On Exotic Lagrangian Tori in $\mathbb{CP}^2$

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Abstract

We construct an exotic monotone Lagrangian torus in $\mathbb{CP}^2$ using techniques motivated by mirror symmetry. We show that it bounds 10 families of Maslov index 2 holomorphic discs, and it follows that this exotic torus is not Hamiltonian isotopic to the known Clifford and Chekanov tori.

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1 Introduction

Using Darboux’s theorem, it is very easy to find Lagrangian tori inside a symplectic manifold, since any open subset of $\mathbb{C}^n$ contains many. On the other hand the only known monotone Lagrangian tori in $\mathbb{C}^n$, up to Hamiltonian isotopy, were the product $(S^1(r))^n \subset \mathbb{C}^n$, the so called Clifford Tori, until 1995, when Chekanov introduced in his paper [4] the first examples of tori not Hamiltonian isotopic to the product tori.

The Clifford torus can be symplectically embedded into the complex projective space $\mathbb{CP}^n$ and the product of spheres $\times_n \mathbb{CP}^1$, giving monotone tori. Each one of these is also known as Clifford torus. Chekanov’s monotone tori were also known to give rise to exotic monotone Lagrangian tori in these spaces. But it was only much later that Chekanov and Schlenk, in [5], described in detail their family of exotic monotone Lagrangian tori, where by exotic we mean not Hamiltonian isotopic to the Clifford torus.

In [1], Auroux studied the SYZ mirror dual (a “Landau-Ginzburg model”) of a singular special Lagrangian torus fibration given on the complement of an anticanonical divisor in $\mathbb{CP}^2$. Such a fibration interpolates between the Clifford torus and the slightly modified version of the Chekanov torus described by Eliashberg and Polterovich in [8]. This construction explains how the count of holomorphic Maslov index 2 discs (described by the superpotential of the Landau-Ginzburg model) changes from the Clifford torus to the Chekanov torus. The key point is that, in the presence of the singular fiber, some other fibers bound Maslov index 0 discs. Such fibers form a “wall” on the base of the fibration, separating the Clifford type torus fibers and the Chekanov type torus fibers, and accounting for differences in the count of Maslov index 2 discs between the two sides of the wall.

In this paper, we modify Auroux’s example described in [1] to prove:

**Theorem 1.1.** There exists a monotone Lagrangian torus in $\mathbb{CP}^2$ endowed with the standard Fubini-Study form bounding 10 families of Maslov index 2 holomorphic discs, that is not Hamiltonian isotopic to the Clifford and Chekanov tori.

Before constructing the exotic torus, which we call the the $Che^2$ torus, and proving the right count of Maslov index 2 discs using purely symplectic geometry techniques, we show how to predict it existence and properties using the wall-crossing formulas. Even though these formulas are believed to hold for the almost toric case, they are not yet completely proven rigorously, and neither is the relation between holomorphic discs and to tropical curves.
upon degeneration to a ‘large limit’ almost complex structure. We emphasize that, even though the motivation comes from mirror symmetry, we give a complete self-contained proof of Theorem 1.1 purely in the language of symplectic topology.

The rest of this paper is organized as follows.

In section 2 we review mirror symmetry in the complement of an anti-canonical divisor, Landau Ginzburg models, wall-crossing phenomena and Auroux’s example we mentioned above, following the approach in [1], [2].

In section 2.3 we review almost toric fibrations and in section 2.4 we explain the relationship between holomorphic discs and tropical discs in almost toric fibrations, working it out for the Example in section 2.1. Even though the approach is not totally rigorous, in section 3 we use tropical discs and wall-crossing formulas for an almost toric fibration to predict the existence of the Che^2^ torus and the number of Maslov index 2 discs it bounds, by computing the superpotential in an informal manner.

In section 4 we relate our work with known results about degenerations of CP^2, which allow us to predict the existence of an infinite series of exotic Lagrangian tori in CP^2. We then define the Che^2^ type torus and set the conditions for computing the discs it bounds.

In section 5 we compute first the homology classes and then the actual holomorphic discs the Che^2^ type torus bounds. We also prove regularity and orient the moduli space of holomorphic discs in each of the classes in order to determine the correct signed count for the superpotential.

In section 6 we consider the monotone Che^2^ torus and prove that it is not symplectomorphic to the known Clifford and Chekanov tori. Finally, in section 7 we repeat the techniques of sections 3 and 4 to conjecture the existence of an exotic monotone torus in CP^1\times CP^1, bounding 9 families of Maslov index 2 holomorphic discs.

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2 Motivation: Mirror symmetry

This section is a summary of the introduction to mirror symmetry in the complement of a anti-canonical divisor explained in [1], [2]. Mirror symmetry has been extended beyond the Calabi-Yau setting by considering Landau-Ginzburg models. More precisely, it is conjectured that the Mirror of a Kähler manifold \((X, \omega, J)\), with respect to a effective anticanonical divisor \(D\), is a Landau-Ginzburg model \((X^{\vee}, W)\), where \(X^{\vee}\) is a mirror of the almost Calabi-Yau \(X \setminus D\) in the SYZ sense, i.e. a (corrected and completed) moduli space of special Lagrangian tori in \(X \setminus D\) equipped with rank 1 unitary local systems \((U(1)\) flat connections on the Lagrangian), and the superpotential \(W : X^{\vee} \to \mathbb{C}\) given by Fukaya-Oh-Ohta-Ono’s \(m_0\) obstruction to Floer homology, which is a holomorphic function defined by a count of Maslov index 2 holomorphic discs with boundary on the Lagrangian, see [1], [2]. Kontsevich’s homological mirror symmetry conjecture predicts that the Fukaya category of \(X\) is equivalent to the derived category of singularities of the mirror Landau-Ginzburg model \((X^{\vee}, W)\).

In order to apply the SYZ construction to \(X \setminus D\), we have to represent it as a (special) Lagrangian fibration over some base. Also, to ensure that the count of Maslov index 2 holomorphic discs is well defined, one asks \(L\) to satisfy some assumptions. More precisely, we require:

1. there are no non-constant holomorphic discs of Maslov index 0 in \((X, L)\);
2. holomorphic discs of Maslov index 2 in \((X, L)\) are regular;
3. there are no non-constant holomorphic spheres in \(X\) with \(c_1(TX) \cdot [S^2] \leq 0\).

In this case one defines the superpotential \(W = m_0 : X^{\vee} \to \mathbb{C}\) by

\[
\begin{align*}
m_0(L, \nabla) &= \sum_{\beta, \mu(\beta) = 2} n_{\beta}(L) \exp\left(-\int_{\beta} \omega \text{hol}_{\nabla}(\partial \beta)\right) \quad (2.1)
\end{align*}
\]

where \(n_{\beta}(L)\) is the (algebraic) count of holomorphic discs in the class \(\beta\) whose boundary passes through a generic point \(p \in L\). More precisely, considering \(\mathcal{M}(L, \beta)\) the oriented (after a choice of spin structure for \(L\)) moduli space of holomorphic discs with boundary in \(L\) representing the class \(\beta\), \(n_{\beta}(L)\) is the degree of its push forward under the evaluation map at a boundary marked point as a multiple of fundamental class \([L]\), i.e., \(ev_*[\mathcal{M}(L, \beta)] = n_{\beta}(L)[L]\).
In principle one does not know if the series (2.1) converge. Thus, it is preferable to replace the exponential by a formal parameter, usually denoted by $T$, and the superpotential then takes values in the Novikov field. Nevertheless, all the superpotentials computed in this paper are given by a finite sums, and we use the exponential for consistency with [1].

For each $\beta \in H_2(X, L, \mathbb{Z})$, with $\partial \beta \neq 0 \in H_1(L, \mathbb{Z})$, we can define a holomorphic function $z_\beta : X^\vee \to \mathbb{C}^*$ by

$$z_\beta(L, \nabla) = \exp(-\int_\beta \omega) \text{hol}_\nabla(\partial \beta)$$

(2.2)

see Lemma 2.7 in [1].

**Remark 2.2.** Actually, the function $z_\beta$ is only defined locally, for we have to keep track of the relative class $\beta$ under deformations of $L$. In the presence of non-trivial monodromy, which appears when we allow the fibration to have singular fibers, the function becomes multivalued.

In some cases, including the Lagrangian fibrations considered in this paper, the map $H_1(L) \to H_1(X)$ induced by inclusion is trivial, and then we can get a set of holomorphic coordinates $z_j = z_{\beta_j}$ by considering relative classes $\beta_j$ so that $\partial \beta_j$ forms a basis of $H_1(L)$. Then our superpotential can be written as a Laurent series in terms of such holomorphic coordinates.

In many cases we consider Lagrangian fibrations with singular fibers, and some of the Lagrangian fibers bound Maslov index 0 holomorphic discs, passing through the singular point. The projection of such Lagrangians forms “walls” in the base, dividing it into chambers. The count of Maslov index 2 holomorphic discs bounded by Lagrangian fibers can vary for different chambers. This is called “wall-crossing phenomenon”, see section 2.2 and section 3 of [1]. Nevertheless, one can still construct the mirror by gluing the various chambers of the base using instanton corrections, see Proposition 3.9 and Conjecture 3.10 in [1].

The example below not only illustrates wall-crossing, but also serves as the main model for the rest of the paper. For a more detailed account see section 5 of [1] or section 3 of [2].

**2.1 A motivating example**

The following example is taken from [1], section 5. We will describe it in detail because our main construction, given in section 4, can be thought as a further development of the same ideas.
Consider \( \mathbb{CP}^2 \), equipped with the standard Fubini-Study Kähler form, and the anticanonical divisor \( D = \{(x : y : z); (xy - cz^2)z = 0\} \), for some \( c \neq 0 \). We will construct a family of Lagrangian tori in the complement of the divisor \( D \). For this we look at the pencil of conics defined by the rational map \( f : (x : y : z) \mapsto (xy : z^2) \). We will mostly work with \( f \) in the affine coordinate given by \( z = 1 \), as a map from \( \mathbb{C}^2 \) to \( \mathbb{C} \), \( f(x, y) = xy \). The fiber of \( f \) over any non-zero complex number is then a smooth conic, while the fiber over 0 is the union of two lines, and the fiber over \( \infty \) is a double line.

There is a \( S^1 \) action on each fiber of \( f \) given by \( (x, y) \mapsto (e^{i\theta}x, e^{-i\theta}y) \). Recall that the symplectic fibration \( f \) carries a natural connection induced by the symplectic form, whose horizontal distribution is the symplectic orthogonal to the fiber. Our family of tori will consist then of parallel transports of each \( S^1 \) orbit, along circles in the base of the fibration, centered at \( c \in \mathbb{C} \).

We say that the height of an \( S^1 \) orbit is the value of \( \mu(x, y) = \frac{1}{2} \frac{|x|^2 - |y|^2}{1 + |x|^2 + |y|^2} \), which is the negative of moment map of the \( S^1 \) action. The moment map remains invariant under parallel transport and hence we get that our family of Lagrangian tori is given by

\[
T^c_{r,\lambda} = \{(x : y : z); \|f(x : y : z) - c\| = r; \mu(x : y : z) = \lambda\}
= \{(x, y); |xy - c| = r; |x|^2 - |y|^2 = 2\lambda(1 + |x|^2 + |y|^2)\} \tag{2.3}
\]

![Diagram](https://via.placeholder.com/150)

**Figure 1:** The special Lagrangian torus \( T^c_{r,\lambda} \) in \( \mathbb{C}^2 \setminus D \) (from [1])

**Remark 2.4.** All the pairs consisting of a symplectic fibration together with a map from the symplectic manifold to \( \mathbb{R} \) (real data) used to define the Lagrangian fibrations considered in this paper form pseudotoric structures as defined by Tyurin, see [16].
Note that actually $T_{c,0}^c$ is a singular torus, pinched at $(0,0)$, so varying $r$ and $\lambda$ give us a singular toric fibration. If $r > |c|$, we say that $T_{r,\lambda}^c$ is of Clifford type, and if $r < |c|$, of Chekanov type. The motivation for this terminology is that in the first case we can deform the circle centered at $c$ with radius $r$ in the base to a circle centered at the origin, without crossing it, and with it we obtain a Lagrangian isotopy from $T_{r,0}^c$ to a Clifford torus $S^1(\sqrt{r}) \times S^1(\sqrt{r})$. Not crossing the origin implies that no torus in the deformation bounds Maslov index 0 discs, hence the count of Maslov index 2 discs remains the same, see section 5.2 in [1]. On the other hand, for $r < |c|$, $T_{r,0}^c$ is the Eliashberg-Polterovich version of the so-called Chekanov torus, see [8].

To compute the Maslov index of discs in terms of their algebraic intersection number with the divisor $D$, one can prove that these Lagrangian tori are special with respect to the holomorphic 2-form $\Omega(x, y) = (xy - c)^{-1} dx \wedge dy$. In general, we can associate to an anticanonical divisor $D$ a nonvanishing holomorphic n-form $\Omega$ on the complement $X \setminus D$ given by the inverse of a section of the anticanonical bundle that defines $D$. Recall the following definition:

**Definition 2.5.** A Lagrangian submanifold $L$ is said to be special Lagrangian, with respect to $\Omega$ and with phase $\phi$, if $\text{Im}(e^{-i\phi}\Omega)|_L = 0$.

For a proof that $T_{r,\lambda}^c$ are special Lagrangian with respect to $\Omega$ see proposition 5.2 of [1]. The following is Lemma 3.1 of [1].

**Lemma 2.6.** If $L \subset X \setminus D$ is special Lagrangian, then for any relative homotopy class $\beta \in \pi_2(X, L)$ the Maslov index of $\beta$, $\mu(\beta)$, is equal to twice the algebraic intersection number $\beta \cdot [D]$.

It can also be shown that $T_{r,\lambda}^c$ bounds Maslov index 0 holomorphic discs if and only if $r = |c|$. So we see that $r = |c|$ creates a wall in the base of our Lagrangian fibration given by pairs $(r, \lambda)$. Then we need to treat the cases $r > |c|$ and $r < |c|$ separately.

For $r > |c|$, we argue that $T_{r,\lambda}^c$ is Lagrangian isotopic to a product torus $S^1(r_1) \times S^1(r_2)$, without altering the disc count throughout the deformation. Denote by $z_1$ and $z_2$ respectively the holomorphic coordinates on the mirror associated to the relative homotopy classes $\beta_1$ and $\beta_2$ of discs parallel to the $x$ and $y$ coordinate axes in $(\mathbb{C}^2, S^1(r_1) \times S^1(r_2))$. Namely, $z_i = \exp(-\int_{\beta_i} \omega)\text{hol}_{\mathbb{C}}(\partial \beta_i)$. We get from Proposition 4.3 on [1] that the superpotential recording the counts of Maslov index 2 holomorphic discs bounded by $T_{r,\lambda}^c$ for $r > |c|$ is given by
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\[ W = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2}, \quad (2.4) \]

where $\Lambda = \int_{[\mathbb{CP}^1]} \omega$. The term $\frac{e^{-\Lambda}}{z_1 z_2}$ corresponds to discs that project via $f$ to a double cover of $\mathbb{C}^2 \setminus \Delta$ branched at infinity lying in the class $[\mathbb{CP}^1] - \beta_1 - \beta_2 \in \pi_2(\mathbb{CP}^2, T_{c,0}^c)$. The other terms $z_1$ and $z_2$ of the superpotential correspond to sections of $f$ over the disc $\Delta$ centered at $c$ with radius $r$, intersecting respectively the components $\{x = 0\}$ and $\{y = 0\}$ of the fiber $f^{-1}(0)$.

Now we look at the case $r < |c|$, and consider the special case $\lambda = 0$, the Chekanov torus considered by Eliashberg-Polterovich in [8]. One family of Maslov index 2 holomorphic discs lies over the disc $\Delta$ centered at $c$ with radius $r$, given by the intersection of $f^{-1}(\Delta)$ with the lines $x = e^{i\theta}y$. We denote by $\beta$ their relative class in $\pi_2(\mathbb{CP}^2, T_{c,0}^c)$. The other discs are harder to construct. Consider the class $\alpha$ of the Lefschetz thimble associated with the critical point of $f$ at the origin and the vanishing path $[0, c - re^{\arg(c)}i]$. One can see that $\alpha$, $\beta$ and $H = [\mathbb{CP}^1]$ form a basis of $\pi_2(\mathbb{CP}^2, T_{c,0}^c)$. The following Lemma and Proposition, due to Chekanov-Schlenk [5], have their proofs sketched in [1].

**Lemma 2.7** (Chekanov-Schlenk [5]). The only classes in $\pi_2(\mathbb{CP}^2, T_{r,0}^c)$ which may contain Maslov index 2 holomorphic discs are $\beta$ and $H - 2\beta + k\alpha$ for $k \in \{-1, 0, 1\}$.

**Proposition 2.8** (Chekanov-Schlenk [5]). The torus $T_{r,0}^c$ bounds a unique $S^1$ family of holomorphic discs in each of the classes $\beta$ and $H - 2\beta + k\alpha$ for $k \in \{-1, 0, 1\}$. These discs are regular, and the corresponding algebraic count is 2 for $H - 2\beta$ and 1 for the other classes.

Since deforming $\lambda$ to 0 yields a Lagrangian isotopy from $T_{r,\lambda}^c$ to $T_{r,0}^c$ in the complement of $f^{-1}(0)$, so without encountering any Maslov index 0 holomorphic discs, the disc count remains the same and we have that for $r < |c|$ the superpotential is given by

\[ W = u + \frac{e^{-\Lambda}}{u^2 w} + 2 \frac{e^{-\Lambda}}{u^2} + \frac{e^{-\Lambda} w}{u^2} = u + \frac{e^{-\Lambda} (1 + w)^2}{wu^2} \quad (2.5) \]

where $u$ and $w$ are the holomorphic coordinates on the mirror associated to the class $\beta$ and $\alpha$. 

2.2 Wall-crossing

In this section we explain the wall-crossing phenomenon. Then we see how it happens in Example 2.1 and explain the relation between the two formulas for the superpotential in terms of the wall-crossing at \( r = |c| \), still following section 5 of [1].

![Wall-crossing for Maslov index 2 holomorphic discs](from [1])

If one follows a Maslov index 2 holomorphic disc in a class \( \gamma' \) through a Lagrangian deformation of the fibers crossing a wall (formed by projection of fibers bounding Maslov index 0 discs), and assuming the given disc continues to exist throughout the deformation, the following phenomenon typically happens: if the boundary of such a disc intersects that of a Maslov index 0 holomorphic disc in a class \( \alpha \) while on the wall, they can be glued into another Maslov index 2 disc, in the class \( \gamma = \gamma' + \alpha \), on the other side of the wall, besides the deformation that passes through, in the “same” class \( \gamma' \), without attaching the Maslov index 0 disc. Conversely, a Maslov index 2 holomorphic disc in a class \( \gamma \) can split into a Maslov index 2 holomorphic disc in a class \( \gamma' \) and a Maslov index 0 holomorphic disc in a class \( \gamma \), while on the wall, and then disappear after the Lagrangian passes through, see Figure 2.

We see how this phenomenon appears in the Example 2.1. Begin considering the case where \( \lambda > 0 \), so \( T^{c}_{r,\lambda} \) lies in the region where \( |x| > |y| \). Then when \( r = |c| \) the torus intersects \( \{ y = 0 \} \) in a circle bounding a Maslov index 0 disc, \( u_0 \). This disc represents the class \( \alpha \), on the Chekanov side, and \( \beta_1 - \beta_2 \), on the Clifford side. As \( r \) decreases through \( |c| \), the family of holomorphic discs in the class \( \beta_2 \) on the Clifford side become the family of discs on the class \( \beta \) on the Chekanov side, and the discs in the class \( H - \beta_1 - \beta_2 \) on the Clifford side becomes the discs in the class \( H - 2\beta - \alpha \) on the Chekanov side. Since a disc in the class \( H - 2\beta - \alpha \), bounded by a torus over the wall \( r = |c| \), intersects \( u_0 \) in \( [H - 2\beta - \alpha] \cdot [\alpha] = 2 \) points, new discs in the classes \( H - 2\beta \) and \( H - 2\beta + \alpha \) arise from attaching \( u_0 \) to a disc in the class \( H - \beta_1 - \beta_2 = H - 2\beta - \alpha \) at one or both points where their boundaries intersect. Conversely, discs in the class \( \beta_1 \) break into a disc in
the class $\beta_2$ and the disc $u_0$, see figures 5, 6, 7 (in these figures, discs are represented tropically). Taking the wall-crossing into account the correct change of coordinates in the mirror is given as follows:

| Homology Classes | Coordinates |
|------------------|-------------|
| $\alpha \rightleftharpoons \beta_1 - \beta_2$ | $w \rightleftharpoons \frac{z_1}{z_2}$ |
| $\beta \rightleftharpoons \{\beta_1, \beta_2\}$ | $u \rightleftharpoons z_1 + z_2$ |
| $H - 2\beta + \{-1, 0, 1\} \alpha \rightleftharpoons H - \beta_1 - \beta_2$ | $e^{-\Lambda (1+w)^2 \over u^2 w} \rightleftharpoons e^{-\Lambda \over z_1 z_2}$ |

It is then easy to check that the formulas (2.4) and (2.5) for the superpotential do match up. One can think that the “naive” formula $u = z_2$ is modified by a multiplicative factor of $1 + w$, i.e., $u = (1 + w)z_2 = z_1 + z_2$, as predicted in Proposition 3.9 of [1].

For $\lambda < 0$, when $r = |c|$ the torus intersects $\{x = 0\}$ in a circle bounding a Maslov index 0 disc in the class $\beta_2 - \beta_1 = -\alpha$. As $r$ decreases through $|c|$, the families of holomorphic discs that survive the deformation through the wall are in the classes $\beta_2$ and $H - \beta_1 - \beta_2$ on the Clifford side, becoming $\beta$ and $H - 2\beta - \alpha$ on the Chekanov side. As before, two new families of discs are created in the classes $H - 2\beta$ and $H - 2\beta + \alpha$, while discs in the classes $\beta_1$ disappear, after wall-crossing. Therefore the correct change of coordinates is $u = z_1(1 + w^{-1}) = z_1 + z_2$, $w^{-1} = z_2/z_1$, which is the same as for $\lambda > 0$.

The difference between the “naive” gluing formulas for $\lambda > 0$ and $\lambda < 0$, is due to the monodromy of the Lagrangian fibers $T_{r,\lambda}^c$ around the nodal fiber $T_{|c|,0}^c$, which is explained in the next section. However, the wall-crossing corrections take care of this discrepancy and yield a single consistent gluing for both halves of the wall.

### 2.3 Almost toric manifolds

The aim of this section is to explain the geometry of almost toric fibrations and use it for a better understanding of the singular Lagrangian fibration in the previous example. Most importantly, we can use it to construct other fibrations and predict the superpotential on each of the chambers divided by the walls. This way we can predict existence of exotic Lagrangian tori in almost toric manifolds, and in particular the torus in $\mathbb{CP}^2$ that appears in Theorem 1.1. For a more detailed explanation of almost toric fibrations see [15].
Definition 2.9. A non-degenerate Lagrangian fibration \( \pi : (M, \omega) \to B \) of a symplectic four manifold is an almost toric fibration if it is a non-degenerate topologically stable fibration with no hyperbolic singularities.

The regular fibers of a Lagrangian fibration are tori, see Theorem 2.3, due to Arnold and Liouville, in \([15]\). So, an almost toric fibration admits only elliptic and nodal singularities, i.e., near each point there is a Darboux neighborhood such that \( \pi = (\pi_1, \pi_2) \) with

\[
\pi_j(x, y) = x_j \quad \text{or} \quad \pi_j(x, y) = x_j^2 + y_j^2 \quad j \in \{1, 2\}, \quad (\text{elliptic})
\]
or

\[
\pi(x, y) = (x_1 y_1 + x_2 y_2, x_1 y_2 - x_2 y_1). \quad (\text{nodal})
\]

We call the image of each nodal singularity a node.

Recall that a Lagrangian fibration yields an integer affine structure, called symplectic, on the complement of the singular values on the base, i.e. each tangent space contains a distinguished lattice. These lattices are given by the isotropy subgroups of a natural action of \( T^*B \) on \( M \), that goes as follows. Take \( \xi \in T^*B \) and consider the vector field \( V_\xi \) defined by \( \omega(., V_\xi) = \pi^* \xi \). Set \( \xi \cdot x = \phi_\xi(x) \), where \( \phi_\xi \) is the time-one flow of \( V_\xi \).

Call \( \Lambda^* \) the isotropy subgroup of the action, which is a lattice such that \( (T^*B/\Lambda^*, d\alpha_{\text{can}}) \) and \( (M, \omega) \) are locally fiberwise symplectomorphic (here, \( \alpha_{\text{can}} \) is induced by the canonical 1-form of \( T^*B \)). This induces two other lattices, the dual lattice, \( \Lambda \) given by \( \Lambda_b = \{ u \in T^*_b B \mid v^* u \in \mathbb{Z}, \forall v^* \in \Lambda^*_b \} \), inside \( TB \), and the vertical lattice, \( \Lambda^{\text{vert}} \) given by \( \Lambda^{\text{vert}} = \{ V_\xi \mid \xi \in \Lambda^* \} \), inside the vertical bundle in \( TM \).

For an almost toric 4 manifold, the affine structure defined by the lattice above, completely determines \( M \) up to symplectomorphism, at least when the base is either non-compact or compact with non-empty boundary, see Corollary 5.4 in \([15]\). Also, since \( (T^*B/\Lambda^*, d\alpha_{\text{can}}) \) and \( (M, \omega) \) are locally fiberwise symplectomorphic, a basis of the lattice is in correspondence with a basis of the first homology of the fiber over a regular point, \( H_1(F_b) \). Therefore, the topological monodromy around each node is equivalent to the affine monodromy. The neighborhood of a nodal fiber is symplectomorphic to a standard model, see section 4.2 in \([15]\), and the monodromy around a singular fiber (of rank 1) is given by a Dehn twist, which in suitable coordinates is represented by the matrix:

\[
A_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
A change of basis of $H_1(F_b)$ gives a conjugate of $A_{(1,0)}$, which is, in terms of its eigenvector $(a,b)$:

$$A_{(a,b)} = \begin{pmatrix} 1 - ab & a^2 \\ -b^2 & 1 + ab \end{pmatrix}$$

Due to the monodromy, one cannot find an affine embedding of the base of an almost toric fibration with nodes into $\mathbb{R}^2$ equipped with its standard affine structure of $\mathbb{R}^2$. However, after removing rays in the base starting at each one of the nodes it becomes possible to define such an embedding. The affine direction(s) of the image of such a ray in $\mathbb{R}^2$ determine the monodromy around the corresponding node. For instance, the leftmost picture of Figure 3 represents the Lagrangian fibration seen in Example 2.1. The ray represented by dashed lines in the direction $(1,1)$ is an eigenvector of the monodromy, which hence is given by $A_{(1,1)}$. The rightmost picture represents the same almost toric fibration, since the two pictures are related by transformations in the affine linear group $AGL(2,\mathbb{Z})$, as illustrated by the middle pictures. These pictures also describe the degeneration of $\mathbb{C}P^2$ into $\mathbb{C}P(1,1,4)$ explained in section 4.

Figure 3: The leftmost and rightmost diagrams are almost toric pictures of $\mathbb{C}P^2$ equipped with the fibration given in Example 2.1. The middle picture represents how to get from one to the other by applying transformations of $AGL(2,\mathbb{Z})$.

**Remark 2.10.** The presence of monodromy in the affine structure on $B$ implies the existence of monodromy in the affine structure induced on the
mirror $X^\lor$. This explains the discrepancy between the uncorrected coordinate changes across the two halves of the wall in Example 2.1. See remark 5.11 in [1].

2.4 Holomorphic discs viewed from almost toric fibrations

In this section we use almost toric pictures to describe a limit affine structure of the fibration for which holomorphic curves converge to tropical curves. We illustrate the Maslov index 2 tropical discs given in this limit affine structure for the almost toric fibration considered of Example 2.1. This section is not intended to contain a rigorous approach to the correspondence between tropical curves and holomorphic discs in an almost toric setting.

![Figure 4: After a deformation of the almost complex structure, $J$-holomorphic discs project to amoebas eventually converging to tropical curves in the large complex structure limit.](image)

Assume one has an almost toric fibration with special Lagrangian fibers with respect to $\Omega$, a holomorphic 2-form with poles on the divisor $D$ that projects to the boundary of the base $B$. Then the interior of $B$ carries a second affine structure, sometimes called complex. The lattice which describes this affine structure is given by identifying $T_b B \simeq H^1(L_b, \mathbb{R})$, via the flux of the imaginary part of $\Omega$ and via Poincaré duality with $H_1(L_b, \mathbb{R}) \supset H_1(L_b, \mathbb{Z})$. More precisely, for each vector $v \in T_b B$ one gets the element of $H^1(L_b, \mathbb{R})$ given by the homomorphism

$$[\gamma] \in H_1(L_b, \mathbb{R}) \mapsto \left. \frac{d}{dt} \right|_{t=0} \int_{\Gamma_t} Im(\Omega)$$

where $\Gamma_t$ is given by any parallel transport of $\gamma$ over a curve $c(t)$ on the base, with $c(0) = b$, $c'(0) = v$. Since $Im(\Omega)$ is a closed form, vanishing on
the fibers, the above is independent of $c(t)$ and $\Gamma_t$, and hence well defined. A fiber over the boundary of $B$ is infinitely far from a given fiber over an interior point, since $\Omega$ has a pole on the divisor $D$.

In general, the projections to $B$ of holomorphic curves, called amoebas, can be fairly complicated. However, it is expected that under a suitable deformation of the almost complex structure $J$ towards a ‘large limit’ (where the base directions are stretched), the amoebas converge to tropical curves, see Figure 4. Also, the wall generated by the singular fiber converges to a straight line with respect to this affine structure, since it is the projection of a holomorphic curve containing Maslov index zero discs bounded by the fibers. Moreover, since the boundary of such a disc represents the vanishing cycle in the neighborhood of the nodal fiber, its homology class is fixed by the monodromy. Hence the straight line corresponding to the wall is in the direction of the eigenvector of the affine monodromy. In a neighborhood of a fiber away from the singular ones the almost toric fibration are expected to approach $TB/\epsilon\Lambda$ with $\epsilon \to 0$ at the limit. This way, the change of coordinates and monodromy for this ‘large limit’ complex affine structure is given by the transpose inverse of the symplectic affine structure define in section 2.3.

This principle is illustrated for Example 2.1, see figures 5 and 6. In these two figures:

- The Lagrangian torus under consideration is the fiber over the thick point.
- The dashed lines represent the walls (long dashes) and the cuts (short dashes), and ‘x’ represents the node (singular fiber).
- A tropical disc is a tree whose edges are straight lines with rational slope in $B$, starting at the torus and ending on the nodes or perpendicular to the boundary at infinity. The internal vertices satisfy the balancing condition that the primitive integer vectors entering each vertex of the tree, counted with multiplicity, must sum to 0.
- The Maslov index of the disc equals twice the number of intersection with the boundary at infinity, i.e., the divisor.
- The multiplicity of each edge is depicted by the numbers of lines on figure 6, but on some other figures the multiplicities are represented by the thickness of the line, for visual purposes (they can be computed taking into account the balancing condition).

The vanishing cycle is represented by $(-1, 1)$ on the lattice $H_1(L_b, \mathbb{Z}).$ The relation between these pictures and the formulas in section 2.3 is as fol-
On Exotic Lagrangian Tori in $\mathbb{CP}^2$

\[ z_1, z_2 \text{ are the coordinates on the Clifford side associated with the vectors } (1, 0) \text{ and } (0, 1), \text{ respectively, and } u, w \text{ are coordinates on the Chekanov side associated with the vectors } (1, 0) \text{ and } (-1, 1), \text{ respectively, when } \lambda < 0. \]

The direction of the edge leaving the torus can be read off from the superpotential and is the negative of the vector representing the exponents of the corresponding monomial. For instance, the disc associated with the monomial $\frac{e^{-\lambda}}{z_1 z_2}$ in (2.4) leaves the torus with tangent vector $(1, 1)$ in Figure 5, while the disc associated with the term $\frac{e^{-\lambda}}{u^2}$ in (2.5) has tangent vector $(2, 0)$ (multiplicity 2) in Figure 6. We call this vector the “class” of the tropical disc.

Thus, the tropical discs in Figure 6 can be used to determined heuristically that the equation (2.5) should contain monomials with exponents $u, u^{-2}, u^{-2} w$ and $u^{-2} w^{-1}$. The exact coefficient can be determined by applying the wall-crossing formula to the Clifford-side superpotential, see section 3.

When passing through the wall, the tropical disc in the class $(1, 1)$ breaks into three other tropical discs, while the tropical disc in the class $(0, -1)$ disappears, exactly as shown for the respective holomorphic discs in Example 2.1, see Figure 7.
3 Predicting the existence of the $Che^2$ torus

In this section we apply the same ideas as in the previous section to another almost toric fibration shown on Figure 8, to predict the super potential of the $Che^2$ type torus, obtained from the previous Chekanov torus after another wall-crossing, see section 4.

Figure 8 represents another almost toric fibration on $\mathbb{CP}^2$ containing two singular fibers of rank one. It arises by ‘smoothing’ the corner of the rightmost picture in Figure 3 in the same fashion as we did coming from the standard toric fibration of $\mathbb{CP}^2$ to the almost toric fibration of Example 2.1, see left-most picture in Figure 3. More precisely we smooth the double point $(0 : 1 : 0)$ of the divisor $D = \{(x : y : z); (xy - cz^2)z = 0\}$ of Example 2.1. This fibration is expected to contain $Che^2$ type torus fibers (see section 4), including the monotone one if the degeneration is ‘large enough’.

We assume the Lagrangian fibers are special with respect to some 2-form $\Omega$ with poles on the divisor, and that in a ‘large limit’ almost complex structure, pseudo-holomorphic curves project to tropical curves. We will start the description of the superpotential in the chambers where the fibers are of Clifford type and successively cross two walls in order to arrive at a tentative formula for the superpotential in the chamber where the fibers are exotic $Che^2$ type tori. The construction of the $Che^2$ torus is described in detail in section 4 and the count of Maslov index 2 holomorphic discs is verified rigorously in section 5 using only symplectic geometry techniques.
On Exotic Lagrangian Tori in $\mathbb{CP}^2$

Figure 8: The almost toric picture of $\mathbb{CP}^2$ which is expected to have the Che$^2$ torus defined in section 4 as a fiber.

The matrix of the affine transformation illustrated in Figure 3 on the $\lambda < 0$ part has inverse transpose given by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, hence we perform a change of coordinates $z_1 = \hat{z}_1$, $z_2 = \hat{z}_1^3 \hat{z}_2$, so that a tropical disc in the class $(p, q)$ in Figure 9 corresponds to a monomial with exponents $\hat{z}_1^p \hat{z}_2^q$. After this change of coordinates, the invariant direction at the singularity is $(2, 1)$ and the monodromy is given by $A_{(2,1)}$, see Figure 9. Therefore on the $\lambda < 0$ part of the chamber corresponding to the Clifford-type tori the superpotential is given by

$$W_{Clif} = \hat{z}_1 + \hat{z}_1^3 \hat{z}_2 + e^{-\Lambda} \frac{1}{\hat{z}_1 \hat{z}_2} \quad (3.1)$$

As the vanishing vector of the first wall is $(2, 1)$, the vanishing class is represented by the coordinate $\tilde{w} = \tilde{z}_1^2 \tilde{z}_2 = \tilde{z}_1^3 \tilde{z}_2$, where $\tilde{z}_1$, $\tilde{z}_2$ are the coordinates corresponding to the standard basis on the Chekanov side. The intersection numbers of the classes $(1, 0)$ and $(0, 1)$ represented by the coordinates $\hat{z}_1$, $\hat{z}_2$ with the one represented by $\tilde{w}$ are $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$, $\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2$. Therefore the wall-crossing correction is given by $\tilde{z}_1 = \hat{z}_1 (1 + \tilde{w})$, $\tilde{z}_2 = \hat{z}_2 (1 + \tilde{w})^{-2}$. Applying this transformation to (3.1), the superpotential in the Chekanov region is
On Exotic Lagrangian Tori in $\mathbb{CP}^2$

\[ W_{\text{Che}^1} = \tilde{z}_1 + e^{-\Lambda} (1 + \tilde{w})^2 = \tilde{z}_1 + e^{-\Lambda} \tilde{z}_2 + 2 \frac{e^{-\Lambda}}{\tilde{z}_1^2} + \frac{e^{-\Lambda}}{\tilde{z}_1^4 \tilde{z}_2}. \quad (3.2) \]

We now cross the second wall to obtain what we call the Che type torus. Since the second wall has vanishing vector $(-1, 1)$ and \[
\begin{vmatrix}
1 & -1 \\
1 & 1
\end{vmatrix} = 1,
\]
the change of coordinate is of the form $u_1 = \tilde{z}_1(1 + w)$, $u_2 = \tilde{z}_2(1 + w)$. Here the monomial corresponding to the vanishing class is $w = e^{-\Lambda} \tilde{z}_2 = e^{-\Lambda} u_2 / u_1$, where the factor $e^{-\Lambda}$ is present because the class of the Maslov index 0 disc is $-\tilde{\beta}_1 + \tilde{\beta}_2 + [\mathbb{CP}^1] \in \pi_2(X, L)$, where $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are the classes associated with the coordinates, $\tilde{z}_1, \tilde{z}_2$. Indeed, knowing the boundary of $w$ represents the class $(-1, 1)$, we get the first two coefficients of $\tilde{\beta}_1$ and $\tilde{\beta}_2$. To obtain the coefficient of $[\mathbb{CP}^1]$ we compute the Maslov index. We have that $\mu([\mathbb{CP}^1]) = 6$, see Lemma 2.6 and $\tilde{z}_1$ and $e^{-\Lambda} \tilde{z}_2$ are terms in $W_{\text{Che}^1}$, hence $\mu(\tilde{\beta}_1) = 2$ and $\mu(\tilde{\beta}_2) = -4$. In order to have Maslov index 0, the coefficient of $[\mathbb{CP}^1]$ must be 1. Finally, applying the change of coordinates to $W_{\text{Che}^1}$ we get that
\[ W_{\text{Che}^2} = u_1 + 2 \frac{e^{-\Lambda}}{u_1^2}(1 + w)^2 + \frac{e^{-\Lambda}}{u_1^4 u_2}(1 + w)^5 \]

\[ = u_1 + 2 \frac{e^{-\Lambda}}{u_1^2} + 4 \frac{e^{-\Lambda} u_2}{u_1^3} + 2 \frac{e^{-\Lambda} u_2^2}{u_1^4} + 5 \frac{e^{-\Lambda} u_2^3}{u_1^5} + 10 \frac{e^{-\Lambda} u_2^4}{u_1^6} + 10 \frac{e^{-\Lambda} u_2^5}{u_1^7} + 2 \frac{e^{-2\Lambda}}{u_1^2}(1 + w)^2 + \frac{e^{-2\Lambda}}{w^5 w}(1 + w)^5. \]  

(3.3)

The last formula is a more simplified expression in terms of the coordinates \( u = u_1 \) and \( w \). The expanded version in coordinates \( u_1, u_2 \) makes it easier to visualize the class of each disc. Figure 11 illustrates a Che\(^2\) type torus, predicted to bound 10 different families of holomorphic discs, corresponding to the 10 terms in this expression.

Even though our approach in this section was not completely rigorous, it points toward the existence of such an exotic torus bounding 41 discs, if we count with multiplicity (sum the coefficients of each monomial). The theory for proving the correspondence between tropical curves on the base and holomorphic curves on the total space is not fully developed yet, so the actual proof in section 5 will use a different approach.

**Remark 3.1.** The bottom most region on Figures 9-11 is known to have infinitely many walls, since it can have Maslov index 0 discs ending in both nodes with different multiplicities. This can be detected by the need for consistency of the changes of coordinates due to wall-crossing when we go around the point where the walls intersect. This phenomenon is called scattering, first described by M. Kontsevich and Y. Soibelman in [13]. See also M. Gross [11].

4 Degenerations of \( \mathbb{CP}^2 \) and the exotic torus

This section is devoted to the actual construction of the exotic torus. The idea is to use a degeneration of \( \mathbb{CP}^2 \) into the weighted projective space \( \mathbb{CP}(1, 1, 4) \), as seen in figure 3. So we can ‘smooth a corner’ in the moment polytope of \( \mathbb{CP}(1, 1, 4) \) to get an almost toric fibration in which some fibers are what we would call a Chekanov type torus in \( \mathbb{CP}(1, 1, 4) \). Then we could use the degeneration to bring this torus back to \( \mathbb{CP}^2 \) to get an exotic torus. The degeneration of \( \mathbb{CP}^2 \) to other projective spaces, see [12], can potentially
Figure 11: A $Che^2$ type torus bounding 10 families of Maslov index 2 holomorphic discs. The superpotential is given by $W_{Che^2} = u + 2e^{-\Lambda}u^2(1 + w)^2 + e^{-2\Lambda}u^5(1 + w)^5$.

give an infinite range of exotic Lagrangian tori in $\mathbb{CP}^2$, as conjectured by Galkin-Usnich in [10], where they also explain how to predict the superpotential related to each one of the conjectured tori by applying successive ‘mutations’ to (2.4).

Indeed, the projective plane degenerates to weighted projective spaces $\mathbb{CP}(a^2, b^2, c^2)$, where $(a, b, c)$ is a Markov triple, i.e., satisfies the Markov equation

$$a^2 + b^2 + c^2 = 3abc.$$ 

All Markov triples are obtained from $(1, 1, 1)$ by a sequence of ‘mutations’ of the form

$$(a, b, c) \rightarrow (a, b, c' = 3ab - c)$$

A deformation from $\mathbb{CP}(a^2, b^2, c^2)$ to $\mathbb{CP}(a^2, b^2, c'^2)$ can be seen explicitly inside $\mathbb{CP}(a^2, b^2, c, c')$ via the equation $z_0z_1 - (1 - t)z_2' - tz_3' = 0$. 
We are going to work only with \( \mathbb{CP}^2 = \mathbb{CP}(1, 1, 1) \) and \( \mathbb{CP}(1, 1, 4) \) inside \( \mathbb{CP}(1, 1, 1, 2) \). For \( t \in [0, 1] \), let \( X_t \) be the surface \( z_0 z_1 - (1 - t)z_2^2 - tz_3 = 0 \). Explicit embeddings are

\[
\begin{align*}
\mathbb{CP}(1, 1, 1) & \rightarrow \mathbb{CP}(1, 1, 2) \\
(x : y : z) & \mapsto (x : y : z : \frac{xy - (1 - t)z^2}{t}) \quad \text{for } t \neq 0, \quad (4.1)
\end{align*}
\]

\[
\begin{align*}
\mathbb{CP}(1, 1, 4) & \rightarrow \mathbb{CP}(1, 1, 2) \\
(\tilde{x} : \tilde{y} : \tilde{z}) & \mapsto (\tilde{x}^2 : \tilde{y}^2 : \tilde{x}\tilde{y} : \tilde{z}) \quad \text{for } t = 0. \quad (4.2)
\end{align*}
\]

We now consider a similar fibration to the one in the example 2.1 from \( \mathbb{CP}(1, 1, 1, 2) \) to \( \mathbb{CP}^1 \), and restrict it to \( X_t \), for \( t > 0 \).

\[
f : X_t \setminus \{(1 : 0 : 0 : 0)\} \simeq \mathbb{CP}^2 \setminus \{(1 : 0 : 0)\} \rightarrow \mathbb{CP}^1
\]

\[
f(x : y : z) = \frac{z_2 z_3}{z_1^3} = \frac{z(xy - (1 - t)z^2)}{y^4} \quad (4.3)
\]

Also consider the divisor \( D = f^{-1}(c) \), where we take \( c \) to be a positive real number, thought of as a smoothing of \( f^{-1}(0) = \{z(xy - (1 - t)z^2) = 0\} \). We think of \( t \) as a small real number, i.e., we consider an embedding of \( \mathbb{CP}^2 \) close to that of \( \mathbb{CP}(1, 1, 4) \). In section \[ \text{[3]} \] we will choose an appropriate value of \( c \) in order to obtain a monotone torus.

Set \( \xi = \frac{xy - (1 - t)z^2}{t} \) and then define the tori:

**Definition 4.1.** Given \( c > r > 0 \) and \( \lambda \in \mathbb{R} \),

\[
T^c_{r, \lambda} = \left\{ (x : y : z) \mid \frac{z_2 z_3}{z_1^3} - c = r; \left| \frac{z_2}{z_1} \right|^2 - \left| \frac{z_3}{z_1^2} \right|^2 = \lambda \left( 1 + \left| \frac{z_2}{z_1} \right|^2 + \left| \frac{z_3}{z_1^2} \right|^2 \right) \right\}
\]

\[
= \left\{ (x : y : z) \mid \frac{z_2 \xi}{y^3} - c = r; \left| \frac{z}{y} \right|^2 - \left| \frac{\xi}{y^2} \right|^2 = \lambda \left( 1 + \left| \frac{z}{y} \right|^2 + \left| \frac{\xi}{y^2} \right|^2 \right) \right\} \quad (4.4)
\]

We will choose a symplectic form such that this torus is Lagrangian. For that we consider the 2-form equal to \( \frac{i}{4} \partial \bar{\partial} \log(1 + |z|^2 + |\xi|^2) \), in the coordinate chart \( y = 1 \). In homogeneous coordinates this form is given by
On Exotic Lagrangian Tori in \( \mathbb{CP}^2 \)

\[
\tilde{\omega} = \frac{i}{4} \partial \bar{\partial} \log \left( 1 + \left| \frac{z}{y} \right|^2 + \left| \frac{\xi}{y^2} \right|^2 \right) = \frac{i}{4} \partial \bar{\partial} \log (|y|^4 + |z|^2 |y|^2 + |\xi|^2) \quad (4.5)
\]

The second expression is well defined on \( \mathbb{CP}^2 \setminus (1 : 0 : 0) \), and equal to the first one since \( \partial \bar{\partial} \log (|y|^4) = 0 \). A calculation in the affine chart \( x = 1 \) shows that, along the complex line \( y = 0 \), it becomes \( dy \wedge d\bar{y} / (1 - t)^2 \). So we see that \( \tilde{\omega} \) is well defined and nondegenerate away from \( y = 0 \), but it is degenerate along the line \( y = 0 \), and also is singular at \((1 : 0 : 0)\). Let’s then consider a nearby symplectic form

\[
\omega = \frac{i}{4} \partial \bar{\partial} \log (|y|^4 + |z|^2 |y|^2 + |\xi|^2 + s^2 \eta(\rho)(|x|^2 + |y|^2 + |z|^2)^2) \quad (4.6)
\]

where \( \rho = \frac{|z\xi|}{y^2} \), \( \eta \) is a cut off function that is zero for \( \rho < R \) and one for \( \rho > 2R \), and \( R \) is chosen so that \( c + r < R \), while \( s \) is a very small constant. The constants \( c, r, t, R \) and \( s \) are chosen in this order and from now on we consider \( Che^2 \) type torus \( T^c_{r,\lambda} \), which fits inside the region where \( \omega \) and \( \tilde{\omega} \) agree, and hence is Lagrangian with respect to \( \omega \).

We see that this form is an interpolation between \( \tilde{\omega} \) and the Kähler form \( \omega_{FS} \)

\[
\omega_{FS} = \frac{i}{4} \partial \bar{\partial} \log (|y|^4 + |z|^2 |y|^2 + |\xi|^2 + s^2(|x|^2 + |y|^2 + |z|^2)^2)
\]

which is \( 1/2 \) of the pullback of the Fubini-Study form on \( \mathbb{CP}^{11} \) via the embedding

\[
t : \mathbb{CP}^2 \rightarrow \mathbb{CP}^{11} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \;

Proposition 4.2. For \( s > 0 \) sufficiently small, keeping fixed the other parameters \( c, r, t, R \), \( \omega \) is a well defined nondegenerate Kähler form. Moreover, \( \omega \) lies in the same cohomology class as the Fubini-Study form \( \omega_{FS} \).

Proof. We note that for \( y \neq 0 \),

\[
\omega = \frac{i}{4} \partial \bar{\partial} \log ((1 - \eta)\varphi_1 + \eta\varphi_2) = \frac{1}{2} dd^c \log ((1 - \eta)\varphi_1 + \eta\varphi_2),
\]
where
\[ \varphi_1 = \frac{|y|^4 + |z|^2|y|^2 + |\xi|^2}{|y|^4}, \]
\[ \varphi_2 = \frac{|y|^4 + |z|^2|y|^2 + |\xi|^2 + s^2(|x|^2 + |y|^2 + |z|^2)^2}{|y|^4}, \]
and on a neighborhood of \( y = 0 \), \( \omega \) is equal to \( \omega_s \), hence it is Kähler. We already know \( \omega \) is nondegenerate for \( \rho < R \) and \( \rho > 2R \). Since \( \omega_s \) converges to \( \tilde{\omega} \) uniformly on the compact set \( \rho \leq 2R \) (where \( \tilde{\omega} \) is nondegenerate) as \( s \to 0 \), there is a small enough \( s \) making \( \omega \) nondegenerate.

To determine the cohomology class of \( \omega \), it is enough to compute \( \int_{\mathbb{CP}^1} \omega \).
Considering \( \mathbb{CP}^1 = \{ y = 0 \} \) we see that \( \int_{\mathbb{CP}^1} \omega = \int_{\mathbb{CP}^1} \omega_s \) and so \( \omega = \omega_s = \frac{1}{2} t^* \omega_{\mathbb{CP}^1} = [\omega_{FS}] \).

The space of Kähler forms in the same cohomology class is connected. Hence, by Moser’s theorem, \( (\mathbb{CP}^2, \omega) \) and \( (\mathbb{CP}^2, \omega_{FS}) \) are symplectomorphic. After applying such a symplectomorphism, we get Lagrangian tori in \( (\mathbb{CP}^2, \omega_{FS}) \) with the same properties as the ones we are considering in \( (\mathbb{CP}^2, \omega) \).

We now consider the holomorphic 2-form on \( \mathbb{CP}^2 \), the quotient of \( \Omega_{\mathbb{C}^3} = \frac{dx \wedge dy \wedge dz}{t(\xi z - c)} \) defined on \( \mathbb{C}^3 \), which has poles on the divisor \( D \). On the complement of \( \{ y = 0 \} \), taking \( y = 1 \), it is given by
\[ \Omega = \frac{dx \wedge dz}{t(\xi z - c)} = \frac{d\xi \wedge dz}{\xi z - c} \quad (4.7) \]
Here \( \xi = \frac{x - (1-t)z^2}{t} \).

For the sake of using lemma 2.6, we prove

**Proposition 4.3.** The tori described in the \( (\xi, z) \) coordinate chart by \( T_{r,\lambda}^c = \{(\xi, z); |\xi z - c| = r; |z|^2 - |\xi|^2 = \lambda\} \) are special Lagrangian with respect to \( \Omega \).

**Proof.** Take \( V_H \) the Hamiltonian vector field of the Hamiltonian \( H(\xi, z) = |\xi z - c|^2 \), i.e., defined via \( \omega(V_H, \cdot) = dH \). Since \( H \) is constant on the Lagrangian \( T_{r,\lambda}^c \) and on each symplectic fiber of \( f(\xi, z) = \xi z \), \( V_H \) is symplectically orthogonal to both, hence tangent to the Lagrangian \( T_{r,\lambda}^c \) and not tangent to the symplectic fibers of \( f \). Consider the vector field \( \vartheta = (i\xi, -iz) \), tangent to the fibers and the Lagrangian torus, as they intersect along circles.
of the form \((e^{i\theta}\xi_0, e^{-i\theta}z_0)\). So, \(\{\vartheta, V_H\}\) form a basis for the tangent space of \(T^{c_r,\lambda}\). Now note that
\[
\iota_{\vartheta}\Omega = i\xi dz + izd\xi = id\log(\xi z - c)
\]
Therefore,
\[
\text{Im}(\Omega)(\vartheta, V_H) = d\log|\xi z - c|(V_H) = 0
\]

5 Computing holomorphic discs in \(\mathbb{CP}^2\) bounded by \(T^{c_r,0}\)

In this chapter we focus on the case \(\lambda = 0\) and show that, at least for small enough \(t\) with respect to \(r\) and \(c\), it bounds the expected 10 different families of Maslov index 2 holomorphic discs (with the expected multiplicity modulo signs). We often use the coordinates \((z_0 : z_1 : z_2 : z_3)\), but restricted to \(\mathbb{CP}^2 \simeq X_t\) via the embedding \((4.1)\).

5.1 The homology classes

Recall \(f(x : y : z) = \frac{z_2}{y}\) mapping \(\mathbb{CP}^2\) minus \((1 : 0 : 0)\) to \(\mathbb{CP}^1\).

**Proposition 5.1.** There is only one family of holomorphic discs, up to reparametrization, in \(\mathbb{CP}^2\) with boundary on \(T^{c_r,0}\) that is mapped injectively to the disc \(|w - c| \leq r\) by \(f\), where \(w\) is the coordinate in \(\mathbb{C}\).

**Proof.** Let \(u : \mathbb{D} \to \mathbb{CP}^2\) be such a disc so, up to reparametrization, \(f \circ u(w) = \Psi(w) = rw + c\). The map \(u\) can be described using coordinates \(y = z_1(w) = 1, z = z_2(w)\) and \(\xi = z_3(w)\), so \(z_2(w)z_3(w) = \Psi(w)\), hence \(z_2\) and \(z_3\) have no zeros or poles on the disc. At the boundary of the disc, mapped by \(u\) to \(T^{c_r,0}\), \(|z_2| = |z_3|\). So, by holomorphicity, \(z_2(w) = e^{i\theta}z_3(w) = e^{i\theta}\sqrt{\Psi(w)}\), for some choice of square root and some constant \(\theta\).

Call \(\beta\) the homology class of the above family of discs, \(\alpha\) the class of the Lefschetz thimble associated to the critical point of \(f\) at the origin lying above the segment \([0, c - r]\) (oriented to intersect positively \(\{z = 0\}\)) and \(H = [\mathbb{CP}^1]\) the image of the generator of \(\pi_2(\mathbb{CP}^2)\) in \(\pi_2(\mathbb{CP}^2, T^{c_r,0})\). One checks that \(\alpha, \beta, H\) form a basis of \(\pi_2(\mathbb{CP}^2, T^{c_r,0})\).

In order to understand what homology classes are allowed to have Maslov index 2 holomorphic discs, we analyze their intersection with some other
complex curves, for instance the line over \(\infty\), \(y = 0\), the line and conic over \(0, z = 0\) and \(D_2 : xy - (1 - t)z^2 = 0\). Another curve we use is a quintic, \(D_5\) that converges to \((\bar{x}\bar{z} - c\bar{y}^5)^2 = 0\) on \(\mathbb{CP}(1, 1, 4)\). It is given by

\[
D_5 : z_0^2 z_3^2 - 2cz_1^2 z_2 z_3 + c^2 z_1^5 = x\xi^2 - 2cy^2 z\xi + c^2 y^5 = 0
\]

For \(z \neq 0\) and setting \(f = z\), we can write this equation as

\[
y^5(c^2 - 2cf + \frac{xy}{z^2}f^2) = 0 \tag{5.1}
\]

**Lemma 5.2.** For fixed \(c\) and \(r\), and for \(t\) sufficiently small, the classes \(\alpha\), \(\beta\) and \(H\) intersect the varieties \(\{ z = 0 \}\), \(\{ y = 0 \}\), \(D_3 = f^{-1}(c) \cup \{(0:0:1)\}\), \(D_2 : xy - (1 - t)z^2 = 0\), \(D_5\) and have Maslov index, according to the table below.

| Class | \(z = 0\) | \(y = 0\) | \(D_3\) | \(D_2\) | \(D_5\) | Maslov index \(\mu\) |
|-------|-----------|-----------|--------|--------|--------|----------------|
| \(\alpha\) | 1         | 0         | 0      | -1     | 0      | 0              |
| \(\beta\)  | 0         | 0         | 1      | 0      | 2      | 2              |
| \(H\)     | 1         | 1         | 3      | 2      | 5      | 6              |

**Proof.** In order to use these curves to compute intersection numbers, we first need to ensure they don’t intersect \(T^c_{r,0}\). This is clear for \(z = 0\), \(y = 0\), \(D_3\), and \(D_2\). Also, the intersections with \(H\) follow from Bezout’s Theorem, and for \(\alpha\) and \(\beta\), the only intersection that is not straightforward is with \(D_5\).

To understand the intersection of \(D_5\) with the torus \(T^c_{r,0}\), \(\alpha\) and \(\beta\) we look at the family of conics \(\mathcal{C} = \{ z = e^{i\theta} \xi; \theta \in [0, 2\pi]\}\) containing \(T^c_{r,0}\), the thimble representing the class \(\alpha\) and the discs representing the class \(\beta\) computed in Proposition [5.1]. We use the coordinate chart \(y = 1\).

On \(\mathcal{C}\), \(z = e^{i\theta} \xi = e^{i\theta} \frac{x - (1 - t)z^2}{f}\), \(f = e^{-i\theta} z^2\), so \(x = t e^{-i\frac{\theta}{2}} f^{1/2} + (1 - t) e^{i\theta} f\), for some square root of \(f\). Then, using [5.1], the points of \(D_5 \cap \mathcal{C}\) are those where

\[
c^2 - 2cf + e^{-i\theta} f (te^{-i\frac{\theta}{2}} f^{1/2} + (1 - t) e^{i\theta} f) = (f - c)^2 + t f^{3/2} (e^{-i\frac{3\theta}{2}} - f^{1/2}) = 0
\]

For \(t\) small enough, for each value of \(\theta\), this equation all the solutions of this equation lies in the region \(|f - c| < r\). From this we can conclude that \(D_5 \cap T^c_{r,0} = 0\), \(D_5 \cap \alpha = 0\), since, in \(T^c_{r,0}\), \(|f - c| = r\) and the thimble representing \(\alpha\) lies over \([0, c - r]\). Now, a holomorphic disc representing the class \(\beta\) is given by \(z = \xi = f^{1/2}\) and \(Re(z) > 0\), see Proposition [5.1]. This means that this disc intersects \(D_5\) in exactly two points, namely the two solutions of the above equation where \(z\) is close to \(\sqrt{c}\). As both are complex...
curves, the intersections count positively, so the intersection number between $D_5$ and $\beta$ is equal to 2.

Finally, from lemma 2.6 we see that the Maslov index is twice the intersection with the divisor $D_3$.

\[
\text{Lemma 5.3.} \quad \text{The only classes in } \pi_2(\mathbb{CP}^2, T_{r,0}) \text{ which may contain holomorphic discs of Maslov index 2 are } \beta, H - 2\beta + m\alpha, -1 \leq m \leq 2 \text{ and } 2H - 5\beta + k\alpha, -2 \leq k \leq 4.
\]

\[
\text{Proof.} \quad \text{To have Maslov index 2 the class must have the form } \beta + l(H - 3\beta) + k\alpha. \text{ Considering positivity of intersections with } y = 0 \text{ we get } l \geq 0, \text{ with } z = 0 \text{ and } D_2 \text{ we get } -l \leq k \leq 2l, \text{ and finally with } D_5, l \leq 2. \quad \Box
\]

### 5.2 Discs in classes $H - 2\beta + m\alpha$

\[
\text{Theorem 5.4.} \quad \text{There are no Maslov index 2 holomorphic discs in the class } H - 2\beta - \alpha; \text{ there are one-parameter families of holomorphic discs in the classes } H - 2\beta \text{ and } H - 2\beta + 2\alpha, \text{ with algebraic counts equal to 2 up to sign in both cases, and a one-parameter family of holomorphic discs in the class } H - 2\beta + \alpha, \text{ with algebraic count equal to 4 up to sign.}
\]

This is precisely what we expect from the term $2e^{-\Lambda u^2}(1 + w)^2$ in $W_{Che^2}$, see equation (3.3).

\[
\text{Proof.} \quad \text{We will try to find holomorphic discs } u : (\mathbb{D}, S^1) \to (\mathbb{CP}^2, T_{r,0}^0) \text{ in the class } H - 2\beta + m\alpha, -1 \leq m \leq 2. \text{ Recall } f : \mathbb{CP}^2 \setminus (0 : 0 : 1) \to \mathbb{CP}^1, f(x : y : z) = \frac{x}{y}, \text{ and set } \Psi = f \circ u : \mathbb{D} \to \mathbb{CP}^1. \text{ Since } u \text{ has Maslov index 2 it doesn’t go through } (0 : 0 : 1) \text{ where } D_3 \text{ has a self intersection, so } \Psi \text{ is well-defined.}
\]

Hence the map $\frac{\Psi(w) - c}{r}$ maps the unit circle to the unit circle, has a pole of order 3, since our disc intersects $y = 0$ at 1 point, and a simple zero, since the disc intersects the divisor $D_3$ at one point. Then it has the form

\[
\frac{\Psi(w) - c}{r} = \frac{\tau_{w_0}(w)}{\tau_{w_1}(w)} e^{i\phi}, \quad \text{where} \quad \tau_a(w) = \frac{w - a}{1 - \bar{a}w} \quad (5.2)
\]

for some $w_0, w_1$ in $\mathbb{D}$ and $e^{i\phi} \in S^1$.

Let $\eta_j$ be so that $\Psi(\eta_j) = 0$, so $u(\eta_j)$’s are the three points of intersection with $D_2 \cup \{z = 0\}$, $j = 0, 1, 2$. The integer $m$ determines how many times the disc $u$ intersects $D_2$, and we consider a set $I \subset \{0, 1, 2\}$ with that number.
of elements. Writing \( z_1 = y, z_2 = z \) and \( z_3 = \xi \), and \( \tau_j = \tau_{\eta_j} \), we see that the map \( u \) can be expressed in the form

\[
z_1(w) = \tau_{w_1}(w), \quad z_2(w) = e^{-i\theta} h_2(w) \prod_{j \notin I} \tau_j(w), \quad z_3(w) = e^{i\theta} h_3(w) \prod_{j \in I} \tau_j(w)
\]

(5.3)

where \( h_3(w) \) and \( h_2(w) \) are nonvanishing holomorphic functions and \( e^{i\theta} \in S^1 \). A suitable scaling of the homogeneous coordinates eliminates the need for such a multiplicative factor in the expression for \( z_1(w) \). At the boundary, \( \left| \frac{z_2}{z_1} \right| = \left| \frac{z_3}{z_1} \right| = 1 \), and \( |h_2| = |h_3| \) on the unit circle, therefore \( h_3 = e^{i\theta} h_2 \) for some constant \( \theta' \). Note that we can absorb \(-\theta'/2\) in \( \theta \) and assume that \( h_3 = h_2 = h \). Since \( \Psi(w) = \frac{z_2(w) z_3(w)}{z_1(w)} = \frac{h^3(w) \prod \tau_j(w)}{\tau_{w_1}(w)} \), we get \( h(w) = \left( \frac{\Psi(w) w_3}{\prod \tau_j(w)} \right)^{1/2} \), for some choice of square root. The other choice is absorbed by the parameter \( \theta \).

We can use automorphisms of the disc to assume \( w_1 = 0, \phi = 0 \) and write \( w_0 = a \). In this case

\[
w^3 \Psi(w) = r \tau_a(w) + c w^3 = h^2(w) \tau_0(w) \tau_1(w) \tau_2(w)
\]

(5.4)

There is also an extra equation coming from the fact that \( \xi = -\frac{(1-t)}{t} z^2 \) when \( y = 0 \):

\[
\frac{z_3(0)}{z_2(0)} = -\frac{1-t}{t} = \frac{e^{3i\theta} \prod_{j \in I} \tau_j(0)}{h(0) \prod_{j \notin I} \tau_j^2(0)} = \frac{e^{3i\theta} (-1)^{|I|} \prod_{j \in I} \eta_j}{h(0) \prod_{j \notin I} \eta_j^2} \quad (5.5)
\]

Since (5.4) implies that \(-ar = -h^2(0) \eta_0 \eta_1 \eta_2\), (5.5) can be rewritten as

\[
ar \left( \frac{1-t}{t} \right)^2 \prod_{j \notin I} \eta_j^3 = e^{6i\theta} \prod_{j \in I} \eta_j^3 \quad (5.6)
\]

Note that solving (5.6) amounts to a solution of (5.5) for some choice of square root for \( h(w) = \left( \frac{\Psi(w) w^3}{\prod \tau_j(w)} \right)^{1/2} \). To analyze the existence of \( a \) solving this equation we consider the polynomial

\[
\Xi(w) = w^3(1-\bar{a}w) \Psi(w) = r(w-a) + c w^3(1-\bar{a}w) = -c \bar{a}(w-\zeta)(w-\eta_0)(w-\eta_1)(w-\eta_2)
\]

for some \( \zeta \), with \( |\zeta| > 1 \). Assume \( |\eta_0| \leq |\eta_1|, |\eta_0| \leq |\eta_2| \), and write
\[(w - \eta_0)(w - \eta_1)(w - \eta_2) = w^3 - \sigma w^2 + qw - p\]

By comparing coefficients, we get

\[p\zeta = \frac{ra}{ca} \quad (5.7)\]

\[1 = \bar{a}(\zeta + \sigma) \quad (5.8)\]

Since \(h^2(0) = \frac{\alpha}{p}\) and by equations (5.7) and (5.8), \(1 = |a||\zeta + \sigma| \leq 4|a\zeta| = \frac{4\alpha r}{cp}\), we get \(|h(0)| \geq \frac{\sqrt{c}}{2}\). Looking at equation (5.5), we get that at least one \(\eta_j\) must be in the denominator. More precisely \(\prod_{j \notin I} \eta_j = O\left(\frac{t^1}{2}\right)\) as \(t \to 0\). As the other \(\eta_j\)'s lie in the unit disc, \(p = O\left(\frac{t^1}{2}\right) \to 0\) and by (5.8), as \(\sigma\) is bounded, we get that \(a \to 0\), in fact \(a = O\left(\frac{t^1}{2}\right)\).

Therefore,

\[\Xi(w) = w(cw^2 + r) + w^4O(t^{1/2}) + O(t^{1/2})\]

and we see that \(\eta_0 = O(t^{1/2})\) and \(\eta_1\eta_2 \to \frac{r\sqrt{c}}{c}, \eta_1 \mapsto +\sqrt{\frac{c}{c}}, \eta_2 \mapsto -\sqrt{\frac{c}{c}}\).

Looking again at (5.5) we conclude that \(0 \notin I\). Also, \(p\zeta = \zeta_0\eta_0\eta_2 = \zeta_0\left(\frac{r\sqrt{c}}{c} + O(t^{1/2})\right) = \frac{ra}{ca}, \text{ hence } \zeta_0 = \frac{a}{r} + O(t^{1/2})\).

Now we need to analyze the cases \(I = \emptyset, \{1\}, \{2\}, \{1, 2\}\). Note that since \(|I| \neq 3\), there are no holomorphic discs for \(m = -1\).

Case \(I = \emptyset, m = 2\):

By (5.6), (5.7):

\[(\bar{a}\zeta)^3 = a^4 \left(\frac{1 - t}{t}\right)^2 \frac{e^{-66i}r^4}{c^3} = a^4 K \quad (5.9)\]

where \(K = \left(\frac{1 - t}{t}\right)^2 \frac{e^{-66i}r^4}{c^3}\). But also, by (5.8)

\[(\bar{a}\zeta)^3 = 1 + O(t^{1/2}) \quad (5.10)\]

Combining (5.9) with (5.10) we see that for \(g(a) = 1 - (\bar{a}\zeta)^3\)

\[a^4K - 1 + g(a) = 0 \quad (5.11)\]

One sees that, for sufficiently small \(t\), there are 4 solutions of such equation since \(g(a)\) and \(g'(a)\) are \(O(t^{1/2})\). (To see that \(g'(a) = O(t^{1/2})\), we
use that \( g(a) = \tilde{g}(a, \bar{a}) \), where \( \tilde{g}(a, b) \) is a holomorphic function, and using Cauchy’s differentiation formula, we get that each partial derivative is \( O(t^{1/2}) \).

So for each \( \theta \) there are four solutions for \( a \), each of them close to a fourth root of \( K^{-1} \). Naming these solutions, as varying continuously with \( \theta \), in counter-clockwise order, \( a_1(\theta), a_2(\theta), a_3(\theta), a_4(\theta) \), we see from (5.9) that \( a_j(\theta + \pi/3) = a_{j+1}(\theta) \). So one can parametrize the moduli space of holomorphic discs in the class \( H = 2\beta + 2\alpha \) using only holomorphic discs \( u^\theta_{a_1(\theta)} \) for \( \theta \in [0, 4\pi] \). The notation \( u^\theta_{a_1(\theta)} \) means that we are considering \( u^\theta \) for a given value of \( \theta \) and the other parameters determined by \( a_1(\theta) \). But note that solutions are counted twice as the disc automorphism \( w \mapsto e^{\phi'}w \) amounts to \( a \mapsto e^{-\phi'}a \) and \( 0 = \phi \mapsto \phi - 2\phi' = -2\phi' \) in (5.2), so \( \pm a \) gives the same holomorphic disc modulo reparametrization. Hence we see that \( u^\theta_{a_1(\theta)} = u^\theta_{-a_1(\theta)} \) is the same up to reparametrization. Therefore the map \( \theta \mapsto u^\theta_{a_1(\theta)} \), from \( S^1 = [0, 2\pi]/0 \sim 2\pi \) to the moduli space \( \mathcal{M}(T^c, H = 2\beta + 2\alpha) \) gives a diffeomorphism.

In order to compute \( n_{H-2\beta+2\alpha}(T^c) \) we need to look at \( ev_*[\mathcal{M}(T^c; H = 2\beta + 2\alpha)] = n_{H-2\beta+2\alpha}(T^c) \). The boundary of each holomorphic disc lies in the class \( 2(\partial \alpha - \partial \beta) \), and the parameter \( \theta \) comes from the action \( e^{i\theta} : (\xi, z) = (e^{i\theta} \xi, e^{-i\theta} z) \), described in coordinates \( (\xi, z) \) for \( y = 1 \), whose orbits are in the class of the thimble, i.e., \( \partial \alpha \). Therefore, \( n_{H-2\beta+2\alpha}(T^c) = \pm 2 \).

Case I = \{2\} (similarly \( I = \{1\} \), \( m = 1 \):

By (5.6), (5.7):

\[
\eta_0^3 = \left( \frac{t}{1-t} \right)^2 \frac{e^{\phi_1}}{ar} \eta_1^2 = \left( \frac{t}{1-t} \right)^2 \frac{e^{\phi_1}}{ar} (-1 + O(t^{1/2}))
\]

(5.12)

But since \( \zeta \eta_0 = \frac{a}{t} + O(t^{1/2}) \), and \( a = O(t^{1/2}) \)

\[(\bar{a}\zeta \eta_0)^3 = [a(1+O(t^{1/2}))]^3 = a^3(1+O(t^{1/2})) = (\bar{a}\zeta)^3 \left( \frac{t}{1-t} \right)^2 \frac{e^{\phi_1}}{ar} (-1 + O(t^{1/2})).\]

Using \( (\bar{a}\zeta)^3 = 1 + O(t^{1/2}) \), we get

\[Ka^4 + O(t^{1/2}) = -1 \]

(5.13)

where now \( K = \left( \frac{1-t}{\bar{a}} \right)^2 re^{-\phi_1} \).
Using the same argument as before we get four solutions for \(a, a_1(\theta), a_2(\theta), a_3(\theta), a_4(\theta)\), varying continuously with \(\theta\), ordered in the counterclockwise direction. Again \(a_j(\theta + \pi/3) = a_{j+1}(\theta)\), but now the disc automorphism \(w \mapsto -w\), not only switches \(a \mapsto -a\) but also \(\eta_1 \leftrightarrow \eta_2\), which accounts for the case \(I = \{1\}\). Therefore the moduli space \(M(T_{r,0}^c, H - 2\beta + \alpha)\) is given by \(\{u_\theta^a; \theta \in [0, 4\pi]\}\), and hence \(n_{H - 2\beta + \alpha}(T_{r,0}^c) = \pm 4\).

The case \(I = \{1, 2\}, m = 0\), works in a totally analogous way, with \(n_{H - 2\beta}(T_{r,0}^c) = \pm 2\).

### 5.3 Discs in classes \(2H - 5\beta + k\alpha\)

**Theorem 5.5.** There are no Maslov index 2 holomorphic discs in the class \(2H - 5\beta - 2\alpha\), and one-parameter families of holomorphic discs in the classes \(2H - 5\beta + k\alpha\), \(k = -1, 0, 1, 2, 3, 4\), with algebraic counts equal to 1, 5, 10, 10, 5, 1, up to sign, respectively.

This is precisely what we expect from the term \(e^{-2\Lambda u^5w(1+w)^5}\) in \(W_{Che^2}\), see equation (3.3).

**Proof.** Following the same reasoning and similar notation as in the previous subsection, we consider a holomorphic map \(u : (\mathbb{D}, S^1) \rightarrow (\mathbb{C}P^2, T_{r,0}^c)\) in the class \(2H - 5\beta + k\alpha\), \(-2 \leq k \leq 4\), and \(\Psi(w) = f \circ u(w)\). Analyzing intersection numbers with divisors we get

\[
\frac{\Psi(w) - c}{r} = \frac{\tau_{w_0}(w)}{\tau_{w_1}(w)\tau_{w_2}(w)} e^{i\phi}
\]

and denote by \(\eta_0, \ldots, \eta_5\) the zeros of \(\Psi(w)\). Then, the holomorphic disc can be described by

\[
z_1(w) = \tau_{w_1}(w)\tau_{w_2}(w) , \quad z_2(w) = e^{-i\theta}h(w) \prod_{j \notin I} \tau_j(w) , \quad z_3(w) = e^{i\theta}h(w) \prod_{j \in I} \tau_j(w)
\]

where \(h(w) = \left(\frac{\Psi(w)\tau_{w_1}(w)\tau_{w_2}(w)}{\prod_{j} \tau_j(w)}\right)^{1/2}\), and \(I \subset \{0, 1, 2, 3, 4, 5\}\).

Again using automorphisms of the disc we can choose \(w_1 = 0\) and \(\phi = 0\) and also rename \(w_2 = \nu\) and \(w_0 = b\). In the same way we get a pair of equations.
\[ z_3(0) = \frac{1-t}{t} = \frac{e^{3i\theta} \prod_{j \in I} \tau_j(0)}{h(0) \prod_{j \notin I} \tau_j^2(0)} = \frac{e^{3i\theta}(-1)^{|I|} \prod_{j \in I} \eta_j}{h(0) \prod_{j \notin I} \eta_j^2} \]  

(5.16)

\[ z_3(\nu) = \frac{1-t}{t} = \frac{e^{3i\theta} \prod_{j \in I} \tau_j(\nu)}{h(\nu) \prod_{j \notin I} \tau_j^2(\nu)} = \frac{e^{3i\theta} \prod_{j \in I} q_j}{h(\nu) \prod_{j \notin I} q_j^2} \]  

(5.17)

where we write \( q_j = \tau_j(\nu) \). Consider

\[ \Xi(w) = w^3(1 - \bar{b}w)(w - \nu)^3 \Psi(w) = r(w - b)(1 - \bar{nu})^3 + cw^3(1 - \bar{b}w)(w - \nu)^3 = -\bar{c}b(w - \zeta) \prod_j (w - \eta_j) \]

where \(|\zeta| > 1\), and write

\[ \prod_j (w - \eta_j) = w^6 - \sigma_1 w^5 + \sigma_2 w^4 - \sigma_3 w^3 + \sigma_4 w^2 - \sigma_5 w + p \]

Comparing the coefficients of 1 and \( w^6 \), we get:

\[ p\zeta = -\frac{rb}{\bar{c}b} \]  

(5.18)

\[ 1 + 3\nu \bar{b} = \bar{b}(\zeta + \sigma_1) \]  

(5.19)

For the following we recall that \( h^2(w) = \frac{\Psi(w)w^3 \tau_j(w)}{\prod \tau_j(w)} \), in particular, by (5.14), \( h^2(0) = -\frac{rb}{p} \) and \( h^2(\nu) = \frac{r\bar{b}(\nu)}{\prod q_j} \).

By (5.18), (5.19), we get that

\[ 1 = |b|\zeta + \sigma_1 - 3\nu| \leq 10|\bar{b}||\zeta| = 10 \frac{|h(0)|^2}{c} \]

So \( |h(0)|^2 \geq \frac{\zeta}{10} \). We see from (5.16) that \( \prod_{j \notin I} \eta_j^2 = O(t) \) and hence \( p = O(t^{3/2}) \to 0 \) as \( t \to 0 \). Also \( \zeta \to \infty \), \( b = O(t^{1/2}) \to 0 \) and \( \bar{b} \zeta = 1 + O(t^{3/2}) \to 1 \).

Now look at

\[ \Xi(\nu) = r(1 - |\nu|^2)^3(\nu - b) = (1 - |\nu|^2)^3(1 - \nu b)r\tau_0(\nu) = -\bar{c}b(\nu - \zeta) \prod_j (\nu - \eta_j) = -\bar{c}b(\nu - \zeta) \prod_j q_j(1 - \eta_j \nu) \]

(5.20)
Using that \( \bar{b} \zeta = 1 + O(t^{1/2}) \) and \( \bar{b} \nu = O(t^{1/2}) \) we see that

\[
h^2(\nu) = \frac{r \tau_b(\nu)}{\prod_j q_j} = \frac{r(\nu - b)}{\prod_j q_j} \frac{1}{1 - \nu b} = \frac{c \prod_j (1 - \bar{\eta}_j \nu)}{(1 - |\nu|^2)^3(1 - \nu b)(1 + O(t^{1/2}))}
\]

is bounded away from zero. Indeed, if \( \frac{c \prod_j (1 - \bar{\eta}_j \nu)}{(1 - |\nu|^2)^3(1 - \nu b)} \) approaches zero, then, since \( \bar{c} \bar{b} (\nu - \zeta) \prod_j q_j \) is bounded, we get by (5.20) that \( b - \nu \to 0 \) and hence \( \nu \to 0 \). But in this case we see that \( \prod_j (1 - \bar{\eta}_j \nu) \to 1 \), not 0.

Then we see from (5.17) that \( \prod_{j \neq 1} q_j^2 = O(t) \), hence \( \prod_j q_j = O(t^{1/2}) \). We want to show that \( b - \nu \to 0 \) and hence \( \nu \to 0 \). For that to follow from (5.20), we need to see that \( |\nu| \) does not approach 1 as \( t \to 0 \).

**Lemma 5.6.** As \( t \to 0 \), \( |\nu| \) is bounded by a constant strictly smaller than 1.

**Proof.** Let’s look again to \( \Xi(w) \), knowing that \( b = O(t^{1/2}) \):

\[
\Xi(w) = rw(1 - \bar{\nu} w)^3 + cw^3(w - \nu)^3 + w^7O(t^{1/2}) + O(t^{1/2})
\]

Then we see that the roots \( \eta_j \) lying inside the disc are very close to the solutions of \( rw(1 - \bar{\nu} w)^3 + cw^3(w - \nu)^3 = 0 \), for \( t \) very small. The non-zero solutions satisfy:

\[
|w^2 \tau_\nu(w)^3| = \frac{r}{c} < 1.
\]

Now, assume there is a sequence of values of \( t \) tending to 0 and holomorphic discs such that \( \nu \to \nu_0 \), \( |\nu_0| = 1 \). From equation (5.21), we conclude that three of the roots \( \eta_j \) of \( \Xi(w) \) converge to \( \nu_0 \), say \( \eta_1, \eta_2, \eta_3 \), one converges to 0, say \( \eta_4 \), and the other two solutions, \( \eta_4 \) and \( \eta_5 \) converge to square roots of \( \frac{r}{cw_0} \), by (5.22) (for values of \( w \) lying outside a neighborhood of \( \nu_0 \)),

\[
\tau_\nu(w) = \frac{w - \nu}{1 - \nu w} = -\nu \frac{1 - \nu^{-1} w}{1 - \nu w} \to -\nu_0
\]

Let \( \epsilon_j \) be such that \( \bar{\eta}_j = \nu - \epsilon_j \), for \( j = 1, 2, 3 \). By (5.20), recalling that \( b = O(t^{1/2}) \), \( \prod_j q_j = O(t^{1/2}) \) and \( \bar{b} \zeta = 1 + O(t^{1/2}) \) we see that \( (1 - |\nu|^2)^3 = K \epsilon_1 \epsilon_2 \epsilon_3 \), where \( K \to \frac{r}{\nu_0} - \frac{1}{\nu_0} \) is bounded above and below.

Since \( \prod_j q_j = O(t^{1/2}) \), for \( j = 1, 2, 3 \), some \( q_j \to 0 \). Up to relabeling, assume \( q_1 \to 0 \). We have that, for \( j = 1, 2, 3 \):

\[
q_j = \tau_j(\nu) = \frac{\nu - \bar{\eta}_j}{1 - \bar{\eta}_j \nu} = \frac{\epsilon_j}{1 - |\nu|^2 + \nu \epsilon_j}.
\]
So
\[
\frac{1}{q_1} = \left( \frac{K_1 e_2}{e_1} \right)^\frac{1}{3} + \frac{\nu_1}{e_1} \to \infty.
\] (5.24)

Passing to a subsequence if needed, we may assume that \(|\epsilon_1| \leq |\epsilon_2| \leq |\epsilon_3|\).

So by (5.24), \(\frac{\nu_1}{\epsilon_3} \to 0\). Therefore,
\[
\left| \frac{1}{q_3} \right| = \left| \left( \frac{K_1 e_2}{e_1} \right)^\frac{1}{3} + \frac{\nu_3}{\epsilon_3} \right| \to |\nu_0| = 1.
\] (5.25)

But, by (5.22) and \(|\tau_{\eta_j}(\nu)| = |\tau_{\nu}(\eta_j)|\), \(|\eta_3| q_3^3 \to \zeta\), which gives a contradiction since \(|\eta_3| \to |\nu_0| = 1\) and \(|q_3| \to 1\). \(\square\)

Using Lemma 5.6 equation (5.20), \(b = O(t^{1/2})\) and \(\prod_j q_j = O(t^{1/2})\), we see that \(\nu = O(t^{1/2})\).

So \(\Xi(w) = w(r + cw^5) + w^7 O(t^{1/2}) + O(t^{1/2})\) and, assuming \(|\eta_0| \leq |\eta_j| \forall j\), we get that for \(j \neq 0\), \(\eta_j \to -\left(\frac{\nu}{\epsilon_3}\right)e^{\frac{2\pi i}{3}}\), while \(\eta_0 = O(t^{1/2})\). Therefore, by \(5.16\), \(0 \not\in I\). In particular there is no holomorphic disc representing the class \(2\widetilde{H} - 5\beta + k\alpha\), for \(k = -2\).

To prove the existence of such discs for \(-1 \leq k \leq 4\) with the right count, we look at the limit \(t = 0\), i.e., in \(\mathbb{CP}(1 : 1 : 4)\). These six families of discs are expected to ‘survive’ in the limit and not pass through the orbifold point \(\mathbb{CP}(1, 1, 4)\), since, as represented on the almost toric picture, for \(t > 0\) they don’t touch the singular fiber that collapses into the singular orbifold point when \(t = 0\), see Figure 11. We show that this is the case for the limits of the above families of holomorphic discs, argue that the disc counts are the same for \(X_0\) and \(X_t\) for a sufficiently small \(t\), and in section 5.4 we prove regularity.

In the limit \(t = 0\), taking into account that \(b = O(t^{1/2})\), \(\nu = O(t^{1/2})\), \(\eta_0 = O(t^{1/2})\) and \(\eta_j \to -\left(\frac{\nu}{\epsilon_3}\right)e^{\frac{2\pi i}{3}}\), we have that \(\Psi(w)\), \(\prod_{j=1}^{5} \tau_j(w)\), and \(h^2(w)\) thought as maps form \(\mathbb{D}\) to \(\mathbb{CP}^1\), uniformly converge to
\[
\Psi(w) = \frac{r + cw^5}{w^5}; \quad \prod_{j=1}^{5} \tau_j(w) = \frac{cw^5 + r}{rw^5 + c}; \quad h^2(w) = rw^5 + c.
\]

So, for instance, in the case \(I = \{1, 2, 3, 4, 5\}\) we get that \(z_1(w)\), \(z_2(w)\), \(z_3(w)\) uniformly converge to
\[
z_1(w) = w^2; \quad z_2(w) = e^{-i\theta} w \sqrt{rw^5 + c}; \quad z_3(w) = e^{i\theta} \frac{cw^5 + r}{rw^5 + c} \sqrt{rw^5 + c};
\]
On Exotic Lagrangian Tori in $\mathbb{CP}^2$

$$z_0(w) = \frac{z_2(w)}{z_1(w)} = e^{-2i\theta}(rw^5 + c).$$

Hence using the $(\tilde{x} : \tilde{y} : \tilde{z})$ coordinates of $\mathbb{CP}(1,1,4)$ we have

$$\tilde{x}(w) = e^{-i\theta}\sqrt{rw^5 + c}; \quad \tilde{y}(w) = w; \quad \tilde{z}(w) = e^{i\theta}\sqrt{rw^5 + c}.$$

In general, for each $I$, the holomorphic discs uniformly converge to discs $u^\theta_I$ given by

$$\tilde{x}(w) = e^{-i\theta}\sqrt{rw^5 + c}; \quad \tilde{y}(w) = w; \quad \tilde{z}(w) = e^{i\theta}\sqrt{rw^5 + c}.$$

(5.26)

Note that none of these discs pass through the singular point $(0 : 0 : 1)$ of $\mathbb{CP}(1,1,4)$.

As before, we have extra automorphisms of the disc, given by $w \mapsto e^{ik\frac{2\pi}{5}}w$ that don’t change (5.14) and we need to quotient out by this action of $\mathbb{Z}/5\mathbb{Z}$.

We get that $\mathbb{Z}_5$ acts on $u^\theta_I$ as follows:

| $|I|$ | 0 | 1 | 2 | 3 | 4 | 5 |
|------|---|---|---|---|---|---|
| $u^\theta_I \mapsto$ | $u^\theta_0$ | $u^\theta_{I-k}$ | $u^\theta_{I-k}$ | $u^\theta_{I-k}$ | $u^\theta_{I-k}$ | $u^\theta_I$ |

where $I - k = \{j - k : j \in I\}$.

We also note that, for fixed $I$, varying $\theta \in [0,2\pi]$ and looking at the boundary of the discs, the 2-cycle swept by $\partial u^\theta_I$ is $[\partial u^\theta_I] = \pm 5[T_{r,0}^c]$. Therefore, in the case $|I| = 0$ or 5, after quotienting by $\mathbb{Z}/5\mathbb{Z}$, the algebraic count is $\pm 1$. In the cases $|I| = 1$ or 4, the action of $\mathbb{Z}/5\mathbb{Z}$ permutes the indices, so the moduli space of holomorphic discs is given by $\{u^\theta_I; \theta \in [0,2\pi]\}$, where $I = \{1\}$, respectively $I = \{2,3,4,5\}$, hence the algebraic count is $\pm 5$. Similarly for $|I| = 2$ or 3, the action of $\mathbb{Z}/5\mathbb{Z}$ permutes the indices, so the moduli space of holomorphic discs is given by $\{u^\theta_I; \theta \in [0,2\pi]\} \cup \{u^\theta_{I'}; \theta \in [0,2\pi]\}$, where $I = \{1,2\}$ and $I' = \{1,3\}$, respectively $I = \{3,4,5\}$, $I' = \{2,4,5\}$, hence the algebraic count is $\pm 10$.

**Lemma 5.7.** Assuming regularity, each of the above families of discs in $X_0$ has a corresponding family in $X_t$, for a sufficiently small $t$. 
Proof. We consider the 3-dimensional complex hypersurface $X$ inside $\mathbb{C} \times (\mathbb{C}P(1, 1, 1, 2) \setminus (0 : 0 : 0 : 1))$ defined by the equation

$$X : z_0z_1 - (1 - t)z_2^2 - tz_3 = 0$$

(5.27)

containing

$$\mathcal{L} = \left\{ (t, (z_0 : z_1 : z_2 : z_3)) ; t \in \mathbb{R} \text{ and } \left| \frac{z_2z_3}{z_1^3} - c \right| = r ; \left| \frac{z_2}{z_1} \right| = \left| \frac{z_3}{z_1^2} \right| \right\}$$

(5.28)

as a totally real submanifold.

Then we consider $\mathcal{M}(X, \mathcal{L})$ the moduli space of Maslov index 2 holomorphic discs in $X$ with boundary on $\mathcal{L}$. By applying the maximum principle to the projection on the first factor, we see that such holomorphic discs lie inside the fibers $X_t$, for $t \in \mathbb{R}$. Let’s consider discs that stay away from the singular point in $X_0$, such as those computed above.

Assuming the discs above are regular in $X_0$ implies they are regular as discs in $X$. This follows from the splitting $u^*TX = u^*TX_0 \oplus \mathbb{C}$ and the $\bar{\partial}$ operator being surjective onto 1-forms with values in $u^*TX_0$, by the assumed regularity, and onto 1-forms with values in $\mathbb{C}$, by regularity of holomorphic discs in $\mathbb{C}$ with boundary in $\mathbb{R}$.

Hence $\mathcal{M}(X, \mathcal{L})$ is smooth near the solutions for $t = 0$ given above and the map $\mathcal{M}(X, \mathcal{L}) \to \mathbb{R}$, which takes a disc in the fiber $X_t$ to $t$, is regular at 0. Therefore for a small $t$, all the Maslov index 2 holomorphic discs in $X_0$ computed above deform to holomorphic discs in $X_t$.

The regularity of the discs above is proven in Theorem 5.11.

The families of discs in the classes $2H - 5\beta + k\alpha$, given by (5.15) have been shown to converge uniformly to the corresponding ones in $\mathbb{C}P(1, 1, 4)$ given by (5.26). Smoothness of the moduli space $\mathcal{M}(X, \mathcal{L})$ near the families of discs given by (3.26) in $X_0$ guarantees that each family $\{u^\theta_\theta ; \theta \in [0, 2\pi]\}$ has a unique family in $X_t$ converging to it, for all $t$ sufficiently small. Hence the counts of Maslov index 2 holomorphic discs in the classes $2H - 5\beta + k\alpha$ for $X_t$ are the same as the ones computed in $X_0$. This finishes the proof of Theorem 5.5.

5.4 Regularity

In order to prove regularity, we consider the following two lemmas.

Lemma 5.8. Let $u_\theta$ be a one parameter family of Maslov index 2 holomorphic discs in a Kähler 4 dimensional manifold $X$ with boundary on a
Lagrangian. Set \( u = u_0 \) and \( V = \frac{\partial}{\partial \theta} u_0 \) |_{\theta=0} a vector field along \( u \), tangent to \( TL \) along the boundary of \( u \).

If \( V \) is nowhere tangent to \( u(D) \) and \( u : D \to X \) is an immersion, then \( u \) is regular.

**Proof.** As \( u \) is an immersion, we can consider the splitting \( u^*TX \cong T\mathbb{D} \oplus \mathcal{L} \) as holomorphic vector bundles, where \( \mathcal{L} \) is the trivial line bundle generated by \( V \). Also \( u^*_S TL \cong TS^1 \oplus \text{Re}(\mathcal{L}) \), where \( \text{Re}(\mathcal{L}) = \text{span}_\mathbb{R}\{V\} \) and \( S^1 \cong \partial \mathbb{D} \).

So a section \( \zeta \in \Omega^0|_{S^1} TL(D, u^*TX) \) of \( u^*TX \) that takes values in \( u^*_S TL \) along the boundary splits as \( \zeta_1 \oplus \zeta_2 \in \Omega^0|_{S^1} (\mathbb{D}, T\mathbb{D}) \oplus \Omega^0|_{S^1} (\mathbb{D}, \mathcal{L}) \). Since \( J \) is an integrable complex structure, the kernel of the linearized operator \( D\bar{\partial} \) is given by

\[
\{ \zeta \in \Omega^0|_{S^1} TL(D, u^*TX); \bar{\partial}\zeta = 0 \}
\]

which is isomorphic to

\[
\{ \zeta_1 \in \Omega^0|_{S^1} (\mathbb{D}, T\mathbb{D}); \bar{\partial}\zeta_1 = 0 \} \oplus \{ \zeta_2 \in \Omega^0|_{S^1} (\mathbb{D}, \mathcal{L}); \bar{\partial}\zeta_2 = 0 \}
\]

\[
\cong TId\text{Aut} (\mathbb{D}) \oplus \text{Hol}((\mathbb{D}, S^1), (\mathbb{C}, \mathbb{R}))
\]

The last term on the right comes from \( \mathcal{L} \) being trivial. \( \text{Aut}(\mathbb{D}) \) is known to be 3 dimensional, while \( \text{Hol}((\mathbb{D}, S^1), (\mathbb{C}, \mathbb{R})) \) is the space of real-valued constant functions. Therefore,

\[
\text{DimKer}(D\bar{\partial}) = 4 = 2 \cdot \chi(\mathbb{D}) + \mu(u^*TX, u^*_S TL) = \text{index}(D\bar{\partial})
\]

The following lemma sets a sufficient condition for \( V \), as given in the previous lemma, not to be tangent to \( u(\mathbb{D}) \).

**Lemma 5.9.** Let \( u : \mathbb{D} \to X \) be a Maslov index 2 holomorphic disc in a Kähler 4-manifold \( X \) with boundary on a Lagrangian \( L \) such that \( u_S : S^1 \to L \) is an immersion, and \( V \) a holomorphic vector field on \( X \) along \( u \), tangent to \( L \) at the boundary. If \( V \) is not tangent to \( u(\mathbb{D}) \) at the boundary, then \( u \) is an immersion and \( V \) is nowhere tangent to \( u(\mathbb{D}) \).

**Proof.** Suppose that either \( du(x) = 0 \) or \( V \) is tangent to \( u(\mathbb{D}) \) at a point \( u(x) \); up to reparametrizing the disc we may assume that \( x \neq 0 \). Consider another holomorphic vector field \( W = du(\frac{\partial}{\partial \theta}) \) given by an infinitesimal rotation, which is tangent to \( u(\mathbb{D}) \), has a zero at 0 and is also tangent to \( L \) at
the boundary. Then the Maslov index can be computed using $\det^2(W \wedge V)$, so the number of zeros of $\det(W \wedge V)$ is $\mu(u^*TX, u|_{\partial D}TL)/2 = 1$. But $W$ vanishes at $u(0)$ and either vanishes or is parallel, as a complex vector, to $V$ at $u(x)$. Since the zeros of $W \wedge V$ always occur with positive multiplicity, as the vector fields are holomorphic, we get a contradiction.

Now we are ready to prove

**Theorem 5.10.** The holomorphic discs representing the classes $\beta$ and $H - 2\beta + m\alpha$ in $X_t$ computed on proposition 5.7 and theorem 5.4 are regular, for small $t$.

**Proof.** By Lemmas 5.8, 5.9, we only need to notice that for each of the holomorphic discs $u^0_I$ considered, the vector field $V(w) = \frac{\partial}{\partial \theta} u^0_I(w)$ is not tangent to $u^0_I(\partial D)$. We note that in the limit $t = 0$, we have that $u^0_I$ uniformly converge, in a compact neighborhood of the boundary, to a holomorphic disc given by:

$$z_1(w) = w; \quad z_2(w) = e^{-i\theta} \sqrt{rw^2 + c} \prod_{j \notin I} \tau_{\eta_j}(w); \quad z_3(w) = e^{i\theta} \sqrt{rw^2 + c} \prod_{j \in I} \tau_{\eta_j}(w)$$

where $\eta_1 = i \sqrt{r/c}$, $\eta_2 = -i \sqrt{r/c}$. So we see that in the limit $t = 0$, $V(w) = \frac{\partial}{\partial \theta} u^0_I(w)$ is parallel to the fibers of $f(\tilde{x}, \tilde{z}) = \tilde{x} \tilde{z}$. Hence, by lemmas 5.8, 5.9, these discs are regular.

**Theorem 5.11.** The holomorphic discs in $X_0$ computed in Theorem 5.5 are regular. By Lemma 5.4, for small $t$, the corresponding holomorphic discs in the classes $2H - 5\beta + m\alpha$ in $X_t$ are also regular.

**Proof.** Similar to the other cases, for each considered holomorphic disc $u_0$, we have the vector field $V(w) = \frac{\partial}{\partial \theta} u_0(w) = (i\tilde{x}(w), -i\tilde{z}(w))$ in coordinates $(\tilde{x}, \tilde{z})$, for $\tilde{y} = 1$ on $\mathbb{C}\mathbb{P}(1, 1, 4)$ along the boundary. These vectors are not tangent to $u_0(\partial D)$ along the boundary, since they are nonvanishing and parallel to the fibers of $f(\tilde{x}, \tilde{z}) = \tilde{x} \tilde{z}$. Hence, by lemmas 5.8, 5.9 these discs are regular.
5.5 Orientation

The choice of orientation of the moduli space of holomorphic discs is determined by a choice of spin-structure on the Lagrangian, see [6] section 5. In this section we choose a spin structure on our Lagrangian $\text{Che}^2$ torus and argue that, under the choice of orientations made in [9], see also section 7 of [6], the evaluation map from each of the moduli spaces of Maslov index 2 holomorphic discs considered in this section to the $\text{Che}^2$ torus is orientation preserving. We use the same definition of spin-structure given by C. Cho in section 6 of [6]:

**Definition 5.12.** A spin structure on an oriented vector bundle $E$ over a manifold $M$ is a homotopy class of a trivialization of $E$ over the 1-skeleton of $M$ that can be extended over the 2-skeleton.

In case of surfaces, it’s enough to consider a stable trivialization of the tangent bundle. We see that $\partial \alpha$ and $\partial \beta$ form a basis of $H_1(T^c_{r,0}, \mathbb{Z})$ and hence they induce a trivialization of the tangent bundle of $T^c_{r,0}$ oriented as $\{\partial \alpha, \partial \beta\}$.

The orientation of the moduli space at a disc $u : (\mathbb{D}, \partial \mathbb{D}) \to (X^{2n}, L^n)$ is then given by the orientation of the index bundle of the linearized operator $D\bar{\partial}u$ that is induced by the chosen trivialization of the tangent bundle $TL$ along $\partial \mathbb{D}$, as described in [9].

The rough idea is that we extend the trivialization of the tangent bundle of the Lagrangian to a neighborhood of $\partial \mathbb{D}$, then take a concentric circle contained in it, and pinch it to a point $O \in \mathbb{D}$, the part of the disc inside the circle becoming a $\mathbb{CP}^1$. The trivialization of $TL$ along the pinched neighborhood gives a trivialization of its complexification $TX$. This way, considering the isomorphisms given by the trivializations, the linearized operator is homotopic to a $\bar{\partial}$ operator on $\mathbb{D} \cup \mathbb{CP}^1$, whose kernel consists of pairs $(\xi_0, \xi_1)$ where: $\xi_0$ is a holomorphic section of the trivial bundle $\mathbb{C}^n$ over the disc, with boundary on the trivial subbundle $\mathbb{R}^n$, i.e, a constant maps into $\mathbb{R}^n$; and $\xi_1$ is a holomorphic section of the bundle induced by $u^*TX$ over $\mathbb{CP}^1$, which we denote by $TX|_{\mathbb{CP}^1}$. These sections must match at $O \in \mathbb{D}$ and the ‘south pole’ $S \in \mathbb{CP}^1$. In other words, Fukaya, Oh, Ohta, Ono show that the index of the linearized operator (seen as a virtual vector space $\text{Ker}D\bar{\partial}u - \text{CoKer}D\bar{\partial}u$) is isomorphic to the kernel of the homomorphism:

\[(\xi_0, \xi_1) \in \text{Hol}(\mathbb{D}, \partial \mathbb{D} : \mathbb{C}^n, \mathbb{R}^n) \times \text{Hol}(\mathbb{CP}^1, TX|_{\mathbb{CP}^1}) \to \xi_0(O) - \xi_1(S) \in \mathbb{C}^n \cong TX|_S \]

(5.29)
Now the kernel can be oriented by orienting $\mathbb{R}^n \cong Hol(\mathbb{D}, \partial \mathbb{D} : \mathbb{C}^n, \mathbb{R}^n)$ (which is essentially the trivialization of the tangent space of the Lagrangian), since $Hol(\mathbb{CP}^1, TX|_{\mathbb{CP}^1})$ and $\mathbb{C}^n$ carry complex orientations. For a detailed account of what we just discussed, see Chapter of [9] Part II, also Proposition 5.2 in [6].

Denote by $\tilde{M}(\gamma)$ the space of holomorphic discs on $\mathbb{CP}^2$ with boundary on $T_{c,0}$, in the class $\gamma$, not quotiented out by $Aut(\mathbb{D})$. By the same argument as in section 8 of [6], the factor $\mathbb{R}^n \cong Hol(\mathbb{D}, \partial \mathbb{D} : \mathbb{C}^n, \mathbb{R}^n)$ in (5.29) corresponds to the subspace of $T_u \tilde{M}(\gamma)$ given by the deformations of $u$ which correspond to translations along the boundary of $T_{c,0}$, i.e., generated by $V = \frac{\partial}{\partial \theta} u_\theta$ and by infinitesimal rotations in $Aut(\mathbb{D})$. This way, we orient the moduli space of discs accordingly with our chosen orientation $\{\partial \alpha, \partial \beta\}$. In particular, $\tilde{M}(\beta)$, which consists of one-parameter family of discs $u_\theta$ described in Proposition 5.1 is oriented in the positive direction of $\theta$, since $\frac{\partial u_\theta}{\partial \theta}$ and the tangent vector to the boundary of $u_\theta$ form a positive oriented basis of $TT_{c,0}$; while the other moduli spaces $\mathcal{M}(H - 2\beta + m\alpha)$ and $\mathcal{M}(2H - 5\beta + k\alpha)$ are oriented in the negative direction of the parameter $\theta$, since in these cases $\frac{\partial u_\theta}{\partial \theta}$ and the tangent vector to the boundary of $u_\theta$ form a negative oriented basis.

**Proposition 5.13.** The evaluation maps from $\mathcal{M}_1(\beta)$, $\mathcal{M}_1(H - 2\beta + m\alpha)$ and $\mathcal{M}_1(2H - 5\beta + k\alpha)$ to $T_{c,0}$ are all orientation preserving.

Here the subscript 1 refers to the moduli space with one marked point at the boundary. The proof of the proposition above follows from the same argument as in Proposition 8.2 in [6].

As a corollary of all we have done in this section, we get

**Theorem 5.14.** In the region corresponding to Che$^2$ tori, the mirror superpotential is given by (3.3):

$$W_{Che^2} = u + 2\frac{e^{-\Lambda}}{uw}(1 + w)^2 + \frac{e^{-2\Lambda}}{uw^2}(1 + w)^5$$

### 6 The monotone torus

In this section we show that we can modify our symplectic form in a neighborhood of $D_5$ to a new one for which $T_{c,0}$ is Lagrangian monotone. Recall that a Lagrangian $L$ in a symplectic manifold $(X, \omega)$ is called monotone if there exists a constant $M_L$ such that for any disc $u$ in $\pi_2(X, L)$ satisfies

$$\int u^*\omega = M_L u_L(u)$$
where $\mu_L$ is the Maslov class.

Since $\mu(H) = 6$, $\mu(\beta) = 2$ and $\mu(\alpha) = 0$ and a disc in the class $\alpha$ is given by the Lefschetz thimble over the interval $[0, c - r]$ with respect to the symplectic fibration $f$, so $\int \omega = 0$, we see that $L = T^c_{r,0}$ satisfies the monotonicity condition if and only if $[\omega] \cdot \beta = \int_\beta \omega = \Lambda/3$, where $\Lambda = \int_{\mathbb{C}P^1} \omega$.

**Proposition 6.1.** There is a Kähler form $\omega$ for which $T^c_{r,0}$ is Lagrangian monotone. Moreover, $\omega$ agrees with $\omega$ away from a neighborhood of $D_5$ that is disjoint from $T^c_{r,0}$.

**Proof.** By direct computation one can see that $[\omega] \cdot \beta < \Lambda/3$, for all $r$ and $c$ (if it weren’t the case we could simply choose parameters $c$ and $r$ so that $[\omega] \cdot \beta = \Lambda/3$). So in order to make $T^c_{r,0}$ monotone we perform a Kähler inflation in a neighborhood of the quintic $D_5$ (see section 5.1) to achieve $[\omega] \cdot \beta = \int_{\mathbb{C}P^1} \omega/3$, keeping $[\omega] \cdot \alpha = 0$.

Take a small neighborhood $\mathcal{N}$ of $D_5 = \{s_5 = 0\}$ not intersecting $T^c_{r,0}$, where $s_5 = x^2 - 2cy^2z + c^2y^5$. Take a cutoff function $\chi$ such that $\chi(\{s_5\})$ is equal to 1 in a neighborhood of $D_5$ and is equal to 0 in the complement of $\mathcal{N}$. We then define $\hat{\omega} = \omega + K\sigma$ for

$$\sigma = \frac{i}{2} \partial \bar{\partial} \log \left( |s_5|^2 + \varepsilon \chi(|s_5|^2)(|x|^2 + |y|^2 + |z|^2)^5 \right)$$

where $K$ and $\varepsilon$ are constants to be specified. We use the fact that $\partial \bar{\partial} \log(|f|^2) = 0$ for a holomorphic function $f$, to note that the expression for $\sigma$ is the same for the homogeneous coordinates $(1 : \frac{x}{z} : \frac{y}{z})$, $(\frac{z}{x} : 1 : \frac{y}{z})$ and $(\frac{z}{x} : \frac{y}{z} : 1)$, therefore $\sigma$ defines a 2-form on $\mathbb{C}P^2$, and also to note that $\sigma = \partial \bar{\partial} \log(|s_5|^2) = 0$ outside $\mathcal{N}$, so $T^c_{r,0}$ is Lagrangian with respect to $\hat{\omega}$.

**Lemma 6.2.** $[\sigma] = 5[\omega_{FS}]$ is independent of $\varepsilon$ and the cutoff function $\chi$.

**Proof of Lemma.** To determine the cohomology class of $\sigma$, it is enough to compute $\int_{\mathbb{C}P^1} \sigma$. For this we consider $[\mathbb{C}P^1] = \{x = 0\}$, and write $\sigma = \frac{1}{4} d\bar{d} \log \psi_j$, where

$$\psi_1 = \frac{|s_5|^2 + \varepsilon \chi(|s_5|^2)(|x|^2 + |y|^2 + |z|^2)^5}{|y|^{10}},$$

$$\psi_2 = \frac{|s_5|^2 + \varepsilon \chi(|s_5|^2)(|x|^2 + |y|^2 + |z|^2)^5}{|z|^{10}},$$

are homogeneous functions respectively defined on $\{y \neq 0\}$, $\{z \neq 0\}$, such that
\[ \psi_1 \psi_2 = \frac{|z|^{10}}{|y|^{10}} \]  

(6.1)

We then divide \([\mathbb{CP}^1] = \{x = 0\}\) into two hemispheres \(H_+, H_-\), contained in \(\{y \neq 0\}, \{z \neq 0\}\), respectively, to compute

\[
\int_{[\mathbb{CP}^1]} \omega = \frac{1}{4} \int_{H_+} d\mathcal{F} \log \psi_1 + \frac{1}{4} \int_{H_-} d\mathcal{F} \log \psi_2 = \frac{1}{4} \int_{\partial H_+} d\mathcal{F} \log \psi_1 + \frac{1}{4} \int_{\partial H_-} d\mathcal{F} \log \psi_2 = \frac{1}{4} \int_{\partial H_+} d\mathcal{F} \log \psi_1 - \frac{1}{4} \int_{\partial H_+} d\mathcal{F} \log \psi_2 = \frac{5}{4} \int_{\partial H_+} d\mathcal{F} \log \left( \frac{|z|^{10}}{|y|^{10}} \right) \]  

(6.2)

which by comparison with the same calculation for \(\omega_{FS}\) is 5 times the area of \([\mathbb{CP}^1]\) with respect to the Fubini-Study form \(\omega_{FS}\).

In particular, taking \(\varepsilon \to 0\) we get that \(\sigma\) converges to a distribution supported at \(D_5 = \{s_5 = 0\}\). Now considering \(\alpha, \beta, H = [\mathbb{CP}^1]\) as cycles in \(H_2(\mathbb{CP}^2, \mathbb{CP}^2 \setminus N)\) and \([\sigma] \in H^2(\mathbb{CP}^2, \mathbb{CP}^2 \setminus N)\) we see that their \(\sigma\)-areas are a constant \(\pi\) times their intersection number with \(D_5\), i.e., \(\int \alpha \cdot \sigma = \pi \alpha \cdot [D_5] = 0\), \(\int \beta \sigma = \pi \beta \cdot [D_5] = 2\pi\) and \(\int_H \sigma = \pi [\mathbb{CP}^1] \cdot [D_5] = 5\pi\).

Then, since the ratio between the \(\sigma\)-area of \(\beta\) and \(H\) is \(2/5 > 1/3\), we can choose a constant \(K\), so that \([\hat{\omega}] \cdot \beta = [\hat{\omega}] \cdot H/3\). Given this value of \(K\), we can choose \(\varepsilon\) small enough to ensure that \(\hat{\omega}\) is nondegenerate and hence a Kähler form for which \(T^c_{c,0}\) is monotone Lagrangian.

\[ \square \]

### 6.1 The monotone \(\text{Che}^2\) torus is exotic

We are now going to prove that the count of Maslov index 2 holomorphic discs is an invariant of monotone Lagrangian submanifolds. We will see that it suffices to show it is an invariant under deformation of the almost complex structure. Let \(L\) be a monotone Lagrangian submanifold of a symplectic manifold \((X, \omega)\) and \(J_s, s \in [0, 1]\) a path of almost complex structures such that \((L, J_0), (L, J_1)\) are regular, i.e., for \(k = 0, 1\), Maslov index 2 \(J_k\)-holomorphic discs are regular. Note that since \(L\) is monotone there are no \(J_s\)-holomorphic discs of nonpositive Maslov index for any \(s \in [0, 1]\).

Consider then, for \(\beta \in \pi_2(X, L), \mu(\beta) = 2\), the moduli spaces \(\mathcal{M}(\beta, J_s)\) of \(J_s\)-holomorphic discs representing the class \(\beta\), modulo reparametrization.
We choose the path \( J_s \) generically so that the moduli space
\[
\tilde{\mathcal{M}}(\beta) = \bigsqcup_{s=0}^{1} \mathcal{M}(\beta, J_s)
\]
is a smooth manifold, with
\[
\partial \tilde{\mathcal{M}}(\beta) = \mathcal{M}(\beta, J_0) \cup \mathcal{M}(\beta, J_1).
\]

**Lemma 6.3.** If \( L \) is monotone, the classes and algebraic count of Maslov index 2 \( J \)-holomorphic discs with boundary on \( L \) are independent of \( J \), as long as \( (L, J) \) is regular.

**Proof.** By connectedness of the space of compatible almost complex structures we can consider \( J_s, s \in [0, 1] \) a generic path of almost complex structures such that \((L, J_0), (L, J_1)\) are regular as above. For \( L \) monotone, the Maslov index of a disc is proportional to its area. In particular, there are no \( J_s \)-holomorphic discs of Maslov index 0 or less, and the result follows immediately from the cobordism \( \tilde{\mathcal{M}}(\beta) \) between \( \mathcal{M}(\beta, J_0) \) and \( \mathcal{M}(\beta, J_1) \).

**Theorem 6.4.** If \( L_0 \) and \( L_1 \) are symplectomorphic monotone Lagrangian submanifolds of a symplectic manifold \((X, \omega)\), with an almost complex structure \( J \) so that \((L_0, J)\) and \((L_1, J)\) are regular, then algebraic counts of Maslov index 2 \( J \)-holomorphic discs, and in particular the numbers of different classes bounding such discs, are the same.

**Proof.** Let \( \phi : X \to X \) be a symplectomorphism with \( \phi(L_1) = L_0 \). Apply Lemma 6.3 with \( L = L_0, J_0 = J, J_1 = \phi_* J \).

**Corollary 6.5.** The monotone \( \text{Che}^2 \) torus is not symplectomorphic to either the monotone Chekanov torus or the monotone Clifford torus.

**Remark 6.6.** We can try to find an exotic torus in \( \mathbb{C}^2 \) by considering the \( \text{Che}^2 \) torus in affine charts. If we restrict to the coordinate charts \( \{y \neq 0\} \) or \( \{z \neq 0\} \), only the discs in the class \( \beta \) remain. Hence we cannot distinguish the \( \text{Che}^2 \) torus, considered in the charts \( \{y \neq 0\} \) or \( \{z \neq 0\} \), from the usual Chekanov torus, which also bounds a single family of holomorphic discs in \( \mathbb{C}^2 \). In the \( \{x \neq 0\} \) coordinate chart, another family of holomorphic discs in the class \( 2H - 5\beta - \alpha \) remains present, besides the one in the class \( \beta \). This can be checked directly or just by observing that the intersection numbers of the complex line \( \{x = 0\} \) with \( H, \beta, \alpha \) are 1, 0 and 2, respectively. Therefore our methods cannot distinguish the \( \text{Che}^2 \) torus, considered in the
chart \( \{ x \neq 0 \} \), from the usual Clifford torus in \( \mathbb{C}^2 \), which also bounds two families of Maslov index 2 holomorphic discs in \( \mathbb{C}^2 \), whose boundaries also generate the first homology group of the torus.

6.2 Floer Homology and non-displaceability

The modern way to show that a Lagrangian submanifold \( L \) of a symplectic manifold \( X \) is non-displaceable by Hamiltonian diffeomorphisms is to prove that its Floer homology \( HF(L, L) \) is non-zero. The version of Floer Homology we use in this section to prove that \( HF(T^c_{r,0}) \neq 0 \) (for some choice of local system) is the Pearl Homology, introduced by Oh in [14]. Here we will follow the definitions and notation given in [3].

Fig. 12: A trajectory contributing to the differential of the pearl complex

Let \( (X, \omega) \) be a symplectic manifold, \( J \) a generic almost complex structure compatible with \( \omega \), and \( L \) a monotone Lagrangian submanifold, with monotonicity constant \( M_L \). We also choose a \( \mathbb{C}^* \) local system over \( L \) (we don’t need to use the Novikov ring because \( L \) is monotone, so the area of holomorphic discs is proportional to the Maslov index) and a spin structure to orient the appropriate moduli spaces of holomorphic discs. To define the pearl complex we fix a Morse function \( f : L \rightarrow \mathbb{R} \) and a metric \( \rho \) so that \((f, \rho)\) is Morse-Smale and we denote the gradient flow by \( \gamma \).

The pearl complex \( \mathcal{C}(L; f, \rho, J) = (\mathbb{C}[q, q^{-1}](\text{Crit}(f)), d) \) is generated by the critical points of \( f \), and the differential counts configurations consisting of gradient flow lines of \( \gamma \) together with \( J \)-holomorphic discs as illustrated in Figure 12. More precisely:

\[
dx = \sum_{\text{ind}_f(y) = \text{ind}_f(x) + \mu(\lambda) - 1} \# \mathcal{P}(x, y, \lambda) \text{hol}_\nabla(\partial\lambda)q^{\mu(\lambda)} \cdot y \quad (6.3)\]

where \( \lambda \in \pi_2(M, L), \nabla \) is the chosen local system and \( \mathcal{P}(x, y, \lambda) \) is the the moduli space of “pearly trajectories”, whose elements are gradient flow lines of \( f \) from \( x \) to \( y \) when \( \lambda = 0 \), and otherwise tuples \((u_0, t_1, u_1, \cdots, t_k, u_k)\), \( k \in \mathbb{Z}_{\geq 0} \) so that:
i. For $0 \leq j \leq k$, $u_j$ is a non-constant $J$-holomorphic disc with boundary on $L$, up to reparametrization by an automorphism of the disc fixing $\pm 1$.

ii. $\sum_j [u_j] = \lambda$.

iii. For $1 \leq j \leq k$, $t_j \in (0, +\infty)$, and $\gamma_{t_j}(u_{j-1}(1)) = u_j(-1)$.

iv. $\gamma_{-\infty}(u_0(-1)) = x$, $\gamma_{+\infty}(u_k(1)) = y$.

The choice of spin structure on $L$ gives an orientation for the moduli space of holomorphic discs, and together with the orientation of the ascending and descending manifolds of each critical point of $f$, one can get a coherent orientation for $\mathcal{P}(x, y, \lambda)$ (at the present time, only the $\mathbb{Z}_2$ version is written up).

We have a filtration given by the index of the critical point. For simplicity we write $C_\ast(L; f, \rho, J)$ for $C_\ast(L; f, \rho, J)$. Note that $d = \sum_{j \geq 0} \delta_{2j}$, where $\delta_0 : C_\ast(L) \to C_{\ast-1}(L)$ is the Morse differential of the function $f$ and $\delta_{2j} : C_\ast(L) \to C_{\ast-2+2j}(L)$ considers only configurations for which the total Maslov index is $2j$. This gives a spectral sequence (the Oh spectral sequence), converging to the Pearl homology, for which the second page is the singular homology of $L$ with coefficients in $\mathbb{C}[q, q^{-1}]$.

Let $\mathcal{L}$ be the space of $\mathbb{C}^\ast$ local systems in $L$ and consider the ‘superpotential’ function $W : \mathcal{L} \to \mathbb{C}[q, q^{-1}]$,

$$W(\nabla) = \sum_{\beta, \mu(\beta) = 2} n_\beta q^{2 \text{hol}_\nabla(\partial \beta)} = \sum_{\beta, \mu(\beta) = 2} n_\beta z_\beta(\nabla) \quad (6.4)$$

where $z_\beta(\nabla) = q^{\text{hol}_\nabla(\partial \beta)} = q^{2 \text{hol}_\nabla(\partial \beta)}$ and $n_\beta$ is the count of Maslov index two $J$-holomorphic discs bounded by $L$ in the class $\beta$.

Assume also that the inclusion map $H_1(L) \to H_1(X)$ is trivial, so we have that the ring of regular functions on the algebraic torus $\mathcal{L} \cong \text{hom}(H_1(L), \mathbb{C}^\ast)$ is generated by the coordinates $z_j = z_\beta_j(\nabla)$ for relative classes $\beta_j$ such that $\partial \beta_j$ generates $H_1(L)$.

The following result is the analogue of Proposition 11.1 of [7] (see also section 12) in the pearly setting:

**Proposition 6.7.** Let $f$ be a perfect Morse function. Denote by $p$ the index 0 critical point, by $q_1, \cdots, q_k$ the index 1 critical points, by $\Gamma_1, \cdots, \Gamma_k$, the closure of the stable manifold of $q_1, \cdots, q_k$, respectively, and by $\gamma_1, \cdots, \gamma_k$ the closure of the respective unstable manifolds. Set $z_j = z_{\gamma_j}$. Then

$$\delta_2(p) = \sum_j z_j \frac{\partial W}{\partial z_j} q_j$$
In particular $\delta_2(p) = 0$ precisely for the local systems corresponding to the critical points of $W$.

**Proof.** We note that the only possible pearly trajectories contributing to the coefficient of $q_j$ in $\delta_2(p)$ consist of a holomorphic discs $u$ with $u(-1) = p$ together with a flow line from $u(1)$ ending in $q_j$, i.e., $u(1) \in \Gamma_j$. Hence,

$$\delta_2(p) = \sum_{\beta, \mu(\beta) = 2} n_{\beta} z_\beta(\nabla) \sum_j ([\partial \beta] \cdot [\Gamma_j]) q_j \quad (6.5)$$

Since $[\gamma_1], \ldots, [\gamma_k]$ form a basis for $H_1(L)$, we can write $[\partial \beta] = \sum_j a_j [\gamma_j]$, where $a_j = [\partial \beta] \cdot [\Gamma_j]$. So $z_\beta(\nabla)$ is a constant multiple of $\prod_j z_j^{a_j}$, therefore (6.5) gives precisely $\delta_2(p) = \sum_j z_j \partial W / \partial z_j q_j$.

**Corollary 6.8.** Consider the monotone Che$^2$ torus $T^c_{r,0}$, endowed with the standard spin structure and local system $\nabla$ such that $\text{hol}_\nabla(\partial \beta) = \frac{9}{4} e^{k \frac{2\pi}{3} i}$, for some $k \in \mathbb{Z}$, and $\text{hol}_\nabla(\partial \alpha) = \frac{1}{8}$, where $\alpha$ and $\beta$ are as defined in section 5.1. Then the Floer homology $HF(T^c_{r,0}, \nabla)$ is non-zero. Therefore $T^c_{r,0}$ is non-displaceable.

**Proof.** Since $T^c_{r,0}$ has dimension 2, all the boundary maps $\delta_2$ are zero for $j \geq 2$. Hence the pearl homology $HF(T^c_{r,0}, \nabla)$ is the homology of $(H_*(T^c_{r,0}) \otimes \mathbb{C}[q,q^{-1}], \delta_2)$. Writing $u = z_\beta$ and $w = z_\alpha$, the ‘superpotential’ is given by

$$W_{\text{Che}^2} = u + 2 \frac{q^6}{w^2}(1 + w)^2 + \frac{q^{12}}{w^5 w}(1 + w)^5$$

The result follows from computing the critical points of $W_{\text{Che}^2}$ which are $w = \frac{1}{8}$, $u = \frac{9}{4} e^{k \frac{2\pi}{3} i} q^2$.

**Remark 6.9.** It can be shown that in fact for any monotone Lagrangian torus $\delta_2 = 0$ for the local systems $\nabla$ which are critical points of $W$, so $HF(T^c_{r,0}, \nabla) \cong H_*(T^c_{r,0}) \otimes \mathbb{C}[q,q^{-1}]$.

### 7 Prediction for $\mathbb{CP}^1 \times \mathbb{CP}^1$

In this section we apply the same techniques of sections 3, 4 to predict the existence of an exotic monotone torus in $\mathbb{CP}^1 \times \mathbb{CP}^1$ bounding 9 families of Maslov index 2 holomorphic discs, hence not symplectomorphic to the Clifford or Chekanov ones.
We use coordinates $((x : w), (y : z))$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$. We smooth two corners of the divisor $\{xwyz = 0\}$. First considering the fibration given by $f_0((x : w), (y : z)) = \frac{x}{w}$ and $D_2 = \{\xi = 0\}$, for $\xi = \frac{yz - wz}{2t}$ and a positive real number, we smooth the corner of $\{xy = 0\}$ to get a new divisor $D_2 \cup \{wz = 0\}$. Then using $f = z\xi wy^2$, we smooth the corner of $D_2 \cup \{z = 0\}$ and get a new anticanonical divisor $D = f^{-1}(c) \cup \{w = 0\}$, for $c$ a positive real number. Then we consider a similar singular Lagrangian torus fibration on the complement of $D$. In particular we define the Chekanov type torus:

**Definition 7.1.** Given $c > r > 0$ and $\lambda \in \mathbb{R}$,

$$T_{r,\lambda}^c = \left\{ ((x : w), (y : z)) ; \left| \frac{z\xi}{wy^2} - c \right| = r; \left| \frac{z}{y} \right|^2 - \left| \frac{\xi}{wy} \right|^2 = \lambda \right\},$$

(7.1)

which is Lagrangian for a symplectic form similar to (4.6).

**Remark 7.2.** The role of the parameter $t$ in the definition of $\xi$ is less obvious as in the case of $\mathbb{CP}^2$, since here it amounts to a single rescaling. However as in the case of $\mathbb{CP}^2$, its presence is motivated by considerations about degenerations. More precisely the choice of $\xi = \frac{yz - wz}{2t}$ is based on a degeneration of $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}(1, 1, 2)$, which can be embedded inside $\mathbb{CP}^3$. See Proposition 3.1 in [2].

We then proceed as in section 3 to predict the number of families of Maslov index 2 holomorphic discs this torus should bound, at least for some values of $t$, $c$ and $r$.

It is known that the Clifford torus bounds four families of Maslov index 2 holomorphic discs in the classes $\beta_1, \beta_2, H_1 - \beta_1, H_2 - \beta_2$, where $\beta_1 = [D \times \{1\}]$ and $\beta_2 = \{(1) \times D\}$ seen in the coordinate chart $y = 1$, $w = 1$ and $H_1 = [\mathbb{CP}^1] \times \{pt\}$ and $H_2 = \{pt\} \times [\mathbb{CP}^1]$. On the almost toric fibration illustrated on Figure 13, it is located in the top chamber and has superpotential given by

$$W_{Clif} = z_1 + z_2 + \frac{e^{-A}}{z_1} + \frac{e^{-B}}{z_2},$$

(7.2)

where $z_1, z_2$ are the coordinates associated with $\beta_1, \beta_2$, $A = \int_{[\mathbb{CP}^1] \times \{pt\}} \omega$, $B = \int_{\{pt\} \times [\mathbb{CP}^1]} \omega$. (For a monotone symplectic form $A = B$.)

The first wall-crossing towards the Chekanov type tori gives rise to the change of coordinates $z_1 = v_1(1 + \tilde{w})$, $z_2 = v_2(1 + \tilde{w})^{-1}$, where $\tilde{w} = e^{-A}/z_1 z_2 = e^{-A}/v_1 v_2$. Hence the superpotential becomes
Figure 13: A $Che^2$ type torus in $\mathbb{CP}^1 \times \mathbb{CP}^1$ bounding 9 families of Maslov index 2 holomorphic discs. The superpotential is given by $W_{Che^2} = u_2 + e^{-A} + e^{-B} + e^{-A}u_1 + e^{-B}u_1 + 3 e^{-A-B}u_1 u_2 + 3 e^{-A-B}u_1 + e^{-A-B}u_1^2$. 

$W_{Che} = v_2 + v_1 (1 + \tilde{w}) + e^{-B}(1 + \tilde{w}) \frac{v_2}{v_2} = v_1 + v_2 + \frac{e^{-A}}{v_2} + \frac{e^{-B}}{v_2} + \frac{e^{-A-B}}{v_1 v_2^2}$. (7.3)

Crossing now the other wall towards the $Che^2$ type tori, we get the change of coordinates $u_1 = v_1(1 + w)$, $u_2 = v_2(1 + w)$, $w = v_1/v_2 = u_1/u_2$. The superpotential is then given by
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\[ W_{\text{Che}^2} = u_2 + (e^{-A} + e^{-B}) \frac{1 + w}{u_2} + e^{-A-B}(1 + w)^3 \]

\[ = u_2 + \frac{e^{-A}}{u_2} + \frac{e^{-B}}{u_2} + \frac{e^{-A}u_1}{u_2} + \frac{e^{-B}u_1}{u_2} + \frac{e^{-A-B}u_2}{u_1u_2} + 3 \frac{e^{-A-B}u_1}{u_2} + 3 \frac{e^{-A-B}u_1}{u_2} + \frac{e^{-A-B}u_1}{u_2}. \] (7.4)

7.1 The homology classes

We consider the torus $T_{r,0}^c$. Using notation similar to that in section 5, let’s call $\beta$ the class of the Maslov index 2 holomorphic disc lying on the conic $z = \xi$ that projects into the region $|f - c| \leq r$, and $\alpha$ the Lefschetz thimble associated to the critical point of $f$ at the origin lying above the segment $[0, c - r]$ (oriented to intersect positively with $\{z = 0\}$).

As before we use positivity of intersection with some complex curves to restrict the homology classes.

Lemma 7.3. For fixed $c$ and $r$, for $t$ sufficiently small, the intersection number of the classes $\alpha$, $\beta$, $H_1$ and $H_2$ with the varieties $\{x = 0\}$, $\{y = 0\}$, $\{w = 0\}$, $\{z = 0\}$, $D_3 = f^{-1}(c) \cup \{(0 : 1), (1 : 0)\}$, $D_5 = \{z = 0\}$, $D_6 = D_3 \cup D_3$ (all of them disjoint from $T_{r,0}^c$) and their Maslov indeces $\mu$, are as giving in the table below:

| Class | $x = 0$ | $y = 0$ | $w = 0$ | $z = 0$ | $D_2$ | $D_3$ | $D_3'$ | $D_6$ | $\mu$ |
|-------|---------|---------|---------|---------|-------|-------|-------|-------|-------|
| $\alpha$ | 1       | 0       | 0       | 1       | -1    | 0     | 0     | 0     | 0     |
| $\beta$  | 0       | 0       | 0       | 0       | 0     | 1     | 1     | 2     | 2     |
| $H_1$    | 1       | 0       | 0       | 1       | 0     | 1     | 1     | 2     | 3     |
| $H_2$    | 0       | 1       | 0       | 1       | 1     | 2     | 1     | 4     | 4     |

Proof. The intersection numbers of $H_1$ and $H_2$ with the given complex curves are computed using Bezout’s theorem, and the Maslov index is twice the intersection number with the anticanonical divisor $D = D_3 \cup \{w = 0\}$.

By construction, the intersection of $\alpha$ with $\{z = 0\}$ is one, with $D_2$ is negative one and with $D_3$, $\{y = 0\}$ and $\{w = 0\}$ it is clearly zero, as well as the intersection of $\beta$ and $T_{r,0}^c$ with $\{ywz = 0\}$ and $D_2$. Also clear is the intersection of $\beta$ with $D_3$.

To understand the intersection of the torus $T_{r,0}^c$, $\alpha$ and $\beta$ with $\{x = 0\}$ and $D_3'$, we look at the family of conics $C = \{z = e^{i\theta}\xi; \theta \in [0, 2\pi]\}$ containing
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$T_{r,0}^c$, the thimble representing the class $\alpha$ and holomorphic discs representing
the class $\beta$, similar to the ones in Proposition (5.1) (for instance one where
$z = \xi$, and $\text{Re}(z) > 0$). We use the coordinate chart $y = 1, w = 1$.

For $\{x = 0\} \cap \mathcal{C}$, we have $z = e^{i\theta} \xi = -e^{i\theta} z/2t$, so the intersection is only
at $z = 0$, for $t$ small enough, and with same sign as for $\{z = 0\}$, since $x$ is a
multiple of $z$ along $\mathcal{C}$.

For $D'_3 \cap \mathcal{C}$, we note that $x = z(2te^{-i\theta} + 1)$ along $\mathcal{C}$. Considering
$f = z\xi$, we have

\[ 0 = x\xi - c = z(2te^{-i\theta} + 1)\xi - c = f(2te^{-i\theta} + 1) - c \]

So, $f = \frac{c}{2te^{-i\theta} + 1}$, so we see that for $t$ very small, $D'_3 \cap \mathcal{C}$ intersects in a
circle projecting via $f$ inside the region $|f - c| < r$, therefore $D'_3$ intersects
$T_{r,0}^c$, $\alpha$ and $\beta$ respectively at 0, 0, and 1 point (counting positively as
$D'_3$ and our representative of the $\beta$ class are complex curves).

**Remark 7.4.** $D_6$ was found by considering the degeneration of $\mathbb{CP}^1 \times \mathbb{CP}^1$
to $\mathbb{CP}(1,1,2)$ in a similar manner as in section 5.1 $D_5$ was found using the
degeneration of $\mathbb{CP}^2$ to $\mathbb{CP}(1,1,4)$. Here it turns out that $D_6 = D_3 \cup D'_3$.

**Lemma 7.5.** The only classes in $\pi_2(\mathbb{CP}^1 \times \mathbb{CP}^1, T_{r,0})$ which may contain
holomorphic discs of Maslov index 2 are $\beta, H_1 - \beta, H_2 - \beta, H_1 - \beta + \alpha, \nH_2 - \beta + \alpha$ and $H_1 + H_2 - 3\beta + k\alpha$, $-1 \leq k \leq 2$.

**Proof.** Maslov index 2 classes must be of the form $\beta + k\alpha + m(H_1 - 2\beta) + \nn(H_2 - 2\beta)$. Considering positivity of intersections with complex curves the
proof follows from the inequalities for $k, m$ and $n$ given by the table:

| Curve | $x = 0$ | $y = 0$ | $w = 0$ | $z = 0$ | $D_2$ | $D_3$ | $D'_3$ |
|-------|--------|--------|--------|--------|------|------|------|
| Inequality | $-m \leq k$ | $0 \leq n$ | $0 \leq m$ | $-n \leq k$ | $k \leq m + n$ | $m \leq 1$ | $n \leq 1$ |

**7.2 The monotone torus**

In order to make $T_{r,0}^c$ a monotone Lagrangian torus, we deform our sym-
plectic form using Kähler inflation in neighborhoods of complex curves that
don’t intersect $T_{r,0}^c$ in a similar way as we did in section 6. First one can inflate along $\{y = 0\}$ or $\{w = 0\}$ to get a monotone Kähler form $\tilde{\omega}$ for
$\mathbb{CP}^1 \times \mathbb{CP}^1$, i.e., $\int_{H_1} \tilde{\omega} = \int_{H_2} \tilde{\omega}$, for which $\int_\alpha \tilde{\omega} = 0$. In order for $T_{r,0}^c$ to be monotone, we need a Kähler form $\tilde{\omega}$, satisfying the same conditions as
$\tilde{\omega}$ plus $\int_{H_2} \tilde{\omega} = 2\int_\beta \tilde{\omega}$. Noting that the intersection numbers of $D_6$ with $\alpha,$
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$\beta$, $H_1$, $H_2$ are $0, 2, 3$ and $3$, respectively, we can get $\hat{\omega}$ by adding a specific multiple of a 2-form supported on a neighborhood of $D_6$ to $\tilde{\omega}$ as in Proposition 6.1 so as to satisfy $\int_{\alpha} \hat{\omega} = 0$ and $\int_{H_1} \hat{\omega} = \int_{H_2} \hat{\omega} = 2 \int_{\beta} \hat{\omega}$. The last equality can be achieved because the ratio between the intersection numbers $[D_6] \cdot H_1 = [D_6] \cdot H_2$ and $[D_6] \cdot \beta$ is $2/3$ which is greater than $1/2$.

Therefore one only need to compute the expected Maslov index 2 holomorphic discs in the classes $\beta, H_1 - \beta, H_2 - \beta, H_1 - \beta + \alpha, H_2 - \beta + \alpha$ and $H_1 + H_2 - 3\beta + k\alpha, -1 \leq k \leq 2$ to prove:

**Conjecture 7.6.** There is a monotone $\text{Che}^2$ torus, of the form $T_{r,0}^{c,r}$, in $\mathbb{CP}^1 \times \mathbb{CP}^1$, bounding 9 families of Maslov index 2 holomorphic discs, that is not symplectomorphic to the monotone Chekanov torus nor to the monotone Clifford torus.

**References**

[1] D. Auroux. *Mirror symmetry and T-Duality on the complement of an anticanonical divisor*, J. Gkova Geom. Topol. GGT 1 (2007), 51-91.

[2] D. Auroux. *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, 1-47, Surv. Differ. Geom., 13, Int. Press, Somerville, MA, 2009.

[3] P. Biran, O. Cornea. *Quantum structures for Lagrangian submanifolds*, arXiv:math/0708.4221v1 [math.SG], 2007.

[4] Y. Chekanov. *Lagrangian tori in a symplectic vector space and global symplectomorphisms*, Math. Z. 223 (1996), 547-559.

[5] Y. Chekanov, F. Schlenk. *Notes on monotone Lagrangian twist tori*, Electron. Res. Announc. Math. Sci. 17 (2010), 104-121.

[6] C. Cho. *Holomorphic discs, spin structures and Floer cohomology of the Clifford torus*, Int. Math. Res. Not. 2004, no. 35, 1803-1843.

[7] C. Cho. Y.-G. Oh *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. 10 (2006), no. 4, 773-814.

[8] Y. Eliashberg, L. Polterovich. *The problem of Lagrangian knots in four-manifolds*, Geometric Topology (Athens, 1993), AMS/IP Stud. Adv. Math., Amer. Math. soc., 1997, pp. 313-327.
[9] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono. Lagrangian intersection Floer theory: anomaly and obstruction, Part I and II, AMS/IP Studies in Advanced Mathematics, 46.1. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009. xii+396 pp. ISBN: 978-0-8218-4836-4.

[10] Galkin S., Usnich A., Mutations of potentials, Preprint IPMU 10-0100, 2010.

[11] M. Gross, B. Siebert. An invitation to toric degenerations, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, 43-78, Surv. Differ. Geom., 16, Int. Press, Somerville, MA, 2011.

[12] P. Hacking, Yu. Prokhorov. Smoothable del Pezzo surfaces with quotient singularities, Compos. Math. 146 (2010), no. 1, 169-192.

[13] M. Kontsevich, Y. Soibelman. Affine structures and non-archimedean analytic spaces, The unity of mathematics (P. Etingof, V. Retakh, I.M. Singer, eds.), 321-385, Progr. Math. 244, Birkhäuser 2006.

[14] Y.-G. Oh. Relative Floer and quantum cohomology and symplectic topology of Lagrangian submanifolds, C.B. Thomas, editor, Contact and symplectic geometry, volume 8 of Publications of Newton institute, pages 201-267. Cambridge Univ. Press, Cambridge, 1996.

[15] M. Symington, Four dimensions from two in symplectic topology, in Proceedings of the 2001 Georgia International Topology Conference, Proceedings of Symposia in Pure Mathematics, 2003, 153-208.

[16] N. A. Tyurin Exotic Chekanov tori in toric symplectic varieties, IOP Publishing, Journal of Physics: Conference Series 411 (2013) 012028