Soliton solution of stationary discrete nonlinear Schrödinger equation with the cubic-quintic nonlinearity

H Qausar¹, M Ramli²*, S Munzir², M Syafwan³, D Fadhiliani¹

¹Graduate School of Mathematics and Applied Sciences, Universitas Syiah Kuala, 23111, Indonesia
²Department of Mathematics, Universitas Syiah Kuala, Banda Aceh, 23111, Indonesia
³Department of Mathematics, Universitas Andalas, Padang, 25163, Indonesia

*E-mail: marwan.math@unsyiah.ac.id

Abstract. This research discusses stationary discrete nonlinear Schrödinger equation with cubic-quintic nonlinearity. This equation is interesting to study because it has a unique solution known as a soliton. This solution has a fixed profile and speed when propagating and in the context of applications in the optical field, soliton can also be engineered as a carrier of information that can propagate on media with very long distances without experiencing significant interference. This paper only focuses on on-site type soliton (soliton that peak in the middle one site). The method of determining solution on stationary discrete nonlinear Schrödinger equation with cubic-quintic nonlinearity is divided into two cases. The first case for the value of parameter is zero and the soliton solution is determined analytically. In this case the soliton solution can be stated explicitly, therefore the soliton solution will be displayed and also the boundaries on the parameters that make the solution in the form of on-site soliton. The second case for the value of parameter is not zero and the soliton solution is determined using a numerical approach namely Trust Region Dogleg Method. In this case the soliton solution cannot be stated explicitly, therefore only boundaries of the parameters that make the solution in the form of on-site soliton will be displayed.

1. Introduction
Discrete nonlinear schrödinger's equation (DNLS) is one of the basic equations in the nonlinear lattice dynamics model [1]. This equation has a wide range of applications including electrical circuits [2], denaturation and phase transformation associated with double strands (DNA) [3], biomolecular chain [4], propagation of the optical beam on a nonlinear waveguide [5], and material form (Bose-Einstein condensation on an optical lattice) [6]. The Bose-Einstein condensation (KBE) is the fifth form of matter in a particle when it approaches 0 Kelvin, which was theoretically predicted by S.N Bose and A. Einstein in 1925 [7]. The first successful experiments to prove the existence of KBE were carried out by Wolfgang Ketterle, Eric Cornell and Carl Wienmann. For this achievement, they were awarded the Nobel Prize in physics in 2001.
The DNLS equation is interesting to study because it has a special solution known as a soliton. This solution has a fixed profile and speed when propagating [8]. In the context of applications in the field of optics, solitons can also be engineered as information carriers that can propagate on media over long distances without experiencing significant disturbances. This is considered very important in the development of information and communication technology in the future [5]. This paper focuses only on on-site solitons (solitons peaking in the middle at one site). This is due to the results of research conducted by Eilbeck and Johansson [9] which states that the on-site type solitons tend to be stable.

The general form of the DNLS equation can be written as follows

\[ i\psi_n = -C(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + F(\psi_{n+1}, \psi_n, \psi_{n-1}) \]  

(1)

with \( \psi_n \equiv \psi_n(t) \in \mathbb{C} \) is a wave function with respect to time \( t \in \mathbb{R}^+ \) dan \( n \in \mathbb{Z} \), \( \psi_n \) denotes the derivative of the function \( \psi_n \) respect to \( t \), \( C \) denotes the coupling parameter between two adjacent sites and \( F \) is a nonlinear term. The nonlinear term in the (1) equation has several forms, namely [9]

1. Cubic type nonlinearity \( :|\psi_n|^2\psi_n \)
2. Quintic type nonlinearity \( :|\psi_n|^4\psi_n \)
3. Ablowitz-Ladik type nonlinearity \( :\frac{1}{2}|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \).

Based on the three nonlinear terms, the nonlinear terms of the cubic and quintic types were selected, so that the (1) equation could be written

\[ i\psi_n = -C(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - B|\psi_n|^2\psi_n + Q|\psi_n|^4\psi_n \]  

(2)

with \( B \) and \( Q \) respectively denote the cubic and quintic nonlinear coefficients. This selection is based on the research of Chong and Pelinovsky [10] which states that the nonlinear effect on some waveguide materials is better modeled by using the cubic-quintic type nonlinearity, precisely when \( B > 0 \) and \( Q > 0 \). Soliton solutions in the Equation (2) can be determined semi-analytically by various methods, some of which are the variational approximation method, anti-continuity principle, center manifold reduction and the Nehari manifold approach [11]. Often the soliton solution calculated by the semi-analytic method is confirmed using the soliton solution of the stationary equation that is determined by numerical methods, as done by Carretero et al [12] using the Newton method. In addition, setting limits on parameter values is also important to finding soliton solutions for stationary DNLS equations [13]. In a previous research, Qausar et al [14] determined nontrivial onsite soliton produced by cubic-quintic discrete nonlinear Schrödinger using the Trust Region-Dogleg method and taking the parameter values \( 0.1 \leq w \leq 0.4 \) and \( 0.16 \leq C \leq 9.3 \). In this paper, we will determine soliton solutions of stationary DNLS equations using different parameter values and also comparisons of solitons based on differences in selecting the coefficient of cubic nonlinearity \( B \) and quintic \( Q \).

2. Materials and Methods

2.1. Stationary DNLS Equation with Cubic-Quintic Nonlinearity

Stationary DNLS form with Cubic-Quintic Nonlinearity can be obtained by performing the substitution \( \psi_n(t) = u_n e^{-i\omega t} \) to the equation (2) with \( \omega \) as the propagation parameter is obtained.

\[ wu_n + C(u_{n+1} - 2u_n + u_{n-1}) + B|u_n|^2u_n - Q|u_n|^4u_n = 0. \]  

(3)

In general, the solution \( u_n \) is a complex value, however the soliton solution must satisfy \( u_n \to 0 \) when \( n \to \pm\infty \), therefore the equation (3) can be simplified without loss of generality by considering only real value solutions. So that the equation becomes:
\[ w u_n + C (u_{n+1} - 2u_n + u_{n-1}) + B u_n^3 - Q u_n^5 = 0 \]  

(4)

with \( B \) is the cubic constant, \( Q \) is the quintic constant [9]. A soliton is said to be on-site when it peaks in the middle and meets \( u_n = u_{n-1} \) [5].

2.2. Trust Region Method

A nonlinear system of equations is given which can be written as follows

\[ F(u) = 0 \]
\[ u = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m \]

(5)

with \( F : \mathbb{R}^m \to \mathbb{R}^m \) is a mapping in the form of \( F(u) = (F_1(u), F_2(u), \ldots, F_m(u))^T \) and for every \( n \in \{1, 2, \ldots, m\} \) applies \( F_n : \mathbb{R}^m \to \mathbb{R} \) continuous and differentiable. In the context of the stationary DNLS equation with cubic-quintic nonlinearity, define

\[ F_n(u) = w u_n + C (u_{n+1} - 2u_n + u_{n-1}) + B u_n^3 - Q u_n^5 \]

(6)

The vector \( u^* \), which satisfies (5) is called the solution or root of the equation (5). Finding a solution to (5) can be done by finding the root of the nonlinear least-squares problem with no constraints as follows.

\[ \min_{u \in \mathbb{R}^m} f(u) = \frac{1}{2} \| F(u) \|^2, \]

(7)

with \( \| \cdot \| \) is the Euclidean norm. In order to solve the problem (7) the Trust Region Method is used. The first thing the Trust Region Method does in determining a solution is to construct a model function that can approach \( F \) around point \( u_k \in \mathbb{R}^m \) as follows

\[ M_k(u) = F(u_k) + J(u_k)(u - u_k), \]

(8)

with \( J(u_k) \) is the Jacobian matrix of \( F \) at point \( u_k \). The problem of determining a solution to the model function \( M_k(u) \) is done by applying (7) so that the problem becomes

\[ \min_{u \in \mathbb{R}^m} m_k(u) = \frac{1}{2} \| M_k(u) \|^2 \]
\[ = \frac{1}{2} \| F(u_k) + J(u_k)(u - u_k) \|^2 \]
\[ = \frac{1}{2} F(u_k)^T F(u_k) + (u - u_k)^T J(u_k)^T F(u_k) + \frac{1}{2} (u - u_k)^T J(u_k)^T J(u_k)(u - u_k). \]

(9)

with \( J(u_k)^T F(u_k) \) is a gradient of \( F \) at point \( u_k \).

Define the trust region \( R_k \) as follows

\[ R_k = \{ u \in \mathbb{R}^m : \| u - u_k \| \leq \Delta_k \}. \]

(10)

with \( \Delta_k > 0 \) is called the trust region radius. After creating the model function (9), then determine \( u \in R_k \) such that \( m_k(u) \) is minimum, mathematically denoted as

\[ v_{k+1} = \arg \min_{u \in R_k} m_k(u) \]

(11)

equation (11) often referred as Trust Region Subproblem [15].
2.2.1. New Point Selection Provisions
Suppose point $v_{k+1}$ satisfies the Trust Region Subproblem (11). Basically, this is the point that should be selected as the next iteration, namely $u_{k+1}$. However, because $v_{k+1}$ is calculated based on the model function $m_k$, this causes the objective function value $f$ in $v_{k+1}$ to have two possibilities, increasing or decreasing. Therefore, a provision is needed in the selection of $u_{k+1}$ and in this paper the provisions for selection $u_{k+1}$ that have been defined by Nocedal and Wright [16] are used, as follows

$$
u_{k+1} = \begin{cases} 
u_{k+1} \text{ jika } \frac{f(u_k) - f(v_{k+1})}{m_k(u_k) - m_k(v_{k+1})} > \eta, \\ u_k \text{ lainnya,} \end{cases}$$ (12)

with $\eta \in (0, 1/4)$ fixed. Note that refusing to select a new point does not stop the algorithm, because by decreasing the trust region radius, $v_{k+2} \neq v_{k+1}$ will be obtained.

2.2.2. Updating the Trust Region
The selection of the new trust region $R_{k+1}$ around $u_{k+1}$ also uses the provisions defined by Nocedal and Wright [16], as follows

$$\Delta_{k+1} = \begin{cases} \frac{\Delta_k}{4} \text{ jika } \frac{f(u_k) - f(v_{k+1})}{m_k(u_k) - m_k(v_{k+1})} < \frac{1}{4}, \\ \min(2\Delta_k, \Delta_{\text{max}}) \text{ jika } \frac{f(u_k) - f(v_{k+1})}{m_k(u_k) - m_k(v_{k+1})} > \frac{3}{4}, \\ \Delta_k \text{ lainnya.} \end{cases}$$ (13)

This provision is made so that $\Delta_k$ does not pass through $\Delta_{\text{max}}$, and also calculate the ratio between the decrease value of the objective function $f(u_k) - f(v_{k+1})$ and the model function $m_k(u_k) - m_k(v_{k+1})$.

Trust Region Algorithm
- (Initialization) Select $k = 0, \Delta_{\text{max}} = 0, \Delta_k \in (0, \Delta_{\text{max}}), \eta \in (0, 1/4), u_k \in \mathbb{R}^m, \epsilon > 0$.
- While $\|\nabla f(u_k)\| \geq \epsilon$ repeat
  - Calculate $v_{k+1}$ as the minimum point on (11).
  - Determine $u_{k+1}$ using (12).
  - Compute $\Delta_{k+1}$ using (13).
  - Establish new functions model $m_{k+1}(u)$.
  - $k \leftarrow k + 1$.
- Done
- Take value $u_k$.

2.3. Dogleg Method
The dogleg method is a method used to find solutions in the Trust Region Subproblem. This method combines the steepest descent method through the Cauchy point and the Quasi-Newton method through the Quasi-Newton point in a point selection that makes (9) a minimum. Another interesting point is that this method works with two lines reminding people of the dog's paw, and the knee is at its minimal point in the steepest descent direction which is $v_k^u = u_k - \alpha_k^u J(u_k)^T F(u_k)$, through $v_k^u$ the dogleg trajectory continues to the Quasi-Newton point $v_k^{qn}$ [17]. Therefore the dogleg path can be described as follows

$$v(\tau) = \begin{cases} u_k + \tau(v_k^u - u_k) \quad \text{for } \tau \in [0, 1], \\ v_k^u + (\tau - 1)(v_k^{qn} - v_k^u) \quad \text{for } \tau \in [1, 2]. \end{cases}$$ (14)

Dogleg Algorithm
- Calculate $v_k^u$
• If $|v^u_k - u_k| \geq \Delta_k$ stop and take value $v_{k+1} = u_k + \frac{\Delta_k}{|v^u_k - u_k|} (v^u_k - u_k)$

• Calculate $v^q_k$

• If $|v^q_k - u_k| \leq \Delta_k$ stop and take value $v_{k+1} = v^q_k$

• Others
  - Determine $\tau^u$ s.t. $|v^u_k + \tau^u (v^q_k - v^u_k) - u_k| = \Delta_k$
  - Take value $v_{k+1} = v^u_k + \tau^u (v^q_k - v^u_k)$

• Done

3. Results and Discussion

3.1. Selecting $B$ and $Q$
This research will determining the limits of the parameter values that produce the on-site soliton in cases $B = 1$ and $Q = 0.5$, in addition to determining the limits of the parameters for cases $B = 2$ and $Q = 1$ as a comparison.

3.2. Creating the Model Function
It should be noted that the stationary DNLS equation with cubic-quintic nonlinearity (4) is computed in the domain $n \in [-N, -N + 1, \ldots, N]$, with $N \in \mathbb{Z}$ and $u_{-N-1} = u_{N+1} = 0$. Therefore $F(u) = (F_{-N}(u), F_{N+1}(u), \ldots, F_N(u))^T$ with $u = (u_{-N}, u_{-N+1}, \ldots, u_N) \in \mathbb{R}^{2N+1}$ and make (5) and (6) into a nonlinear system of equations with $2N + 1$ equations which can be written as follows

$$F(u) = \begin{bmatrix} F_{-N}(u) \\ F_{N+1}(u) \\ F_{-1}(u) \\ F_0(u) \\ F_1(u) \\ \vdots \\ F_{N-1}(u) \\ F_N(u) \end{bmatrix} = \begin{bmatrix} wu_N + C(u_{-N+1} - 2u_{-N} + 0) + Bu_N^3 - Qu_N^5 \\ wu_{-N+1} + C(u_{-N+2} - 2u_{-N+1} + u_{-N}) + Bu_{-N+1}^3 - Qu_{-N+1}^5 \\ \vdots \\ wu_{-1} + C(u_0 - 2u_{-1} + u_{-2}) + Bu_0^3 - Qu_0^5 \\ wu_0 + C(u_1 - 2u_0 + u_{-1}) + Bu_0^3 - Qu_0^5 \\ wu_1 + C(u_2 - 2u_1 + u_0) + Bu_1^3 - Qu_1^5 \\ \vdots \\ wu_{N-1} + C(u_N - 2u_{N-1} + u_{N-2}) + Bu_N^3 - Qu_{N-1}^5 \\ wu_N + C(0 - 2u_N + u_{N-1}) + Bu_N^3 - Qu_N^5 \end{bmatrix} = 0$$

(15)

and the Jacobian matrix at point $u_k$ can be written as

$$J(u_k) = \begin{bmatrix} \frac{\partial F_{-N}(u_k)}{\partial u_{-N}} & \frac{\partial F_{-N}(u_k)}{\partial u_{-N+1}} & \ldots & \frac{\partial F_{-N}(u_k)}{\partial u_N} \\ \frac{\partial F_{N+1}(u_k)}{\partial u_{-N}} & \frac{\partial F_{N+1}(u_k)}{\partial u_{-N+1}} & \ldots & \frac{\partial F_{N+1}(u_k)}{\partial u_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_N(u_k)}{\partial u_{-N}} & \frac{\partial F_N(u_k)}{\partial u_{-N+1}} & \ldots & \frac{\partial F_N(u_k)}{\partial u_N} \end{bmatrix}$$

(16)

and for $i, j \in [-N, N]$ applies

$$\frac{\partial F_i}{\partial u_j} = \begin{cases} w - 2C + 3Bu_i^2 - 5Qu_i^4 & \text{if } i = j, \\ 0 & \text{if } i = j + 1 \text{ or } i = j - 1, \end{cases} \quad \text{others.}$$

(17)
Function model (8) and (9) can be created from (15) and (16).

3.3. Exact On-site Soliton Solution for $C = 0$

Note that if the value is $C = 0$ then the equation (15) becomes:

$$
F(u) = \begin{bmatrix}
    F_{-N}(u) \\
    F_{-N+1}(u) \\
    \vdots \\
    F_{N-1}(u) \\
    F_N(u)
\end{bmatrix} = \begin{bmatrix}
    wu_{-N} + Bu_{-N}^3 - Qu_{-N}^5 \\
    wu_{-N+1} + Bu_{-N+1}^3 - Qu_{-N+1}^5 \\
    \vdots \\
    wu_{N-1} + Bu_{N-1}^3 - Qu_{N-1}^5 \\
    wu_N + Bu_N^3 - Qu_N^5
\end{bmatrix} = 0
\tag{18}
$$

Solving (18) can be done by finding a solution for each $F_n$ because for every $i, j \in [-N, N]$ and $i \neq j$, the functions $F_i$ and $F_j$ do not have interrelated variables. Rewrite $F_n$ as follows

$$
wu_n + Bu_n^3 - Qu_n^5 = 0
\tag{19}
$$

- For $B = 2$ and $Q = 1$

  $$
  u_n \in \left\{ 0, \pm \sqrt{1 \pm \sqrt{1 + w}} \right\} \text{ with } -1 < w < 0
  $$

  so that the following on-site soliton solutions can be formed

  $$\begin{cases}
    u_0 = \sqrt{1 + \sqrt{1 + w}}, \\
    u_{-1} = u_1 = \sqrt{1 - \sqrt{1 + w}}, \\
    u_i = 0, \quad i = \pm 2, \pm 3, \ldots, \pm N.
  \end{cases}
  $$

- For $B = 1$ and $Q = 0.5$

  $$
  u_n \in \left\{ 0, \pm \sqrt{1 \pm \sqrt{1 + 2w}} \right\} \text{ with } -0.5 < w < 0
  $$

  so that the following on-site soliton solutions can be formed

  $$\begin{cases}
    u_0 = \sqrt{1 + \sqrt{1 + 2w}}, \\
    u_{-1} = u_1 = \sqrt{1 - \sqrt{1 + 2w}}, \\
    u_i = 0, \quad i = \pm 2, \pm 3, \ldots, \pm N.
  \end{cases}
  $$

3.4. On-site Soliton Solution Comparison

Based on research conducted by Carretero et al [12], Chong et al [10] and Kusdiantara [13], soliton solutions in the stationary DNLS equation with cubic-quintic nonlinearity can be obtained in

$$
C > 0 \quad \text{ and } \quad -1 < w < 0.
\tag{20}
$$

However, the resulting solitons are not necessarily on-site. Therefore, to find a soliton solution of the on-site type and its constraints, we will examine $C$ and $w$ that meet the (20) with the following steps

1. Selecting pair $(B, Q)$

   In this research, the value was selected $(B, Q) = (1, 0.5)$ or $(B, Q) = (2, 1)$.
2. Selects C and w values on the constraint (20)
   A point is selected along C and w with a difference of 0.1 so that it can be written
   \[ C = 0.1, 0.2, \ldots \text{ and } w = -0.9, -0.8, \ldots, -0.1. \]
3. Enter value \( B, Q, C, w \) to \( F \)
4. Determine \( u^* \) on \( F \) using Trust Region Dogleg Method
   The calculation is done by using the MATLAB software program and using the \texttt{fsolve}
   subroutine and selecting \( u_0 = (\text{sech}(-N), \text{sech}(-N + 1), \ldots, \text{sech}(N)) \) as the initial value.
5. The \( u^* \) value is obtained and then the on-site soliton criterion is checked.
6. The values of \( C \) and \( w \) are taken as parameter limits if \( u^* \) is an on-site soliton.

Carry out the steps as above, which means that at first the values of \( (B, Q) = (1, 0.5) \), \( C = 0.1 \) are
selected and the solution is sought at all values of \( w = -0.9, -0.8, \ldots, -0.1 \) so that the solution graph is
obtained as follows.

![Figure 1](image1.png)

**Figure 1.** Solutions for value \( (B, Q) = (1, 0.5), C = 0.1 \) and \( w = -0.9, -0.8, \ldots, -0.1 \)

In Figure 1 it can be seen that the solution that forms the on-site soliton is only at the value of
\( w = -0.6, -0.5, -0.4, -0.3 \), therefore the values of \( C \) and \( w \) are taken and entered into Table 1. Furthermore, still with the value of \( (B, Q) = (1, 0.5) \), but the value of \( C \) is increased to be \( C = 0.2 \) and the solution is determined at all values of \( w = -0.9, -0.8, \ldots, -0.1 \) in order to obtain the following solution graph.

![Figure 2](image2.png)

**Figure 2.** Solutions for value \( (B, Q) = (1, 0.5), C = 0.2 \) and \( w = -0.9, -0.8, \ldots, -0.1 \)

In Figure 2 it can be seen that the solution that forms the on-site soliton is only at the value of
\( w = -0.6, -0.5, \ldots, -0.1 \), therefore the values of \( C \) and \( w \) are taken and entered into Table 1. These steps continue until no \( C \) and \( w \) are found to form the on-site soliton. The same is done for the value of \( (B, Q) = (2, 1) \). After the simulation with these steps, the on-site soliton limits are obtained as follows.
Table 1. Limitation of on-site soliton parameters for $B = 1$ and $Q = 0.5$

| $C$   | $w$  |
|-------|------|
| min  | max  | min | max |
| 0.1  | 0.1  | -0.6| -0.3 |
| 0.2  | 0.2  | -0.6| -0.1 |
| 0.3  | 0.6  | -0.5| -0.1 |
| 0.7  | 1.3  | -0.4| -0.1 |
| 1.4  | 2.5  | -0.3| -0.1 |
| 2.6  | 3.3  | -0.2| -0.1 |
| 3.1  | 3.7  | -0.1| -0.1 |
| 3.8  | 5.1  | -0.2| -0.1 |
| 5.2  | 12   | -0.1| -0.1 |

3.5. On-site Soliton Solution Comparison

The following is a graph of the on-site soliton solutions searched using the Trust Region Dogleg Method and will be compared based on the selection of constants $B, Q$ and parameter $w, C$. For convenience, solutions with $B = 1, Q = 0.5$ are labeled with DNLS1 and solutions with $B = 2, Q = 1$ are labelled with DNLS2.

Table 2. Limitation of on-site soliton parameters for $B = 2$ and $Q = 1$

| $C$   | $w$  |
|-------|------|
| min  | max  | min | max |
| 0.1  | 0.1  | -0.9| -0.8 |
| 0.2  | 0.2  | -0.9| -0.6 |
| 0.3  | 0.5  | -0.9| -0.3 |
| 0.6  | 1.8  | -0.9| -0.1 |
| 1.9  | 2.6  | -0.8| -0.1 |
| 2.7  | 3.6  | -0.7| -0.1 |
| 3.7  | 5.1  | -0.6| -0.1 |
| 5.2  | 7.2  | -0.5| -0.1 |
| 7.3  | 7.4  | -0.3| -0.1 |
| 7.5  | 10.3 | -0.4| -0.1 |
| 10.4 | 11.2 | -0.3| -0.1 |
| 11.3 | 13.9 | -0.2| -0.1 |
| 14   | 15.7 | -0.3| -0.1 |
| 15.8 | 18.7 | -0.2| -0.1 |
| 18.8 | 27.6 | -0.1| -0.1 |

4. Conclusions

Based on the results of the research and discussion that has been described, it can be concluded that:

1. The on-site soliton solution for $C = 0$ can be constructed with the following conditions:
   - DNLS2 ($B = 2, Q = 1$) with parameter value constraints $-1 < w < 0$
   
   $$u_n = \begin{cases} 
   u_0 = \sqrt{1 + \sqrt{1 + w}}, \\
   u_{n-1} = u_1 = \sqrt{1 - \sqrt{1 + w}}, \\
   u_i = 0, & i = \pm 2, \pm 3, \ldots, \pm N.
   \end{cases}$$
• DNLS1 ($B = 1, Q = 0.5$) with parameter value constraints $-0.5 < w < 0$

$$u_n = \begin{cases} u_0 = \sqrt{1 + 2w}, \\
u_{-1} = u_1 = \sqrt{1 - 2w}, \\
u_i = 0, & i = \pm 2, \pm 3, \ldots, \pm N. \end{cases}$$

2. The on-site soliton solution for $\mathcal{C} \neq 0$ can be found using the parameter constraints listed in Table 2 for DNLS2 and using the parameter constraints shown in Table 1 for DNLS1.

3. In case $\mathcal{C} \neq 0$, the maximum limit value of $\mathcal{C}$ on DNLS2 which can construct on-site solitons is up to $\approx 2x$ greater than that of DNLS1.

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