Error Bounds for Augmented Truncations of Discrete-Time Block-Monotone Markov Chains under Geometric Drift Conditions

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Abstract

This paper studies the augmented truncation of discrete-time block-monotone Markov chains under geometric drift conditions. We first present a bound for the total variation distance between the stationary distributions of an original Markov chain and its augmented truncation. We also obtain such error bounds for more general cases where an original Markov chain itself is not necessarily block-monotone but is block-wise dominated by a block-monotone Markov chain. Finally we discuss the application of our results to GI/G/1-type Markov chains.

Keywords: Augmented truncation; block-monotonicity; block-wise dominance; pathwise ordering; geometric drift condition; level-dependent QBD; M/G/1-type Markov chain; GI/M/1-type Markov chain; GI/G/1-type Markov chain

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1 Introduction

Various semi-Markovian queues and their state-dependent extensions can be analyzed through block-structured Markov chains characterized by an infinite number of block matrices, such as level-dependent quasi-birth-and-death processes (LD-QBDs), M/G/1-, GI/M/1- and GI/G/1-type Markov chains (see, e.g., [8]).

For LD-QBDs, there exist some numerical procedures based on the $RG$-factorization, though their implementation requires the truncation of the infinite sequence of block matrices in a heuristic way [2] [4] [19]. Such “truncation in implementation” is also necessary for level-independent M/G/1- and GI/M/1-type Markov chains (see, e.g., Section 4 in [20]) and thus for GI/G/1-type ones. As far as we know, there is no study on the computation of the stationary distributions of level-dependent M/G/1- and GI/M/1-type Markov chains and more general...
ones. For these Markov chains, the $RG$-factorization method does not seem effective in developing numerical procedures with good properties, such as space- and time-saving and guarantee of accuracy, because the resulting expression of the stationary distribution is characterized by an infinite number of $R$- and $G$-matrices [24]. As for the transient distribution, Masuyama and Takine [16] propose a stable and accuracy-guaranteed algorithm based on the uniformization technique (see, e.g., [22]).

As mentioned above, it is challenging to develop a numerical procedure for computing the stationary distributions of block-structured Markov chains characterized by an infinite number of block matrices. A practical and simple solution for this problem is to truncate the transition probability matrix so that it is of a finite dimension. The stationary distribution of the resulting finite Markov chain can be computed by a general purpose algorithm, in principle. However, the obtained stationary distribution includes error caused by truncating the original transition probability matrix. Therefore from a practical point of view, it is significant to estimate “truncation error”.

Tweedie [23] and Liu [13] study the estimation of error caused by truncating (stochastically) monotone Markov chains (see, e.g., [6]). Tweedie [23] presents error bounds for the last-column-augmented truncation of a monotone Markov chain with geometric ergodicity. The last-column-augmented truncation is constructed by augmenting the last column of the northwest corner truncation of a transition probability matrix so that the resulting finite matrix is stochastic. On the other hand, Liu [13] assumes that a monotone Markov chain is subgeometrically ergodic and then derives error bounds for the last-column-augmented truncation.

Unfortunately, block-structured Markov chains are not monotone in general. Li and Zhao [12] extend the notion of monotonicity to block-structured Markov chains. The new notion is called “(stochastic) block-monotonicity”. Block-monotone Markov chains (BMMCs) arise from queues in Markovian environments, such as queues with batch Markovian arrival process (BMAP) [14]. Li and Zhao [12] prove that if an original Markov chain is block-monotone, then the stationary distributions of its augmented truncations converge to that of the original Markov chain, which motivates this study.

In what follows, we give an overview of Li and Zhao [12]’s work. To this end, we introduce some notations. Let $\mathbb{N} = \{1, 2, 3, \ldots \}$. Let $\mathbb{Z}_{\leq n} := \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$ and $\mathbb{Z}_{\leq \infty} := \mathbb{Z}_{\leq n} \cup \{\infty\}$, where $\mathbb{D} = \{1, 2, \ldots, d\}$. Further let $\mathbb{F}_{\leq n} = \mathbb{Z}_{\leq n} \times \mathbb{D}$ for $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, where $\mathbb{D} = \{1, 2, \ldots, d\}$. For simplicity, we write $\mathbb{F}$ for $\mathbb{F}_{\leq \infty}$.

The following is the definition of block monotonicity for stochastic matrices.

**Definition 1.1 (Definition 2.5 in [12])** For any $n \in \mathbb{N}$, a stochastic matrix $S = (s(k, i; l, j))_{(k, i), (l, j) \in \mathbb{F}_{\leq n}}$ and a Markov chain characterized by $S$ are said to be (stochastically) block-monotone with block size $d$ if for all $k \in \mathbb{Z}_{\leq n}^{\leq n-1}$ and $l \in \mathbb{Z}_{\leq n}$,

$$\sum_{m=l}^{n} s(k, i; m, j) \leq \sum_{m=l}^{n} s(k + 1, i; m, j), \quad i, j \in \mathbb{D}.$$  

We denote by $\text{BM}_d$ the set of block-monotone stochastic matrices with block size $d$.  

Let \( P = (p(k, i; l, j))_{(k, i), (l, j) \in \mathbb{F}} \) denote a stochastic matrix. Let \( \{(X_\nu, J_\nu); \nu \in \mathbb{Z}_+\} \) denote a bivariate Markov chain with state space \( \mathbb{F} \) and transition probability matrix \( P \). The following result is obvious from the definition. We thus omit the proof.

**Proposition 1.1** If \( P \in \text{BM}_d \), then \( \psi(i, j) := \sum_{l=0}^{\infty} p(k, i; l, j) \) \( (i, j \in \mathbb{D}) \) is constant with respect to \( k \in \mathbb{Z}_+ \) and \( \{J_\nu; \nu \in \mathbb{Z}_+\} \) is a Markov chain whose transition probability matrix is given by \( \Psi := (\psi(i, j))_{i, j \in \mathbb{D}} \), i.e., \( \psi(i, j) = P(J_{\nu+1} = j | J_{\nu} = i) \) for \( i, j \in \mathbb{D} \).

Proposition 1.1 implies the pathwise ordered property of BMMCs (see Lemma A.1): If \( P \in \text{BM}_d \), then there exist two BMMCs \( \{(X_\nu', J_\nu'); \nu \in \mathbb{Z}_+\} \) and \( \{(X_\nu'', J_\nu''); \nu \in \mathbb{Z}_+\} \) with transition probability matrix \( P \) on a common probability \( (\Omega, \mathcal{F}, P) \) such that \( X_\nu' \leq X_\nu'' \) and \( J_\nu' = J_\nu'' \) for all \( \nu \in \mathbb{N} \) if \( X_0' \leq X_0'' \) and \( J_0' = J_0'' \).

Let \( (n)P_* = \left((n)p_*(k, i; l, j))_{(k, i), (l, j) \in \mathbb{F}} \right) \) \( (n \in \mathbb{N}) \) denote a stochastic matrix such that for \( i, j \in \mathbb{D} \),

\[
(n)p_*(k, i; l, j) \geq p(k, i; l, j), \quad k \in \mathbb{Z}_+, \ l \in \mathbb{Z}_+^{\leq n},
\]

\[
(n)p_*(k, i; l, j) = 0, \quad k \in \mathbb{Z}_+, \ l \in \mathbb{Z}_+ \setminus \mathbb{Z}_+^{\leq n},
\]

\[
\sum_{l=0}^{n} (n)p_*(k, i; l, j) = \sum_{l=0}^{\infty} p(k, i; l, j), \quad k \in \mathbb{Z}_+.
\]

The stochastic matrix \( (n)P_* \) is called a block-augmented first-\( n \)-block-column truncation (for short, block-augmented truncation) of \( P \).

**Remark 1.1** The block-augmented truncation \( (n)P_* \) can be partitioned as

\[
(n)P_* = \begin{pmatrix}
\mathbb{F}^{\leq n} & \mathbb{F} \setminus \mathbb{F}^{\leq n} \\
\mathbb{F} \setminus \mathbb{F}^{\leq n} & (n)P_*^{\leq n} \setminus \mathbb{F}^{\leq n} & O \\
O & O
\end{pmatrix},
\]

where \( (n)P_*^{\leq n} \) is equivalent to the block-augmented truncation defined in Li and Zhao [12]. Our definition facilitates the algebraic operation for the original stochastic matrix \( P \) and its block-augmented truncation \( (n)P_* \) because they are of the same dimension.

Throughout this paper, unless otherwise stated, we assume that \( P \) is irreducible and positive recurrent and then denote its unique stationary probability vector by \( \pi = (\pi(k, i))_{(k, i) \in \mathbb{F}} > 0 \) (see, e.g., Theorem 3.1 in Section 3.1 of [3]). However, \( (n)P_* \) may have more than one positive recurrent (communication) class in \( \mathbb{F}^{\leq n} \).

Let \( (n)\pi_* = \left((n)\pi_*(k, i))_{(k, i) \in \mathbb{F}} \right) \) \( (n \in \mathbb{N}) \) denote a stationary probability vector of \( (n)P_* \). Equation (1.1) implies that \( (n)\pi_*(k, i) = 0 \) for all \( (k, i) \in \mathbb{F} \setminus \mathbb{F}^{\leq n} \) (see, e.g., Theorem 1 in Section 1.7 of [5]) and \( (n)\pi_*^{\leq n} := \left((n)\pi_*(k, i))_{(k, i) \in \mathbb{F}^{\leq n}} \right) \) is a solution of \( (n)\pi_*^{\leq n}(n)P_*^{\leq n} = (n)\pi_*^{\leq n} \) and \( (n)\pi_*^{\leq n}e = 1 \), where \( e \) denotes a column vector of ones with an appropriate dimension. It is also known that if \( P \in \text{BM}_d \), then \( \lim_{n \to \infty} (n)\pi_* = \pi \), where the convergence is element-wise (see Theorem 3.4 in Li and Zhao [12]).
Let \( (n)P_n = ((n)p_n(k; i, l, j))_{(k, i), (l, j) \in F} \) \( (n \in \mathbb{N}) \) denote a block-augmented truncation of \( P \) such that for \( i, j \in D \),
\[
(n)p_n(k, i; l, j) = \begin{cases}
p(k, i; l, j), & k \in \mathbb{Z}_+, l \in \mathbb{Z}^{\leq n-1}, \\
\sum_{m=n}^{\infty} p(k, i; m, j), & k \in \mathbb{Z}_+, l = n, \\
0, & \text{otherwise},
\end{cases}
\]
which is called the last-column-block-augmented first-\( n \)-block-column truncation (for short, the last-column-block-augmented truncation). Let \( (n)\pi_n = ((n)\pi_n(k, i))_{(k, i) \in F} \) \( (n \in \mathbb{N}) \) denote a stationary probability vector of \( (n)P_n \), where \( (n)\pi_n(k, i) = 0 \) for all \( (k, i) \in F \setminus F^{\leq n} \). We then have the following result.

**Proposition 1.2 (Theorem 3.6 in [12])** If \( P \in \text{BM}_d \) and \( (n)\pi_n \) is the unique stationary distribution of \( (n)P_n \), then there exists an infinite increasing sequence \( \{n_k \in \mathbb{N}; k \in \mathbb{Z}_+\} \) such that for all \( k \in \mathbb{Z}_+ \),
\[
0 \leq \sum_{l=0}^{n_k} \sum_{i \in D} ((n)\pi_n(l, i) - \pi(l, i)) \leq \sum_{l=0}^{n_k} \sum_{i \in D} ((n)\pi_*(l, i) - \pi(l, i)).
\]

Based on Proposition 1.2 Li and Zhao [12] state that the last-column-block-augmented truncation \( (n)P_n \) is the best approximation to \( P \) among the block-augmented truncations of \( P \), though they do not estimate the distance between \( (n)\pi_n \) and \( \pi \).

In this paper, we consider some cases where \( P \) satisfies the geometric drift condition (see Section 15.2.2 in [17]) but may be periodic. We first assume \( P \in \text{BM}_d \) and present a bound for the total variation distance between \( (n)\pi_n \) and \( \pi \), which is expressed as follows:
\[
\| (n)\pi_n - \pi \| := \sum_{(k, i) \in F} |(n)\pi_n(k, i) - \pi(k, i)| \leq C_m(n),
\]
where \( C_m \) is some function on \( \mathbb{Z}_+ \) with a supplementary parameter \( m \in \mathbb{N} \) such that \( \lim_{m \to \infty} \lim_{n \to \infty} C_m(n) = 0 \). The bound presented in this paper is a generalization of that in Tweedie [23] (see Theorem 4.2 therein). We also obtain such error bounds for more general cases where \( P \) itself is not necessarily block-monotone but is block-wise dominated by a block-monotone stochastic matrix.

The rest of this paper is divided into four sections. Section 2 provides preliminary results on block-monotone stochastic matrices. The main result of this paper is presented in Section 3 and some extensions are discussed in Section 4. As an example, these results are applied to GI/G/1-type Markov chains in Section 5.

## 2 Preliminaries

In this section, we first introduce some definitions and notations, and then provide some basic results on block-monotone stochastic matrices.
2.1 Definitions and notations

Let $I$ denote an identity matrix whose dimension depends on the context (we may write $I_m$ to represent the $m \times m$ identity matrix). For any square matrix $M$, let $M^0 = I$. Let $T_d$ and $T_d^{-1}$ denote

\[
T_d = \begin{pmatrix}
I_d & O & O & O & \cdots \\
I_d & I_d & O & O & \cdots \\
I_d & I_d & I_d & O & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}, \quad T_d^{-1} = \begin{pmatrix}
I_d & O & O & O & \cdots \\
-I_d & I_d & O & O & \cdots \\
O & -I_d & I_d & O & \cdots \\
O & O & -I_d & I_d & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

where $T_d T_d^{-1} = T_d^{-1} T_d = I$. Let $T_d^{\leq n}$ ($n \in \mathbb{N}$) denote the $|F^{\leq n}| \times |F^{\leq n}|$ northwest corner truncation of $T_d$, where $| \cdot |$ denotes set cardinality. Note that $T_d = T_d^{\leq \infty}$ and $(T_d^{\leq n})^{-1}$ ($n \in \mathbb{N}$) is equal to the $|F^{\leq n}| \times |F^{\leq n}|$ northwest corner truncation of $T_d^{-1}$.

We now introduce the following definitions.

**Definition 2.1 (Definition 2.1 in [12])** For $n \in \mathbb{N}$, let $f = (f(k,i))_{(k,i) \in F^{\leq n}}$ denote a column vector with block size $d$. The vector $f$ is said to be block-increasing if $(T_d^{\leq n})^{-1} f \geq 0$, i.e., $f(k,i) \leq f(k+1,i)$ for all $(k,i) \in Z_{d-1}^{\leq n} \times \mathbb{D}$. We denote by $\text{Bl}_d$ the set of block-increasing column vectors with block size $d$.

**Definition 2.2** For $n \in \mathbb{N}$, let $\mu = (\mu(k,i))_{(k,i) \in F^{\leq n}}$ and $\eta = (\eta(k,i))_{(k,i) \in F^{\leq n}}$ denote probability vectors with block size $d$. The vector $\mu$ is said to be (stochastically) block-wise dominated by $\eta$ (denoted by $\mu \prec_d \eta$) if $\mu T_d^{\leq n} \leq \eta T_d^{\leq n}$.

**Definition 2.3** For $n \in \mathbb{N}$, let $P_h = (p_h(k,i;l,j))_{(k,i),(l,j) \in F^{\leq n}}$ ($h = 1,2$) denote a stochastic matrix with block size $d$. The matrix $P_1$ is said to be (stochastically) block-wise dominated by $P_2$ (denoted by $P_1 \prec_d P_2$) if $P_1 T_d^{\leq n} \leq P_2 T_d^{\leq n}$.

**Remark 2.1** The columns of $T_d^{\leq n}$ are linearly independent vectors in $\text{Bl}_d$, and thus every vector $f \in \text{Bl}_d$ is expressed as a linear combination of columns of $T_d^{\leq n}$. Therefore $\mu \prec_d \eta$ (resp. $P_1 \prec_d P_2$) if and only if $\mu f \leq \eta f$ (resp. $P_1 f \leq P_2 f$) for any $f \in \text{Bl}_d$. According to this equivalence, we can define the block-wise dominance relation “$\prec_d$” (see Definitions 2.2 and 2.7 in [12]).

2.2 Basic results on block-monotone stochastic matrices

In this subsection, we present three propositions. The first two of them hold for any $|F^{\leq n}| \times |F^{\leq n}|$ ($n \in \mathbb{N}$) stochastic matrix $S = (s(k,i;l,j))$ in $\text{BM}_d$. The first proposition is immediate from Definition [1.1] and thus its proof is omitted. The second one is an extension of Theorem 1.1 in [10].

**Proposition 2.1** $S \in \text{BM}_d$ if and only if $(T_d^{\leq n})^{-1} S T_d^{\leq n} \geq O$. 

**Definition 2.2** Denote
Proposition 2.2 The following are equivalent:

(i) $S \in \text{BM}_d$.

(ii) $\mu S \prec_d \eta S$ for any two probability vectors $\mu$ and $\eta$ such that $\mu \prec_d \eta$.

(iii) $Sf \in \text{Bl}_d$ for any $f \in \text{Bl}_d$.

Remark 2.2 The equivalence of (a) and (c) is shown in Theorem 3.8 in [12].

Proof. [Proof of Proposition 2.2] (a) ⇒ (b): We assume that $S \in \text{BM}_d$ and $\mu \prec_d \eta$. It then follows from Proposition 2.1 and Definition 2.2 that $(T_d^{\leq n})^{-1}ST_d^{\leq n} \geq O$ and $\mu T_d^{\leq n} \leq \eta T_d^{\leq n}$. Thus we have

$$\mu ST_d^{\leq n} = \mu T_d^{\leq n} \cdot (T_d^{\leq n})^{-1}ST_d^{\leq n} \leq \eta T_d^{\leq n} \cdot (T_d^{\leq n})^{-1}ST_d^{\leq n} = \eta ST_d^{\leq n},$$

which shows $\mu S \prec_d \eta S$.

(b) ⇒ (a): For $(k, i) \in \mathbb{F}^{\leq n}$, let $\xi_{(k, i)} = (\xi_{(k, i)}(l, j))_{(l, j) \in \mathbb{F}^{\leq n}}$ denote a $1 \times |\mathbb{F}^{\leq n}|$ unit vector whose $(k, i)$th element is equal to one. Let $\eta = \xi_{(k, i)}$ and $\mu = \xi_{(k-1, i)}$ for any fixed $(k, i) \in (\mathbb{Z}^{\leq n}_+ \setminus \{0\}) \times \mathbb{D}$. It then follows that $\mu \prec_d \eta$ and thus condition (b) yields $(\eta - \mu)ST_d^{\leq n} \geq 0$, where $\eta - \mu$ is equal to the $(k, i)$th row of $(T_d^{\leq n})^{-1}$. Further $\xi_{(0, i)}ST_d^{\leq n} \geq 0$ $(i \in \mathbb{D})$, where $\xi_{(0, i)}$ is equal to the $(0, i)$th row of $(T_d^{\leq n})^{-1}$. As a result, we have $(T_d^{\leq n})^{-1}ST_d^{\leq n} \geq O$, i.e., $S \in \text{BM}_d$ (see Proposition 2.1).

(a) ⇒ (c): According to Definition 2.1 $(T_d^{\leq n})^{-1}f \geq 0$ for any $f \in \text{Bl}_d$. Combining this with $(T_d^{\leq n})^{-1}ST_d^{\leq n} \geq O$ (due to condition (a)), we obtain

$$(T_d^{\leq n})^{-1}Sf = (T_d^{\leq n})^{-1}ST_d^{\leq n} \cdot (T_d^{\leq n})^{-1}f \geq 0,$$

and thus $Sf \in \text{Bl}_d$.

(c) ⇒ (a): Fix $f$ to be a column of $T_d^{\leq n}$. Since $f \in \text{Bl}_d$, it follows from condition (c) that $Sf \in \text{Bl}_d$, i.e., $(T_d^{\leq n})^{-1}Sf \geq 0$. Therefore $(T_d^{\leq n})^{-1}ST_d^{\leq n} \geq O$. 

The last proposition is a fundamental result for any two $|\mathbb{F}^{\leq n}| \times |\mathbb{F}^{\leq n}|$ $(n \in \mathbb{N})$ stochastic matrices $P_1 = (p_1(k, i; l, j))$ and $P_2 = (p_2(k, i; l, j))$ such that $P_1 \prec_d P_2$, which is an extension of Lemma 1 in [7].

Proposition 2.3 If $P_1 \prec_d P_2$ and either $P_1 \in \text{BM}_d$ or $P_2 \in \text{BM}_d$, then the following statements hold:

(i) For all $k \in \mathbb{Z}^{\leq n}_+$ and $i, j \in \mathbb{D}$,

$$\sum_{l \in \mathbb{Z}^{\leq n}_+} p_1(k, i; l, j) = \sum_{l \in \mathbb{Z}^{\leq n}_+} p_2(k, i; l, j),$$

which is constant with respect to $k$.

(ii) $P_1^m \prec_d P_2^m$ for all $m \in \mathbb{N}$. 


(iii) Suppose that \( P_2 \) is irreducible. If \( P_2 \) is recurrent (resp. positive recurrent), then \( P_1 \) has exactly one recurrent (resp. positive recurrent) class that includes the states \( \{(0, i); i \in \mathbb{D}\} \), which is reachable from all the other states with probability one. Thus if \( P_2 \) is positive recurrent, then \( P_1 \) and \( P_2 \) have the unique stationary distributions \( \pi_1 \) and \( \pi_2 \), respectively, and \( \pi_1 \prec_d \pi_2 \).

Proof. We consider only the case of \( P_1 \in \text{BM}_d \) because the case of \( P_2 \in \text{BM}_d \) is discussed in a very similar way. We first prove statement (a). It follows from \( P_1 \in \text{BM}_d \) and Proposition 1.1 that \( \sum_{l \in \mathbb{Z}_+^n} p_1(k, i; l, j) \) is constant with respect to \( k \) for each \( (i, j) \in \mathbb{D}^2 \), which is denoted by \( \psi_1(i, j) \). Further from \( P_1 \prec_d P_2 \), we have

\[
\psi_1(i, j) = \sum_{l \in \mathbb{Z}_+^n} p_1(k, i; l, j) \leq \sum_{l \in \mathbb{Z}_+^n} p_2(k, i; l, j), \quad k \in \mathbb{Z}_+^n, i, j \in \mathbb{D}. \tag{2.1}
\]

Since \( P_1 \) and \( P_2 \) are stochastic matrices, \( \sum_{j \in \mathbb{D}} \psi_1(i, j) = \sum_{j \in \mathbb{D}} \sum_{l \in \mathbb{Z}_+^n} p_2(k, i; l, j) = 1 \) for all \( (k, i) \in \mathbb{F}^{\leq n} \). From this and (2.1), we obtain \( \psi_1(i, j) = \sum_{l \in \mathbb{Z}_+^n} p_2(k, i; l, j) \) for all \( k \in \mathbb{Z}_+^n \) and \( i, j \in \mathbb{D} \).

Next we prove statement (b) by induction. Suppose that for some \( m \in \mathbb{N}, P_1^m \prec_d P_2^m \), i.e., \( P_1^m T_d^{\leq n} \leq P_2^m T_d^{\leq n} \) (which is true at least for \( m = 1 \)). Combining this with \( (T_d^{\leq n})^{-1} P_1 T_d^{\leq n} \geq O \) (due to \( P_1 \in \text{BM}_d \)) yields

\[
P_1^{m+1} T_d^{\leq n} = P_1^m T_d^{\leq n} \cdot (T_d^{\leq n})^{-1} P_1 T_d^{\leq n} \\
\leq P_2^m T_d^{\leq n} \cdot (T_d^{\leq n})^{-1} P_1 T_d^{\leq n} = P_2^m \cdot P_1 T_d^{\leq n} \\
\leq P_2^m \cdot P_2 T_d^{\leq n} = P_2^{m+1} T_d^{\leq n},
\]

and thus \( P_1^{m+1} \prec_d P_2^{m+1} \). Therefore statement (b) is true.

Finally we prove statement (c). Note that there exist two Markov chains characterized by \( P_1 \) and \( P_2 \), called Markov chains 1 and 2, which are pathwise ordered by the block-wise dominance of \( P_2 \) over \( P_1 \) (see Lemma A.2). Since \( P_2 \) is irreducible and recurrent, Markov chain 2 and thus Markov chain 1 can reach any state \((0, i) (i \in \mathbb{D})\) from all the states in the state space \( \mathbb{F}^{\leq n} \) with probability one and the mean first passage time to each state \((0, i) (i \in \mathbb{D})\) is finite if \( P_2 \) is positive recurrent. These facts show that the first part of statement (c) holds. Finally we prove \( \pi_1 \prec_d \pi_2 \). Note here that \((I + P_h)/2 (h = 1, 2)\) is aperiodic and has the same stationary distribution as that of \( P_h \). Thus we assume without loss of generality that \( P_h \) \((h = 1, 2)\) is aperiodic. It then follows from statement (b) and the dominated convergence theorem that \( e \pi_1 T_d^{\leq n} \leq e \pi_2 T_d^{\leq n} \) (see Theorem 4 in Section 1.6 of [5]) and thus \( \pi_1 T_d^{\leq n} \leq \pi_2 T_d^{\leq n} \). \( \square \)

3 Main result

This section presents a bound for \( \| (n) \pi_n - \pi \| \), which is the main result of this paper. To establish the bound, we use the \( \nu \)-norm, where \( \nu = (\nu(k, i))_{(k, i) \in \mathbb{F}} \) is any nonnegative column
vector. The \( v \)-norm is defined as follows: For any \( 1 \times |\mathbb{F}| \) vector \( \mathbf{x} = (x(k, i))_{(k, i) \in \mathbb{F}} \),

\[
\| \mathbf{x} \|_v = \sup_{|g| \leq v} \left| \sum_{(k, i) \in \mathbb{F}} x(k, i)g(k, i) \right| = \sup_{0 \leq g \leq v} \sum_{(k, i) \in \mathbb{F}} |x(k, i)|g(k, i),
\]

where \( |g| \) is a column vector obtained by taking the absolute value of each element of \( g \). By definition, \( \| \cdot \|_e = \| \cdot \|_1 \), i.e., the \( e \)-norm is equivalent to the total variation norm.

We need some further notations. For \( m \in \mathbb{Z}_+ \) and \( (k, i) \in \mathbb{F}, \) let \( p^m(k, i) = (p^m(k, i; l, j))_{(l, j) \in \mathbb{F}} \) and \( (n)P^m_n(k, i) = ((n)P^m_n(k, i; l, j))_{(l, j) \in \mathbb{F}} \) denote probability vectors such that \( p^m(k, i; l, j) \) and \( (n)P^m_n(k, i; l, j) \) represent the \((k, i; l, j)\)th elements of \( P^m \) and \( (n)P^m_n \), respectively (when \( m = 1 \), the superscript “1” may be omitted). Clearly, \( p^m(k, i; l, j) = P(X_m = l, J_m = j \mid X_0 = k, J_0 = i) \) for \((k, i) \times (l, j) \in \mathbb{F}^2 \).

Let \( \varpi(i) = \sum_{k=0}^{\infty} \pi(k, i) > 0 \) for \( i \in \mathbb{D} \). Note here that if \( P \in \text{BM}_d \), then \( \varpi = (\varpi(i))_{i \in \mathbb{D}} \) is the stationary distribution of \( \Psi \) (and thus the Markov chain \( \{J_n\} \); see Proposition 1.1). Note also that if \( P \in \text{BM}_d \), then \( (n)P_n \prec_d P \) and thus \((n)\pi_n \prec_d \pi \) (due to Proposition 2.3(c)), which implies that for all \( n \in \mathbb{N} \),

\[
\sum_{k=0}^{\infty} (n)\pi_n(k, i) = \sum_{k=0}^{\infty} \pi(k, i) = \varpi(i), \quad i \in \mathbb{D}. \tag{3.1}
\]

For any function \( \varphi(\cdot, \cdot) \) on \( \mathbb{F} \), let \( \varphi(k, \varpi) = \sum_{i \in \mathbb{D}} \varpi(i)\varphi(k, i) \) for \( k \in \mathbb{Z}_+ \).

In what follows, we estimate \( \| (n)\pi_n - \pi \| \). By the triangle inequality, we have

\[
\| (n)\pi_n - \pi \| \leq \| p^0(0, \varpi) - \pi \| + \| (n)P^0_n(0, \varpi) - (n)\pi_n \| + \| (n)P^0_n(0, \varpi) - p^0(0, \varpi) \|. \tag{3.2}
\]

The third term on the right hand side of (3.2) is bounded as in the following lemma, which is proved without \( P \in \text{BM}_d \).

**Lemma 3.1** For all \( m \in \mathbb{N} \),

\[
\| (n)P^m_n(k, i) - p^m(k, i) \| \leq \sum_{h=0}^{m-1} \sum_{(l, j) \in \mathbb{F}} (n)p^h(k, i; l, j)\Delta_h(l, j), \quad n \in \mathbb{N}, \ (k, i) \in \mathbb{F}, \tag{3.3}
\]

where

\[
\Delta_h(l, j) = \| P(l, j) - (n)p_n(l, j) \| = 2 \sum_{l' > n, l' \in \mathbb{D}} p(l, j; l', j'), \quad (l, j) \in \mathbb{F}. \tag{3.4}
\]

**Proof.** Clearly (3.3) holds for \( m = 1 \). Note here that for \( m, n \in \mathbb{N} \),

\[
(n)P_n^{m+1} - P^{m+1} = (n)P_n \cdot [(n)P_n^m - P^m] + (n)P_n - P)P^m.
\]
It then follows that for \( m = 2, 3, \ldots \),
\[
\| (n) P_n^{m+1}(k, i) - P^{m+1}(k, i) \|
\leq \sum_{(l, j) \in F} (n) p_n(k, i; l, j) \| (n) P_n^m(l, j) - P^m(l, j) \|
\quad + \sum_{(l, j) \in F} |(n) p_n(k, i; l, j) - p(k, i; l, j)| \sum_{(l', j') \in F} p^m(l, j'; l', j')
\quad = \sum_{(l, j) \in F} (n) p_n(k, i; l, j) \| (n) P_n^m(l, j) - P^m(l, j) \| + \Delta_n(k, i),
\]
where the last equality is due to \( \sum_{(l', j') \in F} p^m(l, j'; l', j') = 1 \). Thus if (3.3) holds for some \( m \geq 2 \), then (3.5) yields
\[
\| (n) P_n^{m+1}(k, i) - P^{m+1}(k, i) \|
\leq \sum_{(l, j) \in F} (n) p_n(k, i; l, j) \left[ \sum_{h=0}^{m-1} \sum_{(l', j') \in F} (n) p_n^h(l, j; j', j') \Delta_n(l', j') + \Delta_n(k, i) \right]
\quad = \sum_{h=0}^{m-1} \sum_{(l', j') \in F} \left( \sum_{(l, j) \in F} (n) p_n(k, i; l, j) p_n^h(l, j; l', j') \right) \Delta_n(l', j') + \Delta_n(k, i)
\quad = \sum_{h=0}^{m-1} \sum_{(l', j') \in F} (n) p_n^{h+1}(k, i; l', j') \Delta_n(l', j') + \Delta_n(k, i) = \sum_{h=0}^{m} \sum_{(l, j) \in F} (n) p_n^h(k, i; l, j) \Delta_n(l, j).
\]

The following lemma implies that the first two terms on the right hand side of (3.2) converge to zero as \( m \to \infty \) without the aperiodicity of \( P \).

**Lemma 3.2** Let \( \kappa \) denote the period of \( P \). If \( P \in \text{BM}_d \) and \( P \) is irreducible, then the following hold:

(i) There exist disjoint nonempty sets \( D_0, D_1, \ldots, D_{\kappa-1} \) such that \( D = \bigcup_{h=0}^{\kappa-1} D_h \) and
\[
\sum_{(l, j) \in \mathbb{Z}_+ \times D_h} p(k, i; l, j) = 1, \quad (k, i) \in \mathbb{Z}_+ \times D_h, \quad h \in \mathbb{Z}_+^{\kappa-1},
\]
where \( D_{h'} = D_h \) if \( h' \equiv h \pmod{\kappa} \).

(ii) \( \kappa \leq d = |D| \). Thus every irreducible monotone stochastic matrix (which is in \( \text{BM}_1 \)) is aperiodic.

(iii) If \( P \) is positive recurrent, then for \( k \in \mathbb{Z}_+ \),
\[
\lim_{m \to \infty} P^m(k, \omega) = \pi, \quad \lim_{m \to \infty} (n) P^m_n(k, \omega) = (n) \pi_n, \quad n \in \mathbb{N}. \tag{3.6}
\]
We prove statement (a) by contradiction. Proposition 5.4.2 in [17] shows that there exist disjoint nonempty sets \( F_0, F_1, \ldots, F_{k-1} \) such that \( F = \bigcup_{h=0}^{k-1} F_h \) and
\[
\sum_{(l,j) \in F_{h+1}} p(k; i; l, j) = 1, \quad (k, i) \in F_h, \quad h \in \mathbb{Z}_+^{\leq k-1},
\] (3.7)
where \( F_{h'} = F_h \) if \( h' \equiv h \pmod{\kappa} \). We suppose that there exist some \((k_*, i_*) \in \mathbb{N} \times \mathbb{D} \) and \( h_* \in \mathbb{Z}_+^{\leq k-1} \) such that \((0, i_*) \in F_{h_*} \) and \((k_*, i_*) \notin F_{h_*} \). We now consider coupled Markov chains \( \{(X'_\nu, J'_\nu); \nu \in \mathbb{Z}_+\} \) and \( \{(X''_\nu, J''_\nu); \nu \in \mathbb{Z}_+\} \) with transition probability matrix \( P \), which are pathwise ordered such that \( X'_\nu \leq X''_\nu \) and \( J'_\nu = J''_\nu \) for all \( \nu \in \mathbb{N} \) if \( X'_0 \leq X''_0 \) and \( J'_0 = J''_0 \) (see Lemma A.1). We also fix \((X'_0, J'_0) = (0, i_*) \in F_{h_*} \) and \((X''_0, J''_0) = (k_*, i_*) \notin F_{h_*} \). It then follows from (3.7) that
\[
(X'_\nu, J'_\nu) \in F_h \text{ implies } (X''_\nu, J''_\nu) \notin F_h \quad \text{for all } \nu \in \mathbb{N}.
\] (3.8)
Further since \( P \) is irreducible, there exists some \( \nu_* \in \mathbb{N} \) such that \((X''_{\nu_*}, J''_{\nu_*}) = (0, i_*) \) and thus \((X'_0, J'_0) \in \mathbb{N} \times \{i_*\} \) due to (3.8). This conclusion, however, contradicts the pathwise ordering of \( \{(X'_\nu, J'_\nu)\} \) and \( \{(X''_\nu, J''_\nu)\} \), i.e., \( X'_\nu \leq X''_\nu \) and \( J'_\nu = J''_\nu \) for all \( \nu \in \mathbb{N} \). As a result, statement (a) holds, and statement (b) is immediate from statement (a).

Next we prove statement (c). Fix \( k \in \mathbb{Z}_+ \) arbitrarily. Let \( q : \mathbb{D} \to \mathbb{Z}_+^{\leq k-1} \) denote a surjection function such that \( i \in \mathbb{D}_{q(i)} \). It then follows from Theorem 4 in Section I.6 of [5] that for \( h \in \mathbb{Z}_+^{\leq k-1} \),
\[
\lim_{m' \to \infty} p^{m' \kappa + h}(k; i; l, j) = \mathbb{I}_{\{h \equiv q(j) - q(i) \pmod{\kappa}\}} \cdot \kappa \pi(l, j), \quad (l, j) \in F,
\] (3.9)
where \( \mathbb{I}_{\{\cdot\}} \) denotes a function that takes value one if the statement in the braces is true and takes value zero otherwise. From (3.9), we have for \( h \in \mathbb{Z}_+^{\leq k-1} \) and \((l, j) \in F\),
\[
\lim_{m' \to \infty} \sum_{i \in \mathbb{D}} \varpi(i) p^{m' \kappa + h}(k; i; l, j) = \lim_{m' \to \infty} \sum_{h'=0}^{k-1} \sum_{i \in \mathbb{D}_{h'}} \varpi(i) p^{m' \kappa + h}(k; i; l, j)
= \kappa \sum_{h'=0}^{k-1} \sum_{i \in \mathbb{D}_{h'}} \varpi(i) \mathbb{I}_{\{h \equiv q(j) - q(i) \pmod{\kappa}\}} \cdot \pi(l, j)
= \kappa \sum_{h'=0}^{k-1} \sum_{i \in \mathbb{D}_{h'}} \varpi(i) \mathbb{I}_{\{h \equiv q(j) - h' \pmod{\kappa}\}} \cdot \pi(l, j)
\] (3.10)
where the last equality is due to \( q(i) = h' \) for \( i \in \mathbb{D}_{h'} \). Note here that \( \sum_{i \in \mathbb{D}_{h'}} \varpi(i) = \sum_{(k,i) \in F_{h'}} \pi(k, i) = 1/\kappa \) for any \( h' \in \mathbb{Z}_+^{\leq k-1} \) (see Theorem 1 in Section I.7 of [5]). Note also that for any \( h \in \mathbb{Z}_+^{\leq k-1} \) and \( j \in \mathbb{D} \) there exists the unique \( h' \in \mathbb{Z}_+^{\leq k-1} \) such that \( h \equiv q(j) - h' \pmod{\kappa} \). From (3.10), we then obtain for \( h \in \mathbb{Z}_+^{\leq k-1} \),
\[
\lim_{m' \to \infty} \sum_{i \in \mathbb{D}} \varpi(i) p^{m' \kappa + h}(k; i; l, j) = \pi(l, j), \quad (l, j) \in F,
\]
which leads to the first limit in (3.6). Further since \( (n)P_n \overset{d}{\rightarrow} P \in \text{BM}_d \), it follows from Proposition 2.3 (c) that \( (n)P_n \) has the unique positive recurrent class. As a result, we can prove the second limit in (3.6) in the same way as the proof of the first one. 

To estimate the first two terms on the right hand side of (3.2), we assume the geometric drift condition for geometric ergodicity:

**Assumption 3.1** There exists a column vector \( \mathbf{v} = (v(k,i))_{(k,i) \in \mathcal{F}} \in \mathbb{B}_d \) such that \( \mathbf{v} \geq \mathbf{e} \) and for some \( \gamma \in (0, 1) \) and \( b \in (0, \infty) \),

\[
P \mathbf{v} \leq \gamma \mathbf{v} + b \mathbb{1}_K,
\]

where \( \mathbb{1}_K = (1_K(k,i))_{(k,i) \in \mathcal{F}} \) denotes a column vector such that \( 1_K(k,i) = 1 \) for \( (k,i) \in \mathcal{F} \cup K \) and \( 1_K(k,i) = 0 \) for \( (k,i) \in \mathcal{F} \setminus \mathcal{F} \cup K \).

**Remark 3.1** Suppose that \( P \) is irreducible. Since the state space \( \mathcal{F} \) is countable, every subset of \( \mathcal{F} \) includes a small set and thus petite set (see Theorem 5.2.2 and Proposition 5.5.3 in [17]). Therefore if the irreducible \( P \) is aperiodic and Assumption 3.1 holds, then there exist \( r \in (1, \infty) \) and \( C \in (0, \infty) \) such that \( \sum_{m=1}^{\infty} r^m \| p^m(k,i) - \pi \|_v \leq C \nu(k,i) \) for all \( (k,i) \in \mathcal{F} \), which shows that \( P \) is \( \nu \)-geometrically ergodic (see Theorem 15.0.1 in [17]).

The following lemma is an extension of Theorem 2.2 in [15] to discrete-time BMMCs.

**Lemma 3.3** Suppose that \( P \in \text{BM}_d \) and \( P \) is irreducible. If Assumption 3.1 holds, then for all \( k \in \mathbb{Z}_+ \) and \( m \in \mathbb{N} \),

\[
\| p^m(k, \omega) - \pi \|_v \leq 2 \gamma^m [v(k, \omega)(1 - 1_0(k, \omega)) + b/(1 - \gamma)], \tag{3.12}
\]

\[
\| (n)P_n^m(k, \omega) - (n)\pi_n \|_v \leq 2 \gamma^m [v(k, \omega)(1 - 1_0(k, \omega)) + b/(1 - \gamma)], \quad \forall n \in \mathbb{N}. \tag{3.13}
\]

**Proof.** We first prove (3.12). To do this, we consider three copies \( \{ (X^{(0)}_\nu, J^{(0)}_\nu); \nu \in \mathbb{Z}_+ \} \) \((h = 0, 1, 2)\) of the BMMC \( \{ (X, J); \nu \in \mathbb{Z}_+ \} \), which are defined on a common probability space in such a way that

\[
(X^{(0)}_0, J^{(0)}_0) = (0, J), \quad (X^{(1)}_0, J^{(1)}_0) = (k, J), \quad (X^{(2)}_0, J^{(2)}_0) = (X, J),
\]

where \( k \in \mathbb{Z}_+ \) and \( (X, J) \) denotes a random vector distributed with \( P(X = l, S = j) = \pi(l, j) \) for \( (l,j) \in \mathcal{F} \). According to the pathwise ordered property of BMMCs (see Lemma A.1), we assume without loss of generality that

\[
X^{(0)}_\nu \leq X^{(1)}_\nu, \quad X^{(0)}_\nu \leq X^{(2)}_\nu, \quad J^{(0)}_\nu = J^{(1)}_\nu = J^{(2)}_\nu, \quad \forall \nu \in \mathbb{Z}_+. \tag{3.14}
\]

For simplicity, let

\[
E_{(k,i)}[\cdot] = E[\cdot | X_0 = k, J_0 = i], \quad (k,i) \in \mathcal{F},
\]

\[
E_{(k,i);(0,j)}[\cdot] = E[\cdot | (X^{(h)}_0, J^{(h)}_0) = (k,i), (X^{(0)}_\nu, J^{(0)}_\nu) = (0,j)], \quad (k,i) \in \mathcal{F}, j \in \mathbb{D},
\]
where \( h = 1, 2 \). Further let \( g = (g(l, j))_{(l, j) \in \mathbb{F}} \) denote a column vector satisfying \( |g| \leq v \), i.e., \( |g(l, j)| \leq v(l, j) \) for \((l, j) \in \mathbb{F}\). It then follows that for \( m = 1, 2, \ldots \),

\[
p^m(k, \varpi)g = \sum_{i \in \mathbb{D}} \varpi(i) \sum_{(l, j) \in \mathbb{F}} p^m(k, i; l, j)g(l, j) = E[E_{(k, J)}[g(X_m, J_m)]] = \pi g = \pi P^m g = \sum_{(k, i) \in \mathbb{F}} \pi(k, i) \sum_{(l, j) \in \mathbb{F}} p^m(k, i; l, j)g(l, j) = E[E_{(X, J)}[g(X_m, J_m)]].
\]

Thus by the triangle inequality, we obtain

\[
|p^m(k, \varpi)g - \pi g| \\
= |E[E_{(k, J)}[g(X_m, J_m)]] - E[E_{(X, J)}[g(X_m, J_m)]]| \\
\leq |E[E_{(k, J)}(0, J)[g(X_m^{(1)}, J_m^{(1)})]] - E[E_{(k, J)}(0, J)[g(X_m^{(0)}, J_m^{(0))})]]| \\
+ |E[E_{(X, J)}(0, J)[g(X_m^{(2)}, J_m^{(2)})]] - E[E_{(X, J)}(0, J)[g(X_m^{(0)}, J_m^{(0))})]]|.
\]

(3.15)

Let \( T_h = \inf \{ m \in \mathbb{Z}_+; X^{(h)}_\nu = X^{(0)}_\nu, \forall \nu \geq m \} \) for \( h = 1, 2 \). We then have

\[
g(X^{(1)}_\nu, J^{(1)}_\nu) = g(X^{(0)}_\nu, J^{(0)}_\nu), \quad \nu \geq T_1, \tag{3.16}
g(X^{(2)}_\nu, J^{(2)}_\nu) = g(X^{(0)}_\nu, J^{(0)}_\nu), \quad \nu \geq T_2. \tag{3.17}
\]

Applying (3.16) and (3.17) to (3.15) and using \( |g| \leq v \) (but not \( P \in \text{BM}_d \)) yield

\[
|p^m(k, \varpi)g - \pi g| \\
\leq |E[E_{(k, J)}(0, J)[g(X_m^{(1)}, J_m^{(1)}) - g(X_m^{(0)}, J_m^{(0)})] \cdot \mathbb{I}\{ T_1 > m \}]| \\
+ |E[E_{(X, J)}(0, J)[g(X_m^{(2)}, J_m^{(2)}) - g(X_m^{(0)}, J_m^{(0)})] \cdot \mathbb{I}\{ T_2 > m \}]| \\
\leq |E[E_{(k, J)}(0, J)[v(X_m^{(1)}, J_m^{(1)})] \cdot \mathbb{I}\{ T_1 > m \}]| \\
+ |E[E_{(X, J)}(0, J)[v(X_m^{(2)}, J_m^{(2)})] \cdot \mathbb{I}\{ T_2 > m \}]|.
\]

(3.18)

Combining (3.14) with (3.18) and \( v \in \text{Bl}_d \), we obtain for all \( |g| \leq v \),

\[
|p^m(k, \varpi)g - \pi g| \leq 2E[E_{(k, J)}(0, J)[v(X_m^{(1)}, J_m^{(1)}) \cdot \mathbb{I}\{ T_1 > m \}] + 2E[E_{(X, J)}(0, J)[v(X_m^{(2)}, J_m^{(2)}) \cdot \mathbb{I}\{ T_2 > m \}]. \tag{3.19}
\]

Further it follows from (3.14) that \( X^{(h)}_\nu = 0 \) \((h = 1, 2)\) implies \( X^{(h)}_\nu = X^{(0)}_\nu \) for all \( \nu \geq m \), which leads to \( T_h \leq \inf \{ \nu \in \mathbb{Z}_+; X^{(h)}_\nu = 0 \} \) \((h = 1, 2)\). Thus we have

\[
\begin{align*}
E[E_{(k, J)}(0, J)[v(X_m^{(1)}, J_m^{(1)}) \cdot \mathbb{I}\{ T_1 > m \}] & \leq E[E_{(k, J)}[v(X_m, J_m) \cdot \mathbb{I}\{ T_0 > m \}]], \\
E[E_{(X, J)}(0, J)[v(X_m^{(2)}, J_m^{(2)}) \cdot \mathbb{I}\{ T_2 > m \}] & \leq E[E_{(X, J)}[v(X_m, J_m) \cdot \mathbb{I}\{ T_0 > m \}]],
\end{align*}
\]

(3.20) (3.21)

where \( T_0 = \inf \{ \nu \in \mathbb{Z}_+; X_\nu = 0 \} \). Substituting (3.20) and (3.21) into (3.19) yields

\[
\|p^m(k, \varpi) - \pi\|_v \leq 2E[E_{(k, J)}[v(X_m, J_m) \cdot \mathbb{I}\{ T_0 > m \}]] + 2E[E_{(X, J)}[v(X_m, J_m) \cdot \mathbb{I}\{ T_0 > m \}]]. \tag{3.22}
\]
Let \( M_m = \gamma^{-m} v(X_m, J_m) \mathbb{I}_{\{\tau_0 > m\}} \) for \( m \in \mathbb{Z}_+ \). If \( \tau_0 \leq m \), \( M_{m+1} = M_m = 0 \). On the other hand, suppose that \( \tau_0 > m \) and thus \( (X_m, J_m) = (k, i) \in \mathbb{N} \times \mathbb{D} \) (due to \( \{\tau_0 > m\} \subseteq \{X_m \in \mathbb{N}\} \)). We then have for \((k, i) \in \mathbb{N} \times \mathbb{D}\),

\[
E[M_{m+1} \mid (X_m, J_m) = (k, i), \tau_0 > m] = \sum_{(i, j) \in \mathbb{N} \times \mathbb{D}} p(k, i; l, j) \gamma^{-m-1} v(l, j)
\leq \sum_{(i, j) \in \mathbb{F}} p(k, i; l, j) \gamma^{-m-1} v(l, j) \leq \gamma^{-m} v(k, i),
\]

where the last inequality follows from (3.11). Thus \( \{M_m\} \) is a supermartingale.

Let \( \{\theta_{\nu}; \nu \in \mathbb{Z}_+\} \) denote a sequence of stopping times for \( \{M_m; m \in \mathbb{Z}_+\} \) such that \( 0 \leq \theta_1 \leq \theta_2 \leq \cdots \) and \( \lim_{\nu \to \infty} \theta_{\nu} = \infty \). Note that for any \( m' \in \mathbb{Z}_+ \), \( \min(m', \theta_{\nu}) \) is a stopping time for \( \{M_m; m \in \mathbb{Z}_+\} \). It then follows from Doob’s optional sampling theorem that for \((k, i) \in \mathbb{F}, E_{(k,i)}[M_{\min(m, \theta_{\nu})}] \leq E_{(k,i)}[M_0] \), i.e.,

\[
E_{(k,i)}[\gamma^{-\min(m, \theta_{\nu})} v(X_{\min(m, \theta_{\nu})}, J_{\min(m, \theta_{\nu})}) \mathbb{I}_{\{\tau_0 > \min(m, \theta_{\nu})\}}] \leq v(k, i)(1 - 1_0(k, i)).
\]

Thus letting \( \nu \to \infty \) and using Fatou’s lemma, we have

\[
E_{(k,i)}[v(X_m, J_m) \mathbb{I}_{\{\tau_0 > m\}}] \leq \gamma^m v(k, i)(1 - 1_0(k, i)), \tag{3.23}
\]

which leads to

\[
E[E_{(k,j)}[v(X_m, J_m) \mathbb{I}_{\{\tau_0 > m\}}]] = \sum_{i \in \mathbb{D}} \pi(i) E_{(k,i)}[v(X_m, J_m) \mathbb{I}_{\{\tau_0 > m\}}]
\leq \gamma^m v(k, \pi)(1 - 1_0(k, \pi)), \tag{3.24}
\]

where we use \( 1_0(k, i) = 1_0(k, \pi) \) for all \( i \in \mathbb{D} \). Note here that pre-multiplying both sides of (3.11) by \( \pi \) yields \( \pi v \leq b/(1 - \gamma) \), from which and (3.23) we obtain

\[
E[E_{(X,j)}[v(X_m, J_m) \cdot \mathbb{I}_{\{\tau_0 > m\}}]] \leq \gamma^m \sum_{(k, i) \in \mathbb{F}} \pi(k, i) v(k, i) \leq \gamma^m \frac{b}{1 - \gamma}. \tag{3.25}
\]

Substituting (3.24) and (3.25) into (3.22) yields (3.12).

Next we consider (3.13). Since \( P \in \text{BM}_d \), we have \( (n)P_n \in \text{BM}_d \) and \( (n)P_n \prec_d P \). Thus since \( P \) is irreducible and positive recurrent, Proposition 2.3 (c) implies that \( (n)P_n \) has the unique positive recurrent class, which includes the states \( \{(0, i); i \in \mathbb{D}\} \). Further it follows from \( v \in \text{Bl}_d \), (3.11) and Remark 2.1 that

\[
(n)P_n v \leq P v \leq \gamma v + b 1_0.
\]

Therefore we can prove (3.13) in the same way as the proof of (3.12).

Combining (3.2) with Lemmas 3.1 and 3.3 we obtain the following theorem.
Theorem 3.1 Suppose that $P \in BM_d$ and $P$ is irreducible. If Assumption [3.1] holds, then for all $n \in \mathbb{N},$

$$
\| (n)\pi_n - \pi \| \leq 4\gamma^n \frac{b}{1 - \gamma} + 2m \sum_{i \in D} (n)\pi_n(n, i), \quad \forall m \in \mathbb{N}, \quad (3.27)
$$

$$
\| (n)\pi_n - \pi \| \leq \frac{b}{1 - \gamma} \left( 4\gamma^n + 2m \sum_{i \in D} \frac{1}{v(n, i)} \right), \quad \forall m \in \mathbb{N}. \quad (3.28)
$$

Remark 3.2 If $d = 1$, Theorem [3.1] is reduced to Theorem 4.2 in [23].

Proof. [Proof of Theorem 3.1] From (3.2) and Lemma 3.3, we have

$$
\| (n)\pi_n - \pi \| \leq \sum_{i \in D} \| (n)p_{(i)}^n(0, \pi) - p^n(0, \pi) \|. \quad (3.29)
$$

From Lemma [3.1] (which does not require $P \in BM_d$), we obtain for $m \in \mathbb{N},$

$$
\| (n)p_{(i)}^n(0, \pi) - p^n(0, \pi) \| \leq \sum_{i \in D} \pi(i) \| (n)p_{(i)}^n(0, i) - p^n(0, i) \|

\leq \sum_{h=0}^{m-1} \sum_{(l,j) \in F} \left( \sum_{i \in D} p_{(i)}(0, l, j) \right) \Delta_n(l, j). \quad (3.30)
$$

It follows from (3.1) and $(n)P_n \in BM_d$ that $(\pi, 0, 0, \ldots) \prec_d (n)\pi_n$ and $(n)P_n^h \in BM_d$ for $h \in \mathbb{N}.$ Thus Proposition 2.2 yields

$$
(\pi, 0, 0, \ldots)(n)P_n^h \prec_d (n)\pi_n(n)P_n^h = (n)\pi_n. \quad (3.31)
$$

In addition, $P \in BM_d$ and (3.4) imply that a column vector $\tilde{\delta}_n := (\Delta_n(l, j))_{(l,j) \in F}$ with block size $d$ is block-increasing, i.e., $\tilde{\delta}_n \in Bl_d.$ Combining this and (3.31) with Remark 2.1 we have

$$
(\pi, 0, 0, \ldots)(n)P_n^h \tilde{\delta}_n \leq (n)\pi_n \tilde{\delta}_n.
$$

Applying (3.4) to the right hand side of the above inequality, we obtain

$$
\sum_{(l,j) \in F} \left( \sum_{i \in D} \pi(i)p_{(i)}^n(0, l, j) \right) \Delta_n(l, j)

\leq 2 \sum_{(l,j) \in F} (n)\pi_n(l, j) \sum_{l' > n, j' \in D} p(l, j; l', j')

\leq 2 \sum_{(l,j) \in F} (n)\pi_n(l, j) \sum_{j' \in D} (n)p_{(j')}n(l, j; n, j') = 2 \sum_{j' \in D} (n)\pi_n(n, j'), \quad (3.32)
$$

where the second inequality follows from (1.2) and the last equality follows from $(n)\pi_n \cdot (n)P_n = (n)\pi_n.$ Substituting (3.32) into (3.30) yields

$$
\| (n)p_{(n)}^n(0, \pi) - p^n(0, \pi) \| \leq 2m \sum_{j' \in D} (n)\pi_n(n, j'),
$$
from which and (3.29) we have (3.27).

Next we prove (3.28). Pre-multiplying both sides of (3.26) by \((n)\pi_n\) and using \((n)\pi_n \cdot (n)P_n = (n)\pi_n\), we obtain \((n)\pi_n v \leq b/(1 - \gamma)\), which leads to

\[
(n)\pi_n(n, i) \leq \frac{b}{1 - \gamma} \frac{1}{v(n, i)}, \quad i \in \mathbb{D}.
\]

Substituting this inequality into (3.27) yields (3.28).

\[\square\]

## 4 Extensions of main result

In this section, we do not necessarily assume that \(P\) (i.e., Markov chain \(\{(X_\nu, J_\nu); \nu \in \mathbb{Z}_+\}\)) is block-monotone, but assume that \(P\) is block-wise dominated by an irreducible and positive recurrent stochastic matrix in \(\text{BM}_d\), which is denoted by \(\tilde{P} = (\tilde{p}(k; i, l, j))(k, i, l, j) \in \mathbb{F}\). Let \(\pi = (\pi(k; i))_{(k, i) \in \mathbb{F}}\) denote the stationary probability vector of \(\tilde{P}\). It follows from \(P \prec_d \tilde{P} \in \text{BM}_d\) and Proposition 2.3(c) that \(\pi \prec_d \tilde{\pi}\) and thus

\[
\sum_{k=0}^{\infty} \pi(k, i) = \sum_{k=0}^{\infty} \pi(k, i) = \omega(i), \quad i \in \mathbb{D}.
\]

Let \(\{(\tilde{X}_\nu, \tilde{J}_\nu); \nu \in \mathbb{Z}_+\}\) denote a BMMC with state space \(\mathbb{F}\) and transition probability matrix \(\tilde{P}\). Since \(P \prec_d \tilde{P} \in \text{BM}_d\), we can assume (without loss of generality) that the pathwise ordering of \(\{(\tilde{X}_\nu, \tilde{J}_\nu)\}\) and \(\{(X_\nu, J_\nu)\}\) holds, i.e., if \(X_0 \leq \tilde{X}_0\) and \(J_0 = \tilde{J}_0\), then \(X_\nu \leq \tilde{X}_\nu\) and \(J_\nu = \tilde{J}_\nu\) for all \(n \in \mathbb{N}\) (see Lemma A.2).

The following result is an extension of Theorem 5.1 in [23].

**Theorem 4.1** Suppose that (i) \(\tilde{P} \in \text{BM}_d\) and \(\tilde{P}\) is irreducible; (ii) \(P \prec_d \tilde{P}\); and (iii) there exists a column vector \(v = (v(k, i))_{(k, i) \in \mathbb{F}} \in \text{Bl}_d\) such that \(v \geq e\) and

\[
\tilde{P}v \leq \gamma v + b\mathbb{1}_0,
\]

for some \(\gamma \in (0, 1)\) and \(b \in (0, \infty)\). Under these conditions, (3.28) holds for all \(n \in \mathbb{N}\).

**Proof.** We first prove the two bounds (3.12) and (3.13). Let \((X, J)\) and \((\tilde{X}, \tilde{J})\) denote two random vectors on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(\mathbb{P}(X = k, J = i) = \tilde{\pi}(k, i)\) for \((k, i) \in \mathbb{F}\). Note here that since \(\pi \prec_d \tilde{\pi}\), \(\sum_{i=k}^{\infty} \pi(l, i)/\omega(i) \leq \sum_{i=k}^{\infty} \tilde{\pi}(l, i)/\omega(i)\) for \((k, i) \in \mathbb{F}\). According to this inequality and (4.1), we can assume that \(X \leq \tilde{X}\) and \(J = \tilde{J}\) (see Theorem 1.2.4 in [18]). We then introduce the copies \(\{(\tilde{X}_\nu^{(h)}, \tilde{J}_\nu^{(h)})\}\) and \(\{(X_\nu^{(h)}, J_\nu^{(h)})\}\) \((h = 0, 1, 2)\) of the Markov chains \(\{(\tilde{X}_\nu, \tilde{J}_\nu)\}\) and \(\{(X_\nu, J_\nu)\}\), respectively, on the common probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where

\[
(\tilde{X}_0^{(0)}, \tilde{J}_0^{(0)}) = (0, \tilde{J}), \quad (\tilde{X}_0^{(1)}, \tilde{J}_0^{(1)}) = (k, \tilde{J}), \quad (\tilde{X}_0^{(2)}, \tilde{J}_0^{(2)}) = (\tilde{X}, \tilde{J}),
\]
\[
(X_0^{(0)}, J_0^{(0)}) = (0, J), \quad (X_0^{(1)}, J_0^{(1)}) = (k, J), \quad (X_0^{(2)}, J_0^{(2)}) = (X, J).
\]
From the pathwise ordering of \( \{(\tilde{X}_\nu, \tilde{J}_\nu)\} \) and \( \{(X_\nu, J_\nu)\} \), we have for \( h = 0, 1, 2 \),
\[
X_\nu^{(h)} \leq \tilde{X}_\nu^{(h)}, \quad J_\nu^{(h)} = \tilde{J}_\nu^{(h)}, \quad \forall \nu \in \mathbb{Z}_+.
\] (4.3)

In addition, by the pathwise ordered property of \( \tilde{P} \in \text{BM}_d \) (see Lemma A.1), we assume that
\[
\tilde{X}_\nu^{(0)} \leq \tilde{X}_\nu^{(1)}, \quad \tilde{X}_\nu^{(0)} \leq \tilde{X}_\nu^{(2)}, \quad \tilde{J}_\nu^{(0)} = \tilde{J}_\nu^{(1)} = \tilde{J}_\nu^{(2)}, \quad \forall \nu \in \mathbb{Z}_+.
\] (4.4)

Let \( g = (g(l,j))_{(l,j) \in \mathbb{F}} \) denote a column vector satisfying \( |g| \leq \nu \). It then follows that (3.18) holds under the assumptions of Theorem 4.1 because (3.18) does not require that \( \{(X_\nu, J_\nu)\} \) is block monotone. Further applying (4.3), (4.4) and \( \nu \in \text{Bl}_d \) to (3.18), we obtain for all \( |g| \leq \nu \),
\[
|p^m(k, \varpi)g - \pi g| \leq 2E_{(k,\tilde{j});(0,\tilde{j})}[v(\tilde{X}_m^{(1)}, \tilde{J}_m^{(1)}) \cdot II_{\{T_1 > \tilde{m}\}}] + 2E_{(\tilde{X},\tilde{j});(0,\tilde{j})}[v(\tilde{X}_m^{(2)}, \tilde{J}_m^{(2)}) \cdot II_{\{T_2 > \tilde{m}\}}],
\] (4.5)
where \( T_h = \inf\{m \in \mathbb{Z}_+: \tilde{X}_\nu^{(h)} = X_\nu^{(0)} (\forall \nu \geq m)\} \) for \( h = 1, 2 \).

It follows from (4.3) and (4.4) that for each \( h \in \{1, 2\}, \tilde{X}_\nu^{(h)} = 0 \) implies \( X_\nu^{(h)} = X_\nu^{(0)} = 0 \) and thus \( \tilde{X}_\nu^{(h)} = X_\nu^{(0)} \) for all \( \nu \geq m \), which leads to \( T_h \leq \inf\{\nu \in \mathbb{Z}_+: \tilde{X}_\nu^{(h)} = 0\} \). Therefore from (4.5), we can obtain the following inequality (see the derivation of (3.22) from (3.19)):
\[
\|p^m(k, \varpi) - \pi\|_\nu \leq 2E_{(k,\tilde{j})}[v(\tilde{X}_m, \tilde{J}_m) \cdot II_{\{\tilde{m}_0 > \tilde{m}\}}] + 2E_{(\tilde{X},\tilde{j})}[v(\tilde{X}_m, \tilde{J}_m) \cdot II_{\{\tilde{m}_0 > \tilde{m}\}}],
\] (4.6)
where \( \tilde{m}_0 = \inf\{\nu \in \mathbb{Z}_+: \tilde{X}_\nu = 0\} \). Further, following the discussion after (3.22), we can show that for all \( k \in \mathbb{Z}_+ \) and \( m \in \mathbb{N} \),
\[
\|p^m(k, \varpi) - \pi\|_\nu \leq 2\gamma^m [v(k, \varpi) (1 - 1_0(k, \varpi)) + b/1(1 - \gamma)],
\]
\[
\|p^m(k, \varpi) - (n)\pi_n\|_\nu \leq 2\gamma^m [v(k, \varpi) (1 - 1_0(k, \varpi)) + b/1(1 - \gamma)], \quad \forall n \in \mathbb{N}.
\]
Consequently, we obtain the two bounds (3.12) and (3.13).

It remains to prove that
\[
\|p^m(0, \varpi) - p^m(0, \varpi)\| \leq \frac{2mb}{1 - \gamma} \sum_{i \in \mathbb{D}} \frac{1}{v(n, i)}.
\] (4.7)

Let \( \Delta_n(l, j) = 2 \sum_{l', j' \in \mathbb{D}} \tilde{p}(l, j; l', j') \) for \( (l, j) \in \mathbb{F} \). Since \( P \prec_d \tilde{P} \), we have \( \Delta_n(l, j) \leq \Delta_n(l, j) \) for \( (l, j) \in \mathbb{F} \). Note here that (3.30) still holds and thus
\[
\|p^m(0, \varpi) - p^m(0, \varpi)\| \leq \sum_{h=0}^{m-1} \sum_{(l,j) \in \mathbb{F}} \left( \sum_{i \in \mathbb{D}} \varpi(i) \pi_n(h, i; l, j) \right) \Delta_n(l, j).
\] (4.8)

We now define \( (n)\tilde{P}_n \) as the last-column-block-augmented first-\( n \)-block-column truncation of \( \tilde{P} \) and \( (n)\pi_n = \{(n)\varpi_n(k, i)\}_{(n,i) \in \mathbb{F}} \) as the stationary distribution of \( (n)\tilde{P}_n \). We also define
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\( (n)\tilde{P}_n^m(k, i) = ((n)\tilde{P}_n^m(k, i; l, j))_{(l,j)\in F} \) as a probability vector such that \( (n)\tilde{P}_n^m(k, i; l, j) \) represents the \((k, i; l, j)\)th element of \( (n)\tilde{P}_n^m \). It then follows from \( (n)P_n \prec_d (n)\tilde{P}_n \) and Proposition 2.3(b) that \( ((n)P_n)^h \prec_d ((n)\tilde{P}_n)^h \) for \( h \in \mathbb{N} \). Therefore Remark 2.1 and \( (\tilde{\Delta}_n(l, j))_{(l,j)\in F} \in Bl_d \) (due to \( \tilde{P} \in BM_d \)) yield

\[
\sum_{(l,j)\in F} (n)P^h_n(0, i; l, j)\tilde{\Delta}_n(l, j) \leq \sum_{(l,j)\in F} (n)P^h_n(0, i; l, j)\Delta_n(l, j). \tag{4.9}
\]

Substituting (4.9) into (4.8), we have

\[
\| (n)P^m_n(0, \omega) - P^m(0, \omega) \| \leq \sum_{h=0}^{m-1} \sum_{(l,j)\in D} \left( \sum_{i\in \mathbb{D}} \omega(i)(n)\tilde{P}_n^h(0, i; l, j) \right) \Delta_n(l, j).
\]

In addition, since \( (n)\tilde{P} \prec_d \tilde{P} \in BM_d \), Proposition 2.3(c) implies that \( (n)\tilde{P}_n \prec_d \tilde{\pi} \) and thus \( \sum_{k=0}^{\infty} (n)\tilde{\pi}_n(k, i) = \sum_{k=0}^{\infty} \tilde{\pi}(k, i) \) for \( i \in \mathbb{D} \). Combining this with (4.1), we have \( \omega(i) = \sum_{k=0}^{\infty} (n)\tilde{\pi}_n(k, i) \) for \( i \in \mathbb{D} \). As a result, according to the discussion following (3.30) in the proof of Theorem 3.1, we can prove that

\[
\| (n)P^m_n(0, \omega) - P^m(0, \omega) \| \leq 2m \sum_{i\in \mathbb{D}} (n)\tilde{\pi}_n(n, i) \leq 2mb \frac{1}{1 - \gamma} \frac{1}{\nu(n, i)}.
\]

We can relax (4.2) if the direct path to the states \( \{(0, i); i \in \mathbb{D}\} \) is enough “large”.

**Theorem 4.2** Suppose that conditions (i) and (ii) of Theorem 4.1 are satisfied. Further suppose that there exists a column vector \( \nu' = (\nu'(k, i))_{(k,i)\in F} \in Bl_d \) such that \( \nu' \geq e \) and for some \( \gamma' \in (0, 1), b' \in (0, \infty) \) and \( K \in \mathbb{Z}_+ \),

\[
\tilde{P}\nu' \leq \gamma'\nu' + b'K; \tag{4.10}
\]

\[
\tilde{P}(K; 0)e > 0, \tag{4.11}
\]

where \( \tilde{P}(k; l) \) \((k, l \in \mathbb{Z}_+)\) denotes a \( d \times d \) matrix such that \( \tilde{P}(k; l) = (\tilde{p}(k, i; l, j))_{(i,j)\in \mathbb{D}} \). Under these conditions, (3.28) holds for all \( n \in \mathbb{N} \), where

\[
\gamma = \frac{\gamma' + B}{1 + B}; \tag{4.12}
\]

\[
b = b' + B; \tag{4.13}
\]

\[
\nu(k, i) = \begin{cases} 
\nu'(0, i), & k = 0, \ i \in \mathbb{D}; \\
\nu'(k, i) + B, & k \in \mathbb{N}, \ i \in \mathbb{D}; 
\end{cases} \tag{4.14}
\]

and

\[
B \in (0, \infty) \text{ such that } B \cdot \tilde{P}(K; 0)e > b'e. \tag{4.15}
\]

**Remark 4.1** The condition (4.11) ensures that there exists some \( B \in (0, \infty) \) satisfying (4.15). Further since \( \tilde{P} \in BM_d \), (4.11) implies \( \tilde{P}(k; 0)e > 0 \) for all \( k = 0, 1, \ldots, K \).
Proof. [Proof of Theorem 4.2] According to Theorem 4.1, it suffices to prove that (4.2) holds for some \( \gamma \in (0, 1) \), \( b \in (0, \infty) \) and \( v \in Bl_d \) with \( v \geq e \). Let \( v(k) \) and \( v'(k) \) \( (k \in \mathbb{Z}_+) \) denote \( d \times 1 \) vectors such that \( v(k) = (v(k, i))_{i \in \mathbb{D}} \) and \( v'(k) = (v'(k, i))_{i \in \mathbb{D}} \). Clearly, \( v = (v(0)^T, v(1)^T, \ldots)^T \) and \( v' = (v'(0)^T, v'(1)^T, \ldots)^T \), where the superscript “\( T \)” represents the transpose operator. Thus (4.10), (4.13) and (4.14) yield

\[
\sum_{l=0}^{\infty} \tilde{P}(0; l) v(l) \leq \sum_{l=0}^{\infty} \tilde{P}(0; l) v'(l) + Be \leq \gamma' v'(0) + (b' + B)e
\]

where the last inequality follows from \( \gamma \geq \gamma' \) (due to (4.12)).

Further since \( \tilde{P} \in BM_d, \sum_{l=1}^{\infty} \tilde{P}(k; l) \leq \sum_{l=1}^{\infty} \tilde{P}(K; l) \) for \( k = 1, 2, \ldots, K \). From this and (4.14), we have for \( k = 1, 2, \ldots, K \),

\[
\sum_{l=0}^{\infty} \tilde{P}(k; l) v(l) \leq \sum_{l=0}^{\infty} \tilde{P}(k; l) v'(l) + B \sum_{l=1}^{\infty} \tilde{P}(K; l) e
\]

Applying (4.10) and (4.15) to the right hand side of (4.17), we obtain for \( k = 1, 2, \ldots, K \),

\[
\sum_{l=0}^{\infty} \tilde{P}(k; l) v(l) \leq \gamma' v'(k) + Be + \{b'e - B\tilde{P}(K; 0)e\} \leq \gamma' v'(k) + Be. \tag{4.18}
\]

Note here that (4.12) implies \( \sup_{x \geq 1}(\gamma' x + B)/(x + B) = \gamma \). Thus since \( v' \geq e \), we have

\[\gamma' v'(k, i) + B \leq \gamma(v'(k, i) + B).\]

Combining this with (4.14) yields

\[
\gamma' v'(k) + Be \leq \gamma(v'(k) + Be) = \gamma v(k), \quad k \in \mathbb{N}. \tag{4.19}
\]

Substituting (4.19) into (4.18), we have

\[
\sum_{l=0}^{\infty} \tilde{P}(k; l) v(l) \leq \gamma v(k), \quad k = 1, 2, \ldots, K. \tag{4.20}
\]

Similarly, for \( k = K + 1, K + 2, \ldots, \)

\[
\sum_{l=0}^{\infty} \tilde{P}(k; l) v(l) \leq \sum_{l=0}^{\infty} \tilde{P}(k; l) v'(l) + Be \leq \gamma' v'(k) + Be \leq \gamma v(k), \tag{4.21}
\]

where the last inequality is due to (4.19). Finally, (4.16), (4.20) and (4.21) yield (4.2). \( \Box \)
5 Applications

In this section, we discuss the application of our results to GI/G/1-type Markov chains. To this end, we make the following assumption.

**Assumption 5.1** (i) \( P \) is of the following form:

\[
P = \begin{pmatrix}
B(0) & B(1) & B(2) & B(3) & \cdots \\
B(-1) & A(0) & A(1) & A(2) & \cdots \\
B(-2) & A(-1) & A(0) & A(1) & \cdots \\
B(-3) & A(-2) & A(-1) & A(0) & \cdots \\
& & & & \ddots
\end{pmatrix},
\]

(5.1)

where \( A(k) \) and \( B(k) \) \((k = 0, \pm 1, \pm 2, \ldots)\) are \( d \times d \) matrices; (ii) \( P \in \text{BM}_d \); (iii) \( P \) is irreducible and positive recurrent; (iv) \( A := \sum_{k=-\infty}^{\infty} A(k) \) is irreducible and stochastic; and (v) \( r_{A+} = \sup\{ z > 0; \sum_{k=0}^{\infty} z^k A(k) \text{ is finite} \} > 1 \).

It follows from conditions (i), (ii) and (iv) of Assumption 5.1 and Proposition 1.1 that \( \Psi = \sum_{k=0}^{\infty} B(k) = B(-k) + \sum_{l=-k+1}^{\infty} A(l) \) for all \( k \in \mathbb{N} \), which implies that \( \lim_{k \to \infty} B(-k) = O \) and thus \( A = \Psi \).

Let \( \hat{A}(z) \) denote

\[
\hat{A}(z) = \sum_{k=-\infty}^{\infty} z^k A(k), \quad z \in (1/r_{A-}, r_{A+}) \cap \{1\} =: \mathcal{I}_A,
\]

(5.2)

where \( r_{A-} = \sup\{ z > 0; \sum_{k=1}^{\infty} z^k A(-k) \text{ is finite} \} \geq 1 \). Let \( \delta_A(z) (z \in \mathcal{I}_A) \) denote the real and maximum-modulus eigenvalue of \( \hat{A}(z) \) (see, e.g., Theorems 8.3.1 and 8.4.4 in [9]). Let \( \mu_A(z) = (\mu_A(z,i))_{i \in \mathbb{D}} \) and \( v_A(z) = (v_A(z,i))_{i \in \mathbb{D}} \) \((z \in \mathcal{I}_A)\) denote left- and right-eigenvectors of \( \hat{A}(z) \) corresponding to eigenvalue \( \delta_A(z) \), i.e.,

\[
\mu_A(z) \hat{A}(z) = \delta_A(z) \mu_A(z), \quad \hat{A}(z) v_A(z) = \delta_A(z) v_A(z),
\]

(5.3)

which are normalized such that \( \mu_A(z) v_A(z) = 1 \) and \( v_A(z) \geq e \) for \( z \in \mathcal{I}_A \). We then have \( \delta_A(z) = \mu_A(z) \hat{A}(z) v_A(z) \). It also follows from \( A = \Psi \) and condition (iv) of Assumption 5.1 that \( \delta_A(1) = 1 \), \( \mu_A(1) = c \varpi \) and \( v_A(1) = c^{-1} e \) for some \( c \in (0, 1] \).

**Lemma 5.1** Under Assumption 5.1 there exists an \( \alpha \in (1, r_A) \) such that \( \delta_A(\alpha) < 1 \).

**Proof.** Since \( \delta_A(1) = 1 \) and \( \delta_A(z) \) is differentiable for \( z \in \mathcal{I}_A \) (see Theorem 2.1 in [11]), it suffices to show that \( \delta_A'(1) < 0 \). Indeed, \( \delta_A'(1) = \mu_A(1) \sum_{k=-\infty}^{\infty} k A(k) v_A(1) = \varpi \sum_{k=-\infty}^{\infty} k A(k) e \), which is equal to the mean drift of the process \( \{X_\nu; \nu \in \mathbb{Z}_+\} \) away from the boundary and is strictly negative under Assumption 5.1 (see, e.g., Proposition 2.2.1 in [11]). \( \square \)

We now define \( P(k; l) (k, l \in \mathbb{Z}_+) \) as a \( d \times d \) matrix such that \( P(k; l) = (p(k, i; j))_{i,j \in \mathbb{D}} \). We also fix \( v' = (v'(0)^T, v'(1)^T, \ldots)^T \) such that

\[
v'(k) = \alpha^k v_A(\alpha), \quad k \in \mathbb{Z}_+,
\]

(5.4)
which leads to \( v' \in \text{Bl}_d \). It then follows from (5.1) and (5.4) that
\[
\sum_{l=0}^{\infty} P(l; l) v'(l) = \sum_{l=0}^{\infty} a^l B(l) \cdot v_A(\alpha) =: w(0),
\]
(5.5)
\[
\sum_{l=0}^{\infty} P(k; l) v'(l) = B(-k) v_A(\alpha) + \alpha^k \sum_{l=-k+1}^{\infty} a^l A(l) \cdot v_A(\alpha) =: w(k), \quad k \in \mathbb{N},
\]
(5.6)
where \( w(0) \leq w(0) \leq w(1) \leq \cdots \) because \( P \in \text{BM}_d \) and \( v' \in \text{Bl}_d \) (see Proposition 2.2). Further, using (5.2) and (5.3), we can estimate the right hand side of (5.6) as follows:
\[
\sum_{l=0}^{\infty} P(k; l) v'(l) \leq B(-k) v_A(\alpha) + \alpha^k \tilde{A}(\alpha) v_A(\alpha)
\]
\[
= B(-k) v_A(\alpha) + \alpha^k \delta_A(\alpha) v_A(\alpha) < \infty, \quad k \in \mathbb{N},
\]
(5.7)
which shows that \( w(k) \) is finite for all \( k \in \mathbb{Z}_+ \). Combining (5.7), \( \lim_{k \to \infty} B(-k) = 0 \), \( v_A(\alpha) \geq e \) and Lemma 5.1, we can show that there exist some \( \gamma' \in (0, 1) \) and \( k_* \in \mathbb{N} \) such that
\[
\sum_{l=0}^{\infty} P(k; l) v'(l) \leq \gamma' \alpha^k v_A(\alpha) = \gamma' v'(k), \quad \forall k \geq k_*,
\]
(5.8)
where the last equality is due to (5.4).

**Theorem 5.1** Suppose that Assumption 5.1 holds and fix \( \gamma' \in (0, 1) \) and \( k_* \in \mathbb{N} \) satisfying (5.8). Further if \( B(-K) e > 0 \) for some nonnegative integer \( K \geq k_* - 1 \), then the bound (3.28) holds for \( \gamma \in (0, 1) \), \( b \in (0, \infty) \) and \( v \in \text{Bl}_d \) such that (4.12)–(4.15) are satisfied, where \( v' \) is given by (5.4), \( P(K; 0) = B(-K) \) and
\[
b' = \inf \{ x > 0; x e \geq w(k) - \gamma' \alpha^k v_A(\alpha) \} \quad (0 \leq \forall k \leq K)\]
(5.9)

**Proof.** Fix \( \tilde{P} = P \in \text{BM}_d \). From (5.4)–(5.6) and (5.9), we then have
\[
\sum_{l=0}^{\infty} \tilde{P}(k; l) v'(l) = \gamma' v'(k) + \{ w(k) - \gamma' \alpha^k v_A(\alpha) \} \leq \gamma' v'(k) + b' e, \quad k = 0, 1, \ldots, K.
\]
This inequality and (5.8) yield (4.10). Further (4.11) holds because \( \tilde{P}(K; 0) e = B(-K) e > 0 \). As a result, all the conditions of Theorem 4.2 are satisfied and thus the bound (3.28) holds. \( \square \)

Finally, we consider a special case where \( B(-k) = A(-k) = 0 \) for \( k \geq 2 \), \( B(-1) = A(-1) \) and \( B(k) = A(k-1) \) for \( k \in \mathbb{Z}_+ \), i.e.,
\[
P = \begin{pmatrix}
A(-1) & A(0) & A(1) & A(2) & \cdots \\
A(-1) & A(0) & A(1) & A(2) & \cdots \\
O & A(-1) & A(0) & A(1) & \cdots \\
O & O & A(-1) & A(0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
(5.10)
which is block-monotone with block size $d$. Note that $P$ in (5.10) is an M/G/1-type transition probability matrix and appears in the analysis of the stationary queue length distribution in the BMAP/GI/1 queue (see [21]).

**Theorem 5.2** Suppose that Assumption 5.1 holds. Further if $B(-k) = A(-k) = O$ for $k \geq 2$, $B(-1) = A(-1)$ and $B(k) = A(k - 1)$ for $k \in \mathbb{Z}_+$, then the bound (3.28) holds for $\gamma = \delta_A(\alpha)$, $b = (\alpha - 1) \max_{i \in D} \nu_A(\alpha, i)$ and $\nu = \nu'$ given in (5.4).

**Proof.** Fixing $\nu = \nu'$ and applying (5.2)–(5.4), Lemma 5.1 and the conditions on $\{B(k)\}$ to (5.5) and (5.6), we obtain

$$
\sum_{l=0}^{\infty} P(0; l) \nu(l) = \alpha \delta_A(\alpha) \nu_A(\alpha) \leq \nu(0) + (\alpha - 1) \nu_A(\alpha),
$$

$$
\sum_{l=0}^{\infty} P(k; l) \nu(l) = \alpha^k \delta_A(\alpha) \nu_A(\alpha) = \delta_A(\alpha) \nu(k), \quad k \in \mathbb{N},
$$

which imply that all the conditions of Theorem 3.1 hold. Thus we have (3.28). \qed

## A Pathwise ordering

This section presents two lemmas on the pathwise ordering associated with BMMCs. As in the previous sections, we use $P = (p(k, i; l, j))_{(k, i), (l, j) \in \mathbb{F}}$ and $P = (\tilde{p}(k, i; l, j))_{(k, i), (l, j) \in \mathbb{F}}$ to represent $|\mathbb{F}| \times |\mathbb{F}|$ stochastic matrices, though they are not necessarily assumed to be irreducible or recurrent in this section.

Let $\{U_{\nu}; \nu \in \mathbb{N}\}$ and $\{S_{\nu}; \nu \in \mathbb{N}\}$ denote two independent sequences of independent and identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $U_{\nu}$ and $S_{\nu}$ are uniformly distributed in $(0, 1)$. Let $J_{\nu}^0$ denote a $\mathbb{D}$-valued random variable on the probability space $(\Omega, \mathcal{F}, P)$, which is independent of both $\{U_{\nu}; \nu \in \mathbb{N}\}$ and $\{S_{\nu}; \nu \in \mathbb{N}\}$. Further let $J_{\nu}^* = G^{-1}(S_{\nu} | J_{\nu-1}^*)$ for $\nu \in \mathbb{N}$, where

$$
G^{-1}(s \mid i) = \inf \left\{ j \in \mathbb{D}; \sum_{j'=1}^{j} \psi(i, j') \geq s \right\}, \quad 0 < s < 1, \ i \in \mathbb{D}.
$$

It then follows that $\{J_{\nu}^*; \nu \in \mathbb{Z}_+\}$ is a $\mathbb{D}$-valued Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ such that $P(J_{\nu+1}^* = j \mid J_{\nu}^* = i) = \psi(i, j)$ for $i, j \in \mathbb{D}$ and $\nu \in \mathbb{Z}_+$, where $\psi(i, j)$ is defined in Proposition [1.1]

**Lemma A.1 (Pathwise ordered property of BMMCs)** Suppose $P \in \text{BM}_d$. Let $X_{\nu}^0$ and $X_{\nu}''$ denote nonnegative integer-valued random variables on the probability space $(\Omega, \mathcal{F}, P)$, which are independent of both $\{U_{\nu}; \nu \in \mathbb{N}\}$ and $\{S_{\nu}; \nu \in \mathbb{N}\}$. Further let $X_{\nu} = F^{-1}(U_{\nu} \mid X_{\nu-1}^0, J_{\nu-1}^*, J_{\nu}^*)$
and $X''_\nu = F^{-1}(U_\nu \mid X''_{\nu-1}, J''_{\nu-1}, J''_\nu)$ for $\nu \in \mathbb{N}$, where $F^{-1}(u \mid k, i, j)$ ($0 < u < 1$, $k \in \mathbb{Z}_+, i, j \in \mathbb{D}$) is defined as

$$F^{-1}(u \mid k, i, j) = \inf \left\{ l \in \mathbb{Z}_+; \sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)} \geq u \right\}. \quad (A.1)$$

Under these conditions, $\{(X'_\nu, J'_\nu); \nu \in \mathbb{Z}_+\}$ and $\{(X''_\nu, J''_\nu); \nu \in \mathbb{Z}_+\}$ are Markov chains with transition probability matrix $P$ on the probability space $(\Omega, \mathcal{F}, P)$ such that $X'_\nu \leq X''_\nu$ for all $\nu \in \mathbb{N}$.

**Proof.** Suppose that $X'_\nu \leq X''_\nu$ for some $\nu \in \mathbb{Z}_+$. Then it follows from $P \in \text{BM}_d$ that

$$\sum_{m=0}^{l} p(X'_\nu, J'_\nu; m, J'_{\nu+1}; m, J_{\nu+1}) \geq \sum_{m=0}^{l} p(X''_\nu, J''_\nu; m, J''_{\nu+1}), \quad l \in \mathbb{Z}_+.$$ 

Thus from the definition of $\{X'_\nu\}$ and $\{X''_\nu\}$, we have

$$X''_{\nu+1} = \inf \left\{ l \in \mathbb{Z}_+; \sum_{m=0}^{l} \frac{p(X''_\nu, J''_\nu; m, J''_{\nu+1})}{\psi(J''_\nu, J''_{\nu+1})} \geq U_{\nu+1} \right\}$$

$$\geq \inf \left\{ l \in \mathbb{Z}_+; \sum_{m=0}^{l} \frac{p(X'_\nu, J'_\nu; m, J'_{\nu+1})}{\psi(J'_\nu, J'_{\nu+1})} \geq U_{\nu+1} \right\}$$

$$= F^{-1}(U_{\nu+1} \mid X'_\nu, J'_\nu, J'_{\nu+1}) = X'_{\nu+1}.$$ 

Therefore it is proved by induction that $X'_\nu \leq X''_\nu$ for all $\nu \in \mathbb{N}$.

Next we prove that the dynamics of $\{(X'_\nu, J'_\nu); \nu \in \mathbb{Z}_+\}$ is determined by $P$. Let $\sigma(\cdot)$ denote the sigma-algebra generated by the random variables in the parentheses. From the definition of $\{(X'_\nu, J'_\nu)\}$, we then have for $\nu \in \mathbb{N},$

$$\sigma(X'_0, X'_1, \ldots, X'_{\nu-1}, J'_0, J'_1, \ldots, J'_{\nu-1})$$

$$\subseteq \sigma(X'_0, J'_0, U_1, U_2, \ldots, U_{\nu-1}, S_1, S_2, \ldots, S_{\nu-1}) =: \mathcal{G}_{\nu-1}. \quad (A.2)$$

Note here that for $(k, i) \in \mathbb{F}$ and $j \in \mathbb{D},$

$$\mathcal{G}_{\nu-1} \cap \{X'_\nu = k, J'_\nu = i, J'_{\nu+1} = j\} \subseteq \sigma(X'_0, J'_0, U_1, U_2, \ldots, U_{\nu}, S_1, S_2, \ldots, S_{\nu+1}),$$

which implies that $U_{\nu+1}$ is independent of both $\mathcal{G}_{\nu-1}$ and $\{X'_\nu = k, J'_\nu = i, J'_{\nu+1} = j\}$ for $(k, i) \in \mathbb{F}$ and $j \in \mathbb{D}$. Thus it follows from the definition of $\{X'_\nu\}$ that

$$P(X'_{\nu+1} \leq l \mid \mathcal{G}_{\nu-1}, X'_\nu = k, J'_\nu = i, J'_{\nu+1} = j)$$

$$= P \left( \sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)} \geq U_{\nu+1} \mid \mathcal{G}_{\nu-1}, X'_\nu = k, J'_\nu = i, J'_{\nu+1} = j \right)$$

$$= P \left( \sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)} \geq U_{\nu+1} \right) = \sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)}, \quad (k, i) \times (l, j) \in \mathbb{F}^2. \quad (A.3)$$
Note also that \( S_{\nu+1} \) is independent of \( \mathcal{G}_\nu \supseteq \mathcal{G}_{\nu-1} \cap \{ X_\nu' = k, J_\nu^* = i \} \) for \((k, i) \in \mathbb{F}\). Therefore from the definition of \( \{ J_\nu^* \} \), we have for \((k, i) \in \mathbb{F}\) and \(j \in \mathbb{D}\),

\[
P(J_{\nu+1}^* = j \mid \mathcal{G}_{\nu-1}, X_\nu' = k, J_\nu^* = i) = P \left( \sum_{j'=1}^{j-1} \psi(i, j') < S_{\nu+1} \leq \sum_{j'=1}^{j} \psi(i, j') \mid \mathcal{G}_{\nu-1}, X_\nu' = k, J_\nu^* = i \right)
\]

\[
= P \left( \sum_{j'=1}^{j-1} \psi(i, j') < S_{\nu+1} \leq \sum_{j'=1}^{j} \psi(i, j') \right) = \psi(i, j).
\]

(A.4)

Combining (A.3) and (A.4) yields

\[
P(X_{\nu+1}' \leq l, J_{\nu+1}^* = j \mid \mathcal{G}_{\nu-1}, X_\nu' = k, J_\nu^* = i) = P(X_{\nu+1}' \leq l \mid \mathcal{G}_{\nu-1}, X_\nu' = k, J_\nu^* = i)P(J_{\nu+1}^* = j \mid \mathcal{G}_{\nu-1}, X_\nu' = k, J_\nu^* = i)
\]

\[
= \sum_{m=0}^{l} p(k, i; m, j), \quad (k, i) \times (l, j) \in \mathbb{F}^2,
\]

which shows that \( \{(X_{\nu}', J_{\nu}^*); \nu \in \mathbb{Z}_+\} \) is a Markov chain with transition probability matrix \( P \) on the probability space \((\Omega, \mathcal{F}, P)\). The same argument holds for \( \{(X_{\nu}', J_{\nu}^*); \nu \in \mathbb{Z}_+\} \). We omit the details.

Lemma A.2 (Pathwise ordering by the block-wise dominance) Suppose \( \mathbf{P} \prec_d \mathbf{\tilde{P}} \) and either \( \mathbf{P} \in \text{BM}_d \) or \( \mathbf{\tilde{P}} \in \text{BM}_d \). Let \( X_0^\nu \) and \( \tilde{X}_0^\nu \) denote nonnegative integer-valued random variables on the probability space \((\Omega, \mathcal{F}, P)\), which are independent of both \( \{U_\nu; \nu \in \mathbb{N}\} \) and \( \{S_\nu; \nu \in \mathbb{N}\} \). Further let \( X_{\nu}^* = F^{-1}(U_\nu \mid X_{\nu-1}^*, J_{\nu-1}^*, J_\nu^*) \) and \( \tilde{X}_{\nu}^* = \tilde{F}^{-1}(U_\nu \mid \tilde{X}_{\nu-1}^*, J_{\nu-1}^*, J_\nu^*) \) for \( \nu \in \mathbb{N} \), where \( F^{-1}(u \mid k, i, j) (0 < u < 1, k \in \mathbb{Z}_+, i, j \in \mathbb{D}) \) is defined in (A.1) and \( \tilde{F}^{-1}(u \mid k, i, j) (0 < u < 1, k \in \mathbb{Z}_+, i, j \in \mathbb{D}) \) is defined as

\[
\tilde{F}^{-1}(u \mid k, i, j) = \inf \left\{ l \in \mathbb{Z}_+; \sum_{m=0}^{l} \tilde{p}(k, i; m, j) \geq u \right\}.
\]

Under these conditions, \( \{(X_{\nu}', J_{\nu}^*); \nu \in \mathbb{Z}_+\} \) and \( \{(\tilde{X}_{\nu}', J_{\nu}^*); \nu \in \mathbb{Z}_+\} \) are Markov chains with transition probability matrices \( \mathbf{P} \) and \( \mathbf{\tilde{P}} \), respectively, on the probability space \((\Omega, \mathcal{F}, P)\) such that \( X_{\nu}^* \leq \tilde{X}_{\nu}^* \) for all \( \nu \in \mathbb{N} \) if \( X_0^\nu \leq \tilde{X}_0^\nu \).

Proof. Proposition 2.3(a) shows that for all \( k \in \mathbb{Z}_+ \) and \( i, j \in \mathbb{D} \),

\[
\psi(i, j) = \sum_{l=0}^{\infty} p(k, i; l, j) = \sum_{l=0}^{\infty} \tilde{p}(k, i; l, j).
\]

Therefore, following the proof of Lemma A.1, we can prove that \( \{(X_{\nu}', J_{\nu}^*); \nu \in \mathbb{Z}_+\} \) and \( \{(\tilde{X}_{\nu}', J_{\nu}^*); \nu \in \mathbb{Z}_+\} \) are Markov chains with transition probability matrices \( \mathbf{P} \) and \( \mathbf{\tilde{P}} \), respectively, on the probability space \((\Omega, \mathcal{F}, P)\). Similarly we can prove by induction that if \( X_0^* \leq \tilde{X}_0^* \), then \( X_{\nu}^* \leq \tilde{X}_{\nu}^* \) for all \( \nu \in \mathbb{N} \). We omit the details.
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