Implementation of many-qubit Grover search with trapped ultracold ions

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We propose a potentially practical scheme for realization of an n-qubit (n \geq 2) conditional phase flip (CPF) gate and implementation of Grover search algorithm in the ion-trap system. We demonstrate both analytically and numerically that, our scheme could be achieved efficiently to find a marked state with high fidelity and high success probability. We also show the merits of the proposal that the increase of the ion number can improve the fidelity and the success probability of the CPF gate. The required operations for Grover search are very close to the capabilities of current ion-trap techniques.

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I. INTRODUCTION

Grover’s quantum search algorithm [1] has been considered to be an efficient amplitude-amplification process for quantum states by exploiting the parallelism of quantum mechanics. As a remarkable idea in quantum computation, Grover search algorithm could effectively exemplifies the potential speed-up offered by quantum computers. Recently, many proposals have focused on the search algorithm by the adiabatic evolution method [2-4] or by nonadiabatic scenario [5]. Although achievement of these schemes needs stringent conditions and demanding techniques [6], they are really wonderful ideas. On the other hand, some authors had addressed the effect of decoherence [7], gate imperfection or errors [8], and noise [9] on the efficiency of quantum algorithms. We also noticed that, there had been intensive interests in achieving Grover search algorithm theoretically and experimentally by using NMR systems [10], linear optical elements [11,12], trapped ions [13-15], cavity quantum electrodynamics (QED) [16-19], and superconducting mesocircuits [20].

Trapped ions have been considered to be a promising candidate for quantum-information processing (QIP), due to long coherence time of qubits, full controllability of operations and high efficiency of detection. In [13], a many-qubit Grover search algorithm based on the hot-ion quantum computing [21] in decoherence-free subspace was proposed, where collective dephasing errors could be kept away from the qubit encoding. Ref. [14] is an alternative for simulating Grover search algorithm in an ion-trap system, in which the search could be carried out by more than two qubits. However, in order to achieve the conditional phase flip (CPF) gate, each ion should be illuminated by two lasers with different polarizations. In addition, the proposal needs not only elaborately designed sequences of lasers, but also the auxiliary states of the ions, which would result in considerable difficulty in experimental realization with growing number of the qubits.

In this paper, we propose a potentially practical scheme for implementing n-qubit (n \geq 3) Grover search in an ion trap. Compared to above mentioned schemes, our scheme includes following merits: We carry out the CPF gates with nearly unity success probability and fidelity in a straightforward way, which could relax the rigid requirement on accurate sequences of laser manipulations in previous work, such as sequences for one-qubit and two-qubit gates [12,22] or swap gates [23]. Moreover, the increase of qubits in our scheme could improve the fidelity and the success probability of the CPF gate, which are favorable for a scalable Grover search.

Section II describes the general method for a CPF gate in a linear ion trap. We then describle, in Sec. IV, the implementation of n-qubit Grover search algorithm based on the proposed CPF gates, which is almost within the reach of the present technology and extendable to other QIP candidate systems. We conclude with a discussion in Sec. V.

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II. N-QUBIT CPF GATE

We consider $n$ identical three-level ions confined in a linear ion trap, as shown in Fig. 1 where the ions are individually irradiated with traveling wave laser fields tuned to the first lower vibrational (i.e., red) sideband. The three-level ionic states under our consideration are denoted by $|f_j⟩$, $|g_j⟩$, and $|e_j⟩$, with $|f_j⟩$ and $|g_j⟩$ being states lower than $|e_j⟩$. Because the resonant transition happens between $|g_j⟩$ and $|e_j⟩$ by the laser field, $|f_j⟩$ is not involved in the interaction throughout our scheme. In the rotating wave approximation, the Hamiltonian in units of $\hbar = 1$ reads,

$$H = \sum_{j=1}^{n} \{i\omega_a a + \omega_0 \sigma_{z,j} + [\lambda E^+(r, t) \sigma^+_j + H.c.] \},$$

where $E^+(r, t) = E_0 \exp[-i(kz - \omega_0 t + \phi)]$ is the positive frequency part of the driving laser with amplitude $E_0$ and frequency $\omega = \omega_0 - \nu$. $a^+$ ($a$) is the creation (annihilation) operator of the center-of-mass vibrational mode commonly owned by all the ions, $\sigma^+_j = |e_j⟩⟨g_j|$, $\sigma^-_j = |g_j⟩⟨e_j|$, and $\sigma_{z,j} = \frac{1}{2}(|e_j⟩⟨e_j| - |g_j⟩⟨g_j|)$ are the raising, lowering and inversion operators for the $j$th trapped ion, respectively. $\nu$, $\omega_0$ and $\lambda$ are the trap frequency, the atomic transition frequency and dipole matrix element (assumed to be real as convention), respectively. In the resolved sideband limit and in the interaction picture, the Hamiltonian of Eq. (1) excluding the terms associated with carrier transition can be simplified to [24],

$$H'_{i} = \sum_{j=1}^{n} \sum_{\kappa=0}^{\infty} \Omega_j(t)e^{-\eta^2/2\nu} \sigma^+_j + \frac{(i\nu)^{2\kappa+1}}{\kappa!(\kappa+1)!}(a^+)^{\kappa}a^{\kappa+1} + H.c.,$$

where $\eta = k/\sqrt{2\nu \hbar}$ is the Lamb-Dicke parameter with $\hbar$ being the mass of the ion. We assume the laser pulses to be time-dependent in Gaussian type, and the time varying Rabi frequency $\Omega_j(t)$ of the laser field is given by

$$\Omega_j(t) = \Omega_{\text{max}} \exp\{-(-t - t_0)^2/2\tau_j^2\},$$

with $\tau_j$ being the duration of the Gaussian shaped pulse irradiating on the $j$th ion. In the Lamb-Dicke regime, the Hamiltonian can be approximated by the expansion to the first order in $\eta$,

$$H_1 = \sum_{j=1}^{n} \frac{i\eta \Omega_j(t)(a^+ \sigma^-_j - a \sigma^+_j e^{i\phi_j})}{\sqrt{\nu}}.$$  

Furthermore, by assuming $\phi_j = \pi/2$, we reduce Eq. (4) to

$$H_2 = \sum_{j=1}^{n} \eta \Omega_j(t)(a^+ \sigma^-_j + a \sigma^+_j).$$

We first assume that the center-of-mass mode of the ions is initially in the vacuum state $|0⟩$, and except the last ion (i.e., the $n$th ion) initially prepared in the excited state $|e_n⟩$ and the $k$th ion in the state $|g_k⟩$, other ions are initially prepared in the state $|f⟩$. So we only have the $n$th and the $k$th ions interacting with the vibrational mode by following evolution,

$$\prod_{j=1, j\neq k}^{n-1} |f_j⟩ |g_j⟩ |e_n⟩ |0⟩ \rightarrow \exp(-i \int \mathcal{L}_2 dt) \prod_{j, j\neq k}^{n-1} |g_k⟩ |f_j⟩ |e_n⟩ |0⟩ \rightarrow \frac{\partial^2}{\partial \theta_k^2} \frac{\cos(\eta \theta_k')}{\theta_k^2} \times \prod_{j, j\neq k}^{n-1} |g_k⟩ |f_j⟩ |e_n⟩ |0⟩ + \frac{\partial \theta_k}{\partial \theta_k^2} [\cos(\eta \theta_k') - 1] |e_k⟩ |g_n⟩ \prod_{j, j\neq k}^{n-1} |f_j⟩ |0⟩ - i \frac{\partial n}{\partial \theta_k} \sin(\eta \theta_k') |g_k⟩ |g_n⟩ \prod_{j, j\neq k}^{n-1} |f_j⟩ |1⟩, \quad (6)$$
where \( \vartheta_k' = \sqrt{\vartheta_k^2 + \vartheta_n^2} \), and

\[
\vartheta_j = \int_{t_0}^{2t_0} \Omega_{\text{max}}^j \exp\left\{-\left(t - t_0\right)^2/\tau_j^2\right\} dt \\
= \int_{-t_0}^{t_0} \Omega_{\text{max}}^j \exp\left(-t^2/2\tau_j^2\right) dt \\
= \Omega_{\text{max}}^j \sqrt{2\pi \tau_j} \text{erf}[t_0/\sqrt{2}\tau_j] \\
\approx \Omega_{\text{max}}^j \sqrt{2\pi \tau_j},
\]

(7)

where \( \text{erf}[z] = (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt \) is the error function, and \( j = 1, 2, \cdots, n \). In concrete calculations, we assume that \( \text{erf}[t_0/\sqrt{2}\tau_j] \rightarrow 1 \). Next, we consider another situation, that is, the last ion is initially in the excited state \( |e_n\rangle \), and for other ions some of which are initially in the state \( |g\rangle \) and the rest are in the state \( |f\rangle \), then we can acquire the corresponding time evolution,

\[
\prod_{j, l = 1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |e_n\rangle |0\rangle \longrightarrow \exp(-i \int H_2 dt) \prod_{j, l = 1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |e_n\rangle |0\rangle \\
= \frac{\vartheta^2}{\sqrt{2}} \cos(\eta \vartheta) + \frac{\vartheta^2 - \vartheta_n^2}{\sqrt{2}} \times \prod_{j, l = 1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |e_n\rangle |0\rangle \\
+ \frac{\vartheta_n}{\sqrt{2}} \left[ \cos(\eta \vartheta) - 1 \right] \sum_{k=1}^{n} \vartheta_k |e_k\rangle |g_n\rangle \prod_{j, l = 1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |0\rangle \\
- i \frac{\vartheta_n}{\vartheta} \sin(\eta \vartheta) \prod_{j, l = 1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |g_n\rangle |1\rangle ,
\]

(8)

where \( \vartheta = \sqrt{\vartheta_k^2 + \sum_{j=1}^{s} \vartheta_j^2} \) with \( s \) the number of the ions initially in the state \( |g\rangle \) and other denotations are defined as the same as above. To achieve our aim, we need to consider another case: if the ions are initially in the state \( \prod_{j = 1}^{n-1} |f_j\rangle |e_n\rangle |0\rangle \), then the ions, except the last one, do not interact with the vibrational mode. The evolution of the system is as follows,

\[
\prod_{j = 1}^{n-1} |f_j\rangle |e_n\rangle |0\rangle \longrightarrow \exp(-i \int H_2 dt) \prod_{j = 1}^{n-1} |f_j\rangle |e_n\rangle |0\rangle \\
= \left[ \cos(\eta \vartheta_n) |e_n\rangle |0\rangle - i \sin(\eta \vartheta_n) |g_n\rangle |1\rangle \right] \prod_{j = 1}^{n-1} |f_j\rangle .
\]

(9)

Based on Eqs. (6)-(9), we can construct an n-qubit CPF gate. In our proposal, the qubit definitions are the same for other ions, except the last ion, i.e., the logic state \( |1\rangle \) \((|0\rangle)\) of the \( i \)th qubit is denoted by \(|f_i\rangle \) \(|g_i\rangle\) of the \( i \)th ion with \( i = 1, 2, \cdots, n-1 \), whereas the logic state \( |1\rangle \) \(|0\rangle\) of the \( n \)th qubit is represented by \(|e_n\rangle \) \(|g_n\rangle\) of the \( n \)th ion. By considering the quantum information encoded in the subspace spanned by the states \(|g_1\rangle, |f_1\rangle, |g_2\rangle, |f_2\rangle, \cdots, |g_{n-1}\rangle, |f_{n-1}\rangle, |g_n\rangle, |e_n\rangle\), we have Eq. (9) in the case of \( \eta \vartheta_n = \pi \) to be,

\[
\prod_{j = 1}^{n-1} |f_j\rangle |e_n\rangle |0\rangle \longrightarrow - \prod_{j = 1}^{n-1} |f_j\rangle |e_n\rangle |0\rangle .
\]

(10)

Furthermore, we assume that coupling strengths satisfy the following condition,

\[
\vartheta_1 = \vartheta_2 = \cdots = \vartheta_{n-1} \gg \vartheta_n.
\]

(11)

After inserting Eq. (7) into Eq. (11), we have,

\[
m = \frac{\vartheta_n}{\vartheta_i} = \frac{\Omega_{\text{max}}^n \tau_n}{\Omega_{\text{max}}^i \tau_i} \ll 1,
\]

(12)

with \( i = 1, 2, \cdots, n-1 \), which implies that the condition in Eq. (11) could be met by adjusting the pulse widths \( \tau_i \) and the maximum coupling strength \( \Omega_{\text{max}}^i \). To keep the ions in the vacuum center-of-mass mode, we assume that
each laser pulse width has the identical value ($\tau_j = \tau_0$), but we have different $\Omega_{\text{max}}^i$ by setting $m = \frac{\Omega_{\text{max}}^i}{\Omega_{\text{max}}^j} \ll 1$ to meet the requirements. Then Eqs. (6) and (8) can be reduced to

$$\prod_{j=1, j \neq k}^{n-1} |g_k\rangle |f_j\rangle |e_n\rangle |0\rangle \longrightarrow \beta \times \prod_{j=1, j \neq k}^{n-1} |g_k\rangle |f_j\rangle |e_n\rangle |0\rangle \quad (13)$$

and

$$\prod_{j=1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |e_n\rangle |0\rangle \longrightarrow \alpha_s \times \prod_{j=1, j \neq l}^{n-1} |g_j\rangle |f_l\rangle |e_n\rangle |0\rangle , \quad (14)$$

where $\alpha_s = [m^2 \cos(\sqrt{m^2 + s \pi / m}) + s] / (m^2 + s)$ and $\beta = [m^2 \cos(\sqrt{m^2 + 1 \pi / m}) + 1] / (m^2 + 1) = \alpha_1$. So if the requirement of Eq. (12) is met, we have $\beta \approx \alpha_s \approx 1$ and thereby obtain an $n$-qubit CPF gate in our computational subspace, where the ions from the first to the $(n - 1)$th represent the control qubits, and the last ion represents the target qubit. We take $n = 4$ as an example below. From Eqs. (10)–(14), an approximate four-qubit CPF gate can be reached as follows:

$$U^{(4)}_{\text{CPF}} = \text{diag} \{ 1, 1, \alpha_3, \alpha_2, 1, 1, \alpha_2, \beta, 1, 1, \alpha_2, \beta, 1, 1, \beta, -1 \}, \quad (15)$$

in the computational subspace spanned by $\{ |g_1g_2g_3g_4\rangle, |g_1g_2g_3e_4\rangle, |g_1g_2f_3g_4\rangle, |g_1f_2g_3g_4\rangle, |g_1f_2g_3e_4\rangle, |g_1f_2f_3g_4\rangle, |g_1f_2f_3e_4\rangle, |f_1g_2g_3g_4\rangle, |f_1g_2g_3e_4\rangle, |f_1g_2f_3g_4\rangle, |f_1g_2f_3e_4\rangle, |f_1f_2g_3g_4\rangle, |f_1f_2g_3e_4\rangle, |f_1f_2f_3g_4\rangle, |f_1f_2f_3e_4\rangle \}$, where $\alpha_2 = 0.99952$, $\alpha_3 = 0.99516$, $\beta = 0.99988$ in the case of $m = 0.1$.

We now turn to calculation of the fidelity and the success probability according to the relations $F = \langle \Psi_0 | U^{(n)}_{\text{CPF}} \Psi_i \rangle / \langle \Psi_i | U^{(n)}_{\text{CPF}} \Psi_0 \rangle [25]$ and $P = \langle \Psi_f | \Psi_f \rangle$, where $\Psi_f$ is the final state after the $n$-qubit CPF gate has been made. The overline indicates average over all possible components in $\Psi_0$. In the case of $n$ qubits, we set $|\Psi_0\rangle = \frac{1}{\sqrt{2^n}} (|g_1\rangle + |f_1\rangle) (|g_2\rangle + |f_2\rangle) \cdots (|g_{n-1}\rangle + |f_{n-1}\rangle) (|g_n\rangle + |e_n\rangle)$, the infidelity and the success probability which gives a general assessment for our gateing are obtained straightforwardly by

$$\text{Infidelity} = 1 - F = 1 - \frac{[\sum_{s=2}^{n-1} C^{(n)}_s \alpha_s + (n - 1) \beta + 2^{n-1} + 1]^2}{2^n [\sum_{s=2}^{n-1} C^{(n)}_s \alpha_s + (n - 1) \beta^2 + 2^{n-1} + 1]}, \quad (16)$$

and

$$P = \frac{[\sum_{s=2}^{n-1} C^{(n)}_s \alpha_s + (n - 1) \beta^2 + 2^{n-1} + 1]}{2^n}, \quad (17)$$

where we denote the number of $\alpha_s$ ($s = 2, 3, \ldots, n - 1$) by $C^{(n)}_s$ with $n$ the number of the qubits, and $C^{(n)}_s$ fulfilling the equations,

$$C^{(n)}_s + C^{(n)}_{s+1} = C^{(n+1)}_{s+1}, \quad (18)$$

and

$$\sum_{s=2}^{n-1} C^{(n)}_s = 2^{n-1} - n. \quad (19)$$

We have listed some examples in Table I for the values of $C^{(n)}_s$.

| Table I. List of the values of $C^{(n)}_s$. |
|---|---|---|---|---|---|---|---|---|
| qubit number $n$ | number of $\alpha_2$ | number of $\alpha_3$ | number of $\alpha_4$ | number of $\alpha_5$ | number of $\alpha_6$ | $\cdots$ | number of $\alpha_{n-1}$ | $\sum_{s=2}^{n-1} C^{(n)}_s$ |
| $n = 4$ | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 4 |
| $n = 5$ | 6 | 4 | 1 | 0 | 0 | 0 | 0 | 11 |
| $n = 6$ | 10 | 10 | 5 | 1 | 0 | 0 | 0 | 26 |
| $n = 7$ | 15 | 20 | 15 | 6 | 1 | 0 | 0 | 57 |

We have plotted the infidelity and the success probability versus $m = \Omega^i_{\text{max}} / \Omega^j_{\text{max}}$ in Figs. 2 and 3, which clearly indicate that our proposed $n$-qubit CPF gate is of high fidelity and high success probability as long as the value of $m$ is small enough. The figures also show the increase of $F$ and $P$ with the number of the ions. To obtain the maximal fidelity and success probability, provided that $\Omega^i_{\text{max}} = \Omega_m$, we have made some numerical calculations, which show the values of $m$ to be $m = \Omega^i_{\text{max}} / \Omega_m = 0.0122, 0.0122$ and 0.0147 corresponding to $n = 3, 6$ and 9, respectively. One can see from the figures that the implementation of the $n$-qubit CPF gate with high fidelity and high success probability can be realized by suitable laser-ion coupling strength ratio of the last ion to the others.
In this section, we will implement an n-qubit Grover algorithm by our gating discussed above. One of the key steps in Grover search algorithm is to find a marked element in an unsorted database of size N. Generally speaking, the Grover search algorithm consists of three kinds of operations [13]. The first one is to prepare a superposition state \( |\Psi_0\rangle = (\frac{1}{\sqrt{2^n}}) \sum_{i=0}^{2^n-1} |i\rangle \) using Hadamard gates \( H^\otimes n \) (\( n \) being the number of qubits). The second is for an iteration \( Q(n) \) including following two operations: (a) Inverting the amplitude of the marked state \( |\rho\rangle \) using an operator \( J_\rho = I - 2 |\rho \rangle \langle \rho | \) with \( I \) the identity matrix; (b) Inversion about the average of the amplitudes of all states using the diffusion transform \( \hat{D}(n) \), with \( \hat{D}_{ij}(n) = \frac{2}{N} - \delta_{ij} \) \((i, j = 1, 2, 3, \cdots, N)\) and \( N = 2^n \). This step should be carried out by at least \( \pi \sqrt{N}/4 \) times to maximize the probability for finding the marked state. Finally, a measurement of the whole system is made to get the marked state.

As defined above, the logic state \( |1\rangle \) \((|0\rangle)\) of the \( i \)th qubit is denoted by \( |f_i\rangle \) \((|g_i\rangle)\) of the \( i \)th ion, with \( i = 1, 2, \cdots, n-1 \), whereas the logic state \( |1\rangle \) \((|0\rangle)\) of the \( n \)th qubit is represented by \( |e_n\rangle \) \((|g_n\rangle)\) of the \( n \)th ion. By considering the quantum information encoded in the subspace spanned by \(|g_1\rangle, |g_2\rangle, |f_2\rangle, \cdots, |g_{n-1}\rangle, |f_{n-1}\rangle, |g_n\rangle, |e_n\rangle\), we have following transformation,

\[
Q(n) = W^\otimes n J_{00\cdots 0}^{(n)} W^\otimes n J_\rho = W^\otimes n J_{g_1g_2\cdots g_n}^{(n)} W^\otimes n J_\rho = -\hat{D}(n) J_\rho,
\]

where \( J_{00\cdots 0}^{(n)} = \text{diag}\{1, 1, \cdots, 1\} = I^{(n)} - 2 |00\cdots 0\rangle \langle 00\cdots 0 | \) (the number of 0 is \( n \)) and the Hadamard gate in our whole computational subspace is given by

\[
W^\otimes n = \prod_{i=1}^{n} W_i = \left( \frac{1}{\sqrt{2}} \right)^n \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

In the case of \( n \) qubits, Eq. (20) implies that the diffusion transform \( \hat{D}(n) = -W^\otimes n J_{g_1g_2\cdots g_n}^{(n)} W^\otimes n \) is always unchanged no matter which state is to be searched. The only change is the operator \( J_\rho \) for different marked states. Based on the diffusiongate constructed in last section, we have \( J_{11\cdots 1}^{(n)} \approx \text{diag}\{1, 1, \cdots, 1\} = U_{CPF}^{(4)} \), from which we could achieve the gate \( J_{00\cdots 0}^{(n)} \) and other relevant operations,

\[
J_{00\cdots 0}^{(n)} = \sigma_x,n S_{x,n-1} \cdots S_{x,1} J_{11\cdots 1}^{(n)} S_{x,1} \cdots S_{x,n-1} \sigma_x,n,
\]

and

\[
\hat{D}(n) = W^\otimes n J_{00\cdots 0}^{(n)} W^\otimes n,
\]

where \( S_{x,i} = |f_i\rangle \langle g_i| + |g_i\rangle \langle f_i| \) with \( j \neq i \), and \( \sigma_x,n = |e_n\rangle \langle g_n| + |g_n\rangle \langle e_n| \). To achieve \( Q(n) \), we will construct the diffusiongate \( J_\rho = I - 2 |\rho \rangle \langle \rho | \) for \( \rho \) in the case of \( n \), the number of possible quantum states is \( 2^n \), and the operation is to label a marked state by \( J_\rho \) with \( \rho \) one of the states from \( \{0000\}, \{0001\}, \{0010\}, \cdots, \{1111\}\}. To carry out the four-qubit Grover search, we need two four-qubit Hadamard gates \( W^\otimes 4 \). Based on the gate \( J_{1111} = U_{CPF}^{(4)} \) (See Eq. (15)) marking the state \( |f_1f_2f_3e_4\rangle \), we could construct other fifteen gates for the marking job as,

\[
\begin{align*}
J_{g_1g_2g_3g_4} &= J_{0000} = \sigma_x,4 S_{x,3} S_{x,2} S_{x,1} J_{1111} S_{x,1} S_{x,2} S_{x,3} \sigma_x,4, \quad J_{g_1g_2g_3e_4} = J_{0001} = S_{x,3} S_{x,2} S_{x,1} J_{1111} S_{x,1} S_{x,2} S_{x,3}, \\
J_{g_1g_2f_3g_4} &= J_{0010} = \sigma_x,4 S_{x,2} S_{x,1} J_{1111} S_{x,1} S_{x,2} \sigma_x,4, \quad J_{g_1g_2f_3e_4} = J_{0011} = S_{x,2} S_{x,1} J_{1111} S_{x,1} S_{x,2}, \\
J_{g_1f_2g_3g_4} &= J_{0100} = \sigma_x,4 S_{x,3} S_{x,1} J_{1111} S_{x,1} S_{x,2} \sigma_x,4, \quad J_{g_1f_2g_3e_4} = J_{0101} = S_{x,3} S_{x,1} J_{1111} S_{x,1} S_{x,2}, \\
J_{g_1f_2f_3g_4} &= J_{0110} = \sigma_x,4 S_{x,3} S_{x,2} J_{1111} S_{x,1} S_{x,2} \sigma_x,4, \quad J_{g_1f_2f_3e_4} = J_{0111} = S_{x,2} J_{1111} S_{x,1}, \\
J_{f_1g_2g_3g_4} &= J_{1000} = \sigma_x,4 S_{x,3} S_{x,2} J_{1111} S_{x,2} S_{x,3} \sigma_x,4, \quad J_{f_1g_2g_3e_4} = J_{1001} = S_{x,3} S_{x,2} J_{1111} S_{x,2} S_{x,3}, \\
J_{f_1g_2f_3g_4} &= J_{1010} = \sigma_x,4 S_{x,3} J_{1111} S_{x,1} S_{x,2} \sigma_x,4, \quad J_{f_1g_2f_3e_4} = J_{1011} = S_{x,3} J_{1111} S_{x,1}, \\
J_{f_1f_2g_3g_4} &= J_{1100} = \sigma_x,4 S_{x,3} J_{1111} S_{x,2} S_{x,3} \sigma_x,4, \quad J_{f_1f_2g_3e_4} = J_{1101} = S_{x,3} J_{1111} S_{x,3}, \\
J_{f_1f_2f_3g_4} &= J_{1110} = \sigma_x,4 J_{1111} S_{x,3} \sigma_x,4, \quad J_{f_1f_2f_3e_4} = J_{1111} = \sigma_x,4 J_{1111} \sigma_x,4.
\end{align*}
\]

So with a state marked, and the four-qubit diffusion transform \( \hat{D}(4) \) which is generated by combining two Hadamard gates \( W^\otimes 4 \) with \( J_{0000} \), a full Grover search for four qubits is available. In principle, if each component of the design is available, our scheme would be achievable experimentally.

Taking the marked state \( |f_1g_2g_3e_4\rangle \) as an example, a standard quantum circuit for the Grover search algorithm for \( N = 4 \) entries is shown in Fig. 4. The procedure of the Grover search is accomplished with
and the transition between $|g\rangle$ and $|e\rangle$ respectively. The lifetime of the $(z$-axis) trap frequency $\nu$ and maximal Rabi frequency $\Omega_m$ in current ion trap experiments, all the above-mentioned relations could be met in the weak-excitation regime $\Omega_m \ll \eta \nu$ (which implies a negligible AC Stark shift induced by the radiation), where one may expect the unwanted off-resonant transition to be disregarded on the precondition that the laser field cannot be made too intense at any rate, which thereby is regarded as the dominating limiting factor on the resulting relatively-low gate speed [28].

We briefly address the experimental feasibility of our scheme. We may employ $S_{1/2}(m_j = 1/2)$, $S_{1/2}(m_j = -1/2)$, and $D_{5/2}(m_j = -1/2)$ of $^{40}\text{Ca}^+$ [27] as the states $|f\rangle$, $|g\rangle$ and $|e\rangle$, respectively. The lifetime of $D_{5/2}$ of the $^{40}\text{Ca}^+$ ion is longer than 1 sec, and the transition between $|f\rangle$ and $|g\rangle$ is dipolar forbidden. To carry out the CPF gate in the present scheme, the condition $\eta \Omega_m^\nu \sqrt{2\pi} \tau_n = \pi$ and the approximation relation $\text{erf}(t_0 / \sqrt{2} \tau_\nu) \rightarrow 1$ should be well satisfied. By choosing suitable values of the $(z$-axis) trap frequency $\nu$ and maximal Rabi frequency $\Omega_m$ in current ion trap experiments, all the above-mentioned relations could be met in the weak-excitation regime $\Omega_m \ll \eta \nu$ (which implies a negligible AC Stark shift induced by the radiation), where one may expect the unwanted off-resonant transition to be disregarded on the precondition that the laser field cannot be made too intense at any rate, which thereby is regarded as the dominating limiting factor on the resulting relatively-low gate speed [28]. Taking our proposed n-qubit CPF gate as an example, for a typical value $\eta = 0.1$ of the Lamb-Dicke parameter, we have listed the required time $t_0$ for CPF gate operation in Table II, which clearly indicate that the values of $t_0$ are prolonged with the growth of the number of ions. In addition, the switching rate of the CPF gate will be of order $\lesssim \nu / 1000$, which, from the viewpoint of heating effect, will inevitably bring out disadvantaged impact on the construction of CPF gate. Alternatively, in what follows we will take into account another different regimes $\Omega_m \ll \nu / \eta$, where the only new requirement we should make is that the laser should take a fixed intensity satisfying the resonance condition [28] $\Omega_m = \nu / 2$, which implies an improvement by two orders of magnitude with respect to the CPF gating time in the case of weak-excitation. So to reduce the implementation time $t_0$, we may choose such a stronger radiation with $\Omega_m = \nu / 2$ for a carrier transition. Our direct calculation based on the model in [28] shows that we could also have Eq. (5) due to AC Stark shift and we could should satisfy $t_0 \geq 0.574 ms, 0.766 ms, 0.957 ms$ in the case of $n = 3, 4, 5$, respectively, to achieve our CPF gate. As the heating time of the ground vibrational state of the ions in the linear trap is about $4 \text{ millisecond}$ [29], the implementation time in the case of strong radiation seems better in view of avoiding heating. Note also that single-qubit operation takes negligible time in comparison with that for many-qubit phase gating, so direct calculation shows that one iteration of our proposed Grover search would take $2t_0 \approx 1 ms$, which is shorter than the the heating time of the ions. In this sense, the decoherence originating from the heating effect is not a big obstacle for the simulation of Grover search by our proposal when the number of qubit is small.

TABLE II. List of the required time $t_0$ for CPF gate and the values of $\nu$ and $\Omega_m$ ($\eta = 0.1$).
| regime | qubit number $n$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ |
|--------|----------------|--------|--------|--------|--------|--------|--------|--------|
| $\nu$ (MHz/2$\pi$) | 0.250 | 0.167 | 0.125 | 0.100 | 0.083 | 0.071 | 0.063 |
| $\Omega_m (kHz/2\pi)$ | 2.500 | 1.670 | 1.250 | 1.000 | 0.830 | 0.710 | 0.630 |
| $t_0$ (ms) | 19.14 | 28.71 | 38.29 | 47.86 | 57.43 | 67.00 | 76.57 |

It is generally considered that the computing operation on trapped ions in a linear ion trap would be more and more intricate with the increase of the number of ions. Because of the decrease of the spatial separation of the trapped ions, individual manipulation is more and more difficult, and meanwhile the vibrational mode spectrum becomes more and more unresolved. As a result, the extension of quantum computation from a few qubits to large numbers of qubits is quite technically challenging. In this sense, although our proposal is in principle scalable, the currently technical level regarding the linear trap restricts the application of our scheme. Nevertheless, eight ultracold ions in the linear trap could be individually addressed and entangled [30]. Therefore, if we apply our scheme to these eight ions, a Grover search with $2^8$ states could be achieved in a simpler way.

Alternatively, we may consider the application of our scheme in a multi-zone trap, in which we may carry out our scheme on few ions in separate zones, respectively, and then entangle a large number of ions by moving the ions between different zones [31]. This is a possible way to a large-scale Grover search implementation.

We have reiterated our scheme to achieve a CPF gate in one step, which could save the implementation time and steps compared to conventional methods. Moreover the most impressive feature of our scheme is the higher fidelity and higher success rate with more qubits involved, which favors large-scale QIP. Nevertheless, the more qubits, the more steps necessary for an optimal Grover search. As the CPF gate proposed here is intrinsically imperfect, a decrease of the fidelity is inevitable with more qubits involved, even if we neglect the unpredictable imperfection and decoherence in actually experimental situation. Anyway, the current experimental progress has shown some efficient ways to deal with the imperfection [32]. Other basic operations are also available because the qubit encoding in the first n-1 ions is like that done in Oxford’s group [26] and the qubit encoding in the last ion is employed in Innsbruck’s group [27]. So we believe that our proposal for Grover search should be available with current or near-future techniques.

In summary, a potentially practical scheme for performing $n$-qubit CPF gate as well as the many-qubit Grover search algorithm has been proposed in the ion-trap system. We have demonstrated both analytically and numerically that, our scheme could be achieved efficiently to find a marked state with high fidelity and high success probability. Our proposal could be employed in both the linear ion trap and the multizone trap, and may also be applied to other QIP tasks, such as preparation of cluster states for one-way quantum computation [33]. Therefore, we argue that our idea in the present paper is not only practical, but also simple and experimentally feasible, which would be helpful for working in large-scale QIP devices.

V. ACKNOWLEDGMENTS

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Captions of Figures

FIG. 1. Schematic setup for implementing n-qubit CPF gate and Grover search in a linear trap, where the inset shows the ionic level configuration, with bold lines for the states encoding qubits and the arrows for the coupling of the lasers to the ions.

FIG. 2. Infidelity versus the Rabi frequency ratio of the last ion to the others, where the solid, dashed-dotted and dashed curves represent the case of $n = 3, 6$ and $9$, respectively.

FIG. 3. Success probability versus the Rabi frequency ratio of the last ion to the others, where the solid, dashed-dotted and dashed curves represent the case of $n = 3, 6$ and $9$, respectively.

FIG. 4. Quantum circuit of one iteration of the four-qubit Grover search for the marked state $|f_1 g_2 g_3 e_4 \rangle$, where $W$, $J_{1111}^{(4)}$ and $S_x (\sigma_x)$ are the Hadamard gate, four-qubit controlled phase gate and single-qubit NOT gate, respectively. The state of ions is initially prepared in the average superposition state $|\Psi_I \rangle = \frac{1}{\sqrt{4}}(|g_1 \rangle + |f_1 \rangle)(|g_2 \rangle + |f_2 \rangle)(|g_3 \rangle + |f_3 \rangle)(|g_4 \rangle + |e_4 \rangle)$. The operations in the dashed boxes could be reduced to the transforms $R_i$ and $R'_i$ with $i = 1, 2, 3, 4$ denoting the $i$th ion, which is helpful for experimental implementation. To maximize the search probability, we should implement the circuit repeatedly for several times.

FIG. 5. The search probability for the marked states $|f_1 g_2 g_3 e_4 \rangle$ and $|f_1 g_2 g_3 e_4 e_5 \rangle$ versus the number of the iterations in the case of $n = 4$ and $5$. 

Nature (London) 438, 643-646 (2005).
The iteration number

| Probability | n=4, m=0.0122 | n=4, m=0.0928 | n=5, m=0.0122 | n=5, m=0.0928 |
|-------------|----------------|----------------|----------------|----------------|
| 0           |                |                |                |                |
| 0.1         |                |                |                |                |
| 0.2         |                |                |                |                |
| 0.3         |                |                |                |                |
| 0.4         |                |                |                |                |
| 0.5         |                |                |                |                |
| 0.6         |                |                |                |                |
| 0.7         |                |                |                |                |
| 0.8         |                |                |                |                |
| 0.9         |                |                |                |                |
| 1           |                |                |                |                |

The iteration number