Energy-Based In-Domain Control of a Piezo-Actuated Euler-Bernoulli Beam

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Abstract

The main contribution of this paper is the extension of the well-known boundary-control strategy based on structural invariants to the control of infinite-dimensional systems with in-domain actuation. The systems under consideration, governed by partial differential equations, are described in a port-Hamiltonian setting making heavy use of the underlying jet-bundle structure, where we restrict ourselves to systems with 1-dimensional spatial domain and 2nd-order Hamiltonian. To show the applicability of the proposed approach, we develop a dynamic controller for an Euler-Bernoulli beam actuated with a pair of piezoelectric patches and conclude the article with simulation results.

1 Introduction

For the description of finite-dimensional systems, due to the illustration of the underlying physical effects, the port-Hamiltonian (pH) system representation has proven to be an adequate framework. A major advantage is the close relation between the total energy of the system and the corresponding evolution equations, providing an insight into the energy flows within the system and with the system environment. In particular the fact that so-called energy-ports appear, makes the pH-system representation interesting for control-engineering applications, see [1, 2] for instance. The basic idea of such appropriate control strategies is to design a closed-loop system exhibiting a desired behaviour. Therefore, the objective is to shape the total energy of the system and increase the dissipation rate, or even modify the structure of the system dynamics.

Some of these well-known control strategies, like the so-called energy-Casimir method, have already been extended to systems governed by partial differential equations (PDEs), where for the most parts the occurrence of boundary-energy ports is exploited. It must be mentioned that the pH-system representation in the infinite-dimensional scenario is not unique, see [3] for a comparison of the Stokes-Dirac approach and an approach based on jet-bundle structures by means of a mechanical example. As a consequence, the generation of the boundary ports strongly depends on the chosen approach. In [4, 5, 6], the boundary-energy flow for infinite-dimensional systems formulated within the Stokes-Dirac scenario is investigated, while in [7, 8, 9] the generation of boundary ports for the jet-bundle approach is discussed. With regard to control by interconnection based on structural invariants (Casimir functionals), these boundary ports shall be used to couple the infinite-dimensional plant to a dynamic controller. This control strategy provides the opportunity to inject additional damping into the closed loop, and moreover, the mentioned Casimir functionals are used to relate (some) of the controller states to the plant in order to shape the energy of the system, see, e.g., [10, 11] for the controller design in the Stokes-Dirac framework and [12, 13, 14] for the jet-bundle approach. From a mathematical point of view, an essential feature is that a system governed by PDEs is coupled at (a part of) the boundary with a system that is described by ordinary differential equations (ODEs).

It is worth stressing that the application of boundary-control schemes constitutes a restriction with regard to the energy flows. Furthermore, there are many actuators, like, e.g., piezoelectric actuators, which do not operate on the boundary but within the spatial domain. In view of these facts, it seems natural to extend the known boundary-control schemes to the control of infinite-dimensional systems with in-domain actuation.

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There, a distinction needs to be drawn. On the one hand, we consider systems with lumped inputs that may act on a part of the spatial domain. For this system class, the interconnection of the system with a dynamic controller corresponds to a coupling of a PDE and an ODE-system within the spatial domain of the infinite-dimensional system. On the other hand, the scenario of systems with distributed input densities would call for infinite-dimensional controllers, which represents the coupling of a PDE with a PDE. For stability investigations of infinite-dimensional systems, usually functional-analytic methods are used, see e.g. [13]. This framework also allows to investigate the well-posedness of a problem. However, in this contribution we assume well-posedness and confine ourselves on energy considerations. Thus, no detailed stability investigations are carried out.

To demonstrate the proposed control strategy, we consider pH-systems with 2nd-order Hamiltonian on 1-dimensional spatial domains. For such systems, in [14] a dynamic boundary controller has been derived. Now, we intend to develop an in-domain control strategy for this system class. Therefore, the main contributions of this paper are as follows. First, we state a proper pH-system representation of a piezo-actuated Euler-Bernoulli beam in Section 3. Furthermore, we derive an appropriate control methodology for infinite-dimensional pH-systems with lumped inputs that may act on a part of the spatial domain and show the capability of the approach by means of simulation results for the piezo-actuated Euler-Bernoulli beam, see Section 4.

2 Notation and Preliminaries

Throughout this paper, we make heavy use of differential-geometric methods, where the notation is similar to that of [10]. Formulas are kept short and readable by applying tensor notation and using Einstein’s convention on sums. However, the ranges of the used indices are not indicated when they are clear from the context. We use the standard symbols ∧, | and d denoting the exterior (wedge) product, the natural contraction between tensor fields and the exterior derivative, respectively. To avoid exaggerated notation, the use of pull-back bundles is omitted. Furthermore, the expression $C^\infty(M)$ denotes the set of all smooth functions on a manifold $M$.

In this contribution, we investigate systems governed by PDEs in a pH-setting. Therefore, we introduce some geometrical structures and begin with defining a so-called bundle $\pi : E \rightarrow B$, which allows a clear distinction between dependent and independent coordinates. Here, the base manifold $B$ is equipped with the independent coordinate $z^1$ as we confine ourselves to 1-dimensional spatial domains. Consequently, the boundary $\partial B$ is zero-dimensional and the restriction of a mathematical expression to $\partial B$ is indicated with $(\cdot)|_{\partial B}$. As the total manifold $E$ comprises the dependent coordinates $x^\alpha$, with $\alpha = 1, \ldots, n$, as well, it is equipped with $(z^1, x^\alpha)$. Moreover, $\pi$ is a surjective submersion from the total manifold $E$ to the base manifold $B$ and is called projection. Next, we consider (higher-order) jet manifolds to be able to introduce derivative coordinates (jet variables). For instance, the 4th jet manifold $J^4(E)$ possesses the coordinates $(z^1, x^\alpha, z^1_1, x^\alpha_1, z^1_2, x^\alpha_2, \ldots, z^1_n, x^\alpha_n)$, where, exemplarily, $x^\alpha_1$ denotes the 2nd-order derivative coordinate, i.e. the 2nd derivative of $x^\alpha$ with respect to the independent coordinate $z^1$.

Furthermore, we introduce the so-called tangent bundle $\tau_E : T(E) \rightarrow E$ equipped with the coordinates $(\dot{z}^1, x^\alpha, \dot{z}^1_1, \dot{x}^\alpha_1)$, where the abbreviations $\bar{\partial}_1 = \partial/\partial z^1$ and $\partial^\alpha = \partial/\partial x^\alpha$ denote the fibre bases of the bundle. An important subbundle of $\tau_E$ is the vertical tangent bundle $\nu_E : V(E) \rightarrow E$, which possesses the coordinates $(\dot{z}^1, x^\alpha, \dot{x}^\alpha)$. Since the relation $\dot{z}^1 = 0$ holds, the vertical vector field $\nu = v^\alpha \partial_\alpha$ is tangent to the fibres of $E$. Furthermore, the 2nd prolongation of a vertical vector field $\nu$ is given by $j^2(\nu) = \bar{\partial}_1(\nu^\alpha)\partial^1_\alpha + \partial_1(\partial^1_\alpha)\partial^1_\alpha$ and makes use of the total derivative $d_1 = \bar{\partial}_1 + x^\alpha_1 \partial^\alpha + x^\alpha_2 \partial^1_\alpha + x^\alpha_3 \partial^2_\alpha + \ldots$ together with the abbreviations $\partial^2_{\alpha} = \partial/\partial x^\alpha_1$ and $\partial^3_{\alpha} = \partial/\partial x^\alpha_2$.

Further important bundles are the cotangent bundles $\tau_E^* = T^*(B) \rightarrow B$ and $\tau_E^* = T^*(E) \rightarrow E$ possessing the coordinates $(z^1, \dot{z}_1)$ and $(z^1, x^\alpha, \dot{z}_1, \dot{x}_\alpha)$, respectively, where the holonomic bases are denoted by $dz^1, dx^\alpha$. These bundles allow to locally define one-forms according to $\varpi = \varpi_1 dz^1$ and $\omega = \omega_1 dz^1 + \omega_2 dx^\alpha$, with $\varpi_1 \in C^\infty(B)$ and $\omega_1, \omega_2 \in C^\infty(E)$. In what follows, we are interested in one-forms with coefficients depending on derivative variables. More precisely, we focus on (Hamiltonian) densities $\mathcal{H} = \mathcal{H}(\dot{z}_1, \dot{x}_\alpha)$, where $\mathcal{H} \in C^\infty(J^2(E))$, i.e. on densities that may depend on 2nd-order derivative coordinates. Here, $\Omega = dz^1$ denotes the volume element on $B$ and $\Omega_1 = \bar{\partial}_1 dz^1$ the boundary-volume form. The integrated quantity of $\mathcal{H}$, which is given by $\mathcal{H} = \int_B \mathcal{H} \Omega$, is called the Hamiltonian functional. The bundle structure $\pi : E \rightarrow B$ allows to construct some
3 Infinite-Dimensional PH-Systems

This section deals with the pH-system representation based on jet-bundle structures for systems with 1-
dimensional spatial domain and 2nd-order Hamiltonian, see, e.g. [14]. The framework has its origin in
[7, 17], and is mainly based on a certain power-balance relation, which allows us to introduce (power) ports
distributed over the domain as well as on the boundary. In this paper, we focus on systems with in-domain
actuation and, therefore we include the term 

$$H \in C^\infty(J^2(\mathcal{E}))$$

Let $\mathcal{F}$ be a 2nd-order Hamiltonian, i.e. $\mathcal{F} \in C^\infty(J^2(\mathcal{E}))$, then a pH-system formulation including in- and
outputs on the domain is given by

$$\dot{x} = (J - R)(\delta \mathcal{F}) + u|\mathcal{G},$$

$$y = \mathcal{G}^*|\delta \mathcal{F},$$

(1)

together with appropriate boundary conditions.

Remark 1. It should be noted that in [18, 13, 14], only systems with boundary in- and outputs have been
investigated. In this contribution, we focus our interests on systems with in-domain actuation, and, therefore
we include the term $u|\mathcal{G}$ in our system representation.

In (1), the interconnection map $J$, which describes the internal power flow, and the dissipation map $R$
take the form of $J, R : T^*(\mathcal{E}) \land T^*(\mathcal{B}) \rightarrow V(\mathcal{E})$. Furthermore, $J$ is skew-symmetric, i.e. the coefficients
meet $J^{\alpha\beta} = -J^{\beta\alpha} \in C^\infty(J^4(\mathcal{E}))$, and $R$ is symmetric and positive semidefinite, implying $R^{\alpha\beta} = R^{\beta\alpha} \in C^\infty(J^4(\mathcal{E}))$ and $[R^{\alpha\beta}] \geq 0$ for the coefficient matrix. For 2nd-order Hamiltonian densities, the variational
derivative corresponds to $\delta \mathcal{F} = \partial_\alpha H dx^\alpha \land \Omega$ with $\partial_\alpha (\cdot) = \partial_\alpha (\cdot) - d_1(\partial_\alpha (\cdot)) + d_11(\partial_{11}(\cdot))$. Due to the fact
that we intend to develop in-domain control strategies in this paper, the terms including the external inputs
and collocated outputs in (1) are of particular interest. It is worth stressing that both, the coefficients $\mathcal{G}_\xi^\alpha$
of the input map $\mathcal{G} : \mathcal{U} \rightarrow V(\mathcal{X})$ as well as the input coordinates $u^\xi \in \mathcal{U}$, may depend (amongst others) on
the spatial variable $z^1$. Based on the duality of the input- and the output-bundle, see [7, Section IV], we are
able to deduce the important relation

$$(u|\mathcal{G})\delta \mathcal{F} = u|(\mathcal{G}^*|\delta \mathcal{F}) = u|y.$$

(2)

With regard to the control-engineering purposes of the following section, it is of particular interest how
the Hamiltonian functional $\mathcal{H}$ evolves along solutions of the system (1) (well-posedness provided). If $\mathcal{H}$
corresponds to the total energy of the system, the formal change, which can be given as

$$\dot{\mathcal{H}} = -\int_B R(\delta \mathcal{F})|\delta \mathcal{F} + \int_B u|y + (\dot{x}|\delta^{0,1} \mathcal{F} + \dot{x}_1|\delta^{0,2} \mathcal{F})|_{\partial \mathcal{B}}$$

(3)

by means of (2), states a power-balance relation and comprises dissipation and collocation on the domain.
The collocation term $\int_B u|y$ can be used to define power ports distributed over the spatial domain allowing for
a non-zero power flow. Furthermore, (2) enables us to introduce boundary-power ports, where we basically
exploit the boundary operators $\delta^{0,1} \mathcal{F} = (\partial_\alpha^1 H - d_1(\partial_{11}^1 H)) dx^\alpha \land \Omega_1$ and $\delta^{0,2} \mathcal{F} = \partial_{11}^2 H dx_1^1 \land \Omega_1$, locally given as

$$\delta^{0,1}_\alpha H = \partial_\alpha^1 H - d_1(\partial_{11}^1 H), \quad \delta^{0,2}_\alpha H = \partial_{11}^2 H.$$

(4)

Worth stressing is the fact that for the system class under investigation (1-dimensional spatial domain and
2nd-order Hamiltonian), the formal change (3) can be determined by integration by parts. However, for
pH-systems with 2nd-order Hamiltonian and higher-dimensional spatial domain, the calculation of $\dot{\mathcal{H}}$ is a
non-trivial task, which is treated in [19].

A local system representation of (1) can be given as

$$\dot{x}_\alpha = (J^{\alpha\beta} - R^{\alpha\beta}) \delta_\beta H + \mathcal{G}_\xi^\alpha u^\xi,$$

$$y_\xi = \mathcal{G}_\xi^\alpha \delta_\alpha H,$$

(5)
with \( \alpha, \beta = 1, \ldots, n \) and \( \xi = 1, \ldots, m \). Henceforth, as we focus on systems actuated solely within the spatial domain, we suppose that no power exchange takes place through the boundary \( \partial \mathcal{B} = \{0, L\} \), i.e. \((x^\alpha \delta_0^\alpha H)|_{\partial \mathcal{B}} = 0\) as well as \((x^\alpha \delta_0^\alpha H)|_{\partial \mathcal{B}} = 0\). Consequently, the power-balance relation \( (6) \) follows to
\[
\nabla = - \int_{\mathcal{B}} \delta_\alpha (H) \mathcal{R}^{\alpha \beta} \delta_\beta (H) dz^1 + \int_{\mathcal{A}} w^\xi y^\xi dz^1
\]
in local coordinates.

As an example, an Euler-Bernoulli beam actuated by one pair of piezoelectric macro-fibre composite (MFC) patches is studied. To this end, we summarise the derivation of the equation of motion for the transversal deflection \( \mathcal{W} \) of the beam, which is given in detail in [20]. Furthermore, we aim to find a pH-system representation being suitable for the control strategy presented in Section 4.

**Example 1** (Piezo-actuated Euler-Bernoulli beam). We consider an Euler-Bernoulli beam actuated by one pair of piezoelectric patches with two symmetrically placed actuators on the upper and lower side of the beam. Furthermore, we assume that the beam is clamped at the position \( z = 0 \), i.e. \( \mathcal{W}(t, 0) = 0, \dot{\mathcal{W}}(t, 0) = 0 \) and \( \mathcal{W}(t, 0) = 0 \), while the other end \( z = L \) is free, implying that the shear force and the bending moment vanish.

First, we derive the equation of motion in a Lagrangian framework by exploiting the calculus of variations, see [21] for instance. In this setting, the time \( t \) and the spatial coordinate \( z^1 \) are used as independent variables. To begin with, we state the energy densities of the system under investigation, where we first focus on the beam. Furthermore, we assume that the beam is clamped at the position \( z = 0 \), i.e. \( \mathcal{W}(t, 0) = 0, \dot{\mathcal{W}}(t, 0) = 0 \) and \( \mathcal{W}(t, 0) = 0 \), while the other end \( z = L \) is free, implying that the shear force and the bending moment vanish.

To mathematically describe the position of the piezoelectric pair, we introduce the spatial actuator characteristic
\[
\Gamma(z^1) = h(z^1 - z^1_p) - h(z^1 - z^1_p - L_p),
\]
where \( h(\cdot) \) denotes the Heaviside function and \( z^1_p \) the position where the MFC patches of the length \( L_p \) are attached meeting \( 0 < z^1_p < z^1_p + L_p < L \). Consequently, the kinetic-energy density of the MFC patches follows to
\[
\mathcal{T}_p = \rho_p A_p \Gamma(z^1)(\mathcal{W})^2,
\]
where the corresponding cross section and mass density are denoted by \( A_p \) and \( \rho_p \), respectively. To keep the complexity as low as possible, we assume the following simplifications regarding the MFC patches. First, we suppose a perfect compensation of all actuator nonlinearities as well as an uniaxial state of stress. Furthermore, we describe the electric field between the electrodes by an exclusive field component \( E_1 \), i.e. \( E_2 = E_3 = 0 \), and additionally neglect the self-generated electric field stemming from the direct piezoelectric effect as it is irrelevant compared to \( E_1 \).

Consequently, if we use linear constitutive relations for the MFC patches, see [20] equ. (7), and take the preceding assumptions into account, the potential-energy density follows to
\[
\mathcal{V}_p = \Gamma(z^1)(\Theta_p(\mathcal{W})^2 + 2 \Delta_p \mathcal{W} u_{in}),
\]
where the (constant) material parameters of the MFC patches are hidden in the abbreviations \( \Theta_p \) and \( \Delta_p \). Furthermore, \( u_{in} \) denotes the input voltage of the piezoelectric actuators. To derive the equation of motion together with the boundary conditions, we use the 2nd-order Lagrangian density
\[
\mathcal{L} = \mathcal{T}_b + \mathcal{T}_p - \mathcal{V}_b - \mathcal{V}_p,
\]
which can be given as the difference of the total-kinetic energy density
\[
\mathcal{T} = \frac{1}{2} \kappa(z^1)(\mathcal{W})^2
\]
and the total-potential energy density
\[ \mathcal{V} = \frac{1}{2} \Theta(z^1)(w_{11})^2 + 2\Gamma(z^1)\Delta_p w_{11} u_{in} \] (10)
by means of the spatially varying parameters \( \kappa(z^1) = \rho_0 A_0 + 2\rho_p A_p \Gamma(z^1) \) and \( \Theta(z^1) = EI + 2\Theta_p \Gamma(z^1) \). For the system under consideration – 1st-order derivative variable with respect to \( t \) and (solely) 2nd-order derivative variables with respect to \( z^1 \) –, the Euler-Lagrange operator corresponds to
\[ \delta_{\omega}(\cdot) = \partial_{\omega}(\cdot) - d_i(\partial_{\omega}^i(\cdot)) + d_{11}(\partial_{\omega}^{11}(\cdot)), \] (11)
and the boundary operators are
\[ \delta^b_{\omega}^{,1}(\cdot) = \partial^b_{\omega}^{,1}(\cdot) - d_i(\partial^b_{\omega}^{i,1}(\cdot)), \] (12a)
\[ \delta^b_{\omega}^{,2}(\cdot) = \partial^b_{\omega}^{,2}(\cdot), \] (12b)
as in mechanics it is common to allow for no variation on the time boundary. If we apply the domain operator \( \mathcal{L} \) and the boundary operators \( \mathcal{B} \) to the Lagrangian density \( \mathcal{S} \), due to the requirements \( \delta_{\omega} \mathcal{L} = 0 \) as well as \( (\dot{\omega} \delta^b_{\omega}^{,1} \mathcal{L})|_{\partial B} = 0 \), \( (\dot{\omega} \delta^b_{\omega}^{,2} \mathcal{L})|_{\partial B} = 0 \), we obtain the equation of motion
\[ \kappa(z^1)w_{tt} = -\Theta(z^1)w_{1111} - 4\Theta_p \partial_1(\Gamma(z^1))w_{11} - 2\Theta_p \partial_{11}(\Gamma(z^1))w_{11} - 2\Delta_p \partial_{11}(\Gamma(z^1))u_{in}, \] (13)
overall with the boundary conditions
\[ (\dot{\omega}EI w_{1111})|_{\partial B} = 0, \] (14a)
\[ (\dot{\omega}EI w_{11})|_{\partial B} = 0. \] (14b)
It should be noted that the spatial derivatives of \( \Gamma(z^1) \) occurring in \( (13) \) would cause some problems regarding the formulation of the equation of motion. To avoid this problem, we approximate the discontinuous characteristic \( (1) \) by the spatially differentiable function
\[ \Gamma(z^1) = \frac{1}{2} \tanh(\sigma(z^1 - z^1_p)) - \frac{1}{2} \tanh(\sigma(z^1 - z^1_p - L_p)) \]
with the scaling factor \( \sigma \in \mathbb{R}_+ \).

Next, we are interested in a proper pH-system representation for the system under investigation. It should be noted that in the pH-setting the time \( t \) plays the role of an evolution parameter, i.e. \( t \) is no coordinate any more and, hence, the exclusive independent variable is the spatial coordinate \( z^1 \). Consequently, we consider the bundle \( \pi : \mathcal{E} \to \mathcal{B} \) with the independent coordinate \( (z^1) \) for \( \mathcal{B} \) and coordinates \( (z^1, w, p) \) for \( \mathcal{E} \) in the following, where we have introduced the generalised momenta \( p = \kappa(z^1)\dot{w} = \kappa(z^1)w_{11} \). In principle, the Hamiltonian density is chosen as the sum of potential- and kinetic-energy density according to \( \mathcal{H} = \mathcal{T} + \mathcal{V} \).

However, to obtain an appropriate pH-system formulation, we set
\[ \mathcal{H} = -\frac{1}{2\Delta_p} \Gamma(z^1)w_{11}u_{in}^2 + \frac{1}{2} \Theta(z^1)(w_{11})^2, \]
where we intentionally omit the term \( 2\Delta_p \Gamma(z^1)w_{11}u_{in} \) of \( (13) \), as by using the calculus of variations it generates the input part in \( (13) \), which is hidden in \( g(z^1) = -2\Delta_p \partial_{11}(\Gamma(z^1)) \) in the pH-system representation
\[ \begin{bmatrix} \dot{w} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_w \mathcal{H} \\ \delta_p \mathcal{H} \end{bmatrix} + \begin{bmatrix} 0 \\ g(z^1) \end{bmatrix} u_{in}, \]
\[ y = \begin{bmatrix} 0 & g(z^1) \end{bmatrix} \begin{bmatrix} \delta_w \mathcal{H} \\ \delta_p \mathcal{H} \end{bmatrix} = g(z^1)\dot{w}. \] (15)
Moreover, we are able to deduce the formal change of the Hamiltonian functional, which follows to
\[ \delta \mathcal{H} = \int_{\mathcal{B}} g(z^1)\dot{w}u_{in} dz^1 \] (16)
as the boundary terms vanish due to the boundary conditions \( (14) \). It is worth stressing that \( (11) \) corresponds to an electrical power-balance relation as the unit of the distributed output density \( (15) \) is \( \frac{\Delta}{m} \).

Ex. \( (1) \) highlights that external inputs together with the collocated outputs generate power ports which may be distributed over (a part of) the spatial domain. As distributed ports allow for a non-zero power flow over the domain, we use them to couple an infinite-dimensional system to a dynamic pH-controller in the following section.
4 In-Domain Control by means of Structural Invariants

This section deals with the extension of the energy-Casimir method to infinite-dimensional pH-systems with in-domain actuation. Here, we confine ourselves to systems with lumped inputs that may act distributed over a part of the spatial domain. This has the consequence that the collocated outputs can be interpreted as distributed output densities. However, a certain interconnection allows for the use of a finite-dimensional dynamic controller. An advantage of the proposed control strategy is the applicability to piezo-actuated beams, which is presented at the end of this section.

4.1 Interconnection (Infinite-Finite)

In the following, we aim at stabilising pH-systems of the form (5), where the lumped inputs \( u^\xi \) may act distributed over a part of the spatial domain due to the input-map components \( G^\xi_\alpha \), cf. Ex. 1. To this end, we are interested in a power-conserving interconnection of the infinite-dimensional plant (5) and a finite-dimensional pH-controller, given in local coordinates as

\[
\begin{align*}
\dot{x}^\alpha_{c}\ &= \ (J^\alpha_{c,\beta_{c}} - R^\alpha_{c,\beta_{c}})\partial_{\beta_{c}}H_{c} + G^\alpha_{c,\xi}u^\xi_{c}, \\
y^\xi_{c,\xi} & = G^\alpha_{c,\xi}\partial_{\alpha_{c}}H_{c},
\end{align*}
\]

with \( \alpha_{c}, \beta_{c} = 1, \ldots, n_c \) and \( \xi = 1, \ldots, m \). There, one must take account of the fact that the outputs of (5) are considered as distributed output densities. Thus, to enable a coupling with the finite-dimensional controller, the output densities must be integrated over \( B \) and therefore, we choose a power-conserving interconnection of the form

\[
u^\xi \int_{B} y^\xi_{\xi} d\xi + u^\xi_{c,\xi} = 0.
\]

To obtain a power-conserving interconnection which meets (18), we couple the infinite-dimensional plant and the finite-dimensional controller according to

\[
u^\xi = K^{\xi\eta} \int_{B} y^\eta d\eta , \quad u^\xi_{c} = -K^{\xi\eta_{c}}y^\eta_{c},
\]

with \( K^{\xi\eta} \) denoting the components of an appropriate map \( K \). It is worth stressing that the closed-loop system, which is a result of the interconnection (19), still possesses a pH-structure, with the closed-loop Hamiltonian

\[
\mathcal{H}^{cl} = \int_{B} \mathcal{H} d\xi + H_{c}.
\]

As we consider systems where no power exchange takes place at the boundary \( \partial B \), i.e. \( (\dot{x}^\alpha_{c} \delta^\alpha_{c} \mathcal{H} + \dot{\zeta}^\alpha_{c} \delta^\alpha_{c} \partial \mathcal{H})|_{\partial B} = 0 \) is valid, the formal change of the closed-loop Hamiltonian can be deduced to

\[
\dot{\mathcal{H}}^{cl} = -\int_{B} \delta_{\alpha}(\mathcal{H})R^{\alpha\beta} \delta_{\beta}(\mathcal{H}) d\xi - \partial_{\alpha_{c}}(H_{c})R^{\alpha_{c}\beta_{c}} \partial_{\beta_{c}}(H_{c})
\]

because of the coupling (19).

Having defined the coupling of the plant (5) and the controller (17) by means of (19), we are interested in structural invariants of the closed-loop system aiming at relating some of the controller states to the plant. By means of these controller states, we partially shape \( \mathcal{H}^{cl} \), whereas the controller states that are not related to the plant shall be used for the damping injection and thus for the purpose of stabilisation.

Remark 2. At this point, it should be mentioned again that stability investigations for systems governed by PDEs usually require functional-analytic methods. In this contribution, the focus is on a formal approach exploiting geometric system properties and consequently, no detailed stability investigations will be carried out. However, worth stressing is the fact that \( \mathcal{H}^{cl} > 0 \) and \( \dot{\mathcal{H}}^{cl} \leq 0 \) can be used for stability investigations in the sense of Lyapunov.

4.2 Structural Invariants

Motivated by the form of the plant (distributed) and the controller (lumped), we consider Casimir-functionals according to

\[
\mathcal{C}^{\lambda} = x^{\lambda}_{c} + \int_{B} C^{\lambda} d\xi ,
\]

\( \mathcal{C}^{\lambda} \).
with $\lambda = 1, \ldots, n \leq n_c$. The functionals (20) must be constant along solutions of the closed loop, i.e. $\dot{c}^\lambda = 0$ is valid independently of $\mathcal{H}$ and $H_c$, in order to serve as structural invariants.

**Proposition 1.** Consider the closed-loop system which stems from the coupling of (5) and (17) via the interconnection (19). Then, the functionals (20) are structural invariants of the closed loop, iff the conditions

\begin{align}
(J^\lambda_{\beta\epsilon} - R^\lambda_{\beta\epsilon}) &= 0 \\
\delta_\alpha c^\lambda (J^\lambda_{\beta} - R^\lambda_{\beta}) + G^\lambda_{\epsilon,\xi} K^{\xi\eta} G^\eta_{\gamma,\eta} &= 0 \\
\delta_\alpha c^\lambda G^\lambda_{\xi,\eta} K^{\xi\eta} G^\eta_{\gamma,\eta} &= 0 \\
(\dot{x}^\alpha \delta^{\alpha.1}_\alpha c^\lambda + \dot{x}^\alpha \delta^{\alpha.2}_\alpha c^\lambda)|_{\partial B} &= 0
\end{align}

are fulfilled.

**Proof.** We begin by calculating the formal change of (20) along trajectories of the closed loop, which follows to

$$\dot{c}^\lambda = \dot{x}^\lambda_c + \int_B \dot{x}^\alpha \delta_\alpha c^\lambda d\gamma + (\dot{x}^\alpha \delta^{\alpha.1}_\alpha c^\lambda + \dot{x}^\alpha \delta^{\alpha.2}_\alpha c^\lambda)|_{\partial B},$$

and take into account the requirement $\dot{c}^\lambda = 0$ independently of $\mathcal{H}$ and $H_c$. If we substitute the system equations of the controller and the plant (17) and (5), respectively, as well as the relations (19), then we obtain

$$\dot{c}^\lambda = (J^\lambda_{\beta\epsilon} - R^\lambda_{\beta\epsilon}) \partial_\beta H_c + \int_B (\delta_\alpha c^\lambda (J^\lambda_{\beta} - R^\lambda_{\beta}) + G^\lambda_{\epsilon,\xi} K^{\xi\eta} G^\eta_{\gamma,\eta}) \delta_\beta H d\gamma + \ldots$$

$$- \int_B \delta_\alpha c^\lambda G^\lambda_{\xi,\eta} K^{\xi\eta} G^\eta_{\gamma,\eta} \partial_\alpha H_c d\gamma + (\dot{x}^\alpha \delta^{\alpha.1}_\alpha c^\lambda + \dot{x}^\alpha \delta^{\alpha.2}_\alpha c^\lambda)|_{\partial B}$$

and the proof follows immediately.

Now, it is interesting to interpret the results of Prop. 1 and draw some conclusions to the results and findings of [14 Prop. 1] for the boundary control of 1-dimensional pH-systems with 2nd-order Hamiltonian. In particular, the condition (21b) is of special interest as it allows to relate the plant within the domain to $\lambda = 1, \ldots, n \leq n_c$ controller states, which is not possible with the Casimir condition [14 Eq. (17b)].

Unfortunately, we are not able to relate every system state to the plant, since the condition (21c) describes the fact that we cannot find any Casimir functions depending on variables where the inputs of the plant appear in the corresponding system equations. In [14], the boundary of the infinite-dimensional system is divided into an actuated and an unactuated part, where the actuated boundary is used to relate the plant to the controller, see [14 Eqs. (17c) and (17d)]. In contrast, as we solely consider a coupling of the plant within the spatial domain, the impact of the Casimir functionals at the boundary must vanish, cf. (21d), like in [14 Eq. (17e)] at the unactuated boundary.

Having the preceding findings at hand, the piezo-actuated beam of Ex. 1 serves us to demonstrate the applicability of the proposed control scheme.

**Example 2** (Energy-Casimir controller for Ex. 2). In this example, we develop a finite-dimensional pH-controller for the piezo-actuated Euler-Bernoulli beam of Ex. 1 aiming at stabilising the equilibrium

$$w^d = \begin{cases} 
0 & \text{for } 0 \leq z^1 < z^1_p, \\
(a(z^1 - z^1_p)^2 & \text{for } z^1_p \leq z^1 < z^1_p + L_p, \\
b(z^1 - z^1_p - L_p) + a(L_p)^2 & \text{for } z^1_p + L_p \leq z^1_p \leq L.
\end{cases}$$

(23)

To this end, we relate one controller state to the plant, and two controller states shall be used for the damping injection, i.e., the dimension of the controller follows to $n_c = 3$. By means of a proper choice of the initial controller states, the Casimir function $C^1 = -g(z^1)w$, which satisfies the conditions (21) if we set $G^1_c = 1$ and $K = 1$, yields the important relation

$$x^1_c = \int_B g(z^1)wdz^1.$$
Further, Fig. 1 shows that the proposed controller stabilises the desired equilibrium (23) with $b = 0.1436$. Here, all system parameters of the beam are set to 1 and the piezoelectric patches with the length $L_p = 0.2$ are placed at $z_p^1 = 0.2$. The remaining degrees of freedom for the pH-controller are chosen according to $J_{c,22}^3 = 1$, $R_{c,22}^3 = 3$, $R_{c,23}^3 = -1$, $R_{c,33}^3 = 1.5$, $M_{c,22} = 85$, $M_{c,23} = 0$, $M_{c,33} = 60$, $G_c^2 = G_c^3 = 1.7$ and $c_1 = 0.1$.

5 Conclusion and Outlook

In this article, a control strategy for infinite-dimensional pH-systems with in-domain actuation yielding a dynamic controller has been developed. Here, we focused on systems with lumped inputs that act distributed over a part of the spatial domain and demonstrated the applicability of the control approach by means of a piezo-actuated Euler-Bernoulli beam. Since the distributed output density of the system under investigation cannot be measured, for the implementation of the controller an observer would be required, which is part of future research. Furthermore, we aim to extend the proposed control strategy to infinite-dimensional systems with distributed inputs requiring an infinite-dimensional controller.
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