Geometric Understanding of the Stability of Power Flow Solutions

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Abstract

A grand challenge for power grid management lies in how to plan and operate with increasing penetration of distributed energy resources (DERs), such as solar photovoltaics and electric vehicles, which disturb the power grid stability. Existing approaches are unable to verify if a point is on a loadability boundary or characterize all loadability boundary points exactly. This inability leads to a poor understanding of locational hosting capacity for accommodating distributed resources. To solve these problems, we compare existing approaches and propose a rectangular coordinate-based analysis, which drew less attention in the past. We demonstrate that such a coordinate (1) provides an integrated geometric understanding of active and reactive power flow equations, (2) enables linear representation of elements in the Jacobian matrix, (3) verifies if an operating point is on the loadability boundary and what is the margin, and (4) characterizes the power flow feasibility boundary points. Finally, IEEE standard test cases demonstrate the capability of the new method.

Index Terms

Power flow, loadability, rectangular coordinate, geometric understanding, maximum power output

I. INTRODUCTION

The power system is under increasing pressure from societal demands to become greener and more efficient. At the same time, the grid must maintain its reliability in the face of new generation resources like solar and new loads like electrical vehicles. To maintain reliability in the presence of uncertainties in both load and generation, operating points of the system must remain feasible under large perturbations. In this paper, we study and characterize the feasibility and stability of power flow solutions in the system.

These types of power flow analysis have long been an integral part of both planning and operation for power systems. In the planning phase, power flow analysis can be used to determine the need for shunt compensation, new transmission lines [1], spinning and non-spinning reserve [2]–[8], and other system additions [9]. In the operation phase [10], a load flow solution can be used to find stability margins in preventing voltage collapses caused by large load variation and saddle-node bifurcation [11], [12]. Further control action can be subsequently enforced if the distance between an operation point and the stability boundary can be measured [13]. In this paper, we use the terms power flow solutions and operating points interchangeably.

The stability of power flow solutions can be visualized via the power-voltage (PV) curve for a 2-bus system, where the maximum loading occurs at the tip of the curve [14], [15]. However, for larger systems or systems with

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both real and reactive power constraints, the stability of solutions are not so easily visualized. Instead, algebraic techniques have been used to study their properties. A popular approach is to study the eigenvalues of the power flow Jacobian [16], although the singularity of the Jacobian is necessary but not sufficient to conclude that the solution is close to the boundary. In [17], [18], the authors studied the stability of an operating point by choosing a load increasing direction and repeated solving the power flow equation until no solutions can be found. However, this type of numerical searches does not provide much intuitive understanding of the system. Also, the non-convergence of power flow solvers does not necessarily imply that there are no solutions. Other studies used DC approximation to successively linearize around an operating point, but are sensitive to the parameterizations and cannot directly handle reactive power [19]. Finally, polynomial homotopy continuation methods can be employed to find all the power flow solutions, although often at prohibitively high computational costs [20].

In our work, we address two questions left open by previous research: 1) Is there a way to extend the geometrical intuition in the PV curve to more than 2 buses and include reactive power? 2) Can the stability of an operating point be characterized without solving nonlinear/nonconvex equations or making DC approximations? In this paper, we provide positive results to both questions.

The starting point of our analysis is using rectangular state coordinates to represent complex voltages. We then define the stability boundary of a system as the set of points on the Pareto-Front of active powers [21], where the load on one bus cannot be strictly increased without decreasing the load at some other buses. Therefore, this Pareto-Front represents the limit of operating a system, and an operating point becomes less stable as it approaches this front. By conducting stability analysis in the rectangular coordinate system, we make the following contributions:

1) We develop a geometric view for systems with more than 2 buses that integrates real and reactive powers.
2) We show that the eigenvalues of the power flow Jacobian matrix are insufficient in describing the stability boundary of a system. Instead, we present a linear programming approach to test whether an operating point is on the boundary.
3) Based on the linear programming approach and using rectangular coordinates, we characterize the stability margins of operating points.
4) In addition to showing the margin, we also formulate a linear programming problem to characterize the power flow feasibility boundary points.

We validate our approaches by simulations on 36 different transmission grids (e.g., 14, 300, and 13659-bus) and 2 distribution grids (e.g., 8 and 123-bus).

The rest of the paper is organized as follows: Section II motivates the rectangular coordinate-based analysis and provides an integrated geometric view of active/reactive power flow equations. Section III shows the linear Jacobian matrix and its application for security boundary point verification. Section IV quantifies margins of points that are not on the boundary. Section V shows how to search for all of the boundary points. Section VI evaluates the performance of the new method and Section VII concludes the paper.
II. RECTANGULAR COORDINATES AND GEOMETRY OF POWER FLOW

The steady-state model for a power system is usually described in the polar coordinate system \[22\],
\[
p_d = \sum_{k=1}^{n} |v_d||v_k| (g_{dk} \cos \theta_{dk} + b_{dk} \sin \theta_{dk}), \quad (1a)
\]
\[
q_d = \sum_{k=1}^{n} |v_d||v_k| (g_{dk} \sin \theta_{dk} - b_{dk} \cos \theta_{dk}), \quad (1b)
\]
where \(n\) is the bus number of the electrical system; \(p_d\) and \(q_d\) are the active and reactive power injections at bus \(d\); \(v_d\) is the complex phasor at bus \(d\) and \(|v_d|\) is its voltage magnitude; \(\theta_{dk} = \theta_k - \theta_d\) is the phase angle difference between bus \(k\) and bus \(d\); \(g_{dk}\) and \(b_{dk}\) are the electrical conductance and susceptance between bus \(d\) and bus \(k\).

Together, \(y_{dk} = g_{dk} + j \cdot b_{dk}\) forms the admittance, where \(j\) is the imaginary unit.

The power flow equations in (1) involve two types of functions—sinusoids and polynomials—that interact nonlinearly, which makes the power flow equations difficult to analyze. For a 2-bus system, the nonlinearity of the power flow equations can be visualized geometrically by the so-called power-voltage (PV) curve as seen in Fig. 2. There is no natural extension of the PV curve for larger systems. In fact, if reactive powers are included, it is not clear how to visualize even the complex power flow between two buses.

Recently, a popular method to deal with the non-convexity of the power flow equations is to “lift” the voltage from polar coordinates to a higher dimensional space, either through SDP or SOCP relaxations \[21\], \[23\], \[24\]. However, this transformation necessarily loses some structure of original equations and it is not always possible to recover a physically meaningful answer from the relaxed equations.

A. Rectangular Coordinate-based Power Flow

Despite losing information, a key benefit of analyzing the power flow equation in the lifted space comes from the reduction in function types as shown in Fig. 1. Instead of requiring both sinusoidal and polynomial functions, relaxed power flow equations can be described as a linear function using a single (SDP) matrix variable. In this paper, we use an intermediate rectangular state space between the polar state space and the lifted state space. Let \(\boldsymbol{v}\) be the complex bus voltages, then we think of the real and imaginary parts of \(\boldsymbol{v}\) as the state \(\boldsymbol{x}\) of the system: \(\boldsymbol{x} \triangleq (\text{Re}(\boldsymbol{v}))^T, (\text{Im}(\boldsymbol{v}))^T\). This choice of state variables retains full information about the power flow equations, and makes their analysis significantly easier in the rest of the paper. Of course, rectangular coordinates have been used before to improve the computation of power flow and OPF \[25\], but our focus here is to use them to understand some of the geometric features of the feasible power flow region.

Let \(v_{d,r} \triangleq \text{Re}(v_d)\) and \(v_{d,i} \triangleq \text{Im}(v_d)\) be the real and imaginary parts of the complex voltage at bus \(d\), respectively. Then, the power flow equations in (1) becomes
\[
p_d = t_{d,1} \cdot v_{d,r}^2 + t_{d,2} \cdot v_{d,r} + t_{d,1} \cdot v_{d,i}^2 + t_{d,3} \cdot v_{d,i}, \quad (2a)
\]
\[
q_d = t_{d,4} \cdot v_{d,r}^2 - t_{d,3} \cdot v_{d,r} + t_{d,4} \cdot v_{d,i}^2 + t_{d,2} \cdot v_{d,i}, \quad (2b)
\]
in rectangular coordinates, where
\[
t_{d,1} = - \sum_{k \in \mathcal{N}(d)} g_{kd}, \quad t_{d,2} = \sum_{k \in \mathcal{N}(d)} (v_{k,r} g_{kd} - v_{k,i} b_{kd}), \quad (3a)
\]
\[
t_{d,3} = \sum_{k \in \mathcal{N}(d)} (v_{k,r} b_{kd} + v_{k,i} g_{kd}), \quad t_{d,4} = \sum_{k \in \mathcal{N}(d)} b_{kd}, \quad (3b)
\]
Fig. 1: A representation of power flow equations with different function types. The standard voltage/angle formulation requires two nonlinear functions with two variables, whereas the relaxed formulation with an SDP matrix $W$ is linear in a single variable but is inexact. In this paper, we use rectangular coordinates, where the power flow equations can be described using only polynomial functions.

and \( \mathcal{N}(d) \) means the neighbor bus set of the bus \( d \). The detailed derivations from (1) to (2) are given in the Appendix.

**B. Visualization of Complex Power Flow**

One benefit of using (2) comes from its capability of visualizing active and reactive power flow equations in the same space. For fixed constants \( t_{d,1}, t_{d,2}, t_{d,3}, t_{d,4} \), (2a) and (2b) describe two circles in the \( v_{d,r} \) and \( v_{d,i} \) space. The next lemma characterizes the centers and radii of the active and reactive power flow “circles”:

**Lemma 1.** The centers and radii: The coordinates of circle center $E$ for the active power flow are \( \left( -\frac{t_{d,2}^2 - t_{d,1}}{2t_{d,4}}, -\frac{t_{d,3} - t_{d,1}}{2t_{d,4}} \right) \) for bus $d$. Its radius decreases when $p_d$ increases. The coordinates of circle center $D$ for the reactive power flow are \( \left( \frac{t_{d,2}}{2t_{d,4}}, -\frac{t_{d,3}}{2t_{d,4}} \right) \) for bus $d$. Its radius decreases when $q_d$ increases.

To illustrate this visualization capability, Fig. 2a shows a geometric illustration of a 2-bus system. For this figure, the admittance of the connecting branch is set at \( 0.5 - 0.5j \) per unit (p.u.). Bus 1 is the reference bus with \( v_{1,r} = 1 \).
and $v_{1,i} = 0$. Bus 2 is the load bus with $p_2 = 0.1$ and $q_2 = 0.05$. Plugging these values into (2) leads to the following equations:

\[(v_{2,r} - 0.5)^2 + (v_{2,i} + 0.5)^2 = 0.55^2 \quad (4a)\]
\[(v_{2,r} - 0.5)^2 + (v_{2,i} - 0.5)^2 = 0.63^2. \quad (4b)\]

The equations in (4) represent two circles shown in Fig. 2a. The circles are centered at point $E = (0.5, -0.5)$ and $D = (0.5, 0.5)$, respectively. The two circles intersect with each other at point $A$ and $B$. Such intersections represent the operation points when both active and reactive powers are balanced at the same time. Any other points in the rectangular coordinate are infeasible for the power flow equations.

Notably, Fig. 2a is a generalization of the well-studied concept of PV curve for power systems in Fig. 2b. The voltage solution point $A$ and point $B$ in the PV curve are the same as the two intersection points $A$ and $B$ in Fig. 2a. As load $p_2$ is increased with a fixed power factor, the two circles will depart each other until there is only one intersection point $C$ in Fig. 2a. This point is commonly known as the nose point of the PV curve in Fig. 2b, indicating the loadability of this system.

Similar intuition can be gained for systems with more than two buses. For example, consider the 3-bus network in Fig. 3 where bus 1 is the slack bus, and bus 2 and 3 are PQ load buses. Let the admittances be $1 - 0.5j$ for all the lines, $p_2 = 0.7$, $p_3 = 0.95$, power factor= 0.95. Fig. 3(b) and 3(c) shows the circles formed by (2) for bus 2 and bus 3, respectively. In this case, the intersection points between the two circles in Fig 3(b) are far apart, whereas the two points on the intersection of the two circles in Fig. 3(c) are close together. This suggests that the system is operating close to its limit and small changes may lead to insolvability of the power flow equations at bus 3.

![Fig. 3: Three bus network and the circles formed at buses 2 and 3. The closeness of the intersections at bus 3 indicates that the system is operating close to the boundary of the feasible region.](attachment:fig3.png)
III. BOUNDARY OF THE POWER FLOW REGION: BEYOND THE JACOBIAN

The pictures in Figs. 2 and 3 show a way to visualize operating conditions in power systems. The natural next step would be to algebraically characterize the operating points, especially for those on or close to the boundary of the feasible power flow region.

Commonly, these types of analysis are done through the power flow Jacobian, and in particular the singularity of the Jacobian matrix has long been used to characterize the solvability and stability of power flow solutions [26]. In this section, we show that the Jacobian is not always sufficient to identify the boundary of the power flow region, and we propose a different method of identifying whether a solution is on the boundary by using a linear program in the rectangular coordinates.

A. Power Flow Jacobian

The power flow Jacobian matrix, $J$, is normally defined by the first order partial derivatives of active and reactive powers with respect to the state variables. In our analysis, these partial derivatives are taken with respect to the real and imaginary parts of the bus voltages:

$$J = \begin{bmatrix} \frac{\partial p}{\partial v_r} & \frac{\partial q}{\partial v_r} \\ \frac{\partial p}{\partial v_i} & \frac{\partial q}{\partial v_i} \end{bmatrix},$$

where the elements are given by

$$\frac{\partial p_d}{\partial v_{k,r}} = \begin{cases} 2t_{d,1}v_{d,r} + t_{d,2}, & \text{if } d = k, \\ g_{kd}v_{d,r} + b_{kd}v_{d,i}, & \text{if } d \neq k. \end{cases}$$  \hspace{1cm} (5a)

$$\frac{\partial p_d}{\partial v_{k,i}} = \begin{cases} 2t_{d,1}v_{d,i} + t_{d,3}, & \text{if } d = k, \\ -b_{kd}v_{d,r} + g_{kd}v_{d,i}, & \text{if } d \neq k. \end{cases}$$  \hspace{1cm} (5b)

$$\frac{\partial q_d}{\partial v_{k,r}} = \begin{cases} 2t_{d,4}v_{d,r} - t_{d,3}, & \text{if } d = k, \\ -b_{kd}v_{d,r} + g_{kd}v_{d,i}, & \text{if } d \neq k. \end{cases}$$  \hspace{1cm} (5c)

$$\frac{\partial q_d}{\partial v_{k,i}} = \begin{cases} 2t_{d,4}v_{d,i} + t_{d,2}, & \text{if } d = k, \\ -g_{kd}v_{d,r} - b_{kd}v_{d,i}, & \text{if } d \neq k. \end{cases}$$  \hspace{1cm} (5d)

For the above partial derivatives, $t_{d,1}$ and $t_{d,4}$ are both constant given network parameters, and $t_{d,2}$ and $t_{d,3}$ are linear in the variables. Therefore, each of the partial derivatives in (5) are linear in the state variables $v_{d,r}$ and $v_{d,i}$.

The Jacobian is normally used via the inverse function theorem, which states that the power flow equations stabilize around an operating voltage if the Jacobian is non-singular. This condition is necessary since every stable point must have a nonsingular Jacobian. However, it is insufficient. As the next example will show, a singular Jacobian does not imply that the operating voltage is unstable or at the boundary of the power flow feasibility region.

Again, consider the 3-bus network in Fig. 3. For simplicity, we assume the lines are purely resistive (all line admittances are 1s) and only consider active powers. Let bus 1 be the slack bus fixed and buses 2 and 3 be load buses consuming positive amount of active powers. In this case, the Jacobian becomes:

$$J = \begin{bmatrix} 1 - 4v_2 + v_3 \\ -v_2 \\ v_3 \end{bmatrix},$$

June 23, 2017  \hspace{1cm} (6)
where $v_2$ and $v_3$ are the voltages at bus 2 and bus 3, respectively. Fig. 4 shows the feasible power flow region of power consumptions at buses 2 and 3. The red lines show the points where the Jacobian is singular. Interestingly, singularity points of the Jacobian consist of two parts that are qualitatively different. Points like $A$ are on the boundary of the feasibility region, while points like $B$ are in the strict interior of the region. Therefore, if we are interested in finding whether a point is on the boundary or characterizing its stability margin, just looking at the determinant of the Jacobian is insufficient.

Fig. 4: Feasible power flow region for the 3-bus network. The red points denote operating points when the Jacobian is singular. They separate into two parts: the boundary of the region (e.g., point $A$) and points in the strict interior (e.g., point $B$).

### B. Power Flow Boundary

To isolate just the points on the boundary, we need to look deeper into the power flow equations than just the determinant of the Jacobian. Here, we focus on a network where the buses are loads. Geometrically, a point is on the loadability boundary (or simply boundary) if there does not exist another point that can consume more power:

**Definition 1.** Let $v$ be the complex voltages and $p = (p_1, \cdots, p_n)$ be the corresponding bus active powers. We say this operating point is on the loadability boundary if there does not exist another operating point $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_n)$ such that $\hat{p}_k \geq p_k$ for all $k$ and $\hat{p}_d > p_d$ for at least one $d$.

The above definition coincides with the definition of the Pareto-front since we are modeling each bus (except the slack) as a load bus. It can be easily extended to a network where some buses are generators by changing the direction of inequalities in the above definition.

Instead of looking at the determinant of the Jacobian, the next theorem gives a linear programming condition for the points on the boundary:

**Theorem 2.** Checking whether an operating point is on the boundary of the feasible power flow region is equivalent to solving a linear programming problem.
Let $h_d = [\frac{\partial p_d}{\partial v_1}, \frac{\partial p_d}{\partial v_2}, \frac{\partial p_d}{\partial v_3}, \cdots, \frac{\partial p_d}{\partial v_n}]^T \in \mathbb{R}^{2n}$ be the gradient of $p_d$ with respect to all the state variables. Therefore, $h_d$ is the transpose of the $d^{th}$ row of the Jacobian matrix. Let $z \in \mathbb{R}^{2n}$ be a direction, towards which we move the real and imaginary part of the voltages. Then, by Definition 1, a point is on the boundary if there does not exist a direction to move where the consumption of one bus is increased without decreasing the consumption at other buses.

We can encode this condition in a linear programming (LP) feasibility problem. Suppose that $h_1, \cdots, h_n$ are given. Then, we solve the following:

$$\min_y 1$$

s.t. $y^T h_d \geq 0$, for all $d = 1, \cdots, n$,  \hspace{1cm} (7a)

$$\sum_{d=1}^n y^T h_d = 1.$$

In this optimization problem, the objective is irrelevant since we are only interested in whether the problem is feasible. The constraint (7b) specifies that moving in the direction $y$ cannot decrease any of the active powers. The constraint (7c) is equivalent to that at least one bus’ active power must strictly increase. This comes from the fact that $y$ is not a constraint. Therefore, as long as $y^T h_d > 0$ for some $d$, the sum $\sum_{d=1}^n y^T h_d$ can be scaled to be 1. Finally, an operating point is on the boundary if and only if the problem in (7) is infeasible.

For the example in Fig. 4, point $A$ is given by $v_2 = v_3 = 0.5$ and point $B$ is given by $v_2 = v_3 = 0.25$. Using (5), we have $h_2 = [-0.5, 0.5]^T$ and $h_3 = [0.5, -0.5]^T$ at point $A$, and $h_2 = h_3 = [0.25, 0.25]^T$ at point $B$. It is easy to check that a $y$ feasible for (7) exists at point $B$. However, it is impossible to find such a $y$ at point $A$. Therefore, we can conclude that $A$ is on the boundary whereas $B$ is not, even though both have singular Jacobians.

**Adding Constraints.** Box and linear constraints on the active power can be easily added to (7) by restricting the direction of movements. Given an operating point, for the constraints on active power that are tight, we can add constraints into (7) such that $y$ must move the active powers to stay within the constraint.

**IV. Stability Margins**

In addition to asking whether a point is on the boundary, we are sometimes more interested in how close a point is to the boundary. In this section, we make a simple modification to (7) and use the modified optimization for measuring the distance of an operating point to the boundary.

In the optimization problem (7), we check if it is possible to move the operating point in a direction such that the active powers can be increased. For a point close to the boundary, we are interested in how much a point can be moved before reaching the boundary. Therefore, we use the optimal objective value of the optimization problem in (8) to measure the stability margin of an operating point.

Again, let $h_d^T$ be the $d^{th}$ row of the Jacobian matrix at an operating point of interest. Then, we solve

$$m = \max_y \sum_{d=1}^n y^T h_d$$

s.t. $y^T h_d \geq 0$, for all $d = 1, \cdots, n$,

$$||y||_2 = 1.$$
In this problem, we look for a unit vector $\mathbf{y}$ such that the sum of the active powers can be increased by the maximum amount. The value of the optimization problem is denoted by $m$, which we think as the margin or the distance to the boundary. Note that the constraint in (8c) may seem non-convex, but for any point that is not on the boundary, (8c) can be relaxed to $||\mathbf{y}||_2 \leq 1$ without changing the objective value. Therefore, (8) can be easily solved by standard solvers.

Continuing with the example 3-bus network in Fig. 3 with purely resistive line admittances being 1s, we consider the line in the feasible region (see Fig. 4) connecting the origin, point B and point A. Along this line, both bus 2 and bus 3 have the same active power, varying from 0 to 0.25. Fig. 5 plots the value $m$ along the line. As expected, the margin decreases as the point moves closer to the boundary of the feasible region.

![Fig. 5: The margin of the operating point as it moves from the origin to the point A in Fig. 4. The margin decreases steadily as the point gets closer to the boundary.](image)

V. FINDING THE PARETO-FRONT

In the last two sections, we have explored how to determine whether a point is on the Pareto-front and its “distance” to the front. In this section, we ask the question of whether we can determine the Pareto-front itself. To answer this question, we observe that by definition, for any point $\mathbf{p}$ on the Pareto-front, there exists a vector $\mathbf{z}$ such that $\mathbf{z}^T \mathbf{p}$ is the maximum among all possible active power vectors. Conversely, by varying $\mathbf{z}$ and maximizing over $\mathbf{p}$, we can find the Pareto-front.

Of course, for a large system, exhaustively varying $\mathbf{z}$ is impractical. However, in many cases, there are a few $\mathbf{z}$s of special interest. For example, if $\mathbf{z}$ is the all-ones vector $\mathbf{1}$, we are looking for the maximum sum power that the network can support. In other settings, there are a few classes of loads, and $\mathbf{z}$ has only a few distinct values.

To find an active power vector $\mathbf{p}$ such that $\mathbf{z}^T \mathbf{p}$ is the maximum, we again look at the partial derivatives of active power with respect to the voltages. On the Pareto-front and given a $\mathbf{z}$ tangent to it, the gradients of active power with respect to voltages are orthogonal to $\mathbf{z}$. Let $\mathbf{h}_{d}^T$ be the $d^{th}$ row of the Jacobian, which we decompose into two parts: $\mathbf{h}_{d,r}^T$ is the first $n$ components corresponding to the $d^{th}$ row of $\frac{\partial \mathbf{p}}{\partial \mathbf{v}_c}$ and $\mathbf{h}_{d,i}^T$ is the last $n$ components corresponding to the $d^{th}$ row of $\frac{\partial \mathbf{p}}{\partial \mathbf{v}_i}$. We then look for operating points such that $\mathbf{z}^T \mathbf{h}_{d,r} = 0$ and $\mathbf{z}^T \mathbf{h}_{d,i} = 0$ for all $d = 1, \cdots, n$. 

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Rectangular coordinates make this problem much easier to solve. As shown in (5), the elements of the Jacobian are linear in the real and imaginary parts of the bus voltages. Therefore, for a given \( z \), the system of equations \( z^T h_{d,r} = 0 \) and \( z^T h_{d,i} = 0 \) for all \( d = 1, \ldots, n \) becomes a system of linear equations:

\[
(2t_{d,1}v_{d,r} + t_{d,2})z_d + \sum_{k \neq d} (g_{kd}v_{d,r} + b_{kd}v_{d,i})z_k = 0, \tag{9a}
\]

\[
(2t_{d,1}v_{d,i} + t_{d,3})z_d + \sum_{k \neq d} (-b_{kd}v_{d,r} + g_{kd}v_{d,i})z_k = 0, \tag{9b}
\]

\( d = 1, \ldots, n. \)

Recall that \( t_{d,1} = -\sum_{k \in \mathbb{N}(d)} g_{kd} \) and every equation is linear in \( v_{d,r} \) and \( v_{d,i} \) once \( z \) and the network are given.

Fig. 6 shows the solution of (9) for the 3-bus network in Fig. 3 with all line admittances being 1s and \( z = [1, 1]^T \). In this case, the network is purely resistive and its Jacobian is given in (6). Solving (9) gives \( v_2 = 0.5 \) and \( v_3 = 0.5 \), which corresponds to the point \( A (p_2 = 0.25, p_3 = 0.25) \) on the boundary in Fig. 4.

Fig. 6: A point on the Pareto-front for the three bus network in Fig. 3 obtained from solving (9) with \( z = [1, 1]^T \). This point maximizes the sum power \( P_1 + P_2 \).

**Remark** So far, we considered two bus types in our modeling: the reference bus and the load bus. We did not consider the generator bus type because the solar generator in the distribution grid can be modeled as a PQ bus, [27], [28].

VI. SIMULATIONS

In this section, we perform extensive simulations on both transmission grids and distribution grids. The transmission grid cases include the standard IEEE transmission benchmarks (4, 9, 14, 30, 39, 57, 118, 300-bus networks) and the MATPOWER test cases (3, 5, 6, 24, 89, 145, 1354, 1888, 1951, 2383, 2736, 2737, 2746, 2848, 2868, 2869, 3012, 3120, 6468, 6470, 6495, 6515, 9241, 13659-bus networks) [29], [30]. The distribution grids include standard IEEE 8 and 123-bus networks. The goal is to illustrate the three applications: 1) verifying if a point is on the loadability boundary, 2) measuring the stability margins if the points are not on the boundary, and 3) locating boundary points. As our proposed methods are based on linear programming and convex optimization, the
computation time scales quite well with the size of the networks. The results on most of the test cases are similar to each other and we provide several representative examples in the followings.

### A. Boundary and Stability Margins

First of all, we use (5) to calculate the partial derivatives and then use (7) to test whether the operating points contained in the test cases are on the loadability boundary. Not surprisingly, none of the points are on the boundary as shown in Table I. From the computation times in Table I we can see that for systems with thousands of buses, the condition in (7) can be checked in around 10 seconds (using the CVX package in Matlab [31], [32]).

| No. of Buses | 3   | 4   | 5   | 6   | 9Q  | 9target | 14  | 24  | 30  | 30pwl | 30Q  | 39  | 57  | 89  |
|---------------|-----|-----|-----|-----|-----|---------|-----|-----|-----|-------|-----|-----|-----|-----|
| On Boundary?  | No  | No  | No  | No  | No  | No      | No  | No  | No  | No    | No  | No  | No  | No  |
| Time(s)       | 1.5 | 1.6 | 1.6 | 1.6 | 1.6 | 1.7     | 1.9 | 1.6 | 2.0 | 1.8   | 1.7 | 1.6 | 1.7 | 1.7 |

| No. of Buses | 118 | 145 | 300 | 1354 | 1888 | 1951 | 2383 | 2736 | 2737 | 2746wp | 2746wp | 2848 | 2868 | 2869 |
|---------------|-----|-----|-----|------|------|------|------|------|------|--------|--------|------|------|------|
| On Boundary?  | No  | No  | No  | No   | No   | No   | No   | No   | No   | No     | No     | No   | No   | No   |
| Time(s)       | 1.6 | 1.6 | 1.8 | 3.2  | 5.4  | 5.0  | 6.4  | 9.7  | 9.0  | 8.7    | 8.9    | 11.8 | 11   | 12.7 |

In this table, we check if the current operating points of different networks are on the feasibility boundary. Not surprisingly, none of the points are actually on the boundary. Since our approach is based on solving an LP, the computational time scales well with the network size.

Next, we check the stability margin of the operating points and these are recorded in Table II. Note that some systems are actually operating with fairly small margins, for example, the 2746-bus, 9241-bus, and 13659-bus systems. This means that they may not be robust under perturbations. The computational time again scales quite well with the size of the network.

### B. Going Towards the Boundary

Here, we move the operating points of a network in a direction until it hits the Pareto-front. We first use the operating point calculated by running power flow from Matpower. Then, we use the obtained voltages, namely the operating point, as a direction in the state space to find the boundary point. Then, we change the voltage step by step from the Matpower-based voltage state to the boundary point that we obtain. The x-axis of Fig. VII shows the normalized active power as we successively increase the load in the IEEE 14-bus system, and the y-axis plots the change in the margin. As expected, the margin decreases when the point is moving towards the boundary point. When it is on the boundary point, the margin becomes zero.
TABLE II: Stability Margins

| No. of Buses | 3   | 4   | 5   | 6   | 9Q  | 9target | 14  | 24  | 30  | 30pwl | 30Q  | 39  | 57  | 89  |
|-------------|-----|-----|-----|-----|-----|---------|-----|-----|-----|-------|------|-----|-----|-----|
| Margin      | 13.5| 27  | 156.5| 6.4 | 7.4 | 14.0    | 7.7 | 38.7| 6   | 6     | 6    | 20  | 20  | 30  |
| Time(s)     | 3.3 | 3.3 | 3.4 | 3.4 | 3.3 | 3.3     | 3.4 | 3.3 | 3.6 | 3.5   | 3.4  | 3.3 | 3.4 | 3.4 |

| No. of Buses | 118 | 145 | 300 | 1354| 1888| 2383    | 2736| 2737| 2746wop| 2746wp| 2848| 2868| 2869 |
| Margin      | 8.6 | 1606| 69.9| 32.1| 360.2| 345.5   | 21.0| 14.6| 14.1  | 10.9 | 11.0| 679.9| 672.6| 35.8 |
| Time(s)     | 3.5 | 3.5 | 3.7 | 5.1 | 7.5 | 7.4     | 10.7| 10.5| 10.7  | 10.4 | 10.4| 11.3 | 11.6 | 12.7 |

| No. of Buses | 3012| 3120| 6468| 6470| 6495| 9241    | 13659| 8 (Dist.)| 123 (Dist.) |
| Margin      | 16.4| 15.7| 1741.5| 1769.3| 1754.0| 1756.6  | 23.1 | 17.1 | 79   | 424  |
| Time(s)     | 11.7| 12.4| 45.5 | 50.6 | 42.4 | 44.8    | 102.1| 281.9| 1.5  | 2.0  |

For the operating points in Table II, we check their stability margin. Note that some systems are actually operating with fairly small margins, for example the 2746-bus, 9241-bus and 13659-bus systems. This means that they may not be robust under perturbations.

For the operating points in Table II, we check their stability margin. Note that some systems are actually operating with fairly small margins, for example the 2746-bus, 9241-bus and 13659-bus systems. This means that they may not be robust under perturbations.

Fig. 7: Margins vs. increasing active power for the IEEE 14-bus system. Here, we successively increase the load on the buses (proportionally) to observe the change with respect to the margin.

C. Locating Boundary Points

Given a network and by varying the search direction \( z \), we can use (9) to find boundary points. Fig. 8 shows the result for the network behind Fig. 4. By varying \( z \), we successfully locate the boundary points in Fig. 8 when comparing to Fig. 4.

VII. CONCLUSION

In general, power flow problems are hard to solve. We propose to use rectangular coordinate, which not only provides an integrated geometric understanding of active and reactive powers simultaneously but also a linear Jacobian matrix for loadability analysis. By using such properties, we proposed three optimization-based approaches for (1) loadability verification, (2) calculating the margin of an operating point, and (3) calculating the boundary...
Fig. 8: Points on the Pareto-front for the 3-bus network behind Fig. 4. They are obtained from solving (9) with different zs.

points. Numerical results demonstrate the capability of the new method. Future work includes applying the geometric understanding to other analysis in power systems.

REFERENCES

[1] E. B. Shand, “The limitations of output of a power system involving long transmission lines,” *Transactions of the American Institute of Electrical Engineers*, vol. 43, p. 59, Jan. 1924.

[2] P. Lajda, “Short-term operation planning in electric power systems,” *Journal of the Operational Research Society*, vol. 32, no. 8, p. 675, Aug. 1981.

[3] N. A. E. R. Corporation, “Planning reserve margin.” [Online]. Available: http://www.nerc.com/pa/RAPA/rePages/PlanningReserveMargin.aspx

[4] J. Bebic, “Power system planning: emerging practices suitable for evaluating the impact of high-penetration photovoltaics,” *National Renewable Energy Laboratory*, 2008.

[5] J. P. Pfeifenberger, K. Spees, K. Carden, and N. Wterminental, “Resource adequacy requirements: Reliability and economic implications,” *The Brattle Group*, 2013.

[6] V. Ajjarakp and C. Christy, “The continuation power flow: a tool for steady state voltage stability analysis,” *IEEE Transactions on Power Systems*, vol. 7, no. 1, p. 416, Feb. 1992.

[7] K. Iba, H. Suzuki, M. Egawa, and T. Watanabe, “Calculation of critical loading condition with nose curve using homotopy continuation method,” *IEEE Transactions on Power Systems*, vol. 6, no. 2, p. 584, May 1991.

[8] V. A. Venikov, V. A. Stroev, V. I. Idelchick, and V. I. Tarasov, “Estimation of electrical power system steady-state stability in load flow calculations,” *IEEE Transactions on Power Apparatus and Systems*, vol. 94, no. 3, p. 1034, May 1975.

[9] B. K. Johnson, “Extraneous and false load flow solutions,” *IEEE Transactions on Power Apparatus and Systems*, vol. 96, no. 2, pp. 524–534, Mar 1977.

[10] T. V. Cutsem, “A method to compute reactive power margins with respect to voltage collapse,” *IEEE Transactions on Power Systems*, vol. 6, no. 1, pp. 145–156, Feb. 1991.

[11] T. van Cutsem and C. Vournas, *Voltage Stability of Electric Power Systems.* Springer, 1998.

[12] J. Baillieur and C. Byrnes, “Geometric critical point analysis of lossless power system models,” *IEEE Transactions on Circuits and Systems*, vol. 29, no. 11, pp. 724–737, 1982.
A. Derivation of (2)

The active and reactive powers in the polar coordinate are

\[ p_d = \sum_{k=1}^{n} |v_d||v_k| \left( g_{dk} \cos \theta_{dk} + b_{dk} \sin \theta_{dk} \right) , \]

\[ q_d = \sum_{k=1}^{n} |v_d||v_k| \left( g_{dk} \sin \theta_{dk} - b_{dk} \cos \theta_{dk} \right) . \]
Let $v_{d,r} = |v_d| \cos \theta_d$ and $v_{d,i} = |v_d| \sin \theta_d$, we have

$$p_d = v_{d,r} \sum_{k \in \mathcal{N}(d)} v_{k,r} g_{kd} + v_{d,i} \sum_{k \in \mathcal{N}(d)} v_{k,i} g_{kd}$$

$$- v_{d,r}^2 \sum_{k \in \mathcal{N}(d)} g_{kd} - v_{d,i}^2 \sum_{k \in \mathcal{N}(d)} g_{kd}$$

$$+ v_{d,i} \sum_{k \in \mathcal{N}(d)} v_{k,r} b_{kd} - v_{d,r} \sum_{k \in \mathcal{N}(d)} v_{k,i} b_{kd}$$

$$= \left[ - \sum_{k \in \mathcal{N}(d)} g_{kd} \cdot v_{d,r}^2 + \sum_{k \in \mathcal{N}(d)} v_{k,r} g_{kd} \cdot v_{d,r} - \sum_{k \in \mathcal{N}(d)} v_{k,i} b_{kd} \cdot v_{d,r} + \sum_{k \in \mathcal{N}(d)} v_{k,r} b_{kd} \cdot v_{d,i} - \sum_{k \in \mathcal{N}(d)} g_{kd} \cdot v_{d,i}^2 \right]$$

$$= t_{d,1} \cdot v_{d,r}^2 + t_{d,2} \cdot v_{d,r} + t_{d,3} \cdot v_{d,i}^2 + t_{d,4} \cdot v_{d,i}.$$  (10)

Similarly, expanding the equation for $q_d$ and collecting terms gives

$$q_d = \left( v_{d,r}^2 + v_{d,i}^2 \right) \sum_{k \in \mathcal{N}(d)} b_{kd} - \left[ \sum_{k \in \mathcal{N}(d)} (v_{k,r} b_{kd} + v_{k,i} g_{kd}) \right] v_{d,r}$$

$$+ \left[ \sum_{k \in \mathcal{N}(d)} (v_{k,r} g_{kd} - v_{k,i} b_{kd}) \right] v_{d,i}$$

$$= t_{d,2} v_{d,r}^2 + t_{d,3} v_{d,r} + t_{d,4} v_{d,i}^2 + t_{d,4} v_{d,i}.$$  (11)

where $t_{d,1}, t_{d,2}, t_{d,3}, t_{d,4}$ are given in (3).

**B. Proof for Lemma 7**

From (10), we can complete the square to write it as an equation of a circle:

$$\frac{p_d}{t_{d,1}} + \frac{t_{d,2}^2}{4t_{d,1}^2} + \frac{t_{d,3}^2}{4t_{d,1}^2} = \left( v_{d,r} + \frac{t_{d,2}}{2t_{d,1}} \right)^2 + \left( v_{d,i} + \frac{t_{d,3}}{2t_{d,1}} \right)^2.$$  

Therefore, the center of the circle is \left( -\frac{t_{d,2}}{2t_{d,1}}, -\frac{t_{d,3}}{2t_{d,1}} \right) with the radius being the square root of \frac{p_d}{t_{d,1}} + \frac{t_{d,2}^2}{4t_{d,1}^2} + \frac{t_{d,3}^2}{4t_{d,1}^2}.

Interestingly, since $t_{d,1} = -\sum_{k \in \mathcal{N}(d)} g_{kd}$ is always negative, as the load $p_d$ increases, the radius approaches 0.

This places an upper bound on the possible active power consumption at a bus.

Similarly, we can write (11) as

$$\frac{q_d}{t_{d,2}} + \frac{t_{d,3}^2}{4t_{d,4}^2} + \frac{t_{d,2}^2}{4t_{d,4}^2} = \left( v_{d,r} - \frac{t_{d,3}}{2t_{d,4}} \right)^2 + \left( v_{d,i} + \frac{t_{d,2}}{2t_{d,4}} \right)^2,$$

which is a circle with center \left( \frac{t_{d,3}}{2t_{d,4}}, -\frac{t_{d,2}}{2t_{d,4}} \right) with the radius being the square root of \frac{q_d}{t_{d,2}} + \frac{t_{d,3}^2}{4t_{d,4}^2} + \frac{t_{d,2}^2}{4t_{d,4}^2}.

Again, since $t_{d,4} = \sum_{k \in \mathcal{N}(d)} b_{kd}$ is always negative (assuming transmission lines are inductive), the radius decreases as $q_d$ increases.