PLATEAU’S PROBLEM WITH ČECH HOMOLOGICAL CONDITIONS ON $C^2$ MANIFOLD

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Abstract. Let $\Omega$ be a $n$ dimensional complete Riemannian manifold of class $C^2$, $d$ a positive integer between 1 and $n − 1$. We will solve Plateau’s problem in dimension $d$ on $\Omega$ with Čech homology conditions.

1. Introduction and notation

Plateau’s problem raised from 18th century, flourish since the middle 20th century. The first solution to the Plateau’s problem in the classical form came in the early of 1930’s by Douglas [4] and Radó [12] independently. In 1960’s, Federer and Fleming [9] introduced currents, and solved the Plateau’s problem as a mass minimizing problem of currents. There are simultaneously raised the size minimizing problem of currents. However, the problem is till open. Contemporaneously, Riefenberg [13] considered the Plateau’s problem involving homological boundary conditions, and demonstrated the existence of solutions in euclidean spaces when the coefficient group is compactly abelian and when the support of the algebraic boundary is compact and one-dimension lower.

Decades have passed since Riefenberg proposed Plateau’s problem with Čech homological conditions, which was developed by many, and there are indeed a plenty of existence results in standard euclidean spaces, but little is known when we put it on more complicated ambient space, such as euclidean spaces with holes, manifolds or Banach spaces etc. There are several difficulties obstruct its development. When we consider a minimizing sequence, if it converges in Hausdorff distance to a set, the limit set is indeed a competitor, but we do not know its mass for the sake of lack of lower semi-continuity for Hausdorff measure in general, so we do not know that the limit set is whether or not a minimizer. If we consider that minimal sequence as a sequence of Radon measures, and suppose that it converges to a Radon measure, we automatically get the lower semi-continuity of the total mass, but the hard part is to get the density no less that one for the limit measure.

In this paper, we will consider the Plateau’s problem with homological conditions on a manifold of class $C^2$. By taking profit from the celebrated Nash’s embedding theorem that every Riemannian manifold can be isometrically embedded in some euclidean space, it remains to consider that which is on a submanifold in an euclidean space.

Theorem 1.1. Suppose $1 \leq d < m \leq n$. Let $\Omega \subseteq \mathbb{R}^n$ be an $m$-dimensional closed submanifold of class $C^2$, let $B_0 \subseteq \Omega$ be a compact subset. Let $G$ be an abelian group. Suppose that $\partial \Omega \setminus B_0 = \emptyset$ and $L \subseteq \mathcal{H}_{d−1}(B_0; G)$ is a subgroup. If there exits a uniformly bounded minimizing sequence which spans $L$, then there exits at least one minimizer.

Let $(X, \rho)$ be a metric space, $Y \subseteq X$ a subset. We denote by $\overline{Y}$, $\text{int}(Y)$ and $\text{diam}(Y)$ the closure, the interior and the diameter of $Y$ respectively. For any $x_0 \in X$ and $r > 0$, we denote by $U(x_0, r)$, $B(x_0, r)$ and $\partial B(x_0, r)$ the open ball, the closed ball and the sphere, which are centered at $x_0$ and of radius $r$.

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respectively. We denote by \( \mathcal{C}_c(X, \mathbb{R}) \) the set of continuous functions form \( X \) to \( \mathbb{R} \) which have compact support; and denote by \( \mathcal{R}(X) \) the set of all Radon measures on \( X \), which is always endowed with weak topology given by saying that \( \mu_k \to \mu \) if and only if \( \mu_k(f) \to \mu(f) \) for any \( f \in \mathcal{C}_c(X, \mathbb{R}) \). It is quite easy to see that if \( X \) is locally compact, \( \mu, \mu_k \) are Radon measures with \( \mu_k \to \mu \), then for any bounded compactly supported upper semi-continuous function \( g : X \to \mathbb{R} \) and lower semi-continuous function \( f : X \to \mathbb{R} \), it hold that
\[
\int_X f \, d\mu \leq \liminf_{k \to \infty} \int_X f \, d\mu_k \quad \text{and} \quad \int_X g \, d\mu \geq \limsup_{k \to \infty} \int_X g \, d\mu_k.
\]

For any positive integer \( n \geq 2 \) and positive integer \( d \) between 1 and \( n-1 \), for any \( d \)-plane \( P \) in \( \mathbb{R}^n \), we denote by \( P_k \) the orthogonal projection from \( \mathbb{R}^n \) onto \( P \), by abuse notation, we also denote it by \( P \). We denote by \( \mathcal{G}(n, d) \) the Grassmannian manifold which consists of \( d \)-planes in \( \mathbb{R}^n \) through \( 0 \) equipped with metric \( \rho(T, P) = \| T_k - P_k \| \). For any closed submanifold \( \Omega \) in \( \mathbb{R}^n \), we also denote by \( \partial \Omega \) the boundary of the manifold \( \Omega \). A Radon measure on \( \Omega \times \mathcal{G}(n, d) \) is called a varifold, we denote by \( \mathcal{V}_d(\Omega) \) the set of varifolds on \( \Omega \), see Definition 3.1 in [1]. Let \( \mathcal{M}(\Omega) \) be the collection of \( \mathcal{H}^d \)-measurable sets in \( \Omega \). Let \( \nu : \{ E \in \mathcal{M}(\Omega) : \mathcal{H}^d(E) < \infty \} \to \mathcal{V}_d(\Omega) \) be a mapping such that for any \( d \)-rectifiable \( E \) and \( \varphi \in \mathcal{C}_c(\Omega \times \mathcal{G}(n, d), \mathbb{R}) \),
\[
\nu(E)(\varphi) = \int_E \varphi(x, \tan(E, x))\mathcal{H}^d(x),
\]
and there is a constant \( 0 < \alpha < \infty \) such that for any purely \( d \)-unrectifiable set \( E \)
\[
\| \nu(E) \| \leq \alpha \mathcal{H}^d L E.
\]
A varifold \( V \in \mathcal{V}_d(\Omega) \) is called rectifiable if it can be express as a countable sum \( V = \sum c_i \nu(E_i) \) such that \( c_i \in (0, \infty) \) and \( E_i \) are \( d \)-rectifiable; furthermore, if \( c_i \) are positive integers, we call it integral varifold. We denote by \( \mathbf{RV}_d(\Omega) \) and \( \mathbf{IV}_d(\Omega) \) the sets of rectifiable varifolds and integral varifolds respectively.

**Theorem 1.2.** Suppose \( 1 \leq d < m \leq n \). Let \( \Omega \subseteq \mathbb{R}^n \) be an \( m \)-dimensional closed submanifold of class \( C^2 \). Let \( U \subseteq \mathbb{R}^n \) be an open set such that \( U \cap \Omega \neq \emptyset \) and \( U \cap \partial \Omega = \emptyset \). Let \( E_k \subseteq \Omega \cap U \) be a sequence of \( d \)-sets. Suppose that \( \nu(E_k) \to V \in \mathcal{V}_d(\Omega) \) and
\[
\lim_{k \to \infty} \inf_{\varphi} \left( \mathcal{H}^d(E_k) - \mathcal{H}^d(\varphi(E_k)) \right) = 0,
\]
where the infimum is taken over \( \varphi : \Omega \to \Omega \) ranging over all Lipschitz mappings which are homotopic to \( \text{id}_\Omega \) and whose support is compact and contained in \( U \). Then there is a \( d \)-rectifiable set \( E \subseteq \Omega \cap U \), which is a minimal set in \( U \cap \Omega \), such that
\[
\nu(L U \times \mathcal{G}(n, d)) = \nu(E).
\]

2. Deformation theorem

In this section, we will develop a deformation theorem. By a \( d \)-set, we mean a set which has positive finite many \( \mathcal{H}^d \) measure. For any cube \( \Delta = a + [-r, r]^k \) in \( \mathbb{R}^k \) and \( \eta > 0 \), we denote by \( \eta \Delta \) the cube \( a + [-\eta r, \eta r]^k \), and denote by \( \ell(\Delta) \) the sidelength of \( \Delta \). For any \( x \in \frac{1}{2} \Delta \), let \( p_{\Delta, x} : \mathbb{R}^n \setminus \{ x \} \to \partial \mathcal{B}(x, \sqrt{n} \ell(\Delta)) \) be the mapping defined be
\[
p_{\Delta, x}(z) = x + \frac{\sqrt{n} \ell(\Delta)}{|z - x|} (z - x), \forall z \neq x,
\]
let \( \Pi_{\Delta,x} : \Delta \setminus \{ x \} \to \partial \Delta \) be the mapping defined by

\[
\Pi_{\Delta,x}(z) = \{ x + t(z - x) : t \geq 0 \} \cap \partial \Delta.
\]

**Lemma 2.1.** Let \( X \subseteq \mathbb{R}^n \) be a Borel set, \( E \subseteq X \) a \( \mathcal{H}^d \)-measurable set, \( \Delta \subseteq \mathbb{R}^n \) a \( k \)-cube with \( d + 1 \leq k \leq n \), \( \varphi : \Delta \to \Delta \) a continuous mapping, and \( f : X \to [0,\infty) \) a \( \mathcal{H}^d \)-measurable function. Suppose that \( \mathcal{H}^d \llcorner E \) is locally finite, \( \Delta \) is centered at \( x_0 \) with sidelength \( r > 0 \). Then for any \( 0 < \beta < 1 \), there is a set \( Y = Y_{\Delta,E,\beta} \subseteq B(x_0, r/4) \cap \Delta \) such that \( \mathcal{H}^k(Y) \geq (1 - \beta)\omega_k r^k \), and for any \( x \in Y \)

\[
\int_{z \in E} \| (Dp_{\Delta,x})(\varphi(z)) \|^d f(z) \mathcal{H}^d(z) \leq \frac{4d^k \sqrt{k} \sigma_{k-1}}{(k - d) \omega_k \beta} \int_{z \in E} f(z) \mathcal{H}^d(z). \tag{2.1}
\]

**Proof.** We see that for any unit vector \( v \in \mathbb{R}^k \),

\[
Dp_{\Delta,x}(z)v = \frac{4r \sqrt{k}}{|z - x|} \left( v - \left( \frac{z - x}{|z - x|}, v \right) \frac{z - x}{|z - x|} \right)
\]

and \( \| Dp_{\Delta,x}(z) \| = r \sqrt{k}|z - x|^{-1} \). Thus

\[
\int_{x \in B(x_0, r/4)} \int_{z \in E} \| (Dp_{\Delta,x})(\varphi(z)) \|^d f(z) \mathcal{H}^d(z) \, d\mathcal{H}^k(x) = \int_{z \in E} \int_{x \in B(x_0, r/4)} \frac{(r \sqrt{k})^d f(z)}{|\varphi(z) - x|^d} \, d\mathcal{H}^k(x) \, d\mathcal{H}^d(z)
\]

\[
\leq \frac{k^{d/2} \sigma_{k-1}^k}{(k - d) \omega_k} \int_{z \in E} f(z) \mathcal{H}^d(z).
\]

By Chebyshev’s inequality, there is a set \( Y \subseteq B(0, r) \) such that \( \mathcal{H}^k(Y) \geq (1 - \beta)\omega_k r^k \) and (2.1) hold for any \( x \in Y \).

**Lemma 2.2.** For any \( k \)-cube \( \Delta \subseteq \mathbb{R}^k \), \( \mathcal{H}^d \)-measurable set \( E \subseteq \Delta \) and \( \mathcal{H}^d \)-measurable function \( g : E \to [0,\infty) \). If \( \mathcal{H}^d \llcorner E \) is locally finite and \( \mathcal{H}^{d+1} \llcorner E \) is purely \( d \)-unrectifiable, then we can find \( x_0 \in \frac{1}{2}\Delta \setminus \overline{E} \) and a \( C^\infty \) mapping \( q_{\Delta,x_0} : \mathbb{R}^k \to B(x, \sqrt{\mathcal{H}^k}(\Delta)) \) such that, by setting \( E = E^{rec} \cup E^{irr} \) and \( B = B(x_0, \text{dist}(x_0, E)/2) \), \( q_{\Delta,x_0}(E^{irr}) \) is purely \( d \)-unrectifiable, \( q_{\Delta,x_0}\llcorner \Delta \setminus B = p_{\Delta,x_0}\llcorner \Delta \setminus B \) and

\[
\int_{z \in E} \| Dp_{\Delta,x}(z) \|^d g(z) \mathcal{H}^d(z) \leq \frac{4^{d+1} k^{d/2} \sigma_{k-1}}{(k - d) \omega_k} \int_{z \in E} g(z) \mathcal{H}^d(z). \tag{2.2}
\]

**Proof.** Let \( \kappa : \mathbb{R} \to [0,1] \) be a \( C^\infty \) function such that \( \kappa(t) = 0 \) for any \( t \leq 0 \) and \( \kappa(t) = 1 \) for any \( t \geq 1 \). Applying Lemma 2.1 with \( \beta = 1/4 \), we can find \( Y_0 \subseteq \frac{1}{2}\Delta \) such that \( \mathcal{H}^k(Y_0) > 0 \) and (2.2) hold for any \( x \in Y_0 \). By Lemma 2.2 in [6], we get that for \( \mathcal{H}^k \)-a.e. \( x \in Y_0 \), \( \Pi_{\Delta,x}(E^{irr}) \) is purely \( d \)-unrectifiable, denote by \( \tilde{Y}_0 \) the set of such points \( x \). Since \( \mathcal{H}^{d+1}(\Delta \setminus \overline{E}) = 0 \), we have that \( \mathcal{H}^k(\tilde{Y}_0 \setminus \overline{E}) > 0 \), pick one point \( x_0 \in \tilde{Y}_0 \setminus \overline{E} \), put \( r_0 = \text{dist}(x_0, \overline{E}) \), and define the \( C^\infty \) mapping \( q_{\Delta,x_0} : \mathbb{R}^n \to B(x, \sqrt{\mathcal{H}^k}(\Delta)) \) by

\[
q_{\Delta,x_0}(z) = \begin{cases} \left( x + \kappa \left( 2r_0^{-1}(z - x) \right) \right) \left( p_{\Delta,x}(z) - x \right), & z \neq x, \\ x, & z = x. \end{cases}
\]

Then \( q_{\Delta,x_0}\llcorner \Delta \setminus B = p_{\Delta,x_0}\llcorner \Delta \setminus B \) and \( q_{\Delta,x_0}(E^{irr}) = p_{\Delta,x_0}(E^{irr}) \) is purely \( d \)-unrectifiable.

**Lemma 2.3.** Let \( \Delta \subseteq \mathbb{R}^k \) be a \( k \)-cube. For any \( \mathcal{H}^d \)-measurable set \( E \subseteq \Delta \) and \( \mathcal{H}^d \)-measurable function \( g : \Delta \to \mathbb{R} \), if \( \mathcal{H}^d \llcorner E \) is locally finite, then

\[
\int g \, d\mathcal{H}^d \llcorner \Pi_{\Delta,x}(E) \leq 2^{d} k^{d/2} \int \left( g \circ \Pi_{\Delta,x} \right) \| Dp_{\Delta,x} \|^d \, d\mathcal{H}^d \llcorner \Delta \setminus E
\]
Proof. Let $f_{\Delta,x} : \partial B(x, \sqrt{\ell} \Delta) \to \partial \Delta$ be the mapping defined by

$$f_{\Delta,x}(z) = \{x + t(z-x) : t \geq 0\} \cap \partial \Delta, \forall z \in \partial B(0, \sqrt{\ell} \Delta).$$

Then $f_{\Delta,x}$ is bi-Lipschitz and $f_{\Delta,x} \circ p_{\Delta,x} = \Pi_{\Delta,x}$. Since $B(x, \ell(\Delta)/2) \subseteq \Delta \subseteq B(x, \sqrt{\ell} \Delta)$, we get that Lip($f_{\Delta,x}$) $\leq 2\sqrt{\ell}$. Since $\mathcal{H}^d(f_{\Delta,x}(Z)) \leq \text{Lip}(f_{\Delta,x})^d \mathcal{H}^d(Z)$ for any set $Z \subseteq \partial B(x, \sqrt{\ell} \Delta)$, we get that

$$\int g \, d\mathcal{H}^d \left| f_{\Delta,x}(Z) \right| \leq \text{Lip}(f_{\Delta,x})^d \int g \circ f_{\Delta,x} \, d\mathcal{H}^d \left| L \cdot Z \right|.$$

Since $p_{\Delta,x}$ is a $C^\infty$ mapping and $\Pi_{\Delta,x} = f_{\Delta,x} \circ p_{\Delta,x}$, we get that

$$\int g \, d\mathcal{H}^d \left| \Pi_{\Delta,x}(E) \right| \leq \text{Lip}(f_{\Delta,x})^d \int g \circ f_{\Delta,x} \, d\mathcal{H}^d \left| p_{\Delta,x}(E) \right| \leq \text{Lip}(f_{\Delta,x})^d \int (g \circ f_{\Delta,x} \circ p_{\Delta,x}) \|Dp_{\Delta,x}\|^d \, d\mathcal{H}^d \left| L \cdot E \right| \leq 2^d d^d / 2 \int (g \circ \Pi_{\Delta,x}) \|Dp_{\Delta,x}\|^d \, d\mathcal{H}^d \left| L \cdot E \right|. $$

$\square$

Lemma 2.4. Let $X \subseteq \mathbb{R}^m$ be any set. Let $Y \subseteq \mathbb{R}^k$ be a compact set, $\tilde{f} : X \to \partial Y$ a Lipschitz mapping. Then we can find Lipschitz mapping $f : \mathbb{R}^m \to Y$ such that $f|_X = \tilde{f}$ and Lip($f$) = Lip($\tilde{f}$). Moreover any Lipschitz mapping, which is defined on the $k$-skeleton of an $n$-cube $\Delta \subseteq \mathbb{R}^n$, and which maps each $k$-faces of $\Delta$ to itself, admit a Lipschitz extension which maps each $i$-faces of $\Delta$ to itself, $k \leq i \leq n$.

Proof. Let $g : \mathbb{R}^m \to \mathbb{R}^k$ be a Lipschitz extension of $\tilde{f}$ such that Lip($g$) = Lip($\tilde{f}$). Since $Y$ is convex, for any $z \in \mathbb{R}^k$, there is a unique point $x_z \in Y$ such that $\text{dist}(z, Y) = |z - x_z|$. We denote by $\rho : \mathbb{R}^k \to Y$ the mapping defined by $\rho(z) = x_z$, and then define

$$f = \rho \circ g.$$

Since Lip($\rho$) = 1 and $\rho|_Y = \text{id}_Y$, we see that $f$ is a Lipschitz extension of $\tilde{f}$ with the same Lipschitz constant. The moreover part then follows from induction.

$\square$

Lemma 2.5. Let $\Delta \subseteq \mathbb{R}^k$ be a $k$-cube. For any $x \in \frac{1}{2} \Delta$, setting $B = B(x, \sqrt{k} \Delta)$, there exits a Lipschitz mapping $f_{\Delta,x} : B \to \Delta$ such that $f_{\Delta,x}|_{\partial B} : \partial B \to \partial \Delta$ is bi-Lipschitz, $f_{\Delta,x} \circ p_{\Delta,x}|_{\partial \Delta} = \Pi_{\Delta,x}$ and

$$\text{Lip}(f_{\Delta,x}) = \text{Lip}(f_{\Delta,x}|_{\partial B}) \leq 2^d k^{d/2}. \quad (2.3)$$

Proof. For any $z \in \partial B$, we define

$$h(z) = \{x + t(z-x) : t \geq 0\} \cap \partial \Delta.$$ 

Then $g : \partial B \to \partial \Delta$ is a Lipschitz mapping with Lip($h$) $\leq 2^d k^{d/2}$. Let $f_{\Delta,x} : B \to \Delta$ be a Lipschitz extension of $h$ with Lip($\tilde{h}$) = Lip($h$) as in Lemma 2.4. Then we see that $f_{\Delta,x} \circ p_{\Delta,x}|_{\partial \Delta} = h \circ p_{\Delta,x}|_{\partial \Delta} = \Pi_{\Delta,x}$ and (2.3) holds.

$\square$

Definition 2.6. Let $\mathbb{F}$ be a collection of finitely many $n$-cubes in $\mathbb{R}^n$, let $\mathbb{F}_m$ be the collection all $m$-faces of cubes in $\mathbb{F}$. We say that $\mathbb{F}$ is admissible if

- for any $\Delta_0, \Delta_1 \in \mathbb{F}_m$, either $\Delta_0 \cap \Delta_1 = \emptyset$, $\Delta_0 \subseteq \Delta_1$ or $\Delta_1 \subseteq \Delta_0$, where $\Delta_i = \Delta_i \setminus \cup \mathbb{F}_{-1}$, and
- for any $\Delta_0, \Delta_1 \in \mathbb{F}_m$, if $\Delta_1 \subseteq \Delta_0$, then there exist $\{\Delta_i\}_{2 \leq i \leq 1} \subseteq \mathbb{F}_m$ such that $\Delta_0 = \cup_{1 \leq i \leq 1} \Delta_i$ and $\Delta_i \cap \Delta_j = \emptyset$ for any $1 \leq i < j \leq 1$.
Theorem 2.7 (Deformation theorem). Let $\mathcal{F}$ be a collection of $n$-cubes in $\mathbb{R}^n$ which is admissible. Set $D = \bigcup \mathcal{F}$ and $\mathcal{F}'_m = \{ \Delta \in \mathcal{F}_m : \Delta \subseteq \text{int}(D) \}$ for $0 \leq m \leq n$. There exists $c_0 = c_0(n, d) \geq 1$ such that for any $d$-set $E \subseteq \mathbb{R}^n$ with $\mathcal{H}^{d+1}(E \cap \text{int}(D)) < \infty$, setting $E \cap \text{int}(D) = E^{\text{rec}} \cup E^{\text{irr}}$ such that $E^{\text{rec}}$ is $d$-rectifiable and $\mathcal{H}^d$-measurable, $E^{\text{irr}}$ is purely $d$-unrectifiable and $\mathcal{H}^d$-measurable, we can find a Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, an upper semi-continuous function $\lambda : \mathbb{R}^n \to [0, \infty)$, $\delta \subseteq \mathcal{F}'_d$ and an open set $W \subseteq \mathbb{R}^n$ such that

- $\phi(\Delta) \subseteq \Delta$ for any $\Delta \in \mathcal{F}_m$, $\phi|_{\mathbb{R}^n \setminus D} = \text{id}_{\mathbb{R}^n \setminus D}$, $\overline{E} \subseteq W$,
- $(\cup \mathcal{H}) \cap \text{int}(D) \subseteq \phi(E) \cap \text{int}(D) \subseteq \phi(W) \cap \text{int}(D) \subseteq \mathcal{F}_d \cup \mathcal{H} \subseteq \cup \mathcal{F}'_{d-1}$,
- $\mathcal{H}^d(\phi(E^{\text{irr}}) \cap \text{int}(D)) = 0$, $\lambda(x) = 1$ for any $x \in \mathbb{R}^n \setminus D$,
- for any $\mathcal{H}^d$-measurable $d$-set $Z \subseteq W$ and $\mathcal{H}^d$-measurable function $g : \mathbb{R}^n \to [0, \infty)$,

$$\int g \, d\mathcal{H}^d\mathcal{L}(\phi(Z)) \leq \int (g \circ \phi) \cdot \lambda \, d\mathcal{H}^d\mathcal{L}(Z),$$

(2.4)

- for any $\mathcal{H} \subseteq \mathcal{F}_n$, setting $K = \cup \mathcal{H}$ and $A = \cup \{ \Delta \in \mathcal{F}_n : \Delta \cap \partial D \neq \emptyset \}$, we have that

$$\mathcal{H}^d(\phi(E \cap K)) \leq \int_{\mathcal{H} \cap K} \lambda(x) \, d\mathcal{H}^d\mathcal{L}(x) \leq c_0 \left( \mathcal{H}^d(E^{\text{rec}} \cap K) + \mathcal{H}^d(E^{\text{irr}} \cap K \cap A) \right).$$

(2.5)

In particular, if $E \cap \text{int}(D)$ is relatively closed in $\text{int}(D)$ and

$$\mathcal{H}^d(E^{\text{rec}} \cap \text{int}(D)) \leq c_0^{-1} \min \{ \ell(\Delta)^d : \Delta \in \mathcal{F}'_n \},$$

(2.6)

then

$$\phi(W) \cap \text{int}(D) \subseteq \cup \mathcal{F}'_{d-1}.$$  

(2.7)

Proof. For any $\Delta \in \mathcal{F}_n$, let $x_\Delta$ and $q_{\Delta,x_\Delta}$ be as in Lemma 2.2, let $f_{\Delta,x_\Delta}$ be as in Lemma 2.5 with $x = x_\Delta$. We define the mapping $\phi_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi_1 : \mathbb{R}^n \setminus \{ x_\Delta : \Delta \in \mathcal{F}_n \} \to \mathbb{R}^n$ by

$$\phi_1(z) = \begin{cases} f_{\Delta,x_\Delta} \circ q_{\Delta,x_\Delta}(z), & z \in \Delta, \Delta \in \mathcal{F}_n, \\ z, & \text{otherwise}, \end{cases}$$

and

$$\psi_1(z) = \begin{cases} f_{\Delta,x_\Delta} \circ p_{\Delta,x_\Delta}(z), & z \in \Delta \setminus \{ x_\Delta \}, \Delta \in \mathcal{F}_n, \\ z, & \text{otherwise}. \end{cases}$$

Then we see that $\phi_1$ is Lipschitz, and by setting $r_\Delta = \frac{1}{2} \text{dist}(x_\Delta, E)$, $\psi_1$ coincide with $\phi_1$ on the set

$$U_1 = \mathbb{R}^n \setminus \bigcup_{\Delta \in \mathcal{F}_n} B(x_\Delta, r_\Delta).$$

Put $E_0 = E$, $E_1 = \psi_1(E_0)$, and $E_1 \cap \text{int}(D) = E_1^{\text{rec}} \cup E_1^{\text{irr}}$. Since $\phi_1$ is Lipschitz, we see that

$$\overline{E}_1 = \overline{\psi_1(E)} = \overline{\phi_1(E)} = \phi_1\left(\overline{E}\right),$$

and $\mathcal{H}^{d+1}(E_1) = 0$. Similarly, for any $\Delta \in \mathcal{F}_{n-1}$ with $\Delta \subseteq \text{int}(D)$, we can find $x_\Delta \in \frac{1}{2}\Delta \setminus \overline{E_1}$, such that $\Pi_{\Delta,x_\Delta}(E_1^{\text{irr}} \cap \Delta)$ is purely $d$-unrectifiable. Define the mapping $\psi_2$ by

$$\psi_2(z) = \begin{cases} f_{\Delta,x_\Delta} \circ p_{\Delta,x_\Delta}(z) = \Pi_{\Delta,x_\Delta}(z), & z \in \Delta \setminus \{ x_\Delta \}, \Delta \in \mathcal{F}'_{n-1}, \\ z, & \text{in } \mathbb{R}^n \setminus \text{int}(D). \end{cases}$$

We see that $\psi_2$ is not defined on whole space $\mathbb{R}^n$, which is only defined on

$$\text{dmn}(\psi_2) = (\mathbb{R}^n \setminus \text{int}(D)) \cup \bigcup \{ \Delta \setminus \{ x_\Delta \} : \Delta \in \mathcal{F}'_{n-1} \},$$
but fortunately it is well defined on $\psi_1(U_1)$ at least, and $\text{im}(\psi_2) \cap \text{int}(D) \subseteq \cup\{\Delta : \Delta \in \mathcal{F}_{n-1}\}$. Put

$$r_\Delta = \frac{1}{2} \text{dist}(x_\Delta, E_1) \text{ and } U_2 = (\mathbb{R}^n \setminus \text{int}(D)) \cup \bigcup \{\Delta \setminus \mathcal{B}(x_\Delta, r_\Delta) : \Delta \in \mathcal{F}_{n-1}'\}.$$

We see that $\psi_2|_{U_2}$ is Lipschitz. Let $\phi_2 : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz extension of $\psi_2|_{U_2}$, whose existence is guaranteed by Lemma 2.5 and Lemma 2.4, such that $\psi_2(\Delta) \subseteq \Delta$ for any $\Delta \in \mathcal{F}_n$.

By induction, we can define $\psi_i$, $U_i$, $E_i$ and $\phi_i : \mathbb{R}^n \to \mathbb{R}^n$ for $1 \leq i \leq n-d$ in a similar way, which indeed are satisfying that

$$\text{dmm}(\psi_i) = (\mathbb{R}^n \setminus \text{int}(D)) \cup \bigcup \{\Delta \setminus \{x_\Delta\} : \Delta \in \mathcal{F}_{n-i+1}'\},$$

$$\psi_i(z) = \begin{cases} z, & \Delta \in \mathcal{F}_{n-i+1}', \\ \Pi_{\Delta,x_\Delta}(z), & \Delta \in \mathcal{F}_n, \end{cases}$$

$$E_i = \psi_i(E_{i-1}), \quad r_\Delta = \text{dist}(x_\Delta, E_{i-1}),$$

$$U_i = (\mathbb{R}^n \setminus \text{int}(D)) \cup \bigcup \{\Delta \setminus \mathcal{B}(x_\Delta, r_\Delta) : \Delta \in \mathcal{F}_{n-i+1}'\},$$

$\psi_i(E_{i-1}' \cap \Delta)$ is purely $d$-unreconstructible for any $\Delta \in \mathcal{F}_{n-i+1}$, $\psi_i|_{U_i}$ is Lipschitz, and $\phi_i$ is a Lipschitz extension of $\psi_i|_{U_i}$ such that $\phi_i(\Delta) \subseteq \Delta$ for any $\Delta \in \cup_{n-i+1 \leq j \leq n} \mathcal{F}_j$. Since $E_{n-d} = \psi_{n-d}(E_{n-d-1})$ and $\text{im}(\psi_{n-d}) \cap \text{int}(D) \subseteq \cup\{\Delta : \Delta \in \mathcal{F}_d\}$, we get that

$$E_{n-d} \cap \text{int}(D) \subseteq \cup\{\Delta : \Delta \in \mathcal{F}_d\}.$$

For any $\Delta \in \mathcal{F}_d$, if $\Delta \setminus \mathcal{E}_{n-d} \neq \emptyset$, then we pick one point $x_\Delta \in \Delta \setminus \mathcal{E}_{n-d}$. Let $\phi_{n-d+1} : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz mapping such that $\phi_{n-d+1}|_\Delta = \text{id}_\Delta$ if $\Delta \in \mathcal{F}_d$ and $\Delta \setminus \mathcal{E}_{n-d} = \emptyset$ or $\Delta \subseteq \partial D$; $\phi_{n-d+1}|_\Delta = q_{\Delta,x_\Delta}$ if $\Delta \setminus \mathcal{E}_{n-d} \neq \emptyset$. Put $E_{n-d+1} = \phi_{n-d+1}(E_{n-d})$, $r_\Delta = \text{dist}(x_\Delta, E_{n-d})$, and

$$U_{n-d+1} = (\mathbb{R}^n \setminus \text{int}(D)) \cup \bigcup \mathcal{F}_d' \setminus \bigcup \{\mathcal{B}(x_\Delta, r_\Delta) : \Delta \in \mathcal{F}_d' \cap \mathcal{E}_{n-d} \neq \emptyset\}.$$

Put $W_{n-d+1} = U_{n-d+1}$, and $W_i = U_i \cap \phi_{i-1}^{-1}(W_{i+1})$ for $1 \leq i \leq n-d$. Since $U_i$ is open in $(\mathbb{R}^n \setminus \text{int}(D)) \cup \bigcup \mathcal{F}_{n-i+1}'$, $1 \leq i \leq n-d+1$, we get that $W_i$ is open in $U_i$. Since $U_1$ is open in $\mathbb{R}^n$, we get that $W_1$ is open in $\mathbb{R}^n$. Put $\lambda_{n-d+1} \equiv 1$, and define $\lambda_i : \cup \mathcal{F}_{n-i+1} \to \mathbb{R}$, $2 \leq i \leq n-d$, by

$$\lambda_i(z) = \begin{cases} 2^d(n-i+1)^{d/2} \|Dq_{\Delta,x_\Delta}(z)\|^d \cdot \lambda_{i+1}(\phi_{n-i+1}(z)), & z \in \hat{\Delta} \Delta \in \mathcal{F}_d', \\ 2^d(n-i+1)^{d/2} \cdot 4^d(n-i+1)^{d/2} \cdot \lambda_{i+1}(z), & z \in \Delta \setminus \hat{\Delta}, \Delta \in \mathcal{F}_d, \\ 1, & z \in \cup \mathcal{F}_{n-i+1} \setminus \cup \mathcal{F}_{n-i+1}' . \end{cases}$$

Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be define by

$$\lambda_1(z) = \begin{cases} 2^d n^{d/2} \|Dq_{\Delta,x_\Delta}(z)\|^d \cdot \lambda_2(\phi_1(z)), & z \in \hat{\Delta} \Delta \in \mathcal{F}_n, \\ 2^d n^{d/2} \cdot 4^d n^{d/2} \cdot \lambda_2(z), & z \in \Delta \setminus \hat{\Delta}, \Delta \in \mathcal{F}_n, \\ 1, & z \in \mathbb{R}^n \setminus D. \end{cases}$$

Define $\phi = \phi_{n-d+1} \circ \phi_{n-d} \circ \cdots \circ \phi_1$, $W = W_1$ and $\lambda = \lambda_1$. Then $\phi|_{\mathbb{R}^n \setminus D} = \text{id}_{\mathbb{R}^n \setminus D}$, $\phi(\Delta) \subseteq \Delta$ for any $\Delta \in \mathcal{F}_n$, $0 \leq m \leq n$, $E \subseteq W$, $\phi(E) = E_{n-d+1}$ and $\phi(E) = E_{n-d+1}$. Put $\delta = \{\Delta \in \mathcal{F}_d : \Delta \setminus \mathcal{E}_{n-d+1} = \emptyset\}$. Then $\cup \delta \subseteq \mathcal{F}_{n-d+1}$ and $\phi(W) \subseteq \phi_{n-d+1}(U_{n-d+1})$, thus $\phi(W) \cap \text{int}(D) \subseteq \cup \mathcal{F}_d'$ and $\phi(W) \cap \text{int}(D) \subseteq \cup \mathcal{F}_d'$. By our construction of $\psi_i$, we see that $\psi_{n-d+1} \circ \cdots \circ \psi_1(E_{\text{irr}})$ is purely $d$-unreconstructible, but $\text{int}(D) \cup \psi_{n-d+1} \cdots \circ \psi_1(E_{\text{irr}})$ is contained in $\cup \mathcal{F}_d'$, thus $\text{int}(D) \cap \psi_{n-d+1} \cdots \circ \psi_1(E_{\text{irr}})$ must be a set of $\mathcal{F}_d'$ measure 0. So we get that $\mathcal{F}_d'(\text{int}(D) \cap \phi(E_{\text{irr}})) = 0$. Indeed, $\lambda|_{\mathbb{R}^n \setminus D} \equiv 1$ and
(2.4) clearly follows from the construction of $\lambda$ and Lemma 2.3. For any $\Delta \in F_n'$, $2 \leq m \leq n - d$, since $x_\Delta \in \frac{1}{2}\Delta$, we see that
\[ \|Dq_{\Delta,x_\Delta}(z)\| = \frac{\sqrt{m}(\Delta)}{|z - x_\Delta|} \leq 4\sqrt{m}, \quad \forall z \in \Delta \setminus \Delta, \]
thus the functions $\lambda_i$, $1 \leq i \leq n - d$, are upper semi-continuous. In particular, $\lambda = \lambda_1$ is upper semi-continuous. For any $1 \leq i \leq n - d$ and $\Delta \in F_{n-i+1}'$, setting $\phi_0 = \text{id}_{2^n}$, since $\mathcal{H}^d(E^{rr} \cap \Delta) = 0$, by Lemma 2.1, there is a constant $\zeta_i > 0$, which only depends on $n - i + 1$ and $d$, such that
\[ \int_{\Delta \cap E} \lambda_i(\phi_{i-1} \circ \cdots \circ \phi_0(z)) d\mathcal{H}^d(z) = \int_{\Delta \cap E^{rr}} \lambda_i(\phi_{i-1} \circ \cdots \circ \phi_0(z)) d\mathcal{H}^d(z) \leq \zeta_i d^{\mathcal{H}^d(E \cap \Delta)}; \]
if $\Delta \in F_{n-i+1} \setminus F_{n-i+1}'$, by Lemma 2.1, we only get that
\[ \int_{\Delta \cap E} \lambda_i(\phi_{i-1} \circ \cdots \circ \phi_0(z)) d\mathcal{H}^d(z) \leq \zeta_i d^{\mathcal{H}^d(E \cap \Delta)}, \]
hence (2.5) holds with $c_0 = \max\{1, \zeta_1\}$. \hfill \Box

**Lemma 2.8.** Let $c_0 = c_0(n, d) \geq 1$ be the constant in Theorem 2.7. There is a constant $c_1 = c_1(n, d) > 0$ such that for any $d$-set $E \subset \mathbb{R}^n$ and $0 < r < \rho$, there is a Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(U(x, \rho)) \subset U(x, \rho)$, $\phi|_{\mathbb{R}^n \setminus U(x, \rho)} = \text{id}_{\mathbb{R}^n \setminus U(x, \rho)}$.
\[ \mathcal{H}^d(\phi(E \cap U(x, \rho))) \leq c_0 \left( \mathcal{H}^d(E \cap A(x, r, \rho)) + \mathcal{H}^d(E^{rec} \cap U(x, \rho)) \right), \] (2.8)
and
\[ \mathcal{H}^d(\phi(E \cap U(x, \rho))) \leq c_0 \mathcal{H}^d(E \cap A(x, r, \rho)) + \frac{c_1\rho^n}{(\rho - r)^{d-n}}. \] (2.9)
Moreover, if $r > \rho/4$ and
\[ \mathcal{H}^d(E^{rec} \cap U(x, \rho)) < c_0^{-1}(3\sqrt{n})^{d-n}(\rho - r)^d, \] (2.10)
then we have that
\[ \mathcal{H}^d(\phi(E \cup U(x, \rho))) \leq c_0 \mathcal{H}^d(E \cap A(x, r, \rho)). \] (2.11)

**Proof.** Put $\delta = (\rho - r)/(3\sqrt{n})$. For any $i_1, \ldots, i_n \subset \mathbb{Z}$, we denote by $C_{i_1, \ldots, i_n}$ the cube defined by $(\delta i_1, \ldots, \delta i_n) + [0, \delta]^n$. Let $F$ be the collection of all cubes $C_{i_1, \ldots, i_n} \subset U(x, \rho)$, and let $K = \{ \Delta \in F : \Delta \cap B(x, r) \neq \emptyset \}$. By Theorem 2.7, we can find Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{H} \subset F'$ such that $\phi(\Delta) \subset \Delta$ for any $\Delta \in F$, $\phi(x) = x$ for any $x \in \mathbb{R}^n \setminus \cup F$, $\phi(E) \cap \text{int}(\cup F_n) \subset \cup F_d$, $\phi(E) \cap \text{int}(\cup F_n) \subset \cup F_d$, and
\[ \mathcal{H}^d(\phi(E \cap (\cup K))) \leq c_0 \mathcal{H}^d(E^{rec} \cap (\cup K)). \]

Immediately, we get that (2.8) holds since
\[ \mathcal{H}^d(\phi(E \cap U(x, \rho) \setminus \cup K)) = c_0 \mathcal{H}^d(E \cap U(x, \rho) \setminus \cup K) \leq c_0 \mathcal{H}^d(E \cap A(x, r, \rho)). \]
Since $\phi(E \cup B(x, r)) \subset \cup F_n'$, and the number of cubes in $F$ is no more than $\omega_n\rho^n/\delta^n$, we get that (2.9) hold for some constant $c_1$ which only depends on $n$ and $d$. Since $\phi(E) \cap (\cup F_n') \subset \cup S$ and $\mathcal{H}^d(\phi(E) \cap (\cup F_n')) \leq c_0 \mathcal{H}^d(E^{rec} \cap U(x, \rho)) < \delta^n$, but each element of $S$ has $\mathcal{H}^d$ measure 0 or $\delta^d$, we get that $\mathcal{H}^d(\phi(E) \cap (\cup F_n')) = 0$, and (2.11) holds. \hfill \Box
3. Quasiminimal sets on \( \mathbb{R}^n \)

**Definition 3.1.** For any set \( \Omega \subseteq \mathbb{R}^n \) and any open set \( U \subseteq \mathbb{R}^n \), if \( \Omega \cap U \) is nonempty and relatively closed in \( U \), we denote by \( \mathcal{D}(\Omega, U) \) the collection all of Lipschitz mappings \( \varphi : \Omega \to \Omega \) which is homotopic to \( \text{id}_\Omega \) and \( \{ x \in \Omega : \varphi(x) \neq x \} \) is relatively compact in \( U \).

**Definition 3.2.** Let \( \Omega, U \) be as in Definition 3.1. For any nondecreasing functions \( M : [0, +\infty) \to [1, +\infty) \) and \( \varepsilon : [0, +\infty) \to [0, +\infty) \), we define \( \mathcal{Q}\mathcal{M}(\Omega, U, M, \varepsilon) \) to be the collection of sets \( E \subseteq \Omega \) such that \( \mathcal{H}^{d}(E) \) is locally finite, \( E \cap U \) is relatively closed in \( U \), and for any \( \varphi \in \mathcal{D}(\Omega, U) \),

\[
\mathcal{H}^{d}(E \cap W_\varphi) \leq M(r)\mathcal{H}^{d}(\varphi(E \cap W_\varphi)) + \varepsilon(r),
\]

where \( r = \text{diam}(W_\varphi \cap \varphi(W_\varphi)) \).

If \( M \equiv 1 \) and \( \varepsilon \equiv 0 \), then any elements in \( \mathcal{Q}\mathcal{M}(\Omega, U, M, \varepsilon) \) is called minimal in \( \Omega \cap U \).

If \( M \equiv 1 \), and \( \varepsilon(r) = h(r)r^d \), where \( h \) is a gauge function, then elements in \( \mathcal{Q}\mathcal{M}(\mathbb{R}^n, U, M, \varepsilon) \) are called almost minimal sets in \( U \) with gauge function \( h \), see Definition 4.3 in [3].

If there is a \( \delta > 0 \) such that \( M(r) \equiv M \geq 1 \) and \( \varepsilon(r) \equiv 0 \), then any element in \( \mathcal{Q}\mathcal{M}(\mathbb{R}^n, U, M, \varepsilon) \) is called \((U, M, \varepsilon)\)-quasiminimal, see Definition 2.4 in [2].

If there is a \( \delta > 0 \) and a constant \( h \in (0, 1) \) such that \( M(r) \equiv M \geq 1 \) and \( \varepsilon(r) = h \varepsilon(r)^d \), then any element in \( \mathcal{Q}\mathcal{M}(\mathbb{R}^n, U, M, \varepsilon) \) is called \((U, M, \varepsilon)\)-quasiminimal, see Definition 2.10 in [3].

If \( \delta > 0 \) and \( \varepsilon : (0, \delta) \to [0, +\infty) \) is a nondecreasing function such that \( \varepsilon(0) = 0 \), then any element in \( \mathcal{Q}\mathcal{M}(\mathbb{R}^n, U, 1 + \varepsilon(r), 0) \) is called \((M, \varepsilon, \delta)\)-minimal set, see for example Chapter 11 in [11].

We do not assume previously \( \varepsilon(0+) = 0 \) in our definition. It is easy to see from the definition that

\[
\mathcal{Q}\mathcal{M}(\Omega, U, M, \varepsilon) \subseteq \mathcal{Q}\mathcal{M}(\Omega', U', M', \varepsilon'),
\]

if \( \Omega' \subseteq \Omega, U' \subseteq U, M \leq M' \) and \( \varepsilon \leq \varepsilon' \).

**Lemma 3.3.** For any \( E \in \mathcal{Q}\mathcal{M}(\mathbb{R}^n, U, M, \varepsilon) \) and open set \( O \subseteq \mathbb{R}^n \), we have that

\[
\mathcal{H}^{d}(E^{\text{irr}} \cap O) \leq \varepsilon(\text{diam}(O)).
\]

**Proof.** Write \( E \cap U = E^{\text{irr}} \cup E^{\text{rec}} \), where \( E^{\text{rec}} \) is \( d \)-rectifiable and \( \mathcal{H}^{d} \)-measurable, \( E^{\text{irr}} \) is purely \( d \)-unrectifiable and \( \mathcal{H}^{d} \)-measurable. Then for \( \mathcal{H}^{d} \)-a.e. \( x \in E^{\text{irr}} \),

\[
\Theta^{d}(E^{\text{rec}}, x) = 0 \text{ and } \Theta^{d}(E^{\text{irr}}, x) \geq 2^{d},
\]

and we denote by \( E_0 \) the collection of such points. For any \( \tau > 0 \), and \( x \in E_0 \), we can find a sequence of decreasing positive numbers \( \{ \rho_{x,m} \} \) such that \( \rho_{x,m} \to 0 \) as \( m \to \infty \), \( \mathcal{H}^{d}(E \cap \partial B(x, \rho_{x,m})) = 0 \),

\[
\frac{\mathcal{H}^{d}(E^{\text{rec}} \cap B(x, r))}{\omega_d r^d} \leq \tau, \forall 0 < r \leq \rho_{x,1}
\]

and

\[
\frac{\mathcal{H}^{d}(E^{\text{irr}} \cap B(x, r))}{\omega_d r^d} > \frac{1}{2^{d+1}}.
\]

We see that \( \{ B(x, \rho_{x,m}) : x \in E_0, m \geq 1, \rho_{x,m} < \text{dist}(x, \mathbb{R}^n \setminus O) \} \) is a Vitali covering of \( E_0 \cap O \). By Vitali covering theorem, we can find finite many disjoint balls \( \{ B_i \}_{1 \leq i \leq m} \) such that

\[
\mathcal{H}^{d}(E_0 \setminus \bigcup_{i=1}^{m} B_i) \leq \tau \mathcal{H}^{d}(E_0).
\]

For any \( 0 < r < \rho < \text{dist}(x, \mathbb{R}^n \setminus O) \), by Lemma 2.8, there exits Lipschitz mapping \( \phi_x : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \phi_x(B(x, \rho)) \subseteq B(x, \rho), \phi_x(y) = y \) for \( y \notin U(x, \rho) \) and

\[
\mathcal{H}^{d}(\varphi(E \cap U(x, \rho)) \leq c_0(\mathcal{H}^{d}(E \cap A(x, r, \rho)) + \mathcal{H}^{d}(E^{\text{rec}} \cap B(x, r))).
\]
Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping given by
\[
\varphi(y) = \begin{cases} 
  y, & y \in \mathbb{R}^n \setminus \bigcup_{i=1}^m B_i \\
  \phi_{x_i}(y), & y \in B_i.
\end{cases}
\]

Then $\varphi$ is Lipschitz, thus for any $0 < \delta < 1/2$, setting $B_i = B(x_i, \rho_i)$, $U_i = U(x_i, \rho_i)$ and $\rho'_i = (1 - \delta)\rho_i$, we get that
\[
J^d_{\omega}(E \cap \bigcup_{i=1}^m U_i) \leq M(\text{diam } O)J^d_{\omega}(\varphi(E \cap \bigcup_{i=1}^m U_i)) + \varepsilon(\text{diam } O)
\]
\[
\leq c_0M(\text{diam } O)\sum_{i=1}^m \left( J^d_{\omega}(E \cap A(x_i, \rho'_i, \rho_i)) + J^d_{\omega}(E^{rec} \cap B(x_i, \rho'_i)) \right) + \varepsilon(\text{diam } O).
\]
By the arbitrariness of $\delta$, we get that
\[
J^d_{\omega}(E \cap \bigcup_{i=1}^m U_i) \leq c_0M(\text{diam } O)\sum_{i=1}^m J^d_{\omega}(E^{rec} \cap U_i) + \varepsilon(\text{diam } O)
\]
\[
\leq c_0M(\text{diam } O)\sum_{i=1}^m 2^{d+1}rJ^d_{\omega}(E^{irr} \cap U_i) + \varepsilon(\text{diam } O),
\]
thus
\[
\left( 1 - 2^{d+1}c_0(\text{diam } O)r \right) \sum_{i=1}^m J^d_{\omega}(E^{irr} \cap U_i) \leq \varepsilon(\text{diam } O).
\]
By the arbitrariness of $\tau$, we get that
\[
J^d_{\omega}(E^{irr} \cap O) \leq \varepsilon(\text{diam } O).
\]

\[\square\]

**Corollary 3.4.** If $\limsup_{r \to 0^+} r^{-d}\varepsilon(r) < 2^{-d}\omega_d$, then every element in $\text{QM}(\mathbb{R}^n, U, M, \varepsilon)$ is rectifiable.

**Proof.** Write $E \cap U = E^{rec} \cup E^{irr}$. Since $\limsup_{r \to 0^+} r^{-d}\varepsilon(r) < 2^{-d}\omega_d$, by Lemma 3.3, we get that $\Theta^{-d}(E^{irr}, x) < 2^{-d}$ for every point $x \in U$. Thus $J^d_{\omega}(E^{irr}) = 0$. \[\square\]

**Lemma 3.5.** Suppose that $E \in \text{QM}(\mathbb{R}^n, U, M, \varepsilon)$. Let $c_1 = c_1(n, d) > 0$ be the constant in Lemma 2.8, and let $\eta > 1$ and $r_0 > 0$ be such that $2(n-d)c_0(\ln \eta)^{-1/2}M(\eta r_0) \leq 1$. Then for any $x \in E \cap U$ and $0 < r < \min\{r_0, \eta^{-1}\text{dist}(x, \mathbb{R}^n \setminus U)\}$, we have that
\[
J^d_{\omega}(E \cap B(x, r)) \leq 2c_1M(2\eta r)(\eta r)^d + \varepsilon(2\eta r).
\]

**Proof.** We take $\alpha_1 = 1$, and
\[
\gamma = \left( 1 - \frac{1}{c_0M(2\eta r)} \right)^{\frac{1}{2n-d}}, \quad \alpha_{m+1} = \left( 1 - \gamma^m \right)^{-1} \alpha_m, \quad m \geq 1.
\]
Since $(1 + x)^\beta \leq 1 + \beta x$ for $\beta \in [0, 1]$ and $x \geq -1$, we have that
\[
\gamma \leq \left( 1 - 2(n-d)(\ln \eta)^{-1/2} \right)^{1/(2(n-d))} \leq 1 - (\ln \eta)^{-1/2}
\]
Since $x/(1 + x) \leq \ln(1 + x) \leq x$ for $-1 < x < 1$, we have that
\[
\ln \alpha_{m+1} = -\sum_{k=1}^m \ln \left( 1 - \gamma^k \right) \leq \sum_{k=1}^m \frac{\gamma^k}{1 - \gamma^k} \leq \sum_{k=1}^m \frac{\gamma^k}{1 - \gamma} \leq \frac{\gamma}{(1 - \gamma)^2} < \ln \eta.
\]
Thus $\alpha_m < \eta$ for any $m \geq 1$. Put $r_1 = r$ and $r_m = \alpha_m r$. By Lemma 2.8, we can find Lipschitz mappings $\phi_m : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi_m(B(x,r_{m+1})) \subseteq B(x,r_m)$, $\phi_m|_{\mathbb{R}^n \setminus B(x,r_{m+1})} = \text{id}_{\mathbb{R}^n \setminus B(x,r_{m+1})}$ and

$$\mathcal{H}^d(\phi_m(E \cap U(x,r_{m+1}))) \leq c_0 \mathcal{H}^d(E \cap A(x,r_{m+1},r_m)) + c_1(\alpha_{m+1} - \alpha_m)^{d-n}(\alpha_{m+1})^n r^d.$$

Thus

$$\mathcal{H}^d(E \cap U(x,r_{m+1})) \leq M(2\eta)\mathcal{H}^d(\phi_m(E \cap U(x,r_{m+1}))) + \varepsilon(2\eta)$$

$$\leq c_0 M(2\eta)\mathcal{H}^d(E \cap A(x,r_{m+1},r_m)) + c_1 M(2\eta)(\alpha_{m+1} - \alpha_m)^{d-n}\alpha_{m+1}^n r^d + \varepsilon(2\eta),$$

and we get that

$$\mathcal{H}^d(E \cap B(x,r_m)) \leq \frac{c_0 M(2\eta) - 1}{c_0 M(2\eta)} \mathcal{H}^d(E \cap U(x,r_{m+1})) + \frac{c_1 M(2\eta)(\alpha_{m+1} - \alpha_m)^{d-n}\alpha_{m+1}^n r^d}{c_0 M(2\eta)} + \varepsilon(2\eta),$$

thus

$$\mathcal{H}^d(E \cap B(x,r)) \leq \left(\frac{c_0 M(2\eta) - 1}{c_0 M(2\eta)}\right)^m \mathcal{H}^d(E \cap U(x,r_{m+1})) + \sum_{k=1}^m \frac{c_1 c_0^{-1} r^d \alpha_{k+1}^n}{(\alpha_{k+1} - \alpha_k)^{n-d}} \left(\frac{c_0 M(2\eta) - 1}{c_0 M(2\eta)}\right)^k + \varepsilon(2\eta).$$

Since $\mathcal{H}^d(E \cap B(x,\eta)) < \infty$, we get that

$$\mathcal{H}^d(E \cap B(x,r)) \leq c_1 c_0^{-1} \sum_{k=1}^m \frac{\alpha_{k+1}^n}{(\alpha_{k+1} - \alpha_k)^{n-d}} \left(\frac{c_0 M(2\eta) - 1}{c_0 M(2\eta)}\right)^k r^d + \varepsilon(2\eta)$$

$$\leq c_1 c_0^{-1} \eta^d \sum_{k=1}^\infty (1 - \alpha_k/\alpha_{k+1})^{d-n} \gamma^{(n-d)k} r^d + \varepsilon(2\eta)$$

$$= c_1 c_0^{-1} \eta^d \sum_{k=1}^\infty \gamma^{(n-d)k} r^d + \varepsilon(2\eta) = c_1 c_0^{-1} \eta^d \frac{\gamma^{n-d}}{1 - \gamma^{n-d}} r^d + \varepsilon(2\eta)$$

$$\leq 2 c_1 \eta^d M(2\eta)r^d + \varepsilon(2\eta).$$

\[\square\]

**Lemma 3.6.** Suppose that $E_k \in \mathbb{Q}(U, M_k, \varepsilon_k)$, $M(r) = \lim sup_k M_k(r)$, $\varepsilon(r) = \lim sup_k \varepsilon_k(r)$. $\mathcal{H}^d L(E_k \cap U) \to \mu$, and $\varepsilon_0 = \lim sup_{r \to 0} r^{-d}\varepsilon(r) < \infty$. Then we have that for any $x \in U \cap \text{spt} \mu$

$$\Theta^d(\mu, x) \leq \left(2 c_1 M(0+)\omega^{-1}_d + 2^d \omega^{-1}_d \varepsilon_0\right) \exp \left(4d(n-d)^2 c_0^2 M(0+)^2\right).$$

**Proof.** Take $\eta > \exp (4(n-d)^2 c_0^2 M(0+))$. Then $2(n-d)c_0(\ln \eta)^{-1/2} M(0+) < 1$, and there is a radious $r_0 > 0$ such that $2(n-d)c_0(\ln \eta)^{-1/2} M(\eta r_0) < 1$, thus $2(n-d)c_0(\ln \eta)^{-1/2} M(\eta r_0) < 1$ for $k$ large enough. For any $x \in U \cap \text{spt} \mu$ and $0 < r < \min\{r_0, \eta^{-1} \text{dist}(x, \mathbb{R}^n \setminus U)\}$, we have that

$$\mathcal{H}^d(E_k \cap B(x,r)) \leq 2 c_1 M_k(2\eta)(\eta r)^d + \varepsilon_k(2\eta),$$

thus

$$\mu(U(x,r)) \leq \liminf_{k \to \infty} \mathcal{H}^d(E_k \cap U(x,r)) \leq 2 c_1 M(2\eta)(\eta r)^d + \varepsilon(2\eta),$$

and

$$\Theta^d(\mu, x) \leq 2 c_1 M(0+)\eta^d \omega^{-1}_d + 2^d \eta^d \omega^{-1}_d \limsup_{r \to 0} r^{-d}\varepsilon(r) \leq \left(2 c_1 M(0+)\omega^{-1}_d + 2^d \omega^{-1}_d \varepsilon_0\right) \eta^d.$$

\[\square\]
Lemma 3.7. There exist constants $c_3 = c_3(n, d) > 0$ and $c_4 = c_4(n) > 0$ such that for any $M, \epsilon, E \in \mathcal{OM}(\mathbb{R}^n, U, M, \epsilon)$, open set $O \subseteq U$, $\delta > 0$ and $0 < \tau < \min\{(10\sqrt{n})^{-d}, c_3^{-1} M \mathrm{diam}(O)^{-1}\}$, by setting
\[ m(O, x, \delta) = \inf \left\{ r^{-d} \mathcal{H}^d(E \cap B(x, r)) : 0 < r < \min\{\mathrm{dist}(x, \mathbb{R}^n \setminus O), \delta\} \right\} \]
and
\[ E(O, \delta, \tau) = \{ x \in E \cap O : m(O, x, \delta) \leq \tau \}, \]
we have that
\[ \mathcal{H}^d(E(O, \delta, \tau)) \leq \frac{c_4 \epsilon \mathrm{diam}(O)}{1 - c_3 M \mathrm{diam}(O) \tau}. \]

Proof. For any $x \in E(O, \delta, \tau)$, we denote $m_x = m(O, x, \delta)$ and choose radius $r_x > 0$ such that $r_x < \min\{\mathrm{dist}(x, \mathbb{R}^n \setminus O), \delta\}$, $\mathcal{H}^d(E \cap \partial B(x, r_x)) = 0$ and
\[ \frac{\mathcal{H}^d(E \cap B(x, r_x))}{r_x^d} < \left( 1 + \left( 6c_0 \sqrt{n} \cdot m_x \right)^{1/d} \right) m_x. \]
Applying Besicovitch’s covering theorem to $E \cap B(x, r_x)$, we can find constant $c_4 = c_4(n)$ of positive integers and balls $B_{i, j} \in J_i$, $1 \leq i \leq c_4$, such that $B_{i, j} \cap B_{i', j'} = \emptyset$ for any $i \neq i'$, and
\[ E(O, \delta, \tau) \subseteq \bigcup_{i=1}^{c_4} \bigcup_{j \in J_i} B_{i, j}. \]
By Lemma 2.8, we can find Lipschitz mappings $\varphi_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi_i(B_{i, j}) \subseteq B_{i, j}$ for $j \in J_i, \varphi(y) = y$ for $y \notin \bigcup_{i \in J_i} B_{i, j}$, and
\[ \mathcal{H}^d(E \cap \bigcup_{j \in J_i} B_{i, j}) \leq c_0 M \mathrm{diam}(O) \mathcal{H}^d(E \cap \bigcup_{j \in J_i} A_{i, j}) + \epsilon \mathrm{diam}(O), \]
where $A_{i, j} = A(x_{i, j}, (1 - \delta_{i, j}) r_{i, j}, r_{i, j})$ and $\delta_{i, j} = (6c_0 \sqrt{n} \cdot m_x)^{1/d}$. Since $m_{i, j} < \tau$ and
\[ \mathcal{H}^d(E \cap A_{i, j}) \leq \left( 1 + \left( 6c_0 \sqrt{n} \cdot m_x \right)^{1/d} \right) m_{i, j} r_{i, j}^d - m_{i, j} (1 - \delta_{i, j})^d r_{i, j}^d \leq (d + 1) \delta_{i, j} m_{i, j} r_{i, j}^d, \]
we get that
\[ \mathcal{H}^d(E \cap \bigcup_{j \in J_i} B_{i, j}) \leq c_0 M \mathrm{diam}(O)(d + 1)(6c_0 \sqrt{n})^{1/d} \tau \sum_{j \in J_i} m_{i, j} r_{i, j}^d + \epsilon \mathrm{diam}(O) \]
\[ \leq c_0 M \mathrm{diam}(O)(d + 1)(6c_0 \sqrt{n})^{1/d} \tau \mathcal{H}^d(E \cap \bigcup_{j \in J_i} B_{i, j}) + \epsilon \mathrm{diam}(O). \]
We take $c_3 = (d + 1) \sqrt{n}/2 c_0^{(d+1)/d}$. Then
\[ \mathcal{H}^d(E \cap \bigcup_{j \in J_i} B_{i, j}) \leq \frac{\epsilon \mathrm{diam}(O)}{1 - c_3 M \mathrm{diam}(O) \tau}. \]
Hence
\[ \mathcal{H}^d(E(O, \delta, \tau)) \leq \sum_{i=1}^{c_4} \mathcal{H}^d(E \cap \bigcup_{j \in J_i} B_{i, j}) \leq \frac{c_4 \epsilon \mathrm{diam}(O)}{1 - c_3 M \mathrm{diam}(O) \tau}. \]

\[ \square \]

Lemma 3.8. Suppose that $E_k \in \mathcal{OM}(\mathbb{R}^n, U, M_k, \epsilon_k)$ and $\mathcal{H}^d \bigcup E_k \to \mu$, $M(r) = \lim sup_k M_k(r), \epsilon(r) = \lim sup_k \epsilon_k(r), \mathcal{H}^d \bigcup (E_k \cap U) \to \mu$ and $\epsilon_0 = \lim sup_{r \to 0} r^{-d} \epsilon(r) < \infty$. Then for any $x \in U \cap \mathrm{spt} \mu$, if $\Theta^d(\mu, x) > 6^{d+1} c_4 \omega_{d}^{-1} \epsilon_0$, then $\Theta^d(\mu, x) \geq (2\omega_d c_0 M(0+))^{-1}$. In particular, if $\epsilon_0 = 0$, then $\Theta^d(\mu, x) \geq (2\omega_d c_0 M(0+))^{-1}$ for $\mathcal{H}^d$-a.e. $x \in U \cap \mathrm{spt} \mu$. 

\[ \square \]
Theorem 3.9. Suppose that $E_k \in \mathcal{QM}(\mathbb{R}^n, U, M_k, \varepsilon_k)$ and $\mathcal{H}^d E_k \rightarrow \mu$. $M(r) = \limsup_k M_k(r)$, $\varepsilon(r) = \limsup_k \varepsilon_k(r)$ and $\mathcal{H}^d E_k \cap U \rightarrow \mu$. If $\lim_{r \to 0} r^{-d} \varepsilon(r) = 0$, then $\mu \ll E$ is rectifiable.

Proof. Write $E_k \cap U = E_k^{rec} \cup E_k^{irr}$, $E_k^{rec}$ is $d$-rectifiable and $\mathcal{H}^d$-measurable, $E_k^{irr}$ is purely $d$-unrectifiable and $\mathcal{H}^d$-measurable. Put $\mu_k = \mathcal{H}^d E_k^{rec}$. By Lemma 3.3, we have that $\mathcal{H}^d E_k^{irr} \rightarrow 0$ and $\mu_k \rightarrow \mu$. Put $E = U \cap \text{spt} \mu$. By Lemma 3.6 and Lemma 3.8, there is a constant $c_2 > 0$ which only depends on $n, d, M(0+)$ such that $1/c_2 \leq \Theta^d_k(\mu, x) \leq \Theta^d(\mu, x) \leq c_2$ for almost every $x \in E$, thus there is a Borel function $\theta : E \rightarrow \mathbb{R}$ such that $\mu = \theta \mathcal{H}^d E$. We assume by contradiction that $E \cap U$ is not $d$-rectifiable, and $E \cap U = E^{rec} \cup E^{irr}$. Then there exists $x \in E^{irr}$ such that

$$\Theta^d(E^{rec}, x) = 0, \quad 2^{-d} < \Theta^d(E^{irr}, x) \leq 1.$$ 

For any $0 < \delta < \min\{1/10, (c_22^{-d+4\sqrt{n}-1}\} and $0 < \tau < (c_0\omega_d)^{-1}(10\sqrt{n})^{-d-n}\delta^d$, we take radius $\rho > 0$ such that $\rho < 1/2 \text{dist}(x, \mathbb{R}^n \setminus U)$,

$$\frac{\mathcal{H}^d(E^{rec} \cap B(x, r))}{\omega_d r^d} \leq \tau \quad \text{and} \quad \frac{\mathcal{H}^d(E^{irr} \cap B(x, r))}{\omega_d r^d} \leq (1 + \tau)\Theta^d(E, x) \text{ for any } 0 < r \leq 2\rho,$$

and

$$\frac{\mathcal{H}^d(E^{irr} \cap B(x, \rho))}{\omega_d \rho^d} \geq (1 - 2\tau)\Theta^d(E, x).$$
For any $i_1, \ldots, i_n \in \mathbb{Z}$, we denote by $C_{i_1, \ldots, i_n}$ the cube given by $(\delta \rho i_1, \ldots, \delta \rho i_n) + [0, \delta \rho]^n$. Let $\mathcal{F}$ be the collection of all cubes $C_{i_1, \ldots, i_n}$ which is contained in $U(x, (1 + 3\sqrt{n}\delta)\rho)$, let $\mathcal{F}' = \{\Delta \in \mathcal{F} : \Delta \subseteq \text{int}(\cup \mathcal{F})\}$. By Theorem 2.7, there exist a Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, an upper semi-continuous function $\lambda : \mathbb{R}^n \to [0, \infty)$ and open set $W \subseteq \mathbb{R}^n$ such that $\phi(x) = x$ for $x \in \mathbb{R}^n \setminus \cup \mathcal{F}$, $\phi(\Delta) \subseteq \Delta$ for any $\Delta \in \mathcal{F}$, $E \subseteq W$, $\phi(W \cap B(x, \rho)) \subseteq \cup \mathcal{F}'$, $\lambda(x) = 1$ for $x \in \mathbb{R}^n \setminus \cup \mathcal{F}$, for any $d$-set $Z \subseteq W$,

$$\mathcal{H}^d(\phi(Z)) \leq \int_{x \in Z} \lambda(x) \, d\mathcal{H}^d(x),$$

and for any $\mathcal{K} \subseteq \mathcal{F}$

$$\mathcal{H}^d(\phi(E \cap (\cup \mathcal{K}))) \leq \int_{E \cap (\cup \mathcal{K})} \lambda(x) \, d\mathcal{H}^d(x) \leq c_0 \mathcal{H}^d(E \cap (\cup \mathcal{K})).$$

Since $\mathcal{H}^d(\mathcal{L}(E_k \cap U) \to \mu$, $E = U \cap \text{spt} \mu$, $E \subseteq W$ and $W$ is open, we see that

$$\lim_{k \to \infty} \mathcal{H}^d(E_k \cap U \setminus W) = 0,$$

and

$$\lim_{k \to \infty} \mathcal{H}^d(\phi(E_k \cap U \setminus W)) = 0.$$

Setting $D = \cup \mathcal{F}$, $\mathcal{B}_\rho' = \mathcal{B}(x, (1 + 3\sqrt{n}\delta))$ and $\mathcal{B}_\rho = \mathcal{B}(x, \rho)$, by (2.7), we have that $\mathcal{H}^d(\mathcal{B}_\rho \cap W) = 0$, thus

$$\mathcal{H}^d(\phi(E_k \cap \mathcal{B}_\rho \cap W) = 0, \lim_{k \to \infty} \mathcal{H}^d(\phi(E_k \cap \mathcal{B}_\rho' \setminus W)) = 0,$$

and

$$\mathcal{H}^d(\phi(E_k \cap \mathcal{B}_\rho')) \leq \mathcal{H}^d(\phi(E_k \cap \mathcal{B}_\rho' \setminus W)) + \int_{E_k \cap W \cap \mathcal{B}_\rho' \setminus \mathcal{B}_\rho} \lambda(x) \, d\mathcal{H}^d(x).$$

Thus, setting $\rho_1 = (1 + 3\sqrt{n}\delta)\rho$ and $U_\rho = U(x, \rho)$,

$$\limsup_{k \to \infty} \mathcal{H}^d(\phi(E_k \cap \mathcal{B}_\rho')) \leq \limsup_{k \to \infty} \int_{W \cap \mathcal{B}_\rho' \setminus \mathcal{B}_\rho} \lambda \, d\mathcal{H}^d \mathcal{L} E_k \leq \int_{W \cap \mathcal{B}_\rho' \setminus \mathcal{B}_\rho} \lambda \, d\mu$$

$$\leq c_2 \int_{W \cap \mathcal{B}_\rho' \setminus \mathcal{B}_\rho} \lambda \, d\mathcal{H}^d \mathcal{L} E \leq c_2 c_0 \mathcal{H}^d(E \cap W \cap \mathcal{B}_\rho' \setminus U_\rho)$$

$$\leq c_2 c_0 \mathcal{H}^d(E \cap \mathcal{B}_\rho' \setminus U_\rho) \leq \left((1 + \tau)\rho_1^d - (1 - 2\tau)\rho^d\right) \Theta^d(E, x) \omega_d$$

$$\leq \left((2 + 2^d)\tau + 2^{d+1} \sqrt{n}\delta\right) \Theta^d(E, x) \omega_d \rho^d.$$

We have that

$$\mu(U(x, \rho_1)) \leq \liminf_{k \to \infty} \mathcal{H}^d(E_k \cap U(x, \rho_1))$$

$$\leq \liminf_{k \to \infty} \left(M(2\rho_1)\mathcal{H}^d(\phi(E_k \cap \mathcal{B}(x, \rho_1))) + \epsilon(2\rho_1)\right) \leq 2^{d+2} \sqrt{n}\omega_d \delta \rho^d.$$

But

$$\mu(U(x, \rho_1)) \geq \frac{1}{c_2} \mathcal{H}^d(E \cap U(x, \rho_1)) \geq \frac{1}{c_2} (1 - 2\tau) \Theta^d(E, x) \omega_d \rho^d \geq \frac{\omega_d}{2^{d+1}c_2} \rho^d,$$

this is a contradiction. 

\qed

The following corollary is a direct consequence of Lemma 3.6, Lemma 3.8 and Proposition 3.9.
Corollary 3.10. Suppose that $E_k \in \mathcal{Q}(\mathbb{R}^n, U, M_k, \varepsilon_k)$ and $\mathcal{H}^d \mathcal{L}(E_k) \rightarrow \mu$. $M(r) = \limsup_k M_k(r)$, $\varepsilon(r) = \limsup_k \varepsilon_k(r)$ and $\mathcal{H}^d \mathcal{L}(E_k \cap U) \rightarrow \mu$. If $M(0+) < \infty$ and $\lim_{r \to 0} r^{-d} \varepsilon(r) = 0$, then $U \cap \text{spt} \mu$ is $d$-rectifiable and there is a constant $c_2 = c_2(n, d, M(0+)) > 0$ such that for $\mathcal{H}^d$-a.e. $x \in U \cap \text{spt} \mu$,

$$c_2^{-1} \leq \Theta^d(\mu, x) \leq c_2.$$

Definition 3.11. Let $\mu$ be any Radon measure $\mu$ on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ satisfies that $\Theta^d(\mu, x) \in (0, \infty)$, and there exists $T \in \mathcal{G}(n, d)$ such that for any $\tau > 0$,

$$\Theta^d \left( \mu \mathcal{L} \left( \mathbb{R}^n \setminus \mathcal{C}(T, x, \tau) \right) \right) = 0,$$

where $\mathcal{C}(T, x, \tau) = \{ y \in \mathbb{R}^n, \text{dist}(y - x, T) \leq \tau | y - x | \}$, then we call that $T$ an approximate tangent $d$-plane of $\mu$ at $x$. It is quite easy to see the uniqueness if it exists, and we denote it by $\text{Tan}^d(\mu, x)$.

Lemma 3.12. Suppose that $E_k \in \mathcal{Q}(\mathbb{R}^n, U, M_k, \varepsilon_k)$, $M(r) = \limsup_k M_k(r)$, $\varepsilon(r) = \limsup_k \varepsilon_k(r)$ $\mathcal{H}^d \mathcal{L}(E_k \cap U) \rightarrow \mu$ and $x \in U \cap \text{spt} \mu$. If $\lim_{r \to 0} r^{-d} \varepsilon(r) = 0$, $\Theta^d(\mu, x)$ and $\text{Tan}^d(\mu, x)$ exists, then for any decreasing sequence $\{r_m\}$ of positive numbers, setting $B_m = B(x, r_m)$, we have that

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{r_m^d} \left( \mathcal{H}^d \left( E_k^{rec} \cap B_m \right) - M(0+) \mathcal{H}^d \left( \text{Tan}_k^{E_k^{rec}} \cap B_m \right) \right) \leq 0. \quad (3.2)$$

Proof. Without loss of generality, we assume that $x = 0 \in U \cap \text{spt} \mu$, $T = \text{Tan}^d(\|V\|, 0)$. Write $E_k \cap U = E_k^{rec} \cup E_k^{rr}$. By Lemma 3.3, we see that $\mathcal{H}^d \mathcal{L}(E_k^{rr}) \rightarrow 0$, thus $\mathcal{H}^d \mathcal{L}(E_k^{rec}) \rightarrow \mu$.

For any $0 < \delta < 1/10$, taking $\tau = \delta^2$, we see that

$$\lim_{m \to \infty} \frac{\|T\|(B(0, r_m))}{\omega d r_m^d} = \Theta^d(\|T\|, 0) \in [1/c_2, c_2] \quad \text{and} \quad \lim_{m \to \infty} \frac{\|T\|(B(0, r_m) \setminus \mathcal{C}(T, 0, \tau))}{r_m^d} = 0.$$

Put $A_m = (T + B(0, \tau r_m)) \cap B(0, (1 - 2\delta)r_m)$. Then we see that $A_m \subseteq T + B(0, \tau r_m)$ and $A_m + B(0, \delta r_m) \subseteq U(0, r_m)$. By Lemma 2.5 in [6], there is a Lipschitz mapping $\varphi_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\varphi_m(B(0, r_m)) \subseteq B(0, r_m)$, $\varphi_m|\mathbb{R}^n \setminus U(0, r_m) = \text{id}_{\mathbb{R}^n \setminus U(0, r_m)}$, $\varphi_m|A_m = T_k^1|A_m$ and $\text{Lip}(\varphi_m) \leq 3 + \tau/\delta$. Putting $r'_m = (1 - 2\delta)r_m$ and $B'_m = B(0, r'_m)$, we get that

$$\mathcal{H}^d(\varphi_m(E_k \cap B(0, r_m))) \leq \text{Lip}(\varphi_m)^d \mathcal{H}^d(E_k \cap B(0, r_m) \setminus A_m) + \mathcal{H}^d(T_k^1(E_k \cap A_m))$$

$$\leq 4^d \left( \mathcal{H}^d(E_k \cap B_m \setminus B'_m) + \mathcal{H}^d(E_k \cap B_m \setminus \mathcal{C}(T, 0, \tau)) + \mathcal{H}^d(T_k^1(E_k \cap A_m)) \right).$$

Since $\mathcal{H}^d(E_k \cap B(0, r_m)) \leq M_k(2r_m)\mathcal{H}^d(\varphi_m(E_k \cap B(0, r_m))) + \varepsilon_k(2r_m)$, we get that

$$\mathcal{H}^d(E_k \cap B_m) - M_k(2r_m)\mathcal{H}^d(T_k^1(E_k \cap B_m)) \leq 4^d M_k(2r_m) \left( \mathcal{H}^d(E_k \cap B_m \setminus B'_m) + \mathcal{H}^d(E_k \cap B_m \setminus \mathcal{C}(T, 0, \tau)) \right)$$

$$+ \varepsilon_k(2r_m).$$

By Lemma 3.3, we have that $\mathcal{H}^d(E_k^{rr} \cap B_m) \leq \varepsilon_m(2r_m)$, thus

$$\limsup_{k \to \infty} \left( \mathcal{H}^d(E_k^{rec} \cap B_m) - M(2r_m)\mathcal{H}^d(T_k^{E_k^{rec}} \cap B_m)) \right) \leq M(0+) 4^d \left( \omega_d \Theta^d(\|T\|, 0) + 2\varepsilon(2r_m) \right),$$

and

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{r_m^d} \left( \mathcal{H}^d(E_k^{rec} \cap B_m) - M(2r_m)\mathcal{H}^d(T_k^{E_k^{rec}} \cap B_m)) \right) \leq M(0+) 4^d (1 - (1 - 2\delta)) \omega_d \Theta^d(\|T\|, 0).$$

Let $\delta$ tend to $0$, we get that

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{r_m^d} \left( \mathcal{H}^d(E_k^{rec} \cap B_m) - M(2r_m)\mathcal{H}^d(T_k^{E_k^{rec}} \cap B_m)) \right) \leq 0.$$

Since $M(2r_m) \rightarrow M(0+)$, we get that (3.2) holds. \qed
Theorem 3.13. Suppose that $E_k \in \mathcal{QM}(\Omega, U, M_k, \epsilon_k)$, $M(r) = \limsup_k M_k(r)$, $\epsilon(r) = \limsup_k \epsilon_k(r)$, $M(0+) = 1$, $\lim_{r \to 0} r^{-d} \epsilon(r) = 0$ and $\nu(E_k \cap U) \to V$. If $x \in U \cap \text{spt} \|V\|$, $\Theta^d(\|V\|, x)$ and $T = \text{Tan}^d(\|V\|, x)$ exists, then

$$\text{VarTan}(V, x) = \{\nu(T)\}.$$ 

Proof. Without loss of generality, we assume that $x = 0$, $E = U \cap \text{spt} \|V\|$, $0 \in E$, $T = \text{Tan}^d(\|V\|, 0)$. Write $E_k \cap U = E_k^{rec} \cup E_k^{irr}$. By Lemma 3.3, we see that $\mathcal{G}^d \mathbb{L} E_k^{irr} \to 0$, thus $\nu(E_k^{rec}) \to V$.

For any $C \in \text{VarTan}(V, 0)$, there is a sequence $(r_m)$ of decreasing positive numbers such that

$$C = \lim_{m \to \infty} (\mu_{1/r_m})_{\#} V,$$

thus

$$C = \lim_{m \to \infty} \lim_{k \to \infty} \nu(E_k^{rec}) \quad \text{and} \quad \|C\| = \lim_{m \to \infty} \lim_{k \to \infty} \mathcal{G}^d \mathbb{L} \mu_{1/r_m}(E_k^{rec}).$$

For any $\varphi \in \mathcal{C}_c(\mathbb{R}^n \times G(n, d), \mathbb{R})$, we have that

$$\mathcal{C} \mathbb{L} \mathbb{B}(0, 1) \times G(n, d)(\varphi) = \lim_{m \to \infty} \lim_{k \to \infty} \frac{1}{r_m} \int_{E_k^{rec} \cap B_m} \varphi(x, \text{Tan}^d(E_k^{rec}, x)) \mathcal{G}^d(x).$$

Putting $B_m = B(0, r_m)$, by Lemma 3.12, we have that

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{r_m} \int_{E_k^{rec} \cap B_m} \left(1 - J_d(T_k|E_k^{rec})(x)\right) \mathcal{G}^d(x) \leq 0.$$

Putting $q_k(x) = \text{Tan}(E_k^{rec}, x)$, by Lemma 11.4 in [7], we have that

$$\|q_k(x) - T\|^2 \leq 2 \left(1 - \text{ap} J_d(T_k|E_k^{rec})(x)\right).$$

Hence, setting $E_k^* = E_k^{rec}$,

$$\left(\int_{E_k^* \cap B_m} \|q_k(x) - T\| \mathcal{G}^d(x)\right)^2 \leq \int_{E_k^* \cap B_m} \mathcal{G}^d(x) \int_{E_k^* \cap B_m} \|q_k(x) - T\|^2 \mathcal{G}^d(x),$$

and

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{r_m} \int_{E_k^{rec} \cap B_m} \|q_k(x) - T\| \mathcal{G}^d(x) = 0.$$

For any $\varphi \in \mathcal{C}_c(\mathbb{R}^n \times G(n, d), \mathbb{R})$, we see that $\varphi(\cdot, T) \in \mathcal{C}_c(\mathbb{R}^n, \mathbb{R})$, thus

$$\|C\| \mathbb{L} \mathbb{B}(0, 1)(\varphi(\cdot, T)) = \lim_{m \to \infty} \lim_{k \to \infty} \frac{1}{r_m} \int_{E_k^{rec} \cap B_m} \varphi(x, T) \mathcal{G}^d(x),$$

setting $C_1 = \mathcal{C} \mathbb{L} \mathbb{B}(0, 1) \times G(n, d)$ and $T_1 = T \cap \mathbb{B}(0, 1)$, we have that

$$|C_1(\varphi) - \nu(T_1)(\varphi)| = \lim_{m \to \infty} \lim_{k \to \infty} \frac{1}{r_m} \int_{E_k^{rec} \cap B_m} \left|\varphi(x, \text{Tan}^d(E_k^{rec}, x)) - \varphi(x, T)\right| \mathcal{G}^d(x),$$

and

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \frac{1}{r_m} \int_{E_k^{rec} \cap B_m} ||D \varphi|| \|q_k(x) - T\| \mathcal{G}^d(x) = 0.$$

Hence $\mathcal{C} \mathbb{L} \mathbb{B}(0, 1) \times G(n, d) = \nu(T \cap \mathbb{B}(0, 1))$. Since $C$ is a cone, we get that $C = \nu(T)$.
Lemma 3.14. Suppose that $E_k \in QM(\Omega, U, M_k, \varepsilon_k)$, $M(r) = \limsup_k M_k(r)$, $\varepsilon(r) = \limsup_k \varepsilon_k(r)$, $M(0+) < \infty$, $\lim_{r \to 0} r^{-d} \varepsilon(r) = 0$ and $\mathcal{H}^d \bigcap(E_k \cap U) \rightarrow \mu$. If $x \in U \cap \text{spt} \mu$, $\Theta^d(\mu, x) \in (0, \infty)$ and $T = \text{Tan}^d(\mu, x)$ exists, then
\[
\liminf_{r \to 0} \liminf_{k \to \infty} \frac{\mathcal{H}^d(T_k(E_k \cap B(x, r)))}{\omega_d r^d} \geq 1. \tag{3.3}
\]

Proof. For any $0 < \delta < \min\{(10\sqrt{n})^{-1}, (5^d c_0 d M(0))^2\}$, taking $\tau = \delta^2$, setting $r' = (1 - \delta)r$, $B_r = B(x, r)$, $U_r = U(x, r)$, $B_r' = B(x, r')$ and $A_r = B_r \cap (T + B(0, \tau r))$, we see that
\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{\omega_d r^d} = \Theta^d(\mu, x) \text{ and } \lim_{r \to 0} \frac{\mu(B(0, r) \cap \{T(r, x)\})}{r^d} = 0.
\]
Let $\mathcal{F}'$ be the collection of cubes $C_{i_1, \ldots, i_n} = (\tau r i_1, \ldots, \tau r i_n) + [0, \tau r]^{n}$ which is contained in $U(x, r) \setminus A_r$. We have that
\[
\lim_{r \to 0} r^{-d} \mu(\bigcup \mathcal{F}') = 0.
\]
There exists $r_r > 0$ such that $\mu(\bigcup \mathcal{F}') < c_0^{-1}(\tau r)^d$ for any $0 < r \leq r_r$, thus there exist $k, r, \tau$ such that $\mathcal{H}^d(E_k \bigcap \mathcal{F}') < c_0^{-1}(\tau r)^d$ for any $k \geq k, r, \tau$. By Theorem 2.7, there exist $\phi_{r, k} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi_{r, k}(\bigcup \mathcal{F}') \subseteq \bigcup \mathcal{F}'$, $\phi_{r, k}[\mathbb{R}^n \setminus \mathcal{F}'] = \mathbb{R}^n \setminus \mathcal{F}'$, and
\[
\mathcal{H}^d(\phi_{r, k}(E_k \bigcap \mathcal{F}')) \leq c_0 \mathcal{H}^d(E_k \bigcap \mathcal{F}').
\]
Setting $A_r' = B_r' \cap (T + B(0, \tau r))$ and $H_r' = B_r' \cap (T + B(0, 2\sqrt{n} \tau r))$, we have that $B_r' \setminus H_r' \subseteq \mathcal{F}'$ and $\phi_{r, k}[A_r'] = \mathbb{R}^n \setminus \mathcal{F}'$. By Lemma 2.5 in [6], there exist Lipschitz mapping $\varphi_r : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi_r(B_r') \subseteq B_r'$, $\varphi_r[B_r' \setminus \mathcal{F}'] = \mathbb{R}^n \setminus \mathcal{F}'$, and
\[
\text{Lip}(\varphi) \leq 3 + 2\sqrt{n} \tau / \delta \leq 3 + 2\sqrt{n} \delta \leq 5.
\]
Hence
\[
\mathcal{H}^d(\varphi_r \circ \phi_{r, k}(E_k) \setminus T) = 0.
\]
We claim that $\varphi_r \circ \phi_{r, k}(E_k) \cap B_r' \cap T = T \cap B_r'$. Otherwise, we take $y \in T \cap B_r' \setminus \varphi_r \circ \phi_{r, k}(E_k)$, since $\varphi_r \circ \phi_{r, k}(E_k)$ is compact, there is a small ball $U(y, r)$ such that $U(y, r) \cap \varphi_r \circ \phi_{r, k}(E_k) = 0$. Let $p_y$ the be a Lipschitz mapping such that $p_y[\mathbb{R}^n \setminus U(y, r)] = \mathbb{R}^n \setminus U(y, r)$, $p_y(B_r') \subseteq B_r'$ and $p_y[B_r' \setminus U(y, r)]$ is the radial projection onto $\partial B_r'$ centered at $y$. Then
\[
\mathcal{H}^d(p_y \circ \varphi_r \circ \phi_{r, k}(E_k) \cap B_r') = s d c_0 \mathcal{H}^d(E_k \cap B_r \setminus A_r'),
\]
and
\[
\mathcal{H}^d(E_k \cap B_r) \leq M_k(2r) \mathcal{H}^d(p_y \circ \varphi_r \circ \phi_{r, k}(E_k) \cap B_r') + \varepsilon_k(2r) \leq M_k(2r) s d c_0 \mathcal{H}^d(E_k \cap B_r \setminus A_r') + \varepsilon_k(2r).
\]
Hence
\[
\mu(U_r) \leq \liminf_{k \to \infty} \mathcal{H}^d(E_k \cap U_r) \leq s d c_0 M(2r) \mu(B_r \setminus \text{int}(A_r')) + \varepsilon_k(2r)
\]
and
\[
\Theta^d(\mu, x) \leq \lim_{r \to 0} \frac{\mu(U_r)}{\omega_d r^d} \leq s d c_0 M(0+) \limsup_{r \to 0} \frac{1}{r^d} (\mu(B_r) - \mu(U_r')) \leq s d c_0 M(0+) \Theta^d(\mu, x) d \delta,
\]
this is a contradiction, so the claim holds, that is, $\varphi_r \circ \phi_{r, k}(E_k) \cap B_r' \cap T = T \cap B_r'$. Thus
\[
\omega_d r^d \leq s d c_0 \mathcal{H}^d(\phi_{r, k}(E_k) \cap B_r' \cap T) \leq s d c_0 \mathcal{H}^d(E_k \cap B_r \setminus A_r') + \mathcal{H}^d(T_k(E_k \cap A_r')),
\]
we get that
\[
\omega_d (1 - \delta)^d \leq s d c_0 \limsup_{r \to 0} \frac{1}{r^d} (\mu(B_r) - \mu(U_r')) + \liminf_{k \to \infty} \frac{1}{r^d} \mathcal{H}^d(T_k(E_k \cap A_r')).
\]
and
\[
\liminf_{r \to 0} \liminf_{k \to \infty} \frac{\mathcal{H}^d(T_k(E_k \cap B(x, r)))}{\omega_d r^d} \geq (1 - \delta)^d - S_d \epsilon_0 \Theta^d(\mu, x) d\delta.
\]
Let \( \delta \to 0 \), we get that (3.3) holds.

\[\square\]

**Theorem 3.15.** Suppose that \( E_k \in \mathcal{Q}M(\Omega, U, M_\delta, \varepsilon_k) \), \( M(r) = \limsup_k M_k(r) \), \( \varepsilon(r) = \limsup_k \varepsilon_k(r) \), \( M(0+) < \infty \), \( \lim_{r \to 0} r^{-d} \varepsilon(r) = 0 \) and \( \mathbf{v}(E_k \cap U) \rightarrow V \). If \( x \in U \cap \text{spt} \|V\| \), \( \Theta^d(\|V\|, x) = 1 \) and \( T = \text{Tan}^d(\|V\|, x) \) exists, then
\[
\text{VarTan}(V, x) = \{\mathbf{v}(T)\}.
\]

**Proof.** Similar to the proof of Theorem 3.13, we assume that \( x = 0 \), \( E = U \cap \text{spt} \|V\| \), \( 0 \in E \), \( T = \text{Tan}^d(\|V\|, 0) \), \( E_k \cap U = E_k^{\text{rec}} \cup E_k^{\text{irr}} \). For any \( C \in \text{VarTan}(V, 0) \), there is a sequence \( \{r_m\} \) of decreasing positive numbers such that
\[
C = \lim_{m \to \infty} (\mu_{1/r_m})_V.
\]
Since \( \Theta^d(\|V\|, 0) = 1 \), by Lemma 3.14, we have that
\[
1 \leq \liminf_{m \to \infty} \liminf_{k \to \infty} \frac{\mathcal{H}^d(T_k(E_k \cap B(0, r_m)))}{\omega_d r_m^d} \leq \limsup_{m \to \infty} \limsup_{k \to \infty} \frac{\mathcal{H}^d(E_k \cap B(0, r_m))}{\omega_d r_m^d} \leq 1.
\]
Setting \( B_m = B(x, r_m) \), by Lemma 3.3, we have that
\[
\limsup_{m \to \infty} \liminf_{k \to \infty} \frac{1}{r_m^d} \mathcal{H}^d(E_k^{\text{irr}} \cap B_m) = 0.
\]
Since
\[
\mathcal{H}^d(T_k(E_k^{\text{rec}} \cap B_m)) \leq \int_{x \in E_k^{\text{rec}} \cap B_m} \text{ap} J_d(T_k\big|_{E_k^{\text{rec}}}) d\mathcal{H}^d(x),
\]
we have that
\[
\lim_{m \to \infty} \liminf_{k \to \infty} \frac{1}{r_m^d} \int_{E_k^{\text{rec}} \cap B_m} \left(1 - J_d(T_k\big|_{E_k^{\text{rec}}})\right) d\mathcal{H}^d(x) = 0.
\]
The rest of the proof is the same as the proof of Theorem 3.13. \[\square\]

**Theorem 3.16.** Suppose that \( E_k \in \mathcal{Q}M(\mathbb{R}^n, U, M_\delta, \varepsilon_k) \), \( M(r) = \limsup_k M_k(r) \), \( \varepsilon(r) = \limsup_k \varepsilon_k(r) \) and \( \mathcal{H}^d \cup E_k \cap U \rightarrow \mu \). If \( M(0+) < \infty \) and \( \lim_{r \to 0} r^{-d} \varepsilon(r) = 0 \), then we have that for any \( \mathcal{H}^d \)-a.e. \( x \in U \cap \text{spt} \mu \), \( \Theta^d(\mu, x) \) exits and
\[
1 \leq \Theta^d(\mu, x) \leq M(0+).
\]

**Proof.** By Corollary 3.10, there is a constant \( c_2 > 0 \) such that for \( \mathcal{H}^d \)-a.e. \( x \in E \), \( \Theta^d(\mu, x) \) exits and
\[
1/c_2 \leq \Theta^d(\mu, x) \leq c_2.
\]
We take \( x \in U \cap \text{spt} \mu \), such that \( \Theta^d(\mu, x) \in [1/c_2, c_2] \) and \( T = \text{Tan}^d(\mu, x) \) exits. By Lemma 3.12, we get that
\[
\limsup_{r \to 0} \limsup_{k \to \infty} \frac{1}{r^d} \mathcal{H}^d(E_k^{\text{rec}} \cap B(x, r)) \leq M(0+)\omega_d.
\]
Since
\[
\limsup_{k \to \infty} \mathcal{H}^d(E_k^{\text{irr}} \cap B(x, r)) \leq \varepsilon(2r)
\]
and \( \lim_{r \to 0} r^{-d} \varepsilon(2r) = 0 \), we get that

\[
\Theta^d(\mu, x) = \lim_{r \to 0} \frac{\mu(B(x, r))}{r^d} \leq \limsup_{r \to 0} \limsup_{k \to \infty} \frac{1}{\omega_d r^d} \mathcal{H}^d(E_k \cap B(x, r)) \leq M(0+).
\]

By Lemma 3.14, we get that

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{r^d} \geq \liminf_{r \to 0} \limsup_{k \to \infty} \frac{1}{\omega_d r^d} \mathcal{H}^d(E_k \cap B(x, r)) \geq \liminf_{r \to 0} \liminf_{k \to \infty} \frac{1}{\omega_d r^d} \mathcal{H}^d(T_k(E_k \cap B(x, r))) \geq 1.
\]

\[\square\]

**Corollary 3.17** (Theorem 3.4 in [2] and Lemma 3.12 in [3]). Suppose that \( E_k \in \text{QM}(\mathbb{R}^n, U, M_k, \varepsilon_k) \). \( M(r) = \limsup_k M_k(r) \). \( \varepsilon(r) = \limsup_k \varepsilon_k(r) \) and \( \mathcal{H}^d(E_k \cap U) \to \mu \). \( E = \text{spt} \mu \). If \( M(0+) < \infty \) and \( \lim_{r \to 0} r^{-d} \varepsilon(r) = 0 \), then for any open set \( O \subseteq U \),

\[
\mathcal{H}^d(E \cap O) \leq \liminf_{k \to \infty} \mathcal{H}^d(E_k \cap O),
\]

and for any compact set \( H \subseteq U \),

\[
\limsup_{k \to \infty} \mathcal{H}^d(E_k \cap H) \leq M(0+) \mathcal{H}^d(E \cap H).
\]

**Corollary 3.18.** Suppose that \( E_k \in \text{QM}(\mathbb{R}^n, U, M_k, \varepsilon_k) \), \( M(r) = \limsup_k M_k(r) \), \( \varepsilon(r) = \limsup_k \varepsilon_k(r) \) \( \mathcal{V}(E_k \cap U) \to V, E = \text{spt} ||V|| \) and \( \lim_{r \to 0} r^{-d} \varepsilon(r) = 0 \).

- If \( M(0+) = 1 \), then \( \mathcal{V}(U \times G(n, d)) = \mathcal{V}(E \cap U) \).
- If \( M(0+) < \infty \) and there is an open set \( O \subseteq U \) such that \( \mathcal{H}^d(E \cap O) = \lim_{k \to \infty} \mathcal{H}^d(E_k \cap O) \), then \( \mathcal{V}(U \times G(n, d)) = \mathcal{V}(E \cap O) \).

**Proof.** If \( M(0+) = 1 \), then the conclusion directly follows from Theorem 3.13 and Proposition 3.9. If \( M(0+) < \infty \) and \( \mathcal{H}^d(E \cap O) = \lim_{k \to \infty} \mathcal{H}^d(E_k \cap O) \), by Theorem 3.16, we see that \( \Theta^d(\mu, x) = 1 \) for \( \mathcal{H}^d \)-a.e. \( x \in E \cap O \), then by Theorem 3.15 and Proposition 3.9, we get the conclusion. \[\square\]

## 4. Quasiminimal sets on \( C^2 \) submanifold

For any nonempty set \( A \subseteq \mathbb{R}^n \), we denote by \( \delta = \delta_A : \mathbb{R}^n \to \mathbb{R} \) the distance function defined by \( \delta(x) = \text{dist}(x, A) \). Let \( \text{Unp}(A) \) be the set of those points in \( \mathbb{R}^n \) for any such point \( x \) there is a unique point \( \xi_A(x) \in A \) such that \( \delta(x) = |x - \xi_A(x)| \). For any \( z \in A \), we define reach(\( A, z \)) to be the supremum of numbers \( r > 0 \) for which \( B(x, r) \subseteq \text{Unp}(A) \), and define the reach of \( A \) by \( \text{reach}(A) = \inf \{\text{reach}(A, z) : z \in A\} \). By Remark 4.2 in [8], we see that \( \text{reach}(A, \cdot) \) is continuous on \( A \). For any set \( W \subseteq \text{Unp}(A) \), setting \( \rho = \sup \{\text{dist}(x, A) : x \in W\} \) and \( R = \inf \{\text{reach}(A, \xi(x)) : x \in W\} \), by Theorem 4.8 (8) in [8], we have that

\[
\text{Lip}(\xi|_W) \leq \frac{R}{R - \rho}.
\]

**Lemma 4.1.** Suppose \( 1 \leq d < m \leq n \). Let \( \Omega \subseteq \mathbb{R}^n \) be an \( m \)-dimensional closed submanifold of class \( C^2 \). Let \( U \subseteq \mathbb{R}^n \) be a bounded open set with \( U \cap \Omega \neq \emptyset \) and \( \overline{U} \cap \partial \Omega = \emptyset \). If \( E \in \text{QM}(\Omega, U, M, \varepsilon) \), then there exist \( r_0 = r_0(\Omega, U) > 0 \), increasing functions \( M' \) and \( \varepsilon' \) such that \( E \in \text{QM}(\mathbb{R}^n, U, M', \varepsilon') \), \( M'(r) \leq (1 + r/r_0)^d M(r) \) and \( \varepsilon'(r) \leq \varepsilon(r) \) for \( 0 < r \leq r_0 \).
Proof. We take $\xi = \xi_\Omega$ and $r_0 = \frac{1}{4}\inf\{\text{reach}(\Omega, \xi(x)) : x \in U \cap \text{Unp}(\Omega)\}$. Since $\overline{U} \cap \partial \Omega = \emptyset$, and reach(\Omega, \cdot) is continuous on $\Omega$, we see that $r_0 > 0$. For any $\varphi \in \mathcal{D}(\mathbb{R}^m, U)$ with $\text{diam}(W_\varphi \cap \varphi(W_\varphi)) \leq r_0$, we assume $W_\varphi \cap E \neq \emptyset$, otherwise we have nothing to do. Then $W = W_\varphi \cup \varphi(W_\varphi) \subseteq \text{Unp}(\Omega)$ and sup\{dist$(x, \Omega) : x \in W$\} $\leq \text{diam}(W) \leq r_0$, thus we get that

$$\text{Lip}(\xi|W) \leq \frac{2r_0}{2r_0 - \text{diam}(W)} \leq 1 + \frac{\text{diam}(W)}{r_0}.$$ 

Since $\xi \circ \varphi \in \mathcal{D}(\Omega, U)$, we get that

$$\text{H}^d(E \cap W_\varphi) \leq M(\text{diam} W)\text{H}^d(\varphi(E \cap W_\varphi)) + \varepsilon(\text{diam} W)$$

$$\leq M(\text{diam} W)\text{Lip}(\xi|W)\text{H}^d(\varphi(E \cap W_\varphi)) + \varepsilon(\text{diam} W)$$

$$\leq M(\text{diam} W)(1 + \frac{\text{diam}(W)}{r_0})\varepsilon(\text{diam} W) + \varepsilon(\text{diam} W).$$

The conclusion holds with $M'(r) = (1 + r/r_0)M(r)$ and $\varepsilon'(r) = \varepsilon(r)$ for $0 < r \leq r_0$. \hfill $\Box$

**Theorem 4.2.** Suppose $1 \leq d < m \leq n$. Let $\Omega \subseteq \mathbb{R}^n$ be a closed $m$-dimensional submanifold of class $C^2$. Let $U \subseteq \mathbb{R}^m$ be an open set such that $U \cap \Omega \neq \emptyset$ and $U \cap \partial \Omega = \emptyset$. Suppose that $\{E_k\} \subseteq \text{QM}(\Omega, U, M_k, \varepsilon_k)$. Then $M(r) = \limsup_{k \to \infty} M_k(r)$, $\varepsilon(r) = \limsup_{k \to \infty} \varepsilon_k(r)$, $\text{H}^d(E_k \cap U) \to \mu$, $M(0+)= \lim_{r \to 0+} r^d\varepsilon(x)$, and $\text{H}^d(L(x) \cap U) = 0$. Then we have that

- $U \cap \text{spt} \mu$ is $d$-rectifiable, for $\text{H}^d$-a.e. $x \in U \cap \text{spt} \mu$, $\Theta^d(\mu, x)$ exists and $1 \leq \Theta^d(\mu, x) \leq M(0+)$;
- if $O \subseteq U$ is an open set and $\Theta^d(\mu, x) = 1$ for $\mu$-a.e. $x \in O$, then $\nu(O \cap E_k)$ converges to a varifold $V$ with $V \ll O \times G(n, d) = \nu(O \cap \text{spt} \mu)$.

In particular, if $M(0+) = 1$ and $\nu(U \cap E_k) \to V$, then $V \ll U \times G(n, d) = \nu(U \cap \text{spt} ||V||)$.

**Proof.** For any closed ball $B(x, r) \subseteq U$, if $U(x, r) \cap \Omega \neq \emptyset$, by Lemma 4.1, we see that

$$E_k \in \text{QM}(\mathbb{R}^n, U, M'_k, \varepsilon'_k),$$

where $U = U(x, r)$, $M'_k(r) = (1 + r/r_0)M_k(r)$ and $\varepsilon'_k(r) = (1 + r/r_0)\varepsilon_k(r)$ for $0 < r \leq r_0$, $r_0 = r_0(\Omega, U) > 0$. Put $M(r)' = \limsup_{k \to \infty} M'_k(r)$ and $\varepsilon(r)' = \limsup_{k \to \infty} \varepsilon'_k(r)$. Then $M'(0+)= M(0+) = \lim_{r \to 0+} r^d\varepsilon'(r) = \lim_{r \to 0+} r^d\varepsilon(x) = 0$. By Theorem 3.16, we get that $\text{H}^d(U \cap E_k)$ is $d$-rectifiable, for $\text{H}^d$-a.e. $x \in U$, $\Theta^d(\mu, x)$ exists and $1 \leq \Theta^d(\mu, x) \leq M(0+)$. Putting $E_k = E_k \cap O$ for any closed ball $B(x, r) \subseteq O$. By Theorem 3.15, we get that $\nu(O \cap E_k)$ converges to a varifold $V$, which satisfies

$$V \ll O \times G(n, d) = \nu(O \cap \text{spt} ||V||).$$ 

**Proof of Theorem 1.2.** Put $\zeta_k = \inf \{\text{H}^d(E_k) - \text{H}^d(\varphi(E_k)) : \varphi \in \mathcal{D}(\Omega, U)\}$. From (1.1), we get that $\zeta_k \geq 0$ and $\zeta_k \to 0$ as $k \to \infty$. Put $M_k \equiv 1$ and $\varepsilon_k \equiv \varepsilon_k$, $M(r) = \limsup_{k \to \infty} M_k(r)$ and $\varepsilon(r) = \limsup_{k \to \infty} \varepsilon_k(r)$. Then we see that $E_k \in \text{QM}(\Omega, U, M_k, \varepsilon_k)$. Since $M \equiv 1$ and $\varepsilon \equiv 0$, by Theorem 4.2, we get that (1.2) hold. \hfill $\Box$

5. Plateau’s Problem

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty closed subset, $B_0 \subseteq \Omega$ a nonempty compact subset. Let $G$ be an abelian group. Suppose that $L \subseteq \hat{H}_{d-1}(B_0; G)$ is a subgroup. A compact set $E \subseteq \Omega$ is called spanning $L$ (or whose algebraic boundary contains $L$) if $E \supseteq B_0$ and $\hat{H}_{d-1}(i_{B_0,E})(L) = 0$, where $i_{B_0,E} : B_0 \to E$ is the inclusion mapping and $\hat{H}_{d-1}(i_{B_0,E})$ is the homomorphism induced by $i_{B_0,E}$. We denote by $\hat{C} = \hat{C}(\Omega, B_0, G, L)$ the collection of subsets in $\Omega$ which are spanning $L$. It is possible $\hat{C} = \emptyset$, if
we do not presuppose any condition on $\Omega$ and $B_0$. It is quite easy to see that, for any $E \in \mathcal{C}$ and $\varphi \in D(\Omega, \Omega \setminus B_0)$, $\varphi(E) \in \mathcal{C}$.

**Lemma 5.1.** Let $A_1$ and $A_2$ be two compact subsets in $\mathbb{R}^n$. If $\tilde{H}_{d-1}(A_1 \cap A_2; G) = 0$, then the homomorphism $\tilde{H}_{d-1}(i_{A_1 \cup A_2})$ is injective.

**Proof.** By [5, Theorem 15.3 in p.39] and fact that every compact triad is a proper triad [5, p.257], we get that the Mayer-Vietoris sequence of triad $(A_1 \cup A_2; A_1, A_2)$ for Čech homology is exact. That is, it holds the following exact sequence:

$$\cdots \rightarrow \tilde{H}_{d-1}(A_1 \cap A_2; G) \overset{\phi}{\rightarrow} \tilde{H}_{d-1}(A_1; G) \oplus \tilde{H}_{d-1}(A_2; G) \overset{\delta}{\rightarrow} \tilde{H}_{d-1}(A_1 \cup A_2; G) \rightarrow \cdots \rightarrow 0,$$

where homomorphism $\psi$ and $\phi$ are defined by

$$\psi(u) = \tilde{H}_{d-1}(i_{A_1 \cap A_2}, A_1)(u) - \tilde{H}_{d-1}(i_{A_1 \cap A_2}, A_2)(u)$$

and

$$\phi(v_1, v_2) = \tilde{H}_{d-1}(i_{A_1 \cap A_2}, A_1)(v_1) + \tilde{H}_{d-1}(i_{A_1 \cap A_2}, A_2)(v_2).$$

Since $\tilde{H}_{d-1}(A_1 \cap A_2; G) = 0$, we get that $\phi$ is injective. Let $j : \tilde{H}_{d-1}(A_1; G) \rightarrow \tilde{H}_{d-1}(A_1; G) \oplus \tilde{H}_{d-1}(A_2; G)$ be the homomorphism defined by $j(v) = (v, 0)$. Then $\tilde{H}_{d-1}(i_{A_1, A_2, A_2}) = \phi \circ j$ is injective. 

\[\square\]

**Lemma 5.2.** Let $B_0 \subseteq \mathbb{R}^n$ be a nonempty compact subset. Let $G$ be an abelian group. Suppose that $L \subseteq \tilde{H}_{k}(B_0; G)$ is a subgroup. Let $\{E_k\}$ be a sequence of compact subsets in $\mathbb{R}^n$, which is uniformly bounded and each of them is spanning $L$. Suppose that $\mathcal{H}(\bigcup E_k \cap B_0) = \mu$. Then $B_0 \cup \text{spt} \mu$ spans $L$.

**Proof.** We assume $B_0 \cup \bigcup E_k \subseteq B(0, R)$, $R > 0$, put $E = B_0 \cup \text{spt} \mu$ and $U = U(0, R + 1) \setminus E$. Then $E$ is compact and contained in $B(0, R)$, $U$ is open. Let $\mathcal{F}$ be the family of cubes which raise from a Whitney decomposition of $U$ with the following conditions hold:

- $\mathcal{F}$ consists of interior disjoint dyadic cubes and $\cup_{Q \in \mathcal{F}} Q = U$;
- $\sqrt{n}(Q) \leq \text{dist}(Q, \mathbb{R}^n \setminus U) \leq 4\sqrt{n}(Q)$;
- if $Q_1 \cap Q_2 \neq \emptyset$, then $1/4 \leq \ell(Q_1)/\ell(Q_2) \leq 4$.

We put $F_k = E + U(0, 4^{-k+1})$ and let $\Omega^k$ be the collection of cubes $Q$ in $\mathcal{F}$ whose sidelength is no less than $4^{-k}$. By putting $G_k = \bigcup \{Q : Q \in \Omega^k\}$, we have that $G_k \subseteq\text{int}(G_{k+1})$ and $\mathbb{R}^n \setminus \text{int}(G_k) \subseteq F_k$.

Since $\mu(U) = 0$, there exists a increasing sequence $\{m_k\}$ of positive integers such that

$$\mathcal{H}^{d}(E_k \cap G_k) < c_0^{-1}4^{-kd}, \forall m \geq m_k.$$  

We take $E_k' = E_{m_k}$. By Theorem 2.7, there exist Lipschitz mappings $\phi_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\phi_k(Q) \subseteq Q$ for $Q \in \Omega^k$, $\phi_k|_{\mathbb{R}^n \setminus G_k} = \text{id}_{\mathbb{R}^n \setminus G_k}$ and $\text{int}(G_k) \cap \phi_k(E_k') \subseteq \Omega_{d-1}^k$. Since $E_k'$ spans $L$, we get that $\phi_k(E_k')$ spans $L$. Setting $E_k'' = \phi_k(E_k') \cap \text{int}(G_k)$ and $H_k = \text{int}(G_k) \cap \phi_k(E_k')$, we see that $H_k \cap E_k'' \subseteq \Omega_{d-2}^k$ and $E_k'' \cup H_k = \phi_k(E_k')$, thus $\mathcal{H}^{d-1}(H_k \cap E_k'') = 0$. Applying Theorem VII 3 in [10], we get that

$$\tilde{H}_{d-1}(H_k \cap E_k'') = 0.$$

By lemma 5.1, we get that the homomorphism $\tilde{H}_{d-1}(i_{E_k', \phi_k(E_k')})$ is injective. Since

$$i_{B_0, \phi_k(E_k')} = i_{E_k', \phi_k(E_k')} \circ i_{B_0, E_k'},$$

we have that

$$\tilde{H}_{d-1}(i_{B_0, \phi_k(E_k')}) = \tilde{H}_{d-1}(i_{E_k', \phi_k(E_k')}) \circ \tilde{H}_{d-1}(i_{B_0, E_k'}).$$
Since $L \subseteq \ker \left( \mathcal{H}_{d-1}(i_{B_0, \phi_k(E'_k)}) \right)$ and that $\mathcal{H}_{d-1}(i_{B_0, \phi_k(E'_k)})$ is injective, we get that

$$L \subseteq \ker \left( \mathcal{H}_{d-1}(i_{B_0, E''_k}) \right).$$

Since $\mathbb{R}^n \setminus \text{int}(G_k) \subseteq F_k$, we get that $E''_k \subseteq F_k$, thus

$$L \subseteq \ker \left( \mathcal{H}_{d-1}(i_{B_0, E_k}) \right).$$

By Lemma 12.2 in [7], we get that

$$L \subseteq \ker \left( \mathcal{H}_{d-1}(i_{B, E}) \right).$$

Proof of Theorem 1.1. Let $\{E_k\}$ be a minimizing sequence. That is, $E_k \in \mathcal{C}$ and

$$\mathcal{H}^d(E_k) \to \inf \{ \mathcal{H}^d(E \setminus B_0) : E \in \mathcal{C} \}.$$

Without loss of generality, we assume that $0 < \inf \{ \mathcal{H}^d(E) : E \in \mathcal{C} \} < \infty$. Put $U = \Omega \setminus B$, $\zeta_k = \inf \{ \mathcal{H}^d(E_k) - \mathcal{H}^d(F(E_k)) : \varphi \in \mathcal{D}(\Omega, U) \}$, $M_k \equiv 1$, $\varepsilon_k \equiv \zeta_k$, $M(r) = \lim sup_{k \to \infty} M_k(r)$ and $\varepsilon(r) = \lim sup_{k \to \infty} \varepsilon_k(r)$. Then $M \equiv 1$, $\varepsilon \equiv 0$ and $E_k \in \mathcal{Q}(\Omega, U, M_k, \varepsilon_k)$.

Since $\{E_k\}$ is uniformly bounded and $\inf \{ \mathcal{H}^d(E) : E \in \mathcal{C} \} < \infty$, we can find a subsequence $\{E_{k_m}\}$ such that $\mathcal{H}^d\left|_{U \cap E_{k_m}}\right|$ converges to a Radon measure $\mu$. By Theorem 4.2, we get that $\nu(U \cap E_{k_m}) \to V$ and $\nu(U \cap \text{spt } \mu) = \nu(U \cap \text{spt } \mu)$. By Lemma 5.2, we get that $E_{\infty} = B_0 \cup \text{spt } \mu \in \mathcal{C}$. Thus

$$\mathcal{H}^d(U \cap E_{k_m}) = \mathcal{H}^d(U \cap \text{spt } \mu) \leq \lim inf_{m \to \infty} \mathcal{H}^d(U \cap E_{k_m}) = \inf \{ \mathcal{H}^d(E \setminus B_0) : E \in \mathcal{C} \}.$$

Therefore, $E_{\infty}$ is a minimizer.

Lemma 5.3. Let $U \subseteq \mathbb{R}^n$ be an open set. Let $\{E_k\}$ be a sequence of closed subsets such that $U \cap E_k$ is uniformly bounded. Suppose that $\lim sup_{k \to \infty} \mathcal{H}^d(U \cap E_k) < \infty$ and $\mathcal{H}^d\left|_{U \cap E_k}\right. \to \mu$. Then, for any $\tau > 0$, there exist a subsequence $\{E_{k_i}\}$, a sequence of Lipschitz mappings $\{\phi_i\} \subseteq \mathcal{D}(\mathbb{R}^n, U)$ such that $\|\phi_i - \text{id}\| \leq \tau$, $\phi_i(U \cap E_k)$ converges to a compact set $E$ in $U$ in local Hausdorff distance,

$$\lim_{i \to \infty} \mathcal{H}^d(\phi_i(E_{k_i}) \Delta E_{k_i}) = 0, U \cap \text{spt } \mu \subseteq E \text{ and } \mathcal{H}^d(U \cap E \setminus \text{spt } \mu) = 0. \quad (5.1)$$

Proof. We assume that $\tau < 1/10$ and $E_k \cap U \subseteq B(0,R)$ for any $i \geq 1$ and some $R > 0$, put $U = U(0, R+1) \setminus \text{spt } \mu$ and $E = U \cap \text{spt } \mu$. Then $U$ is open. Let $\mathcal{F}$ be the family of cubes which raise from a Whitney decomposition of $U$ with the following conditions hold:

- $\mathcal{F}$ consists of interior disjoint dyadic cubes and $\cup_{Q \in \mathcal{F}} Q = U$;
- $\sqrt{n} \ell(Q) \leq \text{dist}(Q, \mathbb{R}^n \setminus U) \leq 4 \sqrt{n} \ell(Q)$;
- if $Q_1 \cap Q_2 \neq \emptyset$, then $1/4 \leq \ell(Q_1)/\ell(Q_2) \leq 4$.

We construct $\mathcal{F}$ from $\mathcal{F}$ as follows: for any $C \in \mathcal{F}$, if its sidelength is less than $\tau$, then we put $C \in \mathcal{F}$; otherwise, we decompose it into diadic cubes of sidelength $2^{[\ln \tau / \ln 2]}$, then put each smaller cubes into $\mathcal{F}$. We put $F_k = E + U(0, 4^{-k+1})$ and let $\mathcal{Q}^k$ be the collection of cubes $Q$ in $\mathcal{F}$ whose sidelength is no less than $4^{-k}$. By putting $G_k = \bigcup \{Q \in \mathcal{Q} : Q \cap \mathcal{Q} \neq \emptyset \}$, we have that $G_k \subseteq \text{int}(G_{k+1})$ and $\mathbb{R}^n \setminus \text{int}(G_k) \subseteq F_k$.

Since $\mu(U) = 0$, there exists a increasing sequence $\{k_i\}$ of positive integers such that

$$\mathcal{H}^d(E_{m(i)} \cap G_{k_i}) < c_0^{-1} 4^{-id}, \forall m \geq k_i.$$
By Theorem 2.7, there exist Lipschitz mappings $\phi_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi_i(Q) \subseteq Q$ for $Q \in \Omega^i$, $\phi_i|_{\mathbb{R}^n \setminus G_i} = \text{id}_{\mathbb{R}^n \setminus G_i}$ and $\text{int}(G_i) \cap \phi_i(E_{k_i}) \subseteq \Omega^i_{d-1}$. Since $\cup_{i} G_i = U$, $\mathbb{R}^n \setminus \text{int}(G_i) \subseteq E + U(0, 4^{-i+1})$ and $\phi_i(E_{k_i}) \cap \text{int}(G_i)$ is contained in the union of $(d-1)$-faces of cubes in $\mathcal{F}$, we get that $\phi_i(U \cap E_{k_i})$ converges to $E$ in local Hausdorff distance, and (5.1) hold.

**Theorem 5.4.** Suppose $1 \leq d < m \leq n$. Let $\Omega \subseteq \mathbb{R}^n$ be an $m$-dimensional closed submanifold of class $C^2$. Let $U \subseteq \mathbb{R}^n$ be an open set with $U \cap \Omega \neq \emptyset$ and $U \cap \partial \Omega = \emptyset$. Let $\mathcal{E}(\Omega, U)$ be a collection of compact subsets in $\Omega$ such that

- there exists $E \in \mathcal{E}(\Omega, U)$ such that $E \cap U \neq \emptyset$ and $\mathcal{H}^d(E \cap U) < \infty$;
- for any $E \in \mathcal{E}(\Omega, U)$ and $\varphi \in \mathcal{D}(\Omega, U)$ we have that $\varphi(E) \in \mathcal{E}(\Omega, U)$;
- if $\{E_i\} \subseteq \mathcal{E}(\Omega, U)$ is a sequence such that $E_i \cap U$ converges to $E$ in $\Omega$ in local Hausdorff distance, then $E \in \mathcal{E}(\Omega, U)$.

If there is a sequence $\{E_i\} \subseteq \mathcal{E}(\Omega, U)$ such that $\{E_i \cap U\}$ is uniformaly bounded and

$$\mathcal{H}^d(E_i \cap U) \to \inf \{\mathcal{H}^d(S \cap U) : S \in \mathcal{E}(\Omega, U)\},$$

then there exit minimizers, i.e. that exits $E \in \mathcal{E}(\Omega, U)$ such that

$$\mathcal{H}^d(E \cap U) = \inf \{\mathcal{H}^d(S \cap U) : S \in \mathcal{E}(\Omega, U)\}.$$

**Proof.** We assume that $E_i \cap U \subseteq B(0, R)$ for all $i \geq 1$ and some $R > 0$. Since $\Omega$ is submanifold of class $C^2$, we see that $\Omega \cap B(0, R + 1)$ is of positive reach. Let $\xi$ be the projection. By Lemma 5.3, for any $\tau > 0$, we can find a subsequence $\{E_{k_i}\}$ of $\{E_k\}$ and a sequence of Lipschitz mappings $\{\phi_i\} \subseteq \mathcal{D}(\mathbb{R}^n, U)$ such that $\|\phi_i - \text{id}\| \leq \tau, \phi_i(U \cap E_{k_i})$ converges to a set $E$ in $U$ in Hausdorff distance.

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