Galilei invariant theories.
II. Wave equations for massive fields

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Abstract

Galilei invariant equations for massive fields with various spins are found
and classified. They have been obtained directly, i.e., by using requirement of
Galilei invariance and the facts on representations of the Galilei group deduced
in our previous paper de Montigny M, Niederle J and Nikitin A G, J. Phys.
A \textbf{39}, 1-21, 2006 . It is shown that the collection of non-equivalent Galilei-
invariant wave equations for vector and scalar fields is very broad and describes
many physically consistent systems.

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1 Introduction

It is well known that the Galilei group $G(1,3)$ and its representations play the same role in non-relativistic physics as the Poincaré group $P(1,3)$ and its representations do in the relativistic one. In fact the Galilei group and its representations form a group-theoretical basis of classical mechanics and electrodynamics. They replace the Poincaré group and its representations whenever velocities of bodies are much smaller than that of light in vacuum. On the other hand, the structure of subgroups of the Galilei group and of its representations are in many respects more complex than those of the Poincaré group and therefore it is not perhaps so surprising that the representations of the Poincaré group were described in [1] almost 15 years earlier than the representations of the Galilei group [2] in spite of the fact that the relativity principle of classical physics was formulated by Galilei in 1632, about three centuries prior that of relativistic physics by Einstein.

An excellent review of representations of the Galilei group was written by Lévy-Leblond [3]. It appears that, as distinct from the Poincaré group, that the Galilei group has besides ordinary also projective representations (see [2] and [4] respectively). However its subgroup – the homogeneous Galilei group $HG(1,3)$ which plays in non-relativistic physics the role of the Lorentz group in the relativistic case, has a more complex structure so that its finite-dimensional indecomposable representations are not classifiable in general (for details see [5]). And they are the representations which play a key role in formulation of physical models satisfying the Galilei relativity principle!

An important class of indecomposable finite-dimensional representations of $HG(1,3)$ was found and completely classified in [5]. It contains all representations of the homogeneous Galilei group which when restricted to its rotation subgroup, decompose to spin $0$, $1/2$ and $1$ representations. Moreover, a connection of these representations with those of the Lorentz group by means of the Inönü-Wigner contractions was cleared up in [5] and [6] too.

This offers possibilities to construct various quantum mechanical and field theoretical models for interacting particles and fields with spins $0$, $1/2$ and $1$. For instance, the most general Pauli interaction of Galilean spin–$1/2$ particles with an external electromagnetic field can be found in [5].

In the present paper we study the vector and spinor representations of the homogeneous Galilei group in detail and use them for construction various wave equations for particles with spin $0$, $1/2$, $1$ and $3/2$.

There are well-developed theories of wave equations invariant w.r.t. the Poincaré group which can be taken for a start of construction of the Galilei invariant equations. First, we begin with the Bhabha approach which is a direct extension of the method yielding the Dirac equation. The corresponding relativistic wave equations can be written as systems of linear first order partial-differential equations of the form:

$$(\beta_{\mu}p^\mu + \beta_{4}m) \Psi(x, t) = 0,$$

(1)
where \( p^0 = i \frac{\partial}{\partial x^0} \), \( p^a = i \frac{\partial}{\partial x^a} \) (\( a = 1, 2, 3 \)), and \( \beta_\mu \) (\( \mu = 0, 1, 2, 3 \)) and \( \beta_4 \) are square matrices restricted by the condition of the Poincaré invariance. Notice that in the relativistic approach matrix \( \beta_4 \) is usually assumed to be proportional to a unit matrix.

The theory of the Poincaré-invariant equations \([11]\) is clearly explained for instance in the Gel’fand–Minlos–Shapiro book \([7]\). Some particular results related to the Galilei invariant equations \([11]\) can be also found in \([3]\), \([5]\)-\([12]\). Galilean analogues of Bargman-Wigner equations are presented in the recent paper \([13]\). However, these equations became incompatible whenever the minimal interaction with an external e.m. field be introduced \([13]\).

The other approaches make use of the tensor calculus, and the associated equations have the form of covariant vectors or tensors (see, e.g., \([14]\)). Popular examples of these equations are the Proca \([15]\), Rarita-Schwinger \([16]\) and Sign–Hagen \([17]\) equations. Let us mention that all these equations violate causality or predict incorrect values of the gyromagnetic ratio \( g \). The tensor-spinorial equations for particles with arbitrary half-integer spin which are not violating causality and admit the right value for \( g \) have been discussed in detail in \([18]\).

In the present paper we use both above mentioned approaches and derive the Galilei-invariant equations for particles with spins 0, \( \frac{1}{2} \), 1 and \( \frac{3}{2} \). Moreover we present a complete description of Galilei-invariant equations \([11]\) for scalar and vector fields.

### 2 Galilei algebra and Galilei–invariant wave equations

In this section we use a Galilean version of the Bhabha approach and present a complete list of the corresponding Galilei-invariant wave equations.

Equation \([11]\) is said to be invariant w.r.t. the Galilei transformations

\[
\begin{align*}
t &\to t' = t + a, \\
x &\to x' = Rx + vt + b,
\end{align*}
\]

where \( a, b, v \) are real parameters and \( R \) is a rotation matrix, if function \( \Psi \) in \([11]\) cotransforms as

\[
\Psi(x, t) \to \Psi'(x', t') = e^{i f(x, t) T} \Psi(x, t),
\]

i.e., according to a particular representation of the Galilei group. Here \( T \) is a matrix depending on transformation parameters only, \( f(x, t) = m (v \cdot x + tv^2/2 + c) \), \( c \) is an arbitrary constant and \( \Psi'(x', t') \) satisfies the same equation in prime variables as \( \Psi(x, t) \) in the initial ones.

The Lie algebra corresponding to representation \([3]\) has the following generators

\[
\begin{align*}
P_0 &= i \partial_t, & P_a &= -i \partial_a, & M &= m I, \\
J_a &= -i \varepsilon_{abc} x_b \partial_c + S_a, \\
G_a &= -ix_0 \partial_a - mx_a + \eta_a,
\end{align*}
\]
where $S_a$ and $\eta_a$ are matrices which satisfy the following commutation relations:

\begin{align}
[S_a, S_b] &= i\varepsilon_{abc}S_c, \\
[\eta_a, S_b] &= i\varepsilon_{abc}\eta_c, \\
[\eta_a, \eta_b] &= 0
\end{align}

that is, they form a basis of $hg(1,3)$ – of the Lie algebra of the homogeneous Galilei group.

Let us note that special classes of the Galilei invariant equations (1) were described in [10] and [12].

Equation (1) is invariant with respect to the Galilei transformations (3), if their generators (4) transform solutions of (1) into solutions. This requirement together with the existence of the Galilei invariant Lagrangian for (1) yields the following conditions on matrices $\beta_\mu$ ($\mu = 0, 1, 2, 3$) and $\beta_4$ [11]:

\begin{align}
\eta_a^\dagger \beta_4 - \beta_4 \eta_a &= -i\beta_a, \\
\eta_a^\dagger \beta_b - \beta_b \eta_a &= -i\delta_{ab}\beta_0, \\
\eta_a^\dagger \beta_0 - \beta_0 \eta_a &= 0, \quad a, b = 1, 2, 3.
\end{align}

Moreover, $\beta_0$ and $\beta_4$ should to be scalars w.r.t. rotations, i.e., they have to commute with $S_a$.

Thus the problem of classification of the Galilei invariant equations (1) is equivalent to find the matrices $S_a$, $\eta_a$, $\beta_0$, $\beta_a$ and $\beta_4$ satisfying relations (5), (6) and (7). Unfortunately, a subproblem of this problem, i.e., the complete classification of non-equivalent finite-dimensional representations of algebra $hg(1,3)$ appears to be in general unsolvable (that is a ‘wild’ algebraic problem). However, for two important particular cases, i.e., for the purely spinor and vector-scalar representations the problem of finding of all finite-dimensional indecomposable representations of algebra $hg(1,3)$ was completely solved in [5].

### 2.1 Spinor fields and the corresponding wave equations

Let $\tilde{s}$ be the highest value of spin which appears when representation of algebra $hg(1,3)$ is reduced to a direct sum of irreducible representations of its subalgebra $so(3)$ generated by $S_a$ which satisfy (3). Then the corresponding representation space of this representation of $hg(1,3)$ is said to be a space of fields of spin $\tilde{s}$.

As mentioned in [5] there exist only two non-equivalent indecomposable representations of algebra $hg(1,3)$ defined on the fields of spin 1/2. One of them, $D_{\frac{1}{2}}(1)$, when restricted to the subalgebra $so(3)$ remains irreducible while the other one, $D_{\frac{1}{2}}(2)$, decomposes to two irreducible representations $D(1/2)$ of $so(3)$. The corresponding matrices $S_a$ and $\eta_a$ can be written in the following form:

\begin{align}
S_a &= \frac{1}{2}\sigma_a \quad \text{and} \quad \eta_a = 0 \quad \text{for } D_{\frac{1}{2}}(1)
\end{align}

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and
\[ S_a = \frac{1}{2} \left( \begin{array}{cc} \sigma_a & 0 \\ 0 & \sigma_a \end{array} \right) \quad \text{and} \quad \eta_a = \frac{i}{2} \left( \begin{array}{cc} 0 & 0 \\ \sigma_a & 0 \end{array} \right) \quad \text{for} \ D_{1/2}(2). \] (9)

Here \( \sigma_a \) are the Pauli matrices and \( 0 \) is the \( 2 \times 2 \) zero matrix.

Realization (8) with conditions (7) yields equation (1) trivial, i.e., with zero \( \beta \)-matrices.

The elements of the carrier space of representation (9) will be called Galilean bi-spinors. It can be found in [5] how the Galilean bi-spinors transform w.r.t. the finite transformations from the Galilei group (see also equation (11) below).

Solutions of relations (7) with \( S_a, \eta_a \) given by formulae (9) can be written as
\[ \beta_0 = \left( \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right), \quad \beta_a = \left( \begin{array}{cc} 0 & \sigma_a \\ \sigma_a & 0 \end{array} \right), \quad a = 1, 2, 3, 4 \] (10)
where \( I \) and \( 0 \) are the \( 2 \times 2 \) unit and zero matrices respectively, and \( \omega \) and \( \kappa \) are constant multiples.

Notice that parameter \( \kappa \) can be chosen zero since the transformation \( \Psi \rightarrow e^{i\kappa m t} \Psi \) lives equation (1) invariant. Parameter \( \omega \) is inessential too since it can be annulled by the transformation \( \beta_n \rightarrow U^\dagger \beta_n U \), where \( m = 0, 1, 2, 3, 4 \) and
\[ U = \left( \begin{array}{cc} I & i\omega I \\ 0 & I \end{array} \right). \]

Galilean invariance of the equation (1) with \( \beta_n \) given in (10) can be verified by using the explicit transformations (2) and (3) for the space-time variables and wave function respectively, where
\[ T = \exp(i\eta \cdot v) \exp\left( \frac{i}{2} \sigma \cdot \theta \right) = (1 + \eta \cdot v) \left( \cos \frac{\theta}{2} + i \frac{\sigma \cdot \theta}{\theta} \sin \frac{\theta}{2} \right). \] (11)

Here \( \theta = (\theta_1, \theta_2, \theta_3) \) are rotation parameters, \( \theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2} \), and \( \eta \) is a matrix three-vector whose components \( \eta_1, \eta_2, \eta_3 \) are given in (9).

Let us note that if we consider a more general case in which matrices \( S_a \) and \( \eta_a \) are represented by a direct sum of an arbitrary finite number of matrices (9) and solve the related equations (7), then we obtain matrices \( \beta_n \) which can be reduced to direct sums of matrices (10) and zero matrices. In other words, equation (11) with matrices (10) is the only non-decoupled system of the first order equations for spin 1/2 field invariant under the Galilei group.

Equation (11) with matrices (10) and \( \omega = \kappa = 0 \) coincides with the Lévy-Leblond equation in [3].

Let us remark that matrices \( \hat{\gamma}_n = \eta \hat{\beta}_n |_{\kappa = \omega = 0} \), \( n = 0, 1, 2, 3, 4 \) with
\[ \eta = \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right) \] (12)
satisfy the following relations
\[ \hat{\gamma}_n \hat{\gamma}_m + \hat{\gamma}_m \hat{\gamma}_n = 2 \hat{g}_{nm}, \]  
(13)
where \( \hat{g}_{nm} \) is a symmetric tensor whose non-zero components are
\[ \hat{g}_{01} = \hat{g}_{40} = -\hat{g}_{11} = -\hat{g}_{22} = -\hat{g}_{33} = 1. \]  
(14)

In the Galilei–invariant approach tensor (14) plays the same role as the metric tensor for the Minkovski space in the relativistic theory.

The matrices \( \hat{\gamma}_a \) will be used many times later on. Therefore, for convenience, we present them explicitly:
\[ \hat{\gamma}_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\gamma}_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad a = 1, 2, 3, \quad \hat{\gamma}_4 = \begin{pmatrix} 0 & 2I \\ 0 & 0 \end{pmatrix}. \]  
(15)

2.2 Vector fields and the corresponding wave equations
2.2.1 Indecomposable representations for vector fields

A complete description of indecomposable representations of algebra \( hg(1, 3) \) in the spaces of vector and scalar fields is given in [5]. The corresponding matrices \( S_a \) and \( \eta_a \) have the following forms:
\[ S_a = \begin{pmatrix} I_{n \times n} \otimes s_a & B_{n \times m} \otimes k_a^\dagger \\ 0_{m \times m} & C_{m \times n} \otimes k_a \end{pmatrix}, \quad \eta_a = \begin{pmatrix} A_{n \times n} \otimes s_a & B_{n \times m} \otimes k_a^\dagger \\ 0_{m \times m} & C_{m \times n} \otimes k_a \end{pmatrix}, \]  
(16)

where \( I_{n \times n} \) and \( 0_{m \times m} \) are the unit and zero matrices of dimension \( n \times n \) and \( m \times m \) respectively, \( A_{n \times n} \), \( B_{n \times m} \) and \( C_{m \times n} \) are matrices of the indicated dimensions whose forms will be specified later on, \( s_a \) are matrices of spin one with elements \( (s_a)_{bc} = i \varepsilon_{abc} \), \( k_a \) are \( 1 \times 3 \) matrices of the form
\[ k_1 = (i, 0, 0), \quad k_2 = (0, i, 0), \quad k_3 = (0, 0, i). \]  
(17)

Matrices (16) fulfill relations (5) and (6), iff matrices \( A_{n \times n} \), \( B_{n \times m} \) and \( C_{m \times n} \) satisfy the following relations (we omit the related subindices):
\[ AB = 0, \quad CA = 0, \quad A^2 + BC = 0. \]  
(18)

This system of matrix equations appears to be completely solvable, i.e. it is possible to find all non-equivalent indecomposable matrices \( A, B \) and \( C \) which satisfy relations (18). Any set of such matrices generates a representation of algebra \( hg(1, 3) \) whose basis elements are given by equations (16).

According to [5] indecomposable representations \( D(n, m, \lambda) \) of \( hg(1, 3) \) for scalar and vector fields are labelled by integers \( n, m \) and \( \lambda \). They specify dimensions of submatrices in (16) and the rank of matrix \( B \) respectively. As shown in [5], there exist ten
non-equivalent irreducible representations which correspond to matrices $A_{n \times n}$, $B_{n \times m}$ and $C_{m \times n}$ given in the Table 1.

In addition to the scalar representation whose generators are presented in formula (16) and Item 1 of the table, there exist nine vector representations corresponding to matrices enumerated in the Table 1, items 2-10. The related basis elements are the matrices of dimension $(3n + m) \times (3n + m)$ whose explicit forms are given in (16) and Items 1-10 of Table 1.

Table 1. Solutions of equations (18)

| No | $(n, m, \lambda)$ | Matrices $A$, $B$, $C$ |
|----|-------------------|-----------------|
| 1. | (0,1,0)           | $A$, $B$ and $C$ do not exist since $m = 0$ |
| 2. | (1,0,0)           | $A = 0, B$ and $C$ do not exist since $n = 0$ |
| 3. | (1,1,0)           | $A = 0$, $B = 0$, $C = 1$ |
| 4. | (1,1,1)           | $A = 0$, $B = 1$, $C = 0$ |
| 5. | (1,2,1)           | $A = 0$, $B = (1 \ 0)$, $C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |
| 6. | (2,0,0)           | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B$ and $C$ do not exist since $n = 0$ |
| 7. | (2,1,0)           | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C = (1 \ 0)$ |
| 8. | (2,1,1)           | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = (0 \ 0)$ |
| 9. | (2,2,1)           | $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ |
| 10. | (3,1,1)          | $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, $C = (1 \ 0 \ 0)$ |

The finite Galilei transformations of vector fields (which can be obtained by integrating the Lie equations for generators (16)) and examples of such fields can be found in paper [5]. Here we present two examples of Galilei vectors which have been in fact already used in Section 2.

The matrix five-vector $\hat{\gamma} = (\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4)$, whose components are given by equation (15), form a carrier space of representation $D(1, 2, 1)$. Under a Galilei boost its components transform as $\hat{\gamma}_m \to T(0, \mathbf{v}) \hat{\gamma}_m T^{-1}(0, \mathbf{v})$, where $T(0, \mathbf{v})$ are transformation matrices (14) for $\theta \equiv 0$. The explicit form of these transformations is:

$$\hat{\gamma}_0 \to \hat{\gamma}_0, \quad \hat{\gamma}_a \to \hat{\gamma}_a + v_a \hat{\gamma}_0, \quad a = 1, 2, 3, \quad \hat{\gamma}_4 \to \hat{\gamma}_4 + \mathbf{v} \cdot \hat{\gamma} + \frac{v^2}{2} \hat{\gamma}_0. \quad (19)$$
This five–vector $\hat{\gamma}$ is involved in the spinor equation (1) with $\beta_m = \eta \hat{\gamma}_m$. Another five-vector used there has the following form

$$p = (p^0, p^1, p^2, p^3, p^4), \quad (20)$$

where

$$p^0 = i \frac{\partial}{\partial t}, \quad p^a = i \frac{\partial}{\partial x^a}, \quad a = 1, 2, 3, \text{ and } p^4 = m. \quad (21)$$

The Galilei transformation law for $p$ is analogous to (19), namely

$$p^0 \rightarrow \tilde{p}^0 = p^0 + p \cdot \mathbf{v} + \frac{v^2}{2} p^4, \quad p \rightarrow \tilde{p} = p + \mathbf{v} p^4, \text{ and } p^4 \rightarrow \tilde{p}^4 = p^4. \quad (22)$$

Notice that relations (22) are in accordance with (2) and (3), namely, $\tilde{p}^n = \exp(-i f(t', x')) \times p^n \exp(i f(t', x'))$, $n = 0, 1, 2, 3, 4$, where $p^0 = i \frac{\partial}{\partial t'}$ and $p^a = i \frac{\partial}{\partial x^a}$. The convolution $\hat{\gamma}_m p^m$ is a Galilean scalar, and consequently equation (1) with matrices (3) is Galilei–invariant.

### 2.2.2 General wave equations for vector and scalar fields

Let us consider equation (1) and describe all admissible matrices $\beta_4$ compatible with the invariance conditions (7). We shall restrict ourselves to matrices $\eta_a, S_a$ belonging to representations described in Subsection 2.1 (see Table 1) or to direct sums of these representations. Then the general form of matrices $S_a$ and $\eta_a$ is again given by equations (16) where, however, matrices $A, B$ and $C$ can be reducible:

$$A = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & \\ & B_2 & \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & & \\ & C_2 & \end{pmatrix} \quad (23)$$

and are of dimensions $N \times N, M \times N$ and $N \times M$ respectively with $N$ and $M$ being arbitrary integers. The unit and zero matrices in the associated spin operator $S$ defined by equation (16) are of dimension $N \times N$ and $M \times M$ respectively.

The sets of matrices $(A_1, B_1, C_1), (A_2, B_2, C_2), \ldots$ are supposed to be indecomposable sets presented in Table 1. Any of them is labelled by a multiindex $q_i = (n_i, m_i, \lambda_i), \quad i = 1, 2, \ldots$

Matrices $\beta_4$ and $\beta_0$ should commute with $S$ and therefore have the following block diagonal form:

$$\beta_4 = \begin{pmatrix} R_{N \times N} & 0_{N \times M} \\ 0_{M \times N} & E_{M \times M} \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} F_{N \times N} & 0_{N \times M} \\ 0_{M \times N} & G_{M \times M} \end{pmatrix}. \quad (24)$$
Let us denote by \(|q, s, \nu >\) a vector belonging to a carrier space of representation \(D_q\) of algebra \(hg(1,3)\), where \(q = (n, m, \lambda)\) is a multiindex which labels a particular indecomposable representation as indicated in Table 1, \(s\) is a spin quantum number which is equal to 0,1 and index \(\nu\) specifies degenerate subspaces with the same fixed \(s\). Then taking into account that matrix \(\beta_4\) commutes with \(S_a\) its elements can be expressed as

\[
< q, s, \lambda | \beta_4 | q', s', \lambda' > = \delta_{s1} \delta_{s'1} R_{\lambda \lambda'} (q, q') + \delta_{s0} \delta_{s'0} E_{\lambda \lambda'} (q, q').
\]  

(25)

In order to find matrices \(R(q, q')\) and \(E(q, q')\) (whose elements are denoted by \(R_{\lambda \lambda'}\) and \(E_{\lambda \lambda'}\) respectively) expression (25) should be substituted into (7) and matrices \(\eta\) in form (16) together with relations (18) used. As a result we obtain the following condition

\[
(A')^2 R + R(A')^2 = A^\dagger RA' - C^\dagger EC',
\]  

(26)

where \(A, C, (A', C')\) are submatrices used in (18), which correspond to representation \(D_q\) (and \(D'_q\)).

Formulae (25) and (26) express all necessary and sufficient conditions for matrix \(\beta_4\) imposed by the Galilei invariance conditions (7). Suppose a matrix \(\beta_4\) (25) satisfying (26) be known, then the remaining matrices \(\beta_a\) \((a = 1, 2, 3)\) and \(\beta_0\) can be found by a direct use of the first and second relations (7). By this way we obtain

\[
< q, s, \lambda | \beta_0 | q', s', \lambda' > = \delta_{s1} \delta_{s'1} F_{\lambda \lambda'} (q, q') + \delta_{s0} \delta_{s'0} G_{\lambda \lambda'} (q, q'),
\]

\[
< q, s, \lambda | \beta_a | q', s', \lambda' > = i \delta_{s1} \delta_{s'1} H_{\lambda \lambda'} (q, q') s_a + \delta_{s1} \delta_{s'0} M_{\lambda \lambda'} (q, q') k^i_a + \delta_{s0} \delta_{s'1} N(q, q')_{\lambda \lambda'} k_a,
\]

(27)

where \(s_a\) are matrices of spin one, \(k_a\) are matrices (17) and \(F(q, q'), G(q, q'), H(q, q'), M(q, q'), N(q, q')\) are matrices defined by the following relations

\[
H = A^\dagger R - RA', \quad M = C^\dagger E - RB', \quad N = B^\dagger R - EC',
\]

\[
F = C^\dagger EC' + A^\dagger RA', \quad G = 2 B^\dagger RB' - B^\dagger C^\dagger E - EC'B'.
\]

(28)

Thus to derive a Galilei-invariant equation (11) for vector fields it is sufficient to choose a realization of algebra \(hg(1,3)\) from Table 1 or a direct sum of such realizations, and find the associated matrix \(\beta_4\) (25) whose block matrices \(R\) and \(E\) satisfy relations (26). Then the corresponding matrices \(\beta_0\) and \(\beta_a\) (27) can be found from (28).

All non-trivial solutions for matrices \(R\) and \(E\) are specified in the Appendix.

2.2.3 Equations for scalar and vector fundamental particles

In the previous section we have found all matrices \(\beta_m\) for which equation (11) is invariant with respect to vector and scalar representations of the homogeneous Galilei group. However, the Galilei invariance itself guaranties neither the consistency nor
the right number of independent equations. Moreover the spin content of the obtained equations as well as their possibilities to describe fundamental quantum mechanical systems have not been discussed.

A non-relativistic quantum system is said be fundamental if the space of its states forms a carrier space of an irreducible representation of the Galilei group $G(1,3)$. We shall call such systems "non-relativistic particles" or simply "particles".

For the group $G(1,3)$ there exist the following three invariant operators

\[ C_1 = M, \quad C_2 = 2MP_0 - P^2, \quad \text{and} \quad C_3 = (MJ - P \times G)^2. \]  

(29)

Here $P$, $J$ and $G$ are three-vectors whose components are specified by equations (11). Eigenvalues of these operators are associated with mass, internal energy and square of mass multiplied by eigenvalue of total spin operator respectively.

Galilei-invariant equation (11) is said to be consistent and describes a particle with mass $m$, internal energy $\varepsilon$ and spin $s$ if it has non-trivial solutions $\psi$ which form a carrier space for a representation of the Galilei group in which the following conditions

\[ C_1 \psi = m \psi, \quad C_2 \psi = \varepsilon \psi, \quad C_3 \psi = m^2 s(s + 1) \psi \]  

(30)

are true.

We stress that relations (30) generate extra conditions for $\beta$-matrices so that equation (11) with such $\beta_a$ guaranties the validity of equations (30).

Using definitions (11) we find the following forms of the invariant operators

\[ C_1 = Im, \quad C_2 = 2mp_0 - p^2, \]
\[ C_3 = m^2S^2 + m(S \times \eta) \cdot p - m(\eta \times S) \cdot p + p^2 \eta^2 - (p \cdot \eta)^2. \]  

(31)

We see that $C_3$ is a rather complicated second-order differential operator with matrix coefficients. In order to diagonalize this operator, we apply the similarity transformation

\[ \psi \rightarrow \psi' = W \psi, \quad C_a \rightarrow C'_a = WC_aW^{-1}, \quad a = 1, 2, 3, \]  

(32)

with

\[ W = \exp \left( \frac{i}{m} \eta \cdot p \right). \]  

(33)

Since $(\eta \cdot p)^3 = 0$ for representations $D(3,1,1)$ and $D(1,2,1)$ and $(\eta \cdot p)^2 = 0$ for the other representations described in Sections 2.1 and 2.2.1, $W$ is the second- or the first-order differential operator in $x$. Using conditions (17) we find that

\[ C'_1 = C_1, \quad C'_2 = C_2, \quad C'_3 = m^2S^2 \]  

(34)

and consequently for consistent equations $\psi'$ satisfies (30) with the transformed invariant operators (34), i.e.,

\[ (2mp_0 - p^2) \psi' = \varepsilon \psi' \]  

(35)
and
\[ S^2\psi' = s(s + 1)\psi'. \] (36)

In order to see when conditions (35) and (36) are true we transform equation (1) by means of \( W \). We obtain
\[ (\beta_0C_2 - \beta_42m^2)\psi' = 0 \] (37)
since in accordance with (7)
\[ L' = (W^{-1})^\dagger2m(\beta^\mu p_\mu - \beta_4m)W^{-1} = \beta_0(2mp_0 - \mathbf{p}^2) - \beta_42m^2. \]

In order (37) to be compatible with (35) the matrix \( M' = \beta_0\varepsilon - \beta_42m^2 \) (where \( \varepsilon \) is an arbitrary parameter) should be non-regular. Moreover, solutions of equation (37) must also satisfy condition (36) and form a carrier space of irreducible representation \( D(s) \) of the rotation group; consequently equation (37) must have \((2s+1)\) independent solutions.

As shown in Section 2.2, both matrices \( \beta_0 \) and \( \beta_4 \) has the block diagonal form given by equation (24). Thus equation (37) is decoupled to two subsystems
\[ (RC_2 - 2m^2F)\varphi_1 = 0, \] (38)
and
\[ (EC_2 - 2m^2G)\varphi_2 = 0 \] (39)
where the functions \( \varphi_1 \) and \( \varphi_2 \) are columns with \( 3n \) and \( m \) components respectively such that \( \psi' = \text{column}(\varphi_1, \varphi_2) \).

Equations (38) and (39) describe a particle of spin \( s = 1 \) (and internal energy \( \varepsilon \)) provided matrices \( R, F, E \) and \( G \) satisfy the following conditions:
\[ \text{Rank}||\varepsilon R - 2m^2F|| = n - 1, \quad \text{Rank}||\varepsilon E - 2m^2G|| = m. \] (40)

In the case of the equation for particle of spin \( s = 0 \) we have instead of (40) the following relations
\[ \text{Rank}||\varepsilon R - 2m^2F|| = n, \quad \text{Rank}||\varepsilon E - 2m^2G|| = m - 1. \] (41)

Thus to find Galilei invariant equations (11) for particle with a fixed spin we have to take into account equations discussed in the previous section as well as conditions (40) or (41) imposed on block components of matrices \( \beta_0 \) and \( \beta_4 \).

The explicit forms of matrices \( \beta_4 \), \( \beta_0 \) and \( \beta_a \) for the indecomposable representations of algebra \( h\mathfrak{g}(1, 3) \) are presented in the next subsection.
2.2.4 Equations invariant with respect to the indecomposable representations

Let us restrict ourselves to the indecomposable representations of algebra $hg(1, 3)$ specified in equation (16) and Table 1, and find the associated matrices $\beta_4$, $\beta_0$ and $\beta_a$ which appear in the Galilei invariant equations (11). Taking into account that $A' = A$, $B' = B$ and $C' = C$ in (26), where $A$, $B$ and $C$ are matrices given in Table 1, we easily find the associated block matrices $R$, $E$ and consequently all matrices (28). To simplify matrices $\beta_m$ ($m = 0, 1, \cdots, 4$) we used equivalence transformations $\beta_m \to W^\dagger \beta_m W$, where $W$ are invertible matrices commuting with the Galilei boost generators $\eta_a$.

It appears that non-trivial solutions for $\beta_4$ (and consequently also for $\beta_0, \beta_a$) exist only for representations $D(1, 1, 0), D(2, 1, 0), D(2, 2, 1)$ and $D(3, 1, 1)$. They have the following form:

**Representation $D(1, 1, 0)$**:

$$\beta_4 = \begin{pmatrix} I_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 2 \end{pmatrix}, \quad \beta_a = i \begin{pmatrix} 0_{3 \times 3} & -k_a^\dagger \\ k_a & 0 \end{pmatrix},$$  

**Representation $D(2, 1, 0)$**:

$$\beta_4 = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 1 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 2I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 0 \end{pmatrix},$$  

$$\beta_a = i \begin{pmatrix} 0_{3 \times 3} & s_a & k_a^\dagger \\ -s_a & 0_{3 \times 3} & 0_{3 \times 1} \\ -k_a & 0_{1 \times 3} & 0 \end{pmatrix},$$  

**Representation $D(2, 2, 1)$**:

$$\beta_4 = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 0 & 0 \\ 0_{1 \times 3} & 0_{1 \times 3} & 0 & 1 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 2I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 2 & 0 \\ 0_{1 \times 3} & 0_{1 \times 3} & 0 & 0 \end{pmatrix},$$  

$$\beta_a = i \begin{pmatrix} 0_{3 \times 3} & s_a & 0_{3 \times 1} & k_a^\dagger \\ -s_a & 0_{3 \times 3} & -k_a^\dagger & 0_{3 \times 1} \\ 0_{1 \times 3} & k_a & 0 & 0 \\ -k_a & 0_{1 \times 3} & 0 & 0 \end{pmatrix},$$  

**Representation $D(3, 1, 1)$**:

$$\beta_4 = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & \nu I_{3 \times 3} & 0_{3 \times 1} \\ 0_{3 \times 3} & \nu I_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} \\ \nu I_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & -\nu \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0 \end{pmatrix},$$
\[ \beta_a = i \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & s_a & 0_{3 \times 1} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & -k_a^\dagger \\ -s_a & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & k_a & 0_{1 \times 3} & 0 \end{pmatrix} \]  \hspace{1cm} (45)

Here \( I_{k \times k} \) and \( 0_{k \times r} \) are the unit and zero matrices of dimensions \( k \times k \) and \( k \times r \) and \( \nu \) is an arbitrary non-zero parameter.

Thus there exist four equations (1) for spinor and vector fields which are invariant with respect to indecomposable representations of the homogeneous Galilei group. The associated matrices \( \beta_\mu \) and \( \beta_4 \) are given by formulae (42)–(45). All these equations admit a Lagrangian formulation with the corresponding Lagrangian of the following standard form

\[ L = \frac{1}{2} \psi^\dagger (\beta_\mu p^\mu + \beta_4 m) \psi + h.c.. \]  \hspace{1cm} (46)

The equations (1) with \( \beta \)-matrices specified in (42), (43), and (45) are equivalent to ones analyzed in papers [9] and [10]. However, at the best of our knowledge equation (1) with \( \beta \)-matrices (44) is new. Let us remark that these matrices satisfy neither relations (40) nor (41) and the related equation describes a Galilean quantum mechanical system whose spin can take two values: \( s = 1 \) and \( s = 0 \). Notice that matrices \( \beta_a \) satisfying (40) or (41) do not exist for representation \( D(2, 2, 1) \).

There exist also a number of wave equations (1) invariant with respect to decomposable representations of \( hg(1, 3) \). We shall discuss some of them in detail in the sections which follow. A general description of all such equations is given in Sections 2.4, 2.5 and in the Appendix.

3 Galilean analogues of some basic relativistic equations

All relativistic wave equations have their Galilei invariant counterparts. For instance, the Galilean analogue of the Dirac equation is the Lévy-Leblond equation which was discussed in Section 2.1.

In the present section we consider some basic relativistic wave equations for particles with higher spin. Using our knowledge of scalar, spinor and vector representations and of invariants of the homogeneous Galilei group we construct Galilean forms of these equations.

The Galilean analogues of some basic relativistic equations form basic non-relativistic equations. They are, however, not obtained by direct non-relativistic limits of basic relativistic equations (but of some other ones).
3.1 The Galilean second order Proca equation

The relativistic Proca equation for a vector field $\psi^\mu$ can be written as \cite{15}:

$$W^\mu \equiv (p_\nu p^\nu - m^2) \psi^\mu - p^\mu p^\nu \psi_\nu = 0, \quad (47)$$

where $p_\nu = i \frac{\partial}{\partial x^\nu}, \ \nu = 0, 1, 2, 3$, and the raising and lowering of covariant indices $\mu, \nu$ is made by means of the relativistic metric tensor

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1). \quad (48)$$

Reducing the l.h.s. of equation (47) by $p_\mu$ we obtain the following consequence

$$p_\mu \psi^\mu = 0 \quad (49)$$

and then, putting it into (47),

$$\left( p_\nu p^\nu - m^2 \right) \psi^\mu = 0. \quad (50)$$

Consequently, the four-vector $\psi^\mu$ satisfies both the Klein-Gordon equation (50) and the four-divergenceless condition (49). The latest one reduces the number of independent components of $\psi^\mu$ to 3 as it is required for a wave function of a spin-one (vector) particle.

Notice that contracting the representation $D(\frac{1}{2}, \frac{1}{2})$ of the Lorentz group to the representation $D_1(1, 1, 1)$ of the homogeneous Galilei group equation (47) can be contracted to the direct sum of Schrödinger equations for $\psi^1, \psi^2, \psi^3$ while the zero component of $\psi$ be expressed as

$$\psi^0 = \frac{1}{m} p_\alpha \psi^\alpha \quad (51)$$

Such Galilean system is completely decoupled with respect to physical components $\psi^1, \psi^2, \psi^3$ and so is not too interesting.

To construct a consistent Galilean analogue of the Proca equation we start with a wave function which transforms as a five-vector with respect to the Galilei group transformations. Namely, the Galilean Proca equation is found to be of the following form

$$W^m \equiv p_\alpha p_\beta \hat{\psi}^m - p_\alpha p_\beta \hat{\psi}^m + \lambda \delta^{m0} m^4 \hat{\psi}^4 = 0, \quad m, n = 0, 1, 2, 3, 4, \quad (52)$$

where $\lambda$ is an arbitrary parameter and $\delta^{m0}$ is a Kronecker symbol.

Equation (52) has the following features distinct from those of (47):

- the indices $m$ and $n$ take the values 0, 1, 2, 3, 4 while in (47) the indices $\mu$ and $\nu$ run from 0 to 3;

- the relativistic four-gradient $p_\nu$ is replaced by the Galilean five-vector $p$ (20) whose transformation properties are given by equation (21);
raising and lowering of covariant indices $m, n$ is made by using the Galilean metric tensor (14) instead of (48), thus for example $p_m \hat{\psi}^m = p^4 \hat{\psi}^0 + p^0 \hat{\psi}^4 - p^1 \hat{\psi}^1 - p^2 \hat{\psi}^2 - p^3 \hat{\psi}^3$, and $p_m p^n = 2p^0 p^4 - (p^1)^2 - (p^2)^2 - (p^3)^2 = 2mp^0 - p^2$;

- equation (52) is invariant w.r.t. the Galilei transformations (2) provided the wave function $\hat{\psi}^m$ cotransforms as a Galilean five-vector, i.e., $\hat{\psi}^4 \rightarrow \exp(i f(t, x)) \hat{\psi}^4$, $\hat{\psi}^a \rightarrow \exp(i f(t, x))(\hat{\psi}^a + v^a \hat{\psi}^4)$, $\hat{\psi}^0 \rightarrow \exp(i f(t, x))(\hat{\psi}^0 + v^a \hat{\psi}^a + \frac{v^2}{2} \hat{\psi}^4)$, in other words $\hat{\psi}^m$ should transforms as a vector from the carrier space of representation $D(1, 2, 1)$ of the homogeneous Galilei group.

Equation (52) admits, like (47), a Lagrangian formulation and describes a particle with spin 1. The corresponding Lagrangian has the following form:

$$
L = (p_m \hat{\psi}^m - p_n \hat{\psi}^m)^* (p^m \hat{\psi}^m - p^n \hat{\psi}^n) - (p_m \hat{\psi}^m)^* p^m \hat{\psi}^m + (p^m \hat{\psi}^m)^* p_n \hat{\psi}^n - \lambda m^2 \hat{\psi}^4, \tag{53}
$$

where $p^m$ are components of five-vector $p$ (20) and the asterisk is used to denote complex conjugation.

The system of equations (52) is coupled and can be used to describe spin effects if we introduce a minimal interaction with an external field, see Section 4.2. In absence of interaction it is equivalent to Schrödinger equations for three vector components of $\hat{\psi}^m$. Indeed, reducing five-vector $W^m$ (whose components are given by equation (52)) by $p_m$ we immediately obtain the consequence: $\lambda m^2 \hat{\psi}^4 = 0$, i.e., $\hat{\psi}^4 = 0$. Then considering equation (52) for $m = 4$ we conclude that $mp_n \hat{\psi}^n = 0$ and equation (52) is reduced to the following system

$$
(2mp_0 - p^2)\hat{\psi}^m = 0, \\
mp\hat{\psi}^0 + p_a \hat{\psi}^a = 0, \text{ and } \hat{\psi}^4 = 0, \tag{54}
$$

where $a = 1, 2, 3$.

In accordance with (54) the wave function $\hat{\psi}^m$ satisfies the Schrödinger equation componentwise and has three non-zero components in the rest frame, which transform as a three-vector under rotations. Consequently equation (52) describes a Galilean particle with spin $s = 1$.

Notice that the system of equations (52) includes a five-component wave function and thus cannot be obtained as a simple non-relativistic approximation of the Proca equation. This is in accordance with the fact that some of indecomposable representations of the homogeneous Galilei algebra (and representation $D(1, 2, 1)$ in particular) cannot be obtained by contraction of irreducible representations of the Lorentz algebra, see [5] for more details.

3.2 The Galilean Duffin-Kemmer equation

The relativistic Duffin-Kemmer equation can be written as [19]

$$
(\beta \mu p^\mu - \kappa)\psi = 0. \tag{55}
$$
Here $\beta_\mu (\mu = 0, 1, 2, 3)$ are square matrices which satisfy the Duffin-Kemmer-Petiau (DKP) algebra
\[ \beta_\mu \beta_\nu \beta_\sigma + \beta_\sigma \beta_\nu \beta_\mu = 2g_{\mu \nu} \beta_\sigma + 2g_{\sigma \nu} \beta_\mu, \tag{56} \]
where $g_{\mu \nu}$ is a metric tensor given in (48).

The ring of DKP matrices has two non-trivial irreducible representations with matrices of dimension $5 \times 5$ and $10 \times 10$. The associated equations (55) describe relativistic particles with spin 0 and 1 correspondingly.

Considering matrices (45) we conclude that the related matrices
\[ \tilde{\beta}_\mu = \eta \beta_\mu, \quad 0, 1, 2, 3, \quad \text{and} \quad \tilde{\beta}_4 = \eta \beta_4 - \nu I_{10 \times 10}, \tag{57} \]
where $\eta$ is an invertible matrix
\[ \eta = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} \\ 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & -1 \end{pmatrix} \tag{58} \]
satisfying relations (56), where, however, $\mu, \nu, \sigma = 0, 1, 2, 3, 4$ and $g_{\mu \nu}$ has “Galilean form” (14). Following [20] we say that such relations define a Galilean DKP algebra.

Equation (1) with matrices $\tilde{\beta}_\mu, \tilde{\beta}_4$ which satisfy the Galilean DKP algebra is called the Galilean Duffin-Kemmer equation. Notice that this equation coincides with equation (1) multiplied by the invertible matrix $\eta$ provided matrices $\beta_\mu, \beta_4$ have the form (45). For the first time the Galilean Duffin-Kemmer equation was considered in [10].

The Galilean Duffin-Kemmer equation for spin zero particle can also be written in the form (1) where $\beta_\mu$ and $\beta_4$ are $6 \times 6$ matrices satisfying a Galilean DKP algebra. These matrices can be chosen up to equivalence in the form
\[ \beta^0 = -e_{56} - e_{61}, \quad \beta^a = e_{61+a} - e_{1+a,6}, \quad a = 1, 2, 3, \\
\beta^4 = e_{15} - e_{22} - e_{33} - e_{44} + e_{51} + e_{66} + e_{16} + e_{65}, \tag{59} \]
where $e_{ab}$ denotes a $6 \times 6$-dimensional matrix with 1 at the $ab$ entry, and 0 everywhere else.

Notice that matrices (59) multiplied by the following hermitizing matrix
\[ \hat{\eta} = e_{15} - e_{22} - e_{33} - e_{44} + e_{51} + e_{66} \]
became a particular case of our general solution for $\beta$–matrices (presented in equations (25), (27), (28) and in the Appendix) for a direct sum of representations $D(1, 2, 1)$ and $D(0, 0, 0)$.

Let us return to the relativistic Duffin-Kemmer equation (55) and contract it directly to non-relativistic (i.e., Galilei invariant) approximation. It is convenient to start with the following tensorial form of this equation [15]
\[ p^\mu \Psi^\nu - p^\nu \Psi^\mu = \kappa \Psi^{\mu \nu}, \]
\[ p_\mu \Psi^\nu = \kappa \Psi^\mu \tag{60} \]
where $\Psi^\mu$ and $\Psi^{\mu\nu}$ are four-vector and skew-symmetric spinor which transform according to the representation $D(\frac{1}{2}, \frac{1}{2}) \oplus D(1, 0) \oplus D(0, 1)$ of the Lorentz group.

It was shown in papers [5] and [6] how this representation can be contracted to representation $D_1(3, 1, 1)$ of the homogeneous Galilei group. This contraction can be used to reduce equation (60) to a Galilei invariant form. To do this it is necessary:

- To choose the following new dependent variables
  \[ R^a = -\frac{1}{2}(\Psi^{0a} + \Psi^a), \quad N^a = \Psi^{0a} - \Psi^a, \quad W^c = \frac{1}{2}\varepsilon^{abc}\Psi_{bc}, \quad B = \Psi^0, \]
  which, in accordance with (60), satisfy the following equations:

  \[ 2(p^0 - \kappa)R^a + p^a B + \varepsilon^{abc}p_b W_c = 0, \]
  \[ (\bar{p}^0 + \kappa)N^a - \varepsilon^{abc}p_b W_c + p^a B = 0, \]
  \[ \varepsilon^{abc}p_b (R_c + \frac{1}{2}N_c) = \kappa W^a, \]
  \[ \frac{1}{2}p_a N^a - p_a R^a = \kappa B; \]

- To act on variables $R^a, N^a, W^a$ and $B$ by a diagonal contraction matrix. This action yields the change
  \[ R^a = \tilde{R}^a, \quad N^a = \varepsilon^2 \tilde{N}^a, \quad W^a = \varepsilon \tilde{W}^a, \quad B = \varepsilon \tilde{B} \]
  where $\varepsilon$ is a small parameter associated with the inverse speed of light;

- To change relativistic four-momentum $p^\mu$ and mass $\kappa$ by their Galilean counterparts $\bar{p}^a, \bar{p}^0$ and $m$ where
  \[ \bar{p}^a = \varepsilon^{-1}p^a, \quad \bar{p}^0 = p_0 - \kappa \text{ and } m = \frac{1}{2}(p_0 + \kappa)\varepsilon^{-2}; \]

- In each equation in (61) keep only terms which are multiplied by lowest present powers of $\varepsilon$.

As a result we obtain the following equations for $\tilde{R}^a, \tilde{N}^a, \tilde{W}^a$ and $\tilde{B}$:

\[ 2\bar{p}^0 \tilde{R}^a + \bar{p}^a \tilde{B} + \varepsilon^{abc}\bar{p}_b \tilde{W}_c = 0, \]
\[ \varepsilon^{abc}\bar{p}_b \tilde{R}_c = m \tilde{W}^a, \]
\[ \bar{p}_a \tilde{R}^a + m \tilde{B} = 0, \]
\[ 2m \tilde{N}^a = \varepsilon^{abc}\bar{p}_b \tilde{W}_c - \bar{p}^a \tilde{B}. \]

The system (63) is nothing else but the Galilei-invariant equation (11) with matrices (43) written componentwise. Relation (64) expresses the extra component $\tilde{N}^a$ via derivatives of the essential ones, i.e. of $W^a$ and $B$.

Thus the Galilean analogue of the Duffin-Kemmer equation (11), (45) cannot be obtained as a non-relativistic limit of the relativistic Duffin-Kemmer equations (55), (56) but is a specific generalization of it. The relativistic counterpart of equations (11), (45) is a specific generalization of (55), (56) which will be studied in a separate publication.
3.3 The Galilean Rarita-Schwinger equation

Till now we have used our knowledge of indecomposable representations of algebra $hg(1,3)$ for spinor, scalar and vector fields to construct wave equations for fields of spin $\tilde{s} \leq 1$. In this section we derive Galilean invariant equations for the field transforming as a direct product of spin $1/2$ and spin 1 fields. The relativistic analogue of such system is the famous Rarita-Schwinger equation.

The relativistic Rarita-Schwinger equation for a particle with spin $s = \frac{3}{2}$ is constructed by using a vector-spinor wave function $\Psi^\mu_\alpha$, where $\mu = 0, 1, 2, 3$ and $\alpha = 1, 2, 3, 4$ are vector and spinor indices respectively. Moreover, $\Psi^\mu_\alpha$ is supposed to satisfy the equation

$$\left(\gamma^\nu p_\nu - m\right) \Psi^\mu - \gamma^\mu p_\nu \Psi^\nu - p^\mu \gamma_\nu \Psi^\nu + \gamma^\mu \left(\gamma^\rho p^\rho + m\right) \gamma_\sigma \Psi^\sigma = 0,$$  \hspace{1cm} (65)

where $\gamma^\mu$ are the Dirac matrices acting on the spinor index $\alpha$ of $\Psi^\mu_\alpha$ which we do not write explicitly.

Reducing the left hand side of equation (65) by $p_\mu$ and $\gamma^\mu$ we obtain the following expressions:

$$\gamma_\mu \Psi^\mu = 0, \text{ and } p_\mu \Psi^\mu = 0,$$  \hspace{1cm} (66)

which reduce the number of independent components of $\Psi^\mu_\alpha$ to 8 as required for a wave function of a relativistic particle with spin $3/2$.

Using our knowledge of invariants for Galilean vector fields from [5], [6] we can easily find a Galilean analogue of equation (65). Like in the case of the Galilean Proca equation we begin with a five-vector $\hat{\Psi}^m$, $m = 0, 1, 2, 3, 4$ which has, in addition, a bi-spinorial index which we do not write explicitly. Thus our Galilei invariant equation can be written in the following form:

$$\hat{\gamma}^m p^a \hat{\Psi}^a - \hat{\gamma}^m p_n \hat{\Psi}^n - p^m \hat{\gamma}^n \hat{\Psi}^n + \hat{\gamma}^m \hat{\gamma}^n p^a \hat{\gamma}^b \hat{\Psi}^b + \lambda \delta^{mn} m \hat{\Psi}^4 = 0.$$  \hspace{1cm} (67)

Here, $\hat{\gamma}^m$ are the Galilean $\gamma$-matrices (15), $p^a$ is the Galilean “five-momentum” (20), $\lambda$ is an arbitrary non-vanishing parameter and raising and lowering of indices $m$ and $n$ is made by using the Galilean metric tensor (14).

Reducing (67) by $p_m$ we receive $\lambda m^2 \hat{\Psi}^4 = 0$, i.e., $\hat{\Psi}^4 = 0$. Whereas, reducing (67) by $\hat{\gamma}_m$ we obtain

$$p_n \hat{\Psi}^a = \hat{\gamma}_m p^a \hat{\gamma}_m \hat{\Psi}^a.$$  \hspace{1cm} (68)

Finally, comparing (68) with equation (67) for $m = 4$ we find the following consequences of equation (67):

$$\hat{\gamma}_n p^a \hat{\Psi}^\sigma = 0, \quad \sigma = 0, 1, 2, 3,$$  \hspace{1cm} (69)

$$m \hat{\Psi}_0^0 - p^a \hat{\Psi}^a = 0, \quad a = 1, 2, 3,$$  \hspace{1cm} (70)

$$\hat{\gamma}_0 \hat{\Psi}^0 + \hat{\gamma}_a \hat{\Psi}^a = 0, \quad \text{and } \hat{\Psi}^4 = 0.$$  \hspace{1cm} (71)
On the other hand equation (67) follows from (69)–(71), so that equations (67) and (69)–(71) are equivalent.

In accordance with (69) any component of $\Psi^m$ satisfies the Lévy-Leblond equation, compare with Section 2.1. Let us prove that equations (69)–(71) describe indeed a particle with spin $s=3/2$.

It follows from (70)–(71) that $\hat{\Psi}^0 = \hat{\Psi}^4 = 0$ in the rest frame. Using the realization (15) for $\hat{\gamma}$-matrices we conclude that, in accordance with (71), $\hat{\Psi}^a$ satisfies the equation

$$\hat{\sigma}^a \hat{\Psi}^a = 0,$$

where $\hat{\sigma}^a = I_{2 \times 2} \otimes \sigma_a$. (72)

It follows from (72) that function $\hat{\Psi}^a$ satisfies conditions (35) and (36) with $s = 3/2$ since the total spin operator $S$ is a sum of operators of spin one and of spin one–half:

$$S_a = \hat{s}_a + \frac{1}{2} \hat{\sigma}_a, \quad \hat{s}_a = I_{2 \times 2} \otimes s_a.$$ (73)

Hence

$$S^2 = \frac{11}{4} + \hat{s} \cdot \hat{\sigma}. \quad (74)$$

Let $\tilde{\Psi}$ denotes the column $(\hat{\Psi}^1, \hat{\Psi}^2, \hat{\Psi}^3)$. In accordance with (74) the condition $S^2 \tilde{\Psi} = s(s + 1) \tilde{\Psi}$ reduces to the form

$$\hat{\Psi}_a - \frac{i}{2} \varepsilon_{abc} \hat{\sigma}_b \hat{\Psi}_c = 0,$$ (75)

for $s = 3/2$ provided we use the representation with $(s_a)_{bc} = i \varepsilon_{abc}$, where $\varepsilon_{abc}$ is a totally antisymmetric unit tensor.

Comparing (72) with (75) we conclude that these equations are completely equivalent since multiplying (72) by $\hat{\sigma}_a$ we obtain (75) and multiplying (75) by $\hat{s}_a$ and contracting it with respect to index $a$ we receive (72). Thus indeed equation (67) describes a Galilean particle with spin $s = 3/2$.

Like equation (52) the equation (67) admits a Lagrangian formulation. The corresponding Lagrangian can be written as

$$L = \frac{1}{2} \bar{\Psi}_m \hat{\gamma}_n \hat{\gamma}^n \hat{\Psi}_m - \bar{\Psi}_m \hat{\gamma}_n \hat{\gamma}^n \hat{\Psi}_n - \bar{\Psi}_m \hat{\gamma}_n \hat{\gamma}^n \hat{\Psi}_n + \bar{\Psi}_m \hat{\gamma}_n \hat{\gamma}^n \hat{\Psi}_n + \lambda m \bar{\Psi}_0 \hat{\Psi}_4 + h.c., \quad (76)$$

where $\bar{\Psi}_m = \hat{\Psi}_m^\dagger \eta$ and $\eta$ is the hermitizing matrix (12).

We note that it is possible to find a Galilei-invariant equation for particle with spin 3/2 starting with the relativistic equation (65) and making the Inönü-Wigner contraction of the representation of the Lorentz group realized on solutions of the Rarita-Schwinger equation. However the related theory appears to be rather cumbersome in comparison with our Galilei-invariant Rarita-Schwinger equation (67).
4 Equations for charged particles interacting with an external gauge field

4.1 Minimal interaction with an external field

We have described Galilei invariant equations (1) for free particles with spins 0, 1/2, 1 and 3/2. These equations have admitted Lagrangian formulation (46), so that to generalize them to the case of particles interacting with an external field means as usually to apply the minimal interaction principle, i.e., to make the following change in the Lagrangian

\[ p^\mu \rightarrow \pi^\mu = p^\mu - eA^\mu, \quad p^4 \rightarrow \pi^4 = p^4 - eA^4, \]

(77)

where \( A^\mu \) and \( A^4 \) are components of a vector-potential of the external field, \( p^4 = m \) and \( e \) is a particle charge.

Thus we come to the Lagrangian

\[ L = \frac{1}{2} \psi^\dagger \left( \beta_\mu \pi^\mu + \beta_4 \pi^4 \right) \psi(x,t) + h.c. \]

(78)

It is important to note that change (77) is compatible with the Galilei invariance provided the components \((A^0, A^1, A^2, A^3, A^4)\) of the vector-potential transform as a Galilean five-vector, i.e., as

\[ A^0 \rightarrow A^0 + v \cdot A + \frac{v^2}{2} A^4, \quad A \rightarrow A + v A^4, \quad A^4 \rightarrow A^4. \]

(79)

Formula (78) presents the most general Lagrangian for the Galilean Bhabha equation describing a particle interacting with an external gauge field via minimal interaction. On the other hand, in contrast to the relativistic case, there are many other possibilities. Here we shall mention several of them concerning various types of Galilean massless fields.

- First we consider a four-vector potential \( A = (A^0, A^1) \) which transforms according to the representation \( D(1,1,1) \), i.e., via the following expression under Galilei boost:

\[ A^0 \rightarrow A^0 + v \cdot A, \quad A \rightarrow A. \]

(80)

Such potential corresponds to a "magnetic" limit of the Maxwell equations, see [8], [21].

- The second possibility concerns a four-vector potential \( A = (A^1, A^4) \) which transforms according to the representation \( D(1,1,1) \), i.e.,

\[ A^4 \rightarrow A^4, \quad A \rightarrow A + v A^4. \]

(81)

It corresponds to an "electric" limit of the Maxwell equations [21].

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• The third and fourth cases consists of either a three-vector potential $A$ which transforms according to the representation $D(1, 0, 0)$ or the scalar potential $A^4$ which transforms according to the representation $D(0, 1, 0)$. Both these potentials are invariant with respect to Galilei boosts.

• The fifth possibility is formed by the above mentioned potentials restricted by some Galilei invariant conditions. For example it is possible to impose on potential $A = (A, A^4)$ the condition $\nabla A^4 = 0$. The complete description of all Galilean vector-potentials will be given elsewhere.

The corresponding equations for a charged particle interacting with an external field are the Euler equations derived from the Lagrangian (78) in which the non-zero components of $A$ specified in the above Items.

Thus there are many possibilities how to generalize Lagrangians (46) for particles interacting with external fields via minimal interaction. However, it is also possible to introduce interaction via other means, e.g., via the anomalous (Pauli) term.

4.2 The Galilean Proca equation for interacting particles

Let us consider now the Galilean Proca equation for a charged particle interacting with an external e.m. field in magnetic limit. For this purpose we introduce a minimal interaction into the Lagrangian (76), i.e., make the change $p^n \rightarrow \pi^n$ therein.

The corresponding Euler-Lagrange equation is then of the form

$$\hat{W}^n \equiv \pi^a \pi^n \hat{\psi}^m + 2ieF^{an} \hat{\psi}_n - \pi^a \pi_n \hat{\psi}^n + \lambda \delta^m_0 m \hat{\psi}^4 = 0,$$

(82)

where $e F^{an} = -i [\pi^a, \pi^n]$, so that

$$F^{0a} = -F^{a0} = -E_a, \quad F^{ab} = \varepsilon^{ab} c H_c, \quad F^{4\mu} = -F^{\mu4} = 0.$$

(83)

To evaluate equations (82) we change $\hat{\psi}^m \rightarrow \psi^m$, namely:

$$\hat{\psi}^4 = \psi^4, \quad \hat{\psi} = \psi + \pi^4 \hat{\psi}, \quad \hat{\psi}^0 = \psi^0 + \frac{1}{m} \pi \cdot \psi + \frac{\pi^2}{2m^2} \psi^4.$$

(84)

This is nothing else then the specified later similarity transformation $\hat{\psi} \rightarrow \psi = W \hat{\psi}$, where $W$ is operator (100) with the appropriate matrices $\eta$.

Substituting (83) and (84) into (82) we obtain the following system of equations

$$\hat{W}^4 = \frac{1}{2} (2m\pi^0 - \pi^2) \psi^4 - m^2 \psi^0 = 0,$$

$$\hat{W} = (2m\pi^0 - \pi^2)(\psi + \frac{1}{m} \pi \psi^4) + 2ieH \times \psi - (2ieF + e^2 J) \psi^4 - \pi (m\psi^0 + \frac{1}{2} (2m\pi^0 - \pi^2) \psi^4)$$

$$- \pi (m\psi^0 + \frac{1}{2} (2m\pi^0 - \pi^2) \psi^4) = 0,$$

$$\hat{W}^0 = (2m\pi^0 - \pi^2)(\psi^0 + \frac{1}{m} \pi \cdot \psi + \frac{\pi^2}{2m^2} \psi^4) + \lambda m^2 \psi^4 - 2ie E \cdot (\psi + \frac{1}{m} \pi \psi^4) - \pi^0 (m\psi^0 + \frac{1}{2} (2m\pi^0 - \pi^2) \psi^4) = 0$$

(87)
where we use the following notation $\text{div} \mathbf{E} = ej^0$ and $\text{curl} \mathbf{H} = ej$.

In accordance with (85) $\psi^b$ can be expressed via derivatives of $\psi^4$. In this way the following coupled system of equations can be derived from (86) and (87):

\[
\left( \pi^0 - \frac{\pi^2}{2m} + \frac{e}{m} s \cdot \mathbf{H} \right) \psi = \frac{e^4}{2\lambda m^5} j (j \cdot \psi - \frac{1}{m}(j^0 m - j \cdot \pi) \psi^4),
\]

and

\[
(\lambda m^4 + e^2 (j^0 m - j \cdot \pi)) \psi^4 = e^2 m j \cdot \psi.
\]

The r.h.s. of equation (88) includes a very small multiplier $\frac{e^4}{2\lambda m^5}$ and therefore the wave function $\psi$ satisfies the Schrödinger-Pauli equation with a very high accuracy. Moreover, a gyromagnetic ratio (i.e., the coefficient at the term $\frac{e}{2m} s \cdot \mathbf{H}$) has the desired value which is equal to 2.

4.3 Galilean Bhabha equations with minimal and anomalous interactions

Taking Lagrangian (78) we can derive the following equation for a charged particle interacting with an external field

\[
L \Psi \equiv \left( \beta_\mu \pi^\mu + \beta_4 \pi^4 \right) \Psi(x, t) = 0.
\]

Let us consider now equation (90) with general matrices $\beta_\mu$ and $\beta_4$. If we restrict ourselves to a vector-potential of magnetic type, i.e., to $A = (A^0, A, 0)$, then

\[
\pi^0 = p^0 - eA^0, \quad \pi^a = p^a - eA^a, \quad \pi^4 = m.
\]

Like free particle equations (11) the equations (90) are Galilei-invariant provided matrices $\beta_\mu, \beta_4$ satisfy conditions (7). In addition the vector-potential of an external field should transform according to properties (80).

Following Pauli [22] we generalize our equation (90) by adding to it an interaction terms linear in electromagnetic field strengths and consider the equation:

\[
(\beta_\mu \pi^\mu + \beta_4 m + F) \psi = 0, \text{ where } F = \frac{e}{m} (A \cdot \mathbf{H} + G \cdot \mathbf{E}).
\]

Here, $\mathbf{A}$ and $\mathbf{G}$ are matrices determined by requirement of Galilei invariance, i.e., by the conditions that $\mathbf{A} \cdot \mathbf{H}$ and $\mathbf{G} \cdot \mathbf{E}$ should be Galilean scalars.

In paper [5] we find the most general form of the Pauli interaction which can be introduced into the Lévy-Leblond equation for a particle of spin 1/2. Finding the general Pauli interaction for other Galilean particles is a special problem for any equation previously considered. Here we restrict ourselves to a systematic analysis of the Pauli terms which are true for any Galilean Bhabha equation.
First we shall prove the following statement.

**Lemma.** Let $S_a, \eta_a$ be matrices which realize a representation of algebra $hg(1, 3)$, $\Lambda$ be a matrix satisfying the conditions

$$ S_a \Lambda = \Lambda S_a, \quad \eta_a^\dagger \Lambda = \Lambda \eta_a $$  \hspace{1cm} (93)  

and

$$ E_a = -\frac{\partial A_0}{\partial x_a} - \frac{\partial A_a}{\partial t}, H_a = \varepsilon_{abc} \frac{\partial A_b}{\partial x_c} $$

be vectors of the electric and magnetic field strength respectively. Then matrices

$$ F_1 = \Lambda (s \cdot H - \eta \cdot E) $$ \hspace{1cm} (94)  

and

$$ F_2 = \Lambda \eta \cdot H $$  \hspace{1cm} (95)  

are invariant with respect to the Galilei transformations provided the vector-potential $A$ is transformed in accordance with Galilean law (79).

**Proof.** First we note that matrices (94) and (95) are scalars with respect to rotations. Then, starting with transformation laws (2) and (79) we easily find that under a Galilei boost the vectors $E$ and $H$ co-transform as

$$ E \rightarrow E - \mathbf{v} \times H, \quad H \rightarrow H. $$(96)

On the other hand transformation laws for matrices $\Lambda S$ and $\Lambda \eta$ can be found using the exponential mapping of boost generators $\mathbf{G}$ given in equation (4):

$$ S \rightarrow \exp(i \mathbf{G}^\dagger \cdot \mathbf{v}) \Lambda S \exp(-i \mathbf{G} \cdot \mathbf{v}) = \Lambda \exp(i \eta \cdot \mathbf{v}) \exp(-i \eta \cdot \mathbf{v}) = \Lambda (s + \mathbf{v} \times \eta), \quad \eta \rightarrow \exp(i \mathbf{G}^\dagger \cdot \mathbf{v}) \Lambda \eta \exp(i \mathbf{G} \cdot \mathbf{v}) = \Lambda \exp(i \eta \cdot \mathbf{v}) \exp(-i \eta \cdot \mathbf{v}) = \Lambda \eta. $$

(97)

One easily verifies that transformations (96) and (97) leave matrices $F_1$ and $F_2$ invariant. Q. I. D.

In accordance with the Lemma there are many possibilities how to generalize equations previously considered to the case of anomalous interaction. Indeed, for any Galilean Bhabha equation there are matrices $S_a, \eta_a$ and $\Lambda$ for which conditions (93) are satisfied. For example, we can choose $\Lambda = \beta_0$. In addition, for many cases there exist a hermitizing matrix $\eta = \Lambda$ satisfying (93), see, e.g., equations (12) and (58). For particular representations of algebra $hg(1, 3)$ there are also other solutions of equations (93).

Thus the Pauli terms for the Galilean Bhabha equations can be chosen in the form (94) or (95) or, more generally, as a linear combination of both, $F_1$ and $F_2$. As a result we obtain the following equation,

$$ \left( \beta_{\mu} \pi^\mu + \beta_4 m + \lambda_1 \frac{e}{m} \beta_0 \eta \cdot H + \lambda_2 \frac{e}{m} \beta_0 (S \cdot H - \eta \cdot E) \right) \psi = 0, $$

(98)
where $\lambda_1$ and $\lambda_2$ are dimensionless coupling constants.

Let us analyze a physical content of this equation. Since equation (98) is reduced to equation (90) for $\lambda_1 = \lambda_2 = 0$, we shall study the cases with anomalous and with minimal interaction simultaneously by analyzing a more general equation (98).

In order to receive the physical content of these equations it is convenient to apply the transformation

$$
\Psi \rightarrow \Psi' = W^{-1}\Psi, \quad L \rightarrow L' = W^\dagger LW,
$$

(99)

where

$$
W = \exp \left( -i \frac{\eta \cdot \pi}{m} \right)
$$

(100)

and $\eta$ is a vector whose components are the Galilei boost generators (9). For the case $e = 0$ (or $A_\mu = 0$) the operator $W$ reduces to operator $U$ given in equation (33), which was used for our analysis of free particle equations.

Using relations (7) and supposing that the nilpotence index of matrices $\eta \cdot \pi$ is less than 4 we come to the following equation which is equivalent to (98):

$$
L'\Psi' \equiv \left\{ \left( \beta_0 \left( \pi^0 - \frac{\pi^2}{2m} + \frac{e}{m} \eta \cdot F \right) - \frac{e}{2m} \beta \times \eta \cdot H + \beta_4 m - \frac{e}{6m^2} \tilde{Q}_{ab} \frac{\partial H}{\partial x^b} \right) \right. \\
+ \frac{e}{m} \Lambda \left[ \lambda_1 \eta \cdot H + \lambda_2 \left( S \cdot H - \eta \cdot F + \frac{1}{2m} \tilde{Q}_{ab} \frac{\partial H}{\partial x^b} \right) \right] \left\} \Psi' = 0
$$

(101)

where $E = -\nabla A^0 - \frac{\partial A}{\partial t}$ and $H = \nabla \times A$ are vectors of the corresponding electric and magnetic field strength respectively,

$$
F = E + \frac{1}{2m} (\pi \times H - H \times \pi), \quad \tilde{Q}_{ab} = \eta_a \varepsilon_{bcd} \beta_c \eta_d + \varepsilon_{bcd} \beta_c \eta_d \eta_a, \quad \tilde{Q}_{ab} = \eta_a S_b + S_b \eta_a.
$$

(102)

Equation (101) includes the Schrödinger terms $(\pi^0 - \frac{\pi^2}{2m}) \Psi'$ and additional terms which are linear in vectors of the external field strengths and their derivatives.

Notice that if the nilpotence index $N$ of matrices $\eta \cdot \pi$ is smaller than 4, the transformed equation (101) is completely equivalent to initial equation (98). The condition $N < 4$ is fulfilled for all representations of algebra $h(1, 3)$ considered in the present paper, so this equivalence takes place for all equations for vector, scalar and spin 1/2 fields studied in the paper.

### 4.4 Galilean equation for spinor and vector fields with interactions

Let us consider equation (101) for two particular realizations of $\beta$–matrices in more detail. First notice, that our conclusions from equations (90)–(101) are true in general and in particular for the Lévy-Leblond equation, i.e., when $\beta_\mu, \beta_4 4 \times 4$ are matrices
determined by relations (10) with \( \omega = \kappa = 0 \). Then \( \beta_0 \eta_a = 0 \), \( Q_{ab} = Q'_{ab} = 0 \), \( \beta \times \eta = -2\beta_0 \mathbf{S} \), and equation (101) is reduced to the following form:

\[
\left\{ \begin{array}{l}
\beta_0 \left( \pi^0 - \frac{\pi^2}{2m} + \frac{e}{m} \mathbf{S} \cdot \mathbf{H} \right) + \beta_4 m \\
+ \frac{e}{m} \Lambda \left[ \lambda_1 \eta \cdot \mathbf{H} + \lambda_2 \left( \mathbf{S} \cdot \mathbf{H} - \eta \cdot \mathbf{F} \right) \right]
\end{array} \right\} \Psi' = 0
\] (103)

For \( \lambda_1 = \lambda_2 = 0 \) (i.e., when only the minimal interaction is present) equation (103) is reduced to the following system:

\[
\left( \pi_0 - \frac{\pi^2}{2m} + \frac{e}{2m} \mathbf{\sigma} \cdot \mathbf{H} \right) \varphi_1 = 0,
\] (104)

\[m \varphi_2 = 0, \quad \text{or} \quad \varphi_2 = 0,
\] (105)

where \( \varphi_1 = \beta_0 \Psi' \) and \( \varphi_2 = (1 - \beta_0) \Psi' \) are two-component spinors.

Thus introducing the minimal interaction (77) into the Lévy-Leblond equation, we receive the Pauli equation for physical components of the wave function; moreover, the coupling constant (gyromagnetic ratio) for the Pauli interaction \( \frac{e}{2m} \mathbf{s} \cdot \mathbf{H} \), \( \mathbf{s} = \frac{1}{2} \mathbf{\sigma} \) has the same value \( g = 2 \) as in the case of the Dirac equation [3].

Considering now the case with an anomalous interaction we conclude that the general form of matrix \( \Lambda \) satisfying relations (93) is

\[\Lambda = \nu \beta_0 + \mu \eta,\] (106)

where \( \eta \) is hermitizing matrix (12), \( \nu \) and \( \mu \) are arbitrary parameters. Substituting (106) into (103) we obtain the following analogue of system (104):

\[
\left( \pi_0 - \frac{\pi^2}{2m} + \frac{e \lambda_1}{2m} \mathbf{\sigma} \cdot \mathbf{H} - \frac{e \lambda_3}{2m} \mathbf{\sigma} \cdot \mathbf{F} - \frac{\lambda_3^2 e^2}{8 m^2} \mathbf{H}^2 \right) \varphi_1 = 0,
\] (107)

where \( g = 2 + \mu \lambda_1 + \nu \lambda_2 \), \( \lambda_3 = \mu \lambda_2 \) are arbitrary parameters.

We see that the Lévy-Leblond equation with minimal and anomalous interactions reduces to the Schrödinger-Pauli-like equation (107) for two-component spinor \( \varphi_1 \), which, however, includes additional terms linear in strengths of an electric field and linear and quadratic in strengths of a magnetic field. We shall discuss them at the end of this section.

Consider now the Galilean Duffin-Kemmer equation for interacting vector field. The corresponding \( \beta \)-matrices (98) have dimension \( 10 \times 10 \) and are given explicitly by relations (45). Then

\[
\beta \times \eta \cdot \mathbf{H} = - \begin{pmatrix}
0_{3 \times 3} & \mathbf{s} \cdot \mathbf{H} & 0_{3 \times 3} & 2k^\dagger \cdot \mathbf{H} \\
\mathbf{s} \cdot \mathbf{H} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\
2k \cdot \mathbf{H} & 0_{1 \times 3} & 0_{1 \times 3} & 0
\end{pmatrix},
\]

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$$\beta_0 \mathbf{n} = \begin{pmatrix}
\mathbf{s} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & \mathbf{s} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & \mathbf{s} & 0_{3 \times 1} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0
\end{pmatrix}, \quad \beta_0 \mathbf{S} = \begin{pmatrix}
0_{3 \times 3} & \mathbf{s} & 0_{3 \times 3} & 0_{3 \times 1} \\
\mathbf{s} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & \mathbf{s} & 0_{3 \times 1} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0
\end{pmatrix},$$

$$\tilde{Q}_{ab} = -3 \begin{pmatrix}
Q_{ab} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & Q_{ab} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & Q_{ab} & 0_{3 \times 1} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0
\end{pmatrix}, \quad \tilde{Q}_{ab} = \begin{pmatrix}
Q'_{ab} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & Q'_{ab} & 0_{3 \times 3} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & Q'_{ab} & 0_{3 \times 1} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0
\end{pmatrix},$$

where

$$Q_{ab} = s_a s_b + s_b s_a - \frac{4}{3} \delta_{ab}, \quad Q'_{ab} = Q_{ab} + \frac{4}{3} \delta_{ab}. \quad (109)$$

Let us consider the corresponding equation (111) and restrict ourselves to $\Lambda = \beta_0$. Representing $\Psi'$ as a column vector $(\psi_1, \psi_2, \psi_3, \varphi)$, where $\psi_1, \psi_2, \psi_3$ are three-component vector functions and $\varphi$ is a one-component scalar function and using (15), (108) we reduce (111) to the following Pauli-type equation for $\psi_1$:

$$i \frac{\partial}{\partial t} \psi_1 = \tilde{H} \psi_1, \quad (110)$$

where

$$\tilde{H} = \frac{\nu^2}{2} m + \frac{\pi^2}{2m} + e A_0 - \frac{ge}{2m} \mathbf{s} \cdot \mathbf{H} + \frac{ge}{\nu m} \mathbf{s} \cdot \mathbf{E} - \frac{ge}{2\nu m} \mathbf{s} \cdot (\pi \times \mathbf{H} - \mathbf{H} \times \pi) + \frac{e}{2\nu m^2} (1 + \lambda_2) Q_{ab} \frac{\partial H_a}{\partial x_b} + \frac{e^2}{2\nu m^2} (\mathbf{H}^2 - (\mathbf{s} \cdot \mathbf{H})^2), \quad (111)$$

where $g = 1 + 2\lambda_1 + 2\lambda_2$ and $q = 1 - \lambda_2$.

The other components of $\Psi'$ can be expressed by $\psi_1$:

$$\varphi = -\frac{e}{\nu^2 m^2} \mathbf{k} \cdot \mathbf{H} \psi_1, \quad \psi_2 = -\frac{1}{\nu} \psi_1, \quad \psi_3 = -\nu \psi_2 - \frac{1}{m} \left( \pi_0 - \frac{1}{2m} \pi^2 + \frac{e}{2m} \mathbf{s} \cdot \mathbf{H} \right) \psi_1.$$ 

Up to realizations of spin matrices, equation (110) is rather similar to equation (107) describing Galilean particles with spin $1/2$. However, in comparison with (107) the essentially new feature of equation (110) appears, namely, that setting $\lambda_1 = \lambda_2 = 0$ in (111) (i.e., excluding the anomalous interaction) we receive a Hamiltonian which still includes the term $-\frac{e}{\nu m} \mathbf{s} \cdot \mathbf{E}$ describing the coupling of spin with an electric field. We show in the next section that this effectively represents the spin-orbit coupling. The other terms of Hamiltonian (111) (which are placed in the second line of equation (111)) can be neglected starting with a reasonable assumption about the possible values of the magnetic field strength.

However, equation (107) for $\lambda_1 = \lambda_2 = 0$ it reduces to the Schrödinger-Pauli equation (104) which has nothing to do with the spin-orbit coupling.

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4.5 Galilei invariance and spin-orbit coupling

Consider now the first of equations (107) for particular values of arbitrary parameters, namely for \(\tilde{\lambda}_1 + \tilde{\lambda}_2 = -1\):

\[
\dot{L}\varphi_1 \equiv \left( \pi_0 - \frac{\pi^2}{2m} - \frac{e\lambda_3}{2m}\sigma \cdot F - \frac{\lambda_3^2 e^2}{8m^3} H^2 \right) \varphi_1 = 0.
\] (112)

First, let us remind that this equation is a direct consequence of the Galilei invariant Lévy-Leblond equation with anomalous interaction, i.e., of equation (98) where \(\beta_n\) are matrices (10) with \(\kappa = \omega = 0\). Secondly, equation (112) by itself is transparently Galilei invariant since the operator \(\hat{L}\) in (112) is a Galilean scalar provided the value of arbitrary parameter \(\lambda_3\) is finite. We shall assume \(\lambda_3\) to be small.

In order to find out the physical content of equation (112) we transform it to a more transparent form using the operator

\[
U = \exp(-\frac{i\lambda_3}{2m}\sigma \cdot \pi).
\]

Applying this operator to \(\varphi_1\) and transforming \(\hat{L} \rightarrow \hat{L}' = U\hat{L}U^{-1}\) we obtain the equation

\[
L'\varphi'_1 = \left( \pi_0 - \frac{\pi^2}{2m} - eA_0 - \frac{e\lambda_3^2}{8m^2} (\sigma \cdot (\pi \times E - E \times \pi) - \text{div}E) + \cdots \right) \varphi'_1 = 0 \quad (113)
\]

where the dots denote the small terms of orders \(o(\lambda_3^3)\) and \(o(e^2)\).

All terms in square brackets have the exact physical meaning. They include first the Schrödinger terms \(\pi_0 - \frac{\pi^2}{2m} - eA_0\) then the term \(\sim s \cdot (\pi \times E - E \times \pi)\) and \(\sim \text{div}E\) describing the spin-orbit coupling and finally the term \(\sim \text{div}E\), i.e., the Darwin coupling.

Similarly, starting with equation (110), setting \(\lambda_2 = -1\), \(\lambda_1 = \frac{1}{2}\), supposing \(\frac{1}{\nu}\) to be a small parameter and making use of a transformation \(\psi_1 \rightarrow \psi'_1 = \hat{U}\psi_1\) with \(\hat{U} = \exp(-\frac{2i}{\nu m}s \cdot \pi)\) we obtain the equation

\[
\left( \pi_0 - \frac{\nu^2}{2m}m - \frac{\pi^2}{2m} - eA_0 \right.
\left. - \frac{2e}{\nu^2m^2} \left( s \cdot (\pi \times E - E \times \pi) - Q_{ab} \frac{\partial E_a}{\partial x_b} + \frac{4}{3} \text{div}E \right) + \cdots \right) \psi'_1 = 0 \quad (114)
\]

Here the dots denote small terms of the orders \(o(\frac{1}{\nu^3})\) and \(o(e^2)\).

Like relation (113), the equation (114) includes the terms which represent the spin-orbit and Darwin couplings. In addition, there is the term \(\sim Q_{ab} \frac{\partial E_a}{\partial x_b}\) which describes the quadrupole interaction of a charged vector particle with an electric field.

Thus we again come to the conclusion [10] that the spin-orbit and Darwin couplings can be effectively described within framework of a Galilei-invariant approach and so they have not be necessarily interpreted as pure relativistic effects.

Let us note that it is possible to choose parameters \(\lambda_1\) and \(\lambda_2\) in Hamiltonian (111) in such a way that the anomalous interaction with electric field will not be present. Namely, we can set \(\lambda_2 = 1\), \(\lambda_1 = \frac{1}{2}\), and obtain, instead of (114), the following equation:

\[
\left( \pi_0 - \frac{\nu^2}{2m}m - \frac{\pi^2}{2m} - eA_0 + \frac{2e}{2m} s \cdot H + \cdots \right) \psi'_1 = 0 \quad (115)
\]
where \( g = 2 \). In other words, introducing a specific anomalous interaction into the Galilean Duffin-Kemmer equation we can reduce it to the Schrödinger-Pauli equation with the correct value of the gyromagnetic ratio \( g \).

## 5 Discussion

In the present paper we continue the study of the Galilei invariant theories for vector and spinor fields, started in [5]. The peculiarity of our approach is that as distinct to the other approaches (e.g., to [8]-[12], [23], [24]) it enables to find out a complete list of Galilei invariant equations for scalar and vector fields. This is possible due to our knowledge of all non-equivalent indecomposable representations of the Galilei algebra \( hg(1,3) \) that can be constructed on representation spaces of scalar and vector fields described for the first time in paper [5].

Thus using this complete list of representations we have been able to find all possible systems of the first order Galilei-invariant wave equations (1) for scalar and vector fields. All \( \beta \)-matrices for these Galilei-invariant wave equations have been presented in the Appendix. In fact we have described how to construct arbitrary wave equations of finite order invariant with respect to the Galilei group since all of them can be obtained from the first order equations in which various derivatives of fields are considered as new dependent variables.

Then Galilean analogues of some popular relativistic equations for vector particles and particles with spin 3/2 are discussed, in particular, the Galilean second order Proca equation and Galilean first order Rarita-Schwinger equation. However these Galilean equations are not non-relativistic limits of the corresponding relativistic equations Proca and Rarita-Schwinger equation since among other things they have more components. Thanks to that it is possible to obtain equations which keep all the main features of their relativistic analogues. At the best of our knowledge this is done for the first time in the present paper.

We pay a specific attention to description of the Galilean particles interacting with an external electromagnetic field. We study both the minimal interaction as well as anomalous one. The results presented in Sections 4.1 and 4.3 are valid for generic equations describing scalar, spinor or vector fields.

The equation (98) includes a quite general form of an anomalous interaction which satisfies the Galilei invariance condition. The main idea of our analysis of this equation is to transform it to the equivalent form (101) in which all terms in brackets commute with matrix \( \beta_0 \). The related transformation (99) can be treated as a Galilean analogue of the Foldy-Wouthuysen transformation [25].

Notice that the results presented in Sections 4.1 and 4.3 are valid for arbitrary equations (98) invariant with respect to the Galilei group. The results presented in Sections 4.2, 4.4 and 4.5 are restricted to particular equations with anomalous interaction, i.e., to the Galilean Proca equation, generalized Lévy-Leblond equation and to generalized Galilean Duffin-Kemmer equation. We prove that the last two equations
present consistent models of charged particles interacting with an electromagnetic field. In particular, they describe such important physical effect as the spin-orbit coupling which traditionally is interpreted as a pure relativistic phenomenon.

However, it is necessary to fix some difficulties of principal nature appearing in the Galilean approach, like that the Galilei invariance requires that the mass and energy each are separately conserved, and that within the Galilean theories there are not concept of proper time which produces phase effects which do not depend on the velocity of light and so do not dissapear in the non-relativistic limit. Of course, there are obvious restrictions to phenomena which are characterized by velocities much smaller than the velocity of light. In addition, in our approach there are also problems with interpretation of undesired terms $Q_{ab} \partial H_a \partial x_b$ and $s \cdot (\pi \times H - H \times \pi)$ which appear in the Hamiltonian (110). Thanks to appropriate choice of arbitrary parameters $\lambda_1$ and $\lambda_2$ these terms are not present in the effective Hamiltonians (114) and (115) which describe spin-orbit and Pauli couplings respectively. However, if we would like to keep both these couplings, then the undesired terms may appear.

One more problem is connected with the signs for the terms presenting the spin-orbit and Darwin couplings. Comparing (113) with the quasirelativistic approximation of the Dirac equation (see, e.g., ref. [25]) we conclude, that to obtain the correct signs it is necessary to suppose that $\lambda_3$ be purely imaginary. Moreover, for $\lambda_3 = i$ the coupling constants for spin-orbit and Darwin interactions in (113) coincide with the relativistic ones predicted by the Dirac equation.

Notice that the exact equation (112) is much simplier then the approximate equation (113) and can be solved exactly for some particular (e.g., the Coulomb) external fields. However, for $\lambda_3$ imaginary the term $-\frac{e\lambda_3}{2m}\sigma \cdot F$ in equation (112) and the corresponding Hamiltonian $\hat{H} = -L + p_0 = A_0$ are non-hermitean. On the other hand, for $A_0$ and $A$ being even and odd functions of $x$ correspondingly the equation (112) appears to be invariant with respect to the product of the space inversion $P$ and Wigner time inversion $T$, and so presents one more (and as it seemd for us, promising) field for application of tools of $PT$-symmetric quantum mechanics [26].

Any Galilean theory by definition is only an approximation of a relativistic one. The very existence of a physically consistent non-relativistic approximation can serve as a criteria of consistency of a relativistic theory. Thus our study of Galilean wave equations makes a contribution into the theory of relativistic ones, since effectively we have analyzed possible non-relativistic limits of theories for vector and scalar particles.

An intriguing problem is the description of Galilean theories for massless vector and scalar fields. It appears that the fundamental analysis of Galilean limits of Maxwell’s equations presented in papers [8], and [3] can be essentially completed using the list of indecomposable representations of algebra $hg(1,3)$ presented in [3]. This work is in progress.
6 Appendix

Here we present all non-trivial solutions of equations (26) which give rise to explicit descriptions of matrices $\beta_4$ given by equation (25). The related matrices $\beta_0$ and $\beta_a$ are given by equations (27) and (28).

Solving equations (26), where $A, C$ and $A', C'$ are matrices given in Table 1 which correspond to $q = (n,m,\lambda)$ and $q' = (n',m',\lambda')$ respectively we obtain the related matrices $R = R(q,q'), E = E(q,q')$ (which define matrix $\beta_4$ in accordance with equation (25)) in the forms presented in Tables 2, where the Greek letters denote arbitrary real parameters.

Table 3. Submatrices $R$ and $E$ of matrices $\beta_4$

| $m',n',\lambda'$ | $m,n,\lambda$ | 2,1,1 | 2,0,0 | 1,2,1 |
|-------------------|---------------|-------|-------|-------|
| 2,1,1             | $R = \begin{pmatrix} \mu & \nu \\ \alpha & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \omega & \nu \\ \mu & 0 \end{pmatrix}$ | $R = (\mu \nu)$ | $E = (\sigma \alpha)$ |
|                   | $E = \sigma$ | $E$ not existing | $E = (\mu \nu)$ | $E$ not existing |
| 2,0,0             | $R = \begin{pmatrix} \mu & \nu \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$ | $R = (\mu \nu)$ | $E$ not existing |
|                   | $E$ not existing | $E$ not existing | $E = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$ | |
| 1,2,1             | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$ | $R = (\mu \nu)$ | $E = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$ |
|                   | $E = (\sigma \alpha)$ | $E$ not existing | $E = (\mu \nu)$ | $E$ not existing |
| 1,1,0             | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$, $E = \sigma$ | $R = \mu$, $E$ not existing | $R = \mu$, $E = \begin{pmatrix} \nu \\ 0 \end{pmatrix}$ |
| 1,1,1             | $R = \begin{pmatrix} \mu & \nu \\ \nu & 0 \end{pmatrix}$, $E = \sigma$ | $R = \mu$, $E$ not existing | $R = \mu$, $E = \begin{pmatrix} \nu \\ 0 \end{pmatrix}$ |
| 1,0,0             | $R = \begin{pmatrix} \kappa & \sigma \\ \sigma & 0 \end{pmatrix}$ | $R = \mu$, $E$ not existing | $R = \mu$, $E = \begin{pmatrix} \nu \\ \alpha \end{pmatrix}$ |
|                   | $E$ not existing | $E$ not existing | $E = \begin{pmatrix} \nu \\ \alpha \end{pmatrix}$ |
| 0,1,0             | $R$ not existing, $E = \alpha$ | $R$ and $E$ not existing | $R$ not existing, $E = \mu$ |
| $m', n', \lambda'$ | $m, n, \lambda$ | $m, n, \lambda$ | $m, n, \lambda$ |
|-----------------|-----------------|-----------------|-----------------|
| 3,1,1           | $R = \begin{pmatrix} \mu & \nu & \sigma \\ \nu & \alpha & 1 \\ \sigma & 1 & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \nu & \alpha & 0 \\ \mu & \sigma & \omega \end{pmatrix}$ | $R = \begin{pmatrix} \nu & \alpha & \omega \\ \mu & \sigma & 0 \end{pmatrix}$ |
|                 | $E = \alpha - 2\sigma$ | $E = \begin{pmatrix} \kappa \\ \omega - \alpha \end{pmatrix}$ | $E = \kappa$ |
| 2,2,1           | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & \kappa \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & \omega \end{pmatrix}$ |
|                 | $E = (\kappa (\omega - \alpha))$ | $E = \begin{pmatrix} \sigma & \omega \\ \omega & \kappa \end{pmatrix}$ | $E = (\kappa \omega)$ |
| 2,1,0           | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \omega \end{pmatrix}$, $E = \begin{pmatrix} \kappa \\ \omega \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \nu & \kappa \end{pmatrix}$ |
|                 | $E = \kappa$ | $E = \sigma$ | $E = \kappa$ |
| 2,1,1           | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ 0 & \omega \end{pmatrix}$, $E = \begin{pmatrix} \alpha \\ \sigma \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \sigma \\ 0 & \nu \end{pmatrix}$ |
|                 | $E = \omega - \alpha$ | $E = \kappa$ | $E = \kappa$ |
| 2,0,0           | $R = \begin{pmatrix} \mu & \nu \\ \sigma & \alpha \\ \alpha & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu & \nu \\ \sigma & 0 \end{pmatrix}$ |
|                 | $E$ not existing | $E$ not existing | $E$ not existing |
| 1,2,1           | $R = \begin{pmatrix} \mu \\ \nu \\ \alpha \end{pmatrix}$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$, $E = \begin{pmatrix} \mu & \nu \\ \omega & 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ |
|                 | $E = (\omega \alpha)$ | $E = (\sigma 0)$ | $E = (\sigma 0)$ |
| 1,1,0           | $R = \begin{pmatrix} \mu \\ \nu \\ \alpha \end{pmatrix}$, $E = \alpha$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$, $E = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ |
|                 | $E = \sigma$ | $E = \sigma$ | $E = \sigma$ |
| 1,1,1           | $R = \begin{pmatrix} 0 \\ \nu \\ \alpha \end{pmatrix}$, $E = \omega$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$, $E = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ | $R = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$ |
|                 | $E = \omega$ | $E = 0$ | $E = 0$ |
| 1,0,0           | $R = \begin{pmatrix} \mu \\ \alpha \\ 0 \end{pmatrix}$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ | $R = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ |
|                 | $E$ not existing | $E$ not existing | $E$ not existing |
| 0,1,0           | $\ not not existing, $ $E = \alpha$ | $\ not not existing, $ $E = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$ | $E = \alpha$, $\ not not existing$ |
| $m', n', \lambda'$ | 1,1,0 | 1,1,1 | 1,0,0 | 0,1,0 |
|-----------------|-----|-----|-----|-----|
| $R = \mu$ | $R = \mu$ | $R = \mu, E$ not existing | $E = \mu, R$ not existing |
| $E = \nu$ | $E = \nu$ | $E = 0$ | $E = \mu, R$ not existing |
| $R = \mu$, $E = \nu$ | $R = \mu$, $E$ not existing | $R = \mu, E$ not existing | $E = \mu, R$ not existing |
| $E = \mu$, $R$ not existing | $E = \mu, R$ not existing | $R$ and $E$ not existing | $E = \mu, R$ not existing |

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