ABSENCE OF SCALAR HAIR IN SCALAR-TENSOR GRAVITY

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Stationary, asymptotically flat black holes in scalar-tensor theories of gravity are studied. It is shown that such black holes have no scalar hair and are the same as in General Relativity.

Keywords: Black holes, scalar-tensor gravity

1. Introduction

In General Relativity (GR) stationary black holes, which are the endpoint of gravitational collapse, must be axisymmetric and are described by the Kerr-Newman metric. The prototypical alternative theory of gravity is Brans-Dicke theory with (Jordan frame) action

\[ S_{BD} = \int d^4x \sqrt{-\hat{g}} \left[ \varphi \hat{R} - \frac{\omega_0}{\varphi} \hat{\nabla}^{\mu} \varphi \hat{\nabla}_{\mu} \varphi + L_m(\hat{g}_{\mu\nu}, \psi) \right]. \]  

In 1972 Hawking showed that stationary black holes in this theory must be the Kerr-Newman black holes of GR. This result was generalized by Bekenstein to more general scalar-tensor theories, but with the additional assumption of spherical symmetry. Hawking’s result has recently been extended to general scalar-tensor theories with action

\[ S_{ST} = \int d^4x \sqrt{-\hat{g}} \left[ \varphi \hat{R} - \frac{\omega(\varphi)}{\varphi} \hat{\nabla}^{\mu} \varphi \hat{\nabla}_{\mu} \varphi - V(\varphi) + L_m(\hat{g}_{\mu\nu}, \psi) \right] \]  

without any extra assumption of symmetry apart from stationarity. This proof is presented below.

2. The proof

To begin with, we require:
• **Asymptotic flatness:** this requires $V(\varphi_0) = 0$ and $\varphi_0 V'(\varphi_0) = 2V(\varphi_0)$, where $\varphi_0$ is the value the Brans-Dicke scalar field approaches as $r \to +\infty$ (gravitational collapse occurs on scales much smaller than the Hubble scale $H_0^{-1}$, so asymptotic flatness is expected to be an adequate approximation physically).

• **Stationarity:** the black hole is supposed to be the endpoint of collapse.

We map the theory to the Einstein conformal frame according to $\hat{g}_{\mu\nu} \to g_{\mu\nu} = \varphi \hat{g}_{\mu\nu}$, $\varphi \to \phi$, with $d\phi = \sqrt{\frac{2\omega(\varphi)+3}{16\pi}} \frac{d\varphi}{\varphi}$ (for $\omega \neq -3/2$). The action becomes

$$S_{ST} = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi} - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - U(\phi) + L_m(\hat{g}_{\mu\nu}, \psi) \right],$$

(3)

where $U(\phi) = V(\varphi)/\varphi^2$. The field equation for the scalar in vacuo in the Einstein frame is

$$\Box \phi = U'(\phi).$$

(4)

Since the conformal factor of the transformation depends only on the Brans-Dicke field $\varphi$, the Einstein frame symmetries are the same as in the Jordan frame. In particular, there exists a Killing vector $\xi^\mu$ which is timelike at infinity (stationarity). In the Einstein frame and in electrovacuum the theory is essentially GR with a minimally coupled scalar field. So, stationary, asymptotically flat black holes have to be axisymmetric and, hence, there should be a second Killing vector $\zeta^\mu$ which is spacelike at infinity, provided that the stress-energy tensor for $\phi$ satisfies the weak energy condition. Consider, in vacuo, a 4-volume $V$ bounded by the horizon $H$, two partial Cauchy hypersurfaces $S_1, S_2$, and a timelike 3-surface at infinity. Now multiply both sides of eq. (4) by $U'$ and integrate over the 4-volume $V$, obtaining

$$\int_V d^4x \sqrt{-g} U'(\phi) \Box \phi = \int_V d^4x \sqrt{-g} U'^2(\phi).$$

(5)

We can rewrite this equation as

$$\int_V d^4x \sqrt{-g} \left[ U''(\phi) \nabla^\mu \phi \nabla_\mu \phi + U'^2(\phi) \right] = \int_{\partial V} d^3x \sqrt{|h|} U'(\phi)n^\mu \nabla_\mu \phi,$$

(6)

where $n^\mu$ is the normal to the boundary and $h$ is the determinant of the induced metric $h_{\mu\nu}$ on this boundary. Splitting the boundary into its constituent parts one has $\int_{r=\infty} = 0$,

$$\int_{\text{horizon}} d^3x \sqrt{|h|} U'(\phi)n^\mu \nabla_\mu \phi = 0,$$

(7)

because of the spacetime symmetries, and $\int_{S_1} = -\int_{S_2}$ if $S_2$ is obtained by shifting each point of $S_1$ along integral curves of $\xi^\mu$, hence

$$\int_V d^4x \sqrt{-g} \left[ U''(\phi) \nabla^\mu \phi \nabla_\mu \phi + U'^2(\phi) \right] = 0.$$

(8)

$U'^2 \geq 0$, $\nabla^\mu \phi$ (which is orthogonal to both $\xi^\mu$ and $\zeta^\mu$) is spacelike or zero, and with $U''(\phi) \geq 0$ being a local stability condition, one concludes that it must be $\nabla_\mu \phi \equiv 0$.
in $V$ and $U'(\phi_0) = 0$. But for $\phi =$const., to which we have reduced, the scalar-tensor theory reduces to GR and the black hole must be described by the Kerr metric.

Metric $f(R)$ gravity, which has seen much recent attention\cite{4,5} is a special Brans-Dicke theory with parameter $\omega = 0$ and a non-trivial potential $V$ for the Brans-Dicke field $\varphi = f'(R)$. Palatini $f(R)$ gravity, instead, corresponds to an $\omega = -3/2$ Brans-Dicke theory (again, with a potential). The case $\omega = -3/2$ was explicitly excluded in our discussion but $\omega = -3/2$ Brans-Dicke theory reduces to GR in vacuo anyway.

3. Conclusions

The proof presented above extends immediately to electro-vacuum and to any form of conformal matter with trace of the energy-momentum tensor $T = 0$. It implies that asymptotically flat black holes that are the endpoint of collapse in scalar-tensor gravity are described by the Kerr-Newman metric. The assumption of asymptotic flatness is a limitation mathematically, but one expects on physical grounds that the effect of a Friedmann-Lemaître-Robertson-Walker asymptotic structure on astrophysical collapse to be completely negligible (except for primordial black holes for which the collapse and the Hubble scales can be comparable\cite{3}).

There are certain exceptions to the proof, which include: (i) theories in which $\omega \to \infty$ somewhere outside or on the horizon; (ii) theories in which $\varphi \to \infty$ or $\varphi \to 0$ somewhere outside or on the horizon; (iii) theories in which the stress-energy tensor of the Einstein-frame scalar violates the weak energy condition.

It is likely that the majority of these exceptional theories or solutions will be unphysical (e.g., the gravitational coupling in scalar-tensor gravity is inversely proportional to $\varphi$) but interesting exceptions might exist. This issue will be addressed in future work.

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*An example of a solution where $\varphi \to \infty$ on the horizon is that of Bocharova et al.\cite{8} (which is, however, unstable\cite{9}).*
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