Numerical Solution of Fractional Integro-differential Equations with Weakly Singular Kernels via Bernstein Polynomial

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Abstract. This paper is concerned with obtaining approximate numerical solutions of a class of fractional Volterra integro-differential equations with weakly singular kernels which using Bernstein Polynomials as basis in collocation methods (BPCM). The approximate solution is obtained by expanding the functions in terms of Bernstein Polynomial, whose unknown coefficients are determined by solving final system of linear equations. The convergence analysis of the proposed method are taken into consideration and Numerical examples are included to confirm the efficiency and accuracy of the method.

1. Introduction

Fractional order dynamics derives from the physical problems such as electromagnetic waves and dynamics of interfaces between nanoparticles and substrates. Several numerical calculation techniques have been developed for the approximate solutions of these types of equations, such as the spline collocation method proposed by Pedas [1], the piecewise polynomial collocation method proposed by Brunner [2], the second kind of Chebyshev polynomial collocation method proposed by Nemati [3], Yi [7] considered polynomial spline collocation methods to solve linear fractional integro-differential equations of Volterra type. In [8], the authors used a collocation method to obtain approximate solution for nonlinear fractional integro-differential equations, and etc. Although these method has its own advantages, there are still little shortcoming, such as the solution speed is slow or the accuracy is not high, etc. In this paper we consider using Bernstein Polynomials as basis in collocation methods to obtain approximate numerical solutions of a class of fractional Volterra integro-differential equations with weakly singular kernels. Consider the following fractional integro-differential equation which has weakly singular kernel

\[ ^C_D_0^\alpha y(t) = g(t) + p(t)y(t) + \int_0^t (t-s)^{-\beta} y(s)ds, \alpha > 0, f(t), 0 \leq \beta \leq 1, t \in I(T) \]

\[ y^{(i)}(0) = y_0^{(i)}, i = 0, 1, \cdots, n-1, \]

with \( y(t) \) an unknown function, \( g(t) \) and \( p(t) \) continuous functions known on \( I(T) = [0, T] \), \( y_0^{(i)} = (i = 0, 1, \cdots, n-1) \) real numbers, \( n = \lceil \alpha \rceil \) the up rounding of \( \alpha \), and \( ^C_D_0^\alpha \) Caputo
differential operator of order \( \alpha \).

2. Algorithm and Convergence Analysis

First, we introduce an important Lemma.

**Lemma 1** ([4]). Let \( \alpha > 0 \) and \( \alpha \neq N \). If the right-hand term of equation (1) is continuous, the initial value problem (1) is equivalent to the following Volterra integral equation of the second kind

\[
y(t) = f(t) + \int_0^t (t-w)^{\alpha-1} k(t,w)y(w)dw, \quad t \in I(T),
\]

where

\[
f(t) = f_1(t) + I_1^\alpha g(t), \quad f_1(t) = \sum_{i=0}^{[\alpha]-1} y^{(i)}(0) \frac{\zeta}{\alpha}, \quad k(t,w) = \frac{1}{\Gamma(\alpha)} [p(w) + (t-w)^{1-\beta} B(\alpha,1-\beta)],
\]

\[
B(\alpha, b) = \int_0^t (1-s)^{b-1}ds.
\]

Obviously, the equation is transformed into a corresponding linear second kind of weak singular Volterra integral equation, and we will describe the collocation method based on the Bernstein polynomial for solving (2). The Algorithm are described as follows.

**Step 1.** By reconstructing the right end term of equation (2), we can get

\[
\int_0^t (t-w)^{\alpha-1} k(t,w)y(w)dw = \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} p(w)y(w)dw
\]

\[+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1+\beta} B(\alpha,1-\beta)y(w)dw
\]

\[= I_1^\alpha (p(t)y(t)) + \frac{B(\alpha,1-\beta)}{\Gamma(\alpha)} \int_0^t (t-w)^{1+\alpha-\beta-1} y(w)dw
\]

\[= I_1^\alpha (p(t)y(t)) + \frac{B(\alpha,1-\beta)\Gamma(1+\alpha-\beta)}{\Gamma(\alpha)} I_1^{1+\alpha-\beta} y(t)
\]

**Step 2.** let

\[
\chi_{\alpha,\beta} = \frac{B(\alpha,1-\beta)\Gamma(1+\alpha-\beta)}{\Gamma(\alpha)}
\]

then equation (2) can be expressed as

\[
y(t) = f_1(t) + I_1^\alpha g(t) + I_1^\alpha (p(t)y(t)) + \chi_{\alpha,\beta} I_1^{1+\alpha-\beta} y(t),
\]

where

\[
f_1(t) = \sum_{i=0}^{[\alpha]-1} y^{(i)}(0) \frac{\zeta}{\alpha},
\]

\[
I_1^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)d\tau, \alpha > 0,
\]

\[
I_1^\alpha (p(t)y(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} p(\tau)y(\tau)d\tau, \alpha > 0
\]

\[
I_1^{1+\alpha-\beta} y(t) = \frac{1}{\Gamma(1+\alpha-\beta)} \int_0^t (t-\tau)^{\alpha-\beta} y(\tau)d\tau, \alpha > 0, 0 < \beta < 1
\]
Step 3. To find an appropriate solution of (3), $y(t)$ is approximated in the Bernstein polynomial as

$$y_{m,m}(t) = \frac{m!}{i!(m-i)!} t^i (1-t)^{m-i}, 0 \leq i \leq m,$$  

(8)

Step 4. A series of equations (4) - (8) are substituted into (3), and the improved Bernstein collocation method is used to discretize the integral term with the corresponding integral formula. The following linear algebraic equations are obtained

$$A_{n\times n}U_{n\times n} = G_{n\times 1} + A_{n\times n}P_{n\times n}U_{n\times n},$$  

(9)

where $y(t) \approx AU$, $I^0 g(t) \approx G$, $I^0 p(t) \approx P$, $I^0 y(w) \approx AUw$.

Step 5. According to the result of system (9), the numerical solution of the original integro-differential equation is obtained.

We next derive the convergence analysis of Bernstein polynomials for the expansion of functions. Firstly, we have the following lemma.

Lemma 2([6]). Let the kernel function of the integral equation have weak singularity, all the kernel functions are bounded from the second iteration.

Proof. Because of the kernel function satisfies

$$K(M, M') = \frac{H(M, M')}{r^\alpha}, (0 < \alpha < n),$$  

where $M, M'$ are any two points in the m-dimensional finite field $\Omega$, $r$ is the distance between $M, M'$ two points. Then the estimation of kernel function after m times iteration is

$$\left|K_m(M, M')\right| < \begin{cases} C_m, & m\alpha - (m-1)n > 0, \\ C_m, & m\alpha - (m-1)n > 0, \end{cases}$$

where the constant $C_m > 0$. Then, if the integer m satisfies the following inequality $m > n - \frac{n}{n - \alpha}$, then $K_m(M, M')$ is a bounded function.

Theorem 3. Suppose $y(t) \in H^{n+1}(I(T))$ as the analytical solution of (1), let $y_N(t)$ is the expansion of Bernstein polynomials of $y(t)$ and it is the numerical solution of (2). Then

$$\|y(t) - y_N(t)\|_2 \to 0,$$

as $N \to \infty$, for all $t \in I(T)$, where $\|\cdot\|_2$ represents the 2-norm of a vector.

Proof. Substituting $y_N(t)$ into equation (2), we get

$$y_N(t) + \lambda \int_0^1 K_2(t, w)y_N(w)dw = f(t), 0 \leq t \leq 1,$$

then the corresponding error function can be defined as

$$\|y_N\|_2 = \|y_N(t) - y(t)\|_2 + \lambda \int_0^1 \|K_2(t, w)\|_2 \|y_N(w) - y(w)\|_2 dw.$$

Because $y(t) = \lim_{N \to \infty} y_N(t)$, then

$$\lim_{N \to \infty} \|y_N(t) - y(t)\|_2 = 0, \lim_{N \to \infty} \|y_N(w) - y(w)\|_2 = 0.$$
According to Lemma 2, \( \|K_z(t, w)\| \leq C, \quad C \) is a constant, then we obtain

\[
\lim_{N \to \infty} \|x_N\| = \lim_{N \to \infty} \|y_N(t) - y(t)\|_2 + \lim_{N \to \infty} \|\int_0^t c(t, w) y_N(w) - y(w)\|_2 \, dw
\]

\[
\leq \lim_{N \to \infty} \|y_N(t) - y(t)\|_2 + \|c(t) \cdot \lim_{N \to \infty} \|y_N(w) - y(w)\|_2 \, dw
\]

\[
= 0 + \|c(t) \cdot 0\, dt = 0,
\]

which completes the proof of Theorem 3.

3. Numerical Example

In this subsection we present an example to display the accuracy and applicability of the proposed method.

Example 1. Given the following fractional order integro-differential equation with weak singular kernel [3]

\[
\frac{c}{6} D^\frac{3}{6} y(t) = g(t) + p(t) y(t) + \int_0^t (t - s)^{-\frac{1}{2}} y(s) \, ds, t \in [0, 1],
\]

where

\[
g(t) = \frac{6^3}{t^3} + \left( \frac{32}{35} - \frac{1}{t^3} \right) \frac{11}{6} + \frac{7}{3} t, \quad p(t) = -\frac{32}{35} t^2, \quad y(0) = 0.
\]

Notice that the exact solution of this problem is \( y(t) = t^3 + t^2 \).

According to the method described in Section 2, the fractional order integro-differential equation is transformed into the second kind of weak singular Volterra integral equation, to solve which the Bernstein Polynomials is used, and the results are compared with those in reference [3].

In reference [3], the second kind of Chebyshev polynomials after transformation is used to solve the problem, and the results as \( a_0 = 0.6471911, a_1 = 0.477613, a_2 = 0.121475 \).

Then the approximate solution of this problem is

\[
y_2(t) = 0.0563899 - 0.0331502 t + 1.9436 t^2.
\]

The numerical results for the equation are listed in Table 1 and Table 2.

| Nodes | exact solution | reference [3] | BPCM |
|-------|---------------|---------------|------|
| 0     | 0             | 0.0564        | -0.0033 |
| 0.2000| 0.1250        | 0.1275        | 0.1281 |
| 0.4000| 0.3587        | 0.3541        | 0.3579 |
| 0.6000| 0.7221        | 0.7362        | 0.7222 |
| 0.8000| 1.2547        | 1.2738        | 1.2570 |
| 1.0000| 2.0000        | 1.9668        | 2.0000 |

| N    | \( e_N(t) \) | Computing time |
|------|--------------|----------------|
| 3    | 0.0079       | 0.021659       |
| 4    | 0.0007       | 0.021659       |
| 5    | 0.0006       | 0.022659       |

Table 1 shows that the numerical solution by BPCM is effective and converge to its exact solution. We
also applied this method to the problem with different index N and display the numerical result in Table 2 which shows that the proposed method has very high accuracy and the computing times imply that the algorithm is very fast.

4. Conclusion
In this paper, we use Bernstein Polynomials as basis in collocation methods to give the approximate solutions of fractional Volterra integro-differential equations with weakly singular kernels. The approximate solution is obtained by expanding the functions in terms of Bernstein Polynomial, whose unknown coefficients are determined by solving final system of linear equations. The convergence analysis of the proposed method are taken into consideration and Numerical examples are included to confirm the efficiency and accuracy of the method.

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