A quantitative discounted central limit theorem using the Fourier metric

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Abstract

The discounted central limit theorem concerns the convergence of an infinite discounted sum of i.i.d. random variables to normality as the discount factor approaches 1. We show that, using the Fourier metric on probability distributions, one can obtain the discounted central limit theorem, as well as a quantitative version of it, in a simple and natural way, and under weak assumptions.

1 Introduction

Let $X_n$ ($n \geq 0$) be a sequence of i.i.d. real-valued random variables, with

$$\mu = E(X_0), \quad \sigma^2 = Var(X_0) < \infty.$$  \hspace{1cm} (1)

For $a \in [0,1)$, we define the random variable

$$S_a = \sum_{n=0}^{\infty} a^n X_n.$$  \hspace{1cm} (2)

Standard results ensure that (2) converges almost surely (see e.g. [2], Sec. 5.3). $S_a$ can be understood as the present value of a future stream of i.i.d. payments, where $a$ is the discount factor.

Gerber [6] proved, assuming that $X_n$ have finite third moments, that as $a \to 1$, the distribution of $S_a$ approaches a normal distribution: normalizing $S_a$ by setting

$$\hat{S}_a = \frac{S_a - E(S_a)}{\sqrt{Var(S_a)}} = \frac{\sqrt{1-a^2}}{\sigma} \cdot \left( S_a - \frac{\mu}{1-a} \right),$$  \hspace{1cm} (3)

we have

$$\hat{S}_a \overset{D}{\to} N(0,1) \quad \text{as} \quad a \to 1-, \hspace{1cm} (4)$$

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that is, defining the corresponding cumulative distribution functions

\[ F_a(x) = P \left( \hat{S}_a \leq x \right), \quad (5) \]

\[ \Phi(x) = P \left( N(0,1) \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \quad (6) \]

we have, for all \( x \in \mathbb{R} \),

\[ \lim_{a \to 1^-} F_a(x) = \Phi(x). \quad (7) \]

This is the discounted central limit theorem. Gerber also gave a quantitative bound of Berry-Eseen type for this convergence:

\[ \sup_{x \in \mathbb{R}} |F_a(x) - \Phi(x)| \leq C \cdot \frac{E(|X_0 - \mu|^3)}{\sigma^3} \cdot (1 - a)^{\frac{1}{2}}, \quad (8) \]

and one can take \( C = 5.4 \) (we note that the formulation given in [6] is slightly different, but equivalent, because of the different normalization taken there). Subsequent works extended and refined the results of [6] in several directions (see e.g. [3, 8, 9, 10]).

Here we will prove a discounted central limit theorem without any assumption on moments higher than 2, and also give a new and different quantitative bound for the convergence, in the case that some moment of order \( s = 2 + \epsilon \) (\( \epsilon > 0 \)) exists. This bound will be given in terms of a Fourier-based metric, which will be seen to provide a simple and natural approach to the study of discounted sums. A key observation underlying our proofs is that the distribution \( F_a \) can be realized as a fixed point of a mapping on a space of distributions, which is a contraction with respect to this metric. Fourier-based metrics were introduced in connection with study of the Boltzmann equation [5], and have since found many applications (see e.g. [1] for a review). In particular in [7] these metrics have been used to obtain Berry-Esseen type inequalities.

We recall the definition of the Fourier-based metrics. For any real \( s > 0 \), we denote by \( \mathcal{P}^s \) the set of all distribution functions \( G \) on \( \mathbb{R} \) with finite moment of order \( s \), and with expectation 0 and variance 1:

\[ \int_{-\infty}^{\infty} |x|^s dG(x) < \infty, \quad (9) \]

\[ \int_{-\infty}^{\infty} x dG(x) = 0, \quad \int_{-\infty}^{\infty} x^2 dG(x) = 1. \quad (10) \]

To each distribution function \( G \) we associate its characteristic function

\[ C_G(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} dG(x). \]
If $s > 0$, and $G, H$ are probability distributions, their Fourier distance of type $s$ is defined by

$$d_s(G, H) = \sup_{\xi \neq 0} \frac{|C_G(\xi) - C_H(\xi)|}{|\xi|^s}. \quad (11)$$

If $s \in [2, 3]$ and $G, H \in \mathcal{P}^s$, then $d_s(G, H) < \infty$ (see [1], Proposition 2.6).

We prove that

**Theorem 1.** If (11) holds, then

$$\lim_{a \to 1^-} d_2(F_a, \Phi) = 0. \quad (12)$$

(12) implies pointwise convergence of $C_{F_a}$ to $C_{\Phi}$, which, by Levy’s Continuity Theorem (see e.g. [2], Sec 6.3), implies (7). The validity of the discounted central limit theorem (4), without any assumption on moments higher than 2, thus follows from Theorem 1 - we note however that it also follows from previous results such as those in [3].

A quantitative version of Theorem 1 can be obtained if we assume that $X_n$ have a finite $s$-moment for some $s > 2$. Set

$$\hat{X}_0 = \sigma^{-1}(X_0 - \mu),$$

and let $F$ denote its distribution function:

$$F(x) = P\left(\hat{X}_0 \leq x\right). \quad (13)$$

Note that by (11) have $F \in \mathcal{P}^2$.

**Theorem 2.** Assume $F \in \mathcal{P}^s$ where $s \in (2, 3]$. Then, for $a \in (0, 1)$

$$d_2(F_a, \Phi) \leq \left[\frac{(s - 2)(1 - a^2)}{e \cdot a^2}\right]^{(s-2)/2} \cdot d_s(F, \Phi). \quad (14)$$

Note that $s > 2$ implies that the right-hand side of (14) goes to 0 as $a \to 1$, so that (14) implies (12) for $s > 2$ (but not for $s = 2$, which is the reason that Theorem 1 needs a separate proof).

Comparing the bound of Theorem 2 with Gerber’s bound (8), we note several differences.

1. Theorem 2 provides a bound whenever some moment of order $s > 2$ is finite, while (5) requires a finite third central moment for $X_0$.

2. The bound (8) is universal, hence does not take into account the distance between the distribution of $X_0$ and the normal distribution. In (14), the bound becomes small if $X_0$ is close to normal.
A major difference is of course the fact that the distance between distributions is measured differently: while (14) uses a Fourier metric, (8) uses the Kolmogorov metric. In fact it is possible to bound the Kolmogorov metric in terms of the $d_2$ metric: using the Berry-Eseen inequality (see [4], Sec. XVI.4, Lemma 2) we get

$$|F_a(x) - \Phi(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \frac{|C_{F_a}(\xi) - C_\Phi(\xi)|}{|\xi|^2} |\xi| d\xi + \frac{24}{\pi T},$$

and optimizing over $T$ gives

$$\sup_{x \in \mathbb{R}} |F_a(x) - \Phi(x)| \leq \frac{3 \cdot 12^4}{\pi} \cdot \left( d_2(F_a, \Phi) \right)^{\frac{1}{3}},$$

so that convergence in the $d_2$ metric implies convergence in the Kolmogorov metric (as well as in the Wasserstein metric, see [1], Theorem 2.21). However, it should be noted that using this bound together with (14) gives a bound of order $O((1 - a)^{\frac{1}{6}})$ as $a \to 1$ for the convergence of the Kolmogorov metric, which in the case $s = 3$ (which is relevant for this comparison) gives $O((1 - a)^{\frac{1}{6}})$, a weaker convergence rate than the one given by (8).

We thus conclude that none of the inequalities (8) and (14) is a consequence of the other, and each has its advantages. It might be an interesting problem to obtain bounds which combine the advantages of the two inequalities.

2 Proofs of the theorems

Noting that $\hat{S}_a$ does not change if a linear function is applied to all $X_n$’s, there is no loss of generality in proving our results under the normalization

$$E(X_n) = 0, \ Var(X_n) = 1,$$

which we will henceforth assume. Under this assumption the distribution $F$ given by (13) is simply the distribution of the $X_n$’s:

$$F(x) = P(X_0 \leq x),$$

and (3) becomes

$$\hat{S}_a = \sqrt{1 - a^2} \cdot S_a.$$  

For $a \in [0, 1)$, we now define the operator $T_a : \mathcal{P} \rightarrow \mathcal{P}$, whose unique fixed point will later be shown to be the distribution function $F_a$ of $\hat{S}_a$. If $G \in \mathcal{P}$,
let $Y$ be a random variable with distribution function $G$. Let $X$ be a random variable independent of $Y$, with distribution function $F$ given by (15). Then $\mathcal{T}_a[G]$ is defined to be the distribution function of $aY + \sqrt{1-a^2} \cdot X$:

$$
\mathcal{T}_a[G](x) = P\left(aY + \sqrt{1-a^2} \cdot X \leq x\right).
$$

Since

$$
E\left(aY + \sqrt{1-a^2} \cdot X\right) = aE(Y) + \sqrt{1-a^2} \cdot E(X) = 0,
$$

$$
\text{Var}\left(aY + \sqrt{1-a^2} \cdot X\right) = a^2 \text{Var}(Y) + (1-a^2) \text{Var}(X) = a^2 + 1 - a^2 = 1,
$$

we indeed have $\mathcal{T}_a[G] \in \mathcal{P}^2$.

By the above definition, the $n$-fold composition $\mathcal{T}_a^n[G]$ ($n \geq 1$) is the distribution function of the random variable $Y_n$ given by the autoregressive process

$$
Y_{n+1} = aY_n + \sqrt{1-a^2} \cdot X_n, \quad n \geq 0 \tag{17}
$$

when $G$ is the distribution function of $Y_0$.

In terms of characteristic functions we have

$$
C_{\mathcal{T}_a[G]}(\xi) = C_F\left(\sqrt{1-a^2} \cdot \xi\right) \cdot C_G(a\xi). \tag{18}
$$

The following Lemma summarizes properties of the operator $\mathcal{T}_a$.

**Lemma 1.** For $a \in [0, 1)$:

(i) If $G, H \in \mathcal{P}^2$ then for any integer $n \geq 1$

$$
d_2(\mathcal{T}_a^n[G], \mathcal{T}_a^n[H]) \leq a^{2n} \cdot d_2(G, H). \tag{19}
$$

(ii) $F_a$, given by (12), is a fixed point of $\mathcal{T}_a$, and if $a > 0$ it is the unique fixed point.

(iii) For any $G \in \mathcal{P}^2$ we have

$$
d_2(G, F_a) \leq \frac{d_2(G, \mathcal{T}_a[G])}{1-a^2}. \tag{20}
$$

**Proof.** (i) For $n = 1$ we have, using (18) and the fact that $|C_F(\xi)| \leq 1$,

$$
d_2(T_a[G], T_a[H]) = \sup_{\xi \neq 0} \frac{|C_F(\sqrt{1-a^2} \cdot \xi)| \cdot |C_G(a\xi) - C_H(a\xi)|}{\xi^2}
\leq \sup_{\xi \neq 0} \frac{|C_G(a\xi) - C_H(a\xi)|}{\xi^2} = a^2 \cdot \sup_{\xi \neq 0} \frac{|C_G(a\xi) - C_H(a\xi)|}{(a\xi)^2} = a^2 \cdot d_2(G, H).
$$
Proceeding by induction, we have
\[ d_2 (T_n^{a+1} | G], T_n^{a+1} | H]) \leq a^2 \cdot d_2 (T_n^a | G], T_n^a | H]) = a^{2(n+1)} \cdot d_2 (G, H). \]

(ii) We denote equality in distribution of two random variables by \( \mathcal{D} \). Let \( X \) be a random variable with distribution \( F \), independent of \( S_a \). We claim that
\[ a \hat{S}_a + \sqrt{1 - a^2} \cdot X \overset{\mathcal{D}}{=} \hat{S}_a, \tag{21} \]
which implies \( T_n[F_a] = F_a \). To show (21), note that we have
\[ S_a = \sum_{n=0}^{\infty} a^n X_n = X_0 + a \sum_{n=0}^{\infty} a^n X_{n+1} \overset{\mathcal{D}}{=} X + aS_a, \]
so that, using (16),
\[ a \hat{S}_a + \sqrt{1 - a^2} \cdot X = \sqrt{1 - a^2} \cdot [aS_a + X] \overset{\mathcal{D}}{=} \sqrt{1 - a^2} \cdot S_a = \hat{S}_a, \]
so we have (21).

To show uniqueness of the fixed point when \( a > 0 \), assume \( G \in \mathcal{P}^2 \), \( T_n[G] = G \). Using (19),
\[ d_2 (F_a, G) = d_2 (T[F_a], T[G]) \leq a^2 d_2 (F_a, G) \Rightarrow d_2 (F_a, G) = 0 \Rightarrow G = F_a. \]

(iii) By (19) we have
\[ d_2 (T^n_{a}[G], T^{n+1}_{a}[G]) \leq a^{2n} d_2 (G, T^n_{a}[G]), \]
so the triangle inequality gives
\[ d_2 (G, T^n_{a}[G]) \leq \sum_{k=0}^{n-1} d_2 (T^k_{a}[G], T^{k+1}_{a}[G]) \leq \frac{1 - a^{2n}}{1 - a^2} \cdot d_2 (G, T^n_{a}[G]). \tag{22} \]

From (i),(ii) we have
\[ d_2 (T^n_{a}[G], F_a) = d_2 (T^n_{a}[G], T^n_{a}[F_a]) \leq a^{2n} \cdot d_2 (G, F_a). \tag{23} \]

Therefore, using the triangle inequality and (22), (23),
\[ d_2 (G, F_a) \leq d_2 (G, T^n_{a}[G]) + d_2 (T^n_{a}[G], F_a) \leq \frac{1 - a^{2n}}{1 - a^2} \cdot d_2 (G, T^n_{a}[G]) + a^{2n} \cdot d_2 (G, F_a), \]
and taking \( n \to \infty \) we obtain (20).

The following Lemma plays a key role in proving the theorems:
Lemma 2. Let \( \Phi \) be the Normal distribution function \((\text{6})\). Then for \( a \in [0,1) \)
\[
d_2 (F_a, \Phi) \leq \sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} \cdot \left| \frac{C_F(w) - C_\Phi(w)}{w^2} \right| \right].
\]

Proof. Applying \((\text{20})\) with \( G = \Phi \) we have
\[
d_2 (\Phi, F_a) \leq \frac{d_2 (\Phi, T_a[\Phi])}{1 - a^2}. \tag{24}
\]
The characteristic function of the normal distribution \( \Phi \) is given by \( C_\Phi (\xi) = e^{-\frac{\xi^2}{2}} \), so using \((\text{18})\) and the substitution \( w = \sqrt{1-a^2} \cdot \xi \), we have
\[
d_2 (T_a[\Phi], \Phi) = \sup_{\xi \neq 0} \left| \frac{C_{T_a[\Phi]}(\xi) - C_\Phi(\xi)}{(1-a^2)\xi^2} \right| = \sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} \cdot \left| \frac{C_F(w) - C_\Phi(w)}{w^2} \right| \right]. \tag{25}
\]
Combining \((\text{24})\) and \((\text{25})\) we have the result.

We can now give the proofs of the theorems.

Proof of Theorem \(\text{(7)}\). By Lemma \(\text{2}\) it suffices to show that
\[
\lim_{a \to 1} \sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} \cdot \left| \frac{C_F(w) - C_\Phi(w)}{w^2} \right| \right] = 0.
\]
Fix \( \epsilon > 0 \). By the assumption \((\text{11})\), \( C_F \) and \( C_\Phi \) are twice differentiable, with
\[
C_F(0) = C_\Phi(0) = 1, \ C_F'(0) = C_\Phi'(0) = 0, \ C_F''(0) = C_\Phi''(0) = -1
\]
so application of L’Hospital’s rule gives
\[
\lim_{w \to 0} \frac{C_F(w) - C_\Phi(w)}{w^2} = 0.
\]
Therefore we can choose \( \delta > 0 \) so that
\[
|w| < \delta \Rightarrow e^{-\frac{(aw)^2}{2(1-a^2)}} \cdot \left| \frac{C_F(w) - C_\Phi(w)}{w^2} \right| \leq \left| \frac{C_F(w) - C_\Phi(w)}{w^2} \right| < \epsilon. \tag{26}
\]
Using the fact that $|C_F(w)|, |C_\phi(w)| \leq 1$, we have

$$|w| \geq \delta \Rightarrow e^{-\frac{(aw)^2}{2(1-a^2)}} \frac{|C_F(w) - C_\phi(w)|}{w^2} \leq \frac{2}{\delta^2} e^{-\frac{(aw)^2}{2(1-a^2)}} \leq \frac{2}{\delta^2} e^{-\frac{(aw)^2}{2(1-a^2)}} \cdot |w|^2 \leq 2 \delta^2 e^{-\frac{(aw)^2}{2(1-a^2)}}.$$

The right-hand side of the above inequality goes to 0 as $a \to 1$, hence for $a$ sufficiently close to 1 we have

$$|w| \geq \delta \Rightarrow e^{-\frac{(aw)^2}{2(1-a^2)}} \frac{|C_F(w) - C_\phi(w)|}{w^2} \leq \epsilon.$$  \hspace{1cm} (27)

From (26),(27) we have that, for $a$ sufficiently close to 1,

$$\sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} \frac{|C_F(w) - C_\phi(w)|}{w^2} \right] \leq \epsilon,$$

concluding the proof.

**Proof of Theorem 2.** Assume $s > 2$. Using Lemma 2 we have

$$d_2(F_n, \Phi) \leq \sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} \frac{|C_F(w) - C_\phi(w)|}{w^2} \right] = \sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} \cdot |w|^{s-2} \cdot \frac{|C_F(w) - C_\phi(w)|}{|w|^s} \right] \leq d_s(F, \Phi) \cdot \sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} |w|^{s-2} \right]$$  \hspace{1cm} (28)

Computing the supremum on the right-hand side of (28) by elementary calculus we find that it is attained at $w = \pm \sqrt{\frac{(s-2)(1-a^2)}{a}}$, hence

$$\sup_{w \neq 0} \left[ e^{-\frac{(aw)^2}{2(1-a^2)}} |w|^{s-2} \right] = \left[ \frac{(s-2)(1-a^2)}{e \cdot a^2} \right]^\frac{1}{2} \left( \frac{1}{2} (s-2) \right),$$

which gives (14).

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