On the harmonic oscillator properties in a twisted Moyal plane

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Abstract

This work prolongs, using an operator method, the investigations started in our recent paper [\textit{J. Math. Phys.} \textbf{51} 102108] on the spectrum and states of the harmonic oscillator on twisted Moyal plane, where rather a Moyal-star-algebraic approach was used. The physical spectrum and states of the harmonic oscillator on twisted Moyal space, obtained here by solving the corresponding differential equation, are similar to those of the ordinary Moyal space, with different parameters. This fortunately contrasts with the previous study which produced unexpected results, i.e. infinitely degenerate states with energies depending on the coordinate functions.

Keywords Twisted Moyal plane, harmonic oscillator, states and spectrum.

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1 Introduction

The harmonic oscillator is undoubtedly one of the most important model systems in classical and quantum physics, what justifies its intense study in the literature. Referring the reader to our previous work [17] for the motivations of this investigation, we briefly recall here some salient features needed to understand our development in the sequel. The noncommutative (NC) spacetime is described by the commutation relation between coordinate operators as

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\tilde{\Theta}^{\mu\nu}(x). \]  

(1)

In this equation \( \tilde{\Theta}^{\mu\nu} \) is a deformation parameter tensor which should vanish at large distances where one experiences the commutative world and may be determined by experiments in the high energy case. In the simpler case when \( \tilde{\Theta}^{\mu\nu}(x) \equiv \Theta^{\mu\nu} \), where \( \Theta^{\mu\nu} \) is a constant skew-symmetric tensor, the NC spacetime remains flat. Noncommutative field theory defined on this \( \Theta \)-deformed spacetime is extensively studied in the literature [1]-[35]. When the parameter \( \tilde{\Theta}^{\mu\nu}(x) \) depends on the space variables, it can engender a dynamical twisted space and the geometry associated with this deformation becomes curve. Consider, (see [13] and references therein), \( E = \{\hat{x}^\mu, \mu \in [(1, 2, \cdots D)]\} \) and \( \mathbb{C}[[\hat{x}^1, \hat{x}^2, \cdots \hat{x}^D]] \), the free algebra generated by \( E \). Let \( \mathcal{I} \) be the ideal of \( \mathbb{C}[[x^1, x^2, \cdots x^D]] \), engendered by the elements \( \hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu - i\tilde{\Theta}^{\mu\nu} \). The twisted Moyal Algebra \( \hat{\mathcal{A}}_{\tilde{\Theta}} \) is the quotient

\[ \mathbb{C}[[\hat{x}^1, \hat{x}^2, \cdots \hat{x}^D]]/\mathcal{I}. \]  

(2)

Without bothering about the convergence, each element in \( \hat{\mathcal{A}}_{\tilde{\Theta}} \) is a formal power series in the \( \hat{x}^\mu \)'s for which the relation \( [\hat{x}^\mu, \hat{x}^\nu] = i\tilde{\Theta}^{\mu\nu} \) holds. The Moyal algebra can be also defined as the linear space of smooth and rapidly decreasing functions equipped with the NC star product. The algebra of functions of such noncommuting coordinates can be represented by the algebra of functions on ordinary spacetime, equipped with a noncommutative \( \star \)-product, i.e. \( (\hat{\mathcal{A}}_{\tilde{\Theta}}, \star) \approx (\mathcal{A}_{\tilde{\Theta}}, \star) \). It is obvious that this deformation breaks the classical Lorentz symmetry. However, one can construct a deformation of Lorentz symmetry such that it is a symmetry of relation \( [x^\mu, x^\nu]_\star = i\tilde{\Theta}^{\mu\nu}(x) \). The way to do this construction is to use a Lie algebra \( \mathcal{G} \) generated by \( D \) elements \( t^\mu \). The twist element \( \mathcal{F} \in \mathcal{U}_\mathcal{G} \otimes \mathcal{U}_\mathcal{G} \), where \( \mathcal{U}_\mathcal{G} \) is the universal enveloping algebra of \( \mathcal{G} \). \( \mathcal{U}_\mathcal{G} \) is a Hopf algebra and there is a linear map, called coproduct \( \Delta: \mathcal{U}_\mathcal{G} \to \mathcal{U}_\mathcal{G} \otimes \mathcal{U}_\mathcal{G} \) such that \( \Delta(t^\mu) = t^\mu \otimes 1 + 1 \otimes t^\mu \). The main property that \( \mathcal{F} \) has to satisfy is the cocycle condition \( \mathcal{F} \otimes 1(\Delta \otimes \text{id})\mathcal{F} = 1 \otimes \mathcal{F}(\text{id} \otimes \Delta)\mathcal{F} \); it ensures the associativity of the star-product, important in
the construction of this deformation. Aschieri et al [11] showed that, using the commuting vector field \( X_a = e^\mu_a \partial_\mu \in T^{DR}_\Theta \), i.e. \([X_a, X_b] = 0\), the deformation of usual Moyal algebra remains associative, that is not the case in general.

This work aims at deepening, by using an operator method, the investigations started in our recent paper [17] on the spectrum and states of the harmonic oscillator on twisted Moyal plane, where rather a Moyal-star-algebraic approach has been used and has produced a series of new unexpected results. As in the latter study, we deform the Moyal algebra with noncommuting vectors fields \( X_a \), i.e. \([X_a, X_b] \neq 0\), and suitable condition on the parameter \( e^\mu_a \) so that the star-product obeys the associativity property. All the results of our investigation obviously remain valid when the considered vector fields commute. We then solve the harmonic oscillator eigenvalue problem in such a pertinent way to enjoyably produce appropriate physical quantities, i.e. eigen energies and states.

This paper is organized as follows. In section 2, we recall the star-product properties used in the sequel, and deal with a comparative study of the harmonic oscillator in both ordinary and twisted Moyal planes. In section 3, the properties of harmonic oscillator at infinity are given. Section 4 is devoted to final remarks.

2 Harmonic oscillator in twisted Moyal space

2.1 Brief review of useful twisted Moyal product properties

As a matter of completeness, let us immediately provide a quick general survey of useful properties of twisted Moyal product which are used in the sequel.

In the context of a dynamical noncommutative field theory, the vector field can be generalized to take the form \( X_a = e^\mu_a (x) \partial_\mu \), where \( e^\mu_a (x) \) is a tensor depending on the coordinate functions in the complex general linear matrix group of order \( D \) denoted by GL\((D, \mathbb{C})\) [1]. The star product takes the form

\[
(f \star g)(x) = m \left\{ e^{\frac{i e^a_{\mu b}}{2} X_a \otimes X_b} f(x) \otimes g(x) \right\}, \quad x \in \mathbb{R}^D, \forall f, g \in C^\infty(\mathbb{R}^D) \tag{3}
\]

and the vielbeins are given by the infinitesimal affine transformation as

\[
e^\mu_a (x) = \delta^\mu_a + \omega^\mu_{ab} x^b, \tag{4}
\]

where \( \omega_{ab} \in \text{GL}(D, \mathbb{C}) \) is skewsymmetric. Using (4), the non vanishing Lie bracket peculiar to
the non-coordinate base \cite{17}

\[ [X_a, X_b] = e^c_\nu \left[ e^\mu_a \partial_\mu e^\nu_b - e^\mu_b \partial_\mu e^\nu_a \right] X_c = C^c_{ab} X_c \]  
(5)

is here simply reduced to

\[ [X_a, X_b] = \omega^\mu_a \partial_\mu - \omega^\mu_b \partial_\mu = -2\omega^\mu_{ab} \partial_\mu. \]  
(6)

Besides, the dynamical star product \cite{3} can be now expressed as

\[ (f \ast g)(x) = m \left[ \exp \left( \frac{i}{2} \theta e^{-1} e^{\mu \nu} \partial_\mu \otimes \partial_\nu \right) (f \otimes g)(x) \right] \]  
(7)

where \( e^{-1} =: \det(e^\mu_a) = 1 + \omega^1_{12} x^2 - \omega^2_{12} x^1; \) \( e^{\mu \nu} \) is the symplectic tensor in two dimensions, \( (D = 2) \), with components \( e^{12} = -e^{21} = 1, \ e^{11} = e^{22} = 0. \) The coordinate function commutation relation becomes \( [x^\mu, x^\nu]_\ast = i\tilde{\Theta}^{\mu \nu} = i(\Theta^{\mu \nu} - \Theta^{[\mu, \nu]} x^b) \) which can be reduced to the usual Moyal space relation, as expected, by setting \( \omega^\mu_{ab} = [0]. \) One can check that the Jacobi identity is also well satisfied, i.e.

\[ [x^\mu, [x^\nu, x^\rho]_\ast]_\ast + [x^\rho, [x^\mu, x^\nu]_\ast]_\ast + [x^\nu, [x^\rho, x^\mu]_\ast]_\ast = \Theta^{\mu \nu} \Theta^{d[\nu, \omega_{bd}]} = 0 \]  
(8)

conferring a Lie algebra structure to the defined twisted Moyal space. This identity ensures the associativity of the star-product \cite{3} and implies that

\[ \tilde{\Theta}^{\sigma \rho} \partial_\rho \tilde{\Theta}^{\mu \nu} + \tilde{\Theta}^{\nu \rho} \partial_\rho \tilde{\Theta}^{\sigma \mu} + \tilde{\Theta}^{\mu \rho} \partial_\rho \tilde{\Theta}^{\nu \sigma} = 0. \]  
(9)

Therefore the algebra \( \mathcal{A}_\tilde{\Theta} \) with this approach is the associative algebra and its universal algebra \( \mathcal{U}(\mathcal{A}_\tilde{\Theta}) \) is a Hopf algebra. The twisted star-product \cite{3} is then well defined. Remark that with the relation \cite{6}, the requirement that \( \omega_{ab} \) is a symmetric tensor trivially ensures the associativity of the star product. In the interesting particular case addressed in this work, the associativity of the star product \cite{3} is guaranteed even with the non symmetric tensor \( \omega_{ab}. \) See proof in \cite{17}. Besides,

**Proposition 2.1** If \( f \) and \( g \) are two Schwartz functions on \( \mathbb{R}^2_{\tilde{\Theta}} \), then \( f \ast g \) is also a Schwartz function on \( \mathbb{R}^2_{\tilde{\Theta}}. \)

and

**Proposition 2.2** Notwithstanding the condition \( [X_a, X_b] \neq 0 \), i.e. \( \omega^\mu_{ab} \) is skew-symmetric, the defined twisted \( \ast \) product remains noncommutative and associative.
Proof: See [17]. □

The tensor $\tilde{\Theta}^{\mu\nu}$ can be decomposed in our case as:

\[
(\tilde{\Theta})^{\mu\nu} = (\Theta)^{\mu\nu} - (\Theta_a^{[\mu, \omega_{\nu}]} a^b) x^b = \left( \begin{array}{cc} 0 & e^{-1} \\ -e^{-1} & 0 \end{array} \right)
\]

and the twisted Moyal star-product satisfies the useful relation

\[
x^\mu \star f = x^\mu f + i \frac{\Theta}{2} e^\mu e^\rho \partial_\rho f \quad \text{and} \quad f \star x^\mu = x^\mu f - i \frac{\Theta}{2} e^\mu e^\rho \partial_\rho f.
\]

The anticommutator and commutator star brackets of $x^\mu$ and $f$ can be immediately deduced as follows:

\[
\{x^\mu, f\}_\star = 2x^\mu f, \quad [x^\mu, f]_\star = i\Theta e^\mu e^\rho \partial_\rho f.
\]

The relations (11) can be detailed for $x^\mu, \mu = 1, 2$ as:

\[
x^1 \star f = x^1 f + i \frac{\Theta}{2} e^{-1} \partial_2 f \quad \text{and} \quad f \star x^1 = x^1 f - i \frac{\Theta}{2} e^{-1} \partial_2 f \quad \text{(13)}
\]

\[
x^2 \star f = x^2 f - i \frac{\Theta}{2} e^{-1} \partial_1 f \quad \text{and} \quad f \star x^2 = x^2 f + i \frac{\Theta}{2} e^{-1} \partial_1 f \quad \text{(14)}
\]

giving rise to the creation and annihilation functions

\[
a = \frac{x^1 + ix^2}{\sqrt{2}} \quad \text{and} \quad \bar{a} = \frac{x^1 - ix^2}{\sqrt{2}}
\]

with the commutation relation $[a, \bar{a}]_\star = \Theta e^{-1}$. It then becomes a matter of algebra to use the transformations of the vector fields $\partial_1$ and $\partial_2$ into $\partial_a := \frac{\partial}{\partial a}$ and $\partial_{\bar{a}} := \frac{\partial}{\partial \bar{a}}$ and vice-versa to infer

\[
e^{-1} = 1 - a\omega - \bar{a}\bar{\omega} \quad \text{and} \quad e = 1 + a\omega + \bar{a}\bar{\omega},
\]

where

\[
\omega = \frac{\omega_{12}^2 + i\omega_{12}^1}{\sqrt{2}} \quad \text{and} \quad \bar{\omega} = \frac{\omega_{12}^2 - i\omega_{12}^1}{\sqrt{2}}.
\]

There result the useful relations

\[
\frac{\partial e^{-1}}{\partial a} = -\omega, \quad \frac{\partial e^{-1}}{\partial \bar{a}} = -\bar{\omega} \quad \text{and} \quad \text{for } k \in \mathbb{Z}, \quad \omega e^k = \omega, \quad \bar{\omega} e^k = \bar{\omega}.
\]

Expressing the twisted $\star$-product (7) in terms of vector fields $\partial_a$ and $\partial_{\bar{a}}$ as

\[
(f \star g)(a, \bar{a}) = m \left[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \frac{1}{k!(n-k)!} \frac{\Theta}{2} e^{-1} \frac{\partial^n}{\partial a^k \partial_{\bar{a}}^{n-k}} (f \otimes g)(a, \bar{a}) \right]
\]

and using equations (13) and (14) (or independently (18)) yield

\[
a \star f = \left( a + \frac{\theta e^{-1}}{2} \frac{\partial}{\partial a} \right) f \quad \text{and} \quad \bar{a} \star f = \left( \bar{a} - \frac{\theta e^{-1}}{2} \frac{\partial}{\partial \bar{a}} \right) f
\]

\[
f \star a = \left( a - \frac{\theta e^{-1}}{2} \frac{\partial}{\partial a} \right) f \quad \text{and} \quad f \star \bar{a} = \left( \bar{a} + \frac{\theta e^{-1}}{2} \frac{\partial}{\partial \bar{a}} \right) f.
\]
2.2 Physical states and spectrum

From the above derived results, the twisted ho Hamiltonian operator $H = a\tilde{a}$ is defined by the relation

$$H \ast (.) = \frac{1}{2} \left[(x_1)^2 + (x_2)^2 + \left(i\theta e^{-1}x_1 - \frac{\theta^2}{4}\omega_1^2\right)\partial_2 - \left(i\theta e^{-1}x_2 - \frac{\theta^2}{4}\omega_2^2\right)\partial_1 - \frac{\theta^2}{4}e^{-2} (\partial_1^2 + \partial_2^2) \right] \equiv \frac{1}{2}\mu_1$$

with the domain

$$\mathcal{D}(H\ast) = \left\{ f \in L^2(\mathbb{R}_G^2) \mid f, f_{x^1}, f_{x^2} \in AC_{loc}(\mathbb{R}_G^2); \frac{\mu_1}{2} f \in L^2(\mathbb{R}_G^2) \right\}. \tag{22}$$

$AC_{loc}(\mathbb{R}_G^2)$ denotes the set of locally absolutely continuous functions on $\mathbb{R}_G^2$. Similarly,

$$(.) \ast H = \frac{1}{2} \left[(x_1)^2 + (x_2)^2 - \left(i\theta e^{-1}x_1 - \frac{\theta^2}{4}\omega_1^2\right)\partial_2 + \left(i\theta e^{-1}x_2 - \frac{\theta^2}{4}\omega_2^2\right)\partial_1 - \frac{\theta^2}{4}e^{-2} (\partial_1^2 + \partial_2^2) \right] \equiv \frac{1}{2}\mu_2$$

defined in the domain

$$\mathcal{D}(\ast H) = \left\{ f \in L^2(\mathbb{R}_G^2) \mid f, f_{x^1}, f_{x^2} \in AC_{loc}(\mathbb{R}_G^2); \frac{\mu_2}{2} f \in L^2(\mathbb{R}_G^2) \right\}. \tag{24}$$

Setting $\omega_2 = \omega_1^2, \omega_1 = \omega_1^2, e^{-1} = 1 + \omega_2x_2 - \omega_1x_1$, the eigenvalue equation can be written as

$$\left[ x_1^2 + x_2^2 - \frac{\theta^2}{4}\frac{\partial}{\partial x_2} + \frac{\theta^2}{4}\frac{\partial}{\partial x_1} - \frac{\theta^2}{4}e^{-2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \right] f = 2Ef \tag{25}$$

with

$$\left( x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) f = 0 \tag{26}$$

which naturally suggests the use of polar coordinates as follows:

$$x_1 = r \cos \alpha, \quad x_2 = r \sin \alpha, \quad x_1^2 + x_2^2 = r^2. \tag{27}$$

The variable change (27) transforms the equation (25) into the form:

$$\left[ x_1^2 - \frac{\theta^2}{4} \left( \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) + \frac{\theta^2}{4} \left( \cos \alpha \frac{\partial}{\partial r} - \sin \alpha \frac{\partial}{\partial \alpha} \right) - \frac{\theta^2}{4} (1 + 2\omega_2r \sin \alpha - 2\omega_1r \cos \alpha) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} \right) \right] f = 2Ef. \tag{28}$$

Separating the variables in the function $f$ as follows:

$$f(r, \alpha) = \chi(r)g(\alpha) \text{ where } g(\alpha) = e^{i\theta \alpha}, \tag{29}$$
and replacing the result in (28), we get
\[
\begin{align*}
& \left[ r^2 - 2E - \frac{\theta^2 \omega_2}{4} \left( \sin \alpha \frac{\partial}{\partial r} + i k \cos \alpha \frac{\partial}{\partial r} \right) + \frac{\theta^2 \omega_1}{4} \left( \cos \alpha \frac{\partial}{\partial r} - i k \sin \alpha \frac{\partial}{\partial r} \right) 
- \frac{\theta^2}{4} (1 + 2\omega_2 r \sin \alpha - 2\omega_1 r \cos \alpha) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} \right) \right] \chi(r) = 0
\end{align*}
\]
resulting in the unique equation
\[
\begin{align*}
& \left[ r^2 - 2E - \frac{\theta^2 \omega_2}{4} \sin \alpha \frac{\partial}{\partial r} + \frac{\theta^2 \omega_1}{4} \cos \alpha \frac{\partial}{\partial r} 
- \frac{\theta^2}{4} (1 + 2\omega_2 r \sin \alpha - 2\omega_1 r \cos \alpha) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} \right) \right] \chi(r) = 0
\end{align*}
\]
with the constraint relation
\[
\omega_1 \sin \alpha + \omega_2 \cos \alpha = 0. \tag{32}
\]

Finally, introducing the latter in the equation \([31]\) generates the appropriate differential equation:
\[
\begin{align*}
& \left[ r^2 - 2E - \frac{\theta^2 \omega_2}{4} \sin \alpha \frac{\partial}{\partial r} + \frac{\theta^2 \omega_1}{4} \cos \alpha \frac{\partial}{\partial r} 
- \frac{\theta^2}{4} (1 + 2\omega_2 r \sin \alpha - 2\omega_1 r \cos \alpha) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} \right) \right] \chi(r) = 0. \tag{33}
\end{align*}
\]
As a matter of result comparison, let us now search for the solutions of this equation by considering both ordinary and twisted Moyal spaces.

**(B1) Case of the ordinary Moyal space**

It corresponds to \( \omega_1 = 0 \) reducing the equation \([33]\) to
\[
\left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} - \frac{4}{\theta^2} (r^2 - 2E) \right] \chi(r) = 0. \tag{34}
\]
Making the variable change \( r^2 - E = u \Rightarrow 2rdr = du, \)
\[
\frac{r}{\partial r} = 2(u + E) \frac{\partial}{\partial u}, \quad \frac{\partial^2}{\partial r^2} = 4(u + E) \frac{\partial^2}{\partial u^2} + 2 \frac{\partial}{\partial u} \tag{35}
\]
and the equation \([34]\) takes the form
\[
\left[ 4(u + E)^2 \frac{\partial^2}{\partial u^2} + 4(u + E) \frac{\partial}{\partial u} - \frac{4}{\theta^2} (u^2 - E^2 + \frac{\theta^2 k^2}{4}) \right] \chi(u) = 0. \tag{36}
\]
Then, choosing \( u + E = \rho \) further simplifies the expressions to give
\[
\left[ \rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} - \frac{1}{\theta^2} \left( \rho^2 - 2E\rho + \frac{\theta^2 k^2}{4} \right) \right] \chi(\rho) = 0 \tag{37}
\]
with singularities at $\rho = 0$ and at $\rho = \infty$. In the vicinity of $\rho = 0$, we find $\chi(\rho)$ proportional to $\rho^\nu$, and at infinity $\chi(\rho) = e^{-B\rho}$. Therefore, if we write the solution as follows:

$$
\chi(\rho) = \rho^\nu e^{-B\rho}F(\rho), \text{ with } \nu = 1 \pm \sqrt{1 + \frac{k^2}{4}}, \quad B = \frac{1}{\theta}
$$

the resulting differential equation for $F(\rho)$ turns out to be in the form

$$
\left[ \rho \frac{\partial^2}{\partial \rho^2} + (2\nu + 1 - 2B\rho) \frac{\partial}{\partial \rho} + \left( \frac{2E}{\theta^2} - 2\nu B - B \right) \right] F(\rho) = 0
$$

which, with

$$
2B\rho = \rho', \quad a = \frac{1}{2B} \left( \frac{2E}{\theta^2} - 2\nu B - B \right), \quad b = 2\nu + 1,
$$

is transformed into the Kummer confluent hypergeometric equation

$$
\left[ \rho' \frac{\partial^2}{\partial \rho'^2} + (b - \rho') \frac{\partial}{\partial \rho'} + \left( a - \rho' \right) \right] F(\rho') = 0
$$

whose the general solution is given by

$$
F(\rho') = A_1 \Phi(a, b; \rho') + A_2 \rho'^{1-b} \Phi(a - b + 1, 2 - c; \rho'), \quad A_1, A_2 \in \mathbb{R},
$$

where

$$
\Phi(a, b; \rho') = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} \rho'^n, \quad (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1) \cdots (a + n - 1).
$$

The solution of (37) then becomes

$$
\chi(\rho) = \rho^\nu e^{-B\rho} \left[ A_1 \Phi(a, b; 2B\rho) + A_2 (2B\rho)^{1-b} \Phi(a - b + 1, 2 - c; 2B\rho) \right],
$$

where

$$
\nu = 1 \pm \sqrt{1 + \frac{k^2}{4}}, \quad B = \frac{1}{\theta}, \quad 2B\rho = \rho', \quad a = \frac{E}{\theta} - \nu - \frac{1}{2}, \quad b = 2\nu + 1.
$$

The physical states i.e. bounded states occur only for $B > 0$, i.e. $\theta > 0$, since the confluent series $\Phi(a, b; \rho)$, for large values of $\rho$, is proportional to $e^\rho$ so that $\chi$ diverges for $\rho \to \infty$.

Further, the second term of (44) has a regular singularity at $\rho = 0$ if $b > 1$. Hence the physical states (see details in the case (B2) below) are given by

$$
f(r, \alpha) = \sqrt{\frac{2^{2\nu_p + \frac{3}{2}} B^{2\nu_p + \frac{3}{2}} \Gamma(2\nu_p + 1)^2 \Gamma(2\nu_p + 1)}{\pi p! \Gamma(2\nu_p + p + 1) \Gamma(2\nu_p + \frac{3}{2})}} \cdot r^{2\nu_p} e^{-Br^2} \Phi(a, b; 2B r^2) e^{i\alpha k},
$$
corresponding to the eigen-energies
\[ E_{l,k}^\pm = \theta \left( \frac{3}{2} \pm \sqrt{1 + \frac{k^2}{4} - l} \right), \]  
(47)

with \( a = -l, \ l = 0, 1, 2, \ldots \). Finally, in accordance with [14] (and references therein), \( k \) must satisfy the relation
\[ k = \pm \sqrt{(n + l - 1)^2 - 4}, \]  
where \( n \in \mathbb{N} \Rightarrow E_{l,\nu}^+ = E_n = \theta \left( n + \frac{1}{2} \right). \)  
(48)

The solution (46) can be transformed into Laguerre or Hermite polynomials by using a suitable transformation. Further by adopting an appropriate normalization constant, one can show that it is well equivalent to the result given in [14]. See Appendix.

(B2) Case of the twisted Moyal space

Consider now equation (33) in the case when \( \omega_1 \neq 0 \). This equation can be re-expressed as follows:
\[ \left[ r^2 - 2E + \frac{\theta^2 k^2}{4r^2} \left( 1 - \frac{2\omega_1 r}{\cos \alpha} \right) - \frac{\theta^2}{4r} \left( 1 - \frac{3\omega_1 r}{\cos \alpha} \right) \frac{\partial}{\partial r} - \frac{\theta^2}{4} \left( 1 - \frac{2\omega_1 r}{\cos \alpha} \right) \frac{\partial^2}{\partial r^2} \right] \chi(r) = 0. \]  
(49)

As \( \omega_1 \) is an infinitesimal parameter, the equation (49) can be reduced to
\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \left( 1 - \frac{\omega_1 r}{\cos \alpha} \right) \frac{\partial}{\partial r} - \frac{k^2}{r^2} - \frac{4}{\theta^2} \left( 1 + \frac{2\omega_1 r}{\cos \alpha} \right) (r^2 - 2E) \right] \chi(r) = 0. \]  
(50)

Defining \( \chi \) in terms of series, \( \chi(r) = \sum_n a_n r^n \), with \( a_n \) such that \( a_{2p+1} = 0, \forall p \in \mathbb{N} \), then (50) is re-expressed as
\[
\sum_n (n^2 - k^2) a_n r^{n-2} - \frac{\omega_1}{\cos \alpha} \sum_n n a_n r^{n-1} + \frac{8E}{\theta^2} \sum_n a_n r^n + \frac{16E\omega_1}{\theta^2 \cos \alpha} \sum_n a_n r^{n+1} - \frac{4\omega_1}{\theta^2 \cos \alpha} \sum_n a_n r^{n+2} - \frac{8\omega_1}{\theta^2 \cos \alpha} \sum_n a_n r^{n+3} = 0. \]  
(51)

The sequence \( a_n \) satisfies the recurrence relation
\[
\left( (n + 2)^2 - k^2 \right) a_{n+2} - \frac{\omega_1}{\cos \alpha} (n + 1) a_{n+1} + \frac{8E}{\theta^2} a_n + \frac{16E\omega_1}{\theta^2 \cos \alpha} a_{n-1} - \frac{4\omega_1}{\theta^2 \cos \alpha} a_{n-2} - \frac{8\omega_1}{\theta^2 \cos \alpha} a_{n-3} = 0. \]  
(52)

If \( n = 0 \), then \( a_2 = -\frac{8E}{\theta^2} \frac{1}{4-k^2} a_0 \). Similarly, if \( n = 1 \), \( a_2 = \frac{8E}{\theta^2} a_0 \) and we infer that \( k = \pm \sqrt{5} \).

If \( n = 2 \) and \( n = 3 \), then we obtain the values
\[
a_4 = \left[ \left(-\frac{8E}{\theta^2}\right)^2 \frac{1}{2^2 - k^2} \frac{1}{4^2 - k^2} + \frac{4}{\theta^2} \frac{1}{4^2 - k^2} \right] a_0, \quad \text{and} \]

\[ a_4 = \left[ \left( -\frac{16E}{\theta^2} \right)^2 \frac{1}{2} \frac{1}{4} - \frac{8}{\theta^2} \frac{1}{4} \right] a_0, \quad (53) \]

respectively, implying \( k = 3\sqrt{2} \). Continuing this procedure, we succeed in separating the relation (52) into new recurrence relations

\[ \left( (n + 2)^2 - k^2 \right) a_{n+2} + \frac{8E}{\theta^2} a_n - \frac{4}{\theta^2} a_{n-2} = 0 \quad (54) \]

and

\[ - (n + 2) a_{n+2} + \frac{16E}{\theta^2} a_n - \frac{8}{\theta^2} a_{n-2} = 0. \quad (55) \]

which are equivalent if and only if

\[ k = \pm \sqrt{\frac{(n + 2)(2n + 5)}{2}}. \quad (56) \]

As \( n \) is an even integer in the equations (54) and (55), we obtain

\[ k = k_p = \pm \sqrt{(p + 1)(4p + 5)}, \quad \text{where} \quad \frac{n}{2} = p \in \mathbb{N}. \quad (57) \]

**Remark 2.1** The recurrence relation (54) can be also deduced from the case (B1) by solving, with the same series solution method, the equation (33) for \( \omega_1 = 0 \).

The Hilbert space structure on \( S(\mathbb{R}^2_{\tilde{\theta}}) \) is defined by the scalar product

\[ <f, g> = : \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1 dx_2 \int_{0}^{2\pi} d\alpha \int_{0}^{+\infty} dr f(x_1, x_2, r, \alpha) g(x_1, x_2, r, \alpha), \quad f, g \in S(\mathbb{R}^2_{\tilde{\theta}}) \]

\[ = \int_{0}^{2\pi} d\alpha \int_{0}^{+\infty} r^2 dr \int_{0}^{+\infty} f(r, \alpha) g(r, \alpha). \quad (58) \]

There results the normalization condition

\[ \int_{0}^{2\pi} d\alpha \int_{0}^{+\infty} r^2 dr \int_{0}^{+\infty} f_p(r, \alpha) f_q(r, \alpha) = \delta_{pq}. \quad (59) \]

Therefore, the following result is in order.

**Proposition 2.3** The normalised eigenstates and eigenenergies of the harmonic oscillator on twisted Moyal plane are given, respectively, by

\[ f_p(r, \alpha) = \sqrt{\frac{2^{2\nu_p+\frac{3}{2}} B^{2\nu_p+\frac{3}{2}} [(2\nu_p + 1)_{p}]^2 \Gamma(2\nu_p + 1)}{\pi p! \Gamma(2\nu_p + p + 1) \Gamma(2\nu_p + \frac{3}{2})}} r^{2\nu_p} e^{-Br^2} \Phi(-p, 2\nu_p + 1; 2Br^2) e^{iak_p} \]

\[ (60) \]
and
\[ E_p^{(\pm)} = \theta \left( \frac{3}{2} \pm \sqrt{1 + \frac{(p + 1)(4p + 5)}{4} - p} \right), \quad p \in \mathbb{N} \] (61)

with
\[ \nu_p = 1 \pm \sqrt{1 + \frac{k_p^2}{4}}, \quad B = \frac{1}{\theta}, \quad a = \frac{E_p}{\theta} - \nu_p - \frac{1}{2} =: -p, \quad b = 2\nu_p + 1. \] (62)

**Proof:** Using the relation (96) given in Appendix, the identity
\[ \int_0^{+\infty} dz e^{-z} z^{\sigma+\delta} L_n^\sigma(z) L_m^\delta(z) = \delta_{nm} \frac{\Gamma(n + \sigma + 1)\Gamma(\sigma + \delta + 1)}{n!\Gamma(\sigma + 1)} \] (63)

and (60), we arrive at the normalization condition
\[ \int_0^{2\pi} d\alpha \int_0^{+\infty} r^2 dr f_p(r, \alpha)f_q(r, \alpha) = |A_1|^2 \frac{2\pi}{2\sqrt{2}} \left( \frac{1}{2} \right)^{2\nu_p+1} \]
\[ \times B^{-\frac{3}{2} - 2\nu_p} \frac{(p)^2}{[(2\nu_p + 1)p]^2} \frac{\Gamma(p + 2\nu_p + 1)\Gamma(2\nu_p + \frac{3}{2})}{p!\Gamma(2\nu_p + 1)} \delta_{pq} = \delta_{pq}, \] (64)
yielding
\[ A_1^2 = \frac{2^{2\nu_p + \frac{3}{2}} B^{2\nu_p + \frac{3}{2}} [(2\nu_p + 1)p]^2 \Gamma(2\nu_p + 1)}{\pi \Gamma(2\nu_p + p + 1)\Gamma(2\nu_p + \frac{3}{2})p!}, \] (65)

where $\Gamma(.)$ is the gamma function defined by the relation $\Gamma(z) = \int_0^{\infty} dt e^{-t} t^{z-1}$ while $(2\nu_p + 1)p$ is provided by the expression (83) found in Appendix. □

Finally, we conclude that the states of the harmonic oscillator in the ordinary Moyal plane given by (44) are similar to those of the twisted Moyal space by replacing $k$ by $k_p$ found in (57). Figures 1 and 2 illustrate the energy spectrum behaviour versus $p$. $E_p^{(+)}$ decreases from 3 to its asymptotic value 1.5 as $p$ increases. $E_p^{(-)}$ admits a lower limit 0 for $p = 0$ and linearly varies as $-p$ with increasing values of $p$. Therefore one can conclude that $E_p^{(+)}$ and $E_p^{(-)}$ represent the scattering and bound state energies, respectively, of the harmonic oscillator in twisted Moyal plane. This is a novel feature that has not been observed in our previous investigation of the harmonic oscillator in twisted Moyal plane ([17] and references therein).

### 3 Physical properties at infinity

Equation (37) is reduced to
\[ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \left( 1 - \frac{2E}{\rho} + \frac{\gamma^2}{\rho^2} \right) \right] \chi(\rho) = 0 \] (66)
Figure 1: Energy $E_{p}^{(+)}$ versus $p$ for $\theta = 1$.

Figure 2: Energy $E_{p}^{(-)}$ versus $p$ for $\theta = 1$. 

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where $\gamma = \frac{\theta^2 k^2}{4}$ and $\rho = r^2$. The potential $V(\rho) = -\frac{1}{\rho^2} \left(1 - \frac{2E}{\rho} + \frac{\gamma}{\rho^2}\right) \to -\frac{1}{\rho^2}$ if $\rho \to \infty$. In this limit, the equation (66) is written as

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - B^2\right] \chi(\rho) = 0 \quad (67)$$

where $B = \frac{1}{\theta}$. Suppose that $\chi(\rho \to \infty) = \sum_l a_l \rho^l$. Then (67) is re-expressed as

$$\sum_l l^2 a_l \rho^{l-2} - B^2 \sum_l a_l \rho^l = 0 \quad (68)$$

and the series $a_l$ satisfies the recurrence relation

$$(n + 2)^2 a_{n+2} - B^2 a_n = 0 \Rightarrow a_{n+2} = \frac{B^2}{(n + 2)^2} a_n, \quad (69)$$

with

$$a_n = \frac{B^2}{n^2} a_{n-2} = \frac{(B^2)^2}{n^2(n-2)^2} a_{n-4} = \cdots = \frac{(B^2)^l}{n^2(n-2)^2 \cdots (n - 2l + 2)^2} a_{n-2l}. \quad (70)$$

If $l = \frac{n}{2}$ implying that $n$ is an even integer, we get

$$a_n = \frac{B^n}{n^2(n-2)^2 \cdots 2^2} a_0 = \left[\frac{B^{n/2}}{n(n-2) \cdots 2}\right]^2 a_0 = \left[\frac{B^{n/2}}{2^{n/2} (n/2)!}\right]^2 a_0. \quad (71)$$

We then obtain

$$a_{2n} = \frac{B^{2n}}{2^{2n} (n!)^2} a_0 \text{ and } \chi(\rho)_{\infty} = a_0 \sum_{n=0}^\infty \frac{B^{2n}}{2^{2n} (n!)^2} \rho^{2n}. \quad (72)$$

Finally there results the following solution:

$$\chi(\rho) = a_0 e^{-\lambda \rho} \sum_n \frac{B^{2n}}{2^{2n} (n!)^2} \rho^{2n} \quad \lambda > 0. \quad (73)$$

Putting (73) into the equation (66) yields

$$(\lambda^2 - B^2) \sum_n \frac{B^{2n}}{2^{2n} (n!)^2} \rho^{2n} + (2EB^2 - \lambda) \sum_n \frac{B^{2n}}{2^{2n} (n!)^2} \rho^{2n-1}$$

$$- 2\lambda \sum_n \frac{2nB^{2n}}{2^{2n} (n!)^2} \rho^{2n-1} - B^2 \gamma \sum_n \frac{B^{2n}}{2^{2n} (n!)^2} \rho^{2n-2}$$

$$+ \sum_n \frac{2nB^{2n}}{2^{2n} (n!)^2} \rho^{2n-2} + \sum_n \frac{2n(2n-1)B^{2n}}{2^{2n} (n!)^2} \rho^{2n-2} = 0. \quad (74)$$

The parameter $\lambda$ satisfies the equation

$$\lambda^2 - \frac{3B}{\sqrt{\pi}} \lambda + \frac{2E_{0,k} B^3}{\sqrt{\pi}} - \frac{B^4 \gamma}{4} = 0 \quad (75)$$
affording the solution

$$\lambda = \frac{3B}{2\sqrt{\pi}} + \frac{1}{2} \sqrt{\frac{9B^2}{\pi} - \frac{8E_{0,k}B^3}{\sqrt{\pi}}} + \gamma B^4. \quad (76)$$

This relation bounds the energy by

$$E_{0,k} \leq \frac{\sqrt{\pi}}{8} \left( \frac{9}{\pi B} + \gamma B \right) = \frac{\sqrt{\pi} \theta}{4} \left( \frac{9}{\pi} + \frac{k^2}{4} \right). \quad (77)$$

Remark that, for $\rho \to \infty$, $\lambda = 0$ and the energy spectrum of the ground state takes the form

$$E_{0,k}^\infty = \frac{\gamma B\sqrt{\pi}}{8} = \frac{k^2 \theta \sqrt{\pi}}{32}. \quad (78)$$

We finally arrive at the following main result:

**Proposition 3.1** The state of the harmonic oscillator in twisted Moyal space is given by

$$f(r, \alpha) = a_0 e^{-\lambda r^2} \sum_n \frac{B^n}{2^{n/2} \pi^{2n} (n!)^2} r^{2n} e^{ika}, \quad a_0 \in \mathbb{R}, \quad \alpha \in \mathbb{R}, \quad \gamma = \frac{6^2 k^2}{4},$$

$$B = \frac{1}{\theta}, \quad \lambda = \frac{3B}{2\sqrt{\pi}} + \frac{1}{2} \sqrt{\frac{9B^2}{\pi} - \frac{8E_{0,k}B^3}{\sqrt{\pi}}} + \gamma B^4 \quad (79)$$

with the corresponding energy

$$E_{n,k} = \frac{(4n + 3)\lambda}{2B^2} - \frac{(n + \frac{1}{2})^2 \lambda^2}{2(n!)^2 B^3} + \frac{(n + \frac{1}{2})^2 \gamma B}{4(n + 1)!^2}, \quad E_{n,k}^\infty = \frac{(n + \frac{1}{2})^2 \gamma B}{4(n + 1)!^2}. \quad (80)$$

**Proof:** Equation (79) is immediately obtained by substitution of (73) in (29). The expression (80) is the solution of (74) reducible, after some algebra, to

$$\frac{B^3 (n!)^2 E_{n,k}}{(n + \frac{1}{2})^2} + \lambda^2 - \frac{(4n + 3)(n!)^2 \lambda B}{2(n + \frac{1}{2})^2} - \frac{\gamma B^4}{4(n + 1)^2} = 0. \quad \Box \quad (81)$$

### 4 Final remarks

By deepening the analysis of the physical properties of the harmonic oscillator on twisted Moyal plane, this work has proved that one can retrieve useful physical quantities, i.e. physical states with real energies even in such deformed situation, thanks to the efficiency of the used operator approach. The twisted Moyal space favours the appearence of both scattering and bound states for the particle subject to a harmonic potential.

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Appendix B: Confluent hypergeometric function

Related to the hypergeometric functions $F(a, b, c; z)$, an important role is played in the theory of special functions by the function

$$
\Phi(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}, \quad |z| < \infty, \quad b \neq 0, -1, -2, \ldots
$$

(82)

known as the confluent hypergeometric function. Here $z$ is a complex variable, $a$ and $b$ are parameters which can take arbitrary real or complex values (except that $b \neq 0, -1, -2, \cdots$), and

$$
(\lambda)_0 = 1, \quad (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda + 1) \cdots (\lambda + k - 1), \quad k = 0, 1, \cdots.
$$

(83)

The series (82) converges for all finite $z$, and therefore represents an entire function of $z$. To prove this, we use the ratio test. Noting that if

$$
u_k = \frac{(a)_k z^k}{(b)_k k!}, \quad \left| \frac{\nu_{k+1}}{\nu_k} \right| = \left| \frac{a + z}{(b + z)(1 + k)} z \right| \to 0 \text{ as } k \to \infty.
$$

(84)

We can show that

$$
\Phi(a, c; z) = \lim_{b \to \infty} F(a, b, c; z/b).
$$

(85)

One can also easily check that the confluent hypergeometric function $\Phi(a, b; z)$ is a particular solution of the linear differential equation

$$
z f''(z) + (b - z)f'(z) - af(z) = 0.
$$

(86)

In fact, denoting the left-hand side of this equation by $L(f)$, and setting $f(z) = f_1(z) = \Phi(a, b; z)$, we have

$$
L(f_1(z)) = \sum_{k=2}^{\infty} \frac{k(k-1)(a)_k z^{k-1}}{(b)_k k!} + (b - z) \sum_{k=1}^{\infty} \frac{(a)_k z^k}{(b)_k k!} - a \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} z^k
$$

$$
= \left[ \frac{b(a)_1}{(b)_1} - a \right] + \sum_{k=1}^{\infty} \frac{(a)_k z^k}{(b)_k k!} \left[ k + \frac{a + k}{b + k} + \frac{a + k}{b + k} - k - a \right] = 0.
$$

(87)

To obtain a second linearly independent solution of (86), we assume that $|argz| < \pi$ and make the substitution $f(z) = z^{1-b}g(z)$. Then the equation (86) goes into an equation of the same form, i.e.,

$$
z g''(z) + (b' - z)g'(z) - a'g(z) = 0
$$

(88)

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with new parameters $a' = 1 + a - b$, $b' = 2 - b$. It follows that the function

$$f(z) = f_2(z) = z^{1-z} \Phi(1 + a - b, 2 - b; z)$$  \hspace{1cm} (89)$$

is also a solution of (86) if $b \neq 2, 3, \cdots$. Thus, if $b \neq 0, \pm 1, \pm 2, \cdots$, both solutions $f_1$ and $f_2$ are meaningful and are linearly independent, (except for the case $b = 1$ where $f_1 = f_2$), so that the general solution of (86) can be written in the form

$$f(z) = A \Phi(a, b; z) + B z^{1-b} \Phi(1 + a - b, 2 - b; z),$$  \hspace{1cm} (90)$$

where $|\arg z| < \pi$, $b \neq 0, \pm 1, \pm 2, \cdots$.

According to the definition of the Hermite polynomials,

$$H_n(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}$$  \hspace{1cm} (91)$$

the even polynomials can be written in the form

$$H_{2n}(z) = \sum_{k=0}^{n} \frac{(-1)^k (2n)!}{k!(2n-2k)!} (2x)^{2n-2k} = (-1)^n (2n)! \sum_{k=0}^{n} \frac{(-1)^k (2z)^{2k}}{(n-k)!(2k)!} = (-1)^n \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-n)_k (z^2)^k}{(2k)!},$$  \hspace{1cm} (92)$$

since $(2k)! = 2^{2k} \binom{1}{2} k!$, and therefore

$$H_{2n}(z) = (-1)^n \frac{(2n)!}{n!} \Phi(-n, \frac{1}{2}; z^2).$$  \hspace{1cm} (93)$$

For the odd Hermite polynomials, we have the analogous formula

$$H_{2n+1}(z) = (-1)^n \frac{(2n+1)!}{n!} 2z \Phi(-n, \frac{3}{2}; z^2).$$  \hspace{1cm} (94)$$

The even Laguerre polynomials can be written in the form

$$L_n^\sigma(z) = \sum_{k=0}^{n} \frac{\Gamma(n + \sigma + 1)}{\Gamma(k + \sigma + 1) k!(n-k)!} (-z)^k = \frac{(\sigma + 1) n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (-z)^k}{(\sigma + 1) k!},$$  \hspace{1cm} (95)$$

and hence

$$L_n^\sigma(z) = \frac{(\sigma + 1) n}{n!} \Phi(-n, \sigma + 1; z).$$  \hspace{1cm} (96)$$

Note that relation (63) is obtained by using a novel property of Laguerre polynomials given by

$$n \int_0^\infty dz e^{-z} z^\beta [L_n^\sigma(z)]^2 = (n + \alpha) \int_0^\infty dz e^{-z} z^\beta [L_{n-1}^\sigma(z)]^2.$$  \hspace{1cm} (97)$$
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