ASSOCIATION BETWEEN TEMPERATE DISTRIBUTIONS AND ANALYTICAL FUNCTIONS IN THE CONTEXT OF WAVE-FRONT SETS

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Abstract. Let \( \mathcal{B} \) be a translation invariant Banach function space (BF-space). In this paper we prove that every temperate distribution \( f \) can be associated with a function \( F \) analytic in the convex tube \( \Omega = \{ z \in \mathbb{C}^d; | \text{Im} \, z | < 1 \} \) such that the wave-front set of \( f \) of Fourier BF-space types in intersection with \( \mathbb{R}^d \times S^{d-1} \) consists of the points \( (x, \xi) \) such that \( F \) does not belong to the Fourier BF-space at \( x - i\xi \).

0. Introduction

Wave-front sets of Fourier Banach function types where introduced by Coriasco, Johansson and Toft in [1]. Roughly speaking, the wave-front set of Fourier Banach function type, \( \text{WF}_{\mathcal{B}}(f) \), of a distribution \( f \), consists of all pairs \( (x_0, \xi_0) \) such that no localization of the distribution \( f \) at \( x_0 \) belongs to \( \mathcal{B} \) in the direction \( \xi_0 \). Several properties of classical wave-front sets (with respect to smoothness) can be found in Hörmander [12]. One of these are mapping properties for pseudo-differential operators (with smooth symbols) on wave-front sets which were generalized to Fourier Lebesgue type by Pilipovic, Teofanov and Toft in [14]. These properties were also proved to hold for wave-front sets of Fourier Banach function types. (Cf. Coriasco, Johansson and Toft [1].)

In this paper we consider another property of wave-front sets concerning association between a temperate distribution and an analytic function, which was proved for classical wave-front sets by Hörmander in [12]. More precisely, Hörmander showed that every temperate distribution \( f \) can be associated with a function \( F \) analytic in the convex tube \( \{ z \in \mathbb{C}^d; | \text{Im} \, z | < 1 \} \) such that

\[
f = \int_{|\xi|=1} F(\cdot + i\xi) \, d\xi, \tag{0.1}
\]
and

\[(\mathbb{R}^d \times S^{d-1}) \cap \text{WF}_L(f)\]

\[= \{(x,\xi); \, |\xi| = 1, \, F \text{ is not in } C^L \text{ at } x - i\xi\}.\]

Here WF\(_L(f)\) is the wave-front set with respect to a class of smooth functions \(C^L\). (Cf. Section 8.4 in Hörmander [12].)

In this paper we generalize this result to wave-front sets of Fourier Banach function types. We show that for every temperate distribution \(f\) there exists a function \(F\) with the properties given before, satisfying (0.1) and such that

\[(\mathbb{R}^d \times S^{d-1}) \cap \text{WF}_{\mathcal{F}\mathcal{B}}(f)\]

\[= \{(x,\xi); \, |\xi| = 1, \, F \text{ is not in } \mathcal{F}\mathcal{B} \text{ at } x - i\xi\}. \quad (0.2)\]

Since every Lebesgue space is a Banach function space we get by choosing \(\mathcal{B} = L^p\) that the analogous result for wave-front sets of Fourier Lebesgue types is contained in (0.2) as a special case.

As shown later on in this paper, analogous results hold also for the weighted cases as well as inf types and modulation space types of wave-front sets. The latter is a direct consequence of the identification of wave-front sets of Fourier BF-spaces types with wave-front sets of modulation space types.

The modulation spaces were introduced by Feichtinger in [2], and the theory was developed in [1, 4–6, 9]. The modulation space \(M(\omega, \mathcal{B})\), where \(\omega\) is an appropriate weight function (or time-frequency shift) on phase space \(\mathbb{R}^{2d}\), appears as the set of temperate (ultra-)distributions whose short-time Fourier transform belong to the weighted Banach space \(\mathcal{B}(\omega)\). This family of modulation spaces contains the (classical) modulation spaces \(M^p(\mathbb{R}^{2d})\) as well as the space \(W^{p,q}(\mathbb{R}^{2d})\) related to the Wiener amalgam spaces. In fact, these spaces which occur frequently in the time-frequency community are obtained by choosing \(\mathcal{B} = L^p(\mathbb{R}^{2d})\) or \(\mathcal{B} = L^p_2(\mathbb{R}^{2d})\) (see Remark 6.1 in [1]).

The paper is organized as follows. In Section 1 we recall the definitions and some basic properties for translation invariant Banach function spaces (BF-spaces) and Fourier Banach function spaces. In Section 2 we prove that every temperate distribution \(f\) can be associated with a function \(F\) analytic in a convex tube satisfying (0.1) and (0.2). Analogous results are given in Sections 3–5 for the weighted case, inf types and modulation space types, respectively. We use this result in Section 6 to show some further properties of these wave-front sets. In particular we show a result about the relation between wave-front sets of Fourier Banach function types and analytic wave-front sets.
1. Preliminaries

In this section we recall some notations and basic results. The proofs are in general omitted. In what follows we let $\Gamma$ denote an open cone in $\mathbb{R}^d \setminus 0$. If $\xi \in \mathbb{R}^d \setminus 0$ is fixed, then an open cone which contains $\xi$ is sometimes denoted by $\Gamma_\xi$.

Assume that $\omega, v \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ are positive functions. Then $\omega$ is called $v$-moderate if

$$\omega(x + y) \leq C \omega(x)v(y)$$

for some constant $C$ which is independent of $x, y \in \mathbb{R}^d$. If $v$ in (1.1) can be chosen as a polynomial, then $\omega$ is called polynomially moderate.

We let $P(\mathbb{R}^d)$ be the set of all polynomially moderated functions on $\mathbb{R}^d$. We say that $v$ is submultiplicative when (1.1) holds with $\omega = v$.

Throughout we assume that the submultiplicative weights are even.

If $\omega(x, \xi) \in P(\mathbb{R}^{2d})$ is constant with respect to the $x$-variable ($\xi$-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x, \xi)$. In this case we consider $\omega$ as an element in $P(\mathbb{R}^{2d})$ or in $P(\mathbb{R}^d)$ depending on the situation.

For any weight $\omega$ in $\mathcal{P}(\mathbb{R}^d)$ we let $L^p_{(\omega)}(\mathbb{R}^d)$ be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that $f \cdot \omega \in L^p(\mathbb{R}^d)$.

The Fourier transform $\mathcal{F}$ is the linear and continuous mapping on $\mathcal{S}'(\mathbb{R}^d)$ which takes the form

$$\mathcal{F}f(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)}

when $f \in L^1(\mathbb{R}^d)$. We recall that $\mathcal{F}$ is a homeomorphism on $\mathcal{S}'(\mathbb{R}^d)$ which restricts to a homeomorphism on $\mathcal{S}(\mathbb{R}^d)$ and to a unitary operator on $L^2(\mathbb{R}^d)$.

Next we recall the definition of Banach function spaces.

**Definition 1.1.** Assume that $\mathcal{B}$ is a Banach space of complex-valued measurable functions on $\mathbb{R}^d$ and that $v \in \mathcal{P}(\mathbb{R}^d)$ is submultiplicative.

Then $\mathcal{B}$ is called a (translation) invariant BF-space on $\mathbb{R}^d$ (with respect to $v$), if there is a constant $C$ such that the following conditions are fulfilled:

1. $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^d)$ (continuous embeddings);
2. if $x \in \mathbb{R}^d$ and $f \in \mathcal{B}$, then $f(\cdot - x) \in \mathcal{B}$, and

$$\|f(\cdot - x)\|_{\mathcal{B}} \leq C v(x)\|f\|_{\mathcal{B}};$$

3. if $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfy $g \in \mathcal{B}$ and $|f| \leq |g|$ almost everywhere, then $f \in \mathcal{B}$ and

$$\|f\|_{\mathcal{B}} \leq C\|g\|_{\mathcal{B}}.$$
Assume that $\mathcal{B}$ is a translation invariant BF-space. If $f \in \mathcal{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathcal{B}$ and

$$
\|f \cdot h\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{B}}\|h\|_{L^\infty}.
$$

**Remark 1.2.** Assume that $\omega_0, v, v_0 \in \mathcal{P}(\mathbb{R}^d)$ are such $v$ and $v_0$ are sub-multiplicative, $\omega_0$ is $v_0$-moderate, and assume that $\mathcal{B}$ is a translation-invariant BF-space on $\mathbb{R}^d$ with respect to $v$. Also let $\mathcal{B}_0$ be the Banach space which consists of all $f \in L_{1\text{loc}}^1(\mathbb{R}^d)$ such that $\|f\|_{\mathcal{B}_0} \equiv \|f \omega_0\|_{\mathcal{B}}$ is finite. Then $\mathcal{B}_0$ is a translation invariant BF-space with respect to $v_0v$.

For future references we note that if $\mathcal{B}$ is a translation invariant BF-space with respect to the submultiplicative weight $v$ on $\mathbb{R}^d$, then the convolution map $\ast$ on $\mathcal{S}(\mathbb{R}^d)$ extends to a continuous mapping from $\mathcal{B} \times L_{1\text{loc}}^1(\mathbb{R}^d)$ to $\mathcal{B}$, and for some constant $C$ it holds

$$
\|\varphi \ast f\|_{\mathcal{B}} \leq C\|\varphi\|_{L_{1\text{loc}}^1}\|f\|_{\mathcal{B}}, \tag{1.2}
$$

when $\varphi \in L_{1\text{loc}}^1(\mathbb{R}^d)$ and $f \in \mathcal{B}$. In fact, if $f, g \in \mathcal{B}$, then $f \ast g \in \mathcal{B} \subseteq \mathcal{B}$ in view of the definitions, and Minkowski’s inequality gives

$$
\|f \ast g\|_{\mathcal{B}} = \left\| \int f(\cdot - y)g(y)\,dy \right\|_{\mathcal{B}}
$$

$$
\leq \int \|f(\cdot - y)\|_{\mathcal{B}}|g(y)|\,dy \leq C \int \|f\|_{\mathcal{B}}|g(y)v(y)|\,dy = C\|f\|_{\mathcal{B}}\|g\|_{L_{1\text{loc}}^1}.
$$

Since $\mathcal{S}$ is dense in $L_{1\text{loc}}^1$, it follows that $\varphi \ast f \in \mathcal{B}$ when $\varphi \in L_{1\text{loc}}^1$ and $f \in \mathcal{S}$, and that (1.2) holds in this case. The result is now a consequence of Hahn–Banach’s theorem.

From now on we assume that each translation invariant BF-space $\mathcal{B}$ is such that the convolution map $\ast$ on $\mathcal{S}(\mathbb{R}^d)$ is uniquely extendable to a continuous mapping from $\mathcal{B} \times L_{1\text{loc}}^1(\mathbb{R}^d)$ to $\mathcal{B}$, and that (1.2) holds when $\varphi \in L_{1\text{loc}}^1(\mathbb{R}^d)$ and $f \in \mathcal{B}$. We note that $\mathcal{B}$ can be any mixed and weighted Lebesgue space.

In particular we then have that

$$
\left\| \int_{|y|=1} g(\cdot - y)\,dy \right\|_{\mathcal{S}_{\mathcal{B}}} \leq C \left\| g(\cdot - y) \right\|_{\mathcal{S}_{\mathcal{B}}} dy.
$$

Assume that $\mathcal{B}$ is a translation invariant BF-space on $\mathbb{R}^d$ and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then we let $\mathcal{F}\mathcal{B}(\omega)$ be the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\xi \mapsto \hat{f}(\xi)\omega(\xi)$ belongs to $\mathcal{B}$. It follows that $\mathcal{F}\mathcal{B}(\omega)$ is a Banach space under the norm

$$
\|f\|_{\mathcal{F}\mathcal{B}(\omega)} \equiv \|\hat{f}\omega\|_{\mathcal{B}}.
$$

Recall that a topological vector space $V \subseteq \mathcal{S}'(X)$ is called local if $V \subseteq V_{\text{loc}}$. Here $X \subseteq \mathbb{R}^d$ is open, and $V_{\text{loc}}$ consists of all $f \in \mathcal{S}'(X)$ such that $\varphi f \in V$ for every $\varphi \in C_0^\infty(X)$. For future references we note that
if $B$ is a translation invariant BF-space on $\mathbb{R}^d$, then it follows from (1.2) that $F_B$ is a local space, i.e.

$$F_B \subseteq F_B_{\text{loc}} \equiv (F_B)_{\text{loc}}.$$  

Let

$$I(\xi) = \int_{|\omega|=1} e^{-\langle \omega, \xi \rangle} d\omega \quad \text{and} \quad K(z) = (2\pi)^{-d} \int e^{i\langle z, \xi \rangle}/I(\xi) \, d\xi.$$  

These functions will play an important role when proving the main results. We therefore explicitly give properties of these functions. These results can be found in Section 8.4 in Hörmander [12].

Let $I$ be given by (1.3) then we have that

$$I(\xi) = 2 \cosh \xi \quad \text{for} \quad d = 1$$  

and

$$I(\xi) = I_0(\langle \xi, \xi \rangle^{1/2}) \quad \text{for} \quad d > 1.$$  

Here

$$I_0(\rho) = c_{d-1} \int_{-1}^{1} (1 - t^2)^{(d-3)/2} e^{t\rho} \, dt,$$  

where $c_{d-1}$ is the area of $S^{d-2}$. Then $I_0$ is an even analytic function in $\mathbb{C}$ such that for every $\varepsilon > 0$

$$I_0(\rho) = (2\pi)^{(d-1)/2} e^{\rho} \rho^{-(d-1)/2} (1 + O(1/\rho))$$  

if $\rho \to \infty$, $|\arg \rho| < \pi/2 - \varepsilon$. Furthermore there is a constant $C$ such that for all $\rho \in \mathbb{C}$ we have that

$$|I_0(\rho)| \leq C (1 + |\rho|)^{-(d-1)/2} e^{|\text{Re} \rho|}.$$  

The following lemma can be found with proof in [12].

**Lemma 1.3.** $K(z)$ is an analytic function in the connected open set

$$\tilde{\Omega} = \{ z \in \mathbb{C}^d; \langle z, z \rangle \not\in (-\infty, -1] \} \supseteq \Omega.$$  

Here $\Omega = \{ z \in \mathbb{C}; |\text{Im} \, z| < 1 \}$. Furthermore, for any closed open cone $\Gamma \subset \tilde{\Omega}$ such that $\langle z, z \rangle$ is never $\leq 0$ when $z \in \Gamma \setminus 0$ there is some $c > 0$ such that $K(z) = O(e^{-c|z|})$ when $z \to \infty$ in $\Gamma$. We have for real $x$ and $y$ that

$$|K(x + iy)| \leq K(iy) = (d - 1)! (2\pi)^{-d} (1 - |y|)^{-d} (1 + O(1 - |y|)), \quad (1.5)$$  

$|y| \to 1^-$.

Furthermore

$$|D^\beta K(x + iy)| \leq C_{\beta} (1 - |y|)^{-n - |\beta|} e^{-c|x|}, \quad |y| < 1$$  

holds by Cauchy’s inequalities. (Cf. [12].)
2. Analytic functions associated with temperate distributions

In this section we show that (0.2) holds. Assume that $\omega \in \mathcal{P}(\mathbb{R}^d)$. We recall that the wave-front sets of weighted Fourier Banach function types $WF_{\mathcal{B}(\omega)}(f)$ consists of all pairs $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d \setminus 0$ such that

$$|\varphi f|_{\mathcal{B}(\omega, \Gamma_{\xi_0})} \equiv \| \mathcal{F}(\varphi f) \chi_{\Gamma_{\xi_0}} \omega \|_\mathcal{B} = \infty,$$

for every open conical neighbourhood $\Gamma_{\xi_0}$ of $\xi_0$, and $\varphi \in C_0^\infty$ with $\varphi = 1$ in some open neighbourhood $X$ of $x_0$. Here $\chi_{\Gamma_{\xi_0}}$ is the characteristic function of $\Gamma_{\xi_0}$. (Cf. Coriasco, Johansson and Toft [1].)

Let $WF_{\mathcal{B}(\omega)}(f) = WF_{\mathcal{B}}(f)$ if $\omega \equiv 1$.

**Definition 2.1.** Assume that $f \in \mathcal{D}'(\mathbb{R}^d)$, $\mathcal{B}$ is a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then $f \in \mathcal{F}\mathcal{B}(\omega)$ at $x_0$ if and only if there exists $\varphi \in C_0^\infty$ with $\varphi \equiv 1$ in a neighbourhood of $x$ such that $\varphi f \in \mathcal{F}\mathcal{B}(\omega)$.

**Remark 2.2.** For convenience we say that $f \in \mathcal{F}\mathcal{B}$ at $x_0$ if the statement in Definition 2.1 is true for $\omega \equiv 1$.

We note that if $f$ belongs to $\mathcal{F}\mathcal{B}(\omega)$ at $x_0$ then $(x_0, \xi_0) \notin WF_{\mathcal{B}(\omega)}(f)$ for any $\xi_0 \in \mathbb{R}^d \setminus 0$.

**Definition 2.3.** For $f \in \mathcal{D}'(X)$ the singular support $\text{sing supp}_{\mathcal{B}(\omega)} f$ is the smallest closed subset of $X$ such that $f$ is in $\mathcal{F}\mathcal{B}(\omega)$ in the complement.

We use the notation $\text{sing supp}_{\mathcal{B}(\omega)} f = \text{sing supp}_\mathcal{B} f$ when $\omega \equiv 1$.

**Theorem 2.4.** Assume that $f \in \mathcal{D}'(\mathbb{R}^d)$, $\mathcal{B}$ is a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^d)$. The projection of $WF_{\mathcal{B}(\omega)}(f)$ in $X$ is equal to $\text{sing supp}_{\mathcal{B}(\omega)} f$.

**Proof.** (a) Assume that $x_0 \notin \text{sing supp}_{\mathcal{B}(\omega)}(f)$. Then $f$ belongs to $\mathcal{F}\mathcal{B}(\omega)$ at $x_0$. This implies that $(x_0, \xi_0) \notin WF_{\mathcal{B}(\omega)}(f)$, for any $\xi_0 \in \mathbb{R}^d \setminus 0$.

(b) Assume that $(x_0, \xi_0) \notin WF_{\mathcal{B}(\omega)}(f)$ for all $\xi_0 \in \mathbb{R}^d \setminus 0$. Then by the compactness of unit sphere we can choose a neighbourhood $K$ of $x_0$ such that $WF_{\mathcal{B}(\omega)}(f) \cap (K \times \mathbb{R}^d) = \emptyset$. This implies that we can choose a function $\varphi_{x_0} \in C_0^\infty$ which is equal to 1 in a neighbourhood $X$ of $x_0$ such that $\varphi_{x_0} f \in \mathcal{F}\mathcal{B}(\omega)$. Hence $x_0 \notin \text{sing supp}_{\mathcal{B}(\omega)}(f)$.

The next theorem is given without proof since the result follows directly from Theorem 8.4.8 in Hörmander [12] together with the observation that $WF_{\mathcal{B}}(f) \subseteq WF(f)$.

**Theorem 2.5.** Let $X \subseteq \mathbb{R}^d$ be open, $\Gamma$ an open convex cone in $\mathbb{R}^d$ and let

$$Z = \{ z \in \mathbb{C}^d; \ Re \ z \in X, \ Im \ z \in \Gamma, \ |Im \ z| < \gamma \},$$
for some $\gamma > 0$. Also let $F$ be an analytic function in $Z$ such that
\[ |F(z)| \leq C|\text{Im } z|^{-N}, \quad z \in Z. \]

Then $F(\cdot + iy)$ has the limit $F_0 \in \mathcal{D}'^{N+1}(X)$ as $y \in \Gamma$ tends to zero and
\(WF_{\mathcal{F}\mathcal{B}}(F_0) \subset X \times (\Gamma^c \setminus 0),\) where $\Gamma^c$ is the dual cone of $\Gamma$. Furthermore $F = 0$ if $F_0 = 0$.

Next we associate the temperate distribution $f$ with a function $F$ analytic in the convex cone $\Omega = \{z \in \mathbb{C}^d; \ |\text{Im } z| < 1\}$ such that
\[
f = \int_{|\xi| = 1} F(\cdot + i\xi) \, d\xi.
\]
We recall the following result from Hörmander [12, Theorem 8.4.11].

**Theorem 2.6.** Let $K$ be given by (1.3). If $f \in \mathcal{S}'(\mathbb{R}^d)$ and $F = K \ast f$, then $F$ is analytic in $\Omega = \{z; \ |\text{Im } z| < 1\}$ and for some $C, a, b$
\[ |F(z)| \leq C(1 + |z|)^a(1 - |\text{Im } z|)^{-b}, \quad z \in \Omega. \]

The boundary values $F(\cdot + i\xi)$ are continuous functions of $\xi \in S^{d-1}$ with values in $\mathcal{S}'(\mathbb{R}^d)$, and
\[
\langle f, \phi \rangle = \int \langle F(\cdot + i\xi), \phi \rangle \, d\xi, \quad \phi \in \mathcal{S}. \tag{2.2}
\]
Conversely, if $F$ satisfies (2.1), then (2.2) defines a distribution $f \in \mathcal{S}'$ with $F = K \ast f$.

Next we give the main theorem.

**Theorem 2.7.** Assume that $f$ and $F$ satisfies the conditions in Theorem 2.6. Then we have that
\[
(R^d \times S^{d-1}) \cap WF_{\mathcal{F}\mathcal{B}}(f) = \{(x, \xi); \ |\xi| = 1, \ F \text{ is not in } \mathcal{F}\mathcal{B} \text{ at } x - i\xi\}.
\]

We remark that $F$ is in $\mathcal{F}\mathcal{B}$ at $x - i\xi$ if for some neighbourhood $V$ of $(x, \xi)$ there exists some localization $\varphi \in C^\infty_0$ with $\varphi = 1$ in $V$ such that $\varphi f \in \mathcal{F}\mathcal{B}$. Before the proof we note that $C^L$ is a subset of $\mathcal{F}\mathcal{B}$.

**Proof.** First assume that $(x_0, \xi_0) \notin WF_{\mathcal{F}\mathcal{B}}(f)$ and $|\xi_0| = 1$. Then we want to show that $F = K \ast f \notin \mathcal{F}\mathcal{B}$ at $x_0 - i\xi_0$. By the hypothesis there exist $r > 0$ and $\varphi_{x_0} \in C^\infty_0$ such that $\varphi_{x_0}(x) = 1$ if $|x - x_0| < r$ and an open conical neighbourhood $\Gamma_{\xi_0}$ of $\xi_0$ such that
\[
\|\mathcal{F}(\varphi_{x_0}f)\chi_{\Gamma_{\xi_0}}\|_{\mathcal{F}} < \infty.
\]
We also recall that since $\varphi_{x_0}f$ has compact supports it holds
\[
|\mathcal{F}(\varphi_{x_0}f)(\xi)| \leq C(1 + |\xi|)^M, \quad \xi \in \mathbb{R}^d,
\]
for some fixed constants $C, M \geq 0$. Set $f = \varphi_{x_0}f + v$ where $v = f(1 - \varphi_{x_0})$. Then $F = K \ast f = K \ast (\varphi_{x_0}f) + K \ast v$ and
\[
K \ast v(z) = \langle K(z - \cdot), v \rangle.
\]
Now $K(x + iy - t)$ is well-defined when $|y|^2 < 1 + |x - t|^2$, so it is well-defined and rapidly decreasing with all derivatives when $|t - x_0| \geq r$ if

$$|y|^2 < 1 + (r - |x - x_0|)^2, \quad |x - x_0| < r. \quad (2.3)$$

(Cf. Lemma 8.4.10 and Theorem 8.4.11 in [12].) It follows that $K \ast v$ is analytic and bounded in compact subsets of the set defined by (2.3), which is a neighbourhood of $x_0 - i\xi_0$. Then it follows that $K \ast v$ belongs to $\mathcal{FB}$ at $x_0 - i\xi_0$.

Next we consider $K \ast (\varphi_{x_0} f)$. It is left to prove that $K \ast (\varphi_{x_0} f)$ belongs to $\mathcal{FB}$ at $x_0 - i\xi_0$. The Fourier transform of $K \ast (\varphi_{x_0} f)(\cdot + iy)$ is $e^{-\langle y, \xi \rangle} \mathcal{F}(\varphi_{x_0} f)/I(\xi)$. By (8.4.12) in Hörmander [12] it follows that

$$\frac{1}{|I(\xi)|} \leq C e^{-|\xi|}(1 + |\xi|)^{(d-1)/2}.$$ 

Using this we conclude that

$$\|K \ast (\varphi_{x_0} f)\|_{\mathcal{FB}} = \|e^{-\langle y, \cdot \rangle} \mathcal{F}(\varphi_{x_0} f)/I(\cdot)\|_{\mathcal{FB}}$$

$$\leq C_1 \|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{(d-1)/2} \mathcal{F}(\varphi_{x_0} f)\|_{\mathcal{FB}}$$

$$\leq C_2 \|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{(d-1)/2} \mathcal{F}(\varphi_{x_0} f)\|_{\mathcal{FB}}$$

$$+ \|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{(d-1)/2} \mathcal{F}(\varphi_{x_0} f)(1 - \chi_{\Gamma_0})\|_{\mathcal{FB}} \quad (2.4)$$

For the first part in the right-hand side of (2.4) we recognize that for every $y$ such that $|y| < 1$ sup$\xi e^{-\langle y, \xi \rangle - |\xi|}(1 + |\xi|)^{(d-1)/2} < \infty$ and therefore

$$\|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{(d-1)/2} \mathcal{F}(\varphi_{x_0} f)\|_{\mathcal{FB}} < \infty.$$ 

Then for the second part we have that

$$\|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{(d-1)/2} \mathcal{F}(\varphi_{x_0} f)(1 - \chi_{\Gamma_0})\|_{\mathcal{FB}}$$

$$\leq C \|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{M+(d-1)/2}(1 - \chi_{\Gamma_0})\|_{\mathcal{FB}}.$$ 

Choose $\varepsilon > 0$ such that $\langle \xi_0, \xi \rangle < (1 - 2\varepsilon)|\xi|$ when $\xi \notin \Gamma_0$. Then

$$\langle y, \xi \rangle + |\xi| > \varepsilon |\xi|$$

if $\xi \notin \Gamma_0$ and $|y + \xi_0| < \varepsilon$. Hence we obtain

$$\|e^{-\langle y, \cdot \rangle - |\xi|}(1 + |\cdot|)^{M+(d-1)/2}(1 - \chi_{\Gamma_0})\|_{\mathcal{FB}}$$

$$\leq C \|e^{-|\xi|}(1 + |\cdot|)^{M+(d-1)/2}\|_{\mathcal{FB}} < \infty$$

This completes the first part of the proof. \qed

For the second part of the proof we need the following lemma.
Lemma 2.8. Let $d\mu$ be a measure on $S^{d-1}$ and $\Gamma$ an open convex cone such that

$$\langle y, \xi \rangle < 0$$

when $0 \neq y \in \overline{\Gamma}$, $\xi \in \text{supp} \ d\mu$.

If $F$ is analytic in $\Omega$ and satisfies (2.1), then

$$F_1(z) = \int F(z + i\xi) \ d\mu(\xi)$$

is analytic and $|F_1(z)| \leq C'(1 + |\text{Re} \ z|^{a} |\text{Im} \ z|^{-b}$ when $\text{Im} \ z \in \Gamma$ and $|\text{Im} \ z|$ is small enough.

For every measure $d\mu$ on $S^{d-1}$ we have

$$\text{WF}_{F^B}(F_{\mu}) \subset \{(x, \zeta); \ -\zeta/|\zeta| \in \text{supp} \ d\mu \text{ and } F \notin F^B \text{ at } x - i\zeta/|\zeta|\}$$

(2.5)

Here $F_{\mu} = \int F(\cdot + i\xi) \ d\mu(\xi)$.

Proof. The first statement was proved by Hörmander in [12, Theorem 8.4.12]. Let $\Gamma^\circ$ be the dual cone of $\Gamma$. By Theorem 2.5 it follows that

$$\text{WF}_{F^B}(F_{\mu}) \subset \mathbb{R}^d \times \Gamma^\circ.$$

Assume that

$$x_0 \in \{x; \ F \in F^B \text{ at } x + i\xi \text{ for every } \xi \in \text{supp} \ d\mu\}.$$

Then we have that for every $\xi_0 \in \text{supp} \ d\mu$ there exists an open neighbourhood $U_{\xi_0}$ of $(x_0, \xi_0)$ and a function $\varphi_{\xi_0} \in C_0^\infty$ with $\text{supp} \varphi_{\xi_0} \subseteq U_{\xi_0}$ such that $x \mapsto \varphi_{\xi_0} F \in F^B$. Since the set

$$\{x_0 + i\xi; \ \xi \in \text{supp} \ d\mu\}$$

is compact, it follows from arguments about compactness that there exist finitely many points $\xi_j$ such that

$$\{x_0 + i\xi_j; \ \xi_j \in \text{supp} \ d\mu\} \subseteq \bigcup U_{\xi_j}.$$

For every $\xi_j$ we choose an open neighbourhood $V_{\xi_j}$ of $x_0$ and let $X = \bigcap V_{\xi_j}$. Then we can choose $\varphi_0 \in C_0^\infty$ equal to one in the neighbourhood $X$ of $x_0$ such that $x \mapsto \varphi_0 F \in F^B$. Furthermore, we have that there exists $\varphi \in C_0^\infty$, with support in a neighbourhood of $x_0$, such that

$$\|\varphi F_{\mu}\|_{F^B} = \|\int \varphi F(\cdot + i\xi) \ d\mu\|_{F^B} \leq \int \|\varphi F(\cdot + i\xi)\|_{F^B} \ d\mu < \infty.$$

Then $\varphi F_{\mu} \in F^B$ at $x_0$.

From the arguments above it follows that

$$\text{sing supp}_{F^B}(F_{\mu}) \subset \{x; \ F \text{ is not in } F^B \text{ at } x + i\xi \text{ for some } \xi \in \text{supp} \ d\mu\}.$$

Then we may write $d\mu = \sum d\mu_j$ where $\text{supp} \ d\mu_j$ is contained in the intersection of $\text{supp} \ d\mu$ and a narrow open convex cone $V_j$. Applying
the result just proved with $d\mu$ replaced by $d\mu_j$ and $\Gamma$ replaced by the interior of the dual cone $-V_j^o$ we obtain

$$\WF_{\mathcal{FB}}(F_\mu) \subset \bigcup \{(x, \zeta); -\zeta/|\zeta| \in V_j, F \notin \mathcal{FB}\text{ at } x+i\xi \text{ for some } \xi \in V_j\}.$$ 

If $-\zeta/|\zeta| \notin \text{supp } d\mu$ or $F \in \mathcal{FB}$ at $x-i\zeta/|\zeta|$ we can choose the covering so that $-\zeta/|\zeta| \notin V_j$ for every $j$ or for all $j \neq 1$ while $F \in \mathcal{FB}$ at $x+i\xi$ for every $\xi \in V_1$. In both cases it follows that $(x, \xi) \notin \WF_{\mathcal{FB}}(F_\mu)$ which proves (2.5). This completes the proofs of Lemma 2.8 and Theorem 2.7. □

The following Corollary is an analogue to Corollary 8.4.13 in Hörmander [12].

**Corollary 2.9.** Let $\Gamma_1, \ldots, \Gamma_m$ be closed cones in $\mathbb{R}^d \setminus 0$ such that

$$\bigcup_{j=1}^m \Gamma_j = \mathbb{R}^d \setminus 0.$$ 

For every $f \in \mathcal{S}'(\mathbb{R}^d)$ there exists a decomposition $f = \sum_{j=1}^m f_j$, where $f_j \in \mathcal{S}'$ and

$$\WF_{\mathcal{FB}}(f_j) \subseteq \WF_{\mathcal{FB}}(f) \cap (\mathbb{R}^d \times \Gamma_j). \quad (2.6)$$

If there exists another decomposition $f = \sum_{j=1}^m f'_j$ which also satisfies the conditions above, then $f'_j = f_j + \sum_{k=1}^m f_{jk}$ where $f_{jk} \in \mathcal{S}'$, $f_{jk} = -f_{kj}$ and

$$\WF_{\mathcal{FB}}(f_{jk}) \subseteq \WF_{\mathcal{FB}}(f) \cap (\mathbb{R}^d \times (\Gamma_j \cap \Gamma_k)). \quad (2.7)$$

**Proof.** Let $\phi_j$ be the characteristic function on $\Gamma_j \setminus (\Gamma_1 \cup \cdots \cup \Gamma_{j-1})$. Then since supp $\phi_j \cap \text{supp } \phi_k = \emptyset$ for every $j \neq k$ and

$$\bigcup_{j=1}^m \Gamma_j \setminus (\Gamma_1 \cup \cdots \cup \Gamma_{j-1}) = \bigcup_{j=1}^m \Gamma_j = \mathbb{R}^d \setminus 0$$

it follows that $\sum \phi_j = 1$ in $\mathbb{R}^d \setminus 0$. Let $F = K \ast f$ and $F_j = K \ast (f'_j - f_j)$. Then

$$\sum_{j=1}^m F_j = \sum_{j=1}^m (K \ast (f'_j - f_j)) = K \ast (\sum_{j=1}^m f'_j - \sum_{j=1}^m f_j) = 0.$$ 

Let

$$f_j = \int F(\cdot - i\xi)\phi_j(\xi) \, d\xi$$

and

$$f_{jk} = \int F_j(\cdot - i\xi)\phi_k(\xi) \, d\xi - \int F_k(\cdot - i\xi)\phi_j(\xi) \, d\xi. \quad (2.8)$$
Then it follows by straightforward calculations that $f_j' = f_j + \sum_{k=1}^m f_{jk}$ and $f_{jk} = -f_{kj}$. More precisely we have that
\[
\sum_{k=1}^m f_{jk} = \int F_j(\cdot - i\xi) \sum_{k=1}^m \phi_k(\xi) \, d\xi - \int \sum_{k=1}^m F_k(\cdot - i\xi) \phi_j(\xi) \, d\xi
\]
\[= \int F_j(\cdot - i\xi) \, d\xi = f_j' - f_j.
\]
From Theorem 2.7 and Lemma 2.8 it follows that (2.6) holds using that $\phi_j$ has support in $\Gamma_j \cap S^{d-1}$ and letting $d\mu_j(\xi) = \phi_j(\xi) \, d\xi$. Use the measure defined above and treat the integrals on the right-hand side of (2.8) separately. By using the arguments above we see that the wave-front sets of these integrals are contained in
\[
(WF_{\mathcal{FB}}(f_j) \cup WF_{\mathcal{FB}}(f_j')) \cap (\mathbb{R}^d \times \Gamma_k)
\]
and
\[
(WF_{\mathcal{FB}}(f_k) \cup WF_{\mathcal{FB}}(f_k')) \cap (\mathbb{R}^d \times \Gamma_j)
\]
respectively. Now (2.7) follows immediately from this together with the fact that $f_j$ and $f_j'$ satisfies (2.6). □

3. Wave-front sets of weighted Fourier BF-types

In this section we consider weighted Fourier BF-spaces and prove results analogous to the non-weighted case. We start by assuming that $\mathcal{B}$ is a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then let $\mathcal{B}_1 = \mathcal{B}(\omega)$. By the following lemma we see that there is no restriction to assume that $\omega$ is $v_0$-moderated for some $v_0 \in \mathcal{P}(\mathbb{R}^d)$ which is submultiplicative.

**Lemma 3.1.** Assume that $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then there exists $v_0 \in \mathcal{P}(\mathbb{R}^d)$ such that $v_0$ is submultiplicative and
\[
\omega(x) \leq Cv_0(x),
\]
where the constant $C > 0$ is independent of $x \in \mathbb{R}^d$.

**Proof.** Assume that $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then we can choose constants $N$ and $C$ large enough such that
\[
\omega(x) \leq C w(0) \langle x \rangle^N.
\]
Note that $C$ and $N$ do not depend on $x \in \mathbb{R}^d$. From the fact that
\[
\langle x + y \rangle \leq 2 \langle x \rangle \langle y \rangle,
\]
for every $x, y \in \mathbb{R}^d$ it follows that $\langle x \rangle^N$ is submultiplicative and polynomially moderated. □
Lemma 2.8. Also let $d\mu$ be a measure on $S^{d-1}$ and $\Gamma$ an open convex cone such that

$$\langle y, \xi \rangle < 0 \text{ when } 0 \neq y \in \overline{\Gamma}, \xi \in \text{supp } d\mu.$$ 

If $F$ is analytic in $\Omega$ and satisfies (2.1), then

$$F_1(z) = \int F(z + i\xi) \, d\mu(\xi)$$

is analytic and $|F_1(z)| \leq C(|1 + |\text{Re } z||)|\text{Im } z|^{-b}$ when $\text{Im } z \in \Gamma$ and $|\text{Im } z|$ is small enough.

For every measure $d\mu$ on $S^{d-1}$ we have

$$\text{WF}_{\mathcal{B};\omega}(F_\mu) \subset \{(x, \zeta); -\zeta/|\zeta| \in \text{supp } d\mu \text{ and } F \notin \mathcal{B}(\omega) \text{ at } x - i\zeta/|\zeta|\} \quad (3.1)$$

Here $F_\mu = \int F(\cdot + i\xi) \, d\mu(\xi)$.

Corollary 2.9. Let $\mathcal{B}$ and $\omega$, $v$ and $v_0$ be defined as in Theorem 2.7. Also let $\Gamma_1, \ldots, \Gamma_m$ be closed cones in $\mathbb{R}^d \setminus 0$ such that

$$\bigcup_{j=1}^m \Gamma_j = \mathbb{R}^d \setminus 0.$$ 

For every $f \in \mathcal{S}'(\mathbb{R}^d)$ there exists a decomposition $f = \sum_{j=1}^m f_j$, where $f_j \in \mathcal{S}'$ and

$$\text{WF}_{\mathcal{B};\omega}(f_j) \subseteq \text{WF}_{\mathcal{B};\omega}(f) \cap (\mathbb{R}^d \times \Gamma_j). \quad (3.2)$$

If there exists another decomposition $f = \sum_{j=1}^m f'_j$ which also satisfies the conditions above, then $f'_j = f_j + \sum_{k=1}^m f_{jk}$ where $f_{jk} \in \mathcal{S}'$, $f_{jk} = -f_{kj}$ and

$$\text{WF}_{\mathcal{B};\omega}(f_{jk}) \subset \text{WF}_{\mathcal{B};\omega}(f) \cap (\mathbb{R}^d \times (\Gamma_j \cap \Gamma_k)). \quad (3.3)$$
4. Wave-front sets of inf type

In this section we show analogous results for wave-front sets of inf types. We recall the definitions of these types of wave-front sets from Coriasco, Johansson and Toft [1]. Let $\mathcal{B}_j$ be a translation invariant BF-space on $\mathbb{R}^d$ and $\omega_j \in \mathcal{P}(\mathbb{R}^d)$, when $j$ belongs to some index set $J$, and consider the array of spaces, given by

$$ (B_j) \equiv (B_j)_{j \in J}, \text{ where } B_j = \mathcal{F}\mathcal{B}_j(\omega_j), \quad j \in J. \quad (4.1) $$

We recall that the wave-front sets of inf types $WF_{inf}(\mathcal{B}_j)(f) = WF_{inf}(\mathcal{F}\mathcal{B}_j(\omega_j))(f)$ consists of all pairs $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d \setminus 0$ such that for every open conical neighbourhood $\Gamma_{\xi_0}$ of $\xi_0$, every $\varphi \in C_0^\infty$ with $\varphi = 1$ in some open neighbourhood $X$ of $x_0$ and for every $j \in J$ it holds that

$$ |\varphi f|_{\mathcal{F}\mathcal{B}_j(\Gamma_{\xi_0})} \equiv \|\mathcal{F}(\varphi f)\chi_{\Gamma_{\xi_0}}\|_{\mathcal{B}_j(\omega_j)} = \infty. $$

Here $\chi_{\Gamma_{\xi_0}}$ is the characteristic function of $\Gamma_{\xi_0}$.

Before stating analogous results to those for wave-front sets of Fourier BF-spaces we compare the wave-front sets of Fourier BF-spaces with the wave-front sets of inf types defined above.

Since $(x_0, \xi_0) \in WF_{inf}(\mathcal{B}_j)(f)$ if and only if $(x_0, \xi_0) \in WF_{\mathcal{B}_j}(f)$ for every $j \in J$, it follows that

$$ WF_{inf}(\mathcal{B}_j)(f) = \bigcap_j WF_{\mathcal{B}_j}(f) \quad (4.2) $$

Theorem 2.7'. Let $\mathcal{B}_j$ be a translation invariant BF-space on $\mathbb{R}^d$ and $\omega \in \mathcal{P}(\mathbb{R}^d)$ for every $j \in J$. Also let $\mathcal{B}_j$ be defined as in (4.1) and let $f$ and $F$ satisfy the conditions in Theorem 2.6. Then we have that

$$ (\mathbb{R}^d \times S^{d-1}) \cap WF_{inf}(\mathcal{B}_j)(f) = \{(x, \xi); |\xi| = 1, F \text{ is not in } \bigcup_j \mathcal{B}_j \text{ at } x - i\xi\}. $$

Proof. We have that

$$ (\mathbb{R}^d \times S^{d-1}) \cap WF_{inf}(\mathcal{B}_j)(f) = \bigcap_j ((\mathbb{R}^d \times S^{d-1}) \cap WF_{\mathcal{B}_j}(f)) $$

$$ = \bigcap_j \{(x, \xi); |\xi| = 1, F \text{ is not in } \mathcal{B}_j \text{ at } x - i\xi\} $$

$$ = \{(x, \xi); |\xi| = 1, F \text{ is not in } \bigcup_j \mathcal{B}_j \text{ at } x - i\xi\}. \quad (4.3) $$

The proof is complete \qed

Lemma 4.1. Let $d\mu$ be a measure on $S^{d-1}$ and $\Gamma$ an open convex cone such that

$$ \langle y, \xi \rangle < 0 \text{ when } 0 \neq y \in \overline{\Gamma}, \xi \in \text{supp } d\mu. $$
If $F$ is analytic in $\Omega$ and satisfies (2.1), then
\[ F_1(z) = \int F(z + i\xi) \, d\mu(\xi) \]
is analytic and $|F_1(z)| \leq C'(1 + |\text{Re} \, z|) |\text{Im} \, z|^{-b}$ when $\text{Im} \, z \in \Gamma$ and $|\text{Im} \, z|$ is small enough.

For every measure $d\mu$ on $S^{d-1}$ we have
\[ \text{WF}^{\text{inf}}((B_j)(F)) \subset \{(x, \zeta); -\zeta/|\zeta| \in \text{supp } d\mu \text{ and } F \notin \bigcup_j B_j \text{ at } x - i\zeta/|\zeta|\} \] (4.4)

Here $F_\mu = \int F(\cdot + i\xi) \, d\mu(\xi)$.

The following Corollary is an analogue to Corollary 8.4.13 in Hörmander [12].

**Corollary 4.2.** Let $\Gamma_1, \ldots, \Gamma_m$ be closed cones in $\mathbb{R}^d \setminus 0$ such that
\[ \bigcup_{j=1}^m \Gamma_j = \mathbb{R}^d \setminus 0. \]

For every $f \in \mathcal{S}''(\mathbb{R}^d)$ there exists a decomposition $f = \sum_{j=1}^m f_j$, where $f_j \in \mathcal{S}'$ and
\[ \text{WF}^{\text{inf}}(B_j)(f_j) \subset \text{WF}^{\text{inf}}(B_j)(f) \cap (\mathbb{R}^d \times \Gamma_j). \] (4.5)

If there exists another decomposition $f = \sum_{j=1}^m f'_j$ which also satisfies the conditions above, then $f'_j = f_j + \sum_{k=1}^m f_{jk}$ where $f_{jk} \in \mathcal{S}'$, $f_{jk} = -f_{kj}$ and
\[ \text{WF}^{\text{inf}}(B_j)(f_{jk}) \subset \text{WF}^{\text{inf}}(B_j)(f) \cap (\mathbb{R}^d \times (\Gamma_j \cap \Gamma_k)). \] (4.6)

**5. Wave-front sets of modulation space types**

In this section we show that the results obtained for wave-front sets of Fourier BF-space types also hold for wave-front sets of modulation space types.

We start by defining general types of modulation spaces. Let (the window) $\phi \in \mathcal{S}'(\mathbb{R}^d) \setminus 0$ be fixed, and let $f \in \mathcal{S}''(\mathbb{R}^d)$. Then the short-time Fourier transform $V_\phi f$ is the element in $\mathcal{S}'(\mathbb{R}^{2d})$, defined by the formula
\[ (V_\phi f)(x, \xi) = \mathcal{F}(f \cdot \phi(\cdot - x))(\xi). \]

We usually assume that $\phi \in \mathcal{S}(\mathbb{R}^d)$, and in this case the short-time Fourier transform $(V_\phi f)$ takes the form
\[ (V_\phi f)(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y)\phi(y - x)e^{-i\langle y, \xi \rangle} \, dy, \]
when $f \in \mathcal{S}(\mathbb{R}^d)$.

Now let $\mathcal{B}$ be a translation invariant BF-space on $\mathbb{R}^{2d}$, with respect to $v \in \mathcal{S}(\mathbb{R}^{2d})$. Also let $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and $\omega \in \mathcal{S}(\mathbb{R}^{2d})$ be such that
ω is $v$-moderate. Then the modulation space $M(\omega) = M(\omega, \mathcal{B})$ is a Banach space with the norm
\[
\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_0 f \omega\|_{\mathcal{B}}
\] (cf. [8]).

Assume that $\omega \in \mathcal{P}(\mathbb{R}^{2d})$. We recall that the wave-front sets of modulation space types $WF_{M(\omega, \mathcal{B})}(f)$ consists of all pairs $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d \setminus 0$ such that
\[
|\varphi f|_{M(\omega, \mathcal{B}, \Gamma_{\xi_0})} \equiv \|V_0(\varphi f)\chi_{\Gamma_{\xi_0}} \omega\|_{\mathcal{B}} = \infty,
\]
for every open conical neighbourhood $\Gamma_{\xi_0}$ of $\xi_0$, and $\varphi \in C^\infty_0$ with $\varphi = 1$ in some open neighbourhood $X$ of $x_0$. Here $\chi_{\Gamma_{\xi_0}}$ is the characteristic function of $\Gamma_{\xi_0}$. It can also be showed that wave-front sets of modulation space types and wave-front sets of Fourier BF-types coincide. More precisely, let
\[
\mathcal{B}_0 = \{f \in \mathcal{P}'(\mathbb{R}^d) : \varphi \otimes f \in \mathcal{B}\}.
\] (5.2)

Then $\mathcal{B}_0$ is a translation invariant BF-space on $\mathbb{R}^d$, which is independent of the choice of $\varphi$. Furthermore $M(\omega, \mathcal{B})$ and $\mathcal{B}_0$ are locally the same and
\[
WF_{\mathcal{B}_0}(\omega)(f) = WF_{M(\omega, \mathcal{B})}(f).
\]
(Cf. Coriasco, Johansson and Toft [1].)

By using the previous results in combination with this we obtain the following results.

**Definition 5.1.** Assume that $f \in \mathcal{P}'(\mathbb{R}^d)$, $\mathcal{B}$ is a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^{2d})$. Then $f \in M(\omega, \mathcal{B})$ at $x_0$ if and only if there is some neighbourhood $X$ of $x_0$ such that for some $\varphi \in C^\infty_0$ with $\varphi \equiv 1$ in $X$ we have that $\varphi f \in M(\omega, \mathcal{B})$.

We recognize by the arguments before that since the definition above only concerns local properties it holds that $f \in M(\omega, \mathcal{B})$ at $x_0$ if and only if $f \in \mathcal{B}_0(\omega)$ at $x_0$, where $\mathcal{B}_0$ is given by (5.2).

We note that if $f$ belongs to $M(\omega, \mathcal{B})$ at $x_0$ then $(x_0, \xi_0) \notin WF_{M(\omega, \mathcal{B})}(f)$ for any $\xi_0 \in \mathbb{R}^d \setminus 0$.

**Definition 5.2.** For $f \in \mathcal{P}'(X)$ the singular support sing supp$_{M(\omega, \mathcal{B})} f$ is the smallest closed subset of $X$ such that $f$ is in $M(\omega, \mathcal{B})$ in the complement.

**Theorem 5.3.** Assume that $f \in \mathcal{P}'(\mathbb{R}^d)$, $\mathcal{B}$ is a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^{2d})$. The projection of $WF_{M(\omega, \mathcal{B})}(f)$ in $X$ is equal to sing supp$_{M(\omega, \mathcal{B})} f$.

*Proof.* (a) Assume that $x_0 \notin$ sing supp$_{M(\omega, \mathcal{B})} (f)$. Then $f$ belongs to $M(\omega, \mathcal{B})$ at $x_0$. This implies that $(x_0, \xi_0) \notin WF_{M(\omega, \mathcal{B})}(f)$, for any $\xi_0 \in \mathbb{R}^d \setminus 0$.

(b) Assume that $(x_0, \xi_0) \notin WF_{M(\omega, \mathcal{B})}(f)$ for all $\xi_0 \in \mathbb{R}^d \setminus 0$. Then we can choose a neighbourhood $K$ of $x_0$ such that $WF_{M(\omega, \mathcal{B})}(f) \cap (K \times \mathbb{R}^d)$
$\mathbb{R}^d = \emptyset$. This implies that we can choose a function $\varphi_{x_0} \in C_0^\infty$ which is equal to 1 in a neighbourhood $X$ of $x_0$ such that $\varphi_{x_0} f \in M(\omega, B)$. Hence $x_0 \notin \text{sing supp}_{M(\omega, B)}(f)$. □

Next theorem is analogous to Theorem 2.7.

**Theorem 2.7.** Assume that $f$ and $F$ satisfy the conditions in Theorem 2.6. Also let $B$ be a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Then we have that

$$(\mathbb{R}^d \times S^{d-1}) \cap \text{WF}_{M(\omega, B)}(f) = \{(x, \xi); |\xi| = 1, F \text{ is not in } M(\omega, B) \text{ at } x-i\xi\}.$$

We remark that $F$ is in $M(\omega, B)$ at $x-i\xi$ if for some neighbourhood $V$ of $(x, \xi)$ there exists some localization $\varphi \in C_0^\infty$ with $\varphi = 1$ in $V$ such that $\varphi f \in M(\omega, B)$.

**Proof.** Let $B_0$ be defined as before. Then it follows that

$$(\mathbb{R}^d \times S^{d-1}) \cap \text{WF}_{M(\omega, B)}(f) = (\mathbb{R}^d \times S^{d-1}) \cap \text{WF}_{B_0}(\omega)(f).$$

From the result in the previous section we also have that

$$(\mathbb{R}^d \times S^{d-1}) \cap \text{WF}_{B_0}(\omega)(f) = \{(x, \xi); |\xi| = 1, F \text{ is not in } B_0(\omega) \text{ at } x-i\xi\}.$$

Now since the right-hand side only concern local properties and $B_0(\omega)$ and $M(\omega, B)$ are locally the same it follows that

$$\{(x, \xi); |\xi| = 1, F \text{ is not in } B_0(\omega) \text{ at } x-i\xi\}$$

$$= \{(x, \xi); |\xi| = 1, F \text{ is not in } M(\omega, B) \text{ at } x-i\xi\}.$$

This completes the proof. □

By arguments given before it is obvious that Lemma 2.8 and Corollary 2.9 hold also for modulation spaces instead of Fourier BF-spaces. We therefore state the following results without proofs.

**Lemma 2.8.** Let $B$ be a translation invariant BF-space and $\omega \in \mathcal{P}(\mathbb{R}^d)$. Also let $d\mu$ be a measure on $S^{d-1}$ and $\Gamma$ an open convex cone such that

$$\langle y, \xi \rangle < 0 \text{ when } 0 \neq y \in \overline{\Gamma}, \xi \in \text{supp } d\mu.$$

If $F$ is analytic in $\Omega$ and satisfies (2.1), then

$$F_1(z) = \int F(z + i\xi) d\mu(\xi)$$

is analytic and $|F_1(z)| \leq C'(1 + |\text{Re } z|)^a |\text{Im } z|^{-b}$ when $\text{Im } z \in \Gamma$ and $|\text{Im } z|$ is small enough.

For every measure $d\mu$ on $S^{d-1}$ we have

$$\text{WF}_{M(\omega, B)}(F_\mu) \subset \{(x, \xi); -\langle \xi, \xi \rangle \in \text{supp } d\mu \text{ and } F \notin M(\omega, B) \text{ at } x-i\xi \}.$$  

Here $F_\mu = \int F(\cdot + i\xi) d\mu(\xi)$. (5.3)
Corollary 2.9. Let \( \mathcal{B} \) be a translation invariant BF-space and \( \omega \in \mathcal{P}(\mathbb{R}^d) \). Also let \( \Gamma_1, \ldots, \Gamma_m \) be closed cones in \( \mathbb{R}^d \setminus 0 \) such that
\[
\bigcup_{j=1}^m \Gamma_j = \mathbb{R}^d \setminus 0.
\]

For every \( f \in \mathcal{S}'(\mathbb{R}^d) \) there exists a decomposition \( f = \sum_{j=1}^m f_j \), where \( f_j \in \mathcal{S}' \) and
\[
WF_{\mathcal{M}(\omega, \mathcal{B})}(f_j) \subseteq WF_{\mathcal{M}(\omega, \mathcal{B})}(f) \cap (\mathbb{R}^d \times \Gamma_j). \tag{5.4}
\]
If there exists another decomposition \( f = \sum_{j=1}^m f_j' \) which also satisfies the conditions above, then \( f_j' = f_j + \sum_{k=1}^m f_{jk} \) where \( f_{jk} \in \mathcal{S}' \), \( f_{jk} = -f_{kj} \) and
\[
WF_{\mathcal{M}(\omega, \mathcal{B})}(f_{jk}) \subseteq WF_{\mathcal{M}(\omega, \mathcal{B})}(f) \cap (\mathbb{R}^d \times (\Gamma_j \cap \Gamma_k)). \tag{5.5}
\]

6. Some additional properties

In this section we prove some further properties for the wave-front sets of Fourier Banach types using results from the previous section.

Theorem 6.1. Let \( f \in \mathcal{S}'(X) \), \( X \subseteq \mathbb{R}^d \), and \( WF_{\mathcal{F} \mathcal{B}}(f) \subseteq X \times \Gamma^o \), where \( \Gamma^o \) is the dual of an open convex cone \( \Gamma \). If \( X_1 \subseteq X \) and \( \Gamma_1 \) is an open convex with \( \overline{\Gamma_1} \subseteq \Gamma \cup \{0\} \), then there exists a function \( F \) that is analytic in \( \{x + iy; x \in X_1, y \in \Gamma_1, |y| < \gamma\} \), such that
\[
|F(x + iy)| < C|y|^{-N}, \quad y \in \Gamma_1, \quad x \in X_1,
\]
and such that the limit of \( F(\cdot - iy) \) in \( \Gamma_1 \), when \( y \to 0 \), differs from \( f \) by an element in \( \mathcal{F} \mathcal{B}(X_1) \).

Proof. Set \( v = \chi f \) where \( \chi \in C_0^\infty \) is equal to 1 in \( X_1 \). If \( V = K \ast v \) is defined as in Theorem 2.7, then
\[
WF_{\mathcal{F} \mathcal{B}}(v) = WF_{\mathcal{F} \mathcal{B}}(\chi f) \subseteq WF_{\mathcal{F} \mathcal{B}}(f) \subseteq X \times \Gamma^o
\]
gives
\[
\mathcal{C}(X \times \Gamma^o) \subseteq \mathcal{C} WF_{\mathcal{F} \mathcal{B}}(v).
\]
From this follows that
\[
X_1 \times \mathcal{C}\Gamma^o \subseteq \mathcal{C} WF_{\mathcal{F} \mathcal{B}}(v).
\]

Then Theorem 2.7 implies that \( V \in \mathcal{F} \mathcal{B} \) at every point in \( X_1 + i(S^{d-1} \cap \mathcal{C}(-\Gamma^o)) \). Choose an open set \( M \) with \( \Gamma^o \cap S^{d-1} \subseteq M \subseteq S^{d-1} \) and where \( \overline{M} \) belongs to the interior of \( \Gamma^o \). Then \( v = v_1 + v_2 \) where
\[
v_1 = \int_{-\xi \in M} V(\cdot + i\xi) \, d\xi
\]
belongs to \( \mathcal{F} \mathcal{B} \) in \( X_1 \) and \( v_2 \) is the boundary value of the analytic function
\[
F(z) = \int_{-\xi \in M} V(z + i\xi) \, d\xi, \quad \text{Im} \, z \in \Gamma_1, \quad |\text{Im} \, z| < \gamma.
\]
Lemma 2.8 completes the proof.

As mentioned before we have that \( \text{WF}_{\mathcal{B}}(f) \subseteq \text{WF}_{A}(f) \). In the following proposition we describe a relation between the wave-front sets of Fourier Banach function types and analytic wave-front sets.

**Proposition 6.2.** Let \( \mathcal{B} \) be a translation invariant BF-space and \( f \in \mathcal{D}'(\mathbb{R}^d) \). Then

\[
\text{WF}_{\mathcal{B}}(f) = \bigcap_{g \in \mathcal{B}} \text{WF}_{A}(f - g).
\]

For the proof we need the following Lemmas which are extensions of Proposition 1.5 and Lemma 1.6 in [15].

**Lemma 6.3.** Let \( X \subseteq \mathbb{R}^d \) be open and \( \mathcal{B} \) be a translation invariant BF-space. Then the map \((f_1, f_2) \mapsto f_1 f_2\) from \( \mathcal{S} (\mathbb{R}^d) \times \mathcal{S} (\mathbb{R}^d) \) to \( \mathcal{S} (\mathbb{R}^d) \) extends uniquely to continuous mapping from \( \mathcal{B} \mathcal{B}(\mathbb{R}^d) \times \mathcal{F} L_{1,v}^1 (\mathbb{R}^d) \) to \( \mathcal{F} \mathcal{B}(\mathbb{R}^d) \).

**Proof.** (1) Let \( f_1 \in \mathcal{F} \mathcal{B}(\mathbb{R}^d) \) and \( f_2 \in \mathcal{S} (\mathbb{R}^d) \). By Minkowski’s inequality it follows that

\[
\|f_1 f_2\|_{\mathcal{B}} = \|\hat{f}_1 \ast \hat{f}_2\|_{\mathcal{B}} \leq C \|\hat{f}_1\|_{\mathcal{S}} \|\hat{f}_2\|_{L_{1,v}^1}.
\]

The assertion (1) now follows from this estimate and the fact that \( \mathcal{S} (\mathbb{R}^d) \) is dense in \( \mathcal{F} L_{1,v}^1 \). \( \square \)

**Lemma 6.4.** Let \( X \subseteq \mathbb{R}^d \) be open, \( f \in \mathcal{D}'(X) \) and let \( \mathcal{B} \) be a translation invariant BF-space. Also let \((x_0, \xi_0) \in X \times \mathbb{R}^d \setminus 0\). Then the following conditions are equivalent:

1. \((x_0, \xi_0) \not\in \text{WF}_{\mathcal{B}}(f)\);
2. there exists \( g \in \mathcal{F} \mathcal{B}(\mathbb{R}^d) \) \((g \in \mathcal{F} \mathcal{B}_{\text{loc}}(X))\) such that \((x_0, \xi_0) \not\in \text{WF}(f - g)\);
3. there exists \( g \in \mathcal{F} \mathcal{B}(\mathbb{R}^d) \) \((g \in \mathcal{F} \mathcal{B}_{\text{loc}}(X))\) such that \((x_0, \xi_0) \not\in \text{WF}_{A}(f - g)\).

**Proof.** In this proof we use the same ideas as in [13] Proposition 8.2.6 (see also [15] and [16]). We may assume that \( g \in \mathcal{F} \mathcal{B}_{\text{loc}}(X) \) in (2) and (3) since the wave-front sets concern local properties. Assume that (2) holds. We can then find an open subset \( X_0 \) of \( X \) and some open cone \( \Gamma = \Gamma \xi_0 \) and a sequence \( \varphi_N \in C_0^\infty \) such that \( \varphi_N (f - g) = f - g \) on \( X_0 \) and

\[
|\mathcal{F} (\varphi_N (f - g))(\xi)| \leq C_{N, \varphi_N} |\xi|^{-N}, \quad N = 1, 2, \ldots, \xi \in \Gamma. \quad (6.1)
\]

In particular it follows that if \( N_0 \) is chosen large enough, then \( |\varphi_N (f - g)|_{\mathcal{F} \mathcal{B} (\Gamma)} \) is finite for every \( N > N_0 \). Since \( g \in \mathcal{F} \mathcal{B}(\mathbb{R}^d) \), it follows by Lemma 6.3 that \( |\varphi_N g|_{\mathcal{F} \mathcal{B} (\Gamma)} \) is finite for every \( \varphi_N \). Then \( |\varphi_N f|_{\mathcal{F} \mathcal{B} (\Gamma)} \) is finite for every \( N > N_0 \) and (1) holds.
Conversely, if \((x_0, \xi_0) \notin \text{WF}_\mathcal{B}(f)\), then there exist an open neighbourhood \(X_0\) of \(x_0\) and an open conical neighbourhood \(\Gamma\) of \(\xi_0\) such that
\[
|\varphi f|_{\mathcal{B}(\Gamma)} < \infty,
\]
when \(\varphi \in C_0^\infty\), in view of Theorem 3.2 in [1].

Let \(\varphi_1, \varphi \in C_0^\infty(X_0)\) be chosen such that \(\varphi(x_0) \neq 0\) and \(\varphi_1 = 1\) in the support of \(\varphi\). Furthermore let \(\hat{g} = \mathcal{F}(\varphi_1 f)\) in \(\Gamma\) and otherwise 0. Then \(g \in \mathcal{B}(\mathbb{R}^d)\).

By [12, Lemma 8.1.1] and its proof, it follows that
\[
\|\mathcal{F}(\varphi_1(\varphi f - g))(\xi)\|_{\mathcal{B}} < C_N(\xi)^{-N}, \quad N = 0, 1, 2 \ldots,
\]
when \(\xi \in \Gamma\) and \(\Gamma\) is chosen sufficiently small. Since \(\varphi \varphi_1 = \varphi\) we have that (6.1) holds. This implies that \((x_0, \xi_0) \notin \text{WF}(f - g)\). This proves that (1) and (2) are equivalent.

Since \(\text{WF}(f) \subseteq \text{WF}_A(f)\) for each distribution \(f\), it follows that (2) holds if (3) is fulfilled. Assume that (2) holds. Then in view of of the remark before Corollary 8.4.16 in [12] there exists some \(h \in C^\infty(X)\) such that \((x_0, \xi_0) \notin \text{WF}_A(f - g - h)\). Since \(C^\infty \subseteq \mathcal{B}_{\text{loc}}(X)\) it follows that \(g_1 = g + h \in \mathcal{B}_{\text{loc}}(X)\). Hence (3) holds, and the result follows. \(\square\)

**Proof of Proposition 6.2.** We start by showing that
\[
\text{WF}_\mathcal{B}(f) \subseteq \text{WF}_A(f - g),
\]
for every \(g \in \mathcal{B}\). Since \(\text{WF}_\mathcal{B}(f - g) \subseteq \text{WF}_A(f - g)\) it is sufficient to show that
\[
\text{WF}_\mathcal{B}(f) \subseteq \text{WF}_\mathcal{B}(f - g), \tag{6.2}
\]
for every \(g \in \mathcal{B}\).

Assume that \((x_0, \xi_0) \notin \text{WF}_\mathcal{B}(f - g)\). Then there exist \(\varphi_{x_0} \in C_0^\infty\) with \(\varphi_{x_0}(x_0) \neq 0\) and an open conical neighbourhood \(\Gamma = \Gamma_{\xi_0}\) of \(\xi_0\) such that
\[
\|\mathcal{F}(\varphi_{x_0}(f - g))(\xi)\|_{\mathcal{B}} < \infty.
\]
It follows by Lemma 6.3 that
\[
\|\mathcal{F}(\varphi_{x_0} g)\chi_{\Gamma_{\xi_0}}\|_{\mathcal{B}} < \infty
\]
for every \(g \in \mathcal{B}\) and then
\[
\|\mathcal{F}(\varphi_{x_0} f)\chi_{\Gamma_{\xi_0}}\|_{\mathcal{B}} = \|\mathcal{F}(\varphi_{x_0}(f - g))(\xi)\|_{\mathcal{B}} + \|\mathcal{F}(\varphi_{x_0} g)\chi_{\Gamma_{\xi_0}}\|_{\mathcal{B}} < \infty.
\]
This shows that (6.2) holds. In fact, by similar calculations we can show the opposite inclusion and thereby obtain equality in (6.2). We have now shown that
\[
\text{WF}_\mathcal{B}(f) \subseteq \bigcap_{g \in \mathcal{B}} \text{WF}_A(f - g).
\]
We obtain the opposite inclusion by using Proposition 6.1. This completes the proof. \(\square\)
Corollary 6.5. If \( f \in \mathcal{D}'(X) \) where \( X \) is an interval on \( \mathbb{R} \) and if \( x_0 \in X \) is a boundary point of \( \text{supp} \, f \), then \( (x_0, \pm 1) \in \text{WF}_{\mathcal{F}\mathcal{B}}(f) \).

Proof. Assume for example that \( (x_0, -1) \not\in \text{WF}_{\mathcal{F}\mathcal{B}}(f) \). Then we can find \( F \) analytic in \( \Omega = \{z; \text{Im} \, z > 0, |z - x_0| < r\} \) with boundary value \( f \). There is an interval \( I \subseteq (x_0 - r, x_0 + r) \) where \( f = 0 \). By Theorem 3.1.12 and 4.4.1 in Hörmander \[12\] \( F \) can be extended analytically across \( I \) so that \( F = 0 \) below \( I \). Thus the uniqueness of analytic continuation gives \( F = 0 \), hence \( f = 0 \) in \( (x_0 - r, x_0 + r) \). This contradicts that \( x_0 \) is a boundary point of \( \text{supp} \, f \) and proves the corollary. \( \square \)

The result in the following lemma follows directly from Lemma 8.4.17 in Hörmander.

Lemma 6.6. If \( f \in \mathcal{D}' \) then \( \text{WF}_{\mathcal{F}\mathcal{B}}(f) \subseteq \mathbb{R}^d \times F \) where \( F \) is the limit cone of \( \text{supp} \, \hat{f} \) at infinity, consisting of all limits of sequences \( t_j x_j \) with \( x_j \in \text{supp} \, \hat{f} \) and \( 0 < t_j \to 0 \).

Next we give some computational rules for wave-front sets of Fourier Banach function types. We prove that some of the rules that Hörmander obtained for classical and analytical wave-front sets in \[12\] holds also for wave-front sets of Fourier Banach function types. For completeness we give the proofs which are similar to those of the analogous results in \[12\].

Theorem 6.7. Let \( X \subseteq \mathbb{R}^{d_1} \) and \( Y \subseteq \mathbb{R}^{d_2} \) be open. Also let \( f : X \to Y \) be a real analytic map with normal set \( N_f \). Then

\[
\text{WF}_{\mathcal{F}\mathcal{B}}(f^* g) \subseteq f^* \text{WF}_{\mathcal{F}\mathcal{B}}(g), \quad \text{if } g \in \mathcal{D}'(Y), \quad N_f \cap \text{WF}_{\mathcal{F}\mathcal{B}}(g) = \emptyset 
\]

(6.3)

Proof. Assume that there exists an analytic function \( \Phi \) in

\[
\Omega = \{y' + iy''; \, y' \in Y, y'' \in \Gamma, |y''| < \gamma\},
\]

where \( \Gamma \) is an open convex cone, such that

\[
|\Phi(y' + iy'')| \leq C|y''|^{-N} \quad \text{in } \Omega, \quad g = \lim_{\Gamma \ni y'' \to 0} \Phi(\cdot + iy).
\]

Let \( \Gamma^\circ \) be the dual \( \Gamma \). Then by the arguments in the proof of Theorem 8.5.1 in Hörmander \[12\] it follows that \( \text{WF}_A(g) \subseteq Y \times \Gamma^\circ \). If we assume that \( x_0 \in X \) and that \( ^t f'(x_0)\eta \neq 0 \), \( \eta \in \Gamma^\circ \setminus 0 \), then \( ^t f'(x_0)\Gamma^\circ \) is a closed, convex cone and

\[
\text{WF}_A(f^* g)|_{x_0} \subseteq \{(x_0, ^t f'(x_0)\eta); \, \eta \in \Gamma^\circ \setminus 0\}.
\]

Next by using Corollary 2.9 and Theorem 6.1 it follows that any distribution \( g \) can be written as a finite sum \( \sum g_j \) where each term either belongs to \( \mathcal{F}\mathcal{B} \) in a neighbourhood of \( f(x_0) \) of satisfies the hypotheses above with some \( \Gamma_j \) such that \( \Gamma_j^\circ \) is small and intersects \( \text{WF}_{\mathcal{F}\mathcal{B}}(g)|_{f(x_0)} \).
By the hypotheses \( f'(x_0) \eta \neq 0 \) when \((f(x_0), \eta) \in \text{WF}_{\mathcal{FB}}(f)\). We then conclude that
\[
\text{WF}_{\mathcal{FB}}(f^*g)|_{x_0} \subseteq \{(x_0, f'(x_0) \eta), \eta \in \bigcup \Gamma_j^0\}.
\]
This implies (6.3). \(\Box\)

**Theorem 6.8.** Let \( f \in \mathcal{E}'(\mathbb{R}^d) \). Split the coordinates in \( \mathbb{R}^d \) into two groups \( x' = (x_1, \ldots, x_{d_1}) \) and \( x'' = (x_{d_1+1}, \ldots, x_d) \), and set
\[
f_1(x') = \int f(x', x'') \, dx''
\]
Then
\[
\text{WF}_{\mathcal{FB}}(f_1) \subseteq \{(x', \xi'); (x', x'', \xi', 0) \in \text{WF}_{\mathcal{FB}}(f) \text{ for some } x''\}.
\]

**Proof.** We use the same notation as in Theorem 2.7. Then
\[
\langle f, \phi \otimes \psi \rangle = \int_{|\omega|=1} \langle U(\cdot + i\omega), \phi \otimes \psi \rangle \, d\omega,
\]
for \( \phi \in C^\infty_0(\mathbb{R}^{d_1}) \) and \( \psi \in C^\infty_0(\mathbb{R}^{d-d_1}) \). Take \( \psi(x'') = \chi(\delta x'') \) where \( \chi = 1 \) in the unit ball and let \( \delta \to 0 \). \( U \) is decreasing exponentially at infinity and therefore it follows that
\[
\langle f_1, \phi \rangle = \int_{|\omega|=1} \langle U(\cdot + i\omega), \phi \otimes 1 \rangle \, d\omega = \int_{|\omega|=1} \langle U_1(\cdot + i\omega'), \phi \rangle \, d\omega
\]
where
\[
U_1(z') = \int U(z', x'') \, dx'' = \int U(z', x'' + iy'') \, dx'', \quad |\text{Im } z'|^2 + |y''|^2 < 1
\]
is an analytic when \( |\text{Im } z'| < 1 \), which is bounded by \( C(1 - |\text{Im } z'|)^{-N} \).
If \( |\omega'_0| < 1 \) and \((x', x'', \omega'_0) \notin \text{WF}_{\mathcal{FB}}(f) \) for every \( x'' \in \mathbb{R}^{d-d_1} \), then \( u_1 \in \mathcal{FB} \) at \( x' - i\omega'_0 \). Hence Lemma 2.8 implies that \((x', \omega'_0) \notin \text{WF}_{\mathcal{FB}}(f)\). \(\Box\)

**Theorem 6.9.** Let \( X \subseteq \mathbb{R}^{d_1} \) and \( Y \subseteq \mathbb{R}^{d_2} \) be open sets and \( K \in \mathcal{D}'(X \times Y) \) be a distribution such that the projection \( \text{supp } K \rightarrow X \) is proper. If \( f \in \mathcal{FB}(Y) \) then
\[
\text{WF}_{\mathcal{FB}}(K \ast f) \subseteq \{(x, \xi); (x, y, \xi, 0) \in \text{WF}_{\mathcal{FB}}(K) \text{ for some } y \in \text{supp } f\}.
\]
Here \( K \) is the linear operator with kernel \( K \).

**Proof.** Replace \( K \) by \( K(1 \otimes f) \) and assume that \( f = 1 \). Without changing \( K \) over a given compact subset of \( X \) we may replace \( K \) by a distribution of compact support, and then the statement is identical to Theorem 6.8. \(\Box\)
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