MAXIMAL GREEN SEQUENCES OF SKEW-SYMMETRIZABLE 3 × 3 MATRICES

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ABSTRACT. Maximal green sequences are particular sequences of mutations of skew-symmetrizable matrices which were introduced by Keller in the context of quantum dilogarithm identities and independently by Cecotti-Córdova-Vafa in the context of supersymmetric gauge theory. In this paper we study maximal green sequences of skew-symmetrizable 3 × 3 matrices. We show that such a matrix with a mutation-cyclic diagram does not have any maximal green sequences. We also obtain some properties of maximal green sequences of skew-symmetrizable matrices with mutation-acyclic diagrams.

1. INTRODUCTION

Maximal green sequences are particular sequences of mutations of skew-symmetrizable matrices. They were used in [3] to obtain quantum dilogarithm identities. Moreover, the same sequences appeared in theoretical physics where they yield the complete spectrum of a BPS particle, see [2, Section 4.2]. In this paper we study the maximal green sequences of skew-symmetrizable 3 × 3 matrices. We show that those matrices with a mutation-cyclic diagram do not have any maximal green sequences. We also obtain some properties of maximal green sequences of skew-symmetrizable matrices with mutation-acyclic diagrams.

To be more specific, we need some terminology. Let us recall that a skew-symmetrizable matrix $B$ is an $n \times n$ integer matrix such that $DB$ is skew-symmetric for some diagonal matrix $D$ with positive diagonal entries. We consider pairs $(c, B)$, where $B$ is a skew-symmetrizable integer matrix and $c = (c_1, ..., c_n)$ such that each $c_i = (c_1, ..., c_n) \in \mathbb{Z}^n$ is non-zero. Motivated by the structural theory of cluster algebras, we call such a pair $(c, B)$ a $Y$-seed. Then, for $k = 1, \ldots, n$, the $Y$-seed mutation $\mu_k$ transforms $(c, B)$ into the $Y$-seed $\mu_k(c, B) = (c', B')$ defined as follows [4, Equation (5.9)], where we use the notation $[b]_+ = \max(b, 0)$:

- The entries of the exchange matrix $B' = (B'_{ij})$ are given by

$$
B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k; \\
B_{ij} + [B_{ik}]_+ [B_{kj}]_+ - [-B_{ik}]_+ [-B_{kj}]_+ & \text{otherwise.}
\end{cases}
$$

$$
(1.1)
$$

- The tuple $c' = (c'_1, \ldots, c'_n)$ is given by

$$
c'_i = \begin{cases} -c_i & \text{if } i = k; \\
c_i + [\text{sgn}(c_k)B_{k,i}]_+ c_k & \text{if } i \neq k.
\end{cases}
$$

$$
(1.2)
$$

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This transformation is involutive; furthermore, $B'$ is skew-symmetrizable with the same choice of $D$. We also use the notation $B' = \mu_k(B)$ (in (1.1)) and call the transformation $B \mapsto B'$ the \textit{matrix mutation}. This operation is involutive, so it defines a \textit{mutation-equivalence} relation on skew-symmetrizable matrices.

We use the $Y$-seeds in association with the vertices of a regular tree. To be more precise, let $\mathbb{T}_n$ be an $n$-\textit{regular tree} whose edges are labeled by the numbers $1, \ldots, n$, so that the $n$ edges emanating from each vertex receive different labels.

We write $t \xrightarrow{k} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by $k$. Let us fix an initial seed at a vertex $t_0$ in $\mathbb{T}_n$ and assign the (initial) $Y$-seed $(c_0, B_0)$, where $c_0$ is the tuple of standard basis. This defines a \textit{Y-seed pattern} on $\mathbb{T}_n$, i.e. an assignment of a seed $(c_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \xrightarrow{k} t'$ are obtained from each other by the seed mutation $\mu_k$. We write:

$$c_t = c = (c_1, \ldots, c_n), \quad B_t = B = (B_{ij}).$$

We refer to $B$ as the \textit{exchange matrix} and $c$ as the \textit{c-vector} tuple of the $Y$-seed. It is conjectured that c-vectors have the following \textit{sign coherence property}:

$$\text{(1.4)} \quad \text{each vector } c_j \text{ has either all entries nonnegative or all entries nonpositive.}$$

This conjectural property (1.4) has been proved in [3] for the case of skew-symmetric exchange matrices, using \textit{quivers with potentials} and their representations.

We need a bit more terminology. The \textit{diagram} of a skew-symmetrizable $n \times n$ matrix $B$ is the directed graph $\Gamma(B)$ defined as follows: the vertices of $\Gamma(B)$ are the indices $1, 2, \ldots, n$ such that there is a directed edge from $i$ to $j$ if and only if $B_{j,i} > 0$, and this edge is assigned the weight $|B_{j,i}|$. By a subdiagram of $\Gamma(B)$, we always mean a diagram obtained from $\Gamma(B)$ by taking an induced (full) directed subgraph on a subset of vertices and keeping all its edge weights the same as in $\Gamma(B)$. By a cycle in $\Gamma(B)$ we mean a subdiagram whose vertices can be labeled by elements of $\mathbb{Z}/m\mathbb{Z}$, so that the edges between them are precisely $\{i, i + 1\}$ for $i \in \mathbb{Z}/m\mathbb{Z}$. Let us also note that if $B$ is skew-symmetric then it is also represented, alternatively, by a quiver whose vertices are the indices $1, 2, \ldots, n$ and there are $B_{j,i} > 0$ many arrows from $i$ to $j$. This quiver uniquely determines the corresponding skew-symmetric matrix, so mutation of skew-symmetric matrices can be viewed as a "quiver mutation". We call a diagram $\Gamma$ \textit{mutation-acyclic} if it is mutation-equivalent to an acyclic diagram (i.e. a diagram which has no oriented cycles at all); otherwise we call it \textit{mutation-cyclic}.

Now we can recall the notion of a green sequence [5]:

\textbf{Definition 1.1.} Let $B_0$ be a skew-symmetrizable $n \times n$ matrix. A \textit{green sequence} for $B_0$ is a sequence $\mathbf{i} = (i_1, \ldots, i_l)$ such that, for any $1 \leq k \leq l$ with $(c, B) = \mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(c_0, B_0)$, we have $c_{i_k} > 0$; here if $k = 1$, then we take $(c, B) = (c_0, B_0)$.

A green sequence $\mathbf{i} = (i_1, \ldots, i_l)$ is \textit{maximal} if, for $(c, B) = \mu_{i_1} \circ \cdots \circ \mu_{i_l}(c_0, B_0)$, we have $c_k < 0$ for all $k = 1, \ldots, n$.

In this paper, we study the maximal green sequences in the basic case of size 3 skew-symmetrizable matrices. Our first result is the following:

\textbf{Theorem 1.2.} Suppose that $B$ is a skew-symmetrizable $3 \times 3$ matrix. Under the assumption (1.4), if $\Gamma(B)$ is mutation-cyclic, then $B$ does not have any maximal green sequences.
For skew-symmetrizable matrices with mutation-acyclic diagrams, we have the following result:

**Theorem 1.3.** Suppose that \( B \) is a skew-symmetrizable \( 3 \times 3 \) matrix. Suppose also that \( \Gamma(B) \) is mutation-acyclic and \( i = (i_1, \ldots, i_l) \) is a maximal green sequence for \( B \). Let \( B = B_0 \) and, for \( j = 1, \ldots, l \), let \( B_j = \mu_{i_j} \circ \cdots \circ \mu_{i_1}(B) \). Then, under the assumption (1.4), the diagram \( \Gamma(B_j) \) is cyclic for some \( 0 \leq j \leq l \).

We also have the following result, which writes the initial exchange matrix in terms of a \( Y \)-seed:

**Theorem 1.4.** For any skew-symmetrizable \( 3 \times 3 \) matrix \( B \), let \( u = u(B) = (a, b, c) \) be defined as follows: if \( \Gamma(B) \) is cyclic, then \( (a, b, c) = (d_2|B_{2,3}|, d_3|B_{3,1}|, d_1|B_{1,2}|) \); if \( \Gamma(B) \) is acyclic, then \( (a, b, c) = (\pm d_2|B_{2,3}|, \pm d_3|B_{3,1}|, \pm d_1|B_{1,2}|) \) such that the coordinates corresponding to the source and sink have the same sign and the remaining coordinate has the opposite sign.

Assume without loss of generality that (1.4) is satisfied. First we note the following two properties, which can be easily checked using the definitions:

**Proposition 2.1.** Suppose that \((c, B)\) is a \( Y \)-seed (with respect to an initial \( Y \)-seed). Suppose also that the coordinate vector of \( u \) with respect to \( c \) is \((a_1, \ldots, a_n)\). Let \((c', B') = \mu_k(c, B)\) and \((a_1', \ldots, a_n')\) be the coordinates of \( u \) with respect to \( c' \). Then \( a_i = a_i' \) if \( i \neq k \) and \( a_k' = -a_k + \sum a_i[\text{sgn}(c_k)B_{k,i}]_+ \), where the sum is over all \( i \neq k \).

**Proposition 2.2.** Suppose that \( B \) is a skew-symmetrizable \( 3 \times 3 \) matrix \( B \). Let \( u = u(B) = (a, b, c) \) be defined as follows: if \( \Gamma(B) \) is cyclic, then \((a, b, c) = (d_2|B_{2,3}|, d_3|B_{3,1}|, d_1|B_{1,2}|)\); if \( \Gamma(B) \) is acyclic, then \((a, b, c) = (\pm d_2|B_{2,3}|, \pm d_3|B_{3,1}|, \pm d_1|B_{1,2}|)\) such that the coordinates corresponding to the source and sink have the same sign and the remaining coordinate has the opposite sign. Then the vector \( u \) is a radical vector for \( B \), i.e. \( Bu = 0 \).

We also need the following two lemmas to prove our results:

**Lemma 2.3.** Suppose that \((c, B)\) is a \( Y \)-seed (with respect to an initial \( Y \)-seed) such that \( \Gamma(B) \) is cyclic and let \( u \) be the vector whose coordinate vector with respect to the basis \( c \) is \((d_2|B_{2,3}|, d_3|B_{3,1}|, d_1|B_{1,2}|)\). Let \((c', B') = \mu_k(c, B)\). Then we have the following:

- If the diagram \( \Gamma(B') \) is also cyclic, then the coordinate vector of \( u \) with respect to the basis \( c' \) is \((d_2|B'_{2,3}|, d_3|B'_{3,1}|, d_1|B'_{1,2}|)\).
- If the diagram \( \Gamma(B') \) is cyclic, then the coordinate vector of \( u \) with respect to the basis \( c' \) is obtained from \((d_2|B'_{2,3}|, d_3|B'_{3,1}|, d_1|B'_{1,2}|)\) by multiplying the \( k \)-th coordinate by \(-1\). (Note that the vertex \( k \) is neither a source nor a sink in \( \Gamma(B') \).)

**Proof.** Assume without loss of generality that \( \text{sgn}(c_k) = \text{sgn}(B_{k,i}) \). Then \( c_i' = c_i + |B_{k,i}|c_k \), \( c_j' = c_j, \) and \( c_k' = -c_k \). Thus the \( k \)-th coordinate of \( u \) with respect to \( c' \) will
be \(-d_i|B_{i,j}| + |B_{k,i}| |B_{k,j}| = -d_i|B_{i,j}| + |B_{i,k}| |B_{k,j}| = d_i(-|B_{i,j}| + |B_{i,k}| |B_{k,j}|)\) (Proposition 2.1), the other coordinates are the same (note that the \(k\)-th coordinate of \(u\) with respect to \(c\) is \(d_i|B_{i,j}| = d_i|B_{j,i}|\)). Thus, to prove the statement in the first part, it is enough to show \(|B'_{i,j}| = -|B_{i,j}| + |B_{i,k}| |B_{k,j}|\) (Recall that the skew-symmetrizing matrix \(D\) is preserved under mutations). For convenience, we investigate in cases:

Case 1. \(B_{k,i} > 0\). Then \(B_{i,j} > 0\) and \(B_{j,k} > 0\) (so \(B_{i,k} < 0, B_{j,i} < 0, B_{k,j} < 0\)). Then, since \(\Gamma(B')\) is cyclic, we have \(B'_{k,i} < 0, B'_{i,j} < 0, B'_{j,k} < 0\). By (1.1), we have

\[B'_{i,j} = B_{i,j} + [B_{k,i}] + [B_{k,j}] = \begin{vmatrix} B_{i,j} & -B_{k,j} \\ -B_{k,i} & B_{k,j} \end{vmatrix} = B_{i,j} - B_{k,j} - B_{i,k}B_{k,j}.
\]

Since \(B'_{i,j} < 0\) by assumption in this case, we have \(|B'_{i,j}| = -B_{i,j} + B_{k,i}B_{k,j} = -|B_{i,j}| + |B_{k,i}||B_{k,j}|\) (note \(B_{i,j} > 0, B_{i,k}B_{k,j} > 0\) by assumption) as required.

Case 2. \(B_{k,i} < 0\). Then \(B_{i,j} < 0\) and \(B_{j,k} < 0\) (so \(B_{i,k} > 0, B_{j,i} > 0, B_{k,j} > 0\)). Then, since \(\Gamma(B')\) is cyclic, we have \(B'_{k,i} > 0, B'_{i,j} > 0, B'_{j,k} > 0\). By (1.1), we have

\[B'_{i,j} = B_{i,j} + [B_{k,i}] + [B_{k,j}] = \begin{vmatrix} B_{i,j} & B_{k,j} \\ B_{k,i} & -B_{k,j} \end{vmatrix} = B_{i,j} + B_{k,i}B_{k,j}.
\]

Thus \(|B'_{i,j}| = |B_{i,j}| + |B_{k,i}||B_{k,j}|\) as required.

For the second part, suppose \(\Gamma(B')\) is acyclic. We assume, without loss of generality, that \(\text{sgn}(c_k) = \text{sgn}(B_{k,i})\). To prove the statement (of the second part), it is enough to show that \(-|B'_{i,j}| = -|B_{i,j}| + |B_{i,k}||B_{k,j}|\). For convenience, we investigate in cases:

Case 1. \(B_{k,i} > 0\). Then \(B_{i,j} > 0\) and \(B_{j,k} > 0\) (so \(B_{i,k} < 0, B_{j,i} < 0, B_{k,j} < 0\)). Then, since \(\Gamma(B')\) is acyclic, we have \(B'_{k,i} < 0, B'_{i,j} < 0\) but \(B'_{j,k} > 0\). By (1.1), we have

\[B'_{i,j} = B_{i,j} + [B_{k,i}] + [B_{k,j}] = \begin{vmatrix} B_{i,j} & -B_{k,j} \\ -B_{k,i} & B_{k,j} \end{vmatrix} = B_{i,j} - B_{k,j} - B_{i,k}B_{k,j}.
\]

Thus, we have \(-|B'_{i,j}| = -B_{i,j} + B_{k,i}B_{k,j} = -|B_{i,j}| + |B_{k,i}||B_{k,j}|\) as required.

Case 2. \(B_{k,i} < 0\). Then \(B_{i,j} < 0\) and \(B_{j,k} < 0\) (so \(B_{i,k} > 0, B_{j,i} > 0, B_{k,j} > 0\)). Then, since \(\Gamma(B')\) is acyclic, we have \(B'_{k,i} > 0, B'_{i,j} > 0\) but \(B'_{j,k} < 0\). By (1.1),

\[B'_{i,j} = B_{i,j} + [B_{k,i}] + [B_{k,j}] = \begin{vmatrix} B_{i,j} & B_{k,j} \\ B_{k,i} & -B_{k,j} \end{vmatrix} = B_{i,j} + B_{k,i}B_{k,j}.
\]

Thus \(-|B'_{i,j}| = B_{i,j} + B_{k,i}B_{k,j} = -|B_{i,j}| + |B_{k,i}||B_{k,j}|\) as required.

\[\square\]

**Lemma 2.4.** Suppose that \((c, B)\) is a seed such that \(\Gamma(B)\) is acyclic and \(u\) be the vector whose coordinate vector with respect to the basis \(c\) is \((\pm d_2|B_{2,3}|, \pm d_3|B_{3,1}|, \pm d_1|B_{1,2}|)\) such that the coordinates corresponding to the source and sink have the same sign and the remaining coordinate has the opposite sign. Let \((c', B') = \mu_k(c, B)\). Then we have the following:

If \(k\) is a source or sink in \(\Gamma(B)\) (so the diagram \(\Gamma(B')\) is also acyclic), then the coordinate vector of \(u\) with with respect to the basis \(c'\) is obtained from the one for \(c\) by multiplying the \(k\)-th coordinate by \(-1\).

If \(k\) is neither a source nor a sink in \(\Gamma(B)\) (so the diagram \(\Gamma(B')\) is cyclic), then the coordinate vector of \(u\) with with respect to the basis \(c'\) is \((d_2|B'_{2,3}|, d_3|B'_{3,1}|, d_1|B'_{1,2}|)\) or its negative.

**Proof.** Let \(i, j\) be the remaining vertices (so \(\{i, j, k\} = \{1, 2, 3\}\)). For the first part, suppose that \(k\) is a source or sink, so \(\text{sgn}(B_{k,i}) = \text{sgn}(B_{k,j})\). Let us denote the \(i\)-th, \(j\)-th and \(k\)-th coordinates of \(u\) by \(a_i, a_j, a_k\) respectively, so \(|a_i| = d_k|B_{k,i}| = d_j|B_{j,k}|, |a_j| = d_k|B_{k,j}| = d_i|B_{i,k}|\). Then, in particular,
Then, by the condition on the signs, the numbers $a_i$ and $a_j$ have opposite signs, so (*) implies that
\[
(**) \ a_i |B_{k,i}| = -a_j |B_{k,j}|.
\]
Let us assume first that $\text{sgn}(c_k) = \text{sgn}(B_{k,i}) = \text{sgn}(B_{k,j})$. Then $c'_i = c_i + |B_{k,i}|c_k$, $c'_j = c_j + |B_{k,j}|c_k$, $c'_k = -c_k$. Then the $k$-th coordinate of $u$ with respect to $c'$ will be $-a_k$ and the other coordinates are the same because $u = a_k c_k + a_i c_i + a_j c_j$.

Let us now assume that $\text{sgn}(c_k) = -\text{sgn}(B_{k,i}) = -\text{sgn}(B_{k,j})$. Then $c'_k = -c_k$ and $c'_i = c_i$, $c'_j = c_j$. Then the $k$-th coordinate of $u$ with respect to $c'$ will be $-a_k$ and the other coordinates are the same.

For the second part, suppose that $k$ is neither a source nor a sink, so $\text{sgn}(B_{k,i}) = -\text{sgn}(B_{k,j})$. We may assume, without loss of generality, that $\text{sgn}(c_k) = \text{sgn}(B_{k,i})$. Then $c'_i = c_i + |B_{k,i}|c_k$, $c'_j = c_j$, $c'_k = -c_k$. Let us denote the $i$-th,$j$-th and $k$-th coordinates of $u$ (with respect to $c$) by $a_i, a_j, a_k$ respectively, so $a_i = d_k |B_{k,j}| = d_j |B_{k,i}|$, $|a_k| = d_i |B_{i,j}| = d_j |B_{i,i}|$ such that
\[
\text{sgn}(a_i) = \text{sgn}(a_j) = -\text{sgn}(a_k) \quad (**).
\]
Then the $k$-th coordinate of $u$ with respect to $c'$ will be $a'_k = -a_k + a_i |B_{k,i}|$ and the other coordinates $a'_i, a'_j$ are the same because $u = a_k c_k + a_i c_i + a_j c_j = (-a_k + a_i |B_{k,i}|)(-c_k) + a_i (c_i + |B_{k,i}|c_k) + a_j (c_j - c_k)$. Note that $\text{sgn}(-a_k + a_i |B_{k,i}|) = \text{sgn}(a_i) = \text{sgn}(a_j)$ by (**). Thus we may assume, without loss of generality that, $\text{sgn}(a_i) = \text{sgn}(a_j) = +1 = -\text{sgn}(a_k)$, so $a_i = d_k |B_{k,j}| = d_j |B_{j,k}|$, $a_j = d_k |B_{i,k}| = d_i |B_{i,j}|$, $a_k = -d_j |B_{j,i}| = -d_i |B_{i,i}|$ and show that $a'_i = -a_k + a_i |B_{k,i}| = d_i |B_{i,j}| + d_k |B_{k,j}| |B_{k,i}| = d_i |B_{i,j}| + d_i |B_{j,j}| + d_i |B_{i,k}| + d_k |B_{k,i}|$ as required.

Case 1. $B_{k,i} > 0$. Then $B_{j,i} > 0$ and $B_{j,k} > 0$ (so $B_{i,k} < 0, B_{i,j} < 0, B_{j,k} < 0$). Note then that, since $\Gamma(B')$ is cyclic, we have $B'_{k,i} < 0, B'_{i,j} < 0, B'_{j,k} < 0$. By (11),
\[
B'_{ij} = B_{ij} + |B_{ik}| + |B_{kj}| + |B_{kj}| + |B_{k,j}| = B_{ij} - |B_{ik}||B_{kj}|.
\]
Thus $|B'_{ij}| = -B'_{ij} = |B_{ij} + B_{ik}B_{k,j}| = |B_{ij} + B_{ik}||B_{kj}|$ as required.

Case 2. $B_{k,i} < 0$. Then $B_{j,i} < 0$ and $B_{j,k} < 0$ (so $B_{i,k} > 0, B_{i,j} > 0, B_{j,k} > 0$). Then, since $\Gamma(B')$ is cyclic, we have $B'_{k,i} > 0, B'_{i,j} > 0, B'_{j,k} < 0$. By (11),
\[
B'_{ij} = B_{ij} + |B_{ik}||B_{kj}| = B_{ij} - |B_{ik}||B_{kj}| = B_{ij} + B_{ik}B_{k,j},
\]
so $|B'_{ij}| = B_{ij} + B_{ik}B_{k,j} = |B_{ij} + |B_{kj}||B_{ik}|$ as required.

We can now prove our results.

Proof of Theorem 1.2 Suppose that $i = (i_1, \ldots , i_t)$ is a maximal green sequence for $B$. Let $(c', B') = \mu_i \circ \cdots \circ \mu_i$. Then $c'_j < 0$ for all $j = 1, \ldots , n$. Let $u_0$ be the vector (whose coordinate vector with respect to the initial basis $c$) is $(d_2 |B_{2,3}|, d_3 |B_{3,1}|, d_4 |B_{1,2}|)$. Let $(a_1, a_2, a_3)$ be the coordinates of $u_0$ with respect to $c'$. By Lemma 2, the coordinates $a_1, a_2, a_3 > 0$. This implies that, since $u_0 = a_1 c'_1 + a_2 c'_2 + a_3 c'_3$, the coordinates of $u_0$ with respect to $c$ are non-positive; which is a contradiction.
Proof of Theorem 1.3: Suppose that $\Gamma(B_j)$ is cyclic for all $0 \leq j \leq l$. Let $(c', B') = \mu_i \circ \cdots \circ \mu_1(c, B)$. Then $c'_j < 0$ for all $j = 1, \ldots, n$. Let $u_0$ be the vector (whose coordinate vector with respect to the initial basis $c$ is) $(d_2|B_{2,3}|, d_3|B_{3,1}|, d_1|B_{1,2}|)$. Let $(a_1, a_2, a_3)$ be the coordinates of $u_0$ with respect to $c'$. By the first part of Lemma 2.3, the coordinates $a_1, a_2, a_3 > 0$. This implies that, since $u_0 = a_1 c'_1 + a_2 c'_2 + a_3 c'_3$, the coordinates of $u_0$ with respect to $c$ are non-positive; which is a contradiction.

Proof of Theorem 1.4: By Lemmas 2.3 and 2.4, the coordinate vector of $u$ with respect to $c'$ is $u'$ or $-u'$. Then the conclusions follow.

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