RIGIDITY THEOREMS OF $\lambda$-HYPERSURFACES

QING-MING CHENG, SHIHO OGATA AND GUOXIN WEI

Abstract. Since $n$-dimensional $\lambda$-hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ are critical points of the weighted area functional for the weighted volume-preserving variations, in this paper, we study the rigidity properties of complete $\lambda$-hypersurfaces. We give a gap theorem of complete $\lambda$-hypersurfaces with polynomial area growth. By making use of the generalized maximum principle for $L$ of $\lambda$-hypersurfaces, we prove a rigidity theorem of complete $\lambda$-hypersurfaces.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth $n$-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. In [3], Cheng and Wei have introduced notation of the weighted volume-preserving mean curvature flow, which is defined as the following: a family $X(\cdot, t)$ of smooth immersions $X(\cdot, 0) = X(\cdot)$ is called a weighted volume-preserving mean curvature flow if

\begin{equation}
\left(\frac{\partial X(t)}{\partial t}\right) \perp = (-\alpha(t)N(t) + H(t))
\end{equation}

holds, where

$\alpha(t) = \frac{\int_M H(t) \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu},$

$H(t) = H(\cdot, t)$ and $N(t)$ denote the mean curvature vector and the normal vector of hypersurface $M_t = X(M^n, t)$ at point $X(\cdot, t)$, respectively and $N$ is the unit normal vector of $X : M \to \mathbb{R}^{n+1}$. One can prove that the flow (1.1) preserves the weighted volume $V(t)$ defined by

$V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.$

The weighted area functional $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by

$A(t) = \int_M e^{-\frac{|X|^2}{2}} d\mu,$

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where $d\mu_t$ is the area element of $M$ in the metric induced by $X(t)$. Let $X(t): M \to \mathbb{R}^{n+1}$ with $X(0) = X$ be a variation of $X$. If $V(t)$ is constant for any $t$, we call $X(t): M \to \mathbb{R}^{n+1}$ a weighted volume-preserving variation of $X$. Cheng and Wei [4] have proved that $X : M \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations if and only if there exists constant $\lambda$ such that

\[(1.2) \quad \langle X, N \rangle + H = \lambda.
\]

An immersed hypersurface $X(t): M \to \mathbb{R}^{n+1}$ is called a $\lambda$-hypersurface if the equation (1.2) is satisfied.

**Remark 1.1.** If $\lambda = 0$, then the $\lambda$-hypersurface is a self-shrinker of the mean curvature flow. Hence, the $\lambda$-hypersurface is a generalization of the self-shrinker.

**Example 1.1.** The $n$-dimensional sphere $S^n(r)$ with radius $r > 0$ is a compact $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$.

**Example 1.2.** For $1 \leq k \leq n - 1$, the $n$-dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with radius $r > 0$ is a complete and non-compact $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$.

**Example 1.3.** The $n$-dimensional Euclidean space $\mathbb{R}^n$ is a complete and non-compact $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = 0$.

**Definition 1.1.** If $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say that $M$ has polynomial area growth if there exist constant $C$ and $d$ such that for all $r \geq 1$,

\[(1.3) \quad \text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq C r^d,
\]

where $B_r(0)$ is a standard ball in $\mathbb{R}^{n+1}$ with radius $r$ and centered at the origin.

In [4], Cheng and Wei have studied properties of complete $\lambda$-hypersurfaces with polynomial area growth. They have proved that a complete and non-compact $\lambda$-hypersurface $X : M \to \mathbb{R}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$ has polynomial area growth if and only if $X : M \to \mathbb{R}^{n+1}$ is a complete proper hypersurface. Furthermore, there is a positive constant $C$ such that for $r \geq 1$,

\[(1.4) \quad \text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq C r^{n+\frac{k^2}{r} - 2\beta - \inf H^2},
\]

where $\beta = \frac{1}{4} \inf (\lambda - H)^2$.

In this paper, we study the rigidity of complete $\lambda$-hypersurfaces. We will prove the following:

**Theorem 1.1.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete $\lambda$-hypersurface with polynomial area growth in the Euclidean space $\mathbb{R}^{n+1}$. Then $X : M \to \mathbb{R}^{n+1}$ satisfies one of the following:

1. $X : M \to \mathbb{R}^{n+1}$ is isometric to the sphere $S^n(r)$ with radius $r > 0$,
2. $X : M \to \mathbb{R}^{n+1}$ is isometric to the Euclidean space $\mathbb{R}^n$,
3. $X : M \to \mathbb{R}^{n+1}$ is isometric to the cylinder $S^1(r) \times \mathbb{R}^{n-1}$. 


(4) $X : M \to \mathbb{R}^{n+1}$ is isometric to the cylinder $S^{n-1}(r) \times \mathbb{R}$,
(5) there exists $p \in M$ such that the squared norm $S$ of the second fundamental form and the mean curvature $H$ of $X : M \to \mathbb{R}^{n+1}$ satisfy

$$
\left( \sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda| \frac{n - 2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n}(H(p) - \lambda)^2 > 1 + \frac{n\lambda^2}{4(n-1)}. 
$$

**Corollary 1.1.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete $\lambda$-hypersurface with polynomial area growth in the Euclidean space $\mathbb{R}^{n+1}$. If the squared norm $S$ of the second fundamental form and the mean curvature $H$ of $X : M \to \mathbb{R}^{n+1}$ satisfies

$$
\left( \sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n - 2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n}(H - \lambda)^2 \leq 1 + \frac{n\lambda^2}{4(n-1)},
$$

then $X : M \to \mathbb{R}^{n+1}$ is isometric to one of the following:

1. the sphere $S^n(r)$ with radius $0 < r \leq \sqrt{n},$
2. the Euclidean space $\mathbb{R}^n,$
3. the cylinder $S^1(r) \times \mathbb{R}^{n-1}$ with radius $r > 0$ and $n = 2$ or with radius $r \geq 1$
and $n > 2,$
4. the cylinder $S^{n-1}(r) \times \mathbb{R}$ with radius $r > 0$ and $n = 2$ or with radius $r \geq \sqrt{n-1}$ and $n > 2.$

**Remark 1.2.** If $\lambda = 0$, that is, $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional complete self-shrinker, our condition (1.6) becomes $S \leq 1$. Hence, our theorem is a general generalization of Cao and Li [1] and Le and Sesum [11] to $\lambda$-hypersurfaces. On study of complete self-shrinkers, see [2, 3, 4, 6, 7, 8, 9, 10].

**Theorem 1.2.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete $\lambda$-hypersurface with polynomial area growth in the Euclidean space $\mathbb{R}^{n+1}$. If

$$
(H - \frac{\lambda}{2})^2 \geq n + \frac{\lambda^2}{4},
$$

then $(H - \frac{\lambda}{2})^2 \equiv n + \frac{\lambda^2}{4}$ and $M$ is isometric to the sphere $S^n(r)$ with radius $r > 0$. If we do not assume that $X : M \to \mathbb{R}^{n+1}$ has polynomial area growth, we can prove the following:

**Theorem 1.3.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete $\lambda$-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the squared norm $S$ of the second fundamental form and the mean curvature $H$ of $X : M \to \mathbb{R}^{n+1}$ satisfy

$$
\sup\{\sqrt{S - \frac{H^2}{n}} + |\lambda| \frac{n - 2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H - \lambda)^2 < 1 + \frac{n\lambda^2}{4(n-1)},
$$

then $X : M \to \mathbb{R}^{n+1}$ is isometric to one of the following:

1. the sphere $S^n(r)$ with radius $r < \sqrt{n},$
2. the Euclidean space $\mathbb{R}^n.$

**Remark 1.3.** If $\lambda = 0$, that is, $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional complete self-shrinker, our condition (1.8) becomes $\sup S < 1$. Our theorem is a general generalization of Cheng and Peng [2] to $\lambda$-hypersurfaces.
We next give the following:

**Proposition 1.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional compact \( \lambda \)-hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \). If

\[
\left( H - \frac{\lambda}{2} \right)^2 \leq n + \frac{\lambda^2}{4},
\]

then \( (H - \frac{\lambda}{2})^2 \equiv n + \frac{\lambda^2}{4} \) and \( M \) is isometric to the sphere \( S^n(r) \) with radius \( r > 0 \).

### 2. Proofs of theorems for \( \lambda \)-hypersurfaces

In order to prove our theorems, we prepare several fundamental formulas. Let \( X : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional connected hypersurface of the \((n+1)\)-dimensional Euclidean space \( \mathbb{R}^{n+1} \). We choose a local orthonormal frame field \( \{ e_A \}_{A=1}^{n+1} \) in \( \mathbb{R}^{n+1} \) with dual coframe field \( \{ \omega_A \}_{A=1}^{n+1} \), such that, restricted to \( M^n \), \( e_1, \ldots, e_n \) are tangent to \( M^n \). Then we have

\[
dX = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \omega_{i n+1} e_{n+1}.
\]

and

\[
de_{n+1} = \sum_i \omega_{n+1i} e_i.
\]

We restrict these forms to \( M^n \), then

\[
\omega_{n+1} = 0, \quad \omega_{n+1i} = -\sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji},
\]

where \( h_{ij} \) denotes components of the second fundamental form of \( X : M^n \to \mathbb{R}^{n+1} \). \( H = \sum_{j=1}^n h_{jj} \) is the mean curvature and \( II = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j N \) is the second fundamental form of \( X : M^n \to \mathbb{R}^{n+1} \) with \( N = e_{n+1} \). Let

\[
h_{ijk} = \nabla_k h_{ij} \quad \text{and} \quad h_{ijkl} = \nabla_i \nabla_k h_{ij},
\]

where \( \nabla_j \) is the covariant differentiation operator. Gauss equations, Codazzi equations and Ricci formulas are given by

\[
R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},
\]

(2.1)

\[
h_{ijk} = h_{ikj},
\]

(2.2)

\[
h_{ijkl} - h_{ijlk} = \sum_{m=1}^n h_{im} R_{mjkl} + \sum_{m=1}^n h_{mj} R_{mikl},
\]

(2.3)

where \( R_{ijkl} \) is components of the curvature tensor. For a function \( F \), we denote covariant derivatives of \( F \) by \( F_{,i} = \nabla_i F \), \( F_{,ij} = \nabla_j \nabla_i F \). For \( \lambda \)-hypersurfaces, an elliptic operator \( \mathcal{L} \) is given by

\[
\mathcal{L} f = \Delta f - \langle X, \nabla f \rangle,
\]

(2.4)

where \( \Delta \) and \( \nabla \) denote the Laplacian and the gradient operator of the \( \lambda \)-hypersurface, respectively. The \( \mathcal{L} \) operator is introduced by Colding and Minicozzi in [6] for self-shrinkers and by Cheng and Wei [4] for \( \lambda \)-hypersurfaces.
The following lemma due to Colding and Minicozzi [6] is needed in order to prove our results.

**Lemma 2.1.** Let $X : M \to \mathbb{R}^{n+1}$ be a complete hypersurface. If $u, v$ are $C^2$ functions satisfying

\[
\int_M \left( |u \nabla v| + |\nabla u| |\nabla v| + |u \mathcal{L} v| \right) e^{-\frac{|X|^2}{2}} d\mu < +\infty,
\]

then we have

\[
\int_M u(\mathcal{L} v) e^{-\frac{|X|^2}{2}} d\mu = -\int_M \langle \nabla u, \nabla v \rangle e^{-\frac{|X|^2}{2}} d\mu.
\]

**Proof of theorem 1.1.** Since $\langle X, N \rangle + H = \lambda$, one has

\[
H_{,i} = \sum_j h_{ij} \langle X, e_j \rangle,
\]

\[
H_{,ik} = \sum_j h_{ijk} \langle X, e_j \rangle + h_{ik} + \sum_j h_{ij} h_{jk} (\lambda - H).
\]

From the Codazzi equation (2.2), we infer

\[
\Delta H = \sum_i H_{,ii} = \sum_i H_{,i} \langle X, e_i \rangle + H + S(\lambda - H).
\]

Hence, we get

\[
\mathcal{L} H = \Delta H - \sum_i \langle X, e_i \rangle H_{,i} = H + S(\lambda - H),
\]

(2.8)

\[
\frac{1}{2} \mathcal{L} H^2 = |\nabla H|^2 + H^2 + S(\lambda - H) H.
\]

(2.9)

By making use of the Ricci formulas and the Gauss equations and the Codazzi equations, we have

\[
\mathcal{L} h_{ij} = \Delta h_{ij} - \sum_k \langle X, e_k \rangle h_{ijk}
\]

\[
= \sum_k h_{ijkk} - \sum_k \langle X, e_k \rangle h_{ijk}
\]

\[
= (1 - S) h_{ij} + \lambda \sum_k h_{ik} h_{kj}.
\]

Therefore, we obtain

\[
\frac{1}{2} \mathcal{L} S = \frac{1}{2} \left\{ \Delta \sum_{i,j} (h_{ij})^2 - \sum_k \langle X, e_k \rangle \left( \sum_{i,j} h_{ij}^2 \right) _{,k} \right\}
\]

\[
= \sum_{i,j,k} h_{ijk}^2 + (1 - S) \sum_{i,j} h_{ij}^2 + \lambda \sum_{i,j,k} h_{ik} h_{kj} h_{ji}
\]

\[
= \sum_{i,j,k} h_{ijk}^2 + (1 - S) S + \lambda f_3,
\]

where $f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}$.
Taking \( \{e_1, e_2, \cdots, e_n\} \) such that \( h_{ij} = \lambda_i \delta_{ij} \) at a point \( p \) and putting \( \mu_i = \lambda_i - \frac{H}{n} \), we have

\[
f_3 = \sum_i \lambda_i^3 = \sum_i (\mu_i + \frac{H}{n})^3 = B_3 + \frac{3}{n} H B + \frac{1}{n^2} H^3
\]

with \( B = \sum_i \mu_i^2 = S - \frac{H^2}{n} \) and \( B_3 = \sum_i \mu_i^3 \). Thus, we have

\[
\frac{1}{2} \mathcal{L} B = \frac{1}{2} \mathcal{L} S - \frac{1}{2} \mathcal{L} \frac{H^2}{n}
\]

\[
= \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 + (1 - S) S + \lambda f_3 - \frac{H^2}{n} - S(\lambda - H) \frac{H}{n}
\]

\[
= \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 + (1 - B) B - \frac{1}{n} H^2 B + \lambda B_3 + \frac{2}{n} \lambda H B.
\]

Since

\[
\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = B,
\]

it is not hard to prove

\[
|B_3| \leq \frac{n - 2}{\sqrt{n(n-1)}} B^2
\]

and the equality holds if and only if at least, \( n - 1 \) of \( \mu_i \) are equal. Thus, we have

\[
\frac{1}{2} \mathcal{L} B \geq \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2
\]

\[
+ (1 - B) B - \frac{1}{n} H^2 B - |\lambda| \frac{n - 2}{\sqrt{n(n-1)}} B^2 + \frac{2}{n} \lambda H B
\]

\[
= \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 + B \left( \frac{4(n^2 - 1)}{16} - \frac{n - 2}{\sqrt{n(n-1)}} B^2 + \frac{2}{n} \lambda H \right)
\]

\[
= \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 + B \left( 1 + \frac{n \lambda^2}{4(n-1)} - \frac{1}{n} (H - \lambda)^2 - (\sqrt{B} + |\lambda| \frac{n - 2}{2 \sqrt{n(n-1)}})^2 \right).
\]

Since \( X : M \to \mathbb{R}^{n+1} \) has polynomial area growth, according to the results of Cheng and Wei in [4], we can apply the lemma 2.1 to functions 1 and \( B = S - \frac{H^2}{n} \). Hence, we have

\[
0 \geq \int_M \left\{ \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 \right\} e^{-\frac{|x|^2}{2}} d\mu
\]

\[
+ \int_M B \left( 1 + \frac{n \lambda^2}{4(n-1)} - \frac{1}{n} (H - \lambda)^2 - (\sqrt{B} + |\lambda| \frac{n - 2}{2 \sqrt{n(n-1)}})^2 \right) e^{-\frac{|x|^2}{2}} d\mu.
\]
From the Codazzi equations and the Schwarz inequality, we have
\[ \sum_{i,j,k} h_{ijk}^2 = 3 \sum_{i \neq k} h_{ik}^2 + \sum_i h_{iii}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk}, \quad \frac{1}{n} |\nabla H|^2 \leq \sum_{i,k} h_{ik}^2, \]
and the equality holds if and only if \( h_{ijk} = 0 \) for any \( i, j, k \). Therefore, we get
\[ \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 \geq 2 \sum_{i \neq k} h_{ik}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk} \geq 0, \]
and the following generalized maximum principle holds.

**Theorem 2.1.** (Generalized maximum principle for \( \mathcal{L} \)-operator) Let \( X : M^n \to \mathbb{R}^{n+1} \) be a complete \( \lambda \)-hypersurface with Ricci curvature bounded from below. Let \( f \) be any \( C^2 \)-function bounded from above on this \( \lambda \)-hypersurface. Then, there exists a sequence of points \( \{p_k\} \subset M \), such that
\[ \lim_{k \to \infty} f(X(p_k)) = \sup f, \quad \lim_{k \to \infty} |\nabla f|(X(p_k)) = 0, \quad \limsup_{k \to \infty} \mathcal{L} f(X(p_k)) \leq 0. \]

**Proof of Theorem 2.3.** From the proof in the theorem 1.1, we have
\[ \frac{1}{2} \mathcal{L} B \geq \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 + B \left( 1 + \frac{n\lambda^2}{4(n-1)} - \frac{1}{n} (H - \lambda)^2 - \left( \sqrt{B} + |\lambda| \frac{n - 2}{2\sqrt{n(n-1)}} \right)^2 \right) \]
and
\[ \sum_{i,j,k} h_{ijk}^2 - \frac{1}{n} |\nabla H|^2 \geq 2 \sum_{i \neq k} h_{ik}^2 + \sum_{i \neq j \neq k \neq i} h_{ijk} \geq 0. \]
Hence, we obtain

$$
\frac{1}{2} \mathcal{L} B \geq B \left( 1 + \frac{n \lambda^2}{4(n-1)} - \frac{1}{n} (H - \lambda)^2 - \left( \sqrt{B} + \frac{|\lambda|}{2 \sqrt{n(n-1)}} \right)^2 \right).
$$

Since

$$
sup \left\{ \left( \sqrt{S - \frac{H^2}{n}} + \frac{n - 2}{2 \sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H - \lambda)^2 \right\} < 1 + \frac{\lambda^2}{4(n-1)},
$$

we know $H^2$ and $S$ are bounded. Hence, from the Gauss equations, we infer that the Ricci curvature is bounded from below. Applying the generalized maximum principle for $\mathcal{L}$ of $\lambda$-hypersurfaces to function $B$, there exists a sequence of points $\{p_k\} \subset M$ such that

$$
0 \geq sup B \left( 1 + \frac{n \lambda^2}{4(n-1)} - \sup \left\{ \frac{1}{n} (H - \lambda)^2 + \left( \sqrt{B} + \frac{|\lambda|}{2 \sqrt{n(n-1)}} \right)^2 \right\} \right).
$$

Hence, $sup B = 0$, that is, $S \equiv \frac{H^2}{n}$ and $X : M \rightarrow \mathbb{R}^{n+1}$ is isometric to

1. the sphere $S^n(r)$ with radius $0 < r < \sqrt{n}$ or
2. the Euclidean space $\mathbb{R}^n$.

It completes the proof of the theorem 1.3.

\[\square\]

**Proof of Theorem 1.2.** By a direct calculation, one obtains

$$
\frac{1}{2} \Delta |X|^2 = < \Delta X, X > + \sum_i < X_{,i}, X_{,i} >
$$

$$
= H < N, X > + n
$$

$$
= n + \frac{\lambda^2}{4} - (H - \frac{\lambda}{2})^2.
$$

(2.12)

Since the assumption of polynomial area growth, we have

$$
\int_M (|X|^2)e^{-\frac{|X|^2}{2}}d\mu < +\infty, \quad \int_M |\nabla |X|^2|^2e^{-\frac{|X|^2}{2}}d\mu < +\infty,
$$

then we can apply the lemma 2.1 to function 1 and $|X|^2$ and obtain

$$
\frac{1}{4} \int_M |\nabla |X|^2|^2e^{-\frac{|X|^2}{2}}d\mu = \frac{1}{2} \int_M (|X|^2)e^{-\frac{|X|^2}{2}}d\mu = \int_M (n + \frac{\lambda^2}{4} - (H - \frac{\lambda}{2})^2)e^{-\frac{|X|^2}{2}}d\mu.
$$

From $(H - \frac{\lambda}{2})^2 \geq n + \frac{\lambda^2}{4}$, we get

$$
(2.13) \quad (H - \frac{\lambda}{2})^2 = n + \frac{\lambda^2}{4}, \quad < X, X > = r^2,
$$

namely, $M$ is isometric to the sphere $S^n(r)$ with radius $r > 0$. It completes the proof of the proposition 1.2.

\[\square\]
Proof of Proposition 1.1. Integrating (2.12) over $M$ and using the Stokes formula, one concludes

$$\int_M (n + \frac{\lambda^2}{4} - (H - \frac{\lambda}{2})^2) d\mu = 0,$$

then it follows from $(H - \frac{\lambda}{2})^2 \leq n + \frac{\lambda^2}{4}$ that

$$\left( H - \frac{\lambda}{2} \right)^2 = n + \frac{\lambda^2}{4}$$

and $M$ is isometric to the sphere $S^n(r)$ with radius $r > 0$. It completes the proof of the proposition 1.1.

\[\square\]

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