HIDDEN CORRELATIONS AND ENTANGLEMENT
IN SINGLE-QUDIT STATES

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Abstract

We discuss the notion of hidden correlations in classical and quantum indivisible systems along with such characteristics of the correlations as the mutual information and conditional information corresponding to the entropic subadditivity condition and the entropic strong subadditivity condition. We present an analog of the Bayes formula for systems without subsystems, study entropic inequality for von Neumann entropy and Tsallis entropy of the single-qudit state, and discuss the inequalities for qubit and qutrit states as an example.

Keywords: entanglement, hidden quantum correlations, entropic inequalities, information, qudit.

1 Introduction

The important properties of systems with fluctuations in their physical characteristics, in both classical and quantum domains, like positions, momenta, angular momenta, spin, energy, etc., are connected with correlations in the system degrees of freedom. These correlations are usually associated with the dependence of the behavior of the subsystem’s degrees of freedom on the behavior of the other subsystem in composite systems containing two or more subsystems. In classical domain, this means that one describes the states of a composite (divisible) system by a joint probability distribution of several random variables; these aspects are studied in detail in the probability theory [1, 2]. The notion of entropy is a substantial instrument in the probability theory [3–5]. In quantum domain, this means that one describes the states of such a composite quantum system by the density operator defined in the Hilbert space [6], which is the direct product of Hilbert spaces corresponding to the states of the subsystems, with matrix elements depending on all the indices corresponding to the basis vectors of these Hilbert spaces. The mathematical instruments to describe the correlations of the subsystem degrees of freedom are the Shannon entropy and information in classical probability theory [3] and the von Neumann entropy [7] and information in quantum statistics.

The mutual information introduced for classical systems with two subsystems characterizes the degree of correlations of two subsystems of the composite system [2]. The quantum (von Neumann) information of bipartite system, say, the two-qudit system, provides the characteristics of quantum correlations of...
the qudits. For composite systems with three subsystems, the notion of conditional information is used to describe the degree of correlations of the subsystems in the composite system. The mathematical aspects [8–11] of these characteristics are presented in terms of entropy–information inequalities like the subadditivity condition (bipartite systems) and the strong subadditivity condition (tripartite systems). The strong quantum correlations in multipartite systems (qudits) are determined by the entanglement properties [12,13] of the density matrices.

The main idea of our work is to consider noncomposite (nondivisible) systems, both classical and quantum, and show that there exist analogs of all correlation properties discussed and all entropies mentioned above, along with the entropy–information inequalities, for such systems, as well. For example, the notion of entanglement and properties of the density matrices of single-qudit states correspond to specific correlations available in these systems; we call them the hidden correlations.

It is worth noting that the entanglement of single-qudit states was discussed for particular examples of the qudits in the literature [14–17]; also the notion of generalized entanglement was discussed in [18–20].

In this work, we present a systematic consideration of the entanglement of the single-qudit states. Also we consider hidden correlations in qubits and qutrits, using the probability representation of the density matrices of their states.

Our aim is to discuss hidden correlations available in single-qudit states [21–23]. The mathematical tools we are using in this study are based on applying an invertible map of indices 1, 2, ..., N onto a set of natural indices 1, 2, ..., n_1, 1, 2, ..., n_2, ..., 1, 2, ..., n_M, such that N = n_1 n_2 ... n_M.

The map is realized using the set of “functions detecting the hidden correlations” [24]. After applying the map, the possibility arises to interpret an arbitrary probability distribution of one random variable as a joint probability distribution of M random variables. In the case of quantum states, applying this map, one has the possibility to interpret an arbitrary density matrix of the states in N-dimensional Hilbert space as the density matrix of M qudits. Thus, we extend the notion of different kinds of quantum correlations known for the qudit-system (M qudits) states to the case of the single-qudit state.

We employ the probability description of qubit and qutrit states, where the matrix elements of the state density matrices are explicitly expressed in terms of classical probabilities. Then, in view of these expressions, we obtain new entropy–information inequalities for matrix elements of the qubit and qutrit states.

This paper is organized as follows.

In Sec. 2, we describe the invertible map which provides the possibility to map the density matrix of a single qudit to the density matrix of the multiqudit system. In Sec. 3, we study the entanglement of the single-qudit state. In Sec. 4, we present new entropic inequalities for the qubit and qutrit states. In Sec. 5, we discuss hidden correlations for spin states in the tomographic-probability representation. In Sec. 6, we consider an example of the four-level atom and give the conclusions in Sec. 7.

2 Partition Map

In this section, we describe a map of indices corresponding to the partition procedure; this map was named the map detecting hidden correlations in the system [24].

Given a set of integers y = 1, 2, ..., N. Consider the function 1 ≥ P(y) ≥ 0 satisfying the normalization condition \( \sum_{y=1}^{N} P(y) = 1 \). This function can be interpreted as the probability distribution function of a
random variable $y$. If $N = \prod_{k=1}^{M} X_k$, one can introduce $M$ functions $x_1(y), x_2(y), \ldots, x_M(y)$ of the form

$$x_k(y) - 1 = \frac{y - \left( x_1 + \sum_{i=2}^{k-1} (x_i - 1) \prod_{j=1}^{i-1} X_j \right)}{\prod_{i=1}^{k} X_i} \mod X_k, \quad k = 1, \ldots, M, \quad 1 \leq y \leq N. \quad (1)$$

Here, the variable $y$ is the function of $x_1, x_2, \ldots, x_M$,

$$y = y(x_1, x_2, \ldots, x_M) = x_1 + \sum_{k=2}^{M} (x_k - 1) \prod_{j=1}^{k-1} X_j, \quad 1 \leq x_k \leq X_k, \quad k \in [1, M]. \quad (2)$$

If $M = 2$, the general formulas (1) and (2) provide explicit expressions for $y(x_1, x_2), x_1(y), \text{ and } x_2(y)$ presented as (9) (see also (24))

$$y(x_1, x_2) = x_1 + (x_2 - 1)X_1, \quad 1 \leq x_1 \leq X_1, \quad 1 \leq x_2 \leq X_2, \quad (3)$$

$$x_1(y) = y \mod X_1, \quad 1 \leq y \leq N, \quad (4)$$

$$x_2(y) - 1 = \frac{y - x_1(y)}{X_1} \mod X_2, \quad 1 \leq y \leq N. \quad (5)$$

Functions (1) and (2) provide the possibility to interpret the function $P(y)$ as the joint probability distribution $\Pi(x_1, x_2, \ldots, x_M)$ of $M$ random variables. We define this probability distribution by the equality

$$\Pi(x_1, x_2, \ldots, x_M) \equiv P(y(x_1, x_2, \ldots, x_M)), \quad (6)$$

where the argument of the function $P(y)$ is expressed as the function of $M$ random variables $x_1, x_2, \ldots, x_M$.

Recall that for any joint probability distribution, one can introduce marginal probability distributions; for example, if the integer $s < M$, one has the distribution

$$\mathcal{P}(x_1, x_2, \ldots, x_s) = \sum_{x_{s+1}=1}^{X_{s+1}} \sum_{x_{s+2}=1}^{X_{s+2}} \cdots \sum_{x_M=1}^{X_M} \Pi(x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_M). \quad (7)$$

In the case $M = 2$, one has the joint probability distribution of two random variables $\Pi(x_1, x_2)$ with marginal probability distributions $\mathcal{P}_1(x_1)$ and $\mathcal{P}_2(x_2)$, which read

$$\mathcal{P}_1(x_1) = \sum_{x_2=1}^{X_2} P(y(x_1, x_2)), \quad \mathcal{P}_2(x_2) = \sum_{x_1=1}^{X_1} P(y(x_1, x_2)). \quad (8)$$

Now we consider the conditional probability distribution given by the Bayes formula (see, e.g., (2)[27])

$$p_1(x_1 \mid x_2) = \frac{P(y(x_1, x_2))}{\sum_{x_1=1}^{X_1} P(y(x_1, x_2))}. \quad (9)$$

In view of the map of indices, we can interpret the probability distribution of one random variable $P(y)$ as a joint probability distribution of two random variables $x_1$ and $x_2$ and introduce, along with
marginal probability distributions of one random variable $\mathcal{P}_1(x_1)$ and $\mathcal{P}_2(x_2)$ given by \([8]\), two conditional probability distributions $p_1(x_1 \mid x_2)$ given by \([9]\) with the other one written as

$$p_2(x_2 \mid x_1) = \frac{P(y(x_1, x_2))}{\sum_{x_2=1}^{X_2} P(y(x_1, x_2))}. \quad (10)$$

The generalization to the case $M > 2$ is straightforward. One can introduce the conditional probability distribution of $s < M$ random variables as follows:

$$p(x_1, x_2, \ldots, x_s \mid x_{s+1}, x_{s+2}, \ldots, x_M) = \frac{P(y(x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_M))}{\sum_{x_1=1}^{X_1} \sum_{x_2=1}^{X_2} \cdots \sum_{x_s=1}^{X_s} P(y(x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_M))}. \quad (11)$$

The distributions introduced satisfy all the relationships known for joint probability distributions, marginal probability distributions, and conditional probability distributions.

For example, the subadditivity condition for Shannon entropy of bipartite classical system is expressed as the inequality $S_1 + S_2 \geq S(1, 2)$ for entropies

$$S_1 = - \sum_{x_1=1}^{X_1} \mathcal{P}_1(x_1) \ln \mathcal{P}_1(x_1), \quad S_2 = - \sum_{x_2=1}^{X_2} \mathcal{P}_2(x_2) \ln \mathcal{P}_2(x_2),$$

$$S(1, 2) = - \sum_{x_1=1}^{X_1} \sum_{x_2=1}^{X_2} P(y(x_1, x_2)) \ln P(y(x_1, x_2)); \quad (12)$$

in an explicit form, the inequality reads

$$- \sum_{x_1=1}^{X_1} \left[ \sum_{x_2=1}^{X_2} P(y(x_1, x_2)) \ln \left( \sum_{x_2=1}^{X_2} P(y(x_1, x_2)) \right) \right] - \sum_{x_2=1}^{X_2} \left[ \sum_{x_1=1}^{X_1} P(y(x_1, x_2)) \ln \left( \sum_{x_1=1}^{X_1} P(y(x_1, x_2)) \right) \right]$$

$$\geq - \sum_{x_1=1}^{X_1} \sum_{x_2=1}^{X_2} P(y(x_1, x_2)) \ln P(y(x_1, x_2)). \quad (13)$$

Here, the function $y(x_1, x_2)$ is given by \([3]\).

3 Quantum States

Now we apply the partition map to the density matrix $\rho_{yy'}$ ($y, y' = 1, 2, \ldots, N$) of the single-qudit state either to the $N$ level atom or to the spin-$j$ state, where $N = 2j + 1$. The density matrix $\rho_{yy'}$ is a nonnegative Hermitian matrix $\rho = \rho^\dagger$, $\rho \geq 0$, with unit trace $\text{Tr} \rho = 1$. The nonnegativity of the matrix means that the eigenvalues of the matrix $\lambda_1, \lambda_2, \ldots, \lambda_N$ are nonnegative numbers, i.e., $\lambda_k \geq 0$ ($k = 1, 2, \ldots, N$). The density matrix $\rho_{yy'}$ can be considered as the $N \times N$ matrix $R$ with matrix elements defined by the equality

$$R_{x_1x_2, \ldots, x_M; x'_1x'_2, \ldots, x'_M} = \rho_{y(x_1, x_2, \ldots, x_M), y'(x'_1, x'_2, \ldots, x'_M)} \quad \rho_{yy'}.$$  

(14)
Numerically both matrices are identical, but the interpretation of the matrix $R$ is different from the interpretation of the matrix $\rho$.

The matrix $\rho$ is interpreted as the density matrix of the single-qudit state. The density matrix $R$ is interpreted as the density matrix of the multiqudit state. This fact means that the same numerical matrices $R$ and $\rho$ can be interpreted either as the density matrix of the state of a system without subsystems or as the density matrix of the state of a system with $M$ subsystems.

For a composite system with the density matrix $R$, one has the density matrices of the states of the subsystems; these density matrices are obtained by applying the partial tracing procedure. One can apply this tool to associate with the density matrix $\rho$ of the single-qudit state the density matrix $\rho(1)$ with matrix elements defined by an analogous partial tracing tool, in view of the formula

$$\rho_{x_1,x_2,...,x_s,x'_1,x'_2,...,x'_s}(1) = \sum_{X_{s+1}} \sum_{X_{s+2}} \cdots \sum_{X_M} \rho_g(x_1,x_2,...,x_s,x_{s+1},x_{s+2},...,x_M), y'(x'_1,x'_2,...,x'_s,x_{s+1},x_{s+2},...,x_M).$$

(15)

The density matrix $\rho(1)$ is interpreted as the density matrix of an artificial subsystem state of $s$ qudits.

Analogously, one can introduce the density matrix $\rho(2)$ of an artificial subsystem state of $M - s$ qudits using the definition formulated due to the other partial tracing procedure,

$$\rho_{x_{s+1},x_{s+2},...,x_M,x'_1,x'_2,...,x'_M}(2) = \sum_{X_1} \sum_{X_2} \cdots \sum_{X_s} \rho_g(x_1,x_2,...,x_s,x_{s+1},x_{s+2},...,x_M), y'(x_1,x_2,...,x_s,x'_1,x'_2,...,x'_M).$$

(16)

The von Neumann entropy of the state of a noncomposite system with the density matrix $\rho$, i.e., $S = -\text{Tr} \rho \ln \rho \equiv -\text{Tr} R \ln R$, is a nonnegative number $S \geq 0$.

Using the introduced partition map of indices and the introduced subsystem density matrices of artificial subsystem states, one can define the notion of mutual quantum information $I_q$ using the standard relation for composite systems. In the explicit form, the nonnegative mutual quantum information reads

$$I_q = \text{Tr} (\rho \ln \rho) - \text{Tr} (\rho(1) \ln \rho(1)) - \text{Tr} (\rho(2) \ln \rho(2)).$$

(17)

If in artificial subsystems 1 and 2 associated with the single-qudit state with the density matrix $\rho$, there are no correlations, i.e., $\rho = \rho(1) \otimes \rho(2)$, then the mutual information $I_q = 0$. The difference of information from zero corresponds to the presence of correlations of artificial subsystems 1 and 2 in the single-qudit system. We call these correlations “hidden correlations.”

The notion of entanglement for the single qudit state is defined analogously to the case of the entanglement definition in the multiqudit systems. If the density matrix $\rho$ of the single-qudit state can be presented in the form of convex sum,

$$\rho_g(x_1,x_2,...,x_M), y'(x'_1,x'_2,...,x'_M) = \sum_k p_k \rho^{(k)}_{x_1,x_2,...,x_s,x'_1,x'_2,...,x'_s}(1) \times \rho^{(k)}_{x_{s+1},x_{s+2},...,x_M,x'_{s+1},x'_{s+2},...,x'_M}(2),$$

(18)

where $\rho^{(k)}(1)$ and $\rho^{(k)}(2)$ are density matrices of the artificial subsystem states, the single-qudit state is called the separable state. In [18], $1 \geq p_k \geq 0$ and $\sum_k p_k = 1$. If the density matrix $\rho_{yy'}$ cannot
be presented in the form (18), the single-qudit state is called the entangled state. The entangled state of a single qudit interpreted as a collection of multiqubit subsystems is the state with strong quantum correlations of artificial subsystems. For example, if \( M = 2 \), one has the separability condition for the density matrix \( \rho_{y'y'} \) of the single-qudit state as follows:

\[
\rho_{y(x_1,x_2),y'(x'_1,x'_2)} = \sum_k p_k \rho^{(k)}_{x_1,x_2}(1) \times \rho^{(k)}_{x_2,x'_2}(2),
\]

(19)

For the pure state of a single qudit, the entanglement can be characterized by the entropy of an artificial subsystem state analogously to the entanglement of the pure state of the composite system. One has the parameter called the linear entropy,

\[
E = 1 - \text{Tr}(\rho(2))^2;
\]

it is a number equal to zero if there is no hidden correlations of the artificial subsystems.

### 4 New Entropic Inequalities for the Density Matrix Elements of Qubit and Qutrit States

In this section, we present some examples of new entropic inequalities for density matrices of the qubit and qutrit states using the tomographic probabilities determining the density matrices. We express the density matrix element \( \rho_{1/2,-1/2} \) of the qubit state in terms of probabilities \( p_1 \) and \( p_2 \) of the spin projections \( m = 1/2 \) on the \( x \) and \( y \) axes as follows [28,29]:

\[
\rho_{1/2,-1/2} = (1/2 | \hat{\rho} | -1/2) = p_1 - i p_2 - (1 - i)/2,
\]

(21)

and the matrix element \( \rho_{1/2,1/2} \) of this matrix is expressed in terms of the probability of the spin projection \( (m = 1/2) \) equal to \( p_3 \) on the \( z \) axis as \( \rho_{1/2,1/2} = p_3 \).

Since three classical probability distributions \( (p_1,1-p_1) \), \( (p_2,1-p_2) \), and \( (p_3,1-p_3) \) satisfy the condition of nonnegativity of the relative Shannon [3] and Tsallis [5] entropies, the following entropic inequalities for the matrix elements of the density matrix elements hold:

\[
[(1/2) + \text{Re} \rho_{1/2,-1/2} \ln \left( (1/2) + \text{Re} \rho_{1/2,-1/2} \right) \left( \rho_{1/2,1/2}^{-1} \right)] \geq 0,
\]

(22)

\[
[(1/2) + \text{Re} \rho_{1/2,-1/2} \ln \left( (1/2) + \text{Re} \rho_{1/2,-1/2} \right) \left( (1/2) + \text{Im} \rho_{1/2,-1/2} \right)^{-1}] \geq 0.
\]

(23)

These two inequalities are compatible with the nonnegativity condition of the qubit density matrix.

The qutrit-state density matrix obtained in terms of the nine probabilities, satisfying the constrain \( 1 \geq p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_1^{(3)}, p_2^{(3)}, p_3^{(3)} \geq 0 \) and corresponding to three artificial qubits describing the qutrit state, has the matrix elements [29]

\[
\rho_{11} = p_3^{(2)} + p_3^{(1)} - 1, \quad \rho_{22} = 1 - p_3^{(2)}, \quad \rho_{21} = p_1^{(2)} + i p_2^{(2)} - (i + 1)/2.
\]

(24)
Since the matrix elements are expressed in terms of classical probability distributions, the nonnegativity condition for relative entropy yields new inequalities for the density matrix elements of the qutrit state; one of the inequalities reads

\[(\rho_{11} + \rho_{22}) \ln \left\{ (\rho_{11} + \rho_{22}) \left[ \frac{1}{2} + \text{Re} \rho_{13} \right]^{-1} \right\} + \rho_{33} \ln \left\{ (\rho_{33} + \rho_{22}) \left[ \frac{1}{2} - \text{Re} \rho_{13} \right]^{-1} \right\} \geq 0.\]  

(25)

In view of the nonnegativity condition of the Tsallis relative entropy, we obtain a new inequality for the density matrix elements of the qutrit state; for \(q > 1\), it is

\[-(1 - q)^{-1} \left\{ (\rho_{11} + \rho_{22})^q \left[ \frac{1}{2} + \text{Re} \rho_{13} \right]^{-q} + \rho_{33}^q \left[ \frac{1}{2} - \text{Re} \rho_{13} \right]^{-q} - 1 \right\} \geq 0.\]  

(26)

The entropic inequalities obtained reflect the presence of hidden correlations of artificial qubits associated with the qutrit-state density matrix. (Analogous inequalities can be found for qudit states.) The inequalities are compatible with the inequalities like the nonnegativity condition of the qutrit-state density matrix and the nonnegativity of the von Neumann entropy \(S = -\text{Tr} (\rho \ln \rho) \geq 0\). But the inequalities obtained are new; they can be checked in the experiments, where the qutrit-state tomography provides the reconstruction of the matrix elements of the density matrix, e.g., in superconducting circuits based on Josephson junctions [30–37].

5 Hidden Correlations for Tomographic-Probability Distributions of Spin-\(j\) States

Applying the partition map, we obtain new entropic inequalities for tomographic-probability distribution \(w(m | \vec{n})\) [38] describing the single spin-\(j\) states; here, \(m\) is the spin-\(j\) projection on the direction, given by the unit vector \(\vec{n}\), to be equal \(m\). The tomogram is defined [39, 40] (see also [41, 42]) as the diagonal matrix element of the density matrix \(\rho_{mm}(\vec{n})\), with \(\vec{n} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)\), and the matrix element expressed in terms of the spin-state density operator \(\hat{\rho}\) and the unitary operator \(\hat{u}\) of the irreducible representation of the group \(SU(2)\) by the formula

\[w(m | \vec{n}) = \langle m | \hat{u} \hat{\rho} \hat{u}^\dagger | m \rangle,\]  

(27)

where \(| m \rangle\) is the eigenvector of the spin projection operator \(\hat{J}_z\) on the \(z\) direction, i.e., \(\hat{J}_z | m \rangle = m | m \rangle\), and the operator \(\hat{u}\) matrix elements \(\langle m | \hat{u} | m' \rangle = u_{mm'}\) are the functions of the Euler angles [43].

For spin \(j = 1/2\), the \(2 \times 2\) matrix \(u_{mm'}\) reads

\[u_{mm'}(\varphi, \theta, \psi) = \begin{pmatrix} \cos(\theta/2) e^{i(\psi+\varphi)/2} & \sin(\theta/2) e^{i(\psi-\varphi)/2} \\ -\sin(\theta/2) e^{-i(\psi-\varphi)/2} & \cos(\theta/2) e^{-i(\psi+\varphi)/2} \end{pmatrix}.\]  

(28)

In view of the structure of (27), the tomogram \(w(m | \vec{n})\) depends only on two angles \(\varphi\) and \(\theta\) determining the unit vector \(\vec{n}\).

For given vector \(\vec{n}\), the tomogram is the normalized conditional probability distribution satisfying the equality

\[\sum_{m=-j}^{j} w(m | \vec{n}) = 1, \quad m = -j, -j + 1, \ldots, j - 1, j.\]  

(29)
The tomogram determines the density operator $\hat{\rho}$ of a single qudit. It is worth noting that the symplectic tomography for continuous variables was also introduced \[44, 45\].

To obtain a new relationship for tomogram, we introduce the new notation $w(m \mid \vec{n}) = P(y \mid \vec{n})$, where $y(-j) = 1$, $y(-j + 1) = 2$, $y(j - 1) = N - 1$, and $y(j) = N = (2j + 1)$. In view of this change of the variables, we obtain $m = -j \rightarrow 1, m = -j + 1 \rightarrow 2, \ldots, m = j - 1 \rightarrow N - 1, m = j \rightarrow N$.

We consider the spin tomogram as the probability distribution of one random variable $y$ discussed in the previous sections. In this way, we can obtain new entropic inequalities describing hidden correlations for the single qudit (spin-$j$) system. If spin $j$ of the system is such that $2j + 1 = X_1X_2$, i.e., in previous formulas (2) and (8), we have $M = 2, x_1 = 1, 2, \ldots, X_1$, and $x_2 = 2, \ldots, X_2$, one can apply the partition map and introduce two artificial subsystems corresponding to the probability distributions

$$\mathcal{P}_1(x_1 \mid \vec{n}) = \sum_{x_2=1}^{X_2} w(m \rightarrow y(x_1, x_2) \mid \vec{n}), \quad \mathcal{P}_2(x_2 \mid \vec{n}) = \sum_{x_1=1}^{X_1} w(m \rightarrow y(x_1, x_2) \mid \vec{n}). \quad (30)$$

Using the definition of Tsallis entropy \[5\]

$$S_q^{(1)}(\vec{n}_1) = \frac{1}{1 - q} \left[ \sum_{x_1=1}^{X_1} \mathcal{P}_1^q(x_1 \mid \vec{n}_1) - 1 \right], \quad S_q^{(2)}(\vec{n}_2) = \frac{1}{1 - q} \left[ \sum_{x_2=1}^{X_2} \mathcal{P}_2^q(x_2 \mid \vec{n}_2) - 1 \right],$$

$$S_q(\vec{n}) = \frac{1}{1 - q} \left[ \sum_{x_1=1}^{X_1} \sum_{x_2=1}^{X_2} w^q(m \rightarrow y(x_1, x_2) \mid \vec{n}) - 1 \right] = \frac{1}{1 - q} \left[ \sum_{m=-j}^{j} w(m \mid \vec{n})^q - 1 \right]$$

and the known nonnegativity of information along with the conditions for relative Tsallis entropy, we arrive at new inequalities for spin tomograms of quantum states; they read

$$S_q^{(1)}(\vec{n}) + S_q^{(2)}(\vec{n}) \geq S_q(\vec{n}), \quad (32)$$

$$\frac{1}{q - 1} \left[ \sum_{x_k=1}^{X_k} \mathcal{P}_1^q(x_k \mid \vec{n}_1)\mathcal{P}_2^{1-q}(x_k \mid \vec{n}_2) - 1 \right] \geq 0, \quad k = 1, 2, \quad X_1 = X_2. \quad (33)$$

For $q \rightarrow 1$, the Tsallis entropic relations yield the relations for the Shannon entropy. For example, for $q = 1$, Eq. (32) provides the nonnegativity condition for the mutual tomographic information

$$I(\vec{n}) = -\sum_{x_1=1}^{X_1} \left[ \sum_{x_2=1}^{X_2} w(m \rightarrow y(x_1, x_2) \mid \vec{n}) \right] \ln \left[ \sum_{x_2=1}^{X_2} w(m \rightarrow y(x_1, x_2) \mid \vec{n}) \right]$$

$$-\sum_{x_2=1}^{X_2} \left[ \sum_{x_1=1}^{X_1} w(m \rightarrow y(x_1, x_2) \mid \vec{n}) \right] \ln \left[ \sum_{x_1=1}^{X_1} w(m \rightarrow y(x_1, x_2) \mid \vec{n}) \right]$$

$$+\sum_{x_1=1}^{X_1} \sum_{x_2=1}^{X_2} w(m \rightarrow y(x_1, x_2) \mid \vec{n}) \ln w(m \rightarrow y(x_1, x_2) \mid \vec{n}) \geq 0. \quad (34)$$

If the hidden correlations are not present, information $I(\vec{n}) = 0$, and if information $I(\vec{n})$ is large, the hidden correlations in the state with tomogram $w(m \mid \vec{n})$ are strong.
6 Examples of the Spin-3/2 System and the Four-Level Atom

In this section, we study in detail the hidden correlations in the four-level atomic system; this means that we consider also the particle with spin \( j = 3/2 \). The states of this particle (single qudit) are associated with vectors \( | -3/2 \rangle, | -1/2 \rangle, | 1/2 \rangle, \) and \( | 3/2 \rangle \). The states of the four-level atom are associated with vectors \( | 1 \rangle, | 2 \rangle, | 3 \rangle, \) and \( | 4 \rangle \), which are the energy eigenvectors with energies \( E_1, E_2, E_3, \) and \( E_4 \).

The spin-3/2 states are the eigenstates \( | m \rangle \) of the spin-projection operator \( \hat{J}_z \), i.e., \( \hat{J}_z | m \rangle = m | m \rangle \). The spin projection on the \( z \) axis can take values \( m = -3/2, -1/2, 1/2, 3/2 \). The Hilbert space of the discussed system states is four-dimensional. The density matrix \( \rho_{nn'} \) of any state in this space for the spin-3/2 particle reads

\[
\rho = \begin{pmatrix}
\rho_{-3/2,-3/2} & \rho_{-3/2,-1/2} & \rho_{-3/2,1/2} & \rho_{-3/2,3/2} \\
\rho_{-1/2,-3/2} & \rho_{-1/2,-1/2} & \rho_{-1/2,1/2} & \rho_{-1/2,3/2} \\
\rho_{1/2,-3/2} & \rho_{1/2,-1/2} & \rho_{1/2,1/2} & \rho_{1/2,3/2} \\
\rho_{3/2,-3/2} & \rho_{3/2,-1/2} & \rho_{3/2,1/2} & \rho_{3/2,3/2}
\end{pmatrix}.
\]

(35)

For the four-level atom, the same numerical density matrix \( \rho_{nn'} \) is

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix}.
\]

(36)

Thus, in our formalism of the partition map, we have \( N = 4, M = 2, \) and \( X_1 = X_2 = 2 \), along with the numbers \( x_1 = 1, 2 \) and \( x_2 = 1, 2 \).

These different physical systems (four-level atom and spin-3/2 particle) model the qudit with the same numerical density matrix. The invertible map of the spin-density matrix onto the four-level-atom density matrix uses the change of indices \(-3/2 \rightarrow 1, -1/2 \rightarrow 2, 1/2 \rightarrow 3, \) and \( 3/2 \rightarrow 4 \). Formulas (3)–(5) provide the numerical values for the functions \( x_1(y), x_2(y), \) and \( y(x_1, x_2) \); they are

\[
y(1, 1) = 1, \quad y(2, 1) = 2, \quad y(1, 2) = 3, \quad y(2, 2) = 4,
\]

(37)

\[
x_1(1) = 1, \quad x_1(2) = 2, \quad x_1(3) = 1, \quad x_1(4) = 2,
\]

(38)

\[
x_2(1) = 1, \quad x_2(2) = 1, \quad x_2(3) = 2, \quad x_2(4) = 2.
\]

(39)

Thus, for the four-level-atom density matrix (36) and the same numerical elements, we have the expression \( \rho(y(x_1, x_2)y'(x_1', x_2')) = R_{x_1 x_2, x_1' x_2'} \). The form of this matrix coincides with the density matrix of the two-qubit system: The states of the first and second artificial qubits have the density matrices in terms of matrix elements \( \rho_{nn'} \), namely,

\[
\rho(1) = \begin{pmatrix}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44}
\end{pmatrix}, \quad \rho(2) = \begin{pmatrix}
\rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\
\rho_{21} + \rho_{43} & \rho_{22} + \rho_{44}
\end{pmatrix}.
\]

(40)

The two artificial qubit states determine quantum information

\[
I = -\text{Tr} \rho(1) \ln \rho(1) - \text{Tr} \rho(2) \ln \rho(2) + \text{Tr} \rho \ln \rho;
\]

(41)
it is equal to zero if there is no hidden correlations of artificial qubits in the four-level atom. The entangled
states of the spin-3/2 particle can be presented as $|\psi\rangle = 2^{-1/2}(|3/2\rangle + |-3/2\rangle)$. One can check that
for the two-qubit states, this state is the Bell state. The Bell inequality written for two artificial qubits
in this spin-3/2 state is violated.

7 Conclusions

To conclude, we formulate the main results of our work.

We showed that there exist correlations in noncomposite (nondivisible, both classical and quantum)
systems which are known for multipartite systems. We obtained entropy–information inequalities which
are new relations for the systems without subsystems. We formulated the notion of entanglement for
single-qudit states. In view of the partition map of indices labeling the matrix elements of the density
matrices, we described systematically the results obtained. The new entropic inequalities obtained for
the qubit (22), (23) and qutrit (25), (26) states can be checked in experiments with superconducting
circuits based on Josephson-junction devices.

We showed that the known entropic inequalities which are applied to composite systems, both classical
and quantum, can also be applied to the systems without subsystems. In view of the interpretation of
the density matrix of the noncomposite system as the density matrix of an artificial bipartite system, we
obtained a new entropic inequality (34) for the qudit spin tomogram.

In fact, the approach presented provides the possibility to extend all entropic and information relations
known for for classical and quantum composite systems to the case of the systems without subsystems;
these relations reflect the presence of correlations, either classical or quantum, of the system’s degrees of
freedom. The quantum correlations of the single-qudit states [46–55] can be used for quantum technologies
analogously to the employment of entanglement as a quantum resource.

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