Shear viscosity to relaxation time ratio in SU(3) lattice gauge theory

Yasuhiro Kohno, Masayuki Asakawa, and Masakiyo Kitazawa
Department of Physics, Osaka University, Toyonaka 560-0043, Japan
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We evaluate the ratio of the shear viscosity to the relaxation time of the shear flux above but near the critical temperature $T_c$ in SU(3) gauge theory on the lattice. The ratio is related to Kubo’s canonical correlation of the energy-momentum tensor in Euclidean space with the relaxation time approximation and an appropriate regularization. Using this relation, the ratio is evaluated by direct measurements of the Euclidean observables on the lattice. We obtained the ratio with reasonable statistics for the range of temperature $1.3T_c \lesssim T \lesssim 4T_c$. We also found that the characteristic speed of the transverse plane wave in gluon media is almost constant, $v \approx 0.5$, for $T \gtrsim 1.5T_c$, which is compatible with the causality in the second order dissipative hydrodynamics.

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I. INTRODUCTION

One of the highlights at RHIC (the Relativistic Heavy Ion Collider) is a success of relativistic hydrodynamic models for the description of the space-time evolution of the quark-gluon matter created in heavy ion collisions \[1–3\]. This novel feature revealed that the hot quark-gluon matter is a strongly correlated system, which is one of the most significant subjects over broad areas in physics. Heavy ion collisions at LHC (the Large Hadron Collider) have already reported new experimental results. The transverse momentum dependence of the elliptic flow observed at LHC is similar to that at RHIC \[4\]. The experimental result implies that the hydrodynamic description for the quark-gluon matter is as good at LHC as at RHIC.

Recently the importance of dissipative effects in the hydrodynamic models has been recognized \[5, 6\]. The simplest relativistic dissipative hydrodynamics is the first order theory, which in the non-relativistic limit reduces to Navier-Stokes theory \[7, 8\]. The first order theory is, however, accompanied with the acausal problem and instability in numerical simulations \[9\]. One of the strategies to evade these problems is to extend the theory to the second order with regard to dissipative fluxes. In the second order theory, however, there appear many phenomenological parameters as second order transport coefficients in addition to the first order ones. These transport coefficients cannot be determined within the framework of hydrodynamics. Ab initio calculation based on microscopic theory, i.e. Quantum Chromodynamics (QCD) in our case, is required to study and constrain the parameters in dissipative hydrodynamic models.

Since the temperature range realized at RHIC and LHC is not within the reach of perturbative QCD, we need to employ non-perturbative approaches to investigate the transport coefficients. At present, lattice simulation is the only systematic way to calculate physical quantities in such a non-perturbative region. There are several pioneering studies which analyzed transport coefficients on the lattice \[10–14\]. These studies have used Green-Kubo formulas, which relate the transport coefficients to the low energy behavior of the corresponding spectral functions. In this method, one needs to extract the spectral functions from Euclidean correlators obtained on the lattice. This step is, however, nontrivial because it is an ill-posed problem \[15\].

In this paper, we focus on the ratio of the shear viscosity to the relaxation time of the shear flux. In Refs. \[16, 17\], it is argued that the ratio is related to a static fluctuation of the energy-momentum tensor by rewriting Green-Kubo formula for a classical system with the relaxation time approximation. We modify their arguments to treat a quantum system and show that the ratio is related to Kubo’s canonical correlation of the energy-momentum tensor. It enables us to evade the difficulty in the analysis of spectral functions and calculate the ratio on the lattice directly. Whereas the ratio itself is not a transport coefficient, its determination reduces the number of the free phenomenological parameters in hydrodynamic models. We note that a similar argument has been also given in Ref. \[18\].

We need to regularize Kubo’s canonical correlation to extract physical quantities, because it is ultraviolet divergent. We perform the regularization by taking the vacuum subtraction. We argue that one further needs to take care of the contribution arising from contact terms to obtain physically meaningful quantities in addition to this prescription. While Green-Kubo formula does not contain the equal space-time operator by definition, the lattice observable includes its contribution. This contribution is usually removed solely by the vacuum subtraction, because the coefficients of the contact terms are not dependent on the temperature. For the case of the energy-momentum tensor, to the contrary, the coefficients of the contact terms in Kubo’s canonical correlation of the energy-momentum tensor are proportional to the energy-momentum tensor themselves, and their expectation values are dependent on the temperature. We thus have to remove the contribution of the contact terms separately besides the vacuum subtraction.

The structure of this paper is as follows. In Sec. I, we review Israel-Stewart theory \[19\], which is one of the second order dissipative hydrodynamics. We show how to
introduce the relaxation times of the dissipative fluxes to hydrodynamics. In Sec. [II] we relate the ratio of the shear viscosity to the relaxation time to Kubo’s canonical correlation of a spatial component of the energy-momentum tensor. We discuss how to remove the unphysical divergence in the correlator and to extract the ratio from the canonical correlation in Sec. [IV]. In Sec. [IV] we formulate a method to calculate the ratio on the lattice and show numerical results in SU(3) lattice gauge theory. The last section is devoted to conclusions and discussions.

II. ISRAEL-STEWART THEORY AND RELAXATION TIME

In this section, we give a brief review on Israel-Stewart (IS) theory and the role of the relaxation times in this theory. Hydrodynamics consists of equations of motion for conserved currents. The only conserved current in pure gauge theory is the energy-momentum tensor $T^{\mu\nu}$, for which the conservation law reads

$$\partial_\mu T^{\mu\nu} = 0, \quad (1)$$

The energy-momentum tensor is decomposed as

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)(g^{\mu\nu} - u^\mu u^\nu) + \pi^{\mu\nu}, \quad (2)$$

where $\epsilon$ and $P$ are the energy density and pressure in equilibrium, and $\Pi$ and $\pi^{\mu\nu}$ are the dissipative fluxes for the bulk and shear channels, respectively. The metric $g^{\mu\nu}$ is defined by $g^{\mu\nu} = \text{diag}(+,\ldots)$, and $u^\mu$ is proportional to energy-flow with the normalization $u_\mu u^\mu = 1$.

In second order theory, $\Pi$ and $\pi^{\mu\nu}$ are regarded as dynamical variables. One of the arguments to obtain evolution equations for these variables besides Eq. (1) is a phenomenological one [19, 21]. In the derivation, the evolution equations are obtained as a consequence of the second law of thermodynamics

$$\partial_\mu s^\mu \geq 0, \quad (3)$$

where $s^\mu = su^\mu$ and $s$ is the entropy density in the energy frame. By choosing the entropy density to be the equilibrium one, $s = s_{\text{eq}}$, we obtain the first order theory with Eq. (3) and linearity between the dissipative fluxes and thermodynamic forces. To extend the theory to second order, it is assumed that the entropy near equilibrium is given by a function of $\Pi$ and $\pi^{\mu\nu}$, and that it is analytic in equilibrium $\Pi = \pi^{\mu\nu} = 0$ [19]. The entropy density up to the second order in Taylor expansion with respect to $\Pi$ and $\pi^{\mu\nu}$ then reads

$$s \equiv s_{\text{eq}} - \frac{1}{27} \beta_0 \Pi^2 + \beta_2 \pi^{\mu\nu}\pi^{\mu\nu}, \quad (4)$$

where $\beta_0$ and $\beta_2$ are phenomenological parameters. The second law of thermodynamics Eq. (3) with Eq. (4) leads to a set of evolution equations, which are referred to as IS equations. In the shear channel, for example, the evolution equation reads

$$\dot{\pi}^{\mu\nu} = -\frac{1}{\tau_\pi} [\pi^{\mu\nu} + \alpha], \quad (5)$$

where $\pi^{\mu\nu} = d\pi^{\mu\nu}/dt$ and $\alpha$ denotes contributions which vanishes in a uniform medium. Eq. (5) shows that $\pi^{\mu\nu}$ damps with a time scale $\tau_\pi$ in a uniform medium. $\tau_\pi$ therefore is called the relaxation time of the shear flux and is related to $\beta_2$ in Eq. (1) as

$$\tau_\pi = 2\eta\beta_2, \quad (6)$$

where $\eta$ is the shear viscosity [19]. The corresponding first order equation is obtained by taking the limit $\tau_\pi \to 0$ in Eq. (5). Similarly the evolution equation in the bulk channel includes the relaxation time of the bulk flux $\tau_1$, which is related to $\beta_0$ in Eq. (4) as $\tau_1 = \zeta \beta_0$. The second order theory can become causal due to these relaxation times of dissipative fluxes.

The evolution equation Eq. (5) implies that the temporal correlator of the shear flux behaves as

$$\langle \pi^{\mu\nu}(t)\pi^{\mu\nu}(0) \rangle \sim e^{-t/\tau_\pi} \langle \pi^{\mu\nu}(0)\pi^{\mu\nu}(0) \rangle, \quad (7)$$

where $\pi^{\mu\nu}(t) = \frac{1}{V} \int d^3 x \pi^{\mu\nu}(t, \vec{x})$ is the spatial average of $\pi^{\mu\nu}$ in a volume $V$. We note that $\pi^{\mu\nu}(t)$ in Eq. (7) is a classical variable in hydrodynamical equations. We will later argue an extension of this relation to the quantum correlator.

In addition, we introduce the wave front speed (characteristic speed) for the transverse plane wave $v$ defined by [20]

$$v^2 = \frac{\eta}{\tau_\pi (\epsilon + P)} = \frac{\eta}{\tau_\pi T s}. \quad (8)$$

For the causality to be satisfied, $v^2 < 1$ is required.

III. RATIO OF THE SHEAR VISCOSITY TO THE RELAXATION TIME

A. The ratio $\eta/\tau_\pi$

We now relate the ratio $\eta/\tau_\pi$ to Kubo’s canonical correlation of the energy-momentum tensor in Euclidean space. The original idea of the following discussion was proposed within the framework of classical theory in Refs. [16, 17]. We modify their arguments to treat quantum systems.

We start from Green-Kubo formula in the shear channel

$$\eta = \frac{i}{\hbar} \int \frac{d^3 x}{\tau_\pi} \int_0^\infty dt \langle [\pi^{\mu\nu}(t, \vec{x}), \pi^{\mu\nu}(0, \vec{0})] \rangle$$

$$= \frac{i}{\hbar} V \int_0^\infty dt \langle [\pi^{\mu\nu}(t), \pi^{\mu\nu}(0)] \rangle, \quad (9)$$
where \( \langle O \rangle = \text{Tr}[e^{-H/T} O]/\text{Tr}[e^{-H/T}] \) is the statistical average in equilibrium with \( H \) being the hamiltonian. Here we have set Boltzmann constant \( k_B = 1 \). To rewrite this formula, we use the following identities,
\[
[e^{-H/T}, A] = e^{-H/T} \int_0^{1/T} d\lambda e^{\lambda H}[A, H] e^{-\lambda H},
\]
(10)
and
\[
\langle [A, B] \rangle = \text{Tr} \left[ e^{-H/T} [A, B] \right] = i \langle \int_0^{h/T} d\lambda \dot{A} (-i\lambda) B \rangle.
\]
(11)
Substituting Eqs. (10) and (11) to Eq. (9), we obtain
\[
\eta = -\frac{V}{h} \int_0^\infty dt \langle \int_0^{h/T} d\lambda \bar{\pi}_{12}(t-i\lambda) \bar{\pi}_{12}(0) \rangle
\]
\[
\approx \frac{V}{h} \int_0^\infty dt \langle \int_0^{h/T} d\lambda \bar{\pi}_{12}(-i\lambda) \bar{\pi}_{12}(0) \rangle_{\text{reg}}
\]
for \( t \geq 0 \). On the right hand side of Eq. (13), we have put the subscript “reg” to the statistical average, because the expectation value \( \langle \int_0^{h/T} d\lambda \bar{\pi}_{12}(-i\lambda) \bar{\pi}_{12}(0) \rangle \) is in fact ultraviolet divergent and ill-defined, and should be regularized. This expectation value should not contain the contribution arising from the contact term, because the original formula Eq. (9) does not include it. We postpone detailed discussion on these issues to the next section, and for the moment assume that the right hand side of Eq. (13) can be defined appropriately.

In the following, we refer to Eq. (13) as the relaxation time approximation, owing to its analogy to the standard approximation in nonequilibrium statistical mechanics, \( \langle O(t) O(0) \rangle \approx e^{-t/\tau} \langle O(0)^2 \rangle \). Note, however, that Eq. (13) is slightly different from this approximation; because in Eq. (13) the correlator between operators separated along complex time is concerned, while the standard relaxation time approximation is for a real time separation. Since Eq. (13) should reduce to Eq. (7) for the range of \( t \) where the second order hydrodynamics is applicable, the time scale appearing in the exponential function in Eq. (13) is identified to be the relaxation time of the shear flux \( \tau_\pi \) in Eq. 5.

Using Eq. (13), Eq. (12) reduces to
\[
\eta \simeq \frac{V}{h} \int_0^\infty dt \langle \int_0^{h/T} d\lambda e^{-t/\tau_\pi} \bar{\pi}_{12}(-i\lambda) \bar{\pi}_{12}(0) \rangle_{\text{reg}}
\]
\[
= \frac{V}{h} \int_0^{h/T} d\lambda \langle \bar{\pi}_{12}(-i\lambda) \bar{\pi}_{12}(0) \rangle_{\text{reg}}.
\]
(14)
In the local rest frame of a medium, one can replace \( \bar{\pi}_{12} \) with a spatial component of the energy-momentum tensor \( \bar{T}_{12} \). Eq. (13) then means that the ratio \( \eta/\tau_\pi \) is given by
\[
\frac{\eta}{\tau_\pi} = \frac{V}{h} \int_0^{h/T} d\lambda \langle \bar{T}_{12}(-i\lambda) \bar{T}_{12}(0) \rangle_{\text{reg}} = \frac{V}{T} \langle \bar{T}_{12} \rangle_{\text{reg}}.
\]
(15)
Here \( \langle \bar{A}^2 \rangle \) denotes Kubo’s canonical correlation
\[
\langle \bar{A}^2 \rangle = \frac{T}{h} \int_0^{h/T} d\lambda \langle \bar{A}(\lambda) \bar{A}(0) \rangle.
\]
(16)
The same argument applies to the bulk channel, which leads to a formula to represent the ratio \( \zeta/\tau_1 \) with diagonal components of the energy-momentum tensor. We note that Eq. (14) is also derived by the projection operator method [18].

### B. Classical limit

In the classical limit \( h/T \rightarrow 0 \), one can replace the integral with respect to \( \lambda \) in Eq. (14) by \( h/T \). Eq. (15) then reduces to
\[
\frac{\eta}{\tau_\pi} = \frac{V}{T} \langle \bar{T}_{12} \rangle_{\text{reg}}.
\]
(17)
\( \eta/\tau_\pi \) is, therefore, given by a static fluctuation of \( \bar{T}_{12} \) in classical theory. Eq. (17) is the formula derived in the original works Refs. [16, 17].

In a classical system, one can also derive Eq. (17) using Einstein principle and the entropy density in the IS theory. According to Einstein principle, the probability distribution \( P(x) \) of a state variable \( x \) in equilibrium is given by \( P(x) \sim e^{S(x)} \) where \( S(x) \) is the nonequilibrium entropy in a volume \( V \). If one identifies the IS entropy Eq. (4) to be that in Einstein principle, the probability distribution for a state variable \( \bar{\pi}_{12} \) in equilibrium state in a volume \( V \) is given by
\[
P_\pi(\bar{\pi}_{12}) \sim \exp \left[ -\frac{V}{T} \beta_2 \bar{\pi}_{12}^2 \right].
\]
(18)
With this distribution function the fluctuation of \( \bar{\pi}_{12} \) is calculated to be
\[
\langle \bar{\pi}_{12}^2 \rangle = \frac{\int d\pi \pi^2 P_\pi(\pi)}{\int d\pi P_\pi(\pi)} = \frac{T}{2\beta_2 V} = \frac{\eta T}{\tau_\pi V},
\]
(19)
which is equivalent with Eq. (17). It should, however, be remembered that Einstein principle is applicable only to classical systems. In quantum mechanics, when a state variable \( x \) does not commute with the hamiltonian, \( x \) and the energy cannot be determined simultaneously. This means that the entropy cannot be defined as a function of non-conserving state variables in quantum systems [22]. Einstein principle therefore is not applicable for such cases.
IV. REGULARIZATION

From now on we set $\hbar = 1$. On the lattice, $\eta/\tau_\pi$ is obtained by direct measurement of Kubo’s canonical correlation of the spatial off-diagonal component of the energy-momentum tensor

$$\langle \tilde{T}^2_{12} \rangle = \frac{T}{V} \int d^4x \langle T_{12}(x)T_{12}(0) \rangle,$$  \hspace{1cm} (20)

where $x = (\tau, \vec{x})$. This canonical correlation is, however, different from $\langle \tilde{T}^2_{12} \rangle_{\text{reg}}$ in Eq. (15) because of two reasons. First, the canonical correlation Eq. (20) diverges in the ultraviolet limit. This divergence originates from the short distance behavior of the Euclidean correlator of $T_{12}(x)$. Second, Eq. (20) contains the equal space-time term, i.e. the contact term, while Eq. (15) does not.

To evaluate $\eta/\tau_\pi$ on the lattice with the measurements of Eq. (20), one has to take account of these differences.

Since both of these effects originate from the short distance behavior of the Euclidean correlator of $T_{12}(x)$, we examine them by the operator product expansion (OPE). According to the OPE calculation [24], the correlator at short distance behaves as

$$\langle T_{12}(x)T_{12}(0) \rangle \simeq C_T \frac{1}{|x|^8} + \sum_{\mu} C_{\mu} \langle T_{\mu\mu}(0) \rangle \delta^{(4)}(x) + \cdots,$$  \hspace{1cm} (21)

where $C_T$ and $C_{\mu}$ are c-number Wilson coefficients determined by perturbation theory. Because of the dimensional reason, the first term on the right hand side in Eq. (21) is proportional to $|x|^{-8}$. This term leads to the ultraviolet divergence of Eq. (20). This term neither has medium effects nor affects long distance behavior responsible for hydrodynamics. Therefore, this divergence should be eliminated when one discusses transport coefficients. The contribution is removed by subtracting the correlator in the vacuum,

$$\int d^4x \langle T_{12}(x)T_{12}(0) \rangle_0$$

$$\equiv \int d^4x \left( \langle T_{12}(x)T_{12}(0) \rangle - \langle T_{12}(x)T_{12}(0) \rangle_{T=0} \right).$$  \hspace{1cm} (22)

The next terms on the right hand side of Eq. (21) are proportional to dimension four operators. These terms are the contact terms, i.e. they are proportional to $\delta^{(4)}(x)$, as presented in Eq. (21). As discussed already, we need to remove these terms from Eq. (20) when evaluating Eq. (15), i.e.

$$\int d^4x \langle T_{12}(x)T_{12}(0) \rangle_{\text{reg}}$$

$$\equiv \int d^4x \left( \langle T_{12}(x)T_{12}(0) \rangle_0 - \frac{1}{2} \langle \text{contact terms} \rangle \right).$$  \hspace{1cm} (23)

| $\beta$ | $N_x$ | $N_y$ | $a$ [fm] | $L_x$ [fm] |
|--------|------|------|--------|---------|
| 6.499  | 32   | 4, 6, 8, 32 | 0.046  | 1.5     |
| 6.205  | 32   | 4, 6, 8, 32 | 0.068  | 2.2     |
| 6.000  | 32   | 4, 6, 8, 16 | 0.094  | 3.0     |
| 6.000  | 16   | 4, 6, 8, 16 | 0.094  | 1.5     |

The statistical average of the contact terms has temperature dependence, because it is proportional to the energy-momentum tensor. This contribution therefore cannot be removed by the vacuum subtraction Eq. (22). In Eq. (23), we subtracted the contribution of the contact terms with a factor 1/2 because of the following reason. As shown in Eq. (21), the support of the contact terms or $\delta$-functions is located at the origin. The integral interval for the time in Green-Kubo formula Eq. (9) is 0 $\leq t < \infty$, and the same holds for Eq. (20) as well. Thus $\eta/\tau_\pi$ contains the half-contribution from the vicinity of the origin. The higher order terms in Eq. (21) shown by the dots neither yield divergence nor are singular.

In the leading order OPE calculation [24, 25], the contact terms are most conveniently evaluated by taking the high frequency limit of the correlator with zero spatial momentum

$$G_0(q_4 \to \infty, \vec{q} = 0) = \sum_{\mu} C_{\mu} \langle T_{\mu\mu}(0) \rangle_0$$

$$= \frac{2}{3} \langle T_{00} \rangle_0 + \frac{1}{6} \langle F^2 \rangle_0,$$  \hspace{1cm} (24)

where $G_0(q_4, \vec{q}) = \int d^4x \ e^{i(q_4\tau - \vec{q} \cdot \vec{x})} \langle T_{12}(x)T_{12}(0) \rangle_0$ and $\langle F^2 \rangle = \langle F_{\mu\nu} F^{\mu\nu} \rangle$ is the gluon condensate. In the first equality of Eq. (24), the higher order terms in Eq. (21) vanish by Riemann-Lebesgue lemma.

From Eqs. (23) and (24), we obtain

$$\frac{\eta}{\tau_\pi} = V \langle \tilde{T}_{12}^2 \rangle_0 - \frac{1}{3} \langle T_{00} \rangle_0 - \frac{1}{12} \langle F^2 \rangle_0,$$  \hspace{1cm} (25)

in the leading order perturbation theory. The right hand side of Eq. (25) includes only lattice observables. Therefore, we can evaluate $\eta/\tau_\pi$ directly by lattice simulations.

V. LATTICE MEASUREMENTS

A. Formulation on the lattice

In this section, we analyze the ratio $\eta/\tau_\pi$ on the lattice by evaluating three observables on the right hand side of
We have performed the lattice simulations for SU(3) pure gauge theory with the standard Wilson gauge action. In Table I we list lattice parameters used in this work. To investigate the lattice spacing and spatial volume dependences, simulations have been carried out on four isotropic lattices of volume $N^3$. The lattice spacing, $a$, in Table I for each $\beta$ is determined by the string tension $\sqrt{\sigma} = 460\text{MeV}$ and the parametrization of $a\sqrt{\sigma}$ in Ref.\cite{26}. All observables discussed in the next subsection, however, are dimensionless and do not depend on this physical scale. We use $T_c\sqrt{\sigma} = 0.63 \pm 0.03$ and normalize $T$ by $T_c$. The temporal length $aN_t = 1/T$ of each lattice corresponds to the range of temperature $0.5T_c \lesssim T \lesssim 4T_c$, which covers those realized in heavy ion collisions at RHIC and LHC. Gauge configurations were generated 3000 at RHIC and LHC. Gauge configurations were updated by heatbath and overrelaxation algorithms. We have generated $300,000 - 2,000,000$ configurations for each parameter. Statistical errors were estimated by the jackknife method with bin sizes in the range of 50 - 1000.

In the lattice calculations, we employed the traceless operator for the energy-momentum tensor,

$$T_{\mu\nu} \equiv 2\text{Tr} \left[ F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}F_{\mu\sigma} \right]. \quad (26)$$

For the field strength $F_{\mu\nu}$ on the lattice, we have used the clover type plaquette. With this definition, Eq. (26) reads

$$\frac{\eta}{\tau} = \frac{V}{T}\langle \tilde{T}_{12}^2 \rangle_0 - \frac{1}{3}\langle T_{00} \rangle_0 - \frac{1}{6}\langle F^2 \rangle_0$$

$$= \frac{V}{T}\langle \tilde{T}_{12}^2 \rangle_0 + \frac{1}{4}\langle \epsilon + P \rangle - \frac{1}{6}\langle F^2 \rangle_0, \quad (27)$$

where we have replaced the traceless operator $\langle T_{00} \rangle_0$ with $-\frac{1}{4}\langle \epsilon + P \rangle$. The definition Eq. (26) is sufficient to evaluate $\eta/\tau$ as explained below. To perform the vacuum subtraction, we regarded the lattice with the largest $N_t$ for each $\beta$ to be the vacuum one and subtracted the expectation value on this lattice from the one at each $N_t$. In fact, the lattice with the largest $N_t$ for each $\beta$ corresponds to a temperature well below $T_c$, where medium effects on the expectation value are expected to be well suppressed.

We also used the energy-momentum tensor defined above to evaluate $\epsilon + P$. In the calculation, simply their expectation values $-\langle T_{00} \rangle_0 + \langle T_{33} \rangle_0$ were taken. We note that the effect of the trace anomaly is canceled out in this observable, and hence the traceless definition in Eq. (26) is enough for this measurement. While our definition for $\epsilon + P$ is different from that used in the conventional analysis of lattice thermodynamics, the difference should be of higher order in lattice spacing and converge in the continuum limit.

Finally, we have measured the gluon condensate $\langle F^2 \rangle$ from the gauge action using the relation,

$$\frac{\langle F^2 \rangle_0}{T^4} = N^3T\frac{d^3S^G_0}{dT} = -6aN^2\frac{dg^{-2}}{da}S^G_0. \quad (28)$$

FIG. 1: (a) Kubo’s canonical correlation $\langle \tilde{T}_{12}^2 \rangle$ with only vacuum subtraction. Lattice spacing and volume dependences are small. (b) $\langle \epsilon + P \rangle/4$ obtained from direct measurements of $\langle T_{00} \rangle_0$ and $\langle T_{33} \rangle_0$ defined by the field strength constructed from clover type plaquette. The results qualitatively reproduce well-known behavior of previous lattice works. (c) Gluon condensate measured by conventional approach with standard plaquette.

B. Numerical results

In Fig. 1 we show the temperature dependences of each term on the right hand side of Eq. (27), $\langle \tilde{T}_{12}^2 \rangle_0$, $\langle \epsilon + P \rangle/4$, and $\langle F^2 \rangle_0/6$. Fig. 1(a) shows that the canonical correlation $\langle \tilde{T}_{12}^2 \rangle_0$ takes negative values for $\beta = 6.000$ at $T < 2T_c$. While the negative values are also observed for $N_t = 4$ on finer lattices, for $N_t \geq 6$ the values of $\langle \tilde{T}_{12}^2 \rangle_0$ are consistent with zero within statistics. The error bars in Fig. 1(a), however, can give a constraint on the upper limit on the absolute values of $\langle \tilde{T}_{12}^2 \rangle_0$. Thermodynamic quantities presented in the lower two panels of Fig. 1 show that statistics used in the calculation is sufficient for their determination. We have checked that numerical results in Fig. 1(c) reproduces the trace anomaly measured by the conventional procedure well (see, for instance, Ref. [27]). The spatial volume dependence of these results is estimated by comparing numerical results on the two lattices with $\beta = 6.000$. Fig. 1 shows that the spatial volume dependence is almost negligible in our re-
results except in the vicinity of $T_c$.

In Fig. 2, we show the temperature dependence of the ratio $\eta/\tau_\pi$ given by Eq. (5). The figure shows that the results with all sets of configurations coincide within the statistical error, which indicates that the lattice spacing dependence of the present results is small. Comparing the two results with $\beta = 6.000$, we observe that the spatial volume dependence is also small. It seems that $\eta/\tau_\pi$ has a tendency to increase rapidly in the vicinity of the critical temperature, but it is hard to conclude because of the statistics. $\eta/\tau_\pi$ is related to the characteristic speed $v$ defined by Eq. (5). We show the temperature dependence of $v^2$ in Fig. 3. $v^2$ is constantly around one-quarter of the square of the light speed, $v^2 \simeq 0.25$, for $1.5T_c \lesssim T \lesssim 4T_c$. This result indicates that causality is satisfied in the second order hydrodynamics for this mode in this range of temperature. The statistical error, however, increases as the temperature is lowered to $T_c$.

It is of interest to compare the present result on $v^2$ with other theoretical predictions. Grad 14 moment approximation predicts $v^2 \simeq 1/6$ in the high energy limit [19], and the AdS/CFT correspondence $v^2 = 1/(2\ln 2) \approx 0.36$ [24]. The leading order result of the projection operator method provides $v^2 = P/(c + P) \approx 0.25$ in the high temperature region [18]. The measured values of the characteristic speed are close to the value of the projection operator method especially above $2.5T_c$.

Finally, let us consider the implication of our results on the magnitude of the relaxation time $\tau_\pi$. It has been conjectured that the ratio of the shear viscosity to the entropy density $\eta/s$ has the universal lower bound $1/4\pi$, which is the value obtained by the AdS/CFT correspondence [30]. On the other hand, it has been suggested by the comparison between the results of dissipative hydrodynamics and the experiments at RHIC and LHC that the ratio is less than $\sim 5 \times$ the lower bound above $T_c$ [4]. If one substitutes these values in Eq. (5) and use $v^2 = 0.25$, the relaxation time $\tau_\pi$ is roughly estimated to be $\tau_\pi \approx \mathcal{O}(10^{-2}) - \mathcal{O}(10^{-1})$ fm in the temperature range. This estimate is almost ten times smaller than the result for $\tau_\pi$ in the pion gas below $T_c$ [20].

VI. CONCLUSION

In this paper, we have evaluated the ratio of the shear viscosity to the relaxation time of the shear flux $\eta/\tau_\pi$ by measuring Kubo’s canonical correlation of a spatial component of the energy-momentum tensor for the range of temperature $0.5 \lesssim T/T_c \lesssim 4$ with SU(3) lattice gauge simulation. The canonical correlation diverges in the ultraviolet region and we regularized it by subtracting the vacuum contribution. There, however, exists an extra contribution from the contact terms that remain even after the vacuum subtraction due to their temperature dependence. Using the leading order OPE result for the contact terms and measuring the expectation values of the operators on the lattice, we determined their temperature dependence. Then, we have obtained the temperature dependence of $\eta/\tau_\pi$. The number of the free parameters in relativistic dissipative hydrodynamic models was reduced by this analysis. We also investigated the temperature dependence of the characteristic speed of the transverse plane wave in the same temperature range. We have found that the characteristic speed in the gluon media is $v \simeq 0.5$ for $T \gtrsim 1.5T_c$. The fact that the characteristic speed does not exceed the speed of light within statistical error indicates that relativistic hydrodynamic models based on IS theory retain the causality for the transverse modes in the quark-gluon plasma produced in heavy ion collisions at RHIC and LHC. Note, however, that our calculations are in the quenched approximation, i.e. with no dynamical quarks.
The approach in this work is also applicable to the bulk channel. By repeating the same procedure as in Sections 3 and 4, one can obtain the ratio of the bulk viscosity to the relaxation time of the bulk flux, \( \zeta/\tau \), from Kubo’s canonical correlation of diagonal components of the energy-momentum tensor. This is currently in progress.

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