Stochastic Quantization and AdS/CFT

Diego S. Mansi\textsuperscript{1} and Andrea Mauri\textsuperscript{2}

\textit{Dipartimento di Fisica Teorica, Universita degli Studi di Milano}
\textit{Via Celoria, Milano 20133, Italy}

Anastasios C. Petkou\textsuperscript{3}

\textit{Department of Physics, University of Crete, Heraklion 71003, Greece}

Abstract

We argue that there is a relationship between stochastic quantization and AdS/CFT, and we present an explicit calculation to support our claim. In particular, we show that a conformally coupled scalar with $\phi^4$ interaction on AdS\textsuperscript{4} is related, via stochastic quantization as well as via AdS/CFT, to a massless scalar with $\phi^6$ interaction in 3d. We show that our results have an underlying geometric origin, which might help to elucidate further the proposed relationship between stochastic quantization and holography.

1 Introduction

The possibility that stochastic quantization is related to AdS/CFT has been discussed before (e.g. \cite{1,2}), however the discussion has not picked up momentum mainly due to the absence of an explicit example. It is not hard to anticipate such a relationship. In the stochastic quantization scheme of Parisi & Wu (\cite{3}) the correlation functions of an Euclidean $d$-dimensional field theory arise as equilibrium configurations, for large “fictitious” times, of the corresponding correlation functions of a $d+1$-dimensional field theory described by a Fokker-Planck action. On the other hand, in AdS/CFT the generating functional for connected correlation functions of a $d$-dimensional field theory arises as the appropriately renormalized on-shell action of a $d + 1$-dimensional gravitational theory. Therefore, a connection could be established if the stochastically quantized action is somehow related to the boundary action of AdS/CFT and if the Fokker-Planck and the holographic bulk actions are also related. Clearly, we also need to relate the stochastic ”time” to the holographic direction.\textsuperscript{4} Then, such a relationship would imply a profound connection between stochastic processes and gravitation.

\textsuperscript{1}diego.mansi@mi.infn.it
\textsuperscript{2}andrea.mauri@mi.infn.it
\textsuperscript{3}petkou@physics.uoc.gr
\textsuperscript{4}Related ideas have recently appeared in the lattice approach to quantum gravity \cite{4}.
In this work we revisit the idea that stochastic quantization is related to AdS/CFT and we present an explicit example to support it. We start by sketching a simple formal correspondence between stochastic quantization and AdS/CFT. Namely, we show that the partition function of stochastic quantization corresponds to an average over holographic partition functions, if we identify the Fokker-Planck and the bulk actions, and also the initial classical action with the holographic boundary effective action. Our explicit example involves a conformally coupled scalar with $\phi^4$ interaction in fixed $\text{AdS}_4$. We show that the leading terms in a large coupling expansion of the holographic effective action of the model, give the 3d action of a massless scalar with $\phi^6$ interaction written, curiously, in an unconventional manner. Next, starting from the latter 3d action we use stochastic quantization to arrive at its corresponding 4d Fokker-Planck action. The leading terms in a large coupling expansion of that Fokker-Planck action give precisely (i.e. including the numerical coefficients), the initial 4d action of a massless scalar with $\phi^4$ interaction. The latter action is actually equivalent to that of a conformally coupled scalar with $\phi^4$ interaction on $\text{AdS}_4$. Hence, our explicit example demonstrates that the above 3d and 4d field theories are related both via AdS/CFT as well as via stochastic quantization. We consider our results as a strong indication that stochastic quantization and AdS/CFT are intimately related. We then discuss the general conditions under which such a relationship might arise focusing on the role of boundary conditions. Finally, we point out that our results above have a geometric origin. Indeed, both the 4d and 3d actions that are involved in our example are merely disguised 4d and 3d gravitational actions for conformally flat metrics. Then, the boundary condition that enables the calculation of the boundary effective action in AdS/CFT arises as the stationarity condition for a system involving bulk and boundary gravity. This observation might help to elucidate further the relationship between stochastic quantization and AdS/CFT in our specific example.

2 Is Stochastic Quantization related to AdS/CFT?

It is not hard to sketch a formal connection between stochastic quantization (see Appendix A for a condensed review) and AdS/CFT. The Boltzmann weight for a $d$-dimensional Euclidean theory of the scalar field $\phi(\vec{x})$ is (we set henceforth $\hbar = 1$)

$$\mathcal{P}[\phi] \equiv \frac{1}{Z_d} e^{-S_{\text{cl}}[\phi]}, \quad \text{with} \quad Z_d = \int [\mathcal{D}\phi] e^{-S_{\text{cl}}[\phi]}, \quad (1)$$

or equivalently $\int [\mathcal{D}\phi] \mathcal{P}[\phi] = 1$. The extended scalar field $\phi(\vec{x}) \mapsto \phi(t, \vec{x})$ satisfies the Langevin equation

$$\frac{\partial \phi(t, \vec{x})}{\partial t} + \kappa \frac{\delta S_{\text{cl}}[\phi]}{\delta \phi(t, \vec{x})} = \eta(t, \vec{x}), \quad (2)$$
where $\kappa$ is a generic kernel. The source $\eta(t, \vec{x})$ is a “white noise” defined by the following partition function and correlation functions

$$\mathcal{Z} = \int [\mathcal{D}\eta] \exp \left[ -\frac{1}{4\kappa} \int_{-T}^{0} dt \int d^d x \eta^2(t, \vec{x}) \right],$$  \hspace{1cm} (3)

$$\langle \eta(t, \vec{x}) \rangle = 0,$$  \hspace{1cm} (4)

$$\langle \eta(t_1, \vec{x}_1)\eta(t_2, \vec{x}_2) \rangle = 2\kappa \delta^d(\vec{x}_1 - \vec{x}_2) \delta(t_1 - t_2).$$  \hspace{1cm} (5)

To make the connection with AdS/CFT we need to depart slightly from the standard stochastic quantization procedure where the fictitious "time" interval is taken to be $t \in [0, T]$. In that case, one fixes the initial field configurations at $t = 0$ and lets the system evolve in $t$. The crucial point is then [3] that at $T \to \infty$ the fields and their equal "time" correlation functions relax to their equilibrium values - the latter being identified with properly quantized configurations.

Here instead we take the fictitious "time" interval to be $t \in [-T, 0]$ such that starting from any finite initial $t = -T$, the fields evolve via the Langevin towards their values at $t = 0$ which we denote as $\phi_{-T}(0, \vec{x})$. Sending then the initial "time" $-T \to -\infty$, the system reaches thermal equilibrium at $t = 0$ i.e. the field configurations at $t = 0$ are properly quantized. Accordingly, we have to impose an initial distribution for the field $\phi(t, \vec{x})$ at $t = -T$ by means e.g. of a delta function as

$$P_{t=-T}[\phi] = \Pi_{x} \left\{ \delta^{d}[\phi(-T, \vec{x})] \right\}. \hspace{1cm} (6)$$

Here we have chosen a vanishing initial configuration having in mind to take a large T limit. The $t=0$ correlation functions, which are evaluated as stochastic averages over the white noise with Boltzmann weight that of (3), relax into those of the $d$-dimensional theory (1), namely

$$\lim_{T \to \infty} \langle \phi_{-T}(0, \vec{x}_1)\phi_{-T}(0, \vec{x}_2)\ldots\phi_{-T}(0, \vec{x}_n) \rangle_\eta = \langle \phi(\vec{x}_1)\phi(\vec{x}_2)\ldots\phi(\vec{x}_n) \rangle_{S_{cl}}. \hspace{1cm} (7)$$

One can turn the stochastic averages over the white noise into "path integrals" over the scalar fields changing variables $\eta \mapsto \phi$. After a straightforward calculation (see e.g. [5]) and with our choice of initial data we get

$$\mathcal{Z} = \int [\mathcal{D}\phi(0)] e^{-S_{cl}[\phi(0)}/2 \int [\mathcal{D}\phi] e^{-S_{FP}}, \hspace{1cm} (8)$$

where the Fokker-Planck action $S_{FP}$ and the "path integral" measure are

$$S_{FP} = \int_{-T}^{0} dt \int d^d \vec{x} \left[ \frac{1}{4\kappa} \dot{\phi}^2 + \frac{\kappa}{4} \left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 - \frac{\kappa}{2} \delta^2 S_{cl} \right], \hspace{1cm} (9)$$

$$[\mathcal{D}\phi] = \prod_{-T < t < 0} [\mathcal{D}\phi(t)]. \hspace{1cm} (10)$$
To obtain (8), the following (formal) result\(^5\) for the determinant was used
\[
\det \left( \frac{\delta \eta}{\delta \phi} \right) = \exp \frac{\kappa}{2} \left[ \int_{-T}^{0} dt \int d^d \vec{x} \frac{\delta^2 S_{cl}[\phi]}{\delta \phi(t, \vec{x})^2} \right].
\] (11)

Notice that starting with a conventional scalar theory \(S_{cl}\), the Fokker-Planck action \(S_{FP}\) is generically non-relativistic. A recent discussion on this property of stochastic quantization has appeared in [7].

Consider next a \(d+1\)-dimensional scalar theory on a fixed asymptotically AdS background. The implementation of holography in such a simple case requires the evaluation of the following partition function using a semiclassical approximation as
\[
Z_{\text{hol}}[\phi_0] = \int [D\phi_0] e^{-S_{d+1}[\phi]} \equiv e^{W_d[\phi_0]},
\] (12)

where \([D\phi_0]_0\) denotes path integration with Dirichlet boundary conditions \(\phi|_{\partial M} = \phi_0\). Barring important regularization questions, which are however well understood, the path integral yields the generating functional \(W_d[\phi_0]\) of connected correlation functions of a composite operator \(O\) with dimension \(\Delta\) in a \(d\)-dimensional (generically conformal) field theory.

The arbitrary boundary conditions \(\phi_0(\vec{x})\) are external sources for the operator \(O\). It is important that the scaling dimension of \(O\) is above the unitarity bound of a \(d\)-dimensional CFT, namely \(\Delta > d/2 - 1\). Hence, a path integral over \(\phi_0\), corresponding to the quantization of \(\phi_0\), will generically produce inconsistencies such as negative probabilities or negative norm states. Nevertheless, there are known cases where \(\phi_0\) is a normalizable mode as well and hence it can correspond to an operator \(\bar{O}\) with dimension \(\bar{\Delta} = d - \Delta > d/2 - 1\). A well-known example is the conformally coupled scalar field in 4-dimensions. In such cases the Euclidean functional \(W_d[\phi_0]\) itself can be used to construct a well defined Boltzmann weight for a \(d\)-dimensional theory. In other words, \(W_d[\phi_0]\) can be interpreted as an effective action i.e. we can write \(W_d[\phi_0] \equiv \Gamma_d[\phi_0]\). Now, the leading term of the effective action \(\Gamma_d[\phi_0]\) is a classical action that we denote as \(I_d[\phi_0]\). Then, it is natural to take a further step and define (see also [8])
\[
Z' = \int [D\phi_0] e^{-I_d[\phi_0]} Z_{\text{hol}}[\phi_0] = \int [D\phi_0] e^{-I_d[\phi_0]} \int [D\phi] e^{-S_{d+1}[\phi]},
\] (13)
as an average, with weight \(I_d[\phi_0]\), of the holographic partition functions.

We notice now a strong formal similarity between (8) and (13) provided we make the following correspondences:
\[
\begin{align*}
\text{S,Q.} & : \text{AdS/CFT} \\
S_{FP}[\phi] & \leftrightarrow S_{d+1}[\phi] \\
S_{cl}[\phi_0] & \leftrightarrow 2I_d[\phi_0] \\
\text{stochastic “time”} & \leftrightarrow \text{holographic direction}.
\end{align*}
\] (14)

\(^5\)The determinant gives rise generically to infinities of the form \(\delta^d(0)\) that - when properly regularized - act as counterterms to some of the divergences that arise in the perturbative expansion [6].
At a first glance, the above formal similarity may appear too optimistic. As we have previously commented, any conventional $d$-dimensional action $S_d$ would lead to a non-relativistic Fokker-Planck action $S_{FP}$. This seems irreconcilable with the standard AdS/CFT dictionary where both the bulk $S_{d+1}$ and the boundary $I_d$ actions are relativistic. Furthermore, the boundary action is always conformal. This means that the presumed relationship between stochastic quantization and AdS/CFT is non-generic. Indeed, the explicit example we present below is special and has a geometric origin. Nevertheless, we do believe that the relationship between stochastic quantization and AdS/CFT even for such special cases could shed light into certain quantum properties of spacetime.

3 From the bulk to the boundary: AdS/CFT

To give precise meaning to the formal correspondence sketched above we consider the model studied in [9, 10] of a conformally coupled scalar with $\phi^4$ interaction on fixed Euclidean $\text{AdS}_4$

$$I = \int d^4x \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right), \quad x^\mu = (r, \vec{x}), \quad (15)$$

where $\ell$ is the radius of AdS, determining the mass scale $m^2 \ell^2 = -2$. The dimensionless coupling $\lambda$ is kept general. Upon introducing Poincaré coordinates and rescaling the field as

$$ds^2 = \frac{\ell^2}{r^2}(dr^2 + d\vec{x}^2), \quad g_{\mu\nu} = \Omega^{-2}(x) \eta_{\mu\nu}, \quad \Omega(x) = \frac{r}{\ell}, \quad \phi = \Omega(x) f, \quad (16)$$

the action becomes

$$I = I_f + I_{\text{div}} = \int_0^\infty dr \int d^3\vec{x} \left( \frac{1}{2} \eta^{\mu\nu} \partial_\mu f \partial_\nu f + \frac{\lambda}{4} f^4 \right) + \int d^3\vec{x} \left. \frac{f^2}{2r} \right|_0^\infty. \quad (17)$$

The last term is divergent and needs to be renormalized by the addition of appropriate counterterms (see e.g. [11]). Hence, this simple model essentially reduces to a massless theory on "half" 4-dimensional flat space. The asymptotic boundary resides at $r = 0, \infty$ and is isomorphic to $S^3$. As usual in the AdS/CFT [12], we remove the point at $r = \infty$ and we are left with a theory living on $\mathbb{R}^3$ (the space at $r = 0$). This is consistent with conformal invariance.\(^6\) Physically sensible boundary conditions must imply this regularity condition, namely the vanishing of the fields at the "horizon" point $r = \infty$ of AdS. The reverse does not hold in general.

For $\lambda > 0$, the equations of motion for the action (17)

$$- \partial^\mu \partial_\mu f + \lambda f^3 = 0, \quad (18)$$

\(^6\)Under the conformal inversion $x^\mu \mapsto \hat{x} x \equiv x^\mu/x^2$, scalar fields with dimension $\Delta > 0$ behave as $\phi(x) \mapsto x^{-2\Delta} \phi(\hat{x} x)$. Hence, the finiteness of fields in the origin is preserved under conformal transformations if the fields vanish at infinity.
possess the following 5-parameter family of solutions with vanishing stress tensor (hence, they remain exact solutions in the presence of gravity [10])

\[
\hat{f}(r, \vec{x}) = k \frac{b}{-b^2 + (r + r_0)^2 + (\vec{x} - \vec{x}_0)^2}, \quad k = \sqrt{\frac{8}{\lambda}}.
\]  

The instantonic nature of this type of solutions requires \( \lambda \) to be finite. The parameter \( b \) determines the instanton size, while \((-r_0, \vec{x}_0)\) may be viewed as the coordinates of the instanton center.\(^7\) The solution is regular for all \( r > 0 \) when \( r_0 > b > 0 \).

The upshot of holography is the calculation of the renormalized on-shell action with given boundary conditions. In the present case, the general solution of (18) behaves near \( r = 0 \) as

\[
f(r, \vec{x}) = \phi_0(\vec{x}) + r \phi_1(\vec{x}) + O(r^2),
\]

where \( \phi_0(\vec{x}) \) and \( \phi_1(\vec{x}) \) are the two arbitrary data necessary to determine the general solution of the 2nd order differential equation (18). It was shown in [9, 10] that evaluating (17) on-shell as a functional of \( \phi_0(\vec{x}) \) yields the 3-dimensional effective action for a composite scalar operator with dimension \( \Delta = 1 \) as

\[
I_{\text{on-shell}}^{\text{on-shell}}[\phi_0] = -\Gamma_d[\phi_0].
\]  

To achieve that we need to impose boundary conditions relating \( \phi_1(\vec{x}) \) to \( \phi_0(\vec{x}) \). Such conditions can be generically expressed as \( \mathcal{F}(\phi_0, \phi_1) = 0 \) for some functions \( \mathcal{F} \) [11]. There is a general method to evaluate the boundary on-shell action with given boundary conditions, which is essentially the Hamilton-Jacobi method for field theory [13]. However, in our simple model we can take a more direct approach making use of the exact solution (19). Consider the Hamiltonian formulation of our model which arises in a standard and simple way from the finite part in the rhs of (17). In this case we have

\[
I_f = \int_0^\infty dr \int d^3\vec{x} [\pi \partial_r f - \mathcal{H}], \quad \mathcal{H} = \frac{1}{2} \left( \pi^2 - \partial^i f \partial_i f - \frac{\lambda}{2} f^4 \right).
\]  

Suppose now that on-shell the following condition holds

\[
\mathcal{H}_{\text{on-shell}}(\pi, f) = \partial_i \mathcal{V}_i(f),
\]  

for some functional \( \mathcal{V}_i(f) \). This would greatly simplify the calculation of the on-shell value of (22) since, from the one hand it implies that the contribution of the Hamiltonian in the on-shell action is a total spatial derivative and hence vanishes, and from the other hand it gives an on-shell relationship between \( \pi \) and \( f \) such that the kinetic term in (22) can be written (at least term-by-term in some expansion) as a total \( r \)-derivative. Generically, the functional \( \mathcal{V}_i(f) \) in (23) can be calculated in a spatial derivative expansion and the coefficients are fixed requiring consistency with the e.o.m. [13]. Here, we use input from the exact solution (19) on which the Hamiltonian density is

\[
\hat{\mathcal{H}} = \frac{1}{2} \left( \hat{\pi}^2 - \partial^i \hat{\dot{f}} \partial_i \hat{\dot{f}} - \frac{\lambda}{2} \hat{f}^4 \right) = -\frac{1}{6} \partial^i \partial_i \hat{f}^2,
\]

\(^7\)Notice that for a solution of (18) to exist for \( \lambda > 0 \) the instanton center must lie outside the half \( \mathbb{R}^4 \).
where the canonical momentum is defined by $\pi = \partial_r f$. Hence, if we are interested in calculating the on-shell action up to two spatial derivatives we could use (24) for generic solutions of the e.o.m.

As mentioned above, (24) is at the same time a boundary condition for the generic solution (20), since it relates $\phi_1(\vec{x})$ to $\phi_0(\vec{x})$. Explicitly the latter relation yields (we use now unhatted variables denoting a general solution of the e.o.m.)

$$
\pi(r, \vec{x}) = \pm \sqrt{\frac{\lambda}{2} f^4 + \partial^i f \partial_i f - \frac{1}{3} \partial^i \partial_i f^2} = \pi_0(\vec{x}) + r \pi_1(\vec{x}) + O(r^2),
$$

(25)

$$
\pi_0(\vec{x}) = \phi_1(\vec{x}) = \pm \sqrt{\frac{\lambda}{2} \phi_0^4 + \partial^i \phi_0 \partial_i \phi_0 - \frac{1}{3} \partial^i \partial_i \phi_0^2}.
$$

(26)

An alternative but revealing way to express the boundary condition (26) is

$$
\pi_0(\vec{x}) = \pm \sqrt{\frac{\lambda}{2} \phi_0^2(\vec{x}) \left[ 1 + \frac{1}{3 \lambda} R_3 \right]} = \pm \sqrt{\frac{\lambda}{2} \phi_0^2(\vec{x}) \left( 1 + \frac{1}{6 \lambda} R_3 + O(\lambda^{-2}) \right)},
$$

(27)

where

$$
R_3 \equiv R_3 \left[ g = \phi_0^2 \eta \right] = -\phi_0^{-2} \left[ 2 \partial_i \ln \phi_0(\vec{x}) \partial_i \ln \phi_0(\vec{x}) + 4 \partial_i \partial_i \ln \phi_0(\vec{x}) \right],
$$

(28)

is the scalar curvature of a conformally flat 3-dimensional metric $g_{ij}(\vec{x}) = \phi_0^2(\vec{x}) \eta_{ij}$. We term (24) Hamiltonian boundary condition since it implies that the Hamiltonian density retains the form it has on an exact solution i.e. on a solution where both $\phi_0$ and $\phi_1$ are completely fixed. Finally, we can substitute these results in (22) and consider a large $\lambda$ expansion to obtain

$$
I_{on-shell}^{on-shell}[\phi_0] = -\Gamma[\phi_0] = -\sqrt{\frac{1}{18 \lambda}} \int d^3 \vec{x} \left( \lambda \phi_0^3 + \frac{1}{\phi_0} \partial^i \phi_0 \partial_i \phi_0 + \ldots \right),
$$

(29)

where the dots denote terms $O(1/\lambda)$. This effective action has the intriguing property that it is a disguised form of a well-known action in 3d. Indeed, defining $\Phi^2 = \frac{8}{\sqrt{18 \lambda}} \phi_0$ we find from (29)

$$
\Gamma[\phi_0] = \Gamma[\Phi] = \int d^3 \vec{x} \left( \frac{1}{2} \partial^i \Phi \partial_i \Phi + g \Phi^6 + \ldots \right),
$$

(30)

and the coupling constant $g = (\frac{32}{3 \lambda})^2$. The terms shown in (29) and (30) are then the classical action of a 3d conformal theory.

### 4 From the boundary to the bulk: stochastic quantization

In the AdS/CFT example above the bulk theory is holographically related to a boundary action written in terms of elementary fields. The description passes through the non-canonical
3d action (29). We will show now that this property can be understood, in the opposite
direction, in terms of stochastic quantization. Namely, we will associate the Fokker-Plank
action to the action of the bulk theory and the boundary effective action to $S_{cl}$. The Hamilton-
ian boundary conditions (24) will play a crucial part in this correspondence. This way
we will provide an explicit realization of the formal correspondence (14) between stochastic
quantization and AdS/CFT.

To implement our idea we apply the stochastic quantization procedure to the following 3d
classically conformal action

$$S_{cl}[\phi] = \frac{2}{\sqrt{18\lambda}} \int d^3x \left( \frac{1}{\phi} \partial^i \phi \partial_i \phi + \lambda \phi^3 \right). \quad (31)$$

Notice that our starting action is two times the holographic effective action we found in (29),
in accordance with our identification in (14). As explained at the end of the previous Section,
this unconventional action is related to a canonical $\phi^6$ model by a simple field redefinition.
It is then straightforward to compute the 4d Fokker-Plank action inserting (31) in (9):

$$\left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 = \frac{2}{9} \left[ 9\lambda \phi^4 - 12 \phi \Box \phi + 6 \partial_i \phi \partial_i \phi + \ldots \right], \quad (32)$$

$$S_{FP}[\phi] = \int dt \int d^3x \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{\lambda}{4} \phi^4 + \ldots \right]. \quad (33)$$

Here we have chosen for definiteness $\kappa = 1/2$ and the dots stand for subleading terms in a
large $\lambda$ limit. Hence we see that the leading terms for large-$\lambda$ in the stochastic quantization
of $S_{cl}[\phi]$ reproduce precisely (i.e. including numerical coefficients!) the bulk action (17) for
the conformally coupled scalar field, if we identify the stochastic ”time” with the holographic
direction $r$.

Now we would like to better understand why this connection takes place trying to give
a unified picture. In order to do so we consider the general Fokker-Plank system (9) in
Hamiltonian formalism

$$I = \int_{-\infty}^0 dt \int d^3x \left[ \dot{\phi} \pi - H_{FP} \right], \quad H_{FP} = \frac{1}{2} \left( \pi^2 - \frac{1}{4} \left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 + \frac{1}{2} \frac{\delta^2 S_{cl}}{\delta \phi^2} \right), \quad (34)$$

where $\pi = \dot{\phi}$ is the standard canonical momentum and again we chose $\kappa = 1/2$. We notice
at first that the theory lives in ”half” 4-dimensional flat space. In section 3 we have shown
that the conformally coupled scalar model on $AdS_4$ can be reduced to a theory on this same
space. Then the 4-dimensional space generated by the addition of the fictitious time direction
in stochastic quantization exactly reproduces the space-time associated to our initial bulk
system. More generally it is possible to think of adding suitable time direction(s) and allow
for more general fictitious time evolutions to reproduce different space-time structures to be
associated to other gravitational systems. Having this in mind, we would like to perform

---

As mentioned before, the term (11) gives rise to $\delta^d(0)$ infinities in field theory and is irrelevant here.
an holographic analysis of the simple FP model in (34), now considered as our initial bulk theory. Therefore we evaluate the variation of the action on-shell:

\[ \delta I^{\text{os.}} = - \int d^3 \bar{x} \delta \phi_0 \pi_0. \] (35)

Now we consider a solution \( \phi(r, \bar{x}) \) of the Fokker-Plank equations of motion for which

\[ \pi(0, \bar{x}) = - \frac{1}{2} \frac{\delta S_{cl}}{\delta \phi}. \] (36)

It is clear from (35) that this selects a particular class of solutions with specific boundary conditions such that we simply have \( I^{\text{os.}} = \frac{1}{2} S_{cl} \). This is the identification we were looking for in (14), with the correct overall coefficient. In this case the holographic generating functional of connected diagrams for the boundary theory is directly related to the stochastically quantized action. In our scalar model, the generating functional can also be interpreted as an effective action. In any case, it is clear that the specific boundary condition (36) is the key element to obtain an exact correspondence between the action \( S_{cl} \) and the boundary effective action of holography.

It is easy to see that the Hamiltonian boundary condition (24) we had to introduce in our scalar example exactly coincides with (36). In fact we have

\[ \frac{1}{4} \left( \frac{\delta S_{cl}}{\delta \phi} \right)^2 = \frac{1}{18 \lambda} \left( 6 \lambda \partial^i \phi \partial_i \phi - 12 \lambda \phi \Box \phi + 9 \lambda^2 \phi^4 + \ldots \right) \equiv \pi(0, \bar{x})^2 = \dot{\phi}^2, \] (37)

and again we had to consider a large \( \lambda \) limit.

One way to understand the boundary condition (36) from the point of view of stochastic quantization is the following. The field configuration is constrained to satisfy the Langevin evolution (2). After infinite time the field will eventually relax to the equilibrium configuration at the boundary \( t = 0 \). At the equilibrium, the role of quantum oscillation is played by the noise average. Taking the average of the Langevin equation we directly read the boundary conditions in (36), which are now valid for the “quantum” configuration of the field. The key role played by the choice of boundary conditions deserves some more analysis and in the next section we will provide a geometrical interpretation for them from the holographic point of view.

5 Geometric interpretation and Outlook

There is a simple, but possible far reaching geometrical origin behind the relationship between the 4-dimensional \( \phi^4 \) theory and the 3-dimensional \( \phi^6 \) theory. Consider the Euclidean

9In the context of AdS/CFT it can be shown that an action such as (34) can arise considering a non-conformally coupled scalar on fixed AdS_4 and taking the non-relativistic limit.
Einstein-Hilbert action in half $\mathbb{R}_4$

$$I_{EH}^{(4)} = -\frac{1}{16\pi G_4} \int_0^\infty dr \int d^3\vec{x} \sqrt{g} (\mathcal{R} - 2\Lambda_4). \quad (38)$$

It is well-known that in order to setup a proper Dirichlet problem for the metric at the boundary $r = 0, \infty$, i.e. in order that the variation of bulk on-shell action vanishes when we fix the metric at the boundary, we have to add to (38) boundary the Gibbons-Hawking term

$$I_{GH}^{(4)} = \frac{1}{8\pi G_4} \int_{\partial M} d^3\vec{x} \sqrt{g} g^{ij} K_{ij}, \quad (39)$$

where $g_{ij}(\vec{x})$ is the restriction of the bulk metric to the boundary and $K_{ij}(\vec{x})$ is the extrinsic curvature. From now on we take all fields to vanish at $r = \infty$, hence the boundary $\partial M = \mathbb{R}_3$ is at $r = 0$. Consider conformally flat metrics $g_{\mu\nu}(x) = \varphi^2(x) \eta_{\mu\nu}$. An explicit calculation gives

$$I_4 = I_{EH}^{(4)} + I_{GH}^{(4)} = -\frac{3}{4\pi} \int d^4x \left( \frac{1}{2} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{\lambda_4}{6} \varphi^4 \right), \quad (40)$$

where we have defined

$$\varphi(x) = \sqrt{G_4} \phi(x), \quad \lambda_4 = G_4 \Lambda_4 = -\frac{3}{2} \lambda. \quad (41)$$

The GH term cancelled exactly the boundary term arising from $I_{EH}^{(4)}$. As we have mentioned before, (40) is equivalent to the 1st-order action (22). Hence, its on-shell variation yields

$$\delta I_{on-shell}^{(4)} = \frac{3}{4\pi} \int_{\mathbb{R}_3} d^3\vec{x} \delta \phi_0(\vec{x}) \pi_0(\vec{x}), \quad (42)$$

where the boundary values of the canonical variables have been defined in Section 3.

However, if we do not wish to fix the boundary metric, the only way to make (40) stationary is to impose (Neumann) boundary conditions on $\pi_0(\vec{x})$ e.g. the above case we should require $\pi_0(\vec{x}) = 0$. This gives us intriguing possibilities, for example by adding the appropriate boundary functionals we can impose boundary conditions satisfied by exact non-perturbative solutions of the bulk equations of motion.

With this in mind we consider extending (40) by just the 3d gravity in the boundary with action

$$I_{EH}^{(3)} = -\frac{1}{16\pi G_3} \int_{\mathbb{R}_3} d^3\vec{x} \sqrt{\hat{g}} (\mathcal{R}_3 - 2\Lambda_3). \quad (43)$$

Since there are no boundary terms now, the variation of (43) gives

$$\delta I_{EH}^{(3)} = -\frac{1}{16\pi G_3} \int d^3\vec{x} \sqrt{\hat{g}} \delta \hat{g}^{ij} \left( R_{ij} - \frac{1}{2} \hat{g}_{ij} R + \Lambda_3 \hat{g}_{ij} \right). \quad (44)$$

Take now the boundary metric to be conformally flat and related to the bulk one as

$$\hat{g}(\vec{x})_{ij} = \varphi_0^2(\vec{x}) \eta_{ij}, \quad (45)$$
and using (41) we find after some algebra

$$\delta I_{EH}^{(3)} = \frac{3\Lambda_3 G_4}{8\pi} \sqrt{\frac{G_4}{G_3}} \int_{\mathcal{R}_3} d^3\bar{x} \delta\phi_0(\bar{x}) \left( 1 - \frac{1}{6\Lambda_4 G_4} \mathcal{R}_3 \right) \phi_0^2(\bar{x}),$$

(46)

where $\mathcal{R}_3$ is given by (28). Quite remarkably, the Hamiltonian boundary condition (24) arises as the stationarity condition for the total action

$$\mathcal{I} = I_4 + I_{EH}^{(3)},$$

(47)

which is nothing but the sum of bulk and boundary gravity in 4d and 3d. Matching the coefficients gives the following relationships between 4d and 3d quantities

$$\lambda = -\Lambda_3 G_4, \quad \frac{2}{\lambda} G_3^2 = G_4, \quad \Lambda_3 = \frac{2}{3}\Lambda_4,$$

(48)

hence we need a negative cosmological constant in the boundary as well.

To summarize, we presented a rather simple example of an explicit correspondence between a 4d bulk and a 3d boundary theory. We have argued that it gives support to our claim that there is a relationship between stochastic quantization and AdS/CFT, at least under certain conditions. We pointed out that our results have a geometric origin since they can be obtained by the coupling of 4d and 3d conformal gravities.

We need not stress again that our results must be taken only as an incentive to study further the relationship between stochastic quantization and holography\(^\text{10}\). In particular, one would need to understand further the subleading term in $\lambda$ and how they match between 3d and 4d, where we suspect we could find the evidence for the presence of Gaussian noise. Moreover, one could study correlation functions and extend our analysis to gauge fields and gravity. Such issues are currently under study.

Acknowledgments:

The work of ACP was partially supported by the research grant with KA 2745 from the University of Crete. A. C. P. would like to thank G. Kofinas and G. Semenoff, for useful discussions, and J. Ambjorn and P. Damgaard for interesting correspondence.

Appendix

A Stochastic Quantization

In many ways stochastic quantization may be viewed as an application of Stoke’s theorem in field theory [15]. Consider a $(d + 1)$-dimensional manifold $\mathcal{X}$ with a non empty boundary

\(^{10}\)For a related but slightly different approach see [17].
∂X. Then, for any d-form \(Ω_{[d]}\) the following equality holds
\[
\int_X dΩ_{[d]} = \int_{∂X} Ω_{[d]}.
\]
(49)

Now suppose that \(Ω_{[d]}\) is the lagrangian of a d-dimensional Euclidean theory with action \(S_{cl}[φ]\) involving a single scalar field \(φ\),
\[
S_{cl}[φ] = \int_X Ω_{[d]}[φ; \vec{x}],
\]
(50)

where \(\vec{x}\) are the coordinates of the boundary. The above imply that (49) provides an alternative definition of the d-dimensional theory \(S_{cl}[φ]\) using \(d+1\)-forms that depend on the \(d+1\) coordinates \(\{t, \vec{x}\}\). However, despite the fact that we have added a new dimension - the stochastic "time" coordinate \(t\) - the physical content of the theory still resides on the boundary. The latter property implies the presence of a topological invariance in the \((d+1)\)-dimensional description of the theory: the action cannot depend on the specific extension of the field in the bulk.Explicitly, the \(d+1\)-dimensional action in terms of the field \(Φ \in C^\infty(\mathcal{X})\) which is the extension on \(X\) of \(φ \in C^\infty(∂X)\) constrained by \(Φ|_{∂X} = φ\), is defined as
\[
S_{d+1}[Φ] = \int_X dΩ_{[d]}.
\]
(51)

The topological invariance can thus be phrased as \(S_{d+1}[Φ + δΦ] = S_{d+1}[Φ]\) for any variation vanishing on the boundary \(δΦ|_{∂X} = 0\). The idea is to gauge-fix such a symmetry a-la BRST.

To be more concrete consider a manifold \(X\) with a cylindrical structure, \(X = B × [0, T]\) where the base manifold \(B\) is parameterized by the coordinates \(\vec{x}\) and the additional coordinate \(t\) runs from 0 to \(T\). The boundary is given by the \(d\)-chain consisting in the two copies of \(B\) at \(t = 0\) and at \(t = T\) with the correct orientations, \(∂X = B_T - B_0\). Explicitly
\[
S_{d+1}[Φ] = \left(\int_{B_T} - \int_{B_0}\right) Ω_{[d]} = S_{cl}[φ_T] - S_{cl}[φ_0] = \tilde{S}_{cl}[\tilde{Φ}],
\]
(52)

where \(φ_0 = Φ|_{B_0}\) and \(φ_T = Φ|_{B_T}\). For \(Ω_{[d]}\) a pure \(d\)-form on the base manifold we have
\[
Ω_{[d]} = \frac{1}{n!} Ω_{i_1...i_d}(t, \vec{x}) dx^{i_1} ∧ ... ∧ dx^{i_d}, \quad dΩ_{[d]} = dt ∧ \tilde{Ω}_{[d]}.
\]
(53)

But since \(Ω_{[d]}\) depends on \(t\) only through the extension \(Φ\) of the scalar field we simply have that
\[
S_{d+1}[Φ] = \int_X dt ∧ \tilde{Ω}_{[d]} = \int_X dt ∧ \dot{Φ} \frac{δS_{cl}}{δΦ}.
\]
(54)

Next we choose a convenient gauge-fixing condition \(E[Φ; t, \vec{x}]\) corresponding to a particular choice of the extension \(Φ\) of the scalar field. For example, we may consider an instantonic gauge-fixing condition given by the Langevin equation
\[
E[Φ; t, \vec{x}] = \dot{Φ}(t, \vec{x}) \varepsilon_{[d]}(\vec{x}) + κ \frac{δS_{cl}}{δΦ(t, \vec{x})},
\]
(55)
where $\alpha$ is a constant kernel and $\varepsilon_{[d]}$ is the volume form on the boundary. Classically we would like to take $E[\Phi; t, \vec{x}] = 0$ but is not a good quantum condition. Hence the idea is to let it hold only as an average, $\langle E[\Phi; t, \vec{x}] \rangle = 0$ and hence we are led to introduce a white noise $\eta$ as a source for the Langevin equation, $E[\Phi; t, \vec{x}] = \eta(t, \vec{x})\varepsilon_{[d]}(\vec{x})$. We thus introduce the ghost $\psi$ which is an anti-commuting scalar, the corresponding anti-ghost $\bar{\psi}$, a source $\eta$ for the gauge fixing condition which is typically given by a white noise, and a BRST nilpotent operator $Q$ such that

$$Q\Phi = \psi, \quad Q\bar{\psi} = \eta, \quad Q\psi = 0, \quad Q\eta = 0.$$  \hspace{1cm} (56)

It is easy to show that the action $S_{d+1}[\Phi]$ is invariant under the fermionic transformation $\delta_\epsilon \equiv \epsilon Q$ provided that $\psi$ satisfies periodic boundary conditions\(^\text{11}\). We thus add a $Q$-exact term in the action which does not modify the $\delta_\epsilon$ invariance of the theory

$$S_{d+1}[\Phi] \rightarrow S_{d+1}[\Phi] + (QS_d)[\Phi, \eta, \psi, \bar{\psi}], \quad \text{with} \quad S_{d+1}[\Phi, \eta, \psi, \bar{\psi}] = \frac{1}{2\kappa} \int_X \bar{\psi}(t) (\eta \varepsilon_{[d]} - 2E). \hspace{1cm} (57)$$

Doing so we arrive at

$$(QS_d)[\Phi, \eta, \psi, \bar{\psi}] = \frac{1}{2\kappa} \int_X dt \wedge \left[ \eta \left( \eta \varepsilon_{[d]} - 2E \right) + 2\dot{\psi} \frac{\delta E}{\delta \Phi} \right]$$

$$= \frac{1}{\kappa} \int_X dt \wedge \left[ \frac{1}{2} \left( \eta - *_d E \right)^2 + \frac{1}{2} \left( *_d E \right)^2 \right] \varepsilon_{[d]} +$$

$$+ \frac{1}{\kappa} \int_X dt \wedge \bar{\psi}(t, \vec{x}) \left[ \delta^4(\vec{x} - \vec{y}) \varepsilon_{[d]}(\vec{y}) \partial_t + \kappa \frac{\delta^2 S_{d+1}}{\delta \Phi(t, \vec{x}) \delta \Phi(t, \vec{y})} \right] \psi(t, \vec{y}). \hspace{1cm} (58)$$

Consider then the partition function for the theory

$$Z[\phi_0, \phi_T] = \int D\Phi D\eta D\psi D\bar{\psi} e^{-\left\{ S_{d+1}[\Phi] + (QS_d)[\Phi, \eta, \psi, \bar{\psi}] \right\}}. \hspace{1cm} (59)$$

If $\kappa > 0$ we can integrate out the white noise ending with the following effective action

$$S_{\text{eff}}[\Phi, \psi, \bar{\psi}] = \int_X dt \wedge \left[ \dot{\Phi}^2 + \frac{\delta S_{d+1}}{\delta \Phi} + \frac{1}{2\kappa} \left( *_d E \right)^2 \varepsilon_{[d]} + \mathcal{F}[\Phi, \psi, \bar{\psi}] \right], \hspace{1cm} (60)$$

where $\mathcal{F}[\Phi, \psi, \bar{\psi}]$ is given by the last line of (58). Then, using

$$(*_d E)^2 = \dot{\Phi}^2 + 2\kappa \dot{\Phi} *_d \frac{\delta S_{d+1}}{\delta \Phi} \frac{\delta S_{d+1}}{\delta \Phi} + \kappa^2 \left( *_d \frac{\delta S_{d+1}}{\delta \Phi} \right)^2, \hspace{1cm} (61)$$

we obtain

$$S_{\text{eff}}[\Phi, \psi, \bar{\psi}] = \int_X dt \wedge \left[ \frac{1}{2\kappa} \dot{\Phi}^2 \varepsilon_{[d]} + \frac{\delta S_{d+1}}{\delta \Phi} \frac{\delta S_{d+1}}{\delta \Phi} \right] + \mathcal{F}[\Phi, \psi, \bar{\psi}]. \hspace{1cm} (62)$$

\(^\text{11}\)For simplicity we will assume the fields $\psi$ and $\bar{\psi}$ to vanish on the boundary.
The second term is a total derivative contribution which vanishes if we choose periodic boundary conditions for $\Phi$. We thus have

$$S_{\text{eff}}[\Phi, \psi, \bar{\psi}] = \frac{1}{\kappa} \int_X dt \wedge \left[ \frac{1}{2} \dot{\Phi}^2 \varepsilon[\delta] + \frac{\kappa^2}{2} \left( s d \frac{\delta S_{\text{cl}}}{\delta \Phi} \right)^2 \varepsilon[\delta] \right] + \mathcal{F}[\Phi, \psi, \bar{\psi}].$$  \hspace{1cm} (63)

The fermionic part of the action can be formally integrated out to give

$$\int D\psi D\bar{\psi} e^{-\mathcal{F}[\Phi, \psi, \bar{\psi}]} = \det \left( \frac{\delta E}{\delta \Phi} \right) \sim \exp \left[ \kappa \int_X dt \wedge \frac{\delta^2 S_{\text{cl}}}{\delta \Phi(t, \vec{x})^2} \right] - \exp \left[ -\kappa \int_X dt \wedge \frac{\delta^2 S_{\text{cl}}}{\delta \Phi(t, \vec{x})^2} \right].$$  \hspace{1cm} (64)

In this step one must be consistent with the choice of periodicity for the fields on the boundary. Collapsing the forms we can then write the end result as

$$Z = \int D\Phi \left[ e^{-S_{FP}} - e^{-S_{FP}} \right], \quad S_{FP}^\pm[\Phi] = \int d^{d+1} x \left[ \frac{1}{2\kappa} \dot{\Phi}^2 + \frac{\kappa}{2} \left( \frac{\delta S_{\text{cl}}}{\delta \Phi} \right)^2 \pm \kappa \frac{\delta^2 S_{\text{cl}}}{\delta \Phi^2} \right].$$  \hspace{1cm} (65)

It’s important to stress that this derivation of the partition function for the FP system leads to the supersymmetric realization of stochastic quantization [16]. Supersymmetry follows from the choice of periodic boundary conditions for the fields $\psi$ and $\Phi$. It’s straightforward to reintroduce in the partition function the dependence on general choices of boundary values for the fields and consider only forward propagation in time. In this more general setting one is free to fix a given initial configuration for the fields and let them evolve according to the Langevin equation. In Section 2 we explicitly chose an initial condition for the field $\phi$ generally breaking supersymmetry.

References

[1] D. Polyakov, Class. Quant. Grav. 18, 1979 (2001) [arXiv:hep-th/0005094].
[2] G. Lifschytz and V. Periwal, JHEP 0004 (2000) 026 [arXiv:hep-th/0003179].
[3] G. Parisi and Y. s. Wu, Sci. Sin. 24, 483 (1981).
[4] J. Ambjorn, R. Loll, W. Westra and S. Zohren, Phys. Lett. B 680, 359 (2009) [arXiv:0908.4224 [hep-th]].
[5] P. H. Damgaard and H. Huffel, Phys. Rept. 152, 227 (1987).
[6] J. Zinn-Justin, Nucl. Phys. B 275 (1986) 135.
[7] P. Horava, Phys. Rev. D 79 (2009) 084008 [arXiv:0901.3775 [hep-th]].
[8] G. Compere and D. Marolf, Class. Quant. Grav. 25, 195014 (2008) [arXiv:0805.1902 [hep-th]].
[9] S. de Haro and A. C. Petkou, JHEP 0612 (2006) 076 [arXiv:hep-th/0606276].

[10] S. de Haro, I. Papadimitriou and A. C. Petkou, Phys. Rev. Lett. 98 (2007) 231601 [arXiv:hep-th/0611315].

[11] I. Papadimitriou, JHEP 0705 (2007) 075 [arXiv:hep-th/0703152].

[12] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253 [arXiv:hep-th/9802150].

[13] J. Parry, D. S. Salopek and J. M. Stewart, Phys. Rev. D 49 (1994) 2872 [arXiv:gr-qc/9310020].

[14] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15 (1977) 2752.

[15] L. Baulieu and B. Grossman, Phys. Lett. B 212 (1988) 351.

[16] E. Gozzi, Phys. Rev. D 28 (1983) 1922.

[17] R. Dijkgraaf, D. Orlando and S. Reffert, Nucl. Phys. B 824 (2010) 365 [arXiv:0903.0732 [hep-th]].