New class of $G$-Wolfe-type symmetric duality model and duality relations under $G_f$-bonvexity over arbitrary cones

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Abstract
This paper is devoted to theoretical aspects in nonlinear optimization, in particular, duality relations for some mathematical programming problems. In this paper, we introduce a new generalized class of second-order multiobjective symmetric $G$-Wolfe-type model over arbitrary cones and establish duality results under $G_f$-bonvexity/$G_f$-pseudobonvexity assumptions. We construct nontrivial numerical examples which are $G_f$-bonvex/$G_f$-pseudobonvex but neither $\eta$-bonvex/$\eta$-pseudobonvex nor $\eta$-invex/$\eta$-pseudoinvex.

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1 Introduction
It is an undeniable fact that all of us are optimizers as we all make decisions for the sole purpose of maximizing our quality of life, productivity in time, and our welfare in some way or another. Since this is an ongoing struggle for creating the best possible among many inferior designs and is always the core requirement of human life, this fact yields the development of a massive number of techniques in this area, starting from the early ages of civilization until now. The efforts and lives behind this aim dedicated by many brilliant philosophers, mathematicians, scientists, and engineers have brought a high level of civilization we enjoy today. The decision process is relatively easier when there is a single criterion or object in mind. The process gets complicated when we have to make decisions in the presence of more than one criteria to judge the decisions. In such circumstances a single decision that optimizes all the criteria simultaneously may not exist. For handling such type of situations, we use multiobjective programming, also known as multiattribute optimization, which is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints. Multiobjective optimization problems can be found in various fields such as product and process design, finance, aircraft design, the oil and gas industry, automobile design, and other where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives.
Mangasarian [1] was the first who introduced the concept of second-order duality for nonlinear programming. Gulati and Gupta [2] introduced the concept of $\eta_1$-bonvexity/$\eta_2$-boncavity and derived duality results for a Wolfe-type model. The concept of $G$-invex function is given by Antczak [3] and derived some duality results for a constrained optimization problem. Later on, generalizing his earlier work, Antczak [4] introduced $G_f$-invex functions for multivariate models and obtained optimality results for multiobjective programming problems. Liang et al. [5] discussed conditions for optimality and duality in a multiobjective programming problem. Bhatia and Garg [6] discussed the concept of $(V, p)$ invexity for nonsmooth vector functions and established duality results for multiobjective programs. Jayswal et al. [7] discussed multiobjective fractional programming problems involving an invex function. Stefanescu and Ferrara [8] studied new invexities for multiobjective programming problem. Several researchers [9–21] have studied related areas.

This paper is organized as follows. In Sect. 2, we give some preliminaries and definitions used in this paper and also a nontrivial example for such type functions. In Sect. 3, we formulate second-order multiobjective symmetric $G$-Wolfe-type dual programs over arbitrary cones. We prove weak, strong, and converse duality theorems by using $G_f$-bonvexity/$G_f$-pseudobonvexity assumptions over arbitrary cones. Finally, we construct nontrivial numerical examples that are $G_f$-bonvex/$G_f$-pseudobonvex but neither $\eta$-bonvex/$\eta$-pseudobonvex nor $\eta$-invex/$\eta$-pseudoinvex functions.

### 2 Preliminaries and definitions

Let $f = (f_1, f_2, f_3, \ldots, f_k) : X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subseteq R^n$, and let $I_{f_i}(X)$, $i = 1, \ldots, k$, be the range of $f_i$, that is, the image of $X$ under $f_i$. Let $G_f = (G_{f_1}, G_{f_2}, \ldots, G_{f_k}) : R \to R^k$ be a differentiable function such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of $I_{f_i}, i = 1, 2, 3, \ldots, k$.

**Definition 2.1** The positive polar cone $S^*$ of a cone $S \subseteq R^n$ is defined by

\[ S^* = \{ y \in R^n : x^T y \geq 0 \}. \]

Consider the following vector minimization problem:

\[
\begin{align*}
\text{Min.} & \quad f(y) = \begin{bmatrix} f_1(y), f_2(y), \ldots, f_k(y) \end{bmatrix}^T \\
\text{subject to} & \quad S^0 = \{ y \in S \subseteq R^n : h_j(y) \leq 0, j = 1, 2, 3, \ldots, m \}, \quad (\text{MP})
\end{align*}
\]

where $f = (f_1, f_2, \ldots, f_k) : S \to R^k$ and $h = (h_1, h_2, \ldots, h_m) : S \to R^m$ are differentiable functions on $S$.

**Definition 2.2** $\bar{y} \in S^0$ is an efficient solution of (MP) if there exists no other $y \in S^0$ such that $f_r(y) < f_r(\bar{y})$ for some $r = 1, 2, 3, \ldots, k$ and $f_i(y) \leq f_i(\bar{y})$ for all $i = 1, 2, 3, \ldots, k$.

**Definition 2.3** If there exists a function $\eta : S \times S \to R^m$ such that for all $y \in S$,

\[ f_i(y) - f_i(v) \geq \eta^T(y, v) \nabla_y f_i(v) \quad \text{for all } i = 1, 2, 3, \ldots, k, \]

then $f$ is called invex at $v \in S$ with respect to $\eta$. 
**Definition 2.4** If there exist $G_f : I_f(S) \to R$ and $\eta : S \times S \to R^n$ such that for all $y \in S$,

$$\eta^T(y,v)G'_f(f(y))\nabla_y f(y) + \frac{1}{2}p^T G''_f(f(y))\nabla_y f(y)\nabla_y f(y)^T \geq 0$$

then $f_i$ is called $G_f$-pseudoinvex at $u \in S$ with respect to $\eta$.

**Definition 2.5** If there exist $G_f : I_f(S) \to R$ and $\eta : S \times S \to R^n$ such that for all $y \in S$ and $p \in R^n$,

$$G_f(f(y)) - G_f(f(v)) \geq \eta^T(y,v)\left[G'_f(f(v))\nabla_y f(v) \right] + \frac{1}{2}p^T \left[G''_f(f(v))\nabla_y f(v)\nabla_y f(v)^T \right] \geq 0$$

then $f_i$ is called $G_f$-convex at $v \in S$ with respect to $\eta$.

**Definition 2.6** If there exist functions $G_f$ and $\eta : S \times S \to R^n$ such that for all $y \in S$ and $p \in R^n$,

$$\eta^T(y,v)\left[G'_f(f(v))\nabla_y f(v) \right] + \frac{1}{2}p^T \left[G''_f(f(v))\nabla_y f(v)\nabla_y f(v)^T \right] \geq 0$$

then $f_i$ is called $G_f$-pseudobonvex at $v \in S$ with respect to $\eta$.

Now, we discuss nontrivial numerical examples that are $G_f$-convex/$G_f$-pseudobonvex but neither $\eta$-convex/$\eta$-pseudobonvex nor $\eta$-invex/$\eta$-pseudoinvex functions.

**Example 2.1** Let $f : [-1, 1] \to R^4$ be defined as

$$f(y) = \{f_1(y), f_2(y), f_3(y), f_4(y)\},$$

where $f_1(y) = \sqrt[10]{y}$, $f_2(y) = \arcsin y$, $f_3(y) = \arctan y$, and let $G_f = \{G_{f_1}, G_{f_2}, G_{f_3}, G_{f_4}\} : R \to R^4$ be defined as

$$G_{f_i}(t) = t^5 + 5, \quad G_{f_2}(t) = \sin t + 2, \quad G_{f_3}(t) = \tan t + 9, \quad G_{f_4}(t) = \cot t + 2.$$

Let $\eta : [-1, 1] \times [-1, 1] \to R$ be defined as

$$\eta(y, v) = -\frac{1}{9}v^{14} + y + \frac{1}{99}y^{17}v^5 - \frac{1}{7}y^4v^3 + v^3.$$
To show that \( f \) is \( G_f \)-bonvex at \( v = 0 \) with respect to \( \eta \), we have to claim that

\[
\pi_i = G_{f_i}(f_i(y)) - G_{f_i}(f_i(v)) - \eta^T(y,v)[G'_{f_i}(f_i(v))\nabla_x f_i(v)\nabla_x f_i(v)^T + G''_{f_i}(f_i(v))\nabla_{xx}^2 f_i(v)]p
\]

\[
+ \frac{1}{2}p^T[G'_{f_i}(f_i(v))\nabla_x f_i(v)\nabla_x f_i(v)^T + G''_{f_i}(f_i(v))\nabla_{xx}^2 f_i(v)]p
\]

\[
\geq 0, \quad i = 1, 2, 3, 4.
\]

Putting the values of \( f_i, G_{f_i}, i = 1, 2, 3, 4 \), into the last expression, after simplifying at the point \( v = 0 \in [-1,1] \), we clearly see from Fig. 1 that \( \pi_i \geq 0, \quad i = 1, 2, 3, 4, \) for all \( y \in [-1,1] \). Therefore \( f \) is \( G_f \)-bonvex at \( v = 0 \in [-1,1] \) with respect to \( \eta \) and \( p \).

Now, suppose

\[
\xi = f_3(y) - f_3(v) - \eta^T(y,v)[\nabla_x f_3(v) - \nabla_{xx} f_3(v)p] + \frac{1}{2}p^T[\nabla_{xx}^2 f_3(v)]p
\]

or

\[
\xi = \arctan y - \arctan v
\]

\[
- \left( \frac{1}{9}y^{14} + y + \frac{1}{99}y^{17}v^5 - \frac{1}{7}y^4v^3 + v^3 \right) \left[ \frac{1}{1 + v^2} - \frac{2vp}{(1 + v^2)^2} \right] - \frac{vp^2}{(1 + v^2)^2},
\]

\[
\xi = \arctan y + \frac{1}{9}y^{14} - y \quad \text{at} \quad v = 0,
\]

\[
\xi \not\geq 0 \quad \text{(from Fig. 2)}.
\]

Therefore \( f_3 \) is not \( \eta \)-bonvex at \( v = 0 \) with respect to \( p \). Hence \( f \) is not \( \eta \)-bonvex at \( v = 0 \) with respect to \( p \).

Next,

\[
\delta = f_3(y) - f_3(v) - \eta^T(y,v)\nabla_x f_3
\]
Figure 2 $\xi = \arctan y + \frac{1}{9}y^{14} - y \geq 0$, at $v = 0, \forall y, y \in [-1,1]$

or

$$\delta = \arctan y - \arctan v - \left( \frac{1}{9}y^{14} + y + \frac{1}{99}y^{17}v^5 - \frac{1}{7}y^{4}v^3 + v^3 \right) \frac{1}{1 + v^2},$$

$$\delta = \arctan y + \frac{1}{9}y^{14} - y \text{ at } v = 0,$$

$$\delta = \frac{\pi}{4} + \frac{1}{9} - 1 < 0 \text{ at } y = 1 \in [-1,1].$$

Therefore $f_3$ is not $\eta$-invex at $v = 0$. Hence $f$ is not $\eta$-invex at $v = 0$.

**Example 2.2** Let $f : [-2,2] \to R^2$ be defined as

$$f(y) = \{f_1(y), f_2(y)\},$$

where $f_1(y) = \left(\frac{e^{y^2}}{y^2} - 1\right)$, $f_2(y) = y^3$, and $G_f = \{G_{f_1}, G_{f_2} : R \to R^2$ is defined as

$$G_{f_1}(t) = t^2 + 1, \quad G_{f_2}(t) = t^2 + 3.$$  

Let $\eta : [-2,2] \times [-2,2] \to R$ be given as

$$\eta(y, v) = y^6 + v^9y^4 + v^5y + v + 3.$$  

To show that $f$ is $G_f$-pseudobonvex at $v = 0$ with respect to $\eta$, we have to claim that, for $i = 1,2$,

$$\zeta_i = \eta^T(y, v) \left[ G'_{f_i}(f_i(v)) \nabla_x f_i(v) + \left[ G''_{f_i}(f_i(v)) \nabla_x f_i(v)(\nabla_x f_i(v))^T + G'_{f_i}(f_i(v)) \nabla_{xx} f_i(v) \right] \rho \right] \geq 0$$  

$$\Rightarrow \quad G_{f_i}(f_i(y)) - G_{f_i}(f_i(v))$$  

$$+ \frac{1}{2} \rho^T \left[ G''_{f_i}(f_i(v)) \nabla_x f_i(v)(\nabla_x f_i(v))^T + G'_{f_i}(f_i(v)) \nabla_{xx} f_i(v) \right] \rho \geq 0.$$
Let

\[ \phi_1 = \eta^T(y, v) \left[ G_{f_1}^\prime(f_1(v)) \nabla_y f_1(v) + \left\{ G_{f_1}^{\prime\prime}(f_1(v)) \nabla_y f_1(v) \left( \nabla_y f_1(v) \right)^T + G_{f_1}(f_1(v)) \nabla_{xx} f_1(u) \right\} p_1 \right]. \]

Substituting the values of \( \eta \) and \( f_1 \) at the point \( v = 0 \), we get

\[ \phi_1 \geq 0 \quad \text{for all } y \in [-2, 2] \text{ and } p. \]

Next, consider

\[ \phi_2 = G_{f_2}^\prime(f_2(y)) - G_{f_2}(f_2(v)) + \frac{1}{2} p^T \left[ G_{f_2}^{\prime\prime}(f_2(v)) \nabla_y f_2(v) \left( \nabla_y f_2(v) \right)^T + G_{f_2}(f_2(v)) \nabla_{yy} f_2(v) \right] \]

At \( v = 0 \), we get \( \phi_2 \geq 0 \) for all \( y \in [-1, 1] \) and \( p \) (from Fig. 3);

\[ \phi_2 = \eta^T(y, v) \left[ G_{f_2}^\prime(f_2(v)) \nabla_y f_2(v) + \left\{ G_{f_2}^{\prime\prime}(f_2(v)) \nabla_y f_2(v) \left( \nabla_y f_2(v) \right)^T + G_{f_2}(f_2(v)) \nabla_{yy} f_2(v) \right\} p_1 \right], \]

\[ \phi_2 = \left( y^6 + v^9y^4 + v^5y + v + 3 \right) \left( 6v^5 + 30v^5p \right). \]

At the point \( v = 0 \), we have

\[ \phi_2 \geq 0 \quad \text{for all } y \in [-2, 2] \text{ and } p. \]

Also,

\[ \phi_2 = G_{f_2}^\prime(f_2(y)) - G_{f_2}(f_2(v)) + \frac{1}{2} p^T \left[ G_{f_2}^{\prime\prime}(f_2(v)) \nabla_y f_2(v) \left( \nabla_y f_2(v) \right)^T + G_{f_2}(f_2(v)) \nabla_{yy} f_2(v) \right] p, \]

\[ \phi_2 = y^6 - v^6 + 15p^2v^4. \]

At the point \( v = 0 \), we obtain

\[ \phi_2 \geq 0 \quad \text{for all } y \in [-2, 2] \text{ and } p. \]
Hence from the expressions \( \phi_i \) and \( \varphi_i, i = 1, 2 \), we get that \( f \) is \( G_f \)-pseudobonvex at \( v = 0 \) with respect to \( \eta \).

Next, let

\[
\phi_3 = \eta^T (y, v) \left[ \nabla_y f_2(v) + \nabla_{yy} f_2(v) p \right],
\]

\[
\phi_3 = (y^6 + y^3 v + y^3 v + v + 3)[3v^2 + 6vp].
\]

At the point \( v = 0 \), we have

\[
\phi_3 \geq 0 \text{ for all } y \in [-2,2] \text{ and } p.
\]

Further, consider

\[
\varphi_3 = f_2(y) - f_2(v) + \frac{1}{2} p^2 \nabla_{yy} f_2(v),
\]

\[
\varphi_3 = y^3 - v^3 + 3p^2 v.
\]

At the point \( v = 0 \), we obtain

\[
\varphi_3 \geq 0 \text{ for all } y \in [-2,2] \text{ and } p \text{ (from Fig. 4).}
\]

Hence \( f_2 \) is not \( \eta \)-pseudobonvex at \( v = 0 \in [-2,2] \). Therefore \( f = (f_1, f_2) \) is not \( \eta \)-pseudobonvex at \( v = 0 \in [-2,2] \).

Finally,

\[
\phi_4 = \eta^T (y, v) \nabla_y f_2(v),
\]

\[
\phi_4 = 3(y^6 + y^3 v + y^3 v + v + 3)v^2.
\]

At the point \( v = 0 \), we have

\[
\phi_4 \geq 0 \text{ for all } y \in [-2,2] \text{ and } p.
\]
Also,
\[ \varphi_4 = f_2(y) - f_2(v), \]
\[ \varphi_3 = y^3 - v^3. \]

At the point \( v = 0 \), we obtain
\[ \varphi_4 \geq 0 \quad \text{for all} \quad y \in [-2, 2]. \]

Hence \( f_2 \) is not \( \eta \)-pseudoinvex at \( v = 0 \in [-2, 2] \). Hence \( f = (f_1, f_2) \) is not \( \eta \)-pseudoinvex at \( v = 0 \in [-2, 2] \).

3 Second-order multiobjective G-Wolfe-type symmetric dual program

Consider the following pair of second-order multiobjective G-Wolfe-type dual programs over arbitrary cones.

Primal problem (GWP) Minimize
\[
R(y, z, \lambda, p) = (R_1(y, z, \lambda_1, p), R_2(y, z, \lambda_2, p), \ldots, R_k(y, z, \lambda_k, p))^T
\]
subject to
\[
- \sum_{i=1}^{k} \lambda_i \left[ \nabla_{f_i}(y, z) \right] + \left[ \nabla_{f_i}(y, z) \right] + G_i^f(y, z) \nabla_{y^i}(y, z) = 0, \quad i = 1, 2, 3, \ldots, k.
\]
(1)
\[
\lambda_i > 0, \quad \lambda^T e_k = 1, \quad x \in C_1, \quad i = 1, 2, 3, \ldots, k.
\]
(2)

Dual problem (GWD) Maximize
\[
S(v, w, \lambda, q) = (S_1(v, w, \lambda_1, q), S_2(v, w, \lambda_2, q), \ldots, S_k(v, w, \lambda_k, q))^T
\]
subject to
\[
\sum_{i=1}^{k} \lambda_i \left[ \nabla_{f_i}(v, w) \right] + \left[ \nabla_{f_i}(v, w) \right] + G_i^f(v, w) \nabla_{x^i}(v, w) = 0, \quad i = 1, 2, 3, \ldots, k.
\]
(3)
\[
\lambda_i > 0, \quad \lambda^T e_k = 1, \quad v \in C_2, \quad i = 1, 2, 3, \ldots, k.
\]
(4)

where for all \( i = 1, 2, 3, \ldots, k, \)
\[
R_i(y, z, \lambda, p) = G_i^f(f_i(y, z)) - z^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_{f_i}(f_i(y, z)) \right] + \left[ \nabla_{f_i}(f_i(y, z)) \right] + G_i^f(f_i(y, z)) \nabla_{y^i}(f_i(y, z))
\]
\[
- \frac{1}{2} \sum_{i=1}^{k} \lambda_i p^T \left( G_i^f(f_i(y, z)) \nabla_{f_i}(f_i(y, z)) \nabla_{y^i}(f_i(y, z)) + G_i^f(f_i(y, z)) \nabla_{y^i}(f_i(y, z)) \right) p.
\]
\[ S_i(v, w, \lambda, q) = G_i(f_i(v, w)) - v^T \sum_{i=1}^{k} \lambda_i (G_i'(f_i(v, w)) \nabla f_i(v, w) \nabla z f_i(v, w) \lambda) + G_i'(f_i(v, w)) \nabla z f_i(v, w) \lambda q) + G_i'(f_i(v, w)) \nabla z f_i(v, w) \lambda q) \]

and

(i) \( e_k = (1, 1, \ldots, 1) \in \mathbb{R}^k \) and \( \lambda \in \mathbb{R}^k \).

(ii) \( q \) and \( p \) are vectors in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively.

Let \( Y^0 \) and \( Z^0 \) be the sets of feasible solutions of (GWP) and (GWD), respectively.

**Theorem 3.1** (Weak duality) Let \( (y, z, \lambda, p) \in Y^0 \) and \( (v, w, \lambda, q) \in Z^0 \). Suppose that for all \( i = 1, 2, 3, \ldots, k \),

(i) \( f_i(v) \) is \( G_i \)-bonvex at \( v \) with respect \( \eta \),

(ii) \( f_i(x, \cdot) \) be \( G_i \)-boncave at \( y \) with respect \( \eta \),

(iii) \( \eta_1(y, v) + u \in C_1 \) and \( \eta_2(w, z) + y \in C_2 \).

Then the following inequalities cannot hold together:

\[ R_i(y, z, \lambda, p) \leq S_i(v, w, \lambda, q) \quad \text{for all} \quad i = 1, 2, 3, \ldots, k, \quad \text{(5)} \]

and

\[ R_r(y, z, \lambda, p) < S_r(v, w, \lambda, q) \quad \text{for at least one} \quad r \in K. \quad \text{(6)} \]

**Proof** If possible, then suppose inequalities (5) and (6) hold. For \( \lambda > 0 \), we obtain

\[ \sum_{i=1}^{k} \lambda_i \left[ G_i(f_i(y, z)) - z^T \sum_{i=1}^{k} \lambda_i (G_i'(f_i(y, z)) \nabla f_i(y, z) \nabla z f_i(y, z) \lambda) + G_i'(f_i(y, z)) \nabla z f_i(y, z) \lambda q) + G_i'(f_i(y, z)) \nabla z f_i(y, z) \lambda q) \right] \]

\[ - \frac{1}{2} \sum_{i=1}^{k} \lambda_i (p^T [G_i'(f_i(y, z)) \nabla f_i(y, z) \nabla z f_i(y, z) \lambda) + G_i'(f_i(y, z)) \nabla z f_i(y, z) \lambda p) \]

\[ < \sum_{i=1}^{k} \lambda_i \left[ G_i(f_i(v, w)) - v^T \sum_{i=1}^{k} \lambda_i (G_i'(f_i(v, w)) \nabla f_i(v, w) + G_i'(f_i(v, w)) \nabla z f_i(v, w) + (\nabla z f_i(v, w))^T + G_i'(f_i(v, w)) \nabla z f_i(v, w) q) \right] \]

\[ - \frac{1}{2} \sum_{i=1}^{k} \lambda_i (q^T [G_i'(f_i(v, w)) \nabla f_i(v, w) (\nabla z f_i(v, w))^T + G_i'(f_i(v, w)) \nabla z f_i(v, w) q)]. \quad \text{(7)} \]
From assumption (i) we get
\[
G_i(f_i(y, w)) - G_i(f_i(v, w)) \\
\geq \eta_1 x, u^T \left[ G_i'(f_i(v, w)) \nabla_i f_i(v, w) \\
+ \left\{ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right\}^T + G_i'(f_i(v, w)) \nabla_i f_i(v, w) \right\} q \\
- \frac{1}{2} \left[ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right] \nabla_i f_i(v, w) q.
\]

Since \( \lambda > 0 \), this inequality yields
\[
\sum_{i=1}^k \lambda_i [G_i(f_i(y, w)) - G_i(f_i(v, w))] \\
\geq \eta_1^T(x, u) \left\{ \sum_{i=1}^k \lambda_i [G_i'(f_i(v, w)) \nabla_i f_i(v, w) \\
+ \left\{ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right\}^T + G_i'(f_i(v, w)) \nabla_i f_i(v, w) \right\} q \\
- \frac{1}{2} \left[ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right] \nabla_i f_i(v, w) q. \tag{8}
\]

From the dual constraint (3) and assumption (iii) it follows that
\[
[\eta_1(y, v) + v]^T \left\{ \sum_{i=1}^k \lambda_i [G_i'(f_i(v, w)) \nabla_i f_i(v, w) \\
+ \left\{ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right\}^T + G_i'(f_i(v, w)) \nabla_i f_i(v, w) q \right\} \geq 0,
\]
which implies
\[
\eta_1(y, v)^T \left\{ \sum_{i=1}^k \lambda_i [G_i'(f_i(v, w)) \nabla_i f_i(v, w) \\
+ \left\{ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right\}^T + G_i'(f_i(v, w)) \nabla_i f_i(v, w) q \right\} \\
\geq -v^T \left\{ \sum_{i=1}^k \lambda_i [G_i'(f_i(v, w)) \nabla_i f_i(v, w) \\
+ \left\{ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right\}^T + G_i'(f_i(v, w)) \nabla_i f_i(v, w) q \right\}.
\]

Using inequalities (3) and (8), we obtain
\[
\sum_{i=1}^k \lambda_i [G_i(f_i(y, w)) - G_i(f_i(v, w))] \\
+ \frac{1}{2} \left[ \sum_{i=1}^k \lambda_i q^T \left[ G_i''(f_i(v, w)) \nabla_i f_i(v, w) \right] \nabla_i f_i(v, w) q + G_i'(f_i(v, w)) \nabla_i f_i(v, w) q \right].
\]
Using assumption (iv) and primal constraint (1), we get
\[
-\nu^T \left\{ \sum_{i=1}^{k} \lambda_i \left[ G'_{f_i}(f_i(v, w)) \nabla_z f_i(v, w) + G'_{f_i}(f_i(v, w)) \nabla_z f_i(v, w) \right] \right\}.
\]
\[
+ \left\{ G'_{f_i}(f_i(v, w)) \nabla_y f_i(v, w) \right\} (\nabla_y f_i(v, w))^T + G'_{f_i}(f_i(v, w)) \nabla_y f_i(v, w) \right] \right\}.
\]
(9)

Finally, adding inequalities (9) and (10) and using \( \lambda^T e_k = 1 \), we obtain
\[
\sum_{i=1}^{k} \lambda_i \left[ G_{f_i}(f_i(v, w)) - \nu^T \sum_{i=1}^{k} \lambda_i (G'_{f_i}(f_i(v, w)) \nabla_z f_i(v, w)
+ G'_{f_i}(f_i(v, w)) \nabla_y f_i(v, w) \right] \right] \right\}.
\]
\[
+ G'_{f_i}(f_i(v, w)) \nabla_y f_i(v, w) \right\} (\nabla_y f_i(v, w))^T + G'_{f_i}(f_i(v, w)) \nabla_y f_i(v, w) \right] \right\}.
\]
(10)

This contradicts (7). Hence the result. \(\square\)

**Remark 3.1** Since every \( G_f \)-bonvex function is \( G_f \)-pseudobonvex, Theorem 3.1 can also be obtained under \( G_f \)-pseudobonvexity assumptions.

**Remark 3.2** A vector space \( V \) over field \( K \), the span of a set \( S \), may be defined as the set of all finite linear combinations of elements (vectors) of \( S \):
\[
\text{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i : k \in N, u_i \in S, \lambda_i = 1, 2, 3, \ldots, k \right\}.
\]
**Theorem 3.2** (Strong duality) Let \((\bar{y}, \bar{z}, \bar{\lambda}, \bar{p})\) be an efficient solution of \((GWP)\); fix \(\lambda = \bar{\lambda}\) in \((GWD)\) such that

\(\text{(i) for all } i = 1, 2, 3, \ldots, k, \{G_{f_i}^*(f_i(\bar{y}, \bar{z}))\} = 0 \text{ is nonsingular,}\)

\(\text{(ii) } \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x [(G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})]T + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})] \nabla p \in \text{ span}(G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}), \ldots, G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})) \text{ \(\setminus\) \{0\},}\)

\(\text{(iii) the vectors } \{G_{f_i}^*(f_i(\bar{y}, \bar{z}))\} = 0 \text{ are linearly independent,}\)

\(\text{(iv) } \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x [(G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})]T + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})] \nabla p = 0 \text{ implies that } \bar{p} = 0.\)

Then for \(\bar{q} = 0, \) we have \((\bar{v}, \bar{u}, \bar{\lambda}, \bar{p}) = 0 \in \mathbb{Z}^k\) and \(R(\bar{y}, \bar{z}, \bar{\lambda}, \bar{q}) = S(\bar{y}, \bar{z}, \bar{\lambda}, \bar{q}).\) Also, from Theorem 3.1 it follows that \((\bar{v}, \bar{u}, \bar{\lambda}, \bar{p}) = 0\) is an efficient solution for \((GWD)\).

**Proof** By the Fritz–John necessary conditions [22] there exist \(\alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^m, \) and \(\eta \in \mathbb{R}\) such that

\[
(y - \bar{y})^T \left\{ \sum_{i=1}^{k} \alpha_i \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) \right] + \sum_{i=1}^{k} \bar{\lambda}_i \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right] \right\}

\[
+ \left\{ \beta - (\alpha^T e_k) \right\} \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0 \text{ for all } y \in C_1, \quad (11)
\]

\[
\sum_{i=1}^{k} \left( \alpha_i - (\alpha^T e_k) \bar{\lambda}_i \right) \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) \right]

\[
+ \sum_{i=1}^{k} \bar{\lambda}_i \left[(G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right]T + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad (12)
\]

\[
\left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right] \nabla p + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad i = 1, 2, 3, \ldots, k, \quad (13)
\]

\[
G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})] \nabla p + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z})] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad (14)
\]

\[
\left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad (15)
\]

\[
\left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad (16)
\]

\[
\left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad (17)
\]

\[
\left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} \left[ G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_x f_i(\bar{y}, \bar{z}) + G_{f_i}^*(f_i(\bar{y}, \bar{z}))\nabla_{x} f_i(\bar{y}, \bar{z}) \right] \nabla p \right\} \left( \beta - (\alpha^T e_k) \right) \left( \bar{y} + \frac{1}{2} \bar{p} \right) \right\} = 0, \quad (18)
\]
By assumption (i), since $\lambda_i > 0$ for $i = 1, 2, 3, \ldots, k$, (18) gives

$$\beta = (\alpha^T e_k)(\tilde{p} + \tilde{y}), \quad i = 1, 2, 3, \ldots, k. \quad (19)$$

If $\alpha = 0$, then (19) implies that $\beta = 0$. Further, equation (18) gives $\eta = 0$. Consequently, $(\alpha, \beta, \eta) = 0$, which contradicts (17). Hence $\alpha \neq 0$, or $\alpha^T e_k > 0$.

Using (19) and $\alpha^T e_k > 0$ in (12), we get

$$\sum_{i=1}^{k} \tilde{\lambda}_i \left[ \nabla_z \left\{ \left( G_{\beta_i}^{f_i}(\tilde{y}, \tilde{z}) \right) \nabla_z f_i(\tilde{y}, \tilde{z}) \nabla_z f_i(\tilde{y}, \tilde{z})^T + G_{\beta_i}^{f_i}(\tilde{y}, \tilde{z}) \nabla_z f_i(\tilde{y}, \tilde{z}) \right\} \tilde{p} \right] \right]$$

$$= -\frac{2}{\alpha^T e_k} \sum_{i=1}^{k} \left[ G_{\beta_i}^{f_i}(\tilde{y}, \tilde{z}) \nabla_z f_i(\tilde{y}, \tilde{z}) \right] (\alpha_i - (\alpha^T e_k) \tilde{\lambda}_i). \quad (20)$$

It follows from assumption (ii) that

$$\sum_{i=1}^{k} \tilde{\lambda}_i \left[ \nabla_z \left\{ \left( G_{\beta_i}^{f_i}(\tilde{y}, \tilde{z}) \right) \nabla_z f_i(\tilde{y}, \tilde{z}) \nabla_z f_i(\tilde{y}, \tilde{z})^T + G_{\beta_i}^{f_i}(\tilde{y}, \tilde{z}) \nabla_z f_i(\tilde{y}, \tilde{z}) \right\} \tilde{p} \right] = 0. \quad (21)$$

Hence by assumption (iv) we get $\tilde{p} = 0$, and therefore inequality (19) implies

$$\beta = (\alpha^T e_k) \tilde{y}. \quad (22)$$

Now, using $\tilde{p} = 0$ and (20), we obtain

$$\sum_{i=1}^{k} (\alpha_i - (\alpha^T e_k) \tilde{\lambda}_i) \left[ G_{\beta_i}^{f_i}(\tilde{y}, \tilde{z}) \nabla_z f_i(\tilde{y}, \tilde{z}) \right] = 0.$$

Assumption (iii) yields

$$\alpha_i = (\alpha^T e_k) \tilde{\lambda}_i, \quad i = 1, 2, 3, \ldots, k. \quad (23)$$
Using $\alpha^T e_k > 0$ and (21)–(23) in (11), we get

$$
(y - \bar{y})^T \sum_{i=1}^k \lambda_i [G'_i(f_i(\bar{y}, \bar{z})) \nabla z_i f_i(\bar{y}, \bar{z})] \geq 0 \text{ for all } y \in C_1.
$$

Let $y \in C_1$. Then, $y + \bar{y} \in C_1$, and it follows that

$$
y^T \sum_{i=1}^k \lambda_i [G'_i(f_i(\bar{y}, \bar{z})) \nabla z_i f_i(\bar{y}, \bar{z})] \geq 0 \text{ for all } y \in C_1.
$$

Therefore

$$
\sum_{i=1}^k \lambda_i [G'_i(f_i(\bar{y}, \bar{z})) \nabla z_i f_i(\bar{y}, \bar{z})] \in C_1^*.
$$

Also, from (22) we have

$$
\bar{y} = \frac{\bar{\beta}}{\alpha^T e_k} \in C_2.
$$

Hence $(\bar{v}, \bar{w}, \bar{\lambda}, \bar{p} = 0$) satisfies the dual constraints and $Z^0$.

Now, letting $y = 0$ and $y = 2\bar{y}$ in (24), we get

$$
\bar{y}^T \sum_{i=1}^k \lambda_i [G'_i(f_i(\bar{y}, \bar{z})) \nabla z_i f_i(\bar{y}, \bar{z})] = 0.
$$

Using (28) and $\bar{q} = \bar{p} = 0$ completes the proof. \qed

**Theorem 3.3** (Converse duality) Let $(\bar{v}, \bar{w}, \bar{\lambda}, \bar{q})$ be an efficient solution of (GWD). Fix $\lambda = \bar{\lambda}$ in (GWP) such that

(i) for all $i = 1, 2, 3, \ldots, k$, $[G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w})(\nabla z_i f_i(\bar{v}, \bar{w}))^T + G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w})]$ is nonsingular,

(ii) $\sum_{i=1}^k \lambda_i \nabla z_i (G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w})(\nabla z_i f_i(\bar{v}, \bar{w}))^T + G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w})) - \bar{q} \bar{q} \notin \text{span} \{G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w}), \ldots, G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w}) \} \setminus \{0\},$

(iii) the vectors $(G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w}), G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w}), \ldots, G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w}) \}$

are linearly independent,

(iv) $\sum_{i=1}^k \lambda_i \nabla z_i ((G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w})(\nabla z_i f_i(\bar{v}, \bar{w}))^T + G'_i(f_i(\bar{v}, \bar{w})) \nabla z_i f_i(\bar{v}, \bar{w})) \bar{q} \bar{q} = 0 \Rightarrow \bar{q} = 0.$

Then, taking $\bar{p} = 0$, we have that $(\bar{v}, \bar{w}, \bar{\lambda}, \bar{p} = 0) \in Y^0$ and $R(\bar{v}, \bar{w}, \bar{\lambda}, \bar{p}) = S(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p})$. Also, by Theorem 3.1 $(\bar{v}, \bar{w}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution for (GWP).

**Proof** Proof follows the lines of Theorem 3.2. \qed

**4 Concluding remarks**

In this paper, we have formulated a second-order symmetric G-Wolfe-type dual problem for a nonlinear multiobjective optimization problem with cone constraints. A number of duality relations are further established under $G_f$-convexity/$G_f$-pseudobonvexity.
assumptions on the function $f$. We have discussed various numerical examples to show the existence of $G_f$-bonvex/$G_f$-pseudobonvex functions. The question arises whether the duality results developed in this paper hold for $G$-Wolfe- or mixed-type higher-order multiobjective optimization problems. This may be the future direction for the researchers working in this area.

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