The nuclear symmetry energy and stability of matter in neutron star

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It is shown that behavior of the nuclear symmetry energy is the key quantity in the stability consideration in neutron star matter. The symmetry energy controls the position of crust-core transition and also may lead to new effects in the inner core of neutron star.

INTRODUCTION

In neutron star, the prevailing part of its interior is fulfilled with matter which is in the state called the beta equilibrium [1]. Already few meters below neutron star surface, at densities \( \sim 10^7 g/cm^3 \) the degenerated electrons become relativistic and may easily penetrate the nuclei volume. The nucleons, although being confined to nucleus, are subject to the beta equilibrium, e.i. they fulfill the constraint \( \mu_n - \mu_p = \mu_e \). This leads to more and more neutron-rich nuclei and at the density \( \rho_{drip} \sim 10^{11} g/cm^3 \) the neutron drip takes place. Above the neutron drip density one may consider the neutron star matter as a two-phase system in partial phase equilibrium. It is, indeed, because not all Gibbs conditions are fulfilled: \( P^d = P^{nuc} \), \( \mu_e^d = \mu_e^{nuc} \), \( \mu_n^d = \mu_n^{nuc} \) but \( \mu_p^d \neq \mu_p^{nuc} \), where superscripts correspond respectively to the dripped and nuclear phase. At higher density, close to the saturation density the proton drip occurs as well [2]. Above this point \( \mu_p^d = \mu_p^{nuc} \) and one get a real two-phase system with all Gibbs conditions satisfied. The two phases have different properties like baryon density \( \rho \) and proton fraction \( x \). Soon after the proton drip the differences in \( \rho \) and \( x \) vanishes and the star matter represents "npl matter" - a homogeneous nucleon-liquid. (muons start to be produced at slightly higher densities). In the two-phase system, the nuclei form a lattice leading to a matter with solid state properties, unlike the homogeneous system being a liquid. In this way a neutron star has dense, liquid core covered by solid crust. The inclusion of finite-size effects, in some models [3], leads to the presence of "funny phase" (rods, plates) in the transition region between the crust and core. In this short letter we limit ourselves to the bulk approximation and shall find the critical density where homogeneous npl matter splits into two phases not going into details how the phase coexistence looks like. In this way we show that the crust-core transition is directly connected to the behavior of the nuclear symmetry energy. The same analysis applied to higher density region reveal the same kind of instability and suggests repeated solidification in the central part of neutron star core.

STABILITY CONDITIONS

The beta reactions, taking place in neutron star, conserve charge \( Q \) and baryon number \( B \). Such statement means that any thermodynamical state in a volume \( V \) is uniquely described by setting the temperature \( T \) and all conserved quantity, here \( B \) and \( Q \). We neglect the temperature effects which are relevant only for a very hot, proto-neutron star. Then the total energy \( U \) becomes a function of volume and conserved numbers \( U(V, B, Q) \). In order to consider stability of single phase one need to introduce intense (local) quantity \( u = U/B \). The energy per particle \( u \) becomes then a function of other local quantities, taken per baryon number \( v = V/B \) and \( q = Q/B \). The first principle of thermodynamics takes the following form

\[
\quad du = -P\,dv - \mu\,dq
\]

where \( P \) is the pressure and \( \mu \) chemical potential of electric charge. From beta equilibrium one may reads that

\[
\mu = \mu_e = \mu_n = \mu_{nuc} - \mu_p
\]

The minus sign before \( \mu \) in (1) comes from the definition of \( Q = N_p - N_e - N_{mu} \) which is negative for leptons [4]. The stability of any single phase, also called the intrinsic stability, is ensured by the convexity of \( u(v, q) \) [4]. Thermodynamical identities allows to express this requirement in terms of following inequalities [4]

\[
-\left( \frac{\partial P}{\partial q} \right)_v > 0 \quad -\left( \frac{\partial u}{\partial q} \right)_P > 0
\]

Usually, only the positive compressibility is examined, in particular, it is required for locally neutral matter that

\[
-\left( \frac{\partial P}{\partial v} \right)_q = 0 > 0.
\]

However the second inequality in (3) is of the same importance. It concerns the stability of charge fluctuations and as it will be shown later it is connected to the positive value of the screening length in matter. Not all nuclear models ensure the charge fluctuations to be stable. As...
was shown in the case of kaon condensation for wide class of models the system is not stable at any density \[3\]. In this work we would like to show that even such a simple system like \(npl\) matter, without any exotic components, may represent a region of density where instability occurs.

One may find another pair of inequalities which are equivalent to those in \([3]\) and, as it will appear later, are more convenient in further calculations:

\[
-\left(\frac{\partial P}{\partial v}\right)_{\mu} > 0 \quad \text{and} \quad -\left(\frac{\partial \mu}{\partial q}\right)_{v} > 0. \tag{5}
\]

The intrinsic stability is determined by the details of nucleon-nucleon interactions. In order to show that, let’s split the total energy per baryon into the nucleonic and leptonic part

\[u = u^N + u^L.\tag{6}\]

The nucleonic contribution may be always expressed as a function of baryon number density \(n = B/V\) and the proton fraction \(x = N_p/B\). For leptons \(\varepsilon^L\), the energy per volume, is completely determined by their chemical potential \(\mu\). Such decomposition is also true for the total pressure

\[
\begin{align*}
\mu &= -u^N_x. \tag{9}
\end{align*}
\]

Below we show that the equation \([4]\) allows to express the stability conditions \([3]\) or \([6]\) in terms only of the nucleonic contribution \(u^N\) to the total energy \(u\). First, we take the compressibility from the second pair of inequalities \([3]\)

\[
-\left(\frac{\partial P}{\partial v}\right)_{\mu} = n^2 \left(\frac{\partial P^N}{\partial n}\right)_{\mu} = \begin{cases} 
\begin{align*}
\frac{n^2}{u^N_n} \left(\frac{\partial P^N}{\partial n}\right)_{\mu} & \text{for leptons does not contribute when the derivative is taken under } \mu \text{ fixed. From } \tag{10}
\end{align*}
\end{cases}
\]

the leptons does not contribute when the derivative is taken under \(\mu\) fixed. From \([3]\) one get \((\partial x/\partial n)_{\mu} = -u^N_{nx}/u^N_{xx}\) and joining with \([11]\) we obtain

\[
-\left(\frac{\partial P}{\partial v}\right)_{\mu} = n^2 \left(\frac{2 u^N_n}{n} + u^N_{nn} - \frac{(u^N_{nx})^2}{u^N_{xx}}\right) \tag{12}
\]

where for the right-hand side we applied the expression for the nucleonic pressure \(P^N = n^2 u^N_n\). The first two terms in the \([12]\) refers to the nucleonic pressure and compressibility and they are positive from very fundamental reasons. As we show later, the third term may lead to negative contribution and it comes from the leptons presence, it is the leptons that make the matter unstable.

In order to find similar expression for the second inequality in \([3]\) we make use of the expression for the total charge per baryon \(q = x - n_L/n\), where the lepton number density \(n_L\) depends on \(\mu\) only (see Appendix). Differentiation of the equation by \(\mu\) under constant \(n\) we obtain

\[
\left(\frac{\partial q}{\partial \mu}\right)_{n} = \frac{n''(\mu)}{n} \tag{13}
\]

Here one may note that the second stability condition in \([3]\) is directly connected to the stable screening of charged particles. The derivative \(n''_L\) appearing in \([13]\), is proportional to the sum of squared inverse of screening lengths for leptons \([3]\). The first term in \([13]\), however, lacks this simple interpretation, it refers to hadronic interactions concealed by the beta equilibrium constraint. It is useful to call quantities like \(\partial q/\partial \mu\), as the ”electrical capacitance” of matter. It measures the energetic cost of change in electric charge held in matter. For further discussion we introduce the compressibility and electric capacitance as

\[
K_i = -v^2 \left(\frac{\partial P}{\partial v}\right)_i = (\partial P/\partial q)_i, \quad i = q, \mu \tag{14}
\]

\[
\chi_j = -\left(\frac{\partial q}{\partial \mu}\right)_j, \quad j = P, v \tag{15}
\]

By use of the beta equilibrium relation \([9]\) one may find that \((\partial x/\partial n)_{\mu} = -1/u^N_{nx}\) which allows to write the conditions \([3]\) only in terms of nucleonic energy contribution \(u^N\)

\[
K_{\mu} = 2n u^N_n + n^2 u^N_{nn} - \frac{(u^N_{nx})^2}{u^N_{xx}} > 0 \tag{16}
\]

\[
\chi_v = \frac{1}{u^N_{xx}} + \frac{\mu(n^2 + k^2)}{n^2} > 0. \tag{17}
\]

because also the second term in \([17]\), coming from leptons, is determined by the relation \(\mu = -u^N_x\) and nucleonic pressure is \(P^N = n^2 u^N_n\). In the same way we may express the first pair of conditions \([3]\). In order to derive them it is useful to find identities between compressibilities and capacities. Then using standard theorem for implicit functions one may get the following relations

\[
\frac{\chi_P - \chi_v}{\chi_v} = K_{\mu} \frac{\mu^2}{n^2} \tag{18}
\]

\[
\frac{K_q - K_{\mu}}{\chi_v} = \frac{\chi_P}{\chi_v} \tag{19}
\]

\[
\frac{K_q}{K_{\mu}} = \frac{\chi_P}{\chi_v} \tag{20}
\]
Above equations show explicitly the importance of symmetric shape of the function $E$ significantly the slope of isospin diffusion in heavy-ion collisions constrained significantly the slope of isospin diffusion in heavy-ion collisions constrained so the conditions finally take the form

$$K_q = n^2 u_n^N + 2 n u_n^N + \frac{\partial^2 u_n^N}{\partial n^2} > 0 \quad \text{(22)}$$

and thus the quantities $K_\mu$ or $\chi_v$ vanish first.

As was mentioned above the nucleonic contribution $u^N$ to the total energy, is a function of $n$ and $x$. Isospin symmetry allows for the expansion in even powers of $1 - 2x$ but terms higher than quadratic are negligible, then the energy per baryon takes the form

$$u^N(n, x) = V(n) + E_s(n)(1 - 2x)^2, \quad \text{(25)}$$

where $V(n)$ is the isoscalar potential and $E_s(n)$ - the symmetry energy, corresponding to the interactions isovector channel. Applying (23) to (18) we obtain

$$K_\mu = n^2 \left( E_s''(1 - 2x)^2 + V'' \right) + 2 n (E_s'(1 - 2x)^2 + V') - \frac{2(1 - 2x)^2 E_s'^2 n^2}{E_s} > 0 \quad \text{(26)}$$

and thus the quantities $K_\mu$ or $\chi_v$ vanish first.

Above equations show explicitly the importance of symmetry energy $E_s$ in the stability considerations. The concrete shape of the function $E_s(n)$ is not well known. We know only its value at saturation point, $n_0 = 0.16 \text{ fm}^{-3}$, where it takes about 30 MeV. Recent analysis on the isospin diffusion in heavy-ion collisions constrained significantly the slope of $E''_s(n_0)$ and the stiffness $E'_s(n_0)$ at saturation point, however these results does not determine the high density behavior definitely. There are no experimental evidence about values of $E_s$ at very high density which is available in the central parts of a neutron star. In such extrapolations we must rely on the model calculations. For all of them the symmetry energy at saturation point have positive slope but at higher densities, they lead to different conclusions. For most the $E_s$ is monotonically increasing function of $n$ but some models lead to the $E_s$ which saturates at higher densities or even bends down at some point and goes to zero. This kind of behavior is especially interesting as the last term in (26) may then take arbitrary large negative values and lead to instability. From the other side, going to very low density, we encounter uncertainties as well. All models predict $E_s$ decreasing to 0. However recent experiments [14] show that symmetry energy saturates at the level about 10 MeV for very low densities.

**NUCLEAR MODELS**

In order to present the role played by the symmetry energy we apply a set nuclear models. To emphasize the $E_s$ effects, the isoscalar part $V(n)$ is kept the same, whereas the symmetry energy takes different forms. The isoscalar potential $V(n)$, is that one taken from [8] which leads to the compressibility of nuclear matter equal to 240 MeV at saturation point. For $E_s$ we used two family of functions: four at low density and three for high density regime. At lower densities, we use shapes applied by Chen et.al. in [9], there were numbered by a paramater $x = 1, 0, -1, -2$. Here we named them by a,b,c,d to avoid confusion with proton fraction $x$. The shapes of symmetry energy dependence at lower densities are presented in Fig.1. At higher densities, much above $n_0$, we applied a "bent down" function. This type of symmetry energy with low values of $E_s$ at high density is typical for realistic potentials [10, 11]. Also in modern approaches like chiral dynamics [12] and Skyrme effective forces [13] small values of $E_s$ were obtained. Here, for numerical simplicity, we introduced the simple polynomial function
which imitate results of works mentioned above

\[ E_s(n) = E_s^{(0)} \frac{n(n-n_1)(n-n_2)}{n_0(n_0-n_1)(n_0-n_2)}, \]  

(28)

where \( E_s^{(0)} = 30 \) MeV and \( n_i \) are the zeros of \( E_s \) (see Appendix). The shapes of these functions, named A, B, C are shown in the Fig.2.

FIG. 2: Three different shapes of the symmetry energy at densities above saturation point (solid lines). For comparison the results of realistic potentials (dotted lines), the higher: UV14UVII, and the lower: UV14TNI.

LOW DENSITIES REGION

The transition between liquid core and solid crust of the star is strictly connected to the breaking of the conditions $^{[26,27]}$. When at least one of them is broken the matter cannot be homogeneous anymore, it splits into two phases. The Fig.3 shows the compressibility under constant \( \mu \) and its two contribution: "nuclear" $K_{\text{nuc}}^\mu$ - the two first terms in $^{[26]}$ - and "beta" $K_{\beta}^\mu$ - the last term in $^{[26]}$ which comes from the leptons presence. The "beta" contribution is always negative, hence there is always a competition between the positive "nuclear" compressibility and the beta reactions which tends to destabilize the matter. At some critical point, \( n_c \), the compressibility changes its sign and below \( n_c \) the matter cannot exist as a single, neutral phase. The actual splitting into two phases does not occur exactly at \( n_c \), but slightly above \( n_c \), because the system must find a state where the two charged phases may coexist. The correction is expected to be tiny, so the critical point for the vanishing of $K_{\mu}$ may be treated as a good estimation for the boundary of the liquid core in neutron star. The Table I shows the critical density at which $K_{\mu}$ vanishes. It depends strongly on $E_s$ but does not behave monotonically with the $E_s$. For symmetry energy taking both high and low values (a, d case) the \( n_c \) moves to higher density close to saturation point. The lowest values of \( n_c \), are achieved with intermediate $E_s$ (models b and c). There is no simple relation between values of $E_s$ and \( n_c \), because the first and the second derivatives of $E_s$ are essential as well.

TABLE I: The critical density for different models.

| model | a | b | c | d |
|-------|---|---|---|---|
| $n_c$, fm$^{-3}$ | 0.119 | 0.092 | 0.095 | 0.160 |

HIGH DENSITIES REGION

The stability of matter at densities much greater than \( n_0 \) do not need to be obvious. For the symmetry energy increasing in the whole range of density the matter is stable indeed. However, the chosen nuclear models with very low values of $E_s$ lead again to the same kind of instability as occurs in the crust-core transition region. For all presented models A,B,C there is a critical density \( n_c \) where $K_{\mu}$ vanishes. The behavior of the compressibility $K_{\mu}$ for the model B is shown on the Fig.4. It is worth to notice that energy per baryon for neutral matter, $u(n,0)$ does not reveal any pathology, it is monotonically increasing and its convexity, $u_{nn}(n) = K_{q}/n^2$, remains positive. If one look only on the energy per baryon behavior, one may overlook that the matter becomes unstable at some point.

FIG. 3: The compressibility $K_{\mu}$ (thick) and its contributions (thin lines). The dotted line corresponds to the energy per baryon for neutral matter $u(n,0)$.

FIG. 4: The compressibility $K_{\mu}$ (thick) and its contributions (thin lines). The dashed line corresponds to $K_q$ and dotted line energy per baryon for neutral matter, $u(n) \equiv u(n,0)$.

Of course, the instability point has physical meaning only if it is attainable in a neutron star. The Tab.1

FIG. 4: The compressibility $K_{\mu}$ (thick) and its contributions (thin lines). The dotted line corresponds to $K_q$ and dotted line energy per baryon for neutral matter, $u(n) \equiv u(n,0)$.
shows the neutron star properties (the central density \( n_{cen} \) of a star with maximal mass \( M_{max} \) found by solving of TOV equation [17]. Actually, in the case A and B

| TABLE II: The critical density and neutron star properties for different models. |
|-----------------|-----|-----|-----|
|                | A   | B   | C   |
| \( n_{cen}, \text{fm}^{-3} \) | 1.92 | 1.32 | 1.21 |
| \( M_{max}/M_\odot \)          | 1.64 | 1.73 | 1.84 |
| \( n_c, \text{fm}^{-3} \)      | 0.74 | 1.20 | 1.43 |

the \( n_c \) is smaller than the central density of maximal star. It means that for sufficiently massive stars their structure changes essentially in the central part of the star. The homogeneous phase again splits into two-phase system. One may suspect formation of funny phases or solidification of the central part of stellar core.

**SUMMARY**

In this letter we would like to notice the role played by the nuclear energy symmetry \( E_s \) in the stability of dense matter under beta equilibrium. It was shown that relevant quantity in such considerations is \( K_\mu \) - the compressibility under constant chemical potential, rather than \( K_q \) - the compressibility under constant charge. The instability of matter leads to phase separations and corresponds to the transition from the liquid core to the solid crust. In this way one may get simple connection between \( E_s \) behavior at low densities and the size of the star crust, which may be estimated from pulsar glitching [13].

The stability considerations were also carried out at very high density. It was shown that for nuclear models with small values of \( E_s \) it is possible that instability occurs. This may lead to repeated solidification the internal parts of a core. Such solid inner core would have important consequences on the rotation of a star and its magnetic properties.

**APPENDIX**

Here we present formulas mentioned in the regular text, the lepton number density

\[
n_L = \frac{k^3_x}{3\pi^2} + \frac{k^3_q}{3\pi^2} = \frac{\mu^3}{3\pi^2} + \theta(\mu - \mu_q)\frac{(m^2 - \mu^2)^{3/2}}{3\pi^2}
\]

and mixed derivatives appearing in (18) and (19)

\[
\alpha_q = \left( \frac{\partial \mu}{\partial n} \right)_q = \frac{u^N_{xx} n_L c^2 - u^N_{nx}}{1 + u^N_{xx} n_L v}
\]

\[
\alpha_P = \left( \frac{\partial P}{\partial n} \right)_P = \frac{u^N_{xx}(n^2 u^N_{xx} + 2nu^N_{nx}) - (u_{nx}n)^2}{n^2 u^N_{xx} - n_L u^N_{xx}}
\]

The parameterization of \( E_s \) at high density regime:

| model | A  | B  | C  |
|-------|----|----|----|
| \( n_1, \text{fm}^{-3} \) | 1.0 | 1.5 | 1.8 |
| \( n_2, \text{fm}^{-3} \) | 2.3 | 2.5 | 10.0 |

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[17] this convention of charge sign is opposite to that used in [13] and of course more natural
[18] The results of TOV equations are presented under assumption that single phase EOS is stable for all densities up to the max density, also above \( n_c \). It only shows that \( n_c \) is attainable for a given EOS. Above \( n_c \) the EOS should be corrected and \( n_{cen} \) may change but not much.