Hamiltonian mean field model : effect of temporal perturbation in coupling matrix

Nivedita Bhadra and Soumen K Patra
Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata, Mohanpur, Nadia, West Bengal 741246, India

The Hamiltonian mean-field model is a system of fully coupled rotators which exhibits a second order phase transition at some critical energy in its canonical ensemble. We investigate the case where the interaction between the rotors is governed by a time-dependent coupling matrix. Our numerical study reveals a shift in the critical point due to the temporal modulation. The shift in the critical point is shown to be independent of the modulation frequency above some threshold value whereas the impact of the amplitude of modulation is dominant. In the microcanonical ensemble, the system with constant coupling reaches a quasi-stationary state at an energy near the critical point. Our result indicates that the quasi-stationary state subsists in presence of such temporal modulation in the system.

PACS numbers: 05.20.-y, 05.70.Fh, 64.60.F

I. INTRODUCTION

Hamiltonian mean field model (HMF) represents a conservative system with long-range interactions of particles moving on a circle coupled by a repulsive or attractive cosine potential. This coupled rotator system exhibits several unusual properties such as the presence of quasistationary states (QSS) characterized by: anomalous diffusion, vanishing Lyapunov exponents, non-gaussian velocity distributions, aging and fractal-like phase space structure etc[1]. The underlying physics of this coupled rotor system explains several real life phenomena in free electron lasers[2], rarefied plasmas[3, 4], beam particle dynamics, the gravitational many body problem[5]. This system is an example of long-range interaction and shows a second order phase transition from a clustered phase to a gaseous one (where the particles are homogenously distributed on a circle) as a function of energy. Moreover, the system presents interesting features below its critical point. If the particles are prepared in a “water bag” initial state, the relaxation to the equilibrium becomes very slow[6–13]. In our present work, we introduce a temporal perturbation to the coupling and investigate the effect on the critical point and QSS.

In conventional HMF model, the moments of inertia of the rotors are usually considered to be time independent, identical, isotropic and they are equally interacting to all others rotors[14–17]. Such simplified model with uniform constant coupling provided several important insights about systems with long-range interaction. However, interactions are rarely uniform and heterogeneity in the coupling strength is more realistic in real-life for a system with long-range interaction e.g., stars and self-gravitating system has heterogeneous mass distribution, vortices in 2D turbulence have a heterogeneous circulation. The interaction is encoded in a coupling matrix and network topology of this matrix plays a crucial role in determining the critical behaviour of this thermodynamic system. It has been reported that the critical energy depends on the network parameter in such a system[18–20]. Some recent studies have shown that spectral properties of coupling matrix plays significant role in the synchronization of this system[21, 22]. HMF model on Erdös-Renyi networks was studied in[23]. Nigris-Leoncini[20] studied the model with Watt-Strogatz small-world network[24]. Importance of link density in a network to understand such a system with long-range interaction has been addressed in[23, 25, 26].

A natural question is: What happens when the element of the coupling matrix are time dependent? How does it influence the second order phase transition? To address this issue we consider a model where each element of the coupling matrix is temporally and periodically modulated. The coupling is taken in such a way that it is always positive in magnitude. We observe shift in the critical point in the canonical ensemble. We perform numerical analysis to study the system. Our study shows the system is insensitive to the change of the modulation frequency above certain threshold value, whereas, the effect of amplitude is dominant. Our numerical analysis shows that the QSS subsists in presence of such temporal modulation.

The rest of the paper is organized as follows. In Sec[1] we would introduce the standard HMF model and introduce HMF model with periodic modulation. Our numerical results has been discussed in Sec[11]. In Sec[14] we have drawn conclusion from the numerical results.

II. MODEL

The hamiltonian describing the HMF model with constant coupling is given by

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{\epsilon}{2N} \sum_{i,j=1}^{N} (1 - \cos(\phi_i - \phi_j)) = K + V, (1) \]

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{\epsilon}{2N} \sum_{i,j=1}^{N} (1 - \cos(\phi_i - \phi_j)) = K + V, (1) \]
where $\phi_i \in [-\pi, \pi]$ is the angle that particle $i$ makes with a reference axis and $p_i$ stands for its conjugate momentum. The system is basically a system of identical particles on a circle of unit mass or a classical XY rotor system with infinite range coupling. For $\epsilon > 0$ (attractive), the rotor tends to align (ferromagnetic case) whereas for $\epsilon < 0$ (repulsive), spin tends to anti-align (anti-ferromagnetic case). The first term is the kinetic energy $K$ and the second term $V$ is the interaction energy which is rescaled by total number of particle($N$) to make it thermodynamically stable. The $1/N$ factor in the potential energy makes the energy extensive (Kac prescription) and justify the validity of mean field approximation in the limit $N \to \infty$.

To understand the physical meaning of this system, the state of the art method is to consider a mean field vector $M$

$$M = Me^{i\phi} = \frac{1}{N} \sum_{i=1}^{N} m_i, \quad \text{(2)}$$

where, $m_i = (\cos \phi_i, \sin \phi_i)$. $M$ and $\phi$ are modulus and phase of the order parameter which specifies the clustering of particles or for XY model, it is the magnetization. The potential energy can be rewritten as a sum of single particle potentials $v_i$

$$V = \frac{1}{2} \sum_{i=1}^{N} v_i, \quad v_i = 1 - M \cos(\phi_i - \varphi). \quad \text{(3)}$$

This model is exactly solvable at equilibrium, where a second order phase transition is observed. The transition is from a low energy condensed phase or ferromagnetic phase where $M \neq 0$, to a high energy phase or paramagnetic phase with magnetization $M = 0$. The transition can be quantified from the caloric curve

$$U = \frac{\partial(\beta f)}{\partial \beta} = \frac{1}{2\beta} + \frac{\epsilon}{2}(1 - M^2), \quad \text{(4)}$$

where $\beta = 1/K_BT$, $K_B$ being the Boltzmann constant. Considering $\epsilon = 1, \beta = 2$, a transition is found at $U = U_c = 0.75$. At variance with this scenario we observe several departure from equilibrium case if we consider the case when the coupling $\epsilon$ is modulated. We have considered the case when the modulation is periodic and of the form $\epsilon(t) = |\alpha \cos \omega t|$. Hamiltonian governing our problem is given by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{\epsilon(t)}{2N} \sum_{i,j=1}^{N} (1 - \cos(\phi_i - \phi_j)) = K + V(\phi) \quad \text{(5)}$$

Assuming the similar mean filed vector we obtain the hamiltonian for this model of similar form

$$H(t) = \sum_{i=1}^{N} \frac{p_i^2}{2} + \epsilon(t)M \sum_{i=1}^{N} (1 - \cos(\phi_i - \varphi)). \quad \text{(6)}$$

The equation of motion for this system becomes

$$\ddot{\phi}_i + \epsilon(t)M \sum_{i=1}^{N} \cos(\phi_i - \varphi) = 0. \quad \text{(7)}$$

We change both the amplitude and frequency of the modulation and observe the effect. Our numerical analysis shows shift in critical point due to the presence of modulation in the coupling.

### III. NUMERICAL ANALYSIS

We perform N-body numerical simulations to obtain the critical point for this system in its canonical ensemble. The equation of motions are integrated (Eq.6) by adopting RK4 algorithm. The equation of motions are integrated starting from their formulation in terms of the single particle hamiltonian, where the dynamics of each particle is dependent on the meanfield variables (Eq.7). This approach reduces the computational cost to a large extent. This allows to perform simulations with large number of particles, e.g., N=100, 500, 1000, 10000. We took integration step as $dt = 0.01$.

The initial positions of particles are chosen to be uniformly distributed over the circle with a zero mean value. All the momenta are scaled in such a way that the desired initial total energy($U$) can be attained. To avoid transients effect the convergence of energy value has been checked for individual cases. Averaging was done typically over total integration steps $\approx 10^4$. However, the transients time and averaging has been changed for individual cases but the order remains almost the same. We took 20 different initial conditions for each simulation. Fig[4] shows variation of time averaged estimation of magnetization with energy for $\omega = 0, 10, 20$. Fig[2] shows variation of time averaged estimation of kinetic energy with initial energy for $\omega = 0, 10, 20$. We compare these two cases in Fig[4]. We have shown the simulated results for the constant coupling and temporally modulated coupling. We calculated the critical point in the curve by linear fitting. The critical point for the constant coupling case is $\approx 0.76$ and for the modulated case it is $\approx 0.50$.

We estimate the trapping probability of low energy particle for both cases. Trapping probability $P(U)$ was introduced in Ref.[13]. The particle can be divided into two section depending on the values of energy: high energy particles(HEP), having energy $e_i > \epsilon(1 + M)$ and low energy particles(LEP) having energy $e_i < \epsilon(1 + M)$. LEP are trapped particles bounded by the separatix. The trapping probability is defined as the ratio of LEP and total number of particle (LEP+HEP). Fig[6] shows the energy dependence of trapping probability. To estimate trapping probability we have considered particle number as $N=10000$. This represents the energy distribution of the system among the particles. Another statistical distribution of the mean magnetization is obtained for the
estimation of deviation from the constant coupling case. We start with three different initial energy $U = 0.1, 0.5, 1$ and see the distribution of mean magnetization in Fig. 7 for $\omega = 0, 10$ and 20.

The dynamical behaviour of this model can be investigated in the microcanonical ensemble by starting the system with water bag initial condition (WBIC) i.e., $\phi_i = 0$ and velocities uniformly distributed, and integrating the equation of motion Eq 2 for the model. Time evolution of $T = 2\langle K \rangle/N$ is shown in Fig. 8 for $\omega = 0$. Starting with WBIC the system rapidly reaches a quasistationary (QSS) or metastable state. This state needs a long time to relax to the canonical equilibrium state. This canonical inequivalence is observed just below $U_c$ (here we considered initial energy $U=0.69$). We have observed the similar appearance of QSS in presence of temporal periodic modulation. Time evolution of $T$ for $\omega = 10$ is shown in Fig. 8. We performed these simulation keeping the initial energy at $U = 0.4$.

**IV. CONCLUSION**

This HMF model is a limiting case of driven coupled pendula. Several interesting quantum mechanical phenomena can be manifested with the aid of newtonian mechanics of coupled classical harmonic oscillators. For example, quantum spin Hall effect(QSHE) can be exhibited by a two dimensional coupled system of mechanical oscillators. Coupling matrix for such a system plays a significant role to characterize the QSHE. Ref[27] beautifully demonstrates how they achieve mechanical ‘topological insulator’ displaying QSHE. Tunable spin orbit coupling has been realized in a classical system of six coupled pendula arranged in a 2D honeycomb lattice structure and thus showing QSHE[28]. A theoretical study reports dynamical localization in a classical context which is analo-

**FIG. 1**. Modulus of magnetization $M$ as a function of initial energy $U = H/N$ for N=1000. Inset figure shows how the magnetization is deviated in the modulated case from the constant coupling case.

**FIG. 2**. Variation of kinetic energy $T = 2\langle K \rangle/N$ as a function of initial energy $U$ for $N = 1000$. Inset figure shows how kinetic energy is deviated in the modulated case from the constant coupling case.

**FIG. 3**. Variation of critical point $U_c$ with amplitude of modulation $a$ for $\omega = 0$ and $\omega = 10$.

**FIG. 4**. Figure shows variation of the critical point($U_c$) with frequency of modulation.
clear observation of such effects. For example, one dimensional array of Josephson junction can be realised with an atomic BEC in optical lattice\cite{30}. The array is created by a laser standing wave and the condensates are trapped in the valleys of the periodic potential and weakly coupled by the interwell barriers. Periodic modulation of the potential barrier can be introduced in such a system experimentally. There exists huge volume of literature where such kind of periodic drive has been considered while studying BHH model with temporal modulation of the potential. The route to simulating this kind of model encouraged us to consider a system consisting of pendula with temporal and periodic modulation.

In our present work we investigate the phase transition of HMF model in presence of the modulation in the coupling. We observe shift in the critical point when temporal modulation is included. We have performed the numerical simulation for frequency of modulation $\omega$ over a large range and found that the system is insensitive to the change of frequency for $\omega > 2$ in Fig.\ref{fig:5}. The change of the function $\int \epsilon \, dt$ for a range of $\omega = 0$ to 20 is shown in Fig.\ref{fig:5}. The fluctuation of the time averaged value of the periodic coupling $\epsilon$ due to the change in frequency is very small. We performed the same simulations for different kind of periodic modulation and observed shift for those cases as well (we have not presented those results here). Here, critical point $U_c$ for the modulated case comes before the constant coupling case. This can be understood from the trapping probability. As soon as we apply the periodic modulation the fraction of low energy particle diminishes. As a result the system achieves a state when most of the particles are of higher energy than the critical energy $U_c$. It loses its clustered (ferromagnetic) phase and a homogeneous phase (paramagnetic) is achieved at a lower initial energy. Hence, average magnetism reaches zero value much before the constant coupling case. Therefore, by controlling the coupling the critical point for the phase transition critical point $U_c$ can be manipulated. In the microcanonical ensemble we observe similar appearance of QSS for both $\omega = 0$ and $\omega = 10$ cases.

V. ACKNOWLEDGEMENTS

The author would like to thank Anandamohan Ghosh for valuable discussions.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Figure shows how the function $f(t) = \int [a \cos(m \omega t)] \, dt$ changes as we change $\omega$ in the range 0 to 20. Inset shows the function $\varepsilon(t)$ as a function of time($t$). The change in the value of function for $\omega > 2$ is $10^{-2}$ which is negligible compared to the mean value of it.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6}
\caption{Trapping probability as a function of energy $U = H/N$. The horizontal straight segment is drawn to show the regime $P(U) \approx 1$ where most of the particles are trapped. The other segment guides the eyes through decreasing values of $P(U)$.}
\end{figure}

\begin{thebibliography}{9}
\bibitem{1} Vito Latora, Andrea Rapisarda, and Constantino Tsallis. Non-gaussian equilibrium in a long-range hamiltonian system. \textit{Phys. Rev. E}, 64:056134, Oct 2001.
\bibitem{2} A. Antoniazzi, Y. Elskens, D. Fanelli, and S. Ruffo. Statistical mechanics and vlasov equation allow for a simplified hamiltonian description of single-pass free electron lasersaturated dynamics. \textit{The European Physical Journal B - Condensed Matter and Complex Systems}, 50(4):603–611, Apr 2006.
\bibitem{3} Y Elskens and D Escande. Microscopic dynamics of plasmas and chaos. \textit{Plasma Physics and Controlled Fusion}, 45(4):521, 2003.
\bibitem{4} C Benedetti, S Rambaldi, and G Turchetti. Relaxation to boltzmann eqpisardauilibrium of 2d coulomb oscillati-
After initial quick cooling the system reaches a plateau for a long time and then reaches its equilibrium temperature.

FIG. 8. Time evolution of $T = 2⟨K⟩/N$ for 1000 different initial conditions for $ω = 0$. Initial energy is $U = 0.69$. Parameters are: $N = 500$, $a = 1.0$. After initial quick cooling the system reaches a plateau for a long time and then reaches its equilibrium temperature.

FIG. 9. Time evolution of $T = 2⟨K⟩/N$ for the energy density $U = 0.4$ with different initial conditions (we have shown 20 in this figure) when $ω = 10$. Parameters are: $N = 500$, $a = 1.0$. After initial quick cooling the system reaches a plateau for a long time and then reaches its equilibrium temperature.

[11] C. B. Tauro, G. Maglione, and F. A. Tamarit. Relaxation dynamics and topology in the hamiltonian mean field model. *The European Physical Journal Special Topics*, 143(1):9–12, Apr 2007.

[12] Andrea Antoniazzi, Francesco Califano, Duccio Fanelli, and Stefano Ruffo. Exploring the thermodynamic limit of hamiltonian models: Convergence to the vlasov equation. *Physical review letters*, 98(15):150602, 2007.

[13] X. Leoncini, T. L. Van Den Berg, and D. Fanelli. Out-of-equilibrium solutions in the xy-hamiltonian mean-field model. *EPL (Europhysics Letters)*, 86(2):20002, 2009.

[14] Mickael Antoni and Stefano Ruffo. Clustering and relaxation in hamiltonian long-range dynamics. *Phys. Rev. E*, 52:2361–2374, Sep 1995.

[15] Julien BarrÃľ, Freddy Bouchet, Thierry Dauxois, Stefano Ruffo, and Yoshiyuki Y. Yamaguchi. The vlasov equation and the hamiltonian mean-field model. *Physica A: Statistical Mechanics and its Applications*, 364:197–212, 2006.

[5] D. Lynden-Bell. Statistical mechanics of violent relaxation in stellar systems.

[6] Marcelo A Montemurro, Francisco A Tamarit, and Celia Anteneodo. Aging in an infinite-range hamiltonian system of coupled rotators. *Physical Review E*, 67(3):031106, 2003.

[7] Alessandro Pluchino, Vito Latora, and Andrea Rapisarda. Glassy dynamics in the hmf model. *Physica A: Statistical Mechanics and its Applications*, 340(1):187 – 195, 2004. News and Expectations in Thermostatistics.

[8] Andrea Rapisarda and Alessandro Pluchino. Nonextensive thermodynamics and glassy behaviour. *Europhysics News*, 36(6):202–206, 2005.

[9] A. Pluchino, A. Rapisarda, and V. Latora. Metastability and Anomalous Behavior in the Hmf Model: Connections to Nonextensive Thermodynamics and Glassy Dynamics. In C. Beck, G. Benedek, A. Rapisarda, and C. Tsallis, editors, *Complexity, Metastability and Nonextensivity*, pages 102–112, September 2005.
A: Statistical Mechanics and its Applications, 365(1):177 – 183, 2006. Fundamental Problems of Modern Statistical Mechanics.Proceedings of the 3rd International Conference on 'News, Expectations and Trends in Statistical Physics':News, Expectations and Trends in Statistical Physics.

[16] Alessandro Campa, Andrea Giansanti, and Gianluca Morelli. Long-time behavior of quasistationary states of the hamiltonian mean-field model. Phys. Rev. E, 76:041117, Oct 2007.

[17] T Konishi and K Kaneko. Clustered motion in symplectic coupled map systems. Journal of Physics A: Mathematical and General, 25(23):6283, 1992.

[18] Kateryna Medvedyeva, Petter Holme, Petter Minnhagen, and Beom Jun Kim. Dynamic critical behavior of the XY model in small-world networks. Phys. Rev. E, 67:036118, Mar 2003.

[19] Beom Jun Kim, H. Hong, Petter Holme, Gun Sang Jeon, Petter Minnhagen, and M. Y. Choi. Xy. Phys. Rev. E, 64:056135, Oct 2001.

[20] Sarah De Nigris and Xavier Leoncini. Critical behavior of the xy-rotor model on regular and small-world networks. Phys. Rev. E, 88:012131, Jul 2013.

[21] Juan G. Restrepo and James D. Meiss. Onset of synchronization in the disordered hamiltonian mean-field model. Phys. Rev. E, 89:052125, May 2014.

[22] Yogesh S. Virkar, Juan G. Restrepo, and James D. Meiss. Hamiltonian mean field model: Effect of network structure on synchronization dynamics. Phys. Rev. E, 92:052802, Nov 2015.

[23] Antonia Ciani, Duccio Fanelli, and Stefano Ruffo. Long-range Interactions and Diluted Networks, pages 83–132. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[24] Duncan J. Watts and Steven H. Strogatz. Collective dynamics of 'small-world' networks. Nature, 393(6684):440–442, Jun 1998.

[25] Albert Luo and Valentin Afraimovich. Long-range Interactions, Stochasticity and Fractional Dynamics: Dedicated to George M. Zaslavsky (1935–2008). Springer Science & Business Media, 2011.

[26] Sarah De Nigris and Xavier Leoncini. Emergence of a non-trivial fluctuating phase in the xy-rotors model on regular networks. EPL (Europhysics Letters), 101(1):10002, 2013.

[27] Roman Süssstrunk and Sebastian D Huber. Observation of phononic helical edge states in a mechanical topological insulator. Science, 349(6243):47–50, 2015.

[28] Grazia Salerno, Alice Berardo, Tomoki Ozawa, Hannah M Price, Ludovic Taxis, Nicola M Pugno, and Iacopo Carusotto. SpinâŠ–orbit coupling in a hexagonal ring of pendula. New Journal of Physics, 19(5):055001, 2017.

[29] Grazia Salerno and Iacopo Carusotto. Dynamical decoupling and dynamical isolation in temporally modulated coupled pendulums. EPL (Europhysics Letters), 106(2):24002, 2014.

[30] FS Cataliotti, S Burger, C Fort, P Maddaloni, Francesco Minardi, Andrea Trombettoni, Augusto Smerzi, and M Inguscio. Josephson junction arrays with bose-einstein condensates. Science, 293(5531):843–846, 2001.