On two-weight codes

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Abstract

We consider q-ary block codes with exactly two distances: d and d + δ. Several combinatorial constructions of optimal such codes are given. In the linear case, we prove that under certain conditions the existence of such linear 2-weight code with δ > 1 implies the following equality of great common divisors: (d, q) = (δ, q). Upper bounds for the maximum cardinality of such codes are derived by linear programming and from few-distance spherical codes. Tables of lower and upper bounds for small q = 2, 3, 4 and q n < 50 are presented.

Key words. Two-weight codes, Bounds for codes, Linear two-weight codes

AMS subject classification. 94B65, 94B05

1 Introduction

Let $E_q = \{0, 1, \ldots, q - 1\}$, where $q \geq 2$ is a positive integer. Any subset $C \subseteq E_q^n$ is called a code and denoted by $(n, N, d)_q$; i.e., a code of length n, cardinality $N = |C|$ and minimum (Hamming) distance $d$. For linear codes we use notation $[n, k, d]_q$ (i.e., $N = q^k$). An $(n, N, d)_q$ code $C$ is equidistant if for any two distinct codewords $x$ and $y$ we have $d(x, y) = d$, where $d(x, y)$ is the (Hamming) distance between $x$ and $y$. A code $C$ is constant weight and denoted $(n, N, w, d)_q$ if every codeword is of weight $w$. For the binary case, i.e. when $q = 2$ we omit $q$ and use the notations $(n, N, d)$ and $[n, k, d]$.

We consider codes with only two distances $d$ and $d + \delta$. Such codes are classical object in algebraic coding theory during already more than 50 years. A comprehensive survey of such codes can be found in the paper of Calderbank and Kantor [7]. Nevertheless in spite of many infinite classes of two-weight codes the complete classification of such codes is far from to be completed. Our purpose here is to understand the structure of two-weight codes and to consider the general properties of all such codes. We believe that for many possible values of $\delta$ such linear codes of dimensions larger than 2 do not exist. In particular, we prove that if there exist a q-ary linear code $C$ with two distances $d$ and $d + \delta$ where $\delta > 1$, then either $(q, d) = (q, \delta)$ or $(q, d_c) = (q, \delta)$, where $d_c$ is the minimum distance of complementary code $C_c$ with two distances $d_c$ and $d_c + \delta$, which coexists with code $C$. It generalizes previous results of Delsarte for projective codes to arbitrary
linear two-weight codes. The case $\delta = 1$ was considered in our previous paper [4], where we classified all such linear codes with distances $d$ and $d + 1$, derived upper bounds for the maximum possible cardinality in this case and presented tables for the maximal possible cardinality for small alphabets and lengths. Here we also give lower and upper bounds for maximum cardinality of codes with two distances $d$ and $d + \delta$ and give tables of such linear and nonlinear codes.

Denote by $(n, N, \{d, d + \delta\})_q$ an $(n, N, d)_q\text{-code}$ $C \subset E^n_q$ with the property under investigation: for any two distinct codewords $x$ and $y$ from $C$ we have $d(x, y) \in \{d, d + \delta\}$. We are interested in existence, constructions, and classification results and lower and upper bounds on the maximal possible size of $(n, N, \{d, d + \delta\})_q$ codes. If $q$ is a prime power, then we $E_q$ will be the set of the elements of the Galois field $\mathbb{F}_q$. In this case, if an $(n, N, d)_q\text{-code}$ $C$ is a $k$-dimensional subspace of the linear space $E^n_q$, then we use for $C$ the standard notation $[n, k, d]_q$, where $N = q^k$, and a two-weight $(n, N, \{d, d + \delta\})_q\text{-code}$ $C$ will be denoted as $[n, k, \{d, d + \delta\}]_q$.

**Definition 1.** A two weight $(n, N, \{d, d + \delta\})_q\text{-code}$ $C$ is called trivial, if it satisfies one of the following properties:

1. $C$ contains trivial positions, i.e. all its codewords contain the same symbol on some position;
2. $C$ is an equidistant code;
3. $C$ has two distances $d$ and $d + 1$, which is obtained from equidistant codes by adding or deleting one position;
4. $C$ can be presented as a concatenation of several two-weight codes with the same parameters:

$$C = (C_1 | \cdots | C_s) = \{c^{(i)} = (c^{(i)}_1 | \cdots | c^{(i)}_s) : c^{(i)} \in C, c^{(i)}_j \in C_j, j = 1, \ldots, s, i \in \{1, \ldots, N\}\},$$

where the every code $C_j$ is a two-weight code with the same parameters, i.e. $C$ is an $(sn, N, \{sd, sd + s\delta\})_q\text{-code}$, where $C_j$ is an $(n, N, \{d, d + \delta\})_q\text{-code}$ for every $j \in \{1, \ldots, s\}$.

Recall that all linear codes with two distances $d$ and $d + 1$ were described in our recent papers [3, 4] (see also [13]). The mentioned above comprehensive survey of Calderbank and Kantor [7] gives complete state (for that time) of this subject mostly in terms of geometric concepts. Here we are going to show that many classes of optimal $[n, k, \{d, d + \delta\}]_q$ codes can be obtained from two $q$-ary linear equidistant codes $A$ and $B$ with additional property that $B$ is a subcode of $A$ by one of several simple combinatorial constructions. We show that better upper bounds for the maximum cardinality of such codes (with two distances $d$ and $d + \delta$) in comparison with bounds in the case when we know only the minimum distance $d$ of the code are possible. We give some new upper bounds for such codes, based on linear programming arguments and also based on known results for two-distance spherical codes. We present several tables with lower and upper bounds for codes with two weights, obtained by computer search and by direct combinatorial constructions.

## 2 Preliminary results

Denote by supp$(x)$ the set of coordinate positions, where the vector $x = (x_1, \ldots, x_n)$ from $E^n_q$ has nonzero coordinates,

$$\text{supp}(x) = \{j : x_j \neq 0, j = 1, \ldots, n\}.$$

For a subset $X \subseteq E^n_q$ define its support supp$(X)$ as

$$\text{supp}(X) = \{j \in \text{supp}(x) : x \in X\}.$$
For a code $C$ from $E_q^n$ with $\text{supp}(C)$ and any set $S \subseteq \text{supp}(C)$ we say that $C_S$ is a projection of $C$ onto $S$ if
\[ C_S = \{ c_S : c \in C \}, \]
where $x_S$ is a projection of $x$ into $S$, i.e. $x_S$ is a vector of length $|\text{supp}(S)|$ which coincides with $x$ in all positions $i$ from $\text{supp}(S)$.

Recall the following result on existence of a large class of nonlinear equidistant codes (which contains a large class of linear such codes) from [17].

**Lemma 1.** Let $p$ be a prime and let $s, \ell, h$ be any positive integers. Then there exists an equidistant $(n_a, N_a, d_a)_{q_a}$ code $A$ with parameters
\[ q_a = p^{sh}, \quad n_a = \frac{p^{s(h+\ell)} - 1}{p^s - 1}, \quad N_a = p^{s(h+\ell)}, \quad d_a = p^s \cdot \frac{p^{sh} - 1}{p^s - 1}. \]

From the recurrent construction in [17] we obtain immediately the following

**Proposition 1.** 1). If $N = q^s$, i.e. $\ell$ is a multiple of $h$, then the code $A$ is linear. In this case $A$ is a well known equidistant code (dual to $q$-ary Hamming code) with the following parameters (let $q_a = q$, $\ell + 1 = m$):
\[ q = p^s, \quad n_a = \frac{q^m - 1}{q - 1}, \quad N_a = q^m, \quad d_a = q^{m-1}. \]

2). For any $j, j = 1, \ldots, m-1$, the code $A$ has a linear equidistant subcode $B_1(j)$ with parameters
\[ q_b = q, \quad n_b = \frac{q^j - 1}{q - 1}, \quad N_b = q^j, \quad d_b = q^{j-1}. \]

3). For any $i, i = 1, \ldots, m - 1$, the code $A$ has a linear subcode $B_2(i)$ with two distances $d_b$ and $d_b + \delta_b$ with parameters
\[ q_b = q, \quad n_b = q^i, \quad N_b = q^{i+1}, \quad d_b = q^i - q, \quad \delta_b = q^{i-1}. \]

For two codes $A$ and $B = \{ y_j : j = 0, 1, \ldots, N_b - 1 \}$ with parameters $(n_a, N_a, d_a)_{q_a}$ and $(n_b, N_b, d_b)_{q_b}$, such that $E_{q_b} \subseteq E_{q_a}$ and $N_b = q_a$, define a new code $C$ over $E_{q_b}$ (which is called a concatenated code, or a concatenation of $A$ and $B$), such that
\[ C = \{ (y_{x_1}, y_{x_2}, \ldots , y_{x_n}) : x = (x_1, x_2, \ldots , x_n) \in A \}, \]
where the every symbol $i \in E_{q_a}$ of codewords of $A$ we change by codewords $y_i$ of $B$ (with index $i$). The code $C$ has parameters $[n, N, d]_{q_b}$, where
\[ n = n_a n_b, \quad d \geq d_a d_b, \quad N = N_a, \quad q = q_b. \]

**Definition 2.** Let $G$ be an abelian group of order $q$ written additively. A square matrix $D$ of order $q\mu$ with elements from $G$ is called a difference matrix and denoted $D(q, \mu)$, if the component-wise difference of any two different rows of $D$ contains any element of $G$ exactly $\mu$ times.

See [2] for difference matrices. From Lemma [1] we have the following result [17].
Lemma 2. For any prime number $q$ and any natural numbers $\ell$ and $h$ there exists a difference matrix $D(q^\ell, q^h)$.

Clearly the matrix $D(q, \mu)$ induces an equidistant $(q\mu - 1, q\mu, \mu(q - 1))_q$ code and also a non-linear two-weight $(q\mu, q^2\mu, \{\mu(q - 1), q\mu\})_q$ code [17], which is optimal according to the following $q$-ary Gray-Rankin bound [1, 10]. Any $q$-ary $(n, N, \{d, n\})_q$-code, whose codewords can be partitioned into trivial subcodes $(n, q, n)_q$ (we call such codes antipodal), has cardinality $N$ such that

$$\frac{N}{q} \leq \frac{q(qd - (q - 2)n)(n - d)}{n - ((q - 1)n - qd)^2},$$

under condition that $n - ((q - 1)n - qd)^2 > 0$. Note that this bound is a $q$-ary analog of the following classical Gray-Rankin bound for a binary antipodal $(n, N, d)$-code $C$

$$N \leq \frac{8d(n - d)}{n - (n - 2d)^2}.$$

We give also the Griesmer bound, which is very often reached by two-weight linear codes. The minimal possible length $n = n(k, d)_q$ of any linear $q$-ary $[n, k, d]_q$-code satisfies the following inequality (which is called Griesmer bound [11]):

$$n_q(k, d) \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil.$$

Recall also Plotkin upper bound:

$$N_q(n, d) \leq \frac{qd}{qd - (q - 1)n},$$

if $qd > (q - 1)n$.

3 Lower bounds

We formulate several simple (and well known) constructions of two-weight codes. We show here that many classes of optimal such codes can be obtained from two $q$-ary linear equidistant codes $A$ and $B$ with additional property that $B$ is a subcode of $A$ by one of several simple combinatorial constructions. One of our purposes here is to give several infinite families of optimal linear two-weight codes, whose optimality was not mentioned before. Some of these codes are optimal according to well known upper bounds and some according to new upper bounds, derived in the present paper.

3.1 Constructions and examples

The construction of the most of these codes depend on two initial equidistant codes $A$ over $E_{q_a}$ and $B$ over $E_{q_b}$, where $B$ is always a subcode of $A$. For the two first constructions we assume that $q_a = q_b = q$. 

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Construction E.1. We delete the set supp(B) from supp(A), so supp(C) = supp(A) \ supp(B) is a projection of A into support of its subcode B.

Construction E.2. We add the set supp(B) to supp(A), so supp(C) = supp(A) \ supp(B).

Next, we use two types of different concatenation constructions, denoted by E.3 and E.4. In E.3 the outer code A is a two-distance code, and the inner code B is equidistant. In E.4 the code A is equidistant, but B has two weights. Finally, we use all these constructions for the case when the code A is partitioned either into the subcode B, or into translates of B.

We illustrate these constructions with several simple (and known) examples of linear two-weight codes. All codes which we consider here are optimal and the only reason, that we describe these examples, is to show their optimality (which was not done before).

Example 1. (Constructions E.1 and E.2, q = 2) Clearly binary Hadamard (or Reed-Muller of the first order) codes form a family of optimal (by Plotkin bound 3) linear two-distance codes with parameters:

\[ n = 2^m, \ k = m + 1, \ d = 2^{m-1}, \ \delta = 2^{m-1}, \ m = 1, 2, \ldots. \]

By Construction E.1 from the equidistant Hadamard \([2^m - 1, m, 2^{m-1}]\) code A, choosing a subcode B with parameters \([2^r - 1, r, 2^{r-1}]\), where \(r = 2, 3, \ldots, m-2\), we obtain a family of binary linear two-weight codes with parameters

\[ n = 2^m - 2^r, \ k = m, \ d = 2^{m-1} - 2^{r-1}, \ \delta = 2^{r-1}, \ r = 1, 2, \ldots, m - 1 \]  

(4)

(These codes are the well known McDonald codes; the family SU1 in [4]). All these codes are optimal, since they meet the Griesmer bound (and also the bound given in Theorem 1). Then by Construction E.2 we obtain from the same subcodes B the family of binary linear two-weight codes with the following parameters:

\[ n = 2^m + 2^r - 2, \ k = m, \ d = 2^{m-1}, \ \delta = 2^{r-1}, \ r = 1, 2, \ldots, m - 1. \]  

(5)

The small codes from [4] and [5] are found also by our program for random search (subsection 3.2). The \((18, 16, \{8, 10\})\) code from [5] (obtained for \(m = 4\) and \(r = 2\)) is not the best – there is better nonlinear code as our program finds an \((18, 19, \{8, 10\})\) code.

Example 2. (Difference Matrices) By Lemma 2 for any prime \(p\) and any two positive integers \(\ell\) and \(h\), there exists a difference matrix \(\mathbf{D}(p^\ell, p^h)\). This matrix \(\mathbf{D}\) induces an (optimal) \(q\)-ary equidistant code with parameters \(n = s(p^{h+\ell} - 1), \ d = s(p^\ell - 1)p^h, \ N = p^{h+\ell}\), and \(q = p^\ell\), where \(s \geq 1\). In turn \(\mathbf{D}\) induces the following family of optimal two-weight codes [17]:

\[ n = sp^{h+\ell}, \ N = p^{h+2\ell}, \ d = sp^h(p^\ell - 1), \ \delta = sp^h, \ q = p^\ell, \ h, \ell \in \{1, 2, \ldots\}. \]  

(6)

These codes are evidently optimal according to Plotkin bound 4 and also according to the \(q\)-ary Gray-Rankin bound 1 (see 1).

In the smallest cases \(p = 2\) and \(p = 3\) we obtain the following two families of optimal two-weight codes:

\[ n = 2^{\ell+h}, \ N = 2^{2\ell+h}, \ d = 2^h(2^\ell - 1), \ \delta = 2^h, \ h, \ell \in \{1, 2, \ldots\}; \]  

(7)
and
\[ n = 3^{\ell+h}, \quad N = 3^{2\ell+h}, \quad d = 3^h(3^\ell - 1), \quad \delta = 3^h, \quad q = 3^\ell, \quad h, \ell \in \{1, 2, \ldots\}. \tag{8} \]

The small binary codes from (4) and the ternary (9, 27, \{6, 9\}) code from (8) are found by our random search program. For \( q = 4 \) the \((8, 32, \{6, 8\})_4\) code beats the best found by the program by 5 points (as expected, the program is not that strong for larger \( q \)).

For \( q = 2 \) the smallest nontrivial case in Definition (2) is \( q = 2 \) and \( \mu = 4 \). It gives the Hadamard \((8, 16, \{4, 8\})\) code. In the next two smallest cases \( (q = 4, \mu = 2 \) and \( q = 4, \mu = 4 \)) we obtain two optimal quaternary codes: \((8, 32, \{6, 8\})_4\) and \((16, 64, \{12, 16\})_4\).

For \( q = 3 \) and \( \mu = 3 \) it gives an optimal \((9, 27, \{6, 9\})_3\) code. The next two cases \( (q = 3, \mu = 9 \) and \( q = 9, \mu = 3 \) give optimal \((27, 81, \{18, 27\})_3\) and \((27, 243, \{24, 27\})_9\) codes, respectively.

**Example 3.** (Constructions E.1 and E.2) The linear equidistant \([n, k, d]_q\)-code \( A \) (Lemma (4)), which is dual to the \( q \)-ary Hamming code of length \( n \), has the following parameters:
\[ n = \frac{q^m - 1}{q - 1}, \quad k = m, \quad d = q^{m-1}. \]

By Proposition (4) this code contains as a subcode a linear \( q \)-ary equidistant code \( B \) with parameters
\[ n_B = \frac{q^r - 1}{q - 1}, \quad k_B = r, \quad d = q^{r-1}, \quad r = 2, 3, \ldots, m - 1. \]

Taking \( s \) copies of \( A \) and \( h \) copies of \( B \), we obtain by Construction E.1 the following family of linear \( q \)-ary two-weight codes (the family \( SU1 \) in (7)):
\[ n = \frac{s(q^m - 1) - h(q^r - 1)}{q - 1}, \quad k = m, \quad d = sq^{m-1} - hq^{r-1}, \quad \delta = hq^{r-1}, \tag{9} \]

where \( r = 2, \ldots, m - 1 \) and \( 1 \leq h \leq s \). For any \( s \) and \( h \), such that \( s, h \leq q - 1 \) and \( h \leq s \) these codes are optimal, since they meet the Griesmer bound (2) (and also the bound of Theorem 4).

According to (2),
\[ n \geq \sum_{i=0}^{m-1} \left\lceil \frac{sq^{m-i} - hq^{r-i}}{q^i} \right\rceil \]
\[ = (sq^{m-1} - hq^{r-1}) + (sq^{m-2} - hq^{r-2}) + \cdots + \left\lceil \frac{sq^{m-r} - h}{q} \right\rceil + \left\lceil \frac{sq^{m-r-1}}{q} \right\rceil + \cdots + s \]
\[ = \frac{s(q^m - 1) - h(q^r - 1)}{q - 1} = n, \]

indeed, since for \( h \leq q - 1 \) we have that
\[ \left\lceil \frac{sq^{m-r} - h}{q} \right\rceil = sq^{m-r-1}, \]
i.e. we obtain the exact equality in (2). Taking again \( s \) copies of \( A \) and \( h \) copies of \( B \), we obtain by Construction E.2 the following family of linear two-weight codes:
\[ n = \frac{s(q^m - 1) + h(q^r - 1)}{q - 1}, \quad k = m, \quad d = sq^{m-1}, \quad \delta = hq^{r-1}, \quad r = 2, \ldots, m - 1, \quad 1 \leq h \leq s. \tag{10} \]


Example 4. (Constructions E.3 and E.4) Let $q = p^m \geq 4$ be a prime power and $2 \leq r \leq q + 1$. From the outer MDS $[r, 2, \{r - 1, r\}]_q$-code $A$ and the inner equidistant (simplex) code $B$ with parameters

$$n_b = \frac{p^m - 1}{p - 1}, \quad k_b = m, \quad d_b = p^m - 1, \quad q_b = p,$$

we obtain by Construction E.3 the following family of two-weight linear $p$-ary codes with the following parameters (the family SU2 in [4]):

$$n = r \frac{p^m - 1}{p - 1}, \quad k = 2m, \quad d = (r - 1)p^m - 1, \quad \delta = p^m - 1, \quad r = 2, \ldots, q + 1. \quad (11)$$

All these codes are optimal by the bound (28) (see Theorem 4 below). Indeed, since (26) and (27) are satisfied, we have for the denominator of (28)

$$n(q - 1)(nq - n + 1) - q^2(2nd + n\delta - d^2 - d\delta) + nq(2d + \delta) = r^2 - r,$$

which implies the exact equality in (28).

Consider the smallest case $q = 4 = 2^2$, i.e. $p = 2$, $m = 2$. The code $A$ is an $[r, 2, \{r - 1, r\}]_4$ code and $B$ is the equidistant $[3, 2, 2]$-code. As a result, we obtain binary linear $[3r, 4, \{2(r - 1), 2r\}]$-codes for $r = 2, 3, 4$. For $r = 3, 4$ the resulting $[9, 4, \{4, 6\}]$ and $[12, 4, \{6, 8\}]$ codes are optimal according to Theorem 4 and found by our program as well.

In the case $q = 8$, using MDS $[r, 2, \{r - 1, r\}]_8$ code as outer and equidistant (Hadamard) $[7, 3, 4]$ code as inner, we obtain binary linear $[7r, 6, \{4(r - 1), 4r\}]$ codes for $r = 2, 3, \ldots, 8$.

Example 5. (Constructions E.3 and E.4) Consider as $A$ an extended MDS $[q + 1, 3, q - 1]_q$ code for the case $q = 2^m$, $m \geq 2$. This code can be presented as a partition into cosets of the equidistant subcode $B$ with parameters $[q + 1, 2, q]_q$ as follows:

$$A = \bigcup_{j=0}^{q-1} (B + y_j), \quad y_j \in A.$$

Add one more $(q + 2)$-th position (a suffix) with $j$-th element $a_j$ of the field $F_q = \{a_0 = 0, a_1 = 1, a_2, \ldots, a_{q-1}\}$ to all codewords from $B + y_j$,

$$B_j = \{(z, a_j) : z \in (B + y_j)\}.$$

Now take the union of codes $B_j$:

$$A^* = \bigcup_{j=0}^{q-1} B_j.$$

In this way, for any $q = 2^m$, $m \geq 2$, we obtain a family of (optimal) two-weight MDS codes with parameters (the family $TF1$ in [4]):

$$n = q + 2, \quad k = 3, \quad d = q, \quad \delta = 2, \quad q = 2^m, \quad m = 2, 3, \ldots.$$

Using Construction E.4 with inner binary equidistant $[2^m - 1, m, 2^{m-1}]$-code we obtain the following family of two-weight binary linear codes:

$$n = (2^m + 2)(2^m - 1), \quad k = 3m, \quad d = 2^{2m-1}, \quad \delta = 2^m, \quad m = 2, 3, \ldots.$$
For $m = 2$ we obtain an optimal $(18, 64, \{8,12\})_2$ code which is found also by the program. All resulting codes are optimal with respect to the bound of Theorem 4. Indeed, since (26) and (27) are satisfied, we have for the denominator of (28)

$$n(q - 1)(nq - n + 1) - q^2(2nd + n\delta - d^2 - d\delta) + nq(2d + \delta) = 2^m + 2,$$

which implies the exact equality in (28).

### 3.2 Randomly generated codes

We use a simple computer program for random generation of good codes. For fixed length $n$, alphabet size $q$ and distances $d$ and $d + \delta$ the program starts filling into a code $C$ with the zero codeword and the word $(11 \ldots 100 \ldots 0)$ of weight $d$. The search space consists of all vectors of weights $d$ and $d + \delta$, extracted from an initial database of all $q^n$ vectors of length $n$ (the database is generated by standard lexicographic means). During the implementation the program adds randomly suitable vectors until the resulting code is good (i.e., until it has only distances $d$ and $d + \delta$). As it might be expected, this approach works very well for relatively small parameters (an inspection of the tables has to suggest what is meant by "relatively small"), where the best codes (found this way) are obtained quickly. The cardinalities of such random codes are shown in Section 6 together with those of the codes obtained from constructions from the previous subsection. The results show that this approach is good when $d = 1, 2$, when $d$ is close to $n - 1$, and when good linear codes with the same parameters exist. Probably it is not good for $d$ in the mid-range.

### 4 Linear two-weight codes

The natural question for existence of a $q$-ary linear two-weight $[n, k, \{d, d + \delta\}]_q$-code is under which conditions such code exist, if we fix, for example, a prime power $q$, the minimum distance $d$ and dimension $k$. The full answer for this question is open. We give here only partial answers. Recall that $(a, b)$ denotes the great common divisor for natural numbers $a$ and $b$.

A linear code $C$ is called projective if its dual code $C^\perp$ has minimum distance $d^\perp \geq 3$ (i.e., the parity check matrix $H$ of $C$ has no same columns). For projective $[n, k, d]_q$-codes $C$ one can define the concept of complementary code (see, for example, [7]). Let $[C]$ denote the matrix formed by the all codewords of $C$. The code $C_c$ is called a complementary of $C$, if the matrix $[[C] | [C_c]]$ is a linear equidistant code and $C_c$ is of the minimal possible length, which gives such property.

The extension of this well known concept to arbitrary linear two-weight codes is formulated as the following evident lemma.

**Lemma 3.** Let $C$ be a $q$-ary linear two-weight $[n, k, \{d, d + \delta\}]_q$-code and let $\mu_1$ and $\mu_2$ denote the number of codewords of weight $d$ and $d + \delta$, respectively. Then there exist the (complementary) linear two-weight $[n_c, k, \{d_c, d_c + \delta\}]_q$-code $C_c$, where

$$n + n_c = s \frac{q^k - 1}{q - 1}, \quad d + d_c + \delta = sq^{k-1}, \quad s = 1, 2, \ldots,$$

and where $C_c$ contains $\mu_1$ codewords of weight $d_c + \delta$ and $\mu_2$ codewords of weight $d_c$ and where $C_c$ is of minimal possible length, such that the matrix $[[C] | [C_c]]$ is an equidistant code.
Note that the integer $s$ in the Lemma 3 is a maximal multiplicity of same columns in the generator matrix of $C$. For projective two-weight codes (i.e. for the case $s = 1$) the following results are known.

**Lemma 4.** Let $C$ be a 2-weight projective $[n, k; \{w_1, w_2\}]_q$ code over $\mathbb{F}_q$, $q = p^m$, $p$ is prime. Then there exist two integers $u \geq 0$ and $h \geq 1$, such that

$$w_1 = hp^u, \quad w_2 = (h + 1)p^u.$$

For the projective case, we recall the following result (which directly follows from the MacWilliams identities, taking into account that the dual $(C)$ code $C^\perp$ has minimum distance $d^\perp \geq 3$) (see [8]).

**Lemma 5.** Let $C$ be a 2-weight projective $[n, k; \{w_1, w_2\}]_q$ code over $\mathbb{F}_q$, $q = p^m$, $p$ is prime. Denote by $\mu_1$ the number of codewords of $C$ of weight $w_1$ and by $\mu_2$ the number of codewords of weight $w_2$. Then

$$w_1 \mu_1 + w_2 \mu_2 = n(q - 1)q^{k-1},$$

$$w_1^2 \mu_1 + w_2^2 \mu_2 = n(q - 1)(n(q - 1) + 1)q^{k-2}.$$ (12) (13)

Here we generalize Lemma 4 to the case of arbitrary two-weight $[n, k, \{d, d + \delta\}]_q$-codes. Besides, we obtain slightly stronger result for projective such codes. Here we assume that $q = p^m$ where $m \geq 1$ and $p$ is prime. For such given $q = p^m$ and for arbitrary natural number $a$ denote by $\gamma_a \geq 0$ the maximal integer, such that $p^{\gamma_a}$ divides $a$, i.e. $a = p^{\gamma_a} h$, where $h$ and $p$ are co-prime. Let $\gamma_d$, $\gamma_\delta$ and $\gamma_c$ denote such maximal integers for $d$, $\delta$ and $d_c$, respectively.

**Theorem 1.** Let $q = p^m$, where $m \geq 1$ and $p$ prime. Let $C$ be a $q$-ary linear nontrivial two-weight $[n, k, \{d, d + \delta\}]_q$-code of dimension $k \geq 2$ and let $C_c$ be its complementary two-weight $[n_c, k, \{d_c, d_c + \delta\}]_q$-code $C_c$, where

$$d + d_c + \delta = s q^{k-1}, \quad s \geq 1.$$

(i) If $s = 1$ and $k \geq 4$, i.e. $C$ and hence $C_c$ are projective codes, then the following two equalities are satisfied:

$$(q, d) = (q, \delta) \quad \text{and} \quad (q, d_c) = (q, \delta)$$

(14)

(ii) If $s = 1$ and $k = 3$, then both equalities in (14) are satisfied, if at least one of the following two conditions takes place:

$$(d, q)^2 \leq q(n(n - 1), q) \quad \text{or} \quad (d + \delta, q)^2 > q(n_c(n_c - 1), q)$$

(iii) If $s = 1$ and $k \geq 2$, then at least one of the following two equalities is satisfied:

$$\gamma_d = \gamma_\delta, \quad \text{or} \quad \gamma_c = \gamma_\delta.$$

(15)

(iv) If $s \geq 1$ and $k \geq 3$, then at least one of the two equalities in (14) (respectively, in (17)) is valid.
Proof. We start from the statement (iv). Let \( C \) be a \( q \)-ary linear two-weight \([n, k, \{d, d + \delta\}]_q\)-code. Recall that \( \mu_1 \) is the number of codewords of \( C \) of weight \( d \) and \( \mu_2 \) is the number of codewords of weight \( d + \delta \). Then (12) and other evident equality for \( \mu_1 \) and \( \mu_2 \),

\[
\mu_1 + \mu_2 = q^k - 1,
\]

imply that

\[
(q^k - 1)d + \mu_2 \delta = n(q - 1)q^{k-1}.
\]

We deduce (recall that \( k \geq 3 \)) from the equality (17) that \( \gamma_\delta \leq \gamma_d \) and \( \gamma_d \leq \gamma_\mu + \gamma_\delta \) (where \( \mu_2 = hq^{\gamma_\mu} \) and \( q \) and \( h \) are co-prime). If \( \gamma_\mu = 0 \), then we obtain \( \gamma_d = \gamma_\delta \), otherwise we have \( \gamma_d > \gamma_\delta \).

Consider the case \( \gamma_\mu \geq 1 \) (or equivalently, \( (\mu_2, q) > 1 \)). By Lemma 3 the existence of \( C \) implies the existence of the complementary two-weight \([n_c, k, \{d_c, d_c + \delta\}]_q\)-code \( C_c \), containing \( \mu_1 \) codewords of weight \( d_c + \delta \) and \( \mu_2 \) codewords of weight \( d_c \). The equation (17) for the code \( C_c \) looks as

\[
(q^k - 1)d_c + \mu_1 \delta = n_c(q - 1)q^{k-1}
\]

(18)

(indeed, in \( C_c \) the codewords of weight \( d_c + \delta \) occur \( \mu_1 \) times). Taking into account that \( (\mu_2, q) > 1 \) and \( \mu_1 + \mu_2 \equiv -1 \pmod q \) from (16), we deduce that \( (\mu_1, q) = 1 \). Hence from (18) we obtain that

\[
\gamma_c = \gamma_\delta.
\]

(19)

From the equalities in (15) between \( \gamma \)'s of parameters \( d, d_c \) and \( \delta \), we obtain the corresponding equalities in (14) between the corresponding greatest common divisors. This completes the proof of (iv).

For (i), i.e. for the case \( s = 1 \) and \( k \geq 4 \), we already know that one of the equalities in (14) is valid. Let us assume for a contradiction that \( (q, d_c) = (q, \delta) \) but \( (q, d) > (q, \delta) \). The identity (13)

can be written as follows:

\[
(\mu_1 + \mu_2)d^2 + 2\mu_2d\delta + \mu_2\delta^2 = n(q - 1)(n(q - 1) + 1)q^{k-2}.
\]

(20)

Taking into account the equalities (15), (17), and (19), set

\[
(d, q) = p^t, \quad (d_c, q) = (\delta, q) = p^r, \quad (\mu_2, q) = p^u,
\]

where \( u, r \geq 1 \) and \( t = r + u \). Then (20) gives a contradiction modulo \( p^{t+r+1} \) for any \( k \geq 4 \) and any \( t \leq m \) (recall that \( q = p^n \) and \( p \) is prime). Indeed, only the third (from the left) monom in (20) is nonzero modulo \( p^{t+r+1} \).

Consider the case \( k = 3 \), i.e. the statement (ii). Clearly we obtain the same contradiction in the equality (20) (i.e., the same third monom at the left would be only nonzero modulo \( p^{t+r+1} \)) for any \( t \leq m/2 \). Furthermore, because of the following equality for great common divisors,

\[
(n(q - 1)(n(q - 1) + 1), q) = (n(n - 1), q),
\]

we arrive to the same contradiction in (20) for the case when

\[
(d, q)^2 \leq (n(n - 1), q).
\]
Clearly the same idea can be used for the complementary (projective) code \( C_c \). The analog of (20) for \( C_c \) is
\[
(\mu_1 + \mu_2)d_c^2 + 2\mu_1 d_c \delta + \mu_1 \delta^2 = n_c(q - 1)(n_c(q - 1) + 1)q^{k-2}.
\]
We obtain a contradiction, if the following inequality would be valid:
\[
(d_c, q)^2 > q(n_c(n_c - 1), q).
\]
Then the left hand side is divisible by \((d_c, q)^2\) which the right hand side is not (note that \( k = 3 \)). It gives (ii).

For the case (iii), according to Lemma 4 there exist nonnegative integers \( g \) and \( \gamma \) such that
\[
d = gp^\gamma, \quad d + \delta = (g + 1)p^\gamma.
\]
First, assume that \( g = \ell p^\alpha \). Using (14), we have
\[
d_c = sq^{k-1} - \ell p^\alpha p^\gamma - p^\gamma = p^\gamma(sp^{m(k-1)-\gamma} - p^\alpha - 1),
\]
implying for this case that
\[
\gamma_d \neq \gamma_\delta, \quad \text{but} \quad \gamma_c = \gamma_\delta.
\]
For the case \( g + 1 = \ell p^\alpha \) similar arguments imply that
\[
\gamma_d = \gamma_\delta, \quad \text{but} \quad \gamma_c \neq \gamma_\delta.
\]
Now assume that \((g, p) = 1\) and \((g + 1, p) = 1\). We obtain for this case
\[
d_c = p^\gamma(sp^{m(k-1)-\gamma} - (g + 1))
\]
implying that \( \gamma_c = \gamma \). Since \( \gamma_d = \gamma_\delta = \gamma \), we obtain the both equalities:
\[
\gamma_d = \gamma_\delta = \gamma_c.
\]

We illustrate Theorem 1 by two examples.

**Example 6.** Note that the condition \( k \geq 3 \) in Theorem 1 can not be removed. It is easy to construct two-weight \([n, 2, \{d, d + \delta\}]_q\)-code, where \( \delta \) is an arbitrary positive integer. Indeed, extend the equidistant \([q + 1, 2, q]\)-code \( A \) (see Example 3) with generating vectors \( x_1 \) and \( x_2 \) as follows: add the zero vector \( 0 \) of length \( \delta \) to \( x_1 \) and any vector \( z \) of weight \( \delta \) and length \( \delta \) to \( x_2 \). The resulting two new codewords \( y_1 = (x_1 \mid 0) \) and \( y_2 = (x_2 \mid z) \) generate a two-weight \([q + 1, 2, \{q, q + \delta\}]_q\)-code \( C \), where \( \delta \) is an arbitrary positive integer (implying, in particular, that the equality \((d, q) = (\delta, q)\) is almost never valid). For such case check the distance \( d_c \) of complementary code. The code \( C \) has \( s = \delta + 1 \). Indeed, we add \( \delta \) linearly dependent over \( \mathbb{F}_q \) columns to the matrix, which has already one such kind of column (upto multiplying by scalar). Hence, we have for the distance \( d_c \) of complementary code \( C_c \):
\[
d_c = (\delta + 1)q - q - \delta = \delta(q - 1).
\]
So we obtain for such codes that \((d, q) = q \) and \((\delta, q) = (d_c, q)\). Hence, the first equality in (14) is valid only for \( \delta \) multiple to \( q \), and the second equality in (14) is valid always.
Example 7. The \([n, 2m, \{d, d + \delta\}]_p\)-codes from Example 4 with parameters (12) have \(d = (r - 1)p^{m-1}\) and \(\delta = p^{m-1}\) where \(r \leq p^m + 1\). Hence for \(r = p^f + 1 \leq p^m + 1\) we obtain \(\gamma_d = m + \ell - 1\) and \(\gamma_\delta = m - 1\) and the first equality in (13) is not valid. Let us find the parameters of complementary code \(C_c\). Clearly
\[ n_c = \frac{(p^m + 1 - r)}{p - 1} p^m - 1, \quad d_c = \frac{(p^m - r)}{p - 1} p^{m-1},\]
which implies \(d_c = (p^m - p^f - 1) p^{m-1}\) and hence \(\gamma_c = m - 1\). Thus, \(\gamma_c = \gamma_\delta\) and the second equality in (13) is valid.

In some cases the conditions (14) and (15) are also sufficient.

Theorem 2. Let \(q = p^u\) be a prime power, \(\delta = (q, \delta)h\) where \(q\) and \(h\) be mutually prime, and let \(s \geq 1\) be a natural number.

(i) If \(d + \delta = sq^r\) then for any \(\delta = (q, \delta)h\), such that \((q, d) = (q, \delta)\) and \(h \leq s\), there exist a \(q\)-ary linear two-weight \([n, r + 1, \{d, d + \delta\}]_q\) code \(C\) of length
\[ n = s \frac{q^{r+1} - 1}{q - 1} - h \frac{q^{\ell+1} - 1}{q - 1}. \] (21)

If \(h \leq q - 1\), then the code \(C\) is optimal.

(ii) If \(d = sq^r\) then for any \(\delta = (q, \delta)h\), such that \((q, d) = (q, \delta)\), there exist a \(q\)-ary linear two-weight \([n, r + 1, \{d, d + \delta\}]_q\) code \(C\) of length
\[ n = s \frac{q^{r+1} - 1}{q - 1} + h \frac{q^{\ell+1} - 1}{q - 1}. \] (22)

If \(h = s\) there then exist an optimal two-weight \([n, N, \{d, d + \delta\}]_q\) code \(C\) of length \(n = d + \delta\) and cardinality \(N = n\).

(iii) Let \(p\) be any prime and \(t\) be any natural number. If \(\gamma_d = t\) and \(\delta = pt^f\), i.e. \(\gamma_d = \gamma_\delta\), then for any \(d = h p^{\gamma_d}\) where \(h\) is a positive integer co-prime to \(p\), such that \(h \leq p^{\ell+1} + 1\), there exists a \(p\)-ary optimal two-weight \([n, k, \{d, d + \delta\}]_p\)-code.

Proof. (i) Let \(d + \delta = sq^r\), where \(q = p^u\) and \(u \geq 1\), i.e. \(q \geq p\). Under conditions of the theorem, we can set \(\delta = h q^f\), where \(1 \leq \ell \leq r - 1\), and where \((h, q) = 1\) and \(h \leq s\). Consider the codes (9) from Example 3. Taking \(s\) copies of code \(A\) and \(h\) copies of \(B\) we obtain a linear two-weight code \(C\) of length (21), which satisfies the condition of the theorem. If \(h \leq q - 1\), then \(C\) is optimal according to the Griesmer bound (see Example 3). It gives the first statement.

(ii) Consider the case \(d = sq^r\). Assume that \(\delta = h q^f\), where \(1 \leq \ell \leq r - 1\), and where \(h\) is any positive integer co-prime to \(q\). Consider the codes (10) from Example 3. Taking \(s\) copies of code \(A\) and \(h\) copies of \(B\) we obtain a linear two-weight code \(C\) of length (22) which satisfies the condition of the theorem.

For the case when \(d = sq^r\) and \(\delta = sq^t\), where \(1 \leq \ell \leq r - 1\), one can choose the optimal codes (9) from Example 2 which have the minimum possible length \(n = d + \delta\) and cardinality \(N = n\). In this case the resulting code is nonlinear until \(n = q^u\).
(iii) Let \( d = h p^\gamma d = h p^t \). Consider the optimal codes (Example 4) with parameters
\[
n = r \frac{p^m - 1}{p - 1}, \quad k = 2m, \quad d = (r - 1)p^{m-1}, \quad \delta = p^{m-1},
\]
where \( r \leq q + 1 \). Set \( m = t + 1 \) and chose any \( h = r \leq p^m + 1 \), which is mutually prime to \( p \). So, for any such \( h \) these codes have \( d = h p^{m-1} \) and \( \delta = p^{m-1} \), such that \( \gamma_d = \gamma_\delta \).

5 Upper bounds
We are interested in the upper bounds for the quantity
\[
A_q(n;\{d, d + \delta\}) = \max\{|C| : C \text{ is an } (n, |C|, \{d, d + \delta\}) \text{ code}\},
\]
the maximal possible cardinality of a code in \( Q^n \) with two distances \( d \) and \( d + \delta \).

5.1 General linear programming bound
We adapt the Delsarte linear programming bound for \( A_q(n;\{d, d + \delta\}) \). Proofs of such bounds are usually considered as folklore (see, for example, [9, 15]).

For fixed \( n \) and \( q \), the (normalized) Krawtchouk polynomials are defined by
\[
Q_i^{(n,q)}(t) = \frac{1}{r_i} K_i^{(n,q)}(z), \quad z = \frac{n(1-t)}{2}, \quad r_i = (q-1)^i \binom{n}{i},
\]
where
\[
K_i^{(n,q)}(z) = \sum_{j=0}^i (-1)^j (q-1)^{i-j} \binom{z^j}{j} \binom{n-z}{i-j}
\]
are the (usual) Krawtchouk polynomials.

If \( f(t) \in \mathbb{R}[t] \) is of degree \( m \geq 0 \), then it can be uniquely expanded as
\[
f(t) = \sum_{i=0}^n f_i Q_i^{(n,q)}(t), \quad (23)
\]
where, if \( \deg(f) \geq n + 1 \), the polynomial \( f(t) \) is considered modulo \( \prod_{i=0}^n (t - 1 + 2i/n) \).

**Theorem 3.** Let \( n \geq q \geq 2 \) and \( f(t) \) be a real polynomial such that:
(A1) \( f(t) \leq 0 \) for \( t \in \{1-2d/n, 1-2(d+\delta)/n\} \);
(A2) the coefficients in the Krawtchouk expansion (23) satisfy \( f_i \geq 0 \) for every \( i \geq 1 \).

Then
\[
A_q(n;\{d, d + \delta\}) \leq \frac{f(1)}{f_0}. \quad (24)
\]

If an \((n, N, \{d, d + \delta\})_q \text{ code } C\) attains \( (24) \) for some polynomial \( f(t) \), then \( f(1 - 2(d+i)/n) = 0, i = 0, \delta, \) whenever there are points of \( C \) at distance \( d + i, i = 0, \delta, \) and \( f_iM_i(C) = 0, \) where
\[
M_i(C) = \sum_{x,y \in C} Q_i^{(n,q)}(1 - 2d(x,y)/n) = 0 \quad (25)
\]
is the \( i \)-th moment of \( C \).
5.2 Specified linear programming bounds

The degree one polynomial \( f(t) = t - 1 + 2d/n \) gives the Plotkin bound which is attained for many large \( d \). We proceed with degree two polynomials, where the bound produced coincides with the bound by Helleseth-Kløve-Levenshtein \[12\] for the maximal cardinality \(|C|\) of a code \( C \) with given minimum and maximum distances; this is also the bound for \( k = 1 \) of Theorem 5.2 in \[5\]. Here we give a proof which is direct from Theorem 3.

**Theorem 4.** If

\[
q(2d + \delta) \geq 2nq + 2 - 2n - q,
\]

(26)

\[
n(q - 1)(nq - n + 1) + nq(2d + \delta) > q^2(2nd + n\delta - d^2 - d\delta),
\]

(27)

then

\[
A_q(n, \{d, d + \delta\}) \leq \frac{d(d + \delta)q^2}{n(q - 1)(nq - n + 1) - q^2(2nd + n\delta - d^2 - d\delta) + nq(2d + \delta)}.
\]

(28)

If this bound is attained by an \( (n, N, \{d, d + \delta\})_q \) code \( C \), then \( M_2(C) = 0 \) and, moreover, \( M_1(C) = 0 \) whenever \( (26) \) is strict. In the latter case \( C \) is an orthogonal array of strength 2.

**Proof.** Consider the second degree polynomial

\[
f(t) = \left( t - 1 + \frac{2d}{n} \right) \left( t - 1 + \frac{2(d + \delta)}{n} \right).
\]

The condition (A1) is obviously satisfied. For (A2), we find the Krawtchouk coefficients of \( f(t) \) as follows

\[
f_0 = \frac{4(n(q - 1)(nq - n + 1) - q^2(2nd + n\delta - d^2 - d\delta) + nq(2d + \delta))}{n^2q^2},
\]

\[
f_1 = \frac{4(q - 1)(2dq + \delta q + 2n + q - 2nq - 2)}{nq^2},
\]

\[
f_2 = \frac{4(q - 1)^2(n - 1)}{nq^2}.
\]

It is obvious that \( f_2 > 0 \). Further, \( f_1 \geq 0 \) and \( f_0 > 0 \) are equivalent to \( (26) \) and \( (27) \), respectively. Therefore, provided \( (26) \) and \( (27) \), we have

\[
A_q(n, \{d, d + \delta\}) \leq \frac{f(1)}{f_0},
\]

which gives the desired bound. □

If the right hand side of \( (28) \) is integer, we are able to find the distance distribution of \( C \) by solving the system of equations coming from \( A_d + A_{d+\delta} = |C| - 1 \) and \( M_i(C) = 0, i = 1, 2 \) (see \( (25) \)). In the range of the tables for \( q = 2 \) this gives three nonexistence results, proving that \( A_2(12, \{6,10\}) \leq 19 \) instead of 20, \( A_2(20, \{10,14\}) \leq 27 \) instead of 28, and \( A_2(16, \{8,14\}) \leq 27 \) instead of 28 from \( (28) \).
5.3 Upper bounds via spherical codes

There is a natural relation between codes from \( Q^n \) and few-distance spherical codes. First, the alphabet symbols 0, 1, \ldots, \( q - 1 \) are mapped bijectively onto the vertices of the regular simplex in \( \mathbb{R}^{q-1} \). Then the codewords of any code \( C \subset Q^n \) can be send (coordinate-wise) to \( \mathbb{R}^{(q-1)n} \). It is not difficult to see that all obtained vectors have the same length and after a normalization a spherical code \( W \subset S^{(q-1)m-1} \) is formed.

The code \( W \) has the same cardinality as \( C \), i.e., \( |W| = |C| \), and its maximal inner product is equal to \( 1 - 2dq/(q-1)n \), i.e., its squared minimum distance is \( 2dq/(q-1)n \). In our considerations, the \( q \)-ary codes \( W_q \) with distances \( d \) and \( d + \delta \) are mapped to spherical 2-distance codes with squared distances \( 2dq/(q-1)n \) and \( 2(d + \delta)q/(q-1)n \). This implies a upper bound for \( A_q(n, \{d, d + \delta\}) \) as follows.

Theorem 5. Let \( \frac{d}{d + \delta} = \frac{2}{15} \) in lowest terms. If \( s - r \geq 2 \) (in particular, if \( \text{GCD}(d, d + \delta) = 1 \) or \( s = r + 1 \) and \( r > (\sqrt{2(q-1)n} - 1)/2 \), then

\[
A_q(n, \{d, d + \delta\}) \leq 2(q-1)n + 1.
\] (29)

Proof. A classical results by Larman, Rogers, and Seidel [14] states that if the cardinality of a 2-distance set \( W \subset \mathbb{R}^m \) with distances \( a \) and \( b \), \( a < b \), is greater than \( 2m + 3 \), then the ratio \( a^2/b^2 \) is equal to \((k-1)/k\), where \( k \in [2, (\sqrt{2m+1}/2)\] is a positive integer. The restriction \( 2m + 3 \) was moved to \( 2m + 1 \) by Neumaier [16].

For \( W \) as above, we have \( a^2/b^2 = d/(d + \delta) = r/s \) and \( m = (q-1)n \). This immediately implies our claim in the case \( s - r \geq 2 \). If \( s = r + 1 \), we need in addition \( r \notin [1, (\sqrt{2(q-1)n} - 1)/2] \) to have again the required bound.

Corollary 1. In the context of Theorem 5 if \( q, n, d, \delta, \) and \( k \) are such that

\[
2(q-1)n + 1 < q^k,
\]

then there exist no linear codes \( C \subset Q^n \) with distances \( d \) and \( d + \delta \) and dimension at least \( k \).

The bound (29) of Theorem 5 is quite better than the linear programming bound (simplex method) for large \( q \) in the cases when \( s - r \geq 2 \). This is evident already for \( q = 3 \) and \( q = 4 \) in the tables below.

5.4 Some simple cases

In this section we assume (without loss of generality) that codes under consideration possess the zero word. Then all other words have weights \( d \) and \( d + \delta \).

Lemma 6. For \( q = 2 \), if \( d \) and \( d + \delta \) are both odd, then

\[
A_2(n, \{d, d + \delta\}) = 2.
\]

Proof. If \( |C| \geq 3 \) and \( x, y \in C \) are nonzero and distinct, then

\[
d(x, y) = \text{wt}(x) + \text{wt}(y) - 2\text{wt}(x * y)
\]
is even, a contradiction. \( \square \)
Lemma 7. For $q = 2$, if $d < \delta$ is odd and $d + \delta$ is even, then

$$A_2(n, \{d, d + \delta\}) = 1 + A(n, d, d).$$

Proof. If $|C| \geq 3$ and $x, y \in C$ are nonzero and having distinct weights, then

$$d(x, y) = \text{wt}(x) + \text{wt}(y) - 2\text{wt}(x \ast y) = 2d + \delta - 2\text{wt}(x \ast y)$$

is odd, thus equal to $d$. Then $d + \delta = 2\text{wt}(x \ast y) \leq 2 \min \{\text{wt}(x), \text{wt}(y)\} = 2d$, a contradiction. Therefore $C \setminus \{0\}$ is a constant weight code of weight $d$ and minimum distance $d$.

Lemma 8. For $q = 2$, if $d$ is odd and $|C| > 4$, then

$$A_2(n, \{d, 2d\}) \leq 1 + \left\lceil \frac{n}{d} \right\rceil.$$

Proof. Let $A_d$ ($A_{2d}$) be the number of the words of weight $d$ ($2d$). Similarly to above we see that if $\text{wt}(x) = \text{wt}(y) = d$, then $\supp(x) \cap \supp(y) = \emptyset$. This means that $A_d \leq \lceil n/d \rceil$. Moreover, since $\supp(x) \subseteq \supp(y)$ for any two words $x$ and $y$ of weights $d$ and $2d$, respectively, it follows that if $A_d \geq 3$, then $A_{2d} = 0$, if $A_d = 2$, then $A_{2d} = 1$ and $|C| = 4$, and if $A_d = 1$, then the supports of all words of weight $2d$ contain the support of the single word of weight $d$ and therefore $A_{2d} \leq \lfloor n/d \rfloor - 1$. In all cases $|C| \leq 1 + \lfloor n/d \rfloor$. It is obvious from this description how this bound is attained.

Lemma 9. We have $A_3(n, \{1, 3\}) = 6$ for every $n \geq 4$.

Proof. Observe that the ternary code $C = \{000, 100, 211, 212, 222, 221\}$ has distances 1 and 3 and cardinality 6. It can be extended by zero coordinates to any length $n \geq 4$. Therefore $A_3(n, \{1, 3\}) \geq 6$.

Let $C$ be a maximal $(n, N, \{1, 3\})_3$ code. Without loss of generality we may assume that $0$ and $(10\ldots0)$ belong to $C$. Then it is obvious that no more words of weight 1 are possible apart from $(20\ldots0)$ in which case $|C| = 3$ is not maximal. The words of weight 3 can only have 2 as first coordinate. We can assume that $(21100\ldots0) \in C$. If the nonzero coordinates of the remaining words are the first three, then at most 4 words of weight 3 are possible and $|C| \leq 1 + 1 + 4 = 6$. Otherwise, exactly one among the second and third coordinates is nonzero and it easy to see that again at most 4 words are possible.

Lemma 10. We have $A_2(n, \{d, d + \delta\}) = 2$ for every odd $d$ and even $\delta$ such that $n < (3d - \delta)/2$.

Proof. Using, as in Lemma 6, the equality $d(x, y) = \text{wt}(x) + \text{wt}(y) - 2\text{wt}(x \ast y)$ we see that $\text{wt}(w \ast y) \in \{(d - \delta)/2, (d + \delta)/2\}$. Therefore

$$n \geq d + \delta \geq d + (d - \max \text{wt}(x \ast y)) = (3d - \delta)/2,$$

which completes the proof.

The cases covered by Lemmas 6, 7 are excluded from the tables. Other similar cases can be dealt as well (for example, one can prove that $A_4(n, \{1, 3\}) = 12$). We formulate as conjectures a few observations.
Conjecture 1. (i) \( A_2(n, \{2, 4\}) = \binom{n}{2} + 1 \) for every \( n \geq 4 \);
(ii) \( A_2(n, \{2, 2 + \delta\}) = n \) for every \( \delta \geq 3 \) and every \( n \geq 4 \), except for \( A_2(n, \{2, n-1\}) = n + 1 \).

The code consisting of all words of weight 2 and the zero word has distances 2 and 4 and cardinality \( \binom{n}{2} + 1 \). This provides the lower bound for (i). A code which achieves the lower bound in (ii) is given by the zero word and all words of weight 2 with nonzero first coordinate (if \( 2 + \delta = n - 1 \), the word of weight \( n - 1 \) with zero first coordinate can be added). If \( \delta \) is odd, then any two words of weight 2 + \( \delta \) are at even distance. Thus these two words have common \( 1 + \delta \) nonzero coordinates. It is clear now that only one word of weight 2 can be added, so our code has cardinality 4. This proves (ii) for odd \( \delta \).

6 Tables

We present tables with lower and upper bounds for \( A_q(n, \{d, d + \delta\}) \) in the ranges \( 2 \leq \delta \leq 6 \), \( 7 \leq n \leq 20, 14, \) and \( 12 \) for \( q = 2, 3, \) and \( 4 \), respectively. Horizontally we give \( d \), vertically \( n \).

The lower bounds show the better of the computer generated random codes and the constructions from Section 3. All our randomly generated codes are available upon request.

The upper bounds are taken from the best of the linear programming bound obtained by the simplex method (marked with \( lp \) when attained or close, or by \( d2 \) when coming from \( \text{28} \)), the corresponding best known upper bound on \( A_q(n, d) \) from \( \text{6} \) (marked with \( * \)), and the bound from Theorem \( 5 \) (marked with \( sc \)).

Key to the tables:

\( lp \) – upper bound by Theorem \( 3 \) (general simplex method), excluding cases of Theorem \( 4 \);
\( * \) – upper bound (often exact value) from Brouwer’s tables \( \text{6} \);
\( sc \) – upper bound by Theorem \( 5 \) (spherical codes);
\( d2 \) – upper bound by Theorem \( 4 \) (particular case of Theorem \( 3 \));
\( dd \) – contradiction by distance distribution.

\[
q = 2, \delta = 2
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
 n & d & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
\hline
 7 & 22-26 & 8^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
 8 & 29-36 & 10-12^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
 9 & 37-40 & 16^{*} & 4^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
10 & 46-56 & 16^{*} & 6^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
11 & 56^{*} & 17-23^{*} & 12^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
12 & 67-77 & 19-25^{*} & 16^{*} & 4^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
13 & 79-87 & 19-37^{*} & 17-19^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} \\
14 & 92-100 & 17-25^{*} & 17-19^{*} & 8^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
15 & 106-120 & 17-31^{*} & 18-31^{*} & 16^{*} & 4^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
16 & 121-126 & 37-75 & 19-33^{*} & 17-29^{*} & 4^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
17 & 137-154 & 42-91 & 20-35^{*} & 19-23^{*} & 6^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} & 2^{*} \\
18 & 154^{*} & 46-116 & 20-37^{*} & 19-23^{*} & 10^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} \\
19 & 172-189 & 52-123 & 21-39^{*} & 20-35 & 14-20^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} & 4^{*} \\
20 & 191-200 & 58-151 & 22-41^{*} & 20-41^{*} & 19-24^{*} & 6^{*} & 4-3 & 4-3 & 4-3 & 4-3 \\
\end{array}
\]
\( q = 2, \delta = 3 \)

| n/d | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|------|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| 7    | 7-8 | 8".\text{ad} | | | | | | | | | | | | |
| 8    | 8-12 | 8" | 2" | | | | | | | | | | | |
| 9    | 9-14 | 8-10 | 4" | 4" | | | | | | | | | | |
| 10   | 10-18 | 8-16 | 4-5 | 6" | 2" | | | | | | | | | |
| 11   | 11-19 | 8-16 | 4-5 | 12".".\text{ad} | 2" | | | | | | | | |
| 12   | 12-24 | 10-21 | 4-5 | 12" | 4" | 4" | 2" | | | | | | |
| 13   | 13-24 | 12-27" | 4-5 | 14" | 4-5 | 4" | 2" | 2" | | | | | |
| 14   | 14-28 | 14-29" | 4-5 | 16" | 4-5 | 4" | 2" | 2" | 2" | | | | |
| 15   | 15-28 | 14-31" | 7-16 | 14-27 | 6" | 16"." | 4" | 4" | 2" | 2" | 2" | | |
| 16   | 16-32 | 14-33" | 7-16 | 15-34 | 6" | 16" | 4" | 4" | 2" | 2" | 2" | 2" | |
| 17   | 17-33 | 14-35" | 8-18 | 15-50 | 6" | 17-21 | 4-6 | 6" | 2" | 2" | 2" | 2" | |
| 18   | 18-36 | 14-37" | 10-22 | 16-8" | 17-29 | 8" | 10-34" | 4" | 4" | 2" | 2" | 2" | |
| 19   | 19-40 | 14-39" | 13-35 | 16-9-20" | 17-29 | 8" | 20" | 4" | 4" | 2" | 2" | 2" | 2" |
| 20   | 20-40 | 14-41" | 17-41 | 18-11-20" | 20-41" | 8" | 20" | 4-5 | 6" | 2" | 2" | 2" | 2" |

\( q = 2, \delta = 4 \)

| n/d | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|------|---|---|---|---|----|----|----|----|
| 7    | 8" | | | | | | | |
| 8    | 8-12 | 16".".\text{ad} | | | | | |
| 9    | 9-16 | 16" | | | | | |
| 10   | 10-18 | 16" | 6" | | | | |
| 11   | 11-23" | 16-30 | 12".".\text{ad} | | | | |
| 12   | 12-25" | 16-30 | 12-19" | 4" | | | |
| 13   | 13-27" | 32-54 | 15-27" | 4" | | | |
| 14   | 14-29" | 64" | 14-29" | 8" | 2" | | |
| 15   | 15-31" | 64-88 | 16-31" | 16".".\text{ad} | 4" | | |
| 16   | 16-33" | 64-128 | 16-33" | 20-24" | 4" | 2" | |
| 17   | 17-35" | 64-150 | 17-35" | 32-36 | 6" | 2" | |
| 18   | 18-37" | 64-256 | 18-37" | 64" | 10-34" | 4" | 2" | |
| 19   | 19-39" | 64-256 | 20-39" | 80-96" | 20-64" | 4" | 2" | |
| 20   | 20-41" | 64-332 | 20-41" | 20-64" | 80-96" | 20-27" | 6" | 2" | 2" | |

\( q = 2, \delta = 5 \)

| n/d | 2 | 4 | 6 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------|---|---|---|---|---|----|----|----|----|----|----|
| 7    | 7-8 | | | | | | | | | | |
| 8    | 9-10 | | | | | | | | | |
| 9    | 9-10 | 8-10 | | | | | | | | |
| 10   | 10-16 | 8-16 | | | | | | | | |
| 11   | 11-18 | 8-16 | 12" | | | | | | | |
| 12   | 12-24 | 8-25 | 12" | 2-3 | | | | | | |
| 13   | 13-24 | 8-25 | 13-14 | 4" | 4" | | | | | |
| 14   | 14-28 | 10-29" | 13-19 | 4" | 8" | 2-3 | | | | |
| 15   | 15-29 | 14-31" | 14-28 | 4" | 16".".\text{ad} | 2-3 | 4" | | | |
| 16   | 16-32 | 14-33" | 14-53" | 4" | 16" | 4" | 4" | 2" | | |
| 17   | 17-34 | 16-35" | 14-35" | 4" | 16-18" | 4" | 6" | 2-3 | 2" | |
| 18   | 18-36 | 18-37" | 15-37" | 4" | 17-22" | 4-5 | 10" | 2-3 | 4" | 2" | |
| 19   | 19-38 | 18-39" | 15-39" | 4" | 17-33" | 4-5 | 20" | 4" | 4" | 2" | 2" | |
| 20   | 20-40 | 18-41" | 16-41" | 4-6" | 21-41" | 4-5 | 20" | 4" | 6" | 2-3 | 2" | 2" |
### $q = 2, \delta = 6$

| $n/d$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|-------|---|---|---|---|----|----|----|
| 8     | 8-10 |   |   |   |    |    |    |
| 9     | 10-16 |   |   |   |    |    |    |
| 10    | 10-18 | 8-16 |   |   |    |    |    |
| 11    | 11-23 | 8-16 |   |   |    |    |    |
| 12    | 12-25 | 8-19 | 24^d |   |    |    |    |
| 13    | 13-27 | 8-26 | 24^p |   |    |    |    |
| 14    | 14-29 | 8-26 | 24-26 | 8^* |    |    |    |
| 15    | 15-31 | 11-31 | 24-27 | 16^* |    |    |    |
| 16    | 16-33 | 14-33 | 24-29 | 16-27^d | 4-5 |    |    |
| 17    | 17-35 | 14-35 | 24-52 | 17-35 | 6^p |    |    |
| 18    | 18-37 | 16-37 | 24-52 | 17-37 | 10^p | 4^* |    |
| 19    | 19-39 | 18-39 | 28-68 | 17-39 | 29^p | 4^* |    |
| 20    | 20-41 | 20-41 | 48-121 | 17-41 | 20-32 | 8^* | 2^* |

### $q = 3, \delta = 2$

| $n/d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|----|----|----|
| 7     | 22-57 | 13-21 | 19-28 | 8-15^d |   |    |    |    |    |    |    |
| 8     | 29-81 | 13-23 | 19-37 | 17-28 | 9^* |    |    |    |    |    |    |
| 9     | 37-86 | 13-30 | 19-57 | 17-35^d | 16-24^d | 6^* |    |    |    |    |    |
| 10    | 46-111 | 13-33 | 20-81 | 17-41 | 28-36 | 13-21 | 6^* |    |    |    |    |
| 11    | 56-158 | 13-33 | 20-125 | 17-45 | 28-45 | 16-33 | 12* | 4^* |    |    |    |
| 12    | 67-197 | 13-33 | 20-162 | 17-49 | 28-49 | 18-37 | 18-30 | 9^* |    |    |    |
| 13    | 79-204 | 13-33 | 21-259 | 17-53 | 28-53 | 18-53 | 19-40 | 15-25 | 6^* | 7^* |    |
| 14    | 92-249 | 13-33 | 21-275 | 17-57 | 28-57 | 18-57 | 20-46 | 15-38 | 11-15 | 6^* | 3^* |

### $q = 3, \delta = 3$

| $n/d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-------|---|---|---|---|---|---|---|---|---|----|----|
| 7     | 9-20 | 7-29 | 27^p | 9-27 |   |    |    |    |    |    |    |
| 8     | 9-28 | 8-33^e | 81^p | 9-33 | 9-33 |    |    |    |    |    |    |
| 9     | 9-37 | 9-37 | 81^p | 9-37 | 10-37^a | 27^p,^a |    |    |    |    |    |
| 10    | 10-41 | 10-41 | 81^p | 10-41 | 10-41 | 81^p,^d | 10-21 |    |    |    |    |
| 11    | 10-45 | 11-45 | 81-91 | 12-45 | 12-45 | 243^a | 12-45 | 12^* |    |    |    |
| 12    | 12-49 | 12-49 | 81-106 | 12-49 | 13-49 | 243^p | 13-49 | 13-53^e | 9^* |    |    |
| 13    | 12-53 | 12-53 | 81-129 | 12-53 | 13-53 | 243-448 | 13-53 | 15-53 | 15-53 | 24-27 | 6^* |
| 14    | 14-57 | 14-57 | 81-162 | 14-57 | 15-57 | 243-729 | 15-57 | 15-57 | 31-57 | 11-48 | 6^* |

### $q = 3, \delta = 4$

| $n/d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|----|
| 7     | 7-21 | 8-20 | 9-13 |   |    |    |    |    |    |    |
| 8     | 9-22 | 10-31 | 10-21 | 17-25 |   |    |    |    |    |    |
| 9     | 10-29 | 12-37 | 16-32 | 18-51 | 9-53 |    |    |    |    |    |
| 10    | 12-33 | 15-41 | 16-41 | 36-61 | 12-41 | 15-41 |    |    |    |    |
| 11    | 14-36 | 15-43 | 16-45 | 42-144 | 13-45 | 15-45 | 12-45 |    |    |    |
| 12    | 16-39 | 18-49 | 16-49 | 49-195 | 14-49 | 22-49 | 25-49 | 27-49 | 27^* | 27^* |    |
| 13    | 18-33 | 18-53 | 16-53 | 56-317 | 27-53 | 22-53 | 25-53 | 27-85 | 27^* | 27^* |    |
| 14    | 18-35 | 18-57 | 16-57 | 56-557 | 27-57 | 22-57 | 25-57 | 27-108 | 19-57 | 11-15 |    |

### $q = 3, \delta = 5$

| $n/d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|
| 7     | 6-13 | 8-21 |   |   |   |   |    |    |    |
| 8     | 9-23 | 9-33 | 9-15 |   |   |   |    |    |    |
| 9     | 12-34 | 9-37 | 10-25 | 9-29 |   |    |    |    |    |
| 10    | 15-41 | 12-41 | 15-39 | 10-41 | 9-31 |   |    |    |    |
| 11    | 15-45 | 12-45 | 17-45 | 12-45 | 12-45 | 15-45 |    |    |    |
| 12    | 18-49 | 12-49 | 17-49 | 13-49 | 21-75 | 18-45 | 12-49 |    |    |
| 13    | 18-53 | 13-53 | 17-53 | 27-53 | 31-140 | 22-53 | 13-53 | 13-53 |    |
| 14    | 18-57 | 14-57 | 17-57 | 27-57 | 50-271 | 22-57 | 15-57 | 13-7 | 29-57 |
### $q = 3, \delta = 6$

| n|d | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |
|---|-----|------|------|------|------|------|------|------|------|
| 7 | 4-7 |      |      |      |      |      |      |      |      |
| 8 | 6-10| 9-22 |      |      |      |      |      |      |      |
| 9 | 6-17| 10-14| 9-18 |      |      |      |      |      |      |
| 10| 10-41| 10-41*| 10-23| 9-41*|      |      |      |      |      |
| 11| 11-42| 12-45*| 15-32| 10-45*| 8-35 |      |      |      |      |
| 12| 13-49*| 13-49*| 17-49*| 13-49*| 9-49*| 25-45 |      |      |      |
| 13| 13-53*| 13-53*| 27-53*| 13-53*| 10-53*| 25-68| 15-53*|      |      |
| 14| 13-57*| 14-57*| 27-57*| 14-57*| 12-57*| 28-106| 14-57*| 14-57*|      |

### $q = 4, \delta = 2$

| n|d | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10 |
|---|-----|------|------|------|------|------|------|------|------|------|----|
| 7 | 12-28| 22-64| 13-43*| 64*2| 14-32*|      |      |      |      |      |    |
| 8 | 12-28| 24-132| 13-49*| 64-146| 17-49*| 32*4*|      |      |      |      |    |
| 9 | 12-28| 37-179| 14-55*| 64-179| 17-55*| 59-64*| 14-20*|      |      |      |    |
| 10| 12-28| 46-256| 16-61*| 64-290| 17-61*| 59-80| 19-56*2| 16*4 |      |      |    |
| 11| 12-28| 56-320| 16-67*| 64-358| 17-67*| 59-179| 19-56*4| 28-49*2| 12*4 |      |    |
| 12| 12-28| 67-320| 16-73*| 64-526| 17-73*| 59-213| 19-73*2| 37-64*2| 17-44*2| 9*4 |    |

### $q = 4, \delta = 3$

| n|d | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    |    |
|---|-----|------|------|------|------|------|------|------|------|------|----|
| 7 | 16-43*| 12-43*| 36-52| 16-31 |      |      |      |      |      |      |    |
| 8 | 16-49*| 12-49*| 81-113| 18-49*| 16-49*|      |      |      |      |      |    |
| 9 | 16-55| 12-55*| 81-270| 18-55*| 16-55*| 28-76 |      |      |      |      |    |
| 10| 16-61| 13-61*| 81-352| 19-61*| 18-61*| 29-216| 15-61*|      |      |      |    |
| 11| 16-67*| 14-67*| 81-511| 19-67*| 16-67*| 46-520| 16-67*| 16-60*|      |      |    |
| 12| 18-73*| 14-73*| 81-738| 19-73*| 16-73*| 46-779| 17-73*| 16-73*| 9-48*2|      |    |

### $q = 4, \delta = 4$

| n|d | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |    |
|---|-----|------|------|------|------|------|------|------|------|----|
| 7 | 14-43*| 12-43*| 12-40 |      |      |      |      |      |      |    |
| 8 | 17-49*| 14-49*| 17-49*| 32-38 |      |      |      |      |      |    |
| 9 | 17-55*| 14-55*| 17-55*| 64-82| 12-44 |      |      |      |      |    |
| 10| 20-61*| 16-61*| 17-61| 256-298| 17-61*| 16-58 |      |      |      |    |
| 11| 22-67*| 16-67*| 17-67| 256-333| 17-67*| 16-67*| 13-67*|      |      |    |
| 12| 25-73*| 25-73*| 17-73| 256-656| 17-73*| 16-73*| 14-73*| 21-73*|      |    |

### $q = 4, \delta = 5$

| n|d | 1    | 2    | 3    | 4    | 5    | 6    | 7    |    |
|---|-----|------|------|------|------|------|------|------|----|
| 7 | 9-28 | 14-34**|      |      |      |      |      |      |    |
| 8 | 13-49**| 14-49**| 18-39*|      |      |      |      |      |    |
| 9 | 17-55**| 14-55**| 18-55**| 16-55**|      |      |      |      |    |
| 10| 20-61**| 15-61**| 18-61**| 17-61**| 16-82 |      |      |      |    |
| 11| 21-67**| 15-67**| 18-67**| 18-67**| 28-132| 24-29 |      |      |    |
| 12| 22-73**| 18-73**| 18-73**| 18-73**| 30-323| 24-73**| 14-73**|      |    |

### $q = 4, \delta = 6$

| n|d | 1    | 2    | 3    | 4    | 5    | 6    |    |
|---|-----|------|------|------|------|------|------|----|
| 7 | 6-14 |      |      |      |      |      |      |    |
| 8 | 9-24 | 16-49**|      |      |      |      |      |    |
| 9 | 12-55**| 16-55**| 18-55**|      |      |      |      |    |
| 10| 15-61**| 18-61**| 18-61**| 16-61**|      |      |      |    |
| 11| 16-67**| 18-67**| 18-67**| 18-67**| 12-67**|      |      |    |
| 12| 18-73**| 18-73**| 18-73**| 18-73**| 14-73**| 14-73**| 48-152|      |    |

**Acknowledgements.** The first author was partially supported by the National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science,"
Education and Security (ICTinSES)" of the Bulgarian Ministry of Education and Science. The second author was supported by the National Programme "Young Scientists and PostDocs" of the Bulgarian Ministry of Education and Science and by Bulgarian NSF under project KP-06-N32/2-2019. The research of the third and forth authors was carried out at the IITP RAS at the expense of the Russian Fundamental Research Foundation (project No. 19-01-00364).

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