SINGULAR PARABOLIC EQUATIONS WITH INTERIOR DEGENERACY AND NON SMOOTH COEFFICIENTS: THE NEUMANN CASE

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To Angelo on the occasion of his 70th birthday, with esteem

Abstract. We establish Hardy–Poincaré and Carleman estimates for non-smooth degenerate/singular parabolic operators in divergence form with Neumann boundary conditions. The degeneracy and the singularity occur both in the interior of the spatial domain. We apply these inequalities to deduce well-posedness and null controllability for the associated evolution problem.

1. Introduction. This paper deals with a class of degenerate and singular parabolic operators with interior degeneracy and singularity of the form

\[ u_t - (a(x)u_x)_x - \frac{\lambda}{b(x)} u, \]

associated to Neumann boundary conditions and with \((t, x) \in Q_T := (0, T) \times (0, 1), T > 0\) being a fixed number. Here \(\lambda \in \mathbb{R}\) satisfies suitable assumptions and the functions \(a\) and \(b\), that can be non-smooth, degenerate at the same interior point \(x_0 \in (0, 1)\). The fact that both \(a\) and \(b\) degenerate at the same point is actually the most complicated situation. Indeed, if \(a\) and \(b\) degenerated at different points, we could separate the problem in a purely singular one and in a purely degenerate

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one and treat them separately. As in [4] and [7], we shall admit different types of
degeneracy for $a$ and $b$. In all cases, we mean that

- there exists $x_0 \in (0, 1)$ such that $a(x_0) = b(x_0) = 0$ and $a, b > 0$ on $[0, 1] \setminus \{x_0\},$
- there exists $K_1, K_2$ such that $(x - x_0)a' \leq K_1 a$ and $(x - x_0)b' \leq K_2 b$ a.e. in

$[0, 1].$

The different nature of the degeneracy depends on the Sobolev space to which $a, b$
belong and on the values of $K_1$ and $K_2$, according to the following cases:

(WWD): weakly–weakly degenerate case: $a, b \in W^{1,1}(0, 1)$ and $K_1, K_2 \in (0, 1);$  
(SSD): strongly–strongly degenerate case: $a, b \in W^{1,\infty}(0, 1)$ and $K_1, K_2 \in [1, 2);$  
(WSD): weakly–strongly degenerate case: $a \in W^{1,1}(0, 1), b \in W^{1,\infty}(0, 1)$
and $K_1 \in (0, 1), K_2 \in [1, 2);$  
(SWD): strongly–weakly degenerate case (SWD): $a \in W^{1,\infty}(0, 1), b \in W^{1,1}(0, 1),$
and $K_1 \in [1, 2), K_2 \in (0, 1).$

Typical examples are $a(x) = |x - x_0|^{K_1}$ and $b(x) = |x - x_0|^{K_2}$, with $0 < K_1, K_2 < 2$.

The restriction $K_1 < 2$ is essentially related to existence and controllability
problems, see [2], [6], [11].

This paper is in some sense a continuation of the previous works [4] and [7],
where the authors study well-posedness and null controllability for the following problem
via suitable Hardy - Poincaré inequalities and Carleman estimates:

\[
\begin{aligned}
&u_t - Au = h(t, x)\chi_\omega(x), \quad (t, x) \in Q_T, \\
&Bu(t, 0) = Bu(t, 1) = 0, \quad t \in (0, T), \\
&u(0, x) = u_0(x) \in X, \quad x \in (0, 1).
\end{aligned}
\]  

(1)

Here

\[ Au := (au_x)_x + \lambda \frac{u}{b} \quad \text{or} \quad A(u) := au_{xx} + \lambda \frac{u}{b} \]

$X$ is a suitable Banach space, $Bu(t, x) = u(t, x)$ or $Bu(t, x) = u_x(t, x)$ for all $t \in (0, T), \chi_\omega$ is the characteristic function of an open set $\omega \subset (0, 1)$ and $h \in L^2(0, T; X)$. Actually, while in [4] the author considers the operator in non divergence form with
Dirichlet or Neumann boundary conditions, in [7], the authors take into account
the operator in divergence form only with Dirichlet boundary conditions. Hence, this
paper is devoted to complete the study of well-posedness and null controllability for
(1) in the case of Neumann boundary conditions and $A(u) := (au_x)_x + \lambda \frac{u}{b}$. Namely, we consider the evolution system

\[
\begin{aligned}
&u_t - (au_x)_x - \frac{\lambda}{b(x)} u = h(t, x)\chi_\omega(x), \quad (t, x) \in Q_T, \\
&u_x(t, 0) = u_x(t, 1) = 0, \quad t \in (0, T), \\
&u(0, x) = u_0(x), \quad x \in (0, 1),
\end{aligned}
\]  

(2)

where $u_0 \in L^2(0, 1)$ and the control $h \in L^2(Q_T)$ acts on a non empty interval
$\omega \subset (0, 1)$. The main goal of the present paper is to establish Hardy - Poincaré
inequalities and global Carleman estimates for operators of the form given in (2)
in order to obtain well-posedness and null controllability. We recall that (2) is said
globally null controllable if for every $u_0 \in L^2(0, 1)$ there exists $h \in L^2(Q_T)$ such
that the solution \( u \) of (2) satisfies \( u(T, x) = 0 \) for every \( x \in [0, 1] \) and \( \|h\|_{L^2(Q_T)}^2 \leq C\|u_0\|_{L^2(0, 1)}^2 \) for some universal positive constant \( C \).

For a brief history on well-posedness, null controllability and Carleman estimates for (2), also in the case of non divergence problem, we refer to [4] and [7].

The present paper is organized in the following way: in Section 2, we establish some Hardy–Poincaré inequalities and, using them, we study the well-posedness. In Section 3, we prove Carleman estimates and we use them, together with a Caccioppoli type inequality, to prove the following observability inequality: there exists a positive constant \( C_T \) such that every solution \( v \) of the adjoint problem

\[
\begin{aligned}
    v_t + (av_x)_x + \frac{\lambda}{b(x)} v &= 0, & (t, x) \in Q_T, \\
    v_x(t, 0) &= v_x(t, 1) = 0, & t \in (0, T), \\
    v(T, x) &= v_T(x) \in L^2(0, 1),
\end{aligned}
\]

satisfies, under suitable assumptions,

\[\|v(0)\|_{L^2(0, 1)}^2 \leq C_T\|v\chi_\omega\|_{L^2(Q_T)}^2.\]  

As an immediate consequence, one can prove, using a standard technique (e.g., see [9, Section 7.4]), null controllability for the linear degenerate/singular problem (2). We underline that in order to prove (3) the Hardy–Poincaré inequalities proved in Propositions 1 and 2 are essential. They are different from the ones proved in [6], which hold only under Dirichlet boundary conditions. Indeed, as we can see, in the inequalities (5) and (6) there are additional terms with respect to the inequalities in [6] and these additional terms complicate the proof of the observability inequality specially in the case \( \lambda > 0 \). We underline also that (3) is proved only in particular cases. Indeed to obtain this estimate, inequality (6) is crucial, but it is proved only if we are in the (WWD), (SSD) or in the (SWD) case. Observe that, also in [7], null controllability was not proved for (2) with Dirichlet boundary conditions in the (SSD) case for the same reason: the observability inequality was established only in the other three cases. However, using a different technique, in [8] it is proved that null controllability still holds for (2) in the case of Dirichlet boundary conditions also in the (SSD) case, at least when \( \lambda < 0 \).

Finally, notice that the results contained in the present paper generalize the ones obtained in [1] in the case of a divergence operator when \( \lambda = 0 \) (that is, in the purely degenerate case).

A final comment on the notation: by \( C \) we shall denote universal positive constants, which are allowed to vary from line to line.

2. Well-posedness. In this section we will consider (2) and, for the well–posedness of the problem, we start introducing the following weighted Hilbert spaces, which are suitable to study all situations, namely the (WWD), (SSD), (WSD) and (SWD) cases:

\[ H^1_a(0, 1) := \left\{ u \in W^{1,1}(0, 1) : \sqrt{au'} \in L^2(0, 1) \right\} \]

and

\[ H^1_{a,b}(0, 1) := \left\{ u \in H^1_a(0, 1) : \frac{u}{\sqrt{b}} \in L^2(0, 1) \right\}, \]

edowed with the inner products

\[ \langle u, v \rangle_{H^1_a(0, 1)} := \int_0^1 au'v' \, dx + \int_0^1 uv \, dx, \]
and

\[ \langle u, v \rangle_{H^{1}_{a,b}(0,1)} = \int_{0}^{1} a u' v' \, dx + \int_{0}^{1} a u v \, dx + \int_{0}^{1} \frac{uv}{b} \, dx, \]

respectively.

Notice that, if \( u \in H^{1}_{a,b}(0,1) \), then \( au' \in L^{2}(0,1) \). The choice of the notation \( H^{1}_{a,b}(0,1) \) is due to the desire to underline the dependence on \( a \), in coherence with the structure of the operator.

**Remark 1.** Of course, it is possible to make different assumptions on the functional setting, and so on the related domain of the operators. For instance, it would be natural to distinguish the case \( K_{1} \in (0,1) \) from the case \( K_{1} \in [1,2) \), using \( H^{1}(0,1) \) in the former case and \( H^{1}_{loc}((0,1) \setminus \{x_{0}\}) \) in the latter, as done in [2],[5], [6]. However, this choice implies that solutions may be discontinuous at \( x_{0} \). We prefer to avoid such unphysical situations \textit{a priori}, making the set of solutions smaller.

The following properties will be very helpful:

**Proposition 1** (Proposition 2.2, [4]). Assume that \( p \in C([0,1]) \), \( p > 0 \) on \([0,1] \setminus \{x_{0}\}, p(x_{0}) = 0 \) and there exists \( q > 1 \) such that

\[ x \mapsto \frac{p(x)}{|x-x_{0}|^{q}} \text{ is non increasing on the left of } x = x_{0} \]

and non decreasing on the right of \( x = x_{0} \).

Then, there exists a constant \( C > 0 \) such that for any function \( w \), locally absolutely continuous on \([0,x_{0}) \cup (x_{0},1] \), satisfying

\[ \int_{0}^{1} p(x)|w'(x)|^{2} \, dx < +\infty, \]

the following inequality holds:

\[ \int_{0}^{1} \frac{p(x)}{(x-x_{0})^{2}} w^{2}(x) \, dx \leq C \left( \int_{0}^{1} p(x)|w'(x)|^{2} \, dx + w^{2}(1) + w^{2}(0) \right). \]  

(5)

Using the previous result, one can prove new Hardy Poincaré inequalities in the space \( H^{1}_{a}(0,1) \) or \( H^{1}_{a,b}(0,1) \):

**Proposition 2.** The following holds:

- If \( (WWD) \) or \( (SWD) \) with \( K_{1} + K_{2} \leq 2 \) holds, then \( H^{1}_{a,b}(0,1) = H^{1}_{a}(0,1) \);
- if \( (WSD) \) with \( K_{1} + K_{2} \leq 2 \) holds, then \( H^{1}_{a,b}(0,1) \rightarrow H^{1}_{a}(0,1) \).

In both cases, there exists \( C_{H^{1}} > 0 \) such that

\[ \int_{0}^{1} \frac{u^{2}}{b} \, dx \leq C_{H^{1}} \int_{0}^{1} (u^{2} + a(u')^{2}) \, dx \]

(6)

holds for every \( u \in H^{1}_{a,b}(0,1) \).

**Proof.** First case. Take \( u \in H^{1}_{a}(0,1) \) and define \( p(x) := \frac{(x-x_{0})^{2}}{b} \), so that \( p \) satisfies (4) with \( q = 2 - K_{2} > 1 \) by [5, Lemma 2.1]. By Proposition 1, there exists \( C > 0 \) such that

\[ \int_{0}^{1} \frac{u^{2}}{b} \, dx = \int_{0}^{1} \frac{p(x)}{(x-x_{0})^{2}} u^{2} \, dx \leq C \left( \int_{0}^{1} p(x)|u'(x)|^{2} \, dx + u^{2}(1) + u^{2}(0) \right). \]  

(7)
Since $H^1(0,1)$ is continuously embedded in $L^\infty(0,1)$, one has that for all $u \in H^1(0,1)$

$$|u(y_0)| \leq \|u\|_{L^\infty(0,1)} \leq C\|u\|_{H^1(0,1)}, \forall y_0 \in [0,1],$$

for a positive constant $C$. In particular, $u^2(0)$ can be estimated in the following way:

$$u^2(0) \leq C\|u\|^2_{H^1(0,\frac{2}{p})} = C\left(\int_0^{\frac{2}{p}} u^2(x)dx + \int_0^{\frac{2}{p}} (u')^2(x)dx\right)$$

$$\leq C\left(\int_0^{\frac{1}{p}} u^2(x)dx + \frac{1}{\min\{a,\frac{2}{p}\}} \int_0^{\frac{2}{p}} (\sqrt{a}u')^2(x)dx\right)$$

$$\leq C\left(\int_0^{\frac{1}{p}} u^2(x)dx + \int_0^{\frac{1}{p}} (\sqrt{a}u')^2(x)dx\right).$$

(8)

Analogously, one has

$$u^2(1) \leq C\left(\int_0^{\frac{1}{p}} u^2(x)dx + \int_0^{\frac{1}{p}} (\sqrt{a}u')^2(x)dx\right).$$

(9)

Moreover, by [5, Lemma 2.1],

$$p(x) = |x-x_0|^2-K_1-K_2a(x)\frac{|x-x_0|K_1}{a(x)} \frac{|x-x_0|K_2}{b(x)} \leq c(a(x)$$

for some $c > 0$. Hence, by (7)-(9)

$$\int_0^{\frac{1}{p}} \frac{u^2}{b}dx \leq C\left(\int_0^{\frac{1}{p}} u^2(x)dx + \int_0^{\frac{1}{p}} a(x)(u')^2dx\right),$$

for a positive constant $C$ and so $u \in H^1_{a,b}(0,1)$.

**Second case.** Take $u \in H^1_{a,b}(0,1)$, and, fixed $\gamma_i \in (0,1)$, $i = 1,2,3,4$, such that

$$\gamma_1 < \gamma_2 < x_0 < \gamma_3 < \gamma_4,$$

consider the cut off functions $\xi_i : [0,1] \rightarrow [0,1]$, $i = 0,1,2$, such that $\xi_1 \equiv 1$ in $[0,\gamma_1]$, and $\xi_2 \equiv 0$ in $[\gamma_2,1]$. Clearly $\xi_0 = \xi_1 = \xi_2$. Now, define $v_i = \xi_i u$, $i = 0,1,2$. Clearly $v_0(0) = v_0(1) = 0$.

Hence, by [7, Proposition 2.14], we have that there exists $C > 0$ such that

$$\int_0^{\frac{1}{p}} \frac{v_i^2}{b}dx \leq C\int_0^{\frac{1}{p}} a(v_i')^2dx \leq C\int_0^{\frac{1}{p}} (u^2 + a(u')^2)dx.$$  

(10)

Now, since $u$ is a continuous function, $v_1(\gamma_2) = \xi_1(\gamma_2)u(\gamma_2) = 0$. Hence, by definition of $\xi_1$ and the classical Hardy’s inequality, we get

$$\int_0^{\frac{1}{p}} \frac{v_1^2}{b}dx = \int_0^{\gamma_2} \frac{v_1^2}{b}dx \leq C\int_0^{\gamma_2} \frac{v_1^2}{(x-\gamma_2)^2}dx \leq C\int_0^{\gamma_3} (v_1')^2dx$$

$$\leq C\int_0^{\gamma_3} a(v_1')dx \leq C\int_0^{\frac{1}{p}} a(v_1')^2dx \leq C\int_0^{\frac{1}{p}} (u^2 + a(u')^2)dx.$$  

(11)

for a positive constant $C$. Analogously, using the fact that $v_2(\gamma_3) = 0$, one can prove

$$\int_0^{\frac{1}{p}} \frac{v_2^2}{b}dx \leq C\int_0^{\frac{1}{p}} (u^2 + a(u')^2)dx.$$  

(12)

Since $u = v_0 + v_1 + v_2$, by (10)-(12) we can conclude that

$$\int_0^{\frac{1}{p}} \frac{u^2}{b}dx \leq C\int_0^{\frac{1}{p}} \frac{v_0^2}{b} + \frac{v_1^2}{b} + \frac{v_2^2}{b}dx \leq C\int_0^{\frac{1}{p}} (u^2 + a(u')^2)dx.$$
as claimed.

The space we will use is obviously the space where (6) holds; thus in view of the previous Propositions, such a space is
\[ \mathcal{H} := \mathcal{H}_{a,b}^1(0,1). \]

**Remark 2.** When Proposition 2 holds, then the standard norm \( \| \cdot \|_{\mathcal{H}}^2 \) is equivalent to
\[ \| u \|_{\mathcal{H}}^2 := \int_0^1 (u^2 + a(u')^2) \, dx \]
for all \( u \in \mathcal{H} \). Indeed, for all \( u \in \mathcal{H} \), we have
\[ \| u \|_{\mathcal{H}}^2 \leq \| u \|_{\mathcal{H}}^2 = \int_0^1 u^2 \, dx + \int_0^1 \frac{u^2}{b} \, dx + \int_0^1 a(u')^2 \, dx \leq c \int_0^1 (u^2 + a(u')^2) \, dx, \]
for \( c = C_{HP} + 1 \).

When \( \lambda < 0 \), an equivalent norm in \( \mathcal{H} \) is
\[ \| u \|_{\mathcal{H}}^2 := \int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx. \]

From now on, we make the following assumptions on \( a, b \) and \( \lambda \).

**Hypothesis 1.** One of the following conditions holds:
1. One among (WWD) or (WSD), (SWD) with \( K_1 + K_2 \leq 2 \) holds true, and \( \lambda \in \left( 0, \frac{1}{C_{HP}} \right) \);
2. One among (WWD), (SSD), (WSD) or (SWD) holds with \( \lambda < 0 \).

Observe that the assumption \( \lambda \neq 0 \) is not restrictive, since the case \( \lambda = 0 \) is considered in [1].

Thanks to the previous propositions, as in [7, Proposition 2.18], one can prove the next inequality.

**Proposition 3.** Assume Hypothesis 1.1. Then there exists \( \Lambda = 1 - \lambda C_{HP} \) such that for all \( u \in \mathcal{H} \)
\[ \int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx \geq \Lambda \int_0^1 a(u')^2 \, dx - \lambda C_{HP} \int_0^1 u^2 \, dx. \]

**Proof.** By using Proposition 2, one has
\[ \int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx \geq \int_0^1 a(u')^2 \, dx - \lambda C_{HP} \left( \int_0^1 a(u')^2 \, dx + \int_0^1 u^2 \, dx \right) \]
\[ = \Lambda \int_0^1 a(u')^2 \, dx - \lambda C_{HP} \int_0^1 u^2 \, dx, \]
since \( \lambda < 1/C_{HP} \).

We recall the following definition:

**Definition 2.1.** Let \( u_0 \in L^2(0,1) \) and \( h \in L^2(0,T;\mathcal{H}^*) \). A function \( u \) is said to be a (weak) solution of (2) if
\[ u \in \mathcal{U} := L^2(0,T;\mathcal{H}) \cap H^1([0,T];\mathcal{H}^*) \]
and it satisfies (2) in the sense of \( \mathcal{H}^* \)-valued distributions.
Remark 3. Notice that, by [12, Lemma 11.4], any (weak) solution belongs to \( C([0, T]; L^2(0, 1)) \).

Finally, we introduce the Hilbert space
\[
H^2_{\lambda, b}(0, 1) := \left\{ u \in H^1_b(0, 1) : au' \in H^1(0, 1), u'(0) = u'(1) = 0 \text{ and } Au \in L^2(0, 1) \right\},
\]
where
\[
Au := (au')' + \frac{\lambda}{b} u \text{ with } D(A) = H^2_{\lambda, b}(0, 1).
\]

Remark 4. Observe that if \( u \in D(A) \), then \( \frac{u}{\sqrt{b}}, \frac{u}{\sqrt{b}} \in L^2(0, 1) \), so that \( u \in H^1_{\lambda, b}(0, 1) \) and inequality (6) holds.

Theorem 2.2. Assume Hypothesis 1. If \( u_0 \in L^2(0, 1) \) and \( h \in L^2(0, T; \mathcal{H}^*) \), there exists a unique solution of problem (2). Moreover, let \( u_0 \in D(A) \); then
\[
h \in W^{1,1}(0, T; L^2(0, 1)) \Rightarrow u \in C^1(0, T; L^2(0, 1)) \cap C([0, T]; D(A)),
\]
\[
h \in L^2(Q_T) \Rightarrow u \in H^1(0, T; L^2(0, 1)).
\]

Proof. If \( \lambda < 0 \) the proof is similar to the one of [7, Theorem 2.22], so we omit it. If \( \lambda \in (0, 1/C_{HF}) \), by Proposition 3 for all \( u \in D(A) \) we have
\[
-(Au, u)_{L^2(0, 1)} = -\int_0^1 \left((au')' + \frac{\lambda}{b} u \right) u \, dx = \int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx
\]
\[
= \int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx + \Lambda \int_0^1 u^2 \, dx
\]
\[
\geq \Lambda \|u\|_{\mathcal{H}}^2 - C \|u\|_{L^2(0, 1)}^2.
\]
The thesis follows by [3, Proposition 4.1.6], [12, Theorem 11.3], or [10, Theorem 3.4.1 and Remark 3.4.3] as in [7].

3. Carleman estimates for singular/degenerate problems and its application to observability inequality in the Neumann case. In this section we prove an estimate of Carleman type for the adjoint problem of (2), that is for
\[
\begin{cases}
v_t + (av_x)_x + \frac{\lambda}{b(x)} v = h, & (t, x) \in Q_T, \\
v_x(t, 0) = v_x(t, 1) = 0, & t \in (0, T), \\
v(T, x) = v_T(x) \in L^2(0, 1),
\end{cases}
\]
where \( T > 0 \) is given and \( h \in L^2(Q_T) \). As it is well known, to prove Carleman estimates the final datum is irrelevant, only the equation and the boundary conditions are important. For this reason we consider only the problem
\[
\begin{cases}
v_t + (av_x)_x + \frac{\lambda}{b(x)} v = h, & (t, x) \in Q_T, \\
v_x(t, 0) = v_x(t, 1) = 0, & t \in (0, T).
\end{cases}
\]

(13)
3.1. **Carleman estimates.** In order to deal with Carleman estimates for (13), we introduce the function \( \tilde{a} \), which is an extension of \( a \) to the interval \([-1, 2] \):

\[
\tilde{a}(x) := \begin{cases} 
  a(-x), & x \in [-1,0], \\
  a(x), & x \in [0,1], \\
  a(2-x), & x \in [1,2].
\end{cases}
\]

Since \( a, b \) have no regularity property beyond Sobolev regularity, for technical reasons (see [4], [6] and [7]), we make the following assumptions:

**Hypothesis 2.** Hypothesis 1 holds. Moreover, if \( K_1 > \frac{4}{3} \), then there exists a constant \( \theta \in (0, K_1] \) such that

\[
x \mapsto \frac{a(x)}{|x-x_0|^{\theta}} \quad \left\{ \begin{array}{l}
\text{is non increasing on the left of } x = x_0, \\
\text{is non decreasing on the right of } x = x_0.
\end{array} \right.
\]

In addition, when \( K_1 > \frac{3}{2} \) the function in (14) is bounded below away from 0 and there exists a constant \( \Sigma > 0 \) such that

\[
|a'(x)| \leq \Sigma |x-x_0|^{2q-3} \text{ for a.e. } x \in [0, 1].
\]

Moreover, if \( \lambda < 0 \) we require that

\[
(x-x_0)b'(x) \geq 0 \text{ in } [0, 1].
\]

**Hypothesis 3.** Assume (WWD) or (WSD). Suppose that there exist \( B_1 \in (0, x_0), B_2 \in (1, 2-x_0) \), two functions \( g \in L^\infty_c((-x_0, 2-x_0) \setminus \{x_0\}), h(\cdot, B_i) \in W^{1,\infty}_{loc}((-x_0, 2-x_0) \setminus \{x_0\}) \) and two strictly positive constants \( g_0, h_0 \) such that \( g(x) \geq g_0 \) and

\[
-\frac{a'(x)}{2\sqrt{a(x)}} \left( \int_{x}^{B_i} g(t) dt + h_0 \right) + \frac{\sqrt{a(x)}}{a(x)} g(x) = h(x, B_i)
\]

with \( i = 1, 2 \), \(-x_0 < x < B_1 \) or \( x_0 < x < B_2 \).

As in [5] or in [6, Chapter 4], we introduce the functions \( \varphi(t, x) := \Theta(t)\psi(x) \) and \( \Phi_{A,B}(t, x) := \Theta(t)\rho_{A,B}(x) \), where

\[
\Theta(t) := \frac{1}{|t(T-t)|^2}, \quad \psi(x) := c_1 \left[ \int_{x_0}^{x} \frac{y-x_0}{a(y)} dy - c_2 \right], \quad \Phi_{A,B}(t, x) := \Theta(t)\rho_{A,B}(x)
\]

and, for \( A < B, \)

\[
\rho_{A,B}(x):=\begin{cases} 
-\frac{r}{A} \int_{x}^{A} \frac{1}{\sqrt{a(t)}} \int_{t}^{B} g(s) ds dt + \int_{A}^{x} \frac{b(t)}{\sqrt{a(t)}} dt - c, & \text{if } a \in W^{1,1}(0,1), \\
\rho_{A,B}(x) = c, & \text{if } a \in W^{1,\infty}(0,1).
\end{cases}
\]

Here \( c_2 > \sup_{[0,1]} \int_{x_0}^{x} \frac{y-x_0}{a(y)} dy, r, c_1 > 0 \) \( (c_1 \) will be taken sufficiently large for the observability inequality), \( \epsilon > 0 \) is such that \( \max_{[A,B]} \rho_{A,B} < 0 \) and

\[
\zeta_B(x) = \tilde{a} \int_{x}^{B} \frac{1}{\tilde{a}(t)} dt,
\]

with \( \tilde{a} = \|\tilde{a}'\|_{L^\infty(A,B)}. \)

Observe that \( \Theta(t) \to +\infty \) as \( t \to 0^+, T^- \) and by [5, Lemma 2.1], we have that \( -c_1c_2 \leq \psi(x) < 0 \).
Once well-posedness has been established, Carleman estimates are actually an easy consequence of the ones in [1]:

**Theorem 3.1.** Let Hypothesis 2 holds and, if \( K_1 < 1 \) assume also Hypothesis 3. Let \( \omega \subset (0, 1) \) be an open interval containing \( x_0, B_1 \) and \( 2 - B_2 \), or let \( \omega = \omega_1 \cup \omega_2 \), where \( \omega_1 = (\lambda_1, \beta_1) \subset (0, 1), i = 1, 2 \), \( \beta_1 \leq B_1 \) and \( 2 - B_2 \leq \lambda_2 \). Then, there exist two positive constants \( C \) and \( s_0 \) (depending on \( \lambda \)) such that every solution \( v \) of (13) in \( V := L^2(0, T; D(A)) \cap H^1(0, T; \mathcal{H}) \) satisfies, for all \( s \geq s_0 \),

\[
\int_0^1 \int_0^1 \left( s\Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq C \int_0^T \int_\omega v^2 \, dx \, dt \\
+ C \left( \int_0^T \int_0^1 h^2 e^{2s\varphi} \, dx \, dt + \int_0^T \int_0^{B_1} h^2 e^{2s\varphi} \, dx \, dt + \int_0^T \int_{2 - B_2}^1 h^2 e^{2s\varphi} \, dx \, dt \right),
\]

where \( \Phi_1(t, x) := \Theta(t) \beta_{B_1, 0}(x) \) and \( \Phi_2(t, x) := \Theta(t) \beta_{2 - B_2, 0}(x) \).

**Proof.** First, assume that \( x_0, B_1, 2 - B_2 \in \omega \). Then, we can fix two subintervals \( \omega_1 = (\lambda_1, \beta_1) \subset (0, x_0), \omega_2 = (\lambda_2, \beta_2) \subset (x_0, 1) \), with \( \beta_1 = B_1 \) and \( \lambda_2 = 2 - B_2 \), and four points \( \tilde{\lambda}_i, \tilde{\beta}_i \in (\lambda_i, \beta_i), i = 1, 2 \), with \( \tilde{\lambda}_i < \tilde{\beta}_i \) and consider a smooth function \( \xi : [0, 1] \to [0, 1] \) such that

\[
\xi(x) = \begin{cases} 
0 & x \in [0, \tilde{\lambda}_1], \\
1 & x \in [\tilde{\lambda}_1, \tilde{\lambda}_2], \\
0 & x \in [\tilde{\beta}_2, 1], 
\end{cases}
\]

where \( \tilde{\lambda}_i = (\tilde{\lambda}_i + \tilde{\beta}_i)/2, i = 1, 2 \). Then, define \( w := \xi v \), where \( v \) is any fixed solution of (13). Hence \( w \) satisfies

\[
\begin{align*}
  w_t + (aw_x)_x + \frac{\lambda}{b} w &= f_1, & (t, x) &\in Q_T, \\
  w(t, 0) &= w(t, 1) = 0, & t &\in (0, T),
\end{align*}
\]

with \( f_1^2 = (\xi h + (a\xi x v)_x + a\xi x v)^2 \leq Ch^2 + C(v^2 + (v_x)^2) \chi_\omega \), where \( \hat{\omega} = (\tilde{\lambda}_1, \tilde{\lambda}_2) \cup (\tilde{\lambda}_2, \tilde{\beta}_2) \). Applying [7, Theorem 3.3] and [7, Proposition 4.6], we have

\[
\begin{align*}
  &\int_0^T \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} \left( s\Theta a(v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx \, dt \\
  &\leq \int_{Q_T} \left( s\Theta a(w_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx \, dt \leq C \int_{Q_T} f_1^2 e^{2s\varphi} \, dx \, dt \\
  &\leq C \int_{Q_T} h^2 e^{2s\varphi} \, dx \, dt + C \int_0^T \int_\omega (v^2 + (v_x)^2) e^{2s\varphi} \, dx \, dt \\
  &\leq C \int_{Q_T} h^2 e^{2s\varphi} \, dx \, dt + C \int_0^T \int_\omega v^2 \, dx \, dt.
\end{align*}
\]

Now, define a smooth function \( \tau : [-1, 2] \to [0, 1] \) such that

\[
\tau(x) = \begin{cases} 
0 & x \in [-1, -\tilde{\beta}_1] \cup [\tilde{\beta}_1, \tilde{\lambda}_2] \cup [2 - \tilde{\lambda}_1, 2], \\
1 & x \in [-\tilde{\lambda}_1, \tilde{\lambda}_1] \cup [\tilde{\lambda}_2, 2 - \tilde{\lambda}_2],
\end{cases}
\]
and
\[
W(t, x) := \begin{cases} 
  v(t, -x), & x \in [-1, 0], \\
  v(t, x), & x \in [0, 1], \\
  v(t, 2 - x), & x \in [1, 2],
\end{cases}
\]
where \( v \) satisfies (13). Thus \( W \) satisfies the problem
\[
\begin{aligned}
W_t + (\tilde{a}W_x)_x + \frac{\lambda}{b}W &= \tilde{h}, & (t, x) \in (0, T) \times (-1, 2), \\
W_x(t, -1) &= W_x(t, 2) = 0, & t \in (0, T),
\end{aligned}
\]
being
\[
\tilde{h}(t, x) := \begin{cases} 
  h(t, -x), & x \in [-1, 0], \\
  h(t, x), & x \in [0, 1], \\
  h(t, 2 - x), & x \in [1, 2]
\end{cases}
\]
and
\[
\tilde{b}(x) := \begin{cases} 
  b(-x), & x \in [-1, 0], \\
  b(x), & x \in [0, 1], \\
  b(2 - x), & x \in [1, 2],
\end{cases}
\]
Defining \( z := \tau W \), we have that \( z \) satisfies the nondegenerate problems
\[
\begin{aligned}
z_t + (\tilde{a}z_x)_x + \frac{\lambda}{b}z &= f_2, & (t, x) \in (0, T) \times (\lambda_2, 2 - \lambda_2), \\
z(t, \lambda_2) &= z(t, 2 - \lambda_2) = 0, & t \in (0, T),
\end{aligned}
\]
and
\[
\begin{aligned}
z_t + (\tilde{a}z_x)_x + \frac{\lambda}{b}z &= f_2, & (t, x) \in (0, T) \times (-\beta_1, \beta_1), \\
z(t, -\beta_1) &= z(t, \beta_1) = 0, & t \in (0, T),
\end{aligned}
\]
with \( f_2 := \tau \tilde{h} + (\tilde{a}z_x)_x + \tilde{a}z_x W_x \). Thanks to Hypothesis 3, function \( \tilde{a} \) satisfies [6, Hypothesis 3.1] in \( [\lambda_2, 2 - \lambda_2] \) and in \( [-\beta_1, \beta_1] \), hence applying [6, Theorem 3.1] in these intervals\(^1\), we have that there exist \( s_0 \), and \( C > 0 \) such that
\[
\int_0^T \int_{\lambda_2}^{2-\lambda_2} (s \Theta(z_x)^2 e^{2s\Phi_2} + s^3 \Theta^3 z^2 e^{2s\Phi_2}) dx dt \leq C \int_0^T \int_{\lambda_2}^{2-\lambda_2} f_2^2 e^{2s\Phi_2} dx dt,
\]
and
\[
\int_0^T \int_{-\beta_1}^{\beta_1} (s \Theta(z_x)^2 + s^3 \Theta^3 z^2) e^{2s\Phi_1} dx dt \leq C \int_0^T \int_{-\beta_1}^{\beta_1} f_2^2 e^{2s\Phi_1} dx dt
\]
for all \( s \geq s_0 \). Now, as in [6], we can prove that exist two positive constants \( k_i \), \( i = 1, 2 \), such that
\[
\tilde{a}(x) e^{2s\varphi(t,x)} \leq k_1 e^{2s\Phi_1(t,x)} \quad \text{and} \quad \frac{(x - x_0)^2}{\tilde{a}(x)} e^{2s\varphi(t,x)} \leq k_1 e^{2s\Phi_1(t,x)} \quad (17)
\]
\(^1\)Actually, Theorem 3.1 in [6] was proved in absence of lower order terms, but in the considered intervals the problems are nondegenerate, and so the Carleman estimate proved therein still holds true in presence of the regular lower order term \( \lambda u/b \), as a standard procedure shows, see [13, Section 2].
for every \((t, x) \in [0, T] \times [-\beta_1, \beta_1]\) and
\[
\hat{a}(x)e^{2s\varphi(t,x)} \leq k_2e^{2s\Phi_2(t,x)} \quad \text{and} \quad \frac{(x-x_0)^2}{\hat{a}(x)}e^{2s\varphi(t,x)} \leq k_2e^{2s\Phi_2(t,x)} \quad (18)
\]
for every \((t, x) \in [0, T] \times [\lambda_2, 2 - \lambda_2]\). Thus, by definitions of \(\tau, W\) and \(z\), by using (17), we have
\[
\int_0^T \int_0^{\beta_1} \left( s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a}v^2 \right) e^{2s\varphi} dx dt
\]
\[
\leq \int_0^T \int_0^{\beta_1} \left( s\Theta a(z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} \right) e^{2s\varphi} dx dt
\]
\[
\leq k_1 \int_0^T \int_0^{\beta_1} \left( s\Theta(z_x)^2 + s^3\Theta^3 z^2 \right) e^{2s\Phi_1} dx dt \leq C \int_0^T \int_0^{\beta_1} f_2^2 e^{2s\Phi_1} dx dt.
\]
Using the fact that \(\tau_x\) is supported in \([-\beta_1, -\tilde{\lambda}_1] \cup [\tilde{\lambda}_1, \beta_1] \cup [\lambda_2, \tilde{\lambda}_2] \cup [2 - \tilde{\lambda}_2, 2 - \lambda_2]\), the boundedness of \(\tilde{a}'\) (far away from \(x_0\) if \(x \in W^{1,1}(0,1)\)) and the fact that \(\rho_{-B_1, B_1}(x) \leq \rho_{-B_1, B_1}(-x)\) for all \(x \in [0, \beta_1]\), as for (16), by [7, Theorem 4.6], we get
\[
\int_0^T \int_{-\beta_1}^{\beta_1} f_2^2 e^{2s\Phi_1} dx dt \leq C \int_0^T \int_{-\beta_1}^{\beta_1} \tilde{h}^2 e^{2s\Phi_1} dx dt + C \int_0^T \int_0^\omega v^2 dx dt
\]
\[
\leq C \left( \int_0^T \int_0^{B_1} \tilde{h}^2 e^{2s\Phi_1(t,-x)} dx dt + \int_0^T \int_0^\omega v^2 dx dt \right)
\]
and
\[
\int_0^T \int_0^{\beta_1} \left( s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a}v^2 \right) e^{2s\varphi} dx dt
\]
\[
\leq C \left( \int_0^T \int_0^{B_1} \tilde{h}^2 e^{2s\Phi_1(t,-x)} dx dt + \int_0^T \int_0^\omega v^2 dx dt \right)
\]
for all \(s \geq s_0\). Analogously, by using (18), we can choose \(s_0\) so large that, for all \(s \geq s_0\) and for a positive constant \(C\):
\[
\int_0^T \int_{-2-B_2}^{2-B_2} \left( s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a}v^2 \right) e^{2s\varphi} dx dt \leq C \int_0^T \int_{-2-B_2}^{2-B_2} f_2^2 e^{2s\Phi_2} dx dt
\]
\[
\leq C \left( \int_0^T \int_{-2-B_2}^{2-B_2} \tilde{h}^2 e^{2s\Phi_2(t,-x)} dx dt + \int_0^T \int_0^\omega v^2 dx dt \right)
\]
(20)
since \(\rho_{-B_2, B_2}(2-x) \leq \rho_{-B_2, B_2}(x)\) for all \(x \in [2 - B_2, 1]\). Hence, by (16), (19) and (20), we can choose \(s_0\) so large that, for all \(s \geq s_0\) and for a positive constant \(C\),
\[
\int_0^T \int_0^{\beta_1} \left( s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a}v^2 \right) e^{2s\varphi} dx dt
\]
\[
\leq C \int_0^T \int_0^\omega v^2 dx dt
\]
\[
+ C \left( \int_0^T \int_0^{B_1} \tilde{h}^2 e^{2s\Phi_1(t,-x)} dx dt + \int_0^T \int_{-2-B_2}^{2-B_2} \tilde{h}^2 e^{2s\Phi_2(t,-x)} dx dt \right).
\]
Nothing changes in the proof if \( \omega = \omega_1 \cup \omega_2 \) and each of these intervals lies on different sides of \( x_0 \), as the assumption implies.

### 3.2. Application to observability inequality

In this section we consider problem (2) and we assume that the control set \( \omega \) is an interval which contains the degeneracy point or the union of two intervals each of them lying on one side of the degeneracy point, i.e.

- \( \omega = (\alpha, \beta) \subset (0, 1) \) is such that \( x_0 \in \omega \),

or

- \( \omega = \omega_1 \cup \omega_2 \), where

\[
\omega_i = (\lambda_i, \beta_i) \subset (0, 1), \ i = 1, 2, \ \text{and} \ \beta_1 < x_0 < \lambda_2.
\]

**Remark 5.** Observe that, if (21) holds, we can find two subintervals

\[
\omega_1 = (\alpha, x_0); \ \omega_2 = (x_0, \beta) \subset (0, 1)
\]

such that

\[
(\alpha, x_0) \subset \omega \ \text{and} \ (x_0, \beta) \subset \omega \setminus \{x_0\}.
\]

**Hypothesis 4.**

1. One among (WWD), (WSD) or (SWD) with \( K_1 + K_2 \leq 2 \) and \( \lambda < \frac{1}{C_{HP}} \);
2. if \( K_1 > \frac{4}{3} \), then there exists a constant \( \theta \in (0, K_1) \) such that (14) holds, namely

\[
x \mapsto \frac{a(x)}{|x - x_0|^\theta} \begin{cases} \text{is non increasing on the left of } x = x_0, \\ \text{is non decreasing on the right of } x = x_0; \end{cases}
\]
3. if \( K_1 > \frac{3}{2} \) the function in (14) is bounded below away from 0 and there exists a constant \( \Sigma > 0 \) such that

\[
|a'(x)| \leq \Sigma |x - x_0|^{2\theta - 3} \text{ for a.e. } x \in [0, 1];
\]
4. if \( \lambda < 0 \) we require that

\[
(x - x_0)b'(x) \geq 0 \text{ in } [0, 1];
\]
5. Hypothesis 3 holds with \( B_1 = \beta_1 \) and \( B_2 = 2 - \lambda_2 \).

**Remark 6.** Though it seems that Hypothesis 4 is simply a re-listing of previous ones with the additional two last conditions, this is not the case, since in Hypothesis 1.1 we assumed \( \lambda > 0 \), while in Hypothesis 4.1 we don’t. On the other hand, the case \( \lambda < 0 \) in Hypothesis 1 is considered also in the (SSD) case, while in Hypothesis 4, it is not. This is due to the fact that we are dealing with sufficient conditions for controllability, not for existence, i.e. if we have a solution, then it satisfies the observability inequality below.

Now, we associate to (2) the homogeneous adjoint problem

\[
\begin{align*}
    v_t + Av &= 0, \quad (t, x) \in Q_T, \\
    v_\alpha(t, 0) &= v_\alpha(t, 1) = 0, \quad t \in (0, T), \\
    v(T, x) &= v_T(x) \in L^2(0, 1),
\end{align*}
\]

where \( T > 0 \) is given. By the Carleman estimate given in Theorem 3.1, we will deduce the analogous observability and controllability results of [7, Proposition 4.4 and Theorem 4.5] in the Dirichlet case:
Theorem 3.2. Assume Hypothesis 4 and (21) or (22). Then there exists a positive constant $C_T$ such that every solution $v \in L^2(0, T; H) \cap H^1([0, T]; H^*)$ of (23) satisfies
\begin{equation}
\int_0^1 v^2(0, x)dx \leq C_T \int_0^T \int_\omega v^2(t, x)dxdt.
\end{equation}
Hence, given $u_0 \in L^2(0, 1)$, there exists $h \in L^2(Q_T)$ such that the solution $u$ of (2) satisfies
\begin{equation}
0 = \int_0^1 v_0(x)dx, \quad \text{for all } x \in [0, 1].
\end{equation}
Moreover
\begin{equation}
\int_{Q_T} h^2dxdt \leq C \int_0^1 u_0^2dx,
\end{equation}
for some positive constant $C$.

3.3. Proof of Theorem 3.2. As a consequence of the Carleman estimate given in Theorem 3.1, here we will prove the observability inequality (24), the null controllability property following in a standard way.

The proof of (24) starts as the one given in [1]. However, due to the presence of $\int_0^1 u^2dxdt$ in (6) and of $u(0)$ and $u(1)$ in (5), it is more difficult and in some sense different from the one of [1] (see in particular the proof of Lemma 3.4 below).

First of all, we consider the adjoint problem with more regular final–time datum
\begin{equation}
\begin{aligned}
v_t + Av &= 0, \quad (t, x) \in Q_T, \\
v(0, x) &= 0, \quad (t, x) \in Q_T, \\
v(T, x) &= h(x), \quad x \in [0, 1],
\end{aligned}
\end{equation}
where $D(A^2) = \{ u \in D(A) : Au \in D(A) \}$. As usual, we define the following class of functions:
\begin{equation}
\mathcal{V} := \{ v \text{ is a solution of (25)} \},
\end{equation}
which is strictly contained in $C^1([0, T]; H^2_{a,b}(0, 1)) \subset \mathcal{V} \subset \mathcal{U} \subset C([0, T]; L^2(0, 1)) \cap L^2(0, T; H)$. For any solution $v$ of (25), as a corollary of Theorem 3.1, we get the following estimate:

Lemma 3.3. Assume Hypothesis 4 and (21) or (22). Then there exist two positive constants $C$ and $s_0$ such that every solution $v \in \mathcal{V}$ of (25) satisfies, for all $s \geq s_0$,
\begin{equation}
\int_0^T \int_0^1 \left( s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a}v^2 \right) e^{2s\varphi}dxdt \leq C \int_0^T \int_\omega v^2dxdt.
\end{equation}
Here $\Theta$ and $\varphi$ are as in (15), with $c_1$ sufficiently large.

Lemma 3.3 is the key tool to prove the following result, which is essential for proving Theorem 3.2:

Lemma 3.4. Assume Hypothesis 4 and (21) or (22). Then there exists a positive constant $C_T$ such that every solution $v \in \mathcal{V}$ of (25) satisfies (24).

Proof. Multiplying the equation of (25) by $v$ and integrating by parts over $(0, 1)$, one has
\begin{equation}
0 = \int_0^1 (v_t + (av_x)_x + \lambda \frac{v}{b})vdx = \frac{1}{2} \frac{d}{dt} \int_0^1 v^2dx + \int_0^1 a(v_x)^2dx + \lambda \int_0^1 \frac{v^2}{b}dx.
\end{equation}
Hence,
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v^2 \, dx = \int_0^1 a(v_x)^2 \, dx - \lambda \int_0^1 \frac{v^2}{b} \, dx.
\]
(26)
If \( \lambda < 0 \), we have obviously that the function
\[
t \mapsto \int_0^1 v^2(t, x) \, dx
\]
is non decreasing for all \( t \in [0, T] \). In particular,
\[
\int_0^1 v^2(0, x) \, dx \leq \int_0^1 v^2(t, x) \, dx.
\]
(27)
If \( \lambda \in (0, 1/CHP) \), by (26) and Proposition 3, we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v^2 \, dx \geq (1 - \lambda CHP) \int_0^1 a(v_x)^2 \, dx - \lambda CHP \int_0^1 v^2 \, dx \geq -\lambda CHP \int_0^1 v^2; 
\]
i.e.
\[
\frac{d}{dt} \int_0^1 v^2 \, dx + 2\lambda CHP \int_0^1 v^2 \, dx \geq 0.
\]
Thus the function
\[
t \mapsto e^{2\lambda CHP t} \int_0^1 v^2(t, x) \, dx
\]
is non decreasing for all \( t \in [0, T] \). Therefore,
\[
\int_0^1 v^2(0, x) \, dx \leq e^{2\lambda CHP t} \int_0^1 v^2(t, x) \, dx \leq e^{2\lambda CHP T} \int_0^1 v^2(t, x) \, dx.
\]
(28)
From (27) and (28) we conclude that, in every case, there exists a positive constant \( C \) such that
\[
\int_0^1 v^2(0, x) \, dx \leq C \int_0^1 v^2(t, x) \, dx.
\]
(29)
Now, we will concentrate on the last integral and we will prove that
\[
\int_0^1 v^2(t, x) \, dx \leq C \left( \int_0^1 a(v_x)^2(t, x) \, dx + \int_0^1 \frac{|x-x_0|^2}{a} v^2(t, x) \, dx \right),
\]
(30)
for a positive constant \( C \). Indeed, using the Young inequality, we find
\[
\int_0^1 v^2 \, dx = \int_0^1 \left( \frac{a^{1/3}}{|x-x_0|^{2/3}} v^2(t, x) \right)^{3/4} \left( \frac{|x-x_0|^2}{a} v^2(t, x) \right)^{1/4} \, dx 
\leq C \left( \int_0^1 \frac{a^{1/3}}{|x-x_0|^{2/3}} v^2(t, x) \, dx + \int_0^1 \frac{|x-x_0|^2}{a} v^2(t, x) \, dx \right).
\]
(31)
Consider the integral
\[
\int_0^1 \frac{a^{1/3}}{|x-x_0|^{2/3}} v^2(t, x) \, dx.
\]
If \( K_1 \leq \frac{4}{3} \) (where \( K_1 \) is the constant appearing in (WWD) or (SSD) or (WSD)), we introduce the function \( p(x) = |x-x_0|^{4/3} \). Obviously, there exists \( q \in \left( 1, \frac{4}{3} \right) \) such that the function \( x \mapsto \frac{p(x)}{|x-x_0|^q} \) is nonincreasing on the left of \( x = x_0 \) and
nondecreasing on the right of \( x = x_0 \). Thus, using the Hardy-Poincaré inequality (see Proposition 1) and \([5, \text{Lemma 2.1}]\), one has

\[
\int_0^1 \frac{a^{1/3}}{|x - x_0|^{2/3}} v^2(t, x) dx \leq \max_{[0, 1]} a^{1/3} \int_0^1 \frac{1}{|x - x_0|^{2/3}} v^2(t, x) dx
\]

\[
= \max_{[0, 1]} a^{1/3} \int_0^1 \frac{p}{(x - x_0)^2} v^2(t, x) dx
\]

\[
\leq C \left( \int_0^1 p(v_x)^2(t, x) dx + v^2(t, 0) + v^2(t, 1) \right)
\]

\[
= C \left( \int_0^1 a^{1/3} \frac{|x - x_0|^{4/3}}{a} (v_x)^2(t, x) dx + v^2(t, 0) + v^2(t, 1) \right)
\]

\[
\leq C \left( \int_0^1 a(v_x)^2(t, x) dx + v^2(t, 0) + v^2(t, 1) \right).
\] (32)

If \( K_1 > 4/3 \) consider the function \( p(x) = (a(x)|x - x_0|^{4/3})^{1/3} \). It is clear that, setting \( C_1 := \max \left\{ \left( \frac{x_0^3}{a(0)} \right)^{2/3}, \left( \frac{1 - x_0^2}{a(1)} \right)^{2/3} \right\} \), by \([5, \text{Lemma 2.1}]\) we have

\[
p(x) = a(x) \left( \frac{(x - x_0)^2}{a(x)} \right)^{2/3} \leq C_1 a(x) \text{ and } \frac{a^{1/3}}{|x - x_0|^{2/3}} = \frac{p(x)}{(x - x_0)^2}. \]

Moreover, using (14), one has that, if \( q := \frac{4 + \vartheta}{3} > 1 \), the function \( \frac{a(x)|x - x_0|^q}{p(x)} \) is nonincreasing on the left of \( x = x_0 \) and nondecreasing on the right of \( x = x_0 \). Thus, the Hardy-Poincaré inequality (see Proposition 1) implies

\[
\int_0^1 \frac{a^{1/3}}{|x - x_0|^{2/3}} v^2(t, x) dx = \int_0^1 \frac{p}{(x - x_0)^2} v^2(t, x) dx
\]

\[
\leq C \left( \int_0^1 p(x)|v_x(t, x)|^2 dx + v^2(t, 1) + v^2(t, 0) \right)
\] (33)

\[
\leq C \left( \int_0^1 a(x)|v_x(t, x)|^2 dx + v^2(t, 1) + v^2(t, 0) \right).
\]

Hence, from (32) and (33), in every case there exists a positive constant \( C \) such that

\[
\int_0^1 \frac{a^{1/3}}{|x - x_0|^{2/3}} v^2(t, x) dx \leq C \left( \int_0^1 a(x)|v_x(t, x)|^2 dx + v^2(t, 1) + v^2(t, 0) \right). \] (34)

Consider now \( v(t, 0) \) and \( v(t, 1) \). Proceeding as in (8), one has

\[
v^2(t, 0) \leq C \left( \int_0^{x_0} v^2(t, x) dx + \int_0^{x_0} (v_x)^2(t, x) dx \right)
\]

\[
\leq C \left( \max_{[0, x_0]} \frac{a}{|x - x_0|^2} \int_0^{x_0} \frac{|x - x_0|^2}{a} v^2(t, x) dx + \frac{1}{\min_{[0, x_0]} a} \int_0^{x_0} (\sqrt{a} v_x)^2(t, x) dx \right)
\]

\[
\leq C \left( \int_0^1 \frac{|x - x_0|^2}{a} v^2(t, x) dx + \int_0^1 (\sqrt{a} v_x)^2(t, x) dx \right) .
\] (35)
for a positive constant $C$. Analogously, one has

$$v^2(t, 1) \leq C \left( \int_0^1 \frac{|x-x_0|^2}{a} v^2(t, x) \, dx + \int_0^1 (\sqrt{a} v_x)^2(t, x) \, dx \right). \quad (36)$$

Hence by (31), (34), (35) and (36), we have (30).

By (29) and (30), one has

$$\int_0^1 v^2(0, x) \, dx \leq C \left( \int_0^1 a(v_x)^2(t, x) \, dx + \int_0^1 \frac{|x-x_0|^2}{a} v^2(t, x) \, dx \right),$$

Integrating the last inequality over $\left[ \frac{T}{4}, \frac{3T}{4} \right]$, $\Theta$ being bounded therein, and fixed $s \geq s_0$, where $s_0$ is the one given in Lemma 3.3, we find

$$\int_0^1 v^2(0, x) \, dx \leq C_T \int_0^{\frac{3T}{4}} \int_0^1 s \Theta a(v_x)^2 e^{2s\varphi} \, dx \, dt + C_T \int_0^{\frac{3T}{4}} \int_0^1 s^3 \Theta^3 \frac{|x-x_0|^2}{a} v^2 e^{2s\varphi} \, dx \, dt \leq C \int_0^T \int_\omega v^2 \, dx \, dt,$$

by Lemma 3.3. The Lemma is proved.

The proof of the observability inequality given in Theorem 3.2 follows from Lemma 3.4 in a standard way by using a density argument.

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