Quasi-energy function for diffeomorphisms with wild separatrices

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Abstract

According to Pixton [8] there are Morse-Smale diffeomorphisms of $S^3$ which have no energy function, that is a Lyapunov function whose critical points are all periodic points of the diffeomorphism. We introduce the concept of quasi-energy function for a Morse-Smale diffeomorphism as a Lyapunov function with the least number of critical points and construct a quasi-energy function for any diffeomorphism from some class of Morse-Smale diffeomorphisms on $S^3$.

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1 Formulation of results

According to [3], given a closed smooth $n$-manifold $M^n$ and a Morse function $\varphi : M^n \to \mathbb{R}$ is called a Morse-Lyapunov function for Morse-Smale diffeomorphism $f : M^n \to M^n$ if:

1) $\varphi(f(x)) < \varphi(x)$ if $x \notin \text{Per}(f)$ and $\varphi(f(x)) = \varphi(x)$ if $x \in \text{Per}(f)$, where $\text{Per}(f)$ is the set of periodic points of $f$;

2) any point $p \in \text{Per}(f)$ is a non-degenerate maximum of $\varphi|_{W^u(p)}$ and a non-degenerate minimum of $\varphi|_{W^s(p)}$.

Definition 1.1 Given a Morse-Smale diffeomorphism $f : M^n \to M^n$, a function $\varphi : M^n \to \mathbb{R}$ is a quasi-energy function for $f$ if $\varphi$ is a Morse-Lyapunov function for $f$ and has the least possible number of critical points among all Morse-Lyapunov functions for $f$.

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In this paper we consider the class $G_4$ of Morse-Smale diffeomorphisms $f : S^3 \to S^3$ whose nonwandering set consists of exactly four fixed points: one source $\alpha$, one saddle $\sigma$ and two sinks $\omega_1$ and $\omega_2$. It follows from [3] (theorem 2.3), that the closure of each connected component (separatrix) of the one-dimensional manifold $W^u(\sigma) \setminus \sigma$ is homeomorphic to a segment which consists of this separatrix and two points: $\sigma$ and some sink. Denote by $\ell_1, \ell_2$ the one-dimensional separatrices containing the respective sinks $\omega_1, \omega_2$ in their closures. According to [2], $\ell_i, i = 1, 2$ is everywhere smooth except, maybe, at $\omega_i$. So the topological embedding of $\ell_i$ may be complicated in a neighborhood of the sink.

According to [1], $\ell_i$ is called tame (or tamely embedded) if there is a homeomorphism $\psi_i : W^s(\omega_i) \to \mathbb{R}^n$ such that $\psi_i(\omega_i) = O$, where $O$ is the origin and $\psi_i(\ell_i \setminus \sigma)$ is a ray starting from $O$. In the opposite case $\ell_i$ is called wild. It follows from a criterion in [1] that the tameness of $\ell_i$ is equivalent to the existence of a smooth 3-ball $B_i$ around $\omega_i$ in any neighborhood of $\omega_i$ such that $\ell_i \cap \partial B_i$ consists of exactly one point. Using lemma 4.1 from [3] it is possible to make this criterion more precise in our dynamical setting: $\ell_i$ is tame if and only if there is 3-ball $B_{\omega_i}$ such that $\omega_i \in f(B_{\omega_i}) \subset \text{int} B_{\omega_i} \subset W^s(\omega_i)$ and $\ell_i \cap \partial B_{\omega_i}$ consists of exactly one point.

It was proved in [3] that, for every diffeomorphism $f \in G_4$, at least one separatrix ($\ell_1$ say) is tame. It was also shown that the topological classification of diffeomorphisms from $G_4$ is reduced to the embedding classifications of the separatrices $\ell_2$; hence there are infinitely many diffeomorphisms from $G_4$ which are not topologically conjugate.

To characterize a type of embedding of $\ell_2$ we introduce some special Heegaard splitting of $S^3$. Let us recall that a three-dimensional orientable manifold is a handlebody of genus $g \geq 0$ if it is obtained from a 3-ball by an orientation reversing identification of $g$ pairs of pairwise disjoint 2-discs in its boundary. The boundary of such a handlebody is an orientable surface of genus $g$.

Let $P^+ \subset S^3$ be a handlebody of genus $g$ such that $P^- = S^3 \setminus \text{int} P^+$ is a handlebody (necessarily of the same genus as $P^+$). Then the pair $(P^+, P^-)$ is a Heegaard splitting of genus $g$ of $S^3$ with Heegaard surface $S = \partial P^+ = \partial P^-.$

**Definition 1.2** A Heegaard splitting $(P^+, P^-)$ of $S^3$ is said to be adapted to $f \in G_4$, or $f$-adapted, if:

- $a)$ $W^u(\sigma) \subset f(P^+) \subset \text{int} P^+$;
- $b)$ $W^s(\sigma)$ intersects $\partial P^+$ transversally and $W^s(\sigma) \cap P^+$ consists of a unique 2-disc.

An $f$-adapted Heegaard splitting $S^3 = P^+ \cup P^-$ is said to be minimal if its genus is minimal among all $f$-adapted splittings.

For each integer $k \geq 0$ we denote by $G_{4,k}$ the set of diffeomorphisms $f \in G_4$ for which the minimal $f$-adapted Heegaard splitting has genus $k$. It is easily seen that, for each $f \in G_{4,0}, \ell_2$ is tame and, according to [3], $f$ possesses an energy function. Conversely any diffeomorphism in $G_{4,k}$, $k > 0$, has no energy function (see [3]). Figure 1 shows the phase portrait of a diffeomorphism $G_{4,1}$. The main result of this paper is the following.

**Theorem 1** Every quasi-energy function for a diffeomorphism $f \in G_{4,1}$ has exactly six critical points.
2 Recollection of Morse theory

According to Milnor ([6], section 3), we use the following definitions.

A compact \((n + 1)\)-dimensional cobordism is a triad \((W, L_0, L_1)\) where \(L_0\) and \(L_1\) are closed manifolds of dimension \(n\) and \(W\) is a compact \((n + 1)\)-dimensional manifold whose boundary consists of the disjoint union \(L_0 \cup L_1\). It is an elementary cobordism when it possesses a Morse function \(\varphi : W \to [0, 1]\) with only one critical point and such that \(\varphi^{-1}(i) = L_i\) for \(i = 0, 1\). When the index of the unique critical point is \(r\), one speaks of an elementary cobordism of index \(r\).

In this situation, \(L_1\) is obtained from \(L_0\) by a surgery of index \(r\), that is: there is an embedding \(h : S^{r-1} \times D^{n-r+1} \to L_0\) such that \(L_1\) is diffeomorphic to the manifold obtained from \(L_0\) by removing the interior of the image of \(h\) and gluing \(D^r \times S^{n-r}\), or

\[
L_1 \cong D^r \times S^{n-r} \bigcup_{h|_{pr-1 \times S^{n-r}}} L_0 \setminus \text{int} (h(S^{r-1} \times D^{n-r+1})) .
\]

Conversely, the following statement holds (see [6], Theorem 3.12):

**Statement 2.1** If \(L_1\) is obtained from \(L_0\) by a surgery of index \(r\), then there exists an elementary cobordism \((W, L_0, L_1)\) of index \(r\).

On figure 3 it is seen a surgery of index 1 from the 2-sphere to the 2-torus with some level sets of a Morse function on the corresponding elementary cobordism.

Finally, we recall the weak Morse inequalities (see [5], Theorem 5.2).
Statement 2.2 Let $M^n$ be a closed manifold, $\varphi : M^n \to \mathbb{R}$ be a Morse function, $C_q$ be the number of critical points of index $q$ and $\beta_q(M^n)$ be the $q$-th Betti number of the manifold $M^n$. Then $\beta_q(M^n) \leq C_q$ and the Euler characteristic $\chi(M^n) \equiv \sum_{q=0}^{n} (-1)^q \beta_q(M^n)$ equals $\sum_{q=0}^{n} (-1)^q C_q$.

3 Proof of Theorem [1]

Let $f$ be a Morse-Smale diffeomorphism of the 3-sphere belonging to $G_{4,1}$. As the number of critical points of any Morse function on a closed 3-manifold is even (it follows from statement 2.2) and greater than four (as $\text{Per}(f) \subset \text{Cr}(\varphi)$ and $\ell_2$ is wild) then, for proving theorem 1, it is enough to construct a Lyapunov function with six critical points.

3.1 Auxiliary statements

For the proof of the following statements 3.1 and 3.2 we refer to [3], lemma 2.2 and lemma 4.2.

Statement 3.1 Let $p$ be a fixed point of a Morse-Smale diffeomorphism $f : M^n \to M^n$ such that $\dim W^u(p) = q$. Then, in some neighborhood $U_p$ of $p$, there exist local coordinates $x_1, \ldots, x_n$ vanishing at $p$ and an energy function $\varphi_p : U_p \to \mathbb{R}$ such that

$$
\varphi_p(x_1, \ldots, x_n) = q - x_1^2 - \ldots - x_q^2 + x_{q+1}^2 + \ldots + x_n^2
$$

and $(TW^u(p) \cap U_p) \subset Ox_1 \ldots x_q$, $(TW^s(p) \cap U_p) \subset Ox_{q+1} \ldots x_n$. 
**Statement 3.2** Let \( \omega \) be a fixed sink of a Morse-Smale diffeomorphism \( f : M^3 \to M^3 \) and \( B_\omega \) be a 3-ball with boundary \( S_\omega \) such that \( \omega \in f(B_\omega) \subset \text{int} \ B_\omega \subset W^s(\omega) \). Then there exists an energy function \( \varphi_{B_\omega} : B_\omega \to \mathbb{R} \) for \( f \) having \( S_\omega \) as a level set.

**Lemma 3.3** Let \( \omega \) be a fixed sink of a Morse-Smale diffeomorphism \( f : M^3 \to M^3 \) and \( Q_\omega \) be a solid torus such that \( \omega \in f(Q_\omega) \subset \text{int} \ Q_\omega \subset W^s(\omega) \). Then there exists a 3-ball \( B_\omega \) such that \( f(Q_\omega) \subset B_\omega \subset \text{int} \ Q_\omega \).

**Proof:** Let \( D_0 \) be a meridian disk in \( Q_\omega \) such that \( \omega \notin D_0 \). As \( Q_\omega \subset W^s(\omega) \) there is an integer \( N \) such that \( f^n(Q_\omega) \cap D_0 = \emptyset \) for every \( n > N \). We may also assume that \( D_0 \) is transversal to \( G = \bigcup_{n \in \mathbb{Z}} f^n(\partial Q_\omega) \), and hence \( G \cap \text{int} \ D_0 \) consists of a finite family \( C_{D_0} \) of intersection curves. Each intersection curve \( c \in C_{D_0} \) belongs to \( f^k(\partial Q_\omega) \) for some integer \( k \in \{1, \ldots, N\} \). There are two cases: (1) \( c \) bounds a disk on \( f^k(\partial Q_\omega) \); (2) \( c \) does not bound a disk on \( f^k(\partial Q_\omega) \). Let us decompose \( C_{D_0} \) as union of two pairwise disjoint parts \( C^1_{D_0} \) and \( C^2_{D_0} \) consisting of curves with property (1) or (2), accordingly.

Let us show that there is a meridian disk \( D_1 \) in \( Q_\omega \) such that \( D_1 \) is transversal to \( G \) and \( G \cap \text{int} \ D_1 \) consists of family \( C_{D_1} = C^2_{D_0} \) of intersection curves. If \( C^1_{D_0} = \emptyset \) then \( D_1 = D_0 \). In the opposite case for any curve \( c \in C^1_{D_0} \) denote by \( d_c \) the disk on \( f^k(\partial Q_\omega) \) such that \( \partial d_c = c \). Notice that \( d_c \) does not contain a curve from the family \( C^2_{D_0} \). Then there is \( c \in C_{D_1} \) which is innermost on \( f^k(\partial Q_\omega) \) in the sense that the interior of \( d_c \) contains no intersection curves from \( C_{D_0} \). For such a curve \( c \) denote \( e_c \) the disk on \( D_0 \) such that \( \partial e_c = c \). As \( \text{int} \ Q_\omega \setminus D_0 \) is an open 3-ball then \( e_c \cup d_c \) bounds a unique 3-ball \( b_c \subset \text{int} \ Q_\omega \). Set \( D'_c = (D_0 \setminus e_c) \cup d_c \). There is a smooth approximation \( D_c \) of \( D'_c \) such that \( D_c \) is a meridian disk on \( Q_\omega \). \( D_c \) is transversal to \( G \). Moreover \( G \cap \text{int} \ D_c \) consists of a family \( C_{D_c} \) of intersection curves having less elements than \( C_{D_0} \); indeed, \( d_c \) disappeared and also all curves from \( C_{D_0} \) lying in \( \text{int} \ e_c \). We will repeat this process until getting a meridian disk \( D_1 \) with the required property.

Now let \( c \in C_{D_1} \), \( c \in f^k(\partial Q_\omega) \). Denote \( e_c \) the disk that \( c \) bounds in \( D_1 \). Let us choose \( c \) innermost in \( D_1 \) in the sense that the interior of \( e_c \) contains no intersection curves from \( C_{D_1} \). There are two cases: (a) \( e_c \subset f^k(Q_\omega) \) and (b) \( \text{int} \ e_c \cap f^k(Q_\omega) = \emptyset \).

In case (a) \( e_c \) is a meridian disk of \( f^k(Q_\omega) \) and \( D = f^{-k}(e_c) \) is a meridian disks in \( Q_\omega \) such that \( f(Q_\omega) \cap D = \emptyset \). Indeed, by construction \( \text{int} \ e_c \cap G = \emptyset \), hence \( \text{int} \ D \cap G = \emptyset \). Thus we can find the required 3-ball \( B_\omega \) inside \( \text{int} \ Q_\omega \setminus D_1 \).

In case (b) there is a tubular neighborhood \( V(e_c) \subset \text{int} \ Q_\omega \) of the disk \( e_c \) such that \( G \cap \text{int} \ V(e_c) = \emptyset \) and \( B_\omega = f^k(Q_\omega) \cup V(e_c) \) is 3-ball. Then \( f^k(Q_\omega) \subset B_k \subset \text{int} \ f^{k-1}(Q_\omega) \). Thus \( B_\omega = f^{1-k}(B_k) \) is the required 3-ball.

3.2 Construction of a quasi-energy function for a diffeomorphism \( f \in G_{4,1} \)

As a similar construction was done in section 4.3 of [3], we only give a sketch of it below.

1. Construct an energy function \( \varphi_p : U_p \to \mathbb{R} \) near each fixed point \( p \) of \( f \) as in statement 3.1.
2. By definition of the class $G_{4,1}$, for each $f \in G_{4,1}$ there is a solid torus $P^+$ belonging to a Heegaard splitting $(P^+, P^-)$ of $S^3$ and such that:

a) $\overline{W^u(\sigma)} \subset f(P^+) \subset \text{int } P^+$;

b) $W^s(\sigma)$ intersects $\partial P^+$ transversally and $W^s(\sigma) \cap P^+$ consists of a unique 2-disk.

As $S^3 \setminus \overline{W^s(\sigma)}$ is the disjoint union $W^s(\omega_1) \cup W^s(\omega_2)$, then by property b), the disk $P^+ \cap W^s(\sigma)$ is separating in $P^+$. Moreover there exists a neighborhood of $P^+ \cap W^s(\sigma)$, such that after removing it from $P^+$ we get a 3-ball $P_{\omega_1}$ and solid torus $P_{\omega_2}$ with the following properties for each $i = 1, 2$:

i) $\omega_i \in f(P_{\omega_i}) \subset \text{int } P_{\omega_i} \subset W^s(\omega_i)$;

ii) $\partial P_{\omega_i}$ is a Heegaard surface and $\ell_i \cap \partial P_{\omega_i}$ consists of exactly one point.

Due to the $\lambda$-lemma\(^1\) (see, for example,[7]), replacing $P_{\omega_i}$ by $f^{-n}(P_{\omega_i})$ for some $n > 0$ if necessary, we may assume that $\partial P_{\omega_i}$ is transversal to the regular part of the critical level set $C := \varphi^{-1}_\sigma(1)$ of the function $\varphi_\sigma$ and the intersections $C \cap \partial P_{\omega_i}$ consist of exactly one circle. For $\varepsilon \in (0, \frac{1}{2})$ define $H^+_\varepsilon$ as the closure of $\{x \in U_\sigma \mid x \notin (P_{\omega_1} \cup P_{\omega_2}), \varphi_\sigma(x) \leq 1 + \varepsilon\}$ and set $P^+_{\varepsilon} = P_{\omega_1} \cup P_{\omega_2} \cup H^+_\varepsilon$. In the same way as in [8] it is possible to choose $\varepsilon > 0$ such that $\partial P_{\omega_i}$ intersects transversally each level set with value in $[1 - \varepsilon, 1 + \varepsilon]$; this intersection consists of one circle. Taking a smoothing $Q^+$ of $P^+_{\varepsilon}$ we have $f(Q^+) \subset \text{int } Q^+$ and $\Sigma := \partial Q^+$ is a Heegaard surface of genus 1. Let $Q^-$ be the closure of $S^3 \setminus \text{int } Q^+$ (see

\(^1\)The $\lambda$-lemma claims that $f^{-n}(S_{\omega_i}) \cap U_\sigma$ tends to $\{x_1 = 0\} \cap U_\sigma$ in the $C^1$ topology when $n$ goes to $+\infty$.}
3. For each \( i = 1, 2 \), let \( \hat{P}_{\omega_i} \) be a handlebody of genus \( i - 1 \) such that \( f(P_{\omega_i}) \subset \hat{P}_{\omega_i} \subset \text{int } P_{\omega_i} \), \( \partial \hat{P}_{\omega_i} \) intersects transversally each level set with value in \([1 - \varepsilon, 1 + \varepsilon]\) along one circle and \( P_{\omega_i} \setminus \text{int } \hat{P}_{\omega_i} \) is diffeomorphic to \( \partial P_{\omega_i} \times [0, 1] \). Define \( d_i \) as the closure of \( \{ x \in U_\sigma \mid x \in (W^s(\omega_i)) \setminus \hat{P}_{\omega_i}, \varphi_\sigma(x) = 1 - \varepsilon \} \). By construction \( d_i \) is a disk whose boundary curve bounds a disk \( D_i \) in \( \partial \hat{P}_{\omega_i} \). We form \( S_i \) by removing the interior of \( D_i \) from \( \partial \hat{P}_{\omega_i} \) and gluing the \( d_i \). Denote \( P(S_i) \) the handlebody of genus \( i - 1 \) bounded by \( S_i \) and containing \( \omega_i \). As in [3] it is possible to choose \( \varepsilon \) such that \( f(P(S_i)) \subset \text{int } P(S_i) \).

Let \( K \) be the domain between \( \partial Q^+ \) and \( S_1 \cup S_2 \). We introduce \( T^+ \), the closure of \( \{ x \in S^3 \mid x \notin (P_{\omega_1} \cup P_{\omega_2}), 1 - \varepsilon \leq \varphi_\sigma(x) \leq 1 + \varepsilon \} \); observe \( T^+ \subset U_\sigma \). We define a function \( \varphi_\kappa : K \to \mathbb{R} \) whose value is \( 1 + \varepsilon \) on \( \partial Q^+ \), \( 1 - \varepsilon \) on \( S_1 \cup S_2 \), coinciding with \( \varphi_\sigma \) on \( K \cap T^+ \) and without critical points outside \( T^+ \). This last condition is easy to satisfy as the domain in question is a product cobordism. In a similar way to [3], section 4.3, one can check that \( \varphi_\kappa \) is a Morse-Lyapunov function.

4. As \( P(S_1) \) is a 3-ball such that \( \omega_1 \in f(P(S_1)) \subset \text{int } P(S_1) \subset W^s(\omega_1) \), then by statement 3.3 there is an energy function \( \varphi_{P(S_1)} : P(S_1) \to \mathbb{R} \) for \( f \) with \( S_1 \) as a level set with value \( 1 - \varepsilon \).

5. As \( P(S_2) \) is a solid torus such that \( \omega_2 \in f(P(S_2)) \subset \text{int } P(S_2) \subset W^s(\omega_2) \), then according to lemma 3.3 there is a 3-ball \( B_{\omega_2} \) such that \( f(P(S_2)) \subset B_{\omega_2} \subset \text{int } P(S_2) \). As in the previous item, there is an energy function \( \varphi_{B_{\omega_2}} : B_{\omega_2} \to \mathbb{R} \) for \( f \) with \( \partial B_{\omega_2} \) as a level set with value \( \frac{1}{2} \).

6. As \( P(S_2) \) is a solid torus, it is obtained from a 3-ball by an orientation reversing identification of a pair of disjoint 2-discs in its boundary; hence the solid torus is the union of a 3-ball and an elementary cobordism of index 1. Since, up to isotopy, there is only one 3-ball in the interior of a solid torus, then \( (W_{\omega_2}, \partial B_{\omega_2}, S_2) \) is an elementary cobordism of index 1, where \( W_{\omega_2} = P(S_2) \setminus \text{int } B_{\omega_2} \). Hence \( W_{\omega_2} \) possesses a Morse function \( \varphi_{W_{\omega_2}} \) with only one critical point of index 1 and such that \( \varphi_{W_{\omega_2}}(\partial B_{\omega_2}) = \frac{1}{2}, \varphi_{W_{\omega_2}}(S_2) = 1 - \varepsilon \).

7. Define the smooth function \( \varphi^+ : Q^+ \to \mathbb{R} \) by the formula

\[
\varphi^+(x) = \begin{cases}
\varphi_\kappa(x), & x \in K; \\
\varphi_{P(S_1)}(x), & x \in P(S_1); \\
\varphi_{B_{\omega_2}}(x), & x \in B_{\omega_2}; \\
\varphi_{W_{\omega_2}}(x), & x \in W_{\omega_2}.
\end{cases}
\]

Then \( \varphi^+ \) is a Morse-Lyapunov function for \( f|_{Q^+} \) with one additional critical point.

8. By the construction \( Q^- \) is a solid torus such that \( \alpha \in f^{-1}(Q^-) \subset \text{int } Q^- \subset W^u(\alpha) \). Since \( \alpha \) is a sink for \( f^{-1} \) then, as in item 4, there is a 3-ball \( B_\alpha \) such that \( f^{-1}(Q^-) \subset B_\alpha \subset \text{int } Q^- \) and an energy function \( \varphi_{B_\alpha} : B_\alpha \to \mathbb{R} \) for \( f^{-1} \) with \( \partial B_\alpha \) as a level set of value \( \frac{1}{2} \).
9. Similarly to item 5, $\partial Q^-$ is obtained from $\partial B_\alpha$ by a surgery of index 1. Therefore $(W_\alpha, \partial Q^-, \partial B_\alpha)$ is an elementary cobordism of index 1, where $W_\alpha = Q^- \setminus \text{int } B_\alpha$. Hence, $W_\alpha$ possesses a Morse function $\varphi_{w_\alpha}$ with only one critical point of index 1. We may choose $\varphi_{w_\alpha}(\partial B_\alpha) = \frac{1}{2}$, $\varphi_{w_\alpha}(\partial Q^-) = 2 - \varepsilon$.

10. Define the smooth function $\varphi^- : Q^- \to \mathbb{R}$ by the formula

$$
\varphi^-(x) = \begin{cases} 
3 - \varphi_{B_\alpha}(x), & x \in \varphi_{B_\alpha}; \\
3 - \varphi_{W_\alpha}(x), & x \in \varphi_{W_\alpha}.
\end{cases}
$$

Then $\varphi^-$ is a Morse-Lyapunov function for $f|_{Q^-}$ with one additional critical point.

11. The function $\varphi : S^3 \to \mathbb{R}$ defined by $\varphi|_{Q^+} = \varphi^+$ and $\varphi|_{Q^-} = \varphi^-$ is the required Morse-Lyapunov function for the diffeomorphism $f$ with exactly six critical points.

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