ON QUANTUM COMPUTATION, ANYONS, AND CATEGORIES

ANDREAS BLASS AND YURI GUREVICH

Abstract. We explain the use of category theory in describing certain sorts of anyons. Yoneda’s lemma leads to a simplification of that description. For the particular case of Fibonacci anyons, we also exhibit some calculations that seem to be known to the experts but not explicit in the literature.

1. Introduction

This paper attempts to explain the use of category theory in describing certain sorts of anyons. These are rather mysterious physical phenomena which, one hopes, will provide an approach to quantum computing needing far less error correction than other approaches.

The first author of this paper has long been a fan of category theory; even as a graduate student, he was described by one of his professors as “functorized”. The second author has been far more skeptical about the value of category theory in computer science, because of its distance from applications and because of the peril of potential (and in some cases actual) over-abstraction. In 2012, both authors began working with the Quantum Architectures and Computing (QuArC) Group at Microsoft Research and found anyons to be near the top of the group’s agenda. We made rather a nuisance of ourselves by asking different people, on different occasions, what anyons actually are, from a mathematical point of view. Are they Hilbert spaces? Are they vectors in a Hilbert space? Are they something else? It turned out that the only mathematically sound answer in the literature involved a special sort of categories, modular tensor categories. So the second author agreed that categories can be quite relevant to important applications in computer science.

Our purpose in this paper is to describe some of the ideas surrounding categories and anyons in general and the special case of Fibonacci anyons and their category description.

1 Other answers explained the physics, in terms of excitations, but these matters are not the subject of this paper, which is specifically about mathematics except for the introductory material summarized in Section 2.
In Section 2, we give a general introduction to anyons from the point of view of physics and quantum computation. That section is intended to give the reader a rough idea of what anyons are and why researchers in quantum computation would be interested in them. The treatment here is quite superficial, and we give references for more detailed treatments.

In Section 3, we gradually introduce modular tensor categories, and we explain how they are intended to be used to describe anyons. This section closely follows the axiomatization given in [6], but with some modifications and rearrangements.

Section 4 is devoted to an application of one of the central theorems of category theory, Yoneda’s Lemma, to producing a simplified view of modular tensor categories.

Finally, in Section 5, we consider the special case of Fibonacci anyons. This special case is unusually simple in some respects. Nevertheless (or perhaps therefore) it occupies a prominent place in quantum computing research.

Appendix A exhibits some calculations, whose results seem to be well known in the quantum computing community but which we have not been able to find written down in the literature.

2. Quantum theory and anyons

This section is a very superficial summary of a small part of quantum theory and some basic information about anyons. The physics described here is intended merely to provide an orientation for understanding the mathematics in the rest of the paper.

2.1. Quantum Mechanics. In quantum theory, the state of a physical system is represented by a non-zero vector in a complex Hilbert space $\mathcal{H}$, but all non-zero scalar multiples of a vector represent the same state. Thus, the states constitute the projective space associated to $\mathcal{H}$. Because of the freedom to adjust scalar factors, one often imposes the normalization that the vectors representing a state should have norm 1; there still remains a freedom to adjust the phase, i.e., a scalar factor of absolute value 1.

If a system has an observable property with infinitely many possible values, for example position or momentum, then the Hilbert space of its states must be infinite-dimensional. In quantum computing, however, one usually ignores many such properties and concentrates on only a small number (often only one) of properties with only finitely many possible values. As a result, one deals with finite-dimensional
Hilbert spaces. (This simplification is analogous to modeling a classical computer by a configuration of bits, not taking account of its other physical properties, like position or momentum or temperature, unless these threaten to interfere with the bits of interest.)

The automorphisms of a Hilbert space $H$ are the *unitary* transformations, i.e., the linear bijections that preserve the inner product structure. These play several important roles, both in physics and in quantum computation. First, they provide the dynamics of quantum systems. That is, if a system is isolated, then its state will evolve in time by the action of a one-parameter group of unitary operators. Second, if a system has symmetries, i.e., if it is invariant under some transformations, then these transformations are usually modeled by unitary operators. Finally, in quantum computation, the basic steps of a computation transform the state in a unitary way. In this context, unitary transformations are often called *gates*. An important part of quantum algorithm theory is designing particular gates, to transform a state that we know how to produce into a state from which we can extract useful information by a measurement. Another important part is producing such useful gates as a composition of simpler gates that we know how to implement.

Where classical computation uses bits, whose possible values are denoted by 0 and 1, quantum computation uses *qubits*. A qubit has two possible values; it is represented by a 2-dimensional Hilbert space, in which a certain orthonormal basis, usually written $\{|0\rangle, |1\rangle\}$, corresponds to the two values. In contrast to the classical case, though, the Hilbert space structure provides many other states in addition to these two basic ones. Any non-zero linear combination of $|0\rangle$ and $|1\rangle$ represents a possible state of the system. If the state is represented by the unit vector $x|0\rangle + y|1\rangle$, then measuring the qubit in the $\{|0\rangle, |1\rangle\}$ basis will produce the outcome 0 with probability $|x|^2$ and the outcome 1 with probability $|y|^2$. Such a state is a *superposition* of the two basic states. More precisely, it is the superposition, with coefficients $x$ and $y$, of the vectors $|0\rangle$ and $|1\rangle$, respectively.

It is more accurate to speak of superposition of vectors than of superposition of states. The reason is that, although phase factors don’t affect the state represented by a vector, *relative* phases do affect superpositions. Thus, for example, although $|1\rangle$ and $-|1\rangle$ represent the same

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2Here we use the so-called Schrödinger picture of quantum mechanics. A physically equivalent alternative view, the Heisenberg picture, has the states remaining constant in time, while the operators modeling properties of the state evolve by conjugation with a one-parameter group of unitary operators.

3A few discrete symmetries can be modeled by anti-unitary transformations.
state of a qubit, the superpositions \((|0\rangle + |1\rangle)/\sqrt{2}\) and \((|0\rangle - |1\rangle)/\sqrt{2}\) represent quite different states.

It is almost true in general that, for any two states of any quantum system, any superposition of the associated vectors also represents a possible state of that system. The word “almost” in the preceding sentence refers to the possibility of superselection rules. These rules specify that, for certain quantities, like electric charge, it is impossible to superpose two states with different values of those quantities. Thus, when discussing a system for which several values of the electric charge can occur, we are, in effect, dealing with several separate Hilbert spaces, called superselection sectors, one for each value of the charge. One can, and sometimes one does, form the direct sum of these Hilbert spaces to obtain a Hilbert space containing all the possible states of that system, but most of the vectors in that direct sum, involving superpositions with different charges, do not represent physically possible states. We prefer, in this paper, to deal with the separate Hilbert spaces and forgo their direct sum. For more information about superselection rules, see [4].

In reality, there are very few superselection rules — electric charge, baryon number, parity — but in the study and application of anyons one often artificially adds superselection rules. This amounts to deciding not to consider superpositions of vectors from certain Hilbert spaces, i.e., to consider those superselection sectors separately rather than considering their direct sum.

In the presence of superselection rules, the operators that one considers are operators acting on each of the superselection sectors separately. In the case of true superselection rules, the dynamics of the system and any gates that one could construct are given by unitary operators acting on each sector separately. In the case of artificial superselection rules, nature may not cooperate with our artificial rules, and states in one sector may evolve out of that sector. Such evolution interferes with our understanding and intentions; it is often called “leakage” and one strives to avoid it.

2.2. Anyons. To understand anyons, it is useful to recall first that ordinary particles are of two sorts, bosons and fermions. These differ in several respects, beginning with the action of spatial rotations on the corresponding Hilbert spaces. For particles in ordinary 3-dimensional space, the group \(SO(3)\) of Euclidean rotations of that space acts on the states of the particle. (More precisely, the group of all Euclidean motions acts, but we abstract from the particle’s position and consider only its orientation in space; thus we ignore translations and consider
only the group of rotations.) Because the vector representing a state is defined only up to a phase factor, the action of the rotation group is not a representation in the usual sense but a projective representation. This means that if \( \rho \) is the assignment, to each rotation \( g \), of a unitary operator \( \rho(g) \) on Hilbert space, then \( \rho(gh) \) and \( \rho(g)\rho(h) \) need not be equal but can differ by a phase factor. Furthermore, \( \rho \) and \( \rho' \) are considered equivalent representations if, for each rotation \( g \), \( \rho(g)\rho'(g)^{-1} \) is a scalar multiple of the identity operator, i.e., if \( \rho(g) \) and \( \rho'(g) \) always differ by only a phase factor and thus produce the same states. It is reasonable to ask, in this connection, why the operators \( \rho(g) \) need to be unitary or even linear, rather than only linear up to phase factors. The reason is that, unlike mere phases, relative phases are relevant in superpositions, so physical symmetries must preserve them.

It turns out that any projective representation \( \rho \) of \( SO(3) \) is given by a genuine unitary representation \( \tilde{\rho} \) of the universal covering group of \( SO(3) \), namely \( SU(2) \) (see for example [1] and [7]). That is, if \( p : SU(2) \to SO(3) \) denotes the 2-to-1 projection map, we have \( \rho \circ p = \tilde{\rho} \). More concretely, it means that there are two sorts of projective representations of \( SO(3) \), up to equivalence. One sort is the ordinary unitary representations of \( SO(3) \); the other is given by unitary representations of \( SU(2) \) that send the non-trivial element \(-I\) of the kernel of \( p \) to the operator \(-I\). (Throughout this paper, we use \( I \), sometimes with subscripts, to denote identity transformations, functions, morphisms, etc.) The first sort of representation corresponds to bosons, whose state vectors (not merely their states) are unchanged when rotated gradually through a full revolution. The second sort corresponds to fermions, where a rotation through \( 2\pi \) changes the state vector by a sign.

A second distinction between bosons and fermions, more important for our purposes, is the behavior of systems of several identical particles. Because the particles are identical, any permutation of the particles leaves the state unchanged and therefore changes the state vector by at most a phase factor. As a result, we have a one-dimensional projective representation of the symmetric group. Again, it turns out that there are just two possibilities (both of which are actual unitary representations of the symmetric group). Either all permutations leave the state vectors unchanged, or the even permutations leave the state vectors unchanged while the odd permutations reverse the vectors’ signs.

A deep theorem of relativistic quantum field theory, the spin-statistics theorem, says that these two behaviors of multi-particle states under permutations exactly match the two behaviors of single-particle states under rotations. Interchanging two identical bosons leaves the state
vector of the pair unchanged; interchanging two identical fermions reverses the sign of the state vector.

The preceding discussion of bosons and fermions depends crucially on the fact that the particles are in ordinary 3-dimensional space. If particles were confined to a 2-dimensional space, more possibilities would arise.

Specifically, the rotation group in two dimensions, $SO(2)$ has more sorts of projective representations than $SO(3)$ does; the reason is ultimately that the universal covering group of the circle group $SO(2)$ is the additive group of real numbers, and the covering projection is not 2-to-1 but $\infty$-to-1. The result is that a gradual rotation of a particle through $2\pi$ can multiply its state vector by an arbitrary phase factor, not just $\pm 1$. The possibility of getting any phase here led to the name anyon.

Reducing the dimensionality of space from 3 to 2 also affects the possibilities for permuting identical particles. For simplicity, consider the case where there are just two particles, and we interchange them. We can perform the interchange gradually, in the plane, by rotating the 2-particle system counterclockwise by $\pi$ around the midpoint between the particles. Alternatively, we can achieve the same interchange by a clockwise rotation. In 3-dimensional space, these two options are equivalent in the sense that they can be deformed into each other, by rotating the plane of the particles' motion about the line through the particles' initial positions. In 2-dimensional space, there is no such deformation without making the particles collide. With more than two particles, there are many more (in fact infinitely many more) ways to achieve the same permutation by moving the particles around in the plane. As a result, in place of (projective) representations of symmetric groups, we have representations of braid groups. For example, in the case of two particles, in place of the group of two possible permutations of the particles, we have the group of all integers, with integer $n$ representing a counterclockwise rotation by $n\pi$ (and negative $n$ representing clockwise rotations).

The preceding discussion was oversimplified in that (among other things) when moving particles around each other, we ignored any rotation that the individual particles might have undergone during the motion. A more accurate presentation would need to suitably combine the braid and rotation groups.

2.3. Anyons in Reality. As explained above, anyons do not occur in 3-dimensional space; it is necessary to reduce the number of spatial dimensions to 2. Since we live in a 3-dimensional space, will we ever find
anyons? It turns out that anyon-like behavior occurs for certain excitations in materials that are so thin as to be effectively two-dimensional. A detailed discussion of this would take us too far from the purpose of this paper, so we refer the reader to Section 1.1 of [6].

We emphasize, however, that the anyons are not what would ordinarily be called “particles” but rather excitations in some medium, which exhibit particle-like behavior. It should be noted in this connection that it is not unusual, in other contexts, for collective excitations to behave like particles and thus to be analyzed mathematically as if they were particles. For example, vibrational excitations in crystal lattices are treated as particles called phonons. Similarly, photons are excitations of the electromagnetic field.

2.4. Anyons in Quantum Computation. Quantum computation is unpleasantly susceptible to environmental disturbances. Its advantages over classical computation depend on maintaining superpositions of state vectors, with high precision in the coefficients of those vectors. Small disturbances can easily modify those coefficients or, indeed, destroy superpositions altogether. Significant effort must therefore be devoted to error correction, and this makes algorithms slower and harder to design.

It has been suggested that qubits could be more robust, i.e., less susceptible to disturbances, if they were implemented using certain sorts of anyons. For example, if qubits were encoded in the way two anyons wind around each other, then this winding, being a topological property of the system, would be robust. A small disturbance in the actual motion of the anyons would leave the winding number intact.

This hope of reducing the error correction needs of quantum computing has motivated much of the current interest in anyons.

It is worth noting explicitly that, in this picture, a qubit is not encoded in the state of a single anyon but rather in a whole system of several anyons. This feature will be quite prominent in the category picture described in the rest of this paper.

3. Modular tensor categories

In this section we describe the category-theoretic structure that has been developed to support a mathematical theory of anyons. Much of what we describe here is in [6], though we have modified some aspects and rearranged others.

Throughout this section, we let \( \mathcal{A} \) be a category, intended to describe the quantum-mechanical behavior of a system of anyons. \( \mathcal{A} \) will carry several sorts of additional structure, roughly classified as “additive”
and “multiplicative” structure, all subject to various axioms. We describe the structures and the axioms a little at a time. We begin with the additive structure, because this is where Hilbert spaces enter the picture, so it is the basis for the connection with the usual formalism of quantum theory.

The vectors in our Hilbert spaces will be the morphisms of \( \mathcal{A} \). Specifically, for each pair of objects \( X, Y \) of \( \mathcal{A} \), the set \( \text{Hom}(X, Y) \) of morphisms from \( X \) to \( Y \) will have the structure of a Hilbert space. So we have many Hilbert spaces, one for each pair \( X, Y \) of objects. Some of these Hilbert spaces will be mere combinations of others, but there will still be several different “basic” Hilbert spaces. This means physically that we regard the system as being subject to superselection rules, which keep these Hilbert spaces separate.

We assume familiarity with some basic notions of category theory, specifically, the notions of product (including terminal object, which is the product of the empty family), coproduct (including initial object), equalizer, coequalizer, monomorphism, epimorphism, isomorphism, functor, and natural transformation. Definitions and examples can be found in [5] or [3, Chapter 1].

3.1. Additive Structure. We begin by requiring \( \mathcal{A} \) to be an abelian category. This requirement, formulated in detail below, provides a well-behaved addition operation on each of the sets \( \text{Hom}(X, Y) \), although the requirement is formulated in purely category-theoretic terms and does not explicitly mention this addition operation.

**Axiom 1 (Abelian).** \( \mathcal{A} \) is an abelian category. That is

1. There is an object 0 that is both initial and terminal. A morphism that factors through this zero object will be called a zero morphism and denoted by 0. Note that each \( \text{Hom}(X, Y) \) contains a unique zero morphism.
2. Every two objects have a product and a coproduct.
3. For every morphism \( \alpha : X \to Y \), the pair \( \alpha, 0 \) has an equalizer and a coequalizer. These are called the *kernel* and *cokernel* of \( \alpha \).
4. Every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

This axiom has a surprisingly rich collection of consequences, developed in detail in Chapter 2 of [3]. We list here only some of the highlights, which will be important for this paper, and we refer the reader to [3] for the proofs and additional information.

**Proposition 1.** The product and coproduct of any two objects coincide.
That is, given two objects $X$ and $Y$, there is an object $X \oplus Y$ that serves simultaneously as the product of $X$ and $Y$, with projections $p_X : X \oplus Y \to X$ and $p_Y : X \oplus Y \to Y$, and as the coproduct of $X$ and $Y$, with injections $u_X : X \to X \oplus Y$ and $u_Y : Y \to X \oplus Y$. (If $X = Y$, then our notations for the projections and injections become ambiguous, and we use $p_1, p_2, u_1, u_2$ instead.) For brevity, we often refer to $X \oplus Y$ as the *sum* of $X$ and $Y$, rather than as the product or coproduct.

As a product, $X \oplus X$ admits a diagonal morphism $\Delta_X : X \to X \oplus X$, namely the unique morphism whose composites with both projections are the identity morphism $I_X$ of $X$. Dually, as a coproduct, it admits the folding morphism $\nabla_X : X \oplus X \to X$, whose composites with both of the injections are $I_X$. Using the diagonal and folding morphisms, we can define a binary operation, called addition, on $\text{Hom}(X, Y)$ for any objects $X$ and $Y$. Given $f, g : X \to Y$, we define $f + g : X \to Y$ to be the composite

$$X \xrightarrow{\Delta_X} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla_Y} Y,$$

where $f \oplus g$ is obtained from the functoriality of products (or of coproducts — they yield the same result).

**Proposition 2.** *This addition operation makes each $\text{Hom}(X, Y)$ an abelian group, with the zero morphism serving as the identity of the group. Composition of morphisms is additive with respect to both factors.*

**Axiom 2** (Vectors). Each of these abelian groups $\text{Hom}(X, Y)$ carries an operation of multiplication by complex numbers, making $\text{Hom}(X, Y)$ a vector space over $\mathbb{C}$, and making composition of morphisms bilinear over $\mathbb{C}$.

The complex vector spaces $\text{Hom}(X, Y)$ will play the role of quantum-mechanical state spaces. For this purpose, they should also be equipped with inner products, making them Hilbert spaces, but, following [6], we refrain from assuming an inner product structure at this stage of the development. It turns out that much of what we shall do later does not depend on the availability of inner products in the vector spaces $\text{Hom}(X, Y)$.

An object $S$ in the abelian category $\mathcal{A}$ is called *simple* if $S \not\sim 0$ and every monomorphism into $S$ is either a zero morphism or an isomorphism. In other words, $S$ is a non-zero object with no non-trivial

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4In fact, inner products are never explicitly assumed in [6]. They are, however, implicit in the statement, in Section 5.1 of [6], that certain bases “are – of course – related by a unitary transformation”.
subobjects. Because of the abelian structure of $\mathcal{A}$, this definition is equivalent to its dual: A non-zero object is simple if and only if it has no non-trivial quotients, i.e., every epimorphism out of $S$ is either a zero morphism or an isomorphism.

**Axiom 3 (Semisimple).** Every object in $\mathcal{A}$ is a finite sum of simple objects.

This axiom considerably simplifies the structure of the vector spaces $\text{Hom}(X, Y)$. In the first place, as shown in [3, Section 2.3], morphisms from a sum $\bigoplus_j S_j$ to another sum $\bigoplus_k S'_k$ are given by matrices of morphisms between the summands. Specifically, the matrix associated to $f : \bigoplus_j S_j \to \bigoplus_k S'_k$ has as its $a, b$ entry the composite

$$S_b \xrightarrow{u_b} \bigoplus_j S_j \xrightarrow{f} \bigoplus_k S'_k \xrightarrow{v'_a} S'_a.$$

Composition of morphisms in $\mathcal{A}$ corresponds to the usual multiplication of matrices.

Furthermore, when the summands are simple, we have the following additional information about the matrix entries, a generalization of Schur’s Lemma in group representation theory.

**Proposition 3.** If $f : S \to S'$ is a morphism between two simple objects, then $f$ is either the zero morphism or an isomorphism.

**Proof.** The kernel of $f$ is a monomorphism into $S$, and if it is an isomorphism then $f$ is zero. So, by simplicity of $f$, we may assume that the kernel of $f$ is zero and therefore $f$ is a monomorphism. Similarly, by considering the cokernel of $f$ and invoking the simplicity of $S'$, we may assume that $f$ is an epimorphism. But in an abelian category, any morphism that is both mono and epi is an isomorphism. (More generally, in any category, any equalizer that is an epimorphism is an isomorphism.) \[\square\]

The last axiom in this section combines two finiteness assumptions.

**Axiom 4 (Finiteness).**

1. There are only finitely many non-isomorphic simple objects.
2. Each of the vector spaces $\text{Hom}(X, Y)$ is finite-dimensional over $\mathbb{C}$.

The first of these two finiteness requirements is merely a technical convenience. The second, however, gives the following important information about the endomorphisms of simple objects.

**Proposition 4.** If $S$ is a simple object, then $\text{Hom}(S, S) \cong \mathbb{C}$. 
Proof. The operation of composition of morphisms is a multiplication operation that makes the vector space $\text{Hom}(S, S)$ into an algebra over $\mathbb{C}$. Since $S$ is simple, Proposition 3 says that every non-zero element of this algebra is invertible. That is, $\text{Hom}(S, S)$ is a division algebra over $\mathbb{C}$. But $\mathbb{C}$ is algebraically closed, so the only finite-dimensional division algebra over it is $\mathbb{C}$ itself. □

Note that the isomorphism $\text{Hom}(S, S) \cong \mathbb{C}$ in this proposition can be taken, as the proof shows, to be an isomorphism of algebras, not just of vector spaces. In particular, the identity morphism of $S$ corresponds to the number 1.

Combining this proposition with our earlier observations about matrices, we find that any morphism $f : \bigoplus_j S_j \to \bigoplus_k S'_k$ between any two objects in $\mathcal{A}$ is given by a matrix whose entries are complex numbers. Moreover, the $a, b$ entry is 0 unless $S_b \cong S'_a$. From this observation, it easily follows that, when an object $X$ of $\mathcal{A}$ is expressed as a sum $\bigoplus_j S_j$ of simple objects, the isomorphism types of the summands $S_j$ and their multiplicities are completely determined by $X$. That is, the representation of $X$ as a sum of simple objects is essentially unique.

3.2. Multiplicative Structure. In this section, we introduce the multiplicative structure that makes $\mathcal{A}$ a braided monoidal category. The central idea is that, if objects $X$ and $Y$ represent certain anyons, then $X \otimes Y$ should represent a system consisting of both of these anyons. We must, however, remember that the Hilbert spaces that occur in this context are not the objects of $\mathcal{A}$ but the vector spaces of morphisms between the objects.

A system consisting of two anyons of types $X$ and $Y$ would, if measured as a whole, appear as another anyon, whose type might not be entirely determined by the types $X$ and $Y$. Formally, this means that $X \otimes Y$ is a sum of several simple objects. Furthermore, there might be several “ways” for a composite system to appear as having a particular type $Z$, modeled as several morphisms from $X \otimes Y$ to $Z$, and our Hilbert spaces will also contain superpositions of these.

The multiplicative structure will also include a unit object 1; its intended interpretation is the vacuum. Thus, $1 \otimes X$ and $X \otimes 1$ amount to just $X$ because a system consisting of $X$ and nothing is the same as $X$.

The first aspect of multiplicative structure can be stated rather briefly as the following axiom, but we expand it afterward because we shall need the details later.

**Axiom 5 (Multiplication).** $\mathcal{A}$ is a monoidal category.
This means that it is equipped with a “multiplication” functor \( \otimes : A \times A \to A \) and a “unit object” \( 1 \) that satisfy the usual associative and unit laws up to coherent isomorphism. Let us first explain “satisfying the laws up to isomorphism” and then discuss “coherent”.

Associativity would mean that \( A \otimes (B \otimes C) \) is the same as \( (A \otimes B) \otimes C \) for any objects \( A, B, C \) (and similarly for morphisms). Associativity up to isomorphism means that these objects need not be equal but they are isomorphic, and we are given specific isomorphisms

\[
\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)
\]

for all \( A, B, C \), and furthermore these isomorphisms constitute a natural transformation between functors \( A \times A \times A \to A \).

Similarly, the requirement that the object \( 1 \) be a unit up to isomorphism means that we are given natural isomorphisms

\[
\lambda_A : 1 \otimes A \to A \quad \text{and} \quad \rho_A : A \otimes 1 \to A.
\]

As is well-known from classical algebra, the associative law implies associative identities for more than three factors at a time; for example, if \( * \) is an associative operation, then all five of the possible parenthesizations of \( a * b * c * d \) give the same result. The analogous result for categories is that any natural isomorphism \( \alpha \) as above produces natural isomorphisms between any two parenthesizations of \( A \otimes B \otimes C \otimes D \). There is, however, an embarrassment of riches, as we can build, from \( \alpha \) (and its inverse) several isomorphisms between such parenthesizations of four factors. Specifically, the “extreme left” and “extreme right” parenthesizations are connected by a product of three \( \alpha \)'s:

\[
((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C} \otimes \text{id}} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B,C,D}} A \otimes ((B \otimes C) \otimes D) \xrightarrow{I_A \otimes \alpha_{B,C,D}} A \otimes (B \otimes (C \otimes D)).
\]

The same two parenthesizations are connected by a product of two other \( \alpha \)'s:

\[
((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha_{A,B,C,D}} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C,D}} A \otimes (B \otimes (C \otimes D)).
\]

One aspect of “coherence” is that these two transformations must agree, so that there is a single, well-defined way of shifting the parentheses from the left to the right. This requirement is often called the pentagon condition, because the diagram exhibiting these two transformations together has the shape of a pentagon. In this connection, the first composition, involving three morphisms, is sometimes called the “long side” of the pentagon, and the second composition is the “short side”.
Another aspect of coherence is that two ways of simplifying \((A \otimes 1) \otimes B\) should agree, namely \(\rho_A \otimes I_B\) and
\[(A \otimes 1) \otimes B \xrightarrow{\alpha_{A,1,B}} A \otimes (1 \otimes B) \xrightarrow{I_A \otimes \lambda_B} A \otimes B.\]

It is easy to think of other compositions of \(\alpha\)'s, \(\lambda\)'s, and \(\rho\)'s that should agree, for example the many ways of connecting different parenthesizations of five or more factors. Fortunately, all of these requirements can be deduced from the two that we have exhibited here. This is Mac Lane’s coherence theorem, and we refer to Chapter VII of [5] for its precise statement, its proof, and additional information about monoidal categories.

The pentagon condition will play a major role in the rest of this paper, because the associativity isomorphism \(\alpha\) is often nontrivial and of considerable interest. The unit isomorphisms \(\lambda\) and \(\rho\), on the other hand, will play essentially no role, because one can safely identify \(1 \otimes X\) and \(X \otimes 1\) with \(X\) and take \(\lambda_X = \rho_X = I_X\) for all \(X\). From now on, we will make these simplifying identifications.

The idea that \(\otimes\) represents combining two anyons (or two systems of anyons) into a single system suggests that this operation should be commutative, i.e., that \(X \otimes Y\) should be naturally isomorphic to \(Y \otimes X\). The next axiom postulates the existence of such an isomorphism, with good behavior in connection with the associativity isomorphism \(\alpha\).

**Axiom 6 (Braiding).** The monoidal structure on \(\mathcal{A}\) is equipped with a braiding, i.e., a natural isomorphism \(\sigma_{X,Y} : X \otimes Y \to Y \otimes X\) subject to two requirements, first that the following two composite isomorphisms be equal:

\[(A \otimes B) \otimes C \xrightarrow{\sigma_{A,B,C}} A \otimes (B \otimes C) \xrightarrow{\sigma_{A,B,C}} (B \otimes C) \otimes A \xrightarrow{\sigma_{B,C,A}} B \otimes (C \otimes A)\]

and

\[(A \otimes B) \otimes C \xrightarrow{\sigma_{A,B,C}} (B \otimes A) \otimes C \xrightarrow{\sigma_{B,A,C}} B \otimes (A \otimes C) \xrightarrow{I_B \otimes \sigma_{A,C}} B \otimes (C \otimes A),\]

and, second, the analogous equality with each \(\sigma_{X,Y}\) replaced with \(\sigma_{Y,X}^{-1}\).

Recall, from Section 2.2, that anyons inhabit two-dimensional space and therefore, when two of them are interchanged, it is necessary to keep track of how they move around each other. A clockwise rotation by \(\pi\) around the midpoint between them is not the same as, nor even deformable to, a counterclockwise rotation. So we should describe \(\sigma_{X,Y}\) not merely as switching \(X\) with \(Y\) but by doing so with a counterclockwise direction. The choice of direction here is a matter of convention; \(\sigma_{Y,X}^{-1}\) is then the clockwise rotation achieving the same interchange.
Thus, we expect that, in general, $\sigma_{X,Y} \neq \sigma_{Y,X}^{-1}$. (If these two were always equal, then we would have a symmetric monoidal category rather than a braided one.)

A useful picture, often used in connection with braiding, is to imagine the factors in a $\otimes$-product as being lined up from left to right. Then the counterclockwise interchange $\sigma_{X,Y}$ amounts to moving $X$ from the left of $Y$ to the right of $Y$ by passing $X$ in front of $Y$. $\sigma_{Y,X}^{-1}$ also moves $X$ from the left to the right of $Y$, but it does so by passing $X$ behind $Y$.

The equality of the two composite morphisms in the definition of braiding is called the *hexagon condition*. In terms of moving anyons around each other, it expresses the fact that moving $A$ past $B \otimes C$ by passing $A$ in front of $B \otimes C$ is equivalent to first passing $A$ in front of $B$ and then passing $A$ in front of $C$. The hexagon condition for $\sigma_{Y,X}^{-1}$ has a similar pictorial description with “in front of” replaced with “behind”.

The last axiom in this subsection relates the multiplicative structure discussed here with the additive structure from the preceding subsection.

**Axiom 7 (Additive-Multiplicative).**  
(1) The monoidal unit $1$ is simple.

(2) The product operation $\otimes$ is bilinear on morphisms.

In more detail, item (2) here means that the function

$$\text{Hom}(A, B) \times \text{Hom}(C, D) \to \text{Hom}(A \otimes C, B \otimes D)$$

given by the functoriality of $\otimes$ is bilinear with respect to the $\mathbb{C}$-vector space structures of the Hom-sets. It follows from this, via results in [3, Section 2.4], that $\otimes$ distributes over $\oplus$ on objects, i.e., that $X \otimes (Y \oplus Z)$ is canonically isomorphic to $(X \otimes Y) \oplus (X \otimes Z)$.

### 3.3. Duals, Twists, and Modularity.

In this subsection, we collect some additional axioms, which are imposed in [6], but which will not play a role in the computations we do later. We list these axioms, for the sake of completeness, but we make only a few comments about them and refer the reader to [6, Sections 4.3, 4.5, and 4.7] for more thorough explanations.

**Axiom 8 (Antiparticles).** For each object $X$ of $\mathcal{A}$, there is a *dual* object $X^*$, and there are two morphisms $i_X : 1 \to X \otimes X^*$ and $e_X : X^* \otimes X \to 1$, such that the compositions

$$X^* \xrightarrow{i_{X^*} \otimes i_X} X^* \otimes X \otimes X^* \xrightarrow{e_X \otimes i_{X^*}} X^*$$
and
\[
X \xrightarrow{i_X \otimes I_X} X \otimes X^* \otimes X \xrightarrow{I_X \otimes e_X} X
\]
are equal to the identity morphisms \(I_X\) and \(I_X\), respectively. Furthermore, dualization commutes with \(\otimes\) and \(\oplus\) and preserves 1 and 0.

For the sake of readability, we have exhibited the compositions in this axiom without the parentheses and associativity isomorphisms that technically should be there. We follow the same convention for iterated \(\otimes\) below.

The intention behind this axiom is that, if \(X\) represents some particle, then \(X^*\) represents its antiparticle. The morphism \(i_X\) represents creation of a particle-antiparticle pair from the vacuum, and \(e_X\) represents annihilation of such a pair.

The operation of dualization becomes a contravariant functor from \(\mathcal{A}\) to itself if one defines the dual \(f^*\) of a morphism \(f : X \to Y\) to be the composite
\[
Y^* \xrightarrow{I_Y \otimes i_X} Y^* \otimes X \otimes X^* \xrightarrow{I_Y \otimes f \otimes I_{X^*}} Y^* \otimes Y \otimes X^* \xrightarrow{e_Y \otimes I_{X^*}} X^*.
\]

**Axiom 9 (Rotations).** There is a natural isomorphism \(\delta\) with components \(\delta_X : X \to X^{**}\) respecting the monoidal structure and duality in the sense that
\[
\delta_1 = I_1, \quad \delta_{X \otimes Y} = \delta_X \otimes \delta_Y, \quad \text{and} \quad \delta_{A^*} = (\delta_A)^{-1}.
\]

By combining these \(\delta\) isomorphisms with the morphisms \(i\) and \(e\) from duality, one can obtain isomorphisms \(X \to X\) that represent twisting an anyon by \(2\pi\); see [6, Section 4.5] for details.

Monoidal categories satisfying the “Antiparticles” axiom are called **rigid**, and those that also satisfy the “Rotations” axiom are called **ribbon** categories.

**Axiom 10 (Modularity).** For any two simple objects \(X\) and \(Y\), let \(s_{X,Y} : 1 \to 1\) be the morphism
\[
1 = 1 \otimes 1 \xrightarrow{i_X \otimes i_Y} X \otimes X^* \otimes Y \otimes Y^* \xrightarrow{I_X \otimes \sigma_{X,Y} \otimes I_Y} X \otimes Y \otimes X^* \otimes Y^*
\]
\[
\xrightarrow{I_X \otimes \sigma_{Y,X} \otimes I_Y} X \otimes X^* \otimes Y \otimes Y^* \xrightarrow{\delta_X \otimes I_{X^*} \otimes \delta_Y \otimes I_{Y^*}} X^{**} \otimes X^* \otimes Y^{**} \otimes Y^*
\]
\[
\xrightarrow{e_{X^*} \otimes e_{Y^*}} 1 \otimes 1 = 1.
\]

Since \(\text{Hom}(1, 1) = \mathbb{C}\), these morphisms \(s_{X,Y}\) constitute a matrix of complex numbers, with rows and columns indexed by the isomorphism classes of simple objects. This matrix is required to be invertible.
Notice that, if $A$ were not merely braided but symmetric, then the $\sigma$’s and the $\sigma^{-1}$’s in this composite would cancel out, and we would have $s_{X,Y} = t_X t_Y$ where $t_X$ is the composite
\[
1 \xrightarrow{i_X} X \otimes X^* \xrightarrow{\delta_X \otimes I_{X^*}} X^{**} \otimes X^* \xrightarrow{\epsilon_{X^*}} 1,
\]
and similarly for $t_Y$. Thus, the matrix described in the modularity axiom would have rank only 1. By requiring this matrix to be invertible, the axiom says, in effect, that the braiding is as far as possible from being symmetric.

4. **Yoneda simplification**

In this section, we point out a simplification of the additive structure of $A$, based on Yoneda’s Lemma. That lemma (see [5, Section III.2]) says roughly that an object in any category is determined, up to isomorphism, by the morphisms into it. More precisely, any category $C$ is equivalent to a full subcategory of the category $\hat{C}$ of contravariant functors from $C$ to the category of sets. Under this equivalence, any object $X$ of $\hat{C}$ corresponds to the functor $\text{Hom}(-, X)$, i.e., the functor sending each object $U$ of $C$ to the set of morphisms $U \to X$ and sending each morphism $f : U \to V$ to the operation $\text{Hom}(V, X) \to \text{Hom}(U, X)$ of composition with $f$.

In the case of our category $A$, we can greatly simplify $\hat{A}$ while still maintaining the Yoneda equivalence. In the first place, since every object $U$ of $A$ is a finite sum, and thus in particular a coproduct, of simple objects, $U = \bigoplus_{j \in F} S_j$, morphisms $U \to X$ amount to $F$-indexed families of morphisms $S_j \to X$. More precisely, any $f : U \to X$ is determined by the composite morphisms $f \circ u_j : S_j \to X$, and, conversely, any family of morphisms $g_j : S_j \to X$ arises in this way from a unique morphism $U \to X$. Thus, $A$ is equivalent to a full subcategory of the category $\hat{S}$ of set-valued functors on the category $S$ of simple objects in $A$.

Up to equivalence, we need not use all the simple objects; it suffices to have at least one representative from each isomorphism class of simple objects. So we can replace the $S$ of the preceding paragraph by a skeleton of it, i.e., a full subcategory $S_0$ consisting of just one representative per isomorphism class.

The structure of this new, skeletal $S_0$ admits, thanks to the finiteness axiom and Proposition 8 the following description. There are finitely

5There are set-theoretic issues if $C$ is a proper class rather than a set, but these issues need not concern us here. The finiteness conditions imposed on our anyon category $A$ ensure that it is equivalent to a small, i.e., set-sized, category.
many objects. The morphisms from any object to itself form a copy of \( \mathbb{C} \). If \( U \) and \( V \) are distinct objects, then the only morphism from \( U \) to \( V \) is zero.

As a result, the Yoneda embedding, simplified as above, sends each object \( X \) of \( \mathcal{A} \) to a finite family of vector spaces, indexed by the simple objects \( U \) in \( S_0 \), namely the vector spaces \( \text{Hom}(U, X) \). Furthermore, the morphisms \( X \rightarrow Y \) in \( \mathcal{A} \) are given by arbitrary families of linear maps \( g_U : \text{Hom}(U, X) \rightarrow \text{Hom}(U, Y) \) between corresponding vector spaces. The reason for "arbitrary" is that, because of the paucity of morphisms in \( S_0 \), all such families automatically satisfy the commutativity conditions required in order to be natural transformations and thus to be morphisms in the functor category \( \hat{S}_0 \).

Summarizing, we have that, up to equivalence of categories, \( \mathcal{A} \) can be described as the category whose objects (resp. morphisms) are families of finite-dimensional vector spaces (resp. linear maps), indexed by the objects of \( S_0 \). Furthermore, it is easy to check that sums in \( \mathcal{A} \) are given, via this equivalence, by direct sums of vector spaces.

In other words, the additive structure of \( \mathcal{A} \) is trivial. The interesting structure is the monoidal structure, and this can be quite complicated. In particular, the associativity isomorphisms \( \alpha \) and the braiding isomorphisms \( \sigma \), though given (like any morphisms) by linear maps, need not have a particularly simple structure.

The analysis of the multiplicative structure of \( \mathcal{A} \) can be facilitated by taking advantage of the semisimplicity of \( \mathcal{A} \) and the fact that \( \otimes \) distributes over \( \oplus \). If we know how \( \otimes \) acts on simple objects, distributivity determines how it acts on sums of simple objects, and, by semisimplicity, those are all the objects. Moreover, because the associativity and braiding isomorphisms are natural, and thus in particular commute with the injection and projection morphisms of sums, the behavior of these isomorphisms on arbitrary objects is determined by their behavior on simple objects. Better yet, the pentagon and hexagon conditions will be satisfied in general as soon as they are satisfied for simple objects.

Thus, the additive and multiplicative structure of \( \mathcal{A} \) can be completely described by giving

1. a complete list of non-isomorphic simple objects (including the unit or vacuum \( 1 \)),
2. for each pair of objects in this list, their \( \otimes \)-product, expressed as a sum of objects from that list,
3. the associativity isomorphisms \( \alpha_{X,Y,Z} \) for all \( X, Y, Z \) in the list, and
(4) the braiding isomorphism $\sigma_{X,Y}$ for all $X,Y$ in the list, subject to the pentagon and hexagon conditions.

We shall not be concerned here with duality and ribbon structure, but it could also be reduced to a consideration of the simple cases.

Often, items (1) and (2) here determine or at least greatly constrain items (3) and (4) via the pentagon and hexagon conditions. One such situation is the subject of the next section and the appendix. Other examples, both of strong constraints on (3) and (4) and of weak constraints can be found in [2].

5. Fibonacci anyons

In this section, we consider the special case of Fibonacci anyons. These are defined by specifying the category $\mathcal{A}$ as follows. There are just two simple objects, 1 (the vacuum, the unit for $\otimes$) and $\tau$. Each is its own dual. The monoidal structure is given by $\tau \otimes \tau = 1 \oplus \tau$ (plus the fact that 1 is the unit, so $1 \otimes \tau = \tau \otimes 1 = \tau$ and $1 \otimes 1 = 1$).

The terminology “Fibonacci anyon” comes from the fact, easily verified using the distributivity of $\otimes$ over $\oplus$, that iteration of $\otimes$ gives $\tau \otimes^n \tau = f_{n-1} \cdot 1 \oplus f_n \cdot \tau$, where the $f$’s are the Fibonacci sequence defined by $f_{-1} = 1$, $f_0 = 0$, and $f_{n+1} = f_n + f_{n-1}$. Here and below, we use notation the notation $k \cdot S$ to mean the sum of $k$ copies of the object $S$ of $\mathcal{A}$.

As explained in Section 4, we can identify the category $\mathcal{A}$ with the category of pairs $(V_1, V_\tau)$ of finite-dimensional complex vector spaces. Explicitly, an object $X$ is identified with the pair $(\text{Hom}(1, X), \text{Hom}(\tau, X))$. In particular, the unit 1 in $\mathcal{A}$ is identified with $(\mathbb{C}, 0)$, and $\tau$ is identified with $(0, \mathbb{C})$. This identification respects the additive structure: $\oplus$ in $\mathcal{A}$ corresponds to componentwise direct sum of pairs of vector spaces.

The multiplicative structure of $\mathcal{A}$, on the other hand, is quite far from componentwise tensor product of vector spaces, as the latter would make $\tau \otimes \tau = \tau$. Our goal in the rest of this paper is to determine the multiplicative structure in terms of pairs of vector spaces.

The equation $\tau^n = f_{n-1} \cdot 1 \oplus f_n \cdot \tau$ mentioned above already determines that structure as far as the objects are concerned, but there remains much to be said about the morphisms.

A morphism from one pair of vector spaces $(V_1, V_\tau)$ to another such pair $(W_1, W_\tau)$ is a pair of linear transformations $(m_1 : V_1 \to W_1, m_\tau : V_\tau \to W_\tau)$. We can think of it as a pair of matrices, provided we fix bases for all the vector spaces involved here.

The choice of bases involves considerable arbitrariness, but there is a (somewhat) helpful guiding principle, namely that, if we have already
chosen bases for two vector spaces, then the union of those bases serves naturally as a basis for the direct sum of those vector spaces. Some caution is needed, though, because the same vector space can arise as a direct sum in several ways and can thus have several equally natural bases. Indeed, much of our work below will be finding the transformations that relate such bases.

The guiding principle tells us nothing about choosing bases for the one-dimensional spaces in the pairs \( 1 = (\mathbb{C}, 0) \) and \( \tau = (0, \mathbb{C}) \); there isn’t even any non-zero morphism between these simple objects to suggest a correlation between the choice of bases. We might as well identify (as our notation has already implicitly done) these one-dimensional spaces with \( \mathbb{C} \) and use the number 1 as the basis vector in both of them.

Then \( \tau \otimes \tau = 1 \oplus \tau = (\mathbb{C}, \mathbb{C}) \) already has a basis for each of the two vector spaces. Let us turn to the triple product

\[
\tau \otimes (\tau \otimes \tau) = \tau \otimes (1 \oplus \tau) = (\tau \otimes 1) \oplus (\tau \otimes \tau) = \tau \oplus (1 \oplus \tau) = 1 \cdot 1 \oplus 2 \cdot \tau.
\]

As a pair of vector spaces, it is \((\mathbb{C}, \mathbb{C}^2)\), but we have some additional information about it, namely that it was obtained as the sum of \( \tau \otimes 1 = \tau \) and \( \tau \otimes \tau = 1 \oplus \tau \). Our guiding principle thus suggests choosing a basis in \( \mathbb{C}^2 \) that respects this sum decomposition. That is, one of the basis vectors in \( \mathbb{C}^2 \) should come from the first \( \tau \) and the other should come from the second summand, \( 1 \oplus \tau \).

Consider, however, the analogous computation with the other way of parenthesizing the triple product:

\[
(\tau \otimes \tau) \otimes \tau = (1 \oplus \tau) \otimes \tau = (1 \otimes \tau) \oplus (\tau \otimes \tau) = \tau \oplus (1 \otimes \tau) = 1 \cdot 1 \oplus 2 \cdot \tau.
\]

It also leads to the pair of vector spaces \((\mathbb{C}, \mathbb{C}^2)\), and it also provides a suggestion for a basis of \( \mathbb{C}^2 \). There is, however, no guarantee that this suggestion agrees with the one in the preceding paragraph. In fact, we shall see below that the two suggestions are guaranteed to disagree. We have two bases for \( \mathbb{C}^2 \), and there will be a non-trivial matrix transforming the one into the other. In fact, we shall find that this matrix is almost uniquely determined.

In fact, there could, a priori, have been two different natural bases for the \( \mathbb{C} \) component in \( \tau \otimes \tau \), although we shall see that, in this particular situation, they coincide.

These basis transformation matrices, relating the bases that arise from \( \tau \otimes (\tau \otimes \tau) \) and from \( (\tau \otimes \tau) \otimes \tau \), amount to the associativity isomorphism \( \alpha_{\tau, \tau, \tau} \) in the definition of the monoidal category \( \mathcal{A} \).

Recall from Section 4 that all the associativity isomorphisms of \( \mathcal{A} \) are determined by those with simple objects as subscripts. One of these is the \( \alpha_{\tau, \tau, \tau} \) mentioned just above; the others involve one or more
1’s in the subscript. Fortunately, all those others are identity maps, thanks the identification of $1 \otimes X$ and $X \otimes 1$ with $X$. So the entire associativity structure of $\mathcal{A}$ comes down to two matrices, a $2 \times 2$ matrix relating the two bases for $\mathbb{C}^2$ and a number (a $1 \times 1$ matrix) relating the two bases for $\mathbb{C}$. These matrices are subject to the constraint given by the pentagon condition. In the appendix, we shall calculate that constraint explicitly. It will almost uniquely determine $\alpha$.

We shall also calculate the constraint imposed by the hexagon condition on the braiding isomorphisms $\sigma$. Again, the only component that needs to be computed is $\sigma_{\tau,\tau}$. The components where at least one subscript is 1 are trivial, and the components with non-simple objects as subscripts reduce, by distributivity, to ones with simple subscripts.

**APPENDIX A. FIBONACCI CALCULATIONS**

A.1. **Notation.** In order to compute the isomorphisms $\alpha_{\tau,\tau,\tau}$ and $\sigma_{\tau,\tau}$ for Fibonacci anyons, we shall view them as matrices, using suitable bases for the relevant vector spaces, and we shall calculate the constraints imposed on those matrices by the pentagon and hexagon conditions. We begin by setting up a convenient notation for those bases.

The domains and codomains of the morphisms under consideration are obtained from $\tau$ and 1 by iterated $\otimes$. We must, of course, be careful about the parenthesization of such $\otimes$-products because, as we saw in Section 5, different parenthesizations can lead to different bases; indeed, $\alpha_{\tau,\tau,\tau}$ contains exactly the information about how two such bases are related.

In general, given a parenthesized $\otimes$-product of $\tau$’s and 1’s, we can use the defining equations for Fibonacci anyons, particularly $\tau \otimes \tau = 1 \oplus \tau$, and the distributivity of $\otimes$ over $\oplus$, to convert the given product into a sum of $\tau$’s and 1’s. Each summand in that sum arises form the original product as a result of certain choices of 1 or $\tau$ when expanding some occurrences of $\tau \otimes \tau$.

For example, in the equation

$$\tau \otimes (\tau \otimes \tau) = \tau \otimes (1 \oplus \tau) = (\tau \otimes 1) \oplus (\tau \otimes \tau) = \tau \oplus (1 \oplus \tau) = 1 \cdot 1 \oplus 2 \cdot \tau$$

considered in Section [5], the summand 1 on at the right end of the equation arose from the $\tau \otimes (\tau \otimes \tau)$ at the left end by first choosing the summand $\tau$ in the evaluation of $(\tau \otimes \tau)$ at the first step in the equation, and then, after applying the distributive law at the second step, choosing the summand 1 in the evaluation of $\tau \otimes \tau$ at the third step. We shall indicate this sequence of choices by the notation

$$(\tau \cdot (\tau \cdot \tau))_1.$$
Here the three $\tau$’s and the parentheses describe the $\otimes$-product $\tau \otimes (\tau \otimes \tau)$ that we began with, and the symbols under the dots indicate the choice of summand at each step. The inner $\cdot$ indicates that, from the evaluation of the inner $\tau \otimes \tau = 1 \oplus \tau$, we chose the $\tau$ summand. After applying distributivity, that leads us to $\tau \otimes \tau$, from which, as indicated by the outer $\cdot_1$, we chose the summand 1.

The other possible choices during the same evaluation would be written $\tau \cdot_\tau (\tau \cdot_\tau (\tau \cdot_\tau \tau))$ and $\tau \cdot_\tau (\tau \cdot_\tau (\tau \cdot_1 \tau))$.

The first of these indicates that, as before, we chose the $\tau$ summand when evaluating the inner $\oplus$, obtaining, when distributivity is applied, the summand $\tau \otimes \tau = 1 \oplus \tau$, but then we chose the $\tau$ rather than the 1. The second indicates that, when evaluating the inner $\tau \otimes \tau$, we chose the summand 1, so that, after applying distributivity, we got $\tau \otimes 1$. Here, there is no choice remaining to be made; $\tau \otimes 1$ is simply $\tau$.

Nevertheless, we write $\tau$ under the outer dot, to make it obvious that the final result here is $\tau$.

In general, we use this sort of notation, a product of $\tau$’s or 1’s with $\tau$’s or 1’s also written under the dots, to represent specific summands (1 or $\tau$) in the fully distributed expansion of a $\otimes$-product of $\tau$’s and 1’s. To evaluate $\left( X \cdot_1 1 \cdot Y \right)$, first evaluate $X$ and $Y$; then apply $\otimes$ to them; and then take the 1 summand in the result. To evaluate $\left( X \cdot_\tau 1 \cdot Y \right)$ do the same except that you take the $\tau$ summand in the result. These notations will never be used in situations where they would be meaningless because the required summand is not present in the result; that is, we never write $\left( X \cdot_1 1 \right)$ when one of $X, Y$ evaluates to 1 and the other to $\tau$, for then $\oplus$ yields only $\tau$; and we never write $\left( X \cdot_\tau 1 \right)$ when both of $X, Y$ evaluate to 1. As in one of the examples above, we include the subscript under the $\cdot$ even when that subscript is forced because one of the factors evaluates to 1.

Notice that our notation provides symbols, like the three examples above, that denote not only an object 1 or $\tau$ (which can be read off by just looking under the outermost dot in the notation) but also a particular occurrence of that 1 = (C, 0) or $\tau$ = (0, C) as a subspace (direct summand) of a specific $\otimes$-product, namely the product with the same factors and the same parentheses as in our notation.

In other words, if we are given a parenthesized $\otimes$-product of 1’s and $\tau$’s, representing the pair of vector spaces $(V_1, V_\tau)$, then by replacing each $\otimes$ by either $\cdot_1$ or $\cdot_\tau$, we obtain (either a meaningless expression
because some required summand is absent or) a notation for a subspace of $V_1$ or $V_\tau$. It denotes a subspace of $V_1$ (resp. $V_\tau$) just in case the outermost $\oplus$ was replaced by $\cdot$ (resp. $\cdot_\tau$).

Remark 5. The reader with an ample supply of paper may find it useful to replace our notations with their parse trees. These can be read as a sort of Feynman diagram. For example $(\tau \cdot \tau)$ depicts two $\tau$’s fusing to one, while $(\tau_1 \cdot \tau)$ depicts two $\tau$’s fusing to form a vacuum (i.e., annihilating). In fact, we originally did the calculations reported here using trees. The present notation is designed to simulate these trees without taking up so much space.

Our notation provides names for certain summands $1 = (C, 0)$ or $\tau = (0, C)$ of certain objects $(V_1, V_\tau)$ of the Fibonacci category $\mathcal{A}$. We shall also use the same notation for the resulting basis vectors. That is, once we have a copy of, say, $(C, 0)$ in $(V_1, V_\tau)$, the number 1 in $C$ corresponds to some vector in $V_1$, and we shall use the same notation for this vector as for the summand. The same goes for the case of copies of $(0, C)$ in $(V_1, V_\tau)$; they provide vectors in $V_\tau$.

Notice that, if we begin with some parenthesized $\otimes$-product of 1’s and $\tau$’s, with value $(V_1, V_\tau)$ in $\mathcal{A}$, and if we form all possible (meaningful) notations by replacing $\otimes$ by $\cdot_1$ or $\cdot_\tau$, then the resulting vectors, as described in the preceding paragraph, constitute bases for the vector spaces $V_1$ and $V_\tau$. This observation is just a restatement of the fact that the original parenthesized $\otimes$-product is the direct sum of all the simple objects obtainable by making the choices indicated by the subscripts in our notation.

A.2. Associativity. Now that we have a general notation system for the basis vectors in parenthesized $\otimes$-products, we turn to the specific cases involved in associativity and the pentagon condition.

The unique “interesting” component of associativity, $\alpha_{\tau,\tau,\tau}$, which we sometimes abbreviate as simply $\alpha$, is an isomorphism from $(\tau \otimes \tau) \otimes \tau$ to $\tau \otimes (\tau \otimes \tau)$, both of which are, as pairs of vector spaces, a 1-dimensional $V_1$ and a 2-dimensional $V_\tau$. The first parenthesization gives a basis vector

$$(\tau \cdot \tau) \cdot_1 \tau \quad \text{for } V_1$$

and two basis vectors

$$(\tau_1 \cdot \tau) \cdot \tau \quad \text{and} \quad ((\tau \cdot \tau) \cdot \tau) \quad \text{for } V_\tau.$$
The second parenthesization similarly gives a basis vector
\[(\tau_1 \cdot (\tau \cdot \tau))\] for \(V_1\)
and two basis vectors
\[(\tau \cdot (\tau_1 \cdot \tau)) \quad \text{and} \quad (\tau \cdot (\tau_1 \cdot \tau))\] for \(V_\tau\).

Our task is to compute the transformation \(\alpha\) between these bases. Thus, \(\alpha\) consists of a non-zero number \(p\) such that
\[((\tau \cdot \tau)_1 \cdot \tau) = p(\tau_1 \cdot (\tau \cdot \tau))\]
and a non-singular matrix \(\begin{pmatrix} q & r \\ s & t \end{pmatrix}\) such that
\[((\tau \cdot \tau)_1 \cdot \tau) = q(\tau \cdot (\tau_1 \cdot \tau)) + r(\tau \cdot (\tau \cdot \tau))\]
\[((\tau \cdot \tau)_1 \cdot \tau) = s(\tau \cdot (\tau \cdot \tau)) + t(\tau \cdot (\tau \cdot \tau)).\]

Here “non-zero” for \(p\) and “non-singular” for the matrix embody the requirement that \(\alpha\) is an isomorphism.

We shall now investigate the constraints imposed on \(p, q, r, s, t\) by the pentagon condition. That condition involves the \(\otimes\)-product of four \(\tau\)’s, parenthesized in five ways, and we shall need to consider the natural bases for all five parenthesizations. Since \(\tau^{\otimes 4} = (\mathbb{C}^2, \mathbb{C}^3)\), each parenthesization will give two vectors as a basis for the 1 component and three as a basis for the \(\tau\) component. We begin by considering the \(\tau\) components, whose bases are displayed below:

- \(((\tau \cdot \tau)_1 \cdot \tau) \cdot \tau\)
- \(((\tau \cdot \tau)_1 \cdot \tau) \cdot (\tau \cdot \tau)\)
- \(((\tau \cdot \tau)_1 \cdot (\tau \cdot \tau)) \cdot (\tau \cdot \tau)\)
- \(((\tau \cdot \tau)_1 \cdot (\tau \cdot \tau)) \cdot (\tau \cdot (\tau \cdot \tau))\)
- \(((\tau \cdot (\tau \cdot \tau)) \cdot (\tau \cdot (\tau \cdot \tau)) \cdot (\tau \cdot (\tau \cdot \tau)) \cdot (\tau \cdot (\tau \cdot \tau))\)

Each row in this picture is a basis for the 3-dimensional \(V_\tau\); specifically, it is the basis arising from the same parenthesization of \(\tau \otimes \tau \otimes \tau \otimes \tau\) as the parenthesization in our notation.

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\(\text{6}^{\text{We have chosen to regard } V_1 \text{ and } V_\tau \text{ as each being a single space, independent of the parenthesization. The different parenthesizations give (possibly) different bases for these spaces. An alternative view is that each parenthesization gives its own } V_1 \text{ and } V_\tau, \text{ isomorphic to } \mathbb{C}^2 \text{ and } \mathbb{C}^3 \text{ respectively, with their standard bases, while } \alpha \text{ gives an isomorphism between the two } V_1 \text{’s and an isomorphism between the two } V_\tau \text{’s. The two viewpoints are easily intertranslatable and the computations that follow would be the same in either picture.}}\)
When writing transformation matrices between these bases, we must regard each basis as given in a specific order, because rows of a matrix come in an order. We (arbitrarily) choose the orders in which the bases are displayed above.

The five isomorphisms that appear in the pentagon condition amount to five transformations between these bases. Let us consider these one at a time, beginning with the one connecting the first two bases in this table. Here we are dealing with the isomorphism

\( \alpha_{\tau \otimes \tau, \tau} : \((\tau \otimes \tau) \otimes \tau \otimes \tau) \rightarrow ((\tau \otimes \tau) \otimes (\tau \otimes \tau)). \)

The first subscript of this \( \alpha \), namely \( \tau \otimes \tau \), can be decomposed as the sum \( 1 \oplus \tau \), and the naturality of \( \alpha \) then implies that \( \alpha_{\tau \otimes \tau, \tau} \) is the direct sum of \( \alpha_{1, \tau, \tau} \) and \( \alpha_{\tau, \tau, \tau} \). The first of these two summands is the identity, like all associativity isomorphisms where one of the three factors is \( 1 \).

The second summand is given by our matrix \( \begin{pmatrix} 0 & q & r \\ 1 & 0 & 0 \\ 0 & s & t \end{pmatrix} \). As a result, we find that the transformation \( \alpha_{\tau \otimes \tau, \tau} \) connecting the first two bases in our list is (taking into account the order in which the basis vectors are listed)

\[ \alpha_{\tau \otimes \tau, \tau} = \begin{pmatrix} 0 & q & r \\ 1 & 0 & 0 \\ 0 & s & t \end{pmatrix}. \]

An exactly analogous computation gives the isomorphism between the second and the last bases in our list:

\[ \alpha_{\tau, \tau, \tau \otimes \tau} = \begin{pmatrix} q & 0 & r \\ 0 & 1 & 0 \\ s & 0 & t \end{pmatrix}. \]

Multiplying these two matrices, we get the transformation from the first basis (parenthesized to the left) and the last (parenthesized to the right) that corresponds to the “short” side of the pentagon (two morphisms). This product is

\[ \begin{pmatrix} rs & q & rt \\ q & 0 & r \\ st & s & t^2 \end{pmatrix}. \]

Turning to the long side of the pentagon (three morphisms), we find that the middle one, corresponding to rows 3 and 4 in our list of bases, is quite analogous to the two that we have already computed. It is

\[ \alpha_{\tau, \tau \otimes \tau, \tau} = \begin{pmatrix} q & 0 & r \\ 0 & 1 & 0 \\ s & 0 & t \end{pmatrix}. \]
The remaining two isomorphisms for the long side of the pentagon are a bit different, as they involve $\alpha$’s on three of the four factors and an identity map on the remaining factor. Let us consider $\alpha_{\tau,\tau,\tau} \otimes I_\tau$, which connects the first basis in our list to the third. In effect, this ignores the rightmost factor and acts like $\alpha$ on the first three factors. In other words, it is given by the same matrix as the transformation from the basis
\[
((\tau \cdot \tau) \cdot \tau_1) \quad ((\tau \cdot \tau) \cdot \tau) \quad ((\tau \cdot \tau) \cdot \tau)
\]
to the basis
\[
(\tau \cdot (\tau \cdot \tau)) \quad (\tau \cdot (\tau \cdot 1)) \quad (\tau \cdot (\tau \cdot \tau)).
\]
Notice that, in each of these bases the first element is in the $V_1$ component, so that component of $\alpha$, namely $p$, enters the picture. Indeed, the matrix connecting these bases is
\[
\alpha_{\tau,\tau,\tau} \otimes I_\tau = \begin{pmatrix} p & 0 & 0 \\ 0 & q & r \\ 0 & s & t \end{pmatrix}.
\]
Similarly, the remaining isomorphism on the long side of the pentagon is also
\[
I_\tau \otimes \alpha_{\tau,\tau,\tau} = \begin{pmatrix} p & 0 & 0 \\ 0 & q & r \\ 0 & s & t \end{pmatrix}.
\]
Multiplying the three matrices for the long side of the pentagon, and equating, as the pentagon condition requires, the resulting product to the product that we obtained for the short side of the pentagon, we have
\[
\begin{pmatrix} p^2q & prs & p^2rt \\ prs & q^2 + rst & qr + rt^2 \\ pst & QS + ST^2 & RS + T^3 \end{pmatrix} = \begin{pmatrix} rs & q & rt \\ q & 0 & r \\ st & s & t^2 \end{pmatrix}.
\]
This is the $V_\tau$ part of the pentagon condition. Before turning to the $V_1$ part, let us extract as much information as possible from the matrix equation that we have just derived.

Suppose, toward a contradiction, that $p \neq 1$. Then the $(1,3)$ and $(3,1)$ components of our matrix equation give $rt = st = 0$, so either $r = s = 0$ or $t = 0$. If $r = s = 0$, then the $(1,2)$ component of the matrix equation gives that $q = 0$ also, but this contradicts the fact that \[\begin{pmatrix} q \\ r \\ s \\ t \end{pmatrix}\] is non-singular. There remains the case that $t = 0$. Then the $(2,2)$ component says $q = 0$, the $(2,3)$ component says $r = 0,$
and we again contradict the non-singularity of \( \begin{pmatrix} q & r \\ s & t \end{pmatrix} \). So we have contradictions in all cases if \( p \neq 1 \).

So \( p = 1 \). Now the (1,1) entry of the matrix equation gives \( q = rs \). Substituting that into the (2,2) component, we get \( q(q + t) = 0 \), so either \( q = 0 \) or \( q = -t \). The first of these options leads, via the (1,2) entry, to \( rs = 0 \) and thus to a contradiction to non-singularity, as before. Therefore \( q = -t \).

From the (2,3) and (3,2) entries, we get that \( (q + t^2)r = r \) and \( (q + t^2)s = s \). We cannot have both \( r = 0 \) and \( s = 0 \), as that would give \( q = 0 \) in the (1,2) entry and contradict non-singularity. So we must have \( q + t^2 = 1 \). In view of \( q = -t \), this means \( q^2 + q - 1 = 0 \) and therefore

\[
q = -t = -\frac{1 \pm \sqrt{5}}{2}.
\]

This evaluation of \( q \) and \( t \), together with the earlier results

\[ p = 1 \quad \text{and} \quad rs = q, \]

satisfy, as one easily checks, the entire matrix equation above. The least trivial item to check is the (3,3) entry, \( rs + t^3 = t^2 \), which, in view of the equations above, becomes \( q - q^3 = q^2 \), i.e., \( 0 = q(q^2 + q - 1) \), and this is true because \( q \) was obtained as a solution of \( q^2 + q - 1 = 0 \).

All of the preceding calculation was based on the \( V_\tau \) component of the \( \tau^\otimes 4 \); we still have the \( V_1 \) component of the pentagon equation to work out. Again, we have a list of five bases, now for a 2-dimensional space, as follows.

\[
((\tau \cdot \tau) \cdot \tau) \cdot \tau \quad ((\tau \cdot \tau) \cdot \tau) \cdot \tau \\
((\tau \cdot \tau) \cdot (\tau \cdot \tau)) \quad ((\tau \cdot \tau) \cdot (\tau \cdot \tau)) \\
((\tau \cdot (\tau \cdot \tau)) \cdot \tau) \quad ((\tau \cdot (\tau \cdot \tau)) \cdot \tau) \\
(\tau \cdot ((\tau \cdot \tau) \cdot \tau)) \quad (\tau \cdot ((\tau \cdot \tau) \cdot \tau)) \\
(\tau \cdot (\tau \cdot (\tau \cdot \tau))) \quad (\tau \cdot (\tau \cdot (\tau \cdot \tau)))
\]

Computations analogous to (but shorter than) the earlier ones give, for the short side of the pentagon,

\[
\alpha_{\tau \otimes \tau, \tau, \tau} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \quad \text{and} \quad \alpha_{\tau, \tau, \tau \otimes \tau} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.
\]
So the product for the short side is simply \( \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \). For the long side, we get

\[
\alpha_{\tau,\tau,\tau} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}
\]

and

\[
\alpha_{\tau\tau\tau} \otimes I_\tau = I_\tau \otimes \alpha_{\tau\tau\tau} = \begin{pmatrix} q & r \\ s & t \end{pmatrix}.
\]

Equating the product of the long side and the product of the short side, we get

\[
\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} q^2 + prs & qr + ptr \\ qs + pts & rs + pt^2 \end{pmatrix}.
\]

This matrix equation is automatically satisfied because of the equations that we had already derived from the \( V_\tau \) component of the pentagon condition. So there is no new information in the \( V_1 \) component.

We can, however, get some additional information if we impose the requirement that the associativity isomorphisms be unitary transformations. This amounts to requiring the vector spaces of morphisms \( \text{Hom}(X,Y) \) to be Hilbert spaces and requiring our natural bases for them to be orthonormal.

Unitarity tells us nothing new about \( p \), since we already know \( p = 1 \), but unitarity of \( \begin{pmatrix} q & r \\ s & t \end{pmatrix} \) gives the equations

\[
q^2 + |r|^2 = q^2 + |s|^2 = 1 \quad \text{and} \quad q(\bar{s} - r) = q(s - \bar{r}) = 0,
\]

where bars denote complex conjugation and where we used the fact that \( q \) is real. So \( s = \bar{r} \) and, since \( rs = q \), we get first that \( q \) has to be positive,

\[
q = \frac{-1 + \sqrt{5}}{2},
\]

and second that

\[
r = \sqrt{q} e^{i\theta} \quad \text{and} \quad s = \sqrt{q} e^{-i\theta}
\]

for some real \( \theta \). Thus, we finally have, under the assumption of unitarity,

\[
\alpha_{\tau,\tau,\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & \sqrt{q} e^{i\theta} \\ 0 & \sqrt{q} e^{-i\theta} & -q \end{pmatrix}
\]

with \( q = \frac{-1 + \sqrt{5}}{2} \) and \( \theta \) arbitrary. We could eliminate \( \theta \) by redefining the phases of some of our basis vectors.
A.3. **Braiding.** We now turn to the task of computing the braiding $\sigma$ in the Fibonacci anyon category $\mathcal{A}$. The only nontrivial component of the natural isomorphism $\sigma$ is $\sigma_{\tau,\tau}$, because components with a subscript 1 are identity morphisms and components with non-simple subscripts reduce to direct sums of components with simple subscripts.

The nontrivial component $\sigma_{\tau,\tau}$ is an isomorphism from $\tau \otimes \tau = 1 \oplus \tau$ to itself. Representing objects of $\mathcal{A}$ by pairs of vector spaces, $\sigma_{\tau,\tau}$ is an automorphism of $(\mathbb{C}, \mathbb{C})$, so it amounts to two non-zero scalars, $a$ multiplying vectors in the first (1) component and $b$ multiplying vectors in the second ($\tau$) component. These are subject to the hexagon identity, which equates the composites

$$
(\tau \otimes \tau) \otimes \tau \xrightarrow{\alpha_{\tau,\tau}} \tau \otimes (\tau \otimes \tau) \xrightarrow{\sigma_{\tau,\tau} \otimes \tau} (\tau \otimes \tau) \otimes \tau \xrightarrow{\alpha_{\tau,\tau}} \tau \otimes (\tau \otimes \tau)
$$

and

$$
(\tau \otimes \tau) \otimes \tau \xrightarrow{\sigma_{\tau,\tau} \otimes I_\tau} (\tau \otimes \tau) \otimes \tau \xrightarrow{\alpha_{\tau,\tau} \otimes \tau} \tau \otimes (\tau \otimes \tau) \xrightarrow{I_\tau \otimes \sigma_{\tau,\tau}} \tau \otimes (\tau \otimes \tau),
$$

as well as the analogous identity with $\sigma^{-1}$ in place of $\sigma$.

Consider the first (1) component of this equation. In the bottom composition, the $\sigma_{\tau,\tau}$ factors in the first and third morphisms must act on the $\tau$ components so that the $\otimes$-product with $I_\tau$ has a 1 component. So both of these are $b$. The $\alpha$ between them, acting on the 1 component, is an identity map, because our previous calculation gave $p = 1$. So the bottom of the hexagon is $b^2$. In the top, both of the $\alpha$’s are again just 1. The $\sigma$ in the middle of that row is $\sigma_{\tau,1 \oplus \tau}$, i.e., the direct sum of $\sigma_{\tau,1}$ and $\sigma_{\tau,\tau}$. The first of these two summands has no 1 component; the second does, and it is $a$. So the top of the hexagon is just $a$, and the hexagon condition reads $a = b^2$. (The corresponding calculation for $\sigma^{-1}$ gives only $a^{-1} = b^{-2}$, which is no new information.)

Now consider the second ($\tau$) component of the hexagon equation. We do the calculation in matrix form, using the natural bases

$$
((\tau \cdot \tau) \cdot \tau) \text{ and } (\tau \cdot (\tau \cdot \tau) \cdot \tau) \text{ for } (\tau \otimes \tau) \otimes \tau
$$

and

$$
(\tau \cdot (\tau \cdot \tau)) \text{ and } (\tau \cdot (\tau \cdot \tau)) \text{ for } \tau \otimes (\tau \otimes \tau).
$$

With respect to these bases, $\alpha_{\tau,\tau,\tau}$ is given by $\begin{pmatrix} q & r \\ s & t \end{pmatrix}$ as computed earlier. Both $\sigma_{\tau,\tau} \otimes I_\tau$ and $I_\tau \otimes \sigma_{\tau,\tau}$ are given by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, because in each case, $\sigma_{\tau,\tau}$ acts as $a$ on the first basis vector (where it interchanges two $\tau$’s that were combined to 1) and as $b$ on the second
(where it interchanges two $\tau$'s that were combined to $\tau$). Finally, $\sigma_{\tau,\tau\otimes\tau}$ is the direct sum of $\sigma_{\tau,1}$ which is 1 and $\sigma_{\tau,\tau}$ acting on the $\tau$ component, which is $b$; since that direct sum decomposition matches our choice of bases, $\sigma_{\tau,\tau\otimes\tau}$ is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$. Multiplying the matrices for each of the rows, we find that the hexagon identity, in the $\tau$ component, reads

$$\begin{pmatrix} q^2 + brs & (q + bt)r \\ (q + bt)s & rs + bt^2 \end{pmatrix} = \begin{pmatrix} b^4q & b^3r \\ b^3s & b^2t \end{pmatrix}. $$

Since we know, from our associativity calculation, that $r$ and $s$ are not zero, the (1,2) and (2,1) entries of this matrix equation reduce to $q + bt - b^3$, or, since $t = -q$,

$$b^3 = q(1-b).$$

The (1,1) and (2,2) entries give, after we remember that $rs = q$ and cancel a common factor $q$,

$$q + b = b^4 \quad \text{and} \quad 1 + bq + b^2 = 0.$$

The last of these equations, being quadratic in $b$, can be solved explicitly:

$$b = \frac{-q \pm \sqrt{q^2 - 4}}{2}.$$

We note that, since $q = \frac{\sqrt{5} - 1}{2}$ is between 0 and 1, the square root in the formula for $b$ is imaginary, so the two values of $b$ are each other’s complex conjugates. The product of the two values for $b$ is 1, so $b$ is a complex number of absolute value 1 with real part $\frac{-q}{2}$.

The ambiguity in the choice of $b$ is unavoidable in this situation. Replacing one choice by the other just replaces $\sigma$ by its inverse (since $|b| = 1$), and there is nothing in the algebra of $\mathcal{A}$ that distinguishes the counterclockwise motion defining $\sigma$ from the clockwise motion defining $\sigma^{-1}$. To put it another way, the change from one value of $b$ to the other can be exactly compensated by reflecting the orientation of the (2-dimensional) space in which the anyons live.

Although we have now computed $b$ and thus also $a = b^2$, we can get a more useful view of these numbers by manipulating the three equations above that relate $b$ to $q$. Solving the last one for $q$ in terms of $b$, and substituting the result, $q = \frac{b^3 - 1}{b}$ into the other two equations, we obtain from the first equation that

$$b^3 = \frac{b^3 - b^2 + b - 1}{b}, \text{ i.e., } b^4 - b^3 + b^2 - b + 1 = 0,$$
which means that $-b$ is a primitive fifth root of unity and therefore $b$ is a primitive tenth root of unity. The third equation above confirms that by reducing to $b^5 = -1$.

Among the four primitive tenth roots of unity only two, $e^{\pm 3\pi i/5}$, have negative real parts, as $b$ does (recall that its real part is $-q/2$). So we conclude that, up to complex conjugations,

$$b = e^{3\pi i/5} \text{ and therefore } a = e^{6\pi i/5}.$$  

This completes the calculation of the braiding $\sigma$ for Fibonacci anyons.

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Mathematics Department, University of Michigan, Ann Arbor, MI 48109–1043, U.S.A.

E-mail address: abliss@umich.edu

Microsoft Research, One Microsoft Way, Redmond, WA 98052, U.S.A.

E-mail address: gurevich@microsoft.com