Properties of the conformal Yangian in scalar and gauge field theories

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ABSTRACT: Properties of the SO(2, n) Yangian acting on scalar and gauge fields are presented. This differential operator representation of the infinite-dimensional extension of the conformal algebra SO(2, n) is proved to satisfy the Serre relation for arbitrary spacetime dimension n for off-shell scalar theory, but only on shell and for n = 4 in the gauge theory. The SO(2, n) Yangian acts simply on the off-shell kinematic invariants \( (k_I + k_{I+1} + \ldots)^2 \), and it annihilates individual off-shell scalar \( \lambda \phi^3 \) Feynman tree graphs for \( n = 6 \) if the differential operator representation is extended by graph dependent evaluation terms. The SO(2, 4) Yangian level one generators are shown to act in a compact way on pure Yang-Mills gluon tree amplitudes. The action of the Yangian on the scattering polynomials of a CHY formalism is also described.

KEYWORDS: Scale and Conformal Symmetries, Scattering Amplitudes, Field Theories in Higher Dimensions, Global Symmetries

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1 Introduction

In this paper we prove various properties of the Yangian algebra of the conformal group for scalar and gauge field theories. Our focus is on the Yangian extension of SO(2, n) and how this infinite-dimensional algebra acts on the kinematic invariants and on tree level scattering amplitudes for both $\lambda\phi^3$ theory and pure Yang-Mills. The $\lambda\phi^3$ theory of a single scalar field and the non-abelian gauge theory are closely associated in the Cachazo-He-Yuan (CHY) formalism [1–3], since they share the same scattering polynomials. It is well known that the Yangian of the superconformal algebra PSU(2, 2$|$4) is a symmetry of planar $\mathcal{N} = 4$ super Yang-Mills theory [4–7], but less is known about how the infinite-dimensional algebras act in a more realistic theory, or in particular, in non-supersymmetric Yang-Mills. To this end, we present certain features of the SO(2, n) Yangian in non-supersymmetric field theories.

In sections 2 and 3, we discuss the differential operator representation in momentum space of the SO(2, n) Yangian, and describe the Serre condition which this representation must satisfy for a consistent algebraic structure.

In section 3, we prove the Serre relation for both the scalar and gauge field theory representations, denoting what restrictions apply.

In section 4, it is shown that the SO(2, n) Yangian acts in a simple way on the kinematic invariants. The $\lambda\phi^3$ theory is conformally invariant at tree level in $n = 6$ dimensions, so the
level zero generators annihilate both partial amplitudes and individual Feynman graphs. But the higher level Yangian generators only annihilate individual Feynman graphs, and only when the representation of the Yangian generators is extended by so-called evaluation terms. That is to say, the \( \text{SO}(2,6) \) Yangian is not a symmetry of \( n = 6 \lambda\theta^3 \) theory, but it has some structure which may be useful.

In section 5, we look to extend our analysis by presenting the action of the Yangian generators on the off-shell scattering polynomials \([8]\). This will provide the Yangian transformations of the graphs, since the momenta of a scalar tree graph in the CHY formalism occur only in the scattering polynomials, and the Yangian generators act only on the momenta.

In section 6, the action of the \( \text{SO}(2,4) \) Yangian on pure Yang-Mills \( N \)-point tree amplitudes for \( N = 3, 4 \) is shown to have a compact form, albeit non-zero. These expressions can also be expressed as traces of Dirac matrices, which are shown to originate from tree amplitudes with 2 fermions and \( N - 2 \) gluons. This reflects the invariance of pure Yang-Mills gluon tree amplitudes under the \( \text{PSU}(2,2|4) \) Yangian, and may lead to some interpretation of the role of supersymmetry in pure non-supersymmetric gauge theory.

2 \ The SO(2, n) conformal algebra Yangian and its defining relations

The \( \text{SO}(2, n) \) conformal generators \( J^{AB} \) satisfy the Lie algebra commutation relations

\[
[J^{AB}, J^{CD}] = -\eta^{AC} J^{BD} - \eta^{BD} J^{AC} + \eta^{AD} J^{BC} + \eta^{BC} J^{AD},
\]

\[
J^{AB} = -J^{BA}, \quad 0 \leq A, B \leq n + 1, \quad \eta^{AB} = \text{diagonal}(1, -1, -1, \ldots, -1, 1),
\]

\[
g^{\mu\nu} = \eta^{\mu\nu} = \text{diagonal}(1, -1, -1, \ldots, -1), \quad 0 \leq \mu, \nu \leq n - 1 \tag{2.1}
\]

The defining relations for the \( \text{SO}(2, n) \) Yangian Hopf algebra \([9, 10]\) are given in terms of the level zero generators \( J^{AB} \), and the level one generators \( \tilde{J}^{AB} \),

\[
[J^{AB}, J^{CD}] = f^{ABCD}_{\text{EF}} J^{EF}, \quad [J^{AB}, \tilde{J}^{CD}] = f^{ABCD}_{\text{EF}} \tilde{J}^{EF}
\]

\[
[\tilde{J}^{AB}, [\tilde{J}^{CD}, J^{EF}]] + [\tilde{J}^{CD}, [\tilde{J}^{EF}, J^{AB}]] + [\tilde{J}^{EF}, [\tilde{J}^{AB}, J^{CD}]]
\]

\[
= \frac{1}{24} f^{AB}_{\text{GHMN}} f^{CD}_{\text{IJOP}} f^{EF}_{\text{KLQR}} f^{MNOPQR} \{J^{GH}, J^{IJ}, J^{KL}\} \tag{2.2}
\]

where the structure constants of (2.1) can be displayed as

\[
f^{ABCD}_{\text{EF}} = \frac{1}{2} (\eta^{AC} \delta^{BD}_{EF} - \eta^{BD} \delta^{AC}_{EF} + \eta^{AD} \delta^{BC}_{EF} + \eta^{BC} \delta^{AD}_{EF})
\]

\[
+ \eta^{AC} \delta^{BD}_{EF} + \eta^{BD} \delta^{AC}_{EF} - \eta^{AD} \delta^{BC}_{EF} - \eta^{BC} \delta^{AD}_{EF} \tag{2.4}
\]

\[
\text{and the indices are raised and lowered with the metric, } f^{AB}_{\text{GHMN}} = f^{ABG'H'}_{\text{MN}} \eta^{GG'} \eta^{HH'}
\]

\[
f^{MNOPQR}_{Q'R'} \eta^{QQ'} \eta^{RR'} \text{ etc. The symmetrized triple product is defined by}
\]

\[
\{J^{GH}, J^{IJ}, J^{KL}\} = J^{GH} J^{IJ} J^{KL} + J^{IJ} J^{GH} J^{KL} + J^{KL} J^{GH} J^{IJ}
\]

\[
+ J^{KL} J^{IJ} J^{GH} + J^{GH} J^{KL} J^{IJ} + J^{IJ} J^{KL} J^{GH} \tag{2.5}
\]
The co-product construction of the Yangian provides for a multi-site representation of the generators,

\[ J^{AB} = \sum_{i=1}^{N} J^{AB}_i, \quad \tilde{J}^{AB} = \frac{1}{2} f^{AB}_{CD} \sum_{1 \leq i < j \leq N} J^{CD}_i J^{EF}_j \]  

(2.6)

where \( N \) stands for the number of sites. So on a single site the level one generators \( \tilde{J}^{AB} = 0 \).

The relations (2.2) follow from the level zero single site commutation relations.

\[ [J^{AB}_i, J^{CD}_j] = \delta_{ij} \left( -\eta^{AC} J^{BD}_i - \eta^{BD} J^{AC}_i + \eta^{AD} J^{BC}_i + \eta^{BC} J^{AD}_i \right) \]  

(2.7)

The Serre relation (2.3) is satisfied if it holds for a single site, namely

\[ 0 = f^{AB}_{GHMN} f^{CD}_{IJOP} f^{EF}_{KLQR} f^{MNOPQR}_{JGH, JIJ, JKL} \]  

(2.8)

Only certain representations of the single site level zero generators will satisfy (2.8), and thus lead to a consistent set of commutation relations. Once (2.8) is proved, the defining relation (2.3) follows from the co-product. The infinite number of higher level generators of the Yangian algebra can then be derived from commutators of the level one generators [11–13].

In preparation for the proof, we can reexpress the triple product times the conformal structure constants in (2.8) as the sum of three cyclic terms

\[
\begin{align*}
& f^{AB}_{GHMN} f^{CD}_{IJOP} f^{EF}_{KLQR} f^{MNOPQR}_{JGH, JIJ, JKL} \nonumber \\
& = 4 \eta_{WY} \left( \eta^{BD} \left( \left\{ J^{FA}, J^{EW}, J^{YC} \right\} + \left\{ J^{EC}, J^{AW}, J^{YF} \right\} - \left\{ J^{FC}, J^{EW}, J^{YA} \right\} - \left\{ J^{EA}, J^{CW}, J^{YF} \right\} \right) \right. \\
& \left. \quad - \eta^{AD} \left( \left\{ J^{FB}, J^{EW}, J^{YC} \right\} + \left\{ J^{EC}, J^{BW}, J^{YF} \right\} - \left\{ J^{FC}, J^{EW}, J^{YB} \right\} - \left\{ J^{EB}, J^{CW}, J^{YF} \right\} \right) \right. \\
& \left. \quad - \eta^{BC} \left( \left\{ J^{FA}, J^{EW}, J^{YD} \right\} + \left\{ J^{ED}, J^{AW}, J^{YF} \right\} - \left\{ J^{FD}, J^{EW}, J^{YA} \right\} - \left\{ J^{EA}, J^{DW}, J^{YF} \right\} \right) \right. \\
& \left. \quad + \eta^{AC} \left( \left\{ J^{FB}, J^{EW}, J^{YD} \right\} + \left\{ J^{ED}, J^{BW}, J^{YF} \right\} - \left\{ J^{FD}, J^{EW}, J^{YB} \right\} - \left\{ J^{EB}, J^{DW}, J^{YF} \right\} \right) \right) \\
& \quad + (ABCDEF \rightarrow CDEFab) + (ABCDEF \rightarrow EFabCD) \\
& \end{align*}
\]

(2.9)

Then rewrite (2.9) in terms of anti-commutators as

\[
\begin{align*}
& = 4 \left[ \left( \eta^{DB} M^{EFCA} - (A \leftrightarrow B) \right) \left( (C \leftrightarrow D) \right) \right. \\
& \left. - \left( \delta^W - 6 \right) \left( \eta^{EC} J^{FA} + \eta^{FA} J^{EC} - \eta^{EA} J^{FC} - \eta^{FC} J^{EA} \right) \right. \\
& \left. + (ABCDEF \rightarrow CDEFab) + (ABCDEF \rightarrow EFabCD) \right] \\
& \end{align*}
\]

(2.10)

The terms proportional to \( \left( \delta^W - 6 \right) \) will make zero contribution in (2.10) when all the permutations are performed, so we drop them. The identities (2.9) and (2.10) hold for any number of sites, but we are interested to prove they vanish on one site, i.e. the Serre condition (2.8) becomes

\[
\begin{align*}
& \left( \left( \eta^{DB} M^{EFCA} - (A \leftrightarrow B) \right) \left( (C \leftrightarrow D) \right) \right. \\
& \left. + (ABCDEF \rightarrow CDEFab) + (ABCDEF \rightarrow EFabCD) \right) = 0 \\
& \end{align*}
\]

(2.11)

where the single site tensor \( M^{EFCA} \) is defined as

\[
M^{EFCA} = \left( J^{FA}_1 J^{EW}_1 J^{YC}_1 \right) + J^{EC}_1 \left( J^{AW}_1 J^{YF}_1 \right) - J^{FC}_1 \left( J^{EW}_1 J^{YA}_1 \right) - J^{EA}_1 \left( J^{CW}_1 J^{YF}_1 \right) \eta_{WY} \\
\]

(2.12)
3 Proof of the Serre relation for the differential operator representation

We consider the Serre relation for the SO(2, n) Yangian, in the differential operator representation in momentum space, which is relevant for scalar and spin one gauge fields with conformal spin $d$ in $n$ spacetime dimensions. In this section we will prove that (2.11) is satisfied for off-shell scalar fields with arbitrary $n$ and $d$. For gauge fields, we prove the Serre relation (2.11) is only satisfied for $n = 4$, $d = 1$, and on shell. All fields are massless.

The differential operator representation for scalar and gauge fields in momentum space is [14, 15]

$$
P^\mu_i = k^\mu_i, \quad L_i^{\mu\nu} = k^\mu_i \partial^\nu_i - k^\nu_i \partial^\mu_i + \Sigma^\mu\nu, \quad D_i = d + k^\mu_i \partial_\mu,$$

(3.1)

for each site $i$, where

$$
J_i^{\mu\nu} = L_i^{\mu\nu}, \quad J_i^{\mu} = \frac{1}{2} (P_i^\mu - K_i^\mu), \quad J_i^{n+1,\mu} = \frac{1}{2} (P_i^\mu + K_i^\mu), \quad J_i^{n,n+1} = D_i
$$

(3.2)

The representation depends on the momenta $k^\mu_i, 1 \leq i \leq N, 0 \leq \mu \leq n-1$, their derivatives $\partial_\mu = \frac{\partial}{\partial k^\mu}$, and a real free parameter $d$. For scalar fields $\Sigma^{\mu\nu} = 0$. For gauge fields $A^\mu_i$, then $\Sigma^{\mu\nu} = (\delta^\mu_\alpha \delta^\nu_\gamma - \delta^\mu_\gamma \delta^\nu_\alpha)$. So $L_i^{\mu\nu} A^\mu_i = (k^\mu_i \partial^\nu_i - k^\nu_i \partial^\mu_i) A^\mu_i + (\delta^\mu_\alpha \delta^\nu_\gamma - \delta^\mu_\gamma \delta^\nu_\alpha) A^\nu_i$. See also appendix A.

To prove the Serre relation for the representation (3.1), first we write the level zero generators $J^{AB}$ and their anticommutator $S^{AD} \equiv \{J^{AB}, J^{CD}\} \eta_{BC}$, on a single site, in terms of a smaller set of operators, $\kappa^A, V^A$ and $\Sigma^{AB}$,

$$
J^{AB} = \kappa^A V^B - \kappa^B V^A + \Sigma^{AB}
$$

$$
S^{AD} = -2 \kappa^B \kappa_B V^A V^D - 2 d \eta^{AD} - 2 \left(d - \frac{n-2}{2}\right) (\kappa^A V^D + \kappa^D V^A)
$$

$$
- 2 \kappa^B V^A \Sigma^{CD} \eta_{BC} - 2 \kappa^B V^D \Sigma^{CA} \eta_{BC} + \Sigma^{AB} \Sigma^{CD} \eta_{BC} + \Sigma^{CD} \Sigma^{AB} \eta_{BC}
$$

(3.3)

where

$$
\kappa^A = (k^\mu, k^n, k^{n+1}), \quad \kappa^\mu = P^\mu = k^\mu, \quad k^n = -(d + k^\rho \partial_\rho) = -k^{n+1}
$$

$$
V^A = (V^\mu, V^n, V^{n+1}), \quad V^\mu = \partial^\mu, \quad V^n = \frac{1}{2} (1 + \partial^\rho \partial_\rho), \quad V^{n+1} = \frac{1}{2} (1 - \partial^\rho \partial_\rho)
$$

$$
\Sigma^{AB} = \delta_A^\mu \delta_B^\nu \Sigma^{\mu\nu} + \delta_A^\mu (\eta^{Bn} + \eta^{B,n+1}) \Sigma^{\mu\rho} V_\rho - \delta_B^\mu (\eta^{An} + \eta^{A,n+1}) \Sigma^{\nu\rho} V_\rho
$$

(3.4)

satisfy a simpler algebra

$$
[k^A, k^B] = \epsilon^{AB} D^C_k, \quad [V^A, V^B] = 0, \quad [k^A, V^B] = -\eta^{AB} + \epsilon^{AB} D^V,
$$

$$
\epsilon^{AB} D^C = -\delta^A_D (\eta^{Bn} + \eta^{B,n+1}) + \delta^B_D (\eta^{An} + \eta^{A,n+1}) = -\epsilon^{BA} D^C
$$

$$
\epsilon^{AB} D^V = -\delta^A_D (\eta^{Bn} + \eta^{B,n+1}) - \delta^B_D (\eta^{An} + \eta^{A,n+1}) = \epsilon^{BA} D^V
$$

$$
[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = -\eta^{\mu\rho} \Sigma^{\nu\sigma} - \eta^{\mu\sigma} \Sigma^{\nu\rho} + \eta^{\nu\rho} \Sigma^{\mu\sigma} + \eta^{\nu\sigma} \Sigma^{\mu\rho}, \quad [\Sigma^{\mu\nu}, k^A] = [\Sigma^{\mu\nu}, V^A] = 0
$$

(3.5)
To prove (2.11) we consider the scalar and gauge cases separately. For the scalar field, $\Sigma^{AB} = \Sigma^{\mu\nu} = 0$, so (3.3) becomes

$$J^{AB} = \kappa^A V^B - \kappa^B V^A,$$

$$S^{AD} = -2k^R\kappa_B V^A V^D - 2d \eta^{AD} - 2 \left(d - \frac{n}{2}\right) (\kappa^A V^D + \kappa^D V^A) \quad (3.6)$$

where for the remainder of this section all operators are at a single site, but we suppress the single site notation. To construct $M^{EFCA}$, first compute

$$-J^{EA}(\kappa^C V^F + \kappa^F V^C) = -(\kappa^E V^A - \kappa^A V^E) (\kappa^C V^F + \kappa^F V^C)$$

$$= -\kappa^E \kappa^C V^A V^F - \kappa^E \kappa^F V^A V^C + \kappa^A \kappa^C V^E V^F + \kappa^A \kappa^F V^E V^C$$

$$- \kappa^E [V^A, \kappa^C] V^F - \kappa^E [V^A, \kappa^F] V^C$$

$$+ \kappa^A [V^E, \kappa^C] V^F + \kappa^A [V^E, \kappa^F] V^C \quad (3.7)$$

and combining as in the four tensor

$$(-J^{EA}(\kappa^C V^F + \kappa^F V^C) - A \leftrightarrow C) - E \leftrightarrow F$$

$$= -[\kappa^E, \kappa^C] V^A V^F + [\kappa^A, \kappa^E] V^E V^C + [\kappa^E, \kappa^A] V^C V^F - [\kappa^C, \kappa^F] V^E V^A$$

$$- J^{EC} [V^A, \kappa^E] + J^{AF} [V^E, \kappa^C] + J^{EA} [V^C, \kappa^E] - J^{CF} [V^E, \kappa^A]$$

$$= -J^{EC} \eta^{AF} + J^{AF} \eta^{EC} + J^{EA} \eta^{CF} - J^{CF} \eta^{EA}$$

$$+ (\eta^{E,n} + \eta^{E,n+1}) 2J^{AC} V^F - (\eta^{F,n} + \eta^{F,n+1}) 2J^{AC} V^E$$

$$- (\eta^{C,n} + \eta^{C,n+1}) 2J^{FE} V^A + (\eta^{A,n} + \eta^{A,n+1}) 2J^{FE} V^C \quad (3.8)$$

From the first term in $S^{AD}$, we have $\kappa^2 = \kappa_B \kappa_B = k^2$,

$$-J^{EA} \kappa^2 V^C V^F = -\kappa^2 J^{EA} V^C V^F - [J^{EA}, \kappa^2] V^C V^F$$

$$= -\kappa^2 J^{EA} V^C V^F - (\eta^{E,n} + \eta^{E,n+1}) \left(2\kappa^2 V^A + 2 \left(d - \frac{n}{2}\right) \kappa^A\right) V^C V^F$$

$$+ (\eta^{A,n} + \eta^{A,n+1}) \left(2\kappa^2 V^F + 2 \left(d - \frac{n}{2}\right) \kappa^E\right) V^C V^F \quad (3.9)$$

with the permutations

$$(-J^{EA} \kappa^2 V^C V^F - A \leftrightarrow C) - E \leftrightarrow F$$

$$= \left(- [J^{EA}, \kappa^2] V^C V^F - A \leftrightarrow C\right) - E \leftrightarrow F$$

$$= -2 \left(d - \frac{\delta^r}{2}\right) \left(\eta^{E,n} + \eta^{E,n+1}\right) J^{AC} V^F - (\eta^{F,n} + \eta^{F,n+1}) J^{AC} V^E$$

$$- (\eta^{C,n} + \eta^{C,n+1}) J^{FE} V^A + (\eta^{A,n} + \eta^{A,n+1}) J^{FE} V^C \quad (3.10)$$

Then combining (3.6), (3.8), (3.10), we find the tensor

$$M^{EFCA} = -J^{EA} S^{CF} - J^{FC} S^{EA} + J^{EC} S^{AF} + J^{FA} S^{EC}$$

$$= (n - 2) \left(J^{EA} \eta^{CF} + J^{FC} \eta^{EA} - J^{EC} \eta^{AF} - J^{FA} \eta^{EC}\right) \quad (3.11)$$
Substituting this form for $M^{EFC\alpha}$ into (2.11) gives a vanishing result due to the sum over various permutations. So we have proved the Serre relation for scalar fields for arbitrary $k^2$, $d$ and $n$, i.e. off-shell, for any conformal dimension $d$ and spacetime dimension $n$.

For gauge fields, the proof of (2.11) is more complicated. The anti-commutator $S^{\alpha\delta\eta}_{}$ has additional terms, from (3.3), which can be simplified using the commutation relations (3.5),

$$-2\kappa B V^A \Sigma^{CD} \eta_{BC} - 2\kappa B V^D \Sigma^{CA} \eta_{BC} + \Sigma^{AB} \Sigma^{CD} \eta_{BC} + \Sigma^{CD} \Sigma^{AB} \eta_{BC}$$

$$= \left( -2V^A \kappa_{\mu} \left( \delta^D_{\mu} + (\delta^D_{n+1} - \delta^D_{n})V_{\mu} \right) \Sigma^{\mu\rho} + \Sigma^{\alpha\delta} \Sigma^{\nu\eta} \eta_{\mu\nu} \right) + A \leftrightarrow D \quad (3.12)$$

With the $\alpha, \gamma$ subscripts on $\Sigma^{\mu\nu}_{\alpha\gamma}$ displayed explicitly, (3.12) becomes

$$= \left( -2k_{\alpha} V^A (\delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma}) \right.$$

$$+ (n - 4) \left( (\delta^A_{\alpha} + (\eta^A + \eta^{A,n+1})V_{\alpha}) \left( \delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma} \right) \right.$$

$$- 2V^A \left( \delta^D_{\mu} + (\delta^D_{n+1} - \delta^D_{n})V_{\mu} \right) \kappa_{\gamma}$$

$$+ (\delta^A_{\delta} + (\delta^A_{n+1} - \delta^A_{n})V_{\delta}) \left( \delta^D_{\delta'} + (\delta^D_{n+1} - \delta^D_{n})V_{\delta'} \right) \eta^{\delta\delta'} \eta_{\alpha\gamma} \bigg) + A \leftrightarrow D \quad (3.13)$$

where we used the anti-commutator

$$\Sigma^{A\mu} \Sigma^{\nu\delta} \eta_{\mu\nu} + A \leftrightarrow D = 2 \left( (\delta^A_{\alpha} + (\delta^A_{n+1} - \delta^A_{n})V_{\alpha}) \left( \delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma} \right) \right.$$

$$+ (n - 2) \left( (\delta^A_{\alpha} + (\delta^A_{n+1} - \delta^A_{n})V_{\alpha}) \left( \delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma} \right) \right.$$

$$+ (\delta^A_{\alpha} + (\delta^A_{n+1} - \delta^A_{n})V_{\alpha}) \left( \delta^A_{\gamma} + (\delta^A_{n+1} - \delta^A_{n})V_{\gamma} \right) \bigg) \quad (3.14)$$

We reduce (3.13) further using

$$V^A \left( \delta^D_{n+1} - \delta^D_{n} \right) + V^D \left( \delta^A_{n+1} - \delta^A_{n} \right) - (\delta^D_{\alpha} + (\delta^A_{n+1} - \delta^A_{n})V_{\alpha}) \left( \delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma} \right) \eta^{\delta\delta'}$$

$$= -\delta^A_{n+1} \delta^D_{n+1} + \delta^A_{n} \delta^D_{n} - \delta^A_{\delta} \delta^\gamma_{\gamma} \eta^{\delta\delta'} = -\eta^{AD} \quad (3.15)$$

and combine it with the $\Sigma^\gamma$ independent terms in (3.3) to evaluate the anti-commutator for the gauge field representation as

$$S^{\alpha\eta} = \left( -2k_{\alpha} V^A \left( \delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma} \right) \right.$$

$$+ (n - 4) \left( (\delta^A_{\alpha} + (\eta^A + \eta^{A,n+1})V_{\alpha}) \left( \delta^D_{\gamma} + (\delta^D_{n+1} - \delta^D_{n})V_{\gamma} \right) \right.$$

$$+ 2V^A \left( \delta^D_{\alpha} + (\delta^D_{n+1} - \delta^D_{n})V_{\alpha} \right) k_{\gamma}$$

$$+ \eta_{\alpha\gamma} \left( -k^2 V^A V^D - (d - 1) \eta^{AD} - 2 \left( \frac{d - n - 2}{2} \right) \kappa^A V^D \right) \bigg) + A \leftrightarrow D \quad (3.16)$$
To construct the four tensor $M^{EFA}$, we first compute from (3.16) the product,

$$-J^{EA}_2 S^{CF}_\alpha \eta^{\beta'} = -J^{EA}_\alpha S^{CF}_\beta \eta^{\beta'}$$

$$= \left(2 J^{EA}_\alpha k^\beta V_C (\delta^F_\gamma + (C^F_n + \delta^{F,n+1}_\gamma) V_\gamma) - (n - 4) J^{EA}_\alpha \left( \eta^{C\beta'} + (\eta^{C_n + \eta^{C,n+1}} V_\beta') \right) (\delta^F_\gamma + (\eta^{F_n + \eta^{F,n+1}}) V_\gamma) - 2 J^{EA}_\alpha V_C \left( \eta^{E\beta'} + (\eta^{E_n + \eta^{E,n+1}} V_\beta') \right) k_\gamma \right) + (d - 1) J^{EA}_{\alpha\gamma} \eta^{CF} + 2 \left( d - \frac{n - 2}{2} \right) J^{EA}_{\alpha\gamma} \kappa^C V_F^2 + C \leftrightarrow F$$

Finally, with the use of

$$J^{EA}_{\alpha\beta} k_\beta = [J^{EA}_{\alpha\beta}, k^\beta] + k^\beta J^{EA}_{\alpha\beta}$$

$$= k_\alpha \left( (\kappa^E + \eta^{E_n + \eta^{E,n+1}} V^A - (\kappa^A + \eta^{A_n + \eta^{A,n+1}}) V^E \right) + (d + 1 - n) \left( \delta^A_\alpha (\eta^{E_n + \eta^{E,n+1}} - \delta^A_\alpha (\eta^{A_n + \eta^{A,n+1}}) \right)$$

$$J^{EA}_\alpha k^2 = [J^{EA}_\alpha, k^2] + k^2 J^{EA}_\alpha$$

$$= 2(\eta^{E_n + \eta^{E,n+1}} (k_\alpha \delta^A_\gamma - \delta^A_\alpha k_\gamma) - 2(\eta^{A_n + \eta^{A,n+1}} (k_\alpha \delta^E_\gamma - \delta^E_\alpha k_\gamma)$$

$$\eta_{\alpha\gamma} (\eta^{E_n + \eta^{E,n+1}} (\kappa^E A + 2 \left( d - \frac{n - 2}{2} \right) \kappa^A)$$

$$- (\eta^{A_n + \eta^{A,n+1}} (2k^2 V^2 + 2 \left( d - \frac{n - 2}{2} \right) \kappa^E)$$

$$+ k^2 J^{EA}_{\alpha\gamma}$$

$$J^{EA}_{\alpha\gamma} k^2 V^C V^F = k_\alpha 2 \left( \delta_\gamma^A (\eta^{E_n + \eta^{E,n+1}} - \delta_\gamma^E (\eta^{A_n + \eta^{A,n+1}}) \right) V^C V^F$$

$$- 2 \left( \delta_\alpha^A (\eta^{E_n + \eta^{E,n+1}} - \delta_\alpha^E (\eta^{A_n + \eta^{A,n+1}}) \right) V^C V^F k_\gamma$$

$$+ 2 \left( \delta_\alpha^A (\eta^{E_n + \eta^{E,n+1}} - \delta_\alpha^E (\eta^{A_n + \eta^{A,n+1}}) \right) \left( (\delta^C_\gamma + (\eta^{C_n + \eta^{C,n+1}}) V_\gamma \right) V^C \left( \delta^C_\gamma + (\eta^{F_n + \eta^{F,n+1}} V_\gamma \right)$$

$$+ \eta_{\alpha\gamma} (\eta^{E_n + \eta^{E,n+1}} (\kappa^E A + 2 \left( d - \frac{n - 2}{2} \right) \kappa^A)$$

$$- (\eta^{A_n + \eta^{A,n+1}} (2k^2 V^2 + 2 \left( d - \frac{n - 2}{2} \right) \kappa^E)$$

$$+ k^2 J^{EA}_{\alpha\gamma} \right) V^C V^F$$

the product becomes

$$-J^{EA}_2 S^{CF}_\alpha = \left[ k_\alpha 2 \left( (n^E + \eta^{E_n + \eta^{E,n+1}} V^A - (\kappa^A + \eta^{A_n + \eta^{A,n+1}}) V^E \right)$$

$$+ (2(d + 1 - n) + 4) \left( \delta_\alpha^A (\eta^{E_n + \eta^{E,n+1}} - \delta_\alpha^E (\eta^{A_n + \eta^{A,n+1}}) \right)$$

$$- 7 -$$
\[ \cdot \left( (\delta^C + (\eta^C + \eta^{C,n+1}) V_\gamma) V^F + V^C \left( \delta^F + (\eta^F + \eta^{F,n+1}) V_\gamma \right) \right) \\
+ k_\alpha 4 \left( \delta^A (\eta^E + \eta^{E,n+1}) - \delta^E (\eta^A + \eta^{A,n+1}) \right) V^C V^F \\
-4 \left( \delta^A (\eta^E + \eta^{E,n+1}) - \delta^E (\eta^A + \eta^{A,n+1}) \right) V^C V^F \kappa_\gamma \\
+2 \left[ \eta_{\alpha\gamma} \left( (\eta^E + \eta^{E,n+1}) \left( 2k^2 V^A + 2 \left( d - \frac{n-2}{2} \right) \kappa^A \right) \right) \right] \\
- (\eta^A + \eta^{A,n+1}) \left( 2k^2 V^E + 2 \left( d - \frac{n-2}{2} \right) \kappa^E \right) \right) + k^2 J_{\alpha\gamma}^E \right] V^C V^F \\
+ \left[ \left( - (n-4) J_{\alpha\beta}^E \left( \eta^C + (\eta^C + \eta^{C,n+1}) V^C \left( \delta^F + (\eta^F + \eta^{F,n+1}) V_\gamma \right) \right) \right] \\
-2 J_{\alpha\beta}^E V^C \left( \eta^F + (\eta^F + \eta^{F,n+1}) V^F \right) \kappa_\gamma + (d-1) J_{\alpha\gamma}^E \eta^C V^F \\
+2 \left( d - \frac{n-2}{2} \right) J_{\alpha\gamma}^E \kappa^C V^F ) + C \leftrightarrow F \right] \\
\right) \right) (3.19) \]

We see that (3.19) does not lead to a four tensor (2.12) that will satisfy the Serre relation for arbitrary \( k^2, d, n \). But for \( k^2 = 0, d = 1, n = 4 \), the product reduces to

\[ -J^{EA} S^{CF} = 2k_\alpha \left[ \left( \kappa^E + \eta^E + \eta^{E,n+1} \right) V^A - \left( \kappa^A + \eta^A + \eta^{A,n+1} \right) V^E \right] \]

\[ \cdot \left[ \left( \eta^C + (\eta^C + \eta^{C,n+1}) V_\gamma \right) V^F + V^C \left( \eta^F + \eta^{F,n+1} \right) V_\gamma \right] \]

\[ + 4k_\alpha \left[ \delta^A (\eta^E + \eta^{E,n+1}) - \delta^E (\eta^A + \eta^{A,n+1}) \right] V^C V^F \\
-4 \left[ \delta^A (\eta^E + \eta^{E,n+1}) - \delta^E (\eta^A + \eta^{A,n+1}) \right] V^C V^F \kappa_\gamma \\
-2 J_{\alpha\beta}^E \left[ V^C \left( \eta^F + \eta^{F,n+1} \right) V^F \right] + C \leftrightarrow F \right) \kappa_\gamma \right) \]

(3.20)

which we recognize as a gauge transformation.

So \( J^{EA} S^{CF} = 0 \) when acting on on-shell gauge amplitudes since they are gauge invariant. (3.20) is a single site expression, but it holds for any site. The gauge invariance of on-shell amplitudes provides \( k_{\gamma_i} A^{\gamma_1 \cdots \gamma_N}_{\alpha_1 \cdots \alpha_N} (k_1, \ldots, k_N) = 0 \) for any \( i, 1 \leq i \leq N \). For example, if we consider (3.20) at site one, \( k_{\gamma_1} A^{\gamma_1 \cdots \alpha_N}_{\alpha_1 \cdots \gamma_N} (k_1, \ldots, k_N) = 0 \) will cause the terms in (3.20) proportional to \( k_{\gamma_1} \) to vanish. The terms proportional to \( k_{\gamma_1} \) will vanish upon multiplication of \( J^{EA} S^{CF} \) by \( e^\alpha (k_1) \), since the polarizations are transverse \( k \cdot \epsilon (k) = 0 \). That is to say

\[ e^\alpha (J^{EA} S^{CF})_{\alpha\gamma} A^{\gamma_1 \cdots}_{\alpha_N} = 0 \]

(3.21)

which implies \( M^{EFCA} \) vanishes on the amplitudes, and proves the Serre condition (2.11) for the gauge field representation (3.1) of the conformal Yangian on-shell when \( n = 4, d = 1 \).

This restriction to fields satisfying their free field equations of motion for representation theory is familiar from earlier discussions of the conformal group [16].

Proof of the Serre relation in the context of the \( SU(N) \) Yangian and the \( PSU(2,2|4) \) Yangian was given in [5] using tensor operator methods. These methods also occur in [17].
In this paper we emphasize that the SO(2, $n$) Yangian gauge field representation only has a consistent Serre relation for on-shell fields and for $n = 4$, $d = 1$. In contrast, the scalar field representation is consistent off shell, for arbitrary $n$ and conformal dimension $d$.

4 Yangian generators and the scalar tree amplitudes

In this section, we compute how the SO(2, $n$) Yangian generators act on the kinematic invariants and the $\lambda d^3$ Feynman tree graphs. In order to deal with the freedom in the amplitudes introduced by momentum conservation, for arbitrary dimension $n$ we will consider the amplitudes for the scalar theory as functions of the $k^2_{[I,J]}$, for $1 \leq I < J < N$, but not $k^2_{[1,N-1]}$, where the consecutive invariants are $k^2_{[I,J]} \equiv (k_I + k_{I+1} + \ldots + k_N)^2$ as in [8].

This set of invariants transforms in a simple way under the level one generators $\hat{P}^\mu$, that is to say merely with a multiplicative factor,

$$-\hat{P}^\mu = \sum_{1 \leq i < j \leq N} \left( P_i^\mu D_j + P_i^\rho L_j^{\mu \rho} - (i \leftrightarrow j) \right),$$

$$-\hat{P}^\mu \ k^2_{[I,J]} = \left[ -2d \sum_{j=1}^N j k^2_j \right] + 2(k_1^\mu + k_2^\mu + \ldots + k_{j-1}^\mu - k_{j+1}^\mu - k_{j+2}^\mu - \ldots - k_N^\mu) k^2_{[I,J]}$$ (4.1)

where we have assumed momentum conservation $\sum_{i=1}^N k_i^\mu = 0$. Apart from the common term proportional to $d$ that can be extracted straightforwardly, we prove this as follows.

$$\sum_{1 \leq i < j \leq N} (P_i^\mu \ k_j \cdot \partial_j + P_i^\rho L_j^{\mu \rho} - (i \leftrightarrow j)) \ k^2_{[I,J]}$$

$$= \sum_{i=1}^J \left[ (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N)^\mu k_i \cdot \partial_i + (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N)^\rho (k_i^\rho \partial_i^\mu - k_i^\rho \partial_i^\mu) \right] \ k^2_{[I,J]}$$

$$= 2 \sum_{i=1}^J \left[ (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N)^\mu k_i \cdot k_{[I,J]} + k_i^\mu (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N) \cdot k_{[I,J]} - k_i^\mu (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N) \cdot k_i \right]$$ (4.2)

We simplify each of the terms in (4.2), where the third term becomes

$$-2 \sum_{i=1}^J k_i^\mu (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N) \cdot k_i$$

$$= -2 k_{[I,J]}^\mu (k_1 + k_2 + \ldots + k_{j-1} - k_{j+1} - k_{j+2} - \ldots - k_N) \cdot k_{[I,J]}$$ (4.3)
and the first term contains the answer (4.1) plus a remainder,

\[
2 \sum_{i=1}^{J} (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N)^\mu k_i \cdot k_{[I,J]}
\]

\[
= 2(k_1 + k_2 + \ldots + k_{I-1} - k_{J+1} - k_{J+2} - \ldots - k_N)^\mu k_{[I,J]}^2
\]

\[
+ 2(-k_{I+1} - k_{I+2} - \ldots - k_J)^\mu k_I \cdot k_{[I,J]} + \ldots + 2(k_I + \ldots
\]

\[
+ k_{J-1})^\mu k_J \cdot k_{[I,J]}
\]

(4.4)

Adding this remainder to (4.3) and the second term in (4.2), we find the cancellation

\[
(-k_{I+1} - k_{I+2} - \ldots - k_J)^\mu k_I \cdot k_{[I,J]} + (k_I - k_{I+2} - k_{I+3} - \ldots - k_J)^\mu k_{I+1} \cdot k_{[I,J]}
\]

\[
+ \ldots + (k_I + \ldots + k_{J-1})^\mu k_J \cdot k_{[I,J]} + \sum_{i=1}^{J} k_i^\mu (k_1 + \ldots + k_{i-1} - k_{i+1} - \ldots - k_N) \cdot k_{[I,J]}
\]

\[
-k_{[I,J]}^\mu (k_1 + k_2 + \ldots + k_{J-1} - k_{J+1} - k_{J+2} - \ldots - k_N) \cdot k_{[I,J]} = 0
\]

(4.5)

by identifying the coefficient of each \( k_i^\mu, I \leq i \leq J \) to be zero in (4.5). Here the consecutive momenta are \( k_{[I,J]}^\mu = (k_I + k_{I+1} + \cdots + k_J)^\mu \). This proves (4.1) which says that \( \tilde{P}^\mu \) acts on off-shell invariants simply as multiplication by a factor, somewhat similar to how the level zero generator \( P^\mu \) acts.

Since any \( \lambda \delta^3 \) Feynman tree graph is just the inverse product of \((N-3)\) invariants, and the level one generator \( \tilde{P}^\mu \) is a first order differential operator, it is now easy to show how \( \tilde{P}^\mu \) acts on the graphs.

We describe an off-shell \( \lambda \delta^3 \) tree graph by

\[
\delta^\mu(k_1 + \ldots + k_N) \ A_N^\Delta(k_1, \ldots, k_N)
\]

(4.6)

where \( \Delta \) denotes a subset of the off-shell invariants,

\[
A_N^\Delta(k_1, \ldots, k_N) = (-1)^{N+1} \prod_{k_{[I,J]}^2 \in \Delta} \frac{1}{k_{[I,J]}^2}
\]

(4.7)

for example for \( N = 4 \), \( \Delta \) is either \( k_{[1,2]}^2 \) or \( k_{[2,3]}^2 \). For \( N = 5 \), \( \Delta \) is any of the five sets \( k_{[1,2]}^2, k_{[1,3]}^2, k_{[1,4]}^2, k_{[2,3]}^2, k_{[2,4]}^2, k_{[3,4]}^2, k_{[2,5]}^2 \). These correspond to the two \( N = 4 \) graphs \( \frac{1}{s_{12}}, \frac{1}{s_{23}} \) and the five \( N = 5 \) graphs \( \frac{1}{s_{12} s_{13}}, \frac{1}{s_{13} s_{14}}, \frac{1}{s_{23} s_{24}}, \frac{1}{s_{24} s_{25}}, \frac{1}{s_{34} s_{35}} \). We can find the subsets for \( N > 5 \) from the off-shell recurrence relation [8],

\[
A_N^\Delta(k_1, \ldots, k_N) = -\frac{1}{s_{34}} A_{N-1}^\Delta(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)
\]

(4.8)

where the momenta can be cycled to find all the \( \Delta \).
The level one generator $\hat{P}^\mu$ acts as
\begin{align}
-\hat{P}^\mu & \delta^n(k_1 + \ldots + k_N) A_N^\lambda(k_1, \ldots, k_N) \\
& = \delta^n(k_1 + \ldots + k_N) \\
& \cdot \left[ -2d \sum_{j=1}^N j k_j^\mu \right] - 2 \sum_{[i,J]} \left( k_{[i,J]}^\mu - k_{[J+1,N]}^\mu \right) A_N^\lambda(k_1, \ldots, k_N)
\end{align}
(4.9)
where $\hat{P}^\mu$ commutes through $\delta^n(k_1 + \ldots + k_N)$ as in appendix B.

Clearly the level one generator does not annihilate the graph, and therefore the conformal Yangian is not a symmetry of the $\lambda\phi^3$ scalar theory. But the multiplicative factor is a sum of momenta $k_i^\mu$ with various coefficients. So we can define a set of evaluation parameters $c_{N,j}^\Delta$ for each individual graph,
\begin{align}
c_{N,j}^\Delta &= -2d_j - 2 \sum_{[i,J]} \left\{ 1 \quad \text{if } j \in [1,I-1] \\
& \quad -1 \quad \text{if } j \in [J+1,N] \right. \\
\hat{P}^\mu & \equiv \hat{P}^\mu + \sum_{j=1}^N c_{N,j}^\Delta P_j^\mu, \quad \hat{P}^\mu \delta^n(k_1 + \ldots + k_N) A_N^\lambda(k_1, \ldots, k_N) = 0
\end{align}
(4.10)
such that the level one generator shifted by the corresponding level zero generator annihilates the graph. Defining all shifted generators using the same parameters $c_{N,j}^\Delta$
\begin{align}
\hat{J}^{AB} & \equiv \hat{J}^{AB} + \sum_{j=1}^N c_{N,j}^\Delta J_j^{AB}
\end{align}
(4.11)
it is straightforward to show that if the $c_{N,j}^\Delta$ commute with $J^{AB}$, $\hat{J}^{AB}$ (which they do in this case because the parameters are constants), and if $J^{AB}$, $\hat{J}^{AB}$ satisfy the defining relations (2.2), (2.3), then so do $J^{AB}$, $\hat{J}^{AB}$. If we have further that all the level zero generators $J^{AB}$ annihilate the graph, then (4.10) implies
\begin{align}
\hat{J}^{AB} \delta^n(k_1 + \ldots + k_N) A_N^\lambda(k_1, \ldots, k_N) = 0
\end{align}
(4.12)
This follows from the values of the conformal structure constants (2.4). See also [18–20].

In $\lambda\phi^3$ theory, SO(2, $n$) is known to be a symmetry of the classical Lagrangian in six dimensions, $n = 6$, where the field $\phi$ has conformal dimension $d = 2$. We can also check directly using the representation (3.1), (3.2) that all tree graphs satisfy
\begin{align}
J^{AB} \delta^n(k_1 + \ldots + k_N) A_N^\lambda(k_1, \ldots, k_N) = 0
\end{align}
(4.13)
For $P^\mu$, $L^\mu$ this follows from momentum conservation, and the Lorentz scalar nature of the kinematic invariants. For the dilatation generator $D = \sum_{i=1}^N (d + k_i \cdot \partial_i)$, we see from the recurrence relation (4.8) and the explicit forms for $A_4^\Delta$, $A_5^\Delta$, that $\sum_{i=1}^N k_i \partial_i A_N^\lambda(k_1, \ldots, k_N) =$
$$D \delta^n(k_1 + \ldots + k_N) A_N^\lambda(k_1, \ldots, k_N)$$

$= (N(d - 2) - n + 6) \delta^n(k_1 + \ldots + k_N) A_N^\lambda(k_1, \ldots, k_N)$  (4.14)

where $D$ moves through the delta function as $D \delta^n(k_1 + \ldots + k_N) = \delta^n(k_1 + \ldots + k_N)(-n + D)$. Clearly (4.14) selects $n = 6$, $d = 2$ to kill the graphs. Lastly the special conformal transformations

$$K^\mu = \sum_{i=1}^N K_i^\mu = \sum_{i=1}^N (2d\partial_i^\mu + 2k_i \cdot \partial_i^\mu - k_i^\mu \partial_i^\rho \partial_\rho)$$  (4.15)

commute with the delta function as do $P^\mu, L^\mu$. The annihilation $K^\mu A_N^X(k_1, \ldots, k_N) = 0$ results from the recurrence relation as follows. If we can show

$$(K_3^\mu + K_4^\mu) \left( -\frac{1}{s_{34}} \right) A_N^\lambda(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)$$

$= -\frac{1}{s_{34}} K_{3+4}^\mu A_{N-1}^\lambda(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)$  (4.16)

then

$$K^\mu \left( -\frac{1}{s_{34}} \right) A_{N-1}^\lambda(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)$$

$= -\frac{1}{s_{34}} (K_1 + K_2 + K_{3+4} + K_5 + \ldots + K_N)^\mu A_{N-1}^\lambda(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)$  (4.17)

which implies

$$K^\mu A_N^X(k_1, \ldots, k_N) = -\frac{1}{s_{34}} (K_1 + K_2 + K_{3+4} + K_5 + \ldots + K_N)^\mu A_{N-1}^X(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N) = 0$$  (4.18)

whenever $A_{N-1}^X$ is annihilated by its relevant special conformal generators. This is explicitly true for $N = 4, 5$, so the annihilation for higher $N$ follows iteratively.

To show (4.16), where $K_{3+4}^\mu \equiv (2d\partial_{3+4}^\mu + 2(k_3 + k_4) \cdot \partial_{3+4}^\mu - (k_3 + k_4)^\mu \partial_{3+4}^\rho \partial_{\rho,3+4})$, we find, using $\partial_{3+4}^\mu \equiv (d - 4)\partial_{3+4}^\mu - \frac{8d - 2n - 4}{s_{34}} (k_3 + k_4)^\mu \partial_{3+4}^\rho \partial_{\rho,3+4}$

$$(K_3^\mu + K_4^\mu) \left( -\frac{1}{s_{34}} \right) A_{N-1}^\lambda(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)$$

$= -\frac{1}{s_{34}} [(4d - 4)\partial_{3+4}^\mu - \frac{(8d - 2n - 4)(k_3 + k_4)^\mu}{s_{34}} + 2(k_3 + k_4) \cdot \partial_{3+4}^\rho \partial_{\rho,3+4}^{-} - (k_3 + k_4)^\mu \partial_{\rho,3+4}^\rho \partial_{\rho,3+4}] A_{N-1}^\lambda(k_1, k_2, k_3 + k_4, k_5, \ldots, k_N)$  (4.19)

which gives (4.16) for $n = 6, d = 2$.

So any tree graph for $\lambda \phi^3$ theory is annihilated by the SO(2, 6) Yangian where the representation of the level one generators $J^{A\dot{A}}$ depend on the graph. We look next briefly at the CHY formalism for these graphs, where the Yangian generators act solely through the scattering polynomials.
5 Yangian properties of the scattering polynomials

In the CHY formalism [1–3], off-shell tree level amplitudes can be described as contour integrals encircling the zeros of the scattering polynomials [8]. These polynomials are functions of the variables \(z_1, \ldots, z_N\) with coefficients given by the set of off-shell invariants \(k^2_{[I,J]}\), \(1 \leq I < J \leq N\) discussed in the previous section. The momentum dependence of \(\lambda \phi^3\) amplitudes is entirely via the polynomials. Thus the action of the Yangian generators on the scalar amplitudes employs how the Yangian acts on the scattering polynomials.

The Yangian transformation properties of the scattering polynomials could also illuminate further properties of the scattering equations.

The off-shell scattering polynomials \(h_m^N\), \(1 \leq m \leq N - 3\) are [8]

\[
h_m^N = \sum_{J=2}^{N-2} k^2_{[1,J]} (z_J - z_{J+1}) \Pi_{[1,J]}^{m-1} - \sum_{J=3}^{N-1} k^2_{[2,J]} (z_J - z_{J+1}) \Pi_{[2,J]}^{m-1} + \sum_{3 \leq I < J \leq N} k^2_{[I,J]} (z_I - z_{I-1}) (z_J - z_{J+1}) \Pi_{[I,J]}^{m-2},
\]

(5.1)

where \(U \subset A\) and \(n \leq |U|\), \(A \equiv \{1, 2, \ldots, N\}\).

Examples for \(N = 4, N = 5\) are

\[
\begin{align*}
h_1^4 &= s_{12} - z_3(s_{12} + s_{23}) \\
h_1^5 &= s_{12} + z_3(s_{123} - s_{23} - s_{12}) + z_4(s_{23} - s_{123} - s_{234}) \\
h_2^5 &= z_3 s_{123} + z_4(s_{12} - s_{123} - s_{34}) + z_3 z_4(s_{34} - s_{12} - s_{234})
\end{align*}
\]

(5.2)

The level zero generators \(L^{\mu \nu}\), \(D\) act simply,

\[
L^{\mu \nu} h_m^N = 0, \quad D \ h_m^N = (N d + 2) \ h_m^N
\]

(5.3)

The off-shell \(\lambda \phi^3\) partial amplitudes are given by

\[
A^\text{partial}_N (k_1, \ldots, k_N) = \oint \frac{1}{z_{N-1}} \frac{1}{\prod_{m=1}^{N-3} \Pi_{m}^{N}(z,k)} \prod_{a=2}^{N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a z_{a+1}}{(z_a - z_{a+1})^2}
\]

(5.4)

and the individual Feynman graphs are given by similar formulae in terms of cross ratios [8].

All of these contain the momenta through inverse products of the polynomials. So the invariance under \(P^\mu, L^{\mu \nu}, D\) for \(n = 6, d = 4\) follows from

\[
P^\mu \left[ \delta^n \left( \sum_{j=1}^{N} k_j \right) \prod_{m=1}^{N-3} \frac{1}{h_m^N} \right] = L^{\mu \nu} \left[ \delta^n \left( \sum_{j=1}^{N} k_j \right) \prod_{m=1}^{N-3} \frac{1}{h_m^N} \right] = 0
\]

\[
D \left[ \delta^n \left( \sum_{j=1}^{N} k_j \right) \prod_{m=1}^{N-3} \frac{1}{h_m^N} \right] = (N (d - 2) + 6 - n) \delta^n \left( \sum_{j=1}^{N} k_j \right) \prod_{m=1}^{N-3} \frac{1}{h_m^N}
\]

(5.5)
The special conformal generator $K^\mu$ is not first order in the derivative operators. We will show how it acts on the contour integral for the two $N = 4$ Feynman graphs. Its action on higher $N$ involve multivariable contour integrals, but they must also vanish for $n = 6$, $d = 2$, in accordance with (4.18). Let $h = h_1^2 = s_{12} - z(s_{12} + s_{23})$, $z = z_3$,

$$K^\mu \frac{1}{h} = \frac{(-8d + 2n + 4)(k_1 + k_2)\mu}{(h)^2} + \frac{8z(1-z)[(k_2^2 - k_3^2)k_1^\mu + (k_2^2 - k_3^2)k_2^\mu + (k_2^2 - k_3^2)k_3^\mu]}{(h)^3}$$  \hfill (5.6)

The $N = 4$ individual graphs are

$$-\frac{1}{s_{12}} = \oint_{h=0} \frac{dz}{h} \frac{1}{z}, \quad -\frac{1}{s_{23}} = \oint_{h=0} \frac{dz}{h} \frac{1}{(1-z)}$$ \hfill (5.7)

From (5.6) the surviving integrals are

$$K^\mu \oint_{h=0} \frac{dz}{h} \frac{1}{z} = (-8d + 2n + 4)(k_1 + k_2)\mu \oint_{h=0} \frac{dz}{(h)^2} \frac{1}{z}$$

$$= (8d - 2n - 4)(k_1 + k_2)\mu \oint_{z=0} \frac{dz}{(h)^2} \frac{1}{z} = \frac{(8d - 2n - 4)(k_1 + k_2)\mu}{s_{12}^2},$$

$$K^\mu \oint_{h=0} \frac{dz}{h} \frac{1}{(1-z)} = (8d - 2n - 4)(k_2 + k_3)\mu \oint_{h=0} \frac{dz}{(h)^2} \frac{z}{1-z}$$

$$= (-8d + 2n + 4)(k_2 + k_3)\mu \oint_{z=1} \frac{dz}{(h)^2} \frac{1}{1-z} = \frac{(8d - 2n - 4)(k_2 + k_3)\mu}{s_{23}^2}$$ \hfill (5.8)

In the integrands with $\frac{1}{(h)^2}$ the contour encircles all the zeros of the denominator, so those integrals vanish. In (5.8) we can swap the contour around $h = 0$ to minus the contour around $z = 0$, etc. since there is no residue at infinity.

For $N = 5$, acting with $K^\mu$ will result in terms like $\oint_{h_1^2 = h_2} \frac{dz_{12}dz_{23}}{h_1^2h_2} \equiv \text{Res}(h_1^2, h_2)$, which can be evaluated with the global residue theorem, by dividing the factors in the denominator into two disjoint sets, $\{h_1^2, z_4\}$ and $\{h_2\}$, so that $\text{Res}(h_1^2, h_2) = -\text{Res}(z_4, h_2)$.

The level one generators acting on the scattering polynomials are similarly complicated. We give the expression for $\hat{P}^\mu$ on $h = h_1^2$,

$$-\hat{P}^\mu \ h = -2k_3^\mu s_{12}(1-z) - 2k_1^\mu s_{23}z + h (-2k_3^\mu + d (3k_1 + k_2 - k_3 - 3k_4)\mu)$$

$$-\hat{P}^\mu \ h = \frac{1}{h^2} \left[2k_1^\mu s_{23}z + 2k_3^\mu s_{12}(1-z)\right] + \frac{1}{h} \left(2k_1^\mu + d (3k_1 + k_2 - k_3 - 3k_4)\mu\right)$$ \hfill (5.9)

The polynomial does not transform simply. The partial amplitude transforms as expected,

$$-\hat{P}^\mu \oint \frac{dz}{h} \left(\frac{1}{z} + \frac{1}{1-z}\right)$$

$$= 2k_3^\mu s_{12} \oint \frac{dz}{h^2z} + 2k_1^\mu s_{23} \oint \frac{dz}{h^2} \frac{1}{1-z} + 2k_3^\mu \oint \frac{dz}{h} \left(\frac{1}{z} + \frac{1}{1-z}\right) + O(d)$$

$$= -\frac{2k_3^\mu}{s_{12}} + \frac{2k_1^\mu}{s_{23}} + 2k_3^\mu \left(\frac{1}{s_{12}} - \frac{1}{s_{23}}\right) + O(d)$$ \hfill (5.10)
For a more general theory beyond $\lambda \delta^3$, the amplitude (5.4) will acquire a further numerator which may depend on the momenta. In particular for Yang-Mills theory, the CHY integral has also a Pfaffian in the numerator that depends on momenta. For these conformally invariant amplitudes, including the Pfaffian has the effect of dropping the spacetime dimension from six to four. This reflects that the Yangian generators must now act on both the polynomials and the Pfaffian.

\section{SO(2, 4) Yangian level one generator acting on gluon tree amplitudes}

We compute explicitly how the level one SO(2, 4) Yangian generator $\hat{\mathcal{P}}^\mu$ acts on gluon tree amplitudes in pure Yang-Mills theory, using the momentum space differential operator representation of $\hat{\mathcal{P}}^\mu$. Although the generator does not annihilate the gluon amplitudes, we find a somewhat compact form. We show this for the 3-gluon amplitude and the 4-gluon partial amplitude, on shell. This could be useful to understand what role the SO(2, 4) Yangian might play in pure Yang-Mills. Since it does not annihilate the amplitude, it is not a symmetry. But it may serve some function as a spectrum generating algebra would. We will show that our answer can also be written in terms of traces of Dirac gamma matrices, which is motivated by the fact that the supersymmetic PSU(2, 2|4) Yangian does annihilate the superamplitude. In particular, the PSU(2, 2|4) Yangian level one generators annihilate the pure gluon amplitudes, and the SO(2, 4) Yangian level one generator acting on the gluon amplitude changes it into an amplitude involving 2 fermions and 2 less gluons. This may help us to interpret a possible role of supersymmetry in non-supersymmetric gauge theory. But this is beyond the scope of this paper.

The three-gluon tree amplitude is

\begin{equation}
A^{\alpha_1, \beta \alpha_2, \alpha_3} = -g f_{abc} (2\pi)^4 \delta^4(k_1 + k_2 + k_3) \cdot \left( \eta^{\alpha_2 \alpha_3} (k_2 - k_3)^{\alpha_1} + \eta^{\alpha_3 \alpha_1} (k_3 - k_1)^{\alpha_2} + \eta^{\alpha_1 \alpha_2} (k_1 - k_2)^{\alpha_3} \right)
\end{equation}

For $N = 3$ the SO(2, 4) Yangian level one generator for the gauge theory is

\begin{align}
-\hat{\mathcal{P}}^\mu_{\gamma_1 \gamma_2 \gamma_3} &= (P^\mu_1 (D_2 + D_3) + P^\mu_2 D_3 - (P_2 + P_3) D_1 - P^\mu_3 D_2) \delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\alpha_3} \\
&\quad + (P_{\gamma \rho} (L_2 + L_3) \mu^\rho + P_{\gamma \rho} L^\rho_{\alpha} - (P_2 + P_3)_{\gamma} L^\rho_{\alpha} - P_{\gamma \rho} L^\rho_{\alpha}) \delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\alpha_3} \\
&= (2(k_1 - k_3)^{\mu} - (k_2 + k_3)^{\mu} k_1 \cdot \partial_1 + (k_1 - k_3)^{\mu} k_2 \cdot \partial_2 + (k_1 + k_2)^{\mu} k_3 \cdot \partial_3 \\
&\quad - (k_2 + k_3)_{\beta} (k_1^{\mu} \partial_{\beta} - k_1^{\beta} \partial_{\mu}) + (k_1 - k_3)_{\beta} (k_2^{\mu} \partial_{\beta} - k_2^{\beta} \partial_{\mu} ) \\
&\quad + (k_1 + k_2)_{\beta} (k_3^{\mu} \partial_{\beta} - k_3^{\beta} \partial_{\mu}) ) \delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\alpha_3} \\
&\quad - (k_2 + k_3)_{\beta} (\eta^{\alpha_2 \alpha_3} \delta_{\gamma_1}^{\beta} \eta^{\beta \alpha_1} \delta_{\gamma_1}^{\gamma_2} \delta_{\gamma_2}^{\alpha_3} + (k_1 - k_3)_{\beta} (\eta^{\mu \alpha_2} \delta_{\gamma_1}^{\beta} \eta^{\beta \alpha_1} \delta_{\gamma_2}^{\alpha_3} + (k_1 + k_2)_{\beta} (\eta^{\alpha_3 \delta} \delta_{\gamma_1}^{\delta} \delta_{\gamma_2}^{\alpha_3} + (k_1 + k_2)_{\beta} (\eta^{\alpha_2 \delta} \delta_{\gamma_1}^{\delta} \delta_{\gamma_2}^{\alpha_3} \delta_{\gamma_3}^{\alpha_3})
\end{align}
This level one generator moves trivially through the delta function, as shown in appendix B, and we find the compact formula

\[
- \frac{\tilde{P}_{\mu_1\mu_2\alpha_3}}{\tilde{b}_{72}} A_{\alpha_1, \mu_2, c_{73}}
= -g f_{abc} (2\pi)^4 \delta^4 (k_1 + k_2 + k_3)
\cdot \left( 4k_1^\mu \eta^{\alpha_2\alpha_3} k_2^{\alpha_1} + \eta^\alpha k_3^{\alpha_2} + 2\eta^{\alpha_1\alpha_2} k_1^{\alpha_3} \right)
- 4k_3^\mu \left( 2\eta^{\alpha_2\alpha_3} k_2^{\alpha_1} + \eta^\alpha k_3^{\alpha_2} + \eta^{\alpha_1\alpha_2} k_1^{\alpha_3} \right)
+ \eta^{\mu\alpha_1} \left( -4k_1^{\alpha_2} k_2^{\alpha_3} \right) + \eta^{\mu\alpha_2} \left( -4k_2^{\alpha_1} k_3^{\alpha_3} \right) + \eta^{\mu\alpha_3} \left( -4k_3^{\alpha_1} k_1^{\alpha_2} \right)
\right)
\]

(6.3)

Since this expression is not zero, we know the SO(2, 4) Yangian is not a symmetry of pure Yang-Mills theory. But it shows how this Yangian acts in a non-supersymmetric gauge theory. The expression is gauge invariant, i.e. it vanishes when multiplied by any \( k_i^{\alpha_i} \) for \( 1 \leq i \leq 3 \). It is not cyclic invariant since \( \tilde{P}^{\mu} \) is not cyclic. We assume transverse polarizations \( k_i \cdot e_i (k_i) = 0 \), and drop terms in (6.3) proportional to \( k_i^{\alpha_i} \) since they correspond to gauge transformations. The on-shell conditions \( k_i^2 = 0 \) in four spacetime dimensions are required from the Serre relation. The presence of the \( \eta^{\mu\alpha_i} \) is necessary for gauge invariance; and these fixed tensor terms cannot be removed by extending \( \tilde{P}^{\mu} \) by evaluation parameters, as they could in the case of scalar \( \lambda \phi^4 \) theory.

To pursue a simpler form of (6.3), we note it can be streamlined from the equivalence

\[
\delta^4 (k_1 + k_2 + k_3)
\cdot \left( 4k_1^\mu \eta^{\alpha_2\alpha_3} k_2^{\alpha_1} + \eta^\alpha k_3^{\alpha_2} + 2\eta^{\alpha_1\alpha_2} k_1^{\alpha_3} \right)
- 4k_3^\mu \left( 2\eta^{\alpha_2\alpha_3} k_2^{\alpha_1} + \eta^\alpha k_3^{\alpha_2} + \eta^{\alpha_1\alpha_2} k_1^{\alpha_3} \right)
+ \eta^{\mu\alpha_1} \left( -4k_1^{\alpha_2} k_2^{\alpha_3} \right) + \eta^{\mu\alpha_2} \left( -4k_2^{\alpha_1} k_3^{\alpha_3} \right) + \eta^{\mu\alpha_3} \left( -4k_3^{\alpha_1} k_1^{\alpha_2} \right)
\right)
\]

\[
= \delta^4 (k_1 + k_2 + k_3) \left( \text{tr}(\gamma^{\alpha_2\gamma^{\alpha_3} \gamma^{\alpha_1} \gamma^{\mu}) k_1 \omega k_2 \omega \right)
- \text{tr}(\gamma^{\alpha_3\gamma^{\alpha_2} \gamma^{\alpha_1} \gamma^{\mu}) k_1 \omega k_3 \omega) + \text{tr}(\gamma^{\alpha_3\gamma^{\alpha_2} \gamma^{\alpha_1} \gamma^{\mu}) k_2 \omega k_3 \omega)
\right)
\]

(6.4)

Here the Dirac \( \gamma \) matrices are in a Weyl representation,

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (1, 1^t), \quad \bar{\sigma}^\mu = (1, -1^t),
\]

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag} (1, -1, -1, -1)
\]

(6.5)

The equivalence can be checked using standard trace formulae,

\[
\text{tr}(\gamma^{\mu} \gamma^{\nu}) = 4\eta^{\mu\nu}, \quad \text{tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\lambda}) = 4 \left( \eta^{\mu\nu} \eta^{\rho\lambda} - \eta^{\mu\rho} \eta^{\nu\lambda} + \eta^{\mu\lambda} \eta^{\nu\rho} \right)
\]

\[
\text{tr}(\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\lambda} \gamma^{\omega}) = \eta^{\mu\nu} \text{tr}(\gamma^{\rho} \gamma^{\lambda} \gamma^{\omega} \gamma^{\lambda}) - \eta^{\mu\rho} \text{tr}(\gamma^{\nu} \gamma^{\lambda} \gamma^{\omega} \gamma^{\lambda}) + \eta^{\mu\lambda} \text{tr}(\gamma^{\nu} \gamma^{\rho} \gamma^{\omega} \gamma^{\lambda})
- \eta^{\nu\rho} \text{tr}(\gamma^{\mu} \gamma^{\lambda} \gamma^{\omega} \gamma^{\lambda}) + \eta^{\nu\lambda} \text{tr}(\gamma^{\mu} \gamma^{\rho} \gamma^{\omega} \gamma^{\lambda})
\]

(6.6)

which can be extended to the trace of products of any number of \( \gamma \) matrices using the anticommutator. We were motivated to find the identity (6.4) by extending the level one...
generator to include supercharges. In position space we would have
\[ \langle 0 | T A^{\alpha_1}(x_1) A^{\beta_2}(x_2) A^{\gamma_3}(x_3) | 0 \rangle = G^{\alpha_1, \beta_2, \gamma_3}(x_1 x_2 x_3), \] (6.7)
and for PSU(2, 2|4) Yangian invariance of the three-gluon tree amplitude we assume a level one generator of the form
\[ - \hat{G}^{\mu_{a_1} \alpha_2 (a_3)}_{x, S} \gamma_{\gamma_1 \gamma_2 \gamma_3} \, G^{\alpha_1, \beta_2, \gamma_3}(x_1 x_2 x_3) \]
\[ = 0 \]
\[ = \sum_{1 \leq i < j \leq 3} \left( P_{i}^\mu D_j + P_{i}^\mu L_{j}^{\mu \rho} \right) - \frac{1}{4} \bar{\sigma}^{\mu \alpha} Q^{A}_{\alpha 1} \hat{Q}_{\dot{A} a 2} A_{\dot{B} b 2}\]
\[ x \gamma_{\gamma_1 \gamma_2 \gamma_3} \, G^{\alpha_1, \beta_2, \gamma_3}(x_1 x_2 x_3) \]
\[ = \sum_{1 \leq i < j \leq 3} \left( P_{i}^\mu D_j + P_{i}^\mu L_{j}^{\mu \rho} \right) (i \leftrightarrow j) \bar{\sigma}^{\alpha_1 \alpha_2 \alpha_3} \gamma_{\gamma_1 \gamma_2 \gamma_3} \, G^{\alpha_1, \beta_2, \gamma_3}(x_1 x_2 x_3) \]
\[ - \frac{1}{4} \hat{G}^{\mu_{a_1} (a_2 (a_3)}_{x, S} \gamma_{\gamma_1 \gamma_2 \gamma_3} \, G^{\alpha_1, \beta_2, \gamma_3}(x_1 x_2 x_3) \] (6.8)
\[ Q^A, \hat{Q}_{\dot{A} a} \] are the conformal supercharges appearing in the superconformal group PSU(2, 2|4), where \( 1 \leq A \leq 4 \). In this section we distinguish spinor indices \( 1 \leq \alpha, \dot{\alpha} \leq 2 \) from the Lorentz indices \( 0 \leq \mu, \alpha, \gamma \leq 3 \), and \( i \) denotes the site. The color indices \( a, b, c \) run over the dimension of the gauge group. The notation follows [22] where \( \sigma^i \) are the Pauli matrices,
\[ \sigma^\mu_{\alpha \beta} = (1, \sigma^i), \quad \bar{\sigma}^\mu \bar{\alpha} \beta = (1, -\sigma^i), \]
\[ \bar{\epsilon}_{\alpha \beta} = -\epsilon_{\beta \alpha}, \quad \epsilon_{\beta \alpha} = -\epsilon_{\alpha \beta}, \quad \epsilon^{12} = \epsilon^{21} = 1 \quad \text{same for dotted indices} \] (6.9)
The conformal supercharges rotate a gluon into a fermion in the adjoint representation,
\[ Q^A_{\alpha} A^{\mu} \sim \epsilon^A_{\alpha \beta} \sigma^\mu \bar{\psi}_{\dot{\gamma}} A^\alpha, \quad \bar{Q}^{A \mu}_{\dot{\beta}} \sim \epsilon_{\dot{\gamma} \dot{\alpha}} \bar{\sigma}^\mu \bar{\psi}_{\dot{\gamma}} A^\alpha \]
We start by computing the first \( Q \bar{Q} \) term in (6.8), using the Lagrangian coupling fermions and gluons
\[ \mathcal{L} = - \frac{1}{4} F^{\mu \nu a} F_{\mu \nu a} - i g \bar{\psi}^A \sigma^\mu \bar{\psi} A^\nu = \mathcal{L}_0 + \mathcal{L}_I \]
\[ \mathcal{L}_I = - g f_{a b c} A^b_{\mu} A^c_{\nu} \partial^\mu A^{\nu a} - \frac{g^2}{4} f_{a b c} f_{d e f} A^b_{\mu} A^c_{\nu} A^{d \mu} A^{e \nu} - i g f_{d e f} \bar{\psi}^A_{\dot{a}} \sigma^\mu \bar{\psi} f_{\dot{a}} A^e_{\mu} \psi_{\dot{a}}^f \] (6.10)
Then moving from the Heisenberg picture to the interaction picture, and working to first order in the coupling $g$, and suppressing $g$,

\[
\frac{1}{4} \bar{\sigma}^{\mu\alpha\beta}(0) T Q^A_{\alpha\beta}(x_1) \bar{Q}_{\alpha\beta}(x_2) A^{\mu\alpha}(x_3)(0)
\]

\[
= \frac{1}{4} \bar{\sigma}^{\mu\alpha\beta}(0) \langle T \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) \rangle A^{\mu\alpha}(x_3)(0)
\]

\[
= \frac{1}{4} \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0)
\]

\[
= \frac{1}{4} \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) f_{def}
\]

\[
= \frac{1}{4} \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) f_{def}
\]

(6.11)

where the gauge and fermion propagators are found from

\[
\delta^{\alpha\sigma} D^{\alpha\sigma}_F(x_3 - x - z) = \langle 0| T A^{\alpha\sigma}(x_3) A^{\alpha\sigma}_F(x_3)(0) \rangle = \delta^{\alpha\sigma} \int d^4p e^{-ip_1(x_3 - z)} D^{\alpha\sigma}_F(p), \quad \tilde{D}^{\alpha\sigma}_F(p) = -\frac{i}{p^2}
\]

\[
\delta^{\beta\sigma} D^{\beta\sigma}_F(x_2 - z) = \langle 0| T \bar{\psi}_{A^\beta}(x_2) \psi_{A^\sigma}(x_3) \rangle = \delta^{\beta\sigma} \int d^4p e^{-ip_1(x_2 - z)} S^{\beta\sigma}_F(p), \quad \tilde{S}^{\beta\sigma}_F(p) = \frac{i\sigma^{\beta\sigma}p_\omega}{p^2}
\]

(6.12)

Then the Fourier transform of (6.11) is

\[
\bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

\[
= \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

\[
= \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

\[
= (2\pi)^4 \delta^{\beta\sigma}(k_1 + k_2 + k_3) \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

\[
= (2\pi)^4 \delta^{\beta\sigma}(k_1 + k_2 + k_3) \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

\[
= (2\pi)^4 \delta^{\beta\sigma}(k_1 + k_2 + k_3) \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

(6.13)

Since the three-gluon amplitude in (6.1) has had the three external propagators truncated, we now truncate (6.13) by multiplying it by the inverse propagators, $\tilde{D}^{-1} \bar{a} a_1(k_1)$

\[
\cdot \tilde{D}^{-1} \bar{a} a_2(k_2) \tilde{D}^{-1} \bar{a} a_3(k_3) = -\bar{\eta} \bar{\tilde{a}} \bar{\tilde{a}} 2 k_1 k_2 k_3 \tilde{D}^{-1} \bar{a} a_3(k_3), \quad \bar{D}^{-1} \bar{a} a(k) \tilde{D}^a(k) = \delta_1^a
\]

Then (6.13) truncated becomes

\[
-(2\pi)^4 \delta^{\beta\sigma}(k_1 + k_2 + k_3) \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

\[
= -(2\pi)^4 \delta^{\beta\sigma}(k_1 + k_2 + k_3) \bar{\sigma}^{\mu\alpha\beta} \sigma^{\alpha\beta}_D \bar{\psi}_{A^\alpha}(x_1) \psi_{A^\beta}(x_2) A^{\mu\alpha}(x_3)(x_3)(0) \int d^4z_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} e^{ik_3 \cdot x_3}
\]

\[
\cdot \int d^4z_1 D^{\alpha\sigma}_F(x_3 - z_1) S^{\beta\sigma}_F(z_1 - x_1) S^{\alpha\beta}_F(x_2 - z_2) f_{abc}
\]

(6.14)

Here we have used properties of the $\sigma, \bar{\sigma}$ matrices

\[
\epsilon^{\tau\kappa} \sigma^{\mu\alpha}_\kappa \bar{\epsilon}^{\beta\gamma} = -(\sigma^{2\mu} \sigma^2)^{\gamma\tau} = -\bar{\sigma}^{\gamma\tau}
\]

(6.15)
since $\sigma^2\sigma^2 = \tilde{\sigma}^T$. And similarly $\epsilon_{\alpha\delta}\tilde{\epsilon}^{\mu\alpha}\epsilon_{\alpha\tau} = -(\sigma^2\tilde{\sigma}^2)_{\kappa\tau} = -\sigma^\mu_{\kappa\tau}$, since $\sigma^2\tilde{\sigma}^2 = \sigma^\mu T$. The second $Q\tilde{Q}$ term in (6.8) leads to

$$\begin{align*}
-\frac{1}{4}\tilde{\epsilon}^{\mu\alpha\alpha}(0)T\tilde{Q}A(\delta\alpha\alpha_1)(x_1)Q\tilde{A}^\delta(\beta\alpha_2)(x_2)A^\delta(x_3)|0\rangle
&\to - (2\pi)^4\delta^4(k_1 + k_2 + k_3) \langle \sigma^\mu\sigma^\delta|\sigma^\alpha\sigma^\beta|\sigma^\alpha\sigma^\delta\rangle \kappa_k \ k_1\omega k_2\zeta f_{abc} \quad (6.16)
\end{align*}$$

On combining (6.14) and (6.16), the truncated Fourier transform sums to

$$\begin{align*}
-\frac{1}{4}\tilde{\epsilon}^{\mu\alpha\alpha}(0)T\tilde{Q}A(\delta\alpha\alpha_1)(x_1)Q\tilde{A}^\delta(\beta\alpha_2)(x_2)A^\delta(x_3)|0\rangle
&\to (2\pi)^4\delta^4(k_1 + k_2 + k_3) \langle \sigma^\mu\sigma^\delta|\sigma^\alpha\sigma^\beta|\sigma^\alpha\sigma^\delta\rangle \kappa_k \ k_1\omega_k^2\zeta f_{abc}
\end{align*}$$

(6.17)

Here we use further properties of the $\gamma$ matrices in the Weyl representation (6.5)

$$\begin{align*}
\gamma^{\mu\nu\rho\lambda\gamma\omega\zeta} &= \left(\begin{array}{cc}
\left(\sigma^\mu\sigma^\nu\sigma^\rho\sigma^\lambda\sigma^\omega\sigma^\zeta\right)_\kappa & 0
\end{array}\right)_{\nu\rho\lambda\gamma\omega\zeta}, \\
\text{tr}(\gamma^{\mu\nu\rho\lambda\gamma\omega\zeta}) &= (\sigma^\mu\sigma^\nu\sigma^\rho\sigma^\lambda\sigma^\omega\sigma^\zeta)_\kappa + (\sigma^\mu\sigma^\nu\sigma^\rho\sigma^\lambda\sigma^\omega\sigma^\zeta)_\kappa
\end{align*}$$

(6.18)

The contribution of the last four terms of (6.8) can be found by exchanging $b, \alpha_2, k_2 \to c, \alpha_3, k_3$ and from there $a, \alpha_1, k_1 \to b, \alpha_2, k_2$ by inspection, to yield that the truncated Fourier transform of the last six terms of (6.8) is

$$\begin{align*}
-\frac{1}{4}\tilde{\epsilon}^{\mu\alpha\alpha}(0)T\tilde{Q}A(\delta\alpha\alpha_1)(x_1)Q\tilde{A}^\delta(\beta\alpha_2)(x_2)A^\delta(x_3)|0\rangle
&\to (2\pi)^4\delta^4(k_1 + k_2 + k_3) \langle \text{tr}(\gamma^{\alpha_2\gamma\zeta}|\gamma^{\alpha_3\gamma\omega}|\gamma^{\alpha_1\gamma\mu}) \ k_1\omega_k^2\zeta
\end{align*}$$

(6.19)

which with the coupling $g$ reinserted, cancels (6.3). This motivates the identity (6.4).
In summary, we have observed that the action of the $SO(2,4)$ Yangian on a pure gluon amplitude is equivalent to an appropriately truncated amplitude with 2 fermions and two less gluons, as expected from the $PSU(2,2|4)$ Yangian symmetry of the pure gluon amplitude. If we could interpret the fermion amplitude as the effect of some process in pure Yang-Mills theory, we might realize a role for the level one generators of the $SO(2,4)$ Yangian in pure gluon theory. Such an analysis could lead to a deeper understanding of the dynamics of non-supersymmetric non-abelian gauge theory, and help to interpret a non-supersymmetric extension of $[23, 24]$. The physical meaning of our results is they may be evidence for some fermionic structure in Yang-Mills theory.

We give an analogous identity for the four-point partial gluon amplitude, whose derivation illuminates further how to extend these identities for all $N$. We expect the relation between the action of the Yangian on the $N$-gluon tree, and the $(N-2)$-gluon with 2 fermion graphs, to hold for all $N$ due to the known $PSU(2,2|4)$ Yangian invariance of the planar $N = 4$ super Yang-Mills theory. We remark that the $(N-2)$ gluon-2 fermion trees we find differ slightly from standard expressions $[25]$ due to our procedure for truncating the external legs, as explained below (6.20).

The on-shell four-gluon tree amplitude $[26]$ is

$$A^{a_1 a_2, b_3, c_4, d_4} = g^2 (2\pi)^4 \delta^4 (k_1 + k_2 + k_3 + k_4) \left( \frac{c_s n^{a_1 a_2 a_3 a_4}}{s} + \frac{c_t n^{a_1 a_2 a_3 a_4}}{t} + \frac{c_u n^{a_1 a_2 a_3 a_4}}{u} \right)$$

where $s = s_{12}$, $t = s_{23}$, $u = (k_1 + k_3)^2$. We focus on the gauge invariant partial amplitude $A(1234)$, where the polarization vectors have been removed,

$$A(1234) = g^2 (2\pi)^4 \delta^4 \left( \sum k_i \right) \left( \frac{i n_s}{s} - \frac{i n_t}{t} \right),$$

$$i n_s^{a_1 a_2 a_3 a_4} = \left( \eta^{a_1 a_2} k_1 + 2k_1^{a_1}(\eta^{a_3 a_4} k_3 - k_4) \right) \eta^{a_3 a_4} \delta_{a_1 a_4} + \eta^{a_1 a_3} \eta^{a_2 a_4} - \eta^{a_1 a_4} \eta^{a_2 a_3}$$

$$i n_t^{a_1 a_2 a_3 a_4} = \left( \eta^{a_1 a_2} k_2 + 2k_2^{a_1}(\eta^{a_3 a_4} k_3 - k_4) \right) \eta^{a_3 a_4} \delta_{a_1 a_4} + \eta^{a_1 a_3} \eta^{a_2 a_4} - \eta^{a_1 a_4} \eta^{a_2 a_3}$$

$$+ \left( -\eta^{a_2 a_4} \delta_{a_1 a_3} + \eta^{a_2 a_1} \eta^{a_3 a_4} \right) t.$$

Since $\tilde{P}^\mu$ commutes with $\delta^4 (\sum k_i)$, in appendix C we derive

$$- \tilde{P}^\mu \left( \frac{i n_s}{s} - \frac{i n_t}{t} \right) = \left[ \eta^{a_1 a_2} \left( \frac{4k_2^{a_2} k_3^{a_3} k_4^{a_4}}{t} - \frac{8k_1^{a_2} k_1^{a_1} k_2}{t} \right) \right]$$

$$= \frac{8k_1^{a_2} (k_2^{a_3} k_3^{a_4} + k_3^{a_3} k_2^{a_4})}{s} + \frac{8k_1^{a_1} (k_2^{a_2} k_3^{a_3} + k_3^{a_2} k_2^{a_3})}{t}$$

$$+ \eta^{a_2 a_4} \left( -4k_4^{a_3} + \frac{8k_2^{a_3} k_1^{a_1} k_2}{t} \right) + \eta^{a_3 a_4} \left( 4k_4^{a_1} - \frac{8k_2^{a_1} k_1^{a_1} k_4}{s} \right)$$

$$+ \frac{8k_1^{a_1} (k_2^{a_2} k_3^{a_3} + k_3^{a_2} k_2^{a_3})}{s} - \frac{8k_2^{a_1} (k_2^{a_2} k_1^{a_1} + k_3^{a_1} k_3^{a_4})}{t}.$$
\[ +k^\mu_1 \left( 6 \eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} - \eta^{\alpha_1 \alpha_4} \eta^{\alpha_2 \alpha_3} \left( 6 + \frac{8k_1 \cdot k_2}{t} \right) - \eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} \left( 6 + \frac{16k_1 \cdot k_4}{s} \right) \right) \\
+ \eta^{\alpha_1 \alpha_2} \left( 16k_4^2 k_1^2 - 16k_4^2 k_3^2 - 4k_4^2 k_1^2 \right) - \frac{12k_4^2 k_3^2}{t} \right) \\
+ \eta^{\alpha_1 \alpha_2} \left( 8k_2^2 k_1^2 - 16k_2^2 k_4^2 - 4k_2^2 k_1^2 \right) - \frac{12k_2^2 k_4^2}{t} \right) \\
+ \eta^{\alpha_1 \alpha_2} \left( 8k_2^2 k_3^2 + \frac{12k_1^2 k_3^2}{t} \right) + \eta^{\alpha_2 \alpha_4} \left( 16k_2^2 k_4^2 + \frac{12k_2^2 k_4^2}{t} \right) \\
+ \eta^{\alpha_1 \alpha_2} \left( -8k_2^2 k_3^2 - \frac{12k_1^4 k_3^2}{t} \right) \right) \\
+ \eta^{\alpha_2 \alpha_3} \left( -12k_4^2 k_3^2 + 12k_4^2 k_3^2 \right) \right) \right) \\
+ k^\mu_2 \left( 2 \eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} - \eta^{\alpha_1 \alpha_4} \eta^{\alpha_2 \alpha_3} \left( 2 + \frac{2k_1^2}{t} \right) - 2 \eta^{\alpha_1 \alpha_2} \eta^{\alpha_3 \alpha_4} \right) \\
+ \eta^{\alpha_1 \alpha_2} \left( 4k_1^2 k_3^2 + \frac{4k_1^2 k_2^2}{t} \right) + \eta^{\alpha_2 \alpha_4} \left( 4k_2^2 \left( k_3 - k_4 \right)^2 \right) - \frac{12k_2^2 k_4^2}{t} \right) \\
+ \eta^{\alpha_1 \alpha_2} \left( 4k_1^2 k_3^2 + \frac{4k_1^2 k_2^2}{t} \right) + \eta^{\alpha_2 \alpha_4} \left( 4k_2^2 \left( k_3 - k_4 \right)^2 \right) - \frac{12k_2^2 k_4^2}{t} \right) \\
+ \eta^{\alpha_2 \alpha_3} \left( -8k_2^2 k_3^2 + 4k_2^2 k_4^2 - 4k_2^2 k_4^2 + 8k_2^2 k_4^2 \right) \right) \right) \right) \right) \\
- (1 \leftrightarrow 4), (2 \leftrightarrow 3) \text{ simultaneous exchange} \] (6.23)

which can also be expressed in terms of Dirac matrices

\[ = \left( \frac{k_1 + k_2}{k_1 + k_2} \right) \left[ -\text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) k_{12} k_{34} + \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) k_{23} \right] \\
+ \left( \frac{k_2 + k_3}{k_2 + k_3} \right) \left[ -\text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) k_{12} k_{34} - \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) k_{23} \right] \\
- \left[ \frac{k_1 k_2}{k_3 + k_4} \right] \left[ -2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) + 2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) \right] \\
+ \eta^{\alpha_1 \alpha_2} \left( -k_1 + k_2 \right) \left[ -2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) + 2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) \right] \\
+ \eta^{\alpha_1 \alpha_2} \left( -k_1 + k_2 \right) \left[ -2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) + 2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) \right] \\
+ \eta^{\alpha_1 \alpha_2} \left( -k_1 + k_2 \right) \left[ -2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) + 2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) \right] \\
+ \eta^{\alpha_1 \alpha_2} \left( -k_1 + k_2 \right) \left[ -2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) + 2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) \right] \\
+ \eta^{\alpha_1 \alpha_2} \left( -k_1 + k_2 \right) \left[ -2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) + 2k_3^4 \text{tr}(\gamma^{\alpha_1 \alpha_2} \gamma^{\alpha_3 \alpha_4} \gamma^{\alpha_5 \alpha_6}) \right] \]
\[ + \frac{k_1 k_4}{(k_2 + k_3)^2} \left[ -2k_3^a \text{tr}(\gamma^a \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\omega) + 2k_2^a \text{tr}(\gamma^a \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\omega) \right] + \eta^{b_2 a_3} (-k_2 + k_3)_\nu \text{tr}(\gamma^b \gamma^\mu \gamma^\nu \gamma^\omega) \right] \]

(6.24)

The equivalence between (6.23) and (6.24) is motivated in appendix D, and can be checked manually via the trace relations (6.6), or by using computer algebra. Here we have isolated a supercharge contribution (6.24) for a Yangian level one generator without using the spinor helicity and superspace formalism. We display explicitly that the \(SO(2,4)\) Yangian level one generator acts on the four-point pure gluon amplitude by transforming it to another amplitude which could be interpreted to involve fermions, fields not in the pure gauge theory. Also the differential operator representation provides more versatility when comparing how the Yangian acts on scalar and gauge theories.

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**A Level one Yangian generators in component form**

Using (3.2) and the conformal structure constants (2.4), we list expressions for all level one Yangian generators (2.6) in component form

\[ \hat{L}^{\mu \nu} = -\frac{1}{2} \sum_{1 \leq i < j \leq N} \left[ (P_i^{\mu} K_j^{\nu} - P_j^{\mu} K_i^{\nu} + 2g_{\rho \sigma} L_i^{\rho \mu} L_j^{\sigma \nu}) - (i \leftrightarrow j) \right] \]

\[ \hat{P}^{\mu} = -\sum_{1 \leq i < j \leq N} \left[ (P_i^{\mu} D_j + g_{\rho \sigma} P_i^{\rho \mu} P_j^{\sigma}) - (i \leftrightarrow j) \right] \]

\[ \hat{K}^{\mu} = \sum_{1 \leq i < j \leq N} \left[ (-D_i K_j^{\mu} + g_{\rho \sigma} L_i^{\rho \mu} K_j^{\sigma}) - (i \leftrightarrow j) \right] \]

\[ \hat{D} = -\frac{1}{2} \sum_{1 \leq i < j \leq N} \left[ g_{\rho \sigma} P_i^{\rho \mu} K_j^{\sigma} - (i \leftrightarrow j) \right] \]

(A.1)

For the momentum space differential operator representation, \(P_i^{\mu}, L_i^{\mu \nu}, K_i^{\mu}, D_i\) are given by (3.1) at site \(i\).

**B Commutation with the delta function**

We show how \(\hat{P}^{\mu}\) commutes with the delta function

\[ \hat{P}^{\mu} \delta^n \left( \sum_{i=1}^{N} k_i \right) M(k) = \delta^n \left( \sum_{i=1}^{N} k_i \right) \hat{P}^{\mu} M(k) \]

(B.1)
for any function of the momenta $M(k) = M(k_1, \ldots, k_N)$ as follows.

$$- \hat{P}^\mu \delta^n \left( \sum_{i=1}^N k_i \right) M(k)$$

$$= \sum_{1 \leq i < j \leq N} \left[ P_i^\mu D_j + P_{ip} L_j^{\mu p} - (i \leftrightarrow j) \right] \delta^n \left( \sum_{i=1}^N k_i \right) M(k)$$

$$= \sum_{1 \leq i < j \leq N} \left[ k_i^\mu (d + k_i \cdot \partial_j) + k_{ip} (k_j^\mu \partial_p - k_p^\mu \partial_j) + \Sigma^\mu_{jp} - (i \leftrightarrow j) \right] \delta^n \left( \sum_{i=1}^N k_i \right) M(k)$$

$$= \delta^n \left( \sum_{i=1}^N k_i \right) (- \hat{P}^\mu) M(k)$$

$$+ \left( \sum_{1 \leq i < j \leq N} \left[ k_i^\mu k_j^\nu \partial_j + k_{ip} (k_j^\mu \partial_p - k_p^\mu \partial_j) - (i \leftrightarrow j) \right] \delta^n \left( \sum_{i=1}^N k_i \right) \right) M(k)$$

The last term is zero because the delta function is a function of the sum of the momentum $\sum_{i=1}^N k_i$, which implies that all the partial derivatives $\partial_i$ act on it in the same way, $\partial_i \to \frac{\partial}{\partial \sum_{\ell=1}^N k_{\ell \rho}}$.

$$\sum_{1 \leq i < j \leq N} \left[ k_i^\mu k_j^\nu \partial_j + k_{ip} (k_j^\mu \partial_p - k_p^\mu \partial_j) - (i \leftrightarrow j) \right] \delta^n \left( \sum_{i=1}^N k_i \right)$$

$$= \sum_{1 \leq i < j \leq N} \left[ (k_i^\mu k_j^\rho + k_j^\mu k_{ip}) \partial_j - k_{ip} k_j^\rho \partial_j - (i \leftrightarrow j) \right] \delta^n \left( \sum_{i=1}^N k_i \right)$$

$$= \sum_{1 \leq i < j \leq N} \left[ (k_i^\mu k_j^\rho + k_j^\mu k_{ip}) \frac{\partial}{\partial \sum_{\ell=1}^N k_{\ell \rho}} - k_{ip} k_j^\rho \frac{\partial}{\partial \sum_{\ell=1}^N k_{\ell \rho}} - (i \leftrightarrow j) \right] \delta^n \left( \sum_{i=1}^N k_i \right) = 0$$

since the coefficients of $\frac{\partial}{\partial \sum_{\ell=1}^N k_{\ell \rho}}$ or $\frac{\partial}{\partial \sum_{\ell=1}^N k_{\ell p}}$ sum to zero.

C A level one generator on the four-point gauge amplitude

We simplify the expression for $\hat{P}^\mu$ using momentum conservation and $k_i^2 = 0$. For $N = 4$, $d = 1$,

$$- \hat{P}^\mu \delta_1 \delta_2 \delta_3 \delta_4 = \sum_{1 \leq i < j \leq 4} \left( P_i^\mu D_j + P_{ip} L_j^{\mu p} - (i \leftrightarrow j) \right)$$

$$= ((3k_1 + k_2 - k_3 - 3k_4)^\mu$$

$$+ 2k_1^\mu k_1 \cdot \partial_1 + 2(k_1 + k_2)^\mu k_2 \cdot \partial_2 - 2(k_3 + k_4)^\mu k_3 \cdot \partial_3 - 2k_4^\mu k_4 \cdot \partial_4$$

$$+ 2k_1^2 k_1 \cdot \partial_2 - 2k_1^2 k_4 \cdot \partial_2 - 2k_2 \partial_2^2 + 2k_1 \cdot k_2 \partial_2^2 \right)$$

$$\delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\alpha_3} \delta_{\gamma_4}^{\alpha_4}$$

$$+ \eta^{\alpha_1 \alpha_2} (k_1 + k_2)^\gamma \delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\alpha_3} \delta_{\gamma_4}^{\alpha_4}$$

$$+ \eta^{\alpha_1 \alpha_2} (2k_1 + k_2)^\gamma \delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\alpha_3} \delta_{\gamma_4}^{\alpha_4}$$
Then from (6.22) and (4.1),

\[-\hat{\rho}^{\mu} \frac{n_4}{s} = \sum_{1 \leq i < j \leq 4} \left( P_i^{\mu} D_j + P_i^{\mu} L_{j^\rho} - (i \leftrightarrow j) \right)^{a_1 a_2 a_3 a_4} \left( n_{i,j}^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \right) \]

\[-\frac{1}{s} n_{i,j}^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \]

\[-\frac{2}{s} n_{i,j}^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \]

Then from (6.22) and (4.1),

\[-\hat{\rho}^{\mu} \frac{n_4}{t} = \sum_{1 \leq i < j \leq 4} \left( P_i^{\mu} D_j + P_i^{\mu} L_{j^\rho} - (i \leftrightarrow j) \right)^{a_1 a_2 a_3 a_4} \left( n_{i,j}^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \right) \]

\[-\frac{1}{t} n_{i,j}^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \]

\[-\frac{2}{t} n_{i,j}^{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \]

Evaluating (C.2), (C.3) using momentum conservation, dropping terms that are gauge transformations i.e. proportional to \( k_i^{\gamma_1} \), and using the on-shell conditions \( k_i^2 = 0 \), yields (6.23).

\[\]  

### D Supercharge contribution for the four-point gauge amplitude

For the four-point function we motivate the identity between (6.23) and (6.24) by adding supercharges to the level one generator for \( N = 4 \),

\[\langle 0 | T A^{71}(x_1) A^{72}(x_2) A^{73}(x_3) A^{74}(x_4) | 0 \rangle = G^{71727374}(x_1 x_2 x_3 x_4) \]
For super Yangian invariance,
\[-\tilde{p}^{\mu_1\nu_1\alpha_3\alpha_4}_{x,S}\tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]
\[\tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4) \]
\[= 0\]
\[\sum_{1 \leq i,j \leq 4} \left( P^\mu_i D_j + P^\nu_j D_i^\mu - (i \leftrightarrow j) \right)
\[\frac{1}{4} e_{\alpha\beta} e_{\gamma\delta} \tilde{T}_{x,S}^{\alpha\beta\gamma\delta} \tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]
\[\sum_{1 \leq i,j \leq 4} \left( P^\mu_i D_j + P^\nu_j D_i^\mu - (i \leftrightarrow j) \right)
\[\frac{1}{4} e_{\alpha\beta} e_{\gamma\delta} \tilde{T}_{x,S}^{\alpha\beta\gamma\delta} \tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]
\[\sum_{1 \leq i,j \leq 4} \left( P^\mu_i D_j + P^\nu_j D_i^\mu - (i \leftrightarrow j) \right)
\[\frac{1}{4} e_{\alpha\beta} e_{\gamma\delta} \tilde{T}_{x,S}^{\alpha\beta\gamma\delta} \tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]

Then working to second order in the coupling $g$,
\[-\frac{1}{4} \sigma^{\alpha_1\beta_1} \langle 0 | T \tilde{Q}^A_{\alpha_2} A^{\alpha_1}(x_1) \tilde{Q}^{\beta_2} A^{\alpha_2}(x_2) A^{\alpha_3}(x_3) A^{\alpha_4}(x_4) | 0 \rangle \]
\[= \frac{1}{4} \sigma^{\alpha_1\beta_1} \sigma^{\beta_2\gamma_1} \epsilon_{\alpha_2\beta_2} \epsilon_{\gamma_2\gamma_1} \tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]
\[= -\frac{1}{4} \sigma^{\alpha_1\beta_1} \sigma^{\beta_2\gamma_1} \epsilon_{\alpha_2\beta_2} \epsilon_{\gamma_2\gamma_1} \tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]
\[= -\frac{1}{4} \sigma^{\alpha_1\beta_1} \sigma^{\beta_2\gamma_1} \epsilon_{\alpha_2\beta_2} \epsilon_{\gamma_2\gamma_1} \tilde{T}_{1\gamma_12\gamma_23\gamma_34}^{\alpha_1\alpha_2\alpha_3\alpha_4}(x_1x_2x_3x_4)\]

(D.1)
\[ \begin{align*}
&= \frac{1}{2} g^2 \sigma_{\mu}^{\alpha} \sigma_{\alpha}^{\alpha} \epsilon_{\alpha \beta} \epsilon_{\alpha \beta} \tilde{\sigma}^{\rho \delta} \tilde{\sigma}^{\rho \delta} \int d^4 z_1 d^4 z_2 \\
& \cdot \left[ D_{\rho}^{\alpha}(x_3 - z_1) D_{\rho}^{\alpha}(x_4 - z_2) S_{\epsilon \gamma}^{\rho}(z_2 - x_1) S_{\epsilon \gamma}^{\rho}(z_2 - x_1) f_{bch} f_{hda} \\
& + D_{\rho}^{\alpha}(x_3 - z_1) D_{\rho}^{\alpha}(x_4 - z_2) S_{\epsilon \gamma}^{\rho}(z_1 - x_1) S_{\epsilon \gamma}^{\rho}(z_1 - x_1) f_{eca} f_{bdt} \\
& + D_{\rho}^{\alpha}(x_3 - z_2) D_{\rho}^{\alpha}(x_4 - z_1) S_{\epsilon \gamma}^{\rho}(z_2 - x_1) S_{\epsilon \gamma}^{\rho}(z_2 - x_1) f_{bch} f_{eca} \\
& + D_{\rho}^{\alpha}(x_3 - z_2) D_{\rho}^{\alpha}(x_4 - z_1) S_{\epsilon \gamma}^{\rho}(z_2 - x_2) S_{\epsilon \gamma}^{\rho}(z_2 - z_1) f_{eda} f_{bec} \right] \\
& \text{(D.3)}
\end{align*} \]

with \( L_I \) given in (6.10). As in section 6, in this appendix \( 1 \leq \alpha, \bar{\alpha} \leq 2 \) and \( 0 \leq \alpha_i \leq 3 \) for site \( i \). With multiplication by four inverse gluon propagators, the truncated Fourier transform of (D.3) is

\[ -ig^2 (2\pi)^4 \delta^4(k_1 + k_2 + k_3 + k_4) \]

\[ \cdot \left( (-\tilde{\sigma}^{\bar{\alpha}_1} \sigma^{\mu_1} \tilde{\sigma}^{\bar{\alpha}_2} \sigma^{\sigma_1} \tilde{\sigma}^{\bar{\alpha}_3} \sigma^{\sigma_2} \tilde{\sigma}^{\bar{\alpha}_4} \sigma^{\sigma_3}) \frac{1}{(k_2 + k_3)^2} f_{bec} f_{eda} \\
+ (-\tilde{\sigma}^{\bar{\alpha}_1} \sigma^{\mu_1} \tilde{\sigma}^{\bar{\alpha}_2} \sigma^{\sigma_1} \tilde{\sigma}^{\bar{\alpha}_3} \sigma^{\sigma_2} \tilde{\sigma}^{\bar{\alpha}_4} \sigma^{\sigma_3}) \frac{1}{(k_2 + k_3)^2} f_{cae} f_{bde} \right) \]

\[ \text{(D.4)} \]

There is also a contribution to (D.2) from the interaction Lagrangian given by

\[ -\frac{1}{4} \tilde{\sigma}^{\mu \alpha}(0) T Q^A_{\alpha \alpha} A^{\alpha \alpha}(x_1) \tilde{Q} A^{ba}(x_2) A^{ca}(x_3) A^{da}(x_4) f_{0} \]

\[ = -\frac{1}{4} \tilde{\sigma}^{\mu \alpha} \sigma^{\alpha \beta} \epsilon_{\alpha \beta} \epsilon_{\alpha \beta} \tilde{\sigma}^{\alpha \beta} \tilde{\sigma}^{\alpha \beta} \int d^4 z_1 d^4 z_2 (0) T \psi^{\alpha \beta}(x_1) \psi^{\beta \alpha}(x_2) A^{ca}(x_3) A^{da}(x_4) f_{bec} f_{hj} \]

\[ = -ig^2 \frac{1}{4} \tilde{\sigma}^{\mu \alpha} \sigma^{\alpha \beta} \epsilon_{\alpha \beta} \epsilon_{\alpha \beta} \tilde{\sigma}^{\alpha \beta} \tilde{\sigma}^{\alpha \beta} \int d^4 z_1 d^4 z_2 S_{\epsilon \gamma}^{\rho}(x_1 - z_1) S_{\epsilon \gamma}^{\rho}(x_2 - z_1) f_{bec} \]

\[ \cdot \left[ D_{\rho}^{\alpha}(x_3 - z_2) D_{\rho}^{\alpha}(x_4 - z_2) \partial_{\rho}(x_1 - z_1) D_{\rho}(x_2 - z_1) \right] \]

\[ \text{(D.5)} \]
The truncated Fourier transform of (D.5) is

\[-ig^2(2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4) f_{bea}f_{ecd} \]
\[\left[ (\sigma^\dagger_1 \sigma^\mu \sigma^\alpha_2 \sigma^\tau \sigma^\beta_4 \sigma^\omega) \frac{k_{1\omega}k_{2\tau}}{(k_3 + k_4)^2} (-k_3 - 2k_4) \right] \]
\[+ (\sigma^\dagger_1 \sigma^\mu \sigma^\alpha_2 \sigma^\tau \sigma^\beta_3 \sigma^\omega) \frac{k_{1\omega}k_{2\tau}}{(k_3 + k_4)^2} (2k_3 + k_4) \]
\[+ (\sigma^\dagger_1 \sigma^\mu \sigma^\alpha_2 \sigma^\tau \sigma^\beta_4 \sigma^\omega) \frac{k_{1\omega}k_{2\tau}}{(k_3 + k_4)^2} \eta^{\beta_3\beta_4} (-k_3 + k_4) \nu \]

(D.6)

The second supercharge term in (D.1) is evaluated in a similar way, which when added to (D.4) and (D.6) will promote the sigma matrices to Dirac matrices (6.18). The truncated Fourier transform of

\[-\frac{1}{4} \sigma^{\dagger a\alpha a}(0)|TQ^A_{a1}A^{a1}(x_1)\tilde{Q}_{a2}A^{b2}(x_2)A^{c3}(x_3)A^{d4}(x_4)|0 \]
\[\to \frac{1}{4} \sigma^{\dagger a\alpha a}(0)|T\tilde{Q}_{a1}A^{a1}(x_1)Q^A_{a2}A^{b2}(x_2)A^{c3}(x_3)A^{d4}(x_4)|0 \]

is

\[ig^2(2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4) \text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\alpha_2 \gamma^\beta_3 \gamma^\gamma_4 \gamma^\omega) k_{1\omega}k_{2\tau} \frac{(k_2 + k_3)^\tau}{(k_2 + k_3)^2} f_{bec}f_{eda} \]
\[+ ig^2(2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4) \text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\alpha_2 \gamma^\beta_4 \gamma^\gamma_3 \gamma^\gamma_4 \gamma^\omega) k_{1\omega}k_{2\tau} \frac{(k_2 + k_4)^\tau}{(k_2 + k_4)^2} f_{bec}f_{eda} \]
\[- ig^2(2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4) f_{bea}f_{ecd} \]
\[\text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\alpha_2 \gamma^\beta_4 \gamma^\gamma_4 \gamma^\omega) \frac{k_{1\omega}k_{2\tau}}{(k_3 + k_4)^2} (-2k_4^\beta_3) \]
\[+ \text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\alpha_2 \gamma^\gamma_3 \gamma^\beta_4 \gamma^\gamma_4 \gamma^\omega) \frac{k_{1\omega}k_{2\tau}}{(k_3 + k_4)^2} \eta^{\beta_3\beta_4} (-k_3 + k_4) \nu \]

(D.8)

where also we could have used the antisymmetry under \(a \leftrightarrow b, x_1 \leftrightarrow x_2, \tilde{a}_1 \leftrightarrow \tilde{a}_2\), to generate the second term from the first term in (D.7). We have dropped gauge transformations, i.e. those terms proportional to \(k_1^\beta_4\).

Analogous symmetries can be used to generate the remaining ten supercharge terms in (D.1) from the first two (D.8), to find that total supercharge contribution to (D.1) is

\[ig^2(2\pi)^4\delta^4(k_1 + k_2 + k_3 + k_4) \]
\[\left[ \text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\beta_2 \gamma^\beta_3 \gamma^\gamma_4 \gamma^\omega) k_{1\omega}k_{2\tau} \right] \]
\[\text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\beta_4 \gamma^\gamma_3 \gamma^\gamma_4 \gamma^\omega) k_{1\omega}k_{2\tau} \]
\[\text{tr}(\gamma^\dagger a_1 \gamma^\mu \gamma^\beta_4 \gamma^\gamma_3 \gamma^\gamma_4 \gamma^\omega) k_{2\omega}k_{4\tau} \]

- 27 -
\[+ \frac{f_{abefcd}}{(k_1 + k_2)^2} \left( - \text{tr}(\gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2}) k_{1 \omega} k_{3 \omega} + \text{tr}(\gamma^{\alpha_2} \gamma^{\mu_1} \gamma^{\alpha_3} \gamma^{\mu_2}) k_{2 \omega} k_{3 \omega} \right) \\
+ \text{tr}(\gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_2}) k_{1 \omega} k_{4 \omega} - \text{tr}(\gamma^{\alpha_2} \gamma^{\mu_1} \gamma^{\alpha_3} \gamma^{\mu_2} \gamma^{\alpha_1} \gamma^{\mu_2}) k_{2 \omega} k_{4 \omega} \right) \\
+ \frac{f_{acefbd}}{(k_1 + k_3)^2} \left( - 2k_2 \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} - 2k_3 \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_1} \gamma^{\mu_1} \right) \\
+ \frac{f_{acbed}}{(k_1 + k_3)^2} \left( - 2k_2 \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} - 2k_3 \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} \right) \\
+ \frac{f_{acbed}}{(k_2 + k_4)^2} \left( - 2k_3 \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_3} - 2k_3 \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_3} \right) \\
+ \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_1} \gamma^{\mu_1} \right) \\
+ \frac{f_{acbed}}{(k_1 + k_3)^2} \left( - 2k_3 \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_1} \gamma^{\mu_1} \right) \\
+ \frac{f_{acbed}}{(k_2 + k_4)^2} \left( - 2k_3 \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_1} \gamma^{\mu_1} \right) \\
+ \gamma^{\alpha_1} \gamma^{\mu_1} \gamma^{\alpha_2} \gamma^{\mu_2} \gamma^{\alpha_3} \gamma^{\mu_3} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_4} \gamma^{\mu_4} \gamma^{\alpha_1} \gamma^{\mu_1} \right) \\
= g^2(2\pi)^4 \delta^4(k_1 + k_2 + k_3 + k_4) \cdot (f_{abefcd} \hat{P}^\mu \frac{n_s}{s} + f_{acefbd} \hat{P}^\mu \frac{n_t}{t} + f_{acbed} \hat{P}^\mu \frac{n_u}{u}) \\
\tag{D.9}
\]

Then the terms relating to \( \hat{P}^\mu \) \( A(1234) \equiv \hat{P}^\mu (\frac{n_s}{s} - \frac{n_t}{t}) \) can be read off from (D.9), by taking the coefficient of \( f_{abefcd} \) and subtracting from it the coefficient of \( f_{acefbd} \). That will result in \( i \) times the expression (6.24) given in section 6. This motivates the equivalence of (6.23) and (6.24).

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