Nonlinear Asymptotic Stability of the Lane-Emden Solutions for the Viscous Gaseous Star Problem with Degenerate Density Dependent Viscosities

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Abstract

The nonlinear asymptotic stability of Lane-Emden solutions is proved in this paper for spherically symmetric motions of viscous gaseous stars with the density dependent shear and bulk viscosities which vanish at the vacuum, when the adiabatic exponent \( \gamma \) lies in the stability regime \((4/3, 2)\), by establishing the global-in-time regularity uniformly up to the vacuum boundary for the vacuum free boundary problem of the compressible Navier-Stokes-Poisson systems with spherical symmetry, which ensures the global existence of strong solutions capturing the precise physical behavior that the sound speed is \( C^{1/2} \)-Hölder continuous across the vacuum boundary, the large time asymptotic uniform convergence of the evolving vacuum boundary, density and velocity to those of Lane-Emden solutions with detailed convergence rates, and the detailed large time behavior of solutions near the vacuum boundary. The results obtained in this paper extend those in [30] of the authors for the constant viscosities to the case of density dependent viscosities which are degenerate at vacuum states.

1 Introduction

The evolution of a viscous gaseous star in three spatial dimensions with the boundary being the interface between the gas and vacuum can be modeled by the following free boundary problem

\[
\begin{cases}
\rho_t + \text{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega(t), \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \text{div}\mathbf{S} = -\rho \nabla_x \Psi & \text{in } \Omega(t), \\
\rho > 0 & \text{in } \Omega(t), \\
\rho = 0 \text{ and } \mathbf{S}\mathbf{n} = 0 & \text{on } \Gamma(t) := \partial \Omega(t), \\
\mathbf{V}(\Gamma(t)) = \mathbf{u} \cdot \mathbf{n}, & \text{on } \Omega := \Omega(0). \\
\rho = \rho_0, \mathbf{u} = \mathbf{u}_0 & \text{on } \Omega := \Omega(0).
\end{cases}
\]

(1.1)

Here \((x, t) \in \mathbb{R}^3 \times [0, \infty)\), \(\rho, \mathbf{u}, \mathbf{S}\) and \(\Psi\) denote, respectively, the space and time variable, density, velocity, stress tensor and gravitational potential; \(\Omega(t) \subset \mathbb{R}^3\), \(\Gamma(t)\), \(\mathbf{V}(\Gamma(t))\) and \(\mathbf{n}\) represent, respectively, the changing volume occupied by a fluid at time \(t\), moving interface of fluids and vacuum states, normal velocity of \(\Gamma(t)\) and exterior unit normal vector to \(\Gamma(t)\). The gravitational potential is given by

\[
\Psi(x, t) = -G \int_{\Omega(t)} \frac{\rho(y, t)}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}, \text{ satisfying } \Delta \Psi = 4\pi G \rho \text{ in } \Omega(t)
\]

with the gravitational constant \(G\) taken to be unity for convenience. The stress tensor takes the form:

\[
\mathbf{S} = pI_3 - \lambda_1 \left( \nabla \mathbf{u} + \nabla \mathbf{u}^t - \frac{2}{3}(\text{div} \mathbf{u})I_3 \right) - \lambda_2 (\text{div} \mathbf{u})I_3,
\]
where $I_3$ is the $3 \times 3$ identical matrix, $p$ is the pressure of the gas, $\lambda_1$ is the shear viscosity, $\lambda_2$ is the bulk viscosity, and $\nabla u^t$ denotes the transpose of $\nabla u$. We consider the polytropic gases for which the equation of state is given by

$$p = p(\rho) = K \rho^\gamma,$$

where $K > 0$ is a constant set to be unity for convenience, $\gamma > 1$ is the adiabatic exponent.

For a non-rotating gaseous star, the stable equilibrium configurations, which minimize the energy among all possible configurations (cf. [21]), are spherically symmetric, called Lane-Emden solutions. Therefore, spherically symmetric motions for dynamical problems are important to consider. When the viscosities are positive constants, the global-in-time spherically symmetric solution to the free boundary problem (1.1) and its nonlinear asymptotic stability toward the Lane-Emden solution were proved in [30] for $4/3 < \gamma < 2$ (the stable index), by establishing the global-in-time regularity uniformly up to the vacuum boundary of solutions capturing an interesting behavior called the physical vacuum which states that the sound speed $c = \sqrt{p'(\rho)}$ is $C^{1/2}$-Hölder continuous near the vacuum boundary (cf. [2, 3, 13, 16, 23, 25, 42]), as long as the initial datum is a suitably small perturbation of the Lane-Emden solution with the same total mass. The large time asymptotic convergence of the global strong solution, in particular, the convergence of the vacuum boundary and the uniform convergence of the density, to those of the Lane-Emden solution with detailed convergence rates as the time goes to infinity are given in [30] when the viscosities are constant. The aim of this work is to extend those results to the more physically reasonable case that the viscosities depend on the density and vanish at vacuum states. The degeneracy of the viscosities near vacuum is one of the main difficulties which makes such a generalization highly nontrivial. We assume that the viscosities $\lambda_1$ and $\lambda_2$ are solely functions of the density in this paper. For simplicity, we set

$$\lambda_1 = \nu_1 \rho^\theta \quad \text{and} \quad \lambda_2 = \nu_2 \rho^\theta,$$

where $\nu_1$, $\nu_2$ and $\theta$ are positive constants. (1.2)

One may check easily that the theorems proved in this article apply to the case that $\lambda_1$ and $\lambda_2$ are positive for $\rho > 0$ and have the behavior of $\rho^\theta$ as $\rho \to 0$.

In the spherically symmetric setting, that is, $\Omega(t)$ is a ball with the changing radius $R(t)$, $\rho(x,t) = \rho(r,t)$ and $u(x,t) = u(r,t)x/r$ with $r = |x| \in (0, R(t))$; system (1.1) can then be reduced to

\[
\begin{cases}
(r^2 \rho)_t + (r^2 \rho u)_r = 0 \\
\rho(u_t + uu_r) + p_r + 4\pi \rho r^{-2} \int_0^r \rho(s,t)s^2 ds \\
\quad = \left[\left(\frac{4}{3} \lambda_1 + \lambda_2\right) \frac{(r^2 u)_r}{r^2}\right]_r - 4(\lambda_1) \frac{u}{r} \\
\rho > 0 \\
\rho = 0 \\
\dot{R}(t) = u(R(t),t) \quad \text{with} \quad R(0) = R_0, \quad u(0,t) = 0, \\
(p, u) = (p_0, u_0) \quad \text{on} \quad (0, R_0).
\end{cases}
\]
The initial domain is taken to be a ball \( \{0 \leq r \leq R_0\} \). And the initial density is supposed to satisfy the following condition:

\[
\rho_0(r) > 0 \quad \text{for} \quad 0 \leq r < R_0, \quad \rho_0(R_0) = 0 \quad \text{and} \quad -\infty < (\rho_0^{-1})_r < 0 \quad \text{at} \quad r = R_0. \tag{1.4}
\]

So,

\[
\rho_0^{-1}(r) \sim R_0 - r \quad \text{as} \quad r \quad \text{close to} \quad R_0, \tag{1.5}
\]

that is, the initial sound speed is \( C_{1/2} \)-Hölder continuous across the vacuum boundary, which is called the physical vacuum for the compressible inviscid flows (cf. [2, 3, 13, 25, 42]).

The behavior (1.4) near the vacuum boundary captures a very interesting feature of the stationary solution of (1.3), \((\rho, u) = (\bar{\rho}, 0)\), the Lane-Emden solution (cf. [1, 22]), corresponding to a non-rotating gaseous sphere in hydrostatic equilibrium. Here \( \bar{\rho} \) solves

\[
\partial_r (\bar{\rho}^\gamma) + 4\pi r^{-2} \bar{\rho} \int_0^r \bar{\rho}(s)s^2 ds = 0. \tag{1.6}
\]

The solutions to (1.6) can be characterized by the values of \( \gamma \) (cf. [22]) for given finite total mass \( M > 0 \). If \( \gamma \in (6/5, 2) \), there exists at least one compactly supported solution, and every solution is compactly supported and unique for \( \gamma \in (4/3, 2) \). If \( \gamma = 6/5 \), the unique solution admits an explicit expression, and it has infinite support. On the other hand, for \( \gamma \in (1, 6/5) \), there are no solutions with finite total mass. For \( \gamma > 6/5 \), let \( \bar{R} \) be the radius of the stationary star giving by the Lane-Emden solution, then it holds (cf. [22, 31])

\[
\bar{\rho}^{\gamma-1}(r) \sim \bar{R} - r \quad \text{as} \quad r \quad \text{close to} \quad \bar{R}. \tag{1.7}
\]

In both astrophysics and the theory of nonlinear PDEs, the problem of nonlinear asymptotic stability of Lane-Emden solutions is of fundamental importance. As mentioned in [30], the key to this is to establish the the global-in-time regularity of higher-order derivatives of solutions uniformly up to the vacuum boundary, which has been challenging due to the high degeneracy of system (1.3) caused by the behavior (1.5) near the vacuum boundary, even for the local-in-time existence theory in both the inviscid and viscous compressible flows. Indeed, the local-in-time well-posedness of smooth solutions to free boundary problems with physical vacuum was only established recently for compressible inviscid flows (cf. [2, 3, 13, 16], and [29] which proved a local-in-time well-posedness theory in a new functional space for the three-dimensional compressible Euler-Poisson equations in spherically symmetric motions).

For the vacuum free boundary problem (1.3) of the compressible viscous flows featuring the behavior (1.5) near the vacuum boundary, a local-in-time well-posedness theory of strong solutions was established in [14] and [7], for the constant and density dependent viscosities, respectively. In [30], the authors succeeded in establishing such a global-in-time higher-order regularity for the constant viscosities. Compared with the case of constant viscosities, the degeneracy of viscosities at vacuum states makes this a much more challenging task. In addition, the term \( 4(\lambda_1)u/r = 4\nu_1(\rho^\theta)u/r \) in the momentum equation, which does not appear in the constant viscosity case, causes serious difficulties in the analysis for the solutions with the behavior (1.5). Indeed, near the vacuum boundary, \( r = R(t) \), this term involves \( \theta(R(t) - r)^{-1+\theta/(\gamma-1)} \) which is unbounded for \( 0 < \theta < \gamma - 1 \), where \( \theta \) is the constant in (1.2).
This difficulty appears even in the study of local existence of strong solutions. In fact, the local existence result of strong solutions to \((1.3)\) in [7] only holds for \(\theta = 1\), which avoids the unboundedness of \(4\nu_1(\rho^\theta), u/r\) for \(0 < \theta < \gamma - 1\) when \(\gamma < 2\). In the present work, special care is taken to deal with this term near the vacuum boundary by constructing suitable weights to capture the behavior of velocity near the vacuum boundary to resolve this difficulty. In addition to this, we also refine the arguments in [30] substantially in obtaining the \(L^{\infty}\)-bound for the first derivative of velocity, for which the weighted \(L^2\)-estimates of the second derivative are used in [30], by establishing a pointwise estimate for the first derivative of velocity near the vacuum boundary only involving the weighted \(L^2\)-estimates of the first derivative.

When \(3/4 < \gamma < 2\), we extend in this article the nonlinear asymptotic stability results in [30] for the constant viscosities to the case of the density dependent viscosities \((1.2)\) with \(0 < \theta \leq \gamma/2\), by proving the existence of a unique global-in-time strong solution to \((1.3)\) and establishing the global-in-time regularity uniformly up to the vacuum boundary, which ensures the large time asymptotic uniform convergence of the evolving vacuum boundary, density and velocity to those of the Lane-Emden solution with detailed convergence rates, and detailed large time behaviors of solutions near the vacuum boundary. In particular, we show that every spherical surface moving with the fluid converges to the sphere enclosing the same mass inside the domain of the Lane-Emden solution with a uniform convergence rate. Therefore, the convergence of the vacuum boundary \(r = R(t)\) as \(t \to \infty\) to that of the Lane-Emden solution \(r = \bar{R}\) holds as a consequence. This also shows that the large time asymptotic states for the vacuum free boundary problem \((1.3)\) are determined by the initial mass distribution and the total mass. The results obtained in the present work are among few results of global-in-time strong solutions to vacuum free boundary problems of compressible fluids capturing the singular behavior of \((1.3)\).

Extensive works have been done on the studies of the Euler-Poisson and Navier-Stokes-Poisson equations with vacuum, especially in recent years. One may find the study of the stability problem of gaseous stars in astrophysics literatures (cf. [1, 41, 19]). The linear stability of Lane-Emden solutions was studied in [22]. By assuming the existence of global solutions of the Cauchy problem for the three-dimensional compressible Euler-Poisson equations which has been a major challenge in the theory of fluid dynamics equations, a conditional nonlinear Lyapunov type stability theory of stationary solutions for \(\gamma > 4/3\) was established in [36] using a variational approach (the same type of nonlinear stability results for rotating stars were given by [26, 27]). Those nonlinear stability results are in the framework of initial value problems in the entire \(\mathbb{R}^3\) and involve only Lyapunov functionals which are essentially equivalent to \(L^p\)-norms of the difference of solutions, where the vacuum boundary cannot be traced. In the framework of free boundary problems for the Euler-Poisson and Navier-Stokes-Poisson equations, the nonlinear dynamical instability of Lane-Emden solutions for \(\gamma \in (6/5, 4/3)\), was proved in [15] and [17], respectively. In inviscid flows, a nonlinear instability for \(\gamma = 6/5\) was proved in [12]; an instability was identified for \(\gamma = 4/3\) in [4] that a small perturbation can cause part of the mass to go off to infinity.

It should be noted that the existence of global weak solutions was proved in [10] for the initial boundary value problem reduced from the vacuum free boundary problem \((1.3)\) after using the Lagrangian mass coordinates, under the constraint that \(\gamma > 4/3, \theta \in (0, \gamma - 1) \cap (0, \gamma - 1)\).
(0, \gamma/2], \text{ and }
\frac{2(8 + 4\alpha - \alpha^2)}{4 - 4\alpha + \alpha^2} < \frac{\nu_2}{\nu_1} < \frac{2(8 + 4\alpha - \alpha^2)}{4 - 4\alpha + \alpha^2} + \frac{8\sqrt{5 + 4\alpha - \alpha^2}}{4 - 4\alpha + \alpha^2}

with \alpha \in (-1, 1) being a constant. In contrast to the strong stability result shown here, for the global weak solutions obtained in [10], only the uniform convergence of the velocity is proved, due to the lack of regularity near the vacuum boundary, and the uniform convergence of the density, in particular, the convergence of the vacuum boundary which is the most interesting part in the study of asymptotic behavior of the free boundary problem, are missing. Furthermore, our nonlinear asymptotic stability result holds for \(4/3 < \gamma < 2\), \(0 < \theta \leq \gamma/2\), \(\nu_1 > 0\) and \(\nu_2 > 0\), without the restrictions on \(\theta < \gamma - 1\) and the ratio of \(\nu_2/\nu_1\) as in [10]. It should be noted that \(\gamma - 1 < \gamma/2\) for \(\gamma < 2\).

We conclude the introduction by reviewing some previous works on viscous flows. It should be noted that there are also other prior results on free boundary problems involving vacuum for the compressible Navier-Stokes equations besides the ones aforementioned. One may refer to [33, 35, 28, 9, 18, 6, 43, 44, 18, 45] and references therein for the one-dimensional motions concerning global weak solutions. For the spherically symmetric motions, global existence and stability of weak solutions were obtained in [34, 32] for gases surrounding a solid ball (a hard core), restricted to cut-off domains excluding a neighborhood of the origin; a global existence of weak solutions containing the origin was established in [11] for which the density does not vanish on the boundary and the viscosities are density dependent. The readers may refer to [37, 38] for the local-in-time well-posedness results and [40] for linearized stability results of stationary solutions for a class of free boundary problems of the compressible Navier-Stokes-Poisson equations away from vacuum states.

2 Main Results

First, we recall some properties for Lane-Emden solutions. For \(\gamma \in (4/3, 2)\), it is known that for any given finite positive total mass, there exists a unique solution to equation (1.6) whose support is compact (cf. [22]). Without abusing notations and for convenience, we use \(x\) as the variable in the study of Lane-Emden solutions. That means, for any \(M \in (0, \infty)\), there exists a unique function \(\bar{\rho}(x)\) such that

\[
\bar{\rho}_0 := \bar{\rho}(0) > 0, \quad \bar{\rho}(x) > 0 \quad \text{for} \quad x \in (0, \bar{R}), \quad \bar{\rho}(\bar{R}) = 0, \quad M = \int_0^{\bar{R}} 4\pi \bar{\rho}(s)s^2ds; \tag{2.1}
\]

\[-\infty < \bar{\rho}_x < 0 \quad \text{for} \quad x \in (0, \bar{R}) \quad \text{and} \quad \bar{\rho}(x) \leq \bar{\rho}_0 \quad \text{for} \quad x \in (0, \bar{R}); \tag{2.2}\]

\[(\bar{\rho}^\gamma)_x = -x\phi\bar{\rho}, \quad \text{where} \quad \phi := x^{-3}\int_0^x 4\pi \bar{\rho}(s)s^2ds \in \left[M/\bar{R}^3, 4\pi \bar{\rho}_0/3\right]; \tag{2.3}\]

for a certain finite positive constant \(\bar{R}\) (indeed, \(\bar{R}\) is determined by \(M\) and \(\gamma\)). Note that

\[(\bar{\rho}^{\gamma-1})_x = \frac{\gamma-1}{\gamma} \bar{\rho}^{-1}(\bar{\rho}^\gamma)_x = -\frac{\gamma-1}{\gamma} x\phi. \tag{2.4}\]
It then follows from (2.1) and (2.3) that $\bar{\rho}$ satisfies the physical vacuum condition, i.e.,

$$\bar{\rho}^{\gamma - 1}(x) \sim \bar{R} - x \quad \text{as} \quad x \quad \text{close to} \quad \bar{R}.$$ 

Indeed, there exists a constant $C$ depending on $M$ and $\gamma$ such that

$$C^{-1} (\bar{R} - x) \leq \bar{\rho}^{\gamma - 1}(x) \leq C (\bar{R} - x), \quad x \in (0, \bar{R}).$$  \tag{2.5}$$

We adopt a particle trajectory Lagrangian formulation for (1.3) as follows. Let $x$ be the reference variable and define the Lagrangian variable $r(x, t)$ by

$$r_t(x, t) = u(r(x, t), t) \quad \text{for} \quad t > 0 \quad \text{and} \quad r(x, 0) = r_0(x), \quad x \in I := (0, \bar{R}).$$

Here $r_0(x)$ is the initial position which maps $I \to [0, R_0]$ satisfying

$$\int_0^{r_0(x)} \rho_0(s) s^2 ds = \int_0^x \bar{\rho}(s) s^2 ds, \quad x \in I,$$  \tag{2.6}$$

so that

$$\rho_0(r_0(x)) r_0^2(x) r_0'(x) = \bar{\rho}(x) x^2, \quad x \in I.$$  \tag{2.7}$$

(Indeed, (2.6) means that the initial mass in the ball with the radius $r_0(x)$ is the same as that of the Lane-Emden solution in the ball with the radius $x$. Then smoothness of $r_0(x)$ at $x = \bar{R}$ is equivalent to that the initial density $\rho_0$ has the same behavior near $R_0$ as that of $\bar{\rho}$ near $\bar{R}$.) The choice of $r_0$ can be described by

$$r_0(x) = \psi^{-1}(\xi(x)), \quad 0 \leq x \leq \bar{R};$$  \tag{2.8}$$

where $\xi$ and $\psi$ are one-to-one mappings, defined by

$$\xi : (0, \bar{R}) \to (0, M) : \ x \mapsto \int_0^x s \bar{\rho}(s) ds \quad \text{and} \quad \psi : (0, R_0) \to (0, M) : \ z \mapsto \int_0^z s \rho_0(s) ds.$$ 

Moreover $r_0(x)$ is an increasing function and the initial total mass has to be the same as that for $\bar{\rho}$, that is,

$$\int_0^{R_0} 4\pi \rho_0(s) s^2 ds = \int_0^{\bar{r}_0(R)} 4\pi \bar{\rho}(s) s^2 ds = \int_0^{\bar{R}} 4\pi \bar{\rho}(s) s^2 ds = M,$$  \tag{2.9}$$

to ensure that $r_0$ is a diffeomorphism from $I$ to $[0, R_0]$. It follows from (1.3), that

$$\int_0^{r(x, t)} \rho(s, t) s^2 ds = \int_0^{r_0(x)} \rho_0(s) s^2 ds, \quad x \in I.$$  \tag{2.10}$$

Set

$$f(x, t) = \rho(r(x, t), t) \quad \text{and} \quad v(x, t) = u(r(x, t), t).$$

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Then the Lagrangian version of system (1.3)\textsubscript{1,2} can be written on the reference domain \( I \) as

\[
\begin{align*}
&\left\{ \begin{array}{c}
(r^2f)_t + r^2 f \frac{v_x}{r_x} = 0, \\
rv_t + \left( \frac{(\gamma)_x}{r_x} + 4 \pi r^{-2} \int_0^{r_0(x)} \rho_0(s)s^2 ds \right) = \frac{1}{r_x} \left[ \left( \frac{4}{3} \lambda_1 + \lambda_2 \right) \frac{(r^2v)_x}{r^2 r_x} \right]_x - 4(\lambda_1)_x \frac{v}{r_x r}. 
\end{array} \right. \\
&\text{(2.11)}
\end{align*}
\]

Solving (2.11)\textsubscript{1} gives that

\[
f(x, t)r^2(x, t)r_x(x, t) = \rho_0(r_0(x))r_0^2(x)r_0(x), \quad x \in I.
\]

Therefore,

\[
f(x, t) = \frac{x^2 \tilde{\rho}(x)}{r^2(x, t)r_x(x, t)} \quad \text{for} \quad x \in I,
\]

due to (2.7). By using (2.3), the free boundary problem (1.3) is then reduced to the following initial boundary value problem on a fixed interval \( \bar{I} \):

\[
\begin{align*}
&\left\{ \begin{array}{c}
\bar{\rho} \left( \frac{x}{r} \right)^2 v_t + \left[ \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right) \right] + \frac{x^2 \bar{\rho}}{r^4} \int_0^x 4\pi \rho y^2 dy = \mathcal{V}(x, t) \quad \text{in} \quad I \times (0, T], \\
v(0, t) = 0 \quad \text{on} \quad (0, T], \\
(r, v)(x, 0) = (r_0(x), u_0(r_0(x))) \quad \text{on} \quad I \times \{t = 0\},
\end{array} \right. \\
&\text{(2.12)}
\end{align*}
\]

where

\[
\mathcal{V} = \nu \left[ \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\theta \frac{(\gamma^2v)_x}{r^2 r_x} \right]_x - 4\nu_1 \left[ \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\theta \right]_x \frac{v}{r} \quad \text{with} \quad \nu = \frac{4}{3} \nu_1 + \nu_2 > 0.
\]

It should be noticed that \( \mathcal{V} \) can be rewritten as

\[
\mathcal{V} = -\frac{\nu}{\theta} \left( \frac{r}{x} \right)^{\frac{4\nu_1}{\theta}} \left\{ \left( \frac{r}{x} \right)^{\frac{4\nu_1}{\theta}} \left[ \bar{\rho}^\theta \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\theta \right]_x \right\}_t.
\]

A strong solution to problem (2.12) is defined as follows.

**Definition 2.1** \( v \in L^\infty ([0, T]; H^2_{loc}([0, \bar{R}])) \cap L^\infty ([0, T]; W^{1,\infty}(I)) \) with

\[
r(x, t) = r_0(x) + \int_0^t v(x, s)ds \quad \text{for} \quad (x, t) \in I \times [0, T]
\]

satisfying the initial condition (2.12)\textsubscript{3} is called a strong solution of problem (2.12) in \([0, T]\), if

1) \( r_x(x, t) > 0 \) for \( (x, t) \in I \times [0, T] \);
2) \( \tilde{\rho}^{1/2}v \in C^1([0, T]; L^2(I)) \);
3) \( r \in L^\infty ([0, T]; H^2_{loc}([0, \bar{R}])) \) and \( \left( \tilde{\rho}^{1/2}r/2x, \tilde{\rho}^{1/2}r_x, \tilde{\rho}^{1/2}v \right) \in L^\infty ([0, T]; L^2(I)) \);
4) \( v(0, t) = 0 \) holds in the sense of \( W^{1,\infty}\)-trace for \( t \in [0, T] \);
5) (2.12)\textsubscript{1} holds for \( (x, t) \in I \times [0, T], \) a.e.
We are ready to state the main theorem of this paper. Denote
\[ \mathcal{E}(t) = \| (r_x - 1, v_x)(\cdot, t) \|_{L^2}^2 + \| \tilde{\rho}^{1/2} (r/|x|, r_x x_r)(\cdot, t) \|_{L^2}^2 + \| \tilde{\rho}^{1/2} v_t(\cdot, t) \|_{L^2}^2. \] (2.15)

**Theorem 2.2** Let \( \gamma \in (4/3, 2) \), \( 0 < \theta \leq \gamma/2 \), and \( \tilde{\rho} \) be the Lane-Emden solution satisfying (2.1) - (2.3). Assume that the compatibility condition \( v(0, 0) = 0 \) holds and the initial density \( \rho_0 \) satisfies (1.4) and (2.9). There exists a constant \( \bar{\delta} > 0 \) such that if
\[ \mathcal{E}(0) \leq \bar{\delta}, \]
then the problem (2.12) admits a unique strong solution in \( I \times [0, \infty) \) with
\[ \mathcal{E}(t) \leq C \mathcal{E}(0), \quad t \geq 0, \]
for some constant \( C \) independent of \( t \).

It should be noted that \( \| \tilde{\rho}^{1/2} v_t(\cdot, 0) \|_{L^2} \) is given in terms of the initial data \((r_0, u_0)\) by equation (2.12).

For any \( t \geq 0 \), since \( r_x(x, t) > 0 \) for \( x \in \bar{I}, r(x, t) \) defines a diffeomorphism from the reference domain \( \bar{I} \) to the changing domain \( \{0 \leq r \leq R(t)\} \) with the boundary
\[ R(t) = r(\bar{R}, t). \] (2.16)

It also induces a diffeomorphism from the initial domain, \( \bar{B}_{R_0}(0) \), to the evolving domain, \( \bar{B}_{R(t)}(0) \), for all \( t \geq 0 \):
\[ x \neq 0 \in \bar{B}_{R_0}(0) \rightarrow r \left( r_0^{-1}(|x|), t \right) \frac{x}{|x|} \in \bar{B}_{R(t)}(0), \]
where \( r_0^{-1} \) is the inverse map of \( r_0 \) defined in (2.8). Here
\[ \bar{B}_{R_0}(0) := \{ x \in \mathbb{R}^3 : |x| \leq R_0 \} \quad \text{and} \quad \bar{B}_{R(t)}(0) := \{ x \in \mathbb{R}^3 : |x| \leq R(t) \}. \]

Denote the inverse of the map \( r(x, t) \) by \( \mathcal{R}_t \) for \( t \geq 0 \) so that
\[ \text{if } r = r(x, t) \text{ for } 0 \leq r \leq R(t), \text{ then } x = \mathcal{R}_t(r). \]

For the strong solution \((r, v)\) obtained in Theorem 2.2 we set for \( 0 \leq r \leq R(t) \) and \( t \geq 0 \),
\[ \rho(r, t) = \frac{x^2 \tilde{\rho}(x)}{r^2(x, t)r_x(x, t)} \quad \text{and} \quad u(r, t) = v(x, t) \quad \text{with} \quad x = \mathcal{R}_t(r). \] (2.17)

Then the triple \((\rho(r, t), u(r, t), R(t))\) \( (t \geq 0) \) defines a global strong solution to the free boundary problem (1.3). Furthermore, the strong nonlinear asymptotic stability of the Lane-Emden solution can be stated as follows.

For any \( \iota \in (0, (2\gamma - 2 - \theta)/8] \), we set
\[ \alpha = \min\{\gamma - 1 + \theta, 2(\gamma - 1)\} - \iota, \] (2.18)
\[ \beta = 1 + (\alpha - \iota)/(\gamma - \theta), \] (2.19)
\[ \varsigma = \min\{1, 2^{-1}\beta + 2^{-1}(\beta - 1) \min\{1, (\gamma - \theta)/\alpha\}\}. \] (2.20)

It should be noted that \( 1 < \beta < 3 \) and \( 0 < \alpha - \theta < (\gamma - 1) \) for \( 4/3 < \gamma < 2 \) and \( 0 < \theta \leq \gamma/2 \).
Theorem 2.3 Under the assumptions in Theorem 2.2, the triple \((\rho, u, R(t))\) defined by (2.16) and (2.17) is the unique global strong solution to the free boundary problem (1.1) satisfying \(R \in W^{1,\infty}(0, +\infty)\). Moreover, the solution satisfies the following estimates: for any \(t \in (0, (2\gamma - 2 - \theta)/8], \ell \in (0, 1)\) and \(b \in [0, 2 - \gamma]\), there exists positive constants \(C_i\) and \(C_{i,\ell}\) independent of \(x\) and \(t\) such that for all \(t \geq 0\),

\[
(1 + t)\frac{3\gamma - 2 + 2(\alpha - \theta)}{2(\gamma + \alpha - \theta)} \beta |r(x, t) - x|^2 \leq C_i \mathcal{E}(0), \quad x \in I,
\]

\[
(1 + t) \left\{ \frac{\beta}{2} - \frac{\min(0, 3\gamma - 5 + 2\theta)}{2(\gamma - \theta)} \right\} \rho^{-\beta}(x)|\rho(r(x, t), t) - \bar{\rho}(x)|^2 \leq C_i \mathcal{E}(0), \quad x \in I,
\]

\[
(1 + t)^{\beta - 1} \left( |r_x(x, t) - 1|^2 + |x^{-1}r(x, t) - 1|^2 \right) \leq C_{i,\ell} \mathcal{E}(0), \quad x \in [0, \ell],
\]

\[
(1 + t)^{\beta/2} |u(r(x, t), t)|^2 \leq C_i \mathcal{E}(0), \quad x \in I,
\]

\[
(1 + t)^{a} \left( |u_r(r, t)|^2 + |r^{-1}u(r, t)|^2 \right) \leq C_i \mathcal{E}(0), \quad x \in I.
\]

Here \(a > 0\) is given by

\[
a = \min \left\{ \beta - 1, \frac{\gamma - 1 + \alpha - \theta}{\gamma + \alpha - \theta}, \frac{3\beta + \xi}{4} - \frac{\beta - \xi}{4\alpha}, \frac{\beta}{2} - \frac{\beta}{2(\gamma + \alpha - \theta)} \max\{0, 4\theta - 4(\gamma - 1)\} \right\}.
\]

Remark 2.4 The uniform convergence of \(r(x, t)\) to \(x\) is given in (2.21). This implies that every spherical surface moving with the fluid converges to the sphere enclosing the same mass inside the domain of the Lane-Emden solution with a uniform convergence rate, and the large time asymptotic states for the vacuum free boundary problem (1.3) are determined by the initial mass distribution and the total mass. In particular, this also implies the convergence of the vacuum boundary:

\[
|R(t) - \bar{R}| \leq C_i (1 + t)^{-\frac{\gamma - 1 + \alpha - \theta}{2(\gamma + \alpha - \theta)} \beta} \sqrt{\mathcal{E}(0)}.
\]

Moreover, a better decay rate for \(|r - x|\) away from the vacuum boundary can be given by

\[
(1 + t)^{3\gamma - 2 + 2(\alpha - \theta)/2(\gamma + \alpha - \theta)} \beta \frac{1}{\bar{\rho}} |r(x, t) - x|^2 \leq C_{i,\ell} \mathcal{E}(0), \quad x \in [0, \ell], 0 < \ell < 1, \quad t \geq 0.
\]

(See (4.1) for the details.) The uniform convergence of the density and the velocity to those of the Lane-Emden solution with uniform convergence rates are given by (2.22) and (2.24), respectively. The estimate (2.22) yields not only the uniform convergence for large time but also the behavior of the density near the vacuum boundary since \(\gamma < 2\).
3 Proof of Theorem 2.2

In this section, we derive some priori estimates under the following apriori assumptions. Let \( v \) be a strong solution to (2.12) in the time interval \([0, T] \) with

\[
r(x, t) = r_0(x) + \int_0^t v(x, \tau) d\tau, \quad (x, t) \in [0, 1] \times [0, T],
\]
satisfying the following a priori assumption:

\[
|r_x - 1| + |r/x - 1| \leq \epsilon_0 \quad \text{for} \; (x, t) \in I \times [0, T], \tag{3.1}
\]

where \( \epsilon_0 \in (0, 1/2] \) is a sufficiently small but fixed constant (indeed, \( \epsilon_0 \) is required to be less than a constant depending only on \( \gamma \)); and

\[
|v_x| + |v/x| \leq 1 \quad \text{for} \; (x, t) \in I \times [0, T]. \tag{3.2}
\]

In particular, it holds that

\[
1/2 \leq r_x, r/x \leq 3/2 \quad \text{for} \; (x, t) \in I \times [0, T]. \tag{3.3}
\]

(Indeed, the a priori assumption (3.1) and (3.2) will be verified in Lemmas 3.6 and 3.10.)

The following Hardy’s inequalities will be used often to derive the a priori estimates, whose proof can be found in [8] or [15].

**Lemma 3.1** (Hardy’s inequalities) Let \( k > 1 \) be a given real number and \( g \) be a function. If \( g \) satisfies that

\[
\int_0^{1/2} x^{k-2} g^2 dx < \infty,
\]

then it holds that

\[
\int_0^{1/2} x^{k-2} g^2 dx \leq \int_0^{1/2} x^{k} (g^2 + g_x^2) dx < \infty. \tag{3.4}
\]

Similarly, if \( \int_{1/2}^{1} (1-x)^{k} (g^2 + g_x^2) dx < \infty \), then

\[
\int_{1/2}^{1} (1-x)^{k-2} g^2 dx \leq \int_{1/2}^{1} (1-x)^{k} (g^2 + g_x^2) dx < \infty. \tag{3.5}
\]

3.1 Lower-order estimates

To derive the lower-order weighted energy estimates, one rewrites equation (2.12) as

\[
\rho \left( \frac{x^2}{r} \right)^2 v_t + \left[ \left( \frac{x^2 \rho}{r^2} \right) r_x \right] - \frac{x^4}{r^4} (\rho') x = \nu \left[ \left( \frac{x^2 \rho}{r^2} \right) \theta \left( \frac{r^2 v}{r^2 + 2} \right) \right] - 4\nu_1 \left[ \left( \frac{x^2 \rho}{r^2} \right) \theta \right] \frac{v}{r}, \tag{3.6}
\]

where (2.3) has been used. Here \( \nu = 4\nu_1/3 + \nu_2 \).

We outline the analysis for the lower-order estimates here and give some motivations for the proofs. In Lemma 3.2, we derive the decay estimates for the zeroth-order energy:

\[
\int x^2 \left\{ \bar{\rho} v^2 + \bar{\rho} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x, t) dx
\]

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and the boundedness of the weighted energy:

\[ \int x^2 \bar{\rho}^\theta \left[ (r - 1)^2 + \left( \frac{r}{x} - 1 \right)^2 \right] (x, t) \, dx, \]

by constructing suitable nonlinear weighted functionals. In Lemma 3.3, the decay estimates for weighted norms of the higher time-derivatives (than those in Lemma 3.2) are derived. Further regularity near the vacuum boundary and decay estimates are obtained in Lemmas 3.4 and 3.5 by showing the boundedness of

\[ \int x^2 \bar{\rho}^{\theta - \alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + \left( r_x - 1 \right)^2 \right] (x, t) \, dx, \]

and the decay of the following weighted quantities:

\[ \int x^2 \bar{\rho}^{\theta} \left[ \left( \frac{r}{x} - 1 \right)^2 + \left( r_x - 1 \right)^2 \right] (x, t) \, dx, \]

\[ \int x^2 \left\{ \bar{\rho} (v^2 + v_x^2) + \bar{\rho} \gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x, t) \, dx, \]

\[ \int (v^2 + x^2 \bar{\rho}^{\theta} v_x^2) (x, t) \, dx \quad \text{and} \quad \int \bar{\rho}^{\theta - \alpha/2} (x^2 v_x^2 + v^2) (x, t) \, dx, \]

where \( \alpha \) is given by (2.18). Those two lemmas play a crucial role to the global regularity uniformly up to the vacuum boundary. In the proof of these two lemmas, we use the multipliers:

\[ \int_0^x \bar{\rho}^{-\alpha} (y) \left( r^3 - y^3 \right)_y \, dy \quad \text{and} \quad \int_0^x \bar{\rho}^{-\alpha} (y) (r^2 v)_y \, dy. \]

In Lemma 3.5, we also obtain the decay estimates of

\[ \int (r(x, t) - x)^2 \, dx \quad \text{and} \quad \int x^2 (r_x(x, t) - 1)^2 \, dx. \]

With the above lower-order estimates in hand, we can bound

\[ \sup_{x \in I} \left( x^3 | r_x(x, t) - 1 |^2 \right) \quad \text{and} \quad \sup_{x \in I} \left( x^3 | v_x(x, t) |^2 \right), \]

and estimate the decay of

\[ \sup_{x \in I} \left( x | r(x, t) - x |^2 \right) \quad \text{and} \quad \sup_{x \in I} \left( x | v(x, t) |^2 \right) \]

in Lemma 3.6, which in particular give the uniform estimates away from the origin. The key idea is to integrate (3.6) both in \( x \) and \( t \) to derive an ODE for the quantity \( Z \) defined in (3.81) whose leading term is \( r_x - 1 \) to obtain the estimate for \( x^3 | r_x(x, t) - 1 |^2 \). In turn, the leading term for \( \partial_t Z \) is \( v_x \) so that the bound for \( \partial_t Z \) yields the bound for \( x^3 | v_x(x, t) |^2 \). In the proof of this lemma, the balance of the pressure and gravitational field plays an important role.
It worths pointing out the difficulties in the lower-order estimates for the case of density dependent viscosities. For example, in the proof of Lemma 3.2 it is shown that the quantity
\[(1 + t) \int x^2 \left\{ \hat{\rho} v^2 + \hat{\rho} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x, t) dx \]
\[\leq C (\text{initial data} + \int_0^t \int x^2 \hat{\rho} \gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds) . \tag{3.7} \]
In order to obtain the decay of the zeroth-order energy, we show that the double integral on the second line of (3.7) can be bounded by the initial data and the term
\[C \int_0^t \int \left[ \hat{\rho} \left( |r_x - 1| + \frac{r}{x} - 1 \right) |v| + \hat{\rho}^\theta |r - x| |x v_x| \right] dx ds, \tag{3.8} \]
using the multiplier \( r^3 - x^3 \) motivated by the virial equations adopted in the study of stellar dynamics and equilibriums (cf. [20, 39]) to detect the detailed balance between the pressure and the self-gravitation. The term (3.8) appears due to the dependence of the viscosities on the density. This is in sharp contrast to the case of constant viscosities for which the multiplier \( r^3 - x^3 \) matches the viscosities well and the double integral on the second line of (3.7) is bounded directly by the initial data. It is quite subtle to bound (3.8) due to the degeneracy of the viscosities at vacuum. We choose a cut-off function deliberately whose effective length is a small positive number \( \delta \) to localize both near the vacuum boundary and the origin, so that (3.8) can be bounded by
\[C \delta^4 \int_0^t \int x^2 \hat{\rho} \gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \]
and other terms. The desired estimates are then obtained by choosing \( \delta \) small. In the estimates of \( x^3 |r_x(x, t) - 1|^2 \) and \( x^3 |v_x(x, t)|^2 \) in Lemma 3.6 we derive an ODE for the quantity \( Z \) defined in (3.81), while in the case of constant viscosities, an ODE is also derived for the quantity \( (\bar{\rho}(x))^{-1} \rho(r(x, t), t) \) which is simpler than that for \( Z \) and can be solved explicitly in some sense. The ODE for \( Z \) is more involved and harder to solve. We have to identify the leading terms and estimate the error terms.

In the following, we give the details and analysis outlined above.

**Lemma 3.2** Let \( \theta \in (0, 1] \). Suppose that (3.1) holds for suitably small constant \( \epsilon_0 \). Then,
\[
\mathcal{D}(t) + \int_0^t \int x^2 \hat{\rho} \gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds
+ \int_0^t (1 + s) \int \hat{\rho} \gamma (x^2 v_x^2 + v^2) dx ds \leq C \mathcal{D}(0), \quad t \in [0, T],
\tag{3.9}
\]
where
\[
\mathcal{D}(t) = (1 + t) \int x^2 \left\{ \bar{\rho} v^2 + \bar{\rho} \gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x, t) dx
+ \int x^2 \bar{\rho} \gamma \left( r_x - 1 \right)^2 \left( \frac{r}{x} - 1 \right)^2 (x, t) dx.
\]
Proof. The proof consists three steps.

Step 1. In this step, we prove that for $0 < \theta \leq 1$ and $\gamma > 4/3$,

$$(1 + t) \int x^2 \left\{ \bar{\rho} v^2 + \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x,t) dx$$

$$+ \int_0^t (1 + s) \int \bar{\rho}^\theta (x^2 v_x^2 + v^2) \, dx ds$$

$$\leq C \int \eta(x,0) dx + C \int_0^t \int x^2 \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx ds,$$

where

$$\eta(x,t) = \frac{1}{2} x^2 \bar{\rho} v^2 + x^2 \bar{\rho}^\gamma \left[ \frac{1}{\gamma - 1} \left( \frac{r}{x} \right)^{2\gamma - 2} \left( \frac{1}{x} \right)^{\gamma - 1} + \left( \frac{r}{x} \right)^2 r_x - 4 \frac{x}{r} - 4 - 3\gamma \right].$$

It follows from (3.6) and the boundary condition (2.1) that, for any $\ell \geq 0$

$$\frac{d}{dt} \left\{ (1 + t)^\ell \int \eta(x,t) dx \right\} + (1 + t)^\ell \int \left( x^2 \bar{\rho} \right)^\theta \left[ \nu \left( \frac{\nu_2}{x} \right)^2 v_x^2 - 4\nu_1 (r v_x^2) \right] \, dx$$

$$= \ell (1 + t)^{\ell - 1} \int \eta(x,t) dx.$$

Each term in the equation above can be estimated as follows. First, using the Taylor expansion, one may verify that for $\gamma > 4/3$,

$$\eta(x,t) \geq \frac{1}{2} x^2 \bar{\rho} v^2 + \frac{3\gamma - 4}{4} x^2 \bar{\rho}^\gamma \left[ 2 \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right],$$

(3.12)

$$\eta(x,t) \leq \frac{1}{2} x^2 \bar{\rho} v^2 + C(\gamma) x^2 \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right],$$

(3.13)

provided (3.1) holds for a suitably small constant $\varepsilon_0$, where $C(\gamma)$ is a positive constant depending on $\gamma$. Also,

$$\nu \left( \frac{\nu_2}{x} \right)^2 v_x^2 - 4\nu_1 (r v_x^2) \geq 3\sigma \left( \frac{r^2 v_x^2}{r_x} + 2 r_x v^2 \right),$$

where $\sigma = \min \{2\nu_1/3, \nu_2\}$. Integrate (3.1) over $[0,t]$ to obtain, by virtue of (3.12) and (3.13), that for $0 < \theta \leq 1$,

$$(1 + t)^\ell \int x^2 \left\{ \bar{\rho} v^2 + \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x,t) dx$$

$$+ \int_0^t (1 + s)^\ell \int \bar{\rho}^\theta (x^2 v_x^2 + v^2) \, dx ds$$

$$\leq C \int \eta(x,0) dx + C \ell \int_0^t (1 + s)^{\ell - 1} \int x^2 \left\{ \bar{\rho}^\theta v^2 + \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} \, dx ds.$$

(3.14)
Setting $\ell = 0$ in (3.14), leads to

$$\int x^2 \left\{ \bar{\rho} v^2 + \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} \, (x, t) dx$$

$$+ \int_0^t \int \bar{\rho}^\theta \left( x^2 v_x^2 + v^2 \right) \, dx \, ds \leq C \int \eta(x, 0) \, dx.$$  \hspace{1cm} (3.15)

Letting $\ell = 1$ in (3.14) and using (3.15) prove (3.10).

Step 2. In this step, we prove that for $\theta \in (0, 1]$,  

$$\int x^2 \bar{\rho}^\theta \left[ (r_x - 1)^2 + \left( \frac{r}{x} - 1 \right)^2 \right] \, dx + \int_0^t \int x^2 \bar{\rho}^\gamma \left[ 2 \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx \, ds$$

$$\leq C \mathbf{D}(0) + C \int_0^t \int \left[ \bar{\rho}^\theta x \left( \left| r_x - 1 \right| + \left| \frac{r}{x} - 1 \right| \right) \, |v| + \bar{\rho}^\theta |r - x| \, |x v_x| \right] \, dx \, ds.$$  \hspace{1cm} (3.16)

It is easy to check, in view of (3.6) and (2.1), that

$$\int \bar{\rho}^\gamma \left\{ \left[ \frac{x^4}{r^4} (r^3 - x^3) \right]_x \right\} - \left( \frac{x^2}{r^2 r_x} \right)^\gamma (r^3 - x^3)_x \right\} \, dx = - \int x^3 \bar{\rho} v_t \left( \frac{r}{x} - \frac{x^2}{r^2} \right) \, dx$$

$$- \nu \int \bar{\rho}^\theta \left( \frac{x^2}{r^2 r_x} \right)^\theta \frac{(r^2 v)_x}{r^2 r_x} \left( r^3 - x^3 \right)_x \, dx + 4 \nu_1 \int \bar{\rho}^\theta \left( \frac{x^2}{r^2 r_x} \right)^\theta \left[ \frac{v}{r} (r^3 - x^3) \right] \, dx.$$  \hspace{1cm} (3.17)

It yields from simple calculations that

$$x^{-2} \left\{ \left[ \frac{x^4}{r^4} (r^3 - x^3) \right]_x \right\} - \left( \frac{x^2}{r^2 r_x} \right)^\gamma (r^3 - x^3)_x \right\}$$

$$= 3 \left( \frac{x^2}{r^2 r_x} \right)^\gamma - 3 \left( \frac{x^2}{r^2 r_x} \right)^{\gamma - 1} - \left( \frac{x}{r} \right)^2 r_x + 4 \left( \frac{x}{r} \right)^5 r_x - 7 \left( \frac{x}{r} \right)^4 + 4 \frac{x}{r},$$

which, together with Taylor’s expansion, gives

$$\int \bar{\rho}^\gamma \left\{ \left[ \frac{x^4}{r^4} (r^3 - x^3) \right]_x \right\} - \left( \frac{x^2}{r^2 r_x} \right)^\gamma (r^3 - x^3)_x \right\} \, dx$$

$$\geq \frac{3(3\gamma - 4)}{2} \int x^2 \bar{\rho}^\gamma \left[ 2 \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx,$$  \hspace{1cm} (3.18)

provided 3.1 holds for small $\varepsilon_0$. Simple calculations show that for $\theta \in (0, 1]$,

$$\nu \int \bar{\rho}^\theta \left( \frac{x^2}{r^2 r_x} \right)^\theta \frac{(r^2 v)_x}{r^2 r_x} \left( r^3 - x^3 \right)_x \, dx = 3 \nu \frac{d}{dt} \int x^2 \bar{\rho}^\theta \eta_0(x, t) \, dx,$$  \hspace{1cm} (3.19)

where

$$\eta_0(x, t) = \begin{cases} \frac{1}{\theta} \left( \frac{x^2}{r^2 r_x} \right)^{\theta - 1} + \frac{1}{\theta(\theta - 1)} & \text{for } 0 < \theta < 1, \\
\frac{x^2}{r^2 r_x} - \ln \left( \frac{x^2}{r^2 r_x} \right) - 1 & \text{for } \theta = 1. \end{cases}$$
Moreover,

\[
\int x^3 \rho v_t \left(\frac{r}{x} - \frac{x^2}{r^2}\right) dx = \frac{d}{dt} \int x^3 \rho v \left(\frac{r}{x} - \frac{x^2}{r^2}\right) dx - \int x^2 \rho v^2 \left(1 + 2 \frac{x^3}{r^3}\right) dx \quad (3.20)
\]

and

\[
4\nu_1 \int \rho^\theta \left(\frac{x^2}{r^2 r_x}\right)^{\theta} \left[\frac{v}{r} (r^3 - x^3)\right] x dx \\
\leq C \int \rho^\theta x \left(|r_x - 1| + \frac{r}{x} - 1\right) |v| dx + C \int \rho^\theta |r - x| |xv_x| dx. \quad (3.21)
\]

It follows from (3.17)-(3.21) that

\[
\frac{d}{dt} \int \left\{3\nu x^2 \rho^\theta \eta_0 + x^3 \rho v \left(\frac{r}{x} - \frac{x^2}{r^2}\right)\right\} dx \\
+ \frac{3}{2} (3\gamma - 4) \int x^2 \rho^\gamma \left[2 \left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2\right] dx \\
\leq C \int \left\{\rho^\theta x \left(|r_x - 1| + \frac{r}{x} - 1\right) |v| + \rho^\theta |r - x| |xv_x| + x^2 \rho v^2\right\} dx. \quad (3.22)
\]

Using the Taylor expansion, shows that for small \(\epsilon_0\) in (3.1),

\[
\eta_0(x, t) \geq \frac{1}{2} \left(\frac{x^2}{r^2 r_x} - 1\right)^2 - C\epsilon_0 \left(\frac{x^2}{r^2 r_x} - 1\right)^2 \geq \frac{1}{4} \left(\frac{x^2}{r^2 r_x} - 1\right)^2 \\
\geq \frac{1}{4} \left[(r_x - 1) + 2 \left(\frac{r}{x} - 1\right)\right]^2 - C\epsilon_0 \left[(r_x - 1)^2 + \left(\frac{r}{x} - 1\right)^2\right].
\]

Note that

\[
\int x^2 \rho^\theta \left[(r_x - 1) + 2 \left(\frac{r}{x} - 1\right)\right]^2 dx \\
= \int x^2 \rho^\theta \left[(r_x - 1)^2 + 4 \left(\frac{r}{x} - 1\right)^2\right] dx - 2 \int (x \rho^\theta)_x (r - x)^2 dx \\
= \int x^2 \rho^\theta \left[(r_x - 1)^2 + 2 \left(\frac{r}{x} - 1\right)^2\right] dx + \frac{2}{\gamma} \int x^4 \phi \rho^\theta - (\gamma - 1) \left(\frac{r}{x} - 1\right)^2 dx
\]

where (2.3) has been used. We then have, noting (2.3) again, that

\[
C^{-1} \int x^2 \rho^\theta \left[(r_x - 1)^2 + \left(\frac{r}{x} - 1\right)^2\right] dx \leq \int x^2 \rho^\theta \eta_0 dx \\
\leq C \int x^2 \rho^\theta \left[(r_x - 1)^2 + \left(\frac{r}{x} - 1\right)^2\right] dx. \quad (3.23)
\]

Therefore, (3.16) follows from (3.22), the Cauchy inequality and (3.15).

**Step 3.** In this step, we prove that

\[
\mathcal{F}(t) \leq C \mathcal{D}(0) + \int_0^t (1 + s)^{-\frac{\gamma - \theta}{(\gamma - 1)/2}} \int x^2 \rho^\theta \left(|r_x - 1|^2 + \left|\frac{r}{x} - 1\right|^2\right) dx ds, \quad (3.24)
\]

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where

\[ \mathcal{F}(t) = \mathcal{D}(t) + \int_0^t \int x^2 \tilde{\rho}^\gamma \left[ \frac{r}{x} - 1 \right]^2 + (r_x - 1)^2 \ dx \, ds \]

\[ + \int_0^t (1 + s) \int \tilde{\rho}^\theta (x^2 v_x^2 + v^2) \ dx \, ds. \]

If (3.24) is true, then (3.9) follows from Grownwall's inequality.

It follows from (3.10) + k(3.16) with a suitably large constant k that

\[ \mathcal{F}(t) \leq C \mathcal{D}(0) + C \int_0^t \int \left[ \tilde{\rho}^\theta \left( |r_x - 1| + \frac{r}{x} - 1 \right) \right] |v| + \tilde{\rho}^\theta |r - x| |xv_x| \ dx \, ds. \]

(3.25)

Next, we estimate the last term on the right-hand side of (3.25). When \( A \geq 0 \), it follows from Young's inequality that for \( 0 < \omega < (\gamma - \theta)/(\gamma - 1) \),

\[ (1 + s)^{-1} \tilde{\rho}^{\theta + \omega(\gamma - 1)} A = \left\{ (1 + s)^{-1} \left( \tilde{\rho}^\theta A \right)^{\frac{\gamma - \theta - \omega(\gamma - 1)}{\gamma - \theta}} \right\} \left\{ (\tilde{\rho}^\gamma A)^{\frac{\omega(\gamma - 1)}{\gamma - \theta}} \right\} \]

\[ \leq (1 + s)^{-\frac{\gamma - \theta}{\gamma - \theta - \omega(\gamma - 1)}} \tilde{\rho}^\theta A + \tilde{\rho}^\gamma A. \]

Then, for any \( 0 < \omega < (\gamma - \theta)/(\gamma - 1) \),

\[ \int_0^t (1 + s)^{-1} \int x^2 \tilde{\rho}^{\theta + \omega(\gamma - 1)} \left( |r_x - 1|^2 + \frac{r}{x} - 1 \right)^2 \ dx \, ds \]

\[ \leq \int_0^t (1 + s)^{-\frac{\gamma - \theta}{\gamma - \theta - \omega(\gamma - 1)}} \int x^2 \tilde{\rho}^\theta \left( |r_x - 1|^2 + \frac{r}{x} - 1 \right)^2 \ dx \, ds \]

\[ + \int_0^t \int x^2 \tilde{\rho}^\gamma \left( |r_x - 1|^2 + \frac{r}{x} - 1 \right)^2 \ dx \, ds. \]

(3.26)

Let \( \chi \) be a smooth cutoff function satisfying

\[ \chi = 0 \ \text{on} \ [0, 1 - 2\delta], \ \chi = 1 \ \text{on} \ [1 - \delta, 1], \ \text{and} \ 0 \leq \chi \leq 1 \ \text{on} \ [0, 1] \]

(3.27)

for a small constant \( \delta \in (0, 1/4) \) to be determined later. It follows from the Cauchy-Schwarz inequality that

\[ \int \chi \tilde{\rho}^\theta \left[ x \left( |r_x - 1| + \frac{r}{x} - 1 \right) \right] |v| + |r - x| |xv_x| \ dx \]

\[ \leq \delta^{1/4} (1 + t) \int \chi \tilde{\rho}^{\theta - (\gamma - 1)} v^2 dx \]

\[ + C \delta^{-1/4} (1 + t)^{-1} \int \chi x^2 \tilde{\rho}^{\theta + \gamma - 1} \left( |r_x - 1|^2 + \frac{r}{x} - 1 \right)^2 \ dx \]

\[ + \delta^{1/4} (1 + t) \int \chi \tilde{\rho}^\theta x^2 v_x^2 dx + C \delta^{-1/4} (1 + t)^{-1} \int \chi \tilde{\rho}^\theta |r - x|^2 dx. \]

(3.28)

In view of the Hardy inequality (3.5) and (2.5), one gets that

\[ \int \chi \tilde{\rho}^{\theta - (\gamma - 1)} v^2 dx \leq \int_{1/2}^1 \tilde{\rho}^{\theta - (\gamma - 1)} v^2 dx \leq C \int_{1/2}^1 \tilde{\rho}^{\theta + (\gamma - 1)} (v^2 + v_x^2) \ dx \]

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and
\[
\int \chi \rho^\theta |r - x|^2 \, dx \leq \int_{1-2\delta}^1 \rho^{\theta(\gamma-1)} \, dx \leq C \int_{1/2}^1 \rho^{\theta(\gamma-1)} \, dx
\]
\[
\leq C \rho^{\theta(\gamma-1)} (|r - x|^2 + (r_x - 1)^2) \, dx,
\]
which, together with (3.28), imply that
\[
\int \chi \rho^\theta \left[ x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + |r - x| |xv_x| \right] \, dx \leq C \rho^{\theta(\gamma-1)} (|r - x|^2 + (r_x - 1)^2) \, dx.
\]
Integrating as using (3.26) with \( \omega = 1/2 \), give
\[
\int_0^t \int \chi \rho^\theta \left[ x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + |r - x| |xv_x| \right] \, dx \, ds \leq C \rho^{\theta(\gamma-1)} (|r - x|^2 + (r_x - 1)^2) \, dx \, ds.
\]
Using the Cauchy-Schwarz inequality and (2.5) again, one can obtain
\[
\int_0^t \int (1 - \chi) \rho^\theta \left[ x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + |r - x| |xv_x| \right] \, dx \, ds \leq C \rho^{\theta(\gamma-1)} (|r - x|^2 + (r_x - 1)^2) \, dx \, ds.
\]
where (3.15) has been used in the last inequality. So, the proof of (3.24) is completed by choosing \( \delta \) suitably small, and combining (3.25), (3.30) and (3.31) together.
The following Lemma gives the decay estimates of weighted norms for the higher time-derivatives (than those in Lemma 3.2).

**Lemma 3.3** Let $\theta \in (0, 1]$. Suppose that (3.1) and (3.2) hold. Then for $0 \leq t \leq T$,

\[
(1 + t) \int \{ x^2 \bar{\rho} v_t^2 + \bar{\rho}^\gamma (v^2 + x^2 v_x^2) \} (x, t) dx + \int_0^t (1 + s) \int \bar{\rho}^\theta \left( x^2 v_x^2 + v_s^2 \right) dx ds \\
\leq C \mathcal{D}(0) + C \int \{ \bar{\rho} x^2 v_t^2 + \bar{\rho}^\gamma (v^2 + x^2 v_x^2) \} (x, 0) dx.
\]  

(3.32)

Moreover, for $t \in [0, T]$, it holds that

\[
\mathcal{D}_1(t) + \int_0^t \int x^2 \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \\
+ \int_0^t (1 + s) \int \bar{\rho}^\theta \left( x^2 v_x^2 + v^2 + x^2 v_x^2 + v_s^2 \right) dx ds \leq C \mathcal{D}_1(0),
\]  

(3.33)

where

\[
\mathcal{D}_1(t) = \int x^2 \bar{\rho}^\theta \left[ (r_x - 1)^2 + \left( \frac{r}{x} - 1 \right)^2 \right] (x, t) dx + (1 + t) \int x^2 \{ \bar{\rho} (v^2 + v_x^2) \\
+ \bar{\rho}^\gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 + \left( \frac{v}{x} \right)^2 + v_x^2 \right] \} (x, t) dx.
\]

**Proof.** It yields from $\int (r^2 (3.6)) t v_i dx$ that

\[
\frac{d}{dt} \int \Phi(x, t) dx + \int \left\{ \nu \left[ \frac{x^2 \bar{\rho}}{r^2 r_x} \right] \frac{(r^2 v_i)_x}{r^2 r_x} \left[ (r^2 v_i)_x - 4 \nu_1 \frac{x^2 \bar{\rho}}{r^2 r_x} \left[ (rv)_i v_i \right] \right] \right\} dx \\
= - \int \left\{ 2 \nu \frac{x^2 \bar{\rho}}{r^2 r_x} \frac{(r^2 v)_x}{r^2 r_x} - 4 \nu_1 \left[ \frac{x^2 \bar{\rho}}{r^2 r_x} \right] \right\} (rvv_i)_x dx \\
+ \int \left[ \frac{x^2 \bar{\rho}}{r^2 r_x} \right] \left( 2r_x v_x^2 + 2(r_x - 1)v_x^2v_x + \frac{\gamma}{2} \left( \frac{r^2}{r_x} \right)_t v_x^2 \right) dx \\
+ \int \left[ \frac{x^2 \bar{\rho}}{r^2 r_x} \right] \left( 2(2r_x - 1)v_x^2 + 2(r_x - 1)rvv_x + \frac{\gamma r^2}{2 r_x} v_x^2 \right) dx \\
- \int \bar{\rho}^\gamma \left[ \left( 4 \frac{x^3}{r^3} - 3 \frac{x^4}{r^4} r_x \right)_t v^2 + 2 \left( \frac{x^4}{r^3} \right)_t vv_x \right] dx =: \text{RHS},
\]

(3.34)

where

\[
\Phi(x, t) = \frac{1}{2} x^2 \bar{\rho} v_t^2 + \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\gamma \left( 2r_x - 1 \right)v_x^2 + 2(\gamma - 1)vv_x + \frac{\gamma r^2}{2 r_x} v_x^2 \\
- \bar{\rho}^\gamma \left[ \left( 4 \frac{x^3}{r^3} - 3 \frac{x^4}{r^4} r_x \right)_t v^2 + 2 \frac{x^4}{r^3} vv_x \right].
\]
Similar to (3.12) and (3.13), one has
\[
\Phi(x, t) \geq \frac{1}{2} x^2 \bar{\rho} v_t^2 + \frac{3\gamma - 4}{4} x^2 \bar{\rho} \left[ 2 \left( \frac{v}{x} \right)^2 + v_x^2 \right],
\] (3.35)
\[
\Phi(x, t) \leq \frac{1}{2} x^2 \bar{\rho} v_t^2 + C(\gamma) x^2 \bar{\rho} \left[ \left( \frac{v}{x} \right)^2 + (v_x)^2 \right].
\] (3.36)

Using the a priori assumptions (3.1) and (3.2), and Cauchy-Schwarz’s inequality, one can obtain that
\[
\int \left\{ \nu \left[ \frac{x^2 \bar{\rho}}{r^2 r_x} \right]^\theta \frac{(r^2 v)_x}{r^2 r_x} (r^2 v_t)_x - 4\nu_1 \left( \frac{x^2 \bar{\rho}}{r^2 r_x} \right)^\theta [(rv)_t v_t] \right\} dx 
\geq 2\sigma \int \bar{\rho}^\theta (x^2 v_{xt}^2 + 2v_t^2) dx - C \int \bar{\rho}^\theta \left[ \epsilon_0 (x^2 v_{xt}^2 + 2v_t^2) + \sigma^{-1} (x^2 v_x^2 + v^2) \right] dx
\] and
\[
\text{RHS} \leq \frac{\sigma}{2} \int \bar{\rho}^\theta (x^2 v_{xt}^2 + v_t^2) dx + C \sigma^{-1} \int \bar{\rho}^\theta (x^2 v_x^2 + v^2) dx,
\]
where \( \sigma = \min\{2\nu_1/3, \nu_2\} \). Here one has used the fact that \( 0 < \theta \leq 1 < \gamma \). Therefore, it follows from (3.34) that
\[
\frac{d}{dt} \int \Phi(x, t) dx + \int \bar{\rho}^\theta (x^2 v_{xt}^2 + v_t^2) dx \leq C \int \bar{\rho}^\theta (x^2 v_x^2 + v^2) dx,
\] (3.37)
provided that \( \epsilon_0 \) is small. This, together with (3.9), implies
\[
\int \Phi(x, t) dx + \int_0^t \int \bar{\rho}^\theta (x^2 v_{xt}^2 + v_t^2) (x, s) dx ds \leq \int \Phi(x, 0) dx + C \Phi(0),
\] (3.38)
which further implies (3.32), by using (3.37), (3.9), (3.35), (3.36) and the fact \( 0 < \theta \leq 1 < \gamma \). Finally, (3.33) is a consequence of (3.9) and (3.32).

In the following lemma, we show the improved regularity near the vacuum boundary and decay.

**Lemma 3.4** Suppose that (3.1) and (3.2) hold. Let \( \theta \in (0, 1] \), and \( \alpha \) and \( \beta \) be given respectively in (2.18) and (2.19). If \( \iota \in (0, (\gamma - 1)/4] \), then for \( 0 \leq t \leq T \),
\[
\mathcal{D}_2(t) + \int_0^t \int x^2 \bar{\rho}^{\gamma - \alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds 
+ \int_0^t (1 + s)^{\beta - 1} \int x^2 \bar{\rho}^{\gamma} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds 
+ \int_0^t (1 + s)^{\beta} \int \bar{\rho}^\theta (x^2 v_x^2 + v^2) (x, s) dx ds \leq C \iota \left( \Phi(1)(0) + \| \nu_0 \|_{L^\infty} \right),
\] (3.39)
where

\[
\mathcal{D}_2(t) = \int x^2 \rho^{\theta-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] (x,t) \, dx \\
+ (1 + t)^{\beta-1} \int x^2 \rho^{\theta} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] (x,t) \, dx \\
+ (1 + t)^{\beta} \int x^2 \left\{ \rho v^2 + \dot{\rho} \gamma \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \right\} (x,t) \, dx.
\]

Here \( C \) is a constant continuously depending on \( \nu \), but not on \( t \).

**Proof.** The proof consists of three steps. Recall that \( \alpha \) and \( \beta \) are given respectively in (2.18) and (2.19), that is,

\[
\alpha = \min\{\gamma - 1 + \theta, 2(\gamma - 1)\} - \nu \quad \text{and} \quad \beta = 1 + (\alpha - \nu)/(\gamma - \theta).
\]

**Step 1.** In this step, we prove that for any \( \omega > 0 \) and \( \kappa > 1 \),

\[
\begin{align*}
& \int x^2 \rho^{\theta-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] (x,t) \, dx \\
& + \int_0^t \int x^2 \rho^{\gamma-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx \, ds \\
\leq & \ C \left( \mathcal{D}_1(0) + \|r_0\|_{L^\infty}^2 \right) + C \omega \int_0^t (1 + s)^{\kappa} \int \rho^\theta (v^2 + x^2 v_x^2) \, dx \, ds \\
& + C \omega^{-1} \int_0^t (1 + s)^{-\kappa} \int x^2 \rho^{\theta-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx \, ds.
\end{align*}
\]

(3.40)

It should be noted that \( \kappa > 1 \), which ensures the last line in (3.40) behaves well according to the Gronwall inequality.

Multiplying (3.6) by \( \int_0^x \rho^{-\alpha}(y) (r^3 - y^3)_y \, dy \) and integrating the resulting equation with respect to the spatial variable, one has, with the help of the integration by parts and boundary condition (2.1), that

\[
\int \rho^\gamma \left\{ \left[ \frac{x^4}{r^4} \int_0^x \rho^{-\alpha}(y) (r^3 - y^3)_y \, dy \right] - \left( \frac{x^2}{r^2 r_x} \right)^\gamma \rho^{\gamma-\alpha} (r^3 - x^3)_x \right\} \, dx \\
= - \int \rho \left( \frac{x}{r} \right)^2 v_t \int_0^x \rho^{-\alpha}(y) (r^3 - y^3)_y \, dy \, dx - \nu \int \rho^{\theta-\alpha} \left( \frac{x^2}{r^2 r_x} \right)^\theta \frac{(r^2 v)_x}{r^2 x} (r^3 - x^3)_x \, dx \\
+ 4 \nu_t \int \rho^\theta \left( \frac{x^2}{r^2 r_x} \right)^\theta \frac{v}{r} \int_0^x \rho^{-\alpha}(y) (r^3 - y^3)_y \, dy \, dx.
\]

As the derivation of (3.16), one can get

\[
\begin{align*}
& \int x^2 \rho^{\theta-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx + \int_0^t \int x^2 \rho^{\gamma-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] \, dx \, ds \\
\leq & \ C \int x^2 \rho^{\theta-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] (x,0) \, dx + C \sum_{i=1}^3 \int_0^t |L_i| \, ds,
\end{align*}
\]

(3.41)
where

\[ L_1 = -\int \frac{x^2}{r^2} v_t \left( \int_0^x \varrho^{-\alpha} (r^3 - y^3) dy \right) dx, \]

\[ L_2 = \int \varrho^\alpha \left( \frac{x^4}{r^3} \right) x \left[ \varrho^{-\alpha} (r^3 - x^3) - \int_0^x \varrho^{-\alpha} (r^3 - y^3) dy \right] dx, \]

\[ L_3 = 4\nu_1 \int \varrho^\alpha \left( \frac{x^2}{r^2 r_x} \right) \left[ \frac{v}{r} \int_0^x \varrho^{-\alpha} (y^3 - y) dy \right] dx. \]

In the above integration by parts, we should notice that \( 0 < \alpha < \theta + (\gamma - 1) \) which ensures that \( \varrho^{\theta - \alpha} \) is integrable on \([0, 1]\) due to (2.5), so that the above integrations by parts are legitimate.

Next, we estimate \( L_1, L_2 \) and \( L_3 \). For \( L_1 \), it follows from the Cauchy inequality that for any \( \omega > 0 \),

\[ |L_1| \leq C \omega^{-1} \int \varrho^{\theta - (\gamma - 1)} v^2_t dx + \omega \int \varrho^{1 - \theta + \gamma} \left| \int_0^x \varrho^{-\alpha} y^2 (|r/y - 1| + |r_y - 1|) dy \right|^2 dx. \]  

(3.42)

Due to (2.5) and (3.5), one has

\[ \int \varrho^{\theta - (\gamma - 1)} v^2_t dx \leq C \int_0^{1/2} \varrho^\theta v^2_t dx + C \int_{1/2}^1 \varrho^{\theta + (\gamma - 1)} (v^2_t + v^2_{tx}) dx \]

\[ \leq C \int_0^{1/2} \varrho^\theta v^2_t dx + C \int_{1/2}^1 \varrho^\theta (v^2_t + x^2 v^2_{tx}) dx \leq C \int \varrho^\theta (v^2_t + x^2 v^2_{tx}) dx. \]  

(3.43)

Due to Hölder’s inequality, \( \theta \leq 1 \) and \( \alpha < 2(\gamma - 1) \) (which implies \( \alpha + \theta < 2\gamma - 1 \)), and (2.5), one obtains

\[ \int \varrho^{1 - \theta + \gamma} \left| \int_0^x \varrho^{-\alpha} y^2 (|r/y - 1| + |r_y - 1|) dy \right|^2 dx \]

\[ \leq \int \varrho^{1 - \theta + \gamma} \left( \int_0^x \varrho^{-\gamma - \alpha} y^2 dy \right) \left( \int_0^1 \varrho^{\gamma - \alpha} y^2 (|r/y - 1| + |r_y - 1|) dy \right) dx \]

\[ \leq C \int_0^1 \varrho^{\gamma - \alpha} y^2 (|r/y - 1| + |r_y - 1|) dy. \]  

(3.44)

It yields from (3.42)-(3.44) that for any \( \omega > 0 \),

\[ \int_0^t |L_1| ds \leq C \omega^{-1} \int_0^t \varrho^\theta (v^2_s + x^2 v^2_{sx}) dx ds \]

\[ + C\omega \int_0^t \int x^2 \varrho^{\gamma - \alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds. \]  

(3.45)
Rewrite $L_2$ as

$$L_2 = \int_0^{1/2} \rho^{\gamma} \left( \frac{x^4}{r^4} \right) \int_0^x (\rho^{-\alpha}) y (r^3 - y^3) dy dx$$

$$+ \int_1^{1/2} \rho^{\gamma} \left( \frac{x^4}{r^4} \right) \left[ \rho^{-\alpha} (r^3 - x^3) - \int_0^x \rho^{-\alpha} (r^3 - y^3) dy \right] dx = : L_{21} + L_{22}.$$  

It follows from that (2.3), and the Cauchy and Hőlder inequalities that

$$|L_{21}| \leq C \int_0^{1/2} x^{-1} \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) \left| \int_0^x y^3 r - y dy \right| dx$$

$$\leq C \int_0^{1/2} x^2 \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx + C \int_0^{1/2} x^{-4} \left( \int_0^x y^6 dy \right) \left( \int_0^x \left| r - y \right|^2 dy \right) dx$$

(3.46)

$$\leq C \int_0^{1/2} x^2 \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx \leq C \int_0^{1/2} x^2 \rho^{\gamma} \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx,$$

and for any $\omega > 0,$

$$|L_{22}| \leq C \int_0^{1/2} \rho^{\gamma-\alpha} \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) \left| r - x \right| dx$$

$$+ C \int_0^{1/2} \rho^{\gamma} \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) \left( \int_0^x \rho^{-\alpha} y^2 \left( |r_y - 1| + \left| \frac{r}{y} - 1 \right| \right) dy \right) dx$$

$$\leq \omega \int_0^{1/2} \rho^{\gamma-\alpha} \left( \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right) dx + C \omega^{-1} \int_0^{1/2} \rho^{\gamma-\alpha} (r - x)^2 dx$$

(3.47)

$$+ C \omega^{-1} \int_0^{1/2} \rho^{\gamma+\alpha} \left( \int_0^x \rho^{-\gamma-2\alpha} y^2 dy \right) \left( \int_0^1 \rho^{\gamma} y^2 \left( |r_y - 1|^2 + \left| \frac{r}{y} - 1 \right|^2 \right) dy \right) dx.$$

By virtue of (2.3), (3.3) and $\alpha < 2(\gamma - 1),$ one has

$$\int_0^1 \rho^{\gamma-\alpha} (r - x)^2 dx \leq C \int_0^1 \rho^{\gamma+\alpha+2(\gamma-1)} \left[ (r - x)^2 + (r_x - 1)^2 \right]$$

$$\leq C \int_0^1 \rho^{\gamma} \left[ (r - x)^2 + x^2(r_x - 1)^2 \right]$$

and

$$\int_0^1 \rho^{\gamma+\alpha} \left( \int_0^x \rho^{-\gamma-2\alpha} y^2 dy \right) dx \leq C.$$

It then yields from (3.46) and (3.47) that for any $\omega > 0,$

$$\int_0^t |L_2| ds \leq C \omega \int_0^t \int x^2 \rho^{\gamma-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds$$

$$+ C \omega^{-1} \int_0^t \int x^2 \rho^{\gamma} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds.$$  

(3.48)
For $L_3$, it holds that

$$|L_3| \leq C \int \tilde{\rho}^{\theta - \alpha} \frac{|v|}{r} |(r^3 - x^3)| dx + C \int \tilde{\rho}^{\theta} \left| \frac{(v)}{r} \right| \int_0^x \tilde{\rho}^{-\alpha} (r^3 - y^3) dy dx$$

$$\leq C \int \tilde{\rho}^{\theta - \alpha} |v| (x|r_x - 1| + |r - x|) dx$$

$$+ C \int \tilde{\rho} x^{-2} (x|v_x| + |v|) \left| \int_0^x \tilde{\rho}^{-\alpha} (r^3 - y^3) dy \right| dx =: L_{31} + L_{32}. \quad (3.49)$$

It follows from (3.21), (3.34), (2.3), $\theta - \alpha > 1 - \gamma$ and $\alpha < 2(\gamma - 1)$ that

$$\int \tilde{\rho}^{\theta - \alpha} v^2 dx \leq C \int_0^{1/2} v^2 dx + C \int_{1/2}^{1} \tilde{\rho}^{\theta - \alpha} v^2 dx \leq C \int_0^{1/2} x^2 (v^2 + v_x^2) dx$$

$$+ C \int_{1/2}^{1} \tilde{\rho}^{\theta - \alpha + 2(\gamma - 1)} (v^2 + v_x^2) dx \leq C \int \tilde{\rho} (v^2 + x^2 v_x^2) dx, \quad (3.50)$$

which, together with the Cauchy inequality, gives that for any $\omega > 0$,

$$L_{31} \leq C \omega (1 + t)^{\kappa} \int \tilde{\rho}^{\theta} (v^2 + x^2 v_x^2) dx$$

$$+ C \omega^{-1} (1 + t)^{-\kappa} \int x^2 \tilde{\rho}^{\theta - \alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx. \quad (3.51)$$

Clearly, $L_{32}$ can be bounded by

$$L_{32} \leq C \int_0^{1/2} x^{-2} (x|v_x| + |v|) \left| \tilde{\rho}^{-\alpha} (r^3 - x^3) - \int_0^x (\tilde{\rho}^{-\alpha})_y (r^3 - y^3) dy \right| dx$$

$$+ C \int_{1/2}^{1} \tilde{\rho}(x|v_x| + |v|) \left| \int_0^x \tilde{\rho}^{-\alpha} y^2 \left( |r_y - 1| + \left| \frac{r}{y} - 1 \right| \right) dy \right| dx =: L_{321} + L_{322}. \quad (3.52)$$

For $L_{321}$, it follows from (2.3) and the Cauchy and Hölder inequalities that

$$L_{321} \leq C \int_0^{1/2} (x|v_x| + |v|)|r - x| dx + C \int_0^{1/2} x^{-2} (x|v_x| + |v|) \left| \int_0^x y^3 |r - y| dy \right| dx$$

$$\leq C \int_0^{1/2} \left[ \tilde{\rho} (x^2 v_x^2 + v^2) + \tilde{\rho}^\gamma (r - x)^2 \right] dx + C \int_0^{1/2} x^{-4} \left( \int_0^x y^6 dy \right) \left( \int_0^x (r - y)^2 dy \right) dx$$

$$\leq C \int_0^{1/2} \left[ \tilde{\rho} (x^2 v_x^2 + v^2) + \tilde{\rho}^\gamma (r - x)^2 \right] dx. \quad (3.53)$$

For $L_{322}$, it follows from the Cauchy and Hölder inequalities that for any $\omega > 0$,

$$L_{322} \leq \omega (1 + t)^{\kappa} \int_{1/2}^{1} \tilde{\rho}^{\theta} (x^2 v_x^2 + v^2) dx$$

$$+ C \omega^{-1} (1 + t)^{-\kappa} \int_0^{1} \tilde{\rho}^{\theta - \alpha} y^2 \left( |r_y - 1|^2 + |r/y - 1|^2 \right) dy. \quad (3.54)$$
Here one has used the fact that $\alpha < 2(\gamma - 1)$, which implies

$$\int_{1/2}^{1} \rho^{\delta} \left( \int_{0}^{x} \rho^{-\theta - \alpha} y^{2} dy \right) dx \leq C.$$ 

So, it holds that for any $\omega > 0$,

$$L_{32} \leq \omega(1 + t)^{\kappa} \int \rho^{\theta}(v^{2} + x^{2}v_{x}^{2}) dx + C \int [\rho^{\theta}(x^{2}v_{x}^{2} + v^{2}) + \rho^{\gamma}(r - x)^{2}] dx$$

$$+ C \omega^{-1}(1 + t)^{-\kappa} \int x^{2}\rho^{\theta - \alpha} \left[ (\frac{r}{x} - 1)^{2} + (r_{x} - 1)^{2} \right] dx.$$

This, together with (3.49) and (3.51), implies

$$\int_{0}^{t} |L_{3}| ds \leq C \omega \int_{0}^{t} (1 + s)^{\kappa} \int \rho^{\theta}(v^{2} + x^{2}v_{x}^{2}) ds$$

$$+ C \int_{0}^{t} \int [\rho^{\theta}(x^{2}v_{x}^{2} + v^{2}) + \rho^{\gamma}(r - x)^{2}] dx ds$$

$$+ C \omega^{-1} \int_{0}^{t} (1 + s)^{-\kappa} \int x^{2}\rho^{\theta - \alpha} \left[ (\frac{r}{x} - 1)^{2} + (r_{x} - 1)^{2} \right] dx ds. \tag{3.52}$$

The estimate (3.40) follows from (3.45), (3.48), (3.52), (3.33), and the fact that $r(0, t) = 0$ (which implies $\|r - x\|_{L^{\infty}} \leq \|r_{x} - 1\|_{L^{\infty}}$).

**Step 2.** In this step, we prove that for any $\kappa > 1$,

$$\mathcal{F}_{1}^{\kappa}(t) \leq C D_{1}(0) + C \int_{0}^{t} (1 + s)^{-1 + \kappa - (1 - \kappa)(\gamma - \theta - \delta)} \int x^{2}\rho^{\theta - \alpha} \left[ (\frac{r}{x} - 1)^{2} + (r_{x} - 1)^{2} \right] dx ds$$

$$+ C \int_{0}^{t} (1 + s)^{-\gamma - \theta - (\gamma - 1)/2}(1 + s)^{-\kappa} \int x^{2}\rho^{\theta} \left[ |r_{x} - 1|^{2} + |\frac{r}{x} - 1|^{2} \right] dx ds$$

$$+ C(1 + t)^{\kappa - 1} \int x^{2}\rho v^{2}(x, t) dx + C \int_{0}^{t} (1 + s)^{-\kappa - 1} \int \rho^{\theta}(x^{2}v_{x}^{2} + v^{2}) dx ds, \tag{3.53}$$

where

$$\mathcal{F}_{1}^{\kappa}(t) = (1 + t)^{\kappa} \int x^{2} \left\{ \bar{\rho}v^{2} + \rho^{\gamma} \left[ (\frac{r}{x} - 1)^{2} + (r_{x} - 1)^{2} \right] \right\} (x, t) dx$$

$$+ (1 + t)^{\kappa - 1} \int x^{2}\rho^{\theta} \left[ (\frac{r}{x} - 1)^{2} + (r_{x} - 1)^{2} \right] (x, t) dx$$

$$+ \int_{0}^{t} (1 + s)^{\kappa} \int \rho^{\theta}(x^{2}v_{x}^{2} + v^{2}) dx ds$$

$$+ \int_{0}^{t} (1 + s)^{\kappa - 1} \int x^{2}\rho^{\gamma} \left[ (\frac{r}{x} - 1)^{2} + (r_{x} - 1)^{2} \right] dx ds.$$
It follows from a suitable combination of \( \int_0^t (1 + s)^{\gamma-1} \) with \( \ref{3.14} \) for \( \ell = \kappa \) that

\[
\mathcal{F}_1^\kappa(t) \leq C \mathcal{D}_1(0) + C \int_0^t (1 + s)^{\gamma-2} \int x^2 \rho^\theta \left[ \left( \frac{\rho}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \\
+ C \int_0^t (1 + s)^{\kappa-1} \int \left[ \hat{\rho}^\theta x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + \hat{\rho}^\theta |r - x| |x v_x| \right] dx ds \\
+ C(1 + t)^{\kappa-1} \int x^2 \rho \nu^2(x,t) dx + C \int_0^t (1 + s)^{\kappa-1} \int \rho^\theta \nu^2 dx ds, \tag{3.54}
\]

where \( \ref{3.23} \) and \( \ref{3.33} \) have been used. Each term on the right-hand side of \( \ref{3.54} \) can be estimated as follows. It follows from the Young inequality that for any \( \omega > 0 \),

\[
\int_0^t (1 + s)^{\kappa-2} \int x^2 \rho^\theta \left[ \left( \frac{\rho}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \\
\leq \omega \int_0^t (1 + s)^{\kappa-1} \int x^2 \rho^\gamma \left[ \left( \frac{\rho}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \\
+ C_\omega \int_0^t (1 + s)^{-1 + (\kappa-1) - \frac{\omega}{\gamma-\alpha}} \int x^2 \rho^{\theta-\alpha} \left[ \left( \frac{\rho}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds, \tag{3.55}
\]

due to

\[
(1 + s)^{\kappa-2} \rho^\theta = \left( (1 + s)^{\frac{\omega}{\gamma-\theta+\alpha}}(\kappa-1)^{-1} \rho^{\frac{\omega}{\gamma-\theta+\alpha}}(\theta-\alpha) \right) \left( (1 + s)^{\frac{\omega}{\gamma-\theta+\alpha}}(\kappa-1) \rho^{-\alpha} \right).
\]

Let \( \chi \) be a smooth cutoff function satisfying \( \ref{3.27} \). Similar to \( \ref{3.29} \) and \( \ref{3.26} \), one obtains

\[
(1 + t)^{\kappa-1} \int \chi \rho^\theta \left[ x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + |r - x| |x v_x| \right] dx \\
\leq C \delta^{1/4}(1 + t)^{\kappa} \int \rho^\theta \left( v^2 + x^2 v_x^2 \right) dx \\
+ C \delta^{1/4}(1 + t)^{\kappa-2} \int x^2 \rho^{\theta+(\gamma-1)/2} \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx \tag{3.56}
\]

and

\[
(1 + t)^{\kappa-2} \int x^2 \rho^{\theta+(\gamma-1)/2} \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx \\
\leq (1 + t)^{\kappa-1 - \frac{\gamma-1}{\gamma-\theta+\alpha}} \int x^2 \rho^\theta \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx \\
+ (1 + t)^{\kappa-1} \int x^2 \rho^\gamma \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx. \tag{3.57}
\]
It thus yields from (3.56) and (3.57) that

\[
\int_0^t (1 + s)^{1-\kappa} \int \chi \rho^\theta \left[ x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + |r - x| |xv_x| \right] dx ds \\
\leq C \delta^{1/4} \int_0^t (1 + s)^{\kappa} \int \rho^\theta \left( v^2 + x^2 v_x^2 \right) dx ds \\
+ C \delta^{1/4} \int_0^t (1 + s)^{\kappa - 1} \int x^2 \rho^\gamma \left( |r_x - 1|^2 + \left| \frac{r}{x} - 1 \right|^2 \right) dx ds
\] (3.58)

Similar to (3.31), one can get

\[
\int_0^t (1 + s)^{\kappa - 1} \int (1 - \chi) \rho^\theta \left[ x \left( |r_x - 1| + \left| \frac{r}{x} - 1 \right| \right) |v| + |r - x| |xv_x| \right] dx ds \\
\leq C \delta^{\frac{1-\kappa}{\kappa}} \int_0^t (1 + s)^{\kappa - 1} \int \rho^\theta \left( x^2 v_x^2 + v^2 \right) dx ds \\
+ C \delta^{\frac{1}{\kappa}} \int_0^t (1 + s)^{\kappa - 1} \int x^2 \rho^\gamma \left( \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right) dx ds
\] (3.59)

Now, the proof of (3.53) is completed by choosing \( \omega \) and \( \delta \) suitably small, and combining (3.54), (3.55), (3.58) and (3.59).

**Step 3.** In this step, we prove that

\[
\mathbb{F}_2^\beta(t) \leq C \left( \mathbb{D}_1(0) + |r_{0x} - 1|_{L^\infty}^2 \right),
\] (3.60)

where \( \beta \) is defined by (2.19), and

\[
\mathbb{F}_2^\kappa(t) = \mathbb{F}_2^\ast(t) + \int x^2 \rho^{-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] (x,t) dx \\
+ \int_0^t \int x^2 \rho^{-\gamma} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \text{ for any } \kappa > 1.
\]

It follows from (3.40) and (3.53) that for any \( \kappa > 1,

\[
\mathbb{F}_2^\kappa(t) \leq C \left( \mathbb{D}_1(0) + |r_{0x} - 1|_{L^\infty}^2 \right) + C \int_0^t \left[ (1 + s)^{-1+\left( \kappa - 1 - \frac{\alpha}{\gamma} \right)} \right] \left[ \frac{1}{(1 + s)^{-\kappa}} \right] \int x^2 \rho^{-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \\
+ C \int_0^t (1 + s)^{-\kappa} \int x^2 \rho^{-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds \\
+ C \int_0^t (1 + s)^{-\kappa} \int x^2 \rho^{-\alpha} \left[ \left( \frac{r}{x} - 1 \right)^2 + \left| \frac{r}{x} - 1 \right|^2 \right] dx ds
\] (3.61)
When $\beta \leq 2$, (3.60) follows from (3.61) with $\kappa = \beta$, (3.33), $\theta \leq 1$, and the Gronwall inequality. When $\beta > 2$, we have

$$1 - \alpha/(\gamma - \theta) < -\ell/(\gamma - \theta).$$

Then, it follows from (3.61) with $\kappa = 2$, (3.33), $\theta \leq 1$, and the Gronwall inequality that

$$\mathcal{F}_2^2(t) \leq C \left( \mathcal{D}_1(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right).$$

(3.62)

Due to $\theta \leq 1$, one has

$$\beta = 1 + \frac{\alpha - \ell}{\gamma - \theta} < 1 + \frac{\alpha}{\gamma - 1} < 3.$$

So, (3.60) follows from (3.61) with $\kappa = \beta$, (3.62), $\theta \leq 1$, and the Gronwall inequality.

As an immediate consequence of (3.39), we have the following Lemma.

**Lemma 3.5** Suppose that (3.1) and (3.2) hold. Let $\theta \in (0, \gamma/2]$, and $\alpha$, $\beta$ and $\varsigma$ be given respectively in (2.18), (2.19) and (2.20). If $\ell \in (0, (2\gamma - 2 - \theta)/8]$, then for $0 \leq t \leq T$,

$$\begin{align*}
(1 + t)^{2\gamma - 2 + \alpha - \theta/\gamma} & \gamma x + \alpha - \theta} \int (r(x, t) - x)^2 dx + (1 + t)^{\alpha - \theta/\gamma} \int x^2 (r_x(x, t) - 1)^2 dx \\
+ (1 + t)^{\beta} & \int \left( x^2 \rho v_t^2 + \dot{\rho} \left( x^2 v_x^2 + v^2 \right) \right) (x, t) dx \\
+ \int_0^t & (1 + s)^{\beta} \int \rho^\varsigma (x^2 v_x^2 + v^2) dx ds \leq C_i \left( \mathcal{D}_1(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right),
\end{align*}$$

(3.63)

$$\begin{align*}
(1 + t)^{\beta} & \int \left( v^2 + x^2 \rho v_x^2 \right) (x, t) dx + (1 + t)^{\beta - \varsigma/2} \int \rho^\varsigma (x^2 v_x^2 + v^2)(t, x) dx \\
\leq C_i \left( \mathcal{D}_1(0) + \|r_{0x} - 1\|_{L^\infty}^2 + \|v_x(\cdot, 0)\|_{L^\infty}^2 \right).
\end{align*}$$

(3.64)

**Proof.** The proof consists of three steps.

**Step 1.** In this step, we prove that

$$\begin{align*}
(1 + t)^{2\gamma - 2 + \alpha - \theta/\gamma} & \gamma x + \alpha - \theta} \int (r(x, t) - x)^2 dx \leq C_i \left( \mathcal{D}_1(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right),
\end{align*}$$

(3.65)

$$\begin{align*}
(1 + t)^{\alpha - \theta/\gamma} & \gamma x + \alpha - \theta} \int x^2 (r_x(x, t) - 1)^2 dx \leq C_i \left( \mathcal{D}_1(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right),
\end{align*}$$

(3.66)

$$\begin{align*}
(1 + t)^{\beta} & \int \left( x^2 \rho v_t^2 + \dot{\rho} \left( x^2 v_x^2 + v^2 \right) \right) (x, t) dx \\
+ \int_0^t & (1 + s)^{\beta} \int \rho^\varsigma (x^2 v_x^2 + v^2) dx ds \leq C_i \left( \mathcal{D}_1(0) + \|r_{0x} - 1\|_{L^\infty}^2 \right).
\end{align*}$$

(3.67)
It follows from Hardy’s inequalities (3.4) and (3.5), (2.5), \( \theta < \alpha \) and the Hölder inequality that
\[
\int (r - x)^2 dx \leq C \int x^2 \rho^2(\gamma - 1) ((r - x)^2 + (r_x - 1)^2) dx \\
\leq C \left( \int x^2 \rho^\theta - \alpha ((r - x)^2 + (r_x - 1)^2) dx \right)^{\frac{2-\gamma}{\gamma + \alpha - \theta}} \times \left( \int x^2 \rho^\gamma ((r - x)^2 + (r_x - 1)^2) dx \right)^{\frac{2\gamma - 2 - \alpha - \theta}{\gamma + \alpha - \theta}}.
\]
Thus, (3.65) follows from (3.39). Clearly, (3.66) follows from (3.39) and
\[
\int x^2 (r_x - 1)^2 dx \leq C \left( \int x^2 \rho^\theta - \alpha (r_x - 1)^2 dx \right)^{\frac{2-\gamma}{\gamma + \alpha - \theta}} \left( \int x^2 \rho^\gamma (r_x - 1)^2 dx \right)^{\frac{2\gamma - 2 - \alpha - \theta}{\gamma + \alpha - \theta}}.
\]
With (3.39), one may easily obtain (3.67) by virtue of (3.37), (3.35), (3.36) and (3.33).

**Step 2.** In this step, we prove that
\[
(1 + t)^\varsigma \int \rho^{-\alpha} [(x^2 (r_x - 1)^2 + (r - x)^2)] (x,t) dx \\
+ \int_0^t (1 + s)^\varsigma \int \rho^\theta - \alpha (x^2 v_x^2 + v^2) dx ds \leq C, (\mathcal{D}_1(0) + \| r_0x - 1 \|^2_{L^\infty}).
\] (3.68)

Multiplying equation (3.66) by \( \int_0^x \rho^{-\alpha}(y)(r^2v)_ydy \) and integrating the product with respect to spatial variable, one obtains, following the derivation of (3.40), that
\[
(1 + t)^\varsigma \int x^2 \rho^\theta - \alpha \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] (x,t) dx \\
+ \int_0^t (1 + s)^\varsigma \int \rho^\theta - \alpha (x^2 v_x^2 + v^2) dx ds \\
\leq C \| r_0x - 1 \|^2_{L^\infty} + C \sum_{i=1}^3 \int_0^t (1 + s)^\varsigma |K_i| ds + C \int_0^t \int x^2 \rho^\theta - \alpha \left[ \left( \frac{r}{x} - 1 \right)^2 + (r_x - 1)^2 \right] dx ds,
\] (3.69)

where
\[
K_1 = \int \frac{x^2}{r^2} v_x \left( \int_0^x \rho^{-\alpha} (r^2v)_y dy \right) dx,
K_2 = \int \rho^\gamma \left( \frac{x^4}{r^4} \right) \left[ \rho^{-\alpha} r^2 v - \int_0^x \rho^{-\alpha} (r^2v)_y dy \right] dx,
K_3 = 4\nu \int \left( \frac{x^4}{r^4} \right) \left( \frac{v_x}{r} \right) \left[ \left( \int_0^x \rho^{-\alpha} (r^2v)_y dy \right) - \rho^{-\alpha} r^2 v \right] dx.
\]
The estimate for \( K_i \) can be obtained in a similar way as the derivation of (3.45), (3.48) and (3.52). Clearly,
\[
\int_0^t (1 + s)^\varsigma |K_1| \leq C \omega^{-1} \int_0^t (1 + s)^\varsigma \int \rho^\theta v_x^2 dx ds \\
+ C \omega \int_0^t (1 + s)^\varsigma \int \rho^\theta (v^2 + x^2 v_x^2) dx ds.
\] (3.70)
for any $\omega > 0$. To deal with $K_2$, one notes that

$$
(1 + t)^\beta \int_{1/2}^1 \bar{\rho}^{\gamma - \alpha} (x|\rho - 1| + |r - x|) |v| dx
\leq (1 + t)^\beta \int_{1/2}^1 \rho^{\theta - \alpha} v^2 dx + (1 + t)^{2\gamma - \beta} \int_{1/2}^1 \bar{\rho}^{2\gamma - \alpha - \theta} (x^2|\rho - 1| + |r - x|^2) dx
\leq C(1 + t)^\beta \int_{1/2}^1 \rho^{\theta}(x^2v_x^2 + v^2) dx + (1 + t)^{2\gamma - \beta} \int_{1/2}^1 \bar{\rho}^{2\gamma - \alpha - \theta} (x^2|\rho - 1| + |r - x|^2) dx.
$$

This implies that if $\theta + \alpha \leq \gamma$,

$$
(1 + t)^\beta \int_{1/2}^1 \bar{\rho}^{\gamma - \alpha} (x|\rho - 1| + |r - x|) |v| dx
\leq C(1 + t)^\beta \int \rho^{\theta}(xv_x^2 + v^2) dx + C(1 + t)^{\beta - 1} \int \bar{\rho}^{\gamma} (x^2|\rho - 1| + |r - x|^2) dx;
$$

and if $\theta + \alpha > \gamma$,

$$
(1 + t)^\beta \int_{1/2}^1 \bar{\rho}^{\gamma - \alpha} (x|\rho - 1| + |r - x|) |v| dx
\leq C(1 + t)^\beta \int \rho^{\theta}(xv_x^2 + v^2) dx + C \int \bar{\rho}^{\gamma - \alpha} (x^2|\rho - 1| + |r - x|^2) dx
+ C(1 + t)^{\beta - 1} \int \bar{\rho}^{\gamma} (x^2|\rho - 1| + |r - x|^2) dx,
$$

due to

$$
\frac{\alpha}{\gamma - \theta}(2\gamma - \beta) \leq \beta - 1.
$$

Then, one can obtain

$$
\int_0^t (1 + s)^\zeta |K_2| ds \leq C \int_0^t (1 + s)^\beta \int \rho^{\theta}(xv_x^2 + v^2) dxds
+ C \int_0^t \int \bar{\rho}^{\gamma - \alpha} (x^2|\rho - 1| + |r - x|^2) dxds
+ C \int_0^t (1 + s)^{\beta - 1} \int \bar{\rho}^{\gamma} (x^2|\rho - 1| + |r - x|^2) dxds.
$$

(3.71)

Note that for any $\omega > 0$,

$$
\int_0^t (1 + s)^\zeta |K_3| ds \leq C \omega^{-1} \int_0^t (1 + s)^\zeta \int \rho^{\theta}(xv_x^2 + v^2) dxds
+ \omega \int_0^t (1 + s)^\zeta \int \rho^{\theta - \alpha}(xv_x^2 + v^2) dxds.
$$

(3.72)

So, (3.68) follows from (3.33), (3.39) and (3.69)-(3.72).
Step 3. In this step, we prove that

\[(1 + t)^{\frac{\beta + \varsigma}{2}} \int \rho^{\theta - \alpha/2} (x^2 v_x^2 + v^2)(x,t) dx \leq C_i \left( \mathcal{O}_1(0) + \| r_{0x} - 1 \|_{L^\infty}^2 + \| v_x(\cdot,0) \|_{L^\infty}^2 \right), \quad (3.73)\]

\[(1 + t)^{\beta} \int (v^2 + x^2 \rho^{\theta} v_x^2)(x,t) dx \leq C_i \left( \mathcal{O}_1(0) + \| r_{0x} - 1 \|_{L^\infty}^2 + \| v_x(\cdot,0) \|_{L^\infty}^2 \right). \quad (3.74)\]

Notice that

\[(1 + t)^{\frac{\beta + \varsigma}{2}} \int \rho^{\theta - \alpha/2} (x^2 v_x^2 + v^2)(x,t) dx
= \int \rho^{\theta - \alpha/2} (x^2 v_x^2 + v^2)(x,0) dx + \frac{\beta + \varsigma}{2} \int_0^t (1 + s)^{\frac{\beta + \varsigma}{2} - 1} \int \rho^{\theta - \alpha/2} (x^2 v_x^2 + v^2) dx ds
+ 2 \int_0^t (1 + s)^{\frac{\beta + \varsigma}{2}} \int \rho^{\theta - \alpha/2} (x^2 v_x v_{sx} + vv_s) dx ds
\leq C \| v_x(\cdot,0) \|_{L^\infty}^2 + C \int_0^t (1 + s)^{\beta} \int \rho^{\theta} (x^2 v_x^2 + v^2 + x^2 v_{sx}^2 + v_s^2) dx ds
+ C \int_0^t (1 + s)^{\varsigma} \int \rho^{\theta - \alpha} (x^2 v_x^2 + v^2) dx ds.\]

Then, (3.73) follows from (3.39), (3.67) and (3.68). Similarly, one can show that

\[(1 + t)^{\beta} \int \rho^{\theta} (x^2 v_x^2 + v^2)(x,t) dx \leq C_i \left( \mathcal{O}_1(0) + \| r_{0x} - 1 \|_{L^\infty}^2 + \| v_x(\cdot,0) \|_{L^\infty}^2 \right).\]

This, together with (2.5), (3.4), (3.5) and \(2(\gamma - 1) > \gamma/2 \geq \theta\), gives (3.74).

Suppose that (3.1) and (3.2) hold. Let \(\theta \in (0, \gamma/2]\), and \(\alpha\) and \(\beta\) be given respectively in (2.18) and (2.19). Then, for any \(\iota \in (0, (2\gamma - 2 - \theta)/8]\) and \(l \in (0, 1)\), one has

**Lemma 3.6** Suppose that (3.1) and (3.2) hold. Let \(\theta \in (0, \gamma/2]\), and \(\alpha\), \(\beta\) and \(\varsigma\) be given respectively in (2.18), (2.19) and (2.20). If \(\iota \in (0, (2\gamma - 2 - \theta)/8]\), then for \((x, t) \in I \times [0, T]\),

\[
(1 + t)^{\frac{\gamma - 1 + \alpha - \theta}{2 + \alpha - \theta}} x |r(x, t) - x|^2 + (1 + t)^{\frac{\beta + \varsigma}{2} - \frac{\beta + \varsigma}{\alpha + \theta}} \max\{0, 4^{\theta - 4(\gamma - 1) - \alpha}\} x v^2(x, t)
+ x^3 |r_x(x, t)| - 1|^2 + x^3 |v_x(x, t)|^2 \leq C_i \left( \mathcal{O}_1(0) + \| r_{0x} - 1 \|_{L^\infty}^2 + \| v_x(\cdot,0) \|_{L^\infty}^2 \right). \quad (3.75)\]

**Proof.** The proof consists of three steps.

**Step 1.** In this step, we prove

\[(1 + t)^{\frac{\gamma - 1 + \alpha - \theta}{2 + \alpha - \theta}} x |r(x, t) - x|^2 \leq C_i \left( \mathcal{O}_1(0) + \| r_{0x} - 1 \|_{L^\infty}^2 \right), \quad (3.76)\]

\[(1 + t)^{\frac{\beta + \varsigma}{2} - \frac{\beta + \varsigma}{\alpha + \theta}} \max\{0, 4^{\theta - 4(\gamma - 1) - \alpha}\} x v^2(x, t) \leq C_i \left( \mathcal{O}_1(0) + \| r_{0x} - 1 \|_{L^\infty}^2 + \| v_x(\cdot,0) \|_{L^\infty}^2 \right). \quad (3.77)\]
Clearly, (3.76) follows from (3.63) and
\[ x(r(x,t) - x)^2 = \int_0^x (y(r(y,t) - y)^2) dy \]
\[ \leq \int (r - x)^2(x,t)dx + 2 \left( \int (r - x)^2 dx \right)^{1/2} \left( \int x^2(r_x - 1)^2 dx \right)^{1/2}, \quad x \in I. \]

Similarly, for \( x \in I \), it holds that
\[ x v^2(x,t) = \int_0^x (y v^2) dy \leq \int v^2(x,t)dx + 2 \left( \int \bar{\rho}^{\alpha/2} v^2 dx \right)^{1/2} \left( \int \rho^{\theta - \alpha/2} x^2 v_x^2 dx \right)^{1/2}. \]

It follows from (2.5) and (3.5) that
\[ \int \bar{\rho}^{\alpha/2} v^2 dx \leq C \int_0^{1/2} v^2 dx + C \int_1^{1/2} \bar{\rho}^{\alpha/2} v^2 dx \]
\[ \leq C \int v^2 dx + C \left( \int x^2 \bar{\rho}^{2} v_x^2 dx \right)^{\frac{2\alpha}{2\alpha + 4\theta + 4(\gamma - 1)}} \left( \int x^2 \bar{\rho}^{2} v_x^2 dx \right)^{\frac{4\theta - 4(\gamma - 1) - \alpha}{\alpha}}, \]
if \( \theta > \gamma - 1 + \alpha/4 \), and if \( \theta \leq \gamma - 1 + \alpha/4 \),
\[ \int \bar{\rho}^{\alpha/2} v^2 dx \leq C \int v^2 dx + C \int x^2 \bar{\rho}^{2} v_x^2 dx. \]

Then, (3.77) follows from (3.64).

Step 2. In this step, we prove
\[ x^3 |r_x(x,t) - 1|^2 \leq C, \quad (D_1(0) + \|r_0x - 1\|^2_{L^\infty} + \|v_x(\cdot, 0)\|^2_{L^\infty}). \]

Rewrite equation (3.6) as
\[ \frac{\nu}{\theta} \left\{ \left( \frac{r}{x} \right)^{\frac{4\nu \theta}{\nu - \theta}} \left[ \bar{\rho}^{\theta} \left( \frac{x^2}{r^2 r_x} \right) \right]^\theta \right\}_t = - \left( \frac{r}{x} \right)^{\frac{4\nu \theta}{\nu - \theta}} \left\{ \bar{\rho} \left( \frac{x^2}{r^2 r_x} \right)^\gamma \right\}_t - \frac{x^4}{r^4} (\bar{\rho}^\gamma)_t \].

Set
\[ Z(x,t) = \frac{\nu}{\theta} \left( \frac{r}{x} \right)^{\frac{4\nu \theta}{\nu - \theta}} \left( \frac{x^2}{r^2 r_x} \right)^\theta - \frac{\nu}{\theta} - \bar{\rho}^{\theta} (x) \int_x^1 \bar{\rho}(y) \left( \frac{r}{y} \right)^{\frac{4\nu \theta}{\nu - \theta} - 2} v(y,t) dy. \]

Integrate equation (3.80) over \([x, 1]\) to get
\[ \partial_t Z + \frac{\theta}{\nu} \bar{\rho}^{\gamma - \theta}(x) \left( \frac{x^2}{r^2 r_x} \right)^{\gamma - \theta} Z + \bar{\rho}^{\gamma - \theta}(x) \left[ \left( \frac{x^2}{r^2 r_x} \right)^{\gamma - \theta} - \left( \frac{r}{x} \right)^{\frac{4\nu \theta}{\nu - \theta} - 4} \right] \]
\[ = - \partial_t Y(x,t) - \sum_{i=1}^2 \mathcal{L}_i(x,t), \]
where \( Y \) and \( \mathcal{L}_i \) satisfies the following estimates:

\[
Y(x, t) = \frac{\nu}{\theta} \rho^{-\theta}(x) \int_x^1 \tilde{\rho}^\theta(y) \left( \frac{y^2}{r^2 r_y} \right) \left( \frac{r}{y} \right)^{\frac{4r_1 \theta}{\nu} - 4} dy,
\]

\[
\mathcal{L}_1(x, t) = \tilde{\rho}^{-\theta}(x) \int_x^1 \tilde{\rho}^\gamma(y) \left\{ \left[ \left( \frac{r}{y} \right)^{\frac{4r_1 \theta}{\nu} - 4} \right] - \left[ \left( \frac{r}{y} \right)^{\frac{4r_1 \theta}{\nu} - 2} \right] \right\} dy,
\]

\[
|\mathcal{L}_2(x, t)| \leq C \tilde{\rho}^{-2\theta}(x) \int_x^1 \tilde{\rho}(y) v(y, t) dy + C \tilde{\rho}^{-\theta}(x) \int_x^1 y^{-1} \tilde{\rho}(y) v^2(y, t) dy.
\]

Rewrite \( Z \) as

\[
Z(x, t) = \frac{\nu}{\theta} (r_x^{-\theta} - 1) + Z_1(x, t),
\]

where

\[
Z_1(x, t) = \frac{\nu}{\theta} \left[ \left( \frac{r_x}{x} \right)^{\frac{4r_1 \theta}{\nu} - 2} - 1 \right] r_x^{-\theta} - \tilde{\rho}^{-\theta}(x) \int_x^1 \tilde{\rho}(y) \left( \frac{r}{y} \right)^{\frac{4r_1 \theta}{\nu} - 2} v(y, t) dy.
\]

Under the a priori assumptions (3.1), we can see that the leading term of \( Z \) is \( \nu(1 - r_x) \). Note that

\[
\tilde{\rho}^{-\theta}(x) \left[ \left( \frac{x^2}{r^2 r_x} \right)^{\gamma - \theta} - \left( \frac{r}{x} \right)^{\frac{4r_1 \theta}{\nu} - 4} \right] = \tilde{\rho}^{-\theta}(x) \left( r_x^{\theta - \gamma} - 1 \right) + \mathcal{L}(x, t),
\]

where

\[
|\mathcal{L}(x, t)| \leq C x^{-1} \tilde{\rho}^{-\theta}(x)|r(x, t) - x|.
\]

Then, (3.1) and the Taylor expansion imply that

\[
\partial_t Z + \tilde{\rho}^{-\theta}(x) a(x, t) Z = -\partial_t Y(x, t) - \sum_{i=1}^3 \mathcal{L}_i(x, t),
\]

where

\[
\gamma/(2\nu) \leq a(x, t) \leq 2\gamma/\nu,
\]

\[
|\mathcal{L}_3(x, t)| \leq C x^{-1} \tilde{\rho}^{-\theta}(x)|r(x, t) - x| + C \tilde{\rho}^{-2\theta}(x) \int_x^1 \tilde{\rho}(y) v(y, t) dy,
\]

due to the smallness of \( \epsilon_0 \). Integrate (3.84) on \([0, t]\) to get

\[
Z(x, t) \leq Z(x, 0) + C \sup_{s \in [0, t]} |Y(x, s)| + \sum_{i=1}^3 \int_0^t \exp \left\{ -\tilde{\rho}^{-\theta}(x) \int_s^t a(x, \tau) d\tau \right\} |\mathcal{L}_i(x, s)| ds.
\]

(3.85)
Next, one needs to estimate the terms on the right-hand side of (3.85). It follows from \(\alpha > \theta, \theta \leq \gamma/2\) and (2.5) that

\[ |Y(x,t)| \leq C\bar{\rho}^{-\theta}(x) \int_x^1 y^{-2}\bar{\rho}^{\theta}(y) (y|r_y-1| + |r-y|) \, dy \]

\[ \leq C\bar{\rho}^{-\theta}(x) \left( \int_x^1 y^{-4}\bar{\rho}^{\theta+\alpha}(y) \, dy \right)^{1/2} \left( \int_0^1 \bar{\rho}^{\theta-\alpha}(y) (y^2|r_y-1|^2 + |r-y|^2) \, dy \right)^{1/2} \]  \quad (3.86)

\[ \leq Cx^{-3/2} \left( \int_0^1 \bar{\rho}^{\theta-\alpha}(y) (y^2|r_y-1|^2 + |r-y|^2) \, dy \right)^{1/2}, \]

\[ |L_1(x,t)| \leq C\bar{\rho}^{-\theta}(x) \int_x^1 y^{-2}\bar{\rho}^{\gamma}(y) (y|r_y-1| + |r-y|) \, dy \]

\[ \leq C\bar{\rho}^{-\theta}(x) \left( \int_x^1 y^{-4}\bar{\rho}^{2\gamma-\theta+\alpha}(y) \, dy \right)^{1/2} \left( \int_0^1 \bar{\rho}^{\theta-\alpha}(y) (y^2|r_y-1|^2 + |r-y|^2) \, dy \right)^{1/2} \]  \quad (3.87)

\[ \leq Cx^{-3/2}\bar{\rho}^{\gamma-\theta} \left( \int_0^1 \bar{\rho}^{\theta-\alpha}(y) (y^2|r_y-1|^2 + |r-y|^2) \, dy \right)^{1/2}, \]

\[ |L_2(x,t)| \leq C\bar{\rho}^{-\theta}(x) \left[ x^{-1/2} \left( \int_0^1 y^2\bar{\rho}^{\alpha} v^2 \, dy \right)^{1/2} + x^{-1} \| (xv^2)(\cdot, t) \|_{L^\infty} \right], \]  \quad (3.88)

\[ |L_3(x,t)| \leq C\bar{\rho}^{-\theta}(x) \left[ x^{-1}|r(x,t) - x| + x^{-1/2} \left( \int_0^1 y^2\bar{\rho}^{\alpha} v^2 \, dy \right)^{1/2} \right]. \]  \quad (3.89)

Notice that

\[ \bar{\rho}^{\gamma-\theta}(x) \int_0^t \exp \left\{ -\bar{\rho}^{\gamma-\theta}(x) \int_x^t a(x,\tau) \, d\tau \right\} \, ds \leq C \]

and

\[ |Z(x,t)| \leq C|\bar{r}_x(x,t) - 1| + Cx^{-1}|r(x,t) - x| + Cx^{-1/2} \left( \int_0^1 y^2\bar{\rho}^{\alpha} v^2 \, dy \right)^{1/2}. \]  \quad (3.90)

Then, (3.79) follows from (3.85), (3.86), (3.88), (3.91), (3.76) and (3.77).

**Step 3.** In this step, we prove

\[ x^3|v_x(x,t)|^2 \leq C \left( \mathcal{D}_1(0) + \|r_{0x} - 1 \|_{L^\infty}^2 + \|v_x(\cdot, 0)\|_{L^\infty}^2 \right). \]  \quad (3.91)

It follows from (3.83), (3.1) and (3.2) that

\[ |v_x(x,t)| \leq C|\partial_t Z(x,t)| + C|\partial_t Z_1(x,t)| \]

and

\[ |\partial_t Z_1(x,t)| \leq Cx^{-1}|v(x,t)| + C\epsilon_0|v_x(x,t)| + C\bar{\rho}^{-\theta}(x) \int_x^1 \bar{\rho}(y) (|v_t(y,t)| + |v(y,t)|) \, dy \]

\[ \leq Cx^{-1}|v(x,t)| + C\epsilon_0|v_x(x,t)| + Cx^{-1/2} \left( \int_0^1 y^2\bar{\rho}(v^2 + v_t^2) \, dy \right)^{1/2}. \]
Due to the smallness of $\epsilon_0$, one has

$$|v_x(x,t)| \leq C|\partial_t Z(x,t)| + Cx^{-1}|v(x,t)| + Cx^{-1/2} \left( \int_0^1 y^2 \bar{\rho} (v^2 + v_t^2) dy \right)^{1/2}. \quad (3.92)$$

Note that

$$|\partial_t Y(x,t)| \leq C\bar{\rho}^{-\theta} \int_x^1 y^{-2\theta} \bar{\rho}(y) (y|v_y| + |v|) dy \leq C\bar{\rho}^{-\theta} \left( \int_x^1 y^{-\theta} dy \right)^{1/2} \left( \int_0^1 \bar{\rho}^{-\gamma} (y^2 v_y^2 + v_t^2) dy \right)^{1/2} \quad (3.93)$$
due to $\theta \leq \gamma/2 < 2(\gamma - 1)$. Then, (3.91) follows from (3.92), (3.84), (3.87)-(3.90), (3.76), (3.77), (3.39), (3.33) and (3.64).

### 3.2 Higher-order estimates

For the higher-order estimates, we set

$$\Omega = \frac{x^2}{r^2 r_x} - 1. \quad (3.94)$$

Then for any positive constant $k$,

$$\left[ \bar{\rho}^k \left( \left( \frac{x^2}{r^2 r_x} \right)^k - 1 \right) \right]_x = [\bar{\rho}^k ((1 + \Omega)^k - 1)]_x = k\bar{\rho}^k (1 + \Omega)^{k-1} \Omega_x + \mathcal{P}_k(x,t); \quad (3.95)$$

where $\mathcal{P}_k$ satisfies the following estimate:

$$|\mathcal{P}_k(x,t)| \leq Cx \bar{\rho}^{k-(\gamma - 1)} (|r_x - 1| + |r/x - 1|), \quad (3.96)$$
due to (2.3). Using above notations and (2.3), one can rewrite (3.6) as

$$\frac{\nu}{\theta} \left[ \left( \frac{r}{x} \right)^{\frac{4\nu}{\gamma} + \theta} \left( \theta \bar{\rho}^\theta (1 + \Omega)^{\theta-1} \Omega_x + \mathcal{P}_\theta \right) \right]_t + \left( \frac{r}{x} \right)^{\frac{4\nu}{\gamma} + \theta} \left[ \gamma \bar{\rho}^{-\gamma} (1 + \Omega)^{\gamma-1} \Omega_x + \mathcal{P}_\gamma \right]. \quad (3.97)$$

The principal part on the left-hand side of (3.97) is

$$\nu \bar{\rho}^\theta \Omega_{xt} + \gamma \bar{\rho}^{-\gamma} \Omega_x = \nu (\bar{\rho}^\theta \Omega_x)_t + \gamma \bar{\rho}^{-\gamma} (\bar{\rho}^\theta \Omega_x),$$
which can be understood as a damped transport operator for $\bar{\rho} \Omega_x$ with a degenerate damping coefficient $\bar{\rho}^{\gamma-\theta}$. This is an interplay between the viscosity and the pressure. This structure leads to desirable estimates on the derivatives of $\Omega$, for example, the bound for

$$
\int \bar{\rho}^{2\gamma-1}(x)\Omega_x^2(x, t)dx + \int_0^t \int \bar{\rho}^{3\gamma-1-\theta}\Omega_x^2dxds.
$$

The bounds for the weighted norms of the derivatives of $\Omega$ in turn yield the bounds for those of the derivatives of $r$ and $v$, which are given by Lemma 3.7 stated below.

In Lemma 3.8, one can use Lemma 3.7 to bound

$$
\int \bar{\rho}^{2\gamma-1} [(r/x)_x^2 + (v^2)_xx] (x, t)dx
$$

and to estimate the decay of

$$
\int_0^t [(r/x)_x^2 + (v^2)_xx] (x, t)dx, \quad 0 < t < 1.
$$

Based on this and other estimates, we can estimate the decay of

$$
\int (\bar{\rho}v_t^2) (x, t)dx \quad \text{and} \quad \int_0^t [(v/x)_x^2 + v^2_{xx}] (x, t)dx, \quad 0 < t < 1.
$$

Putting those together yields the desired decay estimate for

$$
\|(v_x, v/x, r_x - 1, r/x - 1)(\cdot, t)\|_{H^1([0,l])}, \quad 0 < t < 1,
$$

which, together with the supreme norm estimates for $r_x - 1$ and $v_x$ away from the origin given in Lemma 3.6 verifies the a priori assumptions (3.1) and (3.2) and closes the bootstrap argument.

We now carry out the strategy outlined above.

**Lemma 3.7** Suppose that (3.1) and (3.2) hold. Let $w(x)$ be a smooth function on $[0,1]$ satisfying $w(x) \geq 0$ and $w'(x) \leq 0$ on $[0,1]$, and $w(1) = 0$. Then,

$$
\int w (4(r/x - 1)^2 + (r_x - 1)^2) dx \leq 2 \int w \Omega^2 dx,
$$

$$
\int w [(v/x)^2 + (v_x)^2] dx \leq C \int w (\Omega_x)^2 dx,
$$

$$
\int w [((r/x)_x)^2 + (r_{xx})^2] dx \leq C \int w (\Omega_x)^2 dx,
$$

$$
\int w [((v/x)_x)^2 + (v_{xx})^2] dx \leq C \int w [(\Omega_{xt})^2 + (\Omega_x)^2] dx.
$$

**Proof.** It follows from (3.1) that

$$
\Omega = -2(r/x - 1) - (r_x - 1) + O(1)\epsilon_0 (|r/x - 1| + |r_x - 1|),
$$

35
which, together with the integration by parts, \( r(0, t) = 0 \) and \( w'(x) \leq 0 \), gives
\[
\int w \Omega^2 \, dx \geq \int w \left[ 4\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2 \right] \, dx - 2 \int (x^{-1}w)(r - x)^2 \, dx \\
- C\epsilon_0 \int w \left[ (r/x - 1)^2 + (r_x - 1)^2 \right] \, dx \\
\geq \frac{1}{2} \int w \left[ 4\left(\frac{r}{x} - 1\right)^2 + (r_x - 1)^2 \right] \, dx,
\]
due to \( (x^{-1}w)_x \leq 0 \) and the smallness of \( \epsilon_0 \). This proves (3.98). Similarly, (3.99) follows from
\[
\Omega_t = -(1 + \Omega)(2v/r + v_x/r_x).
\]
(3.103)
For (3.100), note that
\[
\Omega_x = -(1 + \Omega)\left[ 2(r/x)(r/x)_x + r_{xx}/r_x \right].
\]
(3.104)
Thus,
\[
\int w(\Omega_x)^2 \geq 2c \int w \left[ 2(r/x)_x + r_{xx} \right]^2 \, dx - C\epsilon_0 \int w \left[ ((r/x)_x)^2 + (r_{xx})^2 \right] \, dx \\
\geq c \int w \left[ ((r/x)_x)^2 + (r_{xx})^2 \right] \, dx,
\]
due to the smallness of \( \epsilon_0 \), and
\[
4 \int w(r/x)_x r_{xx} \, dx = 8 \int w ((r/x)_x)^2 \, dx - 2 \int (xw)_x ((r/x)_x)^2 \, dx \\
= 6 \int w ((r/x)_x)^2 \, dx - 2 \int xw_x ((r/x)_x)^2 \, dx \geq 0.
\]
Similarly, one can prove (3.101).

\[ \square \]

**Lemma 3.8** Suppose that (3.1) and (3.2) hold. Let \( \theta \in (0, \gamma/2] \), and \( \alpha \) and \( \beta \) be given respectively in (2.18) and (2.19). If \( t \in (0, (2\gamma - 2 - \theta)/8] \) and \( l \in (0, 1) \), then for \( t \in [0, T] \),
\[
\int \rho^{2\gamma - 1} \left[ ((r/x)_x)^2 + r_{xx}^2 \right] (x, t)dx \leq C\mathcal{E}(0),
\]
(3.105)
\[
(1 + t)^{3 - 1} \int_0^t \left[ ((r/x)_x)^2 + r_{xx}^2 \right] (x, t)dx \leq C_t\mathcal{E}(0),
\]
(3.106)
\[
\int_0^t (1 + s)^{\beta - 1} \int \rho^\theta \left( v_x^2 + (v/x)^2 \right) dxds \leq C\mathcal{E}(0).
\]
(3.107)

**Proof.** The proof consists three steps.
Step 1. In this step, we prove
\[
\int \hat{\rho}^{2\gamma-1}(x) \Omega_x^2(x,t) dx + \int_0^t \int \hat{\rho}^{3\gamma-1-\theta} \Omega_x^2 dx ds \leq C\mathcal{E}(0). \tag{3.108}
\]
This, together with (3.100), gives (3.105).

It follows from (3.97), (2.3) and the Cauchy inequality that
\[
\int \hat{\rho}^{2\gamma-1}|\Omega_x(x,t)|^2 dx + \int_0^t \int \hat{\rho}^{3\gamma-1-\theta}|\Omega_x|^2 dx \leq C\mathcal{E}(0) + C \int \hat{\rho}(x) \left[ x^2(r_x - 1)^2 + (r - x)^2 \right] (x,t) dx + C \int_0^t \int \hat{\rho}^{\theta-(\gamma-1)} v^2 dx ds \tag{3.109}
\]
\[
+ C \int_0^t \int \hat{\rho}^{\gamma+1-\theta} [x^2(r_x - 1)^2 + (r - x)^2 + v_x^2] dx ds.
\]

Notice that
\[
\int \hat{\rho}^{\theta-(\gamma-1)} v^2 dx \leq C \left[ \int_0^{1/2} x^2(v^2 + v_x^2) + C \int_1^{1/2} \hat{\rho}^{\theta-(\gamma-1)}(v^2 + v_x^2) dx \right] \leq C \int \hat{\rho}^2(v^2 + x^2v_x^2) dx,
\]
where (3.4), (3.5) and (2.5) have been used. Then, (3.108) follows from (3.33) and \( \theta \leq 1 < \gamma \).

Step 2. In this step, we prove (3.106). Let \( \psi \) be a smooth cut-off function defined on \([0, 1]\) satisfying
\[
\psi = 1 \text{ on } [0, 1-l], \; \psi = 0 \text{ on } [1-l/2, 1] \text{ and } -8/l \leq \psi'(x) \leq 0 \text{ on } [0, 1] \tag{3.110}
\]
for any fixed constant \( l \in (0, 1) \). As shown in Step 1, one can obtain
\[
\frac{\nu}{\theta} \frac{d}{dt} \int \psi \hat{\rho}^{2\gamma-1-2\theta} \left[ \frac{\psi}{x} \hat{\rho}^{\theta} (1 + \Omega)^{\theta-1} \Omega_x + \mathcal{R}_\theta \right] \frac{d}{dx} + \int \psi \hat{\rho}^{3\gamma-1-\theta}|\Omega_x|^2 dx \leq C \int \hat{\rho}^2(v^2 + x^2v_x^2 + v_x^2) dx + C \int \hat{\rho}^{\gamma} [x^2(r_x - 1)^2 + (r - x)^2] dx, \tag{3.111}
\]
which, together with (3.39), (3.63) and (3.108), implies that
\[
(1 + t)^{\beta-1} \int \psi \hat{\rho}^{2\gamma-1}(x) \Omega_x^2(x,t) dx + \int_0^t (1 + s)^{\beta-1} \int \psi \hat{\rho}^{3\gamma-1-\theta} \Omega_x^2 dx ds \leq C_{t,l}\mathcal{E}(0). \tag{3.112}
\]
(If \( \beta > 2 \), we first show the time decay with rate \(-1\). With this, we can then show the time decay with rate \( 1 - \beta \). Indeed, this technique has been used in Step 3 of the proof of Lemma 3.4)

Squaring (3.97), multiplying the resulting equation by \( \psi \hat{\rho}^{3\gamma-1-3\theta} \), and integrating the product over \([0, 1]\), one obtains that
\[
\int_0^t (1 + s)^{\beta-1} \int \psi \hat{\rho}^{3\gamma-1-\theta} \Omega_x^2 dx ds \leq C \int_0^t (1 + s)^{\beta-1} \int \psi \hat{\rho}^{3\gamma-1-\theta} \Omega_x^2 dx ds
\]
\[
+ C \int_0^t (1 + s)^{\beta-1} \int \psi \hat{\rho}^{\gamma+1-\theta} [x^2(r_x - 1)^2 + (r - x)^2 + v_x^2 + x^2v_x^2 + v_x^2] dx ds, \tag{3.113}
\]
\[37\]
where (3.2) has been used. This, together with (3.112), (3.39), (3.63) and \( \theta \leq 1 < \gamma \), gives

\[
\int_0^t (1 + s)^{\beta - 1} \int \psi \bar{\rho}^{3\gamma - 1 - \theta} Q_{xs}^2 dx ds \leq C_{l,E}(0).
\] (3.114)

So, it follows from (3.100), (3.101), (3.112) and (3.114) that

\[
(1 + t)^{\beta - 1} \int \psi \bar{\rho}^{2\gamma - 1} \left[ (r/x)^2 + r_{xx}^2 \right] (x, t) dx
+ \int_0^t (1 + s)^{\beta - 1} \int \psi \bar{\rho}^{3\gamma - 1 - \theta} \left[ (v/x)^2 + v_{xx}^2 \right] dx ds \leq C_{l,E}(0).
\] (3.115)

**Step 3.** In this step, we prove (3.107). Choose \( l = 1/2 \) in (3.115) to get

\[
\int_0^t (1 + s)^{\beta - 1} \int_0^{1/2} \left[ (v/x)^2 + v_{xx}^2 \right] dx ds \leq C_{l,E}(0).
\] (3.116)

Note that

\[
\int_0^{1/2} \left( v_x^2 + (v/x)^2 \right) dx \leq C \int_0^{1/2} x^2 \left[ v_x^2 + v_{xx}^2 + (v/x)^2 + ((v/x)_x)^2 \right]
\leq C \int_0^{1/2} \left[ x^2 v_x^2 + v^2 + v_{xx}^2 + ((v/x)_x)^2 \right],
\] (3.117)
due to (3.4). Then, (3.107) follows from (3.39) and (3.116).

**Lemma 3.9** Suppose that (3.1) and (3.2) hold. Let \( \theta \in (0, \gamma/2] \), and \( \alpha \) and \( \beta \) be given respectively in (2.18) and (2.19). If \( \iota \in (0, (2\gamma - 2 - \theta)/8] \) and \( l \in (0, 1) \), then for \( t \in [0, T] \),

\[
(1 + t)^{\beta - 1} \int (\bar{\rho}_t^2) (x, t) dx + \int_0^t (1 + s)^{\beta - 1} \int \bar{\rho}^{\theta} \left( v_{xs}^2 + (v_s/x)^2 \right) dx ds \leq C_{l,E}(0),
\] (3.118)

\[
(1 + t)^{\beta - 1} \int_0^t \left[ (v/x)^2 + v_{xx}^2 \right] (x, t) dx \leq C_{l,E}(0).
\] (3.119)

**Proof.** The proof consists of two steps. We prove (3.118) and (3.119) in Steps 1 and 2, respectively.

**Step 1.** In this step, we prove that

\[
(1 + t)^{\beta - 1} \int (\psi \bar{\rho}_t^2) (x, t) dx + \int_0^t (1 + s)^{\beta - 1} \int \psi \bar{\rho}^{\theta} \left( v_{xs}^2 + (v_s/x)^2 \right) dx ds \leq C_{l,E}(0),
\] (3.120)

where \( \psi \) is a smooth cut-off function defined on \([0, 1] \) satisfying (3.110). Then, (3.118) follows from (3.63) and (3.120).
Differentiating (3.6) with respect to $t$ yields that

$$
\rho \left( \frac{x}{r} \right)^2 v_{tt} - 2 \rho \left( \frac{x}{r} \right) \frac{3}{x} v_t - \gamma \left[ \rho \left( \frac{2}{r} + \frac{v_x}{r_x} \right) \right] + 4 \left( \frac{x}{r} \right)^5 \frac{v}{x} (\rho^\gamma)_x \\
= \nu \left[ \rho \left( \frac{v_{xt}}{r_x} + 2 \frac{v_t}{r} \right) \right]_x - \nu \left[ \rho \left( \frac{v_x}{r_x} + 2 \frac{v_x^2}{r_x^2} \right) \right]_x - \theta \nu \left[ \rho \left( \frac{v_x}{r_x} + 2 \frac{v_x^2}{r_x^2} \right)^2 \right]_x \\
+ 4 \nu_1 \theta \left[ \rho \left( \frac{v_x}{r_x} + 2 \frac{v_x^2}{r_x^2} \right) \right]_x - 4 \nu_1 (\rho^\theta)_x \left( \frac{v_t}{r} - \frac{v^2}{r^2} \right),
$$

(3.121)

where $\rho = \rho(r(x,t), t) = \bar{\rho} r_x^{-1}(r/x)^{-2}$. Multiplying equation (3.121) by $\psi v_t$, integrating the product with respect to the spatial variable, and noting that $v_t(0,t) = v(0,t) = 0$, one has

$$
\frac{d}{dt} \int \frac{1}{2} \rho \psi \left( \frac{x}{r} \right)^2 v_t^2 dx + \nu \int \rho \left( \frac{v_{xt}}{r_x} + 2 \frac{v_t}{r} \right) (\psi v_t)_x dx = \sum_{k=1}^{5} J_k,
$$

(3.122)

where

$$
J_1 = \int \frac{\rho}{x} \psi \left( \frac{x}{r} \right)^3 v_t^2 dx + 4 \int \left( \frac{x}{r} \right)^5 \nu \phi \psi v_t dx \leq C \int \rho \left( v^2 + v_t^2 \right) dx,
$$

$$
J_2 = - \gamma \int \rho \left( \frac{2}{r} + \frac{v_x}{r_x} \right) (\psi v_t)_x dx,
$$

$$
J_3 = \nu \int \rho \left[ \frac{v_x}{r_x} + 2 \frac{v_x^2}{r_x^2} + \theta \left( \frac{v_x}{r_x} + 2 \left( \frac{v_x}{r_x} \right) \right)^2 \right] (\psi v_t)_x dx,
$$

$$
J_4 = - 4 \nu_1 \theta \int \rho \left( \frac{v_x}{r_x} + 2 \frac{v_x^2}{r_x^2} \right) (\psi v_t)_x dx,
$$

$$
J_5 = 4 \nu_1 \int \rho \left( \frac{v_t}{r} - \frac{v^2}{r^2} \right) \psi v_t + \left( \frac{v_t}{r} - \frac{v^2}{r^2} \right) (\psi v_t)_x dx
$$

In view of (3.95), $\psi'(x) \leq 0$ and $(\rho^\theta)_x \leq 0$, the second term on the left-hand side of (3.122) can be estimated as follows:

$$
\int \rho \left( \frac{v_{xt}}{r_x} + 2 \frac{v_t}{r} \right) (\psi v_t)_x dx
$$

$$
= \int \rho \psi \frac{v_{xt}}{r_x} dx + 2 \int \rho \psi \psi v_t v_{xt} dx + \int \rho \psi (\frac{v_{xt}}{r_x} + 2 \frac{v_t}{r}) v_t dx
$$

$$
= \int \rho \psi \left( \frac{v_{xt}}{r_x} + r_x \frac{v_x^2}{r_x^2} \right) dx - H(t) + \int \rho \psi \left( \frac{v_{xt}}{r_x} + 2 \frac{v_t}{r} \right) v_t dx,
$$

where

$$
H(t) = \int \psi \frac{\psi'}{r} v_t^2 dx + \int \psi \left[ (\rho^\theta)_x + W_x \right] v_t^2 dx \leq \int \psi \frac{1}{r} W_x v_t^2 dx,
$$

and

$$
W_x = \left[ \rho \left( \left( \frac{x^2}{r^2 \rho_x} \right)^\theta - 1 \right) \right] = 0 \rho^\theta (1 + \Omega)^{\theta-1} \Omega_x + \mathcal{P}_\theta(x,t).
$$
It follows from (3.108), (3.9), and the Hölder and Cauchy-Schwarz inequalities that for any $\omega > 0$,

$$\begin{align*}
H(t) & \leq \frac{1}{4} \int \frac{\psi \rho^3 r_x}{r^2} v^2_x dx + C \int \psi W^2_x v^2_x dx \\
& = \frac{1}{4} \int \frac{\psi \rho^3 r_x}{r^2} v^2_t dx + 2C \int_0^{1-\gamma/2} \psi W^2_x \left( \int_0^x v_t v_y dy \right) dx \\
& = \frac{1}{4} \int \frac{\psi \rho^3 r_x}{r^2} v^2_t dx + C \left( \int \psi W^2_x dx \right) \left( \int_0^{1-\gamma/2} y^2 v^2_t dy \right)^{1/2} \left( \int_0^{1-\gamma/2} v^2_y dy \right)^{1/2} \\
& \leq \frac{1}{4} \int \frac{\psi \rho^3 r_x}{r^2} v^2_t dx + C_l \omega \int_0^{1-\gamma/2} \psi v^2_x dx + C_l \omega \int_0^{1-\gamma/2} v^2_t dx + C_l \omega^{-1} \int_0^{1-\gamma/2} x^2 v^2_t dx \\
& \leq \frac{1}{2} \int \frac{\psi \rho^3 r_x}{r^2} v^2_t dx + C_l \int \tilde{\rho} (x^2 v^2_x + v^2_t) dx.
\end{align*}$$

This, together with (3.123), implies

$$\int \tilde{\rho} \left( \frac{v}{x} + 2 \frac{v_t}{r} \right) (\psi v_t) dx \geq c \int \psi \left( v^2_x + \frac{v^2_t}{x^2} \right) dx - C_l \int \tilde{\rho} (x^2 v^2_x + v^2_t) dx. \quad (3.124)$$

Thus, by (3.122), one has

$$\frac{d}{dt} \int \frac{\rho(x)}{r} v^2_x dx + \int \psi \left( v^2_x + \frac{v^2_t}{x^2} \right) dx \leq C_l \int \tilde{\rho} (x^2 v^2_x + v^2_t) dx + \sum_{k=2}^6 J_k. \quad (3.125)$$

It therefore follows easily from the Cauchy-Schwarz inequality that

$$\frac{d}{dt} \int \frac{1}{2} \rho(x) \left( \frac{x}{r} \right)^2 v^2_t dx + \int \psi \tilde{\rho} \left( v^2_x + \frac{v^2_t}{x^2} \right) dx \leq C_l \int \tilde{\rho} (x^2 v^2_x + v^2_t) dx. \quad (3.125)$$

So, (3.120) follows from (3.125), (3.63) and (3.107).

Step 2. In a similar way to deriving (3.114), one can use (3.112), (3.120), (3.39) and (3.63) to obtain

$$(1 + t)^{\beta-1} \int \psi(x) \rho^{2\gamma-1}(x) \Omega_{T}^{2}(x,t) dx \leq C_{\omega} \mathcal{E}(0).$$

This, together with (3.101) and (3.112), gives (3.119).

\[\square\]

**Lemma 3.10** Suppose that (3.1) and (3.2) hold. Let $\theta \in (0, \gamma/2]$, and $\alpha$ and $\beta$ be given respectively in (2.18) and (2.19). If $t \in (0, (2\gamma - 2 - \theta)/8]$ and $l \in (0, 1)$, then for $t \in [0, T]$,

$$\begin{align*}
(1 + t)^{\beta-1} \|(v_x, v/x, r_x - 1, r/x - 1)(\cdot, t)\|_{H_1([0, l])}^2 & \leq C_{\omega} \mathcal{E}(0).
\end{align*}$$

(3.126)
Proof. In a similar way as for (3.117), one can use (3.39), (3.63), (3.106) and (3.119) to obtain
\[(1 + t)^{\beta - 1} \int_0^{1/2} \left[ v_x^2 + (v/x)^2 + (r_x - 1)^2 + (r/x - 1)^2 \right] (x, t) \, dx \leq C_\epsilon \mathcal{E}(0),\] (3.127)
which implies, by using (3.39) and (3.63) again, that for any \(l \in (0, 1),\)
\[(1 + t)^{\beta - 1} \int_0^l \left[ v_x^2 + (v/x)^2 + (r_x - 1)^2 + (r/x - 1)^2 \right] (x, t) \, dx \leq C_{l, \epsilon} \mathcal{E}(0).\] (3.128)
This, together with (3.106) and (3.119), gives (3.126).
\[\Box\]

3.3 Global existence of strong solutions

In this subsection, we prove Theorem 2.2 by first verifying the a priori assumptions (3.1) and (3.2). Choose \(\iota = (2\gamma - 2 - \theta)/8\) and \(l = 1/2\) in (3.126) to get
\[\| (v_x, v/x, r_x - 1, r/x - 1)(\cdot, t) \|^2_{H^1([0,1/2])} \leq C \mathcal{E}(0), \quad t \in [0, T],\]
which implies, using \(\| \cdot \|_{L^\infty} \leq \| \cdot \|_{H^1}\), that
\[\| (v_x, v/x, r_x - 1, r/x - 1)(\cdot, t) \|^2_{L^\infty([0,1/2])} \leq C \mathcal{E}(0), \quad t \in [0, T].\]
It follows from choosing \(\iota = (2\gamma - 2 - \theta)/8\) in (3.75) that
\[\| (v_x, v/x, r_x - 1, r/x - 1)(\cdot, t) \|^2_{L^\infty([1/2,1])} \leq C \mathcal{E}(0), \quad t \in [0, T].\]
This verifies the a priori assumption for small \(\mathcal{E}(0)\).

The local existence of strong solutions for the problem (2.12) can be obtained easily by combining the approximation techniques in [14] and the a priori estimates obtained in Sections 3.1 and 3.2, at least for small \(\mathcal{E}(0)\). Indeed, the a priori estimates obtained in Sections 3.1 and 3.2 are also sufficient for the local existence theory, for small \(\mathcal{E}(0)\). Therefore, a standard continuation argument proves Theorem 2.2 with the estimates obtain in Sections 3.1 and 3.2.

Remark 3.11 The local existence theory in [14] which is for the case of \(\theta = 1\), does not apply to our case.

4 Nonlinear Asymptotic Stability

This section is devoted to proving Theorem 2.3.

First, it follows from the fact that \(\| g \|^2_{L^\infty} \leq \| g \|_{L^2} \| g_x \|_{L^2}\) for any function \(g\), (3.63), (3.64), (3.126) and (3.76) that
\[(1 + t)^{\min\left\{ \frac{3\gamma + 2 + 2(\alpha - \theta)}{2(\gamma + \alpha - \theta)} \beta - \frac{1}{2}, \frac{\gamma - 1 + \alpha - \theta}{\gamma + \alpha - \theta} \beta \right\}} |r(x, t) - x|^2 \leq C_\epsilon \mathcal{E}(0), \quad x \in I,\] (4.1)
\[
\left[(1 + t)^\frac{\alpha}{2} + x(1 + t)^{\frac{\alpha + \beta}{4}} \max\{0, 4\theta - 4(\gamma - 1) - \alpha\}\right] |u(r(x, t), t)|^2 \leq C_\iota \mathcal{E}(0), \quad x \in I. \quad (4.2)
\]

Since for small \(\iota\),

\[
\frac{3\gamma - 2 + 2(\alpha - \theta)}{2(\gamma + \alpha - \theta)} \beta - \frac{1}{2} \geq \frac{\gamma - 1 + \alpha - \theta}{\gamma + \alpha - \theta} \beta
\]

then we have (2.21) and (2.22).

Due to the fact that \(\|\cdot\|_{L^\infty} \leq \|\cdot\|_{H^1}\) and (3.126), we have for any fixed \(l \in (0, 1)\),

\[
(1 + t)^{\beta - 1} (|v_\zeta(x, t)|^2 + |x^{-1}v(x, t)|^2 + |r_\zeta(x, t) - 1|^2 + |x^{-1}r(x, t) - 1|^2)
\leq C_{\iota, l} \mathcal{E}(0), \quad x \in [0, l],
\]

which implies (2.23). To prove (2.22), we notice that for any \(b \in [0, 2 - \gamma]\),

\[
x^3 \bar{\rho}^{-b}(x) |\rho(r(x, t), t) - \bar{\rho}(x)|^2 = x^3 \bar{\rho}^{-b}(x) \mathcal{Q}^2(x, t)
= \int_0^x (y^3 \bar{\rho}^{-b}(x) \mathcal{Q}^2)_y dy \leq 3 \int_0^x y^2 \bar{\rho}^{-b} \mathcal{Q}^2 dy + 2 \int_0^x y^3 \bar{\rho}^{-b} \mathcal{Q} \mathcal{Q}_y dy
\leq C \int_0^1 y^2 \bar{\rho}^{-b}(y^2 |y - 1|^2 + |r - y|^2) dy
+ C \left(\int \bar{\rho}^{2\gamma - 1} \mathcal{Q}_y^2 dy\right)^{1/2} \left(\int y^2 \bar{\rho}^{5 - 2\gamma - 2b} \mathcal{Q}^2 dy\right)^{1/2},
\]

where the first inequality is due to (2.3); and if \(\gamma > (5 - 2b)/3\)

\[
\int y^2 \bar{\rho}^{5 - 2b - 2\gamma} \mathcal{Q}^2 dy \leq \left(\int y^2 \bar{\rho}^{\gamma}(y^2 |y - 1|^2 + |r - y|^2) dy\right) \left(\int y^2 \bar{\rho}^{\gamma}(y^2 |y - 1|^2 + |r - y|^2) dy\right)^{\frac{5 - 2\gamma - 2b}{\gamma - \theta}} \times \left(\int y^2 \bar{\rho}^{\gamma}(y^2 |y - 1|^2 + |r - y|^2) dy\right)^{\frac{\gamma - 5 + 2b}{\gamma - \theta}}.
\]

Then, it follows from (3.39) and (3.108) that for any \(b \in [0, 2 - \gamma]\),

\[
x^3 \bar{\rho}^{-b}(x)(1 + t)^{\beta - 1} \frac{\max\{0, 3\alpha - 5 + 2b\}}{2(\gamma - \theta)} |\rho(r(x, t), t) - \bar{\rho}(x)|^2 \leq C \mathcal{E}(0), \quad x \in [0, 1].
\]

Due to (4.3) and \(b < 1\), we have

\[
(1 + t)^{\beta - 1} \bar{\rho}^{-b}(x) |\rho(r(x, t), t) - \bar{\rho}(x)|^2 \leq C_{\iota, l} \mathcal{E}(0), \quad x \in [0, l].
\]

So, (2.22) is a consequence of (1.1) and (1.5).

It remains to proving (2.25). We use (3.92), (3.84), (3.88)-(3.90), (3.93) and

\[
|\mathcal{Q}_1(x, t)| \leq C x^{-3/2} \left(\int_0^1 \bar{\rho}^{2\gamma - 2\theta + (\gamma - 1)}(y) (y^2 |y - 1|^2 + |r - y|^2) dy\right)^{1/2}
\leq C x^{-3/2} \left(\int_0^1 \bar{\rho}^{\gamma}(y) (y^2 |y - 1|^2 + |r - y|^2) dy\right)^{1/2}
\]
to get
\[ x^3 |v_x(x, t)|^2 \leq C x^3 \rho^{2\gamma-2\theta}(x) |r_x(x, t)|^2 + C \int y^2 \rho^{2-1} v_y^2(y, t) dy \]
\[ + C_i \left[ (1 + t)^{-\frac{\gamma+\alpha-\theta}{\alpha} \beta} + (1 + t)^{-\frac{2\gamma-4(\gamma-1)-\alpha}{4\alpha} \max\{0, 4\theta-4(\gamma-1)-\alpha\}} \right] \mathcal{E}(0). \]

(4.6)

Here (3.63), (3.64) and (3.39) have been used. In view of (3.64), we see that if \( \gamma - 1 \geq \theta \),
\[ \int y^2 \rho^{2-1} v_y^2(y, t) dy \leq C_i (1 + t)^{-\beta} \mathcal{E}(0); \]
and if \( \gamma - 1 < \theta \),
\[ \int y^2 \rho^{2-1} v_y^2(y, t) dy \leq C \left( \int y^2 \rho^{2-1/2} v_y^2(y, t) dy \right)^{\alpha-2\theta+2(\gamma-1)} \left( \int y^2 \rho^{2-\alpha/2} v_y^2(y, t) dy \right)^{2\theta-2(\gamma-1)\alpha^{\max\{0, 2\theta-2(\gamma-1)\}} \mathcal{E}(0). \]

That means
\[ \int y^2 \rho^{2-1} v_y^2(y, t) dy \leq C_i (1 + t)^{-\beta+\frac{\alpha-2\theta+2(\gamma-1)}{2\alpha}} \max\{0, 2\theta-2(\gamma-1)\} \mathcal{E}(0). \]

(4.7)

In a similar way to deriving (4.4), we have
\[ x^3 \rho^{2\gamma-2\theta}(x) |r_x(x, t)|^2 \leq C_i (1 + t)^{-\frac{\alpha-2\theta+2(\gamma-1)}{2\alpha} \max\{0, 4\theta-4(\gamma-1)\}} \mathcal{E}(0). \]

(4.8)

So, (2.25) is a conclusion of (4.6)-(4.8) and (4.3).

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