1 Black-Hole Entropy

According to the thermodynamical analogy in black hole physics, the entropy of a black hole in the Einstein theory of gravity is

\[ S_{BH} = A_H / (4l_P^2), \]  

(1.1)

where \( A_H \) is the area of a black hole surface and \( l_P = (\hbar G/c^3)^{1/2} \) is the Planck length [1, 2]. In black hole physics the Bekenstein-Hawking entropy \( S_{BH} \) plays essentially the same role as in the usual thermodynamics. In particular it allows to estimate what part of the internal energy of a black hole can be transformed into work. Four laws of black hole physics that form the basis in the thermodynamical analogy were formulated in [3]. According to this analogy the entropy \( S \) is defined by the response of the free energy \( F \) of the system containing a black hole to the change of its temperature:

\[ dF = -SdT. \]  

(1.2)

The generalized second law [1, 2, 4] (see also [5, 6, 7, 8] and references therein) implies that when a black hole is a part of the thermodynamical system the total entropy (i.e. the sum of the entropy of a black hole and the entropy of the surrounding matter) does not decrease.

The Euclidean approach [9, 10, 11] provides a natural way to derive black hole thermodynamical properties. Doing a Wick’s rotation \( t \to -i\tau \) in the Schwarzschild metric one gets the metric with the Euclidean signature. The corresponding manifold with the Euclidean metric is regular if and only if the imagine time \( \tau \) is periodic and the period is \( 2\pi/\kappa \), where \( \kappa \) is the surface gravity of the black hole. The corresponding
regular Euclidean metric is known as the Gibbons-Hawking instanton. The period \(2\pi/\kappa\) of the imaginary time \(\tau\) is naturally identified with the inverse temperature of the black hole, while the Euclidean action, calculated for the Gibbons-Hawking instanton, is directly related with its free energy. York and collaborators \[12, 13\] made an important observation that the formal derivation of the black hole entropy using the Euclidean approach requires certain modifications for its consistency. Namely, to get a well-defined canonical ensemble, one needs to consider a black hole in a box of finite size and fix the corresponding thermodynamical quantities (temperature) at the boundary. Box filled with thermal radiation and a black hole in it in thermal equilibrium with radiation is thermodynamically stable if the radius of a spherical box \(r\) is less than \(3M\). If one changes the temperature of the box, the mass of a black hole inside the box also changes, as required by the conditions of the thermal equilibrium. As the result of this process the free energy \(F\) of the system is changed. One can use the relation (1.2) and single out the contribution to the entropy due to the black hole. In the classical approximation the so defined thermodynamical entropy coincides with the Bekenstein-Hawking entropy \(S_{BH}\).

The simple relation between the thermodynamical entropy of a black hole and its surface area is characteristic for Einstein theory. The definition of the black hole entropy can be generalized to non-Einstein versions of gravitational theory, provided they allow existence of black holes. Wald \[14\] showed that in the general case the entropy of a black hole is defined by Noether charge related with the Killing vector. (For application and developing this idea, see \[15, 16, 17\].) Recently the interest to the problem of black hole entropy was increased by observation that the entropy of extremal black hole might vanish \[18, 19\].

The success of the thermodynamical analogy in black hole physics allows one to hope that this analogy may be even deeper and it is possible to develop statistical-mechanical foundation of black hole thermodynamics. It is worthwhile to remind that the thermodynamical and statistical-mechanical definitions of the entropy are logically different. *Thermodynamical entropy* \(S_{TD}\) is defined by the response of the free energy \(F\) of a system to the change of its temperature:

\[
dF = -S_{TD}dT.
\]

(This definition applied to a black hole determines its Bekenstein-Hawking entropy.)

*Statistical-mechanical entropy* \(S_{SM}\) is defined as

\[
S_{SM} = -\text{Tr}(\hat{\rho} \ln \hat{\rho}),
\]

where \(\hat{\rho}\) is the density matrix describing the internal state of the system under consideration. It is also possible to introduce the *informational entropy* \(S^I\) by counting different possibilities to prepare a system in a final state with given macroscopical parameters from different initial states

\[
S^I = -\sum_n p_n \ln p_n,
\]
with $p_n$ being the probabilities of different initial states. In standard case all three definitions give the same answer.

Is the analogy between black holes thermodynamics and the 'standard' thermodynamics complete? Are there internal degrees of freedom of a black hole responsible for its entropy? Is it possible to apply the statistical-mechanical and informational definitions of the entropy to black holes and how are they related with the Bekenstein-Hawking entropy? These are the questions that are to be answered.

Historically first attempts of the statistical-mechanical foundation of the entropy of a black hole were connected with the informational approach \[3, 20\]. According to this approach the black hole entropy is interpreted as "the logarithm of the number of quantum mechanically distinct ways that the hole could have been made" \[5, 20\]. The so-defined informational entropy of a black hole is simply related to the amount of information lost by stretching the horizon, and as was shown by Thorne and Zurek it is equal to the Bekenstein-Hawking entropy \[3, 20\]. Quite interesting results relating informational and statistical-mechanical entropies can be obtained in a special model proposed by Bekenstein and Mukhanov \[21, 22\]. According to this model a black hole is identified with a system having discrete internal states, so that an absorption or emission of a particle by a black hole is accompanied by the transition from one state to another. In such model the mass of a black hole is quantized\[1\]. Unfortunately the physical origin of internal degrees of freedom of a black hole in the model and their discreteness is not derived but postulated.

The dynamical origin of the entropy of a black hole and the relation between the statistical-mechanical and Bekenstein-Hawking entropy have remained unclear. In the present talk I describe some recent results obtained in this direction.

2 Dynamical Degrees of Freedom of a Black Hole

The problem of the dynamical origin of the black hole entropy was intensively discussed recently. The proposed basic idea is to relate the dynamical degrees of freedom of a black hole with its quantum excitations. This idea has different realizations\[2\].

In the framework of the membrane paradigm the dynamical degrees of freedom are identified with different possible states of thermal atmosphere of a black hole, while the entropy of a black hole is identified with the amount of information about the state of thermal atmosphere, which is lost by stretching the horizon\[3, 20\].

In his brick wall model ’t Hooft \[25\] proposed to consider a mirror-like boundary, located outside a black hole at the close distance to its horizon. He assumed that outside the boundary there exist thermal radiation with temperature equal to the Hawking temperature. He has shown that the entropy of such thermal radiation is

---

1 Another approach to the quantization of the mass of a black hole can be found in \[23\].

2 For recent review of the problem of the dynamical origin of the entropy of a black hole, see also \[24\].
of the same order of magnitude as the Bekenstein-Hawking entropy, provided the distance of the mirror-like boundary from the horizon is chosen to be of order of Planck’s length.

Bombelli et al [26] attracted attention to the fact that even in a flat spacetime in a Minkowski vacuum state one can obtain a non-vanishing entropy if one restricts his observations to the spatially bounded part of the space. The corresponding entanglement entropy arises due to the presence of correlations of those modes of zero-point fluctuations which are propagating in the vicinity of the boundary of chosen spatial region. It was proposed to relate black hole entropy with the entanglement entropy related with the presence of the horizon [26, 27].

We (with Igor Novikov) arrived to the similar idea independently by analyzing the gedanken experiment proposed in our earlier paper [28]. Namely we assumed that there exist a traversable wormhole, and its mouths are freely falling into a black hole. If one of the mouths crosses the gravitational radius earlier than the other, then rays passing through the first mouth can escape from the region lying inside the gravitational radius. Such rays would go through the wormhole and enter the outside region though the second mouth. As the result during the period of time when the first of the mouth is inside and the other mouth is outside the gravitational radius the surface area of the horizon decreases. If we assume that the black hole entropy is related with the surface area of a black hole, then the only possibility to escape contradictions with the second law is to assume that during such process the decrease of entropy is related with the possibility to get access to some new information concerning black hole internal states. At first sight it looks like a puzzle. We know that (at least in the classical General Relativity) a black hole at late time is completely specified by finite number of parameters. For a non-rotating uncharged black hole one need to know only one parameter (mass M) to describe all its properties. It is true not only for the exterior where this property is the consequence of no-hair theorems, but also for the black hole interior [29]. The reason why it is impossible for an isolated black hole at late time to have non-trivial classical states in its interior in the vicinity of the horizon is basically the same as for the exterior regions. These states might be excited only if a collapsing body emits the pulse of fields or particles immediately after it crosses the event horizon. Due to red-shift effect the energy of the emitted pulse must be exponentially large in order to reach the late time region with any reasonable energy. After short time (say 100M) after the formation of a black hole it is virtually impossible because it requires the energy of emission much greater than the black hole mass M.

In quantum physics the situation is completely different. The above mentioned puzzle can be solved if we remember that any quantum field has zero-point fluctuations. To analyze states of a quantum field it is convenient to use its decomposition into modes. Besides the positive-frequency modes which have positive energy, there exist also positive-frequency modes with negative energy. In a non-rotating uncharged black holes such modes can propagate only inside the horizon where the Killing vector
used to define the energy is spacelike. It is possible to show that at late time for any regular initial state of the field these states are thermally excited and the corresponding temperature coincides with the black hole temperature $T_H = (8\pi M)^{-1}$. (For more details see, e.g.\cite{30}.) In principle by using a traversable wormhole described above one can get information concerning these internal modes propagating near the horizon and change their states.

In the framework of this *dynamical-black-hole-interior* model we proposed to identify the dynamical degrees of freedom of a black hole with the internal modes of all physical fields. The set of the fields must include the gravitational one. It can be shown that for any chosen field the number of the modes infinitely grows as one considers the regions located closer and closer to the horizon. For this reason the contribution of a field to the statistical-mechanical entropy of a black hole calculated by counting the internal modes of a black hole is formally divergent. In order to make it finite one might restrict himself by considering only those modes which are located at the proper distance from the horizon greater than some chosen value $l$. For this choice of the cut-off the contribution of a field to the statistical-mechanical entropy of a black hole is

$$S^{SM} = \alpha \frac{A}{l^2}, \quad (2.1)$$

where $A$ is the surface area of a black hole, and a dimensionless parameter $\alpha$ depends on the type of the field.

### 3 No-Boundary Wave Function of a Black Hole

The calculation of the black hole entropy in the dynamical-black-hole-interior model is made by counting the number of thermally excited internal modes existing at given moment of time (more accurately on the chosen spacelike surface, crossing the horizon). This calculations can be simplified by using the following trick. Consider an eternal version of a black hole, i.e. an eternal black hole with the same mass $M$ as the original black hole formed as the result of collapse. At late time the geometry of both holes are identical. One can trace back in time all the perturbations, propagating at late time in the geometry of the eternal version of a black hole. As the result one can relate perturbations at late time in a spacetime of a real black hole with initial data on the Einstein-Rosen bridge (spatial slice of $t = \text{const}$) of the eternal black hole geometry (see Fig. 1).

We denote the 3-surface of the Einstein-Rosen bridge by $\Sigma$. This surface has the topology $S^2 \times R^1$. The 2-surface $S$ of the horizon $r = 2M$ splits it into two isometric parts: 'external' $\Sigma_+$ and 'internal' $\Sigma_-$. In a spacetime of the eternal black hole the Killing vector $\xi$ which is used to define energy is future directed on $\Sigma_+$ and past directed on $\Sigma_-$. For this reason initial date having a support located on $\Sigma_+$ correspond to the field configurations having positive energy, while the energy of the field configurations with the initial data on $\Sigma_-$ possess negative energy. The
former describes external degrees of freedom of a black hole, while the latter describes the internal ones. The set of fields representing the degrees of freedom of a black hole contains the gravitational perturbations. For given initial values of fields and gravitational perturbations on $\Sigma$ the gravitational constraint equations determine the deformation of the 3-geometry of the Einstein-Rosen bridge (see Fig. 2). We shall use the notion 'deformation' in order to describe not only deformed geometry of the Einstein-Rosen bridge, but also the physical fields on it. By using this terminology we can say that the states of a black hole at late time are uniquely related with deformations of the Einstein-Rosen bridge.

It was proposed in [31] to introduce a wave function of a black hole as the functional over the space of deformations of the Einstein-Rosen bridge. In this approach the wave function of a black hole depends on data located on both parts of the Einstein-Rosen bridge: an external $\Sigma_+$ (external degrees of freedom) and an inner $\Sigma_-$ one (internal degrees of freedom).

Certainly there exist infinite number of different wavefunctions of a black hole. Our aim is to get a useful tool for the description of the canonical ensemble of black holes inside the cavity restricted by a spherical boundary of the radius $r_B$ and with fixed inverse temperature $\beta$ on it. For this reason only very special wavefunctions will be important for us. Here we present a modified version of the no-boundary approach of Ref. [31], which is analogous to the 'no-boundary ansatz' in quantum cosmology [36]. This ansatz singles out a set of no-boundary wavefunctions which is convenient for our purpose.

Instead of the complete Einstein-Rosen bridge we consider its part $\Sigma'$ lying between two spherical 2-dimensional boundaries $S_\pm$ located from both sides of $S$ at

Figure 1: Embedding diagram for a two-dimensional $\theta = \text{const}$ section of the Einstein-Rosen bridge.
the radius \( r = r_B \). \( \Sigma' \) has the topology \( S^2 \times I \), where \( I \) is the unit interval \([0,1] \). We denote by \( M_\beta \) a Euclidean manifold with a boundary \( \partial M_\beta \), which consists of two parts: \( \Sigma' \) and another 3-surface \( \Sigma^B \) with the same topology \( S^2 \times I \), which intersects \( \Sigma' \) at \( S^+ \) and \( S^- \), and which represents the Euclidean evolution of the external boundary \( B \).

We define the no-boundary wavefunction depending on one parameter \( \beta \) by the following path integral

\[
\Psi_\beta(y(x), \varphi(x)) = \int D^4y \ D\phi \ e^{-I[4y, \phi]}.
\] (3.1)

Here \( I[4y, \phi] \) is the Euclidean gravitational action. The integral is taken over Euclidean 4-geometries and matter-field configurations on a spacetime \( M_\beta \) with a boundary \( \partial M_\beta \equiv \Sigma' \cup \Sigma^B \). The integration variables are subject to the conditions

\[
(\bar{y}(x), \varphi(x)), \ x \in \partial M_\beta,
\]

– the collection of 3-geometry and boundary matter fields on \( \partial M_\beta \), which are just the argument of the wavefunction (3.1).

We assume that the 3-metric on the boundary is of the form

\[
ds_{\Sigma'}^2 = d\tau^2 + r_B^2 d\omega^2 + \ldots, \quad \tau \in (-\beta/4, \beta/4),
\] (3.2)

\[
ds_{\Sigma^B}^2 = (1 - r_+/r)^{-1}dr^2 + r^2d\omega^2 + \ldots, \quad r \in [r_+, r_B],
\] (3.3)

where \( r_+ \equiv 2M \), \( d\omega^2 \) is the line element on a unit sphere, and dots indicate omitted terms describing perturbations of the metric.
If \((g_0, \phi_0)\) is a point of the extremum of the action \(I\), then we can write
\[
g = g_0 + \tilde{g}, \quad \phi = \phi_0 + \tilde{\phi},
\]
(3.4)
\[
I[g, \phi] = I_0[g_0, \phi_0] + I_2[\tilde{g}, \tilde{\phi}] + \ldots.
\]
(3.5)

In accordance with this decomposition the no-boundary wavefunction (3.1) in the quasiclassical approximation reads
\[
\Psi_\beta(3g(x), \varphi(x)) = \Psi_\beta^0(3g_0(x), \varphi_0(x)) \cdot \Psi_\beta^1(3\tilde{g}(x), \tilde{\varphi}(x)),
\]
(3.6)
where
\[
\Psi_\beta^0(3g_0(x), \varphi_0(x)) = e^{-I_0[g_0, \phi_0]},
\]
(3.7)
is a classical (tree-level) contribution, and
\[
\Psi_\beta^1(3\tilde{g}(x), \tilde{\varphi}(x)) = \int D\tilde{g} D\tilde{\varphi} e^{-I_2[\tilde{g}, \tilde{\varphi}]},
\]
(3.8)
is a one-loop part.

We consider a theory for which \(\phi_0 = 0\), so that \(g_0\) is a solution of the vacuum Einstein equations. The corresponding Euclidean solution is a part \(M_\beta\) of the Gibbons-Hawking instanton (see Fig. 3), i.e. the Euclidean Schwarzschild solution
\[
ds^2 = F d\tau^2 + F^{-1} dr^2 + d\omega^2, \quad F = 1 - r_+/r,
\]
(3.9)
with \(\tau \in (-\frac{1}{4}\beta_\infty, \frac{1}{4}\beta_\infty)\), where \(\beta_\infty = (F(r_B))^{-1/2}\beta\) is the inverse temperature at infinity. (For a special choice of \(\beta_\infty = 8\pi M\) this part is a half of the instanton.)

To calculate the Euclidean Einstein-Hilbert action
\[
I_0[g_0] = -\frac{1}{16\pi} \int_{M_\beta} R\sqrt{g}d^4x - \frac{1}{8\pi} \int_{\partial M_\beta} K\sqrt{h}d^3x - \frac{1}{8\pi} \int_{\partial M_\beta} K_0\sqrt{h}\sqrt{d^3x}.
\]
(3.10)
for \(M_\beta\) we note that for the vacuum solution \(R = 0\), so that only the surface terms of the action contribute. The calculation of the trace of the extrinsic curvature \(K\) for \(\Sigma^B\) is straightforward and gives
\[
K = \frac{1}{2} \frac{r_+}{r_B^2} \frac{1}{\sqrt{F(r_B)}} + \frac{2\sqrt{F(r_B)}}{r_B}.
\]
(3.11)
By using this expression we get
\[
\frac{1}{8\pi} \int_{\Sigma^B} K\sqrt{h}d^3x = \frac{1}{2} \beta r_B \sqrt{F(r_B)} + \frac{\beta r_+}{8\sqrt{F(r_B)}}.
\]
(3.12)
Figure 3: Part $M_{\beta}$ of the Gibbons-Hawking instanton, which gives the main contribution into the no-boundary wavefunction $\Psi_{\beta}$ in the quasiclassical approximation.

The extrinsic curvature vanishes identically everywhere on the boundary $\Sigma'$ except the point $r = r_+$, where it has $\delta$-type singularity. The corresponding contribution is

$$\frac{1}{8\pi} \int_{\Sigma'} K \sqrt{h} d^3x = \frac{\pi}{2} r_+^2 \left( 1 - \frac{\beta}{4r_+ \sqrt{F(r_B)}} \right). \tag{3.13}$$

There is a well known ambiguity in the term $\frac{1}{8\pi} \int_{\partial M_{\beta}} K \sqrt{h} d^3x$ which is to be subtracted and which depends on the choice of a reference space. Here we fix this ambiguity simply by subtracting from $\frac{1}{8\pi} \int_{\partial M_{\beta}} K \sqrt{h} d^3x$ its value for $M = 0$. Finally we have

$$I_0[g_0] = \frac{1}{2} \beta r_B \left( 1 - \sqrt{F(r_B)} \right) - \frac{\pi}{2} r_+^2. \tag{3.14}$$

Now we calculate the one-loop contribution to the no-boundary wavefunction of a black hole. First of all we note that each of the fields (including the gravitational perturbations) give independent contribution to $I_2$. It means that $\Psi_{\beta}^1$ is a product of wavefunctions $\Psi_{\beta}[\varphi]$ depending on only one particular type of field $\varphi$. (We remind that $\phi_0 = 0$ and hence the value $\varphi$ on the boundary coincides with its perturbation $\tilde{\phi}$.) The Gaussian integral (3.8) in the definition of $\Psi_{\beta}[\varphi]$ can be easily calculated. Really, let us denote by $\phi(\varphi)$ a solution of field equations for the action $I_2[\phi]$ obeying the boundary conditions $\phi|_{\partial M_{\beta}} = \varphi$. Then

$$\Psi_{\beta}[\varphi] = C e^{-I_2[\varphi]}, \tag{3.15}$$

We assume that $\varphi$ on $\Sigma^B$ does not depend on $\tau$. In this case a solution $\phi(\varphi)$ obeying
boundary conditions
\[
\phi(x) \big|_\Sigma^\pm = \phi(\pm \beta_\infty/4, x) = \varphi_\pm(x),
\] (3.16)
can be written as a decomposition
\[
\phi(\tau, x) = \sum_\lambda \left\{ \varphi_{\lambda, \pm}(\tau, x) + \varphi_{\lambda, \mp}(\tau, x) \right\}
\] (3.17)
in the basis functions of the field equation
\[
u_{\lambda, \pm}(\tau, x) = \frac{\sinh(\beta_\infty/4 \mp \tau)}{\sinh(\beta_\infty/2)} R_\lambda(x),
\] (3.18)
where \(R_\lambda(x)\) is a complete set of spatial harmonics on \(M_\beta^B\) with a chosen boundary conditions on \(\Sigma^B\). The coefficients \(\varphi_{\lambda, \pm}\) in (3.17) are just the decomposition coefficients of the fields (3.16) in the basis of spatial harmonics
\[
\varphi_{\pm}(x) = \sum_\lambda \varphi_{\lambda, \pm} R_\lambda(x).
\] (3.19)

Substituting (3.17) into \(I_2[\varphi]\), integrating by parts with respect to the Euclidean time and spatial coordinates and taking into account the equations of motion, one finds that the Euclidean action reduces to the following quadratic form in \(\varphi_{\lambda, \pm}\):
\[
I_2[\varphi_+, \varphi_-] = \sum_\lambda \left\{ \frac{\omega_\lambda \cosh(\beta_\infty \omega_\lambda/2)}{2 \sinh(\beta_\infty \omega_\lambda/2)} (\varphi_{\lambda, +}^2 + \varphi_{\lambda, -}^2) - \frac{\omega_\lambda}{\sinh(\beta_\infty \omega_\lambda/2)} \varphi_{\lambda, +} \varphi_{\lambda, -} \right\},
\] (3.20)
This action is a sum of Euclidean actions for quantum oscillators of frequency \(\omega_\lambda\) for the interval \(\beta_\infty\) of the Euclidean time with the initial value of its amplitude \(\varphi_-\) and the final value \(\varphi_+\).

To summarize we obtain the following expression for the no-boundary wavefunction of a black hole in the semiclassical approximation
\[
\Psi_\beta[M, \varphi_+, \varphi_-] = N e^{-1/2 \beta r_B [1 - (1 - 2M/r_B)^{1/2}]} + 2\pi M^2 - \sum I_2[\varphi_+, \varphi_-],
\] (3.21)
Here \(N\) is a normalization constant. The symbol of summation in the exponent indicates that the additional a summation over all physical fields must be done. The square of this wavefunction gives the probability to find a given configuration in the state determined by the parameter \(\beta\). For large \(M\) \((M \gg m_P)\) this probability is a sharp peak with width \(\approx m_P\) located near the value of \(M = M_\beta \equiv \beta_\infty/8\pi\). For \(\beta_\infty = 8\pi M\) and \(r_B \to \infty\) this wavefunction coincides with a no-boundary wavefunction obtained in [31].

For fixed \(M\) the density matrix for internal variables \(\varphi_-\) of a black hole is defined as
\[
\hat{\rho}_\beta[\varphi_-, \varphi_-'] = \int D\varphi_+ \Psi_\beta[\varphi_+, \varphi_-] \Psi_\beta[\varphi_+, \varphi_-'],
\] (3.22)
and it is of the form

\[ \hat{\rho}[\varphi_-, \varphi'_-] = P' e^{-\tilde{I}_2[\varphi_-, \varphi'_-]}, \]

where \( \tilde{I}_2 \) is given by the expression (3.20) with \( \beta \) changed by \( 2\beta \). It is easy to show that

\[ \hat{\rho}[\varphi_-, \varphi'_-] = P'' \langle \varphi_-|e^{-\beta \hat{H}}|\varphi'_- \rangle, \]

where \( P'' \) is a normalization constant and \( \hat{H} \) is the Hamiltonian of free fields \( \varphi \) propagating on the Schwarzschild background. The statistical-mechanical entropy \( S^{SM} \) of a black hole obtained by using this density matrix coincides with the expression (2.1).

The statistical-mechanical entropy \( S^{SM} \) in this as well as other ‘dynamical’ approaches possesses the following main properties: (1) \( S^{SM} \sim A \), where \( A \) is the surface area of a black hole; (2) \( S^{SM} \) is divergent and requires regularization: \( S^{SM} \sim A/l^2 \), where \( l \) is the cut-off parameter; (3) \( S^{SM} \) depends on the number of fields, which exist in nature; (4) \( S^{SM} \sim S^{BH} \) for \( l \sim l_P \).

The following two problems are of importance: (1) What is the relation between the statistical-mechanical entropy \( S^{SM} \) introduced by counting the internal degrees of freedom of a black hole and its thermodynamical entropy \( S^{TD} \)? In particular how to explain the universality of the Bekenstein-Hawking entropy \( S^{BH} \), while \( S^{SM} \) is not universal and depends on the number of fields? (2) The formal expression for the statistical-mechanical entropy \( S^{SM} \) contains the Planck scale cut-off. Does it mean that by studying the thermodynamical properties of black holes we can obtain certain conclusions concerning physics at Planckian scales?

In what follows we shall try to clarify these questions.

4 Renormalized Effective Action and Free Energy

The complete information concerning the canonical ensemble of black holes with a given inverse temperature \( \beta \) at the boundary is contained in the partition function \( Z(\beta) \) given by the Euclidean path integral \[ Z(\beta) = \int D[\phi] \exp(-I[\phi]). \]

Here the integration is taken over all fields including the gravitational one that are real on the Euclidean section and are periodic in the imaginary time coordinate \( \tau \) with period \( \beta \). The quantity \( \phi \) is understood as the collective variable describing the fields. In particular it contains the gravitational field. Here \( D[\phi] \) is the measure of the space of fields \( \phi \) and \( I_E \) is the Euclidean action of the field configuration. The action \( I_E \) includes the Euclidean Einstein action. The state of the system is determined by the choice of the boundary conditions on the metrics that one integrates over. For
the canonical ensemble for the gravitational fields inside a spherical box of radius \( r_B \) at temperature \( T \) one must integrate over all the metrics inside \( r_B \) which are periodically identified in the imaginary time direction with period \( \beta = T^{-1} \). Such a partition function must describe in particular the canonical ensemble of black holes. The partition function \( Z \) is related with the effective action \( \Gamma = -\ln Z \) and with the free energy \( F = \beta^{-1} \Gamma = -\beta^{-1} \ln Z \).

By using the stationary-phase approximation one gets
\[
\beta F \equiv -\ln Z = I[\phi_0] - \ln Z_1 + \ldots \tag{4.2}
\]

Here \( \phi_0 \) is the (generally speaking, complex) solution of classical field equations for action \( I[\phi] \) obeying the required periodicity and boundary conditions. Besides the tree-level contribution \( I[\phi_0] \), the expression \( (4.2) \) includes also one-loop corrections \( \ln Z_1 \), connected with the contributions of the fields perturbations on the background \( \phi_0 \), as well as higher order terms in loops expansion, denoted by \( (\ldots) \). The one-loop contribution of a field \( \phi \) can be written as follows \( \ln Z_1 = -\frac{1}{2} \text{Tr} \ln (-D) \), where \( D \) is the field operator for the field \( \phi \) inside the box \( r_B \). The one-loop contribution contains divergences and required the renormalization. In order to be able to absorb these divergences in the renormalization of the coefficients of the initial classical action we chose the latter in the form
\[
I_{cl} = \int d^4x \sqrt{g} L, \tag{4.3}
\]

\[
L = \left[ -\frac{\Lambda_B}{8\pi G_B} - \frac{R}{16\pi G_B} + c_1 R^2 + c_2 R_{\mu\nu}^2 + c_3 R_{\alpha\beta\mu\nu}^2 \right]. \tag{4.4}
\]

By using heat-kernel representation for \( \ln Z_1 \) one can write
\[
-\frac{1}{2} \ln \det (-D) = \frac{1}{2} \int_{s^2}^{\infty} ds \frac{1}{s} \text{Tr} K(s), \tag{4.5}
\]

where \( K(s) \) is the heat-kernel of the operator \( D \) which has the following Schwinger-DeWitt expansion
\[
K(s) = e^{-sD} = \frac{1}{16\pi^2 s^2} \sum a_n s^n, s \to 0. \tag{4.6}
\]

For the particular case of a scalar massless field
\[
a_0 = 1, \quad a_1 = (1/6 - \xi)R, \tag{4.7}
\]
\[
a_2 = \frac{1}{180} \left( R_{\alpha\beta\mu\nu}^2 - \frac{1}{180} R_{\mu\nu}^2 - \frac{1}{6} (\frac{1}{5} - \xi) R + \frac{1}{2} (\frac{1}{6} - \xi)^2 R^2 \right) \tag{4.8}
\]

By substituting this expansion into \( (4.3) \) one can conclude that the one-loop contribution \( \Gamma_1 \) to the effective action can be written in the form
\[
\Gamma_1 = \Gamma_1^{\text{div}} + \Gamma_1^{\text{fin}}. \tag{4.9}
\]
where
\[ \Gamma^\text{div}_1 = -\frac{1}{32\pi^2} \int d^4x \sqrt{g} \left[ \frac{a_0}{2\delta^4} + \frac{a_1}{\delta^2} - 2a_2 \ln(\delta) \right]. \tag{4.10} \]

The divergent part of the one-loop effective action has the same structure as the initial classical action \((4.3)\) and hence one can write
\[ \Gamma = \Gamma^\text{ren}_{\text{cl}} + \Gamma^\text{ren}_1, \tag{4.11} \]
\[ \Gamma^\text{ren}_1 = \Gamma_1 - \Gamma^\text{div}_1 = \Gamma^\text{fin}_1. \tag{4.12} \]
Here \(\Gamma^\text{ren}_{\text{cl}}\) is identical to the initial classical action with the only change that all the bare coefficients \(\Lambda_B, G_B,\) and \(c^i_B\) are substituted by their renormalized versions \(\Lambda^\text{ren},\) \(G^\text{ren},\) and \(c^i\text{ren}\)
\[ \frac{\Lambda^\text{ren}}{G^\text{ren}} = \frac{\Lambda_B}{G_B} + \frac{1}{8\pi\delta^4}, \tag{4.13} \]
\[ \frac{1}{G^\text{ren}} = \frac{1}{G_B} + \frac{1}{2\pi\delta^2} \left( \frac{1}{6} - \xi \right), \tag{4.14} \]
\[ c^i\text{ren} = c^i_B + \alpha^i \ln \delta. \tag{4.15} \]

We shall refer to \((4.11)\) as to the loop expansion of the renormalized effective action. After multiplying the the renormalized effective action by \(\beta^{-1}\) we get the expansion for the renormalized free energy.

The effective action \(\Gamma\) contains the complete information about the system under consideration. In particular the variation of \(\Gamma\) with respect to the metric provides one with the equations for the quantum average metric \(\bar{g} = \langle g \rangle:\)
\[ \frac{\delta \Gamma}{\delta \bar{g}} = 0. \tag{4.16} \]

One usually assumes that quantum corrections are small and solves this equation perturbatively:
\[ \bar{g} = g_{\text{cl}} + \delta g, \tag{4.17} \]
where \(g_{\text{cl}}\) is a solution of the classical equations. At this point we need to make an important remark. In principle, there exist two possibilities: either to begin with the solution of the classical equations for the action \((4.3),\) or its renormalized version \(\Gamma^\text{ren}_{\text{cl}},\) which is written in terms of the renormalized constants. One usually assumes that the renormalized values of \(\Lambda^\text{ren}\) and \(c^i\text{ren}\) vanish \(\Lambda^\text{ren} = c^i\text{ren} = 0.\) It means that in general case their initial values were not vanishing unless one is dealing with some special type of theory (e.g. assuming supersimmetry). In other words the global properties of the solutions for \(I_{\text{cl}}\) and \(\Gamma^\text{ren}_{\text{cl}}\) are generally different. So to provide the condition that \(\delta g\) is small, one is to begin with the metric \(g_{\text{cl}}\) that is an extremum of \(\Gamma^\text{ren}_{\text{cl}}\)
\[ \frac{\delta \Gamma^\text{ren}_{\text{cl}}}{\delta g_{\text{cl}}} = 0. \tag{4.18} \]
We assume that the renormalization of the coupling constants in the classical action is made from the very beginning and we shall assume that the ‘classical’ field $g_{\text{cl}}$ is a solution of the equation (4.18). In our case $g_{\text{cl}}$ is the Euclidean black hole metric, while the metric $\bar{g}$ describes the Gibbons-Hawking instanton deformed due to the presence of quantum corrections to the metric. The quantity $\Gamma[\bar{g}]$ being expressed as the function of boundary conditions (\beta and $r_B$) specifies the thermodynamical properties of a black hole.

5 Thermodynamical Entropy

For the above described canonical ensemble of gravitational fields the leading tree-level contribution to the renormalized effective action is given by the Euclidean gravitational action for the Euclidean black hole solution (the Gibbons-Hawking instanton).

Under the assumption that $\Lambda_{\text{ren}} = 0$ and $c_{\text{ren}}^i = 0$ the tree-level contribution to the renormalized free energy of the black hole is

$$F_{\text{ren}}^0 \equiv \beta^{-1} \Gamma_{\text{ren}}[g_{\text{cl}}] = r_B \left(1 - \sqrt{1 - r_+/r_B}\right) - \pi r_+^2 \beta^{-1}. \quad (5.1)$$

Here $r_+ = 2G_{\text{ren}}M$ is the gravitational radius of a black hole of mass $M$, which for a given temperature $\beta^{-1}$ at the boundary $r_B$ is defined by the relation $\beta = \cdots$
According to definition the thermodynamical entropy of a black hole \( S^{TD}_0 \) is determined by the response of the free energy of a system including a black hole to the change of the temperature. One can easily verify that

\[
S^{TD}_0 = -\frac{dF^\text{ren}_0}{dT} \equiv \beta^2 \frac{dF^\text{ren}_0}{d\beta} = \frac{A_H}{4l_P^2}, \tag{5.2}
\]

and hence it coincides with the Bekenstein-Hawking expression \( S^{BH} \). (It is assumed that \( r_+ \) in \( F^\text{ren}_0 \) is expressed in terms of \( \beta \) and \( r_B \) before differentiation with respect to \( \beta \).) One-loop contribution in Eq. (4.2) describes quantum correction to the entropy of a black hole as well as the entropy of thermal radiation in its exterior. The latter evidently depends on the radius \( r_B \) of the boundary. Since the Euclidean black hole background is regular the corresponding contribution \( F^\text{ren}_1 \) is finite. For this reason the quantum corrections to the Bekenstein-Hawking entropy \( 4\pi M^2 \) are also finite. They are small unless the mass of a black hole \( M \) is comparable with the Planckian mass \([37, 38, 39]\). Due to the presence of the conformal anomalies one might expect that the leading one-loop corrections to \( S^{TD} \) are of the order \( \ln M \) (see, e.g. \([40]\)).

## 6 Statistical-Mechanical Entropy

The derivation of the thermodynamical entropy of a black hole requires the *on-shell* calculations. It means that one uses only a regular Euclidean metric that is solution of the field equations (4.16). The discussion of the relation of the thermodynamical and statistical-mechanical entropy of a black hole requires *off-shell* calculations. The reason for this is quite simple and can be explained, for example, by using the approach based on the no-boundary wavefunction of BFZ \([31]\). The matrix elements in the \( |\varphi_-\rangle \) basis of the operator \( \hat{\rho} \ln \hat{\rho} \) which enters the definition of the statistical-mechanical entropy (1.4) can be obtained by partially differentiating (3.24) with respect to \( \beta \). On the other hand the Hamiltonian \( \hat{H} \) depends on the black-hole geometry, and hence on the mass \( M \) of a black hole. That is why to obtain the expression for the statistical-mechanical entropy one needs to be able to use \( \beta \) and \( M \) as independent parameters.

Strictly speaking for \( \beta \neq \beta_H \equiv 8\pi M \) there are no regular Euclidean solutions with the Euclidean black-hole topology \( R^2 \times S^2 \). Such solutions can be obtained only if one exclude a horizon sphere \( S^2 \). One can also consider the spacetime with included horizon, provided the curvature has there \( \delta \)-like behavior corresponding to the cone-like singularity of the metric. For \( \beta = \beta_H \) the singularity dissapears. In order to be able to discuss the statistical-mechanical entropy and its relation to the thermodynamical entropy one must generalize the calculation of the one-loop contribution to the renormalized free energy to the case of spaces with cone-like singularity. The new feature which arises is that the corresponding renormalized one-loop corrections might contain new type of divergence, which is directly connected with the presence
of cone singularity. In order to make the answer finite one might introduce spatial cut-off in the volume integrals near the cone-singularity. It is convenient to restrict the integration by some proper distance \( l \) from singularity. This cut-off was present in the 't Hooft’s ‘brick wall model’ \([25]\). In the model with ‘dynamical black-hole-interior’ the presence of such a cut-off was connected with the quantum fluctuations of the horizon \([30]\). Similar cut-off naturally arises in the string theory \([33]\). The renormalized one-loop contribution \( F^{\text{ren}}_1 \) to the free energy is of the form

\[
F^{\text{ren}}_1 = F^{\text{ren}}_1[\beta, \beta_H, \varepsilon], \tag{6.1}
\]

where \( \varepsilon = (l/2G^{\text{ren}}_{\text{M}})^2 \) is the dimensionless cut-off parameter. For \( \varepsilon \to 0 \) and \( \beta \neq \beta_H \) the one-loop free energy \( F^{\text{ren}}_1 \) is divergent \( F^{\text{ren}}_1 \sim \varepsilon^{-1} f(\beta, \beta_H) \). For \( \beta = \beta_H \) the divergence dissappears, so that \( F^{\text{ren}}_1[\beta_H, \beta_H, 0] \) is finite. This quantity is directly related with quantum (one-loop) corrections to the thermodynamical entropy of a black hole

\[
S^{\text{T}}_1 = \beta^2_H \frac{d}{d\beta_H} \left[ \frac{\partial F^{\text{ren}}_1[\beta_H, \beta_H, 0]}{\partial \beta} + \beta^2_H \frac{\partial F^{\text{ren}}_1[\beta_H, \beta_H, 0]}{\partial \beta_H} \right]_{\beta = \beta_H}. \tag{6.2}
\]

The expression (6.1) allows one to get the statistical-mechanical entropy \( S^{\text{SM}} \).
Dowker and Kennedy [43] and Allen [44] made an important observation that

\[ F_{\text{ren}}^1 = F_{\text{ren}}^\text{vac} + F_{\text{therm}}^\text{ren}, \quad \frac{\partial F_{\text{ren}}^\text{vac}}{\partial \beta} = 0, \quad (6.3) \]

\[ F_{\text{therm}}^\text{ren} = -\beta^{-1} \ln \text{Tr} \left[ e^{-\beta H} \right] = \ln \left[ \sum \exp(-\beta E_n) \right], \quad (6.4) \]

where \( E_n \) is the energy (eigenvalue of the Hamiltonian \( \hat{H} \) of the field \( \varphi \)). By using the expansion in eigenfunctions one can obtain

\[ F_{\text{therm}}^\text{ren} = \sum_{\lambda} f(\beta \omega_\lambda) = \int d\omega N(\omega|\beta_H, \varepsilon) f(\beta \omega). \quad (6.5) \]

\[ f(\beta \omega) = \beta^{-1} \ln[1 - \exp(-\beta \omega)] \] is free energy of an oscillator of frequency \( \omega \) at inverse temperature \( \beta \), and \( N(\omega|\beta_H) \) is the density of number of states at the given energy \( \omega \) in a spacetime of a black hole of mass \( M \). This density of number of states diverges. In order to make it finite we introduced the cut-off \( \varepsilon \). We include \( \varepsilon \) as the argument of \( N \) in order to remind about this. The expression (6.5) is usually a starting point for 'brick wall' model.

The statistical-mechanical entropy \( S^{\text{SM}} \) is

\[ S^{\text{SM}} = \left[ \frac{\partial F_{\text{ren}}^1}{\partial \beta} \right]_{\beta_H} = \left[ \frac{\partial F_{\text{therm}}^\text{ren}}{\partial \beta} \right]_{\beta_H} \quad (6.6) \]

\[ = \int d\omega N(\omega|\beta_H, \varepsilon) s(\beta \omega) \quad (6.7) \]

Here \( s(\beta \omega) = \beta \omega/(e^{\beta \omega} - 1) - \ln(1 - e^{-\beta \omega}) \) is the entropy of a quantum oscillator of frequency \( \omega \) with inverse temperature \( \beta \). \( S^{\text{SM}} \) is divergent in the limit \( \varepsilon \to 0 \). The divergence is directly related with the divergency of the density of number of states located in the narrow region in the vicinity of the horizon.

By comparing the expressions (6.2) and (6.6) we can conclude that \( S^{\text{TD}} \) and \( S^{\text{SM}} \) differs from one another. It happens for the following two reasons: (1) Vacuum polarization \( (F_{\text{ren}}^\text{vac}) \) depends on \( M \) and hence on \( \beta_H \); (2) \( d/d\beta \) does not commute with \( \text{Tr} \)-operation. In general case one gets

\[ S_1^{\text{TD}} = S^{\text{SM}} + \Delta S. \quad (6.8) \]

In the limit \( \varepsilon \to 0 \) \( \Delta S \neq 0 \) is also divergent, but \( S_1^{\text{TD}} \) remains finite and (for \( M \gg m_P \)) small. The relation (6.8) provides explanation of the entropy renormalization procedure by Thorne and Zurek [20].

Fursaev and Solodukhin [47, 48, 49] recently proposed another approach for off-shell calculations of thermodynamical characteristics of a black hole. Namely, instead of cutting-off the vicinity of a cone singularity, they proposed to calculate the effective action directly on a spacetime with cone-like singularity. In order to make this
mathematically well-defined one might at first consider a manifold with the topology $R^2 \times S^2$ which is smooth at the fixed-point sphere (horizon) and differs from the cone metric only in the very narrow region of size $l$ near the horizon. The effective action must be considered as the function of $l$ and the parameter $l$ must be finally put to zero. The one-loop correction to the free energy $F_{\text{cone}}$ can be divergent in this limit, but it is possible to show that the divergence is proportional to $(\beta - \beta_H)^2$. That is why the corresponding contribution $S_{\text{cone}} = \beta^2 \partial F_{\text{cone}} / \partial \beta |_{\beta = \beta_H}$ to the entropy is finite. This off-shell approach gives the same expression for the thermodynamical entropy and might be considered as useful tools for such calculations. The relation of this off-shell entropy $S_{\text{cone}}$ to the statistical-mechanical entropy $S_{\text{SM}}$ is not clear. Among other approaches to the calculation of the entropy we mention also the approach based on the Pauli-Villars regularization [45].

7 Black Hole Thermodynamics and Physics at Planckian Scales

The expression for the statistical-mechanical entropy (2.1) requires cut-off. The value $l$ of the cut-off parameter is of order of the Planckian length. Does it mean that thermodynamical characteristics of a black hole for their understanding require the knowledge of physics at Planckian scales?

When we are discussing black hole solutions and their properties we use gravitational equations. The coupling constants in these equations are assumed to coincide with 'observable' values. Due to the existence of ultraviolet divergencies the observable coupling constants differ from their initial bare values. Any procedure which gives sense to this renormalization procedure finally must deal with the problems of physics at Planckian scale. But in this sense black hole physics does not differ from the usual Newtonian theory. Calculations in the Newtonian theory also use the renormalized ('observable') gravitational constant, and hence in order to derive the same results in the framework of quantum gravity beginning from some initial background theory one must pass through all the complications connected with the renormalization procedure and redefining the coupling constants. One can do it from very beginning or develop more complicated scheme and made all the renormalizations only at the end. The same is true also for black holes. Besides this in the case of black hole there are situations when quantum gravity becomes really important. It is well known that the final stage of a black hole evaporation as well as the structure near singularity inside a black hole for their consideration require quantum gravity. Quantum gravity might be also useful for study of small quantum corrections to black hole characteristics. But these corrections can essentially change parameters of a black hole when the curvature at the horizon becomes comparable with Planckian curvature, i.e. for black holes of Planckian mass. These remarks are of course trivial. But if we exclude these evident cases do we still need quantum gravity to explain
properties of macroscopic black holes?

This question is not new. The standard derivation of Hawking quantum radiation of a black hole formally requires the integration over all (including much higher than Planckian) frequencies of the initial zero-point-fluctuations of a quantum field. There is a belief that one can escape the formal usage of super-Planckian energies in the calculations. This point of view was supported by recent result by Unruh [46]. He considered a model in which due to the presence of dispersion the frequencies of zero-point-fluctuations are restricted. Unruh has shown that nevertheless the Hawking radiation at late time remains constant and with high accuracy thermal.

A similar problem arises in connection of a black-hole entropy. We saw, for example, that the statistical-mechanical entropy $S^{SM}$ is dependent on the cut-off parameter. One might argue that such a cut-off must be provided by quantum gravity. For this reason $S^{SM}$ is (at least potentially) the quantity which for its knowledge requires Planckian scale physics. Does it mean that the study of the thermodynamical properties might give us information about these scales? The above discussion indicates that it is impossible. In the standard gedanken experiments the observable quantity is $S^{TD}$. $S^{SM}$ (at least in the leading order) does not contribute to $S^{TD}$ and hence one cannot measure it. The statistical-mechanical entropy might be useful for description of excitations in the close vicinity of the horizon (for example of their damping). The main contribution to $S^{SM}$ is given by very high frequency modes inside the gravitational barrier propagating very close to the horizon. It looks like that the only reasonable way to measure $S^{SM}$ is to excite these modes. For example, one can do it by colliding particles of superhigh energy near the horizon. This experiment requires very high (super-Planckian) energies. But having these energies available one can use them for study Planckian physics in usual Minkowski spacetime without any black holes.

To conclude, quantum gravity is required for understanding very fundamental problems of black holes, such as the problem of final state, but it also looks like that the thermodynamics of macroscopical black hole does not provide us with any new powerful tools for verifying the theory of quantum gravity.

8 Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] J. D.Bekenstein, Nuov.Cim.Lett. 4 (1972) 737.
[2] J. D. Bekenstein, Phys.Rev. D7 (1973) 2333.
[3] J. M. Bardeen, B. Carter, and S. W. Hawking, Comm.Math.Phys. 31, (1973) 181.

[4] J. D. Bekenstein, Phys.Rev. D9 (1974) 3292.

[5] K. S. Thorne, W. H. Zurek, and R. H. Price, in *Black Holes: The Membrane Paradigm*, edited by K. S. Thorne, R. H. Price, and D. A. MacDonald (Yale University Press, New Haven, 1986), p.280.

[6] I. Novikov and V. Frolov, *Physics of Black Holes*. (Kluwer Academic Publ., Dordrecht-Boston-London), 1989.

[7] R. W. Wald, In:*Black Hole Physics* (Eds.V. DeSabbata and Z. Zhang), (Kluwer Academic Publ., Dordrecht-Boston-London), 1992).

[8] V. Frolov and D. N. Page, Phys.Rev.Lett. 71 (1993) 3902.

[9] J.B.Hartle and S.W.Hawking, Phys.Rev. D13, 2188 (1976).

[10] G. W. Gibbons and S. W. Hawking, Phys.Rev. D15 (1976) 2752.

[11] S. W. Hawking, In: *General Relativity: An Einstein Centenary Survey*. (eds. S. W. Hawking and W. Israel), Cambridge Univ.Press, Cambridge, 1979.

[12] J. W. York, Phys.Rev. D33 (1986) 2092.

[13] H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. Jork, Phys.Rev. D42 (1990) 3376.

[14] R. M. Wald, Phys.Rev. D48 (1993) 3427.

[15] T. Jacobson and R. C. Myers, Phys.Rev. Lett. 70 (1993) 3684.

[16] V. Iyer and R. M. Wald, Phys.Rev. D50 (1994) 846.

[17] T. Jacobson, G. Kang, and R. C. Myers, Phys.Rev. D49 (1994) 6587.

[18] G. W. Gibbons and R. E. Kallosh, Phys.Rev. D51 (1995) 2839.

[19] S. W. Hawking, G. T. Horowitz, and S. F. Ross, Phys.Rev. D51 (1995) 4302.

[20] W. H. Zurek and K. S. Thorne, Phys.Rev.Lett. 54 (1985) 2171.

[21] V. F. Mukhanov, JETP Lett. 44 (1986) 63.

[22] J. D. Bekenstein and V. F. Mukhanov. ‘Spectroscopy of quantum black holes’ Preprint gr-qc/9505012 (1995).

[23] V. A. Berezin, Phys.Lett., B241 (1990) 194.

[24] J. D. Bekenstein, preprint gr-qc/9409015 (1994).

[25] G. ’t Hooft, Nucl.Phys. B256 (1985) 727.

[26] L. Bombelli, R.K.Koul, J.Lee, and R.Sorkin, Phys.Rev. D34 373 (1986).

[27] M.Srednicki, Phys.Rev.Lett. 71, 666 (1993).

[28] V. Frolov and I. Novikov, Phys.Rev. D48 (1993) 1607.

[29] A. G. Doroshkevich and I. D. Novikov, Soviet Phys. JETP. 63 (1993) 1538.

[30] V. Frolov and I. Novikov, Phys.Rev. D48 (1993) 4545.

[31] A. I. Barvinsky, V. P. Frolov, and A. I. Zelnikov, Phys.Rev.D51 (1995) 1741.

[32] S. Carlip and C. Teitelboim, Preprint gr-qc/93122002 (1993).

[33] H. J. de Vega and N. Sánchez, Nucl.Phys. B299 (1988) 818.

[34] L. Susskind and J. Uglum, Phys.Rev. D50 (1994) 2700.
[35] D. Garfinkle, S. B. Giddings, and A. Strominger, Phys.Rev. D49 (1994) 958.
[36] J.B.Hartle and S.W.Hawking, Phys.Rev. D28, 2960 (1983).
[37] P. R. Anderson, W. A. Hiscock, J. Whitesell, and J. W. York, Phys.Rev. D50 (1994) 6427.
[38] H. W. Braden, J.D. Brown, B. F. Whiting, J. W. York, Jr. Phys.Rev.D42 (1990) 3376.
[39] J. L. Louko and B. F. Whiting , Phys.Rev. D51 (1995) 5583.
[40] D. V. Fursaev, Phys.Rev.D51 (1995) 5352.
[41] S. W. Hawking, Comm.Math.Phys. 43, (1975) 199.
[42] V. Frolov, Phys.Rev.Lett. 74 (1995) 3319.
[43] J.S.Dowker and G.Kennedy, J.Phys. A 11 (1978) 895.
[44] B. Allen, Phys.Rev. D33 (1986) 3640.
[45] J.-G. Demers, R. Lafrance, and R. C. Myers. Phys.Rev.D52 (1995) 2245.
[46] W. G. Unruh, Phys.Rev.D51 (1995) 2827.
[47] D. V. Fursaev and S. N. Solodukhin, On one-loop renormalization of black hole entropy, Preprint E2-94-462, hep-th/9412020 (1994).
[48] S. N. Solodukhin, Phys.Rev.D51 (1995) 609.
[49] D. V. Fursaev and S. N. Solodukhin, Phys.Rev.D52 (1995) 2133.