1. Introduction

This article concerns the finite determinacy of the local structure of degenerating families in a sense that is best explained by the following two examples, for precise statements see [2]. A normal crossing degeneration is a flat family over a disc with coordinate \( t \) that is locally of the form

\[ z_0 \cdot \ldots \cdot z_i = t^N. \]

This family is determined at finite order in the sense that any other flat family over the disc that is isomorphic to this one modulo \( t^k \) for some \( k > N \) is in fact entirely isomorphic locally near the origin. Indeed, by flatness, the new family is also given by a single equation and this is of the form

\[ z_0 \cdot \ldots \cdot z_i = t^N + t^k f(t, z_0, ..., z_n) \]

for \( n \) the dimension of the fibres and \( f(t, z_0, ..., z_n) \) some perturbation term. We can rewrite this as

\[ z_0 \cdot \ldots \cdot z_i = t^N \left( 1 + t^{k-N} f(t, z_0, ..., z_n) \right) \]

and since \( g \) is invertible at the origin, we may absorb \( g \) into one of the variables by a coordinate change of the form \( z_0 \mapsto g z_0 \) to find this new family to be isomorphic to the original one locally at the origin. The purpose of this note is to generalize this result to situations where there are more equations than variables, making a similar ad hoc calculation as above very difficult. The following feature observable in the above example will hold more generally as well: if the initial finite order isomorphism of the two families is modulo \( t^k \) then the final all-order isomorphism of the two families agrees with the initial one modulo \( t^{k-N} \).

A slight modification of the above example makes the finite determinacy fail:

\[ xy = w^M t^N, \]
for some $M \geq 2$. Indeed, adding the perturbation term $t^k$ for some $k > N$ results in having smooth nearby fibres while the original family has an $A_{M-1}$-singularity as nearby fibre. Thus, in proving the finite determinacy, we make the assumption that the nearby fibres are locally rigid (e.g. smooth). We however also provide a result that covers the second example in the following way. If the perturbation term is required to be divisible by $w^l$ for $l \geq M$, i.e. geometrically, if the second family preserves a principal divisor containing the singularity, then finite determinacy holds. Indeed, similar to before,

$$xy = w^Mt^N + t^kw^lf(t, x, y, w) = w^Mt^N(1 + t^{k-N}w^{-M}f(t, x, y, w)).$$

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2. Local uniqueness results

We give three versions of the main result: a formal, an étale and a complex analytic one. Proofs for the following four results are provided in §3. In §4 we prove similar theorems where we allow the nearby fibre to be non-rigid but fix a principal divisor containing the non-rigid locus. As an application, we prove the finite determinacy of the log structures on fibres of log morphisms in §5.

For $B$ a ring, $P = B[z_1, ..., z_n]$, $A = P/J$ for an ideal $J \subset P$ and $M$ and $A$-module, one defines the $A$-module $T^1(A/B, M) = \text{coker}(\text{Der}_B(P, M) \to \text{Hom}_A(J, M))$. The author’s favorite reference for the $T^i$ functors is [Ha10]; the original source is [LS65]. We will give a review of the properties that we are going to use in the proof in §3. Let $\hat{T}^1_{\text{Spec } A/\text{Spec } B}$ denote the coherent sheaf associated to $T^1(A/B, A)$.

Setup 2.1. Let $X \to Y$ be a flat finite type map of Noetherian affine schemes and $t \in \Gamma(Y, \mathcal{O}_Y)$ a non-zero-divisor. Flatness implies that the pull-back of $t$ is also a non-zero-divisor on $X$. Assume that $\mathcal{T}^1_{X/Y}$ is supported in $t = 0$ (i.e. it is annihilated by a power of $t$). We denote by $X_k \to Y_k$ the base change of $X \to Y$ to Spec $\Gamma(Y, \mathcal{O}_Y)/(t^k)$. We denote by $\hat{X} \to \hat{Y}$ the completion in $t$. Let $Z \subseteq X$ be a closed subscheme (possibly empty) so that $t$ is a non-zero-divisor also on $Z$.

Theorem 2.2 (formal). There exists $N > 0$ so that if $X' \to Y$ is another flat map of Noetherian affine schemes, $Z' \subseteq X'$ a closed subscheme and $\varphi : X'_k \cup Z' \to X_k \cup Z$ an isomorphism over $Y$ for some $k > 4N$ that restricts to an isomorphism $Z' \xrightarrow{\sim} Z$, then
There is an isomorphism $\hat{\varphi} : \hat{X}^t \rightarrow \hat{X}$ that commutes with the maps to $\hat{Y}$ and so that the restrictions of $\hat{\varphi}$ and $\varphi$ to $\hat{Z}' \cup X'_{k-2N}$ agree where $\hat{Z}'$ is the completion of $Z'$ in $t$.

The authors of [MvS02] prove a theorem similar to Theorem 2.2 without the statement that the resulting isomorphism is compatible with the maps to $Y$, using rather different methods.

**Theorem 2.3 (étale local).** Assume we work over a field or excellent discrete valuation ring. There exists $N > 0$ so that if $X' \rightarrow Y$ is another flat map of Noetherian affine schemes, $Z' \subseteq X'$ a closed subscheme, $\varphi : X'_k \cup Z' \rightarrow X_k \cup Z$ an isomorphism over $Y$ for some $k > 4N$ that restricts to an isomorphism $Z' \sim Z$ and $x \in X_k$ a point, then there are étale neighbourhoods $U, U'$ of $x$ in $X, X'$ respectively and an isomorphism $\varphi_x : U' \rightarrow U$ that commutes with the maps to $Y$ and so that $\varphi_x$ and $\varphi$ agree when restricting to $U' \times_{X'} (X'_{k-2N} \cup Z')$.

**Theorem 2.4 (analytic).** Let $X \rightarrow Y$ be a flat map of complex analytic spaces, $t \in \Gamma(Y, \mathcal{O}_Y)$ a non-zero-divisor and $Z \subseteq X$ a closed complex analytic subvariety so that $t$ is a non-zerodivisor also on $Z$. Assume that $\mathcal{T}^1_{X/Y}$ is annihilated by a power of $t$. Let $Y_k$ denote the complex analytic space given by $t = 0$ with sheaf of rings $\mathcal{O}_Y/(t^k)$, similarly for $X_k$.

There exists $N > 0$ so that if $X' \rightarrow Y$ is another flat map of complex analytic spaces, $Z' \subseteq X'$ a closed analytic subvariety, $\varphi : X'_k \cup Z' \rightarrow X_k \cup Z$ an isomorphism over $Y$ for some $k > 4N$ that restricts to an isomorphism $Z' \sim Z$ and $x \in X_k$ a point, then there are neighbourhoods $U, U'$ of $x$ in $X, X'$ respectively and an isomorphism $\varphi_x : U' \rightarrow U$ that commutes with the maps to $Y$ and so that $\varphi_x$ and $\varphi$ agree when restricting to $U' \cap (X'_{k-2N} \cup Z')$.

We also prove the following useful criterion for the condition on the support of $\mathcal{T}^1_{X/Y}$ as required in the above theorems.

**Lemma 2.5 (rigid nearby fibres).** Let $X \rightarrow Y$ be a flat map of Noetherian schemes or of complex analytic spaces and $t$ a non-zero-divisor on $Y$. If for all points $y \in U := Y \setminus \{t = 0\}$ we have $\mathcal{T}^1_{X_y/y} = 0$ for $X_y$ the fibre over $y$ then the support of $\mathcal{T}^1_{X/Y}$ is contained in $t = 0$.

**Corollary 2.6 (uniqueness of neighbourhoods of points in singular fibres).** Let $\pi : X \rightarrow Y$ be a morphism of Noetherian schemes (resp. complex analytic varieties) and $x \in X$ so that $y := \pi(x)$ is a regular point, $n := \dim \mathcal{O}_{Y,y}$ and assume in some neighbourhood $V$ of $y$ all fibres over $V \setminus \{y\}$ are locally rigid (i.e. have vanishing $\mathcal{T}^1$). Then there is $N > 0$ so that if $\pi' : X' \rightarrow Y$ is another morphism and for some $k > 4N$,

$$\varphi : X' \times_Y \text{Spec} \mathcal{O}_{Y,y}/m_{Y,y}^nk \rightarrow X \times_Y \text{Spec} \mathcal{O}_{Y,y}/m_{Y,y}^nk$$
an isomorphism, then there is an isomorphism \( \varphi_x : U' \to U \) over \( Y \) for \( U', U \) étale neighbourhoods of \( x \) in \( X', X \) (respectively open analytic neighbourhoods) so that \( \varphi_x \) agrees with \( \varphi \) over \( \text{Spec } O_{Y,y}/m^{k-2N}_{Y,y} \).

Proof. By Lemma 2.5, there is \( N > 0 \) so that \( m^N_{Y,y} \) annihilates \( T^i_{X/Y} \). Assume \( k > 4N \). Let \( m_{Y,y} = (t_1, ..., t_n) \), then \( m^k_{Y,y} \subset (t_1^k, ..., t_n^k) \) and \( (t_1^{k-2N}, ..., t_n^{k-2N}) \subset m^{k-2N}_{Y,y} \). We only do the case \( n = 2 \), the general case is similar. We first apply Theorem 2.3 (resp. Theorem 2.4) to the base change of \( X \) to \( Y \) to the closed subspace \( Y_1 \) of \( Y \) given by the ideal \((t_2^k)\) for \( t = t_1 \) and \( Z = \emptyset \). The result is an isomorphism \( \varphi_1 \) of neighbourhoods of \( x \) in the base changes of \( X' \to Y \) and \( X \to Y \) to \( Y_1 \) that agrees with \( \varphi \) modulo \((t_1^{k-2N}, t_2^k)\). In the second step, we apply the corresponding theorem to \( X' \to Y \) and \( X \to Y \) with \( t = t_2 \) using \( \varphi_1 \) as input and obtaining the desired isomorphism as a result. \( \square \)

3. Proofs of the local uniqueness results

For \( B \to A \) a map of rings and \( M \) an \( A \)-module, there is an \( A \)-module \( T^i(A/B, M) \) for \( i = 0, 1, 2 \) defined in [LS65]. More generally, \( T^i \) is the \( i \)th cohomology module of \( \text{Hom}(L^\bullet, M) \) for \( L^\bullet \) a cotangent complex of \( B \to A \). One sets \( T^i_{A/B} := T^i(A/B, A) \).

We recall some standard properties of these \( T^i \) functors for later use, proofs for these can be found in [LS65] and [Ha10].

3.1. \( T^0 \). For \( B \to A \) a map of rings and \( M \) an \( A \)-module: \( T^0(A/B, M) = \text{Der}_B(A, M) \).

3.2. Smoothness. For \( B \to A \) smooth, one has \( T^i(A/B, M) = 0 \) for all \( M \) and \( i > 0 \).

3.3. Finite generatedness. For \( B \) Noetherian, \( A \) a finitely generated \( B \)-algebra and \( M \) a finitely generated \( A \)-module, \( T^i(A/B, M) \) is finitely generated for \( i = 0, 1, 2 \).

3.4. Closed embeddings. If \( B \to A \) is surjective with ideal \( I \), then for all \( M \),

\[
\begin{align*}
T^0(A/B, M) &= 0, \\
T^1(A/B, M) &= \text{Hom}_A(I/I^2, M),
\end{align*}
\]

and if \( I \) is generated by a regular sequence then \( T^2(A/B, M) = 0 \) for all \( M \).

3.5. Change of rings. If \( A \to B \to C \) are maps of rings and \( M \) is a \( C \)-module, there is a long exact sequence of \( C \)-modules

\[
0 \to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M) \\
\to T^1(C/B, M) \to T^1(C/A, M) \to T^1(B/A, M) \\
\to T^2(C/B, M) \to T^2(C/A, M) \to T^2(B/A, M).
\]
3.6. **Change of modules.** If \( B \rightarrow A \) is a map of rings and \( 0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0 \) an exact sequence of \( A \)-modules, there is a long exact sequence

\[
0 \rightarrow T^0(A/B, M) \rightarrow T^0(A/B, M') \rightarrow T^0(A/B, M'') \\
\rightarrow T^1(A/B, M) \rightarrow T^1(A/B, M') \rightarrow T^1(A/B, M'') \\
\rightarrow T^2(A/B, M) \rightarrow T^2(A/B, M') \rightarrow T^2(A/B, M'').
\]

3.7. **Base change.** Let \( A \) be a flat \( B \)-algebra, \( B' \) any \( B \)-algebra, \( A' = A \otimes_B B' \) and \( M' \) an \( A' \)-module then

\[
T^1(A/B, M') = T^1(A'/B', M').
\]

3.8. **Preparations for proving the statements in 3.2.** Assume now we are in a situation as in Setup 2.1. Set \( B = \Gamma(Y, \mathcal{O}_Y), A = \Gamma(X, \mathcal{O}_X), A' = \Gamma(X', \mathcal{O}'_X) \). Let \( I \subset A \) be the ideal of \( Z \) and \( I' \subset A' \) be the ideal of \( Z' \). Since \( t \) is assumed to be a non-zerodivisor in \( A/I \), we have \( I \cap (t^l) = t^l I \) for every \( l \). Similarly, \( I' \cap (t'^l) = t'^l I' \) for every \( l \). The given isomorphism \( \varphi \) at ring level, we also call \( \varphi : A/t^k I \rightarrow A'/t^k I' \).

Since \( T^1(A/B, I) \) is finitely generated by 3.3 by assumption there is \( N > 0 \) such that \( t^N T^1(A/B, I) = 0 \). Let \( \ker_{t^l} T^2(A/B, I) \) denote the kernel of the endomorphism of \( T^2(A/B, I) \) given by multiplication by \( t^l \). Since also \( T^2(A/B, I) \) is finitely generated by 3.3 the sequence

\[
\ker_{t} T^2(A/B, I) \subseteq \ker_{t^2} T^2(A/B, I) \subseteq \ker_{t^3} T^2(A/B, I) \subseteq \ldots
\]

stabilizes and, by increasing \( N \) if needed, we may assume \( \ker_{t^N} T^2(A/B, I) \) is this stable \( A \)-module.

**Lemma 3.1.** Let \( L, l \geq 0 \) be given, we have that \( I/t^L I \) and \( t^l I/t^{l+L} I \) are isomorphic \( A \)-modules.

**Proof.** Since \( t \) is a non-zerodivisor, multiplication by \( t^l \) is injective in \( I \) and thus yields an isomorphism \( I \cong t^l I \). The first two vertical maps in the following diagram are isomorphisms, hence also the third one is

\[
\begin{array}{ccccccc}
0 & \rightarrow & t^L I & \rightarrow & I & \rightarrow & I/t^L I & \rightarrow & 0 \\
& \downarrow{t^l} & & \downarrow{t^l} & & \downarrow{t^l} & & \\
0 & \rightarrow & t^{l+L} I & \rightarrow & t^l I & \rightarrow & t^l I/t^{l+L} I & \rightarrow & 0.
\end{array}
\]

\[\square\]

By 3.6 the short exact sequence of \( A \)-modules \( 0 \rightarrow I \rightarrow I/t^L I \rightarrow 0 \) induces a long exact sequence

\[
\ldots \rightarrow T^1(A/B, I) \rightarrow T^1(A/B, I/t^L I) \rightarrow T^1(A/B, I/t^{l+L} I) \rightarrow 0.
\]
and thus for $L > N$ a short exact sequence

$$0 \to T^1(A/B, I) \to T^1(A/B, I/t^L I) \to \ker t^N T^2(A/B, I) \to 0.$$ 

Since multiplication by $t^N$ acts on this sequence and is zero on the outer terms, we conclude the following result.

**Lemma 3.2.** For $L \geq N$ we have $t^{2N} T^1(A/B, I/t^L I) = 0$ and hence furthermore, for any $l \geq 2L$, by Lemma 3.1,

$$t^{2N} T^1(A/B, t^{-L} I/t^l I) = 0.$$

Set $A_L = A/t^L A$ and similarly for $A'_L, B_L$. Note that $t^{-k} I/t^L I \cong I/t^L I$ is an $A_L$-module. If we assume that $k \geq L$ then base change 3.7 gives

$$T^1(A/B, t^{-k} I/t^L I) = T^1(A_L/B_L, t^{-L} I/t^L I) \cong T^1(A'_L/B'_L, t^{-L} I'/t^L I') = T^1(A'/B, t^{-L} I'/t^l I'),$$

so we conclude the following lemma.

**Lemma 3.3.** For $k \geq L \geq N$ and $l \geq 2L$, we have

$$t^{2N} T^1(A'/B, t^{-L} I'/t^l I') = 0.$$

Now let

$$(3.3) \quad A = B[z_1, \ldots, z_n]/(f_1, \ldots, f_m)$$

be a presentation as a $B$-algebra. Let $\widehat{B[z_1, \ldots, z_n]}, \widehat{A'}, \widehat{B}$ denote the completions in $t$ of $B[z_1, \ldots, z_n], A', B$ respectively. Let $\hat{z}_i \in \widehat{A'}$ be a lift of the image of $z_i$ under the composition $A \to A/t^L A \to A'/t^L I'$. 

**Lemma 3.4.** The map $\pi : \widehat{B[z_1, \ldots, z_n]} \to \widehat{A'}, z_i \mapsto \hat{z}_i$ is surjective.

**Proof.** The proof hinges on the following property. For any $a' \in A'$, there is $b \in A'$ and a polynomial $F$ in the $\hat{z}_i$ with coefficients in $B$ so that

$$a' - t^k b = F.$$

This property follows from the fact that, by construction, the $\hat{z}_i$ generate $A'_k$ as a $B$-algebra. In fact, by iterating, i.e. reapplying the property to $b$ and so forth, we have the more general property that for any $a' \in A'$ and $s \geq 1$, there is $b \in A'$ and a polynomial $F$ in the $\hat{z}_i$ with coefficients in $B$ so that

$$(3.4) \quad a' - t^{sk} b = F.$$

Let $J$ denote the kernel of $\pi$ and set $J_l := \ker \left( B[z_1, \ldots, z_n]/(t^l) \to A'_l \right)$. We claim that the natural map $J_l \to J_{l-1}$ is surjective for all $l$. Indeed, let $\bar{a} \in J_{l-1}$ be given and represented by $a \in B[z_1, \ldots, z_n]$. Hence, $\pi(a) = t^{l-1} a'$ for some $a' \in A'$. By (3.4),
For some $B \in A'$, so $a$ descends to an element of $J_l$ and this maps to $\bar{a}$.

Since the completion functor is left exact, we have $J = \varinjlim J_l$. If we show the surjectivity of $B[z_1, ..., z_n]/(t^l) \to A'_l$ for all $l \geq 0$, then we have for all $l$ an exact sequence

$$0 \to J_l \to B[z_1, ..., z_n]/(t^l) \to A'_l \to 0.$$ 

Since the inverse system $J_l$ has surjective projections as we just verified, taking $\varinjlim$ on of the exact sequence yields again an exact sequence which is in fact $0 \to J \to B[z_1, ..., z_n] \to \hat{A}' \to 0$ thus the assertion of the Lemma follows. Hence, it suffices to prove that $B[z_1, ..., z_n]/(t^l) \to A'_l$ is surjective for all $l \geq 0$. Now let any $l \geq 0$ and an element $a' \in A'$ be given. Choose $s$ so that $sk \geq l$. By \[3.3\], we find $b \in A'$ and $F$ a polynomial in the $\hat{z}_i$ so that $a' - t^{sk}b = F$. We conclude that the projection of $a'$ in $A_l$ is the image of $F(z_1, ..., z_n) \in B[z_1, ..., z_n]$. This proves the desired surjectivity. \[3.3\]

**Lemma 3.5.** $\hat{B} \to \hat{A}'$ is flat.

**Proof.** By assumption $B \to A'$ is flat and thus the base change $\hat{B} \to A' \otimes_B \hat{B}$ is also flat. Since all rings involved are Noetherian, $A' \otimes_B \hat{B} \to (A' \otimes_B \hat{B})$ is also flat, e.g. by \[3.4\] Proposition 10.14]. Since compositions of flat maps are flat, we are done if we show that $\hat{A}' \cong (A' \otimes_B \hat{B})$. This follows by identifying the inverse systems that give these inverse limits,

$$(A' \otimes_B \hat{B})/t^l(A' \otimes_B \hat{B}) \cong (A' \otimes_B \hat{B}) \otimes_B B/t^lB \cong A' \otimes_B (B/t^lB) \cong A'/t^lA'.$$

**Proposition 3.6 (Lifting equations).** We find $g_1, ..., g_m \in B[z_1, ..., z_n]$, so that

$$\hat{A}' = B[z_1, ..., z_n]/(f_1 + t^k g_1, ..., f_m + t^k g_m).$$

**Proof.** By Lemma 3.4, the map $\hat{B}[z_1, ..., z_n] \to \hat{A}'$, $z_i \mapsto \hat{z}_i$ is surjective, let $J$ be the kernel. By Lemma 3.5, $\hat{A}'$ is flat over $\hat{B}$, hence $\text{Tor}_1^{\hat{B}}(B_k, \hat{A}') = 0$. Tensoring the exact sequence of $\hat{B}$-modules

$$0 \to J \to B[z_1, ..., z_n] \to \hat{A}' \to 0$$

with $B_k$ thus yields again an exact sequence

$$0 \to J/t^k J \to B[z_1, ..., z_n]/(t^k) \to A'_k \to 0.$$ 

From $A'_k \cong A_k$ we conclude

$$(3.5) \quad J + (t^k) = (f_1, ..., f_m) + (t^k)$$
as an equality of ideals of $B[z_1, ..., z_n]$. Hence there exist $g_i$ such that $f_i + t^kg_i \in J$. Set $J' = (f_1 + t^kg_1, ..., f_m + t^kg_m)$. We want to show that $J' = J$. Define $R$ by the exact sequence

$$0 \to J' \to J \to R \to 0$$

of $B[z_1, ..., z_n]$-modules, and we want to show $R = 0$. Tensoring (3.6) with $B_k$ yields

$$J' \otimes B_k \xrightarrow{\alpha} J \otimes B_k \to R \otimes B_k \to 0.$$

Note that by (3.5), the map $\alpha$ is surjective and thus $R \otimes B_k = 0$. Now let $r \in R$ be any element. Since $r \otimes 1$ is zero in $R \otimes B_k$, we have $r = t^kr'$ for some $r' \in R$. Similarly, $r' = t^kr''$ and so forth until we have a sequence $r = t^kr_{(s)}$ for $s \geq 0$. This means $r \in \bigcap_{l \geq 0} t^lR$. Note that $J'$ and $J$ are finitely generated $B[z_1, ..., z_n]$-modules, hence $R$ is finitely generated and hence $t$-adically complete (e.g. by [AM94, Proposition 10.13]), i.e. $R = \varprojlim (R/t^lR)$. However this implies $\bigcap_{l \geq 0} t^lR = 0$, hence $r = 0$.

**Proposition 3.7** (Lifting relations). Given $l > 0$, let $a_1, ..., a_m \in B[z_1, ..., z_n]$ be such that

$$a_1f_1 + ... + a_mf_m \in (t^l).$$

Then there are $a'_i \in B[z_1, ..., z_n]$ such that

$$(a_1 + t^la'_1)f_1 + ... + (a_m + t^la'_m)f_m = 0.$$

A similar statement holds if we replace $B[z_1, ..., z_n]$ by $\widehat{B[z_1, ..., z_n]}$, and also if we then additionally replace $f_i$ by $f'_i := f_i + t^kg_i$ in view of Proposition 3.6.

**Proof.** Consider the exact sequence of $B$-modules $0 \to J \to B[z_1, ..., z_n] \to A \to 0$ since all terms except possibly $J$ are flat, we deduce that also $J$ is a flat $B$-module. Let $... \to R_{-2} \to R_{-1} \to R_0 \to J \to 0$ be a resolution of $J$ by finitely generated free $B[z_1, ..., z_n]$-modules. We assume $R_0$ is freely generated by $f_1, ..., f_m$. All terms in the sequence are flat $B$-modules, hence tensoring the sequence with $B_l$ yields another exact sequence. We have $(a_1, ..., a_m) \in \ker(R_0 \otimes_B B_l \to J \otimes_B B_l)$, so it comes from an element $b \in R_{-1} \otimes_B B_l$. Let $\bar{b}$ be a lift of $b$ in $R_{-1}$. Its image in $R_0$ takes the form $(a_1 + t^la'_1, ..., a_m + t^la'_m)$ and it maps to zero in $J$, so is a relation as desired. \qed

3.9. **Proofs for the statements in §2**. We continue with the setup of the previous paragraph. The first draft of the following main result was inspired by [Ha10, Proof of Prop. 4.4].
Proposition 3.8 (Lifting isomorphisms). Assume \( l \geq k \) and that for some \( h \) we have a commutative diagram of \( \hat{B} \)-algebras

\[
\begin{array}{c}
B[z_1, \ldots, z_n] \xrightarrow{z \mapsto \bar{z}} A'/t^{l+1}I' \xrightarrow{\beta} A'/t^kI' \\
\quad \downarrow h \quad \downarrow \beta \quad \downarrow \varphi^{-1} \\
B[z_1, \ldots, z_n] \xrightarrow{z \mapsto z} A/t^{l+1}I \xrightarrow{\beta} A/t^kI
\end{array}
\]

where the horizontal maps are the natural projections (use Lemma 3.4) and \( \beta \) is an isomorphism. Let \( \hat{\beta} : A'/t^{l-2N}I' \to A/t^{l-2N}I \) denote the induced isomorphism. Then there is a \( \hat{B} \)-linear isomorphism \( \hat{\beta} : A'/t^{l+1}I' \to A/t^{l+1}I \) making the following diagram commutative

\[
\begin{array}{c}
A'/t^{l+1}I' \xrightarrow{\hat{\beta}} A'/t^{l-2N}I' \\
\quad \downarrow \beta \quad \downarrow \beta \\
A/t^{l+1}I \xrightarrow{\hat{\beta}} A/t^{l-2N}I.
\end{array}
\]

Proof. First note that only \( \beta \) is the relevant datum since \( h \) can always be found for any given \( \beta \). We find for all \( t^lI \) that

\[
A/t^lI = A/t^l \times_{A/(t^l+I)} A/I \quad \text{and} \quad A'/t^lI = A'/t^l \times_{A'/((t^l)+I')} A'/I'.
\]

Note that the isomorphism \( \beta \) respects these fibre product decompositions, i.e. \( \beta \) decomposes in isomorphisms \( \beta_1 : A'/t^l \to A/t^l \), \( \beta_2 : A'/((t^l)+I') \to A/((t^l)+I) \) and \( \beta_3 : A'/I' \to A/I \). In the pursuit of producing \( \hat{\beta} \), requiring \( \hat{\beta}_3 = \beta_3 \), it suffices to produce an isomorphism \( \hat{\beta}_1 : A'/t^l+1 \to A/t^l+1 \) that restricts to the isomorphism \( \hat{\beta}_2 : A'/((t^l)+I') \to A/((t^l)+I) \) that is already induced from \( \beta_3 \).

Here is how we achieve this. From now on, we mostly work modulo \( t^{l+1} \), first some notation: set

\[
J' := \ker \left( B[z_1, \ldots, z_n]/t^{l+1} \to A'/t^{l+1} \right),
\]

\[
J := \ker \left( B[z_1, \ldots, z_n]/t^{l+1} \to A/t^{l+1} \right),
\]

so \( J = (f_1, \ldots, f_m) \). By Proposition 3.6, \( J' = (f'_1, \ldots, f'_{m'}) \) with \( f'_i = f_i + t^kg_i \). By abuse of notation, we also denote by \( I \) and \( I' \) the pullback of the respective ideal to \( B[z_1, \ldots, z_n]/t^{l+1} \) and also to \( B[z_1, \ldots, z_n] \). Let \( h_{t+1} : B[z_1, \ldots, z_n]/t^{l+1} \to B[z_1, \ldots, z_n]/t^{l+1} \) be the restriction of \( h \). Since \( \beta \) is an isomorphism, we have that \( h_{t+1} \) as well as \( h \) identifies \( t^{l}I' + J' \) with \( t^{l}I + J \) and also their squares. We thus have the following inclusions of \( B[z_1, \ldots, z_n]/t^{l+1} \)-modules,

\[
J^2 \subset (t^{l}I' + J')^2 \subset J' \subset t^{l}I' + J'
\]

\[
J \subset J \subset t^{l}I + J \subset t^{l+1}N + J.
\]
We observe that $h$ induces a $B[z_1, \ldots, z_n]/t^{l+1}$-module homomorphism

$$
\tilde{h} \in \text{Hom} \left( \frac{J'}{J''2}, \frac{t^{l-2N}I + J}{t^{l+1}I + J} \right) =: H.
$$

**Claim:** $\tilde{h} = t^{2N}\tilde{h}'$ for some $\tilde{h}' \in H$.

Indeed, $h$ sends $f'_i$ to $h(f'_i) = f_i + t^lh_i \equiv t^lh_i$ (modulo $J$) for some $h_i \in I$ (as an element of $B[z_1, \ldots, z_n]$). We define $\tilde{h}'$ via $\tilde{h}'(f'_i) = t^{l-2N}h_i$ and need to show this is well-defined. Since $h_i \in I$, $\tilde{h}'$ does indeed map into $t^{l-2N}I + J$ and we need to check that this preserves relations. Let $0 = \sum a_if'_i$ hold in $B[z_1, \ldots, z_n]/t^{l+1}$, so by Prop. 3.7 we may assume this holds already in $B[z_1, \ldots, z_n]$. Hence, $0 = \sum h(a_if'_i) = \sum h(a_i)(f_i + t^lh_i)$ holds in $B[z_1, \ldots, z_n]$ and thus $0 = t^{2N}\sum h(a_i)(t^{l-2N}h_i)$ holds in $A$ where $t^{2N}$ is a non-zerodivisor. We conclude that $0 = \sum h(a_i)(t^{l-2N}h_i)$ holds in $A$ and then also in $A/t^{l+1}$. Hence $\tilde{h}'$ is well-defined and we verified the claim.

We derive from 3.3, 3.1, 3.2 and 3.4 the following exact sequence of $B[z_1, \ldots, z_n]/t^{l+1}$-modules

$$
(3.7) \quad \text{Hom} \left( \Omega_B[z_1, \ldots, z_n]/t^{l+1}/B_{t^{l+1}}, \frac{t^{l-2N}I + J}{t^{l+1}I + J} \right) \rightarrow H \rightarrow T^1 \left( A'_{t^{l+1}}/B_{t^{l+1}}, \frac{t^{l-2N}I + J}{t^{l+1}I + J} \right) \rightarrow 0.
$$

Note that $\frac{t^{l-2N}I + J}{t^{l+1}I + J}$ is an $A_{2N+1}$-module that is identified with the $A_{2N+1}'$-module $\frac{t^{l-2N}I' + J'}{t^{l+1}I' + J'}$ under $\varphi$. Setting $L = 2N + 1$ in Lemma 3.3 combined with the claim above yields that $\tilde{h}$ maps to zero in $T^1 \left( A'_{t^{l+1}}/B_{t^{l+1}}, \frac{t^{l-2N}I + J}{t^{l+1}I + J} \right)$ and hence lifts to a derivation

$$
\theta \in \text{Hom}_{B[z_1, \ldots, z_n]/t^{l+1}} \left( \Omega_B[z_1, \ldots, z_n]/t^{l+1}/B_{t^{l+1}}, (t^{l-2N}I + J)/(t^{l+1}I + J) \right).
$$

Then let $\hat{\theta}$ refer to the map of $B$-modules $B[z_1, \ldots, z_n]/t^{l+1} \rightarrow A_{t^{l+1}}, q \mapsto \theta(dq)$. We then set $\hat{\beta}_1 = h - \hat{\theta}$. Indeed this gives a map $B[z_1, \ldots, z_n]/t^{l+1} \rightarrow A_{t^{l+1}}$ that sends $J'$ to zero since by construction the image of $J'$ under $h$ and under $\hat{\theta}$ agree. We therefore get a map $\hat{\beta}_1 : A'_{t^{l+1}} \rightarrow A_{t^{l+1}}$ of $B$-modules. First observe that $\hat{\beta}_1$ is a map of $B$-algebras: indeed, we find $(h(a) - \hat{\theta}(a))(h(b) - \hat{\theta}(b)) = h(ab) - \hat{\theta}(ab)$ follows from $\hat{\theta}(a)\hat{\theta}(b) = 0$ since $2(l - 2N) \geq l + 1$. Since $\beta$ restricts to an isomorphism $A'/I' \rightarrow A/I$ and $h$ is a lifted of it, the restriction of $h$ to $A'_{t^{l+1}} \rightarrow A_{t^{l+1}}$ takes $I'$ to $I$. Since $\theta$ has image contained in $I$, we conclude that $\hat{\beta}_1$ induces the restriction $A'/(t^{l+1} + I') \rightarrow A/(t^{l+1} + I)$ which coincides with the restriction of $\beta$. This readily allows us to glue $\hat{\beta}_1$ along $\tilde{\beta}_2$ and $\tilde{\beta}_3$ to a map $\tilde{\beta} : A'/t^{2N}I' \rightarrow A/t^{2N}I$. Furthermore, because $\theta$ has image in $t^{l-2N}I$, $\tilde{\beta}$ fits in the commutative diagram with $\tilde{\beta} : A'/t^{l-2N}I' \rightarrow A/t^{l-2N}I$ as required in the assertion.

It remains to show that $\tilde{\beta}$ is an isomorphism, or equivalently that $\hat{\beta}_1$ is one. We know that modulo $t^k$, the map $\hat{\beta}_1$ agrees with the restriction of $\varphi$. By the definition of $\bar{z}_i$, we thus have $h(z_i) = z_i + t^kz'_i$ and thus $h(z_i) - \bar{\theta}(z_i) = z_i + t^kz'_i + t^lz''_i$ for some $z'_i, z''_i$. We can view $h(z_i) - \bar{\theta}(z_i)$ as a lift of $z_i$ in the sense of Lemma 3.4 which then implies that $\hat{\beta}_1$ is surjective. We still have to show that $\hat{\beta}_1$ is also injective. Let
\[ \hat{\beta}_1 : B[z_1, \ldots, z_n]/t^k \rightarrow B[z_1, \ldots, z_n]/t^{k'} \] be a lift of \( \hat{\beta}_1 \) (which sends \( J' \) to \( J \)). Since \( \beta \) is an isomorphism, an element in the kernel of \( \hat{\beta}_1 \) is represented in the form \( t^f \) for some \( f \in B[z_1, \ldots, z_n]/t^{k+1} \). By flatness (as argued for \( I \) before), \( (t^f) \cap J = t^fJ \), so \( \hat{\beta}_1 \) maps \( t^f \) into \( t^fJ \). But on the other hand, since \( \varphi \) is an isomorphism, modulo \( t \), \( \hat{\beta}_1 \) is identity and \( J'/tJ' = J/tJ \). Thus \( f + tf' \in J' \) for some \( f' \). But then \( t^f(f + tf') = t^f \) in \( A'_{l+1} \), so \( t^f = 0 \) in \( A'_{l+1} \).

**Proof of Theorem 2.2.** Using Proposition 3.8, an induction on \( l \geq k \) with \( l = k \) the base case gives an isomorphism \( \varphi_l : A'_l \rightarrow A_l \) for \( l \geq k \). These isomorphisms are not compatible with the projections \( A'_{l+1} \rightarrow A'_l \) and \( A_{l+1} \rightarrow A_l \), however this is easy to fix. Define \( \psi_l := \varphi_l + 2t^l A'_{l+1} \rightarrow A_l \). Then for any \( l \geq 0 \) we have a commutative diagram

\[
\begin{array}{ccc}
A'_{l+1} & \xrightarrow{\psi} & A_l \\
\downarrow{\psi_{l+1}} & & \downarrow{\psi} \\
A_{l+1} & \xrightarrow{\psi_l} & A_l
\end{array}
\]

where the vertical maps are isomorphisms and the horizontal maps the natural projections. Taking the inverse limit over \( l \) gives the result. \( \square \)

**Proof of Theorem 2.3.** In view of [Ar69, Corollary (2.1)], Theorem 2.3 is essentially a corollary of Theorem 2.2 as follows. Present \( A = B[z_1, \ldots, z_n]/(f_1, \ldots, f_m), A' = B[z_1, \ldots, z_n]/(g_1, \ldots, g_m) \) so that \( x \) is the origin. Consider the system of \( m \) equations

\[(3.8) \quad f_i(a_1, \ldots, a_n) = \sum_{j=1}^m b_{ij} g_j\]

in the variables \( a_i, b_{ij} \). The \( \hat{B} \)-linear isomorphism \( \hat{\varphi} \) of \( \hat{A} \) with \( \hat{A}' \) obtained by Theorem 2.2 provides a formal solution of (3.8) by setting \( a_j = \hat{\varphi}_j(z_1, \ldots, z_n) \) and power series \( b_{ij} \) exist since \( \hat{\varphi} \) takes the ideal \( (f_1, \ldots, f_m) \) to \( (g_1, \ldots, g_m) \). Hence, the étale approximation of the formal solution as of [Ar69, Corollary (2.1)] provides the result to be proven. \( \square \)

**Proof of Theorem 2.4.** This goes precisely as the previous proof except that now we approximate the formal solution by an analytic solution, the corresponding approximation theorem by Artin is [Ar68, Theorem (1.2)]. \( \square \)

**Proof of Lemma 2.5.** Let \( \pi : X = \text{Spec} A \rightarrow Y = \text{Spec} B \) denote the map and let \( U = Y \setminus \{ t = 0 \} \). The sheaf \( T^{1}_{X/Y} \) is coherent. To prove that it restricts to zero over \( f^{-1}(U) \), by Nakayama’s lemma, it suffices to show that \( T^{1}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_{X,x} = 0 \) for all \( x \in X \) with \( \pi(x) \in U \). It suffices to show that if \( y \in U \) is a point with \( m_{y, y} = (t_1, \ldots, t_r) \)

then \(T^1(A/B, A) \otimes_A A/(t_1, \ldots, t_r) = 0\). For \(M\) an \(A\)-module, we get via (3.6) the exact sequence

\[
\cdots \to T^1(A/B, M/\Ann(t_i)) \xrightarrow{j_i^*} T^1(A/B, M) \to T^1(A/B, M/t_iM) \to \cdots
\]

from which we learn that

\[
T^1(A/B, M) \otimes A/(t_i) \subseteq T^1(A/B, M \otimes A/(t_i)).
\]

By the assumption and by (3.6) we have \(0 = T^1(X_y/y) = T^1(A/B, A/(t_1, \ldots, t_r))\).

Now by successively using (3.9) and Nakayama’s Lemma again, we indeed find that \(T^1(A/B, A) \otimes_A A/(t_1, \ldots, t_r) = 0\) as desired. \(\Box\)

4. APPROXIMATIONS WITH NON-RIGID NEARBY FIBRES

We next prove variants of the theorems in [2] where \(T^1_{X/Y}\) is supported in \(\{tw = 0\}\) where \(t \in \Gamma(Y, \mathcal{O}_Y)\) and \(w \in \Gamma(X, \mathcal{O}_X)\), so nearby fibres need not be rigid. To remedy this, in the notation of [2] we require \(I = (w^r)\) for a suitable \(r\).

**Setup 4.1.** Let \(X \to Y\) be a flat finite type map of Noetherian affine schemes. Let \(t \in \Gamma(Y, \mathcal{O}_Y) =: B\) and \(w \in \Gamma(X, \mathcal{O}_X) =: A\) be non-zero-divisors. Let \(t\) be a non-zero divisor in \(A/(w^r)\) for all \(r\). Set \(Z_r := \Spec A/(w^r)\). Assume that \(T^1_{X/Y}\) is supported in \(\{tw = 0\}\). Let \(\widehat{X} \to \widehat{Y}\) be the completion in \(t\).

**Theorem 4.2 (formal with a divisor).** There exist \(N > 0\) and \(M > 0\) so that if \(X' \to Y\) is another flat map of Noetherian affine schemes, \(w' \in \Gamma(X', \mathcal{O}_{X'})\), \(Z'_r \subset X'\) given by \((w')^r = 0\) and \(\varphi : X'_k \cup Z'_r \to X_k \cup Z_r\) an isomorphism over \(Y\) for some \(k > 4N\) and \(r > 4M\) that restricts to an isomorphism \(Z'_r \simto Z_r\) and has \(\varphi^* (w|_{X'_k \cup Z'_r}) = w'|_{X_k \cup Z_r}\), then there is an isomorphism \(\widehat{\varphi} : \widehat{X}' \to \widehat{X}\) that commutes with the maps to \(\widehat{Y}\) and so that the restrictions of \(\widehat{\varphi}\) and \(\varphi\) to \(X'_{k-2N} \cup Z'_{r-2M}\) agree where \(Z'_{r-2M}\) is the completion of \(Z'_{r-2M}\) in \(t\).

**Proof.** For the largest part, the proof coincides with the proof of Theorem 2.2 in [2]. We only address the adaptations. The first relevant observation is that \((t^kw^l)\) is isomorphic to \(A\) as an \(A\)-module for all \(k,l \geq 0\). Similarly, as in Lemma 3.1, for \(r, k \geq 0\), \((t^kw^r)/(t^{k+K}w^{r+L})\) is isomorphic to \(A/(t^kw^L)\) as an \(A\)-module. By the support assumption and since \(A \cong (w^r)\), there are \(N, M, \geq 0\) such that

\[
t^N w^MT^1(A/B, (w^r)) = 0
\]

for all \(r \geq 0\) and furthermore such that \(\ker_1 N w^MT^2_{A/B}\) is stable in the sense of (3.1). Similar to (3.8) from the short exact sequence \(0 \to A \xrightarrow{t^M} (w^M) \to A/(t^l) \to 0\), for
Remark 4.3. The resulting $\hat{\varphi}$ in Theorem 4.2 doesn’t necessarily map $w$ to $(w')^r$. However, it does map $(w)$ to $(w')^r$, so we can only conclude that it maps $w$ to $\varepsilon w'$ for $\varepsilon$ a unit satisfying $1 = \varepsilon|x_{k-2N}\cup Z_{r-2M-1}|$.

Theorem 4.4 (étale with a divisor). Assume we work over a field or excellent discrete valuation ring. There exist $N > 0, M > 0$ with the following property. Let $X' \to Y$ be another flat map of Noetherian affine schemes, $w' \in \Gamma(X', \mathcal{O}_{X'})$, $Z'_r \subset X'$ given by $(w')^r = 0$ and $\varphi : X'_k \cup Z'_r \to X_k \cup Z_r$ an isomorphism over $Y$ for some $k > 4N$ and $r > 4M$ that restricts to an isomorphism $Z'_r \sim \to Z_r$ and has $\varphi^* (w|_{X'_k \cup Z'_r}) = w'|_{X_k \cup Z_r}$. Let $x \in X$ be a point with $t(x) = w(x) = 0$, then there are étale neighbourhoods $U, U'$ of $x$ in $X, X'$ respectively and an isomorphism $\varphi_x : U' \to U$ that commutes with the maps to $Y$ and so that the restrictions of $\varphi_x$ and $\varphi$ to $X'_k \cup Z'_{r-2M}$ agree.

Proof. Given Theorem 4.2, the proof proceeds precisely as the proof of Theorem 2.3 in [3.9] except that we approximate with respect to the ideal $(tw)$ rather than $(t)$.

Theorem 4.5 (analytic with a divisor). Let $X \to Y$ be a flat map of complex analytic spaces and $t \in \Gamma(Y, \mathcal{O}_Y)$ and $w \in \Gamma(X, \mathcal{O}_X)$ be non-zero-divisors. Let $Z, Z_k, X_k \subset X$ be the closed subspaces given by $w = 0, w^k = 0, t^k = 0$ respectively. Assume that $\mathcal{T}^a_{X/Y}$ is supported in $tw = 0$. There exist $N > 0, M > 0$ with the following property. Let $X' \to Y$ be another flat map of complex analytic spaces with
w′ ∈ Γ(X′, ℓO_X′), Z′_r ⊂ X′ given by (w′)^r = 0 and ϕ : X'_k ∪ Z'_r → X_k ∪ Z_r an isomorphism over Y for some k > 4N and r > 4M that restricts to an isomorphism Z'_r ⊃ Z_r and has ϕ^*(w|_{X'_k∪Z'_r}) = w'|_{X_k∪Z_r}. Let x ∈ X be a point with t(x) = w(x) = 0, then there are étale neighbourhoods U, U' of x in X, X' respectively and an isomorphism ϕ_x : U' → U that commutes with the maps to Y and so that the restrictions of ϕ_x and ϕ to (X'_k−2N ∪ Z'_r−2M) ∩ U' agree.

Proof. Given Theorem 4.2, the proof proceeds precisely as the proof of Theorem 2.4 in §3.9 except that we approximate with respect to the ideal (tw) rather than (t). □

5. Finite determinacy of log morphisms

We explain in this section how to deduce from the local uniqueness results in the previous sections a useful fact for fibres of morphisms of log spaces. Let X be either an algebraic space with the étale topology or a complex analytic space with the complex analytic topology. For an effective reduced Weil divisor D ⊂ X let j : X \ D → X denote the open embedding of its complement. The divisorial log structure M_{(X,D)} is defined to be the monoid sheaf

\[ M_{(X,D)} := \frac{j_! \mathcal{O}_{X \setminus D}}{\mathcal{O}_X} \]

together with its natural embedding in \( \mathcal{O}_X \). Note that M_{(X,D)} contains \( \mathcal{O}_{X}^X \).

Lemma 5.1. If X is normal then the sheaf \( \overline{\mathcal{M}}_{(X,D)} := M_{(X,D)}/\mathcal{O}_X^X \) is constructible, i.e. every point has an étale neighbourhood that is stratified by locally closed sets on which the sheaf is constant with finitely generated and integral stalks. Furthermore, the stalks are saturated.

Proof. We only address the situation where X is an algebraic space, because the same reasoning yields the proof in the situation where X is a complex analytic space. Set \( \overline{\mathcal{M}} = \overline{\mathcal{M}}_{(X,D)} \). Given a point x ∈ X, let V be an (étale) neighbourhood of x where every component of D containing x is geometrically unibranch. Let D_1, ..., D_r be these components. Consider the map \( \mathcal{M}_{(X,D),x} \rightarrow \mathbb{N}^r \) given by recording the vanishing order of a section along D_1, ..., D_r. This map factors through \( \overline{\mathcal{M}}_x \). We claim that in fact \( \overline{\mathcal{M}}_x \rightarrow \mathbb{N}^r \) is injective. Indeed, given two functions \( f_1, f_2 \) with the same image, the quotient \( f_1/f_2 \) is invertible on a set whose complement has codimension two, so by the \( S_2 \) property extends to a section of \( \mathcal{O}_X^{X,x} \) and this is trivial in \( \overline{\mathcal{M}}_x \). The group G of Cartier divisors with support in D_1 ∪ ... ∪ D_r is a subgroup of \( \mathbb{Z}^r \) and hence finitely generated and G ∩ \( \mathbb{N}^r \) coincides with the image of \( \overline{\mathcal{M}}_x \) in \( \mathbb{N}^r \), so \( \overline{\mathcal{M}}_x \) is integral, saturated and finitely generated.
We now address the stratification of the chart $V$. This will be a refinement of the stratification by the locally closed sets

$$S := \left\{ \left( \bigcap_{i \in I} D_i \right) \setminus \left( \bigcup_{j \in \{1, \ldots, r\} \setminus I} D_j \right) \mid I \subset \{1, \ldots, r\} \right\},$$

indeed let $\eta$ be the generic point of some irreducible component $E$ of a stratum $S_I \in S$ for some $I \subset \{1, \ldots, r\}$. By what we just said before, $\mathcal{M}_{|E}$ is a subsheaf of the constant sheaf $\mathcal{N}_{|I}$. The stalk $\mathcal{M}_{\bar{\eta}}$ is finitely generated, let $g_1, \ldots, g_n \in \mathcal{M}_{\bar{\eta}}$ descend to a set of generators of $\mathcal{M}_{\bar{\eta}}$. Let $V_i \to V$ be the open set of definition of $g_i$, so we have an isomorphism

$$\pi : \Gamma(V_1 \cap \ldots \cap V_n \cap E, \mathcal{M}) \to \mathcal{M}_{\bar{\eta}}.$$

Let $\bar{y} \in V_1 \cap \ldots \cap V_n \cap E$ be a point, we want to show that also $\rho : \Gamma(V_1 \cap \ldots \cap V_n \cap E, \mathcal{M}) \to \mathcal{M}_{\bar{\eta}}$ is an isomorphism. We know $\rho$ is injective because $\pi$ factors through $\rho$. This implies surjectivity as well because a section at $x$ with certain vanishing orders along the $D_i$ has that same vanishing behaviour at $\bar{\eta}$. Hence $\mathcal{M}$ is constant on $V_1 \cap \ldots \cap V_n \cap E$.

We obtain a stratification into sets like this by induction on dimension (also applying the above to generic points of components of the locus that wasn’t covered in a prior step).

**Remark 5.2.** Note that divisorial log structures are not necessarily coherent in the sense of admitting charts (an assumption often made in the literature). For instance for $X = \text{Spec} \mathbb{Z}[x, y, z, w]/(xy - zw)$ and $D = V(z)$, the sheaf $\mathcal{M}_{(X, D)}$ has rank one at the origin (generated by $z$) but rank two along $x = y = 0$ away from the origin, so $\mathcal{M}_{(X, D)}$ cannot have a chart at the origin.

Let $f : X \to Y$ be a finitely presented flat morphism of either

1. normal complex analytic spaces or
2. normal algebraic spaces defined over a field or excellent discrete valuation ring with $Y$ locally Noetherian.

Let $D \subset Y$ be a non-empty reduced effective Cartier divisor. We define log structures $\mathcal{M}_X := \mathcal{M}_{(X, f^{-1}(D))}$ and $\mathcal{M}_Y := \mathcal{M}_{(Y, D)}$ on $X$ and $Y$ respectively. Furthermore $f$ is compatible with these log structures, i.e. we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{f} & \mathcal{M}_X \\
\uparrow \quad & & \uparrow \\
f^{-1}\mathcal{O}_Y & \xrightarrow{f^{-1}} & f^{-1}\mathcal{M}_Y.
\end{array}$$

Let $f^\dagger : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ denote the induced map of log spaces. Let $0 \in D$ be a point. We denote by $f_0^\dagger : X_0^\dagger \to 0^\dagger$ the base change of $f^\dagger$ to $0$ as a log morphism.
obtained as follows. The underlying morphism is just the fibre $f_0 : X_0 \to 0$. Consider the commutative diagram

$$
\begin{array}{c}
\mathcal{O}_{X_0} \xrightarrow{\alpha} \mathcal{M}_X|_{X_0} \\
f_0^{-1}\mathcal{O}_0 \xrightarrow{f^{-1}\mathcal{M}_Y|_0}
\end{array}
$$

which constitutes a map of pre-log-spaces. We transition to the associated log spaces by taking for $\mathcal{M}_{X_0}$ the pushout $\mathcal{M}_X|_{X_0} \oplus_{\alpha^{-1} (\mathcal{O}_{X_0})} \mathcal{O}_{X_0}$ and similarly for $\mathcal{M}_0$.

In the complex analytic situation, assume that $f$ is proper. In the algebraic space situation, assume that $f$ is finitely presented. In either case, assume that, for every point $y \in Y \setminus D$, the fibre $X_y := f^{-1}(y)$ is locally rigid, i.e. $T^1_{X_y/y} = 0$. Let $t$ be a local equation for $D$ at $0$.

**Theorem 5.3.** There is $k \geq 0$ such that $f_0^\dagger$ only depends on the restriction of $f^\dagger$ to $(D, \mathcal{O}_Y/(t^k))$ in the following sense: let $f' : X' \to Y$ be another flat proper map (respectively finitely presented map) together with an isomorphism $\varphi$ of the base changes of $f$ and $f'$ to $(D, \mathcal{O}_Y/(t^k))$. Let $(f')^\dagger$ and $(f_0^\dagger)^\dagger$ be the corresponding log morphisms obtained analogously as we produced $f^\dagger, f_0^\dagger$, then $(f_0^\dagger)^\dagger$ is canonically isomorphic to $f_0^\dagger$ (by means of $\varphi$).

**Proof.** In the first step, we show that $\mathcal{M}_X|_{X_0}$ and the map $f^{-1}\mathcal{M}_Y|_{X_0} \to \mathcal{M}_X|_{X_0}$ are independent of $X, X'$ (i.e. canonically identified by $\varphi$). By Lemma 5.1, $\mathcal{M}_X$ and $\mathcal{M}_Y$ are constructible sheaves. Let $N'$ be larger than the vanishing orders along components of $D$ and $f^{-1}(D)$ of all generators of all stalks as well as the images of the generators of $\mathcal{M}_Y$ (which is a finite set by constructibleness and compactness/finite type).

By Lemma 2.5 and quasi-compactness, there is $N > 0$ such that $t^N T^1_{X/Y} = 0$. By increasing $N$ further if needed, we may assume that $N$ is larger than $N'$ and that $\ker_{t^N}(T^2_{X/Y})$ is stable in the sense of (3.1). We set $k = 4N + 1$.

If we are given $X' \to Y$ with isomorphism $\varphi$ to $X \to Y$ modulo $t^k$ then, by Theorem 2.3 or Theorem 2.4 and quasi-compactness, we find a finite cover $\{U_\alpha\}_\alpha$ of $X$ and $\{U'_\alpha\}_\alpha$ of $X'$ such that $\varphi_\alpha : U_\alpha \to U'_\alpha$ is an isomorphism over $Y$ restricting to the identity modulo $t^{2N + 1}$.

This readily implies that the restrictions to $X_0$ of $\mathcal{M}_X$ and $\mathcal{M}_{X'}$ are isomorphic by means of $\varphi$ (compatibly with the map from $f_0^{-1}\mathcal{M}_{Y,0}$). Indeed $\varphi_\alpha$ provide isomorphisms locally and to check that these globalize, note that $\varphi_\alpha$ and $\varphi_\beta$ differ (additively) by terms of the form $t^{4N + 1}g$ for $g$ a holomorphic (respectively regular) function, so by our choice of $N$ in particular the vanishing orders of generators of stalks are preserved but then stalks entirely are preserved since stalks embed in the vanishing order monoids by the proof of Lemma 5.1.
In order to produce an isomorphism of the restrictions of $\mathcal{M}_X$ and $\mathcal{M}_{X'}$ to $X_0$ compatibly with the embeddings of $f^{-1}\mathcal{M}_Y$, we show that the restrictions of the local isomorphisms $\varphi_\alpha$ glue. Consider open sets $U_\alpha, U_\beta$ that meet each other, so we have two morphisms

$$\mathcal{M}_{U_\alpha \cap U_\beta} \xrightarrow{\varphi_\alpha} \mathcal{M}_{U'_\alpha \cap U'_\beta}$$

and these are identical modulo $t^{2N}$ and induce the identity on $\overline{\mathcal{M}}_{X_0}$. Composing yields an automorphism of the source $(\varphi_\beta^{-1} \circ \varphi_\alpha)|_{X_0} \in \text{Aut}(\mathcal{M}_{U_\alpha \cap U_\beta}|_{X_0})$ and if we show this is the identity then we are done. Note that both $\varphi_\alpha$ and $\varphi_\beta$ fix $\mathcal{O}_{X_0}^\times$ pointwise. Using constructibleness, by [ScSi06, Proposition 2.2], the natural inclusion

$$\text{Hom}(\overline{\mathcal{M}}_X, \mathcal{O}_X^\times) \to \text{Aut}(\mathcal{M}_X)$$

that sends $h$ to $(m \mapsto m \cdot h(\overline{m}))$ is an isomorphism. The restriction of an automorphism is just the image in $\text{Hom}(\overline{\mathcal{M}}_X|_{X_0}, \mathcal{O}_{X_0}^\times)$. We see now that $(\varphi_\beta^{-1} \circ \varphi_\alpha)|_{X_0}$ restricts to the identity since the images of the generators in $\mathcal{O}_{U_\alpha \cap U_\beta}^\times$ are 1 modulo $t^{2N}$, so in particular 1 when restricting to $U_\alpha \cap U_\beta \cap X_0$. \qed

We finally prove a variant of Theorem 5.3 that makes use of the possibility $Z \neq \emptyset$ in the theorems in [2]. We keep the setup of the map $f : X \to Y$ with divisor $D \subset Y$ with all assumptions as before. In addition, assume to be given an effective Cartier divisor $E$ in $X$, so that $E \cap f^{-1}(D)$ is a Cartier divisor in $E$. We now define the log structure on $X$ to be the divisorial log structure $\mathcal{M}_{(X, E \circ f^{-1}(D))}$ and the one on $Y$ to be $\mathcal{M}_{(Y, D)}$ as before. This upgrades $f$ to a log morphism $f^\dagger : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ similar to before. Again, fix $0 \in D$.

**Theorem 5.4.** There is $k \geq 0$ such that $f_0^\dagger$ only depends on the restriction of $f^\dagger$ to $(D, \mathcal{O}_Y/(t^k)) \cup E$ in the following sense: let $f' : X' \to Y$ be another flat proper map (respectively finitely presented map) together a divisor $E' \subset X'$ and an isomorphism $\varphi : V(t^k) \cup E \to V(t^k) \cup E'$ of the restrictions of $f$ and $f'$ to the non-reduced subspaces $(f^{-1}(D), \mathcal{O}_X/(t^k)) \cup E$ and $((f')^{-1}(D), \mathcal{O}_{X'}/(t^k)) \cup E'$ inducing an isomorphism $E \cong E'$ over $Y$. Let $(f')^\dagger$ and $(f_0^\dagger)^\dagger$ be the corresponding log morphisms, then $(f_0^\dagger)^\dagger$ is canonically isomorphic to $f_0^\dagger$ (by means of $\varphi$).

**Proof.** The proof is literally the same as the proof of Theorem 5.3 except that we use $Z = E$ and $Z' = E'$ now. \qed

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