A pedagogical history of compactness

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Modern mathematics tends to obliterate history: each new school rewrites the foundations of its subject in its own language, which makes for fine logic but poor pedagogy.

R. Hartshorne

1 Introduction

For better or for worse, mathematics tends to be taught and studied ahistorically. We tend to present theorems in an order that makes sense after the fact, valuing brevity over exposition, conciseness over connection, and a certain kind of logical purity at the expense of motivation.

One case in point is the typical treatment of compactness. Compactness has come to be one of the most important and useful notions in advanced mathematics. However, most textbooks do little to elucidate where the notion came from, how the definition developed, or what deep and important questions the idea of compactness attempted to answer.

The treatment of compactness in Rudin is typical. The main findings of compactness are presented as isolated facts, such as:

Theorem 1.1 (Weierstrass) Every bounded infinite subset of $R^k$ has a limit point in $R^k$ [47, p. 40].

Rudin does not help students make the connection between this theorem and the fact that a continuous function achieves a maximum on a closed, bounded interval (which is stated two chapters later). Nor does Rudin explain that the continuous function theorem, originally proved by Weierstrass, was one of the main motivations for the notion of compactness. In fact, the theorem is stated as an application for compactness rather than a motivation.

*This paper is a condensed, revised version of my masters thesis [39]. The research for this project was carried out at UC Berkeley, Umeå University, and Columbia University. I am grateful to the many mathematicians and librarians at these universities, including Hendrik Lenstra and Hans Wallin, who patiently answered my many questions as well as two anonymous reviewers who gave me valuable feedback on an earlier version.
The goal of this paper is to fill in some of the gaps left out in a standard treatment of compactness. In the first section, we discuss the original motivations for the idea. In the second section, we trace the development of what have become the two main characterizations of compactness, in terms of sequences and open covers. In the third section, we discuss how sequential compactness can be generalized, using nets and filters, to make it equivalent to open cover compactness.

1.1 Concept Map
We begin with a map of the history covered in this paper. This map, on the following page and roughly chronological from top to bottom, lists the major landmarks in the development of topology and other related fields and their influences on each other.

1.2 Terminology
We know now that there are many notions related to (but not necessarily equivalent to) compactness. Table 1 contains a list of some of these notions.

We also know now how these compactness notions are related. For instance, compactness implies countable compactness implies limit point compactness. Sequential compactness implies countable compactness. And if we put further restrictions on our spaces we can get implications in the other direction. In $T_1$ spaces, limit-point compactness implies countable compactness. In first countable spaces, countable compactness implies sequential compactness. In second countable spaces, sequential compactness implies compactness. In particular, we know that in metric spaces, which turn out to be second countable, the first four notions of compactness in Table 1 are equivalent.

However, it took some time as compactness was applied to different types of spaces for relationships like these to be worked out. And to make things even more difficult, the names for these notions did not stabilize for some time. Table 2 lists different terms used for compactness-related ideas used by some of the most influential mathematicians in the historical development. In this paper I will use the modern names listed in Table 2 and will focus mostly on open-cover and sequential compactness.

2 Possible motivations for compactness
Compactness grew out of one of the most productive periods of mathematical activity. In middle to late nineteenth century Europe, advanced mathematics began to take the form which we know today. In the background was Cantor's

\[ \text{The third section, which includes a discussion of nets and filters, is more demanding technically than the other two sections. These topics are not always included at the undergraduate level, although one popular textbook [36] has recently included the topic of nets as a supplement to the chapter on compactness.} \]
compact: Every open cover has a finite subcover.
(also called the Borel-Lebesgue property)

sequentially compact: Every sequence has a convergent subsequence.

countably compact: Every countable open cover has a finite subcover.

limit-point compact: Every infinite subset of \( X \) has a limit point in \( X \).
(also called Fréchet compact or the Bolzano-Weierstrass property)

relatively compact: The closure is compact.

quasi-compact: Compact to Bourbaki.

pseudo-compact: Each continuous real valued function on \( X \) is bounded.

finally compact: index of compactness is \( \aleph_0 \).
(also called Lindelöf compact)

Table 1: Flavors of compactness

| Who            | When  | Their term       | Modern term         |
|---------------|-------|------------------|---------------------|
| Fréchet       | 1906  | compact          | relatively compact  |
|               |       | extremal         | sequentially compact|
| Russian School| 1920’s| bicompa ct        | compact             |
| (Alexandroff, etc.) |      | compact          | countably compact   |
| Bourbaki      | 1920’s| quasi-compact    | compact and Hausdorff|
|               |       | compact          |                     |

Table 2: Names of compactness-related terms
work establishing the beginning of a systematic study of set theory and point-set topology \[34\]. Also, many mathematicians—including Cantor, Weierstrass, and Dedekind—were worried about the foundations of mathematics and began to make rigorous many of the ideas that had for centuries been taken for granted\[2\]. While some of the nineteenth century work can be traced to mathematical concerns of the early Greeks, the level of rigor and abstraction provided a sort of revolution of mathematical thought.

It is in this context that we will discuss some specific problems that appear to have motivated the concept of compactness. In particular, we will discuss the influence of the study of properties of closed bounded intervals of real numbers (which I will denote \([a, b]\)), spaces of continuous functions, and solutions to differential equations.

### 2.1 Properties of \([a, b]\)

In mid and late nineteenth century, mathematicians worked out what we now know as properties of the real line. There were two camps that ended up influencing the notion of compactness. One was that of Bolzano and Weierstrass who, roughly half a century apart, studied functions defined on sequences of real numbers. The other camp included mathematicians like Heine, Borel, and Lebesgue, who looked at topological features, such as the covering of sets by open neighborhoods. We will examine both of these camps in more detail.

One of the central questions involved in the study of behavior of sequences concerned the behavior of continuous functions defined on closed, bounded intervals of the real line. Weierstrass proved rigorously in 1877\[3\] the following theorem, corresponding to Figure 2:

**Theorem 2.1 (Weierstrass)** Each function continuous in a limited [equivalent to modern-day “closed and bounded”] interval attains there at least once its maximum \[44, \text{p. 244}\].

Fréchet, who first defined compactness, did so in a paper entitled Généralisation d’un théorème de Weierstrass, in which he explicitly tries to generalize Theorem \[2.1\] to abstract topological spaces \[18, \text{p. 848}\].

To prove his theorem, Weierstrass built on the work of Bolzano, who had proved back in 1817 the following:

**Lemma 2.2 (Bolzano)** If a property \(M\) does not apply to all values of a variable quantity \(x\), but to all those that are smaller than a certain \(u\), there is always

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\[2\] There was some work on foundational questions as early as the 18th century, which arose in response to an attack on mathematical foundations by philosopher Bishop Berkeley and an increased need for mathematicians to teach. However, many 18th century mathematicians produced results without a concern for rigorous foundations. See \[21, \text{p. 23-8}\] for more details.

\[3\] This date refers to one of the earliest publications of the theorem, however it is likely that Weierstrass actually proved it years before. As noted in the introduction, Weierstrass was very concerned about rigor so was reluctant to publish his proofs. However, he was very active and influential, so his ideas were disseminated orally.
a quantity $U$ which is the greatest of those of which it can be asserted that all smaller $x$ possess the property $M$ \[40\], p. 150].

This lemma, known today as the greatest lower bound property for real numbers (any subset of the real line which is bounded below has a greatest lower bound \[38\] p. 426)), was used by Bolzano to prove the Intermediate Value Theorem: if $f$ is continuous on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$, then for some $x$ between $a$ and $b$, $f(x)$ will be exactly $0$.

In particular, Bolzano’s lemma allowed Weierstrass to prove that every bounded infinite set of real numbers has a limit point. And it is this property that Fréchet used when he generalized Weierstrass’s theorem to abstract spaces. We now know this property as the Bolzano-Weierstrass property, or limit-point compactness.

While Bolzano and Weierstrass were trying to characterize properties of the real line in terms of sequences, other mathematicians, such as Borel and Lebesgue were trying to characterize it in terms of open covers. Borel proved the following in his 1894 thesis:

\[\text{There is an error in the proof of the Intermediate Value Theorem, namely that to prove a limit exists, Bolzano assumes a limit to exist. Nonetheless, most of the ideas in this paper are correct, and the paper quite sophisticated for its time. See [29] for an insightful analysis.}\]
Theorem 2.3 (Borel) If there is an infinite number of partial intervals on a line segment, such that every point on the line is in at least one of the intervals, then one can effectively find a finite number of intervals from the given intervals having the same property (any point on the line is in at least one of them.) [9, p. 43].

It turns out that Borel’s approach was similar to the approach Heine used to prove in 1872 that a continuous function on a closed interval was uniformly continuous. Interestingly, Heine doesn’t appear to have recognized how his method of proof could be generalized to prove the covering theorem [10, p. 108]. Moreover Heine’s theorem was first proven by Dirichlet in his lectures of 1852, with a more explicit use of coverings and subcoverings than in Heine’s theorem [10 p. 91]. However Dirichlet’s notes were not published in 1904, which might explain why he does not get credit for the generalized version of the Borel theorem. The reason that Heine’s name is attached to the theorem is that Schönflies, a student of Weierstrass, noticed the connection between Heine’s work and Borel’s. The generalized theorem, which is now commonly called the Heine-Borel theorem, with modern notation, is:

Theorem 2.4 (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded [41, p. 39].

While Heine is credited with a theorem he did not prove, it appears that Cousin was largely overlooked for a theorem he did prove. In 1895, he generalized the Borel Theorem for arbitrary covers:

Theorem 2.5 (Cousin’s Theorem) If to each point of a closed region [equivalent to modern-day “closed and bounded”] there corresponds a circle of finite radius [“neighborhood”], then the region can be divided into a finite number of subregions such that each subregion is interior to a circle of a given set having its center in the subregion [26, p. 29].

Cousin’s theorem is generally attributed to Lebesgue, who was said to be aware of the result in 1898 and published his proof in 1902 [26, p. 29].

While there is some debate over who was really responsible for the ideas and proofs, it was well known in Western mathematics right about the time that Fréchet first defined compactness that any closed bounded subset of $\mathbb{R}^n$ has the open-cover property (sometimes called the Borel-Lebesgue property). And, as we will see, after some deliberation, this open-cover property became accepted as the definition for compactness.

2.2 Spaces of continuous functions

A second motivation for the notion of compactness was the study of abstract topological spaces like spaces of continuous functions, $C^0$. In $C^0$, points are

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5From [12] p. 22.
functions (whereas in \([a,b]\) points are real numbers). The properties of \([a,b]\)
onely might not have been seen as important to generalize if it wasn’t the casethat the properties seemed to be important in more abstract spaces as well.

However it turned out that infinite dimensional spaces like \(C^0\) were not aswell-behaved as finite dimensional spaces like \(R^n\). For instance, closed, boundedsubsets of continuous functions on \(R^n\) do not necessarily have the Bolzano-Weierstrass or open-cover property. The work in this area was done by Ascoliand Arzelà in the last decades of the 1800’s.

The following example \cite[p. 238]{28} illustrates that a closed, bounded subsetof continuous functions on \(R^n\) is not, in our modern language, sequentiallycompact.

Consider \(B\), the set of continuous functions, \(f\), defined on \([0,1]\) with \(\|f\| < 1\).(This is the closed unit ball in \(C^0[0,1]\) and \(\|\|\) is the sup norm.) We will showthat there is a sequence in \(B\) that does not have a convergent subsequence. Let\(f_n(x) = x^n\). It lies in \(B\). But we cannot find a subsequence that convergesto a function in \(C^0\). Suppose \(f\) is such a function. Then

\[ f(x) = \lim_{k \to \infty} f_{n_k}(x) \]

which would imply (by pointwise convergence) that

\[ f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases} .\]

Since \(f\) is a discontinuous function, it is not in \(C^0\). Hence the sequence \(f_n(x)\)has no convergent subsequence.

The problem in this example comes from how the functions converge. Ifconvergence means pointwise convergence, we do not get behavior analogous to that of,say, sequences in closed unit balls of \(R^n\). In order to avoid thisproblem, Ascoli introduced the notion of equicontinuity (Bourbaki \cite{2}, Book10). Equicontinuity requires functions to converge to a limit all at once instead ofpointwise.

**Definition 2.6** A set \(E\) is equicontinuous iff for all \(\epsilon > 0\) there exists an \(\delta > 0\)such that \(|s-t| < \delta\) and \(f \in E\) imply \(|f(s) - f(t)| < \epsilon\).

The Arzelà-Ascoli theorem then states:

**Theorem 2.7** (Arzelà-Ascoli) Any bounded equicontinuous sequence of functions in \(C^0[a, b]\) has a uniformly convergent subsequence \cite[p. 279]{28}.

And using modern terminology we can state a consequence of this theorem,analogous to the Heine-Borel theorem:

**Theorem 2.8** A subset of \(C^0[a, b]\) is compact iff it is closed, bounded andequicontinuous \cite[p. 279]{25}.
Ascoli proved the sufficiency of this condition in 1883 and Arzelà the necessity in 1893 [17, p. 392]. This generalization of Bolzano-Weierstrass (although not stated in terms of compactness) was apparently well known after 1880. Moreover, Hilbert seems to have discovered this property independently and published it in 1900 [13, p. 82].

It is unclear whether Arzelà and Ascoli themselves were aware of how their work was connected with compactness. But it is clear that Fréchet’s work was influenced by theirs [44, p. 255].

2.3 Solutions to differential equations

A third motivation for the notion of compactness came from the desire to find solutions to differential equations. Peano, a contemporary of Arzelà and Ascoli as well as a fellow Italian, realized that the Arzelà-Ascoli theorem might be useful for demonstrating the existence of such solutions. He searched for solutions by making a sequence of approximations. He then used what we now call compactness to show that there will be a subsequence that converges to a limit (which will be the solution to the differential equation).

Peano proved the following theorem in 1890:

**Theorem 2.9 (Peano)** Let \( f(x,y) \) be bounded and continuous on a domain \( G \). Then for each interior point \( (x_0, y_0) \) of \( G \), at least one integral curve of the differential equation

\[
\frac{dx}{dy} = f(x, y)
\]

passes through that point [37, p. 29].

While it is not clear if Fréchet was aware of this application, it is important to note that there already seemed to be applications for the notion of compactness before it was formally defined.

It is also interesting to think about what conditions must be satisfied in order for the conclusion of a theorem to become a definition. One reasonable requirement seems to be that the notion should have practical use. And we have seen now three uses for compactness: on the real line, in spaces of continuous functions, and for showing the existence of solutions to differential equations. In other words, we see that compactness notions already seemed to be important in the fields we now call real analysis, topology, functional analysis, and differential equations. So it seems the time was ripe for someone to propose to give a name to and formally define this notion, as we will see in the next section.

3 Developing the definition

We will trace below the development of the two central notions of compactness discussed above, those stemming from sequences and open covers of real numbers. Again, it is useful to know something about the climate of the mathematics
community at the time of these historical developments. We will focus on the contributions of only a few main characters, but there was a large supporting cast of mathematicians who were developing ideas that are now the foundations for analysis and topology. There were many groups of mathematicians in very close contact with each other, so much so that it is difficult to tease apart their contributions. Among them, in France, were Hadamard, Lebesgue, and Fréchet. In Russia, Alexandrov and Urysohn. In Germany, Hausdorff, Hilbert, Schönhflies, and Cantor. In Hungary, Riesz. In the U.S., Chittenden, Hedrick, and Moore.

We will start with the work of Fréchet, who coined the term “compact” and gave definitions for what we now know as countable and sequential compactness. We will then briefly discuss contributions by Alexandroff and Urysohn who developed and stated what we now call compactness. We will show why these notions of compactness are not equivalent, providing motivation for a generalization of sequential compactness in abstract topological spaces.

3.1 Fréchet: Countable and limit-point compactness

While Fréchet was influenced by many contemporaries and predecessors, it seems he deserves credit as the father of compactness. It was Fréchet who first recognized that the notion deserved a name. Fréchet also defined metric spaces for the first time and made inroads into functional analysis, thus providing a context for which the importance of compactness became clear.

Fréchet was a mathematician of big ideas. He preferred definitions that had an intuitive feel rather than analytic power. This preference can be seen in his 1904 paper where he defined compactness and developed some of the ideas that would end up in his 1906 thesis. He says that he captured the intuitive idea of what we now call countable compactness by using nested intersections.

Definition 3.1 A set $E$ is called compact if, whenever $\{E_n\}$ is a sequence of nonempty, closed subsets of $E$ such that $E_{n+1}$ is a subset of $E_n$ for each $n$, there is at least one element that belongs to all of the $E_n$’s \[44, \text{p. 244}] (translated with modern language).

The exact nature of Fréchet’s intuition for this definition is unclear, but here is a guess. One thing the nested intersection definition allows us to quickly see is that sets that have “holes” are not compact. For instance, we can see that $X = [a, b]$ is compact and $Y = [a, b) \cup (b, c]$ is not.

In the latter case, consider $E_i = [a_i, b) \cup (b, c_i]$ where $a_{i+1} > a_i$ and $a_i \to b$ and $c_i \to b$. These sets are clearly nested and are closed in $Y$, but the infinite intersection of those intervals is empty. Hence $Y$ is not compact.

The nested intersection property also allows us to easily rule out sets that have tails running to $\infty$. For instance, we can show that $R$ is not compact.

\[\text{From the original French: Enfin nous appellerons itensemble compact tout ensemble $E$ tel qu'il existe toujours au moins un élément commun à une suite infinie quelconque d'ensembles $E_1$, $E_2$, ..., $E_n$, ..., contenus dans $E$, lorsque ceux-ci (possédant au moins un élément chacun) sont fermés et chacun contenu dans le précédent \[13\], p. 849].\]
Let $E_n = \left[ n, \infty \right)$. Each $E_n$ is closed because it contains its limit points, and clearly $E_{n+1} \subset E_n$. But the infinite intersection of these intervals is empty, so $R$ is not compact.

While Fréchet preferred his intuitive definition involving nested intersections [38, p. 430], he realized the importance of also providing a more useful, if less intuitive, definition. He claimed that the following definition, which uses the Bolzano-Weierstrass property, is much more useful:

**Definition 3.2** A set $E$ in a space $C$ is [relatively limit-point] compact if and only if every infinite subset of $E$ has at least one limit element in $C$ (but not necessarily in $E$) [44, p. 244].

In Fréchet’s 1906 thesis, this characterization is given as the definition, while the definition from his 1904 paper, in terms of nested intersections, is given as a theorem.

When comparing these two definitions, it is good to keep in mind that Fréchet’s work came before the study of abstract topological spaces was clearly worked out. (Hausdorff’s important book on the subject was not published until 1914.) So while Fréchet was trying to state compactness for abstract topological spaces, he was basically working within metric spaces where limit-point, countable (and sequential) compactness are equivalent.

Another interesting detail in the history of compactness concerns why Fréchet chose the name “compact.” When Fréchet first introduced the term, some mathematicians did not like his choice. For instance, Schönflies suggested that what Fréchet called compact be called something like “lickenlos” (without gaps) or “abschliessbar” (closable) [44] p. 266. One can see here, too, a sign that basic

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7From the original French: Nous dirons qu’un ensemble est ite compact lorsqu’il ne comprend qu’un nombre fini d’éléments ou lorsque toute infinité de ses éléments donne lieu à au moins un élément limite [49] p. 6].
notions of topology were not yet worked out. For instance the term “lückenlos” seems more appropriate for the notion of completeness.

However, despite all of Fréchet’s early concern with intuitive definitions and choice of terminology, it is surprising that at the end of his life, he could not remember why he chose the term:

“... j’ai voulu sans doute éviter qu’on puisse appeler compact un noyau solide dense qui n’est agrémenté que d’un fil allant jusqu’à l’infini. C’est une supposition car j’ai complètement oublié les raisons de mon choix!” [38, p. 440]

[Doubtless I wanted to avoid a solid dense core with a single thread going off to infinity being called compact. This is a hypothesis because I have completely forgotten the reasons for my choice!]

So even in the lifetime of the mathematician who named the concept, the original intuition behind the concept was somewhat lost. And Fréchet’s intuitive nested intersection definition was supplanted by less intuitive but more powerful notions of limit-point, sequential, and—as we will examine next—open-cover compactness.

3.2 Alexandroff and Urysohn: Open-cover compactness

While Fréchet was the first to use the term compactness for the Bolzano-Weierstrass property, his contemporaries in Russia, Alexandroff and Urysohn, are credited with defining a related term, orginally called bicompactness, based on the Borel-Lebesgue property, which has come to be the modern notion of compactness [1, p. 425S]. Alexandroff and Urysohn were actually in close contact with Fréchet [45, pp. 319-357].

At the time that Alexandroff and Urysohn introduced their definition based on the Borel-Lebesgue property, they actually thought that the Bolzano-Weierstrass property was a more important concept. It took some time before it became clear that the Borel-Lebesgue property was more important. Probably what tilted the balance was Tychonoff’s theorem, proved first for a special case of closed unit intervals in 1930 and in general in 1935, which stated that an infinite product of compact sets is compact [10]. Tychonoff’s theorem holds for open-cover compactness, but not for sequential compactness.

Also, though Alexandroff and Urysohn usually get credit for defining open-cover compactness, Fréchet was not unaware of the possibility of using neighborhoods to characterize compactness. In a correspondence in 1905, Hadamard suggested that Fréchet think in terms of neighborhoods to generalize the properties of the real line to abstract topological spaces. And the first definition that Fréchet gave, in terms of nested intersections, is the dual of (and hence logically equivalent to) countable open-cover compactness.

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8Urysohn died in a swimming accident off the coast of France at the age of 26. Much of his work was published posthumously by Alexandroff, who kept up his correspondence with Fréchet after Urysohn’s untimely death.
3.3 Open-cover vs. limit-point compactness

Though Fréchet may have been motivated originally to define compactness for abstract topological spaces, we have seen that it was the open-cover property that eventually took priority over both limit-point compactness and the related notion of sequential compactness. While these notions are equivalent on metric spaces, they are not equivalent on abstract topological spaces. Here we will look at one space in which the notions are not equivalent.

Consider \( S_\Omega = \{ \alpha \mid \alpha \text{ is an ordinal number and } \alpha < \Omega \} \) with the order topology, where \( \Omega \) is the first uncountable ordinal number. (See diagram below. The first infinite ordinal, \( \omega \), is the first ordinal after “exhausting” the natural numbers. The first uncountable ordinal, \( \Omega \), is the ordinal after “exhausting” the countable ordinals.)

![Diagram of S_\Omega](image)

Figure 6: Representation of \( S_\Omega \).

We know that all closed subsets of compact sets are compact (and vice versa). So \( S_\Omega \) is not compact since it is not closed in the compact set \( S_\Omega \cup \{ \Omega \} \).

However, it turns out that \( S_\Omega \) is limit-point compact. To see why this is true, we will use the fact that any countable subset of \( S_\Omega \) has an upper bound in \( S_\Omega \) [35, p. 67]. If we take any infinite subset of \( S_\Omega \), it has a countably infinite subset, which we will call \( X \). Since \( X \) is countable, it has an upper bound, let’s call it \( b \), in \( S_\Omega \).

![Diagram of b in S_\Omega](image)

Figure 7: Illustration of \( b \) in \( S_\Omega \).

But the interval \([1, b]\) is compact since \( S_\Omega \) has the l.u.b. property. So there must be a point in \([1, b]\) which is a limit point (of both \( X \) and any set containing it). Thus, \( S_\Omega \) is limit-point compact. Essentially the same argument shows

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9Unless otherwise stated, we will be dealing with metric spaces on which limit-point and sequential compactness are equivalent. (An infinite subset has a limit point iff there is a subsequence converging to that limit.)
that any sequence in $S_\Omega$ must have a convergent subsequence in $S_\Omega$, so $S_\Omega$ is sequentially compact.

We see then that limit-point/sequential compactness and open-cover compactness do not agree on abstract topological spaces. However, it turns out that there is a way to generalize the notion of sequence, using nets and filters to formulate concepts similar to sequential compactness that do agree. We will define these concepts in the next section.

3.4 Side note: Compactness and finiteness

Before we do so, we take a short digression to point out the connection between compactness and finiteness, which may be obvious to mathematicians but not so obvious to students.

Both sequential and open-cover notions of compactness allow us to see compactness as a generalization of finiteness. From the perspective of sequences: if we take any infinite sequence of elements of a finite set, then one element will repeat infinitely often. In other words, there is a subsequence of that sequence that converges in the most trivial sense; each element has the same value. Sets that are sequentially compact have a similar property. Given any sequence, we can find a subsequence, not that has exactly the same value, but one that converges to some limit. Intuitively, if one squints, then a convergent sequence looks constant.

The open-cover definition can also be seen as a generalization of finiteness. While it is not true that all compact sets have a finite number of elements, it is true that every compact set can be covered with finitely many small neighborhoods. If one imagines a compact set in $\mathbb{R}^3$ in which open sets are little balls, then we can envision a covering of the set by balls which are metaphorically like enlarged points. If you squint, those balls can look like points and the compact set like a finite set. As Herman Weyl joked: A compact city is one that can be guarded by finitely many arbitrarily nearsighted policemen [25, p. 499]!

4 Sequential compactness in abstract topological spaces

In the previous section we saw that the two important properties of compactness, those stemming from the Bolzano-Weierstrass property and the Borel-Lebesgue property, are not equivalent in abstract topological spaces. In this section we will show how the former can be generalized to restore that equivalence. We

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There are several good references for discussions of the relationship between compactness and finiteness. For example, Hewitt gives examples to support his claim that many properties in analysis are trivial for finite sets, true and reasonable for compact sets, and either false or ridiculously hard to prove for non-compact sets. There is also a nice discussion in [20] on the problem of finding eigenvalues of infinite dimensional matrices. It turns out compactness is what takes the place of boundedness for finite spaces. Further, several textbook authors, e.g. [11], use finiteness as a motivation for compactness.
will introduce definitions and theorems about nets and filters, both of which are generalizations of sequences which allow us to restate sequential compactness in a way that is equivalent to open-cover compactness.

4.1 Moore and Smith: Nets

The theory of nets was developed by E. H. Moore and his student H. L. Smith in 1922. As I mentioned in the introduction, it is unclear whether Moore and Smith knew how nets could be used to define compactness. This connection is usually credited to Birkhoff [28, p. 64], but Moore and Smith did generalize some of Fréchet’s compactness results in the same paper in which they defined nets [33, p. 118]. One of the motivations for Moore and Smith’s theory was “to be able to minimize integrals of the form

\[ \int_a^b F(x, y, \frac{dx}{dy})\,dx \]

for a given class of functions \( y(x) \)" [46, p. 113]. Our goal here, though, is to express compactness in terms of nets, so we will use the \( S_\Omega \) example to motivate and illustrate net compactness.

The problem in the \( S_\Omega \) example is that while \( \Omega \) is a limit point of \( S_\Omega \) (any neighborhood of \( \Omega \) contains points of \( S_\Omega \)), no sequence in \( S_\Omega \) converges to \( \Omega \) [28, p. 76]. If we are limited to taking a countable number of elements in the sequence, we will never reach \( \Omega \). Nets provide one way of getting around this problem by allowing us to have something like uncountable sequences.\(^{12}\) In our discussion of both nets and filters, we will consider only topological spaces, on which the notion of neighborhood is defined.

To see how nets are a generalization of sequences, it is useful to think of sequences as functions on the natural numbers.

**Definition 4.1** A sequence, \( \{x_n\}_{n \in \mathbb{N}} = \{x_1, x_2, x_3, \ldots\} \) is a function which assigns to each element \( n \) of the natural numbers, \( \mathbb{N} \), a functional value \( x_n \) in a set \( X \).

We would like to replace \( \mathbb{N} \) with a set that can be uncountable but has an ordering similar to that of \( \mathbb{N} \). In other words, we want to stipulate conditions for an ordering relation on a generic set that generalizes the way \( > \) orders natural numbers. We will call this relation “directs” a given set.

**Definition 4.2** A non-empty set \( D \), with the relation \( \succ \) is called directed iff

(i) if \( d_1, d_2, d_3 \in D \) such that \( d_1 \succ d_2 \) and \( d_2 \succ d_3 \) then \( d_1 \succ d_3 \);

(ii) if \( d_1, d_2 \in D \), then there is a \( d_3 \in D \) such that \( d_3 \succ d_1 \) and \( d_3 \succ d_2 \).

\(^{11}\)Little biographical information about Smith is available. I know that he received his Ph.D. from University of Chicago under Moore and got a job at Louisiana State University. But apparently after his important work on nets and filters, he dropped into obscurity [22, p. 563].

\(^{12}\)This treatment follows [28, pp. 62-70] and [32, pp. 281-283, 286-289]. See also [6].
So the definition for net is simply the definition for sequence with $N$ replaced by the notion of a directed set. From now on, $D$ will stand for a directed set with the relation $\succ$ as defined above.

**Definition 4.3** A net (denoted $\{x_d\}_{d \in D}$ or simply $\{x_d\}$) is a function which assigns to each element $d$ of a directed set $D$ a functional value $x_d$ in a set $X$.

As a simple example of a net, take a non-empty totally ordered directed set. Every function on such a set is a net.

Once we know what a net is, we can state what it means for it to converge. Again we can derive the definition for net convergence and limit point by taking the definitions involving sequences and simply replacing $N$ and $\succ$ with $D$ and $\succ$.

**Definition 4.4** A net $\{x_d\}$ converges to $a \in X$ (denoted $\{x_d\} \rightarrow a$) iff for every neighborhood $U$ of $a$, there is an index $d_0 \in D$ such that if $d \succ d_0$ then $x_d \in U$ (i.e. if the net is eventually in each neighborhood of $a$).

**Definition 4.5** A point $a$ is a limit point of $\{x_d\}$ if every neighborhood of $a$ contains at least one element of the net.

In order to state compactness in terms of nets, we also need the concept of subnet, the analog of subsequence. Part of the definition of subsequence generalizes easily, but the other part requires us to think about subsequences in a slightly different way than we are accustomed. The first defining property of subsequence is that each element of the subsequence can be identified with an element of the sequence. This property is generalized in (i) below. The second defining property requires that the subsequence is ordered in a similar way as the sequence. Usually we require the indices of the subsequence, like the indices of the sequence, to be strictly increasing. In other words, for a subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$, the $n_k$ are positive integers such that $n_1 < n_2 < n_3 \cdots$. But the feature of this condition which turns out to be important is simply the fact that as $k \rightarrow \infty$, so do the $n_k$. This property is generalized in (ii) below.

**Definition 4.6** A subnet of a net $\{x_d\}_{d \in D}$ is a net $\{y_b\}_{b \in B}$ where $B$ is a directed set and there is a function $\varphi : B \rightarrow D$ such that:

(i) $y_b = x_{\varphi(b)}$ and

(ii) $\forall d \in D, \exists b_0 \in B$ such that if $b \succ b_0$ then $\varphi(b) \succ d$.

We now are ready to characterize compactness in terms of nets.

**Theorem 4.7** A topological space $X$ is compact iff . . .

- Every net of points of $X$ has a limit point in $X$.
- Every net of points of $X$ has a convergent subnet in $X$.

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13 Incidentally, Kelley, who first coined the term “net,” had considered using the term “way” so the analog of subsequence would be “subway.” McShane also proposed the term “stream” for net since he thought it was intuitive to think of the relation of the directed set as “being downstream from” [32, p.282].
Notice that these definitions are precisely the same as limit point and sequential compactness for metric spaces with the term “net” substituted for “sequence.”

Applying these definitions to the $S_Ω$ example, we can show why $S_Ω$ is not compact. If we take a net $x_d$ of elements of $S_Ω$, it is no longer the case that there will necessarily be a limit in $S_Ω$. In particular, let $D = S_Ω$ and $x_d = d$. Then $x_d$ converges to $Ω$, which is not in $S_Ω$. Thus no subnet of $x_d$ will converge to a point in $S_Ω$.

4.2 Cartan (and Smith): Filters

Nets are not the only way of generalizing sequences. Another generalization of sequence is filter, a notion suggested by Cartan in 1937. These notions are essentially the same (see [28, p. 83] and [6]), save for a subtle distinction involving a particular type of limit found in the advanced theory of integration [43, p. 371].

Again, we will define the notions we need to state compactness in terms of filters and then apply our compactness result to show $S_Ω$ is not compact. As with nets, we can look at convergence of sequences to motivate the idea of convergence of filters. But whereas with nets, the focus was on the index set, with filters the focus is on neighborhoods.

**Definition 4.8** Let $X$ be a set. A set $Φ$ of subsets of $X$ is called a filter iff

(i) $∅ /∈ Φ$
(ii) $A_1 ⊂ A_2 ⊂ X$ and $A_1 ∈ Φ ⇒ A_2 ∈ Φ$
(iii) $A_1, A_2 ∈ Φ ⇒ A_1 ∩ A_2 ∈ Φ$

(Dixmier, p. 13).

An example of a filter is called the co-finite filter: Let $X$ be an infinite set.

$F = \{U ⊆ X : X \mid U$ is finite $\}.$

As with nets, we should define what it means for a filter to converge:

**Definition 4.9** A filter $Φ$ converges to $a ∈ A$ (denoted $Φ \rightarrow a$) iff each neighborhood of $a$ is a member of $Φ$.

There is a natural way to associate a filter with any sequence. If $x_1, x_2, x_3, \ldots$ is a sequence in $X$, we can associate with this sequence a filter $Φ$ on $X$ such that $∀a ∈ X, \{x_n\} \rightarrow a$ iff $Φ \rightarrow a$. In particular, let $Φ = \{A ⊂ X \mid ∃k_A$ such that $∀i ≥ k_A, x_i ∈ A\}$.

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14There is a lot of interesting history and folklore surrounding nets and filters. It turns out that nets are more popular in the U.S. and filters in Europe [43, p. 114]. One might think that this division comes from the fact that Moore and Smith were American and Cartan was French [28, p. 83]. However, it turns out that Smith actually independently discovered filters as an attempt to explain what was lacking in the theory of nets that he and Moore proposed. Moreover, the idea behind filters was actually foreshadowed by Riesz in 1909 when he provided axioms for topology based on limit points instead of metrics. Riesz defines a concept called an “ideal” which is essentially the same as what we now call an ultrafilter [46, p. 111].
Figure 8: Member of filter with $k_A = 5$.

So the tails of the sequence are contained in neighborhoods which are members of the filter. The condition that each neighborhood of $a$ is in the filter is then equivalent to the condition that the sequence is eventually in any neighborhood of $a$.

In order to define compactness in terms of filters, we need one more notion, that of an ultrafilter.

**Definition 4.10** A filter in $X$ is an ultrafilter iff no filter in $X$ properly contains it [15, p. 42].

The notion of ultrafilter is not exactly analogous to subsequence, but in the formulation of compactness, it serves the same purpose.

**Theorem 4.11** A topological space $X$ is compact iff every ultrafilter on $X$ converges in $X$ [2, Vol. 2, C, p. 238].

Now we can return again to our example and get a sense in terms of filters for why $S_\Omega$ is not compact. We want to show that there is an ultrafilter on $S_\Omega$ that does not converge. Consider all the neighborhoods of $\Omega$ in $S_\Omega \cup \Omega$. Let $\Phi = \{ A \subset S_\Omega \mid \exists \alpha \in S_\Omega \text{ such that } \forall \beta \geq \alpha, \beta \in A \}$.

This clearly satisfies the definition for a filter. Let $\Psi$ be any ultrafilter containing $\Phi$. We claim $\Psi$ does not converge in $S_\Omega$. Suppose it did. Say that $\Psi \rightarrow b$. Now pick some $\alpha > b, \alpha \in S_\Omega$. Then $A^+ = \{ \beta : \beta \geq \alpha \} \in \Psi$ since the proof of this claim, in particular when we assert that there is an ultrafilter containing our filter, actually relies on the axiom of choice.

\[\text{The proof of this claim, in particular when we assert that there is an ultrafilter containing our filter, actually relies on the axiom of choice.}\]
Figure 9: Member of filter on $S_\Omega$.

$A^+ \subset \Phi \subset \Psi$. We also have $A^- = \{\beta : \beta < \alpha\} \in \Psi$ since $A^-$ is an open neighborhood of $b$ (and we claim $\Psi$ converges to $b$).

But $A^+ \cap A^- = \emptyset$, which violates the definition of a filter, so our assumption must be wrong. Thus, $\Psi$ must not converge, and hence $S_\Omega$ is not compact.

5 Conclusion

This paper has traced the history of compactness from the original motivating questions, through the process of being defined, to more contemporary work that restores an equivalence between the two main competing characterizations, the Bolzano-Weierstrass property using sequences and the Borel-Lesbesgue property using open covers. These concepts are not the same—there are some spaces that are compact which are not sequentially compact, and visa versa. However, the two concepts agree on metric spaces, and the former can be modified, using nets and filters, to be equivalent to the latter.

We have seen how the notion of compactness evolved from a more intuitive notion to a less intuitive one. Given the difficulty and subtlety of the open-cover characterization of compactness, it might be useful when teaching compactness to either connect back to the more intuitive notions, or actually develop the material more along the historical lines discussed here.

Finally, the idea for compactness grew out of a rather simple, but ultimately deep question about the real numbers. The fact that Bolzano (and later Cauchy and Weierstrass) saw the need to answer questions about the Intermediate Value Property and Maximum Value Property of continuous functions marks a major innovation in mathematical thought. The opportunity to expose students to
these ideas, which are within reach of an advanced undergraduate student, seems too precious to miss. I hope this paper provides some information, or at least some starting points, for doing so.

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