A brief history of the pion-nucleon coupling constant

E. Matsinos*

*Electronic mail: evangelos (dot) matsinos (at) sunrise (dot) ch

Abstract

This work provides a brief history of determinations of the pion-nucleon (πN) coupling constant from πN and NN data. From a robust analysis of twenty reported values of the charged-pion coupling constant, exhibiting sizeable fluctuation, the result \( f_c^2 = 0.0762^{+0.0015}_{-0.0008} \) is obtained. Similar values are extracted for the other two πN coupling constants, \( f_0^2 \) and \( f_p^2 \), from fewer data. The average values of the various πN coupling constants, extracted in this work, suggest no splitting, in agreement with the thesis of the Nijmegen group. Additional analysis of the \( f_c^2 \) and \( f_0^2 \) values, both reported in four studies, turned to be inconclusive: one of these studies strongly suggests that \( f_0 < f_c \), whereas another slightly favours \( f_0 > f_c \); no significant effects are observed in the remaining two studies. The analysis of the low-energy πN data with the ETH model indicates strong splitting effects and, under certain conditions, it implies that \( f_0 > f_c \). Also discussed in the paper are the electromagnetic corrections, which need to be applied to the strong shift and to the total decay width of the 1s state of pionic hydrogen in order that estimates for the hadronic s-wave πN scattering lengths be obtained; this is a relevant subject as \( f_c^2 \) may be obtained from the isovector scattering length by use of the Goldberger-Miyazawa-Oehme sum rule. Regarding the removal of the electromagnetic effects in the πN system at threshold, my conclusion is that Theory must find a way to provide reliable and accurate corrections, matching the level of accuracy of the experimental results.

PACS 2010: 13.75.Cs; 13.75.Gx; 25.40.Cm; 25.40.Dn; 25.40.Kv; 25.80.Dj; 25.80.Gn; 11.30.-j

Key words: pion-nucleon interaction, pion-nucleon coupling constants, nucleon-nucleon interaction, sum rules, isospin invariance, charge independence

1 Introduction

That the pion-nucleon (πN) coupling constant is fundamental in our understanding of the Cosmos has been adequately emphasised in numerous works. In meson-exchange models of the strong interaction, a significantly weaker
coupling between pions and nucleons would have prevented the neutrons from combining fast with protons in the early Universe; they would have decayed before they had any chance to be enmeshed first in deuterons, then in other light nuclei. According to the Big-Bang Nucleosynthesis, within half an hour of the Big Bang, all existing matter had assumed the form of free electrons, protons, and helium nuclei (as well as traces of other nuclei up to $^7\text{Be}$). On the contrary, a significantly stronger coupling would have resulted in the rapid creation of bound diprotons and would have led to a helium-dominated Universe. It is hard to imagine how life could emerge in such a Universe: typical stars burn hydrogen to helium for about 90% of their lives. Evidently, the stellar evolution in a helium-dominated Universe would have been considerably shorter. Apart from the obvious cosmological implications, the $\pi N$ coupling constant enters a variety of hadronic phenomena, low-energy theorems, useful relations (e.g., the Goldberger-Treiman relation), etc.

Hypothesised as the carrier of the nuclear force by Yukawa in 1935, the pion was discovered in 1947 by means of the revolutionary at that time photographic emulsion technique [1]. Two pion-related Nobel Prizes were awarded in successive years: to Yukawa in 1949 “for his prediction of the existence of mesons on the basis of theoretical work on nuclear forces” and to Powell in 1950 “for his development of the photographic method of studying nuclear processes and his discoveries regarding mesons made with this method.”

The efforts to determine the value of the coupling constant between pions and nucleons date back to almost the time of the discovery of the pion. My aim in this paper is to provide a brief history of determinations of this coupling constant from $\pi N$ and $NN$ data. Each of the values, accepted for analysis in this work, fulfils the following selection criteria.

- The value had been a new result; excluded from this paper are averages appearing in compilations of physical constants or in review works dedicated to the $\pi N$ coupling constant.
- The value had been ‘final’ within a given methodology, employed in one research programme. I believe that it makes no sense to list and/or analyse ‘progress’ values, i.e., those which are routinely obtained during the development phase of each programme.
- The value had appeared in a peer-reviewed journal. Values, reported in unpublished works, may occasionally be quoted, but they will not be included in the statistical analyses pursued in Sections 4.1, 4.2, and 4.3.
- No definite proof exists that the value is not correct.

Since 1990, when I became acquainted with Pion Physics, I have come across papers providing lists of values of the $\pi N$ coupling constant(s) and obtaining recommended averages from these values. I will make no effort to include in this work a list of these papers. It makes no sense to add (at least) ten
papers to an already long reference list, in particular as I have no interest to include/use here any results from those works.

I start with some useful definitions in Section 2. The determinations of the values of the various \( \pi N \) coupling constants are discussed in Section 3. That section is split into three parts: the first part discusses the early determinations, up and including 1980 (the original title of that section was ‘Pre-meson-factory determinations’); the second part deals with the determinations between 1981 and 1997, i.e., the year in which the 7\textsuperscript{th} MENU Conference took place - a decisive moment in Hadronic Physics as the validity of the ‘canonical value’ (details will be given later on), which had routinely been imported into many analyses for nearly two decades, was widely questioned; the last part of Section 3 discusses the determinations after MENU’97, many of which were based on the measurements obtained from pionic hydrogen (and deuterium) at the \( \pi N \) threshold (vanishing kinetic energy of the incident pion). On the basis of the reported values of Section 3, I obtain (what I believe to be) meaningful averages in Section 4, first for the charged-pion coupling constant (most of the reported values relate to this quantity), then for the neutral-pion coupling constant. The values, extracted from low-energy \( \pi N \) data with the ETH model, are discussed in Section 5; only one value from this research programme is included in Section 4.1. A discussion of the main findings of this work and a list of subjects which call for clarification follows in the last section of the paper. Appendix A concerns the determination of the \( \pi N \) coupling constant from the total decay width of the 1\textsuperscript{s} state of pionic hydrogen\textsuperscript{1} by use of the Goldberger-Miyazawa-Oehme (GMO) sum rule. In this context, the corrections, which need to be applied to the two s-wave \( \pi N \) scattering lengths, in order that the effects of electromagnetic (EM) origin be removed, are of pivotal importance. The GMO sum rule is discussed in Appendix B.

\textsuperscript{1} A two-parameter fit to the measurements of the strong shifts of the 1\textsuperscript{s} state in pionic hydrogen and deuterium, as well as of the total decay width of the 1\textsuperscript{s} state of pionic hydrogen, yields a more accurate estimate for the isovector s-wave \( \pi N \) scattering length.
2 Definitions

2.1 The various $\pi N$ coupling constants

The general form of the $\pi NN$ interaction Lagrangian density involves both pseudoscalar and pseudovector vertices:

$$\Delta \mathcal{L}_{\pi N} = -\frac{1}{1 + x}\bar{\psi}\gamma^5 \vec{\tau} \cdot \left( g_{\pi NN}ix\vec{\pi} + f_{\pi NN}\sqrt{4\pi}m_c\gamma^\mu \partial_\mu \vec{\pi} \right)\psi , \quad (1)$$

where $\vec{\pi}$ and $\psi$ respectively stand for the quantum fields of the pion and of the nucleon, $m_c$ for the mass of the charged pion, and $\vec{\tau}/2$ for the isospin operator of the nucleon. The quantities $\gamma^\mu$ ($\mu = 0, 1, 2, 3$) are the Dirac $4 \times 4$ matrices, satisfying the relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I_4$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

The quantity $g_{\pi NN}$ in Eq. (1) is known as ‘pseudoscalar coupling’, whereas $f_{\pi NN}$ is the ‘pseudovector coupling’. The parameter $x$ determines the strength of the pseudoscalar admixture in the $\pi NN$ vertex. Both pure pseudovector ($x = 0$) and pure pseudoscalar ($x \to \infty$) couplings have been used in the past.

The two couplings of Eq. (1) are linked via the equivalence relation:

$$f_{\pi NN}^2 = \left( \frac{m_c}{m_1 + m_2} \right)^2 \frac{g_{\pi NN}^2}{4\pi} , \quad (2)$$

where $m_1$ and $m_2$ stand for the masses of the two nucleons involved in the $\pi NN$ vertex: $m_1$ of the incoming (incident, initial-state) nucleon, $m_2$ of the outgoing (emitted, final-state) nucleon. (Some authors define the two coupling constants differently, e.g., using the transformation $g_{\pi NN} \to g_{\pi NN}\sqrt{4\pi}$ or $f_{\pi NN}\sqrt{4\pi} \to f_{\pi NN}$ in Eqs. (1,2).) The squares of these two coupling constants are usually reported. The pseudoscalar coupling $g_{\pi NN}^2$ has also appeared as $g_{\pi NN}^2$ or simply $g^2$. Similarly, the pseudovector coupling $f_{\pi NN}^2$ has also appeared as $f_{\pi NN}^2$ or simply $f^2$. I will use $g^2$ and $f^2$ in this work\footnote{When addressing issues of the ETH model, I will use $g_{\pi NN}$. This hadronic model contains another coupling constant, $g_{\pi N\Delta}$; to discriminate between the two couplings, it is customary to use the full vertex as subscript.}, identifying them with $g_{\pi NN}^2$ and $f_{\pi NN}^2$ of Eq. (2). As citing both $g_{\pi NN}^2$ and $f_{\pi NN}^2$ values would be impractical, Eq. (2) will be used in transforming any reported $g^2$ values into $f^2$ results. The first question one may pose is: Is only one $f^2$ value to be used in all vertices involving one pion and two nucleons? If the isospin invariance is broken in the $\pi N$ interaction, the answer to this question is negative.

Four coupling constants have been introduced to regulate the strength of the coupling in the various $\pi N$ vertices, see Fig. 1: the first one ($f_+$) is associated.
with the transitions $\pi^+ n \rightarrow p$ and $n \rightarrow \pi^- p$, the second ($f_-$) with $\pi^- p \rightarrow n$ and $p \rightarrow \pi^+ n$, whereas the remaining two enter the interactions of the neutral pion with the proton ($f_p$) and with the neutron ($f_n$), regardless of whether the $\pi^0$ is incoming or outgoing. As a result, the analyses of $\pi^\pm p$ elastic-scattering (ES) measurements determine the product $f_+ f_-$, which is usually denoted as $f_c^2$ or $f_0^2$, and is known as charged-pion coupling constant. The product $f_p f_n$ enters the description of the $np$ scattering data ($f_c^2$ is also involved here): $f_0^2 = f_p f_n$, which is known as neutral-pion coupling constant. The analysis of the $pp$ data determines $f_0^2$. The early studies had been carried out with only one constant (mostly $f_c$, denoted in those early works simply as $f$). Modern analyses distinguish between $f_c$ and $f_0$, and some even determine all three coupling constants: $f_c$, $f_p$, and $f_0$, depending on which databases - henceforth, DBs - are used as input.

### 2.2 The s-wave $\pi N$ scattering lengths

Three s-wave $\pi N$ scattering lengths (simply ‘scattering lengths’ from now on) are defined as appropriate limits of the scattering amplitudes $F$ associated with the three low-energy $\pi N$ reactions which are experimentally accessible: the two ES reactions $\pi^\pm p \rightarrow \pi^\pm p$ and the charge-exchange (CX) reaction $\pi^- p \rightarrow \pi^0 n$. Assuming no inelasticities (which is a good approximation at low energy), each of these scattering amplitudes may be put in the form

$$F = e^{i \delta(q)} \sin(\delta(q)) / q,$$

where the quantity $\delta$ is known as the (energy-dependent) phase shift and $q$ denotes the magnitude of the 3-momentum of the incident pion in the centre-of-mass (CM) coordinate system. The scattering lengths $a$ are defined as follows.

$$a = \lim_{q \rightarrow 0} \frac{\delta(q)}{q}$$

The three scattering lengths, corresponding to the low-energy $\pi N$ reactions, will be denoted as: $a_{\pi^+ p}$ (for the $\pi^+ p$ reaction), $a_{cc}$ (for the $\pi^- p$ ES reaction), and $a_{c0}$ (for the $\pi^- p$ CX reaction); in the context of this work, only $a_{cc}$ and $a_{c0}$ are relevant. Effects of EM origin are present in $a_{\pi^+ p}$, $a_{cc}$, and $a_{c0}$. The removal of these EM contributions leads to the hadronic scattering lengths, which will be denoted here as $\tilde{a}_{\pi^+ p}$, $\tilde{a}_{cc}$, and $\tilde{a}_{c0}$.

The fulfilment of the isospin invariance in the $\pi N$ system implies that the three scattering lengths $a_{\pi^+ p}$, $a_{cc}$, and $a_{c0}$ may be expressed as suitable combinations of two quantities, i.e., of the scattering lengths in isospin (I) basis: $a_3$ for $I = 3/2$ and $a_1$ for $I = 1/2$. (No tilde is placed over $a_3$ and $a_1$, as these quantities are defined within the context of the isospin invariance in the $\pi N$ interaction.)
Fig. 1. The general coupling constants between pions and nucleons. The vertices \((\pi^+ n, p)\) and \((n, \pi^- p)\) - where the first elements indicate incoming and the second outgoing particles - involve the coupling constant \(\sqrt{2} f_+\). The vertices \((\pi^- p, n)\) and \((p, \pi^+ n)\) involve the coupling constant \(\sqrt{2} f_-\). The vertices \((\pi^0 n, n)\) and \((n, \pi^0 n)\) involve the coupling constant \(-f_n\). The vertices \((\pi^0 p, p)\) and \((p, \pi^0 p)\) involve the coupling constant \(f_p\). The coefficients and the signs have been chosen in such a way as to result in the equality of all couplings, i.e., \(f_+ = f_- = f_n = f_p = f\), if the isospin invariance is fulfilled [2].
The relations are: $\tilde{a}_{\pi p} = a_3$, $\tilde{a}_{cc} = (a_3 + 2a_1)/3$, and $\tilde{a}_{c0} = \sqrt{2}(a_3 - a_1)/3$. The s-wave part of the low-energy $\pi N$ scattering amplitude is of the form $b_0 + b_1 \vec{\tau} \cdot \vec{t}$, where $b_0$ and $b_1$ are the isoscalar and isovector scattering lengths, respectively, and $\vec{t}$ is the isospin operator of the pion. The quantity $b_0$ (frequently denoted as $a^+$ or $a^+_{c0}$ in other works) is related to $a_{cc}$ and $a_{c0}$ according to the formula $b_0 = a_{cc} + a_{c0}/\sqrt{2}$, whereas $b_1$ (in several works in the domain of Pion Physics, $-b_1$ is denoted as $a^-$ or $a^-_{c0}$) is simply equal to $a_{c0}/\sqrt{2}$. After removing the EM contributions from $b_0$ and $b_1$, one obtains the isoscalar and isovector hadronic scattering lengths $\tilde{b}_0$ and $\tilde{b}_1$. The relations to the two scattering lengths in isospin basis read as: $\tilde{b}_0 = (2a_3 + a_1)/3$ and $\tilde{b}_1 = (a_3 - a_1)/3$.

3 Determinations of the various $\pi N$ coupling constants

The methods for determining the $\pi N$ coupling constant may be categorised on the basis of the theoretical model, which is used in order to describe the experimental data ($\pi N$, $NN$), and, of course, of the experimental input itself.

- Physical models of the $\pi N$ interaction and the $\pi N$ experimental data (differential and total/partial-total/total-nuclear cross sections, as well as analysing powers).
- Physical (meson-exchange) models of the $NN$ interaction and the $NN$ (e.g., $pp$, $np$, $\bar{p}p$) experimental data. Relevant in this case are Feynman graphs (simply graphs from now on) with exchanged pion(s) between the two interacting nucleons.
- Dispersion-relation analyses, performed on the $\pi N$ and/or $NN$ experimental data.
- Use of Current-Algebra constraints, of the GMO sum rule [3], etc.

3.1 Early determinations

Although the first efforts to determine the $\pi N$ coupling constant took place in the beginning of the 1950s (see Section 2 of Ref. [2]), the estimates were rather inaccurate for at least one decade[3]. One of the very first accurate $f^2_c$ estimates - perhaps, the first one - appeared in Ref. [4]. This is an interesting review paper, also promulgating the use of forward dispersion relations for the $B$ amplitudes as “the most promising method for determining $f^2_n$” (p. 762). From an analysis of $\pi N$ data, the authors obtained $f^2_c = 0.081(3)$.

[3] Section 2 of Ref. [2] provides an extensive list of the determinations of the $\pi N$ coupling constant before 1968; unfortunately, Ref. [4] is not mentioned in that list.
Performing a dispersion-relation analysis of the then available $pp$ data, Bugg determined $f_p^2$ to 0.075(4) [5] in 1968. Two years later, Ebel and collaborators [6] placed $f_c^2$ between 0.076 and 0.082. In a subsequent paper, Brown and collaborators [7] found that their $f_c^2$ estimates, based on Current-Algebra constraints and the Adler-Weisberger theorem [8,9], ranged between 0.075 and 0.080. The authors favoured $f_c^2 = 0.077(3)$, where the uncertainty has been obtained by means of a comparison of their Eqs. (25,34,37,41).

Applying fixed-$t$ dispersion relations to low-energy $\pi N$ differential (DCS) and total cross sections, Bugg and collaborators [10] obtained in 1973 estimates for the $\pi N$ coupling constant, as well as for the two scattering lengths for ES $\pi^\pm p \to \pi^\pm p$. Having been used (as input) in a variety of studies, the result $f_c^2 = 0.0790(10)$ has been one of the most influential in the domain of Hadronic Physics. For several decades, the value of Ref. [10] was acknowledged as ‘canonical’, and (deplorably) still is for some. As de Swart and collaborators [2] remarked: “We were surprised to note the many physicists trying to hold on to the old values.” Yes, abiding by outdated values and practices is a surprising habit. Be that as it may, one cannot but notice the very small $f_c^2$ uncertainty of Ref. [10], which de Swart and collaborators [2] considered as “optimistic”; I endorse their opinion.

The results of the broadly used partial-wave analysis (PWA) of Koch and Pietarinen [11] appeared in 1980. That solution became known as KH80 (the initials ‘KH’ stand for ‘Karlsruhe’ and ‘Helsinki’). In the abstract of their paper, the two authors summarised their work: “An energy-independent partial-wave analysis has been performed on pion-nucleon elastic and charge-exchange differential cross sections and elastic polarizations, for lab. momenta below 500 MeV/$c$ . . . For the pion-nucleon coupling constant the value $f_c^2 = 0.079(1)$ was obtained.”

Regarding the KH80 solution, a number of remarks need to be made. To start with, very few low-energy data had been available at the time when that analysis was performed. The trouble unfolds as one notices that, in the low-energy region, the analysis entirely relied on the $\pi^+ p$ DCSs of Bertin and collaborators [12]. These seven data sets, each comprising ten measurements, have been criticised in all modern analyses of the $\pi N$ data; they prominently stand out from the bulk of the measurements. There are three ways in which one could possibly make use of the BERTIN76 data sets in a phase-shift analysis: a) by implementing a robust-analysis technique, b) by assigning a low weight to these data sets in optimisations featuring the standard $\chi^2$ minimisation function, or c) by using these measurements in conjunction with a plethora of other data, enabling at the same time the rescaling (controlled floating) of the

---

4 These studies are easily recognisable, as - if not directly quoting the $f_c^2$ result of Ref. [10] - they mention the use of $g^2/(4\pi) = 14.28$. 

8
input data sets. One additional objection to the KH80 analysis relates to their omission of the normalisation uncertainties of the input data sets (see p. 336 of Ref. [11]). Many researchers still make use of the KH80 solution without realising (or after turning a blind eye to) these shortcomings. Let me finally comment on the $f^2_c$ estimate of Ref. [11]. Koch and Pietarinen provide some relevant details in Section 4.2 of their paper; as we will shortly see, not everyone agrees that these authors determined $f^2_c$ in Ref. [11]. I have few doubts that, though they attempted to ‘sell’ this $f^2_c$ value as a determination in the abstract of their paper, they adroitly manoeuvred towards the reported value by letting themselves be steered by the result of Ref. [10].

3.2 Determinations between 1981 and 1997

For about one decade, most researchers in the domain of Hadronic Physics believed that the consistency of the results between Refs. [10,11] (which were taken for independent determinations) suggested that the question of the $\pi N$ coupling constant had been resolved and that attention could be diverted to other, more urgent matters, e.g., to the obvious discrepancies between the first modern (meson-factory) measurements of the $\pi^+p$ ES DCSs and the corresponding predictions obtained from the Karlsruhe analyses (i.e., KH80 and, performed by Koch in 1985, KA85). Fortunately, there were also those who had doubts, as (for instance) the case was with the Nijmegen group. Details on the development of the Nijmegen potentials, as well as a list of estimates for the various $\pi N$ coupling constants from their DBs, are given in Sections 3 and 4 of Ref. [2]. I will now attempt to concisely reconstruct the Nijmegen story (all references may be found in Ref. [2]). The description of the experimental data with the Nijmegen hard-core potential of 1975 resulted in $f^2 \approx 0.0741$. The Nijmegen soft-core potential of 1978, along with constraints on the $f^2$ range of permissible values, yielded $f^2 \approx 0.0772$. (At this point, Ref. [2] hints at the extraction of a smaller $f^2$ value, in case that an unconstrained fit had been performed - which, no doubt, would have been the authors’ choice; the constrained fit had prevented the drop of the fitted $f^2$ value ‘beyond reason’.)

After analysing $pp$ data, the Nijmegen group became gradually convinced that $f^2_p$ should be significantly smaller than the canonical value, and announced their suspicions at the 1983 Few-Boby Conference in Karlsruhe. A few years later, an analysis of $pp$ data at 350 MeV resulted in an accurate determination of $f^2_p$ to 0.0725(6), which was updated (around the end of the 1980s) to 0.0749(6). As the group had not extracted themselves an estimate for $f^2_c$ yet, they had to rely on the use of the canonical value in their investigation of the violation of the isospin invariance (the preferred term for ‘isospin invariance’ in the $NN$ sector is ‘charge independence’). Evidently, the comparison between their $f^2_p$ value and the canonical value yielded large isospin-breaking effects; the report of those effects was not received with enthusiasm.
It was in summer 1990 when Arndt and collaborators [13] published an article favouring an $f_c^2$ value which was considerably smaller than the canonical value. After analysing the then available $\pi N$ ES measurements below 2 GeV using fixed-$t$ dispersion relations (see also Ref. [14] for details), the authors reported the result $f_c^2 = 0.0735(15)$ and commented further in the abstract of their paper: “...a value in conflict with the result of Koch and Pietarinen, yet consistent with the value of the $\pi^0pp$ coupling determined in the recent Nijmegen analysis of $pp$ scattering data.” Although it had not really been “the result of Koch and Pietarinen,” the wheels were set in motion.

The first accurate estimates by the Nijmegen group for all $\pi N$ coupling constants appeared in autumn 1991. Klomp and collaborators [15] remarked in the abstract of that paper: “The $NN\pi$ coupling constants are extracted in $NN$ partial-wave analyses. The data base contains all $pp$ and $np$ scattering data below $T_{\text{lab}} = 350$ MeV. Introducing different coupling constants at the different $NN\pi$ vertices, at the pion pole we find for the $pp\pi^0$ coupling $f_p^2 = 0.0751(6)$, for the $nn\pi^0$ coupling $f_n^2 = 0.075(2)$, and for the charged-pion coupling $f_c^2 = 0.0741(5)$. These results allow only small charge-independence-breaking effects in the $NN\pi$ coupling constants. If we assume charge independence, we find $f^2 = 0.0749(4)$.” To the best of my knowledge, that was the first statement by the Nijmegen group on the absence of splitting effects in the $\pi N$ coupling constant. The $f_0^2 = 0.0752(8)$ result of Ref. [15] was quoted in Ref. [2]; I am not aware of a more recent $f_0^2$ result by the Nijmegen group. Additional details on the analysis may be found in Ref. [16], an important paper featuring the precise result $f_c^2 = 0.0748(3)$, the final $f_c^2$ value by the Nijmegen group.

Using fixed-$t$ dispersion relations on $\pi N$ ES data for pion laboratory kinetic energy $T$ between 100 and 310 MeV, Markopoulou-Kalamara and Bugg [17] obtained in 1993 a new accurate estimate for $f_c^2$: 0.0771(14), i.e., a value smaller than (yet not incompatible with) the 1973 result of Ref. [10].

From a PWA of all $\bar{p}p$ scattering data below 925 MeV/c (antiproton laboratory momentum), Timmermans and collaborators [18] obtained $f_c^2 = 0.0732(11)$ in 1994. Also in 1994, Arndt and collaborators [19] performed PWAs of the $\pi N$ ES data up to 2 GeV, using forward and fixed-$t$ dispersion relations, and obtained chi-square maps for various values of $g^2/(4\pi)$ and $b_0$. Their preferred solution for $g^2/(4\pi)$ was 13.75(15), translating into $f_c^2 = 0.0761(8)$. (It needs to be said that Ref. [19] uses another definition of $f^2$, not absorbing in it a factor $4\pi$.) By the end of 1994, two groups of $f_c^2$ values had clearly been established: the pre-meson-factory group, comprising values obtained up to 1980 and centred around the canonical value, and the Nijmegen-VPI group, comprising values obtained in the early 1990s and centred around 0.075. Those of us who portrayed this discrepancy as a ‘disagreement between the outdated and the modern’ learnt in 1995 (the hard way) that the modern is
not necessarily self-consistent.

I believe that, if one year deserves to be chosen as the one which brought true perplexity to those occupying themselves in the domain of Hadronic Physics, then that year should be 1995. Already in January, Bradamante and collaborators [20], using $\bar{p}p \to \bar{n}n$ DCSs from the CERN Low Energy Antiproton Ring (LEAR), reported an unusually low estimate for $f_c^2$; the reported value was equal to $0.071(2)$. The year went on as promisingly as it had started. In August 1995, Ericson and collaborators [21], using $np$ DCSs at 162 MeV, acquired at the neutron beam facility at the Svedberg Laboratory in Uppsala, obtained $f_c^2 = 0.0808 \pm 0.0003 \pm 0.0017$, i.e., a large estimate for the charged-pion coupling constant, in support of the canonical value and in conflict with most of the values obtained from the $\pi N$ sector, as well as with the recent (at that time) result from LEAR.

About one month later, the paper of Bugg and Machleidt [22] appeared, reporting the results of an analysis of high partial waves for $pp$ and $np$ ES between 210 and 800 MeV. The authors remarked in the abstract of their paper: “There are some discrepancies, but sufficient agreement that values of the $\pi NN$ coupling constants $g_0^2$ for $\pi^0$ exchange and $g_c^2$ for charged $\pi$ exchange can be derived. Results are $g_0^2 = 13.94 \pm 0.17 \pm 0.07$ ($pp$) and $g_c^2 = 13.69 \pm 0.15 \pm 0.24$ ($np$), where the first error is statistical and the second is an estimate of the systematic error arising from uncertainties in the normalization of total cross sections and $d\sigma/d\Omega$.” (In this paper, the factor $4\pi$ has been absorbed in the quantities $g_0^2$ and $g_c^2$.) The two results translate into $f_c^2 = 0.0756(22)$ and $f_0^2 = 0.0771(13)$, where the statistical and systematic uncertainties of Ref. [22] have been linearly combined (i.e., summed). The $f_c^2$ estimate of Ref. [22] landed in-between the two earlier results of that year.

The 7$^{th}$ MENU (‘Meson-Nucleon Physics and the Structure of the Nucleon’) Conference took place in Vancouver in summer 1997. Before my contribution, de Swart gave an emotional talk on the status of the $\pi N$ coupling constant [2], one of those talks which are bound to remain in one’s memory; I will shortly comment further on that talk. Timmermans came afterwards [23] with values of the $\pi N$ coupling constant close to the one I was about to report. I followed with the description of a robust analysis of the low-energy $\pi N$ measurements [24,25] and reported: $f_c^2 = 0.0765(14)$. By the end of the conference, there seemed to be a general consensus of opinion that the value of $\pi N$ coupling constant had to be significantly smaller than the canonical value, see also the remarks in Ref. [26].

I shortly return to de Swart’s talk. The abstract of his paper with Rentmeester and Timmermans, which appeared in the proceedings of the conference, is indicative of the atmosphere which the talk itself created. The authors vividly state: “A review is given of the various determinations of the different $\pi NN$
coupling constants in analyses of the low-energy $pp$, $np$, $\bar{p}p$, and $\pi p$ scattering data. The most accurate determinations are in the energy-dependent partial-wave analyses of the $NN$ data. The recommended value is $f^2 = 0.075$. A recent determination of $f^2$ by the Uppsala group from backward $np$ cross sections is shown to be model dependent and inaccurate, and therefore completely uninteresting . . . ” Regarding the KH80 $f^2_c$ determination, the authors clarify: “The outstanding Karlsruhe-Helsinki partial-wave analyses of the $\pi N$ data used the value of Bugg et al. as input. In 1980, Koch and Pietarinen [11] used fixed-$t$ dispersion relations and found again that $f^2_c = 0.079(1)$. However, this is more a consistency check than a real determination, because the value of the coupling constant was used as input in the analyses. Other values of $f^2_c$ were not tried as input.” I must admit that de Swart’s comments, as well as those few lines in Ref. [2], enjoined me to reread Ref. [11], with a more critical eye.

Regarding the Nijmegen analyses, one comment is due. In relation to the $NN$ interaction, the Nijmegen group had created (and were able to analyse) three DBs: $pp$, $np$, and $\bar{p}p$. The analysis of the first reaction yields $f^2_p$, of the second a combination of $f^2_0$ and $f^2_c$, and of the third a combination of $f^2_p$ and $f^2_c$. Had I been a member of the Nijmegen group, I would have pursued a combined analysis of all three reactions; I believe that such an analysis would provide a definite answer regarding the splitting in the $\pi N$ coupling constant in their programme. In relation to the Nijmegen results, two issues must be borne in mind:

- No splitting has been observed in the (values of the) $\pi N$ coupling constant.
- The recommended value for the $\pi N$ coupling constant lies in the vicinity of 0.075, i.e., well below the canonical value.

According to Timmermans [27], the results of Refs. [16,18] for $f^2_c$ and the one of Ref. [28] for $f^2_p$ should be considered as final by the Nijmegen group. I will return to Ref. [28] shortly.

### 3.3 Determinations after 1998

Early in 1998, Gibbs and collaborators [29] obtained $f^2_c = 0.0756(7)$ from an analysis of modern low-energy $\pi^{\pm}p$ ES data. In their determination, the authors made use of the GMO sum rule.

The most recent determination of $f^2_p$ by the Nijmegen group originates from 1999. Studying the long-range properties of the $pp$ interaction in an energy-dependent PWA, Rentmeester and collaborators [28] obtained $f^2_p = 0.0755(7)$.

The next two reports [30,31] may be thought of as a follow-up (but au-
tonomous) work of Ref. [21]. The former paper reports the results of an analysis of the np DCS at 162 MeV between 72° and 180°. The authors stress again that “special attention was paid to the absolute normalization of the data.” They also observe that “in the angular range 150° – 180°, the data are steeper than those of most previous measurements and predictions from energy-dependent partial-wave analyses or nucleon-nucleon potentials. At 180°, the difference is of the order of 10 – 15 %.” The authors finally report: $f^2_c = 0.0803(14)$. Measurements of the np DCS at 96 MeV (within almost the same angular range) were analysed in Ref. [31], and the value $f^2_c = 0.0814(18)$ was obtained. The results from Uppsala [21,30,31] are consistent among themselves.

The subsequent paper comprised the final report of the group which acquired the pioneering measurements of the strong shift $\epsilon_{1s}$ and of the total decay width $\Gamma_{1s}$ in pionic hydrogen and deuterium [32] at the Paul Scherrer Institute (PSI). In that paper, the EM effects in the case of pionic hydrogen were removed after using the results of an earlier work, which had been published by part of the group and Oades in 1996 [33]. Two quantities were imported from that work into Ref. [32], namely $\delta_\epsilon = -2.1(5) \cdot 10^{-2}$ (relating to the correction which one needs to apply to $\epsilon_{1s}$) and $\delta_\Gamma = -1.3(5) \cdot 10^{-2}$ (relating to the removal of the EM effects from $\Gamma_{1s}$).

The trouble with the results of Ref. [33] is that they stemmed from a two-channel calculation ($\pi^- p \rightarrow \pi^- p, \pi^0 n$), along with the phenomenological addition of the contributions of the ‘third’ channel ($\pi^- p \rightarrow \gamma n$). The three-channel calculation [34], performed a few years after Ref. [32] appeared, resulted in a significantly different result for $\delta_\epsilon$. To be fair, I remember that, on a number of occasions even before Ref. [33] was published, Oades had expressed his suspicions that the contributions of the $\gamma n$ channel could be sizeable in the case of $\epsilon_{1s}$. In Appendix A, I present results obtained after the application of different sets of corrections to the experimental results for $\epsilon_{1s}$ and $\Gamma_{1s}$ of Ref. [32].

In Ref. [32], the authors obtained an estimate for the $\pi N$ coupling constant (namely, $g_{\pi N} = 13.21^{+0.14}_{-0.09}$) from the isovector hadronic scattering length $\tilde{b}_1$ by use of the GMO sum rule. To obtain the $\tilde{b}_1$ estimate, Schröder and collaborators combined information from pionic hydrogen and deuterium. In view of the fact that the EM corrections in the former case [33] appear incomplete, I will refrain from including the authors’ $f^2$ value in this report.

Using the results of Ref. [32], two papers appeared by Ericson and collaborators [35,36]. The former paper directly used the result of Ref. [32] for the scattering length $\tilde{a}_{cc}$. In the second paper, the authors turned a critical eye on the EM corrections of Ref. [33]. They comment that the potential-model approach is “model dependent” and, furthermore, they find it inconsistent with their low-energy expansion [35]. The authors derived the EM corrections at threshold on the basis of their own model of the $\pi^- p$ atom; numerical results
may be found in their Table 1. Their $\delta_\epsilon$ value lies in-between the results of Refs. [33,34], slightly closer to the latter result, whereas their $\delta_\Gamma$ value is of opposite sign and strongly disagrees with both works [33,34]. The result of Ref. [36] for $g^2_c/(4\pi)$ was 14.04(17), which translates into $f^2_c = 0.07756(94)$. This value will be included in the analysis presented in Section 4.1.

Performing a re-analysis of $\pi N$ data and dispersion relations for the isoscalar invariant amplitude $B^+$, Bugg obtained in 2004 a new estimate for $g^2_c/(4\pi)$ [37], which translates to $f^2_c = 0.07590(55)$.

In their 2011 paper, Baru and collaborators [38] obtained $g^2_c/(4\pi) = 13.69(20)$ from PSI results on pionic hydrogen and deuterium, using the GMO sum rule. As I have major objections to the authors’ choice of input, I will refrain from including their $g^2_c/(4\pi)$ estimate in this report. To start with, the authors chose to use in their study preliminary $\epsilon_1s$ and $\Gamma_1s$ results from an experiment at PSI, which is a follow-up experiment of the mid-1990s measurements by the ETH-Neuchâtel-PSI Collaboration, rather than the final results of that first experiment [32]. It so happened that the final $\epsilon_1s = -7.0858 \pm 0.0071$(stat.) $\pm 0.0064$(syst.) eV of the follow-up experiment (which is beautifully compatible with the result of Ref. [32], and considerably more accurate) appeared a few years later [39] and was somewhat smaller (in absolute value) than the value used as input in Ref. [38]. (Reference [39] uses another sign convention for $\epsilon_1s$.)

My major objection to Ref. [38] relates to $\Gamma_1s$; to the best of my knowledge, the Pionic-Hydrogen Collaboration have not yet published a final result! In their 2015 paper [40], their estimate for $\Gamma_1s$ of pionic hydrogen was still marked as ‘preliminary’ and was given as $\Gamma_1s = 850_{-50}^{+40}$ meV, i.e., slightly more accurate than the $\Gamma_1s$ result of Ref. [32]. Baru and collaborators [38] had used $\Gamma_1s = 0.823(19)$ eV in their 2011 paper, i.e., a smaller value, accompanied by a largely underestimated uncertainty; as a result, the uncertainty of their $\tilde{b}_1$ value in the case of pionic hydrogen has been underestimated.

Within an $NN$ model based on one-pion exchange, Babenko and Petrov [41] reported $g^2_c/(4\pi) = 14.55(13)$ and $g^2_0/(4\pi) = 13.55(13)$. Their two results suggest the violation of the isospin invariance, which (as we saw earlier) the Nijmegen analyses refute. As the authors remark, their $g^2_c/(4\pi)$ value is consistent with those reported by the Uppsala group [21,30,31]. To the best of my knowledge, the works of Babenko and Petrov are the only ones which point to $f_c > f_0$. The two values translate into $f^2_c = 0.08038(72)$ and $f^2_0 = 0.07496(72)$. They slightly updated their results one year later to $g^2_c/(4\pi) = 14.53(25)$ and $g^2_0/(4\pi) = 13.52(23)$ [42]; the updated values (i.e., $f^2_c = 0.0803(14)$ and $f^2_0 = 0.0748(13)$) will be used here.

In their 2016 paper, Ruiz Arriola and collaborators [43] discussed their recently obtained values of $f^2_v = 0.0759(4)$, $f^2_0 = 0.079(1)$, and $f^2_c = 0.0763(6)$, extracted from a PWA of a DB which the authors call “3σ self-consistent $NN$
database” (Granada-2013) comprising 6713 measurements acquired between 1950 and 2013. The results were slightly updated in 2017 [44], where the authors report the values $f_p^2 = 0.0761(4)$, $f_0^2 = 0.0790(9)$, and $f_c^2 = 0.0772(6)$, obtained from almost the same data as their earlier work; the updated values will be used here. Interestingly, the authors draw attention to the (large) anticorrelation between $f_c^2$ and $f_0^2$ in their analysis. The authors’ estimates for $f_c^2$ and $f_0^2$ do not match with the results of Ref. [42].

4 Determinations of averages for the various $\pi N$ coupling constants

4.1 Determinations of $f_c^2$

The $\pi N$ coupling constant $f_c^2$ is the one with the most determinations (or, better expressed, attempts at a determination); in total, twenty reported values fulfil the criteria put forward in Section 1. Analysed in this section are the ten $f_c^2$ values of Refs. [6,16,18,20,21,22,30,31,42,44] from the $NN$ system and another ten from the $\pi N$ system [4,7,10,13,17,19,25,29,36,37].

Figure 2 contains all the results, separately for the estimates originating from the $\pi N$ and from the $NN$ analyses. Poring over this figure without knowing what has been plotted, one might fail to guess that the data points correspond to estimates for the same physical quantity. On the other hand, the trouble with the fluctuation, present in Fig. 2, evidently lies with the $f_c^2$ estimates obtained from the $NN$ data; those extracted from the $\pi N$ data cluster around their weighted average in an acceptable way. Although it has been suggested in the past that the best determinations of $f_c^2$ should come from the $NN$ sector, Fig. 2 can hardly substantiate such a claim.

For the analyst, Fig. 2 is a nightmare. I had been considering for a while which procedure I could implement in order to obtain a meaningful average from such a wild spread of values (and of associated uncertainties). Because of the one-sided outliers (large-$f_c^2$ data points), it seemed to me reasonable to apply a robust technique. I will first present what I believe to be a meaningful solution, then provide some details as to what else I have tried.

One category of robust optimisations rest upon the use of the standard $\chi^2$ minimisation function and the application of hard or soft weights to the input

---

Using only the $f_c^2$ estimates from the $\pi N$ data, one obtains from a simple $\chi^2$ (one-parameter) fit: $f_c^2 = 0.07639(42)$; the resulting $\chi^2$ of this fit is about 16.95 for 9 degrees of freedom (DoF), corresponding to a p-value of $4.95 \times 10^{-2}$, which exceeds the threshold $p_{min} = 1.00 \times 10^{-2}$, regarded by most statisticians as the outset of statistical significance.
data points. At each (iteration) step in the optimisation scheme, the software application, which drives the function minimisation, varies the fit parameters (according to dedicated algorithms) and passes each new vector of parameter values to the user-defined function which hosts the parametric (theoretical) model. Fitted values (corresponding to the input vector of parameter values at that step) are generated within this function for all input data points. The distance between the input and the fitted values is evaluated for each point. Hard-weight techniques use this distance in order to enable a decision on whether an input data point is an ordinary one or an outlier (at that step). Such an optimisation scheme is dynamical, in that data points which are outliers at one step may become ordinary at the next; similarly, ordinary points may turn into outliers from one step to another. These interchanges are more frequent in the initial phases of the optimisation, when the changes of the parameter values are larger. The essential point to bear in mind is that the distances between the input and the fitted values (or, as the case is with measurements in Physics, the distances divided by the input uncertainties, i.e.,
the normalised residuals \( r_i \) determine whether each data point is an ordinary point or an outlier, and also fix the weights of the \( \chi^2 \) contributions of all input data points at all steps.

Both hard- and soft-weight techniques evaluate a weight for each input data point at each step of the optimisation. Their difference is that the \( \chi^2 \) contributions from the outliers are turned into 0 in the former case; the outliers are excluded. On the other hand, soft-weight techniques apply small, albeit non-zero, weights to the outliers, allowing them to participate at all steps of the optimisation. Hard and soft weights may be continuous or discontinuous. An example of a discontinuous hard-weight scheme would be to apply the weight of 0 to all outliers and that of 1 to all ordinary points. An example of a continuous hard-weight scheme would emerge if the input data points are categorised as follows: of type (a) are the ordinary points which lie within a distance \( \epsilon_1 > 0 \) of the fitted values, i.e., those satisfying \( |r_i| \leq \epsilon_1 \); of type (c) are the outliers, characterised by \( |r_i| \geq \epsilon_2 \), where \( \epsilon_2 > \epsilon_1 \); and of type (b) are the ordinary points which satisfy \( \epsilon_1 < |r_i| < \epsilon_2 \), neither ‘too good’ nor outliers. Weights of 1 could be assigned to type (a), 0 to type (c), and between 0 and 1 to type (b). The weights \( W_i \) in the case of the type-(b) points may be chosen in such a way as to be continuous, monotonic, and fulfilling

\[
\lim_{{|r_i|\to\epsilon_1^+}} W_i = 1
\]

and

\[
\lim_{{|r_i|\to\epsilon_2^-}} W_i = 0.
\]

It is simple to pass into soft-weight schemes by devising a scheme which assigns a small (non-zero), monotonic weight to the outliers.

The approach of the present work relies on the use of a continuous soft-weight robust optimisation. A fit, featuring a logarithmic minimisation function [24,25], was performed on the data displayed in Fig. 2. The linear model involves only one parameter, the level (intercept). For the purpose of the function minimisation, the MINUIT package [45] of the CERN library - FORTRAN version - was used. The fit yielded the result:

\[ f^2_c = 0.0762^{+0.0015}_{-0.0008}. \]  

(3)

It is not difficult to obtain the fitted uncertainties when minimising a logarithmic function.

\[
\chi^2 = \sum_{i=1}^{N} \ln \left(1 + r_i^2 \right) = \sum_{i=1}^{N} \frac{\ln(1 + r_i^2)}{r_i^2} r_i^2 = \sum_{i=1}^{N} W_i(r_i) r_i^2
\]

where \( N \) denotes the number of input data points. Therefore, the results are identical to those emerging from the weighted linear least-squares fit, with
weights assigned to each normalised residual $r_i$ according to the formula:

$$W_i(r_i) = \begin{cases} 
1, & \text{if } r_i = 0 \\
\ln(1 + r_i^2)/r_i^2, & \text{otherwise}
\end{cases}$$

(4)

The uncertainties in Eq. (3) contain the Birge factor (called scale factor by the Particle-Data Group); their asymmetry reflects the presence of more outliers above $\langle f_c^2 \rangle$. The result of Eq. (3) is my recommendation as representative of the $f_c^2$ estimates of Fig. 2.

Other efforts to extract a meaningful $\langle f_c^2 \rangle$ value from the data of Fig. 2 were carried out. A linear (one-parameter) least-squares fit, weighted only with the uncertainties of the input $f_c^2$ values (i.e., using $W_i = 1$), yielded $\langle f_c^2 \rangle = 0.07593$, which is close to the result of Eq. (3); smaller fitted uncertainties than those quoted in Eq. (3) were obtained in this fit. Evidently, the estimate for $\langle f_c^2 \rangle$ from this fit is pulled ‘downwards’ by the $f_c^2 = 0.0748(3)$ result of Ref. [16], which is the one accompanied by the smallest input uncertainty.

My next idea was to perform other robust analyses of the input data. To this end, fits were made using five types of weights in the optimisation: Cauchy, fair, Huber, logistic, and Welsch. The resulting $\langle f_c^2 \rangle$ values lay between 0.07601 and 0.07611 when using the default values (which are different for each of these five types of weights) of the so-called “tuning constant” $k$, i.e., of the largest value of $|r_i|$ receiving the weight of 1. Unfortunately, the choice of $k$ is arbitrary. Although some procedures have been put forward in order to derive optimal $k$ values from the input data (e.g., see Ref. [46]), the superiority of the logarithmic fit relates to the fact that there is no adjustable parameter in the definitions of the weights $W_i$ in Eq. (4). Of course, what is called ‘superiority’ by some may be called ‘inflexibility’ by others.

4.2 Determinations of $f_0^2$

There are only four determinations of $f_0^2$, all from $NN$ measurements, see Refs. [22,2,42,44]. These values are displayed in Fig. 3. The weighted average comes out as

$$f_0^2 = 0.0765(12),$$

(5)

but the $\chi^2$ of the reproduction of these data (by one constant) is poor: $\chi^2 \approx 14.67$ for 3 DoF (corresponding to a p-value of about $2.12 \cdot 10^{-3} < p_{\text{min}}$).

It is also interesting to investigate the difference $\Delta f^2 := f_0^2 - f_c^2$ from the analyses which reported both coupling constants. In the works of Refs. [22,2], there is no indication that $\Delta f^2 \neq 0$: in the former case, $\Delta f^2 = 15(25) \cdot 10^{-4}$; in the latter, $\Delta f^2 = (-3.0 \pm 9.5) \cdot 10^{-4}$. On the other hand, an effect at the level
4.3 Determinations of $f^2_p$ and $f^2_n$

The $f^2_p$ determinations of Refs. [5,28,44] are well compatible. An average of these estimates is $f^2_p = 0.07595(19)$, in good agreement with the weighted average of $f^2_c$ in Eq. (3). Only one $f^2_n$ value is available, that of Ref. [15]; its large uncertainty makes it compatible with all aforementioned averages.
The ETH model of the $\pi N$ interaction (see Ref. [47] and the references therein) is an isospin-invariant hadronic model based on $\sigma$- and $\rho$-meson $t$-channel exchanges, as well as on the $s$- and $u$-channel graphs with $N$ and $\Delta (1232)$ intermediate states. The contributions of all well-established $s$ and $p$ higher resonances with masses below 2 GeV are also analytically included. This model uses no form factors in the hadronic part of the $\pi N$ interaction. The fit to the $\pi N$ measurements in the low-energy region ($T \leq 100$ MeV) involves seven parameters, one of which is the $\pi N$ coupling constant. At this point, it is interesting to examine which of the $\pi N$ coupling constants (or their combinations) are determined in the model fits to the $\pi N$ experimental data.

There is no doubt that the fit to the two ES DBs determines $f_{\pi}^2$. The nucleon $u$-channel graph in the $\pi^+ p$ case involves the vertices $p \rightarrow \pi^+ n$ and $\pi^+ n \rightarrow p$. The former is associated with $f_-$, the latter with $f_+$; therefore, the $\pi^+ p$ scattering amplitude involves the product $f_- f_+ \equiv f_{\pi}^2$. The same applies to $\pi^- p$ ES. On the other hand, the model fits to the CX measurements involve unusual combinations of the coupling constants: the $s$-channel graph involves the combination $f_- f_n$, whereas the $u$-channel graph $f_- f_p$.

For over two decades, fits to the ES data have routinely been performed; occasional fits to the CX DB were also attempted, but they were rarely used because of the strong correlations among the model parameters when the input DB contains measurements of only one reaction. The origin of these correlations is not difficult to identify. In the fits to the two ES DBs, the $\pi^+ p$ data essentially fix the isospin $I = 3/2$ partial-wave amplitudes, whereas the $I = 1/2$ amplitudes are determined from the $\pi^- p$ ES data. (The $\pi^+ p$ scattering amplitudes are pure $I = 3/2$ in nature, whereas the $\pi^- p$ ES amplitudes receive both $I = 1/2$ and $I = 3/2$ contributions.) Of course, the combined fit to the two ES reactions ensures that the finally extracted $I = 3/2$ amplitudes have been adjusted (during the optimisation) in such a way as to describe optimally both ES DBs. The CX reaction also involves a combination of the two isospin amplitudes (a different one to that of the $\pi^- p$ ES). The exclusive fits to the CX DB cannot reliably determine both isospin amplitudes; measurements of another reaction are needed. For over two decades (i.e., between 1990 and 2011), an indirect approach had been followed in the investigation of the violation of the isospin invariance in the low-energy $\pi N$ interaction: the scattering amplitudes, obtained from the model fits to the ES DBs, were used

---

6 In the graphs of the model, the nucleon is assigned the proton mass, whereas the hadronic mass of the pion is taken to be the charged-pion mass. External mass differences are taken care of by the EM corrections applied to the $\pi N$ scattering amplitudes, as well as to the $\pi N$ phase shifts. Regarding Eq. (2), the ETH model uses $m_1 = m_2 = m_p$. 
in order to predict the amplitude of the CX reaction. The reproduction of the CX measurements on the basis of that amplitude was subsequently pursued (and was always found very poor). The 1997 report of the isospin-breaking effects in the low-energy $\pi N$ interaction [25] rested upon this approach.

In 2012, a direct approach in the investigation of the violation of the isospin invariance at low energy was implemented. Since then, two types of fits are being performed: the first one to the ES measurements, whereas the second attempts the simultaneous description of the $\pi^+ p$ and CX DBs. Both isospin amplitudes can be determined in both fits. A comparison between the results of these two fits enables tests of the isospin invariance in the $\pi N$ system.

As aforementioned, the combined fit to the $\pi^+ p$ and CX DBs involves both $f_2^c$ (because of the $\pi^+ p$ reaction), as well as the products $f_- f_n$ and $f_- f_p$ (relevant in the case of the CX reaction). Therefore, it is not possible to associate the results of this fit with any of the standard combinations of Section 2. There is, however, one way in which this analysis is useful. If the isospin invariance is fulfilled in the low-energy $\pi N$ interaction, there should be only one $\pi N$ coupling constant, and, regardless of which input DB is used, the fitted values of the model parameter $g_{\pi NN}$ should come out compatible. In fact, all values between 2012 and the present time strongly suggest that $f_{DB0+} > f_0$, where $f_{DB0+}$ is the result of the fit to the combined $\pi^+ p$ and CX DBs. This result is significant and refutes the possibility that all $\pi N$ data for $T \leq 100$ MeV could be described with one $\pi N$ coupling constant. Provided that $f_p \approx f_n$ and $f_+ \approx f_-$, all the model analyses of the (combined) $\pi^+ p$ and CX DBs after 2012 strongly support $f_0 > f_0$. I will now give the $f_0$ and $f_{DB0+}$ values obtained with the ETH model thus far.

The first fits of the ETH model did not involve genuine measurements. During the first years of development and application of the model, fits were made to phase-shift results of various PWAs, e.g., of KH80, of KA85, of CMU-LBL, etc. From the phase shifts up to the energy corresponding to the position of the $\Delta(1232)$ resonance, the first fitted value of $g_{\pi NN} = 12.95 \pm 0.08(\text{stat.}) \pm 0.03(\text{syst.})$ was obtained in 1993 [48]. Similar values followed in 1994 [49,50].

The first estimate for $f_2^c$ from genuine measurements was reported in the MENU'97 Conference [24] and later on appeared in the first (in this research programme) study of the violation of the isospin invariance in the $\pi N$ system [25]. I will shortly explain why I decided to include only this $f_2^c$ value in Section 4.1. Additional determinations of $f_0$ were made in Refs. [51,52,53,47,54] and of $f_{DB0+}$ in Refs. [53,54]. These recent determinations are all based on the use of the Arndt-Roper formula [55] in the optimisation and, as such, they rest upon the identification (and removal) of the outliers contained in the input DBs. On the other hand, the studies [24,25] featured a robust fit to the input data. No data point had to be removed, as the minimisation function had
Table 1

Values of the coupling constants $f_c^2$ and $f_{DB0+}^2$ obtained with the ETH model of the $\pi N$ interaction between 2006 and the present time.

| Source | $f_c^2$       | $f_{DB0+}^2$ |
|--------|---------------|--------------|
| [51]   | 0.0733(14)    | --           |
| [52]   | 0.0726(14)    | --           |
| [53]   | 0.0726(14)    | 0.0794(13)   |
| [47]   | 0.0723(14)    | --           |
| [54]   | 0.0746(14)    | 0.0805(11)   |

been chosen in such a way as to match the distribution of the normalised residuals: in the framework of Refs. [24,25], one could argue that there are no outliers; in the more conventional language of the $\chi^2$ minimisation function, one could say that any outliers in the input DB are rendered harmless (by applying smaller weights to these points, in comparison to those assigned to the ordinary points). Table 1 provides the list of all $f_c^2$ and $f_{DB0+}^2$ values, obtained with the model between 2006 and the present time.

‘Large’ estimates for $f_c^2$ have never been extracted from the model fits to the low-energy $\pi N$ data. The largest $f_c^2$ value, ever obtained, was the one reported in Refs. [24,25], when the robust fit was carried out and no rescaling of the input data sets was permitted. On the other hand, all $f_{DB0+}^2$ estimates since 2012 have been close to the canonical value.

The difference between the values of (on one hand) Refs. [51,52,53,47] and (on the other) Ref. [54] is predominantly due to the inclusion of the BERTIN76 DCSs [12] in the input $\pi^+ p$ DB. To be able to include these measurements in the fits, normalisation uncertainties needed to be assigned to the seven data sets of Ref. [12]; this was done on the basis of an analysis of the normalisation uncertainties reported in the modern experiments. A simple linear fit to these normalisation uncertainties was performed using $T$ as the independent variable. Finally, the normalisation uncertainty, assigned to each of the BERTIN76 data sets, was set equal to double the fitted uncertainty (result of the fit to the reported normalisation uncertainties of the modern experiments) at the energy corresponding to that data set. The use of such generous uncertainties was not meant as retribution for the lack of proper reporting of these uncertainties in Ref. [12], but as a precaution: it remains unknown whether the normalisation effects were even thought of as potentially important sources of uncertainty back in 1976. Since 2006, a similar procedure had been applied to a few other data sets with unknown normalisation uncertainties. Excluding the BERTIN76 DCSs, on the basis of the argument that these measurements
have been criticised in most modern analyses of the $\pi N$ data, appeared to be arbitrary.

Regarding my reluctance to use any of the $f^2_c$ values of Table 1 in Section 4.1, one word is due. I consider the $f^2_c$ value, obtained in Refs. [24,25], as final in relation to the methodology, as well as to the DB content in the late 1990s. On the other hand, Refs. [51,52,53,47], though published, represent (in my opinion) improvements in the approach set forward for the identification of the outliers in the input DBs. I believe that the approach was perfected in Ref. [54], to the extent that I consider the results of that paper as representing the 'state-of-the-art' for an optimisation resting upon the use of the Arndt-Roper formula. Had it been properly published, I would have included the $f^2_c$ result of Ref. [54] in Section 4.1.

Regarding the model fits to the data, I plan to explore ways to also incorporate the rescaling of the input data sets within the framework of a robust optimisation, along the guidelines of Refs. [24,25]. I also believe that the implementation of a procedure, similar to the one described in Ref. [56], would yield reliable estimates for the uncertainties of the model-parameter values.

6 Conclusions and discussion

Results for the various $\pi N$ coupling constants were listed, discussed, and analysed in this work. Included in the statistical analysis of Section 4.1 were twenty values of the charged-pion coupling constant $f^2_c$, namely those of the results which fulfilled the four selection criteria put forward in Section 1. To the best of my knowledge, only four reports of the neutral-pion coupling constant $f^2_0$ may be found in the literature, three of the coupling of the neutral pion to the proton ($f^2_p$), and one of the coupling of the neutral pion to the neutron ($f^2_n$).

A scatter plot of the values of the charged-pion coupling constant and the year they were reported is displayed in Fig. 2. The $f^2_c$ determinations from the $\pi N$ data cluster around their weighted average in a satisfactory manner; those extracted from the $NN$ data exhibit significant fluctuation. An average was obtained in Section 4.1 from a robust fit to these values: $f^2_c = 0.0762^{+0.0015}_{-0.0008}$.

The fluctuation, observed in Fig. 2, is sizeable. To be able to identify possible sources of this wide spread of values, a short, general outline of the procedure for the extraction of the various $\pi N$ coupling constants from the experimental

\footnote{Of course, this applies to reported values. Estimates for $f_n$ may be obtained from $f_0$ and $f_p$ from the studies where both coupling constants were determined.}
data would be helpful.

- Experimental values of the standard observables (differential and total cross sections, analysing powers, etc.), corrected for all (known) effects relating to beam contamination, target composition, detector efficiency, etc. comprise the input into the analyses. Of importance is the absolute normalisation of the measurements comprising each input data set. The responsibility for the application of these corrections lies with the experimenters who acquired the measurements.
- Removal of the electromagnetic (EM) effects, to obtain hadronic quantities from the measurements. This is predominantly a task for theorists.
- Modelling of the hadronic interaction. Included here are also the details of the methodology, which is followed in the analysis. This is usually a task for theorists or phenomenologists. Relevant at this level is the model dependence of the results.

If I were asked to single out one of these factors as the most probable source of the fluctuation observed in Fig. 2, I would rather opt for the model dependence of the results. Apart from the theoretical background, the methodology matters too. For the sake of example, the various $\pi N$ coupling constants have been obtained in the Nijmegen programme from separate descriptions of the $pp$, $np$, and $\bar{p}p$ databases. This methodology does not seem to yield any difference in the fitted values of $f_2^c$, $f_2^p$, and $f_0^c$. However, we all agree that the isospin invariance is broken in the $NN$ interaction, see Refs. [54,57] for a discussion. If no splitting is found in the extracted values of the various $\pi N$ coupling constants in an analysis of $NN$ data, then the isospin-breaking effects must manifest themselves in another way, e.g., as sizeable differences in the potentials which describe these reactions. This may explain why isospin-breaking effects in the coupling constant are observed in Refs. [41,42,43,44], which attempt the simultaneous description of data belonging to different $NN$ reactions.

In addition, I suspect that the use of different schemes for the removal of the EM effects from the measurements (or from the scattering amplitudes obtained thereof) is another important reason for the fluctuation observed in Fig. 2. I am not aware of comparative studies addressing the differences among the various schemes of application of the EM corrections both in the $\pi N$ and the $NN$ sectors. Regarding the $\pi N$ sector, there are as many such schemes as research programmes and, worst of all, information about how the EM effects are treated in these schemes is either sparse or non-existing. Visual inspection of Table A.1 attests to the lack of a consensus on the EM corrections which one needs to apply even to the measurements obtained at the $\pi N$ threshold. Corrections which should (in principle) be compatible (as those discussed in Sections A.1 and A.2) disagree and even differ in sign. The corrections obtained within the framework of Chiral Perturbation Theory for
the strong shift of the 1s state in pionic hydrogen are large (when compared to
the experimental uncertainties, as well as to the magnitude - absolute value - of
the effects obtained in Sections A.1 and A.2) and, because of the poorly known
low-energy constant $f_1$, very uncertain. To the best of my knowledge, only the
Aarhus-Canberra-Zurich Collaboration has attempted the determination of
the EM corrections for the scattering data (above threshold) and at threshold
in a consistent manner (i.e., using the same potentials). It is my firm opinion
that the problem of the EM corrections in $\pi N$ scattering must be revisited; the
data analysis necessitates the availability of a consistent and reliable set of EM
corrections from threshold up to the energy of a few GeV. I believe that the
work of the NORDITA group during the 1970s must be upgraded, after taking
into account both the theoretical advancements, as well as the entirety of the
experimental information which became available from the meson factories
after 1980. Similar remarks apply to the (more complex) $NN$ case.

My third guess (as to the important reasons for the discrepancies in Fig. 2)
would be the (generally) underestimated systematic uncertainties associated
with the experimental values, i.e., the normalisation uncertainties of each data
set. (One should add here the uncertainties induced by the removal of the EM
effects.) The normalisation uncertainties of the data sets are closely linked
to the uncertainties of the outcome of an analysis of the measurements. My
experience suggests to me that, when providing estimates for the normalisation
uncertainty in their experiments, most experimental groups tend to be
on the optimistic side. One good example may be taken from the analyses
of the low-energy $\pi N$ databases. Scatter plots of the reported normalisation
uncertainty (from one group and type of experiment) and the time (when the
experiment was conducted) are generally expected to have a negative slope: on
average, the experimental group is expected to gain experience with time, per-
fect their techniques, hence have a better grasp on the absolute normalisation
of their data sets. In fact, the opposite tendency is observed on some occa-
sions. This observation suggests to me that new sources of uncertainty surfaced
with evolving time, evidently indicating that the normalisation uncertainties
of the earlier works of the experimental group had been underestimated. As
the experimenters bear the responsibility for updating their results (which,
compared to the past, is straightforward and efficient nowadays), the only
action, left to the analyst, is to wait for that moment to come.

Based on the averages of $f_c^2$ and $f_0^2$, given in Sections 4.1 and 4.2 respectively,
there seems to be no evidence that $f_c \neq f_0$. The paired test, which is described
at the end of Section 4.2, yielded inconclusive results: no splitting was observed
in two studies [22,2], one study reported results marginally compatible with
no splitting [44], whereas a strong effect was observed in the fourth study [42].
It is worth mentioning that the analysis of the low-energy $\pi N$ data with the
ETH model results in a strong splitting in the $\pi N$ coupling constant $g_{\pi NN}$,
see Table 1. However, the strong effect observed in Ref. [42] (namely, $f_c > f_0$)
is opposite to the results extracted from the $\pi N$ data with the ETH model ($f_c < f_0$) provided that $f_p \approx f_n$ and $f_+ \approx f_-$. 

Included in this work are determinations of the $\pi N$ coupling constants fulfilling the selection criteria put forward in Section 1. If a reader is of the opinion that I have been unfair in my judgment when applying these criteria, or if relevant published work on this issue has been overlooked, I would be glad to be notified.

Acknowledgements

I acknowledge a useful discussion with R.G.E. Timmermans; he drew my attention to the results of Refs. [16,18,28], which should be considered as final in the Nijmegen programme. I also acknowledge numerous stimulating discussions over the years with A. Badertscher and, naturally, with G. Rasche, in particular regarding the EM corrections in the $\pi N$ system. The use of the hadronic masses (instead of the physical ones) in the EM corrections, discussed in Section A.7, is not one of my own ideas; it is the outcome of a long-term debate between G. Rasche and W.S. Woolcock. I am grateful to G. Rasche also for drawing my attention to Refs. [58,59].

References

[1] C.M.G. Lattes, H. Muirhead, G.P.S. Occhialini, C.F. Powell, ‘Processes involving charged mesons’, Nature 159 (1947) 694–697. DOI: 10.1038/159694a0

[2] J.J. de Swart, M.C.M. Rentmeester, R.G.E. Timmermans, ‘The status of the pion-nucleon coupling constant’, $\pi N$ Newslett. 13 (1997) 96–107; arXiv:9802084 [nucl-th]

[3] M.L. Goldberger, H. Miyazawa, R. Oehme, ‘Application of dispersion relations to pion-nucleon scattering’, Phys. Rev. 99 (1955) 986. DOI: 10.1103/PhysRev.99.986

[4] J. Hamilton, W.S. Woolcock, ‘Determination of pion-nucleon parameters and phase shifts by dispersion relations’, Rev. Mod. Phys. 35 (1963) 737–787. DOI: 10.1103/RevModPhys.35.737

[5] D.V. Bugg, ‘Meson-nucleon coupling constants from nucleon-nucleon forward dispersion relations’, Nucl. Phys. B 5 (1968) 29–46. DOI: 10.1016/0550-3213(68)90205-8

[6] G. Ebel, H. Pilkuhn, F. Steiner, ‘Compilation of coupling constants and low-energy parameters’, Nucl. Phys. B 17 (1970) 1–26. DOI: 10.1016/0550-3213(70)90400-1
[7] L.S. Brown, W.J. Pardee, R.D. Peccei, ‘Adler-Weisberger Theorem reexamined’, Phys. Rev. D 4 (1971) 2801–2810. DOI: 10.1103/PhysRevD.4.2801

[8] S.L. Adler, ‘Sum rules for the axial-vector coupling-constant renormalization in β decay’, Phys. Rev. 140 (1965) B736; see also Errata Phys. Rev. 149 (1966) 1294, Phys. Rev. 175 (1968) 2224. DOI: 10.1103/PhysRev.140.B736

[9] W.I. Weisberger, ‘Unsubtracted dispersion relations and the renormalization of the weak axial-vector coupling constants’, Phys. Rev. 143 (1966) 1302. DOI: 10.1103/PhysRev.143.1302

[10] D.V. Bugg, A.A. Carter, J.R. Carter, ‘New values of pion-nucleon scattering lengths and f^2’, Phys. Lett. B 44 (1973) 278–280. DOI: 10.1016/0370-2693(73)90225-6

[11] R. Koch, E. Pietarinen, ‘Low-energy πN partial wave analysis’, Nucl. Phys. A 336 (1980) 331–346. DOI: 10.1016/0375-9474(80)90214-6

[12] P.Y. Bertin et al., ‘π+p scattering below 100 MeV’, Nucl. Phys. B 106 (1976) 341–354. DOI: 10.1016/0550-3213(76)90383-7

[13] R.A. Arndt, Zhujun Li, L.D. Roper, R.L. Workman, ‘Determination of the πNN coupling constant from elastic pion-nucleon scattering data’, Phys. Rev. Lett. 65 (1990) 157. DOI: 10.1103/PhysRevLett.65.157

[14] R.A. Arndt, Zhujun Li, L.D. Roper, R.L. Workman, J.M. Ford, ‘Pion-nucleon partial-wave analysis to 2 GeV’, Phys. Rev. D 43 (1991) 2131. DOI: 10.1103/PhysRevD.43.2131

[15] R.A.M. Klomp, V.G.J. Stoks, J.J. de Swart, ‘Determination of the N\bar{N}\pi coupling constants in N\bar{N} partial-wave analyses’, Phys. Rev. C 44 (1991) R1258(R). DOI: 10.1103/PhysRevC.44.R1258

[16] V. Stoks, R. Timmermans, J.J. de Swart, ‘Pion-nucleon coupling constant’, Phys. Rev. C 47 (1993) 512. DOI: 10.1103/PhysRevC.47.512

[17] F.G. Markopoulou-Kalamara, D.V. Bugg, ‘A new determination of the πNN coupling constant f^2’, Phys. Lett. B 318 (1993) 565–567. DOI: 10.1016/0370-2693(93)91556-3

[18] R. Timmermans, Th.A. Rijken, J.J. de Swart, ‘Antiproton-proton partial-wave analysis below 925 MeV/c’, Phys. Rev. C 50 (1994) 48. DOI: 10.1103/PhysRevC.50.48

[19] R.A. Arndt, R.L. Workman, M.M. Pavan, ‘Pion-nucleon partial-wave analysis with fixed-t dispersion relation constraints’, Phys. Rev. C 49 (1994) 2729–2734. DOI: 10.1103/PhysRevC.49.2729

[20] F. Bradamante, A. Bressan, M. Lamanna, A. Martin, ‘Determination of the charged pion-nucleon coupling constant from \bar{p}p \rightarrow \bar{n}n differential cross-section’, Phys. Lett. B 343 (1995) 431–435. DOI: 10.1016/0370-2693(94)01565-T
[21] T.E.O. Ericson et al., ‘πNN coupling from high precision np charge exchange at 162 MeV’, Phys. Rev. Lett. 75 (1995) 1046–1049. DOI: 10.1103/PhysRevLett.75.1046

[22] D.V. Bugg, R. Machleidt, ‘πNN coupling constants from NN elastic data between 210 and 800 MeV’, Phys. Rev. C 52 (1995) 1203. DOI: 10.1103/PhysRevC.52.1203

[23] R.G.E. Timmermans, ‘Novel pion nucleon partial-wave analysis’, πN Newslett. 13 (1997) 80–89.

[24] E. Matsinos, ‘πN scattering below 100 MeV’, πN Newslett. 13 (1997) 132–137.

[25] E. Matsinos, ‘Isospin violation in the πN system at low energies’, Phys. Rev. C 56 (1997) 3014–3025. DOI: 10.1103/PhysRevC.56.3014

[26] G.J. Wagner, ‘Symposium summary’, πN Newslett. 13 (1997) 385–392.

[27] R.G.E. Timmermans, private communication.

[28] M.C.M. Rentmeester, R.G.E. Timmermans, J.L. Friar, J.J. de Swart, ‘Chiral two-pion exchange and proton-proton partial-wave analysis’, Phys. Rev. Lett. 82 (1999) 4992. DOI: 10.1103/PhysRevLett.82.4992

[29] W.R. Gibbs, Li Ai, W.B. Kaufmann, ‘Low-energy pion-nucleon scattering’, Phys. Rev. C 57 (1998) 784–797. DOI: 10.1103/PhysRevC.57.784

[30] J. Rahm et al., ‘np scattering measurements at 162 MeV and the πNN coupling constant’, Phys. Rev. C 57 (1998) 1077–1096. DOI: 10.1103/PhysRevC.57.1077

[31] J. Rahm et al., ‘np scattering measurements at 96 MeV’, Phys. Rev. C 63 (2001) 044001. DOI: 10.1103/PhysRevC.63.044001

[32] H.-Ch. Schröder et al., ‘The pion-nucleon scattering lengths from pionic hydrogen and deuterium’, Eur. Phys. J. C 21 (2001) 473–488. DOI: 10.1007/s100520100754

[33] D. Sigg, A. Badertscher, P.F.A. Goudsmit, H.J. Leisi, G.C. Oades, ‘Electromagnetic corrections to the s-wave scattering lengths in pionic hydrogen’, Nucl. Phys. A 609 (1996) 310–325. DOI: 10.1016/S0375-9474(96)00238-2

[34] G.C. Oades, G. Rasche, W.S. Woolcock, E. Matsinos, A. Gashi, ‘Determination of the s-wave pion-nucleon threshold scattering parameters from the results of experiments on pionic hydrogen’, Nucl. Phys. A 794 (2007) 73–86. 10.1016/j.nuclphysa.2007.07.007

[35] T.E.O. Ericson, B. Loiseau, A.W. Thomas, ‘Determination of the pion-nucleon coupling constant and scattering lengths’, Phys. Rev. C 66 (2002) 014005. DOI: 10.1103/PhysRevC.66.014005

[36] T.E.O. Ericson, B. Loiseau, S. Wycech, ‘A phenomenological π−p scattering length from pionic hydrogen’, Phys. Lett. B 594 (2004) 76–86. DOI: 10.1016/j.physletb.2004.05.009
[37] D.V. Bugg, ‘The pion nucleon coupling constant’, Eur. Phys. J. C 33 (2004) 505–509. DOI: 10.1140/epjc/s2004-01666-y

[38] V. Baru, C. Hanhart, M. Hoferichter, B. Kubis, A. Nogga, D.R. Phillips, ‘Precision calculation of the $\pi^-$-$d$ scattering length and its impact on threshold $\pi N$ scattering’, Phys. Lett. B 694 (2011) 473–477. DOI: 10.1016/j.physletb.2010.10.028

[39] M. Hennebach et al., ‘Hadronic shift in pionic hydrogen’, Eur. Phys. J. A 50 (2014) 190. DOI: 10.1140/epja/i2014-14190-x

[40] D. Gotta et al., ‘Pionic hydrogen and friends’, Hyperfine Interact 234 (2015) 105–111. DOI: 10.1007/s10751-015-1157-5

[41] V.A. Babenko, N.M. Petrov, ‘Study of the charge dependence of the pion-nucleon coupling constant on the basis of data on low-energy nucleon-nucleon interactions’, Phys. Atomic Nuclei 79 (2016) 67–71. DOI: 10.1134/S1063778815090033

[42] V.A. Babenko, N.M. Petrov, ‘Relation between the charged and neutral pion-nucleon coupling constants in the Yukawa model’, Phys. Part Nuclei Lett. 14 (2017) 58–65. DOI: 10.1134/S1547477117010083

[43] E. Ruiz Arriola, J.E. Amaro, R. Navarro Pérez, ‘Three pion nucleon coupling constants’, Mod. Phys. Lett A 31 (2016) 1630027. DOI: 10.1142/S0217732316300275

[44] R. Navarro Pérez, J.E. Amaro, E. Ruiz Arriola, ‘Precise determination of charge dependent pion-nucleon-nucleon coupling constants’, Phys. Rev. C 95 (2017) 064001. DOI: 10.1103/PhysRevC.95.064001

[45] F. James, ‘MINUIT - Function Minimization and Error Analysis’, CERN Program Library Long Writeup D506.

[46] You-Gan Wang, Xu Lin, Min Zhu, Zhidong Bai, ‘Robust estimation using the Huber function with a data-dependent tuning constant’, J. Comput. Graph. Stat. 16 (2012) 468–481. DOI: 10.1198/106186007X180156

[47] E. Matsinos, G. Rasche, ‘Aspects of the ETH model of the pion-nucleon interaction’, Nucl. Phys. A 927 (2014) 147–194. DOI: 10.1016/j.nuclphysa.2014.04.021

[48] P.F.A. Goudsmit, H.J. Leisi, E. Matsinos, ‘A pion-nucleon interaction model’, Phys. Lett. B 299 (1993) 6–10. DOI: 10.1016/0370-2693(93)90875-I

[49] P.F.A. Goudsmit, H.J. Leisi, E. Matsinos, B.L. Birbrair, A.B. Gridnev, ‘The extended tree-level model for the pion-nucleon interaction’, Nucl. Phys. A 575 (1994) 673–706. DOI: 10.1016/0375-9474(94)90162-7

[50] P.F.A. Goudsmit, H.J. Leisi, E. Matsinos, ‘The low-energy pion-nucleon interaction’, Helv. Phys. Acta 67 (1994) 369–391.
[51] E. Matsinos, W.S. Woolcock, G.C. Oades, G. Rasche, A. Gashi, ‘Phase-shift analysis of low-energy $\pi^\pm p$ elastic-scattering data’, Nucl. Phys. A 778 (2006) 95–123. DOI: 10.1016/j.nuclphysa.2006.07.040

[52] E. Matsinos, G. Rasche, ‘Analysis of the low-energy $\pi^\pm p$ elastic-scattering data’, J. Mod. Phys. 3 (2012) 1369–1387. DOI: 10.4236/jmp.2012.310174

[53] E. Matsinos, G. Rasche, ‘Analysis of the low-energy $\pi^- p$ charge-exchange data’, Int. J. Mod. Phys. A 28 (2013) 1350039. DOI: 10.1142/S0217751X13500395

[54] E. Matsinos, G. Rasche, ‘Update of the phase-shift analysis of the low-energy $\pi N$ data’, arXiv:1706.05524 [nucl-th]

[55] R.A. Arndt, L.D. Roper, ‘The use of partial-wave representations in the planning of scattering measurements. Application to 330 MeV $np$ scattering’, Nucl. Phys. B 50 (1972) 285–300. DOI: 10.1016/S0550-3213(72)80019-1

[56] P.J. Huber, ‘Robust regression: asymptotics, conjectures, and Monte Carlo’, Ann. Stat. 1 (1973) 799–821.

[57] E. Matsinos, G. Rasche, ‘Systematic effects in the low-energy behavior of the current SAID solution for the pion-nucleon system’, Int. J. Mod. Phys. E 26 (2017) 1750002. DOI: 10.1142/S0218301317500021

[58] G. Rasche, W.S. Woolcock, ‘The effect of radiative capture on threshold $\pi^- p$ scattering and the theory of the Panofsky ratio’, Helv. Phys. Acta 49 (1976) 557–567.

[59] G. Rasche, W.S. Woolcock, ‘Connection between low-energy scattering parameters and energy shifts for pionic hydrogen’, Nucl. Phys. A 381 (1982) 405–418. DOI: 10.1016/0375-9474(82)90367-0

[60] M. Tanabashi et al. (Particle Data Group), ‘The Review of Particle Physics (2018)’, Phys. Rev. D 98 (2018) 030001.

[61] S. Deser, M.L. Goldberger, K. Baumann, W. Thirring, ‘Energy level displacements in pi-mesonic atoms’, Phys. Rev. 96 (1954) 774–776. DOI: 10.1103/PhysRev.96.774

[62] T.L. Trueman, ‘Energy level shifts in atomic states of strongly-interacting particles’, Nucl. Phys. 26 (1961) 57–67. DOI: 10.1016/0029-5582(61)90115-8

[63] V.E. Lyubovitskij, A. Rusetsky, ‘$\pi^- p$ atom in ChPT: strong energy-level shift’, Phys. Lett. B 494 (2000) 9–18. DOI: 10.1016/S0370-2693(00)01185-0

[64] D. Eiras, J. Soto, ‘Light fermion finite mass effects in non-relativistic bound states’, Phys. Lett. B 491 (2000) 101–110. DOI: 10.1016/S0370-2693(00)01004-2

[65] G. Höhler, ‘Pion Nucleon Scattering. Part 2: Methods and Results of Phenomenological Analyses’, Landolt-Börnstein, Vol. 9b2, ed. H. Schopper, Springer, Berlin, 1983.
[66] J. Gasser, M.A. Ivanov, E. Lipartia, M. Mojžiš, A. Rusetzky, ‘Ground-state energy of pionic hydrogen to one loop’, Eur. Phys. J. C 26 (2002) 13–34. DOI: 10.1007/s10052-002-1013-z

[67] V.E. Lyubovitskij, Th. Gutsche, A. Faessler, R. Vinh Mau, ‘Electromagnetic couplings of the chiral perturbation theory Lagrangian from the perturbative chiral quark model’, Phys. Rev. C 65 (2002) 025202. DOI: 10.1103/PhysRevC.65.025202

[68] V. Baru, C. Hanhart, M. Hoferichter, B. Kubis, A. Nogga, D.R. Phillips, ‘Precision calculation of threshold $\pi^- d$ scattering, $\pi N$ scattering lengths, and the GMO sum rule’, Nucl. Phys. A 872 (2011) 69–116. DOI: 10.1016/j.nuclphysa.2011.09.015

[69] P. Zemp, ‘Pionic Hydrogen in QCD + QED: Decay width at NNLO’, PhD dissertation, University of Bern, 2004.

[70] A. Gashi, E. Matsinos, G.C. Oades, G. Rasche, W.S. Woolcock, ‘Electromagnetic corrections to the phase shifts in low energy $\pi^+ p$ elastic scattering’, Nucl. Phys. A 686 (2001) 447–462. DOI: 10.1016/S0375-9474(00)00603-5

[71] A. Gashi, E. Matsinos, G.C. Oades, G. Rasche, W.S. Woolcock, ‘Electromagnetic corrections for the analysis of low energy $\pi^- p$ scattering data’, Nucl. Phys. A 686 (2001) 463–477. DOI: 10.1016/S0375-9474(00)00604-7

[72] V.V. Abaev, P. Metsä, M.E. Sainio, ‘The Goldberger-Miyazawa-Oehme sum rule revisited’, Eur. Phys. J. A 32 (2007) 321–325. DOI: 10.1140/epja/i2007-10377-6
A The EM corrections at the $\pi N$ threshold

The determination of the $\pi N$ coupling constant from the isovector hadronic scattering length $\bar{b}_1$, by use of the GMO sum rule, gained momentum over the past decades, in parallel to the remarkable enhancement of the low-energy $\pi N$ DB, which the experiments, conducted at the three meson factories (LANL, PSI, TRIUMF) after 1980, attained. The analysis of the low-energy DB enables the extraction of reliable estimates for the $\pi N$ scattering lengths. In addition, the first measurement of the total decay width $\Gamma_{1s}$ of the 1$s$ state of pionic hydrogen in 1995 permitted the direct (i.e., not involving an extrapolation of the $\pi N$ scattering amplitudes to threshold) extraction of $b_1$. Of course, the EM effects need to be removed in both cases, i.e., both from the scattering amplitudes before they can be extrapolated to threshold, as well as from $b_1$ directly obtained from $\Gamma_{1s}$. I decided to include in this work a rather detailed description of the approaches tailored to the removal of the EM effects from the measurements on pionic hydrogen at threshold.

The most recent compilation of the physical constants [60] has been used in extracting the numerical results below. All masses are expressed in energy units. The uncertainties are total, i.e., they include the effects of the variation of all physical ‘constants’ entering each determination, as well as those relating to the variation of the experimental (and, as far as the EM corrections are concerned, theoretical) input. The results have been obtained by means of a Monte-Carlo generation of one billion events. During the calculation, all scattering lengths were expressed exclusively in length units (i.e., fm in this case, not $m^{-1}$). Although it appears ludicrous (to me) to express lengths in units of inverse mass, I felt somewhat compelled to give some of the resulting scattering lengths also in units of $m^{-1}$ in order to facilitate the comparison with other works.

Introduced by Deser and collaborators [61], the first of the Deser formulae\footnote{I prefer this short form as reference to the two important relations developed by Deser, Goldberger, Baumann, and Thirring in 1954, as well as by Trueman in 1961, rather than Deser-Goldberger-Baumann-Thirring, Deser-Trueman, Trueman-Deser relations, or any other combination of these names.} relates the strong shift $\epsilon_{1s}$ in pionic hydrogen with the ‘untreated’ (i.e., containing all EM effects) scattering length $a_{cc}$ which, as already mentioned in Section 2.2, is associated with $\pi^- p$ ES.

\begin{equation}
\epsilon_{1s} = -4 \frac{E_{1s}}{r_B} a_{cc} = -\frac{2\alpha^3\mu^2}{\hbar c} a_{cc} ,
\end{equation}

where $E_{1s} = \alpha^2\mu/2$ is the (point-Coulomb) EM binding energy of the 1$s$ level and $r_B = \hbar c/(\alpha\mu)$ is the Bohr radius; $\alpha$ denotes the fine-structure constant.
and \( \mu \) stands for the reduced mass of the \( \pi^- p \) system. Using the \( \epsilon_{1s} \) value of Ref. [32] as input, namely \(-7.108 \pm 0.013 \text{(stat.)} \pm 0.034 \text{(syst.)} \) eV, and after combining the statistical and systematic uncertainties linearly\(^9\), one obtains from Eq. (A.1): \( a_{cc} = 0.12226(81) \) fm.

The second of the Deser formulae, put into its current form by Trueman [62], enables the extraction (from \( \Gamma_{1s} \)) of the scattering length \( a_{c0} \), associated with the \( \pi^- p \) CX reaction.

\[
\Gamma_{1s} = 8q_0 E_{1s} \frac{E_{1s}}{E_B} \left( 1 + P^{-1} \right) a_{c0}^2 = 4q_0 \frac{\alpha^3 \mu^2}{(\hbar c)^2} \left( 1 + P^{-1} \right) a_{c0}^2 , \tag{A.2}
\]

where \( q_0 \) stands for the magnitude of the 3-momentum of the outgoing \( \pi^0 \) (or neutron) in the CM coordinate system and \( P = 1.546(9) \) is known as Panofsky ratio. The result of Ref. [32] for \( \Gamma_{1s} \) was \( 0.868 \pm 0.040 \text{(stat.)} \pm 0.038 \text{(syst.)} \) eV. From Eq. (A.2), one obtains\(^10\) \( a_{c0} = -0.1784(81) \) fm.

Evidently, the ETH-Neuchâtel-PSI Collaboration delivered \( a_{cc} \) to an accuracy of 0.66 \% and \( a_{c0} \) to an accuracy of 4.5 \%. At this point, corrections need to be applied, in order to rid \( a_{cc} \) and \( a_{c0} \) of the effects of EM origin, and lead to estimates for the corresponding hadronic scattering lengths, \( \tilde{a}_{cc} \) and \( \tilde{a}_{c0} \). The EM corrections are usually expressed in the form of two quantities, \( \delta_e \) for \( a_{cc} \) and \( \delta_{\Gamma} \) for \( a_{c0} \). The two hadronic scattering lengths are obtained from \( a_{cc} \) and \( a_{c0} \) according to the following two definitions.

\[
\tilde{a}_{cc} = a_{cc}/(1 + \delta_e) \tag{A.3}
\]

\[
\tilde{a}_{c0} = a_{c0}/(1 + \delta_{\Gamma}) \tag{A.4}
\]

It is understood that the scattering lengths \( a_{cc} \) and \( a_{c0} \) in Eqs. (A.3,A.4) are associated with the original Deser formulae (A.1,A.2). These formulae represent leading-order (LO) evaluations of \( \epsilon_{1s} \) and \( \Gamma_{1s} \), namely evaluations at \( \mathcal{O}(\alpha^3) \). In several works, the quantities \( a_{cc} \) and \( a_{c0} \) of Eqs. (A.1,A.2) are therefore denoted as \( a_{cc}^{LO} \) and \( a_{c0}^{LO} \). This is done in order to distinguish these scattering lengths from those appearing in the upgraded forms of the Deser formulae, i.e., the expressions obtained at higher orders of \( \alpha \). At the present time, only the next-to-leading-order (NLO) evaluations of \( \epsilon_{1s} \) and \( \Gamma_{1s} \) are available, i.e., the evaluations at \( \mathcal{O}(\alpha^4) \). Some authors denote the scattering lengths, entering the NLO evaluations of \( \epsilon_{1s} \) and \( \Gamma_{1s} \), as \( a_{cc}^{NLO} \) and \( a_{c0}^{NLO} \). In this work, \( a_{cc} \) and \( a_{c0} \) will represent \( a_{cc}^{LO} \) and \( a_{c0}^{LO} \), respectively. I will introduce \( \mathcal{A}_{cc} \) and \( \mathcal{A}_{c0} \) later on, to refer to \( a_{cc}^{NLO} \) and \( a_{c0}^{NLO} \), respectively.

\(^{9}\) The group favour the linear combination of their statistical and systematic uncertainties, see Section 4.1 of Ref. [32].

\(^{10}\) The scattering length \( a_{c0} \) is negative.
Regarding the removal of the EM effects at threshold before the first experiment on pionic hydrogen was conducted at PSI in the mid 1990s, I am only aware of two papers by Rasche and Woolcock [58,59]. The former paper develops the methodology needed for the correct inclusion of the effects of the $\gamma n$ channel at threshold. The second paper presents a method for the determination of the strong shift and total decay width in pionic atoms. The numerical results in Section 3 of Ref. [59] were tailored to the $2p \rightarrow 1s$ transition in pionic hydrogen, which was deemed at the time as the most promising (on the basis of the yield) transition. Extensive comments on the corrections of Ref. [59] may be found in Section 4 of Ref. [33]. The corrections of Ref. [59] were superseded by those of Ref. [34].

Several schemes of EM corrections were developed after the experimental results at threshold became available. Some of these schemes aim at the removal of the ‘trivial’ EM effects, i.e., of those associated with the contributions from the vacuum polarisation, from the extended-charge distributions of the pion and of the proton, as well as from the mass differences of the particles in the initial and final states. In the context of Ref. [34], all these contributions comprise the stage-1 EM corrections. The models of Sections A.1 and A.2 are expected to remove these effects. On the other hand, works carried out within the framework of Chiral Perturbation Theory (ChPT) also attempt the removal of the effects which are associated with the mass difference between the $u$ and $d$ quarks, i.e., effects which belong to the stage-2 EM corrections in the context of Ref. [34]. The models of Sections A.3, A.4, A.5, and A.6 belong to this category. For a meaningful comparison of the results, one needs to bear in mind the distinction between these two categories of corrections.

### A.1 Potential models for the removal of the EM effects

In their assessment of the EM effects at threshold, Refs. [33,34] made use of suitable potentials.

An estimate for $\delta_\epsilon$ was obtained in Ref. [33] by means of a two-channel calculation, along with the phenomenological addition of the effects of the $\gamma n$ channel: $\delta_\epsilon = -2.1(5) \cdot 10^{-2}$. This $\delta_\epsilon$ value was imported into Ref. [32] and yielded $\tilde{a}_{cc} = 0.1249(10)$ fm or, for those who prefer to express the scattering lengths in $m_c^{-1}$, $\tilde{a}_{cc} = 0.08833(74) m_c^{-1}$.

In Ref. [34], the correction $\Delta a_{cc} := a_{cc} - \tilde{a}_{cc}$ was evaluated by means of a three-channel calculation: $\Delta a_{cc} = 0.0008(8)$ fm. Applying this correction to the untreated $a_{cc}$ value, emerging from the $\epsilon_{1s}$ result of Ref. [32], leads to $\tilde{a}_{cc} = 0.1215(11)$ fm or $0.08591(80) m_c^{-1}$. To enable the comparison of the corrections, obtained in the two papers, one may translate the $\Delta a_{cc}$ value of
Ref. [34] into a $\delta_\epsilon$ value; one obtains $\delta_\epsilon = 0.66(66) \cdot 10^{-2}$.

In Ref. [33], the estimate for $\delta_\Gamma$ of $-1.3(5) \cdot 10^{-2}$ had been extracted. On the other hand, the correction of Ref. [34] had been expressed as the difference between the untreated and the corrected scattering lengths for the CX reaction: $\Delta a_{c0} := a_{c0} - \tilde{a}_{c0}$. Expressed as a $\delta_\Gamma$ value, the correction $\Delta a_{c0}$ of Ref. [34] would have been: $\delta_\Gamma = -1.66(33) \cdot 10^{-2}$. One therefore concludes that the two corrections [33,34] agree within the uncertainties in the case of $\Gamma_{1s}$. Evidently, only the correction applied to $a_{cc}$ is sensitive to the treatment of the $\gamma n$ channel. The $\Gamma_{1s}$ result of Ref. [32], along with the EM correction $\Delta a_{c0}$ of Ref. [34], yields $\tilde{a}_{c0} = -0.1814(81) \text{ fm or } -0.1283(57) \text{ m}^{-1}$. The $\tilde{a}_{c0}$ result of Ref. [32]) (i.e., $-0.128(6) \text{ m}^{-1}$, see Eq. (30) therein) was identical in the physical sense.

A.2 The model of Ericson, Loiseau, and Wycech [36]

In 2004, Ericson and collaborators [36] followed a non-relativistic approach using Coulomb wavefunctions, with a short-range hadronic interaction and extended-charge distributions, and treated four sources of EM corrections: the first two originate from the interferences of the vacuum-polarisation potential and of the extended-charge potential with the hadronic potential; the remaining two are renormalisation and gauge contributions.

The estimates of Ref. [36] for the corrections $\delta_\epsilon$ and $\delta_\Gamma$ may be found in their Table 1: $\delta_\epsilon = -0.62(29) \cdot 10^{-2}$ and $\delta_\Gamma = 1.02(23) \cdot 10^{-2}$. The discrepancy in the $\delta_\Gamma$ values between the results of Sections A.1 and A.2 is noticeable. The correction $\delta_\epsilon$ of Ref. [36] lies in-between the results obtained with the two potential models of the previous section, slightly closer to the result of Ref. [34].

Extensive comments on the approach of Ref. [36] may be found in Section 4 of Ref. [34]; there is no point in repeating them here. I will only re-iterate that the uncertainties of $\delta_\epsilon$ and $\delta_\Gamma$ of Ref. [36] appear to be on the optimistic side.

A.3 The Lyubovitskij-Rusetsky correction to $a_{cc}$ [63]

In my opinion, the paper of Lyubovitskij and Rusetsky [63] in 2000 was essential for two reasons.

- The authors presented an evaluation of $\epsilon_{1s}$ at $\mathcal{O}(\alpha^4)$; this is an important upgrade of Eq. (A.1). Part of the effects, which need to be taken care of by the EM corrections in case that Eq. (A.1) is used, are contained in the upgraded expression.
• Their work constituted the first attempt to derive the EM corrections within the (systematic) framework of ChPT.

The relation at $O(\alpha^4)$ between $\epsilon_{1s}$ and the scattering length (denoted as $A$ in Ref. [63], $A_{cc}$ in this work) reads as:

$$\epsilon_{1s} = -\frac{2\alpha^3 \mu^2}{\hbar c} A_{cc} \left( 1 + 2\alpha (1 - \ln \alpha) \frac{\mu A_{cc}}{\hbar c} \right), \quad (A.5)$$

which, after appending the effects due to the interference between the strong interaction and the vacuum polarisation $\phi \approx 0.483 \cdot 10^{-2}$ of Ref. [64] (these effects had not been included in Ref. [63]), may be rewritten as

$$\epsilon_{1s} = -\frac{2\alpha^3 \mu^2}{\hbar c} A_{cc} \left( 1 + \varphi + 2\alpha (1 - \ln \alpha) \frac{\mu A_{cc}}{\hbar c} \right). \quad (A.6)$$

Lyubovitskij and Rusetsky did not identify $A_{cc}$ with $\tilde{a}_{cc}$. Additional isospin-breaking corrections (denoted as $\epsilon$ in Ref. [63], $\Delta A_{cc}$ in this work), to be understood as residual effects of EM origin and contributions originating from the mass difference between the $u$ and $d$ quarks, were evaluated in Ref. [63] at $O(p^2)$ in ChPT. The relation between the quantities $A_{cc}$ and $\tilde{a}_{cc}$ was given in Ref. [63] as:

$$\tilde{a}_{cc} = A_{cc} - \Delta A_{cc}.$$  

The correction $\Delta A_{cc}$ depends on three low-energy constants (LECs) $c_1$, $f_1$, and $f_2$ in the $O(p^2)$ chiral $\pi N$ Lagrangian, one of which ($f_1$) is poorly known. According to Ref. [63]:

$$\Delta A_{cc} = \frac{m_p \hbar c}{2(m_p + m_c) \pi F^2_{\pi}} \left( \frac{2(m_c^2 - m_0^2)}{2} c_1 - \alpha(4f_1 + f_2) \right), \quad (A.7)$$

where $m_0$ is the mass of the neutral pion and $F_{\pi} = 92.28(12)$ MeV is the pion-decay constant.

• For $c_1$, Lyubovitskij and Rusetsky used a result from one of Karlsruhe analyses of the mid 1980s, privately communicated to the authors; that value was equal to $-0.925$ GeV$^{-1}$ (no uncertainty was quoted in Ref. [63]). Also using information from the Karlsruhe programme of the 1980s [65], Gasser and collaborators [66] came up (in 2002) with $c_1 = -0.93(7)$ GeV$^{-1}$. Also in 2002, Lyubovitskij and collaborators [67] imported (from a work of 2001) a different $c_1$ value, namely $c_1 = -1.2(1)$ GeV$^{-1}$. More recent works [38,68] recommend: $c_1 = -1.0(3)$ GeV$^{-1}$.

• Regarding the LEC $f_2$, Ref. [63] used $f_2 = -0.97(38)$ GeV$^{-1}$, which is also the recommended value in Ref. [66].

• As aforementioned, the LEC $f_1$ is poorly known. To derive an estimate for the correction $\delta_\epsilon$, Lyubovitskij and Rusetsky assumed in Ref. [63] that $|f_1| \leq |f_2|$. However, Lyubovitskij and collaborators [67] arrived in 2002 at the sizeably different result for the ratio $f_1/f_2$, namely 2.24(26). The authors
also favoured \( f_1 = -2.29(19) \text{ GeV}^{-1} \), which does not seem to be in line with the other ‘expectations’ for this LEC. Gasser and collaborators [66] mention their “order of magnitude” estimate for \(|f_1|\) at about 1.4 GeV\(^{-1}\).

Be that as it may, Ref. [63] reported a large negative correction: \( \delta_\epsilon = (-4.8 \pm 2.0) \cdot 10^{-2} \), where the uncertainty is dominated by the poor knowledge of \( f_1 \). This correction rests upon the assumption \( |f_1| \leq |f_2| \). I set out to re-evaluate the correction \( \delta_\epsilon \) in the Lyubovitskij-Rusetsky scheme, using Eq. (A.6), rather than Eq. (A.5) which they had used. As Ref. [63] mentions no uncertainty in the LEC \( c_1 \), I first assumed that \( c_1 \) was not varied in their analysis; however, the resulting uncertainty of \( \delta_\epsilon \) turned out to be nearly a factor of 2 smaller than the one quoted in Ref. [63]. Therefore, I concluded that also \( c_1 \) was varied in Ref. [63] and proceeded by changing the assigned \( c_1 \) uncertainty, until the final result for \( \delta_\epsilon \) matched the reported \( \delta_\epsilon \) uncertainty of Ref. [63]. My conclusion is that Lyubovitskij and Rusetsky had most likely used a \( \delta c_1 \) value between 0.2 and 0.3 GeV\(^{-1}\) in their work. In any case, the \( \delta c_1 \) value of 0.3 GeV\(^{-1}\), also recommended in Refs. [38,68], appears to be reasonable and conservative. For the needs of Table A.1, I obtain the correction \( \delta_\epsilon \) using Eq. (A.6) with the \( \varphi \) value of Ref. [64] and \( \delta c_1 = 0.3 \text{ GeV}^{-1} \). The other two LECs are varied according to Ref. [63].

As the models of Sections A.1 and A.2 do not contain any stage-2 EM corrections, the comparison of their \( \delta_\epsilon \) values with the result of this section does not make much sense. On the other hand, one could pose the question whether a comparison could be meaningful if \( \tilde{a}_{cc} \) were identified as the scattering length \( A_{cc} \), obtained from Eq. (A.6). There is no doubt that some of the effects, which are treated by the models of Sections A.1 and A.2, are contained in \( \Delta A_{cc} \) of Eq. (A.7). Unfortunately, because of the mixed term in Eq. (A.7) (last term within the large brackets), one cannot disentangle the EM contributions and those relating to the \( m_u \neq m_d \) effects. It appears to me that there is no guarantee that a comparison of the results of this section with those obtained with the models of Sections A.1 and A.2 is meaningful. Nevertheless, I will also obtain a \( \delta_\epsilon \) value corresponding to the case that \( A_{cc} \) of Eq. (A.6) is identified as \( \tilde{a}_{cc} \). This intermediate result will be helpful later on in assessing the importance of the isospin-breaking effects \( \Delta A_{cc} \) of Eq. (A.7).

In 2002, Lyubovitskij and collaborators [67] provided an update of \( \delta_\epsilon \), on the basis of improved knowledge of \( f_1 \) when employing their “perturbative chiral quark model”; the new value was \( \delta_\epsilon = -2.8 \cdot 10^{-2} \), quoted in Ref. [67] without an uncertainty. Evidently, the updated value of Ref. [67] is not incompatible with the 1996 result extracted with the potential model of Ref. [33].
A.4 *Isospin-breaking corrections evaluated at $O(p^3)$ in ChPT [66]*

An even larger (and more uncertain) correction $\delta_\epsilon$ was extracted in 2002 [66] within a calculation at NLO ($O(p^3)$) in isospin breaking and in the low-energy expansion: $(-7.2 \pm 2.9) \cdot 10^{-2}$.

A.5 *Leading-order correction $\delta_T$ in ChPT [69]*

The LO correction to $a_{c0}$, derived in ChPT in Ref. [69] in 2004, was found small: $\delta_T = 0.6(2) \cdot 10^{-2}$, see Eq. (5.26) therein. One notices that the correction $\delta_T$ from ChPT is more accurate than the correction $\delta_\epsilon$. This is due to the fact that the LEC $f_1$ does not enter the determination of $\delta_T$.

A.6 *The corrections developed by the Bonn-Jülich group*

Between 2005 and 2011, the Bonn-Jülich group developed a correction scheme for the pionic-hydrogen measurements, similar to those detailed in Sections A.3, A.4, and A.5, see Refs. [38,68] and the relevant papers therein. In addition, corrections for the strong shift of 1s state in pionic deuterium were developed.

Regarding $\epsilon_{1s}$ in pionic hydrogen, Ref. [38] uses Eq. (A.6) to extract $A_{cc}$, which the authors call $a_{\pi-p}$ in their paper. They subsequently associate $A_{cc}$ with the difference $b_0 - \tilde{b}_1$.

$$b_0 - \tilde{b}_1 = A_{cc} - \Delta a_{cc} \hbar c,$$  \hspace{1cm} (A.8)

where the isoscalar scattering length is to be thought of as untreated\(^\text{11}\) as the lack of the tilde over it indicates, and $\Delta a_{cc} = (-2.0 \pm 1.3) \cdot 10^{-3} m_c^{-1}$.

For the relation between $\Gamma_{1s}$ of pionic hydrogen and the corresponding scattering length $A_{c0}$, the authors use the expression:

$$\Gamma_{1s} = 4 q_0 \frac{\alpha^3 \mu^2}{(\hbar c)^2} \left(1 + P^{-1}\right) A_{c0}^2 \left(1 + \varphi + 4 \alpha (1 - \ln \alpha) \frac{\mu A_{cc}}{\hbar c} + 2 (m_p + m_c - m_n - m_0) \frac{\mu b_0^2}{(\hbar c)^2}\right),$$  \hspace{1cm} (A.9)

\(^{11}\) The untreated isoscalar scattering length $b_0$ also enters the strong shift $\epsilon_{1s}$ in pionic deuterium. A combined analysis of the $\epsilon_{1s}$ values in pionic hydrogen and deuterium, and of $\Gamma_{1s}$ of pionic hydrogen enables a more accurate determination of the quantities $b_0$ and $\tilde{b}_1$ (compared to the use of the information extracted only from pionic hydrogen), see Fig. 2 of Ref. [38].
where $m_n$ is the mass of the neutron and $\mathcal{A}_{c0} = \sqrt{2}b_1 + \Delta a_{c0}\hbar c$, with $\Delta a_{c0} = 0.4(9) \cdot 10^{-3} m^{-1}$. Equations (A.8,A.9) contain two unknowns: $b_0$ and $\tilde{b}_1$. The quantity $\tilde{b}_1$ may be obtained by use of a simple recursion scheme; the convergence is very fast. The quantity $b_0$ is subsequently obtained via Eq. (A.8). Evident from Refs. [38,68] is that the isospin-breaking effects have a larger impact on the isoscalar part of the $\pi N$ interaction at threshold; for this correction, the authors give the expression:

$$\tilde{b}_0 = b_0 - \frac{m_p\hbar c}{m_p + m_e} \left( \frac{m_e^2 - m_0^2}{\pi F_\pi^2} c_1 - 2\alpha f_1 \right),$$

where the values and uncertainties of the LECs $c_1$ and $f_1$, used in Refs. [38,68], have already been given in Section A.3. Comparison with Eq. (A.7) implies that the correction to $b_1$ reads as

$$b_1 - \tilde{b}_1 = \frac{m_p\hbar c}{m_p + m_e} \frac{\alpha f_2}{2},$$

and comes out equal to $-0.43(17) \cdot 10^{-3} m^{-1}$. Presumably, this correction is contained in $\Delta a_{cc}$ of Eq. (A.8). The corrected $\tilde{a}_{cc}$ may then be obtained as the difference $\tilde{b}_0 - \tilde{b}_1$, whereas $\tilde{a}_{c0} = \sqrt{2}\tilde{b}_1$.

### A.7 A few remarks on the EM corrections at threshold

The important results of the application of the aforementioned correction schemes to $\epsilon_{1s}$ and $\Gamma_{1s}$ in the case of pionic hydrogen [32] are given in Table A.1. The $f_\pi^2$ values, contained in the last column of the table, have been obtained by use of the GMO sum rule. The $f_\pi^2$ value of this work after applying the corrections of Ref. [36] is smaller than the value quoted in Ref. [36] and is accompanied by a considerably larger uncertainty. These changes are due to three reasons: a) The constraint from $\epsilon_{1s}$ in pionic deuterium had also been used in Ref. [36], in order to restrict their estimate for $\tilde{b}_1$; b) Reference [36] combined the statistical and systematic uncertainties of Ref. [32] quadratically; c) The values of the integral $J^-$, used in the GMO sum rule, differ between the two works: Ref. [36] uses the estimate of Ref. [32], whereas this work uses the weighted average of three results, one of which is the estimate of Ref. [35], see Appendix B. To enable a comparison with the results obtained with the models of Sections A.1 and A.2, and also provide an impression of the largeness of the $\mathcal{O}(p^2)$ corrections in Ref. [63], a $\delta_\epsilon$ result was obtained after identifying $\mathcal{A}_{cc}$ with $\tilde{a}_{cc}$ or, equivalently, after ignoring the correction $\Delta \mathcal{A}_{cc}$ of Eq. (A.7). The difference between the corrections $\delta_\epsilon$ between Ref. [63] and the value of Table A.1 is accounted for by the use of Eq. (A.6), instead of Eq. (A.5) which had been used in Ref. [63].
Table A.1

The important results of the application of a few correction schemes to the measurements of $\epsilon_{1s}$ and $\Gamma_{1s}$ of pionic hydrogen [32]. The input, common in all cases, comprises the $a_{cc}$ and $a_{c0}$ results obtained using Eqs. (A.1,A.2). All corrections have been expressed in the form $\delta_\epsilon$ and $\delta_\Gamma$, see Eqs. (A.3,A.4). The $f_c^2$ values of the last column have been obtained by use of the GMO sum rule, see Appendix B. A $\delta_\epsilon$ result was also obtained after identifying the solution $A_{cc}$ of Eq. (A.6) with $\tilde{a}_{cc}$.

| Source | $\delta_\epsilon$ (%) | $\delta_\Gamma$ (%) | $\tilde{a}_{cc}$ (fm) | $\tilde{a}_{c0}$ (fm) | $f_c^2$ |
|--------|-----------------------|---------------------|-----------------------|-----------------------|---------|
| [33]   | −2.1(5)               | −1.3(5)             | 0.1249(10)            | −0.1808(82)           | 0.0780(25) |
| [36]   | −0.62(29)             | 1.02(23)            | 0.12302(89)           | −0.1766(80)           | 0.0768(24) |
| [34]   | 0.66(66)              | −1.66(33)           | 0.1215(11)            | −0.1814(81)           | 0.0782(24) |
| [63], $\tilde{a}_{cc}$ $\equiv A_{cc}$ | 1.1257(42) | − | − | − | − |
| [63]   | −4.3 ± 2.2            | −                   | 0.1278(30)            | −                     | −       |
| [66]   | −7.2 ± 2.9            | −                   | 0.1318(42)            | −                     | −       |
| [69]   | −                   | 0.6(2)              | −                     | −0.1774(80)           | 0.0770(24) |
| [38,68]| −7.2 ± 2.6           | 0.56(72)            | 0.1318(37)            | −0.1774(81)           | 0.0770(24) |

Visual inspection of Table A.1 leads to the following conclusions.

- A consistent picture for the corrections $\delta_\epsilon$ and $\delta_\Gamma$ does not emerge from this table.
- One may argue that the three-channel calculation of Ref. [34] constitutes an improvement over the two-channel evaluation of Ref. [33], and thus proceed to compare the $\delta_\epsilon$ and $\delta_\Gamma$ results of Ref. [34] with those obtained with the only other approach which does not deploy ChPT, namely Ref. [36]. Obviously, there is no matching; the signs are opposite in both corrections $\delta_\epsilon$ and $\delta_\Gamma$. Moreover, the difference of 2.7 % in the two corrections $\delta_\Gamma$ is disturbing.
- The correction $\delta_\epsilon$ of Ref. [34] appears compatible with the result obtained from the upgraded Deser formula for $\epsilon_{1s}$ (see Eq. (A.6)), whereas the corresponding result of Ref. [36] is not. However, it is not clear that such a comparison is meaningful. Part of the EM corrections of Refs. [36,34] are contained in the upgraded Deser formula for $\epsilon_{1s}$; another part is contained in the correction $\Delta A_{cc}$; a third part is not contained in the correction $\Delta A_{cc}$. In addition, the correction $\Delta A_{cc}$ contains effects which go beyond those tackled in Refs. [33,36,34], e.g., effects emanating from the mass difference between the $u$ and $d$ quarks. Therefore, the compatibility between the correction $\delta_\epsilon$ of Ref. [34] with the result obtained from the upgraded Deser formula for $\epsilon_{1s}$ could be coincidental.
- The correction $\delta_\Gamma$ extracted in Ref. [36] is compatible with the two estimates.
obtained within the framework of ChPT in Refs. [69,38,68]. It has been suggested that potential models are prone to yield negative corrections $\delta_T$. Considering the outcome of Refs. [33,34], this might indeed be the case.

- It is time I discussed the corrections obtained within the framework of ChPT. Compared to the experimental uncertainty of $\epsilon_{1s}$, the corrections $\delta$ of Refs. [63,66,38,68] are large and, even worse, poorly known. The large uncertainties are attributable to the poor knowledge of the LEC $f_1$. The essential difference between Refs. [63,66] is that, in the former work, the additional isospin-breaking effects are treated at $\mathcal{O}(p^2)$; in Ref. [66], they are treated at $\mathcal{O}(p^3)$. If, as the result of the application of the correction $\Delta A_{cc}$ of Eq. (A.7), $\delta_\epsilon$ changes by as much as $-5.4\%$ (i.e., from $+1.1\%$ to $-4.3\%$) and the result of the correction at the next order brings another $-2.9\%$, then I do wonder what gifts a calculation at $\mathcal{O}(p^4)$ would bear. If any convergence can be substantiated on the basis of these two numbers, then it ought to be a weak one. Moreover, I hardly see a meaningful use of a procedure which increases the uncertainty of the correction $\delta_\epsilon$ at ‘every next order’ by $1\%$. As a result, I cannot understand at all why the corrections of Refs. [66,38,68] are applied unquestionably to (approximately) twenty times more accurate experimental results as, for instance, the case has been in Ref. [39]. (Incidentally, the value of $\tilde{a}_{cc}$ in Eq. (9) of Ref. [39] is wrong. Using their results (and the quadratic combination of statistical and systematic uncertainties, as the preference of the authors is), the correct outcome, when applying the corrections of Refs. [38,68], is: $\tilde{a}_{cc} = 0.0930(26) m_\pi^{-1}$.)

- The only positive conclusion from the visual inspection of Table A.1 is the overall agreement of Refs. [36,69,38,68] in regard to the correction $\delta_T$; they all agree that this correction is small, below the $1\%$ level. I honestly do not see much else worthy of remembrance in Table A.1.

In order that the $\delta_\epsilon$ value, obtained from Refs. [38,68], approach the results extracted with the models of Sections A.1 and A.2, the LEC $f_1$ needs to be substantially more negative than it is currently allowed ($-1.4\, \text{GeV}^{-1}$). An $f_1$ value in the vicinity of $-4\, \text{GeV}^{-1}$ would lead to vanishing $\delta_\epsilon$.

At this point, I feel that I need to make one statement. Between 1990 and 1995, I had heard at least four prominent theorists lamenting the lack of precise experimental information at threshold. The ETH-Neuchâtel-PSI Collaboration provided $\epsilon_{1s}$ to an accuracy well below $1\%$ in the late 1990s, whereas both statistical and systematic uncertainties, reported by the Pionic-Hydrogen Collaboration, are at the level of or below $0.1\%$. Such accuracy is unprecedented in Pion Physics. After this precise information became available, the theorists discovered that no competitive scheme of corrections, to be applied to the experimental results, had been developed. My opinion is that, regarding the

\[12\] In fact, the lack of such a scheme was the motivation for Sigg and collaborators to set out to investigate the EM effects in pionic hydrogen in Ref. [33].
EM corrections at threshold, Theory needs to find a way to catch up.

I left one subtle subject for the end of this section. The way I understand the issue of the EM corrections is as follows. If complete, an EM correction to a value of a physical quantity in this Universe would translate it into the corresponding value in a Universe where there is no EM interaction (that Universe will be named ‘hypothetical’). All available EM-correction schemes aim at the removal of effects relating to the interaction of the particles involved, but assume that no change is induced on the particles themselves as the result of the absence of the EM interaction. One may pose the question: What happens to the particle ‘proton’ itself when the EM interaction is switched off? The physical mass of the proton surely receives EM contributions; these contributions need to be subtracted in the hypothetical Universe. Therefore, the particle ‘proton’ of this Universe will have another mass in the hypothetical Universe (and, of course, will be neutral). The same applies to all other (charged or composite) particles, e.g., to the ‘neutron’ and to the ‘pions’. If the hadronic interaction does not distinguish between the members of one isospin multiplet, then the hadronic mass of protons and neutrons should be the same. The same applies to the charged and neutral pions, which should share one hadronic mass. Therefore, the four physical masses of this Universe (proton, neutron, charged pion, and neutral pion) would reduce to two hadronic masses in the hypothetical Universe, namely the hadronic mass of the nucleon and that of the pion. The former should be smaller than the physical mass of the proton, whereas the latter should be smaller than or equal to the physical mass of the neutral pion. Some would suggest that the hadronic mass of the pion should be taken to be the physical mass of the neutral pion. That would be a breakthrough (as one hadronic mass in the $\pi N$ interaction would be known), but I believe that one can argue further and refute this possibility. The issue is that a neutral pion consists of $q\bar{q}$ pairs. Therefore, we moved down by one notch, but the question remains: What are the EM contributions to the quark ‘physical’ mass? Therefore, it makes sense to expect that the hadronic mass of the neutral pion should be smaller than its physical mass. At the end of the day, the hadronic mass of the nucleon is unknown, but should be smaller than the physical mass of the proton; the hadronic mass of the pion is unknown, but should be smaller than the physical mass of the neutral pion. We must agree on something before attempting a solution to this problem: Are the EM corrections supposed to also remove the EM contributions to the physical masses, so that the particles could interact via their hadronic masses in the hypothetical Universe?

The current schemes, used in the removal of the EM contributions, assume that the proton in this Universe and the proton in the hypothetical Universe have the same mass, namely the physical mass of the proton. The same applies to the other particles, i.e., to the neutron and to the pions. This is the reason that, since 2006, the authors of Ref. [51] have distinguished between stage-1
and stage-2 EM corrections. The stage-1 corrections provide estimates for the effects of the Coulomb interaction and, in the case of $\pi^-p$ scattering, for the external mass differences and for the $\gamma n$ channel. Assumed in the derivation of the stage-1 corrections was that the hadronic masses of the proton and of the charged pion are equal to their physical masses. The stage-2 EM corrections go one step further; they should take account of graphs with internal photon lines, as well as of the effects relating to the use of the physical masses of the particles in the stage-1 corrections, instead of the hadronic ones.

The three works of the Aarhus-Canberra-Zurich Collaboration on the EM corrections in the $\pi N$ system aimed at the removal of the stage-1 effects in low-energy scattering [70,71], as well as at threshold [34], in a consistent manner. If the EM corrections of Refs. [70,71] are applied to the $\pi N$ scattering data, it would be inconsistent to apply corrections to the scattering lengths $a_{cc}$ and $a_{c0}$ other than those extracted in Ref. [34]. The application of another correction scheme would automatically invalidate any comparison between the corrected values of the scattering lengths (extracted from the measurements at threshold) and those obtained on the basis of an extrapolation from the scattering data. I believe that the treatment of the stage-2 corrections is well beyond the capability of a simple potential model.

There is one last point which I would like to bring up. What is named strong shift $\epsilon_{1s}$ in the two PSI experiments on pionic hydrogen is simply the difference of two energy differences: the first of these differences relates to the EM transition energy between the $3p$ and $1s$ states of pionic hydrogen, the second to the experimentally measured transition energy $E_{3p\rightarrow1s}$. To be able to obtain $\epsilon_{1s}$ an assumption is made: that the hadronic shift in the $3p$ state is negligible. Consequently, one identifies the difference between the aforementioned two energies with the $\epsilon_{1s}$ of the $1s$ state. Although one frequently reads that $\epsilon_{3p}$ is either negligible or “very small” [66], I am not aware of a paper where this question is thoroughly addressed. In addition, a non-negligible $\Gamma_{3p}$ would affect the estimates for $\Gamma_{1p}$, hence also $\tilde{b}_1$ (and $f_c^2$ extracted thereof).

I think that (even if they could be conducted somewhere) new $\pi N$ experiments at low energy would not bring much betterment in our knowledge. An advancement of knowledge in low-energy Pion Physics could only be instigated by a theoretical breakthrough, in particular in relation to the reliable removal of the EM effects from the various $\pi N$ scattering amplitudes. I have my doubts that ChPT is a promising place to look for such a breakthrough. There might be more hope in a non-perturbative approach, such as in Lattice QCD.
Table B.1

Recent estimates for the integral $J^-$ of Eq. (B.1). All values are expressed in mb. A systematic uncertainty has not been reported in Ref. [29].

| Source | $J^-$ | $\delta J^-$ (stat.) | $\delta J^-$ (syst.) |
|--------|-------|----------------------|----------------------|
| [29]   | −1.051 | 0.005                | −                    |
| [35]   | −1.083 | 0.009                | 0.031                |
| [72]   | −1.060 | 0.007                | 0.030                |

B On obtaining an estimate for $f_c^2$ using the Goldberger-Miyazawa-Oehme (GMO) sum rule

The GMO sum rule relates the isovector hadronic scattering length $\tilde{b}_1$ with $f_c^2$ [3]. The relation reads as

$$f_c^2 = -\frac{1}{2} \left( 1 - \left( \frac{m_c}{2m_p} \right)^2 \right) \left( \frac{m_c J^-}{(hc)^2} + \left( 1 + \frac{m_c}{m_p} \right) \frac{m_c \tilde{b}_1}{hc} \right),$$

where $J^-$ is defined as

$$J^- = \frac{1}{4\pi^2} \int_0^\infty \frac{\sigma_T^{\pi^\pm p}(q) - \sigma_T^{\pi^0 p}(q)}{\sqrt{q^2 + m_c^2}} dq; \quad (B.1)$$

the quantities $\sigma_T^{\pi^\pm p}(q)$ denote total cross sections, not containing any EM contributions. Recent estimates for $J^-$ are given in Table B.1. A weighted average was obtained using only the statistical uncertainties of the three entries of this table, and the statistical uncertainty of this average was corrected with the application of the Birge factor. An average systematic uncertainty was obtained from Refs. [35,72] and was quadratically combined with the statistical uncertainty of the weighted average. The $J^-$ value, thus obtained, is equal to $-1.059(32)$ mb; this value is used for the extraction of $f_c^2$ estimates in Table A.1. The large quoted uncertainty of $J^-$ reflects the magnitude of the systematic effects reported in Refs. [35,72]. (The early determinations of $J^-$ from the Karlsruhe analyses, not quoted in this work, are consistent with the values reported in Table B.1, see Refs. [29,35,72] for details.)