Entangling two distant nanocavities via a waveguide

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(Received 24 March 2011; published 8 June 2011)

In this paper, we present a scheme for generating continuous-variable entanglement between two spatially separated nanocavities in photonic crystals, which are mediated by a coupled-resonator optical waveguide. The entanglement degree and purity of the generated states are investigated as varying the cavity frequencies, the cavity-waveguide coupling strength, and the location of the second cavity. It is shown that a steady and pure entanglement between separated nanocavities can be generated only with a weak cavity-waveguide coupling when the cavities are resonant with the band center of the waveguide. Various cases with different cavity frequencies and coupling strengths, which affect the degree of entanglement, are also investigated, and interestingly sudden death and sudden birth of entanglement occur for strong couplings.

DOI: 10.1103/PhysRevA.83.062310 PACS number(s): 03.67.Bg, 42.50.Dv, 03.65.Yz

I. INTRODUCTION

Entanglement of the electromagnetic field with continuous variables [1], which embodies quantum correlations in amplitude and phase quadratures of the field, has been proven to be very important for building up continuous-variable quantum communication networks [2–5] and quantum computations [6–8]. Generally, continuous-variable entanglement is generated via nondegenerate optical parametric oscillation [3,5,9–11] or beam splitters incorporated with squeezed light beams [2,12].

Based on recent achievements in light manipulation with highly controllable photonic crystals [13], we investigate in this paper the generation of continuous-variable entanglement between two spatially separated nanocavities mediated by a coupled-resonator optical waveguide (CROW) in photonic crystals. As we know, the long-distance entanglement is an essential ingredient for transmitting quantum information over long distances in quantum communication networks [14,15]. Very recently, the distant entanglement between two qubits mediated by a plasmonic waveguide [16] or two vibrating trapped ions assisted by a phononic reservoir [17] have been studied. In fact, the entanglement between two qubits can also be created by interaction to a common heat bath [18]. Here we shall focus on the distant continuous-variable entanglement between two nanocavity fields. The main merits of the present scheme are as follows. The nanocavity could be a point defect created in photonic crystals and its frequency can be simply controlled just by changing the size or the shape of the defect [19]. Meanwhile, the waveguide, as the mediator of the two cavities as well as the output channel, can be considered as a set of linearly coupled defects in photonic crystals in which light propagates due to the coupling between the adjacent defects [20,21]. The transmission properties of the CROW can also be easily manipulated by changing the modes of the resonators and the coupling configuration [22]. Furthermore, the coupling of the cavity to the waveguide is also controllable through the change of the distance between the corresponding defects [23]. Besides the above flexible controllability, the possibility of miniaturizing the entanglement setup with solid-state photonic structures is highly desirable for scalable and on-chip photonic quantum information processing [24,25]. Obviously, the bulk setups for producing continuous-variable entangled light in Refs. [2,3,5–12] are not suitable for integration. In addition, compared to qubit or phononic entanglement proposed in Refs. [16,17], optical entanglement with continuous variables can be easily detected and manipulated experimentally, which can make it very promising for the implementation of continuous-variable quantum protocols [2–8].

The effects of various environments on continuous-variable entanglement of optical fields or harmonic oscillators have been extensively investigated, where one is mainly interested in the non-Markovian decoherence dynamics of the entanglement [26–30]. For the present system of two spatially separated nanocavities mediated by a waveguide, the waveguide actually serves as a structured reservoir, which is highly controllable in experiments. We therefore are interested in the entanglement generation and its manipulation through the controls of the waveguide as well as the nanocavity properties. We shall study the exact entanglement dynamics solved from the exact master equation of the nanocavity system coupled to the waveguides that we developed very recently [31–34]. The temporal behavior of the entanglement and purity of the two-mode cavity state is discussed in detail for initially separated squeezing inputs. We find that when the cavity frequencies are resonant with the band center of the waveguide, a stable and pure entangled state of the two distant cavities can be generated with a requirement of only a very weak coupling between the cavities and the waveguide. It shows that the strong cavity-waveguide coupling will lead to the entanglement sudden death and sudden birth [35]. When the frequencies of the cavities locate outside of the waveguide band, the cross frequency shift of the nanocavities induced by the waveguide can optimize the achievable entanglement. In addition, it is also shown that the entanglement can be easily manipulated by changing the cavity frequencies within the waveguide band.

The rest of the paper is organized as follows. In Sec. II, we formulate the entanglement dynamics of two spatially separated nanocavities coupled to a waveguide within the framework of the exact master equation and the correlation
photonic crystals. Then the photon loss into the photonic crystals is totally negligible [36].

matrix, in which the latter determines the continuous-variable entanglement of the two nanocavities through the measure of logarithmic negativity. In Sec. III, the properties of the entanglement and the corresponding purity are investigated in detail. Finally, a conclusion is given in Sec. IV.

II. HAMILTONIAN AND ENTANGLEMENT MEASURE

As schematically shown in Fig. 1, we consider two single-mode nanocavities coupled with a CROW in photonic crystals at different sites. Each cavity is formed by a point defect created in photonic crystals, with the cavity frequency tunable by changing the geometrical parameters of the defect. The waveguide in photonic crystals consists of a series of coupled point defects in which light propagates due to the coupling between the adjacent defects. Experimentally, the large-scale CROW consisting of more than one-hundred coupled resonators has been successfully fabricated [19]. To be specific, let the cavity 1 be coupled to the waveguide at the site \( n_1 \) and the cavity 2 coupled to the waveguide at the site \( n_2 \), as shown in Fig. 1. By treating the waveguide as a tight-binding model, the Hamiltonian of the whole system is given by [32]

\[
H = \sum_{i=1,2} \omega_i a_i^\dagger a_i + \sum_k \omega_k b_k^\dagger b_k + \sum_{i,k} V_{ik} (a_i b_k^\dagger + b_k a_i^\dagger),
\]

where

\[
\omega_k = \omega_0 - 2\xi_0 \cos k, \quad V_{ik} = \sqrt{\frac{2}{\pi}} \xi_i \sin(n_i k),
\]

with \( 0 \leq k \leq \pi \). The operators \( a_i \) and \( a_i^\dagger \) are the annihilation and creation operators of the cavity fields with frequencies \( \omega_i \). The annihilation and creation operators \( b_k \) and \( b_k^\dagger \) describe the Bloch modes \( \omega_k \) of the waveguide, with \( \omega_0 \) being the identical frequency of each resonator in the waveguide. The frequencies \( \omega_k \) and \( \omega_0 \) are tunable by adjusting the geometrical parameters of the corresponding defects. The strength \( \xi_0 \) characterizes the photon hopping between two adjacent resonators in the waveguide and is controllable by changing the corresponding distance between the two defects. The controllable coupling strength \( \xi_i \) is the coupling of the \( i \)th cavity to the waveguide at the sites \( n_i \). We should point out that the frequencies of the two cavities and the waveguide band considered in the above system should lie inside the photonic band gap of the photonic crystals. Then the photon loss into the photonic crystals is totally negligible [36].

We will investigate the generation of the entanglement between the two spatially separated cavity fields through the controllable waveguide, where the two nanocavities are initially prepared in a separated two-mode Gaussian state. For a two-mode Gaussian state, its quantum statistical property is fully determined by the \( 4 \times 4 \) correlation matrix \( \chi \) which is defined by

\[
\chi_{ij} = \frac{1}{2} \langle x_i x_j + x_j x_i \rangle,
\]

with the vector \( x = (X_1, Y_1, X_2, Y_2) \) and the quadrature operators \( X_j = (a_j + a_j^\dagger)/\sqrt{2} \) and \( Y_j = -i(a_j - a_j^\dagger)/\sqrt{2} \). With the correlation matrix \( \chi \), the continuous-variable entanglement between the two cavity fields can be well quantified with the measure of logarithmic negativity. By re-expressing the correlation matrix \( \chi \) in terms of three \( 2 \times 2 \) matrices \( \varrho_1, \varrho_2, \) and \( \varrho_3 \),

\[
\chi = \begin{pmatrix} \varrho_1 & \varrho_3 \\ \varrho_3^\dagger & \varrho_2 \end{pmatrix},
\]

the logarithmic negativity \( E_N(\chi) \) is defined as [37]

\[
E_N(\chi) = \max[0, -\ln(2\lambda)],
\]

where

\[
\lambda = \sqrt{\Delta - \Delta^2 - 4\det(\chi)/4},
\]

and \( \Delta = \det \varrho_1 + \det \varrho_2 - 2\det \varrho_3 \). Thus, the entanglement between the two cavity fields occurs for \( E_N(\chi) > 0 \), i.e., for \( \lambda > \frac{1}{2} \). From the definitions of Eqs. (3) and (4), we see that the matrix elements of \( \chi \) are a linear function of the second-order quantities

\[
n_{ij}(t) = \langle a_i^\dagger(t) a_j(t) \rangle, \quad s_{ij}(t) = \langle a_i^\dagger(t) a_j^\dagger(t) \rangle,
\]

plus its Hermitian conjugate. The entanglement dynamics between the two cavity fields at any time is then completely determined by these time-dependent second-order quantities.

On the other hand, the Hamiltonian of Eq. (1) describes effectively the two spatially separated nanocavities coupled to a common reservoir with a controllable spectral structure. In other words, the waveguide plays a role of a structured reservoir as a mediator between two nanocavities. We can use the exact master equation we developed recently for nanocavities coupled to waveguides in photonic crystals [32-34] to investigate the exact entanglement dynamics of the two nanocavities coupled to the waveguide in photonic crystals. By assuming that two nanocavities are initially uncorrelated to the structured reservoir (i.e., the waveguide) and the waveguide is initially in vacuum, the exact master equation for the density operator \( \rho(t) \) of the cavity system can be obtained through the Feynman-Vernon influence functional approach [38], in the framework of coherent-state path-integral representation [39].

The resulting exact master equation is given by

\[
\frac{d\rho(t)}{dt} = -i[H_{\text{eff}}(t), \rho(t)] + \sum_{ij} \gamma_{ij}(t) \times [2a_j \rho(t) a_i^\dagger - a_i^\dagger a_j \rho(t) - \rho(t) a_i^\dagger a_j],
\]
where $H_{\text{ef}}(t) = \omega_j(t) a_i^\dagger a_j$ is the effective Hamiltonian of the cavity fields with the time-dependent renormalized frequencies $\omega_j(t)$, which results from the back reaction of the waveguide to the cavity system. The time-dependent coefficients $\gamma_j(t)$ describe the dissipation of the cavity system due to the coupling to the waveguide. The coefficients $\omega_j(t)$ and $\gamma_j(t)$ are nonperturbatively determined by

$$\omega_j(t) = \frac{i}{2} [\mu(t) \mu^{-1}(t) - \text{H.c.}]_{jj}, \quad \gamma_j(t) = -\frac{1}{2} [\mu(t) \mu^{-1}(t) + \text{H.c.}]_{jj}, \quad (9a)$$

where $\mu(t)$ is the cavity photon propagating function, which obeys the integrodifferential equation of motion

$$\frac{d}{dt} \mu(t) = -i \tilde{\omega} \mu(t) - \int_0^t g(t - \tau) \mu(\tau) d\tau, \quad (10)$$

subjected to the initial condition $\mu(t_0) = 1$. Here, the frequency matrix $\tilde{\omega} = \text{diag}[\omega_1, \omega_2]$ is a diagonal frequency matrix of the two cavities. The integral kernel in Eq. (10) involves the time-correlation function $g(t - \tau)$, which nonperturbatively characterizes the non-Markovian memory structure between the cavity system and the waveguide.

By introducing the spectral densities of the waveguide, $J_{ij}(\omega) = 2\pi \sum_k V_{ik} V_{jk} b(\omega - \omega_k)$, the time-correlation function can be expressed as

$$g_{ij}(\tau - \tau') = \int \frac{d\omega}{2\pi} J_{ij}(\omega) e^{-i\omega(\tau - \tau')} \quad (11a)$$

Since the waveguide in photonic crystals has a narrow but continuous band structure, the spectral densities become $J_{ij}(\omega) = 2\pi \rho(\omega) V_i V_j$, where $\rho(\omega)$ is the density of states in the waveguide determined by Eq. (2). Explicitly, we have

$$\rho(\omega) = \frac{1}{\sqrt{4\xi_0^2 - (\omega - \omega_0)^2}} \quad (12a)$$

$$V_i(\omega) = \sqrt{\frac{2}{\pi}} \xi_i \sin \left\{ n_i \arcsin \left[ \frac{\sqrt{4\xi_0^2 - (\omega - \omega_0)^2}}{2\xi_0} \right] \right\} \quad (12b)$$

where $0 < 2\xi_0 \leq \omega \leq \omega_0 + 2\xi_0$ is the band of the waveguide.

In Fig. 2, the spectral densities $J_{11}(\omega)$ and $J_{12}(\omega)$ are plotted as a function of $\omega$ with different distances between the two cavities, for the case of the equal cavity frequencies $\omega_1 = \omega_2 = \omega_0$ and the equal cavity-waveguide couplings $\xi_1 = \xi_2 = \xi$. As we will show later in the next section, the entanglement characteristics depend heavily on the spectral structures. In fact, it is the back action through the off-diagonal matrix element $J_{12}(\omega)$ of the spectral density that induces an effective coupling between the two separated nanocavities which leads to the entanglement generation. The environment-assisted continuous-variable entanglement has been investigated in the literature [28,40,41]. However, the entanglement in these investigations is uncontrollable. Here, the waveguide-induced, i.e., a structured-reservoir-induced, entanglement between two spatially separated nanocavities in photonic crystals is fully controllable and is promising for quantum information processing in all-optical circuits.

Now, the exact temporal behavior of the second-order quantities of Eq. (7) can be completely determined by the exact master equation. Explicitly, from the exact master equation (8), it is not too difficult to find that the second-order quantities obey the following equations:

$$\frac{d}{dt} n(t) = [\hat{\mu}(t) \hat{\mu}^{-1}(t)] n(t) + n(t) [\hat{\mu}(t) \hat{\mu}^{-1}(t)]^T, \quad (13a)$$

$$\frac{d}{dt} s(t) = [\hat{\mu}(t) \hat{\mu}^{-1}(t)] s(t) + s(t) [\hat{\mu}(t) \hat{\mu}^{-1}(t)]^T. \quad (13b)$$

The exact solutions of the above equations are found to be

$$n(t) = \mu(t) n(0) \mu^{-1}(t), \quad s(t) = \mu(t) s(0) \mu^T(t). \quad (14)$$

where $n(0)$ and $s(0)$ are the initial second-order quantities. Thus, once the photon propagating function $\mu(t)$ is solved from the integrodifferential equations of motion (10), the above exact solution allows us to investigate the entanglement generation of the two separated nanocavities. Meanwhile, the temporal evolution of the entanglement measure subjected to the non-Markovian dissipation and fluctuation from the structured reservoir can be fully taken into account.
III. RESULTS AND DISCUSSION

Now we are ready to investigate the entanglement generation between the two cavity fields and its temporal evolution for a given initially separated state. We assume that the two cavity fields are initially prepared in a single-mode squeezed state, respectively, i.e.,

$$ |\psi_0(0)\rangle = \exp\left(-\frac{r_i}{2}a_i^2 + \frac{r_j}{2}d_j^2\right)|0_i\rangle, \quad i = 1, 2, \quad (15) $$

where the squeezing parameter $r_i$ is controllable as an input. The preparation of the initial squeezed states can be well accomplished by injecting into the nanocavities squeezed radiation fields produced via degenerate parametric oscillation [42]. For the initial states of Eq. (15), the initial average photon numbers and two-photon correlations are given by

$$ n_{ij}(0) = \sinh^2(r_i)\delta_{ij}, \quad s_{ij}(0) = \sinh(r_i)\cosh(r_j)\delta_{ij}. \quad (16) $$

With the help of the above initial conditions, we can analyze the temporal behavior of the correlation matrix $\chi$ in Eq. (3), and then the logarithmic negativity $E_N$ given by Eq. (5).

In the following, we will discuss the entanglement in three cases: (i) the cavity frequency $\omega_c$ is resonant with the band center $\omega_0 (\omega_c = \omega_0)$, (ii) the cavity frequency $\omega_c$ stays on the outside of the waveguide band ($|\omega_c - \omega_0| \geq 2\xi_0$), and (iii) the cavity frequency $\omega_c$ lies within the waveguide band ($|\omega_c - \omega_0| < 2\xi_0$).

A. $\omega_c = \omega_0$

At first, we consider the situation where the frequency $\omega_c$ of the two cavities is resonant with the band center $\omega_0$ of the waveguide. In this case, the steady-state average values in Eq. (14) in the weak cavity-waveguide coupling can be analytically obtained from Eq. (13) as

$$ \langle a_1^\dagger a_1 \rangle = \gamma_{22} \sinh^2r \gamma_{11}, \quad \langle a_2^\dagger a_2 \rangle = \gamma_{22} \sinh^2r \gamma_{11}, \quad (17a) $$

$$ \langle a_1^\dagger a_2 \rangle = \frac{\gamma_{22} \sinh 2r}{2(\gamma_{11} + \gamma_{22})}, \quad \langle a_2^\dagger a_1 \rangle = \frac{\gamma_{22} \sinh 2r}{2(\gamma_{11} + \gamma_{22})}, \quad (17b) $$

$$ \langle a_1a_2 \rangle = -\frac{\gamma_{22} \sinh 2r}{2(\gamma_{11} + \gamma_{22})}, \quad \langle a_2a_1 \rangle = -\frac{\gamma_{11} \sinh 2r}{2(\gamma_{11} + \gamma_{22})}, \quad (17c) $$

where the subscript $s$ denotes the steady state, $\gamma_{ij} = \gamma_{ij} e^{-2i\omega_0 t}$, and the initial squeezing $r_1 = r_2 = r$ is also assumed. From the above results, the expression of the stationary logarithmic negativity $E_N^s$ can be obtained and the result is rather cumbersome so that we do not give it here explicitly. Nevertheless, it can be seen from Eq. (17) that the waveguide-induced collective effect between the two cavities [characterized by the cross damping factor $\gamma_{12}$ in Eq. (17c)] is crucial for generating the entanglement. Since the cross damping rate $\gamma_{12} = \sqrt{\gamma_{11}\gamma_{22}}$ with $\gamma_{11} = (2\xi_0^2)/(\kappa_1^2)$ in the weak-coupling region, the stationary entanglement cannot be generated if either (or both) of the sites $n_1$ and $n_2$ of the two cavities is even. Only when $n_1$ and $n_2$ are both odd numbers can the cross damping rate not vanish. For the equal couplings $\xi_1 = \xi_2 = \xi$, we have $\gamma_{11} = \gamma_{22} = 2\xi^2/(\kappa_0^2)$. Then, the steady-state logarithmic negativity $E_N^s$ is reduced to

$$ E_N^s = r. \quad (18) $$

This shows that the entanglement degree between the two spatially separated nanocavities can be controlled by the input initial squeezing parameter in the resonant case. Furthermore, from the definition of the purity $P = Tr(\rho^2) = 1 - 2EN$, it is also not difficult to find that steady purity $P_s = 1$ for the steady state of Eq. (17). This can be clearly understood by performing a balanced beam-splitter transformation $d_1 = \sin \theta a_1 + \cos \theta a_2$ and $d_2 = \cos \theta a_1 - \sin \theta a_2$ on the Hamiltonian in Eq. (1), where $\sin \theta = V_{1k}/\sqrt{V_{1d}^2 + V_{1k}^2}$, $\cos \theta = V_{1k}/\sqrt{V_{1d}^2 + V_{1k}^2}$. Then the whole Hamiltonian of Eq. (1) is reduced to $H = \sum_j \omega_j d_j^\dagger d_j + \sqrt{V_{1d}^2 + V_{1k}^2} \left(d_1^\dagger b_1 + d_2^\dagger b_2\right)$. It shows that the waveguide only couples to the collective mode $d_1$, while the other collective mode $d_2$ is totally decoupled from the waveguide. Therefore, the collective mode $d_2$ of the two cavities forms a decoherence-free subspace, which is not subject to any dissipation due to the presence of the structured reservoir. This results in the steady and pure entangled state between the two distant nanocavities, as shown in Fig. 3.

In Fig. 3, the exact temporal evolution of the logarithmic negativity $E_N$ and the purity $P$ of the two-mode cavity states for different values of the site $n_2$ of the second cavity and the cavity-waveguide coupling $\eta$. The cavity frequency is resonant with the band center of the waveguide, i.e., $\omega_c = \omega_0$, the hopping rate $\xi_0 = 0.05\omega_0$, the squeezing parameter $r = 1.0$, and the site of the first cavity $n_1 = 1$.
In Fig. 4, the entanglement and purity are plotted for the enhanced cavity-waveguide couplings $\eta = 0.2$ and $0.4$, with the different locations of the second cavity $n_2 = 5, 15$, and 25. It shows that as the cavity-waveguide coupling increases, the entanglement decreases by accompanying with some oscillations in the time evolution. The oscillation comes from the non-Markovian effect due to the back action between the cavity and the waveguide when the cavity-waveguide coupling increases. Besides, one can also see that the entanglement sudden death and sudden birth occurs in the short-time regime with a relatively large cavity-waveguide coupling ($\eta = 0.4$) and also a relatively long distance between the two cavities; see Figs. 4(b) and 4(c). In addition, as we see, the resulting entangled states usually become a mixed state as the cavity-waveguide coupling increases. By comparing Figs. 4(a)–4(c), one can also find that the entanglement decreases as the distance between the cavities as well as the cavity-waveguide coupling increases. Therefore, to maintain the high entanglement for the two distant cavities, a small cavity-waveguide coupling is more favorable in the resonant case.

Furthermore, as shown in Fig. 5, the entanglement appears in the case of the first cavity siting at $n_1 = 1$ and the second cavity siting at an even number of $n_2$. The exact numerical result shows that the entanglement between the two cavities can be generated under the relatively large coupling in a short-time scale, but as the time progresses, the entanglement soon decays to zero. This is consistent with the analytical solution given by Eq. (17), where it is pointed out that the steady-state entanglement in this case cannot exist if any one (or both) of the sites $n_1$ and $n_2$ of the two cavities is even.

To further investigate the controllability of the entanglement generation of the two spatially separated nanocavities, we consider next the entanglement behavior for the cavity frequency $\omega_c$ outside the waveguide band, i.e., $|\omega_c - \omega_0| \geq 2\xi_0$. Let the first cavity site at $n_1 = 1$ and the second cavity locate at different sites. The temporal behaviors of the entanglement and the purity with different values of the cavity-waveguide coupling are plotted in Fig. 6. It shows that the entanglement exhibits a very regular oscillation with an even stronger entanglement degree, $E_N^{\max} > 1$, for the input squeezing $r = 1$. For example, the maximal entanglement $E_N^{\max} \approx 1.94$ for the coupling $\eta = 0.2$ in the present case; see Fig. 6(a). When the coupling is increased, the entanglement degree oscillates faster but the maximal entanglement is degraded only slightly.

The above phenomenon can be understood as follows: in the weak-coupling region, the damping rates $\gamma_{ij} = 0$. 

$E_N^{\max}$ for the different values of the cavity-waveguide coupling $\eta$. The cavity frequency is resonant with the band center of the waveguide ($\omega_c = \omega_0$), and the other parameters are the same as in Fig. 3.

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since the spectral densities $J_{ij}(\omega_c) = 0$ when the frequency $\omega_c$ of the cavities lies outside the band of the waveguide. However, the cavity frequency shift $\delta \omega = P \int d\omega J^{(\omega)}_{(-)}$ does not vanish in this case. As a result, the reduced density matrix is purely determined by the effective Hamiltonian $H_{\text{eff}} = \sum (\omega_c + \delta \omega_{12}) a_i a_i + \delta \omega_{012}(a_i^† a_j + a_j a_i)$. In other words, an effective beam-splitter-type coupling (determined by the cross frequency shift $\delta \omega_{12}$) between the two cavities is induced by the waveguide, which results in the entanglement for the separated squeezing inputs. With the weak-induced by the waveguide, which results in the entanglement.

However, the cavity frequency shift $\delta \omega_{12}$ is purely determined by the cross frequency shift $\delta \omega_{12}$ and the weak-coupling solution given in the Appendix is obviously with the changes of the distance between the two cavities. Moreover, the average values in Eq. (14) reduce to $\langle a_i^† a_i \rangle = \sinh^2 r \gamma$, $\langle a_i^† a_i^† \rangle = \sinh r \cos r \cos(2\delta \omega_{12}) e^{-2i\delta \omega_{12} t}$, $\langle a_i a_j \rangle = -i \sinh r \cos r \sin(2\delta \omega_{12}) e^{-2i\delta \omega_{12} t}$, and $\langle a_i a_i^† \rangle = 0$. Thus, unlike the resonant case, the entanglement in this case is purely determined by the cross frequency shift $\delta \omega_{12}$. At the times when $\cos(2\delta \omega_{12} t) = 0$, we have the nonzero average values $\langle a_i^† a_i \rangle = \sinh^2 r$ and $\langle a_i a_j \rangle = \sinh r \cos r$, which correspond to a pure two-mode squeezed vacuum state with the squeezing parameter $r$ [43]. Accordingly, the entanglement degree at these times becomes optimal with the maximal logarithmic negativity

$$E_N^\text{max} = 2r.$$  

Figure 6(a) shows that the maximal entanglement approaches the limit $2r$ when the coupling becomes sufficiently weak and the weak-coupling solution given in the Appendix is almost exact. With the cavity-waveguide coupling increase, the maximal entanglement is decreased a little bit, as shown in Fig. 6(a).

Figure 6(b) depicts the corresponding purity of the entangled state. Since the damping rate vanishes in the weak-coupling limit when the frequency of the cavities lies outside the waveguide band, the entangled state should be a pure state. Figure 6(b) shows that for $\eta = 0.2$, the exact numerical solution gives the purity $P \approx 0.97$, which is consistent with the weak-coupling solution. When the cavity-waveguide coupling is increased, the purity of the waveguide-generated entanglement is decreased. Figures 6(c) and 6(d) show further the dependence of the entanglement degree and the purity on the distance between the two nanocavities. Similar behavior of the oscillating entanglement is obtained as the cavity distance changes. This is due to the cross frequency shift $\delta \omega_{12}$ again, which is determined by the cross spectral density $J_{12}(\omega)$ that decreases slightly as $n_2$ increases for the fixed $n_1 = 1$. Besides, it is also found that the maximal entanglement and purity in the long-time region do not change obviously with the changes of the distance between the two nanocavities.

C. $|\omega_c - \omega_0| < 2\Delta_0$

Finally, we shall consider the case where the cavity frequency $\omega_c$ is not resonant with the band center, but still lies inside the waveguide band. In Fig. 7(a), the entanglement and the purity are plotted for the cavity sites $n_1 = 1$ and $n_2 = 5$, with the cavity frequency $\omega_c = 1.03\omega_0$ at which the spectral density $J_{12}(\omega) = 0$ for $\omega = \omega_c$ (see the red dashed line in Fig. 2). In this case, the phenomenon of entanglement sudden death and sudden birth occurs in the weak-coupling region. The existence of the entanglement sudden death and sudden birth originates from the fact that the damping $\gamma_{12} \approx 0$ and also $\gamma_{12} \approx 0$ for $\omega_c = 1.03\omega_0$, so that the entanglement is purely governed by the cross frequency shift $\delta \omega_{12}$, which leads to the entanglement oscillation. In the long-time limit, the entanglement is destroyed through the damping channel $\gamma_{11} \neq 0$. When the cavity-waveguide coupling increases, the entanglement only exists in a very short time and then quickly decays to zero significantly, which is quite different from the resonant case shown in Fig. 4. The inset in Fig. 7(a) is the intracavity average photon numbers $\langle a_i a_i \rangle$, which approach to zero in the long-time limit. It tells that the cavity evolves asymptotically into a vacuum state. This is why the entanglement in the steady state vanishes. The purity shown in Fig. 7(b) approaches to one in the steady-state limit because the steady state is just a trivial vacuum state.

FIG. 7. (Color online) The temporal behavior of the logarithmic negativity $E_n$ and the corresponding purity $P$ of the two-mode cavity states for the cavity-waveguide coupling $\eta = 0.1$ (red dashed line), $0.2$ (green dotted line), and $0.4$ (blue solid line). The cavity frequency $\omega_c = 1.03\omega_0$ within the waveguide band and the site of the second cavity $n_2 = 5$. The inset in (a) depicts the average photon numbers $\langle a_i a_i \rangle$ of the two cavities for the coupling $\eta = 0.4$. The other parameters are the same as in Fig. 3.

FIG. 8. (Color online) The temporal behavior of the logarithmic negativity $E_n$ for the cavity frequency $\omega_c = 1.06\omega_0$ outside the band of the waveguide and the second cavity locating at the site $n_2 = 5$. The other parameters are the same as in Fig. 3.
In Fig. 8, we plot the entanglement for the cavity frequency \( \omega_c = 1.066\omega_0 \), while the spectral density \( I_i(\omega) \neq 0 \) (i = 1, 2) at \( \omega = \omega_c \). Then, both the decay channels \( \gamma_{11} \) and \( \gamma_{22} \) become active, and the entanglement is a combination effect of the nonzero cross damping \( \gamma_{12} \) and the nonzero cross frequency shift \( \delta \omega_{12} \). Compared to that in Fig. 7(a), the entanglement here is enhanced significantly in the long-time regime. The phenomenon of the entanglement sudden death and sudden birth disappears in this case. As a result, we see that the entanglement generation can be easily controlled by changing the cavity frequency within the band of the waveguide.

IV. CONCLUSION

In conclusion, the generation of continuous-variable entanglement between two spatially separated nanocavities mediated by a CROW in photonic crystals is investigated. By solving the exact master equation for the cavity system coupled to the waveguide as a structured reservoir, the entanglement and the purity of the two-mode cavity field with the band center but are still within the waveguide band, the optimal entanglement can be achieved by the cross frequency shift between the two cavities, which is induced by the waveguide. When the cavity frequencies lie outside the waveguide band, the optimal entanglement can be achieved by the cross frequency shift limit is then given by

\[
\mu_{\text{bm},11} = \frac{1}{\gamma_{11} + \gamma_{22}} \left[ \gamma_{22} + \gamma_{11} e^{-i(\gamma_{11} + \gamma_{22})t} \right] e^{-i\omega_c t},
\]

\[
\mu_{\text{bm},22} = \frac{1}{\gamma_{11} + \gamma_{22}} \left[ \gamma_{11} + \gamma_{22} e^{-i(\gamma_{11} + \gamma_{22})t} \right] e^{-i\omega_c t},
\]

\[
\mu_{\text{bm},12} = \mu_{\text{bm},21} = -\frac{\gamma_{12}}{\gamma_{11} + \gamma_{22}} \left[ 1 - e^{-i(\gamma_{11} + \gamma_{22})t} \right] e^{-i\omega_c t},
\]

where the damping rates \( \gamma_i = \frac{2\xi_i}{\hbar} \sin^2 \left( \frac{\eta_i\pi}{2} \right) \) and the collective damping \( \gamma_{12} = \sqrt{\gamma_{11}\gamma_{22}} \).

ACKNOWLEDGMENTS

This work is supported by the National Science Council (NSC) of ROC under Contract No. NSC-99-2112-M-006-008-MY3, the National Center for Theoretical Science of NSC, the National Natural Science Foundation of China (Grants No. 10804035, No. 60878004, and No. 11074087), SRFDP (Grants No. 200805111014 and No. 200805110002), SCRF of CCNU (Grant No. CCNU 09A01023), and the Natural Science Foundation of Hubei Province (Grant No. 2010CDAD075).

APPENDIX: A WEAK-COUPLING ANALYTICAL SOLUTION

It should be pointed out that the master equation of Eq. (8) is exact, far beyond the Born-Markovian approximation, and valid for arbitrary cavity-waveguide coupling. The back action between the cavities and the waveguide is embedded into the time-dependent coefficients \( \omega_i(t) \) and \( \gamma_i(t) \) of Eq. (9), which are in turn determined completely by the propagating function \( \mu(t) \) of Eq. (10). Generally, it is not easy to obtain the analytical propagating function \( \mu(t) \). However, for a weak coupling, the analytical solution of the photon propagating function can be found as

\[
\mu_{\text{bm}}(t) = e^{-(i\gamma + i\tilde{\omega})t},
\]

where the damping rate becomes time independent: \( \gamma = \int \frac{d\omega}{2\pi} J(\omega) \), and the waveguide-induced renormalized frequency is given by \( \tilde{\omega} = \omega_c I + \delta \omega_c \), with \( \delta \omega_P = P \int \frac{d\omega}{2\pi} J(\omega) \).

Here \( P \) denotes the principal value of the integral. When the cavity frequency \( \omega_c \) is resonant with the band center \( \omega_0 \) of the waveguide \( (\omega_c = \omega_0) \), the frequency shift \( \delta \omega = 0 \), and the explicit propagating function \( \mu_{\text{bm}}(t) \) in the weak-coupling limit is then given by

\[
\mu_{\text{bm},11} = \frac{1}{\gamma_{11} + \gamma_{22}} \left[ \gamma_{22} + \gamma_{11} e^{-i(\gamma_{11} + \gamma_{22})t} \right] e^{-i\omega_c t},
\]

\[
\mu_{\text{bm},22} = \frac{1}{\gamma_{11} + \gamma_{22}} \left[ \gamma_{11} + \gamma_{22} e^{-i(\gamma_{11} + \gamma_{22})t} \right] e^{-i\omega_c t},
\]

\[
\mu_{\text{bm},12} = \mu_{\text{bm},21} = -\frac{\gamma_{12}}{\gamma_{11} + \gamma_{22}} \left[ 1 - e^{-i(\gamma_{11} + \gamma_{22})t} \right] e^{-i\omega_c t},
\]

where the damping rates \( \gamma_i = \frac{2\xi_i}{\hbar} \sin^2 \left( \frac{\eta_i\pi}{2} \right) \) and the collective damping \( \gamma_{12} = \sqrt{\gamma_{11}\gamma_{22}} \).
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