Effects of dark energy on $P − V$ criticality and efficiency of charged Rotational black hole

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Abstract

In this paper, we study $P − V$ criticality of Kerr-Newman $AdS$ black hole with a quintessence field. We calculate critical quantities and show that for the equation state parameter $\omega = -\frac{1}{3}$, the obtained universal ratio $(\frac{P}{\eta \nu})$ is quite same as Kerr-Newman $AdS$ black hole without dark energy parameter. We investigate the influence of quintessence field $\alpha$, equation state parameter $\omega$ and angular momentum $J$ on the efficiency $\eta$. We find that $\eta$ is increased by increasing $J$ and $\alpha$ and decreasing charge $Q$ of black hole. We show when $\omega$ increases from $-1$ to $-\frac{1}{3}$ the efficiency decreases. Also we study ratio $\frac{\eta}{\eta_C}$ (which $\eta_C$ is the Carnot efficiency) and see that the second law of the thermodynamics is satisfied by special values of $J$ and $\alpha$ and holds for any value of $Q$. We notice that in this case by increasing $\omega$ from $-1$ to $-\frac{1}{3}$ the range of $J$ and $\alpha$ increases.

Keywords: Phase transition; Quintessence matter; Kerr-Newman-AdS black hole solution; Heat engine.

1 Introduction

As we know black holes play an important role in physics especially quantum gravity. One of the interesting methods to study the quantum gravity is black hole thermodynamics in AdS spacetime [1,2]. First time Hawking and Bekenstein investigated the black hole thermodynamic. They found that all laws of black hole mechanics are similar to ordinary thermodynamics where surface gravity, mass and area of black hole are related to the temperature, energy and entropy respectively [3]. The study of black hole thermodynamic will be interesting with presence of a negative cosmological constant. Because such a cosmological constant play important role in holography and $AdS/CFT$. From $AdS/CFT$ point

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of view, asymptotically AdS black hole spacetimes admit a gauge duality description with

dual thermal field theory. Such theory lead us to interesting phenomenon which is called

phase transition. The phase transition plays a key role to study thermodynamical properties

of a system near critical point [4-8]. For the first time, the phase transition was studied

by Hawking and Page [9]. They found a phase transition between Schwarzschild-AdS black

holes and thermal radiation which is known as Hawking-Page phase transition. After that,

a lot of studies done in context of AdS black hole thermodynamics especially in extended

phase space [10-14]. In AdS black hole thermodynamics, the cosmological constant of the

spacetime treat as pressure and its conjugate quantity as a thermodynamic volume. But in

extended phase space, it appear as a thermodynamical variable in the first law of black hole

thermodynamics. Here, remarkable matter is that in extended phase space the mass of black

hole \( M \) is not related to energy but it is associated with enthalpy by \( M = H \equiv U + PV \)

[15]. So the first Law of thermodynamics is expressed by,

\[
dM = TdS + VdP + \Phi dQ + \Omega dJ. \tag{1}
\]

The thermodynamical properties will be very interesting with presence a dark energy pa-

ter. Recently observations show the accelerating expansion of the universe which may

be due to dark energy [16,17]. The dark energy imply a state with the negative pressure. There are two proposed forms for dark energy. The first candidate which is the simplest

explanation for dark energy is cosmological constant and the second is called quintessence

which is described by the state equation \( \omega = \frac{P}{\rho} \) where \( P \) and \( \rho \) are pressure and energy
density respectively. The state parameter \( \omega \) be restricted \(-1 < \omega < 1\) which the case of

\( \omega = \frac{1}{3} \) represents radiation and case of \( \omega = 0 \) represents dust around black hole. The range

of \(-1 < \omega < -\frac{1}{3}\) causes the acceleration and \( \omega = -1 \) covers the cosmological constant term

[18-23]. As we know [24] dark energy forms 70 percent of the universe and black holes are

also part of this universe. This fact makes a motivation for studying black holes surrounded

by dark energy. Such black holes play the crucial role in cosmology and it is very interesting

to know influence of dark energy on thermodynamical properties of black holes. The first
time in 2003, Kiselev obtained the Einstein equations solution for the quintessence matter

around schwarzschild black hole. This solution is written in terms of \( \omega \) and \( \alpha \) where \( \alpha \) is

a normalization factor that represents the intensity of the quintessence field related to the

black hole [17]. The Kiselev black hole for different situations was studied in Ref. [25-34].
The rotation Kiselev and Kerr-Newman kiselev solution was determine in Ref. [35-38] and

also Kerr- Newman-AdS solution in the quintessence was obtained in [39]. Then some ef-
forts have been made in context of thermodynamic and phase transition for Kiselev and Kerr

Kiselev black hole [18,39].

After studying of phase transition, it is interesting to define classical cycles for black holes

like usual thermodynamic systems. It means that when small black hole translate to large

black hole we need again the large black hole translate to small black hole, in other words

the system must be back to primary state. For the first time the holographic heat engine

has been studied for charge AdS black hole [40]. Thereafter some effort has been made for
different situations [41-56]. Authors investigated the holographic heat engine for charge AdS

black hole with presence dark energy parameter in Ref. [23]. Also rotational black hole as
heat engine was studied in Ref. [57-58]. In this paper, we are going to study holographic heat
engine for charged rotational black hole with presence dark energy parameter and discuss
about the influence of quintessence field on the $P - V$ criticality and efficiency of a such
black hole.

The outline of the paper is as follows. In section II, we investigate $P - V$ criticality of
Kerr-Newman-AdS black hole solution with present dark energy parameter. In section III,
we consider corresponding black hole as heat engine and study the influence of quintessence
field on the efficiency. Finally, in section IV we have some results and conclusion.

2 P-V criticality of Kerr-Newman-AdS black hole with quintessence field

The Kerr-Newman AdS metric in quintessence matter is expressed as follows [28],
\[ ds^2 = \frac{\Sigma^2}{\Delta_r} dr^2 + \frac{\Sigma^2}{\Delta_{\theta}} d\theta^2 + \frac{\Delta_{\theta} \sin^2 \theta}{\Sigma^2} \left( a \frac{dt}{\Xi} - (r^2 - a^2) \frac{d\phi}{\Xi} \right)^2 - \frac{\Delta_r}{\Sigma^2} \left( \frac{dt}{\Xi} - a \sin^2 \theta \frac{d\phi}{\Xi} \right)^2, \]
(2)

where
\[ \Delta_r = r^2 - 2Mr + a^2 + Q^2 + \frac{r^2}{\ell^2} (r^2 + a^2) - \alpha r^{1-3\omega}, \]
\[ \Delta_{\theta} = 1 - \frac{a^2}{\ell^2} \cos^2 \theta \quad \Xi = 1 - \frac{a^2}{\ell^2} \quad \Sigma^2 = r^2 + a^2 \cos^2 \theta, \]
(3)

where mass $M$, charge $Q$, the angular momentum $J$ are related to parameters $m$, $q$, and $a$
respectively by,
\[ M = \frac{m}{\Xi^2}, \quad Q = \frac{q}{\Xi}, \quad J = \frac{am}{\Xi^2} \]
(4)
as pervious mentioned $\alpha$ and $\omega$ are normalization factor and state parameter respectively.
The cosmological constant $\Lambda = -\frac{3}{\ell^2}$ interpret as a thermodynamic pressure $P$
which is given by,
\[ P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi \ell^2}. \]
(5)
The mass of black hole is determined by $\Delta_r(r_+) = 0$, so
\[ M = \frac{(r_+^2 + a^2)(r_+^2 + \ell^2) + Q^2 \ell^2 - \alpha \ell^2 r^{1-3\omega}}{2r_+ \ell^2}, \]
(6)
the area of event horizon is given by,
\[ A = 4\pi \frac{(r_+^2 + a^2)}{\Xi}, \]
(7)
so The Bakenstein-Hawking entropy is expressed by
\[ S = \frac{A}{4} = \pi \frac{(r_+^2 + a^2)}{\Xi}. \]
(8)
The Hawking temperature which is related to surface gravity is defined by,

\[ T_H = \frac{\kappa(r_+)}{2\pi} = \lim_{\theta = 0, r \to r_+} \frac{\partial r \sqrt{g_{tt}}}{2\pi \sqrt{g_{rr}}}, \tag{9} \]

thus, the thermal temperature of Kerr-Newman-AdS black hole in quintessential dark energy is obtained as,

\[ T = \frac{r_+}{4\pi \Xi(r_+^2 + a^2)} \left( \frac{3r_+^2}{\ell^2} + \frac{a^2}{\ell^2} + 1 - \frac{a^2 + Q^2}{r_+^2} \right) + 3\alpha \omega r^{-1-3\omega}. \tag{10} \]

The thermodynamical volume \( V \) conjugated to the pressure \( P \) is given by,

\[ V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,J}. \tag{11} \]

Now we are going to study P-V criticality of corresponding black hole for three different values of \( \omega \). In the first case we consider \( \omega = -\frac{1}{3} \). Using the expressions (4) and (8) and the fact that \( \Delta r_+ = 0 \), one can obtain a generalized Smarr formula for Kerr-Newman-AdS black holes with quintessence as follows,

\[ M = \left( \frac{\pi}{4S} \left( \frac{4SJ^2}{\pi \ell^2} + 4J^2 + \left[ \frac{S^2}{\pi^2 \ell^2} + \frac{S}{\pi} + Q^2 - \frac{\alpha S}{\pi} \right]^2 \right) \right)^{\frac{1}{3}}. \tag{12} \]

The thermodynamical volume is,

\[ V = \frac{2\pi}{3r_+^2 \Xi} \left( \frac{r_+}{r_+^2 + a^2} \right)\left( \frac{2r_+^2 - r_+^2 a^2}{\ell^2} + a^2 (1 - \alpha) + Q^2 a^2 \right) + \frac{4\pi a^2 r_+^3}{3} \left[ \frac{2r_+^4 - r_+^2 a^2}{\ell^2} + \frac{2r_+^2}{\ell^2} - \alpha - \frac{a^2}{\ell^2} \right]. \tag{13} \]

The specific volume of the corresponding fluid which is related to thermodynamical volume \( (v = \frac{2V}{4\pi}) \) will be as,

\[ v = 2r_+ + \frac{12J^2 (8\pi Pr_+^4 + r_+^2 (3 - \alpha) + Q^2)}{r_+ (8\pi Pr_+^4 + 3r_+^2 (1 - \alpha) + 3Q^2)^2} + \frac{16J^2 r_+^2 (64\pi^2 P^2 r_+^4 + 12\pi Pr_+^2 (2 - \alpha) - \frac{9\alpha}{2})}{(8\pi Pr_+^4 + 3r_+^2 (1 - \alpha) + 3Q^2)^3}. \tag{14} \]

The equation of state is given by,

\[ P = \frac{T}{v} + \frac{(\alpha - 1)}{2\pi v^2} + \frac{2Q^2}{\pi v^4} + \frac{48J^2}{\pi v^6} + \frac{6J^2 B}{\pi v^6 (\pi T v^3 + (1 - \alpha) v^2 + 8Q^2)^2}, \tag{15} \]

where

\[ B = 4v^2 (v^2 + 4\alpha v^2 - 2\alpha^2 v^2) + 16Q^2 (9\alpha v^2 - 9v^2 - 30Q^2) + 4\pi T v^3 (3v^2 (1 + \alpha) - 28Q^2 - 4\pi T v^3). \tag{16} \]

We depict the \( P - v \) diagram in Fig. 1 and see the qualitative behavior is similar to the Van der Waals fluid. Critical points occur at stationary points of inflection in the \( P - v \) diagram, where

\[ \frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0, \tag{17} \]
which leads to the following equation,

\[ \upsilon_c = 2\sqrt{\frac{3(Q^2 + \sqrt{Q^4 - 10\alpha J^2 + 10J^2})(1 - \alpha)}{(1 - \alpha)}}. \]  

Due to the complexity of the obtained relations for temperature and pressure, one can determine critical points numerically. We obtain the critical specific volume \( \upsilon_c \), critical temperature \( T_c \), critical pressure \( P_c \) and the universal critical ratio \( \left( \frac{P_c \upsilon_c}{T_c} \right) \) for different values of \( Q, J \) and \( \alpha \) as follows,

| \( \alpha \) | \( J \) | \( Q \) | \( \upsilon_c \) | \( T_c \) | \( P_c \) | \( \frac{P_c \upsilon_c}{T_c} \) |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.05 | 2.0083 | 0.1138 | 0.0235 | 0.4160 |
| 0.5 | 0.1 | 0.05 | 2.3295 | 0.0544 | 0.0544 | 0.4158 |
| 0.1 | 0.2 | 0.05 | 2.8343 | 0.0807 | 0.0097 | 0.4163 |
| 0.1 | 0.1 | 0.2 | 2.1374 | 0.0103 | 0.0118 | 0.4082 |

Table 1: \( \upsilon_c, T_c, P_c \) and their ratio for different value of \( Q, J \) and \( \alpha \).

As we see the critical specific volume and critical pressure are increased and critical temperature is decreased by increasing \( \alpha \). When angular momentum increases \( \upsilon_c \) and \( T_c \) increase and \( P_c \) decreases. But by increasing charge of black hole only \( \upsilon_c \) increases and both \( T_c \) and \( P_c \) decrease. Also table (1) shows that \( \frac{P_c \upsilon_c}{T_c} \) is increased by increasing \( J \) and is decreased by increasing \( \alpha \) and \( Q \).

The second case we consider \( \omega = -\frac{2}{3} \). The generalized Smarr formula for this case is obtained as,

\[
M = \left( \frac{\pi}{4S} \left\{ \frac{4S J^2}{\pi \ell^2} + 4J^2 + \left[ \frac{S^2}{\pi^2 \ell^2} + \frac{S}{\pi} + Q^2 - \alpha \left( \frac{S}{\pi} \right)^{\frac{2}{3}} \right]^2 \right\} \right)^{\frac{1}{2}}. \]  

(19)
The thermodynamical volume is determined as,
\[
V = \frac{2\pi}{3r^2} \left( (r^2 + a^2)(2r - \frac{r^2 a^2}{\ell^2} + a^2 - \alpha \tau^2 + a^2) + Q^2a^2 \right) - \frac{4\pi a^2 r_3}{3} \frac{2r_1}{r^2} - \frac{2r_3^2}{r^2} + \frac{3}{2} \alpha r^2 + \frac{3\alpha r^3}{2\ell^2}.
\]

(20)

The specific volume is obtained as,
\[
v = 2r_+ + \frac{12J^2(3\pi r_+^4 + 3\tau^2 - \alpha \tau^2 + Q^2)}{8J_+^2(128\pi^2 r_+^4 + 48\pi P r_+^4 - \frac{27}{2} \alpha r_+^2 - 36\alpha \pi r_3)} + \frac{8J_+^2(128\pi^2 r_+^4 + 48\pi P r_+^4 - \frac{27}{2} \alpha r_+^2 - 36\alpha \pi r_3)}{(8\pi P r_+^4 + 3\tau^2 - 3\alpha \tau^2 + 3Q^2)^2}.
\]

(21)

By using equations (5), (10) and (21), we can obtain the equation of state as,
\[
P = \frac{T}{\nu} - \frac{1}{2\pi u^2} + \frac{2Q^2}{\pi^2 u^4} + \frac{\alpha}{2\pi u} + \frac{48J^2}{\pi^2 u^6} + \frac{6J^2 B}{\pi^2 u^6(\pi T u^3 + u^2 + 8Q^2 - \frac{\alpha u^4}{2})^2},
\]

where
\[
B = \nu^2(4\nu^2 - 2\alpha \nu^3 + \frac{\alpha^2 \nu^4}{2}) + 4Q^2(13\alpha \nu^3 - 36\nu^2 - 112Q^2) + 4\pi T \nu^3(3\nu^2 - 28Q^2 - 4\pi T \nu^3).
\]

(23)

We have plotted \(P - \nu\) diagram in Fig. 2 which shows the Van der Waals like behavior. By using equations (17) and (22) the critical specific volume is obtained by,
\[
v_c = 2\sqrt{3(Q^2 + \sqrt{Q^4 + 20J^2})}.
\]

(24)

We obtain critical quantities and ratio of them for different values of \(Q\), \(J\) and \(\alpha\) as follows,

| \(\alpha\) | \(J\) | \(Q\) | \(\nu_c\) | \(T_c\) | \(P_c\) | \(P_{\nu_c}/T_c\) |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.05 | 2.3230 | 0.1071 | 0.0244 | 0.5309 |
| 0.3 | 0.1 | 0.05 | 1.9557 | 0.1299 | 0.0520 | 0.7837 |
| 0.1 | 0.2 | 0.05 | 3.2807 | 0.07131 | 0.0123 | 0.5658 |
| 0.1 | 0.1 | 0.2 | 2.4223 | 0.0973 | 0.0211 | 0.5257 |

Table 2: \(\nu_c\), \(T_c\), \(P_c\) and their ratio for different value of \(Q\), \(J\) and \(\alpha\).

Table (2) shows that \(T_c\) and \(P_c\) are increased by increasing \(\alpha\) and decreasing \(J\) but they are decreased by increasing \(Q\). As we see \(\nu_c\) increases when \(J\) and \(Q\) increase and it decreases when \(\alpha\) increases. Also we find that \(P_{\nu_c}/T_c\) is increased by increasing \(\alpha\) and \(J\) and is decreased by increasing \(Q\).

In third case, we assume \(\omega = -1\). In this case the generalized Smarr formula is,
\[
M = \left( \frac{\pi}{4s} \left\{ \frac{4sJ^2}{\pi \ell^2} + 4J^2 + \left[ \frac{S^2}{\pi^2 \ell^2} + \frac{S}{\pi} + Q^2 - \frac{\alpha S^2}{\pi^2} \right] \right\} \right)^{\frac{1}{2}}.
\]

(25)

The thermodynamical volume will be as,
\[
V = \frac{2\pi}{3r^2} \left( (r^2 + a^2)(2r - \frac{r^2 a^2}{\ell^2} + a^2 - \alpha \tau^2 + a^2) + Q^2a^2 \right) - \frac{4\pi a^2 r_3}{3} \frac{2r_1}{r^2} - \frac{2r_3^2}{r^2} + \frac{3}{2} \alpha r^2 + \frac{3\alpha r^3}{2\ell^2}.
\]

(26)
Figure 2: The $P - v$ diagram for $Q = 0.05$, $J = 0.1$, $\alpha = 0.1$ and different values of $T$.

The specific volume of the corresponding is fluid as fellows,

$$v = 2r_+ + \frac{12J^2(8\pi Pr_+^4 + 3r_+^2 - \alpha r_+^4 + Q^2)}{r_+(8\pi Pr_+^4 + 3r_+^2 - 3\alpha r_+^4 + 3Q^2)^2} + \frac{16J^2r_+^3(64\pi^2 P^2 r_+^4 + 24\pi P r_+^2 - 9\alpha r_+^2 - 24\alpha \pi P r_+^4)}{(8\pi P r_+^4 + 3r_+^2 - 3\alpha r_+^4 + 3Q^2)^3}.$$  \hfill (27)

The equation of state is given by,

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4} + \frac{3\alpha}{8\pi} + \frac{48J^2}{\pi v^6} + \frac{6J^2B}{\pi v^6(\pi T v^3 + v^2 + 8Q^2)^2},$$  \hfill (28)

where

$$B = v^2(4v^2 - \alpha v^4 + 8\alpha^2 v^4) + 8Q^2(2\alpha v^4 - 18v^2 - 56Q^2) + \pi T v^3(12v^2 - \alpha v^4 - 112Q^2 - 16\pi T v^3).$$  \hfill (29)

Figure (3) shows qualitative behavior $P$ with respect to $v$ which is same as the Van der Waals fluid. In that case we obtain $v_c$ as follows,

$$v_c = 2\sqrt{3(Q^2 + \sqrt{Q^4 + 10J^2})}.$$  \hfill (30)

We indicate changes of $v_c$, $T_c$ and $P_c$ for different values of $Q$, $J$ and $\alpha$ in table (3). As we see $v_c$ and $\frac{P_c}{T_c}$ are increased but $T_c$ and $P_c$ are decreased by increasing $Q$ and $J$. Also we observe when $\alpha$ increases $v_c$ and $T_c$ don’t change but $P_c$ and $\frac{P_c}{T_c}$ increase. By comparing results we find that $P_c$ and $\frac{P_c}{T_c}$ decreases when $\omega$ increases from $-1$ to $-\frac{4}{3}$. We find that for $\omega = -1$ and $-\frac{2}{3}$ and for $\alpha > 0.4$, $\frac{P_c}{T_c}$ will be greater than one. Also we notice that the obtained result for $\omega = -\frac{1}{3}$ is very similar to case of without dark energy parameter as Ref. [14].
In this section, we are going to investigate the effects of quintessence field on the efficiency of Kerr-Newman AdS black hole. We need to calculate heat capacity under constant pressure and constant volume to obtain efficiency of black hole. For the static holes, the thermodynamic volume and the entropy are not independent. So, the heat capacity in constant volume ($C_V = T\frac{\partial S}{\partial T}|_V$) is zero. It shows that the heat flow occur along the isobars and one can obtain the efficiency by straightforward approach as Ref [40]. But for rotation black holes the thermodynamic volume and the entropy are independent and the heat capacity is non-zero under constant volume. So, heat flow occur along the isobars and isochores. In this case one cannot employ usual methods for obtaining efficiency. Here we use the mentioned approach in Ref. [56] and obtain the efficiency as,

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}, \quad (31)$$

3 The efficiency of heat engine of corresponding black hole

In this section, we are going to investigate the effects of quintessence field on the efficiency of Kerr-Newman AdS black hole. We need to calculate heat capacity under constant pressure and constant volume to obtain efficiency of black hole. For the static holes, the thermodynamic volume and the entropy are not independent. So, the heat capacity in constant volume ($C_V = T\frac{\partial S}{\partial T}|_V$) is zero. It shows that the heat flow occur along the isobars and one can obtain the efficiency by straightforward approach as Ref [40]. But for rotation black holes the thermodynamic volume and the entropy are independent and the heat capacity is non-zero under constant volume. So, heat flow occur along the isobars and isochores. In this case one cannot employ usual methods for obtaining efficiency. Here we use the mentioned approach in Ref. [56] and obtain the efficiency as,

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}, \quad (31)$$
Figure 4: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $J$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $\alpha = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $J$.

Figure 5: Left plot: $\eta$ with respect to $V_R$ for $J = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\alpha$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $J = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\alpha$. 

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Figure 6: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $Q$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $Q$.

Figure 7: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $J$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $\alpha = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $J$. 
Figure 8: Left plot: $\eta$ with respect to $V_R$ for $J = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\alpha$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $J = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\alpha$

Figure 9: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $Q$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $Q$
Figure 10: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $J$; Right plot: $\eta/\eta_C$ with respect to $V_R$ for $\alpha = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $J$.

Figure 11: Left plot: $\eta$ with respect to $V_R$ for $J = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\alpha$; Right plot: $\eta/\eta_C$ with respect to $V_R$ for $J = 0.1$, $Q = 0.05$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\alpha$. 


Figure 12: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $Q$; Right plot: $\frac{\eta}{\eta_C}$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $P_T = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $Q$.

Figure 13: Left plot: $\eta$ with respect to $V_R$ for $\alpha = 0.1$, $J = 0.1$, $Q = 0.05$, $P_B = 0.07$, $P_B = 0.06$, $V_L = 10$ and different values of $\omega$; Right plot: $\eta$ with respect to $P_T$ for $\alpha = 0.1$, $Q = 0.05$, $J = 0.1$, $P_B = 0.05$, $V_R = 20$, $V_L = 10$ and different values of $\omega$. 
where \( Q_H \), \( Q_C \) and \( W \) are net input heat flow, net output heat flow and net output work respectively. In the case of a rectangular cycle for rotating black holes with \( C_V \neq 0 \), \( Q_H \) and \( Q_C \) are expressed by following expression,

\[
Q_C = M(V_R, P_T) - M(V_L, P_B) - \Delta PV_R
\]
\[
Q_H = M(V_R, P_T) - M(V_L, P_B) - \Delta PV_L,
\] (32)

where \( \Delta P = P_T - P_B \). By substituting (32) to (31), the efficiency is determined as,

\[
\eta = \frac{\Delta P \Delta V}{\Delta M_T + \Delta U_L},
\] (33)

where \( \Delta V = V_R - V_L \), \( U = M - PV \) and

\[
\Delta M_T = M(V_R, P_T) - M(V_L, P_T)
\]
\[
\Delta U_L = U(V_L, P_T) - M(V_L, P_B).
\] (34)

The efficiency of engine cannot be larger than the Carnot efficiency. Because in this case the second law will be violated. The efficiency of the Carnot engine is obtained by following,

\[
\eta_C = 1 - \frac{T_C}{T_H} = 1 - \frac{T(P_B, V_L)}{T(P_T, V_R)}.
\] (35)

Here important subject is that how the efficiency changes by varying normalization factor \( \alpha \), state parameter \( \omega \), angular momentum \( J \) and charge of black hole. We have plotted efficiency \( \eta \) and ratio \( \frac{\eta}{\eta_C} \) for \( \omega = -\frac{1}{3} \) in figures (4-6). Figure (4) shows differentiation of \( \eta \) and \( \frac{\eta}{\eta_C} \) with respect to \( V_R \) for different values of \( J \). As we see, \( \eta \) increases by increasing \( J \) but it monotonously decrease with \( V_R \). In other words, when the volume difference between small black hole \( (V_L) \) and large black hole \( (V_R) \) is increased, the efficiency is declined until it reaches to a constant value. Also, we see when the angular momentum rises the diagram of \( \eta \) shifts to a larger \( V_R \). It means that by increasing \( J \) the volume of large black hole increases therefore \( \Delta V \) (the volume difference between small black hole and large black hole) will be larger. The right figure of Fig 4. is similar to the left one. It shows that \( \frac{\eta}{\eta_C} \) monotonously decreases by increasing \( V_R \) which means that the difference between the actual efficiency \( (\eta) \) and \( (\eta_C) \) will be more for larger \( \Delta V \). Also, we notice that \( \frac{\eta}{\eta_C} < 1 \) holds for \( J < 0.5 \) so the second law of the thermodynamics would violate for \( J > 0.5 \).

The Fig. 5 has been plotted for three different normalization factor \( \alpha \). As we see, the curves are very similar to the Fig.4, then we can use same discussions in this case. Also we notice that the length of the defined cycle significantly increases for \( 1 < \alpha < 2 \) i.e. \( \Delta V \) highly rises in this range. The right figure in Fig.5 shows that the second law is satisfied for \( \alpha < 2 \). We have plotted figures of \( \eta \) and \( \frac{\eta}{\eta_C} \) for three values of charge in Fig. 6. The discussions are somewhat similar to figures of (4) and (5) but the difference is that \( \eta \) increases by decreasing \( Q \) and the second law holds for any value of charge. We continue our analysis
for $\omega = -\frac{2}{3}$. The results are almost identical to the previous one. But there is a significant difference in this case. As we notice that in Fig. 7 $\frac{\eta}{\eta_C}$ reduces by decreasing $J$. But for very small values of $J$ the difference between $\eta$ and $\eta_C$ is such that $\frac{\eta}{\eta_C}$ first decreases afterward increases and then it reduces and eventually it reaches to a constant value. One can use similar explanation for right figure of Fig. 9. Also we investigate behavior of $\eta$ and $\frac{\eta}{\eta_C}$ under changes of $J$, $\alpha$ and $Q$ for $\omega = -1$ in figures (10-12). Our analysis is same as case of $\omega = -\frac{2}{3}$. Finally we investigate our the $\eta$ under change of pressure $P_T$. The figure (13) shows that the $\eta$ monotonously increase as $P_T$ rises. And by reducing $\omega$ from $-\frac{1}{3}$ to $-1$ the efficiency increases. Finally, by comparing these figures we notice when the state parameter $\omega$ increases from $-1$ to $-\frac{1}{3}$ range of $J$ and $\alpha$ (which can satisfy the second law) rises.

4 conclusion

In this paper, we studied $P-V$ criticality of Kerr-Newman $AdS$ black hole with a quintessence field. We calculated critical quantities and saw that by reducing the state parameter $\omega$ from $-\frac{1}{3}$ to $-1$ the critical pressure and temperature increases. Also we obtained universal ratio and noticed for the state parameter $\omega = -\frac{1}{3}$, the obtained ratio is quite same as Kerr-Newman $AdS$ black hole without dark energy parameter. Then, we considered corresponding black hole and investigated the influence of quintessence field $\omega$ and $J$ on the efficiency $\eta$. We found that the $\eta$ increases by increasing $J$ and $\alpha$ and reducing charge $Q$ of black hole. Also we saw that the $\eta$ monotonously decrease by rising the volume of large black hole ($V_R$) i.e. when volume difference between small black hole and large black hole ($\Delta V$) grows then the $\eta$ highly reduces. We found that when $J$ and $\alpha$ increases the length of defined cycle increases, in other words $\Delta V$ increases. By studying ratio the $\eta$ and the highest efficiency (Carnot efficiency $\eta_C$) we noticed that the second law of the thermodynamics will be violated for special values of $J$ and $\alpha$ but it holds for any value of $Q$. It means that constraint of $\frac{\eta}{\eta_C} < 1$ is not satisfied for any value of $J$ and $\alpha$. And also by increasing $\omega$ from $-1$ to $-\frac{1}{3}$ the range of $J$ and $\alpha$ which satisfy the second law increases. We showed that when $\omega$ increases from $-1$ to $-\frac{1}{3}$ then the efficiency decreases. Finally, we investigated behavior the $\eta$ under changes of pressure. We saw that by increasing pressure the $\eta$ monotonously increases and the second law is violated by increasing pressure difference between $P_B$ and $P_T$ in other words the second law holds for very small $\Delta P$.

References

[1] J. D. Bekenstein, Phys. Rev. D 7 (1973) 2333.
[2] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31 (1973) 161.
[3] J. D. Bekenstein, Phys.Rev. D 9 (1974) 3292.
[4] P. C. W. Davies, Class. Quant. Grav. 6 (1989) 1909.
[5] P. C. W. Davies, Reports on Progress in Physics, 41 (1978) 1313.

[6] P. C. W. Davies, Proceedings of the Royal Society of London Series A, 353 (1977) 499.

[7] X. Guo, H. Li, L. Zhang and R. Zhao, Phys. Rev. D, 91 (2015) 084009.

[8] Y. Liao, X. B. Gong and J. S. Wu, Astrophysical Journal, Volume 835 (2016) 2.

[9] S. Hawking and D. N. Page, Commun. Math. Phys. 87 (1983) 577.

[10] Y. S. Myung, Y. W. Kim and Y. J. Park, Phys. Rev. D 78 (2008) 084002.

[11] Y. S. Myung, Phys. Lett. B 638 (2006) 515.

[12] B. M. N. Carter and I. P. Neupane, Phys. Rev. D 72 (2005) 043534.

[13] D. Kubiznak and R. B. Mann, J. High Energy Phys. 078 (2012) 033.

[14] S. Gunasekaran, D. Kubiznak and R. B. Mann, J. High Energy Phys. 11 (2012) 110.

[15] D. Kastor, S. Ray, and J. Traschen, Class. Quant. Grav. 26 (2009) 195011.

[16] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517 (1999) 565.

[17] V. V. Kiselev, Class. Quant. Grav. 20 (2003) 1187.

[18] B. Majeed, M. Jamil and P. Pradhan, Adv. High Energy Phys. 2015 (2015) 124910.

[19] Y. Fujii, Phys. Rev. D 26 (1982) 2580.

[20] L. Ford, Phys. Rev. D 35 (1987) 2339.

[21] B. Ratra and P. Peebles, Phys. Rev. D 37 (1988) 3406.

[22] C. Wetterich, Nucl. Phys. B 302 (1988) 668.

[23] H. Liu and X. H. Menga, Eur. Phys. J. C 77 (2017) 556.

[24] S. G. Ghosh, S. D. Maharaj, D. Baboolal and T. H. Lee, Eur. Phys. J. C 78 (2018) 90.

[25] B. Malakolkalami and K. Ghaderi, Astrophys. Space Sci. 357 (2015) 112.

[26] R. Uniyal, N. C. Devi, H. Nandan and K. D. Purohit, General Relativity and Gravitation, 47 (2015) 16.

[27] S. Fernando, Gen. Rel. Gravit. 44 (2012) 1857.

[28] N. Varghese and V. C. Kuriakose, Gen. Rel. Gravit. 41 (2009) 1249.

[29] M. Saleh, B. T. Bouetou and T. C. Kofane, Astrophys. Space Sci. 333 (2011) 449.
[30] H. Nandan and R. Uniyal, Eur. Phys. J. C 77 (2017) 552.

[31] S. G. Ghosh, M. Amir and S. D. Maharaj, Eur. Phys. J. C 77 (2017) 530.

[32] B. Bouetou, T. Saleh and T. C. Kofane, Gen. Rel. Gravit. 44 (2012) 2181.

[33] M.S. Ma, R. Zhao and Y. Q. Ma, Gen. Rel. Gravit. 49 (2017) 79.

[34] K. Ghaderi and B. Malakolkalami, Astrophys. Space Sci. 361 (2016) 161.

[35] S. G. Ghosh, Eur. Phys. J. C 76 (2016) 222.

[36] T. Oteev, A. Abdujabbarov, Z. Stuchlik and B. Ahmedov, Astrophys. Space Sci. 361 (2016) 269.

[37] B. Toshmatov, Z. Stuchlik, and B. Ahmedov, Eur. Phys. J. Plus 132 (2017) 98.

[38] Z. Xu and J. Wang, Phys. Rev. D 95 (2017) 064015.

[39] Z. Xu and J. Wang, arXiv:1610.00376 [gr-qc].

[40] C. V. Johnson, Class. Quant. Grav. 31 (2014) 205002.

[41] C. V. Johnson, Class. Quant. Grav. 33 (2016) 215009.

[42] C. V. Johnson, Class. Quant. Grav. 33 (2016) 135001.

[43] C. V. Johnson, Entropy 18 (2016) 120 arXiv:1602.02838 [hep-th]

[44] Kh. Jafarzade and J. Sadeghi, Int. J. Mod. Phys. D 26 (2017) 1750138.

[45] C. Bhamidipati and P. K. Yerra, Eur. Phys. J. C 77 (2017) 534.

[46] M. Zhang and W. B. Liu, Int. J. Theor. Phys. 55 (2016) 5136.

[47] A. Chakraborty and C. V. Johnson, arXiv:1612.09272 [hep-th].

[48] J. X. Mo, F. Liang and G. Q. Li, JHEP 03 (2017) 010.

[49] C. V. Johnson, arXiv:1703.06119 [hep-th]

[50] S. H. Hendi, B. Eslam Panah, S. Panahiyan, H. Liu and X. H. Meng, arXiv:1707.02231 [hep-th].

[51] J. X. Mo and G. Q. Li, arXiv:1707.01235 [gr-qc].

[52] S. W. Wei and Y. X. Liu, arXiv:1708.08176 [gr-qc].

[53] A. Chakraborty, C. V. Johnson, arXiv:1709.00088 [hep-th].

[54] J. Zhang, Y. Li and H. Yu, arXiv:1801.06811 [hep-th].
[55] Kh. Jafarzade and J. Sadeghi, Int. J. Theor. Phys. 56 (2017) 3387.

[56] K. Fernandes and A. Lahiri, Class. Quant. Grav. 34 (2017) 174001.