On lambda-statistical convergence using $f$-density

Stuti Borgohain$^a, *$

$^a$Department of Mathematics, Institute of Chemical Technology(UDCT), Nathalal Parekh Marg, Matunga, Mumbai: 400019, Maharashtra (India)

Abstract

We study the concept of $f_\lambda$-statistical convergence in a probabilistic normed space, where $f$ is an unbounded modulus function. Also we are trying to investigate some relation between the ordinary convergence and module statistical convergence for every unbounded modulus function. Later on, we study $f_\lambda$-statistical convergence with partial average too.

Keywords: $f$-statistical convergence, Probabilistic Normed Space, $f$-density, modulus function, $\lambda$-statistical convergence, partial average.

2010 MSC: 40A05, 40A35.

1. Introduction and preliminaries

The study of summability theory and convergence of sequences has been one of the most important and active area of research work in Pure mathematics for the last several decades. Its extensive works are also applicable in Topology, Functional Analysis, Fourier Analysis, Measure Theory, Applied Mathematics, Mathematical Modelling, Computer Science etc. In recent years, the concept of statistical convergence of sequences which was first introduced by Fast [8] as an extension of the usual concept of sequential limits, has also been used as a tool by many mathematicians to solve many open problems in the area of sequence spaces and summability theory and some other applications as well. One may refer to ([6],[16],[12],[14],[17],[19],[21],[22],[23],[25],[31],[10]).

In 2014, A. Aizpuru et al [1] introduced a new concept of density for sets of natural numbers with respect to the modulus function. They studied and characterized the generalization of this notion of $f$-density with statistical convergence and proved that ordinary convergence is equivalent to the module statistical convergence for every unbounded modulus function. They also worked on double sequence spaces for the results of $f$-statistical convergence by using unbounded modulus function. Savas and Borgohain [24] introduced some new spaces of lacunary $f_\lambda$-statistical convergent sequences of order $\alpha$.

Menger [13] introduced the notion of metric space under the name of statistical metric space, which is known as probabilistic metric space. The idea of Menger was to use distribution functions instead of nonnegative real numbers. The probabilistic theory has become an area of active research for the last forty years. It has a wide range of applications in functional analysis also[7]. An important family of probabilistic metric spaces are probabilistic normed spaces (briefly, PN-spaces).

*Corresponding author

Email address: stutiborgohain@yahoo.com (Stuti Borgohain)

doi:10.31559/glm2019.7.2.2

Received 6 jun 2019 : Accepted 30 Nov 2019
The notion of probabilistic normed spaces was introduced by Sherstnev [30] in 1963 and later on studied by various authors, see ([2],[3], [4], Savas and Mohiuddine [29]).

2. Preliminary Concepts

A sequence \((x_i)\) of real numbers is statistically convergent to \(L\) if for arbitrary \(\varepsilon > 0\), the set \(K(i) = \{i \leq k : |x_i - L| \geq \varepsilon\}\) has natural density zero, i.e.,

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \chi_{K(i)}(j) = 0,
\]

where \(\chi_{K(i)}\) denotes the characteristic function of \(K(i)\).

By a modulus function, we mean a function \(f : \mathbb{R}^+ \to \mathbb{R}^+\) which satisfies ([15]):

1. \(f(x) = 0\) if and only if \(x = 0\).
2. \(f(x + y) \leq f(x) + f(y)\) for every \(x, y \in \mathbb{R}^+\).
3. \(f\) is increasing.
4. \(f\) is continuous from the right at 0.

Note: A modulus function must be continuous on \(\mathbb{R}^+\). Examples of moduli are \(f(x) = \frac{x}{1+x}\) and \(f(x) = x^p, 0 < p \leq 1\).

Let \(A \subseteq \mathbb{N}\), \(f\)-density of \(A\) is defined as \(\delta_f(A) = \lim_{k \to \infty} \frac{f([A(i)])}{f(k)}\), (in case this limit exists) where \(A(i) = \{k \in A : k \leq i\}\) and \(f\) is an unbounded modulus function.

Let \((x_i)\) be a sequence in \(X\) (\(X\) is a normed space). If for each \(L > 0\), \(A = \{i \leq k : \|x_i - L\| > \varepsilon\}\) has \(f\)-density zero, then it is said that \((x_i)\) is \(f\)-statistically convergent to \(L \in X\), and we write it as \(f - \text{stat}\lim_{k} x_i = L\). Note that \(\delta(A) = 1 - \delta(N\setminus A)\).

A triangular norm \((t\text{-norm})\) is a continuous mapping \(* : [0,1] \times [0,1] \to [0,1]\) such that \(([0,1],\ast)\) is an abelian monoid with unit one and \(c \ast d \geq a \ast b\) if \(c \geq a\) and \(d \geq b\) for all \(a,b,c,d \in [0,1]\). Let \(X\) be a real linear space and \(N : X \to D\), where \(D\) is the set of all distribution functions \(g : \mathbb{R} \to \mathbb{R}^+_0\) such that it is non-decreasing and left-continuous with \(\inf_{t \in \mathbb{R}} g(t) = 0\) and \(\sup_{t \in \mathbb{R}} g(t) = 1\).

The probabilistic norm or \(N\)-norm is a triangular norm satisfying the following conditions:

1. \(N_p(0) = 0\).
2. \(N_p(t) = 1\) for all \(t > 0\) iff \(p = 0\).
3. \(N_{\alpha p}(t) = N_p\left(\frac{t}{\alpha}\right)\) for all \(\alpha \in \mathbb{R}\setminus\{0\}\) and for all \(t > 0\),
4. \(N_{p+q}(s+t) \geq N_p(s) \ast N_q(t)\) for all \(p,q \in X\) and \(s,t \in \mathbb{R}^+_0\);

where \(N_p\) means \(N(p)\) and \(N_p(t)\) is the value of \(N_p\) at \(t \in \mathbb{R}\). \((X,N,\ast)\) is named as a probabilistic normed space, in short PN-space.

Let \(\lambda = (\lambda_i)\) be a non-decreasing sequence of positive numbers such that,

\[
\lambda_1 = 1, \lambda_{i+1} \leq \lambda_i + 1 \text{ and } \lambda_i \to \infty \text{ as } i \to \infty.
\]

Note: The collection of all such sequences \(\lambda\) will be denoted by \(\Lambda\).

If \(\lambda_i \leq \mu_i, \forall \lambda, \mu \in \Lambda\), we have

\[
I_i \subseteq J_i, \text{ where } I_i = [i - \lambda_i + 1, i] \text{ and } J_i = [i - \mu_i + 1, i].
\]

A sequence \((x_i)\) of real numbers is said to be \(\lambda\)-statistically convergent to \(L\) if for any \(\varepsilon > 0\),
\[
\lim_{i \to \infty} \frac{1}{\lambda_i} \left| \{ i \in I_i : |x_i - L| \geq \varepsilon \} \right| = 0,
\]

where \( I_i = [i - \lambda_i + 1, i] \) and \(|A|\) denotes the cardinality of \( A \subseteq N \) (refer [14]).

Savaş (18) studied \( \lambda \)-statistical convergence in random 2-normed spaces and \( I_\lambda \)-statistical convergence of sequences in topological groups. In [4], Alotaibi studied the notion of \( \lambda \)-statistical convergence for single sequences in probabilistic normed spaces. Also Savas and Mohiuddine [29] studied \( \lambda \)-statistically convergent double sequences in probabilistic normed spaces.

In this paper, we study the \( f_\lambda \)-statistical convergence with respect to the probabilistic norm \( N \) in the PN-space \( (X, N, \ast) \). We also investigate some results on the new concept of \( f_\lambda \)-statistical convergence with partial average. Moreover, the concepts of \( f_\lambda \)-statistical limits, \( f_\lambda \)-cluster points and \( f_\lambda \)-equivalence are introduced and try to find out the relations among them.

3. Main Results

**Definition 3.1.** Let \((X, N, \ast)\) be a PN-space. Then a sequence \( x = (x_i) \) is said to be \( f_\lambda \)-statistically convergent to \( L \) with respect to the probabilistic norm \( N \) provided that, for every \( t > 0 \) and \( \varepsilon > 0 \),

\[
\delta_{f_\lambda}(\{|i \in I_i : N_{x_i-L}(t) \leq \varepsilon\}) = 0, \quad \text{where } I_i = [i - \lambda_i + 1, i].
\]

**Definition 3.2.** Let \((X, N, \ast)\) be a PN-space. An element \( L \in X \) is said to be an ordinary limit point of the sequence \( (x_i) \in X \) with respect to the probabilistic norm \( N \) if there is a subsequence of the sequence \( (x_i) \) which also converges to \( L \) with respect to \( N \). We denote,

\[
\Omega_N(x) = \text{set of all ordinary limit points of the sequence w.r.t. } N.
\]

**Definition 3.3.** Let \((X, N, \ast)\) be a PN-space. An element \( L \in X \) is said to be a \( f_\lambda \)-limit point of the sequence \( (x_i) \) w.r.t. \( N \) if there is a subset \( I = \{i_m : i_1 < i_2 < i_3 \ldots \} \subseteq \mathbb{N} \) such that \( \delta_{f_\lambda}(I) = 1 \) and \((x_{i_m})\) is statistically convergent to \( L \) with respect to the probabilistic norm \( N \). We denote it as,

\[
\Omega_{\lambda}^f(x) = \text{set of all } f_\lambda \text{-limit points of } x.
\]

**Definition 3.4.** Let \((X, N, \ast)\) be a PN-space. An element \( L \in X \) is said to be a \( f_\lambda \)-cluster point of \( (x_i) \) w.r.t. \( N \) if for each \( t > 0 \) and \( \varepsilon > 0 \), \( \delta_{f_\lambda}(I) = 1 \) where \( I = \{i \in I_i : N_{x_i-L}(t) \geq 1 - \varepsilon\} \). We define it as,

\[
\Pi_{\lambda}^f(x) = \text{set of all } f_\lambda \text{-cluster points of } x.
\]

Particular case : \( Y_{\lambda}(x) \subseteq Y_{\lambda}^f(x) \) and \( \Pi_{\lambda}(x) \subseteq \Pi_{\lambda}^f(x) \), since \( \delta_f(A) = 0 \) implies \( \delta(A) = 0 \), for some \( A \subseteq \mathbb{N} \).

In the following results, we investigate the relations between \( f \)-limit points, \( f \)-cluster points and ordinary limit points of \((X, N, \ast)\) with respect to the non-decreasing sequences of positive numbers \((\lambda_i)\) and \((\mu_i)\) where \( \lambda_i \leq \mu_i \). Also in Theorem 3.2., we establish the result of \( f_\lambda \)-statistical convergence of \((x_i)\) with respect to \( f_\lambda \)-limit points and \( f_\lambda \)-cluster points of \((X, N, \ast)\).

**Theorem 3.5.** Let \((X, N, \ast)\) be a PN-space. For \( \lambda_i \leq \mu_i \), we have \( f_\mu \)-limit points \( \Rightarrow \) \( f_\lambda \)-cluster points \( \Rightarrow \) ordinary limit points with respect to \( N \) if \( \lim \inf_i \frac{f(\lambda_i)}{f(\mu_i)} > 0 \)

**Proof.** Let \( L \) be a \( f_\mu \)-limit point of \((x_i)\). Then for \( R = \{i_m : i_1 < i_2 < \ldots\} \) such that \( \delta_{f_\mu}(R) = 1 \) and \( \lim m \ x_{i_m} = L \) with respect to \( N \). For each \( t > 0, \varepsilon > 0 \), there exists \( i_0 \in \mathbb{N} \) such that for \( i > i_0, N_{x_i-L}(t) > 1 - \varepsilon \).

Hence, \( \{i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\} \subseteq N \setminus \{i_{i_0+1}, i_{i_0+2}, \ldots\} \) which implies that, \( \delta_{f_\mu}(\{|i \in I_i : N_{x_i-L}(t) \leq 1 - \varepsilon\}) = 0 \), where \( I_i = [i - \mu_i + 1, i] \).
Proof. Let \( \rho \) be a \( f_{\lambda} \)-limit point such that \( \rho \neq L \).

Construct two subsets of \( N \) as \( K = \{ k_m : k_1 < k_2 < k_3 < \ldots \} \) and \( L = \{ l_m : l_1 < l_2 < l_3 \ldots \} \) such that \( \delta_{f_{\lambda}}(K) = 1 \) and \( \delta_{f_{\lambda}}(L) = 1 \). Also \( (x_{k_m}) \) and \( (x_{l_m}) \) are convergent to \( L \) and \( \rho \) respectively with respect to the probabilistic norm \( N \).

We assume that \( \rho \) is the \( f_{\lambda} \)-limit point of \( X \), then for given \( t > 0 \) and \( \varepsilon > 0 \), there exists \( m_0 \in N \) such that \( N_{x_{k_m}-\rho}(t) > 1 - \varepsilon, \forall m \geq m_0 \).

Let

\[
P = \{ l_m \in L : N_{x_{k_m}-\rho}(t) \leq 1 - \varepsilon \} \subset \mathbb{N} \backslash \{ l_{m_0+1}, l_{m_0+2}, \ldots \}
\]

\[\Rightarrow \delta_{f_{\lambda}}(P) = 0\]

Again, construct \( Q = \{ l_m \in L : N_{x_{l_m}-\rho}(t) > 1 - \varepsilon \} \) which implies that \( \delta_{f_{\lambda}}(Q) = 1 \).

Assume, \( L = P \cup Q \), so that if \( \delta_{f_{\lambda}}(Q) = 0 \Rightarrow \delta_{f_{\lambda}}(P \cup Q) = 0 \), which implies that \( \delta_{f_{\lambda}}(L) = 0 \). This leads to a contradiction of the fact that \( \delta_{f_{\lambda}}(L) = 1 \).

Let \( R = \{ i \in I_i : v_{x_{l_m}-\lambda}(t) \leq 1 - \varepsilon \}, \) where \( I_i = [i - \lambda_i + 1, i] \).

Since \( (x_i) \) is convergent to \( L \) with respect to the probabilistic norm \( N \), then \( N_{x_{l_m}-\lambda}(t) > 1 - \varepsilon \). So we have for each \( t > 0 \) and \( \varepsilon > 0 \), the set \( \delta_{f_{\lambda}}(\{ i \in I_i : N_{x_{l_m}-\lambda}(t) \leq 1 - \varepsilon \}) = 0 \), which follows that \( \delta_{f_{\lambda}}(R^c) = 1 \), where \( R^c = \{ i \in I_i : N_{x_{l_m}-\lambda}(t) > 1 - \varepsilon \} \).

If \( \rho \neq L \), then \( Q \cap R^c \) is empty and \( Q \subset R \). Also \( \delta_{f_{\lambda}}(R) = 0 \) implies \( \delta_{f_{\lambda}}(Q) = 0 \), which leads to a contradiction that \( \delta_{f_{\lambda}}(Q) = 1 \). Hence \( L \) is also a \( f_{\lambda} \)-limit point of \( (x_i) \) in \( (X, N_{\ast}) \).

To prove the second part, let us take \( \rho \) be a \( f_{\lambda} \)-cluster point such that \( L \neq \rho \). Also, for each \( t > 0 \) and \( \varepsilon > 0 \), we construct the sets \( A = \{ i \in I_i : N_{x_{l_m}-\lambda}(t) > 1 - \varepsilon \} \) and \( B = \{ i \in I_i : N_{x_{l_m}-\rho}(t) > 1 - \varepsilon \} \) such that \( \delta_{f_{\lambda}}(A) = 1 \) and \( \delta_{f_{\lambda}}(B) = 1 \) respectively.
We assume that \( L \neq \rho \), which implies that \( A \cap B = \emptyset \) and therefore \( B \subset A^c \) where \( A^c = \{ i \in I_i : N_{x_i-L}(t) \leq 1 - \epsilon \} \).

Also, as \((x_i)\) is convergent to \( L \) with respect to \( N \), we have \( \delta_{f_i}(A^c) = 0 \Rightarrow \delta_{f_i}(B) = 0 \), which is a contradiction.

Hence, \( \rho = L \), i.e. \( L \) is also a \( f_\lambda \)-cluster point. This completes the proof. \( \square \)

In this section, we define the concept of \( f_\lambda \)-statistically Cauchy in \((X, N, \ast)\) and establish the relation between \( f \)-statistical Cauchy and \( f_\lambda \)-statistical convergence of \((x_i)\) with respect to \((\lambda_i)\) and \((\mu_i)\) where \( \lambda_i \leq \mu_i \).

**Definition 3.7.** Let \((X, N, \ast)\) be a PN-space. Then \((x_i)\) is said to be \( f_\lambda \)-statistically Cauchy if \( \delta_{f_i}(\|B_N(t)\|) = 0 \) where \( B_N(t) = \{ i, l \in I_i : N_{x_i-x_l}(t) \leq 1 - \epsilon \} \).

**Theorem 3.8.** Let \((X, N, \ast)\) be PN-space. For \( \lambda_i \leq \mu_i \), \( f_\mu \)-statistical Cauchy implies \( f_\lambda \)-statistical convergence if \( \lim_{i} \frac{f(\lambda_i)}{f(\mu_i)} > 0 \).

**Proof.** Let \( B_N(t) = \{ i, l \in I_i : N_{x_i-x_l}(t) \leq 1 - \epsilon \} \) such that \( \delta_{f_i}(\|B_N(t)\|) = 0 \), where \( I_i = [i - \mu_i, i] \).

Assume that \( x = (x_i) \) is not \( f_\mu \)-statistically convergent but convergent to \( L \) with respect to the probabilistic norm \( N \), so \( N_{x_i-L} \left( \frac{t}{2} \right) > 1 - \epsilon \).

For given \( \epsilon > 0, \epsilon > 0 \), let us take \( r > 0 \) such that \((1 - r) \ast (1 - r) \geq 1 - \epsilon \).

Thus, for \( i, l \in I_i = [i - \lambda_i, i] \),

\[
N_{x_i-x_l}(t) = N_{x_i-L + L-x_l} \left( \frac{t}{2} + \frac{t}{2} \right) \\
\geq N_{x_i-L} \left( \frac{t}{2} \right) \ast N_{x_i-L} \left( \frac{t}{2} \right) \\
> (1 - r) \ast (1 - r) \\
> 1 - \epsilon
\]

Hence,

\[
\begin{align*}
\{ i \in I_i : N_{x_i-x_l} \left( \frac{t}{2} \right) > 1 - \epsilon \} & \subset \{ i, l \in I_i : N_{x_i-x_l}(t) > 1 - \epsilon \} \\
\{ i \in I_i : N_{x_i-x_l} \left( \frac{t}{2} \right) > 1 - \epsilon \} & \subset \{ i, l \in I_i : N_{x_i-x_l}(t) > 1 - \epsilon \}, \text{ (by using (1))}
\end{align*}
\]

\[
\Rightarrow \lim_{i} \frac{f(\{ i, l \in I_i : N_{x_i-x_l}(t) \leq 1 - \epsilon \})}{f(\mu_i)} \leq \frac{f(\{ i \in I_i : N_{x_i-L} \left( \frac{t}{2} \right) \leq 1 - \epsilon \})}{f(\mu_i)} f(\lambda_i) \\
\leq \lim_{i} \frac{f(\{ i \in I_i : N_{x_i-L} \left( \frac{t}{2} \right) \leq 1 - \epsilon \})}{f(\mu_i)} f(\lambda_i)
\]

Since \((x_i)\) is not \( f_\lambda \)-statistical convergent, so with the help of \( \lim_{i} \frac{f(\lambda_i)}{f(\mu_i)} > 0 \), we get

\[
\delta_{f_i} \left( \left\{ i \in I_i : N_{x_i-L} \left( \frac{t}{2} \right) \leq 1 - \epsilon \right\} \right) = 1,
\]

which leads to a contradiction that

\[
\delta_{f_i}(\|B_N(t)\|) = 1, \text{ where } B_N(t) = \{ i, l \in I_i : N_{x_i-x_l}(t) \leq 1 - \epsilon \}.
\]

This arrives to the conclusion that \((x_i)\) is \( f_\mu \)-statistically Cauchy implies that it is \( f_\lambda \)-statistically convergent if \( \lim_{i} \frac{f(\lambda_i)}{f(\mu_i)} > 0 \). This completes the proof. \( \square \)
Corollary 3.9. Let \((X, N, \ast)\) be a PN-space. Then if \((x_i)\) is \(f_\lambda\)-statistically Cauchy sequence then it has a Cauchy subsequence with respect to the probabilistic norm \(N\).

We also examine some results on \(f_\lambda\)-statistical equivalence with \(f_\lambda\)-limit points and \(f_\lambda\)-cluster points. Moreover, in Theorem 3.5., the relation between \(f\)-statistical Cauchy, \(f\)-statistical convergence and \(f\)-statistical equivalence of \((x_i) \in (X, N, \ast)\) with respect to \((\lambda_i)\).

Definition 3.10. Let \((X, N, \ast)\) be a PN-space. Then \(x, y \in X\) is said to be \(f_\lambda\)-statistical equivalent if \(\delta_{f_\lambda}(i \in I : x_i \neq y_i) = 0\).

Proofs of the following results are routine works, so ommitted.

Theorem 3.11. Let \((X, N, \ast)\) be a PN-space, If \(x, y \in X\) are \(f_\lambda\)-statistical equivalent then \(f_\lambda\)-limit points and \(f_\lambda\)-cluster points of both \(x\) and \(y\) are same respectively.

Theorem 3.12. Let \((X, N, \ast)\) be a PN-space. Then the following are equivalent to each other:

1. \((x_i)\) is \(f_\lambda\)-statistically Cauchy.
2. \((x_i)\) is \(f_\lambda\)-statistically convergent to \(L\).
3. There exists \(y \in X\) such that \(x\) and \(y\) are \(f_\lambda\)-statistical equivalent and \(y\) is also \(f_\lambda\)-statistically convergent to \(L\).

4. \(f_\lambda\)-statistical convergence with partial average

Partial average/means of sequences has an important role in the theory of ergodic systems [34]. Very recently, Mark Burgin and Oktay Duman [33] studied the statistical convergence of sequences of real numbers in the direction of Statistics. In this section, we define the idea of \(f\)-partially statistical convergence and \(f_\lambda\)-partially statistical convergence of \((x_i)\) with respect to the probabilistic norm \(N\) and investigate some relations between \(f_\lambda\)-partial statistical convergence and \(f_\lambda\)-statistical convergence.

Definition 4.1. Let \(x = (x_i)\) be a sequence of real numbers. The partial average of \(x = (x_i)\) is defined as,

\[
\mu(x) = \left\{ \mu_i : \frac{1}{i} \sum_{k=1}^{i} x_k ; i = 1, 2, 3... \right\}.
\]

Definition 4.2. Let \((X, N, \ast)\) be a PN-space. A sequence \((x_i)\) is said to be partially convergent to \(L\) with respect to the probabilistic norm \(N\), if for \(\varepsilon > 0\) and \(t > 0\), \(\frac{1}{t} \sum_{k=1}^{i} N_{x_k - L}(t) > 1 - \varepsilon\).

Definition 4.3. Let \((X, N, \ast)\) be a PN-space. A sequence \(x = (x_i)\) is said to be \(f_\lambda\)-partially statistical convergent to \(L\) with respect to the probabilistic norm \(N\) if its partial average \(\mu(x)\) is \(f_\lambda\)-statistically convergent to \(L\) with respect to \(N\), i.e. \(\lim_{i} \frac{f(\mu_i(x))}{f(\lambda_i)} = 0\), where \(\mu_i(x) = \{i \in I : \frac{1}{i} \sum_{k=1}^{i} N_{x_k - L}(t) \leq 1 - \varepsilon\}\).

We denote it as \(f_\lambda_{\text{stat}}\lim x_i = L\).

Theorem 4.4. Let \((X, N, \ast)\) be a PN-space. If a bounded sequence \(x = (x_i)\) is \(f_\lambda\)-statistically convergent to \(L\) with respect to the probabilistic norm \(N\) then it is also \(f_\lambda\)-partially statistical convergent to \(L\) with respect to \(N\). But not conversely.
Proof. Let \( x = (x_i) \) be \( f_\lambda \)-statistical convergent to \( L \) with respect to \( N \). Then for any \( t > 0 \) and \( \varepsilon > 0 \),
\[
\lim_{i} \frac{f(|A_i(t)|)}{f(\lambda_i)} = 0 \quad \text{where} \quad A_i(t) = \{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t} \}, \quad \text{where} \quad I_i = [i-\lambda+1, i].
\]
Since \((x_i)\) is a bounded sequence, so there exists \( C > 0 \) such that for \( t > 0 \), \( N_{x_i-L}(t) \geq 1 - C \).
\[
\sum_{i \in I_i} N_{x_i-L}(t) = \sum_{i \in I_i, N_{x_i-L}(t) \leq 1 - \frac{1}{t}} N_{x_i-L}(t) + \sum_{i \in I_i, N_{x_i-L}(t) > 1 - \frac{1}{t}} N_{x_i-L}(t)
\]
\[
\leq C(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t}\}| + \frac{1}{\varepsilon})
\]
Consequently we get,
\[
\frac{1}{i} \sum_{i \in I_i} N_{x_i-L}(t) \leq \frac{1}{i} \{C(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t}\}| + \frac{1}{\varepsilon})\}
\]
Also,
\[
\{i \in I_i : \frac{1}{i} \sum_{i \in I_i} N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\} \leq \frac{C(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t}\}|)}{i}
\]
\[
\leq |\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t}\}|
\]
\[
f(\{i \in I_i : \frac{1}{i} \sum_{i \in I_i} N_{x_i-L}(t) \leq 1 - \frac{1}{\varepsilon}\})
\]
\[
\Rightarrow \frac{f(|\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t}\}|)}{f(\lambda_i)} \leq \frac{f(\{i \in I_i : N_{x_i-L}(t) \leq 1 - \frac{1}{t}\})}{f(\lambda_i)}
\]
By taking \( i \to \infty \), we get \( f_\lambda \)-statistically convergence implies \( f_\lambda \)-partially statistical convergence.

For the converse part, let \( f \) be an unbounded modulus function such as \( f(x) = \log(1 + x) \). Take \((\mathbb{R}, N, *)\) be a PN-space with \( a * b = ab, N_a(t) = \frac{1}{t + |x|} \). Now we define a sequence ,
\[
x_i = \left\{ \begin{array}{ll}
\sqrt{k} & \text{if } i - \sqrt{N_i} + 1 \leq k \leq i; \\
0 & \text{otherwise.}
\end{array} \right.
\]
which shows that \((x_i)\) is \( f_\lambda \)-statistically divergent, but it is \( f_\lambda \)-partially statistical convergent. \( \Box \)

**Theorem 4.5.** Let \((X, N, *)\) be a PN-space. Then the \( f_\lambda \)-partial statistical limit of \((x_i) \in X\) is unique with respect to the probabilistic norm \( N \).

*Proof.* Proof is easy, so omitted. \( \Box \)

**Corollary 4.6.** Let \((X, N, *)\) be a PN-space. If a sequence \((x_i) \in X\) is \( f_\lambda \)-partial statistical convergence then it has also a partial convergent subsequence with respect to \( N \).

**Theorem 4.7.** Let \((X, N, *)\) be a PN-space. If \( \lambda \in \Delta \) with \( \lim \inf_{i} \frac{f(\lambda_i)}{i} > 0 \), then a sequence \((x_i)\) is partially statistical convergent to \( L \) with respect to the probabilistic norm \( N \), then it is \( f_\lambda \)-statistically convergent to \( L \) with respect to \( N \).

*Proof.* Let \((x_i)\) be a partially convergent to \( L \) with respect to the probabilistic norm \( N \), then for \( \varepsilon > 0 \) and \( t > 0 \), we have,
\[
\frac{1}{i} \sum_{k=1}^{i} N_{x_k-L}(t) > 1 - \varepsilon.
\]
We can write,
\[ \sum_{k=1}^{i} N_{x_k-L}(t) \geq f\left( \sum_{k=1}^{i} N_{x_k-L}(t) \right) \geq f\left( \left| \{ k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon \} \right| \varepsilon \right) \geq cf\left( \left| \{ k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon \} \right| f(\varepsilon) \right) \]

Consequently we get,
\[ \frac{1}{i} \sum_{k=1}^{i} N_{x_k-L}(t) \geq \frac{c}{i} f\left( \left| \{ k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon \} \right| \right) f(\varepsilon) \geq \frac{f(\lambda_i)}{i} f\left( \left| \{ k \leq i : N_{x_k-L}(t) \leq 1 - \varepsilon \} \right| \right) f(\lambda_i) \]

Since \( \liminf_{i} \frac{f(\lambda_i)}{i} > 0 \), so it follows that partial convergence imples \( f_{\lambda} \)-statistical convergence with respect to the probabilistic norm \( N \). This completes the proof.

5. Conclusion

Convergence of statistical characteristics such as the average/mean and standard deviation are related to statistical convergence as we see in our results. This concept of convergence with respect to means/averages and standard deviations have been studied by many mathematicians. We have studied the convergence of the average/mean in the propbabilistic normed spaces. It can be further extended in the direction of fuzzy real numbers and can be found very interesting results in intuitionistic fuzzy normed linear spaces too using various aspects.

6. Acknowledgements

The authors acknowledges the reviewes for their valuable comments which helped the author to improve the paper in many ways.

References

[1] M.C.A. Aizpuru, Listán-Garcí and F. Rambla-Barreno, Density by Moduli and Statistical Convergence, Quaestiones Mathematicae, 37 (2014), 525–530.
[2] C. Alsina, B. Schweizer, A. Sklar, On the definition of a probabilistic normed space, Aequationes Math., 46 (1993), 91–98.
[3] C. Alsina, B. Schweizer, A. Sklar, Continuity properties of probabilistic norms, J. Math. Anal. Appl., 208, (1997), 446–452.
[4] Alotaibi, Generalized Statistical Convergence in Probabilistic Normed Spaces, The Open Mathematics Journal, 1 (2008), 82–88.
[5] S. Borgohain, On new f-statistical convergence in probabilistic normed spaces, New Trends in Mathematical Scienes, 6 (3)(2018), 181–188.
[6] J. Connor, E. Savaş, Lacunary statistical and sliding window convergence for measurable functions, Acta Mathematica Hungarica, 145(2)(2015), 416–432.
[7] S.S. Chang, B.S. Lee, Y.J. Cho, Y.Q. Chen, S.M. Kang, J.S. Jung, Generalized contraction mapping principles and differential equations in probabilistic metric spaces, Proc. Amer. Math. Soc., 124(1996), 2367–2376.
[8] H. Fast, Sur la convergence statistique, Colloq.Math., 2 (1951), 241–244.
[9] S. Karakus, Statistical convergence on probabilistic normed spaces, Math. Comm., 12 (2007), 11–23.
[10] E. Kolk, The statistical convergence in Banach spaces, Acta Et Commentationes Univ. Tartuensis, 28 (1991), 41–52.
[11] I.J. Maddox, Statistical convergence in locally convex spaces, Math Proc. Camb. Phil. Soc., 104 (1988), 141–145.
[12] G.D. Maio, L.D.R. Kocinac, Statistical convergence in topology, Topology Appl., 156 (2008), 28–45.
[13] K. Menger, *Statistical metrics*, Proc Nat Acad Sci USA, 28 (1942), 535–537.
[14] M. Mursaleen, *λ-statistical convergence*, Math. Slovaca, 50 (2000), 111–115.
[15] M. Mursaleen, S.A. Mohiuddine *On ideal convergence in probabilistic normed spaces*, Mathematica Slovaca, 62(1)(2012), 49–62. https://doi.org/10.2478/s12175-011-0075-5
[16] H. Nakano, *Concave Modulars*, J. Math. Soc. Japan, 5, (1953), 29–49.
[17] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, 30 (1980), 139–150.
[18] E. Savas, *Generalized statistical convergence in random 2-normed space*, Iran. J. Sci. Technol. Trans. A Sci, 36(2012), 417–423.
[19] E. Savas, *Iλ -double statistically convergent sequences in topological groups*, Mathematics and computing Commun. Comput. Inf. Sci., 655, Springer, Singapore, (2017), 349-357.
[20] E. Savas, *A generalized statistical convergent functions via ideals in intuitionistic fuzzy normed spaces*, Applied and computational mathematics, 16(1)(2017), 31–38.
[21] E. Savas, *Iλ -double statistical convergence of order α in topological groups*, Ukran. Mat. Zh., 68(9) (2016), 1251-1258. 40A35.
[22] E. Savas, *On generalized double statistical convergence in locally solid Riesz spaces*, Miskolc Math. Notes, 17 (1) (2016), 591-603. 40G15 (46A40).
[23] E. Savas, *Iλ -statistically convergent sequences in topological groups*, Acta Et Commentationes Universitatis Tartuensis De Mathematica, Vol. 18, No. 1 (2014), pp. 33-38. https://doi.org/10.12697/acutm.2014.18.04.
[24] E. Savas, S. Borgohain, *On strongly almost lacunary statistical A-convergence and lacunary A-statistical convergence*, Filomat, 30(3) (2016), 689-697.
[25] E. Savas, S. Borgohain, *Some new spaces of lacunary f -statistical A-convergent sequences of order a*, Advancements in Mathematical Sciences, AIP Conf. Proc. 1676, 2015, 020086-1 à”020086-8; doi: 10.1063/1.4930512.
[26] E. Savas, *On lacunary double statistical convergence in locally solid Riesz spaces*, Journal of Inequalities and Applications, 99 (2013) https://doi.org/10.1186/1029-242x-2013-99.
[27] E. Savas, *Strong almost convergence and almost λ-statistical convergence*, Hokkaido Math. J., Vol. 29, (2000), pp. 531- 536. https://doi.org/10.14492/hokmj/1380912989.
[28] E. Savas, *Iλ -statistically convergent sequences in topological groups*, Acta Et Commentationes Universitatis Tartuensis De Mathematica, Vol. 18, No. 1 (2014), pp. 33-38.
[29] E. Savas, S. A. Mohiuddine , *λ-statistically convergent double sequences in probabilistic normed spaces*, Mathematica Slovaca, Vol. 62, No.1 (2012), pp. 99-108. https://doi.org/10.2478/s12175-011-0075-5.
[30] E. Savas, *Iλ -Statistical Convergence of Order α in Topological Groups*, General Letters in Mathematics, 1(2) (2016), doi:10.31559/glm2016.1.2.2. https://doi.org/10.31559/glm2016.1.2.2.
[31] A. N. Sherstnev, *On the notion of a random normed space*, Dokl. Akad. Nauk. SSSR, 149 (1963), 280–283 [English translation in Soviet Math. Dokl 1963;4:388-0].
[32] I. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly , 66 (1959), 361–375. https://doi.org/10.2307/2308747.
[33] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North Holland, New York-Amsterdam-Oxford. (1983).
[34] M. Burgin and O. Duman, *Statistical convergence and convergence in Statistics*, arXiv.org / math /arXiv:math/0612179(2006).
[35] P. Billingsley, *Ergodic Theory and Information*, John Wiley & Sons, New York, (1965).