BOUNDEDNESS OF BILINEAR MULTIPLIERS WHOSE SYMBOLS HAVE A NARROW SUPPORT

By

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Abstract. This work is devoted to studying the boundedness on Lebesgue spaces of bilinear multipliers on \( \mathbb{R} \) whose symbol is narrowly supported around a curve (in the frequency plane). We are looking for the optimal decay rate (depending on the width of this support) for exponents satisfying a sub-Hölder scaling. As expected, the geometry of the curve plays an important role, which is described. This has applications to the bilinear Bochner-Riesz problem (in particular, boundedness of multipliers whose symbol is the characteristic function of a set), as well as to the bilinear restriction-extension problem.

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1  Introduction

1.1  The central question.  Pseudo-products were introduced by Bony [3] and Coifman-Meyer [5]; we define and study them only for bilinear operators. Given a symbol $m(\zeta, \eta)$, the pseudo-product $B_m$, acting on functions over $\mathbb{R}^d$, is
defined by
\[
B_m(f, g)(x) := \frac{1}{(2\pi)^d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\xi \cdot (\xi + \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\eta \, d\xi
\]
\[
= \mathcal{F}^{-1} \left[ \xi \rightarrow \int_{\mathbb{R}} m(\xi - \eta, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta \right](x).
\]
(Notation, in particular the convention used for the Fourier transform, is given in Section 2.) Our aim is to study, for \( d = 1 \), the connection between singularities of \( m \) (on various scales and along any possible smooth geometry) and boundedness properties of \( B_m \).

A relevant model is the following. Consider a smooth curve \( \Gamma \subset \mathbb{R}^2 \), and let \( m_\epsilon \) be a symbol satisfying \( m_\epsilon \leq 1 \) and \( m_\epsilon(y) = 0 \) whenever \( \text{dist}(y, \Gamma) \geq \epsilon \). What can be said about the regularity of \( m_\epsilon \)? A first possibility is simply to ask that it vary on a typical length \( \epsilon \); one can also ask for more smoothness in the direction tangential to \( \Gamma \): see Definitions 1.2 and 1.3 for greater precision.

**Question.** Set \( d = 1 \). For which Lebesgue exponents \((p, q, r) \in [1, \infty]^3\) and for which functions \( \alpha(\epsilon) \) does
\[
\|B_m\|_{L^p \times L^q \to L^r} \lesssim \alpha(\epsilon)
\]
hold; or, written more symmetrically,
\[
|\langle B_m(f, g), h \rangle| \lesssim \alpha(\epsilon) \|f\|_{L_p} \|f\|_{L_q} \|f\|_{L_r}?
\]
(It turns out that, typically, \( \alpha(\epsilon) \) must be a power function with perhaps a logarithmic correction).

As we show, the answer the above question helps answer the following questions.

- (Bilinear Bochner-Riesz) Given a compact domain \( K \) with smooth boundary, for which \( p, q, r, \kappa \) is \( B^\kappa_{m_K} \) bounded from \( L^p \times L^q \) into \( L^r \) if \( m^\kappa_K(\eta, \xi) = \chi_K(\xi, \eta) \text{dist}(\xi, \eta, \partial K)^\kappa \)?
- (Bilinear restriction-extension) Given a curve \( \Gamma \), for which \( p, q, r \) is \( B_{d\sigma_G} \) bounded from \( L^p \times L^q \) into \( L^r \)?

(The quantities \( \chi_K \) and \( d\sigma_G \) are defined in Section 2).

**1.2 Analogy with the linear case.** The above questions have clear analogs in the linear case: the well-known Bochner-Riesz (boundedness between Lebesgue spaces of \( f \mapsto \mathcal{F}^{-1} \left[ \hat{f} \chi_K \text{dist}(\cdot, \partial K)^\kappa \right] \) for \( K \subset \mathbb{R}^d \)), restriction (boundedness of \( f \mapsto \hat{f}|_\Gamma \) for \( \Gamma \) hypersurface of \( \mathbb{R}^d \)), and extension (boundedness of \( f \mapsto \int_{\mathbb{R}^d} \hat{f}d\sigma_G \)).
problems. Notice that combining the restriction and extension problem gives the transformation $f \mapsto \hat{f} d\sigma_\Gamma$, which we call restriction-extension in the bilinear setting.

We do not discuss these standard problems here, but rather refer the reader to Stein [31] or Grafakos [12]. We simply note that while the case $d = 1$ is essentially trivial, a crucial element in the answer in case $d = 2$ is the curvature of $\partial K$ and $\Gamma$.

1.3 Known results for the bilinear case. Much of the research on bilinear operators has focused on boundedness of mappings between Lebesgue spaces at the Hölder scaling: from $L^p \times L^q$ into $L^{r'}$, with $1/p + 1/q = 1/r$.

The first results, obtained by Coifman and Meyer [5], allowed for a singularity localized at a point: the symbol $m$ satisfies a Mikhlin-type condition

$$|\xi^\alpha \eta^\beta m(\xi, \eta)| \lesssim \frac{1}{(|\xi| + |\eta|)^{1/2}}.$$  

If $m$ satisfies this inequality, $B_m : L^p \times L^q \to L^{r'}$ is bounded if $1 = p^{-1} + q^{-1} + r^{-1}$ and $1 < p, q, r < \infty$. For another result on boundedness with a singularity at a point, see Gustafson, Nakanishi and Tsai [17] and the version of their result in Guo and Pausader [16].

The bilinear Hilbert transform corresponds to taking $d = 1$, and the characteristic function $m$ of a (perhaps tilted) half-plane is singular along a line. The celebrated results of Lacey and Thiele [23, 24, 25, 26] give boundedness of $B_m : L^p \times L^q \to L^{r'}$ in the case $1 < p, q < \infty$ and $0 < 1/r' = 1/p + 1/q < 3/2$. These results were later extended by Grafakos and Li [14, 27] to cover the case that $m$ is the characteristic function of a polygon. We refer the reader to [1], where a proof of boundedness for particular square functions built on a covering of the frequency plane with polygons appears.

Finally, we discuss the case that $m$ is the characteristic function of the ball. The singularity is now localized on a curved set. Diestel and Grafakos [7] proved that the characteristic function of the four-dimensional ball $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^{r'}(\mathbb{R}^2)$ is not a bounded bilinear multiplier operator in the outside of $L^2$-case, i.e., when neither $1/p + 1/q + 1/r = 1$ nor $2 \leq p, q, r < \infty$. Conversely, it was shown by Grafakos and Li [15] that the characteristic function of the unit disc in $\mathbb{R}^2$ $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^{r'}(\mathbb{R}^2)$ is a bounded bilinear multiplier in the local $L^2$-case. The corresponding problem in higher dimension remains unresolved.

Let us point out that the easiest case, viz., $p = q = r = 2$, can be studied directly using the Plancherel inequality. This special setting has been involved with the use of $X^{s,b}$ spaces and has also been extended to less regular symbols to the so-called
“multilinear convolution in $L^2$” by Tao; see [32]. In the opposite direction, we emphasize that the “limit situation” obtained for $\epsilon = 0$ and the Hölder scaling is far more difficult and is not studied here.

1.4 Results. The previously cited works deal with the critical (and more complicated) case in which the exponents $p, q, r \in [1, \infty]$ satisfy the Hölder scaling $1 = 1/r + 1/p + 1/q$ and sometimes allow for Lebesgue exponents less than 1. By contrast, we assume that $1 \leq p, q, r \leq \infty$, and focus attention on the “sub-critical” range

$$1 \leq \frac{1}{r} + \frac{1}{p} + \frac{1}{q}. \quad (1.3)$$

What are the important geometric features of $\hat{\Gamma}$ so far as boundedness of $B_{m_\epsilon}$ is concerned? It turns out that the answer depends on whether $\hat{\Gamma}$ has tangents parallel to the axes $\{\zeta = 0\}$, $\{\eta = 0\}$, and $\{\zeta + \eta = 0\}$.

**Definition 1.1.** A **characteristic point** of a smooth curve $\Gamma$ in the $(\zeta, \eta)$-plane is defined as a point where the tangent to $\Gamma$ is parallel to one of the $\{\zeta = 0\}$, $\{\eta = 0\}$, or $\{\zeta + \eta = 0\}$ axes. We call $\Gamma$ characteristic (respectively, non-characteristic) if such points exist (respectively, do not exist).

The best bounds for $B_{m_\epsilon}$ are obtained for non-characteristic $\Gamma$; the next best case occurs when the set of characteristic points is finite with non-zero curvature of $\Gamma$; and the worst case is, of course, when a piece of $\Gamma$ is a segment parallel to one of the $\zeta$, $\eta$, or $\zeta + \eta$ axes.

It is worth noticing that, as opposed to the linear case, the curvature of $\Gamma$ does not play any role per se in obtaining bounds; it only enters in the sense that non-vanishing curvature prevents points close to a characteristic point from being too close to characteristic points. In particular, at non-characteristic points, bounds do not depend on whether or not $\Gamma$ has non-zero curvature.

Before stating our theorems, we define the regularity classes for $m_\epsilon$. The first class only requires $m_\epsilon$ to be supported in an $\epsilon$-neighborhood of $\Gamma$, with derivatives of order $1/\epsilon$.

**Definition 1.2.** The scalar-valued symbol $m_\epsilon$ is said to belong to the class $M^\Gamma_\epsilon$ if

- $\Gamma$ is a smooth curve in $\mathbb{R}^2$,
- $m_\epsilon$ is supported in $B(0, 1)$ and in a neighborhood of size $\epsilon$ of $\Gamma$,
- $|\partial^\alpha \zeta \partial^\beta \eta m_\epsilon(\zeta, \eta)| \lesssim \epsilon^{-|\alpha| - |\beta|}$ for sufficiently many indices $\alpha$ and $\beta$. 
The above class turns out to be too weak in the case $\Gamma$ is characteristic with non-vanishing curvature and nearly Hölder exponents. In this case, tangential smoothness is required. This case motivates our next definition. We could weaken it significantly, but then appropriate conditions would become too technical.

**Definition 1.3.** Close to $\Gamma$, it is possible to define “normal directions” simply by prolonging the normals to $\Gamma$, and “tangential directions” as lines whose tangents are everywhere orthogonal to normal directions. If $\nu(x)$ is the distance of $x$ to $\Gamma$, $\nabla \nu$ can be considered as the direction of the local normal coordinate and $(\nabla \nu)_{\perp}$ as the direction of the local tangential coordinate. We are interested in symbols $m_\epsilon$ having nice behavior in the tangential directions given by $(\nabla \nu)_{\perp}$. Hence, for a vector $X$, we define $\partial_X := X \cdot \nabla$. The scalar-valued symbol $m_\epsilon$ belongs to the class $\mathcal{N}_\epsilon^\Gamma$ if

- $\Gamma$ is a smooth curve in $\mathbb{R}^2$;
- $m_\epsilon$ is supported in $B(0, 1)$ and a neighborhood of size $\epsilon$ of $\Gamma$;
- $|\partial_\alpha \nabla_{(\nabla \nu)}^\beta m_\epsilon(\xi, \eta)| \lesssim \epsilon^{-|\alpha|}$ for sufficiently many indices $\alpha, \beta$.

We now come to our results, which, for the sake of brevity and clarity, we do not always state in optimal fashion. The interested reader is referred to Sections 4, 5, 6, and 7, where more precise statements are given. Section 9 contains extensions of results to rough curves $\Gamma$ (provided they satisfy some rectifiability and Ahlfors-regularity properties).

**1.4.1 The non-characteristic case.**

**Theorem 1.4.** Assume that $\Gamma$ is non-characteristic. Let $m_\epsilon \in \mathcal{M}_\epsilon^\Gamma$ and $p, q, r \in (1, \infty)$.

- If
  
  \[
  1 \leq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 2, \\
  \frac{1}{r} + \frac{1}{q} \leq \frac{3}{2}, \\
  \frac{1}{p} + \frac{1}{q} \leq \frac{3}{2}, \\
  \frac{1}{p} + \frac{1}{r} \leq \frac{3}{2},
  \]

  then $\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{1/p+1/q+1/r-1}$, and this power of $\epsilon$ is optimal.

- If $q \geq 2 \geq p, r$ and $1/p + 1/r \geq 3/2$, then $\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{1/q+1/2}$. This also holds for all permutations of the indices $(p, q, r)$.

- If $p, q, r \leq 2$ and $1/p + 1/q + 1/r \geq 2$, then $\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon$, and this bound is optimal.
The three cases in the above theorem cover the full range $1 < p, q, r < \infty$. The bounds are optimal except in the second case. Optimality extends to symbols in the smoother class $N^\Gamma_\epsilon$; this is also the case in the following theorems.

The proof of Theorem 1.4 can be found in Section 7, where the interpolation between endpoint type results obtained in Sections 4 and 5 is performed. The optimality statements follow from results in Section 3.

### 1.4.2 The non-vanishing curvature case.

**Theorem 1.5.** Assume $\Gamma$ has non-vanishing curvature, and let $m_\epsilon \in \mathcal{M}_\epsilon^\Gamma$ and $p, q, r \in (1, \infty)$.

- If $p, q, r \geq 2$, then $\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{1/p+1/q+1/r-1}$, and this power of $\epsilon$ is optimal.
- If $q > 2$ and $p, r < 2$, then $\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{-1/2+1/q+1/(p+1/r)/2-\delta}$ for all $\delta > 0$. The power of $\epsilon$ is optimal up to the additional $\delta$. This also holds for all permutations of the indices $(p, q, r)$.
- If $1/p + 1/q + 1/r > 5/2$, then $\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{(1/p+1/q+1/r-1/2-\delta)/2}$ for all $\delta > 0$.

Theorem 1.5 only leaves out the case that exactly one of the three Lebesgue indices is less than 2. The bounds stated are optimal except in the last case above; see Section 1.5.2 for some improvements in this direction.

In order to cover the remaining cases, we require more tangential regularity from $m_\epsilon$.

**Theorem 1.6.** Assume $\Gamma$ has non-vanishing curvature, and let $m_\epsilon \in \mathcal{N}_\epsilon^\Gamma$ and $p, q, r \in (1, \infty)$.

- If $p < 2$, $q > r > 2$, and $1/p + 1/r > 1$, then
  $$\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{-1/2+1/q+1/2\left(1 + \frac{1}{r}\right)} - \delta$$
  for all $\delta > 0$, and this estimate is optimal up to the additional $\delta$.
- If $1/p + 1/r < 1$, $1/p + 1/q < 1$, and $1/p + 1/q + 1/r > 1$, then
  $$\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \lesssim \epsilon^{1/p+1/q+1/r - 1 - \delta}$$
  for all $\delta > 0$, and this estimate is optimal up to the additional $\delta$.
- The above statements also hold for all permutations of the indices $(p, q, r)$.
The three cases above do not completely cover all cases in which exactly one of the three Lebesgue indices is less than 2. We refrain from giving bounds for any of \((p, q, r)\), since the formulas obtained become too complicated.

Theorems 1.5 and 1.6 are proved by interpolating between the results of Sections 4, 5, and 6. The optimality statements follow from results in Section 3. Details of the proofs are omitted, as they are very similar to that of Theorem 1.4.

### 1.4.3 The general case.

**Theorem 1.7** (Theorem 7.1). Suppose \(\Gamma\) is an arbitrary curve. Let \(m_\epsilon \in M_{\epsilon}^{\Gamma}\) and \(p, q, r\) in \((1, \infty)\).

- If \(p, q, r \geq 2\) and satisfy (1.3), then \(\|B_{m_\epsilon}\|_{L^p \times L^q \rightarrow L^r'} \lesssim \epsilon^{1/p + 1/q + 1/r - 1}\), and this power of \(\epsilon\) is optimal.
- If at least two of the three indices \((p, q, r)\) are smaller than 2, then

\[
\|B_{m_\epsilon}\|_{L^p \times L^q \rightarrow L^r'} \lesssim \epsilon^{\inf(1/p, 1/q, 1/r)},
\]

and this power of \(\epsilon\) is optimal.

Theorem 1.7 leaves aside, amongst all possible values for \((p, q, r)\), only the case that exactly two of the three indices \((p, q, r)\) are greater than 2. Once again, the result for this case can be obtained by interpolating between the results of Sections 4 and 5 and using the optimality criteria of Section 3.

**Remark 1.8** (Hölder case). Suppose that the exponents \(1 < p, q, r < \infty\) satisfy \(1/p + 1/q + 1/r = 1\). We get the expected bound of order 1 for \(B_{m_\epsilon}\) in two cases: when \(\Gamma\) is non-characteristic, or when \(p, q, r > 2\). We can prove that if \(\Gamma\) has non-zero curvature and \(m_\epsilon \in N_{\epsilon}^{\Gamma}\), then \(B_{m_\epsilon} : L^p \times L^q \rightarrow L^r\) has norm \(\sim \epsilon^\rho\), where \(\rho \rightarrow 0\) as \(1/p + 1/q + 1/r \searrow 1\). We believe that the techniques developed in this paper can lead to the same result for arbitrary curves \(\Gamma\).

### 1.5 Applications and extensions.

**1.5.1 Non-smooth setting.** In Section 9, we discuss to what extent our results can be extended to the case \(\Gamma\) is not necessarily smooth, but is rectifiable and Ahlfors-regular.

**1.5.2 Bilinear restriction-extension inequalities.** Letting \(\epsilon\) tend to 0 and rescaling \(m_\epsilon\) in a natural way transforms the problem into determining for which \((p, q, r)\) the mapping \(B_{\lambda, \sigma, \Gamma}\) is bounded from \(L^p \times L^q\) into \(L^r\). Here, \(\lambda\) is a
smooth, compactly supported function and $d\sigma_{\Gamma}$ is the one-dimensional Hausdorff measure supported on $\Gamma$.

It is easy to see that if (1.1) holds for $p, q, r$ with $\alpha(\epsilon) \lesssim \epsilon$, then $B_{\lambda d\sigma_{\Gamma}}$ is bounded. Thus, we deduce from Theorems 1.4 and 1.5 that if $\Gamma$ is non-characteristic, then $B_{d\sigma_{\Gamma}}$ is bounded for $p, q, r \geq 2$ and $1/p + 1/q + 1/r \geq 2$. If the curvature of $\Gamma$ does not vanish, the condition becomes $1/p + 1/q + 1/r > 5/2$. More refined criteria are derived in Section 8.

1.5.3 Bilinear Bochner-Riesz problem. In Section 10.1, conditions on a compact set $K$ and a real $\lambda > 0$ are deduced which ensure that operators from $L^p \times L^q$ into $L'^r$ with symbol $\chi_K(\xi, \eta)$ or $\chi_K(\xi, \eta)\text{dist}((\xi, \eta), \partial K)^\kappa$ are bounded.

As usual, this has the consequence that if $\text{Int}(K)$ contains 0, $(f, g) \in L^p \times L^q$, and $1/p + 1/q = 1/r'$, then $B^\kappa_{\lambda K}(f, g) \to fg \in L'^r$, where $\lambda K$ denotes the dilation of $K$ around 0 by a factor of $\lambda$.

1.5.4 Singular symbols. In Section 10.2, we give results on boundedness of operators between Lebesgue spaces which have symbols of the type $\Phi_1(\xi, \eta)\text{dist}((\xi, \eta), \Gamma)^{-\alpha}$, where $\Phi \in \mathcal{C}_0^\infty$, $\Gamma$ is a smooth curve, and $\alpha > 0$.

Kenig and Stein [21] derived sharp results in the same spirit for the case in which $\Gamma$ is a line; in this specific situation, the bilinear multiplier $B_m$ can be represented by bilinear fractional integral operators. These authors were able to deal with Lebesgue exponents less than 1.

1.5.5 Dispersive PDEs. In this subsection, we explain briefly a motivation for our study. A more detailed presentation can be found in [2].

Consider the dispersive PDE with real symbol $p$

$$\partial_t u + ip(D)u = B_m(u, u)$$

$$u(t = 0) = u_0,$$

where, as above, $B_m$ denotes the pseudo-product with symbol $m$. In the weakly non-linear setting, the first Duhamel iterate

$$(f, g) \to \int_0^t B_{e^{i\phi}m}(f, g)ds$$

is crucial to understanding this equation. It has symbol $(e^{i\phi} - 1)/i\phi$, where $\phi(\xi, \eta) := p(\xi + \eta) - p(\eta) - p(\xi)$. Under suitable assumptions on $\phi$, the symbol fits into the previous setting, relative to the curve $\Gamma := \phi^{-1}([0])$. Of particular importance is the large time problem $t \to \infty$. It is examined briefly in Section 10.3.
2 Notation and preliminaries

2.1 Some standard notation. We adopt the following notation.

- We write \( A \lesssim B \) if \( A \leq CB \) for some implicit, universal constant \( C \). The value of \( C \) may change from line to line.
- The relation \( A \sim B \) means that \( A \lesssim B \) and \( B \lesssim A \).
- \( \chi_E \) is the characteristic function of the set \( E \).
- The “Japanese brackets” \( \langle \cdot \rangle \) stand for \( \langle x \rangle = \sqrt{1 + x^2} \).
- \( d\sigma_\Gamma \) is the 1-dimensional Hausdorff measure of a rectifiable curve \( \Gamma \) restricted to \( \Gamma \). For \( \epsilon > 0 \), we denote by \( \Gamma_\epsilon \) the \( \epsilon \)-neighborhood of \( \Gamma \),
  \( \Gamma_\epsilon := \{ \xi \in \mathbb{R}^2, d(x, \Gamma) \leq \epsilon \} \).
- \( \mathcal{H}^1 \) denotes the one-dimensional Hausdorff measure.
- The standard \( L^2 \) scalar product is denoted by \( \langle f, g \rangle := \int_{\mathbb{R}^d} f \, \overline{g} \).
- The Fourier transform of a function \( f : \mathbb{R}^d \to \mathbb{R} \), denoted either by \( \hat{f} \) or \( \mathcal{F}(f) \), is given by
  \( \hat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix\xi} f(x) \, dx; \) thus \( f(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix\xi} \hat{f}(\xi) \, d\xi \).
  We systematically drop constants such as \( 1/(2\pi)^{d/2} \) when they are not important.
- The Fourier multiplier with symbol \( m(\xi) \) is defined by \( m(D)f = \mathcal{F}^{-1} [m\mathcal{F}f] \).

2.2 Some harmonic analysis. Let us now recall some useful and well-known operators on \( \mathbb{R} \).

Definition 2.1. For \( s > 0 \), the fractional integral operator of order \( s > 0 \) is defined by
  \( I_s(f)(x) := x \mapsto \int_{\mathbb{R}} f(y) |x - y|^{s-1} \, dy \).
  \( I_0 \) is not defined for \( s = 0 \), since \( 1/|\cdot| \) is not locally integrable. To deal with this case, we consider the Hardy-Littlewood maximal function
  \( \mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{2r} \int_{B(x,r)} |f(z)| \, dz \).
  For \( s \geq 1 \), we set \( \mathcal{M}_s \) for its \( L^s \)-version defined by \( \mathcal{M}_s(f) = [\mathcal{M}(|f|^s)]^{1/s} \).
  These operators are bounded over Lebesgue spaces.
Proposition 2.2. Let \( s \in (0, 1) \) and \( 1 < p < q < \infty \) satisfy \( 1/p - 1/q = s \). Then \( I_s : L^p \to L^q \) is bounded (see [18]). For \( s \geq 1 \) and \( p \in (s, \infty) \), \( \mathcal{M}_s \) is \( L^p \)-bounded.

We recall the boundedness of Rubio de Francia’s square functions (see [30]).

Proposition 2.3. Let \( 2 \leq p < \infty \) and \( I := (I_i)_i \) be a bounded covering of \( \mathbb{R} \). Then the square function

\[
f \to \left( \sum_i |\pi_i(f)|^2 \right)^{1/2}
\]

is \( L^p \)-bounded. The non-smooth truncations \( \pi_i \) can be replaced with smooth truncations.

2.3 Bilinear multipliers. For a symbol \( m \in \mathcal{S}'(\mathbb{R}^2) \), we can define the bilinear multiplier

\[
T_m(f, g)(x) := \int_{\mathbb{R}^2} e^{ix(\xi + \eta)} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) d\xi d\eta
\]

in the distributional sense. Conversely, it is well known that any translation invariant bilinear bounded multiplier \( \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})' \) can be written in this form.

As viewed in physical space, the bilinear operator \( T_m \) can be represented as

\[
(2.1) \quad B_m(f, g)(x) = \sqrt{2\pi} \int \int \hat{m}_\epsilon(y-x, z-x) f(y) g(z) dy dz.
\]

3 Necessary conditions and the specific case \( \Gamma \) is a line

3.1 Necessary conditions. It is well known that multilinear multipliers (the operators commuting with the simultaneous translations) cannot lose integrability. Thus, a bilinear multiplier can be bounded from \( L^p \times L^q \) into \( L^{r'} \) only if \( 1/r' \leq 1/p + 1/q \). This corresponds to (1.3).

Remark 3.1. Observe that since \( \Gamma \) is compactly supported, (1.1) holds for \( (p, q, r, \alpha(\epsilon)) \) if it holds for \( (P, Q, R, \alpha(\epsilon)) \) with \( p \leq P, q \leq Q \) and \( r \leq R \). Thus our task is to push indices up!

Let \( \Gamma \) be a smooth compact curve in \( \mathbb{R}^2 \). Recall that we are looking for exponents \( p, q, r \in [1, \infty] \) that satisfy (1.3) and a function \( \alpha(\epsilon) \) such that for small
enough $\epsilon > 0$, any symbol $m_\epsilon$ in either the class $\mathcal{M}_\epsilon^\Gamma$ or $\mathcal{N}_\epsilon^\Gamma$ gives rise to a bilinear multiplier $B_{m_\epsilon}$ satisfying

$$\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r} \lesssim \alpha(\epsilon).$$

(3.1)

This subsection is devoted to describing some necessary conditions for the existence of such $p$, $q$, $r$, and $\alpha$.

**Proposition 3.2** (Necessary conditions). Assume that $m_\epsilon \in \mathcal{N}_\epsilon^\Gamma$, $m_\epsilon \geq 0$, and $m_\epsilon = 1$ on a (non-empty) curve $\Gamma' \subset \Gamma$. Suppose that $p$, $q$, $r$ satisfy (1.3). Then (3.1) holds only if

(i) $\alpha(\epsilon) \gtrsim \epsilon$,

(ii) $\alpha(\epsilon) \gtrsim \epsilon^{1/p + 1/q + 1/r - 1}$ for $\delta > 0$ (if $\Gamma'$ has non-vanishing curvature and has, in the coordinates $(\xi, \eta)$, a tangent parallel to the $\xi$ axis),

(iii) $\alpha(\epsilon) \gtrsim \epsilon^{1/p}$ (if $m_\epsilon(\xi, \eta) = \chi(\xi/\epsilon)\chi(\eta)$, where $\chi$ is a smooth function equal to 1 on $B(0, 1)$ and vanishing outside of $B(0, 2)$).

**Proof.** Consider a suitable non-negative symbol $m_\epsilon$.

(i). Take $R$ such that $\text{Supp } m_\epsilon \subset B(0, R)$. Then take $f$, $g$, and $h$ in $S(\mathbb{R})$ such that $\hat{f} = \hat{g} = \hat{h} = 1$ on $B(0, 10R)$. A trivial computation using Lemma 9.4 gives $\langle B_{m_\epsilon}(f, g), h \rangle \sim \epsilon$, hence the desired bound on $\alpha$.

(ii). Assume that $\Gamma'$ goes through $(0, 0)$ and take $f$, $g$, $h$ in $S(\mathbb{R})$ such that $\hat{f}, \hat{g}, \hat{h} \geq 0$, and $\text{Supp } \hat{f}, \text{Supp } \hat{g}, \text{Supp } \hat{h} \subset B(0, 1/100)$. Next define

$$f^\epsilon = \epsilon f(\epsilon \cdot), \quad g^\epsilon = \epsilon g(\epsilon \cdot) \quad \text{and} \quad h^\epsilon = \epsilon h(\epsilon \cdot).$$

Obviously, $\langle B_{m_\epsilon}(f^\epsilon, g^\epsilon) \rangle \sim \epsilon^2$, whereas $\|f^\epsilon\|_{L^p} \|g^\epsilon\|_{L^q} \|h^\epsilon\|_{L^r} \sim \epsilon^{3 - 1/p - 1/q - 1/r}$. This gives the desired bound on $\alpha$.

(iii). We treat only the case $p, q, r > 1$; only a small modification is needed if one of the exponents equals 1. With the hypotheses on $\Gamma'$, this curve can be parameterized in some region by $\eta = \phi(\xi)$, where $\phi'$ vanishes at a point, say $\xi_0$. But $\phi''(\xi_0) \neq 0$, since the curvature of $\Gamma'$ does not vanish. For simplicity, assume that $x_0 = 0$, $\phi''(0) = 1$, and consider the test functions

$$\hat{f}(\xi) = \Phi(\xi) \left|\frac{\xi}{\epsilon}\right|^a, \quad \hat{g}(\xi) = \Phi(\xi) \left|\frac{\xi}{\epsilon}\right|^b \quad \text{and} \quad \hat{h}(\xi) = \Phi(\xi) \left|\frac{\xi}{\epsilon}\right|^c,$$

where $\Phi \in C^\infty_0$ is non-negative and does not vanish at 0, and

$$a = 1 - \frac{1}{p} - \frac{\delta}{3}, \quad b = 1 - \frac{1}{q} - \frac{\delta}{3}, \quad \text{and} \quad c = 1 - \frac{1}{r} - \frac{\delta}{3}.$$
for some small $\delta > 0$. It is easy to check that $f$, $g$, and $h$ belong, respectively, to $L^p$, $L^q$, and $L^r$. Next, we want to estimate $\langle B_{m,\epsilon}(f, g), h \rangle$. For some appropriate constant $c_0$,

$$\langle B_{m,\epsilon}(f, g), h \rangle = \int \int m_{\epsilon}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(-\eta - \xi) d\eta d\xi$$

An easy computation then yields

$$\langle B_{m,\epsilon}(f, g), h \rangle \geq \epsilon^{-1/2+1/p+1/r}/2+2\delta,$$

which is the desired result.

(iv). Let $m_{\epsilon} = \chi(\xi/\epsilon)\chi(\eta)$ and $B_{m,\epsilon}(f, g) = \chi(D/\epsilon)f\chi(D)g$. Choosing $f, g, h$ in the Schwartz class with Fourier transforms localized in $B(0, 1/2)$, set $f^\epsilon = f(\epsilon \cdot)$. Then $\langle B_{m,\epsilon}(f, g), h \rangle \to f(0) \int gh$ as $\epsilon \to 0$, whereas $\|f^\epsilon\|_{L^p} \|g\|_{L^q} \|h\|_{L^r} = \epsilon^{-1/p} \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}$. This implies (iv) immediately. □

3.2 Necessary and sufficient conditions; case $m_{\epsilon} = \lambda d\sigma_\Gamma \ast \epsilon^{-1}\chi_{\epsilon}$. In this subsection, we assume that $m_{\epsilon} = \lambda d\sigma_\Gamma \ast \epsilon^{-1}\chi_{\epsilon}$, where $\chi_{\epsilon} = \chi(\epsilon^{-1} \cdot)$ is a smooth, non-negative function supported on $B(0, 2)$, and equal to 1 on $B(0, 1)$. We also assume that $\lambda \in C_0^\infty$ and $d\sigma_\Gamma$ is arc-length measure on the smooth curve $\Gamma$. Observe that this type of $m_{\epsilon}$ belongs to $\mathcal{M}_{\epsilon}$; it also belongs to $\mathcal{N}_{\epsilon}^\Gamma$ if $\lambda = 1$ and $\Gamma = \mathbb{S}^1$.

Proposition 3.3. (i) Let $m_{\epsilon} = d\sigma_\Gamma \ast \epsilon^{-1}\chi_{\epsilon}$. If the curvature of $\Gamma$ does not vanish, then

$$\|B_{m,\epsilon}\|_{L^1 \times L^1 \to L^{p'}} \sim \begin{cases} \epsilon & \text{if } 2 < p' < \infty, \\ \epsilon \sqrt{-\log \epsilon} & \text{if } p' = 2, \\ \epsilon^{1/2+1/p} & \text{if } 1 < p' < 2. \end{cases}$$

(ii) The restricted type estimate at the point $(1, 1, 2)$ holds; namely, for any three sets $F$, $G$, and $H$, $\|B_{m,\epsilon}(\chi_F, \chi_G, \chi_H)\| \lesssim \epsilon \|F\| \|G\| \|H\|^{1/2}$.

(iii) If $\Gamma$ is non-characteristic on $\text{Supp} \lambda$, the only change to (i) is the point $(1, 1, 2)$ which becomes $\|B_{m,\epsilon}\|_{L^1 \times L^1 \to L^2} \sim \epsilon$.

The curvature of the curve $\Gamma$ comes into play in the following result. (We refer the reader to [31, Section 3.1, Chap VIII] for details on this topic.) The proofs rely on properties of oscillatory integrals.
Lemma 3.4. Assume that $\Gamma$ is a smooth curve in $\mathbb{R}^2$ with non-vanishing Gaussian curvature. Then for all compactly supported smooth functions $\psi : \mathbb{R}^2 \to \mathbb{R}$,

$$\left| \int_{\mathbb{R}^2} e^{i(x_1 \xi + x_2 \eta)} \psi(\xi, \eta) d\sigma_\Gamma(\xi, \eta) \right| \lesssim (1 + |(x_1, x_2)|)^{-1/2},$$

where $d\sigma_\Gamma$ is the arclength measure on $\Gamma$.

We now turn to the proof of Proposition 3.3.

Proof. Another expression for $\|B_{m_\eps}\|_{L^1 \times L^1 \to L^p'}$. A straightforward argument left to the reader gives

$$(3.2) \quad \|B_{m_\eps}\|_{L^1 \times L^1 \to L^p'} = \sup_{y, z} \|\hat{m}_\eps(y - x, z - x)\|_{L^{p'}}.$$

Estimates on the convolution kernels. It is well known (see Lemma 3.4) that if the curvature of $\Gamma$ does not vanish, then $|\hat{\lambda d\sigma_\Gamma}(x)| \lesssim 1/\langle x \rangle^{1/2}$ for $x \in \mathbb{R}^2$. Examining further the stationary phase argument which gives the above, we find the following. Suppose that the normals to $\Gamma$ on $\text{Supp } \lambda$ span a subset $[a, \beta]$ of $S^1$. If $(x_1, x_2)/|(x_1, x_2)| \in [a + \epsilon, \beta - \epsilon]$ for a small $\epsilon$, then $|\hat{\lambda d\sigma_\Gamma}(x)| \gtrsim 1/\langle x \rangle^{1/2}$, whereas if $(x_1, x_2)/|(x_1, x_2)| \notin [a - \epsilon, \beta + \epsilon]$ for a small $\epsilon$, then $|\hat{\lambda d\sigma_\Gamma}(x)| \lesssim 1/\langle x \rangle^N$. Finally, recall that $\mathcal{F}[\hat{\lambda d\sigma_\Gamma}] * \epsilon^{-1} \chi_\eps = \mathcal{F}[\hat{\lambda d\sigma_\Gamma}] \epsilon \hat{\chi}(\cdot)$. The above bounds combined with (3.2) give the desired bounds, except for the restricted type estimates to which we now turn.

The restricted type estimate. For these estimates, we argue with the physical space version of $B_{m_\eps}$. Recall that $m_\eps = \chi_\eps * d\sigma_\Gamma$. Thus $\hat{m}_\eps = \epsilon \hat{\lambda d\sigma_\Gamma} \hat{\chi}_\eps$. Since $\chi$ is in the Schwartz class, so is $\hat{\chi}$. It is well known that $|\hat{\lambda d\sigma_\Gamma}(X)| \lesssim 1/\sqrt{\langle X \rangle}$ since $\Gamma$ has non-vanishing curvature. Thus $|\hat{m}_\eps(X)| \lesssim 1/\sqrt{\langle X \rangle}$. This implies

$$|\hat{m}_\eps(x - y, z - x)| \lesssim \frac{\epsilon}{\sqrt{\langle (x - y, z - x) \rangle}} \lesssim \frac{\epsilon}{\sqrt{\langle (x - y, z - y) \rangle}}.$$

Therefore,

$$|\langle B_{m_\eps}(\chi_E, \chi_F), \chi_G \rangle| \lesssim \epsilon \int \int \int \frac{1}{\sqrt{\langle (x - y, z - y) \rangle}} \chi_F(y) \chi_G(z) \chi_H(x) dx dy dz$$

$$\lesssim \epsilon |F| \sup_y \int \int \frac{1}{\sqrt{\langle (x - y, z - y) \rangle}} \chi_G(z) \chi_H(x) dx dz \text{ (3.3)}$$

By symmetry in $y$, the desired estimate is a consequence of the inequality

$$\int \int \frac{1}{\sqrt{\langle (x, z) \rangle}} \chi_G(z) \chi_H(x) dx dz \lesssim |G| |H|^{1/2}.$$
But this inequality follows from
\[
\int \int \frac{1}{\sqrt{(x, z)}} \chi_G(z) \chi_H(x) \, dx \, dz \leq \int \chi_G(z) \int \frac{1}{\sqrt{(x)}} \chi_H(x) \, dx \, dz
\]
\[
\leq |G| \int_0^{|H|} \frac{1}{\sqrt{(x)}} \, dx \lesssim |E||F|^{1/2}.
\]
(3.4) □

3.3 A specific case: \( \Gamma \) is a line. Consider a symbol of the type
\( m_\epsilon(\xi, \eta) = \chi((\xi - \lambda \eta)/\epsilon), \)
where \( \chi \) is a smooth function supported in \( \overline{B(0, 1)} \). Such symbols belong to \( M_\epsilon^\Gamma \) relative to the line \( \Gamma := \{ (\xi, \eta), \xi = \lambda \eta \} \). The degenerate lines are those corresponding to \( \lambda \in \{ 0, -1 \} \).

3.3.1 The non-characteristic case. In this case,
\[
B_{m_\epsilon}(f, g)(x) = \int \epsilon \hat{\chi}(\epsilon y)f(x + y)g(x - \lambda y)dy;
\]
thus
\[
\langle B_{m_\epsilon}(f, g), h \rangle = \int \epsilon \hat{\chi}(\epsilon y)f(x + y)g(x - \lambda y)h(x)dydx.
\]

Proposition 3.5. If \( \lambda \neq 0, -1 \), then
\[
|\langle T_{m_\epsilon}(f, g), h \rangle| \lesssim \epsilon^\rho \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.
\]
holds if and only if \( 1 \leq \rho + 1 = 1/p + 1/q + 1/r \leq 2 \).

Proof. We omit the proof, which follows from elementary considerations.

3.3.2 The characteristic case. Here we consider one of the degenerate lines, for instance, the case \( \lambda = 0 \). We then have
\[
T_{m_\epsilon}(f, g) = \left[ \chi \left( \frac{D}{\epsilon} \right) f \right] g,
\]
and so
\[
\langle T_{m_\epsilon}(f, g), h \rangle = \int \left[ \chi \left( \frac{D}{\epsilon} \right) f \right] gh.
\]

Proposition 3.6. If \( \lambda = 0 \) then
\[
|\langle T_{m_\epsilon}(f, g), h \rangle| \lesssim \epsilon^\rho \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}
\]
holds if and only if \( 1 \leq \rho + 1 = 1/p + 1/q + 1/r \leq 2 \) and \( 1/q + 1/r \leq 1 \).

We omit the proof.
4 The local-$L^2$ case with finite exponents

Let us first study the case in which all three exponents $p$, $q$, $r$ belong to $[2, \infty)$.

**Proposition 4.1.** Let $\Gamma$ be a compact smooth curve. Let $p, q, r \in [2, \infty)$ be exponents satisfying $2s := 1/p + 1/q + 1/r - 1 \geq 0$. Then there exists a constant $C = C(p, q, r)$ such that for $\epsilon > 0$ and symbols $m_\epsilon \in \mathcal{M}_\Gamma^r$,

$$\|T_{m_\epsilon}(f, g)\|_{L^r} \leq C\epsilon^{1/p + 1/q + 1/r - 1}\|f\|_{L^p}\|g\|_{L^q}.$$ 

Moreover, the decay in $\epsilon$ is optimum.

**Proof.** Proposition 3.2 implies that the decay in $\epsilon$ is optimum.

Observe that the domain $\Gamma_\epsilon$ can be covered by balls of radius $\epsilon$, the cover having the bounded overlap property. A partition of the unity subordinate to this covering allows us to write the symbol $m_\epsilon$ as $m_\epsilon = \sum_{i \in \Theta} m_i^\epsilon$, where for each index $i$, $m_i^\epsilon$ is a symbol satisfying the same regularity properties as $m_\epsilon$ and supported in a ball of radius $\epsilon$. Let us denote by $I_1^\epsilon$, $I_2^\epsilon$ and $I_3^\epsilon$ intervals of length comparable to $\epsilon$ such that

$$\{(\zeta, \eta, \zeta + \eta), m_i^\epsilon(\zeta, \eta) \neq 0\} \subset I_1^\epsilon \times I_2^\epsilon \times I_3^\epsilon.$$ 

For $J$ an interval, we denote by $\Delta_J$ the Fourier multiplier associated to the symbol $\phi_J$ (a smooth version of $1_J$ at the scale $|J|$ such that $1_J \leq \phi_J \leq 1_{2J}$). The kernel $\hat{m}_\epsilon$ satisfies $|\hat{m}_\epsilon(y, z)| \lesssim \epsilon^2/(1 + \epsilon|y, z|)^N$ for all non-negative real $N$ (since $m_i^\epsilon$ is supported on a ball of radius $\epsilon$). Hence, $T_{m_i}$ satisfies

$$|\langle T_{m_i}^\epsilon(f, g), h \rangle| \lesssim \int_{\mathbb{R}^3} \frac{\epsilon^2}{(1 + \epsilon|x - y, x - z|)^N} |\Delta_{I_1^\epsilon}f(y)||\Delta_{I_2^\epsilon}g(z)||\Delta_{I_3^\epsilon}h(x)|dxdydz.$$ 

It follows that

$$|\langle T_m(f, g), h \rangle| \lesssim \sum_i \int_{\mathbb{R}^3} \frac{\epsilon^2}{(1 + \epsilon|x - y, x - z|)^N} |\Delta_{I_1^\epsilon}f(y)||\Delta_{I_2^\epsilon}g(z)||\Delta_{I_3^\epsilon}h(x)|dxdydz.$$ 

It suffices to consider only the case that $(I_1^\epsilon)$ and $(I_2^\epsilon)$ form a bounded covering of the real line. Assume that $s > 0$ (we explain at the end of the proof the modifications needed for $s = 0$). Consider non-negative reals $s_p, s_q \in (0, 1)$ such that $2s = s_p + s_q, 1/p - s_p > 0$, and $1/q - s_q > 0$. Such $s_p$ and $s_q$ exist, since $1/r = 1/p + 1/q - 2s > 0$. Using the fact that $(1 + \epsilon|x - y, x - z|) \leq (1 + \epsilon|x - y|)(1 + \epsilon|x - z|)$,
we get for \( N_p = 1 - s_p \) and \( N_q = 1 - s_q \)

\[
|\langle T_m(f, g), h \rangle| \lesssim \sum_i \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{\epsilon}{(1 + \epsilon|x - y|)^{N_p}} |\Delta_{I_i^1} f(y)| dy \right) \times \left( \int_{\mathbb{R}} \frac{\epsilon}{(1 + \epsilon|x - z|)^{N_q}} |\Delta_{I_i^2} g(z)| dz \right) |\Delta_{I_i^3} h(x)| dx
\]

\[
\lesssim \epsilon^{2s} \sum_i \int_{\mathbb{R}} I_{s_p}(|\Delta_{I_i^1} f|)(x) I_{s_q}(|\Delta_{I_i^2} g|)(x) |\Delta_{I_i^3} h(x)| dx
\]

\[
\lesssim \epsilon^{2s} \left( \sum_i I_{s_p}(|\Delta_{I_i^1} f|)(x) \right)^{1/2} \left( \sum_i I_{s_q}(|\Delta_{I_i^2} g|)(x) \right)^{1/2} \sup_i |\Delta_{I_i^3} h(x)| dx,
\]

where \( I_{s_p} \) is the fractional integral operator of order \( s_p \); see Definition 2.1. Bounding the third factor above by the maximal function, we get

\[
|\langle T_m(f, g), h \rangle| \lesssim \epsilon^{2s} \int_{\mathbb{R}} \left( \sum_i I_{s_p}(|\Delta_{I_i^1} f|)(x) \right)^{1/2} \left( \sum_i I_{s_q}(|\Delta_{I_i^2} g|)(x) \right)^{1/2} \mathcal{M}(h)(x) dx.
\]

Then by Hölder’s inequality with the exponents \( p_s, q_s \) such that

\[
\frac{1}{r'} = \frac{1}{p} + \frac{1}{q} - 2s = \left( \frac{1}{p} - s_p \right) + \left( \frac{1}{q} - s_q \right) = \frac{1}{p_s} + \frac{1}{q_s},
\]

it follows (by boundedness of \( \mathcal{M} \) over Lebesgue spaces) that

\[
|\langle T_m(f, g), h \rangle| \lesssim \epsilon^{2s} \left\| \left( \sum_i I_{s_p}(|\Delta_{I_i^1} f|)^2 \right)^{1/2} \right\|_{L^{p_s}} \left\| \left( \sum_i I_{s_q}(|\Delta_{I_i^2} g|)^2 \right)^{1/2} \right\|_{L^{q_s}} \|h\|_{L^{r'}}.
\]

Because of Proposition 2.2, we know that the fractional integral operator \( I_{s_p} : L^p \to L^{p_s} \) (respectively, \( I_{s_q} : L^q \to L^{q_s} \)) is bounded. It thus admits an \( l^2 \)-valued extension; see Theorem [12, 4.5.1] and the original work of Marcinkiewicz and Zygmund [29]. Consequently,

\[
|\langle T_m(f, g), h \rangle| \lesssim \epsilon^{2s} \left\| \left( \sum_i |\Delta_{I_i^1} f|^2 \right)^{1/2} \right\|_{L^p} \left\| \left( \sum_i |\Delta_{I_i^2} g|^2 \right)^{1/2} \right\|_{L^q} \|h\|_{L^{r'}}.
\]

Recall that \( (I_{i}^1) \) and \( (I_{i}^2) \) form a bounded covering. We can thus apply Rubio de Francia’s result (see Proposition 2.3) for \( p, q \geq 2 \) to obtain

\[
|\langle T_m(f, g), h \rangle| \lesssim \epsilon^{2s} \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^{r'}}.
\]

For the limit case \( s = 0 \), argue similarly, replacing the fractional integration operators \( I_{s_p} \) and \( I_{s_q} \) with the Hardy-Littlewood maximal operator \( \mathcal{M} \). We leave it to the reader to check that with this minor modification, everything still works, the key argument coming from the Fefferman-Stein maximal inequality. \( \square \)
Remark 3.1 implies the following result.

**Proposition 4.2.** Let $\Gamma$ be a smooth compact curve and $m_\epsilon \in \mathcal{M}_\epsilon^\Gamma$. For exponents $p, q, r \in [1, \infty)$, there exists a constant $C = C(p, q, r)$ such that

$$\| T_{m_\epsilon}(f, g) \|_{L^r'} \leq C \epsilon^{1/\max\{p, 2\}+1/\max\{q, 2\}+1/\max\{r, 2\}-1} \| f \|_{L^p} \| g \|_{L^q}$$

if $2s := 1/\max\{p, 2\} + 1/\max\{q, 2\} + 1/\max\{r, 2\} - 1 \geq 0$.

Finally, for a curve $\Gamma$ that is non-characteristic, we can improve the decay in $\epsilon$ in the non-local-$L^2$ case.

**Proposition 4.3.** Let $\Gamma$ be a smooth compact curve, which is non-characteristic (see Definition 1.1). For exponents $p, q, r \in (1, \infty)$ satisfying (1.3) and $\min\{p, q, r\} < 2$, there exists a constant $C = C(p, q, r)$ such that

$$\| T_{m_\epsilon}(f, g) \|_{L^r'} \leq C \epsilon^\rho \| f \|_{L^p} \| g \|_{L^q},$$

where

$$\rho := \min \left\{ \frac{1}{\max\{p, 2\}} + \frac{1}{\max\{q, 2\}} + \frac{1}{\max\{r, 2\}} - 1 + \left( \max \left\{ \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right\} - \frac{1}{2} \right), 1 \right\}.$$ 

Since $\min\{p, q, r\} < 2$, $(\max\{1/p, 1/q, 1/r\} - 1/2)$ is non-negative. Thus the new exponent $\rho$ is larger than the one in Proposition 4.2.

**Proof.** First assume that only one of the three exponents $p, q, r$ is less than 2. Since $p, q, r$ play a symmetrical role, assume without loss of generality that $p = \min\{p, q, r\} \in (1, 2)$. The proof is then exactly the same as that of Proposition 4.1. \hfill \Box

### 5 Study of particular points

#### 5.1 The point $(1, 1, 2)$.

**Proposition 5.1.** Let $m_\epsilon \in \mathcal{M}_\epsilon^\Gamma$.

(i) If $\Gamma$ is non-characteristic, then $\| B_{m_\epsilon}(f, g) \|_{L^2} \lesssim \epsilon \| f \|_{L^1} \| g \|_{L^1}$.

(ii) If $\Gamma$ has non-vanishing curvature, then

$$\| B_{m_\epsilon}(f, g) \|_{L^2} \lesssim \epsilon \sqrt{-\log \epsilon} \| f \|_{L^1} \| g \|_{L^1},$$

and the following restricted type inequality holds: for any three sets $F$, $G$, and $H$,

$$| \langle B_{m_\epsilon}(\chi_F, \chi_G), \chi_H \rangle | \lesssim \epsilon | F | | G | | H |^{1/2}. $$
(iii) If $\Gamma$ is arbitrary, then $\|B_{m_*}(f, g)\|_{L^2} \lesssim \sqrt{\epsilon}\|f\|_{L^1}\|g\|_{L^1}$.

Furthermore, the above estimates are optimal in that the powers of $\epsilon$ cannot be improved.

**Proof.** The $TT^*$ argument. As the first step in the proof, we have

$$\|B_{m_*}(f, g)\|_{L^2}^2 = |\mathcal{F}B_{m_*}(f, g)|_{L^2}^2$$

(5.1) \[ = \int \int \int m_*(\xi - \eta, \eta)\hat{f}(\xi - \eta)\hat{g}(\eta)\tilde{m}_*(\xi - \zeta, \zeta)\tilde{f}(\xi - \zeta)\tilde{g}(\zeta)d\eta d\zeta d\xi \]

\[ = \int \int \int f(x^1)g(y^1)\tilde{g}(y^2)K_\epsilon(x^1 - y^1, x^2 - y^2, -x^1 - x^2)dx^1 dx^2 dy^1 dy^2, \]

where $K_\epsilon(a, b, c) = \int \int \int m_*(\xi - \eta, \eta)\tilde{m}_*(\xi - \zeta, \zeta)e^{iax + b\xi + c\eta}d\eta d\zeta d\xi$. In other words, $K_\epsilon = \mathcal{F}^{-1}(m_*(\xi - \eta, \eta)\tilde{m}_*(\xi - \zeta, \zeta))$ if one views $m_*(\xi - \eta, \eta)m_*(\xi - \zeta, \zeta)$ as a function of $\eta, \zeta, \xi$.

Upper bounds for $B_{m_*} : L^1 \times L^1 \to L^2$. It is clear from (5.1) that

$$\|B_{m_*}(f, g)\|_{L^2}^2 \leq \|K_\epsilon\|_{L^\infty(\mathbb{R}^3)}\|f\|_{L^1}\|g\|_{L^1}^2.$$ 

Hence

$$\|B_{m_*}\|_{L^1 \times L^1 \to L^2}^2 \leq \|K_\epsilon\|_{L^\infty(\mathbb{R}^3)} \leq \|m_*(\xi - \eta, \eta)m_*(\xi - \zeta, \zeta)\|_{L^1(\mathbb{R}^3)},$$

\[ = \int \left( \int |m_*(\xi - \eta, \eta)| d\eta \right)^2 d\xi. \]

We now consider three cases.

- If $\Gamma$ is non-characteristic, then $\int m_*(\xi - \eta, \eta) d\eta \lesssim \epsilon$ for any $\xi$; thus

\[ \int \left( \int |m_*(\xi - \eta, \eta)| d\eta \right)^2 d\xi \lesssim \epsilon^2 \]

and $\|B_{m_*}\|_{L^1 \times L^1 \to L^2} \lesssim \epsilon$.

- If $\Gamma$ is arbitrary, then the Cauchy-Schwarz inequality gives

\[ \int \left( \int |m_*(\xi - \eta, \eta)| d\eta \right)^2 d\xi \lesssim \int \int |m_*(\xi - \eta, \eta)|^2 d\eta d\xi \lesssim \epsilon, \]

which implies $\|B_{m_*}\|_{L^1 \times L^1 \to L^2} \lesssim \sqrt{\epsilon}$.

- If $\Gamma$ has non-vanishing curvature, the estimate is slightly more involved. It is easily seen that one can restrict attention to regions where $\Gamma$ is parametrized as $\xi = \Phi(\eta)$. Then $m_*(\xi - \eta, \eta)$ is localized $\epsilon$ away from $\xi = \eta + \Phi(\eta)$. Difficulties appear where $\Phi'(\eta) = -1$. Let us assume, without loss of generality, that $\Phi'(0) = -1$. A short computation shows that for large enough
\[ C_0 \text{ and small enough } \delta, \int m_\epsilon(\xi - \eta, \eta) \, d\eta \lesssim \sqrt{\epsilon} \text{ for } |\xi| \leq C_0 \epsilon, \text{ whereas for } \delta \geq \xi \geq C_0 \epsilon, \int m_\epsilon(\xi - \eta, \eta) \, d\eta \lesssim \epsilon / \sqrt{|\xi|}. \]

Thus

\[ \int_{|\xi| \leq \delta} \left( \int m_\epsilon(\xi - \eta, \eta) \, d\eta \right)^2 \, d\xi = \int_{|\xi| \leq C_0 \epsilon} \epsilon \, d\xi + \int_{\delta \geq \xi \geq C_0 \epsilon} \epsilon \, d\xi \lesssim \epsilon |\log(\epsilon)|. \]

This implies \( \|B_{m_\epsilon}\|_{L^1 \times L^1 \to L^2} \lesssim \sqrt{\epsilon} \sqrt{-\log \epsilon}. \)

Optimality. Optimality follows from Propositions 3.3 and 3.2. \( \Box \)

5.2 The point \((2, 2, 1)\).

**Proposition 5.2.** Let \( m_\epsilon \in M_\Gamma^\epsilon \).

(i) If \( \Gamma \) is non-characteristic, then \( \|B_{m_\epsilon}(f, g)\|_{L^\infty} \lesssim \epsilon \|f\|_{L^2} \|g\|_{L^2} \).

(ii) If \( \Gamma \) has non-vanishing curvature, then \( \|B_{m_\epsilon}(f, g)\|_{L^\infty} \lesssim \epsilon^{3/4} \|f\|_{L^2} \|g\|_{L^2} \).

(iii) If \( \Gamma \) is arbitrary, then \( \|B_{m_\epsilon}(f, g)\|_{L^\infty} \lesssim \sqrt{\epsilon} \|f\|_{L^2} \|g\|_{L^2} \).

Furthermore, the above exponents of \( \epsilon \) are optimal.

**Proof.** Proof of (i): \( \Gamma \) non-characteristic. Let \( \Gamma_\epsilon \) be an \( \epsilon \)-neighborhood of \( \Gamma \). We split \( \Gamma_\epsilon \) into parts by considering its intersection with the strips \( \{(\xi, \eta), n \epsilon < \eta \leq (n + 1)\epsilon\} \), where \( n \in \mathbb{Z} \) - of course, only finitely many of these intersections are non-empty. Since \( \Gamma \) is non-characteristic, it is possible to write

\[ \Gamma_\epsilon \cap \{(\xi, \eta), n \epsilon < \eta \leq (n + 1)\epsilon\} \subset \{(\xi, \eta), n \epsilon < \eta \leq (n + 1)\epsilon \text{ and } x_\epsilon^n - C_0 \epsilon < \xi \leq x_\epsilon^n + C_0 \epsilon\}, \]

where \( (x_\epsilon^n) \) is a family of real numbers and \( C_0 \) is a constant independent of \( \epsilon \); the above decomposition is almost-orthogonal in \( \xi \), as obviously it is in \( \eta \). Namely, there exists a constant \( M \), also independent of \( \epsilon \), such that at most \( M \) intervals \( [x_\epsilon^n - C_0 \epsilon, x_\epsilon^n + C_0 \epsilon] \) have non-empty intersection. Then, by the Cauchy-Schwarz
inequality and the almost-orthogonality property, we obtain
\[
|B_{m_{\epsilon}}(f, g)(x)| = \left| \int \int e^{ix(\xi + \eta)} m_{\epsilon}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\eta d\xi \right|
\]
\[
\lesssim \int \int_{\Gamma_{\epsilon}} |\hat{f}(\xi)||\hat{g}(\eta)| d\eta d\xi
\]
\[
\lesssim \sum_{n} \int_{x_n^\epsilon-CN_{\epsilon}} \int_{n(\eta+1)\epsilon} |\hat{f}(\xi)||\hat{g}(\eta)| d\eta d\xi
\]
\[
= \sum_{n} \int_{x_n^\epsilon-CN_{\epsilon}} |\hat{f}(\xi)| d\xi \int_{n(\eta+1)\epsilon} |\hat{g}(\eta)| d\eta
\]
\[
\lesssim \sum_{n} \sqrt{\epsilon} ||\hat{f}||_{L^2([\xi_n^\epsilon-CN_{\epsilon}, \epsilon])} \sqrt{\epsilon} ||\hat{g}||_{L^2([n\epsilon, (n+1)\epsilon])}
\]
\[
\lesssim \epsilon \left[ \sum_{n} ||\hat{f}||_{L^2([\xi_n^\epsilon-CN_{\epsilon}, \epsilon])} \right]^{1/2} \left[ \sum_{n} ||\hat{g}||_{L^2([n\epsilon, (n+1)\epsilon])} \right]^{1/2}
\]
\[
\lesssim \epsilon ||f||_{L^2} ||g||_{L^2}.
\]

The norm \(\epsilon\) for this bilinear operator is, of course, optimal by Proposition 3.2(i).

Proof of (ii): \(\Gamma\) has non-vanishing curvature. The proof of (i) is valid except where \(\Gamma\), in \((\xi, \eta)\) coordinates, has a tangent parallel to the \(\xi\) or \(\eta\) axes. By symmetry, it suffices to focus on the former possibility and assume that \(\Gamma\) can be parametrized by \(\eta = \phi(\xi)\). The problem is then treating regions where \(\phi'\) vanishes. Without loss of generality, let us assume that \(\phi'\) remains small, say, less than 1/10; this means that \(\Gamma_{\epsilon} \subset \{|\phi(\xi) - \eta| < 3\epsilon\}\). Proceeding as above, we split \(\Gamma_{\epsilon}\) into parts and consider its intersection with strips \([(\xi, \eta), n\epsilon < \eta \leq (n+1)\epsilon]\), where \(n \in \mathbb{Z}\). These intersections can be covered as follows:

\[
\Gamma_{\epsilon} \cap \{n\epsilon < \eta \leq (n+1)\epsilon\}
\]

\[
\subset \{(\xi, \eta), n\epsilon < \eta < (n+1)\epsilon\text{ and } (n-1)\epsilon < \phi(\xi) < (n+2)\epsilon\}
\]

\[
:= (x_n^\epsilon, y_n^\epsilon) \times (n\epsilon, (n+1)\epsilon).
\]

The almost-orthogonality property for the intervals \((x_n^\epsilon, y_n^\epsilon)\) is obvious from their definition. Furthermore, since the curvature of \(\Gamma\) does not vanish, their size is bounded by \(y_n^\epsilon - x_n^\epsilon \lesssim \sqrt{\epsilon}\). It is then easy to follow the proof of (i) and get the desired estimate; optimality follows from Proposition 3.2(iii).

Proof of (iii): \(\Gamma\) arbitrary. By duality, it suffices to prove
\[
||B_{m_{\epsilon}}(f, g)||_{L^2} \lesssim \sqrt{\epsilon} ||f||_{L^2} ||g||_{L^2}.
\]

But this is a simple consequence of the Cauchy-Schwarz, Hausdorff-Young, and
Plancherel inequalities:
\[ \| B_m(f, g) \|_{L^2} \leq \| m(\xi - \eta, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \|_{L^2(\xi)} \]
\[ \leq \| f \|_{L^\infty} \left[ \int m(\xi - \eta, \eta^2) d\eta \right]^{1/2} \| g \|_{L^2} \]
\[ \lesssim \| f \|_{L^1} \| g \|_{L^2} \left[ \int \int m(\xi - \eta, \eta^2) d\eta d\xi \right]^{1/2} \lesssim \sqrt{\epsilon} \| f \|_{L^1} \| g \|_{L^2}. \]

This bound is optimal by Proposition 3.2(iv).

5.3 The point \((\infty, 1, 2)\).

**Proposition 5.3.** Let \( m \in \mathcal{M}_e^\Gamma \).

(i) If \( \Gamma \) is non-characteristic, then \( \| B_m(f, g) \|_{L^2} \lesssim \epsilon^{1/4} \| f \|_{L^\infty} \| g \|_{L^1}. \)

(ii) If \( \Gamma \) has non-vanishing curvature, then
\[ \| B_m(f, g) \|_{L^2} \lesssim \epsilon^{1/4} \sqrt{-\log(\epsilon)} \| f \|_{L^\infty} \| g \|_{L^1}. \]

(iii) If \( \Gamma \) is arbitrary, then \( \| B_m(f, g) \|_{L^2} \lesssim \| f \|_{L^\infty} \| g \|_{L^1}. \)

Furthermore, the bounds in (ii) and (iii) are optimal up to the logarithmic factor.

**Proof.** The TT\(^*\) argument. It follows from (5.1) that
\[ \| B_m(f, g) \|_{L^2}^2 \leq \| f \|_{L^\infty}^2 \int \int |g(y^1)| |g(y^2)| \left| K_\epsilon(x^1 - y^1, y^2 - x^2, x^2 - x^1) \right| dx^1 dx^2 dy^1 dy^2, \]
\[ \lesssim \left( \sup_{y^1, y^2} \int \int |K_\epsilon(x^1 - y^1, y^2 - x^2, x^2 - x^1) dx^1 dx^2 \right) \| f \|_{L^\infty}^2 \| g \|_{L^1}^2. \]

Everything now boils down to estimating \( \int \int |K_\epsilon(x^1 - y^1, y^2 - x^2, x^2 - x^1)| dx^1 dx^2. \) A change of variables gives
\[ K_\epsilon(x^1 - y^1, y^2 - x^2, x^2 - x^1) = \int \int F_\gamma(a, \beta) e^{i\alpha(y^1 - x^1)} e^{i\beta(x^2 - y^2)} d\alpha d\beta = \hat{F}_\gamma(x^1 - y^1, y^2 - x^2), \]
where
\[ F_\gamma(a, \beta) = \int m_\epsilon(\alpha, \xi - \alpha) m_\epsilon(\beta, \xi - \beta) e^{-iy \xi} d\xi \quad \text{and} \quad y = y^1 - y^2. \]

Combining the last few lines, we obtain
\[ \| B_m(f, g) \|_{L^2} \lesssim \left( \sup_y \| \hat{F}_\gamma \|_{L^1} \right) \| f \|_{L^\infty}^2 \| g \|_{L^1}^2. \]
Proof of (i): $\Gamma$ non-characteristic. If $\Gamma$ is non-characteristic, the support of $(\alpha, \xi) \mapsto m_\epsilon(\alpha, \xi - \alpha)$ is contained in the set $\{|\alpha - \phi(\xi)| \leq C_0 \epsilon\}$ for a certain invertible function $\phi$ and constant $C_0$. Given the definition (5.2) of $F_\gamma$, this implies immediately that $|\text{Supp } F_\gamma| \sim \epsilon$ and $|F_\gamma|_{L^\infty(\mathbb{R}^2)} \lesssim \epsilon$. Furthermore, since taking derivatives of the symbol $m_\epsilon$ essentially amounts to multiplying it by $1/\epsilon$, we also obtain $\|\nabla^2_{\alpha, \beta} F_\gamma\|_{L^\infty(\mathbb{R}^2)} \lesssim 1/\epsilon$.

Combining these two estimates with Plancherel’s identity gives

$$
\|F_\gamma\|_{L^1(\mathbb{R}^2)} \lesssim \|F_\gamma\|^{1/2}_{L^2(\mathbb{R}^2)} \cdot \|\nabla F_\gamma\|^{1/2}_{L^2(\mathbb{R}^2)} = \|F_\gamma\|^{1/2}_{L^2(\mathbb{R}^2)} \|\nabla^2_{\alpha, \beta} F_\gamma\|^{1/2}_{L^2(\mathbb{R}^2)} \lesssim |\text{Supp } F_\gamma|^{1/2}_{L^\infty(\mathbb{R}^2)} |\text{Supp } F_\gamma|^{1/2}_{L^\infty(\mathbb{R}^2)} \|\nabla^2_{\alpha, \beta} F_\gamma\|_{L^\infty(\mathbb{R}^2)}^{1/2} \lesssim \sqrt{\epsilon}.
$$

By (5.3), this gives the desired bound.

Proof of (ii): $\Gamma$ has non-vanishing curvature. Consider the curve $\Gamma$ in the coordinates $(\alpha, \xi - \alpha)$. In regions where it can be parametrized by $\alpha = \phi(\xi)$ with $\phi$ smooth and with a smooth inverse, the result follows from (i). Difficulties appear when the parametrization becomes $\alpha = \phi(\xi)$ with $\phi'$ vanishing, or $\xi = \psi(\alpha)$ with $\psi'$ vanishing. From now on, we focus on these two cases. In both cases, we assume, for simplicity, that $\phi'(0) = \psi'(0) = 0$. Since the curvature does not vanish, $\phi''(0)$ and $\psi''(0)$ are both non-zero. Focusing on a small neighborhood $[-\delta, \delta]$ of 0 in both cases, we assume, for simplicity, that $\phi(\xi) = \xi^2$ and $\psi(\alpha) = \alpha^2$; this assumption simplifies notation while retaining all the essential features.

Let us start with the case $\Gamma$ is parametrized by $\alpha = \xi^2$ and restrict $\alpha$ to $[-\delta, \delta]$. The support of $(\alpha, \xi) \mapsto m_\epsilon(\alpha, \xi - \alpha)$ is then contained in $[-\delta, \delta] \cap \{|\alpha - \xi^2| \leq 2\epsilon\}$. For fixed $\alpha$, the set $\{\xi, |\xi^2 - \alpha| < 2\epsilon\}$ has size

$$
|\{\xi, |\xi^2 - \alpha| < 2\epsilon\}| \lesssim \begin{cases} 
\sqrt{\epsilon} & \text{if } |\alpha| < 10\epsilon, \\
\epsilon/\sqrt{|\alpha|} & \text{if } |\alpha| > 10\epsilon.
\end{cases}
$$

This implies

$$
|F_\gamma(\alpha, \beta)| \lesssim \begin{cases} 
\sqrt{\epsilon} & \text{if } |\alpha| < 10\epsilon, \\
\epsilon/\sqrt{|\alpha|} & \text{if } |\alpha| > 10\epsilon.
\end{cases}
$$

Since furthermore $|\alpha - \beta| < 3\epsilon$ on the support of $F_\gamma$, we obtain

$$
\|F_\gamma\|_2^2 = \int \int |F_\gamma(\alpha, \beta)|^2 \, d\alpha d\beta \lesssim \int \epsilon \sup \|F_\gamma(\alpha, \beta)\|^2 \, d\alpha \\
\lesssim \int_{|\alpha| < 10\epsilon} \epsilon^2 \, d\alpha + \int_{10\epsilon < |\alpha| < \delta} \frac{\epsilon}{\sqrt{|\alpha|}} \, d\alpha \lesssim -\epsilon^3 \log(\epsilon).
$$

Let us now consider the case $\Gamma$ is parametrized by $\xi = \alpha^2$. The support of $(\alpha, \xi) \mapsto m_\epsilon(\alpha, \xi - \alpha)$ is then contained in the set $\{\alpha \in [-\delta, \delta] : |\xi - \alpha^2| \leq 2\epsilon\}$. 
An examination of the definition of $F_Y$ reveals that $\|F_Y\|_{L^\infty(\mathbb{R}^2)} \lesssim \epsilon$, and that for fixed $\alpha$, the set $\text{Supp} F_Y(\alpha, \cdot)$ has size

$$|\text{Supp} F_Y(\alpha, \cdot)| \lesssim \begin{cases} \sqrt{\epsilon} & \text{if } |\alpha| < 10\sqrt{\epsilon}, \\ \epsilon/|\alpha| & \text{if } |\alpha| > 10\epsilon. \end{cases}$$

This implies immediately that $|\text{Supp} F_Y| \lesssim -\epsilon \log(\epsilon)$, which gives in turn (recall that $\|F_Y\|_{L^\infty(\mathbb{R}^2)} \lesssim \epsilon$)

$$\|F_Y\|_{L^2(\mathbb{R}^2)} \approx |\text{Supp} F_Y| \|F_Y\|_{L^\infty(\mathbb{R}^2)} \approx -\epsilon^3 \log(\epsilon).$$

Thus we can prove in both cases that $\|F_Y\|_{L^2(\mathbb{R}^2)} \approx -\epsilon^3 \log(\epsilon)$.

Proof of (iii): $\hat{W}$ arbitrary. The $L^2$ norm of $F_Y$ can be estimated using the Cauchy-Schwarz inequality, the result being

$$\|F_Y\|_{L^2(\mathbb{R}^2)}^2 = \int \int ( \int m_\epsilon(\alpha, \xi) m_\epsilon(\beta, \xi - \beta) e^{-iy\xi} d\xi )^2 d\alpha d\beta \leq \left( \int \int |m_\epsilon(\alpha, \xi - \alpha)|^2 d\alpha d\xi \right) \left( \int \int |m_\epsilon(\beta, \xi)|^2 d\beta d\xi \right) \leq \epsilon^2.$$

Recall that taking derivatives of the symbol $m_\epsilon$ essentially amounts to multiplying the symbol by $1/\epsilon$. Therefore, proceeding as above, we obtain

$$\left\| \nabla_{a,\beta}^2 F_Y \right\|_{L^2(\mathbb{R}^2)}^2 \lesssim \epsilon^2 \frac{1}{\epsilon^4} = \frac{1}{\epsilon^2}.$$

Putting these two estimates together and using Plancherel’s identity gives

$$\|\hat{F}_Y\|_{L^1(\mathbb{R}^2)} \lesssim \|\hat{F}_Y\|_{L^1(\mathbb{R}^2)}^{1/2} \cdot \|\nabla_{a,\beta} F_Y\|_{L^2(\mathbb{R}^2)}^{1/2} = \|F_Y\|_{L^2(\mathbb{R}^2)}^{1/2} \|\nabla_{a,\beta} F_Y\|_{L^2(\mathbb{R}^2)}^{1/2} \lesssim 1.$$

By (5.3), this gives the bound. It is optimal by Proposition 3.2(iv). \qed

5.4 The point $(1, 1, \infty)$.

Proposition 5.4. Let $m_\epsilon \in \mathcal{M}_\Gamma^\Gamma$.

(i) If $\Gamma$ is non-characteristic, then $\|B_{m_\epsilon}(f, g)\|_{L^1} \lesssim \sqrt{\epsilon} \|f\|_{L^1} \|g\|_{L^1}$.

(ii) If $\Gamma$ has non-vanishing curvature, then $\|B_{m_\epsilon}(f, g)\|_{L^1} \lesssim \sqrt{\epsilon} \log(1/\epsilon) \|f\|_{L^1} \|g\|_{L^1}$.

(iii) If $\Gamma$ is arbitrary, then $\|B_{m_\epsilon}(f, g)\|_{L^1} \lesssim \|f\|_{L^1} \|g\|_{L^1}$.

Furthermore, the $\epsilon$-dependence of these bounds is optimal up to the logarithmic factor.
Proof. Proof of (i): $\Gamma$ non-characteristic. Equation (2.1) implies
\[
\left\| B_{m_\epsilon}(f, g) \right\|_{L^1} \leq \left\| f \right\|_{L^1} \left\| g \right\|_{L^1} \left( \sup_{y,z} \left\| \widehat{m_\epsilon}(y - \cdot, z - \cdot) \right\|_{L^1} \right).
\]

Thus it suffices to estimate $\sup_{y,z} \left\| \widehat{m_\epsilon}(y - \cdot, z - \cdot) \right\|_{L^1}$. In order to do so, we write $\widehat{m_\epsilon}$ as $\widehat{m_\epsilon}(y - x, z - x) = \int e^{i\xi(x-y)} \int e^{i\eta(x-z)} m_\epsilon(\xi, \eta) d\eta d\xi := \int e^{i\alpha y} F_{y,z}(\alpha) d\alpha$, where $F_{y,z}(\alpha) := \int e^{-i(\xi y + \eta z)} m_\epsilon(\xi, \eta) d\eta d\xi$. Hence $\sup_{y,z} \left\| \widehat{m_\epsilon}(y - \cdot, z - \cdot) \right\|_{L^1} = \sup_{y,z} \left\| F_{y,z} \right\|_{L^1}$. In order to estimate $\widehat{F_{y,z}}$ in $L^1$, we interpolate it between $L^2$ and $L^2(\chi^2 dx)$. Since $\Gamma$ is non-characteristic and bounded,
\[
\left\| \widehat{F_{y,z}} \right\|_{L^2}^2 = \left\| F_{y,z} \right\|_{L^2}^2 \lesssim \left\| F_{y,z} \right\|_{L^\infty}^2 \lesssim \epsilon^2.
\]

Similarly, using the fact that $\nabla m_\epsilon$ has size at most $1/\epsilon$, we obtain
\[
\left\| x \widehat{F_{y,z}}(x) \right\|_{L^2}^2 = \left\| \partial_x F_{y,z} \right\|_{L^2}^2 \lesssim \epsilon^2 \frac{1}{\epsilon^2}.
\]

This gives the estimate, since
\[
\sup_{y,z} \left\| \widehat{m_\epsilon}(y - \cdot, z - \cdot) \right\|_{L^1} \lesssim \sup_{y,z} \left\| F_{y,z} \right\|_{L^1} \lesssim \left\| F_{y,z} \right\|_{L^2}^{1/2} \left\| x \widehat{F_{y,z}}(x) \right\|_{L^2}^{1/2} \lesssim \sqrt{\epsilon}.
\]

Optimality is a consequence of Proposition 3.3.

Proof of (ii): $\Gamma$ has non-vanishing curvature. We proceed as above. Things boil down to estimating $\left\| \widehat{F_t} \right\|_{L^1}$ for every $t = (y, z)$. As above, we obtain this estimate by interpolating $L^1$ between $L^2$ and $L^2(\chi^2 dx)$.

Treating the parts of $\Gamma$ which are non-characteristic can be done using the previous case. We now focus on a part of $\Gamma$ which is characteristic, namely, where its tangent is parallel to the $(\xi - \eta)$ axis. For the sake of simplicity, we consider only the model case in which $\Gamma$ is given (say, in the ball of radius 1) around $(0,0)$ by the equation $(\xi + \eta)^2 = (\xi - \eta)^2$. We denote by $\Gamma_\epsilon$ the set of points which are within $\epsilon$ of $\Gamma$ and by $\partial_{\alpha}$ the line given by the equation $\xi + \eta = \alpha$.

The formula giving $\mathcal{F}_{y,z}$ implies immediately that
\[
|F_{y,z}(\alpha)| \leq \left| \partial_{\alpha} \cup \Gamma_\epsilon \right| \lesssim \left\{ \begin{array}{ll} |\alpha| & \text{if } |\alpha| \leq 100\epsilon, \\
\epsilon/|\alpha| & \text{otherwise.} \end{array} \right.
\]

Thus, by Plancherel’s inequality,
\[
\left\| \widehat{F_{y,z}} \right\|_{L^2}^2 = \left\| F_{y,z}(\alpha) \right\|_{L^2}^2 \leq \int_{|\alpha| \leq 1} \left| \partial_{\alpha} \cup \Gamma_\epsilon \right|^2 d\alpha \lesssim \epsilon^2 \log \epsilon.
\]

One finds, as above, that $\left\| x \widehat{F_t} \right\|_{L^2}^2 = \left\| \partial_x F_t \right\|_{L^2}^2 \lesssim |\log \epsilon|$, and the result follows by interpolation. It is optimal up to the logarithmic factor by Proposition 3.3.
Proof of (iii): arbitrary $\Gamma$. Continuing to follow the above pattern, we obtain, by Cauchy-Schwarz,

$$\|\hat{F}_t\|_{L^2}^2 \lesssim \|F_t\|_{L^2}^2 \leq \int \int |m_\epsilon(z - \eta, \eta)|d\eta \, d\xi \lesssim \int \int |m_\epsilon(z, \eta)|^2d\eta d\xi \lesssim \epsilon.$$ 

Similarly, $\|\hat{F}_t\|_{L^1} \lesssim \|\hat{F}_t\|_{L^2}^{1/2} |\hat{F}_t(\xi)|^{1/2} \lesssim 1$, which is the desired result. Optimality follows from Proposition 3.2 (iv).

5.5 The point $(1, 1, 1)$.

Proposition 5.5. For an arbitrary $\Gamma$, $\|B_{m_\epsilon}(f, g)\|_{L^\infty} \lesssim \epsilon \|f\|_{L^1} \|g\|_{L^1}$ and the $\epsilon$-dependence of the bound is optimal.

Proof. The optimality claim follows from Proposition 3.2(i). To prove that the bound holds, recall that by (2.1),

$$< B_{m_\epsilon}(f, g), h > = \int \int \int \tilde{m_\epsilon}(y - x, z - x)f(y)g(z)h(x)dxdydz.$$

Therefore, $| < B_{m_\epsilon}(f, g), h > | \lesssim \|\tilde{m_\epsilon}\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^1} \|g\|_{L^1} \|h\|_{L^1}$ and $\|\tilde{m_\epsilon}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|m_\epsilon\|_{L^1(\mathbb{R}^2)} \lesssim \epsilon$. □

6 Close to Hölder points, in the non-vanishing curvature case

In this section, we examine the case of Lebesgue exponents $(p, q, r)$, with $1/p + 1/q + 1/r$ close to 1 when $\Gamma$ has non-vanishing curvature. The case that all three exponents are larger than 2 is taken care of by Proposition 4.1, and the assumption $m_\epsilon \in \mathcal{M}_\epsilon$ suffices. To deal with the case that one exponent is less than 2, it seems that more regularity is needed from $m_\epsilon$, namely that it belong to $\mathcal{N}_\epsilon$. We distinguish two cases: $(p, q, r) = (2, \infty, 2)$ and $(p, q, r)$ close to $(\infty, \infty, 1)$. Interpolation then gives all Lebesgue exponents such that $1/p + 1/q + 1/r > 1$, with an arbitrarily small deviation from the optimal bound $\epsilon^{1/p + 1/q + 1/r - 1}$.

6.1 The point $(2, 2, \infty)$.

Proposition 6.1. Assume that $m_\epsilon \in \mathcal{N}_\epsilon$ and that $\Gamma$ has non-vanishing curvature. Then $\|B_{m_\epsilon}(f, g)\|_{L^1} \lesssim \|f\|_{L^2} \|g\|_{L^2}$. 


Proof. Step 1: decomposition of $m_\epsilon$. The proof of the proposition encounters new difficulties when the tangent of $\Gamma$ is parallel to one of the coordinate axes; in other cases, it is possible to rely on Proposition 4.3.

For the sake of simplicity of notation, we treat only a model case; namely, we assume that $\Gamma$ is the circle with radius 1 and center $(\xi = 0, \eta = 1)$, i.e., $\Gamma$ is given by the equation $\xi^2 + (\eta - 1)^2 = 1$. We focus on the point $(\xi = 0, \eta = 0)$ of the circle, where the tangent is parallel to the $\xi$ axis. Thus we assume that $m_\epsilon = 0$ if $|\xi, \eta| \geq 1/20$. Recall that the support of $m_\epsilon$ is contained in a strip of width $2\epsilon$ around $\Gamma$.

Next we split $m_\epsilon$ smoothly into a sum of symbols, each of which is supported on a chord of length $1/\sqrt{\epsilon}$. For the moment, switch to polar coordinates centered at $(\xi = 0, \eta = 1)$ and denote by $\theta$ the angular coordinate, with the convention that $\theta = 0$ corresponds to the $\eta$ axis below $(0, 1)$: $\{\xi = 0, \eta \leq 1\}$. Finally set $m^k_\xi(\xi, \eta) := m_\epsilon(\xi, \eta)\Phi\left(\frac{\theta}{\sqrt{\epsilon} - k}\right)$, so that

$$m_\epsilon(\xi, \eta) = \sum_{|k| \leq 1/10\sqrt{\epsilon}} m^k_\xi(\xi, \eta).$$

(Note that because of our assumption that $m_\epsilon$ vanishes for $|\xi, \eta| \geq 1/20$, the above sum runs over $|k| \leq 1/10\sqrt{\epsilon}$.) Thus each of the symbols is supported on a chord of length $\sim \sqrt{\epsilon}$, thickened to reach a width of length $\sim \epsilon$, and with an angular parameter $\theta \sim k\sqrt{\epsilon}$. Thus it suffices to control $B_{m_\epsilon}(f, g)(x) = \sum_k \int \int \hat{m}^k_\epsilon(y - x, z - x)f(y)g(z)dydz$. Now let $I_k$, respectively $J_k$, denote the intervals given by the projection of the support of $m^k_\xi(\xi, \eta)$ on the $\xi$, respectively $\eta$, axis. It is easy to check that these intervals are almost-orthogonal, i.e., $\sum_k \chi_{I_k}(x) \lesssim 1$ and $\sum_k \chi_{J_k}(x) \lesssim 1$ for all $x$, where the implicit constants do not depend on $k$. Define $f_k := \chi_{I_k}(D)f$ and $g_k := \chi_{J_k}(D)g$. The quantity we want to control can then also be written as

$$B_{m_\epsilon}(f, g)(x) = \sum_k \int \int \hat{m}^k_\epsilon(y - x, z - x)f_k(y)g_k(z)dydz. \tag{6.1}$$

Step 2: examination of the kernels. We claim that the kernels $\hat{m}^k_\epsilon$ are uniformly bounded in $L^1(\mathbb{R}^2)$. By translation and rotation invariance, it suffices to see this for $\hat{m}^0_\epsilon$. Then, with the notation of Definition 1.3, in the chord of length $\sqrt{\epsilon}$ around the point $(0, 0)$, we can write $\partial_\xi = \partial_{(\nu_\epsilon)} + O(\epsilon^{1/2})\partial_{\nu_\epsilon}$. Since $m_\epsilon$ belongs to $N^\Gamma_\epsilon$, we
also have \( \| \partial_x^\alpha \partial_y^\beta m_\epsilon^0 \|_{L^1(\mathbb{R}^2)} \lesssim \epsilon^{3/2} e^{-\frac{\epsilon^2}{2} - \beta} \). This gives, on the Fourier side (keeping in mind that \( \mathcal{F} \) maps \( L^1 \) to \( L^\infty \)),

\[
(6.2) \quad \left| \hat{m}_\epsilon^0(y, z) \right| \lesssim \epsilon^{3/2} \inf \left( 1, \frac{1}{(\sqrt{\epsilon}|y| + \epsilon|z|)^N} \right)
\]

for any natural number \( N \). Obviously, this implies the desired bound.

Step 3: orthogonality. It now suffices to use the fact that the \( (I_k) \) and \( (J_k) \) form a bounded covering of the real line. It follows that

\[
\| B_{m_\epsilon} (f, g) \|_{L^1} \lesssim \sum_k \left\| \int \int \hat{m}_\epsilon^k(y - x, z - x) f_k(y) g_k(z) dy dz \right\|_{L^1(x)}
\]

\[
\lesssim \sum_k \left\| \hat{m}_\epsilon^k \right\|_{L^1(\mathbb{R}^2)} \| f_k \|_{L^2(I_k)} \| g_k \|_{L^2(J_k)}
\]

\[
\lesssim \left( \sum_k \| f_k \|_{L^2(I_k)}^2 \right)^{1/2} \left( \sum_k \| f_k \|_{L^2(J_k)}^2 \right)^{1/2}
\]

\[
\lesssim \| f \|_{L^2} \| g \|_{L^2}.
\]

\( \square \)

6.2 Points close to \((\infty, \infty, 1)\).

**Proposition 6.2.** Assume that \( m_\epsilon \in \mathbb{N}_\epsilon^\Gamma \) and that \( \Gamma \) has non-vanishing curvature. Then for exponents \( p, q \geq 2 \),

\[
\| B_{m_\epsilon} (f, g) \|_{L^\infty} \lesssim \epsilon^{\frac{q}{4} + \frac{1}{2p}} | \log \epsilon | \| f \|_{L^p} \| g \|_{L^q}.
\]

**Remark 6.3.** Proposition 6.2 is interesting in the limit as \( p \) and \( q \) tend to \( \infty \). The point \((p, q, r)\) approaches \((\infty, \infty, 1)\) with a bound \( O(\epsilon^{3/4q + 1/2p}) \), which converges to the optimal bound at the limit point, namely \( O(1) \).

**Proof.** Step 1: decomposition of \( m_\epsilon \). This step is identical to Step 1 of the proof of Proposition 6.1, so we simply adopt the same notation. The only difference is that now (by translation invariance) it suffices to control

\[
(6.3) \quad B_{m_\epsilon} (f, g)(0) = \sum_k \int \int \hat{m}_\epsilon^k(y, z) f_k(y) g_k(z) dy dz.
\]

Step 2: reduction to simpler kernels. The choice of the length scale \( \sqrt{\epsilon} \) ensures that \( \hat{m}_\epsilon^k \) is essentially supported on a rectangle. We establish this fact for \( m_\epsilon^0 \); the general case follows by rotating the plane. Since \( m_\epsilon^0 \in \mathbb{N}_\epsilon^\Gamma \),

\[
\left\| \partial_x^\alpha \partial_y^\beta m_\epsilon^0 \right\|_{L^1(\mathbb{R}^2)} \lesssim \epsilon^{3/2} e^{-\alpha/2 - \beta}.
\]
This gives, on the Fourier side (keeping in mind that $F$ maps $L^1$ to $L^\infty$),

\begin{equation}
|m_0^k(y, z)| \lesssim \epsilon^{3/2} \inf \left( 1, \frac{1}{(\sqrt{\epsilon}|y| + \epsilon|z|)^{3/2}} \right)
\end{equation}

for any natural number $N$, which in turn implies that

\begin{equation}
|m_0^k(y, z)| \lesssim \epsilon^{3/2} \sum_{\ell \in \mathbb{N}} a_\ell F(2^{-\ell} \sqrt{\epsilon}y, 2^{-\ell} \epsilon z),
\end{equation}

where $F$ denotes the characteristic function of the unit cube and the sequence $\{a_\ell\}$ decays very fast. Similarly, $|\hat{m}_1^k(y, z)| \lesssim \epsilon^{3/2} \sum_{\ell \in \mathbb{N}} a_\ell F(R_{k/\epsilon}^\theta(2^{-\ell} \sqrt{\epsilon}y, 2^{-\ell} \epsilon z))$, where $R_\phi$ denotes rotation by angle $\phi$ around $(0, 0)$. We see from (6.3) that

\begin{equation}
|B_m(f, g)(0)| \lesssim \epsilon^{3/2} \sum_k \sum_{\ell} a_\ell \int F \left( R_{k/\epsilon}^\theta(2^{-\ell} \sqrt{\epsilon}y, 2^{-\ell} \epsilon z) \right) |f_k(y)| |g_k(z)| dydz.
\end{equation}

Step 3: the crucial claim and why it implies the proposition. From now on, we abbreviate by setting $F_k^\epsilon(y, z) = F(R_{k/\epsilon}^\theta(\sqrt{\epsilon}y, \epsilon z))$. We prove the following claim, which implies the proposition.

\textbf{Claim 6.4.} For any sequence of functions $\{f_k\}$,

\begin{equation}
\left\| \left[ \sum_k \left| \int F_k^\epsilon(y, z)f_k(y)dy \right|^2 \right]^{1/2} \right\|_{L^{p/(p-1)}(z)} \lesssim \epsilon^{\frac{2}{p} + \frac{1}{p'}} |\log \epsilon| \left\| \left[ \sum_k f_k^2 \right]^{1/2} \right\|_{L^p}.
\end{equation}

Why does this claim imply the proposition? Starting from (6.3) and using successively the Cauchy-Schwarz (in $k$) and H"older (in $z$) inequalities, we obtain from the above claim and Rubio de Francia’s inequality that

\begin{align*}
|B_m(f, g)(0)| & \lesssim \sum_k \sum_{\ell} a_\ell \int \int F \left( R_{k/\epsilon}^\theta(2^{-\ell} \sqrt{\epsilon}y, 2^{-2\ell} \epsilon z) \right) |f_k(y)||g_k(z)| dydz \\
& \lesssim \sum_{\ell} a_\ell \left\| \left[ \sum_k \left| \int F \left( R_{k/\epsilon}^\theta(2^{-\ell} \sqrt{\epsilon}y, 2^{-2\ell} \epsilon z) \right) f_k(y)dy \right|^2 \right]^{1/2} \right\|_{L^{p/(p-1)}(z)} \\
& \quad \left\| \left[ \sum_k g_k^2 \right]^{1/2} \right\|_{L^q} \\
& \lesssim \epsilon^{\frac{2}{p} + \frac{1}{p'}} |\log \epsilon| \left\| \left[ \sum_k f_k^2 \right]^{1/2} \right\|_{L^p} \left\| \left[ \sum_k g_k^2 \right]^{1/2} \right\|_{L^q} \\
& \lesssim \epsilon^{\frac{2}{p} + \frac{1}{p'}} |\log \epsilon| \|f\|_{L^p} \|g\|_{L^q}.
\end{align*}

(6.7)

This is exactly the statement of the proposition.

Step 4: decomposition of $f_k$ along its level sets. We write $f_k(y) = \sum_j f_k^j(y)$, where $f_k^j(y)$ either takes values between $2^{j-1}$ and $2^j$ or is 0. We can assume a
fortiori that \( f^j_k \) takes either the value \( 2^j \) or 0. In other words, we assume that 
\( f^j_k = 2^j \chi_{E^j_k} \) for some set \( E^j_k \). Observe that there exists a constant \( C_0 \) such that
for any \( z \), \( \text{Supp} F^k_\epsilon (\cdot, z) \subset [-C_0 k/\sqrt{\epsilon}, C_0 k/\sqrt{\epsilon}] \). Consequently, we can assume that
\( E^k_j \subset [-k/\sqrt{\epsilon}, k/\sqrt{\epsilon}] \), since the parts of \( E^j_k \) outside of \([ -k/\sqrt{\epsilon}, k/\sqrt{\epsilon}] \) do not contribute to (6.6). We need the following bound on \( \sum_{k=1}^n |E^j_k| \):

\[
\sum_{k=-n}^n |E^j_k| = \sum_{k=-n}^n \int_{-C_0 k/\sqrt{\epsilon}}^{C_0 k/\sqrt{\epsilon}} \chi_{E^j_k}(y) dy = 2^{-2j} \sum_{k=-n}^n \int_{-C_0 k/\sqrt{\epsilon}}^{C_0 k/\sqrt{\epsilon}} f^j_k(y)^2 dy
\]

(6.8)

\[
\leq 2^{-2j} \left( \sum_{k} (f^j_k)^2 \right)^{1/2} \left( \frac{n}{\sqrt{\epsilon}} \right)^{1-2/p}.
\]

Finally, observe that it suffices to prove Claim 6.4 when \( f_k \) is replaced with \( f^j_k \). Indeed, the scales \( j \) such that \( 2^j < \epsilon^{100} \) can be estimated trivially, whereas summing over the other scales simply contributes \( \log \epsilon \). Thus, in order to deduce the claim, it suffices to prove

\[
\left\| \left[ \sum_k \int_{-2/\epsilon}^{2/\epsilon} F^k_\epsilon(y, z) f^j_k(y) dy \right]^2 \right\|_{L^q(z)}^{1/2} \lesssim \epsilon^{\frac{3p}{2p} + \frac{1}{2p}} \left\| \sum_k f^2_k \right\|_{L^p}^{1/2}.
\]

Step 5: discretization of the \( z \) variable. The variable \( z \) in (6.6) can be restricted to \( |z| \in [-2/\epsilon, 2/\epsilon] \), for otherwise \( F^k_\epsilon(y, z) \) vanishes. We now split the interval \([-2/\epsilon, 2/\epsilon]\) into \( 1/\sqrt{\epsilon} \) intervals \([Z/\sqrt{\epsilon}, (Z+1)/\sqrt{\epsilon}] \) \( Z \) is an integer satisfying \( |Z| \leq 2/\sqrt{\epsilon} \).

Observe that if \( z \in [Z/\sqrt{\epsilon}, (Z+1)/\sqrt{\epsilon}] \), then \( \text{Supp} F^k_\epsilon(y, z) \subset I^k_{\epsilon z} \), where

\[
I^k_{\epsilon z} := \left[ \tan(k\sqrt{\epsilon}) \frac{Z}{\sqrt{\epsilon}} - \frac{C_0}{\epsilon}, \tan(k\sqrt{\epsilon}) \frac{Z}{\sqrt{\epsilon}} + \frac{C_0}{\epsilon} \right].
\]

for a sufficiently large constant \( C_0 \). Thus, if \( z \in [Z/\sqrt{\epsilon}, (Z+1)/\sqrt{\epsilon}] \), then

\[
\int_{-2/\epsilon}^{2/\epsilon} F^k_\epsilon(y, z) \chi_E(y) dy \lesssim \epsilon \sqrt{\epsilon} |I^k_{\epsilon z} \cap E^j_k|.
\]

This implies that

\[
\left\| \left[ \sum_k \int_{-2/\epsilon}^{2/\epsilon} F^k_\epsilon(y, z) f^j_k(y) dy \right]^2 \right\|_{L^q(z)}^{1/2} \lesssim \epsilon \sqrt{\epsilon} 2^j \left\| \sum_k \frac{1}{\sqrt{\epsilon}} \left[ \sum_k I^k_{\epsilon z} \cap E^j_k \right]^{2q'/2} \right\|_{L^q(z)}^{1/q'}
\]

(6.9)

\[
= \epsilon^{3/2 - 1/2q'} 2^j \left\| \sum_k I^k_{\epsilon z} \cap E^j_k \right\|_{L^2(k)}^{1/q'}. \]

Step 6: proof of the claim. We bound the above right-hand side of (6.9) by interpolating the \( \ell^2 \) norm between \( \ell^1 \) and \( \ell^\infty \). The \( \ell^\infty \) bound is the easier one, since
the number of indices $Z$ is of the order of $1/\sqrt{\epsilon}$ and the length of $I_{k}^{Z,\epsilon}$ is bounded by $2C_{0}/\sqrt{\epsilon}$. We have

$$
\left[ \sum_{Z} \left\| I_{k}^{Z,\epsilon} \cap E_{k}^{j} \right\|_{L_{\epsilon}^{q}}^{q'} \right]^{1/q'} = \left[ \sum_{Z} \left( \sup_{k} \left\| I_{k}^{Z,\epsilon} \cap E_{k}^{j} \right\|_{L_{\epsilon}^{q}}^{q'} \right) \right]^{1/q'} 
\lesssim \left[ \frac{1}{\sqrt{\epsilon}} \left( \frac{1}{\sqrt{\epsilon}} \right) \right]^{q'} \lesssim \epsilon^{-1/2-1/2q'}.
$$

(6.10)

For the $\ell^{1}$ bound, first use the embedding $\ell^{1} \rightarrow \ell^{q}$ to obtain

$$
\left[ \sum_{Z} \left\| I_{k}^{Z,\epsilon} \cap E_{k}^{j} \right\|_{L_{\epsilon}^{q}}^{q'} \right]^{1/q'} = \left[ \sum_{Z} \left( \sum_{k} \left\| I_{k}^{Z,\epsilon} \cap E_{k}^{j} \right\|_{L_{\epsilon}^{q}}^{q'} \right) \right]^{1/q'} 
\lesssim \sum_{Z} \sum_{k} \left\| I_{k}^{Z,\epsilon} \cap E_{k}^{j} \right\|
$$

(6.11)

Next use the fact that given $k$, a number $x$ can belong to at most $\sim 1/k\sqrt{\epsilon}$ intervals $I_{k}^{Z,\epsilon}$. This implies that $\sum_{k} \left\| I_{k}^{Z,\epsilon} \cap E \right\| \lesssim (1/k\sqrt{\epsilon})|E|$. Thus (6.11) can be bounded with (6.8)) by

$$
(6.11) \lesssim \sum_{|k| \leq \frac{1}{10\sqrt{\epsilon}}} \frac{|E_{k}^{j}|}{k\sqrt{\epsilon}} 
\lesssim \frac{1}{\sqrt{\epsilon}} \sum_{n=0}^{1/10\sqrt{\epsilon}-1} \left( \frac{1}{n} - \frac{1}{n+1} \right) \left( \sum_{k=0}^{n} |E_{k}^{j}| \right) + \sqrt{\epsilon} \sum_{k=0}^{1/10\sqrt{\epsilon}} |E_{k}^{j}| 
\lesssim 2^{-2j} \left( \sum_{k} (f_{k}^{j})^{2} \right)^{1/2} \left[ \frac{1}{\sqrt{\epsilon}} \sum_{n=0}^{1/10\sqrt{\epsilon}-1} \frac{1}{1+n^{2}} \left( \frac{n}{\sqrt{\epsilon}} \right)^{1-2/p} + \left( \frac{1}{\epsilon} \right)^{1-2/p} \right] 
\lesssim 2^{-2j} \left( \sum_{k} (f_{k}^{j})^{2} \right)^{1/2} \epsilon^{-1+1/p}.
$$

Starting with the inequality (6.9), and interpolating between (6.10) and (6.12) gives

$$
\left\| \left[ \sum_{k} \left| \int F_{k}^{\epsilon}(y,z) f_{k}^{j}(y) \, dy \right|^{2} \right]^{1/2} \right\|_{L_{\epsilon}^{p}(z)} \lesssim \epsilon^{3/2-1/2q'2j} \left[ \epsilon^{-1/2-1/2q'} 2^{-2j} \left( \sum_{k} (f_{k}^{j})^{2} \right)^{1/2} \epsilon^{-1+1/p} \right]^{1/2} 
\lesssim \epsilon^{3/4q+1/2p} \left( \sum_{k} (f_{k}^{j})^{2} \right)^{1/2}.
$$

As noticed at the end of Step 4, this inequality implies the claim. \qed
Remark 6.5. Returning to (6.5), observe that it can be written as

\[ |B_m(\epsilon f, g)(0)| \lesssim \sum_{k,l} 2^{2l} \alpha_l \frac{\epsilon^{3/2}}{2^{2l}} \int_{(\sqrt{\epsilon}y, \epsilon z) \in R_{-k\sqrt{\epsilon}}([-2^l, 2^l]^2)} |f_k \otimes g_k(y, z)| dy dz. \]

The set \{(y, z), (\sqrt{\epsilon}y, \epsilon z) \in R_{-k\sqrt{\epsilon}}([-1, 1]^2)\} is a rectangle whose dimensions are \(2^{l+1}\epsilon^{-1/2} \times 2^{l+1}\epsilon^{-1}\) and whose measure is \(2^{2l+2}\epsilon^{-3/2}\). As a consequence,

\[ |B_m(\epsilon f, g)(0)| \lesssim \sum_{k,l} 2^{2l} \alpha_l K_{\epsilon^{-1/2}}(f_k \otimes g_k)(0, 0), \]

where \(K_{\epsilon^{-1/2}}\) is the Kakeya maximal operator on \(\mathbb{R}^2\); see [12, Section 10.3] for a modern review of this subject. Translating in \(x\), we get

\[ |B_m(\epsilon f, g)(x)| \lesssim \sum_k K_{\epsilon^{-1/2}}(f_k \otimes g_k)(x, x). \]

Thus the boundedness of \(B_m\) is closely related to the boundedness of a “bilinear Kakeya operator” (the one corresponding to the restriction of a 2-dimensional linear Kakeya operator to the diagonal).

7 Interpolation of the different results

Theorems 1.4, 1.5, and 1.6 can be obtained by interpolating the results obtained in Sections 4 and Section 5. Here, for the convenience of the reader, we provide the details of one of these interpolation results.

Theorem 7.1. Let \(p, q, r \in [1, \infty]\) satisfy \(1 \leq 1/p + 1/q + 1/r\). For all smooth bounded curves \(\Gamma\), we have

\[ \|T_m\|_{L^p \times L^q \to L^r \prime} \lesssim \epsilon^\rho, \]

where

\[ \rho := \frac{1}{\max\{p, 2\}} + \frac{1}{\max\{q, 2\}} + \frac{1}{\max\{r, 2\}} - 1 \]

in the following cases:

(a) \(1 \leq 1/p + 1/q + 1/r \leq 3/2, p, q, r < \infty, \) and \(\rho \geq 0\);

(b) \(3/2 \leq 1/p + 1/q + 1/r (in this case, one of \(p, q, r\) is infinite and \(\rho \geq 0\)).

Moreover, if \(p, q, r \leq 2\), the exponent \(\rho\) can be improved to

\[ \tilde{\rho} := \min \left\{ \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right\}. \]

(If at least two of the three indices \((p, q, r)\) are less than 2, then \(\rho = \tilde{\rho}\).)
Figure 1. Exponents \((p^{-1}, q^{-1}, r^{-1}) \in [0, 1]^3\).

**Proof.** In the case \(p, q\) and \(r\) are finite, this was proved in Proposition 4.2. So let us consider only case (b) with only one infinite exponent (since we cannot have two infinite exponents). Because of symmetry, there is no loss of generality in assuming \(p = \infty\). Then \(1/q + 1/r \geq 3/2\) implies \(q, r \leq 2\). Proposition 5.3 gives the result for \((q, r) = (1, 2)\) and by symmetry, for \((q, r') = (2, 1)\). Proposition 5.4 gives the result for \((q, r) = (1, 1)\). So by interpolation, we have the desired estimate for all exponents \(q, r\) satisfying \(1/q + 1/r \geq 3/2\). This completes the proof of case (b).

It remains to prove the last claim. From Propositions 4.1, 5.1, and 5.2, we know that the exponent \(\rho = 1/2\) is optimal on the points \((p, q, r) = (2, 2, 2)\), \((2, 2, 1)\), and \((1, 1, 2)\). As a consequence, by symmetry and interpolation, we know that we can obtain the exponent \(1/2\) if \(\max\{p, q, r\} = 2\); this corresponds to \(\rho\) given by (7.1). Since Proposition 5.5 gives the exponent 1 at the point \((1, 1, 1)\), we can interpolate each point \(u := (x, y, z)\) belonging to the cube \(C := [1/2, 1]^3\) by the end-point \((1, 1, 1)\) with another point belonging to the subset
\[ C := \{(p^{-1}, q^{-1}, r^{-1}); \max\{p, q, r\} = 2\}. \] Indeed, if \( x = \min\{x, y, z\} \), then

\[
(x, y, z) = (2x - 1)(1, 1, 1) + 2(z - x) \left( \frac{1}{2}, \frac{1}{2}, 1 \right) + 2(y - x) \left( \frac{1}{2}, 1, \frac{1}{2} \right) + (2 + 2x - y - z) \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)
\]

which, by interpolation, gives the exponent

\[
\tilde{\rho} = (2x - 1) + \frac{1}{2} [2(z - x) + 2(y - x) + (2 + 2x - y - z)] = x.
\]

\[\square\]

8 Bilinear Fourier transform restriction-extension inequalities

**Definition 8.1.** Let \( p, q, r \in [1, \infty] \) satisfy (1.3). We say that a curve \( \Gamma \subset \mathbb{R}^2 \) satisfies a \((p, q, r)\) restriction-extension inequality if the frequency restriction-extension bilinear multiplier \( T_\Gamma \) given by

\[
(f, g) \rightarrow T_\Gamma(f, g)(x) := \int e^{ix(\xi + \eta)} \hat{f}(\xi) \hat{g}(\eta) d\sigma_\Gamma(\xi, \eta)
\]

is bounded from \( L^p(\mathbb{R}) \times L^q(\mathbb{R}) \) into \( L^{r'}(\mathbb{R}) \). Here, \( d\sigma_\Gamma \) is the arc-length measure on \( \Gamma \), is bounded.

Before proceeding, let us first say a few words about the linear theory. We could ask when, for exponents \( p, r \in [1, \infty] \) and a curve \( \gamma \) in \( \mathbb{R}^2 \), the linear operator \( U_\Gamma \) defined by

\[
U_\Gamma(f) := x \rightarrow \int e^{ixx} \hat{f}(\xi) d\sigma_\Gamma(\xi)
\]

is bounded from \( L^p(\mathbb{R}^2) \) into \( L^{r'}(\mathbb{R}^2) \). This operator is a multiplier and corresponds to the convolution operation with “\( d\sigma_\Gamma \)”.

From Lemma 3.4, it follows that for a compact smooth curve \( \Gamma, \tilde{d}\sigma_\Gamma \in L^{4+\epsilon}(\mathbb{R}^2) \) for every \( \epsilon > 0 \). By Young’s inequality, the operator \( U_\Gamma \) is bounded from \( L^p(\mathbb{R}^2) \) into \( L^{r'}(\mathbb{R}^2) \) for all exponents \( p, r \geq 1 \) satisfying \( 1/r' + 1 < 1/4 + 1/p \), which is equivalent to \( 1/p + 1/r > 7/4 \).

Under the same assumption of non-vanishing curvature, we now want to obtain similar results for the bilinear operator \( T_\Gamma \).

**Proposition 8.2** (Bilinear restriction). Let \( \Gamma \) be a compact smooth curve in \( \mathbb{R}^2 \) with non-vanishing Gaussian curvature. For all exponents \( p, q, r \in [1, \infty) \)
satisfying

\[ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > \frac{5}{2}, \]

we have the bound \( \| T^\Gamma (f, g) \|_{L^{r'}} \lesssim \| f \|_{L^p} \| g \|_{L^q} \), so \( \Gamma \) satisfies a \((p, q, r)\) restriction-extension inequality.

**Proof.** Although this follows from Theorem 1.5, we give an alternative proof. Because of the assumption on the curvature, Lemma 3.4 yields

\[ \left| \int e^{i(x_1 \xi + x_2 \eta)} d\sigma^\Gamma(\xi, \eta) \right| \lesssim (1 + |(x_1, x_2)|)^{-1/2}. \]

Consequently, the bilinear kernel \( K \) in \( \mathbb{R}^2 \) of \( T^\Gamma \) belongs to \( L^{4+\epsilon}(\mathbb{R}^2) \) for all \( \epsilon > 0 \). The result now follows from the usual estimates for bilinear convolution (Brascamp-Lieb inequality [4], [28]).

**Proposition 8.3.** Assume that the curve \( \Gamma \subset \mathbb{R}^2 \) satisfies a \((p, q, r)\)-restriction inequality with \( p, q, r \in (1, \infty) \). Then for all smooth symbols \( m \in S(\mathbb{R}^2) \), there exists a constant \( C \) such that \( \| B_{md\sigma^\Gamma} (f, g) \|_{L^{r'}} \leq C \| f \|_{L^p} \| g \|_{L^q} \).

**Proof.** We just develop the smooth symbol \( \sigma \) via Fourier transform. There exists a smooth function \( k \in S(\mathbb{R}^2) \) such that \( \sigma(\xi, \eta) = \int e^{i(\xi y + \eta z)} K(y, z) dydz \). Therefore, \( B_{md\sigma^\Gamma} = \int K(y, z) T_\Gamma (\tau_y f, \tau_z g) dydz \). It follows by Minkowski’s inequality and the facts that translation does not change the Lebesgue norm and that \( K \in L^1(\mathbb{R}^2) \) that \( B_{md\sigma^\Gamma} \) is bounded from \( L^p \times L^q \) into \( L^{r'} \).

We now want to combine the two previous kinds of argument using the decay of the kernel due to the curvature and the orthogonality properties in the frequency space used in Section 5.

**Proposition 8.4.** Let \( \Gamma \) be a smooth curve in \( \mathbb{R}^2 \) with non-vanishing curvature. Then the bilinear multiplier \( T_\Gamma \) associated to the singular symbol \( m(\xi, \eta) := d\sigma^\Gamma(\xi, \eta) \) is bounded from \( L^p(\mathbb{R}) \times L^q(\mathbb{R}) \) into \( L^{r'}(\mathbb{R}) \) if \( p, q, r \in (1, \infty) \) satisfy

\[ \begin{cases} 1/p - 1/q + 1/r \leq 1, \\ -1/p + 1/q + 1/r \leq 1, \\ 1/p + 1/q - 1/r \leq 1, \\ 1/p + 1/q + 1/r > 7/3. \end{cases} \]

Thus \( \Gamma \) satisfies a \((p, q, r)\)-restriction inequality for such exponents.
Proof. The idea is to improve the previous estimates by interpolating with the decay of the kernel (Lemma 3.4). Consider \( K \) the linear kernel in \( \mathbb{R}^2 \) given by

\[
K(x_1, x_2) := \int e^{i(x_1 \xi + x_2 \eta)} d\sigma_\Gamma(\xi, \eta).
\]

We split the kernel in the space variable, using a function \( \Psi_1 \in \mathcal{S} \) (such that \( \Psi_1(0) = 0, \hat{\Psi}_1 \) is compactly supported in \([-1, 1]\), and \( \sum_j \Psi_j(2^j \cdot) = 1 \)) as follows:

\[
K(x_1, x_2) = \sum_{j \geq 0} \Psi(2^{-j}(x_1, x_2))K(x_1, x_2) + \Phi(x_1, x_2)K(x_1, x_2) := \sum_j K_j + K_\Phi,
\]

where \( \Phi \) satisfies \( \Phi(\cdot) = 1 - \sum_{j \geq 0} \Psi(2^j \cdot) \). Since \( K \) satisfies the bound

\[
|K(x_1, x_2)| \lesssim (1 + |(x_1, x_2)|)^{-1/2},
\]

(due to Lemma 3.4), it follows that \( \|K_j\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^{-j/2} \), which gives

(8.3) \[ \|B_{\hat{K}_j}\|_{L^1 \times L^1 \rightarrow L^\infty} \lesssim 2^{-j/2}. \]

Moreover, since \( K \in L^\infty \),

(8.4) \[ \|B_{\hat{K}_\Phi}\|_{L^1 \times L^1 \rightarrow L^\infty} \lesssim 1. \]

In addition, writing the kernel \( K_j \) in the frequency space, we have

\[
\hat{K}_j(\xi_1, \eta_1) = \int 2^{2j} \hat{\Psi}(2^j (\xi_1 - \xi_2, \eta_1 - \eta_2)) d\sigma_\Gamma(\xi_2, \eta_2).
\]

Consequently, the bilinear symbol \( (\xi, \eta) \rightarrow 2^{-j} \hat{K}_j(\xi_1, \eta_1) \) belongs to \( \mathcal{M}_{2^{-j}}^{\Gamma} \).

According to results in Subsection 5.2 (and by changing the role of \( p, q, r \)), the bilinear operator associated to \( 2^{-j} K_j \) is bounded from \( L^2 \times L^2 \) into \( L^\infty \). So we know that

(8.5) \[ \|B_{\hat{K}_j}\|_{L^{p_0} \times L^{q_0} \rightarrow L^{0'}} \lesssim 2^{j/2-3j/4} \lesssim 2^{j/4} \]

for all exponents \( p_0, q_0, r_0 \in [1, 2] \) satisfying

(8.6) \[ \frac{1}{p_0} + \frac{1}{q_0} + \frac{1}{r_0} = 2. \]

Concerning the remainder term \( K_\Phi \), it is clear that

(8.7) \[ \|B_{\hat{K}_\Phi}\|_{L^{p_0} \times L^{q_0} \rightarrow L^{0'}} \lesssim 1. \]

Using real or complex bilinear interpolation, on the one hand between (8.3) and (8.5) and on the other hand, between (8.4) and (8.7), we conclude that for every
“intermediate” triplet \((p, q, r)\) between \((p_0, q_0, r_0)\) and \((1, 1, 1)\) (where \((p_0, q_0, r_0)\) is any triplet of \([1, 2]^3\) satisfying \((8.6)\)),

\[
\|B_{K_0}^\ast\|_{L^p \times L^q \to L^r'} \lesssim 1 \quad \text{and} \quad \|B_{K_j}^\ast\|_{L^p \times L^q \to L^r'} \lesssim 2^{-\epsilon j},
\]

for some \(\epsilon := \epsilon(p, q, r) > 0\) if \(1/p + 1/q + 1/r > 7/3\). Summing over \(j \geq 0\) proves the boundedness of \(T_K\), i.e., \(\|T_K\|_{L^p \times L^q \to L^r'} < \infty\). The range of allowed exponents is exactly as described by \((8.2)\). Indeed the first inequality in \((8.2)\) is given by the plane containing \(p = q = r = 1; p = q = 2, r = 1;\) and \(p = 1, q = 2, r = 1;\) and \(q = 1, p = r = 2,\) and the third inequality is given by the plane containing \(p = q = r = 1; r = q = 2p = 1;\) and \(q = 1, p = r = 2.\) The fourth equation in \((8.2)\) corresponds to the condition required in order to obtain some \(\epsilon > 0\) (due to the strict inequality) such that \((8.8)\) holds.

The set of \((p, q, r)\) given by \((8.2)\) is the tetrahedron built on the points \((1, 1, 1), (1, 3/2, 3/2), (3/2, 1, 3/2),\) and \((3/2, 3/2, 1)\). So by interpolation with Proposition 8.2, we get the following result.

**Theorem 8.5.** Let \(\Gamma\) be a smooth curve in \(\mathbb{R}^2\) with non-vanishing curvature. The bilinear multiplier \(T_\Gamma\) associated to the singular symbol \(m(\xi, \eta) := d\sigma_\Gamma(\xi, \eta)\) is bounded from \(L^p(\mathbb{R}) \times L^q(\mathbb{R})\) into \(L^r'(\mathbb{R})\) if \((p, q, r) \in [1, 2]^3\) belongs to the convex hull of the points

\[
(1, 1, 1), (1, 1, 2), (3/2, 3/2, 1)
\]

and those obtained from them by symmetry; i.e., \(\Gamma\) satisfies a \((p, q, r)\)-restriction inequality for such exponents.

Having obtained some “bilinear Fourier restriction-extension inequalities,” we now return to the smooth symbols \(m_\epsilon\). For a curve \(\Gamma\), it should be reasonable from a bilinear Fourier restriction-extension inequality to prove \((1.1)\) with a decay function \(\alpha(\epsilon) = \epsilon.\) However, to do that, we have to decompose the \(\epsilon\)-neighborhood at the scale \(\epsilon\) and then sum up all these small pieces. Since we start from a global estimate with a symbol carried on the whole curve, we have to do this splitting uniformly “along the tangential variable”. That is why we cannot deduce \((1.1)\) for all symbols \(m_\epsilon\) belonging to the class \(M_\epsilon^{[\Gamma]}\).

Let us assume that \(\Gamma\) is a smooth and compact curve and denote the distance function by \(\nu := d_\Gamma.\) For every \((\xi, \eta) \not\in \Gamma,\) we know that \(|\nabla \nu(\xi, \eta)| = 1.\) With this notation, \(\nabla \nu\) can be considered as the direction of local normal coordinates and \((\nabla \nu)^\perp\) as the direction of the local tangential coordinate. We are interested in
symbols $m_\epsilon$ having nice behavior in the tangential directions $\nabla v^\perp$. More precisely, we are interested in symbols $m_\epsilon$ taking the form

\begin{equation}
(8.10) \quad m_\epsilon := \frac{1}{\epsilon} \int_{\Gamma} \phi \left( \frac{(\xi, \eta) - \lambda}{\epsilon} \right) m(\lambda) d\sigma_\Gamma(\lambda),
\end{equation}

where $m$ is a smooth and compactly supported function on $\Gamma$ and $\phi \in C_0^\infty(\mathbb{R}^2)$ satisfies

\[
\phi(\xi, \eta) = \begin{cases} 
1 & \text{if } |(\xi, \eta)| \leq 1/2, \\
0 & \text{if } |(\xi, \eta)| \geq 1.
\end{cases}
\]

Let us verify that $m_\epsilon$ is regular at the scale $\epsilon$ in the normal direction and at the scale 1 in the tangential direction.

**Proposition 8.6.** A symbol $m_\epsilon$ given by (8.10) satisfies the regularity property

\begin{equation}
(8.11) \quad \|\nabla^\alpha m_\epsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim \epsilon^{-|\alpha|}, \quad \|\langle \nabla v \rangle^\perp, \nabla m_\epsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim 1.
\end{equation}

**Proof.** First observe that for fixed $(\xi, \eta)$,

\[
|m_\epsilon(\xi, \eta)| \leq e^{-1/2} 1_{\Gamma \cap B((\xi, \eta), \epsilon)} \lesssim 1,
\]

\[
\nabla m_\epsilon(\xi, \eta) = \frac{1}{\epsilon^2} \int_{\Gamma} \nabla \phi \left( \frac{(\xi, \eta) - \lambda}{\epsilon} \right) m(\lambda) d\sigma_\Gamma(\lambda).
\]

Iterating, we easily obtain $\|\nabla^\alpha m_\epsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim \epsilon^{-|\alpha|}$. To check the tangential derivative, let $\gamma$ be a normalized parametrization of $\Gamma$, i.e., $\gamma : [0, 1] \to \mathbb{R}^2$ satisfies $|\gamma'(t)| = 1$. Then,

\[
\nabla m_\epsilon(\xi, \eta) = \frac{1}{\epsilon^2} \int_0^1 \nabla \phi \left( \frac{(\xi, \eta) - \gamma(t)}{\epsilon} \right) m(\gamma(t)) dt,
\]

since

\[
\int_0^1 \left< \nabla \phi \left( \frac{(\xi, \eta) - \gamma(t)}{\epsilon} \right), \gamma'(t) \right> m(\gamma(t)) dt = - \int_0^1 \left[ \frac{d}{dt} \phi \left( \frac{(\xi, \eta) - \gamma(t)}{\epsilon} \right) \right] m(\gamma(t)) dt
\]

\[
= \int_0^1 \phi \left( \frac{(\xi, \eta) - \gamma(t)}{\epsilon} \right) \left[ \frac{d}{dt} m(\gamma(t)) \right] dt.
\]

So, as previously,

\[
\left| \int_0^1 \left< \nabla \phi \left( \frac{(\xi, \eta) - \gamma(t)}{\epsilon} \right), \gamma'(t) \right> m(\gamma(t)) dt \right| \lesssim 1.
\]
Consequently,\[
\left| \frac{1}{\epsilon^2} \int_0^1 \left( \frac{\langle \xi, \eta \rangle - \gamma(t)}{\epsilon} \right) m(\gamma(t)) dt \right| \leq 1 + \left| \frac{1}{\epsilon^2} \int_0^1 \left( \frac{\langle \xi, \eta \rangle - \gamma(t)}{\epsilon} \right) \langle \gamma'(t), (\nabla v) \rangle m(\gamma(t)) dt \right|.
\]
However, \(\langle \gamma'(t)^\perp, (\nabla v)^\perp \rangle = \langle \gamma'(t), \nabla v \rangle\); and since \(|(\xi, \eta) - \gamma(t)| \leq \epsilon\), the smoothness of the curve \(\Gamma\) implies that \(|\gamma'(t), \nabla v|| \leq \epsilon\), which concludes the proof of (8.11).

Using similar arguments, we can obtain estimates for the higher order derivatives of these specific symbols.

**Corollary 8.7.** The symbols \(m_\epsilon\) given by (8.10) belong to the class \(\mathcal{N}_{\epsilon}^\Gamma\).

For these specific symbols \(m_\epsilon\), we can prove equivalence between a restriction-extension property and a decay rate in (1.1) with \(\alpha(\epsilon) = \epsilon\) for all exponents \(p, q, r\).

**Proposition 8.8.** Let \(\Gamma\) be a smooth and compact curve in \(\mathbb{R}^2\). Assume that \(\Gamma\) satisfies a \((p, q, r)\)-restriction inequality for some exponents \(p, q, r' \in [1, \infty]\). Then there exists a constant \(c\) such that for all \(\epsilon \leq 1\) and all symbols \(m_\epsilon\) given by (8.10),

\[
\| T_{m_\epsilon} \|_{L^p \times L^q \to L^{r'}} \leq c \epsilon.
\]

**Proof.** A change of variables yields

\[
T_{m_\epsilon}(f, g)(x) := \epsilon^{-1} \int_{\Gamma} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} \phi \left( \frac{\langle \xi, \eta \rangle - \lambda}{\epsilon} \right) m(\lambda) \hat{f}(\xi) \hat{g}(\eta) \, d\xi d\eta d\sigma_{\Gamma}(\lambda)
\]

\[
= \epsilon^{-1} \int_{\Gamma} \int_{\mathbb{R}^2} e^{ix(\xi+\eta+\lambda_1+\lambda_2)} \phi \left( \frac{\langle \xi, \eta \rangle}{\epsilon} \right) m(\lambda) \hat{f}(\xi+\lambda_1) \hat{g}(\eta+\lambda_2) d\xi d\eta d\sigma_{\Gamma}(\lambda).
\]

Hence

\[
T_{m_\epsilon}(f, g)(x) = \epsilon^{-1} \int_{\mathbb{R}^2} \phi \left( \frac{\langle \xi, \eta \rangle}{\epsilon} \right) e^{ix(\xi+\eta)} T_{md\sigma_{\Gamma}}(M_{\xi} f, M_{\eta} g)(x) d\xi d\eta,
\]

where \(M_{\xi}\) and \(M_{\eta}\) are modulation operators. Since \(T_{d\sigma_{\Gamma}}\) is assumed to be bounded and \(m\) is smooth (at the scale 1), \(T_{md\sigma_{\Gamma}}\) is also bounded from \(L^p \times L^q\) into \(L^{r'}\). The proof is concluded with an application of Minkowski’s inequality. \(\square\)
9 The setting of rough curves and extension of the results

In this subsection, we define some notions of rough curve. This allows us to extend the results. We first recall the notions of rectifiable and Ahlfors-regular curves, which support a measure. Then, we make precise some of our results which still hold in this general setting.

Definition 9.1 (Rectifiable curve). A rectifiable curve is the image of a continuous map $\gamma : [0, 1] \to \mathbb{R}^d$ such that

$$\text{length}(\gamma) := \sup_{0 \leq t_0 \leq \ldots \leq t_n \leq 1} \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| < \infty.$$ 

We refer the reader to [6, 8] for more details concerning rectifiable sets. If $\hat{\Gamma}$ is a compact curve, then it is a rectifiable curve if and only if it is connected and has finite one-dimensional Hausdorff measure. It is well known that if $\hat{\Gamma}$ is a rectifiable curve, we can construct a finite measure $d\sigma_{\hat{\Gamma}}$ (corresponding to the measure of the length) on $\hat{\Gamma}$ such that $d\sigma_{\hat{\Gamma}}$ is equivalent to $\mathcal{H}_1|_{\hat{\Gamma}}$ (the restriction of the one-dimensional Hausdorff measure to the set $\hat{\Gamma}$). For any measurable subset $E \subset \hat{\Gamma}$,

$$\sigma_{\hat{\Gamma}}(E) \simeq \mathcal{H}_1|_{\hat{\Gamma}}(E) = \lim_{\delta \to 0} \inf_{E \subset \bigcup_i U_i \text{ diam}(U_i) \leq \delta} \sum_i \text{diam}(U_i),$$

where $U_i$ are open sets.

Definition 9.2 (Ahlfors-regular curve). A continuous curve $\hat{\Gamma} \subset \mathbb{R}^2$ is said to be Ahlfors-regular if

$$(9.1) \sup_{z \in \hat{\Gamma}} \sup_{\epsilon \in (0, 1)} \frac{\mathcal{H}_1(\hat{\Gamma} \cap \overline{B}(z, \epsilon))}{\epsilon} < \infty.$$ 

For an Ahlfors-regular curve $\Gamma$, $\epsilon \leq \text{length}(\Gamma)$, and $z \in \Gamma$, there exists a part of $\Gamma$ of length larger than $\epsilon$ contained in $\overline{B}(z, \epsilon)$. Thus, obviously,

$$(9.2) 1 \leq \inf_{z \in \Gamma} \sup_{\epsilon < \text{length}(\Gamma)} \frac{\mathcal{H}_1(\Gamma \cap \overline{B}(z, \epsilon))}{\epsilon} < \infty.$$ 

We refer the reader to [6] for the analysis of curves satisfying these regularity properties.

Proposition 9.3. Let $\Gamma$ be a continuous Ahlfors-regular curve. There exists a one-parameter family of symbols $\{m_\epsilon\}_{\epsilon > 0}$ such that $m_\epsilon \in M^\Gamma$ for every $\epsilon > 0$ and such that $\epsilon^{-1}m_\epsilon$ converges weakly to $d\sigma_{\Gamma}$ in the distributional sense.
Proof. First observe that it is obvious that since $\hat{W}$ satisfies (9.1), it is a rectifiable curve and so admits a arc-length measure (denoted by $d\sigma_{\hat{W}}$). Moreover, it is easy to see that (9.1) self-improves to

\begin{equation}
\sup_{z \in \mathbb{R}^2} \sup_{\epsilon \in (0, 1/2)} \frac{H^1(\hat{W} \cap B(z, \epsilon))}{\epsilon} < \infty
\end{equation}

Choose a non-negative smooth function $\chi : \mathbb{R}^2 \to \mathbb{R}$ supported on $B(0, 1)$ and satisfying $\int_{\mathbb{R}^2} \chi(x)dx = 1$. Then for $\epsilon \in (0, 1/2)$, it suffices to consider the symbols $m_\epsilon(x) := (1/\epsilon) \int_{\Gamma} \chi((x-y)/\epsilon) d\sigma_{\hat{W}}(y)$. Details are left to the reader. \qed

Lemma 9.4. Let $\Gamma$ be a rectifiable curve of finite length. There exists a constant $c_{\Gamma}$ such that for small enough $\epsilon > 0$, $|\Gamma_\epsilon| \leq c_{\Gamma} \epsilon$.

Proof. Consider $\epsilon \leq c \text{length}(\Gamma)$, where $c$ is a small constant, and choose a non-negative smooth function $\chi : \mathbb{R}^2 \to \mathbb{R}$ supported on $B(0, 1)$ satisfying $\chi(x) = 1$ for $|x| \leq 1$. For all $z \in \Gamma_\epsilon$,

$$\frac{1}{\epsilon} \int_{\Gamma} \chi \left( \frac{z-y}{2\epsilon} \right) d\sigma_{\hat{W}}(y) \geq \frac{H^1(\Gamma \cap B(z, 2\epsilon))}{\epsilon}.$$ 

However, since $z \in \Gamma_\epsilon$, there exists $x \in \Gamma \cap B(z, \epsilon)$; hence

$$H^1(\Gamma \cap B(z, 2\epsilon)) \geq H^1(\Gamma \cap B(x, \epsilon)).$$

Since $\epsilon$ is assumed to be smaller than the length of $\Gamma$, there exists a part of $\Gamma$ of length larger than $\epsilon$ contained in $B(x, \epsilon)$. It follows that

$$\frac{1}{\epsilon} \int_{\Gamma} \chi \left( \frac{z-y}{2\epsilon} \right) d\sigma_{\hat{W}}(y) \geq 1.$$ 

Consequently,

$$|\Gamma_\epsilon| \leq \int_{\mathbb{R}^2} \frac{1}{\epsilon} \int_{\Gamma} \chi \left( \frac{z-y}{2\epsilon} \right) d\sigma_{\hat{W}}(y) dz \lesssim \epsilon \left( \int_{\mathbb{R}^2} \chi(x)dx \right) \left( \int_{\Gamma} d\sigma_{\hat{W}}(y) \right) \lesssim \epsilon,$$

where we have used the fact that $\Gamma$ has a finite length. \qed

To extend our results, we have to perform a suitable decomposition of the curve. To this end, we introduce the two following notions.

Definition 9.5. Let $\Gamma$ be a Ahlfors-regular curve in $\mathbb{R}^2$ and $\pi_1, \pi_2, \pi_3$ be the orthogonal projections, on the degenerate lines $\{ \eta = 0 \}, \{ \xi = 0 \},$ and $\{ \xi + \eta = 0 \}$, respectively. The curve $\Gamma$ is said to have finitely bi-Lipschitz projections if it can be split into pieces $(\Gamma_i)_{i=1,...,N}$, where for every $i$ and at least two indices $k \in \{ 1, 2, 3 \}$, there exists a bi-Lipschitz parametrization of $\pi_k(\Gamma_i)$.
It is known that a general Ahlfors-regular curve can be split into bi-Lipschitz parametrized pieces. Here, we require that these pieces have bi-Lipschitz parametrization through certain projections.

Obviously, smooth curves have this property, as do polygons. On the other hand, there exist Ahlfors-regular curves, even of finite length, which do not have this property (for example, a logarithmic spiral).

**Definition 9.6.** An Ahlfors-regular curve $\Gamma$ in $\mathbb{R}^2$ is said to be **nowhere characteristic** if there exists a constant $c$ such that for all real $t \in \mathbb{R}$, for every characteristic angle $\theta \in \{0, \pi/2, 3\pi/4\}$,

\[(9.4) \quad \mathcal{H}^1(\Gamma \cap \{(\xi, \eta), t \leq \xi - \tan(\theta)\eta \leq t + \epsilon\}) \leq c\epsilon,\]

for small enough $\epsilon > 0$. This assumption describes the situation in which $\Gamma$ has no characteristic tangential directions (Ahlfors-regular curves admit a tangential vector almost everywhere). It is easy to check that a nowhere characteristic curve has finitely bi-Lipschitz projections.

We leave it to the reader with these two notions in hand to check the following possible extensions. (We give only a sample.)

**Proposition 9.7.** Let $\Gamma$ be an Ahlfors-regular curve in $\mathbb{R}^2$.

- If $\Gamma$ has finitely bi-Lipschitz projections (not necessarily bounded), then for $p, q, r \in [2, \infty)$, there exists a constant $c$ such that for all symbols $m_\epsilon \in \mathcal{M}_\epsilon^\Gamma$,

\[\|B_{m_\epsilon}\|_{L^p \times L^q \to L^r'} \leq c\epsilon^{1/p+1/q+1/r-1}.\]

- If $\Gamma$ is nowhere characteristic, then the above result holds with one of the three exponents less than 2.

- If $\Gamma$ is nowhere characteristic, then bilinear Fourier restriction-extension inequalities still hold for $1/p+1/q+1/r > 2$ (since (1.1) holds with $\alpha(\epsilon) \lesssim \epsilon$).

### 10 Applications to bilinear multipliers

**10.1 Non-smooth bilinear multipliers and bilinear Bochner-Riesz means.** This subsection is devoted to the question of whether characteristic functions of a compact set of $\mathbb{R}^2$ give a bounded bilinear multiplier from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^{r'}(\mathbb{R}^2)$. We refer the reader to the introduction for a presentation of works concerning the ball and polygons.

We do not aim at exhaustiveness but give only a sample of results in this direction.
**Theorem 10.1.** Let $K$ be a compact subset of $\mathbb{R}^2$

(i) If $\partial K$ is an Ahlfors-regular curve in $\mathbb{R}^2$ with finitely bi-Lipschitz projections, then for exponents $p, q, r' \in [2, \infty)$,

\[
\|B_{\chi_K}(f, g)\|_{L^{r'}} \lesssim \|f\|_{L^p} \|g\|_{L^q}
\]

if $1/p + 1/q + 1/r > 1$. If $p, q, r' > 1$, then (10.1) still holds if

\[
\frac{1}{\max\{p, 2\}} + \frac{1}{\max\{q, 2\}} + \frac{1}{\max\{r, 2\}} > 1.
\]

If $\partial K$ is smooth and has non-vanishing curvature, then for $p, q, r' \in (1, \infty)$,

\[
\|B_{\chi_K}(f, g)\|_{L^{r'}} \lesssim \|f\|_{L^p} \|g\|_{L^q}
\]

if $1/p + 1/q + 1/r > 1$.

**Proof.** We simply explain the proof of the first statement. Let $\Gamma = \partial K$. Without loss of generality and for convenience, assume that the diameter of $K$ is less than 1.

Next we need a partition of unity $(\chi_n)_{n \geq 0}$ such that

- for all $(\xi, \eta)$ in $K$; $1 = \sum_{n \geq 0} \chi_n(\xi, \eta)$;
- for each integer $n \geq 0$, $\chi_n$ is supported in $\Gamma_{102^{-n}}$;
- for each integer $n \geq 0$ and every multi-index $\alpha$, $\|D^\alpha \chi_n\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^n|\alpha|$.

To construct this partition of unity, consider a Whitney covering of $K^c$ by balls $(O_i = B(x_i, r_i))_i$. Then consider a smooth adapted partition of unity $\chi_{O_i}$ and set $\chi_n := \sum_{2^{-n} \leq d((\xi, \eta), S) < 2^{-n+1}} \chi_{O_i}$. We leave it to the reader to verify that by the definition of Whitney balls, these $\chi_n$ satisfy the desired properties.

We then have $B_{\chi_K}(f, g) = \sum_{n \geq 0} B_{\chi_n}(f, g)$. In addition, the symbol $m_n$ belongs to $\mathcal{M}_\epsilon^\Gamma$ with $\epsilon = 2^{-n}$. Hence, by Proposition 4.1 (with Proposition 9.7),

\[
\|T_{m_n}(f, g)\|_{L^{r'}} \leq C 2^{-2n} \|f\|_{L^p} \|g\|_{L^q}.
\]

Since $s > 0$, we can sum with $n \in \mathbb{N}$. This finishes the proof of the first claim.

The second claim is obtained by the same reasoning as in Proposition 4.2.

For an example, consider $K$ to be a disc, square, or any polygon. For the specific case of a disc, Grafakos and Li have obtained in [15] boundedness in the local $L^2$ case for the bilinear multiplier under the Hölder scaling. Here we have general results for general sets but in the sub-Hölder scaling.

If the exponents $p, q, r$ satisfy Hölder’s relation

\[
1 = \frac{1}{r} + \frac{1}{p} + \frac{1}{q},
\]

Proposition 4.1 allows us to obtain estimates for the bilinear multipliers $m_\epsilon$ without decay. Because of this absence of decay, we cannot sum the different inequalities.
over \( n \geq 0 \). To get around this difficulty, we can proceed as for the Bochner-Riesz means.

Let us recall this phenomenon in the linear setting. The linear operator \( T \) defined by \( \hat{T}(f) = 1_{B(0,1)} \hat{f} \) is bounded on \( L^2(\mathbb{R}^d) \) for every integer \( d \geq 1 \) and is unbounded on \( L^p(\mathbb{R}^d) \) for every \( p \neq 2 \). This is a famous result of Fefferman \cite{9}.

One way around this unboundedness is to add some regularity near the boundary and to study the linear operator

\[
T^\lambda(f)(x) := \int_{\mathbb{R}^n} e^{ix \cdot \xi}(1 - |\xi|^2)_+^\lambda \hat{f}(\xi) d\xi,
\]

where \((1 - |\xi|^2)_+ := (1 - |\xi|^2)1_{|\xi| \leq 1}\). This new symbol corresponds to a regularization of the initial symbol \( 1_{B(0,1)} \) at the boundary.

Note that \( T^\lambda(f) \) converges to \( T(f) \) as \( \lambda \to 0 \) tends to 0. However, the symbol in \( T^\lambda \) is slightly more regular at the boundary \( S^{d-1} \). The main question relies on the range of exponent \( p \) (depending on \( \lambda \) and \( d \)) on which \( T^\lambda \) is \( L^p(\mathbb{R}^d) \)-bounded. The question remains open for arbitrary dimension \( d \) but, as the following result shows, is completely solved for \( d = 2 \).

**Theorem 10.2.** For \( d = 2 \) and \( \lambda \in (0, 1/2] \), \( T^\lambda \) is \( L^p(\mathbb{R}^2) \)-bounded if and only if \( 4/(3 + 2\lambda) < p < 4/(1 - 2\lambda) \).

We refer the reader to \cite[Section 10.2]{12} for a modern review of this subject and only point out the main idea, which is to add regularity on the characteristic function at the boundary of the set in order to gain integrability of the multiplier.

We aim to apply this same idea in our current bilinear setting. To do so, consider \( K \subset \mathbb{R}^2 \) as in Theorem 10.1. The bilinear multiplier associated to the symbol \( 1_K \) may be not bounded from \( L^p \times L^q \) into \( L^r \), so we regularize this symbol at the boundary \( \partial K \) to get boundedness.

**Theorem 10.3.** Let \( K \) be a compact set and \( m(\xi, \eta) := \chi_K(\xi, \eta)d((\xi, \eta), K)^\lambda \) with \( \lambda > 0 \). and let \( (p, q, r) \) satisfy (10.2).

(i) If \( \partial K \) is an Ahlfors-regular curve in \( \mathbb{R}^2 \) with finitely bi-Lipschitz projections, then for exponents \( p, q, r \in [2, \infty) \) satisfying (10.2),

\[
\|T_m(f, g)\|_{L^r'} \leq C\|f\|_{L^p} \|g\|_{L^q}
\]

(ii) If \( \partial K \) is a smooth curve with non-vanishing curvature, then for exponents \( p, q, r \in (1, \infty) \), \( \|T_m(f, g)\|_{L^r'} \leq C\|f\|_{L^p} \|g\|_{L^q} \).

**Proof.** We prove only the first assertion. As previously, we can decompose the symbol \( m_\lambda \) into \( m = \sum_{n \geq 0} m_n \), where each \( m_n \) is a symbol supported in
\(K \cap \{(\xi, \eta), \ 2^{-n-1} \leq d((\xi, \eta), \partial K) \leq 2^{-n+2}\}\) and for all integers \(d \geq 0\),
\[
\|\nabla^d m_n\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^{-\lambda n} 2^{-dn}.
\]

We then deduce from Proposition \(4.1\) that \(T_{m_n}\) is bounded from \(L^p \times L^q\) into \(L^{r'}\); in fact, \(\|T_{m_n}\|_{L^p \times L^q \to L^{r'}} \lesssim 2^{-\lambda n}\). We conclude that \(T\) is bounded by summing over \(n \geq 0\).

We leave it to the reader to obtain the other boundedness properties (taking other exponents \(p, q, r\)) according to the geometrical assumptions of the curve \(\Gamma\).

10.2 Singular symbols. Proceeding pretty much as in Section 10.1, we obtain the following result.

**Theorem 10.4.** Let \(\Phi\) be a smooth compactly supported function, \(\Gamma\) a smooth curve, and let \(m(\eta, \xi) = \Phi(\eta, \xi) \text{dist}((\eta, \xi), \Gamma)^{-\alpha}\) with \(\alpha > 0\). Suppose that \(2 < p, q, r < \infty\) and \(\alpha < 1/p + 1/q + 1/r - 1\). Then \(B_m\) is bounded from \(L^p \times L^q\) into \(L^{r'}\).

We could, of course, obtain corresponding statements for the whole range of exponents \((p, q, r)\), with conditions on \(\alpha\) depending on the properties of \(\Gamma\). We chose to present only the case \(p, q, r > 2\) for the sake of simplicity.

10.3 Bilinear oscillatory integral near a singular domain. Let \(\phi : \mathbb{R}^2 \to \mathbb{R}\) be smooth and let \(m\) be a smooth, compactly supported symbol. Consider the bilinear oscillatory integral
\[
B_t(f, g)(x) := \int_0^t \int_{\mathbb{R}^2} e^{i\phi(\xi, \eta)} f(s, \cdot)(\xi) \overline{g(s, \cdot)(\eta)} m(\xi, \eta) d\xi d\eta ds,
\]
where \(f\) and \(g\) are functions on \(\mathbb{R}^+ \times \mathbb{R}\). Some multilinear integrals appear in the space-time resonances method, as explained in [10, 2].

Integration over \(s\) gives
\[
\int_0^t e^{i\phi(\xi, \eta)} ds = \begin{cases} (e^{i\phi(\xi, \eta)} - 1)/\phi(\xi, \eta) & \text{if } \phi(\xi, \eta) \neq 0, \\ t & \text{if } \phi(\xi, \eta) = 0. \end{cases}
\]
So let us consider the singular set \(S := \phi^{-1}(0) := \{(\xi, \eta), \phi(\xi, \eta) = 0\}\) and assume that \(\nabla\phi\) is not vanishing on \(\Gamma\), so that \(S\) is a smooth submanifold of dimension 1.

**Assumption.** Let us assume that for some exponents \(p, q, r\), there exists \(\rho \in (0, 1]\) such that
\[
\|T_{m_\epsilon}(f, g)\|_{L^{r'}} \lesssim \epsilon^\rho \|f\|_{L^p} \|g\|_{L^q}
\]
for small enough parameter \(\epsilon\) if \(m_\epsilon\) is a symbol in \(\mathcal{M}_\epsilon^\rho\) or \(\mathcal{N}_\epsilon^\rho\).
We then have the following result. In it and what follows, \( L_T^\infty \) denotes the space of \( L^\infty \) functions in the variable \( T \).

**Proposition 10.5.** Assume that the smooth symbol \( m \) is supported on \( S_\varepsilon \) for \( \varepsilon \leq 1 \). If \( T \varepsilon^p \lesssim 1 \), the operator \( B_t \) is uniformly bounded (with respect to \( \varepsilon \) and \( T \)) from \( L_T^\infty L^p \times L_T^\infty L^q \) into \( L_T^\infty L^r \).

**Proof.** Using a partition of unity associated to \( S_\varepsilon \) and covering \( S_\varepsilon \) as in the proof of 10.1 (for \( 2^{-n} \leq \varepsilon \)), define for \( s \in [0, T] \) \( B_s^\varepsilon \) the bilinear multiplier with symbol \( \sigma_n = e^{is\phi(\xi, \eta)} \chi_n(\xi, \eta)m(\xi, \eta) \), satisfying \( \|D^a \sigma_n\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^n|\alpha| \) for each multi-index \( \alpha \). We leave it to the reader check that \( \|D^a \sigma_n\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^n|\alpha| \) for each multi-index \( \alpha \). Indeed, differentiation may make quantities appear bound by \( s^b 2^{nc} \) for \( b + c \leq |\alpha| \); in this case, we use the fact that \( s \leq T \leq \varepsilon^{-p} \lesssim 2^{-np} \lesssim 2^{-n} \), since \( \rho \leq 1 \). So by assumption, we know that \( B_s^\varepsilon : L^p \times L^q \rightarrow L^r \) is bounded with

\[
\|B_s^\varepsilon(f(s, \cdot), g(s, \cdot))\|_{L^r} \lesssim 2^{-\rho n} \|f(s, \cdot)\|_{L^p} \|g(s, \cdot)\|_{L^q}
\]

We can then sum over \( n \geq 0 \) with \( 2^{-n} \lesssim \varepsilon \) to get

\[
\|B_t(f, g)\|_{L^r} \lesssim \int_0^T \left\| \int_{\mathbb{R}^2} e^{is\phi(\xi, \eta)} \hat{f}(s, \cdot)(\xi) \hat{g}(s, \cdot)(\eta)m(\xi, \eta) d\xi d\eta \right\|_{L^r} ds
\]

for all \( t \in [0, T] \). Since this estimate is uniform with respect to \( t \in [0, T] \), the proof is completed by taking the supremum over \( t \).

**Corollary 10.6.** The bilinear multipliers

\[
(f, g) \rightarrow \int_0^t \int_{|\phi| \leq \varepsilon} e^{is\phi(\xi, \eta)} \hat{f}(s, \cdot)(\xi) \hat{g}(s, \cdot)(\eta)m(\xi, \eta) d\xi d\eta ds
\]

are uniformly bounded from \( L_T^\infty L^p \times L_T^\infty L^q \) into \( L_T^\infty L^r \) if \( T \varepsilon^p \lesssim 1 \).

In particular, for time-independent functions \( f, g \), the bilinear multipliers

\[
(f, g) \rightarrow \int_0^t \int_{|\phi| \leq \varepsilon} e^{is\phi(\xi, \eta)} \hat{f}(\xi) \hat{g}(\eta)m(\xi, \eta) d\xi d\eta ds
\]

are uniformly bounded from \( L^p \times L^q \) into \( L^r \) if \( T \varepsilon^p \lesssim 1 \).

**Example 1.** We conclude with an example in the linear theory showing that under assumption (10.4), we cannot improve on Corollary 10.6. Consider the function \( \phi(\xi) = |\xi|^2 \) and corresponding linear operator

\[
T_{\varepsilon, t} := f \rightarrow \int_0^t \int_{|\phi| \leq \varepsilon} e^{is\phi(\xi)} \hat{f}(\xi)m(\xi) d\xi ds
\]
Let us determine when $T$ remains bounded as $t \to \infty$ and $\epsilon \to 0$. Consider a smooth function $f$. By a change of variables,
\[
T_{\epsilon,t}(f)(x) = \int_{|\eta| \leq \sqrt{\epsilon t}} e^{ix\eta/\sqrt{t}} \frac{1 - e^{i\eta^2}}{\eta^2} t \hat{f}(\eta/\sqrt{t}) m(\xi/\sqrt{t}) \frac{d\eta}{\sqrt{t}}
\]
\[
\simeq_{t \to \infty} t \epsilon \hat{f}(0)m(0).
\]
So the limit can be defined in $L^\infty$ only if $t\epsilon$ is bounded and $f \in L^1$ (to give meaning to $\hat{f}(0)$).

However, let us now see when assumption (10.4) is satisfied in this particular setting. Consider a smooth symbol $m_\epsilon$ at the scale $\epsilon$ of $\phi^{-1}(0) = \{0\}$ and estimate the multiplier
\[
m_\epsilon(D)(f)(x) := \int e^{ix\xi} \hat{f}(\xi)m_\epsilon(\xi) d\xi.
\]
Then,
\[
|m_\epsilon(D)(f)(x)| \lesssim \int |f(y)| \frac{\epsilon}{(1 + \epsilon|x - y|)^M} dy
\]
for large enough integer $M$. Consequently, we get that $m_\epsilon(D) : L^1 \to L^\infty$ is bounded with a bound controlled by $\epsilon$. So, in this case, assumption (10.4) is satisfied for $\rho = 1$ with $L^1 \to L^\infty$. Thus, we cannot expect a better result than the one described in Corollary 10.6.

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