The Schrödinger equation for general non-hermitian quantum system

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We derive a new time-dependent Schrödinger equation (TDSE) for quantum models with non-hermitian Hamiltonian. Within our theory, the TDSE is symmetric in the two Hilbert spaces spanned by the left and the right eigenstates, respectively. The physical quantities are also identical in these two spaces. Based on this TDSE, we show that exchanging two quasi-particles in a non-hermitian model can generate arbitrary geometric phase. The system can also violate the Lieb-Robinson bound in non-relativistic quantum mechanics so that an action in one place will immediately cause a change in the distance. We show that the above two surprising behaviors can also appear in anyonic model, which makes us propose that the non-hermitian single particle model may possess many common features with anyonic model.

In quantum mechanics, the time-dependent Schrödinger equation (TDSE):

\[ i\hbar \partial_t \Psi = \hat{H} \Psi, \]

is crucially important in determining the evolution of the wave-function \( \Psi \). But how will this equation be changed when the Hamiltonian \( \hat{H} \) becomes non-hermitian (NH) and explicitly time-dependent, is still under debates [1–12].

Although the discussions on the possibility of NH Hamiltonian started more than half a century ago [13], this vital gap has not been filled. The situation does not get better even when a specific kind of NH systems, \( PT \)-symmetric systems, has been systematically studied both theoretically [14–32] and experimentally [33–49]. The obstacle lies in the fact that NH Hamiltonian \( \hat{H} \) has the right eigenstates \( |n) \) and the left eigenstates \( \langle n| \):

\[ \hat{H}|n) = E_n|n), \]

\[ \langle n|\hat{H} = E_n\langle n|, \]

which are not conjugate to each other: \( \langle n| \neq (|n) \rangle \). So a NH Hamiltonian induces multiple mappings into two Hilbert spaces spanned by \( |n) \) and \( \langle n| \), respectively [4, 50]. Here \( |n) \) and \( \langle n| \) are the hermitian conjugate of \( |n) \) and \( \langle n| \). In either Hilbert space, the inner product of wave functions, \( \langle \Psi|W|\Phi) \) and \( \langle \Psi|W'|\Phi) \), where \( W \) and \( W' \) are called the metric operators in the Hilbert spaces [15, 16, 50, 51], is also distinct from \( \langle \Psi|\Phi) \) in the hermitian case. As the constraints on the metric are only positive-definite and hermitian, different TDSEs are formulated by choosing different metrics. For instance, Faria chose a time-independent metric [1], Fring used the metric obeying his equation Eq. (2.6) in Ref. [11] while Gong chose a metric defined by the eigenstates [9]. Besides that, physicists lack a suitable system to examine these equations. Although the \( PT \) models have been realized in experiments, they cannot be used to inspire the problem for the two reasons. One is that the Hamiltonians in these systems are time-independent. The other is more essential: the \( PT \) systems are mostly governed by the predefined fundamental rules instead of the NH Hamiltonian, e.g., Maxwell equations in optics [34] or Newton’s law in classical acoustic systems [47]. So the requirements on the quantum NH TDSE, such as the evolution should be unitary, may not subject to the experimental systems at all.

In this article, by assuming that the damping rates, which are associated with the imaginary parts of the eigenvalues of the Hamiltonian, are physically meaningful, we propose that the TDSE for a general NH system should read as

\[ i\hbar \partial_t \Psi = \left( \frac{\bar{W}^{-1}\hat{H}^\dagger \bar{W} + \hat{H}}{2} - \frac{i\hbar}{2} \bar{W}^{-1}\bar{W} \right)\Psi, \]

where the instantaneous metric \( \bar{W} \) is defined by the eigenstates of the Hamiltonian and \( \bar{W} \) refers to its time derivative. The Born’s representation on the wave-function, the probability of finding the quasi-particle at the state \( i \), is redefined by a metric connection \( W \) that connecting the instantaneous metrics \( \bar{W} \) from initial time \( t = 0 \). We will show that, due to the extra \( i\hbar/2 \bar{W}^{-1}\bar{W} \) term, a quench in NH quantum system is significantly different from that in the hermitian case. By solving the differential equation, the wave-function infinitely after the quench is distinct.
from the one infinitely before the quench. This effect is “an action in distance” that violates the Lieb-Robinson bound (LRB). Although a similar result has been given in Ref. [12], we want to emphasize that the starting point of the discussions are totally different: one is from the traditional TDSE in Eq. (1) and the other is from the new TDSE in Eq. (4). We also find that, by exchanging two quasi-particles adiabatically in such a NH system, the accumulated geometric phase can be different from π, a phenomenon similar to exchanging anyons [55, 57]. As anyonic system may also possess a similar “an action in distance” during quench, we reveal that a NH quantum system without many-body interaction can share many features with anyonic system.

The time-dependent Schrödinger equation for nonhermitian system.— In hermitian quantum system, the Hamiltonian $\hat{H}$ plays dual roles. On the one side, it determines the evolution of wave function through the TDSE in Eq. (1). On the other side, the eigenvalues of the Hamiltonian are physical meaningful: they are the energy levels of the system. But in NH quantum system, it is commonly accepted that such duality is destroyed [4, 8, 11]. So there are two options: one is by retaining the TDSE in Eq. (1) while abandoning the energy representation of the Hamiltonian and the other is vice versa.

Several authors have adopted the latter option. For instance, by normalizing the total probability to unity, Wieser wrote down a new TDSE [58].

$$i\hbar \partial_t |\Psi(t)\rangle = (\hat{H} + (\Psi(t))(\frac{\hat{H}^\dagger - \hat{H}}{2})|\Psi(t)\rangle)|\Psi\rangle. \quad (5)$$

He had successfully derived the semi-classical Landau-Lifshitz equation from this fully quantum equation. Although we do not agree with him on this equation in general because the metric has not been considered entirely, Wieser’s work implies that the imaginary parts of the eigenvalues of the Hamiltonian, $\Im(E_n)$, are characterizing the damping rates of the corresponding eigenstates. This makes us adopt the same option by keeping the eigenvalues of the Hamiltonian as physically meaningful and modifying TDSE. Besides that, his work also indicates that the overall evolution is nonlinear in the presence of damping. This makes us think how to unify the TDSE with the superposition principle in the NH quantum mechanics.

Gong, in Ref. [9], proposed another TDSE reading as

$$i\hbar \partial_t |\Psi\rangle = (\hat{H} - i\hbar W^{-1}\hat{W})|\Psi\rangle. \quad (6)$$

The effect of the metric has been considered, but the equation can only subject to a special kind of NH models with conserved PT symmetry, $\Im(E_n) = 0$. Inspired by this work, we build up the TDSE in Eq. (4) that can be applied to any NH system. And our equation can be retrieved to Eq. (4) in PT-symmetric cases and to Eq. (5) when the metric is a unit matrix.

As there is biorthonormal relation for the left eigenstates $\langle \langle n |$ and the right eigenstates $|n\rangle \rangle$ [16, 58], $\langle \langle n |m \rangle \rangle = \delta_{nn}$, after proper normalization, we can define the instantaneous metric as

$$\hat{W} = \sum_n \hat{W}_n, \quad \hat{W}_n = |n\rangle \langle n|. \quad (7)$$

Here we have omitted the time-dependent notation for brevity. Each individual $\hat{W}_n$ can help to determine the components of the wave function $|\Psi\rangle$ in the Hilbert space spanned by $|n\rangle, |c_n(t)|^2 \propto \langle \Psi| \hat{W}_n |\Psi\rangle$. The damping (with $\Im(E_n) < 0$) and inflation (with $\Im(E_n) > 0$) of the basis in this space can be absorbed by the definition of a metric connection $\hat{W}$. This is our first ansatz: if the wave-function $|\Psi\rangle$ is found, we can determine its components in each eigenstate by $|c_n(t)|^2 = \frac{1}{2\langle\Psi|\hat{W}_n|\Psi\rangle}$, where $W_n(t) = \hat{W}_n(t)e^{\int_{t_0}^{t} \Im(E_n(t'))dt'}$, $t_0$ is the initial time and the normalizing factor $A(t)$ is also a function of $|\Psi\rangle$, $A(t) = \sum_n |\langle\Psi|\hat{W}_n|\Psi\rangle|$. The observation of an observable operator $\hat{O}$ is $\langle\Psi|\hat{O}\hat{\Psi}\rangle = \frac{1}{2\langle\Psi|\hat{W}^\dagger\hat{W}|\Psi\rangle}e^{\int_{t_0}^{t} \Im(E_n(t'))dt'}((n(t))$. \quad (8)$$

As the damping has been absorbed in the metric connection, we can impose the unitary condition on the wave function when acting with instantaneous metric $\hat{W}$,

$$\frac{d}{dt}(\langle\Psi|\hat{W}^\dagger\hat{W}|\Psi\rangle) = 0. \quad (9)$$

After that, we employ the second ansatz: the TDSE should reads as $i\hbar \partial_t |\Psi\rangle = (\hat{H} + \Lambda)|\Psi\rangle$, where $\hat{H}$ is the Hamiltonian and $\Lambda$ can exchange with $\hat{W}$ like $\hat{W}A = \alpha \Lambda \hat{W}$. Here $\alpha$ is a number to be determined self-consistently. After substituting the above ansatz to the unitary condition in Eq. (9) one can find $\alpha = -1$ and the TDSE in Eq. (4).

We have a few remarks on the equations. Firstly, they implies that the equations of motion for the wave functions $|\Psi\rangle$ and $|\Psi\rangle \rangle$, that lie in the Hilbert spaces spanned by $|n\rangle$ and $|n\rangle \rangle$ respectively, are symmetric. From $|\Psi\rangle \rangle = \hat{W}|\Psi\rangle$, we can find

$$i\hbar \partial_t |\Psi\rangle = \frac{1}{2} [\hat{W}\hat{H}\hat{W}^{-1} + \hat{H}^\dagger - \frac{i\hbar}{2} \hat{W}d\hat{W}^{-1}/dt]|\Psi\rangle. \quad (10)$$

This equation is symmetric to Eq. (4) by exchanging $\hat{H}$ with $\hat{H}^\dagger$ and $\hat{W}$ with $\hat{W}^{-1}$ (here the metric in the latter Hilbert space has changed to $W^{-1}$). This is sound because, in principle, the two Hilbert spaces spanned by the right eigenstates and the left eigenstates of a matrix, are equally weighted. If one insists on the traditional TDSE like Eq. (1) such symmetry will be destroyed. Secondly, although Eq. (4) seems linear, the overall system is nonlinear because the calculations of physical quantities are depending on the metric connection $W$. One can catch this easily in such a demo model. Let $\hat{W}(t) = 1$ at any time and $\hat{H} = |1\rangle E_1|1\rangle + |2\rangle E_2|2\rangle$. Here the left
eigenstate $\langle n |$ retrieves to the hermitian conjugate of the right eigenstate $| n \rangle$ because $\tilde{W} = 1$. For an initial state $| \Psi \rangle = c_1 | 1 \rangle + c_2 | 2 \rangle$, $| c_1(t) |^2$ will change with time as $\frac{1}{A(t)} |c_1|^2 \exp \int_0^t dt' 2\Im \{E_i(t')\}$ where the normalized factor $A(t) = |c_1(t)|^2 + |c_2(t)|^2$ is dependent on $|c_1(t)|^2$ non-linearly when $\Im \{E_i(t')\}$ is not zero. Such kind of equation is equivalent to Eq. (4) in Wieser’s letter. So we can claim that our equation can be retrieved to Wieser’s one when the instantaneous metric $\tilde{W}(t)$ is a unit matrix all the time. Thirdly, to calculate the observation of an operator $\hat{O}$, our equation relies on $\eta$, which is related with the metric connection $W$ by $W = \eta^\dagger \eta$. But a given $W$ can map to infinite $\eta$s because by unitary transforming $\eta \rightarrow U \eta$, $\eta^\dagger \eta$ is still $W$. We would like to emphasize that such a transformation is associated with an unitary translation in the Hilbert spaces so that the operator $\hat{O}$ should also be changed as $\hat{O} \rightarrow U \hat{O} U^\dagger$. So the observation values are not modified by such transformation. Fourthly, when $\mathcal{PT}$-symmetry is reserved, $W^{-1} H W = H$ and $\Im \{E_i\} = 0$. Our equation will retrieve to Gön’s equation in Eq. (6). Fifthly, the extra $W^{-1} W'$ term can trigger many interesting phenomena. We will only cover two of them in this article. One is that $\tilde{W}$ can be discontinuous when quenching a Hamiltonian. This will introduce a $\delta$-like function on the right-hand side of Eq. (6) because of the presence of $\tilde{W}$. By solving this first-order differential equation, one will find that the wave function is also discontinuous. The other interesting phenomenon is the expression of geometric phase is also different from that in the traditional quantum mechanics.

An action in distance.— In hermitian quantum system, a quench is usually employed by suddenly changing the Hamiltonian from $H_0$ to $H_1$ at time $t = 0$. As the time interval of the quench is absolutely zero, the wave functions before and after the quench, $| \Psi(t = 0^-) \rangle$ and $| \Psi(t = 0^+) \rangle$, are identical. But this is not the case for NH system. As the instantaneous metrics, $\tilde{W}_- = \tilde{W}(t = 0^-)$ and $\tilde{W}_+ = \tilde{W}(t = 0^+)$ are different, one cannot solve the TDSE in Eq. (4) at $t = 0$ without properly regulating the differential equation. We can actually solve this by employing the unitary condition in Eq. (4) directly, which gives

$$\langle \Psi(0^-) | \tilde{W}_- | \Psi(0^-) \rangle = \langle \Psi(0^+) | \tilde{W}_+ | \Psi(0^+) \rangle.$$  

By denoting the evolution as $| \Psi(0^+) \rangle = \tilde{L} | \Psi(0^-) \rangle$, we find that the solution to the above equation is $\tilde{L} = (\sqrt{\tilde{W}_+})^{-1} U \sqrt{\tilde{W}_-}$, where $U$ is a unitary matrix. Due to the anti-symmetric relation $\tilde{W} \Lambda = -\Lambda^\dagger \tilde{W}$, we also have $\tilde{W}_+ L = \tilde{L}^\dagger \tilde{W}_+$. After some algebra, we find $U$ can be determined by

$$U(\sqrt{\tilde{W}_-} \tilde{W}_+^{-1} \sqrt{\tilde{W}_-}) U^\dagger = (\sqrt{\tilde{W}_-})^{-1} \tilde{W}_- (\sqrt{\tilde{W}_+})^{-1}. $$

Here $\sqrt{\tilde{W}_\pm}$ is still hermitian by making the square root of the eigenvalues of $\tilde{W}_\pm$.

When representing such equation in real space, we realize that LRB should be violated. LRB states that one can not measure the signal of change outside a cone around the source of change (at where the quench takes place). The slope of the cone is referring to the maximal speed of the signal in the system. But as $L$ is off-diagonal in general, it is implied that a quench at a source can immediately affect the wave function far away from it. This is actually an effect of “an action in distance”. One should note that LRB in non-relativistic quantum system is not protected by any basic principle so that its violation does not implies the fault of the relativistic principle.

The geometric phase.— As the wave function is evolving according to the new TDSE, the form of the geometric phase is also changed. We suppose that the evolution is so slow that the jumps between the eigenstates are ignored. So the evolution of eigenstates can be written as $| \Psi_\eta(t) \rangle = c_\eta(t) | n(t) \rangle$, where $| n(t) \rangle$ is the instantaneous eigenstate. Beside the dynamical phase, $c_\eta(t)$ also possesses a geometric factor like $\exp(i \gamma \eta)$, where $\gamma \eta$ is the geometric phase. After substituting the above wave function to the TDSE, we get the geometric phase as

$$\dot{c}_\eta = i \langle n | \tilde{W} | \eta \rangle + \frac{1}{2} \langle n | \dot{\tilde{W}} | n \rangle.$$  

This form of geometric phase is the same as that in Ref. 9, because the two TDSEs are only distinguished in the dynamical part. One can also express the geometric phase by the left and the right eigenstates like $\dot{c}_\eta = \frac{i}{2} \langle \langle n | \tilde{W} | \eta \rangle \rangle$. Similarly, when expanding the wave function in the Hilbert space spanned by the left eigenstates $| n \rangle$, we can find that the geometric phase in this space is the same. This makes us raise the third an-tatz which is used to check the correction of equations: In NH system, physical quantities should be identical and the formulas should be symmetric in either Hilbert space spanned by the left or by the right eigenstates.

Now we discuss the geometric phase by exchanging two quasi-particles in a NH system. We start with a $2 \times 2$ Hamiltonian so that there are only two eigenstates denoted by 1 and 2 respectively. The eigenstates are represented by the points in the Bloch sphere, $| 1 \rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix}$ and $| 1 \rangle = \begin{pmatrix} \cos(\theta'/2) \\ \sin(\theta'/2) e^{i\phi'} \end{pmatrix}$, where $A = \cos(\theta/2) \cos(\theta'/2) + \sin(\theta/2) \sin(\theta'/2) e^{i(\phi' - \phi)}$ is the factor to normalize $\langle 1 | 1 \rangle$ and $\theta, \phi$ are the polar angles. One should note that $| 1 \rangle$ and $| 1 \rangle$ can refer to different points in the Bloch sphere in NH case. The eigenstate $| 2 \rangle$, which should be orthogonal to $| 1 \rangle$, is get by replac-"
 Ayrıca in NH model, the eigenstates are orthogonal in the sense of traditional inner product, $|1\rangle \neq \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \\ \end{array} \right)$ and $|2\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -1 \\ \end{array} \right)$ are exchanged adiabatically along the traces, the total wave function $|1\rangle \oplus |2\rangle$ is changed to $|2\rangle \oplus |1\rangle$ and the total geometric phase is $\pi$. This is consistent with the anti-symmetry of the fermions. So we call this model as the poor man’s exchange mode because there are only two quasi-particles in the two levels system.

\[H = \sum_{\delta} \delta \left( a_{\delta} a_{\delta}^\dagger + a_{\delta}^\dagger a_{\delta} \right) + H.C., \]
\[a_{\delta} \text{ and } a_{\delta}^\dagger \text{ are anyonic annihilation and creation operators.} \]

One can calculate the anyon density by mapping the Hamiltonian to a fermionic Hamiltonian through a generalized Jordan-Wigner transformation. The technical details of the calculation can be found elsewhere. Here we suppose a quench start from a uniform chain with $t_l = 1$ and suddenly eliminate the hopping at the center $l = N/2$ with $t_{N/2} = 0$. After quench, the system is separated into unrelated two parts and the density is calculated in these parts individually. Based on the same fermionic ground state in the fermionic representation, we calculate the anyonic density and find that the density away from the source of the quench (the center of the chain) is also changed immediately after the quench. This indicates that the anyonic wave function must suffer a sudden change during the quench, which is similar to that in NH system.

**Conclusions and outlooks.**— We find a TDSE for a general NH system based on several reasonable ansatzes. Within our theory, the physical quantities are identical and the formulas are symmetric in the multiple Hilbert spaces spanned by the left eigenstates and the right eigenstates. As the extra time derivative metric term explicitly appears in the equation, a quench in Hamiltonian can cause a discontinuity for the wave functions. This induces “an action in distance”, which will violate the no-signaling condition from special relativity or LRB from non-relativistic quantum mechanics. It will be interesting to embed the idea of the new TDSE to Dirac equation to examine whether the no-signaling condition is really violated in the relativistic NH quantum mechanics. We also find that the geometric phase of exchanging two quasi-particles is path-dependent and can be distinguished from $\pi$. So if there is a possibility to fix the exchanging traces in a special manner, the quasi-particles in NH system should work like anyons. This makes us propose that the NH single particle system shares many features with the anyonic system, and it may be possible...
to mimic one system with the other in experiments. In our discussion, we have supposed that the Hamiltonian is not defective. It will be interesting to extend the study to the defective case in which the metric is not full rank and its inverse is not well defined.

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