroup $K$ of $\text{PSL}(2,3^n)$ consisting of matrices

$$M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in F_3.$$
induced by $K$, or it can consist of at most one element from each of the conjugates of $K$. Since there are $3^n + 1$ such conjugates, it follows that $\omega(\Gamma) \leq 3^n + 1$. Adding the identity, which is not in $\Gamma$, it follows that $\rho(G) \leq (3^n + 2)/3$.

Let $q = 3^n$, and let $K = K_0, K_1, \ldots, K_q$ be the conjugates of $K_0$, where we may assume that $K_1$ consists of lower triangular matrices with 1 on diagonal fixing the projective point corresponding to $\langle [0, 1]^T \rangle$. Suppose that there exists a maximal clique $K$ not contained in $K_i$ for any $i$ and such that $|K| \geq 3^n$. By the argument from the first paragraph, we know that $|K \cap K_i| \leq 1$, for each $i$, and so we may assume that $K \cap K_i = \emptyset$ for at most one $i \in \{0, 1, \ldots, q\}$. Since $\text{PSL}(2, q)$ acts 2-transitively on projective points, we may assume, without loss of generality, that $K \cap K_0 \neq \emptyset$ and that $K \cap K_1 \neq \emptyset$.

Let $A_0 = M_x$. By calculation we see that the only neighbour of $A_0$ in $K_1$ is $A_1 = \begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix}$. Let $A = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix}$ be a common neighbour of $A_0$ and $A_1$. Observe that $\text{Tr}(A_0^2A) = 2cx + 1$ and $\text{Tr}(A_1^2A) = \frac{2b}{x} + 1$. Using the assumption that $A_0^2A$ and $A_1^2A$ are elements of order 3, it follows that their traces must be equal to $\pm 1$, implying that $c = -1/x$ and $b = -x$. Hence

$$A = \begin{bmatrix} a & -x \\ \frac{1}{x} & 1 - a \end{bmatrix}.$$ 

Since $\text{det}(A) = a - a^2 - 1$, and since $A$ is in $\text{PSL}(2, q)$, it follows that $a - a^2 - 1 = \pm 1$. There are at most 4 elements $a$ in $F_q$ that satisfy this equation. It follows that for a fixed $x \in F_q^*$ there are at most four possibilities for the matrix $A$, and therefore $A_0$ and $A_1$ can have at most 4 common neighbours, that is the clique containing the edge $\{A_0, A_1\}$ can have size at most 6. Since $3^n - 1 > 6$, it follows that $\omega(\Gamma) = 3^n - 1$. This shows that $\rho(G) = 3^n/3 = 3^{n-1}$.

Suppose now that $n$ is even. The elements $M_x$ of $K$ split into two conjugacy classes, depending on whether $x$ is a square in $F_{3^n}$ or a non-square. It is easy to see that the subgraph of $\Gamma_G$ induced by $K$ is isomorphic to the Paley graph or its complement. Combining Proposition 2.5 with the fact that the Paley graph and its complement are isomorphic we obtain that a maximum clique of the latter graph $\Gamma_G[K]$ is of size $3^{n/2}$. Using the same argument as for $n$ odd we obtain that a clique in $\Gamma_G$ that is not contained inside $K_i$, $i \in \{0, 1, \ldots, q\}$ can have size at most 6. Since $n \geq 4$, it follows that $3^{n/2} \geq 9$, and so the clique number of $\Gamma_G$ is $3^{n/2}$, implying that $\rho(G) = 3^{n/2-1}$.

As a final remark regarding possible future work, we would like to pose the following problem which generalizes the results of this paper for cyclic groups of orders 2 and 3.

**Problem 7.3** Given a group $H$ determine all possible intersection densities of transitive permutation groups having point stabilizers isomorphic to $H$.

**References**

[1] W. Bosma, J. Cannon, and C. Playoust, The Magma Algebra System I: The User Language, *J. Symbolic Comput.* 24 (1997) 235–265.

[2] I. Broere, D. Döman, J. N. Ridley, The clique numbers and chromatic numbers of certain Paley graphs, *Quaestiones Math.* 11 (1988), no. 1, 91–93.

[3] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser.* 12 (1961), 313–320.
[4] A. Hujdurović, K. Kutnar, D. Marušić and Š. Miklavič, Intersection density of transitive groups of certain degrees, to appear in *Algebraic Combin.*. arXiv preprint [arXiv:2104.04699], 2021.

[5] A. Hujdurović, K. Kutnar, D. Marušić and Š. Miklavič, On maximum intersecting sets in direct and wreath product of groups, [arXiv:2108.03943] [math.CO].

[6] A. Hujdurović, K. Kutnar, B. Kuzma, D. Marušić, Š. Miklavič, and M. Orel, On intersection density of transitive groups of degree a product of two odd primes, *Finite Fields and Their Appl.* 78 (2022), 101975.

[7] C. H. Li, Z. P. Lu and D. Marušić, On primitive permutation groups with small suborbits and their orbital graphs, *J. Algebra* 279 (2004), 749–770.

[8] C. H. Li, S. J. Song, and V. R. T. Pantangi, Erdős-Ko-Rado problems for permutation groups, 2021, [https://arxiv.org/abs/arXiv:2006.10339](https://arxiv.org/abs/arXiv:2006.10339).

[9] P. Lorimer, Vertex-transitive graphs: symmetric graphs of prime valency, *J. Graph Theory* 8 (1984), 55–68.

[10] G. James and M. Liebeck, *Representations and characters of groups - second edition*, Cambridge University Press, 2001.

[11] C. H. Li and J. Pan, Finite 2-arc-transitive abelian Cayley graphs, *European J. Combin.* 29 (2008), 148–158.

[12] A. M. Machbet, Generators of the linear fractional groups, *Proc. Sympos. Pure Math.* 12 (1967), 14–32.

[13] K. Meagher, An Erdős-Ko-Rado theorem for the group PSU(3,q), *Des. Codes Cryptogr.* 87 (2019), no. 4, 717–744.

[14] K. Meagher, A. S. Razafimahatratra, The Erdős-Ko-Rado theorem for 2-pointwise and 2-setwise intersecting permutations, *Electron. J. Combin.* 28 (2021), no. 4, Paper No. 4.10.

[15] K. Meagher, P. Sin, All 2-transitive groups have the EKR-module property, *J. Combin. Theory Ser. A* 177 (2021), Paper No. 105322.

[16] K. Meagher, A. S. Razafimahatratra and P. Spiga, On triangles in derangement graphs, *J. Combin. Theory, Ser. A* 180 (2021), 105390.

[17] C. E. Praeger and M.-Y. Xu, A characterisation of a class of symmetric graphs of twice prime valency, *European J. Combin.* 10 (1989), 91–102.

[18] A. S. Razafimahatratra, On multipartite derangement graphs, *Ars Math. Contemp.* 21 (2021), #P1.07.

[19] P. Spiga, The Erdős-Ko-Rado theorem for the derangement graph of the projective general linear group acting on the projective space, *J. Combin. Theory Ser. A* 166 (2019), 59–90.

[20] M. Suzuki, *Group Theory I*, Springer-Verlag, 1982.