Surfaces of Revolution via the Schrödinger Equation: Construction, Integrable Dynamics and Visualization

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Zusammenfassung

Surfaces of revolution in three-dimensional Euclidean space are considered. Several new examples of surfaces of revolution associated with well-known solvable cases of the Schrödinger equation (infinite well, harmonic oscillator, Coulomb potential, Bargmann potential, etc.) are analyzed and visualized. The properties of such surfaces are discussed. Two types of deformations (evolutions) of the surfaces of revolution, namely 1) preserving the Gaussian curvature and 2) via the dynamics of the Korteweg-de Vries equation are discussed.

1 Introduction

Surfaces and curvature are the key ingredients in a number of problems and phenomena in mathematics and physics. The differential geometry of surfaces in three-dimensional Euclidean space $R^3$ has been completed, in essence, at the end of last and the beginning of this century (see e.g. [1]-[3]). Basic
differential equations which describe surfaces in $R^3$ have been studied from various points of view. A study of the interrelation between special classes of surfaces and linear and nonlinear differential equations (PDE) has been one of the classical problems of differential geometry [1]-[3].

Interfaces, surfaces, fronts and their dynamics are also the key objects in numerous nonlinear phenomena in classical physics, such as surface waves, growth of crystals, propagation of flame fronts, deformations of membranes and many problems of hydrodynamics (see e.g. [4], [5]). In quantum field theory and statistical physics surfaces are of importance, too. Numerous papers have been devoted to the study and applications of integrals over surfaces in gauge field theories, string theory, quantum gravity and statistical physics (see e.g. [6], [7]).

The discovery of the inverse scattering (spectral) transform (IST) method, a powerful tool to construct and solve integrable PDEs (see e.g. [8]-[10]), about 30 years ago, gave a new impulse for the study of the interrelations between nonlinear PDEs and differential geometry of surfaces. On one hand, new nonlinear integrable PDEs (in addition to the sine-Gordon and Liouville equations) with a particular geometrical meaning have been found (see e.g. [11]-[13]). On the other hand, the IST method has been used to construct wide classes of surfaces (see e.g. [14]-[16]).

An alternative use of the IST method in the framework of differential geometry of surfaces has been proposed recently in the papers [17]-[19]. The approach discussed in [17] allows to generate surfaces in $R^3$ (and $R^N$) via two-dimensional linear problems and formulate their integrable dynamics via the associated 2+1-dimensional integrable PDEs. This approach has been applied in 2-dimensional gravity [20] and in some pure mathematical problems of differential geometry [21]. The approach discussed in [18] and [19] is quite different. It is mainly concerned with the intrinsic geometry of surfaces and spaces. In [18] it was noted, in particular, that the recent results on exact solutions of the one-dimensional Schrödinger equation and the Korteweg-de Vries (KdV) equation provide us with the variety of surfaces where metric and Gaussian curvature are given by simple explicit formulas. Surfaces of revolution form a very special and simple class of surfaces. Nevertheless, they are of great importance both in mathematics and in physics. Perhaps, they are the best candidates to deal with in terms of the approach proposed in [18] and [19], also since in the literature only few examples of surfaces of revolution (spherical and pseudospherical surfaces of
constant Gaussian curvature, catenoid) are discussed.

In the present paper we consider surfaces of revolution in three-dimensional Euclidean space $\mathbb{R}^3$ and their integrable deformations (dynamics). Our approach is based on the observation that surfaces of revolution in $\mathbb{R}^3$ are completely governed by the one-dimensional Schrödinger equation in the sense that the Gauss equation for surfaces in terms of geodesic coordinates, i.e. with a metric

$$ds^2 = dx^2 + H^2 d\varphi^2,$$

simply reads

$$H_{xx} + KH = 0$$

where $K$ is the Gaussian curvature. It is used, for instance, to analyze surfaces of constant Gaussian curvature (see e.g. [1]-[3]). The connection of (2) with the standard one-dimensional Schrödinger equation

$$-\Psi_{xx} + u\Psi = E\Psi$$

is established by the simple relations

$$K = E - u, \quad H = Re(\Psi) \quad \text{or} \quad H = Im(\Psi).$$

In this paper we will analyze and visualize the surfaces of revolution associated with well-known solvable cases of equation (3). We treat the infinite well potential, $\delta$-function potential, harmonic oscillator, effective radial Coulomb potential and Bargmann potentials. Such "physically motivated" surfaces of revolution have several properties which distinguish them from the old "mathematically motivated" surfaces, like the catenoid. In particular, surfaces of revolution associated with the bound states are spindle-shaped surfaces which at infinity tend to strings. The surfaces generated by the $n+1$-th excited bound state have $n$ knots (conic points).

We will consider also the deformations (dynamics) of surfaces of revolution of two distinct types. The evolutions of the first type preserve the Gaussian curvature $K(x)$ and change the metric according to

$$\Psi(x) \rightarrow \Psi'(x, t) = A(t)\Psi(x) + B(t)\bar{\Psi}(x)$$

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where $A$ and $B$ are two arbitrary functions of the parameter $t$ and $\tilde{\Psi}(x)$ is the solution of the Schrödinger equation (3) linear independent to $\Psi(x)$. The simplest case $B \equiv 0$ corresponds just to the rescaling of the metric. But even this simple transformation can deform the surface drastically. For instance, for an increasing function $A(t)$ the evolution in $t$ may generate discontinuities on the surface for certain $t$ and, finally, may lead to the splitting of the surface into disconnected parts. The deformations with $B \neq 0$ create edges and singularities in the profile of the surface in $R^3$ immediately, i.e. for any $t \neq 0$. Regular surfaces of revolution associated with the bound states are absolutely instable with respect to such deformations. The particular class of deformations for which

$$A = \frac{1}{\sqrt{-2T_t(t)}}, \quad B = \frac{T(t)}{\sqrt{-2T_t(t)}}$$

(6)

where $T(t)$ is an arbitrary function of $t$ and $T_t$ indicates the derivative with respect to $t$, corresponds to Liouville type evolutions recently discussed in [22]. In this case the function $q = -2 \ln \Psi$ obeys the equation $q_{tt} = \exp q$.

Another type of deformation is of completely different nature. It is given by the KdV equation for the Gaussian curvature

$$K_t + K_{xxx} + 6KK_x - 6EK_x = 0$$

(7)

and by the equation

$$H_t + H_{xxx} - 6EH_x - 3\frac{H_xH_{xx}}{H} = 0$$

(8)

for the metric. The KdV equation is integrable by the IST method and has a number of very interesting properties (see e.g. [8]-[10]). Consequently, the KdV-type transformations of surfaces inherits all these properties. In particular, it preserves an infinite set of integral characteristics of the surface of the type

$$Q_n = 2\pi \int_{-\infty}^{\infty} dx \frac{dx}{C_n(x)}$$

(9)

where the $C_n$ are differential polynomials of $K$ (e.g. $C_1 = K, C_2 = K^2, C_3 = K_x^2 - 2K$). In contrast to the deformations of the first type the KdV-type deformations tend to smooth out the surfaces of revolution. They
described by explicit formulas for the so-called multi-soliton solutions of the KdV equation.

The paper is organized as follows. In section 2 we present the basic formulas for the surfaces of revolution. The following sections 3-8 contain the discussion of the surfaces generated by well-known solvable potentials of the Schrödinger equation. Then the two types of deformations of surfaces of revolution are introduced. Their properties are discussed for the examples treated in the preceding sections.

2 Surfaces of Revolution: Basic Formulas and Inducing via the Schrödinger Equation

Here we will present for convenience some basic formulas for the surfaces of revolution in $\mathbb{R}^3$ (see e.g. [1]–[3]). A surface of revolution is a surface which is obtained by the rotation of a plane curve around an axis belonging to the same plane. Denoting the Cartesian coordinates in $\mathbb{R}^3$ as $X, Y$ and $Z$ one can define a surface of revolution according to

$$X = r \cos \varphi, \quad Y = r \sin \varphi, \quad Z = f(r) \tag{10}$$

where $r^2 = X^2 + Y^2$, $\varphi$ is the angle in the $X$-$Y$ plane and $f$ is an arbitrary function of $r$. The choice of the function $f(r)$ specifies the surface of revolution. The Euclidean metric in $\mathbb{R}^3$ induces on the surface (10) the following metric

$$\Omega = [1 + f^2_r(r)]dr^2 + r^2d\varphi^2 \tag{11}$$

where $f_r = \frac{df}{dr}$. Introducing the variables $x$ and $H$ via

$$dx = \sqrt{1 + f^2_r(r)}dr \tag{12}$$

and

$$r = H(x) \tag{13}$$

one converts the metric (11) into the form

$$\Omega = dx^2 + H^2(x)d\varphi^2. \tag{14}$$
The lines $\varphi = \text{const}$ are the meridians, which are geodesics, and the lines $x = \text{const}$ are the parallels of the surface of revolution. The meridians and parallels are also curvature lines. For surfaces of revolution the metric depends only on $x$. The corresponding Gauss equation reads

$$H_{xx} + KH = 0$$  \hspace{1cm} (15)

where $K(x)$ is the Gaussian curvature. The second fundamental form and other characteristics also have a very simple structure in the geodesic coordinates $x$ and $\varphi$.

Locally any form (14) corresponds to a surface of revolution in $\mathbb{R}^3$. If one considers the global properties of surfaces of revolution, however, it is necessary to pay attention to the following. Equations (12) and (13) imply that

$$\left(1 - H_x^2\right)dx^2 = dZ^2$$  \hspace{1cm} (16)

or

$$dZ = \sqrt{1 - H_x^2}dx$$  \hspace{1cm} (17)

where $H_x = \frac{dH}{dx}$. Given $H(x)$ one gets from (17)

$$Z = \int_{x_0}^{x} dx' \sqrt{1 - H_x^2(x')}.$$  \hspace{1cm} (18)

This formula together with (13) defines the surface of revolution in $\mathbb{R}^3$ according to (10). However, the constraint

$$|H_x| \leq 1$$  \hspace{1cm} (19)

for the values of the coordinate $x$ has to be taken into account. In other words, (19) defines the set of admissible values of $x$ while (13) determines the corresponding set of admissible values of $r$. We will construct the surfaces in such a way that there will be no gaps in the profile of the surface along the $Z$-axis. Thus, in the case when $x$ varies in the set of disconnected intervals we join the surface at the end of the intervals in order to avoid gaps.

It is clear, however, that in general depending on the choice of the explicit form of the metric $H$ the profile of the surface of revolution thus may exhibit discontinuities or singularities connected to the existence of different intervals.
of admissible values of $x$. The weakest irregularity consists of edges which are due to a jump in $H_x$ but still satisfying the constraint (19). The points for which $H_x = \pm 1$ have to be considered carefully. At these points we have

$$\frac{dZ}{dx} = 0$$ \hfill (20)

and

$$\frac{dZ}{dr} = \frac{\sqrt{1 - H_x^2}}{H_x} = 0.$$ \hfill (21)

The points under consideration do not belong to the surface, but if $H$ is finite it is possible to connect the surface by assigning appropriate values. This can give rise to the situation that the tangent plane at these points on the surface is parallel to the $X$-$Y$-plane. If $d^2Z/dx^2$ changes sign at these points the surface of revolution will have cuspidal edges. Furthermore, if $|H_x| \geq 1$ for a whole interval of $x$ the surface will be in general disconnected by a gap parallel to the $X$-$Y$-plane. Note, that we will use the term cuspidal edge also for the boundary of finite surfaces having the property (20).

Well-known examples of surfaces of revolution are the spherical ($K = 1$) and the pseudospherical ($K = -1$) surfaces and the catenoid ($K = -a^2/(a^2 + x^2)^2$, $H^2 = a^2 + x^2$). Some other examples can be found in [23].

In our approach to the surfaces of revolution we will follow the papers [18] and [19]. We start with the presentation of the Gauss equation (15) in the form of the standard one-dimensional Schrödinger equation

$$-\frac{\Psi_{xx}}{x} + u(x) \Psi = E \Psi.$$ \hfill (22)

For a given solution of (22) with fixed $u(x)$, energy $E$ and wavefunction $\Psi(x)$ one gets a solution of the Gauss equation (15) via the identification

$$K = E - u(x), \quad H = Re(\Psi(x)) \quad \text{or} \quad H = Im(\Psi(x)).$$ \hfill (23)

Our selection of surfaces of revolution is motivated by the physics associated with the Schrödinger equation (22). We will consider the exactly solvable cases and mainly the corresponding bound states, i.e. stationary states of the discrete spectrum. Cases when the potential $u(x)$ has singularities will be discussed as well. More complicated problems like the scattering problems...
and surfaces generated by Schrödinger equations with random potentials will be considered elsewhere.

All figures presented here have been made using the plotting facilities of MAPLE. Since the integration in (18) can in general not be calculated in closed form a cubic spline approximation for the integrand has been used to get a polynomial approximation of the function $Z(x)$ in the interval of interest which can always be calculated to arbitrary accuracy [25].

3 Spherical and Pseudospherical Surfaces via Free Motion

We start with the simplest possible case for the Schrödinger equation, i.e. the free motion. For $u \equiv 0$ and positive energy, $E > 0$, the general form of the associated metric is

$$H = A \cos(kx + \alpha),$$

(24)

where $k^2 = E$ and $A$ and $\alpha$ are arbitrary real constants. Without loss of generality one can put $\alpha = 0$. In this case we have

$$Z = \int^x dx' \sqrt{1 - A^2 k^2 \sin^2 (kx)}.$$  

(25)

Therefore three different classes of surfaces are possible.

1. In the case $Ak = 1$ one has

$$Z = \frac{1}{k} \sin (kx) + Z_0, \quad r = H = \frac{1}{k} \cos (kx),$$

(26)

leading to

$$X^2 + Y^2 + (Z - Z_0)^2 = \frac{1}{k^2}$$

(27)

and our surface is the standard sphere of radius $\frac{1}{k}$. Due to the fact that $H$ according to (24) is periodic in $x$ we will get an infinite number of spheres glued to each other along the $Z$-axis (figure 1).
2. For $Ak < 1$ there is also no constraint on the values of $x$. The solution of (25) is given in terms of elliptic functions. The surface looks like an infinite sequence of pressed spheres similar to american footballs, which are depicted in figure 2.

3. If we choose $Ak > 1$ the variable $x$ is allowed to vary within the intervals defined by the inequality $|\sin(kx)| \leq \frac{1}{Ak}$. The surface consists of an infinite sequence of pieces from which a single surface is shown in figure 3.

We see that the form of these surfaces depends crucially on the choice of the normalization constant $A$. In the first and the second case we have a regular surface while in the third case the surface has cuspidal edges which disconnect it into distinct finite pieces.

The situation is quite different for the nonphysical case of negative energy $E = -\lambda^2 < 0$. With this choice there are three different solutions of equation (22), namely

\begin{align}
H_1 &= A_1 \cosh (\lambda x), \\
H_2 &= A_2 \sinh (\lambda x), \\
H_3 &= A_3 \exp (-\lambda x)
\end{align}

where $A_1$, $A_2$ and $A_3$ are arbitrary real constants. In the first and the second case the corresponding surfaces exhibit edges independent of the choice of $A_1$ and $A_2$ at the points $x_1 = \pm \frac{1}{\lambda} \text{arcsinh} \left( 1/A_1 \lambda \right)$ and $x_2 = \pm \frac{1}{\lambda} \text{arccosh} \left( 1/A_2 \lambda \right)$, respectively. They are presented in figure 4 and 5. In the third case, however, the corresponding surface has only one edge at the point $x_3 = \frac{1}{\lambda} \ln |A_3 \lambda|$. Shifting $x \to x + x_3$ one gets the surface defined for $x > 0$ with the cuspidal edge at $x = 0$ and with $H_3 = \frac{1}{\lambda} \exp (-\lambda x)$, which is shown in figure 6. It is nothing but the famous pseudosphere. It is generated by the rotation of the so-called tractrix.

All surfaces presented in this section are well-known. They correspond to spherical ($E > 0$) and pseudospherical ($E < 0$) surfaces of different types, which hence all are generated by the physically simplest case of the Schrödinger equation, namely, free motion.
4 Surfaces of Revolution via the Infinite Well Potential

Let us consider now the simplest nontrivial potential, namely the infinite well potential

\[ u(x) = \begin{cases} 
0, & 0 \leq x \leq \pi a \\
\infty, & x < 0 \text{ and } x > \pi a,
\end{cases} \tag{29} \]

where \( a \) is a constant. In this case one has bound states with energies

\[ E_n = \frac{n^2}{a^2}, \quad n = 1, 2, 3, \ldots \tag{30} \]

and the corresponding wavefunctions \[ \Psi_n(x) = \begin{cases} 
A_n \sin \left( \frac{nx}{a} \right), & 0 \leq x \leq \pi a \\
0, & x < 0 \text{ and } x > \pi a,
\end{cases} \tag{31} \]

where \( A_n \) are arbitrary real constants, since we do not require the normalization of the wavefunction. The choice of \( u(x) \) according to (29) leads to an infinite family of surfaces whose metric is given by

\[ H_n = \begin{cases} 
A_n \sin \left( \frac{nx}{a} \right), & 0 \leq x \leq \pi a \\
0, & x < 0 \text{ and } x > \pi a.
\end{cases} \tag{32} \]

Since \( K = E - u(x) \) our surfaces have constant Gaussian curvature \( K_n = \frac{n^2}{a^2} \) for \( 0 \leq x \leq \pi a \) and infinite negative curvature outside this interval. Therefore within the infinite well they look like the surfaces with positive Gaussian curvature described in the previous section while outside they have the form of infinitely thin tubes, i.e., strings going to infinity.

The ground state \( n = 1 \) gives rise to three different surfaces depending on the choice of \( A_1 \). For \( A_1 = a \) we have the surface which consists of the standard sphere of radius \( a \), connected with the infinities by strings from the south and north poles, respectively (left upper picture in figure 7). If \( A_1 < a \) we have surfaces of the type shown in the left upper picture of figure 8, while
the case $A_1 > a$ is depicted in the left upper part of figure 9. It has to be noted, however, that in this case at the points where $|\Psi_x| = 1$ the modulus of the wavefunction is not equal to zero. Gluing the strings, whose radius is zero, thus introduces a jump of the radius.

For excited states ($n > 1$) the corresponding surfaces of revolution look like a chain of $n$ pieces of the form as for $n = 1$ separated by $n - 1$ knots (conic points) and connected by strings to infinity. If we choose $A_n = \frac{a}{n}$ we get a chain of spheres with radius $\frac{a}{n}$ (figure 7), for $A_n < \frac{a}{n}$ the chain consists of squeezed spheres (figure 8) which are extended to infinity by the strings. The choice $A_n > \frac{a}{n}$ is connected to a sequence of squeezed and cut spheres separated by cuspidal edges. As before, this is accompanied by a jump of the radius to zero for the strings tending to infinity (figure 9).

It should be noted that if one chooses for the amplitude $A_n$ the physical normalization of the function $\Psi_n$ ($\int_{-\infty}^{\infty} dx \Psi_n^2 = 1$), i.e. $A_n = \sqrt{\frac{2}{\pi a}}$, then the $n$-th excited state gives rise to the sequence of spheres if $\pi a^3 = 2n^2$.

5 \hspace{1em} \boldsymbol{\delta}\text{-Function Potentials and Surfaces with Singular Gaussian Curvature}

The next of our examples is determined by the potential

$$u(x) = -2\kappa \delta(x),$$

(33)

where $\kappa$ is a positive real constant and $\delta(x)$ is the Dirac delta-function. There is one bound state with

$$E = -\kappa^2$$

(34)

and the corresponding wavefunction reads [24]

$$\Psi(x) = Ae^{-\kappa|x|}.$$  

(35)

For $A\kappa < 1$ the surface is of the form shown in figure 10. It extends to infinity and has an discontinuity at $x = 0$. If $A\kappa = 1$ it becomes the cuspidal edge ($\frac{dz}{dr} = 0$ at $x = 0$) and the corresponding surface, which can be seen in figure 11, consists of two parts ($x < 0$ and $x > 0$) where both are just the pseudosphere discussed in section 3. In the case of $A\kappa > 1$ the surface of
revolution is built from two disconnected intervals of the parameter \( x \), namely

\[-\infty < x < -\frac{1}{\kappa} \ln (A\kappa) \text{ and } \frac{1}{\kappa} \ln (A\kappa) < x < \infty.\]

They have cuspidal edges at \( x = \pm \frac{1}{\kappa} \ln (A\kappa) \). Each of these parts is nothing but the pseudosphere. For the solutions \( \Psi (x) \) with \( E > 0 \) the corresponding surfaces look similar to the case of free motion with \( E > 0 \).

In a similar manner one can construct surfaces associated with a potential which is the sum of several \( \delta \)-functions. The presence of \( \delta \)-functions allows us to describe surfaces with constant Gaussian curvature except at some points which correspond to edges. The Dirac comb potential

\[
u (x) = \alpha \sum_{n=-\infty}^{\infty} \delta (x-na)
\]

is of particular interest. In solid state physics it is used as a model potential with periodic structure which leads to energy bands for the stationary states. In our approach it allows to describe surfaces with an infinite number of cuspidal edges. If \( \alpha > 0 \) the Dirac comb generates for \( E > 0 \) surfaces which look like pieces of the form as in figure 3 glued together. Between the points \( x = na \) they have constant positive Gaussian curvature while at these points the curvature is negative and infinite.

The case \( \alpha < 0 \) and \( E < 0 \) corresponds to the surface which consists of an infinite sequence of the parts presented in the figures 4 and 5. So we have the ”pseudospherical” surface with an infinite set of edges. Note, that if we exclude the points \( x = na \) then the rest is the disconnected pseudospherical surface.

6 Surfaces Associated with the Harmonic Oscillator

The following example is the famous harmonic oscillator potential, namely

\[
u (x) = x^2.
\]

It has an infinite set of stationary states with the energies

\[
E_n = 2 \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, ...
\]

and the wavefunctions
\[ \Psi_n(x) = A_n \exp \left( -\frac{x^2}{2} \right) H_n(x), \quad (39) \]

where \( H_n \) are the Hermite polynomials and \( A_n \) are arbitrary real constants \[^{24}\]. For the ground state \((n = 0)\), the first \((n = 1)\) and the second \((n = 2)\) excited state the corresponding wavefunctions take the form

\[ \Psi_0(x) = A_0 \exp \left( -\frac{x^2}{2} \right) \quad (40) \]
\[ \Psi_1(x) = A_1 x \exp \left( -\frac{x^2}{2} \right) \quad (41) \]
\[ \Psi_2(x) = A_2 [4x^2 - 2] \exp \left( -\frac{x^2}{2} \right), \quad (42) \]

respectively. In the figures 12, 13 and 14 one can see the induced surfaces for a choice of the constants \( A_n \) which satisfies the constraint \(|\Psi_x| < 1\) for all \(x\). Figure 15 shows a surface connected to the ground state where \(A_0\) was chosen in such a way that at the points \(x = \pm 1\) the tangent plane on the surface is parallel to the \(X-Y\)-plane, i.e. \(|\Psi_x| = 1\).

## 7 Effective One-Dimensional Coulomb Potential

With the help of the substitution

\[ R(r) = \frac{\Psi(r)}{r} \quad (43) \]

the radial Schrödinger equation for the hydrogen atom can be reduced to the one-dimensional equation (\[^{24}\], §36)

\[ -\Psi_{xx} + \left[ \frac{l(l+1)}{x^2} - \frac{2}{x} \right] \Psi = 2E \Psi, \quad (44) \]

where \(l\) is the angular momentum and we substituted \(r\) by \(x\). The functions \(\Psi(x)\) corresponding to the bound states with energies
are given in the form

\[ E_n = -\frac{1}{2n^2} \]  \hspace{1cm} (45)

\[ \Psi_{n,l}(x) = A_n x^{l+1} \exp\left(-\frac{x}{n}\right)L_{n+l}^{2l+1}\left(\frac{2r}{n}\right). \]  \hspace{1cm} (46)

In (46) the functions \( L_{n+l}^{2l+1} \) are the Laguerre polynomials and \( A_n \) are as usual arbitrary real constants. As can be seen from (44) the Gaussian curvature now has pole type singularities. For the ground state \( (n=1, l=0) \) we thus have

\[ K_{1,0}(x) = \frac{2}{x} - \frac{1}{2} \]  \hspace{1cm} (47)

\[ \Psi_{1,0}(x) = A_1 x \exp(-x). \]  \hspace{1cm} (48)

Figure 16 gives an impression of the surface. For \( n=2 \) we have the bound states with \( l=0 \),

\[ K_{2,0}(x) = \frac{2}{x} - \frac{1}{8} \]  \hspace{1cm} (49)

\[ \Psi_{2,0}(x) = A_2 x \left(1 - \frac{x}{2}\right) \exp\left(-\frac{x}{2}\right), \]  \hspace{1cm} (50)

and \( l=1 \) corresponding to

\[ K_{2,1}(x) = \frac{2}{x} - \frac{2}{x^2} - \frac{1}{8} \]  \hspace{1cm} (51)

\[ \Psi_{2,1}(x) = A_2 x^2 \exp\left(-\frac{x}{2}\right). \]  \hspace{1cm} (52)

The figures 17 and 18 show the surfaces of revolution, respectively.
8 Soliton Surfaces via Bargmann Potentials

A wide class of solvable potentials for the Schrödinger equation is given by the multi-Bargmann potentials (see e.g. [8] - [10]). The corresponding surfaces of revolution can be represented as

\[ K(x) = \lambda^2 + 2(\ln \det D)_{xx} \]  \hspace{1cm} (53)

\[ H(x) = \text{Re} \left[ A \exp(-i\lambda x) \left\{ 1 + \sum_{k=1}^{N} \frac{\det D^{(k)} \exp(-\lambda_k x)}{\det D \lambda_k + i\lambda} \right\} \right], \]  \hspace{1cm} (54)

where \( E = \lambda^2 \) and the matrix elements of the \( N \times N \) matrix \( D \) read

\[ D_{kl} = \delta_{kl} + \frac{\mu_k \exp(-[\lambda_k + \lambda_l] x)}{\lambda_k + \lambda_l} \]  \hspace{1cm} (55)

for \( k, l = 1, \ldots, N \). \( N \) is an arbitrary integer, \( \lambda_k \) and \( \mu_k \) are arbitrary real constants and the matrix elements of the matrix \( D^{(k)} \) are given by (55) where the last column is substituted by the column \( -\beta_k \exp(-\lambda_k x) \) \( (k = 1, \ldots, N) \). The \( N \)-Bargmann potential has \( N \) bound states with the energies \( -\lambda_1^2, -\lambda_2^2, \ldots, -\lambda_N^2 \) \( (\lambda_1 < \lambda_2 < \ldots < \lambda_N) \). \( A \) is an arbitrary amplitude.

Let us first consider the simplest Bargmann potential, i.e. \( N = 1 \). Depending on the choice of the parameter \( \lambda \) we get three different types of surfaces. We will only treat the case when \( A \) is chosen in such a way that the condition \( |H_x| < 1 \) is satisfied which corresponds to regular surfaces of revolution.

1. For the bound state we put \( \lambda = i\lambda_1 \) which leads to

\[ K(x) = -\lambda_1^2 + \frac{2\lambda_1^2}{\cosh^2(\lambda_1 x)} \]  \hspace{1cm} (56)

\[ H(x) = \frac{A}{\cosh(\lambda_1 x)}, \]  \hspace{1cm} (57)

where \( A \) is once again an arbitrary real constant. The corresponding surface looks like a bubble which decays exponentially fast into strings tending to infinity. Figure 19 shows the surface.
2. The case of zero energy, i.e. $\lambda = 0$ leads to the corresponding surface of revolution plotted in figure 20. Gaussian curvature and metric take the form

$$K(x) = \frac{2\lambda_1^2}{\cosh^2(\lambda_1 x)}$$

$$H(x) = A \tanh(\lambda_1 x).$$

3. If we want to consider a positive energy we have to choose $\lambda$ real. The associated surface of revolution then represents a scattering state which, up to a slight decrease of the amplitude in the area where the potential has its maximum, is similar to the free motion case. In that case we have

$$K(x) = \lambda^2 + \frac{2\lambda_1^2}{\cosh^2(\lambda_1 x)}$$

$$H(x) = A[\lambda \sin(\lambda x) + \lambda_1 \cos(\lambda x) \tanh(\lambda_1 x)].$$

Figure 21 shows the surface for a special choice of parameters. Note, that far from the potential well the surface has the same periodic form on both sides. This corresponds exactly to the transparency of the Bargmann potential.

In the case $N = 2$ we end up with a potential having two wells separated by a distance which can be adjusted freely simply by fixing the corresponding parameters. We will omit the formulas for Gaussian curvature and the metric here, they can be derived straightforward from (53) and (54), for details we refer to section 10. Two figures might be sufficient to illustrate the situation, namely in figure 22 we see a surface corresponding to the ground state of the 2-soliton potential, whereas figure 23 shows the excited bound state surface.

9 Deformations of Surfaces Preserving the Gaussian Curvature

In this section we will consider deformations of the surfaces of revolution constructed above. The simplest type of deformations is given by rescaling the metric, i.e.
\[ H(x) \rightarrow H'(x) = g(t) H(x), \]  

(62)

where \( g(t) \) is an arbitrary function. The Gaussian curvature is invariant under such deformations, but the shape of the surfaces in \( R^3 \) may change drastically.

Let us start with the surfaces associated with the free motion (section 3) at \( E > 0 \) in the case \( Ak < 1 \). Let \( H_x(t = 0) < 1 \) and \( g(t) \) be a monotonically increasing function with \( g(t = 0) = 1 \). Hence at \( t = 0 \) we have the surfaces of the type depicted in figure 2. For \( t > 0 \) the height of each segment increases and at the moment \( t_0 \) defined by \( Ak g(t_0) = 1 \) the surface becomes the sphere. For \( t > t_0 \) there will be cuspidal edges and the surface changes its shape to that given in figure 3. For monotonically decreasing functions \( g(t) \) the deformations of the surface proceed in the opposite direction. Evidently, for deformations given by functions such that \( |g(t)| < 1 \) for all \( t \) the shape of the surface is changed unessentially.

For the pseudospherical surfaces the deformations (62) look simpler. For the surfaces of figure 4 and 5 this type of deformations changes, in essence, only the distance between the edges. For the pseudosphere of figure 6 the deformation (62) changes just the position of the edge which can always be shifted to the origin by a simple redefinition of \( x \). So, the pseudosphere is, in fact, invariant with respect to deformations (62) for any smooth function \( g(t) \).

The behavior of other surfaces of revolution considered in the previous sections under the deformations of the type (62) is similar. For regular surfaces (\( |H_x| < 1 \)) corresponding to bound states (62) will disconnect the surface. These cuts are developing, evidently, at those points where \( |H_x| \) has maximums. For instance, for the surface generated by the bound state of the 1-soliton potential with \( H = A/\cosh(\lambda_1 x) \) the function \( H_x \) has extrema at the points \( x_{1,2} = \pm \frac{1}{\lambda_1} \arccosh(\sqrt{2}) \). Thus, for a monotonically increasing function \( g(t) \) as \( t \rightarrow t_0 \), where \( t_0 \) is the time when \( |H_x| \) becomes equal to one, the profile of the surface near the points \( x_{1,2} = \pm 1 \) becomes parallel to the X-Y-plane at \( t_0 \) and later the surfaces gets disconnected. Figure 24 shows the behavior for a special choice of the function \( g(t) \). As \( t \rightarrow \infty \) the outer ring becomes thinner and finally disappears whereupon the whole surface is converted into the pseudosphere. Note, that for deformations (62) the splitting of surfaces after some finite time \( \Delta t \) when starting from regular
ones is a general feature. The lap $\Delta t$, of course, is determined by the function $g(t)$ and the smallest difference $|H_x| - 1$.

A quite obvious generalization of the deformations (62) is of the form

$$\Psi(x) \rightarrow \Psi'(x, t) = A(t)\Psi(x) + B(t)\bar{\Psi}(x), \quad (63)$$

where $\bar{\Psi}(x)$ is the linearly independent solution of the Schrödinger equation with respect to $\Psi(x)$ and $A(t)$ and $B(t)$ are two arbitrary functions which satisfy $A(0) = 1$ and $B(0) = 0$. In the general case $\bar{\Psi}(x)$ is related to $\Psi(x)$ according to

$$\bar{\Psi}(x) = \Psi(x) \int_x^\infty \frac{dx'}{\Psi^2(x')} \quad (64)$$

whereupon one can rewrite (63) as

$$\Psi(x) \rightarrow \Psi'(x, t) = A(t)\Psi(x) + B(t)\Psi(x) \int_x^\infty \frac{dx'}{\Psi^2(x')} \quad (65)$$

The deformations (63) also do not change the Gaussian curvature but their action on the profile of the surfaces is stronger than that of the transformations (62).

For the standard sphere (figure 1) the linearly independent solution is $\bar{\Psi}(x) = \frac{1}{k} \sin (kx)$ so (63) reads explicitly

$$\Psi'(x, t) = \frac{A(t)}{k} \cos(kx) + \frac{B(t)}{k} \sin (kx) \quad (66)$$

or

$$\Psi'(x, t) = \frac{\sqrt{A^2(t) + B^2(t)}}{k} \cos[kx - \varphi(t)] \quad (67)$$

where

$$\varphi(t) = \arccos \frac{A(t)}{\sqrt{A^2(t) + B^2(t)}} \quad (68)$$

Thus the deformation of the sphere is given by a motion along $x$ with the velocity $\varphi(t)$ and a change of shape to that type shown in figure 2 as long as $|\Psi_x(x, t)| < 1$ for all $t$. The situation is similar for other spherical surfaces.
For the pseudospherical surfaces of figure 4 and 5 the transformation (65)
is of the form
\[
\Psi'(x, t) = A(t) \cosh(\lambda x) + B(t) \sinh(\lambda x).
\] (69)

Basically, this results again in shifting and rescaling the surface. The only
significant change of the shape concerns the distance between the two edges.

For the pseudosphere the situation is quite different. In this case the
linear independent solution of the Schrödinger equation is \(\tilde{\Psi}(x) = \frac{1}{\lambda} \exp(\lambda x)\)
leading to a deformation of the form
\[
\Psi'(x, t) = A(t) \frac{1}{\lambda} \exp(-\lambda x) + B(t) \frac{1}{\lambda} \exp(\lambda x).
\] (70)

Since now \(H_x\) is growing not only at \(x < 0\) but also for \(x > 0\) there are for
any given \(B(t)\) at any time \(t \neq 0\) two cuspidal edges on the deformed surface.
So the deformation (70) creates immediately after the initial moment \(t = 0\)
an additional edge on the pseudosphere which thereby takes a shape similar
to those of figures 4 and 5 depending on the explicit form of the functions
\(A(t)\) and \(B(t)\).

A similar behavior can be observed for surfaces associated with bound
states except the infinite well potential. For bound states the wavefunctions
are decreasing at \(|x| \to \infty\). This, however, implies that the linearly independent
solution is increasing at infinity. Hence the function \(\Psi(x, t)\) defined by (65)
always has an additional edge for any \(t \neq 0\) for any function \(B(t)\). Therefore,
regular surfaces connected to bound states are unstable with respect to
deformations of the form (65).

A very particular class of deformations (65) can be obtained if we choose
the following form for the functions \(A(t)\) and \(B(t)\)
\[
A = \frac{1}{\sqrt{-2T_t(t)}}, \quad B = \frac{T(t)}{\sqrt{-2T_t(t)}}
\] (71)

where \(T(t)\) is an arbitrary function and \(T_t(t)\) is the derivative of \(T(t)\) with
respect to \(t\). This choice leads to an evolution for the Schrödinger equation
(22) recently discussed in [22] which does not change the potential. In terms
of the variable
\[
q = -2 \ln \Psi
\] (72)
such an evolution in time is governed by the Liouville equation

\[ q_{xt} = e^q. \]  \hspace{1cm} (73)

The well-known general solution of the Liouville equation

\[ q(x,t) = \ln \frac{2T_t(t) S_x(x)}{[T(t) + S(x)]^2} \]  \hspace{1cm} (74)

implies

\[ \Psi(x,t) = \frac{T(t) + S(x)}{\sqrt{2T_t(t) S_x(x)}} \]  \hspace{1cm} (75)

where \( T(t) \) and \( S(x) \) are arbitrary functions. Demanding at \( t = 0 \) that \( T(0) = 0 \) and \( T_t(0) = -1/2 \) one has

\[ \Psi(x,0) = \frac{S(x)}{\sqrt{-S_x(x)}} \]  \hspace{1cm} (76)

which after integration leads to

\[ S(x) = \left( \int^x \frac{dx'}{\Psi^2(x',0)} \right)^{-1}. \]  \hspace{1cm} (77)

Substituting this expression into (73) one gets

\[ \Psi(x,t) = \frac{\Psi(x,0)}{\sqrt{-2T_t(t)}} + \frac{T(t)}{\sqrt{-2T_t(t)}} \Psi(x,0) \int^x \frac{dx'}{\Psi^2(x',0)}, \]  \hspace{1cm} (78)

i.e. nothing but (65) with \( A \) and \( B \) given by (71). Thus the Liouville type deformations have properties similar to those discussed above. Figure 25 illustrates the situation for the ground state of the 1-soliton potential.

## 10 Deformations via the KdV Equation

A completely different class of deformations of surfaces of revolution is given by the KdV equation which governs the isospectral deformations of the Schrödinger equation (22). It reads (see e.g. [8] - [11])
while the wavefunction $\Psi$ evolves according to the linear equation

$$\Psi_t + 4\Psi_{xxx} - 6u\Psi_x - 3u_x\Psi = 0.$$  \hfill (80)

Eliminating $u$ from (80) with the use of (22) one gets the equation for $\Psi$ only, namely

$$\Psi_t + \Psi_{xxx} - 6E\Psi_x - 3\frac{\Psi_x\Psi_{xx}}{\Psi} = 0.$$  \hfill (81)

The KdV equation is integrable by the IST method. Using this method one can analyze this equation in detail (see e.g. [8] - [10]). In particular, one reduces the solution of the nonlinear initial value problem $u(x, 0) \to u(x, t)$ to a set of linear problems. For a wide class of initial data it is possible to calculate exact asymptotics at $t \to \infty$. The IST method provides us with explicit formulas for the multi-soliton solutions. The KdV equation has a number of remarkable properties: infinite symmetry group, Darboux and Bäcklund transformations and an infinite set of integrals of motion given by

$$Q_n = 2\pi \int_{-\infty}^{\infty} dx C_{2n+1}(x),$$  \hfill (82)

where the densities $C_n$ are calculated via the recurrent relations

$$C_{n+1} = C_n + \sum_{k=1}^{n-1} C_k C_{m-k} \ , \ n = 1, 2, ...$$  \hfill (83)

$$C_1 = -u(x, t).$$  \hfill (84)

All $C_{2n}$ are total derivatives. So the nontrivial integrals are given by (82).

The KdV deformations of surfaces of revolution are governed by the equations (79) - (81) with the substitution $K = E - u$, i.e. the Gaussian curvature evolves according to the equation

$$K_t - 6EK_x + K_{xxx} + 6KK_x = 0$$  \hfill (85)

and the metric depends on time via...
\[ H_t - 6EH_x + 4H_{xxx} + 6KH_x + 3K_xH = 0. \] (86)

The KdV deformations of surfaces inherit all remarkable properties of the KdV equation. First, one is able to linearize the initial value problem \( \{K(x,0), H(x,0)\} \rightarrow \{K(x,t), H(x,t)\} \) for the deformations of the surfaces. Moreover, it is possible to find exact asymptotic expressions for the Gaussian curvature and the metric as \( t \to \infty \) and, finally, an infinite set of deformations of surfaces is given in terms of explicit formulas, namely

\[ K(x) = \lambda^2 + 2(\ln \det D)_{xx} \] (87)

\[ H(x) = \text{Re} \left[ A \exp \left(-i\lambda x\right) \left\{ 1 + \sum_{k=1}^{N} \frac{\det D^{(k)} \exp (-\lambda_k x)}{\det D \lambda_k + i\lambda} \right\} \right], \] (88)

where \( E = \lambda^2 \) and the matrix elements of the \( N \times N \) matrix \( D \) read

\[ D_{kl} = \delta_{kl} + \frac{\exp \left(-[\lambda_k + \lambda_l] x + 8\lambda_k^3 t + \gamma_k\right)}{\lambda_k + \lambda_l}. \] (89)

The matrix elements of the matrices \( D^{(k)} \) are given by (89) with the substitution of the last column by the column \( -\exp (-\lambda_k x + 8\lambda_k^3 t + \gamma_k) \) for \( k = 1, \ldots, N \). The parameters \( \lambda_k \) are real constants and \( \gamma_k \) are arbitrary phases. \( A \) is an arbitrary amplitude. According to section 8 where we discussed soliton surfaces we can now investigate their evolution under KdV deformation. To this end, we will restrict ourselves to three examples of regular surfaces, i.e. to an appropriate choice of \( A \).

1) The surface generated by the bound state of the 1-soliton potential evolves in time according to the following Gaussian curvature and metric

\[ K(x) = -\lambda_1^2 + \frac{2\lambda_1^2}{\cosh^2 (\lambda_1 x - 4\lambda_1^3 t + \gamma_1)} \] (90)

\[ H(x) = \frac{A}{\cosh (\lambda_1 x - 4\lambda_1^3 t + \gamma_1)}. \] (91)
The deformation (90) and (91) is nothing but the uniform motion of the 1-soliton surface along the $Z$-axis with the constant velocity $4\lambda_1^3$ without a change of motion (cf. figure 19).

2) In the case of two solitons we can write the potential in the form ($\lambda_1 < \lambda_2$)

$$u(x, t) = \frac{W_1}{W_2}$$ (92)

with

$$W_1 = 2(\lambda_2^2 - \lambda_1^2)[\lambda_2^2 \cosh^2 \xi_1 (\sinh^2 \xi_2 - \cosh^2 \xi_2)$$
$$+ \lambda_1^2 \sinh^2 \xi_2 (\sinh^2 \xi_1 - \cosh^2 \xi_1)]$$ (93)

$$W_2 = [\lambda_2 \cosh \xi_1 \cosh \xi_2 - \lambda_1 \sinh \xi_1 \sinh \xi_2]^2$$ (94)

and

$$\xi_{1,2} = \lambda_{1,2} x - 4\lambda_{1,2}^3 t + \gamma_{1,2}.$$ (95)

Thus for the ground state the Gaussian curvature and the metric take the form

$$K(x) = -\lambda_2^2 - u(x, t)$$ (96)

$$H(x) = A \frac{\lambda_2(\lambda_2^2 - \lambda_1^2) \cosh \xi_1}{\lambda_2 \cosh \xi_1 \cosh \xi_2 - \lambda_1 \sinh \xi_1 \sinh \xi_2}.$$ (97)

while the surface corresponding to first excited state is described by

$$K(x) = -\lambda_1^2 - u(x, t)$$ (98)

$$H(x) = A \frac{\lambda_1(\lambda_1^2 - \lambda_2^2) \sin \xi_2}{\lambda_2 \cosh \xi_1 \cosh \xi_2 - \lambda_1 \sinh \xi_1 \sinh \xi_2}.$$ (99)
For the ground state we see one bubble moving along the \( Z \)-axis which in contrast to the 1-soliton case slightly changes its shape. The first excited state, however, mimics the 2-soliton interaction of the KdV equation. Figures 26 and 27 illustrate the deformation of the two surfaces of revolution.

For the KdV deformations the Gaussian curvature evolves in time but there is an infinite set of integral characteristic of the surfaces which are preserved. They are of the form (82) with the substitution \( u \rightarrow E - K \). For \( N \)-soliton deformations these integral characteristics are (see e.g. [8])

\[
Q_n = 2\pi \frac{2^n}{2n - 1} \sum_{k=1}^{N} \lambda_k^{2n-1}, \quad n = 1, 2, 3, \ldots \tag{100}
\]

It is known that any initial data which obey the constraint (see [8]-[10])

\[
\int_{-\infty}^{\infty} dx \left( 1 + |x| \right) |u(x)| < \infty \tag{101}
\]

decompose asymptotically \( t \to \infty \) under the KdV evolution into pure solitons. In terms of surfaces this means that asymptotically we end up with soliton surfaces. To obtain a regular surface for all times the amplitude \( A \) has to be fixed in an appropriate way, but since during the evolution the wavefunction is bounded it is always possible to adjust \( A \) such that \( |H_x| < 1 \) for all \( t \). Moreover, the KdV deformations tend to smooth out the initial surfaces. Indeed, let us take the \( \delta \)-function as the initial data, i.e. \( u(x,0) = -2k \delta(x) \). The corresponding pseudosphere has an edge at \( x = 0 \). Under the KdV evolution this initial data converts asymptotically into one soliton (see e.g. [10]). Thus, the KdV deformation converts the pseudosphere figure 11 into the soliton bubble, figure 19, as \( t \to \infty \). In contrast, starting from the regular 1-soliton surface the deformations discussed in section 9, namely (62) and (65) create at finite time or immediately singularities, i.e. edges. So, there is a crucial difference between the KdV deformations and those which preserve the Gaussian curvature. The former tend to smooth out the surfaces while the latter create singularities.
11 Acknowledgment

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Figure Captions

Figure 1: Free motion, $\Psi(x) = A \cos(kx)$; $A = 2$, $k = 0.5$

Figure 2: Free motion, $\Psi(x) = A \cos(kx)$; $A = 1.8$, $k = 0.5$

Figure 3: Free motion, $\Psi(x) = A \cos(kx)$; $A = 2.2$, $k = 0.5$

Figure 4: Free motion, $\Psi(x) = A \cosh(\lambda x)$; $A = 0.5$, $\lambda = 1$

Figure 5: Free motion, $\Psi(x) = A \sinh(\lambda x)$; $A = 0.5$, $\lambda = 1$

Figure 6: Free motion, $\Psi(x) = A \exp(-\lambda x)$; $A = 1$, $\lambda = 1$

Figure 7: Infinite well potential, $\Psi(x) = A \sin\left(\frac{nx}{a}\right)$; $a = 1$, $n = 1, \ldots, 4$, $A = \frac{a}{n}$

Figure 8: Infinite well potential, $\Psi(x) = A \sin\left(\frac{nx}{a}\right)$; $a = 1$, $n = 1, \ldots, 4$, $A = \frac{5a}{6n}$

Figure 9: Infinite well potential, $\Psi(x) = A \sin\left(\frac{nx}{a}\right)$; $a = 1$, $n = 1, \ldots, 4$, $A = \frac{6a}{9n}$

Figure 10: $\delta$-function potential, $\Psi(x) = A \exp(-k|x|)$; $A = 0.5$, $k = 1$

Figure 11: $\delta$-function potential, $\Psi(x) = A \exp(-k|x|)$; $A = 1$, $k = 1$

Figure 12: Harmonic oscillator potential, $\Psi(x) = A \exp\left(-\frac{x^2}{2}\right)$; $A = 1$

Figure 13: Harmonic oscillator potential, $\Psi(x) = A x \exp\left(-\frac{x^2}{2}\right)$; $A = 0.8$

Figure 14: Harmonic oscillator potential, $\Psi(x) = A (4x^2 - 2) \exp\left(-\frac{x^2}{2}\right)$; $A = 0.2$

Figure 15: Harmonic oscillator potential, $\Psi(x) = A \exp\left(-\frac{x^2}{2}\right)$; $A = e^{1/2}$

Figure 16: Effective Coulomb potential, $\Psi(x) = A x \exp(-x)$; $A = 0.9$

Figure 17: Effective Coulomb potential, $\Psi(x) = A x \left(1 - \frac{x}{2}\right) \exp\left(-\frac{x}{2}\right)$; $A = 0.5$
Figure 18: Effective Coulomb potential, $\Psi(x) = Ax^2 \exp\left(-\frac{x^2}{2}\right)$; $A = 0.16$

Figure 19: 1-soliton potential, ground state,

$$\Psi(x) = \frac{A}{\cosh(\lambda_1 x)}; \quad A = 1, \lambda_1 = 1$$

Figure 20: 1-soliton potential, zero energy,

$$\Psi(x) = A \tanh(\lambda_1 x); \quad A = 0.5, \lambda_1 = 1$$

Figure 21: 1-soliton potential, positive energy,

$$\Psi(x) = A [\lambda \sin(\lambda x) + \lambda_1 \cos(\lambda x) \tanh(\lambda_1 x)]; \quad A = 0.4, \lambda_1 = 1, \lambda = 1$$

Figure 22: 2-soliton potential, ground state,

$$\Psi(x) = A \frac{\lambda_2 \left(\lambda_2^2 - \lambda_1^2\right) \cosh \xi_1}{\lambda_2 \cosh \xi_1 \cosh \xi_2 - \lambda_1 \sinh \xi_1 \sinh \xi_2},$$

$$\xi_k = \lambda_k x - 4\lambda^2 t + \gamma_k; \quad A = 1, \lambda_1 = 0.5, \lambda_2 = 1, \gamma_1 = 1, \gamma_2 = -1, t = -2$$

Figure 23: 2-soliton potential, excited state,

$$\Psi(x) = A \frac{\lambda_1 \left(\lambda_1^2 - \lambda_2^2\right) \sinh \xi_2}{\lambda_2 \cosh \xi_1 \cosh \xi_2 - \lambda_1 \sinh \xi_1 \sinh \xi_2},$$

$$\xi_k = \lambda_k x - 4\lambda^2 t + \gamma_k; \quad A = 1, \lambda_1 = 0.5, \lambda_2 = 1, \gamma_1 = 1, \gamma_2 = -1, t = -2$$

Figure 24: 1-soliton potential, deformation (62) with $g(t) = 1 + t$;

$$\lambda_1 = 1, t = 0, 1, 1.1, 1.5$$

Figure 25: 1-soliton potential, deformation (78) with $T(t) = -\frac{t}{2}$;

$$\lambda_1 = 1, t = 0, 0.1, 0.5, 1$$

Figure 26: 2-soliton potential, ground state, KdV deformation;

$$t = -1.5, -1, -0.5, 0$$

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Figure 27: 2-soliton potential, excited state, KdV deformation; 
\[ t = -1.5, -1, -0.5, 0 \]
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