Computing directed path-width and directed tree-width of recursively defined digraphs

Frank Gurski\textsuperscript{1} and Carolin Rehs\textsuperscript{1}

\textsuperscript{1}University of Düsseldorf, Institute of Computer Science, Algorithmics for Hard Problems Group, 40225 Düsseldorf, Germany

June 13, 2018

Abstract

In this paper we consider the directed path-width and directed tree-width of recursively defined digraphs. As an important combinatorial tool, we show how the directed path-width and the directed tree-width can be computed for the disjoint union, order composition, directed union, and series composition of two directed graphs. These results imply the equality of directed path-width and directed tree-width for all digraphs which can be defined by these four operations. This allows us to show a linear-time solution for computing the directed path-width and directed tree-width of all these digraphs. Since directed co-graphs are precisely those digraphs which can be defined by the disjoint union, order composition, and series composition our results imply the equality of directed path-width and directed tree-width for directed co-graphs and also a linear-time solution for computing the directed path-width and directed tree-width of directed co-graphs, which generalizes the known results for undirected co-graphs of Bodlaender and Möhring.

Keywords: directed path-width; directed tree-width; directed co-graphs

1 Introduction

Tree-width is a well-known graph parameter \cite{9}. Many NP-hard graph problems admit polynomial-time solutions when restricted to graphs of bounded tree-width using the tree-decomposition \cite{1, 3, 22, 27}. The same holds for path-width \cite{35} since a path-decomposition can be regarded as a special case of a tree-decomposition. Computing both parameters is hard even for bipartite graphs and complements of bipartite graphs \cite{2}, while for co-graphs it has been shown \cite{9, 10} that the path-width equals the tree-width and how to compute this value in linear time.

During the last years, width parameters for directed graphs have received a lot of attention \cite{10}. Among these are directed path-width and directed tree-width \cite{25}. Since for complete bi-oriented digraphs the directed path-width equals the (undirected) path-width of the corresponding underlying undirected graph it follows that determining whether the directed path-width of some given digraph is at most some given value \(w\) is NP-complete. The same holds for directed tree-width. There is an XP-algorithm for directed path-width w.r.t. the standard parameter by \cite{29}, which and implies that for each constant \(w\), it is decidable in polynomial time whether a given digraph has directed path-width at most \(w\). The same holds for directed tree-width by \cite{25}. This motivates to consider the recognition problem restricted to special digraph classes.

We show useful properties of directed path-decompositions and directed tree-decompositions, such as bidirectional complete subdigraph and bidirectional complete bipartite subdigraph lemmas. These results allow us to show how the directed path-width and directed tree-width can be computed for the disjoint union, order composition, directed union, and series composition of two directed graphs. Our proofs are constructive, i.e. a directed path-decomposition and a
directed tree-decomposition can be computed from a given expression. These results imply the equality of directed path-width and directed tree-width for all digraphs which can be defined by the disjoint union, order composition, directed union, and series composition. This allows us to show a linear-time solution for computing the directed path-width and directed tree-width of all these digraphs. Among these are directed co-graphs, which can be defined by disjoint union, order composition, and series composition. Directed co-graphs are useful to characterize digraphs of directed NLC-width 1 and digraphs of directed clique-width 2 and are useful for the reconstruction of the evolutionary history of genes or species using genomic sequence data. Our results imply the equality of directed path-width and directed tree-width for directed co-graphs and a linear-time solution for computing the directed path-width and directed tree-width of directed co-graphs. Since for complete bioriented digraphs the directed path-width equals the (undirected) path-width of the corresponding underlying undirected graph and the directed tree-width equals the (undirected) tree-width of the corresponding underlying undirected graph our results generalize the known results from [9] [10].
2.3 Digraphs

A directed graph or digraph is a pair \( G = (V,E) \), where \( V \) is a finite set of vertices and \( E \subseteq \{(u,v) \mid u,v \in V, u \neq v \} \) is a finite set of ordered pairs of distinct vertices called arcs. A digraph \( G' = (V',E') \) is a subdigraph of digraph \( G = (V,E) \) if \( V' \subseteq V \) and \( E' \subseteq E \). If every arc of \( E \) with both end vertices in \( V' \) is in \( E' \), we say that \( G' \) is an induced subdigraph of digraph \( G \) and we write \( G' = G[V'] \). For some digraph \( G = (V,E) \) its complement digraph is defined by

\[
\overline{G} = (V, \{(u,v) \mid (u,v) \notin E, u,v \in V, u \neq v \})
\]

and its converse digraph is defined by

\[
G^c = (V, \{(u,v) \mid (v,u) \in E, u,v \in V, u \neq v \})
\]

Let \( G = (V,E) \) be a digraph.

- \( G \) is edgeless if for all \( u,v \in V, u \neq v \), none of the two pairs \((u,v)\) and \((v,u)\) belongs to \( E \).
- \( G \) is a tournament if for all \( u,v \in V, u \neq v \), exactly one of the two pairs \((u,v)\) and \((v,u)\) belongs to \( E \).
- \( G \) is semicomplete if for all \( u,v \in V, u \neq v \), at least one of the two pairs \((u,v)\) and \((v,u)\) belongs to \( E \).
- \( G \) is (bidirectional) complete if for all \( u,v \in V, u \neq v \), both of the two pairs \((u,v)\) and \((v,u)\) belong to \( E \).

Omitting the directions For some given digraph \( G = (V,E) \), we define its underlying undirected graph by ignoring the directions of the edges, i.e. \( \text{und}(G) = (V, \{(u,v) \mid (u,v) \in E \text{ or } (v,u) \in E \}) \).

Orientations There are several ways to define a digraph \( G = (V,E) \) from an undirected graph \( G_u = (V,E_u) \). If we replace every edge \( (u,v) \in E_u \) by

- one of the arcs \((u,v)\) and \((v,u)\), we denote \( G \) as an orientation of \( G_u \). Every digraph \( G \) which can be obtained by an orientation of some undirected graph \( G_u \) is called an oriented graph.
- one or both of the arcs \((u,v)\) and \((v,u)\), we denote \( G \) as a biorientation of \( G_u \). Every digraph \( G \) which can be obtained by a biorientation of some undirected graph \( G_u \) is called a bioriented graph.
- both arcs \((u,v)\) and \((v,u)\), we denote \( G \) as a complete biorientation of \( G_u \). Since in this case \( G \) is well defined by \( G_u \) we also denote it by \( \overline{G_u} \). Every digraph \( G \) which can be obtained by a complete biorientation of some undirected graph \( G_u \) is called a complete bioriented graph.

2.4 Recursively defined Digraphs

2.4.1 Operations

The following operations have already been considered by Bechet et al. in [6, 25]. Let \( G_1 = (V_1,E_1), \ldots, G_k = (V_k,E_k) \) be \( k \) vertex-disjoint digraphs.
The disjoint union of \( G_1, \ldots, G_k \), denoted by \( G_1 \odot \ldots \odot G_k \), is the digraph with vertex set \( V_1 \cup \ldots \cup V_k \) and arc set \( E_1 \cup \ldots \cup E_k \).

The series composition of \( G_1, \ldots, G_k \), denoted by \( G_1 \odot \ldots \odot G_k \), is defined by their disjoint union plus all possible arcs between vertices of \( G_i \) and \( G_j \) for all \( 1 \leq i, j \leq k \), \( i \neq j \).

The order composition of \( G_1, \ldots, G_k \), denoted by \( G_1 \odot \ldots \odot G_k \), is defined by their disjoint union plus all possible arcs from vertices of \( G_i \) to vertices of \( G_j \) for all \( 1 \leq i < j \leq k \).

The directed union of \( G_1, \ldots, G_k \), denoted by \( G_1 \oplus \ldots \oplus G_k \), is defined by their disjoint union plus possible arcs from vertices of \( G_i \) to vertices of \( G_j \) for all \( 1 \leq i < j \leq k \).

### 2.4.2 Directed co-graphs

We recall the definition of directed co-graphs from [13].

**Definition 2.2 (Directed co-graphs, [13])** The class of directed co-graphs is recursively defined as follows.

1. Every digraph on a single vertex (\( \{v\}, \emptyset \)), denoted by \( \bullet \), is a directed co-graph.
2. If \( G_1, \ldots, G_k \) are vertex-disjoint directed co-graphs, then
   a. the disjoint union \( G_1 \oplus \ldots \oplus G_k \),
   b. the series composition \( G_1 \odot \ldots \odot G_k \), and
   c. the order composition \( G_1 \odot \ldots \odot G_k \) are directed co-graphs.

By the definition we conclude that for every directed co-graph \( G = (V, E) \) the underlying undirected graph \( \text{und}(G) \) is a co-graph, but not vice versa.

Similar as undirected co-graphs by the \( P_1 \), also directed co-graphs can be characterized by excluding eight forbidden induced subgraphs [13].

Obviously for every directed co-graph we can define a tree structure, denoted as di-co-tree. The leaves of the di-co-tree represent the vertices of the graph and the inner nodes of the di-co-tree correspond to the operations applied on the subexpressions defined by the subtrees. For every directed co-graph one can construct a di-co-tree in linear time, see [13]. The following lemma shows that it suffices to consider binary di-co-trees.

**Lemma 2.3** Every di-co-tree \( T \) can be transformed into an equivalent binary di-co-tree \( T' \), such that every inner vertex in \( T' \) has exactly two sons.

**Proof** Let \( G \) be a directed co-graph and \( T \) be a di-co-tree for \( G \). Since the disjoint union \( \odot \), the series composition \( \odot \), and the order composition \( \odot \) is associative, i.e. \( G_1 \odot \ldots \odot G_k = (G_1 \odot \ldots \odot G_{k-1}) \odot G_k \), we can transform \( T \) recursively into a binary di-co-tree \( T' \) for \( G \). \( \square \)

Using the di-co-tree a lot of hard problems have been shown to be solvable in polynomial time when restricted to directed co-graphs [17]. In [19] the relation of directed co-graphs to the set of graphs of directed NLC-width 1 and to the set of graphs of directed clique-width 2 is analyzed.

By [23, 33] directed co-graphs are very useful for the reconstruction of the evolutionary history of genes or species using genomic sequence data.

**Lemma 2.4** Let \( G \) be some digraph, then the following properties hold.

1. Digraph \( G \) is a directed co-graph if and only if digraph \( \overline{G} \) is a directed co-graph.
2. Digraph \( G \) is a directed co-graph if and only if digraph \( G^c \) is a directed co-graph.

1That is, \( G_1, \ldots, G_k \) are induced subdigraphs of \( G_1 \odot \ldots \odot G_k \) and there is no edge \( (u, v) \) in \( G_1 \odot \ldots \odot G_k \) such that \( v \in V_i \) and \( u \in V_j \) for \( j > i \). The directed union generalizes the disjoint union and the order composition.
2.4.3 Extended directed co-graphs

Since the directed union generalizes the disjoint union and also the order composition we can generalize the class of directed co-graphs as follows.

Definition 2.5 (Extended directed co-graphs) The class of extended directed co-graphs is recursively defined as follows.

(i) Every digraph on a single vertex \((\{v\}, \emptyset)\), denoted by •, is an extended directed co-graph.

(ii) If \(G_1, \ldots, G_k\) are vertex-disjoint extended directed co-graphs, then

(a) the directed union \(G_1 \ominus \ldots \ominus G_k\) and

(b) the series composition \(G_1 \otimes \ldots \otimes G_k\) are extended directed co-graphs.

Also for every extended directed co-graph we can define a tree structure, denoted as \(\text{ex-di-co-tree}\). The leaves of the ex-di-co-tree represent the vertices of the graph and the inner nodes of the ex-di-co-tree correspond to the operations applied on the subexpressions defined by the subtrees. Following Lemma 2.3 it suffices to consider binary ex-di-co-trees.

By applying the directed union which is not a disjoint union and an order composition we can obtain digraphs whose complement digraph is not an extended directed co-graph. An example for this leads the directed path on 3 vertices \(\rightarrow P_3 = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_3)\})\). Thus we only can carry over one of the two results shown in Lemma 2.4 to the class of extended directed co-graphs.

Lemma 2.6 Let \(G\) be some digraph. Digraph \(G\) is an extended directed co-graph if and only if digraph \(G^c\) is an extended directed co-graph.

3 Directed path-width

According to Barát [5], the notation of directed path-width was introduced by Reed, Seymour, and Thomas around 1995 and relates to directed tree-width introduced by Johnson, Robertson, Seymour, and Thomas in [25].

Definition 3.1 (directed path-width) A directed path-decomposition of a digraph \(G = (V, E)\) is a sequence \((X_1, \ldots, X_r)\) of subsets of \(V\), called bags, such that the following three conditions hold true.

\[(dpw-1) \ \ X_1 \cup \ldots \cup X_r = V.\]

\[(dpw-2) \ \ For \ each \ (u, v) \in E \ there \ is \ a \ pair \ i \leq j \ such \ that \ u \in X_i \ and \ v \in X_j.\]

\[(dpw-3) \ \ If \ u \in X_i \ and \ u \in X_j \ for \ some \ u \in V \ and \ two \ indices \ i, j \ with \ i \leq j, \ then \ u \in X_\ell \ for \ all \ indices \ \ell \ with \ i \leq \ell \leq j.\]

The width of a directed path-decomposition \(X = (X_1, \ldots, X_r)\) is

\[\max_{1 \leq i \leq r} |X_i| - 1.\]

The directed path-width of \(G\), \(d-pw(G)\) for short, is the smallest integer \(w\) such that there is a directed path-decomposition of \(G\) of width \(w\).

Lemma 3.2 ([40]) Let \(G\) be some digraph, then \(d-pw(G) \leq pw(u(G))\).

Lemma 3.3 ([5]) Let \(G\) be some complete bioriented digraph, then \(d-pw(G) = pw(u(G))\).

---

2The proofs shown in [40] use the notation of directed vertex separation number, which is known to be equal to directed path-width.
The proof can be done straightforward since a for $G$ of width $k$ leads to a layout for $\overrightarrow{G}$ of width at most $k$ and vice versa.

Determining whether the (undirected) path-width of some given (undirected) graph is at most some given value $w$ is NP-complete \cite{28} even for bipartite graphs, complements of bipartite graphs \cite{2}, chordal graphs \cite{20}, bipartite distance hereditary graphs \cite{30}, and planar graphs with maximum vertex degree 3 \cite{32}. Lemma 3.3 implies that determining whether the directed path-width of some given digraph is at most some given value $w$ is NP-complete even for digraphs whose underlying graphs lie in the mentioned classes. On the other hand, determining whether the (undirected) path-width of some given (undirected) graph is at most some given value $w$ is polynomial for permutation graphs \cite{3}, circular arc graphs \cite{35}, and co-graphs \cite{10}.

While undirected path-width can be solved by an FPT-algorithm \cite{7}, the existence of such an algorithm for directed path-width is still open. The directed path-width of a digraph $G = (V, E)$ can be computed in time $O(|E| |V|^2 \cdot \text{pw}(G))$ by \cite{29} which leads to an XP-algorithm for directed path-width w.r.t. the standard parameter and implies that for each constant $w$, it is decidable in polynomial time whether a given digraph has directed path-width at most $w$.

In order to prove our main results we show some properties of directed path-decompositions. Similar results are known for undirected path-decompositions and are useful within several places.

Lemma 3.4 (\cite{40}) Let $G$ be some digraph and $H$ be an induced subdigraph of $G$, then $d$-pw$(H) \leq d$-pw$(G)$.

Lemma 3.5 (Bidirectional complete subdigraph) Let $G = (V, E)$ be some digraph, $G' = (V', E')$ with $V' \subseteq V$ be a bidirectional complete subdigraph, and $(X_1, \ldots, X_r)$ a directed path-decomposition of $G$. Then there is some $i$, $1 \leq i \leq r$, such that $V' \subseteq X_i$.

Proof We show the claim by an induction on $|V'|$. If $|V'| = 1$ then by (dpw-1) there is some $i$, $1 \leq i \leq r$, such that $V' \subseteq X_i$. Next let $|V'| > 1$ and $v \in V'$. By our induction hypothesis there is some $i$, $1 \leq i \leq r$, such that $V' - \{v\} \subseteq X_i$. By (dpw-3) there are two integers $r_1$ and $r_2$, $1 \leq r_1 \leq r_2 \leq r$, such that $v \in X_j$ for all $r_1 \leq j \leq r_2$. If $r_1 \leq i \leq r_2$ then $V' \subseteq X_i$. Next suppose that $i < r_1$ or $r_2 < i$. If $i < r_1$ we define $j = r_1$ and if $i > r_2$ we define $j = r_2$. We will show that $V' \subseteq X_j$. Let $w \in V' - \{v\}$. Since there are two arcs $(v, w)$ and $(w, v)$ in $E$ by (dpw-2) there is some $r_1 \leq j'' \leq r_2$ such that $v, w \in X_{j''}$. By (dpw-3) we conclude $w \in X_j$. Thus $V' - \{v\} \subseteq X_j$.

Lemma 3.6 Let $G = (V, E)$ be a digraph and $(X_1, \ldots, X_r)$ a directed path-decomposition of $G$. Further let $A, B \subseteq V$, $A \cap B = \emptyset$, and $\{(u, v), (v, u) \mid u \in A, v \in B\} \subseteq E$. Then there is some $i$, $1 \leq i \leq r$, such that $A \subseteq X_i$ or $B \subseteq X_i$.

Proof Suppose that $B \not\subseteq X_i$ for all $1 \leq i \leq r$. Then there are $b_1, b_2 \in B$ and $i_{1,1}, i_{1,2}, i_{2,1}, i_{2,2}$, $1 \leq i_{1,1} < i_{1,2} < i_{2,1} < i_{2,2} < r$, such that $\{i \mid b_1 \in X_i\} = \{i_{1,1}, \ldots, i_{1,2}\}$ and $\{i \mid b_2 \in X_i\} = \{i_{2,1}, \ldots, i_{2,2}\}$ (and both sets are disjoint). Let $a \in A$. Since $(b_2, a) \in E$ there is some $i_{2,1} \leq i \leq r$ such that $a \in X_i$, and since $(a, b_1) \in E$ there is some $1 \leq j \leq i_{1,2}$ such that $a \in X_j$. By (dpw-3) it is true that $a \in X_k$ for every $1 \leq i \leq k \leq i_{2,2}$.

If we suppose $A \not\subseteq X_i$ for all $1 \leq i \leq r$ it follows that $B \subseteq X_k$ for every $1 \leq k \leq i_{2,2}$. \hfill $\Box$

Lemma 3.7 Let $\mathcal{X} = (X_1, \ldots, X_r)$ be a directed path-decomposition of some digraph $G = (V, E)$. Further let $A, B \subseteq V$, $A \cap B = \emptyset$, and $\{(u, v), (v, u) \mid u \in A, v \in B\} \subseteq E$. If there is some $i$, $1 \leq i \leq r$, such that $A \subseteq X_i$, then there are $1 \leq i_1 \leq i_2 \leq r$ such that

1. for all $i$, $i_1 \leq i \leq i_2$ is $A \subseteq X_i$,

2. $B \subseteq \bigcup_{i=i_1}^{i_2} X_i$, and

3. $\mathcal{X}' = (X_1', \ldots, X_r')$ where $X_i' = X_i \cap (A \cup B)$ is a directed path-decomposition of the digraph induced by $A \cup B$.

Proof Let $i_1 = \min\{i \mid A \subseteq X_i\}$ and $i_2 = \max\{i \mid A \subseteq X_i\}$. Since $\mathcal{X}$ satisfies (dpw-3), it holds (1).
Since there is some $i$, $1 \leq i \leq r$, such that $A \subseteq X_i$ we know that $X = (X_1, \ldots, X_r)$ is also a directed path-decomposition of $G = (V, E')$, where $E' = E \cup \{(u, v) \mid u, v \in A, u \neq v\}$. For every $b \in B$ the graph with vertex set $\{b\} \cup A$ is bidirectional complete subdigraph of $G$ which implies by Lemma 3.8 that there is some $i$, $i_1 \leq i \leq i_2$ such that $A \cup \{b\} \subseteq X_i$. Thus there is some $i$, $i_1 \leq i \leq i_2$ such that $b \in X_i$ which leads to (2.).

In order to show (3.) we observe that for the sequence $X' = (X'_1, \ldots, X'_{i_2})$ condition (dpw-1) holds by (1.) and (2.).

By (1.) and (2.) the arcs between two vertices from $A$ and the arcs between a vertex from $A$ and a vertex from $B$ satisfy (dpw-2). So let $(b', b'') \in E$ such that $b', b'' \in B$. By (2.) we know that $b' \in X_i$ and $b'' \in X_j$ for $i_1 \leq i, j \leq i_2$. If $j < i$ then by (dpw-3) for $X$ there is some $X'_{j'}$, $j' > i_2$ such that $b'' \in X'_{j'}$ but by (dpw-3) for $X$ is $b' \in X_i$.

Further $X'$ satisfies (dpw-3) since $X$ satisfies (dpw-3).

\end{proof}

\section*{Theorem 3.8}
Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two vertex-disjoint digraphs, then the following properties hold.

\begin{enumerate}
\item $d\text{-\textit{pw}}(G \oplus H) = \max\{d\text{-\textit{pw}}(G), d\text{-\textit{pw}}(H)\}$
\item $d\text{-\textit{pw}}(G \ominus H) = \max\{d\text{-\textit{pw}}(G), d\text{-\textit{pw}}(H)\}$
\item $d\text{-\textit{pw}}(G \otimes H) = \max\{d\text{-\textit{pw}}(G), d\text{-\textit{pw}}(H)\}$
\item $d\text{-\textit{pw}}(G \oslash H) = \min\{d\text{-\textit{pw}}(G) + |V_H|, d\text{-\textit{pw}}(H) + |V_G|\}$
\end{enumerate}

\section*{Proof}

1. In order to show $d\text{-\textit{pw}}(G \oplus H) \leq \max\{d\text{-\textit{pw}}(G), d\text{-\textit{pw}}(H)\}$ we consider a directed path-decomposition $(X_1, \ldots, X_r)$ for $G$ and a directed path-decomposition $(Y_1, \ldots, Y_s)$ for $H$. Then $(X_1, \ldots, X_r, Y_1, \ldots, Y_s)$ leads to a directed path-decomposition of $G \oplus H$.

Since $G$ and $H$ are induced subdigraphs of $G \oplus H$, by Lemma 3.4 the directed path-width of both digraphs leads to a lower bound on the directed path-width for the combined graph. Thus $d\text{-\textit{pw}}(G \oplus H) \leq \min\{d\text{-\textit{pw}}(G) + |V_H|, d\text{-\textit{pw}}(H) + |V_G|\}$.

2. By the same arguments as used for (1.).

3. By the same arguments as used for (1.).

4. In order to show $d\text{-\textit{pw}}(G \ominus H) \leq d\text{-\textit{pw}}(G) + |V_H|$ let $(X_1, \ldots, X_r)$ be a directed path-decomposition of $G$. Then we obtain by $(X_1 \cup V_H, \ldots, X_r \cup V_H)$ a directed path-decomposition of $G \ominus H$. In the same way a directed path-decomposition of $H$ leads to a directed path-decomposition of $G \ominus H$ which implies that $d\text{-\textit{pw}}(G \ominus H) \leq d\text{-\textit{pw}}(H) + |V_G|$. Thus $d\text{-\textit{pw}}(G \ominus H) \leq \min\{d\text{-\textit{pw}}(G) + |V_H|, d\text{-\textit{pw}}(H) + |V_G|\}$.

For the reverse direction let $X = (X_1, \ldots, X_r)$ be a directed path-decomposition of $G \ominus H$. By Lemma 3.3 we know that there is some $i$, $1 \leq i \leq r$, such that $V_G \subseteq X_i$ or $V_H \subseteq X_i$. We assume that $V_G \subseteq X_i$. We apply Lemma 3.7 using $G \ominus H$ as digraph, $A = V_G$ and $B = V_H$ in order to obtain a directed path-decomposition $X' = (X'_1, \ldots, X'_t)$ for $G \ominus H$ where for all $i$, $i_1 \leq i \leq i_2$, it holds $V_G \subseteq X_i$ and $V_H \subseteq \cup_{i=i_1}^{i_2} X_i$. Further $X'' = (X''_1, \ldots, X''_{i_2})$, where $X''_i = X'_i \cap V_H$ leads to a directed path-decomposition of $H$. Thus there is some $i$, $i_1 \leq i \leq i_2$, such that $|X_i \cap V_H| \geq d\text{-\textit{pw}}(H) + 1$. Since $V_G \subseteq X_i$, we know that $|X_i \cap V_H| = |X_i| - |V_G|$ and thus $|X_i| \geq |V_G| + d\text{-\textit{pw}}(H) + 1$. Thus the width of directed path-decomposition $(X_1, \ldots, X_r)$ is at least $d\text{-\textit{pw}}(H) + |V_G|$. If we assume that $V_H \subseteq X_i$ it follows that the width of directed path-decomposition $(X_1, \ldots, X_r)$ is at least $d\text{-\textit{pw}}(G) + |V_H|$.

\end{proof}

\section*{Lemma 3.9}
Let $G$ and $H$ be two directed co-graphs, then $\text{pw}(\text{und}(G \ominus H)) > d\text{-\textit{pw}}(G \ominus H)$.
Proof Let $G$ and $H$ be two directed co-graphs.

\[
\text{pw}(\text{und}(G \circledcirc H)) = \text{pw}(\text{und}(G) \times \text{und}(H)) \\
= \min\{\text{pw}(\text{und}(G)) + |V_H|, \text{pw}(\text{und}(H)) + |V_G|\} \quad \text{(by [10])} \\
> \min\{\text{pw}(\text{und}(G)) + \text{d-pw}(H), \text{pw}(\text{und}(H)) + \text{d-pw}(G)\} \\
\geq \min\{\text{d-pw}(G) + \text{d-pw}(H), \text{d-pw}(H) + \text{d-pw}(G)\} \\
= \text{d-pw}(G) + \text{d-pw}(H) \\
= \max\{\text{d-pw}(G), \text{d-pw}(H)\} \\
= \text{d-pw}(G \circledcirc H)
\]

\[\square\]

Corollary 3.10 Let $G$ be some directed co-graph, then $\text{d-pw}(G) = \text{pw}(u(G))$ if and only if there is an expression for $G$ without any order operation. Further $\text{d-pw}(G) = 0$ if and only if there is an expression for $G$ without any series operation.

Proof If there is a construction without order operation, then Theorem 3.8 and the results of [10] imply $\text{d-pw}(G) = \text{pw}(u(G))$. If there is a construction using an order operation, Lemma 3.10 implies that $\text{d-pw}(G) \neq \text{pw}(u(G))$.

\[\square\]

4 Directed tree-width

An acyclic digraph (\textit{DAG} for short) is a digraph without any cycles as subdigraph. An out-tree is a digraph with a distinguished root such that all arcs are directed away from the root. For two vertices $u, v$ of an out-tree $T$ the notation $u \leq v$ means that there is a directed path on $\geq 0$ arcs from $u$ to $v$ and $u < v$ means that there is a directed path on $\geq 1$ arcs from $u$ to $v$.

Let $G = (V, E)$ be some digraph and $Z \subseteq V$. A vertex set $S \subseteq V$ is $Z$-\textit{normal}, if there is no directed walk in $G - Z$ with first and last vertices in $S$ that uses a vertex of $G - (Z \cup S)$. That is, a set $S \subseteq V$ is $Z$-normal, if every directed walk which leaves and again enters $S$ must contain only vertices from $Z \cup S$. Or, a set $S \subseteq V$ is $Z$-normal, if every directed walk which leaves and again enters $S$ must contain a vertex from $Z$.

Definition 4.1 (directed tree-width, [25]) A (arboreal) tree-decomposition of a digraph $G = (V_G, E_G)$ is a triple $(T, X, W)$. Here $T = (V_T, E_T)$ is an out-tree, $X = \{X_e \mid e \in E_T\}$ and $W = \{W_r \mid r \in V_T\}$ are sets of subsets of $V_G$, such that the following two conditions hold true.

\begin{enumerate}[\textit{(dtw-1)}]
\item $W = \{W_r \mid r \in V_T\}$ is a partition of $V_G$ into nonempty subsets.
\item For every $(u, v) \in E_T$ the set $\bigcup \{W_r \mid r \in V_T, v \leq r\}$ is $X(u, v)$-normal.
\end{enumerate}

The width of a (arboreal) tree-decomposition $(T, X, W)$ is

$\max_{r \in V_T} |W_r \cup \bigcup_{e \sim r} X_e| - 1$.

Here $e \sim r$ means that $r$ is one of the two vertices of arc $e$. The directed tree-width of $G$, $\text{d-tw}(G)$ for short, is the smallest integer $k$ such that there is a (arboreal) tree-decomposition $(T, X, W)$ of $G$ of width $k$.

Remark 4.2 (Z-normality) Please note that our definition of $Z$-normality slightly differs from the following definition in [25] where $S$ and $Z$ are disjoint. A vertex set $S \subseteq V - Z$ is $Z$-normal, if there is no directed walk in $G - Z$ with first and last vertices in $S$ that uses a vertex of $G - (Z \cup S)$. That is, a set $S \subseteq V - Z$ is $Z$-normal, if every directed walk in $G - Z$ which leaves and again enters $S$ must contain only vertices from $Z \cup S$. Or, a set $S \subseteq V - Z$ is $Z$-normal, if every directed walk which leaves and again enters $S$ must contain a vertex from $Z$, set [7].

Every set $S \subseteq V - Z$ which is is $Z$-normal w.r.t. the definition in [25] is also $Z$-normal w.r.t. our definition. Further a set $S \subseteq V$ which is $Z$-normal w.r.t. our definition, is also $Z - S$-normal w.r.t. the definition in [25]. Thus the directed tree-width of a digraph is equal for both definitions of $Z$-normality.

\[3\]A remarkable difference to the undirected tree-width [36] is that the sets $W_r$ have to be disjoint and non-empty.
Lemma 4.3 ([25]) Let \( G \) be some digraph, then \( d-tw(G) \leq tw(und(G)) \).

Lemma 4.4 ([25]) Let \( G \) be some complete bioriented digraph, then \( d-tw(G) = tw(und(G)) \).

Determining whether the (undirected) tree-width of some given (undirected) graph is at most some given value \( w \) is NP-complete even for bipartite graphs and complements of bipartite graphs [2]. Lemma 4.3 implies that determining whether the directed tree-width of some given digraph is at most some given value \( w \) is NP-complete even for digraphs whose underlying graphs lie in the mentioned classes.

The results of [25] lead to an XP-algorithm for directed tree-width w.r.t. the standard parameter which implies that for each constant \( w \), it is decidable in polynomial time whether a given digraph has directed tree-width at most \( w \).

In order to show our main results we show some properties of directed tree-decompositions.

Lemma 4.5 ([25]) Let \( G \) be some digraph and \( H \) be an induced subdigraph of \( G \), then \( d-tw(H) \leq d-tw(G) \).

Lemma 4.6 (Bidirectional complete subdigraph) Let \((T, X, W), T = (V_T, E_T)\), where \( r_T \) is the root of \( T \), be a directed tree-decomposition of some digraph \( G = (V, E) \) and \( G' = (V', E') \) with \( V' \subseteq V \) be a bidirectional complete subdigraph. Then \( V' \subseteq W_{r_T} \) or there is some \((r, s) \in E_T\), such that \( V' \subseteq W_s \cup X_{(r, s)} \).

Proof First we show the existence of a vertex \( s \) in \( V_T \), such that \( W_s \cap V' \neq \emptyset \) but for every vertex \( s' \) such that \( s < s' \) holds \( W_s \cap V' = \emptyset \). If there is a leaf \( \ell \) in \( T \), such that \( W_\ell \cap V' = \emptyset \), we can choose \( s = \ell \). Otherwise we look for vertex \( s \) among the predecessors of the leaves in \( T \), and so on.

Next we show that \( W_s \) leads to a set which shows the statement of the lemma. If \( s \) is the root of \( T \), then \( W_s \cap V' \neq \emptyset \) for none of its successors \( s' \) in \( T \) i.e. \( W_{s'} \cap V' = \emptyset \) for all of its successors \( s' \) in \( T \), which implies by (dtw-1) that \( V' \subseteq W_s \). Otherwise let \( r \) be the predecessor of \( s \) in \( T \). If \( V' \subseteq W_s \) the statement is true. Otherwise let \( e \in V' - W_s \) and \( c' \in V' \cap W_s \). Then \((e, c') \in E \) and \((c', c) \in E \) implies that \( c \in X_{(r, s)} \) by (dtw-2).

Lemma 4.7 Let \( G = (V, E) \) be some digraph, \((T, X, W), T = (V_T, E_T)\), where \( r_T \) is the root of \( T \), be a directed tree-decomposition of \( G \). Further let \( A, B \subseteq V \), \( A \cap B = \emptyset \), and \( \{(u, v), (v, u) \mid u \in A, v \in B\} \subseteq E \). Then \( A \cup B \subseteq W_{r_T} \) or there is some \((r, s) \in E_T\), such that \( A \subseteq W_s \cup X_{(r, s)} \) or \( B \subseteq W_s \cup X_{(r, s)} \).

Proof Similar as in the proof of Lemma 4.6 we can find a vertex \( s \) in \( V_T \), such that \( W_s \cap (A \cup B) \neq \emptyset \) but for every vertex \( s' \) such that \( s < s' \) holds \( W_s \cap (A \cup B) = \emptyset \).

If \( s \) is the root of \( T \), then \( W_s \cap (A \cup B) \neq \emptyset \) for none of its successors \( s' \) in \( T \), i.e. \( W_{s'} \cap (A \cup B) = \emptyset \) for all of its successors \( s' \) in \( T \), which implies by (dtw-1) that \( A \cup B \subseteq W_s \).

Otherwise let \( r \) be the predecessor of \( s \) in \( T \). If \( A \cup B \subseteq W_s \) the statement is true. Otherwise we know that there is some \( a \in A \) such that \( a \in W_s \) and \( B \nsubseteq W_s \) or some \( b \in B \) such that \( b \in W_s \) and \( A \nsubseteq W_s \). We assume that there is some \( a \in A \) such that \( a \in W_s \) and \( B \nsubseteq W_s \). Let \( b \in B - W_s \) and \( a \in A \cap W_s \). Then \((a, b) \in E \) and \((b, a) \in E \) implies that \( b \in X_{(r, s)} \) by (dtw-2). Thus we have shown \( B \subseteq W_s \cup X_{(r, s)} \).

If we assume that there some \( b \in B \) such that \( b \in W_s \), we conclude \( A \subseteq W_s \cup X_{(r, s)} \).

Lemma 4.8 Let \( G \) be a digraph of directed tree-width at most \( k \). Then there is a directed tree-decomposition \((T, X, W), T = (V_T, E_T)\), of width at most \( k \) for \( G \) such that \( |W_r| = 1 \) for every \( r \in V_T \).

Proof Let \( G = (V, E) \) be a digraph and \((T, X, W), T = (V_T, E_T)\), of directed tree-decomposition of \( G \). Let \( r \in V_T \) such that \( W_r = \{v_1, \ldots, v_k\} \) for some \( k > 1 \). Further let \( p \) be the predecessor of \( r \) in \( T \) and \( s_1, \ldots, s_k \) be the successors of \( r \) in \( T \). Let \((T', X', W')\) be defined by the following modifications of \((T, X, W): We replace vertex \( r \) in \( T \) by the directed path \( P(r) = \{(r_1, r_2), (r_2, r_3), \ldots, (r_{k-1}, r_k)\} \) and replace arc \((p, r) \) by \((p, r_1) \) and the \( \ell \) arcs \((r, s_j)\), \( 1 \leq j \leq \ell \), by the \( \ell \) arcs \((r, s), 1 \leq j \leq \ell \) in \( T' \). We define the sets \( W'_{r_j} = \{v_j\} \)
Let new arc $(r_{i-1}, r_i)$, $1 < i < k$, such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, and $\{u, v, (v, u) | u \in V_1, v \in V_2\} \subseteq E$. Then there is a directed tree-decomposition $(T, X, W)$, $T = (V_T, E_T)$, of width at most $k$ for $G$ such that for every $e \in E_T$ holds $V_1 \subseteq X_e$ or for every $e \in E_T$ holds $V_2 \subseteq X_e$.

**Proof.** Let $G = (V, E)$ be a digraph of directed tree-width at most $k$ and $(T, X, W)$, $T = (V_T, E_T)$, be a directed tree-decomposition of width at most $k$ for $G$. By Lemma 4.8 we can assume that holds: $|W_{t, t'}| = 1$ for every $r \in V_T$.

We show the claim by traversing $T$ in a bottom-up order. Let $t'$ be a leaf of $T$, $t$ be the predecessor of $t'$ in $T$ and $W_{t, t'} = \{v\}$ for some $v \in V_1$. Then the following holds: $V_2 \subseteq X_{(t, t')}$ since $(v, v') \in E$ and $(v', v) \in E$ for every $v' \in V_2$.

If $t'$ is a non-leaf of $T$ and there is a successor $t''$ of $t'$ in $T$ such that $V_1 \subseteq X_{(t', t'')}$ and there is a successor $t'''$ of $t''$ in $T$ such that $V_2 \subseteq X_{(t'', t''')}$. Then the width of $(T, X, W)$ is $|V_1| + |V_2| - 1$ which allows us to insert $V_1$ into every set $X_e$ as well as $V_2$ into every set $X_e$.

Otherwise let $t'$ be a non-leaf of $T$ and $V_2 \subseteq X_{(t', t'')}$ for every successor $t''$ of $t'$. Let $t$ be the predecessor of $t'$ and a non-successor of $t$ in $T$. We distinguish the following two cases.

- Let $V_1 \subseteq \cup_{r < \tilde{t}} W_r$. We replace $X_{(t, t')}$ by $X_{(t, t')} \cup V_2$ in order to meet our claim for edge $(t, t')$.

We have to show that this does not increase the width of the obtained directed tree-decomposition at vertex $t'$ and at vertex $t$.

The value of $|W_{t', t''} \cup \bigcup_{v \neq t} X_v|$ does not change, since $V_2 \subseteq X_{(t', t'')}$ by induction hypothesis and $(t', t'') \sim t'$.

Since $V_1 \subseteq \cup_{r \leq \tilde{t}} W_r$ by (dtw-2) we can assume that $V_1 \cap X_{(s, t)} = \emptyset$. Since all $W_{t'}$ have size one we know that $|W_{t, t'} \cup \bigcup_{v \neq t} X_v| \leq |W_{t'} \cup \bigcup_{v \neq t} X_v|.

- Let $V_1 \subseteq \cup_{r \leq \tilde{t}} W_r$. We distinguish the following two cases.

  - Let $V_2 \cap \cup_{v \leq \tilde{t}} W_v = \emptyset$, then $W_{t, t'} = \{v\}$ for some $v \in V_1$ and thus $V_2 \subseteq X_{(t, t')}$ since $(v, v') \in E$ and $(v', v) \in E$ for every $v' \in V_2$.

  - Let $V_2 \cap \cup_{v \leq \tilde{t}} W_v \neq \emptyset$. Since $\{(u, v), (v, u)| u \in V_1, v \in V_2\} \subseteq E$ the following is true:

    \[ V - \bigcup_{v \leq \tilde{t}} W_v \subseteq (V_1 \cup V_2) - \bigcup_{v \leq \tilde{t}} W_v \subseteq X_{(t, t')}. \] (1)

That is, all vertices of $G$ which are not of one of the sets $W_r$ for all successors $\tilde{t}$ of $t'$ are in set $X_{(t, t')}$. We define $X_{(t, t')} = (V - \cup_{v \leq \tilde{t}} W_v) \cup V_2$ in order to meet our claim for edge $(t, t')$.

We have to show that this does not increase the width of the obtained directed tree-decomposition at vertex $t'$ and at vertex $t$.

The value of $|W_{t'} \cup \bigcup_{v \neq t} X_v|$ does not change, since $V_2 \subseteq X_{(t', t'')}$. We further define the sets $X_{(t, t')} = (V - \cup_{v \leq \tilde{t}} W_v) \cup V_2$ in order to meet our claim for edge $(t, t')$.

We have to show that this does not increase the width of the obtained directed tree-decomposition at vertex $t'$ and at vertex $t$.

The value of $|W_{t'} \cup \bigcup_{v \neq t} X_v|$ does not change, since $V_2 \subseteq X_{(t', t'')}$. We further define the sets $X_{(t, t')} = (V - \cup_{v \leq \tilde{t}} W_v) \cup V_2$ in order to meet our claim for edge $(t, t')$.

Further implies that $X_{(s, t)} \subseteq X_{(t, t')}$ and thus $|W_{t} \cup \bigcup_{v \neq t} X_v| \leq |W_{t'} \cup \bigcup_{v \neq t} X_v|$.

10
Thus if $T$ has a leaf $t'$ such that $W_r = \{v\}$ for some $v \in V_1$ we obtain a directed tree-decomposition $(T, \mathcal{X}, W)$, $T = (V_T, E_T)$, such that $V_2 \subseteq X_e$ for every $e \in E_T$. And if $T$ has a leaf $t'$ such that $W_r = \{v\}$ for some $v \in V_2$ we obtain a directed tree-decomposition $(T, \mathcal{X}, W)$, $T = (V_T, E_T)$, such that $V_1 \subseteq X_e$ for every $e \in E_T$. \hfill \square

**Theorem 4.10** Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two vertex-disjoint digraphs, then the following properties hold.

1. $d-tw(G \circ H) = \max\{d-tw(G), d-tw(H)\}$
2. $d-tw(G \circ H) = \max\{d-tw(G), d-tw(H)\}$
3. $d-tw(G \circ H) = \max\{d-tw(G), d-tw(H)\}$
4. $d-tw(G \circ H) = \min\{d-tw(G) + |V_H|, d-tw(H) + |V_G|\}$

**Proof** Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two vertex-disjoint digraphs. Further let $(T_G, \mathcal{X}_G, W_G)$ be a directed tree-decomposition of $G$ such that $r_G$ is the root of $T_G = (V_{T_G}, E_{T_G})$ and $(T_H, \mathcal{X}_H, W_H)$ be a directed tree-decomposition of $H$ such that $r_H$ is the root of $T_H = (V_{T_H}, E_{T_H})$.

1. We define a directed tree-decomposition $(T_J, \mathcal{X}_J, W_J)$ for $J = G \circ H$. Let $\ell_G$ be a leaf of $T_G$. Let $T_J$ be the disjoint union of $T_G$ and $T_H$ with an additional arc $(\ell_G, r_H)$. Further let $\mathcal{X}_J = \mathcal{X}_G \cup \mathcal{X}_H \cup \{X_{(t_G, r_H)}\}$, where $X_{(t_G, r_H)} = \emptyset$ and $W_J = W_G \cup W_H$. Triple $(T_J, \mathcal{X}_J, W_J)$ satisfies (dtw-1) since the combined decompositions satisfy (dtw-1). Further $(T_J, \mathcal{X}_J, W_J)$ satisfies (dtw-2) since additionally in $J$ there is no arc from a vertex of $H$ to a vertex of $G$. This shows that $d-tw(G \circ H) \leq \max\{d-tw(G), d-tw(H)\}$. Since $G$ and $H$ are induced subgraphs of $G \circ H$, by Lemma 4.5 the directed tree-width of both leads to a lower bound on the directed tree-width for the combined graph.

2. The same arguments lead to $d-tw(G \circ H) = \max\{d-tw(G), d-tw(H)\}$.

3. The same arguments lead to $d-tw(G \circ H) = \max\{d-tw(G), d-tw(H)\}$.

4. In order to show $d-tw(G \circ H) \leq d-tw(G) + |V_H|$ let $T_J$ be the disjoint union of a new root $r_J$ and $T_G$ with an additional arc $(r_J, r_G)$. Further let $\mathcal{X}_J = \mathcal{X}_G \cup \{X_{(r_J, r_G)}\}$, where $X_{(r_J, r_G)} = \emptyset$ and $W_J = W_G \cup W_H$. Then we obtain by $(T_J, \mathcal{X}_J, W_J)$ a directed tree-decomposition of width at most $d-tw(G) + |V_H|$ for $G \circ H$.

In the same way a new root $r_J$ and $T_H$ with an additional arc $(r_J, r_H)$, $\mathcal{X}_J' = \{X_e \cup V_G \mid e \in E_{T_H}\}$, $X_{(r_J, r_H)} = V_G$, $W_{r_J} = V_G$ lead to a directed tree-decomposition of width at most $d-tw(H) + |V_G|$ for $G \circ H$. Thus $d-tw(G \circ H) \leq \min\{d-tw(G) + |V_H|, d-tw(H) + |V_G|\}$.

For the reverse direction let $(T_J, \mathcal{X}_J, W_J)$, $T_J = (V_T, E_T)$, be a directed tree-decomposition of minimal width for $G \circ H$. By Lemma 4.8 we can assume that $V_G \subseteq X_e$ for every $e \in E_T$ or $V_H \subseteq X_e$ for every $e \in E_T$. Further by Lemma 4.8 we can assume that $|W_t| = 1$ for every $t \in V_T$.

We assume that $V_G \subseteq X_e$ for every $e \in E_T$. We define $(T_J', \mathcal{X}_J', W_J')$, $T_J' = (V_T', E_T')$, by $X_{s'} = X_e \cap V_H$ and $W_{s'} = W_s \cap V_H$. Whenever this leads to an empty set $W_{s'}$ where $t$ is the predecessor of $s$ in $T_J$ we remove vertex $s$ from $T_J'$ and replace every arc $(s, t')$ by $(t, t')$ with the corresponding set $X_{(t, t')} = X_{(s, t')} \cap V_H$.

Then $(T_J', \mathcal{X}_J', W_J')$ is a directed tree-decomposition of $H$ as follows.

- $W_J'$ is a partition of $V_H$ into nonempty sets.
- Let $e$ be an arc in $T_J'$ which is also in $T_J$. Since $e \sim s$ implies $W_s = W'_s = \{v\}$ for some $v \in V_H$ normality condition remains true.

Ares $(t, t')$ in $T_J'$ which are not in $T_J$ are obtained by two arcs $(t, s)$ and $(s, t')$ from $T_J$. If $\cup\{W_r \mid r \in V_T', t' \leq t\}$ is $X_{(s, t')}$-normal, then $\cup\{W_r \mid r \in V_T', t' \leq t\}$ is $X_{(s, t')}$-normal since $X_{(s, t')} = X_{(s, t')} \cap V_H$.  

11
The width of \((T_j', X_j', W_j')\) is at most \(d\text{-tw}(G \odot H) - |V_G|\) as follows.

- Let \(s\) be a vertex in \(T_j'\) such that \(W_t \cap V_H \neq \emptyset\) for all \((s, t)\) in \(T_j\).

\[
|W_s' \cup \bigcup_{e \sim s} X_e'| = |(W_s \cap V_H) \cup \bigcup_{e \sim s} (X_e \cap V_H)| \quad \text{by definition} \\
= |(W_s \cup \bigcup_{e \sim s} X_e) \cap V_H| \quad \text{factor out } V_H \\
= |W_s \cup \bigcup_{e \sim s} X_e| - |V_G| \quad \text{since } V_G \subseteq X_e
\]

- Let \(s\) be a vertex in \(T_j'\) such that there is \((s, t)\) in \(T_j\) with \(W_t \cap V_H = \emptyset\).

\[
|W_s' \cup \bigcup_{e \sim s} X_e'| = |(W_s \cap V_H) \cup (X_{(t,u)} \cap V_H) \cup \bigcup_{(s,t) \in E_T \setminus W_t \cap V_H = \emptyset} (X_{(s,t)} \cap V_H)|
\]

\[\bigcup_{(s,t) \in E_T \setminus W_t \cap V_H = \emptyset} (X_{(s,t)} \cap V_H)\]

In order to bound this value we observe that for \(W_t \cap V_H = \emptyset\) the following is true: \(W_t = \{v\}\) for \(v \in V_G\). Then \(X_{(s,t)} = (V_G \cup V_H) - \bigcup_{i \in \pi(W_t)} \cup V_G\) by Lemma 4.9. That is, \(X_{(s,t)}\) consists of all vertices from \(V_G\) and all vertices which are not of one of the sets \(W_t\) for all successors \(t\) of \(t\). Applying this argument to \(X_{(t,u)}\) we only can have \(v\) as an additional vertex. But since \(v \in V_G\) we know that \(v \in X_{(s,t)}\) by our assumption. This implies

\[X_{(t,u)} \subseteq X_{(s,t)}\] for all arcs \((s, t)\) in \(T_j\) such that \(W_t \cap V_H = \emptyset\)

which allows the following estimations:

\[
|W_s' \cup \bigcup_{e \sim s} X_e'| = |(W_s \cap V_H) \cup \bigcup_{e \sim s} (X_e \cap V_H)| \quad \text{by (2) and (3)} \\
= |(W_s \cup \bigcup_{e \sim s} X_e) \cap V_H| \quad \text{factor out } V_H \\
= |W_s \cup \bigcup_{e \sim s} X_e| - |V_G| \quad \text{since } V_G \subseteq X_e
\]

Thus the width of \((T_j', X_j', W_j')\) is at most \(d\text{-tw}(G \odot H) - |V_G|\) and since \((T_j', X_j', W_j')\) is a directed tree-decomposition of \(H\) it follows \(d\text{-tw}(H) \leq d\text{-tw}(G \odot H) - |V_G|\).

If we assume that \(V_H \subseteq X_e\) for every \(e \in E_T\) it follows that \(d\text{-tw}(G) \leq d\text{-tw}(G \odot H) - |V_H|\).

\[\square\]

The proof of Theorem 4.10 even shows that for any directed co-graph there is a tree-decomposition \((T', X', W)\) of minimal width such that \(T\) is a path.

Similar to the path-width results, we conclude the following results.

**Lemma 4.11** Let \(G\) and \(H\) be two directed co-graphs, then \(\text{tw}\left(\text{und}(G \odot H)\right) > d\text{-tw}(G \odot H)\).

**Corollary 4.12** Let \(G\) be some directed co-graph, then \(d\text{-tw}(G) = \text{tw}(u(G))\) if and only if there is an expression for \(G\) without any order operation. Further \(d\text{-tw}(G) = 0\) if and only if there is an expression for \(G\) without any series operation.

## 5 Directed tree-width and directed path-width of special digraphs

For general digraphs the directed tree-width is at most the directed path-width are by the following Lemma.

**Lemma 5.1** Let \(G\) be some digraph, then \(d\text{-tw}(G) \leq d\text{-pw}(G)\).
Directed path-width and directed tree-width can be computed in time $O(n)$.

**Theorem 5.4**

Let $G$ be a directed graph and $d$ a directed tree-width depending on its height. The path-width of perfect $k$-ary trees of height $h$ is $O(h\log h)$ and for $k \geq 3$ the path-width of perfect $k$-ary trees of height $h$ is exactly $h$ by Corollary 3.1 of [17].

**5.1 Directed Co-graphs**

**Theorem 5.3**

For every directed co-graph $G$, it holds that $d$-pw$(G) = d$-tw$(G)$.

**Proof** Let $G = (V, E)$ be some directed co-graph. We show the result by induction on the number of vertices $|V|$. If $|V| = 1$, then $d$-pw$(G) = d$-tw$(G) = 0$. If $G = G_1 \oplus G_2$, then by Theorem 5.3 and Theorem 4.10 follows:

$$d$-pw$(G) = \max\{d$-pw$(G_1), d$-pw$(G_2)\} = \max\{d$-tw$(G_1), d$-tw$(G_2)\} = d$-tw$(G).$$

For the other two operations a similar relation holds.

By Lemma 5.3 and Lemma 4.4 our results generalize the known results from [9, 10] but cannot be obtained by the known results.

**Theorem 5.4**

For every directed co-graph $G = (V, E)$ which is given by a binary di-co-tree the directed path-width and directed tree-width can be computed in time $O(|V|)$.

**Proof** The statement follows by the algorithm given in Fig. 1. Theorem 5.3 and Theorem 4.10. The necessary sizes of the subdigraphs defined by subtrees of di-co-tree $T_G$ can be precomputed in time $O(|V|)$.

**Algorithm Directed Path-width($v$)**

if $v$ is a leaf of di-co-tree $T_G$

then $d$-pw$(G[T_v]) = 0$

else { Directed Path-width($v_\ell$) \hspace{1cm} $\triangleright v_\ell$ is the left successor of $v$

Directed Path-width($v_r$) \hspace{1cm} $\triangleright v_r$ is the right successor of $v$

if $v$ corresponds to a $\oplus$, or a $\ominus$ operation

then $d$-pw$(G[T_v]) = \max\{d$-pw$(G[T_{v_\ell}]), d$-pw$(G[T_{v_r}]\} \}$

der $d$-pw$(G[T_v]) = \min\{d$-pw$(G[T_{v_\ell}]) + |V_{G[T_{v_\ell}]}|, d$-pw$(G[T_{v_r}]) + |V_{G[T_{v_r}]}|\}$

Figure 1: Computing the directed path-width of $G$ for every vertex of a di-co-tree $T_G$.

For general digraphs $d$-pw$(G)$ leads to a lower bound for $pw(und(G))$ and $d$-tw$(G)$ leads to a lower bound for $tw(und(G))$, see [5, 25]. For directed co-graphs we obtain a closer relation as follows.

**Corollary 5.5** Let $G$ be a directed co-graph and $\omega(G)$ be the size of a largest bioriented clique of $G$. It then holds that

$$\omega(G) = d$-pw$(G) - 1 = d$-tw$(G) - 1 \leq pw(und(G)) - 1 = tw(und(G)) - 1 = \omega(und(G)).$$

All values are equal if and only if $G$ is a complete bioriented digraph.

**Proof** The equality $pw(und(G)) - 1 = tw(und(G)) - 1 = \omega(und(G))$ has been shown in [9, 10]. The equality $\omega(G) = d$-pw$(G) - 1 = d$-tw$(G) - 1$ follows by Lemma 5.5 (or Lemma 4.4) and Theorem 5.3. The upper bound follows by Lemma 5.5 or Lemma 4.4.

13
5.2 Extended Directed Co-graphs

Theorem 5.6 For every extended directed co-graph $G$, it holds that $d-pw(G) = d-tw(G)$.

The algorithm shown in Fig. 4 can be adapted to show the following result.

Theorem 5.7 For every extended directed co-graph $G = (V,E)$ which is given by a binary ex-di-co-tree the directed path-width and directed tree-width can be computed in time $O(|V|)$.

In order to process the strong components of a digraph we recall the following definition. The acyclic condensation of a digraph $G$, $AC(G)$ for short, is the digraph whose vertices are the strongly connected components $V_1, \ldots, V_c$ of $G$ and there is an edge from $V_i$ to $V_j$ if there is an edge $(v_i, v_j)$ in $G$ such that $v_i \in V_i$ and $v_j \in V_j$. Obviously for every digraph $G$ the digraph $AC(G)$ is always acyclic.

Lemma 5.8 Every digraph $G$ can be represented by the directed union of its strong components.

Proof Let $G$ be a digraph, $AC(G)$ be the acyclic condensation of $G$, and $v_1, \ldots, v_c$ be a topological ordering of $AC(G)$, i.e. for every edge $(v_i, v_j)$ in $AC(G)$ it holds $i < j$. Further let $V_1, \ldots, V_c$ be the vertex sets of its strong components ordered by the topological ordering. Then $G$ can be obtained by $G = G[V_1] \circ \ldots \circ G[V_c]$.

Theorem 5.9 Let $G$ be a digraph, then it holds:

1. The directed tree-width of $G$ is the maximum tree-width of its strong components.
2. The directed path-width of $G$ is the maximum path-width of its strong components.

Proof Follows by Lemma 5.8 and Theorem 3.8 and Theorem 4.10.

The directed path-width result of Theorem 5.9 was also shown in [30] using the directed vertex separation number, which is equal to the directed path-width.

6 Conclusion and Outlook

In this paper we could generalize the equivalence of path-width and tree-width of co-graphs which is known from [9, 10] to directed graphs. The shown equality also holds for more general directed tree-width definitions such as allowing empty sets $W_r$ in [24].

This is not possible for the directed tree-width approach suggested by Reed in [34], which uses sets $W_r$ of size one only for the leaves of $T$ of a directed tree-decomposition $(T, X, W)$. To obtain a counter-example let $S_{1,n} = (V, E)$ be a star graph on $1 + n$ vertices, i.e. $V = \{v_0, v_1, \ldots, v_n\}$ and $E = \{\{v_0, v_i\} | 1 \leq i \leq n\}$. Further let $G_n$ be the complete bi-orientation of $S_{1,n}$, which is a directed co-graph. Then $tw(S_{1,n}) = 1$ and by Theorem 5.3 and Theorem 1.6 we know $d-pw(G_n) = d-tw(G_n) \leq 1$. Using the approach of [34] in any possible tree-decomposition $(T, X', W')$ for $G_n$ there is a leaf $u$ of $T$ such that $W_u = \{v_0\}$. Further there is some $u' \in V_T$, such that $(u', u) \in E_T$. By normality for edge $(u', u)$ it holds $X_{(u', u)} = \{v_1, \ldots, v_n\}$ which implies that using the approach of [34] the directed tree-width of $G$ is at least $n$.

The approach given in [13, Chapter 6] using strong components within (dtw-2) should be considered in future work. Further research directions should extend the shown results to larger classes as well as consider related width parameters.

The class of directed co-graphs was studied very well in [33]. For the class of extended directed co-graphs it remains to show how to compute an ex-di-co-tree in order to apply Theorem 5.7.

Acknowledgements

The work of the second author was supported by the German Research Association (DFG) grant GU 970/7-1.
References

[1] S. Arnborg. Efficient algorithms for combinatorial problems on graphs with bounded decomposability – A survey. *BIT*, 25:2–23, 1985.

[2] S. Arnborg, D.G. Corneil, and A. Proskurowski. Complexity of finding embeddings in a $k$-tree. *SIAM Journal of Algebraic and Discrete Methods*, 8(2):277–284, 1987.

[3] S. Arnborg and A. Proskurowski. Linear time algorithms for NP-hard problems restricted to partial $k$-trees. *Discrete Applied Mathematics*, 23:11–24, 1989.

[4] J. Bang-Jensen and G. Gutin. *Digraphs. Theory, Algorithms and Applications*. Springer-Verlag, Berlin, 2009.

[5] J. Barát. Directed pathwidth and monotonicity in digraph searching. *Graphs and Combinatorics*, 22:161–172, 2006.

[6] D. Bechet, P. de Groote, and C. Retoré. A complete axiomatisation of the inclusion of series-parallel partial orders. In *Rewriting Techniques and Applications*, volume 1232 of *LNCS*, pages 230–240. Springer-Verlag, 1997.

[7] H.L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM Journal on Computing*, 25(6):1305–1317, 1996.

[8] H.L. Bodlaender, T. Kloks, and D. Kratsch. Treewidth and pathwidth of permutation graphs. In *Proceedings of International Colloquium on Automata, Languages and Programming*, volume 700 of *LNCS*, pages 114–125. Springer-Verlag, 1993.

[9] H.L. Bodlaender and R.H. Möhring. The pathwidth and treewidth of cographs. In *Proceedings of Scandinavian Workshop on Algorithm Theory*, volume 447 of *LNCS*, pages 301–309. Springer-Verlag, 1990.

[10] H.L. Bodlaender and R.H. Möhring. The pathwidth and treewidth of cographs. *SIAM J. Disc. Math.*, 6(2):181–188, 1993.

[11] M. Burlet and J.P. Uhry. Parity graphs. *Annals of Discrete Mathematics*, 21:253–277, 1984.

[12] D.G. Corneil, H. Lerchs, and L. Stewart-Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3:163–174, 1981.

[13] C. Crespelle and C. Paul. Fully dynamic recognition algorithm and certificate for directed cographs. *Discrete Applied Mathematics*, 154(12):1722–1741, 2006.

[14] M. Dehmer and F. Emmert-Streib, editors. *Quantitative Graph Theory: Mathematical Foundations and Applications*. Crc Pr Inc, New York, 2014.

[15] J.A. Ellis, I.H. Sudborough, and J.S. Turner. The vertex separation and search number of a graph. *Information and Computation*, 113(1):50–79, 1994.

[16] R. Ganian, P. Hlinený, J. Kneis, D. Meisters, J. Obdrzálek, P. Rossmanith, and S. Sikdar. Are there any good digraph width measures? *Journal of Combinatorial Theory, Series B*, 116:250–286, 2016.

[17] F. Gurski. Dynamic programming algorithms on directed cographs. *Statistics, Optimization and Information Computing*, 5:35–44, 2017.

[18] F. Gurski and C. Rehs. Directed path-width and directed tree-width of directed co-graphs. In *Proceedings of International Computing and Combinatorics Conference (COCOON)*, *LNCS*. Springer-Verlag, 2018. to appear.

[19] F. Gurski, E. Wanke, and E. Yilmaz. Directed NLC-width. *Theoretical Computer Science*, 616:1–17, 2016.
[20] J. Gusted. On the pathwidth of chordal graphs. *Discrete Applied Mathematics*, 45(3):233–248, 1993.

[21] M. Habib and C. Paul. A simple linear time algorithm for cograph recognition. *Discrete Applied Mathematics*, 145:183–197, 2005.

[22] T. Hagerup. Dynamic algorithms for graphs of bounded treewidth. *Algorithmica*, 27(3):292–315, 2000.

[23] M. Hellmuth, P.F. Stadler, and N. Wieseke. The mathematics of xenology: di-cographs, symbolic ultrametrics, 2-structures and tree-representable systems of binary relations. *Journal of Mathematical Biology*, 75(1):199–237, 2017.

[24] T. Johnson, N. Robertson, P.D. Seymour, and R. Thomas. Addendum to ”Directed tree-width”, 2001.

[25] T. Johnson, N. Robertson, P.D. Seymour, and R. Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82:138–155, 2001.

[26] H.A. Jung. On a class of posets and the corresponding comparability graphs. *Journal of Combinatorial Theory, Series B*, 24:125–133, 1978.

[27] M.A. Kashem, X. Zhou, and T. Nishizeki. Algorithms for generalized vertex-rankings of partial $k$-trees. *Theoretical Computer Science*, 240(2):407–427, 2000.

[28] T. Kashiwabara and T. Fujisawa. NP-completeness of the problem of finding a minimum-clique-number interval graph containing a given graph as a subgraph. In *Proceedings of the International Symposium on Circuits and Systems*, pages 657–660, 1979.

[29] K. Kitsunai, Y. Kobayashi, K. Komuro, H. Tamaki, and T. Tano. Computing directed pathwidth in $O(1.89^n)$ time. *Algorithmica*, 75:138–157, 2016.

[30] T. Kloks, H. Bodlaender, H. Müller, and D. Kratsch. Computing treewidth and minimum fill-in: All you need are the minimal separators. In *Proceedings of the Annual European Symposium on Algorithms*, volume 726 of *LNCS*, pages 260–271. Springer-Verlag, 1993.

[31] H. Lerchs. On cliques and kernels. Technical report, Dept. of Comput. Sci, Univ. of Toronto, 1971.

[32] B. Monien and I.H. Sudborahough. Min cut is NP-complete for edge weighted trees. *Theoretical Computer Science*, 58:209–229, 1988.

[33] N. Nojgaard, N. El-Mabrouk, D. Merkle, N. Wieseke, and M. Hellmuth. Partial homology relations - satisfiability in terms of di-cographs. In *Proceedings of International Computing and Combinatorics Conference (COCOON)*, LNCS. Springer-Verlag, 2018. to appear.

[34] B. Reed. Introducing directed tree width. *Electronic Notes in Discrete Mathematics*, 3:222–229, 1999.

[35] N. Robertson and P.D. Seymour. Graph minors I. Excluding a forest. *Journal of Combinatorial Theory, Series B*, 35:39–61, 1983.

[36] N. Robertson and P.D. Seymour. Graph minors II. Algorithmic aspects of tree width. *Journal of Algorithms*, 7:309–322, 1986.

[37] P. Scheffler. *Die Baumweite von Graphen als Mass für die Kompliziertheit algorithmischer Probleme*. Ph. D. thesis, Akademie der Wissenschaften in der DDR, Berlin, 1989.

[38] K. Suchan and I. Todinca. Pathwidth of circular-arc graphs. In *Proceedings of Graph-Theoretical Concepts in Computer Science*, volume 4769 of *LNCS*, pages 258–269. Springer-Verlag, 2007.

[39] P.D. Sumner. Dacey graphs. *Journal of Aust. Soc.*, 18:492–502, 1974.

[40] B. Yang and Y. Cao. Digraph searching, directed vertex separation and directed pathwidth. *Discrete Applied Mathematics*, 156(10):1822–1837, 2008.