THE USE OF COMPUTER ALGEBRA IN MAXWELL’S THEORY*†

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ABSTRACT

We present a small computer algebra program for use in Maxwell’s theory. The Maxwell equations and the energy-momentum current of the electromagnetic field are formulated in the language of exterior differential forms. The corresponding program can be applied (in the presence of a gravitational field) in curved Riemannian spacetime as well as in flat Minkowski spacetime in inertial or non-inertial frames. Our program is written for the computer algebra system REDUCE with the help of the EXCALC package for exterior differential forms.

The two major advantages of this modern approach to electrodynamics — the natural formulation, free of both metric and coordinates, and the straightforward programming of problems — are illustrated by examining a number of examples ranging from the Coulomb field of a static point charge to the Kerr–Newman solution of the Einstein–Maxwell equations.

1. Introduction

In the lecture of Hartley[1], the effective use of exterior differential forms has been demonstrated convincingly. We were led to our Maxwell program because we were searching for electrically charged solutions in the framework of general relativity — see the talk of Brans[2] in this context — and of alternative gravitational theories[3], and only streamlining our program using exterior differential forms made it a feasible task. We believe that our program, besides being helpful in physics, could also have applications in electrical engineering (see the textbook of Meetz and Eng[4], written for engineering students, inter alia, where the calculus of exterior forms is applied throughout).

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Our axiomatic formulation of Maxwell’s theory is based on three principles standing on firm experimental bases: (i) the conservation of electric charge, (ii) the notion of the Lorentz force, and (iii) the conservation of electromagnetic flux.

2. Maxwell’s Theory in Brief

2.1. Electric Charge Conservation

The inhomogeneous set of Maxwell’s equations, in terms of exterior differential forms, may be written as

\[
\left\{ \begin{array}{l}
\underline{d}H - \dot{\underline{D}} = \underline{j}, \\
\underline{d}D = \rho.
\end{array} \right.
\]

The sources are the electric current density 2–form \( \underline{j} \) and the electric charge density 3–form \( \rho \). The magnetic excitation 1–form is denoted by \( \underline{H} \) and the electric excitation 2–form by \( \underline{D} \). The underline \( \underline{d} \) is the 3–dimensional exterior derivative, and the dot denotes the time derivative.

These equations, reformulated in 4 dimensions, read

\[
dH = J,
\]

provided we introduce the time-space decompositions of the electromagnetic excitation 2–form (\( t = \) time coordinate)

\[
H = -\underline{\mathcal{H}} \wedge dt + \mathcal{D}
\]

and the electric current 3–form

\[
J = -\underline{j} \wedge dt + \rho.
\]
If the spacetime under consideration is contractable, then it can be shown that (2) can be derived from the integral version
\[ \oint_{\partial V} J = \int_{V} dJ = 0 \] (5)
of the conservation law for electric charge.

2.2. Lorentz Force
Postulating the (vectorial) Lorentz-force equation as
\[ f = \rho \wedge E + j \wedge B, \] (6)
we are led in a most natural way to the electrical field strength 1-form \( E \) and the magnetic field strength 2-form \( B \), with \( E = e \wedge E \) and \( B = e \wedge B \), where \( e = \{e_1, e_2, e_3\} \) represents an arbitrary 3-dimensional vector basis (frame), a so-called triad, and \( \wedge \) denotes the interior product. In analogy to (3), the field strength can be collected in the 4-dimensional electromagnetic field strength 2-form
\[ F = -dt \wedge E + B. \] (7)

One possible graphic representation of the sources and fields of Maxwell’s theory is displayed in Fig.1. We find this way of representing exterior differential forms quite intuitive and, for more details, refer to the literature cited in the caption of Fig.1.

2.3. Electromagnetic Flux Conservation
The homogeneous set of Maxwell’s equations
\[ \begin{cases} dE + \dot{B} = 0, \\ dB = 0, \end{cases} \] (8)
can be rewritten by means of (7) as
\[ dF = 0, \] (9)
or, in integral form, as an electromagnetic flux conservation law
\[ \oint_{\partial V} F = \int_{V} dF = 0. \] (10)

*In 4 dimensions, instead of (6), we have \( f_{\alpha} = (e_{\alpha} \wedge F) \wedge J \), with a 4-dimensional frame \( e_{\alpha} = \{e_0, e_1, e_2, e_3\} \).
2.4. Maxwellian Structure and Spacetime

So far we have the two Maxwell equations (2) and (9) and the expression for the Lorentz force (6) interrelating the current and field strength by means of the force concept of mechanics. These equations can be formulated on any 4-dimensional differential manifold, as long as it allows a (1+3)–decomposition. It is worth noting that these Maxwellian structures emerge prior to the introduction of a metric or a connection. The conservation laws (5) and (10) only require the possibility to circumscribe a boundary around an arbitrary 4- or 3-dimensional volume element, and to count the charge or flux units contained therein.

In particular this means that the equations (2), (9), and (6) are valid in this form in a curved and contorted spacetime, i.e. with non-vanishing curvature and torsion, or in a flat Minkowskian spacetime in inertial and non-inertial frames – they always keep this form.

However, in (2) and (9) we have only 6 independent evolution equations, namely (1) and (8), for specifying the $2 \times 6$ independent fields $\{H,F\}$. Something is still missing, namely the constitutive laws, such as the one relating $H$ to $F$, and Ohm’s law.

In a vacuum, the field strength and excitation are related by

$$H = \frac{1}{\mu} F , \quad \text{with} \quad \frac{\text{action}}{\text{charge}^2} = \frac{V}{A} = \Omega, \quad (11)$$

where $*$ is the Hodge star operator, which does depend on the metric — or rather on its determinant.

3. Program

In an interactive Reduce 3.5 session\cite{11–13}, the Excalc package\cite{14} can be loaded by the command\footnote{Soleng\cite{15} has written the Mathematica package CARTAN which should be also useful for programming Maxwell’s theory.}

\begin{verbatim}
load_package excalc$
\end{verbatim}

First Excalc has to be acquainted with the rank of the differential forms to be used in the code. This is achieved by the declaration

\begin{verbatim}
pform pot1=1, \{farad2,excit2\}=2, \{maxhom3,maxinh3\}=3$
\end{verbatim}

where $pot1$ and $farad2$ are the identifiers of the electromagnetic potential and field strength, $excit2$ represents the electromagnetic excitation, and $maxhom3$ and $maxinh3$ denote the left hand sides of the homogeneous and the inhomogeneous Maxwell equations. The numbers after the equal sign specify the rank of the forms.

The mathematical symbols used in exterior calculus, translate into Excalc as listed in table \cite{14}.
Table 1. Translation of mathematical symbols into Excalc.

Now we are ready to write a small prototype program. Suppose we consider Minkowski spacetime and use polar coordinates \( \{x^0, x^1, x^2, x^3\} = \{t, r, \theta, \phi\} \). Then we have to specify a local basis of 1–forms, a coframe, by means of the Excalc declaration

\[
\text{coframe } \sigma(t) = d t, \\
\sigma(r) = d r, \\
\sigma(\theta) = r \ast d \theta, \\
\sigma(\phi) = r \ast \sin(\theta) \ast d \phi
\]

with signature \((1, -1, -1, -1)\);

\[
\text{frame } e;
\]

The command \texttt{frame e;} provides us with the frame \( e_\alpha \) dual to the coframe that we have chosen to call \( \sigma^\alpha \).

3.1. Static Point Charge

As test scenario, let us take the Coulomb field of a static point charge \( q \). The corresponding electromagnetic potential 1–form \( A \) and the field strength 2–form \( F = dA \) in Excalc read

\[
pot1 := -(q/r) \ast d t; \\
farad2 := d pot1;
\]

Now we translate into Excalc the homogeneous Maxwell equation \((9)\)

\[
\text{maxhom3 := d farad2;}
\]

the constitutive law for vacuum \((11)\)

\[
\text{excit2 := (1/mu) \ast \# farad2;}
\]

and the inhomogeneous Maxwell equation \((2)\)

\[
\text{maxinh3 := d excit2;}
\]
The result of these commands, typed in during an interactive Reduce session, is that both left hand sides of Maxwell’s equations are zero; in other words, Maxwell’s equations are fulfilled, and the ansatz for $A$ is justified.

The canonical energy-momentum 3–form of Maxwell’s field is given by

$$\Sigma_\alpha = e_\alpha \mathcal{J}L + (e_\alpha \mathcal{J}F) \wedge H,$$

where $L$ is the Lagrange 4–form of Maxwell’s field:

$$L = -\frac{1}{2} F \wedge H.$$  \hfill (13)

We declare the corresponding forms

\texttt{pform lmax4=4, maxenergy3(a)=3$}

and define the Lagrangian

\texttt{lmax4 := -(1/2) * farad2 ^ excit2;}

Now the energy-momentum 3–form \texttt{(12) can be computed straightforwardly (note that a lower index is always marked with a minus sign in Excalc):}

$$\text{maxenergy3(-a) := e(-a) _\mathcal{J} lmax4 + (e(-a) _\mathcal{J} farad2) ^ excit2;}$$

We find

$$\Sigma_t = -\frac{q^2}{2\mu r^4} \sigma^r \wedge \sigma^\theta \wedge \sigma^\varphi$$

and (up to minus signs depending on the ordering of the basis 1–forms $\sigma^\alpha$) equivalent expressions for the three other components. As expected, energy-momentum turns out to be proportional to $q^2/r^4$.

3.2. Small Ring Current

An equally simple task is the potential of the magnetic dipole moment $\vec{M}$ of a small ring current. From the literature (see Jackson\cite{17}, p.182) we take the vector potential in the Minkowski spacetime as $\vec{A} = \vec{M} \times \vec{r}/r^3$. We evaluate the vector product (see Fig. 4) and find $\vec{A} = \mathcal{M}/r^2 \sin \theta \vec{e}_\varphi$, where $\vec{e}_\varphi$ is the unit vector in $\varphi$–direction. Then the four-potential 1–form reads

$$A = \frac{\mathcal{M}}{r^2} \sin \theta \sigma^\varphi = \frac{\mathcal{M}}{r} \sin^2 \theta d\varphi.$$  \hfill (15)

\footnote{The conventions we use for energy-momentum are as follows: define the energy-momentum tensor $T_{\beta\alpha}$ in terms the energy-momentum 3–form $\Sigma_\alpha$ by $\Sigma_\alpha = T_{\beta\alpha} \eta^\beta$, with $\eta^\beta := * \sigma^\beta$. If this $T_{\beta\alpha}$ is multiplied by $-1$, we have the conventions of Ref.16. This is the reason for the minus sign on the right hand side of \texttt{(17) below.}}
Let us denote the angular momentum of the electrically charged circling particles by \( L \) and the angular momentum per unit mass by \( a := L/m \). Then the gyromagnetic parallelism yields (Jackson, p.183) \( M = g q a/2 \) with \( q = \) charge or, substituted in (15),
\[
A = g \frac{qa}{2r} \sin^2 \theta \, d\varphi. \tag{16}
\]
By \( g \) we denote the \( g \)-factor of the current distribution, which equals 1 for orbital angular momentum (as for a ring current) and 2 for spin angular momentum.

In Excalc we have
\[
\text{pot1} := g \times q \times a / (2 \times r) \times \sin(\text{theta})^2 \times \text{d phi};
\]
and, for field strength and excitation,
\[
\text{farad2} := \text{d pot1}; \\
\text{maxhom3} := \text{d farad2}; \\
\text{excit2} := (1 / \mu) \times \# \text{farad2}; \\
\text{maxinh3} := \text{d excit2};
\]
The Maxwell equations turn out to be fulfilled. We will apply the potential (16) in Sec.4.3 in a less trivial context.

4. Examples

4.1. Electromagnetism in a Rotating Frame

In the flat spacetime of special relativity, inertial frames of reference are realized by (pseudo-)orthonormal tetrads which are all arranged in a parallel fashion — here
parallel is meant in the ordinary (pseudo-)Euclidean way. A corresponding coframe, in polar coordinates, was specified in Sec. 3. If we want to describe non-inertial frames of reference, then we can turn to an accelerated and rotating frame, as used in Ref. 18, for example, (see also Ref. 19), and the parallel arrangement of the frames will be disturbed thereby.

There is a plethora of different books and papers on electrodynamics referred to non-inertial frames, see Refs. 20, 21 and literature given there. Formulated in terms of exterior differential forms, such formalisms simplify appreciably. We will demonstrate this claim by investigating the question of how a rotating observer will see the field of the magnetic dipole as specified in Sec. 3.2.

Let us assume that the axis of rotation coincides with the $z$-axis such that $\omega$ is the only nonvanishing component of the local angular velocity. We use the coframe  as given in Thirring (last equation on p.556) in order to express this configuration in Excalc:

```
coframe
  o(t) = (d t - omega*rho**2* d phi)/sqrt(1 - (omega*rho)**2),
o(rho) = d rho,
o(phi) = (d phi - omega* d t)*rho/sqrt(1 - (omega*rho)**2),
o(z) = d z
```

with signature 1,-1,-1,-1;
frame e;

The dipole potential (16) is expressed in terms of polar coordinates. To put it into our rotating cylindrical coordinates, we would need a pullback operator for forms which will, however, only be available in the next Excalc release. We shall overcome this deficiency by a small trick. Define

```
pform {r,theta}=0$
```

```p
r := sqrt(rho**2 + z**2);
theta := acos(z/sqrt(rho**2 + z**2));
```

If we start with polar coordinates $\{t,r,\theta,\phi\}$, Excalc will then substitute cylindrical coordinates $\{t,\rho,\phi,z\}$ instead. Using this transformation, the dipole potential (16) can be viewed from the rotating system of reference:

```
factor o(0),o(1),o(2),o(3), ^$ off exp$ on gcd$
pform pot1=1, {farad2,excit2}=2, {maxhom3,maxinh3}=3$
```

```
pot1 := g*q*a/(2*r) * sin(theta)**2 * d phi;
fardad2 := d pot1;
```

By using Excalc, one can prove easily that the Riemannian curvature belonging to our new coframe vanishes identically, that is, we are still in a flat Minkowski space.

Before making a new coframe statement, we recommend starting a new Reduce session.
By making use of the factor command we caused Reduce to “factor out” the terms given. Additionally, we turned off and on the two Reduce switches exp (expand) and gcd (greatest common divisor), respectively, in order to simplify the resulting expressions.

Let us have a closer look at the physical situation as seen by the rotating observer. For this purpose, we compute the components of the electric and the magnetic field, respectively (see Ref.19 for the $(3+1)$–decomposition of the electromagnetic field strength 2–form):

\[
\text{indexrange } \{i,j,k\} = \{\text{rho,phi,z}\}$
\[
\text{pform } \{\text{ee}(i),\text{bb}(i,j)\} = 0$
\[
\text{index_symmetries } \text{bb}(i,j) : \text{antisymmetric}$
\[
\text{ee}(-i) := \text{e}(-i) \mid (\text{e}(-t) \mid \text{farad2});$
\[
\text{bb}(-i,-j) := \text{e}(-i) \mid (\text{e}(-j) \mid \text{farad2});$

The indexrange declaration allows to have certain indices not ranging over all space-time dimensions but over a subset that can be chosen arbitrarily. Here we declared the indices $i, j,$ and $k$ to cover only the spatial section of spacetime.

The rotating observer still perceives, up to a Lorentz factor $\gamma = 1/\sqrt{1-(\omega \rho)^2}$, the magnetic dipole field

\[
B_{\phi z} = -\frac{\gamma \, agq}{2} \frac{3\rho z}{(\rho^2 + z^2)^{5/2}} = -\gamma \, M \frac{3\rho z}{(\rho^2 + z^2)^{5/2}}
\]
\[
B_{z \phi} = 0
\]
\[
B_{\rho \phi} = \frac{\gamma \, agq}{2} \frac{(\rho^2 - 2z^2)}{(\rho^2 + z^2)^{5/2}} = \gamma \, M \frac{(\rho^2 - 2z^2)}{(\rho^2 + z^2)^{5/2}}
\]

(we substituted again the magnetic dipole moment $M = agq/2$), but in addition he observes an electric field

\[
E_{\rho} = \frac{\gamma \, agq \, \omega \rho}{2} \frac{(\rho^2 - 2z^2)}{(\rho^2 + z^2)^{5/2}} = \gamma \, M \frac{\omega \rho \, (\rho^2 - 2z^2)}{(\rho^2 + z^2)^{5/2}}
\]
\[
E_{\phi} = 0
\]
\[
E_{z} = \frac{\gamma \, agq}{2} \frac{3\omega \rho^2 z}{(\rho^2 + z^2)^{5/2}} = \gamma \, M \frac{3\omega \rho^2 z}{(\rho^2 + z^2)^{5/2}}
\]

whose components correspond to those of the magnetic field rotated by $\pi/2$ around an axis perpendicular to the $z$–axis and multiplied by $\omega \rho$.

For completeness, we verify that the inhomogeneous Maxwell equation is still satisfied:

\[
\text{excit2} := (1/\mu) * \# \text{farad2};$
\[
\text{maxinh3} := d \text{excit2};$

\[
\text{excit2} := (1/\mu) * \# \text{farad2};$
\[
\text{maxinh3} := d \text{excit2};$

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4.2. Deriving the Reissner–Nordström from the Schwarzschild Solution

To give a less trivial example, we will derive the spacetime of a spherically symmetric mass distribution carrying a net electric charge \( q \). In general relativity, the corresponding solution of the Einstein equation is called the Reissner–Nordström solution.

We start from the uncharged spherically symmetric solution (the Schwarzschild solution). Its spacetime geometry is determined by the coframe that can immediately be expressed in Excalc:

\[
\begin{align*}
\text{pform psi}=0 \quad \text{fdomain psi}=\text{psi}(r) \\
\text{coframe o}(t) &= \text{psi} \quad * \ d \ t, \\
\text{o}(r) &= \left(1/\text{psi}\right) \quad * \ d \ r, \\
\text{o}(\text{theta}) &= \text{r} \quad * \ d \ \text{theta}, \\
\text{o}(\text{phi}) &= \text{r} \ * \ \sin(\text{theta}) \ * \ d \ \text{phi}
\end{align*}
\]

with signature \((1,-1,-1,-1)\)

Note that functions have to be declared as 0–forms also. The \text{fdomain} declaration specifies the independent variable(s) of a given form. The Schwarzschild function \( \psi \) has the form \( \psi = \sqrt{1 - 2m/r} \), where \( m \) is the mass of the object. It can be shown, see below, that this orthonormal coframe satisfies the vacuum Einstein equation indeed.

Now we want to generalize the Schwarzschild solution to a corresponding electrically charged solution. We proceed in exactly the same manner as in Sec.3.1: the Schwarzschild coframe, with arbitrary \( \psi \), is already prescribed (and has been checked), we type in (or read from a corresponding file) the commands for the point charge \( q \):

\[
\begin{align*}
\text{pform pot1}=1, \{\text{farad2,excit2}\}=2, \{\text{maxhom3, maxinh3}\}=3 \\
pot1 &= -(q/r) \ * \ d \ t; \\
\text{farad2} &= \ d \ \text{pot1}; \\
\text{maxhom3} &= \ d \ \text{farad2}; \\
\text{excit2} &= (1/\mu) \ * \ # \ \text{farad2}; \\
\text{maxinh3} &= \ d \ \text{excit2};
\end{align*}
\]

We find that Maxwell’s equations are still fulfilled, even though we did not even specify the function \( \psi \)! Note that this only turns out to be true since we specified the potential \( \text{pot1} \) in terms of \( dt \) and not \( o(t) \). However, the electromagnetic energy–momentum 3–form \( \Sigma_\alpha \) given in (12) acts as a source in the Einstein equation:

\[
G_\alpha = -\kappa \Sigma_\alpha .
\] (17)

The Einstein 3–form \( G_\alpha \) (called \( \text{einstein3} \) in the code) is defined according to (see Ref.23, e.g.):

\[
G_\alpha := (1/2) \ R^{\beta\gamma} \wedge (o_\alpha \wedge o_\beta \wedge o_\gamma) .
\] (18)
Basically we had put a point charge on a prescribed Schwarzschild geometry — and, up to now, we forgot the gravitating character of the electromagnetic field. The obvious consequence is that the corresponding Einstein-Maxwell equation is not satisfied. In order to fix this, we have to “relax” the Schwarzschild spacetime; this is done by reformulating the coframe. Since the charged solution sought should remain spherically symmetric, we try to change the function $\psi$ alone. The most straightforward ansatz that comes to mind is

$$\text{pform unknown}=0 \text{ fdomain unknown}=\text{unknown}(r)$$

$$\psi := \sqrt{1 - 2 * m/r + \text{unknown}};$$

The new function $\text{unknown}$ must be determined from the Einstein-Maxwell equation \[\text{(17)}\].

We first have to calculate the Riemann or Levi–Civita connection 1–form $\Gamma^{\alpha\beta}$ and the curvature 2–form $R^{\beta\gamma}$ (with identifiers $\text{chris1}$ and $\text{curv2}$). The corresponding Excalc code reads

$$\text{pform chris1}(a,b)=1, \text{curv2}(a,b)=2, \text{einstein3}(a)=3$$

$$\text{index_symmetries \ chris1}(a,b), \text{curv2}(a,b):\text{antisymmetric}$$

$$\text{riemannconx christ1}\ chris1(a,b) := \text{christ1}(b,a)$$

$$\text{curv2}(a,b) := d \text{chris1}(a,b) + \text{chris1}(-c,b) ^ \text{chris1}(a,c);$$

$$\text{einstein3}(-a) := (1/2) * \text{curv2}(b,c) ^ \#(\text{o}(-a)^\text{o}(-b)^\text{o}(-c));$$

The Excalc command $\text{riemannconx} <\text{identifier}>$ initiates the calculation of the connection 1–form and stores it in $\text{identifier}$. If $\text{unknown}$ vanishes, we have $G_{\alpha} = 0$, which proves our earlier claim that the original coframe satisfies the Einstein vacuum equation.

Subsequently, we compute the Maxwell energy–momentum, see \[\text{(12)}\]:

$$\text{pform lmax4}=4, \text{maxenergy3}(a)=3$$

$$\text{lmax4} := -(1/2) * \text{farad2} ^ \text{excit2};$$

$$\text{maxenergy3}(-a) := e(-a) _\text{l} \text{lmax4} + (e(-a) _\text{l} \text{farad2})^\text{excit2};$$

If the vacuum Einstein-Maxwell equation is to be satisfied, the sum of the Einstein 3–form and (a constant times) the Maxwell energy–momentum 3–form must vanish:

$$\text{pform ode3}(a)=3$$

$$\text{ode3}(a) := \text{einstein3}(a) + \kappa * \text{maxenergy3}(a)$$

**Note that the old $\text{antisymmetric} <\text{identifier}>$; declaration is unsupported already in Reduce 3.5. It has to be replaced by $\text{index_symmetries} <\text{identifier}(a,b):\text{antisymmetric}$;**

**The transposition according to $\text{chris1}(a,b) := \text{christ1}(b,a)$ is necessary to convert the Excalc conventions for the connection 1–form into our own convention, see Ref.24.**
A typical component of the output of ode3 reads

\[
\text{ode3}^\varphi = \left( -\frac{1}{2} \frac{\partial \text{unknown}}{\partial r^2} - \frac{1}{r} \frac{\partial \text{unknown}}{\partial r} + \frac{\kappa}{2\mu} \frac{q^2}{r^4} \right) o^t \wedge o^r \wedge o^\theta.
\] (19)

Requiring the coefficients to vanish yields a second order differential equation for the function \text{unknown}. It can be solved by means of the ansatz \text{unknown} := a * r^n; A short Reduce calculation yields\[If ode3 was of a more complicated nature, we could feed it into Maple or Mathematica, which have at the moment a more complete ODE solver than Reduce.

\[\text{unknown} := \kappa/(2\mu) * (q/r)^2;\]

This reproduces, when \psi is evaluated, the well known Reissner–Nordström function:

\[\psi := \psi ;\]

\[
\psi := \sqrt{1 - \frac{2m}{r} + \frac{\kappa}{2\mu} \left( \frac{q}{r} \right)^2}.
\] (20)

It is remarkable that, although we are now working on a curved spacetime manifold, \textit{not a single command} of the Maxwell code had to be changed. This should clearly demonstrate the simplifying value of the formulation of Maxwell’s theory presented here.

4.3. Deriving the Kerr-Newman from the Kerr Solution

In analogy to Sec.4.2, we will determine the solution to the Einstein-Maxwell equation of a rotating axial symmetric mass distribution with net charge \(q\). It is called the Kerr-Newman solution.

Again we take as starting point the spacetime of the corresponding \textit{uncharged} solution, the Kerr solution. Using Boyer-Lindquist coordinates \(\{t, \rho, \theta, \varphi\}\), the orthonormal coframe reads:

\[
pform \{rr, delsqr\}=0$
\[
\text{fdomain } rr=rr(\rho, \theta), delsqr=delsqrt(\rho)$
\[
\text{coframe}
\]
\[
o(t) = (delsqrt/rr) * (d t - (a*\sin(\theta))^2) * d \phi),
\]
\[
o(\rho) = (rr/delsqr) * d \rho,
\]
\[
o(\theta) = rr * d \theta,
\]
\[
o(\phi) = 1/rr * \sin(\theta) * (a*d t - (\rho^2 + a^2) * d \phi)
\]
\[
\text{with signature } 1, -1, -1, -1$
\]
\[
\text{frame e}$

\[††If ode3 was of a more complicated nature, we could feed it into Maple or Mathematica, which have at the moment a more complete ODE solver than Reduce.\]
The functions \texttt{delsqrt} and \texttt{rr} depend on the parameters \(m\) (mass) and \(a\) (angular momentum per unit mass). They have the form \(\texttt{delsqrt} = \sqrt{\rho^2 + a^2 - 2m\rho}\) and \(\texttt{rr} = \sqrt{\rho^2 + (a \cos \theta)^2}\). We can prove (in the same way as in Sec. 4.2) that this coframe satisfies the vacuum Einstein equation.

To determine the Kerr-Newman solution, we first have to specify the electromagnetic potential. To start with, we take the potential of a rotating charge distribution which we approximate to lowest order of \(1/\rho\) by a static point charge and a magnetic dipole (see Secs. 3.1 and 3.2):

\[
\text{pot1} = -\frac{q}{\rho} \left( d t - \frac{g}{2} a \sin \theta \sin \phi \right);
\]

Here the \(g\)-factor is to be determined.

In Minkowski spacetime, the homogeneous and inhomogeneous Maxwell equations are satisfied, as we have seen in Secs. 3.1 and 3.2 (a superposition of solutions is again a solution). However, now we have to compute Maxwell’s equations on the curved Kerr spacetime:

\[
\text{farad2} := d \text{pot1};
\text{maxhom3} := d \text{farad2};
\text{excit2} := \frac{1}{\mu} \cdot \# \text{farad2};
\text{maxinh3} := d \text{excit2};
\]

We find that the Hodge star operator spoils (by means of \(\det g\)) the inhomogeneous Maxwell equation. Since we expect that this failure is connected to the \(1/\rho\) behavior of the potential, we change the potential by the following ansatz

\[
\text{pot1} = -\frac{q}{\rho \cdot f} \left( d t - \frac{g}{2} a \sin \theta \sin \phi \right);
\]

Before we repeat the calculation of Maxwell’s equation, we again make use of the Reduce switches \texttt{exp} and \texttt{gcd}:

\[
\text{off exp}\$\text{on gcd}$
\text{farad2} := d \text{pot1};
\text{maxhom3} := d \text{farad2};
\text{excit2} := \frac{1}{\mu} \cdot \# \text{farad2};
\text{maxinh3} := d \text{excit2};
\]

The expression for \texttt{maxinh3} looks rather complicated — if we require it to vanish, we get a system of second order PDEs in two variables. We try solve it by inspection; in a first step first let us specify the \(g\)-factor. There are two values for \(g\) that seem plausible: \(g = 1\) corresponds to a small ring current as considered in Sec. 3.2, and \(g = 2\) is the \(g\)-factor of an electron. As \texttt{maxinh3} contains \((g-2)\) several times as a factor, the choice
\[ g := 2\]

seems promising. Indeed, if we compute the inhomogeneous Maxwell equation again

\[
\text{maxinh3} := \text{d excit2};
\]

the coefficient of \( o^t \wedge o^\rho \wedge o^\theta \) turns out to be quite compact. If it is to vanish, we have to solve

\[
\frac{2 qa}{rr f^3} \left( \frac{\cos \theta \partial f}{\partial \theta} - \rho \sin \theta \frac{\partial f}{\partial \rho} \right) = 0. \quad (21)
\]

Solving this equation by hand, fixes the function \( f \) up to a constant (which we can absorb into the charge \( q \)). We substitute the solution and check the inhomogeneous field equation:

\[
f := 1 + (a/rho * \cos(theta))**2$
\]

\[
\text{maxinh3};
\]

Note that the inhomogeneous Maxwell equation is satisfied irrespective of the explicit form of the functions \( rr \) and \( \text{delsqrt} \).

With this result, the final form of the potential can be checked:

\[
pot1 := \text{pot1};
\]

\[
\text{pot1} := \frac{q \rho rr}{(a^2 \cos^2 \theta + \rho^2) \text{delsqrt}} o^t. \quad (22)
\]

What remains to be found is the solution of the Einstein-Maxwell equation. This is done in much the same manner as in Sec.4.2 by modifying the function \( \text{delsqrt} \). In addition, we now specify \( rr \) without modification:

\[
pform u1=0$ fdomain u1=u1(r,theta)$$
\]

\[
\text{delsqrt} := \text{sqrt(rho**2 + a**2 - 2*m*rho + u1)}$
\]

\[
\text{rr} := \text{sqrt(rho**2 + (a * cos(theta))**2)}$
\]

The field equation must determine the new function \( u1 \). Thus we compute the left hand side (the Einstein 3-form)

\[
\text{riemannconx chris1}$ chris1(a,b) := chris1(b,a)$
\]

\[
\text{pform curv2(a,b)=2, einstein3(a)=3}$
\]

\[
\text{index_symmetries curv2(a,b):antisymmetric}$
\]

\[
\text{curv2(a,b) := d chris1(a,b) + chris1(-c,b) \wedge chris1(a,c);}\]

\[
\text{einstein3(-a) := (1/2) * curv2(b,c) \wedge (#(o(-a)\wedge o(-b))\wedge o(-c))};
\]

and the right hand side (the electromagnetic energy–momentum 3–form)

\[
\text{pform lmax4=4, maxenergy3(a)=3}$
\]

\[
lmax4 := -(1/2) * \text{farad2} \wedge \text{excit2};
\]

\[
\text{maxenergy3(-a) := e(-a) \wedge lmax4 + (e(-a) \wedge \text{farad2})\wedge \text{excit2};}
\]

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of the field equation. These yield a differential equation for the function $u_1$:

```plaintext
pform ode3(a)=3
ode3(a) := einstein3(a) + kappa * maxenergy3(a);
```

This system can be solved most easily by inspection of the $\varphi$–component of $\text{ode3}$:

$$\text{ode3}\varphi = -\frac{a \cos \theta}{2 \text{delsqrt}} \frac{\partial u_1}{\partial \theta} \rho \wedge \theta \wedge \varphi + \frac{\kappa q^2 - 2 \mu u_1}{2 \text{rr}^4} \rho \wedge \theta \wedge \varphi \tag{23}$$

The first term demands that $u_1$ must not dependent on the coordinate $\theta$. The second term vanishes if we substitute

$$u_1 := \frac{\kappa}{2 \mu} q^2$$

Finally, we display the explicit form that $\text{delsqrt}$ evaluates to:

```plaintext
delsqrt := delsqrt;
delsqrt := \sqrt{\rho^2 - 2 m \rho + \frac{\kappa q^2}{2 \mu} + a^2} \tag{24}
```

Again, although we modified the spacetime geometry via the function $\text{delsqrt}$, we do not worry about the inhomogeneous Maxwell equation because it has been found to be satisfied irrespective of the explicit form of the functions contained in the coframe.

This completes our deduction of the Kerr-Newman solution from the Kerr solution and the Einstein-Maxwell equations. This derivation is certainly not very original. However, the charged version of the Kerr solution with dynamic torsion was found by this type of technique.

**Acknowledgment**

We are grateful to David Hartley for helpful comments.

**Appendix: Maxwell program**

```plaintext
% file mustermax.exi, 1995-02-15

load_package excalc$
```

```plaintext
% the Maxwell program is preceded by coframe and frame commands:
% specify the coframe by filling in the exact form of the
% appropriate coframe with your favorite editor, specify possibly
% unknown functions (zero-forms) psi etc. and their domains

pform psi=0$
fdomain psi=psi(r)$
```
coframe o(0) = d t,  
o(1) = d r,  
o(2) = r * d theta,  
o(3) = r * sin(theta) * d phi  
with signature (-1,1,1,1)$
frame e$

% here begins the Maxwell program proper: unknown functions aa0,  
% aa1, etc. are prescribed for the components of the potential  
% 1-form; field strength farad2, excitation excit2, and the left  
% hand sides of the Maxwell equations are defined

pform {aa0,aa1,aa2,aa3}=0, pot1=1, {farad2,excit2}=2,  
{maxhom3,maxinh3}=3$

pot1 := aa0*o(0) + aa1*o(1) + aa2*o(2) + aa3*o(3)$
farad2 := d pot1;
maxhom3 := d farad2;
excit2 := (1/mu) * # farad2;
maxinh3 := d excit2;

% Maxwell Lagrangian and energy-momentum current are assigned

pform lmax4=4, maxenergy3(a)=3$

lmax4 := -(1/2)*farad2^excit2;
maxenergy3(-a) := e(-a) _|lmax4 + (e(-a) _|farad2)^excit2;

end;

% Be sure that you use a blank before the interior product sign!

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