Finite speed of propagation for the 2- and 3-dimensional multiplicative stochastic wave equation

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October 18, 2021

Abstract

We prove finite speed of propagation for the multiplicative stochastic wave equation in two and three dimensions which leads us to a global space-time well-posedness result for the cubic nonlinear equation in the analogue of the energy space.

1 Introduction

The aim of this paper is to solve the cubic multiplicative stochastic wave equation globally in space and time, which is formally

$$\partial_t^2 u - \Delta u - u \cdot \xi = -u^3 \text{ on } \mathbb{R}_+ \times \mathbb{R}^d$$

for $$d = 2, 3$$ and $$\xi$$ being spatial white noise, see Section 2 for a rigorous definition. This continues the investigation of this equation from [9] where the equation was shown to be globally well-posed in the periodic setting with data in the energy space. Moreover, Strichartz estimates were shown to hold in 2 dimensions in [19] (following the work [25] on Strichartz estimates on the Schrödinger analogue of the equation considered here). To be precise, the result from [9] is that until any time $$T > 0$$ one has a unique solution to

$$\partial_t^2 u - Hu = -u^3 \text{ on } [0, T] \times \mathbb{T}^d$$

with continuous dependence on the data. Here $$H$$ denotes the continuous Anderson Hamiltonian on $$\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$$, $$d = 2, 3$$, which is formally

$$H = \Delta + \xi - \infty$$

and $$\mathcal{D}(\sqrt{-H})$$ denotes its form domain, i.e. the functions $$u \in L^2(\mathbb{T}^d)$$ s.t.

$$|\langle u, Hu \rangle_{L^2}| < \infty.$$
finite speed of propagation for this kind of equation, meaning that the solution at a given space-time point will only depend on the “backward light cone” which is well-known for the classical wave equation, see e.g. [7]. This means that it is enough in some sense to solve “locally” which means that the unboundedness of the noise does not play as much of a role. Due to the presence of irregular objects, we use the approach due to Tartar [23] which was applied to somewhat similar situations in the more recent works [3] and [17].

Let us mention here also a couple of somewhat related recent papers: In [24] global space-time solutions to the 2 dimensional cubic additive stochastic nonlinear wave equation (first solved in the periodic setting in [8]) were constructed using the finite speed of propagation of the (classical) wave equation together with an argument based on the I-method; in [3] the two dimensional Schrödinger analogue of the equation considered here (first solved on the torus in [6]) was solved on the full space with some range of power nonlinearities.

We introduce some notation and conventions which will be used frequently throughout the paper. For the majority of the article we will be on the three dimensional euclidean space \( \mathbb{R}^3 \) (noting that the two dimensional case is analogous but simpler) meaning that we often omit it from our notation, meaning we write e.g.

\[
L^p = L^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{s.t.} \|f\|_{L^p} := \left( \int_{\mathbb{R}^d} |f|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}
\]

for the Lebesgue spaces (the case \( p = \infty \) having the usual modification), \( \mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d) \) for the Sobolev spaces and \( B^s_{p,q} = B^s_{p,q}(\mathbb{R}^d) \) for the Besov spaces, see the appendix for the definition of these spaces. Whenever this is not the case, e.g. if we state a result valid on the torus, we demark this explicitly. For the ball of radius \( r > 0 \) around \( x \in \mathbb{R}^3 \) we write \( B(x,r) := \{ y \in \mathbb{R}^3 : |y-x| \leq r \} \) and \( B(r) = B(0,r) \). In general we use the convention that generic constants may change from line to line, we also write \( \lesssim \Leftrightarrow \leq \) up to a generic constant and \( C(X) = C_X \) for a constant depending on the quantity \( X \) and we also frequently write things like

\[
\|f\|_{\mathcal{H}^s - \epsilon} \lesssim 1 \text{ for } \epsilon > 0 \text{ to mean that } \|f\|_{\mathcal{H}^s - \epsilon} \leq C(\epsilon) \text{ with } C(\epsilon) \uparrow \infty \text{ as } \epsilon \to 0.
\]

We also frequently write \( \langle \cdot, \cdot \rangle \) to mean a dual pairing without having to specify exactly in which spaces, for example for \( f \in \mathcal{H}^{-\epsilon} \) for \( \epsilon > 0 \) and \( g \in \mathcal{H}^1 \) we would write

\[
\langle f, g \rangle = \langle f, g \rangle_{\mathcal{H}^{-\epsilon}, \mathcal{H}^1}
\]

etc. In some cases it will be important exactly which pairing it is and then we will use the latter notation. As is customary, we also write \( \chi_A \) as the indicator function of the set \( A \subset \mathbb{R}^d \) i.e.

\[
\chi_A(y) = \begin{cases} 
1 & \text{if } y \in A \\
0 & \text{if } y \notin A
\end{cases}
\]

The paper is organised as follows:

Section 2.1 details how one transforms the equation using the exponential of a stochastic object as was done in [9] in the periodic setting and how it has to be modified on the whole space. In Section 2.2 we recall the localising operators from [10] and the existence and convergence properties of the stochastic objects appearing.

In Section 3 we solve a suitably renormalised and truncated version of (1.1) globally in space-time, which is analogous to how the periodic version (1.2) was solved globally in time in [9]. Thereafter, in Section 4, we prove finite speed of propagation for the linear multiplicative stochastic wave equation which is the main tool of how to globalise the solutions we constructed in Section 3. Our method is based on an approach due to Tartar [23], which we will also recall.

Lastly, in Section 5, we apply the results from Section 4 to the nonlinear equation (1.1) in order to get finite speed of propagation and consequently a global-in-time well-posedness result.
Acknowledgments

The author would like to thank Antoine Mouzard for some helpful comments and Massimiliano Guibinelli for pointing out an alternative approach sketched in Remark 4.

2 The exponential transformation and the noise terms

In [9] the authors used an exponential transform inspired by [6]—which in turn was inspired by [13]—in order to remove the worst part of the irregularity of the noise. It turns out that if one is content with constructing a form-domain (as we are here) this is sufficient, see Section 2.2 in [9]. In Section 2.1 we first recall this construction on the torus and how to extend it to the whole space, or at least a localised version of it. Then in Section 2.2 we recall some results about localising operators from [10] and the existence and regularity properties of the relevant stochastic objects.

2.1 The exponential transform on the torus vs. the whole space

We make a similar computation as in Section 2.2 of [9] in 3 dimensions with the 2 dimensional case being a bit simpler. Initially we recall the computation which works on the torus and then detail how one has to modify it in order to make it work on the whole space.

We start, using the same notation as in [9], with

$$\xi \in C^{-\frac{1}{2}-\varepsilon}(T^3), \varepsilon > 0$$ chosen very small, a spatial white noise on $T^3$, see the appendix for the definition if the Hölder-Besov spaces. Then we formally set (we abuse notation a bit here since we will have to slightly redefine some of the objects later when working on the whole space)

$$X := (1 - \Delta)^{-1} \xi (x) \in C^{\frac{1}{2}+\varepsilon}(T^3)$$ (2.1)

$$X^V := (1 - \Delta)^{-1} |\nabla X|^2 \in C^{1-\varepsilon}(T^3)$$ (2.2)

$$X^\Psi := 2(1 - \Delta)^{-1} \left( \nabla X \cdot \nabla X^\Psi \right) \in C^{\frac{3}{2}-\varepsilon}(T^3)$$ (2.3)

$$W := X + X^V + X^\Psi \in C^{\frac{3}{2}-\varepsilon}(T^3)$$ (2.4)

and make the following ansatz for the form domain of $\Delta + \xi$

$$u = e^W v = e^{X + X^V + X^\Psi} v,$$ (2.5)

where the regularity of $v$ will be specified later. We begin by computing formally

$$\Delta u + u \xi = e^W \left( \Delta v + \Delta W v + \nabla \left( X + X^V + X^\Psi \right) \right)^2 v + 2 \nabla W \cdot \nabla v + v \xi$$

$$= e^W \left( \Delta v + \left( |\nabla X|^2 + |\nabla X^V|^2 + 2 \nabla X \cdot \nabla X^\Psi + 2 \nabla X^V \cdot \nabla X^\Psi - X - X^V - X^\Psi \right) v + 2 \nabla W \cdot \nabla v \right),$$

so we see some cancellations happening from our choice of $W$. However, the right-hand side contains some terms which are undefined (in fact, even the term $|\nabla X|^2$ appearing in (2.2) is not defined), ultimately one is able to probabilistically give a rigorous meaning to

$$: |\nabla X|^2 : = \in C^{-\varepsilon}(T^3)$$

$$: |\nabla X^V|^2 : = \in C^{-\varepsilon}(T^3)$$

$$\text{and} \quad \nabla X \cdot \nabla X^\Psi \in C^{-\varepsilon}(T^3),$$

for $\varepsilon > 0$, where the colons denote Wick ordering which is a kind of renormalisation, see Theorem 1 for a rigorous statement. This is the origin of the formal $-\infty$ appearing in (1.3).
In this way, we can define the operator $\tilde{H}$ as the “correct” renormalised version of $\Delta + \xi$

$$\tilde{H}(e^W v) := e^W \left( \Delta v + \left( \left| \nabla X^\gamma \right|^2 + 2 \nabla X \cdot \nabla X^\gamma + 2 \nabla X^\gamma \cdot \nabla X^\gamma - W \right) v + 2 \nabla W \cdot \nabla v \right),$$

for $W := X + X^\gamma + X^\tilde{\gamma}$ and the “corrected” $X^\gamma := (1 - \Delta)^{-1} : |\nabla X|^2 :$

Now the observation in [9] is that for $v \in \mathcal{H}^1(\mathbb{T}^3)$, one actually has that

$$|(e^W v, \tilde{H}(e^W v))| < \infty \Leftrightarrow v \in \mathcal{H}^1(\mathbb{T}^3), \quad (2.6)$$

meaning that $e^W \mathcal{H}^1(\mathbb{T}^3)$ is the form domain of the operator $\tilde{H}$. In fact, by integrating by parts one sees

$$(e^W v, \tilde{H}(e^W v)) = (e^{2W} v, \Delta v + 2 \nabla W \cdot \nabla v + \tilde{Z} v)$$

and, since one can show $e^{2W} \tilde{Z} \in C^{-\frac{1}{2} - \varepsilon}(\mathbb{T}^3)$ (see Lemma 4.0 in [9]) one has that ($C_\Xi$ denotes a changing constant which depends only on the Hölder norms of the noise objects)

$$|\langle e^W v, \tilde{H}(e^W v) \rangle| \leq \|e^{2W} v\|_{L^\infty(\mathbb{T}^3)} \|\nabla v\|_{L^2(\mathbb{T}^3)} + \|e^{2W} \tilde{Z} v\|_{C^{-\frac{1}{2} - \varepsilon}(\mathbb{T}^3)} \|v\|_{B^0_{1,2}(\mathbb{T}^3)}$$

having used Besov duality, Lemma 4 and interpolation/Young’s inequality as well as the fact that both $e^{\pm W}$ are bounded in $L^\infty$. Analogously one can bound the $\mathcal{H}^1(\mathbb{T}^3)$ norm of $v$ by using

$$(\nabla v, \nabla v)_{L^2(\mathbb{T}^3)} + (v, v)_{L^2(\mathbb{T}^3)} \leq C_{\Xi} (e^W \nabla v, e^W \nabla v)_{L^2(\mathbb{T}^3)} + (e^W v, e^W v)_{L^2(\mathbb{T}^3)}$$

and proceeding as before to bound the term containing $\tilde{Z}$, thus one has shown (2.6) and even norm equivalence.

Now we want to adapt the same approach to the whole space $\mathbb{R}^3$. The thing that makes the problem on the whole space more difficult is that all the quantities are unbounded, i.e. live only in weighted Hölder spaces (see Definition 2 for the definition of weighted Besov spaces of which Hölder spaces are a particular case and Section 2.2 for a discussion of how to define the stochastic objects).

In particular it is not good to have unbounded terms inside the exponential, since we later want to use that the exponential and its inverse are both bounded in $L^\infty$. By making use of the localising operators from [10] whose definition and properties we recall in Section 2.2, we modify (2.6) in the following way

$$u = e^{W_\sigma} v, \quad (2.7)$$

where $W_\sigma$ can be thought of capturing the high frequencies and analogously to the periodic case the stochastic terms are

$$X := (1 - \Delta)^{-1} \xi(x) \in C_\xi^\frac{1}{2}$$

$$X^\gamma := (1 - \Delta)^{-1} : |\nabla X|^2 : \in C_\xi^{\frac{1}{2} - \varepsilon}$$

$$X^\tilde{\gamma} := 2(1 - \Delta)^{-1} \left( \nabla X \cdot \nabla X^\gamma \right) \in C_\xi^{\frac{3}{2} - \varepsilon}$$

$$W := X + X^\gamma + X^\tilde{\gamma} \in C_\xi^{\frac{1}{2} - \varepsilon}$$

for suitable $\sigma > 0$, see Proposition 11.
and we redo the above formal computation instead with the ansatz (2.7)

\[
\Delta u + u\xi = e^{W>}(\Delta W_{>\xi} v + \nabla W_{>\xi}^2 v + \Delta v + 2\nabla W_{>\xi} \cdot \nabla v + v\xi)
\]

\[
= e^{W>}(\Delta W - \Delta W_{\xi}) v + \nabla W - \nabla W_{\xi}^2 v + \Delta v + 2\nabla W_{>\xi} \cdot \nabla v + v\xi
\]

\[
= e^{W>}
\left((W - \Delta W_{\xi} - \xi : |\nabla X|^2 : - 2\nabla X \cdot \nabla X^\xi + |\nabla W|^2 - 2\nabla W \cdot \nabla W_{\xi} + \nabla W_{\xi}^2 + \xi v + \Delta v + 2\nabla W_{>\xi} \cdot \nabla v)\right)
\]

which leads to the formal definition (the difference is that we replace the ill-defined squares by their Wick ordered versions)

\[
H(e^{W>} v) = e^{W>}(\Delta v + Z_{>\xi} v + \nabla W_{>\xi} \cdot \nabla v + v\xi) = e^{W>}(\Delta v + Z_{>\xi} v + Z_{<\xi} v + 2\nabla W_{>\xi} \cdot \nabla v)
\]

having defined

\[
Z := W - \Delta W_{\xi} + \left(\nabla X^\xi \right)^2 + \left(\nabla X^\xi \right)^2 + 2\nabla X \cdot \nabla X^\xi + 2\nabla X^\xi \cdot \nabla X^\xi - 2\nabla W \cdot \nabla W_{\xi} + |\nabla W_{\xi}|^2, \quad (2.13)
\]

which we have split as

\[
\mathcal{C}^{-\frac{1}{2}} e^{-\varepsilon} \ni Z = \mathcal{Z}_{\gamma} + \mathcal{Z}_{\xi} + \mathcal{Z}_{\xi}, \text{ for } \gamma' > \gamma > 0 \text{ dictated by Proposition 11.}
\]

We have thus split our formal operator \(\Delta + \xi\) into a rough operator whose treatment is analogous to the periodic case in [9] and an unbounded but regular part. Since we will be interested in finite speed of propagation which is a local concept, we will consider truncated operators which are defined rigorously on functions of the form (2.7) as follows.

**Definition 1** We define the following operators for smooth functions \(v\) and \(R > 0\)

\[
H_R(e^{W>} v) := e^{W>}(\Delta v + Z_{>\xi} v + \chi_{B(R)} Z_{\xi} v + 2\nabla W_{>\xi} \cdot \nabla v)
\]

\[
H_{>\xi}(e^{W>} v) := e^{W>}(\Delta v + Z_{>\xi} v + 2\nabla W_{>\xi} \cdot \nabla v)
\]

and

\[
H_{\ll\xi}(e^{W>} v) := e^{W>}(\Delta v + Z_{>\xi} v - C_{\ll\xi}(\Xi)v + 2\nabla W_{>\xi} \cdot \nabla v)
\]

\[
= H_R(e^{W>\xi} v) - C_{\ll\xi}(\Xi)e^{W>\xi} v
\]

\[
= H_{R}(e^{W>\xi} v) - \chi_{B(0,R)} Z_{\xi} v - C_{\ll\xi}(\Xi)e^{W>\xi} v
\]

where the constant \(C_{\ll\xi}(\Xi) > 0\) is chosen depending on the norms of the noise terms s.t.

\[
-(e^{W>\xi} v, H_{\ll\xi}(e^{W>\xi} v)) \geq \|e^{W>\xi} v\|^2_{L^2},
\]

this is similar to Proposition 2.53 in [9].

**Lemma 1** The form domain of all three operators in Definition 1 is \(e^{W>\xi}H^1\) and we have the bounds (note that because of (2.19) \(H_{\ll\xi}\) is the only operator that has a definite sign)

\[
|\langle e^{W>\xi} v, H_{>\xi}(e^{W>\xi} v)\rangle| \leq -(e^{W>\xi} v, H_{>\xi}(e^{W>\xi} v)) \leq -(e^{W>\xi} v, H_{>\xi}(e^{W>\xi} v)) + C_{\ll\xi}(\Xi)\|e^{W>\xi} v\|^2_{L^2}
\]

and

\[
|\langle e^{W>\xi} v, H_{R}(e^{W>\xi} v)\rangle| \leq -(2e^{W>\xi} v, H_{R}(e^{W>\xi} v)) + C(R, \Xi)\|e^{W>\xi} v\|^2_{L^2}
\]

\[
-(e^{W>\xi} v, H_{>\xi}(e^{W>\xi} v)) \leq -2(e^{W>\xi} v, H_{>\xi}(e^{W>\xi} v)) + C(R, \Xi)\|e^{W>\xi} v\|^2_{L^2},
\]

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where \( C_\gg(\Xi) > 0 \) is the constant from Definition I and \( C(R, \Xi) > 0 \) is a constant that may depend polynomially on \( R \).

**Proof** This follows from the definitions of the operators as well as the previous discussion on how to bound the \( Z \) term as well as the bound

\[
\|\chi_{B(R)}Z_\xi v\|_{L^2} \leq \|v\|_{L^2} \|\chi_{B(R)}Z_\xi\|_{L^\infty} \leq \|v\|_{L^2} R^\rho \|Z_\xi\|_{L^\infty_{(-\rho)}}; \text{ for } \rho > 0.
\]

which is where the polynomial dependence on \( R \) comes from. \( \Box \)

For the sake of completeness, we give the analogous statements in the two-dimensional setting. It is enough to consider one renormalised quantity, namely we have for the stochastic objects, \( \xi^{(2)} \in C^{0-\varepsilon}((\mathbb{R}^2) \) being the spatial white noise on \( \mathbb{R}^2 \),

\[
X^{(2)} := (1 - \Delta)^{-1}\xi^{(2)} \in C^{1-\varepsilon}_{(-\delta)}(\mathbb{R}^2),
\]

\[
|\nabla X^{(2)}|^2 : \in C^{0-\varepsilon}_{(-\delta)}(\mathbb{R}^2),
\]

for \( \varepsilon, \delta > 0 \), which leads to the exponential ansatz \( u = e^{X^{(2)}_> v} \) with \( v \in H^1(\mathbb{R}^2) \) for which one formally has

\[
\Delta u + u\xi = e^{X^{(2)}_> v} \left( \Delta X^{(2)}_> v + |\nabla X^{(2)}_>|^2 v + \Delta v + 2\nabla X^{(2)}_> \cdot \nabla v + \nu(2) \right)
\]

\[
= e^{X^{(2)}_> v} \left( \left( \Delta X^{(2)}_> - \Delta X^{(2)}_\xi \right) v + |\nabla X^{(2)}_> - \nabla X^{(2)}_\xi|^2 v + \Delta v + 2\nabla X^{(2)}_> \cdot \nabla v + \nu(2) \right)
\]

\[
= e^{X^{(2)}_> v} \left( (X^{(2)}_> - \Delta X^{(2)}_\xi) + |\nabla X^{(2)}_>|^2 - 2\nabla X^{(2)} \nabla X^{(2)}_\xi + |\nabla X^{(2)}_\xi|^2 v + \Delta v + 2\nabla X^{(2)}_> \cdot \nabla v \right)
\]

leading to the rigorous definition

\[
H^{(2)}(e^{X^{(2)}_> v}) := e^{X^{(2)}_> (Z^{(2)}_> v + \Delta v + 2\nabla X^{(2)}_> \cdot \nabla v)},
\]

having defined

\[
Z^{(2)} := (X^{(2)}_> - \Delta X^{(2)}_\xi) + |\nabla X^{(2)}_>|^2 - 2\nabla X^{(2)}_> \nabla X^{(2)}_\xi + |\nabla X^{(2)}_\xi|^2 \quad \in C^{0-\varepsilon}(\mathbb{R}^2),
\]

and some \( \gamma > 0 \) dictated by Proposition II. This leads to the truncated operator

\[
H^{(2)}_R(e^{X^{(2)}_> v}) := e^{X^{(2)}_> (Z^{(2)}_> v + \chi_{B(R)}Z^{(2)}_\xi + \Delta v + 2\nabla X^{(2)}_> \cdot \nabla v)},
\]

and the uniformly positive operator

\[
H^{(2)}_e(e^{X^{(2)}_> v}) := e^{X^{(2)}_> (\Delta v + X^{(2)}_> v - C_\gg(\Xi^{(2)}) v + 2\nabla X^{(2)}_> \cdot \nabla v)}, \quad (2.20)
\]

where \( C_\gg(\Xi^{(2)}) > 0 \) is a constant depending on the norms of the noise terms s.t.

\[
\|e^{X^{(2)}_> v}\|_{L^2(\mathbb{R}^2)} \leq -(e^{X^{(2)}_> v}, H^{(2)}_e(e^{X^{(2)}_> v})).
\]

### 2.2 The localising operators and the stochastic terms

Due to the unbounded nature of \( \xi \) on \( \mathbb{R}^d \), namely it lives only in a weighted space \( C^{0-\varepsilon}_{(-\delta)} \) – see Definition II for the definition of (weighted) Hölder-Besov spaces, we do not proceed directly as on the torus. Instead we use the decomposition from Gubinelli-Hofmanova I of the noise terms into two parts

\[
C_{(-\delta)}^{0-\varepsilon} \ni \Xi = \Xi_\ell + \Xi_\gg \quad \text{where} \quad \Xi_\ell \in L^\infty_{(-\delta'; \sigma')} \text{ and } \Xi_\gg \in C^{0-\sigma'}, \text{for } \sigma' > \sigma > 0 \text{ and some } \delta' > \delta > 0
\]

i.e. we obtain a part \( \Xi_\ell \) which is regular but unbounded and a part \( \Xi_\gg \) which is irregular but “bounded”. We recall the localisation operators \( U_\gg \) and \( U_\ell \) from II defined as

\[
U_\gg f := \sum_k w_k \Delta_\gg L_k f \quad \text{and} \quad U_\ell f := \sum_k w_k \Delta_\ell L_k f,
\]

where \( \{w_k\} \) is a smooth dyadic partition of unity and \( \Delta_\gg L_k \) and \( \Delta_\ell L_k \) are projections on frequencies higher or lower than \( L_k \) respectively. The following is a result from II which we will apply liberally.
Proposition 1 (Localisation operators, [10]) Let $L > 0$, then there exists a choice of parameters $(L_k)$ such that for all $\alpha, \delta, \gamma > 0$ and $a, b \in \mathbb{R}$

$$
\|U_\varepsilon f\|_{C^{\alpha-\delta}} \lesssim 2^{-\delta L} \|f\|_{C^{\alpha-\delta}} \quad \text{and} \quad \|U_\varepsilon f\|_{C^{\alpha-b+\gamma}} \lesssim 2^{\gamma L} \|f\|_{C^{\alpha-b+\gamma}}
$$

We give two remarks that one should keep in mind.

Remark 1 Note that the above result is stated only for $f$ with strictly negative regularity, however we sometimes apply it to stochastic terms with positive regularity (e.g. to $W$ in (2.11)), meaning of course that the decomposition is actually

$$
W = W_\geq + W_\leq
$$

with

$$
W_\geq = (1 - \Delta)^{-1}U_\varepsilon((1 - \Delta)W) \quad \text{and} \quad W_\leq = (1 - \Delta)^{-1}U_\varepsilon((1 - \Delta)W)
$$

which has the desired properties.

Remark 2 The decomposition from Proposition 1 clearly depends on the precise choice of the sequence $(L_k)$ and thus one might wonder whether objects like the truncated operators in Definition 1 are actually well-defined. While changing the sequence $(L_k)$ of course changes the objects appearing in the truncated operators, importantly the form domain $L_\varepsilon$ of size $M > 0$ respectively. Let

$$
X^{(d)} := (1 - \Delta)^{-1}\xi^{(d)}
$$

$$
\xi^{(d)} = \xi^{(d)} * \eta^{(d)}
$$

$$
X_\varepsilon^{(d)} := (1 - \Delta)^{-1}\xi_\varepsilon^{(d)}
$$

$$
X_M^{(d)} := (1 - \Delta)^{-1}\xi_M^{(d)}
$$

$$
\xi_M^{(d)} = \xi_M^{(d)} * \eta^{(d)}
$$

$$
X_\varepsilon^{(d)} := (1 - \Delta)^{-1}\xi_\varepsilon^{(d)}
$$

$$
X_M^{(d)} := (1 - \Delta)^{-1}\xi_M^{(d)}
$$

for some smoothing kernels $\eta^{(d)}$.

i. In the case $d = 2$ there exist random distributions $: |\nabla X^{(2)}_M| :$ and $: |\nabla X^{(2)}| :$ s.t.

$$
|\nabla X^{(2)}_M|_2^2 - a^{(2)}_M \rightarrow |\nabla X^{(2)}_M|_2^2 : \quad \text{a.s. in } C^{-\delta}(\mathbb{T}^2_M) \text{ as } \varepsilon \rightarrow 0
$$

$$
|\nabla X^{(2)}_\varepsilon|_2^2 - a^{(2)}_\varepsilon \rightarrow |\nabla X^{(2)}|_2^2 : \quad \text{a.s. in } C_{(\cdot)-\sigma}(\mathbb{R}^2) \text{ as } \varepsilon \rightarrow 0
$$

and

$$
\nabla X_M \rightarrow \nabla X \text{ and } |\nabla X^{(2)}_M|^2 \rightarrow |\nabla X^{(2)}|_2^2 : \quad \text{a.s. in } C^{-\delta}_{(\cdot)-\sigma}(\mathbb{R}^2) \text{ as } M \rightarrow \infty,
$$

for $\delta, \sigma > 0$ and suitably chosen diverging constants $a^{(2)}_M, a^{(2)}_\varepsilon$.

ii. In the case $d = 3$ there exist random distributions

$$
X^{(3)}_M, X^{(3)}_\varepsilon, : |\nabla X^{(3)}_M|_2^2, : |\nabla X^{(3)}_\varepsilon|_2^2, : |\nabla X^{(3)}_M|_2^2, : |\nabla X^{(3)}_\varepsilon|_2^2
$$

$$
\nabla X^{(3)}_M \rightarrow \nabla X \text{ and } |\nabla X^{(3)}_M|^2 \rightarrow |\nabla X^{(3)}|_2^2 : \quad \text{a.s. in } C_{(\cdot)-\sigma}^{\delta}(\mathbb{R}^3) \text{ as } M \rightarrow \infty,
$$

for $\delta, \sigma > 0$ and suitably chosen diverging constants $a^{(2)}_M, a^{(2)}_\varepsilon$. 

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Lastly we make a simple observation that for functions in weighted\(\L^\rho\) spaces, say \(f \in \L^\rho\) for \(\rho > 0\), one has the following bound for the product with an indicator function of a ball
\[
\|\chi_{B(R)}f\|_{\L^\rho} \leq \|\chi_{B(R)}\|_{\L^\infty} \|f\|_{\L^\rho} \leq R^\rho \|f\|_{\L^\rho}.
\] (2.22)

This allows us to localise the “bulk” terms at the cost of gaining a large constant and will be quite useful later in Gronwall-type arguments.
3 Global space-time solutions for the equation with truncated noise

In this section we prove global-in-time well-posedness for the truncated wave equation

\[
\begin{align*}
\partial_t^2 u(R) - H_R u(R) &= -u(R)|u(R)|^2 \quad \text{on} \quad [0, T] \times \mathbb{R}^3 \\
(u(R), \partial_t u(R)) &= (u_0(R), u_1(R)),
\end{align*}
\]

for \( T > 0 \), where \((u_0(R), u_1(R)) \in D(\sqrt{-H_R}) \times L^2 \). Recall the operators \( H_R \) and \( H_{>0} \) defined in Definition \( \text{(1)} \) and the fact that \( D(\sqrt{-H_{>0}}) = e^{W \cdot H^1} \) as well as the relevant bounds for \( u(R) \in e^{W \cdot H^1} \)

\[
\begin{cases}
\|u(R)\|_{L^2}^2 \leq -2(u(R), H_{>0} u(R)) \leq -2(u(R), H_R u(R)) + C(R, \Xi)\|u(R)\|_{L^2}^2
\end{cases}
\]

from Lemma \( \text{(1)} \).

We firstly observe that the PDE \( (3.1) \) admits a conserved energy (recall that \( H_R \) is self-adjoint) denoted by

\[
E^{(R)}(u(R))(t) := \frac{1}{2}(\partial_t u(R)(t), \partial_t u(R)(t))_{L^2} - \frac{1}{2}(u(R)(t), H_R u(R)(t))_{L^2} + \frac{1}{4} \int_{\mathbb{R}^3} |u(R)(t, x)|^4 dx
\]

\[
= E^{(R)}(u(R))(0)
= E^{(R)}((u_0(R), u_1(R))
:= \frac{1}{2}(u_1(R), u_1(R))_{L^2} - \frac{1}{2}(u_0(R), H_R u_0(R))_{L^2} + \frac{1}{4} \int_{\mathbb{R}^3} |u_0(R)(x)|^4 dx,
\]

see Section 3.3 in \( \text{(9)} \) for a rigorous justification.

Inspired by \( \text{(3.2)} \) we define the “rough part” of the energy to be

\[
E_{>0}(u(R))(t) := \frac{1}{2}(\partial_t u(R)(t), \partial_t u(R)(t))_{L^2} - \frac{1}{2}(u(R)(t), H_{>0} u(R)(t))_{L^2} + \frac{1}{4} \int_{\mathbb{R}^3} |u(R)(t, x)|^4 dx
\]

\[
\overset{\text{(3.3)}}{=} E^{(R)}(u(R))(t) + \frac{1}{2}(u(R)(t), (C_{>0}(R, \Xi) + \chi_{B(R)} Z_{<0}) u(R)(t))_{L^2}
\]

But, importantly, it is positive and controls the energy norm

\[
\|\partial_t u(R)\|_{L^2} + \left\| \sqrt{-H_{>0}} u(R) \right\|_{L^2}
\]

uniformly in time. Evidently \( E_{>0}(u(R)) \) will not be conserved in time, however we get

\[
\frac{d}{dt} E_{>0}(u(R))(t) = \frac{d}{dt} E^{(R)}(u(R))(t) + \frac{1}{2} (\partial_t u(R)(t), \Xi_{<0} u(R)(t))_{L^2}
\]

\[
\overset{\text{(3.2)}}{=} (\partial_t u(R)(t), \Xi_{<0} u(R)(t))_{L^2}
| \ldots | \leq \|\Xi_{<0}\|_{L^\infty} \|u(R)(t)\|_{L^2} \|\partial_t u(R)(t)\|_{L^2}
\leq \frac{1}{2} \|\Xi_{<0}\|_{L^\infty}(\|u(R)(t)\|^2_{L^2} + \|\partial_t u(R)(t)\|^2_{L^2})
\leq C \|\Xi_{<0}\|_{L^\infty} E^{(R)}(u(R))(t),
\]

for some universal constant \( C > 0 \) having used \( \text{(3.2)} \) and \( \text{(3.5)} \) in the last step.

Thus we get an exponential bound for all times by Gronwall, namely

\[
E_{>0}(u(R)(t)) \leq e^{\tilde{C}(\Xi, R)} E_{>0}(u_0(R), u_1(R))
\]

for some constant \( \tilde{C}(\Xi, R) > 0 \), recalling that by \( \text{(2.22)} \) the norm \( \|\Xi_{<0}\|_{L^\infty} \) grows polynomially in \( R \). Clearly this bound blows up if we take \( R \to \infty \) but for finite \( R \) we will see that this is enough to get global-in-time solutions to \( (3.1) \).
Theorem 2 For any $R > 0$ the equation (3.1) is globally well-posed. More precisely, for any $T > 0$ and initial data $(u_0^{(R)}, u_1^{(R)}) \in D \left( \sqrt{-H_{\gg}} \right) \times L^2$ there exists a unique solution to

$$u^{(R)}(t) = \cos(t \sqrt{-H_{\gg}}) u_0^{(R)} + \frac{\sin(t \sqrt{-H_{\gg}})}{-H_{\gg}} u_1^{(R)} + \int_0^t \frac{\sin((t-s) \sqrt{-H_{\gg}})}{\sqrt{H_{\gg}}} (u^{(R)}(s) + u^{(R)}(s) \Xi^{R}_{\ll}) \, ds$$

in $C_{[0,T]} D \left( \sqrt{-H_{\gg}} \right) \cap C^1_{[0,T]} L^2$ which depends continuously on the data.

Proof First of all, the fact that (3.7) is the mild formulation of (3.1) follows simply by recalling that $H_R = H_{\gg} + \Xi^{R}_{\ll}$ and putting the linear term into the nonlinearity of the mild formulation.

Next, we define the operator

$$\Psi(w)(t) := \cos\left(t \sqrt{-H_{\gg}}\right) u_0^{(R)} + \frac{\sin\left(t \sqrt{-H_{\gg}}\right)}{-H_{\gg}} u_1^{(R)} + \int_0^t \frac{\sin\left((t-s) \sqrt{-H_{\gg}}\right)}{\sqrt{H_{\gg}}} (w^3(s) + w(s) \Xi^{R}_{\ll}) \, ds,$$

for which we have the straightforward bounds

$$\|\sqrt{-H_{\gg}} \Psi(w)(t)\|_{L^2} \leq \|\sqrt{-H_{\gg}} u_0^{(R)}\|_{L^2} + \|u_1^{(R)}\|_{L^2} + \int_0^t \|w(s)\|_{L^5} + \|\Xi^{R}_{\ll}\|_{L^\infty} \|w(s)\|_{L^2} \, ds$$

$$\lesssim \|(u_0^{(R)}, u_1^{(R)})\|_{D(\sqrt{-H_{\gg}}) \times L^2} + \int_0^t \|\sqrt{-H_{\gg}} w(s)\|_{L^2}^3 \, ds + t \|\Xi^{R}_{\ll}\|_{L^\infty} \|w\|_{L^2[0,t]}$$

$$\lesssim \|(u_0^{(R)}, u_1^{(R)})\|_{D(\sqrt{-H_{\gg}}) \times L^2} + t \|\sqrt{-H_{\gg}} w(s)\|_{L^3[0,t]}^3 + t \|\Xi^{R}_{\ll}\|_{L^\infty} \|w\|_{L^2[0,t]}$$

having used the embedding

$$D(\sqrt{-H_{\gg}}) \hookrightarrow L^6,$$

which simply follows by noting that for $u = e^{W_{\ll}} v$, $v \in H^1$, we have

$$\|u\|_{L^6} \leq \|e^{W_{\ll}}\|_{L^\infty} \|v\|_{L^6} \lesssim \|v\|_{H^1} \lesssim \|u\|_{D(\sqrt{-H_{\gg}})}.$$

Thus we may bound, using our almost-conserved energy $E^{(R)}_{\gg}$,

$$\sup_{0 \leq t \leq T^*} \|\sqrt{-H_{\gg}} \Psi(w)(t)\|_{L^2} \lesssim \|(u_0^{(R)}, u_1^{(R)})\|_{D(\sqrt{-H_{\gg}}) \times L^2} +$$

$$+ T^* e^{\tilde{C}(\Xi^{R}) T^*} \left( (E^{(R)}_{\gg}(u_0^{(R)}, u_1^{(R)}))^\frac{3}{2} + \|\Xi^{R}_{\ll}\|_{L^\infty} (E^{(R)}_{\gg}(u_0^{(R)}, u_1^{(R)}))^\frac{1}{2} \right)$$

and analogously

$$\|\partial_t \Psi(w)(t)\|_{L^2} \lesssim \|(u_0^{(R)}, u_1^{(R)})\|_{D(\sqrt{-H_{\gg}}) \times L^2} +$$

$$+ \left\| \frac{d}{dt} \int_0^t \sin\left((t-s) \sqrt{-H_{\gg}}\right) (w^3(s) + w(s) \Xi^{R}_{\ll}) \, ds \right\|_{L^2}$$

$$\lesssim \|(u_0^{(R)}, u_1^{(R)})\|_{D(\sqrt{-H_{\gg}}) \times L^2} + \left\| \int_0^t \cos\left((t-s) \sqrt{-H_{\gg}}\right) w^3(s) \, ds \right\|_{L^2}$$

$$\lesssim \|(u_0^{(R)}, u_1^{(R)})\|_{D(\sqrt{-H_{\gg}}) \times L^2} + \int_0^t \|w(s)\|_{L^5}^3 + \|\Xi^{R}_{\ll}\|_{L^\infty} \|w(s)\|_{L^2} \, ds.$$
and a time horizon

\[ T^* = T^*((E_{\gg})(u^{(R)}_0, u^{(R)}_1)), \|\Xi^R\|_{L^\infty}) \]

for which we have

\[
\left\| \sqrt{-H_{\gg}}(\Psi(w) - \Psi(v))(t) \right\|_{L^2} = \int_0^t \sin \left( (t - s)\sqrt{-H_{\gg}} \right) (w^3(s) - v^3(s) + (w(s) - v(s))\Xi^R_s) \right\|_{L^2} \\
\leq CT^* \|\Xi^R_s\|_{L^\infty} \|w - v\|_{L^\infty_{[0,T^*]} L^2} + T^* \|w^3 - v^3\|_{L^\infty_{[0,T^*]} L^2} \\
\leq CT^* \|\Xi^R_s\|_{L^\infty} \|w - v\|_{L^\infty_{[0,T^*]} L^2} + T^* e^{C(\Xi, R)T^*} \|w - v\|_{L^\infty_{[0,T^*]} D(\sqrt{-H_{\gg}})} E_{\gg}(u^{(R)}_0, u^{(R)}_1)) \\
\leq \frac{1}{3} \|w - v\|_{L^\infty_{[0,T^*]} D(\sqrt{-H_{\gg}})}
\]

for \( w, v \) in the ball of radius \( M \) in the space \( C_{[0,T^*]} D(\sqrt{-H_{\gg}}) \cap C^1_{[0,T^*]} L^2 \) and \( 0 < t \leq T^* \). Here we have used

\[
\|w^3(s) - v^3(s)\|_{L^2} = \|(w(s) - v(s))(w^3(s) + w(s)v(s) + v^3(s))\|_{L^2} \\
\leq 2\|w(s) - v(s)\|_{L^6}(\|w(s)\|_{L^6}^2 + \|v(s)\|_{L^6}^2) \\
\leq C\|w - v\|_{L^\infty_{[0,T^*]} D(\sqrt{-H_{\gg}})} e^{C(\Xi, R)T^*} E_{\gg}(u^{(R)}_0, u^{(R)}_1)).
\]

Analogously we prove

\[
\|\partial_t (\Psi(w) - \Psi(v))(t)\|_{L^2} \leq \frac{1}{3} \|w - v\|_{L^\infty_{[0,T^*]} D(\sqrt{-H_{\gg}})}
\]

for \( w, v \) and \( t \) as above.

This gives us a solution to \( (3.7) \) up to time \( T^* \) which lies in \( L^\infty_{[0,T^*]} D(\sqrt{-H_{\gg}}) \cap W^{1,\infty}_{[0,T^*]} L^2 \). In addition, Stone’s theorem (see Theorem VIII.7 in [20]) implies that the solution is even continuous in time and its derivative is continuous in \( L^2 \).

Lastly we want to globalise this solution, which essentially means that we want to resolve the equation on intervals of length \( T^* \) in order to obtain a solution in the entire interval \([0, T]\). In order to do that we need to bound the norm of the solution at time \( t \) by the initial data and the final time \( T \). In fact we get for a solution \( u \) the bound

\[
\left\| \sqrt{-H_{\gg}} u(t) \right\|_{L^2} + \|\partial_t u(t)\|_{L^2} \leq K \left( \|u_0, u_1\|_{D(\sqrt{-H_{\gg}}) \times L^2} \right) + T e^{C(\Xi, R)T} L(E_{\gg}(u^{(R)}_0, u^{(R)}_1), \|\Xi^R\|_{L^\infty})
\]

simply by proceeding as above, here \( K \) and \( L \) denote some constants depending on the data polynomially. This bound allows us to choose a global \( M \) in our fixed point procedure in which we also have a global \( T^* \).

This implies that we can restart the solution until the final time \( T > 0 \), which concludes the proof. \( \square \)

4 Finite speed of propagation

We begin this section by giving first of all the classical proof of the finite speed of propagation for the wave equation which can be found for example in [7]. As we shall see it is not clear whether it can be adapted to our situation, however it turns out that we can adapt a modified approach which goes back to Tartar [23] which we will briefly review.

Consider first the classical linear wave equation (in three dimensions for definiteness)

\[
\partial_t^2 v - \Delta v = 0 \\
(v, \partial_t v)|_{t=0} = (v_0, v_1) \in \mathcal{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),
\]

which has the conserved energy

\[
E(v) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t v|^2 + |\nabla v|^2.
\]
Furthermore, for a space-time point \((t, x)\) we consider the \textbf{backward light cone}

\[
\mathcal{C}_{(t,x)} := \{(s, y) \in \mathbb{R}^3 : 0 \leq s \leq t \text{ and } |y - x| \leq t - s\}.
\]

Now, finite speed of propagation means that the solution \(v\) at the space-time point \((t, x)\) depends \textit{only} on the backward light cone \(\mathcal{C}_{(t,x)}\). In order to make this more quantitative, we define the \textit{local energy}

\[
e_{(t,x)}(s) := \frac{1}{2} \int_{B(x,t-s)} |\partial_t v(s, y)|^2 + |\nabla v(s,y)|^2 dy.
\]  

(4.1)

A simple computation yields

\[
\frac{d}{ds} e_{(t,x)}(s) = -\frac{1}{2} \int_{\partial B(x,t-s)} |\partial_t v(s, y)|^2 + |\nabla v(s,y)|^2 dy + \int_{\partial B(x,t-s)} \partial_t^2 v(s,y)\partial_t v(s,y) + \nabla \partial_t v(s,y)\nabla v(s,y) dy
\]

\[
= -\frac{1}{2} \int_{\partial B(x,t-s)} |\partial_t v(s, y)|^2 + |\nabla v(s,y)|^2 dy + \int_{\partial B(x,t-s)} \partial_t v(s,y)\nabla v(s,y) dy +
\]

\[
+ \int_{B(x,t-s)} \partial_t v(s,y) \left( \partial_t^2 v(s,y) - \Delta v(s,y) \right) dy
\]

\[
\leq -\frac{1}{2} \int_{\partial B(x,t-s)} |\partial_t v(s, y)|^2 + |\nabla v(s,y)|^2 dy + \frac{1}{2} \int_{\partial B(x,t-s)} |\partial_t v(s,y)|^2 + |\nabla v(s,y)|^2 dy
\]

\[=0,
\]

where we have integrated by parts and used Young’s inequality in the second and third step respectively.

Thus we have for \(0 < s < t\)

\[
e_{(t,x)}(s) \leq e_{(t,x)}(0) = \frac{1}{2} \int_{B(x,t)} |v_1|^2 + |\nabla v_0|^2.
\]

In particular this implies that if the initial data \((v_0, v_1)\) are constantly equal to zero in \(B(x, t)\) then the solution \(v\) will also be equal to zero inside the cone \(\mathcal{C}_{(t,x)}\).

We now reformulate the above result in the following way which is due to Tartar [23]: Instead of the definition in (4.1) we make the modification

\[
e_{(t,x)}(s) := \frac{1}{2} \int_{\mathbb{R}^3} \varphi_{t-s,x}(y) \left( |\partial_t v(s,y)|^2 + |\nabla v(s,y)|^2 \right) dy,
\]

(4.2)

where \(\varphi_{t-s,x}\) is a positive radially symmetric bump function approximating \(\chi_{B(x,t-s)}\). We repeat the above computation

\[
\frac{d}{ds} e_{(t,x)}(s) = \frac{1}{2} \int_{\mathbb{R}^3} \frac{d}{ds} \varphi_{t-s,x}(y) \left( |\partial_t v(s,y)|^2 + |\nabla v(s,y)|^2 \right) dy +
\]

\[
+ \int_{\mathbb{R}^3} \varphi_{t-s,x}(y) \left( \partial_t^2 v(s,y)\partial_t v(s,y) + \nabla \partial_t v(s,y)\nabla v(s,y) \right) dy
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^3} \frac{d}{dt} \varphi_{t-s,x}(y) \left( |\partial_t v(s,y)|^2 + |\nabla v(s,y)|^2 \right) dy +
\]

\[
+ \int_{\mathbb{R}^3} \varphi_{t-s,x}(y) \partial_t v(s,y) \left( \partial_t^2 v(s,y) - \Delta v(s,y) \right) dy - \int_{\mathbb{R}^3} \nabla \varphi_{t-s,x}(y) \partial_t v(s,y)\nabla v(s,y) dy
\]

\[
\leq \int_{\mathbb{R}^3} \left( |\nabla \varphi_{t-s,x}(y)| - \frac{d}{dt} \varphi_{t-s,x}(y) \right) \left( |\partial_t v(s,y)|^2 + \frac{1}{2} |\nabla v(s,y)|^2 \right) dy.
\]

(4.3)

So if we want to re-obtain the same result as before, we should choose \(\varphi_{t-s,x}\) s.t.

\[
|\nabla \varphi_{t-s,x}(y)| - \frac{d}{dt} \varphi_{t-s,x}(y) \leq 0.
\]
We make the following choice. Set \( \psi : \mathbb{R} \to \mathbb{R}_+ \) as
\[
\psi(r) = \begin{cases} 
1 & r \in (-\infty, 0] \\
1 - r & r \in [0, 1] \\
0 & r \in [1, \infty)
\end{cases}
\]
which is Lipschitz and a.e. differentiable with \( \psi' \leq 0 \); then we consider
\[
\varphi_{(t,x)}(y, s) := \psi(|y - x| - c(t - s)) \quad (4.4)
\]
which is equal to 1 for \( |y - x| \leq c(t - s) \) and 0 for \( |y - x| - c(t - s) \geq 1 \) and interpolates linearly inbetween. Here \( c > 0 \) is a constant we will choose later; It can be thought of as the speed of propagation. Observe that
\[
|\nabla \varphi_{(t,x)}(y, s)| = \left| \frac{1}{c} \frac{d}{ds} \psi(|y - x| - c(t - s)) \frac{y - x}{|y - x|} \right|
\]
\[
= \frac{1}{c} \left| \frac{d}{ds} \varphi_{(t,x)}(y, s) \right|
\]
\[
= -\frac{1}{c} \frac{d}{ds} \varphi_{(t,x)}(y, s)
\]
\[
= \frac{1}{c} \frac{d}{dt} \varphi_{(t,x)}(y, s)
\]
\[
= -\psi'(|y - x| - c(t - s))
\]
\[
= \chi_{|y-x|-c(t-s)\in[0,1]} \quad (4.5)
\]
because of the choice of \( \psi \). This shows us that the constant \( c \) allows us to make the bound in more negative.

Note also that we could in principle choose the constant \( c \) to depend on other parameters such as the size of the noise, however it appears that is actually sufficient for all our purposes to set
\[
c = 2, \quad (4.6)
\]
although it seems the computations would still be true for any \( c > 1 \).

Another thing to note is that if we add a constant quadratic term to the local energy i.e.
\[
e_{(t,x)}(s) := \frac{1}{2} \int_{\mathbb{R}^3} \varphi_{t-s,x}(y)(|\partial_t v(s, y)|^2 + |\nabla v(s, y)|^2 + K|v(s, y)|^2)dy
\]
for \( K \geq 1 \) this leads to the bound
\[
\frac{d}{ds} e_{(t,x)}(s) = -\frac{1}{2} \int_{\mathbb{R}^3} \frac{d}{dt} \varphi_{t-s,x}(y)(|\partial_t v(s, y)|^2 + |\nabla v(s, y)|^2 + K|v(s, y)|^2)dy +
\]
\[
+ K \int_{\mathbb{R}^3} \varphi_{t-s,x}(y)\partial_t v(s, y)v(s, y) - \int_{\mathbb{R}^3} \nabla \varphi_{t-s,x}(y)\partial_t v(s, y)\nabla v(s, y)dy
\]
\[
\leq -\frac{K}{2} \int_{\mathbb{R}^3} \chi_{|y-x|-2(t-s)\in[0,1]}|v(s, y)|^2dy + \int_{\mathbb{R}^3} \varphi_{t-s,x}(y) \left( \frac{1}{2} |\partial_t v(s, y)|^2 + K^2|v(s, y)|^2 \right)dy
\]
\[
\leq -\frac{K}{2} \int_{\mathbb{R}^3} \chi_{|y-x|-2(t-s)\in[0,1]}|v(s, y)|^2dy + K e_{(t,x)}(s)
\]
meaning there is a trade-off in that we gain a negative term on the right-hand side while paying with a “Gronwall” term.

Note also that this approach has the upside that one does not need to evaluate anything on the boundary of a ball, as one does if one takes the approach \((4.1)\), which is useful since we are dealing with distributions for which it is a priori not at all clear how one would do that.

Now we are ready to state the first new result which extends the approach we just described in order to obtain finite speed of propagation for the linear wave-type equation
\[
\begin{cases}
\partial_t^2 u - Hu = 0 \quad &\text{on } \mathbb{R}_+ \times \mathbb{R}^3 \\
(u, \partial_t u)|_{t=0} = (u_0, u_1)
\end{cases} \quad (4.7)
\]
where $H^\ast = \Delta + \xi$ is the full Anderson Hamiltonian.

Since we do not have a direct way of solving (4.7) (or indeed making sense of it for now), we instead consider the family of solutions $u^{(R)}$ to

$$\begin{cases}
    \partial_t^2 u^{(R)} - H_R u^{(R)} = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^3 \\
    (u^{(R)}, \partial_t u^{(R)})|_{t=0} = (u_0^{(R)}, u_1^{(R)})
\end{cases},$$

(recalling Definition 1 and prove finite speed of propagation for them which will give us inside a space-time region $G_L$ which is increasing in $L$ and tends to $\mathbb{R}_+ \times \mathbb{R}^3$ as $L \to \infty$ as long as their initial data agree.

We introduce a weak formulation for the formal PDE (4.7) and the weak formulation will be a suitably truncated version thereof. We say $u = e^{W}v$ is a weak solution to (4.7) if

$$(v, \partial_t v)|_{t=0} = (e^{-W}u_0, e^{-W}u_1)$$

and

$$(e^W \partial_t^2 v, e^W \phi)_{D(\sqrt{-H_{\infty}})^\ast, D(\sqrt{-H_{\infty}})} = (He^Wv, e^W \phi)_{D(\sqrt{-H_{\infty}})^\ast, D(\sqrt{-H_{\infty}})} - (e^{2W} \nabla v, \nabla \phi)_{L^2} + (e^{2W} Z v, \phi)_{\mathcal{H}^{-1}, \mathcal{H}^1} + (e^{2W} \chi_{B(1)} Z v, \phi)_{L^2}$$

for all compactly supported $\phi \in \mathcal{H}^1$.

Analogously we say that $u^{(R)} = e^{W}v^{(R)}$ is a weak solution to (4.8) if

$$(u^{(R)}, \partial_t v^{(R)})|_{t=0} = (e^{-W}u_0^{(R)}, e^{-W}u_1^{(R)})$$

and

$$(e^W \partial_t^2 v^{(R)}, e^W \phi)_{D(\sqrt{-H_{\infty}})^\ast, D(\sqrt{-H_{\infty}})} = (H_R e^{W}v^{(R)}, e^W \phi)_{D(\sqrt{-H_{\infty}})^\ast, D(\sqrt{-H_{\infty}})} - (e^{2W} \nabla v^{(R)}, \nabla \phi)_{L^2} + (e^{2W} Z v^{(R)}, \phi)_{\mathcal{H}^{-1}, \mathcal{H}^1} + (e^{2W} \chi_{B(1)} Z v^{(R)}, \phi)_{L^2}$$

for all compactly supported $\phi \in \mathcal{H}^1$.

**Remark 3** Note that the space $D(\sqrt{-H_{\infty}})^\ast$, which is the dual of the energy space, is the natural space of the terms $\partial_t^2 u$ and $\partial_t^2 u^{(R)}$ and indeed one can readily show that the solutions from Section 3 satisfy this property.

Now we give the main result which says that the linear equation has finite speed of propagation implying that solutions to (4.10) are actually local solutions to (4.9).

**Theorem 3** Let $u^{(R)}, u^{(L)}$ be solutions to

$$\begin{cases}
    \partial_t^2 u^{(i)} - H_i u^{(i)} = 0 & \\
    (u^{(i)}, \partial_t u^{(i)})|_{t=0} = (u_0^{(i)}, u_1^{(i)})
\end{cases}$$

for $i = R, L$ and $R \geq L > 0$. Moreover we choose the initial data to satisfy $(u_0^{(i)}, u_1^{(i)}) \in D(\sqrt{-H_{\infty}}) \times L^2$ and

$$(u_0^{(R)}, u_1^{(R)}) = (u_0^{(L)}, u_1^{(L)})$$
on $B(2L + 1)$. 

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Then we have
\[ u^{(R)}(t, x) = u^{(L)}(t, x) \text{ for all } (t, x), \text{s.t. } \mathcal{C}_{(t,x)} \subset \left[ 0, \frac{L}{2} \right] \times B(L), \]
where the backward light-cone \( \mathcal{C}_{(t,x)} \) at the space-time point \((t, x)\) is defined as
\[ \mathcal{C}_{(t,x)} := \{ (s, y) \in \mathbb{R} \times \mathbb{R}^3 : 0 \leq s \leq t \text{ and } y \in B(x, 2(t-s)) \}. \]

Moreover, we have the following local bounds for the solutions
\[ e^{(i)}_{(t,x)}(s) \leq C(\Xi) e^{(i)}_{(t,x)}(0) = C(\Xi, i) \int_{\mathbb{R}^d} \varphi_{(t,x)}(0) \frac{1}{2} e^{2W} (|v_1^{(i)}|^2 + |\nabla v_0^{(i)}|^2 + C(\Xi, i) |v_0^{(i)}|^2 - |v_0^{(i)}|^2 Z_>) \]
for suitable constants \(C(i, \Xi), C(\Xi, i) > 0\), having defined the exponentially transformed solution and initial data as
\[ v^{(i)} := e^{-W} u^{(i)} \text{ and } v_j^{(i)} := e^{-W} u_j^{(i)} \text{ for } i = L, R \text{ and } j = 0, 1 \]
and the appropriate local energy quantity as
\[ e^{(i)}_{(t,x)}(s) := \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) \frac{1}{2} e^{2W} (|\partial_t v^{(i)}(s)|^2 + |\nabla v^{(i)}(s)|^2 + C(\Xi, i) |v^{(i)}(s)|^2 - |v^{(i)}(s)|^2 Z_>) \]
for \(i = L, R\). The bump function \(\varphi\) is the one defined in (4.4) setting \(c = 2\).

**Proof** We follow the general method of Tartar, see [23], which was sketched above. Using the exponential transform, we rewrite the equations for \(u^{(R)}\) and \(u^{(L)}\) instead as equations for \(v^{(R)}\) and \(v^{(L)}\) which are given by (4.10) in weak form.

We now consider the local energy quantity (4.14) which is of course inspired by (4.2) and suppress the \((t, x)\) and the \(i\) dependence of \(e\) for ease of notation. Moreover, we have added a (large) \(L^2\) term—namely \(C(\Xi, i)\) — which does not come from the equation but makes it uniformly positive i.e we have
\[ e(s) = 0 \implies v^{(i)} = 0 \text{ in } \supp(\varphi_{(t,x)}(s)), \]
see Definition [1].

Note that this is for now only formal, since the term
\[ \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) \frac{1}{2} e^{2W} |v^{(i)}(s)|^2 Z_> \]
is not an honest integral but rather should be thought of as a pairing like
\[ \left( \varphi_{(t,x)}(s) \frac{1}{2} e^{2W} Z_>, |v^{(i)}(s)|^2 \right)_{B^{\frac{1}{2} - \varepsilon}_{\infty, 1}, B^{\frac{1}{2} + \varepsilon}_{1, 1}} \]
for \(\varepsilon > 0\) small, see Lemma [2] for why the first term is in fact in \(B^{\frac{1}{2} - \varepsilon}_{\infty, \infty} = C^{-\frac{1}{2} - \varepsilon}.\) For the right-hand side we invoke Lemmas [4] and [5] in order to bound
\[ \|v^{(i)}(s)|^2\|_{B^{\frac{1}{2} + \varepsilon}_{1, 1}} \leq C \|v^{(i)}(s)\|_{L^2} \|v^{(i)}(s)\|_{H^{\frac{1}{2} + 2\varepsilon}} \leq C \|v^{(i)}(s)\|_{H^2}^2 \leq \delta \|e^{W} \nabla v^{(i)}(s)\|_{L^2}^2 + C(\delta, \Xi) \|e^{W} v^{(i)}(s)\|_{L^2}^2, \]
for small \(\delta > 0\), interpolating in the \(H^\alpha\)—scale and applying Young’s inequality, showing that this term is not only well-defined but also “lower-order” with respect to the gradient term. In light of this computation, we will continue to make this mild abuse of notation.
Analogously to the strategy above, we compute the derivative $\frac{d}{ds}c(s)$ in order to make a Gronwall argument. This yields

$$
\frac{d}{ds}c(s) = \int_{\mathbb{R}^3} \frac{d}{ds}\varphi(t,x)(s) \, \chi_{[-2(1-s),1]}(s) \left( 2W^2 \left( |\partial_tv_i(s)|^2 + |\nabla v_i(s)|^2 + |v_i(s)|^2 (C(\Xi, i) - Z) \right) + \right.
$$

$$
+ \int_{\mathbb{R}^3} \varphi(t,x)(s) e^{2W} \partial_t v_i(s) \partial t v_i(s) - v_i(s) Z - \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \right.
$$

$$
+ \int_{\mathbb{R}^3} \varphi(t,x)(s) e^{2W} \partial_t v_i(s) \partial t v_i(s) - v_i(s) Z - \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \right.
$$

$$
- \int_{\mathbb{R}^3} \varphi(t,x)(s) e^{2W} \partial t v_i(s) \partial t v_i(s) - v_i(s) Z - \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \right.
$$

$$
= - \int_{\mathbb{R}^3} \varphi(t,x)(s) e^{2W} \partial t v_i(s) \partial t v_i(s) - v_i(s) Z - \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \chi_{B(R)} Z e^{2W} (C(\Xi, i) - Z) + \right.
$$

where we have used (4.14), integration by parts and Young’s inequality.

Now we want to conclude by arguing that the term

$$
\int_{\mathbb{R}^3} \chi_{[-2(1-s),1]}(s) e^{2W} Z \, |v_i(s)|^2(s)
$$

can be absorbed by the other two using the fact that we may freely choose $C(\Xi, i)$ depending on the norm of $\Xi$ by an argument similar to (4.15).

We however need one trick to proceed, since the term in the right-hand side of the bound (4.15) can not be controlled by the terms we have. Recall that we have a bounded restriction and an extension operator on Besov spaces $\|A\|$ and $E_A$ for nice sets $A \subset \mathbb{R}^d$, see Proposition 2.

Generally we may bound the product with an indicator function in the following way using Lemma 4 and Lemma 5 and the restriction/extensions from Proposition 2.

$$
\|A_{f|B^{p,1}_{\rho,1}(\mathbb{R}^d)} \| = \|A_{f|B^{p,1}_{\rho,1}(\mathbb{R}^d)} \|
$$

$$
\|A_{f|B^{p,1}_{\rho,1}(\mathbb{R}^d)} \| + \|E_A f|A|_{L^p(\mathbb{R}^d)} \| \leq \|E_A f|A|_{L^p(\mathbb{R}^d)} \| \leq \|f|A|_{L^p(\mathbb{R}^d)} \|
$$

for

$$
1 = 1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q},
$$

where the terms involving $\chi_A$ are finite for $A \subset \mathbb{R}^d$ with finite perimeter and $\alpha < \frac{1}{p}$ by Lemma 5 and Lemma 6.

In fact, we bound the term (4.17) as follows. We set

$$
A := \{ y \in \mathbb{R}^3 : |y - x| - 2(t - s) \in [0, 1] \}.
$$
and taking suitably small $\varepsilon, \tilde{\varepsilon}, \delta > 0$ we proceed as above

$$
(\chi_A|v^{(i)}(s)|^2, e^{2W > Z_\gamma}) \leq \|e^{2W > Z_\gamma}\|_{B_{\infty,\infty}^{\frac{1}{2} + \frac{\varepsilon}{4}}} \|\chi_A|v^{(i)}(s)|^2\|_{B_{1,1}^{\frac{1}{2} + \frac{\varepsilon}{4}}}
= \|e^{2W > Z_\gamma}\|_{B_{\infty,\infty}^{\frac{1}{2} + \frac{\varepsilon}{4}}} \|\chi_A E_A(|v^{(i)}(s)|^2)|A|\|_{B_{1,1}^{\frac{1}{2} + \frac{\varepsilon}{4}}}
\leq C\|e^{2W > Z_\gamma}\|_{B_{\infty,\infty}^{\frac{1}{2} + \frac{\varepsilon}{4}}} \left(\|\chi_A\|_{B_{\infty,\infty}^{\frac{1}{2} + \frac{\varepsilon}{4}}} + \|E_A(|v^{(i)}(s)|^2)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}} + \|E_A(|v^{(i)}(s)|^2)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}}ight)
+ \|E_A(|v^{(i)}(s)|^2)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}} \|\chi_A\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}}
\leq C(A, \Xi) \left(\|E_A(|v^{(i)}(s)|^2)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}} + \|E_A(|v^{(i)}(s)|^2)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}} \right)
\leq C(A, \Xi) \left(\|v^{(R)}(s)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}} + \|v^{(R)}(s)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}}ight)
\leq C(A, \Xi) \left(\|v^{(i)}(s)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}} + \|v^{(i)}(s)|A|\|_{L_\infty^{\frac{1}{2} + \frac{\varepsilon}{4}}}ight)
\leq \frac{1}{4}\|e^{2W > Z_\gamma}\|_{L_\infty} \int_A |\nabla v^{(i)}(s)|^2 + C(A, \Xi) \int_A |v^{(i)}(s)|^2
\leq \frac{1}{4} \int_A e^{2W > Z_\gamma} |\nabla v^{(i)}(s)|^2 + C(A, \Xi) \int_A e^{2W > Z_\gamma} |v^{(i)}(s)|^2.
\tag{4.19}
$$

Now we observe that the dependence of the constant in $A$ can be chosen uniformly in $s \in [0, t]$ and does not depend on the point $x$ at all. Instead this will result in an $i$ dependent constant. More precisely, by translation invariance one can see that the norms do not depend on the spatial variable $x$ and to see that one may choose it independent of the time integration parameter $s$ one observes that the function $s \mapsto \|\chi_{x - x - 2(t - s)\in[0,1]}\|_{L^{\frac{1}{2} + \frac{\varepsilon}{4}}}^2$ is continuous on $s \in [0, t]$ and bounded at the end points (see Lemma 3) hence is bounded on the whole interval. Lastly, since we only consider times $t \leq i$ we have that the constant can be chosen to depend on $R$ and we relabel it as

$$
e(\Xi, i) = C(A, \Xi).
$$

If we insert the bound (4.19) into (4.16) we get after choosing the constant $C(\Xi, i)$ sufficiently large

$$
\frac{d}{ds} e(s) \leq \int_{\mathbb{R}^d} \chi_{x - x - 2(t - s)\in[0,1]} \frac{1}{2} e^{2W > Z_\gamma} \left(\int_{0}^{1} 4 - \frac{3}{4} \right) \left|\nabla v^{(i)}(s)\right|^2 + \left(\int_{0}^{1} \chi_B(R) \left|Z_{\gamma} + C(\Xi, i)\right| e(s) \right)
\leq \frac{1}{2} \int_{\mathbb{R}^d} \varphi(\tau, x)(0) \frac{1}{2} e^{2W > Z_\gamma} \left|v^{(i)}\right|^2 + \left|\nabla v^{(i)}\right|^2 + C(\Xi, i) \left|v^{(i)}\right|^2 - |v^{(i)}|^2 Z_\gamma.
\tag{4.20}
$$

which by Gronwall implies

$$
e(s) \leq C(i, \Xi) e(0) = C(i, \Xi) \int_{\mathbb{R}^d} \varphi(\tau, x)(0) \frac{1}{2} e^{2W > Z_\gamma} \left|v^{(i)}\right|^2 + \left|\nabla v^{(i)}\right|^2 + C(\Xi, i) \left|v^{(i)}\right|^2 - |v^{(i)}|^2 Z_\gamma.
\tag{4.20}
$$

This in particular implies that $v^{(i)}$ is controlled in points inside the backwards light cone by the initial conditions in the support of $\varphi(\tau, x)(0)$, i.e. a ball around $x$. Moreover this implies for two different parameters

$$0 \ll L \ll R$$
that the solutions \( v^{(L)} \) and \( v^{(R)} \) to (4.8) with the same initial data \( (v_0, v_1) \) actually agree in the backward light-cones which are contained in \([0, \frac{t}{3}] \times B(L)\).

In order to make this precise, we observe that the difference
\[
d := v^{(L)} - v^{(R)}
\]
solves the equation
\[
(\partial_t^2 - H_L)d = \begin{cases} \chi_{B(R)}Z_\xi - \chi_{B(L)}Z_\xi \end{cases} v^{(R)}
\]
\[
(d, \partial_t d)|_{t=0} = (0, 0).
\]
Thus the above argument applied to points \((t, x)\) for which \(\text{supp}(\varphi_{(t,x)}(s)) \subset B_L\) for all \(0 \leq s \leq t\) gives
\[
0 \leq \int_{R^d} \varphi_{(t,x)}(s)e^{2W >} |d(s)|^2
\]
\[
\leq \int_{R^d} \frac{1}{2} e^{2W >} \varphi_{(t,x)}(s) (|\partial_t d(s)|^2 + |\nabla d(s)|^2 + |d(s)|^2 (C(\Xi, R) - Z_>) )
\]
\[
\leq C(\Xi, R) \int_{R^d} \varphi_{(t,x)}(0) (|\partial_t d(0)|^2 + |\nabla d(0)|^2 + (C(\Xi, R) - Z_>) |d(0)|^2 )
\]
\[
\leq 0,
\]
which implies that \(d \equiv 0\) in that region. This finishes the proof. \(\square\)

**Remark 4** Shortly before completion, it was pointed out to the author by Massimilano Gubinelli that one could alternatively prove the finite speed of propagation of the multiplicative stochastic wave equation by approximating the equation by regularising the noise and localising
\[
\partial_t^2 u - H_{\varepsilon}^{\text{loc}} u = 0 \tag{4.21}
\]
\[
(u, \partial_t u)|_{t=0} = (u_0^\varepsilon, u_1^\varepsilon), \tag{4.22}
\]
where \(H_{\varepsilon}^{\text{loc}} = \Delta + \Xi_{\text{smooth}}^{\text{loc}}\) is some suitable regular and localised approximation to the Anderson Hamiltonian which in particular should be self-adjoint and semibounded. This equation then has unit speed of propagation by the classical proof above, meaning for every \(t, \varepsilon > 0\) and \(x \in \mathbb{R}^3\)
\[
\chi_{B(x,t-s)} S_{\varepsilon}^{\text{loc}}(u_0^\varepsilon, u_1^\varepsilon) = S_{\varepsilon}^{\text{loc}}(s)(\chi_{B(x,t)} u_0^\varepsilon, \chi_{B(x,t)} u_1^\varepsilon) \text{ for any } 0 < s < t, \tag{4.23}
\]
where \(S_{\varepsilon}^{\text{loc}}\) is the propagator of the equation (4.21). If one then proves the strong resolvent convergence of the operators \(H_{\varepsilon}^{\text{loc}}\) to a localised version of the Anderson Hamiltonian as \(\varepsilon \to 0\), one gets that the associated propagators converge strongly in \(L^2\), cf Section 3.3 in [9]. This convergence would imply that the identity (4.23) passes to the limit as \(\varepsilon \to 0\) meaning we have unit speed of propagation with a localisation which can be removed since we are interested in a local property.

This approach, however, does not immediately give us bounds on the local energies like in the approach in the current work. Also it is not immediately clear how one would prove the analogous result in the nonlinear case.

## 5 Putting it all together

Finally we want to apply Theorem 3 also to semilinear wave equations which are formally
\[
\begin{cases}
\partial_t^2 u - Hu = -u^3 \text{ on } \mathbb{R}_+ \times \mathbb{R}^3 \\
(u, \partial_t u)|_{t=0} = (u_0, u_1)
\end{cases} \tag{5.1}
\]
whose solutions should be suitable limits of the solutions \(u^{(R)}\) to (4.1).
Analogously to before, we introduce the weak formulation of both the “full” PDE and its truncated version, which we have solved in Theorem 2.

We say \( u = e^{W^t} v \) is a weak solution to (5.1) if
\[
(e^{W^t} \partial_t^2 v, e^{W^t} \phi)_D(\sqrt{-\mathcal{H}_Z})^*, D(\sqrt{-\mathcal{H}_Z}) = (He^{W^t} v, e^{W^t} \phi)_D(\sqrt{-\mathcal{H}_Z})^*, D(\sqrt{-\mathcal{H}_Z}) - (e^{4W^t} v^3, \phi)_L^2 \]
\[
= - (e^{2W^t} \nabla v, \nabla \phi)_{L^2} + (e^{2W^t} Z_{\leq} v, \phi)_{H^{-1}, H^1} + (e^{2W^t} Z_{\geq} v, \phi)_{L^2} - (e^{4W^t} v^3, \phi)_{L^2}
\]
for all compactly supported \( \phi \in \mathcal{H}_1 \).

and
\[
(v, \partial_t v)_{t=0} = (e^{-W^t} u_0, e^{-W^t} u_1)
\]

Analogously we say that \( u^{(R)} = e^{W^t} v^{(R)} \) is a weak solution to (3.1) if
\[
(e^{W^t} \partial_t^2 v^{(R)}, e^{W^t} \phi)_D(\sqrt{-\mathcal{H}_Z})^*, D(\sqrt{-\mathcal{H}_Z}) = (H_{R} e^{W^t} v^{(R)}, e^{W^t} \phi)_D(\sqrt{-\mathcal{H}_Z})^*, D(\sqrt{-\mathcal{H}_Z}) - (e^{4W^t} v^3, \phi)_L^2 \]
\[
= - (e^{2W^t} \nabla v^{(R)}, \nabla \phi)_{L^2} + (Z_{\geq} v^{(R)}, \phi)_{H^{-1}, H^1} + (e^{2W^t} Z_{\leq} v^{(R)}, \phi)_{L^2} - (e^{4W^t} v^3, \phi)_{L^2}
\]
for all compactly supported \( \phi \in \mathcal{H}_1 \).

and
\[
(u^{(R)}, \partial_t v^{(R)})_{t=0} = (e^{-W^t} u_0^{(R)}, e^{-W^t} u_1^{(R)})
\]

We wish to prove an analogous bound to (5.2) for the nonlinear equation. In fact we get the following result which extends the finite speed of propagation argument to the semilinear case. Since the nonlinearity is controlled by the energy, this is essentially like Theorem 3 with some modifications.

Theorem 4 (Finite speed of propagation for the cubic multiplicative stochastic wave equation)

Let \( R \geq L \geq 0 \) and \( u^{(i)} = e^{W^t} v^{(i)} \) be the solutions to
\[
\partial_t^2 u^{(i)} - H_i u^{(i)} = -u^{(i)}|u^{(i)}|^2 \text{ on } [0, T] \times \mathbb{R}^3
\]
\[
(u, \partial_t u) = (u_0, u_1) \in D \left( \sqrt{-\mathcal{H}_Z} \right) \times L^2,
\]
from Theorem 2 for some \( T > 0 \) and \( i = R, L \). We set
\[
e^{(i)}(s) := \int_{\mathbb{R}^d} \varphi(t, x) \left( \frac{1}{2} e^{2W^t} (|\partial_t v^{(i)}(s)|^2 + |\nabla v^{(i)}(s)|^2 + |v^{(i)}(s)|^2 (C(\Xi, i) - Z_>) \right)
\]
and
\[
e(s) := \int_{\mathbb{R}^d} \varphi(t, x) \left( \frac{1}{2} e^{2W^t} (|\partial_t b(s)|^2 + |\nabla b(s)|^2 + |b(s)|^2 (C(\Xi, i) - Z_>) \right),
\]
where \( b := v^{(R)} - v^{(L)} \) for large constants \( C(\Xi, i) > 0 \) chosen below.

Then there exist constants \( c(i, \Xi, u_0^{(i)}, u_1^{(i)}) > 0 \) for which the bounds
\[
e^{(i)}(s) \leq c(i, \Xi, u_0^{(i)}, u_1^{(i)}) e^{(i)}(0)
\]
\[
= c(i, \Xi, u_0^{(i)}, u_1^{(i)}) \int_{\mathbb{R}^d} \varphi(t, x) \left( \frac{1}{2} e^{2W^t} (|v_1^{(i)}|^2 + |\nabla v_0^{(i)}|^2 + |v_0^{(i)}|^2 (C(\Xi, i) - Z_>) \right)
\]
and
\[
e(s) \leq c(\Xi, L, R, u_0^{R}, u_1^{R}, u_0^{L}, u_1^{L}) e(0)
\]
\[
= c(\Xi, L, R, u_0^{L}, u_1^{L}) \left( \frac{1}{2} e^{2W^t} (|b_1|^2 + |\nabla b_0|^2 + |b_0|^2 (C(\Xi, R, L) - Z_>) \right)
\]
hold for
\[
0 \leq s \leq t \text{ and } x \in \mathbb{R}^d \text{ s.t. } \mathcal{C}(t, x) \subset \left[ 0, \frac{L}{2} \right] \times B(L),
\]
where the backward light cone is defined as

\[ \mathcal{C}_{(t,x)} := \{(s, y) \in \mathbb{R} \times \mathbb{R}^3 : 0 \leq s \leq t \text{ and } y \in B(x, 2(t - s))\} \quad (5.7) \]

as in Theorem \( \text{\textsuperscript{3}} \).

**Proof** In this case the difference of the solutions \( u^{(R)} \) and \( u^{(L)} \) with \( R \geq L \) will not solve the same equation as in the linear case. Instead we make the observation that the difference \( d := u^{(R)} - u^{(L)} \) in this case solves the equation

\[
\partial_t^2 d - H_L d = (\chi_{B(R)} Z_\leq - \chi_{B(L)} Z_\leq) u^{(R)} - d(|u^{(R)}|^2 + u^{(L)} u^{(R)} + |u^{(L)}|^2) \\
(d, \partial_t d)|_{t=0} = (0, 0).
\]

For future reference we also give the equation solved by \( b := e^{-W_\sigma} d \) and its “weak” formulation; in analogy to the previous sections we also write \( u^{(i)} = e^{W_i} v^{(i)} \) for \( i = R, L \)

\[
\partial_t^2 b - e^{-W_\sigma} H_L e^{W_\sigma} b = (\chi_{B(R)} Z_\leq - \chi_{B(L)} Z_\leq) e^{2W_\sigma} b(|u^{(R)}|^2 + v^{(L)} v^{(R)} + |v^{(L)}|^2) \\
(d, \partial_t d)|_{t=0} = (0, 0);
\]

\[(\phi, e^{2W_\sigma} \partial_t^2 b - e^{-W_\sigma} H_L e^{W_\sigma} b) = (\phi, (\chi_{B(R)} Z_\leq - \chi_{B(L)} Z_\leq) e^{2W_\sigma} b(|u^{(R)}|^2 + v^{(L)} v^{(R)} + |v^{(L)}|^2))
\]

for all \( \phi \in \mathcal{H}_1 \).

We thus again compute integrating the gradient term as in the proof of Theorem \( \text{\textsuperscript{3}} \)

\[
\frac{d}{ds} e(s) = \int_{\mathbb{R}^d} \frac{d}{ds} \varphi_{(t,x)}(s) \frac{1}{2} e^{2W_\sigma} (|\partial_t b(s)|^2 + |\nabla b(s)|^2 + (C(\Xi, L) - Z_>) b(s))^2 + \]

\[
\quad + \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) e^{2W_\sigma} \partial_t b(s) (\partial_t^2 b(s) + (C(\Xi, L) - Z_>) b(s)) + \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) e^{2W_\sigma} \partial_t \nabla b(s) \nabla b(s) \]

\[
= \int_{\mathbb{R}^d} \frac{d}{ds} \varphi_{(t,x)}(s) \frac{1}{2} e^{2W_\sigma} (|\partial_t b(s)|^2 + |\nabla b(s)|^2 + (C(\Xi, L) - Z_>) b(s))^2 + \]

\[
\quad + \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) e^{2W_\sigma} \partial_t b(s) (\partial_t^2 b(s) - e^{-W_\sigma} H_L e^{W_\sigma} b(s) + e^{2W_\sigma} b(s)|u^{(R)}|^2 + v^{(L)} v^{(R)} + |v^{(L)}|^2)) + \]

\[
\quad + \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) e^{2W_\sigma} \partial_t b(s) (C(\Xi, L) + \chi_{B(L)} Z_\leq) + \]

\[
\quad + \int_{\mathbb{R}^d} \varphi_{(t,x)}(s) e^{4W_\sigma} \partial_t b(s) b(s)|u^{(R)}|^2 + v^{(R)} v^{(L)} + |v^{(L)}|^2). \]

Now we proceed similarly to the proof of Theorem \( \text{\textsuperscript{3}} \). We need, however, one additional ingredient, namely bounds on \( v^{(R)} \) and \( v^{(L)} \) which allow us control the final term. Since they are all bounded analogously we show how to bound one term and the others are analogous. We start by making a simple observation about our bump function \( \varphi_{(t,x)} \), which is that

\[
\varphi_{(t,x)}(s) \leq \chi_{\text{supp} \varphi_{(t,x)}(s)} = \chi_{B(x, 2(t - s))} + \chi_{\text{supp} \varphi_{(t,x)}(s)} \quad (5.8)
\]

\[
\quad = \chi_{B(x, 2(t - s))} + \chi_{|x - x| - 2(t - s) \in [0, 1]} \quad (5.9)
\]

\[
\quad = \chi_{B(x, 2(t - s))} - \varphi_{(t,x)}(s) \quad (5.10)
\]

and

\[
\chi_{B(x, 2(t - s))} \leq \varphi_{(t,x)}(s). \quad (5.11)
\]
For $\varepsilon > 0$ we bound by Young, Hölder and the $\mathcal{H}^1 \rightarrow L^6$ Sobolev embedding

$$\left| \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{4W^s} \partial_t b(s)b(b(s))v^{(R)}(s) \right|^2 \leq$$

$$\leq C(\varepsilon) \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{2W^s} |\partial_t b(s)|^2 + \varepsilon \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{6W^s} |b(s)|^2 v^{(R)}(s)^4$$

$$\leq C(\varepsilon) \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{2W^s} |\partial_t b(s)|^2 + \varepsilon \left( \int_{B(x,2(t-s))} |b(s)|^6 \right)^{\frac{1}{6}} \sup_{0 \leq \tau \leq t} \left( \int_{\mathbb{R}^d} \varphi_{(t,x)}(\tau) |v^{(R)}(\tau)|^6 \right)^{\frac{2}{3}} \|e^{6W^s}\|_{L^\infty}$$

$$\leq C(\varepsilon) \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{2W^s} |\partial_t b(s)|^2 + \varepsilon \left( \int_{B(x,2(t-s))} |b(s)|^6 \right)^{\frac{1}{6}} \sup_{0 \leq \tau \leq t} \left( \int_{\mathbb{R}^d} |v^{(R)}(\tau)|^6 \right)^{\frac{2}{3}} \|e^{6W^s}\|_{L^\infty}$$

$$+ \left( \int_{\text{supp} p'} |b(s)|^6 \right)^{\frac{1}{6}} \sup_{0 \leq \tau \leq t} \left( \int_{\mathbb{R}^d} |v^{(R)}(\tau)|^6 \right)^{\frac{2}{3}} \|e^{6W^s}\|_{L^\infty}$$

$$\leq C(\varepsilon) \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{2W^s} |\partial_t b(s)|^2 + C\varepsilon \left( \int_{B(x,2(t-s))} |\nabla b(s)|^2 + |b(s)|^2 \right) +$$

$$+ \left( \int_{\text{supp} p'} |\nabla b(s)|^2 + |b(s)|^2 \right) \sup_{0 \leq \tau \leq t} \left( \int_{\mathbb{R}^d} |\nabla v^{(R)}(\tau)|^2 + |v^{(R)}(\tau)|^2 \right) \|e^{6W^s}\|_{L^\infty}$$

$$\leq C(\varepsilon) \int_{\mathbb{R}^3} \varphi_{(t,x)}(s)e^{2W^s} |\partial_t b(s)|^2 + \varepsilon C(\Xi) \left( \int_{\mathbb{R}^d} \varphi_{(t,x)}(s)e^{2W^s} |\nabla b(s)|^2 \right)$$

$$- \left| \int_{\mathbb{R}^d} \varphi_{(t,x)}(s)e^{2W^s} |\nabla b(s)|^2 \right| e^{C(\Xi,R)(\varepsilon)}(u_0^R, u_1^R)$$

hence by choosing $\varepsilon$ small enough depending on the norms of the noise terms, the parameter $L$ and the $L_{[0,t]}^\infty \mathcal{H}^1$-norm of $v^{(R)}$, $v^{(L)}$ (which are bounded by the initial data by Theorem 3 and choosing a suitably small $\varepsilon > 0$. Proving the bound (5.3) is analogous.

This finishes the proof.

This leads us to the final result, which tells us that the PDE (5.1) whose weak formulation is (5.2) is globally well-posed in space and time.

**Theorem 5** (Global well-posedness of the cubic multiplicative stochastic wave equation) Let $T > 0$ and initial data $(u_0, u_1) \in e^{W^s} \mathcal{H}^1 \times L^2$. Then there exists a unique solution $u \in C([0,T]; e^{W^s} \mathcal{H}^1_{\text{loc}}) \cap C^1([0,T]; L^2_{\text{loc}})$ to the PDE (5.2) with continuous dependence of the localised norms on the data inside the backward light cones.

**Proof** By Theorem 3 we have local space-time existence for solutions to (5.2) and then Theorem 4 gives us local uniqueness and local continuous dependence on the data in the backward light cone. \(\square\)

### A Appendix: Some results on Besov spaces and weights

**Definition 2** (Weighted Besov spaces, [10]) For $\nu \in \mathbb{R}$ we consider the following class of weights

$$\langle x \rangle^\nu = (1 + |x|^2)^{\frac{\nu}{2}} \quad x \in \mathbb{R}^d$$

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and define the weighted $L^p$ space w.r.t. this weight as

$$L^p_{\langle \cdot \rangle^\nu} := \{ f \in \mathcal{S}'(\mathbb{R}^d) : f \cdot \langle \cdot \rangle^\nu \in L^p \}, p \in [1, \infty],$$

whose norm is defined as

$$\|f\|_{L^p_{\langle \cdot \rangle^\nu}} := \|f \cdot \langle \cdot \rangle^\nu\|_{L^p}.$$

Moreover, for a Littlewood-Paley decomposition $(\Delta_i)_{i \geq 1}$, one defines weighted Besov spaces as

$$B^s_{p,q,\nu} := \{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,q,\nu}} < \infty \}, \text{ with the norm } \|f\|_{B^s_{p,q,\nu}} := \left\| \| \Delta_i f \|_{L^p_{\langle \cdot \rangle^\nu}} \right\|_{L^q}.$$

In particular, for $\nu = 0$ this agrees with the usual unweighted Besov space, moreover we write

$$C^s := B^s_{\infty, \infty, \nu}$$

and refer to it as a weighted Besov-Hölder space.

We note that one can analogously define these spaces on the torus $\mathbb{T}^d$, however one usually does not need weights in that setting due to the compactness of $\mathbb{T}^d$.

**Lemma 3 (Besov regularity of indicator functions, Theorem 2 in [22])** Let $A \subset \mathbb{R}^d$ be a bounded set with finite perimeter, then we have

$$\chi_A \in B^\frac{1}{2}_{p,\infty} \text{ for any } p \in [1, \infty).$$

**Lemma 4 (Fractional Leibnitz for Besov spaces, Proposition A.7 in [18])** For $s > 0$ and $1 \leq p, q, p_1, p_2, p_3, p_4 \leq \infty$ s.t.

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

we have the bound

$$\|f \cdot g\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p_1,q_1}} \|g\|_{L^p} + \|f\|_{L^p} \|g\|_{B^s_{p_4,q}}.$$  

**Lemma 5 (Besov embedding and interpolation)** Let $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$ and $\alpha \in \mathbb{R}$. Then we have the continuous embeddings

$$B^\alpha_{p_1,q_1} \hookrightarrow B^{\alpha - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}_{p_2,q_2}$$

as well as

$$B^{d\left(\frac{1}{2} - \frac{1}{q}\right)}_{p_1} \hookrightarrow L^q \quad \text{for } 1 \leq p \leq q \leq \infty \quad \text{and} \quad B^0_{p_2} \hookrightarrow L^p \quad \text{for } 2 \leq p < \infty.$$

Moreover, for $\alpha_1 < \alpha_2$ and $\theta \in (0, 1)$ we have for any $p, r \in [1, \infty]$ the bounds

$$\|u\|_{B^{\alpha_2 + (1-\theta)\alpha_2}_{p,r}} \lesssim \|u\|_{B^{\alpha_1}_{p,r}} \|u\|_{B^{1-\theta \alpha_2}_{p,r}}^{1-\theta}$$

and

$$\|u\|_{B^{\alpha_2 + (1-\theta)\alpha_2}_{p,1}} \lesssim \|u\|_{B^{\alpha_1}_{p,\infty}} \|u\|_{B^{1-\theta \alpha_2}_{p,\infty}}^{1-\theta}.$$

**Proof** See Propositions 2.18, 2.22 and 2.39 as well as Theorem 2.40 in [2].

**Proposition 2 ([21], Extension and restriction operators on Besov spaces)** Let $\Omega \subset \mathbb{R}^d$ be an open set with Lipschitz boundary, then the Besov space $B^s_{p,q}(\Omega)$ is defined as

$$B^s_{p,q}(\Omega) := \{ f \in D'(\Omega) : \|f\|_{B^s_{p,q}(\Omega)} := \inf\{\|g\|_{B^s_{p,q}(\mathbb{R}^d)} : g \in D'(\mathbb{R}^d) : g|_\Omega = f \} \}. \quad (A.1)$$
Then there exists a bounded extension operator $E_{\Omega}$ s.t. for $f \in B_{p,q}^s(\Omega)$
\[ \|E_{\Omega}f\|_{B_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{B_{p,q}^s(\Omega)}. \]
Moreover, by the definition of the space, one has the bound for the restriction to $\Omega$
\[ \|g|_{\Omega}\|_{B_{p,q}(\Omega)} \lesssim \|g\|_{B_{p,q}^s(\mathbb{R}^d)} \]
for any $g \in B_{p,q}^s(\mathbb{R}^d)$.

**Remark 5** By Proposition 2 the results from Lemma 5 are still true on the space $B_{p,q}^s(\Omega)$.

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