COMBINATORIAL CHARACTERIZATION OF RIGHT-ANGLED HYPERBOLICITY OF 3-ORBIFOLDS

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ABSTRACT. We study the right-angled hyperbolicity of a class of 3-handlebodies with simple facial structures, each of which possesses the property that its nerve is a triangulation of its boundary. We show that such a 3-handlebody admits a right-angled hyperbolic structure if and only if it is flag and contains no □-belts, which is a generalization of Pogorelov’s theorem (resp. the right-angled case of Andreev’s theorem). To make sure that this characterization of right-angled hyperbolicity is of combinatorial nature, we generalize the notions of flag and □-belt in the setting of simple 3-polytopes to the setting of simple 3-handlebodies, with a quite difference.

The basic idea of proof of our main result consists of two aspects. First, we construct the manifold double $M_Q$ of such a 3-handlebody $Q$ by using a basic construction method from Davis; Second, based upon the works of Thurston and Perelman, we reduce the problem to how to characterize the asphericality and atoroidality of $M_Q$ in terms of combinatorics of $Q$. Most of our arguments can actually perform in the case of dimension more than or equal to three. The key point of our arguments is to cut $Q$ into a simple polytope $P_Q$, so that we can give a right-angled Coxeter cellular decomposition of $Q$, and further we can obtain an explicit presentation of $\pi_{orb}^1(Q)$. In particular, this presentation of $\pi_{orb}^1(Q)$ is an iterative HNN-extension over some right-angled Coxeter group associated with $P_Q$.

1. INTRODUCTION

A right-angled Coxeter n-orbifold, introduced by Davis and Januszkiewicz in [12], is locally isomorphic to the n-orbifold $\mathbb{R}^n/(\mathbb{Z}_2)^n$, which is the quotient of the standard $(\mathbb{Z}_2)^n$-action on $\mathbb{R}^n$ by reflections across the coordinate hyperplanes, so it is also regarded as the standard simplicial cone

$$\mathcal{C}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0, 1 \leq i \leq n\}$$

in $\mathbb{R}^n$ topologically and combinatorially. Each point $x$ in $\mathbb{R}^n/(\mathbb{Z}_2)^n$ has the local group $(\mathbb{Z}_2)^{c(x)}$, where $c(x)$ is the number of coordinates of $x$ which are equal to zero in $\mathcal{C}^n$, called the codimension of $x$. Thus, in the viewpoints of topology and combinatorics, each right-angled Coxeter n-orbifold naturally inherits the structure of an n-manifold with corners defined and studied by Davis in [13], where an n-manifold with corners is locally modelled on open subsets of $\mathcal{C}^n$ such that overlap maps are homeomorphisms of preserving codimension. On the other hand, since the topological and combinatorial structure of $\mathcal{C}^n$ is compatible with structure of the right-angled Coxeter orbifold on $\mathbb{R}^n/(\mathbb{Z}_2)^n$, an n-manifold with corners admits the structure of a right-angled Coxeter n-orbifold. All strata in a right-angled Coxeter orbifold bijectively correspond to its all faces as a manifold with corners. A stratum or face of codimension one is called a facet.

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In this paper, we consider a class of right-angled Coxeter $n$-orbifolds, named simple orbifolds, each $Q$ of which satisfies the three conditions:

(a) $|Q|$ is compact and connected with $\partial|Q| \neq \emptyset$ where $|Q|$ denotes the underlying space of $Q$;

(b) The nerve of $Q$, denoted by $\mathcal{N}(Q)$, is a triangulation of the boundary $\partial|Q|$, where $\mathcal{N}(Q)$ is the abstract simplicial complex with a vertex for each facet and a $(k-1)$-simplex for each nonempty $k$-fold intersection;

(c) Each facet in $Q$ is a simple polytope (note that when $n \leq 3$, this condition will be automatically omitted).

Simple polytopes together with the natural structure of right-angled Coxeter orbifold give the canonical examples of simple orbifolds. Similar to simple polytopes, a simple orbifold is completely determined by its underlying space and the combinatorial information on its boundary. We will pay more attention on the case in which the underlying space $|Q|$ of a simple $n$-orbifold $Q$ is an $n$-dimensional handlebody, where an $n$-dimensional handlebody of genus $g \geq 0$ is a tubular neighborhood of the wedge sum of $g$ circles in $\mathbb{R}^n$ (of course, an $n$-dimensional handlebody of genus 0 is exactly an $n$-ball). Such a simple $n$-orbifold is called a simple $n$-handlebody here. In particular, a simple 3-handlebody of genus zero must be a simple 3-polytope. This can be proved by Steinitz Theorem and [25, Proposition 3.4]. However, for $n > 3$, if a simple $n$-handlebody $Q$ has genus zero, then $Q$ may not be a simple polytope in general.

Pogorelov Theorem, which was generalized by Andreev [2] and revisited by Roeder-Hubbard-Dunbar [34], states that a simple convex 3-polytope admits a right-angled hyperbolic structure if and only if it is flag and contains no $\square$-belts, where “right-angled” means that all dihedral angles are $\frac{\pi}{2}$. This gives a combinatorial equivalent description of right-angled hyperbolicity of simple 3-polytopes. A simple convex 3-polytope $P$ with right-angled hyperbolic structure in $\mathbb{H}^3$ can also be presented as a hyperbolic right-angled Coxeter orbifold $\mathbb{H}^3/W_P$ where $W_P$ is the right-angled Coxeter group of $P$. Thus, the right-angled hyperbolicity of simple 3-polytopes as a class of right-angled Coxeter 3-orbifolds (or simple orbifolds) can completely be determined in terms of combinatorics.

The above observation naturally arises the following question, which is a main motivation of this paper.

(Q) whether can right-angled hyperbolic structures of simple 3-orbifolds be characterized in terms of combinatorics on their underlying spaces?

As far as authors know, the existence of the hyperbolicity with arbitrary assigned dihedral angles in $(0, \frac{\pi}{2}]$ of 3-manifolds with corners is still unsolved and open.

We give an answer of the question (Q) in the case in which a simple 3-orbifold is a simple 3-handlebody, and the result is stated as follows:

**Theorem A.** A simple 3-handlebody admits a (right-angled) hyperbolic structure if and only if it is flag and contains no $\square$-belts.

**Remark 1.**

(1) [Theorem A] is a generalization of Pogorelov Theorem (i.e., the right-angled case of Andreev Theorem), and it is of combinatorial nature. The notions of flag and
□-belt in Theorem A are also the generalizations for usual flag and 4-belt in a simple 3-polytope, respectively. However, there is a quite difference. This will be seen in Definition 3.2 and Definition 3.3.

(2) With respect to “right-angled hyperbolicity”, the following two statements are equivalent. Namely, a simple 3-handlebody as a 3-manifold with corners is right-angled hyperbolic if and only if it, as a right-angled Coxeter 3-orbifold, is hyperbolic. Also see subsection 2.5.

(3) By Mostow rigidity theorem, the hyperbolic structure on a hyperbolic simple 3-handlebody is unique up to an isometry.

(4) With a bit additional argument, the “simple” condition in Theorem A can be generalized to the case of a right-angled Coxeter 3-handlebody whose nerve is an ideal triangulation of its boundary, where the concept of ideal triangulation can be referred to [18, Section 2]. In this case, a 3-handlebody with an ideal nerve is hyperbolic if and only if it is very good, flag and contains no □-belts, see subsection 6.3 for details.

(5) For the “non-simple” case, there exists a right-angled hyperbolic 3-handlebody whose faces may not be contractible. In this case, the flag condition and no □-belt condition are not enough to characterize its right-angled hyperbolicity. An example is given in subsection 6.4. Meanwhile, there may exist bad 3-handlebodies, that is, as right-angled Coxeter orbifolds, they cannot be covered by 3-manifolds. So these bad orbifolds cannot admit any hyperbolic metric. See Lemma 6.2.

Our strategy for dealing with the problem will be carried out via the following points:

(I) A point is that a simple orbifold will be associated with a covering space of it. We use a basic construction method from Davis [10, Chapter 5], which tells us that each simple n-orbifold Q can be finitely covered by a closed n-manifold MQ, which is called a manifold double of Q in [14, Proposition 2.4]. Then we will see in Proposition 2.3 that a simple 3-orbifold is (right-angled) hyperbolic if and only if its manifold double is hyperbolic. Based on Perelman’s work, one version of Hyperbolization Theorem says that a closed oriented 3-manifold is hyperbolic if and only if it is aspherical and atoroidal (Theorem 2.3). Hence the question (Q) is reduced to asking how to characterize the asphericity and atoroidality of manifold double of a simple 3-orbifold Q in terms of combinatorics of Q.

(II) Another point is to perform some kind of “cutting surgery” for simple orbifolds, which is analogous to the hierarchy for Haken 3-manifolds [36]. Here we will carry out our work for “special” simple handlebodies in arbitrary dimension, where a simple n-handlebody Q with genus greater than zero is special if there exist some disjoint codimension-one B-belts, named cutting belts, such that Q can be cut into a simple polytope PQ along those cutting belts (for the notion of B-belts, see Definition 3.1), and a simple n-handlebody with genus zero is said to be special if it is a simple n-polytope. We shall show that a simple 3-handlebody is always special (see Proposition 3.1). An example is shown in Figure 1. This cutting surgery allows us to get a presentation of the orbifold fundamental group π1orb(Q) of a special simple n-handlebody Q, which is an iterative HNN-extension over some right-angled Coxeter group associated with
\[ \pi_1^{\text{orb}}(Q) \] will not be the Coxeter group of \( Q \), given by only reflections on facets of \( Q \), and it actually contains torsion-free generators. This is a key step, which plays an important role on our arguments. In particular, this also allows us to carry out some further work in higher-dimensional case.

Now let \( Q \) be a simple \( n \)-orbifold with \( m \) facets, write \( \mathcal{F}(Q) = \{ F_1, \ldots, F_m \} \). Consider a coloring \( \lambda : \mathcal{F}(Q) \rightarrow (\mathbb{Z}_2)^m \) defined by \( \lambda(F_i) = e_i \), where \( \{ e_1, \ldots, e_m \} \) is the standard basis of \( (\mathbb{Z}_2)^m \). This coloring determines a coloring on all faces of \( Q \) in such a way that:

For a \( k \)-face \( f^k \), it is the intersection of \( n - k \) facets, say \( F_{i_1}, \ldots, F_{i_{n-k}} \), and then \( f^k \) is colored by a subgroup \( G_{f^k} \) generated by \( \lambda(F_{i_1}), \ldots, \lambda(F_{i_{n-k}}) \). Note that each \( x \in \partial |Q| \) lies in the relative interior of a unique face \( f \). Then the manifold double of \( Q \) is defined as follows:

\[
U(Q, (\mathbb{Z}_2)^m) = Q \times (\mathbb{Z}_2)^m / \sim
\]

where

\[
(x, g) \sim (y, h) \iff \begin{cases} 
  x = y \text{ and } g = h & \text{if } x \in \text{Int}(|Q|) \\
  x = y \text{ and } gh^{-1} \in G_f & \text{if } x \in f \subset \partial |Q|. 
\end{cases}
\]

Essentially this is a special case of “basic construction” of Davis [10, Chapter 5]. It follows from [10, Proposition 10.1.10] that \( U(Q, (\mathbb{Z}_2)^m) \) is an \( n \)-dimensional closed manifold and naturally admits an action of \( (\mathbb{Z}_2)^m \) with quotient orbifold \( Q \).

Many important works with respect to the topology and geometry of \( U(Q, (\mathbb{Z}_2)^m) \) have been carried out by associating with the topology, geometry and combinatorics of \( Q \), especially for \( Q \) to be a simple polytope (e.g., see [10, 11, 12]).

For a special simple handlebody \( Q \) of arbitrary dimension, we obtain that

**Proposition 1.1.** Let \( Q \) be a special simple handlebody of dimension \( n \geq 3 \), and \( U(Q, (\mathbb{Z}_2)^m) \) be the manifold double over \( Q \). Then the following statements are equivalent.

- \( U(Q, (\mathbb{Z}_2)^m) \) is aspherical;
- \( U(Q, (\mathbb{Z}_2)^m) \) is non-positively curved;
- \( Q \) is flag.
Remark 2. Proposition 1.1 is a generalization of the result of Davis, Januszkiewicz and Scott for small covers in [16, Theorem 2.2.5], and its proof is heavily based upon Gromov Lemma [20, Section 4.2], Cartan-Hadamard Theorem and Davis’s method [10, Chapter 8].

Making use of Tit’s theorem [10, Theorem 3.4.2] of Coxeter groups and the normal form theorem of HNN-extensions [27, Theorem 2.1, Page 182] (also see Theorem 2.2 in this paper), we do not only give a presentation of $\pi^\text{orb}_1(Q)$ for $Q$ to be a special simple handlebody of arbitrary dimension, but also use it to characterize non-existence of rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi^\text{orb}_1(Q)$ in terms of combinatorics of $Q$.

Proposition 1.2. Let $Q$ be a flag special simple handlebody of dimension $n \geq 3$. Then there is no rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi^\text{orb}_1(Q)$ if and only if $Q$ contains no $\Box$-belts.

In the case of dimension 3, together with Proposition 1.1 and the theory of hyperbolic 3-manifolds, we conclude that for a simple 3-handlebody $Q$ with its manifold double $U(Q, (\mathbb{Z}_2)^m)$,

- $U(Q, (\mathbb{Z}_2)^m)$ is aspherical if and only if $Q$ is flag;
- If $Q$ is also flag, then $U(Q, (\mathbb{Z}_2)^m)$ is atoroidal if and only if $Q$ contains no $\Box$-belt.

This characterize the asphericality and atoroidality of $U(Q, (\mathbb{Z}_2)^m)$ in terms of combinatorics of $Q$, implies that Theorem A holds.

This paper is organized as follows. In section 2, we review the notions of (right-angled Coxeter) orbifolds and manifolds with corners. We introduce the right-angled Coxeter cellular decomposition of right-angled Coxeter orbifolds, and discuss their orbifold fundamental groups. In addition, we also give a simple review on hyperbolic geometry. In section 3 we introduce the notion of $B$-belts and study some basic properties. We give a right-angled Coxeter cellular decomposition of a special simple $n$-handlebody $Q$, so that we can explicitly give a presentation of orbifold fundamental group $\pi^\text{orb}_1(Q)$. We show that this presentation of orbifold fundamental group $\pi^\text{orb}_1(Q)$ is an iterative HNN-extension of some right-angled Coxeter group. In section 4 we prove Proposition 1.1. In section 5 we show that the existence of a rank-two free abelian subgroup in the orbifold fundamental group of a flag simple handlebody $Q$ is characterized by an $\Box$-belt in $Q$ (Proposition 1.2). The proof of our main theorem will be given in section 6. In Appendix A we construct the orbifold universal cover of a special simple handlebody $Q$ with the aid of the theory of fundamental domain, and compute the homology groups of the universal cover of $Q$ by Davis method, which are useful in the proof of Proposition 1.1.

2. Preliminaries

2.1. Orbifold. As a generalization of manifolds, an $n$-dimensional orbifold $O$ is a singular space which is locally modelled on the quotient of a finite group acting on an open subset of $\mathbb{R}^n$. For any point $p \in O$, there is an orbifold chart $(U, G, \psi)$ such that $\psi(U)$ is an open set in $O$ that contains $p$, where $U$ is a connected open set in $\mathbb{R}^n$, $G$ is a finite group of linear automorphisms of $U$, and $\psi$ is the quotient map induced by the action of $G$ on $U$. The isotropy group of $p' \in \psi^{-1}(p)$ in $U$ is called the local group at $p$.

Definition 2.1 (Thurston [37, Definition 13.2.2]). A covering orbifold of an orbifold $O$ is an orbifold $\tilde{O}$ with a projection $\pi : \tilde{O} \to O$, satisfying that:
• \( \forall x \in \mathcal{O} \) has a neighborhood \( V \) which is identified with an open subset \( U \) of \( \mathbb{R}^n \) modulo a finite group \( G_x \), such that each component \( V_i \) of \( \pi^{-1}(V) \) is homeomorphic to \( U/\Gamma_i \), where \( \Gamma_i < G_x \) is some subgroup;
• \( \pi|_{V_i}: V_i \to V \) corresponds to the natural projection \( U/\Gamma_i \to U/G_x \).

An orbifold is good (resp. very good) if it can be covered (resp. finitely) by a manifold. Otherwise it is bad. Any orbifold \( \mathcal{O} \) has an universal cover \( \tilde{\mathcal{O}} \), see [37, Proposition 13.2.4]. The orbifold fundamental group of an orbifold is defined as the deck transformation group of its universal cover, see [37, Definition 13.2.5].

For more details with respect to orbifolds, see [1, 7, 8, 35].

2.2. Right-angled Coxeter orbifolds and manifolds with corners. Following [12, 14], a right-angled Coxeter \( n \)-orbifold \( Q \) is a special \( n \)-orbifold locally modelled on the quotient \( \mathbb{R}^n/\langle \mathbb{Z} \rangle^n \) of the standard \( \langle \mathbb{Z} \rangle^n \)-action on \( \mathbb{R}^n \) by reflections across the coordinate hyperplanes. A stratum of codimension \( k \) is the closure of a component of the subspace of \( \{ Q \} \) consisting of all points with local group \( \langle \mathbb{Z} \rangle^k \), where \( |Q| \) denotes the underlying space of \( Q \). It is easy to see that \( \mathbb{R}^n/\langle \mathbb{Z} \rangle^n \) possesses the following properties:

• Topologically and combinatorially, \( \mathbb{R}^n/\langle \mathbb{Z} \rangle^n \) is the standard simplicial cone \( C^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0, 1 \leq i \leq n \} \) in \( \mathbb{R}^n \);
• The local group at \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n/\langle \mathbb{Z} \rangle^n \) is the subgroup \( \langle \mathbb{Z} \rangle^{c(x)} \), where \( c(x) \) is the number of those coordinates \( x_i = 0 \) in \( x \), called the codimension of \( x \);
• For \( 0 \leq k \leq n \), \( \langle \mathbb{Z} \rangle^k \) as a local group determines \( \binom{n}{k} \) strata of codimension \( k \), each of which is isomorphic to \( \mathbb{R}^{n-k}/\langle \mathbb{Z} \rangle^{n-k} \).

Davis in [13, Section 6] (or [10, Chapter 10, Page 180]) defined \( n \)-manifolds with corners, each of which is a Hausdorff space \( X \) together with a maximal atlas of local charts onto open subsets of the standard simplicial cone \( C^n \) such that the overlap maps are homeomorphisms of preserving codimension, where for any chart \( \varphi : U \to C^n \), the codimension of any \( x \in U \) is defined as \( c(\varphi(x)) \), denoted by \( c(x) \), and it is independent of the chart. An open face of codimension \( k \) is a component of \( \{ x \in X | c(x) = k \} \). A face is the closure of such a component.

A right-angled Coxeter orbifold \( Q \) naturally inherits the structure of a manifold with corners. On the other hand, since the topological and combinatorial structure of \( C^n \) is compatible with that of right-angled Coxeter orbifold on \( \mathbb{R}^n/\langle \mathbb{Z} \rangle^n \), an \( n \)-manifold with corners naturally admits a right-angled Coxeter orbifold structure. Furthermore, all strata in a right-angled Coxeter orbifold \( Q \) bijectively correspond to all faces in \( Q \) as a manifold with corners. A stratum or face of codimension one is called a facet.

In this paper we are mainly concerned with a special class of right-angled Coxeter orbifolds, i.e., simple orbifolds, as defined in section 1. Given a simple \( n \)-orbifold \( Q \) with \( m \) facets, we have seen in section 1 that \( Q \) is finitely covered by a closed manifold \( \mathcal{U}(Q, \langle \mathbb{Z} \rangle^m) \) with an action of \( \langle \mathbb{Z} \rangle^m \), so \( Q \) is a very good orbifold.

It should be pointed out that an \( n \)-manifold \( Q \) with corners may allows many other orbifold structures different from that of right-angled Coxeter orbifold. Indeed, making use of the “basic construction” of Davis [10, Chapter 5], we may construct some covering spaces of \( Q \) with actions of different groups, giving different orbifold structures on \( Q \).
Actually, let $Q$ be a nice $n$-manifold with corners with facet set $\mathcal{F}(Q) = \{F_1, ..., F_m\}$, satisfying that $\partial|Q|$ is the union $\bigcup_{i=1}^m F_i$ and each $k$-face is a component of the intersection of some $n-k$ facets. Then one can proceed as follows.

- Let $W = \langle S|R \rangle$ be a Coxeter group where $S$ is the set of generators $s_F$ indexed by $F \in \mathcal{F}(Q)$, and $R$ gives relations that for $F \in \mathcal{F}(Q)$, $s_F^2 = 1$, and for any $F, F' \in \mathcal{F}(Q)$, there is a $m_{FF'} \geq 2$ in $\mathbb{N} \cup \{\infty\}$ such that $(s_Fs_{F'})^{m_{FF'}} = 1$. Such $W$ is not uniquely defined in the above way since $m_{FF'}$ may have many different choices. Since each $k$-face $f$ is the intersection of $n-k$ facets, say $F_{i_1}, ..., F_{i_{n-k}}$, one has that $f$ determines a subgroup of $W$, generated by $s_{F_{i_1}}, ..., s_{F_{i_{n-k}}}$, denoted by $W_f$.
- Define an equivalence relation $\sim$ on $Q \times W$ by $(x, s) \sim (y, s')$ if and only if $x = y$ and $ss'^{-1} \in W_f$ where $x$ is in the relative interior of a face $f$.
- Finally, the required covering space $U(Q, W)$ is the quotient space $U(Q, W) = Q \times W/ \sim$

with a natural action of $W$.

Moreover, the quotient space $U(Q, W)/W$ is just $Q$, on which an orbifold structure is naturally endowed.

An easy argument shows that $U(Q, W)/W$ is a right-angled Coxeter orbifolds if and only if $W$ must satisfy that for any $F, F' \in \mathcal{F}(Q)$,

$$m_{FF'} = \begin{cases} 2 & \text{if } F \cap F' \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

Namely, $W$ is a right-angled Coxeter group (abbreviate as RACG), denoted by $W_Q$. This also gives the reason why an orbifold locally isomorphic to $\mathbb{R}^n/(\mathbb{Z}_2)^n$ is called a right-angled Coxeter orbifold.

Since the abelization of $W_Q$ is exactly $(\mathbb{Z}_2)^n$, $U(Q, W_Q)$ is a covering space of $U(Q, (\mathbb{Z}_2)^m)$. Thus, by Proposition 1.1 we have that

**Corollary 2.1.** Let $Q$ be a special simple handlebody of dimension $n \geq 3$. Then the following statements are equivalent.

- $U(Q, W_Q)$ is aspherical;
- $U(Q, W_Q)$ is non-positively curved;
- $Q$ is flag.

### 2.3. The right-angled Coxeter cellular decomposition.

Now let us introduce the right-angled Coxeter orbifold cellular decomposition for right-angled Coxeter orbifolds, which will play an important role on the calculation of the orbifold fundamental groups and homology groups of right-angled Coxeter orbifolds (also see [26]). The more general notion of cellular decomposition of certain orbifolds are considered as q-cellular complex (or, q-CW complex) in [6, 32].

Let $r_i : \mathbb{R}^n \to \mathbb{R}^n$ be the $i$-th standard reflection defined by $r_i(x_1, \cdots, x_i, \cdots, x_n) = (x_1, \cdots, -x_i, \cdots, x_n)$. All standard reflections in $\mathbb{R}^n$ induce a standard $(\mathbb{Z}_2)^n$-action on the closed unit $n$-ball $B^n$ with a right-angled corner $B^n/(\mathbb{Z}_2)^n$ as its orbit space. Of course, $\text{Int}B^n$ is $(\mathbb{Z}_2)^n$ equivariantly homeomorphic to $\mathbb{R}^n$. 

Definition 2.2 (Right-angled Coxeter cells). Let $\Gamma$ be a group generated by some standard reflections in $\mathbb{R}^n$. Then the quotient $B^n/\Gamma$ is called a right-angled Coxeter $n$-ball, and the quotient $\text{Int}B^n/\Gamma$ is called an open right-angled Coxeter $n$-ball. Note that if $\Gamma$ is not a trivial group, then the right-angled Coxeter $n$-ball $B^n/\Gamma$ is an $n$-orbifold with boundary $\partial B^n/\Gamma$.

If $e^n$ is $\Gamma$-equivariantly homeomorphic to $\text{Int}B^n$, then the quotient $e^n/\Gamma$ is called a right-angled Coxeter $n$-cell, and its closure is call a closed right-angled Coxeter $n$-cell.

For example, a right-angled Coxeter 1-cell is either a connected open interval or a semi-open and semi-closed interval whose closed endpoint gives a local group $\mathbb{Z}_2$. A right-angled Coxeter 2-cell has three kinds of possible types with local group being trivial group, $\mathbb{Z}_2$ and $(\mathbb{Z}_2)^2$ respectively, as shown in Figure 2.

In a similar way as in the construction of CW complexes (see Hatcher [21, Page 5]), a right-angled Coxeter cellular complex $\mathcal{O}$ of dimension $n$ can be constructed by the following procedures:

1. Start with a discrete set $\mathcal{O}^0$, whose points are regarded as (right-angled Coxeter) 0-cells;
2. Inductively, form the $n$-skeleton $\mathcal{O}^n$ from $\mathcal{O}^{n-1}$ by attaching finitely many right-angled Coxeter $n$-cells $e^n_\alpha/\Gamma_\alpha$ via orbifold attaching maps
   $$\phi_\alpha : \partial e^n_\alpha/\Gamma_\alpha \to \mathcal{O}^{n-1},$$
   where each $\phi_\alpha$ preserves the local group of every point in $\partial e^n_\alpha/\Gamma_\alpha$;
3. One can stop this inductive process at a finite stage by setting $\mathcal{O} \cong \mathcal{O}^n$ for some $n < \infty$.

Here the attaching maps $\{\phi_\alpha\}$ of right-angled Coxeter cells with non-trivial local groups are usually a much stronger definition than in CW complexes. Actually, $\phi_\alpha$ preserving local groups implies that singular points and non-singular points of each embedding right-angled Coxeter $n$-cell are still singular and non-singular in $\mathcal{O}$ respectively. Therefore, there is no case where the boundary of a right-angled Coxeter $n$-cell with nontrivial local group is mapped to $\mathcal{O}^{n-2}$.

Remark 3 (Right-angled Coxeter cubical complex). Recall that a cubical complex is a CW complex $X$ whose cells are cubes, with the property that for two cubes $c, c'$ of $X$,
\( c \cap c' \) is a common face of \( c \) and \( c' \); in other words, cubes are glued in \( X \) via combinatorial isometries of their faces. Similarly, a **right-angled Coxeter cubical complex** can be defined in the same way whose cells are all right-angled Coxeter cubical cells, that is, the orbits of standard reflections on an \( n \)-cube \([-1, 1]^n\). For example, the standard cubical decomposition of a simple polytope \( P \) (i.e., the cone of the barycentric subdivision of \( N(P) \)) gives a right-angled Coxeter cubical complex structure of \( P \). Of course, right-angled Coxeter cubical complexes form a special class of right-angled Coxeter cellular complexes.

**Proposition 2.1.** Each special simple handlebody has a finite right-angled Coxeter cellular complex structure.

**Proof.** Let \( Q \) be a special simple \( n \)-handlebody with the associated simple polytope \( P_Q \). Then the standard cubical subdivision of \( P_Q \) induces a right-angled Coxeter cellular decomposition of \( Q \). More details will be shown in section 3. \( \square \)

**Remark 4.** It should be pointed out that each special simple handlebody still has a right-angled Coxeter cubical complex structure. This can be seen in section 4.

In general, a right-angled Coxeter cellular complex is just an orbispace. Its orbifold fundamental group is defined by the homotopy classes of based orbifold loops. For more details, see [7, Section 3]. Although a right-angled Coxeter cell with non-trivial local group is not contractible in the sense of orbifold, all attaching maps \( \{ \phi_i \} \) preserving local groups ensures that the orbifold fundamental group of a right-angled Coxeter cellular complex is isomorphic to the orbifold fundamental group of its 2-skeleton.

**Proposition 2.2.** Let \( O \) be a right-angled Coxeter cellular complex. Then

\[
\pi_1^{\text{orb}}(O^2) \cong \pi_1^{\text{orb}}(O),
\]

where \( O^2 \) is the 2-skeleton of \( O \).

**Proof.** Proposition 2.2 can be proved in a similar way as shown by Hatcher [21, Proposition 1.26]. The only thing to note is that the local group information of each right-angled Coxeter \( n \)-cell can be inherited by the boundary orbifold of its closure in \( O^{n-1} \). \( \square \)

**Remark 5.** We can easily read out the generators and relations of \( \pi_1^{\text{orb}}(O) \cong \pi_1^{\text{orb}}(O^2) \) from the 2-skeleton of a right-angled Coxeter cellular complex \( O \). Let us look at a right-angled Coxeter 2-cell with non-trivial local group in \( O \). Assume that the boundary of a right-angled Coxeter 2-cell with non-trivial local group consists of \( x_1, x_2, \cdots, x_n \), where each \( x_i \) is a closed oriented orbifold loop in \( O \), and only one endpoint of \( x_1 \) and \( x_n \) has non-trivial local group. Regard these closed orbifold loops as generators. Then \( x_1^2 = x_n^2 = 1 \). Moreover, the right-angled Coxeter 2-cell with local group \( \mathbb{Z}_2 \) gives a relation \( x_1 x_2 \cdots x_n \cdot x_{n-1}^{-1} \cdots x_2^{-1} = 1 \), while the right-angled Coxeter 2-cell with local group \( \mathbb{Z}_2^2 \) gives a relation \( (x_1 x_2 \cdots x_n \cdot x_{n-1}^{-1} \cdots x_2^{-1})^2 = 1 \). This can intuitively be seen from Figure 3 when \( n = 3 \).

**Example 2.1.** Let \( P \) be a simple polytope with facet set \( F(P) \). Regard \( P \) as a right-angled Coxeter orbifold. The standard cubical subdivision of \( P \) is a right-angled Coxeter cellular decomposition of \( P \). Calculating the orbifold fundamental group of \( P \) by the 2-skeleton of its right-angled Coxeter cellular decomposition, \( \pi_1^{\text{orb}}(P) \) can be represented...
by the right-angled Coxeter group $W_P$ of $P$:

$$\pi_1^{orb}(P) \cong W_P = \langle s_F, F \in \mathcal{F}(P) | s_F^2 = 1, \text{ for all } F; (s_Fs_{F'})^2 = 1, \text{ for } F \cap F' \neq \emptyset \rangle$$

2.4. **Right-angled Coxeter group and HNN extension.** In this subsection, we refer to [10, Chapter 3] and [27, Chapter 4].

Let $w = s_1s_2 \cdots s_m$ be a word in a right-angled Coxeter group $W = \langle S | R \rangle$. An *elementary operation* on $w$ is one of the following two types of operations:

(i) Length-reducing: Delete a subword of $ss$;

(ii) Braid (commutation): Replace a subword of the form $st$ with $ts$, if $(st)^2 = 1$ in the relations set $R$ of $W$.

A word is *reduced* if it cannot be shorten by a sequence of elementary operations.

**Theorem 2.1** (Tits [10, Theorem 3.4.2]). Two reduced words $x, y$ are the same in a right-angled Coxeter group if and only if one can be transformed into the other by a sequence of elementary operations of type (ii).

**Definition 2.3** (Higman-Neumann-Neumann Extension [27, Page 179]). Let $G$ be a group with presentation $G = \langle S | R \rangle$, and let $\phi : A \to B$ be an isomorphism between two subgroups of $G$. Let $t$ be a new symbol out of $S$. Then the HNN extension of $G$ relative to $\phi$ is defined as

$$G*_{\phi} = \langle S, t | R, t^{-1}gt = \phi(g), g \in A \rangle$$

Let $\omega = g_0t^{\epsilon_1}g_1t^{\epsilon_2} \cdots g_{n-1}t^{\epsilon_n}g_n$ $(n \geq 0)$ be an expression in $G*_{\phi}$, where each $g_i$ is an element in $G$ (probably $g_i$ may be taken as the unit element 1 in $G$), and $\epsilon_i$ is either number $1$ or $-1$. Then $\omega$ is said to be *t-reduced* if there is no consecutive subword $t^{-1}g_it$ or $tg_it^{-1}$ with $g_i \in A$ and $g_j \in B$, respectively.

A *normal form* of an element in $G*_{\phi}$ is a word $\omega = g_0t^{\epsilon_1}g_1t^{\epsilon_2} \cdots g_{n-1}t^{\epsilon_n}g_n$ $(n \geq 0)$ where

(i) $g_0$ is an arbitrary element of $G$;

(ii) If $\epsilon_i = -1$, then $g_i$ is a representative of a coset of $A$ in $G$;

(iii) If $\epsilon_i = +1$, then $g_i$ is a representative of a coset of $B$ in $G$;
There is no consecutive subword $t^\epsilon t^{-\epsilon}$.

**Theorem 2.2** (The Normal Form Theorem for HNN Extensions, [27, Theorem 2.1, Page 182]). Let $G^*_\phi = \langle G, t \mid t^{-1}gt = \phi(g), g \in A \rangle$ be an HNN extension. Then there are two equivalent statements:

(I) The group $G$ is embedded in $G^*_\phi$ by the map $g \mapsto g$. If $\omega = g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n = 1$ in $G^*_\phi$, then $\omega$ is not reduced;

(II) Every element $\omega$ of $G^*_\phi$ has a unique representation $\omega = g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n$ which is a normal form.

A $t$-reduction of $\omega = g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n$ is one of the following two operations.

- replace a subword of the form $t^{-1}gt$, where $g \in A$, by $\phi(g)$;
- replace a subword of the form $tgt^{-1}$, where $g \in B$, by $\phi^{-1}(g)$.

A finite number of $t$-reductions leads from $\omega = g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n$ to a normal form.

2.5. **Hyperbolic geometry** (cf. [23, 28, 33, 37, 38]). A hyperbolic manifold of dimension $n$ is a complete Riemannian $n$-manifold of constant sectional curvature $-1$. The universal cover space of any closed hyperbolic $n$-manifold is isometric to the $n$-dimensional hyperbolic space $\mathbb{H}^n$. Thus any closed hyperbolic $n$-manifold can be realized as a quotient of the action of $\mathbb{H}^n$ by a torsion-free discrete subgroup of $\text{Isom}(\mathbb{H}^n)$. As a generalization of hyperbolic manifolds, an $n$-orbifold is hyperbolic if it is a quotient of $\mathbb{H}^n$ by a discrete subgroup (not necessarily free action) of $\text{Isom}(\mathbb{H}^n)$.

As a generalization of 3-dimensional hyperbolic polyhedra, a 3-manifold with corners is hyperbolic if its interior admit a hyperbolic metric which can extend to the boundary such that its all faces are totally geodesic (or locally convex). Moreover we say a 3-manifold with corners is right-angled hyperbolic if its all dihedral angles are $\frac{\pi}{2}$.

Notice that a hyperbolic 3-manifold with corners is not right-angled in general. As seen before, a 3-manifold with corners can be equipped with many different orbifold structures. A hyperbolic structure on a 3-manifold with corners should be compatible with an orbifold structure on it. Hence there may be different hyperbolic structures on a 3-manifold with corners. However, the hyperbolic structure of a hyperbolic closed 3-orbifold (or 3-manifold) is unique by Mostow Rigidity Theorem [29]. The hyperbolization of 3-manifold with corners corresponds to the generalization of Andreev Theorem. This question is still open now.

Here we mainly consider the right-angled hyperbolicity of simple 3-manifolds with corners, where a simple 3-manifold with corners is given by forgetting the orbifold structure on a simple 3-orbifold. Then one can obtain the same understanding for right-angled hyperbolicity from the following two geometric objects:

1. A right-angled hyperbolic simple 3-manifold with corners;
2. A hyperbolic simple 3-orbifold (as a right-angled Coxeter 3-orbifold).

Thus, a right-angled hyperbolic simple 3-manifold with corners is a hyperbolic simple 3-orbifold, and vice versa.

Together with Perelman’s work, Thurston’s Hyperbolization Theorem implies that a closed 3-manifold is hyperbolic with finite volume if and only if it is irreducible, atoroidal and $\pi_1$-infinite (see Davis [10, Page 105]). And a closed oriented 3-manifold
is irreducible and \( \pi_1 \)-infinite if and only if it is aspherical. This follows from the Sphere Theorem [22, Theorem 4.3]. Hence,

**Theorem 2.3** (Hyperbolization Theorem of closed oriented 3-manifolds). A closed oriented 3-manifold is hyperbolic if and only if it is aspherical and atoroidal.

In addition, Hyperbolization Theorem of 3-orbifolds also tells us that a closed atoroidal Haken 3-orbifold admits a geometrically finite hyperbolic structure, see [5, Theorem 6.5]. Here, there is an approach to consider the hyperbolicity on a simple 3-orbifold by making use of its manifold double.

**Proposition 2.3.** A simple 3-orbifold is hyperbolic if and only if its manifold double is hyperbolic.

**Proof.** Assume that \( Q \) is a simple 3-orbifold with \( m \) facets. Then there is an orbifold covering \( \pi : U(Q, (\mathbb{Z}_2)^m) \rightarrow Q \), which is a manifold double over \( Q \). Write \( M = U(Q, (\mathbb{Z}_2)^m) \). Furthermore, there is the following short exact sequence

\[
1 \rightarrow \pi_1(M) \rightarrow \pi_{1\text{ orb}}(Q) \rightarrow (\mathbb{Z}_2)^m \rightarrow 1.
\]

Now if \( Q \) admits a hyperbolic structure, then \( \pi_{1\text{ orb}}(Q) \) can be viewed as a subgroup of \( \text{Isom}(\mathbb{H}_n) \), so is \( \pi_1(M) \). Hence, that \( Q \) is hyperbolic implies that \( M \) is hyperbolic.

Conversely, suppose that \( M \) admits a hyperbolic structure. Denote the \((\mathbb{Z}_2)^m\)-action of \( m \) diffeomorphism involutions on \( M \) by \( \Phi \). Mostow rigidity theorem (or [17, Theorem H]) implies that the action \( \Phi \) on \( M \) is homotopic to an isometric action \( \Psi \) on \( M \) of the same group \((\mathbb{Z}_2)^m\). Two orbifolds \( M/\Phi \) and \( M/\Psi \) have isomorphic orbifold fundamental groups. By the generalized Johannson-Waldhausen homeomorphism theorem [23, Theorem 6.33], \( M/\Phi \) and \( M/\Psi \) are isomorphic as orbifolds. This gives a hyperbolic structure on \( Q \).

**Remark 6.** In the proof of Proposition 2.3, if \( M \) is a hyperbolic closed 3-manifold, without a loss of generality, one may assume that the action of \((\mathbb{Z}_2)^m\) on \( M \) is isometric. Then by [33, Theorem 13.1.1], \( \pi \) induces an isometry from \( B(x,r)/G_x \) onto \( B(\pi(x),r) \) for a small radius \( r \), where \( G_x \cong (\mathbb{Z}_2)^k \) is the isotropy group at \( x \), generated by reflections. Hence now each face of \( Q \) is totally geodesic, and all dihedral angles are right-angled.

### 3. Special Simple Handlebodies

Let \( Q \) be a simple \( n \)-handlebody, and \( \mathcal{N}(Q) \) be the nerve of \( Q \). Denote \( Q^* \) as the dual handlebody of \( Q \), whose facial structure is given by \( \mathcal{N}(Q) \).

#### 3.1. \( B \)-belts.

**Definition 3.1** (\( B \)-belts). Let \( i : B \hookrightarrow Q \) be an embedding closed simple \( k \)-suborbifold whose underlying space is a \( k \)-ball. We say that \( i(B) \) is an \( B \)-belt of \( Q \) if

- \( i \) preserves codimensions, i.e., \( i \) maps each codimension-\( d \) face \( f \) of \( B \) to a codimension-\( d \) face \( F_f \) of \( Q \);
- The intersection \( \cap f_\alpha = \emptyset \) for some facets \( f_\alpha \) in \( B \) if and only if either \( \cap F_{f_\alpha} = \emptyset \) or \( \cup F_{f_\alpha} \) cannot deformatively retract onto \( B \) in \( |Q| \).
Remark 7. The orbifold embedding $i : B \hookrightarrow Q$ preserving codimension is equivalent to that $i$ restricting on the local group of each point in $B$ induces an identity. The statement that $\bigcup F_{f_{\alpha}}$ cannot deformatively retract onto $B$ in $|Q|$ is equivalent to that there is at least a hole in the area surrounded by $\{ F_{f_{\alpha}} \}$ and $B$.

A simple polytope $P$ itself is a belt. A 2-dimensional $B$-belt in a simple 3-handlebody $Q$ is a $k$-gon. Traditionally, such $B$-belt is also called a $k$-belt of $Q$. In the case of dimension three, any simple 3-polytope except tetrahedron has a 2-dimensional $B$-belt.

A simple handlebody $Q$ is special if it is a simple polytope or there exist finitely many disjoint codimension-one $B$-belts, named cutting belts, such that $Q$ can be cut open into a simple polytope $P_Q$ along those cutting belts. Of course, each cutting belt must be a simple polytope in this case. Here the cutting operation is similar to a hierarchy of Haken 3-manifolds (or 3-orbifolds). Generally a simple 3-handlebody is not Haken except it is flag defined in Definition 3.2.

**Proposition 3.1.** Every simple 3-handlebody is special.

**Proof.** A simple 3-polytope with genus 0 is naturally special. Let $Q$ be a simple 3-handlebody with genus $g > 0$, and $\{ (D_i^2, \partial D_i^2) \hookrightarrow (|Q|, \partial |Q|) \ | i = 1, 2, \ldots, g \}$ be some disjoint compressing 2-disks in $|Q|$ such that $|Q|$ is cut into a connected 3-ball along those compressing 2-disks. Considering the facial structure determined by the triangulation $\mathcal{N}(Q)$ of $\partial |Q|$, we can always do some slight deformations for the boundaries of compressing 2-disks on faces of $Q$, so that $\{ D_i^2 \}$ can be modified into some embedded sub-orbifolds $\{ B_i \}$ of preserving codimension in $Q$. Each $B_i$ is a polygon.

Given a $B_i$, by the definition of $B$-belts, we see that $B_i$ is not an $B$-belt if and only if there must exist two non-adjacent edges $f_1$ and $f_2$ in $B_i$ such that

(i) $F_{f_1} \cap F_{f_2} \neq \emptyset$ (probably $F_{f_1}$ and $F_{f_2}$ can even be the same face of $Q$);

(ii) $F_{f_1} \cup F_{f_2}$ can deformatively retract onto $B_i$ in $|Q|$ (in fact, $F_{f_1} \cup F_{f_2}$ can deformatively retract onto $\partial B$ in $\partial |Q|$).

where $F_{f_1}$ and $F_{f_2}$ are two 2-faces of $Q$ that contain $f_1, f_2$ respectively. So, if $B_i$ is not an $B$-belt, then there is no any hole in the area $A$ in $\partial |Q|$ surrounded by $F_{f_1}, F_{f_2}$ and $B_i$. Then, we can modify the boundary of $B_i$ by pushing the retract of $F_{f_1} \cup F_{f_2}$ into $F_{f_1}, F_{f_2}$ and throwing some edges of $B_i$ away, as shown in [Figure 4] so that one can obtain

![Figure 4. Modifying the boundary of $B_i$.](image-url)
a new $B'_i$ with fewer edges which intersects transversely with $F_{f_1} \cap F_{f_2}$. In particular, if $F_{f_1} = F_{f_2}$, then $f_1$ and $f_2$ will become the same edge in $B'_i$, and if $F_{f_1} \neq F_{f_2}$ then $f_1$ is adjacent to $f_2$ in $B'_i$. In addition, if there is also another sub-orbifold $B_j$ which intersects with the area $A$ in $\partial |Q|$, this means that $B_j$ is not a belt, too. The above "pushing" process will move the boundary of $B_j$ out from the area $A$ and modify $B_j$ into $B'_j$ with fewer edges such that $B'_i \cap B'_j = \emptyset$. Since $B_i$ is a polygon with finite edges, this process can end after a finite number of steps until one has modified $B_i$ into an $B$-belt which does not intersect with other $B_j$.

We can perform the same procedure to other non $B$-belts in $\{B_j\}_{j \neq i}$. Finally one can obtain a set of disjoint cutting belts such that $Q$ is cut open into a simple 3-polytope along those cutting belts, implying that $Q$ is special.

**Definition 3.2.** A special simple handlebody is flag if it contains no $\triangle^k$-belt for any $k \geq 2$.

Recall that a simplicial complex $K$ with vertices set $V$, which is pairwise joined by edges, spans a simplex. Now let $Q$ be a special simple handlebody. We see that some vertices $F_1, F_2, \ldots, F_k$ of $\mathcal{N}(Q)$ span a simplex $\triangle^{k-1}$ in $\mathcal{N}(Q)$ if and only if the associated vertices span a simplex in $\mathcal{N}(P_Q)$, and they span an empty simplex (that is, $\partial \triangle^{k-1} \subset \mathcal{N}(Q)$ but $\triangle^{k-1}$ itself is not in $\mathcal{N}(Q)$) whose interior is contained in the interior of $Q^*$ if and only if associated vertices span an empty simplex in $\mathcal{N}(P_Q)$. Specifically, those empty simplices correspond to some $\triangle^k$-belts in $Q$. Hence, we have the following result.

**Lemma 3.1.** A special simple handlebody $Q$ is flag if and only if the associated simple polytope $P_Q$ is flag (in other words, $\mathcal{N}(P_Q)$ is a flag simplicial complex).

**Remark 8.** Notice that a flag simple handlebody defined above may contain an empty simplex whose interior cannot be embedded in $Q^*$, as shown in Figure 5 for three pairwise intersected faces $F_1, F_2, F_3$ in a flag simple solid torus. Therefore, the statement that $\mathcal{N}(Q)$ is a flag simplicial complex is not equivalent to that $Q$ is a flag simple handlebody.

![Figure 5](image.png)

**Figure 5.** A flag special simple 3-handlebody whose nerve is not a flag simplicial complex.

**Definition 3.3.** Let $Q$ be a flag simple handlebody, and $Q^*$ be its dual. By $\square$-belt we mean a quadrilateral-belt in $Q$. The dual of an $\square$-belt in $Q$ is said to be an $\square$ in $Q^*$. 

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Remark 9. (1) In Gromov’s paper [20, Section 4.2], Siebenmann’s no $\square$-condition for a flag simplicial complex $K$ means no empty square in $K$, where an empty square in $K$ must make sure that neither pair of opposite vertices is connected by an edge, which is a special case in our definition.

(2) When $\dim Q = 3$, an $\square$ in $Q^*$ corresponds to a 4-belt in $Q$. Under an additional condition that $Q$ admits a $(\mathbb{Z}_2)^3$-coloring, Li and Ma in [24] showed that $Q$ is right-angled hyperbolic if and only if each embedding disk in $Q$ intersects with at least 5 edges, except vertex-linking disk or edge-linking disk. So the later statement is equivalent to saying that $Q$ is flag and contains no $\square$-belt according to Theorem A.

(3) A prismatic 3-circuit [34] in a simple 3-polytope $P^3$ determines an $\Delta^2$-belt in $P^3$. If there is no prismatic 3-circuit in $P^3$, then $P^3$ is a flag polytope or a tetrahedron. Similarly for a prismatic 4-circuit [34] in a flag simple polytope, it determines an $\square$-belt in $P$ in our definition.

Let $Q$ be a flag simple handlebody, and $B_{\square}$ be an $\square$-belt in $Q$ with four ordered edges $f_1, f_2, f_3, f_4$, any two of which have a non-empty intersection except for pairs $\{f_1, f_3\}$ and $\{f_2, f_4\}$. Assume that each $f_i$ is contained in a facet $F_i$ of $Q$. Then we may claim that $\{F_i \mid i = 1, 2, 3, 4\}$ must be different from each other. More precisely, we have the following lemma.

**Lemma 3.2.** Let $Q$ be a flag simple handlebody, and $B_{\square}$ be an $\square$-belt in $Q$. Then,

- Two adjacent edges of $B_{\square}$ cannot be contained in the same facet of $Q$;
- Two disjoint edges of $B_{\square}$ cannot be contained in the same facet of $Q$.

**Proof.** Assume that the four edges $\{f_1, f_2, f_3, f_4\}$ of $B_{\square}$ are contained in four ordered facets $\{F_1, F_2, F_3, F_4\}$ of $Q$, respectively. If there are two adjacent edges of $B_{\square}$ contained in the same facet of $Q$. Without loss of generality, suppose that $F_1 = F_2$. Then $f_1 \cap f_2 \neq \emptyset$ implies that $F_1$ has a self-intersection, which is equivalent to that there is a 1-simplex which bounds a single vertex in $\mathcal{N}(Q)$. This contradicts that $Q$ is simple.

Similarly, if there are two disjoint edges of $B_{\square}$ contained in the same facet of $Q$, then one can assume that $F_1 = F_3$. This happens only for the case where the genus of $Q$ is more than zero since $B_{\square}$ is an $\square$-belt in $Q$. Thus there are some holes between $F_1$ and $B_{\square}$. However, $F_2$ is contractible, so this induces that $F_2 \cap F_1$ is disconnected. In other words, there are two 1-simplices which bound the same two vertices in $\mathcal{N}(Q)$. This is also impossible since $Q$ is simple. $\square$

**Lemma 3.2** tells us that in a flag simple handlebody $Q$, an $\square$-belt can be presented as four different vertices $\{F_1, F_2, F_3, F_4\}$ in $\mathcal{N}(Q)$, which satisfies the following two conditions:

(I) $\{F_1, F_2, F_3, F_4\}$ bounds a square with its interior located in the interior of $Q^*$ and with its edges contained in 1-skeleton of $\mathcal{N}(Q)$;

(II) The full subcomplex spanned by $\{F_1, F_2, F_3, F_4\}$ in $\mathcal{N}(Q)$ is either a square or a non-square subcomplex (containing two 2-simplices gluing along an edge).

Here the latter “a non-square subcomplex” may happen only when the genus of $Q$ is more than zero.

**Example 3.1** ($\square$s in the dual of a simple handlebody). Let $Q$ be a simple handlebody, and $Q^*$ be its dual. There are some possible cases of $\square$s and non-$\square$s in $Q^*$, listed in
Figure 6, where all vertices and edges are considered in $\mathcal{N}(Q)$. (a) and (b) are not □ in $Q^*$, while (c) and (d) are. Notice that (d) is not an empty square in $\mathcal{N}(Q)$, which is different from the case of Siebenmann’s no □-condition, as stated in Remark 9(1).

![Figure 6](image_url)

**FIGURE 6.** □s and non-□s

**Lemma 3.3.** Let $B^\square$ be an □-belt in a special simple $n$-handlebody $Q$, and $B$ be a cutting belt of $Q$. Then either $B^\square$ and $B$ can be separated in $Q$, or $B$ intersects transversely with only a pair of disjoint edges of $B^\square$.

**Proof.** Assume that the four ordered edges $f_1, f_2, f_3, f_4$ of $B^\square$ are contained in four facets $F_1, F_2, F_3, F_4$ of $Q$, respectively. Since $B^\square$ and $B$ are contractible, we see that $B$ and $B^\square$ can be separated if and only if their boundaries can be separated.

First we assume that $\partial B$ and $\partial B^\square$ intersect transversely, meaning that $\partial B \cap \partial B^\square$ is a set of isolated points cyclically ordered on the boundary of $B^\square$, which is denoted by $\mathcal{V}$. Then $\mathcal{V}$ contains at least two points if $\mathcal{V}$ is non-empty.

Let $v$ and $v'$ be two adjacent points in $\mathcal{V}$. Then there are the following cases:

(i) $v$ and $v'$ are located in the same edge of $B^\square$;
(ii) $v$ and $v'$ are located in two adjacent edges of $B^\square$;
(iii) $v$ and $v'$ are located in two disjoint edges of $B^\square$.

In the case (i), without loss of generality, suppose that $v, v' \in \text{int}(f_1)$. Now if $v$ and $v'$ are contained in the same connected component of $F_1 \cap B$ (without a loss of generality, assume that $B$ is regarded as $B_1$ of (a) in Figure 7), then we can deform the interior of $f_1$ such that $f_1 \cap \partial B = \emptyset$ will not contain $v$ and $v'$. If $v$ and $v'$ are contained in two connected components of $F_1 \cap B$, without loss of generality, assume that $B$ is regarded as $B_2$ of (a) in Figure 7. Since $B$ is an $B$-belt, there is a hole surrounded by $f_1$ and $B$. This case is allowed (also see (b) and (c) in Figure 7).

In the case (ii), without loss of generality, assume that $B$ intersects with $f_1$ and $f_2$. Now if $B \cap F_1 \cap F_2 \neq \emptyset$ (regard $B$ as $B_3$ of (a) in Figure 7), then we can move vertex $f_1 \cap f_2$ in $F_1 \cap F_2$ such that $\partial B^\square \cap \partial B$ does not contain $v$ and $v'$.
Repeating this operation, we can assume that any two adjacent points $v$ and $v'$ in $V$ cannot remove. This means that $B \cap F_1 \cap F_2 = \emptyset$ in the case (ii), so we may regard $B$ as $B_4$ of (a) in Figure 7. Then by the definition of $B$-belt, there is a hole in the area surrounded by $B, f_1, f_2$ (see (d) in Figure 7). If $|V| = 2$, then $B_\Box$ will not be contractible. This is a contradiction. If $|V| > 2$, let $v''$ be a point after $v'$ by the cyclic order of all isolated points in $V$. If $v'$ and $v''$ belong to the same edge $f$ of $B_\Box$, then there must be a hole surrounded by $f$ and $B$. If $v'$ and $v''$ belong to two adjacent edges $f'$ and $f''$ of $B_\Box$, then there is also a hole surrounded by $f', f''$ and $B$. If $v'$ and $v''$ belong to two disjoint edges $f'$ and $f''$ of $B_\Box$, then there is still a hole surrounded by $f, f''$ and $B$, where $f$ is the edge containing $v$. Whichever of all possible cases above happens implies that $\partial B_\Box$ is not contractible in $|Q|$, but this is impossible.

The case (iii) is allowed, see $B_5$ of (a) in Figure 7. So the conclusion holds. □

We see that if there are some cutting belts that intersect with $B_\Box$, then one can do some deformations such that those cutting belts either do not intersect with $B_\Box$ or intersect transversely with only a pair of disjoint edges of $B_\Box$. In 3-dimensional case, there always exist a set of cutting belts that separate from a fixed □-belt.

**Lemma 3.4.** Let $Q$ be a flag simple 3-handlebody, and $B_\Box$ be an □-belt in $Q$. Then there exist a set of cutting belts, each of which does not intersect with $B_\Box$.

**Proof.** By Lemma 3.3, assume that some cutting belts intersect transversely with two opposite edges $f_1$ and $f_3$ of $B_\Box$. We may push the boundaries of those cutting belts outside of $B_\Box$, so that those modified cutting belts do not intersect with $B_\Box$. However, those modified cutting belts may not be $B$-belts. Of course, they are 2-suborbifolds. See Figure 8. Using the approach used in the proof of Proposition 3.1, we further deform those 2-suborbifolds into required cutting belts. □

**3.2. The right-angled Coxeter cellular decomposition of special simple handlebodies.** Let $Q$ be a special simple $n$-handlebody of genus $g$ with facet set $\mathcal{F}(Q) = \{F_1, ..., F_m\}$. Then we can cut $Q$ into a simple polytope $P_Q$ along $g$ cutting belts $B_1, ..., B_g$, each of which intersects transversely with some facets of $Q$ and is a simple $(n - 1)$-polytope.
Two copies of $B_i$ in $P_Q$, denoted by $B^+_i$ and $B^-_i$, respectively, are two disjoint facets of $P_Q$. Since they share the common belt $B_i$ in $Q$, by $B^+_i \sim B^-_i$ we denote this share between them. The number of facets of $P_Q$ around $B^+_i$ is the same as the number of facets of $P_Q$ around $B^-_i$. In addition, each facet $F$ of $P_Q$ around $B^+_i$ also uniquely corresponds to a facet $F'$ of $P_Q$ around $B^-_i$ such that $F$ and $F'$ share a common facet in $Q$, so by $F \cap B^+_i \sim F' \cap B^-_i$ we mean this share between $F$ and $F'$ via the belt $B_i$ of $Q$.

Let $F(P_Q)$ denote the set of all facets in $P_Q$ and $F_B$ denote the set of those facets in $P_Q$, produced by cutting belts of $Q$, so $F_B$ contains $2g$ facets of $P_Q$, appearing in pairs.

$P_Q$ is viewed as a right-angled Coxeter orbifold with boundary consisting of all facets in $F_B$. By attaching all pairs $B^+_i \sim B^-_i$ in $F_B$ and all corresponding pairs $(F, F')$ with $F \cap B^+_i \sim F' \cap B^-_i$ together, we can recover $Q$ from $P_Q$. Thus $Q$ can be regarded as a quotient $P_Q/\sim$, and we denote the quotient map by

$$q : P_Q \to Q.$$ (3.1)

There is a canonical right-angled Coxeter cellular decomposition $C(P_Q)$ of $P_Q$, whose cells consist of

- all cubes in the standard cubical decomposition of $P_Q$;
- all cubes in the standard cubical decomposition of all boundary components of $P_Q$ in $F_B$.

Moreover, $C(P_Q)$ induces a right-angled Coxeter cellular decomposition on $Q$ by attaching some cubical cells of the copies of $B$-belts. Let $c$ be a $k$-cube in $C(P_Q)$ and $B \in F_B$.

- If $c \cap B = \emptyset$, then we may take $c$ as a right-angled Coxeter cubical cell for $Q$. Such $c$ corresponds to a codimension $k$ face in $P_Q$ which is determined by $k$ facets in $F(P_Q) - F_B$, so $c$ is of the form $e^k/(\mathbb{Z}_2)^k$.
- If $c$ is a $k$-cube in $C(B^+) \subset C(P_Q)$, then there is also another $k$-cube $c' \in C(B^-) \subset C(P_Q)$. Both $c$ and $c'$ are codimension-one faces of two $(k + 1)$-cubes in $C(P_Q)$, respectively. Gluing those two $(k + 1)$-cubes by identifying $c$ with $c'$, we obtain a right-angled Coxeter cubical cell with form $e^{k+1}/(\mathbb{Z}_2)^k$.

Finally, we obtain a right-angled Coxeter cellular decomposition of $Q$, denoted by $C(Q)$, whose cells are right-angled cubes. Of particular note is that $C(Q)$ is not cubical. This is because there exists the cubical cell glued by two cells $c$ and $c'$ in $C(P_Q)$ as above, which has a self-intersection, namely the cone point $x_0$, as shown in Figure 9. The cone
point is the only 0-cell in $\mathcal{C}(Q)$, which will be chosen as the basepoint when we calculate the orbifold fundamental group $\pi_1^{\text{orb}}(Q)$ of $Q$.

![Diagram of an orbifold with a cutting belt](image)

**Figure 9.** The right-angled Coxeter 2-cell nearby $B$-belt.

3.3. **The orbifold fundamental groups of special simple handlebodies.** Following the above notations, by Proposition 2.2 we can directly write out a presentation of orbifold fundamental group of $Q$.

**Proposition 3.2.** Let $Q$ be a special simple handlebody of genus $g$, and $P_Q$ be the associated simple polytope with copies of cutting belts $\mathcal{F}_B$. Then $\pi_1^{\text{orb}}(Q)$ has a presentation with generators $s_F$ indexed by $F \in \mathcal{F}(P_Q)$, satisfying the following relations:

1. $s_F^2 = 1$ for $F \in \mathcal{F}(P_Q) - \mathcal{F}_B$;
2. $t_{B^+}t_{B^-} = 1$ for two $B^+$ and $B^-$ with $B^+ \sim B^-$ in $\mathcal{F}_B$;
3. $(s_Fs_{F'})^2 = 1$ for $F, F' \in \mathcal{F}(P_Q) - \mathcal{F}_B$ with $F \cap F' \neq \emptyset$;
4. $s_Ft_{B^+} = t_{B^+}s_F$ for $B^+ \sim B^-$ in $\mathcal{F}_B$ and $F, F' \in \mathcal{F}(P_Q) - \mathcal{F}_B$ with $F \cap B^+ \sim F' \cap B^-$ where the basepoint of $\pi_1^{\text{orb}}(Q)$ is the cone point $x_0$ in the interior of $Q$.

On the other hand, we show here that $\pi_1^{\text{orb}}(Q)$ is actually an iterative HNN-extension on $W(P_Q, \mathcal{F}_B)$, where $W(P_Q, \mathcal{F}_B)$ is a right-angled Coxeter group determined by facial structure of $P_Q$ by ignoring the facets of $\mathcal{F}_B$:

$$W(P_Q, \mathcal{F}_B) = \langle s_F, \forall F \in \mathcal{F}(P_Q) - \mathcal{F}_B \mid s_F^2 = 1, \forall F \in \mathcal{F}(P_Q) - \mathcal{F}_B; \quad (s_Fs_{F'})^2 = 1, \forall F, F' \in \mathcal{F}(P_Q) - \mathcal{F}(B), F \cap F' \neq \emptyset \rangle$$

which can be regarded as the orbifold fundamental group $\pi_1^{\text{orb}}(P_Q)$ of $P_Q$ as a right-angled Coxeter orbifold with boundary consisting of the disjoint union of all facets in $\mathcal{F}_B$.

Let $B$ be a cutting belt in $Q$, and $B^+, B^- \in \mathcal{F}_B$ are two copies of $B$, denoted $\mathcal{F}^{B^+} = \{F \in \mathcal{F}(P_Q) - \mathcal{F}_B | F \cap B^+ \neq \emptyset \}$, and $\mathcal{F}^{B^-} = \{F \in \mathcal{F}(P_Q) - \mathcal{F}_B | F \cap B^- \neq \emptyset \}$. The associated right-angled Coxeter group $W_{B^+}$ and $W_{B^-}$ are two isomorphic groups since $B^+$ and $B^-$ are homeomorphic as simple polytopes.

**Lemma 3.5.** The maps $i_{B^+} : W_{B^+} \to W(P_Q, \mathcal{F}_B)$ and $i_{B^-} : W_{B^-} \to W(P_Q, \mathcal{F}_B)$ induced by inclusions $i_{B^+} : B^+ \hookrightarrow P_Q$ and $i_{B^-} : B^- \hookrightarrow P_Q$ are monomorphisms.

**Proof.** According to the definition of $B$-belt, $i_{B^+}$ and $i_{B^-}$ are obviously well-defined. There are two group homomorphisms $j_{B^+} : W(P_Q, \mathcal{F}_B) \to W_{B^+}$ and $j_{B^-} : W(P_Q, \mathcal{F}_B) \to W_{B^-}$ which module the normal subgroups generated by facets not in $\mathcal{F}^{B^+}$ and $\mathcal{F}^{B^-}$, such that $j_{B^+} \circ i_{B^+} = id_{W_{B^+}}$ and $j_{B^-} \circ i_{B^-} = id_{W_{B^-}}$. The result follows from this.  \[\square\]
Hence, $W_{B^+}$ and $W_{B^-}$ can also be regarded as two isomorphic subgroups of $W(P_Q, \mathcal{F}_B)$ generated by $s_F, F \in \mathcal{F}_B^+$ and $s_{F'}, F' \in \mathcal{F}_B^-$, respectively. Define $\phi_B : W_{B^-} \to W_{B^+}$ by $\phi_B(s_{F'}) = s_F$ with $F' \cap B^- \sim F \cap B^+$. Then $\phi_B$ is a well-defined isomorphism. Furthermore, attaching two facets on $P_Q$ corresponding to the belt $B$ is equivalent to doing once HNN-extension on its orbifold fundamental group, giving new elements $t_{B^+}, t_{B^-}$ with certain conditions in $\pi_1^{orb}(Q)$. By doing an induction on the genus of $Q$ and repeating the use of the normal form theorem of HNN-extension (Theorem 2.2), the orbifold fundamental group of $Q$ is isomorphic to $g$ times HNN-extensions on the right-angled Coxeter group $W(P_Q, \mathcal{F}_B)$, as shown below:

\[
(Q_g, B_g) \to \cdots \to (Q_1, B_1) \to Q_0 = P_Q
\]

\[
\begin{array}{c}
\text{Cutting} \\
\text{HNN extension}
\end{array}
\]

\[
G_g = \pi_1^{orb}(Q) \leftarrow \cdots \leftarrow G_1 \leftarrow G_0 = W(P_Q, \mathcal{F}_B)
\]

where each $Q_k$ is the simple handlebody of genus $k$ obtained from $Q_{k+1}$ by cutting open along the $(k+1)$-th belt $B_{k+1}$, which is a right-angled Coxeter orbifold with boundary consisting of double copies of $\{B_{k+1}, \ldots, B_g\}$, and each $G_k$ is the orbifold fundamental group of $Q_k$ which is obtained from an HNN extension on $G_{k-1}$.

**Proposition 3.3.** Let $Q$ be a special simple handlebody of genus $g$ with cutting belts $B_1, \ldots, B_g$. Then $\pi_1^{orb}(Q) \cong (\cdots ((W(P_Q, \mathcal{F}_B)^{\phi_{B_1}})^{\phi_{B_2}}) \cdots)^{\phi_{B_g}}$.

Notice that the expression $(\cdots ((W(P_Q, \mathcal{F}_B)^{\phi_{B_1}})^{\phi_{B_2}}) \cdots)^{\phi_{B_g}}$ in Proposition 3.3 is independent of orders of $\phi_{B_i}$. In addition, the presentation of $\pi_1^{orb}(Q)$ in Proposition 3.2 can be simplified by deleting all generators $t_{B^-}$ and relations $t_{B^+}t_{B^-} = 1$, meanwhile, replaced by only all $t_B$. Here the group $\pi_1^{orb}(Q)$ is called a handlebody group. It should be pointed out that the right-angled Coxeter group $W_Q$ determined by facial structure of $Q$ is not a subgroup of $\pi_1^{orb}(Q)$ in general. Actually, $W_Q$ is the quotient group of $\pi_1^{orb}(Q)$ with respect to the normal group generated by all $t_B$.

**Remark 10.** In [15, Theorem 4.7.2], Davis, Januszkiewicz and Scott give a similar form. However, all generators in their paper lifted into the universal space as homeomorphisms onto itself are involutions, i.e., $i_B^2 = 1$. Here, with a little difference, we require that the lifted action of $t_B$ is free. In particular, the last relation in Proposition 3.2 belongs to a kind of Baumslag-Solitar relations, which are related to the HNN-extension; in other words, pasting pairs of facets corresponding to cutting belts of the polytope $P_Q$ can be viewed as a topological explanation for the HNN-extension of their orbifold fundamental groups. More precisely, for a cutting belt $B$, there are two copies $B^+$ and $B^-$ in $P_Q$, and the composite map

\[
W_B \cong W_{B^+} \xrightarrow{i_{B^+}} W(P_Q, \mathcal{F}_B) \xrightarrow{i_1} G_1 \xrightarrow{i_2} \cdots \xrightarrow{i_g} G_g = \pi_1^{orb}(Q)
\]
embeds $W_B$ into $\pi_1^{orb}(Q)$, where $i_k$ is defined by $i_k(h) = h \in G_k$ for $h \in G_{k-1}$. $W_{B+}$ and $W_{B-}$ are linked in $\pi_1^{orb}(Q)$ by an isomorphism and the injectivity of $i_k$ is followed by the normal form theorem of HNN-extension (Theorem 2.2).

4. Proof of Proposition 1.1

A geodesic metric space $X$ is non-positively curved if it is a locally CAT(0) space. The Cartan-Hadamard theorem implies that non-positively curved spaces are aspherical. Cf [3, 10, 20].

**Definition 4.1** (The links in a cubical complex [3, Subsection 7.15] or [10, Page 508]). Let $K$ be a cubical complex. For each vertex $v \in K$, its (geometric) link, denoted by $\text{Lk}(v)$, is a simplicial complex defined by all cubes in $K$ that properly contains $v$ with respect to the conclusion. A $d$-cube $c$ of $K$ that properly contains $v$ determines a $(d-1)$-simplex $s(c)$ in $\text{Lk}(v)$.

**Proposition 4.1** (Gromov [10] Corollary I.6.3). A piecewise Euclidean cubical complex is nonpositively curved if and only if the link of its each vertex is a flag complex.

Let $Q$ be a special simple handlebody of dimension $n \geq 3$ and genus $g$, and $M \rightarrow Q$ be the manifold double over $Q$, as defined in [11]. Let $P_Q$ be the simple polytope obtained from $Q$ by cutting open along $g$ disjoint cutting belts $B_1, ..., B_g$ in $Q$. More precisely, $P_Q$ can be obtained as follows: For each belt $B_i$, choose a regular neighborhood $N(B_i)$ of $B_i$ that is homeomorphic to $B_i \times [-1, 1]$ as manifolds with corners. Clearly $N(B_i)$ is identified with a simple polytope, and it can also be understood as the disk $D^1$-bundle of the trivial normal bundle of $B_i$ in $Q$. Then we get $P_Q$ by removing the interiors of trivial $D^1$-bundles $B_i \times [-1, 1]$ of all $B_i$.

In order to use Gromov Lemma as above, we need a cubical cellular structure of the manifold double $M$ over $Q$. For this, we perform the following procedure:

1. First we decompose $Q$ into more pieces

$$Q = P_Q \bigcup_{i=1}^{g} N^+(B_i) \cup N^-(B_i)$$

where $N^+(B_i) = B_i \times [0, 1]$ and $N^-(B_i) = B_i \times [-1, 0]$ satisfy $N(B_i) = N^+(B_i) \cup N^-(B_i)$.

2. Next, the standard cubical decompositions of $P_Q$ and all $N^\pm(B_i)$ determine a right-angled Coxeter cubical cellular decomposition of $Q$, denoted by $\mathcal{C}(Q)$. Specifically, all cone points of $P_Q$ and all $N^\pm(B_i)$ will be 0-cells with trivial local group in $\mathcal{C}(Q)$. There are two kinds of $k(>0)$-cubes in the cubical decompositions of $P_Q$ and all $N^\pm(B_i)$, each of which either intersects transversely with an $(n-k)$-face $f^k_i = F_{i1} \cap \cdots \cap F_{ik}$ or intersects transversely with an $(n-k)$-face $f^k = F_{i1} \cap \cdots \cap F_{ik-1} \cap B_i$. The first type of cubes determine right-angled Coxeter cubical cells of the form $e^k/\mathbb{Z}^k_2$ in $\mathcal{C}(Q)$, and the second type of cubes determine right-angled Coxeter cubical cells of the form $e^k/\mathbb{Z}^{k-1}_2$. Then, $\mathcal{C}(Q)$ is obtained by attaching each pair associated with $B_i$ of the second type of cubes together. It is clear that $\mathcal{C}(Q)$ is a right-angled Coxeter cubical cellular decomposition of $Q$. 

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(3) Finally, by pulling back $C(Q)$ to $M$ via the covering map $p : M \rightarrow Q$, one can obtain a cubical cellular decomposition of $M$, denoted by $C(M)$, such that each cube in $C(M)$ is a connected component of $p^{-1}(c)$ for $c$ in $C(Q)$. In particular, all vertices in $C(M)$ exactly consist of the lifting of cone points in $C(Q)$.

**Lemma 4.1.** Let $v$ be a vertex in $C(M)$. Then $Lk(v)$ in $C(M)$ is combinatorially isomorphic to one of nerves $\mathcal{N}(P_Q)$, $\mathcal{N}(N^+(B_i))$ and $\mathcal{N}(N^-(B_i))$.

*Proof.* In fact, if $p(v)$ is the cone point of $P_Q$, then each $k(>0)$-cube adjacent to $v$ gives a $(k-1)$-simplex in $\mathcal{N}(P_Q)$, which corresponds to an $(n-k)$-face of $P_Q$. Therefore, $Lk(v) \cong \mathcal{N}(P_Q)$. A same argument can be applied to the case where $p(v)$ is the cone point of $N^+(B_i)$ or $N^-(B_i)$.

Therefore we have that

**Proposition 4.2.** Let $Q$ be a special simple handlebody of dimension $n \geq 3$, and $M$ be the manifold double over $Q$. Then the following conditions are equivalent,

1. $M$ is aspherical;
2. $M$ is non-positively curved;
3. $Q$ is flag.

*Proof.* Gromov’s Lemma ([Proposition 4.1](#)) tells us that $M$ is non-positively curved if and only if the link of each vertex in the cubical cellular decomposition of $M$ is flag. By [Lemma 4.1](#), the latter of the above statement means that $\mathcal{N}(P_Q)$ and all $\mathcal{N}(N^\pm(B_i))$ are flag, so $P_Q$ and all $N^\pm(B_i)$ are flag simple polytopes. This is also equivalent to saying that $Q$ is flag. Thus, a special simple handlebody $Q$ is flag if and only if its manifold double $M$ is non-positively curved. This proves (2) $\iff$ (3).

It follows (2) $\Rightarrow$ (1) by Cartan-Hadamard Theorem.

It remains to prove (1) $\Rightarrow$ (3), saying that $Q$ is flag if $M$ is aspherical. Assume that $q : P_Q \rightarrow Q$ is the quotient map by gluing all paired facets in $\mathcal{F}_B$. Using $\pi^\text{orb}_1(Q)$ with the presentation in [Proposition 3.2](#) and [Lemma A.4](#), the universal cover $\tilde{Q}$ of $Q$ can be defined as follows:

\[(4.1) \quad \tilde{Q} = P_Q \times \pi^\text{orb}_1(Q)/\sim\]

where $(x, g) \sim (y, h)$ if and only if

\[(4.2) \quad \begin{cases} 
    x = y \in F \in \mathcal{F}(P) - \mathcal{F}_B, gs_F = h, \\
    (x, y) \in (B^+, B^-), B^+, B^- \in \mathcal{F}_B, q(x) = q(y), t_B \cdot g = h. 
\end{cases}\]

If $M$ is aspherical, then $\tilde{Q}$ is contractible. Using an idea of Davis in [13](#) Subsection 8.2, we shall show that if $P_Q$ is not flag then $\tilde{Q}$ is not contractible. Indeed, if $P_Q$ is not flag, then $\mathcal{N}(P_Q)$ contains an empty $k$-simplex for $k \geq 2$. The dual of this empty $k$-simplex gives an essential embedding sphere in $\tilde{Q}$. Then the fundamental class of such a sphere is nontrivial in $H_k(\tilde{Q})$, which contradicts that $\tilde{Q}$ is contractible. See [Theorem B](#) in [Appendix A](#) for the calculation of the homology groups of $\tilde{Q}$.

In the case of dimension 3, together with [Proposition 3.1](#) we have the following result.

**Corollary 4.1.** A simple 3-handlebody is flag if and only if its manifold double is aspherical.
5. Proof of Proposition 1.2

In Gromov’s paper [20] (see also Davis [10, Proposition I.6.8]), let $X$ be a finite cubical complex satisfying that the link of each cube in $X$ is flag and contains no Siebenmann’s □, and let us give $X$ a suitable piecewise hyperbolic structure. Then $X$ admits a strict negative curvature in the sense of Alexandrov [3] Definition 2.1 in Chapter II.1. However, we cannot use Gromov’s result directly, since it may produce a new □ when we cut $Q$ along a cutting belt $B$. Let us look at the cubical decomposition of $M$ constructed in section 4. It is obvious that the nerve of $N^+(B)$ or $N^-(B)$ has an □ as long as $B$ is flag, which is a link of some vertex in the cubical decomposition of $M$. In fact, such an □ cannot make sure that there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ in general.

The main purpose of this section is to characterize the rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{orb}(Q)$ in terms of an □-belt in $Q$.

**Proposition 5.1** (□-conditions for handlebody groups). Suppose that $Q$ is a special simple $n$-handlebody. Then there is a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{orb}(Q)$ if and only if $Q$ contains an □-belt.

**Remark 11.** The “simple” condition of a handlebody is necessary in above proposition. In fact, it is easy to see that the orbifold fundamental group of a two-dimensional annulus as a right-angled Coxeter orbifold is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}_2 \ast \mathbb{Z}_2)$, which contains a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$. Consider a right-angled Coxeter 3-handlebody $Q$ with an $\pi_1$-injective annulus-suborbifold $B$ such that $B$ is a $\pi_1$-injective suborbifold, it provides a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in its orbifold fundamental group. Of course, such $Q$ is not simple. All of these results are the generalization of [3, Lemma 5.22] which is related to the Flat Torus Theorem in [3, Chapter II.7].

**Example 5.1** (□’s of Example 3.1). We show that each □ in (c) and (d) of Example 3.1 determines a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{orb}(Q)$, whereas the cases of (a) and (b) do not so.

On (a), the four facets $F_1, F_2, F_3, F_4$ correspond to a suborbifold $B$ which is a quadrilateral in $Q^\ast$, but it is not an □-belt in $Q$. In fact,

$$i_\ast(\pi_1^{orb}(B)) \cong W_\square / \langle (s_1s_3) \rangle \cong (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2 \ast \mathbb{Z}_2) < \pi_1^{orb}(Q)$$

and $s_1s_3, s_2s_4$ generate a subgroup $\mathbb{Z}_2 \oplus \mathbb{Z} in i_\ast(\pi_1^{orb}(B)) < \pi_1^{orb}(Q)$, where $i_\ast : \pi_1^{orb}(B) \rightarrow \pi_1^{orb}(Q)$ is induced by the inclusion $i : B \hookrightarrow Q$. Thus, there is no subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $i_\ast(\pi_1^{orb}(B))$.

On (b), $\{F_1, F_2, F_3, F_4\}$ does not determine a quadrilateral sub-orbifold. Without loss of generality, assume that $\{F_1, F_2, F_3, F_4\}$ bounds only one hole of $Q^\ast$. Then there are at least 5 generators in $\pi_1^{orb}(Q)$ associated to five facets in $P_Q$, denoted by $\{F_1, F_2, F_3, F_4, F'_1\}$ with $F_1 \cap B^+ \sim F'_1 \cap B^-$, where $B$ is the cutting belt of $Q$ and cut $F_1$ into two facets in $P_Q$. Thus, (b) induces a subgroup of $\pi_1^{orb}(Q)$ as follows:

$$W_b := \langle (s_1s_2s_3s'_1t) | (s_1) = 1 \rangle \cong \{s_1s_2s_3s'_1t \mid (s_1s_2s_3s'_1t) = 1\}$$

which contains no subgroup $\mathbb{Z} \oplus \mathbb{Z}$.

On (c) or (d), $\{F_1, F_2, F_3, F_4\}$ determines an □-belt $B_\square$ of $Q$. If $B_\square$ does not intersect with any cutting belt, then $B_\square$ is kept in $P_Q$, so there is a subgroup $\mathbb{Z} \oplus \mathbb{Z} < \pi_1^{orb}(Q)$, but □ cannot make sure that there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ in general.
If there are some cutting belts $B_1, B_2, \ldots, B_k$ intersecting transversely with only a pair of disjoint edges of $B_□$, without loss of generality, assume that $B_1, B_2, \ldots, B_k$ intersect with two disjoint edges $f_1$ and $f_3$ of $B_□$, where some cutting belts may cut $f_1$ and $f_3$ many times, see (c) in Figure 7. Then there is also a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ generated by $s_1s_3$ and $s_2t_1t_2 \cdots t_ks_i^{-1}t_{i+1}^{-1} \cdots t_2^{-1}t_1^{-1}$ where each $t_i$ is one of $\{t_{B_i}^{\pm 1}\}$. Also see the following figure.

**Figure 10. □-belt and $\mathbb{Z} \oplus \mathbb{Z}$.**

5.1. **The special case where $Q$ is a simple polytope.** First let us prove Proposition 5.1 when $Q$ is a simple polytope. This case can be followed by Moussong’s result [30] (see also [10, Corollary 12.6.3] in details). Here we give an alternative proof as follows.

Let $W = \langle S | R \rangle$ be the right-angled Coxeter group associated with a simple polytope $P$. We are going to show that there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $W$ if and only if $P$ contains an □-belt.

Assume that $w = s_1 \cdots s_m$ is a reduced word of length $m$ in $W$, and $t$ is a generator in $S$.

- If the length of $wt$ equals $m - 1$ after a sequence of elementary operations, then we call $t$ **DIE** in $w$, i.e., there is a $s_i = t$ such that $(s_jt)^2 = 1$ ($s_j$ is a $s_j$ with $j > i$);
- If $wt = tw$ is reduced, then we call $t$ **SUCCESS** for $w$, meaning that $t$ can commute with all $s_i$;
- If $wt$ is reduced and $wt \neq tw$, then we call $t$ **FAIL** for $w$; in other words, there is a $s_i$ in $w$ such that $(s_it)^2 \neq 1$ and $(s_jt)^2 = 1$.

**Proof of Proposition 5.1 for $Q$ to be a simple polytope $P$.** Let $W_□$ be the right-angled Coxeter group determined by a quadrilateral. Then two pairs of disjoint edges of □ provide two elements $s_1s_3$ and $s_2s_4$ which generate a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $W_□ < W_P$. Suppose that there is subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $W_P$. Then we need to find a required □-belt in $P$. We proceed as follows.

**Claim-1** There are two generators $x = s_1 \cdots s_m$ and $y = t_1 \cdots t_n$ of $\mathbb{Z} \oplus \mathbb{Z}$ such that all $t_i$ commute with all $s_j$. 


Assume that $x = s_1 \cdots s_n$ and $y = t_1 \cdots t_n$ are arbitrary two reduced expressions, which generate a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $W$. Then $xy = yx$, giving that

$$s_1 \cdots s_m \cdot t_1 \cdots t_n = t_1 \cdots t_n \cdot s_1 \cdots s_m$$

where, without loss of generality, assume $m \geq n$. Tits’ theorem (Theorem 2.1) tells us that the word $w = s_1 \cdots s_m \cdot t_1 \cdots t_n$ can be turned into $t_1 \cdots t_n \cdot s_1 \cdots s_m$ by a series of elementary operations. We perform an induction with $xt_1$ as a starting point. On $xt_1$, there are the following three cases:

(A) $t_1$ is DIE at $s_k$ in $x$. Then we may write $x = t_1 s'_2 \cdots s'_{m-1} t_1$ and $y = t_1 t'_2 \cdots t'_{n-1} t_1$. So we can take two shorter words $x' = s'_2 \cdots s'_{m-1}$ and $y' = t'_2 \cdots t'_{n-1}$ as generators of $\mathbb{Z} \oplus \mathbb{Z}$ as well. This return back to the starting of our argument with two words with shorter word lengths.

(B) $t_1$ is FAIL in $x$. Then we can take $s_1 = t_1$, so $x = t_1 s_2 \cdots s_m$.

(C) $t_1$ is SUCCESS. Then $t_1$ commutates with all $s_i$.

Consider $xt_2$ in the case (B), if $t_2$ is DIE, then $(t_1 t_2)^2 = 1$, and one may write $x = t_1 t_3 s'_3 \cdots s'_{m-1} t_2$ and $y = t_1 t'_3 \cdots t'_{n-1} t_2$. In a similar way to the case (A), set $t_1 s'_3 \cdots s'_{m-1}$ and $t_1 t'_3 \cdots t'_{n-1}$ as new generators of $\mathbb{Z} \oplus \mathbb{Z}$. If $t_2$ is FAIL, then one may write $x = t_1 t_2 s_3 \cdots s_m$. If $t_2$ is SUCCESS, then $t_2$ commutates with all $s_i$ (including $s_1 = t_1$).

Consider $xt_2$ in the case (C), if $t_2$ is DIE, then one can take $s_1 = t_2$, so $x = t_2 s_2 \cdots s_m$ and $(t_1 t_2)^2 = 1$. Moreover, exchanging $t_1$ and $t_2$ in $y$ returns to the case (A), so we can take two shorter words as generators of $\mathbb{Z} \oplus \mathbb{Z}$. If $t_2$ is FAIL, then $x = t_2 s_2 \cdots s_m$. Otherwise, $t_2$ is SUCCESS, too.

The above procedure can always be carried out by inductive hypothesis. We can end this procedure after finite steps of elementary operations until we have obtained a complete analysis for all $t_i$. Actually, each $t_i$ is either FAIL or SUCCESS for the final $x$ and $y$. There are only three possibilities as follows:

(i) All $t_i$ are FAIL. In this case, we may write $x = t_1 t_2 \cdots t_n s_{n+1} \cdots s_m = y s_{n+1} \cdots s_m$, so that we can take $y^{-1} x$ and $y$ as new generators of $\mathbb{Z} \oplus \mathbb{Z}$. Of course, all $t_i$ can commutate with all $s_j$, as desired.

(ii) Some $t_i$’s are FAIL. In this case, we may write $x = t_{i_1} \cdots t_{i_k} s_{k+1} \cdots s_m$ such that each of those $t_{j_a} \neq t_{i_1}, \ldots, t_{i_k}$ commutates with all $s_i$ and $t_{i_1}, \ldots, t_{i_k}$. So one may write $x = t_1 \cdots t_n \cdot t_{j_1} \cdots t_{j_{k-1}} \cdot s_{k+1} \cdots s_m$. Then $y^{-1} x$ removes those FAIL $t_i$’s in $x$. Furthermore, $y^{-1} x$ and $y$ can be chosen as new generators of $\mathbb{Z} \oplus \mathbb{Z}$ as desired.

(iii) All $t_i$ are SUCCESS. In this case, $x$ and $y$ are naturally the required generators of Claim-1.

Thus we finish the proof of Claim-1.

Now choose two generators $x$ and $y$ of $\mathbb{Z} \oplus \mathbb{Z}$ which satisfy the property in Claim-1.

Claim-2: There are two letters $s, s’$ in $x$ and two letters $t, t’$ in $y$, which correspond to four facets of $P$, denoted by $F_s, F_{s’}, F_t, F_{t’}$, that form an $\Box$ in $P^*$. Since $x$ is free, there must exist two letters $s, s’$ in $x$ such that $F_s \cap F_{s’} = \emptyset$. Similarly, there also exist two letters $t, t’$ in $y$ such that $F_t \cap F_{t’} = \emptyset$. If $\{s, s’\} \cap \{t, t’\} = \emptyset$, since $t, t’$ commute with $s, s’$, then clearly $F_s, F_{s’}, F_t, F_{t’}$ determine an $\Box$-belt in $P$. Otherwise, $\{s, s’\} \cap \{t, t’\} \neq \emptyset$. Assume that $s = t$, then $(tt’)^2 = (st’)^2 = 1$. This gives a contradiction since $y$ is a reduced word.
Together with the above arguments, this completes the proof. □

Next let us deal with the case of a simple handlebody. Let \( Q \) be a special simple handlebody of genus \( g \), and \( P_Q \) be the associated simple polytope obtained by cutting \( Q \) open along cutting belts \( \{B_i, i = 1, 2, \cdots, g\} \).

5.2. Proof of the sufficiency of Proposition 5.1. Assume that there is an \( \square \)-belt \( B_\square \) given by \( \{F_1, F_2, F_3, F_4\} \) in \( N(Q) \). After cutting \( Q \) open along cutting belts \( B_i, i = 1, 2, \cdots, g \), by Lemma 3.3, there are the following two cases.

- The \( B_\square \) is still kept in \( P_Q \). Then \( B_\square \) gives a subgroup \( \mathbb{Z}^2 \) in \( W(P_Q, F_B) \). After cutting \( Q \) open along cutting belts \( B_i, i = 1, 2, \cdots, g \), \( \mathbb{Z}^2 \) is generated by \( s_1s_3 \) and \( s_2s_4 \).
- The \( B_\square \) is not kept in \( P_Q \). Then there is only one situation in which some cutting belts \( B_i \) intersect transversely with a pair of disjoint edges of \( B_\square \), say \( F_1 \) and \( F_3 \). If \( B_\square \) intersects transversely with cutting belts \( B_1, B_2, \cdots, B_k \) in turn, then \( s_1s_3 \) and \( s_2t_1 \cdots t_k s_1^{-1} \cdots t_1^{-1} \) generate a subgroup \( \mathbb{Z}^2 \) in \( \pi_1^{orb}(Q) \), as the cases of (c) or (b) on Example 3.1. See also Example 5.1. □

5.3. Proof of the necessity of Proposition 5.1. Cutting \( Q \) open along a cutting belt \( B_i \) we get a simple \( n \)-handlebody of genus \( g - 1 \), denoted by \( Q_{g-1} \). Conversely, \( Q \) can be recovered from \( Q_{g-1} \) by gluing its two disjoint boundary facets, which implies that the orbifold fundamental group of \( Q \) is an HNN-extension on \( \pi_1^{orb}(Q_{g-1}) \). Write \( G_{g-1} = \pi_1^{orb}(Q_{g-1}) \), and let \( W_{B^+} \) and \( W_{B^-} \) be two isomorphic subgroups of \( G_{g-1} \) determined by two copies of \( B_i \). Then we have

\[
\pi_1^{orb}(Q) \cong G_{g-1}*_{\phi} \langle G_{g-1}, t \mid t^{-1}at = \phi(a), a \in W_{B^-} \rangle
\]

where \( \phi : W_{B^-} \to W_{B^+} \) is an isomorphism by mapping \( s' \in W_{B^-} \) into \( s \in W_{B^+} \). Generally, \( \pi_1^{orb}(Q) \) is isomorphic to \( g \) times HNN-extensions on the right-angled Coxeter group \( W(P_Q, F_B) \) as we have seen in the proof of Proposition 3.3.

\[
\pi_1^{orb}(Q) \leftarrow G_{g-1} \leftarrow \cdots \leftarrow G_1 \leftarrow G_0 = W(P_Q, F_B)
\]

where each \( G_k \) is also an HNN-extension over \( G_{k-1} \) for \( 1 \leq k \leq g - 1 \), and \( G_0 = W(P_Q, F_B) \) is a right-angled Coxeter group.

According to the normal form theorem of HNN-extension (Theorem 2.2), each element \( x \) in \( \pi_1^{orb}(Q) \) has a unique iterative normal form. First, write

\[
x = g_0t_1^1g_1t_2^1 \cdots g_{n-1}t_m^ng_n
\]

as a normal form for \( t_g \) where \( g_i \in G_{g-1} \). Next inductively each \( g_i \) is also a normal form in \( G_k \) for \( 1 \leq k \leq g - 1 \). More generally, \( x \) has a unique form

\[
x = g_0t_1g_1 \cdots g_{m-1}t_mg_m
\]

where each \( g_i \) is reduced in \( G_0 = W(P_Q, F_B) \), and each \( t_i \) is one of \( \{t_B^\pm\} \) which determines an isomorphism of \( \{\phi_B^{\pm}\} \) on some subgroups of \( \pi_1^{orb}(Q) \). This expression of \( x \) is a normal form with respect to all possible \( t_B \). The expression in (5.2) is called a reduced normal form of \( x \) in \( \pi_1^{orb}(Q) \). The number \( m \) is called the (total) \( t \)-length of \( x \).

By applying the Tits Theorem (Theorem 2.1) and the Normal Form Theorem of HNN-extension (Theorem 2.2), we have the following conclusion.
Lemma 5.1. Two reduced words $x, y$ are the same in $\pi_{1}^{orb}(Q)$ if and only if one can be transformed into the other by a sequence of commutations of RACG and $t$-reductions of HNN-extension.

Next, we prove two lemmas.

Lemma 5.2. If there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_{1}^{orb}(Q)$, then one generator of $\mathbb{Z} \oplus \mathbb{Z}$ can be presented as a cyclically reduced word in $W(P_{Q}, F_{B})$.

Proof. Assume that there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_{1}^{orb}(Q)$, which is generated by two reduced normal forms as in (5.2):

$$x = g_{0}t_{1}g_{1} \cdots g_{m-1}t_{m}g_{m}$$

and

$$y = h_{0}t'_{1}h_{1} \cdots h_{n-1}t'_{n}h_{n}.$$ 

Then $xy = yx$ in $\pi_{1}^{orb}(Q)$. By Lemma 5.1, $xy$ and $yx$ have the same reduced normal form as in (5.2).

We do $t$-reductions on

$$xy = g_{0}t_{1}g_{1} \cdots g_{m-1}t_{m}g_{m} \cdot h_{0}t'_{1}h_{1} \cdots h_{n-1}t'_{n}h_{n}$$

and

$$yx = h_{0}t'_{1}h_{1} \cdots h_{n-1}t'_{n}h_{n} \cdot g_{0}t_{1}g_{1} \cdots g_{m-1}t_{m}g_{m}.$$

Since $x, y$ are reduced normal forms, $xy$ and $yx$ have the same tails. Without loss of generality, assume that $m \geq n$. Write $\tilde{y} = t'_{1}h_{1} \cdots h_{n-1}t'_{n}h_{n} = h_{0}^{-1}y$. Then $x$ can be written as

$$x = g_{0}t_{1}g_{1} \cdots t_{m-n}g_{m-n} \cdot \tilde{y} = g_{0}t_{1}g_{1} \cdots t_{m-n}g_{m-n} \cdot h_{0}^{-1}y.$$ 

Since $x$ and $y$ generate $\mathbb{Z} \oplus \mathbb{Z}$, both $y$ and $xy^{-1}$ do so. The word $xy^{-1}$ has a shorter $t$-length. We further do $t$-reductions on $xy^{-1}$ to get a normal form, also denoted by $x$.

We can always continue to do this algorithm, so that we can take either $x$ or $y$ from $W(P_{Q}, F_{B})$. Suppose $y = h \in W(P_{Q}, F_{B})$.

Furthermore, we can assume that $h$ is a cyclically reduced word in $W(P_{Q}, F_{B})$. In fact, if $h$ is not cyclically reduced, without the loss of generality, assume that $h$ is of the form $w^{-1}h'w$, where $w$ is an arbitrary word and $h'$ is a cyclically reduced word in $W(P_{Q}, F_{B})$. Then we replace $h$ by $h'$, such that $h'$ and $wxw^{-1}$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_{1}^{orb}(Q)$. This completes the proof.

Lemma 5.3. Let $x = g_{0} \cdot t_{1} \cdots t_{k}$ be a reduced normal form, where $g_{0} \in W(P_{Q}, F_{B})$ and each $t_{i}$ is one of $\{t_{B}^{\pm 1}\}$, and $h$ is a cyclically reduced word in $W(P_{Q}, F_{B})$. Then $x, h$ cannot generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_{1}^{orb}(Q)$.

Proof. If $x, h$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_{1}^{orb}(Q)$, then

$$x \cdot h = g_{0} \cdot t_{1} \cdots t_{k} \cdot h = g_{0}h' \cdot t_{1} \cdots t_{k}$$

where $h' = \phi_{1} \circ \cdots \circ \phi_{k}(h)$ is the image of the composition of some $\phi_{i}$ on $h$.

We first claim that $h'$ is reduced in $W(P_{Q}, F_{B})$, and the word length of $h'$ and $h$ are equal. In fact, for each $i$, $\phi_{i}$ is an isomorphism from some $W_{B^{-}}$ to $W_{B^{+}}$ which maps generators to generators, and all $W_{B^{+}}$ and $W_{B^{-}}$ are subgroups of $W(P_{Q}, F_{B})$.  

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Next, we claim that $h = h'$. In fact, $xh = hx$ implies that $g_0h' = hg_0$, that is $h' = g_0^{-1}hg_0$. Let $s$ be a letter in $g_0$. If $s$ is FAIL in $h$, then the length of $h'$ is greater than the length of $h$, which is a contradiction. If $s$ is DIE in $h$, then $h$ has a form $\bar{s}hs$, which contradicts that $h$ is cyclically reduced. Thus, all letters in $g_0$ is SUCCESS in $h$. In other words, $g_0h = hg_0 = g_0h'$, Thus $h = h' = \phi_1 \circ \cdots \circ \phi_k(h)$.

If $\phi_1 \circ \cdots \circ \phi_k = id$, then the associated sequence $t_1 \cdots t_k = 1$, which contradicts that $x$ is reduced. If $\phi_1 \circ \cdots \circ \phi_k \neq id$ and there is a letter $s$ in $h$ such that $\phi_1 \circ \cdots \circ \phi_k(s) = s_0$, then $s_0, \phi_1(s_0), \cdots, \phi_1 \circ \cdots \circ \phi_{k-1}(s_0)$ determines a non-contractible facet in $Q$, which contradicts that $Q$ is simple. More generally, if $\phi_1 \circ \cdots \circ \phi_k \neq id$, there is a generator $s_1$ as a letter in $h$, such that $s_2 = \phi_1 \circ \cdots \circ \phi_k(s_1) \neq s_1$. Continue this procedure, one can get a sequence $s_1, s_2, s_3, \ldots$, such that each $s_i$ is a generator as a letter in $h$ and $s_i = \phi_1 \circ \cdots \circ \phi_k(s_{i-1})$. However, the word length of $h$ is finite, thus there must be two same elements in the sequence. Geometrically, this means that there is a non-contractible facet in $Q$, which contradicts that $Q$ is simple. This completes the proof.  

Now let us give the proof of the necessity of Proposition 5.1 in the general case.

**Proof of the necessity of Proposition 5.1.** Suppose that there are two elements $x$ and $y$ in $\pi_1^{orb}(Q)$ which generate a rank two free abelian subgroup $\mathbb{Z} \oplus \mathbb{Z}$. Our arguments are divided into the following steps.

**Step-1. Simplify two generators $x, y$ of $\mathbb{Z} \oplus \mathbb{Z}$ by doing $t$-reductions.**

**Lemma 5.2** tells us that one of $x, y$ can be chosen as a cyclically reduced word $h$ in $W(P_Q, \mathcal{F}_B)$, say $y = h$. Now if $x$ is also a word in $W(P_Q, \mathcal{F}_B)$ (i.e., the $t$-length of $x$ is zero), then by **subsection 5.1** there is an $\mathbb{I}$-belt in $P_Q$ which can appear in $Q$, as desired.

Next let us consider the case in which the $t$-length of $x$ is greater than zero. Let

$$x = g_0t_1g_1 \cdots g_{m-1}t_m g_m$$

be a reduced normal form in $\pi_1^{orb}(Q)$. Then $xh = hx$ implies that

- $g_mh = hg_m$; $t_m \cdot h = \phi_m(h) \cdot t_m$;
- $g_{m-1} \cdot \phi_m(h) = \phi_m(h) \cdot g_{m-1}$; $t_{m-1} \cdot \phi_m(h) = \phi_{m-1} \circ \phi_m(h) \cdot t_{m-1}$;
- $\cdots$
- $g_0 \cdot \phi_1 \circ \cdots \circ \phi_m(h) = \phi_1 \circ \cdots \circ \phi_m(h) \cdot g_0$

where each $\phi_i : W_{B_i} \rightarrow W_{B_i}$ is an isomorphism determined by some $B_i \in \mathcal{F}_B$, each $\phi_1 \circ \cdots \circ \phi_m(h)$ is an expression in $W_{B_i} \cap W_{B_{i-1}}$ for $i = 2, \cdots, m$ and $\phi_1 \circ \cdots \circ \phi_m(h) \in W_{B_i}$. $h \in W_{B_m}$. Here two $B_i$ and $B_j$ may correspond to the same $B \in \mathcal{F}_B$.

**Step-2. Find facets $F_1, F_3$ around $B$ or $B'$ in $P_Q$.**

Without loss of generality, $h \in W(P_Q, \mathcal{F}_B)$ is a cyclically reduced word. Since $h$ is a free element in $W_{B_m} \cap W(P_Q, \mathcal{F}_B)$, we can take two generators $s_1$ and $s_3$ in $h$ corresponding to two disjoint facets $F_1$ and $F_3$ of $P_Q$ such that $F_1$ and $F_3$ intersect with $B_m$. In particular, $s_1s_3$ is a free element in $W_{B_m} < W(P_Q, \mathcal{F}_B) = G_0 < \cdots < G_{g-1} < G_g = \pi_1^{orb}(Q)$.

**Step-3. Find the facet $F_2$ which intersects with $F_1, F_3$.**

If $g_m \neq 1$, since $x$ is a normal form, then $g_m$ is a representative of a coset of $W_{B_m}$ in $\pi_1^{orb}(Q)$. Thus there is a generator $s_2 \notin S(W_{B_m})$ in $g_m$ such that $hs_2 = s_2h$, where
$S(W_{B_m})$ is the generators set of $W_{B_m}$. This generator $s_2$ determines a facet $F_2$ in $P_Q$, as desired.

If $g_m = 1$, then
\[
x \cdot h = g_0 t_1 g_1 \cdots t_m \cdot h = g_0 t_1 g_1 \cdots t_{m-1} g_{m-1} \cdot \phi_m(h) \cdot t_m.
\]
A similar argument shows that either there is a $s_2 \notin S(W_{B_{m-1}})$ as desired, or
\[
x \cdot h = g_0 t_1 g_1 \cdots t_{m-1} \cdot \phi_m(h) \cdot t_m = g_0 t_1 g_1 \cdots t_{m-2} g_{m-2} \cdot \phi_{m-1} \circ \phi_m(h) \cdot t_{m-1} t_m.
\]

We can continuously carry out the above procedure. Finally we can arrive at two possible cases:

- There exist some $g_i \neq 1$ for $i > 0$, now there must be a letter $s_2$ in $g_i$ which determines the required $F_2$;
- $x$ is of the form $x = g_0 \cdot t_1 \cdots t_m$, where $g_0 \in W(P_Q, \mathcal{F}_B)$ and $t_1 \cdots t_m$ is a word formed by letters in $\{t^\pm_1\}$. By Lemma 5.3, $x$ and $h$ cannot generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{\#}(Q)$. So $x = g_0 \cdot t_1 \cdots t_m$ is impossible.

Thus, we can always find a facet $F_2$ from a nontrivial $g_i$ in the reduced form (5.2) of $x$ where $i > 0$.

**Step-4. Find a facet $F_4$ such that $F_1, F_2, F_3, F_4$ determine an $\square$-belt in $Q$.**

We proceed our argument as follows.

(I). If there is only one $g_i \neq 1$ (i.e., $g_j = 1$ for any $j \neq i$) in the expression of $x$, then $x = t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m$, where $i$ must be more than zero by Lemma 5.3. Now $xh = hx$ implies that $t_1 \cdots t_i \cdot t_{i+1} \cdots t_m = 1$. Actually, if $t_1 \cdots t_i \cdot t_{i+1} \cdots t_m \neq 1$, then $\phi_1 \circ \cdots \circ \phi_m(h) = h$ implies that there is a non-contractible facet in $Q$, which is impossible (also see the proof of Lemma 5.3). Thus, $t_1 \cdots t_i = (t_{i+1} \cdots t_m)^{-1}$, so $x = t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m = (t_{i+1} \cdots t_m)^{-1} g_i (t_{i+1} \cdots t_m)$. Since $x, h$ generate a $\mathbb{Z} \oplus \mathbb{Z}$, we see that $g_i, \phi_{i+1} \circ \cdots \circ \phi_m(h)$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $W(P_Q, \mathcal{F}_B)$. Then by subsection 5.1, there is an $\square$-belt in $Q$.

(II). If there are at least two nontrivial $g_i, g_j \neq 1$ in $x$ where $0 < j < i \leq m$ but $g_k = 1$ for all $k > j$ and $k \neq i$, then one may write $x = \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m$. So we have
\[
\begin{align*}
xh &= \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m \cdot h \\
&= \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot g_i \cdot h' t_{i+1} \cdots t_m \\
&= \cdots t_j \cdot g_j \cdot t_{j+1} \cdots t_i \cdot h' g_i \cdot t_{i+1} \cdots t_m \\
&= \cdots t_j \cdot g_j \cdot h'' t_{j+1} \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m.
\end{align*}
\]
where $h' = \phi_{i+1} \circ \cdots \circ \phi_m(h)$ and $h'' = \phi_{j+1} \circ \cdots \circ \phi_m(h)$. Since $xh = hx$, we have that $g_j h'' = h'' g_j$, so we can take a generator $s_4$ in $g_j$ (not in $S(W_B)$) such that $h'' s_4 = s_4 h''$. Similarly, here $s_4$ determines a facet $F_4$ of $P_Q$ such that $F_4 \cap F_1'' \neq \emptyset$ and $F_4 \cap F_3'' \neq \emptyset$ where $F_1''$ and $F_3''$ are two facets of $P_Q$ determined by the images of $\phi_{j+1} \circ \cdots \circ \phi_m$ on $s_1, s_3$. In particular, $F_2 \neq F_4$ in $P_Q$. Otherwise, the intersection of $q(F_1)$ and $q(F_2)$ in $Q$ is disconnected where $q : Q \to P_Q$ is defined in (3.1), which contradicts that $Q$ is simple. Hence, we get an $\square$-belt in $Q$.

(III). If there are only $g_0$ and $g_i$ that are non-trivial in $x$ where $i > 0$, then one may write $x = g_0 t_1 \cdots t_i \cdot g_i \cdot t_{i+1} \cdots t_m$. Without a loss of generality, assume that $g_0, g_i$ are two
reduced words in $W(P_Q, F_B)$. Now if $x = g_0 t_1 \cdots t_i g_1 t_{i+1} \cdots t_m = t_1 \cdots t_i g_0' t_{i+1} \cdots t_m$ where $g_0' = \phi_1^{-1} \circ \cdots \circ \phi_1^{-1}(g_0)$, then by the proof of Lemma 5.3, $x h = h x$ implies that $t_1 \cdots t_i t_{i+1} \cdots t_m = 1$. As in the first case (I), $g_0 g_1$ and $h' = \phi_{i+1} \circ \cdots \circ \phi_m(h)$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ in $W(P_Q, F_B)$. Hence we can find an $\square$ in $Q$. If $x = g_0 t_1 \cdots t_i g_1 t_{i+1} \cdots t_m = g_0' t_1 \cdots t_j g_0'' t_{j+1} \cdots t_i \cdots g_0'' t_{i+1} \cdots t_m$ where $g_0''$ cannot cross $t_{j+1}$ and $g_0' = g_0' \phi_1 \circ \cdots \circ \phi_j(g_0') \phi_1 \circ \cdots \circ \phi_j(g_0'')$. As in the second case (II), there is a generator $s_4$ in $g_0'$ which is not in $S(W_{B''_{i+1}})$. Then $s_4$ determines a facet $F_4$ of $P_Q$ such that $F_4$ intersects with $F_1$ and $F_3$ in $Q$. So there is an $\square$-belt in $Q$.

Together with all arguments above, we complete the proof. \hfill $\Box$

5.4. **Atoroidal 3-manifolds.** Now we can consider *when the manifold double $M$ over a flag simple 3-handlebody is atoroidal*.

**Definition 5.1** ([23] Definition 1.18 and 1.19). A closed orientable irreducible 3-manifold $M$ is geometrically atoroidal or topologically atoroidal if there is no incompressible embedding torus in $M$.

A closed irreducible 3-manifold $M$ is homotopically atoroidal or algebraic atoroidal if there is no rank two free abelian subgroup in $\pi_1(M)$.

It is well-known that “homotopically atoroidal” implies “geometrically atoroidal”, but the converse is not quite true (see [23] Example 1.20). However, if $M$ is not a Seifert 3-manifold, then $M$ is homotopically atoroidal if and only if it is geometrically atoroidal.

Here we shall identify “atoroidal” with “algebraic atoroidal”.

**Corollary 5.1.** Let $Q$ be a flag simple 3-handlebody with $m$ facets, and $M$ be the manifold double over $Q$ as defined in (1.7). Then the following statements are equivalent:

1. $M$ is atoroidal;
2. There is no $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1^{3h}(Q)$;
3. $Q$ contains no $\square$-belts.

**Proof.** By Proposition 5.1 it suffices to show that “(1) $\iff$ (3)” holds.

Suppose that there exists an $\square$-belt $B_\square$ in $Q$. By Lemma 3.4, there exists a set of cutting belts that do not intersect with $B_\square$, so $B_\square$ will be kept in $P_Q$. Let the cone point of $P_Q$ be in the interior of $B_\square$. So the cubical decomposition of $B_\square$ is a subcomplex of the cubical decomposition of $P_Q$. Let $\mathcal{C}(M)$ be a cubical decomposition of $M$ constructed in section 4. Then the cubical decomposition of $B_\square$ induces the cubical complex structure of a torus $T^2$ in $M$, denoted by $\mathcal{C}(T^2)$. Thus there is a natural inclusion $i : \mathcal{C}(T^2) \to \mathcal{C}(M)$, satisfying that

- For each vertex $v$ of $\mathcal{C}(T^2)$, the induced map $Lk(i) : Lk(v, \mathcal{C}(T^2)) \to Lk(i(v), \mathcal{C}(M))$ is a simplicial embedding.
- $Lk(i)$ maps $Lk(v, \mathcal{C}(T^2))$ onto a full subcomplex of $Lk(i(v), \mathcal{C}(M))$ in $\mathcal{C}(M)$.

According to [16] Proposition 1.7.1, $\mathcal{C}(T^2)$ is a totally geodesic immersive torus in $M$, which induces a monomorphism on their fundamental groups. Hence $M$ is not atoroidal.
Conversely, assume that $M$ is not atoroidal. Then there is a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$. The orbifold covering $M \rightarrow Q$ gives a short exact group sequence:

$$1 \rightarrow \pi_1(M) \rightarrow \pi_{orb}^1(Q) \rightarrow (\mathbb{Z}_2)^m \rightarrow 1$$

implies that the subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ is also a subgroup in $\pi_{orb}^1(Q)$. Moreover, it follows by Proposition 5.1 that there must be an $\square$-belt in $Q$. □

6. Proof of Theorem A

This section is devoted to the proof of Theorem A. Some examples of simple handlebodies which are hyperbolic or non-hyperbolic are given. In addition, we also consider the (right-angled) hyperbolicity of a right-angled Coxeter 3-handlebody with ideal nerve.

6.1. Proof of Theorem A. First let us show a useful lemma.

**Lemma 6.1.** The manifold double of a special simple handlebody is orientable.

**Proof.** Let $Q$ be a special simple $n$-handlebody with $m$ facets and cutting belts $\mathcal{F}_B$, $P_Q$ be the associated simple polytope, and $M = Q \times (\mathbb{Z}_2)^m / \sim$ be the manifold double over $Q$, as defined in (1.1). Making use of the proof method of Nakayama and Nishimura [31, Theorem 1.7], it suffices to prove that

$$H_n(M; \mathbb{Z}) \cong \mathbb{Z}.$$

The combinatorial structure of $P_Q$ defines a natural cellular decomposition of $M$. We denote by $\{(C_k(M), \partial_k)\}$ the chain complex associated with this cellular decomposition. In particular, $C_n(M)$ and $C_{n-1}(M)$ are the free abelian groups generated by $\{P_Q\} \times (\mathbb{Z}_2)^m = \{(P_Q, g) \mid g \in (\mathbb{Z}_2)^m\}$ and $\mathcal{F}(P_Q) \times (\mathbb{Z}_2)^m / \sim = \{[F, g] \mid F \in \mathcal{F}(P_Q), g \in (\mathbb{Z}_2)^m\}$, respectively, where the equivalence class of $\mathcal{F}(P_Q) \times (\mathbb{Z}_2)^m$ is defined by the equivalence relation

$$\begin{cases}
(F, g) \sim (F, g \cdot e_F) & \text{if } F \in \mathcal{F}(P_Q) - \mathcal{F}_B, \\
(B^+, g) \sim (B^-, g) & \text{if } B^+, B^- \in \mathcal{F}_B.
\end{cases}$$

It should be pointed out that actually there is a coloring $\lambda : \mathcal{F}(Q) \rightarrow (\mathbb{Z}_2)^m$ in the construction of $M = Q \times (\mathbb{Z}_2)^m / \sim$ such that $\{\lambda(F) = e_F \mid F \in \mathcal{F}(Q)\}$ is the standard basis $\{e_i \mid i = 1, ..., m\}$ of $(\mathbb{Z}_2)^m$. For any facet $F'$ in $\mathcal{F}(P_Q) - \mathcal{F}_B$, there must be a facet $F$ in $\mathcal{F}(Q)$ such that $F' = F$ or $F' \not\subset F$, so $F'$ and $F$ are colored by the same element $e_F$ of $(\mathbb{Z}_2)^m$. For any $B$ in $\mathcal{F}_B$, since $\text{Int}B \subset \text{Int}(Q)$, we convention that $B$ is colored by the unit element $e_0$ of $(\mathbb{Z}_2)^m$. In other words, the coloring $\lambda : \mathcal{F}(Q) \rightarrow (\mathbb{Z}_2)^m$ induces a compatible coloring $\lambda' : \mathcal{F}(P_Q) \rightarrow (\mathbb{Z}_2)^m$ such that for any $F' \in \mathcal{F}(P_Q) - \mathcal{F}_B$, $e_{F'} = \lambda'(F') = \lambda(F) = e_F$ where $F \in \mathcal{F}(Q)$ with $F' \subset F$, and for $B$ in $\mathcal{F}_B$, $\lambda'(B) = e_0$.

We give an orientation on each facet $F_i$ and $B^\pm_i$ such that the orientation of $B^+_i$ is exactly the inverse orientation of $B^-_i$, so

$$\partial P_Q = \sum_{F \in \mathcal{F}(P_Q)} F = F_1 + \cdots + F_m' + B^+_1 + \cdots + B^+_g + B^-_1 + \cdots + B^-_g = \sum_{F \in \mathcal{F}(P_Q) - \mathcal{F}_B} F.$$
where \( m' \) is the number of all facets in \( \mathcal{F}(P_Q) - \mathcal{F}_B \).

Let \( c_n = \sum_{g \in (\mathbb{Z}_2)^m} n_g(P, g) \) be an \( n \)-cycle of \( C_n(M) \) where \( n_g \in \mathbb{Z} \). Then

\[
\partial(c_n) = \left[ \sum_{g \in (\mathbb{Z}_2)^m} n_g \sum_{F \in \mathcal{F}(P_Q)} (F, g) \right] = \sum_{[F,g] \in ((\mathcal{F}(P_Q) - \mathcal{F}(B)) \times (\mathbb{Z}_2)^m) / \sim'} (n_g + n_{gF})[F, g] = 0
\]

which induces that \( n_g = -n_{gF} \) for any facet \( F \in \mathcal{F}(P_Q) - \mathcal{F}_B \) and \( g \in (\mathbb{Z}_2)^m \). Let \( l(g) \) denote the word length of \( g \) presented by \( \{e_F\} \). For any \( g \in (\mathbb{Z}_2)^m \), there exists a subset \( I_g = \{F_1, \ldots, F_{i_k}\} \) of \( \mathcal{F}(P_Q) - \mathcal{F}_B \) such that \( g = \prod_{F \in I_g} e_F \). Then we see easily that

\[
n_g = -n_{gF_1} = n_{gF_1} e_{F_2} = \cdots = (-1)^{l(g)} n_{gF_{i_k}} e_{F_{i_k}} = (-1)^{l(g)} n_{e_0}
\]

so \( c_n = n_{e_0} \sum_{g \in (\mathbb{Z}_2)^m} (-1)^{l(g)} (P, g) \). Then we obtain that \( H_n(M; \mathbb{Z}) = \ker \partial_n \cong \mathbb{Z} \) is generated by \( \sum_{g \in (\mathbb{Z}_2)^m} (-1)^{l(g)} (P, g) \), which follows that \( M \) is orientable. \( \square \)

Now let us finish the proof of Theorem A.

**Proof of Theorem A.** Together with Hyperbolization theorem (Theorem 2.3), Proposition 2.3 and Lemma 6.1, we obtain that a simple 3-handlebody is hyperbolic if and only if its manifold double is aspherical and atoroidal. Moreover, by Corollary 4.1 and Corollary 5.2, again, we conclude that a simple 3-handlebody is hyperbolic if and only if it is flag and contains no \( \square \)-belts, as desired. \( \square \)

6.2. Examples from Löbell polytopes. A pentagonal flower is a 2-dimensional combinatorial object consisting of an \( n \)-gon surrounded by \( n \) pentagons. A Löbell polytope, denoted by \( L(n) \), is obtained from two copies of a pentagonal flowers by gluing along their boundaries by isometries. Clearly \( L(n) \) admits a right-angled Coxeter orbifold structure, so it is (right-angled) hyperbolic when \( n \geq 5 \), but is not (right-angled) hyperbolic when \( n = 3 \) and 4.

When \( n \geq 5 \), we can always construct a combinatorial handlebody \( Q \) by attaching some copies of \( L(n) \) together along their \( n \)-gons. If the number of copies is greater than 2, then \( Q \) will become a simple handlebody which meets the condition of Theorem A, so it is hyperbolic. If the number of copies is 1 or 2, then \( Q \) is not simple. In this case, we use at least three copies of \( Q \) to construct the covering space of \( Q \) in such a way that: first we may use a fixed cutting belt \( B \) to cut open each of copies, and then form a connected handlebody \( \hat{Q} \) by attaching them together along those facets produced by \( B \). This connected handlebody is exactly the required covering space of \( Q \). In particular, it is simple and also meets the condition of our theorem, so it is hyperbolic. Hence, \( Q \) is hyperbolic, too.

When \( n = 3 \) or 4, in a similar way to the case of \( n \geq 5 \), we can still construct a combinatorial handlebody \( Q \) by attaching some copies of \( L(n) \) together along their \( n \)-gons. If \( Q \) is not simple, then we can use the same approach as in the case of \( n \geq 5 \) to get a connected covering \( \hat{Q} \) of \( Q \) such that \( \hat{Q} \) is a simple handlebody. However, \( \hat{Q} \) must not admit a right-angled hyperbolic structure. This is because \( \hat{Q} \) is not flag if \( n = 3 \), and it always contains an \( \square \)-belt if \( n = 4 \). Furthermore, \( Q \) does not admit a right-angled hyperbolic structure yet.
6.3. **3-handlebodies with ideal nerve.** We say that $Q$ is a 3-handlebody with ideal nerve if $Q$ is a right-angled Coxeter 3-orbifold such that its underlying space $\partial Q$ is a 3-handlebody and its nerve is an ideal triangulation of the boundary $\partial|Q|$.

Now let $Q$ be a 3-handlebody with ideal nerve. Then, by the definition of ideal triangulations ([18, Definition 2.6]), the interior of each face of $Q$ is also contractible. On the facial structure of $Q$, there are three possible cases:

- Some 2-faces of $Q$ are henagons (i.e., 2-faces with only one point of codimension 3 in $Q$, see (a) in Figure 11) or digons (i.e., 2-faces with only two points of codimension 3 in $Q$, see (b) in Figure 11);
- There may be some 2-faces with self-intersection (see (c) in Figure 11);
- The intersection of two 2-faces may be not connected (see (d) in Figure 11).

If there is a henagon 2-face of $Q$, then it gives a self-folded ideal triangle. For example, see the blue part of (a) in Figure 11. In general, if there is a henagon 2-suborbifold in $Q$, then the nerve of associated faces may give some ideal triangles, such as (e) and (f) in Figure 11. In particular, the nerve of (f) contains only one vertex and two ideal triangles gluing along their three edges as shown in Figure 11. All those cases agree with the definition of ideal triangulations in [18, Definition 2.6].

**Lemma 6.2.** Let $Q$ be a 3-handlebody with ideal nerve. Then $Q$ is very good if and only if it does not contain a henagon 2-suborbifold.

**Proof.** By applying a theorem of Morgan or Kato [23, Theorem 6.14], each compact locally reflective 3-orbifold that contains no bad 2-suborbifolds is very good. This means that if there is no henagon 2-suborbifold in $Q$, then $Q$ is very good. Conversely, if there is a henagon 2-suborbifold in $Q$, then it is obvious that $Q$ is bad. □

Hence, if there is a henagon 2-suborbifold of $Q$, then $Q$ cannot be hyperbolic.

**Figure 11.** Ideal nerves.
Suppose that $Q$ contains no henagon 2-suborbifolds. Then, by Lemma 6.2, $Q$ can be covered finitely by a closed 3-manifold $M$. In general, $Q$ is not nice in the sense of Davis [10, Page 180], thus there is no natural manifold double defined as in (1.1) for $Q$.

A digon 2-suborbifold in $Q$ is said to be essential if its two vertices are not contained in a unique edge of $Q$. If there is an essential digon 2-suborbifold in $Q$, then its nerve $\mathcal{N}(Q)$ will contain two simplices with common vertices. See (d) in Figure 11.

**Lemma 6.3.** Let $Q$ be a 3-handlebody with ideal nerve. Assume that there is no henagon suborbifold in $Q$, and $M$ is a covering manifold over $Q$. If $Q$ contains an essential digon suborbifold, then $M$ is reducible.

**Proof.** Assume that two edges of a digon are contained in two 2-faces $F_1$ and $F_2$ of $Q$. Then we consider the double cover of $Q$, denoted by $D_Q$, which is obtained by gluing two copies of $Q$ along $F_1$. At the same time, two copies of $F_2$ are also glued along $F_1 \cap F_2$, giving an annulus in $D_Q$. Let $M'$ be a manifold cover over $D_Q$. Then $M'$ can be decomposed into the connected sum of some 3-manifolds, which implies that $M'$ is reducible. Hence, $D_Q$ and $Q$ are reducible. So $M$ is reducible. \[\square\]

A digon 2-face of $Q$ will give an essential digon 2-suborbifold in $Q$ unless that $Q$ is a trihedron. Thus in this cases $Q$ is reducible as well. Therefore, if there is a henagon 2-suborbifold or an essential digon 2-suborbifold in $Q$, then $Q$ cannot be hyperbolic.

Next, suppose that $Q$ is not a trihedron and contains no henagon and essential digon 2-suborbifolds. If there are some 2-faces with self-intersection or the intersection of two 2-faces is not connected, then we can always construct some simple orbifold covers of $Q$. In fact, similar to the operation in subsection 6.2, we can use some copies of $Q$ to construct a covering space of $Q$ as follows: first we cut open each of copies by using a fixed 2-suborbifold $B$, and then form a connected handlebody $\hat{Q}$ by attaching them together along those new facets produced by $B$. If necessary, we can choose enough copies of $Q$ so as to make sure that this connected handlebody is simple, and is exactly the required covering space of $Q$. Applying Theorem A gives

**Corollary 6.1.** A 3-handlebody with ideal nerve is hyperbolic if and only if it is not trihedron, tetrahedron and contains no $\Delta^2$, $\square$-belts and no henagon or essential digon 2-suborbifolds.

**Remark 12.** Let $Q$ be a 3-handlebody with ideal nerve. We can define henagon 2-suborbifolds and essential digons 2-suborbifolds in $Q$ as $1-$ and $2$-belts of $Q$, respectively. Then by Lemma 6.2, $Q$ is very good if and only if $Q$ contains no $1$-belts. An easy argument gives that a very good $Q$ is flag if and only if it is not trihedron and tetrahedron (i.e., $S^3/\mathbb{Z}_2^2$ and $S^3/\mathbb{Z}_2$) and contains no $2-$ and $3$-belts (i.e., $\pi_1$-injective $S^2/\mathbb{Z}_2^2$ and $S^2/\mathbb{Z}_2$-suborbifolds). Furthermore, a very good flag $Q$ is hyperbolic if and only if it contains no $4$-belts (i.e., $\pi_1$-injective $T^2/\mathbb{Z}_2$-suborbifolds). Thus, a right-angled Coxeter 3-handlebody with ideal nerve except trihedron and tetrahedron is hyperbolic if and only if it contains no $1, 2, 3, 4$-belts. In other words, in this case the topological conditions in Hyperbolization Conjecture of 3-orbifolds [5, Conjecture 6.8] can also be replaced by combinatorial conditions.

6.4. Example of non-simple 3-handlebody. Let $P$ be the product of a pentagon and $[0,1]$. Gluing two opposite pentagons of $P$ together such that its diagonal vertices coincide with each other gives a right-angled Coxeter 3-orbifold with its underlying
space as a solid torus, denoted by \( Q \). Then \( Q \) is a Seifert 3-orbifold. Thus it cannot be hyperbolic. This is because each embedding annulus 2-facet is an obstruction.

6.5. **Further question.** Finally, we would like to end this section with the following conjecture about the relation of combinatorics and (right-angled) hyperbolicity on simple 3-orbifolds.

**Conjecture.** A simple 3-orbifold \( O \) is hyperbolic if and only if the following conditions hold.

- \( O \) is flag and contains no \( \Box \)-belts;
- \( |O| \) is irreducible;
- there is no rank-two free abelian subgroup in \( \pi_1(|O|) \).

**APPENDIX A. THE UNIVERSAL COVER OF A SPECIAL SIMPLE HANDLEBODY**

In this section, by applying the fundamental domain, we give a construction of the universal cover of a special simple handlebody. Moreover, we calculate the homology of the universal cover by Davis method [10, Chapter 8].

A.1. **Fundamental domain** ([10, Page 64] or [39, Page 159-161]). Suppose that a discrete group \( G \) acts properly on a connected topological space \( X \). A closed subset \( D \subset X \) is a fundamental domain for the \( G \)-action on \( X \) if each \( G \)-orbit intersects \( D \) and if for each point \( x \) in the interior of \( D \), \( G(x) \cap D = \{ x \} \). In other words, \( \{ gD | g \in G \} \) forms a locally finite cover for \( X \), such that no two of \( \{ gD | g \in G \} \) have common interior points. Such \( \{ gD, g \in G \} \) is called a decomposition for \( X \) so \( X = \bigcup_{g \in G} gD \)

and each \( gD \) is called a chamber of \( G \) on \( X \).

Throughout the following, the fundamental domain of \( G \) acting on \( X \) will be taken as a simple convex polytope \( D \). Then each \( g \in G \) gives a self-homeomorphism of \( X \)

\[ \phi_g : X \rightarrow X \]

by mapping chamber \( hD \) to \( g \cdot hD \) for any \( h \in G \). If two chambers \( gD \) and \( hD \) have a nonempty intersection which includes some facets of \( gD \) and \( hD \), then there is a homeomorphism \( \phi_{hg^{-1}} \) that maps \( gD \) to \( hD \). Hence, for two facets \( F \) and \( F' \) from \( gD \) and \( hD \), respectively, that are glued together in \( X \), naturally we can assign \( hg^{-1} \) and \( gh^{-1} \) to \( F \) and \( F' \), respectively. This means that the action of \( G \) on \( X \) gives a characteristic map on the facets set of \( D \):

\[ \lambda : \mathcal{F}(D) \rightarrow G. \]

For each facet \( F \) of \( D \), \( \lambda(F) \in G \) is called a coloring on \( F \). Each \( \lambda(F) \in G \) naturally determines a self-homeomorphism \( \phi_{\lambda(F)} \in \text{Homeo}(X) \), which is called an adjacency transformation on \( X \) with respect to \( F \). Such \( \phi_{\lambda(F)} \) maps each chamber into adjacent chamber such that the facet \( F \) is contained in the intersection of those two chambers. Each adjacency transformation has an inverse adjacency transformation corresponding to a facet \( F' \) of \( D \). Of course, \( F = F' \) is allowed. In this case, we call \( F \) a mirror of \( X \) associated with \( G \), and the corresponding adjacency transformation is called a reflection of \( X \) with respect to \( F \).
Remark 13. It should be pointed out that two adjacency transformations determined by different facets of $D$ are viewed as being different, although they may correspond to the same self-homeomorphism of $X$. The inverse adjacency transformation of an adjacency transformation determined by a facet $F$ is exactly determined by another facet $F'$ which is identified with $F$ in $X$.

All inverse adjacency transformations give an equivalence relation $\sim$ on $\mathcal{F}(D) \times G$, where $(F, g) \sim (F', h)$ if and only if

\[
\begin{align*}
\lambda(F) \cdot g &= h \\
\lambda(F') \cdot h &= g.
\end{align*}
\]

In other words, if two chambers $gD$ and $hD$ are attached together by identifying a facet $F$ of $gD$ with a facet $F'$ of $hD$ in $X$, then $\lambda(F) \cdot \lambda(F') = 1$, which gives a pair relation for $G$. When $F$ is a mirror, the pair relation is $\lambda(F)^2 = 1$.

Remark 14. It is easy to see that the equivalence relation $\sim$ on $\mathcal{F}(D) \times G$ gives an equivalence relation $\sim'$ on $\mathcal{F}(D)$ as follows:

$$ F \sim' F' \iff (F, g) \sim (F', h). $$

Thus, we can obtain a quotient orbifold $D/\sim'$ by attaching some facets on the boundary of $D$ via the equivalence relation $\sim'$ on $\mathcal{F}(D)$.

On the contrary, giving a convex polytope $D$ and a characteristic map satisfying (A.1), we can construct a space $X$ with $G$-action in the following way:

\[
X = D \times G / \sim
\]

where the equivalence relation is defined in (A.1).

The construction of $X$ gives a natural polyhedral cellular decomposition of $X$, denoted by $\mathcal{P}(X)$. The dual complex of $\mathcal{P}(X)$ is denoted by $\mathcal{C}(X)$. If each codimension-$k$ face of $D$ in $X$ intersects with exactly $2^k$ chambers, then each cell of $\mathcal{C}(X)$ is a cube, which is exactly one induced by the standard cubical decomposition of the simple polytope $D$. Furthermore, if $\mathcal{C}(X)$ is a cubical complex, then the link of each vertex in $\mathcal{C}(X)$ is a simplicial complex which is exactly the boundary complex of the dual of $D$. The 1-skeleton of $\mathcal{C}(X)$ is exactly the Cayley graph of $G$ with generator set consisting adjacency transformations determined by all facets of $D$. Therefore, one has that

Lemma A.1 ([39 Page 160]). The group $G$ is generated by all adjacency transformations.

For simplifying notation, denote $\lambda(F_i) = s_i$ or $s_F$ for each $F_i \in \mathcal{F}(D)$. Then for each $g \in G$, $\phi_g$ can be decomposed into the composition of some adjacency transformations:

$$ g = s_{i_1}s_{i_2}\cdots s_{i_k} $$

The relations with form $s_{i_1}s_{i_2}\cdots s_{i_k} = 1$ except pair relations is called Poincaré relations.

Lemma A.2 ([39 Page 161]). The Poincaré relations together with the pair relations form a set of relations of group $G$.

For each codimension 2 face of $D$, there is a Poincaré relation with form $s_{k}s_{k-1}\cdots s_1 = 1$ (alternatively, $s'_{i_1}\cdots s'_{i_k} = 1$, where $s'_{i} = (s_i)^{-1}$ for each $i$).
Define a group $G_D$ with generators consisting of all adjacency transformations determined by $\mathcal{F}(D)$ and relations formed by all pair relations and Poincaré relations determined by all codimension 2 faces in $D$.

$$G_D = \langle s_i; \text{for } F_i \in \mathcal{F}(D) \mid s_i s_{j} = 1, \text{for } F_i \sim^t F_j; \quad s_{i_1} s_{i_2} \cdots s_{i_k} = 1, \text{for each codim-2 face in } D \rangle$$

(A.3)

For the sake of preciseness, suppose again that each codimension-$k$ face of $D$ in $X$ intersects with exactly $2^k$ chambers. Then the cubical subdivision of $D$ induces a right-angled Coxeter cellular decomposition for the quotient orbifold $X/G$. It is not difficult to see that $D/\sim'$ is isomorphic to $X/G$ as orbifolds. According to Proposition 2.2, $G_D$ is isomorphic to the orbifold fundamental group of the quotient group $X/G$. Therefore, we have that

**Lemma A.3.** The orbifold fundamental group of $D/\sim' \cong X/G$ is isomorphic to $G_D$.

There is a natural quotient map $\lambda_* : G_D \rightarrow G$, and the image of $\lambda_*$ on each adjacency transformation $s_F$ is the coloring on corresponding facet $F$. Then the fundamental group of $X$ is isomorphic to the kernel of $\lambda_*$. 

**Proposition A.1.** Let $G$ be a discrete group which acts properly discontinuously on a closed manifold $X$. Suppose $X$ is decomposed into $X = \bigcup_{g \in G} gD = D \times G/\sim$, where $D$ is a simple convex polytope and each codimension-$k$ face of $D$ in $X$ intersects with exactly $2^k$ chambers. Let $G_D$ be the group defined as in (A.3), and let $\lambda_*$ be the quotient map from $G_D$ to $G$ induced by the characteristic map $\lambda : \mathcal{F}(D) \rightarrow G$. Then there is a short exact group sequence,

$$1 \rightarrow \pi_1(X) \rightarrow G_D \xrightarrow{\lambda_*} G \rightarrow 1$$

which is induced by an orbifold covering $\pi : X \rightarrow X/G$.

**Proof.** Refer to Chen ([7, Page 40-49]). Here it is only necessary to show that $G_D \cong \pi_1^{orb}(X/G)$, which is exactly Lemma A.3.

Given a simple convex polytope $D$ and a discrete group $G$, assume that there exists a characteristic map $\lambda : \mathcal{F}(D) \rightarrow G$ such that $X = D \times G/\sim$ is a $G$-manifold, where $(F, g) \sim (F', h)$ for any $F, F' \in \mathcal{F}(D), g, h \in G$ if and only if (A.1) holds. Then, we have the following result.

**Corollary A.1.** Under the assumption of Proposition A.1, $X$ is simply-connected if and only if $G \cong G_D$.

**Example A.1.** Let $P$ be a square with faces $F_1, F_2, F_3, F_4$ colored by $e_1, e_2, e_1, e_2$ respectively, where $e_1, e_2$ are generators of $(\mathbb{Z}_2)^2$. Then $X = P \times (\mathbb{Z}_2)^2/\sim \cong \mathbb{T}^2$ is a small cover over $P ([12])$, and $G_P = \langle s_1, s_2, s_3, s_4 \mid s_i^2 = 1; (s_1 s_2)^2 = (s_2 s_3)^2 = (s_3 s_4)^2 = (s_4 s_1)^2 = 1 \rangle$ is the right-angled Coxeter group determined by $P$. Then $\pi_1(X) \cong \ker \lambda_* = \mathbb{Z}^2$ is a normal subgroup in $G_P$ generated by Poincaré relations $s_1 s_3$ and $s_2 s_4$.

**A.2. Universal cover.** Next, we consider the universal covers of simple handlebodies. Let $Q$ be a simple convex polytope with cutting belts $\{B_1, \cdots, B_q\}$, and $P_Q$ be the simple polytope given by cutting $Q$ with the quotient map $q : P_Q \rightarrow Q$. Let $G$ be the orbifold fundamental group $\pi_1^{orb}(Q)$ with the presentation in Proposition 3.2. Define a characteristic map on the facet set of $P_Q$:

$$\lambda : \mathcal{F}(P_Q) \rightarrow G$$
given by \( \lambda(F) = s_F \) for \( F \in \mathcal{F}(P_Q) - \mathcal{F}_B \), and \( \lambda(B) = t_B \) for \( B \in \mathcal{F}_B \). Then we construct the following space

(A.4) \[ \tilde{Q} = P_Q \times G / \sim \]

where \((x, g) \sim (y, h)\) if and only if

(A.5) \[ \begin{cases} x = y \in F \in \mathcal{F}(P_Q) - \mathcal{F}_B, g s_F = h, \\ (x, y) \in (B, B'), B, B' \in \mathcal{F}_B, q(x) = q(y), t_B \cdot g = h. \end{cases} \]

The orbit space of the action of \( G \) on \( \tilde{Q} \) is \( Q \), so the polytope \( P_Q \) can be viewed as the fundamental domain of \( G \) acting on \( \tilde{Q} \). According to Corollary A.1

**Lemma A.4.** \( \tilde{Q} \) is the universal orbifold cover of \( Q \).

A.3. **Homology groups of \( \tilde{Q} \).** Next, we begin with the calculation of the homology groups of \( \tilde{Q} \) by doing a bit generalization of the method of Davis \[10\], Chapter 8.

Let \( Q \) be a special simple handlebody with nerve \( \mathcal{N}(Q) \), and \( P_Q \) be the associated simple polytope. Let \( G = \pi_1^{orb}(Q) \) be the orbifold fundamental group of \( Q \). We have known that \( G \) is an iterative HNN-extension on a right-angled Coxeter group \( W(P_Q, \mathcal{F}_B) \). Namely

\[ G = \pi_1^{orb}(Q) \cong (\cdots ((W(P_Q, \mathcal{F}_B) *_{\phi_{B_1}}) *_{\phi_{B_2}}) \cdots) *_{\phi_{B_k}} \]

where \( g \) is the genus of \( Q \). For any \( w \in G \), consider the following reduced normal form,

\[ w = g_0 t_1 g_1 \cdots g_{m-1} t_m g_m \]

where each \( g_i \) is reduced in \( W(P_Q, \mathcal{F}_B) \), and each \( t_i \) is one of \( \{ t_B^{\pm 1} \} \) which determines an isomorphism of \( \{ \phi_B^{\pm 1} \} \) on some subgroups of \( \pi_1^{orb}(Q) \). Denote the generator set of \( G \) by \( S = \{ s_F; F \in \mathcal{F}(P_Q) - \mathcal{F}_B \} \cup \{ t_B; B \in \mathcal{F}_B \} \). For any word \( w \in G \), put

\[ S(w) = \{ s \in S \mid l(ws) < l(w) \}, \]

where \( l(w) \) is the word length of the reduced normal form of \( w \) in \( G \) (i.e., the shortest length between 1 and \( w \) in the Cayley graph of \( G \) associated with the generator set \( S \)). For each subset \( T \) of \( S \), let \( P_Q^T \) be the subcomplex of \( P_Q \) defined by

\[ P_Q^T = \bigcup_{t \in T} F_t, \]

where \( F_{s_F} = F \) for \( s_F \in \mathcal{F}(P_Q) - \mathcal{F}_B \) and \( F_{t_B} = B' \) for \( B \in \mathcal{F}_B \) with \( B \sim B' \).

Let \( \tilde{Q} = P_Q \times G / \sim \) be the universal cover of \( Q \) defined as (A.4). Then we have the following conclusion which generalizes the Theorem 8.1.2 in \[10\], Theorem 8.12.

**Theorem B.** The homology of \( \tilde{Q} \) is isomorphic to the following direct product

\[ H_\ast(\tilde{Q}) \cong \prod_{w \in G} H_\ast(P_Q, P_Q^{S(w)}). \]

**Remark 15.** Here \( P_Q \) is not a mirrored space.

**Corollary A.2.** If there is an empty \( k \)-simplex \( \Delta^k \) in \( \mathcal{N}(P_Q) \), then \( H_k(\tilde{Q}) \neq 0 \).
Proof. Assume that the vertices set of the empty $k$-simplex $\triangle^k$ in $\mathcal{N}(P_Q)$ is

$$T = \{F_1, F_2, \cdots, F_{k+1}\}$$

which does not contain the facet in $\mathcal{F}_B$ (in fact, any facet in $\mathcal{F}_B$ is not the vertex of any empty simplex of $\mathcal{N}(P_Q)$. This is guaranteed by the definition of $B$-belt). Then $W_T \cong (\mathbb{Z}_2)^{k+1}$ is generated by $s_1, \cdots, s_{k+1}$. Let $w = s_1s_2\cdots s_{k+1}$. Regard $T$ as $\{s_1, \ldots, s_{k+1}\}$. Then $S(w) = T$. Moreover, $P_1^{S(w)} = P_{1}^{Q} = \cup_{i=1}^{k+1}F_i \cong \partial \triangle^k \cong S^{k-1}$. Since $P_Q$ is a contractible ball, by the long exact homology group sequence of pair $(P_Q, P^T_Q)$, we have

$$H_k(P_Q, P^T_Q) \cong H_{k-1}(P^T_Q) \cong H_{k-1}(S^{k-1}) \neq 0.$$ 

Therefore, $H_k(\tilde{Q}) \neq 0$. □

**Corollary A.3.** A special simple handlebody is aspherical if and only if it is flag.

Now by Corollary A.3, the proof of Proposition 4.2 is complete.

**A.4. Proof of Theorem B.** Before we prove Theorem B, we first give some notations (cf [10]).

A subset $T$ of $S$ is called spherical if the subgroup generated by $T$ is a finite subgroup of $G$. Each $s_F$ in a spherical subset $T$ exactly corresponds to a facet $F \in \mathcal{F}(P) - \mathcal{F}_B$, and $F \cap F' \neq \emptyset$ for any $s_F, s_{F'}$ in spherical set $T$. Let $W_T$ be the group generated by a spherical subset $T$. Then $W_T \cong (\mathbb{Z}_2)^{|T|}$, where $|T|$ denotes the number of all elements in $T$.

If the set $T$ is the union of a spherical set $T_S$ and a $t_B$ for $B \in \mathcal{F}_B$, then

$$W_T = W_{T_S} \cup t_B', W_{T_S},$$

where $B'$ is the facet which is identified with $B$ in $\tilde{Q}$.

**Lemma A.5.** Let $G$ be the orbifold fundamental group of a special simple handlebody with generator set $S$. Then, for each $w \in G$, $S(w)$ is either a spherical subset of $S$ or the union of a $t_B$ and a spherical subset.

**Proof.** Let $w = g_0t_1g_1 \cdots g_{m-1}t_mg_m$ be a reduced normal form in $G$. We might as well assume that this expression of $w$ is a normal form in the opposite direction for each $t_B$, that is, each $g_i$ is a representative of a coset of $W_{B,i+1}$ or $W_{F,i+1}$ in $G$, for $i = 0, \ldots, m-1$. It is easy to see that for $F \in \mathcal{F}(P) - \mathcal{F}_B$, $s_F \in S(w)$ if and only if $s_F \in S(t_mg_m)$. If there is a $B \in \mathcal{F}_B$ such that $t_B \in S(w)$, then $g_mt_B = t_Bg'_m$, where $g'_m = \phi_B(g_m)$, and the last $t_m$ is $t_B^{-1}$. For another $t_B' \neq t_B$, it cannot reduce the length of $w$. Thus the conclusion holds. □

For a spherical set $T = S(w)$, we define an element in $ZW_T \subset ZW(P_Q, \mathcal{F}_B)$ by a formula

$$\beta_T = \sum_{w \in W_T} (-1)^{l(w)}w.$$ 

Consider a natural cellular decomposition of $P_Q$ given by its facial structure. Let $C_*(P_Q)$ and and $C_*(\tilde{Q})$ denote the cellular chain complexes of $P_Q$ and $\tilde{Q}$, and let $H_*(P_Q)$ and $H_*(\tilde{Q})$ be their respective homology groups. Since $G$ acts cellularly on $\tilde{Q}$, $C_*(\tilde{Q})$ is a $\mathbb{Z}(G)$-module.
Let $T$ be a spherical set. Multiplication by $\beta_T$ defines a homomorphism $\beta_T : C_*(P_Q) \to C_*(W_T P_Q)$.

**Lemma A.6.** $C_*(P_Q^T)$ is contained in the kernel of $\beta_T : C_*(P_Q) \to C_*(W_T P_Q)$.

**Proof.** Suppose $\tau$ is a cell in $P_Q^T$. If $T$ is a spherical set, then $\tau$ lies in some $F \in \mathcal{F}(P_Q) - \mathcal{F}_B$ such that $s_F \in T$. Let $B$ be a subset of $W_T$ such that $W_T = B \cup s_F B$, then we can write $\beta_T$ as follows

$$\beta_T = \sum_{w \in W_T} (-1)^{l(w)} w = \sum_{v \in B} (-1)^{l(v)} (v - v s_F).$$

Since $v F$ is identified with $v s_F F$ in $\tilde{Q}$, we have that

$$\beta_T = \sum_{\tau} (-1)^{l(\tau)} (v - v s_F) = \sum_{\tau} (-1)^{l(\tau)} (v - v T) = 0.$$  

Thus, $C_*(P_Q^T) \subseteq \ker \beta_T$. \hfill $\square$

Hence, $\beta_T$ induces a chain map $C_*(P_Q, P_Q^T) \to C_*(W_T P_Q)$, still denoted by $\beta_T$.

For each $w \in G$ satisfying that $T = S(w)$ is a spherical set, we then define a map $\rho^w = w \beta_T : C_*(P_Q, P_Q^T) \to C_*(w W_T P_Q)$. Hence, we have a map $\rho^w : H_*(P_Q, P_Q^T) \to H_*(w W_T P_Q)$.

When $T = S(w) = \{t_B\} \cup T_S$ where $T_S$ is a spherical, $t_B s = s t_B$ for any $s \in T_S$ implies that $W_T S < W_B$, i.e., $B \cap F_s \neq \emptyset$ for any $s \in T_S$. So $B'$ does not intersect any $F_s$ hence for $k > 1$ we have $H_k(P_Q, P_Q^T) \cong H_{k-1}(P_Q^T) \cong H_{k-1}(P_Q^T_B \prod B') \cong H_{k-1}(P_Q^T_S) \cong H_k(P_Q, P_Q^T)$ where $P_Q$ and $B'$ are contractible simple polytopes. Now put

$$\rho^w : H_*(P_Q, P_Q^T) \cong H_k(P_Q, P_Q^T_S) \to H_*(W_T S P_Q) \to H_*(W_T P_Q) \to H_*(w W_T P_Q).$$

Order the elements of $G$,

$$w_1, w_2, \ldots$$

so that $l(w_i) \leq l(w_{i+1})$. For each $n \geq 1$, put

$$X_n = \bigcup_{i=1}^n w_i P_Q.$$  

To simplify notation, set $w = w_n$.

**Lemma A.7.** $X_{n-1} \cap w P = w P^{S(w)}$.

**Proof.** Notice that $X_{n-1}$ contains a subgraph of Cayley graph of $G$ associated with the generator set $S$, where the length between each vertex and the unit element is less than or equal to $l(w)$. Then

$$l(ws) = \begin{cases} l(w) - 1, & \text{if } s \in S(w), \\ l(w) + 1, & \text{if } s \in S - S(w). \end{cases}$$

A chamber $w_i P_Q$ ($i < n$) in $X_{n-1}$ intersects with $w P_Q$ in the facet $w F$ if and only if either $w_i \cdot s_F = w$ or $w_i \cdot t_F = w$; in other words, either $l(ws_F) = l(w) - 1$ or $l(wt_F) = l(w) - 1$. Therefore, $X_{n-1} \cap w P_Q = w P^{S(w)}$. \hfill $\square$
Finally we finish the proof of Theorem B.

**Proof of Theorem B.** We know from Lemma A.7 that $X_{n-1} \cap wP_Q = wP_Q^{S(w)}$. Hence, the excision theorem gives an isomorphism

$$H_*(X_n, X_{n-1}) \xrightarrow{\cong} H_*(wP_Q, wP_Q^{S(w)}).$$

Consider the exact sequence of the pair $(X_n, X_{n-1})$

$$\cdots \to H_*(X_{n-1}) \xrightarrow{j_*} H_*(X_n) \xrightarrow{k_*} H_*(X_n, X_{n-1}) \to \cdots$$

We claim that the map $k_*$ is a split epimorphism, which is equivalent to that the map $k_*^w : H_*(X_n) \to H_*(P_Q, P_Q^{S(w)})$ is a split epimorphism, where $k_*^w$ denotes the composition of $k_*$ with the excision isomorphism and left translation by $w^{-1}$. Consider the map $\rho_*^w$ on $H_*(P_Q, P_Q^{S(w)})$ whose image is contained in $H_*(wW_{S(w)}P_Q)$. For every $v \neq 1$ in $W_{S(w)}$, we have $l(wv) < l(w)$; hence, $wW_{S(w)}P_Q \subset X_n$. Hence the image of $\rho_*^w$ is contained in $H_*(X_n)$. All these can be seen from the following commutative diagram:

$$
\begin{array}{ccc}
H_*(X_n, X_{n-1}) & \xrightarrow{\cong} & H_*(wP_Q, wP_Q^{S(w)}) \\
\downarrow k_* & & \downarrow \times w^{-1} \\
H_*(X_n) & \xrightarrow{k_*^w} & H_*(P_Q, P_Q^{S(w)}) \\
\downarrow i_* & & \downarrow \beta_* \\
H_*(wW_{S(w)}P_Q) & \xrightarrow{\times w} & H_*(W_{S(w)}P_Q)
\end{array}
$$

where $\beta_*$ is induced by multiplication by $\beta_{S(w)}$ when $S(w)$ is a spherical set, and is the composition $\beta_{T_S} \circ i_* : H_k(P_Q, P_T^T) \cong H_k(P_Q, P_Q^{T_S}) \xrightarrow{\beta_{T_S}} H_*(W_{T_S}P_Q) \xrightarrow{i_*} H_*(W_TP_Q)$ when $S(w)$ is the union of a $\{t_B\}$ and a spherical set $T_S$.

Since $\tilde{Q}$ is the universal cover of $Q$, $H_1(\tilde{Q}) \cong 0$. For $* > 1$, it can be seen that $k_*^w \circ \rho_*^w$ is the identity on $H_*(P_Q, P_Q^{S(w)})$ by above diagram. Hence there is the following splitting short exact sequence:

$$0 \to H_*(X_{n-1}) \xrightarrow{j_*} H_*(X_n) \xrightarrow{k_*^w} H_*(P_Q, P_Q^{S(w)}) \to 0.$$

This implies that

$$H_*(X_n) \cong H_*(X_{n-1}) \oplus H_*(P_Q, P_Q^{S(w)})$$

where $H_*(X_1) = H_*(P_Q) = 0$. Since $\tilde{Q}$ is the increasing union of the $X_n$, we have

$$H_*(\tilde{Q}) = \lim_{n \to \infty} H_*(X_n) \cong \prod_{w \in G} H_*(P_Q, P_Q^{S(w)}).$$

This completes the proof. \qed

**References**

[1] Alejandro Adem, Johann Leida and Yongbin Ruan, *Orbifolds and stringy topology*. Cambridge, New York, 2007.

[2] E. M. Andreev, *Convex polyhedra of finite volume in Lobachevskii space*, Mat. Sb. (N.S.) 83 (125) 1970 256–260.
[3] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.

[4] Michel Boileau, Bernhard Leeb and Joan Porti, *Geometrization of 3-dimensional orbifolds*. Ann. of Math. (2) 162 (2005), no. 1, 195–290.

[5] Michel Boileau, Sylvain Maillot and Joan Porti, *Three-dimensional orbifolds and their geometric structures*. Panoramas et Synthèses [Panoramas and Syntheses], 15. Société Mathématique de France, Paris, 2003.

[6] Anthony Bahri, Dietrich Notbohm, Soumen Sarkar and Jongbaek Song, *On integral cohomology of certain orbifolds*, [arXiv:1711.01748](https://arxiv.org/abs/1711.01748).

[7] Weimin Chen, *A homotopy theory of orbispace*, [arXiv:math/0102020](https://arxiv.org/abs/math/0102020).

[8] Weimin Chen and Yongbin Ruan, *A new cohomology theory of orbifold*. Comm. Math. Phys. 248 (2004), no. 1, 1–31.

[9] Michael W. Davis, *Lectures on orbifolds and reflection groups*. Transformation groups and moduli spaces of curves, 63–93, Adv. Lect. Math. (ALM), 16, Int. Press, Somerville, MA, 2011.

[10] Michael W. Davis, *The geometry and topology of Coxeter groups*. London Mathematical Society Monographs Series 32, Princeton Univ. Press (2008).

[11] Michael W. Davis and Tadeusz Januszkiewicz, *Hyperbolization of polyhedra*. J. Differential Geom. 34 (1991), no. 2, 347–388.

[12] Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*. Duke Math. vol. 62 (1991), no. 2, 417–451.

[13] Michael W. Davis, *Group generated by reflections and aspherical manifolds not covered by Euclidean space*. Ann. of Math. (2) 117 (1983), no. 2, 293–324.

[14] Michael W. Davis and Allan L. Edmonds, *Euler characteristics of generalized Haken manifolds*. Algebr. Geom. Topol. 14 (2014), no. 6, 3701–3716.

[15] Michael W. Davis, T. Januszkiewicz and R. Scott, *Fundamental groups of blow-ups*. Adv. Math. 177 (2003), no. 1, 115–179.

[16] Michael W. Davis, T. Januszkiewicz and R. Scott, *Nonpositive curvature of blow-ups*. Selecta Math. (N.S.) 4 (1998), no. 4, 491–547.

[17] Jonathan Dinkelbach and Bernhard Leeb, *Equivariant Ricci flow with surgery and applications to finite group actions on geometric 3-manifolds*. Geom. Topol. 13 (2009), no. 2, 1129–1173.

[18] Sergey Fomin, Michael Shapiro and Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*. Acta Math. 201 (2008), no. 1, 83–146.

[19] Bell Foozwell and Hyam Rubinstein, *Introduction to the theory of Haken n-manifolds*. Topology and geometry in dimension three, 71–84, Contemp. Math., 560, Amer. Math. Soc., Providence, RI, 2011.

[20] M. Gromov, *Hyperbolic groups*. Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[21] Allen Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[22] John Hempel, *3-Manifolds*. Ann. of Math. Studies, No. 86. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976.

[23] Michael Kapovich, *Hyperbolic Manifolds and Discrete Groups*. Progress in Mathematics, 183. Birkhäuser Boston, Inc., Boston, MA, 2001.

[24] Youlin Li and Jiming Ma, *$\mathbb{Z}_2^3$-colorings and right-angled hyperbolic 3-manifolds*. Pacific J. Math. 263 (2013), no. 2, 419–434.

[25] Zhi Lü, *Graphs of 2-torus actions*. Toric topology, 261–272, Contemp. Math., 460, Amer. Math. Soc., Providence, RI, 2008.

[26] Zhi Lü, Lisu Wu and Li Yu, *An integral (co)homology theory of Coxeter orbifolds*. Preprint, 2020.

[27] Roger C. Lyndon and Paul E. Schupp, *Combinatorial Group Theory*. Reprint of the 1977 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[28] William H. Meeks and Peter Scott, *Finite group actions on 3-manifolds*. Invent. Math. 86 (1986), no. 2, 287–346.

[29] G. D. Mostow, *Strong rigidity of locally symmetric spaces*. Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973.

[30] Gabor Moussong, *Hyperbolic Coxeter groups*. Thesis (Ph.D.) The Ohio State University. 1988.
[31] Hisashi Nakayama and Yasuzo Nishimura, *The orientability of small covers and coloring simple polytopes*. Osaka J. Math. 42 (2005), no. 1, 243–256.

[32] M. Poddar and S. Sarkar, *On quasitoric orbifolds*. Osaka J. Math. 47 (2010), no. 4, 1055–1076.

[33] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds (Third Edition)*. Graduate Texts in Mathematics, (2019).

[34] Roland K.W. Roeder, John H. Hubbard and William D. Dunbar, *Andreev’s Theorem on hyperbolic polyhedra*. Ann. Inst. Fourier (Grenoble) 57 (2007), no. 3, 825–882.

[35] Satake, *The Gauss-Bonnet Theorem for V-manifolds*. Journal of the Mathematical Society of Japan, Vol. 9, No. 4 (1957), 464–492.

[36] Waldhausen and Friedhelm, *On irreducible 3-manifolds which are sufficiently large*. Ann. of Math. (2) 87 (1968), 56–88.

[37] William P. Thurston, *Three-dimensional geometry and topology*. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[38] William P. Thurston, *Kleinian groups and hyperbolic geometry*. Bull. Amer. Math. Soc. (N.S.) 6 (1982), no. 3, 357–381.

[39] E.B. Vinberg and O.V. Shvartsman, *Discrete Groups of Motions of Spaces of Constant curvature*. Geometry, II, 139–248, Encyclopaedia Math. Sci., 29, Springer, Berlin, 1993.

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