Can the post-Newtonian gravitational waveform of an inspiraling binary be improved
by solving the energy balance equation numerically?

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The detection of gravitational waves from inspiraling compact binaries using matched filtering
depends crucially on the availability of accurate template waveforms. We determine whether the
accuracy of the templates’ phasing can be improved by solving the post-Newtonian energy balance
equation numerically, rather than (as is normally done) analytically within the post-Newtonian
perturbative expansion. By specializing to the limit of a small mass ratio, we find evidence that
there is no gain in accuracy.

I. INTRODUCTION AND SUMMARY

Several kilometer-scale interferometric gravitational
wave detectors are currently being built, among them the
two American LIGO detectors, the French-Italian
VIRGO detector, the German-British GEO 600 detector
and the Japanese TAMA detector. Gravitational waves
from inspiraling compact binaries are among the most
promising candidates to be detected. In order to extract
a gravitational wave signal from the noisy background the
technique of matched filtering \[1,2\] will be used. One of
the drawbacks of matched filtering is that the theoretical
templates used must be close to the actual gravitational
wave signal in order to detect the signal and estimate its
parameters. In the case of a nearly circular inspiral of two
point masses the expected gravitational wave signal has
the form of a chirp, i.e., a roughly sinusoidal signal with
gradually increasing amplitude and frequency. If such a
signal is to be detected by matched filtering, high accu-


## Question: What is the purpose of this paper?

The purpose of this paper is to investigate yet an-
other possible method of improving the accuracy of post-
Newtonian templates. The basic idea is very simple.
In computing post-2-Newtonian templates, for example, one should strictly speaking discard all terms of post-
2.5-Newtonian order (and higher) everywhere in the cal-
culations. To do otherwise would be inconsistent with
the perturbation expansion method. Yet, there could
be pieces of the calculation for which retaining post-2.5-
Newtonian (and higher) order terms would lead to im-
proved accuracy. For example, the dominant, \( m = 2 \)
piece of the waveform is usually written as

\[
h(t) = A(t) \cos \left[ \phi^{GW}(t) \right],
\]

where both the amplitude \( A(t) \) and the phase
\( \phi^{GW}(t) \) have separate post-Newtonian expansions, \( A = \sum_j \varepsilon^j A^{(j)} \) and \( \phi^{GW} = \sum_j \varepsilon^j \phi^{(j)} \), with \( \varepsilon \) a formal expansion parameter. Now a perturbation theory purist would insist on inserting the expansion for \( \phi^{GW} \) into Eq. (1.1)

and on expanding the cosine using a Taylor expansion.
However, it is well known that the resulting expression is
a much poorer representation of the true signal than the original un-expanded form (1.1).

The question then arises: are there other stages in the construction of post-Newtonian templates where one discards higher order terms, which, if retained, might lead to increased accuracy? A natural possibility is the stage in which one goes from the post-Newtonian formulae for the energy flux $F(f) = -dE/dt(f)$ and orbital energy $E(f)$ as functions of gravitational wave frequency $f$, to the formula for the phase $\phi^{GW}(t)$ of the gravitational waveform. Given analytical formulae for $F(f)$ and $E(f)$ up to some post-Newtonian order, one can either (I) solve analytically for $\phi^{GW}(t)$ within the post-Newtonian approximation, discarding all higher order terms, or (II) one can numerically solve the energy balance equation to obtain $\phi^{GW}(t)$. This second procedure effectively generates and retains terms at all post-Newtonian orders, so is strictly speaking inconsistent, but one might hope that it would lead to increased accuracy. We note that the papers [4,5,11] investigating the accuracy of post-Newtonian templates have generally used method (II), whereas the popular data analysis package GRASP [16] uses method (I). The GRASP manual [16] speculates that method (II) might be more accurate.

In this paper we present evidence, based on the limiting case of binaries with small mass ratios, that numerically solving the energy balance equation does not in fact increase the accuracy. We arrive at this conclusion after checking the accuracies of methods (I) and (II) in three ways. We note that the corrections to the stationary phase approximation are very small, arising only at post-5-Newtonian order [18], so it is sufficient for our purposes to use the expression (2.7). In the restricted post-Newtonian approximation, in which we neglect the $m \neq 2$ multipoles, the gravitational waveform has the form $h(t) = A(t) \cos[\phi^{GW}(t)]$, where $A(t)$ is a slowly varying amplitude. The Fourier transform $\tilde{h}(f)$ of this waveform is $\tilde{h}(f) = B(f)e^{i\psi(f)}$, where $B(f)$ is some frequency dependent prefactor, and where, in the stationary phase approximation, the phase $\psi(f)$ is given by

$$\psi(f) = 2\pi ft(v) - 2\phi(v) - \frac{\pi}{4},$$

(2.6)

Using Eqs. (2.4) and (2.5) gives the frequency domain phase [13]:

$$\psi(f) = 2(t_c/m)v^3 - 2\phi_c - \pi/4 - \frac{2}{m}\int_{v_i}^{v_f} d\tilde{v} (\tilde{v}^3 - \tilde{v}^3) \frac{E'(\tilde{v})}{\tilde{F}(\tilde{v})},$$

(2.7)

We note that the corrections to the stationary phase approximation are very small, arising only at post-5-Newtonian order [13], so it is sufficient for our purposes to use the expression (2.7).

Equation (2.7) is the starting point for our analysis. We will investigate the accuracy with which various approximations reproduce the Fourier-domain phase $\psi(f)$, which is the version of the phase function that is most relevant for matched filtering. The two possible calculational methods we consider are (I) to insert post-Newtonian expressions for the functions $E(v)$ and $F(v)$ into Eq. (2.7), and discard all the higher order post-Newtonian terms generated, and (II) to insert post-Newtonian expressions for the functions $E(v)$ and $F(v)$ into Eq. (2.7) and solve exactly for the phase $\psi(f)$, retaining all the higher order post-Newtonian terms generated.

To assess the accuracy of each of these two methods, we specialize to the limit $m_1m_2/m^2 \rightarrow 0$ for which the functions $E(v)$ and $F(v)$ are known [13,14,19]. We then check the accuracy of method (I) and (II) in three ways.

II. METHOD OF CALCULATION

In order to explain our calculation, we first summarize how the waveform’s phasing can be computed from the energy flux function $F(f)$ and orbital energy function $E(f)$, where $f$ is gravitational wave frequency [13,14]. Let $m_1$, $m_2$ be the masses of the two components of the binary and $m = m_1 + m_2$ be the total mass. Let $\phi(t)$ be the orbital phase of the binary, so that $\phi^{GW}(t) = 2\phi(t)$, where $\phi^{GW}$ is the phase of the dominant, $m = 2$ piece of the waveform. We define the dimensionless variable

$$v = (\pi m f)^{1/3},$$

(2.1)

[Here and throughout we use units with $G = c = 1$.] The orbital phase $\phi(t)$ is derived from the relation

$$\frac{d\phi}{dt} = \pi f,$$

(2.2)

and from the energy balance equation

$$\frac{dE(v)}{dt} = -F(v).$$

(2.3)

Equations (2.1) – (2.3) yield a parametric solution for $\phi(t)$ given by

$$\phi(v) = \phi_c - \frac{1}{m}\int_{v_i}^{v_f} d\tilde{v} \tilde{v}^3 \frac{E'(\tilde{v})}{\tilde{F}(\tilde{v})},$$

(2.4)

and

$$t(v) = t_c - \int_{v_i}^{v_f} d\tilde{v} \frac{E'(\tilde{v})}{\tilde{F}(\tilde{v})},$$

(2.5)

where $\phi_c$, $t_c$ and $v_i$ are constants.
A. Checking the accuracy of methods (I) and (II) by comparing expansion coefficients of $\psi(f)$

The first check is entirely analytical. We expand all the phase functions $\psi(f)$ as post-Newtonian power series in $v$ up to some high order (e.g., post-5.5-Newtonian), and compare the accuracy with which methods (I) and (II) reproduce the coefficients in this power series. While this comparison procedure is less accurate than comparing the phases produced by methods (I) and (II) to the exact numerical phase, it does allow us to check whether there is any indication that method (II) is more accurate than method (I).

In more detail, our comparison procedure works as follows. The post-Newtonian expansions for the functions $E(v)$ and $F(v)$ have the general form

$$
E(v) = -\frac{1}{2} \eta m v^2 \left[ 1 + \sum_{i=1}^{\infty} e_i v^{2i} \right],
$$

$$
F(v) = \frac{32}{5} \eta^2 v^{10} \left[ 1 + \sum_{i=2}^{\infty} f_i v^i + \sum_{i=6}^{\infty} g_i \ln(v) v^i + \ldots \right],
$$

where $\eta = m_1 m_2 / m^2$ is the dimensionless mass ratio. The ellipses in Eq. (2.9) represent possible terms proportional to $(\ln v)^m$ for $m \geq 2$ which could arise at high post-Newtonian orders. The coefficients $e_i$, $f_i$, and $g_i$ in Eqs. (2.8) and (2.9) are functions of the mass ratio $\eta$. For general mass ratios, the coefficients $e_i$ and $f_i$ in are known up to $e_4$ and $f_2$ in $[\text{3}]$, while for $\eta = 0$ all the $e_i$ coefficients are known $[\text{11}]$ and the $f_i$ and $g_i$ coefficients are known up to $f_{11}$ and $g_{11}$ $[\text{3}]$. The known coefficients are tabulated in Appendix A.

If we now insert the expansions (2.8) and (2.9) into the formula (2.7) for the phase $\psi(f)$ we obtain

$$
\psi(f) = \frac{3^{v-5}}{128\eta} \left[ P(v) + \frac{256\eta}{3m} v^8 t_K + \frac{128\eta}{3} v^5 K \right].
$$

Here $t_K$ and $K$ are constants which correspond to the initial time and initial phase, and the function $P(v)$ has the expansion

$$
P(v) = 1 + \sum_{j=2}^{\infty} \left\{ p_j + g_j \ln(v) + r_j [\ln(v)]^2 + \ldots \right\} v^j.
$$

For example, the expressions for the first few $p_j$’s are

$$
p_2 = \frac{20(2e_2 - f_2)}{9},
$$

$$
p_3 = -4f_3,
$$

and

$$
p_4 = 10(f_2^2 - 2e_2 f_2 + 3e_4 - f_4).
$$

Suppose now that the functions $E(f)$ and $F(f)$ are known to post-$N$-Newtonian order. Then the coefficients $e_i$, $f_i$ and $g_i$ are known for $0 \leq i \leq 2N$. If we now follow the usual method (I) to generate the phase function $\psi(f)$, we obtain an expansion of the form (2.11) where the coefficients are given by

$$
(N)p_j = \left\{ \begin{array}{ll}
p_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) & j \leq 2N, \\
0 & j > 2N,
\end{array} \right.
$$

and

$$
(N)q_j = \left\{ \begin{array}{ll}
q_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) & j \leq 2N, \\
0 & j > 2N,
\end{array} \right.
$$

and

$$
(N)r_j = \left\{ \begin{array}{ll}
r_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) & j \leq 2N, \\
r_j(e_1, \ldots, e_{2N}, 0, \ldots, 0, f_1, \ldots, f_{2N}, 0, \ldots, 0, g_1, \ldots, g_{2N}, 0, \ldots, 0) & j > 2N.
\end{array} \right.
$$

Here the superscript (I) means method (I) and the subscript $N$ refers to the post-$N$-Newtonian approximation. On the other hand, if we use instead the method (II) to generate $\psi(f)$, we obtain an expansion with expansion coefficients

$$
(I)p_j = \left\{ \begin{array}{ll}
p_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) & j \leq 2N, \\
p_j(e_1, \ldots, e_{2N}, 0, \ldots, 0, f_1, \ldots, f_{2N}, 0, \ldots, 0, g_1, \ldots, g_{2N}, 0, \ldots, 0) & j > 2N,
\end{array} \right.
$$

and

$$
(I)q_j = \left\{ \begin{array}{ll}
q_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) & j \leq 2N, \\
q_j(e_1, \ldots, e_{2N}, 0, \ldots, 0, f_1, \ldots, f_{2N}, 0, \ldots, 0, g_1, \ldots, g_{2N}, 0, \ldots, 0) & j > 2N.
\end{array} \right.
$$

and

$$
(I)r_j = \left\{ \begin{array}{ll}
r_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) & j \leq 2N, \\
r_j(e_1, \ldots, e_{2N}, 0, \ldots, 0, f_1, \ldots, f_{2N}, 0, \ldots, 0, g_1, \ldots, g_{2N}, 0, \ldots, 0) & j > 2N.
\end{array} \right.
$$
Thus, the two methods agree on \( p_j, q_j \) and \( r_j \) for \( j \leq 2N \), but for \( j > 2N \) method (I) gives expansion coefficients of zero while method (II) yields coefficients of the form
\[
p_j(e_1, \ldots, e_{2N}, 0, \ldots, 0, f_1, \ldots, f_{2N}, 0, \ldots, 0, g_1, \ldots, g_{2N}, 0, \ldots, 0)
\]
which differ somewhat from the true values \( p_j(e_1, \ldots, e_j, f_1, \ldots, f_j, g_1, \ldots, g_j) \) because of having the coefficients \( e_i, f_i \) and \( g_i \) set to zero for \( 2N + 1 \leq i \leq j \). We define
\[
p_{j,k} = \frac{(II)}{k/2} p_j,
\]
and similarly for \( q_j \) and \( r_j \), so that \( p_{k,k} = p_k \).

As an example, suppose that the functions \( E(f) \) and \( F(f) \) were known only up to post-1.5-Newtonian order, so that only the coefficients \( e_2, f_2 \) and \( f_3 \) were known, but not \( e_4 \) and \( f_4 \). Up second post-Newtonian order the expansion (2.11) has the form
\[
P(v) = 1 + p_2 v^2 + p_3 v^3 + p_4 v^4 + O(v^5),
\]
where the coefficients \( p_2, p_3, \) and \( p_4 \) are given in Eqs. (2.15) - (2.17) above. How accurately could we determine the coefficients \( p_2, p_3, \) and \( p_4 \) in this case? Obviously we could compute \( p_2 \) and \( p_3 \) exactly since they do not depend on \( e_4 \) and \( f_4 \). However, the coefficient \( p_4 \) does depend on \( e_4 \) and \( f_4 \), and can be written as \([\text{cf. Eq. (2.17) above}]\)
\[
p_4 = p_{4,3} + \Delta p_{4,3}.
\]
Here
\[
p_{4,3} = \frac{(II)}{1,5} p_4 = 10 (f_2^2 - 2e_2 f_2),
\]
is the piece of \( p_4 \) that can be computed from the post-1.5-Newtonian pieces of \( E(f) \) and \( F(f) \); it is thus nonlinear in the coefficients \( e_2 \) and \( f_2 \). The value (2.27) is the prediction of method (II), while the method (I) gives instead the value \( \frac{(II)}{1,5} p_4 = 0 \). The error term in Eq. (2.26) is given by
\[
\Delta p_{4,3} = 10 (3e_4 - f_4)
\]
and is linear in the post-2-Newtonian coefficients \( e_4 \) and \( f_4 \). Using the values of \( e_2, f_2, e_4 \) and \( f_4 \) listed in Appendix A we find that \( \Delta p_{4,3}/p_4 \approx -1.73 \) for \( \eta = 0 \), which is rather large. Hence in this particular example we do not improve the accuracy in the coefficient \( p_4 \) by using method (II) rather than method (I).

In general, the question we want to address is whether the approximate coefficient \( \frac{(II)}{N} p_j = p_{j,2N} \) is typically significantly closer to the true coefficient \( p_j \) than zero is to \( p_j \), for \( j > 2N \), i.e., whether
\[
\frac{|p_{j,2N} - p_j|}{p_j} < \zeta \quad \text{(a few tens of percent)}
\]
and similarly for \( q_j \) and \( r_j \). In Tables I, II and III below we list the values of the true coefficients \( p_j, q_j \) and \( r_j \) and also the approximate coefficients \( p_{j,k}, q_{j,k} \) and \( r_{j,k} \) for various values of \( k \), computed from the values given in Appendix A using Eqs. (2.8), (2.9) and (2.7). We list the analytic expressions for these approximate coefficients in Appendix B. Examination of Tables I, II and III shows that there is no tendency for the inequality (2.29) to be satisfied.

Therefore method (II) does not seem to lead to a gain in accuracy when compared to method (I) in the test mass case \( (\eta \to 0) \).
TABLE I. The coefficients $p_{j,k}^{(II)}$ as calculated according to method (II). These coefficients are what one obtains if the orbital energy $E(f)$ and gravitational wave luminosity $F(f)$ as functions of frequency $f$ are known only up to post-k/2-Newtonian order. Note that the values of $p_{j,k}$ differ significantly from their true values $p_j = p_{j,j}$ for $k < j$.

| method (II) | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ | $k=9$ | $k=10$ | $k=11$ | true values |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------------|
| $p_{3,k} \times 10^{-3}$ | 0 | -0.0503 | | | | | | | | $p_3 \times 10^{-3}$ | -0.0503 |
| $p_{4,k} \times 10^{-3}$ | 0.0821 | 0.0821 | 0.0301 | | | | | | | $p_4 \times 10^{-3}$ | 0.0301 |
| $p_{5,k} \times 10^{-3}$ | 0 | 0.331 | 0.331 | 0.161 | | | | | | $p_5 \times 10^{-3}$ | 0.161 |
| $p_{6,k} \times 10^{-3}$ | -0.609 | -3.77 | -3.60 | -3.60 | -0.441 | | | | | $p_6 \times 10^{-3}$ | -0.441 |
| $p_{7,k} \times 10^{-3}$ | 0 | 7.59 | 7.52 | 2.98 | 2.98 | 0.954 | | | | $p_7 \times 10^{-3}$ | 0.954 |
| $p_{8,k} \times 10^{-3}$ | -0.502 | -7.26 | -7.19 | -2.91 | 0.828 | 0.828 | 0.995 | | | $p_8 \times 10^{-3}$ | 0.995 |
| $p_{9,k} \times 10^{-3}$ | 1.68 | 43.3 | 46.5 | 15.2 | -1.76 | -12.0 | -11.5 | -8.77 | | $p_9 \times 10^{-3}$ | 4.43 |
| $p_{10,k} \times 10^{-3}$ | 0 | -76.8 | -82.5 | -28.4 | 5.61 | 11.6 | 11.6 | 4.43 | | $p_{10} \times 10^{-3}$ | 12.3 |

TABLE II. The coefficients $q_{j,k}$; see caption of Table I.

| method (II) | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ | $k=9$ | $k=10$ | $k=11$ | true values |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------------|
| $q_{3,k} \times 10^{-3}$ | 0 | 0 | | | | | | | | $q_3 \times 10^{-3}$ | 0 |
| $q_{4,k} \times 10^{-3}$ | 0 | 0 | 0 | | | | | | | $q_4 \times 10^{-3}$ | 0 |
| $q_{5,k} \times 10^{-3}$ | 0 | 0.992 | 0.992 | 0.482 | | | | | | $q_5 \times 10^{-3}$ | 0.482 |
| $q_{6,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | -0.326 | | | | | $q_6 \times 10^{-3}$ | -0.326 |
| $q_{7,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 0 | | | | $q_7 \times 10^{-3}$ | 0 |
| $q_{8,k} \times 10^{-3}$ | 1.51 | 21.8 | 21.6 | 8.74 | -2.91 | -2.91 | -3.18 | | | $q_8 \times 10^{-3}$ | -3.18 |
| $q_{9,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | -4.10 | -4.10 | -4.10 | -2.05 | | $q_9 \times 10^{-3}$ | -2.05 |
| $q_{10,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | 1.95 | 1.95 | 0.702 | 0.702 | 0.235 | $q_{10} \times 10^{-3}$ | 0.235 |
| $q_{11,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | -6.00 | -6.00 | -3.05 | -0.356 | -0.356 | -1.41 | $q_{11} \times 10^{-3}$ | -1.41 |

TABLE III. The coefficients $r_{j,k}$; see caption of Table I. These coefficients vanish for $j \leq 7$.

| method (II) | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ | $k=9$ | $k=10$ | $k=11$ | true values |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------------|
| $r_{8,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | 1.29 | 1.29 | 0.584 | | | | $r_8 \times 10^{-3}$ | 0.584 |
| $r_{9,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | $r_9 \times 10^{-3}$ | 0 |
| $r_{10,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | $r_{10} \times 10^{-3}$ | 0 |
| $r_{11,k} \times 10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $r_{11} \times 10^{-3}$ | 0 |
B. Checking the accuracy of methods (I) and (II) numerically

Next we perform a direct numerical check by comparing the phases produced by methods (I) and (II) to the exact numerical phase. Note that the phase $\psi(f)$ in Eq. (2.7) is not directly observable, since it contains the unknown integration constants $\phi_c$ and $t_c$, i.e. $\psi(f)$ is determined only up to a linear function in $f$. Hence the relevant quantity to consider is

$$\psi''(f) = \frac{d^2}{df^2} \psi(f) = -\frac{2\pi^2 m E'(v)}{3v^2 F(v)}. \quad (2.30)$$

In the test mass case $E'(v)$ and $F(v)$ are known exactly [3], and we can therefore find the exact $\psi''(f)$.

Now suppose $E'(v)$ and $F(v)$ are only known up to post-$k/2$-Newtonian order. If method (I) is used $\psi''(f)$ is found to be

$$\psi''(f) = -\frac{2\pi^2 m [E'(v)]_k}{3v^2 [F(v)]_k}. \quad (2.31)$$

On the other hand method (II) yields

$$\psi''(f) = -\frac{2\pi^2 m E'(v)}{3v^2 F(v)} \quad (2.32)$$

Here $[...]_k$ denotes the powerseries of the expression inside the brackets with terms kept up to order $v^k$.

It is convenient to define the logarithmic relative errors

$$\ln \left| \frac{\psi''(f)}{\psi''(f)} \right| = \ln \left| \frac{[E'(v)]_k F(v) - E'(v) F'(v)}{E'(v) F(v)} \right| \quad (2.33)$$

and

$$\ln \left| \frac{\psi''(f)}{\psi''(f)} \right| = \ln \left| \frac{[E'(v)]_k F(v) - E'(v) F'(v)}{E'(v) F(v)} \right| \quad (2.34)$$

These errors are shown in Figures 1 - 3. It can be seen that there is no systematic tendency for method (II) to perform better than method (I). At post-2.5 and post-4-Newtonian order method (I) does better than method (II) for all $v$, while at post-3 and post-3.5-Newtonian order method (II) is more accurate than method (I). We would expect the same trend to hold for general values of the mass ratio $\eta$.

FIG. 1. The errors in the phase $\psi(f)$ of the Fourier transformed waveform produced by methods (I) and (II) in the test mass limit. Plotted here are the logarithms of the relative errors in the second derivative $\psi''(f)$, for the case when the energy $E(f)$ and gravitational wave luminosity $F(f)$ are known only up to post-2.5-Newtonian order. The horizontal axis is log($\pi m f$)/3 where $m$ is the total mass of the system and $f$ is gravitational wave frequency. The line denoted ISCO indicates the location of the innermost stable circular orbit. It can be seen that method (I) is more accurate for all frequencies $f$.

FIG. 2. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-3-Newtonian order; see caption of Fig. 1. In this case method (II) is overall more accurate.
frequencies of Fig. 1. In this case method (I) is more accurate for all frequencies $f$. 

![Diagram](image1)

FIG. 3. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-3.5-Newtonian order; see caption of Fig. 1. In this case method (II) is more accurate for most frequencies $f$.

![Diagram](image2)

FIG. 4. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-4-Newtonian order; see caption of Fig. 1. In this case method (II) is more accurate for all frequencies $f$.

FIG. 5. The errors in the phase $\psi(f)$ when $E(f)$ and $F(f)$ are known only up to post-4.5-Newtonian order; see caption of Fig. 1. In this case method (II) is more accurate for all frequencies $f$.

C. Overlaps of templates constructed by methods (I) and (II) with the exact signal

So far we have only considered how accurately the phase $\psi(f)$ is generated by methods (I) and (II). In this subsection we use the phases $(I)$ and $(II)$ to construct gravitational wave templates and then compute the templates’ overlap with the exact waveform computed from the exact $\psi(f)$.

We use the restricted post-Newtonian approximation and neglect the $m \neq 2$ multipoles. Thus the Fourier transform $\tilde{h}(f)$ of the exact waveform is given by

$$
\tilde{h}(f) \propto f^{-7/6} e^{i \psi(f)}. \tag{2.35}
$$

Similarly we use methods (I) and (II) to construct the templates

$$
(\text{I}) \frac{k}{2} \tilde{h}(f) \propto f^{-7/6} e^{i \psi(f)} \tag{2.36}
$$

and

$$
(\text{II}) \frac{k}{2} \tilde{h}(f) \propto f^{-7/6} e^{i \psi(f)}. \tag{2.37}
$$

Next we compute Apostolatos’ [21] fitting factor ($FF$) to determine the templates’ accuracy. The fitting factor is the ratio of the signal-to-noise ratio obtained with the imperfect template, to the signal-to-noise ratio that a perfect template would yield. The fitting factor can take values from zero to one, with unity indicating a perfect template. It is obtained from the ambiguity function

$$
A = \frac{(\text{I/II})_{k/2}(h, h)}{\sqrt{(\text{I/II})_{k/2}(h, k/2)_{h}(h, h)}} \tag{2.38}
$$

by maximizing over the template parameters, i.e.

$$
FF = \max_{\phi, t} A. \tag{2.39}
$$

Notice that we hold the masses fixed in the maximization procedure: the templates and signal correspond to binaries of the same two masses. Here we have introduced the inner product

$$
(s, h) = 2 \int_0^{\infty} \tilde{s}(f)^* \tilde{h}(f) + \tilde{s}(f) \tilde{h}(f)^* df, \tag{2.40}
$$

where $S_n(f)$ is the spectral density of the detector noise. The noise curve $S_n(f)$ used here is the Cutler-Flanagan fit [21] for the advanced LIGO. The largest contribution
to the overlaps comes from the frequency band between 40 Hz and 200 Hz.

We compute the fitting factors for several different choices of \( \eta \) in order to get an indication of what might happen for general mass ratios, even though our results apply strictly only to the test mass limit \( \eta \to 0 \).

The resulting fitting factors are listed in Tables IV, V, and VI. Examination shows close agreement with the error plots of \( \psi''(f) \) and \( \psi''(f) \) in Figs. 1 – 5. At post-Newtonian orders where \( (I) \) has an error smaller than the error in \( (II) \) everywhere (e.g. post-2.5), method (I) always wins, and vice versa (e.g. post-4.5). On the other hand, at post-Newtonian orders where the error lines cross (e.g. post-3.5), the method with the smaller error in the \( v \)-region \( v = \left( \frac{\pi m f}{12} \right)^{1/3} \) selected by the sensitive frequency band of the detector and the total mass \( m \) yields a larger fitting factor.

Again we find that there is no systematic tendency for method (II) to be more accurate than method (I). Therefore method (II) does not lead to a gain in accuracy when compared to method (I). Our conclusion applies only to the limit \( \eta \to 0 \), but we do not anticipate a different result for the general case.

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### APPENDIX A: COEFFICIENTS IN EXPANSIONS OF ENERGY AND ENERGY FLUX FUNCTIONS

#### 1. The coefficients \( e_i \) and \( f_i \) up to post-2.5-Newtonian order

The coefficients in Eqs. (2.8) and (2.9) up to 2.5 post-Newtonian order as given by Blanchet [3] are:

\[
e_2 = -\frac{9 + \eta}{12},
\]

\[
e_4 = -\frac{27 - 19\eta + \eta^2/3}{8},
\]

\[
f_2 = -\frac{1247}{336} - \frac{35\eta}{12},
\]

\[
f_3 = 4\pi,
\]

\[
f_4 = -\frac{44711}{9072} + \frac{9271\eta}{504} + \frac{65\eta^2}{18},
\]

and

\[
f_5 = -\left( \frac{8191}{672} + \frac{535\eta}{24} \right) \pi.
\]
2. The coefficients $e_i$, $f_i$ and $g_i$ up to post-5.5-Newtonian order

The remaining coefficients in Eqs. (2.8) and (2.9) have been given up to 5.5 post-Newtonian order in the test mass limit in Ref. [19]. These are

$$e_6 = -\frac{675}{64}, \quad (A7)$$

$$e_8 = -\frac{3969}{128}, \quad (A8)$$

$$e_{10} = -\frac{45927}{512}, \quad (A9)$$

$$f_6 = \frac{6643739519}{69854400} - \frac{1712\gamma}{105} + \frac{16\pi^2}{3} - \frac{3424\log(2)}{105}, \quad (A10)$$

$$f_7 = -\frac{16285\pi}{504}, \quad (A11)$$

$$f_8 = -\frac{323105549467}{3178375200} + \frac{232597\gamma}{4410} - \frac{1369\pi^2}{126} + \frac{39931\log(2)}{294} - \frac{47385\log(3)}{1568}, \quad (A12)$$

$$f_9 = \frac{265978667519}{745113600} - \frac{6848\gamma\pi}{105} - \frac{13696\pi\log(2)}{105}, \quad (A13)$$

$$f_{10} = -\frac{2500861660823683}{2831932303200} + \frac{916628467\gamma}{7858620} - \frac{424223\pi^2}{6804} + \frac{83217611\log(2)}{11226600} + \frac{47385\log(3)}{196}, \quad (A14)$$

$$f_{11} = \frac{8399309759401\pi}{101708006400} + \frac{177293\gamma\pi}{1176} + \frac{8521283\pi\log(2)}{17640} - \frac{142155\pi\log(3)}{784}, \quad (A15)$$

$$g_6 = -\frac{1712}{105}, \quad (A16)$$

$$g_7 = 0, \quad (A17)$$

$$g_8 = \frac{232597}{4410}, \quad (A18)$$

$$g_9 = -\frac{6848\pi}{105}, \quad (A19)$$

$$g_{10} = \frac{916628467}{7858620}, \quad (A20)$$

and

$$g_{11} = \frac{177293\pi}{1176}. \quad (A21)$$

APPENDIX B: ANALYTIC EXPRESSIONS FOR THE COEFFICIENTS IN PHASE EXPANSION IN VARIOUS ORDERS OF APPROXIMATION

1. The coefficients $p_{j,n}$, $q_{j,n}$ and $r_{j,n}$ up to post-2.5-Newtonian order for general mass ratios

Here we list the analytic expressions for the coefficients $p_{j,n}$, $q_{j,n}$ and $r_{j,n}$ up to post-2.5-Newtonian order for $\eta \neq 0$.

$$p_{2,2} = \frac{3715}{756} + \frac{55\eta}{9} \quad (B1)$$

$$p_{3,2} = 0 \quad (B2)$$

$$p_{3,3} = -16\pi \quad (B3)$$

$$p_{4,2} = p_{4,3} = \frac{5 \left(926521 + 1880368\eta + 905520\eta^2\right)}{56448} \quad (B4)$$

$$p_{4,4} = \frac{15293365}{508032} + \frac{27145\eta}{504} + \frac{3085\eta^2}{72} \quad (B5)$$

$$p_{5,2} = 0 \quad (B6)$$

$$p_{5,3} = p_{5,4} = \frac{20 \left(995 + 952\eta\right)\pi}{189} \quad (B7)$$

$$p_{5,5} = \frac{5 \left(7729 + 252\eta\right)\pi}{756} \quad (B8)$$

$$q_{2,n} = q_{3,n} = q_{4,n} = 0 \quad (B9)$$

$$q_{5,2} = 0 \quad (B10)$$

$$q_{5,3} = q_{5,4} = \frac{20 \left(995 + 952\eta\right)\pi}{63} \quad (B11)$$

$$q_{5,5} = \frac{38645\pi}{252} + 5\eta\pi \quad (B12)$$

$$r_{2,n} = r_{3,n} = r_{4,n} = r_{5,n} = 0 \quad (B13)$$
2. The coefficients \( p_{j,n} \), \( q_{j,n} \) and \( r_{j,n} \) up to post-5.5-Newtonian order in the test mass limit

Here we list analytic expressions for the remaining coefficients \( p_{j,n} \), \( q_{j,n} \) and \( r_{j,n} \) up to post-5.5-Newtonian order in the test mass limit.

\[
p_{6,2} = -\frac{5776858435}{9483264} \quad (B14)
\]

\[
p_{6,3} = -\frac{5776858435}{9483264} - 320 \pi^2 \quad (B15)
\]

\[
p_{6,4} = p_{6,5} = -\frac{37674179035}{85349376} - 320 \pi^2 \quad (B16)
\]

\[
p_{6,6} = \frac{10817850546611}{4694215680} - \frac{6848 \gamma}{320 \pi^2} - \frac{13696 \log(2)}{21} \quad (B17)
\]

\[ p_{7,2} = 0 \quad (B18) \]

\[ p_{7,3} = \frac{5680085 \pi}{2352} \quad (B19) \]

\[ p_{7,4} = \frac{152000375 \pi}{63504} \quad (B20) \]

\[ p_{7,5} = p_{7,6} = \frac{241249475 \pi}{254016} \quad (B21) \]

\[ p_{7,7} = \frac{77096675 \pi}{254016} \quad (B22) \]

\[ p_{8,2} = -\frac{7203742468445}{1433895168} \quad (B23) \]

\[ p_{8,3} = -\frac{7203742468445}{1433895168} - \frac{43160 \pi^2}{63} \quad (B24) \]

\[ p_{8,4} = -\frac{499400855271485}{1161434308608} - \frac{43160 \pi^2}{63} \quad (B25) \]

\[ p_{8,5} = -\frac{499400855271485}{1161434308608} - \frac{47570 \pi^2}{189} \quad (B26) \]

\[ p_{8,6} = p_{8,7} = -\frac{35381221594107617}{12775777394688} - \frac{1703440 \gamma}{3969} \]

\[ -\frac{63110 \pi^2}{567} - \frac{3406880 \log(2)}{3969} \quad (B27) \]

\[ p_{8,8} = \frac{2496799162103891233}{830425530654720} - \frac{36812 \gamma}{189} - \frac{90490 \pi^2}{567} \quad (B28) \]

\[ -\frac{3969}{196} + 26325 \log(3) \]

\[ p_{9,2} = 0 \quad (B29) \]

\[ p_{9,3} = -\frac{6756514105 \pi}{1185408} - 640 \pi^3 \quad (B30) \]

\[ p_{9,4} = -\frac{23087048755 \pi}{3556224} - 640 \pi^3 \quad (B31) \]

\[ p_{9,5} = -\frac{971321608855 \pi}{341397504} - 640 \pi^3 \quad (B32) \]

\[ p_{9,6} = \frac{121130241969551 \pi}{18776862720} - \frac{27392 \gamma \pi}{21} - \frac{640 \pi^3}{3} \quad (B33) \]

\[ -\frac{3}{21} \quad (B34) \]

\[ p_{9,7} = p_{9,8} = \frac{157063289889551 \pi}{18776862720} - \frac{27392 \gamma \pi}{21} \]

\[ -\frac{640 \pi^3}{3} - \frac{54784 \pi \log(2)}{21} \quad (B35) \]

\[ p_{10,2} = \frac{1796613371630183}{107062572544} \quad (B36) \]

\[ p_{10,3} = \frac{1796613371630183}{107062572544} + \frac{1240765 \pi^2}{294} \quad (B37) \]

\[ p_{10,4} = \frac{54094086068662461}{28906809458688} + \frac{11956093 \pi^2}{2646} \quad (B38) \]

\[ p_{10,5} = \frac{54094086068662461}{28906809458688} + \frac{458972531 \pi^2}{338688} \quad (B39) \]

\[ p_{10,6} = -\frac{4027802547645341665}{317974904045568} + \frac{650561605 \gamma}{333396} \]

\[ + \frac{312163997 \pi^2}{435456} + \frac{650561605 \log(2)}{166698} \quad (B40) \]

\[ p_{10,7} = -\frac{4027802547645341665}{317974904045568} + \frac{650561605 \gamma}{333396} \]

\[ -\frac{138083683 \pi^2}{435456} + \frac{650561605 \log(2)}{166698} \quad (B41) \]
\[ p_{10,8} = p_{10,9} = \frac{23600127211067107843}{1878942614814720} + \frac{116990189 \gamma}{166698} - \frac{181984501 \pi^2}{3048192} + \frac{228376895 \log(2)}{333396} + \frac{15716025 \log(3)}{21952} \quad (B42) \]

\[ p_{10,10} = -\frac{1412206995432957982751}{12630669795858400} + \frac{6470582647 \gamma}{27505170} + \frac{578223115 \pi^2}{3048192} + \frac{53992839431 \log(2)}{55010340} - \frac{5512455 \log(3)}{21952} \quad (B43) \]

\[ p_{11,2} = 0 \quad (B44) \]

\[ p_{11,3} = -\frac{-40905234824185 \pi}{7169347584} - \frac{358720 \pi^3}{189} \quad (B45) \]

\[ p_{11,4} = -\frac{-1456611391753955 \pi}{193572384768} - \frac{358720 \pi^3}{189} \quad (B46) \]

\[ p_{11,5} = -\frac{-1211268636338065 \pi}{387144769536} - \frac{112990 \pi^3}{189} \quad (B47) \]

\[ p_{11,6} = \frac{-40907663544419749 \pi}{4258592464896} - \frac{7577740 \gamma \pi}{3969} + \frac{115130 \pi^3}{567} - \frac{15155480 \pi \log(2)}{3969} \quad (B48) \]

\[ p_{11,7} = \frac{-50239568645429749 \pi}{4258592464896} - \frac{7577740 \gamma \pi}{3969} + \frac{115130 \pi^3}{567} - \frac{15155480 \pi \log(2)}{3969} \quad (B49) \]

\[ p_{11,8} = \frac{-3146788245124283189 \pi}{276808510218240} - \frac{183628 \gamma \pi}{189} + \frac{94390 \pi^3}{567} - \frac{5572040 \pi \log(2)}{3969} - \frac{26325 \pi \log(3)}{49} \quad (B50) \]

\[ q_{6,2} = q_{6,3} = q_{6,4} = q_{6,5} = 0 \]  
[ q_{6,6} = -\frac{6848}{21} \]  
[ q_{7,n} = 0 \]  
[ q_{8,2} = \frac{7203742468445}{4779565056} \]  
[ q_{8,3} = \frac{7203742468445}{4779565056} + \frac{43160 \pi^2}{21} \]  
[ q_{8,4} = \frac{499400855271485}{387144769536} + \frac{43160 \pi^2}{21} \]  
[ q_{8,5} = \frac{499400855271485}{387144769536} + \frac{47570 \pi^2}{63} \]  
[ q_{8,6} = q_{8,7} = \frac{37208950681636577}{425859246896} + \frac{1703440 \gamma}{1323} + \frac{63110 \pi^2}{189} + \frac{3406880 \log(2)}{1323} \]  
[ q_{8,8} = \frac{2550713843998885153}{276808510218240} + \frac{36812 \gamma}{63} + \frac{90490 \pi^2}{189} + \frac{1011020 \log(2)}{1323} + \frac{78975 \log(3)}{196} \]  
[ q_{9,2} = q_{9,3} = q_{9,4} = q_{9,5} = 0 \]  
[ q_{9,6} = q_{9,7} = q_{9,8} = -\frac{27392 \pi}{21} \]  
[ q_{9,9} = -\frac{13696 \pi}{21} \]  
[ q_{10,2} = q_{10,3} = q_{10,4} = q_{10,5} = 0 \]  
[ q_{10,6} = q_{10,7} = \frac{650561605}{333396} \]  
[ q_{10,8} = q_{10,9} = \frac{116990189}{166698} \]  
[ q_{10,10} = \frac{6470582647}{27505170} \]  
[ q_{11,2} = q_{11,3} = q_{11,4} = q_{11,5} = 0 \]
\[ q_{11,6} = q_{11,7} = \frac{-7577740 \pi}{3969} \tag{B70} \]
\[ q_{11,8} = \frac{-183628 \pi}{189} \tag{B71} \]
\[ q_{11,9} = q_{11,10} = \frac{-449308 \pi}{3969} \tag{B72} \]
\[ q_{11,11} = \frac{-3558011 \pi}{7938} \tag{B73} \]
\[ r_{6,n} = r_{7,n} = 0 \tag{B74} \]
\[ r_{8,2} = r_{8,3} = r_{8,4} = r_{8,5} = 0 \tag{B75} \]
\[ r_{8,6} = r_{8,7} = \frac{1703440}{1323} \tag{B76} \]
\[ r_{8,8} = \frac{36812}{63} \tag{B77} \]
\[ r_{9,n} = r_{10,n} = r_{11,n} = 0 \tag{B78} \]

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