DYNAMIC BEHAVIOR OF A STOCHASTIC PREDATOR-PREY SYSTEM UNDER REGIME SWITCHING

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ABSTRACT. In this paper we deal with regime switching predator-prey models perturbed by white noise. We give a threshold by which we know whenever a switching predator-prey system is eventually extinct or permanent. We also give some numerical solutions to illustrate that under the regime switching, the permanence or extinction of the switching system may be very different from the dynamics in each fixed state.

1. Introduction. Lotka-Volterra equations are a simple model of the population dynamics of species and have been extensively investigated in the literature concerning ecological population modeling. One particularly interesting subclass, where the relationship between species is the predator-prey interaction, represented through the deterministic equations

\[
\begin{align*}
    dX(t) &= X(t)\left(a_1 - b_1 Y(t) - c_1 X(t)\right)dt, \\
    dY(t) &= Y(t)\left(-a_2 + b_2 X(t) - c_2 Y(t)\right)dt,
\end{align*}
\]

with \(X(t)\) and \(Y(t)\) to be respectively the density of prey and predator populations and \(a_i, b_i, c_i\) for \(i = 1, 2\) to be positive constants. However, it is well-recognized that the evolution of population in an eco-system is always subject to unpredictable factors and the problem to learn how randomness effects to the long term behavior of population is interesting. Stochastic factors may intervene to the model under the forms of white noise, color noise or bounded noise (see \([1, 2, 15, 20]\)). In case

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the intrinsic growth rate \( c_1 \) of the prey and \( c_2 \) of the predator are perturbed by Brownian motions \( W_1(t) \), \( W_2(t) \), i.e., these coefficients are replaced by \( c_1 + \sigma W_1(t) \) and \( c_2 + \rho W_2(t) \), respectively, the system \( (1) \) becomes

\[
\begin{align*}
\dot{X}(t) &= X(t)(a_1 - a_1 Y(t) - c_1 X(t))dt + \sigma X^2(t)dW_1(t), \\
\dot{Y}(t) &= Y(t)(-a_2 + b_2 X(t) - c_2 Y(t))dt + \rho Y^2(t)dW_2(t). 
\end{align*}
\]

The existence and uniqueness of the positive solution of \( (2) \) has been considered by X. Mao et al. \cite{20}; the estimations of upper growth and lower growth of the sample processes. In continuing these studies, this paper will focus on the system \( (2) \) with taking Markovian switching into account. Formally, under regime switching, the system \( (2) \) becomes

\[
\begin{align*}
\dot{X}(t) &= X(t)(a_1(r_t) - b_1(r_t)Y(t) - c_1(r_t)X(t))dt + \sigma(r_t)X^2(t)dW_1(t), \\
\dot{Y}(t) &= Y(t)(-a_2(r_t) + b_2(r_t)X(t) - c_2(r_t)Y(t))dt + \rho(r_t)Y^2(t)dW_2(t), 
\end{align*}
\]

where \( \{r_t, t \geq 0\} \) is a right continuous Markov chain taking values in \( S = \{1, 2, \ldots, N\} \).

Our main goal in this paper is to provide a classification for the stochastic predator-prey model under regime switching \( (3) \). Using a new technique we are able to derive a sufficient and almost necessary condition for permanence (as well as ergodicity) and extinction of model via a real value \( \lambda \). Among other things, we can apply this technique to improve some results of other stochastic predator-prey models as in \cite{12, 14, 26}.

The rest of paper is arranged as follows. Section 2 provides some main results of this work. We define a threshold \( \lambda \) and show that if \( \lambda < 0 \), the predator \( Y(t) \) will eventually die out and decay in an exponential rate \( e^{\lambda t} \). In this case we derive that \( X(t) - \bar{X}(t) \) converges almost surely to 0 as \( t \to \infty \), where \( \bar{X}(t) \) is the solution on boundary of the prey (when the predator is absent). Meanwhile, when \( \lambda > 0 \) the Markov process solution \( \{(X(t), Y(t), r_t), t \geq 0\} \) converges in total variation to a stationary distribution concentrated in \( \mathbb{R}_+^{2\sigma} \times S \). It means that the predator-prey model is persistent. Further, the ergodicity of the solution process is also proved. Finally, some discussions are issued and some numerical simulations are carried out to illustrate our results in the last section.
2. **Sufficient and almost necessary conditions for Permanence.** Let \( W_1(t) \) and \( W_2(t) \) be two one-dimensional Brownian motions defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote by \( \Gamma = (\gamma_{ij})_{N \times N} \) the generator of the Markov chain \( \{r_t, t \geq 0\} \). This means that

\[
\mathbb{P}\{r_{t+\delta} = j | r_t = i\} = \begin{cases} 
\gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\
1 + \gamma_{ii}\delta + o(\delta) & \text{if } i = j,
\end{cases}
\]

as \( \delta \to 0 \). Hence, \( \gamma_{ij} \) is the transition rate from \( i \) to \( j \) and \( \gamma_{ij} \geq 0 \) if \( i \neq j \), while

\[
\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.
\]

It is well known that almost every sample path of \( \{r_t, t \geq 0\} \) is a right continuous step function with a finite number of jumps in any finite subinterval of \( \mathbb{R}_+ \). We assume that the Markov chain \( \{r_t, t \geq 0\} \) is irreducible, which means that the system will switch from any regime to any other regime. This is equivalent to the condition that for any \( u, v \in \mathcal{S} \), we can find a finite number of states \( i_1, i_2, \ldots, i_k \in \mathcal{S} \) such that \( \gamma_{ui_1}\gamma_{i_1i_2}\cdots\gamma_{i_{k-1}v} > 0 \). For this condition, the Markov chain \( \{r_t, t \geq 0\} \) has a unique stationary distribution \( \pi = (\pi_1, \pi_2, \ldots, \pi_N) \in \mathbb{R}^N \). By the ergodicity of \( \{r_t, t \geq 0\} \) we define the averaged intrinsic growth rate of the prey population \( A_1 \) (resp. the averaged loss rate of the predator \( A_2 \))

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t a_1(r_s) ds = \sum_{i=1}^N a_1(i) \pi_i =: A_1, \quad (4)
\]

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t a_2(r_s) ds = \sum_{i=1}^N a_2(i) \pi_i =: A_2. \quad (5)
\]

We assume that the Markov chain \( \{r_t, t \geq 0\} \) is independent of the Brownian motions \( W_1(t), W_2(t) \). From [13, Theorem 2.2] we yield that for any positive initial value \((X(0), Y(0)) \in \mathbb{R}_+^2 := \{(x, y) : x \geq 0, y \geq 0\} \), there exists a unique global solution \( \{(X(t), Y(t)), t \geq 0\} \) of Equation (3) that remains in \( \mathbb{R}_+^2 \) with probability.

Further, the set \( \mathbb{R}_+^{2,0} := \{(x, y) : x > 0, y > 0\} \) is also an invariant set, i.e., if \( X(0) > 0, Y(0) > 0 \) then \( X(t) > 0, Y(t) > 0 \ \forall \ t > 0 \).

If the predator is absent, we have the prey equation on the boundary

\[
\begin{align*}
\frac{d\bar{X}(t)}{dt} &= (a_1(r_t)\bar{X}(t) - c_1(r_t)\bar{X}^2(t))dt + \sigma(r_t)\bar{X}^2(t)dW_1(t), \\
\bar{X}(0) &= \bar{X}_0 > 0, t \geq 0.
\end{align*}
\]

From [3, Example 3.1, p. 630], we can show that the pair \( \{(\hat{X}(t), r_t), t \geq 0\} \) is a Markov process that has a stationary distribution \( \mu^*(dx, i) \) concentrated on \((0, \infty) \times \mathcal{S}\). Therefore, by using the Ergodic theorem [27, Theorem 3.16, p. 46], we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t b_2(r_s)\bar{X}(s) ds = \sum_{i=1}^N \int_{\mathbb{R}_+} b_2(i) x \mu^*(dx, i) =: m_1.
\]

\( m_1 \) can be considered as the averaged growth rate of predator populations under regime switching in saturated situation.

By setting \( Z(t) = \frac{1}{\bar{X}(t)}, \bar{Z}(t) = \frac{1}{\bar{X}(t)} \) and applying Itô’s formula we have
Using the comparison theorem \([10, \text{Theorem 1.1, p. 352}]\) yields that \(Z(t) \geq \hat{Z}(t) \forall t \geq 0\), provided \(Z(0) = \hat{Z}(0) > 0, Y(0) \geq 0\). Equivalently,
\[
X(t) \leq \bar{X}(t) \forall t \geq 0, \text{ provided } X(0) = \bar{X}(0) > 0, Y(0) \geq 0.
\]

Similarly, in case there is no prey we get the equation on the boundary of the predator as following:
\[
\begin{aligned}
\{d\hat{Y}(t) &= (-a_2(r_t)\hat{Y}(t) - c_2(r_t)\hat{Y}^2(t))dt + \rho(r_t)\hat{Y}^2(t)dW_2(t), \\
\hat{Y}(0) &= Y_0 \geq 0, t \geq 0.
\end{aligned}
\]

We consider the equation
\[
\begin{aligned}
d\bar{Y} &= \rho(r_t)\bar{Y}^2(t)dW_2(t), \\
\bar{Y}(0) &= \bar{Y}_0 \geq 0, t > 0.
\end{aligned}
\]

The comparison theorem implies that \(Y(t) \geq \bar{Y}(t)\) and \(0 \leq \tilde{Y}(t) \leq \bar{Y}(t) \forall t \geq 0\), a.s. provided \(Y(0) = \tilde{Y}(0) = \hat{Y}(0) \geq 0\), \(X(0) \geq 0\).

The equation (11) satisfies all conditions starting in [13, Theorem 5.37, pp. 208] with \(V(x) = |x|^p\) for a \(0 < p < 1\). Therefore, \(\lim_{t \to \infty} \tilde{Y}(t) = 0\), which implies that \(\lim_{t \to \infty} \tilde{Y}(t) = 0\). Though, \(\tilde{Y}(t)\) can not hit 0 in a finite time by the uniqueness of solution for the equation (10).

We are now in position to state the main results of our paper. Define the threshold
\[
\lambda = \sum_{i=1}^{N} \int_{\mathbb{R}^+} b_2(i)x\mu^*(dx,i) - \sum_{i=1}^{N} a_2(i)c_i = m_1 - A_2.
\]

Then, we have

**Theorem 2.1.** For \(\lambda\) determined in (12), we have

(i) If \(\lambda < 0\), then the predator is eventually extinct. In this case, \(\lim_{t \to \infty} \frac{\ln Y(t)}{t} = \lambda\) a.s. Moreover, for all real number \(\lambda < \hat{\lambda}\):
\[
\lim_{t \to \infty} e^{-\lambda t}(X(t) - \bar{X}(t)) = \lim_{t \to \infty} e^{-\lambda t}\left(\frac{1}{X(t)} - \frac{1}{\bar{X}(t)}\right) = 0, \text{ provided } X(0) = \bar{X}(0) > 0.
\]

(ii) If \(\lambda > 0\), the Markov process \(\{(X(t), Y(t), r_t), t \geq 0\}\) has a unique invariant probability measure \(\psi^*\) concentrated on \(\mathbb{R}^{2,0} \times S\). Moreover, if \(W_1(t), W_2(t), r_t\) are mutually independent, the support of \(\psi^*\) is \(\mathbb{R}^{2,0} \times S\).

To prove Theorem 2.1 we need the following lemma.

**Lemma 2.2.** With \(A_1, A_2\) are defined by (4) and (5) respectively, the following assertions hold for almost surely,

a) \(\limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2}Y^2(s) \right) ds \leq A_2,\)

b) \(\liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2}Y^2(s) \right) ds \geq 0,\)
On the other hand, from this relation and the Borel-Cantelli Lemma, for almost all $A$

Hence, by definition of $A$

By using Itô’s formula we have

Proof. By using Itô’s formula we have

Further, proving of [13, Theorem 7] has shown lim sup

Adding (13) and (14) side by side follows that

Hence, by definition of $A_2$ it yields

For any $\varepsilon > 0$, applying the exponential martingale inequality (see e.g. [18]) we get,

From this relation and the Borel-Cantelli Lemma, for almost all $\omega$, there exists a number $n_0 = n_0(\omega)$ such that for all $k > n_0$ and $0 \leq t \leq k$,

This inequality says that

Adding (13) and (14) side by side follows that

On the other hand,

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t b^2(r_s)X(s) ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t b^2(r_s)\hat{X}(s) ds = m_1.$$
Thus, the item a) has been proved.

Letting $\varepsilon > 0$, it yields

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2}Y^2(s) \right) ds
\leq \frac{1}{1 + \varepsilon} \lim_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)(1 + \varepsilon)}{2}Y^2(s) \right) ds
$$

$$
+ \frac{\varepsilon}{1 + \varepsilon} \lim_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) \right) ds \leq \frac{1}{1 + \varepsilon} A_2 + \frac{\varepsilon}{1 + \varepsilon} m_1.
$$

Letting $\varepsilon \to 0$ gets

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2}Y^2(s) \right) ds \leq A_2.
$$

Thus, the item a) has been proved.

In view of $Y(t) \geq \hat{Y}(t)$ $\forall t \geq 0$ and by applying the Itô’s formula, it yields

$$
\ln Y(t) - \ln \hat{Y}(t) = \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2}Y^2(s) \right) ds
$$

$$
+ \int_0^t \rho(r_s)Y(s)dW_2(s) + \int_0^t \left( c_2(r_s)\hat{Y}(s) + \frac{\rho^2(r_s)}{2}\hat{Y}^2(s) \right) ds
$$

$$
- \int_0^t \rho(r_s)\hat{Y}(s)dW_2(s) \geq 0. \quad (15)
$$

By the same argument as the proof of the first assertion,

$$
\lim_{t \to \infty} \frac{1}{t} \left( \int_0^t \rho(r_s)Y(s)dW_2(s) - \frac{\varepsilon}{2} \int_0^t \rho^2(r_s)Y^2(s)ds \right) \leq 0, \quad (16)
$$

and

$$
\lim_{t \to \infty} \frac{1}{t} \left( \int_0^t -\rho(r_s)\hat{Y}(s)dW_2(s) - \frac{\varepsilon}{2} \int_0^t \rho^2(r_s)\hat{Y}^2(s)ds \right) \leq 0.
$$

Combining these inequalities and (15), it gets

$$
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)(1 - \varepsilon)}{2}Y^2(s) \right)
$$

$$
+ c_2(r_s)\hat{Y}(s) + \frac{\rho^2(r_s)(1 + \varepsilon)}{2}\hat{Y}^2(s) ds \geq 0.
$$

Since $\hat{Y}(t) \to 0$ as $t \to \infty$,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left( c_2(r_s)\hat{Y}(s) + \frac{\rho^2(r_s)(1 + \varepsilon)}{2}\hat{Y}^2(s) \right) ds = 0.
$$

Hence,

$$
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)(1 - \varepsilon)}{2}Y^2(s) \right) ds \geq 0.
$$
Note that if \( \lim_{t \to \infty} \frac{1}{t} \int_0^t \rho^2(r_s) Y^2(s) ds = \infty \) then for \( \varepsilon \in (0, 1) \),

\[
0 \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)(1 - \varepsilon)}{2} Y^2(s) \right) ds
\]

\[
\leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t b_2(r_s)X(s) ds + \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( - \frac{\rho^2(r_s)(1 - \varepsilon)}{2} Y^2(s) \right) ds
\]

\[
\leq m_1 - \frac{(1 - \varepsilon)}{2} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \rho^2(r_s) Y^2(s) ds = -\infty.
\]

That is a contradiction. Thus,

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \rho^2(r_s) Y^2(s) ds < \infty \quad \text{a.s.} \quad (17)
\]

From this property we get

\[
0 \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)(1 - \varepsilon)}{2} Y^2(s) \right) ds
\]

\[
\leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2} Y^2(s) \right) ds
\]

\[
+ \frac{\varepsilon}{2} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \rho^2(r_s) Y^2(s) ds.
\]

Let \( \varepsilon \to 0 \), we obtain the item b). The proof of item c) is similar to the one of item a).

Now we prove the last item. From (6),

\[
\frac{\ln \hat{X}(t)}{t} = \frac{\ln \hat{X}_0}{t} + \frac{1}{t} \int_0^t a_1(r_s) ds - \frac{1}{t} \int_0^t \left( \frac{\sigma^2(r_s)}{2} \hat{X}^2(s) \right) ds
\]

\[
+ \frac{1}{t} \int_0^t \rho(r_s) \hat{X}(s)dW_1(s).
\]

By the same argument as above, we can prove

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \rho(r_s) \hat{X}(s)dW_1(s) = 0.
\]

So that,

\[
\lim_{t \to \infty} \frac{\ln \hat{X}(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t a_1(r_s) ds - \lim_{t \to \infty} \frac{1}{t} \int_0^t \left( c_1(r_s) \hat{X}(s) + \frac{\sigma^2(r_s)}{2} \hat{X}^2(s) \right) ds,
\]

\[
= A_1 - \lim_{t \to \infty} \frac{1}{t} \int_0^t \left( c_1(r_s) \hat{X}(s) + \frac{\sigma^2(r_s)}{2} \hat{X}^2(s) \right) ds. \quad (18)
\]

If \( \lim_{t \to \infty} \frac{\ln \hat{X}(t)}{t} > 0 \), the right hand side of (18) tends to \(-\infty\), meanwhile if \( \lim_{t \to \infty} \frac{\ln \hat{X}(t)}{t} < 0 \) we obtain a contradiction since the left hand of (18) is negative and the right hand is positive. Thus, \( \lim_{t \to \infty} \frac{\ln \hat{X}(t)}{t} = 0 \), which follows the item d). The proof of the Lemma is completed. \( \square \)
Firstly, we prove part (i). Let \( \lambda < \bar{\lambda} \), we show that 
\[
\lim_{t \to \infty} e^{-\lambda t} (Z(t) - \hat{X}(t)) = 0.
\]
Recall \( X(t) \leq \hat{X}(t) \) \( \forall t \geq 0 \) and from Itô’s formula we have 
\[
\frac{\ln Y(t)}{t} \leq \frac{\ln Y(0)}{t} + \frac{1}{t} \int_0^t \left( -a_2(r_s) + b_2(r_s)\hat{X}(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2}Y^2(s) \right) ds \\
+ \frac{1}{t} \int_0^t \rho(r_s)Y(s)dW_2(s).
\]
Therefore, 
\[
\frac{\ln Y(t)}{t} \leq \frac{\ln Y(0)}{t} + \frac{1}{t} \int_0^t b_2(r_s)\hat{X}(s)ds - \frac{1}{t} \int_0^t a_2(r_s)ds \\
+ \frac{1}{t} \int_0^t \rho(r_s)Y(s)dW_2(s) - \frac{1}{2t} \int_0^t \rho^2(r_s)Y^2(s)ds.
\] (19)
Applying (16) with \( \varepsilon = 1 \) yields 
\[
\lim_{t \to \infty} \sup \int_0^t \left( \frac{\rho(r_s)Y(s)dW_2(s)}{t} - \frac{1}{2} \int_0^t \rho^2(r_s)Y^2(s)ds \right) \leq 0, \text{ a.s. (20)}
\]
Combining (19), (20) and by the definition of \( \lambda \), we come to 
\[
\lim_{t \to \infty} \sup \frac{\ln Y(t)}{t} \leq \lambda < 0.
\] (21)
Moreover, in view of item c) of Lemma 2.2 and (21) we have 
\[
\lim \inf \frac{1}{t} \int_0^t \left( c_1(r_s)X(s) + \frac{\sigma^2(r_s)}{2}X^2(s) \right) ds \geq A_1, \text{ a.s.}
\]
Hence, using item d) of Lemma 2.2 and (9) we obtain 
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left( c_1(r_s)X(s) + \frac{\sigma^2(r_s)}{2}X^2(s) \right) ds = A_1, \text{ a.s.}
\]
As a consequence from this equality, the strong law of large numbers for martingales holds 
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma(r_s)X(s)dW_1(s) = 0.
\]
Thus, 
\[
\lim_{t \to \infty} \frac{\ln X(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \left( a_1(r_s) - b_1(r_s)Y(s) - c_1(r_s)X(s) - \frac{\sigma^2(r_s)}{2}X^2(s) \right) ds \\
+ \lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma(r_s)X(s)dW_1(s) = 0, \text{ a.s. (22)}
\]
(21) and (22) yield 
\[
\lim \sup_{t \to \infty} \frac{\ln Y(t) - \ln X(t)}{t} \leq \lambda < 0.
\]
As a result, there is a random variable \( \xi > 0 \) such that, 
\[
\frac{Y(t)}{X(t)} = Y(t)Z(t) \leq \xi e^{\frac{(\lambda + \lambda_0)s}{t}} \forall t \geq 0, \text{ a.s.}
\]
In view of (23), it desires
\[ \frac{d(Z(t) - \dot{Z}(t))}{dt} = -a_1(r_t)(Z(t) - \dot{Z}(t)) + b_1(r_t)Y(t)Z(t) + \sigma^2(r_t)(Z^{-1}(t) - \dot{Z}^{-1}(t)) \]
\[ \leq -a_1(r_t)(Z(t) - \dot{Z}(t)) + b_1(r_t)Y(t)Z(t). \]
Hence,
\[ 0 \leq e^{-\lambda t}(Z(t) - \dot{Z}(t)) \leq e^{-\lambda t}\int_0^t e^{\lambda s}b_1(r_s)Y(s)Z(s)ds, \]
\[ \leq \xi e^{-\lambda t}\int_0^t e^{\lambda s}b_1(r_s)e^{\frac{(\lambda + \lambda_s)}{2}}ds, \]
where \( A(t) = \int_0^t a_1(r_s)ds < \infty. \) Applying L’Hospital’s rule for right hand side of (23), it desires
\[ \lim_{t \to \infty} e^{-\lambda t}(Z(t) - \dot{Z}(t)) = 0. \]
Further, let \( \hat{\lambda} \) be a number satisfying \( \lambda < \hat{\lambda} < \lambda. \) Since \( \lim_{t \to \infty} \frac{\ln \tilde{X}(t)}{t} = 0, \)
\[ \lim_{t \to \infty} e^{(\hat{\lambda} - \lambda)t} = 0. \]
Using (24) with \( \hat{\lambda} \) we obtain
\[ \lim_{t \to \infty} e^{-\lambda t}(\tilde{X}(t) - \dot{X}(t)) = \lim_{t \to \infty} e^{\hat{\lambda} t}(\tilde{X}(t) - \dot{X}(t)) \]
\[ = \lim_{t \to \infty} e^{(\hat{\lambda} - \lambda)t}(\tilde{X}(t) - \dot{X}(t)) = 0. \]
To proceed, we prove \( \lim_{t \to \infty} \frac{\ln Y(t)}{t} = \lambda. \) Indeed, we have
\[ \frac{\ln Y(t)}{t} = \frac{\ln Y(0)}{t} + \frac{1}{t}\int_0^t b_2(r_s)\tilde{X}(s)ds - \frac{1}{t}\int_0^t a_2(r_s)ds - \frac{1}{t}\int_0^t b_2(r_s)(\tilde{X}(s) - X(s))ds \]
\[ - \frac{1}{t}\int_0^t c_2(r_s)Y(s)ds + \frac{1}{t}\int_0^t \rho(r_s)Y(s)dW_2(s) - \frac{1}{2t}\int_0^t \rho^2(r_s)Y^2(s)ds. \]
As a consequence from (17), the strong law of large numbers for martingales holds
\[ \lim_{t \to \infty} \frac{1}{t}\int_0^t \rho(r_s)Y(s)dW_2(s) = 0. \]
Moreover, (21) implies that
\[ \lim_{t \to \infty} \frac{1}{2t}\int_0^t \rho^2(r_s)Y^2(s)ds = 0 \text{ and } \lim_{t \to \infty} \frac{1}{t}\int_0^t c_2(r_s)Y(s)ds = 0. \]
and from (25) we obtained
\[ \lim_{t \to \infty} \frac{1}{t}\int_0^t b_2(r_s)(\tilde{X}(s) - X(s))ds = 0. \]
Thus,
\[ \lim_{t \to \infty} \frac{\ln Y(t)}{t} = \lambda. \]
From the item d) of Lemma 2.2 we see that

\[
A_3 = 4 \quad NGUYEN HUU DU, NGUYEN THANH DIEU AND TRAN DINH TUONG
\]

which follows that

\[
\therefore
\]

Combining this inequality and item c) in the Lemma 2.2 obtains

\[
\therefore
\]

Thus, the item i) is proved.

We now move to the proof of part ii). Denote

\[
m_2 := \lim_{t \to \infty} \frac{1}{t} \int_0^t c_1(r_s) \hat{X}(s) ds = \sum_{i=1}^{N} \int_{\mathbb{R}^+} c_1(i) x \mu^*(dx, i) < \infty,
\]

\[
m_3 := \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{\sigma^2(r_s)}{2} \hat{X}^2(s) ds = \sum_{i=1}^{N} \int_{\mathbb{R}^+} \frac{\sigma^2(i)}{2} x^2 \mu^*(dx, i) < \infty.
\]

From the item d) of Lemma 2.2 we see that \( A_1 = m_2 + m_3 \). Further, since \( X(t) \leq \hat{X}(t) \) for any \( t \geq 0 \),

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( \frac{\sigma^2(r_s)}{2} X^2(s) - m_3 \right) ds \leq 0. \tag{26}
\]

Combining this inequality and item c) in the Lemma 2.2 obtains

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_1(r_s) Y(s) + c_1(r_s) X(s) \right) \geq m_2,
\]

which follows that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_1(r_s) Y(s) - c_1(r_s) (\hat{X}(s) - X(s)) \right) \geq 0 \tag{27}
\]

by definition of \( m_2 \). For any function \( f \) defined on \( S \), set \( \hat{f} := \max_{i \in S} f(i) \) and \( \check{f} := \min_{i \in S} f(i) \). From the item a) in Lemma 2.2 we get

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ b_2(r_s)(\hat{X}(s) - X(s)) + c_2(r_s) Y(s) + \frac{\rho^2(r_s)}{2} Y^2(s) \right] ds \geq \lambda.
\]

Therefore,

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ c_1(r_s)(\hat{X}(s) - X(s)) + \frac{c_1(r_s)}{b_2(r_s)} c_2(r_s) Y(s) + \frac{c_1(r_s) \rho^2(r_s) Y^2(s)}{b_2(r_s)} \right] ds
\]

\[
\geq \frac{\hat{c}_1}{b_2} \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ b_2(r_s)(\hat{X}(s) - X(s)) + c_2(r_s) Y(s) + \frac{\rho^2(r_s)}{2} Y^2(s) \right] ds \geq \frac{\lambda \hat{c}_1}{b_2}.
\]

Thus,

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ c_1(r_s)(\hat{X}(s) - X(s)) + \frac{c_1(r_s)}{b_2(r_s)} c_2(r_s) Y(s) + \frac{c_1(r_s) \rho^2(r_s) Y^2(s)}{b_2(r_s)} \right] ds
\]

\[
\geq \frac{\lambda \hat{c}_1}{b_2}. \tag{28}
\]

Adding (27) to (28) yields

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ (b_1(r_s) + \frac{c_1(r_s)}{b_2(r_s)} c_2(r_s)) Y(s) + \frac{c_1(r_s) \rho^2(r_s)}{b_2(r_s)} Y^2(s) \right] ds \geq \frac{\lambda \hat{c}_1}{b_2},
\]

which follows that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t (Y(s) + Y^2(s)) ds \geq \frac{\lambda \hat{c}_1}{b_2 \Delta}.
\]
where \( \Delta = \max \left\{ \bar{b}_1 + \frac{c_1 c_2}{b_2}, \frac{c_1 \bar{b}^2}{2b_2} \right\} \). By using the inequality \( \int_0^t Y(s)ds \leq \sqrt{t} \int_0^t Y^2(s)ds \), it follows that there exists a positive constant (independent on the initial condition) \( \tilde{m} \) satisfying

\[
\lim \inf_{t \to \infty} \frac{1}{t} \int_0^t Y(s)ds \geq \tilde{m}.
\]

On the other hand, from item b) of Lemma 2.2 we have

\[
0 \leq \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)X(s) - c_2(r_s)Y(s) - \frac{\rho^2(r_s)}{2} Y^2(s) \right) ds
\]

\[
\leq \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \left( b_2(r_s)\bar{X}(s) - \frac{\rho^2(r_s)}{2} \bar{Y}^2(s) \right) ds \leq \lim_{t \to \infty} \frac{1}{t} \int_0^t b_2(r_s)\bar{X}(s) ds
\]

\[
+ \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t - \frac{\rho^2(r_s)}{2} \bar{Y}^2(s) ds = m_1 + \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t - \frac{\rho^2(r_s)}{2} Y^2(s) ds.
\]

Hence,

\[
\lim \sup_{t \to \infty} \frac{1}{t} \int_0^t Y^2(s)ds \leq M := \frac{2m_1}{\rho^2}.
\]  (29)

For \( 0 < \bar{h} < \tilde{m} \), Hölder’s inequality yields that

\[
\frac{1}{t} \int_0^t 1_{(Y(s) > \bar{h})} Y(s)ds \leq \left( \frac{1}{t} \int_0^t 1_{(Y(s) > \bar{h})} ds \right)^{\frac{1}{2}} \left( \frac{1}{t} \int_0^t Y^2(s)ds \right)^{\frac{1}{2}},
\]

which implies

\[
\lim \inf_{t \to \infty} \frac{1}{t} \int_0^t 1_{(Y(s) \geq \bar{h})} ds
\]

\[
\geq \left( \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t 1_{(Y(s) \geq \bar{h})} Y(s)ds \right)^{\frac{1}{2}} \left( \lim \sup_{t \to \infty} \frac{1}{t} \int_0^t Y^2(s)ds \right)^{-\frac{1}{2}}
\]

\[
\geq \left( \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t Y(s)ds - \bar{h} \right)^{\frac{1}{2}} \left( \lim \sup_{t \to \infty} \frac{1}{t} \int_0^t Y^2(s)ds \right)^{-\frac{1}{2}}
\]

\[
\geq \frac{(\tilde{m} - \bar{h})^2}{M} =: \nabla > 0. \quad (30)
\]

Let \( H > \max \left\{ \sqrt{\frac{1M}{\nabla}}, \sqrt{\frac{2m_3}{\bar{h}^2\nabla}} \right\} \). From \( (26) \) and \( (29) \) the following inequalities hold a.s.

\[
\lim \sup_{t \to \infty} \frac{1}{t} \int_0^t 1_{(Y(s) \geq H)} ds \leq \frac{1}{H^2} \lim \sup_{t \to \infty} \frac{1}{t} \int_0^t Y^2(s)ds \leq \frac{M}{H^2} < \frac{\nabla}{4},
\]

\[
\lim \sup_{t \to \infty} \frac{1}{t} \int_0^t 1_{(X(s) \geq H)} ds \leq \frac{1}{H^2} \lim \sup_{t \to \infty} \frac{1}{t} \int_0^t X^2(s)ds \leq \frac{2m_3}{\bar{h}^2H^2} < \frac{\nabla}{4}. \quad (31)
\]

By combining \( (30) \) and \( (31) \), we obtain

\[
\lim \inf_{t \to \infty} \frac{1}{t} \int_0^t 1_{((X(s),Y(s),r_s) \in A)} ds \geq \frac{\nabla}{2} \quad \text{a.s.,}
\]
where \( A = \{(x, y) : 0 \leq x < H, h \leq y < H\} \times \mathcal{S} \). Further, by virtue of Fatou’s Lemma, we have
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t P(s, (x, y, i), A)ds \geq \frac{\nabla}{2} > 0 \forall (x, y, i) \in \mathbb{R}_\infty^2 \times \mathcal{S}.
\]

Since \( \mathcal{M} = \{(x, y) : x \geq 0, y > 0\} \) is invariant under equation (3), we can consider the Markov process \( \{(X(t), Y(t), r_t), t \geq 0\} \) on the state space \( \mathcal{M} \times \mathcal{S} \). As an immediate consequence of [24 Theorem 5.1] we can see that \( \{(X(t), Y(t), r_t), t \geq 0\} \) has the Feller property. Thus, inequality (32) and the compactness of \( A \) in \( \mathcal{M} \times \mathcal{S} \) follow that there is an invariant probability measure \( \psi^* \) on \( \mathcal{M} \times \mathcal{S} \) (see [22]). Since \( Y(t) \) tends to 0, provided \( X(0) = 0 \), \( \lim_{t \to \infty} P(t, (0, y, i), K) = 0 \) for all compact set \( K \subset \mathcal{M} \times \mathcal{S} \). Thus, we must have \( \psi^* \{(x, y) : x = 0, y > 0\} \times \mathcal{S} = 0 \), equivalently \( \psi^* (\mathbb{R}_\infty^2 \times \mathcal{S}) = 1 \). Furthermore, by virtue of the invariance of \( \mathbb{R}_\infty^2 \times \mathcal{S} \), we derive that \( \psi^* \) is an invariant probability measure of \( \{(X(t), Y(t), r_t), t \geq 0\} \) on \( \mathbb{R}_\infty^2 \times \mathcal{S} \). On the other hand, from [22, 11 Lemma 4.1, p. 101] and the independence of \( W_1(t), W_2(t), r_t \), it implies that \( \mathbb{R}_\infty^2 \times \mathcal{S} \) is the support of \( \psi^* \).

3. Discussion and numerical solutions. In this work, we proved that a stochastic population under regime switching controlled by a Markov chain was either stochastically permanent or extinct depending on the sign of the threshold \( \lambda = \sum_{i=1}^{N} b_2(t) \int_{\mathbb{R}^+} x \mu^*(i, dx) - \sum_{i=1}^{N} a_2(t) \pi_i \). The results in the paper can be considered as a continuing and improving the results of these models are proposed and studied in [4, 15, 20]. In some cases, if the dynamics of population nonlinearly depends on some perturbed parameters or the experimental data may follow fractional Brownian motion like distribution, not white or colored Gaussian laws, the model with bounded noise should be considered (for more detailed information please see [6]). The problem of how to construct a threshold \( \lambda \) for the permanence of these cases is interesting.

To confirm the above results, we give two following numerical examples with the help of the method mentioned in [9] and Matlab software. These examples show that the Markovian switching may change systems from a non-permanent situation to a sustainable one and vice-versa. In case there exists \( j \in \mathcal{S} \) such that \( r_t = j \) for all \( t \geq 0 \) (no switching between these systems), it is easy to see that \( \pi_i = 0 \) if \( i \neq j \), \( \pi_j = 1 \). Further,
\[
\mu^*(dx, i) = 0 \forall i \neq j \text{ and } \mu^*(dx, j) = k \exp \left\{ \frac{2c_1(j)}{\sigma^2(j)} x - \frac{a_1(j)}{\sigma^2(j)} x^2 \right\} dx
\]
where
\[
k = \left( \int_0^\infty y^2 \exp \left\{ \frac{2c_1(j)}{\sigma^2(j)} y - \frac{a_1(j)}{\sigma^2(j)} y \right\} dy \right)^{-1}
\]
(see [9] for details). Therefore, the threshold \( \lambda \) with respect to the case \( r_t = j \) for all \( t \geq 0 \) is clearly defined by following formula
\[
\lambda_j = k b_2(j) \int_{\mathbb{R}^+} x \exp \left\{ \frac{2c_1(j)}{\sigma^2(j)} x - \frac{a_1(j)}{\sigma^2(j)} x^2 \right\} dx - a_2(j).
\]

Example 1. In this example we show that the population is permanent in each state. Meanwhile, it is extinct if it switches from one to other state according to the telegraph switching. Indeed, we consider the system [9] with the Markovian switching \( \{r_t, t \geq 0\} \) on the state space \( \mathcal{S} = \{1, 2\} \) with the coefficients in each state are given in Table [1].
Table 1. Values of the coefficients in Ex. 3.1

| States | $a_1$ | $a_2$ | $b_1$ | $b_2$ | $c_1$ | $c_2$ | $\sigma$ | $\rho$ |
|--------|-------|-------|-------|-------|-------|-------|----------|-------|
| 1      | 0.9   | 2.5   | 2     | 2.8   | 0.6   | 5     | 0.6      | 4     |
| 2      | 0.2   | 0.1   | 1     | 4     | 3     | 0.5   | 1.5      | 4     |

If the population is always in state 1, ($r_t = 1 \; \forall t \in \mathbb{R}_+$), by using (33), we can show that $\lambda_1 = 0.4763 > 0$; When $r_t = 2 \; \forall t \in \mathbb{R}_+$ (the population is always in state 2), by the same way we have $\lambda_2 = 0.1602 > 0$. Thus, if there is no switching, the population is permanent in each state. For each state, the density of predator $Y(t)$ is demonstrated by Figure 1.

However, if the switching intensities of the process $\{r_t, t \geq 0\}$ are $\gamma_{12} = 0.2$ and $\gamma_{21} = 0.6$, then $\pi = (0.75, 0.25)$ and we can estimate the value $\lambda$ given by (12) as $\lambda = -0.1555 < 0$ by using the strong law of large numbers. This says that the extinction of the predator happens. The switching trajectories $Y(t)$ are illustrated in Figure 2.
Example 2. The following example shows that under the regime switching, a system may be permanent even that the predator goes extinct in each fixed state. For instance, the data is derived in Table 2.

| States | Coefficients | \( a_1 \) | \( a_2 \) | \( b_1 \) | \( b_2 \) | \( c_1 \) | \( c_2 \) | \( \sigma \) | \( \rho \) |
|--------|--------------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1      |              | 0.2      | 0.45     | 1        | 9.5      | 5        | 1        | 2        | 4        |
| 2      |              | 1        | 0.85     | 0.5      | 3.6      | 4.2      | 2        | 1.5      | 4        |

Table 2. Values of the coefficients in Ex. 3.2

In this situation, we have the threshold for state 1 that is \( \lambda_1 = -0.0760 < 0 \), and state 2, \( \lambda_2 = -0.0442 < 0 \). This means that in each state the predator is eventually extinct.

However, in the case of regime switching with \( \gamma_{12} = 0.2 \) and \( \gamma_{21} = 0.6 \), we can also estimate the value \( \lambda = 0.0637 > 0 \) by using the strong law of large numbers. It means that the population is permanent and there exists a stationary distribution of (3). A sample path of \( Y(t) \) is depicted in Figures 4 and the phase picture and the empirical density of this stationary distribution are shown in Figure 5.

Figure 3. Trajectories of \( Y(t) \) in the first state (blue line) and the second state (red line) respectively in Ex. 2.

Figure 4. A switching trajectory \( Y(t) \) in Ex. 2.
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