Remarks on the Bojowald–Paily paper

Deformed general relativity

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Abstract

A couple of technicalities in the paper M. Bojowald, G. Paily, Phys. Rev. D 87, 044044 (2013) are discussed. The explicit formula given there for the function entering the modified hypersurface-deformed algebra, which presumably originates in loop quantum gravity, seems to be oversimplified, and the embedding of deformed special relativity in deformed general relativity proposed there for spherically symmetric models raises some questions as well.

An interesting problem in quantum gravity is to determine how special relativity is deformed due to quantum fluctuations of spacetime metric. Since deformations of special relativity are restricted by symmetry requirements, predictions obtained for them from a particular candidate for quantum theory of gravitation can provide a test of its viability, or distinguish between different approaches within it. Embedding deformed special relativity in deformed general relativity within the framework of loop quantum gravity (LQG) was examined in the paper by Bojowald and Paily [1]. The authors considered a deformation of hypersurface-deformed algebra encoded in a single function $\beta$, coming either from inverse triad corrections [2, 3] or from holonomy corrections [4, 5], and obtained a Poincaré-type algebra in the linear limit (the limit in which lapse and shift functions depend linearly on spacetime coordinates). The resulting algebra turned out to be incompatible with the much-studied $\kappa$-Poincaré algebra, however, as was shown later [6], this could be cured by considering a more general representation of generators of boosts and translations. The deformation of general relativity by means of the function $\beta$ was used also in other contexts, for example, when studying possible signature change at high densities [7, 8, 9].

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Here we discuss some technical points in section II.C of [1], where the authors consider linear limit of the deformed theory applied to spherically symmetric models.

Function $\beta$. In the modified Poisson bracket between Hamiltonian constraints there appears a function $\beta$, which is, according to [1], in spherically symmetric models given by the function $F(K_\phi)$ replacing $K_\phi$ (angular component of extrinsic curvature) in the Hamiltonian. The expression for $\beta$ is

$$\beta = \frac{1}{2} \frac{d^2 F^2}{dK_\phi^2}.$$  \hspace{1cm} (1)

For example, if we chose $F = \sin(\delta K_\phi)/\delta \equiv F_\delta$, we obtain $\beta = \cos(2\delta K_\phi)$. Equation (1) is surely true as long as the function $F$ does not depend on $E^x$, $E^\phi$ (components of densitized triad). However, the main result of [1], the formula for the deformed Poisson bracket $\{B_x, P_\phi\}$, rests on the assumption that the parameter $\delta$ in $F_\delta$ does depend on $E^x$; namely, that it has the form $\delta \propto |E^x|^{-1/2}$. Let us check whether equation (1), with total derivative replaced by partial one, holds also for such function. For that purpose, we will use a simplified version of the theory explained in Appendix A.3 in [4].

Hamiltonian constraint in a spherically symmetric model is

$$H[N] = -\frac{2}{G} \int N|E^x|^{1/2} \left[ 2K_x K_\phi + \frac{E^\phi}{|E^x|} K_\phi^2 + 2\Gamma_\phi + \frac{E^\phi}{|E^x|} (1 - \Gamma_\phi^2) \right] dx,$$  \hspace{1cm} (2)

where $K_x$ is radial component of extrinsic curvature and $\Gamma_\phi$ is angular component of spin connection, $\Gamma_\phi = -E^x/(2E^\phi)$. (All quantities depend on radial coordinate $x$ only.) By using Poisson brackets between components of extrinsic curvature and densitized triad,

$$\{K_x(x), E^\phi(y)\} = \{K_\phi(x), 2E^\phi(y)\} = 2G\delta(x-y),$$

we find that the Poisson bracket between Hamiltonian constraints can be written as

$$\{H[M], H[N]\} = \frac{1}{G} \int (MN' - NM') \left[ \frac{|E^x|}{E^\phi} (K_\phi' + \Gamma_\phi K_\phi) \right] dx. \hspace{1cm} (3)$$

On the right hand side there appears the diffeomorphism constraint $D[\vec{N}]$, with the shift vector $\vec{N}$ given in terms of lapse functions $M, N$ as $\vec{N} = N^x \partial_x, \vec{N}_x = MN' - NM'$. Now, put instead of $K_\phi$ into the first and second term in $H[N]$ some functions $F_2$ and $F_1$, both depending on $K_\phi$, $E^x$ and $E^\phi$. Thus, replace

$$2K_x K_\phi + \frac{E^\phi}{|E^x|} K_\phi^2 \rightarrow 2K_x F_2 + \frac{E^\phi}{|E^x|} F_1^2.$$  

This yields Poisson bracket

$$\{H[M], H[N]\} = \frac{1}{G} \int (MN' - NM') \left[ \frac{|E^x|}{E^\phi} \left( F_2' + \Gamma_\phi K_\phi \frac{\partial F_2}{\partial K_\phi} \right) + \Gamma_\phi \left( \frac{1}{2} \frac{\partial F_1^2}{\partial K_\phi} - F_2 \right) \right] dx. \hspace{1cm} (4)$$

We wish that the right hand side can be written as $D[\beta \vec{N}]$ with some function $\beta$ depending on $K_\phi$, $E^x$ and $E^\phi$. Clearly, in case both $F_1$ and $F_2$ depend only on $K_\phi$, we achieve this by putting

$$F_2 = \frac{1}{2} \frac{dE_1^2}{dK_\phi}, \hspace{1cm} (5)$$
since then the last two terms in the square brackets cancel and the right hand side acquires the desired form with
\[ \beta = \frac{dF_2}{dK_\phi}. \]  
(6)

After inserting here for \( F_2 \), we obtain equation (1) with \( F = F_1 \).

Suppose now that the functions \( F_1, F_2 \) depend also on \( E^x, E^\phi \). Then, in order to achieve the desired form of the Poisson bracket, two requirements must be satisfied: first, the function \( F_2 \) must not depend on \( E^\phi \) (and, as a result, the function \( F_2^2 \) may contain \( E^\phi \) only additively); and second, equation determining \( F_2 \) in terms of \( F_1 \) must be modified to
\[ F_2 + 2E^x \frac{\partial F_2}{\partial E^x} = \frac{1}{2} \frac{\partial F_2^2}{\partial K_\phi}. \]  
(7)

For the function \( \beta \) we have again equation (6), we just have to replace the total derivative by the partial one,
\[ \beta = \frac{\partial F_2}{\partial K_\phi}. \]  
(8)

The extra term on the left hand side of (7) makes the expression of \( \beta \) in terms of \( F_1 \) more complicated: instead of equation (1) with \( F = F_1 \) we have
\[ \beta = |E^x|^{-1/2} \int_0^{|E^x|^{1/2}} \frac{1}{2} \frac{\partial^2 F_2^2}{\partial K_\phi^2} d(|E^x|^{1/2}). \]  
(9)

where \( \bar{F}_1 = F_1(K_\phi, \bar{E}^x) \). (We have chosen the limits of the integral so that \( \beta \to 1 \) in the limit \( |E^x| \to \infty \), when presumably \( F_1 \to K_\phi \).) In particular, if \( F_1 = F_3 \) with \( \delta \propto |E^x|^{-1/2} \), the expression for \( \beta \) is, rather that \( \beta = \cos u \),
\[ \beta = \int_0^1 \cos(u\xi^{-1})d\xi, \]  
(10)

where \( u = 2\delta K_\phi \). This can be written as \( \beta = \cos u - (\pi/2)|u| + \text{Si}(u)u \), where \( \text{Si} \) is sine integral, \( \text{Si}(u) = \int_0^u \frac{\sin u}{u} du \). The asymptotics of \( \beta \) are \( \beta = 1 - (\pi/2)|u| + (1/2)u^2 \) for \( |u| \ll 1 \) and \( \beta = -\sin u/u \) for \( |u| \gg 1 \); thus, \( \beta \) falls linearly rather than quadratically if \( u \) is close to 0, and undergoes damped oscillations rather than keeping its amplitude constant if \( u \) is far from 0.

**Radial momentum.** According to [1], the momentum stored in the region \( \Sigma \) of the hypersurface of constant time, projected onto a given vector \( v^a \), is
\[ P = 2 \int_{\partial\Sigma} v_a (p^{ab} \rho_b - \bar{p}^{ab} \bar{\rho}_b) d^2 z, \]  
(11)

where \( p^{ab} \) is the momentum canonically conjugated with \( h_{ab} \) (metric tensor on \( \Sigma \)), \( \nu^a \) is normal to \( \partial\Sigma \), \( \bar{p}^{ab} \) and \( \bar{\rho}^a \) is momentum and normal computed for reference metric, and \( (z^1, z^2) \) are coordinates on
∂Σ. For a spherically symmetric metric this yields, according to [1], the formula for radial component of momentum

\[ P_x = \frac{8\pi h_{xx} p_{xx}}{E^\phi |E|^1/2}. \]  

(12)

After inserting here \( h_{xx} = (E^\phi)^2/|E^x| \) and \( 16\pi G p_{xx} = h^{1/2} (K^{xx} - h^{xx} K) = -2h^{1/2} h^{xx} K_{\phi} = -2h^{1/2} \times h^{xx} (h^{\theta \theta})^{1/2} K_{\phi} = -2(|E^x|/E^\phi) K_{\phi} \) (the function \( p_{xx} \) in equation (12) is supposed to be stripped of the factor \( \sin \theta \), therefore we suppressed the factor also in the function \( h^{1/2} \)), we can write

\[ P_x = -\frac{1}{G} \frac{K_{\phi}}{|E^x|^{1/2}}. \]  

(13)

This is used later – albeit in combination with expression (1) for \( \beta \), which is apparently flawed – in the derivation of the deformed Poisson bracket \( \{ B_x, P_0 \} \).

The problem with the expression for \( P_x \) is the denominator of the fraction on the right hand side of equation (12). It is unclear where this denominator, which equals \( h^{1/2} \), comes from. To see that, let us look closer at the definition of \( P \).

The object \( v_a \) appearing in [11] is identified in [1] by an expression for \( v^a \) which clearly represents a vector, hence \( v_a \) is a co-vector; and the object \( r_a \) is called “co-normal” in [1], which suggests that it is a co-vector too. However, if this was the case, the expression for \( P \) could not be correct since it would not be invariant with respect to diffeomorphisms on \( \Sigma \). In [10], a different expression for \( P \) is given (although it is not called so) in which \( v^a \) is identified with shift vector, \( v^a = N^a \). The expression contains an additional factor \( 1/N \) in the integral, with \( N \) previously identified as lapse function,

\[ P = 2 \int_{\partial \Sigma} N_a (p^{ab} r_b - \bar{p}^{ab} r_b) \frac{d^2 z}{N}. \]

This cannot be correct either, since \( N \) is scalar and its presence in the integral does not change transformation properties of \( P \). In order to obtain a meaningful expression, we must replace \( N \to n = dl/dx \), where \( x \) is transversal coordinate (a coordinate varying in the direction orthogonal to \( \partial \Sigma \)) and \( l \) is distance measured along \( x \). Clearly, \( n \) plays the same role in \( 2 + 1 \) decomposition as \( N \) plays in \( 3 + 1 \) decomposition. Thus, instead of (11) we have

\[ P = 2 \int_{\partial \Sigma} v_a (p^{ab} r_b - \bar{p}^{ab} r_b) \frac{d^2 z}{n}. \]  

(14)

Note that one can easily check that this expression is right by computing \( P \) as a surface term in diffeomorphism constraint.

In a spherically symmetric metric, in which we chose \( \Sigma \) as a ball with the center in the center of symmetry, we have \( r_a = h^{1/2} \delta_a^x \) and \( n = h^{1/2} \), hence \( r_a/n = \delta_a^x \); furthermore, the vector \( v^a \) is identified as \( v^a = (\partial/\partial x)^a = \delta_a^x \) in [1], hence \( v_a = h_{ax} = h_{xx} \delta_a^x \). By inserting this into equation (14) and using
\( \hat{p}^{xx} = 0 \), we arrive at an expression for \( P_x \) different from \((13)\),

\[
P_x = 8\pi h_{xx} p^{xx} = -\frac{1}{G} E^\phi K_\phi.
\] (15)

If the previous considerations are true and expression \((13)\) for \( P_x \) is indeed wrong, using it seems to be more serious problem of the theory than just using an oversimplified expression for the function \( \beta \). For the reasoning in \([1]\) it is crucial that the expression for \( P_x \) contains the same combination of the variables \( K_\phi, E^x \) and \( E^\phi \) as appears in \( \beta \); however, as we have seen, whatever the combination is, it cannot include the variable \( E^\phi \). This remains true also for more general modification of the Hamiltonian constraint than considered here, in which one uses, in addition to the functions \( F_1 \) and \( F_2 \), four more functions of the variables \( E^x \) and \( E^\phi \) denoted by \( \alpha, \bar{\alpha}, \alpha_\Gamma \) and \( \bar{\alpha}_\Gamma \) \([4]\). Thus, if the expression \((13)\) has indeed to be replaced by the expression \((15)\), the deformed bracket \( \{B_x, P_0\} \) cannot be given by the simple formula from \([1]\). Perhaps it can be still computed within the framework of \([1]\), but its exact form remains to be found.

**Undeformed bracket \( \{B_x, P_0\} \).** The main point of \([1]\) is that we can test LQG by obtaining a hint from it about how the bracket \( \{B_x, P_0\} \) is deformed due to quantum gravity effects. In order to do so we need to know how the undeformed bracket looks like. According to \([1]\), it holds

\[
\{B_x, P_0\} = P_x,
\] (16)

see equation (18) in section II.C in \([1]\), in which one has to put \( \lambda = 0 \). Thus, the equation relating, via Poisson bracket, radial components of the boost generator \( B \) and the momentum \( P \), is according to \([1]\) the same as the equation relating their Cartesian components.

Equation \((16)\) refers to special relativity; thus, the quantities \( P_0, P_x \) and \( B_x \) appearing there are presumably integral characteristics of a classical system in flat spacetime. However, if this is the case, we must add an extra term proportional to \( \dot{P}_x \) to the right hand side of equation \((16)\),

\[
\{B_x, P_0\} = P_x - \dot{P}_x t.
\] (17)

This is most easily seen if we express \( B_x \) in terms of \( P_0 \) and \( P_x \) and use the fact that for any given dynamical quantity \( f \) it holds \( \{f, P_0\} = -\{f, H\} = -\dot{f} \). Denote the energy density by \( \epsilon \) and the total energy of the system by \( E \), \( E = \int \epsilon d^3x \). Radial component of the boost generator can be written as \( B_x = X P_0 - x_0 P_x = -EX + P_x t \), where \( X \) is the “radial center-of-mass coordinate”,

\[
X = \frac{1}{E} \int x \epsilon d^3x.
\]

With this expression for \( B_x \), the bracket becomes

\[
\{B_x, P_0\} = -EX + \{P_x, P_0\} t = E \dot{X} - \dot{P}_x t.
\]
Furthermore, the energy density satisfies the continuity equation \( \dot{\varepsilon} + \nabla \cdot p = 0 \), where \( p \) is the momentum density, therefore the first term on the right hand side can be written as

\[
E \dot{X} = \int x i d^3 x = - \int x \nabla \cdot p d^3 x = \int \nabla x \cdot pd^3 x = \int p x d^3 x = P_x.
\]

(We have skipped the surface integral since we assume that the momentum density falls off rapidly enough as \( x \) goes to infinity.) As a result, we find that the bracket is given by equation (17).

The time derivative of the Cartesian components of momentum is of course zero, however, for other components such as the radial one this is no longer true. For example, for a system of nonrelativistic particles with radius vectors \( x_\alpha \) and interaction potential \( v(x_\alpha) \) we have

\[
\dot{P}_x = \sum \frac{\partial v}{\partial x_\alpha} \frac{x_\alpha - n_\alpha \cdot n_\beta x_\beta}{x_{\alpha\beta}},
\]

where \( n_\alpha = x_\alpha / x_\alpha \) and \( x_{\alpha\beta} = |x_\alpha - x_\beta| \); and for scalar field \( \phi \) with potential \( V(\phi) \) we have

\[
\dot{P}_x = \int \frac{2l}{x} d^3 x, \quad l = \frac{1}{2} \pi^2 - \frac{1}{2} (\partial_x \phi)^2 - V,
\]

where \( \pi \) is the momentum canonically conjugated with \( \phi \), \( \pi = \dot{\phi} \). (The function \( l \) coincides with Lagrangian density in case the field is spherically symmetric). As we can see, the second term in (17) cannot be ignored, so that a question arises whether LQG can make any predictions concerning its deformation, as it presumably does with respect to the first term.

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