YOUNG TYPE INEQUALITIES FOR WEIGHTED SPACES

JOACHIM TOFT, KAROLINE JOHANSSON, STEVAN PILIPOVIĆ, AND NENAD TEOFANOV

Abstract. We establish sharp convolution and multiplication estimates in weighted Lebesgue, Fourier Lebesgue and modulation spaces. Especially we recover some results in [2, 6].

0. Introduction

The aim of the paper is to establish Hölder-Young type properties for convolution and multiplications on weighted Lebesgue, Fourier Lebesgue and modulation spaces.

A frequently used convolution property concerns Young’s inequality, which in terms of the Young functional

\[ R(p) \equiv 2 - \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{p_2}, \quad p = (p_0, p_1, p_2) \in [1, \infty]^3, \]

asserts that

\[ L_{t_1}^{p_1} * L_{t_2}^{p_2} \subseteq L_{-t_0}^{p_0}, \]

when \( R(p) = 0 \),

\[ t_0 + t_1 \geq 0, \quad t_0 + t_2 \geq 0, \quad \text{and} \quad t_1 + t_2 \geq 0. \]

We note that the latter inequalities imply

\[ t_0 + t_1 + t_2 \geq 0. \]

Here \( L_t^p \) is the weighted Lebesgue space with parameters \( p \) and \( t \), and consists of all measurable functions \( f \) on \( \mathbb{R}^d \) such that \( f \cdot \langle \cdot \rangle^t \in L^p \), where \( \langle x \rangle = (1 + |x|^2)^{1/2} \). Furthermore, if \( p \in [1, \infty] \), then \( p' \in [1, \infty] \) is the conjugate exponent for \( p \), i.e. \( 1/p + 1/p' = 1 \).

Especially we are interested to find conditions on \( t_j, j = 0, 1, 2 \), such that (0.2) holds when \( R(p) \) in (0.1) stays somewhere in the interval \([0, 2^{-1}])\).

A rough estimate is obtained by an appropriate application of Hölder’s inequality on Young’s inequality, here above. More precisely, by rewriting \( f_j(x) \cdot \langle x \rangle^{t_j} \) into \( (f_j(x) \cdot \langle x \rangle^{\sigma_j}) \cdot \langle x \rangle^{t_j-\sigma_j} \), it follows by applying Hölder’s inequality on the \( L_{t_j}^{p_j} \)-norms in Young’s inequality that the following result holds true.

2000 Mathematics Subject Classification. 44A35, 42B05, 46E35, 46F99.

Key words and phrases. Fourier, Lebesgue, modulation, sharpness.
Proposition 0.1. Let \( t_j \in \mathbb{R}, p_j \in [1, \infty], j = 0, 1, 2 \) and let \( R(p) \) be given by (0.1). Also assume that \( 0 < R(p) \leq 1/2 \), (0.3) holds true with at least two inequalities strict, and that
\[
t_0 + t_1 + t_2 > d \cdot R(p) \quad (0.4)^{''}
\]
holds. Then \( L^p_{t_1} * L^p_{t_2} \subseteq L^{p_0}_{-t_0} \).

We remark that Proposition 0.1 holds true also after removing the condition \( R(q) \leq 1/2 \).

In contrast to Young’s inequality here above, the most of the inequalities in (0.3) and (0.4) in Proposition 0.1 are strict, and if it is possible to replace any such strict inequality by a non-strict one, then the situation is improved. On the other hand, it seems not to be possible to perform such improvement by only using Hölder’s inequality in such simple way as described here above.

In this paper we use the framework in Chapter 8 in [2] and Section 3 in [6] to decompose the involved functions in the convolutions in convenient ways. These investigations lead to Theorem 2.2 in Section 2, which in particular gives the following improvement of Proposition 0.1.

Proposition 0.1'. Let \( t_j \in \mathbb{R}, p_j \in [1, \infty], j = 0, 1, 2 \) and let \( R(p) \) be given by (0.1). Also assume that \( 0 < R(p) \leq 1/2 \), (0.3) holds true, and that
\[
t_0 + t_1 + t_2 \geq d \cdot R(p) \quad (0.4)^{''}
\]
holds, with strict inequality in (0.4)'' when \( t_j = d \cdot R(p) \) for some \( j = 0, 1, 2 \). Then \( L^p_{t_1} * L^p_{t_2} \subseteq L^{p_0}_{-t_0} \).

Furthermore, if \( t_j \neq d \cdot R(p) \), then we prove that Proposition 0.1 is optimal in the sense that if (0.3) or (0.4)'' are violated, then \( L^p_{t_1} * L^p_{t_2} \) is not continuously embedded in \( L^{p_0}_{-t_0} \).

Obviously, except for a few cases, the strict inequalities in Proposition 0.1 have been replaced by non-strict ones in Proposition 0.1'. The Hörmander theorem [2, Theorem 8.3.1] on microlocal regularity of a product is obtained by choosing \( p_0 = p_1 = p_2 = 2 \) in Proposition 0.1', and note that in contrast to Proposition 0.1, the latter theorem is not covered by Proposition 0.1'. We remark that the results in [2] are given in the framework of weighted Sobolev spaces of the form \( H^2_s \), and the analysis is based on an intensive use of their Hilbert space structure. On the other hand, here (as well as in [6]) our result considerations include Banach spaces which might not be Hilbert spaces, and thereby use a more sophisticated techniques in the proofs are needed.

Finally we remark that Theorem 2.2 leads to Theorem 2.4 in Section 2 which concerns convolution properties for modulation spaces. In particular, if \( t_j, p_j \) and \( R(p) \) are the same as in Proposition 0.1', and
that
\[ \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2} = 1, \quad \text{and} \quad 0 \leq s_0 + s_1 + s_2, \]
then it follows from Theorem \[2.4\] \( M_{s_1,t_1}^{p_1,q_1} * M_{s_2,t_2}^{p_2,q_2} \subseteq M_{s_0,t_0}^{p_0,q_0}. \)

**ACKNOWLEDGMENT**

This research is supported by Ministry of Education, Science and Technological Development of Serbia through the Project no. 174 024.

1. Preliminaries

In this section we review notions and notation, and discuss basic preliminary results. We put \( \mathbb{N} = \{0, 1, 2, \ldots\} \), and \( A \lesssim B \) to indicate \( A \leq cB \) for a suitable constant \( c > 0 \). Any extension of the \( L^2 \)-scalar product on \( C_0^\infty(\mathbb{R}^d) \) is denoted by \( (\cdot, \cdot)_{L^2} = (\cdot, \cdot) \).

The scalar product of \( x \) and \( \xi \) in \( \mathbb{R}^d \) is denoted by \( \langle x, \xi \rangle \). For \( p \in [1, \infty] \) we let \( p' \in [1, \infty] \) denote the conjugate exponent \( (1/p + 1/p' = 1) \).

The Fourier transform \( \mathcal{F} \) is the operator on \( S_0'((\mathbb{R}^d)) \) which takes the form
\[ (\mathcal{F}f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int f(x)e^{-i\langle x, \xi \rangle} \, dx, \quad \xi \in \mathbb{R}^d, \]
when \( f \in L^1(\mathbb{R}^d) \).

The (weighted) Fourier Lebesgue space \( \mathcal{F}L^q_s(\mathbb{R}^d) \), \( s \in \mathbb{R} \) is the Banach space which consists of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that
\[ \|f\|_{\mathcal{F}L^q_s} \equiv \|\hat{f} \cdot \langle \cdot \rangle^s\|_{L^q} \quad (1.1) \]
is finite. Here and in what follows, \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \).

Let \( X \) be an open set in \( \mathbb{R}^d \). Then the local Fourier Lebesgue space \( \mathcal{F}L^q_{s,loc}(X) \) consists of all \( f \in \mathcal{S}'(X) \) such that \( \varphi f \in \mathcal{F}L^q_s(\mathbb{R}^d) \) for every \( \varphi \in C_0^\infty(X) \). The topology in \( \mathcal{F}L^q_{s,loc}(X) \) is defined by the family of seminorms \( f \mapsto \|\varphi f\|_{\mathcal{F}L^q_s(\varphi)} \), where \( \varphi \in C_0^\infty(X) \).

We note that
\[ \mathcal{F}L^q_s(\mathbb{R}^d) \big|_X \subseteq \mathcal{F}L^q_{s,loc}(X). \quad (1.2) \]

and
\[ \mathcal{F}L^q_{s_1,loc}(X) \subseteq \mathcal{F}L^q_{s_2,loc}(X), \quad \text{when } q_1 \leq q_2 \text{ and } s_2 \leq s_1. \quad (1.3) \]
(See e. g. [5].)

Next we define modulation spaces. Let \( \phi \in \mathcal{S}'(\mathbb{R}^d) \setminus 0 \) be fixed. Then the short-time Fourier transform of \( f \in \mathcal{S}'(\mathbb{R}^d) \) with respect to \( \phi \) is defined by
\[ (V_\phi f)(x, \xi) = \mathcal{F}(f \cdot \overline{\phi(\cdot - x)})(\xi). \]
Here the left-hand side makes sense, since it is the partial Fourier transform of the tempered distribution \( F(x, y) = (f \otimes \bar{\phi})(y, y - x) \) with respect to the \( y \)-variable. We also note that if \( f, \phi \in \mathcal{S}(\mathbb{R}^d) \), then \( V_\phi f \) takes the form
\[
V_\phi f(x, \xi) = (2\pi)^{-d/2} \int f(y) \bar{\phi}(y - x) e^{-i\langle y, \xi \rangle} \, dy.
\]

Let \( s, t \in \mathbb{R} \) and \( p, q \in [1, \infty] \) be fixed. Then the modulation space \( M_{s,t}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that
\[
\|f\|_{M_{s,t}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi) \langle x \rangle^s \langle \xi \rangle^t \, dx \right)^q / \xi \, d\xi \right)^{1/q}
\]
is finite (with obvious interpretation of the integrals when \( p = \infty \) or \( q = \infty \)). In the same way, the modulation space \( W_{s,t}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that
\[
\|f\|_{W_{s,t}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi) \langle x \rangle^s \langle \xi \rangle^t \, dx \right)^p / \xi \, d\xi \right)^{1/p}
\]
is finite.

2. Multiplication and convolution properties

In this section we derive multiplication and convolution results on Lebesgue, Fourier Lebesgue and modulation spaces. In particular, we extend some results in [6]. The proofs of the theorems are postponed to Section 3. Our main results are Theorems 2.2 and 2.4. Here we present sufficient conditions on \( t_j \in \mathbb{R} \) and \( p_j \in [1, \infty] \), \( j = 0, 1, 2 \), to ensure that \( v_1 \ast v_2 \in L_{t_0}^{p_0} \) when \( v_j \in L_{t_j}^{p_j} \), \( j = 1, 2 \), and similarly when the convolution product and Lebesgue spaces are replaced by multiplication and Fourier-Lebesgue spaces. The results also include related multiplication and convolution properties for modulation spaces.

Certain parts of the analysis concern reformulation of \( 0 \leq R(p) \leq 1/2 \) into equivalent statements. For convenience we let
\[
G(x) = G(x_0, x_1, x_2) = 2 - \sum_{j=0}^{2} x_j,
\]
and note that \( R(p) \) is equal to \( G(x) \) when \( x_j = 1/p_j \). We also let
\[
H_0(x) = \max_{\pi \in S_3} \left( \min \left( x_{\pi(0)}, \max \left( \frac{1}{2}, \min(x_{\pi(1)}, x_{\pi(2)}) \right) \right) \right),
\]
\[
H_1(x) = \begin{cases} 
\max(x_0, x_1, x_2), & x_0, x_1, x_2 < \frac{1}{2}, \\
\min(x_0, x_1, x_2), & x_0, x_1, x_2 > \frac{1}{2}, \\
\frac{1}{2}, & \text{otherwise},
\end{cases}
\]
and
\[ H_2(x) = \max \left( \frac{1}{2}, \min(x_0, x_1, x_2) \right), \quad (2.4) \]

Here $S_3$ is the permutations of $\{0, 1, 2\}$. The following lemma justifies the introduction of the functions $H_j(x)$, $j = 0, 1, 2$, in (2.2)–(2.4).

**Lemma 2.1.** Let $x = (x_0, x_1, x_2)$, and let $G(x)$ and $H_l(x)$, $l = 0, 1, 2$, be given by (2.1)–(2.4). Then $H_0(x) = H_1(x)$. Furthermore, if $l \in \{0, 1, 2\}$, then the following conditions are equivalent.

1. $0 \leq G(x) \leq \frac{1}{2}$;
2. $0 \leq G(x) \leq H_l(x)$.

**Proof.** We begin to prove $H_0(x) = H_1(x)$. We have $H_0(x) = \max(y_0, y_1, y_2)$, where

- $y_0 = \min \left( x_0, \max \left( \frac{1}{2}, \min(x_1, x_2) \right) \right)$,
- $y_1 = \min \left( x_1, \max \left( \frac{1}{2}, \min(x_0, x_2) \right) \right)$,
- $y_2 = \min \left( x_2, \max \left( \frac{1}{2}, \min(x_0, x_1) \right) \right)$.

If $x_j \leq 1/2$, then $y_j = x_j$, $j = 0, 1, 2$, giving that
\[ H_0(x) = \max(x_0, x_1, x_2) = H_1(x) \]
in this case.

If instead $x_j \geq 1/2$, then $y_j = \min(x_0, x_1, x_2)$, $j = 0, 1, 2$, giving that
\[ H_0(x) = \min(x_0, x_1, x_2) = H_1(x) \]
in this case as well.

Next assume that $x_j > 1/2$ and $x_k < 1/2$, for some choices of $j, k \in \{0, 1, 2\}$. By reasons of symmetry we may assume that $x_0 = \min(x_0, x_1, x_2) < 1/2$ and $x_1 = \max(x_0, x_1, x_2) > 1/2$. Then
\[ H_1(x) = \frac{1}{2}, \quad y_0 = x_0, \quad y_1 = \frac{1}{2} \text{ and } y_2 = \min \left( x_2, \frac{1}{2} \right) \leq \frac{1}{2}. \]

Hence,
\[ H_0(x) = \max(y_0, y_1, y_2) = \frac{1}{2} = H_1(x). \]

which shows that $H_0(x) = H_1(x)$ for all $x$.

It remains to prove the equivalence between (1) and (2). It is obvious that (2) with $l = 1$ or (1) implies (2) with $l = 2$. Next assume that
(2) with \( l = 2 \) holds but not (1). Then \( G(x) > 1/2 \) and \( H_2(x) > 1/2 \), which implies that \( \min\{x_0, x_1, x_2\} > 1/2 \). This gives

\[
G(x) = 2 - \sum_{j=0}^{2} x_j < 2 - \frac{3}{2} = \frac{1}{2},
\]

which is a contradiction. Hence (2) with \( l = 2 \) implies (1), and we have proved the equivalence between (1) and (2) when \( l = 2 \).

Since \( H_0(x) = H_1(x) = H_2(x) \) when \( x_j \geq 1/2 \) for some \( j = 0, 1, 2 \), it suffices to consider the case \( x_j < 1/2, j = 0, 1, 2 \), when proving that (2) is invariant under the choice of \( l = 0, 1, 2 \). Then by the first part of the proof we have \( H_0(x) = H_1(x) < 1/2, H_2(x) = 1/2 \) and

\[
G(x) = 2 - \sum_{j=0}^{2} x_j > 1/2.
\]

Hence (2) is violated in this case for any \( j \in \{0, 1, 2\} \). This proves the invariance of (2) under the choice of \( j \), and the proof is complete. □

In the main results here below we consider convolutions between elements in weighted Lebesgue and modulation spaces, and multiplications between elements in (weighted) Fourier-Lebesgue spaces. For the convolution results, the parameters on the weights should satisfy

\[
0 \leq t_j + t_k, \quad j, k = 0, 1, 2, \quad j \neq k, \tag{2.5}
\]

\[
0 \leq t_0 + t_1 + t_2 - d \cdot R(p), \tag{2.6}
\]

and

\[
0 \leq s_0 + s_1 + s_2, \tag{2.7}
\]

(Cf. (0.3) and (0.4).) If the convolution is replaced by multiplication, then the roles for \( p_j \) and \( q_j \), and for \( s_j \) and \( t_j \) are interchanged. Therefore, (2.5)–(2.7) should be replaced by

\[
0 \leq s_j + s_k, \quad j, k = 0, 1, 2, \quad j \neq k, \tag{2.5}'
\]

\[
0 \leq s_0 + s_1 + s_2 - d \cdot R(q), \tag{2.6}'
\]

and

\[
0 \leq t_0 + t_1 + t_2, \tag{2.7}'
\]

when the Lebesgue parameters are \( q \) and \( q_j \) instead of \( p \) and \( p_j \), respectively.

**Theorem 2.2.** Let \( s_j, t_j \in \mathbb{R}, p_j, q_j \in [1, \infty], j = 0, 1, 2 \) and let \( R \) be the functional in (0.1). Then the following is true:
(1) Assume that $0 \leq R(p) \leq 1/2$, and that $(2.5)$ and $(2.6)$ hold true with strict inequality in $(2.6)$ when $R(p) > 0$ and $s_j = d \cdot R(p)$ for some $j = 0, 1, 2$. Then the map $(f_1, f_2) \mapsto f_1 \cdot f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends uniquely to a continuous map from $L_{s_1}^{q_1}(\mathbb{R}^d) \times L_{s_2}^{q_2}(\mathbb{R}^d)$ to $L_{s_0}^{q_0}(\mathbb{R}^d)$;

(2) Assume that $0 \leq R(q) \leq 1/2$, and that $(2.5)'$ and $(2.6)'$ hold true with strict inequality in $(2.6)'$ when $R(q) > 0$ and $s_j = d \cdot R(q)$ for some $j = 0, 1, 2$. Then the map $(f_1, f_2) \mapsto f_1 \cdot f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends uniquely to a continuous map from $\mathcal{F} L_{s_1}^{q_1}(\mathbb{R}^d) \times \mathcal{F} L_{s_2}^{q_2}(\mathbb{R}^d)$ to $\mathcal{F} L_{s_0}^{q_0}(\mathbb{R}^d)$.

The following corollary follows immediately from $(1.2)$ and Theorem 2.2.

**Corollary 2.3.** Let the hypothesis in Theorem 2.2 hold true, and let $X \subseteq \mathbb{R}^n$ be open. Then the map $(f_1, f_2) \mapsto f_1 \cdot f_2$ on $C_0^\infty(X)$ extends uniquely to a continuous map from $(\mathcal{F} L_{s_1}^{q_1})_{\text{loc}}(X) \times (\mathcal{F} L_{s_2}^{q_2})_{\text{loc}}(X)$ to $(\mathcal{F} L_{s_0}^{q_0})_{\text{loc}}(X)$.

The next result concerns corresponding properties for modulation spaces.

**Theorem 2.4.** Let the hypothesis in Theorem 2.2 hold true and Then the following is true:

(1) Assume that $0 \leq R(p) \leq 1/2$, $R(q) \leq 1$ and $(2.5)$, $(2.7)$ hold true, with strict inequality in $(2.6)$ when $R(p) > 0$ and $s_j = d \cdot R(p)$ for some $j = 0, 1, 2$. Then the map $(f_1, f_2) \mapsto f_1 \cdot f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends to a continuous map from $M_{s_1,t_1}^{p_1,q_1}(\mathbb{R}^d) \times M_{s_2,t_2}^{p_2,q_2}(\mathbb{R}^d)$ to $M_{s_0,t_0}^{p_0,q_0}(\mathbb{R}^d)$;

(2) Assume that $R(p) \leq 1$ and $0 \leq R(q) \leq 1/2$, and $(2.5)'$, $(2.7)'$ hold true, with strict inequality in $(2.6)'$ when $R(q) > 0$ and $s_j = d \cdot R(q)$ for some $j = 0, 1, 2$. Then the map $(f_1, f_2) \mapsto f_1 \cdot f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends to a continuous map from $M_{s_1,t_1}^{p_1,q_1}(\mathbb{R}^d) \times M_{s_2,t_2}^{p_2,q_2}(\mathbb{R}^d)$ to $M_{s_0,t_0}^{p_0,q_0}(\mathbb{R}^d)$.

The same is true after $M_{s_j,t_j}$ have been replaced by $W_{s_j,t_j}$, $j = 0, 1, 2$.

Furthermore, the extensions of these mappings are unique, except when $p_j$ or $q_j$ are equal to $\infty$ for more than one choice of $j = 0, 1, 2$.

**Remark 2.5.** By letting $x_j = 1/p_j$ in Lemma 2.1, we may replace the condition $0 \leq R(p) \leq 1/2$ in Theorems 2.2, 2.4 with

$$0 \leq R(p) \leq \max \left( \frac{1}{2}, \min \left( \frac{1}{p_0}, \frac{1}{p_1}, \frac{1}{p_2} \right) \right).$$

The following result shows that the conditions $(2.5)$ and $(2.6)$ are also necessary in order for the continuity in Theorem 2.2 to hold true.
Proposition 2.6. Let \( p_j, q_j \in [1, \infty] \) and \( s_j, t_j \in \mathbb{R}, \ j = 0, 1, 2. \) Assume that at least one of the following statements hold true:

1. the map \( (f_1, f_2) \mapsto f_1 \ast f_2 \) on \( \mathcal{F}(\mathbb{R}^d) \) is continuously extendable to a map from \( L_{t_1}^{p_1}(\mathbb{R}^d) \times L_{t_2}^{p_2}(\mathbb{R}^d) \) to \( L_{-t_0}^{q_0}(\mathbb{R}^d); \)
2. the map \( (f_1, f_2) \mapsto f_1 \ast f_2 \) on \( \mathcal{F}(\mathbb{R}^d) \) is continuously extendable to a map from \( M_{s_1, t_1}^{p_1, q_1}(\mathbb{R}^d) \times M_{s_2, t_2}^{p_2, q_2}(\mathbb{R}^d) \) to \( M_{-s_0, -t_0}^{q_0}(\mathbb{R}^d); \)
3. the map \( (f_1, f_2) \mapsto f_1 \ast f_2 \) from \( \mathcal{F}(\mathbb{R}^d) \times \mathcal{F}(\mathbb{R}^d) \) to \( \mathcal{F}(\mathbb{R}^d) \) is continuously extendable to a map from \( W_{s_1, t_1}^{p_1, q_1}(\mathbb{R}^d) \times W_{s_2, t_2}^{p_2, q_2}(\mathbb{R}^d) \)

Then \((2.5)\) and \((2.6)\) hold true.

By Fourier transformation, it follows that Proposition 2.6 is equivalent to the following result.

Proposition 2.7. Let \( p_j, q_j \in [1, \infty] \) and \( s_j, t_j \in \mathbb{R}, \ j = 0, 1, 2. \) Assume that at least one of the following statements hold true:

1. the map \( (f_1, f_2) \mapsto f_1 \cdot f_2 \) on \( \mathcal{F}(\mathbb{R}^d) \) is continuously extendable to a map from \( \mathcal{F}L_{t_1}^{q_1}(\mathbb{R}^d) \times \mathcal{F}L_{t_2}^{q_2}(\mathbb{R}^d) \) to \( \mathcal{F}L_{-t_0}^{q_0}(\mathbb{R}^d); \)
2. the map \( (f_1, f_2) \mapsto f_1 \cdot f_2 \) on \( \mathcal{F}(\mathbb{R}^d) \) is continuously extendable to a map from \( M_{s_1, t_1}^{p_1, q_1}(\mathbb{R}^d) \times M_{s_2, t_2}^{p_2, q_2}(\mathbb{R}^d) \) to \( M_{-s_0, -t_0}^{q_0}(\mathbb{R}^d); \)
3. the map \( (f_1, f_2) \mapsto f_1 \cdot f_2 \) from \( \mathcal{F}(\mathbb{R}^d) \times \mathcal{F}(\mathbb{R}^d) \) to \( \mathcal{F}(\mathbb{R}^d) \) is continuously extendable to a map from \( W_{s_1, t_1}^{p_1, q_1}(\mathbb{R}^d) \times W_{s_2, t_2}^{p_2, q_2}(\mathbb{R}^d) \)

Then \((2.5)’\) and \((2.6)’\) hold true.

Remark 2.8. In the literature, there are several results which are related to Theorems 2.2 and 2.4 (cf. e.g. the first part of Proposition 2.3 in [4] and the references therein). It seems that some of these results contain some mistakes.

More precisely, let \( p_j, q_j \in [1, \infty], \ j = 0, 1, 2, \) and \( s \geq 0 \) be such that \( R(p) = 1 \) and \( R(q) = 0. \) Then it is remarked in [3] that the map \( (f_1, f_2) \mapsto f_1 \cdot f_2 \) on \( \mathcal{F} \) is extendable to a continuous map from \( M_{s, 0}^{p, q} \times M_{0, 0}^{p, q} \) to \( M_{-s, 0}^{q, q}. \) (Cf. Remark 2.4 in [3].) We claim that this is not correct when \( s > 0. \)

In fact, by applying the Fourier transform and using duality, the statement is equivalent to the following statement:

Let \( p_j, q_j \in [1, \infty] \) and \( t_j \in \mathbb{R}, \ j = 0, 1, 2, \) be such that \( R(p) = 0, \) \( R(q) = 1, \) \( t_1 = -t_2 \geq 0 \) and \( t_0 = 0. \) Then the map \( (f_1, f_2) \mapsto f_1 \cdot f_2 \) on \( \mathcal{F} \) is extendable to a continuous map from \( W_{s, t_1}^{p, q} \times W_{0, t_2}^{p, q} \) to \( W_{-s, 0}^{q, q}. \)

The hypothesis in Proposition 2.6 is therefore fulfilled, but \((2.5)’\) is violated. This contradicts Proposition 2.6 and the claim follows.
3. Proofs

In this section we present proofs of the results in Section 2. In Subsection 3.1, we study in details the problem of extensions of an auxiliary three-linear map. In Subsection 3.2, we use the results from Subsection 3.1 to prove Lebesgue norm estimates of the three-linear form on different regions. Finally, in Subsection 3.3, we prove the main results.

3.1. The map $T_F(f, g)$. In this subsection, we introduce and study a convenient bilinear map (denoted by $T_F$ here below when $F \in L^1_{loc}$ is appropriate). We refer to [2] and [6] for similar construction. For $F \in L^1_{loc}(\mathbb{R}^{2d})$ and $p, q \in [1, \infty]$, we set

$$\|F\|_{L^{p,q}_F} \equiv \left( \int \left( \int |F(x, y)|^p \, dx \right)^{q/p} \, dy \right)^{1/q}$$

and

$$\|F\|_{L^{p,q}_2} \equiv \left( \int \left( \int |F(x, y)|^q \, dy \right)^{p/q} \, dx \right)^{1/p},$$

and we let $L^{p,q}_1(\mathbb{R}^{2d})$ be the set of all $F \in L^1_{loc}(\mathbb{R}^{2d})$ such that $\|F\|_{L^{p,q}_F}$ is finite. The space $L^{p,q}_2$ is defined analogously. (Cf. [5, 6].) We also let $\Theta$ be defined as

$$(\Theta F)(x, y) = F(x, x - y), \quad F \in L^1_{loc}(\mathbb{R}^{2d}). \quad (3.1)$$

If $F \in L^1_{loc}(\mathbb{R}^{2d})$ is fixed, then we are especially concerned about extensions of the mappings

$$(F, f, g) \mapsto T_F(f, g) \equiv \int F(\cdot, y) f(y) g(\cdot - y) \, dy \quad (3.2)$$

and

$$(F, f, g) \mapsto T_{\Theta F}(f, g) \equiv \int F(\cdot, y) f(\cdot - y) g(y) \, dy. \quad (3.3)$$

from $C_0^{\infty}(\mathbb{R}^d) \times C_0^{\infty}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$.

The following extend [2] Lemma 8.3.2] and [6] Proposition 3.2.

**Proposition 3.1.** Let $F \in L^1_{loc}(\mathbb{R}^{2d})$, $p_j \in [1, \infty]$, $j = 0, 1, 2$. Also assume that $R(p)$ in (3.1) is non-negative, and let $r = 1/R(p) \in (0, \infty]$. Then the following is true:

1. If $R(p) \leq 1/p_0$, then the mappings (3.2) and (3.3) are continuous from $L^{\infty,r}_2(\mathbb{R}^{2d}) \times L^{p_0}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^{r}_0(\mathbb{R}^d)$. Furthermore,

$$\|T_F(f, g)\|_{L^{r}_0} \lesssim \|F\|_{L^{\infty,r}_2} \|f\|_{L^{p_0}} \|g\|_{L^{p_2}} \quad (3.4)$$

and

$$\|T_{\Theta F}(f, g)\|_{L^{r}_0} \lesssim \|F\|_{L^{\infty,r}_2} \|f\|_{L^{p_0}} \|g\|_{L^{p_2}}. \quad (3.5)$$
and $F$-inequality.

Next we use the assumption

Proof.

(1) We only prove (3.4) and leave (3.5) for the reader.

We note that Proposition 3.1 agrees with [2, Lemma 8.3.2] when $p_1 = p_2 = 2$ and with [6, Proposition 3.2] when $p_1 = p_2 \in [1, \infty]$.

First, assume that $p_1, p_2 < \infty$, and let $f, g \in C_0^\infty(\mathbb{R}^d)$. By Hölder’s inequality we get

$$\left( \int |T_f(f, g)(x)|^p \, dx \right)^{1/p_0} \leq \left( \int \left( \int |T_f(f, g)(x)|^r \, dy \right)^{1/r} \left( \int |f(y)|^r |g(x-y)|^r \, dy \right)^{1/r} \, dx \right)^{1/p_0}.$$  \hspace{1cm} (3.6)

Next we use the assumption $R(q) \leq 1/p_0$, that is $r \geq p_0$ and Young’s inequality to obtain

$$\left( \int |T_f(f, g)(x)|^p \, dx \right)^{1/p_0} \leq \|F\|_{L_2^{r, r'}} \left( \|\|f\|_{L_2^{r, r'}} \right)^{1/r'} \leq \|F\|_{L_2^{r, r'}} \left( \|f\|_{L_2^{r, r'}} \right)^{1/r'},$$  \hspace{1cm} (3.7)

where $r_1 = p_1/r'$ and $r_2 = p_2/r'$. The result now follows from the fact that $C_0^\infty$ is dense in $L^{p_1}$ and $L^{p_2}$ when $p_1, p_2 < \infty$.

Next, assume that $p_1 = \infty$ and $p_2 < \infty$, and let $f \in L^\infty$ and $g \in C_0^\infty$. Then, it follows that $T_f(f, g)$ is well-defined, and that (3.7) still holds. The result now follows from the fact that $C_0^\infty$ is dense in $L^{p_2}$. The case $p_1 < \infty$ and $p_2 = \infty$ follows analogously.

Finally, if $p_1 = p_2 = \infty$, then the assumptions implies that $r = 1$ and $p_0 = \infty$. The inequalities (3.4) and (3.5) then follow by Hölder’s inequality.

(2) First we consider the case $r \geq p_1$. Let $h \in C_0(\mathbb{R}^d)$ when $r < \infty$ and $h \in L^1(\mathbb{R}^d)$ if $r = \infty$. Also let $F \in L_1^{r, \infty}(\mathbb{R}^d)$ and $F_0(y, x) = F(x, y)$ and $g(x) = g(-x)$. By [6, page 354], we have $| \langle T_f(f, g), h \rangle | =$
\[ | \langle T_F(h, \hat{g}), f \rangle | \leq \| T_F(h, \hat{g}) \|_{L^p_1} \| f \|_{L^p_1} \leq \| F_0 \|_{L^\infty} \| f \|_{L^p_1} \| h \|_{L^{p_0}} \| g \|_{L^{p_2}} \leq \| F \|_{L^\infty} \| f \|_{L^p_1} \| h \|_{L^{p_0}} \| g \|_{L^{p_2}}. \]

Next, assume that \( r \geq 2 \) and \( F \in L^r_{1, \infty}(\mathbb{R}^{2d}) \). We will prove the assertion by interpolation. First we consider the case \( r = \infty \). Then \( R(p) = 0 \), and

\[
\left\| \int F(x, y) f(y) g(x - y) \, dy \right\|_{L^r_0} \leq \| F \|_{L^{\infty}} \| f \|_{L^p_1} \| g \|_{L^{p_2}}.
\]

For the case \( r = 2 \) we have \( R(p) = 1/2 \). By letting

\[
M = \| F \|_{L^2_{1, \infty}}, \quad \theta = \frac{\| g \|_{L^{p_2}} \| h \|_{L^{2p_2}}}{\| f \|_{L^{p_1}}}^{1/p_1}, \quad r_1 = p_2/2 \quad \text{and} \quad r_2 = p_0/2,
\]

it follows from Cauchy-Schwartz inequality, the weighted arithmetic-geometric mean-value inequality and Young’s inequality that

\[
| \langle T_F(f, g), h \rangle | \leq \int \left( \int | F(x, y) | g(x - y) | h(x) | \, dx \right) | f(y) | \, dy
\]

\[
\leq M \int \left( \int | g(x - y) |^2 | h(x) |^2 \, dx \right)^{1/2} | f(y) | \, dy
\]

\[
\leq M \int \left( \frac{\theta^{p_1}}{p_1} | f(y) |^{p_1} + \frac{1}{p'_1 \theta^{p'_1}} \left( \int | g(x - y) |^2 | h(x) |^2 \, dx \right)^{p'_1/2} \right) \, dy
\]

\[
= M \left( \frac{\theta^{p_1}}{p_1} \| f \|_{L^{p_1}}^{p_1} + \frac{1}{p'_1 \theta^{p'_1}} | g \|_{L^{p'_1}}^2 | h \|_{L^{2p'_2}}^{p'_2} \right)
\]

\[
\leq M \left( \frac{\theta^{p_1}}{p_1} \| f \|_{L^{p_1}}^{p_1} + \frac{1}{p'_1 \theta^{p'_1}} (\| g \|_{L^{p_1}}^{p_1} \| h \|_{L^{2p_2}}^{p_2}) \right)
\]

\[
= M \left( \frac{1}{p_1} + \frac{1}{p'_1} \right) \| f \|_{L^{p_1}} \| g \|_{L^{p_2}} \| h \|_{L^{p_0}}
\]

\[
= M \| f \|_{L^{p_1}} \| g \|_{L^{p_2}} \| h \|_{L^{p_0}}.
\]

This gives the result for \( r = 2 \).

Since we also have proved the result for \( r = \infty \). The assertion (2) now follows for general \( r \in [2, \infty] \) by multi-linear interpolation, using Theorems 4.4.1, 5.1.1 and 5.1.2 in [1].
The assertion (3) follows by similar arguments as in the proof of (2). The details are left for the reader. The proof is complete.

\[ \square \]

3.2. Some lemmas. Before the proof of Theorem 2.2, we need some preparation, and formulate auxiliary results in the Lemmas.

First, we recall [6, Lemma 3.5] which concerns different integrals of the function

\[ F(x, y) = \langle x \rangle^{-t_0} \langle x - y \rangle^{-t_1} \langle y \rangle^{-t_2}, \quad x, y \in \mathbb{R}^d, \quad (3.8) \]

where \( t_j \in \mathbb{R}, \ j = 0, 1, 2 \). These integrals, with respect to \( x \) or \( y \), are taken over the sets

\[ \Omega_1 = \{ (x, y) \in \mathbb{R}^{2d} ; \langle y \rangle < \delta \langle x \rangle \}, \]

\[ \Omega_2 = \{ (x, y) \in \mathbb{R}^{2d} ; \langle x - y \rangle < \delta \langle x \rangle \}, \]

\[ \Omega_3 = \{ (x, y) \in \mathbb{R}^{2d} ; \delta \langle x \rangle \leq \min(\langle y \rangle, \langle x - y \rangle), \ |x| \leq R \}, \quad (3.9) \]

\[ \Omega_4 = \{ (x, y) \in \mathbb{R}^{2d} ; \delta \langle x \rangle \leq \langle x - y \rangle \leq \langle y \rangle, \ |x| > R \}, \]

\[ \Omega_5 = \{ (x, y) \in \mathbb{R}^{2d} ; \delta \langle x \rangle \leq \langle y \rangle \leq \langle x - y \rangle, \ |x| > R \}, \]

for some positive constants \( \delta \) and \( R \). By \( \chi_{\Omega_j} \), we denote the characteristic function of the set \( \Omega_j, \ j = 1, \ldots, 5 \).

Lemma 3.2. Let \( F \) be given by (3.8) and let \( \Omega_1, \ldots, \Omega_5 \) be given by (3.9), for some constants \( 0 < \delta < 1 \) and \( R \geq 4/\delta \). Also let \( p \in [1, \infty] \) and \( F_j = \chi_{\Omega_j} F, \ j = 1, \ldots, 5 \). Then the following is true:

1. \[ \|F_1(x, \cdot)\|_{L^p} \lesssim \begin{cases} \langle x \rangle^{-t_0-t_1} (1 + \langle x \rangle^{-t_2+d/p}) , & t_2 \neq d/p , \\ \langle x \rangle^{-t_0-t_1} (1 + \log \langle x \rangle)^{1/p} , & t_2 = d/p , \end{cases} \]

2. \[ \|F_2(x, \cdot)\|_{L^p} \lesssim \begin{cases} \langle x \rangle^{-t_0-t_2} (1 + \langle x \rangle^{-t_1+d/p}) , & t_1 \neq d/p , \\ \langle x \rangle^{-t_0-t_2} (1 + \log \langle x \rangle)^{1/p} , & t_1 = d/p , \end{cases} \]

3. \[ \|F_3(\cdot, y)\|_{L^p} \lesssim \langle y \rangle^{-t_1-t_2} ; \]

4. if \( j = 4 \) or \( j = 5 \), then

\[ \|F_j(\cdot, y)\|_{L^p} \lesssim \begin{cases} \langle y \rangle^{-t_0-t_1-t_2+d/p} , & t_0 < d/p , \\ \langle y \rangle^{-t_1-t_2} (1 + \log \langle y \rangle)^{1/p} , & t_0 = d/p , \\ \langle y \rangle^{-t_1-t_2} , & t_0 > d/p . \end{cases} \]
We refer to [6] for the proof of Lemma 3.2.

Next we estimate each of the auxiliary functions $T_{F_j}$, defined by (3.2) with $F$ replaced by $F_j$, $j = 1, \ldots, 5$.

**Lemma 3.3.** Let $R(p)$, $F$ and $T_F$ be given by (0.1), (3.8) and (3.2), respectively, and let $\Omega_1, \ldots, \Omega_5$ be given by (3.9), for some constants $0 < \delta < 1$ and $R \geq 4/\delta$. Moreover, let $F_j = \chi_{\Omega_j}$, $j = 1, \ldots, 5$, and $u_j = \langle \cdot \rangle^{t_j} f_j$, $j = 1, 2$. Then the estimate

$$
\|T_{F_j}(u_1, u_2)\|_{L_{R_0}^p} \lesssim \|f_1\|_{L_{t_1}^{p_1}} \|f_2\|_{L_{t_2}^{p_2}}
$$

holds when:

1. $j = 1, 2$, for $R(p) \leq 1/p_0$, $0 \leq t_0 + t_1$, $0 \leq t_0 + t_2$ and
   $$
   0 \leq t_0 + t_1 + t_2 - d \cdot R(p),
   $$
   where the above inequality is strict when $t_1 = d \cdot R(p)$ or $t_2 = d \cdot R(p)$.
2. $j = 3$, for
   $$
   \begin{cases}
   R(p) \leq \min(1/p_1, 1/p_2) & \text{when } p_1, p_2 < 2, \\
   R(p) \leq 1/2 & \text{when } p_1 \geq 2 \text{ or } p_2 \geq 2,
   \end{cases}
   $$
   and
   $$
   0 \leq t_1 + t_2;
   $$
3. $j = 4$ for $R(p) \leq \max(1/p_2, 1/2)$,
   $$
   0 \leq t_1 + t_2 \quad \text{and} \quad 0 \leq t_0 + t_1 + t_2 - d \cdot R(p),
   $$
   with $0 < t_1 + t_2$ when $t_0 = d \cdot R(p)$;
4. $j = 5$, for $R(p) \leq \max(1/p_1, 1/2)$,
   $$
   0 \leq t_1 + t_2 \quad \text{and} \quad 0 \leq t_0 + t_1 + t_2 - d \cdot R(p),
   $$
   with $0 < t_1 + t_2$ when $t_0 = d \cdot R(p)$.

**Proof.** Let $r = 1/R(p)$.

1. The condition $R(p) \leq 1/p_0$ implies that $r \geq p_0$. By Lemma 3.2 (1) it follows that
   $$
   \|F_1\|_{L_{r}^{2,r}} < \infty \quad (3.10)
   $$
   when $0 \leq t_0 + t_1$ and
   $$
   \begin{cases}
   0 \leq t_0 + t_1 + t_2 - d/r, & \text{for } t_2 \neq d/r \\
   0 < t_0 + t_1, & \text{for } t_2 = d/r.
   \end{cases}
   $$
   Similarly, by Lemma 3.2 (2) it follows that
   $$
   \|F_2\|_{L_{r}^{2,r}} < \infty \quad (3.11)
   $$
when $0 \leq t_0 + t_2$ and
\[
\begin{cases}
0 \leq t_0 + t_1 + t_2 - d/r, & \text{for } t_1 \neq d/r \\
0 < t_0 + t_2, & \text{for } t_1 = d/r.
\end{cases}
\]

This, together with Proposition 3.1 (1) gives
\[
\|T_{F_j}(u_1, u_2)\|_{L^{p_0}_i} \lesssim \|f_1\|_{L^{p_1}_i} \|f_2\|_{L^{p_2}_i}, \quad j = 1, 2.
\]

(2) By Lemma 3.2 (3) we have
\[
\|F_3\|_{L^{p_0}_i} < \infty,
\]
when $t_1 + t_2 \geq 0$ and $r_0 \in [1, \infty]$. In particular, if $r_0 = r = 1/R(p)$ and $r \geq \min(2, \max(p_1, p_2))$, then it follows from Proposition 3.1 (2) and (3) that
\[
\|T_{F_3}(u_1, u_2)\|_{L^{p_0}_i} \leq C\|f_1\|_{L^{p_1}_i} \|f_2\|_{L^{p_2}_i}.
\]

This gives (2).

Next consider $T_{F_4}$ and $T_{F_5}$. By Lemma 3.2 (4) it follows that
\[
\|F_4\|_{L^{p_1}_i} \lesssim \infty \quad \text{and} \quad \|F_5\|_{L^{p_1}_i} < \infty
\]
when
\[
\begin{cases}
-t_0 - t_1 - t_2 + d/r \leq 0, & t_0 < d/r \\
t_1 + t_2 > 0, & t_0 = d/r \\
t_1 + t_2 \geq 0, & t_0 > d/r.
\end{cases}
\]

If $t_0 < d/r$ and $-t_0 - t_1 - t_2 + d/r \leq 0$, then $t_1 + t_2 > 0$. Therefore (3.13) holds when
\[
0 \leq t_1 + t_2
\]
and
\[
0 \leq t_0 + t_1 + t_2 - d/r,
\]
with $0 < t_1 + t_2$ when $t_0 = d/r$. Hence Proposition 3.1 (3) gives
\[
\|T_{F_4}(u_1, u_2)\|_{L^{p_0}_i} \lesssim \|f_1\|_{L^{p_1}_i} \|f_2\|_{L^{p_2}_i}
\]
for $r \geq \min(2, p_2)$, and (3) follows.

Finally, by Proposition 3.1 (2) we get that
\[
\|T_{F_5}(u_1, u_2)\|_{L^{p_0}_i} \lesssim \|f_1\|_{L^{p_1}_i} \|f_2\|_{L^{p_2}_i}
\]
when $r \geq \min(2, p_1)$. This gives (4), and the proof is complete. \qed

In the following lemma we give another view to Lemma 3.3 which will be used for the proof of Theorem 2.2.

**Lemma 3.4.** Let $F$, $F_j$ and $u_j$ be the same as in Lemma 3.3. Furthermore, assume that (2.5), $0 \leq R(p) \leq 1/2$, and (2.6) hold, with strict inequality in (2.6) when $t_1$, $t_2$ or $t_0$ is equal to $d \cdot R(p)$. Then
\[
\|T_{F_j}(u_1, u_2)\|_{L^{p_0}_i} \lesssim \|f_1\|_{L^{p_1}_i} \|f_2\|_{L^{p_2}_i}
\]
holds for every \( j \in \{1, \ldots, 5\} \).

Furthermore, if the conditions in (2.5) and (2.6) are violated, then at least one of the relations in (1)-(5) in Lemma 3.3 is violated.

3.3. Proof of main results. Next we prove Theorems 2.2 and 2.4

**Proof of Theorem 2.2.** First we note that \( 0 \leq R(p) \leq 1/2 \) is not fulfilled when all \( p_j \geq 2 \) and at least one of them is strictly larger than 2. The similar fact is true if the condition \( 0 \leq R(p) \leq 1/2 \) is replaced by

\[
0 \leq R(p) \leq H(p),
\]

where \( H(p) = H_1(1/p_1, 1/p_2, 1/p_3) \) and \( H_1 \) is the same as in Lemma 2.1. Hence, we may replace the condition \( 0 \leq R(p) \leq 1/2 \) by (3.14) when proving the proposition.

First we assume that

\[
R(p) \leq \frac{1}{p_0} \quad \text{and} \quad R(p) \leq \max \left( \frac{1}{2}, \min \left( \frac{1}{p_1}, \frac{1}{p_2} \right) \right),
\]

and that (2.6) holds and \( f_j \in L_{p_j}^{p_j}, j = 1, 2 \). We express \( f_1 * f_2 \) in terms of \( T_F \) given by (3.2) and \( F \) given by (3.8) as follows. Let \( \Omega_j, j = 1, \ldots, 5, \) be the same as in (3.9) after \( \Omega_2 \) has been modified into

\[
\Omega_2 = \{ (\xi, \eta) \in \mathbb{R}^{2d}; |\xi - \eta| < \delta(\xi) \} \setminus \Omega_1.
\]

Then \( \cup \Omega_j = \mathbb{R}^{2d}, \Omega_j \cap \Omega_k \) has Lebesgue measure zero when \( j \neq k, \) and

\[
(f_1 * f_2)(\xi) = \int F(\xi, \eta)u_1(\xi - \eta)u_2(\eta)d\eta = T_F(u_1, u_2)
\]

where \( u_j(\cdot) = \langle \cdot \rangle^{t_j}f_j, j = 1, 2, \) and \( F_j = \chi_{\Omega_j}F, j = 1, \ldots, 5. \)

Now, Lemma 3.4 implies that the \( L_{p_0}^p \) norm of each of the terms \( T_{F_j}, j = 1, \ldots, 5 \) is bounded by \( C\|f_1\|_{L_{t_1}^{p_1}}\|f_2\|_{L_{t_2}^{p_2}} \) for some positive constant \( C \) which is independent of \( f_1 \in L_{t_1}^{p_1}(\mathbb{R}^d) \) and \( f_2 \in L_{t_2}^{p_2}(\mathbb{R}^d). \)

Hence, \( f_1 * f_2 \in L_{t_0}^{p_0} \) when (3.15) holds. By duality, the same conclusion holds when the roles for \( p_j, j = 0, 1, 2 \) have been interchanged. By straightforward computations it follows that (3.14) is fulfilled if and only if (3.15) or one of the dual cases of (3.15) are fulfilled. This gives the result.

The assertion (2) follows from (1) and the relation \( \mathcal{F}(f_1 * f_2) = (2\pi)^{d/2}f_1 \cdot \hat{f}_2. \)

**Proof of Corollary 2.3.** Let \( f_j \in \mathcal{F}L_{s_j, \text{loc}}^{q_j}(X), j = 1, 2 \) and let \( \phi \in C_0^\infty(X). \) Then we choose \( \phi_1 = \phi \) and \( \phi_2 = C_0^\infty(X) \) such that \( \phi_2 = 1 \) on \( \text{supp} \phi. \) Since \( \phi_jf_j \in \mathcal{F}L_{s_j}^{q_j}, \) the right-hand side of

\[
f_1f_2\phi = (f_1\phi_1)(f_2\phi_2)
\]
is well-defined, and defines an element in $\mathcal{F}L_{s-t_0}^{q_0}$, in the view of Theorem 2.2 (2). The corollary now follows from (1,2).

**Proof of Theorem 2.4.** The assertion (1) follows immediately from (2), Fourier’s inversion formula and the fact that the Fourier transform maps $M_{s,t}^{p,q}$ into $W_{t,s}^{p,q}$. Hence it suffices to prove (2).

We first consider the case when $p_j, q_j < \infty$ for $j = 1, 2$. Then $\mathcal{S}$ is dense in $M_{s,t}^{p_j, q_j}$ for $j = 1, 2$. Since $M_{s,t}^{p,q}$ decreases with $t$, and the map $f \mapsto (\cdot)^q f$ is a bijection from $M_{s,t}^{p,q}$ to $M_{s,t}^{q,p}$, for every choices of $p, q \in [1, \infty]$ and $s, t, t_0 \in \mathbf{R}$, it follows that we may assume that $t_j = 0, j = 0, 1, 2$.

We have

$$V_\phi(f_1 f_2)(x, \xi) = (2\pi)^{-d/2} (V_\phi f_1(x, \cdot) * V_\phi f_2(x, \cdot))(\xi),$$

where $\phi = \phi_1 \phi_2$, $\phi_j, f_j \in \mathcal{S}(\mathbf{R}^d)$, $j = 1, 2$, (3.16)

which follows by straight-forward application of Fourier’s inversion formula. Here the convolutions between the factors $(V_\phi f_j)(x, \xi)$, where $j = 1, 2$ should be taken over the $\xi$ variable only.

By applying the $L^{q_0}$ norm with respect to the $x$ variables and using Hölder’s inequality we get

$$\|V_\phi(f_1 f_2)(\cdot, \xi)\|_{L^{q_0}} \leq (2\pi)^{-d/2} (v_1 * v_2)(\xi),$$

where $v_j = \|V_\phi f_j(\cdot, \eta)\|_{L^{p_j}}$. Hence by applying the $L^{q_0}_{s-t_0}$ norm on the latter inequality and using Theorem 2.2 we get

$$\|f_1 f_2\|_{M_{s-2, t_0}^{p,q}} \lesssim \|v_1\|_{L_{s-t_0}^{q_1}} \|v_2\|_{L_{s-t_0}^{q_2}} \leq \|f_1\|_{M_{s,0}^{p_1,q_1}} \|f_2\|_{M_{s,0}^{p_2,q_2}},$$

and (2) follows in this case, since $\mathcal{S}$ is dense in $M_{s,0}^{p_j,q_j}$ for $j = 1, 2$.

For general $p_j$ and $q_j$, (2) follows from the latter inequality and Hahn-Banach’s theorem.

Finally, by interchanging the order of integration it follows that the same is true when $M_{s,t_j}$ is replaced by $W_{s,t_j}^{p_j,q_j}$, $j = 0, 1, 2$. The proof is complete. □

In order to prove Proposition 2.6 we recall some facts concerning compactly supported distributions and modulation spaces. By Proposition 4.1 and Remark 4.6 in [7] we have

$$M_{s,t}^{p,q} \cap \mathcal{E}' = W_{s,t}^{p,q} \cap \mathcal{E}' = \mathcal{F}L_s^q \cap \mathcal{E'},$$

and that for every compact set $K \subseteq \mathbf{R}^d$, then

$$\|f\|_{M_{s,t}^{p,q}} \asymp \|f\|_{W_{s,t}^{p,q}} \asymp \|f\|_{\mathcal{F}L_s^q}, \quad f \in \mathcal{S}'(\mathbf{R}^d), \quad \text{supp} f \subseteq K.$$

In particular,

$$\|f \varphi\|_{M_{s,t}^{p,q}} \asymp \|f \varphi\|_{W_{s,t}^{p,q}} \asymp \|f \varphi\|_{\mathcal{F}L_s^q}, \quad f \in \mathcal{S}'(\mathbf{R}^d), \quad \varphi \in C_0^\infty(K).$$
By applying the Fourier transform, and using the fact that \( \mathcal{F} M_{s,t}^{p,q} = W_{t,s}^{p,q} \), we get

\[
\|f * \varphi\|_{M_{s,t}^{p,q}} \asymp \|f * \varphi\|_{W_{t,s}^{p,q}} \asymp \|f * \varphi\|_{\mathcal{F} L_{s,t}^{q}}.
\]

\( f \in \mathcal{S}'(\mathbb{R}^d), \varphi \in \mathcal{F} C_0^\infty(K). \) \( 3.17 \)

Proof of Proposition 2.6. We only prove the result in the case when \( p'_0 < \infty \). The modifications to the case when \( p'_0 = \infty \) are left for the reader.

First we assume that (1) holds, and prove that (2.5) must hold. By duality it suffices to consider the case \( j = 1 \) and \( k = 2 \). Let \( f_0 \in \mathcal{C}_0^\infty(B_2(0)) \) be such that \( 0 \leq f_0 \leq 1, f_0(x) = 1 \) when \( x \in B_1(0) \). Here \( B_r(a) \) is the open ball centered at \( x = a \) and with radius \( r \). Also let \( x_0 \in \mathbb{R}^d, f_1(x) = f_0(x - x_0) \) and \( f_2(x) = f_0(x + x_0) \). Then it follows by straightforward computations that

\[
f_1 * f_2 = f_0 * f_0
\]

is independent of \( x_0 \), and that

\[
\|f_j\|_{L_{t_j}^{p_j}} \asymp \langle x_0 \rangle^{t_j}.
\]

In particular, \( \|f_1 * f_2\|_{L_{t_0}^{p_0}} > 0 \) is independent of \( x_0 \).

Now if \( (f_1, f_2) \mapsto f_1 * f_2 \) is continuous from \( L_{t_1}^{p_1} \times L_{t_2}^{p_2} \) to \( L_{t_0}^{p_0} \), the inequality \( \|f_1 * f_2\|_{L_{t_0}^{p_0}} \lesssim \|f_1\|_{L_{t_1}^{p_1}} \|f_2\|_{L_{t_2}^{p_2}} \) in combination with the previous estimates imply that

\[
C \leq \langle x_0 \rangle^{t_1 + t_2},
\]

for some constant \( C > 0 \) which is independent of \( x_0 \in \mathbb{R}^d \). By letting \( |x_0| \) tend to infinity, it follows from the latter relation that \( t_1 + t_2 \geq 0 \). This proves that (2.5) holds. If instead (2) or (3) hold, then the same arguments show that (2.5) still must hold.

It remains to prove that (2.6) must be true. Again we first consider the case when (1) is true. By the first part of the proof it follows that at least two of \( t_0, t_1, t_2 \) are non-negative, and we may assume that \( t_1 \geq 0 \), by duality. Let \( \alpha \in (0, 1] \), and let \( f_j(x) = \langle x \rangle^{-t_j} e^{-\alpha |x|^2}, j = 1, 2 \). Then

\[
\|f_j\|_{L_{t_j}^{p_j}} \asymp \alpha^{-d/(2p_j)}.
\] \( 3.18 \)

Furthermore, if

\[
\Omega_x = \{ y \in \mathbb{R}^d ; \langle x \rangle/4 \leq |y| \leq \langle x \rangle/2 \},
\]

then \( |x - y| \leq 3 \langle x \rangle/2 \), giving that

\[
-\alpha |y|^2 \geq -\frac{\alpha}{4} - \frac{\alpha}{4} |x|^2 \quad \text{and} \quad -\alpha |x - y|^2 \geq -\frac{9\alpha}{4} - \frac{9\alpha}{4} |x|^2.
\]
Since $t_1 \geq 0$ and $0 < \alpha \leq 1$ we obtain

$$g(x) \equiv (f_1 * f_2)(x) \geq \int_{\Omega_x} f_1(x - y) f_2(y) \, dy$$

$$\geq \langle x \rangle^{-t_1 - t_2} \int_{\Omega_x} e^{-9\alpha|x|^2/4 - \alpha|x|^2/4} \, dy$$

$$\geq \langle x \rangle^{-t_1 - t_2} e^{-3\alpha|x|^2} \int_{\Omega_x} \, dy \asymp \langle x \rangle^{d - t_1 - t_2} e^{-3\alpha|x|^2}. \quad (3.19)$$

Hence, if $h(x) = \langle x \rangle^{d - t_1 - t_2} e^{-3\alpha|x|^2}$, then $\|g\|_{L^p_{t_0}} \asymp \|h\|_{L^p_{r_0}}$. We need to estimate the right-hand side from below. Let $t = t_0 + t_1 + t_2$. Then the result follows if we prove that $t \geq d \cdot R(p)$.

We have

$$\|h\|_{L^p_{r_0}} \asymp \int (1 + |x|)^{r_0(d - t)} e^{-3\alpha r_0^2 |x|^2} \, dx$$

$$\times \int_0^{\infty} r^{d-1} (1 + r)^{r_0(d - t)} e^{-3\alpha r_0^2 r^2} \, dr$$

$$\asymp \alpha^{-d/2} \int_0^{\infty} r^{d-1} \left(1 + \frac{r}{\alpha^{1/2}}\right)^{r_0(d - t)} e^{-r^2} \, dr$$

$$\geq \alpha^{-d/2} \int_1^{\infty} r^{d-1} \left(1 + \frac{r}{\alpha^{1/2}}\right)^{r_0(d - t)} e^{-r^2} \, dr. \quad (3.20)$$

Since $0 < \alpha \leq 1$ we get $1 + r/\alpha^{1/2} \asymp r/\alpha^{1/2}$ when $r \geq 1$. Hence (3.20) gives

$$\|h\|_{L^p_{r_0}} \asymp \alpha^{-d/2} \int_1^{\infty} r^{d-1} \left(\frac{r}{\alpha^{1/2}}\right)^{r_0(d - t)} e^{-r^2} \, dr$$

$$= \alpha^{-(d(r_0+1) - r_0t)/2} \int_1^{\infty} r^{d-1 + r_0(d - t)} e^{-r^2} \, dr$$

$$\asymp \alpha^{-(d(r_0+1) - r_0t)/2}.$$

That is

$$\|f_1 * f_2\|_{L^p_{r_0}} \asymp \alpha^{-(d(1+1/r_0) - 1/2)} = \alpha^{-(d(2-1/p_0)-1/2)}.$$ \quad (3.21)

By the assumptions we have

$$\|g_1 * g_2\|_{L^p_{r_0}} \lesssim \|g_1\|_{L^p_1} \|g_2\|_{L^p_2},$$

for every $g_1, g_2 \in \mathcal{S}$. Hence (3.18) and (3.21) give

$$\alpha^{-(d(2-1/r_0)-1/2)} \lesssim \alpha^{-(d(1/p_1+1/p_2)/2)}, \quad (3.22)$$
Furthermore, since the desired properties.

Since \( t = t_0 + t_1 + t_2 \), the last relation is the same as (2.6), and the assertion follows.

It remains to prove (2.6) when (2) or (3) hold. We choose \( K = B_1(0) \) and an element \( 0 \leq \varphi \in \mathcal{F}C^\infty_0(K) \) such that \( \varphi(x) > 0 \) when \( x \in K \).

In order to find such element \( \varphi \), we first let \( 0 \leq \psi \in C^\infty_0(B_{1/2}(0)) \) be rotation invariant and such that \( \psi(0) > 0 \). Then \( \varphi = \mathcal{F}(\psi * \psi) \) satisfies the desired properties.

Assume that \((f_1, f_2) \mapsto f_1 * f_2\) is continuous from \( M_{s_1,t_1}^{p_1,q_1} \times M_{s_2,t_2}^{p_2,q_2} \) to \( M_{-s_0,-t_0}^{p_0,q_0} \), or from \( W_{s_1,t_1}^{p_1,q_1} \times W_{s_2,t_2}^{p_2,q_2} \) to \( W_{-s_0,-t_0}^{-p_0,-q_0} \). If \( \varphi_0 = \varphi * \varphi \geq 0 \), then it follows from (3.17) that

\[
\| f_1 * f_2 * \varphi_0 \|_{L_{x_0}^{p_0},t_0} \lesssim \| f_1 * \varphi \|_{L_{t_1}^{p_1}} \| f_2 * \varphi \|_{L_{t_2}^{p_2}}.
\]

(3.24)

Now let \( f_j(x) = \langle x \rangle^{-t_j} e^{-\alpha|x|^2} \) as before. Then (3.18) and Young’s inequality gives

\[
\| f_j * \varphi \|_{L_{t_j}^{p_j}} \lesssim \alpha^{-d/(2p_j)}.
\]

(3.18)′

Furthermore, since \( \psi \geq 0 \) is non-zero in \( B_1(0) \), it follows from (3.19) that

\[
 f_1 * f_2 * \varphi_0 \gtrsim h * \psi_0,
\]

where \( h = \langle . \rangle^{d-t_1-t_2} e^{-3\alpha|.|^2} \) are the same as before, and \( \psi_0 = \varphi_0 \) in \( B_1(0) \) and \( \psi_0 = 0 \) otherwise.

Since \( \langle x - y \rangle^{d-t_1-t_2} \sim \langle x \rangle^{d-t_1-t_2} e^{-3\alpha|x-y|^2} \gtrsim e^{-6\alpha|x|^2} \) when \( |y| \leq 1 \), we get

\[
(f_1 * f_2 * \varphi_0)(x) \gtrsim \langle x \rangle^{d-t_1-t_2} e^{-6\alpha|x|^2}.
\]

By the same arguments as in the proof of (3.21) we now obtain

\[
\| f_1 * f_2 * \varphi_0 \|_{L_{x_0}^{p_0},t_0} \gtrsim \alpha^{-(d(2-1/p_0)-t)/2}.
\]

(3.21)′

A combination of (3.24), (3.18)′ and (3.21)′ now gives (3.22) which in turn lead to (3.23) or equivalently to (2.6). The proof is complete. □

References

[1] J. Bergh and J. Löfström Interpolation Spaces, An Introduction, Springer-Verlag, Berlin Heidelberg New York, 1976.

[2] L. Hörmander Lectures on Nonlinear Hyperbolic Differential Equations, Springer-Verlag, Berlin, 1997.

[3] T. Iwabuchi Navier-Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices, J. Differential Equations 248 (2010), 1972–2002.

[4] Q. Liu, S. Cui Well-posedness for the incompressible magneto-hydrodynamic system on modulation spaces, J. Math. Anal. Appl. 389 (2012), 741–753.
[5] S. Pilipović, N. Teofanov, J. Toft, Micro-local analysis in Fourier Lebesgue, Part I, J. Fourier Anal. Appl. 17 (2011), 374–407.
[6] S. Pilipović, N. Teofanov, J. Toft, Micro-local analysis in Fourier Lebesgue and modulation spaces. Part II, J. Pseudo-Differ. Oper. Appl. 1 (2010), 341–376.
[7] M. Ruzhansky, M. Sugimoto, N. Tomita, J. Toft Changes of variables in modulation and Wiener amalgam spaces, Math. Nachr. 284 (2011), 2078–2092.

DEPARTMENT OF COMPUTER SCIENCE, MATHEMATICS AND PHYSICS, LINNAEUS UNIVERSITY, VÄXJÖ, SWEDEN
E-mail address: joachim.toft@lnu.se

DEPARTMENT OF COMPUTER SCIENCE, MATHEMATICS AND PHYSICS, LINNAEUS UNIVERSITY, VÄXJÖ, SWEDEN
E-mail address: karoline.johansson@lnu.se

DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF NOVI SAD, NOVI SAD, SERBIA
E-mail address: stevan.pilipovic@dmi.uns.ac.rs

DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF NOVI SAD, NOVI SAD, SERBIA
E-mail address: nenad.teofanov@dmi.uns.ac.rs