Abstract—In this paper, the alternating direction method of multipliers (ADMM) is investigated for distributed optimization problems in a networked multi-agent system. In particular, a new adaptive-gain ADMM algorithm is derived in a closed form and under the standard convexity property in order to greatly speed up convergence of ADMM-based distributed optimization. Using Lyapunov direct approach, the proposed solution embeds control gains into weighted network matrix among the agents and uses those weights as adaptive penalty gains in the augmented Lagrangian. It is shown that the proposed closed loop gain adaptation scheme significantly improves the convergence speed of underlying ADMM optimization. Convergence analysis is provided and simulation results are included to demonstrate the effectiveness of the proposed scheme.

Index terms: Distributed optimization, ADMM, gain adaptation, rate of convergence, Lyapunov direct method.

I. INTRODUCTION

In the era of Internet of Things (IoT) and smart agents, the amount of data available in the network explodes in both size and complexity. In such a multi-agent setting, distributed algorithm for network optimization becomes lucrative and practical since it is not always efficient to gather all the information for a centralized computation [1]. The distributed optimization algorithm must be capable of collecting localized data across a connected network of agents, and a common solution must emerge among individual agents without requiring any centralized coordination. In the last decade, the Alternating Direction Method of Multipliers (ADMM) has received much attention due to its ability of decomposing complex optimization problems into a sequence of simpler sub-problems that can be solved asymptotically under certain convex properties [2]. ADMM was first introduced by Glowinski & Marroco [3] and by Gabay & Mercier [4]. Most recently, it has been applied to many applications in such areas as image processing [5], machine learning [6], resource allocation [7], power system optimization [8] etc. These diverse applications also demand a detailed study of ADMM convergence properties [9], [10].

The convergence speed of ADMM relies on the selection of penalty parameters [11], which is often manually chosen by the user for a specific problem setup. Convergence rate of ADMM is studied, and earlier work include [9], [12]. It is now well established in the literature that, if the objective functions are strongly convex and have Lipschitz-continuous gradient, the basic ADMM algorithms have global linear convergence [13], [14]. The strong convexity conditions are relaxed in [15], and a constant $O(1/n)$ convergence rate is achieved under mild convex assumptions. It is shown in [16] that convergence can be achieved in $O(1/n^2)$ time if at least one of the objective functions is strongly convex. These specific results all use constant penalty parameters and, in practical applications, efficiency of ADMM is highly sensitive to parameter choices and could be improved via adaptive penalty selection [17]–[19].

The first approach that comes intuitively is to use different penalty parameters in each iteration. In He et al. [17], an adaptive penalty based on the relative magnitude of primal and dual residuals is proposed to balance their magnitudes. In [11], primal and dual residuals are also used to improve a defined convergence factor while solving a class of quadratic optimization problem using ADMM. In both these cases, the ADMM algorithm is shown to converge, but global computation of primal and dual residuals are required and hence the resulting algorithm is no longer distributed. In [20], distributed ADMM is implemented to minimize locally known convex functions over a network, and the effect of communication weights on the convergence rate is investigated. In [21], the weighted network matrix is adaptively tuned to improve convergence in a consensus-based distributed problem framework using cooperative control. This idea is used in [22] where a consensus based distributed ADMM is formulated with a predefined network structure, for which primal and dual residuals are balanced locally by each agent. However, their adaptive penalty needs to be reset after several iterations to guarantee convergence, which results in much weakened convergence conditions. More recently, adaptive penalty parameters are used [2] to improve convergence speed by estimating the local curvatures of the dual functions. However, as pointed out in [23], an increase in the number of nodes causes the local curvature estimation to be inaccurate and possibly unstable.

In this paper, a Lyapunov-based analytical design methodology is proposed to synthesize adaptive penalty parameters for ADMM to ensure convergence and improve convergence time, all in a multi-agent setting. The proposed distributed ADMM algorithm is designed in four steps. First, distributed control gains are embedded into a row-stochastic weighted network connectivity matrix to ensure consistency of the ADMM between its constraints and network connectivity. Second, the entries of weighted network matrix are embedded as the penalty parameters into the augmented Lagrangian...
for ADMM so they can be adjusted in a distributed manner for each agent to use its local information and optimize its local objective function. Third, utilizing the convex property of the individual agents’ objective functions, the standard ADMM formulation is applied to the newly formulated augmented Lagrangian, the resulting ADMM algorithm with adaptive gains is shown to be asymptotically convergent, and its iterative ADMM updating laws are derived. Fourth, using the Lyapunov direct method, adaptive gain updating laws are analytically synthesized, and the improvement of convergence is proven.

The rest of the paper is organized as follows. In section II, the proposed adaptive-gain ADMM algorithm is developed. In section III, the main results of gain adaptation and convergence improvement are established. Superior performance of the proposed ADMM algorithm is shown through comparative simulation studies in section IV. And, section V contains the conclusions.

II. PROBLEM FORMULATION

In this section, a general multi-agent distributed optimization problem is presented for a network of agents and then converted into the ADMM framework. Based on network connectivity and the convex property of objective functions, optimum search iterates at each agent are derived to solve the ADMM sub-problems. Using these iterates and local communication among agents, an adaptive gain updating algorithm is synthesized to further improve convergence of the ADMM algorithm to an optimal solution.

A. ADMM Based Distributed Algorithm

Let us consider the following distributed optimization problem with a conforming communication topology among agents:

\[
\begin{align*}
\min & \quad \sum_{i \in \mathcal{N}} f_i(x_i) \\
\text{s.t.} & \quad \sum_{j \in \mathcal{N}_i} A_{ij} x_j = 0 \quad \text{for} \quad i \in \mathcal{N},
\end{align*}
\]

where \( \mathcal{N} = \{1, 2, \ldots, N\} \) is the set of agents. For agent \( i \), \( \mathcal{N}_i \) denotes the set of its neighbors including itself, \( x_i \in \mathbb{R}^n \) is its state vector, \( f_i(x_i) \) is its objective function, and \( A_{ij} \) are matrices of appropriate dimensions in the linear constrained equations representing the interconnection of the physical layer. The problem can be perceived as each agent optimizing its own objective function \( f_i(x_i) \) while satisfying the interconnection constraint of (1b). The following assumption is made on the individual objective functions.

**Assumption 1:** Functions \( f_i, i \in \mathcal{N} \), are convex and differentiable, and their gradients denoted by \( \nabla x_i f_i(x_i) \) are locally Lipschitz. The set of optimal solutions to (1) is nonempty, and the corresponding minimum of (1a) is finite.

B. Network of Agents

Local communication in the network is characterized by a bidirectional graph \( \mathcal{G} = (\mathcal{N}, \mathcal{E}) \); specifically, its sense-communication matrix is binary and of form [24]:

\[
S = \begin{bmatrix}
1 & s_{12} & \cdots & s_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
s_{N1} & s_{N2} & \cdots & 1
\end{bmatrix}.
\]

Matrix \( S \) has 1 in the diagonal as every agent knows its own information, and it is equal to the sum of the adjacency matrix and identity matrix. The following assumption ensures conformity and connectivity of the network, and the multi-agent system is visualized in figure 1.

**Assumption 2:** The communication graph conforms with system constraints in the sense that, if \( A_{ij} \neq 0 \) or \( A_{ji} \neq 0 \), \( s_{ij} = s_{ji} = 1 \). And, the communication graph is connected, i.e., matrix \( S \) is irreducible.

C. Adaptive-Gain ADMM

To solve (1) using ADMM, we introduce a set of auxiliary variables, \( z_{ij} \), which are the estimates of agent \( j \)’s variables by agent \( i \) [8]. Then, problem (1) can be restated as

\[
\begin{align*}
\min & \quad \sum_{i \in \mathcal{N}} f_i(x_i) \\
\text{s.t.} & \quad \sum_{j \in \mathcal{N}_i} A_{ij} z_{ji} = 0 \quad \text{for} \quad i \in \mathcal{N},
\end{align*}
\]

where \( z_{ij} \) are relaxation variables used in the standard ADMM [19]. All the agents related to the \( i \)th agent try to make their estimates \( z_{ij} \) of \( x_i \) reach consensus so that a solution to (3) converges to an optimal solution \( x^* \) of the original problem (1). The goal of this reformulation is to solve the optimization problem in a distributed fashion that agent \( i \) solves its own optimization sub-problem by exchanging information with its neighboring nodes in set \( \mathcal{N}_i \). To this end, we form the so-called augmented Lagrangian as:

\[
L_D(x, z, \lambda, \mu) = \sum_{i \in \mathcal{N}} L_i(x_i, z_{ij}, \lambda_{ij}, \mu_i),
\]

where \( \lambda_{ij} \) and \( \mu_i \) \( j \in \mathcal{N}_i, i \in \mathcal{N}_i \), are the Lagrange multipliers (dual variables) associated with the constraints,

\[
L_i = f_i(x_i) + \sum_{j \in \mathcal{N}_i} \left[ \lambda_{ij}^T (x_i - z_{ij}) + \frac{\mu_i}{2} ||x_i - z_{ij}||^2 \right] \\
+ \mu_i^T \sum_{j \in \mathcal{N}_i} A_{ij} z_{ji} + \frac{\mu_i}{2} \sum_{j \in \mathcal{N}_i} ||A_{ij} z_{ji}||^2,
\]
and $d_{ij} \geq 0$ are regularized but time-varying penalty parameters from a row-stochastic gain matrix $D^k$, and $w_{ij} > 0$ is the penalty parameter associated with constraint (5b). The augmented Lagrangian reduces to a standard Lagrangian $L_0$ when the penalty terms are removed (i.e., $d_{ij} = 0$ for all $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$).

To ensure that the proposed adaptive scheme is consistent with local communication network, gain matrix $D^k \in \mathbb{R}^{n \times n}$ are locally calculated (by row) as

$$D^k = \begin{bmatrix} d_{1j}^k \\ \vdots \\ d_{nj}^k \end{bmatrix}, \quad d_{ij}^k = \frac{s_{ij}^k \beta_{ij}^k}{\sum_{l=1}^n s_{il} \beta_{il}^k},$$

(5)

where $k \in \mathbb{N}^+$ is the discrete time step, and $\beta_{ij}^k \geq 0$ as local scalar gains (with initial gain values of $\beta_{ij}^0 > 0$). Entries $d_{ij}^0$ (or equivalently $\beta_{ij}^0$) will be updated real-time according to the proposed design. Clearly, gain matrix $D^k$ is non-negative, row-stochastic and diagonally positive. The entries $d_{ij}$ become a constant penalty parameter $\rho$, the proposed design reduces to the standard ADMM algorithm [19].

The ADMM algorithm consists of a $z$-minimization step, a $z$-minimization step, and an update of dual variables. The proposed ADMM algorithm is obtained by applying these steps to the above reformulation, that is,

1. For any $i \in \mathcal{N}$, $x_i$ is updated according to

$$x_i^{k+1} := \arg \min_{x_i \in \mathbb{R}^n} L_D(x_i, z_i^k, \mu_i^k, \lambda_i^k)$$

(6a)

2. For any $i \in \mathcal{N}$ and for $j \in \mathcal{N}_i$, $z_{ij}$ is solved as

$$z_{ij}^{k+1} := \arg \min_{z_{ij} \in \mathbb{R}^{n \times n}} L_D(x_i^{k+1}, z_i^k, \mu_i^k, \lambda_i^k)$$

(6b)

3. For any $i \in \mathcal{N}$ and for $j \in \mathcal{N}_i$, $\mu_i$ and $\lambda_{ij}$ evolves as

$$\mu_i^{k+1} := \mu_i^k + w_i \sum_{j \in \mathcal{N}_i} A_{ij} z_{ij}^{k+1}$$

(6c)

$$\lambda_{ij}^{k+1} := \lambda_{ij}^k + d_{ij}^k (x_i^{k+1} - z_{ij}^{k+1})$$

(6d)

Convergence property of the proposed ADMM algorithm [6] for primal-dual sequences of $\{x_i^k, z_{ij}^k\}$ and $\{\lambda_{ij}^k, \mu_i^k\}$ is summarized as the following lemma, and its proof included in Appendix extends the existing ADMM results to version [6] with time-varying gains.

**Lemma 1**: Under assumptions 1 and 2, ADMM algorithm [6] is convergent to an optimal solution.

**D. Iterative Laws of ADMM**

Under assumption 1, the $i$th agent can solve the suboptimization problems in [6] iteratively. In particular, the gradient descent technique can be applied to the $z$-minimization step, while the $x$-minimization step has a closed form solution once $x_i^{k+1}$ is determined. Hence, the ADMM iterative algorithm becomes: for agent $i$,

$$\hat{x}_i^{k+1} = \hat{x}_i^k - \alpha_i \left[ \nabla_{\hat{x}_i} f_i(x_i^{k}) + \sum_{j \in \mathcal{N}_i} [\hat{\lambda}_{ij}^k + d_{ij}^k (\hat{x}_i^k - \hat{z}_{ij}^k)] \right]$$

(7a)

$$\hat{z}_{ij}^{k+1} = \hat{z}_{ij}^k + \frac{1}{d_{ij}^k} \left[ \hat{\lambda}_{ij}^k - A_{ij}^{T} \hat{\mu}_i^k - w_i A_{ij}^{T} \sum_{\phi \in \mathcal{N}_i} A_{ij} \hat{z}_{i\phi}^k \right]$$

(7b)

$$\hat{\mu}_i^{k+1} = \hat{\mu}_i^k + w_i \sum_{j \in \mathcal{N}_i} A_{ij} \hat{z}_{ij}^{k+1}$$

(7c)

$$\hat{\lambda}_{ij}^{k+1} = \hat{\lambda}_{ij}^k + d_{ij}^k (\hat{x}_i^{k+1} - \hat{z}_{ij}^{k+1})$$

(7d)

where $0 < \alpha_i << 1$ is the step size. The iterative algorithm [7] is a gradient-based optimization and, under convexity, it is guaranteed that its trajectory moves closer to an optimal solution. Hence, the convergence proof of algorithm [7] is neglected for brevity.

In the implementation, agent $i$ updates not only its own state vector $x_i$ but also estimates $z_{ij}$ of its neighboring agents’ states as well as the associated Lagrange multipliers $\lambda_{ij}$ and $\mu_i$. This information flow for updating the iterates are shown in figure 2.

In most of the existing ADMM literature, penalties $d_{ij}$ are set to be constant and identical [13]–[16]. Advantage of using adaptive penalty is noted in [17]–[19], but those results either require global information or have convergence and scalability issues. These motivate us to develop the proposed adaptive penalty algorithm whose gain matrix is chosen constructively to retain the ADMM’s distributed nature while enhancing its scalability and convergence. Specifically, the $i$th agent dynamically adjusts its penalties (i.e., the $i$th row entries of matrix $D^k$) to improve convergence time of ADMM. This design objective is achieved through making the value of an appropriate Lyapunov function decrease more at each of the iteration steps. This idea was first applied successfully to cooperative control among a network of cooperative agents in [21]. An application of this idea to ADMM is pursued in the next section.

**III. IMPROVEMENT OF CONVERGENCE RATE VIA ADAPTIVE GAIN**

At each of the iteration steps, convergence of ADMM algorithm [7] can be measured using the following Lyapunov...
function by agent $i$:
\[
E_{i}^{k} = \|x_{i}^{k}\|^2 + \|\mu_{i}^{k}\|^2 + \sum_{j \in N_{i}} \left( \|z_{ij}^{k}\|^2 + \|\lambda_{ij}^{k}\|^2 \right),
\]  
where $\tilde{x}^{k} = \tilde{x}^{k} - \tilde{x}^{k-1}$, $\tilde{z}^{k} = \tilde{z}^{k} - \tilde{z}^{k-1}$, $\tilde{\lambda}^{k} = \tilde{\lambda}^{k} - \tilde{\lambda}^{k-1}$ and $\tilde{\mu}^{k} = \tilde{\mu}^{k} - \tilde{\mu}^{k-1}$ are incremental residues of the primal and dual variables. The following theorem provides the proposed distributed adaptive-gain algorithm, and its proof is included in the Appendix.

**Theorem 1:** Convergence of ADMM algorithm \([7]\) is improved if $E_{i}^{k+1}$ is made to be less through locally and adaptively choosing $\beta_{ij}^{k}$. Specifically, for each of $k \in \mathbb{N}^+$, only two of the penalties $d_{il}^{k}$ (equivalently, gains $\beta_{il}^{k}$) are adaptively adjusted as
\[
d_{il}^{k} = d_{il}^{k-1} + \epsilon_{i} \quad d_{im}^{k} = d_{im}^{k-1} - \epsilon_{i},
\]  
where indices $l_i$ and $m_i$ are determined according to
\[
l_i \in N_i \implies [\tilde{x}_i - \tilde{z}_{il}^{k}] = \max_{j \in N_i} [\nabla \hat{f}_i(x_i^{k}) \cdot (\tilde{x}_i^{k} - \tilde{z}_{ij}^{k})],
\]  
\[
m_i \in N_i \implies [\tilde{x}_i - \tilde{z}_{im}^{k}] = \min_{j \in N_i} [\nabla \hat{f}_i(x_i^{k}) \cdot (\tilde{x}_i^{k} - \tilde{z}_{ij}^{k})],
\]  
quantity
\[
h_i(\tilde{x}, \tilde{z}) = 2\alpha_i \tilde{x}_i \left[ (\tilde{x}_i^{k} - \tilde{z}_{il}^{k}) - (\tilde{x}_i^{k} - \tilde{z}_{im}^{k}) \right]
\]  
is calculated using the locally-available information, and adjustment $\epsilon_i$ is chosen to be: for some $0 < \gamma_i < 1$,
\[
\epsilon_i = \begin{cases} 
\gamma_i d_{il}^{k-1} & \text{if } h_i(\tilde{x}, \tilde{z}) > 0 \\
-\gamma_i d_{il}^{k-1} & \text{if } h_i(\tilde{x}, \tilde{z}) < 0 \end{cases}
\]  

### IV. Simulation Results

In this section, the proposed gain adaptation technique is illustrated through simulations and in two parts. First, the time trajectory of convergence error measure under the proposed adaptive-gain ADMM is compared to that under fixed penalties for a 5-agent network. Second, comparative studies are done for scaled-up networks up to 100 agents and for different network topologies so improvements of convergence speed are established together with scalability. In both cases, the error residual in the form of $\max_i \sum_{j \in N_i} \|x_i - z_{ij}\|$ is chosen to measure convergence, and the tolerance threshold of $1 \times 10^{-4}$ is used to either stop the simulation or determine the number of iterations required for convergence when comparing the ADMM algorithms. In the implementation of algorithms, the following choices are made: $\alpha_i = 0.1$, $w_i = 1$, $f_i(x_i)$ is convex and only known to the $i$th agent. As the base case, a ring-like topology where each agent is connected to two other agents is constructed. Then, two other topologies are generated on top of the ring-structure where agents are randomly interlinked up to a maximum of five other agents. For each of the scenarios generated, five sets of simulations with different initial conditions are run.

First, let's begin with the 5-agent ring network (whose connectivity matrix is cyclic). The iterative ADMM algorithm \([7]\) is implemented for each agent, and simulations are run twice: one with fixed gains (in which case $d_{ij}$ is computed using \([5]\) at $k = 0$ and then kept constant), and another with adaptive gains (whose initial values are calculated the same way and then the gains are updated over time according to theorem 1). Comparison of convergence under adaptive-gain ADMM versus fixed-penalty ADMM is shown in figure \([3]\).

![Fig. 3: Convergence errors: fixed gain versus adaptive penalty](image)

In the second set of simulation studies, the network and its scaled-up versions up to 100 agents are simulated for various initial conditions and with different connected network topologies as described above. Again, each simulation setup is repeated twice: one of constant penalty, and another with adaptive gains. For a network of certain agents, convergence times are recorded for different initial conditions and randomly generated network topologies, and their average is recorded. The iteration limit is set to 30,000. Table \([1]\) provides comparative summary results for networks of different sizes.

| Number of agents | Fixed penalty | Adaptive penalty |
|------------------|--------------|------------------|
| 5                | 765          | 471              |
| 10               | 11138        | 573              |
| 25               | 5934         | 1924             |
| 50               | 11908        | 7342             |
| 100              | Exceeds iteration limit | 20369 |

The results of our two-part studies clearly show convergence improvements for the proposed adaptive ADMM algorithm.

### V. Conclusion

In this paper, a distributed multi-agent ADMM algorithm with adaptive gains is developed. The convex properties are utilized to obtain closed form iterative dynamics for the optimization sub-problems. In contrast to the standard ADMM which uses a fixed penalty gain in the augmented Lagrangian, the proposed algorithm embeds control gains into a row-stochastic matrix based on network connectivity, utilizes the matrix coefficients as the penalty parameters in ADMM, and uses information received by each agent from its neighbors to adaptively adjust these penalties. The proposed adaptive algorithm is both distributed and of closed form, and it substantially improves the rate of ADMM.
agents’ convergence to an optimal solution. The improvement is analytically shown by the Lyapunov direct approach. Numerical simulation demonstrates the effectiveness of the proposed adaptive-gain ADMM.

REFERENCES

[1] T.-H. Chang, M. Hong, and X. Wang, “Multi-agent distributed optimization via inexact consensus ADMM,” IEEE Transactions on Signal Processing, vol. 63, pp. 482–497, 2015.
[2] Z. Xu, M. A. Figueiredo, and T. Goldstein, “Adaptive ADMM with spectral penalty parameter selection,” arXiv:1605.07246, 2016.
[3] R. Glowinski and A. Marroco, “Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité d’une classe de problèmes de dirichlet non linéaires,” Revue Française d’Automatique, Informatique, Recherche Opérationnelle. Analyse Numérique, vol. 9, pp. 41–76, 1975.
[4] D. Gabay and B. Mercier, “A dual algorithm for the solution of nonlinear variational problems via piecewise linear approximation,” Computers & Mathematics with Applications, vol. 2, pp. 17-40, 1976.
[5] M. A. Figueiredo and J. M. Bioucas-Dias, “Restoration of poissonian images using alternating direction optimization,” IEEE Transactions on Image Processing, vol. 19, pp. 3133–3145, 2010.
[6] P. A. Forero, A. Cano, and G. B. Giannakis, “Consensus-based distributed support vector machines,” Journal of Machine Learning Research, vol. 11, pp. 1603–1707, 2010.

APPENDIX

A. Proof of Lemma 1

Let’s begin with defining the following error terms: for any \( i \in \mathcal{N} \), for \( j \in \mathcal{N}_i \), and for \( k \in \mathbb{N} \),

\[
r_{ij}^{k+1} = \frac{x_{ij}^{k+1} - z_{ij}^{k+1}}{q_{ij}^{k+1}},
\]

or equivalently,

\[
\sum_{i \in \mathcal{N}} f_i(x_{i}^{k+1}) + \sum_{j \in \mathcal{N}_i} \lambda_{ij}^{k+1} T_{ij}^{k+1} + \sum_{j \in \mathcal{N}_i} \mu_{ij}^{k+1} q_{ij}^{k+1}.
\]

Since the optimal solution satisfies the constraints, we know \( x_{i}^{k+1} - z_{ij}^{k+1} = 0 \) and \( \sum_{j \in \mathcal{N}_i} A_{ij} z_{ji}^{k+1} = 0 \). Hence, the above inequality becomes

\[
p^* \leq p^{k+1} + \sum_{i \in \mathcal{N}} \left( \lambda_{ij}^{k+1} T_{ij}^{k+1} + \mu_{ij}^{k+1} q_{ij}^{k+1} \right),
\]

where

\[
p^* = \sum_{i \in \mathcal{N}} f_i(x_{i}^{k+1}) \quad \text{and} \quad p^{k+1} = \sum_{i \in \mathcal{N}} f_i(x_{i}^{k+1}).
\]

Also, it follows that

\[
\lambda_{ij}^{k+1} - \lambda_{ij}^{k} = \left( \lambda_{ij}^{k} - \lambda_{ij}^{k} \right) - \left( \lambda_{ij}^{k} - \lambda_{ij}^{k} \right),
\]

\[
z_{ij}^{k+1} - z_{ij}^{k} = \left( z_{ij}^{k} - z_{ij}^{k} \right) + \left( z_{ij}^{k} - z_{ij}^{k} \right),
\]

\[
\mu_{ij}^{k+1} - \mu_{ij}^{k} = \left( \mu_{ij}^{k} - \mu_{ij}^{k} \right) - \left( \mu_{ij}^{k} - \mu_{ij}^{k} \right).
\]

It follows from (6a) that, for agent \( i \),

\[
0 = \nabla x_i f_i(x_{i}^{k+1}) + \sum_{j \in \mathcal{N}_i} \lambda_{ij}^{k+1} + \sum_{j \in \mathcal{N}_i} d_{ij}^{k} \left[ x_{ij}^{k+1} - z_{ij}^{k} \right].
\]

Substituting (6d) into the above equation yields

\[
0 = \nabla x_i f_i(x_{i}^{k+1}) + \sum_{j \in \mathcal{N}_i} \lambda_{ij}^{k+1} + \sum_{j \in \mathcal{N}_i} d_{ij}^{k} \left[ x_{ij}^{k+1} - z_{ij}^{k} \right] - \sum_{j \in \mathcal{N}_i} \lambda_{ij}^{k+1} + \sum_{j \in \mathcal{N}_i} d_{ij}^{k} \left[ x_{ij}^{k+1} - z_{ij}^{k} \right].
\]

The above equation implies that \( x_{ij}^{k+1} \) also minimizes

\[
f_i(x_{i}^{k+1}) + \sum_{j \in \mathcal{N}_i} \left[ \lambda_{ij}^{k+1} + d_{ij}^{k} x_{ij}^{k+1} - z_{ij}^{k} \right] T_{ij}^{k+1} x_{i}^{k+1}.
\]
Similarly, it follows from (6b) that
\[ 0 = -\lambda_{ij}^k - \lambda_{ij}^k \left[ x_{ij}^{k+1} - z_{ij}^k \right] + A^T_{ji} \mu_{ji}^{k+1} + w_{ij}^k A^T_{ji} \sum_{\phi \in N_i} A_{\phi} \varepsilon_{i\phi}^{k+1}. \]
Substituting (6c) and (6d) in the above equation yields
\[ 0 = -\lambda_{ij}^k + A^T_{ji} \mu_{ji}^{k+1}. \] (20)

Applying Lagrange duality to (19), we have
\[ \sum_{i \in N} \left[ f_i(x_i^{k+1}) + \sum_{j \in N_i} \left[ \lambda_{ij}^{k+1} + d_{ij}^k (z_{ij}^{k+1} - z_{ij}^k) \right] T x_i^{k+1} \right] \]
\[ \leq \sum_{i \in N} \left[ f_i(x_i^*) + \sum_{j \in N_i} \left[ \lambda_{ij}^{k+1} + d_{ij}^k (z_{ij}^{k+1} - z_{ij}^k) \right] T x_i^* \right]. \]

It follows from (20) that
\[ \sum_{i \in N} \sum_{j \in N_i} \left[ -\lambda_{ij}^{k+1} + A^T_{ji} \mu_{ji}^{k+1} \right] T x_i^{k+1} \]
\[ = \sum_{i \in N} \sum_{j \in N_i} \left[ -\lambda_{ij}^{k+1} + A^T_{ji} \mu_{ji}^{k+1} \right] T x_i^*. \]

Adding the above two expressions together and performing simple manipulations, we obtain
\[ p^{k+1} - p^* \leq \sum_{i \in N} \sum_{j \in N_i} \left[ -\lambda_{ij}^{k+1} r_{ij}^{k+1} - (\mu_{ij}^{k+1})^T A_{ji} z_{ij}^{k+1} \right] \]
\[ \leq -d_{ij}^k (z_{ij}^{k+1} - z_{ij}^k) T (x_{ij}^{k+1} + z_{ij}^k - z_{ij}^*). \] (21)

Combining (14) and (21) together with (13) and multiplying both sides by 2 yield
\[ 0 \geq \sum_{i \in N} \sum_{j \in N_i} \left[ 2(\lambda_{ij}^{k+1} - \lambda_{ij}^*)^T r_{ij}^{k+1} + 2d_{ij}^k (z_{ij}^{k+1} - z_{ij}^*)^T \right] \]
\[ \leq 0 + 2(\mu_{ij}^{k+1} - \mu_{ij}^*)^T A_{ji} z_{ij}^{k+1}. \] (22)

Performing the substitutions indicated by the underbraces above, we have
\[ 0 \geq \sum_{i \in N} \sum_{j \in N_i} \left[ \frac{1}{d_{ij}^k} \right] \left[ (\lambda_{ij}^{k+1} - \lambda_{ij}^*)^T r_{ij}^{k+1} + d_{ij}^k (z_{ij}^{k+1} - z_{ij}^*)^2 \right] \]
\[ + d_{ij}^k (z_{ij}^{k+1} - z_{ij}^*)^2 + 2d_{ij}^k (z_{ij}^{k+1} - z_{ij}^*) T (z_{ij}^{k+1} - z_{ij}^*) \]
\[ + 2d_{ij}^k (z_{ij}^{k+1} - z_{ij}^*) T (z_{ij}^{k+1} - z_{ij}^*) + \frac{2}{w_i} \left( \mu_{ij}^{k+1} - \mu_{ij}^* \right)^T \]
\[ (\mu_{ij}^{k+1} - \mu_{ij}^*). \] (23)

Using the substitutions indicated by the underbraces above, we obtain
\[ 0 \geq \sum_{i \in N} \sum_{j \in N_i} \left[ \frac{1}{d_{ij}^k} \right] \left[ (\lambda_{ij}^{k+1} - \lambda_{ij}^*)^2 - (\lambda_{ij}^k - \lambda_{ij}^k)^2 \right] \]
\[ + \frac{1}{w_i} \left[ (\mu_{ij}^{k+1} - \mu_{ij}^*)^2 - (\mu_{ij}^k - \mu_{ij}^k)^2 \right] \]
\[ + d_{ij}^k (z_{ij}^{k+1} - z_{ij}^*)^2 + \frac{1}{w_i} \left( \mu_{ij}^{k+1} - \mu_{ij}^* \right)^2 \]
\[ \mu_{ij}^k (z_{ij}^{k+1} - z_{ij}^*)^2 + \frac{1}{w_i} \left( \mu_{ij}^{k+1} - \mu_{ij}^* \right)^2 \]. \]

Considering the following Lyapunov function
\[ V(k) = \sum_{i \in N} \sum_{j \in N_i} \left[ d_{ij}^k ||z_{ij}^{k+1} - z_{ij}^*||^2 + \frac{1}{d_{ij}^k} ||\lambda_{ij}^{k+1} - \lambda_{ij}^*||^2 \right] \]
\[ + \frac{1}{w_i} \left( \mu_{ij}^{k+1} - \mu_{ij}^* \right)^2 \], \] (23)
we can rewrite inequality (22) in terms of the Lyapunov function as
\[ V^{k+1} - V^k \leq -\sum_{i \in N} \sum_{j \in N_i} \left[ d_{ij}^k ||r_{ij}^{k+1} + (z_{ij}^{k+1} - z_{ij}^*)||^2 \right] \]
\[ + \frac{1}{w_i} \left( \mu_{ij}^{k+1} - \mu_{ij}^* \right)^2 \]. \] (24)

Inequality (24) shows that consensus (6c) is ensured and that \( \lambda_{ij} \) converges. Convergence of \( \mu \) ensures constraint (5b) is also satisfied. It also follows from (6d) that consensus (3c) implies convergence of \( \lambda_{ij} \). These conclude the proof. \( \blacksquare \)

B. Proof of Theorem 1

It follows from (7f) that, for agent \( i \), adaptive gains \( d_{ij} \) only appear in (7a) but not (7b) or (7c); hence, the impact of \( d_{ij} \) on Lyapunov function (6) can be investigated using its expansion:
\[ E_i^{k+1} = ||\hat{x}_i^k||^2 - 2\alpha_i (\hat{x}_i^k)^T (\nabla_{\hat{x}_i} f_i(\hat{x}_i^k) + \sum_{j \in N_i} \lambda_{ij}^*) \]
\[ -2\alpha_i (\hat{x}_i^k)^T \sum_{j \in N_i} d_{ij}^k (\hat{x}_i^k - \hat{x}_i^*) + ||\hat{\mu}_{i}^{k+1}||^2 \]
\[ + \sum_{j \in N_i} \left( ||z_{ij}^{k+1}||^2 + ||\lambda_{ij}^*||^2 \right) \] (25)
in which \( \alpha_i^2 \) terms are neglected due to 0 < \( \alpha_i << 1 \). In (25), the boxed sum contains all the terms associated with \( d_{ij}^k \). Hence, \( E_i^{k+1} \) can assume two different values: one with \( d_{ij} \) updated according to gain adaptation law (9), and another with no adaptation (i.e., \( d_{ij} = d_{ij}^{k-1} \) for all \( j \in N_i \)). The difference is defined and can be calculated as
\[ \Delta E_i^{k+1} := E_i^{k+1} \bigg| d_{ij}^{k+1} \text{ updated using } (9) - E_i^{k+1} \bigg| d_{ij}^k \text{ not updated} \]
\[ = -h_i(\hat{x}, \hat{z}), \]
where \( h(\cdot) \) is given by (10). Thus, the proof is completed by noting that \( \Delta E_i^{k+1} < 0 \) under choice \( \varepsilon_i \) of (11). \( \blacksquare \)