FEYNMAN DIAGRAMS VIA GRAPHICAL CALCULUS

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1. Introduction

It has been known for a while, in Hopf algebraists’ folklore, that there is a very close connection between the graphical formalism for ribbon categories and Feynman diagrams. Although this correspondence is frequently implied, it seems to have been first explicitly described in the recent [1]. Yet, we know of no systematic exposition in existing literature; the aim of this paper is to provide such an account.

In particular, in deriving Feynman diagrams expansion of Gaussian integrals as an application of the graphical formalism for symmetric monoidal categories, we discuss in detail how different kinds of interactions give rise to different families of graphs and show how symmetric and cyclic interactions lead to “ordinary” and “ribbon” graphs respectively.

Feynman diagrams are usually introduced as a kind of combinatorial bookkeeping device in asymptotic expansion of Gaussian integrals (cf. [2]). Indeed, given any integral

$$\int_{\mathcal{H}} f(X)e^{-S(X)}d\mu(X),$$

where $f$ and $S$ are polynomial functions, $d\mu$ the Gaussian measure over a real Hilbert space $\mathcal{H}$ its asymptotic expansion can be written in terms of “correlator functions”

$$\langle X_1 \cdots X_k \rangle := \int_{\mathcal{H}} X_1 \cdots X_k d\mu(X),$$

where $X_1, \ldots, X_k$ are coordinates of $X$ with respect to a chosen basis of $\mathcal{H}$. The data identifying correlators can be put in a one-to-one correspondence with some combinatorial data describing a graph; vice-versa, a correlator may be reconstructed from a graph by some simple “Feynman rules”. Therefore, the asymptotic expansion of (1.1) can be written as a sum over graphs.

In this paper we take the other way round: we associate an analytic expression to a graph by means of graphical calculus, then we show that the summation of all
these expressions, for some chosen class of graphs, gives the asymptotic expansion of an integral of the type (1.1). A relation between the integrand $f$ in (1.1) and the class of graphs being summed upon is derived, and found to coincide with usual \textquotedblleft Feynman rules\textquotedblright. This point of view has an advantage: one can easily reconstruct a \textquotedblleft path integral\textquotedblright formulation from a set of given Feynman rules.

Section 2 presents a sketchy account of graphical calculus, in its \textquotedblleft ribbon graphs with coupons\textquotedblright flavor, following mainly [3]. We assume the reader is already acquainted with the material presented there, and do not provide any proof nor motivation for this theory. A very readable introduction to this sort of graphical calculus may be found in [4]; for a comprehensive treatment and further references consult [5, 6, 7].

We specialize graphical calculus to the vector spaces category (equipped with the trivial braiding); this would be a very uninteresting choice in the context of knot theory — where graphical calculus was originally developed — since it gives rise to trivial link invariants. However, it is the right way to go here, because we want the analytic expression associated with a diagram to depend only on the topology of the diagram, and not on its immersion in the plane.

In Section 3 the relevant theorems relating graphical calculus and Feynman diagrams expansion of Gaussian integrals are stated and proved. It is \textquotedblleft folklore\textquotedblright material, and we know of no other written reference for it.

Section 4 works out in detail an example of physical interest, namely, the Kontsevich model for 2D quantum gravity (of which the \textquoteleft t Hooft standard matrix model is a particular case), showing how the new notation can be applied to known cases.

Gaussian integrals and Feynman diagrams have been generalized by Robert Oeckl to the wider context of (not necessarily symmetric) braided monoidal categories. In particular, using graphical calculus techniques, he proves that any braided Gaussian integral admits an expansion in braided Feynman diagrams (see [1] for details).

\textit{Notations.} If $A$ is a category, we write $X \in A$ to state that $X$ is an object of $A$. The map notation $f : X \to Y$ will be occasionally used to denote a morphism $f \in A(X, Y)$.

The symbol $\mathfrak{S}_k$ stands for the permutation group on $k$ letters.

There are several classes of graphs appearing in the text: we have reserved the term \textquoteleft ordinary\textquoteright graph for purely 1-dimensional CW-complexes; all other graphs (ribbon, RT, modular) differ by some additional structure on the vertices — precise definitions follow in the body of the text.

Unfortunately, there seems to be no agreement among authors about the naming of objects involved in graphical calculus; our own choice, to the readers’ bewilderment, is a mixture of many naming styles found in the literature, and is not entirely consistent with any of our sources.

2. \textit{Feynman diagrams via graphical calculus}

In this section we recall some basic facts of graphical calculus, as introduced by Reshetikhin-Turaev in [3] and Joyal-Street in [3]; in particular, we state the main result of this theory in the simpler case of ribbon graphs (in the sense of [3]). We make a fundamental simplification in our exposition of this theory, namely, we drop the requirement that graphs edges are equipped with a framing: indeed, since our ground category has trivial balancing (cf. [3] and [4]), we do not need the extra structure given by twists. We stick to the usual \textquoteleft wireframe\textquoteright graphs, which somewhat simplifies definitions.

2.1. \textit{Preliminaries on tensor categories and PROPs}. The notion of monoidal category is well-known and discussed at length in the existing literature; we recall
a few facts and definitions. A precise list of axioms may be found in the already cited sources.

A monoidal category \((A, \otimes, I, a, l, r)\) is given by the following data:

(i) a category \(A\);
(ii) a functor \(\otimes : A \times A \to A\);
(iii) a distinguished object (identity) \(I \in A\);
(iv) a natural transformation \(a : (\cdot \otimes \cdot) \to (\cdot \otimes (\cdot \otimes \cdot))\) of functors \(A \times A \times A \to A\);
(v) a natural transformation \(l : (\cdot \otimes I) \to \text{Id}_A\);
(vi) a natural transformation \(r : (I \otimes \cdot) \to \text{Id}_A\).

These data are required to satisfy some compatibility axioms: roughly speaking, one would consider \(\otimes\) as a “multiplication” of objects in the category \(A\), and \(a, l, r\) are the appropriate “categorizations” of usual conditions expressing associativity of multiplication and existence of a bilateral multiplicative identity \(I\). Indeed, these axioms imply that expressions like \(X_1 \otimes X_2 \otimes \cdots \otimes X_k\) are well-defined (i.e., do not depend on the way we put parentheses in them) up to a natural isomorphism, and that insertion or removal of \(I\) can be neglected, again up to a natural isomorphism.

If \(A\) is an Abelian category we require \(\otimes, a, l, r\) to be linear. Abelian monoidal categories are called tensor categories.

A tensor functor \(F\) is a functor such that \(F(A \otimes B) = F(A) \otimes F(B)\), up to a natural isomorphism.

A braided tensor category is a tensor category equipped with a family of isomorphisms \(\tau_{XY} : X \otimes Y \to Y \otimes X\), natural in both \(X\) and \(Y\), satisfying a certain compatibility diagram (MacLane’s hexagon condition). In a braided tensor category any two expressions \(X_1 \otimes X_2 \otimes \cdots \otimes X_r\) and \(X_{\sigma_1} \otimes X_{\sigma_2} \otimes \cdots X_{\sigma_r}\) (with \(\sigma \in \mathfrak{S}_k\)) are isomorphic, but there are (possibly) many different isomorphisms built from maps \(\tau_{X_i,X_j}\).

A symmetric tensor category \(A\) is a braided category such that \(\tau_{YX} \circ \tau_{XY} = \text{id}_{X \otimes Y}\) for all \(X, Y\) objects of \(A\). This implies that an isomorphism between \(X_1 \otimes X_2 \otimes \cdots \otimes X_r\) and \(X_{\sigma_1} \otimes X_{\sigma_2} \otimes \cdots X_{\sigma_r}\), built only from maps \(\tau_{X_i,X_j}\), depends only on the permutation \(\sigma\).

A braided tensor category \(A\) is said to have right duals if for any object \(X \in \mathcal{A}\) there is an object \(X^\vee\) and morphisms \(\text{ev}_X : X \otimes X^\vee \to k\) and \(\text{coev}_X : k \to X^\vee \otimes X\) such that the composition

\[
X \xrightarrow{\text{id} \otimes \text{coev}} X \otimes X^\vee \otimes X \xrightarrow{\text{ev} \otimes \text{id}} X
\]

is the identity morphism on \(X\). Definition of left duals is completely analogous. A braided tensor category is rigid if it has both left and right duals and they are canonically isomorphic. For any object \(A\) of a rigid tensor category \(A\), put

\[
A^r := \begin{cases} 
A^\otimes r, & \text{if } r > 0, \\
I, & \text{if } r = 0, \\
(A^\vee)^{\otimes (-r)}, & \text{if } r < 0.
\end{cases}
\]

One may check that the usual relation \(A^r \otimes A^s = A^{r+s}\) holds, up to a natural isomorphism.

**Example 2.1.** The category of vector spaces, equipped with the usual tensor product and the obvious \(a, l, r\) is rigid and symmetric.

2.1.1. *Free tensor categories.* For any category \(A\) we can form a monoidal category \(A^\otimes\): objects are finite sequences of objects in \(A\), and morphisms are finite sequences of morphisms from \(A\). Tensor product is given by juxtaposition; the identity object
is the empty sequence; associativity and identity natural transformations are the obvious ones. If $A$ is itself monoidal, then there is an obvious functor $A^\otimes \to A$.

2.1.2. An important symmetric rigid tensor category. Fix a vector space $V$ (over a ground field $k$) and a non-degenerate symmetric inner product $b : V \otimes V \to k$ on it. Build a category $\langle V \rangle$: objects are powers $V^\otimes k$ of $V$, for $k \in \mathbb{N}$, and $I := k \simeq V^\otimes 0$; morphisms are linear maps $V^\otimes r \to V^\otimes s$. This category is symmetric with the usual tensor product of vector spaces.

The object $V$ is self-dual, if we define the morphism $e_V$ to be the inner product $b : V \otimes V \to k$, and $\text{coev}_V$ to be the morphism sending $1 \in k$ to the Casimir element $\sum e_i \otimes e'$, where $\{e_i\}$ is a basis of $V$ and $\{e'\}$ is the dual one. Similarly, one can define morphisms such that any $V^\otimes r$ is left and right dual to itself. As a result, $\langle V \rangle$ is rigid.

2.2. PROPs. Informally speaking, a PROP is a monoidal category whose Hom-spaces are objects of another monoidal category: if $A$ is a category, then, for any two objects $X, Y \in A$, $A(X, Y) := \text{Hom}_A(X, Y)$ is a set, what is more, $\text{Hom} : A^{\text{opp}} \times A \to \text{Set}$ is a functor; a structure of $A_{\text{Hom}}$-PROP over $A_{\text{Ob}}$ is given by a functor $\mathcal{P} : (A_{\text{Ob}})^{\text{opp}} \times A_{\text{Ob}} \to A_{\text{Hom}}$ whose properties generalize those of the Hom-functor. A precise definition of PROP is rather cumbersome, so we consign it to Appendix A; here we shall give only some illustrative examples.

If $X$ and $Y$ are vector spaces, then $\text{Hom}(X, Y)$ is a vector space: if we define $\mathcal{V}(X, Y) := \text{Hom}(X, Y)$, then $\mathcal{V}$ is a natural structure of a $\text{Vect}$-PROP over the category $\text{Vect}$.

Similarly, if $H_1$ and $H_2$ are Hilbert spaces, $\text{Hom}(H_1, H_2)$ is a Banach space: the category of Hilbert spaces is in a natural way a Banach spaces-PROP.

Likewise, every tensor category is a $\text{Vect}$-PROP with the trivial PROP structure given by $\mathcal{P}(X, Y) := \text{Hom}(X, Y)$; by abuse of language we say that $\mathcal{P}(X, Y)$ are the Hom-spaces of the PROP. The PROPs of graphs we are going to introduce are indeed tensor categories, but we are mainly interested in their actions as PROPs.

2.2.1. PROP-algebras. Now, build a category $\mathbb{N}$ which has natural numbers $i \in \mathbb{N}$ as objects, and morphisms given by

$$\mathbb{N}(i, j) := \begin{cases} \{\text{id}_i\} & \text{if } i = j, \\ \emptyset & \text{if } i \neq j; \end{cases}$$

it is a monoidal category with the tensor product

$$n \otimes m := n + m.$$ 

It is trivial to check that 0 is the identity object and that $\mathbb{N}$ is freely generated by the object 1. Fix a vector space $V$. Define a functor $\mathcal{E}_V : \mathbb{N}^{\text{opp}} \times \mathbb{N} \to \text{Vect}$ by

$$\mathcal{E}_V(m, n) := \text{Hom}(V^\otimes m, V^\otimes n).$$

This has an obvious structure of a $\text{Vect}$-PROP on $\mathbb{N}$; it is called the endomorphism PROP of $V$.

What should a morphism between PROPs be? Recall that a functor $f : A' \to A''$, i.e., a morphism of categories, is a pair of maps $(f_{\text{Ob}}, f_{\text{Hom}})$, where $f_{\text{Hom}}$ is a natural transform between $A'(\cdot, \cdot)$ and $A''(f_{\text{Ob}}(\cdot), f_{\text{Ob}}(\cdot))$. Therefore, given two PROPs $\mathcal{P}'$ and $\mathcal{P}''$, define a morphism $\rho : \mathcal{P}' \to \mathcal{P}''$ to be a pair $(\rho_{\text{Ob}}, \rho_{\text{Hom}})$, where:

- $\rho_{\text{Ob}} : A_{\text{Ob}}' \to A_{\text{Ob}}''$ is a functor,
- $\rho_{\text{Hom}}$ is a natural transformation $\rho_{\text{Hom}}(A, B) : \mathcal{P}'(A, B) \to \mathcal{P}''(\rho_{\text{Ob}}A, \rho_{\text{Ob}}B)$, that satisfies conditions that express compatibility with the tensor structure on $A_{\text{Hom}}$. 
If \( \rho : \mathcal{P}' \to \mathcal{P}'' \) is a surjective morphism, then we say that \( \mathcal{P}'' \) is a PROP quotient of \( \mathcal{P}' \). For our purposes, it will always be \( \mathcal{A}'_{\text{Ob}} = \mathcal{A}''_{\text{Ob}} \) and \( \rho_{\text{Ob}} = \text{Id} \); in this cases, PROP quotients are characterized by kernels of maps \( \rho_{\text{Hom}}(A,B) : \mathcal{P}'(A,B) \to \mathcal{P}''(A,B) \).

Now we come to the one example most relevant to this paper. Let \( \mathcal{P}' = \mathcal{P} \) be a PROP of vector spaces over \( \mathbb{N} \); let \( \mathcal{P}'' \) be the category of \( k \)-vector spaces considered as a PROP over itself. Since \( \mathbb{N} \) is generated by 1 (as a monoidal category), for any morphism

\[
\rho : \mathcal{P}' \to \mathcal{P}'',
\]

the image of \( \rho_{\text{Ob}} \) is generated by the vector space \( V = \rho_{\text{Ob}}(1) \). Therefore, the data determining \( \rho \) are a family of morphisms

\[
\rho_{m,n} : \mathcal{P}(m,n) \to \mathcal{E}_V(m,n).
\]

Therefore, \( \rho \) is actually a morphism with values in the endomorphism PROP of \( V \).

**Definition 2.2.** An action of a linear PROP \( \mathcal{P} \) (over \( \mathbb{N} \)) on a linear space \( V \) is a morphism \( \rho : \mathcal{P} \to \mathcal{E}_V \). The space \( V \) endowed with an action of \( \mathcal{P} \) is called a \( \mathcal{P} \)-algebra.

The data of an \( \mathcal{P} \)-algebra can be regarded as a family of elements of \( \text{Hom}(V^\otimes m, V^\otimes n) \) parameterized by the space \( \mathcal{P}(m,n) \).

### 2.3. The PROP of Reshetikhin-Turaev diagrams

Let \( L \) be the infinite strip \( \mathbb{R} \times [0,1] \); for \( j \geq 1 \) let \( s_j, t_j \) be the points \( (j,0) \in L \) and \( (j,1) \in L \), respectively.

**Definition 2.3.** A Reshetikhin-Turaev diagram \( \Gamma \) of type \((p,q)\) is given by a finite set \( \Gamma^{(0)} \) (the set of vertices) and a finite set \( \Gamma^{(1)} \) (the set of edges) such that:

- **RT1)** each vertex \( v \) is a tiny rectangle (“coupon” in Reshetikhin-Turaev’s original wording) contained in the strip \( L \) with two of its edges parallel to the boundary of \( L \)—call them \( \text{Top}(v) \) and \( \text{Bottom}(v) \);
- **RT2)** each edge is a smooth immersion \( \ell : [0,1] \to L \);
- **RT3)** for each \( \ell \in \Gamma^{(1)} \), the endpoints \( \ell(0), \ell(1) \) lie in
  \[
  \{s_1, \ldots, s_p\} \cup \{t_1, \ldots, t_q\} \cup \bigcup_{v \in \Gamma^{(0)}} (\text{Top}(v) \cup \text{Bottom}(v));
  \]
  the points \( \ell(0) \) and \( \ell(1) \) are called the source and target of the edge \( \ell \) respectively;
- **RT4)** no two edges have a common endpoint;

\footnote{“RT-diagram” for short.}
RT5) for every crossing of edges (including self-crossings) an element in the set \
\{\begin{array}{c}
\end{array}\}\n
must be specified, that is, we want to know which of the two crossing arcs “passes under”.

A closed diagram is a diagram of type (0, 0). Adding (or removing) a connected component of type (0, 0) to a given diagram does not change its type.

Example 2.4. A braid on r strands can be seen as a diagram of type (r, r) with no vertices (and vice-versa).

The projection \( z : L \to [0, 1] \) induces a differentiable function \( z \circ \ell : [0, 1] \to [0, 1] \) (the height function) on every edge \( \ell \in \Gamma(1) \).

Definition 2.5. Critical points of an RT-diagram \( \Gamma \) are: (i) vertices, (ii) crossings, (iii) critical values of the height function on every edge.

If \( v \) is a vertex of \( \Gamma \), then let \( \text{Leg}(v) \) be the set of edges of \( \Gamma \) incident to \( v \). Every set \( \text{Leg}(v) \) is divided into two disjoint totally ordered subsets \( \text{In}(v) \) and \( \text{Out}(v) \):

\[\begin{array}{c}
\end{array}\]

A sign \( \varepsilon_x \in \{\pm 1\} \) is given to each non-critical point \( x \) according to whether the height function preserves or reverses orientation in a neighborhood of the preimage of \( x \). This sign is locally constant (on edges except critical points); by extension, a sign is unambiguously defined on each source and target point; if \( p \) is an endpoint, denote its sign by \( \text{sgn}(p) \).

Definition 2.6. The source and the target of an RT-diagram \( \Gamma \) of type \( (p, q) \) are the sequences of \( \pm 1 \) given by

\[\text{Src}(\Gamma) := (\text{sgn}(s_1), \text{sgn}(s_2), \ldots, \text{sgn}(s_p)), \text{Tgt}(\Gamma) := (\text{sgn}(t_1), \text{sgn}(t_2), \ldots, \text{sgn}(t_q))\]

For a vertex \( v \) of an RT-diagram we can define \( \text{Src}(v) \) (resp. \( \text{Tgt}(v) \)) as the sequences of the signs of the endpoints of the edges in \( \text{In}(v) \) (resp. \( \text{Out}(v) \)).

Definition 2.7. Two RT-diagrams \( \Gamma \) and \( \Phi \) are composable iff \( \text{Src}(\Gamma) = \text{Tgt}(\Phi) \). If \( \Gamma \) and \( \Phi \) are composable, we can form a new diagram \( \Gamma \circ \Phi \) by “stacking \( \Gamma \) on top of \( \Phi \)” (see Figure 2 on page 7).

Note that this composition product restricts to the usual braid composition on diagrams corresponding to elements of the braid group.

By the above definition, we can take the linear spans \( D(S, T) \) of diagrams with given source \( S \) and target \( T \), as the Hom-spaces of a suitable PROP.

Definition 2.8. \( D \) is the PROP which has finite sequences of \( \pm 1 \) as objects; the Hom-space \( D(S, T) \) is the linear span of the set of diagrams with source \( S \) and target \( T \).

Composition of morphisms is defined by bilinear extension of the composition product \( \circ \) (see Definition 2.7).
Figure 2. Composition product of graphs.

The tensor product is given on objects by concatenation:
\[(\varepsilon_1, \ldots, \varepsilon_r) \otimes (\varepsilon'_1, \ldots, \varepsilon'_s) = (\varepsilon_1, \ldots, \varepsilon_r, \varepsilon'_1, \ldots, \varepsilon'_s),\]
and on morphisms by juxtaposition of diagrams (see Figure 3 on page 7).

The braiding is given by diagrams corresponding to elements of the braid groups.

Figure 3. Tensor product of graphs.

Any RT-diagram can be considered as the planar projection of a purely 1-dimensional CW-complex with oriented edges, and a partition of half-edges occurring at each vertex and of its endpoints into two totally ordered disjoint subsets. The class of CW-complexes with such an additional structure is the class of RT-graphs. There is a natural map “forgetting the planar immersion” \(\mathcal{D} \to \mathcal{F}\). This forgetful functor is actually a PROP quotient, as the following lemma states.

**Lemma 2.9**. The PROP \(\mathcal{D}\) of Reshetikhin-Turaev diagrams is the free PROP generated by the following elementary pieces (note that an orientation must be added to the strands!)

![RT Diagrams](image)

A piece of type (a) is called a “strand”; those of type (b) and (c) are named “crossings”; (d) and (e) are the “coupling” and the “Casimir”; (f) is, plainly, a “vertex”.

**Remark 2.10**. Lemma 2.9 just states that an RT-diagram is a composition of “rows” made of pieces of type (a)–(f). Generically, such rows will be made of one piece of type (b)–(f) padded with a number of strands (a) on the two sides.
Lemma 2.11 (3). The PROP $\mathcal{T}$ of RT-graphs is the quotient of $\mathcal{D}$ with respect to the following relations:

M1) a finite set of graphical moves, called the Reidmeister-Reshetikhin-Turaev moves (see Figure 4 on page 8 or 3 or 3 for a listing);
M2) the equivalence of undercrossings and overcrossings.

Figure 4. Reidmeister-Reshetikhin-Turaev moves for RT-diagrams. Move (RRT2) is different if the ground category has non-trivial balancing.

Remark 2.12. Notice that the list of Reidmeister-Reshetikhin-Turaev moves is different if the ground category has non-trivial balancing (cf. 5 and 9), indeed one would need graphs with “framed” edges, and the move (RRT2) introduces a twist.

Remark 2.13. Lemma 2.11 states that if $\Gamma_1$ and $\Gamma_2$ are two planar projections that realize an RT-graph $\Gamma$ as an RT-diagram, then one can change $\Gamma_1$ into $\Gamma_2$ by a finite sequence of moves M·M·M.

2.4. Graphical calculus on rigid braided tensor categories. Now let $A$ be a tensor category. Define $A^\cdot$ to be the category whose objects are finite sequences $(A_1, \ldots, A_r; \varepsilon_1, \ldots, \varepsilon_r)$ of objects in $A$ and signs $\pm 1$, whereas a morphism $(A_\ast, \varepsilon_\ast) \to (B_\ast, \delta_\ast)$ is an element $f \in A(A_{\varepsilon_1}^1 \otimes \cdots \otimes A_{\varepsilon_r}^r, B_{\delta_1}^1 \otimes \cdots \otimes B_{\delta_s}^s)$ — recall that $A^1 = A$ and $A^{-1} = A^\ast$. There is an obvious functor $A^\cdot \to A$ defined on objects by $(A_1, \ldots, A_r; \varepsilon_1, \ldots, \varepsilon_r) \to A_{\varepsilon_1}^1 \otimes \cdots \otimes A_{\varepsilon_r}^r$.

Definition 2.14. An $A$-colored RT-diagram $\Gamma$ is an RT-diagram together with

(i) an assignment of an object $A_\ell \in A$ for each $\ell \in \Gamma^{(1)}$;
an assignment of a morphism \( f \in A(\text{Src}(v), \text{Tgt}(v)) \) for each vertex \( v \) of \( \Gamma \), where \( \text{Src}(v) \) and \( \text{Tgt}(v) \) are the sequences \((A_1, \ldots, A_r; \varepsilon_1, \ldots, \varepsilon_r)\) of objects and signs decorating edges in \( \text{In}(v) \) and \( \text{Out}(v) \).

The source and the target of an \( A \)-colored RT-diagram are defined analogously and are denoted \( \text{Src}(\Gamma) \) and \( \text{Tgt}(\Gamma) \) respectively.

It is trivial to generalize Definition 2.8 to \( A \)-colored RT-diagrams; call \( D_A \) the PROP of \( A \)-colored RT diagrams. \( \text{Src}_A(\Gamma) \) and \( \text{Tgt}_A(\Gamma) \) are objects of \( A \) for each \( A \)-colored RT-diagram. Obviously, \( \circ \) and \( \otimes \) are bilinear with respect to the vector space structure on \( D_A \). Any \( A \)-colored RT-diagram can be seen as the planar projection of an \( A \)-colored RT-graph and the following analogue of lemmas 2.9 and 2.11 hold.

**Lemma 2.15.** The PROP \( D_A \) of \( A \)-colored RT-diagrams is the free PROP generated by the following elementary pieces (note that an orientation must be added to the strands!)

\[
\begin{array}{cccccc}
(a) & (b) & (c) & (d) & (e) & (f) \\
\begin{array}{c}
X \\
X
\end{array} & 
\begin{array}{c}
Y \\
X
\end{array} & 
\begin{array}{c}
X \\
Y
\end{array} & 
\begin{array}{c}
X \\
X
\end{array} & 
\begin{array}{c}
Y \\
Y
\end{array} & 
\begin{array}{c}
Y \\
X
\end{array} \\
X & X & Y & X & Y & X
\end{array}
\]

**Lemma 2.16.** The PROP \( \mathcal{D}_A \) of \( A \)-colored RT-graphs is the quotient of \( D_A \) with respect to the relations generated by moves M1–M2.

Since \( \mathcal{D}_A \) is free as a PROP, to give a tensor functor \( D_A \to B \) it suffices to define it on the generators. In particular, taking \( B = A \) we find the following

**Proposition 2.17** (Reshetikhin-Turaev’s graphical calculus [3]). For any rigid braided tensor category \( A \), there is a tensor functor \( Z_A : D_A \to A \), mapping an object \((A_1, \ldots, A_r; \varepsilon_1, \ldots, \varepsilon_k) \in D_A \) to \( A_{\varepsilon_1} \otimes \cdots \otimes A_{\varepsilon_k} \in A \), and defined on generators of morphisms in \( D_A \) as

\[
\begin{array}{cccc}
Y & X & \mapsto & \tau_{XY}, \\
X & Y & \mapsto & \tau_{XY}^{-1}, \\
X & X & \mapsto & \text{ev}_X, \\
X & X & \mapsto & \text{coev}_X, \\
& & & f
\end{array}
\]

where \( \tau_{XY}, \text{ev}_X, \text{coev}_X \) are the structure maps of \( A \), and \( f \) is a morphism in \( A \); take the dual of an object if the sign \( \varepsilon \) on the corresponding edge is \( -1 \).

**Remark 2.18.** By definition of a tensor functor, the following relations hold:

\[
Z_A(\Gamma \circ \Phi) = Z_A(\Gamma) \circ Z_A(\Phi), \quad Z_A(\Gamma \otimes \Phi) = Z_A(\Gamma) \otimes Z_A(\Phi).
\]

Moreover, \( Z_A \) is linear:

\[
Z_A(a\Gamma + b\Phi) = aZ_A(\Gamma) + bZ_A(\Phi).
\]

Now we come to the main result of this section.

**Theorem 2.19** (Reshetikhin-Turaev, [3]). Let \( A \) be a rigid symmetric tensor category. Then the Reshetikhin-Turaev’s graphical calculus induces a tensor functor \( Z_A : \mathcal{D}_A \to A \), that is, graphical calculus for symmetric categories is invariant by moves M1–M2.
2.5. **Graphical calculus for ribbon graphs.** We want now to define a variant of graphical calculus suitable for application to ribbon graphs. Such graphs arise both in Feynman expansions of matrix integrals and in orbifold cellularizations of moduli spaces of curves. Any ribbon graph can be realized (in many different ways) as an RT-graph. So, in order to define a graphical calculus on ribbon graphs it suffices to define it on RT-graphs in a way that is independent of the particular realization chosen.

**Definition 2.20.** A ribbon graph of type \((p,q)\) is a purely 1-dimensional CW-complex \(\Gamma\) with

(i) \(p + q\) endpoints divided into two ordered subsets \(\text{In}(\Gamma)\) and \(\text{Out}(\Gamma)\) with \(|\text{In}(\Gamma)| = p\) and \(|\text{Out}(\Gamma)| = q\);

(ii) a cyclic order on half edges stemming from each vertex.

The linear spans \(\mathcal{R}(p,q)\) of ribbon graphs of type \((p,q)\) define the Hom-spaces of the PROP \(\mathcal{R}\) of ribbon graphs.

Ribbon graphs arose in connection with a certain cellular decomposition of the moduli space of smooth complex curves (see [12, 13]). The connection is, very roughly, the following: choose a ribbon graph \(\Gamma\) of type \((0,0)\); one can use the cyclic order to “fatten” edges into thin ribbons (see Figure 5 on page 10) — so we turn the graph into a compact oriented surface with boundary \(S(\Gamma)\); this construction may be refined to take into account a conformal structure on \(S(\Gamma)\). The boundary components of \(S(\Gamma)\) are called “holes” of the ribbon graph \(\Gamma\); the set of holes of \(\Gamma\) is denoted \(\Gamma^{(2)}\). The number of boundary components \(s\) and the genus \(g\) of the ribbon graph \(\Gamma\) are defined to be those of the surface \(S(\Gamma)\). Notice that it is meaningful to speak of the genus and number of boundary components only for ribbon graphs of type \((0,0)\).

There is a natural forgetful functor \(\mathcal{F} \to \mathcal{R}\) which forgets orientations on edges and at each vertex \(v\) remembers only the cyclic order induced by the total order on \(\text{In}(v)\) and \(\text{Out}(v)\).

![Figure 5. Fattening edges at a vertex with cyclic order.](image)

**Lemma 2.21.** The PROP \(\mathcal{R}\) is the quotient of \(\mathcal{F}\) with respect to relations generated by the following moves

M3) reverse orientation on edges;

---

2 These two subjects are indeed deeply related: see [10, 8, 11].

3 Hence the name “ribbon graph”.
M4) the first (resp. the last) edge in In(v) becomes the first (resp. the last) edge in Out(v) and vice-versa.

Lemma 2.22. In any rigid symmetric tensor category $A$ there are natural isomorphisms

$$A(X \otimes Y, Z) \leftrightarrow A(X, Z \otimes Y^\vee), \quad A(X \otimes Y, Z) \leftrightarrow A(Y^\vee, X \otimes Z),$$

for all $X, Y, Z \in A$.

For instance, the second bijection in the above lemma maps a morphism $f : X \otimes Y \to Z$ to the composition

$$X \xrightarrow{\text{coev}_Y \otimes \text{id}_X} Y \otimes Y \otimes X \xrightarrow{\text{id}_Y \otimes f} Y \otimes Z.$$

In graphical notation, this reads:

Let $V$ be a vector space over $k$ equipped with a symmetric inner product $b : V \otimes V \to k$. The category $(V)$ of Section 2.1.2 is rigid symmetric monoidal; every object in $(V)$ is self-dual. Therefore, bijections from Lemma 2.22 translate into linear isomorphisms:

$$\text{Hom}(V^{\otimes p} \otimes V^{\otimes q}, V^{\otimes r}) \leftrightarrow \text{Hom}(V^{\otimes p} \otimes V^{\otimes r} \otimes V^{\otimes q}), \quad (2.1)$$

$$\text{Hom}(V^{\otimes p} \otimes V^{\otimes q}, V^{\otimes r}) \leftrightarrow \text{Hom}(V^{\otimes p} \otimes V^{\otimes q} \otimes V^{\otimes r}), \quad (2.2)$$

for all $p, q, r \in \mathbb{Z}$.

Definition 2.23. A cyclic algebra structure on $(V, b)$ is a sequence $\{T_r\}_{r \in \mathbb{N}}$ of cyclically invariant linear maps $T_r : V^{\otimes r} \to k$:

$$T_r(X_1 \otimes \cdots \otimes X_{r-1} \otimes X_r) = T_r(X_r \otimes X_1 \otimes \cdots \otimes X_{r-1}).$$

Fix a cyclic algebra structure $\{T_r\}$ on $V$. Each map $T_r$ in turn defines linear maps $T_{p,q} : V^{\otimes p} \to V^{\otimes q}$, for all $p$ and $q$ such that $p + q = r$, via the isomorphisms (2.1) and (2.2): cyclical invariance guarantees that $T_{p,q}$ does not depend on the particular sequence of isomorphisms (2.1) and (2.2): any one yielding the right source and target is good.

Theorem 2.24. Let $V$ be a vector space. Then the following data are equivalent:

(i) cyclic algebra structures on $V$;

(ii) $R$-algebra structures on $V$.
Proof. Let $Z : \mathcal{R} \to \mathcal{E}_V$ be a PROP action. Then

$$b := Z \begin{pmatrix} 1^{in} & 2^{in} \\ \vdots \\ (i+1)^{in} & (r-1)^{in} \\ i^{in} & 2^{in} \\ \vdots \\ i^{in} \end{pmatrix}$$

$$T_r := Z \begin{pmatrix} 1^{in} & 2^{in} \\ \vdots \\ (i+1)^{in} \\ i^{in} \end{pmatrix}$$

defines a cyclic algebra structure on $V$. Vice-versa, let $(V, b, T_1, T_2, \ldots)$ be a cyclic algebra. Pick any ribbon graph $\Gamma$ and realize it as an RT-graph; call this RT-graph $\hat{\Gamma}$. Give $\hat{\Gamma}$ the structure of a $\langle V \rangle$-colored RT-graph by coloring all the edges of $\hat{\Gamma}$ with $V$ and coloring any vertex $v$ with $T_{|\operatorname{In}(v)|} |\operatorname{Out}(v)|$. Denote the $\langle V \rangle$-colored RT-graph obtained this way by $\hat{\Gamma}_V$. Two realizations of $\Gamma$ as an RT-graph differ by a finite sequence of moves M3, M4. Since $V$ is self-dual, the graphical calculus $Z_{\langle V \rangle}(\hat{\Gamma}_V)$ is independent of the orientation on the edges, i.e., it is invariant with respect to the move M3. Moreover relations provided by Lemma 2.22 give the invariance of $Z_{\langle V \rangle}(\hat{\Gamma}_V)$ with respect to moves of type M4:

Therefore, $Z(\Gamma) := Z_{\langle V \rangle}(\hat{\Gamma}_V)$ is well defined and is a PROP action. 

Remark 2.25. The tensors $T_r$ are generators of the $\mathcal{R}$-algebra structure on $V$; as such, they are independent and need not satisfy any further compatibility relation. For instance, cyclic algebras need not be associative.

2.6. Graphical calculus for ordinary graphs. It is now easy to adapt constructions above to ordinary graphs. Note that a graph can be obtained from a ribbon graph by forgetting the cyclic order on the edges incident to any vertex. Two ribbon graphs leading to the same graph differ just by a permutation of the edges incident to some vertices, so, to define a graphical calculus for graphs, it will suffice to have a graphical calculus for ribbon graphs, invariant with respect to the action of symmetric groups.

Definition 2.26. A symmetric algebra structure on $(V, b)$ is a sequence $\{S_r\}$ of linear maps $S_r : V^\otimes r \to k$ such that

$$S_r(X_1 \otimes \cdots \otimes X_r) = S_r(X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(r)}), \quad \forall \sigma \in \mathfrak{S}_r,$$

that is, maps $\{S_r\}$ are invariant with respect to the action of the symmetric group.
Like in a cyclic algebra, the tensors $S_r$ are independent one from the other: they should be regarded as generators of a PROP-algebra structure.

**Definition 2.27.** Denote by $\mathcal{G}$ the PROP of (ordinary topological) graphs: $\mathcal{G}(p, q)$ is the linear span of graphs of type $(p, q)$. It is the quotient of $\mathcal{R}$ by the action of symmetric groups on half-edges stemming from any vertex.

**Theorem 2.28.** Let $V$ be a vector space. Then the following data are equivalent:

(i) symmetric algebra structures on $V$;
(ii) $\mathcal{G}$-algebra structures on $V$.

**Proof.** An action $Z : \mathcal{G} \to \mathcal{E}_V$ endows $V$ with a symmetric algebra structure, as in the proof of Theorem 2.24. Conversely, assume we are given a cyclic algebra structure on $V$. Since the tensors $S_r$ are symmetric, they are, in particular, cyclically invariant: so $(V, b, S_1, S_2, \ldots)$ is a cyclic algebra and an action $Z : \mathcal{G} \to \mathcal{E}_V$ is defined. Since the tensors $\{S_k\}$ are symmetric, this action factors through an action $Z : \mathcal{G} \to \mathcal{E}_V$. 

**Remark 2.29.** So far, we have considered cyclic (resp. symmetric) algebras with only one $r$-ary operation for any $r \in \mathbb{N}$. The graphical calculus formalism immediately generalizes to cyclic (resp. symmetric) algebras with a family of cyclic tensors $\{T_{r,\alpha}\}_{\alpha \in I_r}$ (resp. a family of symmetric tensors $\{S_{r,\alpha}\}_{\alpha \in I_r}$): in fact, we only ought to consider graphs whose $r$-valent vertices are decorated with labels from $I_r$.

The *dual graphs* of stable curves [14] provide an example — they are ordinary graphs, with each vertex $v$ decorated by an integer $g(v)$: it is the genus of an irreducible component of the algebraic curve corresponding to the graph. Such graphs were called *modular graphs* in [15]. One can apply methods of graphical calculus to this class of graphs; we give a specimen in Example 3.10.

**2.7. A sample computation.** Let $(V, b, S_1, S_2, \ldots)$ be a symmetric algebra. We want to compute the operator $Z(\Gamma) := Z$

\[
\begin{pmatrix}
1^m \\
3^m \\
2^m \\
\end{pmatrix}

: V^{\otimes 3} \to k
\]

A realization of the graph $\Gamma$ as an RT-diagram is obtained by hanging it by the vertices:
Splitting the RT-diagram above into elementary pieces and applying Reshetikhin-Turaev rules we find

\[
Z(\Gamma) = Z \left( \begin{array}{c}
S_1 & S_2 & S_3 \\
\end{array} \right) = (S_1 \otimes S_2) \circ (Id_V \otimes Id_V \otimes \text{coev}_V \otimes Id_V \otimes \text{coev}_V) \circ (Id_V \otimes \tau_V \otimes Id_V) \circ (Id_V \otimes \text{coev}_V \otimes Id_V)
\]

If \( \{e_i\} \) is a basis of \( V \) and \( \{e^i\} \) denotes the dual basis with respect to the pairing \( b \), then structure constants of the symmetric algebra \( (V, b, S_1, S_2, \ldots) \) are

\[
g_{ij} = b(e_i, e_j), \quad g^{ij} := b(e^i, e^j),
\]

\[
(S_k)_{i_1, \ldots, i_v}^{j_1, \ldots, j_k} := S_k(e_{i_1} \otimes \cdots \otimes e_{i_v} \otimes e^{j_{k+1}} \otimes \cdots \otimes e^{j_k}).
\]

With these notations

\[
\text{coev}_V = \sum_{i,j} g^{ij} e_i \otimes e_j,
\]

and the operator \( Z(\Gamma) \) acts on basis elements by

\[
Z(\Gamma)(e_\alpha \otimes e_\beta \otimes e_\gamma) = \sum_{\delta, \epsilon, \zeta, \eta, \theta, \iota} g^{\epsilon \delta} g_{\eta \theta} (S_3)_{\alpha \epsilon \delta} (S_3)_{\eta \beta \theta} (S_3)_{\iota \gamma}.
\]

2.8. **On graphical notations used in physics literature.** All the graphs we have considered so far are grouped under the generic name of “Feynman diagrams”; the Casimir element and the tensors decorating the vertices are called “propagator” and “interactions”, respectively. Moreover, different types of lines are used to denote different kinds of particles (see, for instance \([16,17]\)); one can recover these notations as follows.

A graph \( \Gamma \) of type \((r, 0)\) gives a linear operator \( Z(\Gamma) : V^{\otimes r} \to \mathbb{C} \). In graphical notations, the value of this operator on \( v_1 \otimes v_2 \otimes \cdots \otimes v_r \) can be represented by the graph \( \Gamma \) with the \( i \)-th incoming edge decorated by the vector \( v_i \):

\[
Z \left( \begin{array}{c}
2^\text{in} & 3^\text{in} \\
1^\text{in} & 4^\text{in} \\
\end{array} \right) (v_1 \otimes v_2 \otimes v_3 \otimes v_4) =:
\]

The vectors \( v_i \) are called the “incoming states”. If a basis \( \{e_i\} \) (not necessarily orthonormal) is given for \( V \), then an incoming state \( e_i \) will be denoted simply by the index \( i \). If \( \{e^i\} \) is the dual basis of \( \{e_i\} \) with respect to the inner product of \( V \), an incoming state \( e^i \) (or an outgoing \( e_i \)) will be denoted by the index \( i^\vee \). It is

\[\footnote{This introduces a notations clash, since we use indices near edges to indicate the total order of \( \text{In}(\Gamma) \) and \( \text{Out}(\Gamma) \).} \]
customary to write

\[
\begin{pmatrix}
  i_1^{\text{out}} & j_1^{\text{in}} & \cdots & i_r^{\text{out}} \\
  \cdots & \cdots & \cdots & \cdots \\
  i_1^{\text{in}} & j_1^{\text{out}} & \cdots & i_r^{\text{in}}
\end{pmatrix}
\cdot e_{j_1} \otimes \cdots \otimes e_{j_s}.
\]

For example, if \( T : V \otimes^3 \to \mathbb{C} \) is a cyclic or symmetric tensor, we have:

\[
T_{i_1}^{\text{in}} T_{i_2}^{\text{in}} \cdots T_{i_r}^{\text{in}} = \sum_{m} T_{i_1 j_1 m} T_{i_2 j_2 m} \cdots T_{i_r j_r m} = \sum_{m,n} T_{i_1 j_1 m} g^{i_2 j_2 n} T_{i_3 j_3 n} \cdots T_{i_r j_r n}
\]

When the space \( V \) of “physical states” is the direct sum of two subspaces \( V_1, V_2 \), vectors are usually depicted by different types of lines according to the subspace they lie in. For example, it is customary in physical literature to depict fermions by a straight line and bosons by a wavy line\(^5\), so that one encounters diagrams like the following, which depicts a photon exchange between two electrons:

\[
\begin{pmatrix}
  (e^-)^{\text{in}} \\
  \cdots \\
  (e^-)^{\text{in}}
\end{pmatrix}
\begin{pmatrix}
  (e^-)^{\text{out}} \\
  \cdots \\
  (e^-)^{\text{out}}
\end{pmatrix}
\]

**Remark 2.30 (Fields of algebras and the WDVV equation).** Let \( \phi \) be an analytic function defined on a neighborhood \( U \) of \( 0 \in V \). Then, for any \( x \in U \), the derivatives \( D^n \phi|_x \) are a family of symmetric tensors on the tangent spaces \( T_x V \). These tensors define a field of symmetric algebras \( A_x \) on \( U \): \( A_x \) is the symmetric algebra \( (T_x V, (-|-), D\phi|_x, D^2\phi|_x, \ldots) \) — the inner product is induced by the canonical identification of \( V \) with its tangent spaces. The function \( \phi \) is called a potential for the field of algebras \( A_x \). Equivalently, the field of algebras \( A_x \) is the datum of a symmetric algebra structure on the \( C^\infty(U)\)-module \( \mathfrak{X}(U) \) of smooth vector fields on \( U \).

Denote by \( Z_{\phi,x} \) the graphical calculus for \( A_x \). Note that for every graph \( \Gamma \) the map \( \phi \mapsto Z_{\phi,x}(\Gamma) \) is a differential operator — we denote it by the symbol \( D_\Gamma \), i.e.,

\[
D_\Gamma(\phi) := Z_{\phi,x}(\Gamma), \quad \forall \phi \text{ analytic in } U.
\]

\((2.3)\)

Abusing notation, we will occasionally write \( \Gamma(\phi) \) to mean \( D_\Gamma(\phi) \); it comes handy when one is dealing with a field of algebras enjoying some special property that can be described in diagrammatic form.

For example, symmetric associative algebras are described by the associativity equation

\[
\begin{pmatrix}
  y^{\text{in}} \\
  \cdots \\
  y^{\text{in}}
\end{pmatrix}
\begin{pmatrix}
  z^{\text{in}} \\
  \cdots \\
  z^{\text{in}}
\end{pmatrix} = \begin{pmatrix}
  x^{\text{in}} \\
  \cdots \\
  x^{\text{in}}
\end{pmatrix}
\]

\[\forall x, y, z, w \in A.\]

---

\(^5\) The graphical convention above is often refined depicting spin 0 bosons by dashed straight lines and reserving wavy lines for spin 1 bosons (see, for example [18]).
So, the condition $\phi$ must satisfy to define a field of symmetric associative algebras is the Witten-Dijkgraaf-Verlinde-Verlinde equation (WDVV for short)

$$\begin{pmatrix}
X^{in} & Z^{in} \\
X^{in} & W^{in}
\end{pmatrix}
(\phi) =
\begin{pmatrix}
Y^{in} & Z^{in} \\
X^{in} & W^{in}
\end{pmatrix}
(\phi), \quad \forall \ X, Y, Z, W \in \mathfrak{X}(U)$$

In terms of the flat vector fields $\{\partial_i\}$ on $U$ corresponding to a basis $\{e_i\}$ of $V$, the WDVV equation reads

$$\sum_{m,n}(\partial_i \partial_j \partial_m \phi) g^{mn}(\partial_n \partial_k \partial_l \phi) = \sum_{m,n}(\partial_i \partial_l \partial_m \phi) g^{mn}(\partial_n \partial_j \partial_k \phi) \quad (2.4)$$

which is the form one usually finds in the literature (e.g. [19]).

3. Gaussian integrals and Feynman diagrams

In this section we show how a Gaussian integral can be expanded into a sum of Feynman diagrams, to be evaluated according to the rules of graphical calculus. Depending on the nature of the integral, this formula will hold as a strict equality or in the sense of asymptotic expansions. In particle physics, Gaussian integrals and their Feynman diagrams expansions are used to describe bosonic statistics.

3.1. Gaussian measures and the Wick’s lemma. Let $V$ be a finite dimensional Euclidean space, with inner product $(-|-)$. If $\{e_i\}$ is a basis of $V$, we denote the coordinate maps relative to this basis as $e_i : V \to \mathbb{R}$, and write $v_i$ for the pairing $\langle e_i, v \rangle$. The matrix associated to $(-|-)$ with respect to the basis $\{e_i\}$ is given by

$$g_{ij} := \langle e_i | e_j \rangle.$$

As customary, we set $g^{ij} := (g^{-1})_{ij} = (e^i|e^j)$.

Let now $dv$ be a (non trivial) translation invariant measure on $V$. The function $e^{-\frac{1}{2}(v|v)}$ is positive and integrable with respect to $dv$.

**Definition 3.1.** The probability measure on $V$ defined by

$$d\mu(v) := \frac{1}{A} e^{-\frac{1}{2}(v|v)}dv, \quad A = \int_V e^{-\frac{1}{2}(v|v)}dv,$$

is called the Gaussian measure on $V$.

Since a non-trivial translation invariant measure on $V$ is unique up to a scalar factor, $d\mu$ is actually independent of the chosen $dv$.

The symbol $\langle f \rangle_V$ denotes the average of a function $f$ with respect to the Gaussian measure, i.e.,

$$\langle f \rangle_V := \int_V f(v)d\mu(v).$$
Lemma 3.2 (Wick). Polynomial functions of the coordinates $v^i$ are integrable with respect to $d\mu$ and:

\[
\langle v^{i_1} v^{i_2} \cdots v^{i_{2n+1}} \rangle_V = 0, \quad \text{(W1)}
\]
\[
\langle v^i v^j \rangle_V = g^{ij}, \quad \text{(W2)}
\]
\[
\langle v^{i_1} v^{i_2} \cdots v^{i_{2n}} \rangle_V = \sum_{s \in P} g^{i_{s_1} i_{s_2}} g^{i_{s_3} i_{s_4}} \cdots g^{i_{s_{2n-1}} i_{s_{2n}}}, \quad \text{(W3)}
\]

where the sum ranges over all distinct pairings of the set of indices $\{i_1, \ldots, i_{2n}\}$, i.e., over the set of all partitions $\{\{i_{s_1}, i_{s_2}\}, \{i_{s_3}, i_{s_4}\}, \ldots\}$ of $\{i_1, i_2, \ldots, i_{2n}\}$ into 2-element subsets.

For a proof of Wick’s lemma see, for instance, [2].

The inner product $(\cdot|\cdot)$ extends uniquely to a Hermitian product on the complex vector space $V_C := V \otimes \mathbb{C}$. Identify $V$ with the subspace $V \otimes \{1\}$ of real vectors in $V_C$; $\{e_i\}$ is a real basis for the complex vector space $V_C$. Extend $\langle \cdot \rangle_V$ to tensor powers of real vectors by

\[
\langle v^\otimes k \rangle_V := \sum \langle v^{i_1} \cdots v^{i_k} \rangle_V e_{i_1} \otimes \cdots \otimes e_{i_k},
\]

so Wick’s lemma can be recast this way:

\[
\langle v^\otimes (2n+1) \rangle_V = 0, \quad \text{(W1')} \]
\[
\langle v^\otimes 2 \rangle_V = \sum_{i,j} g^{ij} e_i \otimes e_j, \quad \text{(W2')} \]
\[
\langle v^\otimes 2n \rangle_V = \sum_{i_1, \ldots, i_{2n}} \sum_{s \in P} g^{i_{s_1} i_{s_2}} g^{i_{s_3} i_{s_4}} \cdots g^{i_{s_{2n-1}} i_{s_{2n}}} e_{i_{s_1}} \otimes e_{i_{s_2}} \cdots \otimes e_{i_{s_{2n-1}}} \otimes e_{i_{s_{2n}}}, \quad \text{(W3')} \]

where the last sum ranges over all distinct pairings of indices in the set $\{i_1, \ldots, i_{2n}\}$.

The right-hand side of (W2') is the Casimir element $\gamma_{V_C}$ of $\{V_C, (\cdot|\cdot)\}$; in Reshetikhin-Turaev’s graphical notation, we can rewrite (W2) as

\[
\langle v \otimes v \rangle_V = \bigcup
\]

The graphical notation becomes particularly suggestive (and useful) when applied to (W3):

\[
\langle v^\otimes 4 \rangle_V = \bigcup + \bigcup + \bigcup + \bigcup + \cdots
\]

\[
\langle v^\otimes 2n \rangle_V = \left( \bigcup \bigcup \cdots \bigcup \right) + \cdots + \bigcup
\]

the last sum ranging over all possible configurations of $n$ Casimir elements and the braiding being the trivial one: $x \otimes y \mapsto y \otimes x$.

In addition, assume we have, for any $k$, a cyclically invariant $k$-tensor

\[
T_k : V_C^\otimes k \to \mathbb{C};
\]
which has the graphical representation

\[
\begin{array}{c}
 T_k \\
 \cdots \\
 k 
\end{array}
\]

The data \((V, (\cdot | \cdot), T_1, T_2, \ldots)\) define a cyclic algebra structure, so we can use Reshetikhin-Turaev’s graphical calculus \(\Gamma \mapsto Z(\Gamma)\) to compute averages

\[
\langle T_1(v)^{l_1} T_2(v^\otimes 2)^{l_2} \cdots T_k(v^\otimes k)^{l_k} \rangle.
\]

**Lemma 3.3.** Any average \(\langle T_1(v)^{l_1} T_2(v^\otimes 2)^{l_2} \cdots T_k(v^\otimes k)^{l_k} \rangle\) is a linear combination

\[
\sum_{\Gamma} \alpha_{\Gamma} Z(\Gamma)
\]

where \(\Gamma\) runs in the set of ribbon graphs of type \((0,0)\) with \(l_i\) vertices of valence \(i\), for \(i = 1, \ldots, k\).

**Proof.** By linearity of the \(T_k\)’s and (3.1),

\[
\langle T_1(v)^{l_1} \cdots T_k(v^\otimes k)^{l_k} \rangle_V = (T_1^\otimes l_1 \otimes \cdots \otimes T_k^\otimes l_k)(v^\otimes \sum_i u_i)_V.
\]

If \(\sum i l_i\) is odd, \(\langle v^\otimes \sum u_i \rangle_V\) is zero, and the set of graphs considered in the statement is empty. If \(\sum i l_i\) is even, according to graphical calculus rules, \(\bigotimes_{i=1}^k T_i^\otimes l_i\) corresponds to the juxtaposition of \(l_1\) univalent vertices, \(l_2\) bivalent vertices, etc., up to \(l_k\) vertices of valence \(k\). In this case, (3.1) translates \(\langle v^\otimes \sum u_i \rangle_V\) into edges connecting these vertices in all possible ways. \(\square\)

**Example 3.4.** For example,

\[
\langle (T_2(v^\otimes 2))^2 \rangle_V = \begin{array}{c}
 \cdot \\
 \cdot \\
 + 2 \cdot \\
 \end{array}
\]

\[
\langle T_4(v^\otimes 4) \rangle_V = \begin{array}{c}
 \cdot \\
 \cdot \\
 + 2 \cdot \\
 \end{array}
\]

**Lemma 3.5.** The coefficient \(\alpha_{\Gamma}\) appearing in Lemma 3.3 is an integer given by:

\[
\alpha_{\Gamma} = \frac{1^{l_1} l_1! 2^{l_2} l_2! \cdots k^{l_k} l_k!}{|\text{Aut } \Gamma|}.
\]

**Proof.** Let \(X\) be the set of all ribbon graphs obtained by: (i) juxtaposing \(l_1\) vertices of valence 1, \(l_2\) vertices of valence 2, etc., up to \(l_k\) vertices of valence \(k\), and, (ii) connecting them in all possible ways by means of arcs. The constant \(\alpha_{\Gamma}\) counts the number of occurrences of graphs isomorphic to \(\Gamma\) in the set \(X\).

The semi-direct product \(K = \prod_{i=1}^k (\mathbb{S}_l_i \rtimes (\mathbb{Z}/l_i)\mathbb{Z})\) acts on \(X\) as follows: the image of a graph \(\Phi\) is obtained by permuting vertices of the same valence and rotating edges incident to each vertex. Since this action is transitive on isomorphism classes,

\[
\alpha_{\Gamma} = \frac{|K|}{|\text{Stab}_K(\Gamma)|} = \frac{|K|}{|\text{Aut } \Gamma|} = \frac{1^{l_1} l_1! 2^{l_2} l_2! \cdots k^{l_k} l_k!}{|\text{Aut } \Gamma|},
\]

where \(\text{Stab}_K(\Gamma)\) is the stabilizer of \(\Gamma\) under the action of \(K\). \(\square\)
Theorem 3.6 (Feynman-Reshetikhin-Turaev). Let $Z_{x_+}$ be the graphical calculus for the cyclic algebra $A(x_+) := \langle V, (-|-), x_1 T_1, x_2 T_2, \ldots \rangle$, where $x_1, x_2, \ldots$ are complex variables. Then, the following asymptotic expansion holds:

$$ Z(x_+) := \int_V \exp \left\{ \sum_{k=1}^{\infty} \frac{\mu(v) }{k} \right\} d\mu(v) = \sum_{\Gamma \in \mathcal{R}(0,0)} \frac{Z_{x_+}(\Gamma)}{|\text{Aut} \Gamma|}, \quad (3.2) $$

where the sum on the right ranges over the set $\mathcal{R}(0,0)$ of all isomorphism classes of (possibly disconnected) ribbon graphs of type $(0,0)$. The formal series $Z(x_+)$ is called the partition function of the algebra $A(x_+)$. 

Proof. Expand in Taylor series the left-hand side:

$$ \int_V \exp \left\{ \sum_{k=1}^{\infty} \frac{x_k}{k} T_k(v) \right\} d\mu(v) = \cdots $$

by lemmas 3.3 and 3.5.

A similar argument yields:

$$ \int_V T_1(v_1) T_2(v_2) T_k(v_k) \cdots \exp \left\{ \sum_{k=1}^{\infty} \frac{T_k(v_k)}{k} \right\} d\mu(v) = \sum_{\Gamma \in \mathcal{R}(0,0)} \frac{Z_{x_+}(\Gamma)}{|\text{Aut} \Gamma|}, \quad (3.3) $$

where the sum in the right-hand side ranges over all ribbon graphs having $l_i$ “special” i-valent vertices (for $i = 1, \ldots, k$), and $\text{Aut}(\Gamma)$ is the group of automorphisms that map the set of special vertices to itself. In (3.3), the graphical calculus $Z_{x_+}$ interprets each i-valent special vertex as the operator $T_i$ and each ordinary i-valent vertex as the operator $x_i T_i$. This is the same as considering graphs with two sorts of vertices (see Remark 2.29), one decorated by operators $T_i$ (call them “special vertices”), and the other decorated by $x_i T_i$ (call them “ordinary”).

Theorem 3.7 can be straightforwardly adapted to a symmetric algebra $\langle V, (-|-), S_1, S_2, \ldots \rangle$.

Theorem 3.7 (Feynman-Reshetikhin-Turaev). Let $Z_{x_+}$ be the graphical calculus for the symmetric algebra $\langle V, (-|-), x_1 S_1, x_2 S_2, \ldots \rangle$, where $x_1, x_2, \ldots$ are complex variables. The following asymptotic expansion holds:

$$ Z(x_+) := \int_V \exp \left\{ \sum_{k=1}^{\infty} \frac{x_k S_k(v)}{k!} \right\} d\mu(v) = \sum_{\Gamma \in \mathcal{R}(0,0)} \frac{Z_{x_+}(\Gamma)}{|\text{Aut} \Gamma|}, $$

where the sum on the right ranges over all isomorphism classes of (possibly disconnected) ordinary graphs of type $(0,0)$. 

Remark 3.8 (Generalized Gaussian integrals). Denote by $Q(v)$ the Gaussian weight $e^{-1/2(v,v)}$, and let $g^2 : V \to V^\vee$ be the isomorphism induced by the non-degenerate pairing $g : V \otimes V \to k$. Moreover, let $m$ be the usual multiplication on the space $K := k[[V^\vee]]$ of formal power series on $V$. In [1], Robert Oeckl shows that the graphical version of Wick’s lemma is a formal consequence of the following properties:

(G1) a “braided Leibniz rule” for derivations:
$$\partial_w \circ m = m(\partial_w \otimes \text{Id}) + \tau_{K,K}^{-1} \circ (\partial_w \otimes \text{Id}) \circ \tau_{K,K},$$
for any $w \in V$ and any $\phi, \psi \in K$;

(G2) $\partial_w Q = -g^2(w) \cdot Q$, for all $w \in V$;

(G3) $g^2$ is an isomorphism and $\text{ev}_V \circ (\text{Id}_V \otimes g^2) = \text{ev}_V \circ (\text{Id}_V \otimes g^\#) \circ \tau_{V,V}$;

(G4) $\int \partial_w (\phi \cdot Q) d\mu = 0$, for any $w \in V$ and any polynomial $\phi \in K$;

Equations (G1-G4) can be used to define Gaussian integrals in the context of arbitrary braided monoidal categories. This has been done by R.Oeckl, with the development of “Braided QFT” [1]; since equation (3.1) is a formal consequence of (G1-G4), the whole machinery of Feynman diagrams expansions will be available for these generalized Gaussian integrals, too. A remarkable by-product of this general theory is that the Berezin super-integrals of fermionic statistics are simply obtained as “braided Gaussian integrals” for a vector space $V$ endowed with the non-trivial braiding $x \otimes y \mapsto -y \otimes x$ (the pairing $g$ must, consequently, be antisymmetric). Therefore, in the particular case of the symmetric category of super vector spaces (with the usual graded symmetric braiding), “braided Gaussian integrals” provide an unified language for both statistics encountered in standard quantum field theory: bosons correspond to even vectors and fermions correspond to odd vectors [20, Sections 3.3 and 3.4].

Example 3.9. Let $\phi$ be an analytic function defined in a neighborhood of $0 \in V$. The Taylor expansion formula
$$\phi(v) = \phi(0) + \sum_{n=1}^\infty \frac{D^n \phi(0)(v \otimes n)}{n!},$$
together with Theorem 3.7 gives
$$\int_V e^{\phi(v)} d\mu(v) = e^{\phi(0)} \cdot \left( \sum_{\Gamma} \frac{D_\Gamma(\phi)}{|\text{Aut } \Gamma|_0} \right). \quad (3.4)$$

Example 3.10. We want to show how this machinery can be used to derive a graph expansion formula cited in the introduction to [15]. If we are given several symmetric tensors $\{S_{k,\alpha_k}\}_{\alpha_k \in I_k}$ for any index $k$, the formula in Theorem 3.7 generalizes to
$$\int_V \exp \left\{ \sum_{k=1}^\infty \sum_{\alpha_k \in I_k} x_k y_{\alpha_k} \frac{S_{k,\alpha_k}(v \otimes k)}{k!} \right\} d\mu(v) = \sum_{\Gamma \in \mathcal{G}/(0,0)} \frac{\text{Z}_{x,y,\Gamma}(\Gamma)}{|\text{Aut } \Gamma|}. \quad (3.5)$$
where $\mathcal{G}/(0,0)$ denotes the set of isomorphism classes of (possibly disconnected) graphs whose $k$-valent vertices are decorated by elements of the set $I_k$.

In particular, if we set, for any $k$,
$$I_k = \mathbb{N}, \quad x_k = 1, \quad y_g = h^2, \quad S_{k,g} = 0 \quad \text{if} \quad 3g - 3 + k \leq 0,$$
and rescale the inner product on \( V \) by a factor \( 1/\hbar \) (i.e., we take \((1/\hbar)(-|-)\) instead), then:

\[
\int_V \exp \left\{ \frac{1}{\hbar} \sum_{k=1}^{\infty} \sum_{g=0}^{\infty} \hbar^g \frac{S_{k,g}(v^\otimes k)}{k!} \right\} \, d\mu_\hbar(v) = \sum_{g=0}^{\infty} \hbar^g \sum_{\Gamma \in M(g,0)} \hbar^{-b_0(\Gamma)} \frac{Z(\Gamma)}{|\text{Aut} \Gamma|}.
\]

(3.6)

where \( M(g,0) \) denotes the set of isomorphism classes of (possibly non connected) genus \( g \) modular graphs with no legs. The \textit{genus} of the modular graph \( \Gamma \) is the integer \( g(\Gamma) := \sum_v g(v) + \dim H^1(\Gamma) \) and \( b_0(\Gamma) = \dim H_0(\Gamma) \).

A similar formula holds for modular graphs with legs. A leg can be considered as an edge ending in a univalent vertex of a distinguished kind, so, fix a linear operator \( \zeta : V \to \mathbb{C} \) and extend graphical calculus so to evaluate univalent vertices to \( \zeta \):

\[
\int_V \exp \left\{ \frac{1}{\hbar} \left( \zeta(v) + \sum_{k=1}^{\infty} \sum_{g=0}^{\infty} \hbar^g \frac{S_{k,g}(v^\otimes k)}{k!} \right) \right\} \, d\mu_\hbar(v) = \sum_{g,n=0}^{\infty} \hbar^g \sum_{\Gamma \in M(g,n)} \hbar^{-b_0(\Gamma)} \frac{Z(\Gamma)}{|\text{Aut} \Gamma|}.
\]

(3.7)

\textbf{Example 3.11} (Asymptotic expansion of the “free energy” functional). The logarithm of the partition function \( Z(x_*) \) is called the \textit{free energy} of the cyclic (resp. symmetric) algebra, and is denoted by \( F(x_*) \); it admits a graph expansion, too. However, expansion of the partition function is a sum over all graphs, whereas expansion of the free energy involves only \textit{connected} ones.

\textbf{Lemma 3.12.} The free energy \( F(x_*) := \log Z(x_*) \) admits a Feynman-Reshetikhin-Turaev expansion in ribbon (resp. ordinary) graphs, given by:

\[
F(x_*) = \sum_{\Gamma \text{ connected}} \frac{Z_{x_*}(\Gamma)}{|\text{Aut} \Gamma|}.
\]

(3.8)

\textit{Proof.} Exponentiate

\[
\sum_{\Gamma \text{ connected}} \frac{Z_{x_*}(\Gamma)}{|\text{Aut} \Gamma|}
\]

to find:

\[
\exp \left\{ \sum_{\Gamma} \frac{Z_{x_*}(\Gamma)}{|\text{Aut} \Gamma|} \right\} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\Gamma_1,\ldots,\Gamma_k} \frac{Z_{x_*}(\Gamma_1) \cdots Z_{x_*}(\Gamma_k)}{|\text{Aut} \Gamma_1| \cdots |\text{Aut} \Gamma_k|},
\]

where each \( \Gamma_i \) is a connected graph. Now recall that juxtaposition defines a tensor product \( \otimes \) in the category of graphs (cf. Definition 2.8) and that \( Z_{x_*} \) is multiplicative with respect to this structure:

\[
Z_{x_*}(\Gamma_1 \otimes \cdots \otimes \Gamma_k) = Z_{x_*}(\Gamma_1) \otimes \cdots \otimes Z_{x_*}(\Gamma_k).
\]

Therefore,

\[
\exp \left\{ \sum_{\Gamma \text{ connected}} \frac{Z_{x_*}(\Gamma)}{|\text{Aut} \Gamma|} \right\} = \sum_{k=0}^{\infty} \sum_{\Gamma_1,\ldots,\Gamma_k, \text{ connected}} \frac{Z_{x_*}(\Gamma_1 \otimes \cdots \otimes \Gamma_k)}{k!|\text{Aut} \Gamma_1| \cdots |\text{Aut} \Gamma_k|}.
\]

For a graph \( \Phi \) having \( k \) connected components \( \Gamma_1,\ldots,\Gamma_k \). Let \( I_\Phi \) be the set of all possible juxtapositions of \( \Gamma_1,\ldots,\Gamma_k \); all graphs in \( I_\Phi \) are isomorphic to \( \Phi \). The semi-direct product \( K \) of \( \mathfrak{S}_k \) and \( \text{Aut} \Gamma_1 \times \cdots \times \text{Aut} \Gamma_k \) acts transitively on \( I \); the stabilizer of any element is isomorphic to \( \text{Aut} \Phi \). Therefore,

\[
|I_\Phi| = \frac{|K|}{|\text{Stab} \Phi|} = \frac{k!|\text{Aut} \Gamma_1| \cdots |\text{Aut} \Gamma_k|}{|\text{Aut} \Phi|}.
\]
So we reckon:
\[
\exp \left\{ \sum_{\Gamma \text{ connected}} \frac{Z_{x} (\Gamma)}{|\text{Aut } \Gamma|} \right\} = \sum_{\Phi} \frac{Z_{x} (\Phi)}{|\text{Aut } \Phi|} = Z (x). \]

4. The ’t Hooft-Kontsevich model

This last section is devoted to the Kontsevich matrix model for 2D quantum gravity, first defined in [8]; it embodies the “standard matrix model” of ’t Hooft as a particular case. A cyclic algebra structure is introduced on the vector space \( \mathcal{H} \) of \( N \times N \) Hermitian matrices; results from the previous sections apply.

Let \( V \) be an \( N \)-dimensional Hilbert space. The space \( \text{End}(V) \) has a natural Hermitian inner product
\[
(X|Y) := \text{tr}(X^*Y),
\]
which induces the standard Euclidean inner product \( (X|Y) = \text{tr}(XY) \) on the real subspace
\[
\mathcal{H} := \{ X \in \text{End}(V)|X^* = X \}
\]
of Hermitian operators.

For any positive definite Hermitian operator \( \Lambda \), we can define a new Euclidean inner product on \( \mathcal{H} \) by
\[
(X|Y)_{\Lambda} := \frac{1}{2} (\text{tr}(XAY) + \text{tr}(YAX)).
\]

Now, define cyclic tensors
\[
T_{k} : \mathcal{H} \otimes \cdots \otimes \mathcal{H} \to \text{tr}(X_{1}X_{2}\cdots X_{k}) \in \mathbb{C}.
\]

These \( T_{k} \), together with the inner product \( (-|-)_{\Lambda} \), define a cyclic algebra structure on \( \mathcal{H} \) called the Kontsevich model. A graphical calculus \( Z \) for the Kontsevich model is defined on the PROP \( \mathcal{R} \) of ribbon graphs.

**Lemma 4.1.** The following formula holds:
\[
Z(\Gamma) = \sum_{c} \prod_{\ell \in \Gamma^{(1)}} \frac{2}{\Lambda_{c}(\ell^{+}) + \Lambda_{c}(\ell^{-})}, \quad c : \Gamma^{(2)} \to \{1, \ldots, N\}
\]
where \( \Lambda_{1}, \ldots, \Lambda_{N} \) are the eigenvalues of \( \Lambda \), \( c \) runs over all colorings of holes of \( \Gamma \) in \( N \) colors, and \( \ell^{+}, \ell^{-} \) are the two holes bounded by the edges \( \ell \) (they are not necessarily distinct).

**Proof.** To evaluate \( Z(\Gamma) \) we need an explicit expression for the Casimir element \( \text{coev}_{\mathcal{H}, \Lambda}(1) \) of the cyclic algebra \( (\mathcal{H}, (-|-)_{\Lambda}, T_{1}, T_{2}, \ldots) \). Since \( \Lambda \) is Hermitian positive definite, there exists an orthonormal basis \( \{e_{i}\} \) of \( V \) in which
\[
\Lambda = \text{diag}(\Lambda_{1}, \Lambda_{2}, \ldots, \Lambda_{N}),
\]
for some \( \Lambda_{i} \) positive real numbers. Any choice of a like basis induces an identification of \( V \) with \( \mathbb{C}^{N} \), and, consequently, of \( \text{End}(V) \) with the space \( M_{N}(\mathbb{C}) \) of \( N \times N \) complex matrices. Let \( \{E_{ij}\} \) be the canonical basis for \( M_{N}(\mathbb{C}) \):
\[
(E_{ij})_{kl} = \delta_{ik} \delta_{jl}.
\]

A basis for \( \mathcal{H} \) is given by matrices
\[
e_{ij} = \begin{cases} \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}) & \text{if } i < j, \\ E_{ii} & \text{if } i = j, \\ \sqrt{-\frac{1}{2}}(E_{ij} - E_{ji}) & \text{if } i > j. \end{cases}
\]
It is orthonormal with respect to the inner product \((-|-\)\), whereas
\[
(e_{ij}|e_{kl})_{\Lambda} = \frac{\Lambda_i + \Lambda_j}{2} \delta_{ij,kl},
\]
i.e., the matrix of \((-|-\)\) with respect to the basis \(\{e_{ij}\}\) is
\[
g = \text{diag}\left(\{\frac{\Lambda_i + \Lambda_j}{2}\}\right).
\]
So we get the following expression for the Casimir element:
\[
\text{coev}_{\mathcal{H},\Lambda}(1) = \sum_{i,j} \frac{2}{\Lambda_i + \Lambda_j} e_{ij} \otimes e_{ij}.
\]
Rewrite this identity as:
\[
\text{coev}_{\mathcal{H},\Lambda}(1) = \sum_{i} \frac{1}{\Lambda_i} e_{ii} \otimes e_{ii} + \sum_{i<j} \frac{2}{\Lambda_i + \Lambda_j} (e_{ij} \otimes e_{ij} + e_{ji} \otimes e_{ji}),
\]
but, for \(i < j\),
\[
e_{ij} \otimes e_{ij} + e_{ji} \otimes e_{ji} = \frac{1}{2}(E_{ij} \otimes E_{ij} + E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ji}) - \frac{1}{2}(E_{ij} \otimes E_{ij} - E_{ij} \otimes E_{ji} - E_{ji} \otimes E_{ji}) = E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij}.
\]
So,
\[
\text{coev}_{\mathcal{H},\Lambda}(1) = \sum_{i} \frac{1}{\Lambda_i} E_{ii} \otimes E_{ii} + \sum_{i<j} \frac{2}{\Lambda_i + \Lambda_j} (E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij}) = \sum_{i,j} \frac{2}{\Lambda_i + \Lambda_j} E_{ij} \otimes E_{ji}.
\]
According to standard graphical calculus rules, evaluation \(Z(\Gamma)\) is performed through the correspondence
\[
\cdots \leftrightarrow T_k, \quad \bigcup \leftrightarrow \text{coev}_{\mathcal{H},\Lambda}(1).
\]
If we introduce the notation
\[
\bigcup_{i,j}^{i,j} = \frac{2}{\Lambda_i + \Lambda_j} E_{ij} \otimes E_{ji},
\]
then we can depict (4.1) as
\[
\bigcup = \sum_{i,j} \bigcup_{i,j}^{i,j},
\]
which turns \(Z(\Gamma)\) into a sum of ribbon graphs equipped with a number in \(\{1, \ldots, N\}\) on each side of every edge, and operators \(T_k\) on each \(k\)-valent vertex.
The map $T_k$ is the restriction of a map $T_k$ defined on $M_N(\mathbb{C})$, namely, the trace of a $k$-fold product. We have
\[
T_k(E_{i_1,j_1} \otimes E_{i_2,j_2} \otimes \cdots \otimes E_{i_k,j_k}) = \delta_{j_1,i_2} \delta_{j_2,i_3} \cdots \delta_{j_{k-1},i_k} \delta_{j_k,i_1}.
\] (4.2)
Therefore, the only graphs that give non-zero contribution to the sum giving $Z(\Gamma)$ are the ones whose boundary components have the same index on all edges — that is, we need only account for graphs equipped with a map $c : \Gamma(2) \to \{1, \ldots, N\}$. An edge whose sides are indexed $i, j$ brings in a factor $2/(\Lambda_i + \Lambda_j)$; combining this with (4.2), we can conclude the proof.

**Example 4.2** (The standard matrix model). Take $\Lambda = I$; formula (4.1) specializes to
\[
Z(\Gamma) = \sum_{c} \prod_{\ell \in \Gamma(1)} \frac{2}{\Lambda_c(\ell^+) + \Lambda_c(\ell^-)} = \sum_{c} 1 = N^{\vert \Gamma(2) \vert}.
\]
Therefore, according to Theorem 3.6,
\[
\int_{\mathfrak{H}} \exp \left\{ \frac{1}{\hbar} \sum_{j=1}^{\infty} \frac{\operatorname{tr} X^j}{j} \right\} d\mu_I(X) = \sum_{\Gamma} \frac{N^{\vert \Gamma(2) \vert} \hbar^{-\vert \Gamma(0) \vert}}{\vert \text{Aut } \Gamma \vert}.
\] (4.3)
This is known as the “standard matrix model” in physics literature.

**Example 4.3** (The orbifold Euler characteristics of $M^n_g$). The Feynman-Reshetikhin-Turaev expansion formula for the Kontsevich model gives
\[
\int_{\mathfrak{H}} \exp \left\{ \sum_{j=1}^{\infty} x_j \frac{\operatorname{tr} X^j}{j} \right\} d\mu_X(X) = \sum_{\Gamma} Z_{x_1}(\Gamma) \frac{Z_{x_2}(\Gamma)}{\vert \text{Aut } \Gamma \vert},
\] (4.4)
where $\mathfrak{R}(0,0)$ is the set of isomorphism classes of (possibly disconnected) ribbon graphs of type $(0,0)$.

Now set
\[
x_1 = x_2 = 0, \quad x_j = (\sqrt{-1})^{j-1} \frac{\sqrt{2}}{j}, \quad \Lambda = I,
\]
to find the graph enumeration formula [3]:
\[
\log \int_{\mathfrak{H}} \exp \left\{ \frac{1}{t} \sum_{j=3}^{\infty} (\sqrt{-1})^{j} \frac{\operatorname{tr} X^j}{j} \right\} d\mu_I(X) = \sum_{g,n} \sum_{\Gamma \in \mathfrak{R}^n_g} \frac{(-1)^{\vert \Gamma(1) \vert}}{\vert \text{Aut } \Gamma \vert} t^{2g-2+n} N^n,
\] (4.5)
where $\mathfrak{R}^n_g$ denotes the set of isomorphism classes of connected genus $g$ ribbon graphs with $n$ boundary components. It is well known (cf. [2, 3]) that the moduli space $M^n_g$ of genus $g$ smooth complex curves with $n$ (non-ordered) punctures has an orbifold triangulation whose cells are indexed by elements of $\mathfrak{R}^n_g$. Moreover, the local isotropy group of the cell $\Delta_\Gamma$ defined by $\Gamma$ is isomorphic to $\text{Aut } \Gamma$, so the orbifold Euler characteristic of $M^n_g$ can be computed to be
\[
\chi^{\text{orb}}(M^n_g) = \sum_{\Gamma \in \mathfrak{R}^n_g} \frac{(-1)^{\dim \Delta_\Gamma}}{\vert \text{Aut } \Gamma \vert},
\]
see [2]. Since $\dim \Delta_\Gamma = \vert \Gamma(1) \vert$, formula (4.3) above is actually a generating series for the orbifold Euler characteristic of moduli spaces; indeed,
\[
\log \int_{\mathfrak{H}} \exp \left\{ \frac{1}{t} \sum_{j=3}^{\infty} (\sqrt{-1})^{j} \frac{\operatorname{tr} X^j}{j} \right\} d\mu(X) = \sum_{g,n} \chi^{\text{orb}}(M^n_g) t^{2g-2+n} N^n.
\]
Appendix A. PROPs and Operads

The concept and structure of a PROP provides adequate language to state properties of the graph families \( T, \mathcal{R}, \mathcal{I} \) from a categorical point of view. Since the definition of a PROP seems not to be widely known, we recall here its axioms; our version is actually a generalization of the one in [22, pp. 37–44], namely, we allow for the category of indices to be an arbitrary monoidal one. We refer the interested reader to [22, 23, 24, 25] for a more thorough discussion and background.

**Definition A.1.** A (non-symmetric) PROP is the data of \((\text{Ob}_A, \text{Hom}_A, \mathcal{P}, \circ, \otimes, E, \Delta, j, \phi^!, \phi^?)\) where

i) \( \text{Ob}_A \) and \( \text{Hom}_A \) are tensor categories, called respectively the **category of objects** and the **category of morphisms**; moreover \( \text{Hom}_A \) is symmetric;

ii) \( \mathcal{P} : (\text{Ob}_A)^{op} \times \text{Ob}_A \to \text{Hom}_A \) is a functor, called the Hom-space functor;

iii) \( \Delta_{A,B,C} : \mathcal{P}(B,C) \otimes \mathcal{P}(A,B) \to \mathcal{P}(A,C) \) is a natural transform, called the composition map;

iv) \( \otimes_{A,B,C,D} : \mathcal{P}(A,B) \otimes \mathcal{P}(C,D) \to \mathcal{P}(A \otimes C, B \otimes D) \) is a natural transform;

v) \( (E, \Delta) \) is a co-associative co-algebra in \( \text{Hom}_A \);

vi) \( j \) is a functorial map \( j_A : E \to \mathcal{P}(A,A) \), called the identity element;

vii) \( \phi^!_X : X \to E \otimes X \) and \( \phi^?_X : X \to X \otimes E \) are natural transformations; such that all the diagrams of Figure A on page 28 and Figure A on page 30 commute.

We say that \((\text{Ob}_A, \text{Hom}_A, \mathcal{P}, \circ, \otimes, E, \Delta, j, \phi^!, \phi^?)\) is a \( \text{Hom}_A \)-PROP over \( \text{Ob}_A \); we denote it just by the symbol \( \mathcal{P} \).

**Remark A.2.** Unless the contrary is explicitly stated, \( E \) is the unit object of \( \text{Hom}_A \) and \( \Delta, \phi^!, \phi^? \) are defined in terms of the natural transformations \( l \) and \( r \) that are part of the tensor category structure on \( \text{Hom}_A \). By abuse of language, elements of \( \text{Hom}_A(E, \mathcal{P}(A,B)) \) are called elements of \( \mathcal{P}(A,B) \) and we write \( f \in \mathcal{P}(A,B) \) to mean \( f : E \to \mathcal{P}(A,B) \).

The commutativity of \((\mathcal{O}1)\) means that composition of morphisms in the PROP is associative. Diagram \((\mathcal{O}2)\) expresses the condition

\[
(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k).
\]

Similarly, diagrams \((\mathcal{O}3)\) and \((\mathcal{O}4)\) express both the fact that \( j_A \) acts as an identity and the compatibility condition

\[
j_A \otimes j_B = j_{A \otimes B}.
\]

Finally, \((\mathcal{O}5)\) and \((\mathcal{O}6)\) state that \( E \) is an identity element for the tensor product \( \otimes \).

The above requirements, admittedly cumbersome and obscure, may be clarified by the following construction. The coassociativity of the comultiplication \( \Delta \) allows one to define a new category \( A_{\mathcal{P}} \) with:

i) the same objects as \( \text{Ob}_A \);

ii) morphisms given by

\[
A_{\mathcal{P}}(A,B) := \text{Hom}_A(E, \mathcal{P}(A,B)).
\]

Composition of two morphisms \( f \in A_{\mathcal{P}}(B,C) \) and \( g \in A_{\mathcal{P}}(A,B) \) is defined by

\[
E \xrightarrow{\Delta} E \otimes E \xrightarrow{j \otimes g} \mathcal{P}(B,C) \otimes \mathcal{P}(A,B) \xrightarrow{\mu} \mathcal{P}(A,C).
\]

The category \( A_{\mathcal{P}} \) is called the category **underlying** the PROP. By the commutativity of \((\mathcal{O}3)\), \( j_A \in \mathcal{P}(A,A) \) is the identity element of \( A \in A_{\mathcal{P}} \). It is an easy exercise to verify that \( A_{\mathcal{P}} \) is a monoidal category with the tensor product \( \otimes \). Note that every tensor category \( A \) defines a Vect-PROP by setting \( \mathcal{P}_A(X,Y) := \text{Hom}(X,Y) \),
for any two objects $X$, $Y$ of $A$. Since the unit object of $\text{Vect}$ is the base field $k$, the canonical isomorphism
\[
\text{Hom}(k, \mathcal{P}_A(X, Y)) = \text{Hom}(k, \text{Hom}(X, Y)) \cong \text{Hom}(X, Y)
\]
gives a natural identification of $A$ with the underlying category of $\mathcal{P}_A$.

Maps $\varphi^l$ and $\varphi^r$ allow one to look at elements of $\mathcal{P}(B, C)$ as left operators on $\mathcal{P}(A, B)$ (resp. elements of $\mathcal{P}(A, B)$ as right operators on $\mathcal{P}(B, C)$) with values in $\mathcal{P}(A, C)$. Indeed, the maps
\[
A_{\text{Hom}}(E, \mathcal{P}(B, C)) \to A_{\text{Hom}}(\mathcal{P}(A, B), \mathcal{P}(A, C))
\]
and
\[
A_{\text{Hom}}(E, \mathcal{P}(A, B)) \to A_{\text{Hom}}(\mathcal{P}(B, C), \mathcal{P}(A, C))
\]
are defined by
\[
\mathcal{P}(B, C) \ni f \mapsto \{ \mathcal{P}(A, B) \xrightarrow{\varphi^l} E \otimes \mathcal{P}(A, B) \xrightarrow{f \otimes 1} \mathcal{P}(B, C) \otimes \mathcal{P}(A, B) \xrightarrow{\varphi^r} \mathcal{P}(A, C) \}
\]
and
\[
\mathcal{P}(A, B) \ni f \mapsto \{ \mathcal{P}(B, C) \xrightarrow{\varphi^r} \mathcal{P}(B, C) \otimes E \xrightarrow{1 \otimes f} \mathcal{P}(B, C) \otimes \mathcal{P}(A, B) \xrightarrow{\varphi^l} \mathcal{P}(A, C) \}.
\]
The commutativity of diagrams $[\text{O5}]$ expresses compatibility of this action with the composition of morphisms in the PROP; commutativity of $[\text{O6}]$ states that the identity elements act trivially.

**Definition A.3.** A braided (resp. symmetric) PROP is a PROP with the additional datum of a natural morphism $\sigma_{A,B} : E \to \mathcal{P}(A \otimes B, B \otimes A)$ inducing a braided (resp. symmetric) category structure on the category underlying the PROP.

**Example A.4.** Any braided (symmetric) tensor category is a braided (symmetric) $\text{Vect}$-PROP. The category of Hilbert spaces is a symmetric Banach spaces PROP.

Moduli spaces of stable curves are an example of a symmetric algebraic-stacks-PROP over the monoidal category $\mathbb{N}$ of natural numbers.

**Definition A.5.** A PROP $\mathcal{P}$ is **linear** iff its category of Hom-spaces is a subcategory of $\text{Vect}$.

Let $\mathcal{P}$ be a linear PROP.

**Definition A.6.** Let $G$ be a subset of the set $\bigcup_{A,B} \mathcal{P}(A, B)$ of PROP operations. The sub-PROP $\mathcal{P}_G$ generated by $G$ is the smallest sub-PROP of $\mathcal{P}$ containing $G$.

The linear PROP $\mathcal{P}$ is **generated** by $G$ iff $\mathcal{P}_G = \mathcal{P}$.

**Definition A.7.** Two PROPs $\mathcal{P} := (A'_\text{Ob}, A'_\text{Hom}, \ldots)$ and $\mathcal{P}'' := (A''_\text{Ob}, A''_\text{Hom}, \ldots)$ are deemed **comparable** iff there exists a monoidal functor $A'_\text{Hom} \to A''_\text{Hom}$.

**Definition A.8.** Given two comparable PROPs $\mathcal{P}'$ and $\mathcal{P}''$, define a morphism $\rho : \mathcal{P}' \to \mathcal{P}''$ to be a triple $(h, \rho_{\text{Ob}}, \rho_{\text{Hom}})$, where:
- $h : A'_\text{Hom} \to A''_\text{Hom}$ is a monoidal functor,
- $\rho_{\text{Ob}} : A'_\text{Ob} \to A''_\text{Ob}$ is a tensor functor,
- $\rho_{\text{Hom}}$ is a natural transformation between the functors $h \circ \mathcal{P}'(-, -)$ and $\mathcal{P}''(\rho_{\text{Ob}}-, \rho_{\text{Ob}}-)$. 
The triple \((h, \rho_{Ob}, \rho_{Hom})\) must satisfy conditions that express compatibility with the tensor structure on \(A_{Hom}\).

If \(\rho : P' \to P''\) is an epimorphism, then we say that \(P''\) is a PROP quotient of \(P'\).

If \(A'_{Ob} = A''_{Ob}, A'_{Hom} = A''_{Hom}, h = Id\) and \(\rho_{Ob} = Id\), then PROP quotients are characterized by kernels of maps \(\rho_{Hom}(A,B) : P'(A,B) \to P''(A,B)\).

Fix a vector space \(V\).

**Definition A.9.** The functor \(E_V : N^{opp} \times N \to \text{Vect}\) defined by

\[
E_V(m,n) := \text{Hom}(V^\otimes m, V^\otimes n).
\]

gives a structure of a \(\text{Vect}-\text{PROP}\) over \(N\) to the monoidal category of tensor powers of \(V\); it is called the *endomorphism PROP* of \(V\).

**Definition A.10.** Let \(P\) be a linear PROP over \(N\). An action of \(P\) on a linear space \(V\) is a morphism of PROPs \(P \to E_V\), that is a collection of maps \(P(p,q) \to \text{Hom}(V^\otimes p, V^\otimes q)\), satisfying obvious compatibility conditions. A representation of \(P\) is a pair \((V, \rho)\) where \(V\) is a vector space and \(\rho\) is an action of \(P\) on \(V\). If \((V, \rho)\) is a representation of \(P\), then \(V\) is called a \(P\)-algebra.

**Remark A.11.** PROPs are deeply related to operads, see [22, 26]. Indeed, if \(P\) is a symmetric \(A_{Hom}\)-PROP over \(N\), the collection

\[
\mathcal{O}_P(n) := P(n,1)
\]

is a \(A_{Hom}\)-operad. Conversely, if the collection \(\mathcal{O}(n)\) is a May operad,

\[
P_\rho(m,l) := \bigoplus_{m_1, m_2, \ldots, m_l \sum_{i} m_i = m} P(m_1) \otimes \cdots \otimes P(m_l)
\]

defines a symmetric PROP. Notice that \(\mathcal{O}_\rho \circ \rho = \mathcal{O}\), whereas it is only \(\mathcal{P}_{\rho,\rho} \subseteq \mathcal{P}\) in general.

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Table 1. Diagrams expressing compatibility relations of structure maps in a PROP, part I.

\[
\begin{align*}
P(C, D) \otimes (P(B, C) \otimes P(A, B)) & \overset{\alpha}{\longrightarrow} (P(C, D) \otimes P(B, C)) \otimes P(A, B) \\
& \\
P(C, D) \otimes P(A, C) & \overset{\text{Id} \otimes \circ}{\longrightarrow} P(A, C) \otimes P(A, B) \\
& \downarrow \circ \downarrow \circ \downarrow \circ \\
P(A, D) & \overset{\circ \otimes \text{Id}}{\longrightarrow} P(A, D) \\
& \\
(P(A, B) \otimes P(C, D)) \otimes (P(F, A) \otimes P(G, C)) & \overset{\text{shuffle}}{\longrightarrow} (P(A, B) \otimes P(F, A)) \otimes (P(C, D) \otimes P(G, C)) \\
& \\
(P(A \otimes C, B \otimes D) \otimes P(F \otimes G, A \otimes C) & \overset{(- \otimes) \otimes (- \otimes)}{\longrightarrow} (P(A \otimes C, B \otimes D) \otimes P(F \otimes G, A \otimes C) \\
& \downarrow \circ \downarrow \circ \downarrow \circ \\
P(F \otimes G, B \otimes D) & \overset{\circ \otimes \circ}{\longrightarrow} P(F \otimes G, B \otimes D) \\
& \\
\end{align*}
\]
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Table 2. Diagrams expressing compatibility relations of structure maps in a PROP, part II.

\[
\begin{align*}
E & \xrightarrow{\Delta} E \otimes E & E & \xrightarrow{\Delta} E \otimes E \\
\forall f & \downarrow & f \otimes j_A & \downarrow & j_B \otimes f & C_1 \\
\mathcal{P}(A, B) & \xrightarrow{\circ} \mathcal{P}(A, B) \otimes \mathcal{P}(A, A) & \mathcal{P}(A, B) & \xrightarrow{\circ} \mathcal{P}(B, B) \otimes \mathcal{P}(A, B)
\end{align*}
\]

\[
\begin{align*}
E \otimes E & \xrightarrow{\varphi^r} E & E \otimes E & \xrightarrow{\varphi^l} E \\
\downarrow & j_A \otimes j_B & \downarrow & j_A \otimes j_B & \downarrow & j_A \otimes j_B & C_2 \\
\mathcal{P}(A, A) \otimes \mathcal{P}(B, B) & \xrightarrow{\circ} \mathcal{P}(A \otimes B, A \otimes B) & \mathcal{P}(A, A) \otimes \mathcal{P}(B, B) & \xrightarrow{\circ} \mathcal{P}(A \otimes B, A \otimes B)
\end{align*}
\]

\[
\begin{align*}
X & \xrightarrow{\varphi_X} E \otimes X & X & \xrightarrow{\varphi_X} X \otimes E \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
E \otimes X & \xrightarrow{\Delta \otimes \text{Id}_X} E \otimes E \otimes X & X \otimes E & \xrightarrow{\text{Id}_X \otimes \Delta} X \otimes E \otimes E & X \otimes E & \xrightarrow{\text{Id}_X \otimes \Delta} X \otimes E \otimes E
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}(A, B) & \xrightarrow{\varphi^l} E \otimes \mathcal{P}(A, B) & \mathcal{P}(A, B) & \xrightarrow{\varphi^l} \mathcal{P}(A, B) \otimes E \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{P}(A, B) & \xrightarrow{\circ} \mathcal{P}(B, B) \otimes \mathcal{P}(A, B) & \mathcal{P}(A, B) & \xrightarrow{\circ} \mathcal{P}(A, B) \otimes \mathcal{P}(A, A)
\end{align*}
\]

(O3)  
(O4)  
(O5)  
(O6)