EXTREMAL FUNCTIONS OF MOSER-TRUDINGER INEQUALITY INVOLVING FINSLER-LAPLACIAN

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Abstract. In this paper, we investigate the Moser-Trudinger inequality when it involves a Finsler-Laplacian operator that is associated with functionals containing \( F^2(\nabla u) \). Here \( F \) is convex and homogeneous of degree 1, and its polar \( F^o \) represents a Finsler metric on \( \mathbb{R}^n \). We obtain an existence result on the extremal functions for this sharp geometric inequality.

1. Introduction. Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain, \( W^{1,n}_0(\Omega) \) be the completion of \( C_0^\infty(\Omega) \) under the norm \( ||u||_{W^{1,n}_0(\Omega)} = (\int_\Omega |\nabla u|^n dx)^{\frac{1}{n}} \). The Sobolev embedding theorem states that \( W^{1,n}_0(\Omega) \) is embedded in \( L^p(\Omega) \) for any \( p > 1 \), but not in \( L^\infty(\Omega) \). However, as a limit case of the Sobolev embedding, the sharp geometric inequalities and their extremal functions play an important role in analysis and geometry. The study of sharp constant for Moser-Trudinger inequality traces back to 1960s and 1970s. Moser [10] elegantly sharpened the results of Phohozaev [12], and Trudinger [14] established the following so called Moser-Trudinger inequality

\[
\sup_{u \in W^{1,n}_0(\Omega), ||u||_{W^{1,n}_0(\Omega)} \leq 1} \int_\Omega e^{\alpha|u|} \frac{n}{n-1} dx < +\infty. \tag{1}
\]

for any \( \alpha \leq \alpha_n \), where \( \alpha_n = n\omega_{n-1}^{\frac{1}{n}} \) and \( \omega_{n-1} \) is the surface area of the unit sphere in \( \mathbb{R}^n \). For any \( \alpha > \alpha_n \), the inequality (1) is invalid, i.e. there exists a sequence of functions \( \{u_\epsilon\} \) in \( W^{1,n}_0(\Omega) \) with \( ||u_\epsilon||_{W^{1,n}_0(\Omega)} = 1 \) such that

\[
\int_\Omega e^{\alpha|u_\epsilon|} \frac{n}{n-1} dx \to +\infty \quad \text{as} \ \epsilon \to 0.
\]

On the other hand, for any fixed \( u \in W^{1,n}_0(\Omega) \), it is also known that

\[
\int_\Omega e^{\alpha|u|} \frac{n}{n-1} dx < +\infty
\]

for any \( \alpha > 0 \).

Actually, the inequality (1) is viewed as an n-dimensional analog of the Sobolev inequality, and it plays an important role in analytic problems. The constant \( n\omega_{n-1}^{\frac{1}{n}} \)

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is optimal in the sense that if $\alpha > n\omega_{n-1}$ we can find a sequence $\{u_\varepsilon\}$ such that $\int_\Omega e^{\alpha |u_\varepsilon|} dx$ diverges.

In this paper, we will investigate this Moser-Trudinger type inequality when it involves a Finsler-Laplacian operator $Q$ that is associated with functionals containing $F^2(\nabla u)$. Here $F$ is convex and homogeneous of degree 1, and its polar $F^o$ represents a Finsler metric on $\mathbb{R}^n$. In 2012, Guofang Wang and Chao Xia [18] proved the following Moser-Trudinger type inequality

$$
\int_{\Omega} e^{\lambda u} dx \leq C(n)|\Omega| (2)
$$

for all $u \in W^{1,n}_0(\Omega)$ and $\int_{\Omega} F(\nabla u) dx \leq 1$. Here $\lambda \leq \lambda_n = n^{-\frac{n-1}{n}} \kappa_n^{-1}$, $\kappa_n$ is the volume of a unit Wulff ball, i.e. $\kappa_n = |\{x \in \mathbb{R}^n : F^o(x) \leq 1\}|$. $\lambda_n$ is also optimal in the sense that if $\lambda > \lambda_n$ we can find a sequence $\{u_k\}$ such that $\int_\Omega e^{\lambda_k u_k} dx$ diverges.

Another interesting question about Moser-Trudinger inequalities is whether an extremal function exist or not. The first result in this direction is due to Carleson and Chang [2], who proved that supremum in (1) is attained when $\Omega$ is a unit ball in $\mathbb{R}^n$. Then Flucher [4] proved the same result when $\Omega$ is a general bounded smooth domain in $\mathbb{R}^2$. Later, Lin [8] generalized the existence result to a bounded smooth domain in all dimensions. But it is unknown that whether extremal functions for the Moser-Trudinger functional involving Finsler-Laplacian exist or not. So our goal in this paper is to prove the existence of extremal functions for the inequality (2) in dimension two. Our main Theorem is

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a Wulff ball with radius $R$ centered at the origin, and $\kappa$ is the volume of a unit Wulff ball. Then for any $0 < \alpha \leq 4\kappa$, the supremum

$$
\sup_{u \in W^{1,2}_0(\Omega), \int_{\Omega} F^2(\nabla u) dx \leq 1} \int_{\Omega} e^{\alpha u^2} dx < +\infty
$$

(3)

can be attained by some function $u_0$. Here $u_0 \in H \cap C^1_{loc}(\overline{\Omega}\setminus\{0\}) \cap C^0(\overline{\Omega})$ and $\int_{\Omega} F^2(\nabla u_0) dx = 1$.

When $\Omega$ is a general domain in high dimensional space $\mathbb{R}^n$ for $n \geq 2$, for lacking of P. Lions type Lemma (see the Lemma 4.1 below), the situation becomes more subtle. So we will consider it later. For the proof of Theorem 1.1, we use an important tool in geometric analysis, the blow-up analysis. This method is based on two facts. One is that an upper bound of Moser-Trudinger energy in $H$ (see the definition in the section 3) can be obtained from a result of Carleson and Chang [2] under the assumption that certain maximizing sequence blows up. The other is that a sequence of functions $f_\varepsilon$ can be constructed to show that the Moser-Trudinger energy is larger than the above upper bound. This contradiction implies that the considered maximizing sequence doesn’t blow up and consequently this maximizing sequence converges to an extremal function of the considered Moser-Trudinger energy. Though the method we carry out by using the blow-up analysis is routine, we will encounter new difficulties which are caused by Finsler-Laplacian.

We should mention that our method is derived from the convex symmetrization method and the level set method which was used in [17] and [18].

We organize this paper as follows. In section 2, we introduce the Finsler-Laplacian, the convex symmetrization of $u$ with respect to $F$, the corresponding
iso-perimetric inequality and the co-area formula. In section 3, we give the existence of maximizers, denote $u_{\varepsilon}$ in the subcritical case. Some properties of $u_{\varepsilon}$ are also given in section 3. In section 4, we use blow-up analysis to show the blow-up behaviors of $u_{\varepsilon}$ near the blow-up point. In section 5, an upper bound is derived if $u_{\varepsilon}$ blows up, and a sequence of functions $f_{\varepsilon}$ is constructed to reach a contradiction, which completes the proof of Theorem.

2. preliminaries. In this section, we will give the notations and preliminaries. Throughout this paper, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative convex function of class $C^2(\mathbb{R}^2 \setminus \{0\})$ which is even and positively homogenous of degree 1, so that

$$F(t\xi) = |t|F(\xi) \quad \text{for any} \quad t \in \mathbb{R}, \xi \in \mathbb{R}^2.$$ 

A typical example is $F(\xi) = (\sum_i |\xi|^q)^{\frac{1}{q}}$ for $q \in [1, \infty)$. We further assume that $F(\xi) > 0$ for any $\xi \neq 0$.

If we consider the minimization problem

$$\min_{u \in W^{1,2}_0(\Omega)} \int_{\Omega} F^2(\nabla u) \, dx,$$

we know that its Euler equation contains an operator of the form

$$Qu := \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i}(F(\nabla u)F_{\xi_i}(\nabla u)).$$

Note that these operators are not linear unless $F$ is the Euclidean norm. We call this nonlinear operator as Finsler-Laplacian. This operator $Q$ is closely related to a smooth, convex hypersurface in $\mathbb{R}^2$, which is called the Wulff shape of $F$. And this operator $Q$ was studied by many mathematicians, see [16, 5, 18, 1] and the references therein.

Usually, we shall assume that the $\text{Hess}(F^2)$ is positive definite in $\mathbb{R}^2 \setminus \{0\}$. Then for such a function $F$, there exist two constants $0 < a \leq b < \infty$ such that

$$a|\xi| \leq F(\xi) \leq b|\xi|$$

for any $\xi \in \mathbb{R}^2$ and there exist two constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\eta|^2 \leq \frac{\partial(F(\xi)F_{\xi_i}(\xi))}{\partial \xi_j} \eta_i \eta_j \leq \Lambda|\eta|^2$$

in any compact subsets of $\mathbb{R}^2 \setminus \{\xi = 0\}$, i.e. this operator $Q$ is uniformly elliptic operator in the compact subsets of $\mathbb{R}^2 \setminus \{x|\nabla u(x) = 0\}$, see [18] and [19].

Consider the map

$$\phi : S^1 \to \mathbb{R}^2, \phi(\xi) = F_\xi(\xi).$$

Its image $\phi(S^1)$ is a smooth, convex hypersurface in $\mathbb{R}^2$, which is called Wulff shape of $F$. Let $F^0$ be the support function of $K := \{x \in \mathbb{R}^2 : F(x) \leq 1\}$, which is defined by

$$F^0(x) := \sup_{\xi \in K} \langle x, \xi \rangle.$$

It is easy to verify that $F^0 : \mathbb{R}^2 \to [0, +\infty)$ is also a convex, homogeneous function of class of $C^2(\mathbb{R}^2 \setminus \{0\})$. Actually $F^0$ is dual to $F$ in the sense that

$$F^0(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^0(\xi)}.$$
One can see easily that $\phi(S^1) = \{ x \in \mathbb{R}^2 : F^0(x) = 1 \}$. We denote $W_F := \{ x \in \mathbb{R}^2 : F^0(x) \leq 1 \}$ and $\kappa := |W_F|$, the Lebesgue measure of $W_F$. We also use the notion $W_r(x_0) := \{ x \in \mathbb{R}^2 : F^0(x - x_0) \leq r \}$. We call $W_r(x_0)$ a Wulff ball of radius $r$ with center at $x_0$. For later use, we give some simple properties of the function $F$, which follows directly from the assumption on $F$, also see [5, 16]

**Lemma 2.1.** We have

(i) $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)$,
(ii) $\frac{1}{2} \leq |\nabla F(x)| \leq C$, and $\frac{1}{C} \leq |\nabla F^0(x)| \leq C$ for any $x \neq 0$,
(iii) $\langle x, \nabla F(x) \rangle = F(x)$, $\langle x, \nabla F^0(x) \rangle = F^0(x)$,
(iv) $F(\nabla F^0(x)) = 1$.

Next we describe the isoperimetric inequality and co-area formula with respect to $F$. For a domain $\Omega \subset \mathbb{R}^2$, a subset $E \subset \Omega$ and a function of bounded variation $u \in BV(\Omega)$, we define the anisotropic bounded variation of $u$ with respect to $F$ is

$$
\int_{\Omega} |\nabla u|_F = \sup \int_{\Omega} u \cdot \text{div} \sigma dx, \quad \sigma \in C_0^1(\Omega; \mathbb{R}^2), \quad F^0(\sigma) \leq 1.
$$

We set anisotropic perimeter of $E$ with respect to $F$ is

$$
P_F(E) := \int_{\Omega} |\nabla \chi_E|_F,
$$

where $\chi_E$ is the characteristic function of the set $E$. It is well known (also see [11]) that the co-area formula

$$
\int_{\Omega} |\nabla u|_F = \int_0^\infty P_F(|u| > t) dt
$$

and the isoperimetric inequality

$$
P_F(E) \geq 2\kappa^{\frac{1}{2}} |E|^\frac{1}{2}
$$

holds. Moreover, the equality in (5) holds if and only if $E$ is a Wulff ball.

In the sequel, we will introduce the convex symmetrization with respect to $F$. The convex symmetrization generalizes the Schwarz symmetrization. It was defined in [1] and will be an essential tool for this paper. Let us consider a measurable function $u \in \Omega \subset \mathbb{R}^2$. The one dimensional decreasing rearrangement of $u$ is $u^* = \sup \{ s \geq 0 : \{ |x \in \Omega : |u(x)| > s| > t \} \}$, for $t \in \mathbb{R}$.

The convex symmetrization of $u$ with respect to $F$ is defined as

$$
u^*(x) = u^*(\kappa F^0(x)^2), \quad \text{for } x \in \Omega^*.
$$

Here $\kappa F^0(x)^2$ is just the Lebesgue measure of a homothetic Wulff ball with radius $F^0(x)$ and $\Omega^*$ is the homothetic Wulff ball centered at the origin having the same measure as $\Omega$. Throughout the following paper, we take $\Omega = W_R(\Omega)$ and $\Omega^* = \Omega$.

3. **Maximizers for subcritical-Moser-Trudinger functional.** In this section, we will show the existence of the maximizers for Moser-Trudinger functional in the subcritical case. In usual, it is easy to show it. For convenience to the readers, we will give the details here. We also will give some properties of the maximizers. We first present a technical Lemma.

**Lemma 3.1.** For any $p > 1$ and any $u \in W_0^{1,2}(\Omega)$, there holds

$$
\int_{\Omega} e^{pu^2} dx < +\infty.
$$
Proof. Fix $p > 1$ and $u \in W^{1,2}_0(\Omega)$. Since $C^\infty_0(\Omega)$ is dense in $W^{1,2}_0(\Omega)$, we take $u_0 \in C^\infty_0(\Omega)$ such that $||u - u_0||^2_{W^{1,2}_0(\Omega)} < \frac{\epsilon}{p}$. This implies $\int_\Omega F(\nabla u - \nabla u_0)^2 \, dx < \frac{\epsilon}{p}$.

Using an inequality $2st \leq s^2 + t^2$ twice, we have

$$\int_\Omega e^{pu^2} \, dx \leq \int_\Omega e^{4pu(u-u_0)^2} \, dx + \int_\Omega e^{6pu_0^2} \, dx.$$  

By (2), we have

$$\int_\Omega e^{4p(u-u_0)^2} \, dx \leq \int_\Omega e^{\frac{4\kappa_2}{2}(u-u_0)^2} \, dx < +\infty.$$  

Since $u_0$ is bounded in $\Omega$, we have $\int_\Omega e^{6pu_0^2} \, dx < +\infty$. This gives the desired estimate. \hfill \Box

For $u \in C^\infty_0(\Omega)$ and $\int_\Omega F(\nabla u)^2 \, dx = 1$, let $u^*$ be the convex symmetrization of $u$ with respect to $F(x)$. Since

$$\int_\Omega F^2(\nabla u^*) \, dx \leq \int_\Omega F^2(\nabla u) \, dx = 1,$$

$$\int_\Omega e^{(4\kappa-\epsilon)u^2} \, dx = \int_\Omega e^{(4\kappa-\epsilon)u^2} \, dx,$$

and

$$\int_\Omega e^{(4\kappa-\epsilon)u^2} \, dx \leq \int_\Omega e^{\frac{4\kappa_2}{2}(u^*)} \, dx,$$

then we only need to consider non-increase and radially symmetric functions with respect to $F^0(x)$ for our maximizers. Let

$$\Sigma = \{ u : u \in C^\infty_0(\Omega), u(x) = u(r) \text{ with } r = F^0(x), u'(r) \leq 0 \},$$

and $H$ be the closure of $\Sigma$ in $W^{1,2}_0(\Omega)$. By the Sobolev embedding theorem, we get

$$H \subset \cap_{p \geq 1} L^p(\Omega) \text{ and } H \subset C^{0,\frac{1}{2}}_\text{loc}(\Omega \setminus \{0\}). \quad (6)$$

We begin with the following existence proposition of maximizers for the subcritical Moser-Trudinger functional.

**Proposition 1.** For any $\epsilon \in (0, 4\kappa)$, there exist some $u_\epsilon \in H \cap C^1(\overline{\Omega})$ satisfying $||F(\nabla u_\epsilon)||_{L^2(\Omega)} = 1$ and

$$\int_\Omega e^{(4\kappa-\epsilon)u_\epsilon^2} \, dx = \sup_{u \in W^{1,2}_0(\Omega), ||F(\nabla u)||_{L^2(\Omega)} \leq 1} \int_\Omega e^{(4\kappa-\epsilon)u^2} \, dx.$$  

In particular, $u_\epsilon$ satisfies the following equation

$$\left\{ \begin{array}{ll}
-\text{div}(F(\nabla u_\epsilon)F_\xi(\nabla u_\epsilon)) = \frac{u_\epsilon}{\lambda_\epsilon} e^{(4\kappa-\epsilon)u^2_\epsilon} & \text{in } \Omega, \\
u_\epsilon = 0 & \text{on } \partial\Omega,
\end{array} \right.$$  

in the distributional sense, where $\lambda_\epsilon = \int_\Omega u^2_\epsilon e^{(4\kappa-\epsilon)u^2_\epsilon} \, dx$.

**Proof.** For any fixed $\epsilon$, we assume that $u_{\epsilon,j}$ be a maximizing sequence in $H$ such that $||F(\nabla u_{\epsilon,j})||_{L^2(\Omega)} \leq 1$. Then we have

$$u_{\epsilon,j} \rightharpoonup u_\epsilon \text{ weakly in } W^{1,2}_0(\Omega),$$

$$u_{\epsilon,j} \to u_\epsilon \text{ strongly in } L^p(\Omega), \text{ for any } p \geq 1,$$

$$u_{\epsilon,j} \to u_\epsilon \text{ a.e. } \Omega.$$
By the weakly lower semi-continuous property of the norm, we have
\[ \int_{\Omega} F^2(\nabla u_e)dx \leq \liminf_{j \to +\infty} \int_{\Omega} F^2(\nabla u_{e,j})dx \leq 1. \]

Because \( e^{4\kappa - \epsilon} u_{e,j}^2 \) is uniformly bounded in \( L^1(\Omega) \), it follows that \( e^{(4\kappa - \epsilon)} u_{e,j}^2 \) is uniformly bounded in \( L^p(\Omega) \) for some \( p > 1 \). Since
\[ |e^{(4\kappa - \epsilon)} u_{e,j}^2 - e^{(4\kappa - \epsilon)} u_e^2| \leq (4\kappa - \epsilon)(e^{(4\kappa - \epsilon)} u_{e,j}^2 + e^{(4\kappa - \epsilon)} u_e^2) |u_{e,j}^2 - u_e^2| \]
and \( u_{e,j} \to u_e \) strongly in \( L^p(\Omega) \) for any \( p \geq 1 \) as \( j \to +\infty \), we conclude that
\[ \lim_{j \to +\infty} \int_{\Omega} e^{(4\kappa - \epsilon)} u_{e,j}^2 dx = \int_{\Omega} e^{(4\kappa - \epsilon)} u_e^2 dx. \]
Thus we have that \( u_e \) attains the supremum. Clearly \( u_e \neq 0 \). If we suppose that \( ||F(\nabla u_e)||_{L^2(\Omega)} < 1 \), then we have that
\[ \int_{\Omega} e^{(4\kappa - \epsilon)} u_e^2 dx < \int_{\Omega} e^{|\nabla u_e|^2} L^2(\Omega) dx, \]
which is a contradiction to the fact that \( u_e \) is a maximizer. Hence \( ||F(\nabla u_e)||_{L^2(\Omega)} = 1 \). A straightforward calculation shows that \( u_e \) satisfies the Euler-Lagrange equation (7) in the distributional sense. By an inequality \( e^{t^2} \leq 1 + t^2 e^{t^2} \),
\[ |\Omega| < \int_{\Omega} e^{(4\kappa - \epsilon)} u_e^2 dx \leq |\Omega| + 4\kappa \lambda. \]
This leads to \( \liminf_{\epsilon \to 0} \lambda_e > 0 \). Moreover, using the Hölder inequality and Lemma 3.1, we can easily know that \( \frac{u_{e,j}}{\sqrt{\lambda}} e^{(4\kappa - \epsilon)} u_j^2 \) is bounded in \( L^s(B_{2\rho}(0)) \) for some \( s > 1 \) and \( B_{2\rho}(0) \subset \Omega \). Hence by Theorem 2 in [6], we have \( u_e \in L^\infty(B_{\rho}(0)) \). Together with \( u_e \in \mathcal{H} \), we have \( u_e \in L^\infty(\Omega) \). It implies \( \frac{u_{e,j}}{\sqrt{\lambda}} e^{(4\kappa - \epsilon)} u_j^2 \) is bounded in \( L^\infty(\Omega) \). Then by Theorem 1 in [9], we easily get \( u_e \in C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0,1) \), which implies that \( u_e \in C^{1}(\Omega) \).

Since \( u \in \mathcal{H} \) is non-increasing and \( \mathcal{H} \) is continuously embedded in \( L^p(\Omega) \) for any \( p \geq 1 \), then for any bounded sequence \( \{u_e\} \subset \mathcal{H} \), we may assume, by taking a subsequence, that \( u_e \) converges to \( u \) weakly in \( W^{1,2}_0(\Omega) \) and \( a.e \) in \( \Omega \). Therefore to prove our main Theorem, we only need to show that
\[ \sup_{u \in \mathcal{H}, \int_{\Omega} F(\nabla u)^2 dx \leq 1} \int_{\Omega} e^{4\kappa u^2} dx < +\infty \]  \( \cdots (8) \)
can be attained by some function \( u_0 \in \mathcal{H} \cap C^1_{loc}(\overline{\Omega} \setminus \{0\}) \) with \( \int_{\Omega} F(\nabla u_0)^2 dx = 1 \). Thus in the sequel, we always assume that \( u_e \in \mathcal{H} \).

4. Blow-up behaviors of \( u_e \). In this section, we will develop the blow-up analysis when the sequence \( u_e \) blows up as \( \epsilon \to 0 \). Since \( u_e \) is bounded in \( \mathcal{H} \) from the previous section, we can assume without loss of generality
\[ u_e \to u_0 \] weakly in \( W^{1,2}_0(\Omega) \),
\[ u_e \to u_0 \] strongly in \( L^q(\Omega) \), \( \forall q \geq 1 \),
\[ u_e \to u_0 \] a.e. in \( \Omega \).
Clearly, $u_0 \in \mathcal{H}$. Now let $M_\epsilon = \max_{\Omega} u_\epsilon = u_\epsilon(0)$. If $M_\epsilon$ is bounded, then for any $u \in W^{1,2}_0(\Omega)$ with $\int_{\Omega} F^2(\nabla u)dx \leq 1$, by the Lebesgue dominated convergence theorem we have
\[
\int_{\Omega} e^{4\kappa u^2}dx = \lim_{\epsilon \to 0} \int_{\Omega} e^{(4\kappa - \epsilon)u^2}dx \leq \lim_{\epsilon \to 0} \int_{\Omega} e^{(4\kappa - \epsilon)u^2}dx = \int_{\Omega} e^{4\kappa u_0^2}dx.
\]
Hence $u_0$ is the desired maximizer.

In the following, we can assume $M_\epsilon = u_\epsilon(0) \to +\infty$ as $\epsilon \to 0$. We start to establish a version of P. Lions type Lemma.

**Lemma 4.1.** Let $u_\epsilon \in \mathcal{H}$ and $\|F(\nabla u_\epsilon)\|_{L^2(\Omega)} = 1$. We can assume that $u_\epsilon \rightharpoonup u_0$ weakly in $\mathcal{H}$ and $u_0 \not\equiv 0$. Then for all $0 < p < 4\kappa(1 - \|F(\nabla u_0)\|_{L^2(\Omega)})^{-1}$, we have
\[
\limsup_{\epsilon \to 0} \int_{\Omega} e^{pu^2}dx < \infty.
\]

**Proof.** First, notice that
\[
\frac{\nabla F(u_\epsilon)}{|\nabla F(u_\epsilon)|^2} F(\nabla u_0) \in L^2(\Omega)
\]
by Lemma 2.1. Since $u_\epsilon \rightharpoonup u_0$ weakly in $\mathcal{H}$, then we get
\[
\lim_{\epsilon \to 0} \int_{\Omega} (\nabla u_\epsilon - \nabla u_0) \frac{\nabla F^0(x)}{|\nabla F^0(x)|^2} F(\nabla u_0)dx = 0.
\]
Also notice that, for any $u \in \mathcal{H}$, $u = u(r) = u(F^0(x))$ for $r = F^0(x)$ and $\nabla u = u'(r)\nabla F^0(x)$. Then by Lemma 2.1 again, we have
\[
\lim_{\epsilon \to 0} \int_{\Omega} (u_\epsilon'(r) - u_0'(r))u_0'(r)dx = 0,
\]
and
\[
\int_{\Omega} F^2(\nabla u_\epsilon - \nabla u_0)dx = \int_{\Omega} (u_\epsilon'(r) - u_0'(r))^2dx
\]
\[
= \int_{\Omega} (u_\epsilon'(r))^2dx - 2 \int_{\Omega} u_\epsilon'(r)u_0'(x)dx + \int_{\Omega} u_0'(r)^2dx
\]
\[
= \int_{\Omega} F^2(\nabla u_\epsilon)dx - 2 \int_{\Omega} u_\epsilon'(r)^2dx + \int_{\Omega} u_0'(r)^2dx + o_\epsilon(1)
\]
\[
= \int_{\Omega} F^2(\nabla u_\epsilon)dx - \int_{\Omega} F^2(\nabla u_0)dx + o_\epsilon(1),
\]
which together with
\[
\|F(\nabla u_\epsilon)\|_{L^2(\Omega)} = 1, \quad u_0 \not\equiv 0,
\]
leads to
\[
\|F(\nabla u_\epsilon - \nabla u_0)\|_{L^2(\Omega)}^2 \to 1 - \|F(\nabla u_0)\|_{L^2(\Omega)}^2 < \frac{4\kappa}{p}.
\]
Thus, for $\epsilon$ small enough we have
\[
p\|F(\nabla u_\epsilon - \nabla u_0)\|_{L^2(\Omega)}^2 < 4\kappa.
\]
Now choosing $q > 1$ close to 1 and $\delta > 0$ satisfying
\[
qp(1 + \delta^2)\|F(\nabla u_\epsilon - \nabla u_0)\|_{L^2(\Omega)}^2 < 4\kappa.
\]
By (2), we have
\[
\int_{\Omega} e^{qp(1+\delta^2)(u_\epsilon - u_0)^2}dx = \int_{\Omega} e^{qp(1+\delta^2)(\|F(\nabla u_\epsilon - \nabla u_0)\|_{L^2(\Omega)}^2\|F(\nabla u_\epsilon - \nabla u_0)\|_{L^2(\Omega)}^2)^2}dx \leq C.
\]
Moreover, since
\[ pn^2 \leq p(1 + \delta^2)(u_e - u_0)^2 + p(1 + \frac{1}{\delta^2})u_0^2, \]
by Hölder inequality, we have for \( q^* = \frac{q}{q-1} \)
\[
\int_{\Omega} e^{p u^2} \leq \int_{\Omega} e^{p(1+\delta^2)(u_e - u_0)^2} e^{p(1+\frac{1}{\delta^2})u_0^2} \, dx
\leq \int_{\Omega} e^{q* p(1+\delta^2)(u_e - u_0)^2} \, dx \int_{\Omega} e^{q p(1+\frac{1}{\delta^2})u_0^2} \, dx \leq C
\]
for \( \epsilon \) small enough. Thus we get the proof of the result.

Clearly, Lions type result provide more information than Proposition 1. Moreover we can prove the concentration-compactness Theorem for \( u_\epsilon \).

**Lemma 4.2.** \( u_0 \equiv 0 \), and \( F(\nabla u_\epsilon)^2 dx \rightarrow \delta_0 \), where \( \delta_0 \) denotes the Dirac measure centered at the origin.

**Proof.** Suppose \( u_0 \not\equiv 0 \). Notice that \( \lim \inf_{\epsilon \rightarrow 0} \lambda_\epsilon > 0 \). By Lemma 4.1 and by Hölder inequality, we have \( \frac{\lambda_\epsilon}{\epsilon^s} e^{4(4k-\epsilon)u_\epsilon^2} \) is uniformly bounded in \( L^s(B_{2t}(0)) \) for some \( s > 1 \) and \( B_{2t}(0) \subset \Omega \). Then by Theorem 2 in [6], \( u_\epsilon \) is uniformly bounded in \( L^\infty(B_t(0)) \), which contradicts \( M_\epsilon \rightarrow +\infty \) as \( \epsilon \rightarrow 0 \). Hence \( u_0 \equiv 0 \).

Since \( \int_{\Omega} F(\nabla u_\epsilon)^2 dx = 1 \), it is not difficult to see that \( F(\nabla u_\epsilon)^2 dx \rightarrow \delta_0 \). Assume that \( F(\nabla u_\epsilon)^2 dx \rightarrow \mu \) in the sense of measure as \( \epsilon \rightarrow 0 \). If \( \mu \neq \delta_0 \), we claim that there exists a cut-off function \( \phi \in C^0_0(\Omega) \), which is supported in \( B_r(0) \subset \Omega \) for some \( r > 0 \) with \( 0 < \phi < 1 \) in \( B_r(0) \setminus B_{2r}(0) \) and \( \phi(x) = 1 \) in \( B_{2r}(0) \), satisfying
\[
\int_{B_r(0)} \phi F(\nabla u_\epsilon)^2 dx \leq 1 - \eta
\]
for some \( \eta > 0 \) and small enough \( \epsilon \). We prove the claim by contradiction. There exist sequence of \( \eta_\iota \rightarrow 0 \) and \( r_\iota \rightarrow 0 \) as \( \iota \rightarrow \infty \) such that
\[
\int_{B_{r_\iota}(0)} \phi_\iota F(\nabla u_\epsilon)^2 dx > 1 - \eta_\iota,
\]
for every \( \phi_\iota \in C^1_0(B_{r_\iota}(0)) \) and \( \phi_\iota = 1 \) in \( B_{2r_\iota}(0) \). Then when \( \epsilon \) small enough
\[
\int_{B_{2r_\iota}(0)} \phi_\iota F(\nabla u_\epsilon)^2 dx > 1 - 2\eta_\iota.
\]
Taking \( \iota \rightarrow \infty \), the left hand side converges to 0. However, \( 1 - 2\eta_\iota \rightarrow 1 \). This contradiction leads to the claim. Since \( u_\epsilon \rightarrow 0 \) strongly in \( L^q(\Omega) \) for any \( q > 1 \), we may assume that
\[
\int_{B_{r}(0)} F(\nabla(\phi u_\epsilon))^2 dx \leq 1 - \eta,
\]
provided \( \epsilon \) is sufficient small. By (2), \( e^{4k(\phi u_\epsilon)^2} \) is uniformly bounded in \( L^s(B_{r_0}(0)) \) for some \( s > 1 \) and \( 0 < r_0 < r \). Then \( \frac{\lambda_\epsilon}{\epsilon^s} e^{4k(\phi u_\epsilon)^2} \) is uniformly bounded in \( L^q(B_{r_0}(0)) \) for some \( q > 1 \). By Theorem 2 in [6], we have that \( u_\epsilon \) is uniformly bounded in \( L^\infty(B_{2r_0}(0)) \), which contradicts the fact that \( M_\epsilon \rightarrow \infty \). Therefore, \( F(\nabla u_\epsilon)^2 dx \rightarrow \delta_0 \) as \( \epsilon \rightarrow 0 \).
Next we pay our attention to the blow up behaviors of \( u_\epsilon \) at the blow up point. We set \( r_\epsilon^2 = \frac{\lambda_\epsilon e^{-(4\kappa - \epsilon)M_\epsilon^2}}{M_\epsilon^2} \). Since
\[
\lambda_\epsilon \leq M_\epsilon^2 e^{(2\kappa - \epsilon)M_\epsilon^2} \int_\Omega e^{2\lambda_\epsilon w_\epsilon^2} dx \leq CM_\epsilon^2 e^{(2\kappa - \epsilon)M_\epsilon^2},
\]
it follows that
\[
\frac{r_\epsilon^2 M_\epsilon^2}{\lambda_\epsilon} = \frac{\lambda_\epsilon e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)}}{e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)}} \leq CM_\epsilon^2 e^{-2\kappa M_\epsilon^2}.
\]
Hence we have \( \lim_{\epsilon \to 0} r_\epsilon M_\epsilon = 0 \), and \( \lim_{\epsilon \to 0} r_\epsilon = 0 \).

Define
\[
w_\epsilon(x) = \frac{u_\epsilon(r_\epsilon x)}{M_\epsilon}
\]
in \( \Omega_\epsilon = \{ x \in \mathbb{R}^2 : r_\epsilon x \in \Omega \} \). By a direct calculation we obtain that
\[
- \text{div}(F(\nabla v_\epsilon)F_\epsilon(\nabla v_\epsilon)) = \frac{v_\epsilon}{M_\epsilon^2} e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)} \quad \text{in} \quad \Omega_\epsilon.
\]
Since \( 0 \leq v_\epsilon \leq 1 \) and \( \frac{v_\epsilon}{M_\epsilon^2} e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)} \to 0 \) in \( B_\epsilon(0) \) for any \( r > 0 \), which implies \( \frac{v_\epsilon}{M_\epsilon^2} e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)} \) is uniformly bounded in \( L^\infty(B_\epsilon(0)) \), by Theorem 1 in [13], \( v_\epsilon \) is uniformly bounded in \( C^{1,\alpha}(\overline{B_\epsilon(0)}) \). By Ascoli-Arzelà’s theorem, we can find a sequence \( \epsilon_j \to 0 \) such that \( v_{\epsilon_j} \to v \) in \( C^{1}_{loc}(\mathbb{R}^2) \), where \( v \in C^1(\mathbb{R}^2) \) and satisfies
\[
- \text{div}(F(\nabla v)F_\epsilon(\nabla v)) = 0 \quad \text{in} \quad \mathbb{R}^2.
\]
Furthermore, we have \( 0 \leq v \leq 1 \) and \( v(0) = 1 \). The Liouville Theorem (see [7]) leads to \( v \equiv 1 \).

Also we have
\[
- \text{div}(F(\nabla w_\epsilon)F_\epsilon(\nabla w_\epsilon)) = v_\epsilon e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)} \quad \text{in} \quad \Omega_\epsilon.
\]

For any \( r > 0 \), since \( 0 \leq u_\epsilon(r,x) \leq M_\epsilon \) we have \( - \text{div}(F(\nabla w_\epsilon)F_\epsilon(\nabla w_\epsilon)) = O(1) \) in \( W_r \) for small \( \epsilon \). Then due to \( u_\epsilon(0) = 0 \) and Theorem 1 in [13] and Ascoli-Arzelà’s theorem, there exists \( w \in C^{1}(\mathbb{R}^2) \) such that \( w_\epsilon \) converges to \( w \) in \( C^{1}_{loc}(\mathbb{R}^2) \).

Therefore we have
\[
\frac{u_\epsilon(r,x)}{M_\epsilon} \to 0, \quad u_\epsilon^2(r,x) - M_\epsilon^2 = 2w_\epsilon + \frac{w_\epsilon^2}{M_\epsilon^2} \to 2w \quad \text{in} \quad C^{1}_{loc}(\mathbb{R}^2).
\]

By taking \( \epsilon \to 0 \), we know that \( w \) satisfies
\[
- \text{div}(F(\nabla w)F_\epsilon(\nabla w)) = e^{8kw}
\]
in the distributional sense. We also have the facts \( w(0) = 0 = \max_{x \in \mathbb{R}^2} w(x) \).

Moreover, for any \( \tilde{R} > 0 \), we have
\[
1 \geq \lim_{\epsilon \to 0} \int_{W_{\epsilon \tilde{R}}} \frac{u_\epsilon^2 e^{(4\kappa - \epsilon)u_\epsilon^2}}{\lambda_\epsilon} dx = \lim_{\epsilon \to 0} \int_{W_{\epsilon \tilde{R}}} \frac{r_\epsilon^2 v_\epsilon^2 e^{(4\kappa - \epsilon)v_\epsilon^2}}{\lambda_\epsilon} dx
\]
\[
= \lim_{\epsilon \to 0} \int_{W_{\epsilon \tilde{R}}} \frac{u_\epsilon^2(r,x) e^{(4\kappa - \epsilon)(u_\epsilon^2(r,x) - M_\epsilon^2)}}{M_\epsilon^2} dx
\]
\[
= \int_{W_{\epsilon \tilde{R}}} e^{8kw} dx.
\]

Taking \( \tilde{R} \to +\infty \) we have \( \int_{\mathbb{R}^2} e^{8kw} dx \leq 1 \).
On the other hand, we claim
\[ \int_{\mathbb{R}^2} e^{8\kappa w} dx \geq 1. \] (13)

Actually we can prove it by using the level-set method. For \( t \in \mathbb{R} \), let \( \Omega_t = \{ x \in \Omega | w(x) > t \} \) and \( \mu(t) = |\Omega_t| \). By the divergence theorem,
\[
\int_{\Omega_t} -\text{div}(F(\nabla w)F_\xi(\nabla w)) \, dx = \int_{\partial \Omega_t} F(\nabla w) < F_\xi(\nabla w), \frac{\nabla w}{|\nabla w|} > ds \\
= \int_{\partial \Omega_t} \frac{F^2(\nabla w)}{|\nabla w|} \, ds.
\]

By using the isoperimetric inequality (5) and the co-area formula (4), it follows form the Hölder inequality that
\[
2\kappa \frac{1}{2} \mu(t)^{\frac{1}{2}} \leq P_F(\Omega_t) = \int_{\partial \Omega_t} \frac{F(\nabla w)}{|\nabla w|} \, ds \\
\leq (\int_{\partial \Omega_t} \frac{F^2(\nabla w)}{|\nabla w|} \, ds)^{\frac{1}{2}}(\int_{\partial \Omega_t} \frac{1}{|\nabla w|} \, ds)^{\frac{1}{2}} \\
= (\int_{\Omega_t} e^{8\kappa w} \, dx)^{\frac{1}{2}}(-\mu'(t))^{\frac{1}{2}}.
\] (14)

Hence
\[
\int_{\mathbb{R}^2} e^{8\kappa w} \, dx = 8\kappa \int_{-\infty}^{\max w} e^{8\kappa t} \mu(t) \, dt \leq 8\kappa \int_{-\infty}^{\max w} e^{8\kappa t} - \frac{\mu'(t)}{4\kappa} \, dt \\
= \int_{-\infty}^{\max w} \frac{d}{dt}(\int_{\Omega_t} e^{8\kappa w} \, dx)^2 \, dt = (\int_{\mathbb{R}^2} e^{8\kappa w} \, dx)^2,
\]

which implies the claim.

Thus we get that \( \int_{\mathbb{R}^2} e^{8\kappa w} \, dx = 1 \), which implies that the equality holds in the above iso-perimetric inequality. Therefore \( \Omega_t \) must be a wulff ball. In other words, \( w \) is radial symmetric with respect to \( F^0 \). We can immediately get
\[ w(r) = -\frac{1}{4\kappa} \log(1 + \kappa r^2), \text{ where } r = F^0(x). \] (15)

**Lemma 4.3.** For any \( L > 1 \), we set \( u_{\epsilon, L} = \min\{ u_\epsilon, \frac{M_\epsilon}{L} \} \). Then we have
\[
\limsup_{\epsilon \to 0} \int_{\Omega} F^2(\nabla u_{\epsilon, L}) \, dx \leq \frac{1}{L}.
\]

**Proof.** We chose \((u_\epsilon - \frac{M_\epsilon}{L})^+\) as a test function of (7) to get
\[
-\int_{\Omega} (u_\epsilon - \frac{M_\epsilon}{L})^+ \text{div}(F(\nabla u_\epsilon)F_\xi(\nabla u_\epsilon)) \, dx = \int_{\Omega} \frac{(u_\epsilon - \frac{M_\epsilon}{L})^+u_\epsilon}{\lambda_\epsilon} e^{(4\kappa-\epsilon)u_\epsilon} \, dx. \] (16)
For any $\hat{R} > 0$, the estimation of the right hand side of (16) is
\[
\int_{\Omega} \frac{(u_\epsilon - M_\epsilon)^+}{\lambda_\epsilon} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \\
\geq \int_{W_\epsilon,\hat{R}(0)} \frac{(u_\epsilon - M_\epsilon)^+}{\lambda_\epsilon} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \\
= \int_{W_\epsilon(0)} (u_\epsilon(r,x) - \frac{M_\epsilon}{L})^+ \frac{r_\epsilon^2 u_\epsilon(r,x)}{\lambda_\epsilon} e^{(4\kappa - \epsilon)u_\epsilon^2(r,x)} dx \\
= \int_{W_\epsilon(0)} (v_\epsilon - \frac{1}{L})^+ v_\epsilon e^{(4\kappa - \epsilon)(v_\epsilon^2 - M_\epsilon^2)} dx \\
\rightarrow \int_{W_\epsilon(0)} (1 - \frac{1}{L}) e^{8\lambda w} dx.  \tag{18}
\]
In virtue of the divergence theorem and Lemma 2.1, the estimation of the left hand side of (16) is
\[
- \int_{\Omega} (u_\epsilon - \frac{M_\epsilon}{L})^+ \text{div}(F(\nabla u_\epsilon)\xi(\nabla u_\epsilon)) dx \\
= - \int_{\Omega} (u_\epsilon - \frac{M_\epsilon}{L})^+ \text{div}(F(\nabla (u_\epsilon - \frac{M_\epsilon}{L})^+)\xi(\nabla (u_\epsilon - \frac{M_\epsilon}{L})^+)) dx \\
= \int_{\Omega} F^2(\nabla (u_\epsilon - \frac{M_\epsilon}{L})^+) dx.  \tag{19}
\]
Putting (16) (17)(19) together, and taking $\hat{R} \to \infty$, we obtain
\[
\int_{\Omega} F^2(\nabla (u_\epsilon - \frac{M_\epsilon}{L})^+) dx \geq 1 - \frac{1}{L},
\]
and
\[
\int_{\Omega} F^2(\nabla u_\epsilon) dx = \int_{\Omega} F^2(\nabla u_{\epsilon,L}) dx + \int_{\Omega} F^2(\nabla (u_\epsilon - \frac{M_\epsilon}{L})^+) dx.
\]
Thus the conclusion can be obtained due to the fact $\int_{\Omega} F^2(\nabla u_\epsilon) dx = 1$. \hfill \square

**Remark 1.** Using inequality (2) to functions $L^2 u_{\epsilon,L}$, we get
\[
\int_{\Omega} e^{4\lambda u_{\epsilon,L}^2} dx \leq C < +\infty. \tag{20}
\]
For any $L > 1$, since $e^{4\lambda u_{\epsilon,L}^2}$ is uniformly bounded in $L^1(\Omega)$, then $e^{4\lambda u_{\epsilon,L}^2}$ is uniformly bounded in $L^q(\Omega)$ for some $q > 1$. Noticing that $u_{\epsilon,L}$ converges to 0 almost everywhere in $\Omega$, we have $e^{4\lambda u_{\epsilon,L}^2}$ converges to 1 in $L^1(\Omega)$. Thus
\[
\lim_{\epsilon \to 0} \int_{\{u_\epsilon \leq M_\epsilon\}} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \leq \lim_{\epsilon \to 0} \int_{\Omega} e^{4\lambda u_{\epsilon,L}^2} dx = |\Omega|.
\]
and then
\[
\lim_{\epsilon \to 0} \int_{\Omega} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \leq \lim_{\epsilon \to 0} \int_{\{u_\epsilon \leq M_\epsilon\}} e^{(4\kappa - \epsilon)u_\epsilon^2} dx + \lim_{\epsilon \to 0} \int_{\{u_\epsilon > M_\epsilon\}} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \\
\leq |\Omega| + \lim_{\epsilon \to 0} \frac{\lambda_\epsilon L^2}{M_\epsilon^2} \int_{\{u_\epsilon > M_\epsilon\}} u_\epsilon^2 e^{(4\kappa - \epsilon)u_\epsilon^2} dx \\
\leq |\Omega| + \lim_{\epsilon \to 0} \frac{\lambda_\epsilon L^2}{M_\epsilon^2}.  \tag{21}
\]
Taking $L \to 1$, we get

$$|\Omega| < \lim_{\epsilon \to 0} \int_{\Omega} e^{(4k-\epsilon)u^2} dx \leq |\Omega| + \limsup_{\epsilon \to 0} \frac{\lambda_\epsilon}{M_\epsilon}.$$ 

We set

$$T_0 = \lim_{\epsilon \to 0} \int_{\Omega} e^{(4k-\epsilon)u^2}.$$ 

Clearly $T_0 > |\Omega|$, and

$$\lim_{\epsilon \to 0} \frac{M_\epsilon}{\lambda_\epsilon} = 0. \quad (22)$$

Now we claim that

$$\lim_{\epsilon \to 0} \int_{\Omega} \frac{M_\epsilon}{\lambda_\epsilon} u e^{(4k-\epsilon)u^2} dx = 1. \quad (23)$$

To this purpose, we denote $\varphi_\epsilon = \frac{M_\epsilon}{\lambda_\epsilon} u e^{(4k-\epsilon)u^2}$. Clearly

$$\int_{\Omega} \varphi_\epsilon dx = \int_{\{Lu < M_\epsilon\}} \varphi_\epsilon dx + \int_{\{Lu \geq M_\epsilon\} \setminus W_{r,\hat{R}}(x_0)} \varphi_\epsilon dx + \int_{W_{r,\hat{R}}} \varphi_\epsilon dx. \quad (24)$$

We estimate the three integrates on the right hands respectively. By (22) and Lemma 4.3 we have

$$0 \leq \int_{\{Lu < M_\epsilon\}} \varphi_\epsilon dx = \frac{M_\epsilon}{\lambda_\epsilon} \int_{\{Lu < M_\epsilon\}} u e^{(4k-\epsilon)u^2} dx \leq \frac{M_\epsilon}{\lambda_\epsilon} \int_{\Omega} u e^{(4k-\epsilon)u^2} dx = o_\epsilon(1)O\left(\frac{1}{L}\right). \quad (25)$$

Moreover for any $\hat{R} > 0$, we have

$$\int_{\{Lu \geq M_\epsilon\} \setminus W_{r,\hat{R}}(x_0)} \varphi_\epsilon dx \leq \int_{\{Lu \geq M_\epsilon\} \setminus W_{r,\hat{R}}(x_0)} \frac{L}{\lambda_\epsilon} u e^{(4k-\epsilon)u^2} dx \leq \frac{L}{\lambda_\epsilon} \int_{\Omega} u e^{(4k-\epsilon)u^2} dx \to L(1 - \int_{W_{r,\hat{R}}} e^{8sw} dx), \quad (26)$$

and

$$\int_{W_{r,\hat{R}}} \varphi_\epsilon dx = \int_{W_{r,\hat{R}}} \frac{u e(r,\hat{R})}{M_\epsilon} e^{(4k-\epsilon)(u^2 - u^2')} \phi dx \to \int_{W_{r,\hat{R}}} e^{8sw} dx. \quad (27)$$

Putting (25) (26) (27) together, taking $\epsilon \to 0$ and then $\hat{R} \to \infty$, we conclude (23).

Next we set $g_\epsilon = M_\epsilon u_\epsilon$. Clearly $g_\epsilon$ satisfy the following equation in the distributional sense:

$$-\text{div}(F(\nabla g_\epsilon)F_\xi(\nabla g_\epsilon)) = \frac{g_\epsilon}{\lambda_\epsilon} e^{(4k-\epsilon)u_\epsilon^2}. \quad (28)$$

**Lemma 4.4.** $g_\epsilon = M_\epsilon u_\epsilon$ is bounded in $W^{1,q}_0(\Omega)$ for any $1 < q < 2$. Furthermore, $g_\epsilon = M_\epsilon u_\epsilon \to G$ weakly in $W^{1,q}_0(\Omega)$ for any $1 < q < 2$ and $M_\epsilon u_\epsilon \to G$ strongly in $C^1_{\text{loc}}(\Omega \setminus \{0\}) \cap L^r(\Omega)$ for any $r > 1$, where $G$ satisfies

$$\left\{ \begin{array}{ll}
-\text{div}(F(\nabla G)F_\xi(\nabla G)) = \delta_0 & \text{in } \Omega \\
G|_{\partial \Omega} = 0 & \text{on } \partial \Omega
\end{array} \right. \quad (29)$$
in the distributional sense.

Proof. It is clear that \( g_\epsilon \in W^{1,2}_0(\Omega) \). From Lemma 4.3, (22) and (23) we have

\[
\int_\Omega -Qg_\epsilon \phi dx = \int_\Omega \frac{g_\epsilon}{\lambda_\epsilon} e^{(4\kappa-\epsilon)u_\epsilon^2} \phi dx \to \phi(0), \tag{30}
\]

for any \( \phi \in C_0^\infty(\Omega) \). From Lemma 2.1 we get that

\[
|\nabla g_\epsilon|^2 \leq C_1 F^2(\nabla g_\epsilon) = C_1 F(\nabla g_\epsilon) F_\xi(\nabla g_\epsilon) \nabla g_\epsilon \leq C_1 F(\nabla g_\epsilon) F_\xi(\nabla g_\epsilon)|\nabla g_\epsilon| \leq C_2 |\nabla g_\epsilon|^2.
\]

Hence for any \( 1 < q < 2 \), and \( p := \frac{q}{q-1} > 2 \), we have

\[
|\nabla g_\epsilon|_{L^q} \leq \sup \left\{ \int_\Omega \nabla g_\epsilon \nabla \phi : ||\phi||_{W^{1,p}_0} = 1 \right\} \leq C \sup \left\{ \int_\Omega F(\nabla g_\epsilon) F_\xi(\nabla g_\epsilon) \nabla \phi : ||\phi||_{W^{1,p}_0} = 1 \right\}.
\]

It follows from the Sobolev embedding Theorem to know that \( ||\phi||_{L^\infty(\Omega)} \leq C \). Hence

\[
\int_\Omega F(\nabla g_\epsilon) F_\xi(\nabla g_\epsilon) \nabla \phi dx = \int_\Omega -Qg_\epsilon \phi dx \leq ||\frac{g_\epsilon}{\lambda_\epsilon} e^{(4\kappa-\epsilon)u_\epsilon^2}||_{L^1(\Omega)} ||\phi||_{L^\infty(\Omega)} \leq C.
\]

Therefore \( ||\nabla g_\epsilon||_{L^q(\Omega)} \leq C \) for any \( 1 < q < 2 \). It follows (30) that \( g_\epsilon \to G \) weakly in \( W^{1,q}_0(\Omega) \). By using the Sobolev embedding Theorem again, \( g_\epsilon \) strongly converges to \( G \) in \( L^r \) for any \( r > 1 \). By using Lemma 4.2 and (2), we have \( e^{(4\kappa-\epsilon)u_\epsilon^2} \) is uniformly bounded in \( L^s(\Omega \setminus B_\delta(0)) \) for any \( s > 1 \), then \( \frac{g_\epsilon}{\lambda_\epsilon} e^{(4\kappa-\epsilon)u_\epsilon^2} \) is uniformly bounded in \( L^t(\Omega \setminus B_\delta(0)) \) for some \( t > 1 \). By using Theorem 2 in [6], \( g_\epsilon \) is uniformly bounded in \( L^\infty(\Omega \setminus B_\delta(0)) \). Then by Theorem 1 in [13] and Ascoli-Arzela’s theorem, it follows that \( g_\epsilon \) strongly converges to \( G \) in \( C^1_{loc}(\Omega \setminus \{0\}) \). \( \square \)

Because of \( \Omega = W_R \), we can easily get

\[
G(\rho) = \frac{1}{2\kappa} \log \frac{R}{\rho} = -\frac{1}{2\kappa} \log \rho + \frac{1}{2\kappa} \log R = -\frac{1}{2\kappa} \log \rho + C_G, \tag{31}
\]

here \( \rho = F^0(x) \).

Lemma 4.5. If \( M_\epsilon \to +\infty \), then \( \limsup_{\epsilon \to 0} \frac{M_\epsilon^2}{\lambda_\epsilon} = (T_0 - \kappa R^2)^{-1} \).

Proof. For any \( 0 < r < R \), set \( W_{\rho,r} = W_\rho \setminus W_r \). From (29), clearly

\[
-\text{div}(F(\nabla G)F_\xi(\nabla G)) = 0 \quad \text{in} \ W_{\rho,r}.
\]
Multiplying by $<\nabla G, x>$, and integrating by parts, we have

$$0 = -\int_{\Omega_\rho,x} \text{div}(F(\nabla G)F(x,\nabla G)) <\nabla G, x> dx$$

$$= \int_{\Omega_{\rho,r}} F^2(\nabla G) dx + \int_{\Omega_{\rho,r}} \sum_{i=1}^2 F(\nabla G)F(x,\nabla G) \sum_{j=1}^2 x_j \frac{\partial^2 G}{\partial x_i \partial x_j} dx$$

$$- \int_{\partial\Omega_{\rho,r}} F(\nabla G) < F(x,\nabla G), \nu > = \int_{\partial\Omega_{\rho,r}} F^2(\nabla G) < x, \nu > ds$$

$$- \int_{\partial\Omega_{\rho,r}} F(\nabla G) < F(x,\nabla G), \nu > < x, \nabla G > ds,$$

where $\nu = \frac{\nabla F}{|\nabla F|}$ is the unit outward normal. On $\partial\Omega$, we have

$$F(\nabla G) = F(\frac{1}{2\kappa} \frac{\nabla F}{F^0}) = \frac{1}{2\kappa r},$$

$$< F(x,\nabla G), \nu > = < F(x,\nabla G), \frac{\nabla F}{|\nabla F|} > = -2\kappa F(\nabla G)\frac{\nabla G}{|\nabla F|} >$$

$$= -\frac{1}{|\nabla F|},$$

$$< x, \nabla G > = < x, -\frac{1}{2\kappa} \frac{\nabla F}{F^0} > = -\frac{1}{2\kappa},$$

$$< x, \nu > = < x, \frac{\nabla F}{|\nabla F|} > = \frac{r}{|\nabla F|}.$$  (33)

Inserting (33)-(35) to (32), taking $r \to 0$, we get

$$\int_{\partial\Omega_{\rho}} \frac{1}{2} F^2(\nabla G) < x, \nu > ds - \int_{\partial\Omega_{\rho}} F(\nabla G) < F(x,\nabla G), \nu > < x, \nabla G > ds = -\frac{1}{4\kappa}. $$

Here we have used the fact that $\frac{1}{2} \int_{\partial\Omega_{\rho}} \frac{1}{|\nabla F|} = 2\kappa$. Recalling the equation (28), we multiply by $<\nabla g, x>$ and integrate by parts to get

$$\int_{\partial\Omega_{\rho}} \frac{1}{2} F^2(\nabla g) < x, \nu > ds - \int_{\partial\Omega_{\rho}} F(\nabla g) < F(x,\nabla g), \nu > < x, \nabla g > ds$$

$$= \frac{M^2}{\lambda e} \int_{\Omega_{\rho}} u e^{(4\kappa - \epsilon)u^2} < \nabla u, x > dx$$

$$= \frac{M^2}{\lambda e} \left[ \frac{1}{2} \int_{\partial\Omega_{\rho}} e^{(4\kappa - \epsilon)u^2} < x, \nu > ds - \int_{\partial\Omega_{\rho}} \frac{e^{(4\kappa - \epsilon)u^2}}{4\kappa - \epsilon} dx \right].$$

Taking $\rho \to R$ and $\epsilon \to 0$, and using Lemma 4.4, we have

$$\lim_{\epsilon \to 0} \frac{M^2}{\lambda e}(T_0 - \kappa R^2) = 1.$$

The result holds.
Remark 2. For any $0 < \rho < R$, we have
\[ \int_{\mathcal{W}_\rho} F^2(\nabla u_\epsilon) \, dx = 1 - \frac{1}{M^2_\epsilon} [G(\rho) + o_\epsilon(1)]. \] (36)
In fact,
\[ \int_{\mathcal{W}_\rho} F^2(\nabla g_\epsilon) \, dx = M^2_\epsilon - \int_{\mathcal{W}_\rho} \text{div}(F(\nabla g_\epsilon)F_\epsilon(\nabla g_\epsilon)) \, dx + \int_{\partial \mathcal{W}_\rho} F(\nabla g_\epsilon) g_\epsilon < F_\epsilon(\nabla g_\epsilon), \nu > ds. \]
Since from (23) we have
\[ -\int_{\partial \mathcal{W}_\rho} F(\nabla g_\epsilon) g_\epsilon < F_\epsilon(\nabla g_\epsilon), \nu > ds = -g_\epsilon(\rho) \int_{\mathcal{W}_\rho} \text{div}(F(\nabla g_\epsilon)F_\epsilon(\nabla g_\epsilon)) \, dx \]
\[ = g_\epsilon(\rho) \int_{\mathcal{W}_\rho} g_\epsilon e^{(4\kappa - \epsilon)u^2_\epsilon} \, dx \]
\[ \rightarrow G(\rho), \text{ as } \epsilon \rightarrow 0, \]
and
\[ -\int_{\mathcal{W}_\rho} \text{div}(F(\nabla g_\epsilon)F_\epsilon(\nabla g_\epsilon)) \, dx = o_\epsilon(1), \]
it follows that (36) holds.

5. proof of Theorem. In this section, we will prove our main Theorem. First we give the definition of normalized concentrating sequences in $W^{1,2}_0(\Omega)$, i.e. if $u_\epsilon$ satisfies
\begin{enumerate}
  \item $\int_{\Omega} F^2(\nabla u_\epsilon) \, dx = 1,$
  \item $u_\epsilon \rightharpoonup 0$ weakly in $W^{1,2}_0(\Omega)$,
  \item there exists $x_0 \in \Omega$ such that any $r > 0$, $\int_{\Omega \setminus B_r(x_0)} F^2(\nabla u_\epsilon) \, dx \rightarrow 0$.
\end{enumerate}
Then we call $u_\epsilon$ a normalized concentrating sequences, and $x_0$ is called as a blow up point.

Lemma 5.1. If $u_\epsilon$ is any normalized concentrating sequence in $W^{1,2}_0(\mathcal{W}_\rho)$ with the origin as a blow up point, then
\[ \limsup_{\epsilon \to 0} \int_{\mathcal{W}_\rho} (e^{4\kappa u^2_\epsilon} - 1) \, dx \leq \kappa \rho^2. \] (37)
Proof. Set $F^0(x) = \rho e^{-\frac{x}{4}}$, and $w_\epsilon(t) = 2\kappa^2 \frac{u^*_\epsilon(x)}{\|F(\nabla u^*_\epsilon)\|_{L^2(\mathcal{W}_\rho)}}$, where $u^*_\epsilon$ is the convex symmetrization of $u_\epsilon$ with respect to $F(x)$. By the direct calculations, we have
\[ \int_0^\infty w^2_\epsilon(t) \, dt = 1, \]
and
\[ \int_{\mathcal{W}_\rho} (e^{4\kappa u^2_\epsilon} - 1) \, dx \leq \int_{\mathcal{W}_\rho} (e^{4\kappa u^2_\epsilon} - 1) \, dx = \kappa \rho^2 \int_0^\infty (e^{w^2_\epsilon(t)} - 1) e^{-t} \, dt. \]
From [2] we have
\[ \limsup_{\epsilon \to 0} \int_0^\infty (e^{w^2_\epsilon(t)} - 1) e^{-t} \, dt \leq e. \]
for any normalized concentrating sequence \( \{ w_\epsilon \} \in C^1[0, \infty) \) and any \( A \in (0, +\infty) \)
\[
\int_0^\infty \dot{w}_\epsilon^2(t) dt = 1, \quad \int_0^A \dot{w}_\epsilon^2(t) dt \to 0, \quad \text{with} \ w_\epsilon(0) = 0, \dot{w}_\epsilon(t) \geq 0.
\]
Hence (37) holds.

Define
\[
T_\epsilon = \int_\Omega \sum_{|\nabla u_\epsilon| \leq 1} e^{(4\kappa - \epsilon)u_\epsilon^2} dx, \quad \forall x \in (0, 4\kappa).
\]
It is easy to see that \( T_\epsilon \) is increasing and
\[
T_0 = \lim_{\epsilon \to 0} T_\epsilon = \sup_{u \in \mathcal{H}} \min_{\| u \|_{W_0^{1,2}(\Omega)} \leq 1} \int_\Omega e^{4\kappa u^2} dx.
\]
To get a contradiction, we use the similar arguments in [17] and [2]. We first claim a Carleson-Chang type result.

**Lemma 5.2.** If \( \lim_{\epsilon \to 0} \| u_\epsilon \|_\infty = \infty \), then
\[
T_0 \leq \kappa R^2 + \kappa e^{1+4\kappa C_\varphi}.
\]

**Proof.** From Lemma 4.3 it follows that
\[
\int_{ \{ Lu_\epsilon \leq M \} } e^{(4\kappa-\epsilon)u_\epsilon^2} dx \leq \int_\Omega e^{(4\kappa-\epsilon)u_\epsilon^2} dx \to \kappa R^2, \quad \text{as} \ \epsilon \to 0,
\]
and
\[
\int_{ \{ Lu_\epsilon \geq M \} \setminus \mathcal{W}_R} e^{(4\kappa-\epsilon)u_\epsilon^2} dx \leq \frac{\lambda_\epsilon L^2}{M_\epsilon} \int_{ \{ Lu_\epsilon \geq M \} \setminus \mathcal{W}_R} \frac{u_\epsilon^2 e^{(4\kappa-\epsilon)u_\epsilon^2}}{\lambda_\epsilon} dx
\]
\[
\leq \frac{\lambda_\epsilon L^2}{M_\epsilon} \int_{\mathcal{W}_R} \frac{1}{\lambda_\epsilon} u_\epsilon^2 e^{(4\kappa-\epsilon)u_\epsilon^2} dx
\]
\[
= \frac{\lambda_\epsilon L^2}{M_\epsilon} (1 - \int_{\mathcal{W}_R} e^{8\kappa u_\epsilon^2} dx) - L^2 (T_0 - \kappa R^2) (1 - \int_{\mathcal{W}_R} e^{8\kappa u_\epsilon^2} dx).
\]
Set
\[
\ell_\epsilon(x) = \| F(\nabla u_\epsilon) \|_{L^2(\mathcal{W}_R)}^{-1} [u_\epsilon(x) - u_\epsilon(\rho)]^+.
\]
Clearly, \( \ell_\epsilon(x) \in W_0^{1,2}(\mathcal{W}_R) \), \( \| F(\nabla \ell_\epsilon) \|_{L^2(\mathcal{W}_R)} = 1 \) and \( \ell_\epsilon(x) \) weakly convergence to 0 in \( W_0^{1,2}(\mathcal{W}_R) \). Thus \( \ell_\epsilon(x) \) is a normalized concentrating sequence in \( W_0^{1,2}(\mathcal{W}_R) \). By Lemma 5.1 there holds
\[
\limsup_{\epsilon \to 0} \int_{\mathcal{W}_R} (e^{4\kappa \ell_\epsilon^2} - 1) dx \leq \kappa e^2.
\]
Moreover, from (10) we know $M^{-1}_\epsilon \ell_\epsilon \to 1$ uniformly in $W_{r_\epsilon \hat R}$ since $M^{-1}_\epsilon u_\epsilon \to 1$ uniformly in $W_R$. Then it follows from (36) that

\[ u_\epsilon^2(x) = [\ell_\epsilon(x) + u_\epsilon(x)]^2 \{ F(\nabla u_\epsilon) \}_{L^2(W_\rho)}^2 \]

\[ = [\ell_\epsilon(x) + M^{-1}_\epsilon G(\rho) + o_\epsilon(M^{-1}_\epsilon)^2 \times [1 - \frac{1}{M^2_\epsilon} G(\rho) + o_\epsilon(1)] \]

\[ = \ell_\epsilon^2(x) + 2\ell_\epsilon(x)M^{-1}_\epsilon G(\rho) - \frac{\ell_\epsilon^2(x)}{M^2_\epsilon} G(\rho) + o_\epsilon(1) \]

Hence we have

\[ \limsup_{\epsilon \to 0} \int_{W_{r_\epsilon \hat R}} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \leq \limsup_{\epsilon \to 0} \int_{W_{r_\epsilon \hat R}} (e^{4\kappa u_\epsilon^2} - 1) dx \]

\[ \leq \limsup_{\epsilon \to 0} e^{4\kappa G(\rho)} \int_{W_{r_\epsilon \hat R}} (e^{4\kappa \ell_\epsilon^2} - 1) dx \]

\[ \leq \limsup_{\epsilon \to 0} e^{4\kappa G(\rho)} \int_{W_{r_\epsilon \hat R}} (e^{4\kappa \ell_\epsilon^2} - 1) dx \]

\[ \leq \kappa R^2 e^{1+4\kappa G(\rho)}. \] (40)

Putting (38)-(40) together and letting $\hat R \to \infty$, we conclude that

\[ T_0 = \lim_{\epsilon \to 0} \int_{\Omega} e^{(4\kappa - \epsilon)u_\epsilon^2} dx \leq \kappa R^2 + \kappa R^2 e^{1+4\kappa G(\rho)}. \]

Using the expansion (31) of $G$, taking $\rho \to 0$, we obtain that

\[ T_0 \leq \kappa R^2 + \kappa e^{1+4\kappa C_G}. \]

Next we construct a sequence of functions $\{f_\epsilon\}$ such that $T_0 \geq \|e^{4\kappa f_\epsilon^2}\|_{L^1(\Omega)} > \kappa R^2 + \kappa e^{4\kappa C_G}$. Thus we get a contradiction and consequently we complete the proof of Theorem.

**Lemma 5.3.** There holds

\[ T_0 > \kappa R^2 + \kappa e^{4\kappa C_G}. \]

**Proof.** We can use the Green function $G$ and the solution $w(r)$ to the bubble equation (11) (see also (15)) to construct the following sequence of functions

\[ f_\epsilon(r) = \begin{cases} 
\beta_\epsilon + \frac{w(\epsilon^{-1}r) + \gamma_\epsilon}{\beta_\epsilon} & \text{if } 0 < \epsilon r \leq \epsilon \hat R, \\
\frac{G(r)}{\beta_\epsilon} & \text{if } \epsilon \hat R \leq r \leq \hat R,
\end{cases} \]

where $\beta_\epsilon$ and $\gamma_\epsilon$ are constants to be chosen later, and $\hat R = -\ln \epsilon$. Here and in the sequel we always denote $r$ as $F^0(x)$.

Choose $\gamma_\epsilon$ such that

\[ \frac{G(\epsilon \hat R)}{\beta_\epsilon} = \beta_\epsilon + \frac{w(\epsilon \hat R) + \gamma_\epsilon}{\beta_\epsilon}, \]

which makes $f_\epsilon$ continuous. Using the expansion of $G$, we have

\[ \beta_\epsilon^2 + \gamma_\epsilon = G(\epsilon \hat R) - w(\epsilon \hat R), \]
and
\[ 4\kappa(\beta^2 + \gamma_\varepsilon) = -2 \log(\varepsilon R_\varepsilon) + 4\kappa C_G + \log(1 + \kappa R_\varepsilon^2) = -2 \log \varepsilon + 4\kappa C_G + \log \kappa + O(R_\varepsilon^{-2}). \]  

(41)

Let \(0 < r < \rho < R\). Since \(F^2(\nabla G) = \frac{1}{4\kappa F^0(x)}\), we have
\[ \int_{\mathbb{W}_\rho \setminus \mathbb{W}_r} F^2(\nabla G) dx = G(r) - G(\rho) \leq G(r). \]

Therefore we obtain
\[ \int_{\mathbb{W}_\varepsilon R_\varepsilon} F^2(\nabla f_\varepsilon(F^0(x))) dx = \frac{1}{\beta_\varepsilon^2} \int_{\mathbb{W}_\varepsilon R_\varepsilon} F^2(\nabla G) dx \leq \frac{1}{\beta_\varepsilon^2} G(\varepsilon R_\varepsilon) = \frac{1}{4\kappa \beta^2_\varepsilon}(-2 \log(\varepsilon R_\varepsilon) + 4\kappa C_G). \]

(42)

On the other hand, we have
\[ \int_{\mathbb{W}_{\varepsilon R_\varepsilon}} F^2(\nabla f_\varepsilon) dx = \frac{1}{\beta^2_\varepsilon} \int_{\mathbb{W}_{\varepsilon R_\varepsilon}} F^2(\nabla w(\varepsilon^{-1} r)) dx \]
\[ \leq \frac{1}{\beta^2_\varepsilon} \int_{\mathbb{W}_{\varepsilon R_\varepsilon}} F^2(\nabla w(r)) dx = \frac{1}{4\beta^2_\varepsilon} \int_{\mathbb{W}_{\varepsilon R_\varepsilon}} \frac{F^0(x)^2}{(1 + \kappa F^0(x)^2)^2} dx \]
\[ = \frac{1}{4\beta^2_\varepsilon} \int_{0}^{R_\varepsilon} \frac{\rho^2}{(1 + \kappa \rho^2)^2} d\rho \int_{F^0(x) = \rho} \frac{1}{|\nabla F^0|} ds = \frac{\kappa}{2\beta^2_\varepsilon} \int_{0}^{R_\varepsilon} \frac{\rho^3}{(1 + \kappa \rho^2)^2} d\rho \]
\[ = \frac{1}{4\kappa \beta^2_\varepsilon^2} \int_{0}^{R_\varepsilon} \frac{1 + \kappa \rho^2}{(1 + \kappa \rho^2)^2} d(1 + \kappa \rho^2) - \int_{0}^{R_\varepsilon} \frac{1}{(1 + \kappa \rho^2)^2} d(1 + \kappa \rho^2) \]
\[ = \frac{1}{4\kappa \beta^2_\varepsilon^2} \left[ \log(1 + \kappa R_\varepsilon^2) - 1 + \frac{1}{1 + \kappa R_\varepsilon^2} \right]. \]

(44)

Putting (42) and (43) together, we obtain that
\[ \int_{\Omega} F^2(\nabla f_\varepsilon) dx = \int_{\mathbb{W}_{\varepsilon R_\varepsilon}} F^2(\nabla f_\varepsilon) dx + \int_{\mathbb{W}_{\varepsilon R_\varepsilon}} F^2(\nabla f_\varepsilon) dx \]
\[ \leq \frac{1}{4\kappa \beta^2_\varepsilon}(-2 \log \varepsilon + 4\kappa C_G - 1 + \log \kappa + O(R_\varepsilon^{-2})). \]

(45)

Now we choose \(\beta_\varepsilon > 0\) such that \(\int_{\Omega} F(\nabla f_\varepsilon) dx = 1\). From (45) we have
\[ 4\kappa \beta^2_\varepsilon \leq -2 \log \varepsilon + 4\kappa C_G - 1 + \log \kappa + O(R_\varepsilon^{-2}) \]
\[ = O(\log \varepsilon), \quad \text{as} \quad \varepsilon \to 0. \]
By (41), it follows that
\[ 4\kappa\gamma_{\varepsilon} \geq 1 + O(R_{\varepsilon}^{-2}) \]
as \( \varepsilon \to 0 \). Next we estimate \( \|e^{4\kappa f_{\varepsilon}^2}\|_{L^1(\Omega)} \). It follows from \( \varepsilon' \geq 1 + t \) to get
\[
\int_{\partial B_{R_{\varepsilon}}} e^{4\kappa f_{\varepsilon}^2} \, dx \geq \kappa R^2 - \kappa (\varepsilon R_{\varepsilon})^2 + \frac{4\kappa}{\beta_{\varepsilon}} \int_{\partial B_{R_{\varepsilon}}} G_0^2 \, dx \\
= \kappa R^2 + \frac{4\kappa}{\beta_{\varepsilon}^2} \left[ \int_{\Omega} G^2 \, dx + o(\varepsilon R_{\varepsilon}) - \frac{\varepsilon^2 R_{\varepsilon}^2 \beta_{\varepsilon}^2}{4} \right] \\
= \kappa R^2 + \frac{4\kappa}{\beta_{\varepsilon}^2} \left[ \int_{\Omega} G^2 \, dx + o_{\varepsilon}(1) \right].
\]
Since
\[ 4\kappa f_{\varepsilon}^2(r) = 4\kappa (\beta_{\varepsilon} + \frac{\omega(-1/r) + \gamma_{\varepsilon}}{\beta_{\varepsilon}})^2 \]
\[ \geq 4\kappa (\beta_{\varepsilon}^2 + 8\kappa \gamma_{\varepsilon} + 8\kappa \omega(\varepsilon^{-1}r) \]
\[ = 4\kappa (\beta_{\varepsilon}^2 + \gamma_{\varepsilon}) + 4\kappa \gamma_{\varepsilon} + 8\kappa \omega(\varepsilon^{-1}r) \]
\[ \geq -2 \log \varepsilon + 4\kappa C_G \log \kappa + 1 + 8\kappa \omega(\varepsilon^{-1}r) + O(R_{\varepsilon}^{-2}), \]
then we obtain that
\[
\int_{\partial B_{R_{\varepsilon}}} e^{4\kappa f_{\varepsilon}^2} \geq e^{-2\log \varepsilon + 4\kappa C_G + \log \kappa + 1 + O(R_{\varepsilon}^{-2})} \int_{\partial B_{R_{\varepsilon}}} e^{8\kappa \omega(\varepsilon^{-1}r)} \, dx \\
= \kappa e^{4\kappa C_G + 1 + O(R_{\varepsilon}^{-2})} \int_{\partial B_{R_{\varepsilon}}} e^{8\kappa \omega} \, dx \\
= \kappa e^{4\kappa C_G + 1}[1 + O(R_{\varepsilon}^{-2})].
\]
Noticing that \( \beta_{\varepsilon} = O(||\log \varepsilon||_1^2) \), we can show that
\[
\int_{\Omega} e^{4\kappa f_{\varepsilon}^2} \geq \kappa R^2 + \frac{4\kappa}{\beta_{\varepsilon}^2} \left[ \int_{\Omega} G^2 \, dx + o_{\varepsilon}(1) \right] + \kappa e^{4\kappa C_G + 1}[1 + O(R_{\varepsilon}^{-2})] \\
= \kappa R^2 + \kappa e^{4\kappa C_G + 1} + \frac{4\kappa}{\beta_{\varepsilon}^2} \left[ \int_{\Omega} G^2 \, dx + o_{\varepsilon}(1) \right].
\]
By choosing a small \( \varepsilon > 0 \), we conclude that \( T_0 \geq \|e^{4\kappa f_{\varepsilon}^2}\|_{L^1(\Omega)} > \kappa R^2 + \kappa e^{4\kappa C_G + 1} \). Thus we finish the proof of this Lemma. 

\[ \square \]

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