Irreducible many-body correlations in topologically ordered systems

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Abstract

Topologically ordered systems exhibit large-scale correlation in their ground states, which may be characterized by quantities such as topological entanglement entropy. We propose that the concept of irreducible many-body correlation (IMC), the correlation that cannot be implied by all local correlations, may also be used as a signature of topological order. In a topologically ordered system, we demonstrate that for a part of the system with holes, the reduced density matrix exhibits IMCs which become reducible when the holes are removed. The appearance of these IMCs then represents a key feature of topological phase. We analyze the many-body correlation structures in the ground state of the toric code model in external magnetic fields, and show that the topological phase transition is signaled by the IMCs.

Topologically ordered phases may not be characterized by any local order parameter associated with Landau’s symmetry breaking picture [1]. How to characterize this type of exotic phase is one of the biggest challenges in modern condensed matter physics. These topological phases may be characterized by many distinguished features, including: the degeneracy of ground states depends on the topology of the manifold that supports the system; the existence of anyonic elementary excitations; the existence of edge states on the open boundaries. Moreover, all these properties characterizing topological phase must be stable against local perturbations, making topologically ordered systems promising candidates for fault-tolerant quantum computing [2, 3].

The topological entanglement entropy is firstly proposed to characterize the ground state with topological order by Kitaev–Preskill [4] and Levin–Wen [5], which builds a nontrivial connection between many-body physics and quantum information theory. The underlying picture of the topological entanglement entropy is to ‘retrieve’ a many-body correlation which cannot be built up from its parts. In general, calculating with large enough parts of the system, the entanglement entropy is successfully used to identify topological order in several microscopic models [6–8].

In this paper, we provide a novel perspective to signal this many-body correlation in topologically ordered ground states, building on the concept of irreducible many-body correlation (IMC) [9, 10]. This approach intuitively sounds, as irreducible r-body correlation is nothing but the correlation that cannot be build up from any ≤(r − 1)-body correlations [9, 10]. (See appendix A for the definition of IMC.) In a topologically ordered system, we demonstrate that for a region of the lattice with holes, the reduced density operator exhibits IMCs.

To be more precise, for an n-body quantum state and any r ≤ n, the irreducible r-party correlation (or irreducible correlation of order r) characterizes how much information is contained in the r-particle reduced density matrices (r-RDMs) but not in the (r − 1)-RDMs. For a state σ, denote C(r)(σ) its r-party irreducible correlation. Here the reduced state for a region L ⊆ G is defined by ρL = TrG\Lρ, where ρ is the state on the region G, and G\L is the complement set of L relative to G. The total correlation is then the information contained in the state beyond that in the 1-RDMs. In this sense the irreducible r-party correlations provide a natural hierarchy of correlations in the system—the sum of all the irreducible r-party correlations equals the total correlation [10].
Throughout the paper we consider lattice spin models. For any regions $\mathcal{A}$, $\mathcal{B}$ of the lattice, with $\mathcal{B} \subseteq \mathcal{A}$, one naturally expects that there are more correlations in $\mathcal{A}$ than those in $\mathcal{B}$, as all the particles in $\mathcal{B}$ are contained in $\mathcal{A}$. This is in general also the case for IMCs. However counter-intuitively, in topologically ordered systems, one could have

$$C^r(\rho^A) < C^r(\rho^B),$$

where $\rho^A$ ($\rho^B$) is the reduced state of the region $\mathcal{A}$ ($\mathcal{B}$). The extreme case could be that $C^r(\rho^A) = 0$ but $C^r(\rho^B) > 0$. This may happen when $\mathcal{B}$ and $\mathcal{A}$ have different topology (e.g., $\mathcal{A} \setminus \mathcal{B}$ is a hole). In this case when we consider the correlations in region $\mathcal{A}$, the $r$-party correlation in $\rho^B$ must become reducible, i.e., the information in the $r$-RDMs of the region $\mathcal{B}$ is contained in the information of the $r'$-RDMs of the region $\mathcal{A}$, with $r' < r$. It is worthy to note that the total correlation in the reduced state of the region $\mathcal{A}$ is always not less than that of the region $\mathcal{B}$.

It turns out that the appearance of these IMCs in a region with holes, or the validity of equation (1), represents a key feature of topologically ordered systems. This key feature, the emergence of long-range interaction on subregions, is also known for probability distributions (see for example corollary 7 of [11]). We will analyze these IMC structures in the ground state of the toric code model in an external magnetic field. In addition to demonstrating the appearance of these irreducible correlations and the nontrivial phenomena of equation (1) in these systems, we show that the topological phase transition is also identified by the creation of these IMCs.

**Results**

The Toric code mode—We start with the toric code model [2], which is a spin-$\frac{1}{2}$ model on an $L_w \times L_h$ square lattice, with every edge representing a spin, hence there are total $n = 2L_wL_h$ spins. Without losing of generality, we always assume that $L_w \geq L_h$. The Hamiltonian is

$$H_{\text{tot}} = -\sum_s A_s - \sum_p B_p,$$

where $s$ runs over all vertices (stars) and $p$ runs over all faces (plaquettes). The star operator $A_s = \prod_{\theta \in \partial s} \sigma^z_\theta$, where $\partial s$ is the set of edges surrounding the vertex $s$. The plaquette operator $B_p = \prod_{\theta \in \delta p} \sigma^x_\theta$, where $\delta p$ is the set of edges surrounding the face $p$. In the above formula, the operators $\sigma^x_\theta$ and $\sigma^z_\theta$ are Pauli operators of the $j$th spin.

When the periodic boundary condition is considered (i.e., a torus), the model shows typical features of topological order such as ground state degeneracy. And the degenerate ground state space is a quantum error-correcting code with macroscopic distance (i.e., the distance grows with system size). As the star and plaquette operators commute the ground state space is given by the stabilizer formalism of quantum code

$$A_s |G\rangle = |G\rangle, \quad B_p |G\rangle = |G\rangle.$$  

Furthermore, we have $\prod_s A_s = 1$ and $\prod_p B_p = 1$ for any closed surface, which implies that there exist one dependent star operator and one dependent plaquette operator. The subspace of the ground states is characterized by the two sets of logical operators

$$\sigma^x_\mathcal{C}_v = \prod_{j \in \mathcal{C}_\mathcal{V}} \sigma^x_j, \quad \sigma^z_\mathcal{C}_v = \prod_{j \in \mathcal{C}_\mathcal{V}} \sigma^z_j,$$

$$\sigma^x_\mathcal{C}_h = \prod_{j \in \mathcal{C}_\mathcal{H}} \sigma^x_j, \quad \sigma^z_\mathcal{C}_h = \prod_{j \in \mathcal{C}_\mathcal{H}} \sigma^z_j,$$

where the loop $\mathcal{C}_\mathcal{V}$ ($\mathcal{C}_\mathcal{H}$) is the set of spins on one vertical (horizontal) line of the lattice, and $\mathcal{C}_\mathcal{V}$ ($\mathcal{C}_\mathcal{H}$) is the corresponding one on the dual lattice, which are demonstrated in figure 1. Hence the ground state degeneracy is 4.

IMCs in toric code model—We start from considering the IMC for any ground state $|G\rangle$ of the toric code model shown in figure 2. For the convenience of the following discussion, we use dashed red lines to divide the system into six parts, denoted as $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$, $\mathcal{R}_5$, and $\mathcal{R}_6$ respectively. The state $\rho^R$ is denoted as the RDM of $|G\rangle$ for the region $\mathcal{R}$.

Recall that for any $n$-qubit state $\sigma$, the IMC $C^r(\sigma) = S(\sigma^{(r-1)}) - S(\sigma^{(r)})$, where $\sigma^{(r)}$ is the $r$-qubit state with maximum entropy among the set of the $r$-qubit states that have the same $r$-RDMs as those of $\sigma$, and $S$ is the von Neumann entropy defined as $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ for any quantum state $\rho$, see appendix A for the details of the definitions of irreducible multiparty correlations. And the total correlation $C^r(\sigma) = \sum_n C^r(\sigma) = \sum_{i=1}^n S(\sigma_i) - S(\sigma)$, where $\sigma_i$ is 1-RDM of the $i$th particle.

To analyze the structure of IMCs for any ground state $|G\rangle$ of the toric code model, a basic tool is theorem 2 in [10], which gives the analytical results of IMCs for all the stabilizer states. It states that the degree of $r$-party correlations...
irreducible correlation in an \(n\)-qubit stabilizer state is equal to the number of independent elements with length \(r\) in the state’s stabilizer group when we choose the independent elements with less length as possible.

Since the 1-RDMs of \(G|\psi\rangle\) are maximally mixed, the total correlation of \(G|\psi\rangle\) is \(C_{G|\psi}\) \(= 1\). Furthermore, for any \(4 \leq r < L_0\), the maximally mixed state \(\rho_{\text{mix}}\) supported on the ground-state space has the maximum entropy among all states with the same \(r\)-RDMs as those of \(G|\psi\rangle\). Therefore we have

\[
C^{(i)}(G|\psi\rangle) = \binom{n}{r} - 2, \quad i = 1, 2, \\
C^{(i)\geq L_4}(G|\psi\rangle) = \sum_{i \geq L_4} C^{(i)}(G|\psi\rangle) = 2, \quad i = 3, 4, 5
\]

and the IMCs of all the other orders are 0, see a proof of equation (6) in appendix A.1. This implies that there are 2 bits of IMCs of macroscopic order in any ground state of toric code model on a torus.

This then raises an interesting question: where do these 2 bits of macroscopic correlations come from? To answer this question, we examine in detail the correlations in the local reduced states of the system, which will demonstrate the essential feature of topological order as given in equation (1).

Notice that the RDM of \(G|\psi\rangle\) for a singly connected region of the lattice, e.g., the regions \(R_1\) and \(R_1 \cup R_2\) in figure 2, is independent of the choice of the ground states. Furthermore, the RDM is a stabilizer state, whose
generator elements may all have the shortest length 4, which implies that the reduced state has and only has irreducible 4-party correlations according to theorem 2 in [10]. Therefore the total correlations in the reduced state for a singly connected region, e.g., $\mathcal{R}_1 \cup \mathcal{R}_2$,

$$C^T(\rho^{\mathcal{R}_1 \cup \mathcal{R}_2}) = C^{(4)}(\rho^{\mathcal{R}_1 \cup \mathcal{R}_2}).$$  \hfill (8)$$

However, if the region contains a hole, e.g., the region $\mathcal{R}_3$, then situation could be dramatically different. Because the reduced state $\rho^{\mathcal{R}_3}$ has two generator elements, one is $\prod_j \sigma_j^z$, where $j$ takes over the sites in the loop marked by 14 rectangles, the other is $\prod_k \sigma_k^z$, where $k$ takes over the sites in the loop marked by 10 crosses. According to theorem 2 in [10], we have

$$C^{(10)}(\rho^{\mathcal{R}_3}) = C^{(14)}(\rho^{\mathcal{R}_3}) = 1.$$  \hfill (9)$$

It is worthy to emphasize that equations (8) and (9) can be regarded as a typical example of equation (1) when we take $A = \mathcal{R}_1 \cup \mathcal{R}_2$ and $B = \mathcal{R}_3$.

Intuitively, these IMCs in $\mathcal{R}_3$ are contributed by the correlations in the reduced state of the hole ($\mathcal{R}_4$), which are reduced correlations in the region with the hole ($\mathcal{R}_1 \cup \mathcal{R}_2$). In general, the reduced states for any region in the lattice topologically equivalent to $\mathcal{R}_3$ exhibit IMCs of macroscopic order, which become reducible for the region including the hole. And the existence of these IMCs of macroscopic order in regions with holes is an essential feature of topological order.

When the hole becomes larger and finally encounters the boundary, e.g., the hole $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_s$, the region surrounding the hole splits into two parts, $\mathcal{R}_d^l$ and $\mathcal{R}_d^r$, and 2 bits of irreducible macroscopic correlations emerge in $\mathcal{R}_d^l$ or $\mathcal{R}_d^r$ for the ground state $|G\rangle$ stabilized by $\sigma_i^z$ and $\sigma_j^z$ defined in equations (4) and (5). It is worthy to note that the region $\mathcal{R}_4 = \mathcal{R}_d^l \cup \mathcal{R}_d^r$ is not topologically equivalent to $\mathcal{R}_3$, and the reduced state for the region $\mathcal{R}_4$ depends on the ground state we choose.

**Characterizing topological phase transition**—Based on the discussions above, it is natural to use the IMCs of orders proportional to the boundary length in a region with holes to signal the topologically ordered phase. As a typical example, we consider the toric model in an external magnetic field along the $\vec{n}$ direction [12], with the Hamiltonian

$$H = H_{\text{tor}} - \hbar \sum_i \vec{n} \cdot \vec{\sigma}_i.$$  \hfill (10)$$

Our calculation is based on the numerical exact diagonalization method: to calculate the largest magnitudes eigenvalues, we use the Lanczos algorithm, which is realized in matlab as the function eigs(). Our system consists of 24 spins on a $4 \times 3$ lattice, and the irreducible 6-particle correlation $C^{(6)}(\rho^{\mathcal{R}_3})$ of 6 spins in a region $\mathcal{R}_3$ is studied, see figure 3. Here it is worth noticing that $C^{(6)}(\rho^{\mathcal{R}_3})$ is not a topological invariant, but it does reflect the power of creating higher order correlations from lower order correlations.

In the thermodynamic limit, we expect that $C^{(6)}(\rho^{\mathcal{R}_3}) \approx 0$ for a topological phase, while it is zero for a non-topological phase. This implies that a kind of discontinuity appears in the maximal entropy interference, which is discussed recently, see [13] and references therein. However, the numerical value of correlation might not be topological invariant if calculating with finite size systems. Nevertheless, even for the system of this small size, the rate change of correlation already clearly signals the phase transition. In this sense we suggest to use the maximal changing rate of $C^{(5)}(C^{(5)} = C^{(5)} + C^{(6)})$ with respect to $\hbar$ as an indicator of the phase transition point:

$$\hbar^* = \arg \max_{\hbar} \frac{dC^{(5)}(\rho^{\mathcal{R}_3}(\hbar))}{d\hbar}.$$  \hfill (11)$$

Based on the numerical algorithms proposed in [14, 15], we obtain the results of $C^{(5)}(\rho^{\mathcal{R}_3})$ and its derivatives, for magnetic field along both the $y$ direction and the $x$-direction, as shown in figure 4. When the magnetic field is along the $y$ direction, there is a sharp transition behavior in $C^{(5)}(\rho^{\mathcal{R}_3})$ with $\hbar$, and the phase transition point is $\hbar^* = 0.99$, which is very close to the previous result $\hbar^* = 1$ in [16, 17]. When the magnetic field is along the $x$ direction, $C^{(5)}(\rho^{\mathcal{R}_3})$ shows a smooth transition behavior, and the transition point is

![Figure 3. The region $\mathcal{R}_3$ contains 6 spins labeled by magenta $\oplus$ on a $4 \times 3$ lattice.](image-url)
$h^* = 0.37$, which is close to the previous result $h^* = 0.34$ in [18, 19]. It is worthy to note that since the toric code model only contains $x$ and $z$ terms, the direction of the external magnetic field is symmetric with respect to $\mathbf{e}_x$ and $\mathbf{e}_z$, but is asymmetric for $\mathbf{e}_y$.

From the numerical results we know that the ground state of region $\mathcal{R}_2$, i.e. $\rho^{\mathcal{R}_2}$ contains only 1 bit $C^{(6)}(\mathcal{R}_2)$ without external field $h$, which can also be obtained analytically as given in appendix. While $h$ is infinite, there are no correlations of order greater than 1. The distribution of correlation among the different order correlations of $\rho^{\mathcal{R}_2}$ under different $h$ is showed in figure 5. When $h < h^*$, the total correlation largely stems from from $C^{(6)}$. However, the lower order correlation $C^{(2)}$ contributes to the main part of the total correlation when $h > h^*$.

**General correlation structure in topological order**—Although we discussed the toric code model, a similar correlation structure should also be valid for topologically ordered systems in general. In the thermodynamic limit, the ground states are degenerate and any ground state exhibits the following correlation structure:

$$C^{(r)}(\mathcal{G}) \begin{cases} \geq 0 & \text{if } r \leq r_0, \\ = 0 & \text{if } r_0 < r < L_h, \\ \geq 0 & \text{if } r \geq L_h, \end{cases} \quad (12)$$

where $r_0$ is a positive integer independent of the system size $L_h$. Furthermore, the value of IMCs of macroscopic order $C^{(2)}(\mathcal{G})$ is a topological invariant.

In a finite system, the ground state space is generally unique, which does not exhibit any IMCs of macroscopic order i.e., $C^{(r)}(\mathcal{G}) = 0$ for $r > r_0$. However, there are always IMCs of higher order in the reduced states of $\mathcal{G}$, which is manifested in a region with holes, e.g. $\mathcal{R}_3$ in figure 2, as the IMCs are of order proportional...
to the length of the inner boundary \( L_i \). Here we have

\[
C^{(2)}(\rho^{R_2}) \begin{cases} 
0 & \text{if } r \leq r_0, \\
\approx 0 & \text{if } r_0 < r < L_i, \\
\geq 0 & \text{if } r \geq L_i.
\end{cases}
\]

When the region \( R_2 \) is large enough (typically larger than the correlation length), it then contains the IMCs

\[
C^{(2)}(\rho^{R_2}) = C^{(2)}(\mathcal{G}_1^2),
\]

as it is in the thermodynamic limit. \( C^{(2)}(\rho^{R_2}) \) therefore is the same topological invariant as \( C^{(2)}(\mathcal{G}_1^2) \).

**Discussion**

The relation with topological entanglement entropy—The construction of topological entanglement entropy by Levin–Wen [5], denoted by \( E_{\text{LW}} \), may be regarded as an approximation for obtaining the IMCs of macroscopic order in a large enough region with a hole, e.g., the region \( R_2 \). To calculate \( E_{\text{LW}} \), they divided the region into three parts \( A, B, \) and \( C \) as demonstrated in figure 6, where \( A \) and \( C \) are far apart so they there should be no correlation between them. It is worth pointing out that here we study correlations among \( A, B, \) and \( C \), but not those among single spins as before.

The Levin–Wen topological entropy is then given by the total correlation

\[
C^T(\rho_{ABCD}) = S(\rho_A) + S(\rho_B) + S(\rho_C) - S(\rho_{ABCD})
\]

minus the biparty correlations \( C^T(\rho_{AB}) \) and \( C^T(\rho_{BC}) \). For the correlation structure as shown in equation (13), it is reasonable to believe that only the IMC \( C^{(2)}(\rho^{R_2}) \) contributes to the three-party correlation among parties \( A, B, \) and \( C \). In fact, one can show that the irreducible three-party correlation is upper bounded by such type of topological entropy, i.e.,

\[
C^{(3)}(\rho_{ABC}) \leq E_{\text{LW}}(\rho_{ABC}).
\]

The proof of the inequality and the equality condition are given in appendix C.

The topological entanglement entropy proposed by Kitaev–Preskill [4], denoted by \( E_{\text{KP}} \), is defined from the von Neumann entropy of a region without a hole, e.g., the region \( R_2 \cup R_4 \) in figure 2. The correlation between \( R_2 \cup R_4 \) and \( R_3 \cup R_4 \) comes from the dependence between \( R_2 \) and \( R_3 \) [20], which implies the area law. Notice that for the topological phase, the IMCs of macroscopic order in the region \( R_2 \) will decrease the correlations between \( R_2 \) and \( R_3 \), thus decreases the entanglement entropy compared to the area law, which then gives the topological entanglement entropy.

It is worth mentioning that [21] proved that topological entanglement entropy is equivalent to IMC if the state has zero–correlation length. Hence it gives a strict mathematical reason for the similarities between the topological entanglement entropy and IMC found in our numerical results of appendix B.

Summary—In sum, we use the concept of IMCs to analyze the correlation structure in the ground states of the toric code model. Based on the analysis, we suggest that the appearance of IMCs of macroscopic order in a region with holes represents an essential feature of topological order. For the toric code model in an external magnetic field, we also demonstrate that the power to create IMCs of higher orders for a region \( B \) with holes, from IMCs of lower order for a region \( A \supset B \), signals the topological transition phase transition. Our calculation uses a relatively small system, which clearly indicates the transition. Our concept has intimate relations with the idea of topologically entangled structure and may be applied to study other systems with topologically order, by calculation with relatively small system size. Our work may shed light on a better understanding of the general many-body correlation structure of a quantum state in topologically ordered phase.

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Appendix A. Irreducible multiparty correlations

In this section, we will first briefly review the definition of irreducible multiparty correlations. Then two typical examples, the toric code model without magnetic field, are given to demonstrate how to get the analytical results of irreducible multiparty correlations. In addition, the latter example explain how the higher order correlations are generated from lower order correlations. In fact, the simplest configuration (only seven spins whose correlation structure to be analyzed) for the toric code model suffices to demonstrate this essential feature of topologically ordered phase.

The concept of irreducible multiparty correlation is introduced [9, 10] to classify the total correlations in a multi-party quantum state into bipartite correlations, tripartite correlations, etc. The definition of irreducible multiparty correlations for an n-party quantum state \( \rho^{[n]} \) with \( [n] = \{1, 2, \ldots, n\} \) can be given in three steps. First, we introduce the set of \( n \)-party states that has the same \( k \)-party \( (1 \leq k \leq n) \) reduced density matrices with those of \( \rho^{[n]} \):

\[
\mathcal{A}_k = \{ \sigma^{[a]}: \sigma^a = \rho^a, \ \forall a \subseteq [n] \ \text{with} \ |a| = k \},
\]

where \(|a|\) is the cardinality of the set \( a \). Second, since the set \( \mathcal{A}_k \) is convex, there is a unique state with maximal von Neumann entropy in \( \mathcal{A}_k \):

\[
\rho_k^{[n]} = \arg \max \{ S(\sigma^{[a]}): \sigma^{[a]} \in \mathcal{A}_k \},
\]

where the von Neumann entropy of a state \( \tau \) is defined as

\[
S(\tau) = - \text{Tr}(\tau \log_2 \tau).
\]

Then the irreducible \( k \)-party correlation \( (2 \leq k \leq n) \) for the \( n \)-party state \( \rho^{[n]} \) is defined as

\[
C^{(k)}(\rho^{[n]}) = S(\rho_k^{[n-1]}) - S(\rho_k^{[n]}),
\]

According to the above definition, the irreducible \( k \)-party correlation in an \( n \)-party quantum state \( \rho^{[n]} \) is the information contained in the \( k \)-party reduced states, which is not contained in the \((k - 1)\)-party reduced states.

It can be shown that

\[
\sum_{k=2}^{n} C^{(k)}(\rho^{[n]}) = C^T(\rho^{[n]}),
\]

where the total correlation is equal to the generalized mutual information

\[
C^T(\rho^{[n]}) = \sum_{i=1}^{n} S(\rho^{[i]}) - S(\rho^{[n]}),
\]

where \( \rho^{[1]} \) is the single particle reduced state of the \( i \)th party. This implies that the irreducible \( k \)-party correlations are a complete classification of the total correlation in an \( n \)-party state.

Notice that the analytical results on irreducible multiparty correlations of all the stabilizer states are given in theorem 2 in [10]. Since the ground state and its reduced states for the toric code model without magnetic field are stabilizer states, we can obtain the distribution of irreducible multi-party correlations in these states by directly applying the above-mentioned theorem.

Here we present two examples to demonstrate how to get the analytical results of the IMCs for the toric code model without magnetic field.

A.1. Example 1: Proof of Equation (6)

In this subsection we gives a proof of equation (6). The density matrix of any ground state can be written as

\[
\rho_G = |G\rangle \langle G| = \rho_M \sigma_2,
\]

where

\[
\rho_M = \sum_{a,b} \prod_i A_i^{a_i} \prod_p B_p^{b_p} 2^n,
\]

\[
\sigma_2 = \sum_{c_i,d_i} \sum_{c_6,d_6} r_{c_i,d_i} e^{i d_i + c_6} (\sigma_i^c)^c (\sigma_i^d)^d (\sigma_6^c)^c (\sigma_6^d)^d.
\]
state $\rho_0$ has the same $r$-RDMs ($1 \leq r < L_0$) with the state $\rho_{H}$, Notice that the state $\rho_{H}$ is a stabilizer state with generators from independent elements from $\{A_i, B_i\}$. Since the number of independent elements of the stabilizer is $n - 2$, and the lengths of the elements are 4, theorem 2 in [10] implies that it has only $n - 2$ bits of irreducible 4-party correlation. Therefore the state $\rho_0$ has the same irreducible $r$-party correlations $(2 \leq r < L_0)$, which completes the proof of equation (6).

**A.2. Example 2: Irreducible multiparty correlations in regions of $|G\rangle$**

In this subsection, we will try to explain why the concept of IMCs plays a key role in characterizing topological orders using the case as shown in figure 3. That is, why we choose $C^{(6)}$ in our numerical analysis of the topological phase transitions.

We consider the model given by equation (10) with the magnetic field $h = 0$ in the lattice shown in figure 3. To give the details of our calculations, we redraw figures 3–7(a), where the spin labels are given for regions $R_i$ and $R_j$. Notice that the region $R_2$ has a hole, i.e., the region $R_t$ (the spin 0).

Our main results for this example are given as follows. Let us denote the density matrix of one of the ground states as $\rho$. Then

$$C^T(\rho^{R_1 \cup R_j}) = C^{(4)}(\rho^{R_1 \cup R_j}) = 2,$$
$$C^T(\rho^{R_j}) = C^{(6)}(\rho^{R_j}) = 1.$$  \hspace{1cm} (A.10)

Obviously this is a typical example of equation (1), i.e., the 6-party correlation is generated from the 4-party correlations, which is the essential feature of the topological order phase as suggested by us.

Now we come to the calculation details. The generator of the stabilizer reduced state $\rho^{R_1 \cup R_j}$ may be taken as

$$g(\rho^{R_1 \cup R_j}) = \{\sigma^+_1 \sigma^+_2 \sigma^+_6 \sigma^-_6, \sigma^-_1 \sigma^-_2 \sigma^-_6 \sigma^+_6\}.  \hspace{1cm} (A.11)$$

Notice that there are different choices of the generator of $\rho^{R_1 \cup R_j}$, and here the generator is taken such that the lengths of its elements as small as possible.

Similarly, we have

$$g(\rho^{R_j}) = \{\sigma^+_1 \sigma^-_2 \sigma^-_6 \sigma^+_6 \sigma^-_1 \sigma^-_6\}.  \hspace{1cm} (A.12)$$

According to theorem 2 in [10], we obtain the main results in equation \hspace{1cm} (A.10). In particular, we notice that the generator of $\rho^{R_2}$ is the product of the two elements of the generator of $\rho^{R_1 \cup R_j}$, which implies the relation between the IMCs of these two states. In other words, the correlation corresponding to $\sigma^+_1 \sigma^-_2 \sigma^-_6 \sigma^+_6 \sigma^-_1 \sigma^-_6$ is reducible in region $R_1 \cup R_2$ while in region $R_2$ it is irreducible because of the hole, $R_t$. In figure 7(b) we mark out the generator that contributes to the IMC because of the hole. They are 6 rectangles representing $\sigma^+$ around the hole $R$. In general cases there are also $\sigma^-$s around the hole, see the red cross marks around the hole in figure 2.

**Appendix B. Numerical results between irreducible correlations and topological entanglement entropy**

Here we will give more numerical examples to demonstrate the features of the IMCs in ground states with topological order, and also compare them with the corresponding numerical results of the topological entanglement entropy.

First, we consider the toric code model on a $3 \times 4$ square lattice with periodic boundary condition (24 spins) with the external magnetic field in the $x$–$z$ plane. The numerical results of $C^{(2,5)}(\rho^{R_j})$ for different magnetic field is shown in figure 8. With the increasing of the external field, $C^{(2,5)}(\rho^{R_j})$ decreases from 1 to 0 with a transition. The transition line is also determined by equation (13) and labeled by the black lines in figure 8. The phase diagram is similar with the previous one in [18].

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**Figure 7.** (a) The region $R_2$ contains 6 spins labeled by 1–6 respectively, and the region $R_1$ contains 1 spin labeled by 0. (b) Reducible and irreducible multiparty correlation. The rectangles represent the correlation $C^{(4)}(\rho^{R_1 \cup R_j})$ which is irreducible in region $R_2$ and reducible in region $R_1 \cup R_j$.
Second, we study how the size of the lattice affect our results of $C_{r}^{(2)}(\rho^{0})$. The numerical results of $C_{r}^{(2)}(\rho^{0})$ for the square lattices $(2 \times 3, 2 \times 4, 3 \times 3, \text{and} ~ 3 \times 4)$ with the magnetic field along the $y$ direction or the $x$ direction are demonstrated in figures 9 and 10.

In figure 9, there is a peak near the transition point $h_{y} = 1$ for different lattice size in the variant rate of $C_{r}^{(2)}(\rho^{0})$. When the lattice size becomes $3 \times 4$, the transition becomes very sharp, which clearly identify the topological transition.

In figure 10, there is a peak near the transition point in the variant rate of $C_{r}^{(2)}(\rho^{0})$. When the system size becomes larger, the peak becomes sharper, which also identify the correct transition point.

To compare our measure with entanglement of topological entropy, we obtain the numerical results of $E_{LW}(\rho^{0})$ in the above two cases, which are shown in figures 11 and 12. When the magnetic field is along the $y$ direction, we observe that $E_{LW}$ shows almost the same behavior as $C_{r}^{(2)}$, which is consistent with the argument we present in the article. However, when the magnetic field is along the $x$ direction, the behaviors between $E_{LW}$ and $C_{r}^{(2)}$ show obvious differences, particularly in the smaller size of the system. The relation between these two quantities will be studied further in the next section.

Appendix C. When will irreducible correlation coincide with topological entropy

The numerical results demonstrated above implies that in many cases the irreducible multiparty correlation and the topological entanglement entropy proposed by Levin–Wen are very similar, which motivates us to ask when will the irreducible correlations coincide with topological entropy.
Figure 10. $C^{(3)}(\rho^{(1)})$ and its derivative in the magnetic field $h_x$.

Figure 11. $E_{LW}(\rho^{(1)})$ and its derivative in the magnetic field $h_y$.

Figure 12. $E_{LW}(\rho^{(1)})$ and its derivative in the magnetic field $h_x$. 
Let $\rho_{ABC}$ be a tripartite state of systems $A$, $B$, $C$. Define the irreducible correlation of $\rho_{ABC}$ (given $\rho_{AB}$, $\rho_{BC}$) as

$$E_{BC}(\rho_{ABC}) = \max \{ S(\sigma_{ABC}) | \sigma_{AB} = \rho_{AB}, \sigma_{BC} = \rho_{BC} \} - S(\rho_{ABC}).$$

(C.1)

The Levin–Wen topological entropy of $\rho$ is

$$E_{LW}(\rho_{ABC}) = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{B}) - S(\rho_{ABC}).$$

(C.2)

First, we prove that $E_{BC}(\rho_{ABC})$ is upper bounded by $E_{LW}(\rho_{ABC})$. Consider the strong subadditivity for the maximum entropy state $\tilde{\rho}_{ABC}$, we have

$$S(\rho_{AB}) + S(\tilde{\rho}_{BC}) - S(\tilde{\rho}_{B}) - S(\tilde{\rho}_{ABC}) \geq 0.$$  

(C.3)

This reduces to

$$S(\tilde{\rho}_{ABC}) \leq S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{B}),$$

which is exactly

$$E_{BC}(\rho_{ABC}) \leq E_{LW}(\rho_{ABC}).$$

(C.4)

From the above discussion we know that the two quantities coincide if and only if the equality condition for strong subadditivity holds for the maximum entropy state $\tilde{\rho}_{ABC}$.

Following theorem 3 from [22], we know that this happens when the state is a quantum Markov state. In particular, the equation (11) of that reference states that this happens when

$$\tilde{\rho}_{ABC} = (\text{id} \otimes \hat{R})\rho_{AB}.$$  

(C.5)

More explicitly, this means that

$$\tilde{\rho}_{ABC} = (I_A \otimes \rho_{BC}^{1/2})(I_A \otimes \rho_{BC}^{-1/2})\rho_{AB}(I_A \otimes \rho_{BC}^{-1/2}) \otimes I_C] \times (I_A \otimes \rho_{BC}^{1/2}).$$

(C.6)

One can verify that the right hand side is indeed a quantum state whose reduced density matrix on $BC$ is exactly $\rho_{BC}$. But it is not always the case that the reduced density matrix on $AB$ is also $\rho_{AB}$. In fact, the following condition is necessary and sufficient for the equality of $E_{BC}$ and $E_{LW}$:

$$\tilde{\rho}_{AB} = \rho_{AB}.$$

(C.7)

Notice that

$$C^{(3)}(\rho_{ABC}) = \max \{ S(\sigma_{ABC}) | \sigma_{AB} = \rho_{AB}, \sigma_{BC} = \rho_{BC}, \sigma_{AC} = \rho_{AC} \} - S(\rho_{ABC}).$$

Therefore we have

$$C^{(3)}(\rho_{ABC}) \leq E_{BC}(\rho_{ABC}) \leq E_{LW}(\rho_{ABC}).$$

(C.8)

The condition for $C^{(3)}(\rho_{ABC}) = E_{BC}(\rho_{ABC})$ is

$$\tilde{\rho}_{AB} = \rho_{AB}, \quad \tilde{\rho}_{AC} = \rho_{AC}.$$  

(C.9)

In corollary 7 of [22], it is mentioned that a necessary condition for a state $\rho_{ABC}$ satisfies the strong subadditivity with equality, $S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{B}) - S(\rho_{ABC}) = 0$, is that $\rho_{AC}$ is separable, i.e., $\rho_{AC} = \sum_i \rho_i \otimes \rho_i$. But generally the product form $\rho_{AC} = \rho_A \otimes \rho_C$ may be neither sufficient nor necessary.

Notice that the above discussion is valid for all the three-party states of finite dimension. It is certainly correct to the configuration in figure 5 for the Levin–Wen topological entropy.

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