ELLIPITIC AND WEAKLY COERCIVE SYSTEMS
OF OPERATORS IN SOBOLEV SPACES

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Abstract. It is known that an elliptic system \{P_j(x, D)\}_{j=1}^N of order \(l\) is weakly coercive in \(\overset{\circ}{W}_\infty^l(\mathbb{R}^n)\), that is, all differential monomials of order \(\leq l - 1\) on \(C_0^\infty(\mathbb{R}^n)\)-functions are subordinated to this system in the \(L^\infty\)-norm. Conditions for the converse result are found and other properties of weakly coercive systems are investigated.

An analogue of the de Leeuw-Mirkil theorem is obtained for operators with variable coefficients: it is shown that an operator \(P(x, D)\) in \(n \geq 3\) variables with constant principal part is weakly coercive in \(\overset{\circ}{W}_\infty^l(\mathbb{R}^n)\) if and only if it is elliptic. A similar result is obtained for systems \{\(P_j(x, D)\)\}_{j=1}^N with constant coefficients under the condition \(n \geq 2N + 1\) and with several restrictions on the symbols \(P_j(\xi)\).

A complete description of differential polynomials in two variables which are weakly coercive in \(\overset{\circ}{W}_\infty^l(\mathbb{R}^2)\) is given. Wide classes of systems with constant coefficients which are weakly coercive in \(W_\infty^l(\mathbb{R}^n)\), but non-elliptic are constructed.

Bibliography: 32 titles.

1. Introduction

Let \(\Omega\) be an arbitrary domain in \(\mathbb{R}^n\), let \(p \in [1, \infty]\), and let \(l := (l_1, \ldots, l_n)\) be a vector with positive integer components. In \(L^p(\Omega)\) consider a system \{\(P_j(x, D)\)\}_{j=1}^N of differential operators of the form

\[
P_j(x, D) = \sum_{|\alpha| \leq l} a_{j\alpha}(x)D^\alpha, \quad j \in \{1, \ldots, N\},
\]

with measurable coefficients \(a_{j\alpha}(\cdot)\). Further, let \(P^l_j(x, D) := \sum_{|\alpha| = 1} a_{j\alpha}(x)D^\alpha\) be the \(l\)-principal part of the operator \(P_j(x, D)\), and let \(P^l_j(x, \xi) := \sum_{|\alpha| = 1} a_{j\alpha}(x)\xi^\alpha\) be its principal \(l\)-quasihomogeneous symbol. We recall the following definition.

Definition 1.1. (see [1]-[3]) A system of differential operators \{\(P_j(x, D)\)\}_{j=1}^N of the form (1.1) is said to be \(l\)-quasielliptic if

\[
(P^l_1(x, \xi), \ldots, P^l_N(x, \xi)) \neq 0, \quad (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}).
\]

In particular, if \(l_1 = \cdots = l_n = l\), then it is called an elliptic system of order \(l\).

As is known, an elliptic operator of order \(l\) does not exist for every \(l\). Using a result due to Lopatinski \[4\] (see also [5], [6], Ch. 2, \(\S\) 1, [7]), for \(n \geq 3\) an elliptic operator \(P(D)\) is properly elliptic and, in particular, has even order. To the best of our knowledge, a similar problem for \(l\)-quasielliptic operators remains unsolved at present. In \(\S\) 3, using the Borsuk-Ulam theorem (Theorem 2.1), we obtain a complete description of those \(l\) for which \(l\)-quasielliptic systems exist. Namely, the following theorem holds.
Theorem 1.2. Let \( l = (l_1, \ldots, l_n) \in \mathbb{N}^n \) and let \( n \geq 2N + 1 \). Then \( l \)-quasielliptic systems \( \{P_j(x, D)\}_{1}^{N} \) of the form (1.1) exist if and only if the number of odd integers among \( l_1, \ldots, l_n \) does not exceed \( 2N - 1 \).

Let \( \{P_j(x, D)\}_{1}^{N} \) be a system of differential operators of the form (1.1) with coefficients \( a_{j\alpha}(\cdot) \in L_{loc}^{\infty}(\Omega) \). We recall the following notion.

Definition 1.3. (see [1], Ch. 3, §11.1) A system of differential operators \( \{P_j(x, D)\}_{1}^{N} \) of the form (1.1) is said to be coercive in the (anisotropic) Sobolev space \( \tilde{W}_p^l(\Omega) \), \( p \in [1, \infty] \), if the following estimate holds:

\[
\|f\|_{W_p^l(\Omega)} := \sum_{|\alpha| \leq 1} \|D^{\alpha}f\|_{L^p(\Omega)} \leq C_1 \sum_{j=1}^{N} \|P_j(x, D) f\|_{L^p(\Omega)} + C_2 \|f\|_{L^p(\Omega)},
\]

(1.2)

where \( C_1 \) and \( C_2 \) do not depend on \( f \in C_0^\infty(\Omega) \).

It is well known (see [1], [3], [8] and [9]) that, under some constraints on the coefficients \( a_{j\alpha}(\cdot) \) and on the domain \( \Omega \) the system (1.1) is \( l \)-quasielliptic if and only if it is coercive in \( \tilde{W}_p^l(\Omega) \) for \( p \in (1, \infty) \). If \( p = 1 \) or \( \infty \) then the estimate (1.2) does not hold any longer for an \( l \)-quasielliptic system. Namely, the following assertion was proved by one of the authors of this paper.

Proposition 1.4. (see [10]-[12], §5, Theorem 3) Let \( \Omega \) be a domain in \( \mathbb{R}^n \), and let \( Q(x, D) \) and \( \{P_j(x, D)\}_{1}^{N} \) be differential operators of the form

\[
Q(x, D) = \sum_{|\alpha| \leq 1} b_\alpha(x)D^\alpha, \quad P_j(x, D) = \sum_{|\alpha| \leq 1} a_{j\alpha}(x)D^\alpha,
\]

(1.3)

where \( x \in \Omega, \quad j \in \{1, \ldots, N\} \), and the coefficients \( a_{j\alpha}(\cdot), b_\alpha(\cdot) \in L_{loc}^{\infty}(\Omega) \) for \( |\alpha : l| < 1 \) and \( a_{j\alpha}(\cdot), b_\alpha(\cdot) \in C^1(\Omega) \) for \( |\alpha : l| = 1 \). Then the estimate

\[
\|Q(x, D)f\|_{L^p(\Omega)} \leq C_1 \sum_{j=1}^{N} \|P_j(x, D) f\|_{L^p(\Omega)} + C_2 \|f\|_{L^p(\Omega)}, \quad f \in C_0^\infty(\Omega),
\]

(1.4)

for \( p = \infty \) yields the equality

\[
Q^j(x, \xi) = \sum_{j=1}^{N} \lambda_j(x)P_j^j(x, \xi), \quad x \in \Omega, \quad \xi \in \mathbb{R}^n,
\]

(1.5)

in which \( \lambda_j(\cdot) \in C^1(\Omega) \). If the operators \( Q(x, D) \) and \( \{P_j(x, D)\}_{1}^{N} \) have constant coefficients, then the functions \( \lambda_j(x) \) in (1.5) are also constant: \( \lambda_j(x) \equiv \lambda_j \).

A criterion for the system \( \{P_j(x, D)\}_{1}^{N} \) to be coercive in \( \tilde{W}_p^l(\Omega) \) was found in [11], [12] (in the isotropic case it was found earlier in [10]). This criterion yields that an \( l \)-quasielliptic system is coercive in \( \tilde{W}_p^l(\Omega) \) only in exceptional cases. Nevertheless, for an \( l \)-quasielliptic system \( \{P_j\}_{1}^{N} \) the following estimate holds:

\[
\sum_{|\alpha| \leq 1} \|D^{\alpha}f\|_{L^p(\Omega)} \leq C_1 \sum_{j=1}^{N} \|P_j(x, D) f\|_{L^p(\Omega)} + C_2 \|f\|_{L^p(\Omega)}, \quad f \in C_0^\infty(\Omega).
\]

(1.6)
For $p \in (1, \infty)$ this estimate is implied by the estimate (1.2) established in [1] and [8] (see also [3]) and for $p = \infty$ it is proved in [11] and [12]. Note also that the fact that the estimate (1.2) is impossible in the case $p = 1$ follows from a result due to Ornstein [13]. But in the case $p = 1$, the estimate (1.6) was proved for operators with constant coefficients in [14] and [15].

These results suggest the following natural definition introduced in [15].

**Definition 1.5.** A system of differential operators \( \{ P_j(x, D) \}_1^N \) of the form (1.1) is said to be weakly coercive in the anisotropic Sobolev space \( \overset{\circ}{W}_p^l(\Omega) \), $p \in [1, \infty]$, if the estimate (1.6) is valid with $C_1$ and $C_2$ independent of $f$.

In the case of isotropic Sobolev space \( \overset{\circ}{W}_p^l(\Omega) \), that is, for $l_1 = \cdots = l_n = l$, the inequality $|\alpha : l| < 1$ in (1.6) takes the usual form $|\alpha| < l$.

In the case of one operator de Leeuw and Mirkil [16] showed before that for $n \geq 3$ an elliptic operator $P(D) = P_1(D)$ can be characterized by means of a priori estimates in $L^\infty(\mathbb{R}^n)$.

**Theorem 1.6.** (see [16], p. 119) Assume that $n \geq 3$. Then the ellipticity of a differential operator $P(D)$ of order $l \geq 2$ is equivalent to its weak coercivity in $\overset{\circ}{W}_\infty^l(\mathbb{R}^n)$.

The condition $n \geq 3$ is essential in Theorem 1.6. In fact, Malgrange presented an example of a non-elliptic operator $P(D) = (D_1 + i)(D_2 + i)$ that is weakly coercive in $\overset{\circ}{W}_\infty^2(\mathbb{R}^2)$ (see [16], p. 123).

In this paper we mainly consider homogeneous systems \( \{ P_j(x, D) \}_1^N \) of the form (1.1) consisting of operators with homogeneous principal symbols of order $l$. Our investigation of the quasihomogeneous case is postponed till the next publication. To avoid the possibility of repetition here we present only those 'anisotropic' results whose proofs do not differ in practice from the corresponding 'isotropic' ones.

A considerable proportion of our results is relate the de Leeuw-Mirkil Theorem 1.6. Namely, we extend Theorem 1.6 to a system \( \{ P_j(D) \}_1^N \) with constant coefficients (Theorem 4.9) and also prove its analogue for an operator $P(x, D)$ with variable coefficients (Theorem 4.11). To prove the latter we use a new method which is essentially based on Proposition 1.4 and also on some topological concepts (summarized in Proposition 4.1, (iii)). Note that the method in [16] is not applicable to operators with variable coefficients, although in proving Theorem 4.9, which concerns systems with constant coefficients, alongside the topological concepts we use some arguments from [16].

In addition, we present a complete description of weakly coercive operators of two variables in $\overset{\circ}{W}_\infty^l(\mathbb{R}^2)$ (Theorems 5.1 and 5.4). In particular, in doing this we show that the non-trivial zeros of the principal symbol of a weakly coercive operator are simple (Proposition 4.1, (iv)). Note that to prove this last result, as well as in the proof of Theorem 4.11, we use an analogue of Theorem 1.6, an anisotropic version of Proposition 1.4. This application of Proposition 1.4 to the proof of 'isotropic' results is based on the possibility, in principle, of a non-unique selection of the principal part of a differential operator.

Note also that topological arguments are also used in §4, to prove an analogue of Theorem 1.2 in the case of a weakly coercive system (Theorem 4.3). Namely, invoking
Borsuk’s theorem (Theorem 2.3) and degree theory we show that, under some restrictions, the system \( \{ P_j(x, D) \} \) has even order.

It is also worth mentioning that in §6, in the construction of weakly coercive, but non-elliptic systems, new non-symmetric multipliers on \( L^p, p \in [1, \infty] \), arise, which are not traditional in elliptic theory. For instance, it is shown in the proof of Theorem 6.2 that if \( P(\xi) \) is an elliptic polynomial of degree \( l \), then

\[
m(\xi) := \chi(\xi) \frac{\xi^\alpha}{P(\xi) \sum_{k=2}^n (1 + \xi_k^2)} \in \mathcal{M}_1(\mathbb{R}^n) \quad \text{for} \quad |\alpha| \leq l + 1, \quad \alpha_1 \leq l - 1,
\]

that is, \( m(\cdot) \) is a multiplier on \( L^1(\mathbb{R}^n) \), hence a multiplier on \( L^p(\mathbb{R}^n) \) for \( p \in [1, \infty] \). Here \( \chi(\cdot) \) is a suitable ‘cutoff’ function. To verify the inclusion \( m \in \mathcal{M}_p(\mathbb{R}^n) \) for \( p \in (1, \infty) \) we can use the Mikhlin-Lizorkin theorem (see [14], and also [17] and [18]), but this is insufficient for verifying the inclusion \( m \in \mathcal{M}_1 \). To prove the latter we use a result on multipliers from [14].

The paper is organized as follows. In §2 we present auxiliary topological and analytic results necessary in what follows. In §3 we prove the existence criterion for \( l \)-quasielliptic systems (Theorem 1.2) and a stability criterion for systems of order \( l \) under perturbations of order \( \leq l - 1 \) (Proposition 3.10). We devote §4 to properties of weakly coercive systems in the isotropic spaces \( \mathring{W}^l_p(\mathbb{R}^n) \). We also prove there analogues of Theorem 1.6 for the case of a homogeneous system (Theorem 4.9) and that of an operator with variable coefficients (Theorem 4.11). In §5 we give a complete description of operators in two variables that are weakly coercive in \( \mathring{W}^l_\infty(\mathbb{R}^2) \), but are not elliptic (Theorems 5.1 and 5.4). Finally, §6 is devoted to describing wide classes of non-elliptic systems that are weakly coercive in the isotropic space \( \mathring{W}^l_\infty(\mathbb{R}^n) \) (Theorem 6.2).

A part of the results here were announced (without proofs) in [15] and [19].

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2. Preliminaries

We will use the following notation. Let \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \), let \( \mathbb{Z}_+^n := \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+ \) (\( n \) is the number of factors), and \( \mathbb{Z}_2 := \{0, 1\} \). Further, let \( D_k := -i \partial / \partial x_k \) and \( D = (D_1, D_2, \ldots, D_n) \); for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \) we set \( |\alpha| := \alpha_1 + \cdots + \alpha_n \) and \( D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} \). If \( l = (l_1, \ldots, l_n) \in \mathbb{N}^n \) and \( \alpha \in \mathbb{Z}_+^n \), then \( |\alpha : l| := \alpha_1/l_1 + \cdots + \alpha_n/l_n \).

Also let \( |x| := (\sum_{k=1}^n x_k^2)^{1/2} \), \( (x, y) := \sum_{k=1}^n x_k y_k \), where \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), x, y \in \mathbb{R}^n \). Denote by \( S^n_r := \{ x \in \mathbb{R}^{n+1} : |x| = r \} \) the \( n \)-dimensional sphere of radius \( r \) in \( \mathbb{R}^{n+1} \), with \( S^n := S^n_1 \); and by \( B^n_r := \{ x \in \mathbb{R}^n : |x| \leq r \} \) the closed ball of radius \( r \).
We denote by $\mathbb{I} = \mathbb{I}_n$ the identity operator in $\mathbb{R}^n$ and by $\mathcal{M}_p = \mathcal{M}_p(\mathbb{R}^n)$ the algebra of multipliers on $L^p(\mathbb{R}^n)$, $p \in [1, \infty]$.

2.1. **Topological concepts.**

**Theorem 2.1.** (the Borsuk-Ulam theorem; see [20], Ch.5, §8.9) For each continuous mapping $f : S^n \to \mathbb{R}^n$, $n \geq 1$, there is a point $x \in S^n$ such that $f(x) = f(-x)$.

Following [20], Ch. 4, §7, and [21] recall the notion of the degree of a map. As is known, the $n$-dimensional homotopy group of the sphere $S^n$ is isomorphic to $\mathbb{Z}$, $\pi_n(S^n) \simeq \mathbb{Z}$. Each continuous map $f : S^n \to S^n$ induces a group homomorphism $f_\ast : \pi_n(S^n) \to \pi_n(S^n)$, hence $f_\ast : \mathbb{Z} \to k\mathbb{Z}$. The integer $k$ does not depend on the choice of a generator of the group $\pi_n(S^n)$; it is referred to as the *degree* of $f$ and is denoted by $\deg f$.

Since the $n$-dimensional homology group $H_n(S^n; \mathbb{Z}) \simeq \mathbb{Z}$, the degree of a map $f : S^n \to S^n$ can be defined in terms of the homomorphism $f_{*n} : H_n(S^n; \mathbb{Z}) \to H_n(S^n; \mathbb{Z})$. These definitions are equivalent.

Further, homotopic maps have equal degree. The converse also holds (Hopf’s theorem).

Since $\mathbb{R}^{n+1} \setminus \{0\}$ is homotopy equivalent to $S^n$, it follows that $\pi_n(\mathbb{R}^{n+1} \setminus \{0\}) \simeq \pi_n(S^n)$ and so maps $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ have well defined degrees.

We will use the following statements repeatedly.

**Theorem 2.2.** (see [21], §1.4) A continuous map $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ can be extended to a continuous map of the closed ball $B_1^{n+1}$ into $\mathbb{R}^{n+1} \setminus \{0\}$ if and only if $\deg f = 0$.

**Theorem 2.3.** (Borsuk’s theorem on the degree of a map; see [21], §1.7) Let $f$ be an odd map of the sphere $S^n$ into itself: $f(-x) = -f(x)$. Then its degree $\deg f$ is odd.

2.2. **Analytic results.**

**Lemma 2.4.** (Eberlein’s theorem; see [16], p.114) Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$ and $\mathcal{M}$ the Fourier-Stieltjes transform of $\mu$. Then the constant function $c \equiv \mu(0)$ can be uniformly approximated by functions of the form $\pi * \mathcal{M}$, where $\pi$ is a probability measure (that is, $\pi(\mathbb{R}^n) = 1$).

**Proposition 2.5.** (see [16], p. 113, Proposition 1) Let $Q(D)$ and $\{P_j(D)\}_{j=1}^N$ be differential operators of the form (1.3) with constant coefficients. Then the estimate (1.4) for $p = \infty$ and $\Omega = \mathbb{R}^n$ is equivalent to the identity

$$Q(\xi) = \sum_{j=1}^N M_j(\xi)P_j(\xi) + M_{N+1}(\xi), \quad \xi \in \mathbb{R}^n,$$

where the $\{M_j(\cdot)\}_{j=1}^{N+1}$ are the Fourier-Stieltjes transforms of finite Borel measures on $\mathbb{R}^n$.

**Proposition 2.6.** (see [16], p.114) Let $\{P_j(D)\}_{j=1}^N$ be a system satisfying estimate (1.4) with $p = \infty$ and $\Omega = \mathbb{R}^n$, and assume that the principal forms $\{P_j(\xi)\}_{j=1}^N$ are linearly independent. Also let $\{\lambda_j\}_{j=1}^N$ be the coefficients in equality (1.5) and $\{\mu_j\}_{j=1}^N$ be finite Borel measures, $\mu_j = M_j$, where $\{M_j(\cdot)\}_{j=1}^N$ are the functions in (2.1). Then $\lambda_j = \mu_j(0)$, $j \in \{1, \ldots, N\}$.

The next statement is well known to experts. Moreover, it was mentioned (without proof) in [22]. For the sake of completeness we present it here with the proof.
Proposition 2.7. Let \( p \in [1, \infty] \) and let \( Q(D) \) and \( \{ P_j(D) \}_{j=1}^N \) be differential operators of the form (1.3) with constant coefficients. Then the a priori estimate (1.4) implies the following algebraic inequality for the symbols:

\[
|Q(\xi)| \leq C_1' \sum_{j=1}^N |P_j(\xi)| + C_2', \quad \xi \in \mathbb{R}^n. \tag{2.2}
\]

Sketch of the proof. (i) Let \( p \in [1, \infty] \) and let \( \psi \in C_0^\infty(\mathbb{R}^n), \psi \not\equiv 0 \). We set \( \psi_r(x) := \psi(x/r) = \psi(x_1/r, \ldots, x_n/r), r > 0 \). It can be verified directly that

\[
\|D^\alpha \psi_r \cdot P(D) e^{i(x,\xi)}\|_p = r^{-|\alpha|} \cdot r^{n/p} |P(\xi)| \cdot \|D^\alpha \psi_r\|_p. \tag{2.3}
\]

Applying Leibniz’s formula to \( f_r(x) := \psi_r(x)e^{i(x,\xi)} \) (see [9], Ch. II, §2.1) we obtain

\[
P(D)f_r = P(D)(\psi_re^{i(x,\xi)}) = \sum_{\alpha} \left( \frac{1}{|\alpha|!} \right) D^\alpha \psi_r \cdot P^{(\alpha)}(D) e^{i(x,\xi)}, \tag{2.4}
\]

where \( P^{(\alpha)}(D) \) is the operator with symbol \( D^\alpha P(\xi) \). Taking (2.3) and (2.4) into account we obtain

\[
\|P_j(D)f_r\|_p \leq r^{n/p} |P_j(\xi)| \cdot \|\psi\|_p + o\left(r^{n/p}\right),
\]

\[
\|Q(D)f_r\|_p \geq r^{n/p} |Q(\xi)| \cdot \|\psi\|_p + o\left(r^{n/p}\right).
\]

Substituting the above expressions in (1.4) we arrive at the estimate

\[
r^{n/p} |Q(\xi)| \cdot \|\psi\|_p + o\left(r^{n/p}\right) \leq C_1' \sum_{j=1}^N r^{n/p} |P_j(\xi)| \cdot \|\psi\|_p + o\left(r^{n/p}\right) + C_2' r^{n/p} \|\psi\|_p. \tag{2.5}
\]

Dividing both sides of (2.5) by \( r^{n/p} \|\psi\|_p > 0 \) and then passing to the limit as \( r \to \infty \) we obtain (2.2).

(ii) For \( p = \infty \) the proof is similar: it suffices to note that for \( p = \infty \) the factor \( r^{n/p} \) in (2.3) is equal to \( r^{n/\infty} = r^0 = 1 \).

Remark 2.8. (i) Proposition 1.4 (together with its proof) remains valid if all the differential monomials \( D^\alpha \) depend only on the components \( D_1, \ldots, D_m \), in accordance with the decomposition \( \mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m} \), that is, \( D^\alpha = D_1^{\alpha_1} \cdots D_m^{\alpha_m} \oplus I_{n-m} \). In this case equality (1.5) follows from inequality (1.4) with \( f \in C_0^\infty(\Omega_m) \), where \( \Omega_m = \pi_m \Omega \) is the projection of the domain \( \Omega \) onto \( \mathbb{R}^m \).

(ii) Inequality (2.2) is a consequence of the estimate (1.4), but is not equivalent to (1.4). For instance, if \( P(\xi) = \xi_1^2 + \xi_2^2 \) and \( Q(\xi) = \xi_1^2 - \xi_1 \xi_2 + \xi_2^2 \), then \( Q(\xi) < \frac{3}{2} P(\xi) \).

(iii) Proposition 2.7 fails for arbitrary domains \( \Omega \). For instance, if \( \Omega \) is bounded and \( p = 2 \), then, by a theorem of Hörmander’s in [9], §2.3 estimate (1.4) holds for \( P(D) = D_1^2 - D_2^2 \) and \( Q(D) = D_1 \), whereas the inequality \( |\xi_1| \leq C [\xi_1^2 - \xi_2^2 + 1] \) fails.

2.3. Multipliers.

Definition 2.9. (see [23], Ch.IV, §3) Let \( \mathcal{F} \) be the Fourier transform in \( L^2(\mathbb{R}^n) \). A bounded (Lebesgue-)measurable function \( \Phi : \mathbb{R}^n \to \mathbb{C} \) is called a multiplier on \( L^p(\mathbb{R}^n) \), \( p \in [1, \infty] \), if the convolution operator \( f \to T_\Phi f := \mathcal{F}^{-1} \Phi \mathcal{F} f \) takes \( L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) and is bounded in \( L^p(\mathbb{R}^n) \).
A simple description of the spaces $\mathcal{M}_p$ of multipliers on $L^p(\mathbb{R}^n)$ is known only for $p = 1, 2, \infty$. In particular, $\mathcal{M}_1 = \mathcal{M}_\infty$ is the set of images under the Fourier-Stieltjes transform of finite Borel measures on $\mathbb{R}^n$ (see [23], Ch. IV, §3)

$$\Phi \in \mathcal{M}_1 = \mathcal{M}_\infty \iff \Phi(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} d\mu(x), \quad \mu(\mathbb{R}^n) < \infty. \quad (2.6)$$

For other values $p \in (1, \infty)$ only sufficient conditions are known for the inclusion $\Phi \in \mathcal{M}_p$ to hold (see [1], [7], [23]). Note that $\mathcal{M}_1 \subset \mathcal{M}_p$ for $p \in (1, \infty)$ by (2.6).

We shall need the following result of [14] on multipliers on $L^1$ which in appearance is a (fairly rough) analogue of the Mikhlin-Lizorkin theorem (see [1]).

**Theorem 2.10.** (see [14], §3, Theorem 2) Let $\Phi \in C(\mathbb{R}^n)$ and assume that for some constants $\delta \in (0, 1)$ and $A_\delta > 0$, $\Phi$ satisfies the following conditions:

\begin{align*}
(i) \quad & \prod_{j=1}^n (1 + |\xi_j|^\delta |\Phi(\xi)|) \leq A_\delta, \quad \xi \in \mathbb{R}^n; \\
(ii) \quad & \text{for all multi-indices } \alpha, \beta \in \mathbb{Z}_+^n \text{ such that } \alpha + \beta = (1, 1, \ldots, 1), \text{ there exist derivatives } D^\alpha \Phi \text{ and} \\
& \prod_{j \in \mathbb{N}_\alpha} |\xi_j|^{-\delta (1 + |\xi_j|^{2\delta})} \prod_{j \in \mathbb{N}_\beta} (1 + |\xi_j|)^\delta |D_1^{\alpha_1} \cdots D_n^{\alpha_n} \Phi(\xi)| \leq A_\delta, \quad \xi \in \mathbb{R}^n. \quad (2.8)
\end{align*}

Here $\mathbb{N}_\alpha \subset \{1, \ldots, n\}$ is the support of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, that is, the set of subscripts $j \in \{1, \ldots, n\}$ for which $\alpha_j > 0$.

Then $\Phi \in \mathcal{M}_1$, and hence $\Phi \in \mathcal{M}_p$ for $p \in [1, \infty]$.

Note that more general results on multipliers on $L^1(\mathbb{R}^n)$ can be found in [24]-[26], and in [27], Theorem 6.4.2. However, in applications to estimates of differential operators the functions $\Phi$ are usually rational functions of $\xi$ and $\xi$. In this case conditions (2.7) and (2.8) can be verified as readily as the corresponding conditions in the Mikhlin-Lizorkin theorem.

2.4. **Properties of $l$-quasielliptic systems.** The following properties of $l$-quasielliptic systems are well-known (see [1], [2], [28]).

**Proposition 2.11.** Let $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$ and let $\{P_j(D)\}_1^N$ be an $l$-quasielliptic system of the form (1.1) with constant coefficients. Then

(i) the zero set of the system $\{P_j(\xi)\}_1^N$ is compact and hence is contained in some ball $B_r^n$;

(ii) the following two-sided estimate holds:

$$C_1 \sum_{k=1}^n |\xi_k|^{2l_k} \leq \sum_{j=1}^N |P_j(D)\xi|^2 \leq C_2 \sum_{k=1}^n |\xi_k|^{2l_k}, \quad C_1, C_2 > 0, \quad \xi \in \mathbb{R}^n. \quad (2.9)$$

3. **Elliptic and quasielliptic systems**

3.1. **For which values of $l$ do $l$-quasielliptic systems exist?** Here we prove Theorem 1.2 which was stated in the introduction, describing all the sets $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$ for which there exist $l$-quasielliptic systems.
Proof of Theorem 1.2. Necessity. Let \( n = 2N + 1 \).

(i) Assume first that all the \( l_j \) are odd. We claim that \( P_j(x, -\xi) = -P_j(x, \xi) \). Let \( \alpha \in \mathbb{N}^n \) and let \( \alpha_1/l_1 + \cdots + \alpha_n/l_n = 1 \). Since all the \( l_j \) are odd, this equality acquires the form \( \alpha_1 k_1 + \cdots + \alpha_n k_n = k_0 \), where all \( k_j \) are odd, \( k_j = 2k'_j + 1, j \in \{0, 1, \ldots, n\} \). Then \( \alpha_1 + \cdots + \alpha_n = 2k'_0 + 1 - \sum_{j=1}^n 2\alpha_j k'_j = 2p + 1 \). Therefore,

\[
P_j(x, -\xi) = \sum_{|\alpha|=1} a_j(x)(-\xi)^{\alpha} = \sum_{|\alpha|=1} a_j(x)(-1)^{|\alpha|}\xi^{\alpha} = -P_j(x, \xi).
\]

Now choosing fixed \( x_0 \in \Omega \) we consider the map \( T := (T_1, \ldots, T_{2N}) : S^{2N} \to \mathbb{R}^{2N} \), where

\[
T_{2j-1}(\xi) := \text{Re} P_j(x_0, \xi), \quad T_{2j}(\xi) := \text{Im} P_j(x_0, \xi), \quad j \in \{1, \ldots, N\}.
\]

This map is odd: \( T(-\xi) = -T(\xi) \), and by Theorem 2.1 we have \( T(\xi^0) = 0 \) at some point \( \xi^0 \in S^{2N} \). But this contradicts the assumption that the system \( \{P_j(x, D)^{1/2}\}^N_1 \) is \( l \)-quasielliptic.

(ii) Suppose that precisely one \( l_j \) is even; for example, let \( l_1 = 2m l_1' \), and assume that \( l_1' \) and the other \( l_j \) are odd, \( j \in \{2, \ldots, n\} \). We claim that the relation \( |\alpha : l| = 1 \) implies that \( \alpha_1 \) is also divisible by \( 2^m \), that is, \( \alpha_1 = 2^m \alpha_1' \), \( \alpha_1' \in \mathbb{N} \). In fact, \( |\alpha : l| = 1 \) reduces to the equality

\[
\alpha_1 k_1 + 2^m (\alpha_2 k_2 + \cdots + \alpha_n k_n) = 2^m k_0,
\]

where all the \( k_j \) are odd. Hence \( \alpha_1 = 2^m \alpha_1' \).

Further, let \( l' := (l_1', l_2, \ldots, l_n') := (l_1', l_2', \ldots, l_n') \) and let \( |\alpha : l| = 1 \). Then \( \alpha_1 = 2^m \beta_1 \). Setting \( \eta_1 = \xi_1^m, \eta_j = \xi_j \) and \( \beta_j = \alpha_j, j \in \{2, \ldots, n\} \), we write the polynomial \( P_j(\xi) \) in the form

\[
P_j(\xi) = \tilde{P}_j(\eta), \quad \text{where} \quad \tilde{P}_j(\eta) := \sum_{|\beta| = 1} a_{\beta} \eta^{\beta}, \quad j \in \{1, \ldots, N\}.
\]

The system \( \{\tilde{P}_j(\eta)\}^N_1 \) is \( l' \)-quasielliptic. In fact, let \( \tilde{P}_j(\eta^0) = 0, j \in \{1, \ldots, N\}, \eta^0 = (\eta_1^0, \ldots, \eta_n^0) \in \mathbb{R}^n \setminus \{0\} \). Since all the \( l_j' \) are odd, it follows by (i) that the \( \tilde{P}_j''(\eta) \) are odd, \( \tilde{P}_j''(-\eta^0) = -\tilde{P}_j''(\eta^0) = 0 \). Therefore, we may assume that \( \eta_1^0 > 0 \). In this case, setting \( \xi_1^1 := (\eta_1^0)^{1/2^m} \) and \( \xi_j^1 := \eta_j^0 \) for \( j \in \{2, \ldots, n\} \) we obtain \( \tilde{P}_j(\eta^0) = 0, j \in \{1, \ldots, N\} \), so that \( \xi_1^1 = 0 \). The last relation contradicts the assumption that \( \eta^0 \neq 0 \). Thus, the system \( \{\tilde{P}_j(\eta)\}^N_1 \) is \( l' \)-quasielliptic, where all the \( l_j' \) are odd. By (i) this is impossible. It follows that there must be at least two even integers among the \( l_j \), that is, we have proved the theorem for \( n = 2N + 1 \).

(iii) Suppose that \( n > 2N + 1 \), but there are more than \( 2N - 1 \) odd integers among the \( l_j \). We assume without loss of generality that \( l_1, \ldots, l_{2N} \) are odd. It is clear that the ‘restricted’ system \( \{P_j(x, \xi_1, \ldots, \xi_{2N+1}, 0, \ldots, 0)\}^N_1 \) is \( l' \)-quasielliptic, where \( l' = (l_1, \ldots, l_{2N+1}) \). Therefore, by (i) and (ii) we arrive at a contradiction. Thus, there are at most \( 2N - 1 \) odd integers among \( l_1, \ldots, l_n \).

Sufficiency. Let \( n = 2N + 1, l = (l_1, \ldots, l_n) \), where \( l_1, \ldots, l_{n-2} \) are odd and \( l_{n-1}, l_n \) are even. Then the system

\[
P_1(\xi) = \xi_1^{l_1} + i\xi_2^{l_2}, \quad \ldots, \quad P_{N-1}(\xi) = \xi_1^{l_{n-4}} + i\xi_2^{l_{n-3}}, \quad P_N(\xi) = i\xi_1^{l_{n-2}} + \xi_1^{l_{n-1}} + \xi_1^{l_n}
\]

is \( l \)-quasielliptic and precisely two numbers among the \( l_j \) are even. \( \square \)
Corollary 3.1. If \( n \geq 3 \), then \( l \)-quasielliptic operators exist if and only if there is at most one odd integer among \( l_1, \ldots, l_n \).

In the homogeneous case Theorem 1.2 reduces to the following result.

Corollary 3.2. Let \( l_1 = \cdots = l_n = l \) and let \( \{P_j(x, D)\}_{1}^{N} \) be an elliptic system of the form (1.1). If \( n \geq 2N + 1 \), then \( l \) is even.

Remark 3.3. (i) The condition \( n \geq 2N + 1 \) of Theorem 1.2 is sharp. Specifically, for \( n = 2N \) and any \( l = (l_1, \ldots, l_{2N}) \) consider the system of operators

\[
P_1(D) := D_1^{l_1} + iD_2^{l_2}, \quad \ldots, \quad P_N(D) := D_{2N-1}^{l_{2N-1}} + iD_{2N}^{l_{2N}}.
\]

(3.2)

The system (3.2) is \( l \)-quasielliptic. In other words, Theorem 1.2 does not hold for \( n \leq 2N \).

(ii) For \( N = 1 \) there is a stronger result than Theorem 1.2 due to Lopatinski: for \( n \geq 3 \) an elliptic operator \( P(D) \) is properly elliptic; in particular, it has even order (see [4]-[7]).

3.2. Characterization of \( l \)-quasielliptic systems by means of a priori estimates.

We characterize \( l \)-quasielliptic systems with the help of a priori estimates in the isotropic Sobolev spaces \( \overset{\circ}{W}^1_p(\Omega) \). Recall (see [1], [8], [29]) the following coercivity criterion for a system \( \{P_j(x, D)\}_{1}^{N} \) in \( \overset{\circ}{W}^1_p(\Omega) \), \( p \in (1, \infty) \).

Theorem 3.4. (see [1], Ch. III, §11, [8] and [29]) Let \( l = (l_1, \ldots, l_n) \in \mathbb{N}^n \), let \( \Omega \) be a domain in \( \mathbb{R}^n \), and let \( \{P_j(x, D)\}_{1}^{N} \) be a system of differential operators of the form (1.1) in which \( a_{j\alpha}(\cdot) \in L^\infty(\Omega) \) for \( |\alpha : l| \leq 1 \) and \( a_{j\alpha}(\cdot) \in C(\Omega) \) for \( |\alpha : l| = 1 \).

Then a necessary condition for the system (1.1) to be coercive in the anisotropic Sobolev space \( \overset{\circ}{W}^1_p(\Omega), p \in (1, \infty) \), is that it is \( l \)-quasielliptic; if the domain \( \Omega \) is bounded, then this condition is also sufficient.

Remark 3.5. \( N \)-quasielliptic operators \( P(x, D) \) defined in terms of the Newton polyhedron were introduced and studied in the book [3], Ch. I, §4 and Ch. V, §2. In particular, an \( N \)-quasielliptic operator is \( l \)-quasielliptic if and only if for every \( x \in \Omega \) the Newton polyhedron \( N(P(x)) \) is a simplex with vertices at the origin and at the points \( (0, \ldots, 0, l_j, \ldots, 0) \) (here \( l_j \) is the \( j \)th component of the vector). In [3], Ch. VI, §4, \( N \)-quasielliptic operators were characterized by means of the a priori estimate

\[
\sum_{\alpha \in N(P)} \|D^\alpha f\|_{L^2(\Omega)} \leq C_1\|P(x, D)f\|_{L^2(\Omega)} + C_2\|f\|_{L^2(\Omega)}, \quad f \in C_0^\infty(\Omega), \tag{3.3}
\]

which develops the coercivity criterion in \( \overset{\circ}{W}^1_p(\Omega) \) significantly.

A coercivity criterion in \( \overset{\circ}{W}^1_p(\Omega) \) was obtained in [12], §5, Theorem 4. This result (as well as Proposition 1.4) implies that in general an \( l \)-quasielliptic system is not coercive in \( \overset{\circ}{W}^1_p(\Omega) \) for \( p = 1, \infty \). However, it is weakly coercive in \( \overset{\circ}{W}^1_p(\Omega) \) (see [12], §5) and in \( \overset{\circ}{W}^1_p(\Omega) \) (see [14], §4, Theorem 3 and [15], where this was proved for operators with constant coefficients).

Theorem 3.6. (see [1] and [12]) Let \( \Omega \) be a domain in \( \mathbb{R}^n \), let \( l = (l_1, \ldots, l_n) \in \mathbb{N}^n \) and let \( \{P_j(x, D)\}_{1}^{N} \) be an \( l \)-quasielliptic system of operators of the form (1.1). Suppose that
Taking into account the fact that (3.4) holds with \( \sum \) for every \( \varepsilon > 0 \) there is a constant \( C_\varepsilon > 0 \) independent of \( p \in (1, \infty) \) (and in the case of operators with constant coefficients independent of \( p \in [1, \infty] \)) such that the following estimate holds:

\[
\sum_{|\alpha| < l} \| D^\alpha f \|_{L^p(\Omega)} \leq \varepsilon \sum_{j=1}^N \| P_j(x, D) f \|_{L^p(\Omega)} + C_\varepsilon \| f \|_{L^p(\Omega)}, \quad f \in C_0^\infty(\Omega). \quad (3.4)
\]

In particular, an \( l \)-quasielliptic system \( \{P_j(x, D)\}_1^N \) of the form (1.1) is weakly coercive in the space \( \tilde{W}_p^l(\Omega) \) for \( p \in (1, \infty) \) (for \( p \in [1, \infty] \) in the case of operators with constant coefficients).

In the next theorem we show that for every \( \varepsilon > 0 \) and any \( p \in (1, \infty] \) inequality (3.4) characterizes elliptic systems in the class of weakly coercive systems in \( \tilde{W}_p^l(\Omega) \) with constant-coefficient principal parts.

**Proposition 3.7.** Let \( l_1 = \cdots = l_n = l \), let \( \Omega \) be a domain in \( \mathbb{R}^n \), and \( \{P_j(x, D)\}_1^N \) a system of operators of the form (1.1) whose principal parts have constant coefficients, so that \( P_j^l(x, D) = P_j^l(D) \), let \( a_{j\alpha}(\cdot) \in L^\infty(\Omega) \) for \( |\alpha| : l | < 1, \; j \in \{1, \ldots, N\} \), and let \( p \in (1, \infty] \). Then the system of operators \( \{P_j(x, D)\}_1^N \) is elliptic if and only if the estimate (3.4) holds for each \( \varepsilon > 0 \) with some constant \( C_\varepsilon > 0 \).

If the operators \( P_j(x, D), \; j \in \{1, \ldots, N\} \), have constant coefficients, then this criterion also holds for \( p = 1 \).

**Proof.** The necessity follows from Theorem 3.6.

Sufficiency. Suppose the estimate (3.4) holds. Setting

\[
\tilde{P}_j(x, D) := P_j(x, D) - P_j^l(D),
\]

from the triangle inequality we obtain

\[
\varepsilon' \sum \| P_j^l(D) f \|_p + C_\varepsilon' \| f \|_p \geq \varepsilon' \sum \| P_j(x, D) f \|_p - \varepsilon' \sum \| \tilde{P}_j(x, D) f \|_p + C_\varepsilon' \| f \|_p,
\]

\( \varepsilon' := \frac{\varepsilon}{\varepsilon + 1} \in (0, 1). \) (3.5)

Taking into account the fact that (3.4) holds with \( \sum_{j=1}^N \| \tilde{P}_j(x, D) f \|_p \) on the left-hand side, from (3.5) we obtain

\[
\varepsilon' \sum \| P_j^l(D) f \|_p + C_\varepsilon' \| f \|_p \geq \varepsilon' \sum \| P_j(x, D) f \|_p - \varepsilon' \left[ \varepsilon' \sum \| P_j(x, D) f \|_p + C_\varepsilon' \| f \|_p \right] + C_\varepsilon' \| f \|_p = (1 - \varepsilon') \left[ \varepsilon' \sum \| P_j(x, D) f \|_p + C_\varepsilon' \| f \|_p \right] \geq (1 - \varepsilon') \sum_{|\alpha| < l} \| D^\alpha f \|_p. \quad (3.6)
\]

Dividing both sides of (3.6) by \( 1 - \varepsilon' > 0 \) and taking into account the relation \( \varepsilon'/(1 - \varepsilon') = \varepsilon \) we derive the estimate (3.4) with \( P_j^l(D) \) in place of \( P_j(x, D) \):

\[
\sum_{|\alpha| < l} \| D^\alpha f \|_p \leq \varepsilon \sum_{j=1}^N \| P_j^l(D) f \|_p + C_\varepsilon \| f \|_p, \quad f \in C_0^\infty(\Omega). \quad (3.7)
\]
Let $φ ∈ C_0^∞(Ω)$, $φ ≠ 0$. We set $f(x) := φ(x)e^{itξ(x)}$, $x ∈ Ω$, in (3.7). Since

$$(D^γ f)(x) = t|γ|ξ^γ f(x) + t|γ|e^{itξ(x)} \sum_{k=1}^n \frac{∂ξ^γ}{∂ξ_k} D_k φ(x) + o(t|γ|)$$

by Leibniz’s formula (2.4), estimate (3.7) implies the inequality

$$\sum_{|α|<l} ∥t|α|ξ^α f + o(t^{l-1})∥_p - C∥f∥_p$$

by assumption, we have

$$\sum_{|α|<l} ∥t|α|ξ^α f + o(t^{l-1})∥_p - C∥f∥_p$$

Choosing $ε > 0$ small enough, dividing both sides of (3.9) by $t^{l-1}$ and passing to the limit as $t → +∞$ we arrive at a contradiction. Hence the system $\{P_j(x, D)\}_1^N$ is elliptic. □

3.3. Systems of principal type. Let $Ω$ be a domain in $R^n$ and $p ∈ [1, ∞]$. We denote by $L^0_{p,Ω}(P_1, \ldots, P_N)$ the space of all differential operators $Q(x, D)$ subordinated to a system $\{P_j(x, D)\}_1^N$, that is, satisfying the estimate (1.4). Following [9],Ch, II, § 2.7, we recall the definition.

Definition 3.8. A system of differential operators $\{P_j(x, D)\}_1^N$ of the form (1.1), with $L^∞(Ω)$-coefficients is called a system of principal type in $L^p(Ω)$ if $\{P_j(x, D)\}_1^N$ has the same force as an arbitrary system $\{R_j(x, D)\}_1^N$ with $L^∞(Ω)$-coefficients and the same principal part, that is, the spaces $L^0_{p,Ω}(P_1, \ldots, P_N)$ and $L^0_{p,Ω}(R_1, \ldots, R_N)$ coincide.

The definition of systems of principal type readily implies the following simple result.

Proposition 3.9. A system $\{P_j(x, D)\}_1^N$ of differential operators of principal type in $L^p(Ω)$ is weakly coercive in $W^l_p(Ω)$, $p ∈ [1, ∞]$.

Proof. Let $D^α$ be a differential monomial, $|α : l| < 1$, and let $\tilde{P}_1 := P_1 + D^α$. Since

$$P_1 ∈ L^0_{p,Ω}(P_1, \ldots, P_N), \quad \tilde{P}_1 ∈ L^0_{p,Ω}(\tilde{P}_1, \ldots, P_N),$$

by assumption, we have

$$\tilde{P}_1 - P_1 = D^α ∈ L^0_{p,Ω}(P_1, \ldots, P_N).$$

By Theorem 3.6 the ‘force’ of an elliptic system in $L^p(Ω)$, $p ∈ [1, ∞]$, does not change under perturbations by operators of smaller order. It turns out that this property singles out the elliptic systems in $L^p(\mathbb{R}^n)$ among the totality of weakly coercive systems in $W^l_p(\mathbb{R}^n)$ that have principal parts with constant coefficients.
Proposition 3.10. Let \( \{P_j(x, D)\}_1^N \) be a system of order \( l \) with \( L^\infty(\Omega) \) coefficients such that the principal parts of the operators \( P_j(x, D) \) have constant coefficients: \( P_j(x, D) \equiv P_j^l(D) \).

Then the system \( \{P_j(x, D)\}_1^N \) is elliptic if and only if it is a system of principal type in \( L^p(\mathbb{R}^n) \) for \( p \in (1, \infty) \) (or for \( p \in [1, \infty] \) in the case of constant coefficients).

Proof. The necessity follows from Theorem 3.6.

Sufficiency. Suppose that the system \( \{P_j(x, D)\}_1^N \) is of principal type, but not elliptic. We may assume without loss of generality that \( P_j^l(\xi^0) = 0, j \in \{1, \ldots, N\} \), for \( \xi^0 = (1, 0, \ldots, 0) \). Since the system \( \{P_j(x, D)\}_1^N \) is weakly coercive in \( \hat{W}^l_\rho(\mathbb{R}^n) \) (Proposition 3.9), it follows that \( D_1 \in L^0_{\rho, \mathbb{R}^n}(P_1, \ldots, P_N) = L^0_{\rho, \mathbb{R}^n}(P_1', \ldots, P_N') \). By Proposition 2.7, this yields the inequality

\[
|\xi_1| \leq C_1 \sum_{j=1}^N |P_j^l(\xi)| + C_2, \quad \xi \in \mathbb{R}^n,
\]

which fails for \( \xi = \xi^0 t \) and large \( t > 0 \). Hence the system \( \{P_j(x, D)\}_1^N \) is elliptic. \( \square \)

The following assertion is a simple generalization of Hörmander’s result in [9], Ch. II, §2.7, Theorem 2.3 to the case \( N > 1 \).

Proposition 3.11. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). A system of differential polynomials \( \{P_j(D)\}_1^N \) of order \( l \) is a system of principal type in \( L^2(\Omega) \) if and only if

\[
(\nabla P_1^l, \ldots, \nabla P_N^l)(\xi) \neq 0, \quad \xi \in \mathbb{R}^n \setminus \{0\}.
\]

The proof of Proposition 3.11 is similar to Hörmander’s (see [9]). However, the analogue of Hörmander’s theorem for the system \( \{P_j(D)\}_1^N \) must be used in place of the theorem itself (see [30], [11] and also [1], Ch. III, §11).

Remark 3.12. (i) It seems that, as well as Propositions 3.7 and 3.10, Theorem 3.6 remains true for all \( p \in [1, \infty) \) in the case of operators with variable coefficients.

(ii) In the case of \( L^2(\Omega) \), where \( \Omega \) is bounded, Proposition 3.9 also follows from Proposition 3.11 (see [9]).

(iii) In [3] operators \( P(x, D) \) of \( N \)-principal type in \( L^2(\Omega) \) defined in terms of the Newton polyhedron \( N(P) \) were introduced and investigated. Estimates of type (3.3) were obtained for them in [3], Chs. V and VI, such that the sum on the left-hand side extends only to the interior points of the Newton polyhedron \( N(P) \). This is a significant improvement of Hörmander’s result [9] mentioned above.

3.4. On the force of the tensor product of elliptic operators in \( L^\infty \). It follows from Proposition 1.4 and Theorem 3.6 that if \( P(D) \) is an elliptic operator of order \( l \), then \( P^l(D) \) and the monomials \( \{D^\alpha\}_{|\alpha|_l \leq l} \) form a basis of the space \( L^0_{\infty, \mathbb{R}^n}(P) \). Here we describe the structure of the space \( L^0_{\infty, \mathbb{R}^n}(P) \) for the operator \( P(D) = P_1(D') \otimes P_2(D'') \), where \( P_1 \) and \( P_2 \) are elliptic operators acting with respect to different variables.

Proposition 3.13. Let \( P_1(\xi) \) and \( P_2(\eta) \) be elliptic polynomials of degrees \( l \) and \( m \), respectively. Let \( \xi = (\xi_1, \ldots, \xi_p) \in \mathbb{R}^p, \eta = (\eta_1, \ldots, \eta_{p_2}) \in \mathbb{R}^{p_2} \), \( p_1 + p_2 = n \), and assume that \( P_1(\xi) \neq 0, P_2(\eta) \neq 0 \) for all \( \xi \in \mathbb{R}^{p_1}, \eta \in \mathbb{R}^{p_2} \). Then

\[
L^0_{\infty, \mathbb{R}^n}(P_1P_2) = \text{span} \{Q_1Q_2 : Q_k \in L^0_{\infty, \mathbb{R}^{pk}}(P_k), k = 1, 2\}.
\]
Proof. (i) Let \(Q \in \text{span}\{Q_1Q_2 : Q_k \in L^0_{\infty, \mathbb{R}^p}(P_k), \ k = 1, 2\}\), that is,
\[
Q(\xi, \eta) = \sum_{j=1}^{s} Q_{j1}(\xi)Q_{j2}(\eta),
\]  
(3.10)
where \(Q_{jk} \in L^0_{\infty, \mathbb{R}^p}(P_k), \ j \in \{1, \ldots, s\}\), \(k = 1, 2\). By Proposition 2.5 the symbols \(Q_{j1}(\xi)\) and \(Q_{j2}(\eta)\) satisfy the relations
\[
Q_{jk} = M_{jk}P_k + N_{jk} = (M_{jk} + P_k^{-1}N_{jk}) P_k =: \tilde{M}_{jk}P_k,
\]  
(3.11)
where \(M_{jk}, N_{jk} \in \mathcal{M}_1(\mathbb{R}^p), \ j \in \{1, \ldots, s\}\), \(k = 1, 2\). Since \(P_1\) and \(P_2\) are elliptic operators with non-degenerate full symbols, it follows that \(1/P_1(\xi) \in \mathcal{M}_1(\mathbb{R}^p)\) and \(1/P_2(\eta) \in \mathcal{M}_1(\mathbb{R}^p)\) (see [14], §4, Theorem 3). Then \(\tilde{M}_{jk}(\cdot) \in \mathcal{M}_1(\mathbb{R}^p) \subset \mathcal{M}_1(\mathbb{R}^n), \ k = 1, 2\). Therefore, combining (3.10), (3.11) and Proposition 2.5 we arrive at \(Q \in L_{\infty, \mathbb{R}^n}(P_1P_2)\).

(ii) Conversely, assume that \(Q \in L_{\infty, \mathbb{R}^n}(P_1P_2)\). We represent the symbol \(Q(\xi, \eta)\) as a sum (3.10). We will show that (possibly, after some rearrangement of the terms in (3.10)) \(Q_{jk} \in L^0_{\infty, \mathbb{R}^p}(P_k), \ j \in \{1, \ldots, s\}, \ k = 1, 2\).

Let \(\max_{j \in \{1, \ldots, s\}} \deg Q_{j1} = l'\). First, we prove that \(l' \leq l\). Indeed, without loss of generality we may assume that \(\deg Q_{j1} = l'\) for \(j \in \{1, \ldots, s'\}\), \(s' \leq s\), and \(\deg Q_{j1} < l'\) for \(j > s'\). Collecting similar terms in the sum (3.10) if necessary we may treat the polynomials \(Q_{j1}(\xi), \ j \in \{1, \ldots, s'\}\), as linearly independent. Choose a vector \(\eta^0\) such that at least one of the polynomials \(Q_{j2}(\eta), \ j \in \{1, \ldots, s'\}\), does not vanish. If we suppose that \(l' > l\), then setting \(\eta := \eta^0\) in the inequality
\[
|Q(\xi, \eta)| \leq C_1|P_1(\xi)P_2(\eta)| + C_2, \quad \xi \in \mathbb{R}^p_1, \quad \eta \in \mathbb{R}^p_2,
\]  
(3.12)
which follows from the inclusion \(Q \in L_{\infty, \mathbb{R}^n}(P_1P_2)\) (see Proposition 2.7), we arrive at a contradiction. In fact, since the principal parts of the polynomials \(Q_{j1}(\xi), \ j \in \{1, \ldots, s'\}\), do not cancel, we have
\[
\deg Q(\xi, \eta^0) \geq \deg \sum_{j=1}^{s'} Q_{j1}(\xi)Q_{j2}(\eta^0) = l'
\]
on the left-hand side of (3.12). On the right-hand side of (3.12) we obtain
\[
\deg P_1(\xi)P_2(\eta^0) = l
\]
as \(P_2(\eta^0) \neq 0\). Hence \(l' \leq l\).

The proof of the relation \(\max_{j \in \{1, \ldots, s\}} \deg Q_{j2} =: m' \leq m\) is similar.

Further, in view of Proposition 1.4, the inclusion \(Q \in L_{\infty, \mathbb{R}^n}(P_1P_2)\) implies the relation
\[
Q^{l+m}(\xi, \eta) = \sum_{j=1}^{s} Q^{l}_{j1}(\xi)Q^{m}_{j2}(\eta) = cP^{l}_1(\xi)P^{m}_2(\eta), \quad \xi \in \mathbb{R}^p_1, \quad \eta \in \mathbb{R}^p_2.
\]  
(3.13)
As above, we may assume that the polynomials \(\{Q^{m}_{j2}(\eta)\}_1^s\) are linearly independent. Then we can find vectors \(\eta^1, \ldots, \eta^s \in \mathbb{R}^p_2\) such that \(\det (Q^{m}_{j2}(\eta^r)) \neq 0, \ j, \ r \in \{1, \ldots, s\}\) (see [31], Ch.V, §19, Lemma 3). Setting \(\eta = \eta^0\) in (3.13) we solve the system we have obtained with respect to the functions \(Q^{l}_{j1}(\xi)\). This implies the relations \(Q^{l}_{j1}(\xi) = \lambda_jP^{l}_1(\xi), \ j \in \{1, \ldots, s\}\).

Arguing similarly we arrive at the relations \(Q^{m}_{j2}(\eta) = \mu_jP^{m}_2(\eta), \ j \in \{1, \ldots, s\}\).
Finally, since the $P_j$ are elliptic, in view of Propositions 1.4 and 3.10, we have $Q_{jk} \in L^0_{\infty,\mathbb{R}^p}(P_k)$, that is, $Q \in \text{span}\{Q_1Q_2 : Q_k \in L^0_{\infty,\mathbb{R}^p}(P_k), \ k = 1,2\}$. □

The non-degeneracy of the full symbols of the operators $P_1$ and $P_2$ is essential for Proposition 3.13 to hold. The following result shows that even in the case of the product $P_1P_2$ of two homogeneous elliptic operators $P_1$ and $P_2$ acting on different groups of variables, the space $L^0_{\infty,\mathbb{R}^n}(P_1P_2)$ contains no differential monomials.

**Proposition 3.14.** Let $P_1(\xi)$ and $P_2(\eta)$ be homogeneous elliptic polynomials of degrees $l$ and $m$, respectively, and let $\xi = (\xi_1, \ldots, \xi_{p_1}) \in \mathbb{R}^{p_1}, \eta = (\eta_1, \ldots, \eta_{p_2}) \in \mathbb{R}^{p_2}, \ p_1, p_2 > 1, \ p_1 + p_2 = n$. Then the inclusion $D^\alpha \in L^0_{\infty,\mathbb{R}^n}(P_1P_2)$ does not hold for any $\alpha \neq 0$.

**Proof.** Let

$$D^\alpha = D^{\alpha_1}D^{\alpha_2} \in L^0_{\infty,\mathbb{R}^n}(P_1P_2), \quad D' := (D_1, \ldots, D_{p_1}), \quad D'' := (D_1, \ldots, D_{p_2}), \quad \alpha_k \in \mathbb{Z}_+^{n_k}, \quad k = 1,2.$$ 

By Proposition 2.5 the estimate

$$\|D^{\alpha_1}D^{\alpha_2}f\|_{L^\infty(\mathbb{R}^n)} \leq C_1\|P_1(D')P_2(D'')f\|_{L^\infty(\mathbb{R}^n)} + C_2\|f\|_{L^\infty(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (3.14)$$

is equivalent to the relation

$$\xi^{\alpha_1}\eta^{\alpha_2} = M(\xi,\eta)P_1(\xi)P_2(\eta) + N(\xi,\eta), \quad \xi \in \mathbb{R}^{p_1}, \ \eta \in \mathbb{R}^{p_2}, \quad (3.15)$$

where $M, N \in \mathcal{M}_1(\mathbb{R}^n)$. We set $\xi_1 = \xi_2 = \cdots = \xi_{p_1}$ and $\eta_1 = \eta_2 = \cdots = \eta_{p_2}$ in (3.15). Taking into account that the ‘restriction’ of a multiplier on $L^\infty(\mathbb{R}^n)$ to a subspace $E \subset \mathbb{R}^n$ is also a multiplier on $L^\infty(E)$ (see [23], Ch. IV, § 7.5), from Proposition 2.5 we see that the differential monomial with symbol $\xi^{\alpha_1}\eta^{\alpha_2}$ can be estimated in $L^\infty(\mathbb{R}^2)$ in terms of another differential monomial with symbol $c\xi^{m_1}\eta^{m_2}$, and we have $c = P_1(1, \ldots, 1)P_2(1, \ldots, 1) \neq 0$ because $P_1$ and $P_2$ are elliptic polynomials. By Boman’s theorem (see [22], § 5, Theorem 2) for $\alpha \neq 0$ this is possible only in the case of $|\alpha_1| = l, |\alpha_2| = m$. In this case the monomial $\xi^{\alpha_1}\eta^{\alpha_2}$ has degree $l + m$ and by Proposition 1.4 we obtain

$$\xi^{\alpha_1}\eta^{\alpha_2} = CP_1(\xi)P_2(\eta) = CP_1(\xi)P_2(\eta),$$

which yields $P_1(\xi) = C_1\xi^{\alpha_1}$ and $P_2(\eta) = C_2\eta^{\alpha_2}$. However, the polynomials $\xi^{\alpha_1}$ and $\eta^{\alpha_2}$ are not elliptic for $p_1 > 1$ and $p_2 > 1$. Thus, the estimate (3.14) does not hold for any $\alpha \neq 0$. □

4. **Weak coercivity in the isotropic space $\hat{W}^l_p(\mathbb{R}^n)$**

Here we will study properties of weakly coercive systems of order $l$ of the form

$$P_j(x,D) = \sum_{|\alpha| \leq l} a_{j\alpha}(x)D^\alpha, \quad j \in \{1,\ldots,N\}, \quad (4.1)$$

in the isotropic Sobolev space $\hat{W}^l_p(\mathbb{R}^n)$. In particular, for these systems we obtain an analogue of Theorem 1.2.
4.1. Properties of weakly coercive systems in $W^l_p(\mathbb{R}^n)$.

Proposition 4.1. Let $\{P_j(x,D)\}_{j=1}^N$ be a system of operators of the form (4.1) of order $l \geq 2$ with the coefficients $a_{j\alpha}(\cdot) \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq l - 1$ and such that the coefficients of the principal parts are constant. Assume also that the system $\{P_j(x,D)\}_{j=1}^N$ is weakly coercive in the isotropic space $W^l_p(\mathbb{R}^n)$, $p \in [1,\infty]$.

(i) If the operators $P_j(x,D)$, $j \in \{1,\ldots,N\}$, have continuous coefficients, then for any fixed $x_0 \in \mathbb{R}^n$ the zero set

$$\mathcal{N}(x_0,P) := \{\xi \in \mathbb{R}^n : P_1(x_0,\xi) = \cdots = P_N(x_0,\xi) = 0\}$$

of the system of polynomials $\{P_j(x_0,\xi)\}_{j=1}^N$ is compact.

(ii) For any system $\{Q_j(x,D)\}_{j=1}^N$, where the $Q_j(x,D)$ are operators of order $\leq l - 2$ with $L^\infty(\mathbb{R}^n)$-coefficients, the system $\{P_j(x,D) + Q_j(x,D)\}_{j=1}^N$ is also weakly coercive in $W^l_p(\mathbb{R}^n)$.

(iii) Let $\xi^0 \in \mathbb{R}^n \setminus \{0\}$ be a zero of the map $P^l = (P_1^l,\ldots,P_N^l) : \mathbb{R}^n \to \mathbb{R}^{2N}$, that is, $P_j^l(\xi^0) = 0$, $j \in \{1,\ldots,N\}$. If $n \geq 2N + 1$, then the Jacobi matrix of the map $P^l := (P_1^l,\ldots,P_N^l) : \mathbb{R}^n \to \mathbb{R}^{2N}$ at the point $\xi^0$ has rank less than $2N$.

(iv) In addition, let $a_{j\alpha}(\cdot) \in C^l(\mathbb{R}^n)$ for $|\alpha| = l - 1$. Let $N = 1$, $p = \infty$, $n \geq 2$. If $\xi^0 \in \mathbb{R}^n \setminus \{0\}$ is a zero of the polynomial $P^l(\xi)$, then $\nabla P^l(\xi^0) \neq 0$. In particular, if $n = 2$, then the polynomial $P^l(\xi)$ has simple zeros.

Proof. (i) Suppose that the set $\mathcal{N}(x_0,P)$ is not compact for some $x_0 \in \mathbb{R}^n$. Then for some sequence $\{\xi^{(m)}\}_{m=1}^\infty$, $\lim_{m \to +\infty} \xi^{(m)} = \infty$, we have $P_j(x_0,\xi^{(m)}) = 0$, $j \in \{1,\ldots,N\}$ (without loss of generality we can assume that $|\xi^{(m)}| > 1$ for $m \in \mathbb{N}$).

Since the principal parts of the operators $P_j(x,D)$ have constant coefficients, the symbols $P_j(x,\xi^{(m)}) = P_j(x,\xi^{(m)}) - P_j(x_0,\xi^{(m)})$ have degree $\leq l - 1$ (with respect to $\xi$). Then for each $\varepsilon > 0$ there exists a ball

$$B_\delta(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq \delta\}, \quad \delta > 0,$$

such that for $x \in B_\delta(x_0)$ we have

$$P_j(x,\xi^{(m)}) \leq \frac{\varepsilon}{N}|\xi^{(m)}|^{l-1}, \quad \left|\frac{\partial P_j}{\partial \xi_k}(x,\xi^{(m)})\right| \leq C|\xi^{(m)}|^{l-1},$$

where $m \in \mathbb{N}$, $j \in \{1,\ldots,N\}$, $k \in \{1,\ldots,n\}$.

Let $\psi \in C^\infty_c(B_\delta(x_0))$, $\psi \not\equiv 0$. Consider the functions $f_r(x) := \psi_r(x)e^{i(x,\xi^{(m)})}$, where $\psi_r(x) := \psi(x/r)$, $r > 0$. Then (4.2) implies the estimates

$$\|P_j(x,\xi^{(m)})e^{i(x,\xi^{(m)})}\psi_r(x)\|_p \leq \frac{\varepsilon}{N}|\xi^{(m)}|^{l-1} \cdot \|\psi_r\|_p,$$

$$\left\|\frac{\partial P_j}{\partial \xi_k}(x,\xi^{(m)})e^{i(x,\xi^{(m)})}D_k\psi_r(x)\right\|_p \leq C|\xi^{(m)}|^{l-1} \cdot \|D_k\psi_r\|_p.$$
Setting \( f = f_r \) in inequality (1.6) and using Leibniz’s formula (2.4) we obtain
\[
\sum_{|\alpha| \leq l} \left\| \xi^{(m)} \right\|^{\alpha} e^{i(x,\xi^{(m)})} \psi_r(x) + o \left( \|\xi^{(m)}\|^{\alpha} \right) \leq C_1 \sum_{j=1}^{N} \left\| P_j(x, \xi^{(m)}) e^{i(x,\xi^{(m)})} \psi_r(x) \right\|_p + C_2 \|\psi_r\|_p.
\]

Taking account of (4.3) and the obvious inequality \(|\xi^{(m)}|^{l-1} \leq C'|\sum_{k=1}^{n} |\xi_k^{(m)}|^{l-1}|\), from (4.4) we obtain
\[
|\xi^{(m)}|^{l-1} \cdot \|\psi_r\|_p \leq C'C_1|\xi^{(m)}|^{l-1} \left[\varepsilon \|\psi_r\|_p + C \sum_{k=1}^{n} \|D_k \psi_r\|_p \right] + o \left( |\xi^{(m)}|^{l-1} \right). (4.5)
\]

Dividing both sides of (4.5) by \(|\xi^{(m)}|^{l-1}\) and passing to the limit as \( m \to \infty \) we see that
\[
\|\psi_r\|_p \leq C'C_1 \left[\varepsilon \|\psi_r\|_p + C \sum_{k=1}^{n} \|D_k \psi_r\|_p \right]. (4.6)
\]

Now let \( p \in [1, \infty) \). Note that
\[
\|\psi_r\|_p = r^{n/p} \|\psi\|_p, \quad \|D_k \psi_r\|_p = r^{-1} \cdot r^{n/p} \|D_k \psi\|_p.
\]

Cancelling the factor \( r^{n/p} \) in (4.6) and letting \( r \to \infty \) we arrive at the estimate
\[
\|\psi\|_p \leq \varepsilon C'C_1 \|\psi\|_p. \quad \text{(For } p = \infty \text{ we must set here } r^{n/p} = r^{n/\infty} = 1. \text{)}
\]

Finally, taking \( \varepsilon > 0 \) sufficiently small we arrive at a contradiction.

(ii) The embedding theorems (see [1], Ch. III, §9) imply that for any \( \varepsilon > 0 \) there is \( C_\varepsilon > 0 \) such that
\[
\sum_{j=1}^{N} \|Q_j(x, D) f\|_p \leq \varepsilon \sum_{|\alpha| \leq l-1} \|D^\alpha f\|_p + C_\varepsilon \|f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n). (4.7)
\]

Combining (4.7) and (1.6) yields
\[
\sum_{j=1}^{N} \|P_j(x, D) f + Q_j(x, D) f\|_p + \|f\|_p \geq (C_1 - \varepsilon) \sum_{|\alpha| \leq l-1} \|D^\alpha f\|_p - C_\varepsilon \|f\|_p \quad \text{(4.8)}
\]

for all \( f \in C_0^\infty(\mathbb{R}^n) \). We choose \( \varepsilon < C_1 \) in (4.7). Then (4.8) implies the following estimate:
\[
\sum_{|\alpha| \leq l-1} \|D^\alpha f\|_p \leq (C_1 - \varepsilon)^{-1} \sum_{j=1}^{N} \|P_j(x, D) f + Q_j(x, D) f\|_p + (C_\varepsilon + 1) \|f\|_p.
\]

The weak coercivity of the system \( \{P_j(x, D) + Q_j(x, D)\}_{1}^{N} \) follows from this inequality.

(iii) First let \( n = 2N + 1 \). By (ii) we may assume that the operators \( P_j \) have the form
\[
P_j(x, D) = P_j^l(D) + P_j^{l-1}(x, D), \quad \text{where } P_j^{l-1}(x, D) := \sum_{|\alpha| = l-1} a_{j\alpha}(x) D^\alpha, \quad j \in \{1, \ldots, N\}.
\]

Assume the contrary: the rank of the Jacobian matrix of the map \( P^l = (P_1^l, \ldots, P_N^l) : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N} \) at the point \( \xi^0 \) equals \( 2N \):
\[
\text{rank } \frac{\partial(\text{Re } P_l^1, \text{Im } P_l^1, \ldots, \text{Re } P_l^N, \text{Im } P_l^N)}{\partial(\xi_1, \ldots, \xi_{2N+1})}(\xi^0) = 2N. (4.9)
\]
Consider a smooth parametrization of the sphere $\mathbb{S}^N$ in a neighborhood of the point $\xi^0$, that is, a diffeomorphism $\Phi := (\Phi_1, \ldots, \Phi_{2N+1}) : B^2_{\varepsilon} \to V$, where $B^2_{\varepsilon} := \{ \varphi \in \mathbb{R}^2 : |\varphi| < \varepsilon \}$, $V$ is an open neighborhood of the point $\xi^0 \in \mathbb{S}^N$, and $\Phi(0) = \xi^0$. Since $\Phi$ is the diffeomorphism, the Jacobi matrix of $\Phi$ at the origin has rank $2N$:

$$\operatorname{rank} \frac{\partial (\Phi_1, \ldots, \Phi_{2N+1})}{\partial (\varphi_1, \ldots, \varphi_{2N})}(0) = 2N. \quad (4.10)$$

Let $\tilde{T} := P^t \circ \Phi$ be the composition of the maps $P^t$ and $\Phi$, $\tilde{T} : B^2_{\varepsilon} \to \mathbb{R}^{2N}$. Since $\Phi(0) = \xi^0$ and since by assumption $P^t_j(\xi^0) = 0$, $j \in \{1, \ldots, N\}$, it follows that

$$\tilde{T}(0) = 0. \quad (4.11)$$

In addition, it follows from (4.9) and (4.10) that the map $\tilde{T}$ has Jacobian $J_{\tilde{T}}$ distinct from zero at the origin:

$$J_{\tilde{T}}(0) \neq 0. \quad (4.12)$$

By (4.11) and (4.12) the origin is an isolated zero of the map $\tilde{T} : B^2_{\varepsilon} \to \mathbb{R}^{2N}$. Let $U \subset \mathbb{R}^{2N}$ be an open neighborhood of the origin such that $\tilde{T}(\varphi) \neq 0$, $\varphi \in U \setminus \{0\}$, and $J_{\tilde{T}}(\varphi) \neq 0$, $\varphi \in U$. We can assume for simplicity that $U = B^2_{\varepsilon}$. We denote by $\Phi^r$ the map from $U$ to $\mathbb{S}^N$ defined by $\Phi^r(\varphi) := r\Phi(\varphi) := (r\Phi_1(\varphi), \ldots, r\Phi_{2N+1}(\varphi))$ for $r > 0$, and we denote by $\tilde{T}^r := P^t \circ \Phi^r : \tilde{T}^r : U \to \mathbb{R}^{2N}$, the composition of the maps $P^t$ and $\Phi^r$. Since the components of $P^t$ are homogeneous polynomials of degree $l$, we have $\tilde{T}^r = P^t(\Phi^r) = P^t(r\Phi) = r^lP^t(\Phi) = r^l\tilde{T}$. Similarly, $J_{\tilde{T}^r} = r^{2N(l-1)}J_{\tilde{T}}$. It follows that the maps $\tilde{T}^r$ satisfy the same relations as $\tilde{T}$, that is,

$$\tilde{T}^r(\varphi) \neq 0, \quad \varphi \in U \setminus \{0\}, \quad J_{\tilde{T}^r}(\varphi) \neq 0, \quad \varphi \in U. \quad (4.13)$$

By (4.13), for each $r > 0$ the vector field $\tilde{T}^r$ does not vanish on the boundary $\partial U = \mathbb{S}^N$ and it has only one singular point in the interior of $U$.

We may assume without loss of generality that $J_{\tilde{T}}(0) > 0$ in (4.12), and hence $J_{\tilde{T}^r}(\varphi) > 0$ in (4.13). Therefore, the singular point $0$ of the vector field $\tilde{T}^r$ has index $1$, $\operatorname{ind}(0,\tilde{T}^r) = 1$, where, as usual, $\operatorname{ind}(x_0,F)$ is the index of the singular point $x_0$ of the vector field $F$. Since the rotation number $\gamma(\tilde{T}^r, \partial U)$ of the vector field $\tilde{T}^r$ on $\partial U$ is equal to the sum of the indices of singular points of the vector field $\tilde{T}^r$ in the interior of $U$, it follows that

$$\gamma(\tilde{T}^r, \partial U) = \operatorname{ind}(0,\tilde{T}^r) = \operatorname{sign} J_{\tilde{T}}(0) = 1. \quad (4.14)$$

We fix $x_0 \in \mathbb{R}^n$. Since $P^t(\xi) \neq 0$, $\xi \in \partial V$, there exists $r_0 > 0$ such that for $r > r_0$ we have

$$P(x_0, r\xi) = P^t(r\xi) + P^{l-1}(x_0, r\xi) = r^l[P^t(\xi) + r^{l-1}P^{l-1}(x_0, \xi)] \neq 0, \quad \xi \in \partial V. \quad (4.15)$$

Let us introduce the maps $\tilde{P} := P \circ \Phi : U \to \mathbb{R}^{2N}$, where

$$P := (\operatorname{Re} P_1, \operatorname{Im} P_1, \ldots, \operatorname{Re} P_N, \operatorname{Im} P_N) : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N},$$

and $\tilde{P}^r := P \circ \Phi^r : U \to \mathbb{R}^{2N}$. Taking account of (4.15) and the fact that $\Phi$ is a diffeomorphism, we obtain $\tilde{P}^r(\varphi) \neq 0$ for $\varphi \in \partial U$. Hence, for each $r > r_0$, the maps $\tilde{T}^r : \partial U \to \mathbb{R}^{2N} \setminus \{0\}$ and $\tilde{P}^r : \partial U \to \mathbb{R}^{2N} \setminus \{0\}$ are homotopic in the space of continuous maps from $\partial U$ into $\mathbb{R}^{2N} \setminus \{0\}$, and the homotopy is given by

$$\Psi_r(t, \xi) = tP(x_0, \xi) + (1-t)P^r(\xi) = P^t(\xi) + tP^{l-1}(x_0, \xi) : \partial V \to \mathbb{R}^{2N+1} \setminus \{0\}, \quad t \in [0, 1].$$
But homotopic fields have equal rotations. Hence, taking account of (4.14) we obtain

\[ \gamma(\tilde{P}^r, \partial U) = \gamma(\tilde{T}^r, \partial U) = 1. \]  

(4.16)

Thus, the map \( P^r : \partial U \to \mathbb{R}^{2N} \setminus \{0\} \) is homotopically non-trivial, hence by Theorem 2.2 any continuous extension of it into the interior of \( U \) has zeros for each \( r > r_0 \). In particular, for every \( r > r_0 \) there exists \( \varphi^0(r) \in U \) such that \( \tilde{P}^r(\varphi^0(r)) = P(\Phi^r(\varphi^0(r))) = 0 \), where \( \Phi^r(\varphi^0(r)) \in \mathbb{S}_r^{2N} \). This contradicts assertion (i) that the zero set is compact. Thus, the statement is proved for \( n = 2N + 1 \).

Now assume that \( n > 2N + 1 \), while the rank of the Jacobi matrix of \( P^l \) at \( \xi^0 \) is \( 2N \). We choose \( 2N \) columns containing a non-trivial minor and set the remaining \( n - 2N \) columns equal to zero. By (i) the zero set \( \mathcal{N}(x_0, P) \) is compact since the system (4.1) is weakly coercive in \( \tilde{W}_p^l(\mathbb{R}^n) \). This property still holds if we restrict the polynomials to a subspace, hence the proof reduces to the previous case of \( n = 2N + 1 \).

(iv) The proof is based on Proposition 1.4. Assume the contrary, that is, suppose that the operator \( P(x, D) \) is weakly coercive in \( \tilde{W}_\infty^l(\mathbb{R}^n) \), while

\[ \nabla P^l(\xi^0) = \left( \frac{\partial P^l}{\partial \xi_1}, \ldots, \frac{\partial P^l}{\partial \xi_n} \right) (\xi^0) = 0. \]  

(4.17)

After a suitable orthogonal change of the variables \( \xi_1, \ldots, \xi_n \), we may assume that \( \xi^0 = (0, \ldots, 0, 1) \). Then equality (4.17) means that the coefficients of the monomials \( \xi^0_1 \) and \( \xi^{l-1}_j, j \in \{1, \ldots, n-1\} \) in \( P^l(\xi) \) are zero. Consider the smallest \( k \in \mathbb{N} \) such that at least one of the monomials \( \xi^0_\alpha, |\alpha| = l, \alpha_n = l - k \), occurs in \( P^l \) with a non-zero coefficient (such a \( k \) exists since \( P^l \neq 0 \) and \( k \geq 2 \) by (4.17)). Let

\[ l' := (l'_1, \ldots, l'_n) := \left( \frac{k(l-1)}{k-1}, \ldots, \frac{k(l-1)}{k-1}, l-1 \right). \]  

(4.18)

The vector \( l' \) defines a hyperplane \( \pi' : |\alpha : l'| = \sum_{j=1}^{n} j_{l_j}/l_j = 1, \) or

\[ \pi' : (k-1)(\xi_1 + \cdots + \xi_{n-1}) + k\xi_n = (k-1). \]  

(4.19)

Let \( P(x, \xi) = \sum_{|\alpha| \leq l} a_\alpha(x)\xi^\alpha \) be the full symbol of the operator \( P(x, D) \). Clearly, \( (0, \ldots, 0, l-1) \in \pi' \), and \( \alpha \in \pi' \) for \( |\alpha| = l \) and \( \alpha_n = l - k \). We claim that the exponents \( \alpha \) of the other monomials \( \xi^\alpha, |\alpha| \leq l \), lie 'below' the hyperplane \( \pi' \). If \( |\alpha| = l \) and \( \alpha_n < l - k \), then

\[ (k-1)(\alpha_1 + \cdots + \alpha_{n-1}) + k\alpha_n < (k-1). \]  

(4.20)

Finally, if either \( |\alpha| \leq l - 1 \) or \( |\alpha| = l - 1 \), but \( \alpha_n < l - 1 \), then inequality (4.20) also holds. Thus, the exponents of all monomials \( \xi^\alpha \) either lie 'below' the hyperplane \( \pi' \) or belong to it. Therefore, the \( l' \)-principal form \( P''(x, \xi) := \sum_{|\alpha| = l, \alpha_n = l-k} a_\alpha(x)\xi^\alpha \) of the full symbol \( P(x, \xi) \) has the form

\[ P''(x, \xi) = c_0(x)\xi^{l-1} + \sum_{|\alpha| = l, \alpha_n = l-k} a_\alpha(x)\xi^\alpha, \quad c_0(x) := a_{0,...,0,l-1}(x). \]  

(4.21)
Now we apply Proposition 1.4 to the operators $Q(D) := D_n^{l-1}$, $P(x, D)$ and the vector $l'$ of the form (4.18). Clearly, $Q'(D) = Q(D) = D_n^{l-1}$, and taking account of (4.21),

$$\xi_n^{l-1} = \lambda(x) \left[ c_0(x)\xi_n^{l-1} + \sum_{|\alpha| = l} a_\alpha \xi_\alpha \right].$$

From (4.22) we obtain

$$\lambda(x)c_0(x) \equiv 1, \quad \lambda(x)a_\alpha \equiv 0, \quad x \in \mathbb{R}^n, \quad |\alpha| = l, \quad \alpha_n = l - k.$$ 

Hence $a_\alpha = 0$, $|\alpha| = l$, $\alpha_n = l - k$. This contradicts the choice of $k$. Thus, $\nabla P^l(\xi^0) \neq 0$. \qed

**Remark 4.2.** (i) In the case of a weakly coercive system $\{P_j(D)\}_{1}^{N}$ with constant coefficients the compactness of the zero set of the map $P = (P_1, \ldots, P_N) : \mathbb{R}^n \to \mathbb{R}^{2N}$ follows from the algebraic inequality (2.2).

(ii) The condition $n \geq 2N + 1$ in assertion (iii) is sharp. For instance, the Jacobi matrix of the system $\{(\xi_1 + i)(\xi_2 + i), (\xi_3 + i)(\xi_4 + i)\}$ has rank one at the point $(1, 0, 0, 0)$ and rank two at $(1, 0, 1, 0)$.

(iii) In the case of constant coefficients assertion (iv) has significance only for $n = 2$ since, in view of Theorem 1.6, any weakly coercive operator in $\tilde{W}_p^l(\mathbb{R}^n)$ is elliptic for $n \geq 3$.

### 4.2. For which $l$ do weakly coercive systems exist?

In the next theorem we extend Theorem 1.2 to systems of operators with constant coefficients, that are weakly coercive in $\tilde{W}_p^l(\mathbb{R}^n)$, $p \in [1, \infty]$.

**Theorem 4.3.** Let $\{P_j(D)\}_{1}^{N}$ be a system of order $l$ that is weakly coercive in the isotropic Sobolev space $\tilde{W}_p^l(\mathbb{R}^n)$, $p \in [1, \infty]$, and suppose that $n \geq 2N + 1$. If the map

$$P^l := (P_1^l, \ldots, P_N^l) : \mathbb{R}^n \to \mathbb{R}^{2N}$$

has finitely many zeros on the sphere $\mathbb{S}^{2N-1}$, then $l$ is even.

**Proof.** (i) Let $n = 2N + 1$. Since the map $P^l$ has finitely many zeros on $\mathbb{S}^{2N}$, there exists a unit sphere $\mathbb{S}^{2N-1}$ such that the restriction $P^l|\mathbb{S}^{2N-1}$ has no zeros. Here the sign $\bigwedge$ denotes the restriction of a map to the corresponding set. Since all the polynomials $P_j^l(\xi)$ are homogeneous, we can assume without loss of generality that $\mathbb{S}^{2N-1} := \{x \in \mathbb{S}^{2N} : x_n = 0\}$.

As in (3.1), we denote by $T = (T_1, \ldots, T_{2N}) : \mathbb{S}^{2N-1} \to \mathbb{R}^{2N}$ the 'restriction' of the map $P^l$ to the sphere $\mathbb{S}^{2N-1}$, that is,

$$T_{2j-1}(\xi) := \text{Re} P_j^l(\xi)[\mathbb{S}^{2N-1}], \quad T_{2j}(\xi) := \text{Im} P_j^l(\xi)[\mathbb{S}^{2N-1}], \quad j \in \{1, \ldots, N\}. \quad (4.23)$$

Since $P^l|\mathbb{S}^{2N-1} \neq 0$ for all $r > 0$, the map

$$T_r := \frac{(T_1, \ldots, T_{2N})}{\|P^l\|} : \mathbb{S}^{2N-1} \to \mathbb{S}^{2N-1}_r$$

is continuous. If $l$ is odd, then $P^l$ is odd: $P^l(-\xi) = -P^l(\xi)$. Then by Theorem 2.3 the maps $T^r$ have odd degree $\text{deg} T^r = 2k + 1$, and hence are homotopically nontrivial (see [20]).
Consider the restriction of the map $P = (P_1, \ldots, P_N)$ to the sphere $\mathbb{S}^{2N-1}_r$. We denote

$$R_{2j-1}(\xi) := \text{Re} P_j(\xi)|_{\mathbb{S}^{2N-1}_r}, \quad R_{2j}(\xi) := \text{Im} P_j(\xi)|_{\mathbb{S}^{2N-1}_r}, \quad j \in \{1, \ldots, N\}.$$ 

For sufficiently large $r$ the maps

$$T^r : \mathbb{S}^{2N-1}_r \to \mathbb{R}^{2N} \setminus \{0\}, \quad R^r := (R_1, \ldots, R_{2N}) : \mathbb{S}^{2N-1}_r \to \mathbb{R}^{2N} \setminus \{0\}$$

are homotopic in the space of continuous maps from $\mathbb{S}^{2N-1}_r$ into $\mathbb{R}^{2N} \setminus \{0\}$. Indeed, for $\xi = r\eta$, $\eta \in \mathbb{S}^1$, and large $r > 0$ we have

$$R^r(\xi) = R^r(r\eta) = r^k T^1(\eta) + O(r^{l-1}) = r^k T^1(\eta)(1 + O(r^{-1})) \neq 0.$$  

(4.24)

Therefore, the maps $R^r$ and $T^r$ are homotopic in $\mathbb{R}^{2N} \setminus \{0\}$ since by (4.24) the homotopy $tR^r + (1-t)T^r$ does not vanish for large $r > 0$:

$$tR^r + (1-t)T^r : \mathbb{S}^{2N-1}_r \to \mathbb{R}^{2N} \setminus \{0\}.$$ 

Hence the map $R^r$ has the same degree as $T^r$, $\deg R^r = 2k + 1$, and is also homotopically non-trivial. Thus, by Theorem 2.2 any continuous extension of it into the interior of the closed ball $B_r^{2N}$ has a zero. In particular, the map

$$\widetilde{P}^r(\xi') = \widetilde{P}^r(\xi_1, \ldots, \xi_{n-1}) := P\left(\xi_1, \ldots, \xi_{n-1}, \sqrt{r^2 - |\xi'|^2}\right) : B_r^{2N} \to \mathbb{R}^{2N},$$

(4.25)

where $\xi' := (\xi_1, \ldots, \xi_{n-1})$, which is a continuous extension of $R^r : \mathbb{S}^{2N-1}_r \to \mathbb{R}^{2N} \setminus \{0\}$ into $B_r^{2N}$, also has zeros. Since the hemisphere $\mathbb{S}^+_{2N} := \{x \in \mathbb{S}^{2N} : x_n \geq 0\}$ is homeomorphic to the ball $B_r^{2N}$, the maps (4.25) define in a natural way maps $P|\mathbb{S}^{2N}_r = R^r|\mathbb{S}^{2N}_{+r}$ of the hemisphere $\mathbb{S}^{2N}_r$ into $\mathbb{R}^{2N}$. Thus, the maps $P|\mathbb{S}^{2N}_r$ have zeros for large $r$. This contradicts Proposition 4.1, (i).

(ii) Now suppose $n > 2N + 1$. Setting $\xi_k = 0$ for $k \in \{2N+2, \ldots, n\}$ we consider the ‘restricted’ system $\{\widetilde{P}_j(\xi_1, \ldots, \xi_{2N+1}, 0, \ldots, 0)\}_{j=1}^N$. The map $\widetilde{P}_j := (\widetilde{P}_1^j, \ldots, \widetilde{P}_N^j) : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N}$ also has a finite number of zeros on the sphere $\mathbb{S}^{2N}$. Moreover, by Proposition 4.1, (i) the zero set $N(P)$ of the symbols of the system $\{P_j(\xi)\}_{j=1}^N$ is compact since the system $\{P_j(D)\}_{j=1}^N$ is weakly coercive in $\mathcal{W}^l_p(\mathbb{R}^n)$. The zero set $N(\widetilde{P})$ of the restricted system $\{\widetilde{P}_j(\xi)\}_{j=1}^N$ remains compact in $\mathbb{R}^{2N+1}$. To complete the proof it remains to repeat the reasoning in item (i) for the system $\{\widetilde{P}_j(\xi)\}_{j=1}^N$. \hfill \Box

Remark 4.4. (i) We conjecture that the conclusion of Theorem 4.3 that $l$ is even holds without additional assumptions on the system $\{P_j(D)\}_{j=1}^N$.

(ii) The condition $n \geq 2N + 1$ is essential in Theorem 4.3. For instance, if $n = 2N$ then the system

$$P_1(D) := (D_1 + iD_2)^l, \quad P_2(D) := (D_3 + iD_4)^l, \quad \ldots, \quad P_N(D) := (D_{n-1} + iD_n)^l$$

is elliptic for any $l$.

(iii) It is clear from the proof of Theorem 4.3 that the condition that the map $P^l$ has only finitely many zeros on the sphere $\mathbb{S}^{n-1}$ can be relaxed, instead only assuming that there exists a sphere $\mathbb{S}^{n-2}$ free of zeros of $P^l$. However, examples do exist where the latter condition is not fulfilled. For instance, if $N = 2$ and $n = 5$, then the system

$$P_1(D) := (D_1 + i)(D_2 + i), \quad P_2(D) = D_3^2 + D_4^2 + D_5^2$$
is weakly coercive in $\tilde{W}_p^l(\mathbb{R}^5)$, $p \in [1, \infty]$, although the restriction of the map $P^l = (\xi_1 \xi_2, \xi_3^2 + \xi_4^2 + \xi_5^2)$ to any sphere $S^3$ has a zero.

4.3. **A characterization of weakly coercive systems of operators with constant coefficients in $\tilde{W}_\infty^l(\mathbb{R}^n)$**. Here we obtain an analogue of Theorem 1.6 for the case of a homogeneous system of operators with constant coefficients. To this end we will use the procedure, described in the following proposition, of 'restricting' an estimate to a subspace.

**Proposition 4.5.** Let $Q(x, D)$ and $\{P_j(x, D)\}_1^N$ be operators of the form (1.3) with $L^\infty(\Omega)$ coefficients and let $(D', 0) := (D_1, \ldots, D_m, 0, \ldots, 0)$. Then for any $m < n$ the estimate (1.4) with $p = \infty$ and $\Omega = \mathbb{R}^n$ implies the 'restricted' estimate

$$
\|Q(x, D', 0) \tilde{f}\|_{L^\infty(\mathbb{R}^n)} \leq C_1 \sum_{j=1}^N \|P_j(x, D', 0) \tilde{f}\|_{L^\infty(\mathbb{R}^n)} + C_2 \|\tilde{f}\|_{L^\infty(\mathbb{R}^n)}
$$

(4.26)

for all $\tilde{f} \in C_0^\infty(\mathbb{R}^m)$. Moreover, if the operators $Q$ and $P_j$ have constant coefficients, then estimate (1.4) remains valid if all the operators are restricted to an arbitrary subspace $E \subset \mathbb{R}^n$.

**Proof.** Let $\varphi \in C_0^\infty(\mathbb{R}^{n-m})$ be a 'cutoff' function equal to 1 in a neighborhood of the origin. Consider functions $f \in C_0^\infty(\mathbb{R}^n)$ of the following form:

$$
f(x_1, \ldots, x_n) := \tilde{f}(x_1, \ldots, x_m) \varphi(x_{m+1}, \ldots, x_n), \quad \text{where } \tilde{f} \in C_0^\infty(\mathbb{R}^m).
$$

(4.27)

Further, for any $r > 0$ and any function $f$ of the form (4.27) we denote by $f_r$ the function

$$
f_r(x) := f \left( x_1, \ldots, x_m, \frac{x_{m+1}}{r}, \ldots, \frac{x_n}{r} \right) = \tilde{f}(x_1, \ldots, x_m) \varphi \left( \frac{x_{m+1}}{r}, \ldots, \frac{x_n}{r} \right).
$$

(4.28)

We substitute (4.28) into (1.4). For any differential monomial $D^\alpha = D_1^{\alpha_1} \ldots D_m^{\alpha_m}$ we have

$$
D^\alpha f_r = r^{-(\alpha_{m+1} + \ldots + \alpha_n)} (D^\alpha f)_r,
$$

hence in view of the estimates

$$
\|a_{j\alpha}(x) D^\alpha f_r\|_{L^\infty(\mathbb{R}^n)} \leq \|a_{j\alpha}(x)\|_{L^\infty(\mathbb{R}^n)} \|D^\alpha f_r\|_{L^\infty(\mathbb{R}^n)} \leq C r^{-(\alpha_{m+1} + \ldots + \alpha_n)},
$$

passing to the limit as $r \to +\infty$ in the inequality obtained we arrive at (4.26).

If the operators $Q$ and $P_j$ have constant coefficients, then every $L^\infty(\mathbb{R}^n)$-norm in (4.26) is equal to the corresponding $L^\infty(\mathbb{R}^m)$-norm. This proves estimate (1.4) holds after 'restricting' all the operators to the subspace $E = \text{span}\{\xi_1, \ldots, \xi_m\}$. Since an orthogonal change of the variables $\xi_1, \ldots, \xi_n$ preserves the original estimate (1.4), the $m$-dimensional subspace $E$ can be arbitrary.

**Definition 4.6.** A subspace $E \subset \mathbb{R}^n$ is said to be **coordinate** if it has the form $E = \{x = (x_1, \ldots, x_n) : x_{i_1} = \cdots = x_{i_k} = 0\}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

We denote by $P(\xi)[E]$ the restriction of a polynomial $P(\xi)$ to a coordinate subspace $E$ and by $P(D)[E]$ the corresponding operator.

**Corollary 4.7.** If a system $\{P_j(D)\}_1^N$ is weakly coercive in the isotropic space $\tilde{W}_\infty^l(\mathbb{R}^n)$, then the system $\{P_j(D)[E]\}_1^N$ remains weakly coercive in $\tilde{W}_\infty^l(E)$ after restriction to an arbitrary coordinate subspace $E \subset \mathbb{R}^n$. 
Remark 4.8. We emphasize that the coefficients of the restricted operators $Q(x, D)$ and 
\{P_j(x, D)\}_1^N$ depend on all $n$ variables as before, while the differentiation is performed 
only with respect to the first $m$ variables. Note also that functions $f \in C_0^\infty(\mathbb{R}^m)$ are not 
compactly supported in $\mathbb{R}^n$.

The following result, announced in [15], presents an analogue of Theorem 1.6 in the 
case of a homogeneous system of operators.

Theorem 4.9. Suppose $l \geq 2$ and let $\{P_j(D)\}_1^N$ be a system of operators with constant 
coefficients of order $l$ satisfying the following conditions:

(i) $n \geq 2N + 1$;

(ii) the polynomials $\{P_j(l)\}_1^N$ restricted to an arbitrary two-dimensional subspace of 
$\mathbb{R}^n$ remain linearly independent.

Then the system $\{P_j(D)\}_1^N$ is weakly coercive in the isotropic Sobolev space $W^l(\mathbb{R}^n)$ 
if and only if it is elliptic.

Proof. The sufficiency is immediate from Theorem 3.6.

Necessity. Let $\{P_j(D)\}_1^N$ be a weakly coercive system in $W^l(\mathbb{R}^n)$ that is not elliptic, 
that is, $P_j(l) = 0$, $j \in \{1, \ldots, N\}$, for some $\xi^0 = (\xi_1^0, \ldots, \xi_n^0) \in \mathbb{R}^n \setminus \{0\}$. Changing the 
variables $\xi_1, \ldots, \xi_n$ if necessary, we can assume that $\xi^0 = (1, 0, \ldots, 0)$. This means that 
in each $P_j(l)$ the coefficients of $\xi_1^0$ are zero.

Let $P_j^{l-1}(D) := \sum_{|\alpha|=l-1} a_{j\alpha} D^\alpha$, $j \in \{1, \ldots, N\}$. We have $P_j^{l-1}(\xi^0) \neq 0$ for some 
$j \in \{1, \ldots, N\}$ since otherwise, after the substitution $\xi = \xi^0 t$, $t > 0$, in the algebraic inequality (2.2), 
which follows from the estimate (1.4) with $Q = D^{l-1}$ (see Proposition 2.7), we arrive at a contradiction as $t \to +\infty$.

After a linear transformation of the system $\{P_j(l)\}_1^N$ we can assume that 

$$P_1^{l-1}(\xi^0) = 1, \quad P_2^{l-1}(\xi^0) = 0, \quad \ldots, \quad P_N^{l-1}(\xi^0) = 0. \tag{4.29}$$

Since the monomial $\xi_1^0$ is missing from every of polynomials $P_j(l)$ and since these polynomials are homogeneous, it follows that 

$$\frac{\partial P_j(l)}{\partial \xi_1}(\xi^0) = 0, \quad j \in \{1, \ldots, N\}. \tag{4.30}$$

Further, because $n \geq 2N + 1$, by Proposition 4.1, (iii) the Jacobi matrix of the map 

$$P^l = (P_1^l, \ldots, P_N^l) : \mathbb{R}^n \to \mathbb{R}^{2N} \text{ at } \xi^0 \text{ has rank at most } 2N - 1.$$

This means that there exists a vector $\lambda := (0, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \setminus \{0\}$ such that 

$$\sum_{k=2}^n \lambda_k \frac{\partial P_j(l)}{\partial \xi_k}(\xi^0) = 0, \quad j \in \{1, \ldots, N\}.$$

If necessary making an orthogonal change of the variables $\xi_1, \ldots, \xi_n$ of the form 

$$\xi'_k = \sum_{r=1}^n c_{kr} \xi_r, \quad k, r \in \{1, \ldots, n\},$$

where $C := (c_{kr})_{n \times n}$ is an orthogonal matrix with the first two rows consisting of the coordinates of the vectors $e_1 = (1, 0, \ldots, 0)$ and $e_2 = \lambda / |\lambda| = (0, \lambda_2 / |\lambda|, \ldots, \lambda_n / |\lambda|)$, we
obtain
\[
\frac{\partial P^i_j}{\partial e_1}(\xi^0) = \frac{\partial P^i_j}{\partial \xi_1}(\xi^0) = 0, \quad \frac{\partial P^i_j}{\partial e_2}(\xi^0) = \sum_{k=2}^{n} \frac{\lambda_k}{|\lambda_k|} \frac{\partial P^i_j}{\partial \xi_k}(\xi^0) = 0.
\]

In addition to (4.30) we may assume that
\[
\frac{\partial P^i_j}{\partial \xi_2}(\xi^0) = 0, \quad j \in \{1, \ldots, N\}.
\]

Relations (4.30) and (4.31) mean that \( \{P_j(\xi)\}_{1}^{N} \) contains neither monomials \( \xi_1^i \) nor \( \xi_1^{i-1} \xi_2 \).

Consider the 'restriction' of the system \( \{P_j(D)\}_{1}^{N} \) to the subspace \( E = \text{span}\{\xi_1, \xi_2\} \).

By Corollary 4.7 it remains weakly coercive in \( W^l_\infty(\mathbb{R}^2) \). We keep the same notation for the 'restricted' objects. Taking account of relations (4.29)–(4.31) and applying Proposition 2.5 to the operators \( \{P_j(D)\}_{1}^{N} \) 'restricted' to \( E \) we obtain
\[
\xi_1^{i-1} = \sum_{j=1}^{N} M_j(\xi) \left[ a_j \xi_1^{i-2} \xi_2 + \cdots + \delta_j^1 \xi_1^{i-1} + \ldots \right] + M_{N+1}(\xi),
\]
where the \( M_j(\cdot), j \in \{1, \ldots, N+1\} \), are multipliers on \( L^\infty(\mathbb{R}^2) \) and \( \delta_j^1 \) is the Kronecker delta.

Dividing both sides of (4.32) by \( \xi_1^{i-1} \), we arrive at
\[
\lim_{\xi_1 \to +\infty} M_1(\xi_1, \xi_2^{0}) = 1, \quad \xi_2^{0} = \text{const} \in \mathbb{R}.
\]
We claim that (4.33) contradicts Lemma 2.4. Indeed, by assumption the leading forms \( P_j(\xi) \) remain linearly independent after 'restriction' to \( E \); therefore, by Proposition 2.6, \( \mu_j(0) = 0, j \in \{1, \ldots, N\} \), where the \( \mu_j \) are the finite measures in the integral representation (2.6) for the multipliers \( M_j = \hat{\mu}_j \) involved in (4.32). By Proposition 2.4 some convex combinations of 'shifts' of the function \( M_1(\xi) \) converge uniformly to the constant function \( \mu_1(0) = 0 \), that is,
\[
\sum_{k=1}^{m} c_k M_1(\xi - \zeta^{(k)}) \Rightarrow 0;
\]
\[
\sum_{k=1}^{m} c_k = 1, \quad c_k > 0, \quad \zeta^{(k)} \in \mathbb{R}^2, \quad k \in \{1, \ldots, m\}.
\]
It follows from (4.34) that for \( \varepsilon = 1/2 \) there exist \( R > 0 \) and \( a_1, \ldots, a_m \in \mathbb{R} \) such that
\[
\left| \sum_{k=1}^{m} c_k M_1(\xi_1, a_k) \right| \leq \frac{1}{2}, \quad \xi_1 > R.
\]
But inequality (4.35) contradicts relation (4.33). The proof is complete. \( \Box \)

Remark 4.10. (i) Condition (ii) of Theorem 4.9 is essential. For instance, condition (i) holds for the system \( P_1(\xi) := \xi_1^2 + \xi_2^3 + \xi_3^2, P_2(\xi) := (\xi_4 + i)(\xi_5 + i) \) \((n = 2N + 1 = 5)\), but condition (ii) fails: the restrictions of the polynomials \( \{P_j(\xi)\}_{1}^{2} \) to the two-dimensional subspace \( \text{span}\{\xi_1, \xi_2\} \) are linearly dependent. The system \( \{P_j(D)\}_{1}^{2} \) is weakly coercive in \( W^2_\infty(\mathbb{R}^5) \) but not elliptic.
At the same time, condition (ii) in Theorem 4.9 is not necessary. For instance, the system $P_1(\xi) := \xi_1^2 + \xi_2^2$, $P_2(\xi) := \xi_3^2 + \xi_4^2$ is weakly coercive in $W^2_\infty(\mathbb{R}^2)$, although the restrictions of the polynomials $\{P_j(\xi)\}^2_1$ to the subspace span$\{\xi_1, \xi_2\}$ are linearly dependent.

(ii) We do not have any examples of systems of operators which are weakly coercive in $W^2_\infty(\mathbb{R}^n)$, but not elliptic for $N > 1$, for which condition (i) fails but (ii) holds. However, it is easy to construct systems failing both conditions (i) and (ii) of Theorem 4.9. For instance, for $n = 2N$ both conditions (i) and (ii) of Theorem 4.9 fail for the system $P_j(\xi) := (\xi_{2j-1} + i)\xi_{2j} + i$, $j \in \{1, \ldots, N\}$. This system is also weakly coercive in $W^2_\infty(\mathbb{R}^n)$, but not elliptic.

4.4. A generalization of the de Leeuw-Mirkil theorem to operators with variable coefficients. By Theorem 1.6, which is due to de Leeuw and Mirkil, if an operator $P(D)$ is weakly coercive in $W^l_\infty(\mathbb{R}^n)$ for $n \geq 3$, then it is elliptic. The next theorem extends this result to operators with variable coefficients.

**Theorem 4.11.** Suppose that $l \geq 2$, $n \geq 3$, and let $P(x, D)$ be a differential operator of order $l \geq 2$ in which $a_\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq l - 1$, $a_\alpha(\cdot) \in C^1(\mathbb{R}^n)$ for $|\alpha| = l - 1$, and $a_\alpha = \text{const}$ for $|\alpha| = l$, that is, $P^l(x, D) = P^l(D)$.

Then the operator $P(x, D)$ is weakly coercive in the isotropic Sobolev space $W^l_\infty(\mathbb{R}^n)$ if and only if it is elliptic.

**Proof.** The necessity follows from Theorem 3.6.

**Sufficiency.** Suppose that the operator $P(x, D)$ is weakly coercive in $W^l_\infty(\mathbb{R}^n)$, that is,

$$
\sum_{|\alpha| < l} \|D^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leq C_1\|P(x, D)f\|_{L^\infty(\mathbb{R}^n)} + C_2\|f\|_{L^\infty(\mathbb{R}^n)}, \quad f \in C^\infty_0(\mathbb{R}^n). \tag{4.36}
$$

If $P$ is not elliptic, then $P^l(\xi^0) = 0$ for $\xi^0 = (\xi^0_1, \ldots, \xi^0_n) \in \mathbb{R}^n \setminus \{0\}$. Changing the variables $\xi_1, \ldots, \xi_n$ if necessary, we can assume that $\xi^0 = (1, 0, \ldots, 0)$.

By Euler’s identity $\sum_{k=1}^n \xi_k(\partial P^l/\partial \xi_k) = nP^l$ for the homogeneous polynomial $P^l(\xi)$, the condition $P^l(\xi^0) = 0$ implies the relation $(\partial P^l/\partial \xi_k)(\xi^0) = 0$. However, since $n \geq 3$, it follows from Proposition 4.1, (iii) that the Jacobi matrix of the map $P^l = (\text{Re}P^l, \text{Im}P^l) : \mathbb{R}^n \to \mathbb{R}^2$ at the point $\xi^0$ has rank at most 1. Making a suitable linear change of the coordinates $\xi_2, \ldots, \xi_n$ if necessary (see the proof of Theorem 4.9) we can assume that the second column of the Jacobi matrix is zero, that is, $(\partial P^l/\partial \xi_2)(\xi^0) = 0$. Thus, the symbol $P(x, \xi)$ of the operator $P(x, D)$ does not contain the monomials $\xi^0_1$ or $\xi^{l-1}_1\xi_2$.

Further, combining Proposition 4.5 with estimate (4.36) yields the ‘restricted’ estimate

$$
\sum_{\alpha_1 + \alpha_2 < l} \|D^\alpha_1 D^\alpha_2 f\|_{L^\infty(\mathbb{R}^n)} \leq C_1\|P(x, D_1, D_2, 0, \ldots, 0)f\|_{L^\infty(\mathbb{R}^n)} + C_2\|f\|_{L^\infty(\mathbb{R}^n)} \tag{4.37}
$$

for all $f \in C^\infty_0(\mathbb{R}^2)$. Note that $P^l(\xi_1, \xi_2, 0, \ldots, 0) \neq 0$, for otherwise estimate (4.37) (see Proposition 1.4 and Remark 2.8, (i)) implies the relations

$$
\xi^\alpha = \lambda_\alpha(x)P^{l-1}(x, \xi_1, \xi_2, 0, \ldots, 0)
$$

for all $|\alpha| = l - 1$, which is obviously impossible. Hence there exists $k \geq 2$ such that the coefficient of $\xi^{l-k}_1\xi_2^k$ in the polynomial $P^l(\xi_1, \xi_2, 0, \ldots, 0)$ differs from zero.
We take the minimum such \( k \) and draw a line \( \omega \) through the points \((l - 1, 0)\) and \((l - k, k)\). It is not vertical since \( k \geq 2 \). We denote by \( l' := (l'_1, l'_2) \) the vector with components \( l'_1 := l - 1 \) and \( l'_2 := k(l - 1)/(k - 1) \) equal to the lengths of the intercepts of \( \omega \) with the coordinate axes. The ‘restricted’ operator has the following form:

\[
P(x, D_1, D_2, 0, \ldots, 0) = \sum_{\alpha_1/l'_1 + \alpha_2/l'_2 \leq 1} a_{(\alpha_1, \alpha_2)}(x) D_1^{\alpha_1} D_2^{\alpha_2},
\]

that is, \( \omega \) is an \( l' \)-principal line for the operator \( P \). Indeed, there are no terms \( \xi_1 \) or \( \xi_1^{l-1} \xi_2 \) in the symbol \( P(x, \xi_1, \xi_2, 0, \ldots, 0) \) and there are no points with integer coordinates in the strip \( l - 1 \leq x + y \leq l \), \( x, y \geq 0 \), except on the lines \( x + y = l - 1 \) and \( x + y = l \), and the line interval with end-points \((l - 1, 0)\) and \((l - k, k)\) lies entirely in this strip (see Fig. 1).

![Fig. 1](image)

It follows that the \( l' \)-principal part of the operator \( P(x, D_1, D_2, 0, \ldots, 0) \) has the form

\[
P''(x, D_1, D_2, 0, \ldots, 0) = c(x) D_1^{l-1} + b D_1^{l-k} D_2^{k},
\]

where \( c(x) := a_{l-1,0,0,...,0}(x) \) and \( b := a_{l-k,k,0,...,0} \), and where we have \( b \neq 0 \).

Since the estimate (4.37) holds with the operator \( D_1^{l-1} \) on the left-hand side, it follows by Proposition 1.4 and Remark 2.8, (i) that

\[
\xi_1^{l-1} = \lambda(x) \left[ c(x) \xi_1^{l-1} + b \xi_1^{l-k} \xi_2^k \right], \quad x \in \mathbb{R}^n, \quad \xi_1, \xi_2 \in \mathbb{R}.
\]

Relation (4.38) implies that \( \lambda(x) c(x) \equiv 1 \) and \( \lambda(x) b \equiv 0 \), which contradicts the condition \( b \neq 0 \). This contradiction proves that the operator \( P(x, D) \) is elliptic.

\[\square\]

**Remark 4.12.** In the space \( L^p(\Omega) \), \( p \in [1, \infty] \), each differential expression \( P(x, D) \) of the form (1.1) is naturally associated with a minimal and a maximal differential operators \( P_{\min} \) and \( P_{\max} \). Recall (see \[29\], Ch. 2, § 2) that by definition \( P_{\min} \) is the closure in \( L^p(\Omega) \) of the differential operator \( P' = P[C_0^\infty(\Omega)] \) defined originally on the domain \( \text{dom}(P') = C_0^\infty(\Omega) \).

Clearly, \( \tilde{W}_p^l(\Omega) \subset \text{dom}(P_{\min}) \).

In addition, the coercivity criterion in \( \tilde{W}_p^l(\Omega) \) for \( p \in (1, \infty) \) (Theorem 3.4) implies that for an operator \( P(x, D) \) with continuous coefficients in a bounded domain \( \overline{\Omega} \) the relation \( \text{dom}(P_{\min}) = \tilde{W}_\infty^l(\Omega) \) is equivalent to the ellipticity of \( P(x, D) \).

Thus, Theorem 4.11 is equivalent to the following result.

**Corollary 4.13.** Under the assumptions of Theorem 4.11 the inclusion

\[
\text{dom}(P_{\min}) \subset \tilde{W}_\infty^{l-1}(\mathbb{R}^n)
\]

is equivalent to the ellipticity of the operator \( P(x, D) \).
Proof. Since the operator $P_{\min}$ is closed, the inclusion $\text{dom}(P_{\min}) \subset \overset{\circ}{W}^{l-1}_{\infty}(\mathbb{R}^n)$ is equivalent to estimate (4.36), that is, to the weak coercivity of $P_{\min}$. It remains to apply Theorem 4.11. □

Remark 4.14. (i) We emphasize that it is because the selection of a principal part of a differential operator is not unique that we can use the anisotropic version of Proposition 1.4 for the proof of 'isotropic' Theorem 4.11.

(ii) In the case when the operator $P$ has constant coefficients the conditions

$$P^l(1,0,\ldots,0) = \frac{\partial P^l}{\partial \xi_1}(1,0,\ldots,0) = \frac{\partial P^l}{\partial \xi_2}(1,0,\ldots,0) = 0$$

after 'restricting' $P$ to the two-dimensional subspace span\{\xi_1, \xi_2\} mean that $\nabla \tilde{P}^l(1,0) = 0$ ($\tilde{P}$ is the corresponding 'restriction' of the operator $P$). The last condition immediately contradicts Proposition 4.1, (iv), and the final part of the proof of Theorem 4.11 can be omitted.

5. A CHARACTERIZATION OF WEAKLY COERCIVE OPERATORS OF TWO VARIABLES IN $\overset{\circ}{W}^l_p(\mathbb{R}^2)$, $p \in [1, \infty]$

In [16], p. 123 the authors give Malgrange’s example of an operator that is weakly coercive in $\overset{\circ}{W}^2_{\infty}(\mathbb{R}^2)$, but not elliptic: $P(D) = (D_1 + i)(D_2 + i)$.

The following assertion, in particular, gives a complete characterization of weakly coercive operators in the isotropic Sobolev space $\overset{\circ}{W}^l_{\infty}(\mathbb{R}^2)$.

Theorem 5.1. (i) An arbitrary weakly coercive operator $P(D)$ of order $l \geq 2$ in the isotropic space $\overset{\circ}{W}^l_{\infty}(\mathbb{R}^2)$ has the form

$$P(D) = R(D) \prod_{k=1}^{m} (\lambda_k D_1 + \mu_k D_2 + \alpha_k) + Q(D),$$

(5.1)

where $R(D)$ is an elliptic operator of order $l - m$, $Q(D)$ is an operator of order $\leq l - 2$, $\alpha_k \in \mathbb{C} \setminus \mathbb{R}$, $(\lambda_k, \mu_k) \in \mathbb{R}^2$ are pairwise non-collinear vectors, where $k \in \{1, \ldots, m\}$, and $m \leq l$.

(ii) Conversely, any operator of the form (5.1) is weakly coercive in $\overset{\circ}{W}^l_p(\mathbb{R}^2)$, $p \in [1, \infty]$.

Proof. (i) We assume first that $P(\xi)$ is an arbitrary polynomial of order $l$ and $P^{l-1}(\xi) := \sum_{|\alpha|=l-1} a_\alpha \xi^\alpha$ is the $(l-1)$-homogeneous part of the polynomial $P(\xi)$. The principal form $P^l(\xi)$ can be represented as follows:

$$P^l(\xi_1, \xi_2) = \prod_{j=1}^{s} (a_j \xi_1 + b_j \xi_2)^{k_j},$$

where $k_j \geq 1$, $\sum_{j=1}^{s} k_j = l$, and $(a_j, b_j) \in \mathbb{C}^2$ are pairwise non-collinear, where $j \in \{1, \ldots, s\}$. We claim that $P(\xi)$ can be expressed as

$$P(\xi_1, \xi_2) = \prod_{j=1}^{s} [(a_j \xi_1 + b_j \xi_2)^{k_j} + Q_j(\xi_1, \xi_2)] + Q(\xi_1, \xi_2),$$

(5.2)
where \( \deg Q_j < k_j, \ j \in \{1, \ldots, s\} \), \( \deg Q \leq l - 2 \). In fact, the rational fraction \( \frac{P^{l-1}(\xi)}{P^l(\xi)} \), which is the ratio of two homogeneous polynomials of two variables, can be decomposed to a sum of partial fractions:

\[
\frac{P^{l-1}(\xi_1, \xi_2)}{P^l(\xi_1, \xi_2)} = \sum_{j=1}^{s} \frac{Q_j(\xi_1, \xi_2)}{(a_j \xi_1 + b_j \xi_2)^{k_j}}, \quad \text{where} \quad \deg Q_j < k_j.
\]

Clearly, this implies that the homogeneous forms of orders \( l \) and \( l - 1 \) of the polynomial

\[
\tilde{P}(\xi_1, \xi_2) := P^l(\xi_1, \xi_2) \prod_{j=1}^{s} \left[ 1 + \frac{Q_j(\xi_1, \xi_2)}{(a_j \xi_1 + b_j \xi_2)^{k_j}} \right] = \prod_{j=1}^{s} \left[ (a_j \xi_1 + b_j \xi_2)^{k_j} + Q_j(\xi_1, \xi_2) \right]
\]

coincide with \( P^l(\xi) \) and \( P^{l-1}(\xi) \), respectively. Therefore, the difference

\[
Q(\xi) := P(\xi) - \tilde{P}(\xi)
\]

is a polynomial of degree \( \leq l - 2 \), which proves that the representation (5.2) holds.

Now let \( P(D) \) be a weakly coercive operator in \( \dot{W}^l(\mathbb{R}^2) \). By Proposition 4.1, (iv) the polynomial \( P^l(\xi) \) has no multiple real zeros, and hence

\[
P^l(\xi_1, \xi_2) = \prod_{j=1}^{s-m} (a_j \xi_1 + b_j \xi_2)^{k_j} \prod_{j=1}^{m} (\lambda_j \xi_1 + \mu_j \xi_2), \quad k_j \geq 1, \quad \sum_{j=1}^{s-m} k_j = l - m.
\]

Here the vectors \((\lambda_j, \mu_j) \in \mathbb{R}^2, \ j \in \{1, \ldots, m\}\), and \((a_j, b_j) \in \mathbb{C}^2, \ j \in \{1, \ldots, s - m\}\), are pairwise non-collinear. Now we write the decomposition (5.2) for the polynomial \( P(\xi) \), with \( a_{s-m+j} = \lambda_j, b_{s-m+j} = \mu_j \) and \( Q_{s-m+j}(\xi) = \alpha_j, j \in \{1, \ldots, m\} \), since \( k_{s-m+1} = \cdots = k_s = 1 \). To complete the proof it suffices to note that \( \alpha_j \notin \mathbb{R}, \ j \in \{1, \ldots, m\} \), by Proposition 4.1, (i) and to set

\[
R(D) := \prod_{j=1}^{s-m} \left[ (a_j D_1 + b_j D_2)^{k_j} + Q_j(D) \right].
\]

The operator \( R(D) \) has order \( \sum_{j=1}^{s-m} k_j = l - m \) and is elliptic because its principal part

\[
R^{l-m}(\xi) = \prod_{j=1}^{s-m} (a_j \xi_1 + b_j \xi_2)^{k_j} \neq 0 \quad \text{for} \quad \xi \in \mathbb{R}^2 \setminus \{0\}.
\]

Taking (5.2) into account we see that \( P(D) \) has the form (5.1).

(ii) Now we prove that the operator (5.1) is weakly coercive in \( \dot{W}^l_p(\mathbb{R}^2) \) for any \( p \in [1, \infty] \). By Proposition 4.1, (ii) we may assume that \( Q(D) = 0 \). First, using induction on \( m, \ m \in \{0, \ldots, l\} \), we prove that

\[
\Phi^{(m)}(\xi) = \Phi^{(m)}_{\gamma_1, \gamma_2}(\xi) := \chi(\xi) R(\xi) \prod_{k=1}^{m} \frac{\xi_1^{\gamma_1 k} \xi_2^{\gamma_2 k}}{(\lambda_k \xi_1 + \mu_k \xi_2 + \alpha_k)} \in \mathcal{M}_1.
\]

Here \( |\gamma| = \gamma_1 + \gamma_2 < l' + m \), \( R(\xi) \) is an elliptic polynomial of degree \( l' \), and \( \chi(\xi) \) is the corresponding 'cutoff' function.

After an orthogonal change of the variables \( \xi_1, \ldots, \xi_n \) we may assume that \( \mu_m = 0 \). Since this change preserves the non-collinearity of the vectors \((\lambda_k, \mu_k), \ k \in \{1, \ldots, m\}\), we conclude that \( \mu_k \neq 0 \) for all \( k \in \{1, \ldots, m - 1\} \).

For \( m = 0 \) the assertion in question is obvious:

\[
\Phi^{(0)}(\xi) = \chi(\xi) \xi^\gamma (R(\xi))^{-1} \in \mathcal{M}_1 \quad \text{for} \quad |\gamma| < l',
\]

because the polynomial \( R(\xi) \) is elliptic (see [14], §4).
Further, we have $(\lambda_k\xi_1 + \mu_k\xi_2 + \alpha_k)^{-1} \in \mathcal{M}_1$, $k \in \{1, \ldots, m\}$. Indeed, let $\chi_+(\cdot)$ be the Heaviside function and let $\delta(\cdot)$ be the Dirac measure on the line. The Fourier-Stieltjes transform $\hat{\sigma}$ of a finite measure $\sigma$ with density $-i\chi_+(t_1)e^{-t_1} \otimes \delta(t_2)$ is

$$ \hat{\sigma} = -i \int_{\mathbb{R}^2} \chi_+(t_1)\delta(t_2)e^{-t_1}e^{it_1\xi_1}e^{it_2\xi_2} dt_1 dt_2 = -i \int_0^{+\infty} e^{it_1(\xi_1+i)} dt_1 = (\xi_1 + i)^{-1}. \quad (5.4) $$

Making the change of the variables

$$ \eta_1 := \lambda_k\xi_1 + \mu_k\xi_2 + \alpha_k, \quad \eta_2 := -\mu_k\xi_1 + \lambda_k\xi_2 $$
in (5.4), we obtain $(\lambda_k\xi_1 + \mu_k\xi_2 + \alpha_k)^{-1} = \hat{\sigma}_1$, where the finite measure $\sigma_1$ has density

$$ \frac{\text{Im} \alpha_k}{\lambda_k^2 + \mu_k^2} \chi_+(\text{Im} \alpha_k(t_1\lambda_k + t_2\mu_k)) \exp \left[ (i \text{Re} \alpha_k - \text{Im} \alpha_k) \frac{t_1\lambda_k + t_2\mu_k}{\lambda_k^2 + \mu_k^2} \right] \otimes \delta(-t_1\mu_k + t_2\lambda_k) $$
at the point $(t_1, t_2) \in \mathbb{R}^2$. This yields the inclusion $(\lambda_k\xi_1 + \mu_k\xi_2 + \alpha_k)^{-1} \in \mathcal{M}_1$ (see [23], Ch. IV, § 3).

Suppose that (5.3) holds for all the functions $\Phi_{\gamma}(\xi)$ with $t \leq m - 1$. If $\gamma_1 > 0$, then $\Phi_{\gamma_1}(\xi) \in \mathcal{M}_1$ as it is the product of $\Phi_{\gamma_1-1, \gamma_2}$ and $\xi_1(\lambda_m\xi_1 + \alpha_m)^{-1}$. If $\gamma_1 = 0$ and $\gamma_2 < l' + m - 1$, then $\Phi_{\gamma}(\xi) \in \mathcal{M}_1$ as it is the product of $\Phi_{0, l', l'}$ and $\prod_{k=1}^m (\lambda_k\xi_1 + \mu_k\xi_2 + \alpha_k)^{-1}$.

Now let $\gamma_1 = 0, \gamma_2 = l' + m - 1$. Let $R(\xi) = a_0\xi_2 + \ldots, a_0 \neq 0$, where the dots stand for a polynomial of degree less than $l'$ with respect to $\xi_2$. Consider the difference between the function $\Phi_{0, l', l'-1}(\xi)$ and the multiplier $\chi(\xi)[a_0\mu_1 \ldots \mu_{m-1}(\lambda_m\xi_1 + \alpha_m)]^{-1}$:

$$ \chi(\xi) \frac{a_0\mu_1 \ldots \mu_{m-1}\xi_2'}{a_0\mu_1 \ldots \mu_{m-1}R(\xi)} - \frac{a_0\xi_2 + \ldots}{\prod_{k=1}^m (\lambda_k\xi_1 + \mu_k\xi_2 + \alpha_k)} \cdot (5.5) $$

The polynomial in the numerator of (5.5) has degree less than $l' + m - 1$ with respect to $\xi_2$. Therefore, this fraction is a linear combination of functions $\Phi_{\gamma}(\xi)$ for which $\gamma_2 < l' + m - 1$, so that it is a sum of multipliers by the above.

Now the weak coercivity of the operator $P(D)$ in $W^l_p(\mathbb{R}^2)$ follows from the identities

$$ \xi^\gamma \hat{f}(\xi) = \Phi_{\gamma}(\xi) P(\xi) \hat{f}(\xi) + \xi^\gamma(1 - \chi(\xi)) \hat{f}(\xi), \quad f \in C^\infty_0(\mathbb{R}^2), \quad |\gamma| < l, $$
to which we apply the inverse Fourier transform while taking (5.3) into account. \hfill \Box

**Corollary 5.2.** The space $L^0_{\infty, \mathbb{R}^2}(P)$, where $P(D)$ is the operator in (5.1), consists of operators of the following form:

$$ T(D) := cr^{l-m}(D) \prod_{k=1}^m (\lambda_kD_1 + \mu_kD_2) + Q(D), \quad c \in \mathbb{C}, \quad (5.6) $$

where $Q(D)$ is an arbitrary operator of order $\leq l - 1$ and $R^{l-m}(D)$ is the principal part of the operator $R(D)$ (of order $l - m$).

**Proof.** By Proposition 1.4, if $T \in L^0_{\infty, \mathbb{R}^2}(P)$, then

$$ T^l(D) = cP^l(D) = cR^{l-m}(D) \prod_{k=1}^m (\lambda_kD_1 + \mu_kD_2); $$
hence the operator $T(D)$ has the form (5.6).
Conversely, let \( T(D) \) be the operator of the form (5.6). By Theorem 5.1, (ii), \( T(D) \) is weakly coercive in \( \hat{W}^l_{\infty}(\mathbb{R}^2) \), and for \( c = 0 \) we have \( Q = T \in L^0_{\infty, \mathbb{R}^2}(P) \). If \( c \neq 0 \), then the corresponding ‘cutoff’ function satisfies

\[
\chi(\xi) \frac{T(\xi)}{P(\xi)} = \chi(\xi) \frac{cP(\xi) + (T(\xi) - cP(\xi))}{P(\xi)} = \chi(\xi) \left[ c + \frac{T(\xi) - cP(\xi)}{P(\xi)} \right] \in M_1,
\]

because \( T^l = cP^l \). Hence \( \deg(T - cP) \leq l - 1 \), so \( T \in L^0_{\infty, \mathbb{R}^2}(P) \). \( \square \)

**Corollary 5.3.** The product of an elliptic operator of order \( l \) of two variables and a weakly coercive operator of order \( m \) in \( \hat{W}^m_{\infty}(\mathbb{R}^2) \) is weakly coercive in \( \hat{W}^{l+m}_{\infty}(\mathbb{R}^2) \).

**Proof.** Let \( T_1(D) \) be an elliptic operator of order \( l \) and \( T_2(D) \) a weakly coercive operator of order \( m \). By Theorem 5.1, (i),

\[
T_2(D) = R(D) \prod_{k=1}^{s} (\lambda_k D_1 + \mu_k D_2 + \alpha_k) + Q(D),
\]

where \( R(D) \) is an elliptic operator of order \( m - s \), \( \deg Q \leq m - 2 \), \( \alpha_k \in \mathbb{C} \setminus \mathbb{R} \), the \( (\lambda_k, \mu_k) \) are pairwise non-collinear vectors in \( \mathbb{R}^2 \), \( k \in \{1, \ldots, s\} \), \( s \leq m \). Multiplying the (5.7) by \( T_1(D) \) we obtain the representation

\[
T_1(D)T_2(D) = T_1(D)R(D) \prod_{k=1}^{s} (\lambda_k D_1 + \mu_k D_2 + \alpha_k) + T_1(D)Q(D),
\]

which also has the form (5.1). In fact, \( T_1(D)R(D) \) is an elliptic operator of order \( l + m - s \) and \( \deg(T_1 Q) = \deg T_1 + \deg Q \leq l + m - 2 \). \( \square \)

The following theorem provides an algebraic criterion for weak coercivity in \( \hat{W}^l_{\infty}(\mathbb{R}^2) \).

**Theorem 5.4.** Let \( P(D) \) with \( D = (D_1, D_2) \) be an operator of order \( l \), and assume that all the coefficients and the zeros of \( P^l(\xi) \) are real.

(i) If \( P(D) \) is weakly coercive in the isotropic space \( \hat{W}^l_{\infty}(\mathbb{R}^2) \), then the polynomials \( P^l(\xi) \) and \( \text{Im} P^{l-1}(\xi) \) have no common non-trivial real zeros.

(ii) Conversely, if polynomials \( P^l(\xi) \) and \( \text{Im} P^{l-1}(\xi) \) have no common non-trivial real zeros, then the operator \( P(D) \) is weakly coercive in \( \hat{W}^l_p(\mathbb{R}^2) \), \( p \in [1, \infty] \).

**Proof.** By assumption the principal part \( P^l(\xi) \) has the form

\[
P^l(\xi) = \prod_{k=1}^{l} (\lambda_k \xi_1 + \mu_k \xi_2), \quad \text{where} \quad (\lambda_k, \mu_k) \in \mathbb{R}^2, \quad k \in \{1, \ldots, l\}. \quad (5.8)
\]

(i) Suppose that the operator \( P(D) \) is weakly coercive in \( \hat{W}^l_{\infty}(\mathbb{R}^2) \). Then the vectors \((\lambda_k, \mu_k) \) in (5.8) are pairwise non-collinear by Proposition 4.1, (iv). Combining Theorem 5.1, (i) and relation (5.8) shows that \( P(\xi) \) has the form

\[
P(\xi) = \prod_{k=1}^{l} (\lambda_k \xi_1 + \mu_k \xi_2 + \alpha_k) + Q(\xi), \quad (5.9)
\]
where $\alpha_k \in \mathbb{C} \setminus \mathbb{R}$, $k \in \{1, \ldots, l\}$, and $\deg Q \leq l - 2$. It follows from (5.9) that

$$P^{l-1}(\xi) = \sum_{j=1}^{l} \alpha_j \prod_{k \neq j} (\lambda_k \xi_1 + \mu_k \xi_2).$$

(5.10)

Now substituting in (5.10) one of the zeros $(-\mu_r, \lambda_r)$ of the principal part $P^l(\xi)$ we obtain

$$P^{l-1}(-\mu_r, \lambda_r) = \alpha_r \prod_{k \neq r} (\lambda_r \mu_k - \lambda_k \mu_r).$$

(5.11)

Under the assumptions on the numbers $\alpha_k$ and the vectors $(\lambda_k, \mu_k)$ this yields

$$P^{l-1}(-\mu_r, \lambda_r) \in \mathbb{C} \setminus \mathbb{R}.$$  

It follows that $P^l(-\mu_r, \lambda_r) = 0$ and $\Im P^{l-1}(-\mu_r, \lambda_r) \neq 0$ for all $r \in \{1, \ldots, l\}$.

(ii) Conversely, assume that the polynomials $P^l(\xi)$ and $P^{l-1}(\xi)$ have no common non-trivial real zeros, that is, that $\Im P^{l-1}(-\mu_r, \lambda_r) \neq 0$ for all $r \in \{1, \ldots, l\}$. It follows from the proof of Theorem 5.1, (i) that a polynomial $P(\xi)$ with principal part (5.8) can be represented in the form (5.9), where $\deg Q \leq l - 2$, and the $\alpha_k \in \mathbb{C}$ are some numbers. In this case the polynomial $P^{l-1}(\xi)$ is represented by the same formula (5.10). In view of the relations $P^{l-1}(-\mu_r, \lambda_r) \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_r \mu_k - \lambda_k \mu_r \in \mathbb{R} \setminus \{0\}$, $k, r \in \{1, \ldots, l\}$, $k \neq r$, equality (5.11) implies that $\alpha_k \in \mathbb{C} \setminus \mathbb{R}$, $k \in \{1, \ldots, l\}$. Now the weak coercivity of the operator $P(D)$ in $\dot{W}^l_p(\mathbb{R}^2)$ follows from Theorem 5.1, (ii).  

Remark 5.5. (i) In Theorem 5.4 the condition that the coefficients $a_\alpha$ of the polynomial $P^l$ must be real is not restrictive: if $P^l$ has real zeros, then its coefficients have the form $a_\alpha = c a^\prime_\alpha$, where $c \in \mathbb{C}$ and $a^\prime_\alpha \in \mathbb{R}$.

(ii) With the use of the resultant $R[f, g]$ of polynomials $f$ and $g$ the conditions of Theorem 5.4 can be written as $R[P^l, \Im P^{l-1}](\xi) \neq 0$, $\xi \in \mathbb{R}^n$. In this form weak coercivity can be verified without knowing the zeros of the polynomial $P^l(\xi)$.

(iii) Theorem 5.1 in combination with Proposition 4.1 shows that any strictly hyperbolic operator of order $l$ in two variables becomes weakly coercive in $\dot{W}^l_p(\mathbb{R}^2)$ after a perturbation by a suitable operator of order $l - 1$. In general, perturbations of order $l - 2$ cannot produce this result.

(iv) Operators (5.1) remain weakly coercive in $\dot{W}^l_p(\Omega)$ for any domain $\Omega \subset \mathbb{R}^2$ (including bounded domains), but they do not exhaust the entire set of weakly coercive operators in $\dot{W}^l_p(\Omega)$.

(v) Theorem 5.1 supplements the results of [32]. More specifically, by [32], p. 220 the d’Alembertian $\Box := D_1^2 - a^2 D_2^2$ is not weakly coercive in $\dot{W}^2_\infty(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^2$. However, a suitable perturbation of $\Box$ by lower-order terms makes it weakly coercive in $\dot{W}^2_\infty(\mathbb{R}^2)$ and hence in $\dot{W}^2_\infty(\Omega)$.

6. WEAKLY COERCIVE NON-ELLiptIC HOMOGENEOUS SYSTEMS

Here we show that the product of an arbitrary elliptic system and a special weakly coercive system is weakly coercive, but not elliptic. This is not the case for an arbitrary weakly coercive system (see Remark 6.5).
We denote by \( \mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^n) \) the class of multipliers satisfying the conditions of Theorem 2.10. Following [22], §2 we also introduce a partial ordering in the set of multi-indices \( \mathbb{Z}_+^n \): we will write \( \alpha \preceq \beta \) if \( \alpha_j \leq \beta_j \) for all \( j \in \{1, \ldots, n\} \); moreover, \( \alpha < \beta \), if \( \alpha_j < \beta_j \) at least for one \( j \).

In some cases the following result makes it easier to verify the assumptions of Theorem 2.10

**Proposition 6.1.** Let \( \alpha \in \mathbb{Z}_+^n \setminus \{0\} \) and let \( P(\xi) \) be a polynomial of degree \( l \). Suppose that the zero set of \( P(\xi) \) lies in a ball \( B^n_\alpha \). Consider the family of functions

\[
\Phi_\beta(\xi) := \chi(\xi) \frac{\xi^\beta}{P(\xi)}, \quad 0 < \beta \leq \alpha, \quad |\beta| < l,
\]

where \( \chi(\xi) \in C_0^\infty(\mathbb{R}^n), \) \( 0 \leq \chi(\xi) \leq 1, \) is a 'cutoff' function equal to zero in \( B^n_\alpha \) and to one for \( |\xi| \geq r_1 > r \). For \( |\xi| \geq r_1 \) suppose that the following conditions hold:

(i) the functions (6.1) satisfy inequality (2.7);

(ii) the polynomial \( P(\xi) \) satisfies the relations

\[
\prod_{j=1}^n (1 + |\xi_j|)^\gamma_j ||D^\gamma P(\xi)|| \leq C|P(\xi)|, \quad \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_+^n, \quad C > 0.
\]

Then \( \Phi_\beta \in \mathcal{M}_1 \) whenever \( \beta \in \mathbb{Z}_+^n, \) \( 0 < \beta \leq \alpha. \)

**Proof.** Note first that relations (2.7) and (2.8) are met for \( |\xi| \leq R, \) where \( R > 0 \) is arbitrary, by the continuity of the functions \( \Phi_\beta \) and their derivatives. Therefore we shall assume that \( |\xi| \) is sufficiently large (so we do not require the 'cutoff' function \( \chi(\xi) \equiv 1 \) for \( |\xi| \geq r_1 \) in what follows). We also assume that \( 0 < \beta \leq \alpha \) and denote by \( C \) various positive constants.

Consider the case \( n = 2 \) (for \( n \geq 2 \) the proof is similar).

(i) Assume that \( |\xi_1| > 1, \) \( |\xi_2| > 1. \) Then relations (2.7), (2.8) and (6.2) are equivalent, respectively, to the following groups of relations:

\[
|\xi_1\xi_2| |D_1\Phi_\beta| \leq C; \quad (6.3)
\]

\[
|\xi_1| |D_1\Phi_\beta| \leq C, \quad |\xi_2| |D_2\Phi_\beta| \leq C, \quad |\xi_1\xi_2| |D_1D_2\Phi_\beta| \leq C; \quad (6.4)
\]

\[
|\xi_1| P(\xi) \leq |P(\xi)|, \quad |\xi_2| D_2 P(\xi) \leq |P(\xi)|, \quad |\xi_1\xi_2| D_1D_2 P(\xi) \leq |P(\xi)|. \quad (6.5)
\]

In view of Theorem 2.10 it suffices to show that (6.3) and (6.5) imply (6.4).

We find an estimate for \( D_1\Phi_\beta \). We have

\[
|D_1\Phi_\beta| = \left| \frac{\beta_1\xi_1}{P(\xi)} \cdot D_1P \right| \leq C \left| \frac{\beta_1\Phi_\beta}{\xi_1} \cdot D_1P \right| \leq C \left| \frac{\Phi_\beta}{\xi_1} \right| \left( \frac{D_1P}{P} \right) \leq C \left( \frac{\Phi_\beta}{\xi_1} \right). \quad (6.6)
\]

We obtain an estimate for \( D_2\Phi_\beta \) if we interchange \( \xi_1 \) and \( \xi_2 \).

In a similar way we estimate \( D_1D_2\Phi_\beta \). We have

\[
|D_1D_2\Phi_\beta| = \left| \frac{\beta_1\beta_2\xi_1}{\xi_2} \cdot D_2\Phi_\beta \cdot D_1P \right| \leq C \left| \frac{\Phi_\beta}{\xi_2} \right| \left( \frac{D_2P}{P} \right) \leq C \left( \frac{\Phi_\beta}{\xi_2} \right). \quad (6.7)
\]
(ii) Assume that $|\xi_1| \leq 1$ and $|\xi_2| > 1$. Then relations (2.7), (2.8) and (6.2) are equivalent, respectively, to the following groups of relations:

$$|\xi_2|^\beta |\Phi_\beta(\xi)| \leq C; \quad (6.6)$$

$$|\xi_2|^\beta |D_1 \Phi_\beta| \leq C, \quad |\xi_2|^{\beta+1} |D_2 \Phi_\beta| \leq C, \quad |\xi_2|^{\beta+1} |D_1 D_2 \Phi_\beta| \leq C; \quad (6.7)$$

$$|D_1 P| \leq C|P(\xi)|, \quad |\xi_2 \cdot D_2 P| \leq C|P(\xi)|, \quad |\xi_2 \cdot D_1 D_2 P| \leq C|P(\xi)|, \quad (6.8)$$

We shall show that (6.6) and (6.8) imply (6.7). We set $\beta' := (\beta_1 - 1, \beta_2)$ for $\beta_1 > 0$ and $\beta' := \beta$ for $\beta_1 = 0$. We find estimates for $D_1 \Phi_\beta, D_2 \Phi_\beta$ and $D_1 D_2 \Phi_\beta$:

$$|(D_1 \Phi_\beta)(\xi)| = \left| \beta_1 \Phi_{\beta'}(\xi) - \Phi_\beta(\xi) \cdot \frac{D_1 P}{P} \right| \leq C (|\Phi_{\beta'}(\xi)| + |\Phi_\beta(\xi)|) \leq \frac{C}{|\xi_2|^{\beta}};$$

$$|(D_2 \Phi_\beta)(\xi)| = \left| \frac{\beta_2 \Phi_{\beta'}(\xi)}{\xi_2} - \Phi_\beta(\xi) \cdot \frac{(D_2 P)(\xi)}{P(\xi)} \right| \leq C \frac{|\Phi_\beta(\xi)|}{|\xi_2|} \leq \frac{C}{|\xi_2|^{\beta+1}};$$

$$|D_1 D_2 \Phi_\beta| = \left| \beta_1 \frac{\beta_2 \Phi_{\beta'}(\xi)}{\xi_2} - D_2 \Phi_\beta \cdot \frac{D_1 P}{P} - \Phi_\beta \left( \frac{D_2 D_1 P}{P} - \frac{D_1 P}{P} \cdot \frac{D_2 P}{P} \right) \right| \leq \frac{C}{|\xi_2|^{\beta+1}}. \quad (iii)$$

The case of $|\xi_1| > 1, |\xi_2| < 1$ is considered in a similar way. Thus, $\Phi \in \mathcal{M}_1$ by Theorem 2.10.

The following theorem describes wide classes of non-elliptic systems that are weakly coercive in the isotropic space $\tilde{W}_p^s(\mathbb{R}^n)$, $p \in [1, \infty]$. More specifically, the condition $n \geq 2N + 1$ of Theorem 4.9 fails for these systems.

**Theorem 6.2.** Let $\{P_j(D)\}_{j=1}^N$ be an elliptic system of order $l$ and let

$$R_{uv}(D) := (D_u + i)(D_v + i), \quad D_k := -i \frac{\partial}{\partial x_k}, \quad (6.9)$$

Then the system of operators

$$S_{juv}(D) := P_j(D) R_{uv}(D), \quad j \in \{1, \ldots, N\}, \quad u, v \in \{1, \ldots, n\}, \quad u > v, \quad (6.10)$$

is weakly coercive in $\tilde{W}_p^{l+2}(\mathbb{R}^n)$ for $p \in [1, \infty]$, but not elliptic.

**Proof.** The system (6.10) is not elliptic because the system $S_{juv}^l(\xi) = \xi_u \xi_v P_j^l(\xi)$ of its $(l + 2)$-principal parts has a common non-trivial zero at $\xi^0 = (0, \ldots, 0, 1)$.

Further, we choose a monomial $D^\alpha$ such that $0 \leq |\alpha| \leq l + 1$. Since the variables $\xi_1, \ldots, \xi_n$ have ‘equal weight’ in (6.9) and (6.10), we can assume without loss of generality that $\alpha_1 > 0$. We claim that the following more stronger estimate holds in place of the weak coercivity inequality:

$$\| D^\alpha f \|_{L^p(\mathbb{R}^n)} \leq C_1 \sum_{j=1}^N \sum_{v=2}^n \| S_{juv}(D) f \|_{L^p(\mathbb{R}^n)} + C_2 \| f \|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (6.11)$$

where only operators containing $D_1$ are present in the right-hand side. To prove (6.11) it suffices to show that the functions

$$\Phi_{\alpha jv}(\xi) := \chi(\xi) \left( \frac{\xi^\alpha S_{j1v}(\xi)}{\sum_{q=1}^N \sum_{s=2}^n |S_{qs}(\xi)|^2} = \chi(\xi) \left( \frac{\xi^\alpha (\xi_v - i) P_j(\xi)}{(\xi_1 + i) \sum_{q=1}^N |P_q(\xi)|^2 \sum_{s=2}^n (\xi_s^2 + 1) \right) \right) \quad (6.12)$$

are multipliers on $L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, whenever $|\alpha| \leq l + 1, j \in \{1, \ldots, N\}, v \in \{2, \ldots, n\}$.

Here $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \chi(\xi) \leq 1$, is a ‘cutoff’ function equal to one for sufficiently large
\[ |\xi| \text{ and to zero in a ball } B^n_1 \text{ containing the compact zero set of the elliptic system } \{ P_j(D) \}_1^N \] (see Proposition 2.11, (i)). In fact, if we prove that \( \Phi_{\alpha jv} \in \mathcal{M}_1 \), then by applying the inverse Fourier transform to the equalities

\[ \xi^\alpha \hat{f}(\xi) = \sum_{j=1}^{N} \sum_{v=2}^{n} \Phi_{\alpha jv}(\xi) S_{jiv}(\xi) \hat{f}(\xi) + \xi^\alpha (1 - \chi(\xi)) \hat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^n), \]

we obtain the desired estimates (6.11).

Assume first that \( |\alpha| < l + 1 \) and that, as mentioned above, \( \alpha_1 > 0 \). Then all the following rational fractions belong to \( \mathcal{M}_1 \):

\[ \frac{\xi_1}{\xi_1 + i}, \quad \frac{\chi(\xi) \xi_1^{\alpha_1 - 1} \xi_2 \cdots \xi_n P_j(\xi)}{\sum_{q=1}^{N} |P_q(\xi)|^2 \sum_{s=2}^{n} (\xi^2_s + 1)}, \quad \frac{\xi_v - i}{\sum_{s=2}^{n} (\xi^2_s + 1)} \]

In fact, \( \xi_1(\xi_1 + i)^{-1} - 1 - i(\xi_1 + i)^{-1} \in \mathcal{M}_1 \) by formula (5.4). It is clear that the two remaining fractions belong to \( \mathcal{M}_1 \) by Theorem 2.10 (or Proposition 6.1), because their denominators are elliptic polynomials of their variables. Finally, since \( \mathcal{M}_1 \) is an algebra, it follows that \( \Phi_{\alpha jv} \in \mathcal{M}_1 \).

Now, consider the case of \( |\alpha| = l + 1 \). Clearly,

\[ \Phi_{\alpha jv}(\xi) = \chi(\xi) \frac{\xi_1}{\xi_1 + i} \frac{\xi_2}{\sum_{q=1}^{N} |P_q(\xi)|^2 \sum_{s=2}^{n} (\xi^2_s + 1)} \]

where \( \beta := (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n) \), \( |\beta| = l \). As mentioned above, \( \xi_1(\xi_1 + i)^{-1} \in \mathcal{M}_1 \).

Therefore, it suffices to show that \( \Psi_{\beta jv} \in \mathcal{M}_1 \).

Let \( \varkappa \) be the exponent of \( \xi_1 \) in the product \( \xi^\beta P_j(\xi) \), \( \varkappa \leq 2l \). We consider two cases.

(i) Suppose \( \varkappa < 2l \). Clearly, the functions \( \Psi_{\beta jv} \) are sums of functions of the form

\[ \Phi_{\gamma}(\xi) := \chi(\xi) \frac{\xi^\gamma}{G(\xi) \sum_{s=2}^{n} (\xi^2_s + 1)}, \quad |\gamma| \leq 2l + 1, \quad \gamma_1 \leq 2l - 1, \quad (6.13) \]

where \( G(\xi) := \sum_{q=1}^{N} |P_q(\xi)|^2 \) is an elliptic polynomial of degree \( 2l \). We will verify the assumptions of Proposition 6.1 for functions (6.13). First, we verify (2.7).

By Proposition 2.11, (ii), if \( |\xi| \) is large enough, then

\[ C_1 |\xi|^{2l} \leq |G(\xi)| \leq C_2 |\xi|^{2l}, \quad C_1, C_2 > 0. \quad (6.14) \]

Now, the inequality between the geometric mean and mean square yields

\[ \prod_{j=1}^{n} (1 + |\xi_j|)^\delta \leq C|\xi|, \quad \text{where } \delta := 1/n, \quad (6.15) \]

for large \( |\xi| \). Since \( |\gamma| \leq 2l + 1 \) and \( \gamma_1 \leq 2l - 1 \), it follows that \( \gamma = \gamma' + \tilde{\gamma} \), where \( \gamma' \) and \( \tilde{\gamma} \) are multi-indices such that \( \gamma_1 = \gamma'_1 \leq 2l - 1 \), \( |\gamma'| \leq 2l - 1 \) and \( |\tilde{\gamma}| \leq 2, \tilde{\gamma}_1 = 0 \). Then

\[ |\xi'| \leq |\xi|^{|\gamma'|} \leq |\xi|^{2l-1}, \quad |\xi^{\tilde{\gamma}}| \leq \sum_{k=2}^{n} (\xi^2_k + 1) \quad (6.16) \]

for \( |\xi| > 1 \). Multiplying inequalities (6.15) and (6.16) and taking (6.14) into account, for large \( |\xi| \) we arrive at relation (2.7) for the function \( \Phi_{\gamma}(\xi) \).
Leibniz’s formula (2.4) implies inequalities (6.2) for the polynomial
\[ G(\xi) \sum_{s=2}^{n} (1 + \xi_s^2) \]
because they hold for the elliptic polynomial \( G(\xi) \) (see [14], §4) and they obviously hold for \( \sum_{s=2}^{n} (1 + \xi_s^2) \).

Thus, \( \Phi_\gamma \in \mathcal{M}_1 \) by Proposition 6.1 and hence \( \Psi_{\beta jv} \in \mathcal{M}_1 \).

(ii) Let \( \alpha = 2l \). Then \( \xi^\beta = \xi_1^\beta \). Also let \( P_j(\xi) = c_j \xi_1^j + \ldots \), where the dots stand for a sum of monomials containing \( \xi_1 \) with exponent \( < l \). Then \( G(\xi) = \sum_{q=1}^{n} |c_q|^2 \xi_1^{2q} + \ldots \), and we have \( \sum_{q=1}^{n} |c_q|^2 \neq 0 \). In view of Theorem 2.10, the function
\[ \Psi_{\beta jv}(\xi) := \chi(\xi) \frac{-c_j \xi_{v-i}}{\sum_{q=1}^{n} |c_q|^2 \xi_s^2 + 1} \]
is a multiplier on \( L^1(\mathbb{R}^n) \), \( \Psi_{\beta jv} \in \mathcal{M}_1 \). Moreover, by step (i),
\[ \Psi_{\beta jv}(\xi) - \Psi'_{\beta jv}(\xi) = \chi(\xi) \frac{\xi_1^j (\tau_j \xi_1^j + \ldots) - \tau_j \sum_{q=1}^{n} |c_q|^2 \xi_1^{2q} - G(\xi) (\xi_{v-i})}{G(\xi) \sum_{s=2}^{n} \xi_s^2 + 1} \in \mathcal{M}_1, \]
because the factor in the square brackets contains no monomials with \( \xi_1^q \).

**Corollary 6.3.** Let \( P(D) \) be an elliptic operator of order \( l \) and let \( R_{uv}(D) \) be operators of the form (6.9). Then the system \( \{P(D)R_{uv}(D)\}_{u>v} \) is weakly coercive in \( \dot{W}^{l+2}_p(\mathbb{R}^n) \), \( p \in [1, \infty] \), but is not elliptic.

Next we show that for \( p = \infty \) the number of operators \( R_{uv}(D) \) in Theorem 6.2 cannot be reduced even if \( N = 1 \).

**Proposition 6.4.** Suppose that \( n \geq 3 \), let \( P(D) \) be an elliptic operator of order \( l \) and \( R_{uv}(D) \) be operators of the form (6.9). Then if an arbitrary operator is removed from the system \( \{P(D)R_{uv}(D)\}_{u>v} \) the rest is no longer weakly coercive in \( \dot{W}^{l+2}_\infty(\mathbb{R}^n) \).

**Proof.** Without loss of generality we may assume that the operator \( P(D)R_{12}(D) \) is removed from the system. Assume that the system
\[ \{P(D)R_{uv}(D)\}_{u>v, \ u+v>3} \]
remains weakly coercive in \( \dot{W}^{l+2}_\infty(\mathbb{R}^n) \). Consider the ‘restricted’ system
\[ \{P(D)R_{uv}(D)[E]\}_{u>v, \ u+v>3}, \]
where \( E := \text{span}\{\xi_1, \xi_2\} \). It has order \( l + 1 \) and by Corollary 4.7 is weakly coercive in \( \dot{W}^{l+2}_\infty(\mathbb{R}^n) \). In particular, this system estimates the operator \( D_1^{l+1} \). By Proposition 1.4 we obtain
\[ \xi_1^{l+1} = \sum_{u>v, \ u+v>3} \lambda_{uv} (P(\xi)R_{uv}(\xi)[E])^{l+1} = (P(\xi)[E]) \left[ \sum_{u>v, \ u+v>3} \lambda_{u1} \xi_1 + \lambda_{u2} \xi_2 \right]. \]
The polynomial \( P(\xi)[E] \) is elliptic and therefore is not a multiple of \( \xi_1 \). Hence \( \xi_1^{l+1} \) must divide the polynomial in the square brackets. However, this contradicts the relation \( l \geq 1 \). \( \Box \)
Remark 6.5. Proposition 6.4 shows that Corollary 5.3 does not hold for $N > 1$ in the general case. For example, the system $\{(D_1 + i)(D_2 + i), (D_3 + i)(D_4 + i)\}$ is weakly coercive in $W_\infty^2(\mathbb{R}^4)$, but the system
$$\left(D_1^2 + \cdots + D_4^2\right)(D_1 + i)(D_2 + i), \quad \left(D_1^2 + \cdots + D_4^2\right)(D_3 + i)(D_4 + i)$$
is not weakly coercive in $W_\infty^4(\mathbb{R}^4)$.

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