Photon directional profile from stimulated decay of axion clouds with arbitrary axion spatial distributions

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(Dated: April 29, 2020)

We model clusters of axions with spherically symmetric momentum but arbitrary spatial distributions and study the directional profile of photons produced in their evolution through spontaneous and stimulated decay of axions via the process \(a \rightarrow \gamma + \gamma\). Several specific examples are presented.

INTRODUCTION

Axions are copiously produced at the QCD phase transition. A possible way to detect these cosmological axions is through the observation of lasing axion clouds (clumps). If axions are a component of the cold dark matter (CDM), they can form density perturbations in the early Universe. If the over dense regions have high enough number density, then ambient photons from the cosmic microwave background (CMB) or from spontaneous axion decays, can induce stimulated axion decay within the clumps, i.e., the axions can lase \(^1\).

Besides the initial clumps, other axion structures can form. The initial density perturbations can infall and evolve to form caustics \(^2\) which have complicated geometries. Yet another possibility is that axions can be produced after the formation of primordial black holes (PBHs). Such black holes can be the results of various early universe processes, from cosmic string or domain wall singularities to density perturbations. However they for, if they have sufficient angular, either initially or from mergers, then superradiance can occur causing axions to populate an \(n, l, m = 2, 1, 1\) hydrogen-like orbit around them if the axion Compton wavelength is comparable to the PBHs’ radius. If the axion density is high enough they can lase \(^3\). The process can saturate, stop and then repeat in what is similar to what has been seen for fast radio bursts (FRBs).

Lasing in the PBH superradiance case has so far only been approximated using the spherically symmetric model \(^4\). In this work and in \(^5\) we point the way to an improving this approximation using multipoles expansions of the spatial and momentum space distributions to more closely represent the physical axion distributions expected around a PBH.

PHOTON ANGULAR DISTRIBUTION

In \(^1\) nonrelativistic axions of mass \(m_a\) were contained in a ball of radius \(R\), with a maximum momentum value of \(p_{\text{max}} \approx m_a \beta\). Here we allow a non spherically symmetric spatial distribution \(X(\theta, \phi)\) to modify the axion clouds model previously studied, with the aim of finding the angular distribution \(Y(\theta, \phi)\) of photons resulted from decays of axions, providing that there is some outside constraint (e.g., a gravitational field or self interactions) that can keep the axions in the initial spatial distribution. For such an axion distribution, assuming it factorizes, the occupation number \(f_a(p, r, \theta, t)\) and number densities \(n_a(r, \theta, t)\) can be written

\[
f_a(p, r, \theta, t) = f_{\text{ac}}(t) \Theta(p_{\text{max}} - p) \Theta(R - r) X(\theta)
\]

and

\[
n_a(r, \theta, t) = \int \frac{d^3p}{(2\pi)^3} f_a(p, r, \theta, t)
\]

\[
= \frac{m_a^3 \beta^3}{6\pi^2} f_{\text{ac}}(t) \Theta(R - r) X(\theta)
\]

\[
= n_{\text{ac}}(t) \Theta(R - r) X(\theta)
\]

where we can translate between the two with

\[
f_{\text{ac}}(t) = \frac{6\pi^2}{m_a^3 \beta^3} n_{\text{ac}}(t).
\]

Here and elsewhere we use the short hand notation \(X(\theta)\) for \(X(\theta, \phi)\), likewise for \(Y, f\) and \(n\).

The photons are contained in a ball of radius \(R\), a momentum spherical shell of inner and outer radius \(k_- = m_a \beta^2 (1 - \beta)\) and \(k_+ = m_a \beta^2 (1 + \beta)\) respectively \(^5\), where we use \(\beta = v/c\).

\[
f_{\lambda}(k, r, \theta, t) = f_{\text{ac}}(t) \Theta(k_+ - k) \Theta(k - k_-) \Theta(R - r) Y(\theta)
\]

and

\[
n_{\lambda}(r, \theta, t) = \int \frac{d^3k}{(2\pi)^3} f_{\lambda}(k, r, \theta, t)
\]

\[
= f_{\text{ac}}(t) \Theta(R - r) Y(\theta) \frac{V_k}{8\pi^3}
\]

\[
= n_{\text{ac}}(t) \Theta(R - r) Y(\theta)
\]

with

\[
f_{\text{ac}}(t) = \frac{8\pi^2}{m_a^3 \beta^3} n_{\text{ac}}(t)
\]
where $f_\lambda(k, r, \theta, t)$ and $n_\lambda(r, \theta, t)$ are the photon occupation number and photon number density, of helicity $\lambda = \pm 1$ respectively, which are related by eq. (3) and $V_k$ is the volume of the momentum spherical shell

$$V_k = \int_{k_-}^{k_+} d^3k = 4\pi \left(\frac{m_\gamma}{2}\right)^2 (k_+ - k-) = \pi m_\gamma^2 \beta \gamma^3 \approx \pi m_\gamma^3 \beta.$$

We assume that the number density of each helicity state is the same, so the total photon number density $n_\gamma$ can be written as

$$n_\gamma(r, \theta, t) = n_{\gamma c}(t)\Theta(R - r)Y(\theta) = n_+(r, \theta, t) + n_-(r, \theta, t) = [n_{+c}(t) + n_{-c}(t)]\Theta(R - r)Y(\theta)$$

$$n_{+c}(t) = n_{-c}(t) = 2n_{\gamma c}(t)$$

which defines $n_{\gamma c}$. Hence the coefficient of the total photon number density is just 2 times that of photon number density of each helicity state.

The evolution relation between axion and photon occupation numbers is (see equation (13) of [2])

$$\frac{df_\lambda(k)}{dt} = \frac{m_\lambda \Gamma_a}{k^2} \int_{\frac{k}{2}}^{\frac{k}{2}} dk_1 \times \{ f_\lambda(k + k_1)[1 + f_\lambda(k)] - f_\lambda(k)f_\lambda(k_1) \}$$

where $f_\lambda(k)$ and $f_\lambda(k_1)$ are photon occupation numbers of momentum $k$ and $k_1$, respectively. Other variables in $f_\lambda(k)$ and $f_\lambda(k_1)$, i.e. $r, \theta, t$, are the same since they share the same spacetime. $f_\lambda(k + k_1)$ is the axion occupation number of momentum $k + k_1$. $\Gamma_a$ is the spontaneous axion decay rate.

This evolution equation can be integrated over $k$ and $k_1$ phase space to yield (see the Appendix)

$$\frac{dn_\lambda}{dt} = \Theta(R - r)\frac{m_\lambda^3 \Gamma_a}{8\pi^2} \left\{ [f_{ac}(x + 2f_{\lambda c} xy) - f_{\lambda c} y^2] \times [2\gamma^2 \beta - \ln \left( \frac{1 + \beta}{1 - \beta} \right) - f_{\lambda c} y^2 \times (2\gamma^2 \beta^2) \right\}$$

(9)

Now we employ the nonrelativistic approximation ($\beta \ll 1$).

$$2\gamma^2 \beta = \frac{2\beta}{1 - \beta^2} \approx 2\beta(1 + \beta^2) = 2\beta + 2\beta^3$$

$$\ln \left( \frac{1 + \beta}{1 - \beta} \right) \approx 2\beta + \frac{2\beta^3}{3}$$

$$2\gamma^2 \beta^2 \approx 2\beta^2$$

to arrive at

$$\frac{dn_\lambda}{dt} = \Theta(R - r)\frac{m_\lambda^3 \Gamma_a \beta^2}{6\pi^2} \left\{ f_{ac}(X + 2f_{\lambda c} XY)Y(\theta) - (\beta + \frac{3}{2})f_{\lambda c} Y^2 \right\}$$

Substituting the derived relations (3) and (9) into (4)

$$\frac{dn_\lambda}{dt} = \Gamma_a \Theta(R - r)$$

$$\times \left[ n_{ac}(X + \frac{16\pi^2 n_{\lambda c}}{\beta m_\lambda^3} XY) - \frac{32\pi^2 n_{\lambda c}^2}{3m_\lambda^3} (\beta + \frac{3}{2})Y^2 \right].$$

Taking into consideration photon surface loss

$$\left( \frac{dn_\lambda}{dt} \right)_{\text{surface loss}} = -\frac{3cn_{\lambda c}}{2R} = -\frac{3c}{2R} n_{\lambda c} \Theta(R - r)Y(\theta),$$

we have an equation which gives the number density for each helicity state

$$\frac{dn_\lambda}{dt} = \Theta(R - r) \times \left[ \frac{n_{ac}}{\tau_a} X(\theta) + \frac{16\pi^2 n_{ac} n_{\lambda c}}{\beta m_\lambda^3 \tau_a} X(\theta)Y(\theta) \right.$$

$$- \frac{32\pi^2 n_{\lambda c}^2}{3m_\lambda^3 \tau_a} (\beta + \frac{3}{2})Y^2(\theta) - \left. \frac{3cn_{\lambda c}}{2R} Y(\theta) \right].$$

where we are assuming, as was shown in (7), that total number density of photon is twice that of the individual helicity states. Therefore the rate of change of total number density of photon is

$$\frac{dn_\gamma}{dt} = \frac{dn_{\gamma c}}{dt} \Theta(R - r)Y(\theta),$$

if we drop the step function $\Theta(R - r)$ we have an equation for the coefficient of total number density of photon

$$\frac{dn_{\gamma c}}{dt} = 2 \frac{n_{ac}}{\tau_a} Y(\theta) + \frac{16\pi^2 n_{ac} n_{\lambda c}}{\beta m_\lambda^3 \tau_a} X(\theta)Y(\theta)$$

$$- \frac{16\pi^2 n_{\lambda c}^2}{3m_\lambda^3 \tau_a} (\beta + \frac{3}{2})Y^2(\theta) - \frac{3cn_{\lambda c}}{2R} Y(\theta).$$

(10)

From the first to the last term on the right hand side(RHS) of the equation, the terms account for spontaneous decay of axions, photon stimulated decay of axions, back reaction of photons, and surface loss of photons, respectively. Following similar approach, we obtain an equation regarding the coefficient of total number density of axions

$$\frac{dn_{ac}}{dt} = - \frac{n_{ac}}{\tau_a} Y(\theta) - \frac{8\pi^2 n_{ac} n_{\lambda c}}{\beta m_\lambda^3 \tau_a} X(\theta) + \frac{8\pi^2 n_{\lambda c}^2 Y^2}{3m_\lambda^3 \tau_a} Y(\theta).$$

(11)

The third term on the RHS of (11) is proportional to $\beta$, while the third term on the RHS of (10) has a factor
of \((\beta + \frac{3}{2})\). Keeping track of two parts of axions generated from the back reacting photons, we find that the \(\frac{3}{2}\) in the third term on the RHS of (10) represents sterile axions and it should have been and was excluded in the derivation of (11).

The left hand sides (LHS) of (10) and (11) have no \(\theta\) dependence, but the RHS does. \(X(\theta) = Y(\theta)\) won’t make (10) and (11) valid simultaneously. So even if there is some outside constraint which can keep the axions in the \(X(\theta)\) distribution fixed, the photons cannot have the same distribution, i.e., \(Y(\theta) \neq X(\theta)\).

There is no simple way to find a closed form for \(Y(\theta)\) because the LHS of the equations (10) and (11) have no \(\theta\) dependence, while the \(\theta\) dependences on the RHS of these equations are different. This suggests the possibility that \(Y(\theta)\) may be found as a series expansion in \(X(\theta)\). As a first test of this idea we replaced the general form \(X(\theta)\) with \(\sin \theta\) to study the distribution with more axions accumulated near the equatorial plane with few near the polar area, aiming at matching orders of \(\sin \theta\) on each side of equations. But this fails as it turns out that \(\sin^n \theta \ (n \in \mathbb{Z})\) is not an orthogonal set of functions and thus the calculation leads to contradictions. Therefore, we must expand the occupation numbers and number density in terms of a full set of orthogonal functions. We do this in the next section where we choose the set to be the real spherical harmonics.

### REAL SPHERICAL HARMONICS EXPANSION

The set-up here is similar to the previous discussion except that the axion and photon occupation numbers and number densities have coefficients labeled by order index \(l\) and \(m\). For the axions

\[
f_a(p, r, \Omega, t) = \sum_{lm} f_{alm}(t) Y_{lm}(\Omega) \Theta(p_{\text{max}} - p) \Theta(R - r)
\]

\[
n_a(r, \Omega, t) = \sum_{lm} n_{alm}(t) Y_{lm}(\Omega) \Theta(R - r)
\]

\[
f_{alm}(t) = \frac{6\pi^2}{m_a^3 \beta^3} n_{alm}(t)
\]

where we have set

\[
f_{ac}(t) X(\theta) = \sum_{lm} f_{alm}(t) Y_{lm}(\Omega)
\]

Note that \(n_a\) can not be any superposition of real spherical harmonics, it has to be real and positive, so it should be put into the form

\[
n_a = \Theta(R - r) \left( \sum_{l'm'} n_{al'l'} Y_{l'm'} \right) \left( \sum_{lm} n_{alm} Y_{lm} \right)
\]

where \(Y_{lm} \) are complex spherical harmonics. This also applies to photons.

\[
f_\lambda(k, r, \Omega, t) = \sum_{lm} f_{\lambda lm}(t) Y_{lm}(\Omega)
\]

\[
\times \Theta(R - r) \Theta(k_+ - k) \Theta(k_-)
\]

\[
n_\lambda(r, \Omega, t) = \sum_{lm} n_{\lambda lm}(t) Y_{lm}(\Omega) \Theta(R - r)
\]

\[
f_{\lambda lm}(t) = \frac{8\pi^2}{m_\lambda^3 \beta^3} n_{\lambda lm}(t)
\]

\[
n_\gamma(r, \Omega, t) = \sum_{lm} [n_+ lm (t) + n_- lm (t)] Y_{lm}(\Omega) \Theta(R - r)
\]

\[
= \sum_{lm} n_{\gamma lm}(t) Y_{lm}(\Omega) \Theta(R - r)
\]

\[
n_+ lm(t) = n_- lm(t)
\]

\[
n_{-lm}(t) = 2n_{\lambda lm}(t)
\]

where similar the the axion case we have set

\[
f_{\lambda c}(t) Y(\theta) = \sum_{lm} f_{\lambda lm}(t) Y_{lm}(\Omega).
\]

Following the steps from the previous general discussion, we have an equation similar to (11) for each choice of \(lm\)

\[
\frac{dn_{\gamma lm}(t)}{dt} = 2n_{alm}(t) + \frac{16\pi^2}{3m_3^3 \tau_a} E_{lm}
\]

\[
- \frac{16\pi^2}{3m_3^3 \tau_a} (\beta + \frac{3}{2}) F_{lm} - \frac{3\pi^2}{2R} n_{-lm}(t)
\]

where \(E_{lm}\) and \(F_{lm}\) are defined through

\[
n_a(\Omega, t) n_\gamma(\Omega, t) = \sum_{l'm'} n_{al'l'} n_{\gamma l'm'} Y_{l'm'} Y_{l''m''}
\]

\[
\sum_{lm} E_{lm} Y_{lm}
\]

and

\[
[n_\gamma(\Omega, t)]^2 = \sum_{l'm'} n_{\gamma l'm'} n_{-l'm'} Y_{l'm'} Y_{l''m''}
\]

\[
= \sum_{lm} F_{lm} Y_{lm}
\]

We also have equations similar to equation (11) for each choice of \(lm\) with regard to the changing number density of axions. The equation includes components representing spontaneous decay, stimulated decay and back reaction with sterile axions excluded.

\[
\frac{dn_{alm}(t)}{dt} = \frac{8\pi^2}{m_\lambda^3 \tau_a} E_{lm} + \frac{8\pi^2 \beta}{3m_3^3 \tau_a} F_{lm}
\]

The sterile axions evolve according to

\[
\frac{dn_{alm}(t)}{dt} = \frac{4\pi^2}{m_\lambda^3 \tau_a} F_{lm}
\]
The rate of change of photon number density component can be expressed in terms of the changing components of normal axion and sterile axion, and the components of surface loss

\[
\frac{dn_{\gamma lm}(t)}{dt} = -2\left[\frac{dn_{atm}(t)}{dt} + \frac{dn_{slm}(t)}{dt}\right] - \frac{3c}{2R} n_{\gamma lm}(t) .
\]  

(17)

We now proceed to explore some example choices of initial axion distributions.

**EXAMPLES**

**Y_{00} distribution**

As a first example we consider the spherical symmetric axion distribution where the only nonzero component of axion number density is \( n_{a00} \).

\[
n_a = \Theta(R - r) n_{a00} Y_{00}(\Omega) ,
\]

then

\[
n_{atm} = 0 \quad (lm \neq 00) .
\]

This simplifies equation (13) to

\[
n_a(\Omega, t) n_\gamma(\Omega, t) = \sum_{lm} n_{a00} Y_{00} n_{\gamma lm} Y_{lm} .
\]

In addition, there is now a relationship between \( E_{lm} \) and \( n_{\gamma lm} \).

\[
E_{lm} = n_{a00} Y_{00} n_{\gamma lm} .
\]  

(18)

Equation (15) is also simplified for \( lm \neq 00 \) to

\[
0 = 0 - \frac{8\pi^2}{\beta m_a^2 r_a} E_{lm} + \frac{8\pi^2}{3m_a^3 r_a} F_{lm} ,
\]

which reduces to

\[
F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 00) .
\]  

(19)

Substitute (18) and (19) into equation (14) gives, upon splitting of 00 pieces, the two forms of (14)

\[
[n_\gamma(\Omega, t)]^2 = F_{00} Y_{00} + \frac{3}{\beta^2} n_{a00} Y_{00} \sum_{lm \neq 00} n_{\gamma lm} Y_{lm}
\]

(20)

and

\[
F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 00) .
\]

(21)

The most conspicuous solution to the equation is

\[
F_{00} = \frac{n_{200} n_{00}}{2\sqrt{\pi}} , \quad n_{\gamma lm} = 0 \quad (lm \neq 00) .
\]

where the only nonzero component of photon number density is \( n_{00} \). So if there is spherical symmetry in the axion distribution, then spherical symmetry also exist in photon distribution.

Now we argue that this is the only solution of finite spherical harmonics series. Suppose that the highest spherical harmonics in the photon number density \( n_\gamma(\Omega, t) \) is \( Y_{00} m_0 \). According to (20) and taking the \( Y_{00} \) as a number, the highest spherical harmonics in \( [n_\gamma(\Omega, t)]^2 \) is also \( Y_{00} m_0 \). However, according to (21), the highest spherical harmonics in \( [n_\gamma(\Omega, t)]^2 \) is going to be \( Y_{20} m_2 \). This contradiction can only be resolved when \( n_{\gamma lm} = 0 \ (lm \neq 00) \), i.e. the photon number density retains spherical symmetry.

The reason why this is the only finite series case is that the \( Y_{00} \) distribution of axions mathematically requires the photons to couple in a specific way that retains the \( Y_{00} \) distribution of axions, as is implied by equations (18) and (19).

Now we know all the coupling coefficients \( E_{lm} \) and \( F_{lm} \),

\[
E_{lm} = \frac{n_{a00} n_{00}}{2\sqrt{\pi}} \delta_{00} \delta_{m0} , \quad F_{lm} = \frac{n_{a00} n_{00}}{2\sqrt{\pi}} \delta_{00} \delta_{m0} .
\]

Equations (12), (15) and (16) reduce to the equations (34'), (37'), (38') in [3] given that

\[
n_{00} = 2\sqrt{\pi} n_\gamma , \quad n_{a00} = 2\sqrt{\pi} n_a ,
\]

because it is the \( n_{00} Y_{00} \) that describes the photon number density. Hence we have checked the spherically symmetric model results given in [3].

**Y_{20} distribution**

For a \( Y_{20} \) axion distribution the only nonzero component of the axion number density is \( n_{a20} \),

\[
n_a = \Theta(R - r) n_{a20} Y_{20}(\Omega) ,
\]

so that

\[
n_{atm} = 0 \quad (lm \neq 20) .
\]

Equation (15) is simplified for \( lm \neq 20 \), to

\[
0 = 0 - \frac{8\pi^2}{\beta m_a^2 r_a} E_{lm} + \frac{8\pi^2}{3m_a^3 r_a} F_{lm} ,
\]

which reduces to

\[
F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 20) .
\]  

(22)
The nonzero component \( n_{a20} \) of axion number density evolves via

\[
\frac{dn_{a20}(t)}{dt} = -\frac{n_{a20}}{\tau_a} - \frac{8\pi^2}{\beta m_a^3 \tau_a} E_{20} + \frac{8\pi^2 \beta}{3m_a^3 \tau_a} F_{20} .
\]

The photon number density component \( n_{\gamma20} \) growth rate is

\[
\frac{dn_{\gamma20}(t)}{dt} = -2\frac{n_{a20}}{\tau_a} + \frac{16\pi^2}{\beta m_a^4 \tau_a} E_{20} - \frac{16\pi^2}{3m_a^4 \tau_a} (\beta + \frac{3}{2}) F_{20} - \frac{3c}{2R} n_{\gamma20}(t) ,
\]

while the other photon number density component \( n_{\gamma lm}(lm \neq 20) \) evolve as

\[
\frac{dn_{\gamma lm}(t)}{dt} = -\frac{8\pi^2}{m_a^3 \tau_a} F_{lm} - \frac{3c}{2R} n_{\gamma lm}(t) .
\]

Since no spontaneous decay from axion feeds into these components, they are negligible. This example is not physical because a density number of the form \( Y_{20} \) becomes negative in some regions. It is included here for demonstration purpose. The next examples is physical and motivated by superradiance.

\[
Y_1^{\pm 1} Y_1^{\pm 1} \sim \sin^2 \theta \text{ distribution}
\]

A \( \sin^2 \theta \) distribution is toroidal and is positive definite everywhere, and hence can represent a physical distribution of particles. For this case the only nonzero components of the axion number density are \( n_{a00} \) and \( n_{a20} \), so we can write \( n_a(r, \theta, t) \) in several useful forms

\[
n_a = \Theta(R - r) n_{a}(t) \sin^2 \theta
\]

\[
= \Theta(R - r) n_{a}(t) \frac{4\sqrt{\pi}}{3} (Y_{00} - \frac{1}{\sqrt{5}} Y_{20})
\]

\[
= \Theta(R - r)(n_{a00}(t) Y_{00} + n_{a20}(t) Y_{20}) .
\]

The relation between \( n_{a00} \) and \( n_{a20} \) is

\[
n_{a20}(t) = -\frac{n_{a00}(t)}{\sqrt{5}} . \tag{23}
\]

Similar to previous examples, we find that for components other than 00 and 20

\[
F_{lm} = \frac{3}{\beta^2} E_{lm} \quad (lm \neq 00, 20) ,
\]

so that the components of photon number density evolve as

\[
\frac{dn_{\gamma lm}(t)}{dt} = -\frac{8\pi^2}{m_a^3 \tau_a} F_{lm} - \frac{3c}{2R} n_{\gamma lm}(t) .
\]

Since no spontaneous decay from axion feeds into these components, they are negligible, as in the previous example. The nonzero axion number density components are given by

\[
\frac{dn_{a00}(t)}{dt} + \frac{n_{a00}}{\tau_a} = -\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{00} + \frac{8\pi^2 \beta}{3m_a^3 \tau_a} F_{00}
\]

\[
\frac{dn_{a20}(t)}{dt} + \frac{n_{a20}}{\tau_a} = -\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{20} + \frac{8\pi^2 \beta}{3m_a^3 \tau_a} F_{20} .
\]

Because of \( \gamma_{a00} \), this leads to the relation

\[
-\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{20} + \frac{8\pi^2 \beta}{3m_a^3 \tau_a} F_{20} = \frac{1}{\sqrt{5}} (\frac{8\pi^2}{\beta m_a^3 \tau_a} E_{00} - \frac{8\pi^2 \beta}{3m_a^3 \tau_a} F_{00}) . \tag{24}
\]

The photon number density component \( n_{\gamma20} \) grows as

\[
\frac{dn_{\gamma20}(t)}{dt} = -2\frac{n_{a20}}{\tau_a} + \frac{16\pi^2}{\beta m_a^4 \tau_a} E_{20} - \frac{16\pi^2}{3m_a^4 \tau_a} (\beta + \frac{3}{2}) F_{20} - \frac{3c}{2R} n_{\gamma20}(t) .
\]

and

\[
\frac{dn_{\gamma00}(t)}{dt} = 2\frac{n_{a00}}{\tau_a} + \frac{16\pi^2}{\beta m_a^4 \tau_a} E_{00} \frac{3c}{2R} n_{\gamma00}(t) .
\]

Because of \( \gamma_{a00} \) and \( \gamma_{a20} \), we can combine the previous two equations and write

\[
\frac{dn_{\gamma00}(t)}{dt} + \frac{3c}{2R} n_{\gamma00}(t) = 2\frac{n_{a00}}{\tau_a} + \frac{16\pi^2}{\beta m_a^4 \tau_a} E_{00} - \frac{16\pi^2 \beta}{3m_a^4 \tau_a} F_{00} \left(\beta + \frac{3}{2}\right) F_{00} ,
\]

\[
\frac{dn_{\gamma20}(t)}{dt} + \frac{3c}{2R} n_{\gamma20}(t) = 2\frac{n_{a20}}{\tau_a} + \frac{16\pi^2}{\beta m_a^4 \tau_a} E_{20} - \frac{16\pi^2 \beta}{3m_a^4 \tau_a} F_{20} .
\]

We observe that if the part of back reaction that results in sterile axions is neglected, then

\[
n_{\gamma00}(t) = \frac{n_{a00}(t)}{\sqrt{5}} ,
\]

so the photons would remain in \( \sin^2 \theta \) distribution.
General distribution

Suppose that we have an axion number density

$$n_a = \Theta(R - r) \sum n_{alm} Y_{lm}(\Omega) ,$$

For $n_{alm} = 0$, then according to (13) this leads to

$$F_{lm} = \frac{3}{\beta^2} E_{lm} \ (n_{alm} = 0)$$

Substituting this condition into equation (12), we have

$$\frac{dn_{\gamma lm}(t)}{dt} = -\frac{16\pi^2}{3m_a^3 \tau_a} \left(\frac{3}{2}\right) F_{lm} - \frac{3c}{2R} n_{\gamma lm}(t)$$

also for $n_{alm} = 0$. Hence there is no source feeding those photon components.

The parts of back reaction that results in sterile axions and surface loss are the only terms that contribute to these components. It is expected that these components die out quickly and thus have no effect on lasing. So

$$n_{\gamma} = \Theta(R - r) \sum n_{\gamma lm}(\Omega) .$$

where

$$n_{\gamma lm} \approx 0 \ (\text{when } n_{alm} = 0) .$$

I.e., the photon field has the same spherical harmonic components as the axion field, as other components die out quickly due to lack of sources. Neither spontaneous decay nor stimulated decay contributes to the harmonic components of photons that are not present in the axions.

Suppose that all the axion components are nonzero, and they are proportional to each other,

$$n_{alm} = \alpha_{lm} n_{alm_0},$$

where $\alpha_{lm}$ are numbers and $n_{alm_0}$ is the fiducial component to which all other components are proportional. Then

$$\frac{8\pi^2}{3m_a^3 \tau_a} F_{lm} = \frac{8\pi^2}{\beta m_a^3 \tau_a} E_{lm} = \frac{dn_{alm}(t)}{dt} + \frac{n_{alm}}{\tau_a}$$

and

$$\frac{dn_{\gamma lm}(t)}{dt} + \frac{3c}{2R} n_{\gamma lm}(t) = 2\frac{n_{alm}}{\tau_a} + \frac{16\pi^2}{3m_a^3 \tau_a} E_{lm} - \frac{16\pi^2}{3m_a^3 \tau_a} (\beta + \frac{3}{2}) F_{lm}$$

If the part of the back reaction that results in sterile axions is neglected, then

$$\frac{dn_{\gamma lm}(t)}{dt} + \frac{3c}{2R} n_{\gamma lm}(t) = -2 \frac{dn_{alm}(t)}{dt} = -2 \alpha_{lm} \frac{dn_{alm_0}(t)}{dt}$$

$$= \alpha_{lm} \left[ \frac{dn_{alm_0}(t)}{dt} + \frac{3c}{2R} n_{\gamma lm_0}(t) \right]$$

$$n_{\gamma lm} = \alpha_{lm} n_{\gamma lm_0}$$

Hence the distribution of photons would keep the same shape as that of the axions if sterile axions were neglected.

**DISCUSSION**

The calculation presented here tells one the initial spatial distribution of photons once the spatial distribution of the axions is given. It does not give direct instructions on how to achieve observable effects from axion cluster lasing. The model does take the mechanism that the stimulated decay of axion produces type of photons that have the same momenta as the photons which induced stimulated decay of axions is neglected, then

$$2k \frac{df_\lambda(\vec{k}, \vec{x}, t)}{dt} = \frac{4m_a \Gamma_a}{\pi} \int d^3k_1 \int d^3p \delta^4(p - k - k_1) \times$$

$$\{ f_a(\vec{p})[1 + f_\lambda(\vec{k}) + f_\lambda(\vec{k}_1)] - f_\lambda(\vec{k}) f_\lambda(\vec{k}_1) \} .$$

However, there is a compromise made here by using this equation. The entire model is a local theory. The photon occupation number here and now depends only on particle occupation numbers here and now. If the cluster in the model is a ball and all the quantities are spherical symmetric, the local theory provides useful predictions about the lasing process. However, if the cluster is of some specific geometrical shape, then the local theory probably won't give pertinent information that reflect the geometry of the cluster. Thus we suggest that a non-local lasing theory which could be governed by the following equation,

$$2k \frac{df_\lambda(\vec{k}, \vec{x}, t)}{dt} = \frac{4m_a \Gamma_a}{\pi} \int d^3k_1 \int d^3p \delta^4(p - k - k_1)$$

$$\times C \int d^3\vec{x}' \left[ f_a(\vec{p}, \vec{x}', t') \int^{t} dt' e^{-\Gamma_\alpha(t-t')} \delta[\vec{x} - \vec{x}' - \frac{\vec{c}k}{k}(t - t')] \right]$$

$$+ f_a(\vec{p}, \vec{x}, t) \int^{t} dt' e^{-\Gamma_\alpha(t-t')} \delta[\vec{x} - \vec{x}' - \frac{\vec{c}k}{k_1}(t - t')] f_\lambda(\vec{k}, \vec{x}', t')$$

$$+ f_a(\vec{p}, \vec{x}, t) \int^{t} dt' e^{-\Gamma_\alpha(t-t')} \delta[\vec{x} - \vec{x}' - \frac{\vec{c}k}{k_1}(t - t')] f_\lambda(\vec{k}_1, \vec{x}', t')$$

$$- f_\lambda(\vec{k}, \vec{x}, t) \int^{t} dt' e^{-\Gamma_\alpha(t-t')} \delta[\vec{x} - \vec{x}' - \frac{\vec{c}k}{k_1}(t - t')] f_\lambda(\vec{k}_1, \vec{x}', t') \} .$$
In the non-local model, the photon occupation number here and now depends on all the past occupation number of events that are casually connected to here and now. The factor \( e^{-\Gamma_a (t-t')} \) takes account the probability that photons propagating from \( \vec{x}' \) to \( \vec{x} \) without stimulating axion or going to annihilation.

**APPENDIX**

Starting from the evolution relation between axion and photon occupation numbers \[3\]

\[
\frac{df_\lambda(k)}{dt} = \frac{m_a \Gamma_a}{k^2} \int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi} \left\{ \right.
\left. [1 + f_\lambda(k)]f_{ac} \Theta(R-r)X(\theta)ight.
\left. + f_{ac} f_{\lambda e} \left[ (R-r)^2 X(\theta) Y(\theta) \right] \right.
\left. \times \Theta(p_{\text{max}} - \sqrt{(k + k_1)^2 - m_a^2}) dk_1 \right.
\left. - f_\lambda(k) f_{\lambda e} \Theta(R-r) Y(\theta) \int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi} \Theta(k_+ - k_1) \Theta(k_1 - k_-) dk_1 \right\}.
\]

The first and second integrals are the same,

\[
\int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi} \Theta(p_{\text{max}} - \sqrt{(k + k_1)^2 - m_a^2}) dk_1
\]

\[
= \int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi} \Theta(p_{\text{max}} - \sqrt{(k + k_1)^2 - m_a^2}) \Theta(k_+ - k_1) \Theta(k_1 - k_-) dk_1
\]

\[
= m_a \gamma - k - \frac{m_a^2}{4k}.
\]

The third integral is related to the back reaction of photons. It is convenient to split it into two parts

\[
\int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi} \Theta(k_+ - k_1) \Theta(k_1 - k_-) dk_1
\]

\[
= \int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi} + \int_{\frac{m_a^2}{4k}} \frac{dk_1}{2\pi}
\]

\[
= (m_a \gamma - k - \frac{m_a^2}{4k}) + (k - k_-).
\]

The first part represents back reaction resulting in axions with energy a less than \( m_a \gamma \) that axions that can again participate in stimulated emission, while the second part gives the back reaction resulting in sterile axions, i.e., where the total energy of the axion \( k + k_1 \) is larger than \( m_a \gamma \).

Moving the step function \( \Theta(R-r) \) in front of the curly brackets and substituting the results of the integrations, we have

\[
\frac{df_\lambda(k)}{dt} = \Theta(R-r) \frac{m_a \Gamma_a}{k^2} \left\{ [1 + f_\lambda(k)]f_{ac} X(\theta) (m_a \gamma - k - \frac{m_a^2}{4k}) \right.
\left. + f_{ac} f_{\lambda e} X(\theta) Y(\theta) (m_a \gamma - k - \frac{m_a^2}{4k}) \right.
\left. - f_\lambda(k) f_{\lambda e} Y(\theta) [(m_a \gamma - k - \frac{m_a^2}{4k}) + (k - k_-)] \right\}.
\]

Collecting terms \( f_\lambda(k) \) can be written

\[
\frac{df_\lambda(k)}{dt} = \Theta(R-r) \Theta(k_+ - k) \Theta(k_- - k) \frac{m_a \Gamma_a}{k^2} \left\{ f_{ac} (X + 2f_{\lambda e}XY) - f_{\lambda e}^2 Y^2 (m_a \gamma - k - \frac{m_a^2}{4k}) \right.
\left. - (k - k_-) \right\).
\]

The rate of change of photon number density is the integration of this equation over \( k \) space

\[
\frac{dn_\lambda}{dt} = \int \frac{df_\lambda(k)}{dt} \frac{dk}{(2\pi)^3}
\]

\[
= \Theta(R-r) \frac{m_a \Gamma_a}{2\pi^2} \left\{ f_{ac} (X + 2f_{\lambda e}XY) - f_{\lambda e}^2 Y^2 \right\} \times
\left\{ \int_{k_-}^{k_+} (m_a \gamma - k - \frac{m_a^2}{4k}) dk - f_{\lambda e}^2 Y^2 \int_{k_-}^{k_+} (k - k_-) dk \right\}
\]

Evaluating the two integrals,

\[
\int_{k_-}^{k_+} (m_a \gamma - k - \frac{m_a^2}{4k}) dk = \frac{m_a^2 \gamma^2 \beta}{2} - \frac{m_a^2}{4} \ln \left( \frac{1 + \beta}{1 - \beta} \right)
\]

\[
\int_{k_-}^{k_+} (k - k_-) dk = \frac{m_a^2 \gamma^2 \beta^2}{2}.
\]

gives

\[
\frac{dn_\lambda}{dt} = \Theta(R-r) \frac{m_a^3 \Gamma_a}{8\pi^2} \left\{ [f_{ac}(x + 2f_{\lambda e}xy) - f_{\lambda e}^2 y^2] \times
\left\{ 2\gamma^2 \beta - \ln \left( \frac{1 + \beta}{1 - \beta} \right) \right\} - f_{\lambda e}^2 y^2 \times (2\gamma^2 \beta^2) \right\}
\]

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