Boost modes for a massive fermion field

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We have shown that Wightman function of a free quantum field generates any complete set of solutions of relativistic wave equations. Using this approach we have constructed the complete set of solutions to 2d Dirac equation consisting of eigenfunctions of the generator of Lorentz rotations (boost operator). It is shown that at the surface of the light cone the boost modes for a fermion field contain δ-function of a complex argument. Due to the presence of such singularity exclusion of even of a single mode with an arbitrary value of the boost quantum number makes the set of boost modes incomplete.

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I. INTRODUCTION

It is well known that quantization procedure in a quantum field theory implies expansion of the field operator in terms of a complete set of modes which are solutions of the corresponding (Dirac or Klein-Fock-Gordon) classical field equation. Therefore finding exact solutions for these equations is one of the central points of the quantum field theory in the presence of a classical background.

One of the most powerful instruments for solving partial differential equations is based on symmetry properties of the physical system described by the equation. According to the Noether’s theorem [1] any differentiable symmetry of the action of a physical system has a corresponding conservation law. One can always choose a set of variables in such a way that a change in one of them corresponds to the symmetry transformation. Since the generator of the symmetry transformation commutes with the differential operator of the equation, the variable corresponding to it can be separated. If there are enough symmetries for the equation, such that their generators commute with each other, the problem of finding the solution can be reduced to solving, generally speaking, of a second order ordinary equation. The set of such solutions is labeled by the eigenvalues of the generators and is complete.

All most important and widely used solutions for relativistic quantum equations in the presence of classical external fields were obtained using this method. We mean the Coulomb field, see, e.g., [2], the constant magnetic [2] and electric [3, 4] fields, the field of a plane electromagnetic wave [5], and some other fields of more sophisticated configurations [6]. In all these cases the solutions were eigenfunctions of linear combinations of generators of time and spatial translations or spatial rotations, and were labeled by values of components of either 4-momentum, or angular momentum respectively. However, the Poincaré group, the group of isometries of Minkowski spacetime (MS), includes also Lorentz rotations (boosts). The boost symmetry was almost never used for field quantization. Certainly, this is because the boost generator does not commute with the Hamiltonian, and thus the boost quantum number and energy cannot be parts of any complete set of observables simultaneously. Nevertheless, the boost modes can be very useful, especially for the quantum field theory in a curved space, or if the symmetry of MS with respect to time and/or space translations is broken in the presence of a classical background. In these cases the boost symmetry can appear to be the sole symmetry for the quantum field. Dilaton gravity in two dimensions [7], as well as Schwarzschild geometry [8], are the examples. Therefore analysis of the properties of boost modes is important.

For the first time the boost modes for a free massive scalar field were discussed in Ref. [9] by W. Unruh, though their explicit form was not specified. Then some properties of scalar boost modes were studied in Refs. [10–12]. In particular, the remarkable properties of boost modes in 2d MS were ascertained in Ref. [12]. It was shown that (i) the zero boost mode of a free massive scalar field coincides up to a trivial constant factor with the positive-frequency Wightman function, and (ii) the boost modes considered as functions of the boost quantum number possess the Dirac δ-function singularity at the surface of the light cone. The properties of the boost modes for the case of a free massless fermion field appeared to be even more interesting. It was shown in Ref. [13] that boost modes of two-dimensional massless fermions on a light cone are expressed in terms of the delta function of a complex argument. Boost modes of a massive fermion field have been first considered in Refs. [14, 15], see also Ref. [16]. However the results of Refs. [14, 15] contain some discrepancies which we discuss in Sec. III of the present paper.

In this paper we consider boost modes for a free massive fermion field. To construct them we start from the Wightman functions. The positive and negative frequency Wightman functions for free fields are explicitly determined (with accuracy to a constant factor) by Lorentz and translational invariance of the theory [17]. And vice versa, once the Wightman functions are known, one can use those symmetries to construct any complete set of positive and negative frequency solutions of Klein-Fock-Gordon (KFG) or Dirac equations. In fact, this is the direct consequence
of the Wightman reconstruction theorem \[18, 19\].

Since the Lorentz rotation singles out a two-dimensional plane in MS, we will discuss the specific properties of boost modes by examples of free 2d KFG and Dirac equations. In the next section we will realize the above formulated approach by the example of a single-component massive neutral field. It is worth noting that, as opposed to the case of plane wave modes, analytic properties of boost modes change dramatically when one passes to multi-component fields which are considered in Sec. III. The discussion of the results and conclusions are given in Sec. IV.

II. BOSONS

In this section we will consider the case of a free neutral massive scalar field in two-dimensional MS. We will start from two-point Wightman function \[18\] which for a free field theory coincides with the positive-frequency part of the commutator of two field operators (Pauli-Jordan function in 4d theory of a scalar field). Positive-frequency Wightman function \(\Delta^{(+)}(x)\), \(x = (t, z)\), for a massive scalar field satisfies KFG equation

\[
K \Delta^{(+)}(x) = 0, \quad K = \partial_t^2 - \partial_z^2 + m^2,
\]

contains only positive frequencies and is invariant with respect to Lorentz rotations (boosts)

\[
B_K \Delta^{(+)}(x) = 0, \quad B_K = i(z \partial_t + t \partial_z).
\]

These conditions determine \(\Delta^{(+)}(x)\) accurate within a constant factor. Indeed, any positive-frequency solution of KFG equation can be written as follows

\[
\Phi^{(+)}(x) = \int d^2 p \phi(p) \delta(p^2 - m^2) \theta(p^0) e^{-ipx},
\]

where \(\phi(p)\) is a certain function of 2-vector \(p = (p^0, p^1)\), \(\theta(p^0)\) is the Heaviside step function. After the change of variables

\[
p^0 = \mu \cosh q, \quad p^1 = \mu \sinh q,
\]

where \(q\) is rapidity, and integration over \(\mu\) we obtain the equation

\[
\Phi^{(+)}(x) = \frac{1}{2} \int dq \phi(m \cosh q, m \sinh q) e^{-im(t \cosh q - z \sinh q)}.
\]

It can be easily seen that condition (2) is satisfied only if \(\phi(q) = \text{const}\). Choosing \(\phi(q) = i/2\pi\) we arrive to the standard representation for \(\Delta^{(+)}(x)\), compare, e.g., \[1, 12\]

\[
\Delta^{(+)}(x) = \frac{i}{4\pi} \int dq e^{-im(t \cosh q - z \sinh q)}.
\]

It is assumed in \[12\] that an infinitely small negative imaginary part is added to \(t\). Another representation for \(\Delta^{(+)}(x)\) reads, see, e.g., \[20\],

\[
\Delta^{(+)}(x) = \frac{i}{4\pi} \int dp e^{-ip(t + ipz)}, \quad \varepsilon_p = \sqrt{p^2 + m^2},
\]

from now on we shall omit the index of the spatial component of the 2-vector \(p\).

According to the Wightman reconstruction theorem \[18\] the two-point Wightman function uniquely determines the quantum theory of a free field. Particularly, it allows to reconstruct any orthonormalized and complete set of solutions.

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1 The natural units \(\hbar = c = 1\) are used throughout this paper.
for the field equation. Indeed, it follows from the translational invariance of the theory that Wightman function with an arbitrarily shifted argument

$$\Delta^{(+)}(x - u), \quad u = \{u^0, u^1\}$$

satisfies KFG equation. Functions (7) constitute an overcomplete set of modes since they are labeled by two independent parameters $u^0, u^1$. Anyway, any complete set of positive-frequency solutions $F_a(x)$

$$\int \, da \, F_a(x') F_a^*(x'') = -i \Delta^{(+)}(x' - x''),$$

orthonormalized by the condition

$$\langle F_a, F_{a'} \rangle = \int_{-\infty}^{\infty} F_a^*(x) \overleftrightarrow{\partial_t} F_{a'}(x) \, dz = \delta(a - a'),$$

can be represented as

$$F_a(x) = \int d^2 u \, f_a(u) \Delta^{(+)}(x - u).$$

However, since the set of solutions (7) is overcomplete, the coefficient functions $f_a(u)$ cannot be determined uniquely. To make the choice of $f_a(u)$ unique one should impose a restriction on $u$ which can be chosen for reasons of symmetry.

First, we will illustrate this procedure for the trivial case of plane waves $\Theta_p(x)$ which are eigenfunctions of the generator of spatial translations

$$-i\partial_z \Theta_p(x) = p \Theta_p(x),$$

In this case it is convenient to limit the family of $\Delta^{(+)}(x - u)$ only by functions spatially shifted with respect to each other. This means that coefficient functions $f_p(u)$ have the form $f_p(u) = \delta(u^0) \vartheta_p(u^1)$, so that

$$\Theta_p(x) = \int_{-\infty}^{\infty} du^1 \vartheta_p(u^1) \Delta^{(+)}(t, z - u^1).$$

Let us substitute this expansion into Eq. (11). Taking into account that Wightman function $\Delta^{(+)}(t, z)$ exponentially tends to zero when $|z| \to \infty$, we can integrate the left-hand side of the resulting equation by part and arrive to the following equation for $\vartheta_p(u^1)$

$$-i \vartheta_p'(u^1) = p \vartheta_p(u^1),$$

so that $\vartheta_p(u^1) = \text{const} \cdot e^{ip \cdot u^1}$. Substituting now $\vartheta_p(u^1)$ of that form into (12) and using (6) we finally obtain after integration over $u^1$

$$\Theta_p(x) = \frac{1}{\sqrt{4\pi \varepsilon_p}} e^{-iz \cdot u^1}. $$

The normalization constant here was determined from the condition (9).

Functions $\Theta_p(x)$ satisfy the relation

$$\int_{-\infty}^{\infty} dp \, \Theta_p(x') \Theta_p^*(x'') = -i \Delta^{(+)}(x' - x''),$$

and thus the set (13) is complete.

We are interested in the set of boost modes $\Psi_\kappa^{(+)}(x)$ which are eigenfunctions of boost operator $B_\kappa$

$$B_\kappa \Psi_\kappa^{(+)}(x) = \kappa \Psi_\kappa^{(+)}(x).$$
It is worth noting from the very beginning that Wightman function $\Delta^{(+)}(x)$ is a zero mode $\Psi^{(+)}_0(x)$ of this set as it follows from Eqs. (15) and (2).

To obtain the boost modes we will confine $u$ in (10) to one of the orbits of the restricted Lorentz group

$$u^2 = u^0 u^0 = \pm u^2, \quad v = \text{const},$$

(16)

so that $f(u) \sim \delta(u^2 \mp v^2)$. Choosing the upper sign in (16) for definiteness, we put

$$u \equiv u_q = (v \cosh q, v \sinh q),$$

(17)

and rewrite Eq. (10) in the form

$$\Psi^{(+)}(x) = \int_{-\infty}^{\infty} dq \zeta_\kappa(q) \Delta^{(+)}(x - u_q).$$

(18)

Substituting (18) into Eq. (15) and using the relation

$$B K \Delta^{(+)}(x - u_q) = -i \frac{\partial}{\partial q} \Delta^{(+)}(x - u_q),$$

(19)

which can be straightforwardly obtained with the help of representation (5), we get the following equation for $\zeta_\kappa(q)$

$$i \frac{\partial \zeta_\kappa(q)}{\partial q} = \kappa \zeta_\kappa(q).$$

Hence, $\zeta_\kappa(q) = \text{const} e^{-i\kappa q}$, and we finally get from (18) with due regard for condition (9) the following representation for the positive frequency boost modes

$$\Psi^{(+)}_\kappa(x) = \frac{1}{2^{3/2}} \int_{-\infty}^{\infty} dq e^{-im(t \cosh q - z \sinh q) - i\kappa q},$$

(20)

which was earlier obtained in Refs. [10–12] by another method. The negative frequency boost modes $\Psi^{(-)}_\kappa(x)$ are defined [12] as

$$\Psi^{(-)}_\kappa(x) = \Psi^{(+)*}_\kappa(x).$$

(21)

Boost modes (20), (21) are distributions with respect to both $x = (t, z)$ and spectral parameter $\kappa$. They are defined on the class of smooth functions of rapid enough descent.

The modes (20), (21) constitute a complete set since they satisfy the condition

$$\int_{-\infty}^{\infty} d\kappa \Psi^{(+)}_\kappa(x') \Psi^{(+)*}_\kappa(x'') = \mp i \Delta^{(\pm)}(x' - x''),$$

(22)

and hence can be used as a basis for quantization of a neutral scalar field, see Ref. [12]. Since the sign of a particle energy is Lorentz invariant, the vacuum states in the boost and plane-wave quantization schemes are identical. Hence, these two quantization schemes are unitary equivalent, see Ref. [12].

The most remarkable property of boost modes is their behavior at the light cone. It is easily seen from (20), (21) that at the vertex of the light cone $\Psi^{(\pm)}_\kappa(x)$ possess the Dirac $\delta$-function singularity

$$\Psi^{(\pm)}_\kappa(0) = \frac{1}{\sqrt{2}} \delta(\kappa).$$

(23)

It was shown in Ref. [12] that $\Psi^{(\pm)}_\kappa(x)$ possess a $\delta$-function singularity at the lines $x_\pm \equiv t \pm z = 0$ as well. Therefore contribution of the single spectral point $\kappa = 0$ to physical quantities can be finite. We will illustrate this by the example of Wightman function.
Taking into account the property of translational invariance we can rewrite Eq. (22) in the form
\[ \Delta^{(+)}(x' - x'') = \int_{-\infty}^{\infty} d\kappa \Psi^{(+)}_{\kappa}(x') \Psi^{(+)}_{\kappa}(x'') = \int_{-\infty}^{\infty} d\kappa \Psi^{(+)}_{\kappa}(x' - x'') \Psi^{(+)}_{\kappa}(0). \] (24)

Using now Eq. (23) we obtain
\[ \Delta^{(+)}(x' - x'') = \frac{i}{\sqrt{2}} \Psi^{(+)}_{0}(x' - x''), \] (25)
i.e. the integral over \( \kappa \) in (24) is determined by the point \( \kappa = 0 \) entirely\(^2\).

This result means that the point \( \kappa = 0 \) cannot be deleted from the spectrum, or in other words the integral over \( \kappa \) in (24) cannot be changed by its principal value
\[ \int_{-\infty}^{\infty} d\kappa \neq \text{P.v.} \int_{-\infty}^{\infty} d\kappa. \] (26)

Thereby, the family of boost modes does not constitute a complete set in MS after excluding the zero mode. However, the equality in Eq. (26) could be restored if we cut the light cone out of MS, compare [12]. This is because all the points where boost modes possess \( \delta(\kappa) \) singularity are located just at the light cone.

Furthermore, the Wightman function Eq. (24) cannot be represented in the form
\[ \tilde{\Delta}^{(+)}(x' - x'') = \int_{0}^{\infty} d\kappa \{ \Psi^{(+)}_{\kappa}(x') \Psi^{(+)}_{\kappa}(x'') + \Psi^{(+)}_{-\kappa}(x') \Psi^{(+)}_{-\kappa}(x'') \}. \] (27)

If it were so, we could rewrite (24), due to translational invariance of the Wightman function and the boost mode property (23), as follows
\[ \tilde{\Delta}^{(+)}(x' - x'') = \frac{i}{\sqrt{2}} \int_{0}^{\infty} d\kappa \{ \Psi^{(+)}_{\kappa}(x' - x'') + \Psi^{(+)}_{-\kappa}(x' - x'') \} \delta(\kappa). \] (28)

But Eq. (28) is evidently meaningless. Indeed, the distribution \( \delta(\kappa) \) is defined on the class of functions continuous at the interval including the point \( \kappa = 0 \). Therefore Eq. (28) should be understood as
\[ \tilde{\Delta}^{(+)}(x' - x'') = \frac{i}{\sqrt{2}} \int_{0}^{\infty} d\kappa \{ \Psi^{(+)}_{\kappa}(x' - x'') + \Psi^{(+)}_{-\kappa}(x' - x'') \} \theta(\kappa) \delta(\kappa), \] (29)

where \( \theta(\kappa) \) is the Heaviside step function. However, the product of two distributions \( \theta(\kappa)\delta(\kappa) \) is not defined.

The authors of Ref. [21] do not agree with this statement. Discussing their Eq. (2.129), which in our notations coincides with (27), they admit that this expression is undefined if either of the two points \( x', x'' \) is located on the light cone. They think that smearing of the distribution \( \Delta^{(+)}(x' - x'') \) with compactly supported functions \( f(x') \) and \( g(x'') \) improve the situation and the smeared "Wightman function" (2.131) is well defined on the whole MS. Our analysis shows however that this is not correct. Indeed, if we use Eq. (27) in the form (28), we see that this expression is undefined for arbitrary \( x', x'' \). Moreover, smearing cannot improve the situation since it does not influence the \( \delta \)-function in (28). The correct expression for the smeared Wightman function can be easily obtained from Eq. (24). It reads
\[ \Delta^{(+)}(f, g) = \int d^2x' d^2x'' f^*(x') g(x'') \Delta^{(+)}(x' - x'') = \int_{-\infty}^{\infty} d\kappa \Psi^{(+)}_{\kappa}(f, g) \Psi^{(+)}_{\kappa}(0) = \frac{i}{\sqrt{2}} \Psi^{(+)}_{0}(f, g), \] (30)

\(^2\) We have mentioned already that Eq. (26) directly follows from Eqs. (2) and (15) accurate within a constant factor. Then Eq. (28) could be derived from Eqs. (29), (24) even not using the representation (20).
Fig. 1: Classical trajectories of massive boost particles.

\[ \Psi^{(+)}(\kappa)(f,g) = \int d^2x' d^2x'' f^*(x')g(x'')\Psi^{(+)}(x' - x'') , \]

(31)
is the smeared boost mode. We see that again only the single spectral point \( \kappa = 0 \) contributes to the integral (30) for the smeared Wightman function.

Note, that we could try to use expression (27) directly, not applying the property of translational invariance to it. In that case we should attach exact mathematical meaning to (27) first. Namely, we should write it in the form

\[ \tilde{\Delta}^{(+)}(x', x'') = i \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} d\kappa \{ \Psi^{(+)}_\kappa(x')\Psi^{(+)}_\kappa(x'') + \Psi^{(+)}(-\kappa)(x')\Psi^{(+)}(-\kappa)(x'') \} . \]

(32)

But it can be easily seen that function \( \tilde{\Delta}^{(+)}(x', x'') \) in (32) is not translationally invariant and thus has nothing to do with Wightman function for a free field in MS. Indeed, if it were translationally invariant, it could be written down as

\[ \tilde{\Delta}^{(+)}(x', x'') = \frac{i}{\sqrt{2}} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} d\kappa \{ \Psi^{(+)}_\kappa(x' - x'') + \Psi^{(+)}(-\kappa)(x' - x'') \} \delta(\kappa) , \]

(33)

and would be equal to zero identically.

Singular behavior of the boost mode at the point \( \kappa = 0 \) can be understood in terms of classical trajectories of free particles with a given value of the boost parameter \( \kappa \)

\[ \kappa = \varepsilon p - t p, \]

(34)

where \( p = m\dot{z}/\sqrt{1 - \dot{z}^2} \) is the momentum, \( \varepsilon_p = m/\sqrt{1 - \dot{z}^2} \) - the energy of a particle, \( \dot{z} = dz/dt \). Let us rewrite Eq. (34) in the the form

\[ z = \dot{z}t + \frac{\kappa}{m} \sqrt{1 - \dot{z}^2} . \]

(35)

After differentiating both sides of Eq. (35) with time one can easily find its regular

\[ z = vt + \frac{\kappa}{m} \sqrt{1 - v^2} , \quad v = \text{const} , \quad |v| < 1 , \]

(36)

and singular

\[ z = \text{sgn}(\kappa) \sqrt{t^2 + \frac{\kappa^2}{m^2}} , \]

(37)
solutions.

When $\kappa \neq 0$ the singular solution is represented by two branches of hyperbola (37). The branch with $\kappa > 0$ is located in the right wedge of MS, while the branch with $\kappa < 0$ in the left one, see Figs. 1a,b. Free boost particles with the given value of $\kappa$ propagate along the regular world lines which are different straight lines tangent to the corresponding branches of hyperbola (37). It is worth noting that classical particles with positive values of the boost parameter $\kappa$ cannot penetrate to the left wedge, as well as particles with $\kappa < 0$ cannot penetrate to the right one.

The world lines of boost particles with $\kappa = 0$ are given by

$$z = vt.$$  \hspace{1cm} (38)

All of them cross the vertex of the light cone $h_0$ and any world line crossing $h_0$ belongs to the family of trajectories with $\kappa = 0$, see Fig. 1c. Hence, elimination of the point $\kappa = 0$ from the spectrum is equivalent to prohibition for particles to cross the point $h_0$, or to pricking it out of MS. Besides, deleting the point $\kappa = 0$ from the spectrum, we delete a bunch of infinite number of trajectories, thus losing a substantial part of degrees of freedom. This means that the point $\kappa = 0$ is of nonzero measure at $h_0$ and explains the $\delta$-function singularity of the boost mode there.

The points lying at the light cone surface need a separate consideration. The lines $z = \pm t$ cannot be trajectories of massive particles. But the branches of hyperbola (37) degenerate into lines constituting the surface of the light cone in our 2d problem when $\kappa \to 0$. As it can easily be seen from Figs. 1a,b, all regular trajectories except those which are tangent to the hyperbola branches at the point $t = 0$ also tend to the lines $z = \pm t$ in the limit $\kappa \to 0$. Hence, these lines $z = \pm t$ are the lines of condensation for regular trajectories with $\kappa \to 0$. Possibly this explains the existence of $\delta$-function singularity of boost modes at the light cone surface.

It is worth noting that there exist another representation for the boost modes (20), (21). Keeping in mind that an infinitely small negative imaginary part $-i\sigma$ is added to $t$ in (20), after the change of the variable of integration $q = lu$ we obtain for $\Psi_{\kappa}^{(\pm)}(x)$ the following expression

$$\Psi_{\kappa}^{(\pm)}(x) = \frac{1}{2\sqrt{2}} \int_0^\infty d\nu \nu^{-\kappa - 1/2} e^{-\frac{x^2}{4
u}}.$$  \hspace{1cm} (39)

with $\beta = \frac{m}{2}(\sigma \pm ix_\pm)$ and $\gamma = \frac{\pi}{2}(\sigma \pm ix_-)$. Using now formula 3.471(9) of Ref. 22 we get

$$\Psi_{\kappa}^{(\pm)}(x) = \frac{1}{\pi \sqrt{2}} \left( \frac{x_+ \mp i\sigma}{x_\pm \pm i\sigma} \right)^{i\pi/2} K_{i\kappa}(w_\pm), \hspace{1cm} w_\pm = m \sqrt{e^{\pm 2\pi}(x_+ \mp i\sigma)(x_- \mp i\sigma)},$$  \hspace{1cm} (40)

where $K_{i\kappa}(w_\pm)$ is the Macdonald function, and distributions $\lambda^\pm = (\xi \mp i\sigma)^\lambda$ should be understood as

$$\lambda = \lambda^\pm \theta(\xi) + e^{\mp i\pi\lambda}(\xi \mp \lambda)\theta(\xi).$$  \hspace{1cm} (41)

see Ref. 23. Using Eq. (41) we obtain

$$\Psi_{\kappa}(x) = \theta(-x_+)\theta(-x_-)\Psi_{\kappa}^{(P)}(x) + \theta(x_+)\theta(-x_-)\Psi_{\kappa}^{(R)}(x) + \theta(x_+)\theta(x_-)\Psi_{\kappa}^{(F)}(x) + \theta(-x_+\theta(x_-)\Psi_{\kappa}^{(L)}(x),$$  \hspace{1cm} (42)

see 11, 12, where

$$\Psi_{\kappa}^{(P)}(x) = \frac{ie^{-\pi\kappa/2}}{2\sqrt{2}} \left( \frac{-x_-}{x_+} \right)^{i\pi/2} H_{i\kappa}^{(1)} \left( m \sqrt{(x_+)(-x_-)} \right), \hspace{1cm} \Psi_{\kappa}^{(R)}(x) = \frac{e^{\pi\kappa/2}}{2\sqrt{2}} \left( \frac{-x_-}{x_+} \right)^{i\pi/2} K_{i\kappa}(m \sqrt{x_+(-x_-)}),$$

$$\Psi_{\kappa}^{(F)}(x) = \frac{-ie^{\pi\kappa/2}}{2\sqrt{2}} \left( \frac{x_+}{x_-} \right)^{i\pi/2} H_{i\kappa}^{(2)} \left( m \sqrt{x_+x_-} \right), \hspace{1cm} \Psi_{\kappa}^{(L)}(x) = \frac{e^{-\pi\kappa/2}}{2\sqrt{2}} \left( \frac{x_+}{-x_-} \right)^{i\pi/2} K_{i\kappa}(m \sqrt{(-x_+)(-x_-)}),$$  \hspace{1cm} (43)

are the expressions for the boost modes in the past (P), right (R), future (F) and left (L) wedges of MS respectively. Here $H_{i\kappa}^{(1,2)}(w)$ are Hankel functions.

Putting $\kappa = 0$ in Eq. (43) and taking into account Eq. (24) one can easily reproduce the Wightman result (20) for $\Delta^{(\pm)}(x)$. Applying the same procedure to Eq. (40) we get

$$\Delta^{(\pm)}(x) = \pm \frac{i}{2\pi} K_0(w_\pm).$$  \hspace{1cm} (44)

This is a new compact representation for the Wightman function in 2d scalar theory.
III. FERMIONS

A set of orthonormalized positive (negative) frequency solutions \( \{ F_a^{(\pm)}(x) \} \) of Dirac equation in 2d MS

\[
\mathcal{D}_- \psi(x) = 0, \quad \mathcal{D}_\pm = \begin{pmatrix} i\gamma^0 \partial_t + i\gamma^1 \partial_z \pm m \end{pmatrix},
\]

(\( \gamma^{0,1} \) - two-dimensional Dirac matrices, \( \gamma^0 \equiv \beta, \gamma^0 \gamma^1 = -\gamma^1 \gamma^0 \equiv \alpha, \alpha^2 = \beta^2 = 1 \)) is complete if

\[
\int da F_a^{(\pm)}(x') F_a^{(\pm)*}(x'') \gamma^0 = -iS^{(\pm)}(x' - x''),
\]

where the positive (+) and negative (−) frequency Wightman functions for the fermion field are equal to

\[
S^{(\pm)}(x) = \mathcal{D}_\pm \Delta^{(\pm)}(x), \quad \Delta^{(-)}(x) = \Delta^{(+)*}(x),
\]

and \( \Delta^{(\pm)}(x) \) is defined in \((47)\).

Any set \( \{ F_a^{(\pm)}(x) \} \) can be constructed of the set of matrices \( \{ S^{(\pm)}(x - u) \} \) in the perfect analogy with the boson case. Taking into account that the Dirac operators \( \mathcal{D}_\pm \) commute with the fermion boost operator \( \mathcal{B}_D \),

\[
[\mathcal{D}_\pm, \mathcal{B}_D] = 0, \quad \mathcal{B}_D = \mathcal{B}_K - i\gamma^0 \gamma^1 = i \left( z\partial_t + t\partial_z - \frac{1}{2}\alpha \right),
\]

it is easy to check that functions

\[
\psi^{(\pm)}(x) = \int dq S^{(\pm)}(x - u_q) f_\kappa(q)
\]

(\( u_q \) are defined in \((17)\)) are the eigenfunctions of this operator if

\[
f_\kappa(q) = e^{-i\kappa q + \frac{1}{2}\alpha q} C,
\]

where \( C \) is an arbitrary constant column. Substituting \((50)\) into Eq. \((49)\) we easily get with the account of definition \((47)\) the following expression for the normalized boost modes

\[
\psi^{(\pm)}(x) = \frac{1}{2\pi} \sqrt{\frac{m}{2}} \int dq e^{\pm im|t\mp i\sigma|\cosh q - z\sinh q - i\kappa q + \frac{1}{2}\alpha q} (1 \pm \beta)\eta.
\]

Here the arbitrary constant column \( \eta \) satisfies the relation

\[
\eta^+ (1 \pm \beta)\eta = 1.
\]

It is difficult to compare representation \((51)\) with the results of Refs. \([14, 15]\) directly since the authors of these works by unknown reasons use in 2d spacetime 4 \times 4 matrices which constitute a reducible representation of the Dirac matrices. The passage to their representation may be realized as follows

\[
\alpha \rightarrow \alpha_3 = \alpha \otimes \sigma_3, \quad \beta \rightarrow \beta \otimes I, \quad \eta \rightarrow \eta \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

where

\[
\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

In particular, using the notations of Refs. \([14, 15]\)

\[
v = \frac{1}{2} \ln \frac{x_+}{-x_-}, \quad u = \sqrt{x_+(-x_-)}, \quad \kappa \equiv \omega, \quad \Phi^{(+)}_\omega = H_{\omega \pm 1/2}(imu),
\]

(\( \omega \equiv \sqrt{x_+(-x_-)} \))
for the positive frequency solution in the Rindler wedge of 2d MS we obtain from Eq. (51) with account of (53), (51)

\[
\psi_\omega^+(x_+)\theta(-x_-) = \frac{i}{4}\sqrt{me^{-i\omega v}} \left( e^{\frac{i}{2}\kappa \phi_0^-} + e^{-\frac{i}{2}\kappa \phi_0^+} \right) 0
\theta(x_+)\theta(-x_-).
\]

(56)

It is worth noting that the results of Refs. [14–16], see Eqs. 21.64, 21.65, 21.77 and 21.78 in [15], differ from (56) by the absence of factors \( e^{\pm\frac{i}{2}\kappa} \) in front of the functions \( \phi_0^\pm \) in (56). In the framework of the method used by the authors of Refs. [14–16], the latter should have appeared as a result of change of Rindler \{u, v\} by Cartesian coordinates

\[
\psi(t, z) = e^{\frac{1}{2}a_v}\psi(u, v),
\]

(57)

see, e.g., [24].

Besides, the modes in Refs. [14–16] differ from (56) by additional normalization factor \( 1/\sqrt{3} \). This discrepancy has arisen due to incorrect method of normalization used by the authors of Refs. [14–16]. Indeed, they have represented the normalization integral in 2d MS as a sum of four terms each equal to the normalization integral in one of four wedges of MS. However, the correct normalization procedure implies integration over some Cauchy surface in MS. The most convenient surface for this purpose is the surface \( t = 0 \), compare, e.g., [3, 12]. Then only the solutions in \( R \) and \( L \) wedges of MS contribute to the normalization integral and the normalization factor coincides with that one in (56).

From now on, we will use a representation of Dirac matrices different from (53),

\[
\gamma^0 \equiv \beta = \sigma_1, \gamma^1 = i\sigma_2, \gamma^0 \gamma^1 \equiv \alpha = -\sigma_3,
\]

(58)

where \( \sigma_i, i = 1, 2, 3 \) are Pauli matrices. Use of representation (58) allows essentially simplify the form of solutions of Eq. (53). We will see below that it also is very convenient for the procedure of transition to the limit \( m \to 0 \). With this set of Dirac matrices the representation (51) for the boost modes reduces to

\[
\psi_{\kappa}^{(\pm)}(x) = \frac{1}{2\pi} \sqrt{\frac{m}{2}} \int_{-\infty}^{\infty} dq e^{i m((l \mp i\sigma) \cosh q - z \sinh q) - in_q} \left( \pm e^{-q/2} \right),
\]

(59)

Using exactly the same procedure as for the scalar boost modes [20] we can transform Eq. (59) to the form

\[
\psi_{\kappa}^{(\pm)}(x) = \frac{1}{2\pi} \sqrt{\frac{m}{2}} \left( \frac{x_+ \pm i\sigma}{x_+ \mp i\sigma} \right)^{\frac{i}{2} + i\kappa} \pm \frac{1}{2} \left( K_{-1/2+i\kappa}(w_\pm) \right),
\]

(60)

where again \( w_\pm = m\sqrt{e^{\pm i\pi}(x_+ \mp i\sigma)(x_+ \mp i\sigma)} \).

In perfect analogy with Eq. (12) the positive frequency boost mode (60) can be rewritten in the form

\[
\psi_\kappa^{(+)}(x) = \theta(-x_+)\theta(-x_-)P\psi_\kappa^{(+)}(x) + \theta(x_+\theta(-x_-)R\psi_\kappa^{(+)}(x) + \theta(x_+\theta(x_-)F\psi_\kappa^{(+)}(x) + \theta(-x_+)\theta(x_-)L\psi_\kappa^{(+)}(x),
\]

(61)

where in \( P \) wedge

\[
P_{\psi_\kappa^{(+)}(x)} = \frac{i}{2\pi} \sqrt{\frac{m}{2}} \left( \frac{-x_-}{-x_+} \right)^{\frac{i}{2} + i\kappa} \left( H_{1/2+i\kappa}(m\sqrt{(-x_+)(-x_-)}) - i \right),
\]

(62)

in \( R \) wedge

\[
R_{\psi_\kappa^{(+)}(x)} = \frac{1}{\pi} \sqrt{\frac{m}{2}} \left( \frac{-x_-}{-x_+} \right)^{\frac{i}{2} + i\kappa} \left( K_{1/2+i\kappa}(m\sqrt{(-x_+)(-x_-)}) i \right),
\]

(63)
in $F$ wedge

$$F_{\psi_{\kappa}^{(+)}}(x) = -\frac{i}{2} \sqrt{\frac{m}{2}} e^{-\frac{2\pi}{\kappa}} \left( \frac{x_-}{x_+} \right)^{\nu+\kappa} \left( \frac{1}{i} \frac{x_-}{x_+} \right)^{-\nu+\kappa} \frac{H_{1/2+i\kappa}^{(2)}(m \sqrt{x_+ x_-})}{H_{-1/2+i\kappa}^{(2)}(m \sqrt{x_+ x_-})}, \quad (64)$$

and finally in $L$ wedge

$$L_{\psi_{\kappa}^{(+)}}(x) = \frac{1}{\pi} \sqrt{\frac{m}{2}} e^{-\frac{2\pi}{\kappa}} \left( \frac{1}{i} \frac{x_-}{x_+} \right)^{\nu+\kappa} \left( \frac{x_-}{x_+} \right)^{-\nu+\kappa} \frac{K_{1/2+i\kappa}(m \sqrt{(-x_+ - x_-)})}{K_{-1/2+i\kappa}(m \sqrt{(-x_+ - x_-)})}. \quad (65)$$

A similar representation is valid for the negative frequency modes $\psi_{\kappa}^{(-)}(x)$ as well. The boost modes (59), (60) are orthonormalized by the condition

$$\langle \psi_{\kappa'}^{(\pm)}, \psi_{\kappa}^{(\pm)} \rangle = \int_{-\infty}^{\infty} dz \psi_{\kappa'}^{(\pm)}(x) \psi_{\kappa}^{(\pm)}(x) = \delta(\kappa - \kappa'), \quad (66)$$

constitute a complete set in the sense (46), and thus can serve a basis for quantization of the fermion field in MS,

$$\Psi(x) = \int_{-\infty}^{\infty} d\kappa \left( b_\kappa \psi_{\kappa}^{(+)}(x) + \tilde{b}_\kappa^\dagger \psi_{\kappa}^{(-)}(x) \right), \quad (67)$$

where $b_\kappa$ and $\tilde{b}_\kappa$ are annihilation operators for boost fermion particles and antiparticles respectively, which obey the standard commutation relations. By the same reasons as in the boson case, boost quantization of the fermion field is unitary equivalent to quantization in the plane wave basis

$$\Psi(x) = \int_{-\infty}^{\infty} dp \left( a_p \psi_{\kappa}^{(+)}(x) + \tilde{a}_p^\dagger \psi_{\kappa}^{(-)}(x) \right), \quad (68)$$

where

$$\psi_{\kappa}^{(\pm)}(x) = \left( \sqrt{\epsilon_p - p} \pm \sqrt{\epsilon_p + p} \right) e^{\pm i \epsilon_p t \pm ipz} / \sqrt{4\pi \epsilon_p}, \quad \epsilon_p = \sqrt{p^2 + m^2}. \quad (69)$$

It is not difficult to ascertain that the boost modes (59) are connected with the plane waves (69) through the following integral transformation

$$\psi_{\kappa}^{(\pm)}(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi \epsilon_p}} e^{-i q \kappa} \psi_{\kappa}^{(\pm)}(x), \quad q = \text{arcsinh} \left( \frac{p}{m} \right). \quad (70)$$

The modes $\psi_{\kappa}^{(\pm)}(x)$ and $\psi_{\kappa}^{(\pm)}(x)$ are distributions with respect to variables $p$ and $\kappa$. This means that they define linear functionals

$$\tilde{\mathfrak{F}}^{(\pm)}(\mathfrak{g}, x) = \int_{-\infty}^{\infty} dp \psi_{\kappa}^{(\pm)}(x) \mathfrak{g}(p), \quad (71)$$

$$\tilde{\mathfrak{F}}^{(\pm)}(\mathfrak{h}, x) = \int_{-\infty}^{\infty} d\kappa \psi_{\kappa}^{(\pm)}(x) \mathfrak{h}(\kappa), \quad (72)$$
on some sets of test functions $g(p)$, $h(\kappa)$ respectively. Such functionals naturally appear when we calculate, e.g.,
matrix elements of the field operators $\langle 1, 2 | \cdots | 0 \rangle$ between the vacuum state $|0_M \rangle$ and the states which are particle (or antiparticle) wave packets,

$$
|1_g\rangle = \int dp \, g(p) a_p^+ |0_M \rangle, \quad |1_h\rangle = \int d\kappa \, h(\kappa) b_\kappa^+ |0_M \rangle.
$$

(73)

Physical one-particle states must be normalized,

$$
\langle 1_g | 1_g \rangle = \int dp |g(p)|^2 = 1, \quad \langle 1_h | 1_h \rangle = \int d\kappa |h(\kappa)|^2 = 1.
$$

(74)

Hence, $g(p)$, $h(\kappa)$ must be square-integrable functions of $p$ and $\kappa$ respectively. Then it is easy to see that matrix elements (71), (72) are square-integrable functions of $x$. It is clear that functionals (71), (72) are the elements of one and the same space of functions. Hence, there exists one to one correspondence between the test functions $g(p)$ and $h(\kappa)$:

$$
h(\kappa) = \int_{-\infty}^{\infty} \frac{e^{i\kappa p}}{\sqrt{2\pi p}} g(p) dp,
$$

(75)

where $q = \text{arsinh}(p/m)$ is rapidity. For the sake of convenience we will confine ourselves to $g(p)$ belonging to the class of continuous piecewise smooth functions descending faster than $|p|^{-1}$ at $|p| \to \infty$. This requirement guarantees finiteness of the mean value of the energy of the one-particle state $|1_g\rangle$ (73),

$$
E(1) = \int_{-\infty}^{\infty} dp \, \varepsilon_p \, \langle 1_g | a_p^+ a_p | 1_g \rangle = \int_{-\infty}^{\infty} dp \, \varepsilon_p \, |g(p)|^2, \quad \varepsilon_p = \sqrt{p^2 + m^2},
$$

(76)

To ascertain the properties of test functions $h(\kappa)$ we will use the relation (76).

First, we will illustrate by a simple example that, unlike the case of the functional (71), we need know the behavior of test functions $h(\kappa)$ for the functional (72) not only on the real axis but also in the complex plane. Consider

$$
g(p) = \frac{e^{-p/\alpha}}{(2\pi \alpha \varepsilon_p^2)^{1/4}} e^{-\frac{\varepsilon_p^2}{2\alpha} + i\beta}, \quad \alpha > 0,
$$

(77)

which evidently satisfies the above formulated requirements to functions $g(p)$. Using Eq. (75) we obtain

$$
h(\kappa) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha \kappa^2 + i\beta \kappa}.
$$

(78)

Direct calculation of the integral (72) at the point $x = 0$ then yields

$$
F(\pm)(\kappa, 0) = \sqrt{\frac{m}{2}} \left(\frac{2\alpha}{\pi}\right)^{1/4} \left(\pm e^{\frac{\alpha}{4} + i\beta} \right) = \sqrt{\frac{m}{2}} \left(\pm h(i/2) \right).
$$

(79)

So we see that the value of the functional (72) at the vertex of the light cone is determined by the values of the test function $h(\kappa)$ at imaginary points $\kappa = \pm i/2$.

Let us now revert to Eq. (75). After the change of the variable of integration $p = m \sinh q$ it can be written as a Fourier transform of function $\tilde{g}(q) = \sqrt{\frac{m \cosh q}{2\pi}} g(m \sinh q)$

$$
h(\kappa) = \int_{-\infty}^{\infty} dq e^{i\kappa q} \tilde{g}(q).
$$

(80)
Under our assumptions
\[
\tilde{g}(q) = O \left( e^{-\left(\frac{1}{2} + \epsilon\right)|q|} \right), \quad \epsilon > 0,
\]
as \(|q| \to \infty\). Then, according to the Paley-Wiener theorem [25] the test functions \(\mathfrak{h}(\kappa)\) are analytic in the strip
\[-1/2 - \epsilon < \text{Im} \kappa < 1/2 + \epsilon, \quad \epsilon > 0.\]

Let us calculate the integral in Eq. (80) by parts. Since the test functions \(\tilde{g}(q)\) are continuous and descend exponentially as \(|q| \to \infty\), see Eq. (81), the first integrated term is equal to zero. Hence, the functions \(\mathfrak{h}(\kappa)\) in their domain of analyticity (82) descend at least as \(\kappa^{-2}\) when \(|\kappa| \to \infty\). This means that the path of integration in (72) can be shifted in the range of the strip (82).

Consider the functional (72) at the point \(x = 0\). Shifting the path of integration for the upper component upward, and for the lower one downward, by \(i/2\), we obtain
\[
F^{(\pm)}(h, 0) = 1/2 \pi \int_{-\infty}^{\infty} d\kappa \mathfrak{h}(\kappa) = \frac{1}{2} \pi \int_{-\infty}^{\infty} d\kappa \delta(\kappa) \mathfrak{h}(\kappa) = \frac{1}{2} \pi \delta(\kappa),
\]
(83)
Eq. (83) generalizes formula (79) to the class of functions \(\mathfrak{h}(\kappa)\) analytic in the strip (82) and descending rapidly enough when \(|\kappa| \to \infty\).

The obtained result makes it possible to write down the fermion boost mode in the vertex of the light cone in terms of \(\delta\)-functions of complex argument
\[
\psi^{(\pm)}(0) = \sqrt{\frac{m}{2}} \left( \pm \delta(\kappa - i/2) \right).
\]
(84)
The functionals \(\tilde{F}^{(\pm)}(h, 0)\) can be also written as integrals around the closed contours represented in Figs. 2a,b
\[
\tilde{F}^{(\pm)}(h, 0) = \oint_{C_{a,b}} d\kappa \psi^{(\pm)}(0) \mathfrak{h}(\kappa) = \frac{1}{2} \pi \int_{-\infty}^{\infty} d\kappa \mathfrak{h}(\kappa) \delta(\kappa).
\]
(85)
Indeed, the explicit form for \(\psi^{(\pm)}(0)\) can be easily derived from Eq. (60). Putting \(x_+ = x_- = 0\) there and using the ascending series for \(K_{\nu}(z)\), we get
\[
\psi^{(\pm)}(0) = \lim_{\sigma \to 0} \frac{1}{2\pi} \int_{m}^{-i\kappa} \Gamma(1/2 + i\kappa) \left( \pm \delta(\kappa - i/2) \right).
\]
Then, taking into account that the integrand in (85) has simple poles at \(\kappa = \pm i/2\) and using the Cauchy’s residue theorem, we immediately reproduce the result (83). Since the integrals taken along the vertical segments of the contours \(C_{a,b}\) are evidently equal to zero, this means, in particular, that the integrals along the upper (lower) piece of the contour \(C_a\) (\(C_b\)) is also equal to zero in the limit \(\sigma \to 0\). The latter statement can be easily checked by direct calculation.
δ-function of complex argument was first introduced by Gel’fand and Shilov in Ref. [23] on the basis of Cauchy residue theorem,

\[
(\delta(\kappa - \kappa_0), f(\kappa)) = \frac{1}{2\pi i} \oint_L \frac{f(\kappa) d\kappa}{\kappa - \kappa_0} = f(\kappa_0),
\]

(87)

where \( L \) is a contour enclosing an arbitrary complex point \( \kappa_0 \). The distribution \( [\delta(\kappa - \kappa_0)] \) was defined in [23] on some class of entire functions \( f(\kappa) \), the so-called Z-class. Our class of test functions analytic in the strip \( \mathcal{S} \) differs from \( Z \). Therefore some properties of the Gel’fand’s δ-function [23] cannot be applied to our one. However, the opposite statement is true: all properties of our δ-function hold valid for the Gel’fand’s one.

So, on the class of test functions analytic in the strip \( \mathcal{S} \), as well as for the Z-class, the following equality of distributions is valid [13]

\[
\lim_{\sigma \to + \frac{1}{2} i} \frac{\Gamma(1/2 \pm i\kappa)}{\sigma^{1/2 \pm i\kappa}} = \delta (\kappa \mp i/2).
\]

(88)

Let us show now that the modes \( \psi_{\kappa}^{(\pm)} \) possess δ-function singularities not only at the vertex of the light cone but at lines \( x_\pm = 0 \) as well. Indeed, at the surface of the light cone the arguments of Macdonald functions in (60) are small, \(|x_\pm| \ll 1\). Therefore using the ascending series for Macdonald functions [22] we get

\[
\psi_{\kappa}^{(\pm)}(x) = e^{\pm \pi i/2 \mp i\kappa/4} \left( \frac{m}{2} \right)^{-i\kappa} \Gamma(1/2 + i\kappa) \left( x_\mp \mp i\sigma \right)^{-1/2 + i\kappa} \left( \frac{m}{2} \right)^{i\kappa} \Gamma(1/2 - i\kappa) \left( x_\pm \mp i\sigma \right)^{-1/2 - i\kappa} + \left( \frac{m}{2} \right)^{-i\kappa + 1} \Gamma(-1/2 - i\kappa) \frac{\Gamma(1/2 + i\kappa)}{(x_\mp + i\sigma)^{-1/2 + i\kappa}} \left( \frac{m}{2} \right) \Gamma(-1/2 + i\kappa) \frac{\Gamma(1/2 - i\kappa)}{(x_\pm + i\sigma)^{-1/2 - i\kappa}}. \]

(89)

So we see that at the surface of the light cone \( \psi_{\kappa}^{(\pm)}(x) \) contains \( \delta(\kappa - i/2) \) at \( x_+ \) = 0 in the upper, and \( \delta(\kappa + i/2) \) at \( x_- = 0 \) in the lower component.

It was shown in the preceding section that in the case of a scalar field the family of boost modes does not constitute a complete set in MS after excluding the zero mode. This is because the scalar boost modes possess a δ-function singularity at the surface of the light cone, and hence the point \( \kappa = 0 \) gives finite contribution to physical quantities. Some of them, e.g., Wightman function, are determined by the spectral point \( \kappa = 0 \) entirely. There is a question whether it is possible to delete the point \( \kappa = 0 \) from the spectrum in the fermion case. To clarify this issue we will consider the integral

\[
\tilde{\mathcal{F}}^{(\pm)}(\hbar, 0) = \lim_{\delta \to 0} \left( \int_{-\infty}^{\kappa_0 - \delta} + \int_{\kappa_\alpha + \delta}^{\kappa_\alpha + \delta} \right) d\kappa \psi_{\kappa}^{(\pm)}(0) \hbar(\kappa) = \delta^{(\pm)}(\hbar, 0) - \lim_{\delta \to 0} \int_{-\infty}^{\kappa_\alpha - \delta} d\kappa \psi_{\kappa}^{(\pm)}(0) \hbar(\kappa), \]

(90)

where \( \kappa_0 \) is an arbitrary real number.

Consider the upper component of the second term in the RHS of Eq. (90)

\[
\frac{1}{2\pi} \lim_{\sigma \to 0} \lim_{\delta \to 0} \left( \int_{\kappa_0 - \delta}^{\kappa_0 + \delta} d\kappa \left( \frac{m}{2} \right)^{-i\kappa} \Gamma(1/2 + i\kappa) \frac{\Gamma(1/2 - i\kappa)}{\sigma^{1/2 + i\kappa}} \hbar(\kappa) \right). \]

(91)

The integral in (91) can be easily calculated in the limit \( \sigma \to 0 \) and we obtain

\[
\lim_{\delta \to 0} \left( \int_{\kappa_0 - \delta}^{\kappa_0 + \delta} d\kappa \left[ \psi_{\kappa}^{(\pm)}(0) \right] \hbar(\kappa) \right) = \frac{i}{2\pi} \left( \frac{m}{2} \right)^{-i\kappa_0} \lim_{\delta \to 0} \lim_{\sigma \to 0} \frac{\sigma^{-i\kappa_0 - 1/2}}{\ln \sigma} \left[ \left( \frac{m}{2} \right)^{-i\delta} \Gamma(1/2 + i\kappa_0 + i\delta) \hbar(\kappa_0 + \delta) \sigma^{-i\delta} - \left( \frac{m}{2} \right)^{i\delta} \Gamma(1/2 + i\kappa_0 - i\delta) \hbar(\kappa_0 - \delta) \sigma^{i\delta} + O \left( \frac{1}{\ln \sigma} \right) \right]. \]

(92)

Eq. (92) evidently shows that the result of calculation for \( \tilde{\mathcal{F}}^{(\pm)}(\hbar, 0) \) strongly depends on the sequence of limit processing \( \delta \to 0 \) and \( \sigma \to 0 \). If we changed the sequence of limit processing in (91), (92), we would return from \( \tilde{\mathcal{F}}^{(\pm)}(\hbar, 0) \) to \( \tilde{\mathcal{F}}^{(\pm)}(\hbar, 0) \) (72). Not surprisingly, Eq. (90) reproduces the result (83) in this case. The accepted sequence which corresponds to calculation of the functional \( \tilde{\mathcal{F}}^{(\pm)} \) leads to a meaningless result. This means that pricking of any point out of the real axis \( \kappa \) is inadmissible in the fermion case. In other words, the set of boost modes
impossibility of deleting an arbitrary point \( \kappa \) from the spectrum is especially simple. Indeed, the Gel’fand’s \( \delta \)-function is an analytic distribution in the whole complex plain \( \kappa \). Thus it can be expanded in a Taylor series around any point \( \kappa - \kappa_0 \),

\[
\delta \left( \kappa \mp \frac{i}{2} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \kappa_0 \mp \frac{i}{2} \right)^n \delta^{(n)}(\kappa - \kappa_0),
\]

(93)

and the radius of convergence of the series \( |\kappa| > \kappa_0 \) is equal to infinity \( \kappa_0 \). Here \( \delta^{(n)}(\kappa - \kappa_0) \) denotes the \( n \)-th derivative of the \( \delta \)-function at the point \( \kappa - \kappa_0 \). As a result,

\[
\lim_{\delta \to 0} \left( \begin{array}{c} \int_{-\infty}^{\kappa_0} + \int_{\kappa_0 + \delta}^{\infty} \end{array} \right) \delta \kappa \delta(\kappa \pm \frac{i}{2}) b(\kappa) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \kappa_0 \pm \frac{i}{2} \right)^n \lim_{\delta \to 0} \left( \begin{array}{c} \int_{-\infty}^{\kappa_0 - \delta} + \int_{\kappa_0 + \delta}^{\infty} \end{array} \right) \delta \kappa \delta^{(n)}(\kappa - \kappa_0) b(\kappa) = 0,
\]

(94)

and hence, on the class \( Z \) of test functions, deleting of an arbitrary spectral point \( \kappa_0 \) from the spectrum leads to vanishing of functionals \( S^{(\pm)}(p, 0) \) at the vertex of the light cone, and its finite variation at other points of the cone surface. However, since the Gel’fand’s \( \delta \)-function is analytic on the whole plane of complex \( \kappa \), it can be expanded in a Taylor series around another real point \( \kappa - \kappa_1, \kappa_1 \neq \kappa_0 \). If we calculate integral \( \theta \) using such representation of the \( \delta \)-function, we will see that deleting of the spectral point \( \kappa_0 \) from the spectrum does not influence the result of integration. Thus, since the value of the matrix element \( S^{(\pm)} \) depends on the way of calculation, we conclude that the operation of exclusion of an arbitrary point from the spectrum is meaningless in agreement with the previous consideration, see discussion of Eq. \( (90) \) in the preceding paragraph.

As it was shown in the preceding section, Wightman functions \( \Delta^{(\pm)}(x) \) of a massive scalar field are determined by the zero boost modes \( \Psi_0^{(\pm)}(x) \), see Eq. \( (24) \). This is a consequence of translational invariance of \( \Delta^{(\pm)}(x) \) and the presence of Dirac \( \delta \)-function singularity of \( \Psi_0^{(\pm)}(x) \) at the light cone. A similar result is valid also for the Wightman functions of a massive fermion field. Indeed, due to the property of translational invariance we can write, compare \( (40) \),

\[
S^{(\pm)}(x', x'') = i \int_{-\infty}^{\infty} d\kappa \psi^{(\pm)}_{\kappa}(x) \psi^{(\pm)}_{\kappa}^{\dagger}(0) \gamma_0, \quad x = x' - x''.
\]

(95)

Using now Eq. \( (84) \) we get the following result for matrix elements \( S^{(\pm)}_{\alpha \beta}(x) \) of Wightman function \( (93) \),

\[
S^{(\pm)}_{\alpha \beta}(x) = \pm i \frac{m}{2} \psi^{(\pm)}_{\alpha}(x), \quad S^{(\pm)}_{\alpha \beta}(x) = \pm i \frac{m}{2} \psi^{(\pm)}_{\alpha \beta}(x), \quad \alpha, \beta = 1, 2,
\]

(96)

where \( \psi^{(\pm)}_{\alpha}(x) \) are the \( \alpha \)-components of the boost modes \( (59), (60) \). Thereby, we see that the matrix elements of \( \Delta^{(\pm)}(x) \) are determined by only two ”spectral points” \( \kappa = \pm i/2 \). Exactly as in the scalar case, it is easy to ascertain that this result holds valid for the smeared Wightman functions as well.

Using Eq. \( (50) \) one can express the Wightman functions \( (93) \) in the form, compare \( (41) \),

\[
S^{(\pm)}(x) = \frac{m}{2\pi} \left( \begin{array}{c} \pm K_0(w_{\pm}) - \frac{i}{2} h_{\pm} \\ \frac{x_+ + i\sigma}{x_- + i\sigma} \end{array} \right) K_1(w_{\pm}),
\]

(97)

where \( w_{\pm} \) is the same as in Eq. \( (40) \).

Consider now the case of a massless fermion field first studied in Ref. \( (13) \). The Wightman function of the massless field \( S^{(\pm)}_0(x) \) can be obtained easily by passage to the limit \( m \to 0 \) in Eq. \( (77) \),

\[
S_0^{(\pm)}(x) = \lim_{m \to 0} S^{(\pm)}(x) = \pm \frac{1}{2\pi} \left( \frac{1}{x_+ + i\sigma} - \frac{1}{x_- + i\sigma} \right), \quad \sigma \to 0.
\]

(98)
Just as in the cases considered earlier, one can use $S_0^{(\pm)}(x - \alpha_q)$ to obtain any complete set of orthonormalized solutions of Dirac equation. In particular, for the boost modes we have,

$$\chi^{(\pm)}(x) = \int dq S_0^{(\pm)}(x - \alpha_q)e^{-i\kappa q - \Phi q^3}C,$$

(99)

where $C = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ is an arbitrary two-component column, compare (49), (50). We see that at $m = 0$ the Wightman function (58) is antidiagonal. Owing to this property the upper and the lower components of the boost mode (99) become independent:

$$\chi^{(\pm)}(x) = \tilde{c}_1 \chi^{(\pm)}_+(x)e_+ + \tilde{c}_2 \chi^{(\pm)}_-(x)e_-,$$

(100)

where $e_\pm$ are the eigenvectors of the Pauli matrix $\sigma_3$, $\sigma_3 e_\pm = \pm e_\pm$. This is because there appears a new conservation law for the massless case, conservation of chirality, see, e.g., [26]. Thereby solutions of the Dirac equation (45) are labeled by the additional quantum number $\tau = \pm$, chirality. The coefficients $\tilde{c}_1$ and $\tilde{c}_2$ in (100) are independent and can be determined by the normalization condition for solutions $\chi^{(\pm)}_+(x)e_+$ and $\chi^{(\pm)}_-(x)e_-$ separately. Note that in the massive case, since chirality is not conserved, the components of the column $C$ (50) cannot be determined separately. In that case their combination forms a monomial factor of the solution $\psi^{(\pm)}_\kappa(x)$ (61).

For the normalized positive frequency functions $\chi^{(\pm)}_\kappa$ we have

$$\chi^{(\pm)}_{\kappa\tau}(x) = \frac{e^{\mp i\lambda \frac{\pi}{2}}\Gamma(\lambda)}{2\pi(x_\tau + i\sigma)^\lambda} \lambda = \frac{1}{2} + i\kappa, \quad \sigma \to 0,$$

(101)

where the phase factors were chosen for the sake of convenience. It is worth noting that the negative frequency functions $\chi^{(-)}_{\kappa\tau}$ come out of (101) by complex conjugation and the change $\kappa \to -\kappa$.

Plane waves are also labeled by the quantum number $\tau$ in the massless case,

$$\varphi^{(\pm)}_{\rho\tau}(x) = \frac{1}{\sqrt{2\pi}} e^{\mp ipx_\tau}, \quad 0 < p < \infty.$$

(102)

They are linked to the boost modes $\chi^{(\pm)}_{\kappa\tau}(x)e_\tau$ through the Mellin transform

$$\chi^{(\pm)}_{\kappa\tau}(x_\tau) = \int_0^\infty \frac{dp}{\sqrt{2\pi}} p^{\lambda - 1} \varphi^{(\pm)}_{\rho\tau}(x_\tau).$$

(103)

This means that, if we assume that distributions $\varphi^{(\pm)}_{\rho\tau}(x_\tau)$ are defined on the same class of test functions $g(p)$ as in the massive case, the distributions $\chi^{(\pm)}_{\kappa\tau}(x_\tau)$ will be defined on the class of test functions $b(\kappa)$ analytic in the strip $\mathbb{C}$ and descending at $|\kappa| \to \infty$ in this strip.

Then taking into account Eq. (88) we conclude that boost modes of a massless fermion field at the surface of the light cone are $\delta$-functions of a complex argument, compare [13],

$$\chi^{(\pm)}_{\kappa\tau}(x_\tau = 0) = \delta \left( \kappa - \frac{i}{\tau} \right).$$

(104)

Now we will show how the modes (101) can be obtained from Eq. (60) by limit processing $m \to 0$. Using the ascending series [22] for functions $K_{1/2 \pm i\kappa}(w_{\pm})$ we get

$$\psi^{(\pm)}_\kappa(x) = \left( \frac{m}{2} \right)^{-ik} \chi^{(\pm)}_{\kappa+} e_+ + \left( \frac{m}{2} \right)^{ik} \chi^{(\pm)}_{\kappa-} e_-,$$

(105)

i.e., representation (100) with coefficients $\tilde{c}_1$ and $\tilde{c}_2$ containing singular at $m \to 0$ phase factors. However these factors do not influence the normalization constants, have no impact on any physical quantities and hence can be omitted.
IV. CONCLUDING REMARKS

We have shown that Wightman function of a free quantum field generates any complete set of solutions of relativistic wave equations. Using this approach we have constructed the complete sets of solutions to KFG and Dirac equations consisting of eigenfunctions of the generator of Lorentz rotations (boost operator).

Boost modes are used as a basis for field quantization very rarely. Till now they were exploited only for analysis of the so-called “Unruh effect” [2, 12] and at attempts to quantize a charged massive scalar field in the presence of an external constant electric field [27, 28]. However, there are many problems, especially in the quantum field theory in a curved space, where the boost symmetry may appear to be the only symmetry for the quantum field, and thus using it for separation of variables in a classical field equation is the only instrument to find solutions for such equations.

The specific feature of the boost modes is that, taken at the surface of the light cone, they as functions of the boost argument was first introduced by Gel’fand and Shilov in Ref. [23], and was defined on the class consisting of eigenfunctions of the generator of Lorentz transformations at $v = c$.

Our $\delta$-function is defined on the class of test functions analytic in the strip (82). Actually, the width of the strip $|b| = \kappa$ means that $\Psi_0(x)$ is a Lorentz invariant positive frequency solution of KFG equation, i.e. the Wightman function for the quantum field coinciding with the positive frequency part of the commutator of two scalar field operators. Hence the exclusion of the zero boost mode results in a ”quantum” theory with commuting field operators.

The singularities of the fermion modes are even stronger. It is shown in the present paper that at the surface of the light cone they possess $\delta$-function of a complex argument $\psi_\kappa(x)$ analytic in the strip (82). Actually, the width of the strip is determined by physical requirements. Our choice provides square integrability of one-particle wave packets and finiteness of their energy. If we require finiteness of the squared energy we should narrow the class of test functions and extend the width of the strip to

$$-3/2 - \epsilon < \text{Im}\kappa < 3/2 + \epsilon, \quad \epsilon > 0.$$  

Further toughening of requirements to physical states will lead to subsequent extension of the width of the strip of analyticity of the test functions. Therefore it is reasonable to generalize the concept of $\delta$-function of a complex argument and introduce the distribution $\delta_s(\kappa)$

$$(\delta_s(\kappa - \kappa_0), f(\kappa)) = f(\kappa_0),$$  

defined on the class $Z_s$ of test functions $f(\kappa)$ analytic in the strip

$$-s - \epsilon < \text{Im}\kappa < s + \epsilon, \quad s > 0, \quad \epsilon > 0,$$

and $\kappa_0$ belong to this strip. In such notation the $\delta$-function introduced in Sec. III will look as $\delta_{1/2}(\kappa)$, the Gel’fand $\delta$-function as $\delta_\infty(\kappa)$ and the standard Dirac $\delta$-function as $\delta_0(\kappa)$.

It is worth noting that for $\delta$-function defined on the class $Z_s$ the following integral representation is valid

$$\delta_s(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{-i\nu q}, \quad \nu = \kappa \mp is, \quad s > 0,$$

which at $s = 0$ is a straightforward generalization of the standard representation for the Dirac $\delta$-function. Besides, the following relations are valid,

$$\delta_s(\nu) = \frac{i}{2} H^{(1)}_{\nu}(0) = -\frac{i}{2} H^{(2)}_{\nu}(0) = \frac{1}{\pi} K_{\nu}(0), \quad \nu = \kappa \mp is, \quad s > 0,$$

compare Ref. [24] for the case $s = 0$.

The presence of a $\delta$-function of a complex argument in a boost mode at the surface of the light cone does not allow to exclude any point from the boost spectrum. We have shown this for the case $s = 1/2$, see Eqs. (90)–(94), and this statement proving does not change for an arbitrary value of $s$. Another way to prove this statement was used for the distribution $\delta_\infty(\kappa)$ based on its analyticity in the whole complex plane. The analogous proof could be given for the $\delta_s$-function as well. In this case, due to the finite value of radius of convergence $R = s + \epsilon$, the expansion (83) can be applied only for real $\kappa_0 = b_1$, $|b_1| < \epsilon$. As a next step the Dirac $\delta$-function $\delta_0(\kappa - b_1)$ and every its derivative can be
represented as a Taylor series of the type \(\delta\) centered at the point \(b_2\) on the real axis, \(|b_2 - b_1| < s + \epsilon\). So, \(\delta_0(\kappa - is)\) will be expanded in a 2-multiple series of the Dirac \(\delta_0(\kappa - b_2)\)-function and its derivatives. After a finite number of steps \(N\) we can reach an arbitrary spectral point \(\kappa_0\) and hence obtain a representation of \(\delta_0(\kappa - is)\) in the form of a \(N\)-multiple series of the Dirac \(\delta_0(\kappa - \kappa_0)\)-function and its derivatives. It is clear that the point \(\kappa_0\) cannot be excluded from the spectrum then.

The latter reasoning clearly explains why the scalar case, when we have the only one distinguished point \(\kappa = 0\) which cannot be excluded from the spectrum, differs drastically from the fermion case. Indeed, the strip of analyticity for \(\delta_0(\kappa)\) degenerates into the real axis of the \(\kappa\) complex plane at \(s = 0\), so that the radius of convergence for the corresponding Taylor series becomes equal to zero, or in other words the Dirac \(\delta_0(\kappa)\)-function is not an analytic distribution. Thus the procedure discussed above cannot be realized.

To conclude, it is worth emphasizing that we have shown explicitly that smearing of boost modes, or Wightman functions does not change our results, see also Ref. [30].

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[1] N.N. Bogoliubov, D.V. Shirkov, \textit{Introduction to the theory of quantized fields}, NY.: John Wiley (1980).
[2] A.I. Akhiezer, V.B. Berestetskii, \textit{Quantum electrodynamics}, NY: Interscience Publishers (1965).
[3] A. I. Nikishov, Zh. Eksp. Teor. Fiz. \textbf{57}, 1210 (1969) [Sov. Phys. JETP, \textbf{30}, 660 (1969)].
[4] N. B. Narozhny, A. I. Nikishov, Theor. Math. Phys. \textbf{26}, 9 (1976); Issues in Intense-Field Quantum Electrodynamics: Proceedings of the Lebedev Physics Institute of the Academy of Sciences of the USSR: v.168, ed. by V.L. Ginzburg (Nova Science Publishers, New York, 1989) p. 226.
[5] D.M. Volkov, Z. Phys., \textbf{94}, 250 (1935).
[6] P. J. Redmond, J. Math. Phys., \textbf{6}, 1163 (1965).
[7] D. Grumiller, W. Kummer, D. V. Vassilevich, Phys.Rept. \textbf{369}, 327 (2002).
[8] S.V. Christensen, S.A. Fulling, Phys. Rev. D \textbf{15}, 2088 (1977).
[9] W.G. Unruh, Phys. Rev. D \textbf{14}, 870 (1976).
[10] A.I. Nikishov, V.I. Ritus, Zh. Eksp. Teor. Fiz. \textbf{94}, 31 (1988) [Sov. Phys. JETP, \textbf{67}, 1313 (1988)].
[11] U. Gerlach, Phys. Rev. D \textbf{38}, 514 (1988).
[12] N.B. Narozhny, A.M. Fedotov, B.M. Karnakov, V.D. Mur, and V.A. Belinskii, Phys. Rev. D \textbf{65}, 025004 (2002).
[13] A. M. Fedotov, N. B. Narozhny, V. D. Mur and E. G. Gelfer, JETP Letters \textbf{89}, 385 (2009).
[14] M. Soffel, B. Müller, W. Greiner, Phys. Rev. D \textbf{22}, 1935 (1980).
[15] W. Greiner, B. Müller, J. Rafelski, \textit{Quantum Electrodynamics of Strong Fields}, (Springer-Verlag, 1985)
[16] D. McMahon, P. M. Asling, and P. Embid, arXiv:gr-qc/061010
[17] R. Jost, \textit{The General Theory of Quantized Fields}, (AMS, Providence, 1965).
[18] R.F. Streater, A.S. Wightman, \textit{PCT, Spin and Statistics and All That}, (Princeton University Press, Princeton, 2000).
[19] N.N. Bogoliubov, A.A. Logunov, I.T. Todorov, \textit{Introduction to axiomatic quantum fields theory}, (W.A. Benjamin, Inc., New York, 1975).
[20] A.S. Wightman,\textit{Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields}, (Princeton University Press, Princeton, 1964).
[21] L.C.B. Crispino, A. Higuchi, G.E.A. Matsas, Rev. of Mod. Phys. \textbf{80}, 788 (2008).
[22] I. S. Gradstein and I. M. Ryzhik, \textit{Table of Integrals, Series, and Products} (Academic Press, New York, 1975).
[23] I.M. Gel’fand, G.E. Shilov, \textit{Generalized functions, Volume 1}, (Academic Press, New York, 1964).
[24] V.A. Fock, Zeit. Phys. 57, 261 (1929).
[25] R.E.A.C. Paley and N. Wiener, \textit{Fourier Transforms in the Complex Domain}, (American Mathematical Society, New York, 1934).
[26] V. N. Krasnikov, V. A. Matveev, V. A. Rubakov, A. N. Tavkhelidze, V. F. Tokarev, Theor. Math. Phys. \textbf{45}, 1048 (1980)
[27] C. Gabriel and P. Spindel, Annals Phys. \textbf{284}, 263 (2000).
[28] N.B. Narozhny, V.D. Mur and A.M. Fedotov, Phys. Lett. A \textbf{315}, 169 (2003).
[29] D.V. L’vov, A.S. Shelepin, L.A. Shelepin, Yad. Fiz. \textbf{57}, 1147 (1994).
[30] N.B. Narozhny, A.M. Fedotov, B.M. Karnakov, V.D. Mur, and V.A. Belinskii, Phys. Rev. D \textbf{70}, 048702 (2004).