A GLOBAL WELL-POSEDNESS AND ASYMPOTIC DYNAMICS OF THE KINETIC WINFREE EQUATION

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Abstract. We study a global well-posedness and asymptotic dynamics of measure-valued solutions to the kinetic Winfree equation. For this, we introduce a second-order extension of the first-order Winfree model on an extended phase-frequency space. We present the uniform(-in-time) \( \ell_p \)-stability estimate with respect to initial data and the equivalence relation between the original Winfree model and its second-order extension. For this extended model, we present uniform-in-time mean-field limit and large-time behavior of measure-valued solution for the second-order Winfree model. Using stability and asymptotic estimates for the extended model and the equivalence relation, we recover the uniform mean-field limit and large-time asymptotics for the Winfree model.

1. Introduction. Collective behaviors of many-body systems are often observed in our nature, for example, flocking of birds, aggregation of bacteria, herding of sheep and synchronous flashing of fireflies, etc. [1, 3, 5, 9, 12, 20, 24, 29, 30, 35]. Among them, our main interest lies on the synchronization phenomenon appearing in the phase models of pacemaker cells and fireflies, arrays of Josephson junctions in superconductors. Despite of its ubiquity in our nature, its systematic study based

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on model was begun by Arthur Winfree only half-century ago in [34], and after Win-
free’s seminal work, Y. Kuramoto [21, 22] introduced an analytically manageable
model from the space-homogeneous Ginzburg-Landau system with a linear coupling.
Although the Winfree model was the first mathematical model for synchronization,
there were very few literature on its emergent dynamics, compared to the vast lit-
erature for the Kuramoto model (see survey articles [1, 12, 17]). This scarce study
on the Winfree model is mostly due to the absence of conservation laws except for
the total number of oscillators. In fact, this lack of conservation laws causes lots
of mathematical difficulties to analyze the model. In contrast, it presents various
interesting asymptotic patterns such as complete and partial oscillator deaths, the
existence of chimera states, etc. Recently, there were a surge of extensive studies on
the emergent properties of synchronization models from analytical and numerical
view points in [4, 6, 7, 8, 11, 10, 13, 14, 16, 19, 20, 25, 2, 28, 31, 32]. In this paper,
we continue the study begun in a series of papers [16, 19, 27, 28] on the emergent
dynamics of the Winfree model in mesoscopic regime where the number of oscilla-
tors is sufficiently large so that the corresponding kinetic equation can describe the
evolution of macroscopic observables of the large oscillator system.

Next, we briefly discuss the Winfree model and our main questions on it. Let
$\theta_i(t)$ be the phase of the $i$-th Winfree oscillator. Then, the dynamics of $\theta_i$ is
governed by the system of first-order ordinary differential equations (ODEs):

$$
\dot{\theta}_i = \Omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} S(\theta_i)I(\theta_j), \quad t > 0, \quad i = 1, 2, \ldots, N, \tag{1}
$$

where $\Omega_i$ and $\kappa$ are the natural (intrinsic) frequency of the $i$-th oscillator and non-
negative coupling strength, respectively, and $S$ and $I$ denote the sensitivity and influence functions (see Section 2.1 for details). Recently, the first author and his collaborators [16] provided several sufficient frameworks for the complete oscillator death and partial phase-locking in terms of system parameters such as natural frequencies, coupling strength and initial configurations (see Section 2 for details) and derived several quantitative estimates such as the structure of equilibria, collisions in the relaxation dynamics and uniform stability estimates for the Winfree model (1). More precisely, in this paper we are interested in the following issues:

- (Q1): Can we derive the mean-field kinetic equation from the finite-dimensional system (1) uniformly in time?
- (Q2): Is the derived mean-field equation well-posed in a suitable space?
- (Q3): Can the measure-valued solution exhibit asymptotic collective dynamics?

For a large particle system $N \gg 1$, it is well known in kinetic theory that the corresponding kinetic equation can effectively describe the evolution of macroscopic observables such as local mass, momentum and energy densities. Now, we describe the kinetic equation which can be obtained from (1) in large $N$-oscillator limit. Let $f = f(\theta, \Omega, t)$ be the one-oscillator probability density function of the ensemble of Winfree oscillators with natural frequency $\Omega$, at phase $\theta$ at time $t$. Then, via the standard BBGKY hierarchy, the formal mean-field equation can be written as a continuity equation with nonlocal flux:

$$
\partial_t f + \partial_\theta (V[f]f) = 0, \quad (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, \quad t > 0,
$$
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\[ V[f](\theta, \Omega, t) := \Omega + \kappa \int_{T \times \mathbb{R}} S(\theta) I(\theta_*) f(\theta_*, \Omega_*, t) d\Omega_* d\theta_* . \tag{2} \]

Note that the variable \( \Omega \) acts like a parameter. Throughout the paper, we consider a special choice of sensitivity and influence functions:

\[ S(\theta) = -\sin \theta \quad \text{and} \quad I(\theta) = 1 + \cos \theta, \tag{3} \]

which has been employed in physics literature \([3, 2, 31, 32]\). For general pair of sensitivity and influence functions discussed in Section 2.1, the same arguments can be made. Thus, our choice (3) should be understood without loss of generality.

The main results of this paper are two-fold. First, we derive the uniform-in-time mean-field limit in a large coupling strength regime. As mentioned before, the formal BBGKY hierarchy based on the molecular chaos assumption yields the kinetic equation (2) and the rigorous justification of mean-field limit can be made using the particle-in-cell method in any finite-time interval as in the Kuramoto model \([23, 26]\), roughly speaking, empirical measures generated by the solution of ODE system converge to \( f \) in suitably weak sense. Our first result says that the mean-field limit can be made \textit{uniformly in time}. The mean-field limit argument based on the particle-in-cell method yields an estimate

\[ \text{distance between empirical measure and } f \lesssim \frac{e^T}{\sqrt{N}}, \quad t \in [0, T), \ T < \infty. \]

Note that the R.H.S. blows up as \( T \to \infty \). Thus, this result cannot be used in the study of large-time dynamics of the measure-valued solution. Thus, in order to derive uniform mean-field limit, we need to incorporate a new component which uses the exponential relaxation estimate for the Winfree model. More precisely, our main strategy is to use three new components; the second-order extension of the Winfree model, the finite-time mean-field limit result and asymptotic property of the kinetic equation for this extended model in a large coupling regime (see Theorem 3.4). Then we use the equivalence relation between the Winfree model and the second-order extension of it to derive the uniform stability estimate of the kinetic Winfree equation (2) and the emergence of asymptotic phase-locked state in an asymptotic limit. Second, we provide the large-time asymptotics of the measure-valued solution toward a unique equilibrium measure. The difficulty to verify this is that the natural frequency \( \Omega \) is just a parameter, not an independent variable, thus we cannot use Lyapunov functional approach to derive relaxation dynamics of (2) unlike to the kinetic Cucker-Smale equation where the frequency is an independent variable. This is why we introduce the second-order extension of the Winfree model and apply Lyapunov functional approach for the extended kinetic equation to derive the asymptotic estimate of the kinetic Winfree model (2). Then, using the equivalence relation between the extended model and the Winfree model, we derive the asymptotic estimate for the kinetic Winfree equation (2) (see Theorem 3.5).

The rest of this paper is organized as follows. In Section 2, we briefly discuss the Winfree model and review a sufficient framework in \([19]\) for the emergence of the complete oscillator death state (i.e., equilibrium state) for the Winfree model (1) in terms of the initial data, coupling strength and structure conditions of \( S \) and \( I \). For the special choice (3), our sufficient framework covers all initial data confined in \( \theta_{i0} \in (-\pi, \pi) \) as long as the coupling strength is sufficiently large. In Section 3, we summarize our main results on the global existence of measure-valued solutions and their asymptotic dynamics for the kinetic equation (2) in a large-coupling strength...
regime. In Section 4, we present an equivalence relation between Winfree model (1) and its second-order extension. Then we will study the uniform-in-time stability of the solution of (1) with respect to the initial configuration in a large coupling strength regime. In Section 5, we apply the uniform-in-time stability estimate in Section 3 to obtain the uniform-in-time mean-field limit of the Winfree model (1), and show that the kinetic equation (2) has similar dynamical features as the model (1). In Section 6, we provide the proof of Theorem 5.4. Finally, Section 7 is devoted to the brief summary of our main results and some open questions.

Notation: For $Z = (z_1, \ldots, z_N) \in \mathbb{R}^N$, we set

$$
\|Z\|_p := \left( \frac{1}{N} \sum_{i=1}^{N} |z_i|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
$$

2. Preliminaries. In this section, we briefly review the Winfree model (1) and its emergent property in a large coupling strength regime.

2.1. The Winfree model. Let $\theta_i(t)$ be the phase of the $i$-th oscillator. In the absence of mutual couplings between oscillators, the instantaneous phase velocity $\dot{\theta}_i$ is given by the natural (or intrinsic) frequency $\Omega_i$ which is a time-independent random variable with the probability density function $g(\Omega)$:

$$
\dot{\theta}_i = \Omega_i, \quad i = 1, \ldots, N. \tag{4}
$$

On the other hand, in the presence of the mutual couplings, we need to employ the effect of phase couplings in a suitable way. Regarding this, Arthur Winfree proposed the following simplified assumptions:

- (Mean-field effect of neighboring oscillators): For a given test oscillator $i$, the stimulus $I_c$ on the $i$-th oscillator by neighboring field oscillators is given by the average of influences by neighboring oscillators:

$$
I_c(\Theta) := \frac{1}{N} \sum_{j=1}^{N} I(\theta_j).
$$

More precisely, three tacit assumptions are made implicitly. First, the influences of individuals are assumed to be propagated without attenuation over distance, in a time much shorter than the average period of the oscillators. Second, they are additive in their effect, and third, time-delay effects are not considered.

- (Modeling of frequency fluctuation): The frequency perturbation $\hat{\Omega}_i$ of the $i$-th oscillator is proportional to the product of the sensitivity $S(\theta_i)$ and the average stimulus $I_c(\Theta)$:

$$
\hat{\Omega}_i = \kappa S(\theta_i) I_c(\Theta) = \frac{\kappa}{N} \sum_{j=1}^{N} S(\theta_i) I(\theta_j),
$$

where $\kappa$ is a nonnegative constant, and $\hat{\Omega}_i$ is additively added to the natural frequency in (4).

Based on the above assumptions, Arthur Winfree introduced the first synchronization model for limit-cycle oscillators:
\[
\dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} S(\theta_i) I(\theta_j), \quad i = 1, \ldots, N.
\]

Next, we briefly discuss the connection of the Winfree model and the Kuramoto model (5):
\[
\dot{\theta}_i = \Omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{5}
\]

As noted in [34], by expanding \(\sin(\theta_j - \theta_i)\) using the additive law of \(\sin\) function, the Kuramoto model cast into a generalized Winfree model with two pairs of response and influence functions:
\[
\dot{\theta}_i = \Omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} \cos \theta_i \sin \theta_j + \frac{\kappa}{N} \sum_{j=1}^{N} \sin \theta_i \times (-\cos \theta_j).
\]

Then, by setting
\[
(S_1(\theta), I_1(\theta)) := (\cos \theta, \sin \theta), \quad (S_2(\theta), I_2(\theta)) := (\sin \theta, -\cos \theta),
\]

system (5) can be rewritten as a generalized Winfree model:
\[
\dot{\theta}_i = \Omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} S_1(\theta_i) I_1(\theta_j) + \frac{\kappa}{N} \sum_{j=1}^{N} S_2(\theta_i) I_2(\theta_j).
\]

Note that the pairs of sensitivity and influence functions are orthogonal in the standard \(\ell^2\)-inner product in \(\mathbb{R}^2\):
\[
\langle (S_1(\theta), I_1(\theta)), (S_2(\theta), I_2(\theta)) \rangle = 0,
\]

where \(\langle \cdot \rangle\) is the standard \(\ell^2\)-inner product in \(\mathbb{R}^2\).

2.2. **Emergent dynamics.** In this subsection, we review rigorous emergent properties of the Winfree model (1). Note that for sufficiently large coupling strength \(\kappa \gg 1\), it is known in [19] that all oscillators’ frequency becomes to zero, i.e., oscillators do not rotate and stop. This kind of state is called a complete oscillator death state. More rigorous definition will be given in the following definition in terms of rotation number \(\rho_i\):
\[
\rho_i := \lim_{t \to \infty} \frac{\theta_i(t)}{t}, \quad \text{if the R.H.S. exists.}
\]

**Definition 2.1.** Let \(\Theta(t) = (\theta_1(t), \ldots, \theta_N(t))\) be a time-dependent phase vector. Then, \(\Theta(t)\) approach to the complete oscillator death state, if the rotations numbers \(\rho_i\) are zero:
\[
\rho_i = 0, \quad i = 1, \ldots, N. \tag{6}
\]

**Remark 1.** The equilibria of the Winfree model (1) clearly correspond to the complete oscillator state.

Now, we describe a sufficient framework leading to the complete oscillator death. For the emergence of the complete oscillator death (COD) state, we make structural assumptions on the sensitive function and the influence function as follows.

- **(A1):** The sensitivity function \(S\) is periodic, odd, and the influence function \(I\) is periodic and even: for \(\theta \in \mathbb{R}\),
\[
S(\theta + 2\pi) = S(\theta), \quad S(-\theta) = -S(\theta), \quad I(\theta + 2\pi) = I(\theta), \quad I(-\theta) = I(\theta). \tag{7}
\]
• (A2): The sensitivity and influence function satisfy some geometric conditions: there exist positive constants \( \theta_s \) and \( \theta^* \) such that

\[
0 < \theta_s < \theta^* < 2\pi,
\]

and

\[
S \leq 0 \text{ on } [0, \theta^*] \text{ and } S' \leq 0, S'' \geq 0 \text{ on } [0, \theta_s],
\]

\[
I \geq 0, I' \leq 0 \text{ on } [0, \theta^*], \text{ and } I'' \leq 0 \text{ on } [0, \theta_s],
\]

\[
(SI)' < 0 \text{ on } (0, \theta_s), (SI)' > 0 \text{ on } (\theta_s, \theta^*).
\]

Now for a given \( \alpha \in (0, \theta^*) \), consider the following equation on the interval \([0, \theta^*]\):

\[
(SI)(x) = (SI)(\alpha), \quad x \in [0, \theta^*].
\]

From the assumptions 7 and 8 we can have the following property of \( SI \):

\[
(SI)(0) = 0, \quad \theta^* = \arg\min_{0 \leq \theta \leq \theta^*} (SI)(\theta),
\]

\[
(SI)(\theta) < 0 \text{ on } (0, \theta^*), \quad (SI)(\theta^*) \leq 0.
\]

Thus the equation (9) has a unique solution \( \alpha^\infty \):

\[
(SI)(\alpha^\infty) = (SI)(\alpha),
\]

and we set

\[
\kappa_e(\alpha^\infty) := -\frac{\Omega^\infty}{S(\alpha^\infty)I(\alpha^\infty)}, \quad \Omega^\infty = \max_{1 \leq i \leq N} |\Omega_i| \quad \text{and}
\]

\[
\mathcal{R}(\alpha) := \{ \Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N \mid \theta_i \in (-\alpha, \alpha), \quad i = 1, \ldots, N \}.
\]

Now we have the following proposition.

**Proposition 1.** [19] Suppose that the conditions (7) and (8) hold. The parameters \( \alpha \) and \( \kappa \) satisfy

\[
\alpha \in (0, \theta^*), \quad \kappa > \kappa_e(\alpha^\infty),
\]

and let \( \{(\theta_i, \omega_i)\} \) be a solution to (14). Then, the following assertions hold.

1. The set \( \mathcal{R}(\alpha^\infty) \) is positively invariant along the Winfree flow (1):

\[
\Theta_0 \in \mathcal{R}(\alpha^\infty) \implies \Theta(t) \in \mathcal{R}(\alpha^\infty), \quad t \in (0, \infty).
\]

2. There exists \( \beta \in (0, \theta_s) \) and \( t_* \geq 0 \), such that

\[
\theta(t) \in \mathcal{R}(\beta) \text{ for } t > t_*.
\]

**Remark 2.** For the choice (3), we can choose

\[
\theta_s = \frac{\pi}{3}, \quad \theta^* = \pi, \quad \alpha \in (0, \pi), \quad \alpha^\infty \in (0, \frac{\pi}{3}).
\]

Next, we recall the result on the emergence of complete oscillator death as follows.

**Theorem 2.2.** [19] Suppose that the conditions (7) and (8) hold, and for \( \alpha \in (0, \theta^*) \), let \( \Theta = \Theta(t) \) be a global smooth solution for the system (1), satisfying

\[
\Theta_0 \in \mathcal{R}(\alpha) \quad \text{and} \quad \kappa > \kappa_e(\alpha^\infty).
\]

Then, \( \Theta(t) \) converges to a unique equilibrium state \( \Phi = (\phi_1, \ldots, \phi_N) \) in the region \( \mathcal{R}(\alpha^\infty) \):

\[
\Omega_i + \frac{\kappa}{N} S(\phi_i) \sum_{j=1}^{N} I(\phi_j) = 0, \quad \lim_{t \to \infty} \Theta(t) = \Phi.
\]
3. **Description of main results.** In this section, we present our main results on the uniform mean-field limit from system (1) to the Vlasov type equation (2), and study the large-time behavior of measure-valued solution. For the simplicity of presentation, we take the particular pair of sensitivity and influence functions (3). However, our argument also works for the general ansatz satisfying the conditions (7) - (9).

Consider the Winfree model:
\[
\dot{\theta}_i = \Omega_i - \frac{\kappa}{N} \sin \theta_i \sum_{j=1}^{N} (1 + \cos \theta_j).
\] (10)

Below, we present a measure-theoretic framework and introduce the concept of measure-valued solutions and present main results on the global existence of measure-valued solution and its asymptotic behavior.

### 3.1. A measure-theoretic framework.

Let \( M(T \times \mathbb{R}) \) be the set of nonnegative Radon measures and for \( \nu \in M(T \times \mathbb{R}) \), we set
\[
\langle \nu, h \rangle := \int_{T \times \mathbb{R}} h(\theta, \Omega) \nu(d\theta, d\Omega), \quad h \in C^0_0(T \times \mathbb{R}),
\]
where \( C^0_0 \) denotes the set of continuous functions vanishing at infinity. Then we denote the space of all weakly continuous time dependent measures by \( C^w(T \times \mathbb{R}) \), i.e., for any \( \mu_t \in C^w([0, T); M(T \times \mathbb{R})) \) we have
\[
\langle \mu_t, h \rangle \text{ is continuous as a function of } t, \quad \forall h \in C^0_0(T \times \mathbb{R}).
\]

Next, we present a definition of measure-valued solution to (2) as follows.

**Definition 3.1.** [18, 23] For \( T \in [0, \infty) \), \( \mu_t \in L^\infty([0, T); \mathcal{P}(T \times \mathbb{R})) \) is a measure-valued solution to (2) with initial datum \( \mu_0 \in \mathcal{P}(T \times \mathbb{R}) \) if the following three conditions hold:

1. Total mass is normalized: \( \langle \mu_t, 1 \rangle = 1 \).
2. \( \mu \) is weakly continuous in \( t \):
   \[
   \langle \mu_t, f \rangle \text{ is continuous in } t, \quad \forall f \in C^0_0(T \times \mathbb{R} \times [0, T]).
   \]
3. \( \mu \) satisfies the equation (2) in a weak sense: for \( \forall \varphi \in C^1_0(T \times \mathbb{R} \times [0, T]) \),
   \[
   \langle \mu_t, \varphi(\cdot, \cdot, t) \rangle - \langle \mu_0, \varphi(\cdot, \cdot, 0) \rangle = \int_0^t \left< \mu_s, \partial_s \varphi + V[f] \partial_\theta \varphi \right> ds,
   \]
   where \( \varphi \) is a test function in \( C^0_0(T \times \mathbb{R} \times [0, T]) \) and
   \[
   V[f](\theta, \Omega, t) := \Omega - \kappa \sin \theta \int_{T \times \mathbb{R}} (\cos \theta_\ast + 1) f(\theta_\ast, \Omega_\ast, t) d\theta_\ast d\Omega_\ast.
   \]

**Remark 3.** Note that for a solution \{\( \theta_i \)\} to (1), the empirical measure \( \mu^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\theta_i} \), is a measure-valued solution in the sense of Definition 3.1. Thus, ODE solution to (1) can be understood as a measure-valued solution for (2). Hence, we can treat the Winfree model and its mean-field equation in a common framework. Likewise, the classical solution is also a measure-valued solution.
By the same arguments as in [23], we can derive a mean-field limit in any finite-time interval. First, we consider the vector field:

$$V[\mu] := \left( \Omega - \kappa \int_{\mathbb{T} \times \mathbb{R}} \sin \theta (1 + \cos \theta_s) \mu_s (d\theta_s, d\Omega_s) \right).$$  \hspace{1cm} (11)

**Theorem 3.2.** [23] Suppose that $V[\mu]$ in (11) satisfies the following two conditions:

1. $V[\mu]$ is a continuous function on $\mathbb{T} \times \mathbb{R} \times [0, T]$ and satisfies the uniform Lipschitz continuity:
   $$\|V[\mu](\theta_1, \Omega_1, t) - V[\mu](\theta_2, \Omega_2, t)\| \leq L \| (\theta_1, \Omega_1) - (\theta_2, \Omega_2) \|$$
   for all $t \in [0, T]$, $(\theta_1, \Omega_1)$, $(\theta_2, \Omega_2)$ $\in$ $\mathbb{T} \times \mathbb{R} \times [0, T]$ and $\mu \in \mathcal{C}_w$.

2. For each fixed $(\theta, \Omega)$ and $t$, $V[\mu]$ as a function on $\mathcal{C}_w$ is also Lipschitz continuous:
   $$\|V[\mu](\theta, \Omega, t) - V[\mu](\theta, \Omega, t)\| \leq M d(\mu, \nu).$$

Then, we have the following assertions:

1. For every $\mu_0$ $\in$ $\mathcal{M}$, the kinetic equation (2) has a unique measure-valued solution $\mu$ $\in$ $\mathcal{C}_w$.

2. If $\mu_0$ is absolutely continuous with density $f_0$ $\in$ $L^1(\mathbb{T} \times \mathbb{R})$, then $\mu_t$ is also absolutely continuous, and its density $f(\cdot, \cdot, t)$ is a weak solution to
   $$\partial_t f + \text{div}(V[f]f) = 0, \quad f(\theta, \Omega, 0) = f_0(\theta, \Omega).$$

3. If $\mu$, $\nu$ $\in$ $\mathcal{C}_w$ are the corresponding measure-valued solutions to equation (2) with initial data $\mu_0$, $\nu_0$ $\in$ $\mathcal{M}$, then we have
   $$d(\mu_t, \nu_t) \leq e^{(L+M)t} d(\mu_0, \nu_0), \quad t \in [0, T),$$
   where $L$ and $M$ are Lipschitz constants in conditions.

**Remark 4.** The second result of Theorem 3.2 provides a weak solution of the kinetic Winfree model (2). Moreover, if $\mu_0$ is absolutely continuous and the correspondent initial density $\rho_0$ and $S(\theta)$, $I(\theta)$ are sufficiently smooth (say, $C^1$), then the weak solution $f$ of equation (2) is smooth.

**3.2. Main results.** Before we state our main results, we recall Wasserstein metric on Radon measures to measure the distance between the solutions of (1) and (2) by equipping a metric to the probability measure space $\mathcal{P}(\mathbb{R} \times \mathbb{R})$, and the concept of local-in-time mean-field limit. In fact, we can endow Wasserstein $p$-distance $W_p$ in the space $\mathcal{P}(\mathbb{T} \times \mathbb{R})$ as follows.

**Definition 3.3.** [26, 33]

1. For $p \in \mathbb{N}$, let $\mathcal{P}_p(\mathbb{T} \times \mathbb{R})$ be a collection of all probability measures with finite $p$-th moment: for some $z_0$ $\in$ $\mathbb{T} \times \mathbb{R}$
   $$\langle \mu, d(z, z_0)^p \rangle < +\infty.$$  

   Then, Wasserstein $p$-distance $W_p(\mu, \nu)$ is defined for any $\mu, \nu$ $\in$ $\mathcal{P}_p(\mathbb{T} \times \mathbb{R})$ as
   $$W_p(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \left( \int_{\mathbb{T}^2 \times \mathbb{R}^2} d(z, z^*)^p \, d\gamma(z, z^*) \right)^{\frac{1}{p}},$$
   where $\Gamma(\mu, \nu)$ denotes the collection of all probability measures on $\mathbb{T}^2 \times \mathbb{R}^2$ with marginals $\mu$ and $\nu$.

2. If $\lim_{p \to \infty} W_p(\mu, \nu)$ exists, then we define $W_\infty(\mu, \nu)$ metric as the limit.
3. For any $T \in (0, \infty]$, the kinetic equation (2) is derivable from the phase model (1) in $[0, T)$, or equivalent to say the mean-field limit from the oscillator system (1) to the kinetic equation (2), which is valid in $[0, T)$, if for every solution $\mu_t$ of the kinetic equation (2) with initial data $\mu_0$, the following condition holds: for some $p \in \mathbb{N}$ and $t \in [0, T)$,
\[
\lim_{N \to +\infty} W_p(\mu^N_0, \mu_0) = 0 \iff \lim_{N \to +\infty} W_p(\mu^N_t, \mu_t) = 0,
\]
where $\mu^N_t$ is a measure-valued solution of the phase model (1) with initial datum $\mu^N_0$.

Remark 5. For a later use, we quote a result on the approximation of a measure by empirical measures and mean-field limit in any finite time interval without a proof. The construction of the approximation can be followed by the method of Theorem 6.18 in the book [33] by finding a sequence of atomic measures $\sum_{j=1}^N a_j \delta_j$ with rational numbers $a_j$ such that $\sum_{j=1}^N a_j = 1$.

In the following, we will assume $d(x, y) := ||x - y||_p$, and we are ready to state our two results on the uniform mean-field limit and asymptotic behavior of measure-valued solution to (2).

**Theorem 3.4.** Suppose that $\alpha, \kappa$ and initial data $\mu_0$ satisfy
\[
\alpha \in (0, \pi), \quad \kappa > \kappa_c(\alpha^\infty), \quad \int_{\mathbb{T} \times \mathbb{R}} (1 + |\theta|^p + |\Omega|^p) \mu_0(d\theta, d\Omega) < \infty, \quad \text{for some } p \in \mathbb{N}.
\]
(12)

Then, the following assertions hold.

1. There exists a unique measure-valued solution $\mu \in L^\infty([0, \infty); \mathcal{P}(\mathbb{T} \times \mathbb{R}))$ to (2) with initial data $\mu_0$: $\mu_t$ is approximated by empirical measure $\mu^N_t$ in Wasserstein-$p$ distance uniformly in time:
\[
\lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_p(\mu^N_t, \mu_t) = 0,
\]
where $\mu^N_t$ is the solution with initial data $\mu^N_0$ and $\lim_{N \to +\infty} W_p(\mu^N_0, \mu_0) = 0$.

2. If $\nu_t$ is another measure-valued solution to (2) with initial datum $\mu_0$ satisfying the conditions (12), there exists a nonnegative constant $G_0$ independent of $t$ such that
\[
W_p(\mu_t, \nu_t) \leq G_0 W_p(\mu_0, \nu_0), \quad t \in [0, \infty).
\]

**Theorem 3.5.** Suppose that $\alpha, \kappa$ and initial data $\mu_0$ satisfy the relations (12), and let $f$ be a measure-valued solution to (2) whose existence is guaranteed by Theorem 3.4. Then, there exists positive constants $G_1$ and $\Lambda$ such that
\[
W_p(\mu_t, \mu_\infty) \leq G_1 e^{-\kappa \Lambda t}, \quad p \in \mathbb{N},
\]
where $\Lambda := p \kappa(\cos \beta + \cos 2\beta)$ is a positive constant, and $\beta$ is the value appearing in Proposition 1.
4. **A second-order extension of the Winfree model.** In this section, we present a second-order extension of system (1), and study its emergent dynamics and uniform $\ell_0$-stability with respect to initial data. We also use the duality relation between the Cauchy problems for the first-order model and second-order model and emergence and stability estimates for the second-order model, we derive corresponding results for the first-order model as a direct application of the results for the second-order model.

4.1. **A second-order formulation.** Consider Cauchy problem for the first-order Winfree model:

\[
\begin{aligned}
\dot{\theta}_i &= \Omega_i - \frac{\kappa}{N} \sin \theta_i \sum_{j=1}^{N} (1 + \cos \theta_j), \\
\theta_i(0) &= \theta_{i0}, \quad i = 1, \cdots, N, \ t > 0.
\end{aligned}
\]  

(13)

To embed the first-order system (13) into the second-order model, we introduce a new frequency variable $\omega_i = \dot{\theta}_i$, and differentiate (13) and obtain a second-order model:

\[
\begin{aligned}
\dot{\theta}_i &= \omega_i, \quad i = 1, \cdots, N, \ t > 0, \\
\dot{\omega}_i &= -\frac{\kappa}{N} \left[ \cos \theta_i \left( \sum_{j=1}^{N} 1 + \cos \theta_j \right) \omega_i + \sin \theta_i \sum_{j=1}^{N} (-\sin \theta_j) \omega_j \right], \\
(\theta_i, \omega_i)(0) &= (\theta_{i0}, \omega_{i0}).
\end{aligned}
\]  

(14)

Next, we discuss the equivalence relation between the first-order model (13) and the second-order model (14) on a submanifold.

- (From first-order model to second-order model): Let $\Theta(t)$ be a solution of the first-order model (13) with initial data $\Theta_0$. Then, we set

\[
\omega_{i0} := \Omega_i - \frac{\kappa}{N} \sin \theta_{i0} \sum_{j=1}^{N} (1 + \cos \theta_{j0}), \quad \omega_i := \dot{\theta}_i, \quad i = 1, \cdots, N.
\]

Then, it is easy to see that $(\theta_i(t), \omega_i(t))$ satisfies (14).

- (From second-order model to first-order model): Suppose that we have a solution $\{(\theta_i(t), \omega_i(t))\}$ of system (14) with initial data $\{(\theta_{i0}, \omega_{i0})\}$. Then, we integrate the equation for $\omega_i$ to obtain

\[
\begin{aligned}
\omega_i(t) &= \omega_{i0} - \frac{\kappa}{N} \int_0^t \left[ \cos \theta_i \left( \sum_{j=1}^{N} 1 + \cos \theta_j \right) \omega_i + \sin \theta_i \sum_{j=1}^{N} (-\sin \theta_j) \omega_j \right] ds \\
&= \omega_{i0} + \frac{\kappa}{N} \sin \theta_{i0} \sum_{j=1}^{N} (1 + \cos \theta_{j0}) - \frac{\kappa}{N} \sin \theta_i(t) \sum_{j=1}^{N} (1 + \cos \theta_j(t)).
\end{aligned}
\]  

(15)

Thus, we use the relation $\omega_i = \dot{\theta}_i$ and (15) to see that $\theta_i$ satisfies the first-order Winfree model with natural frequency given by the initial data $\{(\theta_{i0}, \omega_{i0})\)
\[
\begin{cases}
\dot{\theta}_i = \Omega_i - \frac{\kappa}{N} \sin \theta_i \sum_{j=1}^{N} (1 + \cos \theta_j), \\
\Omega_i := \omega_{i0} + \frac{\kappa}{N} \sin \theta_{i0} \sum_{j=1}^{N} (1 + \cos \theta_{j0}), \\
\theta_i(0) = \theta_{i0}
\end{cases}
\tag{16}
\]

Hence, we can realize the first-order model (13) as a second-order model (14) on a special submanifold.

4.2. **Emergence of phase-locked states.** In this subsection, we study emergent dynamics for the Cauchy problem (14). The strategy for this can be summarized as follows: Let initial data \{((\theta_{i0}, \omega_{i0})\} be given.

- **(Step A):** We solve the system of second-order model (14) with given initial data, and obtain a solution \{((\theta_i, \omega_i)\}.

- **(Step B):** By the duality relation mentioned in Section 4.1, \(\theta_i\) is a solution to the Cauchy problem (13) with the natural frequency \(\Omega_i\):

\[
\Omega_i := \omega_{i0} + \frac{\kappa}{N} \sin \theta_{i0} \sum_{j=1}^{N} (1 + \cos \theta_{j0}).
\]

- **(Step C):** We use the emergent estimate for the Cauchy problem (13) with natural frequency defined in Step B to derive the estimate for (13).

By following the discussion in Section 2 for the first-order Winfree model, we consider the following trigonometric equation on the interval \([0, \frac{\pi}{3}]\):

\[
\sin \theta (\cos \theta + 1) = \sin \alpha (\cos \alpha + 1), \quad \theta \in [0, \frac{\pi}{3}]. \tag{17}
\]

From the properties of trigonometric functions, we can have the following property:

\[
\sin \theta (\cos \theta + 1) \big|_{\theta = 0} = 0, \quad \frac{\pi}{3} = \arg \min_{0 \leq \theta \leq \pi} (-\sin \theta (\cos \theta + 1)),
\]

\[- \sin \theta (\cos \theta + 1) < 0 \quad \text{on} \quad \theta \in (0, \pi), \quad \sin \theta (\cos \theta + 1) \big|_{\theta = \pi} = 0.
\]

Thus the equation (17) has a unique solution \(\alpha^\infty\). For such \(\alpha^\infty\), we denote

\[
\kappa_e(\alpha^\infty) := \frac{\Omega^\infty}{\sin \alpha^\infty (\cos \alpha^\infty + 1)},
\]

\[
\mathcal{R}(\alpha) := \{\Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N \mid \theta_i \in (-\alpha, \alpha), \ i = 1, \ldots, N\}.
\]

As a direct application of Proposition 1 to (16), we obtain the following estimate.

**Lemma 4.1.** Suppose that \(\alpha\) and \(\kappa\) satisfy the following relations:

\[
\alpha \in (0, \pi) \quad \text{and} \quad \kappa > \kappa_e(\alpha^\infty).
\]

Then, the following assertions hold.

1. The set \(\mathcal{R}(\alpha^\infty)\) is positively invariant along the flow (14) in the sense that

\[
\Theta_0 \in \mathcal{R}(\alpha^\infty) \quad \Rightarrow \quad \Theta(t) \in \mathcal{R}(\alpha^\infty), \quad t \in (0, \infty).
\]

2. There exists \(\tilde{\beta} \in (0, \frac{\pi}{3})\) and \(t_* \geq 0\) independent of \(N\) such that

\[
\Theta(t) \in \mathcal{R}(\tilde{\beta}) \quad \text{for} \quad t > t_*.
\]

**Proof.** The proof follows from the result of Proposition 1 to the system (16). \(\square\)
For notational simplicity, we set a frequency vector \( W \):
\[
W := (\omega_1, \cdots, \omega_N).
\]

**Proposition 2.** (Synchronization estimate) Suppose that \( \alpha \) and \( \kappa \) satisfy the following relations:
\[
\alpha \in (0, \pi) \quad \text{and} \quad \kappa > \kappa_c(\alpha^\infty).
\]
Then, for any solution \( \{ (\theta_i, \omega_i) \} \) to (14) and for \( p \in \mathbb{N} \), there exist positive constants \( t_* \) and \( \Lambda := pk(\cos \beta + \cos 2\beta) \) such that
\[
||W(t)||_p \leq e^{-\kappa \Lambda(t-t_*)}||W(t_*)||_p, \quad t \geq t_* \quad i = 1, \cdots, N.
\]

**Proof.** Let \( \{ (\theta_i, \omega_i) \} \) be a solution to the Cauchy problem (14). Then, we multiply \( 2\omega_i \) to (14) and sum it over all \( i \) to obtain
\[
\frac{d}{dt} \sum_{i=1}^{N} |\omega_i|^p = \sum_{i=1}^{N} p \omega_i^{p-2} \frac{d}{dt} |\omega_i|^2 = p \sum_{i=1}^{N} |\omega_i|^{p-2} \omega_i \frac{d}{dt} \omega_i
\]
\[
= -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i|^{p-2} \left[ \cos \theta_i(1 + \cos \theta_j)|\omega_i|^2 - \sin \theta_i \sin \theta_j |\omega_i| \omega_j \right]
\]
\[
\leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i|^{p-2} \left[ \cos \theta_i(1 + \cos \theta_j)|\omega_i|^2 - \sin \theta_i \sin \theta_j |\omega_i| \omega_j \right]
\]
\[
= -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i|^p \left[ \cos \theta_i(1 + \cos \theta_j) - |\sin \theta_i \sin \theta_j| \right]
\]
\[
- \frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i|^{p-1} |\sin \theta_i \sin \theta_j| \left[ |\omega_i| - |\omega_j| \right].
\]
\[(18)\]

We exchange the indices \( i \) and \( j \) in the second summation to attain
\[
\frac{d}{dt} \sum_{i=1}^{N} |\omega_i|^p \leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i|^p \left[ \cos \theta_i(1 + \cos \theta_j) - |\sin \theta_i \sin \theta_j| \right]
\]
\[
- \frac{pk}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\sin \theta_i \sin \theta_j| (|\omega_i| - |\omega_j|) (|\omega_i|^{p-1} - |\omega_j|^{p-1}).
\]
\[(19)\]

Since \( (|\omega_i| - |\omega_j|) (|\omega_i|^{p-1} - |\omega_j|^{p-1}) \geq 0 \) for all \( i \) and \( j \), the relation (19) implies
\[
\frac{d}{dt} \sum_{i=1}^{N} |\omega_i|^p \leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i|^p \left[ \cos \theta_i(1 + \cos \theta_j) - |\sin \theta_i \sin \theta_j| \right]
\]
\[
:= -pk \sum_{i=1}^{N} A_i |\omega_i|^p,
\]
\[(20)\]

where the time depending coefficient \( A_i \) is given by the following relation:
\[
A_i(t) := \frac{1}{N} \sum_{j=1}^{N} \left[ \cos \theta_i(t) + \cos \theta_i(t) \cos \theta_j(t) - |\sin \theta_i(t) \sin \theta_j(t)| \right].
\]
Then, the term $A_i$ can be estimated as follows. We use
\[
\cos \theta_i \cos \theta_j - |\sin \theta_i \sin \theta_j| = \begin{cases} 
\cos(\theta_i + \theta_j) & \text{if } \sin \theta_i \sin \theta_j \geq 0, \\
\cos(\theta_i - \theta_j) & \text{if } \sin \theta_i \sin \theta_j < 0.
\end{cases}
\]
and the result in Lemma 4.1:
\[
\Theta(t) \in \mathcal{R}(\hat{\beta}), \quad t \geq t_* \quad \text{for all } i = 1, \ldots, N,
\]
we obtain
\[
\cos \frac{\pi}{3} < \cos \hat{\beta} \leq \cos \theta_i \leq 1, \quad \cos \frac{2\pi}{3} < \cos 2\hat{\beta} \leq \cos(\theta_i + \theta_j) \leq 1,
\]
\[
\cos \frac{2\pi}{3} < \cos 2\hat{\beta} \leq \cos(\theta_i - \theta_j) \leq 1, \quad t \geq t_*.
\]
These yield
\[
A_i(t) \geq \cos \hat{\beta} + \cos 2\hat{\beta}, \quad t \geq t_*, \quad i = 1, \ldots, N. \tag{21}
\]
In (18), we combine (20) and (21) to find
\[
\frac{d}{dt} \sum_{i=1}^{N} |\omega_i|^p \leq -p\kappa(\cos \hat{\beta} + \cos 2\hat{\beta}) \sum_{i=1}^{N} |\omega_i|^p =: -\kappa \Lambda \sum_{i=1}^{N} |\omega_i|^p, \quad t \geq t_*,
\]
which yields the desired estimate. \hfill \Box

As a direct application of the result in Proposition 2, we can find the unique asymptotic phase locked state which emerges from the initial data \{$(\theta_{i0}, \omega_{i0})$\}. Moreover, \(\Theta(t)\) will tend to \(\Theta_\infty\) exponentially fast.

**Corollary 1.** (Emergence of COD state) Suppose that \(\alpha\) and \(\kappa\) satisfy the following relations:
\[
\alpha \in (0, \pi) \quad \text{and} \quad \kappa > \kappa_c(\alpha^\infty),
\]
and let \{$(\theta_i, \omega_i)$\} be a solution to (14). Then, there exists a unique COD state \(\Theta^\infty := (\theta_1^\infty, \ldots, \theta_N^\infty)\) such that
\[
|\theta_i(t) - \theta_i^\infty| \leq C e^{-\kappa \Lambda^* (t - t_*)}, \quad t \geq t_*, \quad i = 1, \ldots, N. \tag{22}
\]
**Proof.** Let \{$(\theta_i, \omega_i)$\} be a solution to system (14) with initial data \{$(\theta_{i0}, \omega_{i0})$\}. Then, for \(i = 1, \ldots, N\), we apply Proposition 2 to obtain
\[
|\theta_i(\hat{t}) - \theta_i(t)| = \left| \int_{\hat{t}}^{t} \omega_i(s) \, ds \right| \leq e^{-\kappa \Lambda^* \|W(t_*)\|_p} \int_{\hat{t}}^{t} e^{-\kappa \Lambda^* s} \, ds \leq \frac{e^{-\kappa \Lambda^* \|W(t_*)\|_p}}{\kappa \Lambda} \left( e^{-\kappa \Lambda t} - e^{-\kappa \Lambda \hat{t}} \right). \tag{23}
\]
Then for any \(\varepsilon > 0\), we can find a positive time \(M\) such that if \(\hat{t} \geq M\) and \(t \geq M\), then
\[
|\theta_i(\hat{t}) - \theta_i(t)| < \varepsilon.
\]
This immediately implies that there exists a unique asymptotic limit \(\theta^\infty_i\) such that
\[
\lim_{\hat{t} \to \infty} \theta_i(\hat{t}) = \theta^\infty_i. \tag{24}
\]
In (23), we let \(\hat{t} \to \infty\) and use the relation (24) to derive the estimate:
\[
|\theta_i(t) - \theta^\infty_i| \leq \frac{e^{-\kappa \Lambda^* \|W(t_*)\|_p}}{\kappa \Lambda} e^{-\kappa \Lambda t}, \quad t \geq t_*. \tag{22}
\]
We set \(C = \frac{\|W(t_*)\|_p}{\kappa \Lambda}\) to derive the desired estimate (22). \hfill \Box
4.3. Uniform $\ell_p$-stability. In this subsection, we study the uniform $\ell_p$-stability of (14). Before we provide stability results, we recall two lemmas.

**Lemma 4.2.** [15] Suppose that two nonnegative Lipschitz functions $X$ and $V$ satisfy the coupled differential inequalities:

$$
\left| \frac{dX}{dt} \right| \leq V, \quad \frac{dV}{dt} \leq -\alpha V + \gamma e^{-\alpha t} X, \quad \text{a.e. } t > 0,
$$

where $\alpha$ and $\gamma$ are positive constants. Then, $X$ and $V$ satisfy the uniform bound and decay estimates:

$$
X(t) \leq \frac{2M}{\alpha} (X(0) + V(0)), \quad V(t) \leq M(X(0) + V(0)) e^{-\frac{\alpha t}{2}}, \quad t \geq 0,
$$

where $M$ is given by,

$$
M := \max \left\{ 1, \frac{2\gamma}{\alpha e} \right\} + \frac{8\gamma}{\alpha^3 e^3}.
$$

**Proof.** We refer to [15] for a detailed proof. \qed

**Remark 6.** The result of Lemma 4.2 implies

$$
\sup_{0 \leq t < \infty} (X(t) + V(t)) \leq M \left( \frac{2}{\alpha} + 1 \right) (X(0) + V(0)).
$$

(25)

**Lemma 4.3.** Suppose that $\alpha$ and $\kappa$ satisfy the following relations:

$$
\alpha \in (0, \pi) \quad \text{and} \quad \kappa > \kappa_c(\alpha^\infty).
$$

Let $\{ (\theta_i, \omega_i) \}$ and $\{ (\tilde{\theta}_i, \tilde{\omega}_i) \}$ be two solutions of system (14), respectively. Then, for $p \in \mathbb{N}$ we have

$$
\frac{d}{dt} \| \Theta - \tilde{\Theta} \|_p \leq \| W - \tilde{W} \|_p, \quad \text{a.e., } t > 0,
$$

$$
\frac{d}{dt} \| W - \tilde{W} \|_p \leq -\kappa \Lambda \| W - \tilde{W} \|_p + 10\kappa \| \tilde{W}(t_*) \|_p e^{-\kappa \Lambda (t-t_*)} \| \Theta - \tilde{\Theta} \|_p.
$$

(26)

**Proof.** (i) The estimate for the first differential inequality in (26) follows trivially.

(ii) It follows from Lemma 4.1 (ii) that all oscillators enter the invariant set $\mathcal{R}(\tilde{\beta})$ after finite time $t_*$. Thus, without loss of generality, we may assume that the initial data has compact support in the invariant set initially, i.e., $t_* = 0$. By direct calculation, we have

$$
\sum_{i=1}^{N} \frac{d}{dt} |\omega_i - \tilde{\omega}_i|^p
$$

$$
= \frac{p}{N} \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-2} (\omega_i - \tilde{\omega}_i) \frac{d}{dt} (\omega_i - \tilde{\omega}_i)
$$

$$
= \frac{-p\kappa}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-2} (\omega_i - \tilde{\omega}_i)
$$

$$
\times \left[ \cos \theta_i (1 + \cos \theta_j) \omega_j - \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) \tilde{\omega}_j - \sin \theta_i \sin \theta_j \omega_j + \sin \tilde{\theta}_i \sin \tilde{\theta}_j \tilde{\omega}_j \right]
$$

$$
= \frac{-p\kappa}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-2} (\omega_i - \tilde{\omega}_i)
$$
Below, we estimate the term $\mathcal{I}_{1i}$, $i = 1, \ldots, 4$.

**Case A.1 (Estimate of $\mathcal{I}_{11} + \mathcal{I}_{13}$):** By direct estimate, we have

$$
\mathcal{I}_{11} + \mathcal{I}_{13} = -\frac{p_k}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \omega_i - \tilde{\omega}_i \right|^{p-2} (\omega_i - \tilde{\omega}_i) \\
\times \left[ \left( \cos \theta_i (1 + \cos \theta_j) - \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) \right) \omega_i + \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) (\omega_i - \tilde{\omega}_i) \right] \\
- (\sin \theta_i \sin \theta_j - \sin \tilde{\theta}_i \sin \tilde{\theta}_j) \omega_j - \sin \tilde{\theta}_i \sin \tilde{\theta}_j (\omega_j - \tilde{\omega}_j) \right]
$$

(27)

On the other hand, note that

$$
|\cos \theta_i - \cos \tilde{\theta}_i| \leq |\theta_i - \tilde{\theta}_i| \quad \text{and} \quad |\sin \theta_i - \sin \tilde{\theta}_i| \leq |\theta_i - \tilde{\theta}_i|.
$$

It follows from Proposition 2 that we have

$$
|\omega_i(t)| \leq \|W(t)\|_p \leq \|W_0\|_p e^{-\kappa M}.
$$

In (27), we use the above estimate to get

$$
\mathcal{I}_{11} + \mathcal{I}_{13} \leq p_k \|W_0\|_p e^{-\kappa M} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( |\omega_i - \tilde{\omega}_i|^{p-1} + |\omega_j - \tilde{\omega}_j|^{p-1} \right) \left( 3|\theta_i - \tilde{\theta}_i| + 2|\theta_j - \tilde{\theta}_j| \right)
$$

(28)
By Hölder’s inequality, we have
$$
\sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-1} |\theta_i - \tilde{\theta}_i| = N \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-1} |\theta_i - \tilde{\theta}_i|
$$
\[\leq N \left( \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i|^p \right)^{\frac{1}{p}} = N \|W - \tilde{W}\|^\frac{2}{p} \|\Theta - \tilde{\Theta}\|_p \tag{29}\]

where $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, we use Jensen’s inequality to obtain
$$
\sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-1} |\theta_j - \tilde{\theta}_j| = \left( \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-1} \right) \left( \sum_{j=1}^{N} |\theta_j - \tilde{\theta}_j| \right)
$$
\[\leq N \left( \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{N} |\theta_j - \tilde{\theta}_j|^p \right)^{\frac{1}{p}} = N \|W - \tilde{W}\|^\frac{2}{p} \|\Theta - \tilde{\Theta}\|_p. \tag{30}\]

We use (28), (29) and (30) to obtain
$$
\mathcal{I}_{11} + \mathcal{I}_{13} \leq 10pk\|W_0\|_p e^{-\kappa Nt} \|W - \tilde{W}\|^\frac{2}{p} \|\Theta - \tilde{\Theta}\|_p. \tag{31}\]

- Case A.2 (Estimate of $\mathcal{I}_{12} + \mathcal{I}_{14}$): By direct calculation, we have
$$
\mathcal{I}_{12} + \mathcal{I}_{14} = -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-2} (\omega_i - \tilde{\omega}_i)
$$
\[\times \left[ \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) (\omega_i - \tilde{\omega}_i) - \sin \tilde{\theta}_i \sin \tilde{\theta}_j (\omega_j - \tilde{\omega}_j) \right]
\leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p-1} \left[ \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) |\omega_i - \tilde{\omega}_i| - \sin \tilde{\theta}_i \sin \tilde{\theta}_j |\omega_j - \tilde{\omega}_j| \right]
\leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p} \left( \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) - \sin \tilde{\theta}_i \sin \tilde{\theta}_j \right)
\leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\sin \tilde{\theta}_i \sin \tilde{\theta}_j| |\omega_i - \tilde{\omega}_i|^{p-1} |\omega_j - \tilde{\omega}_j|
\leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p} \left( \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) - \sin \tilde{\theta}_i \sin \tilde{\theta}_j \right)
\leq -\frac{pk}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\sin \tilde{\theta}_i \sin \tilde{\theta}_j| |\omega_i - \tilde{\omega}_i| |\omega_j - \tilde{\omega}_j| |\omega_i - \tilde{\omega}_i|^{p-1} |\omega_j - \tilde{\omega}_j|^{p-1}
\leq -\frac{pk}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\omega_i - \tilde{\omega}_i|^{p} \left( \cos \tilde{\theta}_i (1 + \cos \tilde{\theta}_j) - \sin \tilde{\theta}_i \sin \tilde{\theta}_j \right)
\leq -pkA\|W - \tilde{W}\|_p^p. \tag{32}\]

Finally, we combine (27), (31) and (32) to obtain (26).

Now we are ready to provide our first main result in the following theorem.
Theorem 4.4. Suppose that $\alpha$ and $\kappa$ satisfy the following relations:
\[ \alpha \in (0, \pi) \quad \text{and} \quad \kappa > \kappa_c(\alpha^\infty), \]
and let \( \{\theta_i, \omega_i\} \) and \( \{\tilde{\theta}_i, \tilde{\omega}_i\} \) be two solutions of system (14) with initial data \( \{(\theta_{i0}, \omega_{i0})\} \) and \( \{(\tilde{\theta}_{i0}, \tilde{\omega}_{i0})\} \) respectively. Then, there exists a positive constant $G$ independent of $t$ such that
\[
\|\Theta(t) - \tilde{\Theta}(t)\|_p + \|W(t) - \tilde{W}(t)\|_p \leq G \left[ \|\Theta_0 - \tilde{\Theta}_0\|_p + \|W_0 - \tilde{W}_0\|_p \right], \quad t \geq 0.
\]
Proof. In Lemma 4.3, we set \( X := \|\Theta - \tilde{\Theta}\|_p, \ V := \|W - \tilde{W}\|_p, \ \alpha := \kappa \Lambda, \ \gamma = 10\kappa\|\tilde{W}(t_*)\|_p). \)
Note that in this setting, we have
\[
G := \max \left\{1, \frac{20\|\tilde{W}(t_*)\|_p}{\Lambda e} \right\} + \frac{80\|\tilde{W}(t_*)\|_p}{\kappa^2\Lambda^3e^3} \left(\frac{2}{\kappa \Lambda} + 1\right),
\]
and (25) yields the desired estimate. \(\square\)

4.4. Application to the Winfree model. As mentioned in Introduction, it is very difficult to state the frequency decay estimate using the phase information to the first-order model. This is why we introduced a second-order extension. In the previous subsection, we have studied the emergence of COD state and uniform stability for the second-order extension. Now we can use those estimates for the second-order extension and duality relation discussed in Section 4.1 to derive corresponding estimates for the Winfree model.

As a direct application of Theorem 4.4, we have the emergence of COD state for the Winfree model which provides a quantitative estimate for the earlier result in [19] (see Theorem 2.2).

Corollary 2. Suppose that $\alpha$ and $\kappa$ satisfy the following relations:
\[ \alpha \in (0, \pi) \quad \text{and} \quad \kappa > \kappa_c(\alpha^\infty). \]
Then, the following assertions hold.

1. For a solution $\Theta = \Theta(t)$ to the Winfree model (13), there exists a unique COD state $\Theta^\infty := (\theta_1^\infty, \cdots, \theta_N^\infty)$ and $\Lambda > 0$ such that
\[
|\theta_i(t) - \theta_i^\infty| \leq Ce^{-\kappa \Lambda (t-t_*)}, \quad \text{for all} \ t \geq t_*.
\]

2. Let $\Theta$ and $\tilde{\Theta}$ be two solutions to (13) with natural frequencies $\{\Omega_i\}$ and $\{\tilde{\Omega}_i\}$ and the same initial data $\Theta_0$ and $\tilde{\Theta}_0$, respectively. Then, there exists a positive constant $G$ independent of $t$ such that
\[
\|\Theta(t) - \tilde{\Theta}(t)\|_p \leq G \left( \|\Theta_0 - \tilde{\Theta}_0\|_p + \left( \sum_{i=1}^{N} |\Omega_i - \tilde{\Omega}_i| p \right)^{\frac{1}{p}} \right), \quad t \geq 0.
\]
Proof. (i) Let $\Theta(t)$ be a smooth solution to the first-order model (13) with $\Theta_0$.
Then, for such $\Theta(t)$, we set
\[
\omega_i(t) := \hat{\theta}_i(t), \quad \omega_{i0} := \frac{\kappa}{N} \sin \theta_{i0} \sum_{j=1}^{N} (1 + \cos \theta_{j0}), \quad i = 1, \cdots, N.
\]
Then, \((\theta_i, \omega_i)\) satisfies system (14) and a setting depicted in Corollary 1. Thus, and we apply the result of Corollary 1 to derive the desired result. (ii) For \(i = 1, \cdots, N\), we set
\[
\omega_i(t) := \dot{\theta}_i(t), \quad \omega_{i0} := \Omega_i - \frac{\kappa}{N} \sin \theta_{i0} \sum_{j=1}^{N} (1 + \cos \theta_{j0}),
\]
\[
\dot{\omega}_i(t) := \ddot{\theta}_i(t), \quad \dot{\omega}_{i0} := \dot{\Omega}_i - \frac{\kappa}{N} \sin \dot{\theta}_{i0} \sum_{j=1}^{N} (1 + \cos \dot{\theta}_{j0}).
\] (34)

Next, we estimate the term \(||W_0 - \tilde{W}_0||_p\) as follows. First, we use mean-value theorem to find
\[
|\sin \theta_{i0}(1 + \cos \theta_{j0}) - \sin \tilde{\theta}_{i0}(1 + \cos \tilde{\theta}_{j0})| = |\sin \theta_{i0} - \sin \tilde{\theta}_{i0}|(1 + \cos \theta_{j0}) + |\sin \tilde{\theta}_{i0}| |\cos \theta_{j0} - \tilde{\theta}_{j0}| \leq 2|\theta_{i0} - \tilde{\theta}_{i0}| + |\theta_{j0} - \tilde{\theta}_{j0}|
\] (35)

Then, (34) and (35) implies that there exists a positive constant \(C(p, \kappa)\) only depending on \(p\) and \(\kappa\) such that
\[
||W_0 - \tilde{W}_0||_p \leq C(p, \kappa) \left( \left( \sum_{i=1}^{N} |\Omega_i - \tilde{\Omega}_i|^p \right)^{\frac{1}{p}} + ||\Theta_0 - \tilde{\Theta}_0||_p \right).
\] (36)

Now, we use the result of Theorem 4.4 to see that there exist positive constant \(\tilde{G}\) only depending on \(p\) and \(\kappa\) such that
\[
||\Theta(t) - \tilde{\Theta}(t)||_p \leq ||\Theta(t) - \tilde{\Theta}(t)||_p + ||W(t) - \tilde{W}(t)||_p \leq \tilde{G} \left( ||\Theta_0 - \tilde{\Theta}_0||_p + ||W_0 - \tilde{W}_0||_p \right)
\]
\[
\leq \tilde{G} \left( ||\Theta_0 - \tilde{\Theta}_0||_p + \left( \sum_{i=1}^{N} |\Omega_i - \tilde{\Omega}_i|^p \right)^{\frac{1}{p}} \right), \quad t \geq 0,
\]
which yields the desired estimate.

5. A uniform mean-field limit and large-time dynamics. In this section, we consider kinetic Winfree equations for both first and second-order Winfree models. The rigorous derivation of kinetic equations can be made from phase models via the particle-in-cell method in any finite-time interval. Below, we will show that mean-field limit holds for a whole time interval by incorporating finite-time mean-field limit and decay estimate of phase models in a proper metric space.

5.1. An invariant set for the Winfree model. First, we present an invariant set of the kinetic Winfree model. For \(t \geq 0\), we set \(R(t)\) and \(P(t)\) be the projections of \(\text{supp} \mu_t\) to \(\Theta\) and \(\Omega\) spaces, respectively:
\[
R(t) := \mathbb{P}_\Theta \text{supp} (\mu_t) = \{ \theta \in [0, 2\pi) : (\theta, \Omega) \in \text{supp} (\mu_t) \},
\]
\[
P(t) := \mathbb{P}_\Omega \text{supp} (\mu_t) = \{ \Omega \in \mathbb{R} : (\theta, \Omega) \in \text{supp} (\mu_t) \}.
\] (37)

In the sequel, we will provide an invariant set for the kinetic Winfree equation, we first quote the approximation of a probability measure by a sequence of empirical measures in the following proposition.
Proposition 3. [33] For any given $p \in \mathbb{N}$ and $\mu \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R})$ with compact support, there exists a sequence of empirical measures $\mu^N \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R})$ such that

$\mu^N$ has the common compact support with $\mu$ and $\lim_{N \to +\infty} W_p(\mu^N, \mu) = 0$.

In the sequel, we present two estimates on $R(t)$ in the following two lemmas.

**Lemma 5.1.** Suppose that $\alpha$ and $\kappa$ satisfy

$\alpha \in (0, \pi)$ and $\kappa > \kappa_c(\alpha^\infty)$,

and let $\mu \in L^\infty([0, T); \mathcal{P}(\mathbb{T} \times \mathbb{R})$ be a measure-valued solution to (2) with initial datum $\mu_0 \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$. Then the arc $(-\alpha, \alpha)$ is an invariant set for (2):

$$R(0) \subseteq (-\alpha, \alpha) \implies R(t) \subseteq (-\alpha, \alpha).$$

**Proof.** It follows from Proposition 3 and Theorem 2.2 that for any initial data $\mu_0 \in \mathcal{M}(0, 2\pi) \times \mathbb{R}$, we can find a sequence of discrete empirical measures $\{\mu_0^N\}$ such that

$$\text{supp}(\mu_0^N) \subseteq \text{supp}(\mu_0), \quad W_p(\mu_0^N, \mu_0) \to 0, \quad \text{as } N \to \infty. \quad (38)$$

Moreover, we can see that the stability estimate in $W_p$ and (38) yields that the measure-valued solution to (2) with the discrete initial datum $\mu_0^N$ would be an approximation of the measure-valued solution $\mu_t$ to (2) with initial datum $\mu_0$: for

$$W_p(\mu_t^N, \mu_t) \to 0, \quad \text{as } N \to \infty. \quad (39)$$

Then, $\mu_t^N$ solves the first-order Winfree model (1), and it follows from Lemma 4.1 and Lemma 4.2 that we have

$$\mathbb{P}_{\theta}\text{supp}(\mu_t^N) \subseteq (-\alpha, \alpha).$$

Now, we claim:

$$R(t) \subseteq (-\alpha, \alpha). \quad (40)$$

**Proof of claim (40):** Suppose not, i.e., there exists some $t$ such that

$$R(t) \cap (-\alpha, \alpha)^c \neq \emptyset.$$

Then, we can construct some special test function $h$ such that

$$\mathbb{P}_{\theta}\text{supp}(h) \subseteq (-\alpha, \alpha)^c, \quad \text{and } \langle \mu_t, h \rangle \neq 0.$$

Thus we have

$$W_p(\mu_t^N, \mu_t) \geq |\langle \mu_t, h \rangle| > 0, \quad \text{for } \forall N,$$

which contradicts to (39). Hence, (40) holds. \hfill \Box

By the similar arguments for Lemma 5.1, we can conclude the following lemma.

**Lemma 5.2.** Suppose that $\alpha$ and $\kappa$ satisfy

$\alpha \in (0, \pi)$ and $\kappa > \kappa_c(\alpha^\infty)$,

and let $\mu \in L^\infty([0, T); \mathcal{P}(\mathbb{T} \times \mathbb{R})$ be a measure-valued solution to (2) with initial datum $\mu_0 \in \mathcal{M}(\mathbb{T} \times \mathbb{R})$. Then the arc $(-\beta, \beta)$ with $\beta \in (0, \pi)$ is an invariant set for (2):

$$R(0) \subseteq (-\beta, \beta) \implies R(t) \subseteq (-\beta, \beta).$$

Moreover, for any initial measure $\mu_0$ such that $R(0) \subseteq (-\alpha, \alpha)$, there exists a positive constant $t_*$ such that

$$R(t) \subseteq (-\beta, \beta), \quad \text{for } t > t_*.$$
Remark 7. Note that for $\kappa \gg 1$ and $\alpha \in (\frac{\pi}{3}, \pi)$, we have

$$0 < \beta < \frac{\pi}{3}.$$ 

Since we focus on the large-time behavior, we will consider initial data confined inside the invariant set $(-\beta, \beta)$. 

5.2. A second-order Winfree model. In this subsection, we consider the uniform-in-time mean-field limit of the second-order Winfree model (14):

$$\dot{\theta}_i = \omega_i, \quad i = 1, \ldots, N, \quad t > 0,$$

$$\dot{\omega}_i = -\frac{\kappa}{N} \left[ \cos \theta_i \left( \sum_{j=1}^{N} 1 + \cos \theta_j \right) \omega_i + \sin \theta_i \sum_{j=1}^{N} (-\sin \theta_j) \omega_j ) \right].$$

Let $h = h(\theta, \omega, t)$ be a one-oscillator probability density function for the ensemble whose dynamics is governed by the above system. Then, we can formally derive the kinetic equation for $h := h(\theta, \omega, t)$:

$$\begin{cases}
    \partial_t h + \omega \partial_{\theta} h + \partial_{\omega} \left( L(h) h \right) = 0, & (\theta, \omega) \in \mathbb{T} \times \mathbb{R}, \\
    L(h) := -\kappa \omega \cos \theta - \kappa \omega \int_{\mathbb{T} \times \mathbb{R}} \cos(\theta + \theta_s) h(\theta_s, \omega_s, t) \, d\theta_s \, d\omega_s \\
    + \kappa \sin \theta \int_{\mathbb{T} \times \mathbb{R}} \sin \theta_s (\omega_s - \omega) h(\theta_s, \omega_s, t) \, d\theta_s \, d\omega_s, 
\end{cases} \quad (41)$$

Next, we briefly summarize the measure-theoretic framework by reviewing several basics of measure-valued solution to (14) as in Section 3.1. Let $\mathcal{P}(\mathbb{T} \times \mathbb{R})$ be the set of all probability measures on the phase space $\mathbb{T} \times \mathbb{R}$, which can be understood as normalized nonnegative bounded linear functionals on $C_0(\mathbb{T} \times \mathbb{R})$. For a probability measure $\mu \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$, we use a standard duality relation:

$$\langle \mu, h \rangle = \int_{\mathbb{T} \times \mathbb{R}} h(\theta, \omega) \, d\mu(\theta, \omega), \quad h(\theta, \omega) \in C_0(\mathbb{T} \times \mathbb{R}).$$

Below, we recall several definitions to be used in the next subsection.

Definition 5.3. [18] For $T \in [0, \infty)$, $\mu \in L^\infty([0, T); \mathcal{P}(\mathbb{T} \times \mathbb{R}))$ is a measure-valued solution to (41) with initial data $\mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ if the following three assertions hold:

1. Total mass is normalized: $\langle \mu_t, 1 \rangle = 1$.
2. $\mu$ is weakly continuous in $t$:

$$\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle \in C_0(\mathbb{T} \times \mathbb{R} \times [0, T)).$$

3. $\mu$ satisfies the equation (41) in weak sense: for $\forall \varphi \in C_0^1(\mathbb{T} \times \mathbb{R} \times [0, T))$,

$$\langle \mu_t, \varphi(\cdot, \cdot, t) - \langle \mu_0, \varphi(\cdot, \cdot, 0) \rangle \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + \omega \partial_{\theta} \varphi + F \partial_{\omega} \varphi \rangle \, ds,$$

where $\varphi$ is a test function in $C_0(\mathbb{T} \times \mathbb{R} \times [0, T))$.

Recall that the solution of the phase model (14) can be viewed as a measure-valued solution of the kinetic equation (41) (see Remark 3). Moreover, we can apply the Wasserstein metric defined in Definition 3.3 to measure the distance between the solutions of (14) and (41). Then we can apply the method in [15], Lemma 4.3 and Remark 7 to conclude the following theorems.
**Theorem 5.4.** Suppose that $\alpha$ and $\kappa$ satisfy
$$\alpha \in (0, \pi), \quad \kappa > \kappa_c(\alpha^\infty) \quad \text{and} \quad p \in \mathbb{N},$$
and initial datum $\mu_0$ has finite $p$-th moments:
$$\int_{\mathbb{T} \times \mathbb{R}} (1 + |\theta|^p + |\omega|^p) \mu_0(d\theta, d\omega) < \infty. \quad (42)$$
Then, the following assertions hold.

1. There exists a unique measure-valued solution $\mu \in L^\infty([0, \infty); \mathcal{P}(\mathbb{T} \times \mathbb{R}))$ to (41) with initial data $\mu_0$: $\mu$ is approximated by empirical measure $\mu^N$ in Wasserstein-$p$ distance uniformly in time:
$$\lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_p(\mu^N_t, \mu_t) = 0.$$

2. If $\nu_t$ is the measure-valued solution to (41) with initial measure $\nu_0$ which has compact support in the invariant set. Moreover $\nu_0$ has finite moments (42).
Then there exists nonnegative constant $G$ independent of $t$ such that
$$W_p(\mu_t, \nu_t) \leq G W_p(\mu_0, \nu_0), \quad t \in [0, \infty). \quad (43)$$

**Proof.** We use Remark 7 and Lemma 4.3 to see that the interaction force is always attractive in the invariant set. Thus, the second-order model can be treated like the Cucker-Smale type model. Then, we can apply the same method in [15] for the second-order Winfree model to derive the desired estimate. Since the details are rather lengthy, we leave the detailed argument in Section 6. \hfill \Box

**Theorem 5.5.** Suppose that $\alpha$ and $\kappa$ satisfy
$$\alpha \in (0, \pi), \quad \kappa > \kappa_c(\alpha^\infty) \quad \text{and} \quad p \in \mathbb{N},$$
and initial datum $\mu_0$ has finite $p$-th moments:
$$\int_{\mathbb{T} \times \mathbb{R}} (1 + |\theta|^p + |\omega|^p) \mu_0(d\theta, d\omega) < \infty.$$
Moreover, we also assume that the support of $\mu_0$ is in the invariant set in Section 5.1. Then, for the measure-valued solution $\mu$ to (41), there exists a positive constant $\Lambda$ such that
$$\left( \int_{\mathbb{T} \times \mathbb{R}} |w|^p \, d\mu_t \right)^{\frac{1}{p}} \leq Ce^{-\kappa \Lambda t} \left( \int_{\mathbb{T} \times \mathbb{R}} |w|^p \, d\mu_0 \right)^{\frac{1}{p}}, \quad t \geq 0. \quad (44)$$

**Proof.** Note that for empirical measure $\mu^N$, the estimate (44) holds due to Proposition 2:
$$\left( \int_{\mathbb{T} \times \mathbb{R}} |w|^p \, d\mu^N_t \right)^{\frac{1}{p}} \leq Ce^{-\kappa \Lambda t} \left( \int_{\mathbb{T} \times \mathbb{R}} |w|^p \, d\mu^N_0 \right)^{\frac{1}{p}}, \quad t \geq 0. \quad (45)$$
On the other hand, letting $N \to \infty$, the uniform mean-field limit in Theorem 5.4 guarantees that the measure valued solution $\mu$ of kinetic model (41) satisfies
$$\left( \int_{\mathbb{T} \times \mathbb{R}} |w|^p \, d\mu_t \right)^{\frac{1}{p}} \leq Ce^{-\kappa \Lambda t} \left( \int_{\mathbb{T} \times \mathbb{R}} |w|^p \, d\mu_0 \right)^{\frac{1}{p}}.$$
\hfill \Box
5.3. The first-order Winfree model. In this subsection, we provide the proofs of Theorem 3.4 and Theorem 3.5. Consider a kinetic equation for Winfree model:

\[
\begin{aligned}
\partial_t f + \partial_\theta \left(V[f] f\right) &= 0, \quad (\theta, \Omega) \in T \times \mathbb{R}, \\
V[f] &:= \Omega - \kappa \sin \theta \int_0^{2\pi} \int_\mathbb{R} (\cos \theta_* + 1) f(\theta_*, \Omega_*, t) \, d\Omega_* \, d\theta_*, \\
f(\theta, \Omega, 0) &= f_0(\theta, \Omega).
\end{aligned}
\]  
(46)

Note that the density function \(g = g(\Omega)\) appears in the relation:

\[
\int_0^{2\pi} f(\theta, \Omega, t) \, d\theta = g(\Omega), \quad t \geq 0,
\]

and the solution of the Winfree model (1) can be regarded as a measure-valued solution to the kinetic equation (2). Then, we can apply the Wasserstein metric in Definition 3.3 to measure the distance between two measure-valued solutions. Based on Theorem 5.4, we can provide the proof of Theorem 3.4 as follows.

Proof of Theorem 3.4: (1) We follow the same structure of the proof for Theorem 5.4. Thus, instead of the repeated details, we briefly provide the outline of the proof. First of all, we assume that the support of initial configuration is confined in an invariant set as aforementioned in Remark 7. Since the natural frequency \(\Omega\) is not a variable but a parameter, the distribution of natural frequency does not change over time. Hence, the variance of \(\Omega\) between two measure valued solution \(\mu\) and \(\nu\) is constant, i.e.,

\[
\inf_{\gamma \in \Gamma(\mu_0, \nu_0)} \int_{T^2 \times \mathbb{R}^2} |\Omega - \bar{\Omega}|^p \gamma(z, \bar{z}) = \inf_{\gamma \in \Gamma(\mu_n, \nu_n)} \int_{T^2 \times \mathbb{R}^2} |\Omega - \bar{\Omega}|^p \gamma(z, \bar{z}).
\]

It suffices to estimate the variance with respect to \(\theta\) variable. Let \(\mu_0^N\) be an approximation of \(\mu_0\) satisfying

\[
\lim_{N \to +\infty} W_p(\mu_0^N, \mu_0) = 0.
\]

Then, for any \(\varepsilon > 0\), we can choose a positive constant \(N\) such that

\[
W_p(\mu_0^n, \mu_0^m) < \varepsilon, \quad \text{for} \quad m, n > N.
\]

Due to Corollary 2, we can attain

\[
W_p(\mu_t^n, \mu_t^m) < G' W_p(\mu_0^n, \mu_0^m),
\]

for some constant \(G'\), which yields

\[
\lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_p(\mu_t^n, \mu_t^m) = 0.
\]

(2) For a given \(\varepsilon > 0\), we can choose an integer \(N_0 \in \mathbb{N}\) such that

\[
W_p(\mu, \mu^n) < \frac{\varepsilon}{2}, \quad W_p(\nu, \nu^n) < \frac{\varepsilon}{2}, \quad n \geq N_0.
\]

Then, we obtain

\[
W_p(\mu, \nu) \leq \left(W_p(\mu_t^n, \mu_t^n) + W_p(\nu_t^n, \nu_t^n) + W_p(\nu_t^n, \nu_t^n)\right)^p
\]
\[
\leq \left(\varepsilon + W_p(\mu_t^n, \nu_t^n)\right)^p
\]
\[
\leq 2^{p-1} \left(\varepsilon^p + W_p(\mu_t^n, \nu_t^n)\right)^p
\]
\[
\leq 2^{p-1} \left(\varepsilon^p + G' W_p(\mu_0^n, \nu_0^n)\right)^p
\]
By letting $n \to \infty$ and $\varepsilon \to 0$, we get
\[ W_p(\mu_t, \nu_t) < 2^{\frac{1}{p-1}} G' W_p(\mu_0, \nu_0). \]

The proof of Theorem 3.5 is directly obtained from Corollary 2 and Theorem 3.4.

Proof of Theorem 3.5: Consider an empirical measure $\mu^n_t$. According to Corollary 2, for each $\mu^n_t$, we have a unique asymptotic equilibrium $\mu^n_\infty$. The uniform stability in Theorem 3.4 implies that the sequence \{ $\mu^n_\infty$ \} is a Cauchy sequence and thus generates a unique limit measure $\mu_\infty$. Moreover, due to Corollary 2, we obtain
\[ W_p(\mu^n_t, \mu^n_\infty) \leq C e^{-\kappa \Lambda t}. \]
We note that the $p$-th moment of $\Omega$ would be canceled because $\mu^n_N$ and $\mu^n_\infty$ has the same natural frequency distribution. For any $\varepsilon > 0$, we can find $N_0 \in \mathbb{N}$ such that, for $n \geq N_0$ we attain
\[ W_p(\mu_t, \mu_\infty) \leq W_p(\mu_t, \mu^n_t) + W_p(\mu^n_t, \mu^n_\infty) + W_p(\mu^n_\infty, \mu_\infty) \leq 2\varepsilon + C e^{-\kappa \Lambda t}. \]
Since $\varepsilon$ is arbitrary, we have
\[ W_p(\mu_t, \mu_\infty) \leq C e^{-\kappa \Lambda t}. \]

6. Proof of Theorem 5.4. We will show the detailed proof of Theorem 5.4, which is similar to the proof in [15] for the Cucker-Smale model.

6.1. The first assertion. In this subsection, we present the uniform-in-time mean-field limit result generalizing the earlier result in [18] as a direct application of uniform stability result in Section 4.

Let $\mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ be a probability measure with a compact support. Then, we claim that there exists a measure-valued solution $\mu_t \in L^\infty ([0, +\infty); \mathcal{P}(\mathbb{T} \times \mathbb{R}))$ to (14) with initial data $\mu_0$:

1. $\mu_t$ is approximated by $\mu^n_t$ in Wasserstein $(q, p)$-- distance uniformly in time:
\[ \lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_{q,p}(\mu^n_t, \mu_t) = 0. \]

2. $\mu_t$ is unique in the class of measure valued solution with initial data $\mu_0$ and has uniformly compact support.

For the simplicity of presentation, we split its proof into several steps. Let $\mu_0$ be a measure in $\mathcal{P}(\mathbb{T} \times \mathbb{R})$. Then the proof consists of four steps.

- Step A (Extraction of Cauchy approximation for $\mu_0$ in $\mathcal{W}_p$): Let $\mu^n_0$ be an approximation of $\mu_0$ satisfying
\[ \lim_{N \to +\infty} W_p(\mu^n_0, \mu_0) = 0. \]  
(47)

The existence of such approximation is guaranteed by Proposition 3. Then, thanks to (47), for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that
\[ W_p(\mu^n_0, \mu^n_0) < \varepsilon, \quad \text{for } n, m > N(\varepsilon). \]
Moreover, we have the diameter of $\mu^n_t$ is uniformly bounded by the diameter of $\mu_t$. 

\[ W_p(\mu_t, \mu_\infty) \leq C e^{-\kappa \Lambda t}. \]
• Step B (Calculation of $\mathcal{W}_p(\mu^n_0, \mu^m_0)$): Since the empirical measures $\mu^n_0$ and $\mu^m_0$ obtained in Step A are both concentrated at finite points, we denote them by

$$
\mu^n_0 := \frac{1}{n} \sum_{i=1}^{n} \delta(\theta_{i0}, \omega_{i0}), \quad \mu^m_0 := \frac{1}{m} \sum_{j=1}^{m} \delta(\bar{\theta}_j, \bar{\omega}_j).
$$

Then, we can find an optimal plan $(a_{ij})$ whose entries are nonnegative real numbers and satisfying

$$
\mathcal{W}_p^n(\mu^n_0, \mu^m_0) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \left( \|\theta_{i0} - \bar{\theta}_j\|_p^p + \|\omega_{i0} - \bar{\omega}_j\|_p^p \right),
$$

where $a_{ij}$ satisfies the constraints:

$$
\sum_{i=1}^{n} a_{ij} = n, \quad \sum_{j=1}^{m} a_{ij} = m.
$$

• Step C (Approximation of $\mathcal{W}_p(\mu^n_0, \mu^m_0)$): To associate (48) with $\ell_{p,p}$-distance between $\{\theta_{i0}, \omega_{i0}\}$ and $\{\bar{\theta}_j, \bar{\omega}_j\}$, we approximate (48) with rational coefficients $r_{ij}$ instead of real ones $a_{ij}$ with some small error as follows. More precisely, we find proper rational numbers $\bar{r}_{ij}$ such that they have the same denominator $D_{mn}$ and

$$
|r_{ij} - a_{ij}| \leq \frac{\varepsilon^p}{d_{\theta}(0)^p + d_{\omega}(0)^p}, \quad \sum_{i=1}^{n} r_{ij} = n, \quad \sum_{j=1}^{m} r_{ij} = m.
$$

where the nonnegative numbers $d_{\theta}(0)$ and $d_{\omega}(0)$ are given as follows.

$$
d_{\theta}(0) := \max_{i,j} \|\theta_{i0} - \bar{\theta}_j\|_p \quad \text{and} \quad d_{\omega}(0) := \max_{i,j} \|\omega_{i0} - \bar{\omega}_j\|_p.
$$

Then, it follows from (48) and (49) that we have

$$
\left| \mathcal{W}_p^n(\mu^n_0, \mu^m_0) - \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \left( \|\theta_{i0} - \bar{\theta}_j\|_p^p + \|\omega_{i0} - \bar{\omega}_j\|_p^p \right) \right|
\leq \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij} - r_{ij}| \left( \|\theta_{i0} - \bar{\theta}_j\|_p^p + \|\omega_{i0} - \bar{\omega}_j\|_p^p \right)
\leq \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij} - r_{ij}|
\leq \varepsilon^p,
$$

where we used (14) and the fact that $\{\theta_{i0}, \omega_{i0}\}$ and $\{\bar{\theta}_j, \bar{\omega}_j\}$ have common support and

$$
\left( \|\theta_{i0} - \bar{\theta}_j\|_p^p + \|\omega_{i0} - \bar{\omega}_j\|_p^p \right) \leq d_{\theta}(0)^p + d_{\omega}(0)^p
$$

We set

$$
r_{ij} := \frac{N_{ij}}{D_{mn}}, \quad N_{ij} \in \mathbb{Z}^+, \quad M_{mn} := D_{mn}mn.
$$
Then, we can rewrite
\[
\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} \left( \|\theta_{i0} - \bar{\theta}_{j0}\|_p + \|\omega_{i0} - \bar{\omega}_{j0}\|_p \right)
\]
\[
= \frac{1}{M_{mn}} \sum_{i=1}^{n} \sum_{j=1}^{m} N_{ij} \left( \|\theta_{i0} - \bar{\theta}_{j0}\|_p + \|\omega_{i0} - \bar{\omega}_{j0}\|_p \right)
\]
\[
= \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} \left( \|\theta_{k0} - \bar{\theta}_{k0}\|_p + \|\omega_{k0} - \bar{\omega}_{k0}\|_p \right),
\]
by reindexing for each summand. We combine (50) and (51) to obtain
\[
\left| W_p^p(\mu_{i0}^n, \mu_{00}^m) - \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} \left( \|\theta_{k0} - \bar{\theta}_{k0}\|_p + \|\omega_{k0} - \bar{\omega}_{k0}\|_p \right) \right| \leq \varepsilon_p.
\] (52)

**Step D** (Lifting the information at 0 to t > 0): We use the uniform $\ell_p$ stability to show that $\mu_t$ is Cauchy. Note that $\sum_{i,j}^n N_{i,j} = M_{mn}$, and $\{(\theta_{i0}, \omega_{i0})\}$ and $\{(\theta_{j0}, \omega_{j0})\}$ are 2d-vectors chosen from $\{(\theta_{i0}, \omega_{i0}) | 1 \leq i \leq n\}$ and $\{(\theta_{j0}, \omega_{j0}) | 1 \leq j \leq m\}$, respectively. On the other hand, we consider the term
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{N_{ij}}{M_{mn}} \left( \|\theta_{i0} - \bar{\theta}_{j0}\|_p + \|\omega_{i0} - \bar{\omega}_{j0}\|_p \right), \quad \sum_{i=1}^{n} \frac{N_{ij}}{D_{mn}} = n, \quad \sum_{j=1}^{m} \frac{N_{ij}}{D_{mn}} = m.
\]
It actually corresponds to a plan between $\mu_{i0}^n$ and $\mu_{00}^m$, which we denote by $\gamma(\mu_{i0}^n, \mu_{00}^m)$.

\[
\int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \left( \|\theta - \bar{\theta}\|_p + \|\omega - \bar{\omega}\|_p \right) \gamma(\mu_{i0}^n, \mu_{00}^m)
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{N_{ij}}{M_{mn}} \left( \|\theta_{i0} - \bar{\theta}_{j0}\|_p + \|\omega_{i0} - \bar{\omega}_{j0}\|_p \right)
\]
\[
= \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} \left( \|\theta_{k0} - \bar{\theta}_{k0}\|_p + \|\omega_{k0} - \bar{\omega}_{k0}\|_p \right).
\]

Then, by Theorem 4.4 and Jensen’s inequality, we have
\[
W_p^p(\mu_{i0}^n, \mu_{00}^m) \leq \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \left( \|\theta - \bar{\theta}\|_p + \|\omega - \bar{\omega}\|_p \right) \gamma(\mu_{i0}^n, \mu_{00}^m)
\]
\[
= \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} \left( \|\theta_{k0} - \bar{\theta}_{k0}\|_p + \|\omega_{k0} - \bar{\omega}_{k0}\|_p \right)
\]
\[
\leq \frac{G_{p}^p}{M_{mn}} \sum_{k=1}^{M_{mn}} \left( \|\theta_{k0} - \bar{\theta}_{k0}\|_p + \|\omega_{k0} - \bar{\omega}_{k0}\|_p \right)^p
\]
\[
\leq \frac{2p-1}{M_{mn}} \sum_{k=1}^{M_{mn}} \left( \|\theta_{k0} - \bar{\theta}_{k0}\|_p + \|\omega_{k0} - \bar{\omega}_{k0}\|_p \right)^p
\]
\[
\leq 2p^{-1} G_{p}^p \left( W_p^p(\mu_{i0}^n, \mu_{00}^m) + \varepsilon_p \right) \leq 2p^p \varepsilon_p.
\]
Now for any $\varepsilon > 0$, we can find a positive integer $L$ such that for any $n, m > L$, we have

$$W_p(\mu^n_t, \mu^m_t) \leq 2G\varepsilon.$$ 

This shows that the sequence $\mu^n_t$ is a Cauchy sequence in $W_p$ metric, thus we can find a limit measure $\mu_t$. We next apply similar arguments in [18] and show that the limit $\mu_t$ is the unique measure-valued solution of the kinetic equation (14) with initial data $\mu_0$. Moreover, because our estimates (53), we can conclude that for any $\varepsilon$, there exists a positive constant $L$, such that

$$\sup_{t \in [0, +\infty)} W_p(\mu^n_t, \mu^n_t) \leq 4G\varepsilon, \quad \text{for} \quad n > L.$$ 

This yields

$$\lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_p(\mu^N_t, \mu_t) = 0.$$ 

The uniform compact support of $\mu_t$ follows this uniform convergence. \[\square\]

6.2. The second assertion. In this part, we show the uniform $W_p$-stability of the measure-valued solutions whose existences are guaranteed in Section 5. For measures $\mu_0$ and $\nu_0$ in $\mathcal{P}(\mathbb{R}^{2d})$, let $\mu$ and $\nu$ be measure-valued solutions to (41). Then, it follows from (47) that for any $\varepsilon \ll 1$, there exists $N_0(\varepsilon) \in \mathbb{N}$ such that

$$W_p(\mu^n_t, \mu^n_t) < \frac{\varepsilon}{2}, \quad W_p(\nu^n_t, \nu^n_t) < \frac{\varepsilon}{2} \quad n \geq N_0(\varepsilon).$$

Then, we use the above estimates and (53) to obtain

$$W_p^p(\mu_t, \nu_t) \leq \left( W_p(\mu_t, \mu^n_t) + W_p(\mu^n_t, \nu^n_t) + W_p(\nu^n_t, \nu_t) \right)^p \leq \left( \varepsilon + W_p(\mu^n_t, \nu^n_t) \right)^p \leq 2^{p-1} \left( \varepsilon^p + W_p^p(\mu^n_t, \nu^n_t) \right) \leq 2^{p-1} \left( 2\varepsilon^p + G^p W_p^p(\mu_0^n, \nu_0^n) \right).$$

Letting $n \to \infty$, we have

$$W_p^p(\mu_t, \nu_t) \leq 2^p \varepsilon^p + 2^{p-1} G^p W_p(\mu_0, \nu_0).$$

Since $\varepsilon$ was arbitrary, we have the uniform $W_p$-stability:

$$W_p(\mu_t, \nu_t) \leq 2^{\frac{p-1}{p}} GW_p(\mu_0, \nu_0), \quad t \geq 0.$$ \[\square\]

7. Conclusion. In this paper, we studied the uniform mean-field limit in the Winfree phase model and asymptotic properties of the kinetic Winfree equation. For this, we have introduced a second-order extension of the Winfree model which coincides with the first-order Winfree model on some low dimensional submanifold. The second-order extension has an advantage which is more convenient to deal with frequency estimates. For example, the decay estimate of frequencies can be verified using the energy type estimate in a large coupling regime. Thus, in a large coupling regime, the frequencies tend to zero exponentially fast so that the phase configuration leads to the complete oscillator death, and we also provided some uniform stability estimate. There are still many open questions in relation with the Winfree phase model and its corresponding kinetic equation, e.g., stability of asymptotic patterns such as partial death, partial and complete locking, incoherent state and

chimera states are completely open. This interesting issue will be treated in future works.

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