Asymptotic Behavior of the Newton-Boussinesq Equation in a Two-Dimensional Channel

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Abstract
We prove the existence of a global attractor for the Newton-Boussinesq equation defined in a two-dimensional channel. The asymptotic compactness of the equation is derived by the uniform estimates on the tails of solutions. We also establish the regularity of the global attractor.

Key words. Newton-Boussinesq equation, global attractor, asymptotic compactness.

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1 Introduction

In this paper, we investigate the asymptotic behavior of solutions of the Newton-Boussinesq equation defined in an unbounded domain. Let $\Omega = (0,L) \times \mathbb{R}$ where $L$ is a positive number. Consider the system of equations defined in $(x,y) \in \Omega$ and $t > 0$:

$$\partial_t \xi + u \partial_x \xi + v \partial_y \xi = \Delta \xi - \frac{Ra}{Pr} \partial_x \theta + f(x,y),$$  \hspace{1cm} (1.1)

$$\Delta \Psi = \xi, \quad u = \Psi_y, \quad v = -\Psi_x,$$ \hspace{1cm} (1.2)

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\[
\partial_t \theta + u \partial_x \theta + v \partial_y \theta = \frac{1}{Pr} \triangle \theta + g(x, y),
\]
where \( \vec{u} = (u, v) \) is the velocity vector of the fluid, \( \theta \) is the flow temperature, \( \Psi \) is the flow function, \( \xi \) is the vortex. The positive constants \( Pr \) and \( Ra \) are the Prandtl number and the Rayleigh number, respectively. The external terms \( f \) and \( g \) are given in \( L^2(\Omega) \).

The Newton-Boussinesq equation describes many physical phenomena such as Benard flow, see, [7, 8] and the references therein. If the domain is bounded, the existence, uniqueness and the asymptotic behavior of solutions of system (1.1)-(1.3) have been studied by several authors, see, e.g., [9, 10, 11, 12]. In this paper we will examine the dynamical behavior of the solutions when the system is defined in the unbounded two-dimensional channel \( \Omega \). More precisely, we will prove the existence of a global attractor for the system. Note that the unboundedness of the domain \( \Omega \) introduces a major difficulty for proving the existence of a global attractor because Sobolev embeddings are no longer compact in this case, and hence the asymptotic compactness of the solution operator cannot be obtained by a standard method. Several approaches have been developed to overcome this difficulty. The energy equation method is one way to prove the asymptotic compactness of equations defined in unbounded domains. This idea was first developed by Ball in [5, 6] to deal with the compactness of the wave equation and the Navier-Stokes equation in bounded domains, and then extended by other authors in [14, 16, 20] to the Navier-Stokes equation in unbounded domains. Note that the energy equation of the Navier-Stokes equation in \( L^2(\Omega) \) does not contain the nonlinear term. This fact together with the weak compactness can be used to prove the strong asymptotic compactness in \( L^2(\Omega) \) (see, e.g., [16, 20]). However, in our case, the energy equation for system (1.1)-(1.3) in \( L^2(\Omega) \) does contain the nonlinear term, and hence the energy equation approach does not apply. In this paper, we will employ the techniques of uniform estimates on the tails of solutions to establish the asymptotic compactness of the Newton-Boussinesq equation. This idea was develop in [24] for proving the asymptotic compactness of the Reaction-Diffusion equation in unbounded domains, and later used by several authors in [1, 2, 3, 15, 17, 19, 22].

This paper is organized as follows. In the next section, we derive uniform estimates for the
solutions of the system (1.1)-(1.3) when \( t \to \infty \), which are necessary for proving the existence of a bounded absorbing set and the asymptotic compactness of the equation. In Section 3, we first establish the asymptotic compactness of system (1.1)-(1.3) by uniform estimates on the tails of solutions, and then prove the existence of a global attractor. The regularity of the global attractor is given in the last section.

In the sequel, we adopt the following notations. The norm of \( L^2(\Omega) \) is denoted by \( || \cdot || \) which is defined by mean of the usual inner product \( (\cdot,\cdot) \). The norm of any Banach space \( X \) is written as \( || \cdot ||_X \). In particular \( || \cdot ||_p \) represents the norm of \( L^p(\Omega) \). The letter \( C \) is a generic positive constant which may change its value from line to line.

Throughout this paper, we frequently use the following inequality

\[
||u||_4 < C ||u||_{H^1(\Omega)}^{1/2} ||u||^{1/2}, \quad \forall u \in H^1(\Omega),
\]

(1.4)

and the Poincare inequality

\[
||u|| \leq \lambda ||\nabla u|| \quad \forall u \in H^1_0(\Omega),
\]

(1.5)

where \( \lambda \) is a positive constant.

2 Uniform Estimates of Solutions

In this section, we derive uniform estimates for the solutions of the system (1.1)-(1.3) for large time. We also prove that the tails of solutions are uniformly small when space and time variables are sufficiently large.

Notice that system (1.1)-(1.3) can be rewritten as follows: for every \((x,y) \in \Omega\) and \( t > 0 \),

\[
\frac{\partial \xi}{\partial t} - \Delta \xi + J(\Psi, \xi) + \frac{R_a \partial \theta}{Pr} \frac{\partial \theta}{\partial x} = f(x,y), \quad (2.1)
\]

\[
\Delta \Psi = \xi, \quad (2.2)
\]

\[
\frac{\partial \theta}{\partial t} - \frac{1}{Pr} \Delta \theta + J(\Psi, \theta) = g(x,y), \quad (2.3)
\]
with the boundary conditions
\[ \xi|_{\partial \Omega} = 0, \quad \theta|_{\partial \Omega} = 0, \quad \Psi|_{\partial \Omega} = 0, \quad (2.4) \]
and the initial conditions
\[ \xi(x, y, 0) = \xi_0(x, y), \quad \theta(x, y, 0) = \theta_0(x, y), \quad (2.5) \]
where the functional \( J \) is given by
\[ J(u, v) = u_y v_x - u_x v_y. \quad (2.6) \]
It is easy to verify that \( J \) satisfies:
\[ \int_{\Omega} J(u, v) v \, dxdy = 0, \quad \text{for all} \quad u \in H^1(\Omega), \; v \in H^2(\Omega) \cap H^1_0(\Omega), \quad (2.7) \]
\[ ||J(u, v)|| \leq C ||u||_{H^2} ||v||_{H^2}, \quad \text{for all} \quad u \in H^2(\Omega), \; v \in H^2(\Omega), \quad (2.8) \]
\[ ||J(u, v)|| \leq C ||u||_{H^3} ||\nabla v||, \quad \text{for all} \quad u \in H^3(\Omega), \; v \in H^1(\Omega). \quad (2.9) \]
It is standard to prove that problem (2.1)-(2.5) is well posed in \( L^2(\Omega) \times L^2(\Omega) \) (see, e.g., [9]). More precisely, for every \( (\xi_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega) \), system (2.1)-(2.5) has a unique solution \( (\xi, \theta) \) such that for every \( T > 0 \),
\[ \xi \in C^0([0, \infty), L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \]
\[ \theta \in C^0([0, \infty), L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)). \]
Therefore, we can define a semigroup \( \{S(t)\}_{t \geq 0} \) such that for every \( t \geq 0 \), \( S(t) \) maps \( L^2(\Omega) \times L^2(\Omega) \) into itself and \( S(t)(\xi_0, \theta_0) = (\xi(t), \theta(t)) \). We now start to derive uniform estimates for the dynamical system \( \{S(t)\}_{t \geq 0} \).

**Lemma 2.1.** Suppose that \( (\xi_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega) \). Then for every \( T > 0 \), there is a constant \( C > 0 \) such that the solution \( (\xi, \theta) \) of system (2.1)-(2.5) satisfies
\[ ||\xi(t)|| + ||\theta(t)|| \leq C \quad \forall t \in [0, T], \]
where \( C \) depends only on the data \( (\Omega, P_r, R_a) \), \( T \) depends on the data \( (\Omega, P_r, R_a) \) and \( R \) when \( ||\xi_0|| \leq R \) and \( ||\theta_0|| \leq R \).
Proof. We first consider equation (2.3). By taking the inner product of (2.3) with $\theta$ in $L^2(\Omega)$ and using relation (2.7) we obtain

$$\frac{1}{2} \frac{d}{dt}||\theta||^2 + \frac{1}{P_r}||\nabla \theta||^2 = \int \Omega g\theta \, dx \, dy .$$

(2.10)

By Poincaré inequality (1.5) we find that

$$\frac{1}{2} \frac{d}{dt}||\theta||^2 + 1 + 2 \frac{1}{P_r}||\nabla \theta||^2 \leq \int \Omega g\theta \, dx \, dy .$$

(2.11)

Notice that the right-hand side is bounded by

$$\int \Omega g\theta \, dx \, dy \leq ||g|| \, ||\theta|| \leq C ||g||^2 + \frac{1}{4\lambda^2 P_r} ||\theta||^2 .$$

(2.12)

By (2.10)-(2.12) we find that

$$\frac{d}{dt}||\theta||^2 + 1 + 2 \frac{1}{P_r}||\nabla \theta||^2 + \frac{1}{2\lambda^2 P_r} ||\theta||^2 \leq C , \forall t \geq 0 ,$$

(2.13)

which implies that

$$\frac{d}{dt}||\theta||^2 + C_1 ||\theta||^2 \leq C , \forall t \geq 0 .$$

(2.14)

Let $T > 0$ be fixed and take $t \in [0, T]$, integrating (2.14) over $(0, t)$ we obtain

$$||\theta(t)||^2 \leq C_1 t + ||\theta(0)||^2 \leq C_1 t + R^2 \leq C , \forall t \in [0, T] .$$

(2.15)

We now consider equation (2.1). By taking the inner product of (2.1) with $\xi$ in $L^2(\Omega)$ and using relation (2.7) we obtain

$$\frac{1}{2} \frac{d}{dt}||\xi||^2 + ||\nabla \xi||^2 + \frac{R_a}{P_r} \int \Omega \theta \xi \, dx \, dy = \int \Omega f \xi \, dx \, dy .$$

(2.16)

We notice that the following inequalities hold

$$\frac{R_a}{P_r} \int \Omega \theta \xi \, dx \, dy = \frac{R_a}{P_r} \int \Omega \xi \theta \, dx \, dy \leq \frac{R_a}{P_r} ||\nabla \xi|| ||\theta|| \leq \frac{1}{4} ||\nabla \xi||^2 + C ||\theta||^2 ,$$

(2.17)

and

$$\int \Omega f \xi \, dx \, dy \leq ||f|| ||\xi|| \leq \lambda ||f|| ||\nabla \xi|| \leq \frac{1}{4} ||\nabla \xi||^2 + C .$$

(2.18)
By (2.15), (2.17) and (2.18), it follows from (2.16) that
\[
\frac{d}{dt}||\xi||^2 + ||\nabla \xi||^2 \leq C, \quad \forall t \in [0, T].
\] (2.19)

Then Poincaré inequality implies that
\[
\frac{d}{dt}||\xi||^2 + C_1||\xi||^2 \leq C, \quad \forall t \in [0, T].
\] (2.20)

Integrating (2.20) on \((0,t)\) we obtain
\[
||\xi(t)||^2 \leq CT + ||\xi(0)|| \leq C, \quad \forall t \in [0, T].
\] (2.21)

Combining (2.15) and (2.21) we conclude that
\[
||\theta|| + ||\xi(t)|| \leq C, \quad \forall t \in [0, T].
\] (2.22)

The proof is complete. \(\square\)

**Lemma 2.2.** Suppose that \((\xi_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega)\). Then for the solution \((\xi, \theta)\) of system (2.1)-(2.5) we have
\[
||\xi(t)|| + ||\theta(t)|| \leq M_1, \quad \forall t \geq t_1,
\]
and
\[
\int_t^{t+1} ||\nabla \xi(\tau)||^2 d\tau + \int_t^{t+1} ||\nabla \theta(\tau)||^2 d\tau \leq M_2, \quad \forall t \geq t_1,
\]
where \(M_1\) and \(M_2\) are constants depending only on the data \((\Omega, P_r, R_a)\), \(t_1\) depends on the data \((\Omega, P_r, R_a)\) and \(R\) when \(||\xi_0|| \leq R\) and \(||\theta_0|| \leq R\).

**Proof.** By (2.14) and Gronwall inequality we infer that
\[
||\theta(t)||^2 \leq e^{-C_1t}||\theta(0)||^2 + C_2 \leq e^{-C_1t}R^2 + C_2 \leq 2C_2, \quad \forall t \geq t_1^*,
\] (2.23)
where \(t_1^* = \frac{1}{C_1} \ln \left(\frac{R^2}{C_2}\right)\).
Moreover by (2.17), (2.18) and (2.23) we get from (2.20) that
\[
\frac{d}{dt}||\xi||^2 + C_1||\xi||^2 \leq C, \quad \forall t \geq t_1^*.
\] (2.24)
By Lemma 2.1 and Gronwall inequality we have
\[
\|\xi(t)\|^2 \leq e^{-C_1(t-t_1)}\|\xi(t_1)\|^2 + \frac{C}{C_1}, \quad \forall t \geq t_2^* + 1\ln\left(\frac{C_1C_2}{C}\right),
\]
(2.25)
Combining (2.23) and (2.25) we find that
\[
\|\theta(t)\| + \|\xi(t)\| \leq C, \quad \forall t \geq t_1,
\]
(2.26)
where \(t_1 = \max\{t_1^*, t_2^*\}\). By (2.13) we obtain that
\[
\frac{d}{dt}\|\theta\|^2 + C\|\nabla \theta\|^2 \leq C_1, \quad \forall t \geq t_1.
\]
(2.27)
Integrating (2.27) on \((t, t+1)\), by (2.26) we have that
\[
\int_t^{t+1} \|\nabla \theta(\tau)\|^2 d\tau \leq C, \quad \forall t \geq t_1.
\]
(2.28)
By (2.16) and (2.25) we also have
\[
\frac{d}{dt}\|\xi\|^2 + C\|\nabla \xi\|^2 \leq C_1, \quad \forall t \geq t_1.
\]
(2.29)
Integrating (2.29) on \((t, t+1)\), by (2.26) we get
\[
\int_t^{t+1} \|\nabla \xi(\tau)\|^2 d\tau \leq C, \quad \forall t \geq t_1.
\]
(2.30)
Then Lemma 2.2 follows from (2.26), (2.28) and (2.30).

We now derive uniform estimates in \(H^1(\Omega)\).

**Lemma 2.3.** Suppose that \((\xi_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega)\). Then for the solution \((\xi, \theta)\) of system (2.1)-(2.5) we have
\[
\|\nabla \xi(t)\| + \|\nabla \theta(t)\| \leq M_3 \quad \forall t \geq t_3,
\]
\[
\int_t^{t+1} (\|\Delta \xi(t)\|^2 + \|\Delta \theta(t)\|^2) dt \leq M_3, \quad \forall t \geq t_3,
\]
where \(M_3\) is a constant depending only on the data \((\Omega, P, R)\), \(t_3\) depends on the data \((\Omega, P, R)\) and \(R\) when \(\|\xi_0\| \leq R\) and \(\|\theta_0\| \leq R\).
Proof. Taking the inner product of (2.1) with $\triangle \xi$ in $L^2(\Omega)$ we get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \xi \|^2 + \| \triangle \xi \|^2 = \int_{\Omega} J(\Psi, \xi) \triangle \xi \, dx \, dy + \frac{R_a}{P_r} \int_{\Omega} \theta_x \triangle \xi \, dx \, dy - \int_{\Omega} f \triangle \xi \, dx \, dy .
\] (2.31)
Notice that the first term on the right-hand side of (2.31) is given by
\[
\int_{\Omega} J(\Psi, \xi) \triangle \xi \, dx \, dy = \int_{\Omega} \Psi_y \xi_x \triangle \xi \, dx \, dy + \int_{\Omega} \Psi_x \xi_y \triangle \xi \, dx \, dy .
\] (2.32)
We now estimate the first term on the right-hand side of (2.32). By (1.4) and Lemma 2.2 we have the following estimates for $t \geq T$,
\[
\int_{\Omega} \Psi_y \xi_x \triangle \xi \, dx \, dy \leq \| \Psi_y \|_4 \| \xi_x \|_4 \| \triangle \xi \| \leq C \| \Psi_y \|_2 \| \xi_x \|_3 \| \triangle \xi \| \leq \frac{1}{8} \| \triangle \xi \|^2 + C \| \nabla \xi \|^2 .
\] (2.33)
Similarly for the second term on the right-hand side of (2.32) we have
\[
\int_{\Omega} \Psi_x \xi_y \triangle \xi \, dx \, dy \leq \frac{1}{8} \| \triangle \xi \|^2 + C \| \nabla \xi \|^2 .
\] (2.34)
It follows from (2.32) and (2.33) that
\[
\int_{\Omega} J(\Psi, \xi) \triangle \xi \, dx \, dy \leq \frac{1}{4} \| \triangle \xi \|^2 + C \| \nabla \xi \|^2 .
\] (2.35)
Note that the last two terms on the right-hand side of (2.31) are bounded by
\[
C \left( \int_{\Omega} \theta_x \triangle \xi \, dx \, dy + \int_{\Omega} f \triangle \xi \, dx \, dy \right) \leq \frac{1}{4} \| \triangle \xi \|^2 + C \| \nabla \xi \|^2 + C .
\] (2.36)
From (2.31) and (2.35)–(2.36) we have
\[
\frac{d}{dt} \| \nabla \xi \|^2 + \| \triangle \xi \|^2 \leq C \left( \| \nabla \xi \|^2 + \| \nabla \theta \|^2 \right) + C , \ \forall t \geq T .
\] (2.37)
Taking the inner product of (2.3) with $\triangle \theta$ we get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \theta \|^2 + \| \triangle \theta \|^2 = \int_{\Omega} J(\Psi, \theta) \triangle \theta \, dx \, dy + \int_{\Omega} g \triangle \theta \, dx \, dy .
\] (2.38)
By arguments similar to (2.35) and (2.36), we obtain that
\[
\frac{d}{dt} \| \nabla \theta \|^2 + \| \triangle \theta \|^2 \leq C \left( \| \nabla \xi \|^2 + \| \nabla \theta \|^2 \right) + C , \ \forall t \geq T .
\] (2.39)
Let $\alpha = \min \left\{ 1, \frac{1}{2r^2} \right\}$. Then from (2.37) and (2.39), we have

$$\frac{d}{dt} \left( \|\nabla \xi\|^2 + \|\nabla \theta\|^2 \right) + \alpha \left( \|\Delta \theta\|^2 + \|\Delta \xi\|^2 \right) \leq C \left( \|\nabla \xi\|^2 + \|\nabla \theta\|^2 \right) + C, \quad \forall t \geq T. \quad (2.40)$$

By the uniform Gronwall inequality and Lemma 2.2, we find from (2.40) that

$$\|\nabla \xi(t)\| + \|\nabla \theta(t)\| \leq C, \quad \forall t \geq T + 1. \quad (2.41)$$

Integrating (2.40) on $(t, t+1)$, by (2.41) we get

$$\int_t^{t+1} \left( \|\Delta \xi(t)\|^2 + \|\Delta \theta(t)\|^2 \right) dt \leq C, \quad \forall t \geq T + 1. \quad (2.42)$$

Then Lemma 2.3 follows from (2.41)-(2.42).

Next we establish the uniform estimates on the tails of solutions which are crucial for proving the asymptotic compactness of the solution operator. Given $k > 0$, we denote by $\Omega_k$ the set $\Omega_k = \{ (x, y) \in \Omega : |y| \leq k \}$ and $\Omega \setminus \Omega_k$ the complement of $\Omega_k$. For our purpose, we choose a smooth cut-off function $\phi$ such that $0 \leq \phi(s) \leq 1$ and

$$\begin{cases} 
\phi(s) = 0 & \text{if } |s| < 1 \\
\phi(s) = 1 & \text{if } |s| > 2.
\end{cases} \quad (2.43)$$

Then, we have the following Poincaré type of inequality

**Lemma 2.4.** Let $v \in H_0^1(\Omega)$ and $\phi$ be given as above. Then $\exists \alpha > 0$ and $\beta > 0$ such that $\forall k > 0$:

$$\int_\Omega \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla v|^2 dx dy \geq \alpha \int_\Omega \phi^2 \left( \frac{|y|^2}{k^2} \right) v^2 dx dy - \frac{\beta}{k^2} \int_\Omega v^2 dx dy. \quad (2.44)$$

**Proof.** By Poincaré inequality (1.5), we have

$$\int_\Omega \phi^2 \left( \frac{|y|^2}{k^2} \right) v^2 dx dy \leq \lambda^2 \int_\Omega \left| \nabla \left( \phi \left( \frac{|y|^2}{k^2} \right) v \right) \right|^2 dx dy, \quad (2.44)$$

Notice that

$$\int_\Omega \left| \nabla \left( \phi \left( \frac{|y|^2}{k^2} \right) v \right) \right|^2 dx dy \leq \int_\Omega \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla v|^2 dx dy + 4 \int_\Omega \left( \phi' \left( \frac{|y|^2}{k^2} \right) \right)^2 \frac{y^2}{k^4} v^2 dx dy,$$
\[
\leq \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla v|^2 dxdy + 4 \int_{k \leq |y| \leq \sqrt{2}k} \left( \phi' \left( \frac{|y|^2}{k^2} \right) \right)^2 \frac{y^2}{k^4} v^2 dxdy
\leq \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla v|^2 dxdy + \frac{C}{k^2} \int_{k \leq |y| \leq \sqrt{2}k} |v|^2 dxdy
\leq \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla v|^2 dxdy + \frac{C}{k^2} \int_{\Omega} |v|^2 dxdy.
\tag{2.45}
\]

From (2.44) and (2.45) it follows that
\[
\int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) v^2 dxdy \leq \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla v|^2 dxdy + \frac{C}{k^2} \int_{\Omega} |v|^2 dxdy,
\tag{2.46}
\]
which implies Lemma 2.4. The proof is complete. \qed

**Lemma 2.5.** Given \(\epsilon > 0\), then there exist \(t_3 > 0\) and \(k_0 > 0\) such that the solution \((\xi, \theta)\) of system (2.1)-(2.5) with the initial condition \((\xi_0, \theta_0)\) satisfies
\[
\int_{\Omega \setminus \Omega_{k_0}} (|\xi(t)|^2 + |\theta(t)|^2) dxdy \leq \epsilon, \quad \forall t \geq t_3,
\]
where \(k_0\) depends only on the data \((\Omega, P_r, R_a)\) and \(\epsilon\), \(t_3\) depends only on \((\Omega, P_r, R_a)\), \(\epsilon\) and \(R\) when \(||\xi_0|| \leq R\) and \(||\theta_0|| \leq R||\).

**Proof.** Multiplying (2.3) by \(\phi^2 \left( \frac{|y|^2}{k^2} \right) \theta(x, y, t)\) and then integrating the resulting identity over \(\Omega\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \frac{1}{P_r} \int_{\Omega} (\theta \Delta \theta) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
= \int_{\Omega} g \theta \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \int_{\Omega} J(\Psi, \theta) \theta \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy. \tag{2.47}
\]
We now estimate every term in (2.47). We first have, by Lemma 2.4
\[
- \frac{1}{P_r} \int_{\Omega} (\theta \Delta \theta) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
= \frac{1}{P_r} \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla \theta|^2 dxdy + \frac{4}{P_r} \int_{\Omega} \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \theta \theta_y \frac{y}{k^2} dxdy
\geq \frac{\alpha}{P_r} \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\theta|^2 dxdy - \frac{C}{k^2} ||\theta||^2 + \frac{4}{P_r} \int_{\Omega} \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \theta \theta_y \frac{y}{k^2} dxdy. \tag{2.48}
\]
For the last term on the right-hand side of (2.47) we obtain, by integration by parts,

\[
\int \Omega J(\Psi, \theta) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy = \int \Omega \Psi_y \theta_x \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \int \Omega \Psi_x \theta_y \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
\]

\[
= \int \Omega \Psi_y \left( \frac{\partial^2}{\partial x^2} \right) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \int \Omega \Psi_x \left( \frac{\partial^2}{\partial y^2} \right) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
\]

\[
= -\frac{1}{2} \int \Omega \Psi_y \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{1}{2} \int \Omega \Psi_x \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + 2 \int \Omega \Psi_x \phi' \left( \frac{|y|^2}{k^2} \right) \phi \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} dxdy
\]

\[
= 2 \int \Omega \Psi_x \phi' \left( \frac{|y|^2}{k^2} \right) \phi \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} dxdy. \tag{2.49}
\]

It follows, from (2.47) through (2.49) that

\[
\frac{1}{2} \frac{d}{dt} \int \Omega |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{\alpha}{P_r} \int \Omega \phi^2 \left( \frac{|y|^2}{k^2} \right) |\theta|^2 dxdy
\]

\[
= \int \Omega g \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \frac{4}{P_r} \int \Omega \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \theta \theta_y \frac{y}{k^2} dxdy - 2 \int \Omega \Psi_x \phi' \left( \frac{|y|^2}{k^2} \right) \phi \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} dxdy + \frac{C}{k^2} |\theta|^2. \tag{2.50}
\]

Note that the first term on the right-hand side of (2.50) is bounded by

\[
\left| \int \Omega g \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy \right| = \left| \int \Omega_{|y| \geq k} g \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy \right|
\]

\[
\leq \left( \int \Omega_{|y| \geq k} g^2 dxdy \right)^{\frac{1}{2}} \left( \int \Omega_{|y| \geq k} \phi^4 \left( \frac{|y|^2}{k^2} \right) \theta^2 dxdy \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int \Omega_{|y| \geq k} g^2 dxdy \right)^{\frac{1}{2}} \left( \int \Omega \phi^2 \left( \frac{|y|^2}{k^2} \right) \theta^2 dxdy \right)^{\frac{1}{2}}
\]

\[
\leq C \int \Omega_{|y| \geq k} g^2 dxdy + \frac{\alpha}{2P_r} \int \Omega |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy. \tag{2.51}
\]

For the second term on the right-hand side of (2.50) we have

\[
\frac{4}{P_r} \left| \int \Omega \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \theta \theta_y \frac{y}{k^2} dxdy \right|
\]

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\[
\frac{4}{\mathcal{P}_r} \left| \int_{k \leq |y| \leq \sqrt{2k}} \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \theta \theta_y \frac{y}{k^2} \, dx \, dy \right| \leq \frac{C}{k} \int_{k \leq |y| \leq \sqrt{2k}} |\theta| |\theta_y| \, dx \, dy \leq \frac{C}{k} |\theta|||\nabla \theta|| \leq \frac{C}{k},
\]

where the last inequality is obtained by Lemmas (2.2) and (2.3). The third term on the right-hand side is bounded by

\[
2 \left| \int_{k \leq |y| \leq \sqrt{2k}} \Psi_x \theta^2 \phi' \left( \frac{|y|^2}{k^2} \right) \phi \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} \, dx \, dy \right| \leq \frac{C}{k} \int_{k \leq |y| \leq \sqrt{2k}} |\Psi_x||\theta|^2 \, dx \, dy \leq \frac{C}{k} \int_{\Omega} |\Psi_x||\theta|^2 \, dx \, dy \leq \frac{C}{k} |\Psi|||\theta||_6||\theta|| \leq \frac{C}{k} .
\]

It follows, from (2.50) through (2.53) that for \( k \geq 1 \)

\[
\frac{d}{dt} \int_{\Omega} |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx \, dy + \frac{\alpha}{\mathcal{P}_r} \int_{\Omega} |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx \, dy \leq \frac{C'}{k} + C_1 \int_{|y| \geq k} |g|^2 \, dx \, dy.
\]

Now, since \( g \in L^2(\Omega) \), given \( \epsilon > 0 \), there exists \( k_1 > 0 \) such that

\[
C_1 \int_{|y| \geq k_1} |g|^2 \, dx \, dy \leq \frac{\epsilon}{2}, \quad \forall k \geq k_1(\epsilon).
\]

Let \( k_2 = \max\{k_1, \frac{2C'}{\epsilon} \} \), then by (2.54) and (2.55) we obtain that for all \( k \geq k_2 \) and \( t \geq T \),

\[
\frac{d}{dt} \int_{\Omega} |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx \, dy + \frac{\alpha}{\mathcal{P}_r} \int_{\Omega} |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx \, dy \leq \epsilon.
\]

Applying Gronwall lemma to (2.56), by Lemma 2.2 we find that, for all \( k \geq k_2 \),

\[
\int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\theta|^2 \, dx \, dy \leq e^{-\frac{\mathcal{P}_r(t-T)}{\mathcal{P}_r}} \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\theta(T)|^2 \, dx \, dy + \frac{\epsilon \alpha}{\mathcal{P}_r} \leq e^{-\frac{\mathcal{P}_r(t-T)}{\mathcal{P}_r}} |\theta(T)||\theta||^2 + \frac{\epsilon \alpha}{\mathcal{P}_r} \leq \frac{2\epsilon \alpha}{\mathcal{P}_r},
\]

(2.57)
for all \( t \geq T_1 = T - \frac{P_r}{\alpha} \ln \left( \frac{\alpha}{CT_r} \right) \). We now estimate \( \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\xi(x, y, t)|^2 dxdy \). Multiplying (2.1) by \( \phi^2 \left( \frac{|y|^2}{k^2} \right) \xi(x, y, t) \) and then integrating by parts we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \int_{\Omega} (\xi \Delta \xi) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \int_{\Omega} J(\Psi, \xi) \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
\]

\[
= \int_{\Omega} f \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{R_a}{P_r} \int_{\Omega} \theta_x \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy .
\]

(2.58)

For the second term on the left-hand side of (2.58), by Lemma 2.4 we have

\[
-\int_{\Omega} (\xi \Delta \xi) \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
\]

\[
= \int_{\Omega} \nabla \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + 4 \int_{\Omega} \xi \xi_y \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} dxdy
\]

\[
\geq \frac{1}{2} \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\nabla \xi|^2 dxdy + \frac{\alpha}{2} \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\xi|^2 dxdy
\]

\[
- \frac{C}{k^2} |\xi|^2 + 4 \int_{\Omega} \xi \xi_y \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} dxdy .
\]

(2.59)

By (2.58) and (2.59) we find that the following inequality holds

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{1}{2} \int_{\Omega} |\nabla \xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{\alpha}{2} \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) |\xi|^2 dxdy
\]

\[
\leq \int_{\Omega} f \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy - \int_{\Omega} J(\Psi, \xi) \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{R_a}{P_r} \int_{\Omega} \theta_x \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy
\]

\[
- 4 \int_{\Omega} \xi \xi_y \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \frac{y}{k^2} dxdy + \frac{C}{k^2} |\xi|^2 .
\]

(2.60)

Note that the third term on the right-hand side of (2.60) is bounded by

\[
\frac{R_a}{P_r} \left| \int_{\Omega} \theta_x \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy \right| \leq \frac{R_a}{P_r} \left| \int_{\Omega} \theta \xi_x \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy \right|
\]

\[
\leq \frac{1}{2} \int_{\Omega} \xi_x^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + \frac{R^2}{2P_r} \int_{\Omega} |\theta|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right)
\]

\[
\leq \frac{1}{2} \int_{\Omega} |\nabla \xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) dxdy + C\epsilon ,
\]

(2.61)
where the last inequality is obtained by (2.57). It follows, then, from (2.60) and (2.61) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy + \frac{\alpha}{2} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy \\
\leq \int_{\Omega} f \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy - \int_{\Omega} J(\Psi, \xi) \xi \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy \\
- 4 \int_{\Omega} \xi \xi_{y} \phi \left( \frac{|y|^2}{k^2} \right) \phi' \left( \frac{|y|^2}{k^2} \right) \frac{y}{k} \, dx dy + \frac{C}{k^2} ||\xi||^2 + C_1 \epsilon .
\]

(2.62)

By similar arguments used in (2.50), after detailed calculations, we find that for \( k \geq k_2 \) and \( t \geq T_1 \), the right-hand side of (2.62) is bounded by

\[
C \epsilon + \frac{C_1}{k} + C_2 \int_{|y| \geq k} |f|^2 \, dx dy + \frac{\alpha}{4} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy ,
\]

and hence there is \( k_3 > 0 \) such that for all \( k \geq k_3 \) and \( t \geq T_1 \),

\[
\frac{d}{dt} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy + \frac{\alpha}{2} \int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy \leq C \epsilon .
\]

(2.63)

By Gronwall lemma, we find that for any \( k \geq k_3 \),

\[
\int_{\Omega} |\xi|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy \leq e^{-\frac{\alpha}{2}(t-T_1)} \int_{\Omega} |\xi(T_1)|^2 \phi^2 \left( \frac{|y|^2}{k^2} \right) \, dx dy + C \epsilon \\
\leq e^{-\frac{\alpha}{2}(t-T_1)} ||\xi(T_1)||^2 + C \epsilon \leq C_1 e^{-\frac{\alpha}{2}(t-T_1)} + C \epsilon \leq 2C \epsilon ,
\]

(2.64)

for any \( t \geq T_2 = T_1 - \frac{2}{\alpha} \ln \frac{C_1}{C} \). By (2.57) and (2.63) we see that for any \( k \geq k_3 \) and \( t \geq T_2 \),

\[
\int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) (|\theta|^2 + |\xi|^2) \, dx dy \leq C \epsilon ,
\]

(2.64)

and hence for all \( k \geq k_3 \) and \( t \geq T_1 \),

\[
\int_{|y| \geq \sqrt{2k}} (|\theta|^2 + |\xi|^2) \, dx dy \leq \int_{\Omega} \phi^2 \left( \frac{|y|^2}{k^2} \right) (|\theta|^2 + |\xi|^2) \, dx dy \leq C \epsilon ,
\]

(2.65)

which implies Lemma 2.5. The proof is complete. \qed

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3 Existence of Global Attractors

In this section, we prove the existence of global attractors for problem (2.1)-(2.3) in $L^2(\Omega) \times L^2(\Omega)$. To this end, we need to establish the asymptotic compactness of the solution operator which is stated as follows.

**Lemma 3.1.** The dynamical system $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $L^2(\Omega) \times L^2(\Omega)$, i.e., if $t_n \to \infty$ and $\{(\xi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty}$ is bounded in $L^2(\Omega) \times L^2(\Omega)$, then the sequence $\{S(t_n)(\xi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty}$ has a convergent subsequence.

**Proof.** Since $\{(\xi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty}$ is bounded in $L^2(\Omega) \times L^2(\Omega)$, there is $R > 0$ such that
\[
||\xi_{0,n}|| + ||\theta_{0,n}|| \leq R, \quad \forall n \in \mathbb{Z}^+.
\] (3.1)

By Lemma 2.3, there is a positive number $M$, depending on $(\Omega, P_r, R_a)$, such that for every $(\xi_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega)$ with $||\xi_0|| + ||\theta_0|| \leq R$, the following holds
\[
||S(t)(\xi_0, \theta_0)||_{H_0^1(\Omega) \times H_0^1(\Omega)} \leq M, \quad \forall t \geq T_1,
\] (3.2)

where $T_1$ depends on $(\Omega, P_r, R_a)$ and $R$. Since $t_n \to \infty$, there is $N_1 > 0$ such that $t_n \geq T_1$ for all $n \geq N_1$. Therefore we have, for $n \geq N_1$,
\[
||S(t_n)(\xi_{0,n}, \theta_{0,n})||_{H_0^1(\Omega) \times H_0^1(\Omega)} \leq M.
\] (3.3)

By (3.3) we find that there is $(\xi, \theta) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that, up to a subsequence,
\[
S(t_n)(\xi_{0,n}, \theta_{0,n}) \rightharpoonup (\xi, \theta) \quad \text{in} \quad L^2(\Omega) \times L^2(\Omega) \quad \text{and} \quad H_0^1(\Omega) \times H_0^1(\Omega).
\] (3.4)

Given $\epsilon > 0$, by Lemma 2.5, there are positive numbers $k_1$ and $T_2$ such that for any $k \geq k_1$ and $t \geq T_2$, $S(t)(\xi_0, \theta_0)$, with $||\xi_0, \theta_0|| \leq R$, satisfies
\[
\int_{\Omega \setminus \Omega_k} (|S(t)\xi_0|^2 + |S(t)\theta_0|^2) \, dx \, dy \leq \frac{\epsilon}{5}.
\] (3.5)

Let $N_2$ be large enough such that $t_n \geq T_2$ for all $n \geq N_2$. Then by (3.5) we obtain, for $n \geq N_2$,
\[
\int_{\Omega \setminus \Omega_k} (|S(t_n)\xi_{0,n}|^2 + |S(t_n)\theta_{0,n}|^2) \, dx \, dy \leq \frac{\epsilon}{5}.
\] (3.6)
Notice that (3.3) implies that the sequence \( \{S(t_n)(\xi_{0,n}, \theta_{0,n})\}_{n=1}^{\infty} \) is bounded in \( H^1(\Omega_k) \times H^1(\Omega_k) \) and hence precompact in \( L^2(\Omega_k) \times L^2(\Omega_k) \). Therefore, there is \( (\tilde{\xi}, \tilde{\theta}) \in L^2(\Omega_k) \times L^2(\Omega_k) \) such that, up to a subsequence,

\[
S(t_n)(\xi_{0,n}, \theta_{0,n}) \longrightarrow (\tilde{\xi}, \tilde{\theta}) \quad \text{in} \quad L^2(\Omega_k) \times L^2(\Omega_k).
\] (3.7)

By (3.4) and (3.7), we find that

\[
(\tilde{\xi}, \tilde{\theta}) = (\xi, \theta)|_{\Omega_k},
\]

which means that for every \( k \geq k_1 \),

\[
S(t_n)(\xi_{0,n}, \theta_{0,n})|_{\Omega_k} \longrightarrow (\xi, \theta)|_{\Omega_k} \quad \text{in} \quad L^2(\Omega_k) \times L^2(\Omega_k) .
\] (3.8)

In other words, for the given \( \epsilon > 0 \), there is \( N_3 > 0 \) such that for all \( k \geq k_1 \) and \( n \geq N_3 \),

\[
\int_{\Omega_k} \left( |S(t_n)\xi_{0,n} - \xi|^2 + |S(t_n)\theta_{0,n} - \theta|^2 \right) dxdy \leq \frac{\epsilon}{5}.
\] (3.9)

Since \( \xi \) and \( \theta \) are in \( L^2(\Omega) \), there is \( k_2 > 0 \) such that for all \( k \geq k_2 \),

\[
\int_{\Omega \setminus \Omega_k} \left( |\xi|^2 + |\theta|^2 \right) dxdy \leq \frac{\epsilon}{5}.
\] (3.10)

Let \( k_0 = \max\{k_1, k_2\} \) and \( N_0 = \max\{N_1, N_2, N_3\} \), then for all \( n \geq N \), we have

\[
\int_{\Omega} \left( |S(t_n)\xi_{0,n} - \xi|^2 + |S(t_n)\theta_{0,n} - \theta|^2 \right) dxdy
= \int_{\Omega_{k_0}} \left( |S(t_n)\xi_{0,n} - \xi|^2 + |S(t_n)\theta_{0,n} - \theta|^2 \right) dxdy \\
+ \int_{\Omega \setminus \Omega_{k_0}} \left( |S(t_n)\xi_{0,n} - \xi|^2 + |S(t_n)\theta_{0,n} - \theta|^2 \right) dxdy \\
\leq \int_{\Omega_{k_0}} \left( |S(t_n)\xi_{0,n} - \xi|^2 + |S(t_n)\theta_{0,n} - \theta|^2 \right) dxdy \\
+ 2 \int_{\Omega \setminus \Omega_{k_0}} (|S(t_n)\xi_{0,n}|^2 + |S(t_n)\theta_{0,n}|^2) dxdy \\
+ 2 \int_{\Omega \setminus \Omega_{k_0}} (|\xi|^2 + |\theta|^2) dxdy \leq \epsilon,
\] (3.11)

where the last inequality is obtained by (3.6), (3.9) and (3.10). Notice that (3.11) shows that

\[
S(t_n)(\xi_{0,n}, \theta_{0,n}) \longrightarrow (\xi, \theta) \quad \text{in} \quad L^2(\Omega) \times L^2(\Omega),
\]

and hence \( \{S(t)\}_{t \geq 0} \) is asymptotically compact. The proof is complete. \( \square \)
We are, now, ready to prove the existence of a global attractor for problem (2.1)-(2.3).

**Theorem 3.2.** Problem (2.1)-(2.3) has a global attractor \( A \) in \( L^2(\Omega) \times L^2(\Omega) \), which is compact, invariant and attracts every bounded set with respect to the norm of \( L^2(\Omega) \times L^2(\Omega) \).

**Proof.** By Lemma 2.2, the dynamical system \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set in \( L^2(\Omega) \times L^2(\Omega) \), and by Lemma 3.1 \( \{S(t)\}_{t \geq 0} \) is asymptotically compact. Then the existence of a global attractor follows immediately from the standard attractor theory (see e.g., [4, 5, 13, 21, 23]). \( \square \)

4 Regularity of Global Attractors

In this section, we investigate the regularity of the global attractor obtained in Theorem 3.2. We will show that the global attractor \( A \) is actually contained in a bounded subset of \( H^2(\Omega) \times H^2(\Omega) \).

We start with the following lemma

**Lemma 4.1.** Suppose that \((\xi_0,\theta_0) \in L^2(\Omega) \times L^2(\Omega)\). Then the solution \((\xi,\theta)\) of problem (2.1)-(2.3) satisfies

\[
\left\| \frac{d\xi}{dt} \right\| + \left\| \frac{d\theta}{dt} \right\| \leq M, \quad \forall t \geq T ,
\]

where \( M \) depends only on the data \((\Omega, P_r, R_a)\), \( T \) depends on the data \((\Omega, P_r, R_a)\) and \( R \) when \( \|\xi_0\| \leq R \) and \( \|\theta_0\| \leq R \).

**Proof.** By (2.8) and (2.11) we find that

\[
\left\| \frac{d\xi}{dt} \right\| \leq \|\Delta \xi\| + \|J(\psi,\xi)\| + C\|\nabla \theta\| + \|f\| \\
\leq \|\Delta \xi\| + C\|\xi\|\|\Delta \xi\| + C_1\|\nabla \theta\| + C_2 \leq C\|\Delta \xi\| + C_1, \quad \forall t \geq T ,
\]

where the last inequality is obtained by Lemma 2.3. By (4.1) and Lemma 2.3 again we get, for \( t \geq T \),

\[
\int_t^{t+1} \left\| \frac{d\xi}{dt} \right\|^2 dt \leq C \int_t^{t+1} \|\Delta \xi\|^2 dt + C_1 \leq C .
\]

Similarly, by (2.3), we find that, for \( t \geq T \),

\[
\left\| \frac{d\theta}{dt} \right\| \leq C\|\Delta \theta\| + \|J(\psi,\xi)\| + \|g\| \leq C\|\Delta \theta\| + C_1 ,
\]




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which along with Lemma 2.3 implies that, for \( t \geq T \),

\[
\int_t^{t+1} \left\| \frac{d\theta}{dt} \right\|^2 dt \leq C \int_t^{t+1} \left\| \Delta \theta \right\|^2 dt + C_1 \leq C .
\]  

(4.4)

Let \( \tilde{\xi} = \frac{d\xi}{dt} \) and \( \tilde{\theta} = \frac{d\theta}{dt} \). Then it follows from (4.2) and (4.4) that, for \( t \geq T \),

\[
\int_t^{t+1} \left( \left\| \tilde{\xi}(t) \right\|^2 + \left\| \tilde{\theta}(t) \right\|^2 \right) dt \leq C .
\]  

(4.5)

We now differentiate (2.1) and (2.3) with respect to \( t \) to obtain

\[
\frac{\partial \tilde{\xi}}{\partial t} - \Delta \tilde{\xi} + J(\Psi_t, \xi) + J(\Psi, \tilde{\xi}) + \frac{R_a}{P_r} \frac{\partial \tilde{\theta}}{\partial x} = 0 ,
\]  

(4.6)

and

\[
\frac{\partial \tilde{\theta}}{\partial t} - \frac{1}{P_r} \Delta \tilde{\theta} + J(\Psi_t, \theta) + J(\Psi, \tilde{\theta}) = 0 .
\]  

(4.7)

Taking the inner product of (4.6) with \( \tilde{\xi} \) in \( L^2(\Omega) \), we find that

\[
\frac{1}{2} \frac{d}{dt} \left\| \tilde{\xi} \right\|^2 + \left\| \nabla \tilde{\xi} \right\|^2 + \left( J(\Psi_t, \xi), \tilde{\xi} \right) + \frac{R_a}{P_r} \left( \frac{\partial \tilde{\theta}}{\partial x}, \tilde{\xi} \right) = 0 .
\]  

(4.8)

By (2.8) we have

\[
\left| \left( J(\Psi_t, \xi), \tilde{\xi} \right) \right| \leq \left\| J(\Psi_t, \xi) \right\| \left\| \tilde{\xi} \right\| \leq C \left\| \tilde{\xi} \right\|^2 \left\| \Delta \xi \right\| \leq \left\| \tilde{\xi} \right\|^4 + C \left\| \Delta \xi \right\|^2 .
\]  

(4.9)

We also have the following inequality

\[
\left| \frac{R_a}{P_r} \left( \frac{\partial \tilde{\theta}}{\partial x}, \tilde{\xi} \right) \right| \leq C \left\| \nabla \tilde{\xi} \right\| \left\| \tilde{\xi} \right\| \leq \frac{1}{2P_r} \left\| \nabla \tilde{\theta} \right\|^2 + C \left\| \tilde{\xi} \right\|^2 \leq \frac{1}{2P_r} \left\| \nabla \tilde{\theta} \right\|^2 + \left\| \tilde{\xi} \right\|^4 + C_1 .
\]  

(4.10)

It follows from (4.8)-(4.10) that

\[
\frac{d}{dt} \left\| \tilde{\xi} \right\|^2 + \left\| \nabla \tilde{\xi} \right\|^2 \leq C \left\| \tilde{\xi} \right\|^4 + C_1 \left\| \Delta \xi \right\|^2 + \frac{1}{2P_r} \left\| \nabla \tilde{\theta} \right\|^2 + C_2 .
\]  

(4.11)

Now, by taking the inner product of (4.7) with \( \tilde{\theta} \) in \( L^2(\Omega) \) we get

\[
\frac{1}{2} \frac{d}{dt} \left\| \tilde{\theta} \right\|^2 + \frac{1}{P_r} \left\| \nabla \tilde{\theta} \right\|^2 = - \left( J(\Psi_t, \theta), \tilde{\theta} \right) .
\]  

(4.12)

By an argument similar to (4.9), the right-hand side of (4.12) is bounded by

\[
\left| \left( J(\Psi_t, \theta), \tilde{\theta} \right) \right| \leq \left\| \tilde{\theta} \right\|^4 + \left\| \tilde{\theta} \right\|^4 + C \left\| \Delta \theta \right\|^2 .
\]  

(4.13)
By (4.12) and (4.13) we find that
\[
\frac{d}{dt} ||\tilde{\theta}||^2 + \frac{1}{P_r} ||\nabla \tilde{\theta}||^2 \leq 2||\tilde{\xi}||^4 + 2||\tilde{\theta}||^4 + C||\Delta \theta||^2 .
\] (4.14)

By (4.11) and (4.14) it follows that
\[
\frac{d}{dt} \left( ||\tilde{\xi}||^2 + ||\tilde{\theta}||^2 \right) + ||\nabla \tilde{\xi}||^2 + \frac{1}{2P_r} ||\nabla \tilde{\theta}||^2 \leq C \left( ||\tilde{\xi}||^4 + ||\tilde{\theta}||^4 \right) + C_1 \left( 1 + ||\Delta \xi||^2 + ||\Delta \theta||^2 \right) ,
\] (4.15)

which implies that
\[
\frac{d}{dt} \left( ||\tilde{\xi}||^2 + ||\tilde{\theta}||^2 \right) \leq C \left( ||\tilde{\xi}||^4 + ||\tilde{\theta}||^4 \right) \left( ||\tilde{\xi}||^2 + ||\tilde{\theta}||^2 \right) + C_1 \left( 1 + ||\Delta \xi||^2 + ||\Delta \theta||^2 \right) .
\] (4.16)

By Lemma 2.3, (4.5) and the uniform Gronwall inequality we finally obtain that
\[
||\tilde{\xi}(t)||^2 + ||\tilde{\theta}(t)||^2 \leq C , \quad \forall t \geq T + 1 ,
\] (4.17)

which concludes the proof.

**Lemma 4.2.** Suppose that \((\xi_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega)\). Then the solution \((\xi, \theta)\) of problem (2.1)-(2.3) satisfies
\[
||\xi(t)||_{H^2} + ||\theta(t)||_{H^2} \leq M , \quad \forall t \geq T ,
\]

where \(M\) depends only on the data \((\Omega, P_r, R_a)\), \(T\) depends on the data \((\Omega, P_r, R_a)\) and \(R\) when \(||\xi_0|| \leq R\) and \(||\theta_0|| \leq R\).

**Proof.** By (2.1) we have the following inequality
\[
||\Delta \xi|| \leq \left|\left| \frac{\partial \xi}{\partial t} \right|\right| + ||J(\psi, \xi)|| + C||\nabla \theta|| + ||f|| \leq C + C_1||\Psi||_{H^3}||\nabla \xi|| \leq C + C_1||\nabla \xi||^2 \leq C ,
\] (4.18)

where we have used (2.9) and Lemmas 2.3 and 4.1. Similarly, by (2.1) we see that
\[
||\Delta \theta|| \leq \left|\left| \frac{\partial \theta}{\partial t} \right|\right| + ||J(\psi, \xi)|| + ||g|| \leq C .
\] (4.19)

From (4.18) and (4.19), Lemma 4.2 follows.
We are now in position to prove the regularity of the global attractor in $H^2(\Omega) \times H^2(\Omega)$.

**Theorem 4.3.** The global attractor $A$ obtained in Theorem 3.2 is bounded in $H^2(\Omega) \times H^2(\Omega)$, i.e., there is a positive constant $M$ such that

$$\|(|\xi, \theta|)\|_{H^2(\Omega) \times H^2(\Omega)} \leq M, \quad \forall (\xi, \theta) \in A.$$

**Proof.** Since $A$ is bounded in $L^2(\Omega) \times L^2(\Omega)$, by Lemma 4.2, there is a bounded set $E \subseteq H^2(\Omega) \times H^2(\Omega)$ and $T > 0$ such that

$$S(t)A \subseteq E, \quad \forall t \geq T.$$  

But $S(t)A \subseteq A$ and hence $A \subseteq E$. The proof is complete. \hfill \square

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