MIXED LEBESGUE SPACE NORM STRICHARTZ TYPE
ESTIMATION FOR SOLUTION OF INHOMOGENEOUS
PARABOLIC EQUATION,
with constants evaluation.

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Abstract.

We give an optimal in mixed (anisotropic) Strichartz type Lebesgue space-time norm estimates for the solution of linear parabolic inhomogeneous initial problem, with are exact or exact up to multiplicative constant coefficient evaluation.

Keywords and phrases: Multivariate Parabolic PDE equations, scaling method, dilation operator, density of Gaussian distribution, transstable density, Lebesgue-Riesz spaces, initial value problem, mixed (anisotropic) norms and spaces, Strichartz estimates, permutation inequality, Riesz potential, Riemann’s fractional integral, factorable function, Cesaro-Hardy operator estimates, Young inequality, Marcinkiewicz integral triangle inequality, Beckner’s constant, Grand Lebesgue Spaces.

2000 AMS Subject Classification: Primary 37B30, 33K55, 35Q30, 35K45; Secondary 34A34, 65M20, 42B25.

1 Notations. Statement of problem.

Statement of problem.

We consider in this article the classical initial value problem for the function $u = u(x, t)$, $t \in T = (0, I)$, $0 < I \leq \infty$, of a multivariate linear Parabolic PDE equations in whole Euclidean space $x \in X = \mathbb{R}^d$ with density of external force $F(x, t)$ and initial condition $f(x)$ of a form

$$
\partial_t u = 0.5 \Delta u + F(t, x), \; u(x, 0+) = f(x).
$$

1.1

We will update after the statement of this problem: for given $p \in (1, \infty)$
\[
\lim_{t \to 0^+} |u(\cdot, t) - f(\cdot)|_{L_p(R^d)} = 0.
\]

Let us introduce the following notations.

\[
x = \bar{x} = \{x_1, x_2, \ldots, x_d\} \in R^d \Rightarrow |x| = \sqrt{(x, x)} = \sqrt{\sum_{i=1}^{d} x_i^2},
\]

\[
w_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0.
\]

This function is the fundamental solution of classical heat equation ("heat potential").

Further, we denote as usually the convolutions

\[
F * G(x) := \int_{\mathcal{X}} F(x - y) G(y) \, dy, \quad x, y \in R^d;
\]

("space" convolution); and at the same notation will be used for a "time" convolution:

\[
f * g(t) := \int_{0}^{t} f(t - s) \, g(s) \, ds, \quad s, t \geq 0.
\]

Authors hope that this definitions not lead to the confusion.

The unique "regular" solution of the problem (1.1) may be written under known natural conditions imposed on the data \(f, F\) (measurability, boundedness, belonging to and or other Banach space etc.) in explicit view as follows; \(u(x, t) = u_0(x, t) + u_1(x, t)\), where

\[
u_0(x, t) = w_t * [f](x) = \int_{\mathcal{X}} w_t(x - y) \, f(y) \, dy,
\]

(1.2)

\[
u_1(x, t) = [w_t * F](x, t) = \int_{0}^{t} ds \int_{\mathcal{X}} w_{t-s}(x - y) \, F(y, s) \, dy.
\]

(1.3)

Further, denote as ordinary

\[
|f|_r = |f|_{r,X} = \left[ \int_{\mathcal{X}} |f(x)|^r \, dx \right]^{1/r}, \quad r = \text{const} \geq 1;
\]

\[
|g|_q = |g|_{q,T} = \left[ \int_{T} |g(t)|^q \, dt \right]^{1/q}, \quad q = \text{const} \geq 1;
\]

The so-called mixed \(L_{p,q} = L_{p,X;q,T}\) norm of a function of "two" variables \(F = F(x, t)\) is defined by equality:

\[
|F|_{p,X;q,T} := \left\{ \int_{0}^{\infty} \left[ \int_{\mathcal{X}} |F(x, t)|^p \, dx \right]^{q/p} \, dt \right\}^{1/q}, \quad 1 \leq p, q < \infty,
\]

(1.4)

and analogously
\begin{equation}
|F|_{q,T,p,X} := \left\{ \int_X \left[ \int_0^T |F(x,t)|^q \ dt \right]^{p/q} \ dt \right\}^{1/p}, \ 1 \leq p, q < \infty, \quad (1.4a)
\end{equation}

Not to be confused with the standard Lorentz norm!

The space consisting on all the (common measurable) function \( F(x,t) \) with finite mixed norm \( |F(\cdot,\cdot)|_{p,q} \) is said to be anisotropic, or equally Bochner space and denoted similar \( L_{p,q} = L_{p,X,q,T} \).

These spaces appear in an article of Benedek A. and Panzone R. [49] and was completely investigated in the classical monograph of Besov O.V., Ilin V.P., Nikol’skii S.M. [50].

It is known, see [1], [50], p. 7, 25, that if \( p_1 \leq p_2 \)

\[ |F|_{p_1,p_2} \leq |F|_{p_2,p_1}, \]

"permutation inequality".

Note that in general case \( |F|_{p,q} \neq |F|_{q,p} \), but \( |F|_{p,p} = |F|_p \).

Observe also that if \( F(x,t) = G_1(x) \cdot G_2(t) \), (condition of factorization), then \( |F|_{p,q} = |G_1|^p \cdot |G_2|^q \), (formula of factorization).

These spaces arise in the Theory of Approximation, Functional Analysis, theory of Partial Differential Equations, theory of Random Processes etc. Consider for instance the linear integral operator \( U \) acting on the functions defined on the measurable space \((Y,B,\nu)\) onto another measurable space \((X,A,\mu)\) with common measurable kernel \( K(x,y) :\)

\[ U[f](x) = \int_Y K(x,y) \ f(y) \ \mu(dy). \]

then we conclude by means of Hölder’s inequality

\[ |U[f]|_{p,X} \leq |K|_{p,q'} \cdot |f|_{q,Y}; \quad q' \overset{def}{=} q/(q - 1), \quad 1 < q \leq \infty, \]

or equally

\[ |U|_{q\to p} \overset{def}{=} \sup_{f \not= 0, \ f \in L_q(Y)} \left[ \frac{|U[f]|_{p,X}}{|f|_{q,Y}} \right] \leq |K|_{p,q'}. \]

Our purpose is to estimate both the functions \( u_0 \) and \( u_1 \) in the mixed norms \( |u_0|_{r,T,q,X}, \ |u_1|_{p,X,q,T} \) via correspondingly also mixed norms of the data \( |f|_{r,X} \) and \( |F|_{r,X,k,T} \).

We simplify known proofs and estimates, write down the asymptotically optimal estimates of the constants and prove its exactness.

Previous works: [48], [60] - [62], [63], [64] - [65], [45] etc. The applications of these estimates in the theory of non-linear evolutionary PDE, for example, equations of Navier - Stokes see in [3] - [46].
The paper is organized as follows. In the second section we obtain the mixed norm estimates for the first component of solution \( u_0 = u_0(x,t) \), in the third we obtain ones for the second component \( u_1 = u_1(x,t) \).

In the fourth section we will prove the non-refinements of our conditions, where we prove simultaneously the exactness of conditions in the Young inequality convolution. The next section is devoted to common, or equally united estimates. We consider in the sixth section some generalizations of obtained estimates into the so-called Grand Lebesgue Spaces.

The 7th section contains the weight estimates for the classical inhomogeneous parabolic initial value problem. The penultimate section described the mixed norm estimates for solution of non-local evolution initial value problem with fractional Laplace’s operator.

By tradition, the last section contains some concluding remarks.

2 Influence of initial condition.

We intend to obtain the estimation of a form

\[
|u_0|_{p_0,T;0_0,x} = |w_t * f|_{p_0,T;0_0,x} \leq K_0(d; p_0, q_0, r_0) |f|_{p_0}. \tag{2.1}
\]

Notations and restrictions:

\[ p_0, q_0, r_0 \in (1, \infty), \quad d \ r_0 > 2; \quad z = |x - y| > 0; \quad q_0 \in (r_0d/(r_0d - 2), \infty); \]

\[
B(z) = B_r(z) := |w_t(z)|_{r,T} = (2\pi)^{-d/2} \sqrt{\int_{R_+} t^{-dr/2} e^{-r z^2/2t} dt} = \pi^{d/2} z^2/r-d \ r^{1/r-d/2} \ \Gamma^{1/r}(dr/2 - 1) := D(d,r) |x - y|^{2r-d},
\]

where

\[
D(d,r) = \pi^{d/2} r^{1/r-d/2} \ \Gamma^{1/r}(dr/2 - 1).
\]

Note that as \( r \to d/2 + 0 \)

\[
D(d,r) \sim (2\pi)^{d/2} (rd - 2)^{-d/2}.
\]

Further, put

\[
V(p,r) := \frac{(d - 2/r)^{-1}}{[(p - 1)(dr/2 - p)]^{\kappa}};
\]

\[
\kappa := 1 - \frac{2}{dr} \in (0,1); \quad 1 + \frac{1}{q} = \frac{1}{p} + \kappa \iff \frac{1}{q} = \frac{1}{p} - \frac{2}{rd}; \tag{2.2}
\]

Let us introduce the following family of domains \( G_0(p) = G_0(d;p), \ p > 1 \) on the plane \( (q,r) \):
Theorem 2.0a. Let \( f \in L_{p_0}(\mathbb{R}^d) \) for some \( p_0 > 1 \). There holds under described restrictions:

\[
(q_0, r_0) \in G_0(d; p_0)
\]

the following estimate:

\[
|u_0|_{r_0, T; q_0, X} \leq C_0(d; p_0) D(d, r_0) V(p_0, r_0) \left| f \right|_{p_0}.
\] (2.3)

Remark 2.1. We can adopt as the capacity of the value \( C_0(d; p_0) \) its minimal value, indeed:

\[
C_0(d; p) := \sup_{(q, r) \in G_0(p)} \sup \left\{ \left| u_0 \right|_{r, T; q, X} : \left| f \right|_{p} |f|_{p} \right\} < \infty, \quad 1 < p < \infty.
\]

Proof. We can and will suppose without loss of generality that \( f(x) \geq 0 \). We deduce using Marcinkiewicz integral triangle inequality writing locally \((p, q, r)\) instead \((p_0, q_0, r_0)\) :

\[
|u_0|_{r, T} \leq \int_X |w_t(x - y)|_{r, T} f(y) \, dy = D(d, r) \int_X \frac{f(y) \, dy}{|x - y|^{d-2/r}}.
\]

Notice that the integral in the right - hand side is the well - known fractional Riesz potential:

\[
\int_X \frac{f(y) \, dy}{|x - y|^{d-2/r}} = \int_{\mathbb{R}^d} \frac{f(y) \, dy}{|x - y|^{d-2/r}} = I_{2/r}[f](x) = I_{2/r}^{(d)}[f](x).
\]

The using for us exact up to multiplicative constants Lebesgue - Riesz estimates for this operator are obtained in the article [70]:

\[
|I_{2/r}^{(d)}[f](\cdot)|_{q, X} \leq C_0(p, r) V(p, r) |f|_{p}, \quad p, q, r \in (1, \infty), \quad \frac{1}{q} = \frac{1}{p} - \frac{2}{dr}.
\] (2.4)

This completes the proof of theorem 2.0.

Remark 2.2. The condition (2.2) coincides with ones in proposition 1.2 in the articles [30], [31], [45], where is consider also the case of fractional Laplace operator \((-\Delta)^{\alpha}\) instead the classical operator \((-\Delta)\), but without constants estimation.

The method used in [45] is different on our way; for instance, we do not use the theory of interpolation of operators.

Theorem 2.0b. Let \( f \in L_{p_0}(\mathbb{R}^d) \) for some \( p_0 > 1 \). There holds the following estimate:

\[
|u_0|_{q_0, X} \leq K_W(m) K_B(m, p_0) \left| f \right|_{p_0, X} t^{-d/2(1/p_0 - 1/q_0)},
\] (2.5)
where

\[ q_0 \geq p_0, \quad 1 + 1/q_0 = 1/m + 1/p_0, \quad q, p_0, m > 1. \]  

(2.6)

Moreover, the relation (2.6) is necessary for the inequality of a form (2.5).

Besides, the asymptotical equality in (2.5) as \( t \to \infty \) is attained iff \( f_0 \) is density of Gaussian centered non-trivial distribution:

\[ f_0(x) = (2\pi)^{-d/2} \sigma^{-d} \exp \left( -|x|^2/(2\sigma^2) \right), \sigma = \text{const} > 0. \]

**Proof.** We have: \( u_0(x,t) = w_t \ast [f](x) \). We apply the Young’s - Beckner’s inequality:

\[ |u_0|_{q_0,X} \leq K_B(d; m, p_0) |w_t|_{m,X} |f|_{p_0,X} = K_B(d; m, p_0) K_w(m) |f|_{p_0} t^{-d/2(1/p_0 - 1/q_0)}. \]

The necessity of equality (2.6) may be easily proved by means of scaling method. The last proposition about asymptotical equality as \( t \to \infty \) is in fact proved by W. Beckner in [2].

Note that the inequality (2.5) is well-known, see for example, [21]; we write only the constants estimates.

### 3 Influence of right-hand side.

Some new notations and restrictions (conditions). Let four numbers \( (p, q, r, k) \in (1, \infty) \) be given. Denote

\[ Q = Q(p, r) : \frac{1}{p} + 1 = \frac{1}{Q} + \frac{1}{r}; \quad \theta = \theta(p, r) = \frac{d}{2} \left( 1 - \frac{1}{Q} \right), \]

\[ k_1 = 1, \quad k_+ = \frac{d}{d - \theta}, \quad q_- = \frac{q}{\lambda} = \frac{2}{1/r - 1/p}, \quad q_+ = \infty, \]

\[ \beta = \frac{\theta}{d} = \frac{1}{2} \left( \frac{1}{r} - \frac{1}{p} \right). \]  

(3.0)

Further, we denote the \( L_Q \) norms of heat potential:

\[ K_w = K_w(d, Q) := |w_1|_Q = |w_1|_{Q,X} = (2\pi)^{d(1-Q)/(2Q)} Q^{-d/2Q}; \quad Q \geq 1. \]

As a consequence:

\[ |w_t|_Q = (2\pi)^{d(1-Q)/(2Q)} t^{0.5d(1/Q-1)} Q^{-d/2Q} = K_w(d, Q) t^{0.5d(1/Q-1)}, \quad t > 0. \]

We introduce also the following family of domains \( G_1(k, r) = G_1(d; k, r), \quad k, r > 1 \) on the plane \( (p, q) : \)
\( G_1(k, r) = G_1(d; k, r) = \{(p, q) : p > r, q > 1, 1/q + d/2p = 1/k + d/(2r) - 1\}. \quad (3.1) \)

**Theorem 3.1a.** Let again the four numbers \((p_1, q_1, r_1, k_1) \in (1, \infty)\) be a given. Suppose \(F(\cdot, \cdot) \in L(r_1, X; k_1, T)\) for some \(k_1, r_1 > 1\). If \((q_1, r_1) \in G_1(d; k_1, r_1)\), and \(k_1 \in (1, k_+)\), then

\[
|u_1(\cdot, \cdot)|_{p_1, X; q_1, T} \leq K_1(p_1, q_1, r_1, k_1) \|F(\cdot, \cdot)\|_{r_1, X; k_1, T}, \quad (3.2)
\]

where

\[
K_1(p_1, q_1, r_1, k_1) = \frac{C_1(d; r_1, k_1)}{|k_1 - k_-|^p}. \quad (3.3)
\]

Conversely, the equality \((q_1, r_1) \in G_1(d; k_1, r_1)\) is necessary for the estimation of the form (3.2).

Moreover, the estimation (3.2) is exact up to multiplicative constant.

**Remark 3.1.** We can adopt as before as the capacity of the value \(C_1(d; r, k)\) its minimal value, namely:

\[
C_1(d; r, k) := \sup_{(p, q) \in G_1(k, r)} \left[ \frac{|k - k_-|^\beta}{\|F(\cdot, \cdot)\|_{r, X; k, T}} \right] < \infty, \quad r, k > 1.
\]

**Remark 3.2.** Note that the norms in the proposition (3.2) follows in reverse order as in the assertion of theorem 2.0.

**Proof.** We get using again Marcinkiewicz (triangle) inequality writing again \((p, q, r, k)\) instead \((p_1, q_1, r_1, k_1)\):

\[
|u_1(\cdot, \cdot)|_{p, X} = \left| \int_0^t ds \int_X w_{t-s}(x-y) F(y, s) dy \right|_{p, X} \leq \int_0^t ds \int_X |w_{t-s}(x-y)|_{p, X} \cdot |F(\cdot, s)|_{r, X} \, ds.
\]

We apply the Young’s inequality for the space convolution:

\[
|u_1(\cdot, \cdot)|_{p, X} \leq K_B(d; Q, r) \int_0^t |w_{t-s}(\cdot)|_{Q, X} \cdot |F(\cdot, s)|_{r, X} \, ds = K_B(d; Q, r) K_w(d, Q) \int_0^t (t - s)^{0.5d(1/Q - 1)} \|F(\cdot, s)\|_{p, X} \, ds, \quad (3.4)
\]

where \(K_B(d; p, q) < 1\) is the famous Beckner’s constant:

\[
A(p) := \left[ \frac{p^{1/p}}{p^{1/p}} \right]^{1/2}, \quad p > 1, \quad p' := p/(p - 1);
\]

\[
K_B = K_B(p, q) = K_B(d; p, q) := (A(p) A(q) A(r))^d, \quad 1 + 1/r = 1/p + 1/q.
\]
W. Beckner in [2] proved that
\[ |f \ast g|_r \leq K_B(d; p, q) |f|_p |g|_q, \]
where again
\[ 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \]  

(3.5)

See also [51].

The integral in the right-hand (3.4) is so-called Riemann’s fractional integral or equally Cesaro-Hardy average operator. The Lebesgue - Riesz \( L^q(T) \) norm estimates for this operator with exact up to multiplicative constants evaluating are obtained in [71], see also [70]. Indeed, if
\[ U_\theta[h](\cdot) = \int_0^t (t - s)^{-\theta} h(s) \, ds, \quad \theta = \text{const} \in (0, 1), \]
then
\[ |U_\theta[h](t)|_{q,T} \leq K \cdot |h|_{k,T}, \]
where
\[ 1 + \frac{1}{q} = \frac{1}{k} + \theta. \]

Here we must substitute
\[ \theta = \frac{d}{2} \left( 1 - \frac{1}{Q} \right) \in (0, 1). \]

It remains to use this estimations.
This completes the proof of theorem 3.1a.

**Remark 3.3.** The condition (3.1) coincides with ones in equality (1.8) in the article [45], where is consider also the case of fractional Laplace operator \((-\Delta)^\alpha\) instead the classical operator \((-\Delta)\), but *without constants estimation*.

The method used in [45] is different on our way.

We investigate now the inverse order of the powers \((p, q)\).

**Theorem 3.1b.** Let the constants \((h, k, q, r)\) be such that

\[ 1 < h, k, q, r < \infty, \quad \frac{1}{m} = \frac{1}{h} - \frac{2}{dr}, \quad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{k}. \]  

(3.6)

Then
\[ |u_1|_{q,T;m,X} \leq D(d, r) K_B(1; r, k) C_0(h, r) V(h, r) |F|_{k,T;h,X}, \]  

(3.7)

Conversely, the equalities (3.6) is also necessary for the estimation of the form (3.7).

Moreover, the estimation (3.7) is exact up to multiplicative constant.
Proof is alike to ones in theorem 2.0a. We use again the Marcinkiewicz’s and Young’s-Beckner’s inequalities:

\[ |u_{1,q,T}| \leq \int_X dy \int_0^t |w_{t-s}(x-y) F(y,s)|_{q,T} ds \leq \]

\[ K_B(1; r, k) \int_X |w(x-y)|_{r,T} |F(y,\cdot)|_{k,T} dy = \]

\[ K_B(1; r, k) D(d, r) \int_X \frac{|F(y,\cdot)|_{k,T} dy}{|x-y|^{d-2/r}} = K_B(1; r, k) D(d, r) I_{2/r}[F(y,\cdot)|_{k,T}]. \]

It remains to use the estimate (2.4) for Riesz’s fractional potential.

The last propositions of considered theorem will be proved further, in the section 8.

4 Necessity of our conditions.

We intend to prove in this section that our estimates are essentially non-improvable.

We will use the so-called dilation, or equally scaling method, see [40], chapter 9, [58]. Recall the definition: the operator \( T_{\lambda}[f] \), \( \lambda \in (0, \infty) \), (more exactly, the family of operators) of a form

\[ T_{\lambda}[f](x) = f(\lambda x), \quad x \in X = \mathbb{R}^d \] (4.0)

is said to be the dilation operator. Here \( f(\cdot) \) belongs to a certain class of functions, for instance, \( L_p(X) \) space or Schwartz set \( S(\mathbb{R}^d) \) etc. Obviously, if \( f \in S(\mathbb{R}^d) \), then \( T_{\lambda}[f] \in S(\mathbb{R}^d) \).

1. Young’s inequality.

We begin our considerations of this section from the famous Young’s inequality for convolution. Namely,

\[ |f * g|_r \leq K_B(p, q) |f|_p |g|_q, \quad p, q, r \in (1, \infty), \] (4.1)

where \( K_B(p, q) = K_B(d; p, q) < 1 \) is also the Beckner’s constant.

Theorem 4.1. Suppose that there exists a finite constant \( K = K_B(d; p, q) \) for which the inequality (4.1) is satisfied for arbitrary pair of functions \( (f, g) \) from the Schwartz space \( S(\mathbb{R}^d) \). Then the triple \( (p, q, r) \) satisfies the equality (3.5).

Proof. Assume the inequality (4.1) is satisfied for any pair of functions from the Schwartz space; then for arbitrary number \( \lambda \in (0, \infty) \) and some \( f, g \in S(\mathbb{R}^d) \), \( f, g \neq 0 \)

\[ |T_\lambda f * T_\lambda g|_r \leq K_B(d; p, q) |T_\lambda f|_p |T_\lambda g|_q, \quad p, q, r \in (1, \infty), \] (4.3)
Note that
\[ |T_\lambda g|_q = \lambda^{-d/q} |g|_q, \quad |T_\lambda f|_p = \lambda^{-d/p} |f|_p, \]
\[ |T_\lambda f \ast T_\lambda g|_r = \lambda^{-d-d/r} |f \ast g|_r. \]

We deduce substituting into (4.3)
\[ \lambda^{-d-d/r} \leq C \cdot \lambda^{d/p-d/q}, \quad C \in (0, \infty), \quad (4.4) \]
where \( C \) does not depend on the variable \( \lambda \).

Since the number \( \lambda \) is arbitrary positive, we conclude from (4.4) \( 1 + 1/r = 1/p + 1/q \), Q.E.D.

The multivariate version of Young’s inequality has a form, see [50], chapter 2, formula (2.14):
\[ |f \ast g|_{\vec{r}} = \left| \int_{\mathbb{R}^d} f(x - y) g(y) \right|_{\vec{r}} \leq |f|_{\vec{p}} |g|_{\vec{q}}, \]
where
\[ \forall j = 1, 2, \ldots, d \Rightarrow 1 + \frac{1}{r_j} = \frac{1}{p_j} + \frac{1}{q_j} \]
and wherein the last relation is necessary for the multivariate version of Young’s inequality.

This proposition may be proved by means of consideration of the so-called factorable functions
\[ f(x) = \prod_{j=1}^{d} f_j(x_j), \quad g(x) = \prod_{j=1}^{d} g_j(x_j). \]

2. Initial condition.

**Theorem 4.2.** Suppose that there exists a finite constant \( K^{(0)} = K_B^{(0)}(d; p, q) \) for which the inequality (2.3) is satisfied for arbitrary functions \( f \) from the Schwartz space \( S(\mathbb{R}^d) \):
\[ |u_0|_{r,T;q,X} \leq K_B^{(0)}(d; p, q) |f|_p, \quad p, q, r = \text{const} \in \mathbb{R}. \quad (4.5) \]

Then the triple \( (p, q, r) \) satisfies the relation
\[ \frac{1}{q} = \frac{1}{p} - \frac{2}{dr}. \quad (4.6) \]

**Proof** is completely analogous to ones in the theorem 4.1. Suppose that the inequality (4.5) is valid for some functions \( f \neq 0 \) from the Schwartz space; then for arbitrary number \( \lambda \in (0, \infty) \)
Note that
\[ w_t(x/\lambda) = \lambda^d w_{t;\lambda^2}(x), \quad t > 0, \quad x \in \mathbb{R}^d, \]
and we find as before after simple computations using identity (4.8)
\[ |w_t * T_\lambda f|_{r,q,X} = |w_t f|_{r,T;\lambda^q,X} \cdot \lambda^{-d/q-2/r}. \] (4.9)
We deduce substituting into (4.7)
\[ \lambda^{-d/q-2/r} \leq C_0 \cdot \lambda^{-d/p}, \quad C_0 \in (0, \infty), \] (4.10)
where \( C_0 \) does not dependent on the variable \( \lambda \). We conclude equating the exponents by \( \lambda \):
\[ -d/q - 2/r = -d/p, \quad \Leftrightarrow \quad 1/q = 1/p - 2/(dr). \]

3. Right - hand side. Analogously may be proved the following result.

**Theorem 4.3.** Suppose that there exists finite constant \( K^{(1)} = K^{(1)}_B(d; p, q, r, k) \) for which the inequality (3.1) is satisfied for arbitrary functions \( F = F(x,t) \) from the Schwartz space \( S(R^{d+1}) \):
\[ |u_1(\cdot, \cdot)|_{p,q,X,T} \leq K^{(1)}_B(d; p, q, r, k) \ |F(\cdot, \cdot)|_{r,X,k,T}. \] (4.11)
Then the tuple of the four numbers \((p, q, r, k)\) satisfies the equality
\[ 1 + \frac{1}{q} - \frac{1}{k} = \frac{d}{2} \left( \frac{1}{r} - \frac{1}{p} \right). \] (4.12)

**Proof.** We can use for convenient as a capacity of the test function \( F \) the factorable expression:
\[ F_0(x,t) = g(x) \ h(t) \overset{\text{def}}{=} [g \otimes h](x,t). \] (4.13)
We introduce for these functions a following (linear) operator, more exactly, the family of operators (generalized dilation):
\[ S_\lambda[g \otimes h](x,t) := g(\lambda x) \cdot h(\lambda^2 t) = T_\lambda g(x) \cdot T_{\lambda^2} h(t). \]
If the inequality (4.11) is true for arbitrary non - zero functions \( g, h \) from the correspondent Schwartz space, then
\[ |W_t * S_\lambda F_0|_{p,q,X,T} \leq K^{(1)}_B(d; p, q, r, k) \ |T_\lambda g|_{r,X} \cdot |T_{\lambda^2} h|_{k,T}. \] (4.14)
We have consequently:
\[ |T_\lambda g|_{r,X} \cdot |T_{\lambda^2} h|_{k,T} = |g|_{r,X} \cdot |h|_{k,T} \lambda^{-d/r-2/k}, \]
\[ |W_t \ast S \lambda F_0|_{p,X;q,T} = |W_t \ast F_0|_{p,X;q,T} \lambda^{-2-d/p-2/q}. \]

We conclude as before after substituting into (4.14):

\[-2 - d/p - 2/q = -d/r - 2/k.\]

This completes the proof of theorem 4.3.

5 United estimations.

Recall that the solution \( u = u(x,t) \) of the source equation (1.1) with correspondent initial condition may be represented as a sum \( u(x,t) = u_0(x,t) + u_1(x,t) \). We deduce as a slight consequence synthesizing the assertions of theorems 2.0a and 3.1b:

**Proposition 5.1.** Let the tuple \((q_1, m_1, k_1, h_1)\) satisfies simultaneously the conditions (2.2) and (3.6).

Let also \( f \in L_{p_0}(R^d), F \in L_{k_1,T;h_1,X}((0,T) \otimes R^d) \). Then

\[ |u|_{q_1,T;m_1,X} \leq C_0(d, p_0) D(d, q_1) V(p_0, q_1) |f|_{p_0} + C_0(h_1, r_1) D(d, r_1) V(h_1, r_1) K_B(1; r_1, k_1) |F|_{k_1,T;h_1,X}. \tag{5.1} \]

Let us consider more general case. Recall previously that the direct sum

\[ M_{r_0,q_0,q_1,m_1} = M_{r_0,q_0,q_1,m_1}((0,T) \otimes R^d) = \]

\[ L_{r_0,q_0}((0,T) \otimes R^d) \oplus L_{q_1,m_1}((0,T) \otimes R^d) \]

of two Banach spaces \( L_{r_0,q_0}((0,T) \otimes R^d) \) and \( L_{q_1,m_1}((0,T) \otimes R^d) \) is defined as a set of all the functions \( v = v(x,t), x \in R^d, t \in (0,T) \) of a form

\[ v(x,t) = v_0(x,t) + v_1(x,t), \tag{5.2} \]

where \( v_0 \in L_{r_0,q_0}((0,T) \otimes R^d), v_1 \in L_{q_1,m_1}((0,T) \otimes R^d) \), equipped with the norm

\[ |v|M_{r_0,q_0,q_1,m_1} = \inf \left[ |v_0|L_{r_0,q_0}((0,T) \otimes R^d) + |v_1|L_{q_1,m_1}((0,T) \otimes R^d) \right], \tag{5.3} \]

where " \( \inf \) " in (5.3) is calculated over all the functions \( v_0, v_1 \) satisfying the representation (5.2).

The space \( M_{r_0,q_0,q_1,m_1}((0,T) \otimes R^d) \) relative the norm (5.3) is also the complete Banach space.

It follows immediately from theorems 2.0a and 3.1b the following assertion:

**Proposition 5.2.** Let the tuple \( r_0,q_0 \) satisfies the conditions (2.2) and let the tuple \( (q_1,m_1,k_1,h_1) \) satisfies the conditions (3.6). Then
\[ |u| M_{0,q_0,q_1,m_1} \leq C_0(d; p_0) D(d, r_0) V(p_0, r_0) |f|_{p_0} + D(d, r_1) K_B(1; r_1, k_1) C_0(h_1, r_1) V(h_1, r_1) |F|_{k_1,T,h_1,X}. \] (5.4)

6 Generalized Grand Lebesgue Spaces estimations.

We assume in this section that the initial condition \( f = f(x) \) belongs to some Grand Lebesgue Space \( G(\psi) = G(\psi, X) \). This imply by definition that the following norm is finite:

\[ ||f||G(\psi) := \sup_{p \in (a,b)} \left[ \frac{|f|_p}{\psi(p)} \right] < \infty, \ 1 \leq a < b \leq \infty. \] (6.0)

Here \( \psi = \psi(p) \) is some positive continuous in open interval \((a, b)\) such that \( \inf_{p \in (a,b)} \psi(p) > 0 \).

The detail investigation of these spaces see in [59], [52], [53], [54], [55], [56], [67], [68] etc.

Denote by \( M_0(p) = M_0(p; q, r) \), \( p \in (a, b) \), \((q, r) \in G_0(p)\) the minimal value of the coefficient in the inequality (2.3):

\[ M_0(p; q, r) := \sup_{0 \neq f \in L_p(X)} \left[ \frac{|u_0|_{r,T,q,X}}{C_0(d; p) D(d, r) V(p, r) |f|_p} \right], \] (6.1)

we know that \( M_0(p; q, r) \leq 1 \). Denote

\[ H_0 = \{(q, r) : \exists p \in (a,b) \Rightarrow (q, r) \in G_0(p); \} \]

\[ \nu_0 = \nu_0(d; r, q) = \inf_{p \in (a,b)} [M_0(p; q, r) C_0(d; p) D(d, r) V(p, r) \psi(p)], \] \((q, r) \in H_0, \)

and define the following ”two - dimensional” generalization of the Grand Lebesgue Norm:

\[ ||u_0||G_{\nu_0} \overset{def}{=} \sup_{(q, r) \in H_0} \left[ \frac{|u_0|_{r,T,q,X}}{\nu(q, r)} \right]. \] (6.2)

Theorem 6.1.

\[ ||u_0||G_{\nu_0} \leq 1 \cdot ||f||G(\psi), \] (6.3)

wherein the constant ”1” in (6.3) is the best possible.

Proof. The upper estimate. Let \( f \in G(\psi) \); we can and will conclude without loss of generality \( ||f||G(\psi) = 1 \), therefore
\[ |f|_p \leq \psi(p), \ p \in (a, b). \]

It follows from theorem 2.0

\[ |u_0|_{r,T,q,X} \leq M_0(p; q, r) \psi(p), \]

which is equivalent to (6.3) after minimisation over \( p \).

The exactness of the constant "1" is in fact proved in the article [75], where is considered the one-dimensional case; the multivariate version is completely analogous.

Denote by

\[ M_1(p, q, r, k) := \sup_{0 \neq F \in L_{r,X,k,T}} \left[ \frac{|u_1|_{r,T,q,X}}{K_1(p, q, r, k) |F|_{r,k}} \right], \quad (6.4) \]

we know that \( M_1(p, q, r, k) \leq 1 \).

Let \( Y = \{r, k\}, \ r, k > 1 \) be some open non-empty domain in the positive quarter plane \( \mathbb{R}^2 \) and let \( \zeta = \zeta(r, k) \) be continuous in the set \( Y \) positive function such that

\[ \inf_{(r,k) \in Y} \zeta(r,k) > 0. \]

Define as before the two-dimensional Grand Lebesgue Norm

\[ ||F(\cdot, \cdot)||_{G\zeta} = \sup_{(r,k) \in Y} \left[ \frac{|F|_{r,X,k,T}}{\zeta(r,k)} \right]. \quad (6.5) \]

Denote

\[ H_1 = \{(p,q): \exists (r,k): \Rightarrow (p,q,r,k) \in G_1(p,q)\}; \]

\[ \tau_1 = \tau_1(d;p,q) = \inf_{(r,k) \in G_1(p,q)} [M_1(p,q,r,k) \zeta(r,k)], \ (q,r) \in H_1. \]

**Theorem 6.2.**

\[ ||u_1||_{G\tau_1} \leq 1 \cdot ||F||_{G(\zeta)}, \quad (6.7) \]

wherein the constant "1" in (6.6) is the best possible.

**Proof** is at the same as in theorem 6.1 through theorem 3.1 and may be omitted.
7 Weight estimates.

Denote

\[ f(y) = g(y) \, |y|^{-a}, \quad v_0 = v_0(x, t) = u_0(x, t) \, |x|^{-b} \, t^{-\theta}, \quad a, \ b = \text{const} \geq 0. \]  

(7.0)

We try to obtain in this subsection first of all the weight estimations of a form

\[ |v_0|_{r,T,X} \leq K_{a,b,\theta}(p,q,r) \, |g|_{p,X}, \]  

(7.1)

i.e. the weight generalization of theorem 2.0.

**New notations and restrictions:**

\[ p_- = \frac{d}{d - a} \geq 1, \quad p_+ = \frac{d}{2/r - a - 2\theta} > 0, \]

(7.2)

\[ \kappa := \frac{a + b + d - 2\theta - 2/r}{d} \in (0, 1), \]

\[ p,q,r > 1, \quad p \in (p_-, p_+), \]

\[ a + 2\theta < \frac{2}{r} < d + 2\theta, \]

\[ 1 + \frac{1}{q} = \frac{1}{p} + \kappa. \]  

(7.3)

It is easy to see that the new designations agreed with the old in the case when \( a = b = 0 \).

**Theorem 7.0.** We conclude under our restrictions (7.2) - (7.3) that the "constant" \( K_{a,b,\theta}(p,q,r) \) in is finite:

\[ K_{a,b,\theta}(p,q,r) \leq \frac{C_2(d; a, b, \theta) \, D(d + 2\theta, r)}{|p - p_-|^\kappa}. \]  

(7.4)

Moreover, if \( K_{a,b,\theta}(p,q,r) \) is finite, then the triple \((p,q,r)\) satisfies the equality (7.3) and \( p \in (p_-, p_+) \).

**Proof** is the same as one in theorem 2.0. We start from the inequality (2.4):

\[ |v_0|_{r,T} \leq |x|^{-b} \int_X |t^{-\theta} w_t(x - y)|_{r,T} \, |g(y)| \, |y|^{-a} \, dy = \]

\[ |x|^{-b} D(d + 2\theta, r) \int_X \frac{|g(y)| \, |y|^{-a} \, dy}{|x - y|^{d + 2\theta - 2/r}} = D(d + 2\theta, r) \, I_{a,b,2/r-2\theta}[g](x). \]  

(7.5)

It remains to use again the main results of the articles [70], [71].
The *necessity* of the conditions (7.2) - (7.3) may be proved as before by means of the scaling method. By our opinion, the scaling method we used is somewhat simpler than in [77].

Note that the weight generalization of Young’s convolution inequality was considered at first by R.A.Kerman in [77]; see also the articles [57], [79], [81] etc.

8 Fractional Laplace operator.

Let us consider the linear heat type initial value problem with fractional power of Laplace operator:

\[ u_t + (-\Delta)^\alpha u = F(x,t), \quad u(x,0+) = f(x), \quad \alpha = \text{const} > 0, \quad \alpha \neq 1, 2, 3, \ldots \]  

(8.0)

We intend to generalize weight obtained estimations and some estimates of the works [30], [31], [45] on the function \( u^{(\alpha)}(\cdot, \cdot) = u(\cdot, \cdot) \).

Denote

\[ Z(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x,\xi) - |\xi|^{2\alpha}} d\xi, \quad x \in \mathbb{R}^d; \quad (8.1) \]

\[ Z(x) = Z^{(0)}(|x|), \quad Z_t(x) = t^{-d/(2\alpha)} \cdot Z \left( \frac{x}{t^{1/(2\alpha)}} \right), \quad t > 0. \quad (8.2) \]

The ordinary solution of equation (8.0) has a view as before

\[ u^{(\alpha)}(x,t) = \int_{\mathbb{R}^d} Z_t(x-y) f(y) \, dy + \int_0^t \int_{\mathbb{R}^d} Z_{t-s}(x-y) F(y,s) \, ds = \]

\[ u_0^{(\alpha)}(x,t) + u_1^{(\alpha)}(x,t), \quad \text{where} \quad u_0^{(\alpha)}(x,t) = \int_{\mathbb{R}^d} Z_t(x-y) f(y) \, dy, \quad (8.3a) \]

\[ u_1^{(\alpha)}(x,t) = \int_0^t \int_{\mathbb{R}^d} Z_{t-s}(x-y) F(y,s) \, ds. \quad (8.3b) \]

Recall that the function

\[ q_d(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x,\xi) - |\xi|^{2\alpha}} d\xi \]

is named *transstable density*, and appears in the Probability Theory, see, e.g. [80], [76]. Evidently, \( Z(x) = (2\pi)^{d/2} q_d(x, 2\alpha) \).

The asymptotic expression as \(|x| \to \infty\) for the transstable density is calculated in the book of Uchaikin V.V. and Zolotarev V.M. [80], p. 212 - 215; see also [76]: as \(|x| \to \infty\)

\[ |Z(x)| \sim C_{U,Z}(d, \alpha) \cdot (1 + |x|)^{-d-2\alpha}, \quad x \in \mathbb{R}^d. \quad (8.4a) \]

The non-asymptotic estimation for \( Z(x) \) is calculated by Changxing Miao, Baoquan Yuan, and Bo Zhang in [30]:
\[ |Z(x)| \leq C_{M,Y,Z}(d, \alpha) \left(1 + |x|\right)^{-d-2\alpha}, \quad x \in \mathbb{R}^d. \quad (8.4b) \]

Obviously, the function \( Z = Z(x) \) is radial, symbolically: \( Z = Z(|x|) \).

It follows from \((8.4a)\) that the last estimation \((8.4b)\) is asymptotically exact as \(|x| \to \infty\) up to multiplicative constant.

We find by direct calculation from \((8.4b)\)
\[ |Z_t|_{p,X} \leq C_1^{1/p} M,Y,Z(d,\alpha) \omega^{1/p}(d) \frac{t^{d(1+1)/p - 2\alpha}}{B(1/p - d + 2\alpha)}, \quad p \geq 1; \quad (8.5) \]
\[ d - \alpha p > 0, \quad (p - 1)d + 2\alpha p > 0; \]
\[ |Z_t|_{q,T} \leq (2\alpha C_{M,Y,Z}(d,\alpha))^{1/q} |x|^{2\alpha/q - d} B^{1/q}(dq - 2\alpha; q(d + 2\alpha) - d), \quad q \geq 1, \quad (8.6) \]
where \( B(\cdot, \cdot) \) is ordinary Beta-function and
\[ dq - 2\alpha > 0, q(d + 2\alpha) - d > 0, \]
where
\[ \omega(d) = \frac{2 \pi^{d/2}}{\Gamma(d/2)} \]
is an area of unit sphere in the Euclidean space \( \mathbb{R}^d \).

We used the elementary integral
\[ \int_0^\infty \frac{x^{\gamma_1}}{(1 + x)^{\gamma_2}} \, dx = B(\gamma_1 + 1, \gamma_2 - \gamma_1 - 1), \quad (8.7) \]
\[ \gamma_1, \gamma_2 = \text{const}, \quad \gamma_1 > -1, \quad \gamma_2 - \gamma_1 > 1. \]

Moreover, let us denote also
\[ Z_t^{(\theta,\tau)}(x) = |x|^{-\theta} \, t^{-\tau} \, Z_t(x), \quad \theta, \tau = \text{const}. \quad (8.8) \]

We have using \((8.7)\) the following weight estimates:
\[ |Z_t^{(\theta,\tau)}(\cdot)|_{r,T} \leq (2\alpha C_{M,Y,Z}(d + 2\alpha\tau), \alpha))^{1/r} |x|^{2\alpha/r - d - \theta} \times \]
\[ B^{1/q}((d + 2\alpha\tau)q - 2\alpha, q(d + 2\alpha\tau + 2\alpha) - d), \quad r \geq 1, \quad (8.9) \]
if of course
\[ (d + 2\alpha\tau)r - 2\alpha > 0, \quad r(d + 2\alpha\tau + 2\alpha) - d > 0. \quad (8.10) \]

Analogously
\[ |Z_t^{\theta,\tau}(\cdot)|_{p,X} \leq C^{1/p}_{M,Y,Z}(d, \alpha) \omega^{1/p}(d) \ t^{-\tau+\sigma d/(1-p)(2\alpha)-\theta/(2\alpha)} \times B^{1/p}(d-\alpha p, dp + 2\alpha p - d + \theta p), \]  

(8.11)

under restrictions

\[ d - \alpha p > 0, \ dp + 2\alpha p - d + \theta p > 0. \]  

(8.12)

Denote for brevity

\[ \lambda_1 = \lambda_1(r; d, \alpha, \theta, \tau) = -2\alpha/r + d + \theta, \]  

(8.13)

\[ \lambda_1^0 = \lambda_1(r; d, \alpha, \theta, 0), \]  

(8.13a)

\[ \lambda_2 = \lambda_2(p; d, \alpha, \theta, \tau) = \tau - d(1/p - 1)/(2\alpha) + \theta/(2\alpha), \]  

(8.14)

\[ \lambda_2^0 = \lambda_2(p; d, \alpha, 0, \tau) = \tau - d(1/p - 1)/(2\alpha). \]  

(8.14a)

Then the estimates (8.9) and (8.11) may be rewritten in more simple form under restrictions (8.10) and (8.12)

\[ |Z_t^{(\theta,\tau)}(\cdot)|_{q,T} \leq C_q \ |x|^{-\lambda_1}, \ |Z_t^{(\theta,\tau)}(\cdot)|_{p,X} \leq C_p \ t^{-\lambda_2}. \]  

(8.15)

Remark 8.1. Evidently, the estimates (8.15) are exact up to multiplicative constant on the whole real axis (semi-axis):

\[ |Z_t^{(\theta,\tau)}(\cdot)|_{q,T} \geq \tilde{C}_q \ |x|^{-\lambda_1}, \ |Z_t^{(\theta,\tau)}(\cdot)|_{p,X} \geq \tilde{C}_p \ t^{-\lambda_2}. \]  

(8.15a)

Remark 8.2. More surprisingly is the circumstance that the estimates (8.15) and (8.15a) are true even for the integer values \( \alpha \), for instance for the value \( \alpha = 1 \), in which \( Z_t(x) = w_t(x) \).

This is all the more surprising that the tail behavior as \( |x| \to \infty \) of the function \( w_1(x) \) strikingly different on the ones for the function \( Z_1(x) \).

We note passing to the descriptions of next results first of all that the decay estimates as \( t \to \infty \) for solution \( u = u(x,t) \) of linear and nonlinear nonlocal heat equations, for more general pseudo-differential equation, of a form

\[ |u|_{p,X} \leq C \ t^{-[(d/\gamma-1/q-1/p)]} |f|_{q,X}, \ 1 \leq q < p, \ \gamma = \text{const} \in (0, \infty), \]  

(in our notations), was obtained in recent article [47]; see also reference therein.

Let us consider now the inverse order of the spaces. A new notations and restrictions:

\[ v_0^{(\alpha)}(x,t) = v_0(x,t) = u_0^{(\alpha)}(x,t) \ |x|^{-b} t^{-\tau}, \ f(x) = g(x) \ |x|^{-\alpha}, \]
\[ a, b, \tau = \text{const} \geq 0, \ \kappa_0 := (a + b + \lambda_0^0)/d \in (0, 1); \]  \hspace{1cm} \text{(8.16)}

\[ p_+ = \frac{d}{d - a} \geq 1, \ p_+ = \frac{d}{d - a - \lambda_0^1} > 0. \]  \hspace{1cm} \text{(8.17)}

\[ d + \frac{d}{q} = \frac{d}{p} + (a + b + \lambda_0^0), \ q = q(p). \]  \hspace{1cm} \text{(8.18)}

**Theorem 8.0.** We conclude under conditions (8.10), (8.16), (8.17) and (8.18) for the values \( p \) from the interval \( p \in (p_-, p_+) \) and for all the positive values \( \alpha \), including the integer numbers, for example \( \alpha = 1 \), in which \( Z_t(x) \) is the classical solution of the heat equation,

\[
|v_0^{(\alpha)}|_{r,T} \leq \frac{C(d,a,b,\tau,\alpha)}{|(p-p_-)(p_+-p)|\kappa_0} |f|_p,
\]

if, of course, \( f \in L_p(R^d) \).

Conversely, the equality (8.18) is necessary for the estimation of the form (8.19). Moreover, the estimation (8.19) is exact up to multiplicative constant.

**Proof.**

\[
v_0^{(\alpha)}(x,t) = |x|^{-b} \int_X t^{-\tau} Z_t(x-y) g(y) \ |y|^{-a} \ dy =
\]

\[
|x|^{-b} \int_X Z_t^{(0,\tau)}(x-y) g(y) \ |y|^{-a} \ dy.
\]

We have using again Marcinkiewicz integral-triangle inequality:

\[
|v_0^{(\alpha)}(x,t)|_{r,T} \leq \ |x|^{-b} \int_X |Z_t^{(0,\tau)}(x-y)|_{r,T} |g(y)| \ |y|^{-a} \ dy.
\]

We use now the estimate (8.15):

\[
|v_0^{(\alpha)}(x,t)|_{r,T} \leq |x|^{-b} \int_X C_p \ \frac{|y|^{-a} |g(y)| \ dy}{|x-y|^{\lambda_0^0}}.
\]

The integrals of a form (8.20) are called weight Hardy-Littlewood operators or equally weight fractional integral operators. The using for us exact up to multiplicative constant \( L_q(R^d) \) estimates for them are obtained in the article [70].

The necessity of the equality (8.18) for the assertion of considered theorem may be proved as ordinary by means of scaling method.

This completes the proof of theorem (8.1).

We will deduce now he weight estimates for the second term in the representations (8.3a), i.e. for the function \( u_1^{(\alpha)}(x,t) \).

Some new notations and restrictions.

The first weight function \( v_1^{(\alpha)}(x,t) \) for the part of solution \( u_1^{(\alpha)}(x,t) \) is defined as follows:
\( v_1^{(\alpha)}(x,t) := |x|^{-b} u_1^{(\alpha)}(x,t) = |x|^{-b} \int_0^t \int_X Z(x-y,t-s) \, dy \, ds \), \quad (8.21a)

where \( H(y,s) = |y|^a \cdot F(y,s) \) is the first weight function for the right-hand side \( F = F(x,t) \). Here \( a, b = \text{const} \geq 0 \).

The second weight function \( v_2^{(\alpha)}(x,t) \) for the part of solution \( u_1^{(\alpha)}(x,t) \) is defined as follows:

\[ v_2^{(\alpha)}(x,t) := t^{-\tau} u_1^{(\alpha)}(x,t) = t^{-\tau} \int_0^t \int_X Z(x-y,t-s) \, dy \, ds, \quad (8.21b) \]

where \( R(y,s) = s^{-\beta} \cdot F(y,s) \) is the second weight function for the right-hand side \( F = F(x,t) \). Here \( \tau, \beta = \text{const} \geq 0 \).

Other notations and restrictions.

\[
1 + \frac{1}{r_1} = \frac{1}{Q_1} + \frac{1}{m_1}, \quad r_1, Q_1, m_1 \in (1, \infty),
\]

\[
a, b = \text{const} \geq 0, \quad \kappa_1 = \frac{a + b + d - 2\alpha}{d} \in (0, 1),
\]

\[
1 + \frac{1}{q_1} = \frac{1}{p_1} + \frac{a + b + d - 2\alpha}{d} = \frac{1}{p_1} + \kappa_1, \quad p_1, q_1 \in (1, \infty),
\]

\[
q_1 = q_1(p_1), \quad p_1^- = \frac{d}{d - a} > 1, \quad p_1^+ = \frac{d}{2\alpha/Q_1 - a} > 0,
\]

\[
p_1 \in (p_1^-, p_1^+). \]

**Theorem 8.1a** We propose under formulated assumptions

\[
|v_1^{(\alpha)}(\cdot, \cdot)|_{r_1,T; q_1,X} \leq \frac{C_1(d; a, b)}{[(p_1^- - p_1^-)(p_1^+ - p_1^-)]^{\kappa_1}} |H(\cdot, \cdot)|_{m_1,T; p_1,X}. \quad (8.23)
\]

Conversely, the estimation (8.23) is exact up to multiplicative constant and the relations (8.22a), (8.22b) are necessary for the inequality of a form (8.23).

**Proof.** We have from (8.21) using again the Marcinkiewicz integral triangle inequality and Beckner’s convolution constants:

\[
|v_1^{(\alpha)}(x,\cdot)|_{r_1,T; q_1,X} \leq |x|^{-b} \int_X |y|^{-a} K_B(1; Q, m) |Z(x-y,t-s)|_{Q_1,T} |H|_{m_1,T} dy =
\]

\[
C \cdot K_B(1; Q_1, m_1) \cdot \int_X \frac{|x|^{-a} |y|^{-b}}{|x-y|^{d-2\alpha/Q_1}} |H(y,\cdot)|_{m_1,T} dy.
\]

It remains to apply the inequality (2.4) for the weight Hardy-Riesz transform.
The exactness of the inequality (8.23) follows immediately from the main result of the report [70] by means of consideration of the factorable function $F(x, t) = F_0(t) F_1(x)$.

The necessity of the conditions (8.22a) and (8.22b) for (8.23) may be elementary provided by the known scaling method, where the dilation operator $T_{\lambda, \mu}$ is defined as follows

$$T_{\lambda, \mu}[F_0 F_1](t, x) \overset{\text{def}}{=} F_0(\lambda t) F_1(\mu x), \quad \lambda, \mu \in (0, \infty).$$

Let us consider the inverse order of norms. Notations and restrictions:

$$1 + \frac{1}{q_2} = \frac{1}{Q_2} + \frac{1}{p_2}, \quad \quad \quad (8.24a)$$

$$0 \leq \beta, \tau < 1, \quad \zeta_2 := d(1 - 1/Q_2)/(2\alpha) \in (0, 1), \quad \beta + \tau + \zeta_2 < 1,$$

$$1 + \frac{1}{r_2} = \frac{1}{k_2} + (\beta + \tau + \zeta_2), \quad r_2 = r_2(k_2), \quad k_2, r_2, q_2, Q_2, p_2 \in (1, \infty), \quad \quad \quad (8.24b)$$

$$\kappa_2 := \beta + \tau + \zeta_2, \quad p_2^{(2)} = \frac{1}{1 - \beta}, \quad p_+^{(2)} = \frac{1}{1 - \beta - \tau} > 0,$$

$$p_2 \in (p_-, p_+).$$

**Theorem 8.1b** We propose under formulated assumptions

$$|v_2^{(\alpha)}(\cdot, \cdot)|_{q_2, X, r_2, T} \leq \frac{C_2 K_B(d; Q_2, p_2)}{(p_2 - p_2^{(2)})^{\kappa_2}} |R(\cdot, s)|_{p_2, X; k_2, T}. \quad (8.25)$$

As before, the estimation (8.25) is exact up to multiplicative constant and the relations (8.24a), (8.24b) are necessary for the inequality of a form (8.25).

**Proof.** Note that

$$v_2^{(\alpha)}(x, t) = t^{-\tau} \int_0^t s^{-\beta} ds \int_X Z(x - y, t - s) R(y, s) dy =$$

$$t^{-\tau} \int_0^t s^{-\beta} ds \cdot Z_{t-s} * [R](x).$$

Therefore

$$|v_2^{(\alpha)}(x, t)|_{q_2, X} \leq t^{-\tau} \int_0^t s^{-\beta} ds \cdot |Z_{t-s} * [R](x)|_{q_2, X} \leq$$

$$C_2 t^{-\tau} \int_0^t s^{-\beta} ds \cdot K_B(d; Q_2, p_2) (t - s)^{-(1-1/p)/(2\alpha)} |R(\cdot, s)|_{p_2, X}.$$

The proposition of theorem 8.1b follows immediately from the main result of an article [71].
9  Concluding remarks.

**A. General elliptic operator.**

Our results may be easily generalized on the initial value problem for parabolic PDE of a form

\[
\frac{\partial u}{\partial t} = 0.5 \sum_{k=1}^{d} \sum_{i=1}^{d} a_{i,k}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u + F(x,t), \tag{9.1}
\]

\[u(x,0+) = f(x), \quad x \in \mathbb{R}^d, \quad t > 0, \tag{9.2}\]

if for example the functions \(a_{i,k}(x,t)\) are symmetrical \(a_{i,k}(x,t) = a_{k,i}(x,t)\), uniform positive definite:

\[
\sum_{k=1}^{d} \sum_{i=1}^{d} a_{i,k}(x,t) \xi_i \xi_k \geq \lambda \sum_{k=1}^{d} \xi_k^2, \quad \lambda = \text{const} > 0;
\]

all the functions \(a_{i,k}(x,t), b_i(x,t), c(x,t)\) are bounded and satisfy the Hölder’s conditions with positive power.

It is well-known that under these conditions the solution of problem (9.1)-(9.2) may be written as follows:

\[
u(x,t) = \int_{\mathbb{R}^d} \int_0^t G(x,t,y,s) F(y,s) \, dy \, ds + \int_{\mathbb{R}^d} G(x,t,y,0) f(y) \, dy,
\]

where the function \(G(x,t,y,s)\) allows an estimation

\[
G(x,t,y,s) \leq C_1(t-s)^{-d/2} \exp \left( -C_2|x-y|^2/(t-s) \right), \quad 0 \leq s < t < \infty.
\]

**B. Arbitrary convolution kernel.**

Our considerations may be easily generalized on the function of a form

\[
z(x,t) = \int_X M(x-y,t) \, f(y) \, dy + \int_0^t \int_X M(x-y,t-s) \, F(y,s) \, ds \, dy,
\]

where the kernel function \(M(x,t)\) has the form

\[
M(x,t) = t^{-a}M_1(|x|/t^b), \quad M_1(\cdot) \in L_p(\mathbb{R}^d).
\]
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