The Initial Value Problem in Light of Ashtekar’s Variables

**Abstract**

The form of the initial value constraints in Ashtekar’s hamiltonian formulation of general relativity is recalled, and the problem of solving them is compared with that in the traditional metric variables. It is shown how the general solution of the four diffeomorphism constraints can be obtained algebraically provided the curvature is non-degenerate, and the form of the remaining (Gauss law) constraints is discussed. The method is extended to cover the case when matter is included, using an approach due to Thiemann. The application of the method to vacuum Bianchi models is given. The paper concludes with a brief discussion of alternative approaches to the initial value problem in the Ashtekar formulation.

**1 Introduction**

It is with great pleasure that we dedicate this paper to Dieter Brill, our teacher, advisor, and colleague, on the occasion of his 60th birthday. Our contribution concerns the initial value problem for general relativity, which...
is amongst Dieter’s many areas of expertise. As is the case with almost all
research activity developed around the general relativity group at the Uni-
versity of Maryland, the ideas we will present have benefitted from Dieter’s
always kind and sometimes maddening insightful questioning. Of course it
is our wish that this paper will prompt some more such questioning.

General relativity is invariant under four dimensional diffeomorphisms. In
the hamiltonian formulation, this invariance manifests itself in the presence
of constraints on the canonical variables. If the constraints are satisfied at
one instant of time, they continue to hold at all times. The initial value
problem is the problem of giving a construction for a parametrization of the
general solution of the constraints.

The most well-developed approach to the initial value problem is based on
the “conformal technique” (see for example [1] and references therein). In this
approach, the phase space variables for general relativity are the 3-metric $q_{ab}$
and its conjugate momentum $p^{ab}$ which is a tensor density of weight 1, closely
related to the extrinsic curvature of the spatial hypersurface in a solution to
the field equations. There are four constraints, consisting of one Hamiltonian
constraint, which generates diffeomorphisms normal to the spacelike surface
on which the initial data are given, and three momentum constraints, which
generate spatial diffeomorphisms. The Hamiltonian constraint is viewed as
a (quasi-linear, elliptic) equation determining the conformal factor of the
3-metric. The momentum constraints determine the “longitudinal” piece of
$p^{ab}$. In this approach, the freely specified data are, in principle, the confor-
mal equivalence class of the 3-metric, and the transverse traceless part of the
momentum $p^{ab}$.

The conformal technique is well suited to addressing questions of existence
and uniqueness of solutions to the field equations. However, for two reasons
it is not suited to reduction of the phase space to the unconstrained degrees
of freedom. First, the “freely specifiable” transverse traceless tensor densities
are not known for an arbitrary conformal 3-metric. Second, the solutions to
the constraint equations determining the conformal factor and longitudinal
part of the momentum are not given explicitly.

An alternative representation for hamiltonian general relativity was intro-
duced in the mid-eighties by Ashtekar [2]. The Ashtekar representation may
be understood (and was discovered originally) as resulting from a complex
canonical transformation that goes from the (triad-extended) ADM grav-
itational phase space variables to a new set of variables. The new variables
are a spatial SO(3,C) connection, and an so(3,C)-valued vector density as its conjugate momentum. The major benefit of this representation is that the constraints take a simpler form in terms of these variables than in the ADM representation. First, the constraints turn out naturally to be polynomial of low order in the Ashtekar phase space variables. Moreover, for the vacuum case, the scalar constraint is homogenous in the Ashtekar canonical momenta. This simplification has allowed considerable progress in many questions of gravitational physics. Perhaps the most impressive results have been obtained in the quantum theory, where this formulation has made it possible to perform the first few steps in the Dirac quantization of general relativity [3, 4, 5].

Given that the Ashtekar representation simplifies the form of the constraints, it is natural to reconsider the classical initial value problem in this context. As it turns out, generically in phase space, it is possible to obtain algebraically the general solution to the Ashtekar version of the diffeomorphism constraints of general relativity! [6, 7] However, this does not give an easy solution of the initial value problem. The reason is that, due to the covariance of the Ashtekar formalism under local SO(3,C) rotations, there are three additional constraints whose form is identical to the non-abelian form of Gauss’ law familiar from Yang-Mills theories. Once the diffeomorphism contraints are solved by the method to be described below, one still has to face the Gauss law, which has now become a non-linear first order partial differential equation on the remaining variables. In addition, the reality conditions restricting to the phase space of real, Lorentzian GR need to be enforced.

Still, one can hope that the relocation of the difficulties afforded by the Ashtekar approach may prove to be fruitful in some applications. This already seems to be the case in the context of GR reduced by symmetry conditions. In addition, the interplay between “gauge-fixing” and solving the constraints is different in the Ashtekar approach, which is a fact that remains to be fully explored.

The rest of this paper is organized as follows. The form of the initial value constraints in Ashtekar’s hamiltonian formulation of general relativity is recalled in section 2, and in section 3 it is shown how the general solution of the four diffeomorphism constraints in vacuum (or with a cosmological constant) can be obtained algebraically provided the curvature is non-degenerate. The form of the remaining (Gauss law) constraints is also discussed, as is the
relation between the solution given and the structure of the 4-dimensional curvature. In section 4 the method is extended to cover the case when matter is included, using an approach due to Thiemann. The application of the method to vacuum Bianchi models is given in section 5, and the paper concludes in section 6 with a brief discussion of alternative approaches to the initial value problem in the Ashtekar formulation.

2 Ashtekar’s variables

Ashtekar’s representation of Hamiltonian general relativity has been the subject of extensive reviews (cf. [5, 8]), so we shall just recall its main features.

The Ashtekar canonical coordinates are an SO(3,C) spatial connection, \( A^i_a \), and an \( so(3,C) \)-valued vector density of weight 1, \( E^a_i \). The fundamental Poisson bracket is given by

\[
\{ A^i_a(x), E^b_j(y) \} = i\delta^i_j\delta^a_b\delta^3(x,y). \tag{1}
\]

(Our notation is as follows. Latin letters from the beginning of the alphabet denote spatial indices, \( e.g. \ a, b, ... = 1, 2, 3 \). Latin letters from the middle of the alphabet are SO(3,C) indices, \( i, j, ... = 1, 2, 3 \). They are raised and lowered with the Kronecker delta \( \delta^i_j \) and \( \delta_{ij} \). We will also use the totally antisymmetric symbol \( \epsilon_{ijk} \), with \( \epsilon_{123} = 1 \). We use units with \( c = G = 1 \).)

The vector density \( E^a_i \) may be identified with the densitized spatial triad \( \sqrt{q}e^a_i \), with the contravariant spatial metric given by \( q^{ab} = e^a_i e^b_i \). In turn, the SO(3,C) connection may be identified in a solution with the spatial pullback of the self-dual part of the spin-connection.

These variables parametrize the phase space of complex general relativity. A real metric with Lorentzian signature may be recovered by imposing appropriate reality conditions (cf. [3]). These conditions amount to the requirement that \( E^a_i E^{bi} \) be real, and that its time derivative be real. If these conditions are satisfied initially then the dynamical evolution will preserve them in time. A metric of Euclidean signature is obtained by simply taking \( A^i_a \) and \( E^a_i \) real.

In terms of these variables, the constraints of (complex) general relativity take the form

\[
\varepsilon_{abc}\epsilon^{ijk}E^a_i E^b_j B^c_k = -\rho \tag{1}
\]
\[
\varepsilon_{abc}E^b_i B^c_i = -iJ_a \tag{2}
\]
\[
D_a E^a_i = K_i. \tag{3}
\]
Here $\varepsilon_{abc}$ is the standard Levi-Civita tensor density. $B_a^i$ is the “magnetic field” of the connection $A_i^a$, defined by $B_a^i := \varepsilon^{abc} F_{bci}$, and $F_{ab}^i := \partial_b A_a^i - \partial_a A_b^i + \epsilon_i^{jkl} A_j^a A_k^b$. $D_a$ is the SO(3)-covariant derivative determined by $A_i^a$. $ho/\sqrt{q}$ and $J_a$ are (up to coefficients) the matter energy and momentum densities respectively. $K_i$ is a spin density, present only for half-integer spin matter fields.

In the following, we will call the constraints (1) and (2) “diffeomorphism constraints”. The first generates diffeomorphisms normal to the spatial hypersurface $\Sigma$ together with some SO(3) rotation, so it takes the place in Ashtekar’s formalism of the ADM hamiltonian constraint. It will be called here the “scalar constraint”. The second generates diffeomorphisms tangential to $\Sigma$ together with some SO(3) rotation, so it takes the place of the ADM momentum constraint. It will be called here the “vector constraint”.

There are 3 additional constraints, (3), of the Gauss-law type familiar from Yang-Mills theories. They generate local SO(3,C) rotations, under which the Ashtekar formalism is covariant.

The constraints are polynomial in the gravitational phase space variables and for scalar and spin-1/2 matter. For Yang-Mills type fields, polynomiality is maintained only if one multiplies the scalar constraint (1) through by $\det q = \det E^a_i$, which is cubic in $E^a_i$. Moreover, it is remarkable that the gravitational contribution to the scalar constraint (1) is homogenous in the momenta $E^a_i$. This should be compared with the ADM Hamiltonian constraint, where the term quadratic in the canonical momenta must be balanced by a “potential” term given by the 3D Ricci scalar times the determinant of the 3-metric. Note that while the vector and Gauss constraints (2) and (3) are densities of weight 1, the scalar constraint (1) is of weight 2.

### 3 Algebraic solution of the vacuum diffeomorphism constraints

We now proceed to show how one can use the simplification of the constraints for general relativity provided by the Ashtekar formalism to tackle the initial value problem. In particular, we will show how the general solution of the the scalar and vector diffeomorphism constraints can be obtained by algebraic methods. We originally discovered this solution in the context of
a Lagrangian pure spin-connection formulation of GR, in which one solves for the metric variables (self-dual 2-forms) in terms of the connection \( \mathcal{H} \). However, it is unnecessary to view the technique in that context.

The first step is to assume that the magnetic field \( B^a_i \) is non-degenerate as a 3 \times 3 matrix. In a generic real, Lorentzian spacetime, the real and imaginary parts of the complex equation \( \det B = 0 \) will define two surfaces, and their intersection will give a one-dimensional submanifold on which \( B^a_i \) is degenerate. That is, generically \( B^a_i \) is non-degenerate except on a set of measure zero. (The points in phase space with \( \det B = 0 \) everywhere form a set of measure zero in phase space. The identity of these points is best seen in the covariant formalism (cf. [6, 9]). It turns out that they correspond to space-time metrics of Petrov type 0, (4), (3,1), when \( E^a_i \) is non-degenerate.) We will ignore here any problems that might arise as a consequence of degeneracy of \( B^a_i \).

Now, since \( B^a_i \) is assumed to be non-degenerate, one may use it as a basis for the space of SO(3,C) vector densities. In particular, one can expand the momentum \( E^a_i \) with respect to it,

\[
E^a_i = P_{ij} B^{aj} \quad (4)
\]

for some (non-degenerate) 3 \times 3 matrix \( P_{ij} \).

We first consider the vacuum case, \( \rho = 0, J_a = 0, K_i = 0 \). It is easy to see that the vector constraint (2) implies that \( P_{ij} \) must be symmetric. The scalar constraint (1) implies that \( P_{ij} \) must satisfy a quadratic algebraic condition,

\[
(P^i_i)^2 - P_{ij} P^{ij} = 0 \quad (5)
\]

This condition fixes one of the six components of \( P_{ij} \) with respect to the others.

We know of two methods for solving the scalar constraint (5). In the first method, one solves (3) for the trace of \( P \). Decomposing \( P_{ij} \) into the trace \( TrP \) and tracefree part \( \tilde{P}_{ij} := P_{ij} - \frac{1}{3} TrP \delta_{ij} \), (4) becomes the condition

\[
TrP = \pm \left( \frac{3}{2} Tr\tilde{P}^2 \right)^{1/2} \quad (6)
\]

In the second method for solving (5), one notes that the characteristic equation for \( P \) implies that \( TrP^{-1} = [TrP^2 - (TrP)^2]/2\det P \). Thus (5) is equivalent to the tracelessness of \( P^{-1} \), provided \( P \) is invertible (which
it must be if both $B_i^a$ and the 3-metric are nondegenerate). Let $\psi$ denote $P^{-1}$. Then the characteristic equation for $\psi$, assuming $Tr\psi = 0$, reads $\psi^3 - \frac{1}{2}(Tr\psi^2)\psi - det\psi I = 0$, giving $\psi^{-1} = [\psi^2 - \frac{1}{2}(Tr\psi^2)I]/det\psi$. Thus the general solution to (5) for invertible $P$ can be explicitly written in terms of the 5 independent components of a tracefree $3 \times 3$ matrix $\phi$ in the form

$$P = \phi^2 - \frac{1}{2}(Tr\phi^2)I.$$  

(7)

One is left with the Gauss constraint (3), which becomes

$$B_a^i D_a P_{ij} = 0,$$

(8)

where we have used the Bianchi identity $D_a B_i^a = 0$. In view of (5) or (7), the Gauss constraint becomes a non-linear equation in $\hat{P}_{ij}$ or $\phi$ respectively. It is here, and in the reality conditions, that the initial value problem now resides.

We can consider the Gauss law constraint as 3 conditions on the 5 independent components of $P_{ij}$, for a fixed $A_i^a$. Then there are presumably 2 free functions worth of solutions for this equation, yielding the two independent metric degrees of freedom of GR. Up to this stage the connection $A_i^a$ has remained entirely arbitrary. By use of the 4 parameter diffeomorphism and 3 parameter SO(3) transformations, one could now fix all but 2 of the 9 components of the connection, yielding the other half of the coordinates on the reduced phase space.

If a spherically symmetric ansatz is assumed, then the above method of solving the constraints can be carried out entirely, and the Gauss constraint reduces to an ordinary differential equation that can be solved [10].

The existence in general of solutions to the Gauss law constraint (8) with $P_{ij}$ restricted by (3) or (7) is a problem that has not been addressed. To indicate just one complication that can arise, consider the asymptotically flat case. If $A_i^a$ is asymptotically the spin-connection corresponding to a negative-mass spacetime, then we know by the positive energy theorem that it must be impossible to find a regular solution to (8) for $P_{ij}$, subject to the reality conditions.

The geometrical interpretation of the matrix $P_{ij}$ is easily seen from the covariant point of view [11]. The densitized triad $E_{ai}$ of the Ashtekar formalism is related to the triad of anti-self-dual 2-forms by $E_{ai} = \epsilon^{abc} \Sigma_i^{bc}$ [12]. Now the curvature 2-form $R^i$ of the spin-connection can be expanded in terms of
the self-dual and anti-self-dual 2-forms as as \( R_i = \psi_{ij} \Sigma^j + \frac{1}{3} \Lambda \Sigma_i + \Phi_{ij} \Sigma^j \). \( \psi_{ij} \) is symmetric and tracefree and is just the Weyl spinor in SO(3) notation. \( \Lambda \) is proportional to the Ricci scalar, and \( \Phi_{ij} \) is equivalent to the tracefree part of the Ricci tensor. Thus in vacuum, \( \Lambda \) and \( \Phi_{ij} \) vanish, and one can solve for the anti-self-dual 2-forms in terms of the curvature as \( \Sigma_i = (\psi^{-1})^{ij} R_j \). The dual of the spatial pullback of this equation yields immediately the general solution given above for the four diffeomorphism constraints in Ashtekar’s formalism, with \( P^{ij} \) identified with \( (\psi^{-1})^{ij} \).

Let us now consider the addition of a cosmological constant \( \Lambda \). This modifies only the scalar constraint, which becomes

\[
\varepsilon_{abc} \varepsilon^{ijk} E^a_i E^b_j B^c_k - (1/3) \Lambda \varepsilon_{abc} \varepsilon^{ijk} E^a_i E^b_j E^c_k = 0 .
\]

One may now proceed as in the vacuum case. The only difference is that the algebraic condition (5) is replaced by

\[
(P_i^i)^2 - P_{ij} P^{ij} = 2\Lambda (\text{det} P) .
\]

This can be regarded as a cubic equation for \( \text{Tr} P \). It is equivalent to the statement that the inverse of \( P \) has trace equal to \( \Lambda \). In this case, \( P \) corresponds to the inverse of the matrix \( (\psi_{ij} + \frac{1}{3} \Lambda \delta_{ij}) \).

An interesting special solution of the constraints in the case of a non-vanishing \( \Lambda \) is given by the Ansatz \( P_{ij} = (3/\Lambda) \delta_{ij} \), or

\[
E^a_i = (3/\Lambda) B^a_i ,
\]

which solves automatically all of the constraints. (The Gauss constraint is satisfied as a consequence of the Bianchi identity \( D_a B^a_i = 0 \).) This Ansatz was first introduced by Ashtekar and Renteln \[13\], who observed that it gives rise to self-dual solutions of the Einstein equation with a non-vanishing \( \Lambda \).

## 4 Matter couplings

It is possible to extend the method just described to solve the diffeomorphism constraints in the presence of matter using an approach that has recently been brought to our attention by Thomas Thiemann \[7\]. Thiemann’s method proceeds as follows.
One begins with the constraints in the form \(1\), \(2\), \(3\) \cite{14}. Then expanding \(E_i^a = P_{ij} B^{aj}\) as in \(4\), one finds that the vector constraint \(2\) implies that the anti-symmetric part of \(P_{ij}\) no longer vanishes, but is given instead by

\[
A_{ij} := P_{[ij]} = \frac{i}{2B} \varepsilon_{ijk} B^a_k J_a,
\]

where \(B := det B^a_i\). This is an explicit expression for \(P_{[ij]}\) provided \(J_a\) does not depend upon \(P_{[ij]}\) as well. Since \(J_a\) is just the generator of spatial diffeomorphisms for the matter variables, it does not depend on the gravitational field variables for the case of integer spin matter. In the spin-1/2 case, the diffeomorphism is accompanied by an SO(3) rotation, which involves the spin connection but not \(E_i^a\). In all cases therefore, \(J_a\) is independent of \(P_{ij}\).

(In particular, in the scalar, spin-1/2, and Yang-Mills cases, the structure of \(J_a\) is \(\pi \partial_a \phi\), \(\pi^A D_a \psi_A\), and \(e^{bl} f_{abl}\) respectively, where \(e^{bl}\) and \(f_{abl}\) are the Yang-Mills-electric and (dual of) magnetic fields.)

Now turning to the scalar constraint, we begin by noting that when \(P_{ij}\) is decomposed into its symmetric and antisymmetric parts \(P_{ij} = S_{ij} + A_{ij}\), the left hand side of the scalar constraint \(1\) is given by

\[
\varepsilon_{ijk} \varepsilon_{abc} E_i^a E_i^b B^{ck} = det B \left[(Tr S)^2 - Tr S^2 - Tr A^2 \right].
\]

\[
(13)
\]

Let us first consider the case of a single scalar field. Then we have

\[
\rho = \pi^2 + E_i^a E^b_i \partial_a \phi \partial_b \phi + \varepsilon_{ijk} \varepsilon_{abc} E_i^a E_i^b E_i^{ck} V(\phi).
\]

\[
(14)
\]

Now as in the vacuum case there are two approaches to solving the scalar constraint. In the first method, one regards the constraint as a cubic equation on the trace of \(S_{ij}\), which can be solved (albeit messily) in closed form. In the second approach, due to Thiemann \cite{7}, one notes that every term of the constraint is either independent of the scalar field momentum \(\pi\) or depends on \(\pi\) quadratically! To see why, observe that \(J_a\) is linear in \(\pi\), and therefore according to \(12\), so too is \(A_{ij}\). Thus the gravitational part of the constraint \(13\) is quadratic in \(\pi\). As for \(\rho\) \(14\), the second (gradient) term is independent of \(\pi\) because, as one easily sees using \(12\), \(A_{ij}\) drops out of the expression \(E_i^a \partial_a \phi = S^{ij} B^{aj} \partial_a \phi\). Moreover, in the third term one can show that \(det E\) involves \(A_{ij}\), and therefore \(\pi\), also only quadratically.
Thus one can simply solve the constraint for $\pi$ in closed form. Even for the case $V(\phi) = 0$, the resulting expression is fairly complicated:

$$\pi = \pm \left[ \left( (TrS)^2 - TrS^2 + (S\xi)^2 \right) / \left( \xi^2 - 1 \right) \right]^{1/2}, \quad (15)$$

where $\xi^i := B^{ai} \partial_a \phi$.

For the real theory, there is also the constraint that the argument of the square root be a positive real number. When $V(\phi) \neq 0$, the solution for $\pi$ becomes significantly more complicated. It is unlikely that this is of any practical use in general. However, Thiemann’s intended application is quantization of the spherically symmetric scalar-gravitational system, for which the resulting expressions may be more useful.

Note that even when the argument of the square root in (15) is real, the solution for $\pi$ has a sign ambiguity. This can lead to problems if one wishes to pass to the reduced phase space for the purpose of quantization, since it is difficult to eliminate the momentum and not lose part of the reduced phase space[15].

For other types of matter coupling the situation becomes yet more complicated. For a massless spin-1/2 field, the scalar constraint remains quadratic (but not homogeneous) in the two components $\pi^A$ of the spinor field momentum, so one can in principle solve for one of these components. In addition, the Gauss constraint is augmented by a spinor field term in this case. When Yang-Mills fields are included, the scalar constraint remains polynomial only if it is multiplied through by $det E$, resulting in a 4th order polynomial in the scalar field momentum.

5 Bianchi models

In this section, we apply to spatially homogenous vacuum space-times, i.e. Bianchi models, the method for the solution of the constraints illustrated in section 3 for the vacuum case. The hope is to gain some insight on how to deal with the remaining, Gauss constraint in the form [8]. In any case, this analysis may be of use in a minisuperspace approximation of vacuum general relativity.

For Bianchi models, as follows from the assumption of spatial homogeneity, this last constraint reduces to an algebraic condition. There are some
simplifications, but the condition is still rather complicated. We record it here for the benefit of inventory. One interesting feature is that only the traceless part of $P_{ij}$ enters in it.

The formulation of Bianchi models in terms of Ashtekar variables was first worked out by Kodama [16]. Here we follow the approach of Ashtekar and Pullin [17].

As is familiar from the ADM treatment of Bianchi models (see e.g. [18, 19]), we consider a kinematical triad of vectors, $X^a_I$, which commute with the three Killing vectors on the spatial hypersurface $\Sigma$. (Capital latin letters $I,J,K,...$ will be used to label the triad vectors.) The triad satisfies

$$[X_I,X_J]^a = C^a_{IJ} X^K_K, \quad (16)$$

where $C^a_{IJ}$ denote the structure constants of the Bianchi type under consideration. The basis dual to $X^a_I$, $\chi_I^a$, satisfies,

$$2\partial_a \chi^I_b = -C^a_{JK} \chi^J_a \chi^K_b. \quad (17)$$

Without loss of generality, one may set

$$C^a_{IJ} = \epsilon_{IJL} S^{LK} + 2\delta^K_{[I} V_{J]} \quad (18)$$

with $S^{IJ}$ symmetric. (This $S^{IJ}$ has nothing to do with the symmetric part $S^{ij}$ of $P^{ij}$ referred to earlier.) From the Jacobi identities, it follows that $S^{IJ} V_I = 0$. The Bianchi classification may be described in terms of the vanishing or not of $V_I$, and the signature of $S^{IJ}$, subject to this condition. The most popular models are Bianchi I, selected by $C^i_{IJ} = 0$, and Bianchi IX, selected by $V_I = 0$ and $S^{IJ} = \delta^{IJ}$.

The Ashtekar gravitational phase space variables may be expanded with respect to the kinematical triad $X^a_I$, and $\chi^I_a$, as

$$A^i_a = A^i_M \chi^M_a \quad (19)$$

$$E^a_i = det \chi E^M_i X^a_M \quad (20)$$

where $det \chi$ denotes the determinant of $\chi^I_a$, which is introduced in order to de-densitize $E^a_i$. Similarly, the magnetic field may be expanded as

$$B^a_i = det \chi B^M_i X^a_M, \quad (21)$$
and $B_i^M$ is given by

$$B_i^M = -\epsilon^{MNP} C_{NP}^Q A_{Qi} + \epsilon^{MNP} \epsilon_{ijk} A^i_M A^j_P .$$  \hfill (22)

The gravitational phase space has thus been reduced to the matrices $\{A^i_M, E_i^M\}$, \textit{i.e.} from 18 degrees of freedom per space point to only 18 in total.

The constraints for vacuum reduce to

$$\epsilon_{MNP} \epsilon^{ijk} E_i^M E_j^N B_k^P = 0 \quad \hfill (23)$$

$$\epsilon_{MNP} E_i^N B^P_i = 0 \quad \hfill (24)$$

$$C_{KM}^K E_i^M + \epsilon_{ijk} A^j_M E^M_k = 0 \quad \hfill (25)$$

where $B_i^M$ is given by (22).

We can follow now the footsteps of section 3, for the solution of these constraints. Assuming that $B_i^M$ is non degenerate, we expand $E_i^M$ in terms of it:

$$E_i^M = P_{ij} B^{Mj} .$$  \hfill (26)

From (24), we find that $P_{ij}$ must be symmetric, and from (23) that it must satisfy the algebraic condition (5). We arrive at the last constraint, which now takes the form

$$C_{KM}^K B^{Mj} P_{ij} + \epsilon_{ijk} A^j_M B^M_i P^{lk} = 0 .$$  \hfill (27)

Using (22) and the Jacobi identities, after some algebraic manipulations, this may be written in the form

$$W^i \dot{P}_{ij} = Z_{ijk} \dot{P}^{jk} ,$$  \hfill (28)

where $\dot{P}_{ij}$ corresponds to the traceless part of $P_{ij}$, and

$$W^i := 3 V_Q e^{QMN} \epsilon^{jkl} A_{Mk} A_{Ni}$$

$$Z_{ijk} := -2 \epsilon_{ijkl} A^l_M S^{MN} A_{Nk} .$$  \hfill (29)

Note that this condition involves only the traceless part of $P^{ij}$.

We can now go through the Bianchi classification for some understanding of this condition. For a Bianchi model of type I, defined with $C_{IJ}^K = 0$, this condition is trivially satisfied. If $S^{IJ} = 0$, which defines a model of type V, the condition reduces to the requirement that $W^i$ be a null eigenvector for
\[ \dot{P}_{ij} \] If \( V_Q = 0 \), one can look at the simple case in which the connection matrix \( A_{Mi} \) is assumed to be diagonal (see Ref. [17]). Then, recalling that without loss of generality \( S^{IJ} \) can be put in diagonal form with \( \pm 1 \) or 0 entries, and letting \( S^{IJ} = \text{diag}(s_1, s_2, s_3) \), the condition takes the form

\[
[s_1(A_{11})^2 - s_2(A_{22})^2]\dot{P}_{12} = 0 ,
\]

(31)
together with its cyclic permutations. An obvious solution is obtained if \( \dot{P}_{ij} \) itself is diagonal.

6 Other Approaches

The introduction of Ashtekar's variables has generated other approaches to the initial value problem in general relativity. One is the recent proposal by Newman and Rovelli [20]. For a Yang-Mills theory they use a Hamilton-Jacobi technique to solve Gauss' law. The Yang-Mills physical degrees of freedom are coded in one pair of scalar functions per dimension of the gauge group, together with conjugate momenta. Each pair of scalar fields defines a congruence of lines ("generalized lines of force") by the intersections of their level surfaces. The method can also be applied to gravity in the Ashtekar formulation, where Newman and Rovelli solve not only the Gauss constraint, but also the vector constraint, by the device of using three of the scalar fields as spatial coordinates and letting the remaining fields be given as functions of these. It is not known at present if also the scalar constraint can be solved using this technique.

Another potential avenue to the initial value problem is the application of the Goldstone-Jackiw solution [21] of the Gauss constraint in the SU(2) Yang-Mills theory to gravity in the Ashtekar formulation [22]. In this approach, one would start by solving the Gauss law. The solution could then be used in the diffeomorphism constraints. For Yang-Mills, this procedure is not particularly useful, because it results in a complicated hamiltonian. For gravity, the jury is still out.

Finally, Thiemann [7] has just introduced a new approach that enables him to solve all of the constraints. The key idea is to start with the ansatz \( E^{ai} = \varepsilon^{abc}D_{[b}v^i_{c]} \). This ansatz involves no loss of generality provided the curvature is non-degenerate, or provided the spin density \( K_i \) vanishes (so that the Gauss constraint implies \( D_a E^{ai} = 0 \)). Using this ansatz, the Gauss
constraint becomes a linear condition on $v^i_a$. The vector and scalar constraints are then solved in the presence of matter by solving for some of the matter momenta.

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