Polynomials Associated to Integer Partitions

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Abstract

Integer partitions express the different ways that a positive integer may be written as a sum of other positive integers. Here we explore the analytic properties of a polynomial $f_\lambda(x)$ that we call the partition polynomial for the partition $\lambda$, with the hope of learning new properties of partitions. We prove a recursive formula for the derivatives of $f_\lambda(x)$ involving Stirling numbers of the second kind, show that the set of integrals from 0 to 1 of a normalized version of $f_\lambda(x)$ is dense in $[0, 1/2]$, pose a few open questions, and formulate a conjecture relating the integral to the length of the partition. We also provide specific examples throughout to support our speculation that an in-depth analysis of partition polynomials could further strengthen our understanding of partitions.

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The first author’s surname is Dawsey.
1 Introduction and statement of results

A partition $\lambda$ of a nonnegative integer $n$ is a non-increasing sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of positive integers, called the parts of the partition, which sum to $n$. The sum $n$ is called the size of the partition. By convention, the only partition of size $n = 0$ is the empty partition with no parts. Although partitions are classical objects and simple to define, in some respects they are complex additive structures with elusive analytic properties and are still a significant topic of study in number theory today (for more details, see [1, 2, 5]). Hardy, Rademacher, Ramanujan, and many other well known number theorists of the twentieth century made major breakthroughs in our understanding of the size of a partition via the generating function of $p(n)$, the partition function ([8], A000041), which counts the number of partitions of size $n$. These results include the famous Hardy-Ramanujan asymptotic formula which was proven analytically using the circle method [4]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{n}{3}}} \text{ as } n \to \infty;$$

and the Ramanujan congruences which can be proven using the theory of modular forms [10, 11, 12]:

$$p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7}, \quad \text{and } p(11n + 6) \equiv 0 \pmod{11} \text{ for all } n.$$

Rademacher’s main contribution was an exact formula for $p(n)$ as a convergent infinite series involving Kloosterman sums and the $I$-Bessel function [9]. Most of the discoveries that have been made about the size of a partition so far have resulted from studying $p(n)$.

Recent work by the first author, Just, and Schneider [3] takes a different approach to studying partitions. They define a map from the set of all partitions to the set of natural numbers called the supernorm of a partition (previously called the Heinz number: [8], A305078), which sheds more light on properties of partitions by relating them to the multiplicative structure of prime factorizations of integers. For notational convenience, their work utilizes the multiplicative partition notation $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle$, where $m_i$ is the multiplicity of the part $i$ in $\lambda$ and $k$ is the largest part of $\lambda$, as opposed to the traditional additive notation. In an attempt to further understand the parts of a partition $\lambda$ and their multiplicities, the length $\ell(\lambda)$ (the number of parts), the size $|\lambda|$, the norm $N(\lambda)$ (the product of the parts), and the supernorm $\hat{N}(\lambda)$, among other partition statistics, Just [6] defined a polynomial to which one can apply calculus and analysis in order to analyze properties of partitions. To define this polynomial, let $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle$ be a partition written in multiplicative notation. The partition polynomial $f_\lambda$ is defined by

$$f_\lambda(x) = \sum_{i=1}^{k} m_i x^i. \quad (1)$$

Differentiation and integration of the partition polynomial both have the potential to reveal certain properties of partitions. The most basic observations one can make in the
direction of differentiation are that \( f_\lambda(1) = \ell(\lambda) \) and \( f'_\lambda(1) = |\lambda| \). It is natural to then ask what the higher derivatives of \( f_\lambda \) tell us about \( \lambda \) when we evaluate at \( x = 1 \). Does \( f''_\lambda(1) \) give some generalization of the size or the length of \( \lambda \)? We first determine what the higher derivatives look like in general, and then we make some progress toward answering this question by calculating a few specific examples which lead to a few open questions.

**Theorem 1.** Given a partition \( \lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle \), the \( d \)th derivative of its partition polynomial is
\[
f^{(d)}_\lambda(x) = \sum_{i=1}^{k} i^d m_i x^{i-d} - \sum_{j=0}^{d-1} \binom{d}{j} x^{j-d} f^{(j)}_\lambda(x) \tag{2}
\]
for all \( 0 \leq d \leq k \), where \( \{ \frac{d}{j} \} \) is the Stirling number of the second kind.

The \( d = 0 \) derivative \( f^{(0)}_\lambda \) denotes the partition polynomial itself. Note that the second sum is empty when \( d = 0 \) and that \( f^{(d)}_\lambda(x) = 0 \) for all \( d > k \). For background on Stirling numbers of the second kind, see [14] and [8], A008277, for example. We prove Theorem 1 in Section 2.

It is clear that \( f^{(d)}_\lambda(1) \geq 0 \) for all \( d \geq 0 \), since all of the parts of partitions are positive and all of their multiplicities are nonnegative. It is also clear that evaluating the first few derivatives at \( x = 1 \) yields
\[
\begin{align*}
f^{(0)}_\lambda(1) &= \ell(\lambda), \\
f^{(1)}_\lambda(1) &= |\lambda|, \\
f^{(2)}_\lambda(1) &= \sum_{i=1}^{k} i^2 m_i - \binom{2}{1} |\lambda|, \\
f^{(3)}_\lambda(1) &= \sum_{i=1}^{k} i^3 m_i - \binom{3}{2} \sum_{i=1}^{k} i^2 m_i - \left( \binom{3}{1} - \binom{3}{2} \binom{2}{1} \right) |\lambda|, \\
f^{(4)}_\lambda(1) &= \sum_{i=1}^{k} i^4 m_i - \binom{4}{3} \sum_{i=1}^{k} i^3 m_i - \left( \binom{4}{2} - \binom{4}{3} \binom{3}{2} \binom{2}{1} \right) \sum_{i=1}^{k} i^2 m_i - \left( \binom{4}{1} - \binom{4}{2} \binom{2}{1} - \binom{4}{3} \binom{3}{1} + \binom{4}{3} \binom{3}{2} \binom{2}{1} \right) |\lambda|. 
\end{align*}
\]

It would be a natural extension to formulate a nice combinatorial interpretation of the \( d \)th derivative of the partition polynomial at \( x = 1 \), as this would serve as a bridge between the derivatives and the partition itself. As a result, properties of the derivatives could potentially provide new information about the partition. A good start would be to find an explicit formula for \( f^{(d)}_\lambda(1) \) for all \( 0 \leq d \leq k \) that is more enlightening than simply plugging \( x = 1 \) into (2).

**Question 2.** Is there a nice combinatorial interpretation of \( f^{(d)}_\lambda(1) \)?
We now provide an explicit example of the derivatives for two specific partitions of the same length and size.

**Example 3.** Let \( \lambda_1 = \langle 1^1, 2^2, 3^0, 4^0, 5^1 \rangle \) and \( \lambda_2 = \langle 1^1, 2^1, 3^1, 4^1 \rangle \). Here are the partition polynomial, its derivatives, and their evaluations at \( x = 1 \) for \( \lambda_1 \):

\[
\begin{align*}
  f^{(0)}_{\lambda_1}(x) &= x^5 + 2x^2 + x \\
  f^{(1)}_{\lambda_1}(x) &= 5x^4 + 4x + 1 \\
  f^{(2)}_{\lambda_1}(x) &= 20x^3 + 4 \\
  f^{(3)}_{\lambda_1}(x) &= 60x^2 \\
  f^{(4)}_{\lambda_1}(x) &= 120x \\
  f^{(5)}_{\lambda_1}(x) &= 120
\end{align*}
\]

Here are the partition polynomial, its derivatives, and their evaluations at \( x = 1 \) for \( \lambda_2 \):

\[
\begin{align*}
  f^{(0)}_{\lambda_2}(x) &= x^4 + x^3 + x^2 + x \\
  f^{(1)}_{\lambda_2}(x) &= 4x^3 + 3x^2 + 2x + 1 \\
  f^{(2)}_{\lambda_2}(x) &= 12x^2 + 6x + 2 \\
  f^{(3)}_{\lambda_2}(x) &= 24x + 6 \\
  f^{(4)}_{\lambda_2}(x) &= 24
\end{align*}
\]

Although \( \lambda_1 \) and \( \lambda_2 \) share the same length \( \ell(\lambda_1) = \ell(\lambda_2) = 4 \) and the same size \( |\lambda_1| = |\lambda_2| = 10 \), the evaluations of their partition polynomial derivatives at \( x = 1 \) yield different numbers starting at the second derivative.

Obviously the evaluation of the partition polynomial at \( x = 1 \) can distinguish any two partitions of different lengths, and the evaluation of its first derivative can distinguish any two partitions of the same length and different sizes. A natural open question that arises from Example 3 is whether the evaluation of its second derivative can distinguish any two unequal partitions, even if they have the same length and size. Answering this question could possibly shed more light on a combinatorial interpretation of the higher derivatives, or vice versa.

**Question 4.** If \( \lambda, \lambda' \) are any two unequal partitions, is it true that \( f^{(2)}_{\lambda}(1) \neq f^{(2)}_{\lambda'}(1) \)?

One specific direction in which to investigate Question 4 is to search for a counterexample: a pair of unequal partitions \( \lambda, \lambda' \) for which \( f^{(2)}_{\lambda}(1) = f^{(2)}_{\lambda'}(1) \). More generally, one could ask how many derivatives are necessary in order to distinguish any two unequal partitions.

Another interesting observation is that taking the derivative of the partition polynomial of one partition yields the partition polynomial of a different partition. Explicitly, if we define the partition \( \lambda = \lambda^{(0)} = \langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle \), then \( f^{(1)}_{\lambda}(x) \) is the partition polynomial of the
new partition $\lambda^{(1)}$ defined by $\lambda^{(1)} = \langle 1^{2m_2}, 2^{3m_3}, \ldots, (k-1)^{km_k} \rangle$. Continuing in this way, we obtain the following (finite) sequence of partitions $(\lambda^{(d)})_{0 \leq d < k}$ whose partition polynomials are related by differentiation:

$$\lambda^{(d)} = \langle 1^{(d+1)!m_{d+1}/1!}, 2^{(d+2)!m_{d+2}/2!}, \ldots, (k-d)^{!m_k/(k-d)!} \rangle \quad \text{for all } 0 \leq d < k. \quad (3)$$

Strictly speaking, these partitions also have parts of size zero which we exclude, and parts of negative size which occur with multiplicity zero. We have that for all $0 \leq d < k$,

$$\ell \left( \lambda^{(d)} \right) = \sum_{i=d+1}^{k} \frac{i!}{(i-d)!} m_i \quad \text{and} \quad |\lambda^{(d)}| = \sum_{i=d+1}^{k} \frac{i!}{(i-d-1)!} m_i. \quad (4)$$

Therefore, the process of differentiation guarantees that the following relationship between the sizes and lengths of the partitions in the sequence (3) holds for all $1 \leq d \leq k$:

$$|\lambda^{(d-1)}| = \ell \left( \lambda^{(d)} \right) + d!m_d, \quad (5)$$

where $\ell \left( \lambda^{(k)} \right) = 0$ since $\lambda^{(k)}$ is the empty partition.

**Example 5.** Let $\lambda = \langle 1^1, 2^0, 3^3, 4^1 \rangle$. Then we have that

$$\lambda^{(0)} = \langle 1^1, 2^0, 3^3, 4^1 \rangle,$$
$$\lambda^{(1)} = \langle 1^0, 2^9, 3^4 \rangle,$$
$$\lambda^{(2)} = \langle 1^{18}, 2^{12} \rangle,$$
$$\lambda^{(3)} = \langle 1^{24} \rangle.$$

We also see that $\ell \left( \lambda^{(0)} \right) = 5$, $|\lambda^{(3)}| = 4!m_4 = 24$, and

$$|\lambda^{(0)}| = \ell \left( \lambda^{(1)} \right) + 1!m_1 = 14,$$
$$|\lambda^{(1)}| = \ell \left( \lambda^{(2)} \right) + 2!m_2 = 30,$$
$$|\lambda^{(2)}| = \ell \left( \lambda^{(3)} \right) + 3!m_3 = 42.$$

It would be interesting to investigate other relationships among partitions in the sequence (3) which are related by differentiation of partition polynomials, such as patterns in the sequence of lengths or the sequence of sizes from (4) or (5), the zeros of the polynomials, relationships among the polynomials associated to other sequences of partitions, etc.

In addition to studying derivatives, one may also attempt to understand partitions further by studying integrals. Before taking integrals of partition polynomials, we first normalize them and restrict their domain as follows in order to more easily compare partitions. The **normalized partition polynomial** $\hat{f}_\lambda : [0, 1] \rightarrow \mathbb{R}$ of the partition $\lambda = \langle 1^{m_1}, 2^{m_2}, \ldots, k^{m_k} \rangle$ is defined by

$$\hat{f}_\lambda(x) = \frac{1}{\ell(\lambda)} \sum_{i=1}^{k} m_i x^i. \quad (6)$$
Figure 1: Some normalized partition polynomials.

Figure 1 shows the graphs of a few normalized partition polynomials.

It is straightforward to see that the integral of a partition polynomial must have a value between 0 and 1/2 (see Section 3 for more details). Knowing this, it is natural to ask whether one can find partition polynomials with arbitrary integral value in the interval [0, 1/2]. The answer is yes, as shown in the following theorem.

**Theorem 6.** Let $c \in [0, 1/2]$. There exists a sequence of partitions $(\delta_s)$ such that

$$
\lim_{s \to \infty} \int_0^1 \hat{f}_{\delta_s}(x) \, dx = c.
$$

We provide more basic results on integrals of partition polynomials in Section 3, and we prove Theorem 6 in Section 4. The remaining question is what the value of the integral tells us about the partition, and this question is still open, for the most part.

**Question 7.** Is there a combinatorial interpretation of $\int_0^1 \hat{f}_\lambda(x) \, dx$?

Question 7 leads to many other related open questions. For example, it is natural to study integrals of all partition polynomials for partitions of fixed size $n$, but it is still an open problem to obtain a readily accessible combinatorial interpretation of these integral values. This turns out to be difficult, but one could obtain results about partitions of large size $n$ which rely on asymptotic results. It would be desirable to prove a result for all partition sizes $n$ that does not rely on asymptotics.
One could also ask whether certain types of partitions of fixed size \( n \) generally yield larger or smaller integral values. Some computation indicates that although fewer parts do not always correspond to smaller integrals, this trend may hold on average.

**Conjecture 8.** Let \( n \in \mathbb{N} \), and let \( \text{Avg}(n, \ell) \) denote the average integral over all partitions of size \( n \) into \( \ell \) parts. Then we expect that

\[
\text{Avg}(n, 1) \leq \text{Avg}(n, 2) \leq \cdots \leq \text{Avg}(n, n).
\]

Numerical computation confirms that Conjecture 8 holds for \( n \leq 50 \). We partially prove the conjecture by proving the first inequality \( \text{Avg}(n, 1) \leq \text{Avg}(n, 2) \) for all \( n \) and the second inequality \( \text{Avg}(n, 2) \leq \text{Avg}(n, 3) \) in an asymptotic sense for large values of \( n \).

**Theorem 9.** Let \( n \in \mathbb{N} \). Then \( \text{Avg}(n, 1) \leq \text{Avg}(n, 2) \).

Note that \( \text{Avg}(n, \ell) \) may be computed by averaging all of the integrals for partitions of size \( n \) with \( \ell \) parts, but alternatively it may be computed by evaluating the integral of the sum of all partitions of size \( n \) with \( \ell \) parts (see Subsection 3.2 for more details). Instead of considering each individual integral, we can evaluate a single integral involving all of the parts. Although this method uses and produces the same information, it simplifies the estimation. The larger, combined partition has length \( \ell \) times the number of partitions of size \( n \) into \( \ell \) parts.

The proof of Theorem 9 requires one to know the exact form of all partitions of size \( n \) and length \( \ell \) when \( \ell = 1 \) and \( \ell = 2 \). This is a difficult problem in general, so we instead rely on asymptotic approximations for the second inequality. For this reason, the result is only guaranteed for sufficiently large \( n \).

**Theorem 10.** For sufficiently large \( n \), \( \text{Avg}(n, 2) \leq \text{Avg}(n, 3) \).

Proving Conjecture 8 in general seems difficult. More progress could possibly be made in an asymptotic sense using the same method as the proof of Theorem 10. To fully generalize these results, one may need to find a general formula for the number of parts of size \( i \) among all partitions of size \( n \), or among all partitions of size \( n \) and length \( \ell \).

We prove Theorems 9 and 10 in Section 5.

## 2 Proof of Theorem 1

Let \( \lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k}) \). We will prove the derivative formula (2) for \( f_\lambda \) by strong induction. First note that when \( d = 0 \), (2) is trivially true. When \( d = 1 \), we have that

\[
f_\lambda^{(1)}(x) = \sum_{i=1}^{k} im_ix^{i-1},
\]
which satisfies (2) since \( \left\{ \frac{d}{0} \right\} = 0 \). Next, when \( d = 2 \), we have that

\[
\begin{align*}
  f^{(2)}_\lambda(x) &= \sum_{i=2}^{k} i(i-1)m_i x^{i-2} \\
  &= \sum_{i=2}^{k} i^2 m_i x^{i-2} - \sum_{i=2}^{k} i m_i x^{i-2} \\
  &= \sum_{i=2}^{k} i^2 m_i x^{i-2} - \sum_{i=1}^{k} i m_i x^{i-2} + 1m_1 x^{-1} \\
  &= \sum_{i=1}^{k} i^2 m_i x^{i-2} - x^{-1} \sum_{i=1}^{k} i m_i x^{i-1}.
\end{align*}
\]

The second sum above contains the first derivative \( f^{(1)}_\lambda(x) \), so we have that

\[
f^{(2)}_\lambda(x) = \sum_{i=1}^{k} i^2 m_i x^{i-2} - x^{-1} f^{(1)}_\lambda(x),
\]

which also satisfies (2). Now, let \( N \in \mathbb{N} \) and suppose (2) holds for all \( 0 \leq d \leq N \). We rewrite the \( N \)th derivative as follows:

\[
f^{(N)}_\lambda(x) = \sum_{i=N}^{k} i^N m_i x^{i-N} + \sum_{i=1}^{N-1} i^N m_i x^{-(N-i)} - \sum_{j=0}^{N-1} \left\{ \frac{N}{j} \right\} x^{j-N} f^{(j)}_\lambda(x).
\]

Using the same process as above, we take the derivative to obtain

\[
f^{(N+1)}_\lambda(x) = \sum_{i=N+1}^{k} i^N (i-N) m_i x^{i-N-1} - \sum_{i=1}^{N-1} i^N (N-i) m_i x^{-(N-i)-1} - \sum_{j=0}^{N-1} \left\{ \frac{N}{j} \right\} x^{j-N} f^{(j)}_\lambda(x) - \sum_{j=0}^{N-1} \left\{ \frac{N}{j} \right\} x^{j-N} f^{(j+1)}_\lambda(x).
\]

We rewrite the first sum in (7) as

\[
\sum_{i=N+1}^{k} i^N (i-N) m_i x^{i-N-1} = \sum_{i=N+1}^{k} i^{N+1} m_i x^{i-(N+1)} - N x^{-1} f^{(N)}_\lambda(x) - N \sum_{i=1}^{N-1} i^N m_i x^{-(N-i)-1} + N \sum_{j=0}^{N-1} \left\{ \frac{N}{j} \right\} x^{j-N} f^{(j)}_\lambda(x) + N^{N+1} m_N x^{-1}.
\]
Distributing in the second and third sums of (7) and shifting the fourth sum, we now have

\[
f_{\lambda}^{(N+1)}(x) = \sum_{i=N+1}^{k} i^{N+1} m_i x^{i-(N+1)} + \left( N^{N+1} m_N x^{-1} + \sum_{i=1}^{N-1} i^{N+1} m_i x^{-(N+1-i)} \right) \]

\[
+ \left( -N \left\{ \frac{N}{N} \right\} x^{-1} f_{\lambda}^{(N)}(x) - \sum_{j=0}^{N-1} j \left\{ \frac{N}{j} \right\} x^{j-N-1} f_{\lambda}^{(j)}(x) \right) \]

\[
+ \left( N \sum_{i=1}^{N-1} i^{N} m_i x^{-(N+1-i)} - N \sum_{j=0}^{N-1} j \left\{ \frac{N}{j} \right\} x^{j-N-1} f_{\lambda}^{(j)}(x) \right) \]

\[
- N \sum_{i=1}^{N-1} i^{N} m_i x^{-(N+1-i)} + N \sum_{j=0}^{N-1} j \left\{ \frac{N}{j} \right\} x^{j-N-1} f_{\lambda}^{(j)}(x) \]

\[
- \sum_{j=1}^{(N+1)-1} \left\{ \frac{N}{j} \right\} x^{j-N} f_{\lambda}^{(j)}(x). \]

Combining the terms in the grouped expressions above, we obtain

\[
f_{\lambda}^{(N+1)}(x) = \sum_{i=N+1}^{k} i^{N+1} m_i x^{i-(N+1)} + \sum_{i=1}^{(N+1)-1} i^{N+1} m_i x^{i-(N+1)} \]

\[
- \sum_{j=0}^{(N+1)-1} j \left\{ \frac{N}{j} \right\} x^{j-N-1} f_{\lambda}^{(j)}(x) - \sum_{j=1}^{(N+1)-1} \left\{ \frac{N}{j-1} \right\} x^{j-N} f_{\lambda}^{(j)}(x). \]

We combine the first two sums and the last two sums as follows:

\[
f_{\lambda}^{(N+1)}(x) = \sum_{i=1}^{k} i^{N+1} m_i x^{i-(N+1)} - \sum_{j=0}^{(N+1)-1} \left( j \left\{ \frac{N}{j} \right\} + \left\{ \frac{N}{j-1} \right\} \right) x^{j-N-1} f_{\lambda}^{(j)}(x) \]

\[
= \sum_{i=1}^{k} i^{N+1} m_i x^{i-(N+1)} - \sum_{j=0}^{(N+1)-1} \left\{ \frac{N+1}{j} \right\} x^{j-(N+1)} f_{\lambda}^{(j)}(x), \]

where this last equality uses a classical recurrence relation for Stirling numbers of the second kind (for example, see [14]). This completes the proof of Theorem 1.

### 3 Partition polynomials and their integrals

As Figure 1 suggests, there are a few properties that all normalized partition polynomials have in common. Namely, they all start at the point \((0,0)\), end at the point \((1,1)\), and are bounded above by the line \(y = x\).
Proposition 11. Let $\hat{f}_\lambda$ be a partition polynomial. Then following properties hold.

1. $\hat{f}_\lambda(0) = 0$ and $f_\lambda(1) = 1$.

2. $\hat{f}_\lambda(x) \leq x$.

Additionally, we can characterize completely the cases when $\hat{f}_\lambda$ has more than two fixed points.

Proposition 12. Let $\hat{f}_\lambda$ be a partition polynomial. Then $\hat{f}_\lambda(x) = x$ if and only if $m_1 = \ell(\lambda)$; that is, $\lambda$ only contains parts of size 1.

The proofs of Propositions 11 and 12 follow by direct computation. Therefore, if a partition only contains parts of size 1, then $\hat{f}_\lambda(x) = x$ and every point is fixed. Otherwise, $x = 0$ and $x = 1$ are the only fixed points, and $\hat{f}_\lambda(x) < x$ for all $x \in (0, 1)$.

3.1 Integration basics

In this subsection, we show some basic integration results for normalized partition polynomials. Integration on the interval $[0, 1]$ may be viewed as a finite sum of “harmonic-like” numbers. This connection, though straightforward, is fundamental to our work, and the resulting formula will be used extensively throughout this section.

Theorem 13. Let $\hat{f}_\lambda$ be a partition polynomial. Then we have that

$$\int_0^1 \hat{f}_\lambda(x) \, dx = \frac{1}{\ell(\lambda)} \sum_{i=1}^k \frac{m_i}{i+1}.$$  

Proof. We integrate directly to obtain

$$\int_0^1 \hat{f}_\lambda(x) \, dx = \int_0^1 \frac{1}{\ell(\lambda)} \sum_{i=1}^k m_i x^i = \frac{1}{\ell(\lambda)} \sum_{i=1}^k m_i \left. \left( \frac{x^{i+1}}{i+1} \right) \right|_0^1 = \frac{1}{\ell(\lambda)} \sum_{i=1}^k \frac{m_i}{i+1}. $$

We now obtain an elementary bound for integrals of partition polynomials using Theorem 13 and Proposition 11.

Proposition 14. Let $\hat{f}_\lambda$ be a partition polynomial. Then

$$0 < \int_0^1 \hat{f}_\lambda(x) \, dx \leq \frac{1}{2}.$$
Proof. To see the left inequality, we use Theorem 13 and notice the integral is equal to a sum of positive real numbers. Proposition 11 then yields

$$\int_0^1 \hat{f}_\lambda(x) \, dx \leq \int_0^1 x \, dx = \frac{1}{2},$$

and the result follows.

Proposition 12 and the discussion afterwards show that the right inequality in Proposition 14 is strict if a partition contains parts of size greater than 1. Despite this, one can find a sequence of partition polynomials for partitions with parts greater than 1 whose integrals approach \(1/2\). We discuss this further in Section 4.

To make the distinction between partitions with parts greater than 1 and those which only have parts of size 1, we call the former non-trivial partitions.

3.2 Operations on partitions

In this subsection, unless otherwise stated, \(\lambda\) and \(\gamma\) are partitions whose multiplicities are given by the sequence \((a_i)\) and \((b_i)\) respectively. We denote the largest part of \(\lambda\) by \(k\lambda\) and the largest part of \(\gamma\) by \(k\gamma\). We define the sum \(\lambda \oplus \gamma\) as the partition with multiplicities \(a_i + b_i\) and largest part \(\max\{k\lambda, k\gamma\}\). In other words, we obtain the sum \(\lambda \oplus \gamma\) by combining all of the parts of \(\lambda\) and \(\gamma\) into one combined partition. This operation \(\oplus\) on partitions has been previously defined by Schneider [13, Def. 1.2.2] as the product of two partitions.

The partition polynomial of \(\lambda \oplus \gamma\) is given by

$$\hat{f}_{\lambda \oplus \gamma}(x) = \frac{1}{\ell(\lambda) + \ell(\gamma)} \left( \sum_{i=1}^{k\lambda} a_i x^i + \sum_{i=1}^{k\gamma} b_i x^i \right).$$

The following theorem shows what happens when we integrate the partition polynomial of the sum of two partitions.

**Proposition 15.** We have that

$$\int_0^1 \hat{f}_{\lambda \oplus \gamma}(x) \, dx = \frac{\ell(\lambda)}{\ell(\lambda) + \ell(\gamma)} \int_0^1 \hat{f}_\lambda(x) \, dx + \frac{\ell(\gamma)}{\ell(\lambda) + \ell(\gamma)} \int_0^1 \hat{f}_\gamma(x) \, dx.$$

**Proof.** We proceed by direct computation.

$$\int_0^1 \hat{f}_{\lambda \oplus \gamma}(x) \, dx = \frac{1}{\ell(\lambda) + \ell(\gamma)} \left( \sum_{i=1}^{k\lambda} a_i x^i + \sum_{i=1}^{k\gamma} b_i x^i \right)$$

$$= \frac{1}{\ell(\lambda) + \ell(\gamma)} \left( \ell(\lambda) \int_0^1 \hat{f}_\lambda(x) \, dx + \ell(\gamma) \int_0^1 \hat{f}_\gamma(x) \, dx \right).$$

The result follows.
A few important corollaries result from this theorem.

**Corollary 16.** We have that
\[ \int_0^1 \hat{f}_{\lambda \boxplus \lambda}(x) \, dx = \int_0^1 \hat{f}_\lambda(x) \, dx. \]

**Proof.** By Theorem 15, we have that
\begin{align*}
\int_0^1 \hat{f}_{\lambda \boxplus \lambda}(x) \, dx &= \frac{\ell(\lambda)}{\ell(\lambda) + \ell(\lambda)} \int_0^1 \hat{f}_\lambda(x) \, dx + \frac{\ell(\lambda)}{\ell(\lambda) + \ell(\lambda)} \int_0^1 \hat{f}_\gamma(x) \, dx \\
&= \frac{1}{2} \int_0^1 \hat{f}_\lambda(x) \, dx + \frac{1}{2} \int_0^1 \hat{f}_\gamma(x) \, dx \\
&= \int_0^1 \hat{f}_\lambda(x) \, dx.
\end{align*}

\[ \square \]

**Corollary 17.** The sum of two partitions of the same length has integral equal to the average of the individual integrals. Explicitly, if \( \ell(\lambda) = \ell(\gamma) \), then we have that
\[ \int_0^1 \hat{f}_{\lambda \boxplus \gamma}(x) \, dx = \frac{1}{2} \left( \int_0^1 \hat{f}_\lambda(x) \, dx + \int_0^1 \hat{f}_\gamma(x) \, dx \right). \]

The proof of Corollary 17 is similar to the proof of Corollary 16. These results will be used to prove Theorem 6 in Section 4 through a binary search type argument.

### 4 Proof of Theorem 6

We first provide constructive proofs of the \( c = 0 \) and \( c = 1/2 \) cases of Theorem 6.

**Lemma 18.** There exists a sequence of non-trivial partitions \( (\alpha_s) \) such that
\[ \lim_{s \to \infty} \int_0^1 \hat{f}_{\alpha_s}(x) \, dx = 0. \]

**Proof.** Define the partition \( \alpha_s \) by \( (1^1, 2^0, 3^0, \ldots, s^{s-1}) \). Then we observe that
\[ \hat{f}_{\alpha_s}(x) = \frac{1}{s} (x + (s - 1)x^s), \]
and
\[ \lim_{s \to \infty} \int_0^1 \hat{f}_{\alpha_s}(x) \, dx = \lim_{s \to \infty} \frac{1}{s} \left( \frac{1}{2} + \frac{s - 1}{s + 1} \right) = 0. \]

\[ \square \]

A similar result holds for non-trivial partitions whose integral approaches 1/2.
Lemma 19. There exists a sequence of non-trivial partitions \((\beta_s)\) such that

\[
\lim_{s \to \infty} \int_0^1 \hat{f}_{\beta_s}(x) \, dx = 1/2.
\]

Proof. Define the partition \(\beta_s\) by \((1^{s-1}, 2^0, 3^0, \ldots, (s-1)^0, s^1)\). Then we have that \(\hat{f}_{\beta_s}(x) = \frac{1}{s}((s-1)x + x^s)\), and

\[
\lim_{s \to \infty} \int_0^1 \hat{f}_{\beta_s}(x) \, dx = \lim_{s \to \infty} \frac{1}{s} \left( \frac{s-1}{2} + \frac{1}{s+1} \right) = 1/2.
\]

The following theorem shows that a weaker version of Theorem 6 holds under special conditions.

Theorem 20. Suppose that \(\alpha\) and \(\beta\) are partitions such that

\[
\int_0^1 \hat{f}_\alpha(x) \, dx = a \quad \text{and} \quad \int_0^1 \hat{f}_\beta(x) \, dx = b.
\]

If \(0 < a < b < 1/2\) and \(\ell(\alpha) = \ell(\beta)\), then for every \(c \in (a, b)\), there exists a sequence \((\delta_s)\) of partitions such that

\[
\lim_{s \to \infty} \int_0^1 \hat{f}_{\delta_s}(x) \, dx = c.
\]

Proof. Let \(\epsilon > 0\) be given. We construct \((\delta_s)\) recursively by first constructing two sequences \((\beta_s)\) and \((\alpha_s)\) of partitions such that each \(\beta_s\) approximates \(c\) from above and each \(\alpha_s\) approximates \(c\) from below. Define \(\beta_1 = \beta\), \(\alpha_1 = \alpha\), and \(\delta_1 = \alpha_1 \oplus \beta_1\). The sequence \((\delta_s)\) will be defined by \(\delta_s = \alpha_s \oplus \beta_s\).

Let \(r \geq 1\). If \(c = \int_0^1 \hat{f}_{\delta_r}(x) \, dx\), then define \(\delta_s = \delta_r\). Otherwise, we have one of the following two cases.

Case 1: \(c < \int_0^1 \hat{f}_{\delta_r}(x) \, dx\). Define \(\beta_{r+1} = \delta_r\) and \(\alpha_{r+1} = \alpha_r \oplus \alpha_r\). Since \(\ell(\beta_{r+1}) = 2^{r+1}\ell(\alpha) = \ell(\alpha_{r+1})\), we see that

\[
\int_0^1 \hat{f}_{\delta_{r+1}}(x) \, dx = \frac{1}{2} \left( \int_0^1 \hat{f}_{\beta_{r+1}}(x) \, dx + \int_0^1 \hat{f}_{\alpha_{r+1}}(x) \, dx \right).
\]

Case 2: \(c > \int_0^1 \hat{f}_{\delta_r}(x) \, dx\). Define \(\beta_{r+1} = \beta_r \oplus \beta_r\) and \(\alpha_{r+1} = \delta_r\). Since \(\ell(\beta_{r+1}) = 2^{r+1}\ell(\alpha) = \ell(\alpha_{r+1})\), we see that

\[
\int_0^1 \hat{f}_{\delta_{r+1}}(x) \, dx = \frac{1}{2} \left( \int_0^1 \hat{f}_{\beta_{r+1}}(x) \, dx + \int_0^1 \hat{f}_{\alpha_{r+1}}(x) \, dx \right).
\]
Note that by definition,
\[ \int_0^1 \hat{f}_{\delta_s}(x) \, dx = \frac{1}{2} \left( \int_0^1 \hat{f}_{\beta_s}(x) \, dx + \int_0^1 \hat{f}_{\alpha_s}(x) \, dx \right). \]

Since the integral for $\alpha_s$ is less than the integral for $\beta_s$, we obtain the following inequality:
\[ \int_0^1 \hat{f}_{\alpha_s}(x) \, dx < \int_0^1 \hat{f}_{\delta_s}(x) \, dx < \int_0^1 \hat{f}_{\beta_s}(x) \, dx. \]

Now let $N > \log_2 \left( \frac{b-a}{\epsilon} \right)$. The previous inequality, along with the fact that $c$ is an upper bound for the integral for $\alpha_s$, yields the following for all $s \geq N$:
\[ \left| \int_0^1 \hat{f}_{\delta_s}(x) \, dx - c \right| < \int_0^1 \hat{f}_{\beta_s}(x) \, dx - \int_0^1 \hat{f}_{\alpha_s}(x) \, dx \leq \frac{b-a}{2^s} \leq \frac{b-a}{2^N} < \epsilon. \]

Therefore, $\lim_{s \to \infty} \int_0^1 \hat{f}_{\delta_s}(x) \, dx = c$ as desired. \(\square\)

We are now ready to prove Theorem 6.

**Proof of Theorem 6.** The edge cases $c = 0$ and $c = 1/2$ follow from Lemma 18 and Lemma 19, respectively. Thus, it suffices to show that the result holds for $c \in (0, 1/2)$. Define the partitions
\[ \alpha_s = \langle 1^1, 2^0, 3^0, \ldots, (s-1)^0, s^{s-1} \rangle \quad \text{and} \quad \beta_s = \langle 1^{s-1}, 2^0, 3^0, \ldots, (s-1)^0, s^1 \rangle. \]

Then we have that
\[
\int_0^1 \hat{f}_{\alpha_s}(x) \, dx = \frac{1}{s} \left( \frac{1}{2} + \frac{s-1}{s+1} \right), \]
\[
\int_0^1 \hat{f}_{\beta_s}(x) \, dx = \frac{1}{s} \left( \frac{s-1}{2} + \frac{1}{s+1} \right).
\]

Since $\ell(\alpha_s) = \ell(\beta_s)$ for all $s \in \mathbb{N}$, Theorem 20 shows that the result holds for all $c$ with
\[ \int_0^1 \hat{f}_{\alpha_s}(x) \, dx < c < \int_0^1 \hat{f}_{\beta_s}(x) \, dx. \]

For increasing $s$, these are nested intervals. The union of all such intervals is the set $(0, 1/2)$. This proves Theorem 6. \(\square\)
5 Proofs of Theorems 9 and 10

Proof of Theorem 9. Recall that one may compute Avg(n, ℓ) by evaluating the integral of the sum of all partitions of n with ℓ parts. It is straightforward to enumerate all partitions of n into one part and all partitions of n into two parts. Adding all of the parts together for ℓ = 1 and ℓ = 2 gives the partition λ₁ = ⟨1⁰, 2⁰, . . . , n¹⟩ for all n, and the partitions

\[ \lambda_2^{\text{odd}} = ⟨1¹, 2¹, . . . , (n − 1)¹, n⁰⟩ \]

if n is odd or

\[ \lambda_2^{\text{even}} = ⟨1¹, 2¹, . . . , [(n/2) − 1]¹, (n/2)², [(n/2) + 1]¹, . . . , (n − 1)¹, n⁰⟩ \]

if n is even. Let \( H_n \) denote the nth Harmonic number. Then the averages are

\[
\text{Avg}(n, 1) = \int_0^1 \hat{f}_{\lambda_1}(x) \, dx = \frac{1}{n + 1}
\]

for all n; and

\[
\text{Avg}(n, 2) = \int_0^1 \hat{f}_{\lambda_2^{\text{odd}}}(x) \, dx = \frac{1}{2 \cdot \left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) = \frac{1}{2 \cdot \left\lfloor \frac{n}{2} \right\rfloor} (H_n - 1)
\]

if n is odd, or

\[
\text{Avg}(n, 2) = \int_0^1 \hat{f}_{\lambda_2^{\text{even}}}(x) \, dx = \frac{1}{2 \cdot \left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n/2} \right) = \frac{1}{2 \cdot \left\lfloor \frac{n}{2} \right\rfloor} \left( H_n - 1 + \frac{2}{n} \right)
\]

if n is even. Noting that \( \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \), we see that

\[
\text{Avg}(n, 2) \geq \frac{1}{2} \cdot \frac{1}{\left\lfloor \frac{n}{2} \right\rfloor} (H_n - 1) \geq \frac{1}{n + 1} (H_n - 1) \geq \text{Avg}(n, 1)
\]

for all \( n \geq 4 \). One can check by hand that the result also holds for \( n < 4 \). Thus, we have that \( \text{Avg}(n, 1) \leq \text{Avg}(n, 2) \) for all \( n \in \mathbb{N} \).

Note in the above proof that we have the asymptotic expression \( \text{Avg}(n, 2) \sim \ln(n)/n \) as \( n \to \infty \), since the nth harmonic number is asymptotic to \( \ln(n) \).

Proof of Theorem 10. We will show that \( \text{Avg}(n, 3) \) is asymptotically bounded below by \( 2 \ln(n)/n \). Let \( n \in \mathbb{N} \), and let \( \lambda_3 \) be the sum of all partitions of n into 3 parts. To get a lower bound on the multiplicity of each part in \( \lambda_3 \), let \( 1 \leq i \leq n - 2 \), and note that the following sums give partitions of n into 3 parts:

\[
1 + i + (n − (i + 1)) = 2 + i + (n − (i + 2)) = \cdots = (i − 1) + i + (n − (2i − 1)),
\]

\[
i + i + (n − 2i) = i + (i + 1) + (n − (2i + 1)) = \cdots = i + \left\lfloor \frac{n − i}{2} \right\rfloor + \left\lceil \frac{n − i}{2} \right\rceil.
\]
Counting the number of times $i$ appears, there are at least $i - 1$ instances in the first row and at least $\left\lfloor \frac{n - i}{2} \right\rfloor - i + 1$ in the second row. This gives a lower bound of

$$i - 1 + \left\lfloor \frac{n - i}{2} \right\rfloor - i + 1 = \left\lfloor \frac{n - i}{2} \right\rfloor - \frac{n - i}{2} - 1$$

parts of size $i$ among all partitions of $n$ into 3 parts. Define the function $g : [0, n - 2] \rightarrow \mathbb{R}$ by $g(x) = \frac{n}{2} - 1 - \frac{n/2}{n - 2}x$. We use this function to obtain a lower bound for the integral of $f_{\lambda_3}$, as it always lies below the bound achieved above. We see that

$$\text{Avg}(n, 3) = \int_0^1 \hat{f}_{\lambda_3}(x) \, dx$$

$$\geq \frac{1}{\ell(\lambda_3)} \int_0^{n-2} \frac{g(x)}{x+1} \, dx$$

$$= \frac{1}{\ell(\lambda_3)} \cdot \left( \left( \frac{n}{2} - 1 \right) \ln(n - 1) - (n - 2) - \ln(n - 1) \right) \cdot \frac{n/2}{n - 2}$$

$$\sim \frac{1}{\ell(\lambda_3)} \cdot \frac{n}{2} \ln(n)$$

as $n \rightarrow \infty$. Also, as $n \rightarrow \infty$ we have that $\ell(\lambda_3) \sim 3 \cdot \frac{n^2}{12}$, by a known asymptotic formula for the number of partitions of size $n$ into 3 parts (for example, see [7]). This yields

$$\text{Avg}(n, 3) \geq \frac{1}{\ell(\lambda_3)} \cdot \frac{n}{2} \ln(n) \sim \frac{1}{3} \cdot \frac{n^2}{12} \cdot \frac{n}{2} \ln(n) = \frac{2 \ln(n)}{n} \sim 2\text{Avg}(n, 2)$$

as $n \rightarrow \infty$. Therefore, $\text{Avg}(n, 3) \geq \text{Avg}(n, 2)$ for sufficiently large $n$, as desired. \[\square\]

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