Problems with mean curvature-like operators and three-point boundary conditions

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Abstract

In this paper we study the existence of solutions for a new class of nonlinear differential equations with three-point boundary conditions. Existence of solutions are obtained by using the Leray-Schauder degree.

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1 Introduction

The purpose of this article is to obtain some existence results for nonlinear boundary value problems of the form

\[
\begin{align*}
(\varphi(u'))' &= f(t, u, u') \\
l(u, u') &= 0,
\end{align*}
\]

(1.1)

where \(l(u, u') = 0\) denotes the boundary conditions \(u(0) = u'(0) = u'(T)\) or \(u(0) = u(T) = u'(T)\) on the interval \([0, T]\), \(\varphi : \mathbb{R} \to (-a, a)\) is a homeomorphism such that \(\varphi(0) = 0\), \(f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a continuous function, and \(a\) and \(T\) are positive real numbers. The course, a solution of (1.1) is a function \(u : [0, T] \to \mathbb{R}\) of class \(C^1\), satisfying the boundary conditions, such that \(\varphi(u')\) is continuously differentiable and verifies \((\varphi(u'(t)))' = f(t, u(t), u'(t))\) for all \(t \in [0, T]\).

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Several papers have been recently devoted to the study of nonlinear ordinary differential equations of the form \((1.1)\), where \(l(u, u') = 0\) denotes the periodic, Neumann or Dirichlet boundary conditions. In particular, for \(\varphi(s) = s/\sqrt{1 + s^2}\) and Dirichlet conditions, one can consult \([5, 6, 7, 10]\).

In \([2]\), the authors have studied the problem \((1.1)\), where \(f: [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is a Carathéodory function, \(\varphi: \mathbb{R}^n \to B_1(0) \subset \mathbb{R}^n\), and \(l(u, u') = 0\) denotes the periodic boundary conditions. They obtained the existence of solutions by means of the Leray-Schauder degree theory. The interest in this class of nonlinear operators \(u \mapsto (\varphi(u'))'\) is mainly due to the fact that they include the mean curvature operator

\[
\begin{align*}
    u \mapsto \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).
\end{align*}
\]

In 2006, C. Bereanu and J. Mawhin \([4]\), using the Leray-Schauder degree theory, studied the nonlinear problems of the form

\[
\begin{align*}
    \left\{ \begin{array}{l}
    (\varphi(u'))' = f(t, u, u') \\
    u(0) = 0 = u(T)
    \end{array} \right.
\end{align*}
\]

and

\[
\begin{align*}
    \left\{ \begin{array}{l}
    (\varphi(u'))' = f(t, u, u') \\
    u'(0) = 0 = u'(T),
    \end{array} \right.
\end{align*}
\]

where \(f: [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a continuous function and \(\varphi: \mathbb{R} \to (-a, a)\) is a homeomorphism such that \(\varphi(0) = 0\). They obtained the following existence theorems, respectively.

**Theorem 1.1.** If the function \(f\) satisfies the condition

\[
\exists c > 0 \text{ such that } |f(t, x, y)| \leq c < \frac{a}{2T}, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},
\]

the Dirichlet problem has at least one solution.

**Theorem 1.2.** Let \(f\) be continuous. Assume that \(f\) satisfies the following conditions.

1. There exists \(c \in C\) such that \(\|c^-\|_{L^1} < \frac{a}{2}\) and \(f(t, x, y) \geq c(t)\) for all \((t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}\).

2. There exist \(R > 0\) and \(\epsilon \in \{-1, 1\}\) such that

\[
\epsilon \int_0^T f(t, u(t), u'(t)) dt \begin{cases} > 0 & \text{if } u_m \geq R, \quad \|u'_\infty\| \leq M, \\ < 0 & \text{if } u_M \leq -R, \quad \|u'_\infty\| \leq M, \end{cases}
\]
where \( L = \max \left\{ \varphi^{-1}(2 \| c^- \|_{L^1}), \varphi^{-1}(-2 \| c^- \|_{L^1}) \right\} \). Then the Neumann problem has at least one solution.

Inspired by those results, we study the problems (1.1) by using similar topological methods based upon Leray-Schauder degree. The main contribution of this paper is the extension of some results above cited to a more general type of boundary conditions.

The paper is organized as follows. In Section 2, we establish the notation, terminology, and various lemmas which will be used throughout this paper. Section 3 is devoted to the study of existence of solutions for (1.1) with boundary conditions of type \( u(0) = u'(0) = u'(T) \). In Section 4, for \( u(0) = u(T) = u'(T) \) boundary conditions, we investigate the existence of at least one solution for (1.1). Such problems do not seem to have been studied in the literature. In the present paper generally we follow the ideas of Bereanu and Mawhin [1, 4].

2 Notation and preliminaries

We first introduce some notation. For fixed \( T \), we denote the usual norm in \( L^1 = L^1([0, T], \mathbb{R}) \) for \( \| \cdot \|_{L^1} \). Let \( C = C([0, T], \mathbb{R}) \) denote the Banach space of continuous functions from \([0, T]\) into \( \mathbb{R} \), endowed with the uniform norm \( \| \cdot \|_{\infty} \), \( C^1 = C^1([0, T], \mathbb{R}) \) denote the Banach space of continuously differentiable functions from \([0, T]\) into \( \mathbb{R} \), equipped with the usual norm \( \| u \|_1 = \| u \|_{\infty} + \| u' \|_{\infty} \), and \( B_\rho(0) \) the open ball of center 0 and radius \( \rho \) in any normed space.

We introduce the following applications:

the Nemytskii operator \( N_f : C^1 \to C \),

\[
N_f(u)(t) = f(t, u(t), u'(t)),
\]

the integration operator \( H : C \to C^1 \),

\[
H(u)(t) = \int_0^t u(s) ds,
\]

the following continuous linear applications:

\[
K : C \to C^1, \quad K(u)(t) = -\int_t^T u(s) ds,
\]

\[
Q : C \to C, \quad Q(u)(t) = \frac{1}{T} \int_0^T u(s) ds,
\]

\[
S : C \to C, \quad S(u)(t) = u(T),
\]

\[
P : C \to C, \quad P(u)(t) = u(0).
\]
For $u \in C$, we write

$$u_m = \min_{[0,T]} u, \quad u_M = \max_{[0,T]} u, \quad u^+ = \max \{ u, 0 \}, \quad u^- = \max \{ -u, 0 \}.$$ 

For the convenience of the reader we recall some results, which will be crucial in the proofs of our results. The following results are taken from [8] (see also [3, 11], respectively). The first one is needed in the construction of the equivalent fixed point problem.

**Lemma 2.1.** Let $B = \{ h \in C : \| h \|_\infty < a/2 \}$. For each $h \in B$, there exists a unique $Q_\varphi = Q_\varphi(h) \in \text{Im}(h)$ (where $\text{Im}(h)$ denotes the range of $h$) such that

$$\int_0^T \varphi^{-1}(h(t) - Q_\varphi(h)) \, dt = 0.$$ 

Moreover, the function $Q_\varphi : B \rightarrow \mathbb{R}$ is continuous and sends bounded sets into bounded sets.

The second one is an extension of the homotopy invariance property for Leray-Schauder degree.

**Lemma 2.2.** Let $X$ be a real Banach space, $V \subset [0,1] \times X$ be an open, bounded set and $M$ be a completely continuous operator on $V$ such that $x \neq M(\lambda, x)$ for each $(\lambda, x) \in \partial V$. Then the Leray-Schauder degree

$$\deg_{LS}(I - M(\lambda, \cdot), V_\lambda, 0)$$

is well defined and independent of $\lambda$ in $[0,1]$, where $V_\lambda$ is the open, bounded (possibly empty) set defined by $V_\lambda = \{ x \in X : (\lambda, x) \in V \}$.

## 3 Problems with bounded homeomorphisms

In this section we are interested in boundary value problems of the type

$$\begin{cases} (\varphi(u'))' = f(t, u, u') \\ u(0) = u'(0) = u'(T), \end{cases}$$

where $\varphi : \mathbb{R} \rightarrow (-a, a)$ is a homeomorphism, $\varphi(0) = 0$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. In order to apply Leray-Schauder degree theory to show the existence of at least one solution of (3.2), we consider for $\lambda \in [0,1]$, the family of boundary value problems

$$\begin{cases} (\varphi(u'))' = \lambda N_f(u) + (1 - \lambda)Q(N_f(u)) \\ u(0) = u'(0) = u'(T). \end{cases}$$

Notice that (3.3) coincide, for $\lambda = 1$, with (3.2). Now, we introduce the set
\[ \Omega = \{(\lambda, u) \in [0, 1] \times C^1 : \|\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u))\|_\infty < a\}, \]

where clearly \( \Omega \) is an open set in \([0, 1] \times C^1\), and is nonempty because \( \{0\} \times C^1 \subset \Omega \).

Introduce also the operator \( M : \Omega \to C^1 \) defined by
\[
M(\lambda, u) = P(u) + Q(N_f(u)) + H(\varphi^{-1}(\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u)))) \quad (3.4)
\]

Here \( \varphi^{-1} \) with an abuse of notation is understood as the operator \( \varphi^{-1} : B_{a}(0) \subset C \to C \) defined by \( \varphi^{-1}(v)(t) = \varphi^{-1}(v(t)) \). The symbol \( B_{a}(0) \) denoting the open ball of center 0 and radius \( a \) in \( C \). It is clear that \( \varphi^{-1} \) is continuous and sends bounded sets into bounded sets.

When the boundary conditions are periodic or Neumann, an operator has been considered by Bereanu and Mawhin \( [4] \).

The following lemma plays a pivotal role to study the solutions of the problem \( (3.3) \).

**Lemma 3.1.** The operator \( M : \Omega \to C^1 \) is well defined and continuous. Moreover, if \( (\lambda, u) \in \Omega \) is such that \( M(\lambda, u) = u \), then \( u \) is solution of \( (3.3) \).

**Proof.** Let \( (\lambda, u) \in \Omega \). We show that in fact \( M(\lambda, u) \in C^1 \). The continuity of \( M(\lambda, u) \) is a straightforward consequence of the fact that this map is a composition of continuous maps. In addition
\[
(M(\lambda, u))' = \varphi^{-1}(\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u))).
\]

That is, \( (M(\lambda, u))' \) is a composition of continuous operators and thus \( M(\lambda, u) \in C^1 \).

The continuity of \( M \) follows by the continuity of the operators which compose it \( M \).

Now suppose that \( (\lambda, u) \in \Omega \) is such that \( M(\lambda, u) = u \). It follows from \( (3.4) \) that
\[
u(t) = u(0) + Q(N_f(u))(t) + H(\varphi^{-1}(\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u))))(t) \quad (3.5)
\]
for all \( t \in [0, T] \). Then, taking \( t = 0 \) we get
\[
Q(N_f(u)) = 0. \quad (3.6)
\]

Differentiating \( (3.5) \), we obtain that
\[
u'(t) = \varphi^{-1}(\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u)))(t)
= \varphi^{-1}(\lambda H(N_f(u) - Q(N_f(u)))(t) + \varphi(u(0))).
\]

In particular,
\[
u(0) = u'(0) = u'(T).
\]
Applying $\varphi$ to both of its members, differentiating again and using (3.6), we deduce that

$$(\varphi(u'(t)))' = \lambda N_f(u) + (1 - \lambda)Q(N_f(u))(t)$$

for all $t \in [0, T]$. Thus, $u$ satisfies problem (3.3). This completes the proof. \qed

**Remark 3.2.** Note that for $\lambda \in [0, 1]$, if $u$ is a solution of (3.3), then $Q(N_f(u)) = 0$.

The following lemma gives a priori bounds for the possible fixed points of $M$.

**Lemma 3.3.** Assume that $f$ satisfies the following conditions.

1. There exists $M_1 < M_2$ such that for all $u \in C^1$,
   $$\int_0^T f(t, u(t), u'(t))dt \neq 0 \quad \text{if} \quad u'_m \geq M_2,$$
   $$\int_0^T f(t, u(t), u'(t))dt \neq 0 \quad \text{if} \quad u'_M \leq M_1.$$  

2. There exists $c \in C$ such that
   $$f(t, x, y) \geq c(t) \quad \text{and} \quad L + 2\|c^-\|_{L^1} < a$$
   for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ and $L = \max \{|\varphi(M_2)|, |\varphi(M_1)|\}$.

If $(\lambda, u) \in \Omega$ is such that $u = M(\lambda, u)$, then

$$\|\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u))\|_{\infty} < L + 2\|c^-\|_{L^1} \quad \text{and} \quad \|u\|_1 < r(2 + T),$$

where

$$r = \max \{|\varphi^{-1}(L + 2\|c^-\|_{L^1})|, |\varphi^{-1}(-L - 2\|c^-\|_{L^1})|\}.$$

**Proof.** Let $(\lambda, u) \in \Omega$ be such that $u = M(\lambda, u)$. Using Lemma 3.1, $u$ is a solution of problem (3.3), then

$$Q(N_f(u)) = 0,$$  \hspace{1cm} (3.7)

and thus $u$ solves the problem

$$(\varphi(u'))' = \lambda N_f(u), \quad u(0) = u'(0) = u'(T).$$

Hence

$$\varphi(u'(t)) = \lambda H(N_f(u) - Q(N_f(u)))(t) + \varphi(u(0)) \quad (t \in [0, T]).$$  \hspace{1cm} (3.8)

On the other hand using hypothesis 1, we have that
\[ u'_m < M_2 \quad \text{and} \quad u'_M > M_1. \]

It follows that there exists \( \omega \in [0, T] \) such that \( M_1 < u'(\omega) < M_2 \) and

\[
\int_\omega^t (\varphi(u'(s)))' \, ds = \lambda \int_\omega^t N_f(u(s)) \, ds,
\]

which implies that

\[
|\varphi(u'(t))| \leq |\varphi(u'(\omega))| + \int_0^T |f(s, u(s), u'(s))| \, ds,
\]

where

\[
\int_0^T |f(s, u(s), u'(s))| \, ds \leq \int_0^T f(s, u(s), u'(s)) \, ds + 2 \int_0^T c^-(s) \, ds.
\]

Hence

\[
|\varphi(u'(t))| < L + 2 \|c^-\|_{L^1},
\]

where \( L = \max \{|\varphi(M_2)|, |\varphi(M_1)|\} \) for \( t \in [0, T] \). Using the equality above (3.8), we have

\[
\|\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u))\|_{\infty} < L + 2 \|c^-\|_{L^1} < a,
\]

which provides

\[
\|u'\|_{\infty} < r,
\]

where \( r = \max \{|\varphi^{-1}(L + 2 \|c^-\|_{L^1})|, |\varphi^{-1}(-L - 2 \|c^-\|_{L^1})|\} \). Because \( u \in C^1 \) is such that \( u'(0) = u(0) \) we have that

\[
|u(t)| \leq |u(0)| + \int_0^T |u'(s)| \, ds < r + rT \quad (t \in [0, T]).
\]

So, we obtain that \( \|u\|_1 = \|u\|_{\infty} + \|u'\|_{\infty} < r + rT + r = r(2 + T) \). This completes the proof of Lemma 3.3. \( \square \)

Let \( \rho, \kappa \in \mathbb{R} \) be such that \( L + 2 \|c^-\|_{L^1} < \kappa < a \), \( \rho > r(2 + T) \) and consider the set

\[ V = \{ (\lambda, u) \in [0, 1] \times C^1 : \|\lambda H(N_f(u) - Q(N_f(u))) + \varphi(P(u))\|_{\infty} < \kappa, \|u\|_1 < \rho \}. \]

Since the set \( \{0\} \times \{u \in C^1 : \|u\|_1 < \rho, \|\varphi(P(u))\|_{\infty} < \kappa \} \subseteq V \), then we deduce that \( V \) is nonempty. Moreover, it is clear that \( V \) is open and bounded in \([0, 1] \times C^1\) and \( V \subset \Omega \). On the other hand using an argument similar to the one introduced in the proof of Lemma 3.1, it is not difficult to see that \( M : V \to C^1 \) is well defined and continuous. Furthermore, using Lemma 3.3, we have that

\[ u \neq M(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in \partial V. \]

**Lemma 3.4.** The operator \( M : V \to C^1 \) is completely continuous.
Let in addition (3.9), we have that there exists a constant $M$ of $\delta > 0$ such that
\[
\|(\lambda, u)\| = \max \{\|\lambda\|, \|u\|_1\} \leq \delta.
\] (3.9)

Let us show that $\lambda_n H(Nf(u_n)) - Q(Nf(u_n))) + \varphi(P(u_n)))$ is bounded in $C$. Moreover, for any $t, t_1 \in [0, T]$ and for all $n \in \mathbb{N}$ we have
\[
|\lambda_n H(Nf(u_n)) - Q(Nf(u_n)))| f(t, u(t)) dt = \int_0^t f(s, u(s)) ds - \int_0^t Q(Nf(u_n))(s) ds
\]
\[
\leq |t - t_1| \|Nf(u_n)\|_\infty + |t - t_1| \|Q(Nf(u_n))\|_\infty
\]
\[
\leq L_1 |t - t_1| + L_1 |t - t_1|
\]
\[
\leq 2L_1 |t - t_1|
\]
which implies that $(\lambda_n H(Nf(u_n)) - Q(Nf(u_n))) + \varphi(P(u_n)))$ is equicontinuous. Thus, by the Arzelà-Ascoli theorem there is a subsequence of $(\lambda_n H(Nf(u_n)) - Q(Nf(u_n))) + \varphi(P(u_n)))$ which we call $(\lambda_n H(Nf(u_j)) - Q(Nf(u_j))) + \varphi(P(u_j)))$ which is convergent in $C$. Using that $\varphi^{-1} : B_1(0) \subset C \mapsto C$ is continuous it follows from
\[
(M(\lambda_{n_j}, u_{n_j}))' = \varphi^{-1} [\lambda_n H(Nf(u_j)) - Q(Nf(u_j)) + \varphi(P(u_j))]
\]
that the sequence $((M(\lambda_{n_j}, u_{n_j}))')_j$ is convergent in $C$. Then, passing to a subsequence if necessary, we obtain that $(v_{n_j})_j = (M(\lambda_{n_j}, u_{n_j}))_j$ is convergent in $C^1$. Finally, let $(v_n)_n$ be a sequence in $M(\Lambda)$. Let $(z_n)_n \subset M(\Lambda)$ be such that
\[
\lim_{n \to \infty} \|z_n - v_n\|_1 = 0.
\]

Let in addition $(z_{n_j})_j$ be a subsequence of $(z_n)_n$ that converges to $z$. Therefore, $z \in M(\Lambda)$ and $(v_{n_j})_j$ converge to $z$. This concludes the proof.

\[\square\]

### 3.1 Main result

In this subsection, we present and prove an existence theorem for (3.2). We denote by $deg_B$ the Brouwer degree and for $deg_{LS}$ the Leray-Schauder degree, and define the mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ by
\[ G : \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (-\frac{1}{T} \int_0^T f(t, x + yt, y) dt, y - x). \]

**Theorem 3.5.** Let \( f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be continuous and satisfy condition (1) and (2) of Lemma 3.3. Then, for all \( \rho > r(2 + T) \) and for all \( \kappa \in \mathbb{R} \) with \( L + 2 \| c^- \|_{L^1} < \kappa < a \), we have

\[
\text{deg}_{LS}(I - M(1,\cdot), V_1, 0) = \text{deg}_B(G, \Delta, 0),
\]

where \( V_1 \) is the open, bounded set defined by \( V_1 = \{ u \in C^1 : (1,u) \in V \} \) and \( \Delta = B_\rho(0) \cap \mathbb{R}^2 \cap \{(x,y) \in \mathbb{R}^2 : |\varphi(x)| < \kappa \} \). If furthermore

\[
\text{deg}_B(G, \Delta, 0) \neq 0,
\]

problem (3.2) has at least one solution.

**Proof.** Let \( M \) be the operator given by (3.4). Using Lemma 2.2, we deduce that

\[
\text{deg}_{LS}(I - M(0,\cdot), V_0, 0) = \text{deg}_{LS}(I - M(1,\cdot), V_1, 0),
\]

On the other hand, we have that

\[
\text{deg}_{LS}(I - M(0,\cdot), V_0, 0) = \text{deg}_{LS}(I - (P + Q N_f + HP), V_0, 0).
\]

But the range of the mapping

\[
u \mapsto P(u) + Q(N_f(u)) + H(P(u))
\]

is contained in the subspace of related functions, isomorphic to \( \mathbb{R}^2 \). Using homotopy invariance and reduction properties of Leray-Schauder degree [9], we obtain

\[
\text{deg}_{LS}(I - (P + Q N_f + HP), V_0, 0) = \text{deg}_B \left( I - (P + Q N_f + HP) \bigg| \Delta, \Delta \right) = \text{deg}_B(G, \Delta, 0) \neq 0.
\]

Then, \( \text{deg}_{LS}(I - M(1,\cdot), V_1, 0) \neq 0 \). This implies that, there exists \( u \in V_1 \) such that \( M(1,u) = u \), which is a solution for (3.2). \( \square \)

**Remark 3.6.** Using the family of boundary value problems

\[
\begin{cases}
(\varphi(u'))' = \lambda N_f(u) + (1 - \lambda)Q(N_f(u)) \\
u(T) = u'(0) = u'(T)
\end{cases}
\]

which gives the completely continuous homotopy \( \widetilde{M} \) defined by

\[
\widetilde{M}(\lambda, u) = S(u) + Q(N_f(u)) + K(\varphi^{-1} [\lambda H(N_f(u) - Q(N_f(u)) + \varphi(S(u))]),
\]

9
and similar a priori bounds as in the Lemma 3.3, it is not difficult to see that (3.10) has a solution for \( \lambda = 1 \).

Let us give now an application of Theorem 3.5.

**Example 3.7.** Consider the boundary value problem

\[
\begin{cases}
\left( \frac{u'}{1 + u'^2} \right)' = e^{4u'} - e \\
u(0) = u'(0) = u'(T).
\end{cases}
\] (3.11)

Let \( u \in C^1, M_1 = 0 \) and \( M_2 = \frac{1}{2} \). If we suppose that \( u'' \geq M_2 \) and \( u' \leq M_1 \), then

\[
\int_0^T (e^{4u'(t)} - e) dt \geq (e^2 - e) T > 0, \quad \int_0^T (e^{4u'(t)} - e) dt \leq (1 - e) T < 0.
\]

Let \( c(t) = -3 \) for all \( t \in [0, T] \), and let \( L = \frac{1}{\sqrt{5}} \). If \( L + 6T < \kappa = 0, 9 < 1 \) and \( \rho > r(2 + T) = \frac{L + 6T}{\sqrt{1 - (L + 6T)^2}} (2 + T) \), then the equation

\[
G(x, y) = \left( -\frac{1}{T} \int_0^T f(t, x + yt, y) dt, y - x \right) = (0, 0)
\]

\[
= \left( -\frac{1}{T} \int_0^T (e^{4y} - e) dt, y - x \right) = (0, 0)
\]

\[
= (e^{4y} + e, y - x) = (0, 0)
\]

has no solution on \( \partial \Delta \), and hence the Brouwer degree \( \deg_B(G, \Delta, (0, 0)) \) is well defined. So, using the properties of the Brouwer degree, we have

\[
\deg_B(G, \Delta, (0, 0)) = \sum_{(x, y) \in G^{-1}(0, 0)} \text{sgn} J_G(x, y) \neq 0,
\]

where \( (0, 0) \) is a regular value of \( G \) and \( J_G(x, y) = \det G'(x, y) \) is the Jacobian of \( G \) at \( (x, y) \). Therefore, the problem (3.11) has at least one solution.

**4 Existence results for problems with bounded homeomorphisms**

In this section we study the existence of at least one solution for nonlinear problems of the form

\[
\begin{cases}
(\varphi(u'))' = f(t, u, u') \\
u(T) = u(0) = u'(T),
\end{cases}
\] (4.12)
where $\varphi : \mathbb{R} \to (-a, a)$ is a homeomorphism, $\varphi(0) = 0$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(t, x, y)| \leq c < \frac{a}{2T} \quad \text{for all } (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}. \quad (4.13)$$

Now, using Lemma 2.1 and (4.13) we introduce the operator $M_1 : C^1 \to C^1$ defined by

$$M_1(u) = \varphi^{-1}(-Q\varphi(K(N_f(u)))) + H \left( \varphi^{-1} [K(N_f(u)) - Q\varphi(K(N_f(u)))] \right).$$

The following results are taken from [8].

**Lemma 4.1.** If $u \in C^1$ is such that $u = M_1(u)$, then $u$ is a solution of (4.12).

**Lemma 4.2.** The operator $M_1 : C^1 \to C^1$ is completely continuous.

In order to apply Leray-Schauder degree to the fixed point operator $M_1$, we introduce, for $\lambda \in [0, 1]$, the family of boundary value problems

$$\begin{cases}
(\varphi(u'))' = \lambda f(t, u, u') \\
u(T) = u(0) = u'(T).
\end{cases} \quad (4.14)$$

Notice that (4.14) coincide with (4.12) for $\lambda = 1$. For each $\lambda \in [0, 1]$, we can define on $C^1$ the operator $M(\lambda, \cdot)$, where $M$ is defined on $[0, 1] \times C^1$ by

$$M(\lambda, u) = \varphi^{-1}(-Q\varphi(\lambda K(N_f(u)))) + H \left( \varphi^{-1} [\lambda K(N_f(u)) - Q\varphi(\lambda K(N_f(u)))] \right). \quad (4.15)$$

Using the Arzelà-Ascoli theorem it is not difficult to see that $M$ is completely continuous.

**Lemma 4.3.** If $(\lambda, u) \in [0, 1] \times C^1$ is such that $u = M(\lambda, u)$, then $u$ is a a solution of (4.14).

**Proof.** Let $(\lambda, u) \in [0, 1] \times C^1$ be such that $u = M(\lambda, u)$. Then

$$u(t) = \varphi^{-1}(-Q\varphi(\lambda K(N_f(u)))) + H \left( \varphi^{-1} [\lambda K(N_f(u)) - Q\varphi(\lambda K(N_f(u)))] \right) (t) \quad (4.16)$$

for all $t \in [0, T]$. Since $\int_0^T \varphi^{-1} [\lambda K(N_f(u))(t) - Q\varphi(\lambda K(N_f(u)))] \, dt = 0$, therefore, we have that $u(0) = u(T)$. Differentiating (4.16), we obtain that

$$u'(t) = \varphi^{-1} [\lambda K(N_f(u)) - Q\varphi(\lambda K(N_f(u)))] (t).$$

In particular,

$$u'(T) = \varphi^{-1}(0 - Q\varphi(\lambda K(N_f(u)))) = \varphi^{-1}(-Q\varphi(\lambda K(N_f(u)))) = u(0).$$

11
Applying $\varphi$ to both members and differentiating again, we deduce that
\[
(\varphi(u'(t)))' = \lambda N_f(u)(t), \quad u(0) = u(T), \quad u'(0) = u'(T)
\]
for all $t \in [0, T]$. This completes the proof. \hfill \Box

**Remark 4.4.** Note that the reciprocal of Lemma 4.3 is true because we can guarantee that $\|\lambda K(N_f(u))\|_\infty < a/2$ for every solution $u$ of (4.14).

Now we show the existence of at least one solution for problem (4.12) by means of Leray-Schauder degree. This result is inspired from works by Bereanu and Mawhin [4].

**Theorem 4.5.** Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. If $f$ satisfies the condition (4.13), then the problem (4.12) has at least one solution.

**Proof.** Let $(\lambda, u) \in [0, 1] \times C^1$ be such that $u = M(\lambda, u)$. Then
\[
u(t) = \varphi^{-1}(-Q\varphi(\lambda K(N_f(u)))) + H(\varphi^{-1}[\lambda K(N_f(u)) - Q\varphi(\lambda K(N_f(u)))])(t)
\]
for all $t \in [0, T]$. Differentiating, we obtain that
\[
u'(t) = \varphi^{-1}[\lambda K(N_f(u)) - Q\varphi(\lambda K(N_f(u)))](t)
\]
Applying $\varphi$ to both of its members we have that
\[
\varphi(u'(t)) = \lambda K(N_f(u))(t) - Q\varphi(\lambda K(N_f(u))) \quad (t \in [0, T]).
\]
Using the fact that $f$ is bounded, we deduce the elementary inequality
\[
\|\varphi(u')\|_\infty \leq 2cT < a.
\]
Hence,
\[
\|u'\|_\infty \leq L,
\]
where $L = \max \{|\varphi^{-1}(2cT)|, |\varphi^{-1}(-2cT)|\}$. Because $u \in C^1$ is such that $u(0) = u'(T)$ we have that
\[
|u(t)| \leq |u(0)| + \int_0^T |u'(s)| ds \leq L + LT \quad (t \in [0, T]),
\]
and hence
\[
\|u\|_1 = \|u\|_\infty + \|u'\|_\infty \leq L + LT + L = L(2 + T).
\]
Let $M$ be operator given by (4.15) and let $\rho > L(2 + T)$. Using the homotopy invariance of the Leray-Schauder degree, we have
\[
\deg_{LS}(I - M(0, \cdot), B_\rho(0), 0) = \deg_{LS}(I - M(1, \cdot), B_\rho(0), 0).
\]

12
On the other hand, we have that
\[ \text{deg}_{LS}(I - M(0, \cdot), B_\rho(0), 0) = \text{deg}_{LS}(I, B_\rho(0), 0) = 1. \]
Then, from the existence property of Leray-Schauder degree [9], there exists \( u \in B_\rho(0) \) such that \( u = M(1, u) \), which is a solution for (4.12).

**Example 4.6.** Consider the boundary value problem
\[
\begin{align*}
\left( \frac{u'}{\sqrt{1+u'^2}} \right)' &= \beta \cos u \\
u(0) &= u(T) = u'(T).
\end{align*}
\] (4.17)

So, we can choose \( \beta < \frac{1}{2\pi} \) to see Theorem 4.5 holds and so the problem 4.17 has at least one solution.

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