VARIATIONAL STRUCTURE OF THE $\nu_2$-YAMABE PROBLEM

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Abstract. We define a formal Riemannian metric on a conformal class in the context of the $\nu_2$-Yamabe problem. We also give a new variational description of this problem, and show that the associated functional is geodesically convex. Formal properties of the negative gradient flow are also described. These results parallel our work in two dimensions on the Liouville energy and the uniformization of surfaces [14], and our work in four dimensions on the $\sigma_2$-Yamabe problem [15].

1. Introduction

In [14] and [15] we defined a formal Riemannian metric on the space of conformal metrics satisfying some notion of ‘positivity’. In the case of surfaces this condition corresponded to positive Gauss curvature. In four dimensions, given a conformal class $[g_0]$ we considered the subset

$$ C^+ = C^+([g_0]) = \{ g_u = e^{-2u}g_0 : A_u \in \Gamma_2^+ \}, \quad (1.1) $$

where $A_u$ is the Schouten tensor of the metric $g_u$ and $\Gamma_2^+$ is the positive 2-cone. Recall the Schouten tensor is defined by

$$ A = \frac{1}{n-2}(Ric - \frac{1}{2(n-1)}Rg), \quad (1.2) $$

where $Ric$ and $R$ are the Ricci and scalar curvatures of $g$, and

$$ A_g \in \Gamma_k^+ \iff \sigma_1(g^{-1}A_g) > 0, \ldots, \sigma_k(g^{-1}A_g) > 0, $$

where $\sigma_k(\cdot)$ is the elementary symmetric polynomial of degree $k$ of the endomorphism $g^{-1}A$ (see the Introduction of [15]). To simplify notation in what follows we will suppress the inverse of the metric $g_u$ and only write $A_g, A_u$, etc. Assuming $C^+$ is non-empty we defined the Riemannian metric on $C^+$ by

$$ \langle \phi, \psi \rangle_u = \int_M \phi \psi \sigma_2(A_u) dV_u, \quad (1.3) $$

Here we are using the natural identification of the tangent space to $C^+$ at any point with $C^\infty(M)$. Endowed with this metric $C^+$ enjoys a number of nice formal properties; e.g., $C^+$ has non-positive sectional curvature.

Of primary interest in our analysis were the variational properties of the functional $F : C^+ \to \mathbb{R}$ introduced by Chang-Yang [7], whose critical points are conformal metrics satisfying

$$ \sigma_2(A_u) = \text{const.}, \quad (1.4) $$

i.e., $g_u$ is a solution of the $\sigma_2$-Yamabe problem. In particular, we showed that the functional $F$ is geodesically convex and used this fact to prove a remarkable geometric consequence: solutions of (1.4) are unique, unless $(M^4, g_0)$ is conformally equivalent to the sphere. This is a surprising departure from the classical Yamabe problem, where explicit examples of non-uniqueness are known.
To extend these results to higher dimensions \( n \geq 6 \), it would seem natural to consider the set of conformal metrics \( \mathcal{C}^+ = \{ g_u = e^{-2u}g_0 : A_u \in \Gamma_{n/2}^+ \} \), and define the inner product on the tangent space by

\[
\langle \phi, \psi \rangle_u = \int_M \phi \psi \sigma_{n/2}(A_u) dV_u.
\]

However, defining the metric in this way introduces a number of technical issues which can be traced back to the fact that in dimensions \( n \geq 6 \), the quantity \( \sigma_{n/2}(A_u) \) lacks the kind of ‘divergence structure’ that it enjoys in dimension four, or the Gauss curvature enjoys in dimension two. More concretely, the integral

\[
\int \sigma_{n/2}(A_g) \, dV_g
\]

is conformally invariant when \( n = 4 \), but not in general when \( n \geq 6 \). Indeed, this lack of divergence structure is a fundamental difficulty in the study of the \( \sigma_k \)-Yamabe problem, which asks whether it is possible to find a conformal metric \( g_u = e^{-2u}g \) for which \( \sigma_k(A_u) = \text{const.} \), assuming \( A_g \in \Gamma_k^+ \).

If \( (M^n, g) \) is locally conformally flat (LCF) then the integral (1.6) is conformally invariant (3, 21). However, it follows from the work of Guan-Viaclovsky (13) that when \( A_g \in \Gamma_{n/2}^+ \), then the Ricci curvature of \( g \) is positive. Consequently, by Kuiper’s Theorem, in the LCF setting the space of conformal metrics \( g_u = e^{-2u}g \) with \( A_u \in \Gamma_{n/2}^+ \) will be non-empty only when \( (M^n, g) \) is conformally equivalent to the round sphere (or real projective space). Consequently, imposing the LCF condition for metrics whose Schouten tensor is in \( \Gamma_{n/2}^+ \) is too restrictive.

What is needed in higher dimensions is a conformally invariant quantity of the correct weight which does not require the LCF condition. Such a quantity appears in the consideration of the renormalized volume of Poincare-Einstein manifolds, see [10]. To explain this, we briefly recall some definitions.

Let \( X \) be the interior of a compact manifold with boundary \( \overline{X} \) of dimension \( n + 1 \), and let \( M = \partial X \) denote the boundary. A metric \( g_+ \) defined on \( X \) is said to be conformally compact if there is a defining function \( r \in C^\infty(X) \) with \( r > 0 \) and \( dr \neq 0 \) on \( \partial X \), such that \( r^2 g_+ \) extends to a metric \( \overline{g} \) on \( \overline{X} \). Since we can multiply \( r \) by any smooth positive function on \( \overline{X} \), a conformally compact metric naturally defines a conformal class of metrics \( [g = \overline{g}|_M] \) on \( M = \partial X \), called the conformal infinity of \( (X, g_+) \). If in addition \( g_+ \) satisfies the Einstein condition, which we normalize by

\[
\text{Ric}(g_+) = -ng_+,
\]

then we say that \( (X, g_+) \) is a Poincaré-Einstein (P-E) manifold. If \( r \) is a defining function such that \( |dr|_g = 1 \) on \( M = \partial X \) (referred to as a special defining function), then \( g_+ \) can be written

\[
g_+ = r^{-2}(dr^2 + g_r),
\]

where \( g_r \) is a 1-parameter family of metrics on \( M \) with \( g_0 = g \). When \( n \) is even, Graham [10] showed that

\[
g_r = g^{(0)} + g^{(2)} r^2 + \cdots + g^{(n)} r^n + hr^n \log r + \cdots ,
\]

where \( g^{(0)} = g \) is the induced metric on \( M \), the coefficients are formally determined by the conformal representative up to order \( n - 2 \), and \( h \) is also formally determined. Using this expansion, Graham gave an expansion for the volume form

\[
\left( \frac{\det(g_r)}{\det g} \right) \sim 1 + \sum_{k \geq 1} v_k r^k ,
\]
where \( v_k = v_k(g) \) are defined for \( 1 \leq k \leq n/2 \) in general, but are defined for all \( k \geq 1 \) when \((M, g)\) is LCF. When \( k = 1, 2 \), then \( v_k(g) = \sigma_k(A) \), while in the LCF case this holds for \( k \geq 3 \). Moreover, when \( k = n/2 \), then

\[
v = \int v_{n/2}(g)dV_g
\]

is a conformal invariant. Noting these parallels, Chang-Fang [3] proposed the study of the functionals \( g \mapsto \int v_k(g)dV_g \) for \( k < n/2 \) as the natural generalization of the functionals given by the integrals of \( \sigma_k(A) \) in the non-LCF setting. This leads to another generalization of the Yang-m-Yamabe problem, the \( v_k \)-Yamabe problem, to find (under suitable conditions) in a given conformal class a critical point of the functional \( g \mapsto \int v_k(g)dV_g \); i.e., a conformal metric for which \( v_k \) is constant. Later Graham showed [11] further structure of these quantities via their relationship to “extended obstruction tensors.”

These results suggest the following natural extension of the formal Riemannian structure given in our earlier work: Let \((M^{2m}, g)\) be a closed Riemannian manifold of even dimension \( n = 2m \). Given a conformal metric \( g_u = e^{-2u}g \), define the inner product

\[
\langle \alpha, \beta \rangle_u = \int_{M^{2m}} \alpha \beta v_m(g_u)dV_u.
\]

For the inner product above to be positive definite, it is clear that positivity of \( v_k(g_u) \) is required, hence we restrict our conformal metrics to the set

\[
C^+_m := \{ u \in C^\infty(M) \mid v_m(g_u) > 0, \ L > 0 \}.
\]

Here \( L \) is the principal symbol of the linearization of \( v_m \) (see [11] Theorem 1.5, recorded as Theorem 2.1 below). Although the addition of this assumption may seem superfluous, it will be important when verifying certain properties of the metric defined by (1.11). Moreover, it is easy to see that when the dimension is four this set corresponds to the positive cone defined in (1.1).

With these definitions we can now state the objectives of the paper. Our first goal is to verify some of the basic formal properties of the metric defined in (1.11), and to write down the induced connection. This is carried out in [2]. For these results we only need to assume that our conformal metrics are in the set \( C^+_m \). Next, we prove the existence of a functional \( F : [g] \rightarrow \mathbb{R} \) generalizing the functional of Chang-Yang, whose critical points correspond to conformal metrics satisfying

\[
v_m(g_u) = \text{const.}
\]

(compare with (1.4)). Our construction is an adaptation of the method of Brendle-Viaclovsky [4], who gave another proof of the Chang-Yang construction in [7]. In analogy with our work in low dimensions, we would like to verify that \( F \) is geodesically convex.

Here we encounter the crucial point that to understand the variational properties of \( F \), we need to restrict to a smaller set of conformal metrics: even in the LCF case, \( C^+_m \) is strictly larger than the positive \( n/2 \)-cone, so we do not expect \( F \) to have nice properties on \( C^+_m \). A shift in perspective is warranted, and instead of defining cones by imposing positivity conditions on various curvature quantities, we instead consider metrics which verify an Andrews-type inequality,

\[
n \left[ \int_M \phi^2 dV_g - V_g^{-1} \left( \int_M \phi dV_g \right)^2 \right] \leq \int_M \frac{1}{v_m(g)} L^{ij} \nabla_i \phi \nabla_j \phi dV_g, \quad (A)
\]

where \( V_g \) is the volume of \( g \), and equality holds if and only if \( \phi = 0 \) or \((M^{2m}, g)\) is conformally equivalent to the round sphere and \( \phi \) is a first-order spherical harmonic. When the dimension \( n = 4 \) and \( A_g \in \Gamma^+_4 \), this inequality is a consequence of the sharp Poincaré-type inequality of Andrews [2], and played a key role in our previous work. In dimensions \( n = 2m \geq 6 \), given a conformal class \([g]\) we define the set

\[
\mathcal{C}^+_A = \{ g_u = e^{-2u}g \in \mathcal{C}^+_m : (A) \text{ holds for } g_u \}.
\]
Although this set seems unrelated to any ‘cone of ellipticity’ for the equation (1.13), we will see that it carries all the information needed to verify the desired variational properties of $F$. In addition, when the manifold is LCF we will show that (A) holds for any metric in the positive $n/2$-cone (see Proposition 4.5 below). This is further evidence that this condition is natural.

To summarize, our first main result is the following:

**Theorem 1.1.** Let $(M^{2m}, g)$, be a closed, even-dimensional Riemannian manifold.

(i) There exists a functional $F : [g] \to \mathbb{R}$ such that the critical points of $F$ satisfy (1.13).

(ii) $F : C^+_C(A) \to \mathbb{R}$ is geodesically convex with respect to the Riemannian structure defined by (1.11).

As discussed above, in the LCF setting if the cone $\Gamma^+_{n/2}$ is non-empty then (up to a double cover) the manifold is the round sphere. In this case we show that $\Gamma^+_{n/2}$ lies in $C^+_C(A)$, yielding a new perspective on the uniqueness of solutions originally established by Viaclovsky [20]:

**Theorem 1.2.** Let $(S^{2m}, g_0)$ be the round sphere of dimension $2m$. Then

\begin{equation}
\Gamma^+_m([g_0]) = \{g_u = e^{-2u}g_0 : A_u \in \Gamma^+_m\} \subseteq C^+_C(A).
\end{equation}

In particular, (A) holds and $F : \Gamma^+_m([g_0]) \to \mathbb{R}$ is geodesically convex. Also, critical points of $F$ are given by the round metric and its image under the conformal group [20].

The formal framework given by Theorem 1.1 naturally suggests that solutions to (1.13) are unique up to scaling:

**Conjecture 1.3.** Let $(M^{2m}, g)$, be a closed, even-dimensional Riemannian manifold such that $C^+_C(A) \neq \emptyset$. If $(M^{2m}, g)$ is not conformally equivalent to $(S^{2m}, g_0)$, then there exists a unique $u \in C^\infty(M)$ such that $\int_M udV_g = 0$ and $v_m(e^{-2u}g) = \overline{v}$.

To verify this along the lines of [14 15] would require existence results for the geodesic problem, which we do not address here.

As in our previous work, we also consider the gradient flow of $F$ with respect to the metric (1.11). Using properties of $F$ which follow from its construction, one can show that the negative gradient flow (written as an evolution equation for the conformal factor) is given by

\begin{equation}
\frac{\partial}{\partial t} u = 1 - \frac{\overline{v}}{v_m(g_u)}.
\end{equation}

For simplicity we will refer to this as the inverse $v_m$-flow. In [3] we observe a number of formal properties for solutions of this flow in the set $C^+_C(A)$. These are in line with the properties established for Calabi flow in relation to the Mabuchi metric in Kähler geometry [5 9 16 19]:

**Theorem 1.4.** Let $(M^{2m}, g)$ be a closed, even-dimensional Riemannian manifold, and suppose $u = u(t)$ is a solution to inverse $v_m$-flow with $g_u = e^{-2u}g$ and $g_u \in C^+_C(A)$.

(1) Then $F$ is convex along the flow:

\[ \frac{d^2}{dt^2} F[u] \geq 0. \]

(2) The following entropy estimate holds along the flow:

\[ \frac{d}{dt} \int_M v_m \log v_m dV_u \leq 0. \]
Despite the excellent formal properties of the (negative) gradient flow, we lack a short-time existence result for solutions. This is related to the more general issues involved in the study of the $v_k$-Yamabe problem for $k \geq 3$, and are discussed in [6] and [11]. A fundamental difficulty is identifying a useful notion of ellipticity. For example, there is no obvious way to conclude that the linearized operator $L$ is positive definite by imposing sign conditions on the $v_k$’s. In the case where $k = m = n/2$, it would be interesting to see whether the validity of the Andrews-type inequality (A) could be used in place of an algebraic condition. Indeed one of the motivations for this paper is to attempt to put the $v_k$-Yamabe problem (at least for $k = n/2$) in the framework of a convex variational problem defined on a metric space, with the eventual goal of proving the existence of critical points without resorting to elliptic theory (i.e., ‘pointwise’ methods).

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2. Metric and connection

We begin with the following variational formula for $v_m$ shown by Graham [11]. Note that we have changed convention from that paper and parameterize conformal metrics via $g_u = e^{-2u}g$.

**Theorem 2.1.** ([11] Theorem 1.5) Given $(M^{2m}, g)$ be a closed, even-dimensional Riemannian manifold, and let $u = u(t)$ be a one-parameter family of conformal factors such that $u(0) = 0$, and

$$d \frac{d}{dt}u \bigg|_{t=0} = \dot{u}.$$  

There exists a natural tensor $L$ such that

$$\frac{d}{dt}v_m(e^{-2u(t)}g) \bigg|_{t=0} = n\dot{u}v_m(g) + \nabla_i \left( L^{ij} \nabla_j \dot{u} \right).$$  

**Corollary 2.2.** Given $(M^{2m}, g)$ as above, the quantity

$$v := \int_M v_m(g)dV$$

is a conformal invariant.

The tensor $L$ above is derived in [11] using the “ambient metric construction,” and the reader should consult that work for full details. The relevant point for us is that the linearization is a divergence-form operator. Moreover, in the case that the metric $g$ is locally conformally flat, $L$ is the $k - 1$ Newton transformation of the Schouten tensor.

We can now record some basic formal properties of the inner product defined by (1.11).

**Definition 2.3.** Let $(M^{2m}, g)$ be a closed Riemannian manifold of dimension $2m$. The $v_m$-metric is the formal Riemannian metric defined for $u \in C^+_m$, $\alpha, \beta \in T_u C^+_m \cong C^\infty(M)$ via

$$\langle \alpha, \beta \rangle_u = \int_M \alpha \beta v_m(g_u)dV_u.$$  

Moreover, given a path $u = u(t)$ in $C^+_m$ and a one-parameter family of tangent vectors $\alpha = \alpha(t)$ with $\alpha(t) \in T_{u(t)} C^+_m$, let

$$\frac{D}{dt} \alpha := \frac{\partial}{\partial t} \alpha - v_m^{-1} \left( L_i \nabla \alpha \otimes \nabla \frac{\partial}{\partial t} \right)$$

denote the directional derivative along the path $u(t)$.

**Lemma 2.4.** The connection defined by (2.3) is metric compatible and torsion free.
Proof. First we check metric compatibility. Using Theorem 2.1 we compute
\[
\frac{d}{dt} \langle \alpha_t, \beta_t \rangle_{u_t} = \frac{d}{dt} \int_M \alpha \beta v_m(g_u) dV_u
\]
\[
= \left( \frac{\partial}{\partial t} \alpha, \beta \right) + \left( \alpha, \frac{\partial}{\partial t} \beta \right) + \int_M \alpha \beta \left(L^{ij} \nabla^j \frac{\partial u}{\partial t}\right) dV_u
\]
\[
= \left( \frac{\partial}{\partial t} \alpha, \beta \right) + \left( \alpha, \frac{\partial}{\partial t} \beta \right) - \int_M \left(L, (\alpha \nabla \beta + \beta \nabla \alpha) \otimes \nabla \frac{\partial u}{\partial t}\right) dV_u
\]
\[
= \left( \frac{D}{\partial t} \alpha, \beta \right) + \left( \alpha, \frac{D}{\partial t} \beta \right).
\]

Next, to compute the torsion, let \( u = u(s, t) \) be a two-parameter family of conformal factors. Then
\[
\frac{D}{\partial s} \frac{\partial u}{\partial t} - \frac{D}{\partial t} \frac{\partial u}{\partial s} = \frac{\partial^2 u}{\partial t^2} - v^{-1} \left(L, \nabla \frac{\partial u}{\partial s} \otimes \nabla \frac{\partial u}{\partial t}\right) = 0.
\]
The lemma follows. \( \square \)

Next, we observe some properties of lengths of curves and distances in the \( v_m \)-metric.

**Definition 2.5.** Given a path \( u : [a, b] \to C^+_m \), the length of \( u \) is
\[
\ell[u] := \int_a^b \langle \alpha, \beta \rangle^{\frac{1}{2}} dt = \int_a^b \left( \int_M \left( \frac{\partial u}{\partial t}\right)^2 v_m(g_u) dV_u \right)^{\frac{1}{2}} dt.
\]

A curve is a geodesic if it is a critical point for \( \ell \).

**Lemma 2.6.** A curve \( u = u(t) \in C^+_m \) is a geodesic if and only if
\[
\frac{\partial^2 u}{\partial t^2} - v^{-1} \left(L, \nabla \frac{\partial u}{\partial s} \otimes \nabla \frac{\partial u}{\partial t}\right) = 0.
\]

**Proof.** Formally, by Lemma 2.4 the connection is indeed the Riemannian connection and so a curve is a geodesic if and only if
\[
0 = \frac{D}{\partial t} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} - v^{-1} \left(L, \nabla \frac{\partial u}{\partial s} \otimes \nabla \frac{\partial u}{\partial t}\right).
\]
This can also be derived by directly taking the first variation of the length functional. \( \square \)

**Remark 2.7.** There is a canonical isometric splitting of the tangent space at each \( g_u \in C^+_m \) with respect to the \( v_m \)-metric. In particular, the real line \( \mathbb{R} \subset T_u C^+_m \) given by constant functions is orthogonal to
\[
T^0_u C^+_m := \left\{ \alpha \mid \int_M \alpha v_m dV_u = 0 \right\}.
\]
In the next lemma we show two basic properties of geodesics, namely that they preserve this isometric splitting, and are automatically parameterized with constant speed.

**Lemma 2.8.** Let \( u = u(t) \) be a solution to (2.5). Then
\[
\frac{d}{dt} \int_M \left( \frac{\partial u}{\partial t}\right)^2 v_m dV_u = 0,
\]
\[
\frac{d}{dt} \int_M \frac{\partial u}{\partial t} v_m dV_u = 0.
\]
Proof. First we differentiate
\[
\frac{d}{dt} \int_M \frac{\partial u}{\partial t} v_m dV_u = \int_M \left( \frac{\partial^2 u}{\partial t^2} v_m + \frac{\partial u}{\partial t} \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial t}) \right) dV_u
\]
\[
= \int_M \left( \frac{\partial^2 u}{\partial t^2} u - v_m^{-1} \left( L_i \nabla^i u \otimes \nabla \frac{\partial u}{\partial t} \right) \right) v_m dV_u
\]
\[
= 0.
\]
Next
\[
\frac{d}{dt} \int_M \left( \frac{\partial u}{\partial t} \right)^2 v_m dV_u = \int_M \left[ 2 v_m \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} + \left( \frac{\partial u}{\partial t} \right)^2 \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial t}) \right] dV_u
\]
\[
= 2 \int_M v_m \frac{\partial u}{\partial t} \left[ \frac{\partial^2 u}{\partial t^2} - v_m^{-1} \left( L_i \nabla^i \frac{\partial u}{\partial t} \otimes \nabla \frac{\partial u}{\partial t} \right) \right] dV_u
\]
\[
= 0.
\]
\[\square\]

One expects other formal aspects of the metric space structure established in [15] to extend to this setting as well. For instance, it is natural to expect nonpositive curvature of this metric. Moreover, formal arguments suggest that the distance function induced by this Riemannian structure should be nondegenerate. Proofs of these statements should follow along similar lines to [15] but as we have no concrete application we do not pursue this here.

3. The functional \( F \) and geodesic convexity

In this subsection we generalize Brendle-Viaclovsky’s derivation [4] of the conformal primitive for the equation
\[
v_m(u) = \text{const}.
\]
To begin we define a one-form \( \alpha_u : T_u [g] \to \mathbb{R} \) on our given conformal class via
\[
\alpha_u(\phi) := \int_M \phi v_m(u) dV_u.
\]

Lemma 3.1. The 1-form \( \alpha \) is exact.

Proof. Suppose that \( u = u(s, t) \) is a two-parameter family of conformal factors, and compute
\[
\frac{d}{ds} \alpha \left( \frac{\partial u}{\partial t} \right) = \int_M \frac{\partial^2 u}{\partial s \partial t} v_m(u) dV_u + \int_M \frac{\partial u}{\partial t} \frac{\partial}{\partial s} (v_m(u) dV_u)
\]
\[
= \int_M \frac{\partial^2 u}{\partial s \partial t} v_m(u) dV_u + \int_M \frac{\partial u}{\partial t} \nabla_i \left( L^{ij} \nabla_j \frac{\partial u}{\partial s} \right) dV_u
\]
\[
= \int_M \frac{\partial^2 u}{\partial s \partial t} v_m(u) dV_u - \int_M L^{ij} \nabla_i \frac{\partial u}{\partial s} \otimes \nabla_j \frac{\partial u}{\partial t} dV_u.
\]
This expression is manifestly symmetric in \( s \) and \( t \), thus \( \alpha \) is closed. Since the space of conformal factors is contractible, this implies that \( \alpha \) is exact. \[\square\]

Proposition 3.2. Let \((M^{2m}, g)\) be a closed, even-dimensional Riemannian manifold. Then there is a functional \( F : [g] \to \mathbb{R} \) such that if \( u = u(t) : (-\epsilon, \epsilon) \to [g] \) is a path with \( u(0) = u \) and \( \frac{d}{dt} u(t)|_{t=0} = \dot{u} \), then
\[
\frac{d}{dt} F[u(t)]|_{t=0} = \int_M \dot{u} \left[ v_m(u) + \overline{v} \right] dV_u.
\]
Proof. Since the 1-form \( \alpha \) is exact by Lemma 3.1, there exists a function \( E : [g] \to \mathbb{R} \) such that \( dE = \alpha \). We thus set \( F[u] = E[u] - \frac{m}{n} \log \int_M dV_u \), and the result follows. \( \square \)

**Proposition 3.3.** Let \((M^{2m}, g)\) be a closed, even-dimensional Riemannian manifold. Then \( F : \mathcal{C}_+(A) \to \mathbb{R} \) is geodesically convex.

Proof. Let \( u = u(t) \) be a geodesic. Using Lemma 2.8 and the inequality (A) we have

\[
\frac{d^2}{dt^2} F[u(t)] = \frac{d}{dt} \int_M \frac{\partial u}{\partial t} [-v_m + \nabla \phi] dV_u
\]

\[
= v \frac{d}{dt} \int_M (\frac{\partial}{\partial t}) V_u^{-1} dV_u
\]

\[
= v \int_M \left[ \frac{\partial^2 u}{\partial t^2} V_u^{-1} + V_u^{-2} \frac{\partial u}{\partial t} \left( \int_M n \frac{\partial u}{\partial t} dV_u \right) - n V_u^{-1} \left( \frac{\partial u}{\partial t} \right)^2 \right] dV_u
\]

\[
= v V_u^{-1} \left[ \int_M v_m^{-1} \left\langle L, \nabla \frac{\partial u}{\partial t} \otimes \nabla \frac{\partial u}{\partial t} \right\rangle dV_u - n \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 dV_u - V_u^{-1} \left( \int_M \frac{\partial u}{\partial t} dV_u \right)^2 \right) \right]
\]

\[
\geq 0.
\]

\( \square \)

**Proof of Theorem 1.1.** Part (i) follows from Proposition 3.3 while part (ii) follows from Proposition 3.3. \( \square \)

### 4. The Locally Conformally Flat Case

As we pointed out in the Introduction, it follows from Kuiper’s Theorem and the work of Guan-Viaclovky [13] that if \((M^{2m}, g)\) is a closed, even-dimensional LCF manifold with

\[
\Gamma^+_m([g]) = \{ g_u = e^{-2u} g : A_u \in \Gamma^+_m \} \neq \emptyset,
\]

then \((M^{2m}, g)\) must be conformally equivalent to the round sphere or real projective space. In this section we show that for the round sphere \((S^{2m}, g_0)\) the cone \( \Gamma^+_m([g_0]) \) is contained in \( \mathcal{C}_+(A) \); i.e., the Andrews-type inequality holds. As an immediate consequence, \( F : \Gamma^+_m([g_0]) \to \mathbb{R} \) is geodesically convex. It was shown by Viaclovsky [20] that all solutions of \( \sigma_m(A_u) = \text{const.} \) are given by conformal metrics with \( g_u = \varphi^* g_0 \) for some conformal transformation \( \varphi : S^{2m} \to S^{2m} \), therefore giving us a complete variational description in this case.

The proof is an application of a closely related inequality of Andrews [2], exploiting certain inequalities relating elementary symmetric polynomials. We begin with the inequality of Andrews:

**Proposition 4.1.** (Andrews [2], cf. [8] pg. 517) Let \((M^n, g)\) be a closed Riemannian manifold with positive Ricci curvature. Given \( \phi \in C^\infty(M) \) such that \( \int_M \phi dV = 0 \), then

\[
\frac{n}{n-1} \int_M \phi^2 dV \leq \int_M (Rc_0)^{-1} ij \nabla_i \phi \nabla_j \phi dV,
\]

with equality if and only if \( \phi \equiv 0 \) or \((M^n, g)\) is isometric to the round sphere.

To show how this inequality implies (A) for metrics in \( \Gamma^+_m([g]) \), we use an argument that was shown to us by Petrov [17]. Given \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), let

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},
\]

\[
\sigma_k^A(\lambda) = \sigma_k(\lambda)|_{\lambda_i = 0},
\]

\[
\bar{\sigma}_k(\lambda) = \binom{n}{k}^{-1} \sigma_k(\lambda_1, \ldots, \lambda_n).
\]
Lemma 4.2. (cf. [1]) With the notation above one has

$$\sigma_k(\lambda) = \sigma_k; \lambda + \lambda_i \sigma_{k-i}(\lambda)$$

(4.1)

$$\left[ \left( \frac{n-1}{k} \right)^\frac{1}{n} \sigma_k \right]^\frac{1}{n} \leq \left[ \left( \frac{n-1}{l} \right)^\frac{1}{n} \sigma_l \right]^\frac{1}{n}, \quad k \geq l \geq 1$$

Lemma 4.3. Given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, let $F(x) = \prod_{i=1}^n (x - \lambda_i)$, and suppose $F'(x) = n \prod_{i=1}^{n-1} (x - \mu_i)$. Then

$$\tilde{\sigma}_k(\lambda_1, \ldots, \lambda_n) = \tilde{\sigma}_k(\mu_1, \ldots, \mu_{n-1})$$

Proof. This follows directly from the classical Vieta formulas. \hfill \Box

Proposition 4.4. ([17]) Fix $n = 2m$. Given $\lambda = (\lambda_1, \ldots, \lambda_n)$, let

$$A_\lambda = \frac{1}{n-2} \left[ \lambda - \frac{\sigma_1(\lambda)}{2n-1}(1, \ldots, 1) \right]$$

Given $\lambda$ such that $A_\lambda \in \Gamma^+_m$, for all $i$ one has that

$$(n-1)\sigma_m(A_\lambda) \leq \lambda_i \sigma_{m-i}(A_\lambda).$$

(4.3)

Proof. Let $A_\lambda = (a_1, \ldots, a_n)$. We fix $i = n$ for convenience, no ordering on the $\lambda_i$ is assumed. Note that

$$\lambda_n = (n-2)a_n + (a_1 + \cdots + a_n) = (n-1)a_n + (a_1 + \cdots + a_{n-1}).$$

Applying this and (4.1) we see that the required inequality is equivalent to

$$(n-1)\sigma_m(A_\lambda) \leq \lambda_n \sigma_{m-1,n}(A_\lambda)$$

$$= [(n-1)a_n + \sigma_1(a_1, \ldots, a_{n-1})] \sigma_{m-1}(a_1, \ldots, a_{n-1})$$

$$= (n-1) [\sigma_m(a_1, \ldots, a_n) - \sigma_m(a_1, \ldots, a_{n-1})] + \sigma_1(a_1, \ldots, a_{n-1})\sigma_{m-1}(a_1, \ldots, a_{n-1}),$$

hence we see that it suffices to show that

$$(n-1)\sigma_m(a_1, \ldots, a_{n-1}) \leq \sigma_1(a_1, \ldots, a_{n-1})\sigma_{m-1}(a_1, \ldots, a_{n-1}).$$

Written in terms of normalized functions, since $m = \frac{2}{2}$ this is equivalent to

$$(4.4) \quad \tilde{\sigma}_m(a_1, \ldots, a_{n-1}) \leq \tilde{\sigma}_1(a_1, \ldots, a_{n-1})\tilde{\sigma}_{m-1}(a_1, \ldots, a_{n-1}).$$

Now let $f(x) = \prod_{i=1}^{n-1} (x - a_i)$, and let $g(x) = f^{(m-1)}(x)$. Note that $g$ is a polynomial of degree $m$ with real roots $b_1, \ldots, b_m$. By Lemma 4.3 we see that (4.3) is equivalent to

$$\sigma_m(b_1, \ldots, b_m) = \tilde{\sigma}_m(b_1, \ldots, b_m)$$

$$\leq \tilde{\sigma}_1(b_1, \ldots, b_m)\tilde{\sigma}_{m-1}(b_1, \ldots, b_m)$$

$$= m^{-2}\sigma_1(b_1, \ldots, b_m)\sigma_{m-1}(b_1, \ldots, b_m).$$

If all $b_i > 0$, this follows directly from Maclaurin’s inequality, (4.2).

Now suppose that $b_i \geq 0$. Observe that $f(x)(x - a_n)$ has positive coefficients, and $f(x)$ has only real roots. It follows from Rolle’s theorem that $(f(x)(x - a_n))^{(m)}$ has only real roots and they must be positive. Hence $h(x) := (x - a_n)g'(x) + mg(x)$ has $m$ positive roots. Thus $(g(x)(x - a_n)^m)'$ has a root $a_n$ of multiplicity $m - 1$ and $m$ positive roots. If $b_n \leq 0$, then by Rolle’s theorem $(g(x)(x - a_n)^m)'$ has a root between $a_n$ and $b_m$ (or has a root $a_n$ of multiplicity at least $m$ if $a_n = b_m$), which contradicts the above discussion. Analogously, if $a_n \geq 0$ but $b_i \leq 0$ for some $i \neq m$, we get a negative root of $(g(x)(x - a_n)^m)'$, which is again a contradiction. So, $a_n > 0$ and $b_1, \ldots, b_{m-1} > 0$. Since $h(x)$ has $m$ positive roots and positive leading coefficient, we have $h(0)(-1)^m > 0$. On the other hand, $g(0)(-1)^m = b_1 \cdots b_m < 0$. Thus $(-1)^{m-1} a_n g'(0) > 0$, i.e. $\sigma_{m-1}(b_1, \ldots, b_m) > 0$. It follows that $b_1 + \cdots + b_m > 0$, thus $(b_1 + \cdots + b_m)\sigma_{m-1}(b_1, \ldots, b_m) > 0 > m^2 b_1 \cdots b_m$ as desired.
Proposition 4.5. Let \((M^{2m}, g)\) be a closed, even-dimensional Riemannian manifold such that \(A_g \in \Gamma_m^+\). Given \(\phi \in C^\infty(M)\) one has

\[
\int_M \phi^2 dV_g - V_g^{-1} \left( \int_M \phi dV_g \right)^2 \leq \int_M \frac{1}{\sigma_m(A_g)} T_{m-1}(A_g)^{ij} \nabla_i \phi \nabla_j \phi dV_g.
\]

Proof. Choosing a point \(p \in M\) choose a basis for \(T_p M\) such that the Ricci tensor is diagonalized, with eigenvalues \(\lambda_i\). Since, in the notation of Proposition 4.4, the matrix \(A_\lambda\) corresponds to the Schouten tensor, which is also diagonalized, the inequality (4.3) can be rearranged to imply the matrix inequality

\[
(n - 1) \text{Rc}^{-1} \leq \frac{1}{\sigma_m(A_g)} T_{m-1}(A_g).
\]

The proposition thus follows from Proposition 4.1.

Proof of Theorem 1.2. In the LCF setting it was shown by Graham-Juhl [12] that \(\sigma_m(A_g) = v_m(g)\) and consequently \(T_{m-1}(A) = L\). Therefore, Theorem 1.2 follows from Proposition 4.5 and Theorem 1.1.

5. The gradient flow of the functional \(F\)

In this section we derive the \(v_m\)-metric gradient flow of the \(F\) functional and many formal properties of it related to the \(v_m\)-metric, culminating in the proof of Theorem 1.4.

Lemma 5.1. With respect to the \(v_m\)-metric, one has

\[
[\text{grad } F]_u = -1 + \frac{v}{v_m}.
\]

Proof. Using Proposition 3.2 and the definition of the \(\sigma_k\) metric we have

\[
\frac{d}{dt} F[u] = \int_M \frac{\partial}{\partial t} u \left[ -v_m(g_u) + \frac{v}{v_m} \right] dV_u
\]

\[
= \int_M \frac{\partial}{\partial t} u \left[ -1 + \frac{v}{v_m(A_u)} \right] v_m(g_u) dV_u
\]

\[
= \left\langle \frac{\partial}{\partial t} u, -1 + \frac{v}{v_m(g_u)} \right\rangle_u.
\]

Definition 5.2. We say that the path \(u = u(t)\) of conformal factors is a solution to inverse \(v_m\)-flow if

\[
g_u = e^{-2u} g \in C^+(g)
\]

\[
\frac{\partial}{\partial t} u = -[\text{grad } F]_u = 1 - \frac{v}{v_m(g_u)}.
\]

We now establish a number of monotonicity properties for the inverse \(v_m\)-flow. In Proposition 5.3 we show that the \(F\) functional is convex along a flow line. We then prove the monotonicity of entropy (Proposition 5.4). Finally, in Proposition 5.5 we establish that the length of curves is monotone nonincreasing when flowed along inverse \(v_m\) flow.

Proposition 5.3. Given \(u = u(t)\) a solution to the inverse \(v_m\) flow, one has

\[
\frac{d^2}{dt^2} F[u_t] \geq 0.
\]
Proof. Using (3.1) we have for a solution to inverse $u_m$ flow

$$\frac{d}{dt} F = - \int_M \left(1 - \frac{\varphi}{u_m} \right)^2 u_m dV_u$$

$$= - \int_M \left[ 1 - \frac{2\varphi}{u_m} + \frac{\varphi^2}{u_m^2} \right] u_m dV_u$$

$$= - v + 2v - \varphi^2 \int_M \frac{1}{u_m} dV_u.$$

Hence using the variational formula (2.1) we have

$$\frac{d^2}{dt^2} F = \frac{d}{dt} \left[ - \frac{v^2}{V_u^2} \int_M \frac{1}{u_m} dV_u \right]$$

$$= \frac{v^2}{V_u^2} \left[ \frac{d}{dt} u_m \int_M \frac{1}{u_m} dV_u - u_m \frac{d}{dt} \int_M \frac{1}{u_m} dV_u \right]$$

$$= \frac{v^2}{V_u^2} \left[ 2 \left( -n u_m + n\varphi \int_M \frac{1}{u_m} dV_u \right) \int_M \frac{1}{u_m} dV_u \right.$$

$$- V_u \int_M \left( - \frac{n \varphi}{u_m} - \frac{\varphi}{u_m^2} - \frac{n\varphi^2}{u_m^3} \right) dV_u$$

$$- V_u \int_M \left( - \frac{n \varphi}{u_m} + \frac{n\varphi}{u_m^2} + \frac{n\varphi^2}{u_m^3} \right) \left( \nabla_i (L^{ij} \nabla_j \frac{1}{u_m}) \right) dV_u$$

$$= \frac{v^2}{V_u^2} \left[ 2n\varphi \left( \int_M \frac{1}{u_m} dV_u \right)^2 - 2n v \int_M \frac{1}{u_m} dV_u - v \int_M \frac{1}{u_m^2} \nabla_i (L^{ij} \nabla_j \frac{1}{u_m}) dV_u \right]$$

$$= \frac{v^2}{V_u^2} \left[ 2n\varphi \left( \int_M \frac{1}{u_m} dV_u \right)^2 - 2n v \int_M \frac{1}{u_m} dV_u + 2v \int_M \frac{1}{u_m} \left( L, \nabla \frac{1}{u_m} \otimes \nabla \frac{1}{u_m} \right) dV_u \right]$$

$$= \frac{2v^3}{V_u^3} \left[ -n \int_M \left( \frac{1}{u_m} - V_u^{-1} \int_M \frac{1}{u_m} \right)^2 + \int_M \frac{1}{u_m} \left( L, \nabla \frac{1}{u_m} \otimes \nabla \frac{1}{u_m} \right) dV_u \right]$$

$$\geq 0,$$

where the last line follows from applying (A).

The next result is a monotonicity result for the entropy along the flow:

**Proposition 5.4.** Given $u = u(t)$ a solution to inverse $v_m$-flow, one has

$$\frac{d}{dt} \int_M v_m \log v_m dV_u = - \varphi \int_M \left( L, \nabla \frac{1}{v_m} \otimes \nabla \frac{1}{v_m} \right) \leq 0.$$
Proof. Again using the variational formula for $v_m$,

\[ \frac{d}{dt} \int_M v_m \log v_m dV_u \]
\[ = \int_M \frac{\partial}{\partial t} (v_m \log v_m) dV_u + \int_M v_m \log v_m \frac{\partial}{\partial t} dV_u \]
\[ = \int_M [1 + \log v_m] \frac{\partial}{\partial t} v_m dV_u - \int_M v_m \log v_m (n \frac{\partial}{\partial t} u) dV_u \]
\[ = \int_M [1 + \log v_m] \left[ -\nabla_i (L^{ij} \nabla_j \frac{\nabla}{v_m}) - n [v_m - \overline{v}_m] \right] + n \int_M v_m \log v_m \left[ 1 - \frac{\overline{v}_k}{v_m} \right] \]
\[ = -\overline{v} \int_M \left< L, \nabla \frac{1}{v_m} \otimes \nabla \frac{1}{v_m} \right> . \]

□

Proof of Theorem 1.4. This follows immediately from the preceding two propositions. □

We include a final proposition since it provides an interesting parallel with a result of Calabi-Chen [5] on the length of paths under the Calabi flow in Kähler geometry:

Proposition 5.5. Let $u = u(s, t)$ be a two-parameter family of conformal factors such that $g_u = e^{-2u} g \in C_+(A)$ for $s \in [0, 1], t \in [0, T)$. We also assume that for all $s \in [0, 1]$, the path $t \mapsto u(\cdot, t)$ is a solution to inverse $v_m$-flow. Then

\[ \frac{d}{dt} \ell(u(\cdot, t)) \leq 0. \]

Proof. We directly compute

\[ \frac{d}{dt} \ell(u(\cdot, t)) = \frac{d}{ds} \int_0^1 \left[ \int_M \left( \frac{\partial u}{\partial s} \right)^2 v_m(g_u) dV_u \right]^{\frac{1}{2}} ds \]
\[ = \frac{1}{2} \int_0^1 \frac{\partial u}{\partial s} \left| v_m \right|^{-1} \int_M \left[ 2 \frac{\partial^2 u \partial u}{\partial s \partial t} \partial s v_m + \left( \frac{\partial u}{\partial s} \right)^2 \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial t}) \right] dV_u ds. \]

Now we compute

\[ \frac{\partial^2 u}{\partial s \partial t} = \frac{\partial}{\partial s} \left[ 1 - \frac{\overline{v}}{v_m} \right] \]
\[ = \overline{v} \left[ \frac{1}{V_u v_m} \frac{\partial}{\partial s} V_u + \frac{1}{v_m^2} \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial s}) + n \frac{\partial u}{\partial s} \frac{1}{v_m} \right] \]
\[ = \overline{v} \left[ \frac{1}{V_u v_m} \frac{\partial}{\partial s} V_u + \frac{1}{v_m^2} \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial s}) + n \frac{\partial u}{\partial s} \frac{1}{v_m} \right] . \]
Hence
\[ \int_M 2 \frac{\partial^2 u}{\partial s \partial t} \frac{\partial u}{\partial s} v_m dV_u \]
\[ = 2\pi \int_M \left[ -nV_u^{-1} v_m - \int_M \frac{\partial u}{\partial s} dV_u + v_m^2 \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial s}) + nV_u^{-1} \frac{\partial u}{\partial s} v_m dV_u \right] \]
\[ = 2n\pi V_u^{-1} \left\{ \int_M \frac{\partial u}{\partial s}^2 dV_u - \left[ \int_M \frac{\partial u}{\partial s} dV_u \right]^2 \right\} + 2\pi \int_M v_m^{-1} \frac{\partial u}{\partial s} \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial s}) dV_u \]
\[ = 2n\pi V_u^{-1} \left\{ \int_M \frac{\partial u}{\partial s}^2 dV_u - \left[ \int_M \frac{\partial u}{\partial s} dV_u \right]^2 \right\} - 2\pi \int_M \left[ \left\langle L, v_m^{-1} \nabla \frac{\partial u}{\partial s} \otimes \nabla \frac{\partial u}{\partial s} + \frac{\partial u}{\partial s} \nabla \frac{\partial u}{\partial s} \otimes \nabla v_m^{-1} \right\rangle \right] dV_u. \]

Also
\[ \int_M \frac{\partial u}{\partial s}^2 \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial t}) dV_u = -2 \int_M \frac{\partial u}{\partial s} \left\langle L, \nabla u_t \otimes \nabla \frac{\partial u}{\partial s} \right\rangle dV_u \]
\[ = \int_M \frac{\partial u}{\partial s}^2 \nabla_i (L^{ij} \nabla_j \frac{\partial u}{\partial t}) dV_u = -2 \int_M \frac{\partial u}{\partial s} \left\langle L, \nabla u_t \otimes \nabla \frac{\partial u}{\partial s} \right\rangle dV_u \]
\[ = 2\pi \int_M \left\langle L, \frac{\partial u}{\partial s} \nabla \frac{\partial u}{\partial s} \otimes \nabla v_m^{-1} \right\rangle dV_u. \]

Collecting these calculations yields
\[ \frac{d}{dt}((u(\cdot, t))) = - \int_0^1 \left[ \int_M v_m^{-1} \left\langle L, \nabla \frac{\partial u}{\partial s} \otimes \nabla \frac{\partial u}{\partial s} \right\rangle dV_u - n \int_M \left( \frac{\partial u}{\partial s} \right)^2 + nV_u^{-1} \left[ \int_M \left( \frac{\partial u}{\partial s} \right)^2 dV_u \right]^2 \right] ds \]
\[ \leq 0, \]
where the last line follows from (A) (since \( g_u \in C^+_A \)).

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