Wrap Groups of Non-Archimedean Fiber Bundles.

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Abstract

Fiber bundles over infinite fields with non-trivial ultra-norms are considered. For them geometric wrap groups are defined and investigated. Besides fields also Cayley-Dickson algebras over fields of characteristic not equal to two are taken into account. For fibers over them wrap groups are introduced and their structure is investigated. Different classes of smoothness for wrap groups are used. It is demonstrated that generally such groups are infinite dimensional over the corresponding field and totally disconnected groups. That is, they are continuous or differentiable non-archimedean differentiable uniform spaces and the composition \((f, g) \mapsto f^{-1}g\) is continuous or

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differentiable depending on a class of smoothness of groups. Skew products of wrap groups are studied as well.

1 Introduction.

Groups of geometric loops and paths for real manifolds were introduced in the 1930-th and they are very important in differential geometry, algebraic topology and theoretical physics [4, 29, 37]. Possibly the first author was S. Lefshetz who studied them over real manifolds. He used families of continuous mappings, that was rather restrictive and led him to the necessity to combine a geometric construction with an additional algebraic construction with some elements of free groups. Then more natural approach was proposed by J. Milnor in the class of Sobolev mappings. Later on they were generalized for real and complex fiber bundles [4]. Previously loop groups were considered as classes of mappings from the unit circle $S^1$ into a real or complex fiber bundle with a parallel transport structure. For spheres iterated loop groups were considered using reduced products of copies of circles. Recently they were generalized as wrap groups for rather common manifolds and fibers over the real and complex fields, the quaternion skew field and the octonion algebra [27, 25, 22, 23], where also some new structural theorems were proved. In the latter work Sobolev classes of smoothness were used. For general manifolds different from circles and spheres earlier geometric interpretation is already lost, so such groups were called wrap groups.

For manifolds over non-archimedean fields of zero characteristic loop groups were defined and investigated in [18, 19, 26]. But for non-archimedean
fiber bundles they were not yet studied.

In this paper wrap groups are defined and studied for fiber bundles. These fiber bundles are considered over infinite fields with non-trivial ultra-norms. Then Cayley-Dickson algebras over such fields are also considered. Fiber bundles over them are introduced. For such fiber bundles geometric wrap groups are constructed as well. Their structure is investigated. Such investigation is motivated by the following reasons. Geometric loops are used in quantum field theory and they were introduced by Wilson and his followers in physics, for example, to describe confinement of quarks [40, 8, 12].

Non-archimedean functional analysis and quantum mechanics develop intensively in recent times [32, 38]. This is stimulated by several problems. One of them consists in the divergence of some important integrals and series in the real or complex cases and their convergence in the non-archimedean case. Therefore, it is important to consider non-Archimedean wrap semigroups and groups, that are new objects. There are many principal differences between classical functional analysis (over the fields $\mathbb{R}$ or $\mathbb{C}$) and the non-archimedean one [32, 34, 39].

The notions of wrap groups and semigroups in the non-archimedean case are used here in analogy with the case of manifolds over the real field $\mathbb{R}$, but their meaning is quite different, because non-archimedean manifolds $M$ modeled on ultra-normed spaces are totally disconnected with the small inductive dimension $\text{ind}(M) = 0$ (see §6.2 and Chapter 7 in [7]) and real manifolds are locally connected with $\text{ind}(M) \geq 1$. In the real case loop and wrap groups $G$ are locally connected for $\dim_{\mathbb{R}} M < \dim_{\mathbb{R}} N$, but in the non-archimedean case they are zero-dimensional with $\text{ind}(G) = 0$, where $1 \leq \dim_{\mathbb{R}} N \leq \infty$ is
the dimension of the tangent Banach space $T_xN$ over $\mathbb{R}$ for $x \in N$.

In this article wrap groups and semigroups are considered. The wrap semigroups of manifolds are quotients of families of mappings $f$ from one non-archimedean manifold $M$ into another $N$ with $\lim_{x \to s_0} \Phi^v f(x) = 0$ for $0 \leq v \leq t$ by the corresponding equivalence relations, where $s_0$ and $y_0 = 0$ are marked points in $\bar{M}$ and $N$ respectively, $\bar{M} = M \setminus \{s_0\}$, $\Phi^v f$ are continuous extensions of the partial difference quotients $\Phi^v f$. Besides locally compact manifolds also non-locally compact manifolds $M$ and $N$ are considered. More generally differentiable spaces modeled on locally convex non-archimedean spaces are also considered. Then differentiable fiber bundles with a parallel transport structure are introduced and for them wrap groups are defined and investigated. Particularly groups of continuous wraps are also considered. We consider fibers over different infinite ultra-normed locally compact and non locally compact fields of zero and positive characteristics.

In this article over non-archimedean fields apart from works of others authors over the fields of $\mathbb{R}$ or $\mathbb{C}$ we do not impose an additional condition on an operator of a parallel transport structure related with tangent vectors. The latter was used that to bind the parallel transport with the covariant differentiation on a classical manifold. It was useful for geometric interpretations and for physical applications. In the present work we elaborate more general construction without such restriction using an abstract parallel transport structure taking values from a structure group of a fiber bundle. This produces wider family of considered objects of wrap groups. In the future work we plan to bind this with geometry and physical applications over infinite fields with non-archimedean multiplicative norms imposing additional
conditions on the parallel transport structure.

It is demonstrated that for differentiable spaces wrap groups are commutative. For fiber bundles with non commutative structure groups wrap groups are generally non commutative.

Semigroups and groups of wraps are investigated in §3.

The wrap groups are generally non locally compact and have a structure of differentiable groups modeled on differentiable spaces, which may be in particular continuous spaces and groups. The main results of this paper are obtained for the first time and are given in Theorems 3.3, 3.6, 3.11, Proposition 3.13 and Corollaries 3.5, 3.8 and 3.9.

2 Non-archimedean fiber bundles.

To avoid misunderstandings we first present our definitions and notations.

1. Remark. Let $A$ be an algebra with unit 1 over a field $K$ and $\delta$ be some element of $K$ different from zero. Suppose that $A$ is supplied with a $K$ linear mapping $x \mapsto x^*$ being an involution such that

\[(1) \ (x^*)^* = x, \ x + x^* = tr(x) \in K, \ xx^* = n(x) \in K.\]

We take the direct sum of $K$ linear spaces $A_1 := A \oplus A$ and define the multiplication:

\[(2) \ (a_1, b_1)(b_1, b_2) = (a_1b_1 - \delta b_2a_2^*, a_1^*b_2 + b_1a_2)\]

for each $a_1, a_2, b_1, b_2 \in A$. This multiplication supplies $A_1$ with the algebraic structure. Certainly, the initial algebra $A$ is embedded as the subalgebra into $A_1$,

\[(3) \ A \ni x \mapsto (x, 0) \in A_1, \ (a_1, a_2)^* = (a_1^*, -a_2);\]
(4) \(tr(a_1, a_2) = tr(a_1), \ n(a_1a_2) = n(a_1) + \delta n(a_2).\)

This doubling procedure applied by induction gives a sequence of embedded subalgebras \(K =: A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \ldots\). Henceforward, we consider commutative fields \(K\). Cayley-Dickson algebras \(A_r\) with \(r \leq 3\) are alternative:

(5) \(a(bb) = (ab)b\) and \(b(ba) = (bb)a\) for all \(a, b \in A_3\), moreover, the quadratic form \(n(x)\) is multiplicative

(6) \(n(ab) = n(a)n(b)\) for all \(a, b \in A_3\).

In this case the algebra \(A_1(\alpha)\) is commutative, the algebra \(A_2(\alpha, \beta)\) is associative and is called the algebra of (generalized) quaternions. The Cayley-Dickson algebra \(A_3(\alpha, \beta, \gamma)\) is alternative and simple with the center \(Z(A_3) = K\). But generally the Cayley-Dickson algebra \(A_3\) is non-associative.

It is possible to take as \(A_1\) an algebra with a basis \(\{1, u\}\) and with the multiplication \(u^2 = u + \alpha\), where \(\alpha \in K, 4\alpha + 1 \neq 0\), and with the involution \(1^* = 1, \ u^* = 1 - u\).

We consider fields \(K\) of characteristic \(\text{char}(K) \neq 2\). In this case the Cayley-Dickson algebra \(A_3\) has a basis of generators \(\{1, u_1, \ldots, u_7\}\) so that

(7) \(1^* = 1, \ u_j^* = -u_j\) for every \(j = 1, \ldots, 7\), \(u_1^2 = -q_1, u_2^2 = -q_2, u_3^2 = -q_3, u_4^2 = -q_1q_2, u_5^2 = -q_1q_3, u_6^2 = -q_2q_3, u_7^2 = -q_1q_2q_3, u_1u_2 = u_3, u_1u_4 = -u_5, u_2u_4 = -u_6, u_1u_6 = u_3u_4 = -u_2u_5 = -u_7, 1u_j = u_j, 1 = u_j\) and \(u_ju_k = -u_ku_j\) for each \(1 \leq j \neq k\), where \(q_1, q_2, q_3 \in K\) (see [6, 33]). In more details this algebra is denoted by \(A_3(q_1, q_2, q_3)\) and its subalgebras are denoted by \(A_2(q_1, q_2), A_1(q_1)\). The Cayley-Dickson algebra \(A_3\) has the division property \(ab \neq 0\) for each non-zero elements \(a, b \in A_3 \setminus \{0\}\) if and only if the quadratic form \(n(b)\) is non-zero on \(A_3 \setminus \{0\}\).
Henceforth, we consider the Cayley-Dickson algebras $A_3(q_1, q_2, q_3)$ so that they are alternative with the division property if something other is not specified. This implies that $G := A_3(q_1, q_2, q_3) \setminus \{0\}$ has the properties ($G1 – G4$):

1. ($G1$) there is a binary operation $ab \in G$ for all $a, b \in G$;
2. ($G2$) $1b = b1 = b$ for all $b \in G$, that is $1 = e$ is the unit element;
3. ($G3$) each $a \in G$ has the inverse element $a^{-1}$ so that $a^{-1}a = aa^{-1} = e$;
4. ($G4$) $(ab)b = a(bb)$ and $b(ba) = (bb)a$, $(ab)b^{-1} = a$ and $b^{-1}(ba) = a$ for all $a, b \in G$.

Property ($G4$) is called an alternativity, or a weak associativity. For usual groups the axiom ($G4$) is replaced on the associativity:

5. ($G5$) $(ab)c = a(bc)$ for all $a, b, c \in G$.

We shall call $G$ an alternative (or weak associative) group, when it satisfies Conditions ($G1 – G4$). A usual group satisfies ($G1 – G3, G5$), so that Condition ($G4$) follows from ($G5$). For short we call a group $G$ in both cases, when a situation is clear.

It is necessary to note that an object is known which is called a path group (a group of paths) at first introduced in physical literature and then in mathematical. This term does not mean an algebraic group or a topological group, because compositions are defined not for all elements, that is only properties ($G2, G3$) are fulfilled for the path group while ($G1, G5$) may be satisfied for definite combinations of elements only. Wrap groups compose the main subject of this paper such that they will satisfy ($G1 – G5$) or ($G1 – G4$).

2. Definitions. We consider an infinite field $K$ with a non trivial non archimedean normalization. We suppose also that $X$ and $Y$ are topological
vector spaces over $K$ and $U$ is an open subset in $X$. For a function $f : U \rightarrow Y$ we consider the associated function

$$f^{[1]}(x, v, t) := \frac{f(x + tv) - f(x)}{t}$$
on a set $U^{[1]}$ at first for $t \neq 0$ such that $U^{[1]} := \{(x, v, t) : (x, v, t) \in X^2 \times K; x \in U, x + tv \in U\}$. If the function $f$ is continuous on $U$ and $f^{[1]}$ has a continuous extension on $U^{[1]}$, then we say, that $f$ is continuously differentiable or belongs to the class $C^1$. The $K$-linear space of all such continuously differentiable functions $f$ on $U$ is denoted $C^{[1]}(U,Y)$. Then we define by induction functions $f^{[n+1]} := (f^{[n]})^{[1]}$ and their spaces $C^{[n+1]}(U,Y)$ for $n = 1, 2, 3, ..., $ where $f^{[0]} := f$, $f^{[n+1]} \in C^{[n+1]}(U,Y)$ has as the domain $U^{[n+1]} := (U^{[n]})^{[1]}$, $f^{[n]} =: Y^n f$.

At the same time a differential $df(x) : X \rightarrow Y$ is defined as $df(x)v := f^{[1]}(x, v, 0)$.

Define also partial difference quotient operators $\Phi^n$ by variables corresponding to $x$ only such that

$$\Phi^1 f(x; v; t) = f^{[1]}(x, v, t)$$
at first for $t \neq 0$ and if $\Phi^1 f$ is continuous for $t \neq 0$ and has a continuous extension on $U^{[1]} =: U^{(1)}$, then we denote it by $\Phi^1 f(x; v; t)$. We define by induction

$$\Phi^{n+1} f(x; v_1, ..., v_{n+1}; t_1, ..., t_{n+1}) := \Phi^1(\Phi^n f(x; v_1, ..., v_n; t_1, ..., t_n))(x; v_{n+1}; t_{n+1})$$
at first for $t_1 \neq 0, ..., t_{n+1} \neq 0$ on $U^{(n+1)} := \{(x; v_1, ..., v_{n+1}; t_1, ..., t_{n+1}) : x \in U; v_1, ..., v_{n+1} \in X; t_1, ..., t_{n+1} \in K; x + v_1 t_1 \in U, ..., x + v_1 t_1 + ... + v_{n+1} t_{n+1} \in U\}$. If $f$ is continuous on $U$ and partial difference quotients $\Phi^1 f, ..., \Phi^{n+1} f$ have continuous extensions denoted by $\Phi^1 f, ..., \Phi^{n+1} f$ on $U^{(1)}, ..., U^{(n+1)}$ respectively, then we say that $f$ is of class of smoothness $C^{n+1}$. The $K$ linear
space of all $C^{n+1}$ functions on $U$ is denoted by $C^{n+1}(U, Y)$, where $\Phi^0 f := f$, $C^0(U, Y)$ is the space of all continuous functions $f : U \to Y$.

Then the $n$-th differential is given by the equation $d^n f(x; (v_1, ..., v_n)) := n! \Phi^n f(x; v_1, ..., v_n; 0, ..., 0)$, where $n \geq 1$, also denote $D^n f = d^n f$. Shortly we shall write the argument of $f^{[n]}$ as $x^{[n]} \in U^{[n]}$ and of $\Phi^n f$ as $x^{(n)} \in U^{(n)}$, where $x^{[0]} = x^{(0)} = x$, $x^{[1]} = x^{(1)} = (x, v, t)$, $v^{[0]} = v^{(0)} = v$, $t_1 = t$, $x^{[k]} = (x^{[k-1]}, v^{[k-1]}, t_k)$ for each $k \geq 1$; $x^{(k)} := (x; v_1, ..., v_k; t_1, ..., t_k)$.

We denote by $C^n_b(U, Y)$ or $C^{[n]}_b(U, Y)$ respectively subspaces of uniformly $C^n$ or $C^{[n]}$ bounded continuous functions together with $\Phi^k f$ or $\Upsilon^k f$ on bounded open subsets of $U$ and $U^{(k)}$ or $U^{[k]}$ for $k = 1, ..., n$. These spaces of differentiable functions were investigated in details in [28, 17].

We denote by $L(X, Y)$ the space of all continuous $K$-linear mappings $A : X \to Y$. By $L_n(X^\otimes n, Y)$ we denote the space of all continuous $K$-linear mappings $A : X^\otimes n \to Y$, particularly, $L(X, Y) = L_1(X^\otimes 1, Y)$. If $X$ and $Y$ are normed spaces, then $L_n(X^\otimes n, Y)$ is supplied with the operator norm: $\|A\| := \sup_{h_1 \neq 0, ..., h_n \neq 0, h_1, ..., h_n \in X} \|A(h_1, ..., h_n)\|_Y / (\|h_1\|_X ... \|h_n\|_X)$.

3. Definitions. Suppose that $M$ is a manifold modeled on a topological vector space $X$ over $K$ such that its atlas $At(M) := \{(U_j, M\phi_j) : j \in \Lambda_M\}$ is of class $C^\alpha_{\beta'}$, that is the following four conditions are satisfied:

(M1) $\{U_j : j \in \Lambda_M\}$ is an open covering of $M$, $U_j = M U_j$;

(M2) $\bigcup_{j \in \Lambda_M} U_j = M$;

(M3) $M\phi_j := \phi_j : U_j \to \phi_j(U_j)$ is a homeomorphism for each $j \in \Lambda_M$, $\phi_j(U_j) \subset X$, every $\phi_j(U_j)$ is open in $X$;

(M4) $\phi_j \circ \phi_i^{-1} \in C^\alpha_{\beta'}$ on its domain for each $U_i \cap U_j \neq \emptyset$,
where $\Lambda_M$ is a set, $C^\infty := \bigcap_{l=1}^\infty C^l_\beta$, $C^{[\infty]}_\beta := \bigcap_{l=1}^\infty C^[l]_\beta$, $\alpha' \in \{n, [n] : 1 \leq n \leq \infty\}$, $\beta \in \{\emptyset, b\}$, $C^\alpha_{\emptyset} := C^\alpha$.

Supply $C^\alpha_\beta(U,Y)$ with the bounded-open $C^\alpha_\beta$ topology denoted by $\tau_{\alpha,\beta}$ generally or $\tau_{\alpha}$ for $\beta = \emptyset$ (or for compact $U$) with the base $(B1) W(P,V) = \{f \in C^\alpha_\beta(X,Y) : S^k f|_P \in V, k = 0, \ldots, n\}$ of neighborhoods of zero, where $P$ is bounded and open in $U \subset X$, $P \subset U$, $V$ is open in $Y$, $0 \in V$, $S^k = \Phi^k$ or $S^k = \Upsilon^k$ for $\alpha = n$ or $\alpha = [n]$ respectively, $v_1, \ldots, v_n \in (P - y_0)$, $v[k]_l \in (P - y_0)$ for each $k, l$ for some marked $y_0 \in P$ and $|t_j| \leq 1$ for every $j$.

If $M$ and $N$ are $C^\alpha_\beta$ manifolds on topological vector spaces $X$ and $Y$ over $K$ respectively, then let us consider the uniform space $C^\alpha_\beta(M,N)$ of all mappings $f : M \to N$ such that $f_{j,i} \in C^\alpha_\beta$ on its domain for each $j \in \Lambda_N$, $i \in \Lambda_M$, where $f_{j,i} := N \phi_j \circ f \circ M^{-1} \phi_i$ is with values in $Y$, $\alpha \leq \alpha'$. The uniformity in $C^\alpha_\beta(M,N)$ is inherited from the uniformity in $C^\alpha_\beta(X,Y)$ with the help of charts of atlases of $M$ and $N$. If $M$ is compact, then $C^\alpha_\beta(M,N)$ and $C^\alpha(M,N)$ coincide.

The family of all homeomorphisms $f : M \to M$ of class $C^\alpha_\beta$ is denoted by $Diff^\alpha_\beta(M)$.

Let $\gamma$ be a set, then we denote by $c_0(\gamma, K)$ the normed space consisting of all vectors $x = \{x_j \in K : j \in \gamma, \text{ for each } \epsilon > 0 \text{ the set } \{j : |x_j| > \epsilon\} \text{ is finite}\}$, where

$$(N1) \|x\| := \sup_{j \in \gamma} |x_j|.$$ In view of the Kuratowski-Zorn lemma it is convenient to consider $\gamma$ as an ordinal. Henceforth, suppose that $X = c_0(\gamma_X, K)$ and $Y = c_0(\gamma_Y, K)$. 10
As a generalization let \( X = c_0(\gamma, \mathcal{A}_r) \) be a normed space over \( \mathcal{A}_r = \mathcal{A}_r(q_1, \ldots, q_r) \) consisting of all vectors of the form \( x = \{x_j \in \mathcal{A}_r : j \in \gamma, \ \text{for each} \ \epsilon > 0 \ \text{the set} \ \{j : |x_j| > \epsilon\} \ \text{is finite} \} \), where

\[
\|x\| := \sup_{j \in \gamma} |x_j|, \text{where } 1 \leq r \leq 3 \ (\text{see } \S 1),
\]

\[
|x_j| := |x_j| := \max_{k=0,...,2^r-1} |k x_j|, \ x_j = 0 x_j u_0 + \ldots + 2^{r-1} x_j u_{2^r-1},
\]

\( k x_j \in \mathbf{K} \) for all \( k, j, u_0 = 1 \), \( \{u_0, \ldots , u_{2^r-1}\} \) is a basis of generators of \( \mathcal{A}_r \).

By a vector space \( X \) over \( \mathcal{A}_r \) we shall undermine the direct sum \( X = \bigoplus \mathcal{A}_r \mathcal{X}_0 u_0 \oplus \ldots \oplus \mathcal{A}_r \mathcal{X}_{2^r-1} \), where \( \mathcal{X}_0, \ldots , \mathcal{X}_{2^r-1} \) are pairwise isomorphic linear spaces over a field \( \mathbf{K} \), \( 1 \leq r \leq 3 \). That is the following conditions (\( L_1 - L_4 \)) are satisfied:

\[
(L1) \ X \text{ is the additive commutative group,}
\]

\[
(L2) \ a(x + y) = ax + ay \text{ and } (x + y)a = xa + ya,
\]

\[
(L3) \ (a + b)x = ax + bx \text{ and } x(a + b) = xa + xb \text{ for all } a, b \in \mathcal{A}_r \text{ and } x, y \in X,
\]

\[
(L4) \ (ab)_0 x = a(b_0 x) \text{ and } _0 x(ab) = (_0 x a) b \text{ for all } a, b \in \mathcal{A}_r \text{ and } _0 x \in _0 X.
\]

We consider a vector space \( X \) over \( \mathcal{A}_r \) supplied with a topology \( \tau \) with jointly continuous operations of addition of vectors \( X^2 \ni (x, y) \mapsto x + y \in X \) and their multiplication \( \mathcal{A}_r \times X \ni (a, x) \mapsto ax \in X \) and \( \mathcal{A}_r \times X \ni (a, x) \mapsto xa \in X \) on scalars from \( \mathcal{A}_r \) relative to \( \tau \) and the norm topology \( |a| = |a|_r \) in \( \mathcal{A}_r \). Such \( X \) is called a topological vector space.

The algebras \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are associative and for them Conditions (\( L1, L3, L4 \)) imply that \( (ab)x = a(bx) \) and \( x(ab) = (xa)b \) for all \( x \in X \) and \( a, b \) in \( \mathcal{A}_1 \) or \( \mathcal{A}_2 \) respectively, when \( 1 \leq r \leq 2 \). Moreover, over \( \mathcal{A}_1 \) the multiplication on scalars is commutative so that we can consider the equality \( ax = xa \) for all \( a \in \mathcal{A}_1 \) and \( x \in X \), when \( r = 1 \). Thus each vector space over \( \mathcal{A}_r \) with
1 ≤ r ≤ 3 is also a vector space over the initial field $\mathbf{K} = \mathcal{A}_0$.

We mention also that algebras $\mathcal{A}_r$ are finite dimensional over $\mathbf{K}$ and additions and multiplications and taking of the conjugate $a \mapsto a^*$ in them are continuous relative to the norm $|a| = |a|^r$. If $r \leq 3$, then by our convention of §1 $n(a) = aa^* = a^*a \in \mathbf{K}$ and $a^{-1} = a^* / n(a)$ for $a \neq 0$, since $\alpha^* = \alpha$ for each $\alpha \in \mathbf{K}$. This implies that for the considered Cayley-Dickson algebras the inversion $\mathcal{A}_3 \setminus \{0\} \ni a \mapsto a^{-1} \in \mathcal{A}_3 \setminus \{0\}$ is also continuous.

The family of all continuous $\mathcal{A}_r-$additive $\mathbf{K}$-linear mappings $A : X \to Y$ for topological vector spaces $X$ and $Y$ over $\mathcal{A}_r$ we denote by $K_q(X,Y)$, $1 \leq r \leq 3$. Their subfamily of right $\mathcal{A}_r-$linear mappings $A(0,xb) = (A_0x)b$ for all $0x \in 0X$ and $b \in \mathcal{A}_r$ we denote by $K_r(X,Y)$ or $L(X,Y)$. The subfamily of all left $\mathcal{A}_r-$linear mappings $A(b_0x) = b(A_0x)$ for all $0x \in 0X$ and $b \in \mathcal{A}_r$ will be denoted by $K_l(X,Y)$. Certainly that over $\mathcal{A}_1$ these spaces coincide $K_r(X,Y) = K_l(X,Y) = K_q(X,Y)$, since the algebra $\mathcal{A}_1$ is commutative and associative.

Each topological vector space $X$ over $\mathcal{A}_r$ with $1 \leq r \leq 3$ is a topological vector space $X_\mathbf{K}$ over a field $\mathbf{K}$ as well. So by a manifold $M$ modeled on $X$ we shall undermine the corresponding manifold $M_\mathbf{K}$ modeled on $X_\mathbf{K}$ such that a class of smoothness $C^a_\beta$ of $M$ is that of $M_\mathbf{K}$, but with one additional condition:

(M5) each differential $d(\phi_j \circ \phi_i^{-1})$ is a right $\mathcal{A}_r-$linear operator, that is, it belongs to $L(X,X)$ for each $x$ in its domain for each $U_i \cap U_j \neq \emptyset$.

Therefore we obtain also as in §3 above uniform spaces $C^a_\beta(M,N)$ for manifolds $M$ and $N$ modeled on topological vector spaces $X$ and $Y$ over $\mathcal{A}_r$. 

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with $1 \leq r \leq 3$. Certainly the uniformity in $C^\alpha_\beta(M, N)$ is inherited from the uniformity in $C^\alpha_\beta(X, Y)$ with the help of charts of atlases of $M$ and $N$.

4. Plots. Since the Cayley-Dickson algebra $\mathcal{A}_3$ is non-associative, we consider a non-associative subgroup $G$ of the family $\text{Mat}_q(O)$ of all square $q \times q$ matrices with entries in $\mathcal{A}_3$. More generally $G$ is a group which has a $C^\alpha_\beta$ manifold structure over $\mathcal{A}_r$ and group’s operations are $C^\alpha_\beta$ mappings (see also §1), where $0 \leq r \leq 3$. Such groups $G$ we call also a $C^\alpha_\beta$ Lie group over $\mathcal{A}_r$. In particular, it may be a multiplicative or and additive subgroup of $L(X, X)$ (see §3).

As a generalization of manifolds we use the following (over $\mathbb{R}$ and $\mathbb{C}$ see [9, 35]). We adopt that a subset $C$ of a vector space $X$ over $\mathcal{A}_r$ is called $\mathcal{A}_r$ absolutely convex if $ax + by$ and $xa + yb \in C$ for all $x, y \in C$ and $a, b \in \mathcal{A}_r$ with $|a| \leq 1$ and $|b| \leq 1$. Translates $(z + C)$ of absolutely $\mathcal{A}_r$ convex sets $C$ are called $\mathcal{A}_r$ convex, where $z \in X$. A topological vector space $X$ over $\mathcal{A}_r$ is called locally $\mathcal{A}_r$ convex if it has a base of $\mathcal{A}_r$ convex neighborhoods of zero.

Let $X, Y$ be topological vector spaces over $\mathcal{A}_r$, $0 \leq r \leq 3$, particularly $\mathcal{A}_0 = \mathbb{K}$. Suppose that $M$ is a Hausdorff topological space supplied with a family $\mathcal{P}_M := \{h : U \to M\}$ of the so called plots $h$ which are continuous maps satisfying conditions $(D1 - D5)$:

$(D1)$ each plot has as a domain an $\mathcal{A}_r$ convex subset $U = _hU$ in $X$;

$(D2)$ if $h : U \to M$ is a plot, $V$ is an $\mathcal{A}_r$ convex subset in $Y$ and $g : V \to U$ is an $C^\alpha_\beta$ mapping, then $h \circ g$ is also a plot;

$(D3)$ every constant map from an $\mathcal{A}_r$ convex set $U$ in $X$ into $M$ is a plot;

$(D4)$ if $U$ is an $\mathcal{A}_r$ convex set in $X$ and $\{U_j : j \in J\}$ is a covering of $U$ by $\mathcal{A}_r$ convex sets in $X$, each $U_j$ is open in $U$, $h : U \to M$ is such that each
its restriction $h|_{U_j}$ is a plot, then $h$ is a plot. We suppose of course that

(D5) the family of subsets $h_k(U_k)$ which are ranges of plots, $k \in \lambda_M$ for all plots $h_k$ of $M$ forms a covering of $M$, but not necessarily open, where $\lambda_M$ is a set.

Then $M$ is called a $C^\alpha_\beta$-differentiable space.

A mapping $f : M \to N$ between two $C^\alpha_\beta$-differentiable spaces $M$ and $N$ is called $C^\alpha_\beta$ differentiable if it continuous and for each plot $h : U \to M$ the composition $f \circ h : U \to N$ is a $C^\alpha_\beta$ plot of $N$.

To supply a family of $C^\alpha_\beta(M,N)$ mappings between $C^\alpha_\beta$ differentiable spaces with a uniformity we use the following particular case. We suppose that $M$ and $N$ have families of plots $\mathcal{T}_M = \{h_j^M \in \mathcal{P}_M : j \in \Lambda_M\}$ and $\mathcal{T}_N = \{h_j^N \in \mathcal{P}_N : j \in \Lambda_N\}$ correspondingly, so that domains of plots $h_j^M(U_j^M)$ and $h_k^N(U_k^N)$ for $j \in \Lambda_M$ and $N$ with $k \in \Lambda_N$ respectively form coverings of $M$ and $N$ satisfying Conditions (D6 – D8):

(D6) $h_j^M : U_j^M \to h_j^M(U_j^M)$ and $h_k^N : U_k^N \to h_k^N(U_k^N)$ are bijective for all $j \in \Lambda_M$ and $k \in \Lambda_N$, where $\Lambda_M \subset \lambda_M$ and $\Lambda_N \subset \lambda_N$ are subsets;

(D7) transition mappings $(h_i^M)^{-1} \circ h_j^M$ and $(h_s^N)^{-1} \circ h_k^N$ are $C^\alpha_\beta$ differentiable mappings on their domains for each $h_i^M(U_i^M) \cap h_j^M(U_j^M) \neq \emptyset$ and $h_s^N(U_s^N) \cap h_k^N(U_k^N) \neq \emptyset$ with $i, j \in \Lambda_M, s, k \in \Lambda_N$;

(D8) supply $M$ and $N$ with topologies $\tau_{p,M}$ and $\tau_{p,N}$ having bases $h_j^M(V)$ for each $V$ open in $U_j^M$ and every $j \in \Lambda_M$ for $M$, while $h_k^N(V)$ for each $V$ open in $U_k^N$ and every $k \in \Lambda_N$ for $N$ correspondingly.

Then we form atlases for $(M, \tau_{p,M})$ and $(N, \tau_{p,N})$ with (generalized) charts $(h_j^M(U_j^M), (h_j^M)^{-1})$ and $(h_k^N(U_k^N), (h_k^N)^{-1})$ with $j \in \Lambda_M$ and $k \in \Lambda_N$ respectively.
Thus we get the uniform space $C^\alpha_\beta(M, N)$ using transition mappings between (generalized) charts as in §3. That is $C^\alpha_\beta(M, N)$ consists of all mappings $f : M \to N$ such that $f_{k,j} \in C^\alpha_\beta$ on its domain for each $k \in \Lambda_N, j \in \Lambda_M$, where $f_{k,j} := (h^N_k)^{-1} \circ f \circ h^M_j$ is with values in $Y, \alpha \leq \alpha'$. Here $U^M_j$ is considered as open in the vector space $X_j := \text{span}_{\mathcal{A}_r} U^M_j$. For this we supply $X_j$ with a base of topology generated by neighborhoods of zero $\lambda(W-x)$ for each $x \in U^M_j$ so that $U^M_j$ is absolutely convex in $X$, $W$ is open and bounded in $U^M_j$ relative to the topology in $U^M_j$ inherited from $X$, $\lambda \in \mathcal{A}_r$ with $0 < |\lambda| < 1$. The uniformity in $C^\alpha_\beta(M, N)$ is inherited from the uniformity in $C^\alpha_\beta(X, Y)$ with the help of (generalized) charts of atlases of $M$ and $N$. This means that a base of entourages of the diagonal (see also Chapter 8 in [7]) in $C^\alpha_\beta(M, N)$ is formed by sets $\{f, g \in C^\alpha_\beta(M, N) : (f_{l,j} - g_{l,j}) \in W(P, V) \forall l\}$, where $P$ is a bounded open subset in $U^M_j$ and $V$ is an open neighborhood of zero in $Y$, $j \in \Lambda_M$ and $l \in \Lambda_N$. In $C^\alpha_\beta(U, Y)$ the base $W(P, V)$ was defined in (B1) §3.

A topological group $G$ is called an $C^\alpha_\beta$-differentiable group if its group operations are $C^\alpha_\beta$-differentiable mappings.

5. Transformation groups. A pair $(G, F)$ is called a $C^\alpha_\beta$ transformation group acting from the left if the following four conditions are satisfied:

(T1) $G$ is a $C^\alpha_\beta$ differentiable group over $\mathcal{A}_r$, where a number $r$ is supplied, $0 \leq r \leq 3$;

(T2) $F$ is a $C^\alpha_\beta$ differentiable space over $\mathcal{A}_r$;

(T3) a $C^\alpha_\beta$ mapping $(g, x) \in G \times F \to gx \in F$ is given;

(T4) for each $g \in G$ a $C^\alpha_\beta$ mapping $l_g : F \to F$ is defined so that $l_g(x) = gx$ and $l_g \in Diff^\alpha_\beta(F), l_{gh} = l_g \circ l_h$. 

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If \( l_{gh} = l_h \circ l_g \), then \( G \) acts on \( F \) from the right. In particular, if \( G \) is a \( C^\alpha_\beta \) Lie group and \( N \) is a \( C^\alpha_\beta \) manifold, then \( (G, F) \) is called a \( C^\alpha_\beta \) transformation Lie group.

A transformation group \( (G, F) \) with \( l_g = id_F \) for each \( g \in G \) is called trivial, where \( id_F(y) = y \) for each \( y \in F \). Contrary, when \( l_g = id_F \) if and only if \( g \) is a unit element of \( G \), \( g = e \), the transformation group \( G \) is called effective.

6. Fiber bundles.

Let \( E, N, F \) be all either \( C^\alpha'_\beta \)-manifolds or \( C^\alpha'_\beta \)-differentiable spaces over \( \mathcal{A}_r \) with \( 0 \leq r \leq 3 \). Certainly a manifold is a particular case of a differentiable space. Let also \( G \) be a \( C^\alpha'_\beta \) group over \( \mathcal{A}_r \), \( \alpha \leq \alpha' \leq \infty \). We suppose that a projection \( \pi : E \to N \) is given together with an atlas \( \Psi = \{\psi_j\} \) of \( E \) so that

(F1) to each chart \( \psi_j \in \Psi \) an open subset \( V_j \) in \( N \) is counterposed and

(F2) the mapping \( \psi_j : \pi^{-1}(V_j) \to V_j \times F \) is the \( C^\alpha'_\beta \) diffeomorphism so that \( \psi_j(\pi^{-1}(x)) = \{x\} \times F \):

\[
pr_{V_j} \circ \psi_j = \pi|_{\pi^{-1}(V_j)},
\]

where \( pr_{V_j}(x \times y) = x \) for each \( x \in V_j \) and \( y \in F \);

(F3) a system of open subsets \( \{V_j : j \in J\} \) forms a covering of \( N \). We get from (F1 − F3) that \( \pi : E \to N \) is open and surjective. Moreover,

(F4) the mapping \( \psi_{j,x} = pr_F \circ \psi_j|_{\pi^{-1}(x)} : \pi^{-1}(x) \to F \) defines the \( C^\alpha'_\beta \) diffeomorphism of the fiber \( F_x := \pi^{-1}(x) \) on the typical fiber \( F \), where \( pr_F : (x \times y) = y \) for all \( x \in V_j \) and \( y \in F \).

Using restrictions of mappings \( \psi_j \) we can choose \( V_j \) as domains in \( N \) either \( h_j(U_j) \) of plots from \( \mathcal{T}_N \) (see §4) in the case of the differentiable space or of charts in the case of the manifold. Thus either plots or charts on \( N \times F \) are
transferred by $\psi^{-1}_j$ onto plots or charts respectively on $\pi^{-1}(V_j)$.

Let $\psi_j, \psi_l \in \Psi$ and $V_j \cap V_k \neq \emptyset$. In view of (F4)

(F5) for each $x \in V_j \cap V_k$ the mapping is defined: $g_{j,k} : V_j \cap V_k \ni x \mapsto g_{j,k}(x) = \psi_{k,x} \circ \psi^{-1}_{j,x} \in \text{Diff}_{\alpha'}^\beta(F)$. That is to each point $x \in V_j \cap V_k$ a diffeomorphism of $F$ corresponds. Moreover, these mappings $g_{j,k}$ satisfy the following conditions:

(F6) $g_{j,k}(x) = (g_{k,j}(x))^{-1}$, $g_{j,j}(x) = \text{id}_F$, where $\text{id}_F(y) = y$ for each $y \in F$, $g_{l,j}(x) = g_{l,k}(x) \circ g_{k,j}(x)$ for each $x \in V_l \cap V_j \cap V_k$;

(F7) a $C^\alpha_{\beta'}$ differentiable transformation group $(G,F)$ is given so that $g_{j,k}(x) \in G$ for each $x \in V_j \cap V_k$, when $V_k \cap V_j \neq \emptyset$, $g_{j,j}(x) = e \in G$, where $e$ denotes the unit element in $G$, also $l_{g_{j,k}(x)} = \psi_{j,x} \circ \psi^{-1}_{k,x}$ (see §5 as well).

If Conditions (F1 – F7) are satisfied then $E(N,F,G,\pi,\Psi)$ is called a fiber bundle with a fiber space $E$, a base space $N$, a typical fiber $F$, projection $\pi$ and a structural group $G$ over $A_r$, and an atlas $\Psi$, while the mappings $g_{j,k}$ are called the transition functions.

Local trivializations $\phi_j \circ \pi \circ \Psi^{-1}_k : V_k(E) \rightarrow V_j(N)$ induce the $C^\alpha_{\beta'}$-uniformity in the family $\mathcal{W}$ of all principal $C^\alpha_{\beta'}$-fiber bundles $E(N,F,G,\pi,\Psi)$, where $V_k(E) = \Psi_k(U_k(E)) \subset X(N) \times X(F)$, $V_j(N) = \phi_j(U_j(N)) \subset X(N)$, where $X(F), X(G)$ and $X(N)$ are $A_r$-vector spaces on which $F, G$ and $N$ are modeled, $(U_k(E),\Psi_k)$ and $(U_j(N),\phi_j)$ are either ranges of plots or charts of atlases of $E$ and $N$, $\Psi_k = \Psi_k^E$, $\phi_j = \phi_j^N$ (see also §4).

If $G = F$ and $G$ acts on itself by left shifts, then a fiber bundle is called the principal fiber bundle and is denoted by $E(N,G,\pi,\Psi)$. As an example a multiplicative group $G = A^*_r$ may be, where $A^*_r$ denotes the multiplicative group $A_r \setminus \{0\}$. If $G = F = \{e\}$, then $E$ reduces to $N$. 

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3 Wrap groups and semigroups.

1. Parallel transport structure. Let $cl_X(A)$ denote the closure of a subset $A$ in $X$. Let $\hat{M}$ and $\bar{M}$ be two $C^\alpha_\beta'$ differentiable spaces over $A_r$ (see Conditions 2.4(D1 – D8) above), $0 \leq r \leq 3$, and with marked points $\{\hat{s}_{0,q} \in \hat{M}_f : q = 1, ..., 2k\}$ and a $C^\alpha_\beta'$-mapping $\Xi : \hat{M} \to \bar{M}$, where $\hat{M}_f$ is a subset defined below, such that the following conditions ($S1$ – $S5$) are satisfied:

(S1) $\Xi^{-1}(x)$ consists of one or at most finite number of distinct points for each $x \in \bar{M}$, we denote by $\hat{M}_f$ the set of all $y \in \hat{M}$ so that $y \in \Xi^{-1}(x)$ for some $x \in \bar{M}$ with $\Xi^{-1}(x)$ consisting of a finite number of distinct points more than one;

(S2) $\Xi$ is surjective and bijective from $\hat{M} \setminus \hat{M}_f$ onto $\bar{M} \setminus \bar{M}_f$ open in $\bar{M}$, $\Xi(\hat{s}_{0,q}) = \Xi(\hat{s}_{0,k+q}) = s_{0,q}$ for each $q = 1, ..., k$, where $\hat{M}_f := \Xi(\hat{M}_f)$;

(S3) for each point $x \in \bar{M}$ or $y \in \hat{M}$ there exist ranges of plots $h_{\bar{M}}(V)$ and $h_{\hat{M}}(U)$ being open neighborhoods of $x$ and $y$ respectively in $\bar{M}$ and $\hat{M}$ so that $V$ and $U$ are $A_r$ convex in a topological vector space $X$ over $A_r$ on which $\bar{M}$ and $\hat{M}$ are modeled so that $h_{\bar{M}} : V \to h_{\bar{M}}(V)$ and $h_{\hat{M}} : U \to h_{\hat{M}}(U)$ are bijective;

(S4) the closure $cl_{\bar{M}}\bar{M} = \bar{M}$ of $\bar{M}$ in $\bar{M}$ is the entire $\bar{M}$, where $\bar{M} := \{x \in \bar{M} : x \in h_{\bar{M}}(V) \text{ for some plot } h \in T_{\bar{M}} \text{ of } \bar{M} \text{ with } V \text{ open in } X\}$;

(S5) $\hat{M} \subset \bar{M} \setminus \hat{M}_f$.

By $\text{span}_{A_r}V$ we denote a vector space consisting of all finite $A_r$ vector combinations of vectors from $V$ and with multiplication on constants from $A_r$, where the multiplications may be on both sides, when $2 \leq r \leq 3$. 
Mention that $A_r$ with $r \leq 3$ are division algebras and for matrices with entries in $A_r$ the Gauss’ algorithm is accomplished, so they have ranks by rows and columns which coincide. This means that an $A_r$ vector independence and a dimension over $A_r$ are well-defined as it was outlined already by Dickson [6].

Put $M := \bar{M} \setminus \{s_0, q : q = 1, \ldots, k\}$. Here particularly $C^a_\beta$ manifolds $\hat{M}$ and $\bar{M}$ may be as well.

A parallel transport structure on a $C^a_{\beta'}$-differentiable principal $G$-bundle $E(N, G, \pi, \Psi)$ for $C^a_{\beta'}$ differentiable spaces $\bar{M}$ and $\hat{M}$ as above over the same field or an algebra $A_r$, $0 \leq r \leq 3$, with $\alpha' \geq \alpha$ assigns to each $C^a_{\beta}$ mapping $\gamma$ from $\bar{M}$ into $N$ and points $u_1, \ldots, u_k \in E_{y_0}$, where $y_0$ is a marked point in $N$, $y_0 = \gamma(s_0, q)$, $q = 1, \ldots, k$, a unique $C^a_{\beta}$ mapping $P_{\hat{\gamma}, u} : \hat{M} \to E$ satisfying conditions $(P1 - P4)$:

$(P1)$ take $\hat{\gamma} : \hat{M} \to N$ such that $\hat{\gamma} = \gamma \circ \Xi$, then $P_{\hat{\gamma}, u}(\hat{s}_{0, q}) = u_q$ for each $q = 1, \ldots, k$ and $\pi \circ P_{\hat{\gamma}, u} = \hat{\gamma}$

$(P2)$ $P_{\hat{\gamma}, u}$ is the $C^a_{\beta}$-mapping by $\gamma$ and $u$;

$(P3)$ for each $x \in \hat{M}$ and every $\phi \in Diff^a_{\beta}(\hat{M}, \{\hat{s}_{0,1}, \ldots, \hat{s}_{0,2k}\})$ the equality $P_{\hat{\gamma}, u}(\phi(x)) = P_{\gamma \circ \phi, u}(x)$ is satisfied, where $Diff^a_{\beta}(\hat{M}, \{\hat{s}_{0,1}, \ldots, \hat{s}_{0,2k}\})$ denotes the group of all $C^a_{\beta}$ homeomorphisms of $\hat{M}$ preserving marked points $\phi(\hat{s}_{0,q}) = \hat{s}_{0,q}$ for each $q = 1, \ldots, 2k$;

$(P4)$ $P_{\gamma, u}$ is $G$-equivariant, which means that $P_{\gamma, uz}(x) = P_{\gamma, u}(x)z$ for every $x \in \hat{M}$ and each $z \in G$.

Two $C^a_{\beta'}$-differentiable principal $G$-bundles $E_1$ and $E_2$ with parallel transport structures $(E_1, P_1)$ and $(E_2, P_2)$ are called isomorphic, if there ex-
exists an isomorphism $h : E_1 \to E_2$ such that $P_{2,\gamma,u}(x) = h(P_{1,\gamma,h^{-1}(u)}(x))$
for each $C^\alpha_{\beta}$-mapping $\gamma : \bar{M} \to N$ and $u_q \in (E_2)_{y_0}$, where $q = 1, \ldots, k$, $h^{-1}(u) = (h^{-1}(u_1), \ldots, h^{-1}(u_k))$.

2. Subspaces. For $M = \bar{M} \setminus \{s_0,q : q = 1, \ldots, k\}$ either ranges $h_j(W_j) = U_j$ of plots for a $C^\alpha_{\beta}'$ differentiable space $\bar{M}$ or an atlas $At(\bar{M})$ of a $C^\alpha_{\beta}'$ manifold $\bar{M}$ with charts $(U_j, \phi_j), j \in \Lambda_M$ we put

\[(1) \quad U_l = \bar{U}_l \setminus \{s_0,q : q = 1, \ldots, k\}\]

for each $l = 1, \ldots, k$ so that $s_0,q \in \bar{U}_q$ for every $q = 1, \ldots, k$ and either

\[(2) \quad h_q = \bar{h}_q|_{W_q} \text{ with } W_q = \bar{W}_q \setminus h_q^{-1}(s_0,q) \text{ or } \phi_l = \bar{\phi}_l|_{U_l} \text{ for all } q,l = 1, \ldots, k;\]

\[(3) \quad U_j = \bar{U}_j \text{ and } \phi_j = \bar{\phi}_j \text{ for each } j > k, \{s_0,q : q = 1, \ldots, k\} \cap \bar{U}_j = \emptyset \text{ for each } j > k; j \in \Lambda_M = \Lambda_{\bar{M}}, \text{ where due to the Kuratowski-Zorn theorem [7] we can consider, that } \Lambda_M \text{ is an ordinal}.\]

Let the spaces be the same as in §2.3, 4 with the covering of $M$ defined by Conditions (1 – 3). Suppose that $M$ is modeled on $X$ and $N$ on $Y$, where $X$ and $Y$ are $A_r$ vector spaces, $0 \leq r \leq 3$. Then we consider their subspaces of mappings preserving marked points relative to a given mapping $\theta \in C^\alpha_{\beta}(M,N)$:

\[(4) \quad C^\alpha_{\beta,0}(\bar{M}, \{s_0,q : q = 1, \ldots, k\}; (N,y_0)) := \{f \in C^\alpha_{\beta}(\bar{M}, N) : \lim_{|t_1|+\ldots+|t_m|\to 0} S^m(f_{t,j} - \theta_{t,j})(w^m_q(t_1, \ldots, t_m)) = 0 \text{ for each } m \in \{0, 1, \ldots, n\}, \forall j \in \Lambda_M, \forall l \in \Lambda_N, \forall q = 1, \ldots, k\},\]

where either $S^m = \Phi^m$ or $S^m = \Upsilon^m$ for $\alpha = n$ or $\alpha = [n]$ respectively, an argument is either $w^m_q(t_1, \ldots, t_m) = x^m_q \in U^{(m)}_{l,j}$ for $S^m = \Phi^m$ or $w^m_q(t_1, \ldots, t_m) = \ldots$
\[ x_q^{[m]} \in U_{l,j}^{[m]} \] for \( S^m = \Gamma^m \), \( f^{[m]} = \Gamma^mf \), where \( t_1, \ldots, t_m \in K \), \( x_q^{[0]} = x_q^{(0)} = x_q = s_{0,q} \), \( x_q^{[1]} = x_q^{(1)} = (x_q, v, t) \), \( v^{[0]} = v^{(0)} = v \), \( t_1 = t \), \( x_q^{[m]} = (x_q^{[m-1]}, v^{[m-1]}, t_m) \) for each \( m \geq 1 \), \( x^{(m)} := (x_q; v_1, \ldots, v_m; t_1, \ldots, t_m) \) so that \( w_q^m \) is in a domain either \( U_{l,j}^{(m)} \) of \( \Phi^m(f_{l,j} - \theta_{l,j}) \) or \( U_{l,j}^{[m]} \) of \( \Psi^m(f_{l,j} - \theta_{l,j}) \) correspondingly (see also §§2.2,3). For \( \alpha = \infty \) Condition (4) is imposed for each natural value of \( m \). When points \( s_{0,q} \) and \( y_0 \) are specified we can write shortly \( C_{\alpha,\beta}^{\infty}(M, N) \) instead of \( C_{\alpha,\beta}^{\infty}(M, \{s_{0,q} : q = 1, \ldots, k\}; (N, y_0)) \).

As usually a diffeomorphism group \( Diff_{\alpha,\beta}^\infty(\bar{M}) \) of the differentiable space \( \bar{M} \) consists of all surjective bijective mappings \( f \) from \( \bar{M} \) onto \( \bar{M} \) with \( f \) and \( f^{-1} \) belonging to the differentiability class \( C_{\alpha,\beta}^{\infty} \). We consider the following subgroup also:

\[
(5) Diff_{\beta,0}^\alpha(M) := \{ f \in Diff_{\beta}^\alpha(\bar{M}) : f(s_{0,q}) = s_{0,q} \ \forall q = 1, \ldots, k \}.
\]

We introduce also \( Diff_{\beta,0}^\alpha(M) \) of all continuous mappings \( f \) from \( \bar{M} \) into \( \bar{M} \) so that the restriction \( f|_{\bar{M}} \) on \( \bar{M} \) is bijective and surjective from \( \bar{M} \) onto \( \bar{M} \) so that \( f \) and \( f^{-1} \) belong to the class \( C_{\beta}^{\alpha} \) and \( f(s_{0,q}) = s_{0,q} \) for each \( q = 1, \ldots, k \) (see also §1, (S4, S5)). That is if \( f \in Diff_{\beta,0}^\alpha(M) \), then \( f|_{\bar{M}} \in Diff_{\beta}^\alpha(\bar{M}) \).

We call such \( f \) a (generalized) diffeomorphism of a \( C_{\beta}^{\alpha} \) differentiable space \( \bar{M} \), while \( Diff_{\beta,0}^\alpha(M) \) we call a group of (generalized) diffeomorphisms of a differentiable space \( \bar{M} \) preserving marked points \( s_{0,q}, q = 1, \ldots, k \).

It is worth to mention that in the particular case when \( \bar{M} \) is a \( C_{\beta}^{\alpha'} \) manifold with \( \alpha \leq \alpha' \) the family \( Diff_{\beta,0}^\alpha(M) \) coincides with \( Diff_{\beta,0}^\alpha(M) \), since for the manifold \( \bar{M} \) all charts \( \phi_j(U_j) \) are open in \( X \), so the set \( \bar{M} \) is the entire \( M \) (see also §2.3).
The action of $Di^\alpha_{\beta,0}(M)$ on $M$ induces isomorphism classes of $C^\alpha_\beta$ principal $G$ fiber bundles with parallel transport structure for which mappings $\gamma : \bar{M} \to N$ belong to the uniform space $C^{\alpha,\alpha_0}_{\beta,0}((M, \{s_{0,q} : q = 1, .., k\}); (N, y_0))$, where a mapping $\theta = w_0$ is constant: $w_0(\bar{M}) = \{y_0\}$ (see §1 and §2.6). We denote by $(S^M E)_{\alpha,\beta} := (S^M, \{s_{0,q} : q = 1, .., k\}) E; (N, y_0), G, P)_{\alpha,\beta}$ a set of $C^\alpha_\beta$-closures of all such isomorphism classes.

Recall that a subset $A$ of a topological space $B$ so that $A$ is dense in itself and closed in $B$ is called a perfect set $A$ [7].

A topological space $X$ is called a $T_0$ space if for each $x \neq y \in X$ there exists an open subset containing only one of these two points. A topological space $X$ is called a $T_1$-space if for each pair of distinct points $x \neq y \in X$ an open subset $U$ of $X$ exists so that $x \in U$ and $y \notin U$.

3. Theorems. 1. A uniform space $(S^M E)_{\alpha,\beta}$ from §2 exists and it has a structure of a topological $T_1$ alternative monoid with a unit and with a cancelation property and a multiplication operation of $C^\beta_\alpha$ class with $l = \alpha' - \alpha$ $(l = \infty$ for $\alpha' = \infty)$. If $M$, $N$ and $G$ are separable, then $(S^M E)_{\alpha,\beta}$ is separable. If $N$ and $G$ are complete, then $(S^M E)_{\alpha,\beta}$ is complete.

2. If $G$ is associative, then $(S^M E)_{\alpha,\beta}$ is associative. If $G$ is commutative, then $(S^M E)_{\alpha,\beta}$ is commutative. If $G$ is a Lie group, then $(S^M E)_{\alpha,\beta}$ is a Lie monoid.

3. The $(S^M E)_{\alpha,\beta}$ is non-discrete, totally disconnected and infinite and non locally compact for non degenerate $N$. Moreover, if $M$ and $N$ and $E$ are dense in themselves, then the $(S^M E)_{\alpha,\beta}$ is topologically dense in itself and has the cardinality $\text{card}[(S^M E)_{\alpha,\beta}] \geq c := \text{card}(Q_p)$.

Proof. We remind the following. Let $Q$ be a set and $T$ be a subset in
$Q \times Q$. Then $T$ is called a relation on the set $Q$. If $T$ satisfies Conditions $(E1 - E3)$:

(E1) $xT x$ for each $x \in Q$,

(E2) from $xTy$ there follows $yTx$,

(E3) $xTy$ and $yTz$ imply $xTz$, then $T$ is called an equivalence relation.

We mention that each equivalence relation $T$ on $Q$ defines some partition of $Q$ into non-intersecting subsets $A_x := \{ y \in Q : xTy \}$ being equivalence classes in $Q$ relative to $T$. Thus

(E4) $Q = \bigcup_{s \in S} A_s$ with $A_s \cap A_v = \emptyset$ for each $s \neq v \in S$, where $S$ is the corresponding set, $S \subset Q$. Moreover, $x$ and $y \in A_s$ if and only if $xTy$. Vice versa if a partition $\{A_s : s \in S\}$ of $Q$ into pairwise disjoint subsets $A_s$ is given, $A_s \cap A_v = \emptyset$ for each $s \neq v \in S$, then it induces an equivalence relation $T$ on $Q$ so that $xTy$ if and only if $x$ and $y \in A_s$ (see also [7]).

If $Q$ is a topological space and $T$ is some equivalence relation on $Q$, then $Q/T$ denotes a set of all equivalence classes in $Q$ relative to $T$. Then a mapping $q : Q \to Q/T$ exists posing to each point $x \in Q$ its equivalence class $A_x$. This mapping $q$ is called the quotient mapping. In a class of all topologies on $Q/T$ relative to which the quotient mapping is continuous a finest exists: it is a family $\tau_{Q/T}$ of all subsets $U$ in $Q/T$ for which $q^{-1}(U)$ is open. This topology $\tau_{Q/T}$ is called the quotient topology. Moreover, $(Q/T, \tau_{Q/T})$ is called the quotient space, $q$ is also called the natural quotient mapping or shortly the natural mapping (see also §2.4 [7]).

Let $Q$ and $R$ be two topological spaces and let $f : Q \to R$ be a continuous epimorphic mapping. It defines an equivalence relation $T(f)$ on $Q$ generated by a partition $\{f^{-1}(y) : y \in R\}$. Then the mapping $f$ can be presented as a
composition \( f = \bar{f} \circ q \), where \( q : Q \to Q/T(f) \) is the natural mapping, while \( \bar{f} \) is a mapping from \( Q/T(f) \) on \( R \) prescribed by the formula: \( \bar{f}(f^{-1}(y)) = y \) for each \( y \in R \). Evidently \( \bar{f} \) is the bijective continuous mapping, but generally it need not be a homeomorphism. A continuous epimorphic mapping \( f : Q \to R \) is called a quotient mapping, if it is a composition of a natural mapping \( q : Q \to Q/T \) and some homeomorphism, that is an equivalence relation \( T \) on \( Q \) exists and a homeomorphism \( h : Q/T \to R \) so that \( f = h \circ q \).

The following proposition 2.4.3 [7] is useful. For a mapping \( f \) of a topological space \( A \) on a topological space \( B \) the following conditions are equivalent:

(Q1) a function \( f \) is a quotient mapping,
(Q2) a set \( f^{-1}(U) \) is open in \( A \) if and only if \( U \) is open in \( B \),
(Q3) a set \( f^{-1}(C) \) is closed in \( A \) if and only if \( C \) is closed in \( B \),
(Q4) a mapping \( \tilde{f} : A/T(f) \to B \) is a homeomorphism.

We remind that a set \( \lambda \) is directed if it is supplied with a relation \( \leq \) satisfying the following three conditions:

(i) if \( x \leq y \) and \( y \leq z \), then \( x \leq z \);
(ii) \( x \leq x \) for each \( x \in \lambda \);
(iii) for each pair \( x, y \in \lambda \) an element \( z \in \lambda \) exists such that \( x \leq z \) and \( y \leq z \).

A subset \( A \) in a directed set \( \lambda \) is called cofinal if for each \( x \in \lambda \) an element \( a \in A \) exists so that \( x \leq a \). A set \( \lambda \) is ordered if (i, ii) are satisfied and the following:

(iv) if \( x \leq y \) and \( y \leq x \), then \( x = y \).

An element \( y \) of an ordered set \( \lambda \) is called maximal if from \( y \leq x \) the equality \( x = y \) follows. If \( \lambda \) is a set and \( \mathcal{P} \) is some property of its subsets,
then \( P \) is of finite character if the void set \( \emptyset \) has it, and a subset \( A \subset \lambda \) possesses it if and only if each finite subset of \( A \) possesses it.

At first we consider trivial bundles with \( G = \{e\} \). So the equivalence relation introduced in \( \S 2 \) we denote by \( K_{\alpha,\beta} \) and it takes the form:

\[ fK_{\alpha,\beta}g \text{ if and only if there exist nets } \]
\[ \{\psi_n \in D_{\beta,0}^\alpha(M) : n \in \Omega\}, \]
\[ \{f_n \in C_{\beta,0}^{\alpha,w_0}(M,N) : n \in \Omega\} \text{ and } \]
\[ \{g_n \in C_{\beta,0}^{\alpha,w_0}(M,N) : n \in \Omega\} \text{ such that } \]

\[(1) \ f_n(x) = g_n(\psi_n(x)) \text{ for each } x \in M \text{ and } \lim_n f_n = f \text{ and } \lim_n g_n = g, \]

where \( f, g \in C_{\beta,0}^{\alpha,w_0}(M,N) \) and the convergence is considered in this space, \( \Omega \) is a directed set. Due to Condition (1) these equivalence classes \( <f>_{K,\alpha,\beta} \) are closed in \( C_{\beta,0}^{\alpha,w_0}(M,N) \). Then for \( g \in <f>_{K,\alpha,\beta} \) we write \( gK_{\alpha,\beta}f \) also. The quotient space \( C_{\beta,0}^{\alpha,w_0}(M,N)/K_{\alpha,\beta} \) we denote by \( (S^M,N)_{\alpha,\beta} \), where \( w_0(M) = \{y_0\}, \theta = w_0 \).

Now we consider the wedge product \( A \vee B := \rho(\mathcal{Z}) \) be the wedge sum of pointed spaces \((A, \{a_{0,q} : q = 1, \ldots, k\}) \) and \((B, \{b_{0,q} : q = 1, \ldots, k\}) \), where \( \mathcal{Z} := [A \times \{b_{0,q} : q = 1, \ldots, k\} \cup \{a_{0,q} : q = 1, \ldots, k\} \times B] \subset A \times B, \) \( \rho \) is a continuous quotient mapping such that \( \rho(x) = x \) for each \( x \in \mathcal{Z} \setminus \{a_{0,q} \times b_{0,j} : q, j = 1, \ldots, k\} \) and \( \rho(a_{0,q}) = \rho(b_{0,q}) \) for each \( q = 1, \ldots, k, \) where \( A \) and \( B \) are topological spaces with marked points \( a_{0,q} \in A \) and \( b_{0,q} \in B, \) \( q = 1, \ldots, k. \)

Then the composition \( g \circ f \) of two elements \( f, g \in C_{\beta,0}^{\alpha,w_0}(M,N) \) is defined on the domain

\[(W1) \ M \vee M \setminus \{s_{0,q} \times s_{0,q} : q = 1, \ldots, k\} =: M \vee M. \]
Let $M = \bar{M} \setminus \{ s_{0,q} : q = 1, \ldots, k \}$ be as in §1. In view of Conditions 2.4(D1 – D8) we can choose a refinement of an initial covering.

We shall use the Teichmüller-Tukey’s lemma [7]. If $\lambda$ is a set, while $\mathcal{P}$ is a property of a finite order, then each subset $A \subset \lambda$ having this property $\mathcal{P}$ is contained in a set $B$ also satisfying $\mathcal{P}$ and $B$ is a maximal element in an ordered by inclusion family of all subsets of $\lambda$ having the property $\mathcal{P}$.

The topological space $(M, \tau_{p,M})$ is totally disconnected and it is not compact. By its construction the set $\tilde{M}$ is open in $M$, since $\tilde{M} = \bigcup \{ V : h_{\tilde{M}}(V) \text{ is open in } X \}$ for some plot $h \in \mathcal{P}_{\tilde{M}}$, where $\mathcal{P}_{\tilde{M}}$ is a family of plots defining a $C_{\beta}^{0'}$ differentiable structure of $\tilde{M}$ (see also §3.1 above). Put $A_x = \{ x \}$ for each $x \in \tilde{M} \setminus \{ s_{0,q} : q = 1, \ldots, k \}$ and $\bigcup_{q=1}^k A_{s_{0,q}} = \tilde{M} \setminus (\tilde{M} \setminus \{ s_{0,q} : q = 1, \ldots, k \})$, each $A_{s_{0,q}}$ is clopen in $\tilde{M} \setminus (\tilde{M} \setminus \{ s_{0,q} : q = 1, \ldots, k \})$, $A_{s_{0,q}} \cap A_{s_{0,t}} = \emptyset$ for each $1 \leq q \neq t \leq k$. Such partition $\{ A_x \}$ of $\tilde{M}$ induces an equivalence relation $T$ in $\tilde{M}$ (see above). Using the family of (generalized) diffeomorphisms $D_{i_{\beta,0}}^2(M)$ and the quotient space $\tilde{M}/T$ in case of necessity we can reduce our proof to the case, when classes of equivalent mappings are considered on $\tilde{M}$ satisfying the condition

\[(W2) \quad \tilde{M} \setminus (\tilde{M} \setminus \{ s_{0,q} : q = 1, \ldots, k \}) = \{ s_{0,q} : q = 1, \ldots, k \}.\]

We consider families which are bases of $\tau_{p,\tilde{M}}$ open neighborhoods $W_{v,q}$ of marked points $s_{0,q}$ in $\tilde{M}$, that is $s_{0,q} \in W_{v,q}$. Each family $\{ W_{v,q} : v \in \lambda_{q,M} \}$ is ordered by inclusion: $W_{u,q} \leq W_{v,q}$ if and only if $W_{v,q} \subset W_{u,q}$. Each finite intersection of open sets is open. So we fix an infinite (generalized) atlas

\[(2) \quad \hat{\mathcal{A}}'(M) := \{ (\tilde{U}_j', \phi_j') : j \in \lambda \} \text{ such that } \phi_j' : \tilde{U}_j' \to B_j \text{ are homeomorphisms on bounded } \mathcal{A}_r \text{ convex subsets } B_j \text{ in } X, \text{ each } \tilde{U}_j' \text{ is clopen in } (M, \tau_{p,M}), \lambda \text{ is a directed set. Moreover},\]
(3) for each \( q \in \{1, \ldots, k\} \) and every \( u \in \lambda_{q,M} \) there exists \( v \in \lambda \) so that for each \( v \leq j \in \lambda \) either
\[ \bar{U}_j^q \subset W_{u,q} \text{ or } \bar{U}_j^q \cap W_{u,q} = \emptyset. \]
Then also

(4) for each \( q \) a subset \( \omega_q \subset \lambda \) exists so that \( cl_{\bar{M}}[\bigcup_{j \in \omega_q} \bar{U}_j^q] \) is a clopen neighborhood of \( s_{0,q} \) in \( \bar{M} \), where \( cl_{\bar{M}}A \) denotes the closure of a subset \( A \) in \((\bar{M}, \tau_{p,\bar{M}})\). Property (4) follows from (2, 3). The topological space \((M, \tau_{p,M})\) is not compact, hence

(5) \( \text{card}(\lambda) \geq \text{card}(\mathbb{Z}) = \aleph_0. \)

In the wedge product \( M \lor M \) we choose the following atlas

(6) \( \bar{\Lambda}'(M \lor M) = \{(W_l, \xi_l) : l \in \mu\} \) such that \( \xi_l : W_l \to C_l \) are homeomorphisms, \( C_l = C_{l,1} \lor' C_{l,2} \), each \( C_{l,1} \) and \( C_{l,2} \) are bounded \( \mathcal{A}_r \) convex subsets in \( X \), where we denote \([ (C_{l,1} \cup \{x_{0,q} : q = 1, \ldots, k\}) \lor (C_{l,2} \cup \{x_{0,q} : q = 1, \ldots, k\}) \) \( \setminus \{x_{0,q} \times x_{0,q} : q = 1, \ldots, k\} =: C_{l,1} \lor' C_{l,2} \) for suitable distinct marked points \( x_{0,q} \) in \( X \) corresponding to \( s_{0,q} \), \( \mu \) is a directed set; also

(7) for each \( q_1, q_2 \in \{1, \ldots, k\} \) and every \( u_1 \in \lambda_{q_1,M} \) and each \( u_2 \in \lambda_{q_2,M} \) there exists \( v \in \mu \) so that for each \( v \leq l \in \mu \) either
\[ W_l \subset W_{u_1,q_1} \lor W_{u_2,q_2} \text{ or } W_l \cap (W_{u_1,q_1} \lor W_{u_2,q_2}) = \emptyset. \]
Therefore,

(8) for each \( q_1, q_2 \in \{1, \ldots, k\} \) a subset \( \nu_{q_1,q_2} \subset \mu \) exists so that \( cl_{\bar{M} \lor \bar{M}}[\bigcup_{l \in \nu_{q_1,q_2}} W_l] \) is a clopen neighborhood of \( s_{0,q_1} \lor s_{0,q_2} \) in \( \bar{M} \lor \bar{M} \), where \( cl_{\bar{M} \lor \bar{M}}A \) denotes the closure of a subset \( A \) in \((\bar{M} \lor \bar{M}, \tau_{p,\bar{M}} \lor \tau_{p,\bar{M}})\). We get property (8) from Conditions (6, 7). Since the topological space \((M \lor M, \tau_{p,M} \lor \tau_{p,M})\) also is not compact (see \( W_1, W_2 \) above), then the cardinalities of \( \lambda \) and \( \mu \) are the same, so we can choose \( \omega_q \) and \( \nu_{q_1,q_2} \) so that

(9) \( \text{card}(\lambda \setminus \bigcup_{q=1}^k \omega_q) = \text{card}(\lambda \setminus \omega_{q_1}) = \text{card}(\mu \setminus \bigcup_{q_2 \in \{1, \ldots, k\}} \nu_{q_1,q_2}) = \text{card}(\mu \setminus \nu_{q_1,q_2}) \geq \aleph_0, \) also \( \text{card}(\nu_{q_1,q_2}) = \text{card}(\omega_{q_1}) = \text{card}(\omega_{q_2}) \geq \aleph_0 \) for all
\(q_1, q_2 \in \{1, \ldots, k\}\), where the product topology \(\tau_{p,M} \times \tau_{p,M}\) on \(M \times M\) induces the corresponding topology on the subset \(M \vee M\). We denote this topology on \(M \vee M\) by \(\tau_{p,M} \times \tau_{p,M}\) or \(\tau_{p,M} \vee M\). Due to the Zermelo’s Theorem [7] we can consider sets \(\omega_q\) and \(\nu_{q_1,q_2}\) as ordinals \([\omega_q]\) and \([\nu_{q_1,q_2}]\) of the same type \([\omega_q] = [\nu_{q_1,q_2}]\) for all \(q_1, q_2 \in \{1, \ldots, k\}\).

In view of the Teichmüller-Tukey’s lemma we can choose \(W_l\) consistent with \(\tilde{U}_{j_1} \vee \tilde{U}_{j_2}\) such that to fix a (generalized) \(C^\alpha_{\beta,0}\)-diffeomorphisms \(\chi: M \vee M \to M\) satisfying the following conditions (10 – 12):

1. \(\chi(W_l) = \tilde{U}'_l\) for each \(l \in [\nu_{q_1,q_2}] \forall q_1, q_2 \in \{1, \ldots, k\}\) and
2. \(\chi(W_l) = \tilde{U}'_{\kappa(l)}\) for each \(l \in (\mu \setminus \bigcup_{q_1,q_2 \in \{1, \ldots, k\}} \nu_{q_1,q_2}\), where
3. \(\kappa: (\mu \setminus \bigcup_{q_1,q_2 \in \{1, \ldots, k\}} \nu_{q_1,q_2}) \to (\lambda \setminus \bigcup_{\eta \in \{1, \ldots, k\}} \omega_{\eta})\)

is a bijective mapping (see also §2 above). This induces the continuous injective homomorphism

\[
\chi^*: C_{\beta,0}^{\alpha,\text{uw}}(\{s_0,q_1 \times s_0,q_2 : q_1, q_2 \in \{1, \ldots, k\}\}; (N, y_0)) \to C_{\beta,0}^{\alpha,\text{uw}}(\{s_0,q : q = 1, \ldots, k\}; (N, y_0))
\]

such that

\[
\chi^*(g \vee f)(x) = (g \vee f)(\chi^{-1}(x))
\]

for each \(x \in M\), where \((g \vee f)(y) = f(y)\) for each \(y \in M_2\) and \((g \vee f)(y) = g(y)\) for every \(y \in M_1\), \(M_1 \vee M_2 = M \vee M\), \(M_i = M\) for \(i = 1, 2\) are two copies of \(M\). Therefore

\[
g \circ f := \chi^*(g \vee f)
\]

may be considered as defined on \(M\) also, that is, to \(g \circ f\) there corresponds the unique element in \(C_{\beta,0}^{\alpha,\text{uw}}(\{s_0,q : q = 1, \ldots, k\}; (N, y_0))\).
We have \( f(\psi) \in C^{\alpha_{\beta_0},w_0}_\beta(M, \{s_{0,q} : q = 1, \ldots, k\}; (N, y_0)) \) for each \( f \in C^{\alpha_{\beta_0},w_0}_\beta(M, \{s_{0,q} : q = 1, \ldots, k\}; (N, y_0)) \) and \( \psi \in D^{\alpha_{\beta_0}}(M) \) due to Lemma 9 and Corollary 10 in [28] applied uniformly by finite dimensional over \( K \) embedded into \( \bar{M} \) differentiable subspaces with the corresponding (generalized) atlases (see also [21, 24]). The diffeomorphism \( \chi : M \vee M \to M \) is of class \( C^2_{\beta_0} \), \( \alpha' \geq \alpha \), and from Conditions 2(4) for \( f_i \in C^{\alpha_{\beta_0},w_0}_\beta((M, \{s_{0,q} : q = 1, \ldots, k\}); (N, y_0)) \) it follows that for \( f = \chi^*(f_1 \vee f_2) \) also Condition 2(4) is satisfied, since \( \chi \) fulfils Conditions (10 - 14). Moreover, \( < f >_{K,\alpha,\beta} \circ < g >_{K,\alpha,\beta} = < f \vee g >_{K,\alpha,\beta} \) for each \( f \) and \( g \in C^{\alpha_{\beta_0},w_0}_\beta((M, \{s_{0,q} : q = 1, \ldots, k\}); (N, y_0)) \), since if \( f_n(x) = f_n(\eta_n(x)) \) and \( g_n(x) = g_n(\zeta_n(x)) \) for each \( x \in M \), then \( (f_n \vee g_n)(x) = (f_n(\eta_n(\chi(x))))(x) \) where \( \eta_n \) and \( \zeta_n \in D^{\alpha_{\beta_0}}(M) \).

Hence the composition is continuous for the quotient space.

In view of Conditions 2(4) for each \( f \in C^{\alpha_{\beta_0},w_0}_\beta((M, \{s_{0,q} : q = 1, \ldots, k\}); (N, y_0)) \) there exist nets \( \{\psi_n : n \in \Omega\}, \{\eta_n : n \in \Omega\} \) and \( \{\zeta_n : n \in \Omega\} \) in \( D^{\alpha_{\beta_0}}(M) \), \( \{f_n : n \in \Omega\}, \{w_{0,n} : n \in \Omega\} \) and \( \{g_n : n \in \Omega\} \) in \( C^{\alpha_{\beta_0},w_0}_\beta((M, \{s_{0,q} : q = 1, \ldots, k\}); (N, y_0)) \) such that \( [w_{0,n} \vee f_n](\psi_n \vee \eta_n(\chi(x))) = g_n(\zeta_n(\chi(x))) \), where

\[
\begin{align*}
(17) \lim_n f_n &= f, \quad \lim_n g_n = g, \\
(18) \lim_n w_{0,n} &= w_0, \quad f_n(x) \neq y_0 \text{ for each } x \in M,
\end{align*}
\]

\( \Omega \) is a directed set. On the other hand, from \( \lim_n(f_n \vee g_n) = f \vee g \) it follows that \( \lim_n f_n = f \) and \( \lim_n g_n = g \). We choose

\[
(19) \{\zeta_n : n \in \Omega\} \text{ so that for each open neighborhood } U \text{ of } \{s_{0,q} : q = 1, \ldots, k\} \text{ in } (\bar{M}, \tau_p,M) \text{ there exists } m \in \Omega \text{ for which } \zeta_n(\chi([M \times \{s_{0,q} : q = 1, \ldots, k\}] \cap [M \vee M])) \subset U \text{ for every } n \geq m \text{ in } \Omega.
\]
Using Formulas (10–15) and Conditions 2(4–6) we get $< w_0 \circ f >_{K,\alpha,\beta} = < f >_{K,\alpha,\beta}$ and $< w_0 >_{K,\alpha,\beta} = e$ is the unit element in $(S^M N)_{\alpha,\beta}$, since $< f >_{K,\alpha,\beta} \circ < g >_{K,\alpha,\beta} = < f \lor g >_{K,\alpha,\beta}$ for each $f$ and $g \in C^\alpha_{\beta,00}((M, \{s_0, q : q = 1, ..., k\}); (N, y_0))$.

Let us consider now the general case of fiber bundles. If a homomorphism $\theta : G \to F$ of $C^\alpha_{\beta}'$-differentiable groups exists, then an induced principal $F$ fiber bundle $(E \times^\theta F)(N, F, \pi^\theta, \Psi^\theta)$ is given with the total space $(E \times^\theta F) = (E \times F)/\mathcal{Y}$, where $\mathcal{Y}$ is the equivalence relation such that $(v g, f) \mathcal{Y}(v, \theta(g) f)$ for each $v \in E$, $g \in G$, $f \in F$. Then the projection $\pi^\theta : (E \times^\theta F) \to N$ is defined by the equation $\pi^\theta([v, f]) = \pi(v)$, where $[v, f] := \{(w, b) : (w, b) \mathcal{Y}(v, f), w \in E, b \in F\}$ denotes the equivalence class of $(v, f)$.

This implies that each parallel transport structure $\mathbf{P}$ on the principal $G$ fiber bundle $E(N, G, \pi, \Psi)$ induces a parallel transport structure $\mathbf{P}^\theta$ on the induced bundle by the formula $\mathbf{P}^\theta_{\hat{\gamma}, [u, f]}(x) = [\mathbf{P}_{\hat{\gamma}, u}(x), f]$.

We define multiplication with the help of certain embeddings and isomorphisms of spaces of functions. Mention that for each two $C^\alpha_{\beta}'$ diffeomorphic $C^\alpha_{\beta}'$ differentiable spaces $A$ and $B$ in a topological vector space $X$ over $\mathcal{A}^r$, the spaces $C^\alpha_{\beta}(A, Y)$ and $C^\alpha_{\beta}(B, Y)$ are isomorphic as topological $\mathcal{A}^r$ vector spaces, where $Y$ is also a topological vector space over $\mathcal{A}^r$, $\alpha \leq \alpha'$, consequently, $C^\alpha_{\beta}(A, N)$ and $C^\alpha_{\beta}(B, N)$ are isomorphic as uniform spaces. Naturally we consider the space

$C^\alpha_{\beta}(M, \{s_0,1, ..., s_0,k\}; \mathcal{W}, \{y_0\}) := \{(E, f) : E = E(N, G, \pi, \Psi) \in \mathcal{W}, f = \mathbf{P}_{\hat{\gamma}, y_0} \in C^\alpha_{\beta} : \pi \circ f(s_0, q) = y_0 \forall q = 1, ..., k; \pi \circ f = \hat{\gamma}, \gamma \in C^\alpha_{\beta,00}(M, N)\}$

which is the space of all $C^\alpha_{\beta}'$ principal $G$ fiber bundles $E$ with their parallel
transport \( C^\alpha_\beta \)-mappings \( f = P_{\gamma, y_0} \) in accordance with \( \S 2 \), where \( \mathcal{W} \) is the same family of all principal \( C^\alpha_\beta \)-fiber bundles \( E(N, G, \pi, \Psi) \) as in \( \S 2.6 \). Put \( \omega_0 = (E_0, P_0) \) be its element such that \( \gamma_0(M) = \{ y_0 \} \), where \( e \in G \) denotes the unit element, \( E_0 = N \times G, \pi_0(y, g) = y \) for each \( y \in N, g \in G, P_{\gamma_0, u} = P_0 \).

The mapping \( \Xi : \hat{M} \rightarrow M \) from \( \S 1 \) induces the embedding
\[
\Xi^* : C^\alpha_\beta(M, \{ s_{0,1}, ..., s_{0,k} \}; \mathcal{W}, y_0) \hookrightarrow C^\alpha_\beta(\hat{M}, \{ \hat{s}_{0,1}, ..., \hat{s}_{0,2k} \}; \mathcal{W}, y_0).
\]
We consider the wedge product \( g \vee f \) of two elements \( f, g \in C^\alpha_\beta((M, \{ s_{0,1}, ..., s_{0,k} \}); (N, y_0)) \) which is defined on the domain \( M \vee M \), where to \( f, g \) two mappings \( f_1, g_1 \in C^\alpha_\beta((\hat{M}, \{ \hat{s}_{0,1}, ..., \hat{s}_{0,2k} \}); (N, y_0)) \) correspond such that \( f_1 = f \circ \Xi \) and \( g_1 = g \circ \Xi \).

Suppose that \( (E_j, P_{\gamma_j, u_j}) \in C^\alpha_\beta(M, \{ s_{0,1}, ..., s_{0,k} \}; \mathcal{W}, y_0), j = 1, 2 \), then we take their wedge product \( P_{\gamma, u_1} := P_{\gamma_1, u_1} \vee P_{\gamma_2, u} \) on \( M \vee M \) with \( v_q = u_q g_{2,q} g_{1,q+k} \) for each \( q = 1, ..., k \) due to the alternativity of \( G \), \( \gamma = \gamma_1 \vee \gamma_2 \), where \( P_{\gamma_j, u_j} (\hat{s}_{j,0,q}) = y_0 \times g_{j,q} \in E_{y_0} \) for every \( j \) and \( q \). For each \( \gamma_j : M \rightarrow N \) a mapping \( \tilde{\gamma}_j : M \rightarrow E_j \) exists such that \( \pi \circ \tilde{\gamma}_j = \gamma_j \).

We denote by \( \mathbf{m} : G \times G \rightarrow G \) the multiplication operation in the group \( G \).

The wedge product \( (E_1, P_{\gamma_1, u_1}) \vee (E_2, P_{\gamma_2, u_2}) \) is the principal \( G \) fiber bundle \( (E_1 \times E_2) \times \mathbf{m} G \) with the parallel transport structure \( P_{\gamma_1, u_1} \vee P_{\gamma_2, u_2} \).

Due to Conditions \( (10 - 14) \) and \( 2(4, 5) \) we get the following embedding \( \chi^* : C^\alpha_\beta(M \vee M, \{ s_{0,q} \times s_{0,q} : q = 1, ..., k \}; \mathcal{W}, y_0) \hookrightarrow C^\alpha_\beta(M, \{ s_{0,q} : q = 1, ..., k \}; \mathcal{W}, y_0) \). Therefore, \( g \circ f := \chi^*(f \vee g) \) is the composition in \( C^\alpha_\beta(M, \{ s_{0,q} : q = 1, ..., k \}; \mathcal{W}, y_0) \).

Generalizing the beginning of this section we define the following equivalence relation \( K_{\alpha, \beta} \) in \( C^\alpha_\beta(M, \{ s_{0,q} : q = 1, ..., k \}; \mathcal{W}, y_0) : fK_{\alpha, \beta} \) if and only
if nets \( \eta_n \in Di^\alpha_{\beta,0}(M) \), also \( f_n \) and \( h_n \in C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \) with \( \lim_n f_n = f \) and \( \lim_n h_n = h \) such that \( f_n(x) = h_n(\eta_n(x)) \) for each \( x \in M \) and \( n \in \omega \), where \( \omega \) is a directed set and convergence is considered in \( C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \).

Thus the following quotient uniform space
\[
C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0)/K_{\alpha,\beta} =: (S^M E)_{\alpha,\beta}
\]
exists.

We consider an element \( f = P_{\hat{\gamma},u} \) as \( f \circ \Xi^{-1} \) on \( \tilde{M} \setminus \tilde{M}_f \), where \( \pi \circ f = \hat{\gamma} \), \( \hat{\gamma} = \gamma \circ \Xi \). We denote \( f \circ \Xi^{-1} \) also by \( f \). If \( M \) and \( N \) and \( G \) are separable, then \( C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \) is separable, consequently, \( (S^M E)_{\alpha,\beta} \) is also separable.

By our construction each equivalence class \( \langle f \rangle_{K,\alpha,\beta} \) is closed in \( C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \). Therefore, each point \( g \) in \( (S^M E)_{\alpha,\beta} \) is closed in it. A topological space \( S \) is \( T_1 \) if and only if each singleton (one-pointed set) \( \{g\} \) is closed in it (see §1.5 [7]). Thus the topological space \( (S^M E)_{\alpha,\beta} \) possesses the \( T_1 \) separability axiom.

The uniform space \( C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \) is complete due to Theorem 12.1.4 [31], when \( N \) and \( G \) are complete. Each class of \( K_{\alpha,\beta} \)-equivalent elements is closed in it. Consider reparametrizations of elements \( f \) of \( C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \) relative to the action \( f \mapsto f \circ \psi \), \( \psi \in Di^\alpha_{\beta,0}(M) \), of the family \( Di^\alpha_{\beta,0}(M) \) on \( \tilde{M} \). Then to each Cauchy net in \( (S^M E)_{\alpha,\beta} \) there corresponds a Cauchy net in \( C^\alpha_{\beta}(M,\{s_{0,q} : q = 1,...,k\};W, y_0) \). Hence \( (S^M E)_{\alpha,\beta} \) is complete, if \( N \) and \( G \) are complete.

If \( f, g \in C^\alpha_{\beta}(M, X) \) and \( f(M) \neq g(M) \), then
\[
\langle f \circ \psi - g \rangle_{K,\alpha,\beta} \neq \langle w_0 \times e \rangle_{K,\alpha,\beta} \quad \text{for each } \psi \in Di^\alpha_{\beta,0}(M).
\]
Thus equivalence classes \( \langle f \rangle_{K,\alpha,\beta} \) and \( \langle g \rangle_{K,\alpha,\beta} \) are different.
The uniform space $C^\alpha_{\beta}(M, \{s_{0,q} : q = 1, \ldots, k\}; W, y_0)$ is totally disconnected and dense in itself, when $M$ and $N$ and $E$ are dense in themselves, since $C^\alpha_{\beta}(M,Y)$ is such for each topological vector space $Y$ over $\mathcal{A}_r$. Thus, the uniform space $(S^M E)_{\alpha,\beta}$ is non-discrete and dense in itself.

Take a restriction of $E$ for a chosen (generalized) chart $(\phi^U, U)$ of $E$. In accordance with conditions on $E$ there exists $u \in U$ such that $V := \phi^U(U) - u$ is absolutely $\mathcal{A}_r$ convex, where $X_E$ is a topological $\mathcal{A}_r$ vector space on which $E$ is modeled, $\phi^U : U \to X_E$. The subspace $X_{E,u} := cl_{X_E}(span_{\mathcal{A}_r} V)$ we call the (generalized) tangent space at $u \in U$ to $E$. Then we can choose $U$ so that $\pi(U) =: U_N$ and $(\phi^U_N, U_N)$ is the chart of $N$, hence $X_{N,y} := cl_{X_N}(span_{\mathcal{A}_r} V_N)$ is the (generalized) tangent space at $y = \pi(u) \in U_N$ to $N$, where $V_N := \phi^U_N(U_N) - y$ is absolutely $\mathcal{A}_r$ convex in a topological vector space $X_N$ on which $N$ is modeled, $\pi : E \to N$ is the projection mapping of the fiber bundle. For non degenerate $N$ the space $X_{N,y}$ has a dimension over $\mathcal{A}_r$ not less than one. When $\alpha' \geq \alpha + 1$ the tangent bundle $TC^\alpha_{\beta}(M, E_U)$ is isomorphic with $C^\alpha_{\beta}(M, TE_U)$, where $TE_U$ is the $C^{\alpha'-1}_{\beta}$ fiber bundle. There is an infinite family of $f_\alpha \in C^\alpha_{\beta}(M, TE_U)$ with pairwise distinct images in $TE_U$ for different $\alpha$ such that $f_\alpha(M)$ is not contained in $\bigcup_{\beta < \alpha} f_\beta(M)$, $\alpha \in \Lambda$, where $\Lambda$ is an infinite ordinal. Therefore, $T(S^M E_U)_{\alpha,\beta}$ is an infinite dimensional fiber bundle due to (ii). We say that $(S^M E)_{\alpha,\beta}$ is infinite dimensional over $\mathcal{A}_r$ if for each (generalized) chart $U$ of $E$ the fiber bundle $T(S^M E_U)_{\alpha,\beta}$ is infinite dimensional over $\mathcal{A}_r$. For two fiber bundles $E_1$ and $E_2$ which are $C^\alpha_{\beta}$ isomorphic the uniform spaces $(S^M E_1)_{\alpha,\beta}$ and $(S^M E_2)_{\alpha,\beta}$ are $C^\alpha_{\beta}$ isomorphic. Thus $(S^M E)_{\alpha,\beta}$ is infinite and non locally compact for each $\alpha' \geq \alpha$, since $(S^M E_U)_{\alpha,\beta}$ is infinite dimensional over $\mathcal{A}_r$ and there is an embedding.
\((S^M E_U)_{\alpha,\beta} \leftrightarrow (S^M E)_{\alpha,\beta}\).

Evidently, if \(f \lor g = h \lor g\) or \(g \lor f = g \lor h\) for \(\{f, g, h\} \subset C^0(M, \{s_{0,q} : q = 1, \ldots, k\}; W, y_0)\), then \(f = h\). Thus \(\chi^*(f \lor g) = \chi^*(h \lor g)\) or \(\chi^*(g \lor f) = \chi^*(g \lor h)\) is equivalent to \(f = h\) due to the definition of \(f \lor g\) and the definition of equal functions, since \(\chi^*\) is the embedding. Using the equivalence relation \(K_{\alpha,\beta}\) gives \(< f >_K_{\alpha,\beta} \circ < g >_{K,\alpha,\beta} = < h >_{K,\alpha,\beta}\) or \(< g >_{K,\alpha,\beta} \circ < f >_{K,\alpha,\beta} = < g >_{K,\alpha,\beta} \circ < h >_{K,\alpha,\beta}\) is equivalent to \(< h >_{K,\alpha,\beta} = < f >_{K,\alpha,\beta}\). Therefore, \((S^M E)_{\alpha,\beta}\) has the cancelation property.

The group \(G\) is alternative, so \(a_{2,q}[a_{2,q}(a_{2,q+k}(a_{2,q}(a_{1,q+k})))] = a_{2,q+k}(a_{2,q}(a_{1,q+k}))\), hence \((P_1 \lor (P_2 \lor P_2) = (P_1 \lor P_2) \lor P_2; \text{ also } a_{2,q}[a_{2,q}(a_{1,q+k}(a_{1,q}(a_{1,q+k})))] = a_{1,q+k}(a_{1,q}(a_{1,q+k}))\), consequently, \((P_1 \lor (P_1 \lor P_2) = (P_1 \lor P_1) \lor P_2\) and inevitably for equivalence classes \((aa)b = a(ab)\) and \(b(aa) = (ba)a\) for each \(a, b \in (S^M E)_{\alpha,\beta}\). Thus the \((S^M E)_{\alpha,\beta}\) is alternative.

Evidently \(M \lor (M \lor M)\) is (generalized) \(C^0_{\beta,0}\)-diffeomorphic with \((M \lor M) \lor M\) (see 2(6)).

If \(G\) is associative, then the parallel transport structure gives \((f \lor g) \lor h = f \lor (g \lor h)\) on \(M \lor M\) for each \(\{f, g, h\} \subset C^0_{\beta}(M, \{s_{0,q} : q = 1, \ldots, k\}; W, y_0)\). Applying the embedding \(\chi^*\) and the equivalence relation \(K_{\alpha,\beta}\) we get, that \((S^M E)_{\alpha,\beta}\) is associative \(< f >_{K,\alpha,\beta} \circ (< g >_{K,\alpha,\beta} \circ < h >_{K,\alpha,\beta}) = (< f >_{K,\alpha,\beta} \circ < g >_{K,\alpha,\beta}) \circ < h >_{K,\alpha,\beta}\).

Let \(w_0\) be a mapping \(w_0 : M \rightarrow W\) such that \(w_0(M) = \{y_0 \times e\}\). Consider \(w_0 \lor (E, f)\) for some \((E, f) \in C^0_{\beta}(M, \{s_{0,q} : q = 1, \ldots, k\}; W, y_0)\). Suppose \((E, f) \in C^0_{\beta}(M, \{s_{0,q} : q = 1, \ldots, k\}; W, y_0)\). A net \(U_n\) of open or canonical closed subsets in \(M\) exists such that \(\bigcap U_n = \{s_{0,q} : q = 1, \ldots, k\}\). We mention that for each marked point \(s_{0,q}\) in \(M\) there exists a neighborhood \(U\) of \(s_{0,q}\) in
such that for each \( \gamma_1 \in C^0_{\alpha}(\gamma_1); (N, y_0) \) there exists \( \gamma_2 \in C^0_{\beta} \) such that they are \( K_{\alpha, \beta} \) equivalent and \( \gamma_2|_U = y_0 \) due to Conditions 2(4 – 6) and (10 – 15).

A net \( \eta_n \in Di^0_{\alpha, \beta}(M) \) exists together with \( w_n, f_n \in C^0_{\alpha}(M, \{s_0, q : q = 1, ..., k\}; \mathcal{W}, y_0) \) with

\[
(20) \lim_{n} f_n = f, \lim_{n} w_n = w_0 \quad \text{and} \quad \lim_{n} \chi^{\ast}(f_n \vee w_n)(\eta_n^{-1}) = f \quad \text{due to} \quad \pi \circ f(s_0) = s_0 \quad \text{in the formula of difference quotients of compositions of functions (see also (17 – 19) above). Indeed, we can apply Lemma 9 and Corollary 10 in [28] uniformly by finite dimensional over K embedded into \( \bar{M} \) differentiable subspaces with the corresponding (generalized) atlases.}

Therefore, \( w_0 \vee (E, f) \) and \( (E, f) \) belong to the equivalence class < \( (E, f) >_{K, \alpha, \beta} := \{ g \in C^0_{\alpha}(M, \{s_0, q : q = 1, ..., k\}; \mathcal{W}, y_0) : (E, f)K_{\alpha, \beta}g \} \) due to (20). Thus, \( < w_0 >_{K, \alpha, \beta} \circ < g >_{K, \alpha, \beta} = < g >_{K, \alpha, \beta} \).

The \( C^0_{\alpha'} \) differentiable space \( (M \vee M) \setminus \{s_0, q \times s_0, q : q, j = 1, ..., k\} \) is open in \( \bar{M} \) and has the \( C^0_{\beta} \)-diffeomorphism \( \psi \) such that \( \psi(x, y) = (y, x) \) for each \( (x, y) \in ((M \times M) \setminus \{s_0, q \times s_0, q : q, j = 1, ..., k\}). \) Suppose now, that \( G \) is commutative. Then \( (f \vee g) \circ \psi|_{(M \times M \setminus \{s_0, q \times s_0, q : q, j = 1, ..., k\})} = g \vee f|_{(M \times M \setminus \{s_0, q \times s_0, q : q, j = 1, ..., k\})}. \) On the other hand, \( < f \vee w_0 >_{K, \alpha, \beta} = < f >_{K, \alpha, \beta} \circ < w_0 >_{K, \alpha, \beta} = < w_0 \circ < f >_{K, \alpha, \beta}, \) hence, \( < f \vee g >_{K, \alpha, \beta} = < f >_{K, \alpha, \beta} \circ < g >_{K, \alpha, \beta} = < w_0 \circ < f >_{K, \alpha, \beta} \circ < w_0 \vee g >_{K, \alpha, \beta} = < f \vee w_0 >_{K, \alpha, \beta} \circ < w_0 \vee g >_{K, \alpha, \beta} = < (f \vee w_0) \vee (w_0 \vee g) >_{K, \alpha, \beta} = < (w_0 \vee g) \vee (f \vee w_0) >_{K, \alpha, \beta} \) due to the existence of the unit element \( < w_0 >_{K, \alpha, \beta} \) and due to the properties of \( \psi. \) Indeed, take a net \( \psi_n \) as above. Therefore, the parallel transport structure gives \( (g \vee f)(\psi(x, y)) = (g \circ f)(y, x) \) for each \( x, y \in M, \) consequently,
\((f \circ g)K_{\alpha,\beta}(g \circ f)\) for each \(f, g \in C^\alpha_{\beta}(M, \{s_{0,q} : q = 1, ..., k\}; \mathcal{W}, y_0)\). The using of the embedding \(\chi^*\) gives that \((S^ME)_{K,\alpha,\beta}\) is commutative, when \(G\) is commutative.

The mapping \((f, g) \mapsto f \lor g\) from \(C^\alpha_{\beta,0}(M, \{s_{0,q} : q = 1, ..., k\}; \mathcal{W}, y_0)^2\) into \(C^\alpha_{\beta,0}(M \lor M \setminus \{s_{0,q} \times s_{0,j} : q, j = 1, ..., k\}; \mathcal{W}, y_0)\) is of class \(C^\alpha_{\beta}\). Since the mapping \(\chi^*\) is of class \(C^\alpha_{\beta}\), then \((f, g) \mapsto \chi^*(f \lor g)\) is the \(C^\alpha_{\beta}\)-mapping. The quotient mapping from \(C^\alpha_{\beta}(M, \{s_{0,q} : q = 1, ..., k\}; \mathcal{W}, y_0)\) into \((S^ME)_{K,\alpha,\beta}\) is continuous and induces the quotient uniformity. On the other hand, \(T^b(S^ME)_{K,\alpha,\beta}\) has embedding into \((S^MT^bE)_{\alpha,\beta}\) for each \(1 \leq b \leq \alpha' - \alpha\), when \(\alpha' > \alpha\) is finite, for every \(1 \leq b < \infty\) if \(\alpha' = \infty\), since \(E\) is the \(C^\alpha_{\beta}\) fiber bundle, \(T^bE\) is the fiber bundle with the base space \(N\). Hence the multiplication \((< f >_{K,\alpha,\beta}, < g >_{K,\alpha,\beta}>) \mapsto < f >_{K,\alpha,\beta} \circ < g >_{K,\alpha,\beta} = < f \lor g >_{K,\alpha,\beta}\) is continuous in \((S^ME)_{\alpha,\beta}\) and is of class \(C^l_{\beta}\) with \(l = \alpha' - \alpha\) for finite \(\alpha'\) and \(l = \infty\) for \(\alpha' = \infty\).

The topological spaces \(E\) and \(N\) are of cardinalities not less than \(c\), hence \(\text{card}[(S^ME)_{\alpha,\beta}] \geq c\).

4. **Definition.** The object \((S^ME)_{\alpha,\beta}\) from Theorem 3 we call the wrap monoid.

5. **Corollary.** Let \(\phi : \bar{M}_1 \to \bar{M}_2\) be a surjective \(C^\alpha_{\beta}\)-mapping of \(C^\alpha_{\beta}\) differentiable spaces over the same Cayley-Dickson algebra \(A_r\), \(1 \leq r \leq 3\), or a field \(K = A_0\), such that \(\phi(s_{1,0,q}) = s_{2,0,a(q)}\) for each \(q = 1, ..., k_1\), where \(\{s_{j,0,q} : q = 1, ..., k_j\}\) are marked points in \(\bar{M}_j\), \(j = 1, 2, 1 \leq a \leq k_2, l_1 \leq k_2, l_1 := \text{card} \phi(\{s_{1,0,q} : q = 1, ..., k_1\})\). Then there exists an induced homomorphism of topological monoids \(\phi^* : (S^{M_2}E)_{\alpha,\beta} \to (S^{M_1}E)_{\alpha,\beta}\). If \(l_1 = k_2\), then \(\phi^*\) is the embedding.

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Proof. Take \( \Xi_1 : \hat{M}_1 \to \hat{M}_1 \) with marked points \( \{ \hat{s}_{1,0,q} : q = 1, \ldots, 2k_1 \} \) as in §1, then take \( \hat{M}_2 \) the same \( \hat{M}_1 \) with additional \( 2(k_2 - l_1) \) marked points \( \{ \hat{s}_{2,0,q} : q = 1, \ldots, 2k_3 \} \) such that \( \hat{s}_{1,0,q} = \hat{s}_{2,0,q} \) for each \( q = 1, \ldots, k_1, k_3 = k_1 + k_2 - l_1, \) then \( \phi \circ \Xi_1 := \Xi_2 : \hat{M}_2 \to \hat{M}_2 \) is the desired mapping inducing the parallel transport structure from that of \( M_1 \). Therefore, each \( \hat{\gamma}_2 : \hat{M}_2 \to N \) induces \( \hat{\gamma}_1 : \hat{M}_1 \to N \) and to \( P_{\hat{\gamma}_2, u^2} \) there corresponds \( P_{\hat{\gamma}_1, u^1} \) with additional conditions in extra marked points, where \( u^1 \subset u^2 \). The equivalence class \( < (E_2, P_{\hat{\gamma}_2, u^2}) >_{K, \alpha, \beta} \in (SM^2E)_{K, \alpha, \beta} \) gives the corresponding elements \( < (E_1, P_{\hat{\gamma}_1, u^1}) >_{K, \alpha, \beta} \in (SM^1E)_{K, \alpha, \beta} \), since \( D_{\hat{\gamma}_2,0}^{\alpha}(\hat{M}_1, \{ \hat{s}_{0,q} : q = 1, \ldots, 2k_2 \}) \subset D_{\hat{\gamma}_1,0}^{\alpha}(\hat{M}_1, \{ \hat{s}_{0,q} : q = 1, \ldots, 2k_3 \}) \). Then \( \phi^*(< (E_2, P_{\hat{\gamma}_2, u^2}) >_{K, \alpha, \beta} \) = \( \phi^*(< (E_1, P_{\hat{\gamma}_1, u^1}) >_{K, \alpha, \beta}) \) since \( f_2 \circ \phi(x) \) for each \( x \in \Xi_1(\hat{M}_1 \setminus \hat{M}_f) \) coincides with \( f_1(x) \), where \( f_j \) corresponds to \( P_{\gamma_j, g_0 \times e} \) (see also the beginning of §3).

If \( l_1 = k_2 \), then \( \hat{M}_1 = \hat{M}_2 \) and the family \( D_{\hat{\gamma}_2,0}^{\alpha}(\hat{M}_1) \) is the same for two cases, hence \( \phi^* \) is bijective and inevitably \( \phi^* \) is the embedding.

6. Theorems. 1. An alternative topological Hausdorff group \((W^M E)_{\alpha, \beta}\) exists containing the monoid \((S^M E)_{\alpha, \beta}\) and its group operation of \( C^1_{\beta} \) class is with \( l = \alpha' - \alpha \) (\( l = \infty \) for \( \alpha' = \infty \)). If \( M \) and \( N \) and \( G \) are separable, then \((W^M E)_{\alpha, \beta}\) is separable. If \( N \) and \( G \) are complete, then \((W^M E)_{\alpha, \beta}\) is complete.

2. If \( G \) is associative, then \((W^M E)_{\alpha, \beta}\) is associative. If \( G \) is commutative, then \((W^M E)_{\alpha, \beta}\) is commutative. If \( G \) is a Lie group, then \((W^M E)_{\alpha, \beta}\) is a Lie group.

3. The group \((W^M E)_{\alpha, \beta}\) is non-discrete, totally disconnected and non locally compact for non degenerate \( N \). Moreover, if there exist two different
sets of marked points \( s_{0,q,j} \) in \( \bar{M}_f \), \( q = 1, \ldots, k \), \( j = 1,2 \), then two groups \((W^M E)_{\alpha,\beta,j}\), defined for \( \{s_{0,q,j} : q = 1, \ldots, k\} \) as marked points, are isomorphic.

4. The \((W^M E)_{\alpha,\beta}\) has a structure of an \( C_\beta^0 \)-differentiable manifold over \( \mathcal{A}_r \).

**Proof.** If \( \gamma \in C_\beta^0((\bar{M}, \{s_{0,q} : q = 1, \ldots, k\}); (N, y_0)) \), then for \( u \in E_{y_0} \) there exists a unique \( h_q \in G \) such that \( P_{\gamma,u}(s_{0,q+k}) = u_q h_q \), where \( h_q = g_q^{-1} g_{q+k} \), \( y_0 \times g_q = P_{\gamma,u}(s_{0,q}), g_q \in G \). Due to the equivariance of the parallel transport structure \( h \) depends on \( \gamma \) only and we denote it by \( h^{(E,P)}(\gamma) = h(\gamma) = h, h = (h_1, \ldots, h_k) \). The element \( h(\gamma) \) is called the holonomy of \( P \) along \( \gamma \) and \( h^{(E,P)}(\gamma) \) depends only on the isomorphism class of \((E,P)\) due to the use of the family \( D\iota_{\beta,0}(\bar{M}) \) and boundary conditions on \( \hat{\gamma} \) at \( \hat{s}_{0,q} \) for \( q = 1, \ldots, 2k \).

Therefore, \( h^{(E_1,P_1)(E_2,P_2)}(\gamma) = h^{(E_1,P_1)}(\gamma)h^{(E_2,P_2)}(\gamma) \in G^k \), where \( G^k \) denotes the direct product of \( k \) copies of the group \( G \). Hence for each such \( \gamma \) there exists the homomorphism \( h(\gamma) : (S^M E)_{\alpha,\beta} \to G^k \), which induces the homomorphism \( h : (S^M E)_{\alpha,\beta} \to C^0(C_\beta^0((\bar{M}, \{s_{0,q} : q = 1, \ldots, k\}); (N, y_0)); G^k) \), where \( C^0(A,G^k) \) is the space of continuous maps from a topological space \( A \) into \( G^k \) and the group structure \( (hb)(\gamma) = h(\gamma)b(\gamma) \) (see also \( [9] \) for \( S^\alpha \)).

We construct now \((W^M N)_{\alpha,\beta}\) from \((S^M N)_{\alpha,\beta}\). In view of Theorem 3 we have the commutative monoid \((S^M N)_{\alpha,\beta}\) with the unit and the cancelation property. Algebraically a group associated with this monoid is the quotient group \( F/B \), where \( F \) is the free commutative group generated by \((S^M N)_{\alpha,\beta}\), while \( B \) is the minimal closed subgroup in \( F \) generated by all elements of the form \([f+g] - [f] - [g]\) with \( f \) and \( g \in (S^M N)_{\alpha,\beta} \), \([f]\) denotes the element in \( F \)
corresponding to \( f \) (see also about such abstract Grothendieck’s construction in [15, 36]).

In accordance with Theorem 3 the monoid \((S^M N)_{\alpha,\beta}\) is the topological \( T_1 \)-space. In view of Theorem 2.3.11 [7] the product of \( T_1 \)-spaces is the \( T_1 \)-space. On the other hand, for a topological group \( G \) from the separation axiom \( T_0 \) it follows, that \( G \) is the Tychonoff space (see Theorems 4.2 and 8.4 in [13] and also [7]). The latter means that for a topological group being \( T_0 \) or \( T_1 \) or Hausdorff or Tychonoff is equivalent.

At the same time the natural mapping \( \eta : (S^M N)_{\alpha,\beta} \to (W^M N)_{\alpha,\beta} \) is injective. We supply \( F \) with the topology inherited from the topology of the Tychonoff product \((S^M N)_{\alpha,\beta}^Z\), where each element \( z \) in \( F \) has the form \( z = \sum_f n_{f,z}[f] \), \( n_{f,z} \in Z \) for each \( f \in (S^M N)_{\alpha,\beta}, \sum_f |n_{f,z}| < \infty \). By the construction \( F \) and \( F/\mathcal{B} \) are \( T_1 \)-spaces, consequently, \( F/\mathcal{B} \) is the Tychonoff space. In particular, \([n.f] - n[f] \in \mathcal{B}\). We deduce that \( \eta \) is the topological embedding, since \( \eta(f+g) = \eta(f) + \eta(g) \) for each \( f, g \in (S^M N)_{\alpha,\beta}, \eta(e) = e \), since \((z+B) \in \eta(S^M N)_{\alpha,\beta} \), when \( n_{f,z} \geq 0 \) for each \( f \), and inevitably in the general case \( z = z^+ - z^- \), where \((z^+ + B) \) and \((z^- + B) \in \eta(S^M N)_{\alpha,\beta} \). The uniform space \((W^M E)_{\alpha,\beta} \) has embedding as the closed subset into \([((S^M E)_{\alpha,\beta}]^2 \). Thus if \( N \) and \( G \) are complete, then \((W^M E)_{\alpha,\beta} \) is the complete topological group, since the product of complete uniform spaces is complete (see Theorem 8.3.9 [7]) and \((S^M E)_{\alpha,\beta} \) is complete by Theorem 3 above.

Using plots and \( C^\alpha_{\beta'} \) transition mappings of (generalized) charts of \( N \) and \( E(N, G, \pi, \Psi) \) and equivalence classes relative to \( D_{\beta,0}^\alpha(M) \) we get, that \((W^M E)_{\alpha,\beta} \) has the structure of the \( C^\alpha_{\beta'} \)-differentiable manifold, since \( \alpha' \geq \alpha \).

The rest of the proof and the statements of Theorems 6(1-4) follows from
this and Theorems 3(1-3) and [18, 23].

The monoid \((S^M E_U)_{\alpha,\beta}\) is infinite dimensional over \(A_r\) due to Theorem 3.3, consequently, \((W^M E_U)_{\alpha,\beta}\) is infinite dimensional over \(A_r\), when \(\dim_{A_r}(X_U) \geq 1\) that is the case, since \(N\) is non degenerate. Thus \((W^M E)_{\alpha,\beta}\) is non locally compact.

7. Definition. The object \((W^M E)_{\alpha,\beta} = (W^M,\{s_{0,q} : q = 1, \ldots, k\}; N, G, P)_{\alpha,\beta}\) from Theorem 6.1 we call the wrap group.

8. Corollary. There exists the group homomorphism \(h : (W^M E)_{\alpha,\beta} \to C^0(C^\beta_{\alpha}(M, \{s_{0,q} : q = 1, \ldots, k\}; N, y_0), G^k)\).

The proof follows from §6 and putting \(h^f(\gamma) = (h^f(\gamma))^{-1}\).

9. Corollary. If \(M_1\) and \(M_2\) and \(\phi\) satisfy conditions of Corollary 5, then there exists a homomorphism \(\phi^* : (W^M_2 E)_{\alpha,\beta} \to (W^M_1 E)_{\alpha,\beta}\). If \(l_1 = k_2\), then \(\phi^*\) is the embedding.

10. Remark. Each \(C^\alpha_\beta\) manifold is a \(C^\alpha_\beta\) differentiable space. Above differentiable spaces or manifolds modeled on topological \(A_r\) vector spaces were considered. As a particular case of a topological vector spaces \(X\) may be a locally \(A_r\) convex vector space. It is well-known that in this case its topology is equivalently characterized by a family of continuous ultra-pseudo-norms.

We recall that a pseudo-norm \(v\) on \(X\) is called an ultra-pseudo-norm, if instead of the triangle inequality it satisfies the stronger condition: \(v(x + y) \leq \max(v(x), v(y))\) for all \(x, y \in X\). A locally \(A_r\) convex space \(X\) is complete if and only if it is a projective limit of Banach spaces over \(A_r\), since \(X = 0 X u_0 \oplus \ldots \oplus 2^{r-1} X u_{2^{r-1}}\) for each \(1 \leq r\), where \(0 X, \ldots, 2^{r-1} X\) are pairwise isomorphic locally \(K\) convex spaces (see [31, 32]). For \(r = 0\) these spaces
are usual $K$-linear spaces. We mention also that in an ultra-normed space $X$ each two balls $B(X,x,b) := \{z \in X : \|z - x\| \leq b\}$ and $B(X,y,c)$ either do not intersect or one of them is contained in another, where $0 < b, c < \infty$.

Above different classes $C^\alpha_\beta$ of smoothness were considered. In particular for $\alpha = 0$ this simply reduces to the class $C^0_\beta$ of continuous mappings. For $G = \{e\}$ there may $\alpha' = \alpha$ also be.

11. **Theorem.** For a wrap group $W = (W^ME)_{\alpha,\beta}$ (see Definition 7 above) there exists a skew product $\hat{W} = W \hat{\otimes} W$ which is an $C^l_\beta$ alternative group and there exists a group embedding of $W$ into $\hat{W}$, where $l = \alpha' - \alpha$ ($l = \infty$ for $\alpha' = \infty$). $E = E(N,G,\pi,\Psi)$ is a principal $G$-bundle of class $C^\alpha_{\alpha'}$ with $\alpha' \geq \alpha \geq 0$. If $G$ is associative, then $\hat{W}$ is associative.

**Proof.** Suppose that $\hat{W}$ is a set of all elements $(g_1 a_1 \otimes g_2 a_2) \in (W \otimes B)^2$, where $B$ is a free non-commutative associative group with two generators $a, b$, $ab \neq ba$, $g_1, g_2 \in W$. Take in $\hat{W}$ the equivalence relation: $g_1 g_2 a \otimes g_2 b \equiv g_1 e_B \otimes ee_B$, for each $g_1, g_2 \in W$, where $e$ and $e_B$ denote the unit elements in $W$ and in $B$. Define in $\hat{W}$ a multiplication by the formula:

$$(g_1 a_1 \otimes g_2 a_2) \hat{\otimes} (g_3 a_3 \otimes g_4 a_4) := ((g_1 g_3)(a_1 a_3) \otimes (g_4 g_2)(a_1^{-1} a_4 a_1) a_2)$$

for each $g_1, g_2, g_3, g_4 \in W$ and every $a_1, a_2, a_3, a_4 \in B$, hence

$$(e \otimes g_1 a_1) \hat{\otimes} (e \otimes g_2 a_2) = e \otimes (g_2 g_1)(a_2 a_1),$$

$$(g_1 a_1 \otimes e) \hat{\otimes} (g_2 a_2 \otimes e) = (g_1 g_2)(a_1 a_2) \otimes e,$$

$$(g_1 a_1 \otimes e) \hat{\otimes} (e \otimes g_4 a_4) = g_1 a_1 \otimes g_4(a_1^{-1} a_4 a_1),$$

$$(e \otimes g_4 a_4) \hat{\otimes} (g_1 a_1 \otimes e) := g_1 a_1 \otimes g_4 a_4.$$ 

Thus this semidirect product $\hat{W}$ of groups $(W \otimes B) \otimes^s (W \otimes B)$ is non-commutative, since $b^{-1} a b a^{-1} \neq e$, where $e := e \times e_B$, $\otimes^s$ denotes the semidirect product, $\otimes$ denotes the direct product.
We consider the minimal closed subgroup $A$ in the semidirect product $\hat{W}$ generated by elements $(g_1g_2a \otimes g_2b) \tilde{\otimes} (g_1e_B \otimes ee_B)^{-1}$, where $B$ is supplied with the discrete topology and $\hat{W}$ is supplied with the product uniformity. Then put $\hat{W} := \hat{W}/A =: W \tilde{\otimes} W$ and denote the multiplication in $\hat{W}$ as in $\tilde{W}$. We get for $W$ the group embedding $\theta : g \mapsto (ge_B \otimes e)$ into $\hat{W}$ and the multiplication $m[(g_1e_B \otimes e), (g_2e_B \otimes e)] = (g_1e_B \otimes e) \tilde{\otimes} (g_2e_B \otimes e)$.

On the other hand, $(ga_1 \otimes e) \tilde{\otimes} (e \otimes g_{a_2}a_2^{-1}) = ga_1 \otimes g_{a_2} = (e \otimes e) =: \tilde{e}$, $\tilde{e} = \tilde{e}A = A$ is the unit element in $\hat{W}$ and $(e \otimes g_{a_2}a_2^{-1}) = (ga_1 \otimes e)^{-1}$ is the inverse element of $(ga_1 \otimes e)$, where $a_2 \in B$ is such that $(a_1 \otimes a_2) \tilde{\otimes} A = (e \otimes e) \tilde{\otimes} A = A$ in $\hat{W}$, $a_1 = ea_1$, that is $a_1 \otimes a_2 \tilde{\otimes} e \otimes e$ in $\hat{W}$. The preceding formulas mean that $\hat{W}$ is noncommutative and alternative.

Moreover, $\hat{W}$ is the quotient of a $C^\alpha_\beta$ differentiable space or a manifold $W^2$ by the $C^\alpha_\beta$ equivalence relation $K_{\alpha, \beta}$, hence $\hat{W}$ is the $C^\alpha_\beta$ differentiable space, since Conditions $(D1 – D8)$ of §2.4 are satisfied. The group operation and the inversion in $\hat{W}$ combine the product in $W$ and the inversion with the tensor product and the equivalence relation, hence they are $C^\alpha_\beta$ differentiable with $l = \alpha' - \alpha, \ l = \infty$ for $\alpha' = \infty$, (see §§1.11, 1.12, 1.15 in [35] and Theorem 6 above).

Then
\[
((g_1 \otimes g_2) \tilde{\otimes} (g_3 \otimes g_4)) \tilde{\otimes} (g_5 \otimes g_6) := ((g_1g_3)g_5 \otimes g_6(g_4g_2)) \quad \text{and}
\]
\[
(g_1 \otimes g_2) \tilde{\otimes} ((g_3 \otimes g_4) \tilde{\otimes} (g_5 \otimes g_6)) := (g_1(g_3g_5) \otimes (g_6g_4)g_2).
\]
Therefore, $\hat{W}$ is alternative, since $W$ is alternative (see Theorem 6) and $B$ is associative. If $G$ is associative, then $W$ is associative and $\hat{W}$ is associative.

Let us consider the commutator
\[
[(g_1a_1 \otimes g_2a_2) \tilde{\otimes} (g_3a_3 \otimes g_4a_4)] \tilde{\otimes} [(g_1a_1 \otimes g_2a_2)^{-1} \tilde{\otimes} (g_3a_3 \otimes g_4a_4)^{-1}] = \{((g_1g_3)(a_1a_3) \otimes (g_4g_2)((a_1^{-1}a_4a_1)a_2)) \tilde{\otimes}
\]

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\[(g_1^{-1}a_1^{-1} \otimes g_2^{-1}(a_1a_2^{-1}a_1^{-1})) \otimes (g_3^{-1}a_3^{-1} \otimes g_4^{-1}(a_3a_4^{-1}a_3^{-1}))\]

\[= ((g_1g_3)(a_1a_3) \otimes (g_4g_2)((a_1^{-1}a_4a_1a_2) \otimes ((g_1^{-1}g_3^{-1})(a_1^{-1}a_3^{-1}) \otimes (g_1^{-1}g_2^{-1})\]

\[= (a_1(a_3a_4^{-1}a_3^{-1})a_1^{-1})(a_1a_2^{-1}a_1^{-1}))) = (((g_1g_3)(g_1^{-1}g_3^{-1}))(a_1a_3a_4^{-1}a_3^{-1}) \otimes \]

\[(g_1^{-1}g_2^{-1})(g_4g_2)(((a_1a_3)^{-1}((a_1a_3)^{-1}(a_1a_3)^{-1})(a_1a_3))((a_1^{-1}a_3a_4)a_2)\].

By the definition a minimal closed subgroup generated by products of such elements is the (topological) commutant \(\tilde{W}_c\) of \(\tilde{W}\). The group \((W^MN)_{\alpha,\beta}\) is commutative (see Theorem 6(2)). We have \(B/B_c = \{e\}\), the quotient group \(G/G_c = G_{ab}\) is the abelianization of \(G\), particularly if \(G\) is commutative, then \(G_{ab} = G\), where \(G_c\) denotes the (topological) commutant subgroup of \(G\). Therefore, we infer that

\[(W^M; N, G, P)_{\alpha,\beta}/[(W^M; N, G, P)_{\alpha,\beta}]_c = (W^M; N, G_{ab}, P)_{\alpha,\beta}\]

and inevitably we get \(\tilde{W}/\tilde{W}_c = (W^M; N, G_{ab}, P)_{\alpha,\beta}\).

12. **Remark.** We consider the group \(B^2 \otimes B^2/\mathcal{E}\), where an equivalence relation \(\mathcal{E}\) is induced by that of in \(\tilde{B}^2\) as in \(\tilde{W}\): \((a \otimes b) \approx (e \otimes e)\), the group \(B\) is the same as in §11 with two generators \(a, b\). Then this gives the equivalences: 

\[[(a \otimes b) \otimes (a \otimes b)] \mathcal{E} [(e \otimes e) \otimes (e \otimes e)] \mathcal{E} [(e \otimes b) \otimes (a \otimes e)] \otimes [(e \otimes b) \otimes (a \otimes e)] \mathcal{E} [(e \otimes b) \otimes [(a \otimes e) \otimes (e \otimes b)] \mathcal{E} (e \otimes e) \mathcal{E} (e \otimes a^{-1}ba) \otimes (a \otimes e) \mathcal{E} [(e \otimes ab) \otimes (ba \otimes e)] in B^2 \otimes B^2, since B^4 is the associative group. This implies the commutativity of the iterated skew product wrap group, when \(G\) is commutative, that is \((\tilde{W}^M(\tilde{W}^M E)_{\alpha,\beta})_{\alpha,\beta} = (W^M(W^M E)_{\alpha,\beta})_{\alpha,\beta}, G = G_{ab}\). In particular, \((\tilde{W}^M(\tilde{W}^M N)_{\alpha,\alpha})_{\alpha,\beta} = (W^M(W^M N)_{\alpha,\beta})_{\alpha,\beta}, where G = \{e\}.

13. **Proposition.** If there exists an \(C_{\beta}^{\alpha'}\)-diffeomorphism \(\eta : N \to N\) such that \(\eta(y_0) = y_0'\), where \(\alpha \leq \alpha'\) then wrap groups \((W^M E; y_0)_{\alpha,\beta}\) and
(W^M; y_0; y_0')_{\alpha,\beta} defined with marked points y_0 and y_0' are C^l_\beta-isomorphic as C^l_\beta-differentiable groups, where l = \alpha' - \alpha for finite \alpha', l = \infty for \alpha' = \infty.

Proof. Suppose that f \in C^\alpha_\beta(\tilde{M}, E), then \eta \circ \pi \circ f(s_0, q) = \eta(y_0) = y_0' for each marked point s_0, q in \tilde{M}, where \pi : E \to N is the projection, \pi \circ f = \gamma, \gamma is a wrap, that is an C^\alpha_\beta-mapping from \tilde{M} into N with \gamma(s_0, q) = y_0 for q = 1, ..., k. The differentiable space N is totally disconnected together with E and G in accordance with conditions imposed in Section 2. Consider the C^\alpha_\beta'-diffeomorphism \eta \times e of the principal bundle E. Then \Theta : C^\alpha_\beta(\tilde{M}, W) \to C^\alpha_\beta(\tilde{M}, W) is the induced isomorphism such that \pi \circ \Theta(f) := \eta \circ \pi \circ f : \tilde{M} \to N and (\eta \times e) \circ f = \Theta(f) for f \in C^\alpha_\beta(\tilde{M}, E). The mapping \Theta is C^l_\beta differentiable by f, hence it gives the C^l_\beta isomorphism of the considered C^l_\beta-differentiable wrap groups (see Theorem 6(1)).

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