LARGE DEVIATIONS FOR MARKOVIAN NONLINEAR HAWKES PROCESSES

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ABSTRACT. In the 2007 paper, Bordenave and Torrisi [1] proves the large deviation principles for Poisson cluster processes and in particular, the linear Hawkes processes. In this paper, we prove first a large deviation principle for a special class of nonlinear Hawkes process, i.e., a Markovian Hawkes process with nonlinear rate and exponential exciting function, and then generalize it to get the result for sum of exponentials exciting functions. We then provide an alternative proof for the large deviation principle for linear Hawkes process. Finally, we use an approximation approach to prove the large deviation principle for a special class of nonlinear Hawkes processes with general exciting functions.

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1. INTRODUCTION

Let $N$ be a simple point process on $\mathbb{R}$ and let $F_t := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset \mathbb{R}^+)$ be an increasing family of $\sigma$-algebras. Any nonnegative $F_t$-progressively measurable process $\lambda_t$ with

$$E [N(a, b) | F_a] = E \left[ \int_a^b \lambda_s ds | F_a \right]$$

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a.s. for all intervals \((a, b]\) is called an \(\mathcal{F}_t\)-intensity of \(N\). We use the notation \(N_t := N(0, t]\) to denote the number of points in the interval \((0, t]\).

A general Hawkes process is a simple point process \(N\) admitting an \(\mathcal{F}_t\)-intensity
\[
\lambda_t := \lambda \left( \int_0^t h(t-s)N(ds) \right),
\]
where \(\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+, \ h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+\) and we always assume that \(\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty\). In the literatures, \(h(\cdot)\) and \(\lambda(\cdot)\) are usually referred to as exciting function and rate function respectively.

Let \(Z_t = \sum_{0<\tau_j<t} h(t-\tau_j)\). where \(\tau_j\) is the \(j\)-th arrival time of the process for \(j \geq 1\). Thus we can write \(\lambda_t = \lambda(Z_t)\).

This is known as the nonlinear Hawkes process. (See Brémaud and Massoulié [2].) When the exciting function \(h(\cdot)\) is exponential or a sum of exponentials, the process is Markovian and we name it a Markovian nonlinear Hawkes process.

When \(\lambda(\cdot)\) is linear, this is known as the (linear) Hawkes process, which was introduced in Hawkes [10]. If \(\lambda(\cdot)\) is linear and \(h(\cdot)\) is exponential or a sum of exponentials, the (linear) Markovian Hawkes process is sometimes referred to as Markovian self-exciting processes, see for example Oakes [17]. You can think of the arrival times \(\tau_j\) as “bad” events, which can be the arrivals of claims in insurance literature or the time of defaults of big firms in the real world. Hawkes process captures both the self-exciting property and the clustering effect, which explains why it has wide applications in cosmology, ecology, epidemiology, seismological applications, neuroscience applications and DNA modeling. For a list of references to these applications, see Bordenave and Torrisi [1].

Hawkes process has also been applied in finance. Empirical comparisons suggest that Hawkes processes have some of the typical characteristics of financial time series. Financial data have been analyzed using Hawkes processes. Self-exciting processes are used for the calculation of conditional risk measures, such as the Value-at-Risk. Another area of finance where Hawkes processes have been considered is credit default modeling. Hawkes processes have been proposed as models for the arrival of company defaults in a bond portfolio. For a list of references to the applications in finance, see Liniger [15].

For a short history of Hawkes process, we refer to Liniger [15]. For a survey on Hawkes processes and related self-exciting processes, Poisson cluster processes, marked point processes etc., we refer to Daley and Vere-Jones [4].

When \(\lambda(\cdot)\) is linear, say \(\lambda(z) = \nu + z\), then one can use immigration-birth representation, also known as Galton-Watson theory to study it. Under the immigration-birth representation, if the immigrants are distributed as Poisson process with intensity \(\nu\) and each immigrant generates a cluster whose number of points is denoted by \(S\), then \(N_t\) is the total number of points generated in the clusters up to time \(t\). If the process is ergodic, we have
\[
\lim_{t \to \infty} \frac{N_t}{t} = \nu \mathbb{E}[S], \quad \text{a.s.}
\]

Bordenave and Torrisi [1] proves that if \(0 < \mu = \int_0^\infty h(t)dt < 1\) and \(\int_0^\infty th(t)dt < \infty\), then \(\left( \frac{N_t}{t} \right)\) satisfies the Large Deviation Principle (LDP) with the good rate function \(I(\cdot)\), i.e. for any closed set \(C \subset \mathbb{R},\)
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(N_t/t \in C) \leq - \inf_{x \in C} I(x),
\]
and for any open set $G \subset \mathbb{R}$,
\begin{equation}
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}(N_t \in G) \geq - \inf_{x \in G} I(x),
\end{equation}
where
\begin{equation}
I(x) = \begin{cases} 
  x\theta_x + \nu - \frac{\nu x}{\nu + \mu x} & \text{if } x \in [0, \infty) \\
  +\infty & \text{otherwise}
\end{cases}
\end{equation}
where $\theta = \theta_x$ is the unique solution in $(-\infty, \mu - 1 - \log \mu)$ of $E[e^{\theta S}] = \frac{x}{\nu + \mu}$, $x > 0$. It is well known that, for instance, see page 39 of Jagers [12], for all $\theta \in (-\infty, \mu - 1 - \log \mu]$, $E[e^{\theta S}]$ satisfies
\begin{equation}
E[e^{\theta S}] = e^{\theta} \exp \{ \mu(E[e^{\theta S}] - 1) \}.
\end{equation}
See Dembo and Zeitouni [5] for general background regarding large deviations and the applications. Also Varadhan [19] has an excellent survey article on this subject.

Once the LDP for $\sum_{i=1}^{d} (N_i(t) - N_i) \epsilon_i$ is established, it is easy to study the ruin probability. Stabile and Torrisi [18] considered risk processes with non-stationary Hawkes claims arrivals and studied the asymptotic behavior of infinite and finite horizon ruin probabilities under light-tailed conditions on the claims.

We are interested in general non-linear $\lambda(\cdot)$. If $\lambda(\cdot)$ is nonlinear, then the usual Galton-Watson theory approach no longer works. If the exciting function $h$ is exponential or a sum of exponentials, the process is Markovian and there exists a generator of the process. The difficulty arises when $h$ is not exponential or a sum of exponentials in which case the process is non-Markovian. Another possible generalization is to consider $h$ to be random. Then, we will get a marked point process. (For a discussion on marked point processes, see Cox [3].)

When $\lambda(\cdot)$ is nonlinear, Brémaud and Massoulié [2] proves that under certain conditions, there exists a unique stationary version of the nonlinear Hawkes process and Brémaud and Massoulié [2] also proves the convergence to equilibrium of a non-stationary version, both in distribution and in variation.

In this paper, we will prove the large deviation when $h$ is exponential and $\lambda$ is nonlinear first. Then, we will generalize the proof to the case when $h$ is a sum of exponentials. We will use that to recover the result proved in Bordenave and Torrisi [1]. Finally, we will prove the result for a special class of nonlinear $\lambda$ and general $h$.

\section{An Ergodic Lemma}

Let us prove an ergodic theorem first. Assume $h(t) = \sum_{i=1}^{d} a_i e^{-b_i t}$. Here $b_i > 0$, $a_i \neq 0$ might be negative but we assume that $h(t) > 0$ for any $t \geq 0$. $Z_t = \sum_{\tau_j < t} h(t - \tau_j) = \sum_{i=1}^{d} Z_i(t)$, where $Z_i(t) = \sum_{\tau_j < t} a_i e^{-b_i (t - \tau_j)}$. The domain for $(Z_1(t), \ldots, Z_d(t))$ is $\mathcal{Z} := \mathbb{R}^{\epsilon_1} \times \cdots \times \mathbb{R}^{\epsilon_d}$, where $\mathbb{R}^{\epsilon_i} := \mathbb{R}^+$ or $\mathbb{R}^-$ depending on whether $\epsilon_i = +1$ or $-1$, where $\epsilon_i = +1$ if $a_i > 0$ and $\epsilon_i = -1$ otherwise.

The generator $\mathcal{A}$ for $(Z_1(t), \ldots, Z_d(t))$ is given by
\begin{equation}
\mathcal{A}f = -\sum_{i=1}^{d} b_i z_i \frac{\partial f}{\partial z_i} + \lambda(z_1, \ldots, z_d)[f(z_1 + a_1, \ldots, z_d + a_d) - f(z_1, \ldots, z_d)].
\end{equation}

We want to prove the existence and uniqueness of the invariant probability measure for $(Z_1(t), \ldots, Z_d(t))$. Here the invariance is in time.
The lecture notes [9] by Martin Hairer gives the criterion for the existence and uniqueness of the invariant probability measure for Markov processes.

Suppose we have a jump diffusion process with generator \( \mathcal{L} \). If we can find \( u \) such that \( u \geq 0, \mathcal{L}u \leq C_1 - C_2u \) for some constants \( C_1, C_2 > 0 \), then, there exists an invariant probability measure. We thereby have the following lemma.

**Lemma 1.** Consider \( h(t) = \sum_{i=1}^{d} a_i e^{-b_it} > 0 \). Let \( \epsilon_i = +1 \) if \( a_i > 0 \) and \( \epsilon_i = -1 \) if \( a_i < 0 \). Assume \( \lambda(z_1, \ldots, z_n) \leq \sum_{i=1}^{d} a_i |z_i| + \beta \), where \( \beta > 0 \) and \( a_i > 0 \), \( 1 \leq i \leq d \), satisfies \( \sum_{i=1}^{d} \frac{|a_i|}{b_i} \alpha_i < 1 \). Then, there exists a unique invariant probability measure for \( (Z_1(t), \ldots, Z_d(t)) \).

**Proof.** Try \( u(z_1, \ldots, z_d) = \sum_{i=1}^{d} \epsilon_i c_i z_i \geq 0 \), where \( c_i > 0 \), \( 1 \leq i \leq d \). Then,

\[
Av = -\sum_{i=1}^{d} b_i \epsilon_i c_i z_i + \lambda(z_1, \ldots, z_d) \sum_{i=1}^{d} a_i \epsilon_i c_i
\]

\[
\leq -\sum_{i=1}^{d} b_i c_i |z_i| + \sum_{i=1}^{d} \alpha_i |z_i| \sum_{i=1}^{d} |a_i| c_i + \beta \sum_{i=1}^{d} |a_i| c_i.
\]

Take \( c_i = \frac{b_i}{a_i} > 0 \), we get

\[
Av \leq -\left( 1 - \sum_{i=1}^{d} \frac{|a_i| \alpha_i}{b_i} \right) \sum_{i=1}^{d} \alpha_i |z_i| + \beta \sum_{i=1}^{d} |a_i| \alpha_i + \beta \sum_{i=1}^{d} |a_i| \alpha_i.
\]

Next, we will prove the uniqueness of the invariant probability measure. Let us consider the simplest case \( h(t) = ae^{-bt} \). To get the uniqueness of the invariant probability measure, it is sufficient to prove that for any \( x, y > 0 \), there exists some \( T > 0 \) such that \( \mathcal{P}^x(T, \cdot) \) and \( \mathcal{P}^y(T, \cdot) \) are not mutually singular. Here \( \mathcal{P}^x(T, \cdot) = \mathbb{P}(Z_T^x \in \cdot) \), where \( Z_T^x \) is \( Z_T \) starting at \( Z_0 = x \), i.e. \( Z_T^x = xe^{-bT} + \sum_{\tau_j \leq T} ae^{-b(T-\tau_j)} \).

Let us assume that \( x > y > 0 \). Conditional on the event that \( Z_T^x \) and \( Z_T^y \) have exactly one jump during the time interval \( (0, T) \) respectively, the law of \( \mathcal{P}^x(T, \cdot) \) and \( \mathcal{P}^y(T, \cdot) \) are absolutely continuous with respect to some probability measures with positive density on the sets

\[
((a + x)e^{-bT}, xe^{-bT} + a) \quad \text{and} \quad ((a + y)e^{-bT}, ye^{-bT} + a)
\]

respectively. Choose \( T > \frac{1}{b} \log(\frac{x-y+a}{a}) \), we have

\[
((a + x)e^{-bT}, xe^{-bT} + a) \cap ((a + y)e^{-bT}, ye^{-bT} + a) = \emptyset,
\]

which implies that \( \mathcal{P}^x(T, \cdot) \) and \( \mathcal{P}^y(T, \cdot) \) are not mutually singular.

Similarly, we can prove the uniqueness of the invariant probability measure for the multidimensional case. We need to conditional on the event that we have exactly \( d \) jumps during the time interval \( (0, T) \) for both \( Z_T^x \) and \( Z_T^y \), where \( x, y \in \mathbb{Z}_d \). Then,

\[
Z_T^x = (Z_t^x, \ldots, Z_t^x) \in \mathbb{Z}_d,
\]

where \( Z_t^x = xe^{-bT} + \sum_{\tau_j \leq T} a_i e^{-b(t-\tau_j)} \), \( 1 \leq i \leq d \). Then, \( \mathcal{P}^x(T, \cdot) \) and \( \mathcal{P}^y(T, \cdot) \) are not mutually singular for sufficiently large \( T \). □
3. LDP for Markovian Nonlinear Hawkes Processes with Exponential Exciting Function

We assume first that \( h(t) = ae^{-bt} \), where \( a, b > 0 \), i.e. the process \( Z_t \) jumps upwards an amount \( a \) at each point and decays exponentially between points with rate \( b \). In this case, \( Z_t \) is Markovian.

Notice first that \( Z_0 = 0 \) and

\[
(3.1) \quad dZ_t = -bZ_t dt + adN_t,
\]
which implies that \( N_t = \frac{1}{a}Z_t + \frac{b}{a} \int_0^t Z_s ds \).

We prove first the existence of the limit of the logarithmic moment generating function of \( N_t \).

**Theorem 2.** Assume that \( \lim_{t \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot) \) is bounded below by some positive constant, then, we have

\[
(3.2) \quad \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] = \Gamma(\theta),
\]
where \( \Gamma(\theta) \) is defined as

\[
(3.3) \quad \Gamma(\theta) = \sup_{(\hat{\lambda}, \hat{\pi}) \in Q_c} \left\{ \int \frac{\theta b}{a} z\hat{\pi}(dz) + \int (\hat{\lambda} - \lambda)\hat{\pi}(dz) - \int \left( \log(\hat{\lambda}/\lambda) \right) \hat{\lambda}\hat{\pi}(dz) \right\},
\]
where \( Q_c \) is defined as

\[
(3.4) \quad Q_c = \left\{ (\hat{\lambda}, \hat{\pi}) \in Q : \hat{\lambda} \text{ has unique invariant probability measure } \hat{\pi} \right\},
\]
where

\[
(3.5) \quad Q = \left\{ (\hat{\lambda}, \hat{\pi}) : \hat{\pi} \in \mathcal{M}(\mathbb{R}^+), \int z\hat{\pi} < \infty, \hat{\lambda} \in L^1(\hat{\pi}), \hat{\lambda} > 0 \right\},
\]
and for any \( \hat{\lambda} \) such that \( (\hat{\lambda}, \hat{\pi}) \in Q \), we define the generator \( \hat{\mathcal{A}} \) as

\[
(3.6) \quad \hat{\mathcal{A}} f(z) = -bz \frac{\partial f}{\partial z} + \hat{\lambda}(z)[f(z + a) - f(z)].
\]
for any \( f : \mathbb{R}^+ \to \mathbb{R} \) that is \( C^1 \), i.e. continuously differentiable.

**Proof.** Note that for any real \( \theta \), we have \( \mathbb{E}[e^{\theta N_t}] < \infty \) (That is because we can dominate \( \lambda(z) \) by \( \nu_\epsilon + \epsilon z \) for arbitrarily small \( \epsilon > 0 \). When \( \lambda(z) = \nu_\epsilon + \epsilon z \), i.e. in the linear case, \( \mathbb{E}[e^{\theta N_t}] < \infty \) for any \( \theta \leq C(\epsilon) \), where \( C(\epsilon) \to \infty \) as \( \epsilon \to 0 \) and \( C(\epsilon) \) does not depend on \( \nu_\epsilon \) (we refer to Bordenave and Torrisi [11]).) and

\[
(3.7) \quad \mathbb{E}[e^{\theta N_t}] = \mathbb{E}\left[ e^{\frac{\theta}{a} \int_0^t Z_s ds} \right].
\]

By Dynkin’s formula, for any \( u \) which satisfies \( \mathcal{A}u + Vu \leq Mu \), we have

\[
(3.8) \quad \mathbb{E}\left[ u(Z_t) e^{\int_0^t V(Z_s) ds} \right] = u(Z_0) + \int_0^t \mathbb{E}\left[ (\mathcal{A}u(Z_s) + V(Z_s)u(Z_s)) e^{\int_0^s V(Z_r) dr} \right] ds \leq u(Z_0) + M \int_0^t \mathbb{E}\left[ u(Z_s) e^{\int_0^s V(Z_r) dr} \right] ds,
\]
which implies by Gronwall’s lemma that

\[
(3.9) \quad \mathbb{E}\left[ u(Z_t) e^{\int_0^t V(Z_s) ds} \right] \leq u(Z_0)e^{Mt} = u(0)e^{Mt}.
\]
In our case, \( V(z) = \frac{\theta b}{a} z \). Now for any \( u(z) \geq c_1 e^{\frac{\theta}{a} z} \), we have

\[
\mathbb{E} \left[ e^{\theta N_t} \right] \leq \frac{1}{c_1} \mathbb{E} \left[ u(Z_t) e^{\int_0^t \frac{\theta b}{a} Z_s \, ds} \right] \leq \frac{1}{c_1} u(0) e^{M_t}.
\]

Therefore, we get

\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\theta N_t} \right] \leq \inf_{u \in \mathcal{U}_0} \sup_{z \geq 0} \frac{A u(z) + \frac{\theta b}{a} z u(z)}{u(z)}.
\]

where

\[
\mathcal{U}_0 = \left\{ u \in C^1(\mathbb{R}^+, \mathbb{R}^+) : u(z) = e^{f(z)}, \text{ where } f \in \mathcal{F} \right\},
\]

where

\[
\mathcal{F} = \left\{ f : f(z) = K z + g(z) + L, K > \frac{\theta}{a}, K, L \in \mathbb{R}, g \text{ is } C_1 \text{ with compact support} \right\}.
\]

Define the tilted probability measure \( \hat{P} \) as

\[
\frac{d\hat{P}}{dP} \biggr|_{\mathcal{F}} = \exp \left\{ \int_0^t (\lambda(Z_s) - \hat{\lambda}(Z_s)) \, ds + \int_0^t \log \left( \frac{\hat{\lambda}(Z_s)}{\lambda(Z_s)} \right) \, dN_s \right\}.
\]

Notice that \( \hat{P} \) defined in (3.14) is indeed a probability measure by Girsanov formula. (For the theory of absolute continuity for point processes and its Girsanov formula, we refer to Lipster and Shiryaev [16].)

Now, by Jensen’s inequality, we have

\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\theta N_t} \right]
\]

\[
= \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ \exp \left\{ \theta N_t - \log \frac{d\hat{P}}{dP} \right\} \right]
\]

\[
\geq \liminf_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \theta N_t - \frac{1}{t} \log \frac{d\hat{P}}{dP} \right]
\]

\[
= \liminf_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \theta N_t - \frac{1}{t} \int_0^t (\lambda(Z_s) - \hat{\lambda}(Z_s)) \, ds - \int_0^t \log \left( \frac{\hat{\lambda}(Z_s)}{\lambda(Z_s)} \right) \, dN_s \right].
\]

Since \( N_t - \int_0^t \hat{\lambda}(Z_s) \, ds \) is a (local) martingale under \( \hat{P} \), we have

\[
\hat{E} \left[ \int_0^t \log \left( \frac{\hat{\lambda}(Z_s)}{\lambda(Z_s)} \right) (dN_s - \hat{\lambda}(Z_s) \, ds) \right] = 0.
\]

Therefore, by the usual ergodic theorem, (for a reference, see Chapter 16.4 of Koralov and Sinai [14]), for any \( (\hat{\lambda}, \hat{\pi}) \in \mathcal{Q}_c \), we have,

\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\theta N_t} \right]
\]

\[
\geq \liminf_{t \to \infty} \hat{E} \left[ \frac{1}{t} \theta N_t - \frac{1}{t} \int_0^t (\lambda(Z_s) - \hat{\lambda}(Z_s)) \, ds - \int_0^t \log \left( \frac{\hat{\lambda}(Z_s)}{\lambda(Z_s)} \right) \hat{\lambda}(Z_s) \, ds \right]
\]

\[
= \int \frac{\theta b}{a} \hat{\pi}(dz) + \int (\hat{\lambda} - \lambda) \hat{\pi}(dz) - \int \left( \log(\hat{\lambda}) - \log(\lambda) \right) \hat{\lambda}(dz).
\]
Hence, we have

$$
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}]
\geq \sup_{(\hat{\lambda}, \hat{\pi}) \in Q_e} \left\{ \int \frac{\theta b}{a} z \hat{\pi} + \int (\hat{\lambda} - \lambda) \hat{\pi} - \int \left( \log(\hat{\lambda}) - \log(\lambda) \right) \hat{\lambda} \right\}.
$$

Recall that

$$
\mathcal{F} = \left\{ f : f(z) = K z + g(z) + L, K > \frac{\theta}{a}, K, L \in \mathbb{R}, g \text{ is } C_1 \text{ with compact support} \right\}.
$$

We claim that

$$
\inf_{f \in \mathcal{F}} \left\{ \int \hat{\mathcal{A}} f(z) \hat{\pi}(dz) \right\} = \begin{cases} 0 & \text{if } (\hat{\lambda}, \hat{\pi}) \in Q_e, \\ -\infty & \text{if } (\hat{\lambda}, \hat{\pi}) \in Q \setminus Q_e. \end{cases}
$$

It is easy to see that for $(\hat{\lambda}, \hat{\pi}) \in Q_e$, $g \in C_1$ with compact support, $\int \mathcal{A} g \hat{\pi} = 0$. Next, we can find a sequence $f_n(z) \to z$ pointwisely and $|f_n(z)| \leq \alpha z + \beta$, for some $\alpha, \beta > 0$, where $f_n(z)$ is $C_1$ with compact support. But by our definition for $Q$, $\int z \hat{\pi} < \infty$. So by dominated convergence theorem, $\int \mathcal{A} z \hat{\pi} = 0$. The nontrivial part is to prove that if for any $g \in \mathcal{G} = \{g(z) + L, g \text{ is } C_1 \text{ with compact support}\}$ such that $\int \hat{\mathcal{A}} g \hat{\pi} = 0$, then $(\hat{\lambda}, \hat{\pi}) \in Q_e$. We can easily check the conditions in Echeverría [6]. (For instance, $\mathcal{G}$ is dense in $C(\mathbb{R}^+)$, the set of continuous and bounded functions on $\mathbb{R}^+$ with limit that exists at infinity and $\hat{\mathcal{A}}$ satisfies the minimum principle, i.e. $\hat{\mathcal{A}} f(z_0) \geq 0$ for any $f(z_0) = \inf_{z \in \mathbb{R}^+} f(z)$. This is because at minimum, the first derivative of $f$ vanishes and $\hat{\lambda}(z_0)(f(z_0 + a) - f(z_0)) \geq 0$. The other conditions in Echeverría [6] can also be easily verified.) Thus, Echeverría [6] implies that $\hat{\pi}$ is an invariant measure. Now, our proof in Lemma [1] shows that $\hat{\pi}$ has to be unique as well. Therefore, $(\hat{\lambda}, \hat{\pi}) \in Q_e$. This implies that if $(\hat{\lambda}, \hat{\pi}) \in Q \setminus Q_e$, there exists some $g \in \mathcal{G}$, such that $\int \hat{\mathcal{A}} g \hat{\pi} \neq 0$. Now, any constant multiplier of $g$ still belongs to $\mathcal{G}$ and thus $\inf_{g \in \mathcal{G}} \int \hat{\mathcal{A}} g \hat{\pi} = -\infty$ and hence $\inf_{f \in \mathcal{F}} \int \hat{\mathcal{A}} f \hat{\pi} = -\infty$ if $(\hat{\lambda}, \hat{\pi}) \in Q \setminus Q_e$.

Therefore, we have

$$
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] \geq \sup_{(\hat{\lambda}, \hat{\pi}) \in Q} \inf_{f \in \mathcal{F}} \left\{ \int \frac{\theta b}{a} z \hat{\pi} - \hat{H}(\hat{\lambda}, \hat{\pi}) + \int \hat{\mathcal{A}} f \hat{\pi} \right\}
$$

$$
\geq \sup_{(\hat{\lambda}, \hat{\pi}) \in \mathcal{R}} \inf_{f \in \mathcal{F}} \left\{ \int \frac{\theta b}{a} z \hat{\pi} - \hat{H}(\hat{\lambda}, \hat{\pi}) + \int \hat{\mathcal{A}} f \hat{\pi} \right\},
$$

where $\mathcal{R} = \{ (\hat{\lambda}, \hat{\pi}) : (\hat{\lambda}, \hat{\pi}) \in Q \}$ and

$$
\hat{H}(\hat{\lambda}, \hat{\pi}) = \int \left[ (\hat{\lambda} - \lambda) + \log \left( \frac{\hat{\lambda}}{\lambda} \right) \hat{\lambda} \right] \hat{\pi}.
$$

Define

$$
F(\hat{\lambda}, \hat{\pi}, f) = \int \frac{\theta b}{a} z \hat{\pi} - \hat{H}(\hat{\lambda}, \hat{\pi}) + \int \hat{\mathcal{A}} f \hat{\pi}
$$

$$
= \int \frac{\theta b}{a} z \hat{\pi} - \hat{H}(\hat{\lambda}, \hat{\pi}) - \int b z \frac{\partial f}{\partial z} \hat{\pi} + \int \left( f(z + a) - f(z) \right) \hat{\lambda} \hat{\pi}.
$$
Notice that \( F \) is linear in \( f \) and hence convex in \( f \) and also

\[
\hat{H}(\hat{\lambda}, \hat{\pi}) = \sup_{f \in C_b(\mathbb{R}^+)} \left\{ \int [\hat{\lambda} f + \lambda (1 - e^f)] \hat{\pi} \right\},
\]

where \( C_b(\mathbb{R}^+) \) denotes the set of bounded functions on \( \mathbb{R}^+ \). Inside the bracket above, it is linear in both \( \hat{\pi} \) and \( \hat{\lambda} \hat{\pi} \). Hence \( \hat{H} \) is weakly lower semicontinuous and convex in \( (\hat{\lambda}, \hat{\pi}) \). Therefore, \( F \) is concave in \( (\hat{\lambda}, \hat{\pi}) \). Furthermore, for any \( f = Kz + g + L \in \mathcal{F} \),

\[
F(\hat{\lambda}, \hat{\pi}, f) - \int \left( \frac{\theta}{a} - K \right) bz \hat{\pi} - \hat{H}(\hat{\lambda}, \hat{\pi}) - \int \frac{bz}{\partial z} \hat{\pi} + \int (g(z + a) - g(z)) \hat{\lambda} \hat{\pi} + Ka \int \hat{\lambda} \hat{\pi}.
\]

If \( \lambda_n \pi_n \to \gamma_\infty \) and \( \pi_n \to \pi_\infty \) weakly, then, since \( g \) is \( C_1 \) with compact support, we have

\[
- \int \frac{bz}{\partial z} \pi_n + \int (g(z + a) - g(z)) \lambda_n \pi_n + Ka \int \lambda_n \pi_n \to - \int \frac{bz}{\partial z} \pi_\infty + \int (g(z + a) - g(z)) \gamma_\infty + Ka \int \gamma_\infty,
\]

as \( n \to \infty \). Moreover, in general, if \( P_n \to P \) weakly, then, for any \( f \) which is upper semicontinuous and bounded from above, we have \( \limsup_n \int f dP_n \leq \int f dP \). Since \( \left( \frac{\theta}{a} - K \right) bz \) is continuous and nonpositive on \( \mathbb{R}^+ \), we have

\[
\limsup_{n \to \infty} \int \left( \frac{\theta}{a} - K \right) bz \pi_n \leq \int \left( \frac{\theta}{a} - K \right) bz \pi_\infty.
\]

Hence, we conclude that \( F \) is upper semicontinuous in the weak topology.

In order to switch the supremum and infimum in (3.22), since we have already proved that \( F \) is concave, upper semicontinuous in \( (\hat{\lambda}, \hat{\pi}) \) and convex in \( f \), it is sufficient to prove the compactness of \( \mathcal{R} \) to apply Ky Fan’s minmax theorem (see Fan [7]). Indeed, Joó developed some level set method and proved that it is sufficient to show the compactness of the level set (see Joó [13] and Frenk and Kassay [8]). In other words, it suffices to prove that, for any \( C \in \mathbb{R} \) and \( f \in \mathcal{F} \), the level set

\[
\left\{ (\hat{\lambda}, \hat{\pi}) \in \mathcal{R} : \hat{H} + \int \frac{bz}{\partial z} \hat{\pi} - Kz \hat{\pi} - \hat{\lambda} [f(z + a) - f(z)] \hat{\pi} \leq C \right\}
\]

is compact.

Fix any \( f = Kz + g + L \in \mathcal{F} \), where \( K > \frac{\theta}{a} \) and \( g \) is \( C_1 \) with compact support and \( L \) is some constant, uniformly for any pair \((\hat{\lambda}, \hat{\pi})\) that is in the level set of
there exists some $C_1, C_2 > 0$ such that

(3.30)  
\[
C_1 \geq \hat{H} + \left( K - \frac{\theta}{a} \right) b \int z\hat{\pi} - C_2 \int \hat{\lambda}\hat{\pi} 
\]
\[
\geq \int_{\lambda \leq \epsilon \lambda + \lambda} \left[ \lambda - \hat{\lambda} + \hat{\lambda} \log(\hat{\lambda}/\lambda) \right] \hat{\pi} + \left( K - \frac{\theta}{a} \right) b \int z\hat{\pi} 
\]
\[
- C_2 \int_{\lambda \leq \epsilon \lambda + \lambda} \hat{\lambda}\hat{\pi} - C_2 \int_{\lambda < \epsilon \lambda + \lambda} \hat{\lambda}\hat{\pi} 
\]
\[
\geq \left[ \min_{z \geq 0} \frac{cz + \ell}{\lambda} - 1 - C_2 \right] \int_{\lambda \leq \epsilon \lambda + \lambda} \hat{\lambda}\hat{\pi} + \left[ -c \cdot C_2 + \left( K - \frac{\theta}{a} \right) b \right] \int z\hat{\pi} - \ell C_2.
\]

We choose the parameters such that $0 < c < \left( K - \frac{\theta}{a} \right) \frac{b}{C_2}$ and $\ell$ large enough such that $\min_{z \geq 0} \frac{cz + \ell}{\lambda} - 1 - C_2 > 0$, where we used the fact that $\lim_{z \to \infty} \frac{\lambda(z)}{z} = 0$ and $\min_{z} \lambda(z) > 0$. Hence, we proved that

(3.31)  
\[
\int z\hat{\pi} \leq C_3, \quad \int_{\lambda \leq \epsilon \lambda + \lambda} \hat{\lambda}\hat{\pi} \leq C_4,
\]

where

(3.32)  
\[
C_3 = \frac{C_1 + \ell C_2}{-c \cdot C_2 + \left( K - \frac{\theta}{a} \right) b}, \quad C_4 = \frac{C_1 + \ell C_2}{\min_{z \geq 0} \frac{cz + \ell}{\lambda(z)} - 1 - C_2}.
\]

Therefore, we have

(3.33)  
\[
\int \hat{\lambda}\hat{\pi} = \int_{\lambda \leq \epsilon \lambda + \lambda} \hat{\lambda}\hat{\pi} + \int_{\lambda < \epsilon \lambda + \lambda} \hat{\lambda}\hat{\pi} \leq C_4 + c \cdot C_3 + \ell,
\]

and hence

(3.34)  
\[
\hat{H}(\hat{\lambda}, \hat{\pi}) \leq C_1 + C_2 \left( C_4 + c \cdot C_3 + \ell \right) < \infty.
\]

Therefore, for any $(\lambda_n, \pi_n, \pi_n) \in R$, we get

(3.35)  
\[
\lim_{\ell \to \infty} \sup \int_{z \geq \ell} \pi_n \leq \lim_{\ell \to \infty} \sup \frac{1}{\ell} \int z\pi_n \leq \lim_{\ell \to \infty} \frac{C_3}{\ell} = 0,
\]

which implies the tightness of $\pi_n$. By Prokhorov's Theorem, there exists a subsequence of $\pi_n$ which converges weakly to $\pi_\infty$. We also want to show that there exists some $\gamma_\infty$ such that $\lambda_n \pi_n \to \gamma_\infty$ weakly (passing to a subsequence if necessary). Here, $\pi_n$ may be a subsequence of the original sequence if necessary. It is enough to show that

(i) $\sup_n \int \lambda_n \pi_n < \infty$.

(ii) $\lim_{\ell \to \infty} \sup_n \int_{z \geq \ell} \lambda_n \pi_n = 0$.

(i) and (ii) will give us tightness of $\lambda_n \pi_n$ and hence implies the weak convergence for a subsequence.

Now, let us prove statements (i) and (ii).

To prove (i), notice that

(3.36)  
\[
\sup_n \int \lambda_n \pi_n = \sup_n \int \frac{b}{a} z\pi_n \leq \frac{b}{a} [C_4 + c \cdot C_3 + \ell] < \infty.
\]
To prove (ii), notice that \((\lambda - \lambda_n) + \lambda_n \log(\lambda_n/\lambda) \geq 0\). That is because \(x - 1 - \log x \geq 0\) for any \(x > 0\) and hence
\[
\lambda - \hat{\lambda} + \hat{\lambda} \log(\hat{\lambda}/\lambda) = \hat{\lambda} \left[\frac{(\lambda/\hat{\lambda}) - 1 - \log(\lambda/\hat{\lambda})}{\hat{\lambda}}\right] \geq 0.
\]
Notice that
\[
\lim_{\ell \to \infty} \sup_n \int_{z \geq \ell} \lambda_n \pi_n \leq \lim_{\ell \to \infty} \sup_n \int_{\lambda_n < \sqrt{\lambda_n z \geq \ell}} \lambda_n \pi_n + \lim_{\ell \to \infty} \sup_n \int_{\lambda_n > \sqrt{\lambda_n z \geq \ell}} \lambda_n \pi_n.
\]
For the first term, since \(\sup_n \int z \pi_n < \infty\) and \(\lim_{z \to \infty} \frac{\lambda(z)}{z} = 0\),
\[
\lim_{\ell \to \infty} \sup_n \int_{\lambda_n < \sqrt{\lambda_n z \geq \ell}} \lambda_n \pi_n \leq \lim_{\ell \to \infty} \sup_n \int_{z \geq \ell} \sqrt{\lambda_n z} \pi_n = 0.
\]
For the second term, since \(\lim_{z \to \infty} \frac{\lambda(z)}{z} = 0\),
\[
\lim_{\ell \to \infty} \sup_n \int_{\lambda_n > \sqrt{\lambda_n z \geq \ell}} \lambda_n \pi_n \leq \lim_{\ell \to \infty} \sup_n H(\lambda_n, \pi_n) \sup_{\lambda_n \geq \sqrt{\lambda_n z \geq \ell}} \frac{\lambda_n}{\lambda - \lambda_n + \lambda_n \log(\lambda_n/\lambda)} = 0.
\]
Therefore, passing to some subsequence if necessary, we have \(\lambda_n \pi_n \to \gamma_\infty\) and \(\pi_n \to \pi_\infty\) weakly. Since we proved that \(F\) is upper semicontinuous in the weak topology, hence the level set is compact in the weak topology. Therefore, we can switch the supremum and infimum in (3.22) and get
\[
\lim_{\ell \to \infty} \frac{1}{\ell} \log E [e^{\theta N_\ell}]
\]
\[
\geq \inf_{f \in F} \sup_{\hat{\lambda} < \lambda < \infty} \sup_{\lambda \in L^1(\hat{\lambda})} \left\{ \int \frac{\theta b}{a} z \hat{\lambda} \hat{\pi} - \log(\hat{\lambda}/\lambda) \hat{\lambda} \hat{\pi} + \hat{A} f \hat{\pi} \right\}
\]
\[
= \inf_{f \in F} \sup_{\hat{\lambda} < \lambda < \infty} \int \left\{ \frac{\theta b}{a} z + \lambda(z)(e^{f(z+a)} - f(z) - 1) - b z \partial f \partial z \right\} \hat{\pi}(dz)
\]
\[
= \inf_{f \in F} \sup_{\lambda \geq 0} \left\{ \frac{\theta b}{a} z + \lambda(z)(e^{f(z+a)} - f(z) - 1) - b z \partial f \partial z \right\}
\]
\[
\geq \inf \sup_{u \in L^1} \left\{ \frac{A u}{u} + \frac{\theta b}{a} z \right\}.
\]
We need some justifications. Define \(G(\hat{\lambda}) = \hat{\lambda} - \log(\hat{\lambda}/\lambda) + \hat{A} f\). The supremum of \(G(\hat{\lambda})\) is achieved when \(\frac{dG}{d\hat{\lambda}} = 0\) which implies \(\hat{\lambda} = \lambda e^{f(z+a) - f(z)}\). Notice that for a given \(f \in F\), the optimal \(\hat{\lambda} = \lambda e^{f(z+a) - f(z)}\) satisfies \(\int \hat{\lambda} \hat{\pi} < \infty\) since \(\int \lambda \hat{\pi} < \infty\) and \(\int z \hat{\pi} < \infty\). This gives us (3.43). Notice that for any \(f = Kz + g + L \in F\),
\[
\frac{\theta b}{a} z + \lambda(z)(e^{f(z+a)} - f(z) - 1) - b z \partial f \partial z
\]
\[
= \frac{b(\theta - Ka)}{a} z + \lambda(z)(e^{Ka+g(z+a)} - g(z) - 1) - b z \partial g \partial z,
\]
whose supremum is achieved at some finite \( z^* > 0 \) since \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \), \( K > \frac{\theta}{\rho} \) and \( g \in C^1 \) with compact support. Hence \( \int z \hat{\lambda} < \infty \) is satisfied for the optimal \( \hat{\pi} \). This gives us (3.44). Finally, for any \( f \in \mathcal{F}, u = e^f \in \mathcal{U}_0 \), which implies (3.46). □

Now, we are ready to prove the large deviations result.

**Theorem 3.** Assume \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot) \) is bounded below by some positive constant. Then, \( \left( \frac{N_t}{t} \right) \) satisfies the large deviation principle with the rate function \( I(\cdot) \) as the Fenchel-Legendre transform of \( \Gamma(\cdot) \),

\[
I(x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma(\theta) \}.
\]

**Proof.** If \( \limsup_{z \to \infty} \frac{\lambda(z)}{z} = 0 \), then the forthcoming Lemma 5 implies that \( \Gamma(\theta) < \infty \) for any \( \theta \). Thus, by Gärtner-Ellis Theorem, we have the upper bound. For Gärtner-Ellis Theorem and a general theory of large deviations, see for example [5]. To prove the lower bound, it suffices to show that for any \( x > 0, \epsilon > 0 \), we have

\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{N_t}{t} \in B_\epsilon(x) \right) \geq -\sup_{\theta} \{ \theta x - \Gamma(\theta) \},
\]

where \( B_\epsilon(x) \) denotes the open ball centered at \( x \) with radius \( \epsilon \). Let \( \hat{\mathbb{P}} \) denote the tilted probability measure with rate \( \hat{\lambda} \) as defined in Theorem 2. By Jensen’s inequality, we have

\[
\frac{1}{t} \log \mathbb{P} \left( \frac{N_t}{t} \in B_\epsilon(x) \right) \geq \frac{1}{t} \log \int_{\frac{N_t}{t} \in B_\epsilon(x)} \frac{d\mathbb{P}}{d\hat{\mathbb{P}}} d\hat{\mathbb{P}} = \frac{1}{t} \log \hat{\mathbb{P}} \left( \frac{N_t}{t} \in B_\epsilon(x) \right) + \frac{1}{t} \log \int_{\frac{N_t}{t} \in B_\epsilon(x)} \frac{1}{\hat{\mathbb{P}} \left( \frac{N_t}{t} \in B_\epsilon(x) \right)} \frac{d\mathbb{P}}{d\hat{\mathbb{P}}} d\hat{\mathbb{P}} \]
\[
= \frac{1}{t} \log \hat{\mathbb{P}} \left( \frac{N_t}{t} \in B_\epsilon(x) \right) - \frac{1}{t} \hat{\mathbb{E}} \left[ 1_{\frac{N_t}{t} \in B_\epsilon(x)} \log \frac{d\mathbb{P}}{d\hat{\mathbb{P}}} \right].
\]

By ergodic theorem, we get

\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{N_t}{t} \in B_\epsilon(x) \right) \geq -\Lambda(x),
\]

where \( \Lambda(x) \) is defined as

\[
\Lambda(x) = \inf_{(\hat{\lambda}, \hat{\pi}) \in \mathcal{Q}^x_\epsilon} \left\{ \int (\lambda - \hat{\lambda}) \hat{\pi} + \int \log(\lambda/\hat{\lambda}) \hat{\lambda} \hat{\pi} \right\},
\]

where \( \mathcal{Q}^x_\epsilon \) is defined as

\[
\mathcal{Q}^x_\epsilon = \left\{ (\hat{\lambda}, \hat{\pi}) \in \mathcal{Q}_\epsilon : \int \hat{\lambda}(z) \hat{\pi}(dz) = x \right\}.
\]
Notice that

\begin{equation}
\Gamma(\theta) = \sup_{(\hat{\lambda}, \hat{\pi}) \in Q_e} \left\{ \int \theta \hat{\lambda} \hat{\pi} + \int (\hat{\lambda} - \lambda) \hat{\pi} - \int \log(\hat{\lambda}/\lambda) \hat{\lambda} \hat{\pi} \right\}
\end{equation}

\[= \sup_{x} \sup_{(\hat{\lambda}, \hat{\pi}) \in Q_e^x} \left\{ \int \theta \hat{\lambda} \hat{\pi} + \int (\hat{\lambda} - \lambda) \hat{\pi} - \int \log(\hat{\lambda}/\lambda) \hat{\lambda} \hat{\pi} \right\}
\]

\[= \sup_{x} \{\theta x - \Lambda(x)\}.
\]

We prove in Lemma 4 that \(\Lambda(x)\) is convex in \(x\), identifying it as the convex conjugate of \(\Gamma(\theta)\) thus concluding the proof.

**Lemma 4.** \(\Lambda(x)\) in (3.52) is convex in \(x\).

**Proof.** Define

\begin{equation}
\hat{H}(\hat{\lambda}, \hat{\pi}) = \int (\lambda - \hat{\lambda}) \hat{\pi} + \int \log(\hat{\lambda}/\lambda) \hat{\lambda} \hat{\pi}.
\end{equation}

Then, we have

\begin{equation}
\Lambda(x) = \inf_{(\hat{\lambda}, \hat{\pi}) \in Q_e^x} \hat{H}(\hat{\lambda}, \hat{\pi}).
\end{equation}

We want to prove that \(\Lambda(\alpha x_1 + \beta x_2) \leq \alpha \Lambda(x_1) + \beta \Lambda(x_2)\) for any \(\alpha, \beta \geq 0\) and \(\alpha + \beta = 1\). For any \(\epsilon > 0\), we can choose \((\hat{\lambda}_k, \hat{\pi}_k) \in Q_e^x\) such that \(\hat{H}(\hat{\lambda}_k, \hat{\pi}_k) \leq \Lambda(x_k) + \epsilon/2\), for \(k = 1, 2\). Set

\begin{equation}
\hat{\pi}_3 = \alpha \hat{\pi}_1 + \beta \hat{\pi}_2, \quad \hat{\lambda}_3 = \frac{d(\alpha \hat{\pi}_1)}{d(\alpha \hat{\pi}_1 + \beta \hat{\pi}_2)} \hat{\lambda}_1 + \frac{d(\beta \hat{\pi}_2)}{d(\alpha \hat{\pi}_1 + \beta \hat{\pi}_2)} \hat{\lambda}_2.
\end{equation}

Then, for any test function \(f\), we have

\begin{equation}
\int \hat{A}_3 f \hat{\pi}_3 = \alpha \int \hat{A}_1 f \hat{\pi}_1 + \beta \int \hat{A}_2 f \hat{\pi}_2 = 0,
\end{equation}

which implies that \((\hat{\lambda}_3, \hat{\pi}_3) \in Q_e^x\). Furthermore,

\begin{equation}
\int \hat{\lambda}_3 \hat{\pi}_3 = \alpha \int \hat{\lambda}_1 \hat{\pi}_1 + \beta \int \hat{\lambda}_2 \hat{\pi}_2 = \alpha x_1 + \beta x_2.
\end{equation}

Therefore, we have \((\hat{\lambda}_3, \hat{\pi}_3) \in Q_e^{\alpha x_1 + \beta x_2}\). Finally, since \(x \log x\) is a convex function and if we apply Jensen’s inequality, we get

\begin{equation}
\hat{H}(\hat{\lambda}_3, \hat{\pi}_3) = \int \left[ (\lambda - \hat{\lambda}_3 - \hat{\lambda}_3 \log \lambda) + \hat{\lambda}_3 \log \hat{\lambda}_3 \right] \hat{\pi}_3
\end{equation}

\[\leq \int \left[ (\lambda - \hat{\lambda}_3 - \hat{\lambda}_3 \log \lambda) + \alpha \frac{d\hat{\lambda}_1}{d\hat{\pi}_3} \hat{\lambda}_1 \log \hat{\lambda}_1 + \beta \frac{d\hat{\lambda}_2}{d\hat{\pi}_3} \hat{\lambda}_2 \log \hat{\lambda}_2 \right] \hat{\pi}_3
\]

\[= \alpha \hat{H}(\hat{\lambda}_1, \hat{\pi}_1) + \beta \hat{H}(\hat{\lambda}_2, \hat{\pi}_2).
\]

Therefore, we conclude that

\begin{equation}
\Lambda(\alpha x_1 + \beta x_2) \leq \hat{H}(\hat{\lambda}_3, \hat{\pi}_3) \leq \alpha \hat{H}(\hat{\lambda}_1, \hat{\pi}_1) + \beta \hat{H}(\hat{\lambda}_2, \hat{\pi}_2) \leq \alpha \Lambda(x_1) + \beta \Lambda(x_2) + \epsilon.
\end{equation}
Lemma 5. If \( \limsup_{z \to \infty} \frac{\lambda(z)}{b z} < \frac{1}{a} \), then, for any
\[
\theta < \log \left( \frac{b}{a \limsup_{z \to \infty} \frac{\lambda(z)}{z}} \right) - 1 + \frac{a}{b} \limsup_{z \to \infty} \frac{\lambda(z)}{z},
\]
we have \( \Gamma(\theta) < \infty \). If \( \limsup_{z \to \infty} \frac{\lambda(z)}{z} = 0 \), then \( \Gamma(\theta) < \infty \) for any \( \theta \in \mathbb{R} \).

Proof. For \( K \geq \frac{\theta}{a} \), we have \( e^{Kz} \in U_0 \) and
\[
\Gamma(\theta) \leq \inf_{g \in \mathcal{U}_0} \sup_{z \geq 0} \frac{Ag(z) + \frac{ab}{a} z g(z)}{g(z)} \leq \sup_{z \geq 0} \left\{ \frac{A e^{Kz}}{e^{Kz} + \frac{\theta}{a} z} \right\}
\]
\[
= \sup_{z \geq 0} \left\{ - \left( bK - \frac{\theta}{a} \right) z + \lambda(z) (e^{Kz} - 1) \right\}.
\]
Define the function
\[
F(K) = -K + \limsup_{z \to \infty} \frac{\lambda(z)}{b z} \cdot (e^{Kz} - 1).
\]
Then \( F(0) = 0 \), \( F \) is convex and \( F(K) \to \infty \) as \( K \to \infty \) and its minimum is attained at
\[
K^* = \frac{1}{a} \log \left( \frac{b}{a \limsup_{z \to \infty} \frac{\lambda(z)}{z}} \right) > 0,
\]
and \( F(K^*) < 0 \). Therefore, \( \Gamma(\theta) < \infty \) for any
\[
\theta < -a \min_{K > 0} \left\{ -K + \limsup_{z \to \infty} \frac{\lambda(z)}{b z} \cdot (e^{Kz} - 1) \right\}
\]
\[
= \log \left( \frac{b}{a \limsup_{z \to \infty} \frac{\lambda(z)}{z}} \right) - 1 + \frac{a}{b} \limsup_{z \to \infty} \frac{\lambda(z)}{z} < K^* a.
\]
If \( \limsup_{z \to \infty} \frac{\lambda(z)}{z} = 0 \), trying \( e^{Kz} \in U_0 \) for any \( K > \frac{\theta}{a} \), we have \( \Gamma(\theta) < \infty \) for any \( \theta \).

4. LDP for Markovian Nonlinear Hawkes Processes with Sum of Exponentials Exciting Function

Now, let \( h \) be a sum of exponentials, i.e. \( h(t) = \sum_{i=1}^{d} a_i e^{-b_i t} \) and let
\[
Z_i(t) = \sum_{\tau_j < t} a_i e^{-b_i(t-\tau_j)}, \quad 1 \leq i \leq d,
\]
and \( Z_t = \sum_{i=1}^{d} Z_i(t) = \sum_{\tau_j < t} h(t-\tau_j) \). It is easy to see that \((Z_1, \ldots, Z_d)\) is Markovian in \( \mathbb{R}^d \) and its generator is given by
\[
Af = -\sum_{i=1}^{d} b_i z_i \frac{\partial f}{\partial z_i} + \lambda \left( \sum_{i=1}^{d} z_i \right) \cdot [f(z_1 + a_1, \ldots, z_d + a_d) - f(z_1, \ldots, z_d)],
\]
based on any \( 1 \leq i \leq d \). But \( a_i \) can be negative, as long as \( h(t) = \sum_{i=1}^{d} a_i e^{-b_i t} > 0 \).
In particular, \( h(0) = \sum_{i=1}^{d} a_i > 0 \). If \( a_i > 0 \), then \( Z_i(t) \geq 0 \) almost surely; if \( a_i < 0 \), then \( Z_i(t) \leq 0 \) almost surely.
Theorem 6. Assume \( \lim_{z \to \infty} \frac{\Lambda_2}{z} = 0 \). Then,

\[
(4.3) \quad \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] = \inf_{u \in \mathcal{U}_d} \sup_{(z_1, \ldots, z_d) \in Z} \left\{ \frac{Au}{u} + \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d b_iz_i \right\},
\]

where \( Z = \{(z_1, \ldots, z_d) : a_iz_i \geq 0, 1 \leq i \leq d\} \) and

\[
(4.4) \quad \mathcal{U}_d = \{ u \in C_1(\mathbb{R}^d, \mathbb{R}^+), u = e^f, f \in \mathcal{F} \},
\]

where

\[
(4.5) \quad \mathcal{F} = \left\{ f = g + \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d z_i + L, L \in \mathbb{R}, g \in \mathcal{G} \right\},
\]

where

\[
(4.6) \quad \mathcal{G} = \left\{ \sum_{i=1}^d K_{\epsilon_i}z_i + g, K > 0, g \text{ is } C_1 \text{ with compact support} \right\}.
\]

Proof. Notice that

\[
(4.7) \quad dZ_i(t) = -b_iz_i(t)dt + a_idN_i, \quad 1 \leq i \leq d.
\]

Hence, \( a_iN_i = Z_i(t) - Z_i(0) + \int_0^t b_iZ_i(s)ds \) and

\[
(4.8) \quad \mathbb{E}[e^{\theta N_t}] = \mathbb{E}\left[ \exp\left\{ \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d Z_i(t) - Z_i(0) + \frac{\theta}{\sum_{i=1}^d a_i} \int_0^t \sum_{i=1}^d b_iZ_i(s)ds \right\} \right].
\]

Therefore, by Feynman-Kac formula, we obtain the upper bound

\[
(4.9) \quad \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] \leq \inf_{u \in \mathcal{U}_d} \sup_{(z_1, \ldots, z_d) \in Z} \left\{ \frac{Au}{u} + \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d b_iz_i \right\}.
\]

As before, we can obtain the lower bound

\[
(4.10) \quad \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}]
\]

\[
\geq \sup_{(\lambda, \hat{\lambda}) \in \mathcal{Q}, \hat{\lambda} > \lambda} \int \left[ \frac{\theta}{\lambda} - \lambda + \hat{\lambda} - \hat{\lambda} \log \left( \hat{\lambda}/\lambda \right) \right] \hat{\pi}(dz_1, \ldots, dz_d)
\]

\[
\geq \sup_{(\lambda, \hat{\lambda}) \in \mathcal{Q}, \hat{\lambda} > \lambda} \inf_{f \in \mathcal{F}} \int \left[ \theta \frac{\lambda}{\lambda} - \lambda + \hat{\lambda} - \hat{\lambda} \log \left( \hat{\lambda}/\lambda \right) + \hat{\lambda}g \right] \hat{\pi}
\]

\[
= \sup_{(\lambda, \hat{\lambda}) \in \mathcal{Q}, \hat{\lambda} > \lambda} \inf_{f \in \mathcal{F}} \int \left[ \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d b_iz_i - \lambda + \hat{\lambda} - \hat{\lambda} \log \left( \hat{\lambda}/\lambda \right) + \hat{\lambda}f \right] \hat{\pi}.
\]

The last line above is by taking \( f = g + L + \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d z_i \in \mathcal{F} \) for \( g \in \mathcal{G} \), where

\[
(4.11) \quad \mathcal{G} = \left\{ \sum_{i=1}^d K_{\epsilon_i}z_i + g, K > 0, g \text{ is } C_1 \text{ with compact support} \right\}.
\]

Here, \( \epsilon_i = a_i/|a_i|, 1 \leq i \leq d \). Define

\[
(4.12) \quad F(\hat{\lambda}, \hat{\pi}, f) = \int \left[ \frac{\theta}{\sum_{i=1}^d a_i} \sum_{i=1}^d b_iz_i + \hat{\lambda}f \right] \hat{\pi} - \hat{H}(\hat{\lambda}, \hat{\pi}).
\]
F is linear in f and hence convex in f. Also $\hat{H}$ is weakly lower semicontinuous and convex in $(\hat{\lambda}, \hat{\pi})$. Therefore, F is concave in $(\hat{\lambda}, \hat{\pi})$. Furthermore, for any $f = \frac{\theta}{\sum_{i=1}^{d} a_i} \sum_{i=1}^{d} \epsilon_i z_i + g + L \in \mathcal{F}$,

$$F(\hat{\lambda}, \hat{\pi}, f) = \left[ \theta + \sum_{i=1}^{d} K \epsilon_i a_i \right] \hat{\lambda} - \sum_{i=1}^{d} K \epsilon_i b_i z_i - \hat{H}(\hat{\lambda}, \hat{\pi}) + \int \hat{\lambda} \pi.$$

If $\lambda_n \pi_n \to \gamma_\infty$ and $\pi_n \to \pi_\infty$ weakly, then, since $g$ is $C_1$ with compact support, we have

$$\int \left[ \theta + \sum_{i=1}^{d} K \epsilon_i a_i \right] \lambda_n \pi_n + \int \hat{\lambda} \pi_n \to \int \left[ \theta + \sum_{i=1}^{d} K \epsilon_i a_i \right] \gamma_\infty + \int \hat{\lambda} \pi_\infty.$$

Since $- \sum_{i=1}^{d} K \epsilon_i b_i z_i$ is continuous and nonpositive on $Z$, we have

$$\limsup_{n \to \infty} \int \left[ - \sum_{i=1}^{d} K \epsilon_i b_i z_i \right] \pi_n \leq \int \left[ - \sum_{i=1}^{d} K \epsilon_i b_i z_i \right] \pi_\infty.$$

Hence, we conclude that $F$ is upper semicontinuous in the weak topology.

In order to apply the minmax theorem, we want to prove the compactness of the level set in the weak topology

$$\left\{(\hat{\lambda}, \hat{\pi}) : \int \left[ - \frac{\theta}{\sum_{i=1}^{d} a_i} \sum_{i=1}^{d} \epsilon_i \lambda_i - \hat{\lambda} \right] \hat{\pi} + \hat{H}(\hat{\lambda}, \hat{\pi}) \leq C \right\}.$$

For any $f = \frac{\theta}{\sum_{i=1}^{d} a_i} \sum_{i=1}^{d} \epsilon_i z_i + g + L \in \mathcal{F}$, where $g$ is $C_1$ with compact support etc., there exist some $C_1, C_2 > 0$ such that

$$C_1 \geq \hat{H} + \sum_{i=1}^{d} K b_i \epsilon_i \int z_i \hat{\pi} - C_2 \int \hat{\lambda} \hat{\pi}.$$

$$\geq \int_{\lambda \geq \sum_{i=1}^{d} c_i z_i + \ell} \left[ \lambda - \hat{\lambda} + \lambda \log(\lambda/\lambda) \right] \hat{\pi} + \sum_{i=1}^{d} K b_i \epsilon_i \int z_i \hat{\pi}$$

$$- C_2 \int_{\lambda \geq \sum_{i=1}^{d} c_i z_i + \ell} \hat{\lambda} \hat{\pi} - C_2 \int_{\lambda \geq \sum_{i=1}^{d} c_i z_i + \ell} \hat{\lambda} \hat{\pi}$$

$$\geq \left[ \min_{(z_1, \ldots, z_d) \in Z} \log \frac{c_1 z_1 + \cdots + c_d z_d + \ell}{\lambda (z_1 + \cdots + z_d)} - 1 - C_2 \right] \int_{\lambda \geq \sum_{i=1}^{d} c_i z_i + \ell} \hat{\lambda} \hat{\pi}$$

$$+ \sum_{i=1}^{d} [-c_i \cdot C_2 + K b_i \epsilon_i] \int z_i \hat{\pi} - \ell C_2.$$

If $a_i > 0$, then $\epsilon_i > 0$, pick up $c_i > 0$ such that $-c_i \cdot C_2 + K b_i \epsilon_i > 0$. If $a_i < 0$, then $\epsilon_i < 0$, pick up $c_i$ such that $-c_i \cdot C_2 + K b_i \epsilon_i < 0$. Finally, choose $\ell$ big enough such that the big bracket above is positive. Therefore, we have

$$\int |z_i| \hat{\pi} \leq C_3, \quad \int_{\lambda \geq \sum_{i=1}^{d} c_i z_i + \ell} \hat{\lambda} \hat{\pi} \leq C_4.$$
Hence, \( \int \hat{\lambda} \hat{\pi} \leq C_5 \) and \( \hat{H} \leq C_6 \). We can use the similar method as in the proof of Theorem \( \text{2} \) to show that

\[
\lim_{\ell \to \infty} \sup_{n} \int_{|z_i| > \ell} \lambda_n \pi_n = 0, \quad 1 \leq i \leq d.
\]

For any \( (\lambda_n \pi_n, \pi_n) \in \mathcal{R} \), we can find a subsequence that converges in the weak topology by Prokhorov’s Theorem. Therefore,

\[
\lim_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{E}[e^{\theta N_t}]
\]

\[
\geq \sup_{(\hat{\lambda}, \hat{\pi}) \in \mathcal{Q}} \inf_{f \in \mathcal{F}} \int \left[ \frac{\theta \sum_{i=1}^{d} b_i z_i}{\sum_{i=1}^{d} a_i} - \lambda + \hat{\lambda} - \hat{\lambda} \log \left( \frac{\hat{\lambda}}{\lambda} \right) + \hat{\lambda} f \right] \hat{\pi}
\]

\[
= \inf_{f \in \mathcal{F}} \sup_{\hat{\pi}} \sup_{\hat{\lambda}} \int \left[ \frac{\theta \sum_{i=1}^{d} b_i z_i}{\sum_{i=1}^{d} a_i} - \lambda + \hat{\lambda} - \hat{\lambda} \log \left( \frac{\hat{\lambda}}{\lambda} \right) + \hat{\lambda} f \right] \hat{\pi}
\]

\[
= \inf_{f \in \mathcal{F}} \sup_{(z_1, \ldots, z_d) \in \mathcal{Z}} \theta \sum_{i=1}^{d} b_i z_i + \lambda \left( e^{f(z_1 + a_1 + \ldots + a_d)} - f(z_1 + \ldots + z_d) - 1 \right) - \sum_{i=1}^{d} b_i z_i \frac{\partial f}{\partial z_i}
\]

That is because optimizing over \( \hat{\lambda} \), we get \( \hat{\lambda} = \lambda e^{f(z_1 + a_1 + \ldots + a_d) - f(z_1 + \ldots + z_d) - 1} \) and finally for each \( f \in \mathcal{F} \), \( u = e^f \in \mathcal{U}_0 \). \( \square \)

**Theorem 7.** Assume \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot) \) is bounded below by some positive constant. Then, \( \left( \frac{\lambda}{z} \right) \) satisfies the large deviation principle with the rate function \( I(\cdot) \) as the Fenchel-Legendre transform of \( \Gamma(\cdot) \),

\[
I(x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma(\theta) \},
\]

where

\[
\Gamma(\theta) = \sup_{(\hat{\lambda}, \hat{\pi}) \in \mathcal{Q}_+} \int \left[ \theta \hat{\lambda} - \lambda + \hat{\lambda} - \hat{\lambda} \log \left( \frac{\hat{\lambda}}{\lambda} \right) \right] \hat{\pi}.
\]

**Proof.** The proof is the same as in the case of exponential \( h(\cdot) \). \( \square \)

### 5. LDP for Linear Hawkes Processes: An Alternative Proof

In this section, we use our method to recover the result proved in Bordenave and Torrisi \( \text{[1]} \). We prove the existence of the limit of logarithmic moment generating function first. The strategy is to use the tilting method to prove the lower bound. That requires an ergodic lemma, which we state as Lemma \( \text{[5]} \). For the upper bound, we can optimize over a special class of testing functions for the linear rate with sum of exponential exciting function \( h_n \). Any continuous and integrable \( h \) can be approximated by a sequence \( h_n \). By a coupling argument, we can use that to approximate the upper bound for the logarithmic moment generating function when the exciting function is \( h \). Finally, by tilting argument for the lower bound and Gärtner-Ellis theorem for the upper bound, we can prove the large deviations for the linear Hawkes processes.
Lemma 8. Assume $\lambda(z) = \alpha + \beta z$ and $\mu = \int_0^\infty h(t)dt < \infty$. If $\beta\mu < 1$, then there exists a stationary and ergodic probability measure $\pi$ for $Z_t$ and $\int z\pi = \frac{\alpha\mu}{1 - \beta\mu}$.

Proof. The ergodicity is a well known result for linear Hawkes process. (See Hawkes and Oakes [11].) Let $\pi$ be the invariant probability measure for $Z_t$, then

(5.1) \[ \lim_{t \to \infty} \frac{N_t}{t} = \int \lambda(z)\pi(dz) = \alpha + \beta \int z\pi(dz). \]

If $Z_t$ is invariant in $t$, taking expectations to $Z_t = \int_{-\infty}^t h(t-s)dN_s$,

(5.2) \[ E[Z_t] = \int z\pi(dz) = \int \lambda(z)\pi(dz) \int_{-\infty}^t h(t-s)ds = \mu \int \lambda(z)\pi(dz), \]

which implies that $\int z\pi = \frac{\alpha\mu}{1 - \beta\mu}$. \hfill \Box

Remark 9. In Lemma 8, we assumed that $\lambda(z) = \alpha + \beta z$ and $\beta\|h\|_{L^1} < 1$. However, when do the LDP for linear Hawkes process and when we prove Theorem 11, we assume that $\lambda(z) = \nu + z$ since $\lambda(z) = \nu + \beta z$ is equivalent to the case $\lambda(z) = \nu + z$ if we change $h(\cdot)$ to $\beta h(\cdot)$. The reason we used $\lambda(z) = \alpha + \beta z$ in Lemma 8 is because we need to use that when we tilt $\lambda(z) = \nu + z$ to $K\lambda(z) = K\nu + Kz$ in the proof of lower bound in Theorem 11.

Lemma 10. If $h(t) > 0$, $\int_0^\infty h(t)dt < \infty$, $h(\infty) = 0$ and $h$ is continuous, then $h$ can be approximated by a sum of exponentials both in $L^1$ and $L^\infty$ norms.

Proof. The Stone-Weierstrass Theorem says that if $X$ is a compact Hausdorff space and suppose $A$ is a subspace of $C(X)$ with the following properties. (i) If $f, g \in A$, then $f \times g \in A$. (ii) $1 \in A$. (iii) If $x, y \in X$ then we can find an $f \in A$ such that $f(x) \neq f(y)$. Then $A$ is dense in $C(X)$ in $L^\infty$ norm. Consider $X = \mathbb{R}^+ \cup \{\infty\} = [0, \infty]$ and $C[0, \infty]$ consists of continuous functions vanishing at $\infty$ and the constant function 1.

By Stone-Weierstrass Theorem, the linear combination of $1, e^{-t}, e^{-2t}$ etc. is dense in $C[0, \infty]$. In other words, for any continuous function $h$ on $C[0, \infty]$, we have

(5.3) \[ \sup_{t \geq 0} |h(t) - \sum_{j=0}^n a_j e^{-jt}| \leq \epsilon. \]

In fact, since $h(\infty) = 0$, we get $|a_0| \leq \epsilon$. Thus

(5.4) \[ \sup_{t \geq 0} |h(t) - \sum_{j=1}^n a_j e^{-jt}| \leq 2\epsilon. \]

However, $\sum_{j=1}^n a_j e^{-jt}$ may not be positive. We can approximate $\sqrt{h(t)}$ first by a sum of exponentials and then approximate $h(t)$ by the square of that sum of exponentials, which is again a sum of exponentials but positive this time.

Indeed, we can approximate $h(t)$ by the sum of exponentials in $L^1$ norm as well. Suppose $\|h - h_n\|_{L^\infty} \to 0$, where $h_n$ is a sum of exponentials. Then, by Dominated Convergence Theorem, for any $\delta > 0$, $\int |h - h_n|e^{-\delta t}dt \to 0$ as $n \to \infty$. Thus, we can find a sequence $\delta_n > 0$ such that $\delta_n \to 0$ as $n \to \infty$ and $\int |h - h_n|e^{-\delta_n t}dt \to 0$. By Dominated Convergence Theorem again, $\int h(1 - e^{-\delta_n t})dt \to 0$. Hence, we have $\int |h - h_n e^{-\delta_n t}|dt \to 0$ as $n \to \infty$, where $h_ne^{-\delta_n t}$ is a sum of exponentials.
We will show that $h_n e^{-\delta_n t}$ converges to $h$ in $L^\infty$ as well.

\begin{equation}
\|h - h_n e^{-\delta_n t}\|_{L^\infty} \leq \|h - h_n\|_{L^\infty} + \|h_n - h_n e^{-\delta_n t}\|_{L^\infty}.
\end{equation}

Notice that $(1 - e^{-\delta_n t})h_n \leq (1 - e^{-\delta_n t})(h(t) + \epsilon)$. Since $h(\infty) = 0$, there exists some $M > 0$, such that for $t > M$, $h(t) \leq \epsilon$ so that $(1 - e^{-\delta_n t})h_n \leq 2\epsilon$ for $t > M$. For $t \leq M$, $(1 - e^{-\delta_n t})h_n \leq (1 - e^{-\delta_n M})(\|h\|_{L^\infty} + \epsilon)$ which is small if $\delta_n$ is small. \qed

**Theorem 11.** Assume $\lambda(\cdot) = \nu + z$, $\nu > 0$. $h(\cdot)$ satisfies the assumptions in Lemma 10 and $\int_0^\infty h(t)dt < 1$. We have

\begin{equation}
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] = \nu(x - 1),
\end{equation}

where $x$ is the minimal solution to $x = e^{\theta + \mu(x - 1)}$, where $\mu = \int_0^\infty h(t)dt$.

**Proof.** By Lemma 8, we have

\begin{equation}
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] \geq \sup_{(\lambda, z) \in \mathcal{Q}_s} \int \left[ \theta \hat{\lambda} + \hat{\lambda} - \hat{\lambda} \log \left( \hat{\lambda}/\lambda \right) \right] \cdot \hat{\pi}
\end{equation}

\begin{align*}
&\geq \sup_{(K\lambda, \hat{\lambda}) \in \mathcal{Q}_s, K \in \mathbb{R}^+} \int \left[ \theta \hat{\lambda} + \hat{\lambda} - \hat{\lambda} \log \left( \hat{\lambda}/\lambda \right) \right] \cdot \frac{1}{K} \hat{\pi}
&\geq \sup_{0 < K < \frac{1}{\hat{\mu}}} \left[ \theta + 1 - \frac{1}{K} - \log K \right] \cdot \frac{K \nu}{1 - K \mu}
&= \begin{cases} \nu(x - 1) & \text{if } \theta \in (-\infty, \mu - 1 - \log \mu] \\ +\infty & \text{otherwise} \end{cases},
\end{align*}

where $x$ is the minimal solution to $x = e^{\theta + \mu(x - 1)}$.

By Lemma 10, we can find a sequence of $h_n$, where $h_n(t) = \sum_{i=1}^n a_i e^{-h_i t}$ such that $h_n \to h$ as $n \to \infty$ in both $L^1$ and $L^\infty$ norms. Let $h_e(t) = |h(t) - h_n(t)|$. Then $0 \leq h_n - h_e \leq h \leq h_n + h_e$.

Let $D_1$ be the set of points generated by the Hawkes process with intensity $\lambda(\sum_{\tau \in D_1, \tau < t} h_n(t - \tau))$ and then conditional on $D_1$, let $D_2$ be the set of points generated by the point process with intensity $\lambda(\sum_{\tau \in D_1, \tau < t} h_n(t - \tau)) - \lambda(\sum_{\tau \in D_1, \tau < t} h_n(t - \tau))$ and then iteratively, conditional on $D_1, \ldots, D_{j-1}$, let $D_j$ be the set of points generated by the point process with intensity $\lambda(\sum_{\tau \in \cup_{i=1}^{j-1} D_i, \tau < t} (h_n + h_e)(t - \tau)) - \lambda(\sum_{\tau \in \cup_{i=1}^{j-1} D_i, \tau < t} (h_n + h_e)(t - \tau))$, for any $j \geq 3$. Let $D_j(t)$ correspond to the number of points in $D_j$ by time $t$. Therefore, $\sum_{j=1}^\infty D_j(t)$ equals the number of points generated by Hawkes process with intensity $\lambda(\sum_{\tau < t} (h_n + h_e)(t - \tau))$. Our coupling argument is essentially the same as the one used in Brémaud and Massoulie [2]. For a more formal treatment, one can use Poisson canonical space and Poisson embeddings, we refer to Brémaud and Massoulie [2] for the details.

Assume that $\theta > 0$, we therefore have

\begin{equation}
\mathbb{E}[e^{\theta N_t}] \leq \mathbb{E} \left[ e^{\theta \sum_{j=1}^\infty D_j(t)} \right].
\end{equation}
Now, for any $N \in \mathbb{N}$,

\begin{equation}
E \left[ e^{\theta \sum_{j=1}^{N} D_j(t)} \right]
= E \left[ e^{\theta \sum_{j=1}^{N-1} D_j(t)} e^{e^{\theta - 1} f_j^*(\lambda \sum_{\tau \in U_{j-1}} (h_\tau + h_\tau) (s - \tau)) - \lambda \sum_{\tau \in U_{j-1}} (h_\tau + h_\tau) (s - \tau) ds} \right]
\leq E \left[ e^{\theta \sum_{j=1}^{N-2} D_j(t)} e^{(e^{\theta - 1} - 1)\|h_\tau + h_\tau\|_{L^1} + \theta) D_{N-1}(t)} \right]
\leq \ldots \ldots \leq E \left[ e^{\theta D_1(t) + f_{N-1}(\theta) D_2(t)} \right]
\leq E \left[ e^{\theta D_1(t) + (e^{f_{N-1}(\theta)} - 1)\|h_\tau + h_\tau\|_{L^1} D_1(t)} \right],
\end{equation}

where $f_j(\theta) = (e^{f_j(\theta)} - 1)\|h_\tau + h_\tau\|_{L^1} + \theta$, for $j \geq 2$ and $f_1(\theta) = \theta$. Thus, for any $\theta \leq \|h_\tau + h_\tau\|_{L^1} - 1 - \log((\|h_\tau + h_\tau\|_{L^1}))$, $e^{f_{N-1}(\theta)}$ converges to $y_n$ as $N \to \infty$, where $y_n$ is the minimal solution to $y_n = e^{\theta + \|h_\tau + h_\tau\|_{L^1} (y_n - 1)}$. Since $D_1(t)$ is the Hawkes process with exciting function $h_\tau$,

\begin{equation}
\lim \sup_{t \to \infty} \frac{1}{t} \log E[e^{\theta N_t}] \leq \Gamma_n(p_n \theta),
\end{equation}

where $p_n = 1 + y_n\|h - h_\tau\|_{L^1}$. For $\Gamma_n(p_n \theta)$, we have

\begin{equation}
\Gamma_n(p_n \theta) = \inf_{u \in [0, \theta]} \sup_{(z_1, \ldots, z_n) \in \mathbb{Z}} \left\{ \frac{A u}{u} + \frac{p_n \theta}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} b_i z_i \right\}
\leq \inf_{u \in [0, \theta]} \sup_{(z_1, \ldots, z_n) \in \mathbb{Z}} \left\{ \frac{A u}{u} + \frac{p_n \theta}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} b_i z_i \right\}
= \inf_{c_1, \ldots, c_n \in \mathbb{Z}} \sup_{(z_1, \ldots, z_n) \in \mathbb{Z}} \left\{ - \sum_{i=1}^{n} b_i c_i z_i + (\nu + z_1 + \cdots + z_n) \left( e^{\sum_{i=1}^{n} c_i a_i} - 1 \right)
+ \frac{p_n \theta}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} b_i z_i \right\}
= \nu \left( e^{\sum_{i=1}^{n} c_i a_i} - 1 \right) = \nu (x_n - 1),
\end{equation}

where $c_i$ satisfies $-b_i c_i + \sum_{i=1}^{n} c_i a_i - 1 + \frac{p_n \theta}{\sum_{i=1}^{n} a_i} b_i = 0$, for each $1 \leq i \leq n$. By some computations, it is not hard to see that $x_n = e^{\sum_{i=1}^{n} c_i a_i}$ satisfies

\begin{equation}
x_n = \exp \left\{ p_n \theta + \sum_{i=1}^{n} \frac{a_i}{b_i} (x_n - 1) \right\}
= \exp \left\{ (1 + y_n\|h - h_\tau\|_{L^1}) \theta + (x_n - 1) \int_{0}^{\infty} h_n(t) dt \right\}.
\end{equation}

Since $h_n \to h$ in $L^1$ norm, it is not hard to see that $x_n$ converges to the minimal solution of $x = e^{\theta \|h - h_\tau\|_{L^1} (x - 1)}$ as $n \to \infty$. If $\theta < 0$, consider $h \geq h_n - h_\tau \geq 0$ and the argument is similar.

**Theorem 12.** Assume $\lambda(z) = \nu + z$ and $\mu = \int_{0}^{\infty} h(t) dt < \infty$, $h > 0$ and $h$ is continuous. Then, $(N_t / t \in \cdot)$ satisfies a large deviation principle with the rate
function $I(x)$ given by
\begin{equation}
I(x) = \begin{cases} 
  x \log \left( \frac{x}{\nu + x \mu} \right) - x + \mu x + \nu & \text{if } x \in [0, \infty) \\
  +\infty & \text{otherwise}
\end{cases}.
\end{equation}

Proof. For the upper bound, apply Gärtner-Ellis theorem. For the lower bound, use the tilting method and identify $I(x)$ as the Fenchel-Legendre transform of $\Gamma(\theta)$. \hfill \Box

Remark 13. In Bordenave and Torrisi \cite{1}, their $I(x)$ has the form
\begin{equation}
I(x) = \begin{cases} 
  x \theta_x + \nu - \frac{\nu x}{\nu + x \mu} & \text{if } x \in [0, \infty) \\
  +\infty & \text{otherwise}
\end{cases},
\end{equation}
where $\theta = \theta_x$ is the unique solution in $(-\infty, \mu - 1 - \log \mu]$ of $\mathbb{E}[e^{\theta S}] = \frac{x}{\nu + x \mu}, x > 0$. Here, $\mathbb{E}[e^{\theta S}]$ satisfies the equation
\begin{equation}
\mathbb{E}[e^{\theta S}] = e^\theta \exp \left\{ \mu(\mathbb{E}[e^{\theta S}] - 1) \right\},
\end{equation}
which implies that $\theta_x = \log \left( \frac{x}{\nu + x \mu} \right) - \mu \left( \frac{x}{\nu + x \mu} - 1 \right)$. Substituting into the formula, their rate function is the same as what we got.

6. LDP for a Special Class of Nonlinear Hawkes Processes: An Approximation Approach

In this section, we prove the large deviation results for $(N_t/t, t \in \cdot)$ for a very special class of nonlinear $\lambda(\cdot)$ and $h(\cdot)$ that satisfies the assumptions in Lemma 10.

Let $P_n$ denote the probability measure under which $N_t$ follows the Hawkes process with exciting function $h_n = \sum_{i=1}^n a_i e^{-b_i t}$ such that $h_n \to h$ as $n \to \infty$ in both $L^1$ and $L^\infty$ norms. Let us define
\begin{equation}
\Gamma_n(\theta) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_{P_n}[e^{\theta N_t}] .
\end{equation}
We have the following results.

Lemma 14. For any $K > 0$ and $\theta_1, \theta_2 \in [-K, K]$, there exists some constant $C(K)$ only depending on $K$ such that for any $n$,
\begin{equation}
|\Gamma_n(\theta_1) - \Gamma_n(\theta_2)| \leq C(K) |\theta_1 - \theta_2|.
\end{equation}

Proof. Without loss of generality, assume that $\theta_2 > \theta_1$ such that
\begin{equation}
\Gamma_n(\theta_1) \leq \Gamma_n(\theta_2)
= \sup_{(\lambda, \hat{\pi}) \in \mathcal{Q}_x^*} \int (\theta_2 - \theta_1) \lambda \hat{\pi} + \theta_1 \lambda \hat{\pi} - \hat{H}(\lambda, \hat{\pi})
\leq \sup_{(\lambda, \hat{\pi}) \in \mathcal{Q}_x^*} \int (\theta_2 - \theta_1) \lambda \hat{\pi} + \Gamma_n(\theta_1),
\end{equation}
where
\begin{equation}
\mathcal{Q}_x^* = \left\{ (\lambda, \hat{\pi}) \in \mathcal{Q}_x : \int \theta_1 \lambda \hat{\pi} - \hat{H}(\lambda, \hat{\pi}) \geq \Gamma_n(\theta_1) - 1 \right\} .
\end{equation}
The key is to prove that $\sup_{(\lambda, \pi) \in \mathcal{Q}_e^*} \int \hat{\lambda} \hat{\pi} \leq C(K)$ for some constant $C(K) > 0$ only depending on $K$. Define $u(z_1, \ldots, z_n) = e^{\sum_{i=1}^n c_i z_i}$, where
\begin{equation}
(6.5) \quad c_i = \frac{3K}{\sum_{i=1}^n \frac{a_i}{b_i}} \cdot \frac{1}{b_i}, \quad 1 \leq i \leq n.
\end{equation}
Define $V = -\frac{A}{n}$ such that
\begin{equation}
(6.6) \quad V(z_1, \ldots, z_n) = \frac{3K}{\sum_{i=1}^n \frac{a_i}{b_i}} \sum_{i=1}^n z_i - \lambda(z_1 + \cdots + z_n)(e^{3K} - 1).
\end{equation}
Notice that $\int \hat{A} f \hat{\pi} = 0$ for any test function $f$ with certain regularities. If we try $f = \frac{a_i}{b_i}$, $1 \leq i \leq n$, we get
\begin{equation}
(6.7) \quad - \int z_i \hat{\pi} + \frac{a_i}{b_i} \int \hat{\lambda} \hat{\pi} = 0, \quad 1 \leq i \leq n.
\end{equation}
Summing over $1 \leq i \leq n$, we get
\begin{equation}
(6.8) \quad \int \hat{\lambda} \hat{\pi} = \frac{1}{\sum_{i=1}^n \frac{a_i}{b_i}} \int \sum_{i=1}^n z_i \hat{\pi}.
\end{equation}
Notice that $\sum_{i=1}^n \frac{a_i}{b_i} = \|h_\pi\|_{L^1}$ which is approximately $\|h\|_{L^1}$ when $n$ is large. Since $\lim \sup_{z \to \infty} \frac{\lambda(z)}{z} = 0$ and $\sum_{i=1}^n z_i \geq 0$, we have
\begin{equation}
(6.9) \quad \theta_1 \int \hat{\lambda} \hat{\pi} \leq K \int \hat{\lambda} \hat{\pi} = \frac{K}{\sum_{i=1}^n \frac{a_i}{b_i}} \int \sum_{i=1}^n z_i \hat{\pi} \leq \frac{1}{2} \int V \hat{\pi} + C_{1/2}(K),
\end{equation}
where $C_{1/2}(K)$ is some positive constant only depending on $K$.

We claim that $\int V(z) \hat{\pi} \leq \hat{H}(\hat{\pi})$ for any $\hat{\pi} \in \mathcal{Q}_e^*$. Let us prove it. By ergodic theorem and Jensen inequality, we have
\begin{equation}
(6.10) \quad \int V(z) \hat{\pi} = \lim_{t \to \infty} \mathbb{E}\pi \left[ \frac{1}{t} \int_0^t V(Z_s) ds \right] \leq \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}\pi \left[ e^{\int_0^t V(Z_s) ds} \right] + \hat{H}(\hat{\pi}).
\end{equation}
Next, we will show that $u \geq 1$. That is equivalent to proving that $\sum_{i=1}^n \frac{a_i}{b_i} \geq 0$. Consider the process
\begin{equation}
(6.11) \quad Y_t = \sum_{i=1}^n \frac{Z_i(t)}{b_i} = \sum_{\tau_j < t} \sum_{i=1}^n \frac{a_i}{b_i} e^{-b_i(t - \tau_j)} = \sum_{\tau_j < t} g(t - \tau_j),
\end{equation}
where $g(t) = \sum_{i=1}^n \frac{a_i}{b_i} e^{-b_i t}$. Notice that $g(t) = \int_t^\infty h(s) ds > 0$. Therefore, $Y_t \geq 0$ almost surely and thus $\sum_{i=1}^n \frac{a_i}{b_i} \geq 0$. Since $\frac{A}{n} + V = 0$ and $u \geq 1$, by Feynman-Kac formula and Dynkin’s formula, we have
\begin{equation}
(6.12) \quad \mathbb{E}\pi \left[ e^{\int_0^t V(Z_s) ds} \right] \leq \mathbb{E}\pi \left[ u(Z_t) e^{\int_0^t V(Z_s) ds} \right] \leq u(Z_0) + \int_0^t \mathbb{E}\pi \left[ (A u(Z_s) + V(Z_s) u(Z_s)) e^{\int_0^s V(Z_u) du} \right] ds
\end{equation}
$$= u(Z_0),$$
and therefore $\int V(z) \hat{\pi} \leq \hat{H}(\hat{\pi})$ for any $\hat{\pi} \in \mathcal{Q}_e^*$. Hence,
\begin{equation}
(6.13) \quad \theta_1 \int \hat{\lambda} \hat{\pi} \leq \frac{1}{2} \int V(z) + C_{1/2}(K) \leq \frac{1}{2} \hat{H} + C_{1/2}(K).
\end{equation}
Notice that
\begin{equation}
-\infty < \Gamma_n(\theta_1) - 1 \leq \theta_1 \int \lambda \tilde{\pi} - \tilde{H} \leq \Gamma_n(\theta_1) < \infty.
\end{equation}
Hence, we have
\begin{equation}
\Gamma_n(\theta_1) - 1 + \frac{1}{2} \tilde{H} \leq \theta_1 \int \lambda \tilde{\pi} - \frac{1}{2} \tilde{H} \leq C_{1/2}(K),
\end{equation}
which implies that \( \tilde{H} \leq 2C_{1/2}(K) - \Gamma_n(\theta_1) + 1 \). Hence,
\begin{equation}
\int \lambda \tilde{\pi} - \frac{1}{2} \tilde{H} \leq \frac{1}{2K} \int V_{1/2}(K) \leq \frac{1}{K}(C_{1/2}(K) - \Gamma_n(\theta_1) + 1) + \frac{1}{K}C_{1/2}(K).
\end{equation}
Finally, notice that when \( h \to \hat{h} \) in both \( L^1 \) and \( L^\infty \) norms, we can find a function \( g \) such that \( \sup_n h_n \leq g \) and \( \|g\|_{L^1} < \infty \) and thus
\begin{equation}
\Gamma_n(\theta_1) \geq \Gamma_n(-K) \geq \Gamma_{\hat{g}}(-K),
\end{equation}
where \( \Gamma_g \) denotes the case when the rate function is still \( \lambda(\cdot) \) but the exciting function is \( g(\cdot) \) (instead of \( h_n(\cdot) \)). Notice that here \( \|g\|_{L^1} < \infty \) but it may not be less than 1. It is still well defined because of the assumption \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \). Indeed, we can find \( \lambda(z) = \nu_\epsilon + \epsilon z \) that dominates the original \( \lambda(\cdot) \) for \( \nu_\epsilon > 0 \) big enough and \( \epsilon > 0 \) is small enough such that \( \epsilon \|g\|_{L^1} < 1 \). Now, we have \( \Gamma_{\hat{g}}(-K) \geq \Gamma_{\nu_\epsilon}(-K) \) which is finite (see Theorem 11), where \( \Gamma_{\nu_\epsilon}(-K) \) corresponds to the case when \( \lambda(z) = \nu_\epsilon + \epsilon z \). Hence, we conclude that
\begin{equation}
\sup_{(\lambda, \pi) \in \mathcal{Q}_2} \int \lambda \tilde{\pi} \leq C(K),
\end{equation}
for some \( C(K) > 0 \) only depending on \( K \).

\begin{lemma}
Assume that \( \lambda(\cdot) \geq c \) for some \( c > 0 \), \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot)^\alpha \) is Lipschitz with constant \( L_\alpha \) for any \( \alpha \geq 1 \). For any \( K > 0 \), \( \Gamma_n(\theta) \) is Cauchy with \( \theta \) uniformly in \([−K, K] \).
\end{lemma}

\begin{proof}
Let us write \( H_n(t) = \sum_{\tau_j < t} h_n(t - \tau_j) \). Observe first that, for any \( q \),
\begin{equation}
\exp \left\{ q \int_0^t \log \left( \frac{\lambda(H_m(s))}{\lambda(H_n(s))} \right) dN_s - \int_0^t \left( \frac{\lambda(H_m(s))}{\lambda(H_n(s))} \right)^q dN_s - \lambda(H_n(s)) \right\} \end{equation}
is a martingale under \( P_n \). By Hölder’s inequality, for any \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), we get
\begin{equation}
\mathbb{E}^P_n [e^{\theta N_t}] = \mathbb{E}^P_n \left[ e^{\theta N_t} \frac{dP_m}{dP_n} \right] \\
= \mathbb{E}^P_n \left[ e^{\theta N_t - \int_0^t (\lambda(H_m(s)) - \lambda(H_n(s))) ds} - \int_0^t \log \left( \frac{\lambda(H_m(s))}{\lambda(H_n(s))} \right) dN_s \right] \\
\leq \mathbb{E}^P_n \left[ e^{\theta N_t - p \int_0^t (\lambda(H_m(s)) - \lambda(H_n(s))) ds} \right]^{1/p} \mathbb{E}^P_n \left[ e^{q \int_0^t \log \left( \frac{\lambda(H_m(s))}{\lambda(H_n(s))} \right) dN_s} \right]^{1/q}.
\end{equation}
By Cauchy-Schwarz inequality, we get

\begin{align}
\mathbb{E} P_n \left[ e^{\frac{1}{q} \int_0^1 \log \left( \frac{\lambda(H_m(s))}{\lambda(H_{m,n}(s))} \right) dN_s} \right]^{1/q} & \leq \mathbb{E} P_n \left[ e^{\int_0^1 \left( \frac{\lambda(H_m(s)) - \lambda(H_{m,n}(s))}{\lambda(H_{m,n}(s))} \right) dN_s} \right]^{\frac{1}{q}} \\
& \leq \mathbb{E} P_n \left[ e^{C \frac{L_{2q} \int_0^1 \sum_{\tau<\tau_0} |h_m(s)-h_{m,n}(s)| ds}{2q}} \right]^{\frac{1}{q}} \\
& \leq \mathbb{E} P_n \left[ e^{\frac{1}{p} L_{2q} \|h_m-h_{m,n}\|_{L^1}} \right]^{\frac{1}{q}}.
\end{align}

We also have

\begin{align}
\mathbb{E} P_n \left[ e^{\int_0^1 \lambda(H_{m,n}(s)) dN_s} \right]^{1/p} & \leq \mathbb{E} P_n \left[ e^{\int_0^1 \lambda(H_m(s)) dN_s} \right]^{1/p}.
\end{align}

Therefore, by Lemma 13 and the fact \( \Gamma_n(0) = 0 \) for any \( n \), we have

\begin{align}
\Gamma_m(\theta) - \Gamma_n(\theta) & \leq \frac{1}{p} \Gamma_n \left( p\theta + pL_1 \epsilon_{m,n} \right) + \frac{1}{2q} \Gamma_n \left( \frac{L_{2q} \epsilon_{m,n}}{c^{2q-1}} \right) - \Gamma_n(\theta) \\
& \leq C(K) L_1 \epsilon_{m,n} + \frac{C(K)}{2q} \frac{L_{2q} \epsilon_{m,n}}{c^{2q-1}} + \frac{1}{p} \Gamma_n(p\theta) - \frac{1}{p} \Gamma_n(\theta) + \left( 1 - \frac{1}{p} \right) |\Gamma_n(\theta)|, \\
& \leq C(K) L_1 \epsilon_{m,n} + \frac{C(K)}{2q} \frac{L_{2q} \epsilon_{m,n}}{c^{2q-1}} + \frac{C(K)(p-1)K}{p} + \left( 1 - \frac{1}{p} \right) C(K) K,
\end{align}

where \( \epsilon_{m,n} = \|h_m-h_{m,n}\|_{L^1} \). Hence, we have

\begin{align}
\limsup_{m,n \to \infty} \left\{ \Gamma_m(\theta) - \Gamma_n(\theta) \right\} & \leq 2 \left( 1 - \frac{1}{p} \right) C(K) K, \\
\text{which is true for any } p > 1. \text{ Letting } p \downarrow 1, \text{ we get the desired result.} \quad \square
\end{align}

\textbf{Remark 16}. If \( \lambda(\cdot) \geq c > 0 \) and \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) for any \( \alpha > 0 \), then \( \lambda(\cdot)^\alpha \) is Lipschitz for any \( \sigma \geq 1 \). For instance, \( \lambda(z) = [\log(z+c)]^\beta \) satisfies the conditions, where \( \beta > 0 \) and \( c > 1 \).

\textbf{Theorem 17}. Assume that \( \lambda(\cdot) \geq c > 0 \) for some \( c > 0 \), \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot)^\alpha \) is Lipschitz with constant \( L_\alpha \) for any \( \alpha \geq 1 \). We have \( \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] \) exists for any \( \theta \in \mathbb{R} \) and

\begin{align}
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t}] = \Gamma(\theta),
\end{align}

where \( \Gamma(\theta) = \lim_{n \to \infty} \Gamma_n(\theta) \).

\textit{Proof}. By Lemma 15, we have that \( \Gamma(\theta) = \lim_{n \to \infty} \Gamma_n(\theta) \) exists and \( \Gamma_n(\theta) \) converges to the limit uniformly on any compact set \([-K,K]\). Since \( \Gamma_n(\theta) \) is Lipschitz by Lemma 13, it is continuous and the limit \( \Gamma \) is also continuous. Let \( \epsilon_n = \|h_n-h\|_{L^1} \leq \epsilon \). As in the proof of Lemma 15 for any \( \theta \in [-K,K], p, q > 1, \)
\[ \frac{1}{p} + \frac{1}{q} = 1, \text{ we get} \]

\begin{equation}
(6.26) \quad \limsup_{t \to \infty} \frac{1}{t} \log E[e^{\theta N_t}]
\end{equation}

\[ \leq \Gamma_n(\theta) + C(K)L_{1\epsilon_n} + \frac{C(K)}{2q} \cdot \frac{L_{2q\epsilon_n}}{e^{2q-1}} + 2 \left( 1 - \frac{1}{p} \right) C(K)K. \]

Letting \( n \to \infty \) first and then \( p \downarrow 1 \), we get

\[ \limsup_{t \to \infty} \frac{1}{t} \log E[e^{\theta N_t}] \leq \Gamma(\theta). \]

Similarly, for any \( p', q' > 1, \frac{1}{p'} + \frac{1}{q'} = 1, \)

\begin{equation}
(6.27) \quad \Gamma_n(\theta) \leq \liminf_{t \to \infty} \frac{1}{pt} \log E[e^{(p\theta + pL_{1\epsilon_n})N_t}] + \liminf_{t \to \infty} \frac{1}{2qt} \log E \left[ \frac{L_{2q\epsilon_n}}{e^{2q-1}} \right] \quad N_t \]

\[ \leq \liminf_{t \to \infty} \frac{1}{pp't} \log E[e^{p\theta N_t}] + \liminf_{t \to \infty} \frac{1}{pq't} \log E[e^{q'pL_{1\epsilon_n}N_t}] \]

\[ + \liminf_{t \to \infty} \frac{1}{2q't} \log E \left[ \frac{L_{2q\epsilon_n}}{e^{2q-1}} \right]. \]

Since we can dominate \( \lambda(\cdot) \) by the linear case \( \lambda(z) = \nu + z \) and under linear case the limit of logarithmic moment generating function \( \Gamma(\theta) \) is continuous in \( \theta \), letting \( n \to \infty \), we have

\begin{equation}
(6.28) \quad \Gamma(\theta) \leq \liminf_{t \to \infty} \frac{1}{pp't} \log E[e^{p\theta N_t}] . \end{equation}

This holds for any \( \theta \) and thus

\begin{equation}
(6.29) \quad \liminf_{t \to \infty} \frac{1}{t} \log E[e^{\theta N_t}] \geq pp' \Gamma \left( \frac{\theta}{pp'} \right). \end{equation}

Letting \( p, p' \downarrow 1 \) and using the continuity of \( \Gamma(\cdot) \), we get the desired result. \( \square \)

**Theorem 18.** Assume that \( \lambda(\cdot) \geq c \) for some \( c > 0 \), \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot)^{\alpha} \) is Lipschitz with constant \( L_\alpha \) for any \( \alpha \geq 1 \). We have \( (N_t/t \in \cdot) \) satisfies the large deviation principle with the rate function

\begin{equation}
(6.30) \quad I(x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma(\theta) \}. \end{equation}

**Proof.** For the upper bound, apply Gärtner-Ellis Theorem. Let us prove the lower bound. Let \( B_r(x) \) denote the open ball centered at \( x \) with radius \( r > 0 \). By Hölder’s inequality, for any \( p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \), we have

\begin{equation}
(6.31) \quad P \left( \frac{N_t}{t} \in B_r(x) \right) \leq \left\| \frac{dP_n}{dP} \right\|_{L^p(\mathbb{P})} \mathbb{P} \left( \frac{N_t}{t} \in B_r(x) \right)^{1/q}. \end{equation}

Therefore, letting \( t \to \infty \), we have

\begin{equation}
(6.32) \quad \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma_n(\theta) \} = \lim_{t \to \infty} \frac{1}{t} \log P \left( \frac{N_t}{t} \in B_r(x) \right)
\end{equation}

\[ \leq \frac{1}{p} \Gamma(pp' L_{1\epsilon_n}) + \frac{1}{2pq'} \Gamma \left( \frac{L_{2q'\epsilon_n}}{e^{2q'-1}} \right) + \frac{1}{q} \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{N_t}{t} \in B_r(x) \right), \]

where \( \epsilon_n = \| h_n - h \|_{L^1} \). Hence, letting \( n \to \infty \), we have

\begin{equation}
(6.33) \quad \frac{1}{q} \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{N_t}{t} \in B_r(x) \right) \geq \limsup_{n \to \infty} \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma_n(\theta) \}. \end{equation}
We know that \( \Gamma_n(\theta) \to \Gamma(\theta) \) uniformly on any compact set \( K \), thus

\[
\sup_{\theta \in K} \{ \theta x - \Gamma_n(\theta) \} \to \sup_{\theta \in K} \{ \theta x - \Gamma(\theta) \},
\]

as \( n \to \infty \) for any compact set \( K \). Notice that \( \lambda(\cdot) \geq c > 0 \) and recall that the limit for the logarithmic moment generating function with parameter \( \theta \) for Poisson process with constant rate \( c \) is \((e^\theta - 1)c\). Hence

\[
\liminf_{\theta \to +\infty} \frac{\Gamma_n(\theta)}{\theta^\alpha} \geq \liminf_{\theta \to +\infty} \frac{(e^\theta - 1)c}{\theta} = +\infty,
\]

which implies that \( \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma_n(\theta) \} \to \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma(\theta) \} \). Therefore,

\[
\frac{1}{q} \liminf_{r \to \infty} \frac{1}{t} \log \mathbb{P}\left( \frac{N_t}{t} \in B_r(x) \right) \geq \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma(\theta) \}.
\]

Letting \( q \downarrow 1 \), we get the desired result. \( \square \)

**Remark 19.** The class of nonlinear Hawkes process with general exciting function \( h \) for which we proved the large deviation principle here is unfortunately a big too special. It works for the rate function like \( \lambda(z) = (\log(c + z))^3 \) for example but does not work for \( \lambda(\cdot) \) that has sublinear polynomial growth. In fact, by the coupling argument we used in the proof of the case of linear \( \lambda(\cdot) \) in Theorem 11, we can prove that in the case when \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \) and \( \lambda(\cdot) \) is \( \alpha \)-Lipschitz and \( \lambda(\cdot) \geq c > 0 \), \( \Gamma(\theta) = \lim_{n \to \infty} \Gamma_n(\theta) \) for \( \theta \leq \mu - 1 - \log \mu \), where \( \mu = \int_0^\infty h(t)dt \) and \( \Gamma \) and \( \Gamma_n \) are the limit of logarithmic moment generating functions when the exciting functions are \( h \) and \( h_n \) respectively and \( h_n \to h \) in \( L^1 \). For the linear case, since \( \Gamma(\theta) = \infty \) for \( \theta \geq \mu - 1 - \log \mu \), the coupling argument is good enough. However, for the sublinear \( \lambda(\cdot) \), \( \Gamma(\theta) < \infty \) for any \( \theta \) and the coupling argument is not enough. In fact, it will appear in Zhu [20] that under the condition that \( \lim_{z \to \infty} \frac{\lambda(z)}{z} = 0 \), \( \lambda(\cdot) \) is positive, increasing, \( \alpha \)-Lipschitz and \( \lambda(\cdot) \geq c > 0 \) and \( h(\cdot) \) is positive, decreasing, and \( \int_0^\infty h(t)dt < \infty \), there is a level-3 large deviation principle from which we can use the contraction principle to get the level-1 large deviation principle for \( (N_t/t \in \cdot) \). Therefore, we conjecture that in the sublinear case, \( \Gamma(\theta) = \lim_{n \to \infty} \Gamma_n(\theta) \) for any \( \theta \) and \( (N_t/t \in \cdot) \) satisfies the large deviation principle with rate function \( \Gamma(x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Gamma(\theta) \} \). The advantage of approximating the general case by the case when \( h \) is a sum of exponentials is that \( \Gamma_n(\theta) \) can be evaluated as an optimization problem, which should be computable by some numerical schemes.

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