Towards an open set of regular cosmological models

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The possibility of obtaining an open set of regular cosmological models is discussed. Cylindrical stiff perfect fluid cosmologies are studied in detail. The condition for geodesic completeness is easy to check. A large family of non-singular models is found therein.

1 Introduction

Just to settle the issue from the beginning, by a regular cosmological model we consider a perfect fluid cosmological model which is causally geodesically complete, that is, every lighlike or timelike geodesic can be extended from minus infinity to infinity in the affine parametrization.

These models have been neglected in the past since the powerful singularity theorems by Hawking, Penrose and Tipler (see, for instance, [1,2]) seemed to point out that they would violate some physical requirements (causality conditions, energy conditions). For instance, it was thought that they would enclose closed timelike curves or negative energy densities. However, twelve years ago Senovilla published the first known regular model for a cylindrical universe with a radiation fluid as matter content [3].

These models did exist then, but they were considered an issue of luck. Further results seemed to support these thoughts, since the list of regular models is rather sparse.

For instance, in the Ruiz-Senovilla family, regular models are a just a zero-measure set [4].

Stiff perfect fluids are an excellent arena for checking conjectures of this sort, since Einstein equations can be integrated almost to the end.

2 Inhomogeneous stiff fluid cosmologies

We shall consider inhomogeneous spacetimes with cylindrical symmetry and, as a matter of convenience for integration, we choose ignorable coordinates for the
Killing fields and isotropic coordinates for the rest. Metric functions depend just on \( t \) and \( r \) then.

\[ ds^2 = e^{2K}(-dt^2 + dr^2) + e^{-2U}dz^2 + \rho^2 e^{2U}d\phi^2, \quad (1) \]
\[-\infty < t, z < \infty, \quad 0 < r < \infty, \quad 0 < \phi < 2\pi. \quad (2) \]

Einstein equations are written in a comoving frame for the fluid, \( u \propto \partial_t \).

\[ T^{\mu\nu} = \mu u^\mu u^\nu + p(\mathcal{g}^{\mu\nu} + u^\mu u^\nu), \quad 0 \leq \mu, \nu \leq 3, \quad u^\mu u_\mu = -1, \quad (3) \]

where the energy density, \( \mu \) and the pressure, \( p \), are equal for a stiff perfect fluid.

The only assumption we make, since every regular cosmological model in the literature has a spacelike transitivity surface element, \( \rho \), is imposing that \( \rho \) must be orthogonal to the velocity of the fluid \( u \).

Rescaling the coordinates we may take \( \rho = r \) and the system is quite simple,

\[ U_{tt} - U_{rr} - \frac{U_r}{r} = 0, \quad (4) \]
\[ K_t = U_t + 2rU_tU_r, \quad (5) \]
\[ K_r = U_r + r(U_t^2 + U_r^2) + pre^{2K}, \quad (6) \]
\[ K_{rr} - K_{tt} + \frac{U_r}{r} + U_t^2 - U_r^2 = pe^{2K}, \quad (7) \]
\[ K_r + \frac{pr}{2p} = 0, \quad (8) \]
\[ K_t + \frac{pt}{2p} = 0. \quad (9) \]

The energy-momentum conservation equations can be integrated and we are left with a 2D wave equation and a quadrature for \( K \).

\[ p = \alpha e^{-2K}, \quad (10) \]
\[ K_t = U_t + 2rU_tU_r, \quad (11) \]
\[ K_r = U_r + r(U_t^2 + U_r^2) + \alpha r, \quad (12) \]
\[ U_{tt} - U_{rr} - \frac{U_r}{r} = 0. \quad (13) \]

Notice that nice combinations of \( K_t \) and \( K_r \) yield squares of the derivatives of \( U \).

The solution to the 2D wave equation initial data problem provides a compact form for writing the general solution,
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\[ g(r) = U(r, 0), \quad f(r) = U_t(r, 0), \quad \text{(14)} \]

\[ U(r, t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^1 d\tau \frac{\tau}{\sqrt{1 - \tau^2}} \left\{ t g(v) + f(v) + t f'(v) \frac{t\tau^2 + r\tau \cos \phi}{v} \right\}, \quad \text{(15)} \]

where \( v = \sqrt{r^2 + t^2 \tau^2 + 2rt \cos \phi} \), choosing the origin of the polar angle at the angle for \((x, y)\).

There is no need to impose cylindrical symmetry, since every solution has an axis provided we define accordingly the angle coordinate.

### 3 Geodesic completeness

Provided we use regular initial data, the solution has regular components, but this does not guarantee geodesic completeness, as we know.

We make use of a theorem devised by us that relates geodesic completeness to bounds on the derivatives and the metric functions \([5]\):

**Theorem:** A cylindrically symmetric diagonal metric in the form (11) with \(C^2\) metric functions \(f, g, \rho\) is future causally geodesically complete provided that along causal geodesics:

1. For large values of \(t\) and increasing \(r\),
   - (a) \(K_r + K_t \geq 0\), and either \(K_r \geq 0\) or \(|K_r| \lesssim K_r + K_t\).
   - (b) \((K + U)_r + (K + U)_t \geq 0\), and either \((K + U)_r \geq 0\) or \(|(K + U)_r| \lesssim (K + U)_r + (K + U)_t\).
   - (c) \((K - U - \ln \rho)_r + (K - U - \ln \rho)_t \geq 0\), and either \((K - U - \ln \rho)_r \geq 0\) or \(|(K - U - \ln \rho)_r| \lesssim (K - U - \ln \rho)_r + (K - U - \ln \rho)_t\).

2. For \(t\), constant \(b\) exists,
   \[
   \begin{aligned}
   &\quad \frac{K(t, r) - U(t, r)}{2K(t, r)} + \frac{U(t, r) + \ln \rho(t, r)}{K(t, r)} \\
   &\geq -\ln |t| + b.
   \end{aligned}
   \]

And similar conditions for past causally geodesically complete spacetimes.

These conditions look formidable, but the first set is always satisfied for a stiff perfect fluid model \([6]\).

The second set can be seen to reduce to one single condition on the axis, namely,

\[ U(0, t) = \int_0^1 d\tau \frac{\tau}{\sqrt{1 - \tau^2}} \left\{ t g(|t|\tau) + f(|t|\tau) + |t|\tau f'(|t|\tau) \right\} \geq -\frac{1}{2} \ln |t| + b. \quad \text{(16)} \]
4 Explicit models

It is not difficult to find examples of functions that fulfill the aforementioned condition. Consider for instance polynomials. At $r = 0$ the equation for $U$ can be integrated. For $f(r) = r^n$, $g(r) = r^m$ we obtain,

$$U_f(t) = \frac{n!!}{(n-1)!!} \left(\frac{\pi}{2}\right)^{1+(1-n)/2} |t|^n,$$

$$U_g(t) = \frac{m!!}{(m+1)!!} \left(\frac{\pi}{2}\right)^{1+(1-m)/2} |t|^m t,$$

after separating the terms in $f$ and $g$. There are two possibilities for a regular model:

1. If $f, g$ are polynomials in $r$ respectively of degree $n, m$ and $n > m + 1$, we have a non-singular model if $a_n$ is positive.

2. If $f, g$ are polynomials in $r$ respectively of degree $n, n - 1$, $U_f$ and $U_g$ at the axis are polynomials of degree $n$ and we have a non-singular model if

$$\left(n + \frac{1}{2}\right) a_n > |b_{n-1}|.$$

The set of regular models is certainly not negligible. If we restrict ourselves to the space of functions which are polynomials in $t$ at $r = 0$, we find that the subset of complete models encloses an open set, according to the last equation.

5 Final remarks

In this talk a large set of cylindrical stiff perfect fluid cosmological models has been rederived. An easy sufficient condition has been implemented to check whether these models are complete or not. Surprisingly the amount of regular models is far larger than expected. These results need be extended to more general models, but they may point out that complete cosmological models cannot be neglected as isolated points in a set of solutions of the Einstein equations.

Pressure seems determinant to avoid singularities in these models. The stiff fluid case is a extreme situation where the velocity of sound equals the speed of light. On the other hand, pressureless models, dust, are known to be singular due to Raychaudhuri equation. Barotropic fluids where $p = \gamma \mu$ and $\gamma \in (0, 1)$ need be explored in more detail.

Nonseparability of the metric functions seems determinant to avoid singularities. Separable stiff fluid models had been studied before $[7]$ and all of them were found to be singular. The non-separable model in $[8]$ already pointed in that direction.
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