SHOKUROV’S BOUNDARY PROPERTY

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Abstract. For a birational analogue of minimal elliptic surfaces $f : X \to Y$, the singularities of the fibers define a log structure $(Y, B_Y)$ in codimension one on $Y$. Via base change, we have a log structure $(Y', B_{Y'})$ in codimension one on $Y'$, for any birational model $Y'$ of $Y$. We show that these codimension one log structures glue to a unique log structure, defined on some birational model of $Y$ (Shokurov’s BP Conjecture).

We have three applications: inverse of adjunction for the above mentioned fiber spaces, the invariance of Shokurov’s FGA-algebras under restriction to exceptional lc centers, and a remark on the moduli part of parabolic fiber spaces.

0. Introduction

Recall Kodaira’s canonical bundle formula for a minimal elliptic surface $f : S \to C$ defined over the complex number field:

$$K_S = f^*(K_C + B_C + M_C).$$

The moduli part $M_C$ is a $\mathbb{Q}$-divisor such that $12M_C$ is integral and $\mathcal{O}_C(12M_C) \simeq J^*\mathcal{O}_{\mathbb{P}^1}(1)$, where $J : C \to \mathbb{P}^1$ is the $J$-invariant function. The discriminant $B_C = \sum P b_P P$, supported by the singular locus of $f$, is computed in terms of the local monodromies around the singular fibers $S_P$: $b_P$ equals $\frac{m-1}{m}$, $\frac{1}{2}$, $\frac{5}{6}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, depending on whether $S_P$ is of type $mI_b, I_b^*, II, II^*, III, III^*, IV, IV^*$, where $mI_b$ is a multiple fibre of multiplicity $m$, and $b \geq 0$. Kawamata [12, 13] proposed an equivalent definition, which does not require classification of the fibers: $1 - b_P$ is the log canonical threshold of the log pair $(S, S_P)$ in a neighborhood of the fiber $S_P$. The minimality may also be removed: the birational changes of $S$ over $C$ are controlled by a log pair structure on $S$.

The birational analogue consists of data $f : (X, B) \to Y$, where $f$ is a surjective morphism between proper normal varieties and $(X, B)$ is a log pair ($B$ is a $\mathbb{Q}$-Weil divisor such that $K + B$ is $\mathbb{Q}$-Cartier), satisfying the following properties:

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(1) \((X, B)\) has Kawamata log terminal singularities over the generic point of \(Y\).

(2) rank \(f_*\mathcal{O}_X([\mathbb{A}(X, B)]) = 1\).

(3) There exist a positive integer \(r\), a rational function \(\varphi \in k(X)^\times\) and a \(\mathbb{Q}\)-Cartier divisor \(D\) on \(Y\) such that

\[
K + B + \frac{1}{r}(\varphi) = f^*D.
\]

We note here that \(B\) need not be effective. However, if \(B\) is effective over the generic point of \(Y\), then the technical assumption (2) is implied by (1). The discriminant on \(Y\) of the log divisor \(K + B\) is the \(\mathbb{Q}\)-Weil divisor \(B_Y := \sum P b_P P\), where \(1 − b_P\) to be the maximal real number \(t\) such that the log pair \((X, B + tf^*(P))\) has log canonical singularities over the generic point of \(P\). The sum runs after all codimension one points of \(Y\), but it has finite support. The moduli part is the unique \(\mathbb{Q}\)-Weil divisor \(M_Y\) on \(Y\) satisfying

\[
K + B + \frac{1}{r}(\varphi) = f^*(K_Y + B_Y + M_Y).
\]

If the base space \(Y\) is a curve, we recover Kodaira’s formula (except for effective freeness):

**Theorem 0.1.** Let \(f: (X, B) \to Y\) be a fiber space satisfying (1),(2),(3) above, with \(\dim(Y) = 1\). Then the moduli \(\mathbb{Q}\)-divisor \(M_Y\) is semi-ample, i.e. there exists a positive integer \(k\) such that \(kM_Y\) is Cartier and the linear system \(|kM_Y|\) is base point free.

If the base has dimension at least two, the linear system \(|kM_Y|\) is expected to have no fixed components for \(k\) large and divisible. However, \(|kM_Y|\) might have base points, as Prokhorov has pointed out. To remove the non-determinacy locus of the expected rational map, we should replace \(f: (X, B) \to Y\) by a birational base change: if \(f': X' \to Y'\) is a fiber space induced via a birational base change \(\sigma: Y' \to Y\), we have an induced data \(f': (X', B_{X'}) \to Y'\), where \(B_{X'}\) is defined by \(\mu^*(K + B) = K_{X'} + B_{X'}\):

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{\mu} & (X', B_{X'}) \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\sigma} & Y'
\end{array}
\]

We denote by \(B_{Y'}\) and \(M_{Y'}\) the discriminant and moduli part of \(f': (X', B_{X'}) \to Y'\), respectively. The stabilisation of the moduli part after a certain blow-up is our main result:
Theorem 0.2. Let \( f: (X, B) \to Y \) be a fiber space satisfying the above properties (1),(2),(3). Then there exists a proper birational morphism \( Y' \to Y \) with the following properties:

(i) \( K_{Y'} + B_{Y'} \) is a \( \mathbb{Q} \)-Cartier divisor, and \( \nu^*(K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''} \) for every proper birational morphism \( \nu: Y'' \to Y' \).

(ii) \( M_{Y'} \) is a nef \( \mathbb{Q} \)-Cartier divisor and \( \nu^*(M_{Y'}) = M_{Y''} \) for every proper birational morphism \( \nu: Y'' \to Y' \).

The first part is the positive answer to Shokurov’s BP Conjecture [18, page 92]. Prokhorov and Shokurov [17] proved (i), by a different method, in a special case when \( X \) is a 3-fold and \( Y \) is a surface (they also obtain an explicit description of \( Y' \)). Modulo (i), the second part is a result of Kawamata [13, Theorem 2]. It is expected that \( |kM_{Y'}| \) is base point free for a positive integer \( k \). This is known if the generic fibre \( F \) of \( f \) is a curve and \( B|_F \) is effective [10, 12]. We have three applications of Theorem 0.2:

(A) By the definition of the discriminant, \( (X, B) \) and \( (Y, B_Y) \) have “similar singularities” (inverse of adjunction holds) only outside a closed subset of \( Y \) of codimension at least two. We show that inverse of adjunction extends to the whole variety \( Y \), provided that \( Y \) is high enough (Theorem 3.1).

(B) Shokurov [18] has reduced the existence of flips to the finite generatedness of certain (FGA) algebras which are asymptotically saturated with respect to a Fano variety [18, Conjecture 4.39]. We obtain a descent property for asymptotic saturation of algebras (Proposition 6.3). To descend the numerical assumptions of the FGA/0LP Conjecture, the semi-ampleness of the moduli \( \mathbb{Q} \)-divisor \( M_{Y'} \) is required (higher codimensional adjunction is expected to hold for the same reason). However, the nef property of the moduli part is sufficient for some applications to the Fano case: we show the invariance of (FGA) algebras under restriction to exceptional log canonical centers (Theorem 6.5).

(C) The moduli part of a parabolic fiber space has the expected Kodaira dimension, provided that the geometric generic fiber has a good minimal model (Theorem 7.2).

The proof of Theorem 0.2 is based on some of the methods developed for the proof of Iitaka’s Conjecture \( C_{n,m} \), especially [10] (see [15] for an excellent survey). The essential ingredient is the universal base change for relative canonical divisors, in the codimension one semi-stable case. Some of the applications in (B) are explained conjecturally in [18].

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1. Preliminary

A **variety** is a reduced and irreducible scheme of finite type, defined over an algebraically closed field of characteristic zero. An open subset $U$ of a variety $X$ is called *big* if $X \setminus U \subset X$ has codimension at least two. A *contraction* is a proper morphism $f : X \rightarrow Y$ such that $O_Y = f_*O_X$.

Let $\pi : X \rightarrow S$ be a proper morphism from a normal variety $X$, and let $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. A $K$-Weil divisor is an element of $Z^1(X) \otimes_{\mathbb{Z}} K$. The *round up (down) divisors* $\lceil D \rceil$ ($\lfloor D \rfloor$) are defined componentwise. Two $\mathbb{R}$-Weil divisors $D_1, D_2$ are $K$-linearly equivalent, denoted $D_1 \sim_K D_2$, if there exist $q_i \in K$ and rational functions $\varphi_i \in k(X)^\times$ such that $D_1 - D_2 = \sum_i q_i(\varphi_i)$. An $\mathbb{R}$-Weil divisor $D$ is called

(i) *$K$-Cartier* if $D \sim_K 0$ in a neighborhood of each point of $X$.
(ii) *relatively nef* if $D$ is $\mathbb{R}$-Cartier and $D \cdot C \geq 0$ for every proper curve $C$ contracted by $\pi$.
(iii) *relatively free* if $D$ is a Cartier divisor and the natural map $\pi^*\pi_*O_X(D) \rightarrow O_X(D)$ is surjective.
(iv) *relatively ample* if $\pi$ is a projective morphism and $D$ belongs to the real cone generated by relatively ample Cartier divisors.
(v) *relatively semi-ample* if there exists a contraction $\Phi : X \rightarrow Y/S$ and a relatively ample $\mathbb{R}$-divisor $H$ on $Y$ such that $D \sim_\mathbb{R} \Phi^*H$. If $D$ is rational, this is equivalent to $mD$ being relatively free for sufficiently large and divisible positive integers $m$.
(vi) *relatively big* if there exists $C > 0$ such that $\text{rank } \pi_*O_X(mD) \geq Cm^d$ for $m$ sufficiently large and divisible, where $d$ is the dimension of the generic fibre of $\pi$.

A divisor $D$ has **simple normal crossings** if it is reduced and its components are non-singular divisors intersecting transversely, in the smooth ambient space $X$.

**Definition 1.1.** (V.V. Shokurov) A $K$-b-divisor $D$ of $X$ is a family $\{D_{X'}\}_{X'}$ of $K$-Weil divisors indexed by all birational models $X'$ of $X$, such that $\mu_* (D_{X''}) = D_{X'}$ if $\mu : X'' \rightarrow X'$ is a birational contraction.

Equivalently, $D = \sum_E \text{mult}_E(D)E$ is a $K$-valued function on the set of all (geometric) valuations of the field of rational functions $k(X)$, having finite support on some (hence any) birational model of $X$. 
Example 1. (1) Let $\omega$ be a top rational differential form of $X$. The associated family of divisors $K = \{(\omega)_{X'}\}_{X'}$ is called the canonical $b$-divisor of $X$.

(2) A rational function $\varphi \in k(X)^*_{X}$ defines a $b$-divisor $(\varphi) = \{(\varphi)_{X'}\}_{X'}$.

(3) An $\mathbb{R}$-Cartier divisor $D$ on a birational model $X'$ of $X$ defines an $\mathbb{R}$-b-divisor $D$ such that $(D)_{X''} = \mu^*D$ for every birational contraction $\mu: X'' \to X'$.

(4) For an $\mathbb{R}$-b-divisor $D$, the round up (down) b-divisors $[D]$ ($[D]$) are defined componentwise.

An $\mathbb{R}$-b-divisor $D$ is called $K$-$b$-Cartier if there exists a birational model $X'$ of $X$ such that $D_{X'}$ is $K$-Cartier and $D = D_{X'}$. In this case, we say that $D$ descends to $X'$. The relative Kodaira dimension $\kappa(X/S, D)$ of a $K$-Cartier $b$-divisor $D$ is defined as the relative Kodaira dimension of $D_{X'}$, where $X'/S$ is a model where $D$ descends.

An $\mathbb{R}$-b-divisor $D$ is $b$-nef/$S$ ($b$-free/$S$, $b$-semi-ample/$S$, $b$-big/$S$) if there exists a birational contraction $X' \to X$ such that $D = D_{X'}$, and $D_{X'}$ is nef (free, semi-ample, big) relative to the induced morphism $X' \to S$.

To any $\mathbb{R}$-b-divisor $D$ of $X$, there is an associated $b$-divisorial sheaf $\mathcal{O}_X(D)$. If $U \subset X$ is an open subset, then $\Gamma(U, \mathcal{O}_X(D))$ is the set of rational functions $\varphi \in k(X)$ (including 0) such that $\text{mult}_{E}((\varphi)+D) \geq 0$ for every valuation $E$ with $c_{X}(E) \cap U \neq \emptyset$.

A log pair $(X, B)$ is a normal variety $X$ endowed with a $\mathbb{Q}$-Weil divisor $B$ such that $K + B$ is $\mathbb{Q}$-Cartier. A log variety is a log pair $(X, B)$ such that $B$ is effective. A relative log pair (variety) $(X/S, B)$ consists of a proper morphism $\pi: X \to S$ and a log pair (variety) structure $(X, B)$. The discrepancy $b$-divisor of a log pair $(X, B)$ is

$$A(X, B) = K - K + B.$$ 

For a valuation $E$ of $k(X)$, the log discrepancy of $E$ with respect to $(X, B)$ is $a(E; X, B) := 1 + \text{mult}_{E}(A(X, B))$. The minimal log discrepancy of $(X, B)$ in a proper closed subset $W \subset X$, is

$$a(W; X, B) := \inf_{c_{X}(E) \subseteq W} a(E; X, B).$$ 

The lc places of $(X, B)$ are valuations $E$ such that $a(E; X, B) = 0$, and their centers on $X$ are called lc centers. We say that $(X, B)$ has log canonical (Kawamata log terminal) singularities if $a(E; X, B) \geq 0$ ($a(E; X, B) > 0$) for every valuation $E$. The non-klt locus $\text{LCS}(X, B)$ (non-log canonical locus $(X, B)_{-\infty}$) is the union of all centers $c_{X}(E)$ of valuations $E$ with $a(E; X, B) \leq 0$ ($a(E; X, B) < 0$) (see [3] for some
We also denote \( A^*(X, B) = A(X, B) + \sum_{\alpha(E; X, B) = 0} E. \)

A relative generalized log Fano variety is a relative log variety \((X/S, B)\) such that \(-(K + B)\) is ample \(\text{}/S\).

2. THE DISCRIMINANT AND MODULI B-DIVISORS

Definition 2.1. A \(K\)-trivial fibration \(f: (X, B) \to Y\) consists of contraction of normal varieties \(f: X \to Y\) and a log pair \((X, B)\), satisfying the following properties:

1. \((X, B)\) has Kawamata log terminal singularities over the generic point of \(Y\).
2. \(\text{rank } f_*\mathcal{O}_X(\lceil A(X, B) \rceil) = 1\).
3. There exist a positive integer \(r\), a rational function \(\varphi \in k(X)^*\) and a \(\mathbb{Q}\)-Cartier divisor \(D\) on \(Y\) such that

\[ K + B + \frac{1}{r}(\varphi) = f^*D. \]

Remark 2.2. The property \(\text{rank } f_*\mathcal{O}_X(\lceil A(X, B) \rceil) = 1\) holds in the following examples:

(a) \(f\) is birational to the Iitaka fibration of a functional algebra \(\mathcal{L}\) which is \(A(X, B)\)-saturated (Lemma 6.2).

(b) \((F, B|_F)\) has Kawamata log terminal singularities and \(B|_F\) is effective, where \(F\) is the generic fiber of \(f\).

(c) Let \(W\) be the normalization of an exceptional log canonical centre of a log variety \((X, B)\), and let \(h: E \to W\) be the unique lc place over \(W\). By adjunction, there exists a \(\mathbb{Q}\)-divisor \(B_E\) such that \(h: (E, B_E) \to W\) is a \(K\)-trivial fibration (see [3]).

Define \(B_Y = \sum_{P \subseteq Y} b_P P\), where the sum runs after all prime divisors of \(Y\), and

\[ 1 - b_P = \sup\{t \in \mathbb{R}; \exists U \ni \eta_P, (X, B + tf^*(P)) \text{ lc sing}/U\}. \]

The coefficients \(b_P\) are well defined, since \((X, B)\) has at most log canonical singularities over the general point of \(Y\), and each prime divisor is Cartier in a neighborhood of its general point. It is easy to see that the sum has finite support, so \(B_Y\) is a well defined \(\mathbb{Q}\)-Weil divisor on \(Y\). By (3), there exists a unique \(\mathbb{Q}\)-Weil divisor \(M_Y\) such that the following adjunction formula holds:

\[ K + B + \frac{1}{r}(\varphi) = f^*(K_Y + B_Y + M_Y). \]
Definition 2.3. [1] The $\mathbb{Q}$-Weil divisors $B_Y$ and $M_Y$ are the discriminant and moduli part of the $K$-trivial fibration $f : (X, B) \to Y$. Note that $K_Y + B_Y + M_Y$ is $\mathbb{Q}$-Cartier.

The adjunction formula gives a one-to-one correspondence between the choices of $M_Y$ and rational functions with $\mathbb{Q}$-coefficients $r\phi$ such that $K_F + B_F + \frac{1}{r}(\varphi|_F) = 0$, where $F$ is the general fibre of $f$. If $M_Y$ and $M'_Y$ correspond to $\frac{1}{r}\varphi$ and $\frac{1}{s}\varphi'$, respectively, then there exists a rational function $\theta \in k(Y)^\times$ such that $\varphi' = \varphi f^*\theta$ and $rM'_Y = (\theta) + rM_Y$. The smallest possible value of $r$ is $b = b(F, B_F)$, uniquely defined by

$$\{m \in \mathbb{N}; m(K_F + B_F) \sim 0\} = b\mathbb{N}.$$

Unless otherwise stated, we assume that $\frac{1}{r}\varphi$ is fixed, and $r = b(F, B_F)$.

According to the following lemma, $B_Y$ and $M_Y$ are independent of the choice of a crepant model of $(X, B)$ over $Y$:

Lemma 2.4. Let $\sigma : X - \to X'$ be a birational map defined over $Y$, and let $f' : X' \to Y$ be the induced morphism. Then there exists a unique $\mathbb{Q}$-Weil divisor $B_{X'}$ such that $\sigma : (X, B) - \to (X', B_{X'})$ is a crepant birational map. Moreover, $(X, B)$ and $(X', B_{X'})$ induce the same discriminant on $Y$.

Proof. There exists a common normal birational model of $X$ and $X'$ which makes following diagram commute:

Let $K_{X''} + B_{X''} = \mu^*(K + B)$ be the log pullback. Since $K_{X''} + B_{X''} + \frac{1}{r}(\varphi) = \mu^{**}(f^{**}D)$ and $\mu'$ is birational, we have $K_{X''} + B_{X''} = \mu'^*(K_{X'} + B_{X'})$, where $B_{X'} := \mu'_*(B_{X''})$. Therefore there exists a crepant log structure on $X'$. The uniqueness of $B_{X'}$ is clear.

Finally, note that $\mu^*(K + B + tf^*(P)) = K_{X''} + B_{X''} + t(f \circ \mu)^*(P) = \mu'^*(K_{X'} + B_{X'} + tf'^*(P))$. Therefore the thresholds $1 - b_P$ induced by $K + B$ and $K_{X'} + B_{X'}$ coincide. \qed

Let $\sigma : Y' \to Y$ be a birational contraction from a normal variety $Y'$. Let $X'$ be a resolution of the main component of $X \times_Y Y'$ which dominates $Y'$. The induced morphism $\mu : X' \to X$ is birational, and
let \((X', B_{X'})\) be the crepant log structure on \(X'\), i.e. \(\mu^*(K + B) = K_{X'} + B_{X'}\):

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{\mu} & (X', B_{X'}) \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\sigma} & Y'
\end{array}
\]

We say that the \(K\)-trivial fibration \(f': (X', B_{X'}) \to Y'\) is induced by base change. Let \(B_{Y'}\) be the discriminant of \(K_{X'} + B_{X'}\) on \(Y'\). Since the definition of the discriminant is divisorial and \(\sigma\) is an isomorphism over codimension one points of \(Y\), we have \(B_{Y'} = \sigma^*(B_{Y'})\). This means that there exists a unique \(\mathbb{Q}\)-b-divisor \(B\) of \(Y\) such that \(B_{Y'}\) is the discriminant on \(Y'\) of the induced fibre space \(f': (X', B_{X'}) \to Y'\), for every birational model \(Y'\) of \(Y\). We call \(B\) the discriminant \(\mathbb{Q}\)-b-divisor induced by \((X, B)\) on the birational class of \(Y\). Similarly, if we fix the \(\mathbb{Q}\)-rational function \(\frac{1}{r^*}\varphi\), there exists a unique \(\mathbb{Q}\)-b-divisor \(M\) of \(Y\) such that

\[
K_{X'} + B_{X'} + \frac{1}{r^*}\varphi = f^*(K_{Y'} + B_{Y'} + M_{Y'})
\]

for every \(K\)-trivial fibration \(f': (X', B_{X'}) \to Y'\) induced by base change on a birational model \(Y'\) of \(Y\). We call \(M\) the moduli \(\mathbb{Q}\)-b-divisor of \(Y\), induced by \(f: (X, B) \to Y\). We restate Theorem 0.2 in terms of b-divisors:

**Theorem 2.5.** Let \(f: (X, B) \to Y\) be a \(K\)-trivial fibration, and let \(\pi: Y \to S\) be a proper morphism. Let \(B\) and \(M\) be the induced discriminant and moduli \(\mathbb{Q}\)-b-divisors of \(Y\). Then

1. \(K + B\) is \(\mathbb{Q}\)-b-Cartier.
2. \(M\) is b-nef/S.

We expect Theorem 2.5 to hold if we allow \(\mathbb{R}\)-boundaries and \(\mathbb{R}\)-linear equivalence instead of \(\mathbb{Q}\)-boundaries and \(\mathbb{Q}\)-linear equivalence in Definition 2.1, or if \((X, B)\) has log canonical singularities over the generic point of \(Y\). In the latter case, the assumption rank \(f_*\mathcal{O}_X([A(X, B)]) = 1\) should be replaced by rank \(f_*\mathcal{O}_X([A^*(X, B)]) = 1\).

### 3. Inverse of adjunction

Let \(f: (X, B) \to Y\) be a \(K\)-trivial fibration. The \(\mathbb{Q}\)-b-divisor of \(Y\)

\[
A_{\text{div}} := -B
\]

is called the *divisorial discrepancy b-divisor* [18, page 92]. Theorem 2.5(1) is equivalent to the following property: there exists a birational model \(Y''\) of \(Y\) such that \(A_{\text{div}} = A(Y'', B_{Y''})\) for every birational model \(Y''\).
which dominates $Y'$. As a corollary, inverse of adjunction holds for $f: (X, B) \rightarrow (Y, B_Y)$, after a sufficiently high birational base change:

**Theorem 3.1.** (Inverse of adjunction) Let $f: (X, B) \rightarrow Y$ be a $K$-trivial fibration such that $\mathbf{A}_{\text{div}} = \mathbf{A}(Y, B_Y)$. Then there exists a positive integer $N$ such that

$$\frac{1}{N} a(f^{-1}(Z); X, B) \leq a(Z; Y, B_Y) \leq a(f^{-1}(Z); X, B).$$

for every closed subset $Z \subset Y$, where $a(Z; Y, B_Y)$ and $a(f^{-1}(Z); X, B)$ are the minimal log discrepancies of $(Y, B_Y)$ in $Z$, and $(X, B)$ in $f^{-1}(Z)$ respectively.

In particular, $(Y, B_Y)$ has Kawamata log terminal (log canonical) singularities in a neighborhood of a point $y \in Y$ if and only if $(X, B)$ has Kawamata log terminal (log canonical) singularities in a neighborhood of $f^{-1}(y)$.

**Proof.** The assumption $\mathbf{A}_{\text{div}} = \mathbf{A}(Y, B_Y)$ means that the Base Change Conjecture [1, Section 3] holds for $f: (X, B) \rightarrow Y$. The claim is proved in [1, Proposition 3.4], but with $N$ depending on $Z$. The possible values for minimal log discrepancies of a fixed log pair are finite [2, Theorem 2.3], hence a maximal value $N = \max_{Z \subset Y} N(Z)$ exists. □

**Lemma 3.2.** Let $f: (X, B) \rightarrow Y$ be a $K$-trivial fibration such that $\mathbf{A}_{\text{div}} = \mathbf{A}(Y, B_Y)$. Assume moreover that $X, Y$ are non-singular varieties, and the divisors $B, B_Y$ have simple normal crossings support.

Then $f_* \mathcal{O}_X([−B]) = \mathcal{O}_Y([−B_Y])$.

**Proof.** By (i), $f_* \mathcal{O}_X([A(X, B)]) = \mathcal{O}_X([−B])$. Since $B$ has Kawamata log terminal singularities over the generic point of $Y$, we have a natural inclusion

$$\mathcal{O}_Y|_V \subseteq f_* \mathcal{O}_X([-B])|_V$$

for some open subset $V \subset Y$. Since rank $f_* \mathcal{O}_X([A(X, B)]) = 1$, the above inclusion is an equality, after possibly shrinking $V$. Thus we identify $f_* \mathcal{O}_X([-B])$ with a subsheaf of the constant sheaf $k(Y)$. We first show that $f_* \mathcal{O}_X([-B]) \subseteq \mathcal{O}_Y([-B_Y])$. Let $\varphi$ be a rational function of $Y$ such that $(f^* \varphi) + [-B] \geq 0$, and let $P$ be a prime divisor of $Y$. We may replace $X$ by some resolution, so that there exists a prime divisor $Q$ of $X$ such that $f(Q) = P$ and

$$1 - \text{mult}_P(B_Y) = \frac{1 - \text{mult}_Q(B)}{m_{Q/P}},$$

where $m_{Q/P}$ is the multiplicity of $f^*(P)$ at $Q$. By assumption, we have

$$\text{mult}_Q(f^* \varphi) + 1 - \text{mult}_Q(B) > 0.$$
hence \( \text{mult}_P(\varphi) + 1 - \text{mult}_P(B_Y) > 0 \). Therefore \( (\varphi) + [-B_Y] \) is effective at \( P \).

Conversely, assume \( (\varphi) + [-B_Y] \) is effective, and fix a prime divisor \( Q \) of \( X \). There exists a birational base change 

\[
\begin{array}{c}
(X,B) \\ f \\
\downarrow \\
Y
\end{array} \quad \begin{array}{c}
(X',B_{X'}) \\ f' \\
\downarrow \\
Y'
\end{array}
\]

such that \( P := f(Q) \) is a prime divisor of \( Y' \). We have \( \sigma^*(K_Y + B_Y) = K_{Y'} + B_{Y'} \) by \( A_{\text{div}} = A(Y,B_Y) \). Furthermore, the simple normal crossings assumption implies \( \sigma_*\mathcal{O}_{Y'}([-B_{Y'}]) = \mathcal{O}_Y([-B_Y]) \). Therefore \( (\varphi) + [-B_{Y'}] \geq 0 \), hence \( \text{mult}_P(\varphi) + 1 - \text{mult}_P(B_{Y'}) > 0 \).

Since 

\[
1 - \text{mult}_P(B_Y) \leq \frac{1 - \text{mult}_Q(B_{X'})}{m_{Q/P}},
\]

we infer \( \text{mult}_Q(f^*\varphi) + 1 - \text{mult}_Q(B_{X'}) > 0 \), i.e. \( (f^*\varphi) + [-B] \) is effective at \( Q \).

\[ \square \]

**Remark 3.3.** Let \( f: (X,B) \to Y \) be a \( K \)-trivial fibration. If \( L \) is a \( \mathbb{Q} \)-Cartier divisor on \( Y \), let \( B' := B + f^*L \). Then \( f: (X,B') \to Y \) is a \( K \)-trivial fibration, with moduli \( \mathbb{Q} \)-b-divisor \( M' = M \), and discriminant \( \mathbb{Q} \)-b-divisor \( B' = B + \overline{L} \).

4. **Covering tricks and base change**

**Theorem 4.1.** \([8]\) Let \( X \) be a non-singular quasi-projective variety endowed with a divisor \( D \) with simple normal crossings singularities, and let \( N \) be a positive integer. Then there exists a finite Galois covering \( \tau: \tilde{X} \to X \) satisfying the following conditions:

1. \( \tilde{X} \) is a non-singular quasi-projective variety, and there exists a simple normal crossings divisor \( \Sigma_X \) such that \( \tau \) is étale over \( X \setminus \Sigma_X \), and \( \tau^{-1}(\Sigma_X) \) is a divisor with simple normal crossings.

2. The ramification indices of \( \tau \) over the prime components of \( D \) are divisible by \( N \).

**Sketch of proof.** We may assume that \( X \) is projective (by Hironaka’s resolution of singularities, we can compactify to complement of snc in projective, construct the cover, and then restrict back to the original variety). Let \( A \) be a very ample divisor such that \( NA - D_i \) is very ample for each component \( D_i \) of \( D \). Let \( n = \dim(X) \). There exists \( H^{(i)}_1, \ldots, H^{(i)}_n \in |NA - D_i| \) for every \( D_i \), such that \( \Sigma_X := D + \sum_{i,j} H^{(i)}_j \).
is a divisor with simple normal crossings. Let $X = \cup U_\alpha$ be an affine cover, and let $D_i + H_j^{(i)} = (\varphi_{j\alpha}^{(i)})$ on $U_\alpha$. The field extension $L := k(X)[(\varphi_{j\alpha}^{(i)})^{1/\alpha}; i, j]$ is independent of the choice of $\alpha$. Let $\tilde{X}$ be the normalization on $X$ in $L$. Then $\tilde{X}$ is non-singular and $\tau$ is a Kummer cover, étale outside $\Sigma_X$, and $\tau^{-1}(\Sigma_X)$ has simple normal crossings. □

**Remark 4.2.** In the above notations, assume that $\varrho: Y \to X$ is a surjective morphism from a non-singular quasi-projective variety $Y$ such that $\varrho^{-1}(D)$ has simple normal crossings. Then we may assume that $\tau: \tilde{X} \to X$ fits into a commutative diagram

\[
\begin{array}{c}
\tilde{X} \\
\downarrow\tau \\
X
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\nu \\
Y
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\varrho \\
Y
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\tau_1 \\
Y_1
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow\varrho \\
X
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\tau_1 \\
Y_1
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\varrho \\
Y
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow\tau \\
X
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\nu \\
Y
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\varrho \\
Y
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\tau_1 \\
Y_1
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow\tau \\
X
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\nu \\
Y
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\varrho \\
Y
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow\tau \\
X
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\nu \\
Y
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow\varrho \\
Y
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow\tau \\
X
\end{array}

satisfying the following properties:

1. $\nu$ is a finite covering and $g$ is a projective morphism.
2. $\tilde{Y}$ is non-singular quasi-projective.
3. There exists a simple normal crossings divisor $\Sigma_Y$ such that $\nu$ is étale over $Y \setminus \Sigma_Y$, $\nu^{-1}(\Sigma_Y)$ has simple normal crossings, and $\varrho^{-1}(\Sigma_X) \subseteq \Sigma_Y$.

**Proof.** In the proof Theorem 4.1, we may choose the divisors $H_j^{(i)}$ so that $\varrho^{-1}(D + \sum_{i,j} H_j^{(i)})$ is a divisor with simple normal crossings on $Y$. Let $\tau_1: \tilde{Y} \to Y$ be the normalization of the main component of the pull back of $\tau$ to $Y$.

Then $\tau_1$ is a finite cover whose ramification locus is contained in the simple normal crossings divisor $\varrho^{-1}(\Sigma_X)$. Let $N'$ be the least common multiple of its ramification indices, and construct by Theorem 4.1 a finite cover $\pi: Y_1 \to Y$ with respect to $\varrho^{-1}(\Sigma_X)$ and $N'$. Let $\tilde{Y}/Y_1$ be the normalization of the main component of the pull back of $\tau_1$ to $Y_1$. The induced map $\nu: \tilde{Y} \to Y$ is a finite cover. By construction, $\tilde{Y}/Y_1$ is étale, hence $\tilde{Y}$ is non-singular. There exists a simple normal crossings divisor $\Sigma_Y$ containing $\varrho^{-1}(\Sigma_X)$ such that $\pi$ is étale over $Y \setminus \Sigma_Y$ and $\pi^{-1}(\Sigma_Y)$ has simple normal crossings. Therefore $\nu^{-1}(\Sigma_Y)$ has simple normal crossings. □
**Theorem 4.3.** \[14, 19\] (Semi-stable reduction in codimension one) Let $f : X \to Y$ be a surjective morphism of non-singular varieties. Assume $\Sigma_X, \Sigma_Y$ are simple normal crossings divisors on $X$ and $Y$ respectively, such that $f^{-1}(\Sigma_Y) \subseteq \Sigma_X$ and $f$ is smooth over $Y \setminus \Sigma_Y$. Then there exists a positive integer $N$ such that the following hold:

Let $\pi : Y' \to Y$ be a finite covering from a nonsingular variety $Y'$ such that $\Sigma_{Y'} := \pi^{-1}(\Sigma_Y)$ has simple normal crossings and $N$ divides the ramification indices of $\pi$ over the prime components of $\Sigma_{Y'}$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\pi} & Y'
\end{array}
\]

with the following properties:

(a) $X'$ is non-singular and $\Sigma_{X'} := \pi'^{-1}(\Sigma_X)$ has simple normal crossings, where $\pi' : X' \to X$ is the induced projective morphism.

(b) $p$ is projective, and is an isomorphism above $Y' \setminus \Sigma_{Y'}$. In particular, $f'$ is smooth over $Y' \setminus \Sigma_{Y'}$.

(c) $f'$ is semi-stable in codimension one: the fibers over (generic) codimension one points of $Y'$ have simple normal crossings singularities.

**Sketch of proof.** Let $f^*(\Sigma_X) = \sum n_i E_i$, and let $N$ be the least common multiple of the $n_i$’s corresponding to components $E_i$ which dominate some component of $\Sigma_Y$. Consider a finite base change $Y' \to Y$ as above. Then [14, IV] shows that over the generic point of each prime component $Q$ of $\Sigma_{Y'}$, $X \times_Y Y'$ admits a resolution with the desired properties. Therefore there exists a closed subscheme $B \subset X \times_Y Y'$, supported over $\Sigma_{Y'}$, and a closed subset $Z \subset \Sigma_{Y'}$ with $\text{codim}(Z, Y') \geq 2$, such that the blow-up of $X \times_Y Y'$ in $B$ has the desired properties over $Y' \setminus Z$. Then we may take $X'$ to be any resolution of the blow-up, which is an isomorphism outside its singular locus, and such that (a) holds.

**Theorem 4.4.** \[6, 8\] Let $f : X \to Y$ be a projective morphism of non-singular algebraic varieties. Assume $f$ is semi-stable in codimension one, and there exists a simple normal crossings divisor $\Sigma_Y$ such that $f$ is smooth over $Y \setminus \Sigma_Y$. Then the following properties hold:

(1) $f_* \omega_{X/Y}$ is a locally free sheaf on $Y$. 

\[\square\]
(2) $f_*\omega_{X/Y}$ is semi-positive: let $\nu: C \to Y$ be a proper morphism from a non-singular projective curve $C$, and let $L$ be an invertible quotient of $\nu^*(f_*\omega_{X/Y})$. Then $\deg(L) \geq 0$.

(3) Let $\varrho: Y' \to Y$ be a projective morphism from a non-singular variety $Y'$ such that $\varrho^{-1}(\Sigma_Y)$ is a simple normal crossings divisor. Let $X' \to (X \times_Y Y')_{\text{main}}$ be a resolution of the component of $X \times_Y Y'$ which dominates $Y'$, and let $h: X' \to Y'$ be the induced fibre space:

$$
\begin{array}{ccc}
X & \xleftarrow{f} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \leftarrow & Y'
\end{array}
$$

Then there exists a natural isomorphism $\varrho^*(f_*\omega_{X/Y}) \sim f'_*\omega_{X'/Y'}$, which extends the base change isomorphism over $Y \setminus \Sigma_Y$.

Sketch of proof. By the Lefschetz principle and flat base change, we may assume $k = \mathbb{C}$. Let $Y_0 = Y \setminus \Sigma_Y$, $X_0 = f^{-1}(Y_0)$, and let $d = \dim(X/Y)$. The locally free sheaf $H_0 := R^d f_* \mathcal{Q}_{X_0} \otimes \mathcal{Q}_{Y_0} \mathcal{O}_{Y_0}$ is endowed with the integrable Gauss-Manin connection and is the underlying space of a variation of Hodge structure of weight $d$ on $Y_0$, with $F^d H_0 = f_*\omega_{X_0/Y_0}$. Since $f$ is semi-stable in codimension one, $H_0$ has unipotent local monodromies around the components of $\Sigma_Y$. Let $H$ be the canonical extension of $H_0$. By Schmid’s asymptotic behaviour of variations of Hodge structure, the natural inclusion

$$
f_*\omega_{X/Y} \to j_*(F^d H_0) \cap H
$$

is an isomorphism and $f_*\omega_{X/Y}$ is locally free [8]. The semi-positivity follows from unipotence and Griffiths’ semi-positivity of the curvature of the last piece of a variation of Hodge structure [8, 8].

For base change, the sheaf $f'_*\omega_{X'/Y'}$ is independent of birational changes in $X'$ over $Y'$. Thus we may assume that $X' \to X \times_Y Y'$ is an isomorphism above $Y' \setminus \Sigma_{Y'}$, where $\Sigma_{Y'} = \varrho^{-1}(\Sigma_Y)$. Let $H'_0$ be the variation of Hodge structure on $Y' \setminus \Sigma_{Y'}$, induced by $f'$, and let $H'$ be its canonical extension to $Y'$. Since $H'_0$ has unipotent local monodromies around the components of $\Sigma_{Y'}$, the canonical extension is compatible with base change [10, Proposition 1]:

$$
H' \sim \varrho^* H.
$$

This isomorphism preserves the extensions of the Hodge filtration, hence it induces an isomorphism $\varrho^*(f_*\omega_{X/Y}) \sim f'_*\omega_{X'/Y'}$. □
Theorem 4.5. \cite{7, 9} Let \( f : X \to Y \) be a contraction from a non-singular projective variety \( X \) to a projective curve \( Y \), and let \( E \) be a quotient locally free sheaf of \( f_*\omega_{X/Y} \). If \( \deg(\det(E)) = 0 \), then \( \det(E)^{\otimes m} \simeq \mathcal{O}_Y \) for some positive integer \( m \).

Sketch of proof. By \cite{7}, \( E \) is a local system which is a direct summand of \( f_*\omega_{X/Y} \). Since \( E|_{Y_0} \) is a local subsystem of the variation of Hodge structure \( H^0 \), \( \det(E|_{Y_0})^{\otimes m} \simeq \mathcal{O}_{Y_0} \) for some positive integer \( m \) \cite{4}. By flatness, \( \det(E)^{\otimes m} \simeq \mathcal{O}_Y \). \( \square \)

5. An auxiliary relative 0-log pair

We prove Theorems 2.5, 0.1 in this section. The following finite base change formula is essential:

Lemma 5.1. \cite{1}, Theorem 3.2] Consider a commutative diagram of normal varieties

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{\nu} & (X', B_{X'}) \\
f \downarrow & & f' \downarrow \\
Y & \xrightarrow{\tau} & Y'
\end{array}
\]

with the following properties:

1. \((X, B)\) is a log pair with log canonical singularities over the generic point of \( Y \).
2. \( \tau \) is a finite morphism and \( \nu \) is generically finite, \( f, f' \) are proper surjective.
3. \( \nu^*(K + B) = K_{X'} + B_{X'} \).

Let \( B_Y \) and \( B_{Y'} \) be the discriminants of \( K + B \) and \( K_{X'} + B_{X'} \) on \( Y \) and \( Y' \) respectively. Then \( \tau^*(K_Y + B_Y) = K_{Y'} + B_{Y'} \) (pull back of \( \mathbb{Q} \)-Weil divisors under a finite morphism).

The category of \( K \)-trivial fibrations is closed under generically finite base changes. In order to normalize the discriminant \( B_Y \) and the moduli part \( M_Y \), we have to replace the generic fibre of \( X/Y \) by a generically finite cover. The property rank \( f_*\mathcal{O}_X([A(X, B)]) = 1 \) is not invariant under this operation, thus we will consider an auxiliary fibre space (cf. \cite{10, 15}). Throughout this section we consider the following set-up:

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{\pi} & \tilde{X} & \xleftarrow{d} & (V, B_V) \\
f \downarrow & & f \downarrow & & h \downarrow \\
Y & & & & \tilde{Y}
\end{array}
\]
Let \( f : (X, B) \to Y \) be a \( K \)-trivial fibration, \( b = b(F, B_F) \) and
\[
K + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)
\]
where \( \pi : \tilde{X} \to X \) is the normalization of \( X \) in \( k(X)(\varphi^{1/b}) \) and \( d : V - \to \tilde{X} \) is a birational map from a non-singular variety \( V \). The induced rational map \( g : V - \to X \) is generically finite, so there exists a unique log structure \((V, B_V)\) such that \( g : (V, B_V) - \to (X, B) \) is crepant. We assume the following properties hold:

(i) \( X, V, Y \) are non-singular quasi-projective varieties endowed with simple normal crossings divisors \( \Sigma_X, \Sigma_V, \Sigma_Y \) on \( X, V \) and \( Y \) respectively.

(ii) \( f \) and \( h \) are projective morphisms.

(iii) \( f \) and \( h \) are smooth over \( Y \setminus \Sigma_Y \), and \( \Sigma_X^h / Y \) and \( \Sigma_V^h / Y \) have relative simple normal crossings over \( Y \setminus \Sigma_Y \).

(iv) \( f^{-1}(\Sigma_Y) \subseteq \Sigma_X, f(\Sigma_X^h) \subseteq \Sigma_Y \) and \( h^{-1}(\Sigma_Y) \subseteq \Sigma_V, h(\Sigma_V^h) \subseteq \Sigma_Y \).

(v) \( B, B_V \) and \( B_Y, M_Y \) are supported by \( \Sigma_X, \Sigma_V \) and \( \Sigma_Y \), respectively.

In this context, the properties (1) and (2) in the definition of the \( K \)-trivial fibration \( f : (X, B) \to Y \) are equivalent to the following properties:

- \([−B_F] \) is an effective divisor and \( \dim_k H^0(F, [−B_F]) = 1 \).

**Lemma 5.2.** The following properties hold for the above set-up:

1. The extension \( k(V)/k(X) \) is Galois and its Galois group \( G \) is cyclic of order \( b \). There exists \( \psi \in k(V)^* \) such that \( \psi^b = \varphi \) and a generator of \( G \) acts by \( \psi \mapsto \zeta \psi \), where \( \zeta \in k \) is a fixed primitive \( b \)-th root of unity.

2. The relative log pair \( h : (V, B_V) \to Y \) satisfies all properties of a \( K \)-trivial fibration, except that \( \text{rank } f_*O_X([A(V, B_V)]) \) might be bigger than one.

3. Both \( f : (X, B) \to Y \) and \( h : (V, B_V) \to Y \) induce the same discriminant and moduli part on \( Y \).

4. The group \( G \) acts naturally on \( h_*O_Y(K_V/Y) \). The eigen-sheaf corresponding to the eigenvalue \( \zeta \) is \( \mathcal{L} := f_*O_X([-B + f^*B_Y + f^*M_Y]) \cdot \psi \).

5. Assume that \( h : V \to Y \) is semi-stable in codimension one. Then \( M_Y \) is an integral divisor, \( \mathcal{L} \) is semi-positive and \( \mathcal{L} = O_Y(M_Y) \cdot \psi \).

**Proof.** (2) We have \( K_V + B_V + (\psi) = h^*(K_Y + B_Y + M_Y) \), and clearly \( (V, B_V) \) has Kawamata log terminal singularities over the generic point of \( Y \). The generic fibre \( H \) of \( h \) is a non-singular birational model of
the normalization of $k(F)$ in $k(F)((\varphi|_F)^{1/2})$. Since $b$ is minimal with $b(K_F + B_F) \sim 0$, $H$ is connected. Therefore $\mathcal{O}_Y = h_*\mathcal{O}_V$, i.e. $h$ is a contraction.

(3) It follows from (2) and Lemma 5.1. Note that the assumption rank $f_*\mathcal{O}_X([A(V, B_V)]) = 1$ is not required in the definition of the discriminant and moduli part.

(4) The group $G$ acts on $\tilde{f}_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/Y})$. Its decomposition into eigen-sheaves is

$$\tilde{f}_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/Y}) = \bigoplus_{i=0}^{b-1} f_*\mathcal{O}_X([((1-i)K_{\tilde{X}/Y} - iB + if^*B_Y + if^*M_Y)] \cdot \psi^i$$

Since $B - f^*(B_Y + M_Y)$ is supported by the simple normal crossings divisor $\Sigma_X$, $\tilde{X}$ has rational singularities. In particular, $h_*\mathcal{O}_V(K_{V/Y}) = \tilde{f}_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/Y})$ is independent of the choice of $V$.

(5) By the semi-stable assumption, there exists a big open subset $Y^+ \subseteq Y$ such that $(-B_Y + h^*B_Y)|_{h^{-1}(Y^+)}$ is effective and supports no fibres of $h$. Since $(\psi|_F) + K_H = -B_H \geq 0$, $\psi$ is a rational section of $h_*\mathcal{O}_V(K_{V/Y})$. Furthermore, $\psi \mapsto \zeta \psi$ implies that $\psi$ is a rational section of $\mathcal{L}$. Therefore $\mathcal{L} \subseteq k(Y)\psi$, since $\mathcal{L}$ has rank one by (vi) and (4). We have $(h^*a \cdot \psi) + K_{V/Y} = h^*((a) + M_Y) + (-B_Y + h^*B_Y)$.

Since $-B_Y + h^*B_Y$ is effective over $Y^+$, we infer that $\mathcal{O}_Y((M_Y)\psi)|_{Y^+} \subseteq h_*\mathcal{O}_V(K_{V/Y})|_{Y^+}$. Therefore $\mathcal{O}_Y(M_Y)|_{Y^+} \subseteq \mathcal{L}|_{Y^+}$. Conversely, let $h^*a \cdot \psi$ be a section of $\mathcal{L}$. Then $h^*a \cdot \psi$ is a section of $h_*\mathcal{O}_V(K_{V/Y})$, i.e. $(h^*a \cdot \psi) + K_{V/Y} \geq 0$. Since $-B_Y + h^*B_Y$ contains no fibres over codimension one points of $Y$, this implies $(a) + M_Y \geq 0$. In particular, $\mathcal{L} \subseteq \mathcal{O}_Y(M_Y)\psi$. Therefore $\mathcal{O}_Y(M_Y)|_{Y^+} = \mathcal{L}|_{Y^+}$. Since $Y^+ \subseteq Y$ is a big open subset, this implies $\mathcal{L}^{**} = \mathcal{O}_Y(M_Y)\psi$. By Theorem 1.4, $h_*\mathcal{O}_V(K_{V/Y})$ is locally free and semi-positive. Its direct summand $\mathcal{L}$ is locally free and semi-positive as well, hence the conclusion.

Finally, over each prime divisor $P$ of $Y$ there exists a prime divisor $Q$ of $X$ such that $h(Q) = P$ and $\text{mult}_Q(-B_V + h^*B_Y) = 0$. We infer from (2) that $\text{mult}_Q h^*(M_Y) \in \mathbb{Z}$. But $\text{mult}_Q h^*(P) = 1$, therefore $M_Y$ is an integral Weil divisor.

**Remark 5.3.** Let $\gamma : Y' \to Y$ be a generically finite morphism from a non-singular quasi-projective variety $Y'$. Assume there exists a simple normal crossings divisor $\Sigma_{Y'}$ which contains $\gamma^{-1}(\Sigma_Y)$ and the locus where $\gamma$ is not étale. By base change, there exists a commutative
such that \((V', B_{V'}) \rightarrow (X', B_{X'}) \rightarrow Y'\) satisfies the same properties (i)-(v). Here \(B_{X'}\), \(B_{V'}\) are induced by crepant pull back, \(\Sigma_{X'} \supseteq \sigma^{-1}(\Sigma_{X})\), \(\Sigma_{V'} \supseteq \nu^{-1}(\Sigma_{V})\) and \(\varphi' = \sigma' \varphi \in k(X')^*\). We say that the set-up \((V', B_{V'}) \rightarrow (X', B_{X'}) \rightarrow Y'\) is induced by \((V, B_{V}) \rightarrow (X, B) \rightarrow Y\) via the base change \(\gamma: Y' \rightarrow Y\).

**Proposition 5.4.** There exists a finite Galois cover \(\tau: Y' \rightarrow Y\) from a non-singular variety \(Y'\) which admits a simple normal crossings divisor supporting \(\tau^{-1}(\Sigma_{Y})\) and the locus where \(\tau\) is not étale, and such that \(h': V' \rightarrow Y'\) is semi-stable in codimension one for some set-up \((V', B_{V'}) \rightarrow (X', B_{X'}) \rightarrow Y'\) induced by base change.

**Proof.** Let \(N\) be the positive integer associated to \(V \rightarrow Y\) by Theorem 4.3. By Theorem 4.1, there exists a finite Galois cover \(\tau: Y' \rightarrow Y\) such that \(\tau^*(\Sigma_{Y})\) is divisible by \(N\) and there exists a simple normal crossings divisor \(\Sigma_{Y'}\) containing \(\tau^{-1}(\Sigma_{Y})\) and the locus where \(\tau\) is not étale. By Theorem 4.3, there exists an induced set-up \((V', B_{V'}) \rightarrow (X', B_{X'}) \rightarrow Y'\) induced by base change, so that \(h': V' \rightarrow Y'\) is semi-stable in codimension one.

**Proposition 5.5.** Let \(\gamma: Y' \rightarrow Y\) be a generically finite projective morphism from a non-singular variety \(Y'\). Assume there exists a simple normal crossings divisor \(\Sigma_{Y'}\) on \(Y'\) which contains \(\gamma^{-1}(\Sigma_{Y})\), and the locus where \(\gamma\) is not étale. Let \(M_{Y'}\) be the moduli part of the induced set-up \((V', B_{V'}) \rightarrow (X', B_{X'}) \rightarrow Y'\). Then \(\gamma^*(M_{Y}) = M_{Y'}\).

**Proof.** Step 1: Assume that \(V/Y\) and \(V'/Y'\) are semi-stable in codimension one. In particular, \(M_{Y}\) and \(M_{Y'}\) are integral divisors. Since \(h\) is semi-stable in codimension one, Theorem 4.3 implies

\[
h^*_h \mathcal{O}_{V'}(K_{V'/Y'}) \cong h^*_h \mathcal{O}_V(K_{V/Y}).\]

This isomorphism is natural, hence compatible with the action of the Galois group \(G\). We have an induced isomorphism of eigensheaves corresponding to \(\zeta: \gamma^* \mathcal{O}_Y(M_Y) \cong \mathcal{O}_Y(M_{Y'})\). Therefore \(\gamma^* M_Y - M_{Y'}\) is linearly trivial, and is exceptional over \(Y\). Thus \(\gamma^* M_Y = M_{Y'}\).
Step 2: By Theorem 4.3 and Theorem 4.1, we can construct a commutative diagram

\[
\begin{array}{ccc}
\bar{Y} & \xleftarrow{\gamma'} & \bar{Y}' \\
\tau' \downarrow & & \downarrow \\
Y & \xleftarrow{\gamma} & Y'
\end{array}
\]

as in Remark 4.2, so that $\bar{V}/\bar{Y}$ is semi-stable in codimension one for an induced set-up $(\bar{V}, B_{\bar{Y}}) \to (\bar{X}, B_{\bar{X}}) \to \bar{Y}$.

By Theorem 4.3 and Theorem 4.1, we replace $\bar{Y}'$ by a finite covering so that $\bar{V}'/\bar{Y}'$ is semi-stable in codimension one for an induced set-up $(\bar{V}', B_{\bar{Y}'}) \to (\bar{X}', B_{\bar{X}'}) \to \bar{Y}'$. By Step 1, we have $M_{\bar{Y}'} = \gamma'^* (M_{\bar{Y}})$.

Since $\tau$ and $\tau'$ are finite coverings, Lemma 5.1 implies $\tau^* (M_{\bar{Y}}) = M_{\bar{Y}}$ and $\tau'^* (M_{\bar{Y}'}) = M_{\bar{Y}'}$. Therefore $\tau'^* (M_{\bar{Y}'} - \gamma^* (M_{\bar{Y}})) = 0$, which implies $M_{\bar{Y}'} = \gamma^* (M_{\bar{Y}})$. \hfill \Box

Proof. (of Theorem 2.5) Let $f : (X, B) \to Y$ be a $K$-trivial fibration with $b = b(F, B_F)$ and

\[ K + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y). \]

We replace $X$ by a resolution, so that $X$ is non-singular, quasi-projective, and $B - f^*(B_Y + M_Y)$ is supported by a simple normal crossings divisor $\Sigma_X$. Let $V$ be a resolution of the normalization of $X$ in $k(X)(\varphi^{\frac{1}{b}})$ such that $B_V$ has simple normal crossings support. We may assume that $f, h$ are projective morphisms, after a birational base change. Then there exists a closed subvariety $\Sigma_f \subset Y$ such that $(V, B_V) - \to (X, B) \to Y$ satisfies the assumptions of the set-up in the beginning of this section, except that $\Sigma_f$ may not be the support of a simple normal crossings divisor. Let $\sigma : Y' \to Y$ be an embedded resolution so that $\Sigma_{Y'} := \sigma^{-1}(\Sigma_f)$ is a divisor with simple normal crossings. There exists an induced set-up $(V', B_{V'}) - \to (X', B') \to Y'$.

We claim that $\sigma^* (M_{Y'}) = M_{Y''}$ and $\sigma^* (K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''}$ for every birational contraction $\sigma : Y'' \to Y'$. By Hironaka’s resolution of singularities, there exists a diagram of birational morphisms

\[
\begin{array}{ccc}
Y'' & \xleftarrow{\sigma''} & Y''' \\
\sigma' \downarrow & & \downarrow \\
Y' & \xleftarrow{\sigma'} & Y''
\end{array}
\]

such that $Y'''$ is a non-singular quasi-projective variety admitting a simple normal crossings divisor which supports $\sigma'^{-1}(\Sigma_{Y'})$ and the exceptional locus of $Y'''/Y'$. By Proposition 5.5, $\sigma'^* (M_{Y'}) = M_{Y'''}$ and
consequently $\sigma^*(K_Y + B_Y) = K_{Y''} + B_{Y''}$. Since $Y''/Y'$ is a birational morphism, the claim follows.

Let $\tau: \tilde{Y}' \to Y'$ be a covering given by Proposition 5.4. By Lemma 5.2, $M_{Y'}$ is a Cartier divisor and $\mathcal{O}_{\tilde{Y}'}(M_{Y'})$ is a semi-positive invertible sheaf. In particular, $M_{Y'}$ is nef. But $\tau^*(M_{Y'}) = M_{Y'}$ according to Lemma 5.1, hence $M_{Y'}$ is nef. □

Proof. (of Theorem 0.1) By the Lefschetz principle, we may assume $k = \mathbb{C}$. After a finite base change (Lemma 5.1), we may assume that the induced root fiber space $h: V \to Y$ is semi-stable. By construction, the invertible sheaf $\mathcal{L} := \mathcal{O}_Y(M_Y) \subset h^*\omega_{V/Y}$ is a direct summand.

We know that $M_Y$ is a nef Cartier divisor on the curve $Y$. If $\deg(M_Y) > 0$, then $M_Y$ is ample, in particular semi-ample. If $\deg(M_Y) = 0$, Theorem 4.5 implies $\mathcal{L} \otimes m \cong \mathcal{O}_Y$, so $M_Y$ is semi-ample. □

6. Asymptotically saturated algebras

We first recall some terminology from [18]. Let $\pi: X \to S$ be a proper morphism. A normal functional algebra of $X/S$ is an $\mathcal{O}_S$-algebra of the form

$$\mathcal{L} = \mathcal{R}_{X/S}(M_\bullet) = \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(M_i),$$

where $\{M_i\}$ is a sequence of b-free/S b-divisors of $X$ such that $M_i + M_j \leq M_{i+j}$ for every $i$ and $j$. The sequence of $\mathbb{Q}$-b-divisors $D_i = \frac{1}{i}M_i$ is called the characteristic sequence of $\mathcal{L}$. The algebra $\mathcal{L}$ is bounded if there exists an $\mathbb{R}$-b-divisor $D$ of $X$ such that $D_i \leq D$ for every $i$. By the Limiting Criteria [18, Theorem 4.28], the $\mathcal{O}_S$-algebra $\mathcal{L}$ is finitely generated if and only if the characteristic sequence $D_\bullet$ is constant up to a truncation. For an $\mathbb{R}$-b-divisor $A$, the algebra $\mathcal{L}$ is asymptotically $A$-saturated, if there exists a positive integer $I$ such that

$$\pi_* \mathcal{O}_X([A + jD_\bullet]) \subseteq \pi_* \mathcal{O}_X(M_j) \text{ for } I|i,j.$$

The Kodaira dimension of $\mathcal{L}$ is $\kappa(\mathcal{L}) := \max_i \kappa(X/S, M_i)$. We say that $\mathcal{L}$ is a big algebra if $\kappa(\mathcal{L}) = \dim(X/S)$.

Definition 6.1. Let $\mathcal{L}$ be a normal functional algebra of $X/S$. There exists a unique rational map with connected fibers $f: X \to Y/S$ and a normal functional algebra $\mathcal{L}'$ of $Y/S$, such that $f^*: \mathcal{L}' \to \mathcal{L}$ is a quasi-isomorphism and $\mathcal{L}'$ is a big algebra. We say that $(f, \mathcal{L}')$ is the Iitaka fibration of $\mathcal{L}$.

Proof. [18, Lemma 6.22] Let $\mathcal{L} = \mathcal{R}_{X/S}(M_\bullet)$. Since $M_i + M_j \leq M_{i+j}$ and the $M_i$’s are b-free, there exists $I \in \mathbb{N}$ and a rational map
Let \( f : X \rightarrow Y/S \) which is the Iitaka contraction of \( M_i \) for every \( i \) divisible by \( I \). Up to a quasi-isomorphism, we may assume that the \( b \)-free \( b \)-divisors \( M_i \) are effective. Since \( f \) has connected fibers, there exists a convex sequence \( M'_i \) such that \( M_i = f^*(M'_i) \) for every \( I| i \). In particular, \( \mathcal{L} \) is quasi-isomorphic to the big algebra \( \mathcal{L}' := \mathcal{R}_{Y/S}(M'_*) \). □

**Lemma 6.2.** Let \( (f : X \rightarrow Y/S, \mathcal{L}') \) be the Iitaka fibration of a normal functional algebra \( \mathcal{L} \). If \( \mathcal{L} \) is asymptotically \( \mathbb{A} \)-saturated, then rank \( f_*\mathcal{O}_X([A]) \leq 1 \), where \( f' : X' \rightarrow Y \) is a regular representative of the rational function \( f \).

**Proof.** We may assume that \( f' = f \) and \( \mathcal{L} = \mathcal{R}_{X/S}(f^*M'_*), \) where \( \mathcal{L}' = \mathcal{R}_{Y/S}(M'_*) \) is the induced big algebra. By assumption, there exists \( i \) such that \( D_i \) is \( b \)-big/\( S \). After passing to higher models, we may assume that \( D_i \) descends to \( Y \). There exists a birational contraction \( \mu : Y \rightarrow Z/S \) and an ample/\( S \) \( \mathbb{Q} \)-divisor \( H \) on \( Z \) such that \( (D_i)_Y \sim \mu^*H \). For \( j \) sufficiently large and divisible, the \( \mathcal{O}_Z \)-sheaf

\[
\mu_* (f_*\mathcal{O}_X([A + jf*D_i])) = \mu_* f_*\mathcal{O}_X([A]) \otimes \mathcal{O}_Z(jH)
\]

is \( \pi \)-generated. Therefore \( f_*\mathcal{O}_X([A + jf*D_i]) \) is generically \( \pi \)-generated. Asymptotic saturation implies that \( f_*\mathcal{O}_X([A + jf*D_i]) \) is contained in the \( b \)-divisorial sheaf \( \mathcal{O}_Y(M_j) \) on an open subset of \( Y \). The latter has rank one, hence \( f_*\mathcal{O}_X([A]) \) has rank at most one. □

**Proposition 6.3.** (cf. [18, Proposition 4.50]) Consider a commutative diagram

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{f} & Y \\
\downarrow \pi & & \downarrow \sigma \\
S & & \\
\end{array}
\]

and a normal functional algebra \( \mathcal{L} = \mathcal{R}_{X/S}(M_*) \) with the following properties:

(a) \( f : (X, B) \rightarrow Y \) is a \( K \)-trivial fibration.

(b) \( \mathcal{L} \) is bounded and asymptotically \( \mathbb{A}(X, B) \)-saturated.

(c) There exist \( b \)-divisors \( M'_i \) of \( Y \) such that \( M_i = f^*(M'_i) \) for all \( i \).

Then \( \mathcal{L}' := \mathcal{R}_{Y/S}(M'_*) \) is a normal bounded functional algebra of \( Y/S \), which is asymptotically \( \mathbb{A}_{div} \)-saturated. Moreover, the natural map \( f^* : \mathcal{L}' \rightarrow \mathcal{L} \) is an isomorphism of \( \mathcal{O}_S \)-algebras.

**Proof.** It is clear that \( \mathcal{L}' \) is a functional algebra of \( Y/S \), and \( f^* : \mathcal{L}' \rightarrow \mathcal{L} \) is an isomorphism of \( \mathcal{O}_S \)-algebras. The algebra is normal since each \( M'_i \) is \( b \)-free. Let \( D'_i = \frac{1}{I}M'_i \) be the characteristic sequence of \( \mathcal{L}' \).
We first check that $L'$ is bounded. After passing to higher models, we may assume that there exists an effective Cartier divisor $E$ on $X$ such that $f^*D'_i = D_i \leq \overline{E}$. Let $E'$ be the divisorial support of $f(\text{Supp}(E^v)) \subset Y$. For each $i$, we can find a birational model $X'/Y'$ of $X/Y$, fitting in the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{} & Y'
\end{array}
$$

such that $D'_i$ and $D_i$ descend on $Y'$ and $X'$ respectively. In particular, $f'^*((D'_i)_Y) \leq h^*E$. Since $D'_i$ is effective and $Y'/Y$ is an isomorphism over a big open subset of $Y$, we conclude that $(D'_i)_Y$ is supported by $E'$. This holds for every $i$, hence $D'_\bullet$ is bounded.

It remains to check asymptotic $A_{\text{div}}$-saturation. Fix two integers $i, j$ which are divisible by $I$. By Theorem 2.5, we may assume the following properties hold (after a birational base change):

1. $X, Y$ are non-singular.
2. $D'_i, D'_j$ descend to $Y$ (in particular $D_i, D_j$ descend to $X$). Denote $D'_i = (D'_i)_Y$ and $D'_j = (D'_j)_Y$.
3. $\text{Supp}(B) \cup \text{Supp}(f^*D'_i)$ and $\text{Supp}(B_Y) \cup \text{Supp}(D'_i)$ are simple normal crossings divisors on $X$ and $Y$, respectively.
4. $A_{\text{div}} = A(Y, B_Y)$.

Under these assumptions, the saturation for $i, j$ means

$$
\pi_*O_X([-B + jf^*D'_i]) \subseteq \pi_*O_X(jf^*D'_j)
$$

By Lemma 3.2 and Remark 3.3,

$$
f_*O_X([-B + jf^*D'_i]) = O_Y([-B_Y + jD'_j]).
$$

Since $\pi_*O_X(jf^*D'_j) = \sigma_*O_Y(M'_j)$, we infer

$$
\sigma_*O_Y([-B_Y + jD'_j]) \subseteq \sigma_*O_Y(jD'_j).
$$

Therefore $\sigma_*O_Y([-A_{\text{div}} + jD'_j]) \subseteq \sigma_*O_Y(M'_j)$, by (i)-(iv) again. \hfill \Box

**Example 6.4.** (Reduction to big algebras) Let $(X/S, B)$ be a relative log pair, and let $L$ be a normal, bounded functional algebra with Iitaka fibration $(f : X \rightarrow Y/S, L')$, satisfying the following properties:

1. $K_X + B_X \sim_\mathbb{Q} f^*D$, where $f' : X' \rightarrow Y$ is a regular model of $f$ and $B_{X'}$ is a crepant boundary ($A(X, B) = A(X', B_{X'})$).
2. $(X', B_{X'})$ has klt singularities over the generic point of $Y$.
3. $L$ is asymptotically $A(X, B)$-saturated.
By Lemma 6.2, \( f' : (X', B_{X'}) \to Y \) is a \( K \)-trivial fibration. By Theorem 2.5 and Proposition 6.3, we may replace \( Y \) by a higher birational model so that the following properties hold:

(a) \( A_{div} = A(Y, B_Y) \).
(b) \( K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*(K_Y + B_Y + M_Y) \).
(c) \( \mathcal{L}' \) is normal, bounded and asymptotically \( A(Y, B_Y) \)-saturated.

The above example is a first step towards a reduction of (0LP) algebras \([18, \text{Remark 4.40}]\) to the big case. To complete the reduction, we need to know that the moduli b-divisor \( M \) is b-semi-ample. However, the b-nef property of the moduli b-divisor is enough for some applications to the Fano case. We show that the restriction of an FGA algebra to an exceptional log canonical centre is again an FGA algebra (cf. \([18, \text{Proposition 4.50}]\) for lc centers of codimension one):

**Theorem 6.5.** Let \( (X/S, B) \) be a relative generalized Fano log variety, let \( \nu : W \to X \) be the normalization of an exceptional lc centre of \((X, B)\) and let \( \mathcal{L} = \mathcal{R}_{X/S}(M_\bullet) \) be a normal bounded functional algebra of \( X/S \) such that the following hold:

(i) \( \mathcal{L} \) is asymptotically \((A(X, B) + E)\)-saturated, where \( E \) is the unique lc place over \( W \).

(ii) There exists an open subset \( U \subseteq X \) such that \( U \cap \nu(W) \neq \emptyset \), \( D_i|_U = \overline{D_i|_U} \) \( \forall i \) for some \( \mathbb{Q} \)-Cartier divisor \( D \) on \( X \).

(ii\( ^* \)) \( U \) contains \((X, B)_{-\infty} \cap \nu(W)\) and \( C \cap \nu(W) \), for every lc centre \( C \neq \nu(W) \) of \((X, B)\).

Then there exists a well defined restricted algebra \( \mathcal{L}|_W \) of \( W/S \), with the following properties:

1. \( \mathcal{L}|_W = \mathcal{R}_{W/S}(M'_\bullet) \) is a normal, bounded functional algebra.
2. \( \mathcal{L}|_W \) is \( A(W, B_W) \)-saturated, where \((W/S, B_W)\) is a relative generalized Fano log variety.
3. \( \text{LCS}(W, B_W) \subset U' := U|_W \) and \( D'_i|_{U'} = \overline{D'_i|_{U'}} \) for every \( i \).
4. The \( \mathcal{O}_S \)-algebras \( \mathcal{L}|_W \) and \( \mathcal{L}|_E \) are quasi-isomorphic.

**Proof.** Let \( H \) be an ample \( /S \) \( \mathbb{Q} \)-divisor on \( X \) such that \(- (K + B + H)\) is ample \( /S \). Then \([3, \text{Theorem 4.9}]\) constructs an effective \( \mathbb{Q} \)-divisor \( B_W \) on \( W \) such that \((W/S, B_W)\) is a relative generalized Fano log variety with \((K + B + H)|_W \sim_{\mathbb{Q}} K_W + B_W \) and \( \text{LCS}(W, B_W) \) is contained in the union of \((X, B)_{-\infty} \) and all lc centres of \((X, B)\) different than \( \nu(W) \). In particular, \( \text{LCS}(W, B_W) \subset U' \). Consider the induced diagram:

\[
\begin{array}{ccc}
(E, B_E) & \subset & (X', B_{X'}) \\
\downarrow & & \downarrow \\
W & \to & (X, B)
\end{array}
\]
By adjunction and Kawamata-Viehweg vanishing, \( \mathcal{L}_{i,E} \) is asymptotically \( A(E, B_E) \)-saturated [13, Proposition 4.50]. By (ii), there exist \( b \)-free/\( S \) \( b \)-divisors \( M_i' \) of \( W \) such that \( M_i' = h^*(M_i') \) for every \( i \). By construction, \( (E, B_E) \rightarrow W/S \) is a \( K \)-trivial fibration for which Proposition [13] applies. Therefore \( \mathcal{L}_{i,W} := \mathcal{R}_{W/S}(M_i') \) is quasi-isomorphic to \( \mathcal{L}_{i,E} \), it is normal, bounded and asymptotically \( A_{\text{div}} \)-saturated. From the construction of \( B_W \) (choosing \( W' \) high enough so that \( A_{\text{div}} = A(W', B_{W'}) \), in the proof of [3, Theorem 4.9]), we have \( A(W, B_W) \leq A_{\text{div}} \). Therefore \( \mathcal{L}_{i,W} \) is asymptotically \( A(W, B_W) \)-saturated.

Finally, \( D_i|_U = D|_U \) implies \( D_i'|_{U'} = D|_{U'} \). □

7. Parabolic fiber spaces

A parabolic fiber space is a contraction of non-singular proper varieties \( f: X \rightarrow Y \) such that the generic fiber \( F \) has Kodaira dimension zero. Let \( b \) be the smallest positive integer with \( |bK_F| \neq \emptyset \). We fix a rational function \( \varphi \in k(X)^* \) such that \( K + \frac{1}{b}(\varphi) \) is effective over the generic point of \( Y \).

Definition-Proposition 7.1. (cf. [3, Corollary 2.5]) Let \( f: X \rightarrow Y \) be a parabolic fiber space with a choice of a rational function \( \varphi \), as above. There exists a unique \( b \)-nef \( \mathbb{Q} \)-b-divisor \( M = M(f, \varphi) \) of \( Y \) satisfying the following properties:

1. Let \( \varphi: Y' \rightarrow Y \) be a surjective proper morphism, and let \( f': X' \rightarrow Y' \) be an induced parabolic fiber space:

\[
\begin{array}{ccc}
X & \xleftarrow{\nu} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\varphi} & Y'
\end{array}
\]

Then \( \varphi^* M(f, \varphi) \sim_{\mathbb{Q}} M(f', \nu^* \varphi) \). Moreover, \( \varphi^* M(f, \varphi) = M(f', \nu^* \varphi) \) if \( \varphi \) is generically finite.

2. If \( f \) is semi-stable in codimension one, then

\[
\bigoplus_{i} f_* \mathcal{O}_X(iK_X)^{**} = \bigoplus_{i} \mathcal{O}_Y(i(K_Y + M_Y)) \cdot \varphi^i.
\]

We say that \( M = M(f, \varphi) \) is the moduli \( \mathbb{Q} \)-b-divisor associated to the parabolic fiber space \( f \). If \( \varphi' \) is another choice of the rational function, then \( bM \sim bM' \). Therefore \( bM \) is uniquely defined up to linear equivalence.
Proof. There exists [3, Proposition 2.2] a unique \( \mathbb{Q} \)-divisor \( B_X \) on \( X \) satisfying the following properties:

(i) \( K + B_X + \frac{1}{b}(\varphi) = f^*D \) for some \( D \in \text{Div}(Y)_{\mathbb{Q}} \).

(ii) There exists a big open subset \( Y^+ \subseteq Y \) such that \( -B_X|_{f^{-1}(Y^+)} \) is effective and contains no fibers of \( f \) in its support.

One can easily check that \( f : (X, B_X) \to Y \) is a \( K \)-trivial fibration.

Let \( B, M \) be the induced discriminant and moduli b-divisors of \( Y \).

Note that if \( X \to X' \) is a birational map over \( Y \), then \( (X, B_X) \to (X', B_{X'}) \) is crepant, i.e. \( A(X, B) = A(X', B_{X'}) \). Also, \( |B| = 0 \).

We claim that \( M = M(f, \varphi) \) (the uniqueness is clear by semi-stable reduction in codimension one). Since \( \kappa(F) = 0 \), and \( K + B_X + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y) \),
we infer that \( f_*O_X(iK)^{**} = O_Y(i(K_Y + B_Y + M_Y)) \cdot \varphi^i \) for \( b|i \). If \( f \) is semi-stable in codimension one, then \( B_Y = 0 \). Therefore (2) holds.

By Theorem [2.3] \( M \) is b-\( \mathbb{Q} \)-nef. We may replace \( f \) by a birational base change so that \( M \) descends to \( Y: M = M_Y \). If \( g \) is generically finite, (1) holds by inversion of adjunction for finite morphisms (cf. [5, Proposition 4.7]). For general \( g \), we imitate the proof of Proposition [5.3] with the following simplifications: the root fiber space \( h : V \to Y \) is parabolic as well; if \( h \) has simple normal crossings degeneration and is semi-stable in codimension one, then \( M_Y \) is an integral divisor and \( h_*O_V(K_{V/Y}) = O_Y(M_Y) \cdot \varphi \).

A non-singular projective variety \( X \) has a good minimal model if it is birational to a normal projective variety \( Y \) such that \( Y \) has terminal singularities and \( K_Y \) is semi-ample. This holds if \( \dim(X) \leq 3 \), by the Minimal Model Program and Abundance (see [10]). The following is a generalization of [10, Theorem 13]:

**Theorem 7.2.** Let \( f : X \to Y \) be a parabolic fiber space such that its geometric generic fibre \( \bar{F} = X \times_Y \text{Spec}(k(Y)) \) has a good minimal model over \( k(Y) \). Then there exists a diagram

\[
\begin{array}{ccc}
X & \xymatrix{ & X' \ar[d]^{f'} } & \\
Y & \xymatrix{ \bar{Y} & Y' \ar[l]_{f} \ar[u]_{\phi} } & \\
& f & \\
\end{array}
\]

such that the following hold:

1. \( \bar{f} \) and \( f' \) are parabolic fiber spaces.
2. \( \tau \) is generically finite, and \( \phi \) is a proper dominant morphism.
3. \( \bar{f} \) is birationally induced via base change by both \( f \) and \( f' \).
(4) $\tau^* M = M \sim_{Q} g^* M'$, where $M, M, M'$ are the corresponding moduli $Q$-$b$-divisors.

(5) $M'$ is $b$-nef and big and $\text{Var}(f') = \dim(Y')$.

In particular, $\kappa(M) = \text{Var}(f)$, where $\text{Var}(f)$ is the variation of the fiber space $f$.

Proof. By the definiton of the variation of a fibre space, there exists a diagram as above, satisfying (1), (2),(3) and such that $\dim(Y') = \text{Var}(f) = \text{Var}(f')$. After a generically finite base change, we may also assume that $M'$ descends to $Y'$, and $f'$ is semi-stable in codimension one.

By (3) and Definition-Proposition 7.1, (4) holds. In particular, $\kappa(M) = \kappa(M')$. Since $\bar{F}$ has a good minimal model, Viehweg’s $Q(f')$ Conjecture holds \cite{11}. Theorem 1.1.(i), that is the sheaf $(f'_* \omega_{X'/Y'}^i)^{*\star}$ is big for $i$ large and divisible. But $(f'_* \omega_{X'/Y'}^i)^{*\star} \simeq \mathcal{O}_{Y'}(iM'_{Y'})$ for $b|i$, since $f'$ is semi-stable in codimension one. Equivalently, $\kappa(Y', M'_{Y'}) = \dim(Y')$. Therefore $M'$ is b-nef and big.  

\[\square\]

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