Physical Principles and Properties of Unstable States *

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Abstract

The main subject of the paper is the description of unstable states in quantum mechanics and quantum field theory. Unstable states in quantum field theory can only be introduced as the intermediate states and not as asymptotic states. The absence of the intermediate unstable states from the asymptotic states is compatible with unitarity. Thus the concept of an unstable state is not introduced in quantum field theory despite the fact that an unstable state has well defined linear momentum, angular momentum and other intrinsic quantum numbers. In the rigged Hilbert space quantum mechanics one can define vectors that correspond to the unstable states. These vectors are the generalized eigenvectors (kets in the rigged Hilbert space) with complex eigenvalues of the self-adjoint Hamiltonian. The real part of the eigenvalue corresponds to the mass of an unstable state and the imaginary part is one half of the total width. Such vectors form the minimally complex semigroup representation of the Poincaré transformations into the forward light cone.

1 Introduction

Conventional quantum mechanics can be applied to the physical systems that consist of stable particles. For such systems the space of the states is the

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Hilbert space and the observables are represented by the self-adjoint operators in the Hilbert space. This mathematical representation is supplemented by the probabilistic interpretation of quantum mechanics.

The necessity of including also the unstable states in quantum mechanics appeared from the very beginning in the relation with the phenomenon of the radioactivity and in particular $\alpha$ decay. The description of $\alpha$ decay was first formulated in 1928 by George Gamow [1] and was based on the calculation of the transition probability of a potential barrier and led to the appearance of the eigenfunctions of the Hamiltonian with complex eigenvalues. This problem can also be formulated in the language of the quasi-stationary states in scattering theory [2] where the $S$-matrix has simple poles on the second sheet of the complex energy [3, 4]. The most important feature of such a formulation is that the partial cross section in the vicinity of the pole is universal and mostly independent of the details of the interaction potential and depending predominantly on the position of the $S$-matrix pole. This cross section is given by the Breit-Wigner formula which depends on two parameters: $E_0$ and $\Gamma$ and a slowly varying background function. $E_0$ is the energy of the resonance state and $\Gamma$ is the width of the resonance state.

The vectors that correspond to the resonance states do not belong to the Hilbert space. In the case of the complex energy eigenstate the time dependent position wave function diverges for $r \rightarrow \infty$ and cannot be normalized [5, 4]. The $S$-matrix in quantum mechanics is used for the description of scattering. If in a given physical system there are bound states then the $S$-matrix has simple poles for imaginary momenta $\Im(k) > 0$ and the bound state energy is equal to

$$E = \frac{\hbar^2 k^2}{2m}. \quad (1)$$

Unstable states correspond to poles of the $S$-matrix but they are situated in the lower semi plane of complex momentum in the vicinity of the imaginary axis. For the complex energy the $S$-matrix bound state poles are on the first, physical sheet and the resonance poles are on the second sheet below and above the cut along the positive real axis [3, 4].

The eigenvalues of the Hamiltonian on the negative real axis correspond to the bound states. The complex eigenvalues on the second sheet correspond to the unstable states. In the Hilbert space the Hamiltonian is self-adjoint and cannot have complex eigenvalues. It thus follows that the description of the unstable states by eigenvectors in Hilbert space is not possible. Thus the inclusion of the unstable states calls for a modification of quantum theory.
In my talk I will consider three kinds of description of unstable states

1. Unstable states in quantum mechanics–Gamow states.

2. Unstable states in quantum field theory.

3. Rigged Hilbert space of unstable states.

I will present the main assumptions of each method and their main conclusions and will show the relation between the first two approaches and the rigged Hilbert space description.

2 Unstable states in quantum mechanics–Gamow states

The quasi-stationary states in quantum mechanics appears when one considers, e.g., the potentials that have finite width. If one considers the three dimensional potential well shown in Fig. 1 then for energies inside the potential well there may exist bound states for energies $E < 0$ and quasi-stationary states for $0 < E < V_0$. The eigenvalues for the quasi-stationary states are obtained from the condition that the wave function which is the solution of the Schrödinger equation has the following asymptotic behavior

$$
\psi(r) \xrightarrow{r \to \infty} e^{ikr} \quad r.
$$

This produces discrete complex eigenvalues of the Hamiltonian. The wave function that corresponds to this eigenvalue cannot be normalized so it does not belong to the Hilbert space, but it has several properties that permit us to interpret them as the wave functions of the unstable state. The lifetime of the unstable state is related to the imaginary part of the complex eigenvalue $E_0 - i\Gamma/2$ in the following way

$$
\tau = \frac{\hbar}{\Gamma}.
$$

Another approach is to consider the scattering on the potential given in Fig. 1 Then one imposes the following asymptotic condition that corresponds to the stationary state

$$
\psi(r) \xrightarrow{r \to \infty} \sqrt{\frac{2}{\pi}} \sin(kr + \delta)
$$
and then from the condition that the wave function is regular inside the well one obtains the $S$-matrix of the problem which has a simple pole at a complex momentum at exactly the same value as in the first approach. In this case the cross section for the scattering has the Breit-Wigner form

$$\sigma \sim \frac{\Gamma^2/4}{(E - E_0)^2 + \Gamma^2/4}.$$  

(5)

When $\Gamma \neq 0$ it is frequently stated that the quasi-stationary state does not have the definite value of the energy. It should however be stressed that in each single event the energy is well determined. The meaning of the indefiniteness of the energy of an unstable state is that in the range of the energies $|E - E_0| \lesssim \Gamma$ the creation of the long lived unstable state is possible. The other important point is the scattering amplitude is given by the Breit-Wigner formula in the vicinity of the pole which depends only on the position of the pole and is not sensitive to other properties of the potential. This property holds only for the energies close to the position of the pole. For the energies $|E - E_0| > \Gamma$ the former statement does not hold and one has to take into account other properties of the potential. Also one has to say that there is no precise division between the ranges of the energies where
the Breit-Wigner amplitude is sufficient for the description of the process and where it is not, since there are always non-resonant contributions in the process.

3 Unstable states in quantum field theory

In quantum mechanics the unstable state corresponds to a state of the Hamiltonian that is “almost” bound. In other words the unstable state has parts. In quantum field theory the unstable states may be elementary. Let us consider the model presented by Veltman [7] in which there are two scalar particles $A$ and $\phi$ whose masses fulfill the condition $M > 2m$ ($M$ and $m$ are the masses of the $A$ and the $\phi$ fields, respectively). The interaction Lagrangian of these fields is

$$ L_I = \frac{g^2}{2} [\phi^2(x) \cdot A(x) + A(x) \cdot \phi^2(x)]. $$

(6)

From the interaction Lagrangian it follows that the interactions permit the transition

$$ A \rightarrow \phi + \phi $$

(7)

and from the condition on the masses $M > 2m$ it follows that the decay of the particle $A$ is kinematically possible so the particle $A$ is unstable. Since $A$ is unstable it cannot appear as an asymptotic state. Exclusion of some states from the asymptotic states can lead to the breakdown of the unitarity of the $S$ matrix.

The danger is real. Veltman shows that there is no violation of the unitarity of this theory if one includes only the states of the $\phi$ field as the asymptotic states. The idea of the proof is the following.

In the lowest order of the perturbation theory the propagators of the $\phi$ and $A$ fields are

$$ \Delta_\phi(x_i - x_j) = -\frac{1}{(2\pi)^4} \int d^4k \ e^{i k (x_i-x_j)} \frac{1}{k^2 - m^2 + i\varepsilon}. $$

(9)

$$ \Delta_A(x_i - x_j) = -\frac{1}{(2\pi)^4} \int d^4k \ e^{i k (x_i-x_j)} \frac{1}{k^2 - M^2 + i\varepsilon}. $$

(10)
The propagator of $A$ in the next order is given by the diagram in Fig. 2 and the analytic expression corresponding to this diagram has the form

$$\Delta_A(x_i - x_j) = \frac{1}{(2\pi)^4} \int d^4k \ e^{ik(x_i-x_j)} \left\{ (k^2 - M^2)^2 R_2(k^2) + i\theta(k^2 - 4m^2)I_2(k^2) \right\}. \quad (11)$$

This diagram, after iteration, gives the geometric series with the following factor

$$\frac{(k^2 - M^2)^2 R_2(k^2) + i\theta(k^2 - 4m^2)I_2(k^2)}{k^2 - M^2 + i\varepsilon} \quad (12)$$

which has the singularity for $k^2 \approx M^2$ and it appears because of the condition $M > 2m$. We thus see that the series is divergent for any value of the coupling constant $g$ and the perturbation method fails. The way out of this difficulty is to find the propagator for such values of $k^2$ where the perturbation series is convergent and then analytically continue it to the vicinity of the point $k^2 = M^2$. As the result one obtains the following expressions for the propagators

$$\Delta_\phi(k^2) = \frac{1}{k^2 - m^2 - (k^2 - m^2)^2 R_\phi(k^2) + i\theta(k^2 - 9m^2)I_\phi(k^2)}, \quad (13)$$

$$\Delta_A(k^2) = \frac{1}{k^2 - M^2 - (k^2 - M^2)^2 R_A(k^2) + i\theta(k^2 - 4m^2)I_A(k^2)}, \quad (14)$$

where $m$ and $M$ are the physical masses. As can be seen the propagator $\Delta_\phi(k^2)$ has a pole for $k^2 = m^2$ and $\Delta_A(k^2)$ does not have a pole for $k^2 = $
$M^2$. The fact that $\Delta_A(k^2)$ is regular at the point $k^2 = M^2$ is crucial for the demonstration of the unitarity of the $S$-matrix after the exclusion of the particles $A$ from the initial and final asymptotic states. From these consideration we obtain the following picture concerning the unstable states in quantum field theory

- Unstable states should not be included as the initial and final asymptotic states.
- $S$-matrix is unitary.
- Unstable states appear only as the intermediate states and their propagators have the form dictated by the Dyson summation formula.

We thus see that the notion of the unstable state in quantum field theory is realized through the elimination of the unstable states from the asymptotic spaces and the modification of the propagators. In such a way the space of the physical states is richer than the space of the asymptotic states. The propagator of an unstable state does not have a pole at the point $k^2 = M^2$ but has a pole for a complex value of $k^2$.

We thus see that in quantum field theory the vectors of the unstable states are not introduced at all and unstable states are included as the intermediate states with the special form of the propagators. Despite this fact one can draw some conclusions about the properties of the intermediate states:

1. The momentum of the unstable states is well defined

From the general rules of quantum field theory the momentum is conserved at each vertex (see e.g., Fig. 3) so the momentum of an unstable particle is well defined and is real. Later we will discuss the problem of the mass of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Production of an unstable state $A$ by two particles $\phi$.}
\end{figure}
an unstable state and we will see that the mass of an unstable state can be for many purposes considered as the complex quantity.

2. Gauge invariance

Gauge invariance is very important in elementary particle physics. Quantum electrodynamics – the most precise existing theory and the standard model are examples of the theories with gauge symmetry. For this reason the gauge symmetry for the unstable states has to be considered.

In the nineties the problem of the gauge invariance was widely discussed in relation with the extremely precise measurement of the $Z_0$ mass \[8, 9\]. This discussion proliferated the theoretical knowledge of unstable relativistic states and the understanding of gauge invariance for such states. The accuracy of the measurement of the $Z_0$ mass exceeded the accuracy of the theoretical calculations of the next to the leading order in perturbation theory and the next to the next to the leading order calculations of mass and width were not gauge invariant. It is clear that such a situation was very unsatisfactory both from the theoretical and practical point of view, especially because the gauge dependent corrections were not bounded and for some exotic choice of the gauge they could be relatively large. The proposed solution of this problem is to use for the mass definition the $S$-matrix element of the corresponding process since this has been proven to be gauge invariant \[10\]. Therefore the pole of the $S$-matrix should be used for the determination of the mass and width of the resonances.

The problem of the gauge invariance and the correct form of the propagator of an unstable state arose also for many processes involving the gauge bosons $Z_0$ and $W^\pm$, where the propagators appeared. The propagator of a stable vector particle is given by

$$D_{\mu\nu}(q) = \frac{i \left( -g_{\mu\nu} + (1 - \xi) \frac{q_{\mu}q_{\nu}}{q^2 - \xi M^2} \right)}{q^2 - M^2 + i\varepsilon}$$

where $\xi$ is the gauge parameter. This propagator has a pole for $q^2 = M^2$. For an unstable state the following substitution is made in the denominator

$$q^2 - M^2 \rightarrow q^2 - M^2 + iM\Gamma$$

and the following propagator is obtained

$$D'_{\mu\nu}(q) = \frac{i \left( -g_{\mu\nu} + (1 - \xi) \frac{q_{\mu}q_{\nu}}{q^2 - \xi M^2} \right)}{q^2 - M^2 + iM\Gamma}$$

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where $\Gamma$ was called the width of the unstable state. It turns out that such a propagator does not fulfill the Ward identity so it is incompatible with the gauge invariance. The correct form of the propagator must be the following

$$\tilde{D}_{\mu\nu}(q) = \frac{i \left( -g_{\mu\nu} + (1 - \xi) \frac{q_{\mu}q_{\nu}}{q^2 - \xi(M^2 - iM\Gamma)} \right)}{q^2 - M^2 + iM\Gamma}$$

(18)

Moreover it is not sufficient to insert the complex mass in the propagator but it must also be inserted in all the vertices with the $Z_0$ and $W^\pm$ bosons. This rule gives very important information about the properties of the unstable states and is a strong indication that the complex mass is the intrinsic property of the unstable states and not only a mathematical trick. The practical implementation of this idea is given in the rigged Hilbert space quantum mechanics for the unstable states.

4 Rigged Hilbert space of unstable states

From the previous considerations we have seen that unstable states appear both in non relativistic quantum mechanics and in the relativistic quantum field theory. The description in these two cases is drastically different.

In non relativistic quantum mechanics the unstable states are long lived states described either as the eigenstates of the Hamiltonian with complex eigenvalue of energy or as the stationary eigenstates of the Hamiltonian for the scattering on the spherical potential well. In both cases the eigenfunctions do not belong to the Hilbert space so some of the axioms of the orthodox quantum mechanics have to be relaxed.

In relativistic quantum field theory of the situation is completely different. The unstable particle or the resonance is elementary and its field appears in the Lagrangian of the system. The kinematical mass relation allows the decay of the unstable elementary state. The elementary unstable particles are eliminated from the set of asymptotic states. Such a theory remains unitary provided the modification of the propagator of the unstable state is done according to the Dyson expansion. It should be noted that in such an approach the notion of a vector for an unstable state does not exist in quantum field theory: the unstable states appear only as intermediate states and one has to use the right form of the propagator for them. This is the only way how the unstable states appear in the theory. The important
feature of this approach is that the kinematical mass of an unstable state is complex and such a mass should be used without exception not only in the construction of the propagators but also in the vertices. The complex mass is taken from the position of the pole of the $S$-matrix.

It is the natural now to ask the question: does there exist a unified formalism that is mathematically precise and has the physical properties that were required by the two approaches discussed earlier? The formalism with such properties is implemented by the rigged Hilbert space quantum mechanics whose principal properties are the following [11]:

- The linear space for states and observables is provided by the rigged Hilbert space which is a triplet of the spaces, one of which is the ordinary Hilbert space of the system, the two others is a (dense) subspace of it and the dual thereof containing the (Dirac) kets and other generalized vectors like Gamow states.

- The dynamical differential equations of motion are identical to those in conventional quantum mechanics but their boundary conditions are not the Hilbert space conditions but given by the dense subspace.

- The conventional quantum mechanics is contained as a “limiting” case within the rigged Hilbert space quantum mechanics.

The rigged Hilbert space quantum mechanics which is an extension of the conventional quantum mechanics describes the wider class of the physical phenomena and also adds mathematical precision to the conventional quantum mechanics. The main important results of the rigged Hilbert space quantum mechanics are [11, 12, 13]:

- Dirac formalism of bras and kets.

- Precise meaning of the Lippmann-Schwinger kets.

- Description of the unstable states.

I will briefly discuss here only the problem of the unstable states. The unstable states appear only in scattering as intermediate states. In terms of the energy wave function the rigged Hilbert space consists of the following three spaces: the Hilbert space $\mathcal{H}$ is the space $L^2$ of the Lebesgue square integrable functions. For the definition of the rigged Hilbert space one has
to choose a linear space $\Phi$ with stronger than Hilbert space convergence (topology) which is dense in the Hilbert space. Then together with the dual space they form the rigged Hilbert space

$$\Phi \subset \mathcal{H} \subset \Phi^\times.$$  \hspace{1cm} (19)

Usually, for Dirac kets, one chooses for $\Phi$ the Schwartz space. For unstable states the choice of the RHS is $\Phi_+ \subset \mathcal{H} \subset \Phi^\times_+$ where the space $\Phi_+$ is in the energy representation the space $\mathcal{H}^2_+ \cap \mathcal{S}|_{\mathbb{R}_+}$ where $\mathcal{H}^2_+$ is the space of the Hardy class functions in the upper complex half plane and $\mathcal{S}$ is the Schwartz function space.

The state of the unstable particle is defined in terms of the Lippmann-Schwinger kets $|[j, s_R]b^−⟩ \in \Phi^\times_+$

$$|[j, s_R]b^−⟩ = \frac{1}{2\pi} \int_{-\infty}^{\infty} |[j, s]b^−⟩ \frac{1}{s - s_R}.$$  \hspace{1cm} (20)

Here $|[j, s_R]b^−⟩$ denotes the state vector of the unstable particle (Gamow ket) and $|[j, s]b^−⟩$ is the Lippmann-Schwinger ket, $j$ is the spin, $s$ the energy square and $b$ are other quantum numbers. The complex value $s_R$ is the position of the $S$-matrix pole on the second sheet of the lower complex energy plane. The state (20) may be considered as the generalization of the non-relativistic wave function $\Psi$ and its definition here is fully relativistic. The vector defined by (20) is a (generalized) eigenvector of the selfadjoint operator $M^2 = P^\mu P_\mu$ with $s_R$ as its eigenvalue

$$P^\mu P_\mu |[j, s_R]b^−⟩ = s_R |[j, s_R]b^−⟩$$  \hspace{1cm} (21)

which means that the square of the mass of the unstable state is a complex number – the position of the pole of the $S$-matrix. The situation here is identical as in the case of quantum field theory. The property (21) is a justification of the quantum field theory rule (16).

The important property of the Gamow vectors (20) is its time evolution. Here it turns out that the time evolution of the vector (20) is permitted only for $t \geq 0$ \hspace{1cm} (12) and is given by

$$e^{-iH^\times t}|[j, s_R]b^−⟩ = e^{-iE_R t} e^{-\Gamma_R t/2} |[j, s_R]b^−⟩ \text{ for } t \geq 0.$$  \hspace{1cm} (22)

where $E_R$ and $\Gamma_R$ are the following parameterizations of the complex $s_R$ in terms of two real numbers

$$s_R = \left( E_R - i\frac{\Gamma_R}{2} \right)^2.$$  \hspace{1cm} (23)
This is a different parameterization of the complex mass than the one used in (16)–(18). From (22) follows that the decay probability of the Gamow ket is exponential and the lifetime of this exponential decay is \( \tau = \frac{1}{\Gamma_R} \). Therefore (23) is a better parameterization than (16). The vectors (20) form the semigroup representation of the Poincaré group transformations into the forward light cone, they are classified by \([j, s_R]\) [14]. This representation is the minimally complex representation where the momentum is complex but the velocity is real. The transformation property under the Poincaré semigroup transformation, of which Eq. (22) is the special case, introduces a new quantum mechanical arrow of time which distinguishes the semigroup time evolution of scattering and decay theory based on (19) from the unitary Poincaré group evolution for the asymptotic states.

5 Conclusions

I have presented here the original quantum mechanical and field theoretic approaches to the problem of unstable states and compared them to the rigged Hilbert method. My discussion can be summarized as follows.

1. The vectors of the unstable states used in the conventional textbook treatments of this subject do not belong to the Hilbert space.

2. In quantum field theory there are no vectors corresponding to the unstable states. Resonances are included in the theory only as intermediate states with the special form of the propagator that corresponds to the complex mass.

3. The rigged Hilbert space quantum mechanics gives the precise mathematical meaning to the vectors of the unstable states in quantum mechanics.

4. The RHS vectors for the unstable states have a complex mass eigenvalue. This explains the field theoretic rule

\[
M^2 \rightarrow M^2 - iM\Gamma = \left(M_R - i\frac{\Gamma_R}{2}\right)^2.
\]

5. The rigged Hilbert space quantum mechanics predicts new phenomena like the new quantum mechanical arrow of time.

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