EVERY FLAT SURFACE IS BIRKHOFF AND OSCELEDETS
GENERIC IN ALMOST EVERY DIRECTION

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1. Introduction

Flat surfaces and strata. Suppose \( g \geq 1 \), and let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a partition of \( 2g - 2 \), and let \( \mathcal{H}(\alpha) \) be a stratum of Abelian differentials, i.e. the space of pairs \((M, \omega)\) where \( M \) is a Riemann surface and \( \omega \) is a holomorphic 1-form on \( M \) whose zeroes have multiplicities \( \alpha_1 \ldots \alpha_n \). The form \( \omega \) defines a canonical flat metric on \( M \) with conical singularities at the zeros of \( \omega \). Thus we refer to points of \( \mathcal{H}(\alpha) \) as flat surfaces or translation surfaces. For an introduction to this subject, see the survey \[Zo2\].

Affine measures and manifolds. Let \( \mathcal{H}_1(\alpha) \subset \mathcal{H}(\alpha) \) denote the subset of surfaces of (flat) area 1. An affine invariant manifold is a closed subset of \( \mathcal{H}_1(\alpha) \) which is invariant under the \( SL(2, \mathbb{R}) \) action and which in period coordinates (see \[Zo2\, Chapter 3\]) looks like an affine subspace. Each affine invariant manifold \( \mathcal{M} \) is the support of an ergodic \( SL(2, \mathbb{R}) \) invariant probability measure \( \nu_\mathcal{M} \). Locally, in period coordinates, this measure is (up to normalization) the restriction of Lebesgue measure to the subspace \( \mathcal{M} \), see \[EM\] for the precise definitions. It is proved in \[EMM\] that the closure of any \( SL(2, \mathbb{R}) \) orbit is an affine invariant manifold.

The most important case of an affine invariant manifold is a connected component a stratum \( \mathcal{H}_1(\alpha) \). In this case, the associated affine measure is called the Masur-Veech or Lebesgue measure \[Mas1, \, Ve1\].

The Teichmüller geodesic flow. Let

\[
g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

The element \( r_\theta \in SL(2, \mathbb{R}) \) acts by \((M, \omega) \rightarrow (M, e^{i\theta} \omega)\). This has the effect of rotating the flat surface by the angle \( \theta \). The action of \( g_t \) is called the Teichmüller geodesic flow. The orbits of \( SL(2, \mathbb{R}) \) are called Teichmüller disks.

A variant of the Birkhoff ergodic theorem. We use the notation \( C_c(X) \) to denote the space of continuous compactly supported functions on a space \( X \).

One of our main results is the following:
Theorem 1.1. Suppose \( x \in H_1(\alpha) \). Let \( \mathcal{M} = \text{SL}(2, \mathbb{R})x \) be the smallest affine invariant manifold containing \( x \). Then, for any \( \phi \in C_c(\mathcal{H}_1(\alpha)) \), for almost all \( \theta \in [0, 2\pi) \), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(g_tr_{\theta}x) \, dt = \int_{\mathcal{M}} \phi \, d\nu_{\mathcal{M}},
\]

where \( \nu_{\mathcal{M}} \) is the affine measure whose support is \( \mathcal{M} \).

Remark. The fact that (1.1) holds for almost all \( x \) with respect to the Masur-Veech measure is an immediate consequence of the Birkhoff ergodic theorem and the ergodicity of the Teichmüller geodesic flow \([\text{Mas}1]\), \([\text{Ve}1]\). The main point of Theorem 1.1 is that it gives a statement for every flat surface \( x \). This is important e.g. for applications to billiards in rational polygons (since the set of flat surfaces one obtains from unfolding rational polygons has Masur-Veech measure 0).

Remark. The proof of Theorem 1.1 is based on the results of \([\text{EM}]\) and \([\text{EMM}]\) and the strong law of large numbers. One complication is controlling the visits to neighborhoods of smaller affine submanifolds, which we do using the techniques of \([\text{EMM}]\), \([\text{A}]\), \([\text{EMa}]\) and which were originally introduced by Margulis in \([\text{EMaMo}]\).

The Kontsevich-Zorich cocycle. We consider the Hodge bundle whose fiber above the point \( (M, \omega) \) is \( H^1(M, \mathbb{R}) \). If we choose a fundamental domain for the action of the mapping class group \( \Gamma \), then we have the cocycle \( \tilde{A} : \text{SL}(2, \mathbb{R}) \times H_1(\alpha) \to \Gamma \) where for \( x \) in the fundamental domain, \( \tilde{A}(g, x) \) is the element of \( \Gamma \) needed to return the point \( gx \) to the fundamental domain. Then, we define the Kontsevich-Zorich cocycle \( A(g, x) \) by

\[
A(g, x) = \rho(\tilde{A}(g, x)),
\]

where \( \rho : \Gamma \to Sp(2g, \mathbb{Z}) \) is the homomorphism given by the action on homology. The Kontsevich-Zorich cocycle can be interpreted as the monodromy of the Gauss-Manin connection restricted to the orbit of \( \text{SL}(2, \mathbb{R}) \), see e.g. \([\text{Zo}2]\) page 64.

A variant of the Oseledets multiplicative ergodic theorem.

Theorem 1.2. Fix \( x \in H_1(\alpha) \), and let \( \mathcal{M} = \text{SL}(2, \mathbb{R})x \) denote the smallest affine manifold containing \( \mathcal{M} \). Then

1. If \( \psi_1(t, \theta) \leq \cdots \leq \psi_{2g}(t, \theta) \) are the eigenvalues of the matrix \( A^*(g_t, r_{\theta}x)A(g_t, r_{\theta}x) \), then for almost all \( \theta \in [0, 2\pi) \), we have

\[
\lim_{t \to \infty} \frac{1}{t} \log \psi_i(t, \theta) = 2\lambda_i.
\]

Here the numbers \( \lambda_1 \geq \cdots \geq \lambda_{2g} \) depend only on \( \mathcal{M} \). They are called the Lyapunov exponents of the Kontsevich Zorich cocycle on \( \mathcal{M} \).
II. For almost all $\theta$, the limit

$$\lim_{t \to \infty} (A^t(g_t, r\theta x) A(g_t, r\theta x))^{\frac{1}{t}} \equiv \Lambda(x, \theta)$$

exists. Moreover, the eigenvalues of the matrix $\Lambda(x, \theta)$, taken with their multiplicities, coincide with the numbers $e^{\lambda_i}$. Furthermore,

$$\lim_{t \to \infty} \frac{1}{t} \log \| A(g_t, r\theta x) A^{-t}(x, \theta) \| = \lim_{t \to \infty} \frac{1}{t} \log \| \Lambda^n(x, \theta) A^{-1}(g_t, r\theta x) \| = 0.$$

III. Let $\alpha_1 < \cdots < \alpha_s$ denote the distinct Lyapunov exponents $\lambda_i$. Let $\mathcal{U}_i(x, \theta) \subset H^1(M, \mathbb{R})$ denote the corresponding eigenspaces of $\Lambda(x, \theta)$. We set $\mathcal{V}_0(x, \theta) = \{0\}$ and $\mathcal{V}_i(x, \theta) = \mathcal{U}_1(x, \theta) \oplus \cdots \oplus \mathcal{U}_i(x, \theta)$. Then, for almost all $\theta$, and for any $v \in \mathcal{V}_i(x, \theta) \setminus \mathcal{V}_{i-1}(x, \theta)$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log \| A(g_t, r\theta x) v \| = \alpha_i.$$

Remark. The fact that the conclusions of Theorem 1.2 hold for almost all $x$ with respect to the affine measure $\nu_M$ (or in particular with respect to the Masur-Veech measure) is just the classical Osceledets multiplicative ergodic theorem. The main point of Theorem 1.2 is that the conclusion holds for all $x \in H^1(\alpha)$. This has some applications which partly motivated this paper, in particular in connection to the wind-tree model [DHL], [FU] and earlier results on IETs [Zo], [MMY]. By the arguments in [DHL] Theorem 1.2 strengthens [DHL, Theorem 1 part 2] to apply to all obstacles. It is likely that the arguments [FU, Theorem 1.2] to apply for all obstacles with the input of Theorem 1.1 and 1.2 Theorems 1.1 and 1.2 extend [MMY] to apply to a full measure subset of the one parameter family of IETs coming first return to a transversal on any flat surface. Th eorem 1.2 implies condition (a). In particular, an open set in the space of flat surfaces (in a fixed affine invariant submainfold) can be related to the IETs coming from it having $\gamma(n)$ taking all possible names in fixed bounded time. Condition (b) holds by [Fo]. Condition (c) holds by Theorem 1.2. Theorem 1.2 extends [Zo] to apply to a full measure subset of the one parameter family of IETs coming first return to a transversal on any flat surface. The question of whether Theorem 1.2 is true was raised in [Fo2].

It is well known that parts II and III of Theorem 1.2 follow from part I by an argument which does not involve any ergodic theory (see [GM], from which our statement of the multiplicative ergodic theorem was taken). It is thus enough to show that part I holds for all $x$ and almost all $\theta$. Our proof of I is based on the same ideas as the proof of Theorem 1.1 namely the results of [EM], [EMM] and the strong law of large numbers. However, we also need another important input: the theorem of Filip stated as Theorem 1.6 below. This complication can be traced back to the fact that the Kingman and Osceledets ergodic theorems can fail to hold at some points even for uniquely ergodic systems (see [Fu] and references therein).
Definition 1.3 (\(\nu\)-measurable almost invariant splitting). Let \(X\) be a space on which \(G = SL(2,\mathbb{R})\) acts, preserving a measure \(\nu\). Suppose \(V\) is a real vector space, and suppose \(A : G \times X \to SL(V)\) is a cocycle. We say that \(A\) has an almost invariant splitting if there exists \(n > 1\) and for a.e \(x\) there exist nontrivial subspaces \(W_1(x), \ldots, W_n(x) \subset V\) such that \(W_i(x) \cap W_j(x) = \{0\}\) for \(1 \leq i < j \leq n\) and also for a.e \(g \in G\) and \(\nu\)-a.e. \(x \in X\),

\[
A(g, x)W_i(x) = W_j(gx) \quad \text{for some } 1 \leq j \leq n.
\]

The map \(x \to \{W_1(x), \ldots, W_n(x)\}\) is required to be \(\nu\)-measurable. The splitting is called non-degenerate if all the \(W_i(x)\) have distinct top Lyapunov exponents.

Definition 1.4 (Strongly irreducible, weakly irreducible). A cocycle \(A\) is strongly irreducible with respect to the measure \(\nu\) if it does not admit a \(\nu\)-measurable almost invariant splitting. A cocycle \(A\) is weakly irreducible with respect to the measure \(\nu\) if it does not admit a non-degenerate almost invariant splitting.

In this paper, we prove the following:

Theorem 1.5. Fix \(x \in \mathcal{H}_1(\alpha)\), and let \(M = \overline{SL(2,\mathbb{R})x}\) be the smallest affine invariant manifold containing \(x\). Let \(V\) be \(SL(2,\mathbb{R})\) invariant subbundle of (some exterior power of) the Hodge bundle which is defined and is continuous on \(M\). Let \(A_V : SL(2,\mathbb{R}) \times M \to V\) denote the restriction of (some exterior power of) the Kontsevich-Zorich cocycle to \(V\), and suppose that \(A_V\) is weakly irreducible with respect to the affine measure \(\nu_M\) whose support is \(M\). Then, for almost all \(\theta \in [0, 2\pi)\),

\[
\lim_{t \to \infty} \frac{1}{t} \log \|A_V(g_t, r_\theta x)\|
\]

exists and coincides with the top Lyapunov exponent of \(A_V\).

The main additional input needed for the proof of Theorem 1.2 is the following:

Theorem 1.6 (\([E3]\)). Let \(A(\cdot, \cdot)\) denote (some exterior power of) the Kontsevich-Zorich cocycle restricted to an affine invariant submanifold \(M\). Let \(\nu_M\) be the affine measure whose support is \(M\), and suppose \(A\) has a \(\nu_M\)-measurable almost-invariant non-degenerate splitting. Then, the subspaces \(W_i(x)\) in Definition 1.3 can be taken to depend continuously on \(x \in M\).

Proof of Theorem 1.2 from Theorem 1.5 and Theorem 1.6. Let \(A(\cdot, \cdot)\) denote the Kontsevich-Zorich cocycle restricted to an affine invariant submanifold \(M\). Then by \([EM, \text{Theorem A.6}]\) \(A(\cdot, \cdot)\) is semisimple, in the sense that (after passing to some finite cover) for \(\nu_M\)-almost all \(x \in M\) there is a \(\nu_M\)-measurable direct sum decomposition

\[
H^1(M, \mathbb{R}) = \bigoplus_{i=1}^n V_i(x),
\]
where all the subbundles $V_i$ are $\nu_M$-measurable, $SL(2, \mathbb{R})$-invariant and strongly irreducible. This remains true when passing to any exterior power, see [E]. We now combine all the $V_i$ with the same top Lyapunov exponent to obtain a direct sum decomposition

$$H^1(M, \mathbb{R}) = \bigoplus_{i=1}^{n'} W_i(x),$$

where each subbundle $W_i$ is weakly irreducible (see Definition 1.4). By Theorem 1.6 the $W_i(x)$ can be taken to depend continuously on $x$. Then, by Theorem 1.5 it follows that the top Lyapunov exponent on each $W_i$ is defined for almost all $\theta$. (To connect the conclusion of Theorem 1.5 with (2.4), note that the top eigenvalue of $A_{\nu}(g_t, r_\theta x)^*A_{\nu}(g_t, r_\theta x)$ is $\|A_{\nu}(g_t, r_\theta x)\|^2$).

To get that the rest of the Lyapunov exponents are defined for almost all $\theta$ it suffices to repeat the argument for the cocycle acting on the exterior powers of the Hodge bundle. (Note that the norm of $A_{\nu}(g_t, r_\theta x)^*A_{\nu}(g_t, r_\theta x)$ acting on $\bigwedge^d V$ is the product of the top $d$ eigenvalues of $[A_{\nu}(g_t, r_\theta x)^*A_{\nu}(g_t, r_\theta x)]^{1/2}$ acting on $V$). This proves statement I of Theorem 1.2 and then statements II and III of Theorem 1.2 follow as in [GM].

Remark 1.7. For the case of a two-dimensional continuous subbundle $V$ of the Hodge bundle, Theorem 1.2 follows from Theorem 1.5 (without the need for Theorem 1.6). Indeed, by [AEM, Theorem 1.4] any $SL(2, \mathbb{R})$-invariant measurable subbundle of the Hodge bundle is symplectic, and thus even dimensional. Thus, the restriction of the cocycle to a two-dimensional subbundle is automatically strongly irreducible. (This is the case which arises in [DHL, F]).

2. Random walks

To provide intuition, we first prove versions of Theorem 1.1 and Theorem 1.5 for random walks. We use the following setup. Let $\mu$ be an $SO(2)$-bi-invariant compactly supported measure on $SL(2, \mathbb{R})$ which is absolutely continuous with respect to Haar measure. We consider the random walk on $SL(2, \mathbb{R})$ whose transition probabilities are given by $\mu$. This also defines a random walk on $H_1(\alpha)$, via the $SL(2, \mathbb{R})$ action. (The trajectories of this random walk stay in Teichmüller disks).

Let $\bar{g} = (g_1, \ldots, g_2, \ldots)$ denote an element of $SL(2, \mathbb{R})^N$. Let $\mu^N$ denote the product measure on $SL(2, \mathbb{R})^N$. It follows from the Osceledets multiplicative ergodic theorem that for $\mu^N$-almost-all $\bar{g}$, the trajectory

$$g_1, g_2 g_1, \ldots, g_{n-1} \cdots g_1, g_n g_{n-1} \cdots g_1$$

tracks, up to sublinear error, a geodesic of the form $\{g_t r_{\theta} : t \in \mathbb{R}\}$ with respect the the right-invariant metric on $SL(2, \mathbb{R})$. (This will be made more precise in [4]). The angle $\theta$ depends on $\bar{g}$, but as we show in [4] the distribution of $\bar{\theta}$'s induced by $\mu^N$ is uniform. Thus, we expect to have analogues of Theorem 1.1 and Theorem 1.2 (and Theorem 1.5) in the random walk setup, where the clause “for almost all $\theta$”
is replaced by the clause “for almost all \( \bar{g} \)”. This is indeed the case, and we find the proofs of the random walk versions, namely Theorem 2.1 and Theorem 2.6 a bit cleaner and easier to follow. Also we will see below that Theorem 1.5 follows formally from its random walk version Theorem 2.6.

2.1. A Birkhoff type theorem for the random walk.

**Theorem 2.1.** Suppose \( x \in H_1(\alpha) \). Let \( M = SL(2, \mathbb{R})x \) be the smallest affine invariant manifold containing \( x \). Then, for any \( \phi \in C_c(H_1(\alpha)) \), for \( \mu^N \)-almost all \( \bar{g} \in SL(2, \mathbb{R})^N \), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \phi(g_n \ldots g_1 x) = \int_M \phi \, d\nu_M,
\]

where \( \nu_M \) is the affine measure whose support is \( M \).

**Corollary 2.2.** Suppose \( x \in H_1(\alpha) \). Let \( M = SL(2, \mathbb{R})x \) be the smallest affine invariant manifold containing \( x \). Let \( U \) be an open subset of \( M \). Then, for \( \mu^N \)-almost all \( \bar{g} \in SL(2, \mathbb{R})^N \), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_U(g_n \ldots g_1 x) = \nu_M(U),
\]

where \( \nu_M \) is the affine measure whose support is \( M \).

Our proof of Theorem 2.1 follows [BQ]. Let \( x, M \) and \( \nu_M \) be as in Theorem 2.1. We begin with the following:

**Lemma 2.3.** For almost every \( \bar{g} \in SL(2, \mathbb{R})^N \), if \( \tilde{\nu} \) is a weak*-limit point of

\[
\frac{1}{N} \sum_{n=1}^{N} \delta_{g_n \ldots g_1 x}
\]

then \( \tilde{\nu} \) is \( \mu \)-stationary (i.e. \( \mu \ast \tilde{\nu} = \tilde{\nu} \)).

**Proof.** It suffices to check a countable subset of \( C_c(M) \), so it suffices to have the result for each fixed function in \( C_c(M) \). We follow [BQ] Lemma 3.2. Let \( \phi \in C_c(M) \) be a test function. Let

\[
f_n(x, \bar{g}) = \phi(g_n g_{n-1} \ldots g_1 x) - \int_{SL(2, \mathbb{R})} \phi(h g_{n-1} \ldots g_1 x) \, d\mu(h).
\]

By definition \( \int_{A_1 \ldots A_{n-1}} f_n(x, \bar{g}) \, d\mu^N = 0 \) for any \( n \in \mathbb{N} \) and any subsets \( A_1, \ldots, A_{n-1} \) of \( \mathbb{R} \). Additionally, \( \|f_n\|_\infty \leq 2\|\phi\|_\infty \). So by the strong law of large numbers

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(x, \bar{g}) = 0 \quad \text{for a.e.} \ \bar{g}.
\]
Thus $\tilde{\nu}$ is $\mu$-stationary almost everywhere.

We also use the following (which is the main technical result of [EMM]):

**Proposition 2.4** (see [EMM] Proposition 2.13, [EMM] Lemma 3.2). Let $\mathcal{N} \subset \mathcal{H}_1(\alpha)$ be an affine submanifold. (In this proposition $\mathcal{N} = \emptyset$ is allowed). Then there exists an $SO(2)$-invariant function $f_{\mathcal{N}} : \mathcal{H}_1(\alpha) \to [1, \infty]$ with the following properties:

(a) $f_{\mathcal{N}}(x) = \infty$ if and only if $x \in \mathcal{N}$, and $f_{\mathcal{N}}$ is bounded on compact subsets of $\mathcal{H}_1(\alpha) \setminus \mathcal{N}$. For any $\ell > 0$, the set $\{x : f(x) \leq \ell\}$ is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{N}$.

(b) There exists $b > 0$ (depending on $\mathcal{N}$) and for every $0 < c < 1$ there exists $n_0 > 0$ (depending on $\mathcal{N}$ and $c$) such that for all $x \in \mathcal{H}_1(\alpha)$ and all $n > n_0$,

$$\int_{SL(2,\mathbb{R})} f_{\mathcal{N}}(x) \, d\mu^{(n)}(x) \leq cf_{\mathcal{N}}(x) + b.$$ 

Here $\mu^{(n)}$ denotes the convolution $\mu \ast \cdots \ast \mu$ ($n$ times).

(c) There exists $\sigma > 1$ such that for all $g \in SL(2,\mathbb{R})$ with $\|g\| \leq 1$ and all $x \in \mathcal{H}_1(\alpha)$,

$$\sigma^{-1} f_{\mathcal{N}}(x) \leq f_{\mathcal{N}}(gx) \leq \sigma f_{\mathcal{N}}(x).$$

**Lemma 2.5** ([BQ, Proposition 3.9]). Suppose $f_{\mathcal{N}}$ is a function satisfying the conditions of Proposition 2.4. Then, for any $0 < c < 1$ any $M > 0$ and $\mu^{\mathcal{N}}$-almost-all $\bar{g} \in SL(2,\mathbb{R})^\mathcal{N}$, we have, for all sufficiently large $n$,

$$\frac{1}{n} \left| \left\{0 < k < n : f_{\mathcal{N}}(\bar{g}_k \cdots \bar{g}_1 x) > M \right\} \right| \leq \frac{C}{(1-c)M},$$

where $C$ depends only on the constants $n_0$, $b$ and $\sigma$ of Proposition 2.4.

**Proof of Theorem 2.1**. Let $\tilde{\nu}$ be any weak-* limit point of $\frac{1}{N} \sum_{n=1}^{N} \delta_{g_n \cdots g_1 x}$. By Lemma 2.3, for almost all $\bar{g}$, $\tilde{\nu}$ is $\mu$-stationary. By [EM, Theorem 1.6], any $\mu$-stationary measure (such as $\tilde{\nu}$) is $SL(2,\mathbb{R})$-invariant.

By [EM] Theorem 1.4, any ergodic $SL(2,\mathbb{R})$-invariant measure is affine. Therefore, since $\tilde{\nu}$ is supported on $\mathcal{M}$, $\tilde{\nu}$ has can be decomposed into ergodic components as

$$\tilde{\nu} = \sum_{\mathcal{N} \in \mathcal{M}} a_{\mathcal{N}} \nu_{\mathcal{N}},$$

where $a_{\mathcal{N}} \in [0, 1]$ and the sum is over the affine invariant submanifolds $\mathcal{N}$ contained in $\mathcal{M}$. (Here $\mathcal{N} = \emptyset$ is allowed). By [EMM] Proposition 2.16 this is a countable sum. By applying (2.2) for the case $\mathcal{N} = \emptyset$ we get that for $\mu^{\mathcal{N}}$-almost all $\bar{g}$, $\tilde{\nu}$ is a probability measure. Then, by applying (2.2) again with $\mathcal{N}$ any affine invariant submanifold properly contained in $\mathcal{M}$, we see that for $\mu^{\mathcal{N}}$-almost-all $\bar{g}$, $\tilde{\nu}(\mathcal{N}) = 0$. Thus $a_{\mathcal{N}} = 0$ for $\mathcal{N}$ properly contained in $\mathcal{M}$. Since $\tilde{\nu}$ is a probability measure, this forces $\tilde{\nu} = \nu_{\mathcal{M}}$, completing the proof of Theorem 2.1. \qed
2.2. An Osceledeets type theorem for the random walk.

**Theorem 2.6.** Fix \( x \in H_1(\alpha) \), and let \( \mathcal{M} = \overline{SL(2, \mathbb{R}) x} \) be the smallest affine invariant manifold containing \( x \). Let \( V \) be \( SL(2, \mathbb{R}) \) invariant subbundle of (some exterior power of) the Hodge bundle which is defined and is continuous on \( \mathcal{M} \). Let \( A_V : SL(2, \mathbb{R}) \times \mathcal{M} \to V \) denote the restriction of (some exterior power of) the Kontsevich-Zorich cocycle to \( V \), and suppose that \( A_V \) is weakly irreducible (see Definition 1.4) with respect to the affine measure \( \nu_{\mathcal{M}} \) whose support is \( \mathcal{M} \). Then, for \( \mu^N \)-almost-all \( \bar{g} = (g_1, \ldots, g_n, \ldots) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \| A_V(g_n \ldots g_1, x) \| = \lambda_1
\]

where \( \lambda_1 \) is the top Lyapunov exponent of \( A_V \) restricted to \( \mathcal{M} \) (and depends only on \( \mu, V \) and \( \mathcal{M} \)).

Let \( m = \dim(V) \). We recall the statement of the Oseledets multiplicative ergodic theorem from e.g. [GM] in this setting:

**Theorem 2.7.** For \( \nu_{\mathcal{M}} \)-almost all \( y \in \mathcal{M} \) and \( \mu^N \)-almost-all \( \bar{g} \in SL(2, \mathbb{R})^N \), the following hold:

I. Let \( \psi_1(n, \bar{g}, y) \leq \cdots \leq \psi_m(n, \bar{g}, y) \) denote the eigenvalues of the matrix

\[
A_V^t(g_n \ldots g_1, y)A_V(g_n \ldots g_1, y).
\]

Then for \( 1 \leq i \leq m \),

\[
\lim_{n \to \infty} \frac{1}{t} \log \psi_i(n, \bar{g}, y) = 2\lambda_i.
\]

Here the numbers \( \lambda_1 \geq \cdots \geq \lambda_m \) depend only on \( \nu_{\mathcal{M}} \) and \( V \). They are the Lyapunov exponents of the cocycle \( A_V \) on \( \mathcal{M} \).

II. The limit

\[
\lim_{n \to \infty} (A_V^*(g_n \ldots g_1, y)A_V(g_n \ldots g_1, y))^{\frac{1}{n}} \equiv \Lambda(y, \bar{g})
\]

exists. Moreover, the eigenvalues of the matrix \( \Lambda(y, \bar{g}) \), taken with their multiplicities, coincide with the numbers \( e^{\lambda_i} \). Furthermore,

\[
\lim_{n \to \infty} \frac{1}{n} \log \| A_V(g_n \ldots g_1, y) \Lambda^{-n}(y, \bar{g}) \| =
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \log \| \Lambda^n(y, \bar{g})A_{V}^{-1}(g_n \ldots g_1, y) \| = 0.
\]

III. Let \( \alpha_1 < \cdots < \alpha_s \) denote the distinct Lyapunov exponents \( \lambda_i \). Let \( U_i(y, \bar{g}) \subset V \) denote the corresponding eigenspaces of \( \Lambda(y, \bar{g}) \). We set \( V_0(y, \bar{g}) = \{0\} \) and \( V_i(y, \bar{g}) = U_i(y, \bar{g}) \oplus \cdots \oplus U_i(y, \bar{g}) \). Then, for almost all \( y, \bar{g}, \) and for any \( v \in V_i(y, \bar{g}) \setminus V_{i-1}(y, \bar{g}) \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \| A_V(g_n \ldots g_1, y)v \| = \alpha_i.
\]
Remark 2.8. As was done \cite{Filip}, one can use the theorem of Filip Theorem 1.6 and Theorem 2.6 to show that the conclusions of Theorem 2.7 hold for all \(y\) (and almost all \(\bar{g}\)) provided \(\mathcal{M}\) is the smallest affine invariant manifold containing \(y\) (or equivalently \(\mathcal{M} = SL(2, R)y\)).

Notation. For \(L \in \mathbb{N}\), let \(\mu^L\) denote the measure on \(SL(2, \mathbb{R})^L\) given by \(\mu \times \mu \times \ldots \times \mu\) (\(L\) times).

The set \(E_{\text{good}}(\epsilon, L)\). Suppose \(\epsilon > 0\), \(L \in \mathbb{N}\). Let \(E_{\text{good}}(\epsilon, L)\) denote the set of \(y \in \mathcal{M}\) such that for each \(v \in V\) there exists a subset \(H(v) \subset SL(2, \mathbb{R})^L\) so that

\[(2.6) \quad \mu^L(H(v)) > 1 - \epsilon,\]

and for all \((h_1, \ldots, h_L) \in H(v),\)

\[(2.7) \quad (\lambda_1 - \epsilon)^L \leq \frac{\|A_V(h_L \ldots h_1, y)v\|}{\|v\|} \leq \|A_V(h_L \ldots h_1, y)\| \leq (\lambda_1 + \epsilon)^L.\]

The following Lemma is a key step in our proof.

Lemma 2.9. For any fixed \(\epsilon > 0,\)

\[\lim_{L \to \infty} \nu_{\mathcal{M}}(E_{\text{good}}(\epsilon, L)) = 1.\]

2.2.1. Proof of Lemma 2.9. Fix \(1 \leq s \leq m\), and let \(Gr_s(V)\) denote the Grassmanian of \(s\)-dimensional subspaces in \(V\). Let \(\hat{\mathcal{M}} = \mathcal{M} \times Gr_s(V)\). We then have an action of \(SL(2, \mathbb{R})\) on \(\hat{\mathcal{M}}\), by

\[g \cdot (x, W) = (gx, A_V(g, x)W).\]

Let \(\hat{\nu}_{\mathcal{M}}\) be an ergodic \(\mu\)-stationary measure on \(\hat{\mathcal{M}}\) which projects to \(\nu_{\mathcal{M}}\) under the natural map \(\mathcal{M} \to \mathcal{M}\). (Note there is always at least one such: see e.g. \cite[\S C.2]{EM}). We may write

\[d\hat{\nu}_{\mathcal{M}}(x, U) = d\nu_{\mathcal{M}}(x) d\eta_x(U),\]

where \(\eta_x\) is a measure on \(Gr_s(V)\).

For \(v \in V\), let

\[I(v) = \{U \in Gr_s(V) : v \in U\}.\]

Lemma 2.10 (\cite[Lemma C.9(i)]{EM}). Suppose the cocycle \(A_V\) is strongly irreducible with respect to \(\nu_{\mathcal{M}}\). Then for almost all \(y \in \mathcal{M}\), for any \(v_y \in V\), \(\eta_y(I(v_y)) = 0\).

Proof. The proof is given in \cite[Appendix C]{EM}. The essential idea is that if conclusion of Lemma 2.10 is false, then the cocycle would have to permute some finite collection of subspaces, contradicting the strong irreducibility assumption. \(\square\)
Corollary 2.11 (c.f. [EM, Lemma 14.4]). Suppose the cocycle $A_V$ is weakly irreducible (see Definition 1.4) with respect to $\nu_M$. Then for almost all $y \in M$, for any $w_y \in V$,

\[
\mu_N \left( \left\{ \bar{g} \in SL(2, \mathbb{R})^N : w_y \in V_{s-1}(y, \bar{g}) \right\} \right) = 0.
\]

Proof of Corollary 2.11. Since $V$ is weakly irreducible, we may write $V(x) = \bigoplus_{i=1}^{n} W_i(x)$, where each $W_i(x)$ is strongly irreducible, and all the $W_i$ have the same top Lyapunov exponent. Then,

$V_{s-1}(y, \bar{g}) = \bigoplus_{i=1}^{n} V_{s-1}^{(i)}(y, \bar{g})$,

where $V_{s-1}^{(i)}$ is the analogue of $V_{s-1}$ for $W_i$ in place of $V$. Now, (2.8) follows from the corresponding statement for each $W_i$. Thus, without loss of generality, we may assume that $V$ is strongly irreducible.

For $F \subset Gr_{s-1}(V)$ (the Grassmanian of $s-1$ dimensional subspaces of $V$) let

$\hat{\nu}_x(F) = \mu_N \left( \left\{ \bar{g} \in SL(2, \mathbb{R})^N : V_{s-1}(y, \bar{g}) \in F \right\} \right)$,

and let $\hat{\nu}$ denote the measure on the bundle $\mathcal{M} \times Gr_{s-1}(V)$ given by

$d\hat{\nu}(x, W) = d\nu_M(x) d\hat{\nu}_x(W)$.

Then, $\hat{\nu}$ is a stationary measure for the random walk. Let

$Z = \{ y \in \mathcal{M} : \hat{\nu}_y(I(w)) > 0 \text{ for some } w \in \mathbb{P}^1(V) \}$,

Suppose $\nu_M(Z) > 0$. Let $r \geq 1$ be maximal so that for a positive measure subset $Z_r \subset \mathcal{M}$ and all $y \in Z_r$, there exists an $r$-dimensional subspace $U_y$ with $\hat{\nu}_y(I(U_y)) > 0$. Here

$I(U_y) = \{ W \in Gr_{s-1}(V) : W \supset U_y \}$.

Since $r$ is maximal, for almost all $y$, the distinct $U_y$ for which $\hat{\nu}_y(I(U_y)) > 0$ are disjoint. Thus, the set of $U_y$ for which $\hat{\nu}_y(I(U_y))$ is maximal is finite, and by ergodicity, the cardinality of this set is constant almost everywhere. Then, for each $x \in Z_r$ we can measurably choose $U_x \in \mathbb{P}^1(V)$ such that $\hat{\nu}_x(I(U_x)) > 0$. Then,

$\hat{\nu} \left( \bigcup_{x \in Z_r} \{ x \} \times I(U_x) \right) > 0$.

We now measurably choose $w_x \in U_x$. Then,

\[
\hat{\nu} \left( \bigcup_{x \in Z_r} \{ x \} \times I(w_x) \right) > 0.
\]
Therefore, (2.9) holds for some ergodic component of \( \hat{\nu} \). However, this contradicts Lemma 2.10, since the action of the cocycle on \( V \) is strongly irreducible. Thus, \( \nu(Z) = 0 \) and \( \nu(Z^c) = 1 \). By definition, for all \( y \in Z^c \) and all \( w_y \in V \), (2.8) holds.

The following lemma is a consequence of Corollary 2.11.

**Lemma 2.12** (c.f. [EM, Lemma 14.4]). For every \( \delta > 0 \) and every \( \epsilon > 0 \) there exists \( E_{\text{good}} \subset \mathcal{M} \) with \( \nu_M(E_{\text{good}}) > 1 - \delta \) and \( \sigma = \sigma(\delta, \epsilon) > 0 \), such that for any \( y \in E_{\text{good}} \) and any vector \( w \in \mathbb{P}^1(V) \),

\[
\mu^N(\{g : d(w, V_{s-1}(y, g)) > \sigma\}) > 1 - \epsilon
\]

(In (2.10), \( d(\cdot, \cdot) \) is some distance on the projective space \( \mathbb{P}^1(V) \)).

**Proof.** We reproduce the proof from [EM, Lemma 14.4] for the convenience of the reader. By Corollary 2.11 there exists \( Z \subset \mathcal{M} \) with \( \nu_M(Z) = 0 \) such that for all \( y \in Z^c \) and all \( w_y \in \mathbb{P}^1(V) \), (2.8) holds.

Fix \( y \in Z^c \). Then, for every \( w_y \in \mathbb{P}^1(V) \) there exists \( \sigma_0 = \sigma_0(y, w_y, \epsilon) > 0 \) such that

\[
\mu^N(\{g \in SL(2, \mathbb{R})^N : d(V_{s-1}(y, g), w_y) > 2\sigma_0(y, w_y, \epsilon)\}) > 1 - \epsilon.
\]

Let \( U(y, w) = \{z \in \mathbb{P}^1(V) : d(z, w) < \sigma_0(y, w, \epsilon)\} \). Then the \( \{U(y, w)\}_{w \in \mathbb{P}^1(V)} \) form an open cover of the compact space \( \mathbb{P}^1(V) \), and therefore there exist \( w_1, \ldots, w_n \) with \( \mathbb{P}^1(V) = \bigcup_{i=1}^n U(y, w_i) \). Let \( \sigma_1(y, \epsilon) = \min_i \sigma_0(y, w_i, \epsilon) \). Then, for all \( y \in Z^c \),

\[
\mu^N(\{g \in SL(2, \mathbb{R})^N : d(V_{s-1}(y, g), w) > \sigma_1(y, \epsilon)\}) > 1 - \epsilon.
\]

Let \( E_N(\epsilon) = \{x \in Z^c : \sigma_1(x, \epsilon) > \frac{1}{\epsilon}\} \). Since \( \bigcup_{n=1}^\infty E_N(\epsilon) = Z^c \) and \( \nu(Z^c) = 1 \), there exists \( N = N(\delta, \epsilon) \) such that \( \nu(E_N(\epsilon)) > 1 - \delta \). Let \( \sigma = 1/N \) and let \( E_{\text{good}} = E_N \).

Let \( \mathcal{U}_i(n, y, g) \) denote the direct sum of the eigenspaces of \( A_V(g_n \ldots g_1, y)A_V(g_n \ldots g_1, y) \) which correspond to those eigenvalues which will converge as \( n \to \infty \) to \( 2\alpha_i \). Let \( V_i(n, y, g) = \mathcal{U}_i(n, y, g) \oplus \cdots \oplus \mathcal{U}_i(n, y, g) \). Then, it follows from part II of Theorem 2.7 that for almost all \( y \) and almost all \( \bar{g} \),

\[
\lim_{n \to \infty} \mathcal{U}_i(n, y, \bar{g}) = \mathcal{U}_i(y, \bar{g}) \quad \text{and} \quad \lim_{n \to \infty} V_i(n, y, \bar{g}) = V_i(y, \bar{g}).
\]

**The set** \( F_{\text{good}}(\epsilon, \sigma, L) \). Suppose \( \epsilon > 0, \sigma > 0 \), and \( L \in \mathbb{N} \). Let \( F_{\text{good}}(\epsilon, \sigma, L) \) denote the set of \( y \in \mathcal{M} \) such that for any \( v_y \in V \)

\[
(2.12) \quad \mu^N(\{g : d(v_y, V_{s-1}(L, y, \bar{g})) > \sigma\}) > 1 - \epsilon/2
\]

and also

\[
(2.13) \quad \mu^N(\{g : \|A_V(g_L \ldots g_1, y)\| \in ((\lambda_1 - \epsilon/2)^L, (\lambda_1 + \epsilon/2)^L)\}) > 1 - \epsilon/2.
\]

Since the cocycle \( A_V \) is continuous and both (2.12) and (2.13) depend on \( \bar{g} \) only via \( g_1, \ldots, g_L \), the set \( F_{\text{good}}(\epsilon, \sigma, L) \) is open.
Lemma 2.13. For any fixed $\epsilon > 0$ and $\delta > 0$ there exist $L_0 > 0$ and $\sigma > 0$ such that for all $L > L_0$, $\nu_M(F_{\text{good}}(\epsilon, \sigma, L)) > 1 - \delta$.

Proof of Lemma 2.13. Let $\sigma > 0$ and $E_{\text{good}} \subset M$ be as in Lemma 2.12 with $\delta/4$ and $\epsilon/4$ instead of $\delta$ and $\epsilon$. By (2.11), we can find $L_1 > 0$ and a set $E_1 \subset M$ with $\nu_M(E_1) > 1 - \delta/4$ such that for $y \in E_1$ and $L \geq L_1$.

$$\mu^N\left(\{\bar{g} : d(V_{s-1}(L, y, \bar{g}), V_{s-1}(y, \bar{g})) > \sigma/2\}\right) < \epsilon/4.$$ 

Then, for $y \in E_{\text{good}} \cap E_1$, and $L \geq L_1$, (2.12) holds (with $\sigma$ replaced by $\sigma/2$). Also, by Theorem 2.7, part I, there exists $L_2 > 0$ and a subset $E_2 \subset M$ with $\nu_M(E_2) > 1 - \delta/2$ such that for $y \in E_2$ and $L \geq L_2$ (2.13) holds. Now let $E_{\text{good}}(\epsilon, \sigma, L) = E_{\text{good}} \cap E_1 \cap E_2$ and choose $L_0 = \max(L_1, L_2)$.

We also use the following trivial result:

Lemma 2.14. For any $\sigma > 0$ there is a constant $c(\sigma) > 0$ with the following property: Let $A \in GL(V)$ be a linear map, and let $V \subset V$ denote the subspace spanned by the eigenspaces of all but the top eigenvalue of $A^*A$. Then, for any $v$ with $\|v\| = 1$ and $d(v, V) > \sigma$, we have

$$\|A\| \geq \|Av\| > c(\sigma)\|A\|.$$ 

Proof of Lemma 2.14. Suppose $\epsilon > 0$ and $\delta > 0$ are given, and let $\sigma > 0$ and $L_0 > 0$ be as in Lemma 2.13. Choose $L > L_0$ such that $(\lambda_1 - \epsilon/2)^L c(\sigma) > (\lambda_1 - \epsilon)^L$, where $c(\sigma)$ is as in Lemma 2.14. Pick $v_y \in V$. Then, in view of Lemma 2.14 for all $\bar{g}$ satisfying (2.12) and (2.13),

$$(\lambda_1 + \epsilon/2)^L > \|A_V(g_L \ldots g_1, y)\| \geq \frac{\|A_V(g_L \ldots g_1, y)v_y\|}{\|v_y\|} > (\lambda_1 - \epsilon)^L.$$ 

2.2.2. Proof of Theorem 2.6. In view of Lemma 2.14 we choose $L$ so that $\nu_M(E_{\text{good}}(\epsilon, L)) > 1 - \epsilon$. Pick an arbitrary $v_0 \in V$, and let

$$v_i = v_i(\bar{g}) = A(g_i \ldots g_1, x)v_0.$$ 

Let

$$J(\bar{g}) = \{i \in \mathbb{N} : g_i \ldots g_1 x \in E_{\text{good}}(\epsilon, L)\}$$

where $H(\cdot)$ is as in the definition of $E_{\text{good}}(\epsilon, L)$. By Corollary 2.2 and 2.6, for almost all $\bar{g}$, the lower density of $J(\bar{g})$ is at least $1 - 3\epsilon$. For almost every such $\bar{g}$ we can find a subset $I(\bar{g}) \subset J(\bar{g})$ so that

$$\mathbb{N} = K \sqcup \bigcup_{i \in I(\bar{g})} [i, i + L]$$

where the intervals $[i, i + L]$ are disjoint for $i \in I(\bar{g})$, $g_i, g_{i+1}, \ldots, g_{i+L} \in H(v_i)$ (by the strong law of large numbers) and the upper density of $K$ is at most $4\epsilon$. 

\[\square\]
Now suppose \( n \gg L \). Then,
\[
\log \|v_n\| = \sum_{i=1}^{n} \log \frac{\|v_i\|}{\|v_{i-1}\|} = S_1 + S_2 + S_3.
\]

Let \( C \) be such that for all \( g \) in the support of \( \mu \) and all \( y \in M \), \( \|A(g, y)\| \leq C \). Then, \(|S_3| \leq L \log C\). Also, since the upper density of \( K \) is at most \( 3\epsilon \), \(|S_2| \leq 3\epsilon n \log C\).

However, by (2.7),
\[
S_1 \geq |I(\bar{g}) \cap [1, \ldots, n]| \log(\lambda_1 - \epsilon)L \geq (1 - 4\epsilon)n(\lambda_1 - \epsilon).
\]

Thus, for almost all \( \bar{g} \) and any \( n \gg L \),
\[
\frac{1}{n} \log \|A_V(g_n \ldots g_1, x)\| \geq \frac{1}{n} \log \|v_n\| \geq (1 - 4\epsilon)(\lambda_1 - \epsilon) - 4\epsilon \log C - \frac{L}{n} \log C.
\]

Since \( \epsilon > 0 \) is arbitrary, we get that for almost all \( \bar{g} \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \|A_V(g_n \ldots g_1, x)\| \geq \lambda_1,
\]
which proves the lower bound in (2.3). The proof of the upper bound in (2.3) is similar. Let \( a_0 = 1 \), and \( a_i = \|A_V(g_i \ldots g_1, x)\| \). Then
\[
\log a_n = \sum_{i=1}^{n} \log \frac{a_i}{a_{i-1}} = \sum_{i \in I(\bar{g}) \cap [1, \ldots, n-L]} \frac{a_{i+L}}{a_i} + \sum_{i \in K \cap [1, \ldots, n-L]} \log \frac{a_i}{a_{i-1}} + \sum_{i=n-L}^{L} \log \frac{a_i}{a_{i-1}} = S_1 + S_2 + S_3.
\]

As above, \(|S_2| \leq 4\epsilon n \log C\) and \(|S_3| \leq L \log C\). By (2.7),
\[
S_1 \leq |I(\bar{g}) \cap [1, \ldots, n]| \log(\lambda_1 + \epsilon)L \leq n(\lambda_1 + \epsilon).
\]

Therefore, for almost all \( \bar{g} \),
\[
\lim_{n \to \infty} \frac{1}{n} \log a_n \leq (\lambda_1 + \epsilon) + 4\epsilon C.
\]

Since \( \epsilon > 0 \) is arbitrary this completes the proof of Theorem 1.5. \( \square \)

3. Proof of Theorem 1.1

3.1. An analogue of Lemma 2.3

Let \( \eta_{\tau, \theta} \) denote the measure on \( SL(2, \mathbb{R}) \) given by
\[
\eta_{\tau, \theta}(\phi) = \frac{1}{T} \int_{0}^{T} \phi(g_t r_{\theta}) \, dt.
\]

In this subsection we prove the following:
Proposition 3.1. Fix \( x \in \mathcal{M} \). For almost every \( \theta \in [0, 2\pi] \), if \( \nu_{\theta} \) is any weak-star limit point (as \( T \to \infty \)) of \( \eta_{T, \theta} \ast \delta_x \), then then \( \nu_{\theta} \) is invariant under \( P \), where \( P = \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \subset SL(2, \mathbb{R}) \).

The proof of Proposition 3.1 is based on the strong law of large numbers. In fact, Proposition 3.1 holds for arbitrary measure-preserving \( SL(2, \mathbb{R}) \) actions.

It is clear from the definition, that for any \( \theta \), any \( \ast \)-limit point \( \nu_{\theta} \) is invariant under \( g_t \). Let
\[
\begin{align*}
\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix},
\end{align*}
\]
\[u_{\alpha} = \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \quad \text{and} \quad \bar{u}_{\alpha} = \left( \begin{array}{cc} 1 & 0 \\ \alpha & 1 \end{array} \right).
\]
Hence it is enough to show that \( \nu_{\theta} \) is invariant under \( \bar{u}_{\alpha} \) for every \( \alpha \). Fix \( 0 < \alpha < 1 \).

A simple calculation shows that for \( 0 < \xi < \pi/2 \),
\[
r_{\xi} = \bar{u}_{-\tan \xi} a_{\xi} u_{\tan \xi}, \quad \text{where} \quad a_{\xi} = \left( \begin{array}{cc} \cos \xi & 0 \\ 0 & 1/\cos \xi \end{array} \right).
\]
Then,
\[
g_t r_{\xi} = (g_t \bar{u}_{-\tan \xi} g_t^{-1}) a_{\xi} (g_t u_{\tan \xi} g_t^{-1}) g_t = (\bar{u}_{-e^{-2\tan \xi}} a_{\xi} (u_{e^{2\tan \xi}}) g_t.
\]
Let \( \theta_t \) be defined by the equation
\[
e^{2t} \tan \alpha_t = \alpha.
\]
We claim that Proposition 3.1 follows quickly from the following:

**Proposition 3.2.** Fix \( x \in \mathcal{M} \), and \( 0 < \alpha < 1 \). Let \( \phi \in C_c(\mathcal{M}) \) be a test function. Let
\[
f_t(\theta) = \phi(g_t r_{\theta} x) - \phi(g_t r_{\theta + \alpha_t} x)
\]
where \( \theta_t \) is as in (3.2). Then, for almost every \( \theta \in [0, 2\pi] \),
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f_t(\theta) \, dt = 0.
\]

**Proof that Proposition 3.1 follows from Proposition 3.2.** Let \( x, \phi, \alpha, \theta_t \) be as in Proposition 3.2. We need to prove that for almost all \( \theta \),
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (\phi(u_{\alpha} g_t r_{\theta} x) - \phi(g_t r_{\theta} x)) \, dt = 0.
\]
Since the smooth functions are dense in \( L^1 \), without loss of generality, we may assume that \( \phi \) is smooth. Then, there exists a constant \( M \) such that for \( h \in SL(2, \mathbb{R}) \) near the identity \( I \in SL(2, \mathbb{R}) \) and all \( y \in \mathcal{M} \),
\[
|\phi(h y) - \phi(y)| \leq M \|h - I\|.
\]
We write
\[
(3.7) \quad \phi(u_{\alpha} g_t r_{\theta} x) - \phi(g_t r_{\theta} x) = (\phi(u_{\alpha} g_t r_{\theta} x) - \phi(g_t r_{\theta + \alpha_t} x)) + (\phi(g_t r_{\theta + \alpha_t} x) - \phi(g_t r_{\theta} x)).
\]
Let $J_1$ be the contribution of the first term in parenthesis in (3.7) to (3.5) and let $J_2$ be the contribution of the second term. We have, using (3.1) and (3.2),

$$J_1 = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \phi(u_{\alpha}g_tr_{\theta}x) - \phi((\bar{u}_e^{-2t}\tan \alpha_\theta)a_{\alpha_\theta}u_{\alpha}g_tr_{\theta}x) \, dt$$

$$\leq M \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \|\bar{u}_e^{-2t}\tan \alpha_\theta a_{\alpha_\theta} - I\| \, dt = 0,$$

by (3.6) and $\alpha_t = O(e^{-t})$. Also $J_2 = 0$ by Proposition 3.2. Thus (3.5) holds.

This shows that for any fixed $0 < \alpha < 1$ for almost all $\theta$, the measures $\nu_{\theta}$ of Proposition 3.1 are invariant under $u_{\alpha}$ (as well as $g_t$ for all $t$). We now repeat the proof with two different $\alpha$’s linearly independent over $\mathbb{Q}$. We get that for almost all $\theta$, any limit point of $\eta_{T, g, *} \delta_{\alpha}$ is invariant under a dense subgroup of $P$, hence invariant under all of $P$. This completes the proof of Proposition 3.1.

Note that from (3.3) we have

$$\int_{0}^{2\pi} f_{t}(\theta) \, d\theta = 0.$$  
(3.8)

**Lemma 3.3.** There exist $\lambda > 0$ and $C > 0$ such that

$$\int_{0}^{2\pi} f_{t}(\theta) f_{s}(\theta) \, d\theta \leq Ce^{-\lambda|s-t|}.$$  
(3.9)

**Proof.** Without loss of generality we may assume that (3.6) holds, and that $t > s$. Let $r = (t + s)/2$. Let $A_{\varphi} \subset [0, 2\pi]$ be an interval of the form $[\varphi - e^{-2r}, \varphi + e^{-2r}]$.

Then, for $\theta = \varphi + \eta \in A_{\varphi}$,

$$g_{s}r_{\theta} = g_{s}r_{\eta}r_{\varphi} = (g_{s}r_{\eta}g_{s}^{-1})g_{s}r_{\varphi},$$

and hence, using (3.6),

$$|f_{s}(\theta) - f_{s}(\varphi)| \leq 4M\|g_{s}r_{\eta}g_{s}^{-1} - I\| \leq 4Me^{-2(r-s)}.$$  

Therefore,

$$\frac{1}{|A_{\varphi}|} \int_{A_{\varphi}} f_{t}(\theta) f_{s}(\theta) \, d\theta = f_{s}(\varphi) \frac{1}{|A_{\varphi}|} \int_{A_{\varphi}} f_{t}(\theta) \, d\theta + O(e^{-2(r-s)}).  
(3.10)$$

Now, from the definition (3.3) of $f_{t}$, we have

$$\frac{1}{|A_{\varphi}|} \int_{A_{\varphi}} f_{t}(\theta) \, d\theta = O(e^{-2(t-r)}).  
(3.11)$$

(Essentially the integral cancels except for the contribution of two “boundary regions” each of size $O(\theta_{t}) = O(e^{-2t})$. Since $f_{t}$ is bounded and $|A_{\varphi}| = 2e^{-2r}$, (3.11) follows.)
Now from (3.10) and (3.11) we get that for every $\varphi \in [0, 2\pi]$,
\[
\frac{1}{|A_\varphi|} \int_{A_\varphi} f_t(\theta) f_s(\theta) \, d\theta = O(e^{-2(t-r)}) + O(e^{-2(r-s)}).
\]
Since $r = (s + t)/2$, this immediately implies (3.9).

Now, Proposition 3.2 follows from the following straightforward version of the strong law of large numbers, which we will prove in §3.2 for the interested readers’ convenience:

**Lemma 3.4.** Suppose $f_i : [0, 2\pi] \to \mathbb{R}$ are bounded functions satisfying (3.8) and (3.9) (for some $C > 0$ and $\lambda > 0$). Additionally, assume that $f_i(\theta)$ are $2M$-Lipschitz functions of $t$ for each $\theta$ (3.10). Then, for almost every $\theta \in [0, 2\pi]$, (3.4) holds.

### 3.2. Proof of Lemma 3.4

We recall the following basic facts:

**Lemma 3.5** (Chebyshev inequality). Let $f : \Omega \to \mathbb{R}$ have $\int_{\Omega} f(\omega)^2 \, d\nu \leq C$. Then
\[
\nu(\{\omega : |f(\omega)| > sC\}) \leq \frac{1}{s^2 C}.
\]

**Lemma 3.6** (Borel-Cantelli). Let $A_1, \ldots$ be $\mu$-measurable sets such that $\sum_{i=1}^\infty \mu(A_i) < \infty$. Then $\mu(\cap_{i=1}^\infty \cup_{n=i}^\infty A_n) = 0$.

**Proof of Lemma 3.4.** First, because $f_i(\theta)$ is an $2M$-Lipschitz function of $t$ for each $\theta$ it suffices to show that for any $\epsilon$ and almost every $\theta$ we have:

\[
(3.12) \quad \limsup_{n \to \infty} \left| \frac{1}{n} \sum_{i=1}^n f_{i\epsilon}(\theta) \right| < \epsilon.
\]

We will show that (3.12) follows from (3.9), the Borel-Cantelli lemma, and Chebyshev’s inequality.

To see this, observe that $\int (\sum_{i=1}^n f_{i\epsilon}(\theta))^2 = \int \sum_{i=1}^n f_{i\epsilon}(\theta)^2 + 2C \sum_{i<j<n} e^{-\lambda|\epsilon^{-1}(\epsilon^{-1} - \epsilon)|}$. So, there exists $C'_\epsilon$ such that $\int (\sum_{i=1}^n f_{i\epsilon}(\theta))^2 = nC'_\epsilon \int f^2$. By Chebyshev’s inequality: there exists $C''_\epsilon$ such that $\lambda(\{\theta : \frac{1}{n} \sum_{i=1}^n f_{i\epsilon}(\theta) \geq \frac{\epsilon}{2}\}) \leq \frac{C''_\epsilon}{n}$. By the Borel-Cantelli Lemma it follows that for almost every $\theta$ we have:

\[
\limsup_{n \to \infty} \left| \frac{1}{n^2} \sum_{i=1}^n f_{i\epsilon}(\theta) \right| \leq \frac{\epsilon}{2}.
\]

Notice $(N + 1)^2 - N^2 = 2N + 1$ and so for any $M > 0$ there exists $N \in \mathbb{N}$ such that $0 \leq M - N^2 < 2\sqrt{M}$. It follows that for all large enough $M$

\[
\left| \frac{1}{M} \sum_{i=1}^M f_{i\epsilon}(\theta) \right| \leq \left| \frac{1}{M} \sum_{i=1}^{N^2} f_{i\epsilon}(\theta) + \frac{1}{M} \sum_{i=N^2+1}^M f_{i\epsilon}(\theta) \right| \leq \frac{\epsilon}{2} + \frac{2C'\sqrt{M}}{M}.
\]

This uses that the $f_i$ are uniformly bounded. For all large enough $M$ this is smaller than $\epsilon$ and Lemma 3.4 follows. \qed
3.3. Completion of the proof of Theorem 1.1

Proposition 3.7 ([EMM Proposition 2.13]). Let \( \mathcal{N} \) be any affine submanifold. Then there exists an \( SO(2) \) invariant function \( f_N : \mathcal{H}_1(\alpha) \to \mathbb{R}_+ \), \( c, b \in \mathbb{R} \) such that

1. \( f_N(x) = \infty \) iff \( x \in \mathcal{N} \). Also \( f_N \) is bounded on compact subsets of \( \mathcal{H}_1(\alpha) \setminus \mathcal{N} \).
2. There exists \( b > 0 \) (depending on \( \mathcal{N} \)) and for every \( 0 < c < 1 \) there exists \( t_0 > 0 \) (depending on \( \mathcal{N} \) and \( c \)) such that for all \( x \in \mathcal{H}_1(\alpha) \) and all \( t > t_0 \),

\[
\frac{1}{2\pi} \int_0^{2\pi} f_N(g_r \theta x) \, d\theta \leq cf_N(x) + b,
\]

3. For any \( g \in SL(2, \mathbb{R}) \) and \( \|g\| \leq 1 \) we have \( f_N(gx) \leq \sigma' f_N(x) \).

Theorem 3.8 ([A Theorem 2.3]). Given a function \( f_N \) satisfying (2) and (3) of Proposition 3.7, we have that for any \( 0 < \beta < 1 \) there exist \( M < \infty \) and \( \gamma < 1 \) such that for every \( x \) we have

\[
\lambda(\{\theta : f(g_{\lambda_n} r_\theta x) > M \text{ for at least } \beta \text{-fraction of } t \in [0, T]\}) < \gamma T
\]

for all large enough \( T \).

Proof of Theorem 1.1. Let \( \nu_0 \) be any weak-star limit point of the measures \( \eta_{T, \theta} \ast \delta_x \). By Proposition 3.1 for almost all \( \theta \), \( \nu_0 \) is \( P \)-invariant.

By [EMM Theorem 1.4], any ergodic \( P \)-invariant measure is \( SL(2, \mathbb{R}) \)-invariant and affine. Therefore, since \( \nu_0 \) is supported on \( \mathcal{M} \), it has can be decomposed into ergodic components as

\[
\nu_0 = \sum_{\mathcal{N} \subseteq \mathcal{M}} a_{\mathcal{N}}(\theta) \nu_{\mathcal{N}},
\]

where \( a_{\mathcal{N}}(\theta) \in [0, 1] \) and the sum is over the affine invariant submanifolds \( \mathcal{N} \) contained in \( \mathcal{M} \). (Here \( \mathcal{N} = \mathcal{M} \) is allowed). By [EMM Proposition 2.16] this is a countable sum. By applying Theorem 3.8 for the case \( \mathcal{N} = \emptyset \) we get that for almost all \( \theta \), \( \nu_0 \) is a probability measure. Then, by applying Theorem 3.8 again with \( \mathcal{N} \) any affine invariant submanifold properly contained in \( \mathcal{M} \), we see that for almost all \( \theta \), \( \nu_0(\mathcal{N}) = 0 \). Thus, for almost all \( \theta \), \( a_{\mathcal{N}}(\theta) = 0 \) for any \( \mathcal{N} \) properly contained in \( \mathcal{M} \). Since \( \nu_0 \) is a probability measure, this forces \( \nu_0 = \nu_{\mathcal{M}} \) for almost all \( \theta \), completing the proof of Theorem 1.1.

4. Proof of Theorem 1.5

Let \( \mu \) be as in [Z2]. The following lemma expresses the well known fact that a typical random walk trajectory tracks a geodesic (up to sublinear error).

Lemma 4.1 (Sublinear Tracking). There exists \( \lambda > 0 \) (depending only on \( \mu \)), and \( \mu^N \)-almost all \( \bar{g} = (g_1, \ldots, g_n, \ldots) \in SL(2, \mathbb{R})^N \) there exists \( \bar{\theta} = \bar{\theta}(\bar{g}) \in [0, 2\pi) \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log \| (g_{\lambda_n} r_{\bar{\theta}})(g_n \ldots g_1)^{-1} \| = 0,
\]
where \(g_{\lambda n}\) denotes \(\begin{pmatrix} e^{\lambda n} & 0 \\ 0 & e^{-\lambda n} \end{pmatrix}\). Furthermore, the distribution of \(\bar{\theta}\) is uniform, i.e.

\[
(4.2) \quad \mu \left( \{ \bar{g} \in SL(2, \mathbb{R})^N : \bar{\theta}(\bar{g}) \in [\theta_1, \theta_2] \} \right) = \frac{|\theta_2 - \theta_1|}{2\pi}.
\]

**Proof.** We apply the multiplicative ergodic theorem Theorem 2.7 to the identity cocycle \(\alpha(g, x) = g\) (instead of \(A_V\)). Let \(\Lambda \in SL(2, \mathbb{R})\) be as in II of Theorem 2.7.

Since \(\Lambda\) is symmetric, we may write

\[
\Lambda(\bar{g}) = r_{\bar{\theta}}^{-1} \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} r_{\bar{\theta}}.
\]

Then, (2.5) immediately implies (4.1).

Let \(\sigma\) denote the measure on \([0, 2\pi]\) such that \(\sigma([\theta_1, \theta_2])\) is given by the left-hand-side of (4.2). It is easy to show that \(\sigma\) must be \(\mu\)-stationary, i.e. \(\mu * \sigma = \sigma\). Since \(\mu\) is assumed to be \(SO(2)\)-bi-invariant, this implies that \(\sigma\) is the uniform measure. \(\square\)

**Proof of Theorem 1.5.** By Theorem 2.6 there exists a set \(E\) with \(\mu N(E) = 0\) such that for \(\bar{g} \notin E\), (2.3) holds. By Lemma 4.1, for almost all \(\theta \in [0, 2\pi]\) there exists \(\bar{g} = (g_1, \ldots, g_n, \ldots) \notin E\) so that if we write

\[
g_{\lambda n}r_{\theta} = \epsilon_n g_n \ldots g_1,
\]

then \(\epsilon_n \in SL(2, \mathbb{R})\) satisfies

\[
(4.3) \quad \lim_{n \to \infty} \frac{1}{n} \log \|\epsilon_n\| = 0.
\]

Then, by the cocycle relation,

\[
A_V(g_{\lambda n}, r_{\theta} x) = A_V(\epsilon_n, g_n \ldots g_1 x) A_V(g_n \ldots g_1, x).
\]

By \(\text{[Fo]}\), there exists \(C > 0\) and \(N < \infty\) so that for all \(g \in SL(2, \mathbb{R})\) and all \(x \in H_1(\alpha)\), we have

\[
(4.4) \quad A_V(g, x) \leq C\|g\|^N.
\]

Hence, by (4.3) and (4.4), we have

\[
\log \|A_V(g_{\lambda n}, r_{\theta} x)\| = \log \|A_V(g_n \ldots g_1, x)\| + o(n).
\]

Now the existence of the limit in (1.3) follows from (2.3). \(\square\)
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