Instability of a horizontal water half-cylinder under vertical vibration

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We present the results of an experimental investigation on parametrically driven waves in a water half-cylinder on a rigid horizontal plate, which is sinusoidally vibrated in the vertical direction. As the forcing amplitude is raised above a critical value, stationary waves are excited in the water half-cylinder. These curvature controlled waves are non-axisymmetric and qualitatively different from the axisymmetric Savart-Plateau-Rayleigh waves in a vertical liquid cylinder or jet. Depending on the driving frequency, stationary waves of different azimuthal wave numbers are excited. A simple linear theory is also supplemented, which captures the observed dispersion relations quantitatively.

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The parametric excitation of stationary waves on the free surface of an extended layer of liquid is known since the pioneering work of Faraday [1]. The excited waves are subharmonic, i.e., the frequency of such waves is equal to half the driving frequency. These waves may also be synchronous with the driving in a thin viscous layer [2]. Faraday instability leads to interesting fluid patterns [3]. On the other hand, capillary instability of a vertical liquid cylinder or a jet is known since the beautiful experiments of Savart [4] and Plateau [5]. Rayleigh investigated theoretically the instability of liquid jets [6]. A vertical liquid cylinder or jet develops an axisymmetric bead like structure, which ultimately leads to breaking of the jet into detached liquid drops [7]. Plateau remarked that a liquid jet was stable for all purely non-axisymmetric deformations; but was unstable for axisymmetric varicose deformations with wavelengths exceeding the circumference of the cylinder [8]. Other experiments [9, 10] also confirm the instability of a vertical jet through axisymmetric perturbations (see for a lucid review [11]).

We present, in this article, results of an experiment that allows excitation of only non-axisymmetric waves on a horizontal water half-cylinder under sinusoidal vibration in the vertical direction. This simple and novel experiment combines two fluid instability problems, viz. Faraday instability, which breaks the time reversal symmetry, and features of curvature controlled Savart-Plateau-Rayleigh instability [4–6]. A long water half-cylinder has translational symmetry along one direction and its base sticks to plate due to no-slip condition. It is therefore qualitatively different from the classical Faraday experiment, the Savart-Plateau-Rayleigh problem and a vibrating spherical liquid drop [12, 13] problem. As the amplitude of driving is raised above a critical value in our experiment, depending upon the driving frequency, three different types of subharmonic stationary waves are excited: half-bead like structure, waving half-cylinder and complex knitting patterns. These waves are non-axisymmetric, unlike the excited modes of a vertical liquid cylinder [7, 8]. The dispersion curves show windows of frequencies where stationary waves are not sustained at the primary instability. A simplified linear theory for this problem is also presented explaining the gross features of the observed dispersion curves.

Figure 1. (Color online) Schematic diagram of the experimental setup: (a) Side view seen along the axis of the water half-cylinder. (b) Top view of the half-cylinder [blue (dark gray) color] surrounded by a hydrophobic painted region [yellow-green (gray) color] on the plate. (c) Side view from the direction normal to the cylinder axis.

A schematic diagram of the experimental setup is shown in Fig. 1. The flat surface of a square plexiglass plate (20 × 20 cm²) of thickness 1 cm is coated with a super-hydrophobic paint (Rust-Oleum 275619 Never-Wet Nano) except for a rectangular region in the middle. This plate is then rigidly fixed to the vibrating base of an electromagnetic shaker [Model no. V350, Data Physics Corporation (DPC)]. A vibration controller (SignalStar Scalar Vibration Controller 2.4.998, DPC) connected to a power amplifier (DSA5-2K, DPC), sets the amplitude and frequency of the electromagnetic shaker. A small accelerometer (256HX-10, Isotron) is attached to the base of the plate to monitor its oscillations. Sufficient amount of distilled water was poured on the unpainted region of the plate drop by drop using a syringe so that a horizontal water half-cylinder is formed. To maintain the shape and size of the water half-cylinder against evaporation during the experiment, its shadow length was monitored using a camera (Basler scA1000-30gm) fixed above the water half-cylinder. A few drops of distilled
water were added every 10 minutes to compensate the water evaporation. The temperature of the laboratory during the experiment was maintained at 24 °C ± 1 °C. Diffused LED lighting was used for illumination from two sides of the half-cylinder. The dynamics of fluid patterns was captured using a high speed camera (Chronos 1.4, Kron Technologies, Canada), which has a resolution of 1280 x 360 at the rate of 2999 fps. The camera was so placed that its axis made an angle $\theta = 60^\circ$ or 40° with the horizontal plane. The wavelengths of different stationary waves were extracted from captured images using a software (Tracker 4.11.0).

Experiments were performed for five different water half-cylinders of length $L$ (10.0 cm ≤ $L$ ≤ 15.0 cm) and radius $R$ (2.0 mm ≤ $R$ ≤ 4.0 mm) and repeated at least three times for each case. Qualitatively similar results were obtained in all cases. The left column of Fig. 2 displays experimentally captured images for a water half-cylinder of $L = 10.0 \pm 0.1$ cm and $R = 3.1 \pm 0.1$ mm, while the second column displays computer generated patterns, using the theoretical wavefunctions (see Eq. 3). Viewing angle, $\theta$, was identical for corresponding images of both the columns. The value of $\theta$ for the static case [Fig. 2(a)] and patterns with azimuthal wave number $m = 2$ [Fig. 2(d)-(e)] was 60°, while that for patterns with $m = 1$ [Fig. 2(b)-(c)] and $m = 3$ [Fig. 2(f)-(g)] was 40°. Small dashed lines between the experimental and computer generated images mark the actual boundaries of water surfaces. Fig. 2(a) and (i) show the static water half-cylinder before the plate was subjected to vibration. The acceleration amplitude $a$ of the oscillating plate was slowly raised in small steps at a fixed driving frequency $f$, and the excitation of waves was recorded by the camera. Then the driving frequency was raised in steps of 2 Hz and the same procedure was repeated. The error in $f$ is 0.01 Hz, which is negligibly small.

As $a$ was raised above a critical value $a_c(f)$ for a fixed value of $f$, the static water half-cylinder became unstable. Fig. 2(b)-(g) show the snapshots of a portion of the water half-cylinder for different values of $a$ ($> a_c$) and $f$ (20 Hz ≤ $f$ ≤ 110 Hz). The stroboscopic light determined the frequency of the stationary waves. For a range of frequencies (20 Hz ≤ $f$ ≤ 28 Hz), we observed subharmonic (period, $\tau = 2/f$) excitation of stationary waves in the form of a periodic chain of half-beads (horizontally chopped off), which were invariant under mirror reflection about a vertical plane through the cylinder axis. Fig. 2(b)-(c) show the two phases of a portion of the chain of half-beads at two instants of time separated by half the period of excited waves for $f = 20$ Hz and $a = 0.56 \pm 0.01g$, and Fig. 2(ii)-(iii) display computer generated images for azimuthal wave number $m = 1$. As the base of half-beads is fixed to the plate, the vertical oscillations make the transverse cross-sections of half-beads non-axisymmetric. It is qualitatively different from axisymmetric beads ($m = 0$) observed in a vertical liquid cylinder [8]. The axisymmetric mode ($m = 0$) is never excited in our case. Stationary waves were not observed for 30 Hz ≤ $f$ ≤ 38 Hz at the primary instability, as they did not fit the half-cylinder. The water half-cylinder showed a state of frustration and displayed irregular spatio-temporal dynamics at the primary instability in this case.

As $f$ was raised further, we observed again excitation of stationary waves for 40 Hz ≤ $f$ ≤ 72 Hz. The water half-cylinder started waving subharmonically. The resulting pattern broke the mirror symmetry about the vertical plane through the cylinder axis. The waving cylinder showed glide symmetry, i.e., the fluid pattern was invariant under a translation by $\lambda/2 = \pi/k$ along the cylinder axis followed by a mirror reflection about the vertical plane through the axis. Fig. 2(d)-(e) show the two phases of the waving half-cylinder at an instant and half the period later for $f = 60$ Hz and $a = 0.85 \pm 0.01g$. The
Figure 3. (Color online) Thresholds [(a)-(c)] and dispersion curves [(d)-(f)] for subharmonically excited stationary waves on water half-cylinders. Blue (black), green (light gray) and red (gray) symbols (curves) are experimental (theoretical) data points for waves with \( m = 1, 2 \) and \( 3 \) respectively. Black stars are experimental data points showing irregular spatio-temporal patterns.

corresponding computer generated images [Fig. 2(iv)-(v)] confirm the excitation of waves with \( m = 2 \). Stationary waves were not observed again in a frequency window between 74 Hz to 84 Hz, where the half-cylinder was in a state of frustration at primary instability, as mentioned earlier. As the driving frequency was raised further, we observed subharmonically generated stationary waves in the form of a complex knitting pattern for 86 Hz \( \leq f \leq 110 \) Hz. Fig. 2(f)-(g) show fluid patterns at two instants separated by half the wave period for \( f = 90 \) Hz and \( a = 2.48 \pm 0.01 \)g and Fig. 2(vi)-(vii) display computer generated images for \( m = 3 \). The complex fluid pattern showed mirror symmetry about the vertical plane through the cylinder axis. Computer generated images match nicely with all the three patterns observed on the water half-cylinder.

Figure 3(a)-(c) show the dimensionless threshold \( (a_c/g) \) as a function of driving frequency \( f \) for the instability of horizontal water half-cylinders with different \( L \) and \( R \) combinations. Blue (black) squares, green (light gray) circles and red (gray) triangles are experimental data points for subharmonically excited stationary waves in the form of mirror symmetric half-beads \( (m = 1) \), waving half-cylinder with glide symmetry \( (m = 2) \) and complex knitting patterns with mirror symmetry \( (m = 3) \), respectively. Black stars between \( m = 1 \) and \( m = 2 \) and also between \( m = 2 \) and \( m = 3 \) patterns [Fig. 3(a)-(c)] are experimentally observed data points when the water half-cylinder is in a state of frustration at primary instability. The location and the frequency windows of such states are sensitive to the radius, \( R \), of the half-cylinder. These frustrated spatio-temporal states may bifurcate to an ordered state (stationary waves) at secondary instability at higher values of \( a \) [see the upper set of red (gray) triangles]. These are large amplitude nonlinear states. The fluid patterns generally become irregular at higher values of \( a \). Fig. 3(d)-(f) display the corresponding dispersion relations for subharmonically generated stationary waves for the cases shown in Fig.3(a)-(c), respectively. Similar symbols correspond to the similar fluid patterns and size of symbols include the error bars in both directions.

We now present a linear theory to understand the dispersion relation for excitation of capillary waves on a long horizontal water half-cylinder of radius \( R \), density \( \rho \) (= 1 gm/cc) and surface tension \( \sigma \) (= 72 dynes/cm). As water viscosity is small, it is ignored. The determination of acceleration threshold requires a theory of viscous liquid, but the dispersion relations for low viscosity fluids may be computed using this theory. A cylindrical coordinate system is chosen with the \( z \)-axis coincident with the axis of water half-cylinder with origin at any point on the axis. The angle \( \phi \) is measured from the line of intersection of the \( r\phi \)-plane with the horizontal plate surface in counter-clockwise direction. The pressure \( P_0 \) inside the undeformed vibrating water half-cylinder is given by \( \nabla P_0 (r, \phi, t) = \rho G(t) \hat{y} = -\rho (g + a \cos \omega t) \hat{y} \), where \( \omega = 2\pi f \) and \( \hat{y} \) is a unit vector along the vertical direction. The pressure jump across the static cylindrical fluid surface is equal to \( \sigma / R \). As the flat plate starts oscillating, the cylindrical free surface is deformed. A
liquid particle on the curved surface experiences acceleration along its normal and tangential directions. This makes an accurate analysis of the problem difficult. We follow a simplification used recently for a spherical liquid drop [13] by considering the external acceleration \( \mathbf{G} \) only along the radial direction, i.e., \( \mathbf{G}(t) = G(t) \mathbf{r} \), in the perturbation equations. As soon as waves are excited, the velocity field \( \mathbf{u}(r, \phi, z, t) \) develops in the water half-cylinder. It may be written as \( \mathbf{u} = \nabla \Phi \), where \( \Phi(r, \theta, z, t) \) is the velocity potential. The modified pressure in water half-cylinder is \( P(r, \phi, z, t) = P_0(r, \phi, t) + p(r, \phi, z, t) \), where \( p \) is the deviation in pressure field from \( P_0 \) due to instability. The free surface is now located at \( r_s(\phi, z, t) = R + \zeta(\phi, z, t) \), where \( \zeta(\phi, z, t) \) is the surface deformation. The incompressibility condition of the liquid leads to Laplace equation for \( \Phi \), i.e., \( \nabla^2 \Phi = 0 \). The kinematic condition at the curved surface reads as:

\[
\frac{\partial \zeta}{\partial t} = \mathbf{u}_r|_{r=R} = \frac{\partial \Phi}{\partial r}|_{r=R}. \tag{1}
\]

The pressure jump across the free surface now reads as:

\[
\frac{\partial \Phi}{\partial t}|_{r=R} = G(t) + \frac{\sigma}{\rho} \left( \frac{\zeta}{R^2} + \frac{1}{R^2} \frac{\partial^2 \zeta}{\partial \phi^2} + \frac{\partial^2 \zeta}{\partial z^2} \right). \tag{2}
\]

The velocity potential \( \Phi \) and the surface deformation \( \zeta \) are expanded as:

\[
(\Phi; \zeta) = \sum_{m=1}^{\infty} (\Phi_m(t) I_m(kR); \zeta_m(t)) \sin(m \phi) e^{ikz}, \tag{3}
\]

where \( I_m(kR) \) is the \( m^{th} \) order modified Bessel function of the first kind. In the actual experiment, the no-slip condition at the base and the hydrophobic paint outside the base of the water half-cylinder lead to \( \zeta(0, z, t) = \zeta(\pi, z, t) = 0 \). These requirements force \( \sin(m \alpha) \) term in the expansion. Insertion of Eq. 3 in Eq. 2 and use of Eq. 1 yield a Mathieu equation for \( \zeta_m \):

\[
\ddot{\zeta}_m + \omega_m^2 \left[ 1 + (a/a_m) \cos(\omega t) \right] \zeta_m = 0, \tag{4}
\]

where \( \omega_m^2 = \frac{kI_m(kR)}{I_m(kR)} \left[ g + \frac{\sigma}{\rho R^2} (k^2 R^2 + m^2 - 1) \right], \quad a_m = \frac{I_m(kR)}{kI_m(kR)} \omega_m^2. \) The expression for \( \omega_m \) is exactly the same as the dispersion relation for capillary instability of a vertical liquid cylinder [6] in the absence of gravity. Floquet expansion of \( \zeta_m(t) \) is given as:

\[
\zeta_m(t) = e^{(s+i\omega)t} \sum_{n=-\infty}^{\infty} \zeta_m^{(n)} e^{in\omega t}, \tag{5}
\]

where \( s \) is the growth rate and \( \alpha \) is the Floquet exponent. Insertion of Eq. 5 in the Mathieu equation (Eq. 4) then leads to a difference equation:

\[
A_m^{(n)} \zeta_m^{(n)} = a \left( \zeta_m^{(n-1)} + \zeta_m^{(n+1)} \right), \tag{6}
\]

where

\[
A_m^{(n)} = -\frac{2I_m(kR)}{kI_m^2(kR)} \left[ \omega_m^2 - (n + \alpha)^2 \omega_m^2 \right]. \tag{7}
\]

This difference equation can be converted to an eigenvalue matrix equation [2]. Real and positive eigenvalues of the matrix can be determined as a function of wave number \( k \) for fixed values of \( \omega \) and \( m \). This gives marginal \((a-k)\) curve (the growth rate, \( s = 0 \) ) for given values of the azimuthal wave number \( m \). The stationary waves are excited subharmonically only when \( \omega_m = \omega/2 \).

We set \( \alpha = 1/2 \) and \( s = 0 \), as we are interested in subharmonically excited waves at the instability onset. The minimum of the \( a-k \) curve gives the instability threshold and the critical wave number for subharmonically excited stationary waves for fixed values of \( \omega \) and \( m \). The dispersion curves can be computed by varying \( \omega \) in small steps and finding the critical values of \( k \) for a fixed value of \( m \). Solid (dashed) curves are dispersion relations computed for water half-cylinders of different radii without (with) \( g = 980 \text{ cm/s}^2 \) for different values of \( m \) see, Fig. 3(d)-(f)]. Blue (black), green (light gray) and red (gray) colored curves correspond to \( m = 1, 2 \) and 3, respectively. The primary instability wavelengths match nicely with theoretical predictions even at higher frequencies with the continuous curves computed from the theory without \( g \). The inclusion of gravity in the model makes the fit worse.

The instability observed is therefore primarily curvature controlled. The slight deviations of experimental data points, from the theoretical dispersion curves at higher driving amplitudes \((m = 2 \text{ and } 3)\) are for stationary waves observed at the secondary instability and they naturally do not match so well due to nonlinear effects. This requires a nonlinear theory perhaps with viscosity.

A horizontal water half-cylinder under vertical vibration becomes unstable and leads to only nonaxisymmetric \((m \neq 0)\) subharmonic stationary waves, which are new and qualitatively different from the axisymmetric patterns \((m = 0)\) of Savart-Plateau-Rayleigh instability in vertical liquid columns and jets. These curvature controlled waves possess either mirror symmetry \((m \text{ odd})\) or glide symmetry \((m \text{ even})\), which distinguish them from the waves observed in a planer Faraday system or in spherical liquid drops under vertical driving. Excitation of different azimuthal modes at different values of the driving frequency leads to different fluid patterns. Water half-cylinder oscillates as a periodic chain of halfbeads with mirror reflection \((m = 1)\) at lower frequencies. It becomes a wavy half-cylinder with glide symmetry \((m = 2)\) at slightly higher frequencies. It shows a complex knitting pattern with mirror symmetry \((m = 3)\) at relatively higher frequencies. The water half-cylinder is in a state of frustration, if standing waves do not fit on the half-cylinder of a finite length. This simple problem and its results are of considerable academic interest with a wide range of possible applications in diverse areas [11].
This has possible potential applications in material processing, microfluidic flows, fluid atomization, coating and drug mixing to name a few.

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