A modest improvement on the function $S(T)$

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October 25, 2010

Abstract

This paper concerns the function $S(T)$, the argument of the Riemann zeta-function. Improving on the method of Backlund, and taking into account the refinements of Rosser and McCurley it is hereunder proved that for sufficiently large $T$

$$|S(T)| \leq 0.1013 \log T.$$ 

Theorem 2 makes the above result explicit, viz. it enables one to select values of $a$ and $b$ such that, for $T > T_0$,

$$|S(T)| \leq a + b \log T.$$ 

1 Introduction

Whenever $t$ does not coincide with an ordinate of a zero of $\zeta(\sigma + it)$ one defines the function $S(t)$ as

$$S(t) = \pi^{-1} \arg \zeta(\frac{1}{2} + it),$$

where the argument is determined via continuous variation along the straight lines connecting 2, $2 + it$ and $\frac{1}{2} + it$, with $S(0) = 0$. If $t$ is such that $\zeta(\sigma + it) = 0$ then define $S(t) = \frac{1}{2} \lim_{\epsilon \to 0} \{S(t - \epsilon) + S(t + \epsilon)\}$. Without assuming unproven conjectures (for example the Riemann or Lindelöf hypotheses) the classic estimate of von-Mangoldt, $S(T) = O(\log T)$, has never been improved, except in reducing the size of the implied constant. Backlund \cite{1} showed that, for $T \geq 200$,

$$|S(T)| \leq 0.137 \log T + 0.445 \log \log T + 4.35,$$

where the lower order terms were improved by Rosser \cite{9}, who showed that, for $T \geq 1467$,

$$|S(T)| \leq 0.137 \log T + 0.443 \log \log T + 1.588,$$
and a computational check shows that this remains valid for all $T \geq 3$. Such explicit results are useful when estimating sums over the zeroes of $\zeta(s)$ — see, e.g., [4], [10].

The main idea of Backlund’s Method is to count the number of zeroes of $\Re \zeta(\sigma + iT)$ on the line segment $[\frac{1}{2} + iT, 1 + \eta + iT]$ where $0 < \eta \leq \frac{1}{3}$. Suppose there are $n$ such zeroes, labelled $a_1, \ldots, a_n$. These zeroes partition the line segment $[\frac{1}{2} + iT, 1 + \eta + iT]$ into $n + 1$ intervals. On the interior of each interval, $\arg \zeta(s)$ can change by at most $\pi$, since by construction, $\Re \zeta(s)$ is non-zero on each interior. Thus, as $\sigma$ varies from $\frac{1}{2}$ to $1 + \eta$ then

$$|\Delta \arg \zeta(s)| \leq (n + 1)\pi.$$  

One proceeds to bound $n$ from above using Jensen’s formula on the function

$$f(s) = \frac{1}{2} \left\{ \zeta(s + iT)^N + \zeta(s - iT)^N \right\},$$

for $N$ a natural number$^1$ — thus $f(\sigma) = \Re \zeta(\sigma + iT)^N$. There are two ways to proceed.

Method $\mathcal{A}$ takes account of all the zeroes contained in a circle of radius $r(\frac{1}{2} + \eta)$, centred at $s = 1 + \eta + iT$, for some $r \in (1, 2]$. McCurley [6] follows this line of attack, with $r = 2$. Contrarily, method $\mathcal{B}$ makes use of a clever observation by Backlund, henceforth called ‘Backlund’s trick’.

For any $\delta \in [0, \frac{1}{2} + \eta)$, let $\Delta_1 \arg \zeta(s)$ denote the change in the argument of $\zeta(s)$ as $\sigma$ varies from $\frac{1}{2}$ to $\frac{1}{2} + \delta$. Similarly $\Delta_2 \arg \zeta(s)$ is the change in argument as $\sigma$ varies from $\frac{1}{2}$ to $\frac{1}{2} - \delta$. By estimating the change in argument of $\chi(s)$, where

$$\zeta(s) = \chi(s)\zeta(1 - s) = \pi^{s - \frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \zeta(1 - s),$$

(see, e.g., [11, Ch. II]) Backlund [pp. 355-357, op. cit.] was able to show that for $T > 1$

$$|\Delta_1 \arg \zeta(s) + \Delta_2 \arg \zeta(s)| \leq \frac{8}{T},$$

It follows that there are at least $^2 n - 2$ zeroes of $\Re \zeta(s)$ on the line segment $[-\eta + iT, \frac{1}{2} + iT]$, and so at least $2n - 2$ zeroes of $\Re \zeta(s)$ for $\sigma \in [-\eta, 1 + \eta]$. So one uses Jensen’s formula, with a circle of radius $1 + 2\eta$, centred at $s = 1 + \eta + iT$. McCurley’s argument$^3$ works here as well, and gives [Thm 2.1, op. cit.]

$$|S(T)| \leq 0.115 \log T,$$

$^1$Backlund has $N = 1$. The introduction of the number $N$ and the passing through a sequence of $N$s tending to infinity is due to Rosser. The advantages of this will be made plain on p. 14.

$^2$Alternatively, for large enough $T$ there are at least $n$ zeroes of $\Re \zeta(s)$. This matters precious little, especially in light of the improvements given by Rosser given on p. 14.

$^3$McCurley considers Dirichlet $L$-functions, whence he is unable to make use of Backlund’s trick. Also, he considers $N(T)$ to be those zeroes with imaginary part $\gamma \in [-T, T]$. Thus the upper bound in (3) is one quarter of that which is in [6].
for sufficiently large $T$.

The advantage of $B$ is that one gets ‘2-for-the-price-of-1’ in terms of the number of zeroes of $f(s)$. But the drawback is that one must estimate $|\zeta(s)|$ over the strip $-\eta \leq \sigma \leq 1 + \eta$. With $A$ one begins with fewer zeroes, but for a suitably small $r$, the incursion into the strip $\sigma \leq \frac{1}{2}$ is minimal. This is indeed an amelioration since, by convexity, $|\zeta(s)|$ grows much more quickly to the left of the line $\sigma = \frac{1}{2}$. Method $A$ is that which is outlined in \cite[Ch. XIII, §9]{11}.

It needs must be noted that any detriment from using $B$ is nullified if one uses the convexity bound $|\zeta(\frac{1}{2} + it)| \ll t^{1/4}$. Thus, if method $A$ is of to be of any use to anyone, one must know the value of the constant $K$ for which $|\zeta(\frac{1}{2} + it)| \leq Kt^{\theta}$ where $\theta < \frac{1}{2}$. Cheng and Graham \cite{2} have shown that

$$|\zeta(\frac{1}{2} + it)| \leq 3 t^{1/6} \log t,$$

(4)

for $t > e$, and this will be used in \cite[4.1]{11}.

The remainder of the paper sets out to prove

**Theorem 1** (via Method $B$). *If $t > t_0 > e$ then*

$$|S(t)| \leq 1.998 + 0.17 \log t.$$ 

(5)

It should be noted that the theorem is valid for all $t > t_0 > e$, and the particular choice of coefficients is that which minimises the right-side of (5) when $t_0 = 10^{10}$. Better bounds for larger values of $t_0$ are calculable from \cite[5]{6}. The value of the coefficient of $\log t$ can be diminished further, but the limitations of the theorem show that it can not be taken to be less than 0.1027. Any diminution in this coefficient is at the expense of increasing the constant term.

This paper can be considered a sequel to my paper on Turing’s Method \cite{12}, and indeed many of the calculations involving convexity estimates for bounds on $|\zeta(\frac{1}{2} + it)|$ are similar.

### 2 The requisites for Backlund’s Method

The opening gambits of Backlund and McCurley are essentially the same. One writes

$$\xi(s) = \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma(\frac{1}{2}s)\zeta(s),$$

where $\xi(s)$ is an entire function, whose zeroes coincide with the non-trivial zeroes of $\zeta(s)$. If one writes $N(T)$ as the number of complex zeroes of $\zeta(s)$ with imaginary part $\gamma \in [0, T]$, then it follows from Cauchy’s theorem, the functional equation and the reflection principle that $4\pi N(T) = 4\Delta_R \arg \xi(s)$, where $R$ is the pair of lines connecting the points $1 + \eta, 1 + \eta + iT$ and $\frac{1}{2} + iT$. In calculating the change in argument of $\xi(s)$ one finds a main term and then the term corresponding to $\Delta_R \arg \zeta(s)$ which is $4\pi S(T)$. The vertical piece is easily calculated Cauchy’s theorem and the fact that $\arg \zeta(2) = 0$ to show that calculating $\Delta \arg \zeta(s)$ along the aforementioned lines agrees with the definition of $S(T)$.
handled, since here, \(|\arg \zeta(s)| \leq |\log \zeta(s)| \leq \log(1 + \eta)|. What remains is to estimate \(\Delta_h \arg \zeta(s)\): the change in argument of \(\zeta(s)\) along the line segment \([1 + \eta + iT, \frac{3}{2} + iT]\).

With \(f(s)\) defined as in (11), it follows that
\[
|\Delta_h \arg \zeta(s)| = \frac{1}{N} |\Delta_h \arg \zeta(s)|^N \leq \frac{(n + 1)\pi}{N},
\]
whence
\[
|S(T)| \leq \frac{2}{\pi} \log \zeta(1 + \eta) + \frac{n + 1}{N}. \tag{6}
\]
One can now produce an upper bound on \(n\) courtesy of method \(A\) or \(B\). The proof below is valid for any \(r \in (1, 2]\) and it difference between the two methods will be plainly seen.

3 Bounding \(n\) using method \(A\)

For \(r \in (1, 2]\), Jensen’s formula is applied to the function \(f(s)\) on a circle with radius \(r(\frac{1}{2} + \eta)\) centred at \(s = 1 + \eta + iT\), to give
\[
n \log r \leq \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \log |f(1 + \eta + r(\frac{1}{2} + \eta)e^{i\phi})| \, d\phi - \log |f(1 + \eta)| \tag{7}
\]
whence
\[
|f(s)| \leq N \log \zeta(1 + \eta).
\]
On both \(I_2\) and \(I_4\), \(\Re(s) \geq \frac{1}{3}\), so that it is only on \(I_3\) that \(-\eta \leq \Re(s) \leq \frac{1}{2}\)—and this contribution diminishes as \(r\) is taken closer and closer to unity.

To handle the \(\log |f(1 + \eta)|\) term, one makes use of the trick of Rosser. Write \(\zeta(1 + \eta + iT) = ke^{i\psi}\). Now choose a sequence of \(N\)s tending to infinity such that \(N\psi\) tends to 0 modulo \(2\pi\), whence
\[
\lim_{N \to \infty} \frac{f(1 + \eta)}{|\zeta(1 + \eta + iT)|^N} = 1.
\]
Finally, for \(\sigma > 1\), one can consider the Euler product of \(\zeta(s)\) to show that
\[
|\zeta(s)| \geq \frac{\zeta(2\sigma)}{\zeta(\sigma)^2},
\]
whence the bound
\[
-\log |f(1 + \eta)| \leq N \log \zeta(1 + \eta).
\]
The only terms in (7) left to estimate are \(I_2\) and \(I_3\); a bound for \(I_2\) will serve as a bound for \(I_4\). Estimates of the growth of \(|\zeta(s)|\) for in \(\sigma \in [-\eta, \frac{1}{3}]\) and for \(\sigma \in [\frac{1}{2}, 1 + \eta]\) are given in the following section.
4 Preliminary Results

An explicit version of the Phragmén–Lindelöf theorem is needed and this is given below in

**Lemma 1.** Let $a, b, Q$ and $k$ be real numbers, and let $f(s)$ be regular analytic in the strip $-Q \leq a \leq \sigma \leq b$ and satisfy the growth condition

$$|f(s)| < C \exp\left\{ e^{k|t|} \right\},$$

for a certain $C > 0$ and for $0 < k < \pi/(b - a)$. Also assume that

$$|f(s)| \leq \begin{cases} A |Q + s|^{\alpha} & \text{for } \Re(s) = a, \\ B |Q + s|^{\beta} & \text{for } \Re(s) = b \end{cases}$$

with $\alpha \geq \beta$. Then throughout the strip $a \leq \sigma \leq b$ the following holds

$$|f(s)| \leq A^{\frac{b - \sigma}{b - a}} B^{\frac{\sigma - a}{b - a}} |Q + s|^{\frac{\alpha(b - \sigma)}{(b - a)} + \frac{\beta(\sigma - a)}{(b - a)}}.$$

**Proof.** This is a result of Rademacher and can be found in [8, pp. 66-67].

In order to apply Lemma 1, one needs bounds on $|\zeta(s)|$ on each of the three lines: $\sigma = 1 + \eta$, $\sigma = \frac{1}{2}$, and $\sigma = -\eta$. Trivially,

$$|\zeta(1 + \eta + iT)| \leq \zeta(1 + \eta). \quad (8)$$

The bound of Cheng and Graham [4] may be used on the line $\sigma = \frac{1}{2}$. One can bound $|\zeta(-\eta + iT)|$ by using the functional equation [2], [8] and the following result due to Rademacher.

**Lemma 2.** For $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$,

$$\left| \frac{\Gamma(\frac{1}{2} - \frac{s}{2})}{\Gamma(\frac{1}{2} s)} \right| \leq \left( \frac{|1 + s|}{2} \right)^{\frac{1}{2} - \sigma}.$$

**Proof.** See [7, p. 197].

It follows that

$$|\zeta(-\eta + iT)| \leq \left( \frac{|s + 1|}{2\pi} \right)^{\frac{1}{2} + \eta} \zeta(1 + \eta). \quad (9)$$

The following lemma contains two estimates on the growth of $|\zeta(s)|$ in strips on either side of the critical line.

**Lemma 3.** Suppose there exist constants $B$ and $\theta$ satisfying

$$|\zeta(\frac{1}{2} + it)| \leq B|s + 1|^\theta, \quad (10)$$
for all \( t \). Equations (8) and (10) show that for \( \frac{1}{2} \leq \sigma \leq 1 + \eta \), and for \( t > t_0 > e \)

\[
|\zeta(s)| \leq \left\{ C_1^{\sigma/(1+\eta)} B^{1+\eta} \right\}^{1/(1+\eta)}, \quad (11)
\]

where

\[
C_1 = \sqrt{1 + \left( \frac{2 + \eta}{t_0} \right)^2}.
\]

Also, given (10) and (9) then for \( -\eta \leq \sigma \leq \frac{1}{2} \) and for \( t > t_0 > e \),

\[
|\zeta(s)| \leq \left\{ \begin{array}{c} \zeta(1+\eta) \frac{s}{(2\pi)^{1\eta}} \\ B^{\sigma+\eta} \{ C_2 t \} (1+\eta) (\sigma+\eta) \end{array} \right\}^{1/(1+\eta)} \), \quad (12)
\]

where

\[
C_2 = \sqrt{1 + \frac{1}{t_0^2}}.
\]

The writing of (10) is simply to use a suitable form of (4) in Lemma 1. To prove (11) take \( f(s) = (s-1)\zeta(s), a = \frac{1}{2}, b = 1 + \eta, Q = 1 \), and use (10) and (8). The term \( C_1 \) springs from replacing \(|s-1|\) with \(|s+1|\).

To prove (12) take \( f(s) = \zeta(s), a = -\eta, b = \frac{1}{2}, Q = 1 \), and use (9) and (10). The term \( C_2 \) is obtained by replacing \(|s+1|\) with \( t \).

### 4.1 The value of \( B \)

To arrive at (10) consider (4), viz.

\[
|\zeta(\frac{1}{2} + it)| \leq 3t^{1/6} \log t \leq 3|s + 1|^{1/6} \log t,
\]

for \( t > e \). To accommodate the \( \log t \) term, note that one can choose a small \( \delta \) and hence find a (large) \( A_0 = A_0(\delta, t_0) \) such that \( \log t \leq A_0 t^\delta \leq A_0 |s + 1|^\delta \), for \( t \geq t_0 \). Since the function \( \log t/t^\delta \) never exceeds \((\delta e)^{-1}\), it follows that for all \( t > e \)

\[
|\zeta(\frac{1}{2} + it)| \leq \frac{3}{\delta e} |s + 1|^{1/6 + \delta},
\]

and a computational check shows the above to be valid for all \( t \geq 0 \). Thus we may take

\[
B = B(\delta) = \frac{3}{\delta e}. \quad (13)
\]

However, this presupposes that at a reasonable height for computation one wishes to use the bound (4) as opposed to the ‘ordinary’ convexity estimate

\[
|\zeta \left( \frac{1}{2} + it \right) | \leq 2.53 |1 + s|^4, \quad (14)
\]

Note that Lemma \( \text{[10]} \) cannot be applied directly to \( \zeta(s) \) owing to the pole at \( s = 1 \).
which can be deduced from that in [5]. It is clear that as \( t \) increases one should prefer (4) to (14) but, as will be shown in the next section, this preference is not immutable, particularly for modest values of \( t \). Indeed, the dependence of \( B \) on \( \delta \) is the primary source of frustration in seeking an improvement to Backlund’s method, and it would be of great use to have access to a bound of the type

\[
|\zeta(\frac{1}{2} + it)| \leq Ct^{1/6},
\]

which would be of use even if \( C \) were as large as, say 1000.

5 Computation

Equation (6) is

\[
|S(T)| \leq \frac{2}{\pi} \log \zeta(1 + \eta) + \frac{n + 1}{N},
\]

where \( n \) is bounded by (7). One can now use Lemma 3 in (7) to obtain a bound on \( S(T) \) depending on, inter alia, the variable \( r \) where \( 1 < r \leq 2 \). This general form is bloated with terms involving \( \sin^{-1} \frac{1}{r} \) and the like, and to include it here would be inexcusable. One must decide whether to use Backlund’s trick (i.e. \( r = 2 \) and twice as many zeroes) or to take a smaller value of \( r \).

It can be shown, after a little computation, that the use of Backlund’s trick is the better option. The general bound of (6) is given in an appendix, and hereafter we shall choose \( r = 2 \). For ease of exposition many of the error terms have been estimated — probably not optimally\(^6\) — and the \( r = 2 \) upper bound on (6) is given below in

**Theorem 2.** When \( \theta = \frac{1}{6} + \delta \) choose \( B(\delta) = \frac{3}{2\pi} \) so that (10) holds; if \( \theta = \frac{1}{4} \) choose \( B(\delta) = 2.53 \), whence (10) holds by virtue of (14). In either case, for all \( T > T_0 > 3 \)

\[
|S(T)| \leq a + b \log T,
\]

where

\[
a = a(\delta, \eta, T_0) = 1.85 \log \zeta(1 + \eta) + 0.71 \log B(\delta) - 0.58 + \frac{1}{T_0},
\]

and

\[
b = b(\delta, \eta) = \frac{2\theta(1 + \frac{\pi}{3} - \sqrt{3}) + (\eta + \frac{1}{2})(\sqrt{3} - \frac{\pi}{3})}{2\pi \log 2}.
\]

Equation (17) shows that the size of \( b \) is diminished when \( \eta \) and \( \delta \) are taken closer to zero. Also, (16) shows that \( a \) is increased when either of \( \eta \) or \( \delta \) is diminished. This inverse proportionality occurs similarly in analysis of Turing’s Method [12] and it is herewith treated in like fashion.

\(^6\)For example, using an upper bound \( \eta \leq 1 \), while true, is a weaker estimate for many of the applications. But since many of these terms are suitably small, and since Theorem 2 concisely presents the nature of the upper bound for \( S(T) \), such minute savings have been ignored.
If one wishes to investigate the size of $S(T)$ beyond some large height, then one can afford to take $\delta$ and $\eta$ smaller, so long as the term $b \log T$ in (15) continues to dominate. Indeed, for a given $T_0$, the minimal value of $a + b \log T_0$ is sought. As an example, the Riemann hypothesis has been verified past $T_0 = 10^{10}$ (see, e.g. [13]) so it is beyond this height that explicit bounds on $S(T)$ would be of the greatest use.

As a benchmark, Rosser’s bounds on $|S(T)|$ are, for $T \geq T_0$

$$|S(T)| \leq 1.588 + \left\{ 0.137 + 0.443 \frac{\log \log T_0}{\log T_0} \right\} \log T.$$  
(18)

The following table compares the size of $b$ — the coefficient of $\log T$ in (15) — and the overall bound on $S(T)$, where each is obtained by Rosser’s bound (18), Theorem 2 with $\theta = \frac{1}{4}$, and Theorem 2 with $\theta = \frac{1}{6} + \delta$.

| $T_0$ | $b$ | $S(T)$ | $b$ | $S(T)$ | $b$ | $S(T)$ |
|-------|-----|-------|-----|-------|-----|-------|
| $10^{10}$ | 0.1974 | 6.132 | 0.170 | 5.912 | 0.170 | 7.968 |
| $10^{12}$ | 0.1902 | 6.844 | 0.162 | 6.67 | 0.162 | 8.644 |
| $10^{14}$ | 0.1847 | 7.543 | 0.156 | 7.395 | 0.156 | 9.298 |
| $10^{16}$ | 0.1804 | 8.233 | 0.152 | 8.122 | 0.152 | 9.932 |
| $10^{18}$ | 0.1768 | 8.916 | 0.148 | 8.797 | 0.148 | 10.56 |
| $10^{20}$ | 0.1738 | 9.594 | 0.145 | 9.47 | 0.145 | 11.17 |
| $10^{40}$ | 0.159 | 16.21 | 0.131 | 15.78 | 0.126 | 17.26 |
| $10^{60}$ | 0.153 | 22.70 | 0.126 | 21.69 | 0.119 | 22.44 |

Theorem 1 follows at once from the first row of middle-column, along with the calculation of $a$ from (16). Note that the convexity estimates are marginally superior to Rosser’s bounds in each case. Moreover, the sub-convexity estimates (the right-column) only improve on Rosser’s bounds in the last row. A simple computation shows that the value of $b$ obtained from the sub-convexity estimates, only overtakes that obtained by the middle-column when $T_0 > 10^{26}$.

Finally, note that from [3], the bound $\zeta(\frac{1}{2} + it) \ll t^\theta$, where $\theta = \frac{32}{205}$ and Theorem 2 show that

$$|S(T)| \leq 0.1013 \log T,$$

for $T$ sufficiently large.

6 Conclusion

It is tempting to see what further improvements to Theorem 1 might be possible. One way is to try to combine methods $A$ and $B$. That is, to take some $r < 2$ and to try to replicate Backlund’s trick by showing that there must be some
zeroes of $f(s)$ lying on the segment left of $\frac{1}{2} + iT$, that is, the line connecting $1 + \eta - r(\frac{1}{2} + \eta) + iT$ and $\frac{1}{2} + iT$. Unfortunately such a manoeuvre would require some knowledge of the nature of the horizontal distribution of the zeroes of $\Re \zeta(s)$. If such a result were known it would be natural to suspect some diminution in the constants in Theorem 1.

7 Appendix: the explicit bound of method $A$

For any $r \in (1, 2)$ then

$$|S(T)| \leq \frac{2}{\pi} \log \zeta(1+\eta) + \frac{a_1 + a_2 \log B + a_3 \frac{a}{217} + a_4 \log \zeta(1 + \eta) + a_5 \log t}{\pi \log r},$$

where

$$a_1 = \frac{3\pi}{8\pi_0} - \left(\frac{1}{2} \log 2\pi\right)\frac{3}{2} - \sin^{-1} \frac{1}{r} + r\sqrt{1 - \frac{1}{r^2}},$$
$$a_2 = 2 \left(\frac{3}{2} - \sin^{-1} \frac{1}{r}\right) - 3r\sqrt{1 - \frac{1}{r^2}} + r,$$
$$a_3 = r\theta(2 - \sqrt{1 - \frac{1}{r^2}}) + \sin^{-1} \frac{1}{r} + \left(\frac{1}{2} + \eta - 2\theta\right)(\sin^{-1} \frac{1}{r} - \frac{r}{2} + r\sqrt{1 - \frac{1}{r^2}}),$$
$$a_4 = \frac{3\pi}{2} - \sin^{-1} \frac{1}{r} + 2r\sqrt{1 - \frac{1}{r^2}} + 1 - r,$$
$$a_5 = r\theta + \left(\frac{1}{2} + \eta - 2\theta\right)(\sin^{-1} \frac{1}{r} - \frac{r}{2} + r\sqrt{1 - \frac{1}{r^2}}).$$

The bound in (19) only improves on that in Theorem 1 if $\zeta(\frac{1}{2} + it) \ll t^\theta$, where $\theta < 1/50$.

Acknowledgements

I wish to thank Nathan Ng and Habiba Kadiri who suggested this problem, and Roger Heath-Brown for his helpful suggestions.

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