Proper Time Method for Fermions

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Abstract

We show how Schwinger’s proper time method can be used to evaluate directly the determinant of first order operators associated with fermionic theories. Several examples are worked out in detail.

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1 Introduction

Schwinger’s proper time method has been successfully used, in the past, to evaluate various one loop effects in quantum field theories [1, 2, 3] (see also the books by Greiner and Reinhardt [4] and by Reuter and Dittrich [5] and references therein for more details; an elementary introduction on Schwinger’s proper time method can be found in ref. [6]).

It is a very simple and powerful method where the operator whose determinant is being evaluated is treated as the Hamiltonian for evolution in an extra time direction known as the “proper time”. One can easily evaluate the determinant by solving the “Heisenberg equations” for this quantum mechanical problem. However, when applied directly to the evaluation of determinants of first order operators associated with fermionic theories, this method runs into difficulties in the sense that the “Heisenberg equations” do not lead to convenient solutions which can be used to evaluate the determinant (We will explain this in more detail in the following sections). In even space-time dimensions, one usually remedies this problem by converting the determinant of the first order operator to that of a second order operator through the use of the $\gamma_5$ matrix. However, in odd space-time dimensions, there is no $\gamma_5$ matrix and, therefore, the question of a direct evaluation of the determinant of first order operators becomes quite important.

In this paper, we show systematically, how one can use the proper time method to evaluate directly the determinants of first order operators associated with fermionic theories. In section 2, we evaluate the partition function for the bosonic oscillator which involves evaluating the determinant of a second order operator. We choose this simple example mainly to bring out the reasoning that goes into such an evaluation. In section 3, we show how this method fails when applied directly to the evaluation of the partition function for a free fermionic oscillator which involves a first order operator. We also show, in this simple example, how it can be converted to a problem involving a second order operator and hence can be solved. In section 4, we evaluate the effective action for a fermion interacting with an arbitrary, external electromagnetic field, in $0 + 1$ dimension, within the context of the proper time method. In section 5, we further apply this method
to solve the Schwinger model as well as the general Abelian model. Section 6 contains a brief conclusion.

2 Bosonic oscillator

Let us consider the bosonic oscillator in 0 + 1 dimensions. The partition function for this simple system can be calculated in many different ways. However, we will use here the proper time method for the evaluation of the partition function mainly to establish the steps that go into such an evaluation. The partition function is easily evaluated in the imaginary time formalism \[14\]. The Euclidean Lagrangian for the oscillator is given by (up to a total derivative)

\[
L_E = \frac{1}{2} x \left(-\frac{d^2}{dt^2} + m^2\right) x. \tag{1}
\]

Integrating out the variable \(x\), in the path integral, leads to the partition function for the bosonic oscillator as

\[
Z_B(\beta) = \left[\det \left(-\frac{d^2}{dt^2} + m^2\right)\right]^{-1/2} = \exp\left\{-\frac{1}{2} \text{Tr} \ln \left(-\frac{d^2}{dt^2} + m^2\right)\right\}. \tag{2}
\]

Here the determinant has to be evaluated in a space of functions periodic with an interval \(\beta\) which is the inverse of the temperature (in units of the Boltzmann constant, \(k_B = 1\)). Here “Tr” stands for the trace to be taken over a complete basis.

In the proper time formalism, one represents (up to a constant independent of dynamics) for any (non-negative) operator, \(H\),

\[
\text{Tr} \ln H = - \int_0^\infty \frac{d\tau}{\tau} \text{Tr} \ e^{-\tau H}. \tag{3}
\]

Here \(\tau\) is referred to as the proper time and the evaluation of the trace proceeds as follows. For the present case, we note that we can identify \((p = -i \frac{d}{dt})\)

\[
H = -\frac{d^2}{dt^2} + m^2 =: p^2 + m^2, \tag{4}
\]

and with \(\hbar = 1\),

\[
[t, p] = i. \tag{5}
\]
Consequently, we can write

\[
\Tr e^{-\tau H} = \int dt \langle t|e^{-\tau H}|t\rangle.
\]  

(6)

The trace is evaluated by introducing the (modified) Heisenberg picture

\[
|t, \tau\rangle = e^{\tau H}|t\rangle \\
\langle t, \tau| = \langle t|e^{-\tau H},
\]

(7)

so that

\[
\partial_\tau \langle t, \tau|t', 0\rangle = -\langle t, \tau|H|t', 0\rangle.
\]  

(8)

Furthermore, the (modified) Heisenberg equations give

\[
\frac{dt}{d\tau} = -[t, H] = -[t, p^2 + m^2] = -2ip \\
\frac{dp}{d\tau} = -[p, H] = -[p, p^2 + m^2] = 0,
\]

(9)

which can be solved to give

\[
p(\tau) = p(0) = -\frac{t(\tau) - t(0)}{2i\tau}.
\]  

(10)

This also leads to the commutation relation

\[
[t(\tau), t(0)] = -2\tau.
\]  

(11)

The relations (10) and (11) can be used to write

\[
H = -\frac{(t(\tau) - t(0))^2}{4\tau^2} + m^2 \\
= -\frac{1}{4\tau^2} \left[ (t(\tau))^2 - 2t(\tau)t(0) + (t(0))^2 \right] + \left( m^2 + \frac{1}{2\tau} \right),
\]

(12)

so that we obtain

\[
\partial_\tau \langle t, \tau|t', 0\rangle = -\langle t, \tau|H|t', 0\rangle \\
= \left\{ \frac{1}{4\tau^2} (t - t')^2 - \left( m^2 + \frac{1}{2\tau} \right)^2 \right\} \langle t, \tau|t', 0\rangle.
\]  

(13)
which can be integrated to give

$$\langle t, \tau|t',0 \rangle = \frac{C}{\sqrt{\tau}} e^{-\tau m^2-(t-t')^2/4\tau} .$$  \hspace{1cm} (14)$$

Here $C$ is a constant of integration which can be determined from various consistency conditions as well as the requirement

$$\lim_{\tau \to 0} \langle t, \tau|t',0 \rangle = \delta(t-t') ,$$

so that we can write

$$\langle t, \tau|t',0 \rangle =: K^{(0)}(t-t',\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-\tau m^2-(t-t')^2/4\tau} .$$  \hspace{1cm} (15)$$

So far, we have not worried about the periodicity condition necessary for the evaluation of the determinant. Let us next introduce

$$K(t-t';\tau) = \sum_{n=-\infty}^{\infty} K^{(0)}(t-t'-n\beta;\tau) ,$$

which can be easily seen to satisfy the required periodicity condition, namely:

$$K(t-t'+\beta;\tau) = K(t-t';\tau) .$$  \hspace{1cm} (16)$$

Thus, we can now write

$$\text{Tr} \ln \left( -\frac{d^2}{dt^2} + m^2 \right) = -\int_0^\infty \frac{d\tau}{\tau} \int_0^\beta dt K(0;\tau)$$

$$= \frac{\beta}{\sqrt{4\pi}} \sum_{n=-\infty}^{\infty} \int_0^\infty d\tau \tau^{-3/2} e^{-m^2\tau-n^2\beta^2/4\tau} ,$$

where we used equations (16) and (17). Identifying in the previous equation an integral representation of the modified Bessel function of second kind $K_{-1/2}(n\beta\omega)$ and using the fact that $K_{-1/2}(z) = \sqrt{\pi/2z} e^{-z}$, the trace in (19) can be evaluated in a straightforward manner to give

$$Z_B(\beta) = \exp \left\{ -\frac{1}{2} \text{Tr} \ln \left( -\frac{d^2}{dt^2} + m^2 \right) \right\}$$

$$= \exp \left\{ -\frac{1}{2} 2 \ln [2 \sinh(\beta m/2)] \right\}$$

$$= \frac{1}{2 \sinh(\beta m/2)} .$$  \hspace{1cm} (20)$$
This is indeed the correct partition function for the bosonic oscillator. We emphasize here that, in the proper time formalism, the crucial point lies in the fact that the Heisenberg equations can be solved to express the momentum in terms of the conjugate operators (see eq. (10)). As we will see in the next section, this is not possible for operators which are first order in the momentum.

3 Fermionic Oscillator

Let us next evaluate the partition function for the fermionic oscillator in the proper time formalism. Once again, this can be evaluated in many different ways, but we choose this simple example to bring out the difficulties associated with first order operators in the proper time formalism. Let us start with the Euclidean Lagrangian for the fermionic oscillator,

\[ L_E = \bar{\psi} \left( \frac{d}{dt} + m \right) \psi. \]

(21)

We can integrate out the fermions, in the path integral, to obtain the partition function for the fermionic oscillator as

\[ Z_F(\beta) = \det \left( \frac{d}{dt} + m \right) = \exp \left\{ \text{Tr} \ln \left( \frac{d}{dt} + m \right) \right\}. \]

(22)

Here the determinant is understood to be evaluated in the space of functions anti-periodic with an interval \( \beta \).

Following the discussion of the previous section, we note that we have to solve, in this case,

\[ \partial_\tau \langle t, \tau | t', 0 \rangle = -\langle t, \tau | H | t', 0 \rangle, \]

(23)

with

\[ H = \frac{d}{dt} + m = i p + m. \]

(24)

The (modified) Heisenberg equations, in this case, give

\[ \frac{dt}{d\tau} = -[t, H] = -[t, ip + m] = 1 \]

\[ \frac{dp}{d\tau} = -[p, H] = -[p, ip + m] = 0. \]

(25)
These equations can be solved to give

\[\begin{align*}
p(\tau) &= p(0) \\
t(\tau) &= t(0) + \tau.
\end{align*}\]  

(26)

However, they do not allow us to express the momentum in terms of the conjugate operator. Consequently, the evaluation of the matrix element on the right hand side of (23) becomes extremely difficult. This is the difficulty with the first order operators in the proper time formalism.

In this simple example, we can relate the first order operator to a second order operator in a simple manner and thereby evaluate the partition function easily. Let us note that (up to a total divergence)

\[L_E = \overline{\psi} \left( \frac{d}{dt} + m \right) \psi = \frac{1}{2} \frac{\overline{\psi}}{\psi} \left( \frac{d}{dt} + m \right) \psi + \frac{1}{2} \left( \overline{\psi} \left( \frac{d}{dt} + m \right) \psi \right).\]  

(27)

Thus, introducing

\[\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix},\]  

(28)

we can write

\[L_E = \frac{1}{2} \Psi^T \left( \sigma_1 \frac{d}{dt} - i\sigma_2 m \right) \Psi,\]  

(29)

where the \(\sigma\)'s are the usual Pauli matrices. Integrating out the fermions, we obtain

\[Z_F(\beta) = \left[ \det \left( \sigma_1 \frac{d}{dt} - i\sigma_2 m \right) \right]^{1/2} = \left[ \det \left( \sigma_1 \frac{d}{dt} - i\sigma_2 m \right)^2 \right]^{1/4} = \left[ \det \left( \frac{d^2}{dt^2} - m^2 \right) I \right]^{1/4} = \left[ \det \left( -\frac{d^2}{dt^2} + m^2 \right) \right]^{1/2} = \exp \left\{ \frac{1}{2} \text{Tr} \ln \left( -\frac{d^2}{dt^2} + m^2 \right) \right\}.\]  

(30)
In this way, we have related the determinant (partition function) of the first order operator describing the fermionic oscillator to that of the second order operator of the last section. The only difference is that the determinant, here, has to be evaluated over anti-periodic functions. Thus, we define

$$\tilde{K}(t - t'; \tau) = \sum_{n=-\infty}^{\infty} (-1)^n K^{(0)}(t - t' - n\beta; \tau),$$

where $K^{(0)}(t - t'; \tau)$ is defined in (16). It is easy to check that $\tilde{K}(t - t'; \tau)$ defined above satisfies the required anti-periodic condition, namely,

$$\tilde{K}(t - t' + \beta; \tau) = -\tilde{K}(t - t'; \tau).$$

Therefore, we obtain

$$\text{Tr} \ln \left( -\frac{d^2}{dt^2} + m^2 \right) = -\int_0^\infty d\tau \int_0^\beta dt K(0; \tau),$$

which can be evaluated in a simple manner to give

$$Z_F(\beta) = \exp \left\{ \frac{1}{2} \text{Tr} \ln \left( -\frac{d^2}{dt^2} + m^2 \right) \right\}$$

$$= \exp \left\{ \frac{1}{2} 2\text{Tr} \ln [2 \cosh(\beta m/2)] \right\}$$

$$= 2 \cosh(\beta m/2).$$

This is indeed the correct partition function for the fermionic oscillator.

### 4 Fermion in an External Field

In the last section, we evaluated the partition function for a free fermionic oscillator in the proper time formalism by relating the determinant of a first order operator to that of a second order operator for which the Heisenberg equations can be solved in a required form. However, this cannot always be done and so a direct evaluation of the solution of eq. (8) becomes essential. In this section, we analyze the case of a fermion interacting
with an arbitrary, external gauge field in 0 + 1 dimension. The Lagrangian is given by

\[ L = \bar{\psi}(i\partial_t - m - A)\psi, \]  

(35)

where \( A \) is an arbitrary function of \( t \) and we are interested in evaluating the contribution of the fermions to the effective action which is given by

\[ \Gamma[A] = -i \ln \frac{\det(i\partial_t - m - A)}{\det(i\partial_t - m)} = -i \ln \det (1 - S(p)A) \]

\[ = -i \text{Tr} \ln (1 - S(p)A) \]

\[ = i \int_0^\infty \frac{d\tau}{\tau} e^{-\tau} \text{Tr} e^{\tau S(p)A} + \text{constant}. \]  

(36)

Here, the effective action is normalized so that it vanishes when \( A = 0 \). Furthermore, \( S(p) \) stands for the fermion propagator which, at zero temperature, has the form

\[ S(p) = \frac{1}{p - m + i\epsilon} \]  

(37)

In the present case, we can identify the proper time Hamiltonian as

\[ H = -S(p)A \]  

(38)

and it is straightforward to check that the Heisenberg equations cannot be solved in a manner suitable to evaluate the matrix elements. However, let us note that equation (38), in the present case, takes the form

\[ \partial_\tau \langle t, \tau | t', 0 \rangle = -\langle t, \tau | H | t', 0 \rangle = \langle t, \tau | S(p)A | t', 0 \rangle. \]  

(39)

From the form of the propagator, at zero temperature, given in (37), we obtain

\[ \langle t | S(p) | t' \rangle = -i \theta(t - t') e^{-im(t-t')} \]  

(40)

Therefore, we can also write eq. (39) as

\[ \partial_\tau \langle t, \tau | t', 0 \rangle = \int dt'' \langle t, \tau | t'', 0 \rangle \langle t'', 0 | S(p)A | t', 0 \rangle \]

\[ = -i \int dt'' \langle t, \tau | t'', 0 \rangle \theta(t'' - t') A(t') e^{-im(t'' - t')} \]  

(41)
This equation can be integrated to give

\[
\langle t, \tau | t', 0 \rangle = \delta(t - t') - i \int_0^\tau d\tau' \int dt'' \langle t, \tau' | t'', 0 \rangle \theta(t'' - t') A(t') e^{-im(t'' - t')} .
\] (42)

In deriving the previous equation, we have used the fact that (see eq. (13))

\[
\langle t, 0 | t', 0 \rangle = \delta(t - t')
\] (43)

The equation for the inner product (propagator) is now written as an integral equation which cannot always be solved in a closed form. However, we can solve the equation iteratively leading to

\[
\langle t, \tau | t', 0 \rangle = \delta(t - t') - i \tau \theta(t - t') A(t') e^{-im(t - t')} - \frac{\tau^2}{2} \int dt'' \theta(t - t'') \theta(t'' - t') A(t'') A(t') e^{-im(t - t')} + \ldots
\] (44)

We note that the inner product (propagator) in the last equation contains an infinite number of terms. However, for equal times, \( t = t' \) (which is what we need for the trace), all the terms quadratic and higher in the field variables vanish because of opposing theta functions. Consequently, we have

\[
\Gamma[A] = i \int_0^{\infty} \frac{d\tau}{\tau} e^{-\tau} \int dt \langle t, \tau | t, 0 \rangle + \text{constant}
= \frac{1}{2} \int dt A(t) .
\] (45)

This is indeed the exact effective action for this model [7, 8, 9]. Here, we have derived it, for a first order operator, directly within the proper time formalism by solving the integral equation (42). This is quite useful because for systems without a closed form expression for the effective action, this provides a perturbative expansion of the one loop action. Without going into details, we simply note here that at finite temperature [14]

\[
S(p) = \frac{1}{p - m + i\epsilon} + 2i\pi n_F(m) \delta(p - m) ,
\] (46)

giving

\[
\langle t | S(p) | t' \rangle = -i (\theta(t - t') - n_F(m)) e^{-im(t - t')} .
\] (47)
Using this in the equation (39) and various identities derived in [8, 9], one can also obtain the effective action at finite temperature in a closed form. But more important is the fact that, even when the effective action does not have a closed form, the method gives a perturbative expansion of the effective action.

5 Schwinger Model

As a final example, we solve the Schwinger Model as well as the general Abelian model within the proper time formalism using the method outlined in the earlier section. The Lagrangian for the fermions (in 1 + 1 dimensional Minkowski space) has the form

\[ L_f = \bar{\psi} \gamma^\mu (i \partial_\mu - e A_\mu) \psi. \] (48)

Therefore, we obtain the effective action to be

\[ \Gamma[A] = -i \ln \frac{\det (i \partial - e A)}{\det (i \partial)} \]
\[ = -i \ln \det (1 - e S(p) A) \]
\[ = -i \text{Tr} \ln (1 - e S(p) A) \]
\[ = i \int_0^\infty \frac{d\tau}{\tau} e^{-\tau} \text{Tr} e^{e S(p) A}. \] (49)

In the present case,

\[ S(p) = \frac{\dot{p}}{p^2 + i \epsilon}, \] (50)

and the “Tr” involves a trace over the Dirac indices as well.

Equation (8), in the present case, takes the form

\[ \partial_\tau \langle x, \tau | x', 0 \rangle = -\langle x, \tau | H | x', 0 \rangle, \] (51)

with

\[ H = -e S(p) A. \] (52)

Following the discussion of the last section, we can write

\[ \partial_\tau \langle x, \tau | x', 0 \rangle = e \langle x, \tau | S(p) A | x', 0 \rangle \]
\[ = e \int d^2 x'' \frac{d^2 p}{(2\pi)^2} \langle x, \tau | x'', 0 \rangle \frac{\dot{p} A(x')}{p^2} e^{-ip(x''-x')} \] (53).
Integrating this, as in the last section, we obtain the integral equation
\begin{equation}
\langle x, \tau | x'', 0 \rangle = \delta^2(x - x') + e \int_0^\tau d\tau' d^2 x'' \frac{d^2 p}{(2\pi)^2} \langle x, \tau' | x'' | 0 \rangle \frac{\dot{A}(x')}{p^2} e^{-ip(x'' - x')} \tag{54}
\end{equation}
which can be solved iteratively to give
\begin{equation}
\langle x, \tau | x'', 0 \rangle = \delta^2(x - x') + \tau e \int d^2 p \frac{\dot{A}(x')}{p^2} e^{-ip(x - x')}
+ \frac{\tau^2}{2} e^2 \int d^2 x'' d^2 p \frac{\dot{A}(x'') \dot{A}(x')}{p^2} e^{-ip'(x - x'')} e^{-ip(x'' - x')} + \ldots \tag{55}
\end{equation}

Once again, we see that the proper time propagator involves an infinite number of terms. However, it can be shown easily, following from various identities in 1+1 dimensions (see [11]), that all the higher order terms starting with the cubic do not contribute to the trace. In fact, even the linear term does not contribute to the trace because of the odd nature of the momentum integrand. Thus, we have (“tr” denotes the trace over Dirac indices)
\begin{align}
\text{Tr} \langle x, \tau | x, 0 \rangle &= \frac{\tau^2}{2} e^2 \int d^2 x'' d^2 k \frac{d^2 p}{(2\pi)^2} \frac{1}{(2\pi)^2} \text{tr} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho (p + k)_{\mu} p_{\lambda} A_\nu(x'') A_{\rho}(x) e^{-ik \cdot (x - x'')} \\
&= -\frac{i\tau^2 e^2}{2\pi} \int d^2 x A_\mu \left( \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{2} \right) A_\nu. \tag{56}
\end{align}

Putting this back into eq. (49), we obtain
\begin{align}
\Gamma[A] &= i \int_0^\infty \frac{d\tau}{\tau} e^{-\tau} \text{Tr} \langle x, \tau | x, 0 \rangle \\
&= \frac{e^2}{2\pi} \int d^2 x A_\mu \left( \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{2} \right) A_\nu. \tag{57}
\end{align}

This is indeed the exact effective action for the Schwinger model in a gauge invariant regularization [12] (As is well known, there is a one parameter arbitrariness depending on the choice of the regularization, see ref. [12] and references therein. Here we have used a gauge invariant regularization for simplicity). Let us next comment on the solubility of the general Abelian model in 1+1 dimensions within this framework without going into details.
The Lagrangian, for the general model, has the form \([13]\) (with an arbitrary parameter \(r\))

\[
L_f = \bar{\psi} \gamma^\mu (i \partial_\mu - e(1 + r \gamma_5)A_\mu) \psi
\]

and it reduces to all the 1 + 1 dimensional soluble models under different limits \([13]\). Let us note that in 1 + 1 dimensions, the Dirac matrices satisfy

\[
\gamma_5 \gamma^\mu = \epsilon^{\mu\nu} \gamma_\nu,
\]

so that if we define

\[
B^\mu = (\eta_{\mu\nu} + r \epsilon^{\mu\nu})A_\nu
\]

we can write

\[
L_f = \bar{\psi} \gamma^\mu (i \partial_\mu - eB_\mu) \psi.
\]

This is, in fact, the Schwinger model and the effective action would be identical to \(\text{(57)}\), in terms of the \(B_\mu\) field. Substituting relation \(\text{(60)}\), then, would give the exact effective action for the general Abelian model with a gauge invariant regularization.

6 Conclusion

We have shown how Schwinger’s proper time formalism can be applied directly to fermionic systems with first order operators. Here one solves an integral equation leading (in general) to a perturbative expansion of the effective action. This is, of course, quite useful in odd space-time dimensions where the determinant of a first order operator cannot naturally be related to that of a second order operator. However, it is useful in even dimensions as well for a perturbative expansion of the effective action.

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