TRANSCENDENTAL ENTIRE FUNCTIONS WHOSE JULIA SETS
CONTAIN ANY INFINITE COLLECTION OF
QUASICONFORMAL COPIES OF QUADRATIC JULIA SETS

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(Communicated by Weixiao Shen)

Abstract. We prove that for any infinite collection of quadratic Julia sets, there exists a transcendental entire function whose Julia set contains quasiconformal copies of the given quadratic Julia sets. In order to prove the result, we construct a quasiregular map with required dynamics and employ the quasiconformal surgery to obtain the desired transcendental entire function. In addition, the transcendental entire function has order zero.

1. Introduction. Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. The Fatou set \( \mathcal{F}(f) \) is the set of normality in the sense of Montel for the family \( \{f^n\}_{n=1}^{\infty} \) or
\[
\mathcal{F}(f) = \{ z \in \mathbb{C} : \{f^n(z)\}_{n=1}^{\infty} \text{ is a normal family in some neighbourhood of } z \},
\]
where \( f^n = f \circ \cdots \circ f \) is \( n \) iterates of \( f \). By definition, the Fatou set is open. The escaping set \( \mathcal{I}(f) \) is the set of points that escape to infinity under iteration,
\[
\mathcal{I}(f) = \{ z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \}.
\]
The Julia set \( \mathcal{J}(f) \) is the complement of the Fatou set, \( \mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f) \). The Julia set is closed and is equal to the topological boundary of the escaping set. Moreover, the Julia set is equal to the closure of the set of repelling periodic points. If \( f \) is a polynomial of degree at least two, the complement of the escaping set is called the filled Julia set \( \mathcal{K}(f) \) or
\[
\mathcal{K}(f) = \{ z \in \mathbb{C} : \{f^n(z)\}_{n=1}^{\infty} \text{ is bounded} \}.
\]
In this case, \( \mathcal{J}(f) = \partial \mathcal{K}(f) \) holds and the escaping set is a component of the Fatou set. However, if \( f \) is a transcendental entire function, then \( \mathcal{I}(f) \cap \mathcal{J}(f) \neq \emptyset \). These four sets \( \mathcal{F}(f) \), \( \mathcal{K}(f) \), \( \mathcal{J}(f) \) and \( \mathcal{I}(f) \) are completely invariant under \( f \), namely \( f(\mathcal{F}(f)) \subset \mathcal{F}(f) \) and \( f^{-1}(\mathcal{F}(f)) \subset \mathcal{F}(f) \) and so on. For basic properties of these sets, we refer to \([4, 11]\).

The purpose of this paper is to construct a transcendental entire function whose Julia set contains quasiconformal copies of infinitely many quadratic Julia sets, or Julia sets of quadratic polynomials. In other words, for any sequence \( \{c_j\}_{j=1}^{\infty} \) of complex numbers, we construct a transcendental entire function \( f \) such that there are infinitely many quadratic-like maps \( (f, U_j, V_j) \) and each \( (f, U_j, V_j) \) is hybrid equivalent to the quadratic polynomial \( p_{c_j} \) for \( j \geq 1 \), where \( p_c(z) = z^2 + c \). For

2010 Mathematics Subject Classification. Primary: 37F10; Secondary: 30D05, 37F50.
Key words and phrases. Julia sets, polynomial-like maps, quasiconformal/quasiregular maps.
details about quadratic-like maps and hybrid equivalence, see Definition 2.2 and Theorem 2.3. The Julia set \( J(f) \) of the transcendental entire function \( f \) we construct contains a quasiconformal copy of the Julia set \( J(p_{c_j}) \) of the quadratic polynomial \( p_{c_j} \), or the image of \( J(p_{c_j}) \) under a quasiconformal map. In [9], the author constructed a transcendental entire function whose Julia set contains quasiconformal copies of finitely many quadratic Julia sets. It is a finite version of our purpose.

**Theorem 1.1 ([9, Theorem B]).** Let \( n \geq 2 \) be an integer. For any \( c_1, c_2, \ldots, c_n \in \mathbb{C} \), there exist a transcendental entire function \( f \) and distinct bounded simply connected domains \( U_1, U_2, \ldots, U_n \) and \( V \) satisfying \( U_j \Subset V \) for \( j = 1, 2, \ldots, n \) such that each \( (f, U_j, V) \) is a quadratic-like map and is hybrid equivalent to the quadratic polynomial \( p_{c_j} \) for \( j = 1, 2, \ldots, n \).

The main result of this paper is a generalization of Theorem 1.1.

**Theorem A.** For any sequence \( \{c_j\}_{j=1}^\infty \) of complex numbers, there exist a transcendental entire function \( f \) and two sequences of distinct bounded simply connected domains \( \{U_j\}_{j=1}^\infty \) and \( \{V_j\}_{j=1}^\infty \) satisfying \( U_j \Subset V_j \) such that each \( (f, U_j, V_j) \) is a quadratic-like map and is hybrid equivalent to the quadratic polynomial \( p_{c_j} \) for \( j \geq 1 \).

The idea of the proof of Theorem A is the following. First, we interpolate polynomials to construct a quasiregular map which is a map with required dynamics. Next, we cut off countably many small disks and interpolate the quasiregular map and an infinite collection of quadratic polynomials in the small disks. Finally, we employ the quasiconformal surgery for the constructed quasiregular map to obtain a transcendental entire function with the desired property. While the method of construction in this paper is similar to the one in the author’s previous paper [9], we can not use the same strategy in [9] to construct a transcendental entire function whose Julia set contains quasiconformal copies of infinitely many quadratic Julia sets. Theorem A is the first example of a transcendental entire function whose Julia set with quasiconformal copies of infinitely many quadratic Julia sets. In addition, we can calculate the order of the transcendental entire function.

**Theorem B.** The transcendental entire function \( f \) obtained by Theorem A has order zero.

2. Background. In this section, we review some basic definitions and results. For more details, we refer to [1, 6, 8, 11, 12, 15].

**Definition 2.1** (Quasiconformal maps). Let \( U \) be a domain in \( \mathbb{C} \) and let \( \varphi : U \to \varphi(U) \) be a sense-preserving \( C^1 \) homeomorphism. We define the complex dilatation of \( \varphi \) as

\[
\mu_\varphi = \frac{\varphi_z \overline{\varphi_z}}{\overline{\varphi_z} \varphi_z}
\]

and the dilatation of \( \varphi \) as

\[
D_\varphi = \frac{1 + |\mu_\varphi|}{1 - |\mu_\varphi|}.
\]

We say that \( \varphi \) is a \( K \)-quasiconformal map, \( K \geq 1 \), if \( D_\varphi(z) \leq K \) for all \( z \in U \). Note that a \( C^1 \) homeomorphism is conformal if and only if it is 1-quasiconformal.

**Definition 2.2** (Polynomial-like maps). The triple \((g, U, V)\), consisting of bounded simply connected domains \( U \) and \( V \) such that \( U \Subset V \) and a holomorphic proper
map \( g : U \to V \) of degree \( d \), is called a polynomial-like map of degree \( d \). The filled Julia set \( \mathcal{K}(g,U,V) \) of a polynomial-like map \( (g,U,V) \) is defined as

\[
\mathcal{K}(g,U,V) = \{ z \in U : g^n(z) \in U \text{ for all } n \geq 0 \}
\]

and the Julia set \( \mathcal{J}(g,U,V) \) as \( \mathcal{J}(g,U,V) = \partial \mathcal{K}(g,U,V) \). In the case that \( d = 2 \), the triple \( (g,U,V) \) is called a quadratic-like map.

**Theorem 2.3** (Straightening Theorem [7]). Every polynomial-like map is hybrid equivalent to a polynomial of the same degree. Namely, for any polynomial-like map \( (g,U,V) \) of degree \( d \geq 2 \), there exist a polynomial \( p \) of degree \( d \), a neighborhood \( W \) of \( \mathcal{K}(g,U,V) \) in \( U \) and a quasiconformal map \( \varphi : W \to \varphi(W) \) such that

1. \( \varphi(\mathcal{K}(g,U,V)) = \mathcal{K}(p) \),
2. the complex dilatation \( \mu_\varphi \) of \( \varphi \) is zero almost everywhere on \( \mathcal{K}(g,U,V) \),
3. \( \varphi \circ g = p \circ \varphi \) on \( W \cap g^{-1}(W) \).

If \( \mathcal{K}(g) \) is connected, \( p \) is unique up to conjugation by an affine map.

Now, we define quasiregular mappings and state important results concerning them. Throughout this paper, \( \text{“log” denotes the principal branch of the logarithm.} \)

**Definition 2.4** (Quasiregular maps). Let \( U \) be a domain in \( \mathbb{R}^n \) with \( n \geq 2 \). A mapping \( g : U \to \mathbb{R}^n \) is quasiregular if \( g \) belongs to the Sobolev space \( W^{1,\text{loc}}(U) \) and there exists \( K \geq 1 \) such that

\[
||Dg(x)||^n \leq K J_g(x)
\]

almost everywhere in \( U \), where \( ||Dg(x)|| \) is the operator norm of the derivative and \( J_g(x) \) is the Jacobian determinant of \( g \) at \( x \in U \). The smallest \( K \) for which the above inequality holds is the outer dilatation \( K_\mathcal{O}(g) \) of \( g \). If \( g \) is quasiregular, then we have

\[
J_g(x) \leq K' (\ell(Dg(x)))^n
\]

almost everywhere in \( U \) for some \( K' \geq 1 \), where \( \ell(Dg(x)) = \inf_{|v|=1} |Dg(x)v| \). The smallest \( K' \) for which the above inequality holds is the inner dilatation \( K_\mathcal{I}(g) \) of \( g \). The maximal dilatation of \( g \) is \( K(g) = \max\{K_\mathcal{O}(g), K_\mathcal{I}(g)\} \) and a quasiregular mapping is called \( K \)-quasiregular if \( K(g) \leq K \).

Note that non-constant quasiregular maps are open and discrete. The relationships \( K_\mathcal{O}(g) \leq K \mathcal{I}(g) \) and \( K \mathcal{I}(g) \leq K_\mathcal{O}(g)^{n-1} \) hold as is seen by linear algebra. Therefore \( K \mathcal{O}(g) = K \mathcal{I}(g) \) for \( n = 2 \). A quasiregular homeomorphism is called quasiconformal. By Stoilow’s theorem, a quasiregular mapping \( g : U \to \mathbb{R}^2 \) is always of the form \( g = f \circ \varphi \), where \( \varphi : U \to \varphi(U) \) is quasiconformal and \( f : \varphi(U) \to g(U) \) is analytic. For more information on quasiregular mappings, we refer to [14].

**Lemma 2.5** (Interpolation [10, Lemma 6.2]). Let \( k \in \mathbb{N} \), \( 0 < r_1 < r_2 \) and \( \varphi_j(z) \) be analytic on a neighborhood of \( |z| = r_j \) such that \( \varphi_j||_{|z|=r_j} \) goes around the origin \( k \)-times \( (j = 1,2) \). If

\[
|\log \left( \frac{\varphi_2(r_2 e^{iy})}{r_2^k} \cdot \frac{r_1^k}{\varphi_1(r_1 e^{iy})} \right) | \leq \delta_0 \quad (2.1)
\]

and

\[
|\frac{d}{dz} \log \frac{\varphi_j(z)}{z^k} | \leq \delta_1, \quad z = r_j e^{iy}, \quad j = 1,2 \quad (2.2)
\]
hold for every \( y \in [0, 2\pi] \) and for some positive constants \( \delta_0 \) and \( \delta_1 \) satisfying

\[
C = 1 - \frac{1}{k} \left( \frac{\delta_0}{\log(r_2/r_1)} + \delta_1 \right) > 0,
\]

then there exists a quasiregular map

\[
H : \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \} \to \mathbb{C} \setminus \{0\}
\]

without critical points such that \( H = \varphi_j \) on \(|z| = r_j \) \((j = 1, 2)\) and satisfies

\[
K(H) \leq \frac{1}{C}.
\]

**Theorem 2.6** (Quasiconformal surgery [10, Theorem 3.1]). Let \( g : \mathbb{C} \to \mathbb{C} \) be a quasiregular mapping. Suppose that there are disjoint measurable sets \( E_j \subset \mathbb{C} \) \((j = 1, 2, \ldots)\) satisfying:

1. for almost every \( z \in \mathbb{C} \), the \( g \)-orbit of \( z \) passes \( E_j \) at most once for every \( j \);
2. \( g \) is \( K_j \)-quasiregular on \( E_j \);
3. \( K_\infty = \prod_{j=1}^\infty K_j < \infty \);
4. \( g \) is holomorphic almost everywhere outside \( \bigcup_{j=1}^\infty E_j \).

Then there exists a \( K_\infty \)-quasiconformal map \( \varphi : \mathbb{C} \to \mathbb{C} \) such that \( f = \varphi \circ g \circ \varphi^{-1} \) is an entire function.

### 3. Key Lemma.

The following lemma plays an important role to construct a quasiregular map with required dynamics, which is a seed of a desired transcendental entire function. The key Lemma is an analog of the one introduced by Kisaka and Shishikura in [10].

**Lemma 3.1.** Let \( m \) be a positive integer, \( \tau, \omega \in \mathbb{C} \setminus \{0\} \) and \( \rho^2, \lambda^2, \rho^b, \lambda^b \in \mathbb{R} \) satisfying that \( 0 < \lambda^2 < \rho^2 < 1 < \rho^b < \lambda^b \).

1. Suppose that these constants satisfy

\[
\rho^2 \geq 2|\omega|, \quad \lambda^2 \geq e\rho^b, \quad C^2 = 1 - \frac{1}{2m+2} \left( \frac{|\log \tau|}{\log(\lambda^2/\rho^2)} + \frac{8|\omega|}{\rho^2} \right) > 0.
\]

Then the map \( \tau z^{2m}(z - \omega)^2 \) on \(|z| = \rho^2\) and \( z^{2m+2} \) on \(|z| = \lambda^2\) can be interpolated on \( \rho^2 \leq |z| \leq \lambda^2 \) with a quasiregular map \( g \) by Lemma 2.5 and \( g \) satisfies \( K(g) \leq 1/C^2 \).

2. Suppose that these constants satisfy

\[
|\omega| \geq 2\rho^b, \quad \rho^b \geq e\lambda^b, \quad C^b = 1 - \frac{1}{2m} \left( \frac{|\log(\tau\omega^2)|}{\log(\rho^b/\lambda^b)} + \frac{8\rho^b}{|\omega|} \right) > 0.
\]

Then the map \( z^{2m} \) on \(|z| = \lambda^b\) and \( \tau z^{2m}(z - \omega)^2 \) on \(|z| = \rho^b\) can be interpolated on \( \lambda^b \leq |z| \leq \rho^b \) with a quasiregular map \( g \) by Lemma 2.5 and \( g \) satisfies \( K(g) \leq 1/C^b \).

**Proof.** We apply Lemma 2.5 by taking

\[
r_1 = \rho^2, \quad r_2 = \lambda^2, \quad \varphi_1(z) = \tau z^{2m}(z - \omega)^2, \quad \varphi_2(z) = z^{2m+2}
\]

and changing \( k \) to \( 2m+2 \). Let

\[
\delta_0 = |\log \tau| + \frac{4|\omega|}{\rho^2}, \quad \delta_1 = \frac{4|\omega|}{\rho^b}.
\]
Then the assumption \( \rho^d \geq 2|\omega| \) implies that
\[
\left| \log \left( \frac{\varphi_2(r_2e^{iy})}{r_2^{2m+2}} \cdot \frac{r_2^{2m+2}}{\varphi_1(r_1e^{iy})} \right) \right| \leq |\log \tau| + 2 \left| \log \left( 1 - \frac{\omega}{\rho^d e^{iy}} \right) \right| \leq |\log \tau| + \frac{4|\omega|}{\rho^d} = \delta_0
\]
and
\[
\left| \frac{d}{dz} \log \left( \frac{\varphi_1(z)}{z^{2m+2}} \right) \right| = 2 \left| \frac{1}{z/\omega - 1} \right| \leq \frac{2}{\rho^d/|\omega| - 1} \leq \frac{4|\omega|}{\rho^d} = \delta_1,
\]
where \( z = r_1e^{iy} = \rho^d e^{iy} \). It is clear that the inequality
\[
\left| \frac{d}{dz} \log \left( \frac{\varphi_2(z)}{z^{2m+2}} \right) \right| = \left| \frac{d}{dz} \log 1 \right| = 0 < \delta_1
\]
holds. Moreover, we obtain that
\[
C = 1 - \frac{1}{2m+2} \left( \frac{\delta_0}{\log (r_2/r_1)} + \delta_1 \right)
\]
\[
\geq 1 - \frac{1}{2m+2} \left( \frac{|\log \tau|}{\log (\lambda^d/\rho^d)} + \frac{4|\omega|/\rho^d}{1} + \frac{4|\omega|}{\rho^d} \right)
\]
\[
= C^d \geq 0
\]
by using the assumption \( \lambda^d \geq e\rho^d \). Therefore, by Lemma 2.5, there exists a quasiregular map
\[
g : \{ z \in \mathbb{C} : \rho^d \leq |z| \leq \lambda^d \} \rightarrow \mathbb{C} \setminus \{0\}
\]
without critical points such that \( g(z) = \tau z^{2m}(z - \omega)^2 \) on \( |z| = \rho^d \) and \( g(z) = z^{2m+2} \) on \( |z| = \lambda^d \) with \( K(g) \leq 1/C^d \). We can prove (2) similarly. \( \square \)

4. Proof of Theorem A. In this section, we prove Theorem A. Let \( \text{Ann}(c; r_1, r_2) \) be the open annulus centered at \( c \) of inner radius \( r_1 \) and outer radius \( r_2 \) and let \( \text{Ann}(r_1, r_2) = \text{Ann}(0; r_1, r_2) \). We first construct a quasiregular map \( g : \mathbb{C} \rightarrow \mathbb{C} \) with required dynamics by modifying Bergweiler’s method [5], Osborne’s one [13] or the author’s one [9], which are based on the ideas of Kisaka and Shishikura [10]. The construction uses the sequences \( (R_m), (P_m), (Q_m), (S_m), (T_m), (a_m), (b_m), (r_m) \) and \( (\rho_m) \) below. We choose the sequences \( (R_m), (a_m) \) and \( (b_m) \) such that the map \( \psi_m(z) = a_m z^{2m+2} \) maps \( \text{Ann}(R_m, R_m+1) \) onto \( \text{Ann}(R_{m+1}, R_{m+2}) \) and the map \( \eta_m(z) = b_m z^{2m}(z - R_m)^2 \) maps a critical point \( mR_m/(m + 1) \) of \( \eta_m \) to a critical point \( (m + 1)R_{m+1}/(m + 2) \) of \( \eta_{m+1} \). We define the quasiregular map \( g \) as \( \psi_{m-1} \) on \( \overline{\text{Ann}(T_{m-1}, P_m)} \), \( \eta_m \) on \( \text{Ann}(Q_m, S_m) \) and \( \psi_m \) on \( \overline{\text{Ann}(T_m, P_{m+1})} \). Next, we interpolate the maps \( \psi_{m-1} \) and \( \eta_m \) in the annulus \( \text{Ann}(P_m, Q_m) \), and the maps \( \eta_m \) and \( \psi_m \) in the annulus \( \text{Ann}(S_m, T_m) \) respectively. Moreover, we alter the definition of \( g \) in the small disk \( |z - R_m(j)| \leq r_m(j) \) in \( \text{Ann}(Q_m(j), S_m(j)) \) to a quadratic polynomial which is conjugate to \( p_{c_{ij}} \), where \( m(j) \) is a positive integer depending on the complex number \( c_j \), given in Theorem A. Furthermore, we employ the quasiconformal surgery for the constructed quasiregular map to obtain a transcendental entire function with the desired property.
Notation (Sequences with index $m$). We define sequences $(R_m)$, $(P_m)$, $(Q_m)$, $(S_m)$, $(T_m)$, $(a_m)$, $(b_m)$, $(r_m)$, $(p_m)$ and $(A_m)$ as follows. We set $R_0 = 1$, take $R_1 > 1$ large enough and define
\[
R_{m+1} = \frac{R_m^{2m+1}}{R_m^{2m-1}}
\]
for $m \geq 1$. With $\gamma = \log R_1$, we obtain that
\[
\log \frac{R_{m+1}}{R_m} = 2m \cdot \log \frac{R_m}{R_{m-1}} = \cdots = (2m)! \cdot \log \frac{R_1}{R_0} = \gamma \cdot (2m)!!,
\]
where $(2m)!! = 2^m \cdot m!$. We define $(P_m)$, $(Q_m)$, $(S_m)$ and $(T_m)$ as
\[
\log \frac{T_m}{S_m} = \log \frac{S_m}{R_m} = \log \frac{R_m}{Q_m} = \log \frac{Q_m}{P_m} = \sqrt{\log \frac{R_{m+1}}{R_m}} = \sqrt{\gamma \cdot (2m)!!}.
\]
Taking $R_1 > e$ gives $\gamma > 1$ and thus
\[
\frac{T_m}{S_m} = \frac{S_m}{R_m} = \frac{R_m}{Q_m} = \frac{Q_m}{P_m} = e^{\sqrt{\gamma \cdot (2m)!!} > e}.
\]
If $R_1$ and hence $\gamma$ is sufficiently large, we obtain that
\[
\log \frac{P_{m+1}}{T_m} = -2\sqrt{\gamma \cdot (2m+2)!!} - \gamma \cdot (2m)!! - 2\sqrt{\gamma \cdot (2m)!!} > 0
\]
for all $m \geq 1$. Therefore, the inequality
\[
P_m < Q_m < R_m < S_m < T_m < P_{m+1}
\]
holds for all $m \geq 1$. Further, we define sequences $(a_m)$, $(b_m)$, $(r_m)$, $(p_m)$ and $(A_m)$ as
\[
a_m = \frac{R_{m+1}}{R_{2m+2}}, \quad b_m = \frac{(m+1)^{2m+3}}{(m+2) \cdot m^{2m} a_m},
\]
\[
r_m = \frac{1}{m+1} e^{-\sqrt{\gamma \cdot (2m+2)!!} R_m}, \quad p_m = \frac{1}{2e(m+1)} e^{-\sqrt{\gamma \cdot (2m+2)!!} R_m}
\]
and
\[
A_m = b_m R_m^{2m} = \frac{(m+1)^{2m+3}}{(m+2) \cdot m^{2m}} \cdot \frac{R_{m+1}}{R_m}.
\]
By definition of $(R_m)$ and $(a_m)$, $\psi_m$ maps $\text{Ann}(R_m, R_{m+1})$ onto $\text{Ann}(R_{m+1}, R_{m+2})$. The finite critical points of $\eta_m$ are $0$, $R_m$ and $m R_m / (m+1)$. The origin is a superattracting fixed point of $\eta_m$. It is easy to see that
\[
\eta_m \left( \frac{m}{m+1} R_m \right) = \frac{m+1}{m+2} R_{m+1},
\]
which is a critical point of $\eta_{m+1}$. Let $h_m(z) = \eta_m(z+R_m) - R_m = b_m (z+R_m)^{2m} z^2 - R_m$, which is affine conjugate to $\eta_m$. The finite critical points of $h_m$ are $-R_m$, 0 and $-R_m / (m+1)$.
Notation ( Sequences with index $j$ ). Let the sequence $\{c_j\}_{j=1}^{\infty} \subset \mathbb{C}$ be in Theorem A, $p_c(z) = z^2 + c$ and $q_{c,A}(z) = p_c(Az)/A = Az^2 + c/A$, where $c \in \mathbb{C}$ and $A \in \mathbb{C} \setminus \{0\}$ are constants.

**Lemma 4.1.** Suppose that $\gamma$ is sufficiently large. For the sequence $\{c_j\}_{j=1}^{\infty}$ in Theorem A, there exists a subsequence $(m(j))_{j=1}^{\infty}$ of $(m)_{m=1}^{\infty}$ with $m(j) + 1 > m(j)$ for any $j \geq 1$ such that the maps $h_{m(j)}$ and $q_{c_j,A_{m(j)}}$ can be interpolated on $r_{m(j)} \leq |z| \leq \rho_{m(j)}$ with a quasiregular map $I_j$ by Lemma 2.5 and $I_j$ satisfies $K(I_j) \leq 1 + 1/j^2$.

**Proof.** If $\gamma$ is large enough, the origin is the only critical point of $h_m$ inside $|z| = \rho_m$, and by Rouché’s theorem, $h_m$ has two zeros inside $|z| = \rho_m$ for all $m \geq 1$. A direct calculation shows that there exists a subsequence $(m(j))_{j=1}^{\infty}$ of $(m)_{m=1}^{\infty}$ with $m(j) + 1 > m(j)$ for any $j \geq 1$ such that $q_{c_j,A_{m(j)}}$ has two zeros inside $|z| = r_{m(j)}$. Hence, we apply Lemma 2.5 by taking $r_1 = r_m$, $r_2 = \rho_m$, $\varphi_1 = q_{c_j,A_m}$, $\varphi_2 = h_m$, $\delta_0 = \delta_1 = 1/(j^2 + 1)$ and changing $k$ to 2. First, we check the inequality (2.1) in Lemma 2.5. For $y \in [0, 2\pi)$,

$$
\log \left( \frac{h_m(\rho_m e^{iy})}{\rho_m^2} \right) \cdot \frac{r_m^2}{q_{c_j,A_m}(r_m e^{iy})} = \log \left( \frac{b_m}{A_m} \frac{(\rho_m e^{iy} + R_m)^{2m}}{1 + \frac{c_j e^{-2iy}}{r_m^2 A_m^2}} - \frac{R_m}{\rho_m^2 A_m \left( 1 + \frac{c_j e^{-2iy}}{r_m^2 A_m^2} \right)^2} \right).
$$

Then the following holds as $m$ tends to infinity:

$$
b_m b_m = \frac{(m+1)^2}{m+2} \left( \frac{1}{2c(m+1)} \right)^{m+2} e^{\frac{1}{2} \sqrt{\gamma(2m+2)!!} e^{iy}} + 1\right)^{2m} \rightarrow 1,
$$

where $u_m = 2c(m+1) e^{\frac{1}{2} \sqrt{\gamma(2m+2)!!}}$ and $v_m = \frac{2m}{u_m}$.

$$
r_m A_m = \frac{1}{m+1} e^{-\sqrt{\gamma(2m+2)!!}} R_m \frac{(m+1)^{2m+3}}{m^2} \cdot \frac{R_{m+1}}{R_m^2} \frac{R_{m+1}}{R_m} \frac{1}{m^2 \cdot \gamma(2m+2)!!} \frac{R_m}{R_m} \rightarrow \infty.
$$
Therefore, we obtain that
\[ \frac{R_m}{\rho_m^2 A_m} = R_m \cdot 4e^2 (m + 1)^2 e^{\sqrt{(2m+2)!}} \cdot \frac{1}{R_m^2} (m + 2) \cdot m^{2m} \cdot \frac{R_m^2}{R_{m+1}} \]
\[ = 4e^2 \cdot \frac{m + 2}{m + 1} \cdot \frac{m^{2m}}{(m + 1)^{2m}} \cdot e^{\sqrt{(2m+2)!}} \cdot \frac{R_m}{R_{m+1}} \]
\[ = 4e^2 \cdot \frac{m + 2}{m + 1} \left\{ \left( 1 + \frac{1}{m} \right)^m \right\}^{-2} \cdot e^{\sqrt{(2m+2)!}} - \gamma (2m)! \to 0. \]

(4.3)

Therefore, we obtain that
\[ \left| \log \left( \frac{h_m (\rho_m e^{iy})}{\rho_m^2} \cdot \frac{r_m^2}{q_{c_j, A_m} (r_m e^{iy})} \right) \right| \to 0 \]
as \( m \) tends to infinity. Hence, for \( j \geq 1 \), there exists a subsequence \( (\tilde{m}(j))_{j=1}^\infty \) of
\( (m(j))_{j=1}^\infty \), which we relabel as \( m(j) \), such that
\[ \left| \log \left( \frac{h_{m(j)} (\rho_{m(j)} e^{iy})}{\rho_{m(j)}^2} \cdot \frac{r_{m(j)}^2}{q_{c_j, A_{m(j)}} (r_{m(j)} e^{iy})} \right) \right| \leq \delta_0 = \frac{1}{j^2 + 1}. \]

(4.4)

Next, we check the inequality (2.2) in Lemma 2.5. Since (4.2), if \( m \) tends to infinity,
\[ \left| z \frac{d}{dz} \log q_{c_j, A_m} (z) \right| = \left| \frac{-2c_j}{r_m^2 A_m^2 e^{2iy} + c_j} \right| \to 0 \]
holds. Moreover, we obtain that
\[ z \frac{d}{dz} \log h_m (z) = 2 \cdot \frac{mb_m (z + R_m)^{2m-1} z^3 + R_m}{b_m (z + R_m)^{2m} z^2 - R_m} \]
\[ = 2 \cdot \frac{mb_m (s_m R_m e^{iy} + R_m)^{2m-1} (s_m R_m e^{iy})^3 + R_m}{b_m (s_m R_m e^{iy} + R_m)^{2m} (s_m R_m e^{iy})^2 - R_m} \]
\[ = 2 \cdot \frac{mb_m R_m^{2m+1} (s_m e^{iy} + 1)^{2m-1} (s_m e^{iy})^3 + 1}{b_m R_m^{2m+1} (s_m e^{iy} + 1)^{2m} (s_m e^{iy})^2 - 1} \]
\[ = 2 \cdot \frac{m s_m e^{iy} \Omega_m + 1}{(s_m e^{iy} + 1) \Omega_m - 1} = 2 \cdot \frac{m s_m e^{iy} + \Omega_m^{-1}}{s_m e^{iy} + 1 - \Omega_m}, \]
where
\[ \Omega_m = b_m R_m^{2m+1} (s_m e^{iy} + 1)^{2m-1} (s_m e^{iy})^2 \]
\[ = \frac{(m + 1)^{2m+3}}{(m + 2) \cdot m^{2m}} \cdot \frac{R_m^{m+1}}{R_m} \cdot B_m \cdot \frac{1}{4e^2 (m + 1)^2} e^{-\sqrt{(2m+2)!} \gamma} e^{2iy} \]
\[ = \frac{1}{4e^2} \cdot \frac{m + 1}{m + 2} \left\{ \left( 1 + \frac{1}{m} \right)^m \right\}^2 \cdot B_m \cdot e^{\gamma (2m)!} - \sqrt{(2m+2)!} \gamma e^{2iy}, \]
\[ B_m = \left\{ \left( 1 + \frac{e^{iy}}{u_m} \right)^{u_m} \right\}^{w_m}, \quad s_m = \frac{1}{2e(m + 1)} e^{-\frac{1}{2} \sqrt{(2m+2)!} \gamma}, \]
\[ u_m = 2e(m + 1) e^{\frac{1}{2} \sqrt{(2m+2)!} \gamma}, \quad w_m = 2m - 1 e_m. \]
Since \( B_m \to 1, \Omega_m \to \infty \) and \( s_m \to 0 \) as \( m \) tends to infinity, then

\[
\left| z \frac{d}{dz} \log \frac{h_m(z)}{z^2} \right| \to 0.
\]

Hence, for \( j \geq 1 \), there exists a subsequence \((m^{(j)})_{j=1}^\infty\) of \((m(j))_{j=1}^\infty\), which we relabel as \( m(j) \), such that the inequality (4.4),

\[
\left| z \frac{d}{dz} \log \frac{q_{c_{j},A_{m(j)}}(z)}{z^2} \right| \leq \delta_1 = \frac{1}{j^2 + 1}, \quad z = r_{m(j)} e^{iy}
\]

and

\[
\left| z \frac{d}{dz} \log \frac{h_{m(j)}(z)}{z^2} \right| \leq \delta_1 = \frac{1}{j^2 + 1}, \quad z = \rho_{m(j)} e^{iy}
\]

hold. Finally, we check the inequality (2.3) in Lemma 2.5. If \( \gamma \) is sufficiently large, then the inequality

\[
C_j = 1 - \frac{1}{2} \left( \frac{\delta_0}{\log (\rho_{m(j)}/r_{m(j)})} + \delta_1 \right)
\]

\[
= 1 - \frac{1}{2} \left( \frac{1}{\frac{1}{4} \sqrt{\gamma} \cdot (2m(j) + 2)!} - \log 2e \cdot \frac{1}{j^2 + 1} + \frac{1}{j^2 + 1} \right)
\]

\[
\geq 1 - \frac{1}{2} \left( \frac{1}{j^2 + 1} + \frac{1}{j^2 + 1} \right) = \frac{j^2}{j^2 + 1} > 0
\]

holds for any \( j \geq 1 \). By Lemma 2.5, then there exists a quasiregular map

\[ I_j : \{ z \in \mathbb{C} : r_{m(j)} \leq |z| \leq \rho_{m(j)} \} \to \mathbb{C} \setminus \{0\} \]

without critical points such that \( I_j = q_{c_{j},A_{m(j)}} \) on \( |z| = r_{m(j)} \) and \( I_j = h_{m(j)} \) on \( |z| = \rho_{m(j)} \) and satisfies

\[ K(I_j) \leq \frac{1}{C_j} \leq 1 + \frac{1}{j^2}. \]

\[ \square \]

**Construction of a quasiregular map with required dynamics.** Let \( m \geq 2 \) an integer. First, we interpolate \( \psi_{m-1} \) and \( \eta_m \) for \( m \geq 2 \). We consider the functions

\[
\psi_{m-1}^b(z) = \frac{\psi_{m-1}(R_m z)}{R_{m+1}} = z^{2m} \quad \text{and} \quad \eta_m^b(z) = \frac{\eta_m(R_m z)}{R_{m+1}} = \tau_m z^{2m}(z - 1)^2,
\]

where

\[
\tau_m = \frac{(m + 1)^{2m+3}}{(m + 2) \cdot m^{2m}}.
\]

Note that \( R_{m+1} = R_{m+2}^2/R_{m-1}^2 \). Let

\[
C_m^b = 1 - \frac{1}{2m} \left( \frac{\log (\tau \omega^2)}{\log (\rho^{b}/\lambda^{b})} + \frac{8\rho^{b}}{\omega} \right).
\]

We apply Lemma 3.1 by taking

\[
\lambda^{b} = \frac{P_m}{R_m}, \quad \rho^{b} = \frac{Q_m}{R_m}, \quad \tau = \tau_m, \quad \omega = 1, \quad C^{b} = C_m^b.
\]

By definition,

\[
\rho^{b} = \frac{Q_m}{R_m} = e^{-\sqrt{\gamma(2m)!}}
\]
and 
\[ \lambda^b = \frac{P_m}{R_m} = \frac{P_m}{Q_m} \cdot \frac{Q_m}{R_m} = e^{-2\sqrt{\gamma(2m)!}} = \left( \rho^b \right)^2. \]

Then \( |\omega| \geq 2\rho^b \) and \( \rho^b \geq e\lambda^b \) because
\[ |\omega| = 1 \geq 2e^{-\sqrt{\gamma(2m)!}} = 2\rho^b \]
and
\[ e\lambda^b = e\left( \rho^b \right)^2 = e^{1-\sqrt{\gamma(2m)!}} \rho^b \leq \rho^b. \]

Since
\[ |\log(\tau\omega^2)| = \log \left\{ \frac{(m+1)^2}{m+2} \left( \frac{m+1}{m} \right)^2 \right\} \]
\[ \leq \log \frac{(m+1)^2e^2}{2} = 2\{\log(m+1) + 1\}, \]
we obtain that
\[ C_m^b \geq 1 - \frac{1}{2m} \left( 2\log(m+1) + 1 + 8e^{-\sqrt{\gamma(2m)!}} \right) \]
\[ = 1 - \frac{1}{m} \left( \log(m+1) + 1 + \frac{4}{e^\sqrt{2m-1}} \right) \]
\[ \geq 1 - \frac{1}{(m-1)^2 + 1} = \frac{(m-1)^2}{(m-1)^2 + 1} > 0 \]
for all \( m \geq 1 \) if \( \gamma \) is large enough. Therefore, by part (2) of Lemma 3.1, there exists a quasiregular map
\[ g^b_m : \{ z \in \mathbb{C} : \lambda^b \leq |z| \leq \rho^b \} \rightarrow \mathbb{C} \setminus \{0\} \]
without critical points such that \( g^b_m(z) = \psi_{m-1}(z) = 2^m \) on \( |z| = \lambda^b \) and \( g^b_m(z) = \eta^b_m(z) = \tau_m 2^m(z-1)^2 \) on \( |z| = \rho^b \) with
\[ K(g^b_m) \leq \frac{1}{C_m^b} \leq 1 + \frac{1}{(m-1)^2}. \]

Hence, we define the map \( g \) on \( \text{Ann}(T_{m-1}, S_m) \) as follows:
\[ g(z) = \begin{cases} 
R_{m+1} \psi_{m-1} \left( \frac{z}{R_m} \right) = \psi_{m-1}(z) & \text{on } T_{m-1} \leq |z| \leq P_m, \\
R_{m+1} H_{m} \left( \frac{z}{R_m} \right) = H_{m}(z) & \text{on } P_m \leq |z| \leq Q_m, \\
R_{m+1} \eta_{m} \left( \frac{z}{R_m} \right) = \eta_{m}(z) & \text{on } Q_m \leq |z| \leq S_m. 
\end{cases} \]

Similarly, we interpolate \( \eta_m \) and \( \psi_m \) for \( m \geq 1 \). We consider the functions
\[ \eta^b_m(z) = \frac{\eta_m(R_m z)}{R_{m+1}} = \tau_m 2^m(z-1)^2 \] and \( \psi^b_m(z) = \frac{\psi_m(R_m z)}{R_{m+1}} = 2^{m+2}. \)

Let
\[ C_m^b = 1 - \frac{1}{2m + 2} \left( \frac{|\log \tau|}{\log (\lambda^b/\rho^b)} + \frac{8|\omega|}{\rho^b} \right). \]
We apply Lemma 3.1 by taking
\[ \rho^\sharp = \frac{S_m}{R_m}, \quad \lambda^\sharp = \frac{T_m}{R_m}, \quad \tau = \tau_m, \quad \omega = 1, \quad C^\sharp = C_m^\sharp. \]

By definition,
\[ \rho^\sharp = \frac{S_m}{R_m} = e^{\sqrt{\gamma \cdot (2m)!}} \]
and
\[ \lambda^\sharp = \frac{T_m}{R_m} = \frac{T_m}{S_m} \cdot \frac{S_m}{R_m} = e^{2 \sqrt{\gamma \cdot (2m)!}} = (\rho^\sharp)^2. \]

Then \( \rho^\sharp \geq 2|\omega| \) and \( \lambda^\sharp \geq e\rho^\sharp \) because
\[ \rho^\sharp = e^{\sqrt{\gamma \cdot (2m)!}} > e > 2 = 2|\omega| \]
and
\[ \lambda^\sharp = (\rho^\sharp)^2 = e^{2 \sqrt{\gamma \cdot (2m)!}} > e\rho^\sharp. \]

Since
\[ |\log \tau| = \log \left\{ (m + 1)^2 \cdot \frac{m + 1}{m + 2} \cdot \left( 1 + \frac{1}{m} \right)^{2m} \right\} \]
\[ \leq \log \left\{ (m + 1)^2e^2 \right\} = 2\log(m + 1) + 1, \]
we obtain that
\[ C_m^\sharp \geq 1 - \frac{1}{2m + 2} \left( \frac{2\log(m + 1) + 1}{\sqrt{\gamma \cdot (2m)!}} + 8e^{-\sqrt{\gamma \cdot (2m)!}} \right) \]
\[ = 1 - \frac{1}{m + 1} \left( \frac{\log(m + 1) + 1}{\sqrt{\gamma \cdot 2m^m \cdot m!}} + \frac{4}{e^{\sqrt{\gamma \cdot 2m^m \cdot m!}}} \right) \]
\[ \geq 1 - \frac{1}{m^2 + 1} = \frac{m^2}{m^2 + 1} > 0 \]
for all \( m \geq 1 \) if \( \gamma \) is large enough. Therefore, by part (1) of Lemma 3.1, there exists a quasiregular map
\[ g_m^\sharp : \{ z \in \mathbb{C} : \rho^\sharp \leq |z| \leq \lambda^\sharp \} \to \mathbb{C} \setminus \{0\} \]
without critical points such that \( g_m^\sharp(z) = \eta_m^\sharp(z) = \tau_m z^{2m}(z - 1)^2 \) on \( |z| = \rho^\sharp \) and \( g_m^\sharp(z) = \psi_m^\sharp(z) = z^{2m+2} \) on \( |z| = \lambda^\sharp \) with
\[ K(g_m^\sharp) \leq \frac{1}{C_m^\sharp} \leq 1 + \frac{1}{m^2}. \]
Hence, we define the map \( g \) on \( \text{Ann}(Q_m, P_{m+1}) \) as follows:

\[
g(z) = \begin{cases} 
R_{m+1}^{-1} \eta_m^1 \left( \frac{z}{R_m} \right) = \eta_m(z) & \text{on } Q_m \leq |z| \leq S_m, \\
R_{m+1} g_m^1 \left( \frac{z}{R_m} \right) = H_m^0(z) & \text{on } S_m \leq |z| \leq T_m, \\
R_{m+1} \psi_m^1 \left( \frac{z}{R_m} \right) = \psi_m(z) & \text{on } T_m \leq |z| \leq P_{m+1}.
\end{cases}
\]

Moreover, we define the map \( g \) as \( \gamma_1 \) on \( |z| \leq Q_1 \). Therefore, the map \( g \) is \( K_m \)-quasiregular on \( E_m \), for \( m \geq 1 \), where

\[
E_m = \text{Ann}(S_m, T_m) \cup \text{Ann}(P_{m+1}, Q_{m+1}) \quad \text{and} \quad K_m = 1 + \frac{1}{m^2}.
\]

Finally, for \( z \in \text{Ann}(Q_{m(j)}, S_{m(j)}) \), we redefine the map \( g \) as

\[
g(z) = \begin{cases} 
q_{c, A_m(j)} (z - R_{m(j)}) + R_{m(j)} & \text{on } |z - R_{m(j)}| \leq r_{m(j)}, \\
I_j (z - R_{m(j)}) + R_{m(j)} & \text{on } r_{m(j)} \leq |z - R_{m(j)}| \leq \rho_{m(j)}, \\
h_{m(j)} (z - R_{m(j)}) + R_{m(j)} = \eta_{m(j)}(z) & \text{on } \rho_{m(j)} \leq |z - R_{m(j)}|.
\end{cases}
\]

Then, the map \( g \) is \( K'_j \)-quasiregular on \( D_j \) for \( j \geq 1 \), where

\[
D_j = \text{Ann} \left( R_{m(j)} : r_{m(j)}, \rho_{m(j)} \right) \quad \text{and} \quad K'_j = 1 + \frac{1}{j^2}.
\]

Note that the point \( m(j) R_{m(j)}/(m(j) + 1) \), a critical point of \( \eta_{m(j)} \), does not belong to the disk \( |z - R_{m(j)}| \leq \rho_{m(j)} \) for all \( j \geq 1 \).

**Lemma 4.2.** If \( \gamma \) is sufficiently large, \( g(D_j) \subset \text{Ann}(P_{m(j)+1}, Q_{m(j)+1}) \) for \( j \geq 1 \).

**Proof.** First, we show that \( |g(z)| \leq Q_{m(j)+1} \) on \( D_j \) for \( j \geq 1 \). By the maximum principle, for \( z \in D_j \), the inequality

\[
|g(z)| \leq \max_{|z-R_{m(j)}| = \rho_{m(j)}} \left| \frac{b_{m(j)} z^{2m(j)}}{R_{m(j)}^{2m(j)+2}} \left( z - R_{m(j)} \right)^2 \right|
\]

\[
= \frac{(m(j) + 1)^{2m(j)+3}}{(m(j) + 2) \cdot m(j)^{2m(j)}} \cdot \frac{R_{m(j)+1}}{R_{m(j)}} \cdot \frac{1}{2e(m(j)+1)} \cdot \left\{ \frac{1}{2e(m(j)+1)} e^{-\frac{1}{2} \sqrt{\gamma (2m(j)+2)}} \frac{R_{m(j)} + R_{m(j)}}{R_{m(j)}} \right\}^{2m(j)}
\]

\[
= \frac{(m(j) + 1)^{2m(j)+3}}{(m(j) + 2) \cdot m(j)^{2m(j)}} \cdot R_{m(j)+1} \cdot \left\{ \frac{1}{2e(m(j)+1)} e^{-\frac{1}{2} \sqrt{\gamma (2m(j)+2)}} + 1 \right\}^{2m(j)}
\]

\[
\cdot \frac{1}{4e^2(m(j)+1)^2} e^{-\sqrt{\gamma (2m(j)+2)}}
\]
holds for all $j \geq 1$ if $\gamma$ is large enough.

Next, we show that $|g(z)| \geq P_{m(j)+1}$ on $D_j$ for $j \geq 1$. Since $g(z)$ has no zeros in $D_j$, by the minimum principle, for $z \in D_j$, the inequality

$$|g(z)| \geq \min_{|z-R_{m(j)}|=r_{m(j)}} \left| A_{m(j)}(z-R_{m(j)})^2 + \frac{c_j}{A_{m(j)}} + R_{m(j)} \right|$$

$$= \min_{0 \leq \theta \leq 2\pi} \frac{(m(j) + 1)^2m(j)+3}{(m(j) + 2) \cdot m(j)^2m(j)} \cdot \frac{R_{m(j)}+1}{R_{m(j)}^2} \cdot \left\{ \frac{1}{m(j)+1}e^{-\sqrt{\gamma(2m(j)+2)!}R_{m(j)}e^{i\theta}} \right\}^2 \cdot \frac{c_j \cdot (m(j) + 2) \cdot m(j)^2m(j)}{(m(j) + 1)^2m(j)+3} \cdot \frac{R_{m(j)}^2}{R_{m(j)}+1} + R_{m(j)} \right|$$

$$= \min_{0 \leq \theta \leq 2\pi} R_{m(j)+1} \cdot \left| \frac{m(j) + 1}{m(j) + 2} \cdot (1 + \frac{1}{m(j)})^{2m(j)} e^{-2\sqrt{\gamma(2m(j)+2)!}e^{i2\gamma}} + c_j \cdot \frac{m(j) + 2}{(m(j) + 1)^3} \cdot \left( 1 + \frac{1}{m(j)} \right)^{-2m(j)} \cdot \left( \frac{R_{m(j)}}{R_{m(j)}+1} \right)^2 + R_{m(j)} \right|$$

$$\geq R_{m(j)+1}e^{-2\sqrt{\gamma(2m(j)+2)!}} \left\{ \frac{m(j) + 1}{m(j) + 2} \cdot (1 + \frac{1}{m(j)})^{2m(j)} - |c_j| \cdot \frac{m(j) + 2}{(m(j) + 1)^3} \cdot \left( 1 + \frac{1}{m(j)} \right)^{-2m(j)} \cdot e^{2\sqrt{\gamma(2m(j)+2)!}-2\gamma(2m(j))!!} - e^{2\sqrt{\gamma(2m(j)+2)!}-\gamma(2m(j))!!} \right\}$$

holds. Then, there exists a subsequence $(\tilde{m}(j))_{j=1}^{\infty}$ of $(m(j))_{j=1}^{\infty}$, which we again relabel as $m(j)$, such that the inequalities

$$|c_j| \cdot \frac{m(j) + 2}{(m(j) + 1)^3} \cdot \left( 1 + \frac{1}{m(j)} \right)^{-2m(j)} \cdot e^{2\sqrt{\gamma(2m(j)+2)!}-2\gamma(2m(j))!!} < \frac{2}{3}$$

and

$$e^{2\sqrt{\gamma(2m(j)+2)!}-\gamma(2m(j))!!} < \frac{2}{3}$$

hold for all $j \geq 1$. Therefore, the inequality

$$|g(z)| > P_{m(j)+1} \left( \frac{2}{3} \cdot 4 - \frac{2}{3} - \frac{2}{3} \right) = \frac{4}{3}P_{m(j)+1} > P_{m(j)+1}$$

holds for all $j \geq 1$. \qed
Lemma 4.3. If $\gamma$ is sufficiently large, $g(\text{Ann}(S_m, Q_{m+1})) \subset \text{Ann}(S_{m+1}, Q_{m+2})$ for $m \geq 1$.

Proof. Note that $g(z)$ has no critical points in $\text{Ann}(S_m, Q_{m+1})$ and

$$g(z) = \begin{cases} b_m z^{2m} (z - R_m)^2 & \text{on } |z| = S_m, \\ H_m^1(z) & \text{on } S_m \leq |z| \leq T_m, \\ a_m z^{2m+2} & \text{on } T_m \leq |z| \leq P_{m+1}, \\ H_{m+1}^0(z) & \text{on } P_{m+1} \leq |z| \leq Q_{m+1}, \\ b_{m+1} z^{2m+2} (z - R_{m+1})^2 & \text{on } |z| = Q_{m+1}. \end{cases}$$

First, we show that $|g(z)| \leq Q_{m+2}$ on $\text{Ann}(S_m, Q_{m+1})$ for $m \geq 1$. By the maximum principle, for $z \in \text{Ann}(S_m, Q_{m+1})$, the inequality

$$|g(z)| \leq \max_{|z| = Q_{m+1}} \left| b_{m+1} z^{2m+2} (z - R_{m+1})^2 \right|$$

$$= \frac{(m + 2)^{2m+5}}{(m + 3) \cdot (m + 1)^{2m+2}} \cdot \frac{R_{m+2}}{R_{m+1}^2} \cdot (Q_{m+1})^{2m+2} \cdot (Q_{m+1} - R_{m+1})^2$$

$$= (m + 2)^2 \cdot \frac{m + 2}{m + 3} \cdot \left( 1 + \frac{1}{m + 1} \right)^{m+1} \cdot R_{m+2} \cdot \left( \frac{Q_{m+1}}{R_{m+1}} \right)^{2m+2} \cdot \left( \frac{Q_{m+1}}{R_{m+1}} + 1 \right)^2$$

$$< 4e^2 (m + 2)^3 R_{m+2} e^{-2(m+2)\sqrt{\gamma(2m+2)}} \leq R_{m+2} e^{-\sqrt{\gamma(2m+4)}} = Q_{m+2}$$

holds for all $m \geq 1$ if $\gamma$ is large enough.

Next, we show that $|g(z)| \geq S_{m+1}$ on $\text{Ann}(S_m, Q_{m+1})$ for $m \geq 1$. Since $g(z)$ has no zeros in $\text{Ann}(S_m, Q_{m+1})$, by the minimum principle, for $z \in \text{Ann}(S_m, Q_{m+1})$, the inequality

$$|g(z)| \geq \min_{|z| = S_m} \left| b_m z^{2m} (z - R_m)^2 \right|$$

$$= \frac{(m + 1)^{2m+3}}{(m + 2) \cdot m^{2m}} \cdot \frac{R_{m+1}}{R_{m+2}^2} \cdot S_{m+1}^2 (S_m - R_m)^2$$

$$= \frac{(m + 1)^3}{m + 2} \cdot \left( \left( 1 + \frac{1}{m} \right)^m \right)^2 \cdot R_{m+1} \cdot \left( \frac{S_m}{R_m} \right)^{2m} \cdot \left( \frac{S_m}{R_m} - 1 \right)^2$$

$$\geq \frac{8}{3} \cdot 4 \cdot R_{m+1} e^{2m \sqrt{\gamma(2m)}} \left( e^{\sqrt{\gamma(2m)}} - 1 \right)^2$$

$$\geq R_{m+1} e^{2m \sqrt{\gamma(2m)}} \geq R_{m+1} e^{\sqrt{\gamma(2m+2)}} = S_{m+1}$$

holds for all $m \geq 1$. \qed
Proof of Theorem A. Note that the map \( g \) is defined as

\[
g(z) = \begin{cases} 
\eta_1(z) & \text{on } |z| \leq S_1, \\
H^x_1(z) & \text{on } S_1 \leq |z| \leq T_1, \\
\psi_1(z) & \text{on } T_1 \leq |z| \leq P_2, \\
H^x_2(z) & \text{on } P_2 \leq |z| \leq Q_2, \\
\tilde{\eta}_2(z) & \text{on } Q_2 \leq |z| \leq S_2, \\
H^x_2(z) & \text{on } S_2 \leq |z| \leq T_2, \\
\psi_2(z) & \text{on } T_2 \leq |z| \leq P_3, \\
\vdots & \\
\psi_{m-1}(z) & \text{on } T_{m-1} \leq |z| \leq P_m, \\
H^x_m(z) & \text{on } P_m \leq |z| \leq Q_m, \\
\tilde{\eta}_m(z) & \text{on } Q_m \leq |z| \leq S_m, \\
H^x_m(z) & \text{on } S_m \leq |z| \leq T_m, \\
\psi_m(z) & \text{on } T_m \leq |z| \leq P_{m+1}, \\
\vdots & 
\end{cases}
\]
where \( \tilde{\eta}_m(z) = \eta_m(z) \) if \( m \neq m(j) \) for \( j \geq 1 \) and

\[
\tilde{\eta}_{m(j)}(z) = \begin{cases} 
q_{c_j, A_{m(j)}}(z - R_{m(j)}) + R_{m(j)} & \text{on } |z - R_{m(j)}| \leq r_{m(j)}, \\
I_f(z - R_{m(j)}) + R_{m(j)} & \text{on } r_{m(j)} \leq |z - R_{m(j)}| \leq \rho_{m(j)}, \\
\eta_{m(j)}(z) & \text{on } \rho_{m(j)} \leq |z - R_{m(j)}|.
\end{cases}
\]

We will check that disjoint measurable sets \( E_m \ (m = 1, 2, \ldots) \) and \( D_j \ (j = 1, 2, \ldots) \) defined as

\[
E_m = \text{Ann}(S_m, T_m) \cup \text{Ann}(P_{m+1}, Q_{m+1}) \quad \text{and} \quad D_j = \text{Ann}(R_{m(j)}; r_{m(j)}, \rho_{m(j)})
\]

satisfy the assumption of Theorem 2.6. By Lemma 4.2 and Lemma 4.3, for every \( z \in \mathbb{C} \), the \( g \)-orbit of \( z \) passes \( E_m \) and \( D_j \) at most once for every \( m \) and \( j \). By construction, the map \( g \) is \( K_m \)-quasiregular on \( E_m \) and \( K'_j \)-quasiregular on \( D_j \), where \( K_m = 1 + 1/m^2 \) and \( K'_j = 1 + 1/j^2 \). The constant \( K_\infty = \prod_{m=1}^\infty K_m \cdot \prod_{j=1}^\infty K'_j \) is finite and \( K_\infty = (\sinh \pi/\pi)^2 \). Furthermore, the quasiregular map \( g \) is holomorphic almost everywhere outside \( (\bigcup_{m=1}^\infty E_m) \cup (\bigcup_{j=1}^\infty D_j) \). Therefore, by Theorem 2.6, there exists a \( K_\infty \)-quasiconformal map \( \varphi : \mathbb{C} \to \mathbb{C} \) such that \( f = \varphi \circ g \circ \varphi^{-1} \) is a transcendental entire function. We normalize \( \varphi \) as \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). Let

\[
V_j = \varphi \left( \{ z \in \mathbb{C} : |z| < P_{m(j)+1} \} \right)
\]

and let \( U_j \) be the connected component of \( f^{-1}(V_j) \) containing \( \varphi(R_{m(j)}) \). Since \( g(D_j) \subset \text{Ann}(P_{m(j)+1}, Q_{m(j)+1}) \) by Lemma 4.2, \( U_j \) is contained by the bounded component of \( \mathbb{C} \setminus \varphi(D_j) \). Therefore, we obtain that \( U_j \subset V_j \). Moreover, \( U_j \) contains only one critical point \( \varphi(R_{m(j)}) \) because the critical point \( m(j)R_{m(j)}/(m(j)+1) \) of \( g \) does not belong to the disk \( |z - R_{m(j)}| < \rho_{m(j)} \). Then each triple \((f|_{U_j}, U_j, V_j)\) becomes a quadratic-like map. By the straightening theorem, the quadratic-like map \((f|_{U_j}, U_j, V_j)\) is hybrid equivalent to the quadratic polynomial \( q_{c_j, A_{m(j)}} \), which is affine conjugate to the quadratic polynomial \( p_{c_j} : z \mapsto z^2 + c_j \) for all \( j \geq 1 \). The proof of Theorem A is completed. \( \square \)

5. The order of the entire function obtained by Theorem A. We can calculate the order of the entire function \( f = \varphi \circ g \circ \varphi^{-1} \) obtained by Theorem A, considering the dynamics of the quasiregular map \( g \).

**Definition 5.1** (Order). Let \( f \) be a non-constant entire function. The order \( \rho(f) \) of \( f \) is the limit superior

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r},
\]

where \( M(r, f) = \max_{|z|=r} |f(z)| \) is the maximum modulus function. The order of a quasiregular map can be defined in the same way.

For an ordered quadruple \( a, b, c, d \) of distinct points in \( \overline{\mathbb{R}^3} \), we define the absolute cross-ratio \( |a, b, c, d| \) as

\[
|a, b, c, d| = \frac{q(a, c) q(b, d)}{q(a, b) q(c, d)},
\]

where \( q(x, y) = \max_{t \in [0, 1]} |x - ty| \).
where $q$ is the spherical metric in $\mathbb{R}^n$, namely

$$q(x, y) = \begin{cases} 
\frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}} & \text{if } x, y \in \mathbb{R}^n \\
\frac{1}{\sqrt{1 + |x|^2}} & \text{if } x \in \mathbb{R}^n \text{ and } y = \infty.
\end{cases}$$

A Möbius map of $\mathbb{R}^n$ onto itself preserves the absolute cross-ratio. The absolute cross-ratio depends on the order of the points. Note that $|0, e_1, x, \infty| = |x|$.

**Theorem 5.2 ([2, Theorem 14.7]).** For $K \geq 1$, let $f$ be a $K$-quasiconformal automorphism of the plane $\mathbb{R}^2$. Then

$$\frac{1}{\lambda(K)} \min\left\{ t^{1/K}, t^K \right\} \leq |f(a), f(b), f(c), f(d)| \leq \lambda(K) \max\left\{ t^{1/K}, t^K \right\}$$

for each ordered quadruple of distinct points $a, b, c, d$ in the plane, where $t = |a, b, c, d|$. Moreover, the inequalities are sharp for each $K \geq 1$.

The function $\lambda(K)$ is a distortion function of the theory of plane quasiconformal mappings. For details, refer to [2]. Anderson, Vamanamurthy and Vuorinen [3] proved that the inequality $e^{\pi(K-1)} < \lambda(K) < e^{a(K-1)}$ holds, where $1 < K < \infty$ and $a = 4.37688$.

Since the $K_\infty$-quasiconformal map $\varphi$ obtained in the proof of Theorem A fixes $0, 1$ and $\infty$, then

$$\frac{1}{\lambda(K_\infty)} \min\left\{ |z|^{1/K_\infty}, |z|^{K_\infty} \right\} \leq |\varphi(z)| \leq \lambda(K_\infty) \max\left\{ |z|^{1/K_\infty}, |z|^{K_\infty} \right\}.$$

**Proof of Theorem B.** First, we calculate the order of the quasiregular map $g$. By the definition of the sequences $(R_m)$ and $(Q_m)$,

$$R_m = \frac{R_m}{R_{m-1}} \frac{R_{m-1}}{R_{m-2}} \cdots \frac{R_2}{R_1} \cdot R_1 = \exp \left\{ \gamma \sum_{k=0}^{m-1} (2k)!! \right\}$$

and

$$Q_m = \frac{Q_m}{R_m} \frac{R_m}{R_{m-1}} \cdots \frac{R_2}{R_1} \cdot R_1
= \exp \left\{ -\sqrt{\gamma} \cdot (2m)!! + \gamma \sum_{k=0}^{m-1} (2k)!! \right\}
< \exp \left\{ \gamma \sum_{k=0}^{m-1} (2k)!! \right\}.$$
< \frac{\log \log \exp \left\{ \gamma \sum_{k=0}^{m+1} (2k)!! \right\}}{\log \exp \left\{ \gamma \sum_{k=0}^{m-2} (2k)!! \right\}} = \frac{\log \gamma + \log \sum_{k=0}^{m+1} (2k)!!}{\gamma \sum_{k=0}^{m-2} (2k)!!} \\
\leq \frac{\log \gamma + \log (m+2)(2m+2)!}{\gamma \cdot (2m-4)!} \\
\leq \frac{\log \gamma}{\gamma \cdot (2m-4)!} + \frac{\log (m+2)}{\gamma \cdot (2m-4)!} + \frac{\log (2m+2)!}{\gamma \cdot (2m-4)!} \\
\text{holds for all } m \geq 2. \text{ Therefore, we obtain that} \\
\rho(g) = \limsup_{r \to \infty} \frac{\log \log M(r, g)}{\log r} \leq \lim_{m \to \infty} \frac{\log \log Q_{m+2}}{\log R_{m-1}} = 0.

By Theorem 5.2, if \(|z| = r\) is large enough, then 
\(|\phi(z)| \geq \frac{1}{\lambda(K_\infty)} \cdot r^{1/K_\infty} > r^{1/(2K_\infty)}\).

Hence, if \(r\) is sufficiently large, we obtain that 
\[
\frac{\log \log M(r^{1/(2K_\infty)}, f)}{\log r^{1/(2K_\infty)}} < \frac{\log \log \tilde{M}(r, f)}{\log r^{1/(2K_\infty)}}
\]
\[
= 2K_\infty \cdot \frac{\log \log \max_{z \in \Gamma_r} |\phi \circ g \circ \phi^{-1}(z)|}{\log r}
\]
\[
= 2K_\infty \cdot \frac{\log \log \max_{|w| = r} |\phi \circ g(w)|}{\log r}
\]
\[
< 2K_\infty \cdot \frac{\log \log \max_{|w| = r} \lambda(K_\infty) |g(w)|^{K_\infty}}{\log r}
\]
\[
= 2K_\infty \cdot \frac{\log \log \left\{ \lambda(K_\infty) \left( \max_{|w| = r} |g(w)| \right)^{K_\infty} \right\}}{\log r}
\]
\[
= 2K_\infty \cdot \frac{\log \left\{ \log \lambda(K_\infty) + K_\infty \log \max_{|w| = r} |g(w)| \right\}}{\log r}
\]
\[
< 2K_\infty \cdot \frac{2 \log \log \max_{|w| = r} |g(w)|}{\log r}
\]
\[
= 4K_\infty \cdot \frac{\log \log M(r, g)}{\log r},
\]
where \(\tilde{M}(r, f) = \max_{z \in \Gamma_r} |f(z)|\) and \(\Gamma_r = \varphi \left( \{w \in \mathbb{C} : |w| = r\} \right)\). Therefore,
\[
\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r^{1/(2K_\infty)}, f)}{\log r^{1/(2K_\infty)}} \\
\leq 4K_\infty \limsup_{r \to \infty} \frac{\log \log M(r, g)}{\log r} = 4K_\infty \rho(g) = 0.
\]
Acknowledgments. I would like to thank the referees for their valuable comments and suggestions that have improved the presentation of this paper. This work was supported by Grant-in-Aid for Young Scientists (B) No. 17K14212 of Japan Society for the Promotion of Science.

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Received for publication October 2018.
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