RANSAC Algorithms for
Subspace Recovery and Subspace Clustering

Ery Arias-Castro  Jue Wang
University of California, San Diego

Abstract
We consider the RANSAC algorithm in the context of subspace recovery and subspace clustering. We derive some theory and perform some numerical experiments. We also draw some correspondences with the methods of Hardt and Moitra (2013) and Chen and Lerman (2009b).

1 Introduction
The Random Sample Consensus (with acronym RANSAC) algorithm of Fischler and Bolles (1981), and its many variants and adaptations, are well-known in computer vision for their robustness in the presence of gross errors (outliers). In this paper we focus on the closely related problems of subspace recovery and subspace clustering in the presence of outliers, where RANSAC-type methods are believed to be optimal, yet too costly in terms of computations when the fraction of inliers is small. Although this is a well-understood limitation of the RANSAC, we nevertheless establish this rigorously in the present context. In particular, we derive the performance and computational complexity of RANSAC for these two problems, and perform some numerical experiments corroborating our theory and comparing the RANSAC with other methods proposed in the literature.

1.1 The problem of subspace recovery
Consider a setting where the data consist of $n$ points in dimension $p$, denoted $x_1, \ldots, x_n \in \mathbb{R}^p$. It is assumed that $m$ of these points lie on a $d$-dimensional linear subspace $L$ and that the points are otherwise in general position, which means that the following assumption is in place:

Assumption 1. A $q$-tuple of data points (with $q \leq p$) is linearly independent unless it includes at least $d + 1$ points from $L$.

(We say that points are linearly dependent if they are so when seen as vectors.)

The points on $L$ are called inliers and all the other points are called outliers. This is the setting of subspace recovery without noise. When there is noise, the points are not exactly on the underlying subspace but rather in its vicinity. In any case, the goal is to recover $L$, or said differently, distinguish the inliers from the outliers. See Figure 1a for an illustration in a setting where the subspace is of dimension $d = 2$ in ambient dimension $p = 3$. The goal is to recover the subspace $L$ and/or identify the inlier points.
This problem is intimately related to the problem of robust covariance estimation, which dates back decades (Huber and Ronchetti, 2009; Maronna, 1976; Tyler, 1987), but has attracted some recent attention. We refer the reader to the introduction of (Zhang and Lerman, 2014) for a comprehensive review of the literature, old and new. Subspace recovery in the presence of outliers, as we consider the problem here, is sometimes referred to a robust principal components analysis, although there are other meanings in the literature more closely related to matrix factorization with a low-rank component (Candès et al., 2011; Wright et al., 2009).

(a) Subspace recovery problem. (b) Subspace clustering problem.

Figure 1: An illustration of the two settings considered in the paper.

1.2 The problem of subspace clustering

Consider a setting where the data consist of $n$ points in dimension $p$, denoted $x_1, \ldots, x_n \in \mathbb{R}^p$. It is assumed that $m_k$ of these points lie on a $d_k$-dimensional linear subspace $L_k$, where $k = 1, \ldots, K$ (so that there are $K$ subspaces in total). The remaining points are in general position:

Assumption 2. A $q$-tuple of data points is linearly independent unless it includes at least $d_k + 1$ points from $L_k$ for some $k \in \{1, \ldots, K\}$.

In this setting all the points on one of the subspaces are inliers, and all the other points are outliers. This is the setting of subspace clustering without noise. When there is noise, the inliers are not exactly on the subspaces but in their vicinity. See Figure 1b for an illustration in a setting where there is one subspace of dimension $d_1 = 1$ and two subspaces of dimension $d_2 = d_3 = 2$, in ambient dimension $p = 3$. The goal is to cluster $m_k$ points to their corresponding $L_k$ for all $k \in \{1, \ldots, K\}$.

The problem of subspace clustering has applications in computer vision, in particular, movement segmentation (Vidal, 2011; Vidal et al., 2005).

1.3 Contents

In Section 2, we consider the problem of subspace recovery. In Section 3, we consider the problem of subspace clustering. In both cases, we study a ‘canonical’ RANSAC algorithm, deriving some theory and comparing it with other methods in numerical experiments. We briefly discuss our results in Section 4.

Remark 1 (linear vs affine). Throughout, we consider the case where the subspaces are linear, although some applications may call for affine subspaces. (This is for convenience.) Because of this, we are able to identify a point $x \in \mathbb{R}^p$ with the corresponding vector (sometimes written $x – 0$).
2 Subspace recovery

We consider the setting of Section 1.1 and use the notation defined there. In particular, we work under Assumption 1. We consider the noiseless setting for simplicity.

2.1 RANSAC for subspace recovery

We propose a simple RANSAC algorithm for robust subspace recovery. In the present setting, in particular under Assumption 1, the underlying linear subspace $L$ (which we assumed is of dimension $d$) is determined by any $(d+1)$-tuple that comes from that subspace. The algorithm starts by randomly selecting a $(d+1)$-tuple and checking if this tuple forms a linear subspace of dimension $d$. If so, the subspace is recovered and the algorithm stops. Otherwise, the algorithm continues, repeatedly sampling a $(d+1)$-tuple at random until the subspace is discovered. ( Optionally, the algorithm can be made to stop when a maximum number of tuples has been sampled.) In this formulation, detailed in Algorithm 1, $d$ is known.

| Input : data points $x_1, \ldots, x_n \in \mathbb{R}^p$; dimension $d$ |
| Output: a linear subspace of dimension $d$ containing at least $d+1$ points |
| 1 repeat |
| 2 randomly select a $(d+1)$-tuple of data points |
| 3 until the tuple is linearly dependent; |
| 4 return the subspace spanned by the tuple |

**Algorithm 1: RANSAC (Subspace Recovery)**

By design, the procedure is exact. (Again, we are in the noiseless setting. In a noisy setting, the method can be shown to be essentially optimal.) However, researchers have shied away from a RANSAC approach because of its time complexity. We formalize what is in the folklore in the following.

**Proposition 1.** Algorithm 1 is exact and the number of iterations has the geometric distribution\(^1\) with success probability $\theta_1 := \binom{n}{d+1}/\binom{m}{d+1}$. Thus the expected number of iterations is $1/\theta_1 = \binom{n}{d+1}/\binom{m}{d+1}$, which is of order $O(n/m)^{d+1}$ when $d$ is held fixed.

Note that each iteration requires on the order of $O(pd^2)$ operations as it requires computing the rank of a $p$-by-$(d+1)$ matrix.

**Proof.** The algorithm sample a $(d+1)$-tuple independently and uniformly at random until the tuple is linearly dependent. Because of Assumption 1, a $(d+1)$-tuple is linearly dependent if and only if all the points in the tuple are from $L$. While there are $\binom{n}{d+1}$ $(d+1)$-tuples in total, only $\binom{m}{d+1}$ fit the bill, so that the probability of drawing a suitable tuple is $\theta_1 = \binom{m}{d+1}/\binom{n}{d+1}$. Because the draws are independent, the total number of draws until the algorithm stops has the geometric distribution with success probability $\theta_1$.

We know that the mean of this distribution is $1/\theta_1 = \binom{n}{d+1}/\binom{m}{d+1}$, and when $d$ is assumed fixed, while $n$ and $m$ are large, we have

$$\frac{1}{\theta_1} \sim \frac{n^{d+1}/(d+1)!}{m^{d+1}/(d+1)!} = (n/m)^{d+1}.$$  \(^1\) Here we consider the variant of the geometric distribution that is supported on the positive integers.
(The reader is invited to verify that this still holds true as long as $d = o(m^2)$.)

In applications where the number of outliers is a non-negligible fraction of the sample (meaning that $n/m$ is not close to 1), the RANSAC’s number of iterations depends exponentially on the dimension of subspace. This confirms the folklore, at least in such a setting.

**Remark 2.** For simplicity, we analyzed the variant of the algorithm where the tuples are drawn with replacement, so that the worst-case number of iterations is infinite. However, in practice one should draw the tuples without replacement (which is equally easy to do in the present setting), as recommended in (Schattschneider and Green, 2012). For this variant, the worst-case time complexity is $\binom{n}{d+1} - \binom{n}{d} + 1$. Moreover, Proposition 1 still applies if understood as an upper bound. (The number of iterations has a so-called negative hypergeometric distribution in this case.)

**Remark 3.** If the dimension $d$ is unknown, a possible strategy is to start with $d = 1$, run the algorithm for a maximum number of iterations, and if no pair of points is found to be aligned with the origin, move to $d = 2$, and continue in that fashion, increasing the dimension. If no satisfactory tuple is found, the algorithm would start again at $d = 1$. The algorithm will succeed eventually.

### 2.2 The algorithm of Hardt and Moitra for subspace recovery

As we said above, researchers have avoid RANSAC procedures because of the running time, which as we saw can be prohibitive. Recently, however, Hardt and Moitra (2013) have proposed a RANSAC-type algorithm that strikes an interesting compromise between running time and precision.

Their algorithm is designed for the case where the sample size is larger than the ambient dimension, namely $n > p$. It can be described as follows. It repeatedly draws a $p$-tuple at random until the tuple is found to be linearly dependent. When such a tuple is found, the algorithm returns a set of linearly dependent points in the tuple. See the description in Algorithm 2. A virtue of this procedure is that it does not require knowledge of the dimension $d$ of the underlying subspace.

```
Input : data points $x_1, \ldots, x_n \in \mathbb{R}^p$
Output: a linear subspace
1 repeat
2 randomly select a $p$-tuple of data points
3 until the tuple is linearly dependent;
4 return the subspace spanned by any subset of linearly dependent points in the tuple
```

**Algorithm 2: Hardt-Moitra (Subspace Recovery)**

**Proposition 2.** When $n > p$, Algorithm 2 is exact and its number of iterations has the geometric distribution with success probability $\theta_2 := \sum_{k \geq d+1} \binom{m}{k}(\binom{n-m}{p-k}/\binom{n}{p})$. Thus the expected number of iterations is $1/\theta_2$.

Note that each iteration requires on the order of $O(p^3)$ operations as it requires computing the rank of a $p$-by-$p$ matrix.

**Proof.** With Assumption 1 in place, a $p$-tuple is linearly dependent if and only if it contains at least $d + 1$ points from the subspace $L$. Thus the Repeat statement above stops exactly when it found a $p$-tuple that contains at least $d + 1$ points. Moreover, also because of Assumption 1, the points within that tuple that are linear dependent must belong to $L$. Therefore, the algorithm returns $L$, and is therefore exact.
We now turn to the number of iterations. The number of iterations is obviously geometric and the success probability is the probability that a \( p \)-tuple drawn uniformly at random contains at least \( d + 1 \) points from \( L \). \( \theta_2 \) is that probability. Indeed, it is the probability that, when drawing \( p \) balls without replacement from an urn with \( m \) red balls out of \( n \) total, the sample contains at least \( d + 1 \) red balls. In the present context, the balls are of course the points and the red balls are the points on the linear subspace.

Hardt and Moitra (2013) analyze their algorithm in a slightly different setting and with the goal of finding the maximum fraction of outliers that can be tolerated before the algorithm breaks down in the sense that it does not run in polynomial time. In particular, they show that, if \( m/n \geq d/p \), then their algorithm has a number of iterations with the geometric distribution with success probability at least \( 1/(2p^2n) \), so that the expected number of iterations is bounded by \( 2p^2n \), which is obviously polynomial in \((p,n)\). In fact, it can be better than that. The following is a consequence of Proposition 2.

**Corollary 1.** If, in addition to \( n > p \), it holds that \( m/n \geq d/p \), with \( d/p \leq \tau \) and \( p/n \leq \tau \), for some fixed \( \tau < 1 \), then \( \theta_2 \) is bounded from below by a positive quantity that depends only on \( \tau \). Consequently, Algorithm 2 has expected number of iterations of order \( O(1) \).

**Proof.** Let \( U \) denote a random variable with the hypergeometric distribution with parameters \((p,m,n)\) described above. Then \( \theta_2 = \mathbb{P}(U \geq d + 1) \), and it depends on \((d,p,m,n)\). We show that \( \theta_2 \) is bounded from below irrespective of these parameters as long as the conditions are met. Noting that \( \theta_2 \) is increasing in \( d \) and \( n \), and decreasing in \( p \) and \( m \), it suffices to consider how \( \theta_2 \) varies along a sequence where \( n \to \infty \) and \((d,p,m)\) all varying with \( n \) in such a way that \( m/n \to \tau \) and \( d/p \to \tau \), as this makes the expected number of iterations largest. Define

\[
\mu := \mathbb{E}(U) = p(m/n);
\sigma^2 = \text{Var}(U) = p(m/n)(1 - m/n)(n - p)/(n - 1).
\]

The condition \( m/n \geq d/p \) implies that \( \mu \geq d \), and along the sequence of parameters under consideration, \( \sigma \to \infty \). Moreover, along such a sequence, \( Z := (U - \mu)/\sigma \) is standard normal in the limit, so that

\[
\theta_2 = \mathbb{P}(U \geq d + 1) = \mathbb{P}(Z \geq (d + 1 - \mu)/\sigma)
\geq \mathbb{P}(Z \geq 1/\sigma)
\to \mathbb{P}(N(0,1) \geq 0) = 1/2,
\]

using Slutsky’s theorem in the last line. \( \square \)

### 2.3 Numerical experiments

We performed some small-scale numerical experiments comparing RANSAC (in the form of Algorithm 1), the Hardt-Moitra (HM) procedure (Algorithm 2), and the Geometric Median Subspace (GMS) of (Zhang and Lerman, 2014), which appears to be one of the best methods on the market. (We used the code available on Teng Zhang’s website.)

Each inlier is uniformly distributed on the intersection of the unit sphere with the underlying subspace. Each outlier is simply uniformly distributed on the unit sphere. The result of each algorithm is averaged over 1000 repeats. Performance is measured by the (first principal) angle between the returned subspace and the true subspace. (This is to be fair to GMS, as the other two algorithms are exact.) The results are reported in Table 1.
### Table 1: Numerical experiments comparing RANSAC, HM, and GMS for the problem of subspace recovery.

As in the text, $d$ is the dimension of the subspace, $p$ is the ambient dimension, $m$ is the number of inliers, $m_0$ is the number of outliers (so that $n = m + m_0$ is the sample size).

| Parameters       | Average System Time | Difference in Angle |
|------------------|---------------------|---------------------|
| $(d, p, m, m_0)$ | RANSAC   | HM    | GMS    | RANSAC   | HM    | GMS    |
| (8, 10, 100, 50) | .0051   | .0009 | .0947 | 0        | 0     | .0341 |
| (4, 10, 100, 50) | .0011   | .0006 | .1912 | 0        | 0     | 0     |
| (8, 20, 100, 50) | .0064   | .0002 | .5008 | 0        | 0     | .0184 |
| (6, 10, 100, 20) | .0001   | .0006 | .1918 | 0        | 0     | 0     |
| (9, 10, 100, 50) | .0076   | .0093 | .0117 | 0        | 0     | .0411 |
| (18, 20, 100, 50)| .6123   | .0013 | .0160 | 0        | 0     | .2285 |

We performed another set of experiments to corroborate the theory established in Proposition 1 for the complexity of RANSAC. The results are shown in Figure 2, where each setting has been repeated 1000 times. As expected, as the dimensionality of the problem increases, RANSAC’s complexity becomes quickly impractical.

![Figure 2: Average number of iterations for RANSAC (in the form of Algorithm 1) as a function of the subspace dimension $d$ and the ratio of sample size $n$ to number of inliers $m$. The dashed lines are the averages from our simulation while the lines are derived from theory (Proposition 1).](image)

3  Subspace clustering

We consider the setting of Section 1.2 and use the notation defined there. In particular, we work under Assumption 2. We consider the noiseless setting for simplicity. We also assume that all subspaces are of same dimension, denoted $d$ (so that $d_k = d$ for all $k$).
3.1 RANSAC for subspace clustering

We propose a simple RANSAC algorithm for subspace clustering. As before, any of the linear subspaces is determined by any \((d+1)\)-tuple that comes from that subspace. The algorithm starts by randomly selecting a \((d+1)\)-tuple and checking if this tuple forms a linear subspace of dimension \(d\). If so, one of the subspaces is recovered and all the points on the subspace are extracted from the data. Otherwise, the algorithm continues, repeatedly sampling a \((d+1)\)-tuple at random until that condition is met. The algorithm continues in this fashion until all the \(K\) subspaces have been recovered. In this formulation, detailed in Algorithm 3, both \(d\) and \(K\) are assumed known.

Input : data points \(x_1, \ldots, x_n \in \mathbb{R}^p\); dimension \(d\); number of subspace \(K\)
Output: \(K\) linear subspaces of dimension \(d\), each containing at least \(d+1\) points

\begin{algorithm}
  \begin{algorithmic}
    \For {\(k = 1, \ldots, K\)}
      \Repeat
        \State randomly select a \((d+1)\)-tuple of data points
      \Until {the tuple is linearly dependent;}
      \State return the subspace spanned by the tuple
      \State remove the points on that subspace from the data
    \EndFor
  \end{algorithmic}
\end{algorithm}

Algorithm 3: RANSAC (Subspace Clustering)

Again, the procedure is exact by design, since we are in the noiseless setting. Here too, researchers have not embraced RANSAC approaches because of their running time. We confirm this folklore in the following, where we assume for simplicity that all subspaces have the same number of points \(m\) (so that \(m_k = m\) for all \(k\)).

Proposition 3. Algorithm 3 is exact and the number of iterations is has the distribution of \(I_1 + \cdots + I_K\), where the \(I_i\)'s are independent and \(I_j\) has the geometric distribution with success probability \((K-j+1)/\binom{n-(j-1)m}{d+1}\). This is stochastically bounded by the negative binomial with parameters \((K, \theta_1)\). Thus the expected number of iterations is bounded by \(K/\theta_1\), which is of order \(O(n/m)^{d+1}\) when \(d\) and \(K\) are held fixed. The proof is very similar to that of Proposition 1 and is omitted.

Remark 4. When the dimensions of the subspaces are unknown, a strategy analogous to that described in Remark 3 is of course possible. When the number of subspaces is unknown, a stopping rule can help decide whether there remains a subspace to be discovered. Details are omitted as such an approach, although natural, could prove complicated.

3.2 Adapting the algorithm of Hardt and Moitra for subspace clustering

Algorithm 3 consists in applying Algorithm 1 until a subspace is recovered, removing the points on that subspace, and then continuing, until all \(K\) subspaces are recovered. An algorithm for subspace clustering can be based on the algorithm of Hardt and Moitra (2013) (Algorithm 2) instead. The resulting algorithm is suited for the case where \(n - (m_1 + \cdots + m_K) > p\). Based on the fact that Algorithm 2 has expected number of iterations bounded by \(2p^2n\), the resulting algorithm for subspace clustering has expected number of iterations bounded by \(2Kp^2n\). See Algorithm 4, where we assume that the number of subspaces is known, but do not assume that the dimensions of the subspaces are known (and they do not need to be the same).
Input: data points $x_1, \ldots, x_n \in \mathbb{R}^p$; number of subspace $K$

Output: $K$ linear subspaces each with a number of points exceeding its dimension

for $k = 1, \ldots, K$ do
    repeat
        randomly select a $p$-tuple of data points
    until the tuple is linearly dependent;
    repeat
        find the smallest number of linearly dependent points in the tuple
    return the subspace spanned by these points
    remove the points on that subspace from the data
    until there are no more linearly dependent points in the tuple;
end

Algorithm 4: Subspace Clustering based on the Hardt-Moitra Algorithm

The reason why we extract the smallest number of linearly dependent points at each step is to avoid a situation where a $p$-tuple contains $d_j + 1$ points from $L_j$ and $d_k + 1$ points from $L_k$ (with $j \neq k$), in which case, assuming $d_j + d_k < p$, these points are linearly dependent but do not span one of the subspaces. This particular step is, however, computationally challenging as it amounts to finding the sparsest solution to a $p$-by-$p$ linear system, a problem known to be challenging (Tropp and Wright, 2010, Eq 1). One possibility is to replace this with finding the solution with minimum $\ell_1$ norm (Tropp and Wright, 2010, Eq 8). The use of the $\ell_1$ constraint is central to the method proposed by Elhamifar and Vidal (2009).

3.3 The algorithm of Chen and Lerman

The Spectral Curvature Clustering (SCC) algorithm of Chen and Lerman (2009b) is in fact of RANSAC type. The method was designed for the noisy setting and is therefore more sophisticated.\footnote{Chen and Lerman (2009b) consider the case where the subspaces are affine, but we adapt their method to the case where they are linear.} It is based on a function $A : (\mathbb{R}^p)^{d+1} \rightarrow [0, 1]$ that quantifies how close a $(d+1)$-tuple is from spanning a subspace of dimension $d$ or less. It is equal to 1 when this is the case and is strictly less than 1 when this is not the case. The algorithm draws a number, $c$, of $d$-tuples at random, where the $s$-th tuple is denoted $(x_{1,s}, \ldots, x_{d,s})$, and computes the matrix $W = (W_{ij})$, where

$$W_{ij} = \sum_{s=1}^{c} A(x_i, x_{1,s}, \ldots, x_{d,s}) A(x_j, x_{1,s}, \ldots, x_{d,s}).$$

It then applies a form of spectral graph partitioning algorithm to $W$ closely related to method of Ng et al. (2002). (The method assumes all subspaces are of same dimension $d$, and both $d$ and $K$ are assumed known.)

In the noiseless setting, one could take $A$ to return 1 if the tuple is linearly dependent and 0 otherwise. In that case, $W_{ij}$ is simply the number of $d$-tuples among the $c$ that were drawn with whom both $x_i$ and $x_j$ are linearly dependent. Chen and Lerman (2009a) analyzes their method in a setting that reduces to this situation and show that the method is exact in this case.
| parameters | average system time | rand index |
|------------|---------------------|------------|
| (d, p, K, m, m₀) | RANSAC | SSC | SCC | TSC | RANSAC | SSC | SCC | TSC |
| (4,8,3,50,50) | .0622 | .3188 | .3731 | .1767 | 1 | .8407 | .9997 | .7237 |
| (6,8,3,50,50) | .8810 | .2813 | .8257 | .1815 | 1 | .6490 | .8322 | .5829 |
| (4,8,3,50,100) | .2108 | .3974 | .4563 | .1757 | 1 | .7619 | .9689 | .2849 |
| (4,8,5,50,50) | .3425 | .6163 | .9395 | .3282 | 1 | .9206 | .9548 | .7578 |
| (8,10,3,50,50) | 17.1827 | .3185 | 1.4141 | .2019 | 1 | .6148 | .6904 | .5688 |

Table 2: Numerical experiments comparing RANSAC, SSC, SCC, and TSC for the problem of subspace clustering. As in the text, \( d \) is the dimension of the subspaces (assumed to be the same), \( p \) is the ambient dimension, \( K \) is the number of subspaces, \( m \) is the number of inliers per subspace (assumed to be the same), \( m_0 \) is the number of outliers (so that \( n = Km + m_0 \) is the sample size).

### 3.4 Numerical experiments

We performed some numerical experiments to compare various methods for subspace clustering, specifically, RANSAC, Sparse Subspace Clustering (SSC) (Elhamifar and Vidal, 2009), Spectral Curvature Clustering (SCC) (Chen and Lerman, 2009b), and Thresholding-based Subspace Clustering (TSC) (Reinhard Heckel, 2015).

Each inlier is uniformly distributed on the intersection of the unit sphere with its corresponding subspace. Each outlier is simply uniformly distributed on the unit sphere. The result of each algorithm is averaged over 500 repeats. Performance is measured by the Rand index. The results are reported in Table 2.

### 4 Discussion and conclusion

In our small scale experiments, RANSAC is seen to be competitive with other methods, at least when the intrinsic dimensionality is not too large and when there are not too many outliers (or too many underlying subspaces) present in the data. This was observed both in the context of subspace recovery and in the context of subspace clustering.

### References

Candès, E. J., X. Li, Y. Ma, and J. Wright (2011). Robust principal component analysis? *Journal of the ACM (JACM)* 58(3), 11.

Chen, G. and G. Lerman (2009a). Foundations of a multi-way spectral clustering framework for hybrid linear modeling. *Foundations of Computational Mathematics* 9(5), 517–558.

Chen, G. and G. Lerman (2009b). Spectral curvature clustering (scc). *International Journal of Computer Vision* 81(3), 317–330.

Elhamifar, E. and R. Vidal (2009). Sparse subspace clustering. In *Computer Vision and Pattern Recognition, 2009. CVPR 2009. IEEE Conference on*, pp. 2790–2797. IEEE.

Fischler, M. A. and R. C. Bolles (1981). Random sample consensus: a paradigm for model fitting with applications to image analysis and automated cartography. *Communications of the ACM* 24(6), 381–395.

Hardt, M. and A. Moitra (2013). Algorithms and hardness for robust subspace recovery. In *Conference on Learning Theory (COLT)*, Volume 30, pp. 354–375.
Huber, P. J. and E. M. Ronchetti (2009). *Robust Statistics (2nd edition)*. Wiley.
Maronna, R. A. (1976). Robust M-estimators of multivariate location and scatter. *The Annals of Statistics* 4(1), 51–67.
Ng, A. Y., M. I. Jordan, and Y. Weiss (2002). On spectral clustering: Analysis and an algorithm. In *Advances in neural information processing systems*, pp. 849–856.
Reinhard Heckel, Helmut Bolcskei, D. o. I. . E. E. Z. S. (2015). Robust subspace clustering via thresholding. *IEEE Transactions on Information Theory* 61.
Schattenschneider, R. and R. Green (2012). Enhanced ransac sampling based on non-repeating combinations. In *Proceedings of the 27th Conference on Image and Vision Computing New Zealand*, pp. 198–203. ACM.
Tropp, J. A. and S. J. Wright (2010). Computational methods for sparse solution of linear inverse problems. *Proceedings of the IEEE* 98(6), 948–958.
Tyler, D. E. (1987). A distribution-free m-estimator of multivariate scatter. *The Annals of Statistics*, 234–251.
Vidal, R. (2011). Subspace clustering. *IEEE Signal Processing Magazine* 28(2), 52–68.
Vidal, R., Y. Ma, and S. Sastry (2005). Generalized principal component analysis (gpca). *IEEE Transactions on Pattern Analysis and Machine Intelligence* 27(12), 1945–1959.
Wright, J., A. Ganesh, S. Rao, Y. Peng, and Y. Ma (2009). Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization. In *Advances in neural information processing systems*, pp. 2080–2088.
Zhang, T. and G. Lerman (2014). A novel M-estimator for robust PCA. *The Journal of Machine Learning Research* 15(1), 749–808.