ON THE STRATIFIED VECTOR BUNDLES

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ABSTRACT. The stratified vector bundles on a smooth variety defined over an algebraically closed field \( k \) form a neutral Tannakian category over \( k \). We investigate the affine group–scheme corresponding to this neutral Tannakian category.

1. Introduction

Let \( X \) be an irreducible smooth variety defined over an algebraically closed field \( k \) of positive characteristic. Gieseker in [7] introduced the notion of a stratified vector bundle over \( X \). These objects are analogs of the complex algebraic vector bundles equipped with an integrable connection. More precisely, a stratified vector bundle over \( X \) is a pair \((E, \nabla)\), where \( E \) is an algebraic vector bundle over \( X \), and \( \nabla : D(X) \to \mathcal{E}nd_k(E) \) is an \( \mathcal{O}_X \)-linear ring homomorphism, where \( D(X) \) is the sheaf of differential operators sending \( \mathcal{O}_X \) to itself, and \( \mathcal{E}nd_k(E) \) is the sheaf of \( k \)-linear endomorphisms of \( E \).

The stratified vector bundles over \( X \) form a rigid abelian \( k \)-linear tensor category; this category will be denoted by \( \mathcal{C}(X) \). After we fix a \( k \)-rational point \( x_0 \) of \( X \), this category \( \mathcal{C}(X) \) gets the natural fiber functor that sends any stratified vector bundle \((E, \nabla)\) to the fiber of \( E \) over the point \( x_0 \). Therefore, \( \mathcal{C}(X) \) equipped with this fiber functor is a neutral Tannakian category over \( k \). Consequently, we get an affine group scheme over \( k \), which will be called the stratified group–scheme of \( X \). This stratified group–scheme depends only on the pointed variety \((X, x_0)\), and it is denoted by \( S(X, x_0) \).

The group–scheme \( S(X, x_0) \) has a surjective homomorphism to the étale fundamental group \( \pi^{et}(X, x_0) \) (Proposition 3.2). Hence \( S(X, x_0) \) in general is not of finite type. The Frobenius morphism of the group–scheme \( S(X, x_0) \) turns out to be an automorphism (Corollary 4.2). We also show that \( S(X, x_0) \) is different from the fundamental group–scheme constructed by Nori (the details are in Section 3.2).

If \( X \) and \( Y \) are irreducible smooth proper varieties over \( k \), then we show that

\[
S(X \times Y, (x_0, y_0)) = S(X, x_0) \times S(Y, y_0)
\]

(see Theorem 3.4).

For any nonnegative integer \( n \), let \( X_n \) denote the base change of \( X \) using the homomorphism \( k \to k \) defined by \( \lambda \mapsto \lambda^p^n \), where \( p \) is the characteristic of \( k \). In [7], Gieseker

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defined a flat vector bundle over $X$ to be sequence of pairs $\{E_n, \sigma_n\}_{n \geq 0}$, where $E_n$ is a vector bundle over $X_n$, and

$$\sigma_n : F^*_X E_{n+1} \longrightarrow E_n$$

is an isomorphism of vector bundles, where $F_X$ is the relative Frobenius morphism. As shown in [7], there is a canonical equivalence of categories between the category of flat vector bundles on $X$ and the category $\mathcal{C}(X)$ of stratified vector bundles.

Let $\mathcal{G}$ be an affine group–scheme defined over the field $k$. A flat principal $\mathcal{G}$–bundle over $X$ is a sequence of pairs $\{E_n, \sigma_n\}_{n \geq 0}$, where $E_n$ is a principal $\mathcal{G}$–bundle over $X_n$ and

$$\sigma_n : F^*_X E_{n+1} \longrightarrow E_n$$

is an isomorphism of principal $\mathcal{G}$–bundles. A flat principal $\mathcal{G}$–bundle $\{E_n, \sigma_n\}_{n \geq 0}$ is also called a flat structure on the principal $\mathcal{G}$–bundle $E_0$.

We show that there is a tautological principal $\mathcal{S}(X,x_0)$–bundle $E_{\mathcal{S}(X,x_0)}$ on $X$. Let $\mathcal{S}(X,x_0)$ be the sheaf of differential operators of order at most $n$ mapping $\mathcal{O}_X$ to itself. The direct limit

$$D(X) := \lim_{\rightarrow} D^n(X)$$

is called the sheaf of differential operators on $X$.
We recall from [7] the definition of a stratified sheaf (see [7, page 2, Definition 0.3]).

**Definition 2.1.** A *stratified sheaf* on $X$ is an $\mathcal{O}_X$–coherent sheaf $E$ on $X$ together with a ring homomorphism

$$\nabla : \mathcal{D}(X) \longrightarrow \mathcal{E}nd_k(E)$$

which is $\mathcal{O}_X$–linear.

- A stratified sheaf $(E, \nabla)$ with the $\mathcal{O}_X$–module $E$ locally free is called a *stratified vector bundle*.
- For a stratified sheaf $(E, \nabla)$, the homomorphism $\nabla$ is called a *stratification* on $E$.

**Lemma 2.2.** For any stratified sheaf on $X$, the underlying $\mathcal{O}_X$–coherent sheaf is locally free.

See [4, § 2, page 21, Proposition 2.16] for a proof of the above lemma.

A *homomorphism* from a stratified vector bundle $(E, \nabla)$ to a stratified vector bundle $(E', \nabla')$ is a homomorphism of $\mathcal{O}_X$–coherent sheaves

$$f : E \longrightarrow E'$$

that intertwines the actions of $\mathcal{D}(X)$ on $E$ and $E'$, or in other words, $f(\nabla(D)(s)) = \nabla'(D)(f(s))$, where $D$ is a locally defined section of $\mathcal{D}(X)$, and $s$ is a locally defined section of $E$. (In [7], a homomorphism from $(E, \nabla)$ to $(E', \nabla')$ is also called a *horizontal homomorphism* from $E$ to $E'$.)

Using Lemma 2.2 it is easy to check that the stratified vector bundles on $X$ form an abelian category.

For any nonnegative integer $n$, let $X_n = X \times_k k$ be the base change of the variety $X$ using the homomorphism $k \longrightarrow k$ defined by $t \mapsto t^p$. So in particular $X = X_0$. Since the field $k$ is algebraically closed, in particular it is perfect, $X_n$ is isomorphic to $X$ for all $n$. For any $n \geq 0$, let

$$F_X : X_n \longrightarrow X_{n+1}$$

be the relative Frobenius morphism. We should clarify that this is an abuse of notation since the domain of $F_X$ is not fixed and it depends on $n$. However from the context the domain will generally be clear, and whenever there is any ambiguity, we will clearly specify the domain. For any positive integer $d$, let

$$F_X^d := \underbrace{F_X \circ \cdots \circ F_X}_{d\text{-times}} : X_n \longrightarrow X_{n+d}$$

be the $d$–fold iteration of $F_X$. For notational convenience, $F_X^0$ will denote the identity map of $X_n$. (See [12, Section 4.1] for details.)

We recall from [7] the definition of a flat vector bundle (see [7, page 3, Definition 1.1]).

**Definition 2.3.** A *flat sheaf* over $X$ is a sequence of pairs $\{E_n, \sigma_n\}_{n \geq 0}$, where $E_n$ is an $\mathcal{O}_X$–coherent sheaf on $X_n$ and

$$\sigma_n : F_X^*E_{n+1} \longrightarrow E_n$$
is an isomorphism of $\mathcal{O}_X$–coherent sheaves.

- A flat sheaf $\{E_n, \sigma_n\}_{n \geq 0}$ with all $\mathcal{O}_X$–modules $E_n$ locally free is called a flat vector bundle.
- A flat sheaf $\{E_n, \sigma_n\}_{n \geq 0}$ will also be called a flat structure on $E_0$.

A homomorphism from a flat sheaf $\{E_n, \sigma_n\}_{n \geq 0}$ to a flat sheaf $\{E'_n, \sigma'_n\}_{n \geq 0}$ is a homomorphism of $\mathcal{O}_X$–coherent sheaves $\tau_n : E_n \longrightarrow E'_n$ for each $n \geq 0$ such that the two homomorphisms $\sigma'_n \circ (F^*_X \tau_{n+1})$ and $\tau_n \circ \sigma_n$ from $F^*_X E_{n+1}$ to $E'_n$ coincide. (In [7], a homomorphism from $\{E_n, \sigma_n\}_{n \geq 0}$ to $\{E'_n, \sigma'_n\}_{n \geq 0}$ is called a horizontal homomorphism from $E_0$ to $E'_0$.)

A theorem due to Katz identifies stratified vector bundles with flat vector bundles (see [7, page 4, Theorem 1.3]). Using Lemma 2.2, from the proof of Theorem 1.3 in [7] it follows immediately that if $\{E_n, \sigma_n\}_{n \geq 0}$ is a flat sheaf, then the $\mathcal{O}_X$–coherent sheaf $E_n$ is locally free for each $n$. Combining this together with Lemma 2.2 we conclude that the flat vector bundles over $X$ form an abelian category.

2.2. An affine group–scheme. The tensor product of two stratified vector bundles $(E, \nabla)$ and $(E', \nabla')$ is constructed as follows. Take any locally defined vector field $\xi$ on $X$, and let $u$ and $v$ be locally defined sections of $E$ and $E'$ respectively. Define

$$\hat{\nabla}(\xi)(u \otimes v) := (\nabla(\xi)(v)) \otimes w + v \otimes (\nabla'(\xi)(w)),$$

which is a locally defined section of $E \otimes E'$. This operator $\hat{\nabla}$ extends to an $\mathcal{O}_X$–linear ring homomorphism

$$\hat{\nabla} : \mathcal{D}(X) \longrightarrow \mathcal{E}nd_k(E \otimes E').$$

The tensor product $(E, \nabla) \otimes (E', \nabla')$ is defined to be $(E \otimes E', \hat{\nabla})$.

We note that if $\{E_n, \sigma_n\}_{n \geq 0}$ and $\{E'_n, \sigma'_n\}_{n \geq 0}$ are the flat vector bundle over $X$ corresponding to the stratified vector bundles $(E, \nabla)$ and $(E', \nabla')$ respectively, then the flat vector bundle corresponding to the tensor product $(E, \nabla) \otimes (E', \nabla')$ is

$$\{E_n \otimes E'_n, \sigma_n \otimes \sigma'_n\}_{n \geq 0}.$$

Let $\mathcal{C}(X)$ denote the category of stratified vector bundles on $X$. We already noted that $\mathcal{C}(X)$ is an abelian category. In fact, $\mathcal{C}(X)$ is a rigid abelian $k$–linear tensor category (see [5, page 112, Definition 1.7] for the definition of a rigid abelian $k$–linear tensor category).

Let $\text{Vect}(k)$ denote the category of finite dimensional $k$–vector spaces. Fix a $k$–rational point $x_0 \in X$.

We have a fiber functor

$$T_{x_0} : \mathcal{C}(X) \longrightarrow \text{Vect}(k)$$
that sends a stratified vector bundle \((E, \nabla)\) over \(X\) to its fiber \(E_{x_0}\) over the base point \(x_0\). Equipped with this fiber functor \(T_{x_0}\), the category \(\mathcal{C}(X)\) of stratified vector bundles becomes a neutral Tannakian category over \(k\) (see [5, page 138, Definition 2.19]).

Therefore, the neutral Tannakian category \((\mathcal{C}(X), T_{x_0})\), where \(T_{x_0}\) is defined in Eq. (2.2), produces an affine group–scheme over \(k\) [5, page 130, Theorem 2.11], [11, Theorem 1.1], [13, Theorem 1].

**Definition 2.4.** The group scheme over \(k\) given by the neutral Tannakian category \((\mathcal{C}(X), T_{x_0})\) over \(k\) will be called the **stratified group–scheme** of \(X\).

The stratified group–scheme of \(X\) will be denoted by \(S(X, x_0)\).

Let \(Y\) be an irreducible smooth variety defined over \(k\) and

\[
\varphi : X \longrightarrow Y
\]
a morphism. Set \(y_0 := \varphi(x_0) \in Y\). If \((E, \nabla)\) is a stratified vector bundle over \(Y\), then the pull back \((\varphi^*E, \varphi^*\nabla)\) is a stratified vector bundle over \(X\). It is easy to that if

\[
f : (E, \nabla) \longrightarrow (E', \nabla')
\]
is a homomorphism between stratified vector bundles over \(Y\), then

\[
\varphi^*f : (\varphi^*E, \varphi^*\nabla) \longrightarrow (\varphi^*E', \varphi^*\nabla')
\]
is also a homomorphism between stratified vector bundles. Consequently, the morphism \(\varphi\) from \(X\) to \(Y\) induces a homomorphism of group–schemes

\[
(2.3) \quad \varphi^* : S(X, x_0) \longrightarrow S(Y, y_0).
\]

Assume that \(X\) is proper. Let \((E, \nabla)\) be a stratified vector bundle over \(X \times Z\), where \(Z\) is an irreducible smooth variety defined over \(k\). Let

\[
\psi : X \times Z \longrightarrow Z
\]
be the natural projection. Since \(X\) is proper, the direct image \(R^i\psi_*E\) is a coherent sheaf on \(Z\) for each \(i \geq 0\) [8, page 116, Théorème 3.2.1]. The pull back \(\psi^*\mathcal{D}(Z)\) is canonically a subsheaf of \(\mathcal{D}(X \times Z)\) (see Eq. (2.1) for definition). Using this inclusion of \(\psi^*\mathcal{D}(Z)\) in \(\mathcal{D}(X \times Z)\), the homomorphism

\[
\nabla : \mathcal{D}(X \times Z) \longrightarrow \mathcal{E}nd_k(E)
\]

produces a homomorphism

\[
\nabla' : \mathcal{D}(Z) \longrightarrow \mathcal{E}nd_k(R^i\psi_*E)
\]

for each \(i \geq 0\). It is straightforward to check that

\[
(2.4) \quad (R^i\psi_*E, \nabla')
\]
is a stratified sheaf on \(Z\). From Lemma [2,2] it now follows that the \(\mathcal{O}_Z\)–coherent sheaf \(R^i\psi_*E\) is locally free.
Remark 2.5. Let $X$ be an irreducible smooth variety over $k$ and $U \subset X$ a nonempty Zariski open subset such that the complement $X \setminus U$ is of codimension at least two. Fix a $k$–rational point $x_0$ of $U$. The inclusion map of $(U, x_0)$ in $(X, x_0)$ induces an isomorphism

$$S(U, x_0) \sim \rightarrow S(X, x_0)$$

of stratified group–schemes. This follows immediately from [7, page 21, Theorem 3.14].

3. Some properties of the group–scheme

3.1. The étale fundamental group as a quotient group. Throughout this section we will assume $X$ to be an irreducible smooth variety proper over $k$.

A vector bundle $E$ over $X$ is called étale trivializable if there is an algebraic étale Galois covering

$$\gamma : Y \rightarrow X$$

such that the pull back $\gamma^*E$ is trivializable. Therefore, a vector bundle $E$ of rank $n$ over $X$ is étale trivializable if and only if there is a representation

$$\rho : \pi^{et}(X, x_0) \rightarrow \text{GL}(n, k),$$

where $\pi^{et}(X, x_0)$ is the étale fundamental group of $X$ with base point $x_0$, such that the associated vector bundle $V_\rho$ over $X$ is isomorphic to $E$.

We briefly recall a Tannakian description of $\pi^{et}(X, x_0)$.

The étale trivializable vector bundles over $X$ from a rigid abelian $k$–linear tensor category $\mathcal{E}(X)$. For any $E, E' \in \mathcal{E}(X)$, the homomorphisms from $E$ to $E'$ are defined to be all $\mathcal{O}_X$–linear homomorphisms from the vector bundle $E$ to $E'$. The direct sum, tensor product and duals are defined in the obvious way. We have the fiber functor

$$T^0_{x_0} : \mathcal{E}(X) \rightarrow \text{Vect}(k)$$

that sends any étale trivializable vector bundle $E$ to its fiber $E_{x_0}$ over the point $x_0$. The group–scheme given by the neutral Tannakian category $(\mathcal{E}(X), T^0_{x_0})$ over $k$ is identified with the étale fundamental group $\pi^{et}(X, x_0)$.

Take any representation

$$\rho : \pi^{et}(X, x_0) \rightarrow \text{GL}(n, k).$$

Let $V_\rho$ be the vector bundle of rank $n$ over $X$ associated to $\rho$. This $V_\rho$ is equipped with a canonical stratification (see [7, page 7] for the details).

Lemma 3.1. Let $E$ and $F$ be two étale trivializable vector bundles over $X$. Then any $\mathcal{O}_X$–linear homomorphism from the vector bundle $E$ to $F$ intertwines the $\mathcal{D}(X)$–module structures defining the stratifications on $E$ and $F$.

Proof. Let $\gamma : Y \rightarrow X$ be an algebraic étale Galois covering with $Y$ connected such that $\gamma^*E$ is trivializable. Similarly, take an algebraic étale Galois covering

$$\delta : Z \rightarrow X$$
with $Z$ connected such that $\delta^* F$ is trivializable. Fix a connected component 

$$M \subset Y \times_X Z$$

of the fiber product. Let 

$$\phi : M \to X$$

be the natural projection obtained by restricting the map $\gamma \times \delta$. Since $\gamma^* E$ and $\delta^* F$ are trivializable, it follows immediately that $\phi^* E$ and $\phi^* F$ are also trivializable.

Let 

$$h : E \to F$$

be any $\mathcal{O}_X$–linear homomorphism. Let 

$$\phi^* h : \phi^* E \to \phi^* F$$

be the pull back of $h$ to $M$, where $\phi$ is the projection in Eq. (3.2). We noted earlier that both $\phi^* E$ and $\phi^* F$ are trivializable. Fix trivializations of $\phi^* E$ and $\phi^* F$. With respect to these trivializations, the homomorphism $\phi^* h$ in Eq. (3.3) is given by a morphism from $M$ to the variety of $n \times m$ matrices 

$$H : M \to M_{n \times m}(\mathbb{C}),$$

where $m = \text{rank}(E)$ and $n = \text{rank}(F)$.

The variety $M$ is proper over $k$ because $X$ is proper. Hence the morphism $H$ in Eq. (3.4) must be a constant one. This immediately implies that $h$ intertwines the $\mathcal{D}(X)$–module structures of $E$ and $F$. This completes the proof of the lemma. \hfill \Box

In view of Lemma 3.1, we obtain a functor between the neutral Tannakian categories over $k$

$$\mathcal{F} : (\mathcal{E}(X), T^0_{x_0}) \to (\mathcal{C}(X), T_{x_0}),$$

where $T_{x_0}$ and $T^0_{x_0}$ are constructed in Eq. (2.2) and Eq. (3.1) respectively. This functor $\mathcal{F}$ produces a homomorphism of group–schemes over $k$

$$\theta : \mathcal{S}(X, x_0) \to \pi^{et}(X, x_0),$$

where $\mathcal{S}(X, x_0)$ is constructed in Definition 2.4.

If $\rho' : \pi^{et}(X, x_0) \to \text{GL}(n, k)$ is another representation which is not isomorphic to $\rho$, then the stratified vector bundle $V_{\rho'}$ over $X$ associated to $\rho'$ is not isomorphic to the stratified vector bundle $V_{\rho}$ associated to $\rho$ [4 page 7, Proposition 1.9]. From this it follows that the image $\theta(\mathcal{S}(X, x_0))$ of the homomorphism $\theta$ in Eq. (3.6) is not contained in any normal proper subgroup of $\pi^{et}(X, x_0)$. In fact, a stronger statement holds. The homomorphism $\theta$ is surjective, as shown by the following proposition.

**Proposition 3.2.** The homomorphism $\theta$ in Eq. (3.6) is faithfully flat. In particular, $\theta$ is surjective.
Proof. We will use the criterion in [5, page 139, Proposition 2.21(a)] for a homomorphism of group–schemes to be faithfully flat. The functor $F$ in Eq. (3.5) is evidently fully faithful. Therefore, to complete the proof using the criterion in [5, Proposition 2.21(a)] we need to show the following.

Let $E$ be an étale trivializable vector bundle over $X$. Let

$$\nabla: \mathcal{D}(X) \rightarrow \mathcal{E}nd_k(E)$$

be the canonical stratification on $E$. Take any coherent subsheaf

$$F \subset E$$

preserved by $\nabla$, i.e.,

(3.7) $$\nabla(\mathcal{D}(X))(F) \subset F.$$ Then $F$ is also étale trivializable.

To prove that $F$ is étale trivializable, take an algebraic étale Galois covering

$$\gamma: Y \rightarrow X$$

with $Y$ connected such that $\gamma^*E$ is trivializable. The condition in Eq. (3.7) ensures that the $\mathcal{O}_Y$–coherent subsheaf

$$\gamma^*F \subset \gamma^*E$$

is preserved by the natural connection on the trivializable vector bundle $\gamma^*E$. This implies that

- $\gamma^*F$ is subbundle of $\gamma^*E$, and
- the morphism $Y \rightarrow \text{Gr}(r, \gamma^*E)$ associated to the subbundle $\gamma^*F$ is a constant one, where $r = \text{rank}(F)$, and $\text{Gr}(r, \gamma^*E)$ is the Grassmann bundle over $Y$ parametrizing $r$ dimensional subspaces in the fibers of $\gamma^*E$.

Therefore, the subbundle $\gamma^*F$ is trivializable. In particular, $F$ is étale trivializable. This completes the proof of the proposition. \qed

Remark 3.3. There are examples of stratified vector bundles on smooth projective curves which are not étale trivializable; see [6, page 100]. Therefore, the homomorphism $\theta$ in Eq. (3.6) is not a closed embedding in general.

3.2. An essentially finite vector bundle which is not flat. In [10], Nori introduced the notion of an essentially finite vector bundle over a projective variety $X$. He showed that the essentially finite vector bundles form a neutral Tannakian category once a point $x_0$ of $X$ is fixed. The group–scheme $\pi(X, x_0)$ associated to this neutral Tannakian category given by the essentially finite vector bundles is known as the fundamental group–scheme.

We will give an example of an essentially finite vector bundle that is not flat. Such an example shows that the stratified group–scheme $\mathcal{S}(X, x_0)$ does not coincide with the fundamental group–scheme $\pi(X, x_0)$ in general.
Let $X$ be a supersingular elliptic curve defined over an algebraically closed field of positive characteristic. This means that the pull back homomorphism

$$F_X^* : H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X)$$

vanishes. Fix a nonzero element

(3.8) \hspace{1cm} \omega \in H^1(X, \mathcal{O}_X).

Let

(3.9) \hspace{1cm} 0 \longrightarrow \mathcal{O}_X \longrightarrow E \xrightarrow{f} \mathcal{O}_X \longrightarrow 0

be the extension given by $\omega$.

The cohomology class $F_X^* \omega \in H^1(X, \mathcal{O}_X)$ vanishes because $X$ is supersingular. Hence the short exact sequence in Eq. (3.9) splits. Therefore, $F_X^* E = \mathcal{O}_X \bigoplus \mathcal{O}_X$. This implies that the vector bundle $E$ is essentially finite (see [2, pp. 552–553, Proposition 2.3]).

We will show that there is no vector bundle $V$ over $X$ such that $F_X^* V = E$. To prove this by contradiction, let $V$ be a vector bundle over $X$ such that

(3.10) \hspace{1cm} F_X^* V = E.

Since $E$ is indecomposable of degree zero, from Eq. (3.10) it follows that $V$ is also indecomposable of degree zero. From the classification, due to Atiyah, of indecomposable vector bundles of rank two and degree zero over an elliptic curve we know that $V$ fits in a short exact sequence of vector bundles

(3.11) \hspace{1cm} 0 \longrightarrow L \longrightarrow V \longrightarrow L \longrightarrow 0,

where $L$ is a line bundle over $X$ of degree zero (see [1, page 432, Theorem 5(ii)]). Let

(3.12) \hspace{1cm} 0 \longrightarrow F_X^* L \longrightarrow F_X^* V = E \longrightarrow F_X^* L \longrightarrow 0

be the pull back of the exact sequence in Eq. (3.11) be the Frobenius morphism.

The subbundle $\mathcal{O}_X$ of $E$ in Eq. (3.9) is the unique line subbundle of degree zero. Indeed, if $L'$ is a line subbundle of $E$ of degree zero, then consider the composition

(3.13) \hspace{1cm} L' \hookrightarrow E \xrightarrow{f} \mathcal{O}_X,

where $f$ is the projection in Eq. (3.9). Since both $L'$ and $\mathcal{O}_X$ are of degree zero, any nonzero $\mathcal{O}_X$–linear homomorphism between them must be an isomorphism. Therefore, if $L'$ is different from the subbundle $\mathcal{O}_X$ of $E$ in Eq. (3.9), then the composition in Eq. (3.13) yields a splitting of the short exact sequence in Eq. (3.9). But this short exact sequence does not split because $\omega$ in Eq. (3.8) is nonzero. Hence the subbundle $\mathcal{O}_X$ of $E$ in Eq. (3.9) is the unique line subbundle of degree zero.

Therefore, the subbundle $F_X^* L \subset E$ in Eq. (3.12) coincides with the subbundle $\mathcal{O}_X$ of $E$ in Eq. (3.9). This implies that $\omega$ in Eq. (3.8) coincides with $F_X^* \omega'$, where

$$\omega' \in H^1(X, \mathcal{O}_X)$$
is the extension class for the short exact sequence in Eq. (3.11). But this is impossible because $X$ is supersingular. Therefore, we conclude that the vector bundle $E$ is Eq. (3.9) is not flat.

3.3. Product of varieties. Let $X$ and $Y$ be irreducible smooth proper varieties defined over $k$. Fix a $k$–rational point $x_0$ (respectively, $y_0$) of $X$ (respectively, $Y$). Recall stratified group–schemes constructed in Definition 2.4.

**Theorem 3.4.** There is a natural isomorphism of the stratified group–scheme $S(X \times Y, (x_0, y_0))$ with $S(X, x_0) \times S(Y, y_0)$.

**Proof.** Let

$$q_X : X \rightarrow X \times Y$$

be the morphism defined by $x \mapsto (x, y_0)$. Similarly, define

$$q_Y : Y \rightarrow X \times Y$$

(3.14)

to be the map that sends any $y$ to $(x_0, y)$. Consider the corresponding homomorphisms of group–schemes

$$q_X^* : S(X \times Y, (x_0, y_0)) \rightarrow S(X, x_0)$$

and

$$q_Y^* : S(X \times Y, (x_0, y_0)) \rightarrow S(Y, y_0)$$

(see Eq. (2.3)). We will show that the homomorphism

$$\eta := q_X^* \times q_Y^* : S(X \times Y, (x_0, y_0)) \rightarrow S(X, x_0) \times S(Y, y_0)$$

(3.15)

is an isomorphism.

Let

$$p_X : X \times Y \rightarrow X$$

(3.16)

and

$$p_Y : X \times Y \rightarrow Y$$

(3.17)

be the natural projections. We note that the composition homomorphism

$$q_X^* \circ p_X^* : S(X, x_0) \rightarrow S(X, x_0)$$

is the identity map. Similarly, $q_Y^* \circ p_Y^*$ is also the identity map. These imply in particular that the homomorphism $\eta$ in Eq. (3.15) is surjective.

To complete the proof of the theorem we need to show that $\eta$ is a closed embedding. We will use the criterion in [5, page 139, Proposition 2.21(b)] for a homomorphism of group–schemes to be a closed embedding.

Let $(V, \nabla)$ be any stratified vector bundle over $X \times Y$. To show that $\eta$ in Eq. (3.15) is a closed embedding it suffices to prove that there are stratified vector bundles $(E, \nabla^E)$ and $(F, \nabla^F)$ over $X$ and $Y$ respectively, such that $(V, \nabla)$ is a quotient of the stratified vector bundle $(p_X^* E, p_X^* \nabla^E) \boxtimes (p_Y^* F, p_Y^* \nabla^F)$ over $X \times Y$, where $p_X$ and $p_Y$ are the maps in Eq. (3.16) and Eq. (3.17) respectively. (See [5] page 139, Proposition 2.21(b)].)
To prove the existence of \((E, \nabla^E)\) and \((F, \nabla^F)\) satisfying the above condition, consider the stratified vector bundle

\[(\hat{V}, \hat{\nabla}) := (p_Y^* q_Y^* V, p_Y^* q_Y^* \nabla)\]
on \(X \times Y\), where \(q_Y\) is the map in Eq. (3.14) and \(p_Y\) is the projection in Eq. (3.17). Let

\[(W := p_X^* (V \otimes \hat{V}^*) )\]
be the direct image on \(X\), where \(p_X\) is the projection in Eq. (3.16). The stratification \(\hat{\nabla}\) on \(\hat{V}\) (see Eq. (3.18)) and the stratification \(\nabla\) on \(V\) together define a stratification on \(V \otimes \hat{V}^*\). Therefore, we have an induced stratification

\[(\tilde{\nabla} : \mathcal{D}(X) \longrightarrow \mathcal{E}_{ndk}(W)\]
on the direct image \(W\) constructed in Eq. (3.19) (see Eq. (2.4) for stratification on a direct image).

Let

\[(U, \nabla^U) := (p_X^* W, p_X^* \tilde{\nabla}) \otimes (\hat{V}, \hat{\nabla})\]
be the tensor product of stratified vector bundles on \(X \times Y\), where \((\hat{V}, \hat{\nabla})\) is constructed in Eq. (3.18), and \(\tilde{\nabla}\) is constructed in Eq. (3.20). From the construction of \((U, \nabla^U)\) in Eq. (3.21) it follows that we have a homomorphism of stratified vector bundles

\[\nu : (U, \nabla^U) \longrightarrow (V, \nabla).\]

To explain this with more detail, we note that for any two \(\mathcal{O}_{X \times Y}\)-coherent sheaves \(A\) and \(B\) on \(X \times Y\), we have a natural homomorphism

\[p_X^* p_X^* (B \otimes A^*) \otimes A \longrightarrow B\]
of \(\mathcal{O}_{X \times Y}\)-coherent sheaves which is constructed using the obvious homomorphism

\[p_X^* p_X^* (B \otimes A^*) \longrightarrow B \otimes A^*.\]

Let

\[U = p_X^* p_X^* (V \otimes (p_Y^* q_Y^* V)^*) \otimes p_Y^* q_Y^* V \longrightarrow V\]
be the homomorphism of \(\mathcal{O}_{X \times Y}\)-coherent sheaves in Eq. (3.23) obtained by substituting \(p_Y^* q_Y^* V\) for \(A\) and \(V\) for \(B\). This homomorphism intertwines \(\nabla^U\) (defined in Eq. (3.21)) and \(\nabla\), giving the homomorphism \(\nu\) in Eq. (3.22).

Since \(Y\) is proper, using [7, page 9, Proposition 2.4] it follows immediately that the homomorphism \(\nu\) in Eq. (3.22) is surjective. Therefore, if we set

\[(E, \nabla^E) := (W, \tilde{\nabla})\]
where \(W\) and \(\tilde{\nabla}\) are constructed in Eq. (3.19) and Eq. (3.20) respectively, and also set

\[(F, \nabla^F) := (q_Y^* V, q_Y^* \nabla)\]
(see Eq. (3.18)), then

\[(U, \nabla^U) = (p_X^* E, p_X^* \nabla^E) \otimes (p_Y^* F, p_Y^* \nabla^F).\]
Since the homomorphism $\nu$ in Eq. (3.22) is surjective, we now conclude that the stratified vector bundle $(V, \nabla)$ is a quotient of the tensor product $(p_X^*E, p_X^*\nabla^E) \bigotimes (p_Y^*F, p_Y^*\nabla^F)$. We noted earlier that this implies that the homomorphism $\eta$ in Eq. (3.15) is a closed embedding.

Therefore, $\eta$ is an isomorphism. This completes the proof of the theorem. \[\square\]

4. AUTOMORPHISMS OF THE GROUP–SCHEME

As in Section 2.1, we take $X$ to be an irreducible smooth variety defined over $k$. Since the field $k$ is algebraically closed, in particular $k$ is perfect, the arithmetic Frobenius homomorphism $k \rightarrow k$ is an isomorphism. Hence we can identify all $X_n, n \geq 1$, with $X$ itself. Consequently, the Frobenius map $F_X$ becomes a self–map of $X$.

Given a flat vector bundle $\{E_n, \sigma_n\}_{n \geq 0}$ on $X$ (see Definition 2.3), we may construct a new flat vector bundle on $X$ as follows. For each $n \geq 0$, set $E'_n := E_{n+1}$.

Set $\sigma'_n := \sigma_{n+1} : F_X^*E'_{n+1} = F_X^*E_{n+2} \rightarrow E_{n+1} = E'_n$ to be the isomorphism. Now $\{E'_n, \sigma'_n\}_{n \geq 0}$ is a new flat vector bundle on $X$.

Using [7, page 4, Theorem 1.3], which identifies flat vector bundles on $X$ with stratified vector bundles on $X$, the above construction produces a functor from the category $\mathcal{C}(X)$ of stratified vector bundles on $X$ to itself. In fact, this construction yields a functor from the neutral Tannakian category $(\mathcal{C}(X), T_{x_0})$ over $k$ (see Definition 2.4) to itself. Therefore, we get an endomorphism of the stratified group–scheme

\begin{equation}
\Phi : \mathcal{S}(X, x_0) \rightarrow \mathcal{S}(X, x_0)
\end{equation}

(see Definition 2.4 for $\mathcal{S}(X, x_0)$).

**Lemma 4.1.** The homomorphism $\Phi$ in Eq. (4.1) is an isomorphism.

**Proof.** Let $\{E_n, \sigma_n\}_{n \geq 0}$ be a flat vector bundle on $X$. We will construct another flat vector bundle from it. For $n \geq 0$, set $F_n := F_X^*E_n$.

So for $n \geq 1$, using the isomorphism $\sigma_{n-1}$, the vector bundle $F_n$ gets identified with $E_{n-1}$. For $n \geq 0$, set $\tau_n := F_X^*\sigma_n : F_X^*F_{n+1} = F_X^*(F_X^*E_{n+1}) \rightarrow F_X^*E_n = F_n$.

It is easy to check that $\{F_n, \tau_n\}_{n \geq 0}$ is a flat vector bundle on $X$.

Using the identification between stratified vector bundles on $X$ and flat vector bundles on $X$, the above construction produces a functor from the neutral Tannakian category...
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(\mathcal{C}(X), T_{x_0}) over \( k \) in Definition \ref{2.4} to itself. Consequently, we get a homomorphism of group-schemes

\begin{equation}
(4.2) \quad \Psi : \mathcal{S}(X, x_0) \longrightarrow \mathcal{S}(X, x_0).
\end{equation}

It is now straightforward to check that

\begin{equation}
(4.3) \quad \Psi \circ \Phi = \Phi \circ \Psi = \text{Id}_{\mathcal{S}(X, x_0)}.
\end{equation}

This completes the proof of the lemma.

From the construction of the homomorphism \( \Psi \) in Eq. (4.2) it follows that \( \Psi \) coincides with the Frobenius morphism

\( F_{\mathcal{S}(X, x_0)} : \mathcal{S}(X, x_0) \longrightarrow \mathcal{S}(X, x_0) \)

of the group-scheme \( \mathcal{S}(X, x_0) \) (see \cite{9} page 146–148 for the Frobenius morphism of a group-scheme). We have seen in Eq. (4.3) that \( \Psi \) is an isomorphism. Hence the Frobenius morphism \( F_{\mathcal{S}(X, x_0)} \) is an isomorphism.

We put down this observation as the following corollary.

**Corollary 4.2.** The Frobenius morphism \( F_{\mathcal{S}(X, x_0)} \) of the group-scheme \( \mathcal{S}(X, x_0) \) is an isomorphism.

5. Flat principal bundles

5.1. A tautological principal bundle. Consider the group-scheme \( \mathcal{S}(X, x_0) \) in Definition \ref{2.4}. We will show that there is a tautological principal \( \mathcal{S}(X, x_0) \)-bundle over \( X \).

Let \( \text{Vect}(X) \) denote the category of vector bundles over \( X \). We have a functor

\begin{equation}
(5.1) \quad \mathcal{B} : \mathcal{C}(X) \longrightarrow \text{Vect}(X),
\end{equation}

where \( \mathcal{C}(X) \) as before is the category of stratified vector bundles on \( X \), that sends any stratified vector bundle \( (E, \nabla) \) over \( X \) to the vector bundle \( E \). This functor defines a principal \( \mathcal{S}(X, x_0) \)-bundle over \( X \); see \cite{11} Lemma 2.3, Proposition 2.4] (reproduced as Theorem 2.3 in \cite{3} page 6], \cite{5} page 149, Theorem 3.2]. Let

\begin{equation}
(5.2) \quad E_{\mathcal{S}(X, x_0)} \longrightarrow X
\end{equation}

be the principal \( \mathcal{S}(X, x_0) \)-bundle given by the functor \( \mathcal{B} \) in Eq. (5.1).

Therefore, finite dimensional representations of the affine group-scheme \( \mathcal{S}(X, x_0) \) are stratified vector bundles on \( X \), and furthermore, the vector bundle over \( X \) associated to the principal \( \mathcal{S}(X, x_0) \)-bundle \( E_{\mathcal{S}(X, x_0)} \) in Eq. (5.2) for a representation \((E, \nabla)\) of \( \mathcal{S}(X, x_0) \) is the vector bundle \( E \) itself.
5.2. **Flat principal bundles.** Let \( G \) be an affine group–scheme defined over the field \( k \). Imitating the definition of a flat vector bundle we define:

**Definition 5.1.** A flat principal \( G \)-bundle over \( X \) is a sequence of pairs \( \{ E_n, \sigma_n \}_{n \geq 0} \), where \( E_n \) is a principal \( G \)-bundle over \( X_n \) and

\[
\sigma_n : F_X^* E_{n+1} \longrightarrow E_n
\]

is an isomorphism of principal \( G \)-bundles.

A flat principal \( G \)-bundle \( \{ E_n, \sigma_n \}_{n \geq 0} \) will also be called a flat structure on the principal \( G \)-bundle \( E_0 \).

**Lemma 5.2.** The principal \( S(X, x_0) \)-bundle \( E_{S(X, x_0)} \) in Eq. (5.2) has a natural flat structure.

**Proof.** For any positive integer \( n \), let

\[
\Phi^n := \Phi \circ \cdots \circ \Phi : S(X, x_0) \longrightarrow S(X, x_0)
\]

be the \( n \)-fold iteration of the homomorphism \( \Phi \) constructed in Eq. (4.1), and set \( \Phi^0 \) to be the identity map of \( S(X, x_0) \). For \( n \geq 0 \), let

\[
E^n_{S(X, x_0)} := E_{S(X, x_0)}(\Phi^n) \longrightarrow X
\]

be the principal \( S(X, x_0) \)-bundle over \( X \) obtained by extending the structure group of the principal \( S(X, x_0) \)-bundle \( E_{S(X, x_0)} \) in Eq. (5.2) using the homomorphism \( \Phi^n \) in Eq. (3.3).

Let \( F_{S(X, x_0)} \) be any principal \( S(X, x_0) \)-bundle over \( X \). Let \( F_{S(X, x_0)}(\Psi) \) be the principal \( S(X, x_0) \)-bundle over \( X \) obtained by extending the structure group of \( F_{S(X, x_0)} \) using the homomorphism \( \Psi \) in Eq. (1.2). From the construction of \( \Psi \) it follows that \( F_{S(X, x_0)}(\Psi) \) is canonically identified the pull back \( F_X^* F_{S(X, x_0)} \). Using this identification for the principal bundle

\[
F_{S(X, x_0)} = E^n_{S(X, x_0)}
\]

defined in Eq. (3.4), together with the fact that

\[
\Psi \circ \Phi^{n+1} = \Phi^n,
\]

which follows from the observation in the proof of Lemma (1.1) that

\[
\Psi \circ \Phi = \text{Id}_{S(X, x_0)};
\]

we get an isomorphism of the principal \( S(X, x_0) \)-bundle \( F_X^* E^n_{S(X, x_0)} \) with \( E^n_{S(X, x_0)} \).

Let

\[
\sigma_n : F_X^* E^n_{S(X, x_0)} \longrightarrow E^n_{S(X, x_0)}
\]

be the isomorphism of principal \( S(X, x_0) \)-bundles obtained above. Therefore,

\[
\{ E^n_{S(X, x_0)}, \sigma_n \}_{n \geq 0}
\]

is a flat principal \( S(X, x_0) \)-bundle over \( X \). This completes the proof of the lemma. □
5.3. **Universality of the tautological principal bundle.** Let $G$ be any affine group–scheme defined over $k$. Let $\{E_n, \sigma_n\}_{n \geq 0}$ be a flat principal $G$–bundle over an irreducible smooth variety $X$ defined over $k$; see Definition [5.1] for flat $G$–bundle. Let

$$\text{Ad}(E_0) = E_0(G)$$

be the adjoint bundle of the principal $G$–bundle $E_0$. We recall that the adjoint bundle $\text{Ad}(E_0)$ is the fiber bundle over $X$ associated to $E_0$ for the adjoint action of $G$ on itself. So $\text{Ad}(E_0)$ is a group–scheme over $X$.

Let $\text{Ad}(E_0)_{x_0}$ be the fiber of $\text{Ad}(E_0)$ over the base point $x_0$. So the group–scheme $\text{Ad}(E_0)_{x_0}$ is isomorphic to $G$.

Fix a $k$–rational point

$$\hat{x}_0 \in (E_0)_{x_0}$$

in the fiber of $E_0$ of $x_0$. Since the pull back of any principal bundle to the total space of the principal bundle is canonically trivialized, using the base point $\hat{x}_0$ in Eq. (5.6) we get an isomorphism of group–schemes

$$\rho_0 : \text{Ad}(E_0)_{x_0} \longrightarrow G$$

(see also [3 Section 3]).

**Proposition 5.3.** There is homomorphism of group schemes

$$\rho : S(X,x_0) \longrightarrow \text{Ad}(E_0)_{x_0}$$

canonically associated to the flat principal $G$–bundle $\{E_n, \sigma_n\}_{n \geq 0}$.

The flat principal $G$–bundle $\{E_n, \sigma_n\}_{n \geq 0}$ coincides with the one obtained by extending the structure group of the flat principal $S(X,x_0)$–bundle $\{E_n^{S(X,x_0)}, \sigma_n\}_{n \geq 0}$ (see Lemma [5.2]) using the homomorphism $\rho_0 \circ \rho$, where $\rho_0$ is constructed in Eq. (5.7).

**Proof.** We will first recall a Tannakian description of the group–scheme $\text{Ad}(E_0)_{x_0}$.

Let $\text{Rep}(G)$ denote the category of all finite dimensional representations of the group–scheme $G$. It is a rigid abelian $k$–linear tensor category. We will construct a fiber functor on $\text{Rep}(G)$. For any finite dimensional left $G$–module $V$, let $E_0(V)$ denote the vector bundle over $X$ associated to the principal $G$–bundle $E_0$ for the $G$–module $V$. Let $E_0(V)_{x_0}$ be the fiber of $E_0(V)$ over the base point $x_0$. Now we have a fiber functor

$$T_{E_0} : \text{Rep}(G) \longrightarrow \text{Vect}(k)$$

that sends any $V$ to $E_0(V)_{x_0}$. So the pair $(\text{Rep}(G), T_{E_0})$ defines a neutral Tannakian category over $k$. The corresponding affine group–scheme over $k$ (see [5 page 130, Theorem 2.11]) is identified with $\text{Ad}(E_0)_{x_0}$.

Take any $V \in \text{Rep}(G)$. For each $n \geq 0$, let $E_n(V)$ denote the vector bundle over $X$ associated to the principal $G$–bundle $E_n$ for the $G$–module $V$. The isomorphism of principal $G$–bundles

$$\sigma_n : F_X^* E_{n+1} \longrightarrow E_n$$
induces an isomorphism of associated vector bundles

\[ \hat{\sigma}_n : F_x^* E_{n+1}(V) \longrightarrow E_n(V). \]

It is straightforward to check that

\[ \{E_n(V), \hat{\sigma}_n\}_{n \geq 0} \]

is a flat vector bundle over \( X \). Therefore, \( \{E_n(V), \hat{\sigma}_n\}_{n \geq 0} \) gives a stratified vector bundle over \( X \). Let

\[ (V', \nabla') \in \mathcal{C}(X) \]

be the stratified vector bundles given by the flat vector bundle \( \{E_n(V), \hat{\sigma}_n\}_{n \geq 0} \). Consequently, we have a functor

\[ (5.10) \quad \text{Rep}(G) \longrightarrow \mathcal{C}(X) \]

that sends any \( V \) to \( (V', \nabla') \). Now, comparing the two fiber functors \( T_{x_0} \) and \( T_{E_0} \), defined in Eq. (2.2) and Eq. (5.8) respectively, we see that the functor in Eq. (5.10) actually produces to a functor from the neutral Tannakian category \( (\text{Rep}(G), T_{E_0}) \) over \( k \) to the neutral Tannakian category \( (\mathcal{C}(X), T_{x_0}) \). In view of the above Tannakian description of the group-scheme \( \text{Ad}(E_0)_{x_0} \), this functor between neutral Tannakian categories over \( k \) produces a homomorphism

\[ (5.11) \quad \rho : \mathcal{S}(X, x_0) \longrightarrow \text{Ad}(E_0)_{x_0} \]

of group-schemes over \( k \).

To prove the second part of the proposition, for each \( n \geq 0 \), let

\[ F^n_G := E^n_{\mathcal{S}(X, x_0)}(G) \]

be the principal \( G \)-bundle over \( X \) obtained by extending the principal \( \mathcal{S}(X, x_0) \)-bundle \( E^n_{\mathcal{S}(X, x_0)} \) (see Eq. (5.5)) using the homomorphism \( \rho_0 \circ \rho \), where \( \rho \) is the homomorphism constructed in Eq. (5.11), and \( \rho_0 \) is the homomorphism in Eq. (5.7). Let

\[ (5.12) \quad \tau_n : F_x^* F^n_{G} \longrightarrow F^n_{G} \]

be the isomorphism of principal \( G \)-bundles induced by the isomorphism \( \sigma_n \) in Eq. (5.5). Therefore, \( \{F^n_G, \tau_n\}_{n \geq 0} \) is a flat principal \( G \)-bundle over \( X \).

Take any \( V \in \text{Rep}(G) \). For each \( n \geq 0 \), let

\[ F^n_V := F^n_{G}(V) \]

be the vector bundle over \( X \) associated to the principal \( G \)-bundle \( F^n_G \) for the \( G \)-module \( V \). Let

\[ \hat{\tau}_n : F_x^* F^n_V \longrightarrow F^n_V \]

be the isomorphism of vector bundles induced by the isomorphism \( \tau_n \) in Eq. (5.12). From the construction of the homomorphism \( \rho \) in Eq. (5.11) it follows that the flat vector bundle

\[ \{F^n_V, \hat{\tau}_n\} \]
is naturally identified with the flat vector bundle \( \{ E_n(V), \tilde{\sigma}_n \}_{n \geq 0} \) in Eq. (5.9). From this it follows that the flat principal \( G \)–bundle \( \{ E_n, \sigma_n \}_{n \geq 0} \) coincides with \( \{ F_n^G, \tau_n \}_{n \geq 0} \) (see [5, page 149, Theorem 3.2] [11, Lemma 2.3, Proposition 2.4]). This completes the proof of the proposition. 

\[ \square \]

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