SURFACE FOLIATIONS WITH COMPACT COMPLEX LEAVES ARE HOLOMORPHIC

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Abstract. Let \( X \) be a compact complex surface with a real foliation. If all leaves are compact complex curves, the foliation must be holomorphic.

Soit \( X \) une surface complexe compacte munie d'un feuilletage reel. Supposons que tous les feuilles sont de courbes complexes compactes. Alors le feuilletage est holomorphe.

1. The result

Theorem. Let \( X \) be a compact complex smooth surface, and \( \mathcal{F} \) a real foliation on \( X \) such that all the leaves are compact complex curves.

Then \( \mathcal{F} \) is in fact a holomorphic foliation.

Proof. We use the theory of cycle space resp. Chow schemes.\(^1\) By this theory there exists a “cycle space” \( C \) with countably many irreducible components \( C_i \) all of which are compact, and a universal space \( U \subset C \times X \) such that for every compact complex irreducible subspace \( Z \) of \( X \) there is a point \( t \in C \) such that \( U \cap (\{t\} \times X) = \{t\} \times Z \).

This is classical for projective varieties, but actually works for arbitrary compact complex manifolds by work of Barlet \(^2\). We will need the compactness of the components \( C_i \). This is true for all surfaces as well as for all Kähler manifolds. However, there are higher-dimensional non-Kähler manifolds for which \( C_i \) need not be compact (see e.g. \(^4\), cor. 4.11.3).

Because there are only countably many connected components \( C_i \) there must be a component \( C_0 \) such that the collection of subvarieties parametrized by this \( C_0 \) includes uncountably many of the leaves \( L_\alpha \) of the foliation \( \mathcal{F} \). Let \( U_0 \) denote the corresponding component of \( U \), and let \( p : U_0 \rightarrow C_0 \) and \( \pi : U_0 \rightarrow X \) denote the projections. Because everything is compact, \( \pi(U_0) \) is an irreducible closed analytic subset

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of $X$. Since it contains infinitely many distinct curves, we must have $\pi(U_0) = X$.

Now fix a point $q \in C_0$ such that the parametrized subvariety $Z_q = \pi(p^{-1}(q))$ is one of the leaves of $\mathcal{F}$.

Let $L$ be an arbitrary leaf of $\mathcal{F}$ other than $Z_q$. Then $L \cap Z = \emptyset$ and therefore $L \cdot Z = 0$ in the sense of intersection theory. Now for every $s \in C_0$ the corresponding curve $Z_s$ is numerically equivalent to $Z_q$. Hence $L \cdot Z_s = 0$ for all $s \in C_0$ (and every leaf $L$ of $\mathcal{F}$). On the other hand, there must be a parameter $s \in C_0$ such that $L \cap Z_s \neq \emptyset$, because the union of the $Z_s$ covers $X$ (i.e., $\pi(U_0) = X$). As a consequence, every leaf $L$ must coincide with an irreducible component of $Z_s$ for some $s \in C_0$.

As a next step we want to show that actually all the $Z_s$ are irreducible. There are several ways to do this.

One way is to use the fact that all $Z_s$ are connected, and therefore their irreducible components intersect while the leaves are by definition disjoint.

Here we use a different method based on discussing volumes. We observe that for every Kähler form $\omega$ the volume $\int_{Z_s} \omega$ is locally constant as a function of $s \in C_0$. This implies that for almost all leaves $L$ we have $\int_L \omega = K = \int_{Z_q} \omega$. Now for the foliation $\mathcal{F}$ the volume of the leaves $\int_L \omega$ must vary continuously in dependence of the leaf $L$. Hence it can not jump and therefore it equals $K$ for every leaf $L$. Therefore such a component can not occur as leaf of $\mathcal{F}$. It follows that for a reducible $Z_s$ the equality $Z_s \cdot L = 0$ implies that $Z_s$ has empty intersection with every leaf $L$. Since the union of all leaves covers $X$, this is absurd. Hence there is no reducible $Z_s$.

Thus we have seen: Every $Z_s$ (with $s \in C_0$) is an irreducible curve on $X$ and every leaf $L$ of $\mathcal{F}$ equals one of the curves $Z_s$.

Conversely, assume that $s \in C_0$ and consider the curve $Z_s$: Since the leaves of $\mathcal{F}$ cover all of $X$, there is a leaf $L$ with $Z_s \cap L \neq \emptyset$. With $Z_s \cdot L = 0$ it follows that $Z_s = L$.

We have thus established that there is a one-to-one correspondence between the leaves of $\mathcal{F}$ and the curves $C_s$ parametrized by $s \in C_0$.

It follows that $\mathcal{F}$ is a holomorphic foliation. $\square$

2. Higher dimensions

The result is not valid in higher dimensions. Indeed, let $M$ be a real 4-manifold with an anti-self dual Riemannian metric $g$, e.g. a
compact real 4-dimensional torus with flat metric. Then we have a “twistor space” $X$ with a projection $\pi : X \to M$ such that $X$ is a compact complex three-dimensional manifold and all the fibers of $\pi$ are compact complex curves of genus 0 whose normal bundle is $\mathcal{O}(1)^{\oplus 2}$ ([1]). Because the normal bundle of the fibers is not holomorphically trivial, it is clear that there is no way to define a complex structure on $M$ for which $\pi$ is holomorphic. It follows in this way also that these curves are not leaves of a holomorphic foliation on $X$.

3. Non-compact case

Compactness of the surface is crucial, as can be seen by the following example which is due to J.J. Loeb.

We consider the non-compact surface $X = \mathbb{P}_2(\mathbb{C}) \setminus \mathbb{P}_2(\mathbb{R})$. Each $x \in \mathbb{P}_2(\mathbb{C})$ corresponds to a complex line in $\mathbb{C}^3$. Each vector $v \in \mathbb{C}^3$ decomposes into a real part and an imaginary part: $v = u + iw$, $u, w \in \mathbb{R}^3$. The real vectors $u, w$ are linearly independent unless $x \in \mathbb{P}_2(\mathbb{R})$. Thus we obtain a map from $X$ to $\mathbb{P}_2(\mathbb{R})^*$ by mapping each $x = [u + iw] \in X$ to the real hyperplane of $\mathbb{R}^3$ spanned by $u$ and $w$. This yields a real-analytic map $F : X \to \mathbb{P}_2(\mathbb{R})^*$ whose fibers are complex curves: If $H = \langle u, w \rangle_\mathbb{R} \in \mathbb{P}_2(\mathbb{R})^*$, then

$$F^{-1}(H) = \{v \in \mathbb{P}_2(\mathbb{C}) \setminus \mathbb{P}_2(\mathbb{R}) : v = \langle u, w \rangle_\mathbb{C} \}$$

Let $v \in \mathbb{P}_2(\mathbb{R})$. Then for every real hyperplane $H$ in $\mathbb{R}^3$ containing $v$ we can find a sequence $x_n \in X$ with $F(x_n) = H$ and $\lim x_n = v$. Therefore $F$ can nowhere extended to $\mathbb{P}_2(\mathbb{R})$ as a continuous map.

Now assume that the foliation defined by $F$ is holomorphic. Each $F$-fiber has two components (the fibers are isomorphic to $\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{P}_1(\mathbb{R})$). Hence $F$ lifts to a map $\tilde{F}$ to $S^2$, the 2 : 1-covering of $\mathbb{P}_2(\mathbb{R})$. If $F$ defines a holomorphic foliation, this map $\tilde{F}$ must be holomorphic for some complex structure on $S^2$. But then $\tilde{F}$ would be a meromorphic function, and meromorphic functions extend through totally real sub-manifolds like $\mathbb{P}_2(\mathbb{R})$ in $\mathbb{P}_2(\mathbb{C})$ while we have seen that $F$ and $\tilde{F}$ do not even extend as topological maps.

Thus $F$ defines a foliation on $X$ whose leaves are all isomorphic to $H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, but this foliation is not holomorphic.

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