Comparing second-order gravitational self-force and effective one body waveforms from inspiralling, quasi-circular and nonspinning black hole binaries II: the large-mass-ratio case

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We compare recently computed waveforms from second-order gravitational self-force (GSF) theory to those generated by a new, GSF-informed, effective one body (EOB) waveform model for (spin-aligned, eccentric) inspiralling black hole binaries with large mass ratios. We focus on quasi-circular, nonspinning, configurations and perform detailed GSF/EOB waveform phasing comparisons, either in the time domain or via the gauge-invariant dimensionless function $Q_\omega \equiv \omega^2/\dot{\omega}$, where $\omega$ is the gravitational wave frequency. The inclusion of high-PN test-mass terms within the EOB radiation reaction (notably, up to 22PN) is crucial to achieve an EOB/GSF phasing agreement below 1 rad up to the end of the inspiral for mass ratios up to 500. For larger mass ratios, up to $5 \times 10^4$, the contribution of horizon absorption becomes more and more important and needs to be accurately modeled. Our results indicate that our GSF-informed EOB waveform model is a promising tool to describe waveforms generated by either intermediate or extreme mass ratio inspirals for future gravitational wave detectors.

I. INTRODUCTION

In the next twenty years we will witness the development and the launch of new gravitational wave observatories, both ground-based, like Einstein Telescope (ET) \cite{ET1, ET2} and Cosmic Explorer (CE) \cite{CE1, CE2}, and space-based, like LISA \cite{LISA}, TianQuin \cite{TianQuin} and Taiji \cite{Taiji}. Given their increased sensitivities and the larger frequency range they will cover, these detectors will be able to see many more sources than the compact binary coalescences currently observed by the LIGO-Virgo-KAGRA collaboration. In particular, they are expected to observe both intermediate- and extreme-mass-ratio inspirals (IMRIs and EMRIs, respectively) with mass ratios ranging between $10^{-2} - 10^{-4}$ in the first case, and $\sim 10^{-4} - 10^{-7}$ in the second case. EMRIs in particular are binaries where a stellar-mass compact object inspirals for years around a supermassive black hole. The waveform phenomenology for these binaries can be extremely complicated, as it may involve at the same time both high eccentricity and rapid precession of the orbital plane. Because of the resulting, rich harmonic structure in the waveform, and the large number of orbits near merger, these extreme-mass-ratio inspirals have the potential to unveil and test deep features of strong-field General Relativity \cite{GR}. Although the dynamics of the binary can be seen as a perturbation to the underlying Kerr spacetime, the accurate description of its evolution is challenging, since the self-field of the smaller object cannot be neglected. The accurate description of the action of the self-field of the smaller object on the dynamics is described within the gravitational self-force (GSF) program \cite{GSF1, GSF2}. There are different efforts in developing waveform templates for EMRIs building on GSF results \cite{GSF1, GSF2, GSF3, GSF4}. Much less accurate, though faster schemes, go under the name of kludge waveforms \cite{Kludge1, Kludge2, Kludge3, Kludge4}. The recently released FastEMRIWaveform package \cite{FastEMRI1, FastEMRI2} combines speed and accuracy, but is currently implemented only for eccentric inspirals into a non-rotating black hole. All these approaches are based on expansions in the mass ratio $q$, under the hypothesis that it is small.

The effective one body (EOB) approach \cite{EOB1, EOB2}, on the other hand, is a powerful analytical framework that can describe the dynamics of any binary all over the parameter space, for any value of the mass ratio and for any orientation of the vectorial individual spins. Although the EOB method has been largely exploited in building templates for comparable-mass binaries, it is a natural framework to construct waveforms also for binaries with large mass ratios \cite{EOB3, EOB4, EOB5}, since it builds upon a deformation of the geometry of a Kerr black hole with the symmetric mass ratio as the deformation parameter.

In a recent work \cite{Paper1} (hereafter Paper I), we compared waveforms generated by the state-of-the-art EOB model TEOBResumS \cite{TEOB1, TEOB2}, with those from a complete gravitational self-force model called 1PAT1 \cite{1PAT1}. Although the 1PAT1 model is crucially lacking the transition from inspiral to plunge, it was possible to show that the major
difference between the two models arises from contributions that are linear in the symmetric mass ratio \( \nu \equiv m_1 m_2 / M^2 \), where \( m_{1,2} \) are the masses of the two bodies, \( M \equiv m_1 + m_2 \) and we use the convention \( m_1 \geq m_2 \). This result is not surprising. Since TEOBResumS was originally conceived as a waveform model aiming at primarily generating waveforms for comparable mass binaries \(^1\), a relatively limited amount of linear-in-\( \nu \) (or test-mass) information (both in the conservative and nonconservative sector of the model) was included. The rationale behind this choice was to use some test-mass information to improve the behavior of the model for comparable masses, while avoiding this information becoming dominant in this case. A typical example of this procedure is that TEOBResumS partly relies on a mixed TEOBResumS flux \( \nu \); i.e., full \( \nu \)-dependent terms up to 3PN are hybridized with test-mass terms up to 5PN or 6PN depending on the multipole. These hybrid expressions are further resummed using various choices of Padé approximants. Similarly, the TEOBResumS Hamiltonian only incorporates terms up to 3PN or 5PN, depending on the particular EOB potential.

A step towards incorporating full 1GSF information (i.e., linear in \( \nu \)) in the Hamiltonian was taken by Antonelli et al. \(^{39}\). In particular, they used a post-Schwarzschild Hamiltonian \(^\text{10, 11}\) to overcome the well-known problems related to the presence of the light-ring coordinate singularity in the standard EOB gauge (or Damour-Jaranowski-Schäfer, DJJS hereafter \(^{26, 12}\)). Although promising, the approach of \(^{39}\), that was limited to the case of nonspinning binaries, needs more development to construct a complete model, informed by Numerical Relativity simulations, able to span the full range of mass ratios. Reference \(^{43}\) introduced an alternative strategy, that was however specifically designed for IMRIs and EMRIs (including aligned-spin and eccentricity), where the importance of the merger is practically negligible and the signal-to-noise ratio is dominated by the hundred of thousands of cycles of the inspiral. To target these sources, Ref. \(^{43}\) proposed an EOB model in the DJJS gauge (and thus with the well-known light-ring singularity) but introduced suitable resummation procedures to improve the behavior of the PN-expanded EOB potentials in the strong field. By additionally informing them with exact GSF results \(^{44}\), Ref. \(^{43}\) introduced the first, and so far only, EOB waveform model for eccentric, spin-aligned IMRIs and EMRIs that is informed by GSF numerical results.

The aim of this paper is to test the GSF-informed EOB model of \(^{43}\) against the 2GSF waveforms of \(^{10}\), analogously to what Paper I did using the standard TEOBResumS model. The paper is organized as follows. In Sec. \(^{11}\) we recap the main elements of the model of \(^{43}\), reminding readers of the structure of the Hamiltonian and giving some details about the structure of the radiation reaction, which is a novelty introduced here. In Sec. \(^{11}\) we compare the EOB and GSF models, first performing a waveform alignment in the time domain and then using the same gauge-invariant frequency-domain analysis we exploited in Paper I. Section \(^{11}\) gives a more precise analytic interpretation of the results presented in Sec. \(^{11}\) while Sec. \(^{11}\) focuses on the impact of horizon absorption and on the need to accurately model it within EOB to correctly describe EMRIs. Our conclusions are collected in Sec. \(^{11}\). We use units with \( G = c = 1 \).

II. GSF-INFORMED EOB MODEL

A. The Hamiltonian: a reminder

Let us briefly recall the elements of the GSF-informed EOB model of Ref. \(^{43}\). The model builds upon the spin-aligned, eccentric TEOBResumS model, the TEOBResumS-DALI \(^{15, 47}\), but the low-PN accurate, NR-informed EOB potentials are replaced by the 1GSF-informed ones. More precisely, the potentials \((A, D, Q)\) at linear order in \( \nu \) formally read

\[
A = 1 - 2u + \nu a_{1SF}(u) ,
\]

\[
D = 1 + \nu d_{1SF}(u) ,
\]

\[
Q = \nu q_{1SF}(u) ,
\]

where \( u \equiv 1/R = M/R \) is the dimensionless Newtonian potential. Reference \(^{43}\), building upon 8.5PN results \(^{15}\), showed that the strong-field (i.e., nearby the LSO) behavior of the three functions \((a_{1SF}, d_{1SF},q_{1SF})\) can be improved by implementing a certain factorization and resummation procedure based on Padé approximants. Moreover, these Padé-resummed functions can be modified by a certain flexing factor, which effectively takes into account higher-order corrections and can be informed by fitting to the numerical data of Refs. \(^{42, 44}\). This correcting factor yields GSF-informed analytic potentials that display \( \lesssim 0.1\% \) fractional difference with the numerical data up to \( u = 1/3 \) for \( a_{1SF} \) and up to \( u = 1/5 \) for \( d_{1SF} \) and \( q_{1SF} \); see Figs. 2, 3 and 4 of Ref. \(^{43}\). The potentials then enter the Hamiltonian as described in Sec. II of \(^{43}\).

B. Waveform and radiation reaction

To fix conventions, the strain waveform is decomposed on spin-weighted spherical harmonics as

\[
h_+ - i h_x = \frac{1}{D_L} \sum_{\ell=2}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} h_{\ell m - 2} Y_{\ell m}(\ell, \phi) ,
\]

where \( D_L \) indicates the luminosity distance, \(-2 Y_{\ell m}(\ell, \phi)\) are the \( s = -2 \) spin-weighted spherical harmonics,
$\iota$ is the inclination angle with respect to the orbital plane, and $\phi$ the azimuthal one. In the following, we will also work with the Regge-Wheeler-Zerilli (RWZ) normalization convention and express the waveform as $\Psi_{\ell m} = h_{\ell m} / \sqrt{(\ell + 2)(\ell + 1)(\ell - 1)}$. The RWZ normalized strain quadrupole waveform is then separated into standard 3PN-accurate terms in the tidal normal approximation of the mode [49], normalized angular momentum depending on the parity effective source of the field (effective energy or Newton-normalization convention and express the waveform as a sum of leading-order logarithms, while $\rho_{\ell m}$ is the tail factor, which resums an infinite number of leading-order logarithms, while $\rho_{\ell m}$ and $\delta_{\ell m}$ are the residual amplitude and phase corrections, respectively.

For simplicity, Ref. [43] used the standard radiation reaction implemented in TEOBResumS, with the (resummed) PN orders of the various multipoles chosen as in Refs. [34, 35]. However, Ref. [43] already pointed out that the standard TEOBResumS analytical flux needs to be improved to achieve a faithful representation of the exact flux (obtained numerically) in the model [50]. For simplicity, Ref. [43] used the standard radiation reaction already exploited in Sec. VA of Ref. [37], and $\delta_{\ell m}$ is the effective source of the field (effective energy or Newton-normalization convention and express the waveform as a sum of leading-order logarithms, while $\rho_{\ell m}$ is the tail factor, which resums an infinite number of leading-order logarithms, while $\rho_{\ell m}$ and $\delta_{\ell m}$ are the residual amplitude and phase corrections, respectively.

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As a first attempt, we took precisely the model of Ref. [43], in the quasi-circular limit and performed phasing comparisons (either in the time domain or using the local function) for different mass ratios up to $q = 5000$. The dephasing we found is largely nonnegligible as $q$ increases, as illustrated in Appendix A. This is not surprising, and it is consistent with the relatively poor accuracy of the standard TEOBResumS flux in the test-mass limit, as pointed out in the Appendix of [44]. To overcome this difficulty, we attempt here to use the 3$^{+}$PN radiation reaction already exploited in Sec. VA of Ref. [37]. We remind the reader that the notation 3$^{+}$PN means that the standard 3PN-accurate terms in the $\rho_{\ell m}$‘s (that depend

on $\nu$) are hybridized with test-mass terms (that are $\nu$-independent) so as to achieve global 22PN accuracy for all $\rho_{\ell m}$ functions. For simplicity, we do not attempt any additional resummation (e.g., using Padé approximants) on these resulting hybrid functions, although it might be useful to further improve the behavior of the residual PN series in strong field, especially in the presence of spin [54]. From now on, the 3$^{+}$PN-accurate radiation reaction will become our standard choice and we will generically refer to it as the hybrid flux. We will see in the next section that it is essential to deliver an excellent EOB/GSF phasing agreement for large-mass-ratio binaries.

### III. EOB-GSF Phasing Comparisons

The 1PAT1 model was introduced in Ref. [16] and extensively discussed in Paper I, to which we refer the reader for additional technical details. In this section we consider mass ratios $q = (26, 32, 36, 50, 64, 128, 500)$ and compare EOB and GSF waveform phasings using either time-domain or frequency-domain analyses, analogously to what was done in Paper I.

#### A. Time-domain alignment

We align waveforms in the time domain via minimization of the phase difference on a certain interval as in Paper I (for details see Sec. VA therein), and evaluate the phase difference at the time corresponding to the GSF break-down frequency as defined in Eq. (30) of Paper I. The results are displayed in Table 1 while a representative waveform, for $q = 500$, is shown in Fig. 1. The final dephasings are all positive, which means that the EOB plunge is in advance of the GSF one, and the EOB
evolution is overall faster (less adiabatic) than the GSF one. In fact, when comparing these results to Paper I (Table II therein), we can see that the final phase difference for $q = (32, 64, 128)$ for TEDBResumS is negative, corresponding to a delayed plunge for these mass ratios. The implementation of the GSF-tuned potential and of the hybrid 3+19PN flux are thus effective in allowing for a less adiabatic EOB evolution for large mass ratios.

B. Gauge-invariant analysis

Again we perform the same gauge-invariant analysis of Paper I. We exploit the adiabaticity parameter

$$Q_\omega = \frac{\omega^2}{\omega_2},$$

where $\omega \equiv \omega_{22}$ is the $\ell = m = 2$ waveform frequency. Within the GSF approach, for a fixed value of $\omega$, $Q_\omega$ can be given as an expansion in $\nu$, i.e.

$$Q_\omega(\omega; \nu) = \frac{Q_0^0(\omega)}{\nu} + Q_1^1(\omega) + Q_2^2(\omega)\nu + O(\nu^2).$$

For details on the different nPA contributions $Q^n_\omega$, see Sec. VI in Paper I. Following common nomenclature in the self-force literature, we refer to quantities that make order-$\nu^{n-1}$ contributions to the orbital phase as “nth post-adiabatic order” (nPA).

Given the resummed structure of the EOB Hamiltonian, the actual $Q_\omega^{\text{EOB}}$ has in fact an infinite number of $\nu$-dependent terms and Eq. (8) is formally obtained by expanding in $\nu$. 1PAT1 likewise contains an infinite number of terms in Eq. (8), but it only yields complete information about the first two terms, $Q_0^0$ and $Q_1^1$; higher-order GSF calculations will lead to different results for the higher-order coefficients $Q^n_\omega$ with $n > 1$. Our aim here is to extract the three functions $Q_0^0$, $Q_1^1$ and $Q_2^2$ from 1PAT1 and TEDBResumS and compare them. This will give us a precise quantitative understanding of the differences between the two models in the limit of small $\nu$. For the fit we use the same procedure described in Paper I, using mass ratios $q = 26, 32, 36, 50, 64, 128, 500$, and a range $[\omega_{\text{min}}, \omega_{\text{max}}] = [0.023, 0.09]$ with spacing $\Delta \omega = 0.001$. For each value of $\omega$ we fit $Q_\omega(\omega; \nu)$ using Eq. (8). The obtained coefficients are plotted in Fig. 2 along with the exact GSF results for $Q_0^0$ and $Q_1^1$ (as computed in Paper I). All three contributions to the $Q_\omega$ expansion now show a good EOB/GSF agreement, especially concerning $Q_0^0$ and $Q_1^1$. In Fig. 3 we compare these results to those of Paper I. First, we see how the results concerning the GSF fit and the exact $Q_\omega$ are more consistent here with respect to Paper I. This is due to the different choice for the mass ratios included in the fit, as already pointed out in Paper I (see Fig. 11 therein). Then from Fig. 3 we can infer that the new hybrid 3+19PN flux draws the EOB $Q_\omega^{\text{EOB}}$ nearer to the GSF one, while the GSF-tuned contribution to the EOB potential is responsible for the enhancement in $Q_1^1$. A deeper justification for this will be given in the following, considering analytical expressions for the three coefficients.

To assess how much each term in the expansion of $Q_\omega$ impacts the phasing, we can estimate three contributions to the phase difference on the frequency interval $(\omega_1, \omega_2)$:

$$\Delta \phi_0 = \frac{1}{\nu} \int_{\omega_1}^{\omega_2} \log(\omega) \left( Q_0^{\text{EOB}} - Q_0^{\text{GSF}} \right),$$

$$\Delta \phi_1 = \nu \int_{\omega_1}^{\omega_2} \log(\omega) \left( Q_1^{\text{EOB}} - Q_1^{\text{GSF}} \right),$$

$$\Delta \phi_2 = \frac{1}{\nu} \int_{\omega_1}^{\omega_2} \log(\omega) \left( Q_2^{\text{EOB}} - Q_2^{\text{GSF}} \right),$$

so that the total phase difference between $(\omega_1, \omega_2)$ is

$$\Delta \phi(\omega_1, \omega_2) = \Delta \phi_0 + \Delta \phi_1 + \Delta \phi_2.$$
The result of this calculation over the frequency interval \((\omega_1, \omega_2) = (0.023, 0.09)\) is displayed in Table I. As already stressed in Paper I, the results of the integration of \(Q_\omega\) on a given frequency interval cannot be compared to the phase differences obtained via time-domain alignment of the waveforms. The phase differences here are all negative due to the fact that the GSF evolution is more adiabatic than the EOB one for these mass ratios and in this frequency range (compare with Table IV in Paper I). We also see that the absolute value is decreasing as the mass ratio increases, correspondingly to the fact that \(\Delta \phi_0\) becomes progressively more dominant, while the inverse is true for \(\Delta \phi_2\); when it comes to higher mass ratios the EOB/GSF consistency in \(Q_\omega^2\) is more important than their disagreement in \(Q_\omega^2\).

**IV. UNDERSTANDING THE Q\(\_\) expansion**

The behavior of \(Q_\omega\) and of its three different contributions, \((Q_\omega^0, Q_\omega^1, Q_\omega^2)\), can be understood analytically when working in the circular approximation. Assuming for simplicity that the gravitational wave frequency is \(\omega_{22} = 2\Omega\), where \(\Omega\) is the EOB orbital frequency\(^3\) we have

\[
Q_\omega = 2 \frac{\Omega^2}{\Omega} ,
\]  

(13)

and using the frequency parameter \(x \equiv \Omega^{2/3}\), we have

\[
\tilde{\Omega} = \partial_j \Omega \partial_{\ell} j = \frac{3}{2} x^{1/2} \partial x \tilde{\varphi} ,
\]  

(14)

where \(j \equiv J_{\text{circ}}/\mu M\) is the orbital angular momentum along circular orbits and we replaced \(\partial_j = \tilde{F}_{\varphi}\). The angular momentum flux is written as \(\tilde{F}_{\varphi} = \tilde{F}_{\varphi}^{\text{res}}/\nu = \nu \tilde{F}_{\varphi}^{\text{LSF}} + \nu^2 \tilde{F}_{\varphi}^{\text{DSF}} + \nu^3 \tilde{F}_{\varphi}^{\text{SSF}}\) (i.e., as a 2PA expansion) to meaningfully compare EOB and GSF contributions. Note, however, that the complete EOB flux, which is summed up to \(\ell = 8\), has many more \(\nu\)-dependent terms the contribution to the tail cannot be extracted analytically in closed form; see Appendix E.2 of Ref. [35]. Nonetheless, this approximation does not change the conclusions of our reasoning below.

\[\text{FIG. 3. Comparing the EOB/GSF differences in the coefficients } Q_\omega^0, Q_\omega^1 \text{ and } Q_\omega^2 \text{ found in this paper with respect to Paper I, whose results are labeled as TEOBResumS.}\]

| \(q\) | \(\omega_{22}^{\text{GSF break}}\) | \([\omega_L, \omega_R]\) | \(\Delta \phi_{\text{EOB/GSF}}\) |
|------|-----------------|-----------------|-----------------|
| 26   | 0.1135          | [0.023, 0.026]  | 0.42151         |
| 32   | 0.1146          | [0.023, 0.027]  | 0.36474         |
| 36   | 0.1151          | [0.023, 0.027]  | 0.33283         |
| 50   | 0.1167          | [0.023, 0.027]  | 0.29113         |
| 64   | 0.1178          | [0.023, 0.028]  | 0.25885         |
| 128  | 0.1207          | [0.023, 0.027]  | 0.23042         |
| 500  | 0.1251          | [0.023, 0.027]  | 0.21041         |

**TABLE I.** From left to right, the columns report: the mass ratio \(q\), the GSF breakdown frequency, \(\omega_{22}^{\text{GSF break}}\) as defined in Eq. (30) of Paper I; the alignment interval used in the time-domain phasing; the phase difference, computed up to \(\omega_{22}^{\text{GSF break}}\), using the time-domain alignment.

| \(q\) | \(\Delta \phi_0\) | \(\Delta \phi_1\) | \(\Delta \phi_2\) | \(\Delta \phi_{\text{[1, 2]}}\) |
|------|-----------------|-----------------|-----------------|-----------------|
| 26   | 0.0011045       | -0.20971        | -0.10711        | -0.31572        |
| 32   | 0.0013406       | -0.20971        | -0.088252       | -0.29662        |
| 36   | 0.001498        | -0.20971        | -0.078977       | -0.28719        |
| 50   | 0.0020493       | -0.20971        | -0.057734       | -0.26539        |
| 64   | 0.0026006       | -0.20971        | -0.045494       | -0.2526         |
| 128  | 0.0051215       | -0.20971        | -0.023101       | -0.22769        |
| 500  | 0.019776        | -0.20971        | -0.0059827      | -0.19592        |

**TABLE II.** From left to right, the columns report: the mass ratio \(q\), the phase differences due to the first three term in the expansion of \(Q_\omega\), and the sum of these latter. The \(\Delta \phi\)'s are obtained using the definition [3], integrating on the frequency interval \([\omega_1, \omega_2] = [0.023, 0.09]\).
because it incorporates all the known $\nu$ dependent up to 3PN order. The $Q_\omega$ function can be rewritten as

$$Q_\omega(x) = \frac{4}{3} x^{5/2} \left\{ 1 - \frac{\mathcal{F}^{2SF}_{\varphi}}{\mathcal{F}^{1SF}_{\varphi}} \right. \right. \nonumber \\ \\
- \nu^2 \left. \left[ \frac{\mathcal{F}^{3SF}_{\varphi}}{\mathcal{F}^{1SF}_{\varphi}} - \left( \frac{\mathcal{F}^{2SF}_{\varphi}}{\mathcal{F}^{1SF}_{\varphi}} \right)^2 \right] \right\} \partial_x j. \tag{15}$$

Within the EOB approach, the angular momentum along circular orbits is given by

$$j^2 = -\frac{\partial_u A}{\partial_u(u^2 A)} = -\frac{\partial_u A}{2u A}, \tag{16}$$

where $u = M/r$ is the dimensionless gravitational Newtonian potential and $\tilde{A}(u; \nu) \equiv A(u; \nu) + \frac{\nu}{2} \partial_u A(u; \nu)$. Considering the interbody EOB potential $\tilde{A}$ as a formal expansion up to $\nu^2$,

$$A(u; \nu) = 1 - 2u + \nu a_1(u) + \nu^2 a_2(u) + O(\nu^3), \tag{17}$$

from Eq. (16) one obtains $j$ at 2PA order as

$$j(u) = -\frac{1}{32(1 - 3u)^2} \sqrt{\frac{1}{u - 3u^2}} \left[ \left( -32 + 192u - 288u^2 \right) \right. \nonumber \\ \\
+ \nu \left( \frac{3}{4} \left( 8 - 40u + 48u^2 \right) + a_1(u)(16 - 48u) \right) \nonumber \\ \\
+ \nu^2 \left( \frac{2}{12} \left( 1 - 8u + 12u^2 \right) - 4a_1(u)a_1(u) \right. \nonumber \\ \\
- 12a_1(u)^2 + a_2(u)(8 - 40u + 48u^2) \nonumber \\ \\
+ 16(1 - 3u)a_2(u) \right] + O(\nu^3). \tag{18}$$

To obtain $j(x)$ to complete the expression of $Q_\omega(x)$ in Eq. (15), we take advantage of Eq. (2.21) of Ref. [55], which gives $u(x)$. This relation reads

$$u = x - \nu U_1(x; a_1'(x)) + \nu^2 V_2(x; a_1(x), a_1'(x), a_2(x)) + O(\nu^2), \tag{19}$$

where

$$U_1(x; a_1'(x)) = -\frac{1}{6} \left[ a_1'(x) - 4 \left( 1 - \frac{2x}{\sqrt{1 - 3x}} \right) \right], \tag{20}$$

$$U_2(x; a_1(x), a_1'(x), a_2(x)) = -\frac{1}{3} x \left( \frac{x}{(1 - 3x)^{3/2}} - \frac{1}{3} x \right) a_1'(x) \nonumber \\ \\
- \frac{1}{9}(1 - 3x)^{1/2} + \frac{5}{9}(2 - 7x + 4x^2), \tag{21}$$

$$V_2(x; a_1(x), a_1'(x), a_2(x)) = U_1(x; a_1'(x)) \frac{d}{dx} U_1(x; a_1'(x)) \nonumber \\
- U_2(x; a_1(x), a_1'(x), a_2(x)). \tag{22}$$

By combining Eq. (18) and Eq. (19) we have $j(x)$, and we can finally evaluate explicitly Eq. (15) as a function of ($\mathcal{F}^{1SF}_{\varphi}, \mathcal{F}^{2SF}_{\varphi}, \mathcal{F}^{3SF}_{\varphi}, a_1(x), a_2(x)$), to obtain

$$Q_\omega^0 = \frac{2}{3} \left( \frac{1 - 6x}{(1 - 3x)^{3/2}} \right) 3^{2SF} \mathcal{F}^{1SF}_{\varphi}. \tag{23}$$

$$Q_\omega^1 = -\frac{x}{9(\mathcal{F}^{1SF}_{\varphi})^2(1 - 3x)^3} \left\{ \mathcal{F}^{1SF}_{\varphi} \left[ \sqrt{1 - 3x} \left( \left( -54x^2 + 24x - 2 \right) a_1'(x) + \left( 36x^3 - 24x^2 + 4x \right) a_1''(x) \right) \right. \right. \nonumber \\
\left. + \left( 36x - 3 \right) a_1(x) + 72x^2 - 12x + 2 \right\} + \left( 72x^3 - 72x^2 + 14x - 2 \right) \left. \right\} \mathcal{F}^{2SF}_{\varphi} \sqrt{1 - 3x} \left( -108x^2 + 54x - 6 \right) \tag{24}$$

$$Q_\omega^2 = \frac{x}{108(\mathcal{F}^{1SF}_{\varphi})^3(1 - 3x)^5} \left\{ \left( \mathcal{F}^{1SF}_{\varphi} \right)^2 \left[ \sqrt{1 - 3x} \left( a_1(x) \left( \left( 2268x^3 - 1620x^2 + 324x - 12 \right) a_1'(x) \right. \right. \right. \right. \right. \nonumber \\
+ \left( -648x^4 + 648x^3 - 216x^2 + 24x \right) a_1''(x) - 5184x^3 + 1944x^2 - 108x + 12 \nonumber \\
+ a_1'(x) \left( \left( -324x^5 + 648x^4 - 432x^3 + 120x^2 - 12x \right) a_1''(x) \right. \right. \right. \nonumber \\
\left. + \left( -648x^6 + 864x^5 - 432x^4 + 96x^3 - 8x^2 \right) a_1^{(3)}(x) + 6480x^4 - 4752x^3 + 1080x^2 - 96x + 8 \right. \right. \nonumber \\
\left. + \left( -486x^4 + 351x^3 - 54x^2 - 9x + 2 \right) a_1'(x)^2 + \left( -648x^6 + 864x^5 - 432x^4 + 96x^3 - 8x^2 \right) a_1''(x)^2 \right\} \right\}.
As one should expect, the 0PA term \([23]\) is identical to the formula derived within GSF theory; see Eq. (B6) in Paper I [with (12), (14), and the relation \(F^{(1)} = -\Omega F^{1SF}_{\varphi}\), where \(F^{(1)}\) is the leading-order flux of energy to infinity and into the black hole]. The 1PA term \([24]\) can similarly be compared to the first term in Eq. (B7) [with (13)] of Paper I if we note that \(a_1(x)\) is directly related to the binding energy \(E_{\text{SF}}\) appearing there \([12]\).

We are now in the position of understanding in detail the results of Fig. 2. First, since only \(F^{1SF}_{\varphi}\) contributes to \(Q^0\), the excellent EOB/GSF agreement in this coefficient we obtain here is mostly due to the inclusion of the 3+1PN flux at infinity\([1]\). On the other hand, \(Q^1\) is function of \(F^{1SF}_{\varphi}, a_1\) and \(F^{2SF}_{\varphi}\). The good EOB/GSF consistency of \(Q^1\) suggests that the accurate modelling of \(F^{1SF}_{\varphi}\) and \(a_1\) (as is the case because this function is GSF-informed) is more important than the modelling of \(F^{2SF}_{\varphi}\) (that is different in the two models) to correctly capture this contribution. Finally, we see that \(Q^2\) also depends on \(F^{2SF}_{\varphi}\) and \(a_2\). However, \(a_2\) is zero in both models. The \(Q^2\) differences should then mostly come from \(F^{2SF}_{\varphi}\) and \(F^{2SF}_{\varphi}\) (the latter of which is identically zero in 1PAT1).

The availability of the analytic \((Q^0_\omega, Q^1_\omega, Q^2_\omega)\) allows us to devise a more precise interpretation of the analogous analysis shown in Fig. 10 of Paper I. Contrasting with what we obtain here, one has to keep in mind that: (i) for \(Q^0\), most of the EOB/GSF difference obtained considering the standard TEOBResumS is indeed due to the use of a flux that does not include the same amount of test-mass information used here, as already pointed out in Paper I; (ii) for what concerns \(Q^1_\omega\), Paper I uses a different, non-GSF-informed, but NR-informed, expression for \(a_1\), which explains why the disagreement in \(Q^1_\omega\) is larger than the one shown here; (iii) in addition, in Paper I we also had information beyond \(a_1\), related to \(a_2\) and higher (effective) terms informed by NR simulations, which similarly explains the larger disagreement found there in \(Q^2_\omega\).

This intuitive understanding can be made more quantitative as follows. First, let us recall that the expression for \(A\) used in TEOBResumS stems from the formal 5PN expression

\[
A_{5\text{PN}}(u) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41}{32} \pi^2\right) \nu u^4
\]

\[+ \left(-3888 x^5 + 3888 x^4 - 1296 x^3 + 144 x^2\right) a_1''(x) + \left(2592 x^6 - 3456 x^5 + 1728 x^4 - 384 x^3 + 32 x^2\right) a_1^{(3)}(x)

\[+ \left(-1458 x^2 + 567 x - 27\right) a_1(x)^2 + \left(5832 x^2 - 6480 x^3 + 2592 x^2 - 432 x + 24\right) a'_2(x)

\[+ \left(-3888 x^5 + 5184 x^4 - 2592 x^3 + 576 x^2 - 48 x\right) a'_2(x) + \left(-3888 x^3 + 2916 x^2 - 648 x + 36\right) a_2(x)

\[= 500 q + 3888 x^3 - 3528 x^2 + 676 x - 56\right) + \left(2592 x^5 - 6480 x^4 + 3816 x^3 - 840 x^2 + 88 x - 8\right) a'_1(x)

\[+ \left(-2592 x^6 + 5616 x^5 - 3888 x^4 + 1104 x^3 - 112 x^2\right) a''_1(x)

\[+ \left(5184 x^7 - 9504 x^6 + 6912 x^5 - 2496 x^4 + 448 x^3 - 32 x^2\right) a^{(3)}_1(x)

\[+ \left(2160 x^3 - 576 x^2 - 48 x\right) a_1(x) - 18144 x^5 + 30240 x^4 - 16200 x^3 + 4488 x^2 - 760 x + 56\right]

\[+ \mathcal{F}_{\varphi}^{1SF} \left[\mathcal{F}_{\varphi}^{2SF} \sqrt{1 - 3x} \left(-5832 x^4 + 6480 x^3 - 2592 x^2 + 432 x - 24\right) a'_1(x)

\[+ \left(3888 x^3 - 5184 x^4 + 2592 x^3 - 576 x^2 + 48 x\right) a''_1(x) + \left(3888 x^3 - 2916 x^2 + 648 x - 36\right) a_1(x)

\[+ 7776 x^4 - 6480 x^3 + 1944 x^2 - 288 x + 24\right) + 7776 x^5 - 12960 x^4 + 7560 x^3 - 2088 x^2 + 312 x - 24\right]

\[+ \mathcal{F}_{\varphi}^{3SF} \sqrt{1 - 3x} \left(11664 x^4 - 13608 x^3 + 5832 x^2 - 1080 x + 72\right)

\[+ \left(\mathcal{F}_{\varphi}^{2SF}\right)^2 \sqrt{1 - 3x} \left(-11664 x^4 + 13608 x^3 - 5832 x^2 + 1080 x - 72\right)\right].

(25)
The function $\hat{a}_1$ coming from the expansion in $\nu$ of the NR-informed Padé-resummed $A$ potential entering the standard TEOBResumS.

$$\nu[\hat{a}^5_0(\nu) + a^\ln_5(\nu)u^5] + \nu[a^5_0(\nu) + a^\ln_5(\nu)\ln u]u^6,$$

(26)

where $a^5_0$ plays the role of an effective 5PN parameter that is informed by NR simulations once the expression above is replaced by the Padé resummed potential $A(u; a^5_0(\nu); \nu) \equiv P^1_5[A^{5\nu N}(u)]$, where $P^1_5$ indicates the $(1, 5)$ approximant. In Eq. (26), 2PA terms are explicitly included as

\begin{align}
a^5_0 &= \frac{64}{5}, \\
a^5_0(\nu) &= a^5_{0,0} + \nu a^5_{0,1}, \\
a^\ln_5(\nu) &= -\left(\frac{7004}{105} + \frac{144}{5}\nu\right).
\end{align}

(27)\hspace{1cm}(28)\hspace{1cm}(29)

Although the coefficient $a^5_0(\nu)$ is analytically known at 2PA modulo one coefficient, we keep it here as an unknown function to be informed by NR simulations. In particular, we use the expression of $a^5_0(\nu)$ is given by Eqs. (33)-(38) of Ref. 35. The Padé resummed potential can be expanded in $\nu$ as

$$A(u; \nu) \approx 1 - 2u + \nu a^\text{teob}_1(u) + \nu^2 a^\text{teob}_2(u) + O(\nu^3).$$

(30)

It is then convenient to normalize $a^\text{teob}_1(u)$ as

$$a^\text{teob}_1(u) = \frac{a^\text{teob}_1(u)}{2u^3 E(u)},$$

(31)

where $E(u) = (1 - 2u)/\sqrt{1 - 3u}$ to ease the comparison with the GSF-informed function that diverges at $u = 1/3$.

\section{Improving the EOB/GSF Agreement: The Role of the Horizon Flux}

In the previous section we have only focused on mass ratios up to $q = 500$, which pertains to the lower boundary of the intermediate-mass-ratio regime. Let us now move to considering even larger mass ratios, so as to span up to the extreme-mass-ratio (EMR) regime. Figure 5 shows the EOB/GSF $Q_\omega$ differences for $q = (500, 5000, 50000)$: it illustrates how the EOB/GSF evolution worsens progressively as the mass ratio reaches the EMR regime. The plot shows that this EOB model is close to reaching a faithful evolution for $q = 5000$, given that $Q_\omega^{\text{EOB}} - Q_\omega^{\text{GSF}}$ is of order 1 at $\omega = 0.12$, but is still far from being sufficiently accurate to model EMRs. However, from the previous $Q_\omega$ analysis we learnt that as the mass ratio increases, the 0PA contribution to the dephasing is more and more relevant. Given that $Q^0_\omega$ only depends on $F^{\text{GSF}}$, the inconsistency highlighted in Fig. 5 should be mostly due to differences in $F^{\text{GSF}}$ between the EOB dynamics and 1PAT1. This hypothesis is further supported by the fact that the disagreement appears to grow linearly with $q$ when moving from $q = 5000$ to $q = 50000$ in Fig. 5 as one would expect from a disagreement in $Q^0_\omega$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Comparing the GSF-informed $\hat{a}_1$ function defined in Eq. (11) to the $\hat{a}_1$ coming from the expansion in $\nu$ of the NR-informed Padé-resummed $A$ potential entering the standard TEOBResumS.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{EOB/GSF $Q_\omega$ difference for $q = (500, 5000, 50000)$ binaries, with integrated phase differences on the frequency range $\omega = (0.0224, 0.12)$ of $(0.07, 0.27, 5.88)$ rad respectively. The initial separation is $r = 20$ for each configuration. The EOB/GSF performance progressively worsens as the mass ratio enters into the EMR regime.}
\end{figure}
in the modelization of the 1SF horizon flux. 1PAT1 implements the exact 1SF horizon flux summed up to $\ell = 30$. By contrast, TEOBResumS uses an approximate, though resummed, expression that only includes the $\ell = m = 2$ and $\ell = 2$, $m = 1$ multipoles as discussed in Refs. [51][52]. In particular, the PN information beyond the leading order contributions, for each mode, is collected into the residual amplitude correction functions, called $\rho^H_{\ell m}$, that are the analogous of the $\rho_{\ell m}$’s for the horizon flux. The $\rho^H_{22}$ and $\rho^H_{21}$ functions we are using here are those introduced and discussed in detail in Sec. II of Ref. [52]. They are given by formal 4PN polynomials obtained in the following way: (i) first, one was fitting the (multipolar) horizon fluxes of a test-mass around a Schwarzschild black hole with a rational function and then (ii) this rational function was expanded up to 4PN order in order to hybridize the exact 1PN term with the other three effective terms up to 4PN order [51]. Reference [52] computed corrections to the horizon flux up to $\ell = 4$ (see Table I therein), but it explicitly considered only the quadrupolar contribution, which was deemed sufficient for the purposes of that study. By contrast, here we find that the effect of the higher multipoles is actually nonnegligible and is useful to reduce the EOB/GSF gap, as we will discuss next.

A. Improved horizon flux

To understand the impact of horizon absorption on our current results, let us first remember the structure of the $\rho^H_{\ell m}$ functions, up to $\ell = 4$, and of their approximations, according to [51]. For each multipole $(\ell, m)$, Fig. 6 shows: (i) the exact (numerical) curves for a test-mass on circular orbits around a Schwarzschild black hole, as computed in [51]; (ii) their effective 4PN approximation using the polynomial obtained Taylor-expanding up to formal 4PN the fits of [51] (i.e., using the coefficients listed in Table I of [52]); (iii) the PN-expanded $\rho^H_{\ell m}$ obtained from [56], taken at a PN order that delivers an excellent agreement with (most of) the numerical data. In particular, the picture shows the performance of 10PN for the $\ell = m = 2$ mode; of 15PN for $\ell = 2, m = 1$; of 6PN for all multipoles with $\ell = 3$, and of 12PN for all multipoles with $\ell = 4$. It is interesting to note that for some modes, such as those with $\ell = 3$ and $\ell = 4$, the effective 4PN series obtained by expanding the fit is somehow more robust and accurate, in the strong field, than the high PN expansion.

The impact of these high-order terms on the horizon flux (either the effective ones or the true PN ones) is explored in Fig. 7. The top panel refers to the $q = 5000$ case. The standard curve, with only the $\ell = 2$ horizon flux contributions from Ref. [52], is contrasted with two different ways of incorporating more information in the flux. In the first case, we consider the PN-expanded numerical fits up to $\ell = 4$ (green line). We note that, despite the effective nature of the PN coefficients, this choice can already reduce the EOB/GSF disagreement. In the second case (yellow line) we take advantage of the quality of the PN expansions shown in Fig. 6 and use those except for the modes $\ell = m = 4$ and $\ell = 4, m = 2$; for the latter two modes we prefer to stick to the numerically informed effective PN coefficients due to the qualitatively different behavior of the PN-expanded functions shown in the right panel of Fig. 7.

As a general consideration, the rather erratic behavior of the
FIG. 7. Impact of the additional terms in the horizon flux on the EOB/GSF $Q_\omega$ agreement for $q = 5000$ and $q = 50000$. The initial separation is $r = 20$ for each configuration.

Illustrates that the new, more complete horizon flux low-
ers $\Delta Q^\text{EOBGSF}_\omega \equiv Q^\text{EOB}_\omega - Q^\text{GSF}_\omega$ by approximately an order of magnitude at $\omega = 0.12$. By integrating the $Q_\omega$ difference on the frequency range $\omega = (0.0224, 0.12)$ for $q = 5000$ we find accumulated phase differences of $\sim (0.27, 0.07, -0.01)$ radians for the three approxima-
tions to the horizon flux. In the bottom panel of the figure we see that the effect is even more striking for $q = 50000$. The accumulated phase difference up to fre-
quency $\omega = 0.12$ is halved, from $\sim 5.88$ rad with the standard flux to $\sim 2.94$ rad with the improved flux.

Given that the disagreement still grows with increasing $q$, we can infer that it is still caused by a disagreement in $Q^\omega_0$. However, the fact that we no longer see a roughly linear growth with $q$ when moving from $q = 5000$ to $q = 50000$ suggests that the difference $\Delta Q^\omega_0$ is now sufficiently small that at $q = 5000$ it competes with higher-order $\Delta Q^\omega_n$ terms, particularly $\Delta Q^\omega_1$. The linear growth with $q$ only becomes substantial, and starts to dominate, at high values of $q > 5000$.

VI. CONCLUSIONS

We have presented an extensive comparison between a recently proposed EOB model that incorporates linear-in-$\nu$ EOB potentials informed by GSF data \[55\] and 1PAT1, a state-of-the-art 2GSF waveform model \[10\]. We have mainly focused on the large-mass-ratio regime, so as to investigate the mutual properties of the two mod-
els for IMRIs and EMRIs. This study complements Paper I \[52\], which focused on mass ratios up to $q = 128$. Our main findings are as follows:

(i) We presented EOB/GSF phasing comparisons analogous to those discussed in Paper I. These rely on either time-domain phasing analyses or gauge-invariant phasing analyses based on the $Q_\omega$ function. We have found that the standard azimuthal radiation reaction implemented in TEOBResumS is insufficient and that it is necessary to incorpo-
rate more test-mass terms to achieve an acceptable EOB/GSF waveform agreement. In particular, we work at 3+10\text{PN} order in the residual waveform am-
plitudes $\rho_{\ell m}$, implementing their high-PN expansions as obtained in Ref. \[53\]. For simplicity we do not introduce any further resummation of the $\rho_{\ell m}$'s. Also, we sum up modes up to $\ell = 8$ and exclude the $m = 0$ ones.\footnote{This is different from 1PAT1, that implements flux modes up to $\ell = 30$.} The use of GSF information in both the conservative and nonconservative sectors of the model allows us to build an EOB evolution that is more GSF-faithful for large mass ratios, specifically up to mass ratio $q = 500$. This is confirmed both by a time-domain analysis and by a frequency-domain one. Following the same methodology of Paper I, we have contrasted the coefficients $(Q^0_0, Q^1_0, Q^2_0)$ of the $\nu$-expansion of $Q_\omega$ at 2PA, finding an increased EOB/GSF consistency in all three, though mostly in $Q^0_\omega$ and $Q^1_\omega$.

(ii) To deepen our understanding of the impact of the different contributions to the 2PA $Q_\omega$, we have expanded the EOB $Q_\omega$ analytically in $\nu$ for circular orbits, so as to find how $(F^\text{EOB}_\nu, F^\text{GSF}_\nu, F^\text{EOB}_\nu, a_1, a_2)$ enter the three terms $(Q^0_0, Q^1_0, Q^2_0)$. This further sheds light on the reason behind the increased

PN-expanded $\rho_{\ell m}^H$ indicates that they have to be additionally resummed. This is usually done for the flux at infinity in TEOBResumS, as such as in Ref. [53, 57], but it has never been at-
ttempted for the horizon functions in this form (see, however, Ref. [58]). Given the importance of having analytically accurate horizon fluxes, this will be pursued in future work.
EOB/GSF agreement we obtained with the updated EOB model, which is mostly dominated by the (GSF-informed) \( \mathcal{F}^{1SF}, a_1 \) functions. This also shows that, at least up to \( q = 500 \), the known differences in \( \mathcal{F}^{2SF} \) and \( \mathcal{F}^{3SF} \) between 1PAT1 and the EOB model are not very important, since we find a high degree of consistency also between the respective \( Q^2_\nu \)'s (see in particular the third panel of Fig. 2). As shown in Paper I, 1PAT1 appears to substantially overestimate the true value of \( Q^2_\omega \), suggesting that this consistency in \( Q^2_\omega \) might represent a loss of accuracy in the new EOB model relative to the NR-informed \( Q^2_\omega \) in the standard TEOBResumS; however, this should not be relevant for IMRIs and EMRIs, where \( Q^2_\omega \) makes a very small contribution to the phase.

(iii) When moving to larger mass ratios, from \( q = 5000 \) to \( q = 50000 \), so as to enter the EMR regime, we have highlighted that a precise modelization of the contribution to the EOB radiation reaction due to the black hole horizon absorption is needed to provide an acceptable EOB/1PAT1 consistency.

Our main general conclusion is that, if properly informed by GSF results (either numerically or analytically), the TEOBResumS model can generate waveforms that are highly consistent with those generated by 1PAT1. For mass ratios in the hundreds or thousands, the functions \( \mathcal{F}^{1SF} \) and \( \mathcal{F}^{3SF} \) are highly consistent with those generated by 1PAT1, suggesting that this consistency in \( Q^2_\omega \) might represent a loss of accuracy in the new EOB model relative to the NR-informed \( Q^2_\omega \) in the standard TEOBResumS; however, this should not be relevant for IMRIs and EMRIs, where \( Q^2_\omega \) makes a very small contribution to the phase.

In future work, we will explore the EOB model's \( \nu \) dependence in more detail. The representations of the fluxes are intrinsically different in the two models, because of a different amount of \( \nu \)-dependent information included. 1PAT1 calculates the first two orders in the flux, \( \nu^2 F^{1SF}_\nu + \nu^3 F^{2SF}_\nu \), exactly (up to numerical error), while the EOB model approximates these and also includes higher orders in \( \nu \), though limited by being based on resummed PN series. In this respect, it might be helpful to build a version of the EOB flux that only includes corrections up to \( \mathcal{F}^{2SF} \), though evidently based on resummed PN results up to 3PN. In this way, both 1PAT1 and EOB would be exactly at the same order in \( \nu \), and the only differences should come from the uncertainties in the resummation procedures.

Once their \( \nu \) dependence is clearly delineated, EOB models offer a powerful tool for IMRI and EMRI modelling. An EOB model is clearly very flexible, as it easily incorporates higher-order-in-\( \nu \) terms, either in the radiation reaction or in the Hamiltonian. For example, the current EOB model would easily allow us to test, at least approximately, the impact of the 2SF correction \( a_2 \) or 3SF flux \( \mathcal{F}^{3SF}_\nu \) on the long inspiral of an IMRI or EMRI, ensuring that the impact is sufficiently small to neglect the correction \( a_2 \) is now partly known from PN calculations and is among the main challenges of current GSF research. EOB could also be used to inform GSF models, rather than the converse: while 1PAT1 provides a fast, accurate model for quasicircular, nonspinning binaries, GSF models for eccentric, spinning binaries are more limited. Such GSF models include 0PA and some 1PA terms (specifically, 1SF conservative terms) to very high precision but are missing other 1PA terms (specifically, 2SF dissipative terms); an EOB model can provide approximations to those missing terms in regions of the parameter space where 2SF calculations have not yet been performed. Our results in this paper suggest that such approximate 1PA terms may in fact be sufficiently accurate for most purposes, bypassing the need for expensive 2SF calculations, although further work will be needed to access their accuracy for eccentric, spinning binaries.

It is likely that these mutual synergies between GSF and EOB theory will be essential in the construction of accurate waveform models for the next generation of detectors.

ACKNOWLEDGMENTS

A.A. has been supported by the fellowship Lumina Quaeruntur No. LQ100032102 of the Czech Academy of Sciences. We are grateful to S. Albanesi for discussion.
Appendix A: Inaccuracy of the standard TEOBResumS angular momentum flux

In the main text we have mentioned that the standard TEOBResumS flux, as detailed in Ref. [34, 35], turns out to be inaccurate as the mass ratio increases and this has a nonnegligible impact on the phasing. This is testified by Fig. 5 that shows how the EOB/GSF difference is decreased when substituting the standard TEOBResumS flux with the $3^{+19}$PN flux described above for mass ratios $q = (500, 5000)$. The integrated phase difference up to $\omega = 0.12$ lowers from (4.42, 44.04) to (0.07, 0.27) for $q = (500, 5000)$ respectively.
FIG. 8. EOB/GSF difference in $\Delta Q_{\varphi}$ for $q = (500, 5000)$, both using the standard TEOBResumS flux for the evolution and the $3^{+19\text{PN}}$ flux. Notice how for the second choice the difference starts at zero for both mass ratios.

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