Bi-Hamiltonian Structure of a Three-Component Camassa-Holm Type Equation

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A recently proposed three-component Camassa-Holm equation is considered. It is shown that this system is a bi-Hamiltonian system.

Keywords: bi-Hamiltonian structures; three-component CH equation

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1. Introduction

In 1981, the following integrable equation

$$u_t - u_{xxt} + 3uu_x = 2u_uux + uu_{xxx},$$  \hspace{1cm} (1.1)

was proposed as an abstract equation admitting a bi-Hamiltonian structure [20]. This system, now known as Camassa-Holm (CH) equation, was rediscovered by Camassa and Holm [3] as an approximation to the incompressible Euler equations and its relevance for the shallow water waves modeling was discussed [14, 24–26]. The CH equation has been studied extensively and a number of properties are established for it. Indeed, it is shown that this equation is solvable by inverse scattering transformation [1, 2, 5, 6, 15, 17, 27, 30], possesses a large number of solutions including multi-soliton solutions, algebraic-geometric solutions and multi-peakon solutions [22, 33, 34, 37], can be regarded as the geodesic flow on the diffeomorphism group [29, 35] (see also [12, 13, 28]), to mention just a few (also see [21, 32] and the references there).

Also, it is worthwhile to remark that while the CH equation is reciprocally associated to a particular flow of the celebrated Korteweg-de Vries (KdV) equation [19, 32], there are fundamental differences between the CH equation and the KdV equation. Unlike the KdV equation, the CH equation has breaking wave solutions which develop singularities in finite time [8]. Indeed, the construction of such an equation, which can model both breaking waves and peaked traveling waves, was a long-standing open problem in hydrodynamics [39]. Furthermore, the search of models which have peaked wave solutions is also motivated by the quest for waves of great height (see [7, 9, 10, 38]). It is interesting to note that the peaked solitons or peakons of the CH equation are orbitally stable [16, 31].
The CH equation has several generalizations. A two-component and a three-component extensions were proposed in [4, 36] and in [18] respectively and both extensions are shown to be bi-Hamiltonian systems. Very recently, a new three-component generalization of the CH equation was constructed by Geng and Xue [21]. It reads as

\begin{align}
    u_t &= -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\
    v_t &= 2vq_x + v_x q, \\
    w_t &= vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr),
\end{align}

(1.2)

with

\begin{align}
    u &= p - p_{xx}, \\
    v &= \frac{1}{2}(q_{xx} - 4q + p_{xx} r_x - r_{xx} p_x + 3p_x r - 3p_{xx} r), \\
    w &= r_{xx} - r.
\end{align}

(1.3)

Above system was derived from a $3 \times 3$ matrix spectral problem. It was further shown that (1.2) is a Hamiltonian system with infinite number of conservation laws. The purpose of this paper is to construct two Hamiltonian structures for (1.2) and to prove that it constitutes a bi-Hamiltonian system.

2. Bi-Hamiltonian structure of the three-component Camassa-Holm equation

As demonstrated in [21], the system (1.2) arises as a zero curvature equation

\begin{equation}
    M_t - N_x + [M, N] = 0,
\end{equation}

(2.1)

which is the compatibility condition of the linear system

\begin{equation}
    \varphi_t = M \varphi, \quad \varphi_t = N \varphi.
\end{equation}

(2.2)

where

\[ M = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix}, \]

\[ N = \begin{pmatrix} \frac{1}{2}(-q_x - p_x r_x + pr) \\ \frac{1}{2}(-pr_x + p_x r - q + \lambda q v) \frac{1}{2}(q_x - p_x r_x + pr) \frac{p_x}{r_x} + qu \\ -r + \lambda q v \end{pmatrix}. \]

(2.3)

Now, we construct Hamiltonian structures for the equations associated with this $3 \times 3$ spectral problem. To this end, we take $M$ as above and $N$ as follows

\[ N = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \]

(2.3)

with $A_k, B_k, C_k (k = 1, 2, 3)$ are functions depending on the spectral parameter $\lambda$ and the field variables.
Through tedious calculations, we find that the zero-curvature representation (2.1) yields
\[
\begin{pmatrix}
u \\
w \\
w 
\end{pmatrix}_t = (\lambda^{-1} \mathcal{E} + \mathcal{F}) \begin{pmatrix}
C_2 \\
A_2 \\
\lambda A_3 
\end{pmatrix},
\tag{2.4}
\]
where
\[
\mathcal{E} = \begin{pmatrix}
0 & 0 & \frac{\partial^2 - 1}{1 - \partial^2} \\
\frac{1}{2} u_x + \frac{3}{2} u \partial & 2 \partial - \frac{1}{2} \partial^3 & \frac{1}{2} w_x + \frac{3}{2} w \partial \\
0 & 0 & 0
\end{pmatrix},
\tag{2.5}
\]
\[
\mathcal{F} = \begin{pmatrix}
\frac{3}{2} u \partial^{-1} u & u_x + \frac{3}{2} u \partial & -\frac{3}{2} u \partial^{-1} w - v \\
0 & 0 & v \partial + \partial v \\
-\frac{3}{2} w \partial^{-1} u + v w_x + \frac{3}{2} w \partial & 0 & \frac{3}{2} w \partial^{-1} w
\end{pmatrix}.
\tag{2.6}
\]

While neither \( \mathcal{E} \) nor \( \mathcal{F} \) could serve as Hamiltonian operators since both of them are even not skew-symmetric, we may formulate a operator
\[
\mathcal{R} = \mathcal{F} \mathcal{E}^{-1},
\tag{2.7}
\]
which is the candidate for a recursion operator. To justify this, we notice that a Hamiltonian operator was obtained in \([21]\), which reads as
\[
\mathcal{J} = \begin{pmatrix}
0 & 0 & \partial^2 - 1 \\
0 & -\partial v - v \partial & 0 \\
1 - \partial^2 & 0 & 0
\end{pmatrix}.
\tag{2.8}
\]
Then we define a new operator by
\[
\mathcal{K} = \mathcal{F} \mathcal{E}^{-1} \mathcal{J},
\tag{2.9}
\]
whose explicit form is given by
\[
\mathcal{K} = \mathcal{F} - 2 \mathcal{J} (\partial^3 - 4 \partial)^{-1} (\mathcal{J}^\dagger),
\tag{2.10}
\]
with
\[
\mathcal{J} = \begin{pmatrix}
\frac{1}{2} u \partial + u_x \\
v \partial + \partial v \\
\frac{1}{2} w \partial + w_x
\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix}
\frac{1}{2} u \partial^{-1} u & 0 & -v - \frac{3}{2} u \partial^{-1} w \\
0 & 0 & 0 \\
v - \frac{3}{2} w \partial^{-1} u & 0 & \frac{3}{2} w \partial^{-1} w
\end{pmatrix},
\]
and \( \mathcal{J}^\dagger \) denotes the formal adjoint of the operator \( \mathcal{J} \).

Our claim is that \( \mathcal{J} \) and \( \mathcal{K} \) constitutes a compatible Hamiltonian pair. In fact, we have
Theorem 2.1. The system (1.2) is a bi-Hamiltonian system, namely there exist two compatible Hamiltonian operators $\mathcal{J}$ and $\mathcal{K}$ such that

$$
\begin{pmatrix}
u \\
 v \\
 w
\end{pmatrix} = \mathcal{J} \left( \begin{pmatrix} \frac{\partial H_1}{\partial u} \\
 \frac{\partial H_1}{\partial v} \\
 \frac{\partial H_1}{\partial w}
\end{pmatrix} \right) = \mathcal{K} \left( \begin{pmatrix} \frac{\partial H_0}{\partial u} \\
 \frac{\partial H_0}{\partial v} \\
 \frac{\partial H_0}{\partial w}
\end{pmatrix} \right),
\tag{2.11}
$$

where $\mathcal{J}$ and $\mathcal{K}$ are the operators given by (2.8) and (2.9), and

$$
H_0 = \frac{1}{4} \int (4q^2 - qq_x - p_x^2 + 6pp_x + 3p^2 + \gamma) \, dx,
$$

$$
H_1 = \frac{1}{7} \int (2v + ur_x + wp_x) \, dx.
$$

Proof. The first operator $\mathcal{J}$ is a local and linear operator, which already appeared in [21]. Next, we turn to the more complicated operator $\mathcal{K}$. Also, it is easy to show that this operator is skew-symmetric, therefore we prove that the related Jacobi identity holds. Since

\[
\begin{align*}
< f, \mathcal{K}^*[\mathcal{K} g]h > &+ \text{c.p.} \\
= < f, \mathcal{J}^*[\mathcal{J} g](\partial^3 - 4\partial)\mathcal{K}h - 2\mathcal{J}(\partial^3 - 4\partial)^{-1}(\mathcal{J}^*)\mathcal{K}g > + \text{c.p.}
\end{align*}
\]

Let $A, B, C$ be the rows of $\mathcal{K}$, or $\mathcal{K} = (A, B, C)^T$, we find

\[
\begin{align*}
< f, \mathcal{J}^*[\mathcal{K} g]h > &= + \text{c.p.} \\
= < f, \begin{pmatrix} f_1 \\
 f_2 \\
 f_3 \\
\end{pmatrix}, \begin{pmatrix} 3Ag\partial^{-1}(uh_1 - wh_3) + \frac{3}{2}u\partial^{-1}Agh_1 - Bgh_3 - \frac{3}{2}u\partial^{-1}Cgh_3 \\
 Bgh_1 - \frac{3}{2}Cg\partial^{-1}(uh_1 - wh_3) - \frac{3}{2}w\partial^{-1}Agh_1 + \frac{3}{2}w\partial^{-1}Cgh_3 \\
 0
\end{pmatrix} > + \text{c.p.} \\
= < \frac{3}{2}(f_1Ag - g_1Af + Cfg_3 - f_3Cg)\partial^{-1}(uh_1 - wh_3) + f_3Bgh_1 - f_1Bgh_3 > + \text{c.p.} \\
= < \frac{9}{4}(ug_1\partial^{-1}w_3 + wg_3\partial^{-1}w_3, \partial^{-1}uh_1 > + < ug_1\partial^{-1}uf_1 + wg_3\partial^{-1}uf_1, \partial^{-1}h_3w >) \\
+ 3 < f_1(\frac{3}{2}u\partial + u_x)g_0 - f_3(\frac{3}{2}w\partial + w_x)g_0, \partial^{-1}(uh_1 - wh_3) > \\
+ 2 < (v_x + 2v\partial)h_0, f_1g_3 > - (f \leftrightarrow g) + \text{c.p.} \\
= 3 < f_1u\partial + f_1u_x)g_0 - (\frac{3}{2}f_3w\partial + f_3w_x)g_0, \partial^{-1}(uh_1 - h_3w) > \\
+ 2 < (v_x + 2v\partial)h_0, f_1g_3 > - (f \leftrightarrow g) + \text{c.p.} \\
= \frac{3}{2} < g_0, (f_3w_x + 3f_3w - f_1u_x - 3f_1u)\partial^{-1}(uh_1 - wh_3) > \\
+ 2 < g_0, v_x f_1h_3 + 2v(f_1h_3)_x > - (f \leftrightarrow h) + \text{c.p.},
\end{align*}
\]

(2.12)

where $(f \leftrightarrow g)$ denotes the resulted term from the previous ones by permutating $f$ and $g$, and

\[
\begin{align*}
g_0 &= (\partial^3 - 4\partial)^{-1} \left( \frac{3}{2}u_1g_1 + \frac{1}{2}u_x g_1 + v_x g_2 + 2v_g x + \frac{3}{2}w_3 g_3 + \frac{3}{2}w x g_3 \right),
\tag{2.13}
\end{align*}
\]
and similarly for \( f_0, h_0 \). Also, we have

\[
< f, -2\mathcal{H}'[\mathcal{H} g](\partial^3 - 4\partial^{-1}(\partial^4)h - 2\mathcal{H}'(\partial^3 - 4\partial)^{-1}(\partial^4)'[\mathcal{H} g]h > + \text{c.p.} \\
= 2 < h, \mathcal{H}'[\mathcal{H} f]g_0 > - 2 < h, \mathcal{H}'[\mathcal{H} g]f_0 > + \text{c.p.} \\
= 2 < h^T \mathcal{H}'[\mathcal{H} f]g_0 - f^T \mathcal{H}'[\mathcal{H} h]g_0 > + \text{c.p.} \\
= 2 < \left( \frac{3}{2} h_1 Af \partial + h_1(2f)_x + h_2((Bf)_x + 2Bf \partial) + h_3(\frac{3}{2} Cf \partial + (Cf)_x) \right)g_0 > \\
- (f \leftrightarrow h) + \text{c.p.} \\
= < g_0, f_1(Ah)_x + 3f_1 Ah + 2f_2(Bh)_x + 4f_2 Bh + f_3(Ch)_x + 3f_3 Ch > \\
- (f \leftrightarrow h) + \text{c.p.} \\
= \Omega_1 + \Omega_2, \tag{2.14}
\]

where

\[
\Omega_1 = < g_0, f_1(\frac{3}{2} u \partial^{-1}(uh_1 - wh_3) + \frac{3}{2} u^2 h_1 - v, h_3 - vh_3 - \frac{3}{2} uwh_3) > \\
+ < g_0, 3f_1(\frac{3}{2} u \partial^{-1}(uh_1 - wh_3) - vh_3) + 3f_3(uh_1 - \frac{3}{2} w \partial^{-1}(uh_1 - wh_3)) > \\
+ < g_0, f_3(v, h_1 + vh_3 - \frac{3}{2} w, \partial^{-1}(uh_1 - wh_3) - \frac{3}{2} wuh_1 + \frac{3}{2} w^2 h_3) > \\
- (f \leftrightarrow h) + \text{c.p.} \\
= \frac{3}{2} < g_0, (f_1 u_x + 3f_1 u - 3f_3 w_x - 3f_3 w)\partial^{-1}(uh_1 - wh_3) > \\
+ < g_0, 3h_1 v f_3 - h_1(\frac{3}{2} u f_1 - v, f_3 - \frac{3}{2} u w f_3) > \\
- < g_0, 3h_3 v f_1 + h_3(v, f_1 + v f_1 - \frac{3}{2} w f_1 + \frac{3}{2} w^2 f_3) > (f \leftrightarrow h) + \text{c.p.} \\
= \frac{3}{2} < g_0, (f_1 u_x + 3f_1 u - 3f_3 w_x - 3f_3 w)\partial^{-1}(uh_1 - wh_3) > \\
+ 2 < g_0, v, f_3 h_1 > + 2v(f_3 h_1 + f_3 h_1) > -(f \leftrightarrow h) + \text{c.p.}, \tag{2.15}
\]

and

\[
\Omega_2 = 2 < g_0, (f_1 \partial + 3f_3)(\frac{3}{2} v \partial + u_x)h_0 ((2f_2 \partial + 4f_2) (v, + 2v \partial)h_0 > \\
+ 2 < g_0, (f_3 \partial + 3f_3)(\frac{3}{2} w \partial + w_x)h_0 > -(f \leftrightarrow h) + \text{c.p.}. \tag{2.16}
\]

Taking account of (2.12)-(2.14) with (2.15)-(2.16), we obtain

\[
< f, \mathcal{H}'[\mathcal{H} g]h > + \text{c.p.} = \Omega_2 \\
= 2 < g_0, (f_1(\frac{3}{2} u \partial^2 + \frac{5}{2} u_x \partial + u_x) + 3f_1(\frac{3}{2} u \partial + u_x))h_0 > \\
+ 2 < g_0, (f_3(\frac{3}{2} w \partial^2 + \frac{5}{2} w_x \partial + w_x) + 3f_3(\frac{3}{2} w \partial + w_x))h_0 > \\
+ 4 < g_0, f_2(v, x + 3v, \partial + 2v \partial^2) + 2 f_2(v, x + 2v \partial))h_0 > -(f \leftrightarrow h) + \text{c.p.}
\]
Therefore, $\mathcal{K}$ is a Hamiltonian operator. Finally, we prove that $\mathcal{J}$ and $\mathcal{K}$ form a compatible Hamiltonian pair, that is, we have to show that these operators satisfy the following condition

$$<f, \mathcal{J}'[\mathcal{K}g]h> + <f, \mathcal{K}'[\mathcal{J}g]h> + \text{c.p.} = 0.$$  \hfill (2.17)

Indeed, by direct calculation, we find

$$<f, \mathcal{J}'[\mathcal{K}g]h> + \text{c.p.} = -2 <f_2, ((v_x g_0 + 2 v_0 g_0)_x + 2 (v_x g_0 + 2 v_0 g_0) \partial) h_2 > + \text{c.p.}$$

$$= 2 <v_x g_0 + 2 v_0 g_0, h_2 f_2 - h_2 f_2> + \text{c.p.},$$ \hfill (2.18)

and

$$<f, \mathcal{K}'[\mathcal{J}g]h> + \text{c.p.}$$

$$= <f, \mathcal{J}'[\mathcal{K}g]h - 2 \mathcal{J}'[\mathcal{J}g](\partial^3 - 4 \partial)^{-1}\mathcal{J}^\dagger h - 2 \mathcal{J}'(\partial^3 - 4 \partial)^{-1}(\mathcal{J}^\dagger)'[\mathcal{J}g]h> + \text{c.p.}$$
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\[
\begin{align*}
&\leq \frac{3}{2}(f_1(g_{3xx} - g_3) - f_3(g_1 - g_{1xx}))\partial^{-1}(uh_1 - wh_3) + 2vf_1h_3g_{2x} + v_xf_1g_2h_3 > \\
&+ 2 < g_0, f^T \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} (\mathcal{H} h)_x + f^T \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \mathcal{H} h > -(f \leftrightarrow h) + \text{c.p.}
\end{align*}
\]

Thus making use of (2.18) with (2.19), we obtain

\[
\begin{align*}
&\leq f, \mathcal{H}'[\mathcal{J} g]h > + \leq f, \mathcal{J}'[\mathcal{H} g]h > + \text{c.p.} \\
&= \frac{3}{2}(f_1(g_{3xx} - g_3) - f_3(g_1 - g_{1xx}))\partial^{-1}(uh_1 - wh_3) + 2vf_1h_3g_{2x} + v_xf_1g_2h_3 > \\
&+ 2 < v_xg_0 + 2vg_0x, h_xf_{2x} > + 2 < g_0, f^T \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} (\mathcal{H} h)_x + f^T \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \mathcal{H} h > \\
&- (f \leftrightarrow h) + \text{c.p.}
\end{align*}
\]

Therefore, the Hamiltonian operators \( \mathcal{J} \) and \( \mathcal{H} \) are compatible and the proof of the theorem is completed.

According to the bi-Hamiltonian theory, the operator \( \mathcal{R} \) is a recursion operator with hereditary property, which may be used to generate a hierarchy of commuting flows.

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