MILLIHertz QUASI-PERIODIC OSCILLATIONS FROM MARGINALLY STABLE NUCLEAR BURNING ON AN ACCRETING NEUTRON STAR

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ABSTRACT

We investigate marginally stable nuclear burning on the surface of accreting neutron stars as an explanation for the mHz quasi-periodic oscillations (QPOs) observed from three low-mass X-ray binaries. At local accretion rates close to the boundary between unstable and stable burning, the temperature dependence of the nuclear heating rate and cooling rate almost cancel. The result is an oscillatory mode of burning, with an oscillation period close to the geometric mean of the thermal and accretion timescales for the burning layer. We describe a simple one-zone model that illustrates this basic physics and then present detailed multizone hydrodynamic calculations of nuclear burning close to the stability boundary using the KEPLER code. Our models naturally explain the characteristic 2 minute period of the mHz QPOs and why they are seen only in a very narrow range of X-ray luminosities. The oscillation period is sensitive to the accreted hydrogen fraction and the surface gravity, suggesting a new way to probe these parameters. The accretion rate at which the oscillations appear in the theoretical models, however, is of an order of magnitude larger than the rate implied by the X-ray luminosity when the mHz QPOs are seen if the accreted material covers the entire neutron star surface. Bringing the models and observations into agreement requires that the accreted material covers only part of the neutron star surface at luminosities of $L_X \approx 10^{37}$ erg s$^{-1}$ so that the local accretion rate at the burning depth can be higher than the observed average.

Subject headings: accretion, accretion disks — stars: neutron — X-rays: bursts

Online material: mpeg animations

1. INTRODUCTION

Low-mass X-ray binaries, in which a neutron star or black hole accretes from a low-mass companion, exhibit periodic and quasi-periodic phenomena ranging in frequency from very low frequency (mHz) noise to kHz quasi-periodic oscillations (QPOs; see van der Klis 2005 for a recent review). This variability has mostly been associated with orbiting material in the accretion flow close to the compact object. In the case of a neutron star accretor, an important question is whether any of these phenomena originate from or are associated with the neutron star surface. This is important for identifying the compact object as a neutron star or a black hole and associated with the neutron star surface. This is important for identifying the compact object as a neutron star or a black hole and associated with the neutron star surface. This is important for identifying the compact object as a neutron star or a black hole and associated with the neutron star surface.

Unstable nuclear burning on neutron star surfaces has been studied for many years and is observed as type I X-ray bursts. The accreted hydrogen (H) and helium (He) fuel accumulates on the surface of the star and undergoes a thin-shell instability, giving rise to a $10^2$–$10^3$ s burst of X-rays with a typical energy of $10^{37}$ erg (for reviews, see Lewin et al. 1993, 1995; Strohmayer & Bildsten 2006). Not all sources show type I X-ray bursts, however, and in many sources the bursts are not frequent enough to burn all of the accreted fuel (van Paradijs et al. 1988; in T Zand et al. 2003). Bildsten (1993, 1995) suggested that a different mode of nuclear burning involving slowly propagating fires over the neutron star surface operates at high accretion rates and manifests itself in the power spectrum of the source as very low frequency noise (VLFN). He found an anticorrelation between bursting and VLFN, supporting this picture.

Revnivtsev et al. (2001) discovered a new class of mHz QPOs in three atoll sources, 4U 1608–52, 4U 1636–53, and Aql X-1, which they proposed were from a special mode of nuclear burning on the neutron star surface rather than from the accretion flow. These kHz QPOs have frequencies in the range 7–9 mHz (timescales of 1.9–2.4 minutes). The associated flux variations are at the few percent level and are strongest at low photon energies ($\lesssim 5$ keV). This is in contrast to all other observed QPOs, whose amplitudes generally increase with photon energy. Revnivtsev et al. (2001) showed that the centroid frequency of the kHz QPO was stable on year timescales in a given source and was the same to within tens of percent in all three sources in which it was detected. In addition, the presence of the kHz QPOs is affected by type I X-ray bursts: in 4U 1608–52, the kHz QPOs disappeared immediately following a type I X-ray burst. In 4U 1608–52, a transient source whose luminosity is observed to change by orders of magnitude, the kHz QPO was only present within a narrow range of luminosity, $L_X \approx 0.5–1.5 \times 10^{37}$ erg s$^{-1}$.

The association of the kHz QPOs with a surface phenomenon was strengthened by the results of Yu & van der Klis (2002), who showed that the kHz QPO frequency is anticorrelated with the luminosity variations during the kHz oscillation. This is opposite to the long-term trend, which is that the kHz QPO frequency varies proportionally to the X-ray luminosity, consistent with the inner edge of the accretion disk being pushed inward at higher accretion rates. The anticorrelation observed by Yu & van der Klis (2002) during the kHz QPO cycle suggests that the inner edge of the disk moves outward slightly as the luminosity increases during the cycle, perhaps consistent with enhanced radiation drag as the gas orbiting close to the neutron star is radiated by emission from the neutron star surface.

The fact that the properties of the kHz QPOs suggest that they are associated with the neutron star surface led Revnivtsev et al. (2001) to propose that a special mode of nuclear burning operates at luminosities of $L_X \approx 0.5–1.5 \times 10^{37}$ erg s$^{-1}$. The nature of the
burning and the physics underlying the characteristic timescale of \( \approx 2 \) minutes, however, were unexplained. Bildsten (1993) gave the characteristic timescales of the burning layer that might be associated with subhertz phenomena. The thermal timescale of the layer is \( t_{\text{therm}} \approx 10^2 \) s, the time to accrete the fuel is \( t_{\text{facc}} \approx 10^3 \) s at the Eddington accretion rate, and the time for a nuclear burning “fire” to propagate around the surface is estimated to be \( \approx 10^4 \) s. Bildsten (1993) proposed that if several fires are propagating around the star at a given time, the time signature would be broadband noise with frequencies in the mHz range. None of the timescales that he identified match the \( \approx 2 \) minute mHz QPO period, however.

The luminosity at which mHz QPOs are observed (\( L_X \approx 10^{37} \) erg s\(^{-1}\)) is significant because it is similar to the luminosity at which a transition in burning behavior occurs, from frequent type I X-ray burning at low accretion rates to the disappearance of type I X-ray bursts at high accretion rates. This transition is common to many X-ray bursters (e.g., Cornelisse et al. 2003) and is expected theoretically because at high accretion rates the fuel burns at a higher temperature, reducing the temperature sensitivity of helium burning and quenching the thin-shell instability. An outstanding puzzle is that theory predicts a transition accretion rate close to the Eddington rate (Bildsten 1998), which corresponds to luminosities of \( L_X \approx 10^{40} \) erg s\(^{-1}\), much larger than observed. Paczynski (1983) pointed out that near the transition from instability to stability, oscillations are expected because the eigenvalues of the system are complex. Narayan & Heyl (2003) extended Paczynski’s one-zone analysis, calculating linear eigenmodes of truncated steady state burning models. They too found complex eigenvalues near the stability boundary, and they suggested that this might explain the mHz QPOs observed by Revnivtsev et al. (2001). The oscillation frequencies, however, were an order of magnitude too small.

In this paper, we show that the mHz QPO frequencies are, in fact, naturally explained as being due to marginally stable nuclear burning on the neutron star surface. At the transition between unstable and stable burning, the temperature dependence of the nuclear heating rate and cooling rate almost cancel. The result is an oscillatory mode of burning, with an oscillation period close to the geometric mean of the thermal and accretion timescales for the burning layer, \( (t_{\text{therm}}t_{\text{facc}})^{1/2} \approx 100 \) s. In \( \S \) 2, we describe a simple one-zone model that illustrates this basic physics, and then we present detailed hydrodynamic calculations of nuclear burning close to the stability boundary in \( \S \) 3. We discuss the implications of our results in \( \S \) 4. In particular, if the mHz QPOs are due to marginally stable nuclear burning, the local accretion rate onto the star at burning depth must be close to the Eddington rate, even though the global accretion rate inferred from the X-ray luminosity is 10 times lower; i.e., the accreted material covers only a fraction of the neutron star surface.

2. A ONE-ZONE MODEL

In this section, we discuss a simplified model of the burning layers that illustrates the basic physics underlying the oscillations observed in our multizone numerical simulations. Following Paczynski (1983), we consider a one-zone model of the burning layer. The temperature, \( T \), and thickness of the fuel layer, \( y \) (which we measure as a column depth, in units of mass per unit area), obey (equivalent to eqs. [8] of Paczynski 1983)

\[
\frac{dT}{dt} = \frac{c_P}{\tau} (y - \frac{F}{T}), \quad \frac{dy}{dt} = \frac{\dot{m}}{E_x} y. \tag{1, 2}
\]

Equation (1) describes the heat balance, including heating of the layer by nuclear reactions, \( \epsilon \), and radiative cooling, \( -\nabla \cdot F/\rho = dF/dy \approx F/y \). The heat capacity at constant pressure is \( c_p \), and \( F \) is the outward heat flux. Equation (2) tracks the burning depth, allowing for accretion of new fuel at a rate given by the local accretion rate \( \dot{m} \), as well as burning of fuel on a timescale \( E_x/\epsilon \), where \( E_x \) is the energy per gram released in the burning. Note that the pressure at the base of the layer is \( P = \rho g y \) from hydrostatic balance, where \( g \) is the local gravity. We first study equations (1) and (2) analytically (\( \S \) 2.1), and then we show some numerical integrations (\( \S \) 2.2).

2.1. Analytic Estimates

Equations (1) and (2) constitute a nonlinear oscillator. To understand its behavior, we consider linear perturbations to the steady state solution, which has

\[
\epsilon y = F = \dot{m} E_x. \tag{3}
\]

The nuclear energy generation is generally a strong function of temperature, and we assume \( \epsilon \propto T^\alpha \), where \( \alpha = d \ln \epsilon /d \ln T \). The heat flux is approximately \( F \approx acT^{4/3} \epsilon y \) (e.g., Bildsten 1998). To simplify the algebra in this section, we assume that \( \epsilon \) depends only on temperature and that the opacity \( \kappa \) is a constant. We write the deviations from steady state as \( \delta y \) and \( \delta T \), giving

\[
\frac{c_p}{\tau} \frac{\partial \delta T}{\partial t} = \frac{\delta T}{y} + \frac{2F}{T} \frac{\delta y}{y}, \tag{4}
\]

\[
\frac{\partial \delta y}{\partial t} = -\frac{\epsilon}{E_x} \left(\frac{\delta y + \alpha \delta T}{T}\right). \tag{5}
\]

Defining a thermal timescale for the layer \( t_{\text{therm}} = c_p T/\epsilon = c_p T/y \) and an accretion timescale \( t_{\text{facc}} = y/\dot{m} \) and using the steady state relations of equation (3) gives

\[
\frac{\partial}{\partial t} \left(\frac{\delta T}{T}\right) = \frac{\alpha - 4}{t_{\text{therm}}} \frac{\delta T}{y} + \frac{2}{t_{\text{facc}}} \frac{\delta y}{y}, \tag{6}
\]

\[
\frac{\partial}{\partial t} \left(\frac{\delta y}{y}\right) = -\frac{\alpha}{t_{\text{therm}}} \frac{\delta T}{y} - \frac{\alpha}{t_{\text{facc}}} \frac{\delta y}{y}. \tag{7}
\]

It is useful to write \( \delta T/T = f(t) \exp (-t/t_{\text{therm}}) \) and \( \delta y/y = g(t) \exp (-t/t_{\text{facc}}) \). This simplifies equations (6) and (7), allowing us to combine them into a single differential equation for \( f \),

\[
\frac{\partial^2 f}{\partial t^2} + \left(\frac{4 - \alpha}{t_{\text{therm}}} - \frac{1}{t_{\text{facc}}} \right) \frac{\partial f}{\partial t} + \frac{2 \alpha}{t_{\text{facc}} t_{\text{therm}}} f = 0. \tag{8}
\]

Equation (8) is the equation for a damped simple harmonic oscillator. The solution is \( \delta T \propto \exp (\lambda t) \), with

\[
\lambda = \frac{1}{2} \left(\frac{\alpha - 4}{t_{\text{therm}}} - \frac{1}{t_{\text{facc}}} \right) \pm \left[ \frac{1}{4} \left(\frac{\alpha - 4}{t_{\text{therm}}} + \frac{1}{t_{\text{facc}}} \right)^2 - \frac{2 \alpha}{t_{\text{facc}} t_{\text{therm}}} \right]^{1/2}. \tag{9}
\]

Note that the “damping” term in equation (8) can be positive or negative, depending on the relative temperature sensitivities of the heating and cooling.

The key to understanding the behavior of the burning is to note that the timescales \( t_{\text{therm}} \) and \( t_{\text{facc}} \) are very different. Steady burning at accretion rates near the Eddington rate leads to ignition at a column depth of \( y \approx 10^8 \) g cm\(^{-2}\) and temperatures of \( T \approx 5 \times 10^8 \) K;
The accretion timescale is therefore \( y = 1000 \) s for accretion at \( \dot{m} \approx 10^8 \) g cm\(^{-2}\) s\(^{-1}\) (roughly the Eddington rate). For stable burning, \( F = \dot{m} E_\ast \), giving \( t_{\text{therm}} = c_p T / F = (c_p T / E_\ast)(y / \dot{m}) \approx 0.01 \dot{m}_{\text{accr}} / y \approx 10 \) s, since for an ideal gas, \( c_p T \approx (5/2) (k_b T / m_p) = (7.2 \) keV \( T_\text{d} \) nucleon\(^{-1}\), where \( T_\text{d} \) is the temperature in units of \( 10^8 \) K, whereas \( E_\ast \approx 5 \) MeV nucleon\(^{-1}\) (Schatz et al. 1999).  

Because \( t_{\text{therm}} \ll t_{\text{accr}} \), the damping term is usually much larger than the oscillatory term, and the system is either positively or negatively overdamped depending on the sign of \( \alpha - 4 \). When \( \alpha > 4 \), the damping coefficient is negative, leading to strong exponential growth: the steady state solution is linearly stable. Close to marginal stability, when \( \alpha \approx 4 \), however, the effective thermal timescale \( t_{\text{therm}} / (\alpha - 4) \) becomes much larger than the accretion time, and the inequality is reversed, giving weakly damped or excited oscillations with an oscillation frequency of \( \omega \approx (2 \alpha / \dot{m}_{\text{accr}} t_{\text{therm}})^{1/2} \). The condition to observe oscillations is that the oscillation frequency should be larger than the damping rate:

\[
\left( \frac{2 \alpha}{t_{\text{therm}} / t_{\text{accr}}} \right)^{1/2} \geq \frac{1}{2} \left( \frac{t_{\text{therm}}}{t_{\text{accr}}} + 1 \right).
\]

Note that this condition can be satisfied when \( \lambda \) has either a positive or negative real part, so the oscillations may be excited or damped.

Neglecting the slight modification from the damping term, the oscillation period is \( P_{\text{osc}} = 2\pi / \omega \), or

\[
P_{\text{osc}} \approx \frac{2\pi}{(2\alpha)^{1/2}} \left( \frac{c_p T}{E_\ast} \right)^{1/2} \frac{y}{\dot{m}} \left( \frac{\dot{m}}{\dot{m}_{\text{Edd}}} \right)^{-1},
\]

where we set \( \dot{m} = 4 \) at marginal stability and take \( E_\ast = 5 \) MeV nucleon\(^{-1}\). This estimate is remarkably close to the observed mHz QPO period of 2 minutes.

The physics of the oscillations can be understood by considering equations (6) and (7). For most accretion rates, the effective thermal time is much smaller than the accretion timescale, or \( t_{\text{therm}} / (4 - \alpha) \ll t_{\text{accr}} \). The perturbations are then effectively at constant pressure, or \( \delta y = 0 \), as is commonly assumed (see, e.g., Fujimoto et al. 1981; Bildsten 1998), and equation (6) leads directly to exponential growth or decay, depending on the relative temperature sensitivities of the heating and cooling rates. Close to marginal stability, at \( \alpha - 4 \approx 0 \), however, the effective thermal timescale becomes very long compared to the accretion time. Equations (6) and (7) in this limit are

\[
\frac{\partial}{\partial r} \left( \frac{\delta T}{T} \right) \approx \frac{2}{t_{\text{therm}}} \frac{\delta y}{y},
\]

\[
\frac{\partial}{\partial r} \left( \frac{\delta y}{y} \right) \approx - \frac{\alpha}{t_{\text{accr}} / T} \frac{\delta T}{T}.
\]

These equations nicely summarize the physics of the oscillations. Consider an upward fluctuation in temperature, \( \delta T > 0 \). At the point of marginal stability, the increase in the heating rate almost exactly cancels the \( T^4 \) dependence of the cooling rate.

4 In fact, the gas is partially degenerate at the burning location, with \( k_b T \approx E_F \), and so there will be a small correction factor to the ideal gas expression for \( c_p \).

The main effect is that the hotter temperature leads to faster burning of the accreting fuel and a decrease in thickness of \( \delta y < 0 \) on a timescale of \( \sim t_{\text{accr}} \) (eq. [13]). But a thinner layer cools faster, since \( F / \gamma \propto 1 / \gamma^2 \). Therefore, the temperature fluctuation now begins to decrease, this time on the faster timescale \( \sim t_{\text{therm}} \) (eq. [12]). These changes are out of phase, driving an oscillation on the intermediate timescale of \( \sim (t_{\text{therm}} / t_{\text{accr}})^{1/2} \).

The behavior we describe here is shown by the canonical example of a nonlinear oscillator, the van der Pol oscillator (e.g., Abarbanel et al. 1993). This oscillator consists of an LC circuit with an active element that can behave as a “negative resistor,” originally a vacuum tube. The governing equation is of the form \( \ddot{x} + k(x^2 - 1) \dot{x} + \omega^2 x = 0 \). Depending on the choice of control parameter \( k \), the behavior of this circuit is a limit cycle (relaxation oscillations) in which fast and slow timescales dominate at different parts of the cycle (\( k > 1 \)), a strongly damped system that evolves to a steady state (\( k < -1 \)), or oscillatory with growing or damped oscillations (\( \lvert k \rvert < 1 \)). These three states are analogous to bursting, stable burning, and oscillations near the stability boundary in the one-zone model.

2.2. Numerical Integrations

We have integrated equations (1) and (2) with respect to time to determine the nonlinear evolution of the one-zone model. For the nuclear burning, we use the triple-\( \alpha \) reaction (\( 3 \alpha \rightarrow 12 \)C) rate as given by Fushiki & Lamb (1987). We allow for the presence of hydrogen in the accreted fuel, however, by enhancing the energy release from the triple-\( \alpha \) reaction by a factor of \( E_{\text{nuc}} / E_{\text{he}} \), where \( E_{\text{he}} = 0.606 \) MeV nucleon\(^{-1}\) is the energy release from the triple-\( \alpha \) reaction and \( E_{\text{nuc}} \) is the energy release from burning the accreted mixture of hydrogen and helium to the iron group. We assume that \( E_{\text{nuc}} = 1.6 + 4.9 X_0 \) MeV nucleon\(^{-1}\), where \( X_0 \) is the mass fraction of hydrogen in the accreted layer. This expression for \( E_{\text{nuc}} \) includes an energy loss of 25% from neutrino emission during \( \alpha \)- and \( rp \)-process burning (e.g., Fujimoto et al. 1987) and gives \( E_{\text{nuc}} = 5 \) MeV nucleon\(^{-1}\) for \( X_0 = 0.7 \), in good agreement with the energy release in the steady state burning models of Schatz et al. (1999). We write the flux as \( F = a c T^4 / \delta E_\gamma \) (Bildsten 1998), where the opacity \( \kappa \) is calculated as described by Schatz et al. (1999). In addition, we include a flux heating the layer from below of 0.15 MeV nucleon\(^{-1}\) and a contribution to the heating rate from hot CNO hydrogen burning in the accumulating fuel layer. Neither of these extra contributions to the heat balance make a significant difference to our results. Note that the amount of hydrogen burned by the hot CNO cycle prior to helium ignition is very small at the rapid accretion rates considered here.

Figure 1 shows light curves from the one-zone integrations at different accretion rates. To enable a direct comparison with the multizone simulations discussed in §3, we set the local gravity to be the Newtonian gravity for a 1.4 \( M_\odot \), 10 km neutron star, with \( g = 1.9 \times 10^{14} \) g cm\(^{-2}\) s\(^{-1}\). Throughout this paper, we define the local Eddington accretion rate to be \( \dot{m}_{\text{Edd}} \equiv 8.8 \times 10^8 \) g cm\(^{-2}\) s\(^{-1}\). By coincidence, the stability boundary for this one-zone model is very close to the Eddington accretion rate. In Figure 1, we show light curves at \( \dot{m} = 0.95 \dot{m}_{\text{Edd}}, 0.998 \dot{m}_{\text{Edd}}, \) and 1.05 \( \dot{m}_{\text{Edd}} \). Figure 2 shows the corresponding tracks in the temperature–column depth plane. We start the simulations with the arbitrary conditions \( T = 2 \times 10^8 \) K and \( y = 2 \times 10^8 \) g cm\(^{-2}\). At accretion rates below the boundary, the system evolves quickly into a limit cycle.
corresponding to type I X-ray bursts: slow accumulation of fuel, followed by rapid burning. Close to the stability boundary, the recurrence time is $\frac{t}{C_{25}} \approx 30$ minutes. At $\dot{m} = 0.998 \dot{m}_{\text{Edd}}$, bursts occur with a recurrence time of 34 minutes. At $\dot{m} = 1.05 \dot{m}_{\text{Edd}}$, after a few transient oscillations, the burning evolves to a steady state. The steady state flux is $\dot{m}E_{\text{acc}}$, where $E_{\text{acc}} \approx 5$ MeV nucleon$^{-1}$.

Although the one-zone model is approximate, it is useful because it allows us to investigate how the properties of the oscillations change with parameters such as surface gravity and accreted composition. The accreted composition could differ from system to system due to either metallicity variations or variations in the accreted hydrogen fraction. Intermediate-mass binary evolution models (e.g., Podsiadlowski et al. 2002) predict that the companion star in many systems is hydrogen-deficient, so the hydrogen mass fraction is reduced below the solar composition value of $X_0 \approx 0.7$. The importance of the hydrogen fraction is that the nuclear energy release $E_{\text{acc}}$ changes significantly with only small changes in $X_0$. In contrast, we expect that metallicity will not have a large effect on the transition accretion rate or the oscillation period. This is because the metallicity of the accreted material mainly enters into the one-zone model as hot CNO hydrogen burning in the accumulating layer, but the flux from hot CNO burning, $\epsilon_{\text{CNO}} \approx (5 \times 10^{27} \text{erg cm}^{-2} \text{s}^{-1}) y_8 (Z_{\text{CNO}}/0.01)$, is much smaller than the steady burning flux, $\dot{m}E_{\text{acc}} \approx (5 \times 10^{-3} \text{erg cm}^{-2} \text{s}^{-1})(\dot{m}/\dot{m}_{\text{Edd}})$, where $Z_{\text{CNO}}$ is the mass fraction of CNO elements and $y_8$ is the column depth in units of $10^8$ g cm$^{-2}$.

Figure 4 shows the dependence of the oscillation period on the accretion rate for different choices of gravity and accreted hydrogen fraction. We show results for $g_{14} = 1.9$, where $g_{14}$ is the gravitational acceleration in units of $10^{14}$ cm s$^{-2}$, corresponding to the Newtonian gravity of a 1.4 $M_{\odot}$, 10 km neutron star (or the general relativistic gravity for a 1.4 $M_{\odot}$, 12.3 km neutron star), $g_{14} = 2.45$, which is the gravity of a 1.4 $M_{\odot}$, 10 km neutron star if we take general relativistic corrections into account, and a stronger gravity of $g_{14} = 3.1$ that corresponds to the general relativistic surface gravity for a 2 $M_{\odot}$, 11 km neutron star. We also show a model with a hydrogen fraction below solar, $X_0 = 0.5$. Only the accretion rate range in which oscillations are observed is shown. At each accretion rate, we integrate the one-zone model
for 10^6 s and plot the mean oscillation period after discarding the first 100 minutes of data.

For each choice of \( g_{\odot} \) and \( X_0 \), the pattern is similar. The overall range of accretion rates for which oscillations are seen is very narrow, \( \Delta \dot{m}/\dot{m} \approx 1\% \). For most of this range the oscillations are decaying. As the stability boundary is approached from below, we first see oscillations that reach a steady amplitude (Fig. 4, filled squares) whose frequency drops rapidly with increasing values of \( \dot{m} \). At larger accretion rates (Fig. 4, open squares), the oscillations decay with time on timescales of <1 day and have a frequency that is less sensitive to \( \dot{m} \). The transition from growing to decaying oscillations is very rapid. In each case, the model indicated by the last filled square shows stable oscillations for \( 10^6 \) s, whereas with only a small increment in accretion rate the next model (the first open square) has oscillations that decay on a timescale of \( \sim 10^5 \) s. Most interesting is that the oscillation period is sensitive to \( g_{\odot} \) and \( X_0 \). Increasing gravity or decreasing \( X_0 \) moves the transition from unstable to stable burning to higher accretion rates, where the oscillation period is shorter. As \( X_0 \) decreases, the accretion rate range over which the oscillations are growing rather than decaying is larger.

3. MULTIZONE CALCULATIONS OF BURNING NEAR THE STABILITY BOUNDARY

We now present detailed multizone models of nuclear burning at accretion rates close to the transition from unstable to stable burning. These models are extensions of the calculations presented by Woosley et al. (2004) using the implicit one-dimensional hydrodynamic code KEPLER (Weaver et al. 1978). Woosley et al. (2004) calculated sequences of X-ray bursts at accretion rates \( \dot{m} \) of \( 0.03 \dot{m}_{\text{Edd}} \) and \( 0.1 \dot{m}_{\text{Edd}} \). In this paper we show the first results of an extension of these calculations to higher accretion rates.

Following Woosley et al. (2004), we take the gravitational mass of the neutron star to be \( 1.4 \, M_\odot \) and \( R = 10 \) km, giving a Newtonian gravity of \( g_{\odot} = 1.9 \). The effects of general relativity are not included in the simulations themselves, but because the burning layer is very thin, the effects of general relativity are small over the extent of the simulated burning layer and a local Newtonian

\[ \frac{\dot{m}}{\dot{m}_{\text{Edd}}} \]

\[ g_{\odot} = 1.9 \quad X_0 = 0.7 \]
\[ g_{\odot} = 2.45 \quad X_0 = 0.7 \]
\[ g_{\odot} = 3.1 \quad X_0 = 0.7 \]
\[ g_{\odot} = 1.9 \quad X_0 = 0.5 \]

\[ m/m_{\text{Edd}} \]

\[ \text{Period (minutes)} \]

\[ 1 \quad 1.01 \quad 1.2 \quad 1.25 \quad 1.43 \quad 1.81 \quad 1.82 \]

\[ 0.5 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

Fig. 3.—Trajectory in the temperature–column depth plane for \( \dot{m} = 0.998 \dot{m}_{\text{Edd}} \), as shown in Fig. 2, but zooming in on the oscillations around the steady burning location.

Fig. 4.—Oscillation period as a function of accretion rate in the one-zone model, for different choices of surface gravity and accreted hydrogen fraction. At each accretion rate, we integrate the model for \( 10^6 \) s and plot the mean oscillation period, only including data for \( t > 100 \) minutes. The filled symbols indicate models for which the oscillations grow and reach a steady amplitude; the open symbols indicate models for which the oscillations are damped.
frame is a good approximation. General relativity may be accounted for using appropriate redshift factors, as discussed in § 4.4 of Woosley et al. (2004). The results we provide here do not include these redshift corrections; the simulated conditions apply for different combinations of neutron star radius and mass that give the same surface acceleration in the local frame, using appropriate scalings of the surface area, accretion rate, and luminosity.

Our code includes an adaptive nuclear reaction network that automatically adjusts to include or remove isotopes as needed to follow the details of the nucleosynthesis, out of a reaction rate library of about 5000 nuclei (Rauscher et al. 2002). The calculations presented here use up to 1300 different isotopes. The reaction rate library includes recent measurements and estimates of critical nuclear reaction rates. The effect of uncertainties in these rates are discussed in detail in Woosley et al. (2004). The energy generation from the reaction network is directly and consistently coupled into the implicit hydrodynamic solver to allow the structure of the layers to adjust; in the present simulations, the structure remains close to hydrostatic, but the stratification changes due to burning, heating, and cooling. The numerical grid adaptively refines and derefines the Lagrangian grid to resolve gradients, but in the hydrogen-rich layer, in effect it is essentially at constant mass resolution; i.e., linear in column depth. Accretion is modeled by periodically adding an extra zone at the surface of the star (of column depth \( \frac{1}{20} \) of \( \frac{1}{25} \); Woosley et al. 2004). We follow both the compositional and thermal profiles of the layer, including radiative and convective transport, with a time-dependent mixing-length treatment for convection, semi-convective, and thermohaline convection. We include a base flux of 0.15 MeV nucleon\(^{-1}\).

Here we present results for an accreted material metallicity \( \frac{1}{20} \) of solar. At an accretion rate of \( \dot{m} = 0.1 \dot{m}_{\text{Edd}} \), Woosley et al. (2004; their model zM) found a sequence of regular bursts with recurrence times close to 3 hr. Increasing the accretion rate in our new sequence of models, we find a transition from unstable to stable burning at \( \dot{m} = 0.924 \dot{m}_{\text{Edd}} \). Figure 5 shows the behavior close to the transition accretion rate. The light curves show a progression from regular periodic burning at \( \dot{m} = 0.7 \dot{m}_{\text{Edd}} \) (with recurrence times close to 20 minutes), to a combination of irregular bursts intermixed with oscillations at \( \dot{m} = 0.923 \dot{m}_{\text{Edd}} \), to a regular sequence of oscillations at \( \dot{m} = 0.925 \dot{m}_{\text{Edd}} \), and finally to stable burning at \( \dot{m} = 0.95 \dot{m}_{\text{Edd}} \), with the oscillation amplitude rapidly decreasing as the accretion rate is increased.

The oscillation period at \( \dot{m} = 0.925 \dot{m}_{\text{Edd}} \) is 185 ± 5 s. Figure 6 shows a portion of the light curve at this accretion rate. The oscillations have an asymmetric profile, with the decay lasting twice as long as the rise. Revnivtsev et al. (2001) noted marginal evidence that the peaks of the mHz QPOs were asymmetric, with a steep rise and a shallower decline. A more detailed comparison of our models with observed light curves would be interesting, but clearly there should be significant harmonic components. The peak-to-peak amplitude of the oscillation in Figure 6 is \( \approx 4 \times 10^{23} \text{ erg cm}^{-2} \text{ s}^{-1} \), with a minimum flux of \( \approx 4 \times 10^{23} \text{ erg cm}^{-2} \text{ s}^{-1} \) and a maximum flux of \( \approx 8 \times 10^{23} \text{ erg cm}^{-2} \text{ s}^{-1} \). This is in good agreement with the one-zone model (compare with the middle panel of Fig. 1). For comparison, the steady burning nuclear luminosity is \( \approx 5 \times 10^{23} \text{ erg cm}^{-2} \text{ s}^{-1} \) (see the bottom panel of Fig. 5).

The nuclear burning in the oscillation mode is powered by \( \alpha \)P- and \( \nu \)P-process burning (Wallace & Woosley 1981; Schatz et al. 1999), beginning with seed nuclei produced by breakout reactions from the CNO cycle and terminating at a mass number of \( \approx 80 \) (the most abundant nucleus is \( ^{80}\text{Sr} \)). Figure 6 shows the energy generation as a function of column depth, radius, and time through several oscillation cycles. Figure 7 shows the compositional profile of the layer at different phases of the oscillation cycle. Beneath the hydrogen-burning layer, the composition of the ashes shows periodic variations with a spacing in depth at \( \approx \dot{m}_{\text{Edd}} \), where \( P_{\text{osc}} = 185 \text{ s} \) is the oscillation period. Note, however, that the hydrogen-burning depth is relatively constant during the oscillation cycle. The hydrogen burns at a column depth of \( \approx 10^{8} \text{ g cm}^{-2} \), which is about 7 times larger than the amount of mass accumulated in one oscillation cycle. The underlying ashes record the variations in burning temperature and the resulting oscillation of the rp-process ashes during the oscillation cycle. This is reminiscent of the growth of annual rings in a tree trunk (Schatz et al. 2003). Figure 8 shows that the distribution of nuclei in the ashes by mass number, as well as the slight variation in the composition through the cycle.

At the peak in the oscillation light curve, the increased temperature in the burning region drives heat transport inward, heating the underlying material. This is similar to the substrate heating during type I X-ray bursts discussed by Woosley et al. (2004), and it leads to an increase of burning of the \( ^{4}\text{He} \) remaining in the ashes layer by the \( \alpha \)-process and by alpha-captures. In Figure 6 this can be seen as additional spikes in nuclear energy generation below the main burning band of the hydrogen-rich zone. Of particular interest is the amount of carbon remaining in the ashes. Stable burning of accreted H/He has been suggested as the source of carbon fuel for superbursts (\( ^{4}\text{He} \) Zand et al. 2003; Schatz et al. 2003). Figure 8 shows that the amount of carbon at \( y \approx 2 \times 10^{8} \text{ g cm}^{-2} \) is \( \approx 2 \% \) by mass, which is very close to the asymptotic value of slightly below 2% in the deeper layers where all the helium has been burnt.

The layering of different compositions in the ashes does not persist to great depths. The different layers have different values...
of the number of electrons per baryon, $Y_e$, which determines the specific weight of fluid elements under the degenerate conditions in the ashes layer. The variation in composition is stable to the Rayleigh-Taylor instability because of the thermal buoyancy. Secular doubly diffusive instabilities, however, cannot be suppressed. We find that the thermohaline or salt-finger instability...

Fig. 6.—Detailed light curve (top) and specific nuclear energy generation as a function of time and column depth (second from top). In the bottom three panels, each darker shading of blue corresponds to a value of energy generation that is one order of magnitude higher; see the scale on the right-hand side of the figures. In the second panel from the top we label each depth with a Lagrangian column depth. Following a given column depth to the right shows the evolution of that fluid element in time. The sloping black line indicates the surface of the star (the slope gives the accretion rate). The bottom two panels show the evolution as a function of radius coordinate. Zero is chosen to correspond to the location where hydrogen is depleted. The upper panel of the two gives the specific nuclear energy generation rate (same as the panel above), and the lower panel gives the energy generation rate per unit depth.

Fig. 7.—Snapshots of structure (temperature, thick gray line; density, thick gray dashed line; specific nuclear energy generation, black line) and composition (select isotopes, colored lines) during one oscillation cycle; each panel is advanced in time by $P/4$, where $P$ is the oscillation period; thus, the bottom panel is advanced by one full cycle. The bottom axis for each figure gives the column depth, and the top axis gives the corresponding time since the accretion began. This is the same model as shown in panel $c$ of Fig. 5. The white and gray stripes correspond to one cycle of oscillation each, with the interfaces corresponding to the time of a maximum in the light curve at the time of the accretion of that layer. The small inserts in the upper right corners indicate the position in the light curve (red) cycle of the snapshot (black dot; intentionally aligned with a layer interface). Note that the decreases of some of the radioactive isotopes on the right-hand side of the figure is due to their radioactive decay. [Four animations for this figure are available in the electronic edition of the Journal: an animated version of this exact figure; an animation showing all isotopes, but with the isotopes in this figure shown as thick lines; an animation showing all isotopes as thin lines; and an animation showing all isotopes as thick lines, but with better time resolution and in "real time", i.e., the movie time corresponds to the actual time that the oscillations take in nature.]
burning using a multizone code. The period, amplitude, and shape of the oscillations agree well in both models. Remarkably, the basic physics of the oscillations is the same physics as that underlying a nonlinear relaxation oscillator such as the van der Pol oscillator (e.g., Abarbanel et al. 1993). Usually, the positive or negative damping term dominates, giving rise to the familiar X-ray bursts at low accretion rates or stable burning at a fixed temperature and density at high accretion rates. Close to the marginally stable point, however, the effective thermal timescale is very long, allowing the underlying oscillation period of the system to be seen. This period is close to the geometric mean of the thermal time and accumulation time of the burning layer (eq. [11]). This behavior naturally reproduces three properties of the mHz QPOs observed by Revnivtsev et al. (2001): the observed periods of ≈2 minutes, the fact that mHz QPOs were observed in only a narrow range of luminosities in 4U 1608−52, 0.5−1.5 × 10^{37} erg s^{−1}, and the fractional amplitude of the QPOs, which correspond to flux variations of ≈1%−2% (Revnivtsev et al. 2001; comparable to the ratio of nuclear energy to gravitational energy release).

Identification of the mHz QPOs with marginally stable nuclear burning would for the first time relate a feature of the persistent X-ray emission to the neutron star surface in sources that are not X-ray pulsars. Further, our one-zone models indicate that the oscillation period is very sensitive to the surface gravity and the accreted hydrogen fraction (Fig. 4). One of the difficulties in comparing X-ray burst properties with theoretical models is the uncertain relation between X-ray luminosity and accretion rate (e.g., Cumming 2003). This uncertainty is removed for marginally stable burning because it occurs at a specific accretion rate. The dependencies on surface gravity and hydrogen fraction need to be confirmed with multizone models, although unfortunately scanning the parameter space for the location of the transition is computationally very expensive. Our one-zone model results (Fig. 4) suggest that the observed periods of ≈2 minutes require either $X_0 < 0.7$, as predicted by intermediate-mass evolution models (e.g., Podsiadlowski et al. 2002), or a surface gravity of $g_{14} \approx 3$, such as that corresponding to a 2 $M_\odot$ (gravitational mass) star with $R = 11$ km. This can be seen in Figure 9, which shows the oscillation period as a function of $X_0$ and $g_{14}$. In this figure, we have rescaled the one-zone model results to match the multizone model for $X_0 = 0.7$ and $g_{14} = 1.9$; in addition, we include the gravitational redshift factor. Future comparisons of theoretical models with mHz QPO periods and light curves are potentially sensitive probes of the surface gravity and accreted composition.

Two observed features of mHz QPOs remain to be explained. The first is the $Q$-value of the oscillation, which Revnivtsev et al. (2001) found to be $Q \equiv \nu/\Delta \nu \approx 3−4$. This may be related to the range of accretion rates for which oscillatory nuclear burning can be observed. Revnivtsev et al. (2001) constrained the range of luminosities at which mHz QPOs are present to be 0.5−1.5 × 10^{37} erg s^{−1} in 4U 1608−52. In contrast, the theoretical range in which oscillations are seen is much smaller, within a range of $\Delta m/m \approx 0.01$ around the transition accretion rate. Moreover, for much of this range, the oscillations decay in time. Further observations that constrain the range of luminosities for which mHz QPOs can be observed would be valuable.

Gravitational redshift increases the oscillation periods shown in Fig. 4 by $\approx 30\%$. The numerical results in § 3, however, suggest that the one-zone model overpredicts the oscillation period by a similar factor. Therefore, the oscillation periods from the one-zone model without redshifting are in fact approximately those we expect from multizone models corrected for gravitational redshift.

4. DISCUSSION

The fact that oscillatory burning is naturally expected at the transition from unstable to stable nuclear burning was pointed out by Paczyński (1983). We have investigated the properties of marginally stable burning in this paper with a simplified one-zone model (§ 2) and with detailed multizone simulations (§ 3) using the KEPLER code, extending the type I X-ray burst calculations of Woosley et al. (2004) to higher accretion rates. This is the first detailed study of the transition from unstable to stable

![Figure 8](source URL)
The second puzzle is that our theoretical models, in agreement with previous estimates (Fujimoto et al. 1981; Ayasli & Joss 1982; Bildsten 1995), find that the transition to stable burning occurs at a local accretion rate close to the Eddington rate ($\dot{m}_{\text{Edd}}$) for the model presented in § 3. In contrast, the X-ray luminosity at which the mHz QPOs are observed in 4U 1608−52 is $\approx 0.5 \times 10^{37}$ erg s$^{-1}$, implying a global accretion rate that is only a tenth of the Eddington rate: $\dot{m} \approx L_X/(GM/R) \approx 0.1 \dot{m}_{\text{Edd}}$. Because the burning layer is very thin, the properties of the burning depend only on the local accretion rate, which may vary across the surface of the star. Therefore, a simple explanation for this discrepancy is that the accreted material is confined at the burning depth to only $\approx 10\%$ of the stellar surface, in which case the local accretion rate onto the star could lie close to the Eddington rate, even though the global accretion rate is much lower.

The physics that might cause confinement of the fuel onto a small fraction of the surface of the star is not obvious. The pressure at the base of the burning layer is $P = gy = (10^{22}$ erg cm$^{-3}$)$g_{14} \rho_{25}$, implying that magnetic fields of strength approaching $\approx 8 \times 10^5$ G would be required to confine the fuel. This is much larger than the $\approx 10^8$−$10^9$ G fields assumed for the neutron stars in low-mass X-ray binaries (LMXBs; believed to be the progenitors of the millisecond radio pulsars; Bhattacharya et al. 1995), although small-scale fields of these strengths might exist on the surface. The need to transport angular momentum could also potentially delay spreading of material accreted from a disk onto the equator of the star. Inogamov & Sunyaev (1999) studied this problem with a one-zone model of the spreading layer and a basic prescription for angular momentum transport. They found column depths of $< 10^4$ g cm$^{-2}$ in the spreading layer, much smaller than the burning depth.

The possibility that the covering fraction changes with the accretion rate was suggested previously by Bildsten (2000), but in the opposite sense, with the covering fraction increasing with $\dot{m}$. The motivation was to explain a puzzling change in burst behavior that is observed to occur at a luminosity of $\approx 10^{37}$ erg s$^{-1}$. EXOSAT observations of several atoll sources showed that as X-ray luminosity increased, burst properties changed from regular, frequent bursts ([$t_{\text{rec}}$ $\approx$ hours]) with energetics consistent with burning all of the accreted fuel in bursts, to irregular, infrequent ($t_{\text{rec}}$ $> 1$ day) bursts whose energetics indicate that only a small fraction of the fuel burns in bursts (van Paradijs et al. 1988). RATE and BeppoSAX observations confirmed this result, with particularly good coverage for the transient source KS 1731−260 (Muno et al. 2000; Cornelisse et al. 2003). Cornelisse et al. (2003) found that observations of nine bursters with BeppoSAX were consistent with this pattern of bursting, with a universal transition luminosity of $L_X \approx 2 \times 10^{37}$ erg s$^{-1}$.

This change in bursting behavior is not predicted by the standard theory, in which regular bursting should continue up to the stability boundary at $\dot{m} \approx \dot{m}_{\text{Edd}}$. Several theoretical explanations for the discrepancy were put forward, including mixing by Rayleigh–Taylor (Wallace & Woosley 1984) or shear instabilities (Fujimoto et al. 1987), which might allow more rapid burning of hydrogen, a new mode of burning involving slowly propagating fires at $\dot{m} \approx 0.1 \dot{m}_{\text{Edd}}$ (Bildsten 1993), or that the covering fraction of accreted material increases at higher accretion rates, lowering the accretion rate per unit area and lengthening the time between bursts, giving hydrogen time to burn stably (Bildsten 2000). We also mention here that Narayan & Heyl (2003) calculated linear eigenmodes of truncated steady state burning models and found stability for accretion rates of $\dot{m} > 0.25 \dot{m}_{\text{Edd}}$, more consistent with observations. The lower accretion rate is likely the cause of the small oscillation frequencies that they found for marginally stable burning (period of $\approx 20$ minutes). We find, however, that bursting continues unabated up to $\dot{m} \approx \dot{m}_{\text{Edd}}$. Further work comparing linear stability analysis with numerical calculations seems to be required.

The observations of mHz QPOs at a luminosity close to $\approx 10^{37}$ erg s$^{-1}$ and their interpretation as marginally stable oscillatory burning at a local accretion rate $\dot{m} \approx \dot{m}_{\text{Edd}}$ provide new input for these ideas. As we have discussed, if changes in the covering fraction are responsible, this suggests that the covering fraction decreases rather than increases at this luminosity. The observed type I bursts at $L_X > 10^{37}$ erg s$^{-1}$ could be accommodated if there was a slow “leak” of fuel away from the stably burning region. This fuel would deplete hydrogen as it accumulated, giving occasional short helium-rich bursts. Other mechanisms, such as stable burning driven by mixing of fuel by shear instabilities, might also lead to oscillatory burning. More theoretical work is needed. One clue is that the ultracompact source 4U 1820−30, which most likely accretes pure helium (Bildsten 1995; Cumming 2003), shows a similar transition at a similar luminosity, implying that the nature of the transition does not depend on accreted composition.

This question is also likely to be relevant for superbursts and type I burst oscillations. Superbursts are long-duration, rare, and extremely energetic type I X-ray bursts (up to 1000 times the duration and energy and less frequent than normal bursts) that are believed to be due to unstable carbon ignition (Cumming & Bildsten 2001; Strohmayer & Brown 2002). Superbursts are only observed at luminosities above $\approx 10^{37}$ erg s$^{-1}$ and from sources for which burst energetics indicate that bursts burn only a small
fraction of the accreted fuel (in ’t Zand et al. 2003). This fits nicely with the theoretical result that stable burning is much more efficient than unstable burning at producing carbon fuel (Schatz et al. 2003). How to achieve stable burning theoretically at $\dot{m} < \dot{m}_{\text{Edd}}$, however, has been an open question. Type I burst oscillations are high-frequency oscillations during type I X-ray bursts that are believed to be due to burning asymmetries on the surface. Burst oscillations are preferentially seen at higher accretion rates, in the banana branch of the color–color diagram (e.g., Muno et al. 2000). Incomplete covering of the surface might help to explain the origin of burst oscillations, facilitating inhomogeneous burning when type I X-ray bursts are able to occur.

Finally, we note that atoll sources undergo a transition from the island state to the banana branch in the color–color diagram at a luminosity of $L_X \approx 10^{37}$ erg s$^{-1}$. It is well known for these sources that the X-ray luminosity does not track the accretion rate on short timescales (van der Klis 2001). One explanation for the transition from the island state to the banana branch in atoll sources is that a hot quasi-spherical accretion flow at low rates is replaced by a thin disk at high rates (e.g., Gierliński & Done 2002). If this picture is correct, it could be that the change in accretion geometry affects the distribution of fuel on the neutron star surface. Our results suggest that when the mHz QPOs are observed, the local accretion rate is uniform to within roughly 1 part in 100 over approximately 10% of the neutron star surface. An interesting observational question is whether the appearance of the mHz QPOs is linked to a particular luminosity range or a particular part of the color–color diagram, such as the island to banana transition.

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