The superconducting order parameter and gauge dependence

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Abstract

The gauge dependence of the renormalization group functions of the Ginzburg-Landau model is investigated. The analysis is done by means of the Ward-Takahashi identities. After defining the superconducting order parameter, it is shown that its exponent $\beta$ is in fact gauge independent. This happens because in $d = 3$ the Landau gauge is the only gauge having a physical meaning, a property not shared by the four-dimensional model where any gauge choice is possible. The analysis is done in both the context of the $\epsilon$-expansion and in the fixed dimension approach. It is pointed out the differences that arise in both of these approaches concerning the gauge dependence.

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I. INTRODUCTION

It is often stated in the literature that the superconducting order parameter in a Ginzburg-Landau (GL) model has no physical meaning because it is a non-gauge invariant quantity. This statement is obvious if by order parameter we mean \(< \phi(x) >\), the expectation value of the complex scalar field in the GL model. Since physical quantities should be gauge invariant, \(< \phi(x) >\) cannot be considered as a physical quantity. It is desirable, however, to have a gauge invariant definition of order parameter which characterizes the superconducting phase transition. One such definition exists already and has been proposed by Kennedy and King \[1\] in the context of the lattice superconductor. The order parameter proposed by them is given by \(G = \lim_{|x-y| \to \infty} < \phi_x \phi_y^* >\), where \(\phi_x\) is a lattice field. This order parameter describes a phase transition in the lattice GL model. However, it is not clear that this phase transition coincides exactly with the normal-superconducting transition or, in the language of particle physics, the Higgs transition.

In this paper we will discuss the physical meaning of the superconducting order parameter in the context of a continuum GL model. The aim of this paper is to discuss in a deeper way the questions addressed by one of us in a recent letter \[2\]. It will be shown that a gauge invariant definition of order parameter, consistent with the traditional definition of the critical exponent \(\beta\), is in fact possible. Less obvious is the gauge independence of \(\beta\) in different schemes of renormalization. For instance, the gauge dependence obtained through dimensional continuation, the \(\epsilon\)-expansion, is very different from the one obtained using a fixed dimension approach. This approach, though less controlled, gives better values to the critical exponents. In fact, the renormalization group (RG) calculations based on the \(\epsilon\)-expansion does not give very good results for the superconducting phase transition, specially in what concerns the type II regime. Indeed, this regime is not readily accessible by the \(\epsilon\)-expansion which predicts a weak first order transition whatever the value of the Ginzburg parameter \(\kappa\). As shown by Dasgupta and Halperin \[4\] using duality arguments in a lattice model, the weak first order scenario does not hold in the type II regime where the transition should be expected to be second order. In order to describe correctly the critical behavior of superconductors using the \(\epsilon\)-expansion, it is necessary to use resummations techniques such as the Padé-Borel resummation used by Folk and Holovatch \[5\].

The plan of this paper is as follows. In section II we will discuss the RG equations for the GL model and define the order parameter. In section III we will use the Ward-Takahashi (WT) identities to establish the gauge dependence of the order parameter exactly. It will be shown that in a fixed dimension \(d = 3\) RG only the Landau gauge \(a = 0\) has a physical meaning. This result will be anticipated already in section II by looking the 1-loop approximation to the RG functions. However, it will only be proved in section III to all orders in perturbation theory. Section IV concludes the paper.

II. RENORMALIZATION GROUP IN THE GL MODEL

Let us consider the following bare action for the GL model:

\[
S = \int d^d x \left[ \frac{1}{4} F^2_0 + (D^0_\mu \phi_0)(D^0_\mu \phi_0) + \frac{M^2_0}{2} A^0_\mu A^0_\mu + m^2_0 |\phi_0|^2 + \frac{u_0}{2} |\phi_0|^4 \right] + S_{gf}, \tag{II.1}
\]
where the zeroes denote bare quantities, $F_0^2$ is a short for $F_0^{\mu\nu}F_0^{\mu\nu}$ and $D_\mu^0 = \partial_\mu + i\epsilon_0 A_\mu^0$. The $S_{gf}$ is the gauge fixing part and is given by

$$S_{gf} = \int d^d x \frac{1}{2a_0} (\partial_\mu A_\mu^0)^2. \quad (\text{II.2})$$

We have added a mass to the gauge field in order to regularize the infrared divergences arising from the gauge field propagator. This mass breaks explicitly the gauge invariance (also broken by $S_{gf}$), which is recovered at the critical point. Adding a mass to the gauge field does not affect the renormalizability of the model. This should be contrasted with the non-abelian case, where the only way to provide a mass to the gauge field without destroying renormalizability is through the Higgs mechanism [6].

The renormalized action is obtained from the bare action by rewriting it in terms of renormalized quantities. This is achieved through the introduction of renormalization constants. We introduce the field renormalization in a standard way [6], $A_\mu = Z_{a}^{-1/2} A_\mu^0$ and $\phi = Z_{\phi}^{-1/2} \phi_0$. The renormalized action including the other renormalization constants is given by

$$S = \int d^d x \left[ \frac{Z_{a}}{4} F^2 + Z_{\phi} (D_\mu \phi) (D_\mu \phi)^\dagger + \frac{Z_{M} M^2}{2} A^2 ight.$$ 

$$+ Z_m m^2 |\phi|^2 + \frac{Z_{w} u}{2} |\phi|^4 + \frac{Z_{a}}{2a} (\partial_\mu A_\mu)^2 \right]. \quad (\text{II.3})$$

The renormalization constants $Z_M$ and $Z_a$ above are in fact superflous. Indeed, we have that $Z_M = Z_a = 1$ which means that the corresponding terms in the action, the gauge field mass term and the gauge fixing term, are not renormalized. This result is easily checked by studying the WT identities as follows. By adding sources to the corresponding fields we obtain the following WT identity:

$$\left\{ \left( M^2 - \frac{1}{a} \Delta \right) \frac{\delta}{\delta J_\mu(x)} + ie \frac{\delta}{\delta J^\dagger_\mu(x)} - J_\mu(x) \frac{\delta}{\delta J(x)} \right\} W(J_\mu, J^\dagger, J) = \partial_\mu J_\mu(x), \quad (\text{II.4})$$

where $W = \log Z$, $Z$ being the generating functional of correlation functions defined by

$$Z(J_\mu, J^\dagger, J) = \int D\phi^\dagger D\phi DA_\mu \exp \left[ -S + \int d^d x (J_\mu A_\mu + J^\dagger \phi + \phi^\dagger J) \right]. \quad (\text{II.5})$$

When the sources are zero $Z$ gives the partition function. $W$ is the generating functional of the connected correlation functions. The Legendre transform of $\Gamma$ of $W$ is given by

$$W(J_\mu, J^\dagger, J) + \Gamma(\phi^\dagger, \phi, a_\mu) = \int d^d x (J_\mu a_\mu + J^\dagger \phi + \phi^\dagger J) \quad (\text{II.6})$$

where the functional effective action $\Gamma(\phi^\dagger, \phi, a_\mu)$ is the generator of the 1-particle irreducible functions. It satisfies a WT identity which is the Legendre transform of (II.4):

$$\left( \frac{1}{a} \Delta - M^2 \right) \frac{\delta}{\delta a_\mu(x)} + \partial_\mu \frac{\delta \Gamma}{\delta a_\mu(x)} + ie \left[ \phi(x) \frac{\delta \Gamma}{\delta \phi(x)} - \phi^\dagger(x) \frac{\delta \Gamma}{\delta \phi^\dagger(x)} \right] = 0. \quad (\text{II.7})$$

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Both WT identities (II.4) and (II.7) are linear. Thus, if we perform a loop expansion, 
\[ \Gamma = \sum_l \Gamma_l \]
where \( \Gamma_l \) is the \( l \)-loop correction to the effective action, it follows that each \( \Gamma_l \) satisfy separately a WT identity (II.7). The zero-loop contribution \( \Gamma_0 \) is the classical action \( S \), which satisfies the inhomogeneous WT identity (II.7), that is, the identity containing the term \( (a^{-1} \Delta - M^2) \partial_\mu a_\mu(x) \). The loop corrections, on the other hand, satisfy the homogeneous WT identity. The non-gauge invariant terms are all included in the inhomogeneous WT identity, do not receive any correction due to the fluctuations and are in this way not renormalized, that is, the corresponding counterterms are zero. The only non-gauge invariant terms in the action are the gauge field mass term and the gauge fixing term. As a consequence of these reasonings we obtain that
\[ Z_M = Z_a = 1. \]
This implies that
\[ M_0^2 = Z^2 M_0^2 \]
and
\[ a_0 = Z^{-1} a_0. \]
From (II.3) we deduce also
\[ m_0^2 = Z^{-2} m_0^2, \]
\[ e_0^2 = Z_A e_0^2 \]
and
\[ u_0 = Z_u^2 u_0. \]
It is useful at this point to fix the renormalization conditions necessary to define the corresponding renormalized quantities:
\[ \Gamma^{(2)}_{ij}(0) = m_0^2 \delta_{ij}, \]  
\[ \frac{\partial \Gamma^{(2)}_{ij}}{\partial p^2} \bigg|_{p^2=0} = \delta_{ij}, \]  
\[ \Gamma^{(4)}_{ijkl}(0,0,0,0) = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})u, \]  
\[ \Gamma_{\mu\nu}(0) = M_0^2 \delta_{\mu\nu}, \]  
\[ \frac{\partial}{\partial p^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Gamma^{(2)}_{\mu\nu} \bigg|_{p^2=0} = d - 1, \]  
\[ \frac{\partial}{\partial p^2} \left. \frac{p_\mu p_\nu}{p^2} \Gamma^{(2)}_{\mu\nu} \right|_{p^2=0} = \frac{1}{a}, \]  
where the latin indices represent the components of the scalar field.

We are interested in the behavior when \( m \to 0 \), the infrared behavior. Thus, we assume that \( m_0^2 \propto t, t = (T - T_c)/T_c \) being the reduced temperature, and \( m = \xi^{-1} \propto t^\nu \) as \( t \to 0 \), \( \xi \) being the correlation length. The RG equations will be obtained by differentiating with respect to \( \ln m \) with all bare quantities fixed, except for \( m_0^2 \). Let us define the dimensionless couplings \( f = m^{d-4} e^2, \hat{u} = m^{d-4} u \) and \( v = m/M \). The following exact flow equations are readily obtained:
\[ m \frac{\partial M_0^2}{\partial m} = \eta_A M_0^2, \]  
\[ m \frac{\partial a_0}{\partial m} = -\eta_A a_0, \]  
\[ m \frac{\partial f}{\partial m} = (\eta_A + d - 4) f, \]  
\[ m \frac{\partial v}{\partial m} = \left(1 - \frac{\eta_A}{2}\right) v, \]
where we have introduced the RG function $\eta_A$ which at the fixed point gives the anomalous dimension of the gauge field. It is defined by

$$\eta_A = m \frac{\partial}{\partial m} \log Z_A.$$  

(II.18)

At 1-loop order and in fixed dimension $d = 3$ we obtain

$$\eta_A = \frac{f}{24\pi}.$$  

(II.19)

From Eq. (II.16) we obtain that a charged fixed point should correspond to $\eta_A = 4 - d$. In $d = 4$ this corresponds necessarily to zero charge while in $d = 3$ we have $f_* = 24\pi$, a non-zero charge. The only fixed point to Eq. (II.13) corresponding to a charged fixed point in $d = 3$ is $a_* = 0$, that is, the Landau gauge. Due to the negative sign in (II.13), any $a \neq 0$ in the flow diagram will flow to infinity. Fig. 1 shows the flow diagram in the $fa$-plane in the 1-loop approximation in $d = 3$. Note that the line $a = 0$ contains an attractive flow towards the fixed point $f_* = 24\pi$. Any $a \neq 0$, no matter how small, will flow away. This does not happen at $d = 4$ where $f_* = 0$ and $a = a_*$ arbitrary is a line of fixed points. This 1-loop calculation suggests that in $d = 3$ the Landau gauge is the only physical gauge.

The 1-loop beta function for the coupling $\hat{u}$ is given by

$$\beta_{\hat{u}} = m \frac{\partial \hat{u}}{\partial m} = (2\eta_\phi - 1)\hat{u} + \frac{5}{8\pi} \hat{u}^2 + \frac{v}{2\pi} f^2.$$  

(II.20)

This beta function does not exhibit any gauge dependence, which is cancelled out between the $\eta_\phi \hat{u}$ and $f^2$ term. In (II.20) $\eta_\phi$ is defined as $m \partial \ln Z_\phi / \partial m$, which computed in the Landau gauge gives

$$\eta_\phi = -\frac{2}{3\pi} \frac{fv^2}{(1 + v)^2}.$$  

(II.21)

We will see in section III that the gauge independence of $\beta_{\hat{u}}$ is valid to all orders, though $Z_\phi$ does depend on the gauge.

The flow equations we have obtained are similar to the ones obtained by Herbut [7] in the context of the continuum dual GL model [8-12]. The difference is that in our case the charge $e$ is not meant to be the dual charge and, as a consequence, our mass $M$ has nothing to do with the photon mass (anyway, this is a subtle point, even in the dual approach; see ref. [12]).

Let us discuss the critical behavior that arises from the so defined GL model. From an experimental point of view, presently only the exponents $\alpha$ (specific heat) and $\nu$ (correlation length) are accessible. The order parameter exponent $\beta$ seems not to be directly accessible. This is the main point we would like to discuss. At this point, an important remark is in order. The critical exponents are obtained from the singular behavior of the effective action (the free energy) and correlation functions. The WT identities imply that the singular behavior is the same, whatever $t < 0$ (the broken symmetry regime or ordered phase) or $t > 0$ (symmetric regime or disordered phase). This means that the exponents have the same value below or above $T_c$. This remark is very important concerning the superconductors. The point is that below the transition the photon becomes massive, phenomenon known in particle
physics as the Higgs mechanism \[6\]. This mass is generated spontaneously and not added by hand as our mass \(M\). Since the singular behavior is the same as in the symmetric phase, the renormalization proceeds exactly as in the symmetric phase. Thus, this mechanism is very important in the context of non-abelian gauge theories, where adding a mass by hand destroys renormalizability. The photon mass generated by the Higgs mechanism in the GL model corresponds to the inverse of the penetration depth \(\lambda\). This length is known to scales as the correlation length \(\xi\) \[7,13,12\], that is, \(\lambda \sim \xi \sim |t|^{\nu}\). \(\lambda\) is a quantity that arises only for \(t < 0\) but its exponent, being the same appearing in the scaling of \(\xi\), can be evaluated for \(t > 0\) or even for \(t = 0\) (the critical point), this last case needing renormalization conditions different from the ones we use here due to the infrared divergences \[13,14\].

The \(\nu\) exponent is obtained easily by considering the flow of \(m_0^2\) (remember that \(m_0\) is not kept fixed under RG). From the definition of its renormalized counterpart \(m^2 = Z_m^{-1} Z_\phi m_0^2\), we obtain

\[
\frac{m \partial m_0^2}{\partial m} = (2 + \eta_m - \eta_\phi) m_0^2,
\]

Eq. (II.22) implies that near the phase transition \((m \to 0)\) \(m_0^2 \sim m^2 + ^2\eta_m - \eta\), where \(\eta_m^*\) and \(\eta\) are the fixed point values of \(\eta_m\) and \(\eta_\phi\), respectively. Since by definition \(m_0^2 \propto t\) and \(m \sim t^{\nu}\) we obtain immediately

\[
\nu = \frac{1}{2 + \eta_m^* - \eta}.
\]

The \(\nu\) exponent can be accurately measured through a direct measurement of the penetration depth \[15\]. In the accessible critical region, the fluctuations are governed by the 3D XY fixed point \[16\], which is a neutral fixed point. Then, it results that \(\rho_s \sim \lambda^{-2} \sim t^{2\nu'}\), where \(\rho_s\) is the superfluid density \[12,10\] and we have defined the exponent \(\nu'\) of the penetration depth. Using the Josephson relation \(\rho_s \sim t^{\nu(d-2)}\) \[18\] for the \(d = 3\), we obtain \(\nu' = \nu/2\). The exponent \(\nu'\) obtained experimentally in bulk samples of \(YBa_2Cu_3O_{7-\delta}\) is \(\nu' \approx 1/3\) \[15\] and therefore we obtain \(\nu \approx 2/3\), corresponding in this way to a 3D XY universality class. This very same value of \(\nu\) is expected for the charged transition, the so called “inverted” 3D XY universality class \[4\]. In fact, this is confirmed by recent RG calculations \[2,13,7,9,14\] and Montecarlo simulations \[17\]. Let us evaluate \(\nu\) from Eq. (II.24) in the present framework. We have at 1-loop order that

\[
\eta_m = -\frac{f}{4\pi}.
\]

The infrared stable fixed point corresponds to \(f_s = 24\pi\), \(\hat{u}_s = 8\pi/5\) and \(\nu_s = 0\). This gives \(\eta = 0\) and \(\nu = 0.63\) at 1-loop order. This result can be systematically improved. Indeed, the fixed point \(\hat{u}_s\) will always have a 3D XY value because every power of \(f\) will be multiplied by a function of \(\nu\) which goes to zero as \(m \to 0\), since \(\nu \sim m^{1/2}\). Therefore, we obtain already at 2-loops \(\nu \approx 2/3\). It should be noted that this behavior is obtained because from Eq. (II.14)
we have that \( M^2 \sim m^{4-d} \) near the phase transition. For \( d = 3 \) this means \( M \sim t^{\nu/2} \). Note that this is exactly the scaling behavior of the true photon mass near the neutral 3D XY fixed point! This means that \( M \) can be alternatively regarded as a photon mass describing the crossover regime observed experimentally. This is the case if we interpret the GL model given in (II.1) as a disorder field theory, that is, as a continuum dual GL model [7–12]. In this case, the charge \( e \) should be replaced by \( 2\pi M/q \), where \( q \) is a the dual charge satisfying the Dirac condition \( qe = 2\pi \).

Let us investigate now how the exponent \( \beta \) should be defined. This presupposes a definition of order parameter whose scaling has exponent \( \beta \). This order parameter arises naturally from the definition of the superfluid density \( \rho_s \). This is defined for \( t < 0 \) by

\[
\rho_s = <|\phi|^2> = Z_{\phi}^{-1} <|\phi_0|^2>.
\]

We define the order parameter by \( \Phi \) by

\[
\Phi = \sqrt{<|\phi|^2>}.
\]

The expectation value of a gauge invariant operator is gauge independent [3,20]. Thus, \( \rho_s \) should be independent of \( a \). However, \( Z_\phi \) is gauge dependent (see section III) and we have that \( \Phi \) should in fact depends on \( a \) to cancels the gauge dependence coming from \( Z_{\phi}^{-1} \) in (II.26), otherwise \( \rho_s \) would be gauge dependent. Note that the gauge invariance of \( <|\phi_0|^2> \) implies that \( \Phi \) is independent of \( a_0 \) but not necessarily independent of \( a \). However, its scaling behavior near the phase transition will be shown (section III) to be gauge independent or, more precisely, to be evaluated in the Landau gauge. The same will shown to be true for \( Z_\phi \). For the moment, let us check that the above definition of order parameter works. The exponent \( \beta \) is defined through its behavior near the critical point, \( \Phi \sim |t|^\beta \). Also, we have \( Z_{\phi} \sim m^{\eta} \sim |t|^{\nu \eta} \). Putting all of this together we obtain \( \rho_s \sim |t|^{2\beta-\nu \eta} \), which is the Josephson relation as obtained originally in Josephson’s paper [18]. Using the hyperscaling relation \( d\nu = 2 - \alpha \) together with the combination of the scaling relations \( \alpha + 2\beta + \gamma = 2 \) and \( \gamma = \nu(2 - \eta) \) to eliminate \( \gamma \), we obtain the Josephson relation in the form \( \rho_s \sim |t|^{\nu(d-2)} \). This last form of the Josephson relation has been also obtained directly by us [12], without using the scaling relations \( \alpha + 2\beta + \gamma = 2 \) and \( \gamma = \nu(2 - \eta) \). Therefore,

\[
\beta = \frac{\nu}{2}(d - 2 + \eta).
\]

Thus, a gauge dependence in \( \eta \) would imply a gauge dependence in \( \beta \) if \( \nu \) is gauge independent. Note that the gauge independence of \( \rho_s \) does not ensure the gauge independence of \( \nu \). The point is that the gauge independence of \( |t|^{\nu(d-2)} \) can be a result of compensating gauge dependences arising from \( |t| \) (equivalently, \( m_0^2 \)) and \( \nu \).

III. THE GAUGE DEPENDENCE

In the preceding section we discussed the critical behavior of the GL model while trying to get some insight on the effect of the gauge dependence. Clearly a more careful analysis is needed. In this section gauge dependence always means a dependence on \( a \), the renormalized gauge fixing parameter. Therefore, the other renormalized parameters of the model are
trivially gauge independent. This statement implies the gauge independence of all beta functions. Note that this does not necessarily mean that the bare parameters are gauge independent.

In order to obtain the gauge dependence of $Z_\phi$ we will employ the WT identities. From the WT identity Eq. (II.4) we obtain the following identity:

$$\left( M^2 - \frac{1}{a} \partial^2 \right) \partial_\mu W_\mu^{(2)}(z, x) = i e [\delta(y - z) W^{(2)}(z, x) - \delta(z - x) W^{(2)}(y, z)]. \quad (\text{III.1})$$

By using twice Eq. (III.1) we obtain

$$W_\mu^{(2)}(\partial_\mu A_\mu)^2(p) = 2e^2 \int \frac{d^d k}{(2\pi)^d} \frac{a^2}{(k^2 + aM^2)^2} \left[ W^{(2)}(p + k) - W^{(2)}(p) \right], \quad (\text{III.2})$$

where $W_\mu^{(2)}(\partial_\mu A_\mu)^2(p)$ is the Fourier transform of

$$W_\mu^{(2)}(\partial_\mu A_\mu)^2(x, y) = \int d^d z \left[ <(\partial_\mu A_\mu)^2(z)\phi(x)\phi^\dagger(y)> - <(\partial_\mu A_\mu)^2(z)><\phi(x)\phi^\dagger(y)> \right]. \quad (\text{III.3})$$

Let us denote the bare counterpart of $W_\mu^{(2)}(\partial_\mu A_\mu)^2$ by $W_\mu^{(2)}_{\mu, A_\mu, 0}$. We have that

$$2a_0^2 \frac{\partial W_0^{(2)}}{\partial a_0}(x, y) = W_\mu^{(2)}_{\mu, A_\mu, 0}(x, y), \quad (\text{III.4})$$

where $W_\mu^{(2)}(x, y) = <\phi_0(x)\phi_0^\dagger(y)>$ is the bare 2-point connected correlation function. Eq. (III.2) is valid also if we replace the renormalized correlation functions by the bare ones and the renormalized couplings by their bare counterparts. Using then a bare version of (III.2) and Eq. (III.4), we obtain

$$\frac{\partial W_0^{(2)}}{\partial a_0}(p) = e_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{W_0^{(2)}(p + k) - W_0^{(2)}(p)}{(k^2 + a_0 M_0^2)^2}. \quad (\text{III.5})$$

Eq. (III.5) can be rewritten as

$$\frac{\partial \ln Z_\phi}{\partial a_0} W^{(2)}(p) + \frac{\partial W^{(2)}}{\partial a_0}(p) = e_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{W^{(2)}(p + k) - W^{(2)}(p)}{(k^2 + a_0 M_0^2)^2}, \quad (\text{III.6})$$

out of which we obtain

$$\frac{\partial \ln Z_\phi}{\partial a_0} = -e_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + a_0 M_0^2)^2} = e_0^2 a_0^{d-4} M_0^{d-4} \left( \frac{d}{2} - 1 \right) C_d, \quad (\text{III.7})$$

where $\pi/C_d = (4\pi)^{d/2} \Gamma(d/2) \sin(\pi d/2)$. Note that we have a pole for $d = 4$ in the second line of Eq. (III.7). This is a consequence of the logarithmic divergence for $d = 4$. In the $\epsilon$-expansion the singular part of the different correlation functions is isolated as poles in $1/\epsilon$. 

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with $\epsilon = 4 - d$ and the renormalization constants are written as power series in $1/\epsilon$. This way of doing the things leads to the determination of the critical exponents as power series in $\epsilon$. The physical case of interest in critical phenomena of superfluid and magnetic systems corresponds to $\epsilon = 1$.

By integrating the first line of (III.7) we obtain
\[
\ln Z_\phi(a_0) = \ln Z_\phi(a_0 = 0) - e^2 a_0 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + a_0 M_0^2}.
\]  
(III.8)

Since $e^2 a_0 = e^2 a$ and $a_0 M_0^2 = a M^2$, we can rewrite Eq. (III.8) as
\[
\ln Z_\phi(a) = \ln Z_\phi(a = 0) - e^2 a \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + a M^2}.
\]  
(III.9)

Let us assume that $Z_\phi(a = 0)$ has been evaluated as a power series in $1/\epsilon$. After regularizing dimensionally the integral in (III.9), we obtain
\[
\eta_\phi(a) = \eta_\phi(a = 0) - \frac{af}{2\pi},
\]  
(III.10)

which gives the gauge dependence of $\eta_\phi$ in the framework of the $\epsilon$-expansion. Let us assume that an infrared stable fixed point has been obtained, for instance, by resummation methods \cite{5}. As $m \to 0$, $f \to f_\ast \neq 0$ (if $\epsilon = 1$), but $a$ scales as $m^{-1}$ near the fixed point and any non-zero $a$ runs away as $m \to 0$. Thus, the only safe way towards the charged fixed point is over the line $a = 0$, that is, the Landau gauge. Note that for the case of interest in particle physics, $d = 4$, we obtain the same equation as (III.10). However, for $d = 4$ any gauge choice is possible since $f_\ast = 0$ in this case.

In the fixed dimension approach things work differently. For $d = 3$ the integral in Eq. (III.9) is convergent and we can interchange the differentiation with respect to $m$ with the integral sign. Since $ae^2$ and $a M^2$ are both RG invariants, we obtain
\[
\eta_\phi(a) = \eta_\phi(a = 0),
\]  
(III.11)

and we obtain again that the physical gauge corresponds to $a = 0$. At this point some remarks are in order. First, from Eq. (III.11) we obtain $\partial \eta_\phi / \partial a = 0$ while the same is not true for the $\eta_\phi(a)$ given in Eq. (II.10). Second, Eq. (III.11) can be easily checked at 1-loop order. The renormalization constant is given as a function of $f$, $a$ and $v$ and if we take care of differentiating $a$ when obtaining $\eta_\phi$, the result (III.11) follows and coincides with Eq. (II.21). Concerning the 1-loop example, it is instructive to ask ourselves what happens in other fixed dimension approaches. For instance, we could perform a critical point ($m = 0$) calculation where the renormalization conditions are defined at non-zero external momenta, taking the symmetrical point for functions which depend on more than one momentum variable \cite{3,4}. In this case the photon mass $M$ is unnecessary since the non-zero external momenta take care of infrared divergences \cite{13}. The 1-loop expression for $Z_\phi$ in an arbitrary gauge is in this case rather simple and has been calculated by Schakel \cite{21}. It turns out in this case that $Z_\phi$ is independent of $a$ if $d = 3$.

The gauge dependence of $Z_m$ can be obtained in an analogous way. From the renormalization condition (II.8) and $m^2 = Z_m Z^{-1}_m m_0^2$, we obtain that $W^{(2)}(0) = Z_m/m_0^2$. Using again
the bare version of (III.2), we obtain exactly the same equation as Eq. (III.7) but with $Z_\phi$ replaced by $Z_m$. This means that the gauge dependence of $Z_m$ is the same as for $Z_\phi$. If we use the $\epsilon$-expansion we have that $\eta_m - \eta_\phi$ is gauge independent since the gauge dependence of $\eta_m$ will cancel exactly the gauge dependence of $\eta_\phi$. In fixed dimension $d = 3$, on the other hand, $\eta_m(a) = \eta_m(a = 0)$. It follows that the critical exponent $\nu$ is gauge independent. Since $\eta$ is gauge independent, it follows that $\beta$ is gauge independent and the order parameter $\Phi$ has a true physical meaning.

IV. CONCLUSIONS

In this paper we have shown that the critical exponent $\beta$ of the superconducting order parameter is gauge independent or, stating more correctly, that it must be evaluated in the Landau gauge $a = 0$. We may wonder if it is not a wasting of time to prove a result about something unaccessible experimentally. The point is that it is not sure that $\Phi$ cannot be measured and we hope that the discussion in this paper could stimulate some experimental effort in this sense. Moreover, we have shown that transversality (the Landau gauge) is an intrinsic physical feature of the $d = 3$ GL model, a property not shared by the $d = 4$ model. In $d = 4$ (the case of interest in particle physics) the Landau gauge is used due to its computational simplicity \[22\]. In contrast, the Landau gauge is the only physically meaningful gauge in $d = 3$ \[23\].
REFERENCES

[1] T. Kennedy and C. King, Phys. Rev. Lett. 55, 776 (1985); Commun. Math. Phys. 104, 327 (1986); C. Borgs and F. Nill, *ibidem*, page 349.
[2] F. S. Nogueira, Europhys. Lett. 45, 612 (1999).
[3] B. I. Halperin, T. C. Lubensky and S.-K. Ma, Phys. Rev. Lett. 32, 292 (1974); J.-H. Chen, T. C. Lubensky and D. R. Nelson, Phys. Rev. B 17, 4274 (1978).
[4] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. 47, 1556 (1981).
[5] R. Folk and Y. Holovatch, J. Phys. A 29, 3409 (1996).
[6] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, 2nd edition (Oxford, 1993).
[7] I. F. Herbut, J. Phys. A 30, 423 (1997).
[8] A. Kovner, P. Kurzepa and B. Rosenstein, Mod. Phys. Lett. A 8, 1343 (1993).
[9] M. Kiometzis, H. Kleinert and A. M. J. Schakel, Phys. Rev. Lett. 73, 1975 (1994).
[10] M. Kiometzis, H. Kleinert and A. M. J. Schakel, Fortschr. Phys. 43, 697 (1995) and references therein.
[11] Z. Tesanovic, Phys. Rev. B 59, 6449 (1999).
[12] C. de Calan and F. S. Nogueira, Phys. Rev. B (to be published).
[13] I. F. Herbut and Z. Tesanović, Phys. Rev. Lett. 76, 4588 (1996); I. D. Lawrie, Phys. Rev. Lett., 78, 979 (1997); I. F. Herbut and Z. Tesanović, Phys. Rev. Lett. 78, 980 (1997).
[14] C. de Calan, A. P. C. Malbouisson, F. S. Nogueira and N. F. Svaiter, Phys. Rev. B 59, 554 (1999).
[15] S. Kamal, D. A. Bonn, N. Goldenfeld, P. J. Hirschfeld, R. Liang and W. N. Hardy, Phys. Rev. Lett. 73, 1845 (1994); S. Kamal, R. Liang, A. Hosseini, D. A. Bonn and W. N. Hardy, Phys. Rev. B 58, R8933 (1998).
[16] D. S. Fisher, M. P. A. Fisher and D. A. Huse, Phys. Rev. B 43, 130 (1991).
[17] P. Olsson and S. Teitel, Phys. Rev. Lett. 80, 1964 (1998).
[18] B. D. Josephson, Phys. Lett. 21, 608 (1966).
[19] It is unnecessary for computational purposes; however, an analysis of the gauge dependence to all orders still need the mass $M$ in order to avoid infrared divergences in the WT identities.
[20] J. C. Collins, Renormalization (Cambridge, 1984).
[21] A. M. J. Schakel, cond-mat/9805152 (unpublished).
[22] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
[23] For $d = 4$ the gauge independence of the S-matrix elements and particle mass ratios were established to all orders for the massless scalar QED [24]. This analysis has been done in the theoretical framework of radiatively induced symmetry breaking [22] which describes a first order order phase transition. This type of phase transition can be easily obtained in $d = 3$ by means of a gauge field fluctuation-corrected mean-field theory [3] and is consistent with the weak first-order transition scenario. The same mean-field calculation applies in $d = 4$ [25] and reproduces the Coleman-Weinberg effective potential [22]. Unfortunately this phase transition scenario is incompatible with a type II regime of superconductors and the analysis of the gauge dependence following from it is not appropriate in describing thermal fluctuations associated to a second order phase transition.
[24] J. Iliopoulos and N. Papanicolau, Nucl. Phys. B 105, 77 (1976).
[25] A. P. C. Malbouisson, F. S. Nogueira and N. F. Svaiter, Mod. Phys. Lett. B 11, 749 (1996).
FIG. 1. Flow diagram in the $fa$-plane.