Discrete Symmetries from Broken $SU(N)$ and the MSSM

P. L. White

Physics Department, University of Southampton, Southampton SO9 5NH, UK.

Abstract

In order that discrete symmetries should not be violated by gravitational effects, it is necessary to gauge them. In this paper we discuss the gauging of $\mathbb{Z}_N$ from the breaking of a high energy $SU(N)$ gauge symmetry, and derive consistency conditions for the resulting discrete symmetry from the requirement of anomaly cancellation in the parent symmetry. These results are then applied to a detailed analysis of the possible discrete symmetries forbidding proton decay in the minimal supersymmetric standard model.
1 Introduction

It has now been known for some time that discrete symmetries are likely to be strongly violated by gravitational effects [1]. This is a problem for the minimal supersymmetric standard model (“MSSM”), where the possible renormalisable interaction terms in the Lagrangian are constrained not merely by Lorentz and gauge invariance, but also by the requirement that a discrete symmetry, R parity, is not broken by the Lagrangian. This symmetry is introduced purely in order to avoid the presence of Yukawa couplings which would lead to phenomenologically unacceptable proton decay, and a number of other possible discrete symmetries have also been studied [2]. In order to ensure that such symmetries are respected by the gravitational interactions it is necessary to gauge them [3], [4]. Gauging a discrete symmetry means obtaining it as the residual symmetry after the spontaneous breaking of some high energy gauge symmetry.

Since this high energy symmetry must originally have been anomaly free, it is possible to derive consistency conditions for the discrete symmetry, and this has been done for the case of a $U(1)$ gauge symmetry breaking to a $\mathbb{Z}_N$ discrete symmetry [5]. These results have been applied to the possible symmetries of the MSSM [6], with the result that only a very restricted choice of such symmetries is possible.

In this paper we shall discuss another possibility for the production of discrete symmetries, namely that of the breaking of $SU(N)$ to its $\mathbb{Z}_N$ centre, derive the resulting consistency conditions, and apply our results to the MSSM. The structure of the paper is as follows. Following this introduction, section 2 briefly discusses how this breaking may occur in a non-trivial way. In section 3, we derive the consistency conditions which must be obeyed by the $\mathbb{Z}_N$ charges of the theory. These are then applied to the MSSM in section 4. Section 5 is the conclusion.

2 Symmetry Breaking of $SU(N)$ to its Centre

In this section we shall begin by briefly discussing how an $SU(N)$ symmetry can be broken without breaking its $\mathbb{Z}_N$ centre (consisting of those matrices in $SU(N)$ of form $e^{2\pi i/N} \times 1$ where 1 is the $N \times N$ unit matrix), an idea which was mentioned in reference [4]. The simplest such mechanism is that this breaking could occur through a field in the adjoint representation acquiring a vacuum expectation value (hereafter “vev”), but it is also possible to break the $SU(N)$ by giving a vev to a field in any representation which has zero charge under the $\mathbb{Z}_N$. The $\mathbb{Z}_N$ charge of a field which is in an irreducible representation of $SU(N)$ with $n_u$ upper and $n_l$ lower vector indices is given by $n_l - n_u$, and is defined modulo $N$. Although this gives many possible breaking mechanisms, we shall only describe breaking through the adjoint and $N$ index symmetric representations.

To break $SU(N)$ completely down to $\mathbb{Z}_N$ is trivial, and can be done either by repeated breaking or by choosing a representation of high enough dimension that when it gains a vev the group breaks completely. However, for reasons which will become apparent when we come to discuss consistency conditions which must be satisfied by the discrete symmetry, this leads to extremely tight constraints on the resulting $\mathbb{Z}_N$ and so is not the most interesting case. Another possibility is that breaking occurs to give part or all of the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, either directly from $SU(N)$ or with some non-trivial mixing with other symmetry groups. This can occur in many ways,
given enough effort in constructing the potentials. As an example we construct a model in which $N = 3$ and $SU(3) \times U(1)$ breaks to $SU(2) \times U(1) \times \mathbb{Z}_3$. From this it is obvious how one can go about constructing more elaborate theories.

We begin with a gauge group $SU(3) \times U(1)_X$, which we break by giving a vev to a scalar $A$ in the adjoint representation of zero $X$ charge. The vev will be taken to be of form

$$< A > = \sqrt{3}aT_8 = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (2.1)$$

where $a$ is a constant of mass dimension 1. Thus we have breaking of form

$$SU(3) \times U(1)_X \to SU(2) \times U(1)_8 \times \mathbb{Z}_3 \times U(1)_X \quad (2.2)$$

where $U(1)_8$ is the $U(1)$ symmetry generated by the $SU(3)$ generator $T_8$. We now introduce a field $B_{ijk}$ in the three index symmetric representation with non-zero $X$ charge, and give it a vev in the 333 component. This then leaves the full symmetry breaking

$$SU(3) \times U(1)_X \to SU(2) \times U(1)_8 \times \mathbb{Z}_3 \times U(1)_X$$

$$\to SU(2) \times U(1)_Y \times \mathbb{Z}_3 \quad (2.3)$$

Here $U(1)_Y$ is a linear combination of $U(1)_8$ and $U(1)_X$. The potential which will give this breaking is

$$V(A, B) = \lambda_1 (\text{tr}(A^2) - 6a^2)^2 + \lambda_2 \text{tr}(|(A+2a')_m B_{mjk} + \text{cyclic}|^2)$$

$$+ \lambda_3 \text{tr}(|B|^2 - b^2)^2 \quad (2.4)$$

Here $\lambda_i > 0 \ \forall i$, and at tree level we impose $a = a'$. It is simple to check that this breaking is stable under (small) radiative corrections, even if the equality $a = a'$ is broken. Similarly, choosing another form of the potential will allow us to break $SU(2) \times U(1)_X$ to $U(1)$, as in the standard model.

### 3 $\mathbb{Z}_N$ Consistency Conditions

In this section we derive consistency conditions for the residual $\mathbb{Z}_N$ symmetry. These come from two sources: firstly the requirement that in the original $SU(N)$ gauge theory there were no anomalies; and secondly the observation that for each irreducible representation the total $\mathbb{Z}_N$ charge must be zero mod $N$. The latter condition will be much more restrictive in the case where the breaking of $SU(N)$ is to $\mathbb{Z}_N$ only. Since this discussion is rather elaborate, the reader who is uninterested in the derivation is recommended to skip to the summary of results at the end of the section.

We shall use Dynkin indices [7] to describe representations of $SU(N)$ (for reviews see [8]). For $SU(N)$ the Dynkin indices take the form of a set of $N - 1$ non-negative integers which we shall call $a_n$. Each distinct set of Dynkin indices corresponds to an irreducible representation of $SU(N)$, and such properties as the dimension and Casimir invariants of the representation can be represented as functions of these indices. The Dynkin indices of
the most common representations of $SU(N)$ are then given in Table 1 below. It is clear that for all the representations listed here the $\mathbb{Z}_N$ charge of a particle in an irreducible representation with Dynkin indices $a_n$ is given by the relation

$$Q = \sum_{p=1}^{N-1} pa_p$$  \hspace{1cm} (3.1)

and in fact it is easy to use the rules for finding direct product of representations to check that this is true for all irreducible representations of $SU(N)$.

| Representation                  | Dynkin indices $(a_1 \ldots a_{N-1})$ |
|---------------------------------|---------------------------------------|
| singlet                         | $(0 \ldots 0)$                        |
| fundamental                     | $(10 \ldots 0)$                       |
| 2 index symmetric               | $(20 \ldots 0)$                       |
| 2 index anti-symmetric          | $(010 \ldots 0)$                      |
| $N-1$ index anti-symmetric      | $(0 \ldots 01)$                       |
| adjoint                         | $(10 \ldots 01)$                      |

We now wish to use the anomaly cancellation constraints in the form of Dynkin indices. We firstly have the constraint that $SU(N) \times SU(N) \times U(1)$ anomalies must cancel. This is equivalent to the statement that

$$\sum_{\text{all reps } \Lambda} q_\Lambda I_2(\Lambda) = 0$$  \hspace{1cm} (3.2)

where $q_\Lambda$ is the $U(1)$ charge of the fermions in the representation $\Lambda$, and $I_2(\Lambda)$ is the second order index of $\Lambda$, whose relation to the second order Casimir invariant $C_2(\Lambda)$ is given by

$$I_2(\Lambda) = C_2(\Lambda)D(\Lambda)$$  \hspace{1cm} (3.3)

where $D(\Lambda)$ is the dimension of the representation. From this we may use the standard results (using the notation of Slansky [8]) that, in root space with $\delta$ equal to half the sum of the positive roots,

$$C_2(\Lambda) = (\Lambda + 2\delta, \Lambda)$$  \hspace{1cm} (3.4)

Here the symbol $\Lambda$ is used to represent the vector in weight space corresponding to the highest weight of the representation $\Lambda$. This may be expanded in terms of the Dynkin indices $a_n$ (which are the components of $\Lambda$ in the Dynkin basis) to give for $SU(N)$

$$C_2(\Lambda) = \sum_{m=1}^{N-1} \left\{ N(N-m)m a_m + m(N-m)a_m^2 + \sum_{n=0}^{m-1} 2n(N-m)a_na_m \right\}$$  \hspace{1cm} (3.5)
and similarly
\[
D(\Lambda) = \prod_{\text{positive roots } \alpha} \frac{(\Lambda + \delta, \alpha)}{(\delta, \alpha)}
\]
\[
= \prod_{p=1}^{N-1} \left[ \frac{1}{p!} \prod_{q=p}^{N-1} \left( \sum_{r=q-p+1}^{p} (1 + a_r) \right) \right]
\]
\[
= (1 + a_1)(1 + a_2)(1 + a_3) \ldots (1 + a_{N-1}) \times \left( \frac{2 + a_1 + a_2}{2} \right) \left( \frac{2 + a_2 + a_3}{2} \right) \left( \frac{2 + a_3 + a_4}{2} \right) \ldots \left( \frac{2 + a_{N-2} + a_{N-1}}{2} \right) \times \left( \frac{3 + a_1 + a_2 + a_3}{3} \right) \ldots \left( \frac{3 + a_{N-3} + a_{N-2} + a_{N-1}}{3} \right) \times \ldots \times \left( \frac{N-1 + a_1 + \ldots + a_{N-1}}{N-1} \right) \right] D(a_1 \ldots a_{N-1})
\] (3.6)

The corresponding equation for the cancellation of the $SU(N) \times SU(N) \times SU(N)$ anomaly is that the object $I_3(\Lambda)$ should vanish, where $I_3(\Lambda)$ is given by $C_3(\Lambda)D(\Lambda)$. The expression for $C_3(\Lambda)$ in terms of the Dynkin indices is [9]
\[
C_3(\Lambda) = \sum_{p,q,r=1}^{N-1} d_{pqr} (a_p + 1)(a_q + 1)(a_r + 1)
\] (3.7)

with the totally symmetric tensor $d_{pqr}$ defined by
\[
d_{pqr} := \frac{1}{2} p(N - 2q)(N - r) \quad \text{where } p \leq q \leq r \]
(3.8)

Note that we have changed both the notation and the normalisation of reference [9], since the normalisation is irrelevant for our purposes, and the factor half in (3.8) will simplify later algebra.

We now wish to find how these conditions constrain the possible $\mathbb{Z}_N$ symmetries of the theory. In order to do this, we shall need to derive a number of relations involving the Dynkin indices. We first note that $D(\Lambda) = D(a_1 \ldots a_{N-1})$ is obviously an integer (since it is the dimension of a representation). Similarly, $D(a_1 \ldots a_N)$ is defined as the dimension of a representation of $SU(N+1)$ and so is also an integer, where $a_m$ is taken to be the same in both cases for $m < N$, and $a_N$ is arbitrary. We may now use equation (3.6) to find that
\[
D(a_1 \ldots a_N) = (1 + a_N) \left( \frac{2 + a_{N-1} + a_N}{2} \right) \times \ldots \times \left( \frac{N + a_1 + \ldots + a_N}{N} \right) D(a_1 \ldots a_{N-1})
\]
\[
= \frac{1}{N!} \left[ \prod_{m=1}^{N} (m + a_{N-m+1} + \ldots + a_N) \right] D(a_1 \ldots a_{N-1})
\] (3.9)
A useful definition is then $f_0(a_1\ldots a_N)$ from

$$D(a_1\ldots a_N) =: \frac{1}{N} f_0(a_1\ldots a_N) D(a_1\ldots a_{N-1})$$  \hspace{1cm} (3.10)$$

Now, since $D(a_1\ldots a_N)$ is an integer, it is clear that

$$f_0(a_1\ldots a_N) D(a_1\ldots a_{N-1}) = 0 \pmod{N}$$  \hspace{1cm} (3.11)$$

This equation is true for all $a_N$ (so long as $a_N$ is a non-negative integer) and so it is also true if we replace $a_N$ by $(a_N+1)$. Thus we define

$$f_{i+1}(a_N) = f_i(a_N + 1) - f_i(a_N) = \frac{\partial f_i(a_N)}{\partial a_N} + \frac{1}{2!} \frac{\partial^2 f_i(a_N)}{\partial a_N^2} + \frac{1}{3!} \frac{\partial^3 f_i(a_N)}{\partial a_N^3} + \ldots$$  \hspace{1cm} (3.12)$$

from which it is clear that

$$f_i(a_1\ldots a_N) D(a_1\ldots a_{N-1}) = 0 \pmod{N} \ \forall \ i$$  \hspace{1cm} (3.13)$$

In particular we have, suppressing all the $a_m$ dependence except that on $a_N$,

$$f_N(a_N) = \frac{\partial^N f_0(a_N)}{\partial a_N^N}$$  \hspace{1cm} (3.14)$$

$$f_{N-1}(a_N) = \frac{\partial^{N-1} f_0(a_N)}{\partial a_N^{N-1}} + \frac{(N-1)}{2} \frac{\partial^N f_0(a_N)}{\partial a_N^N}$$  \hspace{1cm} (3.15)$$

$$f_{N-2}(a_N) = \frac{\partial^{N-2} f_0(a_N)}{\partial a_N^{N-2}} + \frac{(N-2)}{2} \frac{\partial^{N-1} f_0(a_N)}{\partial a_N^{N-1}} + \frac{(N-2)(N-3)}{8} \frac{\partial^N f_0(a_N)}{\partial a_N^N}$$  \hspace{1cm} (3.16)$$

These equations may be derived using (3.12) and remembering that

$$\frac{\partial^{N+1} f_0(a_N)}{\partial a_N^{N+1}} = 0$$  \hspace{1cm} (3.17)$$

We may now expand out the explicit form of these three equations, and we discover that after some simplification

$$f_N(a_1\ldots a_N) = N!$$  \hspace{1cm} (3.18)$$

$$f_{N-1}(a_1\ldots a_{N-1}0) = \sum_p p a_p$$  \hspace{1cm} (3.19)$$

$$f_{N-2}(a_1\ldots a_{N-1}0) = \frac{1}{2(N-1)} \sum_p \left( p a_p^2 - p^2 a_p + 2 \sum_{q<p} q a_q a_p \right)$$  \hspace{1cm} (3.20)$$

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We must now use (3.5) and (3.7), together with (3.19) and (3.20) to derive constraints on the \( \mathbb{Z}_N \) charges of the theory. We begin with the requirement that the \( SU(N) \times SU(N) \times U(1) \) anomaly cancels. The contribution to this anomaly of a representation \( \Lambda \) is \( q_\Lambda C_2(\Lambda)D(\Lambda) \) which can be expanded out with (3.5) to give

\[
q_\Lambda C_2(\Lambda)D(\Lambda) = -q_\Lambda \left( \sum_p pa_p \right)^2 D(\Lambda) + q_\Lambda N^2 D(\Lambda) \left( \sum_p a_p \right) \\
+ q_\Lambda N D(\Lambda) \left( \sum_{p=1}^{N-1} (-p^2 a_p + pa_p^2 + \sum_{q<p} 2qa_q a_p) \right)
\]

(3.21)

If we now assume that the \( U(1) \) charges have been normalised so that \( q_\Lambda \) is an integer, we have a contribution to the anomaly

\[
\sum_\Lambda q_\Lambda C_2(\Lambda)D(\Lambda) = -\sum_\Lambda q_\Lambda \left( \sum_p pa_p \right)^2 D(\Lambda) \mod N^2
\]

(3.22)

where we have used (3.20) and (3.13) in showing that the extra terms are zero mod \( N^2 \). Now for each irreducible representation \( \Lambda \) there are \( D(\Lambda) \) particles of charge \( \sum_p pa_p \), and so we see that the requirement that the anomaly vanish can be expressed as

\[
\sum_i q_i Q_i^2 = 0 \mod N^2
\]

(3.23)

where the sum is over all particles labelled \( i \) of \( U(1) \) and \( \mathbb{Z}_N \) charge \( q_i \) and \( Q_i \) respectively.

In the case where the \( U(1) \) symmetry of the standard model arises as an unbroken subgroup of the \( SU(N) \), or as a linear combination of such a subgroup and another \( U(1) \) symmetry, say \( U(1)' \), we can use the traceless of \( SU(N) \) to show that for particles \( i \) in each irreducible representation with charge \( q_i \) of which \( q_i' \) comes from \( U(1)' \)

\[
\sum_i q_i = \sum_i q_i'
\]

(3.24)

and so (3.23) is unchanged.

The consistency condition for the cancellation of the \( SU(N)^3 \) anomaly is derived in much the same way, and after a certain amount of algebra we find that

\[
C_3(\Lambda) = (\sum_p a_p)^3 + \sum_p 4N^2 a_p p(N^2 - 3Np + 2p^2) + \sum p \frac{3N}{2} a_p^2 (p^3 - \frac{4}{3}Np^2 + \frac{1}{3}N^2 p) \\
+ \sum_{q>p} \frac{3N}{2} a_p a_q (p^2 q + q^2 p - Np^2 - 2Npq + N^2 p) \\
+ \sum p a_p^3 (-\frac{4}{3}Np^2 + \frac{4}{3}N^2 p) + \sum_{q>p} a_p a_q^2 (-\frac{2}{3}Npq + \frac{4}{3}N^2 p) \\
+ \sum_{q>p} a_p^2 a_q (-3Np^2 - \frac{4}{3}Npq + \frac{4}{3}N^2 p) + \sum_{p<q<r} a_p a_q a_r (-6Npq - 3Npr + 3N^2 p)
\]

(3.25)
Extensive use of (3.19), (3.20), and (3.13) reduces this to

\[ C_3(\Lambda)D(\Lambda) = (\sum_p p a_p)^3 D(\Lambda) \mod N^2 \] (3.26)

from which we conclude that the anomaly cancellation condition is that

\[ \sum_i Q_i^3 = 0 \mod N^2 \] (3.27)

It should be noted that we are assuming that the $\mathbb{Z}_N$ charge $Q$ of a particle in the representation labelled by $a_p$ is given by $\sum_p p a_p$, although this is in fact only true mod $N$ and some of our equations require definitions to be valid mod $N^2$. However, it is easy to check that it does not matter if we select another value for $Q$ (equal to the first mod $N$) so long as we pick the same value for all fermions in each representation. In practical terms this amounts to selecting one set of numbers to represent the $\mathbb{Z}_N$ charges and sticking to it. We shall use the simplest such set, namely \{-\frac{N-1}{2}, \ldots, \frac{N-1}{2}\} for $N$ odd, and \{-\frac{N}{2} + 1, \ldots, \frac{N}{2}\} for $N$ even.

A further problem is that the $U(1)$ charges are assumed integer. If they are not, then it is necessary to normalise them by multiplying by some overall constant until they are.

In addition to (3.23) and (3.27), we can use (3.19) to obtain the result that for each irreducible representation separately

\[ \sum_i Q_i = 0 \mod N \] (3.28)

Thus if the $SU(N)$ symmetry breaks trivially to $\mathbb{Z}_N$ only, and thus has no mixing with the gauge group of the standard model (or their ancestors) at high energies, this constraint can be applied to each set of fermions with the same quantum numbers (including $\mathbb{Z}_N$ charge) separately (since before $SU(N)$ breaking they must have belonged to different irreducible representations) and so we shall discover that our theory is constrained effectively only to generation symmetries. This however does not apply in the case where we allow a more complicated breaking, and then we may only impose that (3.28) is satisfied if the sum is over all fermions which might originally have been in the same representation, that is those with the same $\mathbb{Z}_N$ charge.

These constraints ignore the possible effects of particles which are not visible at low energies because they have acquired large masses [5]. This can occur through a fermion gaining a Majorana mass, which is only possible if it has $\mathbb{Z}_N$ charge 0 or $\frac{N}{2}$ (the latter only for even $N$ when $Q_i = Q_j = \frac{N}{2}$; we are here using our convention for the $\mathbb{Z}_N$ charges which gives that $-\frac{N}{2} < Q_i \leq \frac{N}{2}$) and no $U(1)$ charge. Fermions of $\mathbb{Z}_N$ charge 0 do not matter for the consistency conditions, and so the only effect is for $N$ even when each such fermion adds $\frac{N^3}{8}$ to the right hand side of equation (3.27), and removes the constraint (3.28) for charge $\frac{N}{2}$.

It is also possible for two fermions, say $i$ and $j$, to combine to obtain a Dirac mass. This can only occur if $Q_i + Q_j = 0$ or $Q_i + Q_j = N$, and if $q_i + q_j = 0$ (remember that we
must not violate the $U(1)$ symmetry). Thus we have two effects. Firstly, the constraint (3.28) is weakened so that the sum now runs not over all fermions of the same \( \mathbb{Z}_N \) charge \( q \) but over all fermions of charge \( q \) and \(-q\). The case where the two fermions both have charge \( \frac{N}{2} \) does not give any further restriction.

We have now finished the derivation of the consistency conditions, and will thus summarise them below. For \( N \) odd we have

\[
Q(n_Q - n_{-Q}) = 0 \mod N \forall Q \tag{3.29}
\]

\[
\sum_i q_i Q_i^2 = 0 \mod N^2 \tag{3.30}
\]

\[
\sum_i Q_i^3 = 0 \mod N^2 \tag{3.31}
\]

while for \( N \) even these become

\[
Q(n_Q - n_{-Q}) = 0 \mod N \forall Q \neq \frac{N}{2} \tag{3.32}
\]

\[
\sum_i q_i Q_i^2 = 0 \mod N^2 \tag{3.33}
\]

\[
\sum_i Q_i^3 = \eta \left( \frac{N}{2} \right)^3 \mod N^2 \tag{3.34}
\]

In all of these equations the sum over \( i \) is the sum over all fermions labelled \( i \) whose \( \mathbb{Z}_N \) charge is \( Q_i \), \( n_Q \) is the number of fermions of charge \( Q \), and \( \eta \) is an arbitrary integer. Note that for any given particle content, these are necessary but not sufficient conditions that the \( \mathbb{Z}_N \) symmetry might have originated in an anomaly free high energy \( SU(N) \) symmetry.

4 Application to the MSSM

In this section we demonstrate the use of our results by applying them to the specific case of the MSSM. Here we are severely restricted in which \( \mathbb{Z}_N \) symmetries are acceptable from the requirement that terms in the Lagrangian leading to excessive proton decay be forbidden. The first part of this section will be essentially a summary of the analysis in [6])

We shall impose a number of restrictions on the possible symmetries which we consider. These are that the symmetries be generation-blind with three generations; that the only singlet be the right-handed neutrino; and that \( N \leq 4 \). The second of these is particularly restrictive, since the inclusion of particles which interact only at high energy through the \( SU(N) \) symmetry, and so only affect the theory below the breaking scale through the \( \mathbb{Z}_N \), can seriously weaken the constraints. This effect is more pronounced than the case where breaking from \( U(1) \) is considered, since then more of the constraints involve mixing with other symmetries, which singlets of course cannot affect. Including such particles is rather complicated, since the exact mechanism by which \( SU(N) \) breaks will affect how many such particles are allowed. For example, if \( SU(N) \) breaks to give the \( SU(2) \) of the standard model, then some of the singlets will have originally come from \( SU(N) \) multiplets which
after breaking give the $SU(2)$ multiplets of the standard model. It is not clear how one might handle such a situation in general, although it would be possible to analyse it for each particular case.

The MSSM, with the inclusion of right-handed neutrinos but no other singlets, has the gauge group $SU(3) \times SU(2) \times U(1)$, and particle content (with gauge couplings indicated):

$$
\begin{align*}
q & \quad (3, 2, \frac{1}{6}) & L & \quad (1, 2, -\frac{1}{2}) \\
d & \quad (3, 1, \frac{1}{2}) & e & \quad (1, 1, 1) \\
u & \quad (3, 1, -\frac{2}{3}) & \nu & \quad (1, 1, 0) \\
u & \quad (1, 2, -\frac{1}{2}) & \bar{H} & \quad (1, 2, \frac{1}{2})
\end{align*}
$$

The gauge symmetry allows a number of Yukawa couplings, including the dimension four terms $LHe, qHd, q\bar{H}u,$ and $\mu H\bar{H}$ all of which must be permitted by the discrete symmetry in order to obtain the correct structure of the standard model, although with the inclusion of extra singlets it is possible to avoid the necessity of allowing $\mu H\bar{H}$. In addition to these, there is the neutrino mass term $L\bar{H}\nu$ which is an acceptable addition to the standard model (although it is interesting that it is possible to construct models where the discrete symmetry prevents neutrino masses), and terms allowing proton decay. These are lepton number violating terms

$$
LLe \quad LQd
$$

and the baryon number violating term

$$
udd.
$$

We ignore a possible lepton number violating term $\mu L\bar{H}$ because with our assumption that the term $\mu H\bar{H}$ is allowed it can always be removed by a field redefinition if it appears. Although one conventionally requires all of these to be banned by the discrete symmetry, it is in fact sufficient that either (4.2) or (4.3) should be banned together with certain higher dimension terms. We shall adopt the view that to be acceptable, a symmetry must either prevent all of (4.2) and (4.3) or else must ban all lepton or baryon number violating terms of dimension four or five.

In terms of the $\mathbb{Z}_N$ charges, the requirement that none of the standard model Yukawa interactions violate the symmetry is

$$
\begin{align*}
Q(H) + Q(\bar{H}) &= 0 \mod N \\
Q(L) + Q(H) + Q(e) &= 0 \mod N \\
Q(q) + Q(H) + Q(d) &= 0 \mod N \\
Q(q) + Q(\bar{H}) + Q(u) &= 0 \mod N
\end{align*}
$$

We now begin by considering the specific case $N = 2$, where our $\mathbb{Z}_2$ symmetry is a relic of a high energy $SU(2)$ symmetry. It is clear that for this case the constraints (3.32)
and (3.34) are trivial (of course we would expect the latter, since there are no $SU(2)^3$ anomalies). (3.33) gives the constraint

$$6Q(q)^2 + 6Q(d)^2 - 18Q(L)^2 + 18Q(e)^2 - 3Q(H)^2 + 3Q(\bar{H})^2 = 0 \mod 4 \quad (4.5)$$

since we must multiply all the charges by 6 to make them integers, and there are three generations. After some rearrangement, and using (4.4) and the fact that each charge can only take the values 0 or 1, we find that this constraint is always satisfied. Thus we find that any $Z_2$ symmetry which bans all the unwanted terms in the Lagrangian is acceptable from the point of view of anomaly constraints. Examples of typical theories from which such a symmetry could appear are the wide range of theories involving $SU(2)_R$ (where the subscript $R$ means “right”), and in fact the usual R-parity could be such a case. To see this, note that all discrete symmetries added to the standard model can only really be defined modulo the discrete symmetries which are already there and gauged, and these are the $Z_2$ centre of $SU(2)_L$, the $Z_3$ centre of $SU(3)_C$, and discrete subgroups of $U(1)_Y$. By this argument, the only inequivalent $Z_2$ symmetries are $Z_{2L}$ (from $SU(2)_L$), R-parity, and lepton number, although it is not easy to see whether it is possible to construct an $SU(2)$ symmetry which breaks to give the last.

The case where $N = 3$ is less trivial. The simplest way of solving for all the possible discrete symmetries is to write down every possible set of charges satisfying (4.4) and then to test them all against the consistency conditions. This can be done by noticing that given (4.4) we need only assign $Z_3$ charges to $q$, $L$, $\nu$, and $H$ to define the symmetry completely, and $\nu$ will usually be given by the consistency conditions. We thus only have $3^3$ symmetries to consider, and many of these are equivalent to one another under reflections (replacing each charge $Q$ with $-Q$). The constraints (3.29) to (3.31) are then entirely described by

$$Q(H) + Q(\bar{H}) = 0$$
$$Q(q)^2 - 2Q(u)^2 + Q(d)^2 = 0 \quad (4.6)$$
$$2Q(q) + Q(d) + Q(u) + 2Q(L) + Q(e) + Q(\nu) = 0 \mod 3$$

where we have simplified using the fact that our charges are all in the set $\{-1, 0, +1\}$. This gives a fairly straightforward result, and rather than going through all the algebra in detail we shall simply give the possible symmetries, which are listed in Table 2. Note that because we have 3 generations, the consistency conditions are much weaker than we might have expected.

Having found all the possible symmetries, we now find whether they give acceptable constraints on the MSSM Lagrangian. B clearly does not prevent any of the proton decay terms and is of no interest to us. It is in fact the $Z_3$ centre of the colour group. None of these symmetries bans both (4.3) and (4.2), but A, C, and D will protect lepton number, and they are all equivalent modulo $Z_3$ colour to the residual $Z_3$ from a generational symmetry.

Thus, there is only one possible anomaly free symmetry for $N = 3$ which will protect the proton from excessive decay, although it is noticable that this analysis would not give any possibilities if we had not included right handed neutrinos, and that there is no reason here why such neutrinos should not gain a Dirac mass.
Table 2: $\mathbb{Z}_3$ charges consistent with the MSSM

| Symmetry | $Q(H)$ | $Q(\bar{H})$ | $Q(q)$ | $Q(u)$ | $Q(d)$ | $Q(L)$ | $Q(e)$ | $Q(\nu)$ |
|----------|--------|-------------|--------|--------|--------|--------|--------|----------|
| A        | 0      | 0           | 0      | 0      | 0      | 1      | -1     | -1       |
| B        | 0      | 0           | 1      | -1     | -1     | 0      | 0      | 0        |
| C        | 0      | 0           | 1      | -1     | -1     | 1      | -1     | -1       |
| D        | 0      | 0           | 1      | -1     | -1     | -1     | 1      | 1        |

Table 3: $\mathbb{Z}_4$ charges consistent with the MSSM

| Symmetry | $Q(H)$ | $Q(\bar{H})$ | $Q(q)$ | $Q(u)$ | $Q(d)$ | $Q(L)$ | $Q(e)$ | $Q(\nu)$ |
|----------|--------|-------------|--------|--------|--------|--------|--------|----------|
| A        | 0      | 0           | 0      | 0      | 0      | 1      | -1     | -1       |
| B        | 0      | 0           | 1      | -1     | -1     | 0      | 0      | 0 or 2   |
| C        | 0      | 0           | 1      | -1     | -1     | 1      | -1     | -1       |
| D        | 0      | 0           | 1      | -1     | -1     | 2      | 2      | 0 or 2   |
| E        | 0      | 0           | 1      | -1     | -1     | -1     | 1      | 1        |
| F        | 0      | 0           | 2      | 2      | 2      | 1      | -1     | -1       |
| G        | 1      | -1          | 1      | 0      | 2      | 0      | -1     | -1       |
| H        | 1      | -1          | -1     | 2      | 0      | 2      | 1      | 1        |
| I        | 2      | 2           | 1      | 1      | 1      | 1      | 1      | 1        |
| J        | 2      | 2           | 1      | 1      | -1     | -1     | -1     | -1       |

We conclude our analysis by studying the case where $N = 4$. This again gives fairly simple results, listed in Table 3.

Note that we have ignored cases where all the charges are zero mod 2, since such symmetries are merely $\mathbb{Z}_2$ symmetries. In general a $\mathbb{Z}_N$ symmetry which is a subgroup of $\mathbb{Z}_M$ (so that $M > N$) obeys less restrictive conditions if it is obtained from a breaking of $SU(M)$ than from $SU(N)$. In this case if we impose the standard model constraints (4.4) and require that the $\mathbb{Z}_4$ charges are all either 0 or 2 (so as to give a $\mathbb{Z}_2$ symmetry), then the consistency conditions are all trivially satisfied, just as for $\mathbb{Z}_2$.

From the table there are seven possible $\mathbb{Z}_4$ symmetries which prevent all the unwanted dimension four terms in the lagrangian, and three more which preserve either lepton or baryon number but not both (although two of those listed are in fact redundant, as they are products of others given here). These symmetries also include some which could prevent
the neutrino gaining a large Dirac mass.

5 Conclusion

We have thus discussed how it is possible to break $SU(N)$ to smaller gauge groups in such a way as to leave its $\mathbb{Z}_N$ centre as a residual symmetry. The requirement that the high energy $SU(N)$ theory be anomaly free then gives restrictive constraints on the resulting discrete symmetry, although these are significantly weakened if we allow the addition to the model of singlets which interact only through the $SU(N)$ and thus carry non-zero charge under $\mathbb{Z}_N$.

These constraints can then be simply solved in the case of the MSSM for $N \leq 4$ with the only singlet being the right-handed neutrino, to find that while $\mathbb{Z}_2$ symmetries are not restricted at all, only a few possible $\mathbb{Z}_3$ and $\mathbb{Z}_4$ symmetries are left.

While the constraints given here can be quite restrictive, it may well be possible to derive further ones from a more detailed analysis of the possible symmetry breaking mechanisms. For example, if $SU(3)_C$ arises from the breaking of $SU(N)$, then each irreducible representation of $SU(N)$ must contain an $SU(3)_C$ multiplet, and so extra singlets can only be introduced in conjunction with $SU(3)_C$ multiplets.

There are a number of interesting possibilities for getting discrete symmetries from $SU(N)$ which we have not mentioned. Apart from the most obvious cases of breaking the $SU(N)$ to $SU(M)$ (for $M < N$) or to $U(1)$, and then breaking the latter to a discrete symmetry as usual, we might consider the possibility that after breaking one might have two distinct discrete symmetries. This is essentially covered by the usual analysis, except that if we have both abelian and non-abelian symmetries which break to discrete remnants $\mathbb{Z}_M$ and $\mathbb{Z}_N$ respectively, then we will have a mixed anomaly cancellation constraint of form

$$\sum_i q_i Q_i^2 = 0 \quad \text{gcd}(N^2, MN)$$

where the $\mathbb{Z}_M$ and $\mathbb{Z}_N$ charges are $q_i$ and $Q_i$, and $\text{gcd}(N^2, MN)$ is the greatest common divisor of $N^2$ and $MN$. This assumes that we select the $\mathbb{Z}_M$ charges to lie in a unique set, as for the $\mathbb{Z}_N$ ones.

Finally, we should mention that there is not any particular reason why discrete symmetries must not be non-abelian, although it is not easy to see how the breaking might occur to give rise to these, and there is no use for such symmetries in the standard model.

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