TUBULAR JACOBIAN ALGEBRAS

CHRISTOF GEISS AND RAÚL GONZÁLEZ-SILVA

ABSTRACT. We show that the endomorphism ring of each cluster tilting object in a tubular cluster category is a finite dimensional, non-degenerate Jacobian algebra which is tame of polynomial growth.

1. Introduction

Tubular cluster algebras were introduced in [1] as a particular class of mutation finite cluster algebras. Their common feature is that they admit an additive categorification by a tubular cluster category of the corresponding type. Recall that tubular cluster categories are by definition of the form $\mathcal{C}_X = \mathcal{D}^b(\text{coh}(X))/\langle \tau^{-1} \rangle$, where $\text{coh} X$ is the category of coherent sheaves on a weighted projective line of tubular weight type i.e. $(2, 2, 2, \lambda), (3, 3, 3), (4, 4, 2)$ and $(6, 3, 2)$, see [10]. It follows from [15] that $\mathcal{C}_X$ is a triangulated 2-Calabi-Yau category which admits a cluster structure [2].

Even stronger, it is not hard to derive from known results that in this situation the endomorphism ring of each cluster tilting object is a (finite dimensional) Jacobian algebra and the mutation of cluster tilting objects is compatible with the mutation of QPs in the sense of [4]. In particular, all these Jacobian algebras are non-degenerate. See 2.5 for details.

Moreover, in these cluster categories the indecomposable rigid objects are in bijection with the positive real Schur roots of the corresponding elliptic root system. Via the cluster character indecomposable rigid objects are in bijection with cluster variables [1].

Finally, recall from [14] Section 2.1] that for a cluster tilting object $T \in \mathcal{C}_X$ we have $\text{End}_{\mathcal{C}_X}(T)^{\text{op}}-\text{mod} \cong \mathcal{C}/\text{add} T$. In particular, the Auslander-Reiten quiver of these algebras consists just of tubular families of the corresponding weight type, with all but finitely many tubes being stable.

Since the tubular cluster categories are orbit categories of derived tame categories in the sense of [11], this suggests strongly that all these algebras are tame. In fact, this is our main result:

**Theorem 1.** Let $K$ be an algebraically closed field, $\mathcal{C}_X$ a tubular cluster category over $K$ as above and $T \in \mathcal{C}_X$ a basic cluster tilting object, then the endomorphism ring $\text{End}_{\mathcal{C}_X}(T)$ is a finite dimensional Jacobian algebra which is tame of polynomial growth.

Our strategy is as follows: all these algebras of a given tubular type are in fact non-degenerate Jacobian algebras and related via QP-mutations, as we noted above. Since the representation type of Jacobian algebras is preserved under mutation [13],

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it is sufficient to show for each tubular type that the endomorphism ring of a single cluster tilting object, see Figure 1, is tame (of polynomial growth). We achieve this by providing for each representative a Galois-cover coming from the natural \( \mathbb{Z} \)-grading, which turns out to be iterated tubular in the sense of de la Peña-Tomé. This provides in each case a quite explicit description of the indecomposables.

In fact, we conjecture that a similar description holds for any tubular Jacobian algebra. This is easy to verify for the tubular type \((2, 2, 2)\), since there are only four families of Jacobian algebras. However, for type \((6, 3, 2)\) there are a priori several thousand cases to be analyzed.

\[
\begin{align*}
W_1 &= cba + fed - hig \\
&\quad -elj - bkj + imj \\
W_2 &= \lambda abc - dgc + dki - afi + jgh \\
&\quad -ebh + elf - jkl \quad (\lambda \in K \setminus \{0, 1\}) \\
W_3 &= cab - cef + hde + ifg \\
&\quad -hjk + nkl - ilm - nop \\
W_4 &= acd - ebc + efg - igh + \\
&\quad ikl - mjk + mno - qop
\end{align*}
\]

**Figure 1.** Representatives of tubular QPs

2. Preliminaries

### 2.1. Grothendieck group

**Definition 1.** Let \( K \) be an algebraically closed field.

Let \( A \) be a tame (representation-infinite) connected and hereditary \( K \)-algebra and \( A^T \) a preprojective tilting module. The algebra \( B := \text{End}(A^T) \) is called a tame concealed algebra.

Let \( B \) a tame concealed algebra, \( K_0(B) \) its Grothendieck group and \( C_B \) its Cartan matrix \([18, 2.2.4]\). We have that \( K_0(B) \cong \mathbb{Z}^n \), where we consider the dimension vectors as row vectors. In this setting, the Cartan matrix \( C_B \) is invertible and we consider Ringel’s form \( \langle \cdot, \cdot \rangle \) on \( K_0(B) \) given by:

\[
\langle x, y \rangle = xC_B^{-T}y^T,
\]

the quadratic form \( \chi_B \), given by \( \chi_B(x) = \langle x, x \rangle \) and the Coxeter transformation, given by the action of the Coxeter matrix \( \Phi = -C_B^{-T}C_B \) on \( K_0(B) \).
Remark 1. Since \( \text{gl.dim}(B) \leq 2 \), the matrix \( C_B^{-T} \) codifies arrows and minimal relations for \( B \), i.e. \( (C_B^{-T})_{i,j} \) is equal to the number of relations starting at \( j \) and ending at \( i \) minus the number of arrows starting at \( j \) and ending at \( i \).

We collect from [18] the following well-known results:

**Proposition 1.** Let \( B \) be a tame concealed algebra, then:

a) \( \chi_B \) has corank 1;

b) If \( h \) is the unique minimal positive radical vector of \( K_0(B) \) with \( h = \dim R \) (for \( R \) an indecomposable regular \( B \)-module) then the indecomposable regular \( B \)-modules are those indecomposable \( B \)-modules \( M \) such that \( \langle h, \dim M \rangle = 0 \).

c) \( \dim(\tau M) = (\dim M)\Phi \), for any indecomposable regular \( B \)-module, where \( \tau \) is the AR-translate.

d) An indecomposable regular \( B \)-module \( M \), is simple regular if
\[
\sum_{i=0}^{n-1} (\dim^{\tau} M) = h, \text{ where } n \text{ is the } \tau\text{-period of } M.
\]

For (a) and (b) see [18, Thm. 4.3 (3)]. (c) follows from [18, 2.4 (4)] since the regular modules over a tame concealed algebra have projective dimension 1 [18, 3.1 (5)]. (d) follows with (c) by tilting from the corresponding statement about regular modules over tame hereditary algebras. In this case it can be verified by direct inspection, see for example [7, sec. 6].

2.2. **Jacobian algebras.** In this subsection, we briefly recall the definition of Jacobian algebras. For more details see [4].

Let \( Q \) be a quiver without loops and oriented 2-oriented cycles, and denote by \( K\langle\langle Q\rangle\rangle \) the complete algebra of the quiver \( Q \). Let \( K\langle\langle Q\rangle\rangle_{\text{cyc}} \) be the completion of the subspace spanned by all the oriented cycles in \( Q \).

A potential in \( Q \), is an element \( W \in K\langle\langle Q\rangle\rangle_{\text{cyc}} \), i.e. a possibly infinite linear combination of cycles.

For any arrow \( a \in Q_1 \), define a continuous linear map \( \partial_a \), called the cyclic derivative, which acts on oriented cycles by:
\[
\partial_a(a_1 \cdots a_d) = \sum_{p:a_p=a} a_{p+1} \cdots a_d a_1 \cdots a_{p-1}
\]

Given a potential \( W \) in \( Q \), the Jacobian ideal, \( J(W) \), of \( W \), is the closure of the ideal in \( K\langle\langle Q\rangle\rangle \) generated by the elements \( \partial_a(W) \), for all \( a \in Q \).

The Jacobian algebra of \( W \), denoted by \( \mathcal{P}(Q,W) \), is the factor algebra \( K\langle\langle Q\rangle\rangle / J(W) \).

2.3. **Tubular algebras.** A tubular algebra is a tubular extension of a tame concealed algebra of extension type \( (2, 2, 2, 2) \), \( (3, 3, 3) \), \( (4, 4, 2) \) or \( (6, 3, 2) \) in the sense of [18, page 230]. Note that the global dimension of a tubular algebra is always 2, see [18].

Let \( T \in \{(2, 2, 2), (3, 3, 3), (4, 4, 2), (6, 3, 2)\} \) and let \( A \) be a tubular algebra of tubular type \( T \). By definition, it can be viewed as an extension of a tame concealed algebra \( A_0 \) and also as a coextension of a tame concealed algebra \( A_{\infty} \) [18, pages 268-269].
Let $h_0$, respectively $h_\infty$, be the positive radical generator of $K_0(A_0)$, respectively $K_0(A_\infty)$. If $M$ is an indecomposable $A$-module, we define:

$$\text{index}(M) = -\langle h_0, \dim M \rangle_A - \langle h_\infty, \dim M \rangle_A \in \mathbb{Q} \cup \{\infty\}$$

For $\gamma \in \mathbb{Q}^+$, let $T_\gamma$ be the module class given by all the indecomposable $A$-modules with index $\gamma$. The module class $T_\gamma$ is a $\mathbb{P}_1$-sincere stable tubular family of type $T$. \cite[Thm. 5.2 (2)]{18}

The following theorem from \cite[Thm. 5.2 (4)]{18}, describes the structure of $A$-mod.

**Theorem 2** (Ringel). Let $A$ be a tubular algebra of type $T$. Then $A$-mod has the following components: a preprojective component $P_0$ (the same as for $A_0$), for each $\gamma \in \mathbb{Q}_0$, a separating (\cite[sec. 3.1]{18}) tubular $\mathbb{P}_1$-family $T_\gamma$, all but $T_0$ and $T_\infty$ being stable of type $T$ and a preprojective component $Q_\infty$.

The following figure helps us to visualize the structure of $A$-mod:

![Diagram]

Here, non-zero maps going only from left to right; given indecomposable modules $X$ and $Y$ with $\text{Hom}(X, Y) \neq 0$, then either $X$ and $Y$ belong to the same component, or else, $X \in P_0$ and $Y \notin P_0$, or else, $X \notin Q_\infty$, or else, $X \in T_\gamma$, $Y \in T_\delta$ and $\gamma < \delta$.

**2.4. Galois Coverings.** In this section we use the functorial approach to representation theory of algebras \cite{9}.

We consider algebras as a basic $K$-categories. Given a basic $K$-category $A$ and a group $G$ acting freely on $A$, denote by $A/G$ the orbit category. In this setting the canonical projection $F : A \to A/G$ is a Galois covering. Associated to $F$, we have the “push-down” functor $F_\lambda : \text{mod} A \to \text{mod}(A/G)$, given by:

$$(F_\lambda M)(a) = \prod_{x/a} M(x)$$

for each object $a$ in $A/G$ and the obvious action of morphisms.

For any $A$-module $M$, we will denote by $\text{supp} M$ the support of $M$, that is, the full subcategory of $A$ formed by all objects $x$ such that $M(x) \neq 0$. For each object $x$ in $A$, denote by $A_x$ the full subcategory of $A$ consisting of all the objects in $\text{supp} M$, where $M$ is any indecomposable module with $x$ in $\text{supp} M$. We say that $A$ is locally support-finite if for every object $x$ in $A$, the number of objects in $A_x$ is finite.

From \cite{8} we have the following:

**Theorem 3** (Dowbor-Skowroński). Let $A$ be a locally support-finite $K$-category and let a free abelian group $G$ acting freely on $A$. Assume that the action of $G$ on the isoclasses of indecomposable finitely generated $A$-modules is free. Then $A/G$ is locally support-finite and the push-down functor $F_\lambda : \text{mod} A \to \text{mod}(A/G)$ induces a bijection between the $G$-orbits of isoclasses of indecomposable objects in $\text{mod} A$. 

and the isoclasses of indecomposable objects in \( \text{mod}(A/G) \). In particular, \( A \) is representation-tame if and only if \( A/G \) is so.

**Remark 2.** The theorem in [8] is more general than the theorem above. We state it in this form which is sufficient for this work.

### 2.5. Tubular Jacobian algebras

Let \( A = KQ/I = \text{End}_X(T)^{\text{op}} \) be a basic tubular algebra with \( T \in \text{coh}(X) \) a tilting sheaf over a weighted projective line of the corresponding tubular type. We may view \( I \) as an admissible ideal generated by a minimal set of relations \( \rho_1, \ldots, \rho_r \). In particular, each \( \rho_i \) is a linear combination of (parallel) paths from \( s_i \) to \( t_i \) in \( Q \). Since \( \text{coh}(X) \) is a hereditary category, we can apply the results from [16, Sec. 6]: Let \( \tilde{Q} \) be the quiver obtained from \( Q \) by adding additional arrows \( a_i: t_i \to s_1 \) for \( i = 1, \ldots, r \) and \( W := \sum_{i=1}^r \rho_i a_i \in K_{\text{cycl}}(Q) \). Then \( \text{End}_{C_X}(\pi(T)) \cong P(\tilde{Q}, W) \).

#### 2.5.1. Type \((2, 2, 2, 2, \lambda)\)

There are only relatively few tubular algebras of type \((2, 2, 2, 2, \lambda)\). They were listed for example in [19, 3.2]. It is straightforward to check that they yield the four quivers with potential described in [12, p.117]. It is an easy exercise to see that the corresponding Jacobian algebras (for \( \lambda \in K \setminus \{0, 1\} \)) are finite-dimensional and weakly symmetric. In fact, the same is true for “fake” Jacobian algebras \( K\tilde{Q}/(\partial a_i W: a \in Q_1) \). Thus, these algebras fulfill trivially the “vanishing condition” of [3, Thm. 5.2]. It follows that this class of Jacobian algebras is closed under (per-) mutations. In particular, the just described QPs are non-degenerate in the sense of [4]. Note however, that the parameter \( \lambda \in K \setminus \{0, 1\} \) may change under mutation.

#### 2.5.2. The types \((3, 3, 3), (4, 4, 2), (6, 3, 2)\)

The cluster categories of this type can be realized as stable categories of the form \( C_w \) for \( w \) an element of an appropriate Coxeter group [1, 3.2]. In fact, the Jacobian algebras of those types, which we discuss below correspond to the one of the initial cluster tilting objects associated to an reduced expression for such \( w \). By [3, Thm. 6.6] in this situation the description of \( \text{End}_{C_w}(T) \) as a Jacobian algebra of the required form for an initial cluster tilting object \( T \) associated to a reduced expression for \( w \) is liftable. Since in the tubular situation the exchange graph of the cluster tilting objects is connected [2, Thm. 8.8] it follows from [3 Cor. 5.4(b)] that we have in this situation also the compatibility between mutation of cluster tilting objects and mutation of QPs.

### 3. Galois Coverings of Tubular Jacobian Algebras

#### 3.1. Case \((3, 3, 3)\)

We consider the quiver with potential \((Q_1, W_1)\) from Figure [4]. The corresponding Jacobian algebra \( P(Q_1, W_1) \) is finite dimensional, the dimension vectors of its indecomposable projective modules are as follows:

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
\]
To study the category of representations of this quiver with potential, we define a grading on $Q_1$:

$$\deg(x) = \begin{cases} 1, & \text{if } x = a, d, g, j \\ 0, & \text{otherwise.} \end{cases}$$

This induces a grading on the complete path algebra $K\langle Q_1 \rangle$. With respect to this grading the potential $W_1$ is homogeneous and thus, the cyclic derivatives $\partial_x(W_1)$ are homogeneous. In particular the Jacobian algebra $P(Q_1, W_1)$ is also $\mathbb{Z}$-graded. This allows us to consider the corresponding Galois covering:

3.2. Case $(2, 2, 2, 2, \lambda)$. We consider the quiver with potential $(Q_2, W_2)$ from Figure 1. An easy calculation shows: every path of length 4 belongs to $J(W_2)$; the same is true for paths of length 3 which are not cycles.

The corresponding Jacobian algebra $P(Q_2, W_2)$ is thus finite dimensional, the dimension vectors of its indecomposable projective modules are as follows:

$$2 1 1, 0 1 1, 1 2 1, 1 0 1, 2 1 1,' 1 1 2, 1 1 0, 1 1 2,'$$

To study the category of representations of this quiver with potential we define a grading on $Q_2$:

$$\deg(x) = \begin{cases} 1, & \text{if } x = c, h, i, l \\ 0, & \text{otherwise.} \end{cases}$$

This induces a grading on the complete path algebra $K\langle Q_2 \rangle$. With respect to this grading the potential $W_2$ is homogeneous and thus, the cyclic derivatives $\partial_x(W_2)$ are homogeneous. In particular the Jacobian algebra $P(Q_2, W_2)$ is also $\mathbb{Z}$-graded. This allows us to consider the corresponding Galois covering:
In which all the squares commute up to possibly a factor $\lambda$.

3.3. **Case** $(4,4,2)$. We consider the quiver with potential $(Q_3, W_3)$ from Figure 1. The corresponding Jacobian algebra $\mathcal{P}(Q_3, W_3)$ is finite dimensional, the dimension vectors of its indecomposable projective modules are as follows:

$$
\begin{align*}
1 & 1 0 1 0 \\
1 & 1 1 1 0 1 0 1 0 \\
1 & 1 1 1 1 1 0 1 0 1 0 1 1 \\
1 & 1 1 0 1 1 0 1 0 0 1 2 1 0 \\
1 & 1 1 0 1 1 0 0 1 0 1 1 \\
1 & 0 1 0 1 0 0 0 \\
0 & 0 0 0 0 0 0 0 \\
0 & 0 1 1 0 0 0 0 0 \\
1 & 0 1 1 0 1 1 0 1 1 1 1 \\
1 & 0 0 1 1 0 0 0 0 \\
1 & 0 0 0 0 0 0 0
\end{align*}
$$

To study the category of representations of this quiver with potential, we define a grading on $Q_3$:

$$
\text{deg}(x) = \begin{cases} 
1, & \text{if } x = c, h, i, n \\
0, & \text{otherwise.}
\end{cases}
$$

This induces a grading on the complete path algebra $K\langle \langle Q_3 \rangle \rangle$. With respect to this grading the potential $W_3$ is homogeneous and thus, the cyclic derivatives $\partial_x(W_3)$ are homogeneous. In particular the Jacobian algebra $\mathcal{P}(Q_3, W_3)$ is also $\mathbb{Z}$-graded. This allows us to consider the corresponding Galois covering:

In this diagram the dotted lines indicate zero relations. In addition to these relations, all the squares commute.

3.4. **Case** $(6,3,2)$. We consider the quiver with potential $(Q_3, W_3)$ from Figure 1. The corresponding Jacobian algebra $\mathcal{P}(Q_4, W_4)$ is finite dimensional, the dimension vectors of its indecomposable projective modules are as follows:

$$
\begin{align*}
0 & 1 0 0 0 1 0 0 0 \\
0 & 1 1 1 1 0 1 1 0 1 0 \\
1 & 0 1 1 1 1 0 1 0 1 0 1 0, \\
1 & 0 1 1 0 1 1 0 1 0 1 1 \\
0 & 0 1 0 0 1 0 0 1 0
\end{align*}
$$
To study the category of representations of this quiver with potential we define a grading on $Q_4$:

$$\deg(x) = \begin{cases} 
1, & \text{if } x = a, e, i, m, q \\
0, & \text{otherwise.}
\end{cases}$$

this induces a grading on the complete path algebra $K\langle \langle Q_4 \rangle \rangle$. With respect to this grading the potential $W_4$ is homogeneous and thus, the cyclic derivatives $\partial_x(W_4)$ are homogeneous. In particular, the Jacobian algebra $\mathcal{P}(Q_4, W_4)$ is also $\mathbb{Z}$-graded. This allows us to consider the corresponding Galois covering:

In this diagram the dotted lines indicate zero relations. In addition to these relations all the squares commute.

3.5. **Iterated tubular coverings.**

**Proposition 2.** The Galois coverings associated to the cases $(2, 2, 2, 2)$, $(3, 3, 3)$, $(4, 4, 2)$ and $(6, 3, 2)$ are iterated tubular algebras (in the sense of de la Peña-Tomé [6]).

**Proof.** **Case** $(3, 3, 3)$: Consider the following tame concealed algebra, which is of tubular type $(2, 2, 2)$, [15] pages 365–366):
Consider the modules with dimension vectors:

\[(3.1)\]
\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0
\end{array}
\]

and

\[
\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0
\end{array}
\]

Applying the Coxeter transformation we see that:

\[
\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix};
\]

\[
\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix};
\]

\[
\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix};
\]

\[
\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix};
\]

Here the vector \(h_1\) is a generator of the radical of the quadratic form of the quiver in Figure 2. By Proposition 1 the modules with dimension vectors in \((3.1)\) are simple regular modules in different tubes of rank 2. Applying one-point extensions with these modules we obtain, by \([18]\), the following tubular algebra of type \((3, 3, 3)\):

Now, in the above figure we find the following tame concealed subcategory, which is of tubular type \((3, 3, 2)\) (tame concealed of type \(\tilde{E}_6\)) \([18]\) pages 365–366):
We consider the (simple) module with dimension vector:

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0
\end{pmatrix}
\]

Applying the Coxeter transformation we see that:

\[
\begin{pmatrix}
0 & 0 \\
1 & 1 \\
0 & 0
\end{pmatrix} \Phi = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix} \Phi = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 \\ 0 \\ 1 & 1 \end{pmatrix} ; \quad h_2 = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}
\]

Here, the vector \( h_2 \) is a generator of the radical of the quadratic form of the quiver in Figure 3. By Proposition 1, the module with dimension vector in (3.2) is a simple regular module in a tube of rank 2. The one-point extension with this module is, by [18], the following tubular algebra of type (3, 3, 3):

Now in the above figure, we find the following tame concealed subcategory, which is of tubular type (2, 2, 2), (in fact, this is an hereditary algebra of type \( \tilde{D}_4 \)) [18, pages 365–366]:

Consider the modules with dimension vectors:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} , \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \Phi = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 \\ 0 \\ 1 & 1 \end{pmatrix} ;
\]

Applying the Coxeter transformation we see that:

\[
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix} \Phi = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} , \quad \begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{pmatrix} \Phi = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 \\ 0 \\ 1 & 1 \end{pmatrix} ;
\]
Here the vector $h_3$ is a generator of the radical of the quadratic form of the quiver in Figure 4. By Proposition 11, the modules with dimension vectors in (3.3) are simple regular modules in different tubes of rank 2. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type $(3,3,3)$:

Now in the above figure we find the following tame concealed subcategory, which is of tubular type $(3,3,2)$, (in fact, this is an hereditary algebra of type $\tilde{E}_6$) [18, pages 365–366]:

Consider the module with dimension vector:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 2 \\ 0 & 1 \end{pmatrix},$$

(3.4)

Applying the Coxeter transformation we see that:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 2 \\ 0 & 1 \end{pmatrix} \Phi = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}; \quad h_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}. $$
Here the vector $h_4$ is a generator of the radical of the quadratic form of the quiver in Figure 5. By Proposition 1, the module with dimension vector in (3.4) is a simple regular module in a tube of rank 2. The one-point extension with this module is, by [18], the following tubular algebra of type (3,3,3):

![Diagram](image)

In the above figure, we find the same subcategory of tubular type (2,2,2,2) that in Figure 2. So, this Galois covering is an iterated tubular algebra in the sense of de la Peña-Tomé [6].

**Case (2,2,2,2,λ):**

Consider the following tame concealed algebra, which is of tubular type (2,2), [18] pages 365–366:

![Diagram](image)

Consider the module with dimension vector:

\[(3.5) \quad u_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\]

Applying the Coxeter transformation we see that: \((u_1)\phi = u_1; \ u_1 = h_5\).

Here, the vector $h_5$ is a generator of the radical of the quadratic form of the quiver in Figure 6. By Proposition 1, the module with dimension vector in (3.5) is a simple regular module in a tube of rank 1. The one-point extension with this module is, by [18], the following tubular algebra of type (2,2,2):

![Diagram](image)
Remark 3. In Figure 7 all the squares commute up to \( \lambda \).

Consider the module with dimension vector:

\[
(3.6) \quad u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & \end{pmatrix}
\]

Applying the Coxeter transformation we see that: \((u_1)\phi = u_1; u_1 = h_6\). Here, the vector \( h_6 \) is a generator of the radical of the quadratic form of the quiver in Figure 7. By Proposition 1 the module with dimension vector in (3.6) is a simple regular module in a tube of rank 1. The one-point extension with this module is, by [18], the following tubular algebra of type \((2, 2, 2, 2)\):

\[
\begin{array}{cccc}
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\end{array}
\]

Remark 4. In the above figure all the squares commute up to \( \lambda \).

In the above figure we find the same subcategory of tubular type \((2, 2)\) that in Figure 6. So, this Galois covering is an iterated tubular algebra in the sense of de la Peña-Tomé [6].

Case \((4, 4, 2)\):

Consider the following tame concealed algebra, which is of tubular type \((4, 2, 2)\), [18, pages 365–366]:

\[
\begin{array}{cccc}
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow \\
\end{array}
\]

Consider the modules with dimension vectors:

\[
(3.7) \quad u_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \end{pmatrix}
\]

Applying the Coxeter transformation we see that:

\((u_1)\phi = v_1; (v_1)\phi = u_1; u_1 + v_1 = h_7\)
Here, the vector $h_7$ is a generator of the radical of the quadratic form of the quiver in Figure 8. By Proposition 1, the modules with dimension vectors in (3.7) are simple regular modules in the same tube of rank 2. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type $(4, 4, 2)$:

![Diagram of tubular algebra](image)

Now, in the above figure we find the following tame concealed subcategory, which is of tubular type $(3, 3, 2)$, [18] pages 365-366):

![Diagram of tubular algebra](image)

**Figure 9.**

Consider the modules with dimension vectors:

\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
\end{pmatrix} u_1 = 1, 0, 1 \quad \text{and} \quad \begin{pmatrix}
0 & 0 \\
1 & 0 \\
\end{pmatrix} v_1 = 1, 0, 1
\]

Applying the Coxeter transformation we see that:

\[
(u_1)\phi = u_2, \quad (u_2)\phi = u_3, \quad (u_3)\phi = u_1;
\]

\[
(v_1)\phi = v_2, \quad (v_2)\phi = v_3, \quad (v_3)\phi = v_1;
\]

\[
u_1 + u_2 + u_3 = h_8, \quad v_1 + v_2 + v_3 = h_8
\]

Here, the vector $h_8$ is a generator of the radical of the quadratic form of the quiver in Figure 9. By Proposition 1, the modules with dimension vectors in (3.8) are simple regular modules in different tubes of rank 3. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type $(4, 4, 2)$:
Now, in the above figure we find the following tame concealed subcategory, which is of tubular type $(3,3)$, [18, pages 365–366]:

Consider the modules with dimension vectors:

$$u_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad w_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Applying the Coxeter transformation we see that:

$$(u_1)\phi = u_1;$$
$$(v_1)\phi = v_2; \quad (v_2)\phi = v_3; \quad (v_3)\phi = v_1;$$
$$(w_1)\phi = w_2; \quad (w_2)\phi = w_3; \quad (w_3)\phi = w_1;$$

$$u_1 = h_9, \quad v_1 + v_2 + v_3 = h_9, \quad w_1 + w_2 + w_3 = h_9$$

Here the vector $h_9$ is a generator of the radical of the quadratic form of the quiver in Figure 10. By Proposition 1 the module with dimension vector $u_1$ (in 3.9) is a simple module in a tube of rank 1, and the modules with dimension vectors $v_1$ and $w_1$ (in 3.9) are simple regular modules in different tubes of rank 3. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type $(4,4,2)$:
Now, in the above figure we find the following tame concealed subcategory, which is of tubular type $(3, 3, 2)$, [18, pages 365–366]:

\[
\begin{array}{cccc}
   & & & \\
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   & & & \\
   & & & \\
   & & & \\
   \end{array}
\]

**Figure 11.**

Consider the modules with dimension vectors:

\[
\begin{align*}
   u_1 &= 0, 1, 1, \\
   v_1 &= 0, 1, 1
\end{align*}
\]

(3.10)

Applying the Coxeter transformation we see that:

\[
\begin{align*}
   (u_1) &\phi = u_2, \\
   (u_2) &\phi = u_3, \\
   (u_3) &\phi = u_1; \\
   (v_1) &\phi = v_2, \\
   (v_2) &\phi = v_3, \\
   (v_3) &\phi = v_1;
\end{align*}
\]

\[
u_1 + u_2 + u_3 = h_{10}, \quad v_1 + v_2 + v_3 = h_{10}.
\]

Here, the vector $h_{10}$ is a generator of the radical of the quadratic form of the quiver in Figure 11. By Proposition 11 the modules with dimension vectors in (3.10) are simple regular modules in different tubes of rank 3. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type $(4, 4, 2)$:

In the above figure, we find the same subcategory of tubular type $(4, 4, 2)$ that in Figure 8. So, this Galois covering is an iterated tubular algebra in the sense of de la Peña-Tomé [6].

**Case** $(6, 3, 2)$:

Consider the following tame concealed algebra, which is of tubular type $(4, 2, 2)$, [18, pages 365–366]:

Consider the modules with dimension vectors:
Applying the Coxeter transformation we see that:

\[(u_1)\phi = u_2, \quad (u_2)\phi = u_1;\]
\[(v_1)\phi = v_2, \quad (v_2)\phi = w_1;\]
\[(w_1)\phi = w_2, \quad (w_2)\phi = v_1;\]

\[u_1 + u_2 = h_{11}, \quad v_1 + v_2 + w_1 + w_2 = h_{11}\]

Here, the vector \(h_{11}\) is a generator of the radical of the quadratic form of the quiver in Figure 12. By Proposition 11 the module with dimension vectors \(u_1\) (in (3.11)) is a simple regular module in a tube of rank 2 and the modules \(v_1\) and \(w_1\) are simple regular modules in the same tube of rank 4. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type \((6,3,2)\):

Now, in the above figure we find the following tame concealed subcategory, which is of tubular type \((4,3,2)\), [18] pages 365–366:

Consider the modules with dimension vectors:

\[(3.12) \quad u_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\]

Applying the Coxeter transformation we see that:
Here, the vector $h_{12}$ is a generator of the radical of the quadratic form of the quiver in Figure 13. By Proposition 11 the modules with dimension vectors in (3.12) are simple regular modules in the same tube of rank 4. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type $(6, 3, 2)$:

![Diagram](image1)

Now, in the above figure we find the following tame concealed subcategory, which is of tubular type $(4, 2, 2)$, [18] pages 365–366):

![Diagram](image2)

Consider the modules with dimension vectors:
Applying the Coxeter transformation we see that:

\[(u_1 \phi = u_2, \quad (u_2 \phi = u_1;\]
\[(v_1 \phi = v_2, \quad (v_2 \phi = v_1;\]
\[(w_1 \phi = w_2, \quad (w_2 \phi = w_1;\]
\[u_1 + u_2 = h_{13}, \quad v_1 + v_2 + w_1 + w_2 = h_{13}.\]

Here, the vector \(h_{13}\) is a generator of the radical of the quadratic form of the quiver in Figure 14. By Proposition 1 the module with dimension vectors \(u_1\) (in (3.13)) is a simple regular module in a tube of rank 2 and the modules \(v_1\) and \(w_1\) are simple regular modules in the same tube of rank 4. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type \((6, 3, 2)\):

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{array}
\]

Figure 15.

Now, in the above figure we find the following tame concealed subcategory, which is of tubular type \((4, 3, 2)\), [18] pages 365–366):

Consider the modules with dimension vectors:

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{array}
\]
Applying the Coxeter transformation we see that:

\[(u_1) \phi = u_2, \quad (u_2) \phi = v_1;\]
\[(v_1) \phi = v_2, \quad (v_2) \phi = u_1;\]
\[v_1 + v_2 + w_1 + w_2 = h_{14}.\]

Here the vector \( h_{14} \) is a generator of the radical of the quadratic form of the quiver in Figure 15. By Proposition 1 the modules with dimension vectors in \((3,14)\) are simple regular modules in the same tube of rank 4. Applying one-point extensions with these modules we obtain, by [18], the following tubular algebra of type \((6,3,2)\):

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{tubular_algebra_diagram.png}
\end{array}
\]

In the above figure, we find the same subcategory of tubular type \((4,2,2)\) that in Figure 12. So, this Galois covering is an iterated tubular algebra in the sense of de la Peña-Tomé [6].

3.6. **Proof of Theorem 1** First, we note the following consequence of our considerations:

**Proposition 3.** The Jacobian algebras associated to the quivers with potentials \((Q_1, W_1), (Q_2, W_2), (Q_3, W_3)\) and \((Q_4, W_4)\) are tame of polynomial growth.

*Proof.* From Proposition 2 we know that each of the Galois coverings is an iterated tubular algebra. This implies, [6, Section 2.4], that each Galois covering is tame. Note that in each of these Galois coverings we have an obvious \( \mathbb{Z} \)-free action. Applying Theorem 3 the Jacobian algebras \( P(Q_i, W_i) \) \((i = 1, 2, 3, 4)\) are tame of polynomial growth.

Now, as explained in Section 2.5, the endomorphism ring of each basic cluster tilting object in a tubular cluster category is a non-degenerate Jacobian algebra. As mentioned in the introduction, for a fixed tubular type all these algebras are related via sequences of QP mutations. By [13, Thm. 3.6] and the above Proposition 3 it follows now that all these algebras are tame of polynomial growth. Note, that from the proof of [13, Thm. 3.6] it is clear that QP-mutation preserves even polynomial growth for tame algebras. This concludes the proof of Theorem 1.

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