Research Article

Qualitative Analysis for Multiterm Langevin Systems with Generalized Caputo Fractional Operators of Different Orders

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In this research work, we study two types of fractional boundary value problems for multi-term Langevin systems with generalized Caputo fractional operators of different orders. The existence and uniqueness results are acquired by applying Sadovskii’s and Banach’s fixed point theorems, whereas the guarantee of the existence of solutions is shown by Ulam–Hyer’s stability. Our reported results cover many outcomes as special cases. An example is provided to illustrate and validate our results.

1. Introduction

Fractional order models are more convenient than integer-order ones as fractional derivatives give superb tools for the portrayal of memory and inherited processes. Fractional differential equations (FDEs) have been applied in numerous fields, such as engineering, physical science, chemistry, financial matters, electrodynamics, aerodynamics, and dynamical systems (see [1–8]).

Some new contributions to Langevin FDEs have been investigated (see [9–13]) and the references referred to in that). There are many definitions of fractional integrals and derivatives, e.g., Riemann–Liouville type, Caputo type, Hadamard type, Hilfer type, and Erdelyi–Kober type, etc. With regard to the exciting development of local fractional calculus, it has been applied to deal with numerous different nondifferentiable problems in many applied fields (see [14–16]).

The generalized Riemann–Liouville definition with respect to another function was first presented by Osler [17]. Then, Kilbas et al. [2] presented important characteristics of this operator. The Caputo version called φ-Caputo fractional derivative has been introduced by Almeida [18]. Some amazing properties and generalized Laplace transform for the same operator were introduced by Jarad and Abdeljawad [19]. This recently defined fractional operator could model more precisely the process utilizing differential kernel. In order to evolve these definitions, special kernels and some kinds of operators are selected to apply on FDEs; for more details, we refer to some recent results associated with this development (see [20–29]).

In 2018, Almeida et al. [30] considered the following φ-Caputo type FDE with initial conditions:

\[
\begin{align*}
C^{\sigma_\alpha}_{a^+} \psi (\rho) &= g (\rho, \psi (\rho)), \quad \rho \in [a, b], \\
\psi (a) &= \psi_0, \quad \psi_k (a) = \psi_0^k, k = 1, \ldots, n - 1.
\end{align*}
\]

Ahmad and Nieto [31] considered the following Caputo-type Langevin FDE with boundary conditions:

\[
\begin{align*}
C^{\sigma_{\alpha}} (C^{\kappa}) + \lambda \psi (\rho) &= g (\rho, \psi (\rho)), \\
\psi (0) &= \gamma_1, \quad \psi (1) = \gamma_2.
\end{align*}
\]

In 2020, Laadjal et al. [32] studied a Caputo-type multiterm Langevin FDE with boundary conditions of the form:
In this work, we study two classes of generalized Caputo Langevin equations with two various fractional orders. This work is inspired by the recent works of Laadja et al. [32] and Almeida et al. [30]. Precisely, we consider the following Langevin-type fractional differential problems (FDPs):

\[
\begin{align*}
\mathcal{C} \mathcal{D}^\alpha \mathcal{D}^\mu \mathcal{L} \nu (r) &= g(r, \nu(r)), \quad \nu(0) = \nu'(0) = 0, \\
\mathcal{C} \mathcal{D}^\alpha \mathcal{D}^\mu \mathcal{L} \nu (r) &= g(r, \nu(r)), \\
\nu(0) = \nu'(0) = 0, \\
\end{align*}
\]

where \( \rho \in \mathcal{O} = [0, 1], \; 0 < k_1 \leq \ldots \leq k_m \leq 1, \; 1 < k \leq 2, \; 0 < \alpha \leq 1, \; 1 \leq k - k_1 < 2, \; (i = 1, 2, \ldots, m), \; m \in \mathbb{N}, \; \xi_i, \mu_i \in \mathbb{R}, \)
the symbol \( \mathcal{C} \mathcal{D}^\alpha \mathcal{D}^\mu \mathcal{L} \) denotes the generalized fractional derivative in the Caputo sense of order \( \theta \in [0, \kappa, k], \)
and \( g: \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R} \) is a given continuous nonlinear function.

Here, we investigate the existence and uniqueness of solutions for FDPs (1.4) and (1.5) involving generalized Caputo fractional derivatives of various orders. Moreover, the guarantee of the existence of solutions is proved by UH stability. The studied results expand and generalize many results by selecting special cases of the \( \psi \) function.

The paper is organized as follows: In Section 2, we give some definitions and lemmas that are used in the research paper. Section 3 derives equivalent fractional integral equations to the linear variants of Langevin FDPs (1.4) and (1.5). Section 4 deals with the qualitative analysis of proposed problems. In Section 5, we give an example to substantiate the main outcomes.

2. Preliminaries and Lemmas

We are beginning this portion by endowment with some essential definitions and results required for forthcoming analysis.

We consider the Banach space \( \mathcal{C}(\mathcal{O}, \mathbb{R}) \) with the norm \( \|v\|_\infty = \max\{|v(r)|, \rho \in \mathcal{O}\} \).

Let \( v: \mathcal{O} \rightarrow \mathbb{R} \) be an integrable function and \( \varphi \in \mathcal{C}^n(\mathcal{O}, \mathbb{R}) \) an increasing function such that \( \varphi' (r) \neq 0 \), for any \( r \in \mathcal{O} \).

**Definition 1** (see [2]). The \( \psi \)-Riemann–Liouville fractional integral of a function \( v \) of order \( \theta \) is described by

\[
\mathcal{I}_a^\psi \mathcal{R}_a^\psi v (r) = \frac{1}{\Gamma(\theta)} \int_a^r \varphi' (c) (\varphi (r) - \varphi (c))^{\psi - 1} v (c) d c.
\]

**Definition 2** (see [2]). The \( \psi \)-Riemann–Liouville fractional derivative of a function \( v \) of order \( \theta \) is described by

\[
\mathcal{D}_a^\theta \psi v (r) = \left( \frac{1}{\varphi' (r)} \frac{d}{d r} \right)^n \mathcal{D}_a^{n-\psi \theta} v (r),
\]

where \( n = [\alpha] + 1, \; n \in \mathbb{N} \).

**Definition 3** (see [18]). The \( \psi \)-Caputo fractional derivative of a function \( v \in \mathcal{C}^n(\mathcal{O}, \mathbb{R}) \) of order \( \theta \) is described by

\[
\mathcal{D}_a^\theta \psi v (r) = \mathcal{I}_a^{\psi - n} \mathcal{D}_a^{\theta - n} \mathcal{I}_a^{n} v (r) - \mathcal{I}_a^{\psi - n} \mathcal{D}_a^{\theta - n} v (r),
\]

where \( v^{[n]} (r) = (1/\varphi' (r)) (d/d \varphi)^n v (r) \) and \( n = [\alpha] + 1, \; n \in \mathbb{N} \).

**Lemma 1** (see [2, 18]). Let \( r_1, r_2 > 0 \). Then,

\[
(1) \; \mathcal{I}_a^{\lambda_1 \psi} ( \varphi (r) - \varphi (a)) \varphi' (r) = \Gamma (r_2) / \Gamma (r_1 + r_2) (\varphi (r) - \varphi (a)) \varphi' (r),
\]

\[
(2) \; \mathcal{I}_a^{\lambda_1 \psi} \mathcal{D}_a^\theta v (r) = \Gamma (r_2) / \Gamma (r_1 + r_2) (\varphi (r) - \varphi (a)) \mathcal{D}_a^\theta v (r),
\]

\[
(3) \; \mathcal{D}_a^\theta \psi \mathcal{D}_a^\theta v (r) = 0, \; \text{for} \; r_1 > k \in \mathbb{N}.
\]

**Lemma 2** (see [18]). If \( v \in \mathcal{C}^n(\mathcal{O}, \mathbb{R}) \) and \( r_1 \in (n - 1, n) \), then

\[
\mathcal{D}_a^\psi \mathcal{R}_a^\psi v (r) = \mathcal{D}_a^\psi \mathcal{D}_a^\psi v (r) = v (r) - \mathcal{D}_a^{\psi - n} (a^n) / \kappa ! (\varphi (r) - \varphi (a))^{\psi - n}.
\]

In particular, if \( r_1 \in (0, 1) \), we have

\[
\mathcal{D}_a^\psi \mathcal{R}_a^\psi v (r) = v (r) - v (a).
\]

If \( r_1 \in (1, 2) \), we have

\[
\mathcal{D}_a^\psi \mathcal{D}_a^\psi v (r) = v (r) - v (a) - v'_a (a) (\varphi (r) - \varphi (a)).
\]

Moreover, if \( v \in \mathcal{C}^n(\mathcal{O}, \mathbb{R}) \), then

\[
\mathcal{D}_a^\psi \mathcal{R}_a^\psi v (r) = v (r).
\]

**Lemma 3** (see [18]). Let \( 0 < \theta < 1 \). Then, the unique solution of the following linear FDP is as follows:

\[
\begin{align*}
\mathcal{C} \mathcal{L} \mathcal{D}^\psi \mathcal{D}^\mu v (r) &= g(r, \varphi), \quad \rho \in [0, \rho], \\
v(0) = v_0, \\
\end{align*}
\]

is obtained as

\[
v(r) = v_0 + \frac{1}{\Gamma (\theta)} \int_0^r \varphi' (c) (\varphi (r) - \varphi (c))^{\psi - 1} g (c) d c.
\]
Lemma 4. Let $1 < \theta < 2$. Then, the unique solution of the following linear FDP is as follows:

$$
\begin{cases}
C^{\theta}\mathcal{D}_0^\theta v(\rho) = g(\rho), & \rho \in [0, \rho], \\
v(0) = v_0, & v_0' = v_1,
\end{cases}
$$

(14)

is obtained as

$$
v(\rho) = c_0 + c_1 (\varphi(\rho) - \varphi(0)) + \frac{1}{\Gamma(\theta - 1)} \int_0^\rho \varphi'(c) (\varphi(\rho) - \varphi(c))^\theta - 2 g(c) dc,
$$

(16)

where $c_0, c_1 \in \mathbb{R}$. By conditions of (14), we get $c_0 = v_0$ and

$$
v_0' = \frac{v'(\rho)}{\varphi'(\rho)} = c_1 + \frac{1}{\Gamma(\theta - 1)} \int_0^\rho \varphi'(c) (\varphi(\rho) - \varphi(c))^\theta - 2 g(c) dc,
$$

(17)

which implies

$$
c_1 = v_1 - \frac{1}{\Gamma(\theta - 1)} \int_0^\rho \varphi'(c) (\varphi(\rho) - \varphi(c))^\theta - 2 g(c) dc.
$$

(18)

Substituting the value of $c_0$ and $c_1$ in (16), we get

$$
v(\rho) = v_0 + (\varphi(\rho) - \varphi(0)) \left[ v_1 - \frac{1}{\Gamma(\theta - 1)} \int_0^\rho \varphi'(c) (\varphi(\rho) - \varphi(c))^\theta - 2 g(c) dc \right]
$$

(19)

$$
+ \frac{1}{\Gamma(\theta)} \int_0^\rho \varphi'(c) (\varphi(\rho) - \varphi(c))^\theta - 1 g(c) dc,
$$

which is identical to (15).

\[\square\]

Theorem 1 (see [33]). Let $X$ be a Banach space. The map $P + Q$ is a $\lambda$-set contraction with $0 \leq \lambda < 1$, and thus also condensing, if (i) $P, Q : D \subseteq X \longrightarrow X$ are operators on $X$; (ii) $P$ is $\lambda$ contractive; (iii) $Q$ is compact.

To apply Sadovskii's and Banach's fixed point theorems, we will suffice herewith reference to [34, 35].

3. The Linear Variant of FDPs (1.4) and (1.5)

This section deals with the linear variant of FDPs (1.4) and (1.5). For simplicity, we denote $C^{\mathcal{D}_0^\alpha}$ and $\mathcal{D}_0^\alpha$ by $C^{\mathcal{D}^{\alpha}}$ and $\mathcal{D}^{\alpha}$, respectively.

Lemma 5. Let $1 < \kappa \leq 2$, $0 < \alpha \leq 1$, and $f \in C(\mathcal{U}, \mathbb{R})$. Then, $v$ is a solution of the following linear Langevin-type FDP:

$$
\begin{cases}
C^{\mathcal{D}^{\alpha}}(C^{\mathcal{D}^{\alpha}} + \mu)v(\rho) = f(\rho), \\
v(0) = v_0', 0 = v(1) = 0,
\end{cases}
$$

(20)

if and only if $v$ satisfies
\[ v(\rho) = \frac{1}{\Gamma(\alpha + \kappa)} \int_0^\rho \phi'(\zeta) (\varphi(\rho) - \varphi(\zeta))^{\alpha - 1} f(\zeta) d\zeta - \frac{\mu}{\Gamma(\kappa)} \int_0^\rho \phi'(\zeta) (\varphi(\rho) - \varphi(\zeta))^{\kappa - 1} v(\zeta) d\zeta \]

\[ + \frac{\mathcal{D}_x^\kappa(\rho)}{\mathcal{D}_x^\kappa(1)} \left( \frac{\mu}{\Gamma(\kappa)} \int_0^1 \phi'(\zeta) (\varphi(1) - \varphi(\zeta))^{\kappa - 1} v(\zeta) d\zeta \right) \]

\[ - \frac{1}{\Gamma(\alpha + \kappa)} \int_0^1 \phi'(\zeta) (\varphi(1) - \varphi(\zeta))^{\alpha - 1} f(\zeta) d\zeta \]  

(21)

where \( \mathcal{D}_x^\kappa(\rho) = [\varphi(\rho) - \varphi(0)]^\kappa \).

**Proof.** Applying the operator \( \mathcal{D}_x^\kappa \) on both sides of the first (20) and using Lemma 2, we get

\[ \left( \mathcal{D}_x^\kappa \varphi \right) \left( \mathcal{D}_x^\kappa f(\rho) + \mathcal{D}_x^\kappa \varphi v(\rho) + \frac{c_0}{\Gamma(\kappa + 1)} \mathcal{D}_x^\kappa f(\rho) \right) = \mathcal{D}_x^\kappa f(\rho) + c_0, \]

(22)

where \( c_0 \in \mathbb{R} \).

Next, applying the operator \( \mathcal{D}_x^\kappa \) on both sides of (22) and using Lemma 2, again, we obtain

\[ v(\rho) = c_1 + c_2 \mathcal{D}_x^\kappa(\rho) + \mathcal{D}_x^\kappa \varphi f(\rho) - \mu \mathcal{D}_x^\kappa \varphi v(\rho) + \frac{c_0}{\Gamma(\kappa + 1)} \mathcal{D}_x^\kappa(\rho), \]

(23)

where \( c_1, c_2 \in \mathbb{R} \). In view of (23), we have

\[ \nu'_x(\rho) = \frac{\nu(\rho)}{\varphi'(\rho)} = c_2 \mathcal{D}_x^\kappa(\rho) + \mathcal{D}_x^\kappa \varphi f(\rho) - \mu \mathcal{D}_x^\kappa \varphi v(\rho) + \frac{c_0}{\Gamma(\kappa + 1)} \mathcal{D}_x^\kappa(\rho), \]

(24)

where we used the fact that

\[ \left( \mathcal{D}_x^{\phi(\rho)}(\phi f(\rho)) \right)' = \left( \frac{1}{\varphi'(\rho)} \frac{d}{d\rho} \left( \mathcal{D}_x^{\phi(\rho)} f(\rho) \right) \right) = \left( \frac{1}{\varphi'(\rho)} \frac{d}{d\rho} \right) \left( \frac{1}{\Gamma(\theta)} \int_0^\rho \mathcal{D}_x^{\phi(\rho)}(\rho, \zeta) f(\zeta) d\zeta \right) = \frac{1}{\Gamma(\theta)} \int_0^\rho \mathcal{D}_x^{\phi(\rho)-2}(\rho, \zeta) f(\zeta) d\zeta = \mathcal{D}_x^{\phi(\rho)-1} f(\rho), \]

(25)

where \( \mathcal{D}_x^{\phi(\rho)}(\rho, \zeta) = \phi'(\zeta) (\varphi(\rho) - \varphi(\zeta))^\phi \). Using the initial conditions of (23), we find that \( c_1 = c_2 = 0 \) and

\[ \nu'_x(\rho) = v(\rho) \]

where \( v(\rho) = \left[ \mathcal{D}_x^\kappa \nu(\rho) + \mu \nu(\rho) - \sum_{i=1}^m \xi_i \mathcal{D}_x^\kappa \nu(\rho) \right] v(\rho) = f(\rho), v(0) = v'_x(0) = v(1) = 0. \]

(28)

if and only if \( v \) satisfies

**Lemma 6.** Let \( 1 < \kappa \leq 2, 0 < \alpha \leq 1, 1 \leq \kappa - \kappa_1 < 2, \) and \( f \in C(U, \mathbb{R}) \). Then, \( v \) is a solution of the following linear Langevin-type multiterm FDP:
\[
v(\rho) = \sum_{i=1}^{m} \frac{\xi_i}{\Gamma(\kappa - \kappa_i)} \int_{0}^{\rho} \phi'(\zeta)(\phi(\rho) - \phi(\zeta))^{\kappa - \kappa_i - 1} v(\zeta) d\zeta \\
\quad + \frac{1}{\Gamma(\alpha + \kappa)} \int_{0}^{\rho} \phi'(\zeta)(\phi(\rho) - \phi(\zeta))^{\alpha + \kappa - 1} f(\zeta) d\zeta \\
\quad - \frac{\sigma}{\Gamma(\kappa)} \int_{0}^{\rho} \phi'(\zeta)(\phi(\rho) - \phi(\zeta))^{\kappa - 1} v(\zeta) d\zeta \\
\quad + \frac{2^{\alpha_{+}}(\rho)}{2^{\alpha_{-}}(1)} \left( \frac{\sigma}{\Gamma(\kappa)} \int_{0}^{1} \phi'(\zeta)(\phi(1) - \phi(\zeta))^{\kappa - 1} v(\zeta) d\zeta - \sum_{i=1}^{m} \frac{\xi_i}{\Gamma(\kappa - \kappa_i)} \int_{0}^{1} \phi'(\zeta)(\phi(1) - \phi(\zeta))^{\kappa - \kappa_i - 1} v(\zeta) d\zeta \right) \\
\quad - \frac{1}{\Gamma(\alpha + \kappa)} \int_{0}^{1} \phi'(\zeta)(\phi(1) - \phi(\zeta))^{\alpha + \kappa - 1} f(\zeta) d\zeta,
\]

where \( \sigma = \mu - \sum_{i=1}^{m} \xi_i \mu_i. \)

**Proof.** Applying the operator \( \mathfrak{F}^{\alpha,\psi} \) on both sides of the first (28) and using Lemma 2, we get

\[
\mathfrak{F}^{\alpha,\psi}[\mathcal{D}^{\alpha_{+}}\left( \mathfrak{D}^{\alpha_{-}} + \mu \right) v(\rho)] - \sum_{i=1}^{m} \xi_i \mathfrak{F}^{\alpha,\psi}[\mathcal{D}^{\alpha_{+}}\left( \mathfrak{D}^{\alpha_{-}} + \mu_i \right) v(\rho)] = \mathfrak{F}^{\alpha,\psi} f(\rho),
\]

where \( c_0 \in \mathbb{R}. \)

Next, applying the operator \( \mathfrak{F}^{\kappa_{+},\psi} \) on both sides of (30) and using Lemma 2, again, we obtain

\[
v(\rho) = c_1 + c_2 [\phi(\rho) - \phi(0)] + \sum_{i=1}^{m} \xi_i \mathfrak{F}^{\kappa_{+},\psi} [v(\rho) - v(0)] \\
\quad + \mathfrak{F}^{\alpha_{+},\psi} f(\rho) - \sigma \mathfrak{F}^{\kappa_{+},\psi} v(\rho) + \frac{c_0}{\Gamma(\kappa + 1)} \mathfrak{F}^{\psi}(\rho),
\]

where \( c_1, c_2 \in \mathbb{R} \). In view of (31), we have

\[
0 = \sum_{i=1}^{m} \xi_i \mathfrak{F}^{\kappa_{+},\psi} v(1) + \mathfrak{F}^{\alpha_{+},\psi} f(1) - \sigma \mathfrak{F}^{\kappa_{+},\psi} v(1) + \frac{c_0}{\Gamma(\kappa + 1)} \mathfrak{F}^{\psi}(1),
\]

which implies

\[
c_0 = \frac{\Gamma(\kappa + 1)}{2^{\alpha_{-}}(1)} \left[ \sigma \mathfrak{F}^{\kappa_{+},\psi} v(1) - \sum_{i=1}^{m} \xi_i \mathfrak{F}^{\kappa_{+},\psi} v(1) - \mathfrak{F}^{\alpha_{+},\psi} f(1) \right].
\]

Substituting the values of \( c_0, c_1, \) and \( c_2 \) in (31), we obtain the solution given by (29). The converse follows by direct calculation. Hence, the proof is achieved. \( \square \)
4. Qualitative Analysis

4.1. Existence and Uniqueness Results. This subsection proves the existence and uniqueness results for FDPs (4) and (5) by applying Sadovskii’s theorem [34] and Banach’s theorem [35].

Theorem 2 (existence). Let $g: \mathbb{U} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. We assume that

\[
\begin{align*}
(f_1\nu)(\rho) &= \sum_{i=1}^{m} \xi_i \mathcal{F}_{\kappa}^{x - \kappa} \nu(\rho) + \mathcal{F}_{\kappa}^{x} \nu(\rho) - \sigma \mathcal{F}_{\kappa}^{x} \nu(\rho) \\
&\quad + \frac{\mathcal{F}_{\kappa}^{x}(\rho)}{\mathcal{F}_{\kappa}^{x}(1)} \left( \sigma \mathcal{F}_{\kappa}^{x} \nu(1) - \sum_{i=1}^{m} \xi_i \mathcal{F}_{\kappa}^{x} \nu(1) - \mathcal{F}_{\kappa}^{x} \nu(1, 1) \right),
\end{align*}
\]

for $\rho \in \mathbb{U}$. Let us define two operators $F_1, F_2: C(\mathbb{U}, \mathbb{R}) \rightarrow C(\mathbb{U}, \mathbb{R})$ by

\[
\begin{align*}
(F_1\nu)(\rho) &= \sum_{i=1}^{m} \xi_i \mathcal{F}_{\kappa}^{x - \kappa} \nu(\rho) - \sigma \mathcal{F}_{\kappa}^{x} \nu(\rho) \\
&\quad + \frac{\mathcal{F}_{\kappa}^{x}(\rho)}{\mathcal{F}_{\kappa}^{x}(1)} \left( \sigma \mathcal{F}_{\kappa}^{x} \nu(1) - \sum_{i=1}^{m} \xi_i \mathcal{F}_{\kappa}^{x} \nu(1) - \mathcal{F}_{\kappa}^{x} \nu(1, 1) \right),
\end{align*}
\]

\begin{align*}
(F_2\nu)(\rho) &= \mathcal{F}_{\kappa}^{x} \nu(\rho) - \frac{\mathcal{F}_{\kappa}^{x}(\rho)}{\mathcal{F}_{\kappa}^{x}(1)} \mathcal{F}_{\kappa}^{x} \nu(1, 1).
\end{align*}

\[\text{(H1): there exists a function } p \in C(\mathbb{U}, \mathbb{R}^+) \text{ such that } |g(\rho, x)| \leq p(\rho), \text{ for } (\rho, x) \in \mathbb{U} \times \mathbb{R}.
\]

\[\text{(H2): } \gamma_i = 2 \left( \sum_{i=1}^{m} |\xi_i| \mathcal{F}_{\kappa}^{x} \nu(1, 1) \right) \left( |\kappa - \kappa + 1| + |\sigma| \mathcal{F}_{\kappa}^{x} \nu(1, 1) \right) < 1.
\]

Then, the FDP (1.5) has at least one solution on $\mathbb{U}$.

Proof. Let $\mathcal{E}_r$ be a closed bounded and convex subset of $C(\mathbb{U}, \mathbb{R})$, where $r$ is a fixed constant. By virtue of Lemma 6, we define an operator $f: C(\mathbb{U}, \mathbb{R}) \rightarrow C(\mathbb{U}, \mathbb{R})$ as follows:

\[
\|F\nu\| \leq \sup_{\rho \in \mathbb{U}} \left\{ \sum_{i=1}^{m} |\xi_i| \mathcal{F}_{\kappa}^{x - \kappa} \nu(\rho) + |\sigma| \mathcal{F}_{\kappa}^{x} \nu(\rho) + \frac{\mathcal{F}_{\kappa}^{x}(\rho)}{\mathcal{F}_{\kappa}^{x}(1)} \left( \sigma \mathcal{F}_{\kappa}^{x} \nu(1) + \sum_{i=1}^{m} |\xi_i| \mathcal{F}_{\kappa}^{x - \kappa} \nu(1) \right) \\
+ |\sigma| \mathcal{F}_{\kappa}^{x} \nu(1, 1) \right\}
\]

\[
\leq 2\|\nu\| \left( \sum_{i=1}^{m} \left| \frac{\xi_i}{\Gamma(\kappa - \kappa + 1)} + |\sigma| \right| \left( \frac{\rho(1) - \rho(0))^{\kappa - 1}}{\Gamma(\kappa - \kappa + 1)} + |\sigma| \left( \frac{\rho(1) - \rho(1))^{\kappa - 1}}{\Gamma(\kappa + 1 + 1)} \right) \right) + 2\|\nu\| \left| \frac{\rho(1) - \rho(0))^{\kappa - 1}}{\Gamma(\alpha + \kappa + 1)} \right|
\]

\[
\leq 2\sum_{i=1}^{m} \left| \frac{\xi_i}{\Gamma(\kappa - \kappa + 1)} \mathcal{F}_{\kappa}^{x} \nu(1) + |\sigma| \frac{\mathcal{F}_{\kappa}^{x}(1)}{\Gamma(\kappa + 1 + 1)} \right| + 2\|\nu\| \frac{\mathcal{F}_{\kappa}^{x}(1)}{\Gamma(\alpha + \kappa + 1)}
\]

\[
= r\gamma_1 + \gamma_2 \leq r\gamma_1 + r(1 - \gamma_1) = r,
\]

which implies that $(F_1 + F_2)\mathcal{E}_r \subset \mathcal{E}_r$. 

We observe that $(F\nu)(\rho) = (F_1\nu)(\rho) + (F_2\nu)(\rho), \rho \in \mathbb{U}$.

In order to show that $F_1 + F_2$ has a fixed point, we prove that $F_1$ and $F_2$ satisfy the hypotheses of Sadovskii’s theorem. This will be provided in several steps: 

Step 1. $F\mathcal{E}_r \subset \mathcal{E}_r$.

Let us select $r \geq \gamma_2/1 - \gamma_1$, where

\[
\gamma_2 = 2\|\nu\| \frac{\mathcal{F}_{\kappa}^{x}(1)}{\Gamma(\alpha + \kappa + 1)}(1 + \gamma_1). 
\]

For any $\nu \in \mathcal{E}_r$, we have
Step 2. $F_2$ is compact. We note that $F_2$ is uniformly bounded from Step 1. Let $\rho_1, \rho_2 \in \mathcal{U}$ with $\rho_1 \leq \rho_2$ and $v \in \mathcal{E}_r$. Then, we obtain

\[
\left| (F_2v)(\rho_2) - (F_2v)(\rho_1) \right| \\
= \left| \frac{1}{\Gamma(\alpha + \kappa)} \int_0^{\rho_1} \left( \mathcal{A}_{\phi}^{\alpha - \kappa - 1}(\rho_2, s)g(s, v(s))ds - \mathcal{A}_{\phi}^{\alpha - \kappa}(\rho_2)(1) \mathcal{A}_{\phi}^{\alpha - \kappa}(1, s)g(s, v(s))ds \right) \right| \\
- \left| \frac{1}{\Gamma(\alpha + \kappa)} \int_0^{\rho_1} \left( \mathcal{A}_{\phi}^{\alpha - \kappa - 1}(\rho_1, s)g(s, v(s))ds + \mathcal{A}_{\phi}^{\alpha - \kappa}(\rho_1)(1) \mathcal{A}_{\phi}^{\alpha - \kappa}(1, s)g(s, v(s))ds \right) \right| \\
\leq \frac{\mathcal{A}_{\phi}^{\alpha - \kappa}(\rho_1) - \mathcal{A}_{\phi}^{\alpha - \kappa}(\rho_2)}{\mathcal{A}_{\phi}^{\alpha - \kappa}(1)} \frac{1}{\Gamma(\alpha + \kappa)} \int_0^{\rho_1} \left( \mathcal{A}_{\phi}^{\alpha - \kappa - 1}(1, s)g(s, v(s))ds \right) \\
+ \frac{1}{\Gamma(\alpha + \kappa)} \int_0^{\rho_1} \left| \mathcal{A}_{\phi}^{\alpha - \kappa - 1}(\rho_1, s) - \mathcal{A}_{\phi}^{\alpha - \kappa - 1}(\rho_2, s) \right| g(s, v(s))ds \\
\leq \frac{\left( \varphi(\rho_1) - \varphi(0) \right) - \left( \varphi(\rho_2) - \varphi(0) \right) \kappa}{\Gamma(\alpha + \kappa + 1)} \left( \varphi(1) - \varphi(0) \right) \kappa \|p\| \\
+ \frac{2\|p\|}{\Gamma(\alpha + \kappa + 1)} \left( \varphi(\rho_2) - \varphi(\rho_1) \right) \kappa \\
\leq \frac{2\|p\|}{\Gamma(\alpha + \kappa + 1)} \left( \varphi(\rho_2) - \varphi(\rho_1) \right) \kappa.
\]

From the continuity of $\varphi$, $|(F_2v)(\rho_2) - (F_2v)(\rho_1)| \to 0$ as $\rho_2 \to \rho_1$. Thus, $F_2$ is equicontinuous. So, by the Arzela–Ascoli theorem, $F_2(\mathcal{E}_r)$ is a relatively compact set.

Step 3. $F_1$ is $\gamma$ contractive. Let $v_1, v_2 \in \mathcal{E}_r$. Then, we have

\[
\|F_1v_1 - F_1v_2\| = \sup_{\rho \in \mathcal{U}} \left\{ m \sum_{i=1}^{m} \xi_i \mathcal{A}_{\phi}^{\alpha - \kappa - \rho}(s) \left| v_1(\rho) - v_2(\rho) \right| - \sigma \mathcal{A}_{\phi}^{\alpha - \kappa}(\rho) \left[ v_1(\rho) - v_2(\rho) \right] \right\} \\
\leq \sum_{i=1}^{m} \xi_i \mathcal{A}_{\phi}^{\alpha - \kappa}(s) \left| v_1 - v_2 \right| + |\sigma| \left( \mathcal{A}_{\phi}^{\alpha - \kappa}(s) \right) \left| v_1 - v_2 \right| \\
+ |\sigma| \left( \mathcal{A}_{\phi}^{\alpha - \kappa}(s) \right) \left| v_1 - v_2 \right| \\
\leq 2 \left( \sum_{i=1}^{m} \xi_i \mathcal{A}_{\phi}^{\alpha - \kappa}(s) \mathcal{A}_{\phi}^{\alpha - \kappa}(1) \Gamma(\kappa - \kappa_i + 1) + |\sigma| \frac{\mathcal{A}_{\phi}^{\alpha - \kappa}(1)}{\Gamma(\kappa + 1)} \right) \left| v_1 - v_2 \right| = \gamma_1 \left| v_1 - v_2 \right|,
\]

where $\gamma_1$ is the $\gamma$ constant of $F_1$. Since $\gamma_1 < 1$, the conclusion follows.
which is $p$ contractive since $\gamma_1 < 1$.

**Step 4.** $F$ is condensing.

Due to the fact that $F_1$ is continuous, a $p$ is a contraction and $F_2$ is compact, it follows from Lemma 1 that $F : \mathcal{S} \rightarrow \mathcal{S}$, with $F = F_1 + F_2$ is a condensing on $\mathcal{S}$. From the previous arguments, we conclude through Sadovskii’s theorem that $F$ has a fixed point. As a result, FDP (5) has a solution on $\mathcal{U}$.

The second result is based on Banach’s fixed point theorem.

**Theorem 3** (uniqueness). Let $g : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and there exists $L_g > 0$ such that

$$ |g(\rho, x) - g(\rho, x^*)| \leq L_g |x - x^*|, \quad \rho \in \mathcal{U}, x, x^* \in \mathbb{R}. $$

(40)

**Proof.** We apply Banach’s theorem to prove that $F$ defined by (35) has a fixed point. For this end, we show that $F$ is a contraction. Let $v, v^* \in C(\mathcal{U}, \mathbb{R})$ and $\rho \in \mathcal{U}$. Then,

\[
|Fv(\rho) - Fv^*(\rho)| \leq \sum_{i=1}^{m} \xi_i |3^{\alpha-k}\rho|^M |v(\rho) - v^*(\rho)| + |3^{\alpha-k}\rho| |v(\rho) - v^*(\rho)| + \frac{2^\alpha \rho(1)}{\Gamma(\alpha + k + 1)} \left| \frac{X^{\alpha-k\rho}}{\Gamma(\alpha-k)} \right| |v(1) - v^*(1)| + \sum_{i=1}^{m} \frac{2^\alpha \rho(1)}{\Gamma(\alpha + k + 1)} \left| \frac{X^{\alpha-k\rho}}{\Gamma(\alpha-k)} \right| |v(1) - v^*(1)|
\]

where

\[
|Fv(\rho) - Fv^*(\rho)| \leq \frac{2^\alpha \rho(1)}{\Gamma(\alpha + k + 1)} \left| \frac{X^{\alpha-k\rho}}{\Gamma(\alpha-k)} \right| |v - v^*| + \frac{2^\alpha \rho(1)}{\Gamma(\alpha + k + 1)} \left| \frac{X^{\alpha-k\rho}}{\Gamma(\alpha-k)} \right| |v - v^*|
\]

\[
= \frac{2^\alpha \rho(1)}{\Gamma(\alpha + k + 1)} \left| \frac{X^{\alpha-k\rho}}{\Gamma(\alpha-k)} \right| |v - v^*|,
\]

which implies

$$ ||Fv - Fv^*|| \leq \sigma ||v - v^*||. $$

(43)

As $\sigma < 1$, it follows that $F$ is a contraction. As a result of Banach’s theorem, there is a unique fixed point $v \in C(\mathcal{U}, \mathbb{R})$ such that $Fv = v$. Therefore, the FDP (5) has a unique solution on $\mathcal{U}$.

**Corollary 1** (existence). Let $g : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and there exists a function $p \in C(\mathcal{U}, \mathbb{R}^+)$ such that $|g(\rho, x)| \leq p(\rho)$, for $(\rho, x) \in \mathcal{U} \times \mathbb{R}$. If $\gamma_3 < 1$, where $\gamma_3 = 2|\mu|X^{\alpha-k\rho}(1)/\Gamma(\alpha-k)$, then the FDP (4) has at least one solution on $\mathcal{U}$.

**Corollary 2** (uniqueness). Let $g : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and there exists $M_g > 0$ such that
\( \varphi(\rho) = \rho^\alpha; \) our problems are reduced to Hadamard-type problems and Katugampola problems, respectively.

4.2. UH Stability Analysis. In this subsection, we discuss the UH stability of the considered problem.

**Definition 4.** FDP (5) is UH stable if there exists a constant \( Y_f > 0 \) such that for each \( \varepsilon > 0 \) and every solution \( \omega \in C(\mathbb{U}, \mathbb{R}) \) of the inequalities

\[
|\omega(\rho) - \varphi(\rho)| \leq Y_f \varepsilon, \quad \rho \in \mathbb{U},
\]

there exists a solution \( v \in C(\mathbb{U}, \mathbb{R}) \) of FDP (5) that satisfies

\[
|\omega(\rho) - v(\rho)| \leq Y_f \varepsilon.
\]

**Remark 4.** \( \omega \in C(\mathbb{U}, \mathbb{R}) \) satisfies the inequality (45) if and only if there exists a function \( \Pi \in C(\mathbb{U}, \mathbb{R}) \) with

(i) \( |\Pi(\rho)| \leq \varepsilon, \rho \in \mathbb{U} \)

(ii) For all \( \rho \in \mathbb{U} \)

\[
C^\alpha \varphi(\rho) - \sum_{i=1}^m \xi_i C^\alpha \varphi(\rho) + \mu_i \omega(\rho) = g(\rho, \omega(\rho)) + \Pi(\rho).
\]

**Lemma 7.** We suppose that \( 1 < \kappa \leq 2, 0 < \alpha \leq 1 \) and \( 1 \leq \kappa - \kappa_i < 2, \) and \( \omega \in C(\mathbb{U}, \mathbb{R}) \) is a solution of the inequality (45). Then, \( \omega \) satisfies

\[
\mathcal{W}(\rho) = \sum_{i=1}^m \xi_i \mathcal{F}^\kappa(\rho) - \sigma \mathcal{F}^\kappa(\rho)
\]

\[
\mathcal{F}^\kappa(\rho) = \frac{\mathcal{F}^\kappa(1)}{\Gamma(\alpha + \kappa + 1)}
\]

**Proof.** We suppose that \( \omega \) is a solution of (45). By Lemma 6 and (ii) of Remark 4, we have

\[
\begin{cases}
C^\alpha \varphi(\rho) - \sum_{i=1}^m \xi_i C^\alpha \varphi(\rho) + \mu_i \omega(\rho) = g(\rho, \omega(\rho)) + \Pi(\rho), \\
\omega(0) = \varphi(0) = \omega(1) = 0.
\end{cases}
\]

Then, the solution of FDP (50) is
\[
\omega(\rho) = \sum_{i=1}^{m} \xi \mathcal{X}_{\kappa,\phi}^{\kappa,\phi} \omega(\rho) + \mathcal{X}_{\phi}^{\kappa,\phi} \left( \frac{\mathcal{X}_{\phi}^{\kappa,\phi}(\rho)}{\mathcal{X}_{\phi}^{\kappa,\phi}(1)} \sigma \mathcal{X}_{\phi}^{\kappa,\phi} \omega(1) - \sum_{i=1}^{m} \xi \mathcal{X}_{\kappa,\phi}^{\kappa,\phi} \omega(1) - \mathcal{X}_{\phi}^{\kappa,\phi} g(1, \omega(1)) \right) + \mathcal{X}_{\phi}^{\kappa,\phi} \Pi(\rho) - \frac{\mathcal{X}_{\phi}^{\kappa,\phi}(\rho)}{\mathcal{X}_{\phi}^{\kappa,\phi}(1)} \mathcal{X}_{\phi}^{\kappa,\phi}(1).
\]

(51)

Again by (i) of Remark 4, it is implied that

\[
|\omega(\rho) - \mathcal{Y}(\rho) - \mathcal{X}_{\phi}^{\kappa,\phi} g(\rho, \omega(\rho))| \leq \mathcal{X}_{\phi}^{\kappa,\phi} \Pi(\rho)| + \frac{\mathcal{X}_{\phi}^{\kappa,\phi}(\rho)}{\mathcal{X}_{\phi}^{\kappa,\phi}(1)} \mathcal{X}_{\phi}^{\kappa,\phi}(1)
\]

(52)

**Theorem 4.** Under the hypotheses of Theorem 3, the solution of the FDP (5) is UH stable.

**Proof.** We suppose that \( \omega \in C(\mathcal{U}, \mathcal{R}) \) is a solution of the inequality (45) and \( v \in C(\mathcal{U}, \mathcal{R}) \) is a unique solution of FDP (5). From Lemma 6, we obtain

\[
v(\rho) = \mathcal{Y}(\rho) + \mathcal{X}_{\phi}^{\kappa,\phi} g(\rho, v(\rho)),
\]

where

\[
\mathcal{Y}(\rho) = \sum_{i=1}^{m} \xi \mathcal{X}_{\kappa,\phi}^{\kappa,\phi} v(\rho) + \mathcal{X}_{\phi}^{\kappa,\phi} g(\rho, v(\rho)) - \sigma \mathcal{X}_{\phi}^{\kappa,\phi} v(\rho)
\]

(53)

\[
|\omega(\rho) - v(\rho)| = |\omega(\rho) - \mathcal{Y}(\rho) - \mathcal{X}_{\phi}^{\kappa,\phi} g(\rho, v(\rho))|
\]

\[
\leq |\omega(\rho) - \mathcal{Y}(\rho) - \mathcal{X}_{\phi}^{\kappa,\phi} g(\rho, \omega(\rho))| + \mathcal{X}_{\phi}^{\kappa,\phi}|g(\rho, \omega(\rho)) - g(\rho, v(\rho))|
\]

\[
\leq 2\epsilon \mathcal{X}_{\phi}^{\kappa,\phi}(1) \frac{\mathcal{X}_{\phi}^{\kappa,\phi}(1)}{\Gamma(\alpha + \kappa + 1)} + \mathcal{X}_{\phi}^{\kappa,\phi} L g |\omega(\rho) - v(\rho)|
\]

(55)

which implies

\[
\left(1 - \frac{\mathcal{X}_{\phi}^{\kappa,\phi}(1) L g}{\Gamma(\alpha + \kappa + 1)}\right) |\omega(\rho) - v(\rho)| \leq 2\epsilon \frac{\mathcal{X}_{\phi}^{\kappa,\phi}(1)}{\Gamma(\alpha + \kappa + 1)}
\]

(56)

From (41), we get \( \mathcal{X}_{\phi}^{\kappa,\phi}(1) L g/\Gamma(\alpha + \kappa + 1) < 1 \). It follows that

\[
|\omega(\rho) - v(\rho)| \leq Y_f \epsilon,
\]

(57)

where

\[
Y_f = \frac{2\mathcal{X}_{\phi}^{\kappa,\phi}(1)}{\Gamma(\alpha + \kappa + 1) - \mathcal{X}_{\phi}^{\kappa,\phi}(1)L g}.
\]

(58)

Hence, the FDP (5) is UH stable in \( C(\mathcal{U}, \mathcal{R}) \).
Moreover, we can find that of advanced fractional calculus and characterizing a fixed associated with linear problems by applying the instruments.

In this paper, we have given some results dealing with the stretch be applied to an assortment of real-world problems.

5. Conclusions

We do not apply any significant bearing to the complex transformations, and our outcomes are characteristic of the integral operators’ theory of such kind. Indeed, our methodology is straightforward and can without much of a stretch be applied to an assortment of real-world problems. For the justification of the main results, we have given an example. As a special case, the reported results are new, and we have generalized many results with various values of $\varphi$ function.

We observe that

$$\left| g(\rho, v_1) - g(\rho, v_2) \right| \leq \frac{1}{100} \tan^{-1} v_1 - \tan^{-1} v_2 \leq \frac{1}{100} |v_1 - v_2| = L_g |v_1 - v_2|,$$

$$\mathcal{L}_\varphi^\kappa (1) = \left[ \varphi (1) - \varphi (0) \right] \kappa = \left[ \frac{11}{3} \right], \quad \sigma - \xi_1 \mu_1 = \frac{2}{3}.$$ 

Example 1. We consider the generalized Caputo-type Langevin FDP:

$$\left\{ \begin{array}{l}
\mathcal{D}^{\frac{1}{2}} \varphi \left( \mathcal{D}^{\frac{7}{4}} \varphi + \frac{5}{6} \right) v(\rho) - \frac{1}{3} \mathcal{D}^{\frac{1}{2}} \varphi \left( \mathcal{D}^{\frac{1}{2}} \varphi + \frac{1}{2} \right) v(\rho) = \frac{1}{100} \left( \cos \rho + \tan^{-1} v(\rho) \right), \quad \rho \in [0, 1], \\
v(0) = v'(0) = v(1) = 0,
\end{array} \right.$$ 

where $\alpha = 1/4, \kappa = 7/4, \mu = 5/6, m = 1, \xi_1 = 1/3, \kappa_1 = 1/5, \mu_1 = 1/2, 1 \leq \kappa - \kappa_1 = 31/20 < 2, \varphi(\rho) = \rho/3, \mathcal{U} = [0, 1],$ and

$$\mathcal{L}_\varphi^\kappa (1) = \left[ \varphi (1) - \varphi (0) \right] \kappa = \left[ \frac{11}{3} \right], \quad \sigma - \xi_1 \mu_1 = \frac{2}{3}.$$ 

We observe that

$$\mathcal{L}_\varphi^\kappa (1) = \left[ \varphi (1) - \varphi (0) \right] \kappa = \left[ \frac{11}{3} \right], \quad \sigma - \xi_1 \mu_1 = \frac{2}{3}.$$ 

Clearly, all the hypotheses of Theorem 3 hold, and hence, FDP (59) has a unique solution on $[0, 1].$ Furthermore, we have

$$|g(\rho, v)| \leq \cos \rho + 1 = \rho(\rho) \in C([0, 1], \mathbb{R}^+),$$

which satisfies the assumption $(H1)$ of Theorem 2. Moreover, we can find that

$$\gamma_1 = \frac{2}{9 \times 3^{17/20}} \Gamma (51/20) + \frac{4}{9 \times 3^{13/4}} \Gamma (11/7) \times 1.$$ 

Consequently, by Theorem 1, the FDP (59) has at least one solution on $[0, 1].$ Moreover, we have

$$\gamma_1 = \frac{200}{1799} > 0.$$ 

From Theorem 3, the FDP (59) is UH stable on $[0, 1].$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest related to this work.

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