On the existence of the Boltzmann-Grad limit for a system of hard smooth spheres

M. Tessarotto\textsuperscript{a,b} and P. Nicolini\textsuperscript{a,b}

\textsuperscript{a}Department of Mathematics and Informatics, University of Trieste, Italy, \textsuperscript{b}Consortium of Magneto-fluid-dynamics, University of Trieste, Italy

(Dated: June 30, 2008)

Abstract

Despite the progress achieved by kinetic theory, its rigorous theoretical foundations still remain unsolved to date. This concerns in particular the search of possible exact kinetic equations and, specifically, the conjecture proposed by Grad (Grad, 1972) and developed in a seminal work by Lanford (Lanford, 1974) that kinetic equations - such as the Boltzmann equation for a gas of classical hard spheres - might result exact in an appropriate asymptotic limit, usually denoted as Boltzmann-Grad limit. The Lanford conjecture has actually had a profound influence on the scientific community, giving rise to a whole line of original research in kinetic theory and mathematical physics. Nevertheless, several aspects of the theory remain to be addressed and clarified. In fact, its validity has been proven for the Boltzmann equation only at most in a weak sense, i.e., if the Boltzmann-Grad limit is defined according to the weak * convergence. While it is doubtful whether the result applies for arbitrary times and for general situations (and in particular more generally for classical systems of particles interacting via binary forces), it remains completely unsolved the issue whether the conjecture might be valid also in a stronger sense (\textit{strong Lanford conjecture}). This paper will point out a physical model providing a counter-example to the strong Lanford conjecture, representing a straightforward generalization of the classical model based on a gas of hard-smooth spheres. In particular we claim that that the one-particle limit function, defined in the sense of the strong Boltzmann-Grad limit, does not generally satisfy the BBGKY (or Boltzmann) hierarchy. The result is important for the theoretical foundations of kinetic theory.

PACS numbers: 47.10.ad,05.20.Dd
I. INTRODUCTION: BASIC MOTIVATIONS

Classical statistical mechanics, and in particular kinetic theory represents, is a sense, one of the unsolved problems of classical mechanics. In fact, although the microscopic statistical description (MSD) of classical dynamical systems formed by $N$-body systems is well known, a complete knowledge of their solutions is generally not achievable. From the mathematical viewpoint it provides an example of axiomatic approach following from first principles and as such it must be considered as an 'ab initio' formulation. Two equivalent treatments of MSD are known, which are based respectively on the introduction of a phase-space distribution function (PSDF) either on the $N$-body phase-space $\Gamma_N$ or, respectively, on the 1-particle phase-space $\Gamma_1$. In the $\Gamma_N$-approach the PSDF is the so-called microscopic PSDF $f_N$. It follows that $f_N$ obeys the Liouville equation, whose characteristics are simply the phase-space trajectories of the same dynamical system, to be identified with a classical $N$-body system \[^1, 6\]. This equation is equivalent to a hierarchy of equations (the so-called BBGKY hierarchy) for a suitable set of $s$-particles distribution functions ($f_s^{(N)}$), obtained letting $s = 1, \ldots, N - 1$, which are uniquely related to the corresponding PSDF. On the other hand, in the $\Gamma_1$-approach the PSDF (the Klimontovich probability density $k^{(N)}$, defined in the $\Gamma_1$-space) evolves in time by means of the Klimontovich equation \[^2\]. Also for this equation the characteristics are just the phase-space trajectories of the $N$-body system, this time - however - projected on the $\Gamma_1$-space. Therefore, in both cases it is actually necessary to determine the phase-space trajectories of all the particle. Hence, for classical systems characterized by a large number of particles ($N \gg 1$), the computational complexity (of this problem) is expected to prevent, in general, any direct calculation of the time-evolution either of the $N$-body or any of the the $s$-body distributions. This has justified the constant efforts placed so-far for the search of 'reduced' statistical descriptions, of which kinetic theory (KT) is just an example. This is intended in order to achieve efficient statistical descriptions especially suitable for complex dynamical systems, including both gases and plasmas. Precisely, the primary goal of KT is the search of statistical descriptions, either exact or in some sense approximate, whereby the whole dynamical system is associated only to the one-particle kinetic distribution function ($f_1$) defined on the one-particle phase-space $\Gamma_1$, without requiring the knowledge of the dynamics of the whole dynamical system. As a consequence in KT-descriptions the evolution equation of the kinetic distribution function,
to be denoted as kinetic equation, is necessarily assumed to depend functionally, in some suitable sense, only on the same distribution function and the one-particle dynamics. In particular, one of the most successful developments of KT is doubtless related to the so-called 'ab initio' approaches. These are to be intended (in contrast to heuristic or model equations) as the KT’s which are obtained deductively - by suitable approximation schemes and assumptions - from the corresponding exact MSD. In traditional approaches usually KT is obtained adopting the \( \Gamma_N \)-approach to MSD \[1,6,10\]. However, also the Klimontovich method (based on the \( \Gamma_1 \)-approach) can be used \[2\], since it is completely equivalent to that based on the \( \Gamma_N \)-approach \[24\]. In all cases KT’s have the goal of determining the evolution of suitable fluid fluid fields, associated to prescribed fluids, which are expressed as \textit{velocity moments of the kinetic distribution function} \( f_1 \) and satisfy an appropriate set of fluid equations, generally not closed, which follow from the relevant kinetic equation. 'Ab initio’ kinetic theories are - however - usually asymptotic in character. Namely, kinetic equations are typically satisfied only in an approximate (and asymptotic) sense and in a finite time interval, under suitable assumptions.

A. Asymptotic kinetic theories

A well-known asymptotic kinetic equation of this type is provided by the Boltzmann kinetic equation for a classical gas formed by \( N \) smooth rigid spheres of diameter \( d \) (Grad,1958 \[1\]), which is obtained from the exact equation of the BBGKY hierarchy for the one-particle kinetic distribution, i.e.,

\[
F_1(r_1,v_1,t) f_1^{(N)} d^2(N-1)C_1 f_2^{(N)},
\]

where \( F_1 \) and \( C_1 \) are respectively the free-streaming operator \( F_1(r_1,v_1,t) = \frac{\partial}{\partial t} + v_1 \cdot \frac{\partial}{\partial r_1} \) and a suitable collision operator \[1,6,10\]. For definiteness, in the remainder we adopt a dimensionless notation whereby all relevant functions (in particular, the Newtonian particle state \( x_1 = (r_1,v_1) \), the time \( t \), the particle diameter \( d \) and the volume of the configuration space \( V \)) are considered non-dimensional. The transition from the 1-particle equation \[1\] can be obtained by adopting a suitable asymptotic approximation and suitable assumptions on the joint probability densities \[1,3\]. These require, in particular, the introduction of the so-called \textit{rarefied gas ordering} (RG ordering), to be meant both in a global and local
sense, for the relevant physical parameters. More precisely, by imposing that \( \varepsilon = 1/N \) is an infinitesimal, the particle diameter \( d \), the volume \( V \) of the configuration space (\( \Omega \)) and the particle mass \( m \); the related global orderings are requiring to satisfy the orderings (Grad,1958 [1])

\[ \begin{align*}
    d &\sim o(\varepsilon^{1/2}), \\
    V &\sim o(\varepsilon^0), \\
    m &\sim o(\varepsilon), \\
    \eta(r,t) &= 4\pi n(r,t)d^3/3V \sim o(\varepsilon^{1/2}).
\end{align*} \]

The last ordering, in particular, prevents the number density \( n(r,t) \) from becoming so large that volume fraction \( \eta(r,t) \) can be locally finite, i.e., of order \( o(\varepsilon^0) \). In fact, it is well-known that if there results locally \( \eta(r,t) \sim o(\varepsilon^0) \) particle correlations (in particular two-particle correlations) may become non-negligible also on the large scale \([1, 4, 5]\). In fact, these correlations, which are not generally expected to decay rapidly in time \([4]\), can be also long range in character \([25]\). Instead, in validity of the RG ordering defined above, uniformly in phase-space and at least in a finite time interval \( I = [t_0, t_1] \), with \( \Delta t = t_1 - t_0 \) such that \( \Delta t \sim o(\varepsilon^0) \), the following conditions are assumed to be satisfied:

- Assumption #1 - in \( \Gamma_s \times I_{o1} \), the approximate (i.e., asymptotic) joint probability densities \( f_s(\varepsilon) \) (for any \( s \in \mathbb{N} \) with \( s \ll N \)) are smooth and bounded ordinary functions defined in \( \Gamma_s \times I_{o1} \), where \( \Gamma_s \) is the \( s \)-particle phase-space;

- Assumption #2 - the asymptotic factorization condition (AFC)

\[ f_s(\varepsilon, x_1, ..x_s, t) = \prod_{i=1,s} f_1(\varepsilon, x_i, t) [1 + \Theta(t - t_0) o(\varepsilon^\alpha)] \]

is satisfied identically for any \( s \in \mathbb{N} \) such that \( s/N \sim o(\varepsilon) \). Here \( f_1(\varepsilon, x_i, t) \) (for \( i = 1, s \)) is the one-particle probability density which satisfies the asymptotic Boltzmann equation

\[ F_1(r_1, v_1, t) f_1(\varepsilon,) = d^2 NC_1 f_2(\varepsilon, ), \]

and \( \Theta(t - t_0) \) is the Heaviside theta function which vanishes for \( t = t_0 \);
If the RG ordering and the previous assumptions hold locally (i.e., in the whole phase-space $\Gamma_1$ and at least in a finite time interval $I_{o1} \equiv [t_0, t_1]$), the Boltzmann equation (5) is expected to be locally valid in the same domain \([17, 19, 20]\) at least in an asymptotic sense. This means, introducing an arbitrary monotonic decreasing sequence of infinitesimal parameters $\{\varepsilon\} \equiv \{\varepsilon_i > 0, i \in \mathbb{N}\}$, that the sequence $\{f_1(\varepsilon, x_1, t)\}$ defined in terms of them is expected to converge in a weak (asymptotic) sense for $\varepsilon \to 0$. In other words the whole domain $\Gamma_1 \times I_{o1}$ [existence domain of $f_1(\varepsilon, x_1, t)$]:

- the asymptotic solution $f_1(\varepsilon, x_1, t)$ differs, by an error infinitesimal of order $o(\varepsilon^{\alpha_1})$ with respect to the exact solution $f_1^{(N)}(x_1, t)$, being $\alpha_1$ is an appropriate strictly positive real number. As a consequence, the error $\Delta f_1 \equiv f_1(\varepsilon) - f_1^{(N)}$, while remaining non zero, can be taken arbitrarily small;

- in $\Gamma_1 \times I_{o1}$ the Boltzmann kinetic equation differs from the exact one-particle BBGKY equation at most by terms of order $o(\varepsilon^{\alpha_2})$, where $\alpha_2$ is an appropriate real number $0 < \alpha_2 \leq 1$ in general different from $\alpha_1$.

Even if the rigorous proof of the global validity of the Boltzmann equation for arbitrary initial and boundary conditions has yet to be reached, its success in providing extremely accurate predictions for the dynamics of rarefied gases and plasmas is well known (see for example, Cercignani, 1969 \[6\]; Frieman, 1974 \[8\]).

### B. Boltzmann-Grad limit and the Lanford conjecture

Basic issues remain to be clarified regarding the rigorous theoretical foundations of KT. One such problem - and the one we want to address in this Note - refers in particular to the search of possible exact kinetic equations and, specifically, the conjecture suggested originally by Grad (Grad, 1972 \[3\]) and investigated by Lanford in a seminal paper (Lanford, 1974 \[7\]; see also Frieman, 1974 \[8\]), that kinetic equations - such as the Boltzmann equation for a gas of classical hard spheres - might result exact in an appropriate asymptotic limit, denoted as *Boltzmann-Grad (B-G) limit*. In other words, according to this conjecture, there should exist a suitable operator $L^*$ (denoted as B-G limit operator) such that the limit functions $f_s \equiv L^* f_s^{(N)}$ of the sequences $\{f_s^{(N)}\}$, to be defined in terms appropriate discrete sets $\{N_i \in \mathbb{N}\}$, should result exact solutions of the equation of the Boltzmann hierarchy.
The B-G limit is customarily intended as the limiting "regime" where the total number of particles \( N \) goes to infinity, while the configuration-space volume \( V \) remains constant, the particle diameter \( d \) goes to zero in such a way that \( N d^2 \) approaches a finite non-zero constant and the average mass density \( N m/V = M/V \) remains finite (Grad, 1972 [3]; Lanford, 1974 [7]; Frieman, 1974 [8]), i.e., there results:

\[
\begin{align*}
\frac{1}{N}, \frac{d, m}{\to 0}, \\
\frac{N d^2}{V} \to k_1, \\
M = \frac{mN}{V} \to k_2,
\end{align*}
\]

where \( k_i \) \((i = 1, 2)\) are prescribed non-vanishing finite constants. In the case of plasmas further analogous requirements must be placed on the total electric charge and current carried by each particle species [8, 23]. In addition, the proper definition must be made for the limit operator \( L^* \). In fact, in order that the sequences \( \{ f_s^{(N)} \} \) converge in some sense it is necessary to determine their time evolution. According to Lanford and previous authors this can be achieved by constructing and explicit solution of the corresponding equation of the BBGKY hierarchy, to be represented explicitly by a time-series expansion for each distribution \( f_s^{(N)} \). [7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] As a consequence, it was found that \( L^* \) can be defined in the sense of weak* convergence for the sequence \( \{ f_s^{(N)} \} \). The proof of the weak convergence of \( \{ f_s^{(N)} \} \) in this sense was first reached in the seminal work of Lanford (Lanford, 1974 [7]) who was able to prove also the local validity of the Boltzmann equation in a finite time interval \( I_{o1} = [t_o, t_1] \) of amplitude \( \Delta t = t_1 - t_o \) smaller than \( 1/5 \) and under the assumption of factorization at the initial time for the joint-particle distribution \( f_2 \). Even if this result does not suffice to justify possible meaningful physical applications, the conjecture has actually had a profound influence on the scientific community, giving rise to a whole line of original research in kinetic theory and mathematical physics. The work was later extended by other authors to include 2D and 3D and global validity for the Boltzmann equation. [7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] However, the validity of the Boltzmann equation for general situations remains dubious.

A key issue, however, is related to the possible validity of the Lanford conjecture in a stronger sense, not just for the Boltzmann equation but also for the BBGKY hierarchy itself, as following by suitable definition of the B-G limit operator \( L^* \) acting on the joint probability
densities \( f_s^{(N)} \). In fact, let us assume that \( L^* \) is defined in the sense of uniform convergence in phase-space of the sequences \( \{ f_s^{(N)} \} \) to the strong limit functions \( f_s \equiv L^* f_s^{(N)} \). In such a case the conjecture can be advanced that the strong limit functions \( f_s \) belong to same functional class of \( \{ f_s^{(N)} \} \) (strong Lanford conjecture). In particular, this means that when applying the operator \( L^* \) term by term to the equation of the BBGKY hierarchy for \( f_1^{(N)} \) [Eq. (1)]

\[
L^* F_1 f_1^{(N)} = L^* \left\{ d^2(N - 1)C_1 f_2^{(N)} \right\},
\]

the limit function \( f_1 \equiv L^* f_1^{(N)} \) should satisfy the corresponding equation of the Boltzmann hierarchy

\[
F_1 f_1 = k_1 C_1 f_2.
\]

For the validity of this limit equation it follows that \( L^* \) should commute with the streaming operator \( F_1 \), in the sense that it should result identically

\[
[L^*, F_1] f_1^{(N)}(x_1, t) \equiv 0,
\]

being \( [L^*, F_1] = L^* F_1 - F_1 L^* \). In the sequel we intend to point out, however, that the Lanford conjecture is not generally valid in this sense, i.e., that the limit functions defined in the strong B-G limit do not actually belong to the same functional class of the sequences \( \{ f_s^{(N)} \} \) and in particular to the solutions of the Boltzmann hierarchy. Nevertheless, weak convergence in the sense indicated above may still be warranted. In order to prove the point, in this paper we intend to propose a counter-example based on the introduction of a modified three-dimensional hard-sphere problem.

II. A COUNTER-EXAMPLE: A MODIFIED 3D HARD-SPHERE SYSTEM

To prove this point, we consider here a system \((S_N)\) of \( N \) partially-impenetrable hard-smooth spherical surfaces (‘spheres’). The other key element of the proof is the adoption of the Klimontovich approach. As indicated elsewhere \[24, 26\] this permits to construct an exact explicit integral representation for the \( s \)-particle distribution functions without recurring to cumbersome time-series representations \[24\]. For definiteness, the system \( S_N \) is defined by requiring that all particles are alike with diameter \( d \) and mass \( m \) and are included in
a bounded and connected 3D configuration space $\Omega$ of $\mathbb{R}^3$ of volume $V(\Omega)$. The particles of $S_N$ can be classified respectively as external and internal, according to the sub-domains of the $S_N$-configuration space to which they belong, denoted respectively as external and internal ($\Omega_{\text{ext}}$ and $\Omega_{\text{int}} = \Omega - \Omega_{\text{ext}}$). It is assumed that the two sub-domains are mutually inaccessible, i.e., particles cannot move from $\Omega_{\text{ext}}$ to $\Omega_{\text{int}}$ or vice versa. As a consequence the numbers of internal and external particles (defined by the occupation numbers $N_{\text{int}}$ and $N_{\text{ext}}$, with $N = N_{\text{int}} + N_{\text{ext}}$) are by assumption constant. External particles are those whose inter-particle distances (distance between the centers of the same spheres) is larger than (or equal) to $d$. Two particles are called mutually internal if their inter-particle distance is smaller than (or equal) to $d$. Internal particles are, therefore, those such that there exists at least another particle of $S_N$ with which they are mutually internal. It is required furthermore that: 1) the occupation numbers $N_{\text{int}}$ and $N_{\text{ext}}$ are both non-zero; 2) particles and the boundary of $\Omega$ are mutually impenetrable; 3) external particles are impenetrable when they collide with another (external or internal) particle; 4) two arbitrary mutually internal particles are, by definition, mutually impenetrable (since the intersection between their boundaries is always non-empty). Particles can undergo interactions either with the boundary (unary interactions) or among themselves (binary interactions), all assumed elastic. For binary interactions, we distinguish between external and internal collisions. In particular, external collisions occur when two particles - initially with an inter-particle distance large than $d$ - touch each other. Instead, internal collisions are defined only among mutually internal particles. The MSD for the system $S_N$, in analogy to the customary hard-smooth sphere problem \[1, 6\], can be achieved in principle in an elementary way by distinguishing between external and internal subsets of phase-space, either $\Gamma_N$ or $\Gamma_1$ (see discussion above). In particular, for example, $\Gamma_{N,\text{ext}}^\text{ext}$ and $\Gamma_{N,\text{int}}^\text{int}$ are respectively the subsets of $\Gamma_N$ in which particles belong respectively to the external and internal sub-domains of the configuration space. In the case of the $\Gamma_N$–phase-space formulation, this implies that the Liouville equation must be satisfied identically by the PSDF $f_N$ in both subspaces ($\Gamma_{N,\text{ext}}^\text{ext}$ and $\Gamma_{N,\text{int}}^\text{int}$). This leaves, nonetheless, a large freedom in the choice of the initial-boundary conditions as well as the functional class of $f_N$. In particular, due to the arbitrariness of $f_N$, it is always possible to invoke the assumptions: Assumption $\alpha$) the PSDF $f_N$ results continuous in the whole set $\Gamma_N$ and in particular on the boundary between external and internal particles ($\delta \Gamma_{N,\text{ext}}^\text{ext} = \delta \Gamma_{N,\text{int}}^\text{int}$); Assumption $\beta$) $f_N$ is a smooth and bounded ordinary
The corresponding $\Gamma_1$–phase-space formulation for $S_N$, is obtained by constructing the corresponding Klimontovich probability density. In $\Gamma_1$ for external particles it reads

$$k^{(N_{ext})}(y,t) = \frac{1}{N_{ext}} \sum_{i=1,N_{ext}} \delta(y - x_i(t))\Theta_i(r,t),$$

where $y = (r,v)$ is an arbitrary state vector of the one-particle phase space $\Gamma_1$. Here $x_i(t) = \chi_i(x_o, t_o, t)$ [for $i = 1, N$], denote the phase-space trajectories of the particles of $S_N$ with initial conditions $x_i(t_o) = x_{oi}$ [for $i = 1, N$]. These trajectories are assumed to be defined uniquely in the set $\Gamma_N \times I$, $I$ being a suitable bounded time interval. and $\Theta_i(r,t)$ is the function $\Theta_i(r,t) = 1 - \sum_{j=1,N,j \neq i} \Theta(d - |r - r_j(t)|)$, while $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0.
\end{cases}$$

As a consequence it follows that in the subset of phase-space $\Gamma_1$ for external particles ($\Gamma_1^{ext}$) the one-particle distribution function when expressed in terms of the initial microscopic PSPD reads $f_1^{(N)}(y,t) = \int_{\Gamma_N} d\chi_o f^{(N)}(x_o, t_o)k^{(N_{ext})}(y,t)$. Invoking the Liouville equation for $f^{(N)}$ this can be prove to imply:

$$f_1^{(N)}(y,t) = \widehat{f}_2^{(N)}(y,t) - \widehat{I}_2^{(N)}(y,t),$$

where

$$\widehat{f}_2^{(N)}(y,t) = \int_{\Gamma_N} d\chi f^{(N)}(\chi,t)\delta(y - x_1(t))$$

and

$$\widehat{I}_2^{(N)}(y,t) \equiv (N - 1)\int_{\Gamma_N} d\chi f^{(N)}(\chi,t)\delta(y - x_1(t))\Theta(d - |r - r_2(t)|).$$

Then the following result can be reached [26]:

**Theorem - Non-existence of the strong B-G limit for $S_N$**

*Let us assume that there is at least a finite time interval $I_{o1} = [t_o, t_1] \subseteq \mathbb{R}$ such that the probability densities $f_s^{(N)}$ ($s = 1, 2$) and their strong limit functions $f_s = L^*f_s^{(N)}$ ($s = 1, 2$) $f_1^{(N)}$ is strictly positive in $\Gamma_1 \times I_{o1}$, so that there results uniformly in $\Gamma_1 \times I_{o1}$ for $S_N : 1)$

$L^*\widehat{f}_2^{(N)}(y,t) = 0; 2)$ the strong limit function $f_1(y,t)$ reads $f_1(y,t) = L^*\widehat{f}_2^{(N)}(y,t); 3)$ $f_1(y,t)$ satisfies identically the homogeneous equation

$$F_1f_1(y,t) = 0.$$  \hfill (12)

As consequence, we conclude that in the B-G limit the strong limit function $f_1(y,t)$ does not generally satisfy equation (12) and hence neither the corresponding equation of the Boltzmann
hierarchy. Hence, at least in the case of the hard-sphere system here considered, the strong Lanford conjecture for the BBGKY hierarchy fails.

III. CONCLUSIONS

In this paper the issue of the validity of the Lanford conjecture in the sense of the strong B-G limit has been investigated. An example case has been formulated based on the analysis of a system of partially impenetrable smooth-hard spheres. We have shown that if the one-particle limit function \( f_1(y,t) \) is intended in the sense of the strong B-G limit it does not generally belong to the functional class \( \{ f_1 \} \) of the solutions of the one-particle limit equation. In other words, in such a case the limit function is neither a solution of the corresponding equation of the Boltzmann hierarchy nor - as a main consequence - of the Boltzmann equation. This result raises obviously the interesting question whether similar conclusions can be reached for the customary smooth-hard sphere system [1, 6] or to more general systems of interacting particles. This problem, together with a detailed analysis of the approach here developed, will be discussed elsewhere [26].

Acknowledgments

Work developed in cooperation with the CMFD Team, Consortium for Magneto-fluid-dynamics (Trieste University, Trieste, Italy). Research developed in the framework of the MIUR (Italian Ministry of University and Research) PRIN Programme: Modelli della teoria cinetica matematica nello studio dei sistemi complessi nelle scienze applicate. The support COST Action P17 (EPM, Electromagnetic Processing of Materials) and GNFM (National Group of Mathematical Physics) of INDAM (Italian National Institute for Advanced Mathematics) is acknowledged.

Notice

\[ \textsuperscript{8} \text{ contributed paper at RGD26 (Kyoto, Japan, July 2008).} \]
[1] H. Grad, *Principles of the Kinetic Theory of Gases*, Hand. der Physik, Vol.12 (Ed. S Flügge, Springer, Berlin, Göttingen, Heidelberg, 1958).

[2] Yu.L. Klimontovich, J.Exp.Teor.Phys. *6*, 753 (1958); *On the kinetic Boltzmann equation for a non-ideal gas*, ZhETP *60*, 1351 (1971).

[3] H. Grad, *Singular Limits of Solutions of Boltzmann’s equation*, Proc. of the 8th International Symposium in Rarefied Gas Dynamics, Stanford, July 1972, Vol. 1, p.37.

[4] H. Grad, *Levels of description in Statistical Mechanics and Thermodynamics*, in Delaware Seminar on the Foundations of Physics, Springer-Verlag, 1967.

[5] S. Tsuge, Phys. Letters *33A*, 145 (1970).

[6] C. Cercignani, *Mathematical Methods in Kinetic Theory*, Plenum Press, New York (1969).

[7] O.E. Lanford Jr.III, *Time evolution of large classical systems*, in Proc. Dynamical Systems, *Theory and Applications, 1974 Battelle Rencontre on Dynamical Systems*, Ed. J. Moser, Lecture Notes in Physics, Vol.*38* (Springer,Berlin, p.1 (1975).

[8] E.A. Frieman, *The present status of plasma kinetic theory*, ibid., p.175.

[9] K. Alexander, Ph.D. thesis, Department of Mathematics, University of California at Berkeley (1975).

[10] C. Cercignani, *Theory and application of the Boltzmann equation*, Scottish Academic Press, Edinburgh, New York (1975).

[11] O.E. Lanford, Soc.Math. de France, Asterisque *40*, 117 (1976).

[12] S. Kaniel and M. Shinbrot, Comm.Math.Phys. *58*, 65

[13] H. Van Beijeren, O.E. Lanford, J.L. Lebowitz e H. Spohn, J. Stat. Physics *22*, 237 (1980).

[14] H. Spohn, J.Stat. Phys. *26*, 285 (1981).(1978).

[15] O.E. Lanford, *The hard sphere gas in the Boltzmann-Grad limit*, Physica *106A*, 70 (1981).

[16] H. Spohn, *Fluctuation theory of the Boltzmann equation*, in *Non-equilibrium Phenomena I: the Boltzmann equation*, Amsterdam 1983.

[17] M. Shinbrot, *Local validity of the Boltzmann equation*, Math.Methods Appl.Sc. *6*, 539 (1984).

[18] H. Spohn, *Boltzmann hierarchy and Boltzmann equation*, in *Kinetic theories and the Boltzmann equation*, ed. C. Cercignani, Springer-Verlag (1984).

[19] R. Illner, M. Pulvirenti, *Global validity of the Boltzmann equation for a two-dimensional rare
gas in vacuum, Comm.Math.Phys. 105, 189 (1986).

[20] M. Pulvirenti, *Global validity of the Boltzmann equation for a three-dimensional rare gas in vacuum*, Comm.Math.Phys. 113, 79 (1987).

[21] R. Illner, M. Pulvirenti, *A derivation of the BBGKY-hierarchy for hard-sphere particle systems*, Transp.Th.Stat.Phys. 16(7), 997 (1987).

[22] V.I. Gerasimenko and D.Ya. Petrina, *Teoreticheskaya i Matematiceskaya Fizika*, Vol.83, No.1, pp.92 (1990).

[23] M. Tessarotto, *The existence of the Boltzmann-Grad limit and the present status of plasma kinetic theory*, Bull. Am.Phys. Soc. 44, 1113 (1999).

[24] M. Pin, *Fondamenti di teoria cinetica dei gas rarefatti*, M.D. Thesis in Theoretical Physics, University of Trieste, Supervisor M. Tessarotto (2001).

[25] J. Pasecki and R. Soto, *Ring approximation at equilibrium: the hard sphere correlation function*, Physica A 379, 409 (2007).

[26] M. Tessarotto, *On the Boltzmann-Grad Limit for the classical hard-spheres system*, to appear (2008)