Variety of scaling behaviors in nanocrystalline plasticity

P. Zhang, O. U. Salman, J. Weiss and L. Truskinovsky

1 State Key Laboratory for Mechanical Behavior of Materials, Xi’an Jiaotong University, Xi’an, 710049, China
2 CNRS, LSPM UPR3407, Paris Nord Sorbonne Université, 93430, Villetaneuse, France
3 ISterre, CNRS/Université Grenoble Alpes, 38401 Grenoble, France
4 PMMH, CNRS UMR 7636, ESPCI ParisTech, 10 Rue Vauquelin, 75005, Paris, France

(Dated: April 21, 2020)

Intermittent elasto-plastic deformation of sub-micron crystals features size-sensitivity of scaling exponents. We use a minimal integer-valued automaton model of crystal plasticity to show that with growing variance of quenched disorder, which can serve in this case as a proxy for increasing size, sub-micron crystals undergo a crossover from spin-glass-type marginality to criticality characterizing extended second order phase transition. We argue that this crossover is behind the non-universality of scaling exponents observed in physical and numerical experiments.

Considerable research efforts were recently focused on the study of mechanical properties of sub-micron crystals [1–5]. It was found that the deformation mechanisms, which habitually associate with dislocation plasticity, change dramatically once the sample size is reduced below the micrometer range. Strength of such crystals was shown to be size dependent [6–8], with stress-strain response exhibiting pronounced intermittency and scale-invariance over a wide range of scales, independently of crystal symmetry [9–16]. Both, measured and computed, scaling exponents appear to be featuring continuous size dependence [17, 18].

Moreover, even though plasticity at macroscale is generally associated with ductility, crystal plasticity at sub-micron scales was shown to exhibit major stress drops or strain bursts reminiscent of brittle fracture [5, 19–22]. Britteness, usually attributed to dislocation-free crystals [23], reappears in nanoparticles and nano-pillars that appear to be ‘breaking’ plastically by generating a large number of globally correlated dislocations [24, 25]. The implied system-size events hinder our ability to control plastic deformation at sub-micron scale and compromise the reliable functioning of ultra-small machinery [10, 11, 26–28].

To study these effects, we conducted a set of experiments on pure Mo sub-micron pillars, choosing intentionally a ‘mild’, in the sense of [29], BCC crystal as the bulk material. In Fig. 1, we juxtaposed the stress-strain curves and the statistics of avalanches for Mo pillars with diameters between 500 nm and 1500 nm; the experimental protocol is detailed in [30]. The main observation is that larger, dislocation-rich sub-micron crystals are weak, ductile, and statistically sub-critical, while smaller, dislocation-starved crystals are strong, brittle, and statistically super-critical. The goal of this Letter is to rationalize the observed behavior.

The main dimensionless parameter of the problem is \( R \equiv L/l \), where \( L \) is the system size and \( l \sim Gb/\sigma_\delta \) is the internal length scale [18]. We denoted by \( G \), the shear modulus, by \( b \), the Burgers vector, and by \( \sigma_\delta \), the effective pinning (immobilization) stress. The distinctly brittle regime would then correspond to \( R \ll 1 \). Given that in our Mo samples \( l \sim 1/\sqrt{\rho} \approx 790 \text{ nm} \), where \( \rho \) is the initial density of forest dislocations, brittleness should be expected for our 500 nm pillars, see Fig.1 and [30]. In the strongly ductile regime we should have \( R \gg 1 \), which is the case for our 1500 nm pillars. Dislocation interaction with obstacles becomes relevant when \( R \sim 1 \) which is, roughly, the case for our 1000 nm pillars, see Fig. 1. In such intermediate regime \( \sigma_\delta L \sim 1 \), and since \( \sigma_\delta \) increases with the variance of disorder, the latter can be also chosen as the crossover parameter.

Figure 1: Compression tests on pure Mo sub-micron pillars: (a) stress - strain curves (shear); (b) cumulative distribution of plastic displacements \( X \) detected over the entire loading process.

The intermittent plastic deformation in crystals has been modeled using molecular dynamics [31], discrete dislocation dynamics [17, 19, 32–34], phase field theories [35] and various meso-scopic approaches [36, 37]. The results of different simulations suggest that scaling exponents cover a broad range of values [18, 38, 39] and here we show that the use of a simple paradigmatic model of crystal plasticity [37, 40, 41] allows one to reconcile the obtained data. We reduced the problem to an integer-valued discrete automaton, which can deal with highly anisotropic single slip yield, can account self consistently for both short-range and long-range elastic interactions, including dislocation nucleation and immobilization, and can tackle computationally large systems. The size effect is simulated through variable quenched disorder [79].

Our main result is that the non-universality of sub-micron plasticity and the inferred brittleness can be conceptualized as a crossover behavior associated with a disorder-induced transition from spin-glass-type marginality, characteristic of very small, almost defect-free crystals, to the criticality of larger crystals associated with a stretched brittle-to-ductile (BD) transition. Analogous results were obtained in the stud-
ies of plasticity of glasses even though the disorder there is particular, and dislocations are absent [42–45].

We assume that the system is oriented for a single slip along the only available slip direction. The crystal is modeled as an $N \times N$ square lattice with the meso-scopic spacing normalized to unity. The energy is taken to be additive $\Phi(u) = \sum_{i,j} f(u_{i,j})$, with the dimensionless energy density being piece-wise quadratic [37] $f = (K/2)\xi^2 + (1/2)(\xi - d(\xi))^2 - h\xi - g\xi$. Here we denoted by $u_{i,j}$ the scalar lattice displacement field: $\xi_{i,j} = u_{i+1,j} - u_{i,j}$ and $\xi_{i,j} = u_{i,j+1} - u_{i,j}$ are longitudinal and shear strains. The plastic slip $d$ is then measured by an integer nearest to $\xi$, so that $d_{i,j} = \lfloor \xi_{i,j} \rfloor$ [40].

The quenched disorder is represented by the random coefficients $h_{i,j}$ and $g_{i,j}$, which mimic two different types of lattice pre-stress. The disorder $g$, representing residual shear stresses [46], acts on the primary order parameter $\xi$ locally as in random field Ising model (RFIM) [47]. Instead, the disorder $h$, mimicking pinning solute atoms with volumetric misfit, acts on $\xi$ nonlocally: after the adiabatic elimination of the linear variable $\zeta$, the energy density acquires the term $A_{i,j,k,l} h_{i,j} \xi_{k,l}$ with elastic propagator $A_{i,j,k,l}$, see [48] for the discussion of the implied geometric compatibility relations.

![Figure 2: The effect of disorder on the average strain-stress curves in simple shear tests. Inset shows the yield strain $\gamma_y$ (points A, B, C, D). The zoom on the 'after dislocation' yield stress configurations is shown in the right column with more details presented in [30]. The dislocations with positive and negative Burgers vector are marked by red and blue. Parameters: $K = 2, N = 1024$.](image)

The model can be reduced to a discrete automaton because the elastic problem $d\Phi(u)/du = 0$ can be solved analytically for the given integer-valued field $d$ [37]. Thus, in the framework of the spectral approach we can use the Fourier transform $\tilde{x}(q) = N^{-2} \sum_{i,j} x_{i,j} e^{-iqr}$ with $r = (i,j)$ and $q = (2\pi k/N, 2\pi l/N)$, to obtain explicit representations for the equilibrium strains, for instance, $\xi(q) = \gamma \delta(q) + \hat{\delta}(q) \h(q) \hat{\delta}(q) / \hat{\lambda}(q)$, where the parameter $\gamma = \langle \xi \rangle$ controls the imposed affine strain. Here we also used the notations $H(q) = \hat{\delta}^2(q) \h(q) + \hat{\delta}(q) \hat{\delta}(q)$ and $\hat{\lambda}(q) = 2K [\cos(q_x) - 1] + \hat{\delta}(q) \hat{\delta}(q)$, with $\hat{\delta}(q) = \pm 1$ for $x = y$. The ensuing sandpile type automaton, see [30] for details, depends on a single fixed coupling parameter $K$; note that in the limits $K \to 0, \infty$ we obtain 1D paramagnetic models [40, 49]. The disorder $h, g$, drawn independently in each lattice cell from Gaussian distributions $p_s(r) = (2\pi \delta^2)^{-1/2} \exp(-r^2/(2\delta^2))$ where $s = q, h$. We mostly focus on the role of the more unusual nonlocal disorder $\delta = \delta_k$; the conventional local disorder $\delta_g$ is only briefly discussed at the end of the Letter.

We start with a dislocation-free crystal ($d = 0$) and drive the system quasistatically through the loading parameter $\gamma$ using the conventional athermal quasi-static protocol (zero-velocity limit of a gradient flow) [49, 50]. In Fig. 2 we illustrate the obtained average stress-strain relations $\sigma(\gamma) = d\Phi/d\gamma$ for different values of $\delta$; the inset shows the dislocation dependence of the effective yield strain $\gamma_y$, where yield is defined schematically as a state where the shear stress reaches its maximum value.

For weak disorder, $\delta \leq 0.3$, mimicking small, almost pure crystals, yielding is abrupt and brittle, accompanied by a macroscopic stress drop and robust strain localization within a shear band (regime $A$). With increased disorder (regime $B$), the first order phase transition eventually terminates at a critical point located around $\delta \sim 0.42$ (regime $C$), see [43, 44] for a similar behavior in amorphous plasticity. For even stronger disorder, $\delta \geq 0.5$, representing bulk samples, yielding is gradual and plasticity is ductile with slip uniformly distributed over the whole crystal (regime $D$).

![Figure 3: (a) Disorder dependence of the stress-resolved scaling exponent $\tau$ for immediately pre-yield and post-yield regimes; (b,c) corresponding avalanche distributions for more than 100 realizations.](image)

Independently of the strength of disorder and over the whole loading range, the driven dynamics proceeds via avalanches reflecting intermittent destruction and rebuilding of dislocation structures. The stress-resolved distributions for the released energy $E$ follow the scaling relation $p(E) \sim E^{-\tau} \exp(-E/E_c)$ where the exponent $\tau$ can be extracted using the maximum likelihood method [51]. Our Fig. 3(a) shows continuous dependence of $\tau$ on the disorder strength $\delta$, see also [34]; the associated distributions before and after yield are illustrated in Fig. 3(b,c).

When disorder is weak ($\delta \sim 0.2$), the pre- and post-yield exponents take almost the same value $\tau \sim 1$ (points $E$ and $E'$ in Fig. 3(a)). In such regimes, homogeneously nucleated dislocations are free to self-organize under the influence of long-range elastic forces [25, 52] and the formation of a shear band only mildly affects global dislocation dynamics.
Notwithstanding suppressed super-criticality, similar behavior was observed in DDD simulations of single slip plasticity and linked to self-induced glassiness [33, 34, 52–54]. Besides, the exponent \( \tau = 1 \) was shown numerically to characterize quasi-elastic regimes in structural glasses [45, 55] and emerged analytically in mean field theory of spin glasses [56]; it was associated with marginal stability in [57] and presented as a signature of archetypically ‘wild’ plasticity in [29].

At the intermediate disorder, \( 0.25 < \delta < 0.35 \), we observe a gap opening between the values of pre- and post-yield exponents (regimes \( F \) and \( F' \) in Fig. 3(a)). In view of the progressive rounding of the stress-strain curve near the strain-controlled spinodal point, where \( d\sigma/d\gamma = -\infty \) [44], one can expect the pre-yield scaling to represent the spinodal nucleation which shows scale-free features for systems with the dominating long-range elastic interactions [42, 47, 58, 59]. In the post-yield regime, the characteristic peak of the pdf corresponds to shear bands nucleation, while the scale-free range can be linked their spreading [34]. An abstract example of a double well system with a long-range spinodal, where nucleation and propagation (depinning) exponents are different, is discussed in [60].

Around \( \delta \sim 0.42 \) the pre- and post-yield exponents collapse again as the system undergoes a critical BD transition (regimes \( G \) and \( G' \) in Fig. 3(a)). The exponent stabilize around \( \tau \sim 1.3 \), which is close to the observed value for slip-size statistics in nano-pillars (both FCC and BCC) [8, 15]; the same value characterizes plastic yield in amorphous solids [61–64]. The criticality emerges within a broad range of \( \delta \), which is not uncommon for systems with long-range correlations, where the presence of rare but strong spatial disorder fluctuations can divide the system into spatial regions which independently undergo the transition [65]. More specifically, for a representative sub-system, characterized by the (average) stress-strain relation \( \sigma(\gamma) \) and effectively loaded through an elastic matrix with stiffness \( \mu \), the condition of criticality is \( d\sigma/d\gamma = -\mu, d^2\sigma/d\gamma^2 = 0 \). Then, while the actual critical point in the strain-controlled system will correspond to \( \mu = \infty \), power law distributed avalanches can be also expected in sub-systems exposed to the environments with various other \( \mu \in (0, \infty) \).

As the strength of the disorder increases beyond \( \delta \sim 0.5 \), plastic hardening starts at almost zero stress and scaling is getting lost. Instead of large-scale heterogeneous avalanches, the model predicts homogeneous proliferation of uncorrelated, ‘mild’ [29] plastic activity, not unlike what we see in experiment on 1500 nm samples, see Fig. 1(b).

Figure 4: (a) The divergence of the cut-off size in the pre-yield regime around the critical value of disorder \( \delta \sim 0.42 \). Inset shows the behavior of the fractal dimension \( d_f \); (b) Disorder dependence of the fractal measures \( D_q \) at \( \gamma = 0.8 \); inset reveals multi-fractality at critical disorder.

Figure 5: Scaling collapse for the two cases: (a) \( \delta = 0.30 \) (tuned spinodal criticality) and (b) \( \delta = 0.46 \) (BD criticality). Insets show schematic stress-strain curves around spinodal \( S \) and critical \( C \) points.

To reveal the fine structure of the BD transition we show in Fig. 4(a) the disorder dependence of the cut-off parameter \( E_c \) characterizing stress-integrated pre-yield avalanches. It peaks at critical value of disorder \( \delta \sim 0.42 \), where it diverges with system size, \( E_c \sim N^{\delta/\nu} \). The fractal dimension of avalanches \( d_f \) jumps from the value \( \sim 1 \), in the brittle phase [11, 33], to the value \( \sim 1.4 \) around the BD transition. In Fig. 4(b) we show the disorder dependence of the generalized fractal dimensions \( D_q \) characterizing the plastic strain pattern [30, 66]. For weak disorder \( D_q \sim 1 \) at \( q \geq 1 \) which signals an extreme localization. After the BD transition the fractal dimensions at all scales jump towards the value \( D_q \sim 2.0 \) which indicates that the strain pattern becomes homogeneous. Around the BD transition (\( \delta \sim 0.42 \)), we see the emergence of a turbulence-type multi-fractal pattern: regions of maximum plastic strain spatially cluster on a set with fractal dimension \( \sim 1.5 \), which is characteristic of scale-free systems [61, 67].

Stress-resolved scaling collapse of pre-yield data shows that the critical and the spinodal avalanche statistics are fundamentally different [30]. Thus, in the interval \( 0.3 < \delta < 0.4 \) our data suggest stress-tuned criticality with \( E_c \sim (\sigma - \sigma_c)^{-1/\nu} \) where \( 1/\nu \sim 1.6 \), see Fig. 5(a). Indeed, our spinodal point is associated not only with a particular strain but also with a particular stress. Instead, the asymptotics \( E_c \sim \exp(\sigma/\sigma_0) \), with some constant \( \sigma_0 \) is recorded around \( \delta \sim 0.42 \): here we deal with criticality in a strain-control ensemble, which makes stress poorly constrained, see Fig. 5(b).

To further characterize the disorder-induced crossover, we computed the distribution of stability measures \( \Delta = \xi - \xi_c \) where \( \xi_c = \xi - d(\xi) \) is the elastic strain and \( \xi_c = 0.5 \) is the relevant stability threshold, see Fig. 6. Of particular interest is the zero stability asymptotics \( p(\Delta) \sim \Delta^\theta \) [68]. When disorder is weak, see Fig. 6(a), the pre- and post-yield exponents \( \theta \) agree, and we obtain a gap-less spectrum with \( \theta \sim 0 \) indicating weak criticality [45, 55, 56, 69, 70]. At an intermediate level of disorder, see Fig. 6(b), we observe the emergence of a pseudo-gap signaling spinodal criticality. The
post-yield regimes in this range are characterized by the near zero value of $\theta$, which is consistent with the paradigm of elastic depinning [71]. Around the critical value of disorder, see Fig. 6(c), the exponent $\theta$ shows non-monotone dependence on the applied strain with a strong maximum around $\gamma_y$. This suggests the development of soft collective modes, and the emergence of global connectedness of the energy landscape [72, 73]. At both smaller and larger strains, we observe the same $\theta \sim 0$ asymptotics with dislocation jamming dominating before yield and depinning replacing it after yield.

![Figure 6: Distribution of strain measures $p(\Delta)$ at three characteristic values of disorder $\delta$.](image)

A simple mean field model [29] can be used to rationalize the observed behavior. Assume that the evolution of the spatially averaged density of mobile dislocations $\rho$ is described (on a stress plateau) by a stochastic kinetic equation $\rho \, d\rho/d\gamma = -c + \sqrt{2D} \eta(\gamma)$, where the local shear strain $\gamma$ serves as a time-like parameter, $c \geq 0$ characterizes the rate of dislocation immobilization. The temperature-like parameter $D$ represents the intensity of the multiplicative mechanical noise, $\langle \eta(\gamma) \rangle = 0$ and $\langle \eta(\gamma_1) \eta(\gamma_2) \rangle = \delta(\gamma_1 - \gamma_2)$. Note that the lack of bulk dislocation sources in sub-micron crystals allowed us to neglect the conventional Kocks-Mecking dislocation nucleation term [74].

The stationary probability distribution in such a model is of a pure power law form $p_*\rho(\rho) \sim \rho^{-\alpha}$ with the exponent $\alpha = 1 + c/D$ and our automaton model suggests that $\alpha = \tau$, see [30] for details. For single slip pure nano-crystals dislocation immobilization can be neglected, so $c/D \ll 1$, and the stochastic evolution of $\rho$ reduces to a logarithmic Brownian motion with $\alpha \sim 1$. In such systems dislocation self-organization is governed exclusively by elastic long-range interactions, and we recover our low-disorder jamming limit $\tau \sim 1$, see also [33, 75].

With increasing disorder, one can expect higher immobilization rate $c$ and, therefore, higher value of $\tau$, which is in agreement with our numerical experiments. The crossover from $D$-dominated brittle regimes ($c \ll D$) to $c$-dominated ductile regimes ($c \gg D$) can be expected where the mechanical agitation is balanced by dislocation self-locking ($c \sim D$).

To link these observations to the system size, we need to obtain a quantitative relation between the ratio $c/D$ and the normalized system size $R$. With a simplifying assumption that only defects with the strength above some threshold $h^*$ matter and that they form a regular lattice with spacing $l$ we obtain

$$R \sim (1 - \text{erf}(h^*/(\sqrt{2}\delta)))^{1/2}$$

and our numerical evaluation of the dislocation mean free path suggests that $h \approx 0.395$, see [30]. Given that we know the relation $\tau(\delta)$ for post-yield regimes from our numerical experiments (see Fig. 3(a)), and using the relation $\alpha = \tau$, we can then compute the desired relation between $c/D$ and $R$, see Fig. 7(a).

![Figure 7: (a) Dependence of the exponent $c/D$ in the post-yield regime on the dimensionless parameter $R = L/l$ for $h^* = 0.395$ and $L = 1$; (b) Effect of the `local' disorder $\delta_g$ on the (integrated) scaling exponent $\tau_{in}$ for the case of cyclic loading.](image)

We can interpret the obtained curve as the description of the crossover from nano-crystalline plasticity to micro-crystalline plasticity. Indeed, it is natural to assume that the effective temperature $D$ does not depend on $L$, while diminishing with increasing locking strength of defects, which means that it increases with $L$. At the same time, it is clear that the rate of dislocation reactions (in particular our $c$ controlling immobilization) increases with $L$ [18]. Therefore, in either very small and/or very weakly disordered samples $c < D$ and the response must be brittle. Conversely, in either bigger or more disordered samples one can expect to reach the ductile phase where $c > D$.

The remaining question concerns the role of the local disorder $\delta_g$. To overcome the effects of original preparation we now focus on the cycle-averaged (integrated) exponents $\tau_{in}$ for the case when strain amplitude reaches beyond the yield. The typical stress-strain curves under such loading protocol, illustrated in [30], show that already after the first load reversal, brittleness disappears even for the case of relatively weak disorder. The originally super-critical distribution involving characteristic events, is then replaced by a near critical distribution which maintains a robust range of scale-free behavior. The disorder dependence of the ensuing exponents $\tau_{in}$ is summarized in Fig. 7(b).

When the local disorder is negligible, $\delta_g \sim 0$, we recover the monotone dependence of the after-yield exponent on the nonlocal disorder $\delta = \delta_h$. Note, however, that the value of the integrated exponent $\tau_{in}$ is systematically larger than the value of the corresponding stress resolved exponent $\tau$ due to the stress dependence of the cut-off parameter $E_c$ [76]. Beyond $\delta_g \sim 0.25$ the dependence of the exponent $\tau_{in}$ on $\delta_h$ disappears and the exponent stabilizes around the value $\sim 1.5$, suggesting mean field criticality [43, 59, 78].

In conclusion, based on the study of the minimal model, we showed the existence of at least two clearly defined universality classes of sub-micron plasticity with non-universality ultimately emerging as a size effect. We also revealed the crucial role in this problem of the non-equivalent types of linear disorder.
[68] S. Karmakar, E. Lerner, and I. Procaccia, Phys. Rev. E Stat. Nonlin. Soft Matter Phys. **82**, 055103 (2010).

[69] H. G. E. Hentschel, P. K. Jaiswal, I. Procaccia, and S. Sastry, Phys. Rev. E Stat. Nonlin. Soft Matter Phys. **92**, 062302 (2015).

[70] E. E. Ferrero and E. A. Jagla, Soft matter **15**, 9041 (2019).

[71] D. S. Fisher, Phys. Rep. **301**, 113 (1998).

[72] M. Müller and M. Wyart, Annual Review of Condensed Matter Physics **6**, 177 (2015).

[73] D. Zhang, K. A. Dahmen, and M. Ostoja-Starzewski, Phys Rev E **95**, 032902 (2017).

[74] U. F. Kocks and H. Mecking, Prog. Mater Sci. **48**, 171 (2003).

[75] J. Weiss, Philos. Trans. A Math. Phys. Eng. Sci. **377**, 20180260 (2019).

[76] K. A. Dahmen, Y. Ben-Zion, and J. T. Uhl, Phys. Rev. Lett. **102**, 175501 (2009).

[77] G. Durin and S. Zapperi, J. Stat. Mech. **2006**, P01002 (2006).

[78] K. Dahmen and J. P. Sethna, Phys. Rev. B Condens. Matter **53**, 14872 (1996).

[79] Our association of the variance of disorder with crystal size concerns generic systems without bulk criticality: one can, of course, manufacture small crystals with strong (dense) quenched disorder [18] or grow almost pure large crystals with very weak (sparse) quenched disorder [75].
Supplemental Material for the paper: “Variety of scaling behaviors in nanocrystalline plasticity”

P. Zhang,1 O.U. Salman,2 J. Weiss,3 and L. Truskinovsky4

1State Key Laboratory for Mechanical Behavior of Materials, Xi’an Jiaotong University, Xi’an, 710049, China
2CNRS, LSPM UPR3407, Paris Nord Sorbonne Université, 93430, Villetaneuse, France
3IsTerre, CNRS/Université Grenoble Alpes, 38401 Grenoble, France
4PMMH, CNRS - UMR 7636 PSL-ESPCI, 10 Rue Vauquelin, 75005 Paris, France

(Dated: April 21, 2020)

Experimental results. A millimeter-sized Mo single crystal was cut from a well-annealed Mo ingot with a high purity (> 99.99%). The dislocation structure inside this single BCC crystal was characterized by transmission electron microscopy (TEM), showing straight screw dislocations along \langle 111 \rangle directions (Fig. 1(a)), with a density \rho \approx 1.6 \times 10^{12} \text{ m}^{-2} measured by the line-intercept method. Using these data we estimated the equidistant dislocation spacing to be \lambda \sim 1/\sqrt{\rho} \approx 790 \text{ nm}.

The \langle 112 \rangle-oriented Mo pillars with diameters from 500 nm to 1500 nm were fabricated on the electropolished surface of the single bulk crystal by using a Ga-operated focused-ion beam (FIB). The height-to-diameter ratio of pillars was kept between 2.5:1 and 3:1, and the taper was \sim 1.7\degree (Fig. 1(b)). A nano-indentation system (Hysitron Ti 950) was then used to compress the pillars at room temperature under controlled displacement, with a strain rate of 2 \times 10^{-3} \text{ s}^{-1}. At least four pillars were tested at each value of diameter.

Figure 1. (a) TEM image showing the Grown-in dislocations in the bulk Mo crystal. (b) SEM image of \langle 112 \rangle-orientated micro-pillar before compression.

The obtained force-plastic displacement curves, illustrated in Fig. 2(a), are characterized by a set of abrupt discontinuities. For the chosen pillar orientation, the slip systems with maximum Schmidt factor \( S \) are (101)<-111> and (011)<1-11>. Accordingly, after deformation, we observed the most significant \{110\} slip traces. Therefore, the resolved shear stress and shear strain shown in Fig.1(a) of main text were computed as \( \sigma = \sigma_n S \) and \( \gamma = \gamma_n / S \), where \( \sigma_n \) and \( \gamma_n \) are the corresponding normal components.

Figure 2. Representative force-plastic displacement curve (a) and cumulative probability distribution of plastic displacements \( X \) detected over the entire deformation path (b) for the Mo pillars with 500 nm diameter.

The complex stress-strain configuration of the observed jumps can be explained by the delayed instrumental response during rapid plastic deformation, see [1] for more detail. Here we only briefly mention that the mechanical loading in such experiments is performed through an auto-regulation system with PID feedback. The loading device adjusts dynamically, and in the case of an avalanche, it usually does not have enough time to respond. As a result, we observe displacement jumps at almost constant force. In fact, the plastic displacement jumps \( X \) were extracted from the recorded force-displacement data by using the post-processing methodology developed in our previous work [1]. More specifically, the size of dislocation avalanches was quantified by the plastic displacement \( X = D_e - D_s + (F_s - F_e) / K_p \), where \( D_s \) and \( D_e \) are the measured displacements values when the jump starts and when it ends, \( F_s \) and \( F_e \) are the corresponding values of the force, and \( K_p \) is the measured stiffness of the pillar. Here it is
anticipated that the computed value of \( X \) scales with the cumulative area swept by dislocations [2]. Then, under the assumption that there is an average dislocation length, \( X \) should scale with the cumulative distance covered by mobile dislocations during an avalanche.

The cumulative probability distributions \( P(X) \) of the displacement bursts for all three sample sizes are shown in Fig. 1(b) of the main text. Here we focus on the super-critical response of the 500 nm sample and show in more detail in Fig. 2(b) the structure of the characteristic (dragon) peak [3] (resulting from experiments on four micro-pillars). The large bursts marked in Fig. 2(b) in violet correspond to brittle-like events marked in Fig. 2(a) by arrows. The largest characteristic events, marked in green, correspond to system size avalanches signaling the “global failure” of the sample (see the event marked by the green arrow in Fig.2(a)). In the meanwhile, as it is typical for spinodal criticality [4], the small events follow a power-law distribution \( P(\tau) \). The corresponding integrated exponent is \( \tau_0 \approx 2.07 \).

Note that in our physical experiments the sub-micron pillars repeatedly almost-yielded during the deformation without entering the post-yield regime. Therefore, the obtained scaling effectively corresponds to pre-yield avalanches. While in our computer experiments, where we used periodic boundary conditions, we observed at small disorder the formation of a system-size slip bands with high dislocation density, in experiment on sub-micron pillars, the dislocations could always annihilate on free surfaces, which was bringing the crystal back into the dislocation starvation state over and over again [5]. Therefore, in physical experiment we observed several large bursts instead of a single one which dominated numerical experiments.

Interestingly, the statistical super-criticality of nano-scale samples was underplayed in the previous studies of the power-law distributed bursts in sub-micron crystals [6–8]. It was reported, however, that in almost pure crystals with negligible number of dislocations, for instance, in sub-micron and nano-particles [5, 9], nanowires [10], or sub-micron and nano-pillars [11], the plastic deformation culminates in a single catastrophic event associated with the formation of a system size slip band. In fact, both phenomena have the same origin and can be ultimately linked to the phenomenon of dislocation starvation [12].

![Figure 3](image.png)

**Figure 3.** Stress-integrated probability distributions of pre-yield avalanches, including the system-size failure-like events, at \( \delta = 0.28 \) (a) and at \( \delta = 0.7 \), \( \gamma = 0.8 \) (b) in the automaton model (computed for 100 realizations of disorder). Insets show the corresponding stress strain curves.

The coexistence of characteristic bursts (SNAP events) with power-law distributed small avalanches is captured by our numerical model which reproduces super-critical behavior and predicts realistic value of the exponent, see Fig. 3(a). In the main text, we showed the corresponding stress resolved exponents; a small difference is due to the fact that we compare the distributions of plastic displacements \( X \) and the emitted energy \( E \). While in our model, the absence of hardening ensures that the dissipated energy and the plastic strain distribution in avalanches are characterized by the same exponent [13], in our physical experiments, where small hardening was present, the perfect agreement could not be expected.

**Reduction to automaton.** We recall that the piece-wise quadratic nature of the energy density \( f \) allows one to solve analytically the elastic problem. In terms of the displacement field \( u_{ij} \) the associated equilibrium equations in the bulk read

\[
K(u_{i+1,j} + u_{i-1,j} - 2u_{ij}) + (u_{i,j+1} + u_{i,j-1} - 2u_{ij}) - (d_{ij} - d_{ij-1}) = 0.
\]

The whole system can be written in matrix form \( Mu = b \), where \( M \) is a pentadiagonal matrix and \( b \) is a vector of size \( N \times N \) incorporating the boundary conditions and the field \( d = [\xi] \).

![Figure 4](image.png)

**Figure 4.** The real-space representation of the dipole Fourier kernel \( \hat{L}(q) \).
For periodic boundary conditions \( u_{0,j} = u_{N-1,j} \) and \( u_{i,0} = u_{i,N-1} + \gamma \), where \( \gamma \) is the loading parameter. Equilibrium equations can be solved analytically in Fourier space and in the main text we presented the solution for the total shear strain \( \hat{\xi}(q) = s(q)\hat{u}(q) \), where \( s(q) = [1 - \cos(q_x) + i\sin(q_x)] \), in the form \( \hat{\xi}(q) = \gamma\hat{\delta}(q) + \hat{L}(q)\hat{\lambda}(q) \). Here the di-polar nature of the Eshelby kernel \( \hat{L}(q) = \frac{\sin^2(q_x/2)}{\sin^2(q_x/2) + \sin^2(q_y/2)} \) reflects the scalar nature of our model. We illustrate the real space form of this kernel in Fig. 4. In the presence of disorder, we need to add to \( \hat{\xi}(q) \) a lattice pre-stress term \( \hat{s}_{\gamma}(q)\hat{H}(q)/\hat{\lambda}(q) \) bringing us to the formulas presented in the main text.

Knowing how to update the elastic fields, we can now formulate the dynamics in the form of a discrete automaton for the integer-valued field \( d \). We recall that our numerical experiments always start with the unloaded (\( \gamma = 0 \)) and dislocation free state \( (d_{i,j} = 0) \). We then advance the loading parameter \( \gamma \) and compute (predict) the elastic field \( u_{i,j} \). The knowledge of the shear strain field \( \hat{s}_{\gamma,i,j} \) allows us to update (correct) the plastic strain field using the relation \( d = [\hat{\xi}] \). We repeat the prediction-correction steps a given \( \gamma \) till the correction stops changing the field \( d_{i,j} \). This means that the stress is globally below the threshold, and the avalanche is over. We then search for the smallest increment of loading parameter \( \delta\gamma \) so that the loading \( \gamma \rightarrow \gamma + \delta\gamma \) destabilizes a single unit (an element appears where \( d_{i,j} \hat{} \neq \hat{}[\hat{\xi}] \)). At that point, we apply our relaxation (prediction-correction) protocol again, initiating an avalanche, and when all the elements are again stable, we advance the loading parameter once more.

A single avalanche transforms a mechanically unstable state \( I \) into a mechanically stable state \( F \). The dissipated energy is calculated as the difference between the total energies before and after the avalanche \( E(I \rightarrow F) = \Phi(I) - \Phi(F) \). The statistics of the avalanche energies \( E \) is used to compute the exponent \( \tau \) and the cut-off energy \( E_c \). The stress resolved quantities were calculated within the strain interval \( 10^{-4} \) and the results were averaged over 100 realizations of disorder.

![Figure 5](image)

To illustrate how dislocations nucleate in our model we show in Fig. 5 two dislocations of opposite signs forming a topologically neutral 'loop'. We assume that \( d_{i,j} = 0 \) everywhere except in the finite region shown in red in Fig. 5(a), where \( d_{i,j} = 1 \). The solution of elastic problem for such distribution of \( d_{i,j} \) and zero disorder is illustrated in Fig. 5(b,c) where we show the axial stress \( \sigma_{xx} \) and the shear stress field \( \sigma_{xy} \). Coloring indicates the level of stress: blue - low; red - high.

To illustrate how dislocations nucleate in our model we show in Fig. 5 two dislocations of opposite signs forming a topologically neutral 'loop'. We assume that \( d_{i,j} = 0 \) everywhere except in the finite region shown in red in Fig. 5(a), where \( d_{i,j} = 1 \). The solution of elastic problem for such distribution of \( d_{i,j} \) and zero disorder is illustrated in Fig. 5(b,c) where we show the axial stress \( \sigma_{xx} \) and the shear stress field \( \sigma_{xy} \). The far-field asymptotics agrees with the classical continuum result \( r^{-1} \), while inside the cores, located around the units where \( d_{i+1,j} - d_{i,j} \neq 0 \), the fields remain finite. In our simulations such topologically neutral dislocation couples nucleated spontaneously in the areas of stress inhomogeneity.

**Simulated strain fields and their fractal structure.** In Fig. 6 we show the shear strain patterns for samples with disorder strength marked in Fig. 2 of the main text by the letters A, B, C and D, where the total number of dislocations was 782, 2600, 5220 and 8862, respectively. In the main text, only a subset of dislocations was shown within a window of the size 280 \( \times \) 200. Here we show the corresponding strain and stress fields together with the blow up view of the marked out windows. In the case A, we see the formation of a narrow shear band with sharp boundaries and some relatively homogeneous dislocation activity outside. In the case B, the shear band is broader, and its boundaries are more diffuse. In the near-critical case C, the shear band loses distinct boundaries and fills the whole domain. Finally, in the case D, we see dislocation activity all over the domain, which is no longer centered around a single maximum of the dislocation density.

To illustrate the intricate dislocation patterning, hidden behind this gradual disorder-induced increase of the homogeneity of the deformation field, we present in Fig. 7 highly magnified images of the axial stress field \( K\zeta_{i,j} \) and the shear stress field \( \xi_{\gamma,i,j} - d_{i,j} \). In these images, the affine component of these fields is subtracted, and the resolution allows one to trace the location of individual dislocations. In particular, we see that when the disorder is weak (case A), the shear band features crack-like correlated arrangement of dislocations. Outside the band, the dislocations distribution is relatively uniform, although one can still trace few incipient shear pre-bands. As the strength of the disorder increases (case B), the dislocation density inside the shear band diminishes as it becomes progressively broader. One can interpret this broadening as a propagation of the shear band boundaries towards the boundaries of the sample. In the near-critical case C, we lose a singular band which is being replaced by a diffuse network of an interconnected pre-bands. Finally, in the ductile phase, case D, the defect micro-structure is so tangled that no obviously coherent pattern can be detected.

In order to quantify these observations, we performed a multi-fractal analysis of the field of plastic strain \( d_{i,j} \). To this end we
Figure 6. Shear strain field $\xi_{ij}$ right after yield at different strengths of disorder: $\delta = 0.28$ (A), $\delta = 0.36$ (B), $\delta = 0.42$ (C) and $\delta = 0.46$ (D). Zoom in to the marked areas is shown on the second column.

covered the deformed lattice with a regular array of boxes of size $L_b$ and summed plastic strain in $m^{th}$ box to obtain $N_m(L_b) = \sum d$, where the sum is taken over the mesoscopic units covered by a given box. Then, the density of plastic strain associated with the $m^{th}$ box is $p_m(L_b) = \frac{M_m(L_b)}{\sum_{l=1}^{n(L_b)} M_l(L_b)}$, where $n(L_b)$ is the number of boxes covering the lattice. The moments of order $q$ of this density distribution are $M_q(L_b) = \sum_{m=1}^{n(L_b)} p_m^q(L_b)$.

If the deformation pattern is self-similar, we should observe the scaling $M_q(L_b) \sim L_b^{(q-1)D_q}$, which defines the set of generalized fractal dimensions $D_q(q \neq 1)$. The singular value $D_1$ would be defined as the proportionality coefficient between $\sum_{m} n(L_b) p_m \log(p_m)$ and $\log(L_b)$, and one would expect that that $D_q(q \neq 1) \rightarrow D_1$ as $q \rightarrow 1$ [14, 15]. To check whether the obtained patterns of plastic strain are self-similar, we focused on a particular value of the loading parameter $\gamma = 0.8$, where the steady flow conditions have been already achieved for all representative values of disorder $\delta$. Our results are summarized in Fig. 8, where the anticipated scaling $M_q(L_b) \sim L_b^{(q-1)D_q}$ is apparent. The obtained values of the exponents $D_q$ are discussed in the main text.

Note that the obtained scaling relations are truncated at some cut-off scales $L_b^*(\delta)$ characterizing the spacing of the slip traces, when disorder is week, and the spacing of the strong dislocation locks, when disorder is strong [14]. The conspicuous absence of the cut-off $L_b^*(\delta)$ at $\delta = 0.42$ indicates that at critical conditions the hierarchical micro-structure extends all the way towards the scale of the lattice elements. Indeed, since the fractal dimensions of a single slip trace and of the global plastic field are the same, the plastic strain filed is self-similar, which is the distinct feature of developed criticality.

 Scaling collapse. Here we consider the avalanche size distribution at different levels of the external stress (below the yield stress). In near critical regimes the stress-resolved energy distribution should be of the form $p(E; \sigma) \sim E^{-\tau} f(-E/E_c(\sigma))$, where the function $f$ is universal while the function $E_c(\sigma)$ depends on the type of criticality. The latter is dictated in our case only by the strength of the disorder (proxy for the system size). To check this hypothesis we re-plot our data using the normalized

\[
p(E; \sigma) \sim E^{-\tau} f(-E/E_c(\sigma))
\]
variables $E' = E/E_c$ and $p' = p(E)E_c^*$, see Fig. 9.

We find two distinct regimes where the data collapse could be achieved. In the interval $0.3 < \delta < 0.4$, our Fig. 9(a) shows the validity of the ansatz in the cutoff region with $E_c(\sigma) \sim (\sigma_y - \sigma)^{-1/v}$. Here $\sigma_y$ is the yield stress at the loading level $\gamma$, and $v$ is a constant. This scaling suggests tuned criticality. From the inset in Fig. 9(a) we conclude that $1/v \approx 1.6$, which is different, for instance, from the value predicted in the theory of mean-field depinning where $1/v = 2.0$ [16]. We recall that at this level of disorder yield is associated with reaching the (strain controlled) spinodal point and it is natural that the associated super-criticality is stress-tuned [4, 17].

The second region of scaling collapse is around $\delta \sim 0.42$, see Fig. 9(b). Here the cutoff follows the asymptotics $E_c(\sigma) \sim \exp(\sigma/\sigma_0)$, where $\sigma_0$ is a constant, see the inset in Fig. 9(b). The fact that the cutoff $E_c$ does not diverge at a particular value of external stress and instead exhibits an exponential stress dependence, suggests that the associated scaling is not of the tuned-type.
In the main text we associate this type of scaling with the fact that the critical point at $\delta \sim 0.42$ is localized in strain but not in stress.

**Dislocation density, avalanche energy and the internal length scale.** We interpret the value of the parameter $\rho(\gamma)$ in the mean field model, formulated in the main text, as the density of mobile dislocation during an avalanche at a given value of the loading $\gamma$. It is then a measure of how many dislocations have moved during an avalanche.

![Figure 9](image_url)

**Figure 9.** Scaling collapse for the two cases: (a) $\delta = 0.30$ (tuned spinodal criticality) and (b) $\delta = 0.46$ (extended BD criticality).

We used the automaton model to study empirically the behavior of the dislocation density defined as $\rho(\gamma) = n(\gamma)/N^2$, where $n(\gamma)$ is the number of dislocations moved at the loading value $\gamma$. In Fig. 10(a) we plotted $\rho(\gamma)$ as a function of the avalanche energy $E(\gamma)$. In view of the almost linear nature of the obtained dependence we conclude that the exponent in the mean field model $\alpha = c/D$ can be identified with the exponent $\tau$ characterizing the distribution of the avalanche energy $E$.

The observed linearity at a given strength of the disorder suggests that at a given system size avalanches vary only by how many dislocations are mobile (at the current value of strain) and not by the area slipped by each dislocation. This observations allows us to extract from our computational data the value of the internal length scale. Indeed, our Fig. 10(b) shows that in the automaton model the avalanche energy $E$ is a disorder independent linear function of the total distance traveled by mobile dislocations during an avalanche $\bar{l}$. Given that $E \sim \rho$ we can conclude that $\bar{l} \sim n$ with the actual dependence shown in Fig. 10. We can then extract the characteristic length, covered by a dislocation before it gets immobilized, from the relation $\bar{l} = \bar{l}/n$. The ensuing dependence on $\delta$, reflecting the fact that the spacing of the locking sites is controlled by the strength of the disorder, is shown in Fig. 10. This figure also shows that the definition of $\bar{l}$ proposed in the main text is compatible with our empirical observations if the stress threshold is chosen at the value $h^* = 0.395$. In other words, our automaton model confirms that the mean free path of dislocations $l$, setting an intrinsic internal length scale, is indeed controlled by the tails of the disorder distribution.

**Cyclic loading.** In the main text, we mainly focused on the mechanical response of nano-crystals to monotone loading. Such response necessarily carries a dependence on the initial state. The chosen preparation was entirely dislocation-free, as the goal was to simulate plastic deformation of ultra-small systems (nano-particles and nano-pillars) [18, 19]. To see the response of the naturally dislocation-rich crystals, we can subject the system to cyclic loading with strain amplitude extending beyond the yield.

![Figure 10](image_url)

**Figure 10.** (a) Relation between the density of mobile dislocations $\rho$ and avalanche energy $E$ for representative disorders $\delta = 0.30, 0.42$; (b) dependence of $E$ on the total distance covered by dislocations during an avalanche $\bar{l}$; (c) dependence of $\bar{l}$ on the number of moving dislocations $n$; (d) Internal length scale $l$ as a function of disorder strength $\delta$. 
In this way we avoid non-generic quasi-elastic periodic regimes.

Figure 11. Strain-stress curves for the crystals loaded six cycles. (a) $\delta_h = \delta = 0.30$, $\delta_g = 0.00$ and (b) $\delta_h = \delta = 0.30$, $\delta_g = 0.30$.

The typical stress-strain curves generated by our automaton model subjected to cyclic loading are illustrated in Fig. 11(a) for $\delta_h = 0.0$ and in Fig. 11(b) for $\delta_h = 0.35$. In the absence of local disorder $\delta_h$, see Fig. 11(a), brittleness disappears already after the first load reversal even for the case of relatively weak non-local disorder $\delta_g$. The effective decrease of the yield stress, which can be interpreted as strain softening, is in stark contrast with the classical strain-hardening observed in the cyclic deformation of various bulk FCC and BCC materials and resulting from dislocations interlocking [20]. Given that brittleness disappears already after the first unloading, suggests that nano-crystals can be ‘pre-trained’ to become ductile, see also [21], which is crucial for applications in view of the vulnerabilities of brittle ultra small structures [1]. When significant local disorder is introduced, see Fig. 11(b), brittleness disappears completely even though the stress-strain curves still show a robust yielding plateau which stabilizes after a finite shakedown transient. This behavior is highly reminiscent of cyclic strain-hardening in bulk FCC and BCC materials.

In Fig. 12(a) we show how a representative avalanche distribution changes after the first cycle for $\delta_h = 0.30$ and $\delta_g = 0.0$ (nonlocal disorder only). One can see that the originally super-critical distribution, involving characteristic system size events, is replaced by a near critical distribution which maintains a robust range of scale-free behavior with roughly the same exponent $\tau$. This distribution remains stable after the shakedown.

In Fig. 12(b) we show the case when sufficiently strong local disorder ($\delta_h \sim 0.30$) suppresses super-criticality. The fact that the cut-off increases for the second cycle suggests self-organization of dislocations towards criticality which was linked in the main text with the mean field RFIM/depinning universality class.

[1] P. Zhang, O. U. Salman, J.-Y. Zhang, G. Liu, J. Weiss, L. Truskinovsky, and J. Sun, Acta Mater. 128, 351 (2017).
[2] R. Maaß, P. M. Derlet, and J. R. Greer, Scr. Mater. 69, 586 (2013).
[3] D. Sornette and G. Ouillon, Eur. Phys. J. Spec. Top. 205, 1 (2012).
[4] H. B. da Rocha and L. Truskinovsky, Phys. Rev. Lett. 124, 015501 (2020).
[5] Z.-J. Wang, Z.-W. Shan, J. Li, J. Sun, and E. Ma, Acta Mater. 60, 1368 (2012).
[6] M. Zaiser, J. Schwerdtfeger, A. S. Schneider, C. P. Frick, B. G. Clark, P. A. Gruber, and E. Arzt, Philos. Mag. 88, 3861 (2008).
[7] Y. Cui, G. Po, P. Srivastava, K. Jiang, V. Gupta, and N. Ghoniem, Int. J. Plast. 124, 117 (2020).
[8] G. Sparks, Y. Cui, G. Po, Q. Rizzardi, J. Marian, and R. Maaß, Phys. Rev. Materials 3, 080601 (2019).
[9] D. Chrobak, N. Tymiak, A. Beaber, O. Ugurlu, W. W. Gerberich, and R. Nowak, Nat. Nanotechnol. 6, 480 (2011).
[10] Y. Lu, J. Song, J. Y. Huang, and J. Lou, Nano Research 4, 1261 (2011).
[11] H. Bei, S. Shim, G. M. Pharr, and E. P. George, Acta Mater. 56, 4762 (2008).
[12] J. R. Greer and W. D. Nix, Phys. Rev. B Condens. Matter 73, 245410 (2006).
[13] O. U. Salman and L. Truskinovsky, Int. J. Eng. Sci. 59, 219 (2012).
[14] J. Weiss, Eng. Fract. Mech. 68, 1975 (2001).
[15] M. A. Lebyodkin and T. A. Lebedkina, Phys. Rev. E Stat. Nonlin. Soft Matter Phys. 73, 036114 (2006).
[16] K. A. Dahmen, Y. Ben-Zion, and J. T. Uhl, Phys. Rev. Lett. 102, 175501 (2009).
[17] I. Procaccia, C. Rainone, and M. Singh, Phys. Rev. E 96, 032907 (2017).
[18] A. Sharma, J. Hickman, N. Gazit, E. Rabkin, and Y. Mishin, Nat. Commun. 9, 4102 (2018).
[19] D. Mordehai, O. David, and R. Kositski, Adv. Mater. 30, 1706710 (2018).
[20] S. Suresh, Fatigue of Materials (Cambridge University Press, 1998).
[21] S. Papanikolaou, Y. Cui, and N. Ghoniem, Modell. Simul. Mater. Sci. Eng. 26, 013001 (2017).