Boundary Stabilization and Observation of an Unstable Heat Equation in a General Multi-dimensional Domain *

Hongyinping Feng† Pei-Hua Lang and Jiankang Liu
School of Mathematical Sciences
Shanxi University, Taiyuan, Shanxi, 030006, P.R. China

Abstract

In this paper, we consider the exponential stabilization and observation of an unstable heat equation in a general multi-dimensional domain by combining the finite-dimensional spectral truncation technique and the recently developed dynamics compensation approach. In contrast to the unstable one-dimensional partial differential equation (PDE), such as the transport equation, wave equation and the heat equation, that can be treated by the well-known PDE backstepping method, stabilization of unstable PDE in a general multi-dimensional domain is still a challenging problem. We treat the stabilization and observation problems separately. A dynamical state feedback law is proposed firstly to stabilize the unstable heat equation exponentially and then a state observer is designed via a boundary measurement. Both the stability of the closed-loop system and the well-posedness of the observer are proved. Some of the theoretical results are validated by the numerical simulations.

Keywords: Dynamics feedback, Multi-dimensional heat equation, Observer, Stabilization, Unstable system.

1 Introduction

Since the backstepping approach was first introduced into the systems described by the partial differential equations (PDEs) in [12], [17], and [18], the landscape of one-dimensional PDEs control has completely changed. This can be seen from its success in stabilizing the unstable [19, 20] or

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†Corresponding author. Email: fhyp@sxu.edu.cn.
even anti-stable wave systems [10], which are almost formidable by other approaches. However, the backstepping approach seems only effective for one-dimensional PDEs and there is a formidable obstacle to applying this approach to multi-dimensional PDEs except for some domains of specific geometry.

Very recently, a new approach, called dynamics compensation approach, has been developed to cope with the actuator and sensor dynamics compensations in the abstract framework ([3], [4]). Interestingly, this approach can also be used to the control of unstable PDEs. As an example, the stabilization and observation of an unstable one-dimensional heat equation have been considered in [3] and [4], respectively. In addition to the existing backstepping method, the dynamics compensation approach gives a new and completely different way to cope with the unstable PDEs. In this paper, we will apply this new approach to the multi-dimensional unstable heat equation in a general domain, which is commonly recognized as a longstanding puzzled problem.

Suppose that $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with $C^2$-boundary $\Gamma$, $\Gamma_1$ is a non-empty connected open set in $\Gamma$, $\Gamma_0 = \Gamma \setminus \Gamma_1$ and $\Gamma_0 \neq \emptyset$. Let $\nu$ be the unit outward normal vector of $\Gamma_1$ and let $\Delta$ be the usual Laplacian which is defined by

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}, \quad \forall \, f \in H^2(\Omega). \quad (1.1)$$

We consider the following heat equation:

$$\begin{cases}
\quad w_t(x,t) = \Delta w(x,t) + \mu w(x,t), \quad x \in \Omega, \ t > 0, \\
\quad w(x,t) = 0, \quad x \in \Gamma_0, \ \frac{\partial w(x,t)}{\partial \nu} = u(x,t), \quad x \in \Gamma_1, \ t \geq 0, \\
\quad y(x,t) = w(x,t), \quad x \in \Gamma_1, \ t \geq 0,
\end{cases} \quad (1.2)$$

where $w(\cdot,t)$ is the state, $\mu > 0$, $u$ is the control and $y$ is the output. System (1.2) is a general heat equation with interior convection. In physics and engineering contexts, it describes the flow of heat in a homogeneous and isotropic medium, with $w(x,t)$ being the temperature at the point $x$ and time $t$. The more detailed physical interpretation of the heat equation can be found in [6].

By a simple computation, we can see that there are some eigenvalues of the open-loop system (1.2) (with $u(\cdot,t) \equiv 0$) in the right-half plane provided $\mu$ is sufficiently large. This shows that the open-loop system (1.2) is unstable for large $\mu$. The lower-order term $\mu w(\cdot,t)$ of (1.2) is usually referred to as source term or unstable term. Heat equations with unstable term or source term have been extensively studied by the method of PDE backstepping. Examples can be found in [1], [13], [7], [8], and [19], to name just a few. The PDE backstepping method is powerful and is still valid to other one-dimensional distributed parameter systems such as the wave equation [10], Schrödinger equation [5], the first order hyperbolic equation [9] as well as some special Euler-Bernoulli beam [21]. However, the application of backstepping method seems to stop in front of unstable PDEs in the general multi-dimensional domain. There still exist formidable obstacles to applying this approach to general multi-dimensional PDEs.
In this paper, we combine the newly developed dynamics compensation approach [3, 4] and the finite-dimensional spectral truncation technique [2, 16] to cope with the unstable system (1.2). The control objective is to stabilize the system exponentially by virtue of the measurement output. Owing to the separation principle of the linear systems, the output feedback will be available once we address the following two problems: (i), stabilize system (1.2) by a full state feedback; (ii), design a state observer in terms of the measurement output. We will consider these two problems separately.

We consider system (1.2) in the state space $L^2(\Omega)$. Let

$$Af = \Delta f, \quad \forall f \in D(A) = \left\{ f \mid f \in H^2(\Omega) \cap H_{10}^1(\Omega), \frac{\partial f}{\partial \nu}\big|_{\Gamma_1} = 0 \right\},$$

(1.3)

where $H_{10}^1(\Omega) = \{ f \in H^1(\Omega) \mid f = 0 \text{ on } \Gamma_0 \}$. Then $A$ generates an exponentially stable analytic semigroup on $L^2(\Omega)$. It is well known (e.g. [11, p.668]) that $D((-A)^{1/2}) = H_{10}^1(\Omega)$ and $(-A)^{1/2}$ is a canonical isomorphism from $H_{10}^1(\Omega)$ onto $L^2(\Omega)$. Moreover, the following Gelfand triple inclusions are valid:

$$H_{10}^1(\Omega) = D((-A)^{1/2}) \hookrightarrow L^2(\Omega) = [L^2(\Omega)]' \hookrightarrow [D((-A)^{1/2})]' = H_{10}^{-1}(\Omega),$$

(1.4)

where $H_{10}^{-1}(\Omega)$ is the dual space of $H_{10}^1(\Omega)$ with the pivot space $L^2(\Omega)$. An extension $\tilde{A} \in \mathcal{L}(H_{10}^1(\Omega), H_{10}^{-1}(\Omega))$ of $A$ is defined by

$$\langle \tilde{A}x, z \rangle_{H_{10}^{-1}(\Omega), H_{10}^1(\Omega)} = -\langle (-A)^{1/2}x, (-A)^{1/2}z \rangle_{L^2(\Omega)}, \quad \forall x, z \in H_{10}^1(\Omega).$$

(1.5)

Since $A$ is strictly negative, self-adjoint in $L^2(\Omega)$, and is the inverse of a compact operator, the operator $A$ has the infinite sequence of negative eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and a corresponding sequence of eigenfunctions $\{\phi_j(\cdot)\}_{j=1}^{\infty}$ that forms an orthonormal basis for $L^2(\Omega)$. Without loss of generality, we always assume that

**Assumption 1.1.** Let the operator $A$ be given by (1.3) and $\mu > 0$. Suppose that the eigenpairs $\{(\phi_j(\cdot), \lambda_j)\}_{j=1}^{\infty}$ of $A$ satisfy

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_k > \cdots \rightarrow -\infty,$$

(1.6)

and

$$\begin{cases}
\Delta \phi_k = \lambda_k \phi_k, & \|\phi_k\|_{L^2(\Omega)} = 1, \\
\phi_k(x) = 0, & x \in \Gamma_0, \\
\frac{\partial \phi_k(x)}{\partial \nu} = 0, & x \in \Gamma_1,
\end{cases} \quad k = 1, 2, \cdots. \quad (1.7)
$$

Suppose that $N$ is an integer that satisfies

$$\lambda_k + \mu < 0, \quad \forall k > N.$$  

(1.8)

Define the Neumann map $\Upsilon \in \mathcal{L}(L^2(\Gamma_1), H^{3/2}(\Omega))$ ([11, p. 668]) by $\Upsilon u = \psi$ if and only if

$$\begin{cases}
\Delta \psi = 0 \text{ in } \Omega, \\
\psi|_{\Gamma_0} = 0, & \frac{\partial \psi}{\partial \nu}|_{\Gamma_1} = u.
\end{cases} \quad (1.9)$$
Using the Neumann map, one can write (1.2) in $H^{-1}_{\Gamma_0}(\Omega)$ as

$$\dot{w}(\cdot, t) = \Delta w(\cdot, t) - \Delta \psi + \mu w(\cdot, t) = \Delta(w(\cdot, t) - \psi) + \mu w(\cdot, t)$$

where $B = \Delta \psi + \mu w(\cdot, t) = (\tilde{A} + \mu)w(\cdot, t) - \tilde{A} \Upsilon u(\cdot, t)$. That is

$$\dot{w}(\cdot, t) = (\tilde{A} + \mu)w(\cdot, t) + Bu(\cdot, t) \quad \text{in} \quad H^{-1}_{\Gamma_0}(\Omega),$$

where $B \in \mathcal{L}(L^2(\Gamma_1), H^{-1}_{\Gamma_0}(\Omega))$ is given by

$$Bu = -\tilde{A} \Upsilon u, \quad \forall \ u \in L^2(\Gamma_1).$$

Define $B^* \in \mathcal{L}(H^1_{\Gamma_0}(\Omega), L^2(\Gamma_1))$ by

$$\langle B^* f, u \rangle_{L^2(\Gamma_1)} = \langle f, Bu \rangle_{H^1_{\Gamma_0}(\Omega), H^{-1}_{\Gamma_0}(\Omega)}, \quad \forall \ f \in H^1_{\Gamma_0}(\Omega), \ u \in L^2(\Gamma_1).$$

Then, for any $f \in D(A) = D(A^*)$ and $u \in L^2(\Gamma_1)$, it follows from (1.5), (1.3), (1.13) and (1.9) that

$$\langle B^* f, u \rangle_{L^2(\Gamma_1)} = \langle f, -\tilde{A} \Upsilon u \rangle_{H^1_{\Gamma_0}(\Omega), H^{-1}_{\Gamma_0}(\Omega)} = \langle A^* f, -\Upsilon u \rangle_{L^2(\Omega)}$$

$$= \langle \Delta f, -\psi \rangle_{L^2(\Omega)} = \langle \nabla f, \nabla \psi \rangle_{L^2(\Omega)} = \int_{\Gamma_1} f(x)u(x)dx,$$

which, together with the denseness of $D(A)$ in $H^1_{\Gamma_0}(\Omega)$, implies that

$$B^* f = f|_{\Gamma_1}, \quad \forall \ f \in H^1_{\Gamma_0}(\Omega).$$

Using the operators $A$, $B$ and $B^*$, the control plant (1.2) can be written abstractly

$$\left\{ \begin{array}{l}
\dot{w}(\cdot, t) = (\tilde{A} + \mu)w(\cdot, t) + Bu(\cdot, t), \quad t > 0, \\
y(\cdot, t) = B^* w(\cdot, t), \quad t \geq 0.
\end{array} \right. \quad (1.16)$$

The rest of the paper is organized as follows: In Section 2, we give a spectral truncation stabilizer that will be used in the full state feedback design in Section 3. The exponential stability of the closed-loop system is also proved in Section 3. Section 4 gives some preliminary results about the observer design. Section 5 is devoted to the observer design and its well-posedness proof. Section 6 presents some numerical simulations, followed up conclusions in Section 7. For the sake of readability, some results that are less relevant to the feedback or observer design are arranged in the Appendix.

Throughout the paper, the identity matrix on the space $\mathbb{R}^n$ will be denoted by $I_n$. The space of bounded linear operators from $X_1$ to $X_2$ is denoted by $\mathcal{L}(X_1, X_2)$. The space of bounded linear operators from $X$ to itself is denoted by $\mathcal{L}(X)$. The spectrum, resolvent set and the domain of the operator $A$ are denoted by $\sigma(A)$, $\rho(A)$, and $D(A)$, respectively.
2 A spectral truncation stabilizer

This section is devoted to the preliminaries on the state feedback design. Suppose that \( p \in L^2(\Gamma_1) \) such that
\[
\int_{\Gamma_1} p(x) \phi_j(x) \, dx \neq 0, \quad j = 1, 2, \ldots, N, \tag{2.1}
\]
where \( \phi_j \) is given by (1.7) and \( N \) is an integer that satisfies (1.8). The existence of such a function \( p \) is trivial and is given by Lemma 8.1 in Appendix. In terms of the function \( p \), we can define the operator \( P_p : \mathbb{R} \to L^2(\Omega) \) by
\[
P_p \theta = \zeta_p, \quad \forall \, \theta \in \mathbb{R}, \tag{2.2}
\]
where \( \zeta_p \) is the solution of the following system:
\[
\begin{cases}
\Delta \zeta_p = \theta \zeta_p \quad \text{in} \quad \Omega, \\
\zeta_p(x) = 0, \ x \in \Gamma_0, \ \frac{\partial \zeta_p(x)}{\partial \nu} = p(x), \ x \in \Gamma_1.
\end{cases} \tag{2.3}
\]

**Lemma 2.1.** In addition to Assumption 1.1, suppose that \( p \in L^2(\Gamma_1) \) satisfies (2.1) and suppose that \( \theta \in \mathbb{R} \) satisfies
\[
\theta \neq \lambda_j, \quad j = 1, 2, \ldots, N. \tag{2.4}
\]
Then, the operator \( P_p \) defined by (2.2) satisfies
\[
\langle P_p \theta, \phi_j \rangle_{L^2(\Omega)} \neq 0, \quad j = 1, 2, \ldots, N. \tag{2.5}
\]

**Proof.** It follows from (1.7), (2.2) and (2.3) that
\[
\theta \int_{\Omega} \zeta_p(x) \phi_j(x) \, dx = \int_{\Omega} \Delta \zeta_p(x) \phi_j(x) \, dx \\
= \int_{\Gamma_1} p(x) \phi_j(x) \, dx - \int_{\Omega} \nabla \zeta_p(x) \nabla \phi_j(x) \, dx \\
= \int_{\Gamma_1} p(x) \phi_j(x) \, dx + \lambda_j \int_{\Omega} \zeta_p(x) \phi_j(x) \, dx,
\]
which yields
\[
\int_{\Omega} \zeta_p(x) \phi_j(x) \, dx = \frac{1}{\theta - \lambda_j} \int_{\Gamma_1} p(x) \phi_j(x) \, dx \neq 0, \quad j = 1, 2, \ldots, N. \tag{2.7}
\]

The proof is complete due to (2.1). \( \square \)

For any \( \theta \in \mathbb{R} \), we consider the stabilization of system \((A + \mu, P_p \theta)\) that is associated with the following system:
\[
\begin{cases}
z_t(x,t) = \Delta z(x,t) + \mu z(x,t) + (P_p \theta)(x) u(t), \quad x \in \Omega, \ t > 0, \\
z(x,t) = 0, \quad x \in \Gamma_0, \ \frac{\partial z(x,t)}{\partial \nu} = 0, \quad x \in \Gamma_1.
\end{cases} \tag{2.8}
\]
where \( p \in L^2(\Gamma_1) \) satisfies (2.1) and \( u \) is a scalar control. Since \( \{\phi_j(\cdot)\}_{j=1}^\infty \) defined by Assumption 1.1 forms an orthonormal basis for \( L^2(\Omega) \), the function \( P_p\theta \) and the solution \( z(\cdot, t) \) of (2.8) can be represented respectively as

\[
P_p\theta = \sum_{k=1}^\infty f_k \phi_k, \quad f_k = \int_\Omega (P_p\theta)(x) \phi_k(x) \, dx, \quad k = 1, 2, \ldots
\]

(2.9)

and

\[
z(\cdot, t) = \sum_{k=1}^\infty z_k(t) \phi_k(\cdot), \quad z_k(t) = \int_\Omega z(x, t) \phi_k(x) \, dx, \quad k = 1, 2, \ldots
\]

(2.10)

Inspired by [2, 16] and similarly to [3], system (2.8) can be stabilized by the finite-dimensional spectral truncation technique. Actually, by a simple computation, it follows that

\[
\dot{z}_k(t) = \int_\Omega z_t(x, t) \phi_k(x) \, dx = \int_\Omega [\Delta z(x, t) + \mu z(x, t) + (P_p\theta)(x)u(t)] \phi_k(x) \, dx
\]

\[
= (\lambda_k + \mu) z_k(t) + f_k u(t).
\]

(2.11)

Since \( z_k(t) \) is stable for all \( k > N \), where \( N \) is given by (1.8), it is therefore sufficient to consider \( z_k(t) \) for \( k \leq N \), which satisfy the following finite-dimensional system:

\[
\dot{Z}_N(t) = \Lambda_N Z_N(t) + F_N u(t), \quad Z_N(t) = (z_1(t), \ldots, z_N(t))^\top,
\]

(2.12)

where \( \Lambda_N \) and \( F_N \) are defined by

\[
\begin{cases}
\Lambda_N = \text{diag}(\lambda_1 + \mu, \ldots, \lambda_N + \mu), \\
F_N = (f_1, f_2, \ldots, f_N)^\top.
\end{cases}
\]

(2.13)

In this way, the stabilization of system (2.8) amounts to stabilizing the finite-dimensional system (2.12).

**Lemma 2.2.** In addition to Assumption 1.1, suppose that \( p \in L^2(\Gamma_1) \) satisfies (2.1) and suppose that \( \theta \in \mathbb{R} \) satisfies (2.4). Then, there exists an \( L_N = (l_1, l_2, \ldots, l_N) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}) \) such that \( \Lambda_N + F_N L_N \) is Hurwitz, where \( \Lambda_N \) and \( F_N \) are defined by (2.13). Moreover, the operator \( A + \mu + (P_p\theta)K \) generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \), where \( P_p\theta \) is given by (2.2) and \( K \) is given by

\[
K : g \to \int_\Omega g(x) \left[ \sum_{k=1}^N l_k \phi_k(x) \right] \, dx, \quad \forall g \in L^2(\Omega).
\]

(2.14)

**Proof.** Owing to (2.1), it follows from Lemma 2.1 that (2.5) holds. By Lemma 8.2 in Appendix, the pair \( (\Lambda_N, F_N) \) is controllable and hence, there exists a vector \( L_N = (l_1, l_2, \ldots, l_N) \) such that \( \Lambda_N + F_N L_N \) is Hurwitz.

Since \( A + \mu \) generates an analytic semigroup \( e^{(A+\mu)t} \) on \( L^2(\Omega) \) and \( (P_p\theta)K \in \mathcal{L}(L^2(\Omega)) \), it follows from [15, Corollary 2.3, p.81] that \( A + \mu + (P_p\theta)K \) also generates an analytic semigroup on \( L^2(\Omega) \). As a result, the proof will be accomplished if we can show that \( \sigma(A+\mu+(P_p\theta)K) \subseteq \{ s \mid \text{Re}(s) < 0 \} \).
For any $\lambda \in \sigma(A + \mu + (P_p\theta)K)$, we consider the characteristic equation $(A + \mu + (P_p\theta)K)g = \lambda g$ with $g \neq 0$.

When $g \in \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\}$, there exist $g_1, g_2, \cdots, g_N \in \mathbb{R}$ such that $g = \sum_{j=1}^{N} g_j \phi_j$. The characteristic equation becomes

$$\sum_{j=1}^{N} g_j (A + \mu) \phi_j + P_p \theta \sum_{j=1}^{N} g_j K \phi_j = \sum_{j=1}^{N} \lambda g_j \phi_j. \quad (2.15)$$

Since $(A + \mu) \phi_j = (\lambda_j + \mu) \phi_j$ and

$$K \phi_j = \int_{0}^{1} \phi_j(x) \left[ \sum_{k=1}^{N} l_k \phi_k(x) \right] dx = l_j, \quad j = 1, 2, \cdots, N, \quad (2.16)$$

the equation (2.15) takes the form

$$\sum_{j=1}^{N} g_j (\lambda_j + \mu) \phi_j + P_p \theta \sum_{j=1}^{N} g_j l_j = \sum_{j=1}^{N} \lambda g_j \phi_j. \quad (2.17)$$

Take the inner product with $\phi_k$, $k = 1, 2, \cdots, N$ on equation (2.17) to obtain

$$g_k (\lambda_k + \mu) + f_k \sum_{j=1}^{N} g_j l_j = \lambda g_k, \quad k = 1, 2, \cdots, N, \quad (2.18)$$

which, together with (2.13), leads to

$$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} = 0. \quad (2.19)$$

Since $(g_1, g_2, \cdots, g_N) \neq 0$, we have

$$\text{Det} (\lambda - \Lambda_N - F_N L_N) = 0. \quad (2.20)$$

Hence, $\lambda \in \sigma(\Lambda_N + F_N L_N) \subset \{ s \mid \text{Re}(s) < 0 \}$, since $\Lambda_N + F_N L_N$ is Hurwitz.

When $g \notin \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\}$, there exists a $j_0 > N$ such that $\int_{0}^{1} g(x) \phi_{j_0}(x) dx \neq 0$. Take the inner product with $\phi_{j_0}$ on equation $(A + \mu + (P_p\theta)K)g = \lambda g$ to get

$$(\lambda_{j_0} + \mu) \int_{0}^{1} g(x) \phi_{j_0}(x) dx = \lambda \int_{0}^{1} g(x) \phi_{j_0}(x) dx, \quad (2.21)$$

which implies that $\lambda = \lambda_{j_0} + \mu < 0$. Therefore, $\lambda \in \sigma(A + \mu + (P_p\theta)K) \subset \{ s \mid \text{Re}(s) < 0 \}$. The proof is complete. \qed
3 State feedback

This section is devoted to the stabilization of system (1.2). Inspired by [3], we consider the following dynamics feedback:

\[
\begin{cases}
  u(x, t) = v(x, t), & x \in \Gamma_1, \\
  v_t(\cdot, t) = -\alpha v(\cdot, t) + B_v u_v(t) & \text{in } L^2(\Gamma_1),
\end{cases}
\]

where \( \alpha > 0 \) is a tuning parameter, \( u_v(t) \in \mathbb{R} \) is a new scalar control to be designed and the operator \( B_v \in \mathcal{L}(\mathbb{R}, L^2(\Gamma_1)) \) is given by

\[
B_v c = cp(\cdot), \quad \forall \ c \in \mathbb{R},
\]

with \( p \in L^2(\Gamma_1) \) satisfying (2.1). Under the controller (3.1), the control plant (1.16), or equivalently (1.2), turns to be

\[
\begin{cases}
  \dot{w}(\cdot, t) = (\hat{A} + \mu)w(\cdot, t) + B_v(\cdot, t) & \text{in } H^{-1}_{\Gamma_0}(\Omega), \\
  v_t(\cdot, t) = -\alpha v(\cdot, t) + B_v u_v(t) & \text{in } L^2(\Gamma_1).
\end{cases}
\]

Since (3.3) is a cascade system, the "v-part" can be regarded as the actuator dynamics of the control plant \( w \)-system. As a result, we can stabilize system (3.3) by the newly developed actuator dynamics compensation approach in [3]. To demonstrate the key idea of controller design clearly, we first consider the following finite-dimensional example.

**Example 3.1.** Consider the following system in the state space \( \mathbb{R}^n \times \mathbb{R} \):

\[
\begin{cases}
  \dot{x}_1(t) = Ax_1(t) + Bx_2(t), \\
  \dot{x}_2(t) = -\alpha x_2(t) + B_2 u(t),
\end{cases}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n, B_2 \in \mathbb{R} \) and \( u(t) \) is the control. By [3], if we choose \( S \) specially such that

\[
AS + \alpha S = B,
\]

then system (3.4) can be decoupled by the block-upper-triangular transformation:

\[
\begin{pmatrix}
  I_n & S \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  A & B \\
  0 & -\alpha
\end{pmatrix}
\begin{pmatrix}
  I_n & S \\
  0 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
  A & 0 \\
  0 & -\alpha
\end{pmatrix}.
\]

Hence, the controllability of the following pairs is equivalent:

\[
\begin{pmatrix}
  A & B \\
  0 & -\alpha
\end{pmatrix}, \quad \begin{pmatrix}
  0 \\
  B_2
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  A & 0 \\
  0 & -\alpha
\end{pmatrix}, \quad \begin{pmatrix}
  SB_2 \\
  B_2
\end{pmatrix}.
\]

Owing to the block-diagonal structure, the stabilization of the second system of (3.7) is much easier than the first one. As a consequence of this fact, the controller \( u(t) \) in (3.4) can be designed by stabilizing system \( (A, SB_2) \):

\[
u(t) = (K, 0)
\begin{pmatrix}
  I_n & S \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix}
= KSx_2(t) + Kx_1(t),
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n, B_2 \in \mathbb{R} \) and \( u(t) \) is the control. By [3], if we choose \( S \) specially such that

\[
AS + \alpha S = B,
\]

then system (3.4) can be decoupled by the block-upper-triangular transformation:

\[
\begin{pmatrix}
  I_n & S \\
  0 & 1
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\begin{pmatrix}
  A & B \\
  0 & -\alpha
\end{pmatrix}
\begin{pmatrix}
  I_n & S \\
  0 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
  A & 0 \\
  0 & -\alpha
\end{pmatrix}.
\]

Hence, the controllability of the following pairs is equivalent:

\[
\begin{pmatrix}
  A & B \\
  0 & -\alpha
\end{pmatrix}, \quad \begin{pmatrix}
  0 \\
  B_2
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  A & 0 \\
  0 & -\alpha
\end{pmatrix}, \quad \begin{pmatrix}
  SB_2 \\
  B_2
\end{pmatrix}.
\]

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\[
u(t) = (K, 0)
\begin{pmatrix}
  I_n & S \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix}
= KSx_2(t) + Kx_1(t),
\]
where \( K \in \mathbb{R}^{1 \times n} \) is chosen to make \( A + SB_2K \) Hurwitz. Under the feedback (3.8), we obtain the closed-loop of system (3.4):

\[
\begin{align*}
\dot{x}_1(t) &= Ax_1(t) + Bx_2(t), \\
\dot{x}_2(t) &= (B_2KS - \alpha)x_2(t) + B_2Kx_1(t),
\end{align*}
\] (3.9)

which is stable due to the Hurwitz matrix \( A + SB_2K \) and the similarity

\[
\begin{pmatrix}
A & B \\
B_2K & B_2KS - \alpha
\end{pmatrix} \sim \begin{pmatrix}
A + SB_2K & 0 \\
B_2K & -\alpha
\end{pmatrix}.
\] (3.10)

To sum up, the feedback of system (3.4) can be designed by the following scheme: (i), solve the equation (3.5) to get \( S \); (ii), choose \( K \) such that \( A + SB_2K \) is Hurwitz; (iii), let \( u(t) = KSx_2(t) + Kx_1(t) \).

Now, we return to the feedback design of system (3.3). Inspired by Example 3.1, the controller can be designed as

\[ u_v(t) = Kw(\cdot, t) + KSv(t), \] (3.11)

where \( S \in \mathcal{L}(L^2(\Gamma_1), L^2(\Omega)) \) solves the Sylvester equation

\[ (\tilde{A} + \mu)S + \alpha S = B, \] (3.12)

and \( K \in \mathcal{L}(L^2(\Omega), \mathbb{R}) \) stabilizes system \( (A + \mu, SB_v) \) exponentially in the sense of [22].

**Lemma 3.1.** Let \( A \) and \( B \) be given by (1.3) and (1.12), respectively. Suppose that \( B_v \in \mathcal{L}(\mathbb{R}, L^2(\Gamma_1)) \) is given by (3.2) with \( p \in L^2(\Gamma_1) \) satisfying (2.1) and suppose that

\[ \alpha + \mu \in \rho(-A). \] (3.13)

Then, the solution of Sylvester equation (3.12) satisfies

\[ Sg = -\varphi_g \in L^2(\Omega), \quad \forall \ g \in L^2(\Gamma_1), \] (3.14)

where \( \varphi_g \) is given by

\[
\begin{align*}
\Delta \varphi_g &= (-\alpha - \mu)\varphi_g \text{ in } \Omega, \\
\varphi_g(x) &= 0, \ x \in \Gamma_0, \ \frac{\partial \varphi_g(x)}{\partial \nu} = g(x), \ x \in \Gamma_1.
\end{align*}
\] (3.15)

Moreover, for any \( c \in \mathbb{R} \), we have

\[ SB_v c = -cP_p \theta \text{ with } \theta = -\alpha - \mu, \] (3.16)

where \( P_p : \mathbb{R} \to L^2(\Omega) \) is given by (2.2).
Proof. Owing to (3.13), we solve (3.12) to get
\[ S = (\alpha + \mu + \tilde{A})^{-1}B. \] (3.17)
By a straightforward computation, it follows that
\[
(\alpha + \mu + \tilde{A})\varphi_g = (\alpha + \mu + \tilde{A})\varphi_g - \tilde{A} \varphi_g + \tilde{A} \varphi_g = (\alpha + \mu + \operatorname{A}(\varphi_g - \varphi_g) + \tilde{A} \varphi_g = (\alpha + \mu + \phi_g + \Delta(\varphi_g - \varphi_g) + \tilde{A} \varphi_g = \tilde{A} \varphi_g = -Bg,
\] (3.18)
which, together with (3.17), leads to (3.14) easily.
By (3.2) and (3.14), we have
\[ SB_v = -c\Phi, \]
where
\[
\begin{cases}
\Delta \varphi = (-\alpha - \mu)\varphi & \text{in } \Omega, \\
\varphi(x) = 0, x \in \Gamma_0, \quad \frac{\partial \varphi(x)}{\partial \nu} = p(x), x \in \Gamma_1.
\end{cases}
\] (3.19)
In view of (2.3) and letting \( \theta = -\alpha - \mu \), we can obtain (3.16) easily. The proof is complete.

By Lemmas 2.2 and 3.1, the operator \(-K \in \mathcal{L}(L^2(\Omega), \mathbb{R}) \) defined by (2.14) stabilizes system \((A + \mu, SB_v)\) exponentially. As a result, the controller (3.11) turns to be
\[
u_t(t) = -\int_\Omega [w(x,t) - \varphi_v(x,t)] \left[ \sum_{k=1}^N l_k \phi_k(x) \right] dx,
\] (3.20)
where
\[
\begin{cases}
\Delta \varphi_v(x) = (-\alpha - \mu) \varphi_v(x) & \text{in } \Omega, \\
\varphi_v(x,t) = 0, x \in \Gamma_0, \quad \frac{\partial \varphi_v(x,t)}{\partial \nu} = v(x,t), x \in \Gamma_1.
\end{cases}
\] (3.21)
By (3.20) and (3.3), we obtain the closed-loop system
\[
\begin{cases}
\dot{w}(\cdot, t) = (\tilde{A} + \mu)w(\cdot, t) + Bv(\cdot, t) & \text{in } \Omega, \\
v_t(\cdot, t) = -\alpha v(\cdot, t) - B_v \int_\Omega [w(x,t) - \varphi_v(x,t)] \left[ \sum_{k=1}^N l_k \phi_k(x) \right] dx & \text{in } \Gamma_1, \\
\Delta \varphi_v(\cdot, t) = (-\alpha - \mu) \varphi_v(\cdot, t) & \text{in } \Omega, \\
\varphi_v(x,t) = 0, x \in \Gamma_0, \quad \frac{\partial \varphi_v(x,t)}{\partial \nu} = v(x,t), x \in \Gamma_1.
\end{cases}
\] (3.22)
Combining (3.2), (1.12) and (1.3), system (3.22) turns to be
\[
\begin{aligned}
w_t(x, t) &= \Delta w(x, t) + \mu w(x, t), \quad x \in \Omega, \\
w(x, t) &= 0, \quad x \in \Gamma_0, \quad \frac{\partial w(x, t)}{\partial \nu} = v(x, t), \quad x \in \Gamma_1, \\
v_t(\cdot, t) &= -\alpha v(\cdot, t) - p(\cdot) \int_{\Omega} [w(x, t) - \varphi_v(x, t)] \left[ \sum_{k=1}^{N} l_k \phi_k(x) \right] \, dx \quad \text{in} \quad \Gamma_1, \\
\Delta \varphi_v(\cdot, t) &= (-\alpha - \mu) \varphi_v(\cdot, t) \quad \text{in} \quad \Omega, \\
\varphi_v(x, t) &= 0, \quad x \in \Gamma_0, \quad \frac{\partial \varphi_v(x, t)}{\partial \nu} = v(x, t), \quad x \in \Gamma_1.
\end{aligned}
\] (3.23)

**Theorem 3.1.** In addition to Assumption 1.1, suppose that \( p \in L^2(\Gamma_1) \) satisfies (2.1) and
\[
\alpha + \mu + \lambda_j \neq 0, \quad j = 1, 2, \cdots, N.
\] (3.24)

Then, there exists an \( L_N = (l_1, l_2, \cdots, l_N) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}) \) such that \( \Lambda_N + F_N L_N \) is Hurwitz, where \( \Lambda_N \) and \( F_N \) are defined by (2.13). Moreover, for any \( (w(\cdot, 0), v(\cdot, 0))^T \in L^2(\Omega) \times L^2(\Gamma_1) \), system (3.23) admits a unique solution \( (w, v)^T \in C([0, \infty); L^2(\Omega) \times L^2(\Gamma_1)) \) that decays to zero exponentially in \( L^2(\Omega) \times L^2(\Gamma_1) \) as \( t \to \infty \). Moreover, if the initial state \( (w(\cdot, 0), v(\cdot, 0))^T \in D(A) \times L^2(\Gamma_1) \), the solution \( (w, v)^T \in C^1([0, \infty); L^2(\Omega) \times L^2(\Gamma_1)) \) is classical.

**Proof.** Notice that (2.14), the closed-loop system (3.22) can be written as the abstract form:
\[
\frac{d}{dt}(w(\cdot, t), v(\cdot, t))^T = A(w(\cdot, t), v(\cdot, t))^T,
\] (3.25)
where the operator \( A : D(A) \subset L^2(\Omega) \times L^2(\Gamma_1) \to L^2(\Omega) \times L^2(\Gamma_1) \) is defined by
\[
A = \begin{pmatrix}
A + \mu & B \\
-B_0 K & -B_0 K S - \alpha
\end{pmatrix}
\text{with} \quad D(A) = D(A) \times L^2(\Gamma_1).
\] (3.26)

As proposed in [3] and similarly to (3.6), we introduce the following transformation:
\[
S(f, g)^T = (f + S g, g)^T, \quad (f, g)^T \in L^2(\Omega) \times L^2(\Gamma_1),
\] (3.27)
where \( S \in \mathcal{L}(L^2(\Gamma_1), L^2(\Omega)) \) solves the Sylvester equation (3.12). By a simple computation, \( S \in \mathcal{L}(L^2(\Omega) \times L^2(\Gamma_1)) \) is invertible and its inverse is
\[
S^{-1}(f, g)^T = (f - S g, g)^T, \quad (f, g)^T \in L^2(\Omega) \times L^2(\Gamma_1).
\] (3.28)
Moreover, (see, e.g., [3, Theorem 5.1])
\[
S A S^{-1} = A_S, \quad D(A_S) = S D(A),
\] (3.29)
where
\[
A_S = \begin{pmatrix}
A + \mu - S B_0 K & 0 \\
B_0 K & -\alpha
\end{pmatrix}.
\] (3.30)
SB_v is given by (3.16) and K is given by (2.14). Since \( SB_v = -P_v \theta \) with \( \theta = -\alpha - \mu \), it follows from Lemma 2.2 that the operator \( A + \mu + (P_v \theta)K = A + \mu - SB_v K \) generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \). Owing to the block-triangle structure and [3, Lemma 3.2], the operator \( A_S \) generates an exponentially stable \( C_0 \)-semigroup \( e^{At} \) on \( L^2(\Omega) \times L^2(\Gamma_1) \). As a result, the operator \( A \) generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \times L^2(\Gamma_1) \) due to the similarity (3.29).

4 Preliminaries on observer design

This section is devoted to the preliminaries on the observer design. Let \( q \in L^2(\Gamma_1) \) satisfy

\[
\int_{\Gamma_1} q(x)\phi_j(x) \neq 0, \quad j = 1, 2, \ldots, N, \tag{4.1}
\]

where \( \phi_j \) is given by (1.7) and \( N \) is an integer that satisfies (1.8). For any \( \gamma \in \mathbb{R} \), define the operator \( J_q : L^2(\Omega) \to \mathbb{R} \) by

\[
J_q(g) = -\int_{\Gamma_1} q(x)\xi_g(x)dx, \quad \forall \ g \in L^2(\Omega), \tag{4.2}
\]

where \( \xi_g \) is given by

\[
\begin{cases}
\Delta \xi_g = \gamma \xi_g + g \quad \text{in} \quad \Omega, \\
\xi_g(x) = 0, \ x \in \Gamma_0, \quad \frac{\partial \xi_g(x)}{\partial \nu} = 0, \ x \in \Gamma_1.
\end{cases} \tag{4.3}
\]

Lemma 4.1. Let \( \{(\phi_j, \lambda_j)\}_{j=1}^{\infty} \) be given by (1.7) and \( N \) be an integer that satisfies (1.8). Suppose that \( q \in L^2(\Gamma_1) \) satisfies (4.1) and suppose that \( \gamma \in \mathbb{R} \) satisfies

\[
\gamma \neq \lambda_j, \quad j = 1, 2, \ldots, N. \tag{4.4}
\]

Then, the operator \( J_q \) defined by (4.2) satisfies

\[
J_q(\phi_j) \neq 0, \quad j = 1, 2, \ldots, N. \tag{4.5}
\]

Proof. Let \( \eta_q \) be a solution of the following system

\[
\begin{cases}
\Delta \eta_q = \gamma \eta_q \quad \text{in} \quad \Omega, \\
\eta_q(x) = 0, \ x \in \Gamma_0, \quad \frac{\partial \eta_q(x)}{\partial \nu} = q(x), \ x \in \Gamma_1.
\end{cases} \tag{4.6}
\]

Then, for any \( g \in L^2(\Omega) \), it follows from (4.3) and (4.6) that

\[
\gamma \langle \eta_q, \xi_g \rangle_{L^2(\Omega)} = \langle \Delta \eta_q, \xi_g \rangle_{L^2(\Omega)} = \int_{\Gamma_1} \frac{\partial \eta_q(x)}{\partial \nu} \xi_g(x)dx - \langle \nabla \eta_q, \nabla \xi_g \rangle_{L^2(\Omega)} = \int_{\Gamma_1} q(x)\xi_g(x)dx + \gamma \langle \eta_q, \xi_g \rangle_{L^2(\Omega)} + \langle g, \eta_q \rangle_{L^2(\Omega)}, \tag{4.7}
\]

which yields

\[
J_q(\phi_j) = \langle \phi_j, \eta_q \rangle_{L^2(\Omega)}, \quad j = 1, 2, \ldots, N. \tag{4.8}
\]
On the other hand,

\[
\lambda_j \langle \phi_j, \eta \rangle_{L^2(\Omega)} = \langle \Delta \phi_j, \eta \rangle_{L^2(\Omega)} = -\langle \nabla \phi_j, \nabla \eta \rangle_{L^2(\Omega)} \\
= - \int_{\Gamma_1} \frac{\partial \eta_j(x)}{\partial \nu} \phi_j(x) dx + \langle \Delta \eta_j, \phi_j \rangle_{L^2(\Omega)} , \ j = 1, 2, \cdots, N. \tag{4.9}
\]

That is

\[
\int_{\Gamma_1} q(x) \phi_j(x) dx = (\gamma - \lambda_j) \langle \eta_j, \phi_j \rangle_{L^2(\Omega)}, \ j = 1, 2, \cdots, N. \tag{4.10}
\]

Combining (4.8), (4.10) and (4.1), we obtain (4.5) easily.

Next, we will find $K$ to detect system $(A + \mu, J_q^\gamma)$ exponentially in the sense of [22]. Define the row vector

\[
J_N = (J_1^q(\phi_1), J_2^q(\phi_2), \cdots, J_N^q(\phi_N)), \tag{4.11}
\]

where $\phi_i$ is given by (1.7), $i = 1, 2, \cdots, N$ and $N$ is an integer that satisfies (1.8). By Lemma 4.1 and Lemma 8.2 in Appendix, the finite-dimensional system $(\Lambda_N, J_N)$ is observable, where $\Lambda_N$ is given by (2.13). As a result, there exists a vector $K_N = (k_1, k_2, \cdots, k_N)^T$ such that $\Lambda_N + K_N J_N$ is Hurwitz.

**Lemma 4.2.** Suppose that the operator $A$ is given by (1.3), the eigenpairs $\{(\phi_j(\cdot), \lambda_j)\}_{j=1}^{\infty}$ satisfy (1.7), $q(\cdot) \in L^2(\Gamma_1)$ satisfies (4.1), the integer $N$ satisfies (1.8) and $\mu > 0$. For any $\gamma \in \mathbb{R}$ satisfying (4.4), let $J_q^\gamma : L^2(\Omega) \to \mathbb{R}$ be given by (4.2) and $J_N$ be given by (4.11). Then, there exists a vector $K_N = (k_1, k_2, \cdots, k_N)^T$ such that $\Lambda_N + K_N J_N$ is Hurwitz, where $\Lambda_N$ is given by (2.13). Moreover, the operator $A + \mu + K J_q^\gamma$ generates an exponentially stable $C_0$-semigroup on $L^2(\Omega)$, where the operator $K : \mathbb{R} \to L^2(\Omega)$ is given by

\[
K c = c \sum_{j=1}^{N} k_j \phi_j(\cdot), \ \forall \ c \in \mathbb{R}. \tag{4.12}
\]

**Proof.** Owing to (4.1), it follows from Lemma 4.1 that (4.5) holds. By Lemma 8.2 in Appendix, the pair $(\Lambda_N, J_N)$ is observable and there exists a vector $K_N = (k_1, k_2, \cdots, k_N)^T$ such that $\Lambda_N + K_N J_N$ is Hurwitz.

Since $A + \mu$ generates an analytic semigroup $e^{(A + \mu)t}$ on $L^2(\Omega)$ and $K J_q^\gamma$ is bounded, it follows from [15, Corollary 2.3, p.81] that $A + \mu + K J_q^\gamma$ also generates an analytic semigroup on $L^2(\Omega)$. The proof will be accomplished if we can show that $\sigma(A + \mu + K J_q^\gamma) \subset \{s \mid \text{Re}(s) < 0\}$. For any $\lambda \in \sigma(A + \mu + K J_q^\gamma)$, we consider the characteristic equation $(A + \mu + K J_q^\gamma)g = \lambda g$ with $g \neq 0$.

When $g \in \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\}$, set $g = \sum_{j=1}^{N} g_j \phi_j$. The characteristic equation becomes

\[
\sum_{j=1}^{N} (\lambda_j + \mu) g_j \phi_j + \sum_{j=1}^{N} g_j J_q^\gamma(\phi_j) \sum_{j=1}^{N} k_j \phi_j = \sum_{j=1}^{N} \lambda g_j \phi_j. \tag{4.13}
\]
Take the inner product with \( \phi_i, i = 1, 2, \cdots, N \) on equation (4.13) to obtain
\[
(\lambda_i + \mu)g_i + k_i \sum_{j=1}^{N} g_j J_q^*(\phi_j) = \lambda g_i, \quad i = 1, 2, \cdots, N,
\]
which, together with (2.13) and (4.11), leads to
\[
(\lambda - \Lambda_N - K_N J_N) \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} = 0. \tag{4.15}
\]
Since \((g_1, g_2, \cdots, g_N) \neq 0\), we have
\[
\text{Det}(\lambda - \Lambda_N - K_N J_N) = 0. \tag{4.16}
\]
Hence, \( \lambda \in \sigma(\Lambda_N + K_N J_N) \subset \{s \mid \text{Re}(s) < 0\} \), since \( \Lambda_N + K_N J_N \) is Hurwitz.

When \( g \notin \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\} \), there exists a \( j_0 > N \) such that \( \int_0^1 g(x)\phi_{j_0}(x)dx \neq 0 \). Take the inner product with \( \phi_{j_0} \) on equation \((A + \mu + K J_q^*)g = \lambda g\) to get
\[
(\lambda_{j_0} + \mu) \int_0^1 g(x)\phi_{j_0}(x)dx = \lambda \int_0^1 g(x)\phi_{j_0}(x)dx,
\]
which, together with (1.8), implies that \( \lambda = \lambda_{j_0} + \mu < 0 \). Therefore, \( \lambda \in \sigma(A + \mu + K J_q^*) \subset \{s \mid \text{Re}(s) < 0\} \). The proof is complete. \( \square \)

5 Observer design

This section is devoted to the observer design by the newly proposed approach in [4]. Instead of the system (1.2), we design the observer for the following system:
\[
\begin{align*}
\begin{cases}
\dot{w}(x,t) &= \Delta w(x,t) + \mu w(x,t), \quad x \in \Omega, \\
w(x,t) &= 0, \quad x \in \Gamma_0, \quad \frac{\partial w(x,t)}{\partial \nu} = u(x,t), \quad x \in \Gamma_1, \\
\dot{v}(x,t) &= -\beta v(x,t) + QB^*w(x,t), \quad x \in \Gamma_1, \\
\dot{y}_v(t) &= \int_{\Gamma_1} v(x,t)dx,
\end{cases}
\end{align*}
\]
where \( \beta > 0 \) is a tuning parameter, \( v(\cdot, t) \) is an extended state, \( y_v \) is a new output, \( B^* \) is given by (1.15) and \( Q \in \mathcal{L}(L^2(\Gamma_1)) \) is given by
\[
(Qg)(x) = q(x)g(x), \quad x \in \Gamma_1, \quad \forall \quad g \in L^2(\Gamma_1)
\]
with \( q \in L^2(\Gamma_1) \) satisfying (4.1). By (1.3) and (1.12), system (5.1) can be written as
\[
\begin{align*}
\begin{cases}
\dot{w}(\cdot, t) &= (\tilde{A} + \mu)w(\cdot, t) + Bu(\cdot, t), \\
\dot{v}(\cdot, t) &= -\beta v(\cdot, t) + QB^*w(\cdot, t), \\
\dot{y}_v(t) &= C_v v(\cdot, t),
\end{cases}
\end{align*}
\]
with \( q \in L^2(\Gamma_1) \) satisfying (4.1). By (1.3) and (1.12), system (5.1) can be written as
where \( C_v : L^2(\Gamma_1) \to \mathbb{R} \) is defined by
\[
C_v h = \int_{\Gamma_1} h(x) \, dx, \quad \forall h \in L^2(\Gamma_1).
\] (5.4)

Now we demonstrate the key idea of the observer design via a simple finite-dimensional example.

**Example 5.1.** Consider the following system in the state space \( \mathbb{R}^n \times \mathbb{R} \):
\[
\begin{cases}
\dot{x}_1(t) = Ax_1(t) + Bu(t), \\
\dot{x}_2(t) = -\beta x_2(t) + QB^*x_1(t), \quad \beta > 0, \\
y(t) = x_2(t),
\end{cases}
\] (5.5)
where \( A \in \mathbb{R}^{n \times n} \) is the system matrix, \( B \in \mathbb{R}^n \) is the control matrix, \( Q \in \mathbb{R} \) is a constant, \( u(t) \) is the control and \( y(t) \) is the measurement. The Luenberger observer of system (5.5) is designed as
\[
\begin{cases}
\dot{\hat{x}}_1(t) = A\hat{x}_1(t) + F_1[x_2(t) - \hat{x}_2(t)] + Bu(t), \\
\dot{\hat{x}}_2(t) = -\beta\hat{x}_2(t) + QB^*\hat{x}_1(t) - F_2[x_2(t) - \hat{x}_2(t)],
\end{cases}
\] (5.6)
where \( F_1 \in \mathbb{R}^n \) and \( F_2 \in \mathbb{R} \) are the gain parameters to be determined. To demonstrate the key idea of the observer design for the infinite-dimensional system (5.1), we will find a new way to choose \( F_1 \) and \( F_2 \) rather than the conventional pole placement theorem. Let
\[
\hat{x}_j(t) = x_j(t) - \hat{x}_j(t), \quad j = 1, 2,
\] (5.7)
then the error is governed by
\[
\begin{cases}
\dot{\hat{x}}_1(t) = A\hat{x}_1(t) - F_1\hat{x}_2(t), \\
\dot{\hat{x}}_2(t) = -\beta\hat{x}_2(t) + QB^*\hat{x}_1(t) + F_2\hat{x}_2(t).
\end{cases}
\] (5.8)
If we pick \( F_1 \) and \( F_2 \) properly such that system (5.8) is stable, then \((x_1, x_2)\) can be estimated in the sense that
\[
\|(\hat{x}_1(t) - x_1(t), \hat{x}_2(t) - x_2(t))\|_{\mathbb{R}^n \times \mathbb{R}} \to 0 \quad \text{as} \quad t \to \infty.
\] (5.9)
Inspired by [4], the \( F_1 \) and \( F_2 \) can be chosen easily by decoupling the system (5.8) as a cascade system. Consider the following transformation
\[
\begin{pmatrix}
I_n & 0 \\
P & 1
\end{pmatrix}
\begin{pmatrix}
A & -F_1 \\
QB^* & F_2 - \beta
\end{pmatrix}
\begin{pmatrix}
I_n & 0 \\
P & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
A + F_1 P & -F_1 \\
P(A + F_1 P + QB^* - (F_2 - \beta)P) & F_2 - \beta - PF_1
\end{pmatrix},
\] (5.10)
where \( P \in \mathbb{R}^{1 \times n} \) to be determined. If we choose
\[
F_2 = PF_1 \quad \text{and} \quad PA + QB^* + \beta P = 0,
\] (5.11)
then the matrix on the right side of (5.10) is reduced to
\[
\begin{pmatrix}
A + F_1 P & -F_1 \\
0 & -\beta
\end{pmatrix},
\]
(5.12)
which is obviously a Hurwitz matrix provided \( A + F_1 P \) is Hurwitz. To sum up, the tuning parameters \( F_1 \) and \( F_2 \) can be chosen by the following scheme: (i), solve the equation \( PA + QB^* + \beta P = 0 \) to get \( P \); (ii), choose \( F_1 \) such that \( A + F_1 P \) is Hurwitz; (iii), let \( F_2 = PF_1 \).

Now, we return to design an observer for system (5.3). Inspired by Example 5.1, the observer of system (5.3) can be designed as
\[
\begin{aligned}
\dot{\hat{w}}(x,t) &= \Delta \hat{w}(x,t) + \mu \hat{w}(x,t) + K[C_v v(\cdot, t) - C_v \hat{v}(\cdot, t)], \quad x \in \Omega, \\
\hat{w}(x,t) &= 0, \quad x \in \Gamma_0, \quad \frac{\partial \hat{w}(x,t)}{\partial \nu} = u(x,t), \quad x \in \Gamma_1, \\
\dot{\hat{v}}(\cdot,t) &= -\beta \hat{v}(\cdot,t) + QB^* \hat{w}(\cdot,t) - L[C_v v(\cdot, t) - C_v \hat{v}(\cdot, t)] \quad \text{in} \ \Gamma_1,
\end{aligned}
\]
(5.13)
where \( K \) and \( L \) are tuning parameters that can be chosen by the following scheme:

- Solve the following equation
\[
\beta P + P(A + \mu) + QB^* = 0
\]
(5.14)
to get \( P \in L(L^2(\Omega), L^2(\Gamma_1)) \);
- Find \( K \) to detect system \( (A + \mu, C_v P) \);
- Let \( L = PK \).

By a straightforward computation, the solution of (5.14) is found to be
\[
P = -QB^*(\beta + \mu + A)^{-1} \in L(L^2(\Omega), L^2(\Gamma_1)).
\]
(5.15)
By (1.15), (5.4), (5.2) and (4.2), we have
\[
C_v P = J_\gamma^\gamma \in L(L^2(\Omega), \mathbb{R}) \quad \text{with} \quad \gamma = -\beta - \mu.
\]
(5.16)
By Lemma 4.2, (4.11) and (2.13), the operator \( K \) can be chosen by (4.12), where \((k_1, k_2, \cdots, k_N)^\top\) is a vector such that \( \Lambda_N + (k_1, k_2, \cdots, k_N)^\top J_N \) is Hurwitz. As a result of (4.12), (5.2) and (5.15),
\[
L = PK = \sum_{j=1}^N k_j P \phi_j = -\sum_{j=1}^N k_j QB^*(\beta + \mu + A)^{-1} \phi_j = -\sum_{j=1}^N k_j q(x) \xi_j(x), \quad x \in \Gamma_1,
\]
(5.17)
where
\[
\begin{aligned}
\xi_j(x) &= 0, \quad x \in \Gamma_0, \quad \frac{\partial \xi_j(x)}{\partial \nu} = 0, \quad x \in \Gamma_1, \\
(\beta + \mu + \Delta) \xi_j &= \phi_j \quad \text{in} \ \Omega, \\
j = 1, 2, \cdots, N.
\end{aligned}
\]
(5.18)
Combining (4.12) and (5.17), the observer (5.13) turns to be

\[
\begin{cases}
\dot{w}_t(x,t) = \Delta \dot{w}(x,t) + \mu \dot{w}(x,t) + [C_v v(\cdot,t) - C_v \hat{v}(\cdot,t)] \sum_{j=1}^{N} k_j \phi_j(x), & x \in \Omega, \\
\dot{w}(x,t) = 0, & x \in \Gamma_0, \\
\frac{\partial \dot{w}(x,t)}{\partial \nu} = u(x,t), & x \in \Gamma_1, \\
\dot{v}_t(x,t) = -\beta \dot{v}(x,t) + QB^* \dot{w}(x,t) + \sum_{j=1}^{N} k_j q(x) \xi_j(x)[C_v v(\cdot,t) - C_v \hat{v}(\cdot,t)], & x \in \Gamma_1,
\end{cases}
\]

where $\xi_j$ is given by (5.18), $j = 1, 2, \ldots, N$. By (4.12) and (5.17), the observer can be written as the abstract form:

\[
\frac{d}{dt}(\dot{w}(\cdot,t), \dot{v}(\cdot,t)) = A(\dot{w}(\cdot,t), \dot{v}(\cdot,t)) + (K, -L)^T C_v v(\cdot,t),
\]

where the operator $A : D(A) \subset L^2(\Omega) \times L^2(\Gamma_1) \rightarrow L^2(\Omega) \times L^2(\Gamma_1)$ is defined by

\[
A = \begin{pmatrix}
A + \mu & -KC_v \\
QB^* & LC_v - \beta
\end{pmatrix}
\]

with $D(A) = D(A) \times L^2(\Gamma_1)$. (5.21)

**Theorem 5.1.** Suppose that the operator $A$ is given by (1.3), $B^*$ is given by (1.15), the eigenpairs $\{ (\phi_j(\cdot), \lambda_j) \}_{j=1}^{\infty}$ are given by (1.7) and $Q$ is given by (5.2) with $q(\cdot) \in L^2(\Gamma_1)$ satisfying (4.1). Let the integer $N$ satisfy (1.8) and $\mu, \beta > 0$ satisfy

\[
-\beta - \mu \neq \lambda_j, & j = 1, 2, \ldots, N.
\]

Then, for any $(w(\cdot,0), v(\cdot,0), \dot{w}(\cdot,0), \dot{v}(\cdot,0)) \in [L^2(\Omega) \times L^2(\Gamma_1)]^2$ and $u \in L^2_{\text{loc}}([0, \infty); L^2(\Gamma_1))$, the observer (5.19) of system (5.1) admits a unique solution $(\dot{w}, \dot{v}) = C([0, \infty); L^2(\Omega) \times L^2(\Gamma_1))$ such that

\[
eomega t \| (w(\cdot,t) - \dot{w}(\cdot,t), v(\cdot,t) - \dot{v}(\cdot,t)) \|^2_{L^2(\Omega) \times L^2(\Gamma_1)} \rightarrow 0 \ \text{as} \ t \rightarrow \infty,
\]

where $\omega$ is a positive constant that is independent of $t$.

**Proof.** For any $(w(\cdot,0), v(\cdot,0)) \in L^2(\Omega) \times L^2(\Gamma_1)$ and $u \in L^2_{\text{loc}}([0, \infty); L^2(\Gamma_1))$, it is well known that the control plant (5.1) admits a unique solution $(w, v) \in C([0, \infty); L^2(\Omega) \times L^2(\Gamma_1))$ such that $y_v \in L^2_{\text{loc}}[0, \infty)$. Let

\[
\begin{cases}
\dot{w}(x,t) = w(x,t) - \dot{w}(x,t), & x \in \Omega, \\
\dot{v}(s,t) = v(s,t) - \dot{v}(s,t), & s \in \Gamma_1,
\end{cases}
\]

then, the errors are governed by

\[
\begin{cases}
\dot{w}_t(x,t) = \Delta \dot{w}(x,t) + \mu \dot{w}(x,t) - C_v \dot{v}(\cdot,t) \sum_{j=1}^{N} k_j \phi_j(x), & x \in \Omega, \\
\dot{w}(x,t) = 0, & x \in \Gamma_0, \\
\frac{\partial \dot{w}(x,t)}{\partial \nu} = 0, & x \in \Gamma_1, \\
\dot{v}_t(x,t) = -\beta \dot{v}(x,t) + QB^* \dot{w}(x,t) - C_v \dot{v}(\cdot,t) \sum_{j=1}^{N} k_j q(x) \xi_j(x), & x \in \Gamma_1,
\end{cases}
\]

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By (5.21), (5.17) and (4.12), system (5.25) can be written abstractly

$$\frac{d}{dt}(\hat{w}(\cdot,t),\hat{v}(\cdot,t))^\top = \mathcal{A}(\hat{w}(\cdot,t),\hat{v}(\cdot,t))^\top.$$ \hfill (5.26)

Inspired by [4], we introduce the following transformation

$$\mathbb{P}(f,g)^\top = (f, g + Pf)^\top, \quad (f, g)^\top \in L^2(\Omega) \times L^2(\Gamma_1),$$ \hfill (5.27)

where \(P \in \mathcal{L}(L^2(\Omega), L^2(\Gamma_1))\) is the solution of system (5.14). Then \(\mathbb{P}\) is invertible and its inverse is given by

$$\mathbb{P}^{-1}(f,g)^\top = (f, g - Pf)^\top, \quad (f, g)^\top \in L^2(\Omega) \times L^2(\Gamma_1).$$ \hfill (5.28)

Moreover, a simple computation shows that (see, e.g., [4, Theorem 6.1])

$$\mathbb{P}\mathcal{A}\mathbb{P}^{-1} = \mathcal{A}_\mathbb{P}, \quad D(\mathcal{A}_\mathbb{P}) = \mathbb{P}D(\mathcal{A}),$$ \hfill (5.29)

where

$$\mathcal{A}_\mathbb{P} = \begin{pmatrix} A + \mu + KC_vP & -KC_v \\ 0 & -\beta \end{pmatrix} \quad \text{with} \quad D(\mathcal{A}_\mathbb{P}) = D(A) \times L^2(\Gamma_1).$$ \hfill (5.30)

By Lemma 4.2 and (5.16), the operator \(A + \mu + KC_vP\) generates an exponentially stable \(C_0\)-semigroup on \(L^2(\Omega)\). Thanks to the block-triangle structure and [3], the operator \(\mathcal{A}_\mathbb{P}\) generates an exponentially stable \(C_0\)-semigroup \(e^{\mathcal{A}_\mathbb{P}t}\) on \(L^2(\Omega) \times L^2(\Gamma_1)\). By virtue of the similarity (5.29), the operator \(\mathcal{A}\) also generates an exponentially stable \(C_0\)-semigroup \(e^{\mathcal{A}t}\) on \(L^2(\Omega) \times L^2(\Gamma_1)\). As a result, the error system with initial state \((\hat{w}(\cdot,0),\hat{v}(\cdot,0))^\top = (w(\cdot,0) - \hat{w}(\cdot,0), v(\cdot,0) - \hat{v}(\cdot,0))^\top \in L^2(\Omega) \times L^2(\Gamma_1)\) admits a unique solution \((\hat{w},\hat{v})^\top \in C([0,\infty); L^2(\Omega) \times L^2(\Gamma_1))\) such that

$$e^{\omega t}\|(\hat{w}(\cdot,t),\hat{v}(\cdot,t))\|_{L^2(\Omega) \times L^2(\Gamma_1)} \to 0 \quad \text{as} \quad t \to \infty,$$ \hfill (5.31)

where \(\omega\) is a positive constant that is independent of \(t\). Let

$$(\hat{w}(\cdot,t),\hat{v}(\cdot,t)) = (w(\cdot,t) - \hat{w}(\cdot,t), v(\cdot,t) - \hat{v}(\cdot,t)).$$ \hfill (5.32)

Then, a straightforward computation shows that such a defined \((\hat{w},\hat{v})^\top \in C([0,\infty); L^2(\Omega) \times L^2(\Gamma_1))\) is a solution of system (5.20) or equivalently, system (5.19). Moreover, (5.23) holds due to (5.31) and (5.24). Owing to the linearity of system (5.19), the solution is unique. \hfill \(\square\)

### 6 Numerical simulations

In this section, we present some numerical simulations for the closed-loop system (3.23) to demonstrate the theoretical results visually. In order to avoid the difficulty of numerical discretization, we consider the unstable heat system in the rectangular domain \(\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < 1\}\).

The actuator is installed on the boundary

$$\Gamma_1 = \{(x,y) \in \mathbb{R}^2 \mid x = 1, 0 \leq y \leq 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid y = 1, 0 \leq x \leq 1\}.$$
The fixed boundary is
\[ \Gamma_0 = \{(x, y) \in \mathbb{R}^2 \mid x = 0, 0 \leq y < 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = 0, 0 \leq x < 1\} .\]

We adopt the finite difference scheme to discretize system (3.23) directly. The numerical results are programmed in Matlab. Inspired by [14] where the uniform exponential decay with respect to the mesh size is obtained by the finite difference method, the space step \( h \) and time step \( \tau \) are taken as \( h = \tau = 0.05 \). The initial state and tuning parameters are chosen as
\[
\begin{align*}
    w(x, y, 0) &= x \sin 2\pi y, \quad v(x, y, 0) = 0, \\
    p(x, y) &= \sin x \sin y, \quad \mu = 6, \quad \alpha = 3, \quad N = 1, \quad l_1 = 15.
\end{align*}
\]

By a simple numerical computation, the largest eigenvalue of the operator (1.3) on \( \Omega \) is \( \lambda_1 \approx -4.6947 \). This implies that \( \lambda_1 + \mu > 0 \) and hence the open-loop system (1.2) is unstable.

The the initial state and the final state of closed-loop system (3.23) are plotted in Figure 1. In order to demonstrate the dynamic evolution of the closed-loop system, the state trace \( w(x, 0.5, t) \) is plotted in Figure 2(a). The same state trace without control is plotted in Figure 2(b) for comparison.
The distributed control traces $v(x, y, t)$ are plotted in Figure 3(a) and 3(b). To demonstrate the decay rate, the logarithmic state norm $\|w(\cdot, t)\|_{L^2(\Omega)}$ decay curve and the curve of the state norm itself are plotted in Figure 4(a) and Figure 4(b), respectively. From these Figures 1-4 we observe

that the state of the control plant are stabilized effectively despite the presence of the unstable source term $\mu w(x, y, t)$. Moreover, the dynamic evolution is smooth. Figure 4 implies that the state norm decays to zero exponentially. So all the convergence in the closed-loop system is very fast.

7 Conclusions

In this paper, we consider the stabilization and observation for the unstable heat equation in a general multi-dimensional domain. The newly developed dynamics compensation approach and the finite-dimensional spectral truncation technique are exploited to treat the difficulties caused by instability. Both the full state feedback law and the state observer are designed. The closed-loop system and the observation error are convergent to zero exponentially as $t \to \infty$. The developed
method in this paper provides a new choice, in addition to the PDE backstepping method, for dealing with unstable PDEs, especially for multi-dimensional unstable PDEs. It is very interesting to extend this new method to other unstable or anti-stable PDEs such as the multi-dimensional wave equation and Euler-Bernoulli beam equation, which are our future works.

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8 Appendix

**Lemma 8.1.** For any positive integer $N$, there exists a function $p \in L^2(\Gamma_1)$ such that (2.1) holds.

*Proof.* Choose $p_1(x) = \phi_1(x)$ for any $x \in \Gamma_1$. Then $\langle p_1, \phi_1 \rangle_{L^2(\Gamma_1)} \neq 0$. If $\langle \phi_2, \phi_1 \rangle_{L^2(\Gamma_1)} \neq 0$, then let $p_2 = p_1$. Otherwise, let

$$p_2(x) = p_1(x) + \phi_2(x).$$

(8.1)

Then, $\langle p_2, \phi_1 \rangle_{L^2(\Gamma_1)} \neq 0$ and $\langle p_2, \phi_2 \rangle_{L^2(\Gamma_1)} \neq 0$. Suppose that we have obtained $p_{N-1}$ such that

$$\langle p_{N-1}, \phi_j \rangle_{L^2(\Gamma_1)} \neq 0, \quad j = 1, 2, 3, \cdots, N - 1. \quad (8.2)$$

If $\langle p_{N-1}, \phi_N \rangle_{L^2(\Gamma_1)} \neq 0$, we choose $p_N = p_{N-1}$. Otherwise,

$$p_N(x) = p_{N-1}(x) + \gamma \phi_N(x),$$

(8.3)
where $\gamma$ small enough such that

$$
\langle p_{N-1}, \phi_j \rangle_{L^2}(\Gamma_1) + \gamma \langle \phi_N, \phi_j \rangle_{L^2}(\Gamma_1) \neq 0, \ j = 1, 2, \cdots, N.
$$

(8.4)

Therefore, the proof is complete due to the mathematical induction. \hfill \Box

**Lemma 8.2.** For any positive integer $N$, define

$$
\Lambda_N = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)
$$

and

$$
B_N = (b_1, b_2, \cdots, b_N)^\top,
$$

where $b_k \neq 0, k = 1, 2, \cdots, N$ and

$$
\lambda_i \neq \lambda_j, \ i \neq j, \ i, j = 1, 2, \cdots, N.
$$

(8.7)

Then, system $(\Lambda_N, B_N)$ is controllable.

**Proof.** By a simple computation, the controllability matrix of system $(\Lambda_N, B_N)$ is

$$
P_c = \begin{pmatrix}
    b_1 & \lambda_1 b_1 & \cdots & \lambda_1^{N-1} b_1 \\
    b_2 & \lambda_2 b_2 & \cdots & \lambda_2^{N-1} b_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    b_N & \lambda_N b_N & \cdots & \lambda_N^{N-1} b_N
\end{pmatrix}.
$$

(8.8)

Furthermore,

$$
|P_c| = b_1 b_2 \cdots b_N \begin{vmatrix}
    1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\
    1 & \lambda_2 & \cdots & \lambda_2^{N-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & \lambda_N & \cdots & \lambda_N^{N-1}
\end{vmatrix} = b_1 b_2 \cdots b_N \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j) \neq 0.
$$

(8.9)

Therefore, the proof is complete due to the Kalman’s controllability rank condition. \hfill \Box