Behavior of null-geodesics in the interior of Reissner-Nordstrom black hole

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Abstract

We show that an incoming null-geodesic belonging to a plane passing through the origin and starting outside the outer horizon crosses the outer and the inner horizons. Then it turns at some point inside the inner horizon and approaches the inner horizon when the time tends to the infinity. We also construct a geometric optics solution of the Reissner-Nordstrom equation that has support in a neighborhood of the null-geodesic.

Keywords. Reissner-Nordstrom black hole, null-bicharacteristics, geometric optics solutions.

1 Introduction

The Reissner-Norstrom metric (cf. [7], [9], [12]) is a spherically symmetric metric having the following form in Cartesian coordinates (cf. [11]):

\[ ds^2 = \left( 1 - \frac{2m}{r} + \frac{e^2}{r^2} \right) dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 - 2 \left( \frac{2m}{r} - \frac{e^2}{r^2} \right) dx_0 dr - \left( \frac{2m}{r} - \frac{e^2}{r^2} \right) dr^2, \]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \).

Let

\[ \Box_g u = 0 \]
be the wave operator corresponding to (1.1).

The symbol (Hamiltonian) of $\Box_g$ has the form

$$H = \xi_0^2 - \sum_{j=1}^{3} \xi_j^2 + \left( \frac{2m}{r} - \frac{e^2}{r^2} \right) \left( -\xi_0 + \sum_{j=1}^{3} x_j \xi_j \right)^2,$$

where $x_0$ is the time coordinate, $x = (x_1, x_2, x_3)$ are the space coordinates, $(\xi_0, \xi_1, \xi_2, \xi_3)$ are dual to $(x_0, x_1, x_2, x_3)$, $r = \sqrt{\sum_{j=1}^{3} x_j^2}$.

Denote

$$f = 1 - \frac{2m}{r} + \frac{e^2}{r^2}.$$

Note that $f = 0$ has two real roots

$$r_+ = m + \sqrt{m^2 - e^2}, \quad r_- = m - \sqrt{m^2 - e^2}, \quad f = \frac{(r - r_+)(r - r_-)}{r^2},$$

assuming that $e^2 < m^2$.

It follows from (1.1) that $r = r_+$ and $r = r_-$ are the outer and inner horizons for the Reissner-Nordström metric.

The Reissman-Nordstrom metric is a solution of the Einstein equations of general relativity in vacuum. It was investigated in many papers with the emphasis on the study of the stability of the metric among the solutions of the Einstein equations (cf. [1], [5], [8], [10]).

We will not use that (1.1) is a solution of the Einstein equations.

The null-geodesics of (1.1) are the projections on the $x$-space of the null-bicharacteristics (see §2 for the details).

We shall give an explicit description of the behavior of null-geodesics of the wave equation $\Box_g u = 0$ and we shall prove that the geometric optics solutions of the wave equation are approaching the inner horizon $r = r_-$ from the inside when $x_0 \to +\infty$.

The plan of the paper is the following:

In §2 we study the behavior of null-geodesics located on a plane passing through the origin. We show that any incoming geodesic starting outside the black hole crosses the outer event horizon $r = r_+$ when the time $x_0$ is increases. Then it crosses also the inner horizon $r = r_-$ and reaches a turning point $r = r_0 < r_-$ where $r_0$ depends on the initial data of the null-bicharacteristic. After this it approaches the inner horizon from the inside when $x_0 \to +\infty$. 

2
In §3 we construct geometric optics solutions $u_N$ of (1.2) depending on a large parameter $k$. This geometric optics solution has a support in a neighborhood of some null-geodesics $\gamma_0$. Thus the time evolution of geometric optics solution follows modulo lower order terms the time evolution of the null-geodesic $\gamma_0$.

In §4 we summarize the results of the paper.

2 The behavior of null-bicharacteristics in the case of Reissner-Nordstrom metric

The equations of null-bicharacteristics in Cartesian coordinates has the form

\begin{align}
\frac{dx_k}{ds} &= \frac{\partial H}{\partial \xi_k}, \quad x_k(0) = y_k, \quad 0 \leq k \leq 3, \\
\frac{d\xi_k}{ds} &= -\frac{\partial H}{\partial x_k}, \quad \xi_k(0) = \eta_k, \quad 0 \leq k \leq 3,
\end{align}

where $(\xi_0, \xi_1, \xi_2, \xi_3)$ are dual coordinates to $(x_0, x_1, x_2, x_3)$, $H$ is the same as in (1.3). We study the restriction of the metric (1.1) to the plane $x_3 = 0$. Consider the null-bicharacteristic such that $x_3(0) = 0, \xi_3(0) = 0$. We have

\begin{align}
\frac{dx_3}{ds} &= \frac{\partial H}{\partial x_3} = -2\xi_3 + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)2\left(-\xi_0 + \sum_{k=1}^{3} x_k \xi_k\right)x_3 \\
\frac{d\xi_3}{ds} &= -\frac{\partial H}{\partial x_3} = -\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)2\left(-\xi_0 + \sum_{k=1}^{3} x_k \xi_k\right)\xi_3 \\
&\quad + \left(\frac{2m}{r^2} x_3 + \frac{2e^2}{r^3} x_3\right)\left(-\xi_0 + \sum_{k=1}^{3} x_k \xi_k\right)^2.
\end{align}

Since the initial data $x_3(0) = 0, \xi_3(0) = 0$, by the uniqueness theorem for the system (2.2), we have that $x_3(s) \equiv 0, \xi_3(s) \equiv 0$, i.e. the null-bicharacteristic stays in the plane $x_3 = 0$ and $\xi_3 = 0$. Therefore we can restrict the null-bicharacteristic to the plane $x_3 = 0$ with $\xi_3 = 0$ and the restricted Hamiltonian has the form

\[ H_0(x_1, x_2, \xi_0, \xi_1, \xi_2) = \xi_0^2 - \sum_{k=1}^{2} \xi_k^2 + \left(\frac{2m}{\rho} - \frac{e^2}{\rho^2}\right)\left(-\xi_0 + \sum_{k=1}^{2} x_k \xi_k\right)^2, \]
where \( \rho = \sqrt{x_1^2 + x_2^2} \). In polar coordinates \((\rho, \varphi)\) the Hamiltonian has the form

(2.3) \[ H_0(\rho, \varphi, \xi, \xi_\rho, \xi_\varphi) = \xi_0^2 - \frac{1}{\rho^2} \xi_\varphi^2 + \left( \frac{2m}{\rho} - \frac{e^2}{\rho^2} \right) (-\xi_0 + \xi_\rho)^2 \]

\[ = (2 - f)\xi_0^2 + 2(f - 1)\xi_0\xi_\rho - f\xi_\rho^2 - \frac{1}{\rho^2} \xi_\varphi^2, \]

where \( f \) is the same as in (1.4). The system of null-bicharacteristics has the form

(2.4) \[ \frac{d\rho}{ds} = \frac{\partial H_0}{\partial \xi_\rho} = 2(f - 1)\xi_0 - 2f\xi_\rho, \quad \rho(0) = \rho_0, \]

\[ \frac{d\varphi}{ds} = \frac{\partial H_0}{\partial \xi_\varphi} = -2\xi_\psi, \quad \varphi(0) = \varphi_0, \quad \frac{d\xi_0}{ds} = -\frac{\partial H_0}{\partial x_0}, \quad \xi_0(0) = \xi_0^{(0)}, \]

\[ \frac{d\xi_\rho}{ds} = -\frac{\partial H_0}{\partial \rho}, \quad \xi_\rho(0) = \xi_\rho^{(0)}, \quad \frac{d\xi_\varphi}{ds} = -\frac{\partial H_0}{\partial \varphi}, \quad \xi_\varphi(0) = \xi_\varphi^{(0)}, \]

\[ \frac{dx_0}{ds} = \frac{\partial H_0}{\partial \xi_0} = 2(2 - f)\xi_0 + 2(f - 1)\xi_\rho, \quad x_0(0) = x_0^{(0)}. \]

Since \( H_0 \) is independent of \( x_0 \) and \( \varphi \) we have that \( \xi_0(s) = \xi_0^{(0)}, \xi_\varphi(s) = \xi_\varphi^{(0)} \) for all \( s \). For the simplicity of notations we shall write \( \xi, \xi_\varphi \) instead of \( \xi_0^{(0)}, \xi_\varphi^{(0)} \). Denote by \( \gamma_0 \) the null-bicharacteristic with the initial data \( \rho_0, \varphi_0, \xi_\rho^{(0)}, \xi_\varphi^{(0)}, \xi_0^{(0)} \). Note that the null-bicharacteristic mean that

(2.5) \[ H_0(\rho(s), \varphi(s), \xi_0(s), \xi_\rho(s), \xi_\varphi(s)) = 0 \quad \text{for all} \quad s \geq 0. \]

Therefore we can find \( \xi_\rho \) from (2.5) (cf. [3]). We get

(2.6) \[ \xi_\rho^\pm = \xi_\rho^\pm(\rho, \xi_0, \xi_\varphi) = \frac{-(f - 1)\xi_0 \pm \sqrt{(f - 1)^2 \xi_0^2 + f \left[ (2 - f)\xi_0^2 - \frac{1}{\rho^2} \xi_\varphi^2 \right]}}{-f} \]

\[ = \frac{-(f - 1)\xi_0 \pm \sqrt{\xi_0^2 - f\xi_\varphi^2}}{-f}. \]

In particular, when \( s = 0 \) we have

(2.7) \[ \xi_\rho^\pm(0) = \xi_\rho^\pm(\rho_0, \xi_0, \xi_\varphi). \]
Taking $x_0$ as a parameter instead of $s$ and substituting (2.6) into (2.4) we obtain (cf. [3])

\[
\frac{d\rho^\pm}{dx_0} = \frac{(f - 1)\xi_0 - f\xi_\rho^\pm}{(2 - f)\xi_0 + (f - 1)\xi_\rho^\pm} = \frac{(f - 1)\xi_0 - (f - 1)\xi_0 \pm \sqrt{\Delta}}{(2 - f)\xi_0 + (f - 1)\frac{(f - 1)\xi_0 \pm \sqrt{\Delta}}{f}}
\]

\[
= \frac{\pm \sqrt{\Delta}(- f)}{-\xi_0 \pm (f - 1)\sqrt{\Delta}} = \frac{\pm \sqrt{\Delta} f}{\xi_0 \mp (f - 1)\sqrt{\Delta}},
\]

where

\[
(2.9) \quad \Delta = \xi_0^2 - f(\rho)\frac{\xi_\phi^2}{\rho^2}.
\]

Analogously

\[
\frac{d\varphi^\pm}{dx_0} = \frac{-\xi_\rho f}{(2 - f)\xi_0 + (f - 1)\xi_\rho^\pm} = \frac{\xi_\rho f}{-\xi_0 \pm (f - 1)\sqrt{\Delta}} = \frac{-\xi_\rho f}{\xi_0 \mp (f - 1)\sqrt{\Delta}}
\]

We assume that $\xi_0 > 0$ and $\Delta > 0$, and we denote by $\rho^\pm(x_0), \varphi^\pm(x_0)$ solutions of (2.8), (2.10) corresponding to the sign $\pm$ in (2.8) or (2.10).

Note that $f(\rho) = \frac{(\rho - r_+)(\rho - r_-)}{\rho^2}$. Thus $f(\rho) < 0$ between $\rho = r_-$ and $\rho = r_+$ and $\Delta > 0$ there. By the continuity $\Delta > 0$ is a small neighborhood of $[r_-, r_+]$.

When $\rho \to 0 \frac{f(\rho)}{\rho^2} \to +\infty$. Let $r_0$ be such that $\Delta = \xi_0^2 - \frac{f(r_0)}{\rho^2} \xi_\phi^2 = 0$. Thus $\Delta > 0$ for $\rho > r_0$ and $\Delta < 0$ for $\rho < r_0$.

Consider the minus null-bicharacteristic

\[
(2.11) \quad \frac{d\rho^-}{dx_0} = \frac{-\sqrt{\Delta} f}{\xi_0 + (f - 1)\sqrt{\Delta}}
\]

with the initial conditions $(\rho_0, \varphi_0, \xi_0, \xi_\rho)$, where $(\rho_0, \varphi_0)$ is the point outside the outer horizon $\rho = r_+$, i.e. $\rho_0 > r_+$. Note that

\[
\xi_0 - \sqrt{\Delta} = \frac{\xi_0^2 - (\xi_0 - f\xi_\rho \frac{\xi_\phi^2}{\rho^2})}{\xi_0 + \sqrt{\Delta}} = \frac{f\xi_\rho^2}{\rho^2(\xi_0 + \sqrt{\Delta})}.
\]

Therefore, cancelling $f$ we obtain

\[
\frac{d\rho^-}{dx_0} = \frac{-f\sqrt{\Delta}}{f\sqrt{\Delta} + \frac{f\xi_\rho^2}{\rho^2(\xi_0 + \sqrt{\Delta})}} = \frac{-\sqrt{\Delta}}{\sqrt{\Delta} + \frac{\xi_0^2}{\rho^2(\xi_0 + \sqrt{\Delta})}}, \quad \xi_0 > 0.
\]
Therefore \( \frac{d\varphi^-}{dx_0} < 0 \), i.e. \( \rho^-(x_0) \) decreases when \( x_0 \) increases. Since \( \frac{d\varphi^-}{dx_0} < 0 \) the null-bicharacteristic \( \rho^-(x_0) \) crosses the outer horizon and the inner horizon when \( x_0 \) increases.

Note that

\[
\frac{d\varphi^-}{dx_0} = -\frac{\xi \varphi f}{\xi_0 - \sqrt{\Delta} + f \sqrt{\Delta}} = -\frac{\xi \varphi f}{f \sqrt{\Delta} + \xi_0 \sqrt{\Delta} + \frac{\xi_0^2}{\xi_0 + \sqrt{\Delta}}/\xi_0 + \sqrt{\Delta}}.
\]

Thus \( \frac{d\varphi^-}{dx_0} > 0 \) if \( \xi \varphi < 0 \) and \( \frac{d\varphi^-}{dx_0} < 0 \) if \( \xi \varphi > 0 \). Also we have

\[
(2.12) \quad \frac{d\varphi^-}{d\rho^-} = \frac{\xi \varphi}{\sqrt{\Delta}}.
\]

Thus \( \frac{d\varphi^-}{d\rho^-} < 0 \) if \( \xi \varphi < 0 \) and \( \frac{d\varphi^-}{d\rho^-} > 0 \) if \( \xi \varphi > 0 \).

Let \( r_0 = r_0(\xi_0^2) \) be the root of

\[
(2.13) \quad \Delta(r_0) = \xi_0^2 - f(r_0) \frac{\xi_0^2}{r_0} = 0.
\]

Thus \( \Delta(\rho) = (\rho - r_0)\Delta_1(\rho^-) \), where \( \Delta_1(\rho) > 0 \). We have near \( \rho = r_0, \rho > r_0 \):

\[
\frac{d\rho^-}{dx_0} = -\sqrt{\rho^- - r_0} \Delta_2(\rho),
\]

where \( \Delta_2(\rho) > 0 \). Therefore

\[
(\rho^-(x_0) - r_0)^{\frac{1}{2}} = C(\rho)(t_0 - x_0) \quad \text{for} \quad x_0 < t_0.
\]

Thus \( \rho^-(x_0) \) reaches \( r_0 \) when \( x_0 \to t_0, x_0 < t_0 \), i.e. \( \rho^-(t_0) = r_0 \). Analogously,

\[
(2.14) \quad \frac{d\varphi^-}{d\rho^-} = \frac{\Delta_3(\rho^-)\xi \varphi}{(\rho^- - r_0)^{\frac{1}{2}}},
\]

where \( \rho > r_0, \Delta_3(\rho) > 0 \). Hence, assuming, for the definitness, that \( \xi \varphi > 0 \), we get, for \( \rho^- > r_0, \varphi^- \leq \theta_0 \),

\[
(2.15) \quad (\rho^- - r_0)^{\frac{1}{2}} = C_3(\rho) (\theta_0 - \varphi^-),
\]

where \( r_0 = \rho^-(\theta_0) \).
Note that $d\rho - (t_0) dx_0 = 0$. We shall show that $(r_0, \theta_0)$ is a turning point of the null-geodesic $\gamma_0$. For $x_0 \geq t_0$ we consider the (+) solution (cf. (2.8))

$$d\rho^{+}(x_0) = \frac{f\sqrt{\Delta}}{\xi_0 + \sqrt{\Delta} - f\sqrt{\Delta}}. \tag{2.16}$$

Note that $\frac{d\rho^+(t_0)}{dx_0} = 0$ and $\frac{d\rho^+(x_0)}{dx_0} > 0$ for $t_0 < x_0$.

Also we have for $x_0 > t_0$

$$\frac{d\varphi^+}{d\rho^+} = -\frac{\xi_\varphi}{\rho^2\sqrt{\Delta}}. \tag{2.17}$$

Hence $(\rho^+ - r_0)^{\frac{1}{2}} = C_3(\rho)(\varphi^+ - \theta_0)$, $\varphi^+ > \theta_0$ (cf. (2.15)). As above, we are assuming that $\xi_\varphi > 0$. Therefore in the two-sided neighborhood of $\theta_0$ we have $\rho - r_0 = C(\rho)(\varphi - \theta_0)^2$.

Thus, $(r_0, \theta_0)$ is a turning point.

When $x_0 > t_0$ is increasing $\rho^+(x_0)$ is also increasing since $\frac{d\rho^+}{dx_0} > 0$. Near the inner horizon we have

$$\frac{d\rho^+}{dx_0} = \frac{f\sqrt{\Delta}}{\xi_0 + \sqrt{\Delta} - f\sqrt{\Delta}} = (\rho^+ - r_-)\Delta_4(\rho^+), \tag{2.18}$$

where $\Delta_4(\rho^+) < 0$. Therefore, $\frac{d\rho^+}{dx_0} \leq -C(\rho^+ - r_-)$ and integrating we get $0 < r_- - \rho^+(x_0) \leq C_3e^{-C_2x_0}$. Thus $\rho^+(x_0)$ tends to $r_-$ when $x_0 \to +\infty$.

More precisely, since $f = \frac{(\rho - r_-)(\rho - r_+)}{\rho^2}$ and $\Delta = \xi_\rho^2 - \frac{f(\rho)\xi_\rho^2}{\rho^2}$ we can rewrite (2.18) in the form

$$\frac{d\rho^+}{dx_0} = \frac{(\rho^+ - r_-)(r_- - r_+)}{2\gamma^2} + O((\rho^+ - r_-)^2). \tag{2.19}$$

Therefore

$$r_- - \rho^+ = e^{-\frac{r^+ - r_- - x_0 + C}{2\gamma^2}}(1 + O(e^{-C_1x_0})). \tag{2.20}$$

We shall summarize the results of this section in the following theorem.

**Theorem 2.1.** Any “minus” null-geodesic $\gamma_0$ starting above the outer horizon $r = r_-$ decreases, i.e. $\frac{d\rho^-}{dx_0} < 0$, when the time $x_0$ increases. It passes the outer and the inner horizons $r = r_+$ and $r = r_-$ until it reaches the turning point $(r_0, \theta_0), r_0 < r_-$. Then it increases when the time is increasing and tends to the inner horizon when $x_0 \to +\infty$ (see Fig.1).
Fig. 1. The “minus” null-geodesics crosses the outer and inner horizons, makes a turn at some point \((r_0, \theta_0)\), \(r_0 < r_-\), then it increases and tends to the inner horizon when \(x_0 \to +\infty\).

3 Geometric optics type solution

The equation \(\Box_g u = 0\) has the following form in Cartesian coordinates (cf. (1.3))

\[
\frac{\partial^2 u}{\partial x_0^2} - \sum_{k=1}^{3} \frac{\partial^2 u}{\partial x_k^2} - \left( -\frac{\partial}{\partial x_0} + \sum_{k=1}^{3} x_k \frac{\partial}{\partial x_k} \right)(f - 1) \left( -\frac{\partial}{\partial x_0} + \sum_{k=1}^{3} x_k \frac{\partial}{\partial x_k} \right) u = 0,
\]

where \(f(r) = 1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\).

Denote by \(\Pi_0\) the plane \(x_3 = 0\). The restriction of (3.1) to the plane \(x_3 = 0\) has the form in polar coordinates \((\rho, \varphi)\) (cf. §2)

\[
\frac{\partial^2 v}{\partial x_0^2} - \frac{\partial^2 v}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2 v}{\partial \varphi^2} - \left( -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial \rho} \right)(f(\rho) - 1) \left( -\frac{\partial}{\partial x_0} + \frac{\partial}{\partial \rho} \right) v = 0,
\]

As in §2 we choose arbitrary point \((\rho_0, \varphi_0)\) in the plane \(x_3 = 0\) where \(\rho_0 > r_+\). Denote by \(\gamma_0 = \gamma_0(\rho_0, \varphi_0, \xi_0, \xi_\varphi)\) the null-characteristic in the plane \(x_3 = 0\) starting at \((\rho_0, \varphi_0, \xi_0^{(0)}, \xi_\varphi)\) where (cf. (2.7))

\[
\xi_\rho^{(0)} = \xi_\rho^{-}(\rho_0, \xi_0, \xi_\varphi).
\]
Let \( V_0 \) be a neighborhood of \((\rho_0, \varphi_0)\) in the plane \( x_3 = 0 \) such that \(|\rho' - \rho_0| < \varepsilon, |\varphi' - \varphi_0| < \varepsilon\) when \((\rho', \varphi') \in V_0\). Denote by \( \gamma'(\rho', \varphi', \xi, \xi_0, \xi_0, \xi_0) \) the null-geodesic starting at \((\rho', \varphi', \xi, \xi_0, \xi_0, \xi_0)\). Let

\[
S_{0}^{-}(\rho', \varphi', \xi_0, \xi_0) = \int_{\rho_{10}}^{\rho'} \xi_0^{-}(\rho_1, \xi_0, \xi_0)d\rho_1 + \varphi'\xi_0,
\]

\(\rho_{10} \geq \rho'\).

Denote by \( \rho = \rho(x_0, \rho', \varphi') \), \( \varphi = \varphi(x_0, \rho', \varphi') \) the solution of the bicharacteristic system \((2.8), (2.10)\) with the initial conditions \(\rho', \varphi', \xi_0^{-}(\rho', \xi_0, \xi_0, \xi_0)\). We have that the Jacobian

\[
J(x_0) = \begin{vmatrix}
\frac{\partial \rho}{\partial \rho'} & \frac{\partial \varphi}{\partial \rho'} \\
\frac{\partial \rho}{\partial \varphi'} & \frac{\partial \varphi}{\partial \varphi'}
\end{vmatrix}
\]

is not zero on \([0, t_0 - \varepsilon]\). Therefore the inverse map \( \rho = \rho(x_0, \rho', \varphi'), \varphi = \varphi(x_0, \rho', \varphi') \) exists. It follows from \([2], \S 64\), that

\[
S^{-}(x_0, \rho, \varphi) = S_{0}^{-}(\rho(x_0, \rho, \varphi), \varphi(x_0, \rho, \varphi), \xi_0, \xi_0)
\]

satisfies the eikonal equation for \((3.2)\)

\[
(2 - f)S_{x_0}^{-} + 2(f - 1)S_{\rho}^{-}S_{x_0}^{-} - f(S_{\rho}^{-})^2 - \frac{1}{\rho^2}(S_{\varphi}^{-})^2 = 0
\]

on \([0, t_0 - \varepsilon]\). Denote by \( \Pi_\alpha \) the plane passing through the axis \(0x_3\) and having angle \(\alpha\) with the plane \( \Pi_0 \), \(\alpha \in (-\delta, \delta)\). The orthogonal transformation \( O_\alpha \)

\[
x_1(\alpha) = x_1 \cos \alpha - x_3 \sin \alpha,
\]

\[
x_2(\alpha) = x_2,
\]

\[
x_3(\alpha) = x_1 \sin \alpha + x_3 \cos \alpha
\]

maps \( \Pi_0 \) onto \( \Pi_\alpha \). Since the Reissner-Nordstrom metric is spherically symmetric, the restriction of \((3.1)\) to \( \Pi_\alpha \) has the form \((3.2)\) in polar coordinates in the plane \( \Pi_\alpha \) as in the case \(\alpha = 0\). If \((\rho'(\alpha), \varphi'(\alpha))\) is the image of \((\rho', \varphi') \in V_0\) then the null-bicharacteristic \( \gamma'_\alpha \) with initial data \((\rho'(\alpha), \varphi'(\alpha), \xi^{-}\rho'(\alpha) \xi_0, \xi_0, \xi_0, \xi_0)\) is the image of \( \gamma'(\rho', \varphi', \xi^{-}\rho'(\rho', \xi_0, \xi_0, \xi_0) \xi_0, \xi_0) \) under the transformation \((3.8)\).
The Cartesian coordinates of the initial point \((\rho', \varphi')\) in the plane \(x_3 = 0\) have the form
\[
y_1^{(0)} = \rho' \cos \varphi', \quad y_2^{(0)} = \rho' \sin \varphi', \quad y_3^{(0)} = 0.
\]

For any \(\alpha\) the image of \((y_1^{(0)}, y_2^{(0)}, y_3^{(0)})\) under the map (3.8) has the form
\[
y_1 = \rho' \cos \varphi' \cos \alpha, \\
y_2 = \rho' \sin \varphi, \\
y_3 = \rho' \cos \varphi \sin \alpha
\]
Note that the Jacobian of the map (3.9) is not zero when \((\rho', \varphi', \alpha)\subset V_0 \times (-\delta, \delta)\). Denote by \(U_0\) the image of \(V_0 \times (-\delta, \delta)\) under the map (3.9).

Let \(S_0^- (\rho', \varphi', \xi_0, \xi_\varphi, \alpha)\) be the function (3.4) where \((\rho', \varphi')\) are polar coordinates on the plane \(\Pi_\alpha\). Using the change of variable (3.9) denote by \(\tilde{S}_0^- (y, \xi_0, \xi_\varphi)\) the function \(S_0^- (\rho', \varphi', \xi_0, \xi_\varphi, \alpha)\) in Cartesian coordinates.

Let
\[
x = x(x_0, y)
\]
is the solution of the system of null-bicharacteristics with the initial data \(x(0, y) = y\). Since the Jacobian
\[
J(x_0) = \det \left[ \frac{\partial x_j}{\partial y_k} \right]_{j,k=1}^3
\]
is not zero on \([0, t_0), y \in U_0\), we have an inverse map
\[
y = y(x_0, x).
\]
As in [2], §64, one can show that
\[
S^-(x_0, x, \xi_0, \xi_\varphi) = \tilde{S}_0^- (y(x_0, x), \xi_0, \xi_\varphi)
\]
is the eikonal function for (3.1), i.e.
\[
\left( \frac{\partial S^-}{\partial x_0} \right)^2 - \sum_{k=1}^3 \left( \frac{\partial S^-}{\partial x_k} \right)^2 + (1 - f(r))( - \frac{\partial S^-}{\partial x_0} + \frac{\partial S^-}{\partial r} )^2 = 0.
\]

Geometric optics solution of (3.1) has the following form form on \([0, t_0 - \varepsilon]\) (cf., for example, [2], §64)
\[
u^-(x_0, x, k) = a^-_N (x_0, x, k) e^{ikS^-(x_0, x, \xi_0, \xi_\varphi)},
\]
where the eikonal $S^-(x_0, x, \xi_0, \xi_\varphi)$ satisfies (3.14), $k$ is the large parameter and

\begin{equation}
(3.16) \quad a_N^-(x_0, x, k) = a_0^-(x_0, x) + \frac{1}{k} a_1^-(x_0, x) + \ldots + \frac{1}{k^n} a_{N0}^-(x_0, k).
\end{equation}

Note that $a_{p_0}^-(x_0, x)$ satisfy some transport equations (cf. [2], §64). In particular,

\begin{equation}
(3.17) \quad H_{0\xi_0}(x, \frac{\partial S^-}{\partial x_0}) \frac{\partial a_0^-}{\partial x_0} + \sum_{j=1}^3 H_{0\xi_j}(x, \frac{\partial S^-}{\partial x_j}) \frac{\partial a_0^-}{\partial x_j} + \left( H_0(x, \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x}) S^- + f'(r) \left( - \frac{\partial S^-}{\partial x_0} + \frac{\partial S^-}{\partial r} \right) \right) a_0^- = 0.
\end{equation}

Here $H_0(x, \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x})$ is the principal part of (3.11), $a_0^-$ satisfies the initial condition

\begin{equation}
(3.18) \quad a_0^-(0, x) = \chi(x)
\end{equation}

where $\chi(x) \in C_0^\infty$, $\text{supp} \chi \subset U_0$. Note that $\text{supp} a_{p_0}^-(x_0, x)$, $0 \leq p \leq N$, is contained in a neighborhood of the null-geodesic $\gamma_0$.

The geometric optics solution has the form (3.15) until the Jacobian (3.11) is not zero. The set $\Sigma$ where (3.11) is zero is called the caustic set.

In our case the caustic set $\Sigma$ is not empty. Its intersection with $\Pi_\alpha$, $\alpha \in (-\delta, \delta)$, consists of the circle (2.13) for each $\alpha \in (-\delta, \delta)$.

Let $x = x(x_0, y)$ be the equation of null-geodesic starting at $y \in \text{supp} \chi$. Let $\hat{a}_0^-(x_0, y) = a_0^-(x_0, x(x_0, y))$. If we substitute $x = x(x_0, y)$ in the equation (3.17) we get an ordinary differential equation for $\hat{a}_0^-(x_0, y)$

\begin{equation}
(3.19) \quad \frac{d\hat{a}_0^-(x_0, y)}{dt} = M(x_0, y)\hat{a}_0^-(x_0, y)
\end{equation}

where

\begin{equation}
(3.20) \quad M(x_0, y) = H_0^{-1}(x(x_0, y), S_0^-, S_x^-) \left( H_0(x(x_0, y), \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x}) S^- - f'(r) \frac{\partial S^-}{\partial x_0} + f'(r) \frac{\partial S^-}{\partial r} \right).
\end{equation}

Let $(\rho', \varphi', \alpha)$ be the pre-image of $y = (y_1, y_2, y_3)$ under the map (3.9). Then the null-bicharacteristic $x = x(x_0, y)$ starting at $y$ is the null-bicharacteristic
\[ \gamma'(\rho', \varphi', \xi^-_\rho (\rho', \xi_0, \xi_\varphi), \xi_0, \xi_\varphi) \] in the plane \( \Pi_\alpha \). Therefore in \((\rho', \varphi')\) coordinates we have \( S^-_{x_0} = \xi_0, S^-_{\varphi} = \xi_{\varphi} \), where \( \xi_0, \xi_{\varphi} \) are constants and \( S^-_{\rho} = \xi^-_\rho = \frac{-(f-1)-\sqrt{\Delta}}{-f} \) (cf. \([2.6]\)). Hence substituting in \((3.20)\) we get on \([0, t_0 - \varepsilon]\)

\[ M = \frac{(-f)(-f \frac{\partial L}{\partial \rho} \xi^-_\rho + f'(\rho) \xi^-_\varphi - f'(\rho) \xi_0)}{-\xi_0 - (f-1)\sqrt{\Delta}}. \] (3.21)

Note that \( x_0 = t_0 \) is a caustic point. The geometric optics solution \( u^-_N \) is valid on the interval \([0, t_0 - \varepsilon]\).

When \( x_0 \in [t_0 - \varepsilon, t_0 + \varepsilon] \), i.e. in the neighborhood of the caustic point, one needs to modify the ansatz \((3.15)\) following Maslov theory (cf. \([6]\), and also \([2]\), §66).

We will look for modified geometric optics solution in the form

\[ u^{(0)}_N = \left( \frac{k}{2\pi} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} a^{(0)}_N(x_0, \eta, \varphi)e^{ik\eta - ikL(x_0, \eta, \varphi, \xi_0, \xi_\varphi)} d\eta, \] where \( L(x_0, \eta, \varphi, \xi_0, \xi_\varphi) \) satisfies an eikonal equation of the form

\[ H_0 \left( -\frac{\partial L}{\partial \eta}, \eta, \frac{\partial L}{\partial x_0}, \frac{\partial L}{\partial \varphi} \right) = 0, \] (3.23)

and

\[ a^{(0)}_N = \sum_{p=0}^{N} \frac{1}{k^p} a^{(0)}_{p0}, \] (3.24)

where \( a^{(0)}_{p0} \) is satisfying some transform equations (cf. \([2]\), §66 for details). Applying the stationary phase method to the integral \((3.22)\) at \( x_0 = t_0 - \varepsilon \) we get a stationary point \( \eta^{(0)} \) satisfying the equation \( \rho - \frac{\partial L(t_0 - \varepsilon, \eta^{(0)}, \varphi, \xi_0, \xi_\varphi)}{\partial \eta} = 0. \)

It can be shown that (see \([2]\), §66)

\[ \rho \eta^{(0)} - L(t_0 - \varepsilon, \eta^{(0)}, \varphi, \xi_0, \xi_\varphi) = S^-(t_0 - \varepsilon, \rho, \xi_0, \xi_\varphi), \] (3.25)

where \( S^-(x_0, x, \xi_0, \xi_\varphi) \) is the same as in \((3.13)\). Therefore one can adjust coefficients \( a^{(0)}_{p0} \) to have that

\[ u^-_N \bigg|_{x_0=t_0-\varepsilon} = u^{(0)} \bigg|_{x_0=t_0-\varepsilon} \] (3.26)
modulo lower order terms in $\frac{1}{k}$.

Analogously applying the stationary phase method to (3.22) at $x_0 = t_0 + \varepsilon$ we get modulo lower order terms in $\frac{1}{k}$ that

\begin{equation}
(3.27) \quad u_N^{(0)} \bigg|_{x_0 = t_0 + \varepsilon} = a^+_N e^{ikS^+(t_0 + \varepsilon, \rho, \varphi, \xi_0, \xi_\varphi)},
\end{equation}

where $S^+(x_0, x, \xi_0, \xi_\varphi)$ is the same as $S^-(x_0, x, \xi_0, \xi_\varphi)$ when $\xi^-_\rho$ is replaced by $\xi^+_\rho$. Thus for $x_0 > t_0 + \varepsilon$ we again are dealing with the geometric optics solutions of the form (3.15).

Substituting in $x_0 > t_0 + \varepsilon$ instead of $\xi^-_\rho$ and using (2.20) we get that $M$ has the form $M = O(e^{-C_1x_0})$ when $x_0 \to +\infty$. Therefore solving the ordinary differential equation (3.19) we get that $a^+_0(x_0, y)$ has a finite limit when $x_0 \to +\infty$ for any fixed $y \in U_0$.

We shall summarize the results of this section in the following theorem

**Theorem 3.1.** The geometric optics solution (3.15) has the support in a neighborhood of the null-geodesoc $\gamma_0$. It has the form (3.15) for $x_0 \leq t_0 - \varepsilon$ before approaching the caustic set. To continue the solution through the caustic set one needs to use the ansatz (3.22). After passing the caustic set one can transform modulo lower order terms in $\frac{1}{k}$ the ansatz (3.22) to the geometric optics ansatz $u_N^+ = a^+_N e^{ikS^+}$ similar to (3.15). When $x_0$ increases $\text{supp} a^+_N$ approaches the inner horizon $r = r_-$ and $\lim_{x_0 \to +\infty} a^+_0(x_0, y)$ exists for each $y \in U_0$.

**Remark 3.1** Let $(\rho^{(0)}, \varphi^{(0)})$ be any point in the plane $x_3 = 0$.

Assign initial conditions $\rho^{(0)}, \varphi^{(0)}, \xi^{(0)}_\rho, \xi^{(0)}_\varphi$ at $x_0 = 0$ for the null-bicharacteristic $\gamma^{(0)}$ where $\xi^{(0)}_\rho = \xi^-_\rho(\rho^{(0)}, \xi_0, \xi_\varphi)$ is the solution of the quadratic equation (2.6) with the initial “energy” $\xi_0 > 0$ large. As it was shown in § 2 $\gamma^{(0)}$ has a turning point $r_0$ when

\begin{equation}
(3.28) \quad \xi_0^2 - f(r_0) \left( \frac{\xi_\varphi}{r_0} \right)^2 = 0.
\end{equation}

When $\xi_0$ is large enough the turning point $r_0$ is small, i.e. close to the origin.

The null-bicharacteristic $\gamma^{(0)}$ exists for all $x_0 > 0$ and we can also construct a geometric optics type solution $u_N$ with the support in a neighborhood of $\gamma^{(0)}$ for all $x_0 > 0$. Therefore we can construct the solution of initial boundary value problem with the support as close to the origin $r = 0$ as we wish. Note that the equation (3.1) is not hyperbolic in a neighborhood of
\( r = 0 \). Thus not any initial-value problem has a solution.

**Remark 3.2**

The geometric optics solution (3.15) is only an approximate solution of the wave equation (3.1) since

\[
\Box_g u_N = t_N e^{ikS(x_0,x,\xi_0,\xi,\phi)},
\]

where \( t_N = O\left(\frac{1}{k}\right) \). It is possible to find the correction \( v_N \) such that

\[
\Box_g v_N = -t_N e^{ikS}
\]

. Then

\[
\Box_g (u_N + v_N) = 0.
\]

To find \( v_N \) one can use again geometric optics construction as in [2], §64 and then apply the Duhamel principle (cf. [4]).

### 4 Summary

Since the Reissner-Nordstrom metric is spherically symmetric any plane \( \Pi_0 \) passing through the origin contains null-geodesics. We fix such plane and study arbitrary null-geodesic \( \gamma_0 \) in this plane. The main feature of such null-geodesic is that it has a turning point inside the inner horizon. After passing the turning point the null-geodesic \( \gamma_0 \) approaches the inner horizon from the inside when \( x_0 \to +\infty \). In §3 we use a family of planes \( \Pi_\alpha, |\alpha| < \delta \), to construct a geometric optics solution \( u_N \) for the wave equation (1.3) (see (3.15)) that has the support in a neighborhood of the null-geodesic \( \gamma_0 \).

There is a complication in constructing the geometric optics solution in a neighborhood of the caustic set that requires to use the Maslov theory. We only sketch the construction refering the details to author’s book [2], §66.

After passing the caustic set the geometric optics solution again can be written in the form (3.15) and it tends to the sphere \( r = r_- \) when \( x_0 \to +\infty \).

### References

[1] V.Carloso, J.Costa, K.Destounin, A.Jansen and P.Hintz, Quasinormal modes end Strong Cosmic Censorship, Phys.Rev. Letters 120, 031103, 2018
[2] G.Eskin, Lectures on Linear Partial Differential Equations, AMS, GSM vol. 123 (2011)

[3] G.Eskin, Superradiance initiated inside the ergoregion, Rev. Math. Phys., vol. 28, No. 10 (2016) 1650025

[4] G.Eskin, The Cauchy problem for hyperbolic systems in convolution, Math. USSR - Sbornik vol. 3 (1967), No 2

[5] A.Hamilton and P.Avelino, The physics of the relativistic counter-streaming instability that drives mass inflation inside black holes, Physics Reports 495, 1-32

[6] V.Maslov and M.Fedoryuk, Semiclassical approximation in Quantum mechanics (Reidel, Dordrecht, 1981)

[7] O.Nordstrom, On the energy of gravitational field in Einstein’s theory, Acad., Amsterdam, 26, 1201-1208 (1918)

[8] E.Poisson and W.Israel, Phys. Rev. D 41, 1706 (1990)

[9] H.Reissner, Ann. Phys. 50, 106 (1916)

[10] M.Simpson and R.Penrose, Internal instability in a Reissner-Nordstrom black hole, Int. J. Theor. Phys. 7 (1973), 183-197

[11] M.Visser, The Kerr space-time: A brief introduction (Cambridge University Press, 2009), pp 3-37

[12] R.Wald, General Relativity, University of Chicago Press (1984)