Adiabatic Particle Production with Decaying $\Lambda$ and Anisotropic Universe

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Here we study an anisotropic model of the universe with constant energy per particle. A decaying cosmological constant and particle production in an adiabatic process are considered as the sources for the entropy. The statefinder parameters $\{r, s\}$ are defined and their behaviour are analyzed graphically in some cases.

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The inclusion of cosmological constant into Einstein field equation was done primarily by Einstein himself with a view to model a static universe. Later he discarded the cosmological constant (and commented [1] “...... the biggest blunder of my life”), on the basis of the observational data supporting the expansion of the universe. Subsequently, it was reintroduced and then omitted to match with the observational evidences.

Recently, cosmological models with a cosmological constant are strong candidates to describe the dynamics of the universe. The observational results [2, 3] from type Ia Supernova suggests that our universe is accelerating. The increase of the expansion velocity can be accounted for by introducing a repulsive force which results a negative pressure in the energy-momentum tensor. Cosmological constant (or term, time dependent) is a good candidate to generate negative pressure and to account the vacuum contribution to the energy-momentum tensor. Further, the present small value of $\Lambda$ [4, 5] suggests that it is not a constant but have a dynamical evolution. As universe expands, the effective cosmological term evolves and decreases to its present value. This is also supported strongly from observational point of view, particularly, the anisotropy of the cosmic microwave background radiation, the supernova experiments and estimated age of the universe.

Moreover, the $\Lambda$-term can be considered as a type of dark non-baryonic matter (or dark matter) in the sense that it is not gravitationally clustered at all scales as the usual matter. Lastly, it is to be noted that in the literature there are cosmological models with particle production which are compatible with experimental evidences.

In this work, using the adiabatic condition, the dissipative pressure is considered due to decaying of a cosmological term and particle production. The thermodynamical condition is necessary for the conformity of the cosmic microwave background radiation due to creation of photons.

In thermodynamics, the energy conservation equation is

$$nTds = d\rho - (\mu + T\sigma)dn$$  \hspace{1cm} (1)

where the chemical potential $\mu$ satisfies the Euler equation

$$\mu = \frac{(\rho + p)}{n} - T\sigma$$  \hspace{1cm} (2)

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So eliminating \( \mu \) between (1) and (2) we have

\[
nT \dot{\sigma} = \dot{\rho} - \frac{(\rho + p)}{n} \dot{n}
\]

(‘ . ’ stands for time derivative)

The number density ‘\( n \)’ satisfies the continuity equation

\[
\dot{n} + n \Theta = \Psi
\]

with \( \Theta \), the expansion scalar and \( \psi \), the source of particles. Now the second law of thermodynamics can be written as

\[
S^\alpha_\alpha = n \dot{\sigma} + \sigma \dot{\Psi}
\]

with \( S^\alpha = n \sigma u^\alpha \) as the entropy four vector.

Now, considering the energy-momentum tensor as

\[
T^{\mu \nu} = (\rho + p_T)u^\mu u^\nu - p_T g^{\mu \nu} + \Lambda g^{\mu \nu}
\]

the energy conservation equation \( (T^{\mu \nu}_{\; ; \nu} = 0) \) results

\[
\dot{\rho} + (\rho + p_T) \Theta = -\frac{\dot{\Lambda}}{8\pi G}
\]

Here \( \rho \) is the matter density, \( p_T \) is the sum of the usual thermodynamical pressure \( p \) and a dissipative pressure \( \Pi \) (i.e., \( p_T = p + \Pi \)), \( u^\mu \) is the four velocity vector and \( \Lambda = \Lambda(t) \), a varying cosmological constant.

Now substituting \( \dot{\sigma} \) and \( \dot{n} \) from equation (3) and (4) and using the energy conservation equation (7) we have from equation (5), the expression for the entropy as

\[
S^\alpha_\alpha = -\frac{n}{T} \left[ \frac{\dot{\Lambda}}{8\pi G} + \frac{(\rho + p)}{n} \Psi + \Pi \Theta \right] + \sigma \dot{\Psi}
\]

Considering the particle production process as an adiabatic process (i.e., \( \dot{\sigma} = 0 \)) one writes

\[
\frac{\dot{\Lambda}}{8\pi G} + \frac{(\rho + p)}{n} \Psi + \Pi \Theta = 0
\]

The line element for the Kantowski-Sachs (KS) model is given by

\[
ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)d\Omega_2^2,
\]

The Einstein field equations for the matter content of the form (6) is given by

\[
\frac{2 \ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{k}{b^2} = -8\pi G (p + \Pi) + \Lambda
\]

\[
\frac{\ddot{a}}{a} + \frac{\dot{b}}{a} + \frac{\dot{a} \dot{b}}{ab} = -8\pi G (p + \Pi) + \Lambda
\]
and
\[ \frac{\dot{b}^2}{b^2} + 2\frac{\dot{a}}{a} \frac{\dot{b}}{b^2} + k \frac{1}{b^2} = 8\pi G \rho + \Lambda \]  
(12)

Assuming barotropic equation of state \( p = \epsilon \rho \) (\( 0 < \epsilon < 1 \)), one can eliminate \( \rho, p \) and \( \Pi \) using the field equations (10)-(12) and equation (9) to obtain
\[ \frac{2\dot{b}^2}{b^2} + \left( \frac{\dot{b}^2}{b^2} + k \frac{1}{b^2} \right) \left\{ (1 + \epsilon) \left( 1 - \frac{\Psi}{n \Theta} \right) \right\} + 2 \frac{\dot{a}}{a} \frac{\dot{b}}{b} \left\{ (1 + \epsilon) \left( 1 - \frac{\Psi}{n \Theta} \right) - 1 \right\} \]
\[ = \frac{\dot{\Lambda}}{\Theta} + \Lambda (1 + \epsilon) \left( 1 - \frac{\Psi}{n \Theta} \right) \]
(13)
with \( \Theta = \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} = 3H \).

In the present model, particle production is possible due to variation of \( \Lambda \) and due to dissipative pressure. We now assume, the contribution due to these terms are proportional namely,
\[ \frac{\dot{\Lambda}}{8\pi G} = \omega (\rho + p) \frac{\Psi}{n} \]
(14)

with \( \omega \) as the constant of proportionality. It is to be noted that vanishing of \( \Psi \) implies both \( \Lambda \) and entropy to be constant.

Combining equations (9) and (14), the dissipative pressure is given by
\[ \Pi = -(1 + \omega)(\rho + p) \frac{\Psi}{n \Theta} \]
(15)

Also taking into account relations (7), (9) and the field equations, one gets
\[ (\rho + p) \Theta \left( 1 - \frac{\Psi}{n \Theta} \right) = \frac{1}{8\pi G} (\Lambda - 3H^2) \]
(16)

Now choosing \( \Lambda = \alpha H^2 \) as proposed by Viswakarma [6] in recent past, we see that the particle source corresponding to the \( \Lambda \) term is
\[ \Psi = \beta n \Theta \]
(17)

where \( \beta \) is a phenomenological constant [7].

Using these relations, (namely, (15)-(17)) in the field equations to eliminate \( \rho, p \) and \( \Pi \), we have the differential equation in the scale factors as
\[ 2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} (1 + \epsilon) \{ 1 - (1 + \omega)\beta \} - 2\frac{\dot{a}}{a} \frac{\dot{b}}{b} \{ 1 - (1 + \epsilon) \{ 1 - (1 + \omega)\beta \} \} + \frac{k}{b^2} (1 + \epsilon) \{ 1 - (1 + \omega)\beta \} \]
\[ - \frac{\alpha}{9} \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right)^2 (1 + \epsilon) \{ 1 - (1 + \omega)\beta \} = 0 \]
(18)
Also the differential equation for $\Lambda$ (i.e., eq.(14)) using equations (15) and (17) can be solved as

$$
\Lambda = \omega \beta (1 + \epsilon) \left( ab^2 \right)^{-\omega \beta (1 + \epsilon)} \int \left( ab^2 \right)^{-\omega \beta (1 + \epsilon)} \left( \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} \right) \left( \frac{\dot{b}^2}{\dot{b}^2} + 2 \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{k}{b^2} \right) dt + \Lambda_0 
$$

Now for solution let us assume a power-law form for $b$ as

$$
b = b_0 t^\nu, \quad b_0, \nu \text{ are constants.} \quad (19)$$

Then from the field equations (10) and (11) the differential equation for ‘$a$’ becomes

$$
t^2 \ddot{a} - \nu (2\nu - 1)a + \nu t \dot{a} - \frac{k}{b_0^2} t^{2(1 - \nu)} a = 0 \quad (20)
$$

For $k \neq 0$, it has simple solution for $\nu = 1$ and $\frac{1}{2}$ as follows:

$$
\nu = 1 : \quad a = a_0 t \sqrt{1 + \frac{\beta}{b_0}}, \quad b = b_0 t \quad (21)
$$

$$
\nu = \frac{1}{2} : \quad a = a_0 \sinh \left( \frac{2 \sqrt{|k|} C t^\nu}{5} \right), \quad b = b_0 t^{\frac{\nu}{2}} \quad (22)
$$

However, for general $\nu$ the solution can be written as

$$
a = t^{\frac{1 - \nu}{2}} \left\{ a_0 I_{(1 - 3\nu)} \left[ \frac{\sqrt{k}}{(1 - \nu)b_0} t^{1 - \nu} \right] + a_1 I_{(1 - \nu)} \left[ \frac{\sqrt{k}}{(1 - \nu)b_0} t^{1 - \nu} \right] \right\} \quad (23)
$$

where $I$ is the usual modified Bessel function.

Note that for the solution with $\nu = 1$ the constraint satisfied by the constants appearing in equation (18) has the simple form

$$
(1 + \epsilon) \left\{ 1 - (1 + \omega) \beta \right\} \left( 3 - \alpha + \frac{k}{b_0^2} \right) = 2 \quad (24)
$$

Further, for $k = 0$, equation (20) has the general solution

$$
a = a_0 t^{1 - \nu} + a_1 t^{1 - 2\nu} \quad (25)
$$

In recent past, Sahni et al [8] proposed statefinder parameters to discriminate between different dark energy models. In FRW model with scale factor $a(t)$, the two parameters are defined as

$$
r = \frac{\ddot{a}}{a H^3}, \quad s = \frac{r - 1}{3 \left( q - \frac{1}{2} \right)} \quad (26)
$$

with $H$ and $q$ as the Hubble parameter and deceleration parameter respectively. These dimensionless parameters can be written in general form for any space-time as

$$
r = 1 + \frac{3 \dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \quad s = \frac{r - 1}{3 \left( q - \frac{1}{2} \right)}, \quad q = -1 - \frac{\ddot{H}}{H^2} \quad (27)
$$
Fig.1 shows the time variation of $q$ while Fig.2 represents the variation of the parameters $r$ and $s$ for $k \neq 0$ and $\nu = \frac{1}{2}$.

Fig.3 shows the time variation of $q$ while Fig.4 represents the variation of the parameters $r$ and $s$ for $k = 0$ and $\nu = \frac{1}{2}$.

In the recent model we shall determine these statefinder parameters for different solutions and examine their behaviour.

For $\nu = 1$, $k \neq 0$, $q$, $r$ and $s$ are all constants and $\lambda \propto t^{-2}$ with expressions:

$$q = \frac{1 - \xi}{2 + \xi}, \quad r = 1 - \frac{9\xi}{(2 + \xi)^2}, \quad s = \frac{6}{2 + \xi}, \quad \xi = \sqrt{1 + \frac{k}{b_0}^2}, \quad \Lambda = \frac{\Lambda_0}{t^2}$$

Note that $s$ is positive definite for all $k$ while the universe will be accelerating for closed model and decelerating for open model. For other $\nu$ or for flat model the expressions of these parameters are very complicated so we have shown their behaviour graphically only for $\nu = \frac{1}{2}$ with $k \neq 0$ or $k = 0$.

In figures 1 and 2, we have shown the time variation of $q$ and $r$-$s$ curve respectively for $\nu = \frac{1}{2}$, $k \neq 0$. The figure for $q$ shows that the universe starts from an decelerating phase (radiation era) to an accelerating phase and finally becomes a $\Lambda$CDM model. This is also reflected in the $r$-$s$ curve. The branch on the r.h.s. of the asymptote (in Fig.2) represents the universe from radiation era to dust phase (when $s$ becomes infinity) while the left portion of the curve continues from the dust phase and goes gradually to $\Lambda$CDM model.

For flat case (i.e., $k = 0$) with $\nu = \frac{1}{2}$ the time variation of $q$ and $r$-$s$ curve are shown in figures 3 and 4. These are not of much interest from observational point of view as the
model does not represent any accelerating phase of the universe (since $q$ is never negative).

Therefore, the present model with non-flat universe and $\nu = \frac{1}{3}$ can describe the evolution of the universe from radiation era to $\Lambda$CDM model but beyond radiation in the past is not describable by the model — probably some quantum theory is necessary to describe the evolution of the early era.

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