Asymptotic analysis of the two matrix model with a quartic potential

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Abstract

We give a summary of the recent progress made by the authors and collaborators on the asymptotic analysis of the two matrix model with a quartic potential. The paper also contains a list of open problems.

1 Two matrix model: introduction

The Hermitian two matrix model is the probability measure

$$\frac{1}{Z_n} e^{-n \text{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} dM_1 dM_2$$

(1.1)

defined on pairs \((M_1, M_2)\) of \(n \times n\) Hermitian matrices. Here \(V\) and \(W\) are two polynomial potentials, \(\tau \neq 0\) is a coupling constant, and

$$Z_n = \int e^{-n \text{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} dM_1 dM_2$$

is a normalization constant in order to make (1.1) a probability measure.

In recent works of the authors and collaborators \([20, 21, 22, 35]\) the model was studied with the aim to gain understanding in the limiting behavior of the eigenvalues of \(M_1\) as \(n \to \infty\), and to find and describe new types of critical behaviors.

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The results should be compared with the well known results for the Hermitian one matrix model

\[
\frac{1}{Z_n} e^{-n \text{Tr}(V(M))} \, dM, \tag{1.2}
\]

which we briefly summarize here. The eigenvalues of the random matrix \(M\) from (1.2) have the explicit joint p.d.f.

\[
\frac{1}{Z_n} \prod_{j<k} (x_k - x_j)^2 \prod_{j=1}^n e^{-nV(x_j)},
\]

which yields that the eigenvalues are a determinantal point process with correlation kernel

\[
K_n(x, y) = \sqrt{e^{-nV(x)} e^{-nV(y)}} \sum_{k=0}^{n-1} p_{k,n}(x)p_{k,n}(y),
\]

where \((p_{k,n})_k\) is the sequence of orthonormal polynomials with respect to the weight function \(e^{-nV(x)}\) on the real line. As \(n \to \infty\) the empirical eigenvalue distributions have an a.s. weak limit\(^1\)

\[
\frac{1}{n} \sum_{j=1}^n \delta_{x_j} \to \mu^*\]

where \(\mu^*\) is a non-random probability measure that is characterized as the minimizer of the energy functional (Coulomb gas picture)

\[
E_V(\mu) = \int \int \log \frac{1}{|x - y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x) \tag{1.3}
\]

when taken over all probability measures on the real line. For a polynomial \(V\) the minimizer \(\mu^*\) is supported on a finite union of intervals \([15]\). In addition there is a polynomial \(Q\) of degree \(\deg V - 2\) such that

\[
\xi(z) = V'(z) - \int \frac{d\mu^*_1(s)}{z - s}
\]

is the solution of a quadratic equation

\[
\xi^2 - V'(z)\xi + Q(z) = 0. \tag{1.4}
\]

From this it follows that \(\mu^*_1\) has a density with respect to Lebesgue measure that is real analytic in the interior of any of the intervals and that can be written as

\[
\rho(x) = \frac{d\mu^*_1(x)}{dx} = \frac{1}{\pi} \sqrt{q^-(x)}, \quad x \in \mathbb{R}
\]

\(^1\)i.e., for any bounded continuous function \(f\), we have \(\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n f(x_j) = \int f d\mu^*\) almost surely.
where \(q^-\) denotes the negative part of the polynomial

\[
q(x) = \left(\frac{V'(x)}{2}\right)^2 - Q(x).
\]

2 Limiting eigenvalue distribution

2.1 Vector equilibrium problem

Guionnet [26] showed that the eigenvalues of the matrices \(M_1\) and \(M_2\) in the two matrix model (1.1) have a limiting distribution as \(n \to \infty\). The results of [26] are in fact valid for a much greater class of random matrix models. The limiting distribution is characterized as the minimizer of a certain functional, which is however very different from the energy functional (1.3) for the one matrix model.

Our aim is to develop an analogue of the Coulomb gas picture for the eigenvalues of the matrices in the two matrix model (1.1). We have been successful in doing this for the eigenvalues of \(M_1\) in the case of even polynomial potentials \(V\) and \(W\) with \(W\) of degree 4. Thus our assumptions are

- \(V\) is an even polynomial with positive leading coefficient,
- \(W(y) = \frac{\alpha}{4} y^4 + \frac{\alpha}{2} y^2\) with \(\alpha \in \mathbb{R}\),
- and \(\tau > 0\) (without loss of generality).

We use the following notions from logarithmic potential theory [38]

\[
I(\mu, \nu) = \int \int \log \frac{1}{|x - y|} d\mu(x) d\nu(y), \quad I(\mu) = I(\mu, \mu),
\]

which define the mutual logarithmic energy \(I(\mu, \nu)\) of two measures \(\mu\) and \(\nu\), and the logarithmic energy \(I(\mu)\) of a measure \(\mu\). Then the limiting mean distribution of the eigenvalues of \(M_1\) is characterized by a vector equilibrium problem for three measures. This involves an energy functional

\[
E(\mu_1, \mu_2, \mu_3) = I(\mu_1) + I(\mu_2) + I(\mu_3)
- I(\mu_1, \mu_2) - I(\mu_2, \mu_3) + \int V_1(x) d\mu_1(x) + \int V_3(x) d\mu_3(x) \quad (2.1)
\]

defined on three measures \(\mu_1, \mu_2, \mu_3\). Note that there is an attraction between the measures \(\mu_1\) and \(\mu_2\) and between the measures \(\mu_2\) and \(\mu_3\),
while there is no direct interaction between the measures $\mu_1$ and $\mu_3$. This type of interaction is characteristic for a Nikishin system [36].

The energy functional (2.1) depends on the external fields $V_1$ and $V_3$ that act on the measures $\mu_1$ and $\mu_3$ in (2.1). The vector equilibrium problem will also have an upper constraint $\sigma_2$ for the measure $\mu_2$. These input data take a very special form that we describe next.

**External field $V_1$:** The external field that acts on $\mu_1$ is defined by

$$V_1(x) = V(x) + \min_{s \in \mathbb{R}}(W(s) - \tau xs),$$

where we recall that $W(s) = \frac{1}{4}s^4 + \frac{\alpha}{2}s^2$. For the case $\alpha = 0$, this is simply $V_1(x) = V(x) - \frac{3}{4}|\tau x|^{4/3}$.

**External field $V_3$:** The external field that acts on the third measure is absent if $\alpha \geq 0$, i.e.,

$$V_3(x) \equiv 0 \quad \text{if } \alpha \geq 0.$$

The function $s \in \mathbb{R} \mapsto W(s) - \tau xs$ has a global minimum at $s = s_1(x)$ and this value plays a role in the definition of $V_1$, see (2.2). For $\alpha < 0$, and $x \in (-x^*(\alpha), x^*(\alpha))$, where

$$x^*(\alpha) = \frac{2}{\tau} \left( \frac{-\alpha}{3} \right)^{3/2}, \quad \alpha < 0.$$

The function $s \in \mathbb{R} \mapsto W(s) - \tau xs$ has another local minimum at $s = s_2(x)$, and a local maximum at $s = s_3(x)$.

Then $V_3$ is defined by

$$V_3(x) = (W(s_3(x)) - \tau xs_3(x)) - (W(s_2(x)) - \tau xs_2(x))$$

if $x \in (-x^*(\alpha), x^*(\alpha))$, and

$$V_3(x) \equiv 0 \quad \text{otherwise.}$$

**Upper constraint $\sigma_2$:** The upper constraint $\sigma_2$ that acts on the second measure is the measure on the imaginary axis with the density

$$\frac{d\sigma_2(z)}{|dz|} = \frac{\tau}{\pi s^3 + \alpha s = \tau z} \max_{\text{Re } s, \ z \in i\mathbb{R}} \text{Re } s,$$

In case $\alpha = 0$ this simplifies to

$$\frac{d\sigma_2(z)}{|dz|} = \frac{\sqrt{\pi}}{2\pi} \tau^{4/3}|z|^{1/3}.$$
If $\alpha < 0$ then the density of $\sigma_2$ is positive and real analytic on the full imaginary axis. If $\alpha > 0$ then the support of $\sigma_2$ has a gap around 0:

$$\text{supp}(\sigma_2) = (-i\infty, -iy^*(\alpha)] \cup [iy^*(\alpha), i\infty),$$

where

$$y^*(\alpha) = \frac{2}{\tau} \left( \frac{\alpha}{3} \right)^{3/2}, \quad \alpha > 0.$$

The following result is Theorems 1.1 in [22].

**Theorem 1.** There is a unique minimizer $(\mu^*_1, \mu^*_2, \mu^*_3)$ of the energy functional (2.1) subject to the conditions

(a) $\mu_1$ is a measure on $\mathbb{R}$ with $\mu_1(\mathbb{R}) = 1$,

(b) $\mu_2$ is a measure on $i\mathbb{R}$ with $\mu_2(i\mathbb{R}) = 2/3$,

(c) $\mu_3$ is a measure on $\mathbb{R}$ with $\mu_3(\mathbb{R}) = 1/3$,

(d) $\mu_2 \leq \sigma_2$,

with input data $V_1$, $V_3$, and $\sigma$ as described above.

The proof of the existence of a minimizer was completed and simplified in [27], see subsection 4.2 below.

Now that we have existence and uniqueness, it is natural to ask about further properties of the minimizer. The three measures $\mu^*_1$, $\sigma - \mu^*_2$ and $\mu^*_3$ are absolutely continuous with respect to the Lebesgue measure with densities that are real analytic in the interior of their supports, except possibly at the origin. Furthermore, denoting by $S(\mu)$ the support of a measure $\mu$, we have

- the support of $\mu^*_1$ is a finite union of bounded intervals on the real line;
- there exist $c_2 \geq 0$ such that $S(\sigma_2 - \mu^*_2) = i\mathbb{R} \setminus (-ic_2, ic_2)$, and if $c_2 > 0$ then the density of $\sigma_2 - \mu^*_2$ vanishes like a square root at $\pm ic_2$;
- there exist $c_3 \geq 0$ such that $S(\mu^*_3) = \mathbb{R} \setminus (-c_3, c_3)$, and if $c_3 > 0$ then the density of $\mu^*_3$ vanishes like a square root at $\pm c_3$.

In a generic situation, the density of $\mu^*_1$ is strictly positive in the interior of its support and vanishes like a square root at endpoints. In addition strict inequality holds in the variational inequality outside the support $S(\mu^*_1)$. Moreover, generically if $c_2 = 0$ the density of $\sigma - \mu^*_2$ is positive at the origin, and likewise if $c_3 = 0$ the density of $\mu^*_3$ is positive at the origin. If we are
in such a generic situation, then we say that \((V, W, \tau)\) is regular. See [22, section 1.5] for more details and a discussion on the singular situations that may occur.

The following is Theorem 1.4 in [22].

**Theorem 2.** Let \(\mu^*\) be the first component of the minimizer in Theorem 1, and assume that \((V, W, \tau)\) is regular, then as \(n \to \infty\) with \(n \equiv 0 \pmod{3}\), the mean eigenvalue distribution of \(M_1\) converges to \(\mu^*_1\).

We are convinced that the theorem is also valid in the singular cases, which correspond to phase transitions in the two matrix model. The condition that \(n\) is a multiple of three is non-essential as well. It is imposed for convenience in the analysis.

In [22] only the convergence of mean eigenvalue distributions was considered, which is a rather weak form of convergence. However, when combined with the results of [26] it will actually follow that the empirical eigenvalue distributions of \(M_1\) tend to \(\mu^*_1\) almost surely.

The analysis of [22] also proves the usual universality results for local eigenvalue statistics in Hermitian matrix ensembles, given by the sine kernel in the bulk of the spectrum and by the Airy kernel at edge points. In non-regular situations one may find Pearcey and Painlevé II kernels, while in multi-critical cases new kernels may appear. This was indeed proved recently in [19], see subsection 4.1 below.

### 2.2 Riemann surface

A major ingredient in the asymptotic analysis in [22] is the construction of an appropriate Riemann surface (or spectral curve), which plays a role similar to the algebraic equation (1.4) in the one matrix model. The existence of such a Riemann surface is implied by the work of Eynard [24] on the formal two matrix model. Our approach is different from the one of Eynard, in that we use the vector equilibrium problem to construct the Riemann surface, and in a next step we define a meromorphic function on it.

The main point is that the supports \(S(\mu_1^*), S(\sigma - \mu_2^*)\) and \(S(\mu_3^*)\) associated to the minimizer in Theorem 1, determine the cut structure of a Riemann surface

\[
\mathcal{R} = \bigcup_{j=1}^{4} \mathcal{R}_j
\]
with four sheets
\[
\mathcal{R}_1 = \mathbb{C} \setminus S(\mu_1^*),
\]
\[
\mathcal{R}_2 = \mathbb{C} \setminus (S(\mu_1^*) \cup S(\sigma - \mu_2^*)),
\]
\[
\mathcal{R}_3 = \mathbb{C} \setminus (S(\sigma - \mu_2^*) \cup S(\mu_3^*)),
\]
\[
\mathcal{R}_4 = \mathbb{C} \setminus S(\mu_3^*).
\]

The sheet \(\mathcal{R}_j\) is glued to the next sheet \(\mathcal{R}_{j+1}\) along the common cut in the usual crosswise manner. The meromorphic function on \(\mathcal{R}\) arises in the following way, see Proposition 4.8 of [22].

**Proposition 3.** The function
\[
\xi_1(z) = V'(z) - \int \frac{d\mu_1^*(x)}{z - x}, \quad z \in \mathcal{R}_1,
\]
extends to a meromorphic function on the Riemann surface \(\mathcal{R}\) whose only poles are at infinity. There is a pole of order \(\deg V\) at infinity on the first sheet, and a simple pole at the other point at infinity.

The proof of Proposition 3 follows from the Euler-Lagrange variational conditions that are associated with the vector equilibrium problem. See section 4.2 of [22] for explicit expressions for the meromorphic continuation of \(\xi_1\) to the other sheets.

It follows from Proposition 3 that \(\xi_1\) is one of the solutions of a quartic equation, which is the analogue of the quadratic equation (1.4) that is relevant in the one matrix model.

### 3 About the proof

We describe the main tools that are used in the proof of Theorem 2.

#### 3.1 Biorthogonal polynomials

We make use of the integrable structure of the two matrix model that is described in terms of biorthogonal polynomials. In this context the biorthogonal polynomials are two sequences of monic polynomials \((p_{j,n})_j\) and \((q_{k,n})_k\) (depending on \(n\)) with \(\deg p_{j,n} = j\) and \(\deg q_{k,n} = k\), that satisfy
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{j,n}(x)q_{k,n}(y)e^{-n(V(x)+W(y)-\tau xy)}dxdy = h_{k,n}\delta_{j,k},
\]
see [5, 6, 7, 23, 25]. These polynomials uniquely exist, have real and simple zeros [23], and in addition the zeros of $p_{j,n}$ and $p_{j+1,n}$ interlace, as well as those of $q_{k,n}$ and $q_{k+1,n}$, see [20].

There is an explicit expression for the joint p.d.f. of the eigenvalues of $M_1$ and $M_2$

$$\frac{1}{(n!)^2} \det \begin{pmatrix} K_n^{(1,1)}(x_i, x_j) & K_n^{(1,2)}(x_i, y_j) \\ K_n^{(2,1)}(y_1, y_j) & K_n^{(2,2)}(y_i, y_j) \end{pmatrix}$$  \hspace{0.5cm} (3.1)

with 4 kernels that are expressed in terms of the biorthogonal polynomials and their transformed functions

$$Q_{k,n}(x) = \int_{-\infty}^{\infty} q_{k,n}(y)e^{-n(V(x)+W(y)−τxy)}dy,$$

$$P_{j,n}(y) = \int_{-\infty}^{\infty} p_{j,n}(x)e^{-n(V(x)+W(y)−τxy)}dx,$$

as follows

$$K_n^{(1,1)}(x_1, x_2) = \sum_{k=0}^{n-1} \frac{1}{h_k^{(2)}} p_{k,n}(x_1)Q_{k,n}(x_2),$$  \hspace{0.5cm} (3.2)

$$K_n^{(1,2)}(x, y) = \sum_{k=0}^{n-1} \frac{1}{h_k^{(2)}} p_{k,n}(x)q_{k,n}(y),$$

$$K_n^{(2,1)}(y, x) = \sum_{k=0}^{n-1} \frac{1}{h_k^{(2)}} P_{k,n}(y)Q_{k,n}(x) − e^{-n(V(x)+W(y)−τxy)},$$

$$K_n^{(2,2)}(y_1, y_2) = \sum_{k=0}^{n-1} \frac{1}{h_k^{(2)}} P_{k,n}(y_1)q_{k,n}(y_2).$$

The joint p.d.f. (3.1) is determinantal, which means that eigenvalue correlation functions have determinantal expressions with the same kernels $K_n^{(i,j)}$, $i, j = 1, 2$. In particular, after averaging out the eigenvalues of $M_2$ we get that the eigenvalues of $M_1$ are a determinantal point process with kernel $K_n^{(1,1)}$.

A natural first step to compute the asymptotic behavior of the polynomials and hence the kernels, is to formulate a Riemann-Hilbert problem (RH problem) for the polynomials. Several different formulations exist in the literature [7, 23, 29, 32]. The analysis in [21, 22, 35] is based on the RH problem in [32] that we will discuss in the next subsection.
3.2 Riemann-Hilbert problem

It turns out that the kernel \((3.2)\) has a special structure which relates it to multiple orthogonal polynomials and the eigenvalues of \(M_1\) (after averaging over \(M_2\)) are an example of a multiple orthogonal polynomial ensemble [30]. This is due to the following observation of Kuijlaars and McLaughlin [32].

**Proposition 4.** Suppose \(W\) is a polynomial of degree \(r + 1\), and let

\[
w_{k,n}(x) = \int_{-\infty}^{\infty} y^k e^{-n(V(x)+W(y)-\tau xy)} dy, \quad k = 0, \ldots, r - 1.
\]

Then the biorthogonal polynomial \(p_{j,n}\) satisfies

\[
\int_{-\infty}^{\infty} p_{j,n}(x) x^l w_{k,n}(x) \, dx = 0, \quad l = 0, \ldots, \left\lfloor \frac{j-k}{r} \right\rfloor - 1, \quad (3.3)
\]

for \(k = 0, \ldots, r - 1\).

The conditions \((3.3)\) are known as multiple orthogonality conditions [3], and they characterize the biorthogonal polynomials.

The advantage of the formulation as multiple orthogonality is that these polynomials are characterized by a RH problem of size \((r + 1) \times (r + 1)\), [40], which we state here for the case \(r = 3\) and for \(j = n\) with \(n\) a multiple of three. Then the RH problem has size 4 \(\times\) 4 and it asks for a 4 \(\times\) 4 matrix valued function \(Y\) on \(\mathbb{C} \setminus \mathbb{R}\) such that

\[(a) \ Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{4 \times 4}\ is\ analytic,\]

\[(b) \ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_{0,n}(x) & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ for \ x \in \mathbb{R},\ where\]

\(Y_+(x) (Y_-(x))\) denotes the limiting value of \(Y(z)\) as \(z \to x\) from the upper (lower) half plane,

\[(c) \ Y(z) = (I_4 + O(\frac{1}{z})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & z^{-n/3} & 0 & 0 \\ 0 & 0 & z^{-n/3} & 0 \\ 0 & 0 & 0 & z^{-n/3} \end{pmatrix} \ as \ z \to \infty.\]

The RH problem has a unique solution which is given by

\[
Y = \begin{pmatrix} p_{n,n} & C(p_{n,n}w_{0,n}) & C(p_{n,n}w_{0,1,n}) & C(p_{n,n}w_{2,1,n}) \\ p_{n,n}^{(0)} & C(p_{n,n}w_{0,n})^{(0)} & C(p_{n,n}w_{1,n})^{(0)} & C(p_{n,n}w_{2,n})^{(0)} \\ p_{n,n}^{(1)} & C(p_{n,n}w_{0,n})^{(1)} & C(p_{n,n}w_{1,n})^{(1)} & C(p_{n,n}w_{2,n})^{(1)} \\ p_{n,n}^{(2)} & C(p_{n,n}w_{0,n})^{(2)} & C(p_{n,n}w_{1,n})^{(2)} & C(p_{n,n}w_{2,n})^{(2)} \end{pmatrix},
\]

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where $p_{n,n}$ is the $n$th degree biorthogonal polynomial, $p_{n,n}^{(0)}, p_{n,n}^{(1)}, p_{n,n}^{(2)}$ are three polynomials of degree $\leq n - 1$ that satisfy certain multiple orthogonal conditions and $Cf$ is the Cauchy transform

$$Cf(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z} dx.$$  

By using the the Christoffel-Darboux formula for multiple orthogonal polynomials [10, 14] the correlation kernel $K_n^{(1,1)}$ for the eigenvalues of $M_1$ can be expressed in terms of the solution of the RH problem as follows

$$K_n^{(1,1)}(x, y) = \begin{pmatrix} 0 & w_{0,n}(y) & w_{1,n}(y) & w_{2,n}(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$  

(3.4)

Multiple orthogonal polynomials and RH problems are also used for random matrices with external source [8, 10] and models of non-intersecting paths [31]. In these cases, correlation kernels for the relevant statistical quantities are also expressed in terms of the corresponding RH problem through (3.4).

### 3.3 Steepest descent analysis

The remaining part of the proof of Theorem 2 is an asymptotic analysis of the RH problem via an extension of the Deift-Zhou steepest descent method [16, 17]. The vector equilibrium problem and the Riemann surface play a crucial role in the transformations in this analysis. For the precise transformations and the many details that are involved we refer the reader to [22]. Following the effect of the transformations on the kernel (3.4), one finds that

$$\lim_{n \to \infty} \frac{1}{n} K_n^{(1,1)}(x, x) = \frac{d\mu_+^*(x)}{dx},$$  

which is what is needed to establish the theorem.

A somewhat similar steepest descent analysis is done in [8] for a random matrix model with external source, where vector equilibrium problems and Riemann surfaces also play an important role.

### 4 Further developments

#### 4.1 Critical behavior in the quadratic/quartic model

For the case $V(x) = \frac{1}{2}x^2$ the spectral curve can be computed and a classification of all possible cases can be made explicitly.
The quadratic/quartic model depends on two parameters, namely the coupling constant $\tau$ and the number $\alpha$ in the quartic potential $W(y) = \frac{y^4}{4} + \alpha \frac{y^2}{2}$. Figure 4.1 (taken from [20]) shows the phase diagram in the $\alpha$-$\tau$ plane. Critical behavior takes place on the curves $\tau^2 = \alpha + 2$ and $\alpha \tau^2 = -1$. On the parabola $\tau^2 = \alpha + 2$ a gap appears around 0 in the support of either $\mu_1^*$ (if one moves from Case I to Case II) or $\mu_3^*$ (if one moves from Case I to Case IV). This is a transition of Painlevé II type which also appears in the opening of gaps in one matrix models [9, 13]. On the curve $\alpha \tau^2 = -1$ a gap appears in the support of either $\mu_1^*$ (if one moves from Case IV to Case III) or $\mu_3^*$ (if one moves from Case II to Case III), while simultaneously the gap in the support of $\sigma_2 - \mu_2^*$ closes. This is a transition of Pearcey type, which was observed before in the random matrix model with external source and in the model of non-intersecting Brownian motions [11, 12, 39].

The phase diagram has a very special point $\alpha = -1$, $\tau = 1$ which is on both critical curves, and where all four regular cases come together. For these special values, the density of $\mu_1^*$ vanishes like a square root at the
origin, which is an interior point of $S(\mu_\ast^*)$. The local analysis at this point was done very recently by Duits and Geudens [19]. They found that in the asymptotic limit, the local eigenvalue correlation kernels around 0 are closely related to the limiting kernels that describe the tacnode behavior for non-intersecting Brownian motions [1, 18, 28]. More precisely, the kernels can be expressed in terms of an extension of the same $4 \times 4$ RH problem in [18]. However, they are constructed in a different way out of this $4 \times 4$ RH problem and, as a result, these kernels are not the same.

4.2 Vector equilibrium problems

The analysis in [22] of the vector equilibrium problem was not fully complete, since the lower semi-continuity of the energy functional (2.1) was implicitly assumed but not established in [22].

In the recent papers [4, 27] the vector equilibrium problem was studied in a more systematic way, in the more general context of an energy functional for $n$ measures

$$E(\mu_1, \ldots, \mu_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} I(\mu_i, \mu_j) + \sum_{j=1}^{n} \int_{\Sigma_j} V_j(x) d\mu_j(x), \quad (4.1)$$

where $C = (c_{ij})_{i,j=1}^{n}$ is a real symmetric positive definite matrix (in [4] also semidefinite interaction matrices are considered). The external fields $V_j : \Sigma_j \to \mathbb{R} \cup \{\infty\}$ are lower semi-continuous with domains $\Sigma_j$ that are closed subsets of $\mathbb{C}$. Let $m_1, \ldots, m_n$ be given positive numbers and assume that for every $i = 1, \ldots, n$,

$$\liminf_{|x| \to \infty} \left( V_i(x) - \sum_{j=1}^{n} c_{ij} m_j \log(1 + |x|^2) \right) > -\infty.$$  

Under these assumptions it is shown in [27] that the energy functional (4.1), restricted to the set of measures with $\mu_j(\Sigma_j) = m_j$ for $j = 1, \ldots, n$,

(a) has compact sub-level sets $\{E \leq \alpha\}$ for every $\alpha \in \mathbb{R}$, (so $E$ is in particular lower semi-continuous), and

(b) is strictly convex on the subset where it is finite.

This guarantees existence and uniqueness of a minimizer of (4.1), provided that $E$ is not identically infinite. Existence and uniqueness of a minimizer readily extends to situations where the domain of $E$ is further restricted by
upper constraints $\mu_j \leq \sigma_j$ for $j = 1, \ldots, n$, again provided that $E$ is not identically infinite on this domain. In particular, this applies to the energy functional (2.1) for the two matrix model with quartic potential with the constraint $\mu_2 \leq \sigma_2$ described in section 2.

4.3 Open problems

There are numerous intriguing questions and open problems that arise out of our analysis.

(a) What is the motivation for the central vector equilibrium problem? In the one-matrix model there is a direct way to come from the joint eigenvalue probability density to the equilibrium problem. We do not have this direct link for the two matrix model.

(b) How is the vector equilibrium problem related to the variational problem from [26]?

(c) A possibly related question: is there a large deviation principle associated with the vector equilibrium problem? See e.g. [2] for the large deviations interpretation of the equilibrium problem for the one matrix model.

(d) Our analysis is restricted to even potentials $V$ and $W$. This restriction provides a symmetry of the problem around zero, which is the reason why the second measure $\mu_2$ in the vector equilibrium problem is supported on the imaginary axis. If we remove the symmetry then probably we would have to look for a contour that replaces the imaginary axis. It is likely that such a contour would be an $S$-curve in a certain external field, but at this moment we do not know how to handle this situation. See [33, 37] for important recent developments around $S$-curves for scalar equilibrium problems.

(e) Extensions to higher degree $W$ is wide open. If $\deg W = d$ then one would expect a vector equilibrium problem for $d - 1$ measures. It may be that $S$-curves are needed for $d \geq 6$, even in the case of even potentials.

(f) Exploration of further critical phenomenon in the two matrix model.
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