LOCAL ILL-POSEDNESS OF THE EULER EQUATIONS IN $B^{1}_{∞,1}$

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Abstract. We show that the incompressible Euler equations on $\mathbb{R}^2$ are not locally well-posed in the sense of Hadamard in the Besov space $B^{1}_{∞,1}$. Our approach relies on the technique of Lagrangian deformations of Bourgain and Li. We show that the assumption that the data-to-solution map is continuous in $B^{1}_{∞,1}$ leads to a contradiction with a well-posedness result in $W^{1,p}$ of Kato and Ponce.

1. Introduction

The study of the Cauchy problem for the Euler equations

\begin{align}
(1.1) & \quad u_t + u \cdot \nabla u + \nabla \pi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n \\
(1.2) & \quad \text{div} \ u = 0 \\
(1.3) & \quad u(0) = u_0
\end{align}

has a long history going back to the works of Gyunter [9], Lichtenstein [11] and Wolibner [19] in the late 1920’s and 1930’s. Tremendous progress has been made since those pioneering papers and we refer to several excellent monographs and surveys for example Majda and Bertozzi [12], Constantin [6] or Bahour, Chemin and Danchin [1] for detailed accounts. Nevertheless, the problems related to the phenomenon of turbulence and persistence of smooth solutions in 3D for all time remain open. Furthermore, despite extensive studies of local well-posedness for the Euler equations our understanding of this problem especially in the cases of important borderline spaces including $C^1$, Lip, $B^{1}_{p,q}$, $W^{∞,p}$ etc. has also remained incomplete. However, this picture is changing fast.

Recall that a Cauchy problem is locally well-posed in a Banach space $X$ (in the sense of Hadamard) if for any initial data in $X$ there exist $T > 0$ and a unique solution which persists in the space $C([0,T],X)$ and which depends continuously on the data. Otherwise, the problem is said to be ill-posed.

A few years ago Bardos and Titi [2] used a shear flow of DiPerna and Majda [7] to construct solutions in 3D with an instantaneous loss of regularity in Hölder $C^α$ and Zygmund $B^{1}_{∞,∞}$ spaces. More precisely, they found $C^α$ initial data for which the corresponding (weak) solution does not belong to $C^β$ for any $1 > β > α^2$ and any $t > 0$. This technique has also been used to obtain similar results in the Triebel-Lizorkin $F^{1}_{∞,2}$ space by Bardos, Lemarie and Titi and in the logarithmic Lipschitz spaces $\log\text{Lip}^α$ by the authors [13].

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More recently, in a breakthrough paper Bourgain and Li \[3\] employed both Lagrangian and Eulerian techniques to obtain strong local ill-posedness results in borderline Sobolev spaces \(W^{n/p+1,p}\) for any \(1 \leq p < \infty\) and in Besov spaces \(B^{n/p+1}_{p,q}\) with \(1 \leq p < \infty\) and \(1 < q \leq \infty\) and \(n = 2\) or \(3\). In \[14\] the authors adapted the approach of \[3\] to settle (in the 2D case) a long standing open question of local ill-posedness in the classical \(C^1\) space by showing that the assumption on continuity of the data-to-solution map in \(C^1\) leads to a contradiction with a results of Kato and Ponce \[10\] for \(W^{1,p}\). Almost simultaneously Elgindi and Masmoudi \[8\] and Bourgain and Li \[4\] produced similar results using different methods. It now seems that a complete resolution of local ill-posedness questions for the Euler equations including various borderline spaces is fully within reach. In fact, the main results of \[4\] show that the Euler equations are ill-posed in the \(C^m\) spaces for any integer \(m \geq 1\) which is surprising in light of the local well-posedness results for the Cauchy problem (1.1)-(1.3) in \(C^{1,\alpha}\) for any \(0 < \alpha < 1\).

The goal of this paper is to settle the end-point case of the Besov space of smoothness order 1 and infinite (primary) integrability index. Recall that according to a recent result of Pak and Park \[15\] the Cauchy problem (1.1)-(1.3) admits a unique solution in \(B^1_{\infty,1}(\mathbb{R}^n)\). It is interesting to observe that in order to establish uniqueness they first show that the data-to-solution map \(u_0 \rightarrow u\) is continuous (even Lipschitz) into \(B^0_{\infty,1}\) (cf. \[15\], Section 4). They do not prove that it is continuous into \(B^1_{\infty,1}\) (and consequently that the Euler equations are locally well-posed in the sense of Hadamard in \(B^1_{\infty,1}\)). Our main result is

**Theorem 1.** The 2D incompressible Euler equations are locally ill-posed in the Besov space \(B^1_{\infty,1}\).

As in our previous paper \[14\] we will work with the vorticity equations. Recall that in two dimensions the vorticity of a vector field \(u\) is a 2-form \(\omega = du^\flat\) which is identified with the function

\[
\omega = \text{rot } u = -\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.
\]

In this case the Cauchy problem (1.4)-(1.5) can be rewritten as

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= 0, & t \geq 0, \ x \in \mathbb{R}^2 \\
\omega(0) &= \omega_0
\end{align*}
\]

where the velocity is recovered from \(\omega\) using the Biot-Savart law

\[
u = K * \omega = \nabla^\perp \Delta^{-1} \omega
\]

with kernel \(K(x) = (2\pi)^{-1}(-x_2/|x|^2, x_1/|x|^2)\) and where \(\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})\) denotes the symplectic gradient of a function.

Our strategy is similar to that adopted for the \(C^1\) case in \[14\]. Namely, following \[3\] we first choose an initial vorticity \(\omega_0\) such that the Lagrangian flow of the corresponding velocity field retains a large gradient on a (possibly short) time interval. We then perturb \(\omega_0\) to get a sequence of initial vorticities in \(W^{1,p}\). Finally, we show that the assumption that the Euler equations are well-posed in \(B^1_{\infty,1}(\mathbb{R}^2)\) (in particular, that the solutions depend continuously on the initial data) leads to a contradiction with the following result of Kato and Ponce.
Theorem (Kato-Ponce [10]). Let \( 1 < p < \infty \) and \( s > 1 + \frac{2}{p} \). For any \( \omega_0 \in W^{s-1,p}(\mathbb{R}^2) \) and any \( T > 0 \) there exists a constant \( K = K(T, \omega, s, p) > 0 \) such that
\[
\sup_{0 \leq t \leq T} \| \omega(t) \|_{W^{s-1,p}} \leq K.
\]

Theorem 1 will be a consequence of the following result

**Theorem 2.** Let \( 2 < p < \infty \). Assume that the vorticity equations \((1.3), (1.5)\) are well-posed in \( B^0_{s,1}(\mathbb{R}^2) \). There exist \( T > 0 \) and a sequence \( \omega_{0,n} \) in \( C^\infty_c(\mathbb{R}^2) \) with the following properties

1. there exists a constant \( C > 0 \) such that \( \| \omega_{0,n} \|_{W^{1,p}} \leq C \) for all \( n \in \mathbb{Z}_+ \)

and

2. for any \( M > 1 \) there is \( 0 < t_0 \leq T \) such that \( \| \omega_n(t_0) \|_{W^{1,p}} \geq M^{1/3} \) for all sufficiently large \( n \) and all \( p \) sufficiently close to 2.

In Section 2 we provide some technical tools and construct an initial vorticity whose Lagrangian flow has a large gradient. Since some of the constructions are analogous to those in [14] some details are omitted. The proof of Theorem 2 is given in Section 3.

**Remark 3.** In this paper we do not employ the "patching" argument of [3] which leaves open the question of strong ill-posedness in \( B^1_{s,1,q} \) in the sense of Bourgain-Li.

**Remark 4.** Since we treat here non-decaying data, an analogous local ill-posedness result in 3D follows immediately from our 2D construction. The details will be elaborated elsewhere.

### 2. Vorticity and the Lagrangian Flow

We first recall some basic harmonic analysis. Let \( \psi \) be a smooth radial bump function on \( \mathbb{R}^2 \) which is supported in the unit ball \( B(0,1) \) and equal to 1 on the ball of radius 1/2. Set \( \psi_{-1} = \psi \) and let
\[
(2.1) \quad \psi_\ell(\xi) = \psi(2^{-\ell} \xi) - \psi(\xi) \quad \text{and} \quad \psi_{\ell}(\xi) = \psi_0(2^{-\ell} \xi) \quad \forall \ell \geq 0.
\]

Each \( \psi_\ell \) is supported in a shell \( \{ 2^{\ell-1} \leq |\xi| \leq 2^{\ell+1} \} \) with \( \psi_\ell(\xi) = 1 \) when \( |\xi| = 2^\ell \).

For any \( f \in \mathcal{S}'(\mathbb{R}^2) \) define the frequency restriction operators by
\[
\hat{\Delta}_\ell \hat{f}(\xi) = \psi_\ell(\xi) \hat{f}(\xi) \quad \forall \ell \geq -1,
\]
to obtain the usual Littlewood-Paley decomposition
\[
f = \sum_{\ell \geq -1} \Delta_\ell f \quad \text{where} \quad \Delta_\ell f(x) = \sum_{\xi \in \mathbb{R}^2} \psi_\ell(\xi) \hat{f}(\xi) e^{i(\xi,x)}, \quad x \in \mathbb{R}^2.
\]

For any \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \) the Besov space \( B^s_{p,q}(\mathbb{R}^2) \) is defined as the set of all \( f \in \mathcal{S}'(\mathbb{R}^2) \) such that the number
\[
(2.2) \quad \| f \|_{B^s_{p,q}} = \left\{ \begin{array}{ll} \left( \sum_{\ell \geq -1} 2^{\ell q} \| \Delta_\ell f \|_{L^p}^q \right)^{1/q} & \text{if} \quad 1 \leq q < \infty \\ \sup_{\ell \geq -1} 2^{\ell s} \| \Delta_\ell f \|_{L^p} & \text{if} \quad q = \infty \end{array} \right.
\]
is finite. Among many special cases of interest are the Sobolev spaces \( W^{s,p} = B^s_{p,p} \) and the Hölder-Zygmund class \( C^s = B^s_{\infty,\infty} \) both defined for any \( 1 \leq p < \infty \) and any non-integer \( s > 0 \).
Next, given a radial bump function \( 0 \leq \varphi \leq 1 \) supported in \( B(0,1) \) define
\[
\varphi_0(x_1,x_2) = \sum_{\varepsilon_1,\varepsilon_2 = \pm 1} \varepsilon_1\varepsilon_2 \varphi(x_1-\varepsilon_1, x_2-\varepsilon_2)
\]
and for a fixed positive integer \( N_0 \in \mathbb{Z}_+ \) and any \( M \gg 1 \), set
\[
\omega_0(x) = \omega_0^{M,N}(x) = M^{-2}N^{-\frac{1}{2}} \sum_{N_0 \leq k \leq N_0 + N} \varphi_k(x), \quad N = 1,2,3 \ldots
\]
where \( 2 < p < \infty \) and where
\[
\varphi_k(x) = 2^{(-1+\frac{p}{2})k} \varphi_0(2^k x).
\]
Observe that by construction \( \varphi_0 \) is an odd function in both \( x_1, x_2 \) and for any \( k \geq 1 \) its support is compact and contained in the set
\[
\text{supp } \varphi_k \subset \bigcup_{\varepsilon_1,\varepsilon_2 = \pm 1} B((\varepsilon_1 2^{-k}, \varepsilon_2 2^{-k}), 2^{-(k+2)}).
\]
Combined with the uniform (in time) \( L^\infty \) control of the vorticity in \( \mathbb{R}^2 \) this ensures the existence of a unique solution of the Cauchy problem (1.4)-(1.5) with the initial data (2.4); e.g., by a result of Yudovich [20], see also Majda and Bertozzi [12].

The construction so far parallels that of our previous paper [14] and therefore the proofs of Lemma 5 and Proposition 6 below will be omitted.

**Lemma 5.** We have
\[
\|\omega_0\|_{W^{1,p}} \lesssim M^{-2}
\]
with the bound independent of \( N > 0 \) and \( 2 < p < \infty \).

**Proof.** See [14]; Lemma 3. \( \square \)

Since \( p > n = 2 \) the results of Kato and Ponce [10] (cf. Lemma 3.1; Thm. III) imply that there exists a unique velocity field \( u \in C^1([0, \infty), W^{2,p}(\mathbb{R}^2)) \) solving the problem (1.4)-(1.5) and whose vorticity function \( \omega \in C([0, \infty), W^{1,p}(\mathbb{R}^2)) \) satisfies the initial condition (2.4).

The associated Lagrangian flow \( \eta(t) \) of \( u = \nabla^+ \Delta^{-1} \omega \) is a solution of the initial value problem
\[
\frac{d}{dt} \eta(t,x) = u(t, \eta(t,x)) = F_u(\eta(t,x))
\]
\[
\eta(0,x) = x
\]
and defines a curve in the group of volume-preserving diffeomorphisms such that \( \omega \circ \eta \in C([0, \infty), W^{1,p}(\mathbb{R}^2)) \), see e.g., [17] or [5]. It can be readily checked that the odd symmetry of \( \omega_0 \) is preserved by \( \eta \) and thus (by conservation of vorticity in 2D) is retained by \( \omega \) for all time. In this case the Biot-Savart law (1.6) implies that the velocity field \( v \) must be symmetric in the variables \( x_1 \) and \( x_2 \) so that both coordinate axes are invariant under \( \eta \) with the origin \( x_1 = x_2 = 0 \) a hyperbolic stagnation point.

Observe that if \( \xi : \mathbb{R}^2 \to \mathbb{R}^2 \) is a volume-preserving diffeomorphism then the Jacobi matrix of its inverse \( \xi^{-1} \) can be computed from
\[
D\xi^{-1} = (D\xi)^{-1} \circ \xi^{-1} = \begin{pmatrix}
\frac{\partial \xi_2}{\partial x_1} \circ \xi^{-1} & -\frac{\partial \xi_1}{\partial x_1} \circ \xi^{-1} \\
-\frac{\partial \xi_2}{\partial x_2} \circ \xi^{-1} & \frac{\partial \xi_1}{\partial x_2} \circ \xi^{-1}
\end{pmatrix}
\]
Consider a smooth bump function $\hat{\chi} (3.2)$ so that for any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ we have
\begin{equation}
\nabla (f \circ \xi^{-1}) = (-\nabla f \circ \xi^{-1} \cdot \nabla \perp \xi_2 \circ \xi^{-1}, \nabla f \circ \xi^{-1} \cdot \nabla \perp \xi_1 \circ \xi^{-1})
\end{equation}
where $\nabla \perp$ is the symplectic gradient as in (1.0).

**Proposition 6.** Let $\eta(t)$ be the flow of the velocity field $u = \nabla \perp \Delta^{-1} \omega$ with initial vorticity given by (2.4). Given $M \geq 1$ we have
\begin{equation}
\sup_{0 \leq t \leq M^{-3}} \|D\eta(t)\|_{\infty} > M
\end{equation}
for any sufficiently large integer $N > 0$ in (2.4) and any $2 < p < \infty$ sufficiently close to 2.

**Proof.** See [14]: Section 3.

In what follows it can be assumed without loss of generality that $2 < p < 3$. In this case all estimates on the flow $\eta$ or its derivative $D\eta$ can be made independent of the Lebesgue exponent $2 < p < \infty$.

We will also need a comparison result for solutions of the Lagrangian flow equations, namely

**Lemma 7.** Let $u$ and $v$ be smooth divergence-free vector fields on $\mathbb{R}^2$ and let $\eta$ and $\xi$ be the solutions of (2.7) - (2.8) with the right-hand sides given by $F_u$ and $F_{u + v}$ respectively. Then
\begin{equation}
\sup_{0 \leq t \leq 1} (\|\xi(t) - \eta(t)\|_{\infty} + \|D\xi(t) - D\eta(t)\|_{\infty}) \leq C \sup_{0 \leq t \leq 1} (\|v(t)\|_{\infty} + \|Dv(t)\|_{\infty})
\end{equation}
for some $C > 0$ depending only on $u$ and its derivatives.

**Proof.** See e.g. [3]: Lemma 4.1.

3. **Proof of Theorem 2**

Let $M \geq 1$ be an arbitrary large number and take $T = 1 \leq M^{-3}$. Recall that by assumption $2 < p < \infty$ and set $s = 2$.

Given the initial vorticity $\omega_0$ defined in (2.4) let $\omega(t)$ be the corresponding solution of the vorticity equations (1.4) - (1.5) and let $\eta(t)$ be the associated Lagrangian flow of $u = \nabla \perp \Delta^{-1} \omega$.

If there exists $0 < t_0 \leq M^{-3}$ such that $\|\omega(t_0)\|_{W^{1,p}} > M^{1/3}$ then there is nothing to prove. We will therefore assume that
\begin{equation}
\|\omega(t_0)\|_{W^{1,p}} \leq M^{1/3}, \quad 0 \leq t_0 \leq M^{-3}.
\end{equation}
Using Proposition 6 we can then find a point $x^* = (x_1^*, x_2^*)$ such that at least one of the entries $\partial \eta_i / \partial x_j$ of the Jacobi matrix (for example, the $i = j = 2$ entry) satisfies $|\partial \eta_2(t_0, x^*)| > M$. Therefore, by continuity, there is a $\delta > 0$ such that
\begin{equation}
\left| \frac{\partial \eta_2}{\partial x_2}(t_0, x) \right| > M \quad \text{for all} \quad |x - x^*| < \delta.
\end{equation}
Consider a smooth bump function $\hat{\chi} \in C_0^\infty (\mathbb{R}^2)$ in Fourier variables with support in the unit ball $B(0, 1)$ and such that $0 \leq \hat{\chi} \leq 1$ and $\int_{\mathbb{R}^2} \hat{\chi}(\xi) \, d\xi = 1$. Let $\xi_0 = (2, 0)$ and define
\begin{equation}
\hat{\rho}(\xi) = \hat{\chi}(\xi - \xi_0) + \hat{\chi}(\xi + \xi_0), \quad \xi \in \mathbb{R}^2
\end{equation}
Observe that the support of \( \hat{\rho} \) is contained in \( B(-\xi_0, 1) \cup B(\xi_0, 1) \) and
\[
\rho(0) = \int_{\mathbb{R}^2} \hat{\rho}(\xi) \, d\xi = 2.
\]
For any \( k \in \mathbb{Z}_+ \) and \( \lambda > 0 \) define
\[
(3.5) \quad \beta_{k,\lambda}(x) = \frac{\lambda^{-1+\frac{2}{p}}}{\sqrt{k}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \rho(\lambda(x - x^*_\varepsilon)) \sin kx_1
\]
where \( x^*_\varepsilon = (\varepsilon_1 x^*_1, \varepsilon_2 x^*_2) \).
To proceed we will need two technical lemmas.

**Lemma 8.** For any \( k \in \mathbb{Z}_+ \) and \( \lambda > 0 \) we have
\[
1. \quad \| \partial_{ij} \Delta^{-1} \beta_{k, \lambda} \|_{L^1} \lesssim k^{-1/2} \lambda^{-2+\frac{2}{p}} \| \hat{\rho} \|_{L^1}
\]
\[
2. \quad \| \partial_i \partial_j \Delta^{-1} \beta_{k, \lambda} \|_{L^1} \lesssim k^{-1/2} \lambda^{-1+\frac{2}{p}} \| \hat{\rho} \|_{L^1}
\]
\[
3. \quad \| \beta_{k, \lambda} \|_{W^{1,p}} \lesssim (k^{-1/2} + k^{1/2} \lambda^{-1} + k^{-1/2} \lambda^{-1}) \| \hat{\rho} \|_{L^{p'}}
\]
where \( i, j = 1, 2 \) and \( 1 < p' < 2 \) is the conjugate exponent of \( 2 < p < \infty \).

**Proof.** Let \( \xi_\pm = (\xi_1 \pm k/2\pi, \xi_2) \) and first compute the Fourier transform
\[
(3.6) \quad \hat{\beta}_{k,\lambda}(\xi) = \lambda^{-1+\frac{2}{p}} \sum_{\varepsilon_1, \varepsilon_2} \frac{\varepsilon_1 \varepsilon_2}{2\pi} \left( e^{-2\pi i(\xi_- \cdot x^*_\varepsilon)} \frac{1}{\lambda^2} \hat{\rho} \left( \frac{\xi_-}{\lambda} \right) - e^{-2\pi i(\xi_+ \cdot x^*_\varepsilon)} \frac{1}{\lambda^2} \hat{\rho} \left( \frac{\xi_+}{\lambda} \right) \right).
\]
Next, using the change of variable formula we estimate
\[
| \partial_j \Delta^{-1} \beta_{k, \lambda}(x) | \simeq \left| \mathcal{F}^{-1} \mathcal{F}(\partial_j \Delta^{-1} \beta_{k, \lambda})(x) \right| \lesssim \int_{\mathbb{R}^2} |\xi|^{-1} |\hat{\beta}_{k, \lambda}(\xi)| \, d\xi
\]
\[
\lesssim k^{-1/2} \lambda^{-1+\frac{2}{p}} \int_{\mathbb{R}^2} |\xi|^{-1} \frac{1}{\lambda^2} \left( |\hat{\rho}(\lambda^{-1} \xi_-)| + |\hat{\rho}(\lambda^{-1} \xi_+)| \right) \, d\xi
\]
\[
\simeq k^{-1/2} \lambda^{-2+\frac{2}{p}} \sum_{j=1,2} \int_{\mathbb{R}^2} |\xi|^{-1} |\hat{\rho}(\xi_1 + \frac{(j-1)\lambda}{2\pi} \lambda^{-1} k, \xi_2)| \, d\xi.
\]
Since by construction for any sufficiently large \( k \gg 10 \) we have
\[
\text{supp} \hat{\rho}(\cdot \pm \frac{1}{2\pi} \lambda^{-1} k, \cdot) \cap B(0, 1) = \emptyset
\]
we can further estimate the above expression by
\[
\simeq k^{-1/2} \lambda^{-2+\frac{2}{p}} \sum_{j=1,2} \int_{|\xi| \geq 1} |\hat{\rho}(\xi_1 + \frac{(j-1)\lambda}{2\pi} \lambda^{-1} k, \xi_2)| \, d\xi \simeq k^{-1/2} \lambda^{-2+\frac{2}{p}} \| \hat{\rho} \|_{L^1}.
\]
For the second assertion we similarly obtain
\[
| \partial_i \partial_j \Delta^{-1} \beta_{k, \lambda}(x) | \lesssim \| \hat{\beta}_{k, \lambda} \|_{L^1} \lesssim k^{-1/2} \lambda^{-1+\frac{2}{p}} \| \hat{\rho} \|_{L^1}.
\]
Finally, using the triangle inequality and the change of variables formula we get
\[
\left\| \frac{\partial \beta_{k, \lambda}}{\partial x_1} \right\|_{L^p} \lesssim \frac{1}{\sqrt{k}} \lambda^{2/p} \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 \frac{\partial \rho}{\partial x_1}(\lambda(\cdot - x^*_\varepsilon)) \left\| \frac{\sqrt{k}}{\lambda} \lambda^{2/p} \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 \rho(\lambda(\cdot - x^*_\varepsilon)) \right\|_{L^p}
\]
\[
\simeq \frac{1}{\sqrt{k}} \left( \int_{\mathbb{R}^2} \left| \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 \frac{\partial \rho}{\partial x_1}(x) \right|^p \, dx \right)^{1/p} + \left( \int_{\mathbb{R}^2} \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 \rho(x) \right)^{1/p}.
\]
and using Hausdorff-Young with $1/p' + 1/p = 1$ we find

$$
\lesssim k^{-1/2} \left\| \frac{\partial \rho}{\partial x_1} \right\|_{L^p} + k^{1/2} \lambda^{-1} \left\| \rho \right\|_{L^p} \lesssim k^{-1/2} \left\| \frac{\partial \rho}{\partial x_1} \right\|_{L^{p'}} + k^{1/2} \lambda^{-1} \left\| \hat{\rho} \right\|_{L^{p'}}
\lesssim (k^{-1/2} + k^{1/2} \lambda^{-1}) \left\| \hat{\rho} \right\|_{L^{p'}}.
$$

Similarly, we also obtain

$$
\left\| \frac{\partial \beta_{k,\lambda}}{\partial x_2} \right\|_{L^p} \lesssim k^{-1/2} \left\| \frac{\partial \rho}{\partial x_2} \right\|_{L^p} \lesssim k^{-1/2} \left\| \hat{\rho} \right\|_{L^{p'}}
$$

and

$$
\left\| \beta_{k,\lambda} \right\|_{L^p} \lesssim \frac{\lambda^{-1}}{\sqrt{k}} \left( \int_{\mathbb{R}^2} \left| \chi/2 \sum_{\varepsilon_1, \varepsilon_2} \varepsilon_1 \varepsilon_2 \rho(\lambda(x - x_*^\varepsilon)) \right|^p dx \right)^{1/p} \lesssim k^{-1/2} \lambda^{-1} \left\| \hat{\rho} \right\|_{L^{p'}}
$$

which combined yield the lemma.

\[ \square \]

Observe that choosing

(3.7) \[ k = \lambda^2 \quad \text{and} \quad \lambda = 3n, \quad n \gg 100 \]

and letting $\beta_n = \beta_{k,\lambda}$ in Lemma 8 we immediately obtain

(3.8) \[ \|\nabla^\perp \Delta^{-1} \beta_n\|_{\infty} \to 0 \quad \text{and} \quad \|D\nabla^\perp \Delta^{-1} \beta_n\|_{\infty} \to 0 \quad \text{as} \ n \to \infty \]

and

(3.9) \[ \|\beta_n\|_{W^{1,p}} \approx \|\beta_n\|_{L^p} + \|\nabla^\perp \beta_n\|_{L^p} \lesssim \|\hat{\rho}\|_{L^{p'}} < \infty \quad \text{for any} \ n \in \mathbb{Z}_+.
]

The second lemma we need is

**Lemma 9.** Let $t_0 > 0$ be as in (3.4) and let $k, \lambda$ and $n$ be as in (3.7). Then

1. $\|\partial^2 \beta_{k,\lambda} \partial_t \eta(t_0)\|_{L^p} \lesssim n^{-1} C_{0,T} \left\| \hat{\rho} \right\|_{L^{p'}} \xrightarrow{n \to \infty} 0$

2. $\|\partial^2 \beta_{k,\lambda} \partial^2 \eta(t_0)\|_{L^p} \gtrsim M(\epsilon \pi + O(n^{-1/2})) - n^{-1} C_{0,T} \left\| \hat{\rho} \right\|_{L^{p'}} \xrightarrow{n \to \infty} M$

where $C_{0,T} = \exp^{T \sup_{t_0} \|D\nabla^\perp \Delta^{-1} \omega(t)\|_{\infty}} < \infty$.

**Proof.** From (3.3) we have

\begin{equation}
\left( \int_{\mathbb{R}^2} \left| \frac{\partial \beta_{k,\lambda}}{\partial x_2}(x) \frac{\partial \eta_2}{\partial x_1}(t_0, x) \right|^p dx \right)^{1/p} \leq \frac{1}{\sqrt{k}} \sum_{\varepsilon_1, \varepsilon_2} \left( \int_{\mathbb{R}^2} \left| \frac{\partial \rho}{\partial x_2}(\lambda(x - x_*^\varepsilon)) \right|^p dx \right)^{1/p} \sup_{x \in \mathbb{R}^2} \left| \frac{\partial \eta_2}{\partial x_1}(t_0, x) \right|.
\end{equation}

Differentiating the flow equations (2.7) - (2.8) in the spatial variable and taking the $L^\infty$ norms we obtain a differential inequality which with the help of Gronwall’s lemma gives

$$
\|D\eta(t)\|_{\infty} \leq e^{L_0 t} \|D\eta(t_0)\|_{\infty} \lesssim C_{0,T}
$$

Using this bound the right hand side of (3.10) can be estimated further by

$$
\lesssim k^{-1/2} \|\partial_1 \eta_2(t_0)\|_{\infty} \|\rho\|_{L^p} \lesssim k^{-1/2} C_{0,T} \left\| \hat{\rho} \right\|_{L^{p'}}.
$$
which gives the first assertion. For the second one we have
\[
\left( \int_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial x_1^2} (x) \frac{\partial^2}{\partial x_2^2} (t_0, x) \right|^p dx \right)^{1/p} = \\
= \left( \int_{\mathbb{R}^2} \left| \frac{1}{\sqrt{k}} \lambda^2 \sum_{\varepsilon_1, \varepsilon_2 \in \mathbb{C}} \varepsilon_1 \varepsilon_2 \left( k \rho (\lambda (x - x^*_\varepsilon_1)) \cos k x_1 + \lambda \frac{\partial \rho}{\partial x_1} (\lambda (x - x^*_\varepsilon_1)) \sin k x_1 \right) \right| dx \right)^{1/p},
\]
\[
\geq \left( \int_{B(x^*, \delta)} \sqrt{k} \lambda^{-1+\varepsilon_2} \cos k x_1 \rho (\lambda (x - x^*)) \frac{\partial^2}{\partial x_2^2} (t_0, x) + \frac{1}{\sqrt{k}} \lambda^2 \sin k x_1 \rho (\lambda (x - x^*)) \frac{\partial^2}{\partial x_2^2} (t_0, x) \right)^{1/p}.
\]
Using the triangle inequality, (3.2) and the change of variables formula we estimate the above integral from below by
\[
\geq M \sqrt{k} \lambda^{-1} \left( \int_{B(x^*, \delta)} \lambda^2 \left| \cos k x_1 \rho (\lambda (x - x^*)) \right|^p dx \right)^{1/p} - \\
- \frac{1}{\sqrt{k}} \left( \int_{B(x^*, \delta)} \lambda^2 \left| \sin k x_1 \rho (\lambda (x - x^*)) \frac{\partial^2}{\partial x_2^2} (t_0, x) \right|^p dx \right)^{1/p},
\]
(3.11)
\[
\geq M \sqrt{k} \lambda^{-1} \left( \int_{B(0, \delta)} \left| \cos (k \lambda^{-1} x_1 + k x_1^* \varepsilon_2) \right|^p \rho (x) ^p dx \right)^{1/p} - \\
- \frac{1}{\sqrt{k}} \| \partial_1 \rho \|_{L^p} \| \partial_2 \rho \|_{L^p} \| (B(x^*, \delta)).
\]
We now focus on the integral term on the right hand side of (3.11). From (3.4) we have \( \rho (0) = 2 \) so that by continuity there exists an \( \epsilon > 0 \) such that
\[ |\rho (x)| \geq 1, \quad \text{for any } x \in B(0, \epsilon). \]
Therefore, if we set \( \delta = \epsilon / \lambda \) then the integral term can be bounded below by
\[
\left( \int_{B(0, \epsilon)} \left| \cos (k \lambda^{-1} x_1 + k x_1^* \varepsilon_2) \right|^p dx \right)^{1/p} \geq \left( \int_{\pi^2} \cos^2 (\lambda x + \lambda^2 x_1^*) dx \right)^{1/2} \\
\simeq \frac{\epsilon \pi}{3 \sqrt{2}} + O (\lambda^{-1/2})
\]
by a straightforward calculation using the assumption on \( p > 2 \) and the choices of parameters made in (3.7). To complete the proof of the lemma it suffice to observe that (3.11) can now be further bounded by
\[
\geq M (\epsilon \pi + O (n^{-1/2})) - \frac{C_0}{n} \| \rho \|_{L^p},
\]
which is the required estimate. \( \square \)

Consider the following sequence of initial vorticities
\[
\omega_{0,n} (x) = \omega_0 (x) + \beta_n (x), \quad n \in \mathbb{Z}_+.
\]
From equations (3.9) and (2.6) of Lemma 5 it follows that \( \omega_{0,n} \) belongs to \( W^{1,p} \) for any \( n \in \mathbb{Z}_+ \). Let \( \omega_n(t) \) be the corresponding solutions of the vorticity equations (1.4)-(1.5). For each \( n \in \mathbb{Z}_+ \) (sufficiently large if necessary) let \( \eta_n(t) \) be the flow of volume-preserving diffeomorphisms of the velocity fields \( u_n = \nabla^\perp \Delta^{-1} \omega_n \).

(A). Let \( 1 \leq q < \infty \) and assume that the data-to-solution map for the Euler equations is continuous from bounded sets in \( B^1_{\infty,1}(\mathbb{R}^2) \) to \( C([0,1],B^1_{\infty,1}(\mathbb{R}^2)) \).

**Lemma 10.** For any \( 1 \leq q < \infty \) we have

\[
\| \nabla^\perp \Delta^{-1} \beta_n \|_{B^q_{\infty,q}} \simeq \| \nabla^\perp \Delta^{-1} \beta_n \|_\infty + \| D\nabla^\perp \Delta^{-1} \beta_n \|_\infty
\]

for any sufficiently large \( n \in \mathbb{Z}_+ \).

**Proof.** It will be sufficient to show the equalities \( \| D^n \nabla^\perp \Delta^{-1} \beta_n \|_{B^q_{\infty,q}} \simeq \| D^n \nabla^\perp \Delta^{-1} \beta_n \|_\infty \) for \( |\alpha| = 0 \) and 1.

As in the proof of Lemma 8 recall that for any large enough integer \( n \in \mathbb{Z}_+ \) we have supp \( \hat{\beta}_n \cap B(0,1) = \emptyset \) so that only need to establish \( \| \beta_n \|_\infty \simeq \| \beta_n \|_{B^q_{\infty,q}} \). Let \( \psi_t \) be the family of Paley-Littlewood functions as in (2.1) supported in the shell \( \{2^{j-1} \leq |\xi| \leq 2^{j+1}\} \). From (3.8) and the definition of \( \rho \) we find that

\[
\text{supp} \hat{\beta}_n = B((2^{2j}, 0), 2^j)
\]

since supp \( \hat{\rho} \subset B(0,3) \), see (3.3). It follows that for any \( j \in \mathbb{Z}_+ \) we can find an \( t_j \in \mathbb{Z}_+ \) such that supp \( \hat{\beta}_n \subset \text{supp} \hat{\psi}_{t_j} \) which implies that

\[
\| \beta_n \|_{B^q_{\infty,q}} = \left( \sum_{j \geq -1} \| \Delta_j \beta_n \|_{L^q} \right)^{1/q} \simeq \| \hat{\psi}_{t_j} * \beta_n \|_\infty \simeq \| \beta_n \|_\infty.
\]

The other equality can be shown analogously.

Combining (3.8), (3.12) and Lemma 10 it follows now from the assumption (A) on the continuity of the solution map that

\[
\sup_{0 \leq t_1 \leq 1} \| \nabla^\perp \Delta^{-1}(\omega_n(t) - \omega(t)) \|_{B^1_{\infty,1}} \to 0 \quad \text{as} \quad n \to \infty
\]

and consequently by an elementary embedding \( B^1_{\infty,1} \subset C^1 \) we also have

\[
\sup_{0 \leq t_1 \leq 1} \| \nabla^\perp \Delta^{-1}(\omega_n(t) - \omega(t)) \|_{C^1} \to 0 \quad \text{as} \quad n \to \infty.
\]

Applying the comparison Lemma 7 we now find

\[
\sup_{0 \leq t_1 \leq 1} \left( \| \eta_n(t) - \eta(t) \|_\infty + \| D\eta_n(t) - D\eta(t) \|_\infty \right) = \theta_n \to 0 \quad \text{as} \quad n \to \infty
\]

where \( \eta(t) \) is the flow of the velocity field \( u = \nabla^\perp \Delta^{-1} \omega \) with the initial vorticity \( \omega_0 \) given by (2.4) as in Proposition 6.

Using conservation of vorticity, formula (2.9) and the invariance of the \( L^p \) norms under volume-preserving Lagrangian flows \( \eta_n(t) \) we have

\[
\| \omega_n(t_0) \|_{W^{1,p}} \geq \| \nabla(\omega_{0,n} \circ \eta_n^{-1}(t_0)) \|_{L^p} \simeq \| d\omega_{0,n} \circ \eta_n^{-1}(t_0) \|_{L^p} + \| d\omega_{0,n} \circ \eta_n^{-1}(t_0)(\nabla^\perp \eta_{n,1}(t_0) \circ \eta_n^{-1}(t_0)) \|_{L^p}
\]

\[
\simeq \| d\omega_{0,n}(\nabla^\perp \eta_{n,2}(t_0)) \|_{L^p} + \| d\omega_{0,n}(\nabla^\perp \eta_{n,1}(t_0)) \|_{L^p} \geq \| d\omega_{0,n}(\nabla^\perp \eta_{n,2}(t_0)) \|_{L^p}.
\]

\[
(3.15)
\]
Since from the comparison estimate \((3.14)\) we have
\[\|d\omega_{0,n}(\nabla^\perp \eta_2 - \nabla^\perp \eta_{n,2})(t_0)\|_{L^p} \lesssim \|D(\eta_2 - \eta_{n,2})(t_0)\|_{L^\infty}\|\nabla \omega_{0,n}\|_{L^p}\]
applying the triangle inequality and \((3.12)\) we can further bound the right side of the expression in \((3.15)\) below by
\[(3.16)\]
\[\|d\omega_{0,n}(\nabla^\perp \eta_2(t_0))\|_{L^p} - \theta_n\|\nabla \omega_{0,n}\|_{L^p} \gtrsim \|d\tilde{\beta}_n(\nabla^\perp \eta_2(t_0))\|_{L^p} - \|d\omega_{0}(\nabla^\perp \eta_2(t_0))\|_{L^p} - \theta_n\|\nabla \omega_{0,n}\|_{L^p}.
\]
Observe that by the assumption \((3.1)\) we can bound the middle term on the right side of \((3.16)\) as in \((3.15)\) above by
\[(3.17)\]
\[\|d\omega_{0}(\nabla^\perp \eta_2(t_0))\|_{L^p} \lesssim \|\nabla \omega_{0} \circ \eta^{-1}(t_0) \cdot D\eta^{-1}(t_0)\|_{L^p} \lesssim \|\omega(t_0)\|_{W^{1,p}} \leq M^{1/3}.
\]
It therefore remains to find a lower bound on the \(\beta\)-term in \((3.10)\). This however follows from the two estimates in Lemma \([3]\). Namely, we have
\[(3.18)\]
\[\|d\beta_n(\nabla^\perp \eta_2(t_0))\|_{L^p} = \|\partial_1 \beta_n \partial_2 \eta_2(t_0) + \partial_2 \beta_n \partial_1 \eta_2(t_0)\|_{L^p} \gtrsim \|\partial_1 \beta_n \partial_2 \eta_2(t_0)\|_{L^p} - \|\partial_2 \beta_n \partial_1 \eta_2(t_0)\|_{L^p} \gtrsim M(\mathcal{E} + \mathcal{O}(n^{-1/2})) - n^{-1}C_0,T\|\tilde{\rho}\|_{L^{p'}} - n^{-1}C_0,T\|\tilde{\rho}\|_{L^{p'}} \gtrsim M
\]
provided that \(n\) is sufficiently large. Theorem \([2]\) follows now by combining \((3.15)\) with \((3.16)\), \((3.17)\) and \((3.18)\).

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