Randomness and Differentiability in Higher Dimensions

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Abstract. We present three theorems concerned with algorithmic randomness and differentiability of functions of several variables. Firstly, we prove an effective form of the Rademacher’s Theorem: we show that computable randomness implies differentiability of computable Lipschitz functions of several variables. Secondly, we use this result to prove that computable randomness implies differentiability of monotone functions in higher dimensions. Finally, we demonstrate that weak 2-randomness is equivalent to differentiability of computable a.e. differentiable functions of several variables.

1. Introduction

1.1. Introduction. The main subject of this paper lies at the interface of computable analysis ([24]) and algorithmic randomness ([14], [7]).

Intuitively, a real number is random if it does not have any exceptional properties. This approach can be formalized via identifying exceptional properties with effective null sets. To different types of effective null sets correspond different notions of algorithmic randomness.

One of the most fruitful areas of research concerned with interconnections between the two subjects is differentiability of effective functions. The main reason for this is that sufficiently well-behaved functions are almost everywhere differentiable. In this case the set of non-differentiability points of an effective function forms an effective null set and thus a test for algorithmic randomness. This makes it possible to characterize different randomness notions in terms of sets of points of differentiability of effective functions. Conversely, sets of points of differentiability for functions of particular classes can be characterised in terms of algorithmic randomness. The results of this kind are particularly compelling, since they show non-trivial connections between two seemingly distant areas of mathematics.

In recent years, a number of results of that kind have been published (for example, see [3,16,19]). Most of them are concerned with functions of one variable. Relatively few results are known about effective functions of several variables.

Our first result is concerned with Lipschitz functions, which are particularly well behaved and enjoy a lot of attentions from mathematicians since they appear naturally in various contexts. The following classical result is called Rademacher’s Theorem (see Section 3.1 in [8]), it states that Lipschitz functions are almost everywhere differentiable.

Theorem 1.1.1 (Rademacher, [20]). Suppose $U$ is an open subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}^m$ is a Lipschitz function. Then there exists a null set, such that $f$ is differentiable outside it.

We prove the following effective form of Rademacher’s Theorem.
**Theorem 2.0.9.** Let \( f : [0, 1]^n \to \mathbb{R} \) be a computable Lipschitz function and let \( z \in [0, 1]^n \) be computably random. Then \( f \) is differentiable at \( z \).

The one dimensional variant of effective Rademacher’s Theorem and its converse have been proved in [9].

**Theorem 1.1.2** (Theorem 4.2 in [9]). A real \( z \in [0, 1] \) is computably random \( \iff \) each computable Lipschitz function \( f : [0, 1] \to \mathbb{R} \) is differentiable at \( z \).

Theorem 2.0.9 generalizes the \( \Rightarrow \) direction of the above result. The other direction does not hold in general, as will be shown using a recent result by Doré and Maleva [6].

The question whether the converse of classical Rademacher’s Theorem holds, that is whether every Lebesgue null-set is contained in a set of non-differentiability points of a Lipschitz function, has been answered very recently after several decades of work by classical analysts (see [18] and [17]). The converse holds when \( m \geq n \) and does not hold otherwise.

Our next result is concerned with differentiability of monotone functions of several variables. Monotone functions are closely related to Lipschitz functions and they play a prominent role in variational analysis (see [21]). It is known that on the unit interval differentiability of computable monotone functions is equivalent to computable randomness.

**Theorem 1.1.3** (Theorem 4.1 in [3]). A real \( x \) is computably random \( \iff f'(x) \) exists for each computable nondecreasing function \( f : [0, 1] \to \mathbb{R} \).

Using Theorem 2.0.9 we prove the following generalization of the \( \Rightarrow \) implication of Theorem 1.1.3.

**Theorem 3.4.1.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a computable monotone function and let \( z \in [0, 1]^n \) be computably random. Then \( f \) is differentiable at \( z \).

To characterise differentiability sets of effective functions in terms of randomness in the usual way, those functions must be differentiable almost everywhere, for otherwise the sets of non-differentiability do not form null sets and cannot be interpreted as exceptional properties. This implies that the broadest possible class of functions in this context is the class of almost everywhere differentiable functions.

For functions of one variable the following result is known:

**Theorem 1.1.4** (Theorem 6.1 in [3]). Let \( z \in [0, 1] \). The following are equivalent:

1. \( z \) is weakly 2-random, and
2. all computable a.e. differentiable functions are differentiable at \( z \).

Our final result is the following generalization of that theorem.

**Theorem 4.0.6.** Let \( z \in [0, 1]^n \), then the following are equivalent:

1. \( z \) is weakly 2-random,
2. all partial derivatives exist for all computable a.e. differentiable \( f : [0, 1]^n \to \mathbb{R} \),
3. each computable a.e. differentiable function is differentiable at \( z \).

1.2. Structure of the paper. In the rest of this section we present relevant definitions and facts and introduce some useful notation.

In Section 2 we prove an effective version of Rademacher’s Theorem. We start the section by recalling some important facts about Lipschitz functions and then
proceed with the proof of the main result. The section ends with a discussion of a relatively recent classical result of Maleva and Doré. We verify that their result implies that weak randomness does not imply differentiability of all computable Lipschitz functions from \([0, 1]^n\) to \(\mathbb{R}\).

In Section 3 we prove that computable randomness implies differentiability of computable monotone functions from \(\mathbb{R}^n\) to \(\mathbb{R}^n\).

In Section 4 we demonstrate that weak 2-randomness characterizes differentiability points of computable a.e. differentiable functions.

The last section discusses some open problems related to this article.

1.3. Preliminaries.

1.3.1. Measure. With the exception of Subsection 3.2 and Subsection 3.3 we work exclusively with Lebesgue measure on \([0, 1]^n\). Slightly abusing notation, we always denote it by \(\lambda\).

1.3.2. Derivatives in higher dimensions. Let \(f : [0, 1]^n \to \mathbb{R}\) be a function and let \(x \in [0, 1]^n\). We say \(f\) is differentiable at \(x\) if for some linear map \(T\) the following holds
\[
\lim_{h \to 0} \frac{f(x + h) - f(x) - T \cdot h}{||h||} = 0.
\]
Then, by definition, \(f'(x) = T\).

Let \(\{e_i : 1 \leq i \leq n\}\) denote the standard basis for \(\mathbb{R}^n\). We denote partial derivatives by \(\partial_i f(x)\), lower and upper partial derivatives by \(\underline{\partial}_i f(x)\) and \(\overline{\partial}_i f(x)\), respectively.

Working with derivatives often means working with slopes. In our proofs we use the following notation.

Fix coordinate \(i\). For \(x \in [0, 1]^n\) and \(h \in \mathbb{R}\), define
\[
\delta_i^+(x, h) = \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h},
\]
and
\[
\delta_i^{1..n}(x, h) = \left[ \delta_i^1(x, h) \quad \ldots \quad \delta_i^n(x, h) \right].
\]

1.3.3. Computable real functions. There are multiple ways of formalizing computability of real functions, most of which turned out to be equivalent. We will rely on the following definitions.

A sequence \((q_i)_{i \in \mathbb{N}}\) of elements of \(\mathbb{R}^n\) is called a Cauchy name if the coordinates of each \(q_i\) are rational, and \(\|q_k - q_n\| \leq 2^{-n}\) for all \(n, k\) with \(k \geq n\). If \(\lim_{n \to \infty} q_n = x\), then we say that \((q_i)_{i \in \mathbb{N}}\) is a Cauchy name for \(x\).

We say \(x \in \mathbb{R}^n\) is computable if there is a computable Cauchy name for \(x\).

Definition 1.3.1. A function \(f : [0, 1]^n \to \mathbb{R}^m\) is computable if:

1. \(f(q)\) is computable (uniformly in \(q\)) where \(q\) has all dyadic rational coordinates, and
2. \(f\) is effectively uniformly computable, that is if there is a computable \(h : \mathbb{N} \to \mathbb{N}\) such that \(\|x - y\| \leq 2^{-h(i)}\) implies \(\|f(x) - f(y)\| \leq 2^{-i}\) for all \(x, y \in [0, 1]^n\) and all \(i \in \mathbb{N}\).

A more intuitive understanding of the above definition is that \(f\) is computable if there is an algorithm that, given a Cauchy name for \(x\), computes a Cauchy name for \(f(x)\).
1.3.4. Algorithmic randomness.
The most common method for defining a randomness notion is via effective null sets. The following two randomness notions are of direct interest to us and are defined in terms of avoidance of effective null sets.

**Definition 1.3.2.** Let $z \in [0, 1]^n$. We say $z$ is **weakly random** if there does not exist a $\Pi_0^1$ null set that contains $z$. Similarly, we say $z$ is **weakly 2-random** if there does not exist a $\Pi_0^1$ null set that contains $z$.

An alternative approach to formalizing randomness notions is via effective betting strategies. An infinite binary string can be thought of as random if no (effective) betting strategy can succeed by betting on bits of that string. Betting strategies are usually formalized as martingales (see [14], Chapter 7).

**Definition 1.3.3.** We say a function $B : 2^{<\omega} \to \mathbb{Q}_+$ is a **martingale** if the following condition holds for all $\sigma \in 2^{<\omega}$:

$$2B(\sigma) = B(\sigma_0) + B(\sigma_1).$$

$B(\sigma)$ can be interpreted as the value of capital after betting on bits of $\sigma$. We say $B$ succeeds on $Z \in 2^{<\omega}$ if $\lim \inf_n B(Z \upharpoonright_n) = \infty$.

**Definition 1.3.4.** We say $Z \in 2^{<\omega}$ is **computably random** if no computable martingale succeeds on $Z$.

We say $z = (0.Z_1, \ldots, 0.Z_n) \in [0, 1]^n$ is computably random if its binary expansion, that is $Z = Z_1 \oplus \cdots \oplus Z_n$, is computably random. Here $0.A$ denotes the real number whose binary expansion is $A \in 2^{<\omega}$.

It is known that weak 2-randomness implies computable randomness and computable randomness implies weak randomness.

1.3.5. Preservation of computable randomness.

**Definition 1.3.5.** (cf. 7.1 in [23])

We say that $\phi : [0, 1]^n \to [0, 1]^n$ is **almost everywhere (a.e.) computable** if there exists a partial computable $F : \mathbb{N}^n \to \mathbb{N}^n$ and a subset $A \subseteq [0, 1]^n$ with $\lambda(A) = 1$ such that:

1. For all $x \in A$, given a Cauchy name of $x$, $F$ computes a Cauchy name for $\phi(x)$, and
2. $x \in A$ iff for all $a, b$, which are Cauchy names for $x$, both $F(a)$ and $F(b)$ are Cauchy names for the same element.

We say that $\phi : [0, 1]^n \to [0, 1]^n$ is an **a.e. computable isomorphism** if there exists $\psi : [0, 1]^n \to [0, 1]^n$ such that $\phi \circ \psi = \text{id}$ and $\psi \circ \phi = \text{id}$ almost everywhere and both $\psi, \phi$ are measure preserving and a.e. computable.

We are interested in the above notions for the following property of computable randomness proven by Rute.

**Theorem 1.3.6** (Theorem 7.9 in [23]). Let $T$ be an a.e. computable isomorphism. Then for all $x \in [0, 1]^n$, $x$ is computably random if and only if $T(x)$ is computably random.
1.3.6. Uniform relative computable randomness.
Both the following definition and theorem are due to Miyabe and Rute ([12]).

**Definition 1.3.7.** A total computable function $m : 2^\omega \times 2^{<\omega} \to \mathbb{R}$ is a uniform computable martingale if $m(Z, \cdot)$ is a martingale for every $Z \in 2^\omega$.

We say $A$ is computably random uniformly relative to $B$ if there is no uniform computable martingale $m$ such that $m(B, \cdot)$ succeeds on $A$.

Note that the above definition works for elements of $[0, 1]^n$ as well.

**Theorem 1.3.8** (Theorem 1.3 in [12]). $A \oplus B$ is computably random if and only if $A$ is computably random uniformly relative to $B$ and $B$ is computably random uniformly relative to $A$.

2. Effective form of Rademacher’s Theorem

In this section we prove a theorem which can be seen as an effective version of Rademacher’s.

**Theorem 2.0.9.** Let $f : [0, 1]^n \to \mathbb{R}$ be a computable Lipschitz function and let $z \in [0, 1]^n$ be computably random. Then $f$ is differentiable at $z$.

**Remark 2.0.10.** An immediate consequence of the above theorem is that computable randomness of $z \in [0, 1]^n$ is sufficient for differentiability of computable Lipschitz functions form $[0, 1]^n$ to $\mathbb{R}^m$ for any $n, m$.

Lipschitz functions are particularly well-behaved and have a number of properties related to differentiability in general and to directional derivatives in particular. Some of those properties will be used by us in the proof of the above theorem and this is why we start this section by recalling some facts about Lipschitz functions and by establishing some useful notation before proceeding to the proof.

2.1. Lipschitz functions.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if there exists $L \in \mathbb{R}^+$ such that

$$||f(x) - f(y)|| \leq L \cdot ||x - y|| \text{ for all } x, y \in \mathbb{R}^n.$$ 

The least such $L$ is called the Lipschitz constant for $f$. We denote it by $\text{Lip}(f)$.

Let $K_n \subset \mathbb{R}^n$ be defined as $K_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \leq n\}$. We say $f : \mathbb{R}^n \to \mathbb{R}$ is $K_n$-increasing if $f(x + k) \geq f(x)$ for all $k \in K_n$. $f$ is called $K_n$-monotone if either $f$ or $-f$ is $K_n$-increasing.

**Remark 2.1.1.** Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is a sum of two $K_n$-monotone functions. To see this, let $m = (\text{Lip}(f), \ldots, \text{Lip}(f)) \in \mathbb{R}^n$ and note that $f = (f + \langle m, x \rangle) - \langle m, x \rangle$, and that both summands are $K_n$-monotone.

2.2. Directional, Gâteaux and Fréchet derivatives.

In order to exploit some of the properties of Lipschitz functions, we need to present a more nuanced view of differentiability in higher dimensions.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, we define the Dini-directional derivatives of $f$ at a point $x \in \mathbb{R}^n$ with respect to a direction $v \in \mathbb{R}^n$ as

$$D_+ f(x; v) = \limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}$$

and

$$D_- f(x; v) = \liminf_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$
When $\overrightarrow{D} f(x; v) = \underline{D} f(x; v)$ is finite, we define one-sided directional derivative by

$$D_+ f(x; v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$  

The two-sided directional derivative $D f(x; v)$ is defined by

$$D f(x; v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$  

To work with directional slopes, we need the following notation. For $x \in [0, 1]^n$, $v \in \mathbb{R}^n$, and $h \in \mathbb{R}$, define

$$\delta_f^v(x, h) = \frac{f(x + hv) - f(x)}{h}.$$  

If all two-sided directional derivatives of $f$ at $x$ exist and the function $T$ given by $T(y) = D f(x; y)$ is linear, then $f$ is said to be Gâteaux-differentiable at $x$. The linear map $T$ is called the Gâteaux derivative of $f$ at $x$. Furthermore, if $f$ is Gâteaux-differentiable at $x$ and if

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - T \cdot h}{||h||} = 0,$$  

then $f$ is said to be Fréchet differentiable at $x$. Thus, Fréchet differentiability is equivalent to the usual differentiability.

The following observation is crucial for the main proof and justifies presenting differentiability in this more elaborate way.

**Remark 2.2.1.** For Lipschitz functions on $\mathbb{R}^n$, Gâteaux and Fréchet differentiability coincide (for example, see Observation 9.2.2 in [19]).

Furthermore, it is known (see [3]) that for a $K_n$-monotone function $f$, both $\overrightarrow{D} f(x; \cdot)$ and $\underline{D} f(x; \cdot)$ are continuous on the interior of $K_n \cup -K_n$. Thus, for a Lipschitz function $f$, both $\overrightarrow{D} f(x; \cdot)$ and $\underline{D} f(x; \cdot)$ are continuous everywhere.

The following property is a direct consequence of the above fact. Let $A$ be a dense subset of $\mathbb{R}^n$, let $f : [0, 1]^n \to \mathbb{R}$ be a Lipschitz function and let $x \in [0, 1]^n$, then:

**(*)** if $v \mapsto D_+ f(x; v)$ is defined and is linear on $A$, then $v \mapsto D_+ f(x; v)$ is defined everywhere and is linear.

This means that in order to show that a Lipschitz function $f$ is differentiable at $x$, it is sufficient to show that $D f(x; \cdot)$ is defined and linear on a dense subset of directions.

### 2.3. Overview of the proof.

The proof consists of three distinct steps.

(1) We show that all partial derivatives of $f$ at $z$ exist. Firstly, we prove an analogous result for $K_n$-monotone functions and then use the fact that every Lipschitz function is a sum of two $K_n$-monotone functions to prove the required result holds for Lipschitz functions. The result for $K_n$-monotone functions is a consequence of the following two facts: (i) a uniform relativization of the $\Rightarrow$ implication of Theorem 1.1.3 and (ii) a form of Van Lambalgen’s Theorem for computable randomness proven by Miyabe and Rute [12].

(2) We use the above fact to show that all one-sided directional derivatives of $f$ at $x$ exist. Since, by Remark 2.2.1, we are only required to show this for a dense
set of directions, we only consider computable directions $v$. Two observations play a crucial role in this step: (a) computable randomness is preserved by computable linear isometries, and (b) linear functions are Lipschitz. The above observations are used to define a computable Lipschitz function $g_\varepsilon : [0, 1]^n \to \mathbb{R}$ such that $D_1 g_\varepsilon (\tilde{z}) = Df (z; v)$ for some computably random $\tilde{z} \in [0, 1]^n$. By the result proven in the first step, $D_1 g_\varepsilon (\tilde{z})$ does exist.

(3) Finally, we show that the function $T(u) = D_+ f (z; u)$ is linear. Again, we consider only computable directions. We show that any point where directional derivative is not linear and the failure of linearity is witnessed by a computable direction, belongs to a $\Pi^0_1$ null set. Since $z$ is computably random, this completes the proof.

Showing the linearity of $D_+ f (z; \cdot)$ is the final step of our proof, because for Lipschitz functions, Gâteaux differentiability implies (full) differentiability.

2.4. Existence of partial derivatives. Firstly, we show that computable randomness is sufficient for all partial derivatives of computable $K_n$-monotone functions to exist. An analogous result for computable Lipschitz functions is a simple corollary of that.

To achieve the required result, we combine a variation of Van Lambalgen’s Theorem for computable randomness, proven by Miyabe and Rute [12], with the following variation of the $\Rightarrow$ implication of Theorem 1.1.3.

**Lemma 2.4.1.** Let $g : 2^\omega \times [0, 1] \to \mathbb{R}$ be a total computable function such that $g(X, \cdot)$ is monotone for all $X \in 2^\omega$ and let $Z, Y \in 2^\omega$. If $Z \oplus Y$ is computably random, then $g'_Y (z)$ exists, where $g_Y = g(Y, \cdot)$ and $z = 0.Z$.

The proof of Lemma 2.4.1 is a modification of the proof of Theorem 4 in [15]. Theorem 4 in [15] is a polynomial version of Theorem 1.1.3 but its proof is somewhat simpler and requires only a few modifications to yield the kind of uniform relativization needed for the proof of Lemma 2.4.1. We will describe the required changes without repeating the whole proof.

**Proof.** The proof is by contraposition. Let $g : 2^\omega \times [0, 1] \to \mathbb{R}$ be a total computable function such that $g(X, \cdot)$ is monotone for all $X \in 2^\omega$. Let $z \in [0, 1]$, let $Z$ be the binary expansion of $z$ and let $Y \in 2^\omega$. Define $g_Y = g(Y, \cdot)$ and suppose $g'_Y (z)$ doesn’t exist. We need to exhibit a uniform computable martingale $d$ such that $d(Y, \cdot)$ succeeds on $Z$.

In the $\Rightarrow$ direction of the original proof, assuming $f'(x)$ does not exists (where $f : [0, 1] \to \mathbb{R}$ is a polynomial time computable monotone function), Nies constructed a (polynomial time) computable martingale that succeeds on the binary expansion of $x$. The assumption that $f$ is polynomial time computable was used to show that the resulting martingale is polynomial time computable. If this assumption is relaxed so that $f$ is assumed to be computable, the resulting martingale ends up being computable, rather than polynomial time computable. We need to verify that a slightly modified proof works for demonstrating that there is a uniform computable martingale $d$ such that $d(Y, \cdot)$ succeeds on $Z$.

Uniform relativization of the $\Rightarrow$ implication of Theorem 4 in [15]. Here we use the combined terminology from the original proof and the terminology required for our proof. For the $\Rightarrow$ direction of the proof of Theorem 4 in [15], Nies had to consider two cases: $\overline{D}_2 f(x) < \overline{D} f(x)$ and $\overline{D} f(x) < D_2 f(x)$. Nies constructed a
pair of computable martingales, \(L\) and \(L'\), corresponding to the above mentioned cases, such that either \(L\) succeeds on the binary expansion of \(x\), or \(L'\) succeeds on the binary expansion of \(x - 1/3\). Both \(L\) and \(L'\) query the same martingale \(M\) defined by \(M(\sigma) = S_f([\sigma])\). Since \(M : 2^\omega \times 2^{<\omega} \to \mathbb{R}\) defined by

\[
M(Y, \sigma) = S_g(Y, \cdot)([\sigma])
\]

is a uniform computable martingale, it can be easily checked that constructions of \(L\) and \(L'\) can be naturally extended to define uniform computable martingales \(L\) and \(L'\) such that either

1. \(L(Y, \cdot)\) succeeds on \(Z\) or
2. \(L'(Y, \cdot)\) succeeds on the binary expansion of \(z - 1/3\) (without loss of generality we may assume that \(z > 1/3\)).

The first case implies that \(Z\) is not computably random uniformly relative to \(Y\) and thus \(Z \oplus Y\) is not computably random.

Note that \((x_1, x_2) \mapsto (x_1, x_2 + 1/3 \mod 1)\) is an a.e. computable isomorphism. And since computable randomness is preserved by a.e. computable isomorphisms, the second case implies that \(Z \oplus Y\) is not computably random.

\[\square\]

Remark 2.4.2. The original proof relied on a different preservation property of computable randomness. It was using the fact that computable randomness is base invariant. We could not use the result about base invariance in our proof immediately (since we have now multiple coordinates instead of one), hence we chose to use another preservation property of computable randomness.

Lemma 2.4.3. Let \(z \in [0, 1]^n\) be computably random and let \(f : [0, 1]^n \to \mathbb{R}\) be a computable \(K_n\)-increasing function. Then all partial derivatives of \(f\) at \(z\) exist.

Proof. Fix \(i \leq n\). The proof is by contraposition: suppose \(D_i f(z)\) does not exist, we will show that \(z\) is not computably random.

Let \(y = z - z_i e_i\) and let \(Y\) be its binary expansion. Define \(g : 2^\omega \times [0, 1] \to \mathbb{R}\) by

\[
g(X, h) = f(0.X + he_i)
\]

and let \(g_y = g(Y, \cdot)\). Then \(g\) satisfies relevant assumptions of Lemma 2.4.1 and \(g_y'(z_i) = D_i f(z)\). Furthermore, we know that \(g_y'(z_i)\) does not exist. To show \(z\) is not computably random, by Theorem 1.3.8 it is sufficient to show that \(Z_i\) is not computably random uniformly relative to \(Y\) (as this implies \(Z_i\) not being computably random uniformly relative to \(\oplus_{j \neq i} Z_j\)). This follows from Lemma 2.4.1

\[\square\]

Lemma 2.4.4. Let \(z \in [0, 1]^n\) be computably random and let \(f : [0, 1]^n \to \mathbb{R}\) be a computable Lipschitz function. Then all partial derivatives of \(f\) at \(z\) exist.

Proof. Similar to the Remark 2.4.1, let \(M = \text{Lip}(f)\) and let \(m = (M, \ldots, M) \in \mathbb{R}^n\), then \(g(x) = f(x) + m \cdot x\) is a \(K_n\)-increasing computable function. Thus all partial derivatives of \(g\) at \(z\) exist, and therefore all partial derivatives of \(f\) at \(z\) exist too. \[\square\]
2.5. **Existence of directional derivatives.** We will use the previously proven fact about existence of partial derivatives of Lipschitz functions to show that, in fact, an analogous result holds for all one-sided directional derivatives. The main idea relies on two simple observations:

1. computable randomness is invariant under computable linear isometries, and
2. linear functions are Lipschitz.

For any \( u, v \in \mathbb{R}^n \) with \( \|v\| = \|u\| = 1 \) and \( u \neq v \), fix (say, via the Gram-Schmidt process) two orthonormal bases \( B_u, B_v \) of \( \mathbb{R}^n \) with \( v \in B_v \) and \( u \in B_u \). Let \( \Theta_{u \to v} : \mathbb{R}^n \to \mathbb{R}^n \) denote a change of basis map (that takes \( B_u \) to \( B_v \)) such that \( \Theta_{u \to v}(u) = v \). This function is a linear isometry and it is computable when \( u, v \) are computable.

The image of the unit cube \([0, 1]^n\) under functions of the form \( \Theta_{u \to v} : \mathbb{R}^n \to \mathbb{R}^n \) is not necessarily contained in \([0, 1]^n\). To deal with this issue, we use the function \( \mathcal{P}_1 : \mathbb{R}^n \to [0, 1]^n \) defined by

\[
\mathcal{P}_1(x_1, \ldots, x_n) = (\min\{1, x_1\}, \ldots, \min\{1, x_n\}).
\]

\( \mathcal{P}_1 \) is a computable Lipschitz function which coincides with the identity map on the unit \( n \)-cube. For any function \( f : [0, 1]^n \to \mathbb{R} \), let \( \hat{f} = f \circ \mathcal{P}_1 \), so that if \( f \) is computable and a.e. differentiable, so is \( \hat{f} \). Moreover, if \( f \) is Lipschitz, so is \( \hat{f} \). Note that \( \hat{f} \) is defined on the whole \( \mathbb{R}^n \).

**Lemma 2.5.1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function, let \( u, v, w \in \mathbb{R}^n, x \in [0, 1]^n \) and let \( \Theta = \Theta_{v \to u} \). Then

\[
D_+ f(x; u) = D_+ g(z; v)
\]

where \( g = f \circ (\Theta + w) \) and \( z = \Theta^{-1}(x - w) \).

**Proof.**
First, note that for any \( t > 0 \),

\[
\frac{g(z + tv) - g(z)}{t} = \frac{f(\Theta(z + tv) + w) - f(\Theta(z) + w)}{t} = \frac{f(x + tu) - f(x)}{t}.
\]

By taking the limits of both sides we get the required equality. \( \square \)

**Lemma 2.5.2.** Let \( f : [0, 1]^n \to \mathbb{R} \) be computable Lipschitz and suppose \( x \in [0, 1]^n \) is computably random. Then \( D_+ f(x; u) \) exists for every \( u \in \mathbb{R}^n \).

**Proof.** It is sufficient to show that \( D_+ f(x; u) \) exists for each computable \( u \) with \( \|u\| = 1 \).

Let \( u \) be computable and let \( v = e_1 \). By density we can find some computable \( w \in \mathbb{R}^n \), so that \( z = \Theta_{e_1 \to u}^{-1}(x - w) \) is contained in \([0, 1]^n\). We apply Lemma 2.5.1 to \( f, v, u, w \) and \( x \), so that

\[
D_+ f(x; u) = D_+ \hat{f}(x; u) = D_+ g(z; v)
\]

where \( g \) is Lipschitz and computable and \( z \in [0, 1]^n \) is computably random (again, we use Theorem 1.3.6 here). The required result follows from the fact that \( D_+ g(z; v) = D_1 g(z) \) and we know that \( D_1 g(z) \) exists. \( \square \)
2.6. Linearity of directional derivatives.
In the last step of the proof, we need to show that $D_+ f (z; \cdot)$ is linear on computable elements (where $f$ is computable Lipschitz and $z$ is computably random).

Let $f : [0, 1]^n \rightarrow \mathbb{R}$ be a function. For $u \in \mathbb{R}^n$, define

$$\mathcal{K}^f_u = \{ z \mid D_+ f (z; u) \text{ exists} \}.$$  

For $q \in \mathbb{Q}^+$ and $u, v \in \mathbb{R}^n$, define $\mathcal{L}^f_{u,v,q}$ to be the set of points where linearity of $D f (z; \cdot)$ fails and the failure is witnessed by $u, v$ and $q$, or, more formally,

$$\mathcal{L}^f_{u,v,q} = \mathcal{K}^f_u \cap \mathcal{K}^f_v \cap \mathcal{K}^f_{u+v} \cap \{ z \mid |D_+ f (z; u + v) - D_+ f (z; u) - D_+ f (z; v)| \geq q \}.$$  

**Lemma 2.6.1.** Let $f : [0, 1]^n \rightarrow \mathbb{R}$ be a computable a.e. differentiable function. Let $v, u \in \mathbb{R}^n$ be computable. Let $z \in \mathcal{L}^f_{v,u,q}$ for some $q \in \mathbb{Q}$. Then there exist a $\Pi^0_1$ null-set that contains $z$.

**Proof.** Since $D_+ f (z; v), D_+ f (z; u)$ and $D_+ f (z; v + u)$ exist, there is $p > 0$ such that $|\delta^+_f (z, h) + \delta^+_f (z, h) - \delta^+_{v+u} (z, h)| \geq q$ for all $h \leq p$. Hence the set of all $x$ such that

$$\forall h \left( h \leq p \Rightarrow |\delta^+_f (x, h) + \delta^+_f (x, h) - \delta^+_{v+u} (x, h)| \geq q \right),$$

where $h$ range over rationals, contains $z$. It is clearly a $\Pi^0_1$ set and it is a null set, since its complement contains all points of differentiability of $f$ and $f$ is a.e. differentiable.

So far, we have shown that computable randomness implies existence of directional derivatives in and weak randomness is sufficient for linearity of directional derivatives. This implies that computable randomness is sufficient for Gâteaux differentiability and this completes the proof of Theorem 2.0.3 since Gâteaux differentiability implies differentiability of Lipschitz functions on $[0, 1]^n$.

2.7. Effective Rademacher does not imply weak randomness. Unlike in the case of functions of one variable, computable randomness is not implied by differentiability in higher dimensions. In fact, a relatively recent classical result can be used to show that even weak randomness is too strong in higher dimensions. Doré and Maleva in \cite{DoreMaleva} constructed a family of compact null sets, every member of which contains points of differentiability of every Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Such sets are called *universal*. The crucial idea in their construction is that a Lipschitz function is differentiable at points where a directional derivative is maximal in some specific sense and that such points can be found on small line segments (see Lemmata 4.2 and 4.3 in \cite{DoreMaleva}). Their sets contain lots of such line segments and this is the reason they are universal.

The construction in \cite{DoreMaleva} is parameterized by two sequences. Below we verify that with a suitable parameters this construction yields a $\Pi^0_1$ null set.

**Construction by Doré and Maleva.**
Let $(N_i)_{i \in \mathbb{N}}$ be a sequence of odd integers such that $N_1 > 1$, $\lim_{i \to \infty} N_i = \infty$ and $\sum \frac{1}{N_i} = \infty$. Let $(p_i)_{i \in \mathbb{N}}$ be a sequence of real numbers with $1 \leq p_i \leq N_i$ and $\lim_{i \to \infty} p_i / N_i = 0$. Let $d_0 = 1$ and for all $i \geq 1$ let $d_i = \prod_{k \leq i} N_k^{-1}$ and define a lattice in $\mathbb{R}^2$

$$C_i = \left( \frac{d_{i-1}}{2}, \frac{d_i}{2} \right) + \mathbb{Z}^2.$$
Finally, define

\[ W = \mathbb{R}^2 \setminus \bigcup_{i \in \mathbb{Z}} B_{\infty}(c, p_i d_i / 2), \]

where \( B_{\infty}(x, r) \) denotes an open ball in \((\mathbb{R}^2, \| \cdot \|_{\infty})\).

\( W \) is a closed null set. Doré and Maleva proved [Corollary 3.2 in [6]] that for any such \( W \), any open neighbourhood of the set \( M = \mathbb{R}^{n-2} \times W \) contains a point of differentiability of every Lipschitz function \( f : \mathbb{R}^n \to \mathbb{R} \). In particular, \( [0, 1]^n \cap M \) contains a point of differentiability of every Lipschitz \( f : [0, 1]^n \to \mathbb{R} \).

It is easy to see that both \((N_i)_{i \in \mathbb{N}}\) and \((p_i)_{i \in \mathbb{N}}\) can be taken to be computable sequences and then \([0, 1]^n \cap M\) is a \( \Pi^0_3 \) null set. (For example, take \((N_i)_{i \in \mathbb{N}}\) to be \(3,3,5,5,5,5,7,7,7,...\) and let \(p_i = 4\) for all \(i\).

This shows that for \(n > 1\), the randomness notion induced by Rademacher’s Theorem is not equivalent to computable randomness.

It is worth noting that differentiability of computable Lipschitz functions implies a notion that is properly weaker than weak randomness. For a given porous \( \Pi^0_3 \) null set \( A \), the function \( f(x) = d(A, x) \) (that is, the distance from \( x \) to \( A \)) is Lipschitz, it is computable (see [4]) and it is not differentiable at any point of \( A \).

3. Differentiability of computable monotone functions from \( \mathbb{R}^n \) to \( \mathbb{R}^n \)

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a function. We say \( f \) is monotone if

\[ (f(x) - f(y), x - y) \geq 0 \text{ for all } x, y \in \mathbb{R}^n. \]

As we will see later in this section, monotone functions are very closely related to Lipschitz functions.

In non-effective setting, a.e. differentiability of monotone function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) has been proven by Mignot [10], who used Rademacher’s Theorem and a fact about monotone functions discovered by Minty [11].

In this section we will show that computable randomness implies differentiability of computable monotone real functions of several variables and our proof follows the same path - using the effective form or Rademacher’s Theorem proven in the previous section and the following correspondence observed by Minty.

3.1. Minty parameterization and overview of the proof. Minty showed that the so called Cayley transformation

\[ \Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \text{ defined by } \Phi(x, y) = \frac{1}{\sqrt{2}}(y + x, y - x) \]

transforms the graph of a monotone function into a graph of a graph of a 1-Lipschitz function. Note that when \(n = 1\) this is a clockwise rotation of \(\pi/4\). We will rely on the following consequence of the above fact.

**Proposition 3.1.1** (cf. Proposition 1.2 in [1]). Let \( u : \mathbb{R}^n \to \mathbb{R}^n \) be monotone. Then \((u + I)\) and \((u + I)^{-1}\) are monotone and \((u + I)^{-1}\) is 1-Lipschitz.

**Proposition 3.1.2** (cf. Theorem 12.65 in [21]). Let \( u : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous monotone function. Let \( z \in \mathbb{R}^n \) and define \( f = (u + I)^{-1} \) and \( \hat{z} = u(z) + z \). The following two are equivalent:

1. \( u \) is differentiable at \( z \), and
2. \( f \) is differentiable at \( \hat{z} \) and \( f'(\hat{z}) \) is invertible.
A good exposition of classical results related to this area can be found in [1] and [21].

Now we are ready to explain our proof.

Overview of the proof

Let \( u : \mathbb{R}^n \to \mathbb{R}^n \) be a monotone computable function and let \( z \in [0,1]^n \) be computably random. Then \( g = u + I \) is monotone and computable and \( f = g^{-1} \) is 1-Lipschitz and computable. If we can show that \( g(z) \) is computably random, then \( g(z) \) is differentiable at \( g(z) \). By Proposition 3.1.2 if the derivative of \( f \) at \( g(z) \) is invertible, then \( g \) is differentiable at \( z \).

From the above description, it is clear that we require the following two ingredients to complete the proof:

(preservation property) we need to show that \( g(z) \) is computably random when \( z \) is, and that

(singularity property) computable randomness of \( g(z) \) implies that \( f'(g(z)) \) is invertible.

In the following two subsections we will prove both of the above.

3.2. Another preservation property. To prove the preservation property mentioned in the previous subsection, we require some terminology and notation from [23].

Firstly, we need to extend the notion of computable randomness to \( \mathbb{R}^n \): we say \( z \in \mathbb{R}^n \) is computably random if its binary expansion (or, equivalently, its fractional part) is computably random. When \( z \in [0,1]^n \), this characterisation is equivalent to Definition 1.3.4. Otherwise, when \( z \notin [0,1]^n \), this characterisation is equivalent to computable randomness on some computable translation of the unit cube equipped with the usual Lebesgue measure.

Notation 3.2.1. For every \( n \geq 1 \), let \( A_n \) be some fixed a.e. decidable cell decomposition of \([0,1]^n\). For the sake of simplifying the notation, in the rest of this section, for all \( n \geq 1 \) and all \( \sigma \in 2^{<\omega} \), we denote the cell \([\sigma]_{A_n}\) by \([\sigma]\).

Definition 3.2.2. A Martin-Löf test is a uniformly computable sequence \((U_i)_{i \in \mathbb{N}}\) of \( \Sigma^0_1 \) subsets of \([0,1]^n\) such that \( \lambda(U_i) \leq 2^{-i} \) for all \( i \). We say \((U_i)_{i \in \mathbb{N}}\) covers \( z \in [0,1]^n \) if \( z \in \bigcap_i U_i \).

We say a Martin-Löf test \((U_i)_{i \in \mathbb{N}}\) is bounded if there is a computable measure \( \nu : 2^{<\omega} \to [0,\infty) \) satisfying
\[
\lambda(U_i \cap [\sigma]) \leq 2^{-i} \nu(\sigma)
\]
for all \( i \in \mathbb{N} \) and \( \sigma \in 2^{<\omega} \).

We require the following characterisation of computable randomness in the unit cube due to Rute:

Proposition 3.2.3 (cf. Theorem 5.3 in [23]). Let \( z \in [0,1]^n \). The following two are equivalent:

1. \( z \) is not computably random, and
2. either \( z \) is an unrepresented point, or there is a bounded Martin-Löf test \((U_i)_{i \in \mathbb{N}}\) that covers \( z \).

Remark 3.2.4. For our considerations it is sufficient to know that if \( z \) is an unrepresented point, then it is not weakly random.

We are now in position to state and to prove the required preservation property for computable randomness.
Lemma 3.2.5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a computable Lipschitz function and suppose $z \in \mathbb{R}^n$ is not computably random. Then $f(z)$ is not computably random either.

Proof. Without loss of generality we assume $z \in [0, 1]^n$, $f(z) \in [0, 1]^n$ and Lip$(f) \leq 1$ (otherwise we may consider $f(x) = A \cdot f(x) + B$ for some suitable computable $A$ and $B$).

Firstly, let’s assume that $z$ is an unrepresented point. Let $P \subset [0, 1]^n$ be a $\Pi^0_1$ null set with $z \in P$. Then $f(z) \in f(P) \cap [0, 1]^n$ and since $f(P) \cap [0, 1]^n$ is also a $\Pi^0_1$ null set, $f(z)$ is not weakly random.

Let $(V_i)_{i \in \mathbb{N}}$ be a bounded Martin-Löf test with $z \in \bigcap_i V_i$ and let $\nu$ be a computable measure such that $\lambda(V_i \cap [\sigma]) \leq 2^{-i} \nu(\sigma)$ for all $i, \sigma$.

Define $U_i = f(V_i) \cap [0, 1]^n$ for all $i$. Since $\lambda(U_i) \leq \lambda(V_i)$ (see Lemma 3.10.12 in [2]), $(U_i)_{i \in \mathbb{N}}$ is a Martin-Löf test.

Define $\nu_f = \nu \circ f^{-1}$. It is a computable measure and for all $i, \sigma$ we have

$$\lambda(U_i \cap [\sigma]) = \lambda(f(V_i) \cap [\sigma]) = \lambda(f(V_i) \cap f^{-1}([\sigma])) \leq 2^{-i} \nu(f^{-1}([\sigma])) = 2^{-i} \nu_f(\sigma).$$

It follows that $(U_i)_{i \in \mathbb{N}}$ is a bounded Martin-Löf test that covers $f(z)$ and thus $f(z)$ is not computably random.

\[ \square \]

3.3. Singularity property. The main result in this subsection, Theorem 3.3.2, can be seen as an effective version of Sard’s Theorem for Lipschitz functions. Its classical version, proven by Mignot ([10], also see Theorem 9.65 in [21]), states that for a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$, the set of its critical values is a null-set.

Lemma 3.3.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function. Suppose $z \in \mathbb{R}^n$ is such that $f'(z)$ is singular. Then for every $\epsilon > 0$, there exists an open neighbourhood of $z$, $O$, such that $\lambda(f(O_\epsilon)) \leq \epsilon \lambda(O_\epsilon)$.

Proof. Fix $\epsilon > 0$ and let $k = \text{Lip}(f)$.

Define $\epsilon' = \frac{\epsilon}{k \cdot 2^{(n-1) \text{dim}(H)^n}}$. Since $f$ is differentiable at $z$, there exists $\delta > 0$ such that

$$|f(x) - f(z) - f'(z)(x - z)| \leq \epsilon'|x - z|$$

for all $x \in \mathbb{R}^n$ with $|x - z| \leq \delta$. There is an open $n$-cube $C$ with side length equal to $s = \frac{\delta}{\sqrt{n}}$ such that $z \in C$ and (1) holds for all $x \in C$.

Let $L$ be the mapping defined by $L(x) = f(z) + f'(z)(x - z)$. Since $f'(z)$ is singular, $L$ is not onto and its range is contained in some hyperplane $H$.

As a consequence of (1) we have $|f(x) - L(x)| \leq \epsilon'\delta$ for all $x \in C$. Thus, $f(C) \subset L(C) + [-\epsilon'\delta, \epsilon'\delta]^n$. Since $L$ is a $k$-Lipschitz mapping, the image of $C$ under $L$ lies in the intersection of $H$ with a closed ball with radius $k\delta$ centered at $f(z)$. Then $L(C)$ is contained in a rotated $(n - 1)$-dimensional cube of side $2k\delta$. This shows that $f(C)$ lies in a rotated box $\hat{C}$ with

$$\lambda(\hat{C}) = (2k\delta)^{n-1}2\epsilon'\delta = 2(2k)^{n-1}\epsilon'(\sqrt{n})^n \left(\frac{\delta}{\sqrt{n}}\right)^n = \epsilon \lambda(C).$$

\[ \square \]

Theorem 3.3.2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a computable Lipschitz function and let $z \in \mathbb{R}^n$. If $f(z)$ is computably random, then $f'(z)$ is not singular.
Proof. Without loss of generality we may assume \( f(z) \in [0, 1]^n \) and \([0, 1]^n \subseteq f([0, 1]^n)\). The proof is by contraposition. Suppose \( f'(z) = 0 \).

Let \( \nu = \lambda \circ f^{-1} \) and for every \( i \in \mathbb{N} \), define \( V_i \subset [0, 1]^n \) as the union of all \( [\sigma] \) such that \( \lambda(\sigma) \leq 2^{-i} \nu(\sigma) \). Note that \( \lambda(V_i) \leq 2^{-i} \) and for every \( \tau \), \( \lambda(V_i \cap [\tau]) = \sum_{[\eta] \subseteq [\tau] \cap V_i} \lambda(\eta) \leq 2^{-i} \sum_{[\eta] \subseteq [\tau] \cap V_i} \nu(\eta) \leq 2^{-i} \nu(\tau) \).

Thus \((V_i)_{i \in \mathbb{N}}\) is a bounded Martin-Löf test and, by Lemma 3.3.1, it covers \( f(z) \).

3.4. Main result. We are now ready to formulate and prove our main result concerning monotone computable functions.

**Theorem 3.4.1.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be an computable monotone function and let \( z \in [0, 1]^n \) be computably random. Then \( f \) is differentiable at \( z \).

Proof. Define \( g = (f + I)^{-1} \), then \( g \) is a computable Lipschitz function with \( \text{Lip}(g) \leq 1 \).
Let \( y = f(z) + z \) so that \( g(y) = z \). By Lemma 3.2.5, \( y \) is computably random and hence \( g \) is differentiable at \( y \) and by Theorem 3.3.2 \( g'(y) \) is invertible. Hence, by Proposition 3.1.2, \( f \) is differentiable at \( z \).

\( \Box \)

4. Characterizing weak 2-randomness in terms of differentiability

This section is devoted to proving Theorem 4.0.6, which characterises weak randomness in terms of differentiability of computable functions of several variables. It is worth pointing out that while our result is a generalization of Theorem 1.1.4, it is “stronger” in the sense that we show equivalence of three conditions, rather than two. Recall that Theorem 1.1.4 shows weak 2-randomness is equivalent to differentiability of computable a.e. differentiable functions. Somewhat surprisingly, in higher dimensions, a seemingly weaker condition, the existence of all partial derivatives for all computable a.e. differentiable functions, is also equivalent to weak 2-randomness.

We start the section with a fact of independent interest.

**Lemma 4.0.2.** Let \( f : [0, 1]^n \to \mathbb{R} \) be a computable function. The set of points at which \( f \) is differentiable is a \( \Pi_3^0 \) set.

Proof. Recall that the definition of the derivative of a function of several variables involves a nested limit. The main idea of this proof is that the set of points of differentiability (for a given function), \( D \), can be written as an intersection of two sets, each of which can be described with only one limit. Specifically, we will show that \( D \) is the intersection of two \( \Pi_3^0 \) sets, \( A \) and \( B \), where \( A \) is the set of points where all partial derivatives of \( f \) exist, and \( B \) is the set consisting of those \( x \) satisfying

\[
\lim_{||h|| \to 0, b \to 0} \frac{f(x + h) - f(x) - \delta_{1^{-n}}(x,b) \cdot h}{||h||} = 0.
\] (2)

**Claim 4.0.3.** \( A \) is a \( \Pi_3^0 \) set that contains all points of differentiability of \( f \).
By density, we can take \( h \) such that for all \( \delta > 0 \):

\[ \delta f(x, h) > p \quad \forall \rho \delta. \]

By definition, \( \lim_{|h| \to 0} f(x + h - f(x) - J_f(x) \cdot h) = 0. \]

To see that \( B \) is a \( \Pi^0_3 \) set, we can rewrite the condition (2) in the following form:

\[ \forall \epsilon > 0 \wedge \exists \delta > 0 \wedge \exists b \in \mathbb{Q} \wedge \exists h \in \mathbb{Q} \left( \epsilon > \frac{\|f(x + h) - f(x) - \delta f(x, b) \cdot h\|}{\|h\|} \implies \|f(x + h) - f(x) - \delta f(x, b) \cdot h\| \leq \epsilon \right) \]

Here \( \epsilon, \delta, h, \) and \( b \) are rationals, and \( h \) has rational coordinates.

Suppose \( x \) is a point at which \( f \) is differentiable. Fix \( \epsilon > 0 \). Let \( \delta \) be sufficiently small that for all \( h \) with \( \|h\| < \delta \),

\[ \frac{\|f(x + h) - f(x) - J_f(x) \cdot h\|}{\|h\|} < \epsilon / 2, \]

and also for all \( b \) with \( |b| < \delta \),

\[ \|\delta f(x, b) - J_f(x)^T\| < \epsilon / 2. \]

Here we treat \( \delta f(x, b) - J_f(x) \) as a row vector. Then for any \( h \) and \( b \) with \( \|h\| < \delta \) and \( |b| < \delta \),

\[ \frac{\|f(x + h) - f(x) - \delta f(x, b) \cdot h\|}{\|h\|} = \frac{\|f(x + h) - f(x) - J_f(x)h + J_f(x)h - \delta f(x, b) \cdot h\|}{\|h\|} \]

\[ \leq \frac{\|f(x + h) - f(x) - J_f(x)h\|}{\|h\|} + \frac{\|J_f(x) - \delta f(x, b) \cdot h\|}{\|h\|} \]

\[ < \epsilon / 2 + \frac{\|J_f(x) - \delta f(x, b)^T\|}{\|h\|} \]

Thus \( B \) contains every point at which \( f \) is differentiable.
Claim 4.0.5. \( f \) is differentiable at all elements of \( A \cap B \).

Proof. Let \( x \in A \cap B \). Fix \( \epsilon > 0 \). Since \( x \in B \), we can find \( \delta \) such that
\[
\forall h \forall b \left( |b| < \delta \land ||h|| < \delta \right) \implies \frac{|f(x + h) - f(x) - \delta^{1..n}(x, b) \cdot h|}{||h||} < \epsilon/2,
\]
and since all partial derivatives of \( f \) at \( x \) exist, we can find some \( b \) with \( |b| < \delta \) such that
\[
\frac{||((\delta^{1..n}(x, b) - J_f(x))^T||}{||h||} < \epsilon/2.
\]

Then, for any \( h \) with \( ||h|| < \delta \),
\[
\frac{|f(x + h) - f(x) - J_f(x) \cdot h|}{||h||} = \left( \frac{|f(x + h) - f(x) - \delta^{1..n}(x, b) \cdot h + \delta^{1..n}(x, b) \cdot h - J_f(x) \cdot h|}{||h||} \right) \leq \frac{|f(x + h) - f(x) - \delta^{1..n}(x, b) \cdot h|}{||h||} + \frac{|(\delta^{1..n}(x, b) - J_f(x)) \cdot h|}{||h||} < \epsilon/2 + \epsilon/2 = \epsilon.
\]

Thus \( f \) is differentiable at \( x \). \( \square \)

Theorem 4.0.6. Let \( z \in [0,1]^n \). The following are equivalent:

1. \( z \) is weakly 2-random,
2. all \( D_i f(z) \) exist for all computable a.e. differentiable \( f : [0,1]^n \to \mathbb{R} \),
3. each computable a.e. differentiable function is differentiable at \( z \).

Proof (1) \( \Rightarrow \) (3). Suppose \( z \) is weakly 2-random and \( f \) is an a.e. differentiable computable function.

Since \( z \) cannot be contained in any \( \Sigma_0^0 \) set of measure 0, it must belong to all \( \Pi_1^0 \) sets of full measure. In particular, by Lemma 4.0.3, \( z \) belongs to the set of differentiable points of \( f \).

Proof (3) \( \Rightarrow \) (2). Trivial.

Proof (2) \( \Rightarrow \) (1). Suppose \( z = (z_1, \ldots, z_n) \in [0,1]^n \) is not weakly 2-random.

We may assume that all coordinates of \( z \) are weakly 2-random, otherwise the required conclusion follows from the one dimensional case. For suppose some \( z_j \) is not weakly 2-random. Then there is a computable a.e. differentiable function \( g : [0,1] \to \mathbb{R} \) such that \( g'(z_j) \) does not exist. Define \( \gamma : [0,1]^n \to \mathbb{R} \) as \( \gamma(x_1, \ldots, x_j, \ldots, x_n) = g(x_j) \). Then \( \gamma \) is a computable a.e. differentiable function such that \( \gamma'(z) \) doesn't exist.

In what follows, we ignore those elements of \( [0,1]^n \) that have at least one of its coordinates rational.

Let \( (G_i)_{i \in \mathbb{N}} \) be a sequence of uniformly \( \Sigma_0^0 \) subsets of \( [0,1]^n \) such that \( G_{i+1} \subseteq G_i \) for all \( i \) and \( G = \bigcap G_i \) is a null-set with \( z \in G \). Since we ignore elements with
dyadic coordinates, we may assume that every $G_i$ is an infinite union of basic dyadic $n$-cubes.

Let $(D_{m,l})_{m,l \in \mathbb{N}}$ be an effective double sequence of (open) basic dyadic $n$-cubes such that $G_m = \bigcup_i D_{m,i}$ for each $m$, and for all $n, k$ there is an $l$ with $D_{n+1,k} \subseteq D_{n,l}$.

**General idea of the proof.** We will construct a computable double sequence $(C_{m,i})_{m,i \in \mathbb{N}}$ of basic dyadic $n$-cubes with certain well-behaved properties. For every $n$-cube $C_{m,i}$ in the sequence we will define a tent function $f_{m,i}$ which is 0 outside $C_{m,i}$ and its graph forms a piecewise linear “tent” at $C_{m,i}$. See figure 1 for an illustration of what a graph of a tent function on $[0,1]^2$ might look like.

$E_{m,i}$ is the subarea of $C_{m,i}$ where $|D_{1}f_{m,i}| \neq \pm 1$. The tent functions are defined in such a way that $z$ belongs to only finitely many of $E_{m,i}$. This is where our assumption that all coordinates of $z$ are not weakly 2-random is used. See figure 2.

Any point belonging to infinitely many $E_{m,i}$, by pigeonhole principle, must have at least one coordinate belonging to the (darker) corner areas (one-dimensional $E_{m,i}^{k}$ sets). Our tent functions are defined in such a way that those areas form $\Pi_{0}^{1}$ null sets.

Then $f : [0,1]^n \to \mathbb{R}$ will be defined as a sum of those $f_{m,i}$ for which we know the first partial derivative on $z$ is equal to $\pm 1$. This is used to show that $D_{1}f(z)$ does not exist. The properties of $(C_{m,i})_{m,i \in \mathbb{N}}$ ensure that $f$ is computable and a.e. differentiable.

**Construction of the double sequence $(C_{m,i})_{m,i \in \mathbb{N}}$.**

Suppose $m = 0$, or $m > 0$ and we have already defined $(C_{m-1,j})_{j \in \mathbb{N}}$. Define $(C_{m,j})_{j \in \mathbb{N}}$ as follows.

Let $N \in \mathbb{N}$ be the greatest number such that we have already defined $C_{m,i}$ for $i \leq N$. When a new $n$-cube $D = D_{m,i}$ is enumerated into $G_m$, if $m > 0$, we wait until $D$ is contained in a union of $n$-cubes $\bigcup_{r \in F} C_{m-1,r}$, where $F$ is finite. This is possible since $D$ is contained in a single cell of the form $D_{m-1,\_}$ that was handled in a previous stage. If $m = 0$, let $\delta = \lambda(D)$, otherwise let

$$\delta = \min\{\lambda(D) , \min\{\lambda(C_{m-1,r}) : r \in F\} \}. $$
If $N = 0$, let $\epsilon = 8^{-m}\delta$, otherwise let $\epsilon = \min\{8^{-m}\delta, \lambda(C_m,N-1)\}$.

Finally, partition $D$ into disjoint basic dyadic $n$–cubes $C_{m,i}$, $i = N+1, \ldots, N' < \infty$, with nonincreasing volume $\lambda(C_{m,i}) \leq \epsilon^n$, so that when $m > 0$, each of the cubes is contained in one of $C_{m-1,r}$ for some $r \in F$.

The following claim summarizes all properties of $(C_{m,i})_{m,i \in \mathbb{N}}$ relevant to our proof.

**Claim 4.0.7.** The double sequence $(C_{m,i})_{m,i \in \mathbb{N}}$ is computable and it verifies the following properties:

i) $G_m = \bigcup_{i \in \mathbb{N}} C_{m,i}$,

ii) $C_{m,i} \cap C_{m,k} = \emptyset$ and $\lambda(C_{m,i}) \geq \lambda(C_{m,k})$ for all $i < k$,

iii) if $B = C_{m,i}$ for $m > 0$, then there is an $n$–cube $A = C_{m-1,k}$ such that

$$B \subseteq A \text{ and } \lambda(B) \leq 8^{-m}\lambda(A),$$

(3)

iv) for all $m, k \in \mathbb{N}$

$$D_{m,k} = \text{some finite union of } n \text{-cubes of the form } C_{m,i}$$

with

$$C_{m,i} \subseteq D_{m,k} \implies d_{m,i} \leq 8^{-m}\lambda(D_{m,k})$$

(4)

where $d_{m,i}$ denotes the length of a side of $C_{m,i}$.

**Proof.** All of the listed properties are straightforward consequences of the construction of $(C_{m,i})_{m,i \in \mathbb{N}}$. \hfill \square

**Tent functions $f_{m,j}$.**

Let $m, j \in \mathbb{N}$. For all $i \in \mathbb{N}$ with $1 \leq i \leq n$, define $a^i_{m,j}, b^i_{m,j}$ so that

$$(a^i_{m,j}, b^i_{m,j}) = \pi_i(C_{m,j}),$$

where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denotes the projection onto the $i$–th coordinate.
Claim 4.0.8. There exists \( m,j \) such that to prove this claim, we will use our assumption that all coordinates of \( x \) are weakly 2-random. Specifically, we will show that if a point belongs to an infinitely many \( E \), then one of its coordinates belongs to a \( \Pi^0_2 \) null-set. Indeed, \((a^i_{m,j}, b^i_{m,j}), x_1)\) denotes the distance from \( x_1 \) to \([0, 1] \times (a^i_{m,j}, b^i_{m,j})\). Note that \( f_{m,j} \) is a computable (uniformly in \( m,j \)) a.e. differentiable function.

Lastly, define \( E_{m,j} \) to be the subset of \( C_{m,j} \) where \(|D_1 f_{m,j}| \neq 1\) whenever \( D_1 f_{m,j} \) exists, that is

\[
E_{m,j} = C_{m,j} \setminus \left( (a^i_{m,j}, b^i_{m,j}) \times \prod_{n \geq 2} (a^i_{m,j} + \epsilon, b^i_{m,j} - \epsilon) \right)
\]

The idea behind such definition of \( f_{m,j} \) functions is that \( \epsilon_{m,j} \) goes to 0 so quickly, that \(|D_1 f_{m,j}(z)| \neq 1\) holds only for finitely many \( m,j \in \mathbb{N} \).

Claim 4.0.8. There exists \( N \in \mathbb{N} \) such that for all \( i \in \mathbb{N} \) and \( m > N \), if \( z \in C_{m,i} \) then \(|D_1 f_{m,i}(z)| = 1\).

Proof. To prove this claim, we will use our assumption that all coordinates of \( z \) are weakly 2-random. Specifically, we will show that if a point belongs to an infinitely many \( E_{m,i} \), then one of its coordinates belongs to a \( \Pi^0_2 \) null set.

For every \( m,i,k \in \mathbb{N} \) with \( 2 \leq k \leq n \), let

\[
E^k_{m,i} = (a^k_{m,i}, a^k_{m,i} + \epsilon_{m,i}) \cup (b^k_{m,i} - \epsilon_{m,i}, b^k_{m,i})
\]

Note the following property of those sets: if \( z \in E_{m,i} \) then for some \( k \), \( z_k \in E^k_{m,i} \).

For every \( m,k \in \mathbb{N} \) with \( n \geq k \geq 2 \), let \( B^k_m = \bigcup_{i \in \mathbb{N}} E^k_{i,j} \). Let’s verify that every \( B^k_m = \bigcap_{i \geq m} B^k_i \) is a \( \Pi^0_2 \) null-set. Indeed, \((B^k_i)_{i \in \mathbb{N}}\) is a uniformly computable sequence of \( \Sigma^0_2 \) sets with \( \lambda(B^k_m) \leq \sum_{i \geq m} \sum_j 2^{-i-j} \cdot d_{i,j} \leq 8^{-m} \) for all \( m,k \).

By the pigeonhole principle, if \( z \) belongs to infinitely many \( E_{m,j} \) (for infinitely many \( m \)), then for some \( k \), \( z_k \) belongs to infinitely many \( E^k_{m,j} \). In that case \( z_k \notin B^k_m \) and we get a contradiction.

Let \( N \) be such that \( z_k \notin B^k_N \) for all \( k \) and the required result follows. \( \square \)

Definition of the function \( f \).

Let

\[
f_m = \sum_{i=0}^{\infty} 4^m f_{m,i}
\]

and

\[
f = \sum_{i > N} f_m.
\]

Claim 4.0.9. \( f \) is computable.
Proof. Fix $m > 0$. Note that every $f_{m,i}$ is bounded from above by $d_{m,i}/2$ and since all $C_{m,i}$ are disjoint, $f_m$ is bounded from above by $4^m 8^{-m}/2 = 2^{-m-1}$ and it follows that $f$ is well defined everywhere.

Firstly, let’s show that $f(q)$ is computable uniformly in rational $q$. Given $m > 0$, since $\lim_{i \to \infty} \lambda(C_{m,i}) = 0$, we can find $i^*$ such that
\[ d_{k,i^*} \leq 8^{-m}/(m+1) \text{ for each } k \leq m. \]

Since the $d_{k,i}$ is nonincreasing in $i$ and $f_{k,i} \leq d_{k,i}/2$, we have
\[ 4^{k} f_{k,i}(q) \leq 2^{-m-1}/(m+1) \text{ for all } k \leq m \text{ and } i \geq i^*. \]

Hence
\[ \sum_{k \leq m} \sum_{i \geq i^*} 4^{k} f_{k,i}(q) \leq 2^{-m-1}. \]

Furthermore,
\[ \sum_{k > m} f_k(q) \leq \sum_{k > m} 2^{-k-1} = 2^{-m-1}. \]

Therefore the approximation of $f(q)$ at stage $i^*$ based only on the $n$–cubes of the form $C_{k,i}$ for $k \leq m$ and $i < i^*$ is within $2^{-m}$ of $f(q)$.

Secondly, we need to verify that $f$ is effectively uniformly continuous. Suppose $\|x - y\| \leq d_{m,1}$ for some $m$. Then for $k < m$, we have $|f_k(x) - f_k(y)| \leq 4^{k}d_{m,1}/2$. For $k \geq m$, we have $f_k(x), f_k(y) \leq 2^{-k-1}$. Thus
\[ |f(x) - f(y)| \leq d_{m,1} \sum_{k < m} 4^{k} + \sum_{k \geq m} 2^{-k} < 2^{-m+2}. \]

Define $h(m) = \lceil -\log_2 d_{m,1} \rceil + 1$ so that $2^{-h(m)} \leq d_{m,1}$. Note that $h$ is a computable order function. Then we get that $\|x - y\| \leq 2^{-h(m)}$ implies $|f(x) - f(y)| \leq 2^{-m+2}$.

\[ \square \]

Claim 4.0.10. $D_1 f(z)$ does not exist.

Proof. For all $m > N$, let $d_m = d_{m,i_m}$ where $i_m$ is such that $z \in C_{m,i_m}$. Note that for all $m > N$ we have either $\delta_{f_m}^1(z, d_m/4) = \pm 4^m$ or $\delta_{f_m}^1(z, -d_m/4) = \pm 4^m$. Without loss of generality we may assume that for infinitely many $m$, we have $\delta_{f_m}^1(z, d_m/4) = 4^m$. Fix one such $m > N$. Note that for for every $k \in \mathbb{N}$ with $N < k < m$ we have $\delta_{f_k}^1(z, d_m/4) = 4^k$. Suppose $k > m$. Then we have
\[ f_k(x) \leq 4^{k} 8^{-k} d_m \frac{d_m}{2} = 2^{-k-1} d_m \]
for all $x \in C_{m,l} \setminus E_{m,l}$ and thus we get
\[ \left| \delta_{f_k}^1(z, d_m/4) \right| \leq 2 \cdot 2^{-k-1} d_m = 2^{-k+2}. \]

Hence, for $m > N$ we have
\[ \left| \delta_{f}^1(z, d_m/4) \right| \geq \left( 4^m - \sum_{N < k < m} 4^k - \sum_{k > m} 2^{-k+2} \right) \geq 4^{m-1} - 4. \]

Therefore $D_1 f(z)$ does not exist.

\[ \square \]
Claim 4.0.11. \( f \) is differentiable almost everywhere.

Proof. Let \( x \in [0,1]^n \). There are three possible cases:

1. \( f'_{m,j}(x) \) does not exist for some \( m,j \),
2. \( x \) belongs to the support of \( f_{m,j} \) for infinitely many \( m,j \), or
3. \( x \) belongs to the support of \( f_{m,j} \) for only finitely many \( m,j \) and all \( f'_{m,j}(x) \) exist. Note that this implies differentiability of \( f \) at \( x \).

The first case corresponds to a null-set, since every \( f_{m,j} \) is a.e. differentiable. The second case corresponds to a null-set too, since it implies \( x \in \bigcap_i G_i \). The last case implies differentiability of \( f \) at \( x \) and it must correspond to a set of full measure since the cases (1) and (2) are captured by null-sets. Thus \( f \) is a.e. differentiable. \( \square \)

5. Conclusion and future directions

Our effective version of Rademacher’s Theorem is not sharp in the following sense. Computable randomness does not characterize differentiability sets of computable Lipschitz functions from \([0,1]^n\) to \( \mathbb{R}^m \) for \( n > 1 \). Finding a satisfactory, from a randomness point of view, characterisation of those sets remains an open question of great interest.

There are quite a few results in classical analysis about differentiability of functions of several variables that exhibit Lipschitz-like behaviour. Naturally, those results are related to Rademacher’s Theorem. Studying effective versions of those will improve our understanding of interplay between computable analysis and algorithmic randomness. Here we mention two such theorems that we feel are of particular importance:

1. Alexandrov’s theorem (see 6.4 in [8]) states that convex functions are twice differentiable almost everywhere. Convex functions and monotone functions are closely related: on the real line, a function is monotone if and only if it is a derivative of a convex function. For functions of several variables, the relation is a bit less straightforward (see [22]). It is not known what randomness notion corresponds to twice differentiability of computable convex functions.

2. It is known that \( K_n \)-monotone functions of several variables are a.e. differentiable (see [5]). Two of the three steps of our proof in Section 2 works for \( K_n \)-monotone functions. The one that doesn’t work is the one in Subsection 2.5. It is not known whether computable randomness implies differentiability of \( K_n \)-monotone computable functions, and what randomness notion is induced by a.e. differentiability of computable \( K_n \)-monotone functions.

On the other hand, our result concerning weak 2-randomness is sharp: weak 2-randomness does characterise differentiability sets of computable a.e. differentiable functions of several variables. There are many other similar results in one dimension that characterise differentiability of effective functions in terms of algorithmic randomness. Generalizing those results to higher dimensions (and, perhaps, to more general spaces) will provide more insight into interactions between computable analysis and algorithmic randomness.
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