Composition operators on Bloch and Hardy type spaces

Shaolin Chen¹,² · Hidetaka Hamada³ · Jian-Feng Zhu⁴

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Abstract
The main purpose of this paper is to discuss Hardy type spaces, Bloch type spaces and the composition operators of complex-valued harmonic functions. We first establish a sharp estimate of the Lipschitz continuity of complex-valued harmonic functions in Bloch type spaces with respect to the pseudo-hyperbolic metric, which gives an answer to an open problem. Then some classes of composition operators on Bloch and Hardy type spaces will be investigated. The obtained results improve and extend some corresponding known results.

Keywords Bloch type space · Complex-valued harmonic function · Composition operator · Hardy type space

Mathematics Subject Classification Primary 31A05 · 47B33; Secondary 30H10 · 30H30

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Shaolin Chen
mathechen@126.com

Hidetaka Hamada
h.hamada@ip.kyusan-u.ac.jp

Jian-Feng Zhu
flandy@hqu.edu.cn

¹ College of Mathematics and Statistics, Hengyang Normal University, Hengyang 421002, Hunan, People’s Republic of China
² Hunan Provincial Key Laboratory of Intelligent Information Processing and Application, Changsha 421002, People’s Republic of China
³ Faculty of Science and Engineering, Kyushu Sangyo University, 3-1 Matsukadai 2-Chome, Higashi-ku, Fukuoka 813-8503, Japan
⁴ School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China
1 Introduction

Recently, the characterizations of composition operators on Bloch and Hardy type spaces have been attracted much attention of many mathematicians (one can see the references [1, 6, 14–16, 19, 20, 24, 29, 33, 34] for more details). This paper is mainly motivated by the results given by Chen et al. [6], Ghatage et al. [14], Huang et al. [21], and Madigan [25]. In order to state our main results, we need to recall some basic definitions and some results which motivate the present work.

Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk, and let \( T = \partial D \) be the unit circle.

For \( z = x + iy \in \mathbb{C} \), the complex formal differential operators are defined by
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

For \( \alpha \in [0, 2\pi] \), the directional derivative of a complex-valued harmonic function \( f \) at \( z \in D \) is defined by
\[
\partial_\alpha f(z) = \lim_{\rho \to 0^+} \frac{f(z + \rho e^{i\alpha}) - f(z)}{\rho} = f_z(z)e^{i\alpha} + f_{\bar{z}}(z)e^{-i\alpha},
\]
where \( f_z =: \partial f/\partial z \), \( f_{\bar{z}} =: \partial f/\partial \bar{z} \) and \( \rho \) is a positive real number such that \( z + \rho e^{i\alpha} \in D \). Then
\[
\Lambda f(z) := \max \{|\partial_\alpha f(z)| : \alpha \in [0, 2\pi]\} = |f_z(z)| + |f_{\bar{z}}(z)|.
\]

It is well-known that every complex-valued harmonic function \( f \) defined in a simply connected domain \( \Omega \) admits the canonical decomposition \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \Omega \) with \( g(0) = 0 \).

Denote by \( \mathcal{A} \) and \( \mathcal{H} \) the set of all analytic functions of \( D \) into \( \mathbb{C} \) and all complex-valued harmonic functions of \( D \) into \( \mathbb{C} \), respectively.

Throughout of this paper, we use the symbol \( C \) to denote the various positive constants, whose value may change from one occurrence to another.

2 Preliminaries and main results

2.1 Hardy type spaces

For \( p \in (0, +\infty) \), the generalized Hardy space \( \mathcal{H}_G^p(D) \) consists of all those measurable functions \( f : D \to \mathbb{C} \) such that, for \( 0 < p < +\infty \),
\[
\|f\|_p := \sup_{0<r<1} M_p(r, f) < +\infty,
\]
and, for \( p = +\infty \),
\[
\|f\|_p := \sup_{z \in D} |f(z)| < +\infty,
\]
where

\[ M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f \left( re^{i\theta} \right) \right|^p \, d\theta \right)^{\frac{1}{p}}. \]

Let \( \mathcal{H}^p_H(\mathbb{D}) = \mathcal{H}^p_G(\mathbb{D}) \cap \mathcal{H} \) be the harmonic Hardy space. The classical Hardy space \( \mathcal{H}^p(\mathbb{D}) \), that is, all the elements are analytic, is a subspace of \( \mathcal{H}^p_H(\mathbb{D}) \) (see [6, 8, 10, 11, 22, 32]).

### 2.2 Bloch type spaces

A continuous increasing function \( \omega : [0, +\infty) \to [0, +\infty) \) with \( \omega(0) = 0 \) is called a majorant if \( \omega(t)/t \) is non-increasing for \( t > 0 \) (see [12, 13, 26]). For \( \alpha > 0, \beta \in \mathbb{R}, 1 \leq p \leq +\infty \) and a majorant \( \omega \), we use \( \mathcal{B}_{\alpha,\beta}^{\omega,p} \) to denote the harmonic Bloch type space consisting of all \( f \in \mathcal{H} \) with the norm

\[ \| f \|_{\mathcal{B}_{\alpha,\beta}^{\omega,p}} = \| f \|_{\mathcal{B}_{\alpha,\beta}^{\omega,p}}(z) < +\infty, \]

where

\[ M^p_{\omega,f}(|z|, \Lambda_f(z)) = M_p(|z|, \Lambda_f(z))\omega \left( (1 - |z|^2)^{\alpha \left( \log \frac{e}{1 - |z|^2} \right)^{\beta}} \right) \]

for \( 1 \leq p < +\infty \), and

\[ \mathcal{B}_{\alpha,\beta}^{\omega,p} \left( z \right) = \Lambda_f(z)\omega \left( (1 - |z|^2)^{\alpha \left( \log \frac{e}{1 - |z|^2} \right)^{\beta}} \right) \]

for \( p = +\infty \). In particular, if \( p = +\infty \), then we let \( \mathcal{B}_{\alpha,\beta}^{\omega} : \mathcal{B}_{\alpha,\beta}^{\omega,p} \), \( \| f \|_{\mathcal{B}_{\alpha,\beta}^{\omega}} = \| f \|_{\mathcal{B}_{\alpha,\beta}^{\omega,p}} \) and \( \mathcal{B}_{\alpha,\beta}^{\omega} \left( z \right) = \mathcal{B}_{\alpha,\beta}^{\omega,p} \left( z \right) \) for \( z \in \mathbb{D} \). Set

\[ \| f \|_{\mathcal{B}_{\alpha,\beta}^{\omega}} = \sup_{z \in \mathbb{D}} \mathcal{B}_{\alpha,\beta}^{\omega,p} \left( z \right) \]

be the semi-norm. For the characterizations of \( \mathcal{B}_{\alpha,\beta}^{\omega,p} \), we refer to the reference [6]. In this paper, we mainly focus on the case \( p = +\infty \). The case of \( p \in [1, +\infty) \) is probably of independent interest.

In particular, if \( \omega(t) = t \) and \( f \in \mathcal{B}_{\alpha,\beta}^{1,0} \) (\( f \in \mathcal{B}_{\alpha,\beta}^{1,0} \) resp.), then we call \( f \) the harmonic \( \alpha \)-Bloch mapping (the harmonic Bloch mapping resp.). Moreover, if \( \omega(t) = t \) and \( f \in \mathcal{B}_{\alpha,\beta}^{1,0} \cap \mathcal{A} \), then we call \( f \) the analytic Bloch mapping.

For \( z, w \in \mathbb{D} \), the pseudo-hyperbolic metric is defined as

\[ \rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|. \]

Colonna [9] proved that if \( f \in \mathcal{H} \) satisfies

\[ \sup_{z, w \in \mathbb{D}, z \neq w} \frac{|f(z) - f(w)|}{\sigma(z, w)} < +\infty, \quad (2.1) \]
then \( f \in \mathcal{B}^{1,0}_{H_0} \), where \( \omega(t) = t \) and
\[
\sigma(z, w) = \frac{1}{2} \log \left( \frac{1 + \rho(z, w)}{1 - \rho(z, w)} \right) = \text{arctanh}(\rho(z, w))
\]
is the hyperbolic distance between \( z \) and \( w \) in \( \mathbb{D} \). It is easy to conclude the converse, so \( f \in \mathcal{H} \) satisfies (2.1) if and only if \( f \in \mathcal{B}^{1,0}_{H_0} \).

For \( \omega(t) = t \) and an analytic Bloch mapping \( f \), Ghatage et al. [14] proved that \( \mathcal{B}^{1,0}_{\omega, f}(z) \) is Lipschitz continuous with respect to the pseudo-hyperbolic metric, which is given as follows.

**Theorem A** ([14, Theorem 1]) Let \( \omega(t) = t \) and \( f \) be an analytic Bloch mapping. Then, for all \( z_1, z_2 \in \mathbb{D} \),
\[
\left| \mathcal{B}^{1,0}_{\omega, f}(z_1) - \mathcal{B}^{1,0}_{\omega, f}(z_2) \right| \leq 3.31 \| f \|_{\mathcal{B}^{1,0}_{H_0}} \rho(z_1, z_2).
\]

In [19], Hosokawa and Ohno showed that if \( f \) is an analytic Bloch mapping, then, for all \( z_1, z_2 \in \mathbb{D} \),
\[
\left| \mathcal{B}^{1,0}_{\omega, f}(z_1) - \mathcal{B}^{1,0}_{\omega, f}(z_2) \right| \leq 20 \| f \|_{\mathcal{B}^{1,0}_{H_0}} \rho(z_1, z_2), \tag{2.2}
\]
where \( \omega(t) = t \). They used (2.2) to discuss the composition operators on the Bloch spaces (see [19, 20]). On some related discussions of (2.2), we refer to [5, 17, 31].

It is natural to ask whether Theorem A also holds for harmonic Bloch mappings.

The following explanations show that the answer to this question is positive. For \( \omega(t) = t \), let \( f = h + g \) be a harmonic Bloch mapping, where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Obviously, \( h \) and \( g \) are analytic Bloch mappings. Then, by Theorem A, we see that there is a positive constant \( C \) such that
\[
\left| \mathcal{B}^{1,0}_{\omega, f}(z_1) - \mathcal{B}^{1,0}_{\omega, f}(z_2) \right| \leq C \| h \|_{\mathcal{B}^{1,0}_{H_0}} \rho(z_1, z_2) + C \| g \|_{\mathcal{B}^{1,0}_{H_0}} \rho(z_1, z_2) 
\]
\[
\leq 2C \| f \|_{\mathcal{B}^{1,0}_{H_0}} \rho(z_1, z_2), \tag{2.3}
\]
which implies that \( \mathcal{B}^{1,0}_{\omega, f}(z) \) is also Lipschitz continuous with respect to the pseudo-hyperbolic metric.

In [21, Theorem 1.1], Huang et al. showed that the Lipschitz constant in (2.3) is \( C = 3\sqrt{3}/2 \), which is as follows.
\[
\left| \mathcal{B}^{1,0}_{\omega, f}(z_1) - \mathcal{B}^{1,0}_{\omega, f}(z_2) \right| \leq 3\sqrt{3} \| f \|_{\mathcal{B}^{1,0}_{H_0}} \rho(z_1, z_2).
\]

Furthermore, Huang et al. [21, Open problems] raised the following open problem.

**Question 2.1** Let
\[
C = \sup_{z_1 \neq z_2, z_1, z_2 \in \mathbb{D}, \| f \|_{\mathcal{B}^{1,0}_{H_0}} = 1} \left| \mathcal{B}^{1,0}_{\omega, f}(z_1) - \mathcal{B}^{1,0}_{\omega, f}(z_2) \right| / \rho(z_1, z_2),
\]
where \( \omega(t) = t \) and \( f \) is a harmonic Bloch mapping. What is the value of \( C \)?

In the following, we give an answer to Question 2.1.
Theorem 2.2 Let $\omega(t) = t$ and $f$ be a harmonic Bloch mapping. Then, for all $z_1, z_2 \in \mathbb{D}$,

$$\left| \mathcal{B}_{\omega, f}^{1, 0}(z_1) - \mathcal{B}_{\omega, f}^{1, 0}(z_2) \right| \leq \frac{3\sqrt{3}}{2} \| f \|_{\mathcal{B}_{\omega}^{1, 0} \cap \mathcal{A}} \rho(z_1, z_2). \quad (2.4)$$

Moreover, the constant $3\sqrt{3}/2$ in (2.4) is sharp.

2.3 Composition operators

Given an analytic self mapping $\phi$ of the unit disk $\mathbb{D}$, the composition operator $C_\phi : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_\phi(f) = f \circ \phi,$$

where $f \in \mathcal{H}$.

In [29], Shapiro obtained a complete characterization of compact composition operators on $\mathcal{H}^2(\mathbb{D})$, with a number of interesting consequences for peak sets, essential norm of composition operators, and so on. Recently, the studies of composition operators on analytic function spaces have been attracted much attention of many mathematicians (see references [1, 6, 14, 16, 19, 20, 24, 33]). In particular, the characterizations of composition operators of the Bloch spaces to the Hardy spaces were investigated by Abakumov and Doubtsov [1], Hosokawa and Ohno [20], and Kwon [24]. However, there are few literatures on the theory of composition operators of complex-valued harmonic functions. In the following, we will discuss the characterizations of composition operators on complex-valued harmonic functions.

As an application of Theorem A, Ghatage et al. [14] obtained the following result.

Theorem B ([14, Theorem 2]) Let $\omega(t) = t$ and $\phi$ be an analytic self mapping of the unit disk $\mathbb{D}$. If for some constants $r \in (0, 1/4)$, and $\epsilon > 0$, for each $w \in \mathbb{D}$, there is a point $z_w \in \mathbb{D}$ such that

$$\rho(\phi(z_w), w) < r, \quad \text{and} \quad \frac{1 - |z_w|^2}{1 - |\phi(z_w)|^2} |\phi'(z_w)| > \epsilon,$$

then, there is a constant $C > 0$ depending only on $r$ and $\epsilon$ such that

$$\| C_\phi(f) \|_{\mathcal{B}_{\omega}^{1, 0} \cap \mathcal{A}} \geq C \| f \|_{\mathcal{B}_{\omega}^{1, 0}} \quad (2.5)$$

In the following, by using Theorem 2.2, we extend Theorem B to complex-valued harmonic functions, and also show that “$r \in (0, 1/4)$” in Theorem B can be replaced by a larger interval “$r \in (0, 2\sqrt{3}/9)$”.

Theorem 2.3 Let $\omega(t) = t$ and $\phi$ be an analytic self mapping of the unit disk $\mathbb{D}$. If for some constants $r \in (0, 2\sqrt{3}/9)$, and $\epsilon > 0$, for each $w \in \mathbb{D}$, there is a point $z_w \in \mathbb{D}$ such that

$$\rho(\phi(z_w), w) < r, \quad \text{and} \quad \frac{1 - |z_w|^2}{1 - |\phi(z_w)|^2} |\phi'(z_w)| > \epsilon,$$

then, there is a constant $C > 0$ depending only on $r$ and $\epsilon$ such that

$$\| C_\phi(f) \|_{\mathcal{B}_{\omega}^{1, 0} \cap \mathcal{A}} \geq C \| f \|_{\mathcal{B}_{\omega}^{1, 0}} \quad (2.5)$$
for all \( f \in \mathcal{B}^{1,0}_{H_\omega} \).

In Chen et al. [6], obtained the following result.

**Theorem C** ([6, Theorem 6]) Let \( \alpha \in (0, +\infty) \), \( \beta \leq \alpha \), \( \omega(t) = t \) and \( \phi : \mathbb{D} \to \mathbb{D} \) be an analytic function. Then the followings are equivalent:

1. \( C_\phi : \mathcal{B}^{\alpha,\beta}_{H_\omega} \cap \mathcal{A} \to \mathcal{H}^2(\mathbb{D}) \) is a bounded operator;
2. \( \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{|\phi'(re^{i\theta})|^2}{(1 - |\phi(re^{i\theta})|)\alpha \left( \log \frac{e}{1 - |\phi(re^{i\theta})|} \right)} \left( 1 - r \right) \, dr \, d\theta < +\infty \).

We improve and generalize Theorem C into the following form.

**Theorem 2.4** Let \( p \in (0, +\infty) \), \( \alpha \in (0, +\infty) \), \( \beta \leq \alpha \), \( \omega \) be a majorant and \( \phi : \mathbb{D} \to \mathbb{D} \) be an analytic function. Then the followings are equivalent:

1. \( C_\phi : \mathcal{B}^{\alpha,\beta}_{H_\omega} \to \mathcal{H}^p(\mathbb{D}) \) is a bounded operator;
2. \( \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^1 \frac{|\phi'(re^{i\theta})|^2}{\omega^2 \left( \left( 1 - |\phi(re^{i\theta})| \right)^\alpha \left( \log \frac{e}{1 - |\phi(re^{i\theta})|} \right)^\beta \right)} \left( 1 - r \right) \, dr \right)^{\frac{p}{2}} \, d\theta < +\infty \);
3. \( C_\phi : \mathcal{B}^{\alpha,\beta}_{H_\omega} \to \mathcal{H}^p(\mathbb{D}) \) is compact.

In the other direction, Madigan [25, Theorem 3.20] proved that, for \( p \in [1, +\infty) \),

\[
C_\phi : \mathcal{H}^p(\mathbb{D}) \to \mathcal{B}^{1,0}_{H_\omega} \cap \mathcal{A}
\]

is a bounded operator if and only if

\[
\sup_{z \in \mathbb{D}} \left\{ \frac{|\phi'(z)| \left( 1 - |z|^2 \right)}{(1 - |\phi(z)|^2)^{1 + \frac{1}{p}}} \right\} < +\infty,
\]

where \( \omega(t) = t \) and \( \phi : \mathbb{D} \to \mathbb{D} \) is an analytic function.

In the following, Pérez-González and Xiao generalized [25, Theorem 3.20] to \( p \in (0, +\infty) \) and also obtained the compactness characterization.

**Theorem D** ([28, Theorem 3.1]) Let \( \omega(t) = t \), \( p \in (0, +\infty) \) and \( \phi : \mathbb{D} \to \mathbb{D} \) be an analytic function. Then

1. \( C_\phi : \mathcal{H}^p(\mathbb{D}) \to \mathcal{B}^{1,0}_{H_\omega} \cap \mathcal{A} \) is a bounded operator if and only if

\[
\sup_{z \in \mathbb{D}} \left\{ \frac{|\phi'(z)| \left( 1 - |z|^2 \right)}{(1 - |\phi(z)|^2)^{1 + \frac{1}{p}}} \right\} < +\infty;
\]

2. \( C_\phi : \mathcal{H}^p(\mathbb{D}) \to \mathcal{B}^{1,0}_{H_\omega} \cap \mathcal{A} \) is a compact operator if and only if

\[
\lim_{\phi(z) \to T} \frac{|\phi'(z)| \left( 1 - |z|^2 \right)}{(1 - |\phi(z)|^2)^{1 + \frac{1}{p}}} = 0.
\]

By analogy with Theorem D, we improve and generalize [25, Theorem 3.20] into the following form.
Theorem 2.5 Let \( p \in (1, +\infty) \), \( \omega \) be a majorant with \( \lim_{t \to 0^+} (\omega(t)/t) < +\infty \) and \( \phi : \mathbb{D} \to \mathbb{D} \) be an analytic function. Assume that \( \alpha = 1 \) and \( \beta \leq 0 \) or \( \alpha > 1 \) and \( \beta \in \mathbb{R} \). Then

1. \( C_\phi : \mathcal{A}_H^p(\mathbb{D}) \to \mathcal{B}_{H_\omega}^{\alpha,\beta} \) is a bounded operator if and only if

\[
\sup_{z \in \mathbb{D}} \left\{ |\phi'(z)|\omega \left( (1 - |z|^2)^\alpha \left( \log \frac{1}{1-|z|^2} \right)^\beta \right) \right\} < +\infty; \tag{2.6}
\]

2. \( C_\phi : \mathcal{A}_H^p(\mathbb{D}) \to \mathcal{B}_{H_\omega}^{\alpha,\beta} \) is a compact operator if and only if

\[
\lim_{\phi(z) \to \infty} \frac{|\phi'(z)|\omega \left( (1 - |z|^2)^\alpha \left( \log \frac{1}{1-|z|^2} \right)^\beta \right)}{(1 - |\phi(z)|^2)^{1+1/\beta}} = 0. \tag{2.7}
\]

The proofs of Theorems 2.2–2.5 will be presented in Sect. 3.

3 Proofs of the main results

Let \( n = 1 \) in [7, Lemma 1.1] and [7, Theorem 1.2], respectively. Then we obtain the following results.

Lemma E For \( x \in [0, 1] \), let

\[
\psi(x) = \sqrt{1 + 2\alpha} \left( \frac{1}{2\alpha} \right)^\alpha (1 - x^2)^\alpha.
\]

and \( a_0(\alpha) = 1/\sqrt{1 + 2\alpha} \), where \( \alpha \in (0, +\infty) \) is a constant. Then \( \psi \) is increasing in \([0, a_0(\alpha)]\), decreasing in \([a_0(\alpha), 1]\) and \( \psi(a_0(\alpha)) = 1 \).

Theorem F Suppose \( \omega(t) = t \) and \( f \in \mathcal{B}_{H_\omega}^{\alpha,0} \) is a holomorphic mapping satisfying \( \|f\|_{\mathcal{B}_{H_\omega}^{\alpha,0}} = 1 \) and \( f'(0) = 0 \in (0, 1) \), where \( \alpha \in (0, +\infty) \) is a constant. Then, for all \( z \) with \( |z| \leq a_0(\alpha) \), we have

\[
\text{Re}(f'(z)) \geq \frac{r_0(m_\alpha(r_0) - |z|)}{m_\alpha(r_0)(1 - m_\alpha(r_0)|z|)^{1+2\alpha}}, \tag{3.1}
\]

where \( m_\alpha(r_0) \) is the unique real root of the equation \( \psi(x) = r_0 \) in the interval \([0, a_0(\alpha)]\) and, \( \psi \) and \( a_0(\alpha) \) are defined as in Lemma E. Moreover, for all \( z \) with \( |z| \leq \frac{a_0(\alpha) - m_\alpha(r_0)}{1 - a_0(\alpha)m_\alpha(r_0)} \), we have

\[
|f'(z)| \leq \frac{r_0(m_\alpha(r_0) + |z|)}{m_\alpha(r_0)(1 + m_\alpha(r_0)|z|)^{1+2\alpha}}. \tag{3.2}
\]

Furthermore, the estimates of (3.1) and (3.2) are sharp.

Remark 3.1 Theorem F also holds by replacing the assumption \( \|f\|_{\mathcal{B}_{H_\omega}^{\alpha,0}} = 1 \) by \( \|f\|_{\mathcal{B}_{H_\omega}^{\alpha,0}} \leq 1 \) as in Bonk et al. [3].
3.1 The proof of Theorem 2.2

Since $\mathbb{D}$ is a simply connected domain, we see that $f$ admits the canonical decomposition $f = h + \bar{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$ with $g(0) = 0$. If $\|f\|_{H^1_{\omega, \bar{t}}} = 0$, then it is easy to conclude (2.4). Without loss of generality, we may assume that $\|f\|_{H^1_{\omega, \bar{t}}} = 1$ and $\mathcal{R}_{\omega, f}(z_1) \leq \mathcal{R}_{\omega, f}(z_2)$. For $z \in \mathbb{D}$, let

$$\varphi(z) = \frac{z_2 - z}{1 - \overline{z}_2 z}, \quad w = \varphi^{-1}(z_1) \quad \text{and} \quad F(z) = f(\varphi(z)) = H(z) + \overline{G}(z),$$

where $H = h \circ \varphi$ and $G = g \circ \varphi$. Then $\|F\|_{H^1_{\omega, t}} = \|f\|_{H^1_{\omega, \bar{t}}} = 1$ and

$$\rho(z_1, z_2) = \rho(\varphi^{-1}(z_1), \varphi^{-1}(z_2)) = \rho(w, 0) = |w|.$$

Elementary calculations lead to

$$\varphi(0) = z_2 \quad \text{and} \quad |\varphi'(w)| = 1 - |\varphi(w)|^2 \quad \text{and} \quad \rho(z_1, z_2) = \rho(\varphi^{-1}(z_1), \varphi^{-1}(z_2)) = \rho(w, 0) = |w|.$$

which imply that

$$\mathcal{R}_{\omega, f}(0) = \mathcal{R}_{\omega, f}(\varphi(0)) = \mathcal{R}_{\omega, f}(z_2) \quad (3.3)$$

and

$$\mathcal{R}_{\omega, f}(w) = (1 - |w|^2) \Lambda_f(\varphi(w)) |\varphi'(w)| = \mathcal{R}_{\omega, f}(z_1). \quad (3.4)$$

**Case 1.** If $\mathcal{R}_{\omega, f}(0) = 0$, then it is easy to conclude (2.4).

**Case 2.** Let $\mathcal{R}_{\omega, f}(0) \neq 0$.

In this case, for $z \in \mathbb{D}$, let $F_{\theta_0}(z) = H(z) + e^{i\theta_0} G(z)$, where $\theta_0 \in [0, 2\pi]$ is a real number such that $|F_{\theta_0}(0)| = \mathcal{R}_{\omega, f}(0)$. Then $\|F_{\theta_0}\|_{H^1_{\omega, t}} \leq 1$. Set $F_{\theta_0}'(0) = \beta e^{i\theta}$, where $\beta = \mathcal{R}_{\omega, f}(0)$. An application of Theorem F and Remark 3.1 to $e^{-i\theta} F_{\theta_0}(z)$ gives that, for $|z| \leq \frac{a_0(1) + m_1(\beta)}{1 + a_0(1)m_1(\beta)}$,

$$\Lambda_f(z) \geq \text{Re}(e^{-i\theta} F_{\theta_0}'(z)) \geq \frac{\beta(m_1(\beta) - |z|)}{m_1(\beta)(1 - m_1(\beta)|z|)^3}, \quad (3.5)$$

where

$$\frac{3\sqrt{3}}{2} m_1(\beta)(1 - m_1^2(\beta)) = \beta. \quad (3.6)$$

**Subcase 2.1.** Suppose that $|w| \leq m_1(\beta)$.

Since

$$|w| \leq m_1(\beta) \leq \frac{a_0(1) + m_1(\beta)}{1 + a_0(1)m_1(\beta)},$$

and

$$1 - |w|^2 \geq 1 - m_1(\beta)|w| \geq (1 - m_1(\beta)|w|)^3,$$
by (3.5), we see that
\[
\beta - \mathcal{B}_{\omega, F}^{1,0}(w) \leq \beta - \frac{(1 - |w|^2)\beta(m_1(\beta) - |w|)}{m_1(\beta)(1 - m_1(\beta)|w|)^3} \\
= \frac{\beta}{m_1(\beta)} \left( m_1(\beta) - \frac{(1 - |w|^2)(m_1(\beta) - |w|)}{(1 - m_1(\beta)|w|)^3} \right) \\
\leq \frac{\beta}{m_1(\beta)} (m_1(\beta) - (m_1(\beta) - |w|)) \\
= \frac{\beta}{m_1(\beta)} |w|.
\] 
(3.7)

It follows from (3.3), (3.4), (3.6) and (3.7) that
\[
\mathcal{B}_{\omega, f}^{1,0}(z_2) - \mathcal{B}_{\omega, f}^{1,0}(z_1) = \mathcal{B}_{\omega, F}^{1,0}(0) - \mathcal{B}_{\omega, F}^{1,0}(w) = \beta - (1 - |w|^2) \Lambda_F(w) \\
\leq \frac{\beta}{m_1(\beta)} |w| = \frac{3\sqrt{3}}{2} |w| \left( 1 - m_1^2(\beta) \right) \\
\leq \frac{3\sqrt{3}}{2} |w|.
\]

**Subcase 2.2.** Suppose that \(|w| \geq m_1(\beta)|.

By (3.3), (3.4) and (3.6), we have
\[
\mathcal{B}_{\omega, f}^{1,0}(z_2) - \mathcal{B}_{\omega, f}^{1,0}(z_1) = \mathcal{B}_{\omega, F}^{1,0}(0) - \mathcal{B}_{\omega, F}^{1,0}(w) = \beta - (1 - |w|^2) \Lambda_F(w) \\
\leq \beta = \frac{3\sqrt{3}}{2} m_1(\beta)(1 - m_1^2(\beta)) \\
\leq \frac{3\sqrt{3}}{2} (1 - m_1^2(\beta)) |w| \\
\leq \frac{3\sqrt{3}}{2} |w|.
\]

Combining Subcases 2.1 and 2.2 gives the desired result.

Now we prove the sharpness part. For any \(\varepsilon \in (0, 3\sqrt{3}/2]\), let
\[
m^* = \min \left\{ a_0(1), \sqrt{2\sqrt{3}\varepsilon/3} \right\}
\] 
(3.8)
and
\[
\beta = \frac{3\sqrt{3}}{2} m^* (1 - (m^*)^2).
\] 
(3.9)

Then \(m_1(\beta) = m^*\) and it follows from (3.8) that
\[
\frac{3\sqrt{3}}{2} \left( 1 - (m_1(\beta))^2 \right) \geq \frac{3\sqrt{3}}{2} - \varepsilon.
\] 
(3.10)

For \(z \in \mathbb{D}\), let
\[
f_\beta(z) = \int_0^z \frac{\beta(m_1(\beta) - \xi)}{m_1(\beta)(1 - m_1(\beta)\xi)^3} d\xi.
\]
which implies that $f'_\beta(0) = \beta$. Next we prove $\|f_\beta\|_{B_{H_0, s}^{1, 0}} = 1$. For $z \in \mathbb{D}$, let

$$
\eta(z) = -\frac{3\sqrt{3}}{4}z^2.
$$

By Lemma E, we obtain

$$
\|\eta\|_{B_{H_0, s}^{1, 0}} = \sup_{z \in \mathbb{D}} B_{H_0, s}^{1, 0}(z) = \sup_{z \in \mathbb{D}} \left\{ \frac{3\sqrt{3}}{2} |z| (1 - |z|^2) \right\} = 1.
$$

(3.11)

For $z \in \mathbb{D}$, let

$$
\phi_a(z) = \frac{m_1(\beta) - z}{1 - m_1(\beta)z},
$$

where $a = m_1(\beta)$. Then we see that

$$
B_{H_0, f_{\beta}}^{1, 0}(z) = \frac{3\sqrt{3}}{2} (1 - |z|^2) m_1(\beta) - z (1 - (m_1(\beta))^2)
$$

$$
|1 - m_1(\beta)z|^3
$$

$$
= (1 - |z|^2) |\eta(\phi_a(z))|'
$$

$$
= B_{H_0, \eta}^{1, 0}(\phi_a(z)),
$$

which, together with (3.11), yields that $\|f_\beta\|_{B_{H_0, s}^{1, 0}} = 1$. Let $z_1 = m_1(\beta)$ and $z_2 = 0$. Then, by (3.9), we have

$$
\left| B_{H_0, f_{\beta}}^{1, 0}(z_2) - B_{H_0, f_{\beta}}^{1, 0}(z_1) \right| = \beta = \frac{3\sqrt{3}}{2} m_1(\beta) (1 - (m_1(\beta))^2)
$$

$$
= \frac{3\sqrt{3}}{2} (1 - (m_1(\beta))^2) \rho(z_2, z_1),
$$

which, together with (3.10), implies that

$$
\left| B_{H_0, f_{\beta}}^{1, 0}(z_2) - B_{H_0, f_{\beta}}^{1, 0}(z_1) \right| \geq \left( \frac{3\sqrt{3}}{2} - \varepsilon \right) \rho(z_2, z_1).
$$

Therefore, the above inequality shows that the constant $3\sqrt{3}/2$ is sharp. The proof of this theorem is finished.

We obtain the following lemma as in [4, Lemma 1] and [18, Lemma 2.14].

**Lemma 3.2** (2.5) holds on $B_{H_0}^{1, 0}$ if and only if there exists a constant $\delta \in (0, 1]$ such that

$$
\|C_\phi(f)\|_{B_{H_0, s}^{1, 0}} \geq \delta \|f\|_{B_{H_0, s}^{1, 0}}, \quad \forall f \in B_{H_0}^{1, 0}.
$$

**Proof** Assume that there exists a $\delta \in (0, 1]$ such that $\|f \circ \phi\|_{B_{H_0, s}^{1, 0}} \geq \delta \|f\|_{B_{H_0, s}^{1, 0}}$ for all $f \in B_{H_0}^{1, 0}$. Let $f \in B_{H_0}^{1, 0}$. Then, we have

$$
|f(0)| \leq |f(\phi(0))| + |f(0) - f(\phi(0))|
$$

$$
\leq |f(\phi(0))| + \|f\|_{B_{H_0, s}^{1, 0}} \sigma(0, \phi(0))
$$

$$
= |f(\phi(0))| + \frac{\|f\|_{B_{H_0, s}^{1, 0}}}{2} \log \frac{1 + |\phi(0)|}{1 - |\phi(0)|}.
$$
Therefore, we have
\[
\| f \|_{\mathcal{B}_{H_0}^{1,0}} \leq |f(\phi(0))| + \frac{\| f \circ \phi \|_{\mathcal{B}_{H_0}^{1,0}}}{\delta} \left\{ 1 + \frac{1}{2} \log \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right\} + \frac{1}{\delta} \left\{ 1 + \frac{1}{2} \log \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right\}.
\]
This implies that (2.5) holds on $\mathcal{B}_{H_0}^{1,0}$.

The converse can be proved by using arguments similar to those in the proof of [4, Lemma 1]. This completes the proof.

\[ \square \]

3.2 The proof of Theorem 2.3

Without loss of generality, we may assume that $\| f \|_{\mathcal{B}_{H_0}^{1,0}} = 1$, where $\omega(t) = t$. Let $a \in (0, 1)$ be arbitrarily fixed. There is a point $w \in \mathbb{D}$ such that
\[
\mathcal{B}_{\omega, f}^{1,0}(w) > 1 - \delta,
\]
where $\delta = a(1 - 3\sqrt{3}r/2)$ and $r < 2\sqrt{3}/9$. From the assumption, we see that there is a point $z_w$ such that
\[
\rho(\phi(z_w), w) < r < \frac{2\sqrt{3}}{9}, \text{ and } \frac{1 - |z_w|^2}{1 - |\phi(z_w)|^2}|\phi'(z_w)| > \epsilon,
\]
which, together with (2.4) and (3.12), imply that
\[
\mathcal{B}_{\omega, f}^{1,0}(\phi(z_w)) \geq \mathcal{B}_{\omega, f}^{1,0}(w) - \frac{3\sqrt{3}}{2}\rho(\phi(z_w), w)
\]
\[
\geq 1 - \delta - \frac{3\sqrt{3}}{2}r
\]
\[
= (1 - a) \left( 1 - \frac{3\sqrt{3}}{2}r \right)
\]
\[
> 0.
\]

From (3.13), we conclude that
\[
\| C_\phi(f) \|_{\mathcal{B}_{H_0}^{1,0}} \geq \mathcal{B}_{\omega, f}^{1,0}(\phi(z_w)) \frac{1 - |z_w|^2}{1 - |\phi(z_w)|^2}|\phi'(z_w)| \geq (1 - a) \left( 1 - \frac{3\sqrt{3}}{2}r \right) \epsilon.
\]

Since $a \in (0, 1)$ is arbitrary, we have
\[
\| C_\phi(f) \|_{\mathcal{B}_{H_0}^{1,0}} \geq \left( 1 - \frac{3\sqrt{3}}{2}r \right) \epsilon,
\]
which, together with Lemma 3.2, gives (2.5). The proof of this theorem is complete. \[ \square \]
The following result is well-known.

**Lemma G** (cf. [6, Lemma 5]) Suppose that \( a, \ b \in [0, +\infty) \) and \( \tau \in (0, +\infty) \). Then
\[
(a + b)^\tau \leq 2^{\max\{\tau - 1, 0\}}(a^{\tau} + b^{\tau}).
\]

Consider a weight \( \varphi : [0, 1) \to (0, +\infty) \), that is, \( \varphi \) is a non-decreasing, continuous, unbounded function. Furthermore, a weight function \( \varphi \) is called doubling if there is a constant \( C > 1 \) such that
\[
\varphi(1 - s/2) < C\varphi(1 - s)
\]
for \( s \in (0, 1) \) (see [1]).

**Lemma 3.3** Let \( \alpha \in (0, +\infty), \beta \leq \alpha \) and \( \omega \) be a majorant. Then \( \varphi(t) = 1/\omega(\chi(t)) \) is a doubling function, where
\[
\chi(t) = (1 - t^2)^\alpha \left( \log \frac{e}{1 - t^2} \right)^\beta
\]
for \( t \in [0, 1) \).

**Proof** Since \( \omega(x)/x \) is non-increasing for \( x > 0 \) and \( \chi(t) \) is decreasing for \( t > 0 \), we see that, for \( s \in (0, 1) \),
\[
\frac{\varphi(1 - s/2)}{\varphi(1 - s)} = \frac{\omega(\chi(1 - s))}{\omega(\chi(1 - s/2))} = \frac{\omega(\chi(1 - s/2))}{\omega(\chi(1 - s/2))} \cdot \frac{\chi(1 - s)}{\chi(1 - s/2)} \leq \frac{\chi(1 - s)}{\chi(1 - s/2)}.
\]
Elementary computations give
\[
\lim_{s \to 0} \frac{\chi(1 - s)}{\chi(1 - s/2)} = 2^\alpha
\]
and
\[
\lim_{s \to 1} \frac{\chi(1 - s)}{\chi(1 - s/2)} = \left( \frac{4}{3} \right)^\alpha \frac{1}{(1 + \log \frac{4}{3})^\beta},
\]
which, together with (3.14) and the continuity of \( \chi(1 - s)/\chi(1 - s/2) \), implies that there is a constant \( C > 1 \) such that
\[
\varphi\left(1 - \frac{s}{2}\right) < C\varphi(1 - s).
\]
Hence \( \varphi \) is a doubling function.

Denote by \( L^p(\mathbb{T}) \) (\( p \in (0, +\infty) \)) the set of all measurable functions \( F \) of \( \mathbb{T} \) into \( \mathbb{C} \) with
\[
\|F\|_{L^p} = \left( \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty.
\]

Given \( f \in \mathscr{A} \), the Littlewood-Paley \( \mathcal{G} \)-function is defined as follows
\[
\mathcal{G}(f)(\zeta) = \left( \int_0^1 |f'(r\zeta)|^2 (1 - r) dr \right)^{\frac{1}{2}}, \quad \zeta \in \partial\mathbb{D}.
\]
It is well known that
\[ f \in \mathcal{H}^p(\mathbb{D}) \text{ if and only if } \mathcal{G}(f) \in L^p(\mathbb{T}) \]
(3.15)
for \( p \in (0, +\infty) \) (see [2, 23, 24, 27]). Moreover, (3.15) also can be rewritten in the following form. There is a positive constant \( C \), depending only on \( p \), such that
\[
\frac{1}{C} \| f \|_p^p \leq |f(0)|^p + \frac{1}{2\pi} \int_0^{2\pi} |\mathcal{G}(f)(e^{i\theta})|^p d\theta \leq C \| f \|_p^p
\]
(3.16)
for \( p \in (0, +\infty) \) (see [2, 30]).

### 3.3 The proof of Theorem 2.4

We first prove \((2) \Rightarrow (1)\). Without loss of generality, let \( f \in \mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta} \) with \( \| f \|_{\mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta}} \neq 0 \). Since \( \mathbb{D} \) is a simply connected domain, we see that \( f \) admits the canonical decomposition \( f = h_1 + \overline{h_2} \), where \( h_1 \) and \( h_2 \) are analytic in \( \mathbb{D} \) with \( h_2(0) = 0 \). Then \( h_1, h_2 \in \mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta} \). Let
\[
\chi(t) = (1 - t^2)^\alpha \left( \log \frac{e}{1 - t^2} \right)^\beta
\]
for \( t \in [0, 1) \). For \( j = 1, 2 \), elementary calculations lead to
\[
|C_{\phi}(h_j)'(z)| = |h_j'(\phi(z))| |\phi'(z)| \leq \| f \|_{\mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta}} \frac{|\phi'(z)|}{\omega(\chi(|\phi(z)|))},
\]
which implies that
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \mathcal{G}(C_{\phi}(h_j))(e^{i\theta}) \right)^p d\theta
\]
\[
= \int_0^{2\pi} \left( \int_0^1 \left( h_j'(\phi(re^{i\theta}))^2 |\phi'(re^{i\theta})|^2 (1 - r) dr \right) \frac{p}{2\pi} d\theta \right)^{\frac{p}{2}}
\]
\[
\leq \| f \|_{\mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta}} \int_0^{2\pi} \left( \int_0^1 \frac{|\phi'(re^{i\theta})|^2 (1 - r) dr}{\omega^2(\chi(|\phi(re^{i\theta})|))} \right)^{\frac{p}{2}} d\theta
\]
\[
< +\infty,
\]
(3.17)
where \( \theta \in [0, 2\pi] \) and \( \zeta = e^{i\theta} \). It follows from (3.15) and (3.17) that \( C_{\phi}(h_j) \in \mathcal{H}^p(\mathbb{D}) \), where \( j = 1, 2 \).

Next, we will show that \( C_{\phi} \) is a bounded operator. Since \( \| f \|_{\mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta}} \neq 0 \), we see that at least one of the two functions \( h_1 \) and \( h_2 \) is not constant. Without loss of generality, we can assume \( h_1 \) and \( h_2 \) are not constant functions.

Since
\[
|C_{\phi}(h_j)(0)| \leq |h_j(0)| + |h_j(\phi(0)) - h_j(0)|
\]
\[
\leq |h_j(0)| + \| h_j \|_{\mathcal{B}_{\mathcal{H}_w}^{\alpha,\beta}} \frac{|\phi(0)|}{\omega(\chi(|\phi(0)|))},
\]

\( \square \) Springer
by using (3.16), we see that there is a positive constant \( C \), depending only on \( p \) and \( |\phi(0)| \), such that
\[
\|C_\phi(h_j)\|^p_p \leq C \left( |C_\phi(h_j)(0)|^p + \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{F}(C_\phi(h_j))(e^{i\theta}))^p \, d\theta \right)
\leq C\|h_j\|^p_{\mathcal{B}^{\alpha,\beta}_{H_\omega}} \left( 1 + \int_0^{2\pi} \left( \int_0^1 \frac{|\phi'(re^{i\theta})|^2(1-r) \, dr}{\omega^2(\chi(|\phi(re^{i\theta})|))} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \right),
\]
which, together with Lemma G, gives that
\[
\|C_\phi(f)\|^p_p \leq 2^{\max\{p-1,0\}} \left( \|C_\phi(h_1)\|^p_p + \|C_\phi(h_2)\|^p_p \right)
\leq 2^{\max\{p-1,0\}}CC^* \left( \|h_1\|^p_{\mathcal{B}^{\alpha,\beta}_{H_\omega}} + \|h_2\|^p_{\mathcal{B}^{\alpha,\beta}_{H_\omega}} \right)
\leq 2^{1+\max\{p-1,0\}}f_p^{\mathcal{B}^{\alpha,\beta}_{H_\omega}} CC^*,
\]
where
\[
C^* = 1 + \int_0^{2\pi} \left( \int_0^1 \frac{|\phi'(re^{i\theta})|^2(1-r) \, dr}{\omega^2(\chi(|\phi(re^{i\theta})|))} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} < +\infty.
\]
Consequently, \( C_\phi : \mathcal{B}^{\alpha,\beta}_{H_\omega} \to \mathcal{H}^p_\omega(\mathbb{D}) \) is a bounded operator.

Next, we prove (1) \( \Rightarrow \) (2). By Lemma 3.3 and [1, Lemma 1], we see that there are two functions \( f_1, f_2 \in \mathcal{B}^{\alpha,\beta}_{H_\omega} \cap \omega \) such that, for \( z \in \mathbb{D} \),
\[
|f_1'(z)| + |f_2'(z)| \geq \frac{1}{\omega(\chi(|z|))},
\]
which, together with Lemma G, implies that
\[
+\infty > \frac{1}{2\pi} \int_0^{2\pi} \left( \mathcal{F}(C_\phi(f_1))(e^{i\theta}) \right)^p \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \left( \mathcal{F}(C_\phi(f_2))(e^{i\theta}) \right)^p \, d\theta
\geq \frac{C_p}{2\pi} \int_0^{2\pi} \left( \int_0^1 |\phi'(re^{i\theta})|^2(1-r) \, dr \right)^{\frac{p}{2}} \frac{d\theta}{2\pi}
\geq \frac{C_p}{2\pi} \int_0^{2\pi} \left( \int_0^1 \frac{|\phi'(re^{i\theta})|^2(1-r) \, dr}{\omega^2(\chi(|\phi(re^{i\theta})|))} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi},
\]
where \( C_p = 2^{-p/2-\max\{p/2-1,0\}} \).

Now we prove (1) \( \Leftrightarrow \) (3). We only need to prove (1) \( \Rightarrow \) (3) because (3) \( \Rightarrow \) (1) is obvious. Let \( C_\phi : \mathcal{B}^{\alpha,\beta}_{H_\omega} \to \mathcal{H}^p_\omega(\mathbb{D}) \) be a bounded operator. Then \( f \circ \phi \in \mathcal{H}^p_\omega(\mathbb{D}) \) for all \( f \in \mathcal{B}^{\alpha,\beta}_{H_\omega} \), and \( |\phi(\zeta)| := \lim_{r \to 1^-} |\phi(r\zeta)| \leq 1 \) exists for almost every \( \zeta \in \mathbb{T} \). Suppose that \( \{f_n\} = \{h_n + \bar{g}_n\} \) is a sequence in \( \mathcal{B}^{\alpha,\beta}_{H_\omega} \) such that \( \|f_n\|_{\mathcal{B}^{\alpha,\beta}_{H_\omega}} \leq 1 \), where \( h_n \) and \( g_n \) are analytic in \( \mathbb{D} \) with \( g_n(0) = 0 \). We are going to prove that \( \{C_\phi(f_n)\} \) has a convergent subsequence in \( \mathcal{H}^p_\omega(\mathbb{D}) \). Elementary calculations give that, for \( z \in \mathbb{D} \),
\[
|f_n(z)| \leq |f_n(0)| + \|f_n\|_{\mathcal{B}^{\alpha,\beta}_{H_\omega}} \int_0^1 \frac{|z|}{\omega(\chi(|z|t))} \, dt \leq 1 + \int_0^1 \frac{|z|}{\omega(\chi(|z|t))} \, dt,
\]
\( \Box \) Springer
which implies that \( \{f_n\} \) forms a normal family, so that there exists a subsequence of \( \{f_n\} \) that converges uniformly on compact subsets of \( \mathbb{D} \) to a complex-valued harmonic function \( f = h + g \) (see [11, p.80]), where \( h \) and \( g \) are analytic in \( \mathbb{D} \) with \( g(0) = 0 \). Without loss of generality, we may assume that the sequence \( \{f_n\} \) itself converges to \( f \). Moreover, it is not difficult to know that \( h_n \to h \) and \( g_n \to g \) as \( n \to +\infty \) (cf. [11, p.82]). Consequently, 

\[
|f(0)| + \Lambda_f(z)\omega(\chi(|z|)) = \lim_{n \to +\infty} \left\{ |f_n(0)| + \Lambda_{f_n}(z)\omega(\chi(|z|)) \right\} \leq 1,
\]

which implies that \( f \in \mathcal{B}_n^{\alpha,\beta} \) with \( \|f\|_{\mathcal{B}_n^{\alpha,\beta}} \leq 1 \). Since (1) implies (2), we see that

\[
\int_0^1 \frac{|\phi'(r\zeta)|^2(1-r)}{\omega^2(\chi(|\phi(r\zeta)|))} \, dr < +\infty \tag{3.18}
\]

a.e. \( \xi \in \mathbb{T} \). By (3.18),

\[
\int_0^1 |(h_n - h)' \circ \phi(r\zeta)|^2 \phi'(r\zeta))^2(1-r) \, dr \leq \|f_n - f\|_{\mathcal{B}_n^{\alpha,\beta}} \int_0^1 \frac{|\phi'(r\zeta)|^2(1-r)}{\omega^2(\chi(|\phi(r\zeta)|))} \, dr \leq 4 \int_0^1 \frac{|\phi'(r\zeta)|^2(1-r)}{\omega^2(\chi(|\phi(r\zeta)|))} \, dr \tag{3.19}
\]

and the dominated convergence theorem, we have

\[
\lim_{n \to +\infty} \int_0^1 |(h_n - h)' \circ \phi(r\zeta)|^2 \phi'(r\zeta))^2(1-r) \, dr = 0
\]

a.e. \( \xi \in \mathbb{T} \). Consequently, (3.19) and the dominated convergence theorem again give that

\[
\lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{T}} \left( \int_0^1 |(h_n - h)' \circ \phi(r\zeta)|^2 \phi'(r\zeta))^2(1-r) \, dr \right) \frac{p}{2} \, |d\zeta| = 0. \tag{3.20}
\]

It follows from (3.16) and (3.20) that

\[
\lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{T}} |(h_n - h) \circ \phi(\zeta)|^p \, |d\zeta| = 0, \tag{3.21}
\]

which yields that \( C_\phi(h) \in \mathcal{H}_P^{\mathbb{D}} \) and \( C_\phi(h_n) \to C_\phi(h) \) in \( \mathcal{H}_P^{\mathbb{D}} \) as \( n \to +\infty \).

By a similar proof process of (3.21), we have \( C_\phi(g) \in \mathcal{H}_P^{\mathbb{D}} \) and \( C_\phi(g_n) \to C_\phi(g) \) in \( \mathcal{H}_P^{\mathbb{D}} \) as \( n \to +\infty \). Hence \( C_\phi(f) = C_\phi(h) + \overline{C_\phi(g)} \in \mathcal{H}_P^{\mathbb{D}} \) and \( C_\phi(f_n) \to C_\phi(f) \) in \( \mathcal{H}_P^{\mathbb{D}} \) as \( n \to +\infty \). This completes the proof. \( \square \)

### 3.4 The proof of Theorem 2.5

(1) Assume that (2.6) holds. Let \( f \in \mathcal{H}_P^{\mathbb{D}} \). Since \( \mathbb{D} \) is a simply connected domain, we see that \( f \) admits the canonical decomposition \( f = h + g \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \) with \( g(0) = 0 \). It follows from [22, Theorem 2.1] that \( h, g \in \mathcal{H}_P^{\mathbb{D}} \). By Cauchy’s integral formula, we have

\[
F(z) = \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{H(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{G(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \mathbb{D},
\]
where \( F(z) = f(rz), H(z) = h(rz) \) and \( G(z) = g(rz) \) for \( r \in [0, 1) \). Elementary calculations lead to

\[
F_z(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{H(\zeta)}{(\zeta - z)^2} d\zeta \quad \text{and} \quad \frac{G(z)}{H_z(z)} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{G(\zeta)}{(\zeta - z)^2} d\zeta.
\]

Since \( p \in (1, \infty) \), using Jensen’s inequality, we obtain

\[
|F_z(z)|^p \leq \frac{1}{(1 - |z|^2)^p} \left( \frac{1}{2\pi} \int_{|\zeta|=1} \frac{(1 - |\zeta|^2)}{|\zeta - z|^2} |H(\zeta)| |d\zeta| \right)^p 
\]

\[
\leq \frac{1}{(1 - |z|^2)^p} \frac{1}{2\pi} \int_{|\zeta|=1} \frac{(1 - |\zeta|^2)}{|\zeta - z|^2} |H(\zeta)|^p |d\zeta| 
\]

\[
\leq \frac{(1 + |z|)^2}{(1 - |z|^2)^{p+1}} \frac{1}{2\pi} \int_{|\zeta|=1} |H(\zeta)|^p |d\zeta|,
\]

which, together with letting \( r \to 1^- \), implies that

\[
|h'(z)|^p \leq \frac{4}{(1 - |z|^2)^{p+1}} \|h\|_p^p. \quad (3.22)
\]

Similarly, we conclude

\[
|g'(z)|^p \leq \frac{4}{(1 - |z|^2)^{p+1}} \|g\|_p^p. \quad (3.23)
\]

By (3.22), (3.23) and [22, Theorem 2.1], we have

\[
\Lambda_f(z) = |h'(z)| + |g'(z)| \leq \frac{4^{\frac{1}{p}} (\|h\|_p + \|g\|_p)}{(1 - |z|^2)^{1+\frac{1}{p}}} \leq C \frac{1}{(1 - |z|^2)^{1+\frac{1}{p}}} \|f\|_p, \quad (3.24)
\]

where \( C \) is a constant which depends only on \( p \). Combining this inequality with (2.6) implies that \( C_\phi : \mathcal{H}_H^p(\mathbb{D}) \to \mathcal{B}_H^{\alpha, \beta} \) is bounded.

Conversely, assume that \( C_\phi : \mathcal{H}_H^p(\mathbb{D}) \to \mathcal{B}_H^{\alpha, \beta} \) is bounded. For \( a \in \mathbb{D} \), let

\[
f_a(z) = \left( \frac{1 - |\phi(a)|^2}{(1 - \phi(a)z)^2} \right)^{\frac{1}{p}}, \quad z \in \mathbb{D}.
\]

Then \( \sup_{a \in \mathbb{D}} \|f_a\|_p < +\infty \) and the boundedness of \( C_\phi \) implies that

\[
\sup_{a \in \mathbb{D}} \left\{ \frac{|\phi(a)||\phi'(a)||\omega\left(\left(1 - |a|^2\right)^\alpha \left(\log \frac{e}{1 - |a|^2}\right)^\beta\right)}{(1 - |\phi(a)|^2)^{1+\frac{1}{p}}} \right\} < +\infty.
\]

If (2.6) does not hold, then there exists a sequence \( \{a_n\} \) in \( \mathbb{D} \) such that

\[
\lim_{n \to +\infty} \frac{|\phi'(a_n)|\omega\left(\left(1 - |a_n|^2\right)^\alpha \left(\log \frac{e}{1 - |a_n|^2}\right)^\beta\right)}{(1 - |\phi(a_n)|^2)^{1+\frac{1}{p}}} = +\infty
\]

and \( \phi(a_n) \to 0 \) as \( n \to +\infty \). Since \( \omega \) is a majorant with

\[
\lim_{t \to 0^+} (\omega(t)/t) < +\infty,
\]

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this implies that
\[
\lim_{n \to +\infty} \frac{|\phi'(a_n)| (1 - |a_n|^2)^\alpha \left( \log \frac{e}{1 - |a_n|^2} \right)^\beta}{(1 - |\phi(a_n)|^2)^{1 + \frac{1}{p}}} = +\infty.
\]
Since \( \phi(a_n) \to 0 \) as \( n \to +\infty \) and \( |\phi'(a_n)| (1 - |a_n|^2) \leq 1 \) for all \( n \geq 1 \), this is a contradiction when \( \alpha = 1 \) and \( \beta \leq 0 \) or \( \alpha > 1 \) and \( \beta \in \mathbb{R} \). Thus, (2.6) holds.

(2) Assume that (2.7) holds. Then, using that \( \omega \) is a majorant with
\[
\lim_{t \to 0^+} \left( \frac{\omega(t)}{t} \right) < +\infty,
\]
it follows that \( C_\phi : \mathcal{H}_H^p(\mathbb{D}) \to \mathcal{B}^{\alpha,\beta}_{H_\omega} \) is bounded. We will show that if \( \{f_n\} \) is a bounded sequence in \( \mathcal{H}_H^p(\mathbb{D}) \) which converges to 0 uniformly on compact subsets of \( \mathbb{D} \), then \( C_\phi(f_n) \to 0 \) in \( \mathcal{B}^{\alpha,\beta}_{H_\omega} \) as \( n \to +\infty \). We may assume that \( \|f_n\|_p \leq 1 \) for all \( n \geq 1 \). Let \( \varepsilon > 0 \) be fixed. Then there exists an \( r \in (0, 1) \) such that
\[
|\phi'(z)| \omega \left( (1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta \right) < \frac{\varepsilon}{C}
\]
for all \( z \in \mathbb{D} \) with \( |\phi(z)| > r \), where \( C \) is the constant in (3.24). Combining (3.24) and (3.25) gives
\[
\Lambda_{C_\phi(f_n)}(z) \omega \left( (1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta \right) < \varepsilon
\]
for all \( z \in \mathbb{D} \) with \( |\phi(z)| > r \) and for all \( n \geq 1 \). Since \( \{f_n\} \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \), by using an argument similar to that in (1), we see that
\[
\Lambda_{C_\phi(f_n)}(z) \omega \left( (1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta \right) \to 0, \quad \text{as } n \to +\infty
\]
uniformly on \( |\phi(z)| \leq r \). Also, \( C_\phi(f_n)(0) \to 0 \) as \( n \to +\infty \). Thus, \( C_\phi(f_n) \to 0 \) in \( \mathcal{B}^{\alpha,\beta}_{H_\omega} \) as \( n \to +\infty \).

Conversely, assume that \( C_\phi : \mathcal{H}_H^p(\mathbb{D}) \to \mathcal{B}^{\alpha,\beta}_{H_\omega} \) is compact. If (2.7) does not hold, then there exist a constant \( \varepsilon_0 > 0 \) and a sequence \( \{a_n\} \) in \( \mathbb{D} \) such that \( |\phi(a_n)| \to 1 \) as \( n \to +\infty \) and
\[
|\phi'(a_n)| \omega \left( (1 - |a_n|^2)^\alpha \left( \log \frac{e}{1 - |a_n|^2} \right)^\beta \right) > \varepsilon_0
\]
for all \( n \geq 1 \). We may assume that \( a_n \to T \) and \( \phi(a_n) \to b \in \mathbb{T} \) as \( n \to +\infty \). Let
\[
f_n(z) = \left( \frac{1 - |\phi(a_n)|^2}{1 - \phi(a_n)z} \right)^{\frac{1}{p}}, \quad z \in \mathbb{D}.
\]
Then \( \{f_n\} \) is a bounded sequence in \( \mathcal{H}_p^\alpha (\mathbb{D}) \) which converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( n \to +\infty \). On the other hand,

\[
\| C_\phi (f_n) \|_{\mathcal{B}^\alpha_\omega, p} \geq \frac{|\phi'(a_n)| \omega \left( \left( 1 - |a_n|^2 \right)^{\beta} \left( \log \frac{e}{1 - |a_n|^2} \right)^{\beta} \right)}{(1 - |\phi(a_n)|^2)^{1 + \frac{1}{\beta}}} > |\phi(a_n)|^{\varepsilon_0}
\]

for all \( n \geq 1 \). This contradicts with the compactness of \( C_\phi \). Thus, (2.7) holds. This completes the proof. \( \square \)

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