Michel Hénon’s contributions to collisional stellar systems

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ABSTRACT

The theory of star cluster dynamics was a major topic in Hénon’s early research career. Here we summarise his contributions under three headings: (i) the Monte Carlo method, (ii) homological evolution of star clusters, and (iii) escape from star clusters. In each case we also trace some aspects of how Hénon’s contributions have been developed or applied in subsequent decades up to the present. We also propose that Hénon’s work be commemorated by adopting the names “Hénon units” and “Hénon’s Principle”.

1 INTRODUCTION

Hénon’s contributions to collisional stellar dynamics are found in less than 20 papers.1 All were published before 1976, but represent almost half of the output in that early period of his research career. Though Hénon abandoned the field thereafter, his contributions were taken up again in subsequent decades by others, sometimes after a long gap. Now they have become established as essential tools in the current vigorous exploration of the dynamical evolution of globular star clusters.

In this contribution we summarise Hénon’s research under three headings, not in chronological order. The last in chronological order was the invention of the Monte Carlo method for the numerical simulation of the evolution, and we discuss this first. It also involves a couple of ideas which we recommend be used to commemorate Hénon’s contributions: “Hénon units” and “Hénon’s Principle”. Then we turn to what was essentially Hénon’s thesis work, on self-similar evolution. Besides the self-similar models themselves, it is a seminal work, foreshadowing so much that was painfully rediscovered or refined by others in subsequent decades. Finally we examine a topic to which Hénon returned from time to time: the escape rate from a star cluster, especially an isolated one. This was a natural conclusion to his earliest collisional research, on the evaluation of diffusion coefficients, but that is the subject of another contribution in this volume.

2 THE MONTE CARLO CODE (1967-1975)

This was a subject which occupied Hénon for about ten years (Hénon 1967a,b, 1971a,b, 1972a,b; Hénon 1973; Aarseth, Hénon, & Wielen 1974; Hénon 1975). In fact Monte Carlo codes were invented independently by Hénon, on the one hand, and by Spitzer and his students on the other, in the years around 1970, and their history is well summarised in Vasiliev (2014). After Hénon himself apparently stopped developing the method in the mid-1970s, there was a gap before it was taken up again and further developed by J. Stodólkiewicz in the 1980s. Sadly Stodólkiewicz died in 1988, but the ideas underlying his and Hénon’s codes were faithfully developed from the 1990s by Stodólkiewicz’s student M. Giersz, working at CAMK in Warsaw. The same thread of ideas was also adopted by a group at Northwestern University under the leadership of F. Rasio.

The theory of star cluster dynamics was a major topic in Hénon’s early research career. Here we summarise his contributions under three headings: (i) the Monte Carlo method, (ii) homological evolution of star clusters, and (iii) escape from star clusters. In each case we also trace some aspects of how Hénon’s contributions have been developed or applied in subsequent decades up to the present. We also propose that Hénon’s work be commemorated by adopting the names “Hénon units” and “Hénon’s Principle”.

2.1 Description of the basic Monte Carlo method

Once Hénon had thought of it, the basic idea of these Monte Carlo codes is quite simple. Assuming that a stellar system is spherical and non-rotating, the dynamics of each star is represented by its (specific) energy $E$ and angular momentum $J$. The values of $E$ and $J$ are given random adjustments with the same statistical properties as those given by the theory of two-body relaxation (including moments $\langle \Delta E \rangle$, $\langle (\Delta E)^2 \rangle$, etc.)

In slightly greater detail, the structure of the code is the following.

1 Select the overall time step $dt$, which for a code of Hénon type is a fraction of a relaxation time;
2 From the initial model (King, Plummer, etc), for each star assign radius $r$ (distance from the cluster centre) and $E$ and $J$.
3 Begin:
   I Order the stars by $r$ and compute the potential
   II For each successive pair of stars

1 We have used the bibliography of ADS in this paper
2 The two papers in Astrophysics and Space Science are reprints of those in the Proceedings of IAU Colloquium No.10
(1) From $E, J$ compute radial and transverse components of velocity of both stars. (The orientation of the transverse component is randomised)

(2) Choose the impact parameter $p$ for an encounter so that the statistical effect of the encounter corresponds to the velocity changes predicted by the theory of relaxation for the time $dt$

(3) Compute the velocity components after the encounter, and the new $E$ and $J$ of both stars

(4) Move each star to a random position (radius) on an orbit of energy $E$ and angular momentum $J$

IV Repeat

Really, anyone with a knowledge of collisional stellar dynamics could use this outline to construct a simple Monte Carlo code in a few hours, except for a few tricky points, where Hénon’s ingenuity shows the way. The shortest introduction to his approach is in Hénon’s papers, or his SAAS-Fee lectures (Hénon 1973), which also include a succinct account of the theory of two-body relaxation. Here we summarise Hénon’s solutions to these awkward points.

(i) Computation of the potential

The first is simple enough. In a spherical system the potential gradient is $d\phi/dr = -GM(r)/r^2$, where $M(r)$ is the mass inside radius $r$. Therefore the potential is

$$\phi = \int_0^r \frac{GM(r)}{r^2} dr = \left[ -\frac{GM(r)}{r} \right]_r^\infty + \int_r^\infty \frac{GM(r)}{r} dr.$$

In the Monte Carlo code, each star is thought of as a spherical shell. Thus let $m_i, r_i, \phi_i$ be the mass, radius and potential of the $i$th star or shell. Then $\phi_i = \sum_{j=1}^i \frac{Gm_j}{r_i} + \sum_{j>i+1}^N \frac{Gm_j}{r_j}$, and this can be computed readily by recurrence.

(ii) Choice of impact parameter

The second issue is by no means as simple. In a single encounter between two stars of velocity $v_1, v_2$, respectively, square of the velocity change of the first star is given by $(\Delta v_1)^2 = \frac{4G^2m_2^2}{p^2|v_1 - v_2|^2}$. We compare this with the mean square change in time $dt$ given by the theory of relaxation, which is $(\langle \Delta v_1^2 \rangle) = \frac{8\pi G^2m_2^2n_2 \ln(\gamma N)dt}{|v_1 - v_2|}$, where $\gamma$ is a constant of order unity. Therefore, in order to mimic relaxation by a single encounter, in a statistical sense, we choose the impact parameter by the formula $p = \frac{1}{\sqrt{2\pi n_2 \ln(\gamma N) dt|v_1 - v_2|}}$. This requires an estimate of the density $n_2$, for which Hénon devised the following procedure. The mean radial separation $dr$ between shells is given by $4\pi n_2 dr^2 dr = 1$, and this could be used as the basis of an estimate of $n_2$. To reduce fluctuations, however, Hénon chose the distance to the 5th nearest neighbour to estimate $n_2$

A couple of comments are due. First, note that choosing encounters between successive pairs of stars automatically ensures that each star encounters stars of mass $m$ in proportion to the local stellar mass function. Second, and more remarkable, is that this procedure ensures not only the correct value of $(\langle (\Delta v_1)^2 \rangle)$, but also the correct value of all other first and second moments of $\Delta v_1$.

(iii) Choice of new position Between encounters, stars move on planar orbits in the continuous spherical potential $\phi(r)$ with energy $E$ and angular momentum $J$. Thus

$$E = \frac{1}{2} \left( v_r^2 + \frac{J^2}{r^2} \right) + \phi(r),$$

where $v_r$ is the radial component of velocity. The probability of finding the star between radii $r$ and $r + dr$ proportional to the time spent there, i.e. $f(r)dr \propto \frac{dr}{v_r}$. To choose $r$ with this probability density, one could in principle use the rejection/acceptance technique. However, this is applicable only when the probability density $f$ is bounded, and here $f(r) \to \infty$ as $v_r \to 0$, i.e. at the extreme radii (peri-centre and apocentre, denoted by $r_{\text{min,max}}$) on the orbit (Fig. 1).

Hénon’s solution to this hurdle has not been superseded. Following him, we choose a new radial variable $s$, whose form is to be determined. Then $f(s) = f(r) \frac{dr}{ds}$. Now we choose $s$ so that $\frac{dr}{ds}$ also vanishes (like $v_r$) at $r_{\text{min}}$ and $r_{\text{max}}$. In fact we seek $r(s)$ so that $\frac{dr}{ds} = 0$ at $s = s_{\text{min}}, s_{\text{max}}$ and such that $r_{\text{min,max}} = r(s_{\text{min,max}})$. Hénon’s choice was to set $s_{\text{min,max}} = \pm 1$ and let $\frac{dr}{ds} = 3A(s + 1)(1 - s)$, where $A$ is constant. Hence $r(s) = B + A(3s - s^3)$, where $B$ is another constant. Finally, choose $A, B$ so that $r(\pm 1) = r_{\text{min,max}}$.

2.2 Comments on the method

The time step $dt$ is of order the relaxation time, and each step takes of order $N \ln N$ operations (because sorting of the radii is required, and computation of the potential at the new position of a star requires $O(\ln N)$ operations). Thus the computational effort is of order $N \ln N$ per relaxation time, by contrast with, for example, a direct $N$-body simulation, where for a relatively homogeneous system the effort is of order $\frac{N^{10/3}}{\ln \gamma N}$ per relaxation time (Makino & Hut 1989).

Clearly the Monte Carlo method wins for the range of $N$ corresponding to the globular star clusters. In fact any
Figure 2. A scatter-plot of the Galactic globular clusters: half-light relaxation time against absolute V-magnitude. The data are taken from the most recent version of the Harris catalogue (Harris 1996). The absolute magnitude is taken as a proxy for \( N \), assuming constant mass-to-light ratio and mean stellar mass. The sloping lines represent systems requiring roughly the same computational effort with a modern Monte Carlo code, assuming the scaling given in the text. Adjacent lines correspond to effort changing by a factor of 10, and lines for 1 and 10 days are labelled. One caveat is that both \( t_{\rm{bh}} \) and \( M_V \) vary throughout the lifetime of the cluster, and it is assumed that the current values may be adopted.

Galactic globular cluster can be modelled in roughly 1 day (Fig.2), whereas with direct \( N \)-body methods the modelling of a given cluster for a Hubble time is limited at present to \( N \lesssim 2 \times 10^5 \) stars.

Some of the merits and limitations of the Monte Carlo method are readily summarised. Besides its relative efficiency, just mentioned, we have the fact that one can ‘readily’ add stellar evolution, and dynamical interactions involving binaries. Each required considerable effort in coding, but the job is done. Of course, these are not advantages relative to the \( N \)-body method, but they do account for the fact that the Monte Carlo method leads the field among the fast methods for the simulation of dynamical evolution of rich star clusters.

Among the difficulties of the Monte Carlo method is the modelling of the Galactic tide. This has been steadily improved, but still \( N \)-body methods are needed for an adequate modelling of such features as the kinematics of escaping stars (e.g. Kipper et al. 2014). It also has to be assumed that the system is spherically symmetric and non-rotating. Triples and higher-order multiples have to be ignored, though further development of the code could quite readily remove this limitation.

Because of the approximations involved, the Monte Carlo code has been repeatedly compared with the results of the virtually assumption-free \( N \)-body method, at least in the range of \( N \) which the latter can reach. Naturally the first of these studies was carried out by Hénon himself (Aarseth, Hénon, & Wieland 1971).

At present the Monte Carlo method is the method of choice for all the large globular clusters of the Galactic system, and has been applied to the modelling of the individual objects \( \omega \) Cen (Giersz & Heggie 2003), M4 (Heggie & Giersz 2008), NGC6397 (Giersz & Heggie 2009), 47 Tuc (Giersz & Heggie 2011) and M22 (Heggie & Giersz 2014). It is also being used to study such topics as the origin and distribution of blue stragglers (Hypki & Giersz 2013; Sills et al. 2013; Chatterjee et al. 2013).

2.3 Digression: \( N \)-body Units?

About 30 years ago this author tried to encourage the community to adopt a common system of units for its work, so that different studies could be more readily compared. It is defined by setting

\[
G = 1, \\
M = 1, \\
R = 1,
\]

where \( G \) is the constant of gravitation, \( M \) is the total mass of the system, and \( R \) is its virial radius. Though some researchers find these units inconvenient for some reason or another, nevertheless they have been taken up widely, partly because major \( N \)-body and Monte Carlo codes adopt them, and perhaps also because they preserve the sense that simple \( N \)-body models (i.e. without the accretion of stellar evolution and so on) are scale-free. As a result the paper in which the author’s recommendation appeared (Heggie & Mathieu 1986) is ranked 5th by number of citations among this author’s papers, even though the publication in which it appeared was almost unobtainable for a long time. Furthermore the units were not even his idea, and so the paper’s prominence has long been a source of mild embarrassment.

Actually, \( N \)-body units originated in Hénon’s papers on the Monte Carlo method (see especially Hénon 1971b, 1972b). This fact was implicitly acknowledged in the author’s paper, which did not even use the term \( N \)-body units. It seems to the author altogether appropriate to recommend that they be referred to in future as Hénon units.

2.4 Hénon’s Principle

This is another opportunity to commemorate Hénon, but the idea which he introduced is more profound than in the case of Hénon units, important though those units are in practice.

Simple collisional stellar systems go through a process of core collapse, and for some time this was an impasse in the study of such systems. In the late 70s the fate of a star cluster after core collapse seemed utterly conjectural. (See, for example, the introduction to Lightman, Press, & Odenwald 1978.)

\footnote{This study used scaled models, i.e. models in which the number of stars is much smaller than that in the actual cluster, though the radius is also changed to ensure that two-body relaxation proceeds at the correct rate.}

\footnote{This was actually done by one speaker at the Gravasco program which ran at the Institut Henri Poincaré from 9 September until 13 December, 2013, a point which prompted the author’s remarks.}
In 1974, post-core-collapse evolution could not be simulated with a Monte Carlo code. Hénon knew, from his thesis work (Sec.3), that binaries were the missing dynamical ingredient, but their introduction into the Monte Carlo code was only accomplished later (Stodłowski 1986). Their role is to act as a “heating” source, giving kinetic energy to the stars with which they interact. Hénon realised, however, that the details of the heating were irrelevant, except for determining the parameters of the core. He argued that the core adjusts to produce the energy required for “balanced evolution” of the system as a whole: if the core is too small and dense, it produces too much energy, and expands, which reduces the energy production, while if the core is too large and dilute, it produces too little energy, and contracts, which increases the energy production. Hénon also knew what the “required” energy was. It is the flux of energy from the centre in Hénon’s homological models (see Sec.3). This idea, that the core responds to the energy requirements of the cluster as a whole, can be referred to as “Hénon’s Principle”. It is analogous to Eddington’s realisation (Eddington 1926, especially ch.1.4) that one can predict the luminosity of a star even without knowing the nature of the source of stellar energy.

Hénon’s Principle has several important applications, and is one of the foundation stones of our understanding and simulation of the long-term evolution of star clusters. To allow a Monte Carlo model to pass through core collapse, Hénon took the bold step (Hénon 1975) of introducing a quite artificial energy-generating mechanism, affecting only the innermost particle in the model, but in a manner mimicking the behaviour of a real core. In this way core parameters will be wrong, but the overall evolution will be correct.

The value of Hénon’s Principle extends well beyond the Monte Carlo model, however. It applies if the density-dependence of the energy-generating mechanism is high enough that almost all energy is generated in the core. For example, three-body binary formation gives a rate of energy production proportional to \(G^3 \rho^3 m^3 / \sigma^3\) per unit mass, where \(\rho, \sigma\) are the density and one-dimensional velocity dispersion, respectively; from which Hénon’s Principle allows one to estimate the radius and density in the core of the cluster (see, for example, Goodman 1987). In a similar way, core parameters may be estimated in terms of the overall structure of the cluster (total and mean particle mass, half-mass radius) if the energy-generating mechanism is dynamical evolution of primordial binaries (Vesperini & Chernoff 1994), stellar interactions with a central black hole (Heggie et al. 2007), and perhaps even stellar evolution (Gieles 2013).

3 THE HOMOLOGICAL MODELS (1961-1965)

3.1 The 1961 paper

We go back about 10 years to two papers which laid the groundwork for Hénon’s Principle, but also did much more (Hénon 1963, 1961). His principal aim was to solve the isotropised Fokker-Planck equation for the evolution of a star cluster, i.e. the equation

\[
Q' \frac{\partial F}{\partial T} - F' \frac{\partial Q}{\partial T} = \frac{\partial}{\partial E} \left[ F \int_{-\infty}^{E} F_{1} Q_1 dE_1 + F' \left( \int_{-\infty}^{E} F_{1} Q_1 dE_1 + Q \right) \right],
\]

where \(F(E,T)\) is the 1-particle distribution function, expressed as a function of particle energy \(E\) and time \(T\), \(Q(E,T)\) is the phase-space volume up to energy \(E\) at time \(T\), \(F'\) denotes the energy-derivative \(\partial / \partial E\) and the variable of integration appears in the forms \(F_1 = F(E_1, T)\) and \(Q_1 = Q(E_1, T)\).

Before tackling this formidable problem head-on, Hénon examined it from several angles. First, assuming that the central density is finite, he computed one time step, and showed that the central density always increased. He inferred that solutions might have infinite central density. Next, his review of the literature pointed out that previous work had either (i) neglected the collision term (the right-hand side), and hence approximated the Fokker-Planck equation with the Collisionless Boltzmann equation; or (ii) assumed a steady spatial structure (e.g. a square well potential) and solved the collisional evolution, thus neglecting a term on the left hand side. Hénon asserted that all terms played a comparable role, and decided to search for self-similar (or homological) solutions. Much later, Lynden-Bell explained why it is that self-similar solutions are of such importance and relevance in such problems (Lynden-Bell & Eggleton 1980; Inagaki & Lynden-Bell 1982). The point is quite subtle, but it least it is easy to see that the solving for such a solution simplifies the task a lot.

Even so, Hénon needed to bring his powerful technique and intuition to bear, especially with regard to the question of boundary conditions. Deep in the cluster, he noted that the collision term, i.e. the right-hand side of eq.(1) must nearly vanish. Exploiting some simple freedom of scaling, this implies that, to lowest order, the distribution function is a Boltzmann distribution, i.e. \(F = e^{-E}\). Next he obtained an improved approximation (still with the assumption that the collision term vanishes) by posing \(F = e^{-E} + \delta F\). In this way he showed that

\[
F = e^{-E} + K e^{-E/2} + K_2 (5 - E) e^{-E/2},
\]

and by examining the mass and energy of part of the system, he showed how the new terms could be interpreted: the term in \(K\) corresponds to an energy flux at the centre (\(R = 0\)), while the term in \(K_2\) corresponds to a mass flux at \(R = 0\). Hénon knew that N-body simulations (von Hoerner 1960) had already revealed the formation of energetic binaries. On such grounds Hénon allowed \(K\) to be non-zero in general, while he assumed that \(K_2 = 0\).

Thus conditions near the centre of the system are fixed. The earlier of the two papers we are discussing (Hénon 1961) dealt with a star cluster immersed in the gravitational field of a parent galaxy, which fixed the boundary conditions at the outside. This left Hénon with the task of solving an integrated form of the isotropised, self-similar Fokker-Planck equation: 

\[
\frac{\partial \tilde{F}}{\partial T} = \nabla \cdot \left( \frac{\nabla \tilde{F}}{\tilde{F}} \right) - \frac{1}{\tilde{F}} \int_{-\infty}^{E} \tilde{F} Q \tilde{F}_1 \, dE_1 + \hat{J},
\]

where the integration is over all velocities, and the term \(\hat{J}\) is the energy flux from the core.
equation, i.e.

\[
\frac{3}{2} \int_{-\infty}^{E} F_1 Q_1' dE_1 - 3bFQ = F \int_{E}^{\infty} F_1 Q_1' dE_1 + F' \left( \int_{-\infty}^{E} F_1 Q_1 dE_1 + Q \int_{E}^{\infty} F_1 dE_1 \right),
\]

where \( b \) is another constant to be determined, which arises from the time-dependent terms in eq. (1). This equation is equivalent to a fourth-order system. The inner boundary conditions are given by the expressions \( F \sim e^{-E} + K e^{-E/2}, E \to -\infty \), and three similar expressions for the other three variables; and the outer boundary conditions are \( F = 0 \) and \( \int_{E}^{\infty} F_1 dE_1 = 0 \) at \( E = 0 \), since \( F = 0 \) for \( E > 0 \), if the potential vanishes at the tidal boundary. (Note that two of the four variables in the equivalent system are undetermined at the outside.) Auxiliary equations to be evaluated or solved are for the density \( \frac{d^2Z}{dt^2} = -D \left( \frac{dZ}{dt} \right)^3 Z^{-4} \) where \( Z = 1/R \); and the phase-space volume \( Q = \frac{1}{2} \int_{-\infty}^{E} (2E - 2U)^{1/2} R^2 dU \). Given a fourth-order system with four unknowns (the constants \( K \) and \( b \), and two outer values), it is reasonable to suppose that an isolated solution might exist, and no doubt this is one reason why Hénon chose to retain one of the terms in eq. (2).

Hénon’s solution method was iterative:

(i) Guess \( F, K, b \)
(ii) Do
   (a) Compute \( \rho, U, Q \)
   (b) Solve the Fokker-Planck equation for \( F \), and “tatonner” \( 6 \) on \( K, b \) to satisfy the outer boundary conditions. This gives new \( F, K, b \).
(iii) Until \( (F \) converges)

Remarkably enough the method did indeed converge. Indeed the changes in the unknowns decreased by a factor of about 5 in each cycle of the main loop. The calculations were performed on an IBM 650 at Meudon, and in 8 hours the solution was obtained with an estimated precision of 1 part in 1000.

Hénon presented his results in the form of both graphs and tables, giving \( F \) and the other useful functions obtained by his computations. Table 1 gives the two essential unknown parameters, \( K \) and \( b \), along with results from a recent study, still ongoing. The latter study used matlab, and also takes a few hours, but it includes a lengthy investigation of the dependence of the results on the tolerances of the numerical methods, and on the choice of the lower end of the range of energy \( E \), as the domain of the mathematical problem is semi-infinite. Examining the convergence of our results as these choices are varied, we conclude that the results are accurate to the last significant figure in the table.

We have only just touched on the numerous topics which Hénon treats in this paper. One of these is the evolution of two-component star clusters, in which he made the approximation of assuming that one component contributes negligibly to the total density. An attempt to remove this approximation was the main motivation for the recent work by Apple et al, discussed above. Actually, it seems unlikely that such a model exists, as all the evidence points to the more rapid escape of low-mass stars across a tidal boundary, whereas in a fully homological model, the ratio of the total mass in both components must be constant. For an isolated Fokker-Planck model, however, the mass is constant (Sec. 4), and so the search for a homological, two-component model seems more promising.

Though the 1961 paper contained such a wealth of new material — and there is much more than we have reviewed here — its reception may have been a disappointment to Hénon. It was scarcely cited in the decade following its publication (see Fig. 3), but thereafter it has taken its place as part of the foundations on which our understanding of star cluster evolution is built.

### 3.2 The 1965 paper

There can hardly be any greater contrast than that between the 1961 and 1965 papers. The earlier paper is very long, but the later paper is terse, to the extent that not even the numerical method is discussed, only the results. At first sight, the later paper deals with a less realistic problem than the 1961 paper, as the star cluster is regarded as isolated. However, we shall see that it is needed for a complete understanding of star cluster evolution, even for star clusters immersed in a parent galaxy.

Among the technical differences between the two cases...
are the details of the Fokker-Planck equation to be solved,
because the time-dependence of the scaling of radius and
mass is different. The inner boundary conditions are as be-
dependent conditions, and so in the isolated case the latter
has to be replaced by another condition, for which Apple et
al take the stated condition

This change of outer boundary data requires a different
solution strategy, which can be summarised as follows:

(i) guess $F, K, b$
(ii) do

(a) compute $\rho, U, Q$
(b) solve the Fokker-Planck equation for $F$, and iterate
on $K$ to satisfy $\int E F(E_1) dE_1 \to 0$ as $E \to 0$—
(c) iterate on $b$
(iii) until $U \to 0$ as $R \to \infty$

With modern software all “tâtonnement” could be
avoided. The results are compared with Hénon’s in Table

3.3 Application to the Globular Clusters of the
Milky Way

To illustrate the influence which Hénon’s papers have en-
joyed in recent years, we summarise here a model for the
evolution of the globular clusters in our own Galaxy. First
of all let us compare the time-dependence of the evolution
of the two models.

We suppose that the initial radius of a cluster is much
less than its tidal radius. Then the cluster expands (much
like Hénon’s isolated model) until its radius becomes com-
parable with the tidal radius. After that it evolves much like
the tidally limited model. In fact it is possible to devise a
unified model which smoothly goes over from one form of
evolution to the other \cite{Gies11}. The only modification we
make is to assume that the evolution is faster than in Hénon’s one component models, because a range of stellar masses is present in more realistic clusters.

To illustrate the consequences of this model, we note
that, in the first (essentially isolated) phase, the relaxation
time behaves as $T_r(T) = T_r(0) + T$ (Table 3
noting the footnote there). If the cluster was initially very compact, $T_r(0) \ll T$, and so $T_r(T) \sim T$. Finally, since all clusters have nearly the same age, it follows that all have the same relaxation time, of order their age.

Let us compare this result with the well known “survival
triangle”, shown in Fig 3 which seeks to establish that the
distribution of cluster parameters is restricted by the
time scales on which various destruction mechanisms act
\cite{Fall77,Fene97}. The usual interpretation of this diagram is that it shows that clusters are not found where the mechanisms of destruction would remove them within their lifetime. While most clusters sit comfortably inside the outermost contour in this diagram, they concentrate to the bottom left, where their distribution appears to be limited by destruction due to two-body relaxation. But one can perhaps better interpret the diagram as showing that most clusters have almost the same relaxation time, as predicted in the model of \cite{Gies11}.

4 ESCAPE (1958-1969)

4.1 Isolated star clusters

This section covers a topic which, unlike the foregoing, prob-
bably does not have any evident application to the star
clusters. It is a question of “pure” stellar dynamics, but a loose
end which Hénon’s work did a lot to expose. It begins with
his work on the diffusion coefficients in the theory of two-
body relaxation (see the contribution in this volume by F.
Namouni). A side-product of this work was an expression for
the probability that, in time interval $dt$, a star experiences
an encounter which changes its velocity from $V$ to $V + e$,

\[
P = \frac{8\pi G^2 m^2 e^2}{e^3} \int_{-\infty}^{\infty} a(v) dv,
\]

(Hénon 1960a), where $v_0 = \frac{1}{e} V e + \frac{M + m}{2m} e^2$, $m, M$
are the stellar masses, and $a(v)$ is the distribution of velocities.
Hénon quite rightly referred to “le résultat final, d’une
remarquable simplicité”. Indeed it is, by comparison with the
intricacies of its derivation, but this author knows no sim-
pler derivation than Hénon’s, and indeed the result seems
unattainable if one does not follow his footsteps closely.

In the above paper, he used this result to obtain expres-
sions for third-order moments of the velocity change $e$, ver-
yfying the known result that these are smaller than the first
and second moments (i.e. those that appear in the Fokker-
Planck equation) by a factor of order the Coulomb logarithm
in $\gamma N$, where $N$ is the total number of stars.

A second application of this result, obtained by inte-
Figure 4. The survival triangle of Galactic globular clusters, from Gnedin & Ostriker (1997). It is a scatter plot of the mass and half-mass radius of the clusters, differentiated by their current galactocentric distance $R$. At lower right the destruction time scale due to disk- and bulge-shocking becomes comparable with the clusters’ age; at the top clusters would be destroyed by dynamical friction in the Galaxy; and at lower left destruction is dominated by two-body relaxation: the slope of the linear part of the outer contour corresponds to a line of constant relaxation time. The smooth corners of the triangle come from adding the destruction rates of the three processes, and different contours are obtained at different $R$.

The rate of energy generation is $\dot{E} \sim -M\phi_E$, where $\phi$ is the central potential (Goodman 1987). Then “Hénon’s Principle” (Sec. 2.1) shows that $\dot{E} \sim -\frac{E}{T_r}$, where now $E < 0$ is the total energy of the cluster, and so $\dot{M} \sim -\frac{E}{\phi E T_r} \propto \frac{M}{T_r}$.

This is not inconsistent with Hénon’s argument that the escape rate vanishes in the Fokker-Planck approximation, as the processes of binary formation and hardening are not included in this equation.

One would have thought that numerical experiments would have settled these questions, if indeed questions remain. Unfortunately they themselves raise fresh questions. Simulations of enormous length are needed, because the escape rate from an isolated system is so low, and the expansion after core collapse slows down all dynamical processes. The expansion also speeds up numerical simulations, however, and a series of such simulations was carried out by H. Baumgardt (Baumgardt, Hut, & Heggie 2002) for $N$ up to a relatively modest 8192 particles, but the longest runs covered no less than about $10^{16}$ Hénon units (Fig. 5).

Fig. 5 looks quite at odds with any of the foregoing theoretical discussion, as it seems that all systems lose about 75% of their mass in the same time, independent of $N$. But two things contribute here: escape of stars, of course, but also the post-collapse expansion, which controls all the time scales. The result in this figure could be understood qualitatively if the large-$N$ systems expand more slowly (with reference to the relaxation time) than small-$N$ systems. Then they spend a longer time at radii where the escape rate is larger. Indeed Baumgardt et al find that this is the case, and suggest that the reason for this is that the clusters of different $N$ are not exactly scaled versions of each other. It is known, for example (Goodman 1987), that the core radius is relatively smaller for larger $N$ in steady post-collapse evolution.

Also, Baumgardt et al point out that there are other escape mechanisms beyond the two-body process discussed by Hénon (1960a) and the three-body process discussed above. Lightly bound outlying stars can become unbound because...
of the recoil of the cluster due to escapers resulting from the processes already mentioned. In addition, these slightly bound outlying stars can escape because the potential well in which they sit becomes shallower, because of escape by any of the other processes already noted.

This problem of escape from an isolated cluster, which initially seems so simple, raises a number of tricky questions. In principle some of them could be answered by studying simulations, provided that there are not systematic (\(N\)-dependent) errors in the latter. Certainly there is something to be understood, because the result of Fig.5 cannot continue for arbitrarily large \(N\); the duration of the first part of the curve appears to be approximately proportional to \(N\), and for large enough \(N\) the curve could not then drop steeply enough to cross the approximate common intersection point of the curves shown.

### 4.2 Escape in a tidal field

When a star cluster moves on a circular galactic orbit, it is immersed in the tidal field of the galaxy, and the basic issues of escape seem clearer. Hénon’s 1961 homological model (Sec.3.4) loses mass on the time scale of the relaxation time, though the tidal radius was treated as a cut-off, and the effect of the tidal field on members of the cluster, or on escapers, was ignored.

If the tidal field is properly modelled (Chandrasekhar 1942), the tidal field and inertial forces enter into the equations of motion. In fact the equations resemble those of Hill’s problem in celestial mechanics, a subject which Hénon also studied as part of his work on (mostly periodic) orbits (Hénon 1971). He was well aware of the possible implications for star clusters. In particular, there is a family, family \(f\), of periodic orbits (Fig.6) which, at large distance from the cluster, are essentially epicycles governed by the tidal and inertial forces. The cluster acceleration is a perturbation, which actually acts to make the orbits stable. Hénon noted that a cluster could be surrounded by such orbits, though he expressed some doubt whether they could be occupied by cluster members in the normal course of cluster evolution.

This family continues at lower “energies” (actually, lower Jacobi constant), and eventually is located inside the cluster, even though the energy is still above the energy of escape. In recent years such stars have become known as potential escapers, though they could only ever become escapers as a result of gravitational encounters, or because the potential well of the cluster changes by other processes. These potential escapers can become a significant population, amounting to 10% for \(N = 16384\), and decreasing with \(N\) only as \(N^{-1/4}\) (Baumgardt 2001), so that they represent several percent even for real star clusters, provided that the approximation of a circular galactic orbit is adequate.
The presence of potential escapers has some notable effects. First, models of star clusters never include this population, and it is not known how this affects inferences made on the basis of such models (e.g., estimates of the total mass). It may well be that potential escapers would look like stars with speeds above the escape speed, if compared with a model in which such stars are ignored. Certainly, they have an important role in shaping the velocity dispersion profile near the tidal radius (Küpper et al. 2010). More indirectly, they also act as a buffer between bound members (inside the cluster, and with energies below the escape energy) and escapers, altering the time scale of escape from about the relaxation time scale $T_r$ to roughly $N^{-1/4}T_r$ (Baumgardt 2001), for clusters filling the roche lobe (bounded by the Lagrange points).

Where the problem of escape still leaves the most tricky theoretical questions is the case of clusters on oval galactic orbits. Apparently, such clusters lose stars at about the same rate as a cluster on a circular orbit at some intermediate radius between the apo- and pericentres of its galactic orbit, at least when the eccentricity of the orbit is not very large, and that the time scale of escape varies with $N$ and $T_r$ in the same way (Baumgardt & Makino 2003). But a theoretical understanding of this problem is lacking.

5 EPILOGUE

I met Hénon at several conferences (Fig.7), but in 1975 I also had the privilege of being hosted by him in my first foreign postdoc. For someone with little French this might have been a challenge, and indeed during the spirited and noisy lunches at the Nice Observatory it was. But Michel’s English was excellent, and scientific exchange with him was straightforward. In fact he told me once that he preferred to use English for this purpose, as he found that some foreign visitors who insisted in trying out their French simply made progress next-to-impossible.

Though I thus returned to the UK knowing even less French than when I set out for Nice, I picked some French up as the years passed, and a significant amount I learned from Hénon’s two papers on the homological models (Sec.3). These seem to me a model of scientific writing in any language. Of course the language is broken by numerous equations, but what makes the papers a pleasure to read is fundamentally the quality of the scientific narrative.

The abiding importance of these papers led to a proposal for their translation into English. This was done in autumn 2010 by Dr Florent Renaud, who had been working with Dr Mark Gieles on an application of the models, and the translations are now in the public domain (Hénon 2011a,b).

Before setting about his work, Florent wrote to Michel, seeking his approval. Here is his response.

Cher collègue,

Je vous remercie de votre proposition de traduction de deux de mes articles, et bien entendu je vous donne mon accord enthousiaste.

L'article de 1961 constituait ma thèse de Doctorat. À l'époque cette thèse devait obligatoirement être d'une pièce et ne contenir que du texte original, pas encore publié (au lieu de consister en une collection d'articles déjà publiés, avec un peu de “liant”, comme cela se fait maintenant). D'autre part ce texte devait être entièrement en français. Cela faisait partie du combat d'arrière-garde mené par la langue française contre l’anglaise! Je dois dire qu’après ma thèse j’ai continué à publier en français pendant encore quelque temps, avant de réaliser que c’était le meilleur moyen de ne pas être lu.

C’est un gros travail que vous vous proposez d’entreprendre, et je vous adresse d’avance tous mes remerciements.

Bien cordialement,

Michel Hénon
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