COMPACTLY GENERATED TENSOR T-STRUCTURES ON THE DERIVED CATEGORY OF A NOETHERIAN SCHEME

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Abstract. We introduce a tensor compatibility condition for t-structures. For any Noetherian scheme $X$, we prove that there is a one-to-one correspondence between the set of Thomason filtrations and the set of aisles of compactly generated tensor compatible t-structures on the derived category of $X$. This generalizes the earlier classification of compactly generated t-structures for commutative rings to schemes. Hrbek and Nakamura have reformulated the famous telescope conjecture for t-structures. As an application of our main theorem, we prove that a tensor version of the conjecture is true for separated Noetherian schemes.

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Introduction

The classification of subcategories in terms of subsets of the ambient space, in the setting of derived categories, first appeared in the work of Hopkins. Inspired by the analogous result in the finite stable homotopy category [HS98, Theorem 7], Hopkins obtained that for a commutative ring $R$ there is a one-to-one correspondence between thick subcategories of perfect complexes on $R$ and the specialization closed subsets of Spec$R$ [Hop87, Theorem 11]. Neeman intrigued by this parallel between the stable homotopy category and the derived category conducted further investigation. He obtained the classification of localizing subcategories of the unbounded derived category

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$\mathbf{D}(R)$, which on restricting to the perfect complexes $\text{Perf}(R)$ provides Hopkins’ theorem \cite[Theorem 1.5]{Nee92}. However, Neeman showed all this is true for Noetherian rings and that extra care is needed for arbitrary rings; see \cite[Example 4.1]{Nee92}.

Thomason took a step further and generalized the results of Hopkins and Neeman to schemes. For similar classification to hold in the case of schemes he observed that a tensor compatible condition on thick subcategories of $\text{Perf}(X)$ has to be considered; a condition which all thick subcategories satisfy in the affine situation. His second important observation was that for the non-Noetherian case specialization closed subsets were to be replaced by subsets stronger than specialization closed; such subsets are now being called Thomason subsets after him; see Definition 3.16. Thus he obtained for a quasi-compact quasi-separated scheme $X$, a bijective correspondence between thick $\otimes$-ideals of $\text{Perf}(X)$ and Thomason subsets of $X$ \cite[Theorem 3.15]{Tho97}, thereby, closing the gap pointed out by Neeman and establishing the correct bijective correspondence for arbitrary commutative rings; see his commentaries \cite[History 3.17]{Tho97}. Similar classifications in the situation of Noetherian formal schemes \cite[Theorem 4.12]{ATJLSS04} and for Noetherian graded rings \cite[Theorem 5.8]{DS13} have been achieved.

All these works consider suitable triangulated subcategories in the derived category or perfect complexes. Another important class of subcategories, not necessarily triangulated, is the class of t-structures introduced in \cite{BBD82} by Beilinson et al. in their study of perverse sheaves. Stanley showed the class of t-structures on $\mathbf{D}(\mathbb{Z})$ is not a set \cite[Corollary 8.4]{Sta10} and therefore it is not feasible to classify all t-structures of the unbounded derived category in terms of subsets of $\text{Spec} R$. However, a subclass of t-structures on $\mathbf{D}(R)$, precisely the compactly generated ones, has been classified in terms of filtrations of specialization closed subsets by Alonso et al. \cite[Theorem 3.11]{ATJLS10}, for $R$ Noetherian. Hrbek has extended this classification to arbitrary commutative rings, \cite[Theorem 5.1]{Hrb20}. Although he obtained a one-to-one bijection between the set of compactly generated t-structures and the set of Thomason filtrations, unlike the Noetherian case, it is not clear that the aisle of such a t-structure is determined by cohomological supports; see the introduction to section 5 of \cite{Hrb20}.

In this article, we generalize the classification of compactly generated t-structures to Noetherian schemes. First, we introduce the notion of tensor t-structures; an almost similar notion has been studied in \cite[§5]{ATJLSS03} with a different goal in mind. Such t-structures localize well to the open subsets. For a Noetherian scheme $X$ we prove:

**Theorem 0.1.** There is a bijective correspondence between the set of aisles of compactly generated tensor t-structures on $\mathbf{D}_{\text{qc}}(X)$ and the set of Thomason filtrations of $X$. (For more details see Theorem 4.11.)

Though our approach is close to Thomason’s proof of \cite[Theorem 3.15]{Tho97} in spirit, our proof differs in two aspects. Since the aisle of a tensor t-structure is rarely a $\otimes$-ideal, one of the key ingredients of \cite[Theorem Nilpotence Theorem or any naive variation of it, is not helpful in our case. Instead, we use a local global principle to obtain Lemma 4.9 the counterpart of \cite[Lemma 3.14]{Tho97}. Another obstacle was extending perfect complexes on an open subscheme to perfect complexes on the ambient scheme along an aisle. Thomason and Trobaugh in their seminal paper \cite{TT90} give a
criterion for the extension of perfect complexes using $K_0$ groups; see [TT90, Lemma 5.6.2]. In the case of t-structures, one needs to consider the supports of cohomology sheaves component-wise, instead of their union as in the stable case. Thus, we needed to have extensions of perfect complexes with some boundedness assumption. This we achieve after closely inspecting and later modifying various results of [TT90, §5]; see section 3.2.

A Bousfield subcategory of a triangulated category is smashing if the corresponding localization functor preserves coproducts. The telescope conjecture asks if all such smashing subcategories are compactly generated. The question was originally asked by Ravenel for the stable homotopy category of spectra [Rav84], in this case, the answer remains elusive. However, in the case of algebraic triangulated categories, there are many positive results. The first one is due to Neeman [Nee92, Corollary 3.4]. For a detailed discussion on the history of the conjecture and various positive results in this direction see the introduction of [Ant14]. Since Bousfield subcategories correspond to stable t-structures, it is natural to seek a reformulation of the conjecture which encompasses t-structures. Hrbek and Nakamura have formulated the telescope conjecture for homotopically smashing t-structures in the language of derivators; see [HN21, Question A.7]. They proved for a commutative Noetherian ring $R$ the telescope conjecture for homotopically smashing t-structures is true, more precisely, any homotopically smashing t-structure on $D(R)$ is compactly generated [HN21, Theorem 1.1]. As an application of Theorem 4.11, we obtain, for a separated Noetherian scheme $X$:

**Theorem 0.2.** Any homotopically smashing tensor t-structure on $D(Qcoh(X))$ is compactly generated. (For more details see §5)

This provides yet another proof of the tensor telescope conjecture in this case; see Remark 5.10 and also [BF11, Corollary 6.8].

In section 1 we discuss the basic facts about t-structures, we recall that a t-structure is completely determined by its aisle. The notion of compactly generated t-structure appears in various guises. Though all these notions are known to be equivalent, a proof seems to be missing in the literature. Hence we give a complete treatment of such t-structures in this section. In section 2 we introduce tensor t-structures and $\otimes$-aisles of a tensor triangulated category and collect some basic facts about them. We show the relation between the new notion of $\otimes$-aisle and the earlier notion of $\otimes$-ideal; see Proposition 2.9. In the third section, for a Noetherian scheme $X$, we prove that the associated subcategory of a Thomason filtration is a compactly generated $\otimes$-aisle of $D_{qc}(X)$; see Theorem 3.20. In section 4 we prove the main classification result. In the last section, we discuss the telescope conjecture for homotopically smashing t-structures and prove that a tensor version of the conjecture is true for separated Noetherian schemes.

1. Preliminaries

**Convention.** We always assume a triangulated category $\mathcal{T}$ has all small coproducts unless otherwise stated.
1.1. Basics on $t$-structures. Let $\mathcal{T}$ be a triangulated category. The notion of $t$-structure is introduced by Beilinson, Bernstein, and Deligne in [BBD82, Définition 1.3.1].

**Definition 1.1.** A $t$-structure on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying the following conditions:

(t1) For $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}[1]$, $\text{Hom}(A, B) = 0$;
(t2) $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}[1] \subset \mathcal{T}^{\geq 0}$;
(t3) For any $T \in \mathcal{T}$, there is a distinguished triangle

$$A \to T \to B \to A[1]$$

such that $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}[1]$.

The triangle in (t3) is unique and we call it the $t$-decomposition triangle of $T$ for $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. The full subcategory $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ of $\mathcal{T}$ is called the heart of the $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. It is an abelian category and there is a natural cohomological functor $H^0 : \mathcal{T} \to \mathcal{H}$.

In [KV88] Keller and Vossieck observed that the pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is completely determined by the subcategory $\mathcal{T}^{\leq 0}$. They characterized such subcategories and termed them as aisles of $\mathcal{T}$. We recall the precise definition of aisle [KV88, 1.1].

**Definition 1.2.** A full subcategory $\mathcal{U}$ is an aisle of $\mathcal{T}$ if it satisfies the following conditions:

(a1) closed under positive shifts, i.e. $\mathcal{U}[1] \subset \mathcal{U}$;
(a2) closed under extensions, i.e. for each distinguished triangle

$$A \to B \to C \to A[1]$$

in $\mathcal{T}$, if $A$ and $C$ are in $\mathcal{U}$ then $B$ is in $\mathcal{U}$;
(a3) the inclusion $\mathcal{U} \to \mathcal{T}$ admits a right adjoint denoted by $\tau_{\mathcal{U}}^{\leq}$.

The functor $\tau_{\mathcal{U}}^{\leq}$ is called the truncation functor associated with $\mathcal{U}$.

For any subcategory $\mathcal{U}$ of $\mathcal{T}$, we denote $\mathcal{U}^{\perp}$ to be the full subcategory consisting of objects $B \in \mathcal{T}$ such that $\text{Hom}(A, B) = 0$ for all $A \in \mathcal{U}$. Analogously we define $\mathcal{U}^{\perp}$ to be the full subcategory of objects $B \in \mathcal{T}$ such that $\text{Hom}(B, A) = 0$ for all $A \in \mathcal{U}$.

The assignments, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \mapsto \mathcal{T}^{\leq 0}$ and $\mathcal{U} \mapsto (\mathcal{U}, \mathcal{U}^{\perp}[1])$ give a mutually inverse bijective correspondence between the aisles of $\mathcal{T}$ and the $t$-structures on $\mathcal{T}$; see [KV88, §1]. The subcategory $(\mathcal{T}^{\leq 0})^{\perp} = \mathcal{T}^{\geq 0}[1]$ is called the coaisle of the $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.

We define a few notions weaker than aisle; these have been introduced in [ATJLSS03, §1].

**Definition 1.3.** A full subcategory $\mathcal{U}$ is a preaisle of $\mathcal{T}$ if $\mathcal{U}$ is closed under positive shifts and extensions, that is, $\mathcal{U}$ satisfies (a1) and (a2). We call a preaisle stable if it is closed under negative shifts. A preaisle is a thick preaisle if it is closed under taking direct summands in $\mathcal{T}$. A preaisle that is closed under taking small coproducts in $\mathcal{T}$ is called a cocomplete preaisle.
A cocomplete preaisle is thick by the Eilenberg swindle argument; see [ATJLSS03, Corollary 1.4.]. By [ATJLSS03, Lemma 1.3.] aisles are cocomplete preaisles.

1.2. **Compactly generated t-structures.** A triangulated subcategory $C$ of $\mathcal{T}$ is thick if it is closed under direct summands. A thick subcategory is localizing if it is closed under taking coproducts in $\mathcal{T}$. If $S \subseteq \mathcal{T}$ is a class of objects, then $\langle S \rangle$ denotes the smallest localizing subcategory of $\mathcal{T}$ containing $S$; see for instance [Nee01, Definition 1.12]. Following similar customs, we denote the smallest cocomplete preaisle containing $S$ by $\langle S \rangle^{\leq 0}$ and call it the cocomplete preaisle generated by $S$.

An object $C \in \mathcal{T}$ is compact if for any collection of objects $\{A_i\}$ in $\mathcal{T}$ the natural map

$$\prod_i \text{Hom}(C, A_i) \to \text{Hom}(C, \prod_i A_i)$$

is an isomorphism. We denote the full subcategory of compact objects of $\mathcal{T}$ by $\mathcal{T}^c$; the subcategory $\mathcal{T}^c$ is a thick subcategory of $\mathcal{T}$. In contrast to our convention $\mathcal{T}^c$ need not have coproducts.

**Definition 1.4.** A preaisle $\mathcal{U}$ is compactly generated if $\mathcal{U} = \langle S \rangle^{\leq 0}$ for a set of compact objects $S$. We say an aisle is compactly generated if it is a compactly generated preaisle. A t-structure is compactly generated if the aisle of the t-structure is compactly generated.

We state a result of Keller and Nicolás.

**Proposition 1.5 ( [KN13, Theorem A.7]).** Let $S$ be a set of compact objects of $\mathcal{T}$. Then,

1. $\langle S \rangle^{\leq 0}$ is an aisle of $\mathcal{T}$.
2. Every object $A$ of $\langle S \rangle^{\leq 0}$ fits in a triangle

$$\prod_{i \geq 0} A_i \to A \to \prod_{i \geq 0} A_i[1] \to \prod_{i \geq 0} A_i[1]$$

where $A_i$ is an $i$-fold extension of small coproducts of non negative shifts of objects of $S$.

Now we bring another auxiliary notion from [ATJLS10, 1.1, page. 316] which will be helpful in characterizing compactly generated preaisles. Let $\mathcal{U}$ be a preaisle of $\mathcal{T}$. We say $\mathcal{U}$ is a total preaisle of $\mathcal{T}$ if $\perp (\mathcal{U}) = \mathcal{U}$. Next, we include some results which are probably well known but have not appeared explicitly in the literature.

**Lemma 1.6.** Aisles are total preaisles.

**Proof.** [ATJLSS03 Proposition 1.1 (i)].

Let $S$ be a class of objects of $\mathcal{T}$. Consider the collection of objects $\{S[n] \mid S \in S$ and $n \in \mathbb{N} \cup \{0\}\}$ we denote it by $S[\mathbb{N}]$.

**Lemma 1.7.** The subcategory $\perp (S[\mathbb{N}] \perp)$ is a cocomplete total preaisle of $\mathcal{T}$.
Proof. For any collection of objects $\mathcal{Y} \subset \mathcal{T}$, we have $\mathcal{Y} \subset ((\perp \mathcal{Y})^\perp)$, replacing $\mathcal{Y}$ by $(\mathcal{Y}^\perp)$ we get $(\mathcal{Y}^\perp) \subset ((\perp (\mathcal{Y}^\perp))^\perp)$. Applying left $\perp$ we get $((\perp (\mathcal{Y}^\perp))^\perp) \subset (\perp (\mathcal{Y}^\perp))$. Again for any collection $\mathcal{Y}$ we have $\mathcal{Y} \subset (\perp (\mathcal{Y}^\perp))$, replacing $\mathcal{Y}$ by $(\perp (\mathcal{Y}^\perp))$ we get $(\perp (\mathcal{Y}^\perp)) \subset (\perp (\mathcal{Y}^\perp))^\perp$. This proves $(\perp (\mathcal{Y}^\perp)) = ((\perp (\mathcal{Y}^\perp))^\perp)$.

Replacing $\mathcal{Y}$ by the class of objects $\mathcal{S}[\mathbb{N}]$ we get $(\perp (\mathcal{S}[\mathbb{N}])) = (\perp (\mathcal{S}[\mathbb{N}]))$. Also note that for a class of objects $\mathcal{Y}$ of $\mathcal{T}$, if $\mathcal{Y}[-1] \subset \mathcal{Y}$ then $\perp \mathcal{Y}$ is a cocomplete preaisle. Since $\mathcal{S}[\mathbb{N}][-1] \subset \mathcal{S}[\mathbb{N}]$ therefore $\perp (\mathcal{S}[\mathbb{N}])$ is a cocomplete total preaisle of $\mathcal{T}$.

Lemma 1.8. The subcategory $\perp (\mathcal{S}[\mathbb{N}])$ is the smallest total preaisle containing $\mathcal{S}$.

Proof. By Lemma 1.7, the subcategory $\perp (\mathcal{S}[\mathbb{N}])$ is a total preaisle. Suppose $\mathcal{U}$ is a total preaisle containing $\mathcal{S}$. As $\mathcal{U}$ is closed under positive shifts $\mathcal{S}[\mathbb{N}] \subset \mathcal{U}$, hence $\perp (\mathcal{S}[\mathbb{N}]) \subset \perp (\mathcal{U}) = \mathcal{U}$. This proves the claim.

Lemma 1.9. Let $\mathcal{S}$ be a set of compact objects of $\mathcal{T}$. Then,

$$\langle \mathcal{S} \rangle^{\leq 0} = \perp (\mathcal{S}[\mathbb{N}]).$$

Proof. Recall that $\langle \mathcal{S} \rangle^{\leq 0}$ is the smallest cocomplete preaisle containing $\mathcal{S}$. Now $\perp (\mathcal{S}[\mathbb{N}])$ is a cocomplete preaisle by Lemma 1.7, hence $\langle \mathcal{S} \rangle^{\leq 0} \subset \perp (\mathcal{S}[\mathbb{N}])$. We know $\langle \mathcal{S} \rangle^{\leq 0}$ is an aisle by Proposition 1.5. Since aisles are total preaisles by Lemma 1.6, $\langle \mathcal{S} \rangle^{\leq 0}$ is a total preaisle containing $\mathcal{S}$. Lemma 1.8 says $\perp (\mathcal{S}[\mathbb{N}])$ is the smallest total preaisle containing $\mathcal{S}$ therefore we get the reverse inclusion $\perp (\mathcal{S}[\mathbb{N}]) \subset \langle \mathcal{S} \rangle^{\leq 0}$. □

Proposition 1.10. Let $\mathcal{U}$ be a cocomplete preaisle of $\mathcal{T}$. The following are equivalent:

1. The cocomplete preaisle $\mathcal{U}$ is compactly generated.
2. There is a set of compact objects $\mathcal{S}$ such that $\mathcal{U} = \langle \mathcal{S} \rangle^{\leq 0}$.
3. There is a set of compact objects $\mathcal{S}$ such that $\mathcal{U} = \perp (\mathcal{S}[\mathbb{N}])$.
4. There is a set of compact objects $\mathcal{S} \subset \mathcal{U}$ such that for any $A \in \mathcal{U}$ if $\text{Hom}(\mathcal{S}, A) = 0$ then $A \cong 0$.
5. The pair $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a compactly generated t-structure.

Proof. (1) $\Leftrightarrow$ (2) follows from Definition 1.4. Lemma 1.9 shows (2) $\Leftrightarrow$ (3). Suppose $A \in \mathcal{U}$ and $\text{Hom}(\mathcal{S}, A) = 0$ for all $S \in \mathcal{S}[\mathbb{N}]$, this means $A \in \mathcal{S}[\mathbb{N}]^{\perp}$. By Lemma [AT1][LSS02, Lemma 3.1] we have $\mathcal{S}[\mathbb{N}]^{\perp} = \mathcal{U}^{\perp}$ hence $A \in \mathcal{U} \cap \mathcal{U}^{\perp}$. So we get $A \cong 0$; this proves (3) $\Rightarrow$ (4).

Next, we prove (4) $\Rightarrow$ (2). Note that $\langle \mathcal{S} \rangle^{\leq 0} \subset \mathcal{U}$ and $\langle \mathcal{S} \rangle^{\leq 0}$ is an aisle by Proposition 1.5. Let $A \in \mathcal{U}$. Consider the t-decomposition triangle of $A$ for $\langle \mathcal{S} \rangle^{\leq 0}$,

$$\tau^\mathcal{S}_A \rightarrow A \rightarrow \tau^\mathcal{S}_A \rightarrow \tau^\mathcal{S}_A [1].$$

As $\mathcal{U}$ is closed under extension $\tau^\mathcal{S}_A$ is in $\mathcal{U}$. Since $\tau^\mathcal{S}_A$ is in $\mathcal{S}[\mathbb{N}]^{\perp}$ by (4) $\tau^\mathcal{S}_A \cong 0$. Therefore the map $\tau^\mathcal{S}_A \rightarrow A$ is an isomorphism and hence $A \in \langle \mathcal{S} \rangle^{\leq 0}$. This proves $\mathcal{U} = \langle \mathcal{S} \rangle^{\leq 0}$.
A compactly generated preaisle $U$ is an aisle by Proposition 1.5. By the bijective correspondence between aisles and t-structures and by Definition 1.4 we get (5) $\iff$ (1).

1.3. Smashing t-structures. A thick subcategory $C$ is Bousfield if the inclusion $C \to T$ admits a right adjoint; see [Kra10, 4.9]. Therefore Bousfield subcategories are precisely the stable aisles. A Bousfield subcategory $C$ is smashing if the subcategory $C^\perp$ is localizing; for other characterizations see [Kra10, Proposition 5.5.1]. In view of this we say an aisle $U$ is smashing if $U^\perp$ is closed under coproducts; also see [SSV17, Definition 7.1]. A t-structure $(U, V[1])$ is called smashing if the coaisle $V$ is closed under taking coproducts.

**Proposition 1.11.** Compactly generated t-structures are smashing t-structures.

**Proof.** Suppose $(U, V[1])$ is a compactly generated t-structure and $U = ^\perp (S[N])$. Let $\{A_i\}$ be a collection of objects in $V$. To show $\coprod_i A_i \in V$ it is enough to show $\Hom(S, \coprod_i A_i) = 0$ for all $S \in S[N]$. Since $S \in S[N]$ is compact we have $\Hom(S, \coprod_i A_i) = \prod_i \Hom(S, A_i)$. As $A_i \in V$ we have $\Hom(S, A_i) = 0$ and this proves the claim. $\square$

2. Tensor t-structures

We recall the definition of tensor triangulated category from [HPS97, Definition A.2.1].

**Definition 2.1.** A tensor triangulated category $(T, \otimes, 1)$ is a triangulated category with a compatible closed symmetric monoidal structure. This means there is a functor $- \otimes -$ : $T \times T \to T$ which is triangulated in both the variables and satisfies certain compatibility conditions. Moreover, for each $B \in T$ the functor $- \otimes B$ has a right adjoint which we denote by $\mathcal{H}om(B, -)$. The functor $\mathcal{H}om(-, -)$ is triangulated in both the variables, and for any $A$, $B$, and $C$ in $T$ we have natural isomorphisms $\Hom(A \otimes B, C) \to \Hom(A, \mathcal{H}om(B, C))$.

Suppose $T$ is given with a preaisle $T^{\leq 0}$ such that $T^{\leq 0} \otimes T^{\leq 0} \subset T^{\leq 0}$ and $1 \in T^{\leq 0}$. We introduce the following definition which is motivated from [ATJLSS03, §5]; a similar situation is also considered in [Nee18, §3].

**Definition 2.2.** A preaisle $U$ of $T$ is a $\otimes$-preaisle (with respect to $T^{\leq 0}$) if $T^{\leq 0} \otimes U \subset U$. An aisle is called a $\otimes$-aisle if it is a $\otimes$-preaisle. A t-structure is a tensor t-structure if the aisle of the t-structure is a $\otimes$-aisle.

**Remark 2.3.** Though we define tensor t-structures in this generality, for the derived categories (example $D_\text{per}(X)$) we always take tensor t-structures with respect to the aisle of the standard t-structure.

**Proposition 2.4.** Let $(U, V[1])$ be a t-structure on $T$. Then the following are equivalent:

1. $(U, V[1])$ is a tensor t-structure.
2. $U$ is a $\otimes$-aisle.
(3) $\mathcal{H}om(T^{\leq 0}, V) \subset V$.

**Proof.** (1)$\Leftrightarrow$(2) is by definition. Let $A \in \mathcal{U}$, $B \in \mathcal{V}$, and $X \in T^{\leq 0}$. Then from the adjunction isomorphism we have

$$\text{Hom}(A \otimes X, B) \cong \text{Hom}(A, \mathcal{H}om(X, B)).$$

This proves (2)$\Leftrightarrow$(3). \hfill\Box

**Lemma 2.5.** Let $T^{\leq 0}$ be generated by a set of objects $K$, that is, $T^{\leq 0} = \langle K \rangle^{\leq 0}$. Then a preaisle $U$ of $T$ is a $\otimes$-preaisle if and only if $K \otimes U \subset U$.

**Proof.** Suppose $K \otimes U \subset U$. We define $B = \{X \in T^{\leq 0} \mid X \otimes U \subset U\}$. If $X \in B$ then $X[1] \in B$, as $U$ is closed under positive shifts. Suppose we have a triangle $X \to Y \to Z \to X[1]$ with $X$ and $Z$ in $B$. Since $U$ is closed under extension, we get $Y \in B$. This proves $B$ is a preaisle. Since $K \subset B$ we get $T^{\leq 0} \subset B$ which proves $U$ is $\otimes$-preaisle. The converse follows easily. \hfill\Box

**Proposition 2.6.** If $T^{\leq 0} = \langle 1 \rangle^{\leq 0}$ then every cocomplete preaisle of $T$ is a $\otimes$-preaisle. In particular, all t-structures are tensor t-structures.

**Proof.** Follows from Lemma 2.5. \hfill\Box

We recall the definitions of $\otimes$-ideal and coideal; see for instance [HPS97, Definition 1.4.3].

**Definition 2.7.** A thick subcategory $C$ of $T$ is a $\otimes$-ideal of $T$ if $T \otimes C \subset C$. We say $C$ is a coideal of $T$ if $\mathcal{H}om(T, C) \subset C$.

Recall that a preaisle is called stable if it is closed under negative shifts. We say a t-structure is stable if the aisle is closed under negative shifts. Note that for a stable t-structure the coaisle is closed under positive shifts.

**Lemma 2.8.** Suppose $\langle T^{\leq 0} \rangle = T$. If a cocomplete $\otimes$-preaisle is stable, then it is a $\otimes$-ideal of $T$.

**Proof.** Let $U$ be a stable cocomplete $\otimes$-preaisle of $T$. We define $B = \{X \in T \mid X \otimes U \subset U\}$. Using similar arguments as in Lemma 2.5 we can observe that $B$ is a localizing subcategory of $T$. Since $B$ contains $T^{\leq 0}$ we get $\langle T^{\leq 0} \rangle = T \subset B$. This proves $U$ is a $\otimes$-ideal. \hfill\Box

**Proposition 2.9.** Let $T$ be a tensor triangulated category with a preaisle $T^{\leq 0}$ such that $\langle T^{\leq 0} \rangle = T$. If $(U, V[1])$ is a stable t-structure on $T$, then the following are equivalent:

1. $(U, V[1])$ is a stable tensor t-structure on $T$.
2. The aisle $U$ is a $\otimes$-ideal of $T$.
3. The coaisle $V$ is a coideal of $T$.

**Proof.** (1)$\Leftrightarrow$(2) follows from Lemma 2.8 (2)$\Leftrightarrow$(3) follows from the adjunction isomorphism - same as the proof of Proposition 2.4. \hfill\Box
Let $\mathcal{T}$ be a tensor triangulated category and $\mathcal{S}$ be a class of objects of $\mathcal{T}$. We denote the smallest cocomplete $\otimes$-preaisle containing $\mathcal{S}$ by $\langle \mathcal{S} \rangle_{\otimes}^{\leq 0}$. The following result is a version of [HR17, Lemma 1.1] in the present context.

**Lemma 2.10.** Let $\mathcal{T}$ and $\mathcal{T}'$ be two tensor triangulated categories with preaisles $\mathcal{T}^{\leq 0}$ and $\mathcal{T}'^{\leq 0}$ respectively. Let $F: \mathcal{T} \to \mathcal{T}'$ be a coproduct preserving tensor triangulated functor. We assume $F$ to be right-t-exact, that is, $F(\mathcal{T}^{\leq 0}) \subset \mathcal{T}'^{\leq 0}$. Then the following hold:

1. If $\mathcal{U}'$ is a cocomplete $\otimes$-preaisle of $\mathcal{T}'$, then the full subcategory $F^{-1}(\mathcal{U}')$ of $\mathcal{T}$ is a cocomplete $\otimes$-preaisle.
2. If $\mathcal{S} \subset \mathcal{T}$ is a class of objects, then $\langle F(\mathcal{S}) \rangle_{\otimes}^{\leq 0} = \langle F(\langle \mathcal{S} \rangle_{\otimes}^{\leq 0}) \rangle_{\otimes}^{\leq 0}$.

**Proof.** Let $\mathcal{U}'$ be a cocomplete preaisle of $\mathcal{T}'$. If $A \in F^{-1}(\mathcal{U}')$, then $F(A) \in \mathcal{U}'$ so $F(A)[1] \in \mathcal{U}'$ and this implies $A[1] \in F^{-1}(\mathcal{U}')$. Given a triangle $A \to B \to C \to A[1]$ in $\mathcal{T}$ with $A$ and $C$ in $F^{-1}(\mathcal{U}')$. We have $F(A) \to F(B) \to F(C) \to F(A)[1]$ in $\mathcal{T}'$ with $F(A)$ and $F(C)$ in $\mathcal{U}'$ hence $F(B) \in \mathcal{U}'$, and we get $B \in F^{-1}(\mathcal{U}')$. This proves $F^{-1}(\mathcal{U}')$ is a preaisle of $\mathcal{T}$. The fact $F$ preserves coproducts will imply that if $\mathcal{U}'$ is cocomplete then $F^{-1}(\mathcal{U}')$ is cocomplete. Let $\mathcal{U}'$ be a cocomplete $\otimes$-preaisle of $\mathcal{T}'$. For any $A \in F^{-1}(\mathcal{U}')$ and $B \in \mathcal{T}^{\leq 0}$, we have $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{U}'$ and hence $A \otimes B \in F^{-1}(\mathcal{U}')$. This shows $F^{-1}(\mathcal{U}')$ is a cocomplete $\otimes$-preaisle.

Now part (2). For any class of objects $\mathcal{S} \subset \mathcal{T}$, we have $\langle F(\mathcal{S}) \rangle_{\otimes}^{\leq 0} \subset \langle F(\langle \mathcal{S} \rangle_{\otimes}^{\leq 0}) \rangle_{\otimes}^{\leq 0}$. By (1), $F^{-1}(\langle \mathcal{S} \rangle_{\otimes}^{\leq 0})$ is a cocomplete $\otimes$-preaisle. Since $\mathcal{S} \subset F^{-1}(\langle \mathcal{S} \rangle_{\otimes}^{\leq 0})$ we have $\langle \mathcal{S} \rangle_{\otimes}^{\leq 0} \subset F^{-1}(\langle \mathcal{S} \rangle_{\otimes}^{\leq 0})$. This implies $\langle \mathcal{S} \rangle_{\otimes}^{\leq 0} \subset \langle F(\mathcal{S}) \rangle_{\otimes}^{\leq 0}$. Therefore, $\langle F(\langle \mathcal{S} \rangle_{\otimes}^{\leq 0}) \rangle_{\otimes}^{\leq 0} \subset \langle F(\mathcal{S}) \rangle_{\otimes}^{\leq 0}$.

Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category in the sense of Balmer [BF11, 1.1]. In particular, $\mathbf{1} \in \mathcal{T}^{\leq 0}$ and the tensor product $\otimes$ restricts to $\mathcal{T}^{c}$. In this case, we can prove a tensor triangulated analogue of Proposition 1.5.

**Proposition 2.11.** Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category with a preaisle $\mathcal{T}^{\leq 0}$. Suppose $\mathcal{T}^{\leq 0}$ is compactly generated and $\mathcal{T}^{c} \cap \mathcal{T}^{\leq 0} = \mathcal{K}$. Let $\mathcal{S}$ be a set of compact objects of $\mathcal{T}$. Then,

1. $\langle \mathcal{S} \rangle_{\otimes}^{\leq 0}$ is an aisle of $\mathcal{T}$.
2. Every object $A$ of $\langle \mathcal{S} \rangle_{\otimes}^{\leq 0}$ fits in a triangle

\[ \prod_{i \geq 0} A_i \to A \to \prod_{i \geq 0} A_i[1] \to \prod_{i \geq 0} A_i[1] \]

where $A_i$ is an $i$-fold extension of small coproducts of non negative shifts of objects of $\mathcal{K} \otimes \mathcal{S}$.

**Proof.** If we show

$\langle \mathcal{K} \otimes \mathcal{S} \rangle_{\leq 0}^{\leq 0} = \langle \mathcal{S} \rangle_{\otimes}^{\leq 0}$

then by Proposition 1.5, our claim follows. Since $\mathcal{K} \otimes \mathcal{S} \subset \langle \mathcal{S} \rangle_{\otimes}^{\leq 0}$, we have $\langle \mathcal{K} \otimes \mathcal{S} \rangle_{\leq 0}^{\leq 0} \subset \langle \mathcal{S} \rangle_{\otimes}^{\leq 0}$.
For convenience, we denote $\langle \mathcal{K} \otimes S \rangle_{\leq 0}^0$ by $\mathcal{U}$. Let $\mathcal{A} = \{ X \in \mathcal{U} \mid \mathcal{K} \otimes X \subset \mathcal{U} \}$. As in the proof of Lemma 2.5, it can be checked that $\mathcal{A}$ is a cocomplete preaisle. Since $\mathcal{K} \otimes \mathcal{K} \subset \mathcal{K}$ we have $\mathcal{K} \otimes S \subset \mathcal{A}$, and this proves $\mathcal{A} = \mathcal{U}$. Which means $\mathcal{K} \otimes \mathcal{U} \subset \mathcal{U}$ now by Lemma 2.5 we get $\mathcal{U}$ is a tensor preaisle. Since $1 \in \mathcal{K}$ we have $\mathcal{S} \subset \mathcal{U}$, this shows $\langle \mathcal{S} \rangle_{\leq 0}^0 \subset \mathcal{U}$.

\section{A class of tensor t-structures in $\mathbb{D}_{qc}(X)$}

Let $X$ be a quasi-compact quasi-separated scheme. We denote the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology by $\mathbb{D}_{qc}(X)$. The derived category $(\mathbb{D}_{qc}(X), \otimes_{\mathcal{O}_X}, \mathcal{O}_X)$ is a tensor triangulated category with the derived tensor product $\otimes_{\mathcal{O}_X}$ and the structure sheaf $\mathcal{O}_X$ as the unit. The full subcategory of complexes whose cohomologies vanish in positive degree $\mathbb{D}^{\leq 0}(X)$ is a preaisle of $\mathbb{D}_{qc}(X)$.

We define the $\otimes$-preaisles of $\mathbb{D}_{qc}(X)$ with respect to the standard preaisle $\mathbb{D}^{0}_{qc}(X)$.

In the affine situation when $X = \text{Spec} R$, note that $R$ generates the preaisle $\mathbb{D}^{\leq 0}(R)$. Hence by Proposition 2.7, every cocomplete preaisle of $\mathbb{D}(R)$ is a $\otimes$-preaisle. This has been well known; see for instance [ATJLS10, Proposition 1.10] also [Hrb20, Proposition 2.2].

The following is a characterization of $\otimes$-preaisles in $\mathbb{D}_{qc}(X)$. For separated schemes with an ample family of line bundles (a.k.a. divisorial), see [ATJLSS03, Proposition 5.1] for a stronger result; note that what we call $\otimes$-aisle has been termed as $0$-rigid in [ATJLSS03, §5].

\begin{lemma}
Let $\mathcal{U}$ be a cocomplete preaisle of $\mathbb{D}_{qc}(X)$. The following are equivalent:
\begin{enumerate}
\item The preaisle $\mathcal{U}$ is a $\otimes$-preaisle.
\item For every $G \in \mathcal{U}$ and $F \in \mathbb{D}_{qc}^{\leq 0}(X)$ we have $F \otimes_{\mathcal{O}_X} G \in \mathcal{U}$.
\item For every $G \in \mathcal{U}$ and $F$ a quasi-coherent $\mathcal{O}_X$-module, we have $F \otimes_{\mathcal{O}_X} G \in \mathcal{U}$.
\item For every $G \in \mathcal{U}$ and $F$ a quasi-coherent $\mathcal{O}_X$-module, we have $F \otimes_{\mathcal{O}_X} G \in \mathcal{U}$.
\end{enumerate}
\end{lemma}

\begin{proof}
(1) $\Leftrightarrow$ (2) follows from the definition. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. We prove (4) $\Rightarrow$ (2). Let $F \in \mathbb{D}_{qc}^{\leq 0}(X)$, we can replace $F$ by a $K$-flat resolution say $E$, such that $E$ is in $\mathbb{D}_{qc}^{\leq 0}(X)$ and each component $E^i$ is a flat $\mathcal{O}_X$-module [LH09, Proposition 2.5.5]. For each integer $n$ we denote $\sigma^{\geq n} E$ the brutal truncation of $E$ from below. Note that $E$ is the colimit of the directed system $\{ \sigma^{\geq n} E \}$ with the obvious maps.

For any $G \in \mathcal{U}$, to show $E \otimes_{\mathcal{O}_X} G \in \mathcal{U}$ it is enough to show for each $n \in \mathbb{N}$, $\sigma^{\geq -n} E \otimes_{\mathcal{O}_X} G \in \mathcal{U}$ since $\mathcal{U}$ is cocomplete. We prove by induction on $n$. We have the following distinguished triangle for each $n$,

$$
\sigma^{\geq -(n-1)} E \rightarrow \sigma^{\geq -n} E \rightarrow E^{-n}[n] \rightarrow \sigma^{\geq -(n-1)} E[1].
$$

Here $E^{-n}[n]$ is the complex which is zero everywhere except at $-n$th entry, where it is precisely $E^{-n}$; the $-n$th component of the complex $E$. As $E^{-n}$ is a flat $\mathcal{O}_X$-module and $\mathcal{U}$ is closed under positive shifts, by (4) we have $E^{-n}[n] \otimes_{\mathcal{O}_X} G \in \mathcal{U}$. Now by
induction hypothesis we assume \( \sigma \geq -(n-1)E \otimes^L_{O_X} G \in \mathcal{U} \), this proves the claim since \( \mathcal{U} \) is closed under extensions.

\[ \square \]

3.1. **Filtrations of supports and associated subcategories.** Let \( E \) be a complex in \( D_{qc}(X) \). The **cohomological support** of \( E \) \cite[Definition 3.2]{Tho97} is defined to be the subspace

\[
\text{Supph}(E) = \bigcup_{n \in \mathbb{Z}} \text{Supp}(H^n(E)).
\]

We introduce an auxiliary notation,

\[
\text{Supph}^{\geq i}(E) = \bigcup_{j \geq i} \text{Supp}(H^j(E)).
\]

**Lemma 3.2.** Let \( A \to B \to C \to A[1] \) be a distinguished triangle in \( D_{qc}(X) \). Then,

\[
\text{Supph}^{\geq i}(B) \subset \text{Supph}^{\geq i}(A) \bigcup \text{Supph}^{\geq i}(C).
\]

**Proof.** From the long exact cohomology sequence we have

\[
\text{Supp}(H^j(B)) \subset \text{Supp}(H^j(A)) \bigcup \text{Supp}(H^j(C)).
\]

Now taking union over \( j \geq i \) the result follows. \( \square \)

**Lemma 3.3.** For any \( E \in D_{qc}(X) \), we have

\[
\text{Supph}^{\geq i}(E[1]) \subset \text{Supph}^{\geq i}(E).
\]

**Proof.** Note that \( \text{Supph}^{\geq i}(E[1]) = \text{Supph}^{\geq i+1}(E) \). And \( \text{Supph}^{\geq i+1}(E) \subset \text{Supph}^{\geq i}(E) \) follows from the definition of \( \text{Supph}^{\geq i}(-) \). \( \square \)

**Lemma 3.4.** Let \( \{E_\alpha\} \) be a set of objects in \( D_{qc}(X) \). Then,

\[
\text{Supph}^{\geq i}(\oplus E_\alpha) = \bigcup_\alpha \text{Supph}^{\geq i}(E_\alpha).
\]

**Proof.** It is enough to show \( \text{Supp}(H^i(\oplus E_\alpha)) = \bigcup_\alpha \text{Supp}(H^i(E_\alpha)) \). As cohomology commutes with direct sums we have \( H^i(\oplus E_\alpha) \cong \oplus H^i(E_\alpha) \) so

\[
\text{Supp}(H^i(\oplus E_\alpha)) = \text{Supp}(\oplus H^i(E_\alpha)) = \bigcup_\alpha \text{Supp}(H^i(E_\alpha)).
\]

\( \square \)

**Lemma 3.5.** Let \( B \in D_{qc}(X) \). For any flat quasi-coherent \( O_X \)-module \( F \) and \( n \geq 0 \), we have

\[
\text{Supph}^{\geq i}(F[n] \otimes^L_{O_X} B) \subset \text{Supph}^{\geq i}(B).
\]
\textbf{Proof.} First we prove for \( n = 0 \). Since \( F \) is a flat \( \mathcal{O}_X \)-module we have \( F \otimes_{\mathcal{O}_X} B = F \otimes_{\mathcal{O}_X} B \). The flatness of \( F \) implies \( H^i(F \otimes_{\mathcal{O}_X} B) \cong F \otimes_{\mathcal{O}_X} H^i(B) \). Now \( \text{Supp}(F \otimes_{\mathcal{O}_X} H^i(B)) \subset \text{Supp}(H^i(B)) \), hence we get
\[
\text{Supp}(H^i(F \otimes_{\mathcal{O}_X} B)) \subset \text{Supp}(H^i(B)).
\]
And this proves
\[
\text{Supp}^{\geq i}(F \otimes_{\mathcal{O}_X} B) \subset \text{Supp}^{\geq i}(B).
\]
Next for \( n > 1 \). By Lemma 3.3 we have
\[
\text{Supp}^{\geq i}(F[n] \otimes_{\mathcal{O}_X} B) = \text{Supp}^{\geq i}((F \otimes_{\mathcal{O}_X} B)[n]) \subset \text{Supp}^{\geq i}(F \otimes_{\mathcal{O}_X} B).
\]
Now, applying \( n = 0 \) case we get
\[
\text{Supp}^{\geq i}(F[n] \otimes_{\mathcal{O}_X} B) \subset \text{Supp}^{\geq i}(B).
\]
\( \square \)

\textbf{Definition 3.6.} A filtration of supports of \( X \) is a function \( \phi \) from \( \mathbb{Z} \) with values in the set of subsets of \( X \) such that \( \phi(i+1) \subset \phi(i) \) for each \( i \in \mathbb{Z} \). Let \( \phi \) be a filtration of supports of \( X \). The subcategory of derive \( X \) associated to \( \phi \) and denoted by \( \mathcal{U}_\phi \) is the full subcategory containing objects \( E \) such that \( \text{Supp}(H^i(E)) \subset \phi(i) \) for each \( i \in \mathbb{Z} \).

\textbf{Proposition 3.7.} Let \( \phi \) be a filtration of supports of \( X \) and \( \mathcal{U}_\phi \) be the associated subcategory of \( \mathbf{D}_{qc}(X) \). Then \( \mathcal{U}_\phi \) is a cocomplete \( \otimes \)-preaisle of \( \mathbf{D}_{qc}(X) \).

\textbf{Proof.} Since \( \phi \) is a decreasing function \( E \in \mathcal{U}_\phi \) if and only if \( \text{Supp}^{\geq i}(E) \subset \phi(i) \) for each \( i \in \mathbb{Z} \). By Lemma 3.3 if \( E \in \mathcal{U}_\phi \), then \( E[1] \in \mathcal{U}_\phi \). Using Lemma 3.2 we can observe that \( \mathcal{U}_\phi \) is closed under extensions, and Lemma 3.4 implies \( \mathcal{U}_\phi \) is cocomplete. Now to show \( \mathcal{U}_\phi \) is a \( \otimes \)-preaisle, by Lemma 3.1 it is enough to show for any flat \( \mathcal{O}_X \)-module \( F \) and any \( B \in \mathcal{U}_\phi \) we must have \( F \otimes_{\mathcal{O}_X} B \in \mathcal{U}_\phi \). This follows from \( n = 0 \) case of Lemma 3.5 \( \square \)

\textbf{Lemma 3.8.} Let \( B \in \mathbf{D}_{qc}(X) \), and \( E \) be a perfect complex in \( \mathbf{D}^{\leq 0}_{qc}(X) \). Then,
\[
\text{Supp}^{\geq i}(E \otimes_{\mathcal{O}_X} B) \subset \text{Supp}^{\geq i}(B).
\]

\textbf{Proof.} First, we observe that the brutal truncation of a perfect complex is perfect. For each positive integer \( n \), we have the triangle coming from brutal truncation,
\[
\sigma^{\geq -(n-1)}E \rightarrow \sigma^{\geq -n}E \rightarrow E^{-n}[n] \rightarrow \sigma^{\geq -(n-1)}E[1].
\]
The \( \mathcal{O}_X \)-module \( E^{-n} \) is flat. Tensoring with \( B \) we get the triangle,
\[
\sigma^{\geq -(n-1)}E \otimes_{\mathcal{O}_X} B \rightarrow \sigma^{\geq -n}E \otimes_{\mathcal{O}_X} B \rightarrow E^{-n}[n] \otimes_{\mathcal{O}_X} B \rightarrow \sigma^{\geq -(n-1)}E[1] \otimes_{\mathcal{O}_X} B.
\]
As \( E \) is perfect and \( X \) is quasi-compact there is an \( N \) such that \( \sigma^{\geq -N}E = E \). In view of Lemma 3.2 and by induction on the length of \( E \), it is now enough to show for any flat \( \mathcal{O}_X \)-module \( F \) and positive integer \( n \), \( \text{Supp}^{\geq i}(F[n] \otimes_{\mathcal{O}_X} B) \subset \text{Supp}^{\geq i}(B) \); which has been shown in 3.5. This proves the claim. \( \square \)
3.2. Extending perfect complexes along a tensor preaisle. Let $X$ be a quasi-compact quasi-separated scheme with an ample family of line bundles. Let $U \subset X$ be a quasi-compact open subset and $Z$ be a closed subset of $X$ with $X \setminus Z$ quasi-compact.

The following proposition is an improvement of [TT90, Proposition 5.4.2], we remove both the Noetherian hypothesis and the boundedness assumption on $F$. Also we observe that the map can be extended along a $\otimes$-preaisle.

**Proposition 3.9.** Let $E \in \text{Perf}(X)$ and $F \in D_{qc}(X)$ and $a : E|_U \to F|_U$ be a map in $D_{qc}(U)$. Then, there exists $E' \in \text{Perf}(X)$, and maps $c : E' \to E$ and $b : E' \to F$ such that $c|_U : E'|_U \to E|_U$ is an isomorphism and $a \circ c|_U = b|_U$. Moreover, if $U$ is a $\otimes$-preaisle and $E \in U$, then $E'$ can be chosen such that $E' \in U$.

**Proof.** First, we note that in the proof of [TT90, Proposition 5.4.2] the Noetherian assumption on $X$ and bounded below assumption on $F$ are being used to make sure the map between $E|_U \to F|_U$ is a strict map of complexes. Once the map $E|_U \to F|_U$ is a strict map then by the second half of the proof of [TT90, Proposition 5.4.2], this improved version follows verbatim.

To have a strict map of complexes it is enough if $F|_U$ is K-injective. Let $j : U \to X$ be the open immersion. The functor $j^*$ takes K-injectives to K-injectives as it has an exact left adjoint. For any $F$ in $D_{qc}(X)$, we can replace $F$ by a K-injective resolution. Hence without of loss of generality, we can assume $F$ is K-injective and therefore $j^*F$ is K-injective.

From the proof of Proposition 5.4.2 [TT90], one can take $E' = K^+ \otimes^L_{O_X} E$ where $K^+$ is a perfect complex in $D_{qc}^{\leq 0}(X)$. The last part of the claim follows since $U$ is a $\otimes$-preaisle.

We use the shorthand notation $\text{Perf}^{\leq N}(X)$ for $\text{Perf}(X) \cap D_{qc}^{\leq N}(X)$. We say a complex $A$ in $\text{Perf}^{\leq N}(U)$ has an extension in $\text{Perf}^{\leq N}(X)$ if there is an object $A' \in \text{Perf}^{\leq N}(X)$ such that $A'|_U \cong A$.

**Lemma 3.10.** Let $A \to B \to C \to A[1]$ be a distinguished triangle in $\text{Perf}^{\leq N}(U)$. If $A$ and $B$ have extensions in $\text{Perf}^{\leq N}(X)$ then $C$ has an extension in $\text{Perf}^{\leq N}(X)$.

**Proof.** Suppose we have $A'$ and $B'$ in $\text{Perf}^{\leq N}(X)$ such that $A'|_U \cong A$ and $B'|_U \cong B$. Note that $D_{qc}^{\leq N}(X)$ is a $\otimes$-preaisle. Using Proposition 3.9 if required replacing $A'$, the map $A \to B$ can be extended to a map $A' \to B'$ in $\text{Perf}^{\leq N}(X)$. The cone of the map $A' \to B'$ is an extension of $C$ in $\text{Perf}^{\leq N}(X)$.

The following lemma is a modification of [TT90, Lemma 5.5.1]. To illustrate the difference we write the proof in a similar fashion as in [TT90].

**Lemma 3.11.** For any $F$ in $\text{Perf}^{\leq N}(U)$ there exists a perfect complex $\bar{F}$ in $\text{Perf}^{\leq N}(U)$ such that $F \otimes \bar{F}$ has an extension in $\text{Perf}^{\leq N}(X)$.

**Proof.** Let $j : U \to X$ be the open immersion. Consider $Rj_*F$ on $X$. This is cohomologically bounded below with quasi-coherent cohomology. The same is true for
\( \tau \leq N R_j * F \), so we can apply \cite{TT90} Corollary 2.3.3, which says, there is a directed system of strict perfect complexes \( \{E_\alpha\} \) bounded above by \( N \) such that

\[
\lim_{\to} E_\alpha \cong \tau \leq N R_j * F.
\]

Applying the exact functor \( j^* \) we get,

\[
\lim_{\to} j^* E_\alpha \cong \tau \leq N j^* R_j * F \cong \tau \leq N j^* R_j * F \cong \tau \leq N F \cong F.
\]

As \( F \) is perfect the isomorphism \( F \to \lim_{\to} j^* E_\alpha \) must factor through some \( j^* E_\alpha \).

Since every monomorphism splits in a triangulated category, there is an object \( \bar{F} \) such that \( F \oplus \bar{F} \cong j^* E_\alpha \). As \( \text{Perf} \leq N (U) \) is a thick preaisle \( \bar{F} \in \text{Perf} \leq N (U) \) and \( E_\alpha \) is an extension of \( F \oplus \bar{F} \) in \( \text{Perf} \leq N (X) \).

\( \square \)

**Lemma 3.12.** For any \( F \) in \( \text{Perf} \leq N (U) \) the perfect complex \( F \oplus F[1] \) has an extension in \( \text{Perf} \leq N (X) \).

**Proof.** By Lemma 3.11, for a given \( F \) we have \( \bar{F} \) such that \( F \oplus \bar{F} \) has an extension in \( \text{Perf} \leq N (X) \). Consider the direct sum of the following distinguished triangles

\[
0 \to F \to F \to 0[1],
\]

\[
\bar{F} \to \bar{F} \to 0 \to \bar{F}[1],
\]

\[
F \to 0 \to F[1] \to F[1].
\]

By Lemma 3.10 \( F \oplus F[1] \) has an extension in \( \text{Perf} \leq N (X) \). \( \square \)

Next, we prove a version of the extension lemma with support conditions. We denote the collection of perfect complexes \( E \in \text{Perf} \leq N (X) \) with \( \text{Supp}(E) \subset Z \) by \( \text{Perf} \leq N Z (X) \).

**Proposition 3.13.** For any \( F \) in \( \text{Perf} \leq N Z \cap U (U) \) the perfect complex \( F \oplus F[1] \) has an extension in \( \text{Perf} \leq N Z (X) \).

**Proof.** We denote \( U \cup (X \setminus Z) \) by \( W \). Take the open immersion \( k : U \to W \). Recall the functor \( k_! \) extending by zero is exact and a left adjoint of \( k^* \). We take \( k_! F \), which is a perfect complex on \( W \), since locally it is perfect: \( k_! F \mid_U = F \) and \( k_! F \mid_{(W \setminus Z)} \cong 0 \).

Now we apply the unrestricted extension Lemma 3.12 to \( k_! F \oplus k_! F[1] \). This shows \( F \oplus F[1] \) has an extension with the support condition. \( \square \)

**Without the divisorial condition.** Next, we proceed to remove the hypothesis of ample family of line bundles, using homotopy pushout construction of \cite{TT90} 3.20.4.

**Proposition 3.14.** Let \( X \) be a quasi-compact quasi-separated scheme. Let \( U \subset X \) be a quasi compact open subset and \( Z \) be a closed subset of \( X \) with \( X \setminus Z \) quasi compact.

For any \( F \) in \( \text{Perf} \leq N Z \cap U (U) \) there is a positive integer \( n \) such that the perfect complex \( \oplus_{i=0}^{n} (F[i] \oplus (\cdot)) \) has an extension in \( \text{Perf} \leq N Z (X) \).
Proof. There exists a finite set \{V_1, \ldots, V_n\} of open affine subsets of X such that \(X = U \cup V_1 \cdots \cup V_n\). We prove the result recursively.

The case \(n = 1\). For \(U \cup V_1\) and \(F \in \text{Perf}_{\leq N}^{Z}(U)\). Consider the complex \(F \oplus F[1]\). Since \(V_1\) is divisorial, by Proposition 3.13 the complex \((F \oplus F[1])|_{U \cup V_1}\) has an extension say \(F_{V_1}\) in \(\text{Perf}_{\leq N}^{Z}(V_1)\). Now we take the homotopy pushout of \(F_{V_1}\) on \(V_1\) and \(F \oplus F[1]\) on \(U\), by the methods of [TT90, 3.20.4], which will provide an extension, say \(F_1\), of \(F \oplus F[1]\) in \(\text{Perf}_{\leq N}^{Z}(U \cup V_1)\).

Next for \(U \cup V_1 \cup V_2\) we repeat the construction of step \(n = 1\) by replacing \(U\) by \(U \cup V_1\), \(V_1\) by \(V_2\) and \(F\) by \(F_1\). If we denote the extension of \(F_1 \oplus F_1[1]\) constructed in this manner by \(F_2\), then \(F_2|_U \cong F \oplus F[1] \oplus F[2]\). Repeating the construction \(n\) times we obtain \(F_n\), which is an extension of \(\oplus_n(F[i])^{\oplus(n)}\) in \(\text{Perf}_{\leq N}^{Z}(X)\).

\[\square\]

**Lemma 3.15.** Let \(X\) be a quasi-compact quasi-separated scheme. Let \(U \subset X\) be a quasi-compact open subset and \(Z\) be a closed subset of \(X\) with \(X \setminus Z\) quasi compact. Let \(F \in \text{D}_{\text{qc}}(X)\). If \(F \in (\text{Perf}_{\leq N}^{Z}(X))^{\perp}\) then \(F|_U \in (\text{Perf}_{\leq N}^{Z}(U))^{\perp}\).

**Proof.** For simplicity, we denote \(\text{Perf}_{\leq N}^{Z}(X)\) by \(S(X)\) and \(\text{Perf}_{\leq N}^{Z}(U)\) by \(S(U)\). There exists a finite set \(\{V_1, \ldots, V_n\}\) of open affine subsets of \(X\) such that \(X = U \cup V_1 \cdots \cup V_n\). We prove the result by induction on the number \(n\) of open affine subsets.

For \(n = 1\). Let \(X = U \cup V\) and \(F \in \text{D}_{\text{qc}}(X)\). If \(F|_U \notin (S(U))^{\perp}\) then there is a non zero map \(E \to F|_U\) for some \(E \in S(U)\). Now we take the map \(E \oplus E[1] \to E \to F|_U\).

Since \(V\) is divisorial by Proposition 3.13 \((E \oplus E[1])|_{U \cap V}\) has an extension say \(E' \in S(V)\). By Proposition 3.9 if needed replacing \(E'\) we can extend the map \((E \oplus E[1])|_{U \cap V} \to F|_{U \cap V}\) to a map \(E' \to F|_{V}\). Now we take the homotopy pushout of the maps \(E \oplus E[1] \to F|_U\) and \(E' \to F|_{V}\) by the methods of [TT90, 3.20.4]. This shows \(F \notin (S(X))^{\perp}\).

Now the induction step. We have \(X = U \cup V_1 \cdots \cup V_n\). Let \(W = U \cup V_1 \cdots \cup V_{n-1}\). By induction hypothesis we have the result for \(W\), that is, if \(F|_W \notin (S(W))^{\perp}\) then \(F|_W \notin (S(W))^{\perp}\). Using the case \(n = 1\) for \(W\) and \(V_n\) we can obtain \(F \notin (S(X))^{\perp}\). \(\square\)

### 3.3. Compact generation of tensor preaisles.

**Definition 3.16.** A subset \(Z\) is a specialization closed subset of \(X\) if for each \(x \in Z\) the closure of the singleton set \(\{x\}\) is contained in \(Z\), that is, \(\{x\} \subset Z\). Note that a specialization closed subset is a union of closed subsets of \(X\). A subset \(Y\) is a Thomason subset of \(X\) if \(Y = \bigcup \alpha Y_{\alpha}\) is a union of closed subsets \(Y_{\alpha}\) such that \(X \setminus Y_{\alpha}\) is quasi-compact. Thomason subsets are specialization closed subsets but the converse need not be true in general. However, if \(X\) is Noetherian then the two notions coincide since every open subset of a Noetherian scheme is quasi-compact.

A Thomason filtration of \(X\) is a map \(\phi : Z \to 2^X\) such that \(\phi(i)\) is a Thomason subset of \(X\) and \(\phi(i) \supset \phi(i+1)\) for all \(i \in \mathbb{Z}\).
Let $R$ be a Noetherian ring and $Z$ be a Thomason subset of $\text{Spec} R$. Suppose $i$ is a fixed integer. Consider the Thomason filtration $\psi$ which is defined as
\[
\psi(j) = Z \quad \text{if} \quad j \leq i; \\
= \emptyset \quad \text{if} \quad j > i.
\]

We denote the subcategory of $D(R)$ associated with $\psi$ by $U^i_Z$. Consider the following set of compact objects of $D(R)$, $K^i_Z = \{ K(a_1, a_2, \ldots, a_n)[-i] \mid \{a_1, a_2, \ldots, a_n\} \subset R \text{ and } V((a_1, a_2, \ldots, a_n)) \subset Z, \}
$ where $K(a_1, a_2, \ldots, a_n)$ denote the Koszul complex associated to the set $\{a_1, a_2, \ldots, a_n\} \subset R$.

Lemma 3.17. The preaisle $U^i_Z$ is compactly generated by $K^i_Z$.

Proof. [ATJLS10 Corollary 3.9].

The following result is well known and first appeared in [ATJLS10 Theorem 3.10]. Since it is a crucial step in achieving Theorem 3.20 we include a proof.

Proposition 3.18. Let $R$ be a commutative Noetherian ring. Let $\phi$ be a Thomason filtration of $\text{Spec} R$. The preaisle $U^i_\phi$ is compactly generated by the following set $K^i_\phi = \bigcup_{i \in \mathbb{Z}} K^i_{\phi(i)}$.

First we prove a lemma.

Lemma 3.19. Let $\phi$ be a Thomason filtration of $\text{Spec} R$. Consider the preaisles $U^i_\phi$ and $U^i_{\phi(i)}$. Then we have, $U^i_\phi = \bigcap_{i \in \mathbb{Z}} (U^i_{\phi(i)})^\perp$.

Proof. From the definition we have $U^i_{\phi(i)} \subset U^i_\phi$ for each $i$ hence $U^i_\phi \subset \bigcap_{i \in \mathbb{Z}} (U^i_{\phi(i)})^\perp$.

For the reverse inclusion, we prove by contradiction. Suppose $A \in \bigcap_{i \in \mathbb{Z}} (U^i_{\phi(i)})^\perp$ and $A \notin U^i_{\phi(i)}$. Then there is an object $B \in U^i_\phi$ such that $\text{Hom}(B, A) \neq 0$. This means there is a non-zero map $f : B \to A$. If $B$ is bounded above, then for some $i$, $B \in U^i_{\phi(i)}$. This is a contradiction as $A \in (U^i_{\phi(i)})^\perp$. Now, suppose $B$ is not bounded above. Consider the map $f_n : \tau^n B \to \tau^n A \to A$ where $\tau^n A \to A$ is the natural inclusion. The map $f$ is a filtered colimit of the sequence of maps $\{f_n\}$, since $f \neq 0$, all $f_n$ cannot be zero. Therefore, there is an $i$ such that $f_i : \tau^i B \to A$ is non-zero. This is again a contradiction as $A \in (U^i_{\phi(i)})^\perp$. □
Proof of Proposition 3.18. Let $A \in \mathcal{U}_\phi$ and $A \in (\mathcal{K}_\phi)\perp$. Note that $A \in (\mathcal{K}_\phi)\perp$ implies $A \in (\mathcal{K}_{\phi(i)})\perp$ for all $i$. From lemma 3.17 we have $A \in (\mathcal{U}_{\phi(i)})\perp$ for all $i$. By lemma 3.19 we have $A \in \mathcal{U}_\phi\perp$, this means $A \in \mathcal{U}_\phi \bigcap \mathcal{U}_\phi\perp$ hence $A \cong 0$. This proves, by Proposition 1.10, $\mathcal{U}_\phi$ is compactly generated by the set $\mathcal{K}_\phi$. □

Theorem 3.20. Let $X$ be a Noetherian scheme. Let $\phi$ be a Thomason filtration of $X$. The $\otimes$-preaisle $\mathcal{U}_\phi$ of $\mathcal{D}_{qc}(X)$ is compactly generated.

Proof. Consider the following essentially small set of compact objects on $X$,

$$S_{\phi} = \bigcup_{i \in \mathbb{Z}} \text{Perf}_{\phi(i)}(X).$$

We will show $\mathcal{U}_\phi = \langle S_{\phi} \rangle_{\leq 0}$. By part (4) of Proposition 1.10 it is enough to show if $A \in \mathcal{U}_\phi$ and $\text{Hom}(S, A) = 0$ for all $S \in S_{\phi}$ then $A \cong 0$. Suppose $A \in \mathcal{U}_\phi$ and $A \in S_{\phi}\perp = \bigcap_i (\text{Perf}_{\phi(i)}(X))\perp$. Let $U$ be an open affine subset of $X$. We denote the restriction of $\phi$ on $U$ by $\phi|_U$. For any $K \in K_{\phi|_U}$ we have $\text{Hom}(K, A|_U) = 0$, by Lemma 3.15. Now by Proposition 3.18 we get $A|_U \cong 0$. Since $U$ is arbitrary we get $A \cong 0$. □

4. The classification theorem

4.1. Graded cohomological support of subcategories.

Definition 4.1. Let $\mathcal{U} \subset \mathcal{D}_{qc}(X)$ be a subcategory. The graded support of $\mathcal{U}$ is a function $\varphi_\mathcal{U}$, defined as

$$\varphi_\mathcal{U}(i) = \{x \in X \mid \exists E \in \mathcal{U} \text{ such that } x \in \text{Supp}(H^i(E))\}.$$

Lemma 4.2. Let $\mathcal{U} = \langle S \rangle_{\leq 0}$ be a $\otimes$-aisle of $\mathcal{D}_{qc}(X)$ generated by a set of compact objects $S$. Then

$$\varphi_\mathcal{U}(i) = \bigcup_{S \in S} \text{Supph}_{\geq i}(S).$$

In other words, the graded support of $\mathcal{U}$ can be computed from a set of compact generators.

Proof. For a preaisle $\mathcal{U}$, since it is closed under positive shifts, we have

$$\varphi_\mathcal{U}(i) = \bigcup_{E \in \mathcal{U}} \text{Supp}(H^i(E)) = \bigcup_{E \in \mathcal{U}} \text{Supph}_{\geq i}(E).$$

Clearly,

$$\bigcup_{S \in S} \text{Supph}_{\geq i}(S) \subset \bigcup_{E \in \mathcal{U}} \text{Supph}_{\geq i}(E).$$

By Proposition 2.11(2) and the support lemmas 3.2, 3.3, 3.4 and 3.8 for any $E \in \mathcal{U}$ we have $\text{Supph}_{\geq i}(E) \subset \bigcup_{S \in S} \text{Supph}_{\geq i}(S)$. This proves our claim. □

Lemma 4.3. Let $X$ be a Noetherian scheme and $\mathcal{U}$ be a compactly generated $\otimes$-aisle of $\mathcal{D}_{qc}(X)$. Then $\varphi_\mathcal{U}$ is a Thomason filtration of $X$. 

Proof. As $X$ is quasi-compact a perfect complex $S$ on $X$ is bounded. Also, the cohomology sheaves of $S$ are finite type $\mathcal{O}_X$-modules. Therefore $\text{Supp}^{>i}(S)$ is a closed subset of $X$; as it is a finite union of closed subsets. By Lemma 4.2 the set $\varphi_{\mathcal{U}}(i)$ is a Thomason subset. Since $\mathcal{U}$ is closed under positive shifts we have $\varphi_{\mathcal{U}}(i+1) \subset \varphi_{\mathcal{U}}(i)$ for each $i$, hence $\varphi_{\mathcal{U}}$ is a Thomason filtration. □

4.2. The classification theorem for Noetherian schemes. Let $X$ be a Noetherian scheme and $U$ be an open affine subset of $X$ and $j : U \to X$ denote the open immersion. Let $\mathcal{U}$ be a $\otimes$-preaisle of $D_{qc}(X)$. We define $\mathcal{U}|_{U} := \langle j^* \mathcal{U} \rangle_{\leq 0}$ - the restriction of $\mathcal{U}$ to the open affine subset $U$. Suppose $(\mathcal{U}, \mathcal{V}[1])$ is a tensor t-structure on $D_{qc}(X)$. We define $\mathcal{V}|_{U} := (\mathcal{U}|_{U})^\perp$.

Remark 4.4. In the earlier manuscript, there was an error in the proof of Lemma 4.5. The proof was based on an erroneous lemma, which was used to give a shorter proof of Lemma 4.5. The error was first pointed out to us by Alexander Clark. Later Suresh Nayak also pointed out the same error. We sincerely thank both of them for bringing our attention to it. In this version, we give a correct proof of the lemma.

Lemma 4.5. Let $\mathcal{U}$ be a $\otimes$-preaisle of $D_{qc}(X)$. If $F \in (\mathcal{U})^\perp$ then $j^* F \in (\mathcal{U}|_{U})^\perp$.

Proof. First we prove the following claim: If $A \notin (\mathcal{U}|_{U})^\perp$ then there is an element $E \in \mathcal{U}$ such that $\text{Hom}(j^* E, A) \neq 0$. Take the collection $\{A[-n] \mid n \geq 0\}$ and denote it by $\mathcal{A}$. Note that $\mathcal{A}^\perp$ is a preaisle. If for all $E \in \mathcal{U}$, $\text{Hom}(j^* E, A) = 0$ then the preaisle $\mathcal{A}^\perp$ contains all $j^* E$ for $E \in \mathcal{U}$. Hence, $\mathcal{A}^\perp$ contains $\mathcal{U}|_{U}$. This is a contradiction since $A \notin (\mathcal{U}|_{U})^\perp$. Now we proceed to prove the lemma.

Suppose there is an $F \in (\mathcal{U})^\perp$ such that $j^* F \notin (\mathcal{U}|_{U})^\perp$ then by our claim there is an $E \in \mathcal{U}$ such that $\text{Hom}(j^* E, j^* F) \neq 0$. By tensor-hom adjunction we get

$$\text{Hom}(j^* E, j^* F) = \text{Hom}(j^* \mathcal{O}_X, \mathcal{R}\mathcal{H}om_U(j^* E, j^* F))$$
$$= \text{Hom}(j^* \mathcal{O}_X, j^* \mathcal{R}\mathcal{H}om_X(E, F)).$$

Now if we write $F' = \mathcal{R}\mathcal{H}om_X(E, F)$ then $\text{Hom}(j^* E, j^* F) \neq 0$ implies the group $\text{Hom}(j^* \mathcal{O}_X, j^* F') \neq 0$. Applying Lemma 3.15 to the preaisle $\text{Perf}^\leq N(X)$ where $N = 0$ and $Z = X$, we get that there is a perfect complex $K \in \text{Perf}^{\leq 0}(X)$ such that $\text{Hom}(K, F') \neq 0$. Again by tensor-hom adjunction

$$\text{Hom}(K, F') = \text{Hom}(K, \mathcal{R}\mathcal{H}om_X(E, F))$$
$$= \text{Hom}(K \otimes E, F) \neq 0,$$

since $\mathcal{U}$ is a tensor preaisle, $K \otimes E \in \mathcal{U}$ and we get a contradiction. □
Lemma 4.6. Let \((U, \mathcal{V}[1])\) be a tensor t-structure on \(D_{qc}(X)\). Then \((U|_U, \mathcal{V}|_{U[1]})\) is a t-structure on \(D_{qc}(U)\). In particular, for any \(A \in D_{qc}(X)\) the triangle
\[
j^* \tau^<_{U|U} A \to j^* A \to j^* \tau^>_U A \to j^* \tau^<_{U|U} A[1],
\]
is a t-decomposition triangle of \(j^* A\) in \(D_{qc}(U)\).

Proof. To show \((U|_U, \mathcal{V}|_{U[1]})\) is a t-structure it is enough to get a t-decomposition triangle for each \(B \in D_{qc}(U)\). For any \(B \in D_{qc}(U)\), take \(j_* B\) and its t-decomposition triangle for \((U, \mathcal{V}[1])\)
\[
\tau^<_{U|U} j_* B \to j_* B \to \tau^>_U j_* B \to \tau^<_{U|U} j_* B[1].
\]
Applying \(j^*\) we get a triangle in \(D_{qc}(U)\). Since \(j^* j_* B \cong B\) we get
\[
j^* \tau^<_{U|U} j_* B \to j^* j_* B \to j^* \tau^>_U j_* B \to j^* \tau^<_{U|U} j_* B[1].
\]

By Lemma 4.5 \(j^* \tau^<_{U|U} j_* B \in \mathcal{V}|_U\) and by definition of \(U|_U\) we have \(j^* \tau^<_{U|U} j_* B \in U|_U\), hence the above triangle gives a t-decomposition of \(B\) for \((U|_U, \mathcal{V}|_{U[1]})\).

Now for any \(A \in D_{qc}(X)\) applying \(j^*\) and the argument as above, we get the second half of the claim. \(\square\)

Lemma 4.7. If \(U\) is a compactly generated \(\otimes\)-aisle of \(D_{qc}(X)\), then the restriction \(U|_U\) is a compactly generated \(\otimes\)-aisle of \(D_{qc}(U)\).

Proof. The restriction of a perfect complex is perfect hence by Lemma 2.10 and Proposition 2.11(1) the claim follows. \(\square\)

Lemma 4.8. Let \(R\) be a Noetherian ring. Let \(A\) and \(B\) be two perfect complexes on \(R\). If \(\text{Supph}^{\leq i}(A) \subset \text{Supph}^{\leq i}(B)\) for each \(i\), then \(A \in \langle B \rangle^{\leq 0}\).

Proof. [Hrb20, Proposition 5.1] \(\square\)

Lemma 4.9. Let \(A\) and \(B\) be two perfect complexes on \(X\). If \(\text{Supph}^{\leq i}(A) \subset \text{Supph}^{\leq i}(B)\) for each \(i\), then \(A \in \langle B \rangle^{\leq 0}\).

Proof. Consider the t-decomposition of \(A\) with respect to the aisle \(\langle B \rangle^{\leq 0}\)
\[
\tau^<_B A \to A \to \tau^>_B A \to \tau^<_B A[1].
\]

Let \(U\) be an open affine subset of \(X\), and \(j : U \to X\) be the open immersion. Applying \(j^*\) to the above triangle we get
\[
j^* \tau^<_B A \to j^* A \to j^* \tau^>_B A \to j^* \tau^<_B A[1].
\]

Now using Lemma 4.6 and the Lemma 4.8 we conclude the map \(j^* \tau^<_B A \to j^* A\) is an isomorphism. As the map \(\tau^<_B A \to A\) is locally an isomorphism on every open affine subset of \(X\), it is an isomorphism. This proves \(A \in \langle B \rangle^{\leq 0}\). \(\square\)

Lemma 4.10. Let \(Z\) be a closed subset of \(X\). Then there is a perfect complex \(E \in \text{Perf}^{\leq 0}_Z(X)\) such that \(\text{Supp}(H^0(E)) = Z\).
Proof. Consider an open affine cover of $X$ say $\{U_\alpha = \text{Spec}R_\alpha\}$. We take a Koszul complex $K_\alpha$ on $R_\alpha$ such that $\text{Supph}(K_\alpha) = Z \cap U_\alpha$. By Proposition 3.14 there is an extension of $\oplus_i (K_\alpha[i])^\oplus(n)$; say $E_\alpha$. Now take $E = \bigoplus_{\alpha} E_\alpha$, it is easy to check $\text{Supp}(H^0(E)) = Z$, and by our construction $E \in \text{Perf}_Z^\otimes(X)$. \qed

We denote the collection of Thomason filtrations of $X$ by $\text{Thom}^\text{fil}(X)$ and the collection of compactly generated $\otimes$-aisles of $\mathcal{D}_{qc}(X)$ by $\text{Aisle}_{\otimes}^{\text{cp}}(X)$.

Theorem 4.11. Let $X$ be a Noetherian scheme. There is a one-to-one correspondence between $\text{Thom}^\text{fil}(X)$ and $\text{Aisle}_{\otimes}^{\text{cp}}(X)$. More precisely, the maps

$$
\Phi : \text{Aisle}_{\otimes}^{\text{cp}}(X) \to \text{Thom}^\text{fil}(X), \quad \mathcal{U} \mapsto \varphi_{\mathcal{U}}, \quad (4.11),
$$

$$
\Psi : \text{Thom}^\text{fil}(X) \to \text{Aisle}_{\otimes}^{\text{cp}}(X), \quad \phi \mapsto \mathcal{U}_\phi, \quad (3.0),
$$

are inverse of each other.

Proof. The maps are well defined by Lemma 4.3 and Theorem 3.20. First, we show $\Phi \circ \Psi = \text{Id}$. Given $\phi \in \text{Thom}^\text{fil}(X)$ we have $\phi(i) = \bigcup \lambda Z_\lambda$ where $Z_\lambda$’s are closed subsets of $X$. For a fixed $Z_\lambda$ by Lemma 4.10 we have a perfect complex $E_\lambda$ and $E_\lambda[i] \in \Psi(\phi)$, this proves the claim.

To show $\Psi \circ \Phi = \text{Id}$, we first observe $\mathcal{U} \subset \Psi \circ \Phi(\mathcal{U})$. For the reverse inclusion, we will show that any compact object $A \in \Psi \circ \Phi(\mathcal{U})$ is in $\mathcal{U}$. By Lemma 4.12 we have compact objects $\{B_\alpha\} \subset \mathcal{U}$ such that $\text{Supph}^{\geq i}(A) \subset \bigcup \text{Supph}^{\geq i}(B_\alpha)$.

Since $A$ is perfect, $\text{Supph}^{\geq i}(A)$ is a closed subset of $X$. As $X$ is Noetherian there are finitely many irreducible components of $\text{Supph}^{\geq i}(A)$ say $Z_1, Z_2, \ldots, Z_n$. Now, we take $B_i$ such that the generating point of $Z_i$ lies in $\text{Supph}^{\geq i}(B_i)$. Therefore, we have a finite collection $\{B_i\}^{i=1}_n$ of $\{B_\alpha\}$ such that $\text{Supph}^{\geq i}(A) \subset \bigcup \text{Supph}^{\geq i}(B_i)$. So $\text{Supph}^{\geq i}(A) \subset \text{Supph}^{\geq i}(\bigoplus^{i=1}_n B_i)$, since $\bigoplus^{i=1}_n B_i \in \mathcal{U}$, by Lemma 4.9 we conclude $A \in \mathcal{U}$. This proves the theorem. \qed

Remark 4.12. From [SP16, Theorem 4.10], it follows that, for a separated Noetherian scheme, there is a natural bijection between the set of thick preaisles of $\mathcal{D}(\text{Qcoh}(X))$ and the set of compactly generated $\otimes$-preaisles on $\mathcal{D}(\text{Qcoh}(X))$. By Proposition 2.11 and its proof it can be easily deduced that the bijection of [SP16] restricts to a bijection between the set of thick $\otimes$-preaisles of $\text{Perf}(X)$ and the set of compactly generated tensor $\otimes$-preaisles on $\mathcal{D}(\text{Qcoh}(X))$.

Remark 4.13. Theorem 4.11 together with Remark 4.12 says there is a bijective correspondence between the set of thick $\otimes$-preaisles of $\text{Perf}(X)$ and the set of Thomason filtrations of $X$. Therefore, Theorem 4.11 can be thought of as a generalization of [Tho97, Theorem 3.15] to thick $\otimes$-preaisle, at least for separated Noetherian schemes.
5. Tensor telescope conjecture for t-structures

Let $\mathcal{G}$ be a Grothendieck abelian category. A subcategory $\mathcal{V}$ of $\mathsf{D}(\mathcal{G})$ is closed under homotopy colimits if for any directed system $\{A_i\}$ in $\mathsf{C}(\mathcal{G})$ with all $A_i$ in $\mathcal{V}$, the colimit of the directed system $\{A_i\}$ in $\mathsf{C}(\mathcal{G})$ belongs to $\mathcal{V}$; see [HN21, Fact 2.1], for the definition in the general setting of derivators see [HN21, A.1]. A t-structure $(\mathcal{U}, \mathcal{V}[1])$ on $\mathsf{D}(\mathcal{G})$ is homotopically smashing if the coaisle $\mathcal{V}$ is closed under homotopy colimits. Recall that we say a t-structure is smashing if the coaisle is closed under coproducts.

**Proposition 5.1.** Every homotopically smashing t-structure on $\mathsf{D}(\mathcal{G})$ is smashing.

**Proof.** [SSV17, Proposition 7.2]. □

**Remark 5.2.** Smashing t-structures in general are not homotopically smashing see [SSV17, Example 8.2]. However, in the case of stable t-structures these two notions coincide; see [HN21, A.5].

**Proposition 5.3.** Every compactly generated t-structure on $\mathsf{D}(\mathcal{G})$ is homotopically smashing.

**Proof.** [SSV17, Proposition 7.2]. □

The telescope conjecture for t-structures asks if every homotopically smashing t-structure on $\mathsf{D}(\mathcal{G})$ is compactly generated, that is, if the converse of Proposition 5.3 is true. When $\mathcal{G}$ is $\text{Mod-}R$ for $R$ Noetherian ring, Hrbek and Nakamura have proved the following:

**Theorem 5.4** ([HN21, Theorem 1.1]). Any homotopically smashing t-structure on $\mathsf{D}(R)$ is compactly generated.

For a separated Noetherian scheme $X$ and the derived category of quasi-coherent sheaves $\mathsf{D}(\text{Qcoh}(X))$, which is equivalent to $\mathsf{D}_{\text{qc}}(X)$, we prove the following:

**Theorem 5.5.** Any homotopically smashing tensor t-structure on $\mathsf{D}(\text{Qcoh}(X))$ is compactly generated.

First, we prove some lemmas. In this section $(\mathcal{U}, \mathcal{V}[1])$ will always mean a homotopically smashing tensor t-structure on $\mathsf{D}(\text{Qcoh}(X))$ and $U$ is an open affine subset of $X$.

**Lemma 5.6.** The restriction of $(\mathcal{U}, \mathcal{V}[1])$ to $U$, that is, $(\mathcal{U}|_U, \mathcal{V}|_U[1])$ is a homotopically smashing t-structure on $\mathsf{D}(\text{Qcoh}(U))$.

**Proof.** Recall $\mathcal{U}|_U = (j^*\mathcal{U})^{\leq 0}$ and $\mathcal{V}|_U = (\mathcal{U}|_U)^\perp$. By Lemma 4.6 $(\mathcal{U}|_U, \mathcal{V}|_U[1])$ is a t-structure on $\mathsf{D}(\text{Qcoh}(U))$. Now we will show $(\mathcal{U}|_U, \mathcal{V}|_U[1])$ is homotopically smashing. Let $\{A_i\}$ be a directed system in $\mathcal{V}|_U$. By adjunction isomorphism, if $A_i \in \mathcal{V}|_U$ then $j_*A_i \in \mathcal{V}$. Consider the directed system $\{j_*A_i\}$ in $\mathcal{V}$, since $\mathcal{V}$ is closed under homotopy colimits we have $\lim j_*A_i \in \mathcal{V}$. Since $j^*$ is an exact functor and $j^*j_*A_i \cong A_i$ the colimit of the system $\{A_i\}$ is $j^*(\lim j_*A_i)$ which by Lemma 4.5 is in $\mathcal{V}|_U$.

□
Lemma 5.7. If the graded support of $U$ is $\phi$, then the graded support of $U|_U$ is $\phi|_U$.

Proof. Consider $U_{(\phi|U)}$ the associated subcategory of the filtration $\phi|_U$ in $D(\text{Qcoh}(U))$, it is a cocomplete preaisle by Proposition 3.7. We have $j^*A \in U_{(\phi|U)}$ for any $A \in U$ therefore $U|_U = (j^*U)^{\geq 0} \subset U_{(\phi|U)}$. So the graded support of $U|_U$ is contained in $\phi|_U$. Next, given $x \in \phi|_U(i)$, we have $x \in \phi(i)$ which means there is an object $E$ in $U$ such that $x \in \text{Supp}(H^i(E))$. As $j^*E \in U|_U$ we get the graded support of $U|_U$ is $\phi|_U$. □

Lemma 5.8. If the graded support of $U$ is $\phi$, then $U|_U = U_{(\phi|U)}$.

Proof. By Lemma 5.6 we know $U|_U$ is homotopically smashing. As $U$ is affine, Theorem 5.4 implies $U|_U$ is compactly generated. By Lemma 5.7 the graded support of $U|_U$ is $\phi|_U$. By the classification of compactly generated t-structures, we conclude $U|_U = U_{(\phi|U)}$. □

Lemma 5.9. If the graded support of $U$ is $\phi$ then $\phi$ is a Thomason filtration.

Proof. By Lemma 5.8 we have $U|_U = U_{(\phi|U)}$ and $U|_U$ is compactly generated. Since the graded support of a compactly generated aisle is a Thomason filtration we get $\phi|_U$ is a Thomason filtration. In the Noetherian case, Thomason subsets and specialization closed subsets are the same. Specialization closed is a local property therefore the graded support of $U$ is a Thomason filtration. □

Proof of Theorem 5.5. We denote the graded support of $U$ by $\phi$ and consider $U_{\phi}$ the subcategory associated with $\phi$. Clearly, $U \subset U_{\phi}$.

For $A \in U_{\phi}$, consider the following t-decomposition triangle for $U$

$$\tau^<_{U}A \to A \to \tau^>_{U}A \to \tau^<_{U}A[1].$$

By Lemma 4.6 restricting to $U$ gives a t-decomposition triangle for $U|_U$

$$j^*\tau^<_{UU}A \to j^*A \to j^*\tau^>_{UU}A \to j^*\tau^<_{UU}A[1].$$

Since $j^*A \in U_{(\phi|U)}$ and $U|_U = U_{(\phi|U)}$ the map $j^*\tau^<_{UU}A \to j^*A$ is an isomorphism. As it is true for each open affine subset of $X$, we get $\tau^<_{UU}A \to A$ is an isomorphism. Thus $A \in U$ and this proves $U = U_{\phi}$. By Lemma 5.9 $\phi$ is a Thomason filtration. Now by Theorem 4.11 the subcategory $U_{\phi}$ is a compactly generated $\otimes$-aisle. Therefore $U$ is compactly generated. □

Remark 5.10. By Remark 5.2 and Proposition 2.9 the aisle of a homotopically smashing stable tensor t-structure on $D(\text{Qcoh}(X))$ is a smashing $\otimes$-ideal of $D(\text{Qcoh}(X))$. Therefore, Theorem 5.5 provides another proof of the tensor telescope conjecture for separated Noetherian schemes.

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