Connectivity concerning the last two subconstituents of a $Q$-polynomial distance-regular graph

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Abstract

Let $\Gamma$ be a $Q$-polynomial distance-regular graph of diameter $d \geq 3$. Fix a vertex $\gamma$ of $\Gamma$ and consider the subgraph induced on the union of the last two subconstituents of $\Gamma$ with respect to $\gamma$. We prove that this subgraph is connected.

1 Introduction

All the graphs considered here will be finite and undirected, with no loops nor multiple edges. We briefly review the key definitions and basic results involving distance-regular graphs. For other notations and definitions, see [5, 6, 14]. Let $\Gamma$ be a connected graph with vertex set $X$. For $x, y \in X$, the distance between $x$ and $y$ is denoted by $\partial(x, y)$, and any path between $x$ and $y$ of length $\partial(x, y)$ is called geodesic. The diameter $\max_{x,y \in X} \partial(x, y)$ of $\Gamma$ is denoted by $d$. For an integer $k \geq 0$, $\Gamma$ is said to be regular with valency $k$ whenever each vertex of $\Gamma$ is adjacent to exactly $k$ vertices of $\Gamma$. The graph $\Gamma$ is called distance-regular whenever for all integers $0 \leq h, i, j \leq d$ there exists a nonnegative integer $p_{ij}^h$ such that for all $x, y \in X$ with $\partial(x, y) = h$,

$$p_{ij}^h = \{|z \in X : \partial(z, x) = i, \partial(z, y) = j\}|.$$

For the rest of this paper we assume that $\Gamma$ is distance-regular of diameter $d \geq 2$. Note that $\Gamma$ is regular with valency $k = p_{11}^0$; to avoid trivialities we always assume $k \geq 3$. Let $A_0, A_1, \ldots, A_d$ denote the distance matrices of $\Gamma$ (see [5, p.127]). Then $A_0, A_1, \ldots, A_d$ form a basis for a commutative semisimple $\mathbb{R}$-algebra $M$ known as the Bose-Mesner algebra of $\Gamma$. The algebra $M$ has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d),$$

$$I = E_0 + \cdots + E_d,$$

$$E_0 = |X|^{-1} J,$$

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where $I$ is the identity matrix and $J$ is the all ones matrix (see [5] Thm 2.6.1)). We refer to $E_0, E_1, \ldots, E_d$ as the primitive idempotents of $\Gamma$. The primitive idempotent $E_0$ is called trivial. The ordering $E_0, E_1, \ldots, E_d$ is said to be $Q$-polynomial whenever for $0 \leq i \leq d$ there exists a polynomial $q_i$ of degree $i$ such that $E_i = q_i(E_1)$ (where the matrix multiplication is done entry-wise). For a primitive idempotent $E$ of $\Gamma$, we say that $\Gamma$ is $Q$-polynomial with respect to $E$ whenever there exists a $Q$-polynomial ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents such that $E = E_1$. The graph $\Gamma$ is called $Q$-polynomial whenever it is $Q$-polynomial with respect to at least one primitive idempotent.

We now recall the antipodal property. Define a binary relation $\sim$ on $X$ such that for all $x, y \in X$, $x \sim y$ whenever $x = y$ or $\partial(x, y) = d$. The graph $\Gamma$ is called antipodal whenever $\sim$ is an equivalence relation. The graph $\Gamma$ is said to be primitive whenever $\Gamma$ is not bipartite nor antipodal (see [5] Thm 4.2.1]). A long-standing conjecture of Bannai and Ito [1, p. 312] states that if $\Gamma$ is primitive and $d$ is sufficiently large, then $\Gamma$ is $Q$-polynomial. For more information about the $Q$-polynomial property, see [1,5] or Chapter 5.

For $0 \leq i \leq d$ and $\gamma \in X$, let $\Gamma_i(\gamma)$ denote the set of vertices in $\Gamma$ at distance $i$ from $\gamma$. The subgraph induced by $\Gamma_i(\gamma)$ is called the $i$-th subconstituent of $\Gamma$ with respect to $\gamma$. Combinatorial and algebraic properties of these subconstituents have been studied by several authors (see [9,10,16] for example). The graph $\Gamma$ is called strongly-regular whenever $d = 2$. If $\Gamma$ is strongly-regular and primitive, then with respect to any vertex the second subconstituent of $\Gamma$ is connected. See [6, p. 126] for an algebraic proof, and [16] for a combinatorial proof. Answering a question of Brouwer [4], Cioabă and Koolen [10] generalized this result in the following way. Consider the dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ for the second largest eigenvalue of $\Gamma$ (see equation (2.1) for a definition). By [5, Ch. 4] there exists a unique integer $s$ $(1 \leq s \leq d)$ such that $\theta_{s-1}^* > 0$ and $\theta_s^* \leq 0$. Then for any vertex $\gamma$ of $\Gamma$ the subgraph induced on $\bigcup_{i=0}^d \Gamma_i(\gamma)$ is connected [10]. In [10] the authors also prove that $s \geq d/2$ and pose the following problem.

**Problem 1.1** (Cioabă-Koolen [10]). Assume that $\Gamma$ is primitive and $d \geq 3$. Is it true that for any vertex $\gamma$, the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected?

In [10], this was shown to be true if $d \in \{3, 4\}$. In this note, we show that it is true for all $d \geq 3$, provided that $\Gamma$ is $Q$-polynomial. We now state our main result.

**Theorem 1.1.** Let $\Gamma$ be a $Q$-polynomial distance-regular graph of diameter $d \geq 3$. Then for any vertex $\gamma$ of $\Gamma$ the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected.

The main tool for our proof is Terwilliger’s balanced set condition (see [21,22] or Theorem 2.1 in the next section). This condition has been used by Lewis [20] to prove that the girth is at most 6 for any $Q$-polynomial distance-regular graph of valency at least 3.

## 2 Proof of the main result

For a primitive idempotent $E$ of $\Gamma$, there exist real numbers $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ (called the dual eigenvalues of $\Gamma$ with respect to $E$) such that

$$E = |X|^{-1} \sum_{h=0}^d \theta_h^* A_h. \quad (2.1)$$
We equip the vector space $\mathbb{R}^X$ with an inner product such that $\langle u, v \rangle = u^t v$ for all $u, v \in \mathbb{R}^X$. For $x \in X$, let $\hat{x}$ denote the vector in $\mathbb{R}^X$ with $x$-coordinate 1 and all other coordinates 0. Equation (2.1) implies that
\[
\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^*,
\]
where $i = \partial(x, y)$. The main tool for our proof is the following theorem.

**Theorem 2.1 (Terwilliger [21,22])**. Let $\Gamma$ be a distance-regular graph with diameter $d \geq 3$, and let $E$ denote a nontrivial primitive idempotent of $\Gamma$ with dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Then $\Gamma$ is $Q$-polynomial with respect to $E$ if and only if $\theta_0^* \notin \{\theta_1^*, \ldots, \theta_d^*\}$ and
\[
\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{u \in \Gamma_j(x) \cap \Gamma_i(y)} E\hat{w} = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_j^*} (E\hat{x} - E\hat{y})
\]  
(2.3)

for all integers $h, i, j$ with $1 \leq h \leq d$ and $0 \leq i, j \leq d$ and all vertices $x, y$ with $\partial(x, y) = h$. Furthermore, if the conditions above hold, then $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct.

The equation (2.3) is usually called the balanced set condition. We are now ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1** Let $E$ be a primitive idempotent of $\Gamma$ with respect to which $\Gamma$ is $Q$-polynomial. We will use a proof by contradiction, and assume that there exists $\gamma \in X$ such that the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is disconnected. Let $C$ be the vertex set of a connected component of the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$. Let the set $\Delta$ consist of the vertices in $X$ that lie on a geodesic from $\gamma$ to $C$. The set $\Delta$ is properly contained in $X$ since $C \neq \Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$. We partition $\Delta = \bigcup_{j=0}^d \Delta_j$ where $\Delta_j = \Delta \cap \Gamma_j(\gamma)$ for $0 \leq j \leq d$. Note that for $0 \leq j \leq d-1$, each vertex in $\Delta_j$ has at least one neighbor in $\Delta_{j+1}$.

A vertex in $\Delta$ will be called a border whenever it is adjacent to a vertex in $X \setminus \Delta$. Since $\Delta \neq X$ and $\Gamma$ is connected, $\Delta$ contains at least one border vertex. Let $t$ denote the maximal integer $j$ ($0 \leq j \leq d$) such that $\Delta_j$ contains a border vertex. By the construction $1 \leq t \leq d-2$.

Pick a border vertex $z \in \Delta_t$. There exists $x \in \Delta_{t+2}$ such that $\partial(x, z) = 2$. Let $y \in X \setminus \Delta$ be a neighbor of $z$. Define $\xi = \partial(\gamma, y)$. By the triangle inequality $\xi \in \{t-1, t, t+1\}$. Note that $\xi \neq t-1$; otherwise $y$ is on a geodesic from $\gamma$ to $C$ passing through $z$, forcing $y \in \Delta$ for a contradiction. Therefore $\xi = t$ or $\xi = t+1$.

We next show that $\partial(x, y) = 3$. Because $\partial(x, z) = 2$ and $\partial(z, y) = 1$, the triangle inequality implies that $\partial(x, y) \leq 3$. By the maximality of $t$ and since $x \in \Delta_{t+2}$, we see that $x$ is not a border and not adjacent to a border. Therefore $\Delta$ contains all the vertices of $\Gamma$ that are at distance at most 2 from $x$. The vertex $y$ is not in $\Delta$, so $\partial(x, y) \geq 3$. We have shown that $\partial(x, y) = 3$.

Note that $\Gamma_1(x) \cap \Gamma_2(y) \subset \Gamma_{t+1}(\gamma)$ and $\Gamma_2(x) \cap \Gamma_1(y) \subset \Gamma_t(\gamma)$. We apply the balanced set condition (2.3) to $x$ and $y$ using $h = 3, i = 1, j = 2$ and then take the inner product of each side with $E\hat{\gamma}$; this gives
\[
p_{12}^3(\theta_{t+1}^* - \theta_j^*) = p_{12}^3 \frac{\theta_1^* - \theta_j^*}{\theta_0^* - \theta_j^*} (\theta_{t+2}^* - \theta_j^*). 
\]  
(2.4)
There exists \( y' \in \Gamma_{t-1}(\gamma) \cap \Gamma_1(z) \). We have \( \partial(x, y') = 3 \) and \( \Gamma_1(x) \cap \Gamma_2(y') \subset \Gamma_{t+1}(\gamma) \) and \( \Gamma_2(x) \cap \Gamma_1(y') \subset \Gamma_t(\gamma) \). We apply the balanced set condition (2.3) to \( x \) and \( y' \) using \( h = 3, i = 1, j = 2 \) and then take the inner product of each side with \( E_{\gamma} \); this gives

\[
p_{12}^2(\theta_{t+1}^* - \theta_t^*) = p_{12}^2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*}(\theta_{t+2}^* - \theta_{t-1}^*). \tag{2.5}
\]

Comparing (2.4) and (2.5) we obtain \( \theta_{t+1}^* = \theta_{t-1}^* \). We have \( \xi = t - 1 \) since \( \theta_0^*, \theta_1^*, \ldots, \theta_d^* \) are mutually distinct. We mentioned earlier that \( \xi \neq t - 1 \), for a contradiction. We conclude that the subgraph induced on \( \Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma) \) is connected. \( \square \)

To see how Theorem 1.1 is best possible, assume that \( \Gamma \) is the Odd graph \( O_{d+1} \) with \( d \geq 3 \). Recall that the vertices of \( \Gamma \) are the \( d \)-subsets of a set \( \Omega \) of size \( 2d + 1 \). Two vertices \( \alpha \) and \( \beta \) are adjacent whenever \( \alpha \cap \beta = \emptyset \). The diameter of \( \Gamma \) is \( d \) and its intersection numbers are known (see [2] or [5, Prop 9.1.7]). For \( 0 \leq h \leq d \), we have \( p_{ih}^h = 0 \) if \( h < d \) and \( p_{ih}^h = \lceil \frac{d+1}{2} \rceil \) if \( h = d \). So with respect to any vertex of \( \Gamma \), the \( h \)-th subconstituent has no edges if \( h < d \) and is regular with valency \( \lceil \frac{d+1}{2} \rceil \) if \( h = d \).

**Lemma 2.2.** Assume that \( \Gamma \) is the Odd graph \( O_{d+1} \) with \( d \geq 3 \). For any \( \gamma \in X \), the number of connected components in the \( d \)-th subconstituent of \( \Gamma \) with respect to \( \gamma \) is equal to \( \frac{(2m)_m}{(m)_m} \), where \( m = d/2 \) if \( d \) is even and \( m = (d+1)/2 \) if \( d \) is odd. Moreover, this \( d \)-th subconstituent is not connected.

**Proof.** From the intersection numbers of \( \Gamma \) we obtain \( |\Gamma_d(\gamma)| = \binom{d}{m} \binom{d+1}{m} \). Using the results of Biggs [2], each connected component of \( \Gamma_d(\gamma) \) is isomorphic to the bipartite double (see [5 Section 1.11]) of \( O_{r+1} \), where \( r = d/2 \) if \( d \) is even and \( r = (d-1)/2 \) if \( d \) is odd. This bipartite double has \( 2 \binom{2r+1}{r} \) vertices. The result follows after some routine algebra. Note that the lemma also follows by observing that \( \Gamma_d(\gamma) \) consists of the vertices at distance \( m \) from \( \gamma \) in the Johnson graph \( J(2d+1, d) \). \( \square \)

Note also that for \( O_{d+1} \) the subgraph induced on \( \Gamma_1(\gamma) \cup \Gamma_2(\gamma) \) is disconnected. Next assume that \( \Gamma \) is the folded \( (2d+1) \)-cube. It has diameter \( d \) and for \( 1 \leq h \leq d - 1 \), the \( h \)-subconstituent of \( \Gamma \) with respect to any vertex has no edges (see [5, p. 264]), and consequently not connected. Gardiner, Godsil, Hensel and Royle [16] proved that the diameter of the second subconstituent of a primitive strongly-regular graph is at most three. It would be interesting to extend this result to distance-regular graphs with diameter \( d \geq 3 \). For example, if \( \Gamma \) is a distance-regular with \( d = 3 \), then what is the diameter of \( \Gamma_3(\gamma) \) when \( \Gamma_3(\gamma) \) is connected ? Another related problem from [10] is to classify the distance-regular graphs \( \Gamma \) of diameter 3 such that \( \Gamma_3(\gamma) \) is disconnected for some vertex \( \gamma \). See [19] for related results.

The vertex-connectivity of a primitive distance-regular graph is equal to its valency, as proved by Brouwer and Mesner [8] for diameter \( d = 2 \), and by Brouwer and Koolen [7] for \( d \geq 3 \). Brouwer and Haemers [6] p. 127] observed that for certain strongly-regular graphs constructed by Haemers [17] p. 76] the vertex-connectivity of their second subconstituent is strictly less than the valency. It would be interesting to determine lower bounds for the vertex-connectivity and edge-connectivity of the subconstituents for a distance-regular graph with \( d \geq 3 \). See [3][11][13][15][18] for related connectivity results concerning distance-regular graphs and association schemes.
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