ON ADDITION CHAINS AND PROGRESS ON THE SCHOLZ CONJECTURE

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Abstract. In this paper, we develop some new classes of methods to study the Scholz conjecture on addition chains. It turns out that the exponents of numbers of the form $2^n - 1$ largely determine the length of the shortest addition chain for number producing $2^n - 1$. Exploiting the notion of carries, we obtain improved upper bounds for the length of the shortest addition chains $\lambda(2^n - 1)$ producing $2^n - 1$. Most notably, we show that if $2^n - 1$ has carries of degree at most

$$\kappa(2^n - 1) = \frac{1}{2}(\lambda(n) - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \sum_{j=1}^{\left\lfloor \frac{n}{2^j} \right\rfloor} \left\{ \frac{n}{2^j} \right\})$$

then the inequality

$$\lambda(2^n - 1) \leq n + 1 + \sum_{j=1}^{\left\lfloor \frac{n}{2^j} \right\rfloor} \left( \frac{n}{2^j} - \xi(n,j) \right) + \lambda(n)$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\lambda(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$, $\{\cdot\}$ denotes the fractional part of $\cdot$ and where $\xi(n,1) := \{\frac{n}{2}\}$ with $\xi(n,2) = \{\frac{1}{2}\} \{\frac{n}{2}\}$ and so on.

1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \ldots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms- not necessarily distinct - in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The length of the chain is determined by the number of entries in the sequence excluding the mandatory first term 1, since it is the only term which cannot be expressed as the sum of two previous terms in the sequence. There are numerous addition chains that result in a fixed number $n$; In other words, it is always possible to construct as many addition chains producing a fixed number positive integer $n$ as $n$ grows in magnitude. The shortest among these possible chains producing $n$ is regarded as the optimal or the shortest addition chain producing $n$. There is currently no efficient method for getting the shortest addition yielding a given number, thus reducing an addition chain might be a difficult task, thereby making addition chain theory a fascinating subject to study. By letting $\lambda(n)$ denotes the length of the shortest addition chain producing $n$, then Arnold Scholz conjectured and alfred Braurer proved the following inequalities
Theorem 1.1 (Brauer). The inequality

\[ m + 1 \leq \iota(n) \leq 2m \]

for \( 2^m + 1 \leq n \leq 2^{m+1} \) holds for \( m \geq 1 \).

Conjecture 1.1 (Scholz). The inequality

\[ \iota(2^n - 1) \leq n - 1 + \iota(n) \]

holds for all \( n \geq 2 \).

It has been shown computationally by Neill Clift, that the conjecture holds for all \( n \leq 5784688 \) and in fact it is an equality for all exponents \( n \leq 64 \) [2]. Alfred Brauer proved the Scholz conjecture for the star addition chain, a special type of addition chain where each term in the sequence obtained by summing uses the immediately subsequent number in the chain. By denoting with \( \iota^*(n) \) as the length of the shortest star addition chain producing \( n \), it is shown that (See [1])

Theorem 1.2. The inequality

\[ \iota^*(2^n - 1) \leq n - 1 + \iota^*(n) \]

holds for all \( n \geq 2 \).

In relation to Conjecture 1.1 Arnold Scholz postulated that Conjecture 1.1 can be improved in general. In particular, Alfred Brauer [1] proved the inequality

\[ \iota(n) < \frac{\log n}{\log 2} \left( 1 + \frac{1}{\log \log n} + \frac{2 \log 2}{(\log n)^{1-\log 2}} \right) \]

for \( 2^m \leq n < 2^{m+1} \) for all sufficiently large \( n \).

Quite a particular special cases of the conjecture has also be studied by many authors in the past. For instance, it is shown in [4] that the scholz conjecture holds for all numbers of the form \( 2^n - 1 \) with \( n = 2^q \) and \( n = 2^s(2^q + 1) \) for \( s, q \geq 0 \). If we let \( \nu(n) \) denotes the number of 1’s in the binary expansion of \( n \) for \( m = 2^n - 1 \), then it is shown in [3] that the Scholz conjecture holds in the case \( \nu(n) = 5 \).

In this paper we study short addition chains producing numbers of the form \( 2^n - 1 \) and the scholz conjecture. We obtain some crude and much more weaker inequalities related to the scholz conjecture.
In this section we introduce the notion of sub-addition chains.

**Definition 2.1.** Let \( n \geq 3 \), then by the addition chain of length \( k - 1 \) producing \( n \) we mean the sequence

\[ 1, 2, \ldots, s_{k-1}, s_k \]

where each term \( s_j \) (\( j \geq 3 \)) in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

\[ 2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n \]

with \( a_{i+1} = a_i + r_i \) and \( a_{i+1} = s_i \) for \( 2 \leq i \leq k \). We call the partition \( a_i + r_i \) the \( i \) th generator of the chain for \( 2 \leq i \leq k \). We call \( a_i \) the determiners and \( r_i \) the regulator of the \( i \) th generator of the chain. We call the sequence \((r_i)\) the regulators of the addition chain and \((a_i)\) the determiners of the chain for \( 2 \leq i \leq k \).

**Definition 2.2.** Let the sequence \( 1, 2, \ldots, s_{k-1}, s_k = n \) be an addition chain producing \( n \) with the corresponding sequence of partition

\[ 2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n. \]

Then we call the sub-sequence \((s_{jm})\) for \( 1 \leq j \leq k \) and \( 1 \leq m \leq t \leq k \) a sub-addition chain of the addition chain producing \( n \). We say it is complete sub-addition chain of the addition chain producing \( n \) if it contains exactly the first \( t \) terms of the addition chain. Otherwise we say it is an incomplete sub-addition chain.

### 2.1. Equivalent addition chains.

In this section we introduce and study the notion of equivalence of an addition chain producing a given number. We launch the following languages.

**Definition 2.3.** Let \( 1, 2, \ldots, s_{k-1}, s_k = n \) be an addition chain producing \( n \) with the corresponding sequence of partition

\[ 2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n \]

where \( a_i \) is the determiner and \( r_i \) the regulator of the \( i \) th generator of the chain. Also let \( 1, 2, \ldots, u_{l-1}, u_l = m \) be an addition chain producing \( m \) with the corresponding sequence of partition

\[ 2 = 1 + 1, \ldots, u_{l-1} = g_{l-1} + h_{l-1}, u_l = g_l + h_l = m \]

where \( g_l \) is the determiner and \( h_l \) the regulator of the \( i \) th generator of the chain. Then we say the addition chain \( 1, 2, \ldots, s_{k-1}, s_k = n \) is equivalent to the addition chain \( 1, 2, \ldots, u_{l-1}, u_l = m \) if there exist a complete sub-addition chain \((u_{i_l})\) of the chain \((u_i)\) such that for each determiner \( g_i \) and the corresponding regulator \( h_i \) in the sub-addition chain there exists some \( v_i, d_i \geq 0 \) not necessarily distinct - with \( v_i, d_i \in \mathbb{Z} \) such that \( g_i = a_i - v_i \) and \( h_i = r_i - d_i \). We call each \( v_i \) and \( d_i \) the stabilizer of the determiner and the regulator of the \( i \) th generator of the chain. We denote the equivalence by \((s_j) \Leftrightarrow (u_i)\).
Proposition 2.1. Let $1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)}+1 = n$ and $1, 2, \ldots, u_{\delta(m)}, u_{\delta(m)}+1 = m$ be addition chains producing $n$ and $m$ with length $\delta(n)$ and $\delta(m)$, respectively. If $(s_j) \Rightarrow (u_i)$, then $\delta(n) \leq \delta(m)$.

Proof. Let $1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)}+1 = n$ and $1, 2, \ldots, u_{\delta(m)}, u_{\delta(m)}+1 = m$ be addition chains producing $n$ and $m$ with length $\delta(n)$ and $\delta(m)$, respectively. Suppose $(s_j) \Rightarrow (u_i)$, then by Definition 2.2 there must exist a complete sub-addition chain $(u_i)$ of the chain $(u_i)$ such that for each determiner $g_i$ and the corresponding regulator $h_i$ in the sub-addition chain there exists some stabilizers $v_i, d_i \geq 0$ - not necessarily distinct - with $v_i, d_i \in \mathbb{Z}$ such that $g_i = a_i - v_i$ and $h_i = r_i - d_i$. Since the length of the complete sub-addition chain $(u_i)$ is at most the length of the addition $(u_i)$, and the number of terms in the complete sub-addition chain corresponds to the number of terms in the chain $(s_j)$, it follows that $\delta(n) \leq \delta(m)$, thereby ending the proof.

\section{2.2. Equivalent addition chains in a fixed base.}
In this section we introduce the notion of equivalence of an addition chain in fixed base.

Definition 2.4. Let $(u_i)$ be an addition chain producing $m$ and $(s_j)$ be an addition chain producing $n$. Then we say the addition chain $(s_j)$ is equivalent to the addition chain $(u_i)$ in base $n$ if there exists a complete sub-addition chain $(s_{j,m})$ of the chain $(s_j)$ such that $(s_{j,m}) \Rightarrow (u_i)$. We denote the length of the chain $(u_i)$ in base $n$ with $\delta_u(m)$ and the length of the shortest of all such chains in base $n$ with $\iota_n(m)$.

\section{3. Addition chains of numbers of special forms}
In this section we study addition chains of numbers of special forms. We examine ways of minimizing the length of addition chains for numbers of the forms $2^n$, $2^n - 1$ and $2^n + 1$. For addition chains producing $2^n$ and $2^n + 1$ the process is natural and trivial as opposed to those producing $2^n - 1$. We launch the following primary results.

Proposition 3.1. Let $\iota(n)$ denotes the length of the shortest addition chain producing $n$. Then $\iota(2^n) = n$ and

$$\iota(2^n + 1) = \iota(2^n) + 1 = n + 1.$$  

Proof. It suffices to construct the corresponding sequence of partition generating the shortest addition producing $2^n$. Let us choose the corresponding sequence of partitions

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n + 1$, so that the choice of the regulator $r_i = 2^{i-2}$ minimizes the length of the chain and hence generates the shortest length of the chain $\iota(2^n) = n$, since there are $n$ regulators counting multiplicity in the chain. The corresponding shortest addition chain producing $2^n + 1$ is also obtained with the corresponding sequence of partition by adjoining the term $2^n + 1$ to the last term.
in the corresponding sequence of partition for the chain producing $2^n$, since every addition chain starts with 1, so that we have the shortest length

$$\iota(2^n + 1) = \iota(2^n) + 1 = n + 1.$$  

\hfill \square

Remark 3.1. We now prove an important result that will have significant impact on the main result in this paper. One could consider this result as a weaker version and a first step to affirming the Scholz conjecture.

**Theorem 3.2.** Let $\iota(n)$ denotes the length of the shortest addition chain producing $n$. Then there exists some $G : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\iota(2^n - 1) = \iota(2^n) + 1 + G(n) = n + 1 + G(n).$$

**Proof.** First, let us construct the shortest addition chain producing $2^n$ as $1, 2, 2^2, \ldots, 2^{n-1}, 2^n$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^4, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n+1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i$th generator of the chain. Next we construct a sub-addition chain of some equivalent addition chain by choosing the stabilizers $v_3 = 0$ and $d_3 = 1$ for the determiner $a_3$ and the regulator $r_3$, respectively, and choose new regulators with determiners $g_i = h_i = 2^{i-2} - 2^{i-4}$ for all $4 \leq i \leq n + 1$. Then we obtain a complete sub-addition chain $1, 2, 3, 6, \ldots, 2^n - 2^n - 2^n - 4, 2^n - 2^n - 2^n - 4, 2^n - 2^n - 2^n - 4, 2^n - 2^n - 2^n - 4$ of some equivalent addition chain producing $2^n - 1$ with the corresponding sequence of partition

$$2 = 1 + 1, 2 + (2 - 1) = 2^2 - 1, (2^2 - 1) + (2^2 - 1) = 2^3 - 2$$

$$\ldots,$$

$$(2^n - 2^n - 3) = (2^n - 2^n - 4) + (2^n - 2^n - 4), (2^n - 2^n - 4) = (2^n - 2^n - 4) + (2^n - 2^n - 4).$$

Appealing to Proposition 2.1 we obtain the relation

$$\iota(2^n) + 1 < \iota(2^n - 1)$$

so that there exists some $G := G(n) \in \mathbb{N}$ such that we can write $\iota(2^n - 1) = \iota(2^n) + 1 + G(n)$.

\hfill \square

**Corollary 3.1.** Let $\iota(n)$ denotes the length of the shortest addition chain producing $2^n - 1$. Then

$$\iota(2^n - 1) \leq n + 1 + G(n)$$

where $G : \mathbb{N} \rightarrow \mathbb{N}$.

**Proof.** By invoking Theorem 3.2 we can write

$$\iota(2^n - 1) \leq \iota(2^n - 1) = n + 1 + G(n)$$

with $G : \mathbb{N} \rightarrow \mathbb{N}$.

\hfill \square
Corollary 3.1 could be considered as a weaker and a crude form of the inequality relating the length of the shortest addition producing $2^n - 1$ to the length of the shortest addition chain producing $n$. In particular, Scholz’s conjecture is the claim that

$$G(n) \leq \iota(n) - 2.$$ 

To this end minimizing the function $G(n)$ is of ultimate goal, which also requires choosing a suitably short addition chain that completes the addition chain producing $2^n - 1$. We prove a much weaker upper bound for the function $G(n)$ with a certain construction for the addition chain completing the addition chain in base $2^n$.

3.1. The symmetry method.

**Theorem 3.3.** The inequality holds

$$\iota_{2^n}(2^n - 1) \leq n + 1 + \left\lfloor \frac{n - 2}{2} \right\rfloor.$$ 

**Proof.** By invoking Proposition 4.3 we can write

$$\iota_{2^n}(2^n - 1) = n + 1 + G(n)$$

where $G : \mathbb{N} \rightarrow \mathbb{N}$ under the complete sub-addition chain $1, 2, 3, 6, \ldots, 2^{n-2} - 2^{n-4}, 2^{n-1} - 2^{n-3}, 2^n - 2^{n-2}$ of some addition chain with corresponding sequence of partitions

$$2 = 1 + 1, 2 + (2 - 1) = 2^2 - 1, (2^2 - 1) + (2^2 - 1) = 2^3 - 2, \ldots,$$

$$(2^{n-1} - 2^{n-3}) = (2^{n-2} - 2^{n-4}) + (2^{n-2} - 2^{n-4}), (2^n - 2^{n-2}) = (2^{n-1} - 2^{n-3}) + (2^{n-1} - 2^{n-3}).$$

We note that we can extend the complete sub-addition chain to the addition chain producing $2^n - 1$ in the following manner

$$1, 2, 3, 6, \ldots, 2^{n-2} - 2^{n-4}, 2^{n-1} - 2^{n-3}, 2^n - 2^{n-2}, 2^n - 2^{n-4}, 2^n - 2^{n-6}, \ldots, 2^n - 2^0 = 2^n - 1$$

by adjoining the corresponding sequence of partition to the sequence of partition of the complete sub-addition chain

$$(2^n - 2^{n-2}) + (2^{n-2} - 2^{n-4}), (2^n - 2^{n-4}) + (2^{n-4} - 2^{n-6}), \ldots, (2^n - 2^2) + (2^2 - 1)$$

we obtain

$$G(n) \leq \left\lfloor \frac{n - 2}{2} \right\rfloor$$

where $G(n)$ counts the number of terms adjoined to the complete sub-addition chain before $2^n - 1$, thereby proving the inequality. □
4. Length of general addition chains

In this section we study Scholz’s conjecture. We prove some an inequality related to the conjecture on additions chains. We begin with the following fundamental result which can be found in [1].

**Lemma 4.1.** Let \( \iota(n) \) denotes the shortest length of an addition chain producing \( n \). Then the lower bound holds
\[
\iota(n) > \frac{\log n}{\log 2} - 1.
\]

**Remark 4.2.** We now obtain an inequality related to Scholz’s conjecture.

**Theorem 4.3.** Let \( \delta(n) \) and \( \iota(n) \) denotes the length of an addition chain and the shortest addition chain producing \( n \), respectively. Then there exists some \( G : \mathbb{N} \to \mathbb{R} \) such that
\[
\delta(2^n - 1) \leq n - 1 + \iota(n) + K(n).
\]

**Proof.** First, let us construct the shortest addition chain producing \( 2^n \) as \( 1, 2, 2^2, \ldots, 2^{n-1}, 2^n \) with corresponding sequence of partition
\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1},
\]
with \( a_i = 2^{i-2} = r_i \) for \( 2 \leq i \leq n + 1 \), where \( a_i \) and \( r_i \) denotes the determiner and the regulator of the \( i \)th generator of the chain. Let us consider only the complete sub-addition chain
\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}.
\]

Next we extend this complete sub-addition chain by adjoining the sequence
\[
2^{n-1} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}, 2^{n-1} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 2^{\left\lfloor \frac{n-1}{2^2} \right\rfloor}, \ldots, 2^{n-1} + 2^{\left\lfloor \frac{n-1}{2^j} \right\rfloor} + 2^{\left\lfloor \frac{n-1}{2^j} \right\rfloor} + \cdots + 2^1.
\]

We note that the adjoined sequence contributes at most
\[
\left\lfloor \frac{\log n}{\log 2} \right\rfloor \leq \iota(n)
\]
terms to the original complete sub-addition chain, where the upper bound follows from Lemma **4.1**. Since the inequality holds
\[
2^{\left\lfloor \frac{n-1}{2^j} \right\rfloor} + 2^{\left\lfloor \frac{n-1}{2^j} \right\rfloor} + \cdots + 2^1 < \sum_{j=0}^{n-1} 2^j
\]
\[
= 2^n - 1
\]
there exists some \( K : \mathbb{N} \to \mathbb{R} \) counting the terms in the remaining terms of the addition chain producing \( 2^n - 1 \). This completes the proof.

Scholz’s conjecture is the claim

**Conjecture 4.1 (Scholz).** The inequality holds
\[
\iota(2^n - 1) \leq n - 1 + \iota(n).
\]
This result would follow from the inequality established if we can take \(\delta(2^n - 1) = \iota(2^n - 1)\) and choose \(K(n) = 0\) in Theorem 4.3. In other words, Scholz’s conjecture can be reformulated in the following way:

**Conjecture 4.2 (Scholz).** If \(\delta(2^n - 1) = \iota(2^n - 1)\), then the inequality holds

\[
\delta(2^n - 1) \leq n - 1 + \iota(n) + K(n)
\]

with \(K(n) = 0\) for all \(n \in \mathbb{N}\).

4.1. Explicit upper bound. In this section, we prove an explicit upper bound for the length of an addition chain - not necessarily the shortest - producing numbers of the form \(2^n - 1\). In particular, we study the problem of constructing an addition chain whose length meets a specified threshold requirement.

**Theorem 4.4.** Let \(\delta(n)\) denotes the length of an addition chain producing \(n\). Then there exists an addition chain producing \(2^n - 1\) such that the inequality holds

\[
\delta(2^n - 1) \lesssim n + \iota(n) + \frac{n}{\log n} + 1.3 \log n \int_2^{\frac{n}{\log n}} \frac{dt}{\log^3 t} + \xi(n)
\]

where \(\xi : \mathbb{N} \to \mathbb{R}\).

**Proof.** First, let us construct the shortest addition chain producing \(2^n\) as \(1, 2, 2^2, \ldots, 2^{n-1}, 2^n\) with corresponding sequence of partition

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}
\]

with \(a_i = 2^{i-2} = r_i\) for \(2 \leq i \leq n + 1\), where \(a_i\) and \(r_i\) denotes the determiner and the regulator of the \(i\)th generator of the chain. Let us consider only the complete sub-addition chain

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}.
\]

Next we extend this complete sub-addition chain by adjoining the sequence

\[
2^{n-1} + 2^{\frac{n-1}{2}}, 2^{n-1} + 2^{\frac{n-1}{2}}, 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}}, \ldots, 2^{n-1} + 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} + \cdots + 2^1.
\]

We note that the adjoined sequence contributes at most

\[
\left\lfloor \frac{\log n}{\log 2} \right\rfloor \leq \iota(n)
\]

terms to the original complete sub-addition chain, where the upper bound follows from Lemma 4.1. Since the inequality holds

\[
2^{n-1} + 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} + \cdots + 2^1 < \sum_{i=1}^{n-1} 2^i = 2^n - 1
\]

we make the substitution

\[
R_2(n) := 2^{n-1} + 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} + \cdots + 2^1
\]

and extend the addition chain by further adjoining the sequence

\[
R_2(n) + 2^{\frac{n-1}{2}}, R_2(n) + 2^{\frac{n-1}{2}}, R_2(n) + 2^{\frac{n-1}{2}}, \ldots, R_2(n) + 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} + \cdots + 2^1.
\]
We note that the adjoined sequence contributes at most
\[ \frac{\log n}{\log 3} \]
terms to the original complete sub-addition chain. Since
\[ R_2(n) + 2^\left\lfloor \frac{n-1}{3} \right\rfloor + 2^\left\lfloor \frac{n-1}{3^2} \right\rfloor + \cdots < 2^n - 1 \]
we continue the extension of the addition chain by using all the primes \( p \leq \frac{n-1}{2} \), so that by induction the number of terms adjoined to the original complete sub-addition chains is the sum
\[
\sum_{p \leq \frac{n-1}{2}} \frac{\log n}{\log p} = \frac{\log n}{\log 2} + \sum_{3 \leq p \leq \frac{n-1}{2}} \frac{\log n}{\log p} \\
\leq \iota(n) + \log n \sum_{3 \leq p \leq \frac{n-1}{2}} \frac{1}{\log p}.
\]
Now, we obtain the upper bound
\[
\sum_{3 \leq p \leq \frac{n-1}{2}} \frac{1}{\log p} = \int_{2}^{\frac{n-1}{2}} \frac{d\pi(u)}{\log u} \\
= \frac{\pi(\frac{n-1}{2})}{\log(\frac{n-1}{2})} - \frac{\pi(2)}{\log 2} + \int_{2}^{\frac{n-1}{2}} \frac{\pi(t)}{t \log^2 t} dt \\
\lesssim \frac{n}{\log^2 n} - \frac{1}{\log 2} + 1.3 \int_{2}^{\frac{n-1}{2}} \frac{1}{\log^3 t} dt.
\]
This completes the proof of the inequality. \( \square \)

**Corollary 4.1.** Let \( \iota(n) \) denotes the length of the shortest addition chain producing \( n \). The the inequality holds
\[
\iota(2^n - 1) \lesssim n + \iota(n) + \frac{n}{\log n} + 1.3 \log n \int_{2}^{\frac{n-1}{2}} \frac{dt}{\log^3 t} + \xi(n)
\]
where \( \xi : \mathbb{N} \rightarrow \mathbb{R} \).

5. **The backtracking method**

In this section we study the length of an addition chain using the backtracking method.

**Theorem 5.1.** There exists an addition chain producing \( 2^n - 1 \) of length \( \delta(2^n - 1) \) satisfying the inequality
\[
\delta(2^n - 1) \leq 2n - 1 - 2 \left\lfloor \frac{n - 1}{2^{\left\lfloor \log_2 2 \right\rfloor}} \right\rfloor + \left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
where \( \lfloor \cdot \rfloor \) denotes the floor function.
Proof. First we construct the sequence
\[
2^{n-1} + 2^{n-2} + \ldots + 2^\left\lfloor \frac{n-1}{2} \right\rfloor = 2^n - 2^\left\lfloor \frac{n-1}{2} \right\rfloor
\]
\[
2^{n-1} - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + (2^\left\lfloor \frac{n-1}{2} \right\rfloor - 1 + \ldots + 2^\left\lfloor \frac{n-1}{2} \right\rfloor) = 2^n - 2^\left\lfloor \frac{n-1}{2} \right\rfloor
\]
\[
\vdots
\]
\[
2^n - 2^\left\lfloor \frac{n-1}{2} \right\rfloor.
\]
We note that there are at most
\[
\left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
terms in this sequence. We consider the following regulators
\[
2^{n-1} - 2^\left\lfloor \frac{n-1}{2} \right\rfloor, 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor.
\]
We also note that there are at most
\[
\left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
such regulators. Next we adjoin it to the previously constructed sequence and we note that it contributes at most
\[
\left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
terms in this sequence. Next we examine how these regulators are produced. We note that we can write
\[
2^{n-1} - 2^\left\lfloor \frac{n-1}{2} \right\rfloor = \left(2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor\right) + \left(2^{n-1} + 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right)
\]
We note that we can recast the latter term as
\[
2^n - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + 1 = 2^n - 2^\left\lfloor \frac{n-1}{2} \right\rfloor.
\]
It is observed that there are \(n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor\) such distinct terms in (5.1). We see that these terms generate
\[
n - 2 - \left\lfloor \frac{n-1}{2} \right\rfloor
\]
distinct sub-addition chains. Let us include \(n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor\) such distinct terms in (5.1) and corresponding
\[
n - 2 - \left\lfloor \frac{n-1}{2} \right\rfloor
\]
terms in the sub-addition chain they produce into the a priori constructed sequence. Again, we can write
\[
2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor = \left(2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor\right) + \left(2^\left\lfloor \frac{n-1}{2} \right\rfloor + 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right).
\]
We note that we can recast the latter term as
\[
2^\left\lfloor \frac{n-1}{2} \right\rfloor + 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + 1 = 2^\left\lfloor \frac{n-1}{2} \right\rfloor - 2^\left\lfloor \frac{n-1}{2} \right\rfloor + 1 + 2^\left\lfloor \frac{n-1}{2} \right\rfloor.
\]
We note that there are
\[ \left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2^2} \right\rfloor \]
such distinct terms in the sum. It is observed that these terms generate
\[ \left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2^2} \right\rfloor - 1 \]
distinct terms of a sub-addition chain. Let us include the
\[ \left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2^2} \right\rfloor \]
such distinct terms in the sum and their corresponding
\[ \left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2^2} \right\rfloor - 1 \]
induced sums, which forms a sub-addition chain into the previously constructed sequence. By iterating, we obtain
\[ 2 \left\lfloor \frac{n - 1}{2k-2} \right\rfloor - 2 \left\lfloor \frac{n - 1}{2k-1} \right\rfloor = 2 \left\lfloor \frac{n - 1}{2k-1} \right\rfloor - 2 \left\lfloor \frac{n - 1}{2k} \right\rfloor + \left( 2 \left\lfloor \frac{n - 1}{2k-2} \right\rfloor + 2 \left\lfloor \frac{n - 1}{2k} \right\rfloor - 2 \left\lfloor \frac{n - 1}{2k-1} \right\rfloor + 1 \right). \]

We note that the latter term can be recast as
\[ 2 \left\lfloor \frac{n - 1}{2k-2} \right\rfloor + 2 \left\lfloor \frac{n - 1}{2k} \right\rfloor - 2 \left\lfloor \frac{n - 1}{2k-1} \right\rfloor + 1 = 2 \left\lfloor \frac{n - 1}{2k-2} \right\rfloor + \ldots + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor + 1 + 2 \left\lfloor \frac{n - 1}{2} \right\rfloor. \]

We note that there are
\[ \left\lfloor \frac{n - 1}{2k-2} \right\rfloor - \left\lfloor \frac{n - 1}{2k-1} \right\rfloor \]
such distinct terms in the sum. It is observed that these terms generate
\[ \left\lfloor \frac{n - 1}{2k-2} \right\rfloor - \left\lfloor \frac{n - 1}{2k-1} \right\rfloor - 1 \]
distinct terms of a sub-addition chain. Let us include the
\[ \left\lfloor \frac{n - 1}{2k-2} \right\rfloor - \left\lfloor \frac{n - 1}{2k-1} \right\rfloor \]
such distinct terms in the sum and their corresponding
\[ \left\lfloor \frac{n - 1}{2k-2} \right\rfloor - \left\lfloor \frac{n - 1}{2k-1} \right\rfloor - 1 \]
induced sums, which forms a sub-addition chain into the previously constructed sequence. In particular, we may obtain the total contribution by adding the numbers in the following chains
\[ n - 1 - \left\lfloor \frac{n - 1}{2} \right\rfloor \]
\[ \left\lfloor \frac{n - 1}{2} \right\rfloor - \left\lfloor \frac{n - 1}{2^2} \right\rfloor \]
\[ \ldots \]
\[ \left\lfloor \frac{n - 1}{2k-1} \right\rfloor - \left\lfloor \frac{n - 1}{2k} \right\rfloor. \]
Since there are at most \( \left\lfloor \log_2 \frac{n}{\log_2 n} \right\rfloor \) regulators of the form \( 2^\left\lfloor \frac{n-1}{2^k} \right\rfloor - 2^\left\lfloor \frac{n-1}{2^k} \right\rfloor \), it follows the total number of terms induced is at most
\[
n - 1 - \left\lfloor \frac{n - 1}{2^\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \right\rfloor.
\]
obtained by adding the numbers in the chain above. Similarly the corresponding number of terms in the sub-addition induced by these
\[
n - 1 - \left\lfloor \frac{n - 1}{2^\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \right\rfloor
\]
terms is at most
\[
n - 1 - \left\lfloor \frac{n - 1}{2^\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor.
\]
Let us introduce the non-existent term \( 2^{n-1} + 2^{n-2} - 2^\left\lfloor \frac{n-1}{2^k} \right\rfloor \) into the sequence - which can be obtained by adding the term \( 2^{n-2} \) in the sequence to the regulator \( 2^{n-1} - 2^\left\lfloor \frac{n-1}{2^k} \right\rfloor \) - then we obtain the term \( 2^n - 2^\left\lfloor \frac{n-1}{2^k} \right\rfloor \) by adding one more time the term \( 2^{n-2} \). It is easy to see that our construction yields an addition chain of length at most
\[
\delta(2^n - 1) \leq 2(n - 1) - 2\left\lfloor \frac{n - 1}{2^\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1 + 2\left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
which completes the proof. \( \square \)

The method of backtracking adopted in the above construction produces (by default) a relatively short tail - of logarithmic order - of an addition chain producing \( 2^n - 1 \) and backtracks by constructing required terms of the addition chain. The resulting upper bound for the length of this addition chain may be miles away from what is believed to be the truth

**Conjecture 5.1** (Scholz). The inequality holds
\[
\iota(2^n - 1) \leq n - 1 + \iota(n).
\]

The method of backtracking may be refined and trimmed down by consolidating with a new idea, which is not the purpose of this paper. At least it yields an explicit upper bound which was not attainable in our previous investigations.

### 6. Filling the Pothole method

In this section we describe the method of filling the potholes which is employed to obtain our upper bound. We lay them down chronologically as follows.

- We first construct a complete sub-addition chain producing \( 2^n - 1 \). For technical reasons which will become clear later, we stop the chain prematurely at \( 2^{n-1} \).
- We extend this addition chain by a length of logarithm order.
- This extension has missing terms to qualify as addition chain producing \( 2^n - 1 \). We fill in the missing terms thereby obtaining what one might refer to as spoof addition chain producing \( 2^n - 1 \).
• Creating this spoof addition chain comes at a cost. The remaining step will be to cover the cost and render an account to obtain the upper bound.

**Theorem 6.1.** There exists an addition chain producing $2^n - 1$ of length $\delta(2^n - 1)$ satisfying the inequality

$$\delta(2^n - 1) \leq 2n - 1 - \left\lfloor \frac{n - 1}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor}} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \iota(n)$$

where $\lfloor \cdot \rfloor$ denotes the floor function and $\iota(n)$ the shortest addition chain producing $n$.

**Proof.** First, let us construct the shortest addition chain producing $2^n$ as $1, 2, 2^2, \ldots, 2^n$, with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n + 1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i^{th}$ generator of the chain. Let us consider only the complete sub-addition chain

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}.$$ 

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{n-1} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}, 2^{n-1} + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}, \ldots, 2^{n-1} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} + \ldots + 2^1.$$

We note that the adjoined sequence contributes at most 

$$\left\lfloor \frac{\log n}{\log 2} \right\rfloor \leq \iota(n)$$

by virtue of Lemma (4.1). Since the inequality holds

$$2^{n-1} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} + \ldots + 2^1 < \sum_{i=1}^{n-1} 2^i = 2^n - 2$$

we insert terms into the sum

(6.1) \quad $$2^{n-1} + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} + \ldots + 2^1$$

so that we have

$$\sum_{i=1}^{n-1} 2^i = 2^n - 2.$$ 

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{n-2} + 2^{n-3} + \ldots + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 1$ into (11.28) and this is comes at the cost of adjoining

$$n - 2 - \left\lfloor \frac{n - 1}{2} \right\rfloor$$

terms to the term in (11.28). The last term of the adjoined sequence is given by

(6.2) \quad $$2^{n-1} + (2^{n-2} + 2^{n-3} + \ldots + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 1) + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} + \ldots + 2^1.$$
Again we have to insert $2^{\lfloor \frac{n-1}{2} \rfloor - 1} + \ldots + 2^{\lfloor \frac{n-1}{2^k} \rfloor + 1}$ into (11.29) and this comes at the cost of adjoining

$$[\frac{n-1}{2}] - [\frac{n-1}{2^2}] - 1$$

terms to the term in (11.29). The last term of the adjoined sequence is given by

$$2^{n-1} + (2^{n-2} + 2^{n-3} + \ldots + 2^{\lfloor \frac{n-1}{2^k} \rfloor + 1}) + 2^{\lfloor \frac{n-1}{2^k-1} \rfloor} + \ldots + 2^{\lfloor \frac{n-1}{2^k} \rfloor + 1}.$$ 

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (11.30) and this comes at the cost of adjoining

$$[\frac{n-1}{2^k-1}] - [\frac{n-1}{2^k}] - 1$$

terms to the term in (11.30) for $1 \leq k \leq \lfloor \frac{\log n}{\log 2} \rfloor$ since we filling in at most $\lfloor \frac{\log n}{\log 2} \rfloor$ blocks. It follows that the contribution of these new terms is at most

$$n - 1 - \left[ \frac{n - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor - 1}} \right] - \left[ \frac{n - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right]$$

obtained by adding the numbers in the chain

$$n - 1 - \left[ \frac{n - 1}{2} \right] - 1$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\left[ \frac{n - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor - 1}} \right] - \left[ \frac{n - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right] - 1.$$ 

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain is at most

$$\delta(2^n - 1) \leq n - 1 - \left[ \frac{n - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right] - \left[ \frac{\log n}{\log 2} \right] + \iota(n)$$

thereby ending the construction. □

Remark 6.2. We obtain a slightly weaker but much more explicit weaker version of Scholz’s conjecture.

Corollary 6.1. Let $\iota(n)$ denotes the length of the shortest addition chain producing $n$. Then the inequality holds

$$\iota(2^n - 1) \leq 2n - 1 - \left[ \frac{n - 1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right] - \left[ \frac{\log n}{\log 2} \right] + \iota(n)$$

where $[\cdot]$ denotes the floor function and $\iota(n)$ the shortest addition chain producing $n$. 
6.1. **Improved pothole method.** In this section we introduce a refined version of the pothole method. This method takes into account the factor method. Instead of applying the pothole method on the number $2^n - 1$, we decompose $2^n - 1$ into two factors and applying an inequality of Alfred Brauer on the factors. This essentially trims down the size of the terms encountered by applying only the pothole method. We employ this type of reasoning in the proof of the following result.

**Lemma 6.3.** Let $\iota(n)$ denotes the shortest addition chain producing $n$. If $a, b \in \mathbb{N}$ then
\[ \iota(ab) \leq \iota(a) + \iota(b). \]

**Proof.** The proof of this Lemma can be found in [1]. □

**Theorem 6.4.** The inequality
\[ \iota(2^n - 1) \leq \frac{3}{2}n - \frac{n - 2}{\lfloor \frac{n}{\log 2} \rfloor + 1} - \frac{\log n}{\log 2} - 1 + 1 \left(1 - (-1)^n\right) + \iota(n) \]
holds for all $n \in \mathbb{N}$ with $n \geq 2$, where $\lfloor \cdot \rfloor$ denotes the floor function and $\iota(\cdot)$ the length of the shortest addition chain.

**Proof.** First, we consider the number $2^n - 1$ and examine the length of the addition chain according to the parity of the exponents $n$. If $n \equiv 0 \mod 2$ then we obtain the factorization
\[ 2^n - 1 = (2^\frac{n}{2} - 1)(2^\frac{n}{2} + 1). \]

By setting $\frac{n}{2} = k$, we construct the addition chain producing $2^k$ as $1, 2, 2^2, \ldots, 2^{k-1}, 2^k$ with corresponding sequence of partition
\[ 2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}, 2^k = 2^{k-1} + 2^{k-1} \]
with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq k + 1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i$th generator of the chain. Let us consider only the complete sub-addition chain
\[ 2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^{k-2} + 2^{k-2}. \]

Next we extend this complete sub-addition chain by adjoining the sequence
\[ 2^{k-1} + 2\left\lfloor \frac{k-1}{2} \right\rfloor, 2^{k-1} + 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2\left\lfloor \frac{k-1}{2} \right\rfloor, \ldots, 2^{k-1} + 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2\left\lfloor \frac{k-1}{2} \right\rfloor + \cdots + 2^1. \]

We note that the adjoined sequence contributes at most
\[ \left\lfloor \frac{\log k}{\log 2} \right\rfloor = \left\lfloor \frac{\log n - \log 2}{\log 2} \right\rfloor < \left\lfloor \frac{\log n}{\log 2} \right\rfloor \leq \iota(n) \]
terms to the original complete sub-addition chain, where the upper bound follows by virtue of Lemma 4.1. Since the inequality holds
\[ 2^{k-1} + 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2\left\lfloor \frac{k-1}{2} \right\rfloor + \cdots + 2^1 < \sum_{i=1}^{k-1} 2^i \]
\[ = 2^k - 2 \]
we insert terms into the sum
\[ 2^{k-1} + 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2\left\lfloor \frac{k-1}{2} \right\rfloor + \cdots + 2^1 \]
so that we have

\[ \sum_{i=1}^{k-1} 2^i = 2^k - 2. \]

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert \( 2^{k-2} + 2^{k-3} + \ldots + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor + 1} \) into (11.28) and this is comes at the cost of adjoining

\[ k - 2 - \left\lfloor \frac{k-1}{2} \right\rfloor \]

terms to the term in (11.28). The last term of the adjoined sequence is given by

\[ 2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor + 1}) + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor} + \ldots + 2^1. \]

Again we have to insert \( 2^{\left\lfloor \frac{k-1}{2} \right\rfloor - 1} + \ldots + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor + 1} \) into (11.29) and this comes at the cost of adjoining

\[ \left\lfloor \frac{k-1}{2} - \left\lfloor \frac{k-1}{2^2} \right\rfloor \right\rfloor - 1 \]

terms to the term in (11.29). The last term of the adjoined sequence is given by

\[ 2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor + 1}) + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor} + (2^{\left\lfloor \frac{k-1}{2} \right\rfloor - 1} + \ldots + 2^{\left\lfloor \frac{k-1}{2^2} \right\rfloor + 1}) + 2^{\left\lfloor \frac{k-1}{2^2} \right\rfloor} + \ldots + 2^1. \]

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (11.30) and this comes at the cost of adjoining

\[ \left\lfloor \frac{k-1}{2^s-1} \right\rfloor - \left\lfloor \frac{k-1}{2^s} \right\rfloor - 1 \]

terms to the term in (11.30) for \( 1 \leq s \leq \left\lfloor \frac{\log n}{\log 2} - 1 \right\rfloor \) since we filling in at most \( \left\lfloor \frac{\log k}{\log 2} \right\rfloor \) blocks with \( k = \frac{n}{2} \). It follows that the contribution of these new terms is at most

\[ \left( k - 1 - \left\lfloor \frac{k-1}{2} \right\rfloor \right) - \left\lfloor \frac{\log k}{\log 2} \right\rfloor - 1 \]

obtained by adding the numbers in the chain

\[ k - 1 - \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \]
\[ \left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k-1}{2^2} \right\rfloor - 1 \]
\[ \left\lfloor \frac{k-1}{2^2} \right\rfloor - \left\lfloor \frac{k-1}{2^3} \right\rfloor - 1 \]
\[ \left\lfloor \frac{k-1}{2^3} \right\rfloor - \left\lfloor \frac{k-1}{2^4} \right\rfloor - 1 \]
\[ \left\lfloor \frac{k-1}{2^4} \right\rfloor - \left\lfloor \frac{k-1}{2^5} \right\rfloor - 1 \]
\[ \left\lfloor \frac{k-1}{2^5} \right\rfloor - \left\lfloor \frac{k-1}{2^6} \right\rfloor - 1 \]

Appealing to Lemma 6.3 the inequality

\[ \iota(2^n - 1) \leq \iota(2^{\frac{n}{2}} - 1) + \iota(2^{\frac{n}{2}} + 1) \leq \delta(2^{\frac{n}{2}} - 1) + \iota(2^{\frac{n}{2}} + 1) \]
holds for even \( n \), where \( \delta(\cdot) \) is the length of the constructed addition chain. By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing \( 2^k - 1 \) with \( k = \frac{n}{2} \) is

\[
\delta(2^k - 1) \leq k + k - 1 - \left\lfloor \frac{k - 1}{2^{\left\lfloor \frac{k}{2} \right\rfloor}} \right\rfloor - \left\lfloor \frac{\log k}{\log 2} \right\rfloor + \iota(n)
\]

\[
= n - 1 - \left\lfloor \frac{n - 2}{2^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + 1} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1 + \iota(n).
\]

Now, we construct an addition chain producing \( 2^k + 1 \). We construct the addition chain producing \( 2^k \) as \( 1, 2, 2^2, \ldots, 2^{k-1}, 2^k \) with corresponding sequence of partition

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} + 2^{k-1} = 2^k - 2^k - 2^k = 2^k - 1 + 2^k - 1
\]

with \( a_i = 2^{i-2} - r_i \) for \( 2 \leq i \leq k + 1 \), where \( a_i \) and \( r_i \) denotes the determiner and the regulator of the \( i \)th generator of the chain. By adding 1 to the last term of the chain, we obtain the addition chain producing \( 2^k + 1 \) of the form \( 1, 2, 2^2, \ldots, 2^{k-1}, 2^k, 2^k + 1 \) of length \( k + 1 = \frac{n}{2} + 1 \). By combining the contribution of the length of the addition chains constructed, we obtain in the case \( n \equiv 0 \pmod{2} \) the inequality

\[
\iota(2^n - 1) \leq \frac{3}{2} n - \left\lfloor \frac{n - 2}{2^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + 1} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1 + \iota(n).
\]

We now examine the case \( n \equiv 1 \pmod{2} \). In this case, we write

\[
2^n - 1 = (2^{n-1} - 1) + (2^{n-1} - 1) + 1
\]

so that we construct an addition chain producing \( 2^{n-1} - 1 \). Once this addition chain is obtained, then we add the term \( 2^{n-1} - 1 \) to itself and finally add 1 to obtain the addition chain producing \( 2^n - 1 \). It will follow from this construction that the length \( \delta(2^n - 1) \) is the sum of the length of the addition chain \( \delta(2^{n-1} - 1) \) and 2. Since \( n - 1 \equiv 0 \pmod{2} \), we can adapt the argument of the even case to obtain the upper bound

\[
\delta(2^{n-1} - 1) \leq \frac{3}{2} (n - 1) - \left\lfloor \frac{n - 3}{2^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + 1} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} - 1 \right\rfloor + \iota(n)
\]

so that the length

\[
\iota(2^n - 1) \leq \delta(2^{n-1} - 1) + 2 = \frac{3}{2} (n - 1) - \left\lfloor \frac{n - 3}{2^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + 1} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} - 1 \right\rfloor + \iota(n)
\]

so that the length

\[
\iota(2^n - 1) \leq \frac{3}{2} n - \left\lfloor \frac{n - 3}{2^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + 1} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} - 1 \right\rfloor + \iota(n) + \frac{1}{2}.
\]

The claimed inequality follows by combining both the even and the odd case. \( \square \)

### 7. Addition chains of fixed degree

In this section, we introduce the notion of addition chains of degree \( d \) and their corresponding sub-addition chains. We first recall the following notion of an addition chain.
**Definition 7.1.** Let \( n \geq 3 \), then by an addition chain of length \( k - 1 \) producing \( n \) we mean the sequence

\[
1, 2, \ldots, s_{k-1}, s_k
\]

where each term \( s_j \) (\( j \geq 2 \)) in the sequence is the sum of two earlier terms in the sequence, with the corresponding sequence of partition

\[
s_j = a_{j-1} + r_{j-1}, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n
\]

with \( a_{i+1} = a_i + r_i \) and \( a_{i+1} = s_i \) for \( 2 \leq i \leq k \). We call the partition \( a_i + r_i \) the \( i^{th} \) generator of the addition chain of degree 2 for \( 2 \leq i \leq k \). We call \( a_i \) the determiner and \( r_i \) the regulator of the \( i^{th} \) generator of the chain. We call the sequence \((r_i)\) the regulators of an addition chain and \((a_i)\) the determiners of the addition chain for \( 2 \leq i \leq k \).

**Remark 7.2.** We now introduce the notion of addition chains of degree \( d \) producing a fixed number \( n \). This type of addition chain can be considered as a generalization of addition chains where we allow each term in the chain to be the sum of at most \( d \) earlier terms in the chain. The notion of an addition is the situation where we take \( d = 1 \). We make this notion formal and apply to some problems in the sequel.

**Definition 7.3.** Let \( n \geq 3 \), then by the addition chain of degree \( d \) and of length \( k - 1 \) producing \( n \) we mean the sequence

\[
1, 2, \ldots, s_{k-1}, s_k
\]

where each term \( s_j \) (\( j \geq 3 \)) in the sequence is the sum of at most \( d \) earlier terms, with the corresponding sequence of partition

\[
2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n
\]

with \( a_{i+1} = a_i + r_i \) and \( a_{i+1} = s_i \) for \( 2 \leq i \leq k \), where

\[
r_i = \sum_{m \in [1,d]} s_m
\]

for \( i \leq d \). We call the partition \( a_i + r_i \) the \( i^{th} \) generator of the chain for \( 2 \leq i \leq k \). We call \( a_i \) the block determiners of length at most \( d+1 \) and \( r_i \) the block regulator of length at most \( d \) of the \( i^{th} \) generator of the chain. We call the sequence \((r_i)\) the block regulators of the addition chain and \((a_i)\) the block determiners of the chain for \( 2 \leq i \leq k \).

The notion above is a generalization of the notion of the regulators and the determiners of an addition to the setting of an addition chain of degree \( d \). In this case, the regulator is viewed as a partition of at most \( d \) previous terms and each determiner the partition of at most \( d+1 \) previous terms in the sequence. In the situation where we allow \( d = 1 \), then the block regulators and the block determiners coincides with regulators and determiners of an addition chain.
Definition 7.4. Let the sequence $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing $n$ with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$ 

Then we call the sub-sequence $(s_j)$ for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a sub-addition chain of degree $d$ of the addition chain of degree $d$ producing $n$. We say it is a complete sub-addition chain of degree $d$ of the addition chain of degree $d$ producing $n$ if it contains exactly the first $t$ terms of the addition chain of degree $d$. Otherwise, we say it is an incomplete sub-addition chain of degree $d$.

7.1. Addition chains of degree $d$ of numbers of special forms. In this section we study addition chains of a fixed degree of numbers of special forms. We examine ways of minimizing the length of the addition chain of degree $\lfloor \frac{n-1}{2} \rfloor$ for numbers of the form $2^n - 1$.

Remark 7.5. We now obtain an analogous inequality for addition chains of degree $\lfloor \frac{n-1}{2} \rfloor$ producing $2^n - 1$ related to the Scholz conjecture.

7.2. Analogous result. In this section, we prove an analogue of the Scholz conjecture on addition chains of degree $\lfloor \frac{n-1}{2} \rfloor$.

Theorem 7.6. Let $\ell(n)$ and $\ell(\lfloor \frac{n-1}{2} \rfloor)(n)$ denotes the length of the shortest addition chain and addition chain of degree $\lfloor \frac{n-1}{2} \rfloor$, respectively, producing $n$. Then the inequality holds

$$\ell(\lfloor \frac{n-1}{2} \rfloor)(2^n - 1) \leq n + \ell(n).$$

Proof. First, let us construct the shortest addition chain producing $2^n$ as $1, 2, 2^2, \ldots, 2^{n-1}, 2^n$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^4, \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-1} = r_i$ for $2 \leq i \leq n+1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i^{th}$ generator of the chain. Let us consider only the complete sub-addition chain $1, 2, 2^2, \ldots, 2^{n-1}$. Next we extend this addition chain by adjoining the sequence

$$2^{n-1} + \cdots + 2^{t_1} + \cdots + 2^{\lfloor \frac{n-1}{2} \rfloor}, 2^{n-1} + \cdots + 2^{t_1} + \cdots + 2^{\lfloor \frac{n-3}{2} \rfloor} + \cdots + 2^{2}, \ldots, 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^1,$$

where $\lfloor \frac{n-1}{2} \rfloor + 1 \leq t_1 \leq n-2$, $\lfloor \frac{n-3}{2} \rfloor + 1 \leq t_2 \leq \lfloor \frac{n-1}{2} \rfloor - 1, \ldots, \lfloor \frac{n-1}{2} \rfloor + 1 \leq t_k \leq \lfloor \frac{n-1}{2} \rfloor - 1$. The terms adjoined to the addition chain constructed can be recast into the forms

$$2^{n-1} + \cdots + 2^{t_1} + \cdots + 2^{\lfloor \frac{n-1}{2} \rfloor} = 2^n - 2^{\lfloor \frac{n-1}{2} \rfloor}$$

and

$$2^{n-1} + \cdots + 2^{t_1} + \cdots + 2^{\lfloor \frac{n-1}{2} \rfloor} + \cdots + 2^{t_2} + \cdots + 2^{\lfloor \frac{n-3}{2} \rfloor} = 2^n - 2^{\lfloor \frac{n-1}{2} \rfloor}.$$
so that by induction we can write

\[ 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^1 = 2^n - 2 \]

and we obtain the addition chain of degree \( \lfloor \frac{n-1}{2} \rfloor \)

\[ 1, 2, 2^2, \ldots, 2^{n-1}, 2^n - 2^{\lfloor \frac{n-1}{2} \rfloor}, 2^n - 2^{\lfloor \frac{n-3}{2} \rfloor}, \ldots, 2^n - 2, 2^n - 1. \]

We note that the adjoined sequence contributes at most

\[ \left\lfloor \frac{\log n}{\log 2} \right\rfloor \leq \iota(n) \]

terms to the original complete addition chain, where the upper bound follows from Lemma 4.1. This completes the proof. \( \square \)

**Lemma 7.7.** Let \( \iota(n) \) and \( \iota^{\left\lfloor \frac{n-1}{2} \right\rfloor}(n) \) denotes the length of the shortest addition chain and addition chain of degree \( \lfloor \frac{n-1}{2} \rfloor \), respectively, producing \( n \). Then the inequality

\[ \iota(2^n - 1) \geq \iota^{\left\lfloor \frac{n-1}{2} \right\rfloor}(2^n - 1) \]

holds.

**Remark 7.8.** We now obtain an analogous inequality for addition chains of degree \( \lfloor \frac{n-1}{2} \rfloor \) producing \( 2^n - 1 \) related to the Scholz conjecture.

**7.3. The lower bound.** In this section, we prove an analogue of the Scholz conjecture on addition chains of degree \( \lfloor \frac{n-1}{2} \rfloor \).

**Theorem 7.9.** Let \( \iota(n) \) and \( \iota^{\left\lfloor \frac{n-1}{2} \right\rfloor}(n) \) denotes the length of the shortest addition chain and addition chain of degree \( \lfloor \frac{n-1}{2} \rfloor \), respectively, producing \( n \). Then the identity

\[ \iota^{\left\lfloor \frac{n-1}{2} \right\rfloor}(2^n - 1) = n + 1 \]

holds.

**Proof.** First, let us construct the shortest addition chain producing \( 2^n \) as \( 1, 2, 2^2, \ldots, 2^{n-1}, 2^n \) with corresponding sequence of partition

\[ 2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{n-1} = 2^n - 2^n, 2^n = 2^n - 1 + 2^n - 1 \]

with \( a_i = 2^{i-2} = r_i \) for \( 2 \leq i \leq n+1 \), where \( a_i \) and \( r_i \) denotes the determiner and the regulator of the \( i^{th} \) generator of the chain. Let us consider only the complete subaddition chain \( 1, 2^2, \ldots, 2^{n-1} \). Next we extend this addition chain by adjoining the sequence

\[ 2^{n-1} + \cdots + 2^i + \cdots + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}, 2^n - 1 + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + \cdots + 2^t + 2^{\left\lfloor \frac{n-1}{2} \right\rfloor} + \cdots + 2^t + \cdots + 2^2 + 2 \]

where \( \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \leq t_1 \leq n - 2, 1 \leq t_2 \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \). We note that only two terms have been adjoined to the addition chain of degree \( \left\lfloor \frac{n-1}{2} \right\rfloor \) to complete the chain. This completes the proof. \( \square \)
Corollary 7.1. Let \( \iota(n) \) denotes the length of the shortest addition chain producing \( n \). Then we have the lower bound
\[
\iota(2^n - 1) \geq n + 1.
\]

Proof. The inequality follows by appealing to Lemma 7.7 in Theorem 7.9. \( \square \)

8. Further remarks

The pothole method combined with the factor method appears to be a viable tool to study the Scholz conjecture on additions. It is to be observed that increasing the factors of numbers of the form \( 2^n - 1 \) has the ultimate potential of reducing the currently improved upper bound. The reader may notice that increasing our two-factor case to the three-factor case reduces the term \( \frac{3}{2}n \) to \( \frac{5}{4}n \). In this regard, the Scholz conjecture may be studied and one could come close to a result by carefully choosing the size of the factors of \( 2^n - 1 \).

9. Progress on the Scholz conjecture

In this section we make progress on the Scholz conjecture on addition chains. We combine the factor method and the "fill in the pothole" method to study short addition chains producing numbers of the form \( 2^n - 1 \) and the Scholz conjecture. Given any number of the form \( 2^n - 1 \), we obtain the decomposition
\[
2^n - 1 = (2^{\frac{n-(1-(1)^n)\frac{1}{2}}{2} - 1})(2^{\frac{n-(1-(1)^n)\frac{1}{2}}{2} + 1}) + \frac{(1 - (-1)^n)}{2}(2^{n-(1-(1)^n)\frac{1}{2}})
\]
which eventually yield the following decomposition \( 2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1) \) in the case \( n \equiv 0 \mod 2 \) and
\[
2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1) + 2^{n-1}
\]
in the case \( n \equiv 1 \mod 2 \). We iterate this decomposition up to a certain desired frequency and apply the factor method on all the factors obtained from this decomposition. We then apply the pothole method to obtain a bound for the shortest addition chains producing the only factor of form \( 2^n - 1 \). The length of the shortest addition chains of numbers of the form \( 2^n + 1 \) is easy to construct, by first constructing the shortest addition chain producing \( 2^n \), adding the first term of the chain to the last term and adjoining to the chain. We combine the method of filling the potholes and the factor method to obtain a general inequality related to the Scholz conjecture on length of addition chain producing \( 2^n - 1 \).

Theorem 9.1. The inequality
\[
\iota(2^n - 1) \leq n + 1 - \sum_{j=1}^{\lfloor \log n \rfloor} \xi(n, j) + 3\left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
holds for all \( n \in \mathbb{N} \) with \( n \geq 64 \), where \( \iota(\cdot) \) denotes the length of the shortest addition chain producing \( \cdot \) and where \( \xi(n, 1) := \{ \frac{n}{2} \} \) with \( \xi(n, 2) = \{ \frac{1}{2}, \{ \frac{n}{2} \} \} \) and so on, with \( \{ \cdot \} \) denoting the fractional part of a real number.
Lemma 6.3, we obtain further the inequality

\[2^n - 1 = \left(2^{\frac{n-1-(-1)^{n+1}}{2}} - 1\right)\left(2^{\frac{n-1-(-1)^{n+1}}{2}} + 1\right) + \frac{(1 - (-1)^n)}{2} (2^{n-1-(-1)^{n+1}} - \frac{1}{2}).\]

It is easy to see that we can recover the general factorization of \(2^n - 1\) from this identity according to the parity of the exponent \(n\). In particular, if \(n \equiv 0 \pmod{2}\), then we have

\[2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)\]

and

\[2^n - 1 = (2^{\frac{n-1}{2}} - 1)(2^{\frac{n-1}{2}} + 1) + 2^{n-1}\]

if \(n \equiv 1 \pmod{2}\). By combining both cases, we obtain the inequality

\[\lambda(2^n - 1) \leq \lambda\left(\left(2^{\frac{n-1-(-1)^{n+1}}{2}} - 1\right)\left(2^{\frac{n-1-(-1)^{n+1}}{2}} + 1\right) + \frac{(1 - (-1)^n)}{2} (2^{n-1-(-1)^{n+1}} - \frac{1}{2})\right) + 2\]

obtained by constructing an addition chain producing \(2^{n-1} - 1\), adding \(2^{n-1} - 1\) to \(2^{n-1} - 1\), adding 1 and adjoining the result in the case \(n \equiv 1 \pmod{2}\). Applying Lemma 6.3 we obtain further the inequality

\[(9.1) \quad \lambda(2^n - 1) \leq \lambda\left(\left(2^{\frac{n-1-(-1)^{n+1}}{2}} - 1\right) + \left(2^{\frac{n-1-(-1)^{n+1}}{2}} + 1\right)\right) + 2\]

Again let us set \(\frac{n-1-(-1)^{n+1}}{2} = k\) in (11.23), then we obtain the general decomposition

\[2^k - 1 = \left(2^{\frac{k-1-(-1)^k}{2}} - 1\right)\left(2^{\frac{k-1-(-1)^k}{2}} + 1\right) + \frac{(1 - (-1)^k)}{2} (2^{k-1-(-1)^k} - \frac{1}{2}).\]

It is easy to see that we can recover the general factorization of \(2^k - 1\) from this identity according to the parity of the exponent \(k\). In particular, if \(k \equiv 0 \pmod{2}\), then we have

\[2^k - 1 = (2^{\frac{k}{2}} - 1)(2^{\frac{k}{2}} + 1)\]

and

\[2^k - 1 = (2^{\frac{k-1}{2}} - 1)(2^{\frac{k-1}{2}} + 1) + 2^{k-1}\]

if \(k \equiv 1 \pmod{2}\). By combining both cases, we obtain the inequality

\[\lambda(2^k - 1) \leq \lambda\left(\left(2^{\frac{k-1-(-1)^k}{2}} - 1\right)\left(2^{\frac{k-1-(-1)^k}{2}} + 1\right) + \frac{(1 - (-1)^k)}{2} (2^{k-1-(-1)^k} - \frac{1}{2})\right) + 2\]

obtained by constructing an addition chain producing \(2^{k-1} - 1\), adding \(2^{k-1} - 1\) to \(2^{k-1} - 1\), adding 1 and adjoining the result in the case \(k \equiv 1 \pmod{2}\). Applying Lemma 6.3 we obtain further the inequality

\[(9.2) \quad \lambda(2^k - 1) \leq \lambda\left(\left(2^{\frac{k-1-(-1)^k}{2}} - 1\right) + \left(2^{\frac{k-1-(-1)^k}{2}} + 1\right)\right) + 2\]

so that by inserting (11.24) into (11.23), we obtain the inequality

\[(9.3) \quad \lambda(2^n - 1) \leq \lambda\left(\left(2^{\frac{n-1-(-1)^n}{2}} - 1\right) + \left(2^{\frac{n-1-(-1)^n}{2}} + 1\right)\right) + 2\]

Next we iterate the factorization up to frequency \(s\) to obtain

\[(9.4) \quad \lambda(2^n - 1) \leq \lambda\left(\left(2^{\frac{n-1-(-1)^n}{2}} - 1\right) + \left(2^{\frac{n-1-(-1)^n}{2}} + 1\right)\right) + 2\]

\[+ \cdots + \lambda\left(2^{\frac{n-1-(-1)^n}{2}} - 1\right) + \lambda\left(2^{\frac{n-1-(-1)^n}{2}} + 1\right) + 2\]
where $0 \leq \xi(n, s) < 1$ for an integer $2 \leq s := s(n)$ fixed to be chosen later. For instance,
\[
\xi(n, 1) = (1 - (-1)^n)^{1/4} < 1
\]
and
\[
\xi(n, 2) = (1 - (-1)^n)^{1/8} + (1 - (-1)^k)^{1/4} < 1
\]
with
\[
k := \frac{n - (1 - (-1)^n)^{1/2}}{2}
\]
and so on. That is, $\xi(n, 1) := \{\xi/2\}$ with $\xi(n, 2) = \{1/2\}$ and so on. Indeed the function $\xi(n, s)$ for values of $s \geq 3$ can be read from exponents of the terms arising from the iteration process. It follows from (11.26) the inequality
\[
\epsilon(2^n - 1) \leq \sum_{v=1}^{s} n \frac{1}{2^v} + 3s - \theta(n, s) + \epsilon(2^{\frac{n}{2}} - \xi(n, s) - 1)
\]
\[
= n(1 - \frac{1}{2^v}) + 3s - \theta(n, s) + \epsilon(2^{\frac{n}{2}} - \xi(n, s) - 1)
\]
for some $0 \leq \theta(n, s) := \sum_{j=1}^{s} \xi(n, j)$ and $2 \leq s := s(n)$ fixed, an integer to be chosen later. It is worth noting that
\[
\theta(n, s) := \sum_{j=1}^{s} \xi(n, j) = 0
\]
if $n = 2^r$ for some $r \in \mathbb{N}$, since $\xi(n, j) = 0$ for each $1 \leq j \leq s$ for all $n$ which are powers of 2. It is also important to note that the $2s$ term is obtained by noting that there are at most $s$ terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other $s$ term comes from summing 1 with frequency $s$ finding the total length of the short addition chains producing numbers of the form $2^r + 1$. Now, we set $k = \frac{n}{2} - \xi(n, s)$ and construct the addition chain producing $2^k$ as $1, 2, 2^2, 2^{k-1}, 2^k$ with corresponding sequence of partition
\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}, 2^k = 2^{k-1} + 2^{k-1}
\]
with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq k + 1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i^{th}$ generator of the chain. Let us consider only the complete sub-addition chain
\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}.
\]
Next we extend this complete sub-addition chain by adjoining the sequence
\[
2^{k-1} + 2^{\lfloor \frac{k-2}{2} \rfloor}, 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-3}{2} \rfloor}, \ldots, 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-3}{2} \rfloor} + \ldots + 2^1.
\]
Since $\xi(n, s) = 0$ if $n = 2^r$ and $0 \leq \xi(n, s) < 1$ if $n \neq 2^r$, we note that the adjoined sequence contributes at most
\[
\left\lfloor \frac{\log k}{\log 2} \right\rfloor = \left\lfloor \frac{\log(2^{\frac{n}{2}} - \xi(n, s))}{\log 2} \right\rfloor = \left\lfloor \frac{\log n - s \log 2}{\log 2} \right\rfloor = \left\lfloor \frac{\log n}{\log 2} - s \right\rfloor
\]
terms to the original complete sub-addition chain. Since the inequality holds
\[
2^{k-1} + 2^\lfloor \frac{k-1}{2} \rfloor + 2^\lfloor \frac{k-3}{2^2} \rfloor + \ldots + 2^1 < \sum_{i=1}^{k-1} 2^i
= 2^k - 2
\]
we insert terms into the sum
\[
2^{k-1} + 2^\lfloor \frac{k-1}{2} \rfloor + 2^\lfloor \frac{k-3}{2^2} \rfloor + \ldots + 2^1
\]
so that we have
\[
\sum_{i=1}^{k-1} 2^i = 2^k - 2.
\]
Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert \(2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor + 1}\) into (11.28) and this comes at the cost of adjoining \(k - 2 - \lfloor \frac{k-1}{2} \rfloor\) terms to the term in (11.28). The last term of the adjoined sequence is given by
\[
2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor + 1}) + 2^{\lfloor \frac{k-3}{2^2} \rfloor} + \ldots + 2^1.
\]
Again we have to insert \(2^{\lfloor \frac{k-1}{2^2} \rfloor - 1} + \ldots + 2^{\lfloor \frac{k-1}{2^2} \rfloor + 1}\) into (11.29) and this comes at the cost of adjoining \(\lfloor \frac{k-1}{2} \rfloor - \lfloor \frac{k-1}{2^2} \rfloor - 1\) terms to the term in (11.29). The last term of the adjoined sequence is given by
\[
2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2^2} \rfloor + 1}) + 2^{\lfloor \frac{k-3}{2^2} \rfloor} + (2^{\lfloor \frac{k-1}{2^2} \rfloor - 1} + \ldots + 2^{\lfloor \frac{k-1}{2^2} \rfloor + 1}) + 2^{\lfloor \frac{k-1}{2^3} \rfloor} + \ldots + 2^1.
\]
By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (11.30) and this comes at the cost of adjoining \(\lfloor \frac{k-1}{2^j} \rfloor - \lfloor \frac{k-1}{2^{j+1}} \rfloor - 1\) terms to the term in (11.30) for \(j \leq \lfloor \frac{\log n}{\log 2} \rfloor - s\), since we are filling in at most \(\lfloor \frac{\log k}{\log 2} \rfloor\) blocks with \(k = \frac{n}{2^s} - \xi(n, s)\). It follows that the contribution of these new terms is at most
\[
k - 1 - \left\lfloor \frac{k-1}{2 \lfloor \frac{\log k}{\log 2} \rfloor} \right\rfloor - \left\lfloor \frac{\log k}{\log 2} \right\rfloor
\]
obtained by adding the numbers in the chain
\[
\begin{align*}
k - 1 - \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \\
\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k-1}{2^2} \right\rfloor - 1 \\
\ldots
\end{align*}
\]
By undertaking a quick book-keeping, it follows that the total number of terms in
the constructed addition chain producing $2^k - 1$ with $k = n^{2^s} - \xi(n, s)$ is
\[
\delta(2^k - 1) \leq k + k - 1 - \left\lfloor \frac{k - 1}{2^{\left\lfloor \frac{\log k}{\log 2} \right\rfloor + 1}} \right\rfloor - \left\lfloor \frac{\log k}{\log 2} \right\rfloor + \left\lfloor \log n \log 2 \right\rfloor - s
\]
(9.10)

By plugging the inequality (11.32) into the inequalities in (11.27) and noting that
\(\iota(\cdot) \leq \delta(\cdot)\), we obtain the inequality
\[
\iota(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + 3s - \theta(n, s) + \iota(2^{\frac{n}{2^s}} - \xi(n, s) - 1)
\]
(9.11)
\[
= n(1 - \frac{1}{2^{s-1}}) + \frac{n}{2^{s-1}} - 1 + 3s - \theta(n, s) - \left\lfloor \frac{n}{2^s} - \xi(n, s) - 1 \right\rfloor \left\lfloor \frac{n}{2^s} \right\rfloor
\]
By taking $2 \leq s := s(n)$ such that $s = \left\lfloor \frac{\log n}{\log 2} \right\rfloor$ and noting that
\[
\left\lfloor \frac{n}{2^s} - \xi(n, s) - 1 \right\rfloor \left\lfloor \frac{n}{2^s} \right\rfloor = 0
\]
we obtained further the inequality
\[
\iota(2^n - 1) \leq n - 1 - \theta(n, \frac{\log n}{\log 2}) + 2 + 3\left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
for $\theta(n, \frac{\log n}{\log 2}) := \sum_{j=1}^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \xi(n, j)$ with $n > 64$ and the claimed inequality follows as
a consequence.

In the sequel we prove a much more general inequality. This inequality would be
useful for our subsequent investigations of the Scholz conjecture, which comes from
the general inequality developed in the proof of Theorem 11.1.

**Proposition 9.1.** There exists an $\left\lfloor \frac{\log n}{\log 2} \right\rfloor \geq s := s(n) \geq 2$ such that the inequality holds
\[
\iota(2^n - 1) \leq n - 1 + (3s - \sum_{j=1}^{s} \left\{ \frac{n}{2^j} \right\}) + \iota(\frac{n}{2^s})
\]
for all $n \geq 2$, where $\{ \cdot \}$ and $\lfloor \cdot \rfloor$ denotes the fractional part and the integer part of
any real number $\cdot$. 

\[\square\]
Proof. We recall the general inequality developed in the proof of Theorem 11.1
\[ \iota(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + 3s - \theta(n, s) + \iota(2^{\frac{n}{s}} - \xi(n, s) - 1) \]
(9.12)
\[ = n(1 - \frac{1}{2^s}) + 3s - \theta(n, s) + \iota(2^{\frac{n}{s}} - \xi(n, s) - 1) \]
for some \( 0 \leq \theta(n, s) := \sum_{j=1}^{s} \xi(n, j) \) and \( 2 \leq s := s(n) \) fixed. It is easy to see that \( \xi(n, j) = \{ \frac{n}{2^j} \} \). It is known that the Scholz conjecture is true for all exponents up to 5784688 and it is also true infinitely often so that we can choose an integer \( s \geq 2 \) such that the \( 2^{\frac{n}{s}} - \xi(n, s) - 1 = 2^{\frac{n}{s}} - \{ \frac{n}{2^s} \} - 1 \) satisfies the Scholz conjecture. That is, we can choose \( s := s(n) \geq 2 \) such that the inequality
\[ \iota(2^{\frac{n}{s}} - \xi(n, s) - 1) = \iota(2^{\frac{n}{s}}) - 1 \leq \frac{n}{2^s} - 1 + \iota(\frac{n}{2^s}) \]
(9.13)
so that by plugging (9.13) into (11.27), the claimed inequality follows as a consequence.
\[ \square \]

10. The notion of carries

We devote this section to the study of the notion of carries and its number theoretic properties. It turns out that this notion plays an important role in controlling the length of an addition for numbers of the form \( 2^n - 1 \). Short addition chains with small carries almost satisfy the Scholz conjecture. We launch the following languages.

Definition 10.1. Consider the decomposition
\[ 2^n - 1 = (2^{\frac{n}{2^s}} - 1)(2^{\frac{n}{2^s}} + 1) + \frac{1 - (-1)^n}{2}(2^{n-(1-(1)^n)} \frac{1}{2}) \]
for \( n \geq 2 \). Then the non-zero remainder
\[ \eta(2^n - 1) := \frac{1 - (-1)^n}{2}(2^{n-(1-(1)^n)} \frac{1}{2}) \]
is the level one carry of \( 2^n - 1 \). We say that \( 2^n - 1 \) is free of level one carries if \( \eta(2^n - 1) = 0 \). By letting
\[ m = \lfloor \frac{n}{2^s} \rfloor \]
then we obtain the decomposition
\[ 2^m - 1 = (2^{\frac{m}{2^s}} - 1)(2^{\frac{m}{2^s}} + 1) + \frac{1 - (-1)^m}{2}(2^{m-(1-(1)^m)} \frac{1}{2}) \]
and we denote the carry with
\[ \eta(2^m - 1) = \frac{1 - (-1)^m}{2}(2^{m-(1-(1)^m)} \frac{1}{2}) \]
and we say it is the level two carry of \( 2^n - 1 \) if \( \eta(2^m - 1) \neq 0 \). In general, we denote the level \( k \) carry of \( 2^n - 1 \) as the remainder
\[ \eta(2^r - 1) = \frac{1 - (-1)^r}{2}(2^{r-(1-(1)^r)} \frac{1}{2}) \]
with \[ r = \left\lfloor \frac{n}{2^k} \right\rfloor. \]

We say that \( 2^n - 1 \) is free of level \( k \) carries if \( \eta(2^r - 1) = 0 \). The number of non-zero levels of carry of \( 2^n - 1 \) for all \( 1 \leq k \leq \left\lfloor \frac{\log n}{\log 2} \right\rfloor \) is the degree of carry of \( 2^n - 1 \).

**Proposition 10.1.** The number \( 2^n - 1 \) \((n \geq 2)\) is free of level one carry if and only if \( n \equiv 0 \pmod{2} \).

**Proof.** Suppose that \( 2^n - 1 \) is free of level one carry, then
\[
\eta(2^n - 1) = \frac{(1 - (-1)^n)}{2} (2^n - (1 - (-1)^n)\frac{1}{2^n}) = 0.
\]
This is only possible with \((1 - (-1)^n) = 0\) and when \( n \equiv 0 \pmod{2} \). Conversely, suppose that \( n \equiv 0 \pmod{2} \) then \( \frac{n}{2} \in \mathbb{N} \) and we can write
\[
2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)
\]
and we see that
\[
\eta(2^n - 1) = 0. \quad \square
\]

Integers of the form \( 2^n - 1 \) with high degrees of carry serve as an obstruction to achieving the inequality
\[
\iota(2^n - 1) \leq n - 1 + \iota(n)
\]
using our current method. At best, avoiding them can yield progress on the conjecture using the current method but only for a specialized set of integers of the form \( 2^n - 1 \) with low degrees of carry. It turns out that the nature of the exponents in large part characterizes integers with high degree (resp. low degree) carries. Encountering integers of the form \( 2^n - 1 \) with exponents giving rise to high degree carries can be controlled in a way to minimize the corresponding length of the addition chain. At the moment we prove that we can obtain a chain of small length for numbers \( 2^n - 1 \) with exponents giving rise to low degree carries.

11. Improved inequality using the method of carries

In this section, we prove an explicit upper bound for the length of the shortest addition chain producing numbers of the form \( 2^n - 1 \). We begin with the following important but fundamental result.

**Theorem 11.1.** If \( 2^n - 1 \) has carries of degree at most
\[
\kappa(2^n - 1) = \frac{1}{2(1 + c)} \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1
\]
for \( c > 0 \) fixed, then the inequality
\[
\iota(2^n - 1) \leq n - 1 + \left( 1 + \frac{1}{1 + c} \right) \left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
holds for all \( n \in \mathbb{N} \) with \( n \geq 4 \), where \( \iota(\cdot) \) denotes the length of the shortest addition chain producing \( \cdot \).
Proof. For a fixed $c > 0$ assume that $2^n - 1$ has at most
\[
\frac{1}{2(1+c)} \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1
\]
degrees of carries. Next decompose the number $2^n - 1$ and obtain the decomposition
\[
2^n - 1 = (2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1) + \eta(2^n - 1)
\]
where
\[
\eta(2^n - 1) := \frac{1 - (-1)^n}{2^{2n-(1-(-1)^n)\frac{n}{2}}}
\]
is the level one carry of $2^n - 1$. It is easy to see that we can recover the general factorization of $2^n - 1$ from this identity according to the parity of the exponent $n$. In particular, if $n \equiv 0 \pmod{2}$, then we have
\[
2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)
\]
and
\[
2^n - 1 = (2^{\frac{n-1}{2}} - 1)(2^{\frac{n-1}{2}} + 1) + 2^{n-1}
\]
if $n \equiv 1 \pmod{2}$. By combining both cases, we obtain the inequality
\[
\iota(2^n - 1) \leq \iota((2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1)) + \eta(2^n - 1).
\]
Applying Lemma 6.3, we obtain further the inequality
\[
(11.1)
\]
Again let us set $\left\lfloor \frac{n}{2} \right\rfloor = k$ in (11.23), then we obtain the general decomposition
\[
2^k - 1 = (2^{\left\lfloor \frac{k}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{k}{2} \right\rfloor} + 1) + \eta(2^k - 1)
\]
where
\[
\eta(2^k - 1) = \frac{1 - (-1)^k}{(2^{k-(1-(-1)^k)\cdot\frac{k}{2}})
\]
is the carry of $2^k - 1$. It is easy to see that we can recover the general factorization of $2^k - 1$ from this identity according to the parity of the exponent $k$. In particular, if $k \equiv 0 \pmod{2}$, then we have
\[
2^k - 1 = (2^{\frac{k}{2}} - 1)(2^{\frac{k}{2}} + 1)
\]
and
\[
2^k - 1 = (2^{\frac{k-1}{2}} - 1)(2^{\frac{k-1}{2}} + 1) + 2^{k-1}
\]
if $k \equiv 1 \pmod{2}$. By combining both cases, we obtain the inequality
\[
\iota(2^k - 1) \leq \iota((2^{\left\lfloor \frac{k}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{k}{2} \right\rfloor} + 1)) + \eta(2^k - 1).
\]
Applying Lemma 6.3, we obtain further the inequality
\[
(11.2)
\]
so that by inserting (11.24) into (11.23), we obtain the inequality
\[
(11.3)
\]
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Next we iterate the factorization up to frequency \( s \) to obtain

\[
\iota(2^n - 1) \leq \iota(2^{\lfloor \frac{n}{2} \rfloor}) + 1 + \eta(2^n - 1) + \iota(2^{\lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor \rfloor} - 1) + \iota(2^{\lfloor \frac{1}{2} \rfloor} + 1) + \eta(2^{\lfloor \frac{1}{2} \rfloor} - 1)
\]

(11.4)

\[
+ \cdots + \iota(2^{\lfloor \frac{n}{2} - \xi(n,s) \rfloor} - 1) + \iota(2^{\lfloor \frac{n}{2} - \xi(n,s) \rfloor} + 1) + \eta(2^{\lfloor \frac{n}{2} - \xi(n,s) \rfloor} - 1)
\]

where \( 0 \leq \xi(n,s) < 1 \) for an integer \( 2 \leq s := s(n) \) fixed to be chosen later. For instance,

\[
\xi(n,1) = (1 - (-1)^n) \frac{1}{4} < 1
\]

and

\[
\xi(n,2) = (1 - (-1)^n) \frac{1}{8} + (1 - (-1)^k) \frac{1}{4} < 1
\]

with

\[
k := \left\lfloor \frac{n}{2} \right\rfloor
\]

and so on. That is, \( \xi(n,1) := \{ \frac{1}{4} \} \) with \( \xi(n,2) = \{ \frac{1}{8}, \frac{1}{4} \} \) and so on. Indeed the function \( \xi(n,s) \) for values of \( s \geq 3 \) can be read from exponents of the terms arising from the iteration process. It follows from \([11, 20]\) the inequality

\[
\iota(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2 \sum_{j=1}^{s} \sum_{\eta(2^{m-1}) \neq 0} \sum_{m=\lfloor \frac{n}{2^v} \rfloor} 1 - \theta(n,s) + \iota(2^{\lfloor \frac{n}{2} \rfloor} - \xi(n,s) - 1)
\]

(11.5)

\[
= n(1 - \frac{1}{2^s}) + s + 2 \sum_{j=1}^{s} \sum_{\eta(2^{m-1}) \neq 0} \sum_{m=\lfloor \frac{n}{2^v} \rfloor} 1 - \theta(n,s) + \iota(2^{\lfloor \frac{n}{2} \rfloor} - \xi(n,s) - 1)
\]

where the term

\[
\sum_{j=1}^{s} \sum_{\eta(2^{m-1}) \neq 0} \sum_{m=\lfloor \frac{n}{2^v} \rfloor} 1
\]

counts the number of all non-zero carry of \( 2^n - 1 \) up to level \( s \) and \( 0 \leq \theta(n,s) := \sum_{j=1}^{s} \xi(n,j) \) and \( 2 \leq s := s(n) \) fixed, an integer to be chosen later. It is worth noting that

\[
\theta(n,s) := \sum_{j=1}^{s} \xi(n,j) = 0
\]

if \( n = 2^r \) for some \( r \in \mathbb{N} \), since \( \xi(n,j) = 0 \) for each \( 1 \leq j \leq s \) for all \( n \) which are powers of 2. It is also important to note that the \( 2s \) term is obtained by noting that there are at most \( s \) terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other \( s \) terms comes from summing 1 with frequency \( s \) finding the total length of the short addition chains producing numbers of the form \( 2^n + 1 \). Now, we set \( k = \frac{n}{2} - \xi(n,s) \) and construct the addition chain producing \( 2^k \) as \( 1, 2, 2^2, \ldots, 2^{k-1}, 2^k \) with corresponding sequence of partition

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}, 2^k = 2^{k-1} + 2^{k-1}
\]

with \( \alpha_i = 2^{i-2} = r_i \) for \( 2 \leq i \leq k+1 \), where \( \alpha_i \) and \( r_i \) denotes the determiner and the regulator of the \( i^{th} \) generator of the chain. Let us consider only the complete sub-addition chain

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}.
\]
Next we extend this complete sub-addition chain by adjoining the sequence
\[ 2^{k-1} + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor}, 2^{k-1} + 2^{\left\lfloor \frac{k-1}{2^2} \right\rfloor}, \ldots, 2^{k-1} + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor} + 2^{\left\lfloor \frac{k-1}{2^j+1} \right\rfloor} + \ldots + 2^1. \]
Since \( \xi(n, s) = 0 \) if \( n = 2^r \) and \( 0 \leq \xi(n, s) < 1 \) if \( n \neq 2^r \), we note that the adjoined sequence contributes at most
\[ \left\lfloor \log k \log 2 \right\rfloor = \left\lfloor \log \left( \frac{n}{2^s} \right) \right\rfloor - s \]
terms to the original complete sub-addition chain. Since the inequality holds
\[ 2^{k-1} + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor} + 2^{\left\lfloor \frac{k-1}{2^2} \right\rfloor} + \ldots + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor} + 2^{\left\lfloor \frac{k-1}{2^j+1} \right\rfloor} + \ldots + 2^1 \leq 2^k - 2 \]
we insert terms into the sum
\[ (11.6) \quad 2^{k-1} + 2^{k-1} + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor} + \ldots + 2^1 \]
so that we have
\[ \sum_{i=1}^{k-1} 2^i = 2^k - 2. \]
Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert \( 2^{k-2} + 2^{k-3} + \ldots + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor + 1} \) into \((11.28)\) and this comes at the cost of adjoining
\[ k - 2 - \left\lfloor \frac{k-1}{2^j} \right\rfloor \]
terms to the term in \((11.28)\). The last term of the adjoined sequence is given by
\[ (11.7) \quad 2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor + 1}) + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor} + \ldots + 2^1. \]
Again we have to insert \( 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor - 1} + \ldots + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor + 1} \) into \((11.29)\) and this comes at the cost of adjoining
\[ \left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{k-1}{2^j+1} \right\rfloor - 1 \]
terms to the term in \((11.29)\). The last term of the adjoined sequence is given by
\[ (11.8) \quad 2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor + 1}) + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor - 1} + \ldots + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor + 1} + 2^{\left\lfloor \frac{k-1}{2^j} \right\rfloor} + \ldots + 2^1. \]
By iterating the process, it follows that we have to insert into the immediately previous term by inserting into \((11.30)\) and this comes at the cost of adjoining
\[ \left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{k-1}{2^j+1} \right\rfloor - 1 \]
terms to the term in \((11.30)\) for \( j \leq \left\lfloor \frac{\log n}{\log 2} \right\rfloor - s \), since we are filling in at most \( \left\lfloor \log k \log 2 \right\rfloor \) blocks with \( k = \frac{n}{2^s} - \xi(n, s) \). It follows that the contribution of these new terms is at most
\[ (11.9) \quad k - 1 - \left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{\log k}{\log 2} \right\rfloor. \]
obtained by adding the numbers in the chain

\[
\begin{align*}
k - 1 - \left\lfloor \frac{k - 1}{2} \right\rfloor - 1 \\
\left\lfloor \frac{k - 1}{2} \right\rfloor - \left\lfloor \frac{k - 1}{2^2} \right\rfloor - 1 \\
\cdots \\
\left\lfloor \frac{k - 1}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1}} \right\rfloor - \left\lfloor \frac{k - 1}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1}} \right\rfloor - 1.
\end{align*}
\]

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing \(2^k - 1\) with \(k = \frac{n}{2^s} - \xi(n, s)\) is

\[
\begin{align*}
\delta(2^k - 1) & \leq k + k - 1 - \left\lfloor \frac{k - 1}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1}} \right\rfloor - \left\lfloor \frac{\log k}{\log 2} \right\rfloor + \left\lfloor \frac{\log n}{\log 2} \right\rfloor - s \\
& \leq \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{n}{2^{s-1}} - \xi(n, s) - 1 \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \left\lfloor \frac{\log n}{\log 2} \right\rfloor - s \\
& = \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{n}{2^{s-1}} - \xi(n, s) - 1 \right\rfloor.
\end{align*}
\]

By plugging the inequality (11.32) into the inequalities in (11.27) and noting that \(\lambda(\cdot) \leq \xi(\cdot)\), we obtain the inequality

\[
\begin{align*}
\lambda(2^n - 1) & \leq \frac{n}{2^s} + s + \frac{n}{2^s} - s + 2 \sum_{j=1}^{s} 1 - \theta(n, s) + \lambda(2^{\frac{n}{2^s} - \xi(n, s)} - 1) \\
& = n(1 - \frac{1}{2^s}) + \frac{n}{2^s} - s + 2 \sum_{j=1}^{s} 1 - \theta(n, s) - \left\lfloor \frac{n}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}} - \frac{n}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}} \right\rfloor \\
& = n - 1 + \frac{n}{2^s} + s + 2 \sum_{j=1}^{s} 1 - \theta(n, s) - \left\lfloor \frac{n}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}} - \frac{n}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}} \right\rfloor.
\end{align*}
\]

where we note that

\[
\sum_{j=1}^{s} \sum_{m=\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}^{\infty} 1
\]

counts the number of non-zero carries up to the \(s\) level for the number \(2^n - 1\). By taking \(2 \leq s := s(n)\) such that \(s = \left\lfloor \frac{\log n}{\log 2} \right\rfloor\) which is the maximum frequency of the iteration, then

\[
\left\lfloor \frac{n}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}} - \frac{n}{2^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1-s}} \right\rfloor = 0.
\]
and we obtained that
\[
\sum_{j=1}^s \sum_{m=\left\lfloor \frac{n}{2^j-1} \right\rfloor}^{n-1} \eta(2^m-1) \neq 0
\]
and the inequality
\[
\iota(2^n-1) \leq n-1 - \theta(n, \frac{\log n}{\log 2}) + \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 2 + \frac{1}{(1+c)} \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 2
\]
for \(\theta(n, \frac{\log n}{\log 2}) := \sum_{j=1}^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \xi(n, j) > 0\) with \(n \geq 4\) and the claimed inequality follows as a consequence. \(\Box\)

Now we show that numbers of the form \(2^n-1\) with low degree carries almost satisfy the Scholz conjecture.

**Theorem 11.2.** If \(2^n-1\) has carries of degree at most
\[
\kappa(2^n-1) := \left(\frac{1}{1+\log n}\right) \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1
\]
then the inequality
\[
\iota(2^n-1) \leq n-1 + (1 + \frac{2}{1+\log n}) \left\lfloor \frac{\log n}{\log 2} \right\rfloor
\]
holds for all \(n \in \mathbb{N}\) with \(n \geq 4\), where \(\iota(\cdot)\) denotes the length of the shortest addition chain producing \(\cdot\).

**Proof.** Let \(2^n-1\) has at most
\[
\frac{1}{1+\log n} \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1
\]
degrees of carries. Next decompose the number \(2^n-1\) and obtain the decomposition
\[
2^n-1 = (2^{\left\lfloor \frac{n}{2^j-1} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2^j-1} \right\rfloor} + 1) + \eta(2^n-1)
\]
where
\[
\eta(2^n-1) := \frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})
\]
is the level one carry of \(2^n-1\). It is easy to see that we can recover the general factorization of \(2^n-1\) from this identity according to the parity of the exponent \(n\). In particular, if \(n \equiv 0\) (mod 2), then we have
\[
2^n-1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)
\]
and
\[
2^n-1 = (2^{\frac{n-1}{2}} - 1)(2^{\frac{n-1}{2}} + 1) + 2^{n-1}
\]
if \(n \equiv 1\) (mod 2). By combining both cases, we obtain the inequality
\[
\iota(2^n-1) \leq \iota((2^{\left\lfloor \frac{n}{2^j-1} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2^j-1} \right\rfloor} + 1)) + \eta(2^n-1).
\]
Applying Lemma 6.3, we obtain further the inequality
\[
(11.12) \quad \iota(2^n-1) \leq \iota(2^{\left\lfloor \frac{n}{2^j-1} \right\rfloor} - 1) + \iota(2^{\left\lfloor \frac{n}{2^j-1} \right\rfloor} + 1) + \eta(2^n-1)
\]
Again let us set \( \lfloor \frac{n}{2} \rfloor = k \) in (11.23), then we obtain the general decomposition

\[
2^k - 1 = (2^{\lfloor \frac{k}{2} \rfloor} - 1)(2^{\lfloor \frac{k}{2} \rfloor} + 1) + \eta(2^k - 1)
\]

where

\[
\eta(2^k - 1) = \frac{(1 - (-1)^k)}{2}(2^{k - (1 - (-1)^k) n})
\]

is the carry of \( 2^k - 1 \). It is easy to see that we can recover the general factorization of \( 2^k - 1 \) from this identity according to the parity of the exponent \( k \). In particular, if \( k \equiv 0 \pmod{2} \), then we have

\[
2^k - 1 = (2^{\lfloor \frac{k}{2} \rfloor} - 1)(2^{\lfloor \frac{k}{2} \rfloor} + 1)
\]

and

\[
2^k - 1 = (2^{\frac{k-1}{2}} - 1)(2^{\frac{k-1}{2}} + 1) + 2^{k-1}
\]

if \( k \equiv 1 \pmod{2} \). By combining both cases, we obtain the inequality

\[
\iota(2^k - 1) \leq \iota((2^{\lfloor \frac{k}{2} \rfloor} - 1)(2^{\lfloor \frac{k}{2} \rfloor} + 1)) + \eta(2^k - 1).
\]

Applying Lemma 6.3, we obtain further the inequality

\[
\iota(2^k - 1) \leq \iota(2^{\lfloor \frac{k}{2} \rfloor} - 1) + \iota(2^{\lfloor \frac{k}{2} \rfloor} + 1) + \eta(2^k - 1)
\]

(11.13)

so that by inserting (11.24) into (11.23), we obtain the inequality

\[
\iota(2^n - 1) \leq \iota(2^{\lfloor \frac{n}{2} \rfloor} - 1) + \iota(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \eta(2^n - 1)
\]

(11.14)

Next we iterate the factorization up to frequency \( s \) to obtain

\[
\iota(2^n - 1) \leq \iota(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \eta(2^n - 1)
\]

(11.15)

\[
+ \cdots + \iota(2^{\frac{n}{2} - \xi(n,s)} - 1) + \iota(2^{\frac{n}{2} - \xi(n,s)} + 1) + \eta(2^{\frac{n}{2} - \xi(n,s)} - 1)
\]

where \( 0 \leq \xi(n,s) < 1 \) for an integer \( 2 \leq s := s(n) \) fixed to be chosen later. For instance,

\[
\xi(n,1) = (1 - (-1)^n)\frac{1}{4} < 1
\]

and

\[
\xi(n,2) = (1 - (-1)^n)\frac{1}{8} + (1 - (-1)^k)\frac{1}{4} < 1
\]

with

\[
k := \lfloor \frac{n}{2} \rfloor
\]

and so on. That is, \( \xi(n,1) := \{ \frac{n}{2} \} \) with \( \xi(n,2) = \{ \frac{1}{2} \lfloor \frac{n}{2} \rfloor \} \) and so on. Indeed the function \( \xi(n,s) \) for values of \( s \geq 3 \) can be read from exponents of the terms arising
from the iteration process. It follows from \(11.20\) the inequality

\[
\ell(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2 \sum_{j=1}^{s} \sum_{m=\lfloor 2^{j-1} \rfloor}^{2^{j-1}-1} 1 - \theta(n, s) + \ell(2^{m^{-\xi(n,s)}} - 1)
\]

\((11.16)\)

\[
= n(1 - \frac{1}{2^n}) + s + 2 \sum_{j=1}^{s} \sum_{m=\lfloor 2^{j-1} \rfloor}^{2^{j-1}-1} 1 - \theta(n, s) + \ell(2^{m^{-\xi(n,s)}} - 1)
\]

where the term

\[
\sum_{j=1}^{s} \sum_{m=\lfloor 2^{j-1} \rfloor}^{2^{j-1}-1} 1
\]

counts the number of all non-zero carry of \(2^n - 1\) up to level \(s\) and \(0 \leq \theta(n, s) := \sum_{j=1}^{s} \xi(n, j)\) and \(2 \leq s := s(n)\) fixed, an integer to be chosen later. It is worth noting that

\[
\theta(n, s) := \sum_{j=1}^{s} \xi(n, j) = 0
\]

if \(n = 2^r\) for some \(r \in \mathbb{N}\), since \(\xi(n, j) = 0\) for each \(1 \leq j \leq s\) for all \(n\) which are powers of 2. It is also important to note that the \(2s\) term is obtained by noting that there are at most \(s\) terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other \(s\) term comes from summing 1 with frequency \(s\) finding the total length of the short addition chains producing numbers of the form \(2^n + 1\). Now, we set \(k = \frac{n}{2} - \xi(n, s)\) and construct the addition chain producing \(2^k\) as \(1, 2, 2^2, \ldots, 2^{k-1}, 2^k\) with corresponding sequence of partition

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}, 2^k = 2^{k-1} + 2^{k-1}
\]

with \(a_i = 2^{i-2} = r_i\) for \(2 \leq i \leq k + 1\), where \(a_i\) and \(r_i\) denotes the determiner and the regulator of the \(i^{th}\) generator of the chain. Let us consider only the complete sub-addition chain

\[
2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}.
\]

Next we extend this complete sub-addition chain by adjoining the sequence

\[
2^{k-1} + 2^{k-1 - 1}, 2^{k-1} + 2^{k-1 - 1} + 2^{k-1 - 1}, \ldots, 2^{k-1} + 2^{k-1 - 1} + 2^{k-1 - 1} + 2^{k-1 - 1} + \ldots + 2^1.
\]

Since \(\xi(n, s) = 0\) if \(n = 2^r\) and \(0 \leq \xi(n, s) < 1\) if \(n \neq 2^r\), we note that the adjoined sequence contributes at most

\[
\left\lceil \frac{\log k}{\log 2} \right\rceil = \left\lceil \frac{\log(n/2 - \xi(n, s))}{\log 2} \right\rceil = \left\lceil \frac{\log n - s \log 2}{\log 2} \right\rceil = \left\lceil \frac{\log n}{\log 2} \right\rceil - s
\]

terms to the original complete sub-addition chain. Since the inequality holds

\[
2^{k-1} + 2^{k-1 - 1} + 2^{k-1 - 1} + \ldots + 2^1 < \sum_{i=1}^{k-1} 2^i
\]

\[
= 2^k - 2
\]
we insert terms into the sum

$$2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-2}{2} \rfloor} + \ldots + 2^1$$

so that we have

$$\sum_{i=1}^{k-1} 2^i = 2^k - 2.$$ 

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert

$$2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor} + 1$$

into (11.28) and this comes at the cost of adjoining

$$k - 2 - \lfloor \frac{k-1}{2} \rfloor$$

terms to the term in (11.28). The last term of the adjoined sequence is given by

$$2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor} + 1) + 2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^1.$$ 

Again we have to insert

$$2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^{\lfloor \frac{k-2}{2} \rfloor} + 1$$

into (11.29) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2} \rfloor - \lfloor \frac{k-2}{2} \rfloor - 1$$

terms to the term in (11.29). The last term of the adjoined sequence is given by

$$2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor} + 1) + 2^{\lfloor \frac{k-1}{2} \rfloor} + (2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^{\lfloor \frac{k-2}{2} \rfloor} + 1) + 2^{\lfloor \frac{k-2}{2} \rfloor} + \ldots + 2^1.$$ 

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (11.30) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2} \rfloor - \lfloor \frac{k-2}{2^j+1} \rfloor - 1$$

terms to the term in (11.30) for $j \leq \lfloor \frac{\log k}{\log 2} \rfloor - s$, since we are filling in at most $\lfloor \frac{\log k}{\log 2} \rfloor$ blocks with $k = \frac{n}{2^s} - \xi(n, s)$. It follows that the contribution of these new terms is at most

$$k - 1 - \left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{\log k}{\log 2} \right\rfloor$$

obtained by adding the numbers in the chain

$$k - 1 - \left\lfloor \frac{k-1}{2} \right\rfloor - 1$$

$$\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k-2}{2^2} \right\rfloor - 1$$

$$\vdots$$

$$\left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{k-1}{2^{j+1}} \right\rfloor - 1.$$
By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing $2^k - 1$ with $k = \frac{n}{\log 2} - \xi(n, s)$ is

$$\delta(2^k - 1) \leq k + k - 1 - \left\lfloor \frac{k - 1}{2^{\left\lfloor \frac{\log k}{2} \right\rfloor + 1}} \right\rfloor - \left\lfloor \frac{\log k}{2} \right\rfloor + \left\lfloor \frac{\log n}{\log 2} \right\rfloor - s$$

$$\leq \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{n}{2^s} - \xi(n, s) - 1}{2^{\left\lfloor \frac{\log n}{2} \right\rfloor + 1-s}} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + s + \left\lfloor \frac{\log n}{\log 2} \right\rfloor - s$$

(11.21)

By plugging the inequality (11.32) into the inequalities in (11.27) and noting that $\nu(\cdot) \leq \delta(\cdot)$, we obtain the inequality

$$\nu(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2 \sum_{j=1}^{\eta(2^m-1)} \sum_{m=1}^{\left\lfloor \frac{\log n}{2^s} \right\rfloor} 1 - \theta(n, s) + \nu(\frac{n}{2^s} - \xi(n, s) - 1)$$

(11.22)

$$= n(1 - \frac{1}{2^s}) + \frac{n}{2^{s-1}} - 1 + s + 2 \sum_{j=1}^{\eta(2^m-1)} \sum_{m=1}^{\left\lfloor \frac{\log n}{2^s} \right\rfloor} 1 - \theta(n, s) - \left\lfloor 2^{\left\lfloor \frac{\log n}{2^s} \right\rfloor + 1-s} \right\rfloor$$

$$= n - 1 + \frac{n}{2^s} + s + 2 \sum_{j=1}^{\eta(2^m-1)} \sum_{m=1}^{\left\lfloor \frac{\log n}{2^s} \right\rfloor} 1 - \theta(n, s) - \left\lfloor 2^{\left\lfloor \frac{\log n}{2^s} \right\rfloor + 1-s} \right\rfloor$$

where we note that

$$\sum_{j=1}^{\eta(2^m-1)} \sum_{m=1}^{\left\lfloor \frac{\log n}{2^s} \right\rfloor} 1$$

counts the number of non-zero carries up to the $s$ level for the number $2^n - 1$. By taking $2 \leq s := s(n)$ such that $s = \left\lfloor \frac{\log n}{\log 2} \right\rfloor$ which is the maximum frequency of the iteration, then

$$\left\lfloor \frac{n}{2^s} - \xi(n, s) - 1}{2^{\left\lfloor \frac{\log n}{2^s} \right\rfloor + 1-s} \right\rfloor = 0$$

and we obtained that

$$\sum_{j=1}^{\eta(2^m-1)} \sum_{m=1}^{\left\lfloor \frac{\log n}{2^s} \right\rfloor} 1 \leq \frac{1}{2(1+c)} \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1$$

and the inequality

$$\nu(2^n - 1) \leq n - 1 - \theta(n, \left\lfloor \frac{\log n}{\log 2} \right\rfloor) + \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \frac{2}{(1 + \log n)} \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 2$$

for $\theta(n, \left\lfloor \frac{\log n}{\log 2} \right\rfloor) := \sum_{j=1}^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \xi(n, j) > 0$ with $n \geq 4$ and the claimed inequality follows as a consequence. \qed
The proofs presented in Theorem 11.1 and 11.2 serve as model for obtaining improved upper bound for the shortest length of addition chains producing numbers of the form $2^n - 1$. Indeed, without using the notion of carries one can obtain the weaker upper bound which holds for all exponents $n \geq 4$.

**Theorem 11.3.**

$$\iota(2^n - 1) \leq n + 1 - \sum_{j=1}^{\lfloor \log n/\log 2 \rfloor} \xi(n, j) + 3\lfloor \log n/\log 2 \rfloor$$

for all $n \in \mathbb{N}$ with $n \geq 4$ for $0 \leq \xi(n, j) < 1$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

It follows similarly from the proofs the following result which holds for numbers of the form $2^n - 1$ with low degree of carries

$$\frac{1}{1 + \log \log n} - 1$$

**Theorem 11.4.** If $2^n - 1$ has carries of degree at most

$$\kappa(2^n - 1) := \left(\frac{1}{1 + \log \log n}\right)\lfloor \log n/\log 2 \rfloor - 1$$

then the inequality

$$\iota(2^n - 1) \leq n - 1 + (1 + \frac{2}{1 + \log \log n})\lfloor \log n/\log 2 \rfloor$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

We obtain the more general theorem

**Theorem 11.5.** If $2^n - 1$ has carries of degree at most

$$\kappa(2^n - 1) := \left(\frac{1}{1 + f(n)}\right)\lfloor \log n/\log 2 \rfloor - 1$$

where $f(n) = o(\log n)$ with $f(n) \to \infty$ as $n \to \infty$, then the inequality

$$\iota(2^n - 1) \leq n - 1 + (1 + \frac{2}{1 + f(n)})\lfloor \log n/\log 2 \rfloor$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$.

The following chain of results we have obtained illustrates that to make progress on the Scholz conjecture, it suffices to study possible way of controlling numbers of the form $2^n - 1$ with high carries. In other words, the degree of carries of numbers of the form $2^n - 1$ determines the quality of the upper bound for its corresponding length of the shortest addition using the current method. We end the paper by
proving using the same method that with the expected number of carries for a
fixed number $2^n - 1$, we can obtain the stronger result:

**Theorem 11.6.** If $2^n - 1$ has carries of degree at most

$$\kappa(2^n - 1) = \frac{1}{2}(\iota(n) - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \sum_{j=1}^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \{ \frac{n}{2^j} \})$$

then the inequality

$$\iota(2^n - 1) \leq n + 1 + \sum_{j=1}^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \left( \{ \frac{n}{2^j} \} - \xi(n, j) \right) + \iota(n)$$

holds for all $n \in \mathbb{N}$ with $n \geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition
chain producing $\cdot$, $\{ \cdot \}$ denotes the fractional part of $\cdot$ and where $\xi(n, 1) := \{ \frac{n}{2} \}$ with
$\xi(n, 2) = \{ \frac{n}{2^2} \}$ and so on.

**Proof.** Suppose that $2^n - 1$ has at most

$$\frac{1}{2}(\iota(n) - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \sum_{j=1}^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \{ \frac{n}{2^j} \})$$

degrees of carries. Next decompose the number $2^n - 1$ and obtain the decomposition

$$2^n - 1 = (2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1) + \eta(2^n - 1)$$

where

$$\eta(2^n - 1) := \frac{1 - (-1)^n}{2}$$

is the level one carry of $2^n - 1$. It is easy to see that we can recover the general
factorization of $2^n - 1$ from this identity according to the parity of the exponent $n$.
In particular, if $n \equiv 0 \pmod{2}$, then we have

$$2^n - 1 = (2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1)$$

and

$$2^n - 1 = (2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1) + 2^n - 1$$

if $n \equiv 1 \pmod{2}$. By combining both cases, we obtain the inequality

$$\iota(2^n - 1) \leq \iota\left( (2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1) \right) + \eta(2^n - 1).$$

Applying Lemma 6.3 we obtain further the inequality

$$\iota(2^n - 1) \leq \iota(2^{\left\lfloor \frac{n}{2} \right\rfloor} - 1) + \iota(2^{\left\lfloor \frac{n}{2} \right\rfloor} + 1) + \eta(2^n - 1)$$

Again let us set $\left\lfloor \frac{n}{2} \right\rfloor = k$ in (11.23), then we obtain the general decomposition

$$2^k - 1 = (2^{\left\lfloor \frac{k}{2} \right\rfloor} - 1)(2^{\left\lfloor \frac{k}{2} \right\rfloor} + 1) + \eta(2^k - 1)$$

where

$$\eta(2^k - 1) = \frac{(1 - (-1)^k)}{2}$$
is the carry of $2^k - 1$. It is easy to see that we can recover the general factorization of $2^k - 1$ from this identity according to the parity of the exponent $k$. In particular, if $k \equiv 0 \pmod{2}$, then we have

$$2^k - 1 = (2^{\frac{k}{2}} - 1)(2^{\frac{k}{2}} + 1)$$

and

$$2^k - 1 = (2^{\frac{k-1}{2}} - 1)(2^{\frac{k-1}{2}} + 1) + 2^{k-1}$$

if $k \equiv 1 \pmod{2}$. By combining both cases, we obtain the inequality

$$\tau(2^k - 1) \leq \tau((2^{\frac{k}{2}} - 1)(2^{\frac{k}{2}} + 1)) + \eta(2^k - 1).$$

Applying Lemma 6.3, we obtain further the inequality

$$\tau(2^k - 1) \leq \tau(2^{\frac{k}{2}} - 1) + \eta(2^{\frac{k}{2}} + 1) + \eta(2^k - 1)$$

(11.24)

so that by inserting (11.24) into (11.23), we obtain the inequality

$$\tau(2^n - 1) \leq \tau(2^{\frac{n}{2}} - 1) + \eta(2^{\frac{n}{2}} + 1) + \eta(2^n - 1)$$

(11.25)

Next, we iterate the factorization up to frequency $s$ to obtain

$$\tau(2^n - 1) \leq \tau(2^{\frac{n}{2}} + 1) + \eta(2^n - 1) + \tau(2^{\frac{n}{2}} + 1) + \eta(2^{\frac{n}{2}} + 1) + \eta(2^n - 1)$$

(11.26)

$$+ \cdots + \tau(2^{\frac{n}{2} - \xi(n,s)} - 1) + \eta(2^{\frac{n}{2} - \xi(n,s)} + 1) + \eta(2^n - 1)$$

where $0 \leq \xi(n,s) < 1$ for an integer $2 \leq s := s(n)$ fixed to be chosen later. For instance,

$$\xi(n,1) = (1 - (-1)^n)\frac{1}{4} < 1$$

and

$$\xi(n,2) = (1 - (-1)^n)\frac{1}{8} + (1 - (-1)^k)\frac{1}{4} < 1$$

with

$$k := \left\lfloor \frac{n}{2} \right\rfloor$$

and so on. Indeed the function $\xi(n,s)$ for values of $s \geq 3$ can be read from exponents of the terms arising from the iteration process. It follows from (11.26) the inequality

$$\tau(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2 \sum_{j=1}^{s} \sum_{m=\lceil \frac{j}{2} \rceil}^{s} 1 - \theta(n,s) + \tau(2^{\frac{n}{2} - \xi(n,s)} - 1)$$

(11.27)

$$= n(1 - \frac{1}{2^n}) + s + 2 \sum_{j=1}^{s} \sum_{m=\lceil \frac{j}{2} \rceil}^{s} 1 - \theta(n,s) + \tau(2^{\frac{n}{2} - \xi(n,s)} - 1)$$

where the term

$$\sum_{j=1}^{s} \sum_{m=\lceil \frac{j}{2} \rceil}^{s} 1$$
counts the number of all non-zero carry of $2^n - 1$ up to level $s$ and $0 \leq \theta(n, s) := \sum_{j=1}^{s} \xi(n, j)$ and $2 \leq s := s(n)$ fixed, an integer to be chosen later. It is worth noting that

$$\theta(n, s) := \sum_{j=1}^{s} \xi(n, j) = 0$$

if $n = 2^r$ for some $r \in \mathbb{N}$, since $\xi(n, j) = 0$ for each $1 \leq j \leq s$ for all $n$ which are powers of 2. It is also important to note that the $2s$ term is obtained by noting that there are at most $s$ terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other $s$ term comes from summing 1 with frequency $s$ finding the total length of the short addition chains producing numbers of the form $2^n + 1$. Now, we set $k = \frac{n}{2} - \xi(n, s)$ and construct the addition chain producing $2^k$ as $1, 2, 2^2, \ldots, 2^{k-1}, 2^k$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}, 2^k = 2^{k-1} + 2^{k-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq k + 1$, where $a_i$ and $r_i$ denotes the determiner and the regulator of the $i^{th}$ generator of the chain. Let us consider only the complete sub-addition chain

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3, \ldots, 2^{k-1} = 2^{k-2} + 2^{k-2}.$$ 

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor}, 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2} \rfloor}, \ldots, 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^1.$$ 

Since $\xi(n, s) = 0$ if $n = 2^r$ and $0 \leq \xi(n, s) < 1$ if $n \neq 2^r$, we note that the adjoined sequence contributes at most

$$\left\lfloor \frac{\log k}{\log 2} \right\rfloor = \left\lfloor \frac{\log (\frac{n}{2}) - \xi(n, s)}{\log 2} \right\rfloor = \left\lfloor \frac{\log n - s \log 2}{\log 2} \right\rfloor = \left\lfloor \frac{\log n}{\log 2} \right\rfloor - s$$

terms to the original complete sub-addition chain. Since the inequality holds

$$2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^1 < \sum_{i=1}^{k-1} 2^i = 2^k - 2$$

we insert terms into the sum

$$\tag{11.28} 2^{k-1} + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^1$$

so that we have

$$\sum_{i=1}^{k-1} 2^i = 2^k - 2.$$ 

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor + 1}$ into \textbf{(11.28)} and this comes at the cost of adjoining

$$k - 2 - \left\lfloor \frac{k-1}{2} \right\rfloor$$

terms to the term in \textbf{(11.28)}. The last term of the adjoined sequence is given by

$$\tag{11.29} 2^{k-1} + (2^{k-2} + 2^{k-3} + \ldots + 2^{\lfloor \frac{k-1}{2} \rfloor + 1}) + 2^{\lfloor \frac{k-1}{2} \rfloor} + 2^{\lfloor \frac{k-1}{2} \rfloor} + \ldots + 2^1.$$
Again we have to insert $2^{\left\lfloor \frac{k-1}{2} \right\rfloor -1} + \cdots + 2^{\left\lfloor \frac{k-1}{2^{s-1}} \right\rfloor +1}$ into (11.29) and this comes at the cost of adjoining

$$\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k-1}{2^2} \right\rfloor - 1$$

terms to the term in (11.29). The last term of the adjoined sequence is given by

$$2^{k-1} + (2^{k-2} + 2^{k-3} + \cdots + 2^{\left\lfloor \frac{k-1}{2} \right\rfloor +1}) + 2^{\left\lfloor \frac{k-1}{2^2} \right\rfloor +1} + (2^{\left\lfloor \frac{k-1}{2^2} \right\rfloor +1} + \cdots + 2^{\left\lfloor \frac{k-1}{2^{s-1}} \right\rfloor +1}) + 2^{\left\lfloor \frac{k-1}{2^{s-1}} \right\rfloor +1}$$

(11.30)

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (11.30) and this comes at the cost of adjoining

$$\left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{k-1}{2^{j+1}} \right\rfloor - 1$$

terms to the term in (11.30) for $j \leq \left\lfloor \frac{\log n \log 2}{\log 2} \right\rfloor - s$, since we are filling in at most $\left\lfloor \frac{\log k \log 2}{\log 2} \right\rfloor$ blocks with $k = \frac{n}{2^s} - \xi(n, s)$. It follows that the contribution of these new terms is at most

$$k - 1 - \left\lfloor \frac{k-1}{2^j} \right\rfloor - \left\lfloor \frac{k-1}{2^{j+1}} \right\rfloor - 1$$

(11.31)

obtained by adding the numbers in the chain

$$k - 1 - \left\lfloor \frac{k-1}{2} \right\rfloor - 1$$

$$\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k-1}{2^2} \right\rfloor - 1$$

$$\cdots$$

$$\left\lfloor \frac{k-1}{2^{\left\lfloor \frac{\log k \log 2}{\log 2} \right\rfloor}} \right\rfloor - \left\lfloor \frac{k-1}{2^{\left\lfloor \frac{\log k \log 2}{\log 2} \right\rfloor +1}} \right\rfloor - 1.$$

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing $2^k - 1$ with $k = \frac{n}{2^s} - \xi(n, s)$ is

$$\delta(2^k - 1) \leq k + k - 1 - \left\lfloor \frac{k-1}{2^{\left\lfloor \frac{\log k \log 2}{\log 2} \right\rfloor +1}} \right\rfloor - \left\lfloor \frac{\log k \log 2}{\log 2} \right\rfloor + \left\lfloor \frac{\log n \log 2}{\log 2} \right\rfloor - s$$

$$\leq \frac{n}{2^{s-1}} - 1 - \left\lfloor \frac{n}{2^s} - \xi(n, s) - 1}{2^{\left\lfloor \frac{\log n \log 2}{\log 2} \right\rfloor +1-s}} \right\rfloor - \left\lfloor \frac{\log n \log 2}{\log 2} \right\rfloor + s + \left\lfloor \frac{\log n \log 2}{\log 2} \right\rfloor - s$$

(11.32)
By plugging the inequality (11.32) into the inequalities in (11.27) and noting that \( \iota(\cdot) \leq \delta(\cdot) \), we obtain the inequality

\[
\iota(2^n - 1) \leq \sum_{v=1}^{s} \frac{n}{2^v} + s + 2 \sum_{j=1}^{s} \sum_{m=\lfloor \frac{n}{2^j} \rfloor}^{\lfloor \frac{n}{2^{j-1}} \rfloor} 1 - \theta(n, s) + \iota(2^{\frac{n}{2^j}} - \xi(n, s) - 1)
\]

(11.33)

\[
= n(1 - \frac{1}{2^s}) + \frac{n}{2^{s-1}} - 1 + s + 2 \sum_{j=1}^{s} \sum_{m=\lfloor \frac{n}{2^j} \rfloor}^{\lfloor \frac{n}{2^{j-1}} \rfloor} 1 - \theta(n, s) - \left[ \frac{n}{2^j} - \xi(n, s) - 1 \right] \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1 - \frac{1}{2} \left( \iota(n) - \left\lfloor \log n/\log 2 \right\rfloor \right)
\]

where we note that

\[
\sum_{j=1}^{s} \sum_{m=\lfloor \frac{n}{2^j} \rfloor}^{\lfloor \frac{n}{2^{j-1}} \rfloor} 1
\]

counts the number of non-zero carries up to the \( s \) level for the number \( 2^n - 1 \). By taking \( 2 \leq s := s(n) \) such that \( s = \left\lfloor \log n/\log 2 \right\rfloor \) which is the maximum frequency of the iteration, then

\[
\left[ \frac{n}{2^j} - \xi(n, s) - 1 \right] \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1 - \frac{1}{2} \left( \iota(n) - \left\lfloor \log n/\log 2 \right\rfloor \right) = 0
\]

and we obtained that

\[
\sum_{j=1}^{s} \sum_{m=\lfloor \frac{n}{2^j} \rfloor}^{\lfloor \frac{n}{2^{j-1}} \rfloor} 1 \leq \frac{1}{2} \left( \iota(n) - \left\lfloor \log n/\log 2 \right\rfloor + \sum_{j=1}^{\left\lfloor \log n/\log 2 \right\rfloor} \{ \frac{n}{2^j} \} \right)
\]

and the inequality

\[
\iota(2^n - 1) \leq n - 1 - \theta(n, \left\lfloor \log n/\log 2 \right\rfloor) + \left\lfloor \log n/\log 2 \right\rfloor + 2 + \iota(n) - \left\lfloor \log n/\log 2 \right\rfloor + \sum_{j=1}^{\left\lfloor \log n/\log 2 \right\rfloor} \{ \frac{n}{2^j} \}
\]

for \( \theta(n, \left\lfloor \log n/\log 2 \right\rfloor) := \sum_{j=1}^{\left\lfloor \log n/\log 2 \right\rfloor} \xi(n, j) > 0 \) with \( n \geq 4 \) and \( 0 \leq \xi(n, j) < 1 \), where \( \{ \cdot \} \) denotes the fractional part of a real number. The claimed inequality follows as a consequence. \( \square \)

It turns out that proving integers of the form \( 2^n - 1 \) has carries of degree at most

\[
\kappa(2^n - 1) = \frac{1}{2} \left( \iota(n) - \left\lfloor \log n/\log 2 \right\rfloor + \sum_{j=1}^{\left\lfloor \log n/\log 2 \right\rfloor} \{ \frac{n}{2^j} \} \right)
\]
would yield the inequality
\[
\iota(2^n - 1) \leq n + 1 + \sum_{j=1}^{\lfloor \log_2 n \rfloor} \left( \frac{n}{2^j} \right) - \xi(n, j) + \iota(n)
\]
for all \( n \in \mathbb{N} \) as shown in the preceding proof, which is slightly short of the original conjecture. Indeed, we expect the degree of carries of all integers to be of the above form since for integers of the form \( n = 2^k \), it matches exactly with degree given by the formula. In particular the number \( 2^{2^k} - 1 \) is always free of carries and we see that
\[
\kappa(2^{2^k} - 1) = \frac{1}{2} \left( \iota(2^k) - \left\lfloor \frac{\log 2^k}{\log 2} \right\rfloor + \sum_{j=1}^{\lfloor \log_2 n \rfloor} \left\{ \frac{n}{2^j} \right\} \right) = 0.
\]
We make the following conjecture

**Conjecture 11.1.** (The carry-addition chain conjecture) Let \( \kappa(\cdot) \) denotes the degree of carry of \( \cdot \). Then we have
\[
\kappa(2^n - 1) = \left\lceil \frac{1}{2} \left( \iota(n) - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \sum_{j=1}^{\lfloor \log_2 n \rfloor} \left\{ \frac{n}{2^j} \right\} \right) \right\rceil
\]
where \( \lceil \cdot \rceil \) denotes the ceiling of \( \cdot \).

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