Dissipative and Hall viscosity of a disordered 2D electron gas

Igor S. Burnistrov,1,2,3,4 Moshe Goldstein,5 Mordecai Kot,5 Vladislav D. Kurilovich,6 and Pavel D. Kurilovich6

1. L. D. Landau Institute for Theoretical Physics, acad. Semenova av. 1-а, 142432 Chernogolovka, Russia
2. Laboratory for Condensed Matter Physics, National Research University Higher School of Economics, 101000 Moscow, Russia
3. Institut für Theorie der Kondensierten Materie, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
4. Institut für Nanotechnologie, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany
5. Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, Tel Aviv 6997801, Israel
6. Departments of Physics and Applied Physics, Yale University, New Haven, CT 06520, USA

We study the dissipative and Hall viscosity of a disordered noninteracting 2D electrons, both analytically and numerically. Analytically, we employ the self-consistent Born approximation, explicitly taking into account the modification of the single-particle density of states and the elastic transport time due to the Landau quantization. The reported results interpolate smoothly between the limiting cases of weak (strong) magnetic field and strong (weak) disorder. In the regime of weak magnetic field, our results describes the quantum (de Haas-van Alphen type) oscillations of the dissipative and Hall viscosity. For strong magnetic field, we computed the dependence of the dissipative and Hall viscosity on disorder broadening of a Landau level. In particular, for the Hall viscosity the effect of the disorder broadening is weak. This theoretical conclusion is in agreement with our numerical results for a few lowest Landau levels, which show that Hall viscosity is robust to disorder.

Introduction. — The notion of viscosity is among cornerstones of hydrodynamics, in which it is responsible for dissipation. Under certain conditions charge transport in an electron system can be dominated by hydrodynamic viscous electron flow 112. Appearance of graphene stimulated theoretical 3411 and experimental 1218 interest in the hydrodynamic description of charge transport.

In absence of the time-reversal symmetry the viscositytensor has non-dissipative antisymmetric components. In the presence of a magnetic field B, this non-dissipative Hall viscosity (ηH) was studied theoretically in the classical limit of high temperature plasmas 1123 and for low temperature electron gas 24. Later, interest in the Hall viscosity was renewed in quantum systems with a gapped spectrum, due to connection of ηH to the geometric response 2521, and its expected quantization in the presence of translational and rotational symmetries 29. It was understood that in addition to the Hall conductivity and Hall viscosity there are additional non-dissipative physical quantities which determines the combined electro-magnetic and geometrical responses of gapped quantum systems 3211. Within the hydrodynamic description of electron transport non-zero Hall viscosity influences significantly the structure of an electron flow 1559 that allows one to access ηH experimentally 51. Also, it was argued that the dissipative and Hall viscosity affect the spectrum of edge magnetoplasmons 5254.

For noninteracting electrons in the absence of disorder each filled Landau level (LL) gives the contribution to the Hall viscosity equal h(n + 1/2)/(4πlB) 25, where n denotes the LL index and lB = ℏc/|eB| stands for the magnetic length. As mentioned above, this result is stable against perturbations of the Hamiltonian which preserve translational and rotational invariance 29. However, the fate of this result in the presence of disorder has not been studied yet. Therefore, it is not clear how the clean result obtained within the quantum treatment of an electron motion in a magnetic field connects to the result ηH = ν0μ2ωcτ21/2/[1 + 4ω2cτ21.2] derived for a classical disordered electron gas 24. Here μ denotes the chemical potential, τ1.2 the second transport time, ωc = eB/(mc) the cyclotron frequency, ν0 = m/(2π) the density of states at B = 0, and m the effective electron mass.

In this Letter we report the results of an analytical and numerical study of the dissipative and Hall viscosity of noninteracting 2D electrons in the presence of disorder. Contrary to previous studies we explicitly take into account the Landau quantization of the electron spectrum. Analytically, within the self-consistent Born approximation (SCBA) 55 we derive expressions for the dissipative and Hall viscosities, which smoothly interpolates between the results known in the literature for classical magnetic field 112315 and for the strong magnetic field in the absence of disorder 25. Since the SCBA is rigorously justified for high LLs only, we perform numerical calculation of the Hall viscosity for a few lowest LLs. The obtained numerical results are in a perfect agreement with the expectations based on our theoretical findings within SCBA.

Model. — Noninteracting electrons confined to a 2D plane are described by the following single-particle Hamiltonian

\[ H = \frac{1}{2m} (-i \nabla - eA)^2 + V(r), \] (1)

where V(r) stands for a random potential and A for the vector potential corresponding to the static perpendicular magnetic field B. In this paper we use the Landau gauge: A_y = −Bx and A_z = Az = 0. We assume that
the random potential has Gaussian distribution with a pair correlation function \( V(\mathbf{r})V(\mathbf{r'}) = W(|\mathbf{r} - \mathbf{r'}|) \) which decays with a typical length scale \( d \).

Kubo formula for the viscosity. — The viscosity tensor can be computed by means of the Kubo formula \([55,56]\):

\[
\eta_{jk,ps}(\omega) = -\frac{ik^{-1}}{\omega} \delta_{jk}\delta_{ps} + \frac{1}{i\omega S} \int \frac{d\Omega d\epsilon}{\pi^2} f_{\epsilon} - f_{\epsilon+i\Omega} \\
\times Tr T_{jk} \text{Im} G_{\epsilon+i\Omega}^{R} \text{Im} G_{\epsilon}^{R}. \tag{2}
\]

Here \( f_{\epsilon} = 1/[1 + \exp((\epsilon - \mu)/T)] \) denotes the Fermi distribution function, \( G_{\epsilon}^{R} = 1/(\epsilon - H - i0) \) the retarded Green’s function, \( T_{jk} = m(v_{j}v_{k} + \eta_{j}\eta_{k})/2 \) the stress tensor, \( S \) the system area, and \( \eta \) the internal compressibility \([60]\). Averaging over the random potential is denoted by an overbar.

Self-consistent Born approximation. — In order to compute the viscosity tensor from Eq. (2) we treat the disorder scattering using the SCBA \([55]\). This approximation holds under the following conditions \([61,62]\):

\[
1/k_F, d \ll l_B, \quad d \ll \nu_F \tau_0. \tag{3}
\]

Here \( l_B = 1/\sqrt{\epsilon_B} \) stands for the magnetic length, while \( k_F = m \nu_F \) and \( \nu_F \) denote the Fermi momentum and velocity, respectively. \( \tau_0 \) is the total elastic relaxation time at zero magnetic field. It can be expressed in terms of the Fourier transform \( \tilde{W}(q) \) of the pair correlation function \( W(\mathbf{r}) \). Furthermore, it is convenient to generalize it to

\[
\frac{1}{\tau_m} = \nu_0 \int_{0}^{2\pi} d\phi \tilde{W}(2k_F \sin \phi/2) \cos m\phi, \tag{4}
\]

where \( m = 0, 1, 2, \ldots \), and \( \nu_0 \) denotes the density of states at \( B = 0 \).

The average density of states \( \nu(\epsilon) \) at non-zero \( B \) is determined by the average retarded Green’s function \( G_{\epsilon}^{R} \).

In the LL representation the average density of states is given as \( \nu(\epsilon) = \sum_n \text{Im} G_n^{R}(\epsilon)/(2\pi^2 l_B^2) \), where the retarded Green’s function satisfies the SCBA equation (\( \epsilon_n = \omega_c(n + 1/2) \)) \([55,61,62]\)

\[
\frac{\partial}{\partial \epsilon} G_n^{R} = (\epsilon - \epsilon_n - \Sigma_c)\frac{1}{\pi \tau_0} \sum_n G_n^{R}. \tag{5}
\]

There are two limiting cases in which the self-consistent Eq. (5) can be easily solved \([55]\). In the regime of overlapping LLs, \( \omega_c \tau_0 \ll 1 \), one can use the Poisson formula for summation over LL index. Then the averaged density of states becomes \( \nu(\epsilon) = \nu_0[1 - 2\delta \cos(2\pi \epsilon/\omega_c)] \). Here \( \delta = \exp(-\pi/\omega_c \tau_0) \ll 1 \) is the Dingle parameter. In the opposite case, when the LLs are well separated, one can restrict the summation in Eq. (5) to the single LL which is closest to the energy of interest, \( |\epsilon - \epsilon_n| < \omega_c/2 \), where \( \epsilon_n = \omega_c(N + 1/2) \). Then the average density of states acquires the semi-circle profile:

\[
\nu(\epsilon) = \nu_0 \sqrt{\omega^2 - (\epsilon - \epsilon_n)^2}, \quad \text{where} \quad \Gamma = 2\omega_c/(\pi \tau_0) \text{ determines the the broadened LL width.}
\]

In the presence of long-range disorder correlations, \( d \gg k_F^{-1} \), it is important to take into account the vertex corrections to the “bubble” contribution in the Kubo formula (2) (see Fig. 1). This implies that in addition to the averaged Green’s function one needs also to know the renormalized vertex, which is the stress tensor in the case of the viscosity. Within the SCBA \( T_{jk} \) can be approximated as a linear combination of operators which change the LL index by 2. Under conditions (3) one can show that an operator \( V_m \), which transfers an electron from the \((n + m)\)-th LL to the \(n\)-th LL, is renormalized by the ladder resummation of the disorder lines as follows \([61,62]\) (see the Supplemental Material for details \([59]\)):

\[
V_m \rightarrow \frac{V_m}{1 - \tau_m^{-1} \Pi_m^{RA}}, \quad \Pi_m^{RA} = \frac{\omega_c}{2\pi} \sum_n \delta_n^{R}(\epsilon) G_n^{A}(\epsilon). \tag{6}
\]

Here \( \Pi_m^{RA} \) is the contribution of the bubble without ladder insertions. Using Eq. (5) it can be rewritten as \( \Pi_m^{RA} = i\nu(\epsilon)/[\nu(\epsilon) \nu_0 - i\nu(\epsilon)] \). Therefore, within the SCBA the vertex corrections are expressed in terms of the average density of states only.

Dissipative viscosity. — Disorder averaging restores 2D rotational symmetry. Hence, the viscosity tensor \( \eta_{jk,ps} \) is characterized by only three parameters:

\[
\eta_{jk,ps} = \eta_s(\delta_{jp} \delta_{ks} + \delta_{js} \delta_{kp}) + (\zeta - \eta_s) \delta_{jk} \delta_{ps} + (\eta_H/2)(\epsilon_{jp} \delta_{ks} + \epsilon_{js} \delta_{kp} + \epsilon_{kp} \delta_{js} + \epsilon_{ks} \delta_{jp}), \tag{7}
\]

where \( \zeta \) and \( \eta_s \) denote the bulk and shear viscosities, respectively. Within the SCBA the bulk viscosity vanishes, \( \zeta = 0 \). Using Eqs. (5) and (6), we find the following result for the shear viscosity at \( \omega = 0 \) \([59]\):

\[
\eta_s = \frac{1}{2} \int d\epsilon (-f'_\epsilon)^2 \nu(\epsilon)^2 \tau_{\nu,2}(\epsilon), \tag{8}
\]

where \( \tau_{\nu,2}(\epsilon) = \tau_{\nu,2}\nu_0/\nu(\epsilon) \) is the renormalized second transport time and \( 1/\tau_{\nu,2} = 1/\tau_0 - 1/\tau_2 \) is the second
transport rate at \( B = 0 \). We note that for \( k_F d \gg 1 \) the second transport time becomes \( \tau_{tr,z} = \tau_0 (k_F d/2)^2 \gg \tau_0 \). We mention that Eq. (8) is analogous to the result for the dissipative conductivity \([63]\).

In the regime of overlapping LLs, \( \omega_c \tau_0 \ll 1 \), the shear viscosity exhibits Shubnikov-de Haas-type oscillations:

\[
\eta_s = 4 \mu \omega_c^2 \tau_{tr,z} \left(1 - \frac{16 \alpha^2}{1 + 4 \omega_c^2} \tau_T \cos \frac{2 \pi \mu}{\omega_c} \right),
\]  

where \( \alpha = \omega_c \tau_{tr,2} \) and \( \tau_T = (2\pi^2 T/\omega_c)/\sinh(2\pi^2 T/\omega_c) \).

The non-oscillatory term in \( \eta_s \) reproduces the classical result for the shear viscosity of an electron gas \([24]\).

In the regime of well separated LLs, \( \omega_c \tau_0 \gg 1 \), one finds from Eq. (6) that the shear viscosity is non-zero when the chemical potential is inside the \( N \)-th broadened Landau level \((\mu - \epsilon_N) \leq \Gamma\):

\[
\eta_s = (N^2 \tau_0)/(8 \pi^2 l_B^2 \tau_{tr,2}) \left[1 - (\mu - \epsilon_N)^2/\Gamma^2 \right].
\]  

For chemical potential at the center of the LL, the shear viscosity is \( 2 \omega_c \tau_0/\pi \) times larger when one naively expects on the basis of purely classical expression. The dependence of the shear viscosity on the chemical potential in comparison with the density of states is shown in Fig. 2.

**Hall viscosity.** — The Hall viscosity can be extracted from the viscosity tensor as \( \eta_H = (\eta_{xy,xx} - \eta_{xy,yy})/2 \). Similar to the Hall conductance, the evaluation of \( \eta_H \) from the Kubo formula \([2]\) is complicated due to contributions which come from all the states below the chemical potential. Therefore, it is convenient to proceed in a way pioneered by Smrčka and Streda \([54]\). Similarly to the Hall conductivity we split the Hall viscosity at \( \omega = 0 \) into two parts, \( \eta_H = \eta_H^I + \eta_H^I \), where \([59]\)

\[
\eta_H^I = \text{Re} \int \frac{d\epsilon}{4\pi S} (-f' \epsilon) \text{Tr} T_{xy} G^R \epsilon (T_{xx} - T_{yy}) G^A \epsilon, 
\]

\[
\eta_H^I = \text{Im} \int \frac{d\epsilon}{2\pi S} (-f' \epsilon) \text{Tr} J_{xy} (T_{yy} - T_{xx}) G^A \epsilon + \text{Im} \int \frac{d\epsilon}{2\pi S} f' \epsilon \text{Tr} [J_{xy}, J_{yy} - J_{xx}] \text{Im} G^R \epsilon.
\]

Here \( J_{jk} \) denotes the stress generators which are related with the stress tensor \( T_{jk} = -i[H, J_{jk}] \) \([50]\). One can evaluate \( \eta_H^I \) in a similar way to \( \eta_s \) \([59]\):

\[
\eta_H^I = \int d\epsilon (-f' \epsilon) \nu(\epsilon) \frac{\nu(\epsilon)e^2}{1 + 4 \omega_c^2 \tau_{tr,2} \epsilon}. 
\]  

The evaluation of \( \eta_H^I \) is more involved. Although one can write down the viscoelastic analog of the Smrčka and Streda formula for the Hall viscosity \([63]\), it does not provide a suitable way for evaluation of \( \eta_H^I \) in the presence of disorder. In order to compute \( \eta_H^I \) one needs to know the expressions for the stress generators. In the absence of disorder they can be easily written down explicitly \([56]\), e.g., \( J_{xy} = (T_{xx} - T_{yy})/(4\omega_c) \) and \( J_{yy} = J_{xx} = T_{xy}/\omega_c \).

In the presence of a random potential the stress generators can be constructed as a series in spatial derivatives of a random potential \( V \) \([59]\). This allows us to evaluate \( \eta_H^I \) within the SCBA \([59]\):

\[
\eta_H^I = \epsilon/(2\omega_c) - \int d\epsilon (-f' \epsilon) \nu(\epsilon) e^2/(4\omega_c),
\]  

where \( \epsilon = \int d\epsilon \nu(\epsilon) f_\epsilon \) stands for the energy density. Combining Eqs. (13) and (14), we obtain

\[
\eta_H = \frac{\epsilon}{2\omega_c} - \frac{1}{4\omega_c} \int d\epsilon (-f' \epsilon) \frac{\nu(\epsilon)e^2}{1 + 4 \omega_c^2 \tau_{tr,2} \epsilon}. 
\]  

In the absence of disorder and for the chemical potential above the \( N \)-th Landau level the energy density at \( T = 0 \) can be computed as \( \epsilon = \sum_{n=0}^N \epsilon_n/(2\pi l_B^2) \), which yields the known result \( \eta_H = \sum_{n=0}^N (n+1)/2)/(4\pi l_B^2) \) \([25]\).

Also, we mention that in the Boltzmann limit, \( T \gg E_F \), the energy density is given by \( \epsilon = n_c T \), where \( n_c \) denotes the particle density, such that the Hall viscosity in the absence of disorder and at \( T \gg E_F \) becomes \( \eta_H = n_c T/(2\omega_c) \), in agreement with Eq. (59.38) of Ref. \([63]\) in which the Hall viscosity is denoted by \( \eta_s \). We note that the structure of Eq. (15) resembles the structure of the result for the Hall conductivity \([67]\).

The appearance of the non-zero \( \eta_H^I \) can be explained on a purely classical level \([27]\). The Hall viscosity describes the response of the \( T_{xx} - T_{yy} \) on a shear velocity profile \( U_x = \eta_y \). In the presence of a magnetic field this velocity can be considered as the result of a non-uniform electric field, \( E_y = -U_x B \). This electric field results not only in a drift of the cyclotron orbit but in its deformation into
an ellipse. To linear order in $u$ the ratio between ellipse axes is equal to $1 + u/(2\omega_c)$. This asymmetry between motion in the $x$ and $y$ direction yields the non-zero ratio $(T_{xx} - T_{yy})/\mu$ in the limit $u \to 0$. Hence, non-zero Hall viscosity arises, which is given by the first term in Eq. (15). We note that an electron moving along an ellipse conserves its energy to the first order in $u$, in agreement with non-dissipative nature of $\eta_H$. In the presence of impurity scattering an electron experiences a friction force corresponding to the electric field $E_x = -U_x/(\epsilon\tau_{tr,2})$. This electric field leads to a velocity component in the $y$ direction, $U_y = U_x/(\omega_c\tau_{tr,2})$. This non-uniform velocity produces additional correction to the difference, $T_{xx} - T_{yy} \sim -u\eta_H/(\omega_c\tau_{tr,2})$. Thus there is an additional correction to the Hall viscosity $\Delta\eta_H = -\eta_H/(\omega_c\tau_{tr,2})$ which corresponds the second term in Eq. (15) in the classical regime.

In the case of overlapping LLs, $\omega_c\tau_0 \ll 1$, from Eq. (15) we obtain the Shubnikov-de Haas oscillations of the Hall viscosity:

$$\eta_H = \frac{\nu_0^2}{1 + 4\alpha^2} \left( 1 + \frac{\delta}{2\alpha^2} + \frac{1 + 12\alpha^2}{4\alpha^2} F \cos \frac{2\pi \mu}{\omega_c} \right).$$

(16)

The non-oscillatory term in $\eta_H$ coincides with the classical result for the Hall viscosity of electron gas [24].

In the case of well-separated LLs, $\omega_c\tau_0 \gg 1$, one finds from Eq. (15) that the Hall viscosity is reduced from the quantized value if the chemical potential lies within the broadened LL, $|\mu - \epsilon_N| \leq \Gamma$:

$$\eta_H = \frac{N^2}{8\pi^3 B} \left[ 1 - \frac{\tau_0^2}{2\pi \omega_c\tau_{tr,2}} \left( 1 - \frac{(\mu - \epsilon_N)^2}{\Gamma^2} \right)^{3/2} \right].$$

(17)

We note that for the long-range-correlated random potential the Hall viscosity dominates the shear viscosity, $\eta_H \gg \eta_s$ (cf. Eqs. (16) and (17)).

The deviation of the Hall viscosity from the clean value is controlled by the small parameter $(\tau_0/\tau_{tr,2})^2/\sqrt{\omega_c\tau_0} \ll 1$. In the case of short range random potential correlations, $\tau_0 = \tau_{tr,2}$, the deviation of $\eta_H$ from its clean value is very small. For the long-range-correlated random potential, $\tau_0 \ll \tau_{tr,2}$, the difference $\eta_H - N^2/(8\pi^3 B)$ is additionally suppressed (see Fig. 3).

Numerical results.— We would now like to explore the quantum Hall regime, where the number of filled LLs is of order unity. Here the SCBA cannot be used anymore, and we resort to a numerical calculation. For this we discretize the system and employ the Hofstadter model with uncorrelated random potential, uniformly distributed between $[-w/2, w/2]$ at each lattice site. We calculate the Hall viscosity using retarded correlation function of discretized stress operators [68], and take both the continuum and thermodynamic limits to extrapolate to the behavior of our model [59]. In the presence of disorder we can take these limits while keeping constant $\omega_c\tau$. The results for the Hall viscosity are plotted in Fig. 4 together with the behavior of the Hall conductivity ($\sigma_H$) at zero wavevector. One sees that, somewhat surprisingly, the Hall viscosity maintains its quantization to the same extent as the Hall conductivity.

Conclusions.— To summarize, we studied the dissipative and Hall viscosity of 2D electron system in the presence of a random potential. Within the self-consistent Born approximation we derived an expressions for both the dissipative and Hall viscosities, which takes into account the modification of the single-particle density of states and the elastic transport time due to the Landau
quantization. Our results smoothly interpolate between the case of weak magnetic field and strong disorder, on the one hand, and the case of strong magnetic field and weak disorder, on the other hand. In the former regime, we derived the expressions for the quantum (Shubnikov-de Haas type) oscillations of the dissipative and Hall viscosity. In the case of strong magnetic field, we found that the disorder broadening of the Landau level does no lead to significant change of the Hall viscosity in comparison with the clean result. Our numerical results for a few filled LLs support this conclusion.

There are various ways to extend our work. In Galilean invariant systems it was proven \( [36, 56] \) that the Hall viscosity can be extracted from the Hall conductivity at finite wave-vector \( q \). This allows one to extract the Hall viscosity from the non-local conductivity. In the absence of Galilean invariance there is no reason to expect that \( \eta_H \) is related with \( \sigma_H(q) \) \( [59, 70] \). Also, the relation between \( \eta_H \) and \( \sigma_H(q) \) can be affected by the presence of lattice \( [68] \) or disorder. However, if one treats disorder on the level of Drude model with classical magnetic field the relation of Ref. \( [56] \) between \( \eta_H \) and \( \sigma_H(q) \) still holds \( [71] \). This fact is not surprising since the Drude model does not properly take into account the LLs, which result in the energy dependence of the density of states and elastic scattering transport time. This simplification can be dangerous since \( \eta_H \) and \( \sigma_H(q) \) have contributions coming from the states well below the Fermi energy. It would therefore be worthwhile to extend the presented analytical and numerical approaches to the conductivity at finite wave vector \( [72] \). Also we mention that our techniques can be applied to calculation of the dissipative and Hall viscosity in graphene, where only the result in the absence of disorder in known \( [73] \).

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ONLINE SUPPORTING INFORMATION
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Igor S. Burmistrov,1, 2, 3, 4 Moshe Goldstein,5 Mordecai Kot,5 Vladislav D. Kurilovich,6 and Pavel D. Kurilovich6

1L. D. Landau Institute for Theoretical Physics, acad. Semenova av. 1-a, 142432 Chernogolovka, Russia
2Laboratory for Condensed Matter Physics, National Research University Higher School of Economics, 101000 Moscow, Russia
3Institut für Theorie der Kondensierten Materie, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
4Institut für Nanotechnologie, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany
5Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, Tel Aviv 6997801, Israel
6Departments of Physics and Applied Physics, Yale University, New Haven, CT 06520, USA

In this notes, we present the details of (i) the vertex renormalization in the self-consistent Born approximation, (ii) the calculation of the shear viscosity, (iii) the calculation of the Hall viscosity, (iv) the derivation of the result for the Hall viscosity in the case of well-separated Landau levels, (v) our numerical procedure for the limit of few filled Landau levels.

S.I. THE SELF-CONSISTENT BORN APPROXIMATION: THE SELF ENERGY AND VERTEX RENORMALIZATION

In this section we present details of derivation for the self energy in the average Green’s function and the vertex renormalization within the self-consistent Born approximation (SCBA) [S1]. As in the main text, in order to justify the SCBA, we assume that the following conditions hold [S2, S3, S4]:

\[ 1/k_F, d \ll l_B, \quad d \ll v_F \tau_0. \]  

(S1)

We work in the Landau gauge, \( A_x = A_z = 0 \) and \( A_y = -Bx \). The corresponding basis of the eigenfunctions of the 2D Landau problem is denoted by \(|nk⟩\), where \( n \) enumerates the Landau levels (LLs), whereas \( k \) denotes the pseudo-momentum. In what follows, we shall use the following identities:

\[
\langle nk|e^{iqr}|n'k'⟩ = \frac{2π}{L_y} \delta (k - k' - q_y)e^{iσ_y(k+k')\xi_n^2/2} \sqrt{\frac{n'!}{n!}} (-i)^{n-n'} e^{-q^2q_r^2/4} \left( \frac{q - l_H}{\sqrt{2}} \right)^{n-n'} L_{n'}^{n-n'} \left( \frac{q^2l_H^2}{2} \right),
\]

(S2)

where \( q_x = q_z = iq_y \) and \( L_n^m(z) \) denotes a generalized Laguerre polynomial [S5].

The equation for the self-consistent Green’s function can be written as \( \epsilon_n = \omega_c (n + 1/2) \)

\[
\Sigma^R_n(\omega) = \frac{Γ \sqrt{2}}{\pi} \int dQ W(Q) \left\{ \sum_{n'} \frac{n'!}{n!} \left[ G_n^{R}(\omega) e^{-z_Q z_Q^2} L_{n'}^{n-n'} (z_Q) L_{n-n'}^{n-n'} (z_Q) \right] \right\}
\]

(S3)

Here and afterwards, \( z_Q = Q^2l_H^2/2 \). The Fourier transform of the pair correlation function of the random potential is denoted by \( W(Q) \). To simplify the expression for the self-energy we can use the following asymptotic form of the Laguerre polynomials \( n \gg a \):

\[
L_n^a(x) = \sqrt{\frac{(n+a)!}{n!}} e^{x/2} x^{-a/2} J_a(\sqrt{2x(n+a+1)}),
\]

(S4)

where \( J_a(x) \) stands for the Bessel function [S5]. We thus find the following expression for the self-energy

\[
\Sigma^R_n(\omega) = \sum_{n'} G_n^{R}(\omega) \int \frac{dQ}{(2\pi)^2} W(Q) J_n^{2}(Ql_H\sqrt{n+n'+1}) \approx \int_0^\infty \frac{dQ}{2\pi^2 R_c} W(Q) \sum_{n'} G_n^{R}(\omega),
\]

(S5)

where we used the asymptotic form of the Bessel function, relying on the condition \( QR_c \gg 1 \), where \( R_c = l_B\sqrt{n+n'+1} \) is the cyclotron radius. Provided that only the LLs which are close to the energy \( \omega \) contribute significantly, we can approximate the cyclotron radius as \( R_c = v_c/\omega_c \), where \( v_c = \sqrt{2e/m} \). Then the self-energy becomes
independent of \( n \). Using the expression for the total scattering rate at zero magnetic field

\[
\frac{1}{\tau_0} = \frac{2\pi}{\nu_0} \int_0^{2\pi} d\phi \ W(2k_c|\sin(\phi/2)|) \approx \frac{1}{\pi\nu_c} \int_0^\infty dQ W(Q) = \frac{2\pi}{\omega_c} \int_0^\infty \frac{dQ}{2\pi^2 R_c} W(Q),
\]

(S6)

where \( \nu_0 = m/(2\pi) \) and \( k_c = m\nu_c \), we obtain the following self-consistent equation for the self energy:

\[
\Sigma^R = \frac{\omega_c}{2\pi\tau_0} \sum_n \mathcal{G}^R_n(\varepsilon).
\]

(S7)

This equation demonstrates that the self-energy is related to the density of states as \( \text{Im} \Sigma^R(\varepsilon) = -\nu(\varepsilon)/(2\pi n_0\tau_0) \).

Within the SCBA one can use the quasiclassical form of the matrix elements for the components of the stress tensor:

\[
T_{xy} = \frac{i\varepsilon}{2}(V_2 - V_{-2}), \quad T_{xx} - T_{yy} = \varepsilon(V_2 + V_{-2}), \quad \langle \pi|V_m|\pi'\rangle = \delta_{kk'}\delta_{\pi',\pi+m}.
\]

(S8)

In order to evaluate the viscosity tensor one needs to know the renormalization of the operator \( V_m \) by the ladder of disorder lines. The corresponding renormalized vertex \( \Gamma_m \) is diagonal in the pseudo-momentum \( k \) and satisfies the following equation:

\[
\Gamma_{RA}^R = V_m + \sum_n \left[ \frac{(n + m)!}{(n_1 + m_1)!n_1!} \right]^{1/2} \mathcal{G}^R_{n_1+m}(\varepsilon + \omega)\mathcal{G}^A_{n_1}(\varepsilon)\Gamma_m \int_0^{2\pi} dQ W(Q) e^{-\varepsilon Q R_n^{a-n} L^{n-n} Q m^{a-n} L^{n-n} Q}.
\]

(S9)

Similar equations hold for \( \Gamma_{RA}^R \), \( \Gamma_{RA}^R \) and \( \Gamma_{AA}^R \). Using the asymptotic expression (S4) we find

\[
\Gamma_{RA}^R = V_m + \sum_n \mathcal{G}^R_{n_1+m}(\varepsilon + \omega)\mathcal{G}^A_{n_1}(\varepsilon) \int dQ W(Q) J_{n_1-n}(Ql_B\sqrt{n + n_1 + 1}) J_{n_1-n}(Ql_B\sqrt{n + n_1 + 1 + 2m})\Gamma_{RA}^R.
\]

(S10)

It is convenient to introduce the quantity

\[
\frac{1}{\tau_m} = \frac{2\pi}{\omega_c R_c} \int_0^{2\pi} dQ W(Q) \cos[QR_c^2 + 2ml^2_H - QR_c] \approx \frac{1}{\tau_0} - \frac{m^2}{2\pi^2 R_c^2} \int_0^{2\pi} dQ W(Q).\]

(S11)

Using the asymptotic expressions for the Bessel functions at \( QR_c \gg 1 \), we find

\[
\frac{1}{\tau_m} = \frac{1}{\pi\omega_c R_c} \int_0^\infty dQ W(Q) \cos[Q R_c^2 + 2ml^2_H - QR_c] \approx \frac{1}{\tau_0} - \frac{m^2}{2\pi^2 R_c^2} \int_0^\infty dQ W(Q).\]

(S12)

We note that the transport elastic rate is given as \( 1/\tau_{tr,1} = 1/\tau_0 - 1/\tau_1 \). Finally, we obtain

\[
\Pi_{RA}^R(\omega) = \frac{\omega c}{2\pi} \sum_n \mathcal{G}^R_{n_1+m}(\varepsilon + \omega)\mathcal{G}^A_{n_1}(\varepsilon) = \frac{\tau_0(\Sigma^R_{\omega+\omega} - \Sigma^A_{\omega})}{m\omega_c + \Sigma^R_{\omega+\omega} - \Sigma^A_{\omega}}.
\]

(S13)

### S.II. EVALUATION OF THE SHEAR VISCOSITY

Using the Kubo formula from the main text we can write the real part of the shear viscosity as

\[
\text{Re } \eta_s(\omega) = \text{Im } \int d\Omega \int \frac{d\varepsilon}{\Omega - \omega - i0} \frac{f_\varepsilon - f_{\varepsilon+\Omega}}{\omega / \pi S} \frac{\varepsilon}{\omega} \text{Tr } T_{xy} \text{Im } G^R_{\varepsilon+\Omega} T_{xy} \text{Im } G^R_{\varepsilon} = \text{Im } \int \frac{d\varepsilon}{\Omega - \omega - i0} \int \frac{d\Omega}{\omega} \frac{f_\varepsilon - f_{\varepsilon+\Omega}}{\omega} \sum_{a,b} |\langle a|T_{xy}|b\rangle|^2 \delta(\varepsilon + \Omega - E_b) \delta(\varepsilon - E_a) = \pi \int \frac{d\varepsilon}{\pi S} \frac{f_\varepsilon - f_{\varepsilon+\Omega}}{\omega} \sum_{a,b} |\langle a|T_{xy}|b\rangle|^2 \delta(\varepsilon + \omega - E_b) \delta(\varepsilon - E_a) = \int \frac{d\varepsilon}{\pi S} \frac{f_\varepsilon - f_{\varepsilon+\Omega}}{\omega} \text{Tr } T_{xy} \text{Im } G^R_{\varepsilon+\Omega} T_{xy} \text{Im } G^R_{\varepsilon}.\]

(S14)
Here $E_a$, $E_b$ and $|a\rangle$, $|b\rangle$ denote the eigenenergies and eigenfunctions of $H$ for a given disorder realization. As one can see, $\eta_s$ is determined by the states which are close to the Fermi energy. Using the quasiclassical result (S8) for the stress tensor and the result (S13) for the vertex we find

$$\text{Re} \eta_s(\omega) = \frac{\nu_0}{4\omega} \int d\epsilon \left( f_\epsilon - f_{\epsilon + \omega} \right) \text{Re} \sum_{\sigma=\pm} \left( \frac{\Pi^{RA}_{2\sigma}(\omega)}{1 - \Pi^{RA}_{2\sigma}(\omega)/\tau_2} - \frac{\Pi^{RR}_{2\sigma}(\omega)}{1 - \Pi^{RR}_{2\sigma}(\omega)/\tau_2} \right).$$  \hspace{1cm} (S15)

Setting the frequency $\omega$ to zero and using the expression (S13) for the polarization operators $\Pi^{RA}_{2\sigma}$ and $\Pi^{RR}_{2\sigma}$, we obtain the result for $\eta_s$ from the main text.

S.III. EVALUATION OF THE HALL VISCOSITY

A. Splitting the Hall viscosity into $\eta_H^I$ and $\eta_H^{II}$

Using the Kubo formula from the main text we can write the Hall viscosity as

$$\eta_H(\omega) = \int \frac{d\Omega}{\Omega - \omega - i0} \int \frac{d\epsilon}{2\pi iS} \frac{f_\epsilon - f_{\epsilon + \Omega}}{\omega} \text{Tr} T_{xy} \text{Im} G^{R}_{\epsilon + \Omega}(T_{xx} - T_{yy}) \text{Im} G^{R}_{\epsilon}.$$

Following Smrčka and Středa [S6], we split the Hall viscosity into two parts $\eta_H(\omega) = \eta_H^I(\omega) + \eta_H^{II}(\omega)$, where

$$\eta_H^I(\omega) = \int \frac{d\Omega}{\Omega - \omega - i0} \int \frac{d\epsilon}{2\pi iS} \frac{f_\epsilon - f_{\epsilon + \Omega}}{\omega} \text{Tr} T_{xy} G^{R}_{\epsilon + \Omega}(T_{xx} - T_{yy}) G^{A}_{\epsilon} - T_{xy} G^{A}_{\epsilon + \Omega} (T_{xx} - T_{yy}) G^{R}_{\epsilon},$$

$$\eta_H^{II}(\omega) = \int \frac{d\Omega}{\Omega - \omega - i0} \int \frac{d\epsilon}{4\pi^2 S} \frac{f_\epsilon - f_{\epsilon + \Omega}}{\omega} \text{Tr} T_{xy} G^{R}_{\epsilon + \Omega}(T_{xx} - T_{yy}) \text{Im} G^{R}_{\epsilon} - T_{xy} \text{Im} G^{R}_{\epsilon + \Omega}(T_{xx} - T_{yy}) G^{R}_{\epsilon}. \hspace{1cm} (S17)$$

1. Evaluation of $\eta_H^I$

It is convenient to rewrite the expression for the $\eta_H^I$ in terms of the exact eigenstates:

$$\eta_H^I(\omega) = \int \frac{d\Omega}{\Omega - \omega - i0} \int \frac{d\epsilon}{8\pi^2 i} \frac{f_\epsilon - f_{\epsilon + \Omega}}{\omega} \sum_{a,b} \frac{\langle a | T_{xy} | b \rangle \langle b | (T_{xx} - T_{yy}) | a \rangle}{(\epsilon + \Omega - E_b + i0)(\epsilon - E_a - i0)} \left| \frac{(a | T_{xy} | b) (b | (T_{xx} - T_{yy}) | a)}{(\epsilon + \Omega - E_b - i0)(\epsilon - E_a + i0)} \right|.$$

Relations (S8) imply that one can rewrite the above expression as

$$\text{Re} \eta_H^I(\omega) = \text{Im} \int \frac{d\Omega}{\Omega - \omega - i0} \int \frac{d\epsilon}{4\pi^2} \frac{f_\epsilon - f_{\epsilon + \Omega}}{\omega} \text{Re} \sum_{a,b} \frac{\langle a | T_{xy} | b \rangle \langle b | (T_{xx} - T_{yy}) | a \rangle}{(\epsilon + \Omega - E_b + i0)(\epsilon - E_a - i0)} \left| \frac{(a | T_{xy} | b) (b | (T_{xx} - T_{yy}) | a)}{(\epsilon + \Omega - E_b - i0)(\epsilon - E_a + i0)} \right|.$$

$$= \text{Re} \int \frac{d\epsilon}{4\pi S} \frac{f_\epsilon - f_{\epsilon + \omega}}{\omega} \text{Tr} T_{xy} G^{R}_{\epsilon + \omega}(T_{xx} - T_{yy}) G^{A}_{\epsilon}. \hspace{1cm} (S19)$$

Therefore, similarly to $\eta_s$, $\eta_H^I$ is determined by the states which are close to the Fermi energy. Therefore, we can compute $\eta_H^I$ in the same way as $\eta_s$. Setting $\omega$ to zero, we obtain the Kubo formula for $\eta_H^I$ given in the main text.

Finally, using the result (S13) for the renormalized vertex and the quasiclassical result (S8) for the stress tensor, we obtain

$$\text{Re} \eta_H^{II}(\omega) = -\frac{\nu_0}{4\omega} \int d\epsilon \left( f_\epsilon - f_{\epsilon + \omega} \right) \epsilon^2 \text{Im} \sum_{\sigma} \frac{\sigma \Pi^{RA}_{2\sigma}(\omega)}{1 - \Pi^{RA}_{2\sigma}(\omega)/\tau_2}. \hspace{1cm} (S20)$$

In the limit $\omega \to 0$, we reproduce the result for $\eta_H^I$ from the main text.
2. Evaluation of $\eta_{II}^H$

For the evaluation of $\eta_{II}^H$ we can rewrite Eq. (S17) as

$$
\eta_{II}^H(\omega) = \frac{4\pi}{4\pi\omega} \int \frac{d\Omega d\epsilon}{d\omega} \left[ \frac{\langle a|T_{xy}|b\rangle(b|T_{xx} - T_{yy}\rangle|a\rangle}{(\Omega - \omega - i\delta)(\Omega + E_a - E_b - i\delta)} \right] \left[ \delta(\epsilon - E_b) - \delta(\epsilon + \Omega - E_b) + \delta(\epsilon - \Omega - E_a) - \delta(\epsilon - E_a) \right]
$$

$$
= - \int \frac{4\pi}{4\pi\omega} f_c \text{Tr} \left[ T_{xy} G^R_{\epsilon + \omega}(T_{xx} - T_{yy}) G^R_{\epsilon} - T_{xy} G^A_{\epsilon + \omega}(T_{xx} - T_{yy}) G^A_{\epsilon} \right].
$$

(S21)

Taking the limit $\omega \to 0$, we obtain

$$
\eta_{II}^H = \int \frac{4\pi}{4\pi\omega} f_c \sum_{a,b} \langle a|T_{xy}|b\rangle(b|T_{xx} - T_{yy}\rangle|a\rangle \left[ \frac{1}{(\epsilon - E_b + i\delta)(\epsilon - E_a - i\delta)} \right]
$$

$$
= - \int \frac{4\pi}{2\pi\omega} \left[ -f'_\epsilon \right] \text{Tr} J_{xy}(T_{xx} - T_{yy}) G^A_{\epsilon} - \text{Im} \int \frac{4\pi}{2\pi\omega} f_c \text{Tr}[J_{xy}, J_{xx} - J_{yy}] \text{Im} G^R_{\epsilon}.
$$

(S22)

Now we can use the quasiclassical approximations for the stress generators:

$$
J_{xy} \approx \frac{T_{xx} - T_{yy}}{4\omega_c}, \quad J_{xx} - J_{yy} \approx \frac{T_{xy}}{\omega_c}.
$$

(S23)

Then, we obtain

$$
\text{Im} \int \frac{4\pi}{2\pi\omega} (-f'_\epsilon) \text{Tr} J_{xy}(T_{xx} - T_{yy}) G^A_{\epsilon} = \text{Im} \int \frac{4\pi}{8\pi\omega_c} (-f'_\epsilon) \text{Tr}(T_{xx} - T_{yy})^2 G^A_{\epsilon}
$$

$$
= \text{Im} \int \frac{4\pi}{8\pi\omega_c} (-f'_\epsilon) \frac{1}{2\pi f_b^2} \sum_n G^A_n(\epsilon) = \int \frac{4\pi}{4\omega_c} (-f'_\epsilon) \nu(\epsilon) \epsilon^2.
$$

(S24)

Next, using the relation

$$
i[T_{xx} - T_{yy}, T_{xy}] = 4\omega_c H_0, \quad H_0 = \frac{1}{2m_e} (p - eA)^2,
$$

(S25)

we find in the quasiclassical approximation:

$$
\text{Im} \int \frac{4\pi}{2\pi\omega} f_c \text{Tr}[J_{xy}, J_{xx} - J_{yy}] \text{Im} G^R_{\epsilon} = \text{Re} \int \frac{4\pi}{2\pi\omega_c} f_c \text{Tr} H_0 \text{Im} G^R_{\epsilon} = \int \frac{4\pi}{2\omega_c} \sum_a E_a \delta(\epsilon - E_a)
$$

$$
= - \frac{1}{2\omega_c} \int d\epsilon \nu(\epsilon) \epsilon f_c.
$$

(S26)

Combining together Eqs. (S24) and (S26), we obtain the result for $\eta_{II}^H$ from the main text.

S.IV. SEPARATED LANDAU LEVELS

In this section we demonstrate in details how the quasiclassical approximation works for the case of well-separated Landau levels, $\omega, \tau_0 \gg 1$. For simplicity we consider the case of short-ranged random potential, $d \ll \lambda_F$. In this case one does not need to renormalize vertex by the disorder ladder. For short-ranged random potential the conditions (S1) are equivalent to the condition $\mu \gg \omega_c$.

We start from the evaluation of $\eta_{II}^H$. Using Eq. (S19) we find

$$
\eta_{II} = \text{Re} \int \frac{4\pi}{4\pi} (-f'_\epsilon) \sum_{n,n'k,k'} \langle nk|T_{xy} n'k'|T_{xx} - T_{yy}|nk\rangle G^R_n(\epsilon) G^A_n(\epsilon)
$$

$$
= - \frac{\mu \omega_c^2}{4\pi^2} \int \frac{4\pi}{4\pi} (-f'_\epsilon) \sum_{n} \left| \epsilon - E_n - \frac{n(n-1)}{\omega_c} \right| \text{Im} \frac{\Sigma^R}{\Sigma^A}^2.
$$

(S27)
Here we used the matrix elements of components of the stress tensor:
\[
\langle nk|T_{xy}|n'k'\rangle = im\langle nk|v^2 + \left(-v^2\right)|n'k'\rangle = \frac{1}{mB}\delta_{k,k'}\left(\sqrt{n'(n'-1)}\delta_{n',n+2} - \sqrt{n(n-1)}\delta_{n,n'+2}\right),
\]
\[
\langle nk|T_{xx} - T_{yy}|n'k'\rangle = 2m\langle nk|v^2 + v^2|n'k'\rangle = \frac{1}{mB}\delta_{k,k'}\left(\sqrt{n'(n'-1)}\delta_{n',n+2} + \sqrt{n(n-1)}\delta_{n,n'+2}\right).
\]
(S28)

In order to proceed further, we need to solve the self-consistent equation (S7) for the self-energy. For well-separated Landau levels, \(\omega_c \gg \Gamma\), we can solve it by expanding the right hand side in powers of \(1/\omega_c\):
\[
\Sigma^R_\varepsilon = \frac{\omega_c}{2\pi\tau_0}\sum_n \frac{1}{\varepsilon - \varepsilon_n - \Sigma^R_\varepsilon} = \frac{\omega_c}{2\pi\tau_0}\frac{1}{\varepsilon - \varepsilon_N - \Sigma^R_\varepsilon} + \frac{\omega_c}{2\pi\tau_0}\sum_{n\neq N} \frac{1}{\varepsilon - \varepsilon_n - \Sigma^R_\varepsilon} = \frac{\Gamma^2}{4} \frac{1}{\varepsilon - \varepsilon_N - \Sigma^R_\varepsilon} + \frac{\Gamma^2}{4} \sum_{n\neq N} \left[\frac{1}{\varepsilon - \varepsilon_n - \Sigma^R_\varepsilon} - \frac{\varepsilon - \varepsilon_N - \Sigma^R_\varepsilon}{(\varepsilon - \varepsilon_n - \Sigma^R_\varepsilon)^2}\right] + \ldots.
\]
(S29)

Here \(N\) is the index of the Landau level which is the closest one to the energy \(\varepsilon\), i.e. \(|\varepsilon - \varepsilon_N| < \omega_c/2\). We note that the conditions of applicability of SCBA implies that \(N \gg 1\). Solving Eq. (S29) we obtain
\[
\Sigma^{R,A}_\varepsilon \approx \frac{1}{2} \left[(1 - \gamma)(\varepsilon + \Delta - \varepsilon_N) + i(1 + \gamma)\sqrt{(1 - \gamma)\Gamma^2 - (\varepsilon - \Delta - \varepsilon_N)^2}\right],
\]
(S30)

where
\[
\Delta = \frac{\Gamma^2}{4} \sum_{n\neq N} \frac{1}{\varepsilon - \varepsilon_n}, \quad \gamma = \frac{\Gamma^2}{4} \sum_{n\neq N} \frac{1}{(\varepsilon - \varepsilon_n)^2}.
\]
(S31)

We note that \(\Delta \propto \Gamma^2/\omega_c\) and \(\gamma \propto \Gamma^2/\omega^2_c\). The result (S30) leads to the following expression for the density of states:
\[
\nu(\varepsilon) = \nu_0\tau_0(1 + \gamma)\tilde{\Gamma}(\varepsilon) = \frac{B}{\pi^2}(1 + \gamma)\frac{\tilde{\Gamma}(\varepsilon)}{\Gamma^2}, \quad \tilde{\Gamma}(\varepsilon) = \sqrt{(1 - \gamma)\Gamma^2 - (\varepsilon - \Delta - \varepsilon_N)^2}.
\]
(S32)

In a similar way, expanding in powers of \(1/\omega_c\) we compute the sum over LLs on the right hand side of Eq. (S27):
\[
\sum_n \frac{n(n-1)}{\varepsilon - \varepsilon_n - \Sigma^R_\varepsilon^2} = \frac{1}{\varepsilon - \varepsilon_N - \Sigma^R_\varepsilon^2} \left(N(N-1) + 2(N+2)(N+1)\right)
+ \frac{N(N+1)}{4\omega^2_c} \left[1 - \frac{\varepsilon - \varepsilon_N}{2\omega_c} + \frac{(\varepsilon - \varepsilon_N)^2 - \Gamma^2/4}{4\omega^2_c}\right] + \frac{1}{\omega^2_c} \sum_{n\neq N,N+2} \frac{n(n-1)}{(N-n)^2(N-n+2)^2}.
\]
(S33)

Using the relation
\[
\sum_{n\neq N,N+2} \frac{n(n-1)}{(N-n)^2(N-n+2)^2} = \frac{2(N^2+N)\gamma\omega^2_c}{\Gamma^2} - \frac{3}{8}(N^2+N) + N/2 + O(1),
\]
we get
\[
\eta''_H = \frac{\omega_c}{4} \int d\varepsilon (-f'_\varepsilon)\nu(\varepsilon) \left[N^2 + N + \frac{\varepsilon - \varepsilon_N}{\omega_c} - \frac{N^2 + N + 2\Gamma^2(\varepsilon)}{4\omega^2_c} + \frac{N\Gamma^2}{4\omega^2_c}\right].
\]
(S35)

The terms proportional to \(N^2\) reproduce the result for \(\eta''_H\) from the main text for well-separated LLs. However, we notice that there is a term of the first order in \(N\), \(N(\varepsilon - \varepsilon_N)/\omega_c\), which could compete with the term \(N^2\Gamma^2/\omega^2_c\) at moderate \(N\). As we shall see below, this term actually cancels with the corresponding term in the expression for \(\eta''_H\).

Now we turn to \(\eta''_H\). In order to perform such a computation one needs to construct the stress generators in the presence of disorder. We do it perturbatively with respect to the spatial derivatives of a random potential:
\[
J_{xy} = \frac{T_{xx} - T_{yy}}{4\omega_c} + \Delta J_{xy}, \quad J_{xx} - J_{yy} = -\frac{T_{xy}}{\omega_c} + \Delta J_\parallel,
\]
(S36)
where $\Delta J_{xy}$ and $\Delta J_{||}$ satisfy the following equations

$$[H_0, \Delta J_{xy}] = \frac{i}{2\omega_c} \left( \partial_y V_{xy} - \partial_x V_{yx} \right) + \frac{1}{4m\omega_c} \left( \partial_y^2 V - \partial_x^2 V \right) + \ldots,$$  \hfill (S37)

$$[H_0, \Delta J_{||}] = -\frac{i}{\omega_c} \left( \partial_y V_{yy} + \partial_y V_{yx} \right) - \frac{1}{m\omega_c} \partial_y \partial_x V + \ldots.$$  \hfill (S38)

Then Eq. (S22) can be written as

$$\eta^I_{II} = - \text{Im} \int \frac{d\epsilon}{8\pi\omega_c S} \left( -f'_c \right) \text{Tr}(T_{xx} - T_{yy})^2 \mathcal{G}_c^A = - \int \frac{d\epsilon}{2\omega_c S} f_c \text{Tr} H \text{Im} \mathcal{G}_c^R$$

$$+ \int \frac{d\epsilon}{2\omega_c S} f_c \text{Tr} V \text{Im} \mathcal{G}_c^R - \text{Im} \int \frac{d\epsilon}{2\pi S} \left( -f'_c \right) \text{Tr} \Delta J_{xy}(T_{xx} - T_{yy}) \mathcal{G}_c^A$$

$$+ \text{Im} \int \frac{d\epsilon}{2\omega_c S} f_c \text{Tr}[\Delta J_{xy}, T_{xy}] \mathcal{G}_c^A \text{Im} \mathcal{G}_c^R - \text{Im} \int \frac{d\epsilon}{8\pi\omega_c S} f_c \text{Tr}(T_{xx} - T_{yy}, \Delta J_{||}) \text{Im} \mathcal{G}_c^R.$$  \hfill (S39)

Here we used the relation (S25). As we shall demonstrate below, the terms on the first line of Eq. (S39) gives contributions of order $N^2$. The terms on the second and third lines of Eq. (S39) are of the order $N$ or less.

Using the matrix elements (S28) we find

$$\text{Im} \int \frac{d\epsilon}{8\pi\omega_c S} \left( -f'_c \right) \text{Tr}(T_{xx} - T_{yy})^2 \mathcal{G}_c^A = \text{Im} \int \frac{d\epsilon}{8\pi\omega_c S} \left( -f'_c \right) \text{Tr}(T_{xx} - T_{yy})^2 \mathcal{G}_c^A = - \frac{m\omega_c^2}{4\pi} \int d\epsilon f_c \sum_n \frac{(n^2 + n + 1)}{|\epsilon - \epsilon_n - \Sigma^R_{\epsilon}|} \text{Im} \Sigma^R_{\epsilon}$$

$$= \frac{(1 - \gamma)\omega_c}{4} \int d\epsilon(-f'_c) \nu(\epsilon) \left[ N^2 + N + \frac{\Gamma^2}{4\omega_c^2} \sum_{n \neq N} \frac{n^2 + n + 1}{(n - N)^2} \right] = \frac{\omega_c}{4} \int d\epsilon(-f'_c) \nu(\epsilon) \left[ N^2 + N + \frac{2N\Delta}{\omega_c} \right].$$ \hfill (S40)

Here the prime on the sum over $n$ indicates a restriction of the sum to the vicinity of $n = N$, due to the dependence of $\text{Im} \Sigma^R_{\epsilon}$ on $\epsilon - \epsilon_n$.

The second term in Eq. (S39) is estimated as

$$\int \frac{d\epsilon}{2\pi\omega_c S} f_c \text{Tr} H \text{Im} \mathcal{G}_c^R = - \int \frac{d\epsilon}{2\omega_c} \sum_a E_a \delta(\epsilon - E_a) = - \int \frac{d\epsilon}{2\omega_c} \nu(\epsilon) f_c.$$ \hfill (S41)

The third term in Eq. (S39) can be approximated as follows:

$$\int \frac{d\epsilon}{2\omega_c S} f_c \text{Tr} V \text{Im} \mathcal{G}_c^R \approx \int \frac{d\epsilon}{2\pi S} f_c \text{Tr} V \mathcal{G}_c^R V \mathcal{G}_c^R = \frac{m^2}{4\pi} \int d\epsilon f_c \sum_n \frac{\Sigma^R_{\epsilon}}{\epsilon - \epsilon_n - \Sigma^R_{\epsilon}} = \frac{m^2\tau_0}{2\pi\omega_c} \int d\epsilon f_c \left[ \Sigma^R_{\epsilon} \right]^2$$

$$= - \frac{1}{\omega_c} \int d\epsilon f_c \nu(\epsilon) \text{Re} \Sigma^R_{\epsilon} = - \frac{m}{2\pi\omega_c} \int d\epsilon f_c \sum_n \frac{\nu(\epsilon)}{\epsilon - \epsilon_n} \sqrt{1 - \frac{(\epsilon - \epsilon_n)^2}{\Gamma^2}}.$$ \hfill (S42)

Here the last partially filled LL contributes only in the sum over $n$ at temperatures $T \ll \omega_c$. This implies that this contribution is of order $\Gamma/|\Omega|\omega_c$, i.e., this term is $O(N^0)$.

Now we consider the fourth term in Eq. (S39):

$$\text{Im} \int \frac{d\epsilon}{2\pi S} \left( -f'_c \right) \text{Tr} \Delta J_{xy}(T_{xx} - T_{yy}) \mathcal{G}_c^A \approx \text{Im} \int \frac{d\epsilon}{2\pi S} \left( -f'_c \right) \text{Tr} \Delta J_{xy}(T_{xx} - T_{yy}) \mathcal{G}_c^A V \mathcal{G}_c^A = - \frac{m\omega_c}{8\pi} \int \frac{d\epsilon}{2\pi} \left( -f'_c \right)$$

$$\times \sum_{n \neq n'} \frac{1}{n - n'} \sqrt{\frac{n!}{n'}!} \int \frac{d^2 Q}{(2\pi)^2} W(Q) e^{-2\pi Q_{\parallel} - 2\pi Q_{\perp}} L_n^{n'-n}(zQ) \left[ \sqrt{n'(n'-1)} G_{n'-2}^A(\epsilon) G_{n-2}^A(\epsilon) \sqrt{\frac{n!}{(n'-2)!}} L_{n'-2}^{n'-n-2}(zQ) \right]$$

$$+ \sqrt{(n'+1)(n'+2)} G_{n'+2}^A(\epsilon) G_{n}^A(\epsilon) \sqrt{\frac{n!}{(n'+2)!}} L_n^{n'-n+2}(zQ)$$

$$\approx \frac{1}{\omega_c} \int \frac{d\epsilon}{2\pi} \left( -f'_c \right) \sum_{n \neq n'} \frac{1}{n - n'} \sqrt{\frac{n!}{n'}!} L_n^{n'-n}(zQ) \left[ \left( n'(n'-1) G_{n'-2}^A(\epsilon) L_n^{n'-n}(zQ) - 2L_{n-1}^{n'-n}(zQ) + L_{n-2}^{n'-n}(zQ) \right) \right]$$

$$\times \left( n'+1)(n'+2) \right) (zQ) - 2(n+1)(n'+2) L_{n+1}^{n'-n}(zQ) + (n+1)(n+2) L_{n+2}^{n'-n}(zQ) \right).$$ \hfill (S43)
Using asymptotic expression (S4) we obtain

\[ \text{Im} \int \frac{de}{2\pi S} (-f'_e) \text{Tr} \Delta J_{xy}(T_{xx} - T_{yy}) G_z \approx -\frac{m\omega_c}{8\pi} \text{Im} \int \frac{de}{2\pi} (-f'_e) \sum_{n \neq n'} J_{n'-n}(Q_{1B} \sqrt{n + n' + 1}) \int \frac{d^2Q}{(2\pi)^2} W(Q)e^{-zQ} \]

\[ \times \left\{ \sqrt{n'(n'-1)} G_{n'-2}^A(e) G_{n}^A(e) \left[ \sqrt{n'(n'-1)} J_{n'-n}(Q_{1B} \sqrt{n + n' + 1}) - 2\sqrt{n'(n'-1)} J_{n'-n}(Q_{1B} \sqrt{n + n' - 1}) + \sqrt{n(n-1)} J_{n'-n}(Q_{1B} \sqrt{n + n' - 3}) \right] + \sqrt{(n'+1)(n'+2)} G_{n'+2}^A(e) G_{n}^A(e) \left[ \sqrt{(n'+1)(n'+2)} J_{n'-n}(Q_{1B} \sqrt{n + n' + 1}) - 2\sqrt{(n'+1)(n'+2)} J_{n'-n}(Q_{1B} \sqrt{n + n' + 3}) + \sqrt{(n+1)(n+2)} J_{n'-n}(Q_{1B} \sqrt{n + n' + 5}) \right] \right\} = -\frac{m\omega_c^2}{16\pi^2} \text{Im} \int \frac{de}{2\pi} (-f'_e) \]

\[ \times \sum_{n \neq n'} \frac{\sqrt{n'(n'-1)}}{n'-n'} \left\{ G_{n'-2}^A(e) G_{n}^A(e) \left[ \frac{\sqrt{n'(n'-1)}}{\tau_1} - \frac{2\sqrt{n'(n'-1)}}{\tau_1} + \frac{\sqrt{n(n-1)}}{\tau_2} + G_{n}^A(e)G_{n'-2}^A(e) \left[ \frac{\sqrt{n'(n'-1)}}{\tau_0} \right] - 2\frac{\sqrt{(n+1)(n'+2)}}{\tau_1} + \frac{\sqrt{(n+1)(n+3)}}{\tau_2} \right\} \approx \frac{m\omega_c N}{8\pi^2\tau_{tr,1}} \text{Im} \int \frac{de}{2\pi} (-f'_e)[G_n^A(e)]^2 = \frac{N}{4} \int d\varepsilon (-f'_e)\nu(\varepsilon)(\varepsilon - \epsilon_N). \quad (S44) \]

Here we took into account that \( \tau_{tr,1} = \tau_0 \) for a short-range-correlated random potential.

Next we consider the fifth term in Eq. (S39),

\[ \text{Im} \int \frac{de}{2\pi\omega_c S} f_e \text{Tr} [\Delta J_{xy}, T_{xy}] G_z \approx \text{Im} \int \frac{de}{2\pi\omega_c S} f_e \text{Tr} [\Delta J_{xy}, T_{xy}][G_R^R G_L^R - G_L^R V G_R^A] = \frac{m}{2\pi\omega_c S} \text{Im} \int d\varepsilon f_\varepsilon \]

\[ \times \text{Tr}[\Delta J_{xy}, \varepsilon^2](G_R^R V G_R^R - G_L^R V G_R^A) \left\{ \frac{n!}{(n'-2)!} L_{n'-n-2}(z_Q) - \frac{(n+1)(n+2)}{n!} L_{n'-n-2}(z_Q) \right\} \]

\[ \times \left\{ \sqrt{n'(n'-1)} G_{n'-2}^R(e) G_{n}^R(e) \sqrt{n!} \frac{(Q_{1B}^L)}{\sqrt{n-l}} L_{n'-n}(z_Q) \right\} = -\frac{m}{16\pi^2} \text{Im} \int d\varepsilon f_\varepsilon \sum_{n \neq n'} \frac{1}{n-n'} \int \frac{d^2Q}{(2\pi)^2} W(Q)e^{-zQ} z_Q^{-n'-n} \]

\[ \times L_{n'-n}(z_Q) \left\{ n'(n'-1) G_{n'-2}^R(e) G_{n}^R(e) \left[ L_{n'-n}^R(z_Q) - 2L_{n-1}^L(z_Q) + L_{n-2}^L(z_Q) \right] - (n+1)(n+2) G_{n}^R(e) G_{n+2}^R(e) \right\} \]

\[ \times \left\{ L_{n'-n}^L(z_Q) - 2L_{n+1}^L(z_Q) + L_{n+2}^L(z_Q) \right\} = -\frac{m}{16\pi^2} \text{Im} \int d\varepsilon f_\varepsilon \sum_{n \neq n'} G_{n-2n}^R n-n' \int \frac{d^2Q}{(2\pi)^2} W(Q) \]

\[ \times \left\{ \sqrt{n'(n'-1)} J_{n'-n}(Q_{1H} \sqrt{n + n' + 1}) - \sqrt{n(n-1)} J_{n'-n}(Q_{1H} \sqrt{n + n' - 3}) \right\} \]

\[ - 2\sqrt{n(n-1)} J_{n'-n}(Q_{1H} \sqrt{n + n' - 3}) + \sqrt{n(n-1)} J_{n'-n}(Q_{1H} \sqrt{n + n' + 5}) \right\} = \frac{m\omega_c}{16\pi^2} \text{Im} \int \frac{d\varepsilon f_\varepsilon}{2\pi} \sum_{n \neq n'} (n' - n - 1) \]

\[ \times G_{n'-2}^R G_{n}^R \left( 1 - \frac{1 - 2\sqrt{n'(n'-1)}}{\sqrt{n(n-1)} + \sqrt{n'(n'-1)}} \right) \approx \frac{m\omega_c}{16\pi^2\tau_0} \text{Im} \int \frac{d\varepsilon f_\varepsilon}{2\pi} \sum_n (2n + 1) (G_n^R(e))^2 \]

\[ = \frac{m}{8\pi^2} \int d\varepsilon f_\varepsilon (\varepsilon - \epsilon_N) \sqrt{1 - (\varepsilon - \epsilon_N)^2 I^2}. \quad (S45) \]

Here we took into account that \( 1/\tau_1 = 0 \) for the case of the short-range-correlated random potential. For temperature \( T \ll \omega_c \) the last partially filled Landau level contributes only to the sum over \( n \). This implies that this contribution is of the order of \( NT/(l_B^2\omega_c) \), i.e., can be safely neglected.
Finally, the last term in Eq. (S39) can be estimated as
\[
\text{Im} \int \frac{d\varepsilon}{8\pi c} f_c \text{Tr}[T_{xx} - T_{yy}, \Delta(J_{xx} - J_{yy})] \text{Im} G_R^2 = \frac{m}{4\pi \omega_c} \Re \int \frac{d\varepsilon}{2\pi} f_c \text{Tr}[\Delta(J_{xx} - J_{yy}), v^2_z ((G_R^V G_R^V - G_A^A V G_A^A))
\]
\[= - \text{Re} \frac{m}{16\pi^2} \int \frac{d\varepsilon}{2\pi} f_c \sum_{n \neq n'} \frac{1}{n - n'} \int \frac{d^2 Q}{(2\pi)^2} W(Q)e^{\varepsilon Q} \frac{n_1}{n_1} L_{n_1} L_{n_1} - 2L_{n_1 - 1} L_{n_1 - 1} - (n + 1)(n + 2) G_{n_1}(\varepsilon) G_{n_1 + 2}(\varepsilon) \left( \frac{1}{\tau_0} - \frac{1}{\sqrt{n(n - 1) + n'(n' - 1)}} \right) \]
\[
\approx \frac{m\omega_c}{16\pi^2} \text{Im} \int \frac{d\varepsilon}{2\pi} f_c \sum_n (2n + 1) \left( G_{n_1}(\varepsilon) \right)^2 = \frac{m}{8\pi^2} \int \frac{d\varepsilon}{2\pi} \left( \varepsilon - \epsilon_n \right) \sqrt{1 - \frac{\left( \varepsilon - \epsilon_n \right)^2}{\Gamma^2}}. (S46)
\]
Therefore, the last term is the same as penultimate one.
Combining together the results (S40)-(S46), we find
\[
\eta_H = \frac{1}{2\omega_c} \int d\varepsilon \nu(\varepsilon) f_c - \frac{\omega_c}{4} \int d\varepsilon (-f'_c) \nu(\varepsilon) \left[ N^2 + \frac{\epsilon - \epsilon_N}{\omega_c} \right]. (S47)
\]
Finally, using Eq. (S35) and (S47), we get
\[
\eta_H = \frac{1}{2\omega_c} \int d\varepsilon \nu(\varepsilon) f_c - \frac{N^2}{16\omega_c} \int \frac{d\varepsilon}{4} (-f'_c) \nu(\varepsilon) \Gamma^2(\varepsilon). (S48)
\]

S.V. NUMERICAL METHOD

In the numerical calculation, we employ the Hofstadter model on a square lattice of size \( N_x \times N_y \) with periodic boundary conditions in the presence of disorder:
\[
\hat{H} = \sum_{n_x=1}^{N_x} \sum_{n_y=1}^{N_y} V_{n_x,n_y} |n_x,n_y\rangle \langle n_x,n_y| - t |n_x, n_y + 1, n_y\rangle \langle n_x, n_y + 1| - t e^{-2i\alpha\pi n_x} |n_x,n_y + 1\rangle \langle n_x, n_y| + \text{H.c.}, (S49)
\]
where \( |n_x,n_y\rangle \) denotes the orbital on site \((n_x,n_y)\), \( V_{n_x,n_y} \) are independent random potential amplitudes, uniformly distributed between \([-w/2, w/2]\), \( t = \hbar^2/2ma^2 \) is the nearest-neighbor hopping amplitude, and \( \alpha = eBa^2/2\pi\hbar c \) is the flux per plaquette in units of the flux quantum, \( a \) being the lattice spacing. We employ a lattice expression for the stress tensor operator, which reduces to the usual expression in the continuum limit [S7]:
\[
\hat{T}_{jk} = -t \left( \hat{D}_i \hat{D}_j + \hat{D}_j \hat{D}_i \right), (S50)
\]
where,
\[
\hat{D}_x = \sum_{n_x,n_y} |n_x + 1, n_y\rangle \langle n_x + 1, n_y| - |n_x, n_y\rangle \langle n_x, n_y|, (S51)
\]
\[
\hat{D}_y = \sum_{n_x,n_y} |n_x,n_y + 1\rangle \langle n_x, n_y + 1| e^{-2i\alpha\pi n_x} - |n_x, n_y\rangle \langle n_x, n_y| e^{2i\alpha\pi n_x}. (S52)
\]

Denoting the single-particle eigenenergies and eigenstates of the Hofstadter Hamiltonian by \( E_{\mu} \) and \( |\mu\rangle \), respectively, the Hall viscosity at zero temperature is then calculated from the Kubo formula as a sum over the occupied and unoccupied eigenstates:
\[
\eta_H = -2m \text{Im} \sum_{E_{\mu_1} < E_{\mu_2} > E_F} \langle \mu_2 | \hat{T}_{xx} | \mu_1 \rangle \langle \mu_1 | \hat{T}_{xy} | \mu_2 \rangle (E_{\mu_1} - E_{\mu_2})^2. (S53)
\]
The Hall conductivity is similarly calculated using its lattice Kubo formula. We then take both the thermodynamic limit, $N_x N_y \alpha^2 / \ell_B^2 \to \infty$, and the continuum limit, $\alpha^2 / \ell_B^2 \to 0$. These translate into $N_x N_y \alpha \to \infty$ and $\alpha \to 0$. In practice we choose a series of increasing values of $(N_x, N_y) = (17, 19), (19, 21), (21, 23), (23, 25), (25, 27), (27, 29), (29, 31)$ with $\alpha = 1/N_y$, and extrapolate the results to $\alpha \to 0$ by a second-order polynomial fit. In the presence of disorder we can either keep $W/t$ constant while scaling, or else keep constant $\omega_c \tau = 96\pi \alpha / (w/t)^2$, that is, take $w \propto \sqrt{\alpha}$. The results for each system size and disorder strength was averages over 1000 realizations of the disorder potential.

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