\textbf{C*-Isomorphisms Associated with Two Projections on a Hilbert C*-Module*}

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Abstract Motivated by two norm equations used to characterize the Friedrichs angle, this paper studies C*-isomorphisms associated with two projections by introducing the matched triple and the semi-harmonious pair of projections. A triple \((P, Q, H)\) is said to be matched if \(H\) is a Hilbert C*-module, \(P\) and \(Q\) are projections on \(H\) such that their infimum \(P \wedge Q\) exists as an element of \(\mathcal{L}(H)\), where \(\mathcal{L}(H)\) denotes the set of all adjointable operators on \(H\). The C*-subalgebras of \(\mathcal{L}(H)\) generated by elements in \(\{P - P \wedge Q, Q - P \wedge Q, I\}\) and \(\{P, Q, P \wedge Q, I\}\) are denoted by \(i(P, Q, H)\) and \(o(P, Q, H)\), respectively. It is proved that each faithful representation \((\pi, X)\) of \(o(P, Q, H)\) can induce a faithful representation \((\tilde{\pi}, X)\) of \(i(P, Q, H)\) such that
\[
\begin{align*}
    \tilde{\pi}(P - P \wedge Q) &= \pi(P) - \pi(P) \wedge \pi(Q), \\
    \tilde{\pi}(Q - P \wedge Q) &= \pi(Q) - \pi(P) \wedge \pi(Q).
\end{align*}
\]
When \((P, Q)\) is semi-harmonious, that is, \(\mathcal{R}(P + Q)\) and \(\mathcal{R}(2I - P - Q)\) are both orthogonally complemented in \(H\), it is shown that \(i(P, Q, H)\) and \(i(I - Q, I - P, H)\) are unitarily equivalent via a unitary operator in \(\mathcal{L}(H)\). A counterexample is constructed, which shows that the same may be not true when \((P, Q)\) fails to be semi-harmonious. Likewise, a counterexample is constructed such that \((P, Q)\) is semi-harmonious, whereas \((P, I - Q)\) is not semi-harmonious. Some additional examples indicating new phenomena of adjointable operators acting on Hilbert C*-modules are also provided.

Keywords Hilbert C*-module, Projection, Orthogonal complementarity, C*-Isomorphism

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1 Introduction

Let \(P\) and \(Q\) be two projections on a Hilbert space \(X\). Their infimum \(P \wedge Q\) is the projection from \(X\) onto \(\mathcal{R}(P) \cap \mathcal{R}(Q)\), which can be obtained by taking the limit of \(\{(PQ)^n\}_{n=1}^\infty\) in the strong operator topology (see [9, Lemma 22]). The cosine of the Friedrichs angle (see [4]) between \(M = \mathcal{R}(P)\) and \(N = \mathcal{R}(Q)\) is denoted by \(c(M, N)\), and can be calculated as
\[
c(M, N) = \| (P - P \wedge Q)(Q - P \wedge Q)\|.
\]
The characterization of $c(M, N) = c(N^\perp, M^\perp)$ given in [3, Section 2] yields
\[
\|(P - P \wedge Q)(Q - P \wedge Q)\| \\
= \||I - Q - (I - Q) \wedge (I - P)][I - P - (I - Q) \wedge (I - P)]\|.
\] (1.1)

The Friedrichs angle has also been studied in the setting of $C^*$-algebras. To deal with the Friedrichs angle associated with two projections $P$ and $Q$ in a $C^*$-algebra $\mathfrak{A}$, one approach employed in [1] is to embed $\mathfrak{A}$ into its enveloping von Neumann algebra $\mathfrak{A}''$ via the universal representation $(\pi_u, H_u)$ of $\mathfrak{A}$, and then to use the universal property of $\mathfrak{A}''$ (see [11, Theorem 3.7.7]). By identifying $\mathfrak{A}$ with $\pi_u(\mathfrak{A})$, $P \wedge Q$ can be obtained in $\mathfrak{A}''$, and it is proved in [1, Proposition 2.4] that for every faithful representation $(\pi, X)$ of $\mathfrak{A}$,
\[
\|(P - P \wedge Q)(Q - P \wedge Q)\| \\
= \||\pi(P) - \pi(P) \wedge \pi(Q))[\pi(Q) - \pi(P) \wedge \pi(Q)]\|.
\] (1.2)

Hilbert $C^*$-modules are natural generalizations of Hilbert spaces, and every $C^*$-algebra can be regarded as a Hilbert $C^*$-module over itself in a natural way. The purpose of this paper is, in the framework of Hilbert $C^*$-modules, to give a deeper understanding of (1.1)–(1.2) via algebraic systems rather than on products of finitely many operators.

It is notable that a closed submodule of a Hilbert $C^*$-module may fail to be orthogonally complemented. In this paper much attention has been paid on this aspect. Let $\mathcal{L}(H)$ be the set of all adjointable operators on a Hilbert $C^*$-module $H$. For two projections $P, Q \in \mathcal{L}(H)$, let
\[
\mathcal{R} = \mathcal{R}(Q) \cap \mathcal{R}(P) \quad \text{and} \quad \mathcal{N} = \mathcal{N}(Q) \cap \mathcal{N}(P).
\] (1.3)

To check the validity of the Halmos’ two projections theorem in the Hilbert $C^*$-module case, the term of the harmonious pair of projections is introduced in [7, Section 4], and it is shown later in [12, Theorem 3.3] that for every pair $(P, Q)$ of projections, the Halmos’ two projections theorem is valid if and only if $(P, Q)$ is harmonious. In view of the conditions stated in [7, Lemma 5.4] and [12, Theorem 3.3], we make a definition as follows.

**Definition 1.1** A pair $(P, Q)$ of projections on a Hilbert $C^*$-module $H$ is said to be semi-harmonious if both $\mathcal{R}(P + Q)$ and $\mathcal{R}(2I - P - Q)$ are orthogonally complemented in $H$. If $(P, Q)$ and $(P, I - Q)$ are both semi-harmonious, then $(P, Q)$ is said to be harmonious.

Let $\mathcal{R}$ and $\mathcal{N}$ be defined by (1.3) for projections $P$ and $Q$ on a Hilbert $C^*$-module $H$. Since $\mathcal{R}(2I - P - Q)^\perp = \mathcal{R}$ and $\mathcal{R}(P + Q)^\perp = \mathcal{N}$, a condition weaker than the semi-harmony of $(P, Q)$ turns out to be the orthogonal complementarity of $\mathcal{R}$ and $\mathcal{N}$, which is necessary and sufficient to make use of the notations $P_\mathcal{R}$ and $P_\mathcal{N}$ (the projections from $H$ onto $\mathcal{R}$ and $\mathcal{N}$, respectively). The example constructed in [7, Section 3] (see also the proof of Theorem 2.2) shows that there exist projections $P$ and $Q$ on certain Hilbert $C^*$-module such that $\mathcal{R} = \mathcal{N} = \{0\}$, whereas $(P, Q)$ is not semi-harmonious. So, generally the meaningfulness of $P_\mathcal{R}$ and $P_\mathcal{N}$ does not imply the semi-harmony of $(P, Q)$.

Next, we introduce the matched triple as follows.

**Definition 1.2** A triple $(P, Q, H)$ is said to be matched if $H$ is a Hilbert $C^*$-module, $P$ and $Q$ are projections on $H$ such that $\mathcal{R}$ defined by (1.3) is orthogonally complemented in $H$. In this case, the projection $P_\mathcal{R}$ is denoted by $P \wedge Q$. The $C^*$-subalgebras of $\mathcal{L}(H)$ generated by elements in $\{P - P \wedge Q, Q - P \wedge Q, I\}$ and $\{P, Q, P \wedge Q, I\}$ are denoted by $i(P, Q, H)$ and $o(P, Q, H)$, and are called the inner algebra and the outer algebra, respectively.
Definition 1.3 Two matched triple \((P_i, Q_i, H_i)\) \((i = 1, 2)\) are said to be innerly unitarily equivalent if there exists a unitary operator \(U : H_1 \to H_2\) such that
\[
U(P_1 - P_1 \wedge Q_1)U^* = P_2 - P_2 \wedge Q_2,
\]
\[
U(Q_1 - P_1 \wedge Q_1)U^* = Q_2 - P_2 \wedge Q_2.
\]

Recall that a pair \((\pi, X)\) is said to be a representation of a \(C^\ast\)-algebra \(\mathcal{A}\) if \(X\) is a Hilbert space and \(\pi : \mathcal{A} \to \mathcal{B}(X)\) is a \(C^\ast\)-morphism, where \(\mathcal{B}(X)\) denotes the set of all bounded linear operators on \(X\).

Definition 1.4 Let \((P, Q, H)\) be a matched triple. A representation \((\pi, X)\) of \(o(P, Q, H)\) is called an outer representation of \((P, Q, H)\). If furthermore a \(C^\ast\)-morphism \(\tilde{\pi} : i(P, Q, H) \to \mathcal{B}(X)\) can be induced such that
\[
\tilde{\pi}(I) = \pi(I),
\]

\[
\tilde{\pi}(P - P \wedge Q) = \pi(P) - \pi(P) \wedge \pi(Q),
\]
\[
\tilde{\pi}(Q - P \wedge Q) = \pi(Q) - \pi(P) \wedge \pi(Q),
\]

then \((\pi, X)\) is called an inner-outer representation of \((P, Q, H)\). When both \(\pi\) and \(\tilde{\pi}\) are faithful, \((\pi, X)\) is called a faithful inner-outer representation of \((P, Q, H)\).

Remark 1.1 Let \((\pi, X)\) be an inner-outer representation of \((P, Q, H)\). It is notable that generally \(\pi(P) \wedge \pi(Q)\) is taken in the von Neumann algebra \([\pi(o(P, Q, H))]^\ast\) rather than in the \(C^\ast\)-algebra \(\pi(o(P, Q, H))\), so it may happen that \(\pi(P \wedge Q) \neq \pi(P) \wedge \pi(Q)\).

With the terms given as above, we list the main results of this paper as follows:

1. There exist projections \(P\) and \(Q\) on certain Hilbert \(C^\ast\)-module \(H\) such that \((P, Q)\) is semi-harmonious, whereas it fails to be harmonious (see Theorem 2.2).
2. For every semi-harmonious pair \((P, Q)\) of projections on a Hilbert \(C^\ast\)-module \(H\), the matched triples \((P, Q, H)\) and \((I - Q, I - P, H)\) are innerly unitarily equivalent (see Theorem 3.1).
3. There exist projections \(P\) and \(Q\) on certain Hilbert \(C^\ast\)-module \(H\) such that the triples \((P, Q, H)\) and \((I - Q, I - P, H)\) are both matched, whereas they are not innerly unitarily equivalent (see Theorem 3.2).
4. Every faithful outer representation of a matched triple is a faithful inner-outer representation (see Theorem 4.1).

An application of Theorems 3.1 and 4.1 will be illustrated in Corollary 4.3. Another application, as has been mentioned earlier, concerns a new insight into (1.1)–(1.2). Let \(P\) and \(Q\) be two projections on a Hilbert \(C^\ast\)-module \(H\) such that \(\mathcal{R}\) and \(\mathcal{N}\) defined by (1.3) are orthogonally complemented in \(H\). By Theorem 4.1, we will see that each faithful unital representation \((\pi, X)\) of \(\mathcal{L}(H)\) can induce unital \(C^\ast\)-isomorphisms \(\pi_1 : i(P, Q, H) \to i(\pi(P), \pi(Q), X)\) and \(\pi_2 : i(I - Q, I - P, H) \to i(I - \pi(Q), I - \pi(P), X)\) such that
\[
\pi_1(P - P \wedge Q) = \pi(P) - \pi(P) \wedge \pi(Q),
\]
\[
\pi_1(Q - P \wedge Q) = \pi(Q) - \pi(P) \wedge \pi(Q),
\]
\[
\pi_2[I - Q - (I - Q) \wedge (I - P)] = I - \pi(Q) - \pi(I - Q) \wedge \pi(I - P),
\]
\[
\pi_2[I - P - (I - Q) \wedge (I - P)] = I - \pi(P) - \pi(I - Q) \wedge \pi(I - P).
\]

Thus, \(\|\pi_1(x)\| = \|x\|\) for every \(x \in i(P, Q, H)\). Specifically, if we put \(x = (P - P \wedge Q)(Q - P \wedge Q)\), then (1.2) is obtained.
Note that \((\pi(P), \pi(Q))\) is a pair of projections acting on a Hilbert space, so it is harmonious. Hence, by Theorem 3.1 there exists a unitary operator \(U \in \mathbb{B}(X)\) such that
\[
\begin{align*}
U[\pi(P) - \pi(P) \wedge \pi(Q)]U^* &= I - \pi(Q) - \pi(I - Q) \wedge \pi(I - P), \\
U[\pi(Q) - \pi(P) \wedge \pi(Q)]U^* &= I - \pi(P) - \pi(I - Q) \wedge \pi(I - P).
\end{align*}
\]
Thus, a \(C^*\)-isomorphism \(\rho : i(P, Q, H) \to i(I - Q, I - P, H)\) can be constructed as
\[
\rho(x) = (\pi_2)^{-1}U\pi_1(x)U^*, \quad \forall x \in i(P, Q, H).
\]
Therefore, \(\|\rho(x)\| = \|x\|\) for every \(x \in i(P, Q, H)\). Likewise, if we take \(x = (P - P \wedge Q)(Q - P \wedge Q)\), then (1.1) is obtained. So, a substantive generalization of (1.1)–(1.2) has been made.

The paper is organized as follows. The main purpose of Section 2 is to construct two projections \(P\) and \(Q\) such that \((P, Q)\) is semi-harmonious, whereas it fails to be harmonious. Section 3 focuses on the construction of the unitary operator \(U\) satisfying (3.3)–(3.4). Section 4 is devoted to the study of the faithful inner-outer representation of a matched triple.

2 Semi-Harmonious Pairs of Projections

Throughout the rest of this paper, \(\mathbb{N}, \mathbb{Z}_+,\) and \(\mathbb{C}\) are the sets of all positive integers, non-negative integers and complex numbers, respectively. Unless otherwise specified, \(\mathfrak{A}\) is a \(C^*\)-algebra, \(E, H\) and \(K\) are Hilbert \(\mathfrak{A}\)-modules (see [5, 10]). The set of all adjointable operators from \(H\) to \(K\) is denoted by \(\mathcal{L}(H, K)\). Given \(A \in \mathcal{L}(H, K)\), the adjoint operator, the range and the null space of \(A\) are denoted by \(A^*\), \(\mathcal{R}(A)\) and \(\mathcal{N}(A)\), respectively. Let \(|A|\) designate the square root of \(A^*A\). In case \(H = K\), \(\mathcal{L}(H, K)\) is abbreviated to \(\mathcal{L}(H)\), whose subset consisting of all positive elements is denoted by \(\mathcal{L}(H)_+\). The unit of \(\mathcal{L}(H)\) (namely, the identity operator on \(H\)) is denoted by \(I_H\), or simply by \(I\) when no ambiguity arises. An operator \(P \in \mathcal{L}(H)\) is said to be a projection if \(P = P^* = P^2\). The set of all projections on \(H\) is denoted by \(\mathcal{P}(H)\).

Let \(M\) be a closed submodule of \(H\). Clearly, there exists at most a projection in \(\mathcal{L}(H)\), written \(P_M\), such that \(\mathcal{R}(P_M) = M\). It can be easily verified that \(P_M\) exists if and only if \(H = M + M^\perp\), where
\[
M^\perp = \{x \in H : \langle x, y \rangle = 0, \forall y \in M\}.
\]
In this case, \(M\) is said to be orthogonally complemented in \(H\).

To construct semi-harmonious pairs of projections, we need a couple of lemmas.

**Lemma 2.1** (see [6, Proposition 2.9]) For every \(T \in \mathcal{L}(H)_+\) and \(\alpha > 0\), we have \(\overline{\mathcal{R}(T)} = \mathcal{R}(T^\alpha)\).

**Lemma 2.2** (see [5, Proposition 3.7]) For every \(T \in \mathcal{L}(H, K)\), we have \(\overline{\mathcal{R}(T)} = \mathcal{R}(TT^*)\).

**Lemma 2.3** (see [6, Proposition 2.7]) Let \(B, C \in \mathcal{L}(E, H)\) be such that \(\overline{\mathcal{R}(B)} = \mathcal{R}(C)\). Then for every \(A \in \mathcal{L}(H, K)\), we have \(\overline{\mathcal{R}(AB)} = \mathcal{R}(AC)\).

An approach to construct semi-harmonious pairs of projections reads as follows.

**Theorem 2.1** For every \(P, Q \in \mathcal{P}(H)\), let \(M \subseteq H\) be defined by
\[
M = \mathcal{R}[(P + Q)(2I - P - Q)].
\]
Then \(P|_M\) and \(Q|_M\) are projections on \(M\) such that \((P|_M, Q|_M)\) is semi-harmonious, where \(P|_M\) and \(Q|_M\) are the restrictions of \(P\) and \(Q\) on \(M\), respectively.
**Proof** To simplify the notation, we put
\[
A = P + Q \quad \text{and} \quad G = \mathcal{R}[A(2I - A)]. \tag{2.2}
\]
Since \(G\) defined as above is the range of an adjointable operator, its closure \(M\) is a Hilbert \(\mathfrak{A}\)-module.

For every \(T \in \mathcal{L}(H)\) and \(X \subseteq H\), let \(TX = \{Tx : x \in X\}\). Then \(TX = \mathcal{R}(T|_X)\), and the boundedness of \(T\) gives \(\overline{TX} = \overline{TX}\). Hence
\[
AM = AG \quad \text{and} \quad (2I - A)M = (2I - AG).
\tag{2.3}
\]
Since \(A\) and \(2I - A\) are positive and commutative, we may combine (2.2)–(2.3) with Lemmas 2.1 and 2.3 to get
\[
\mathcal{R}(P|_M + Q|_M) = \overline{AM} = \overline{AG} = \overline{R[(2I - A)A^2]} = \mathcal{R}[(2I - A)A] = M. \tag{2.4}
\]
Similarly,
\[
\mathcal{R}(2I - P|_M - Q|_M) = \overline{(2I - A)M} = \overline{(2I - A)G} = \mathcal{R}[A(2I - A)^2] = \mathcal{R}[A(2I - A)] = M. \tag{2.5}
\]
Furthermore, direct computations yield
\[
P(2I - A)A = P(I - Q)P = A(2I - A)P, \quad Q(2I - A)A = Q(I - P)Q = A(2I - A)Q,
\]
which lead clearly to \(PM \subseteq M\) and \(QM \subseteq M\). Consequently, \(P|_M\) and \(Q|_M\) are projections on \(M\). In view of (2.4)–(2.5), we conclude that \((P|_M, Q|_M)\) is semi-harmonious.

**Theorem 2.2** There exist projections \(P\) and \(Q\) on certain Hilbert \(C^*\)-module such that \((P, Q)\) is semi-harmonious, whereas it fails to be harmonious.

**Proof** We follow the line initiated in [8, Section 3] and modified in [7, Section 3]. Let \(M_2(\mathbb{C})\) and \(I_2\) be the set of all \(2 \times 2\) complex matrices and the identity matrix in \(M_2(\mathbb{C})\), respectively. Denote by \(\| \cdot \|\) the operator norm on \(M_2(\mathbb{C})\). Let \(\mathfrak{A} = C([0, 1]; M_2(\mathbb{C}))\) be the set of all continuous matrix-valued functions from \([0, 1]\) to \(M_2(\mathbb{C})\). For \(x \in \mathfrak{A}\) and \(t \in [0, 1]\), we put
\[
x^*(t) = (x(t))^* \quad \text{and} \quad \|x\| = \max_{0 \leq s \leq 1} \|x(s)\|.
\]
With the *-operation as above and the usual algebraic operations, \(\mathfrak{A}\) is a unital \(C^*\)-algebra, which is also a Hilbert \(\mathfrak{A}\)-module with the inner-product given by
\[
\langle x, y \rangle = x^*y \quad \text{for} \quad x, y \in \mathfrak{A}.
\]
Let \(e\) be the unit of \(\mathfrak{A}\), that is, \(e(t) = I_2\) for every \(t \in [0, 1]\). It is known that \(\mathfrak{A} \cong \mathcal{L}(\mathfrak{A})\) via \(a \rightarrow L_a\) (see [7, Section 3]), where \(L_a(x) = ax\) for \(a, x \in \mathfrak{A}\). For simplicity, we identify \(\mathcal{L}(\mathfrak{A})\) with \(\mathfrak{A}\) and set
\[
c_t = \cos \frac{\pi}{2}t \quad \text{and} \quad s_t = \sin \frac{\pi}{2}t \quad \text{for} \quad t \in [0, 1]. \tag{2.6}
\]
Let $P, Q \in \mathfrak{A}$ be projections determined by the matrix-valued functions

$$P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q(t) = \begin{pmatrix} c_t^2 & s_t c_t \\ s_t c_t & s_t^2 \end{pmatrix} \quad \text{for } t \in [0, 1].$$

(2.7)

It is shown in [7, Section 3] that

$$\mathcal{R}(P) \cap \mathcal{R}(Q) = \mathcal{R}(P) \cap \mathcal{N}(Q) = \mathcal{N}(P) \cap \mathcal{R}(Q) = \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}. \quad (2.8)$$

Now, let $H = \mathfrak{A}$ and $M$ be defined by (2.1). According to Theorem 2.1, $(P|_M, Q|_M)$ is semi-harmonious. In what follows, we prove that $(P|_M, Q|_M)$ is not harmonious.

Direct computation yields

$$(P + Q)(2I - P - Q) = P + Q - PQ - QP = (P - Q)(P - Q)^*,$$

which leads by (2.1) and Lemma 2.2 to

$$M = \overline{\mathcal{R}(P - Q)}. \quad (2.9)$$

Utilizing (2.7) we obtain $P - Q = au$, where $a, u \in H$ are determined by

$$a(t) = \begin{pmatrix} s_t & 0 \\ 0 & s_t \end{pmatrix}, \quad u(t) = \begin{pmatrix} s_t & -c_t \\ -c_t & -s_t \end{pmatrix}, \quad t \in [0, 1].$$

Since $u$ is a unitary and $M$ can be represented by (2.9), we have $M = \overline{\mathcal{R}(a)}$. Specifically,

$$a = ae \in M, \quad \overline{\mathcal{R}(P|_M + I_M - Q|_M)} = \overline{\mathcal{R}(T)},$$

where $T = (P + I - Q)a \in H$. For any $x \in H$ with $x(t) = (x_{ij}(t))_{1 \leq i, j \leq 2}$, it is easy to verify that

$$(Tx)(1) = \begin{pmatrix} 2x_{11}(1) & 2x_{12}(1) \\ 0 & 0 \end{pmatrix},$$

hence

$$\|Tx - a\| \geq \|(Tx)(1) - a(1)\| = \left\| \begin{pmatrix} 2x_{11}(1) - 1 & 2x_{12}(1) \\ 0 & -1 \end{pmatrix} \right\| \geq 1,$$

which implies that $a \notin \overline{\mathcal{R}(T)}$. Furthermore, by (2.8) we have

$$\overline{\mathcal{R}(P|_M + I_M - Q|_M)} \cap \mathcal{R}(Q|_M) \subseteq \mathcal{N}(P) \cap \mathcal{R}(Q) = \{0\}.$$ 

This shows

$$a \notin \overline{\mathcal{R}(P|_M + I_M - Q|_M)} + \overline{\mathcal{R}(P|_M + I_M - Q|_M)}^\perp,$$

whereas $a \in M$. So $\overline{\mathcal{R}(P|_M + I_M - Q|_M)}$ is not orthogonally complemented in $M$.

**Remark 2.1** It is notable that there exist projections $P$ and $Q$ such that $\overline{\mathcal{R}(P + Q)}$ is orthogonally complemented, whereas $\overline{\mathcal{R}(2I - P - Q)}$ fails to be orthogonally complemented. We provide such an example as follows.

**Example 2.1** Let $\mathfrak{A} = H = C([0, 1]; M_2(\mathbb{C}))$ and $P, Q \in \mathcal{P}(H)$ be as in the proof of Theorem 2.2. Put $H_0 = \overline{\mathcal{R}(P + Q)}$, $P_0 = P|_{H_0}$ and $Q_0 = Q|_{H_0}$. From [7, Lemma 2.3] we have
$H_0 = \overline{\mathcal{R}(P) + \mathcal{R}(Q)}$, which means that $\mathcal{R}(P) \subseteq H_0$, hence $P_0H_0 \subseteq H_0$. Consequently, $P_0$ is a projection on $H_0$. Similarly, we have $Q_0 \in \mathcal{P}(H_0)$. In view of (2.8), we get

$$\mathcal{R}(P_0) \cap \mathcal{R}(Q_0) = \mathcal{N}(P_0) \cap \mathcal{N}(Q_0) = \{0\}. \quad (2.10)$$

According to Lemma 2.1, we have

$$H_0 = \overline{\mathcal{R}{\left[(P + Q)^2\right]}} \subseteq \overline{\mathcal{R}(P_0 + Q_0)} \subseteq H_0.$$  

As a result, we arrive at

$$H_0 = \overline{\mathcal{R}(P_0 + Q_0)} = \overline{\mathcal{R}(P_0 + Q_0) + \mathcal{N}(P_0) \cap \mathcal{N}(Q_0)}.$$  

This shows the orthogonal complementarity of $\overline{\mathcal{R}(P_0 + Q_0)}$ in $H_0$.

Let $F = (2I - P - Q)(P + Q)$. Clearly, $F = (I - P)Q + (I - Q)P$, so by (2.7) we have

$$F(t) = \begin{pmatrix} s_t^2 \\ s_t^2 \end{pmatrix}, \quad \forall t \in [0,1],$$

which implies that for every $x \in \mathfrak{A}$, $(Fx)(0) = F(0)x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence

$$\|P - Fx\| \geq \|P(0) - F(0)x(0)\| = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| = 1.$$  

Due to the definition of $F$ and the observation of (2.10), we conclude that

$$P \notin \overline{\mathcal{R}(F)} = \overline{\mathcal{R}(2I_{H_0} - P_0 - Q_0) + \mathcal{R}(P_0) \cap \mathcal{R}(Q_0)}.$$  

On the other hand, $P = PI \in \mathcal{R}(P) \subseteq H_0$. Therefore, $\overline{\mathcal{R}(2I_{H_0} - P_0 - Q_0)}$ is not orthogonally complemented in $H_0$.

### 3 Unitary Equivalences Associated with Two Projections

In this section, we deal with unitary equivalences associated with two projections. We begin with a known result as follows.

**Lemma 3.1** (see [7, Lemma 4.1]) Let $P, Q \in \mathcal{P}(H)$ be such that $\overline{\mathcal{R}(I - Q + P)}$ is orthogonally complemented in $H$. Then $\overline{\mathcal{R}(QP)}$ is also orthogonally complemented in $H$ such that $P_{\overline{\mathcal{R}(QP)}} = Q - P_{\mathcal{R}(Q) \cap \mathcal{N}(P)}$.

Next, we provide a useful lemma as follows.

**Lemma 3.2** For every $P, Q \in \mathcal{P}(H)$, we have

$$|P(I - Q)| + |(I - P)Q| = |(I - Q)P| + |Q(I - P)|. \quad (3.1)$$

**Proof** For simplicity, we put

$$T_1 = P(I - Q) \quad \text{and} \quad T_2 = (I - P)Q. \quad (3.2)$$

It is clear that

$$T_1^*T_1 = (I - Q)T_1^*T_1(I - Q) \quad \text{and} \quad T_2^*T_2 = QT_2^*T_2Q.$$
which gives (3.1) by taking the square roots of positive operators.

Now, we are in the position to provide the main result of this section.

**Theorem 3.1** Let $P, Q \in \mathcal{P}(H)$ be such that $(P, Q)$ is semi-harmonious. Then there exists a unitary $U \in \mathcal{L}(H)$ such that

\[
U(Q - P_R)U^* = I - P - P_N, \tag{3.3}
\]

\[
U(P - P_R)U^* = I - Q - P_N, \tag{3.4}
\]

where $\mathcal{R}$ and $\mathcal{N}$ are defined by (1.3).

**Proof** Let $T_1$ and $T_2$ be defined by (3.2). Since $\overline{\mathcal{R}(P + Q)}$ is orthogonally complemented in $H$, by Lemma 3.1 both $\overline{\mathcal{R}(T_1^*)}$ and $\overline{\mathcal{R}(T_2)}$ are orthogonally complemented in $H$. Similarly, the orthogonal complementarity of $\overline{\mathcal{R}(2I - P - Q)}$ leads to that of $\overline{\mathcal{R}(T_1^*)}$ and $\overline{\mathcal{R}(T_2^*)}$. So for $i = 1, 2$, the notations $P_{\overline{\mathcal{R}(T_i^*)}}$ and $P_{\overline{\mathcal{R}(T_i^*)}}$ are meaningful. The point is, these projections can be used to obtain the canonical forms of $T_1$ and $T_2$ (see [2]). In fact, in view of (3.2) we have

\[
\overline{\mathcal{R}(T_1)} \subseteq \overline{\mathcal{R}(P)} \quad \text{and} \quad \overline{\mathcal{R}(T_1^*)} \subseteq \overline{\mathcal{R}(I - Q)},
\]

hence

\[
P_{\overline{\mathcal{R}(T_1)}} P = P_{\overline{\mathcal{R}(T_1^*)}} \quad \text{and} \quad (I - Q) P_{\overline{\mathcal{R}(T_1^*)}} = P_{\overline{\mathcal{R}(T_1)}};
\]

which lead to

\[
T_1 = P_{\overline{\mathcal{R}(T_1)}} T_1 P_{\overline{\mathcal{R}(T_1^*)}} = P_{\overline{\mathcal{R}(T_1)}} (I - Q) P_{\overline{\mathcal{R}(T_1^*)}} = P_{\overline{\mathcal{R}(T_1)}} P_{\overline{\mathcal{R}(T_1^*)}}. \tag{3.5}
\]

Replacing $P$ and $Q$ with $I - P$ and $I - Q$, respectively, we obtain

\[
T_2 = P_{\overline{\mathcal{R}(T_2)}} P_{\overline{\mathcal{R}(T_2^*)}}. \tag{3.6}
\]

Taking *-operation, from (3.5)–(3.6) we arrive at

\[
T_1^* = P_{\overline{\mathcal{R}(T_1^*)}} P_{\overline{\mathcal{R}(T_1)}} \quad \text{and} \quad T_2^* = P_{\overline{\mathcal{R}(T_2^*)}} P_{\overline{\mathcal{R}(T_2)}}. \tag{3.7}
\]

To make use of (3.1), we need the polar decompositions of $T_1$ and $T_2$. For $i = 1, 2$, as $\overline{\mathcal{R}(T_i)}$ and $\overline{\mathcal{R}(T_i^*)}$ are both orthogonally complemented in $H$, by [6, Lemma 3.6 and Theorem 3.8] there exists a unique partial isometry $V_i \in \mathcal{L}(H)$ such that

\[
T_i = V_i |T_i|, \quad T_i^* = V_i^* |T_i^*|, \quad V_i^* V_i = P_{\overline{\mathcal{R}(T_i)}}, \quad V_i V_i^* = P_{\overline{\mathcal{R}(T_i)}}. \tag{3.8}
\]

Combining the last two equations in (3.8) with (3.5)–(3.7), we obtain

\[
T_i = V_i V_i^* V_i V_i^* \quad \text{and} \quad T_i^* = V_i^* V_i V_i^*. \]
These two equations together with (3.8), Lemmas 2.1–2.2 yield
\[(V_i^*)^2 V_i = V_i^*(V_i V_i^* V_i) = V_i^* T_i = V_i^* V_i |T_i| = |T_i|,\]
which gives
\[(V_i^*)^2 V_i = |T_i| = V_i^* V_i^2\]
by taking *-operation. Similarly, we have
\[V_i^2 V_i^* = |T_i^*| = V_i (V_i^*)^2.\]
Consequently, (3.1) turns out to be
\[V_1^* V_1^2 + V_2^* V_2^2 = V_1^2 V_1^* + V_2^2 V_2^*.\]  
(3.9)

Now, we are ready to construct the desired unitary operator. Let
\[U_1 = V_1 + P_R, \quad U_2 = V_2 + P_N, \quad U = U_1 - U_2,\]  
(3.10)
where \(R\) and \(N\) are defined by (1.3). Then by Lemma 3.1, (3.2) and (3.8), we have
\[
P = V_1 V_1^* + P_R, \quad I - P = V_2 V_2^* + P_N; \quad \tag{3.11}
\]
\[
Q = V_2 V_2^* + P_R, \quad I - Q = V_1 V_1^* + P_N. \quad \tag{3.12}
\]
It follows from (3.11) that
\[V_1 V_1^* P_R = V_2 V_2^* P_N = V_1 V_1^* V_2 V_2^* = V_1 V_1^* P_N = P_R V_2 V_2^* = P_R P_N = 0,\]
or equivalently,
\[V_1^* P_R = V_2^* P_N = V_1^* V_2 = V_1^* P_N = P_R V_2 = P_R P_N = 0.\]  
(3.13)

Similarly, it can be inferred from (3.12) that
\[V_2 P_R = V_1 P_N = V_2 V_1^* = V_2 P_N = P_R V_1^* = 0.\]  
(3.14)
It follows from (3.10)–(3.14) that
\[
U_1 U_2^* = U_1^* U_2 = 0, \quad U_1 U_1^* = P, \quad U_2 U_2^* = I - P, \quad \tag{3.15}
\]
\[
U_1^* U_1 + U_2^* U_2 = V_1^* V_1 + P_R + V_2^* V_2 + P_N = Q + I - Q = I. \quad \tag{3.16}
\]
Therefore \(UU^* = U^* U = I\), so the operator \(U\) defined by (3.10) is a unitary.

Finally, we check the validity of (3.3)–(3.4). According to (3.11)–(3.12), we have
\[
Q - P_R = V_2 V_2^*, \quad I - P - P_N = V_2 V_2^*, \quad \tag{3.17}
\]
\[
P - P_R = V_1 V_1^*, \quad I - Q - P_N = V_1 V_1^*; \quad \tag{3.18}
\]
and thus
\[V_1 V_1^* + V_2 V_2^* = I - P_R - P_N = V_1^* V_1 + V_2^* V_2.\]
The above equations together with (3.9)–(3.10) and (3.13)–(3.14) yield
\[U (Q - P_R) = (U_1 - U_2) V_2 V_2 = (V_1 + P_R - V_2 - P_N) V_2 V_2 = -V_2\]
where

\[ U(P - P_R) = (V_1 + P_R - V_2 - P_N)V_1^* = V_2^2V_1^* - V_2V_1^* \]

\[ V_1^*V_1^* - (I - P_R - P_N - V_2V_2^*) = V_1^2V_1^* + V_2^2V_2^* - V_2 \]

\[ V_1^*V_1^* + V_2^*V_2^* - V_2 = V_1^*V_1^* + (I - P_R - P_N - V_1^*V_1)V_2 - V_2 \]

\[ = (I - Q - P_N)U. \]

Therefore, (3.3)–(3.4) are satisfied.

**Remark 3.1** Let \( P, Q \in \mathcal{P}(H) \) be such that \((P, Q)\) is harmonious. In this case, the Halmos' two projections theorem (see [12, Theorem 3.3]) indicates that up to unitary equivalence, \( P \) and \( Q \) have the block matrix forms

\[
P = \begin{pmatrix}
I_{H_1} & 0 & 0 \\
0 & I_{H_2} & 0 \\
0 & 0 & I_{H_3}
\end{pmatrix},
\quad Q = \begin{pmatrix}
I_{H_1} & 0 \\
0 & I_{H_3} \\
& & 0
\end{pmatrix},
\]

where

\[
H_1 = \mathcal{R}, \quad H_2 = \mathcal{R}(P) \cap \mathcal{N}(Q), \quad H_3 = \mathcal{N}(P) \cap \mathcal{R}(Q), \quad H_4 = \mathcal{N},
\]

\[
H_5 = \mathcal{R}(P) \oplus (H_1 \oplus H_2), \quad H_6 = \mathcal{N}(P) \oplus (H_3 \oplus H_4),
\]

\[ Q_0 = \begin{pmatrix}
A & U_0A^{\frac{1}{2}}(I_{H_5} - A)^{\frac{1}{2}}U_0^* \\
U_0A^{\frac{1}{2}}(I_{H_5} - A)^{\frac{1}{2}}U_0^*
\end{pmatrix} \in \mathcal{L}(H_5 \oplus H_6), \quad (3.15)
\]

in which \( U_0 \in \mathcal{L}(H_5, H_6) \) is a unitary, \( A \in \mathcal{L}(H_5) \) is a positive contraction such that both \( A \) and \( I_{H_5} - A \) are injective and \( \mathcal{R}(A - A^2) = H_5 \), which implies that

\[
\mathcal{R}(A) = \mathcal{R}(I_{H_5} - A) = H_5.
\]

With the notations as above and that in the proof of Theorem 3.1, we have

\[
P_R = I_{H_5} \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0,
\]

\[
P_N = 0 \oplus 0 \oplus 0 \oplus I_{H_4} \oplus 0 \oplus 0,
\]

\[
T_1 = 0 \oplus I_{H_2} \oplus 0 \oplus 0 \oplus S_1, \quad V_1 = 0 \oplus I_{H_2} \oplus 0 \oplus 0 \oplus S_2,
\]

\[
T_2 = 0 \oplus 0 \oplus I_{H_3} \oplus 0 \oplus S_3, \quad V_2 = 0 \oplus 0 \oplus I_{H_3} \oplus 0 \oplus S_4,
\]

where

\[
S_1 = \begin{pmatrix}
I_{H_5} - A & -A^{\frac{1}{2}}(I_{H_5} - A)^{\frac{1}{2}}U_0^* \\
0 & 0
\end{pmatrix},
\]

\[
S_2 = \begin{pmatrix}
(I_{H_5} - A)^{\frac{1}{2}} & 0 \\
0 & A^{\frac{1}{2}}U_0^*
\end{pmatrix}.
\]


\[ S_3 = \begin{pmatrix}
0 & 0 \\
U_0A^{\frac{1}{2}}(I_{H_5} - A)^{\frac{1}{2}} & U_0(I_{H_5} - A)U_0^*
\end{pmatrix}, \]

\[ S_4 = \begin{pmatrix}
0 & 0 \\
U_0A^{\frac{1}{2}} & U_0(I_{H_5} - A)^{\frac{1}{2}}U_0^*
\end{pmatrix}. \]

In virtue of (3.10), we have

\[ U = V_1 + P_R - V_2 - P_{\mathcal{N}} = \text{diag}(X, Y), \]

where \( X = \text{diag}(I_{H_1}, I_{H_2}, -I_{H_3}, -I_{H_4}) \) and

\[ Y = S_2 - S_4 = \begin{pmatrix}
(I_{H_5} - A)^{\frac{1}{2}} & -A^{\frac{1}{2}}U_0^* \\
-U_0A^{\frac{1}{2}} & -U_0(I_{H_5} - A)^{\frac{1}{2}}U_0^*
\end{pmatrix}. \]

This gives the block matrix form of the unitary operator \( U \) satisfying (3.3)–(3.4).

**Remark 3.2** Since every closed linear subspace of a Hilbert space is orthogonally complemented, Theorem 3.1 is therefore always applicable to every pair of projections on a Hilbert space.

**Theorem 3.2** There exist projections \( P \) and \( Q \) on certain Hilbert \( C^* \)-module such that \( \mathcal{R} = \mathcal{N} = \{0\} \), whereas (3.3)–(3.4) have no common unitary operator solution.

**Proof** Following [8, Section 3], we put \( \mathfrak{B} = C([0, 1]; M_2(\mathbb{C})) \) and set

\[ \mathfrak{A} = \{ f \in \mathfrak{B} : f(0) \text{ and } f(1) \text{ are both diagonal} \}. \]  

As is shown in the proof of Theorem 2.2, \( \mathfrak{A} \) itself is a Hilbert \( \mathfrak{A} \)-module, and we can identify \( \mathcal{L}(\mathfrak{A}) \) with \( \mathfrak{A} \). Let \( H = \mathfrak{A} \) and \( Q \in \mathcal{P}(H) \) be determined by (2.7), and let \( P \in \mathcal{P}(H) \) be changed to

\[ P(t) = \begin{pmatrix}
c_t^2 & -s_tc_t \\
-s_tc_t & s_t^2
\end{pmatrix} \quad \text{for } t \in [0, 1], \]

where \( c_t \) and \( s_t \) are defined by (2.6).

Let \( \mathcal{R} \) and \( \mathcal{N} \) be defined by (1.3), and suppose that \( x \in \mathcal{N} \) is determined by \( x(t) = (x_{ij}(t))_{1 \leq i, j \leq 2} \) for \( t \in [0, 1] \). Utilizing

\[ P(t) + Q(t) = \begin{pmatrix}
2c_t^2 & 0 \\
0 & 2s_t^2
\end{pmatrix} \quad \text{and} \quad [P(t) + Q(t)]x(t) = 0 \]

for \( t \in [0, 1] \), we obtain \( x_{ij}(t) = 0 \) for \( i, j \in \{1, 2\} \) and every \( t \in (0, 1) \), which imply that \( x_{ij} = 0 \) for \( 1 \leq i, j \leq 2 \), since all functions considered are continuous on \([0, 1]\). This shows that \( \mathcal{N} = \{0\} \). In view of \( \mathcal{R} = \mathcal{N}(I - P) \cap \mathcal{N}(I - Q) \), the proof of \( \mathcal{R} = \{0\} \) is similar.

Suppose that \( U \) determined by \( U(t) = (U_{ij}(t))_{1 \leq i, j \leq 2} \) is a unitary in \( H \), which satisfies both (3.3) and (3.4). Due to \( P_{\mathcal{R}} = 0 \) and \( P_{\mathcal{N}} = 0 \), (3.3)–(3.4) are simplified as

\[ UP = (I - Q)U \quad \text{and} \quad UQ = (I - P)U, \]

or equivalently,

\[ U(Q - P) = (Q - P)U \quad \text{and} \quad U(P + Q) = (2I - P - Q)U. \]  

(3.18)
Substituting

\[(Q - P)(t) = \begin{pmatrix} 0 & 2stc_t \\ 2stc_t & 0 \end{pmatrix}, \quad t \in [0, 1] \]

into the first equation in (3.18) yields

\[U_{12}(t) = U_{21}(t), \quad U_{11}(t) = U_{22}(t), \quad t \in [0, 1].\]

Combining the above equations with the expression of \(P(t) + Q(t)\) given in (3.17) and the second equation in (3.18), we arrive at

\[U_{11}(t) (s_t^2 - s_t^2) = 0, \quad \forall t \in [0, 1],\]

hence \(U_{11}(t) \equiv 0\) for \(t \in [0, 1]\) by the continuity of \(U_{11}\). Consequently,

\[U(t) = \begin{pmatrix} 0 & U_{12}(t) \\ U_{12}(t) & 0 \end{pmatrix}.\]

This together with (3.16) yields \(U(0) = U(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). It is a contradiction, since \(U\) is a unitary in \(H\) which ensures that all the \(2 \times 2\) matrices \(U(t)(t \in [0, 1])\) are unitary.

### 4 \(C^*\)-Isomorphisms Associated with Two Projections

Unless otherwise specified, throughout this section \((P, Q, H)\) is a matched triple, \(i(P, Q, H)\) and \(o(P, Q, H)\) are its inner algebra and outer algebra (see Definitions 1.2). It is clear that

\[i(P, Q, H) = \text{span}\{X^{(P; Q, k)} : X \in \{A, B, C, D\}, k \in \mathbb{Z}_+\}, \quad (4.1)\]

where \(R\) (also \(N\)) is defined by (1.3), \(P_R = P \land Q\) and

\[A^{(P, Q, k)} = [(P - P_R)(Q - P_R)]^k, \quad (4.2)\]

\[B^{(P, Q, k)} = A^{(P, Q, k)}(P - P_R), \quad (4.3)\]

\[C^{(P, Q, k)} = (A^{(P, Q, k)})^* = A^{(Q, P, k)}, \quad (4.4)\]

\[D^{(P, Q, k)} = C^{(P, Q, k)}(Q - P_R) = B^{(Q, P, k)} \quad (4.5)\]

with the convention that \(A^{(P, Q, 0)} = I\). For each \(k \geq 1\), by utilizing \(P_R \leq P\) and \(P_R \leq Q\) we obtain

\[A^{(P, Q, k)} = (PQ - P_R)^k = (PQ)^k - P_R, \quad (4.6)\]

which gives

\[A^{(P, Q, k)}(A^{(P, Q, k)})^* = [(PQ)^k - P_R][(QP)^k - P_R] = (PQ)^k(QP)^k - P_R = (PQP - P_R)^{2k-1} - P_R \quad (4.7)\]

It follows that

\[A^{(P, Q, k)}(A^{(P, Q, k)})^* = [A^{(P, Q, 1)}(A^{(P, Q, 1)})^*]^{2k-1}, \quad \forall k \geq 1. \quad (4.8)\]
Similarly,
\[
B^{(P,Q,k)} = [(PQ)^k - P_R(P - P_R) = (PQP)^k - P_R = (PQP - P_R)^k
\]
\[
= [A^{(P,Q,1)}(A^{(P,Q,1)})^*]^k, \quad \forall k \geq 1.
\] (4.9)

Now, let \((\pi, X)\) be a faithful representation of \(o(P, Q, H)\). Replacing \(X\) with \(\pi(I)X\) if necessary, in what follows we always assume that \(\pi\) is unital. The infimum of \(\pi(P)\) and \(\pi(Q)\), and its subtraction by \(\pi(P_R)\) are denoted simply by \(P_{R\pi}\) and \(\tilde{P}_R\), respectively, that is,
\[
P_{R\pi} = P_R(\pi(P))\cap \pi(\pi(Q)), \quad \tilde{P}_R = P_{R\pi} - \pi(P_R).
\] (4.10)

It is clear that \(\pi(P_R) \leq P_{R\pi}\), so \(\tilde{P}_R\) defined as above is a projection. By (4.2) we have \(A^{(\pi(P), \pi(Q), 0)} = I\) and when \(k \geq 1\),
\[
A^{(\pi(P), \pi(Q), k)} = [(\pi(P) - P_{R\pi})(\pi(Q) - P_{R\pi})]^k = [\pi(P)\pi(Q)]^k - P_{R\pi}.
\]
This together with (4.6) and (4.10) gives
\[
\pi(A^{(P,Q,k)}) = A^{(\pi(P), \pi(Q), k)} + \tilde{P}_R.
\]

The derivation above shows that
\[
\pi(X^{(P,Q,k)}) = X^{(\pi(P), \pi(Q), k)} + \tilde{P}_R \quad \text{whenever} \quad X^{(P,Q,k)} \neq I.
\] (4.11)

Note that \(\tilde{P}_R\) is a projection, and
\[
X^{(\pi(P), \pi(Q), k)} \cdot \tilde{P}_R = \tilde{P}_R \cdot X^{(\pi(P), \pi(Q), k)} = 0 \quad \text{whenever} \quad X^{(P,Q,k)} \neq I,
\] (4.12)
so by (4.11) together with the observation \(\|I\| = 1 \geq \|\tilde{P}_R\|\), we arrive at
\[
\|X^{(P,Q,k)}\| = \|\pi(X^{(P,Q,k)})\|
\]
\[
= \max\{\|X^{(\pi(P), \pi(Q), k)}\|, \|\tilde{P}_R\|\}, \quad \forall k \geq 0.
\] (4.13)

We are now ready to derive a couple of norm equations. The first one reads as follows.

**Lemma 4.1** Let \((\pi, X)\) be a faithful representation of \(o(P, Q, H)\). Then for every \(X \in \{A, B, C, D\}\) and \(k \in \mathbb{Z}_+\), we have
\[
\|X^{(P,Q,k)}\| = \|X^{(\pi(P), \pi(Q), k)}\|.
\] (4.14)

**Proof** First, we prove that
\[
\|A^{(P,Q,k)}\| = \|A^{(\pi(P), \pi(Q), k)}\|, \quad \forall k \in \mathbb{Z}_+.
\] (4.15)
The case of \(k = 0\) is trivial, so we start with \(k = 1\). From (4.11), we have
\[
\|A^{(P,Q,1)}\|^2 = \|\pi(A^{(P,Q,1)}))\|^2 = \|\pi(A^{(P,Q,1)})(A^{(P,Q,1)})^*\|
\]
\[
= \|A^{(\pi(P), \pi(Q), 1)} + \tilde{P}_R)(A^{(\pi(P), \pi(Q), 1)})^* + \tilde{P}_R\|
\]
\[
= \|A^{(\pi(P), \pi(Q), 1)}(A^{(\pi(P), \pi(Q), 1)})^* + \tilde{P}_R\|
\]
\[
= \max\{\|A^{(\pi(P), \pi(Q), 1)}(A^{(\pi(P), \pi(Q), 1)})^*\|, \|\tilde{P}_R\|\}
\]
\[
= \max\{\|A^{(\pi(P), \pi(Q), 1)}\|^2, \|\tilde{P}_R\|\},
\] (4.16)
which implies that
\[
\|A^{(P,Q,1)}\| = \|A^{(\pi(P),\pi(Q),1)}\| \tag{4.17}
\]
whenever \(\|A^{(\pi(P),\pi(Q),1)}\| = 1\). Suppose that \(\|A^{(\pi(P),\pi(Q),1)}\| < 1\). Then according to (4.7)–(4.8), we have
\[
\|(\pi(P)\pi(Q))-P_{R\pi}\| = \|A^{(\pi(P),\pi(Q),1)}(A^{(\pi(P),\pi(Q),1)})^*\|^{2k-1}
\geq \|A^{(\pi(P),\pi(Q),1)}\|^2 \to 0 \quad \text{as} \quad k \to \infty.
\]
It follows that \(P_{R\pi}\) is contained in the \(C^*\)-algebra \(\pi(C^*_{(P,Q)})\), so \(\pi^{-1}(P_{R\pi}) \leq \pi^{-1}(\pi(P)) = P\) and \(\pi^{-1}(P_{R\pi}) \leq Q\) as well. Therefore, \(\pi^{-1}(P_{R\pi}) \leq P_{R}\), and thus \(P_{R\pi} \leq \pi(P_{R}) \leq P_{R\pi}\).

By (4.3), both \(B^{(\pi(P),\pi(Q),0)}\) and \(B^{(\pi(P),\pi(Q),0)}\) are projections, and
\[
\|B^{(\pi(P),\pi(Q),0)}\| = 0 \iff P \leq Q \iff \pi(P) \leq \pi(Q) \iff \|B^{(\pi(P),\pi(Q),0)}\| = 0.
\]
This shows the validity of (4.18) for \(k = 0\). Suppose that \(k \geq 1\). Then by (4.9), we can get
\[
\|B^{(P,Q,k)}\| = \|A^{(P,Q,1)}\|^2 \|A^{(\pi(P),\pi(Q),1)}\|^2 \|B^{(\pi(P),\pi(Q),k)}\|.
\]

Exchanging \(P\) with \(Q\), we conclude that for every \(k \in \mathbb{Z}_+\),
\[
\|C^{(P,Q,k)}\| = \|A^{(Q,P,1)}\| = \|A^{(\pi(Q),\pi(P),1)}\| = \|C^{(\pi(P),\pi(Q),1)}\|,
\|D^{(P,Q,k)}\| = \|B^{(Q,P,1)}\| = \|B^{(\pi(Q),\pi(P),1)}\| = \|D^{(\pi(P),\pi(Q),1)}\|.
\]
This completes the proof of (4.14).

**Corollary 4.1** Let \((\pi, X)\) be a faithful representation of \(o(P,Q,H)\). Then for every \(n \in \mathbb{N}\), \(X_i \in \{A, B, C, D\}\) and \(k_i \in \mathbb{Z}_+\) \((1 \leq i \leq n)\), we have
\[
\left\|\prod_{i=1}^{n} X_i^{(P,Q,k_i)}\right\| = \left\|\prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)}\right\|. \tag{4.19}
\]

**Proof** From the definition of \(X^{(P,Q,k)}\) given by (4.2)–(4.5), it is clear that
\[
\prod_{i=1}^{n} X_i^{(P,Q,k_i)} = Z^{(P,Q,k)} \quad \text{and} \quad \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} = Z^{(\pi(P),\pi(Q),k)}
\]
for some \(Z \in \{A, B, C, D\}\) and \(k \in \mathbb{Z}_+\). Due to (4.14), the desired norm equation follows.

We provide a technical lemma as follows.
Lemma 4.2 Let \((P, Q)\) be a harmonious pair of projections on \(H\). Suppose that \(n \in \mathbb{N}\), \(X_i \in \{A, B, C, D\}\) and \(k_i \in \mathbb{Z}_+\) \((1 \leq i \leq n)\) are given such that

\[
\left\| \prod_{i=1}^{n} X_i^{(P, Q, k_i)} \right\| = 1. \tag{4.20}
\]

Then for every \(\lambda_i \in \mathbb{C}\) \((1 \leq i \leq n)\), we have

\[
\left| \sum_{i=1}^{n} \lambda_i \right| \leq \left\| \sum_{i=1}^{n} \lambda_i X_i^{(P, Q, k_i)} \right\|. \tag{4.21}
\]

Proof Denote by \(\lambda = \sum_{i=1}^{n} \lambda_i\). The verification of

\[
|\lambda| \leq \left\| \sum_{i=1}^{n} \lambda_i X_i^{(P, Q, k_i)} \right\|
\]

will be carried out by taking several cases into consideration.

Case 1 \(X_i^{(P, Q, k_i)} \in \{I, P - P_R\}\) for all \(i \in \{1, 2, \cdots, n\}\). If \(X_i^{(P, Q, k_i)} \equiv I\), then (4.21) is obviously satisfied. Otherwise, we have

\[
\prod_{i=1}^{n} X_i^{(P, Q, k_i)} = P - P_R,
\]

so according to (4.20) we obtain \(\|P - P_R\| = 1\). It follows that

\[
\left\| \sum_{i=1}^{n} \lambda_i X_i^{(P, Q, k_i)} \right\| \geq \left\| (P - P_R) \sum_{i=1}^{n} \lambda_i X_i^{(P, Q, k_i)} \right\| = \|\lambda(P - P_R)\| = |\lambda|.
\]

Case 2 \(X_i^{(P, Q, k_i)} \in \{I, Q - P_R\}\) for all \(i \in \{1, 2, \cdots, n\}\). The same verification gives (4.21).

Case 3 There exist \(i_1, i_2 \in \{1, 2, \cdots, n\}\) such that \(X_i^{(P, Q, k_i)} \notin \{I, P - P_R\}\) and \(X_i^{(P, Q, k_i)} \notin \{I, Q - P_R\}\). In this case, firstly we show that

\[
\|A^{(P, Q, 1)}\| = 1. \tag{4.22}
\]

Subcase 1 \(k_{i_1} \neq 0\) or \(k_{i_2} \neq 0\). Without loss of generality, we may assume that \(k_{i_1} \neq 0\). In this subcase,

\[
1 \geq \|A^{(P, Q, 1)}\| = \|C^{(P, Q, 1)}\| \geq \left\| X_i^{(P, Q, k_i)} \right\| \geq \left\| \prod_{i=1}^{n} X_i^{(P, Q, k_i)} \right\| = 1.
\]

Thus, (4.22) is satisfied.

Subcase 2 \(k_{i_1} = k_{i_2} = 0\). In this subcase, we have \(X_i^{(P, Q, k_i)} = Q - P_R\) and \(X_i^{(P, Q, k_i)} = P - P_R\), which mean that

\[
\prod_{i=1}^{n} X_i^{(P, Q, k_i)} = W_1 A^{(P, Q, 1)} W_2 \quad \text{or} \quad \prod_{i=1}^{n} X_i^{(P, Q, k_i)} = W_1 C^{(P, Q, 1)} W_2
\]

for some contractions \(W_1, W_2 \in \mathcal{L}(H)\). As is shown in Subcase 1, (4.22) is also satisfied.
Since \((P, Q)\) is harmonious, the Halmos’ two projections theorem is applicable. Following
the notations as in Remark 3.1, we have
\[
\begin{align*}
P - P_R &= 0 \oplus I_{H_2} \oplus 0 \oplus 0 \oplus I_{H_5} \oplus 0, \\
Q - P_R &= 0 \oplus 0 \oplus I_{H_3} \oplus 0 \oplus Q_0, \\
A^{(P, Q, 1)} &= 0 \oplus 0 \oplus 0 \oplus 0 \oplus S,
\end{align*}
\]
where \(Q_0\) is defined by (3.15) and \(S\) is given by
\[
S = \begin{pmatrix} A & A^* \left( I_{H_5} - A \right)^{\frac{1}{2}} U_0^* \end{pmatrix}.
\]
Based on the above block matrices, we have
\[
\|A\| = \left\| \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|SS^*\| = \|S\|^2 = \|A^{(P, Q, 1)}\|^2 = 1.
\]
This together with the positivity and contraction of \(A\) implies that \(1 \in \text{sp}(A)\), where \(\text{sp}(A)\) denotes the spectrum of \(A\).
It is easy to verify that for every \(X \in \{A, B, C, D\}\) and \(k \in \mathbb{Z}_+\), there exists \(r \in \mathbb{N}\) depending on \(X\) and \(k\) such that
\[
A^{(P, Q, 1)} X^{(P, Q, k)} (P - P_R) = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} A^r & 0 \end{pmatrix}.
\]
Consequently,
\[
\left\| \sum_{i=1}^{n} \lambda_i X^{(P, Q, k_i)} \right\| \geq \left\| A^{(P, Q, 1)} \left( \sum_{i=1}^{n} \lambda_i X^{(P, Q, k_i)} \right) (P - P_R) \right\|
= \left\| \begin{pmatrix} \sum_{i=1}^{n} \lambda_i A^{r_i} \\ 0 \end{pmatrix} \right\| = \left\| \sum_{i=1}^{n} \lambda_i A^{r_i} \right\|\]
for some \(r_i \in \mathbb{N}\) \((1 \leq i \leq n)\). Let \(f(t) = \sum_{i=1}^{n} \lambda_i t^{r_i}\) for \(t \geq 0\). Then
\[
\left\| \sum_{i=1}^{n} \lambda_i A^{r_i} \right\| = \max\{|f(t)|: t \in \text{sp}(A)\} \geq |f(1)| = |\lambda|.
\]
Combining the above inequality with (4.25) gives (4.21).
Along the same line, another technical lemma can be provided as follows.

**Lemma 4.3** Let \((P, Q)\) be a harmonious pair of projections on \(H\) such that \(PQ \neq QP\). Suppose that \(n \in \mathbb{N}\), \(X_i \in \{A, B, C, D\}\) and \(k_i \in \mathbb{Z}_+\) are given such that (4.20) is satisfied and \(X_i^{(P, Q, k_i)} \neq I\) for all \(i \in \{1, 2, \ldots, n\}\). Then for every \(\lambda_i \in \mathbb{C}\) \((0 \leq i \leq n)\), we have
\[
|\lambda_0| \leq \left\| \lambda_0 (I - P_R) + \sum_{i=1}^{n} \lambda_i X_i^{(P, Q, k_i)} \right\|.
\]

\(^1\)In some cases, it may happen that the closed subspaces \(H_5\) and \(H_6\) constructed for the Halmos decomposition are trivial, that is, \(H_5 = H_6 = \{0\}\). Due to (4.22), both \(H_5\) and \(H_6\) are non-trivial, hence \(A^{(P, Q, 1)}\) has the form (4.23).
It follows from (4.27) that

$$\lambda = \sum_{i=0}^{n} \lambda_i, \quad W = \lambda_0(I - P_R) + \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)}. \quad (4.27)$$

Case 1 \( X_i^{(P,Q,k_i)} = P - P_R \) for all \( i \in \{1, 2, \cdots, n\} \). In this case,

$$W = \lambda_0(I - P) + \lambda(P - P_R).$$

Since \( PQ \neq QP \), we have \( I - P \neq 0 \), hence

$$\|W\| \geq \|\lambda_0(I - P)\| = |\lambda_0|.$$  

Case 2 \( X_i^{(P,Q,k_i)} = Q - P_R \) for all \( i \in \{1, 2, \cdots, n\} \). As is shown in the above case, we have \( \|W\| \geq |\lambda_0| \).

Case 3 There exist \( i_1, i_2 \in \{1, 2, \cdots, n\} \) such that \( X_{i_1}^{(P,Q,k_{i_1})} \neq P - P_R \) and \( X_{i_2}^{(P,Q,k_{i_2})} \neq Q - P_R \). Following the notations as in Remark 3.1, we have

$$H = \bigoplus_{i=1}^{6} H_i$$

and up to unitary equivalence, every operator \( Y \in \mathcal{L}(H) \) has the matrix form \( Y = (Y_{ij})_{1 \leq i,j \leq 6} \) with \( Y_{ij} \in \mathcal{L}(H_j, H_i) \). Let the linear map \( \phi: \mathcal{L}(H) \to \mathcal{L}(H_6) \) be defined by \( \phi(Y) = Y_{66} \). According to the definition of \( X^{(P,Q,k)} \) given by (4.2)–(4.5), we have \( \phi(B^{(P,Q,0)}) = 0 \) and

$$\phi(A^{(P,Q,k)}) = \phi(B^{(P,Q,k)}) = \phi(C^{(P,Q,k)}) = 0$$

for every \( k \geq 1 \). Furthermore, direct computations yield

$$\phi(D^{(P,Q,k)}) = U_0(I - A)A^kU_0^*, \quad \forall k \in \mathbb{Z}_+.$$  

It follows from (4.27) that

$$\phi(W) = U_0 \left[ \lambda_0 I + \sum_{j} \lambda_{i_j} (I - A)A^{k_{i_j}} \right] U_0^*,$$

where \( i_j \) is chosen in \( \{1, 2, \cdots, n\} \) whenever \( X_{i_j}^{(P,Q,k_{i_j})} = D^{(P,Q,k_{i_j})} \). Let

$$f(t) = \lambda_0 + \sum_{j} \lambda_{i_j} (1 - t)t^{k_{i_j}}, \quad t \geq 0.$$  

Since \( 0 \leq A \leq I \) and \( 1 \in \text{sp}(A) \) (see (4.24)), we have

$$\|W\| \geq \|\phi(W)\| = \max\{|f(t)|: t \in \text{sp}(A)\} \geq |f(1)| = |\lambda_0|.$$  

This completes the proof of (4.26).

Now, we provide the main result of this section as follows.

**Theorem 4.1** For each faithful representation \((\pi, X)\) of \( o(P,Q,H) \), a faithful representation \((\tilde{\pi}, X)\) of \( i(P,Q,H) \) can be induced such that \( \tilde{\pi}(I) = \pi(I) \), and

$$\tilde{\pi}(P - P_R) = \pi(P) - \pi(P) \wedge \pi(Q), \quad \tilde{\pi}(Q - P_R) = \pi(Q) - \pi(P) \wedge \pi(Q). \quad (4.28)$$

\(^2\)If \( X_i \neq D \) for all \( i \in \{1, 2, \cdots, n\} \), then \( \phi(W) = \lambda_0 I \), so in this case \( \|\phi(W)\| = |\lambda_0| \).
Proof According to (4.1), we need only to prove that $\|T\| = \|T\|$ for every $n \in \mathbb{N}$, $X_i \in \{A, B, C, D\}$, $k_i \in \mathbb{Z}_+$ and $\lambda_i \in \mathbb{C}$ ($0 \leq i \leq n$) such that $X_i^{(P,Q,k_i)} \not= I$ for all $i \geq 1$, where\(^3\)

$$T = \lambda_0 I + \sum_{i=1}^{n} \lambda_i X_i^{(P,Q,k_i)}, \quad \mathcal{T} = \lambda_0 I + \sum_{i=1}^{n} \lambda_i X_i^{(\pi(P),\pi(Q),k_i)}. \tag{4.29}$$

If $\widetilde{P_R} = 0$, then we see from (4.11) that $\pi(T) = \mathcal{T}$, which gives $\|T\| = \|\mathcal{T}\|$ as $\pi$ is faithful. In what follows, we assume that $\widetilde{P_R} \not= 0$. In this case, we have $PQ \not= QP$. Otherwise, $P_R = PQ$ and $P_R\pi = \pi(P)\pi(Q) = \pi(P_R)$, which contradicts the assumption of $\widetilde{P_R} \not= 0$. Let $\lambda = \sum_{i=0}^{n} \lambda_i$. By (4.11) we have

$$\pi(T) = \mathcal{T} + (\lambda - \lambda_0)\widetilde{P_R} = L + \lambda\widetilde{P_R}, \tag{4.30}$$

where

$$L = \lambda_0 (I - \widetilde{P_R}) + \sum_{i=1}^{n} \lambda_i X_i^{(\pi(P),\pi(Q),k_i)} = Z + \lambda_0 \pi(P_R),$$

in which

$$Z = \lambda_0 (I - P_R\pi) + \sum_{i=1}^{n} \lambda_i X_i^{(\pi(P),\pi(Q),k_i)}.$$

Hence

$$\|L\| = \max\{\|Z\|, |\lambda_0| \|\pi(P_R)\|\}, \tag{4.31}$$

$$\|T\| = \|\pi(T)\| = \max\{\|L\|, |\lambda| \|\widetilde{P_R}\|\} = \max\{\|L\|, |\lambda|\}, \tag{4.32}$$

$$\|T\| = \|L + \lambda_0\widetilde{P_R}\| = \max\{\|L\|, |\lambda_0| \|\widetilde{P_R}\|\} = \max\{\|L\|, |\lambda_0|\}. \tag{4.33}$$

Let

$$\alpha = \left\| \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} \right\|.$$  

By (4.19) and (4.11)–(4.12), we have

$$\alpha = \left\| \pi \left( \prod_{i=1}^{n} X_i^{(P,Q,k_i)} \right) \right\| = \left\| \prod_{i=1}^{n} \pi(X_i^{(P,Q,k_i)}) \right\| = \left\| \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} + \widetilde{P_R} \right\| = \max\{\alpha, \|\widetilde{P_R}\|\} = \max\{\alpha, 1\}.$$  

Hence $\alpha = 1$, which apparently gives

$$\left\| \prod_{i=1}^{n} X_i^{(\pi(P),\pi(Q),k_i)} \right\| = 1$$

by setting $X_0^{(\pi(P),\pi(Q),k_0)} = I$. It is notable that $\pi(P)$ and $\pi(Q)$ are projections acting on a Hilbert space, so $(\pi(P), \pi(Q))$ is harmonious. It follows from Lemmas 4.2–4.3 that

$$\|\mathcal{T}\| \geq |\lambda| \quad \text{and} \quad \|Z\| \geq |\lambda_0|.$$

\(^3\)When $\lambda_0 = 0$, the first term in $T$ and $\mathcal{T}$ will disappear.
So by (4.31) we have \( \|L\| \geq \|Z\| \geq |\lambda_0| \), which leads by (4.33) to \( \|T\| = \|L\| \). Combining this equality with \( \|T\| \geq |\lambda| \) and (4.32), we arrive at \( \|T\| = \|L\| \).

Under the restriction of \( \lambda_0 = 0 \) in (4.29), a corollary can be derived immediately as follows.

**Corollary 4.2** Let \( C^*(P, Q, P_R) \) and \( C^*(P - P_R, Q - P_R) \) denote the \( C^* \)-subalgebras of \( \mathcal{L}(H) \) generated by elements in \( \{P, Q, P_R\} \) and \( \{P - P_R, Q - P_R\} \), respectively. Then each faithful representation \( (\pi, X) \) of \( C^*(P, Q, P_R) \) can induce a faithful representation \( (\tilde{\pi}, X) \) of \( C^*(P - P_R, Q - P_R) \) such that (4.28) is satisfied.

In the derivations given as above, we merely consider the meaningfulness of \( P_R \). At this moment, we do not know whether there exist projections \( P \) and \( Q \) such that \( R \) is orthogonally complemented, whereas \( N \) fails to be orthogonally complemented\(^4\). To give a partial answer, we need an auxiliary lemma, whose proof is given for the sake of completeness.

**Lemma 4.4** Let \( P, Q \in \mathcal{P}(H) \) be such that the sequence \( \{(PQP)^n\}_{n=1}^{\infty} \) converges to \( T \in \mathcal{L}(H) \) in norm-topology. Then \( T \) is a projection such that \( \mathcal{R}(T) = \mathcal{R} \), where \( \mathcal{R} \) is defined by (1.3).

**Proof** Clearly, \( T \) is a projection such that \( \mathcal{R} \subseteq \mathcal{R}(T) \) and \( PT = T \). So it needs only to show that \( QT = T \), or equivalently, \( [(I - Q)T]^{\ast}(I - Q)T = 0 \), which can be derived from the equations

\[
(PQP)^n(I - Q)(PQP)^n = (PQP)^{2n} - (PQP)^{2n+1}, \quad \forall n \in \mathbb{N}.
\]

**Corollary 4.3** Let \( (P, Q, H) \) be a matched triple such that \( \|(P - P_R)(Q - P_R)\| \leq 1 \). Then \( (I - Q, I - P, H) \) is also a matched triple.

**Proof** Choose any faithful unital representation \( (\pi, X) \) of \( \mathcal{L}(H) \). Let \( P_{R\pi} \) be defined by (4.10), and put

\[
P_{N\pi} = (\pi(I - Q)) \wedge (\pi(I - P)), \\
S = [I - \pi(Q) - P_{N\pi}][I - \pi(P) - P_{N\pi}], \\
T = [\pi(P) - P_{R\pi}][\pi(Q) - P_{R\pi}], \\
W = (I - P)(I - Q)(I - P).
\]

Then

\[
P_{N\pi} = P_{R(\pi(I - Q)) \cap \mathcal{R}(\pi(I - P))} = P_{\mathcal{R}(I - \pi(Q)) \cap \mathcal{R}(I - \pi(P))} = P_{\mathcal{N}(\pi(P)) \cap \mathcal{N}(\pi(Q))}.
\]

By Theorems 3.1 and 4.1, there exists a unitary \( U \in \mathbb{B}(X) \) such that

\[
\|S\| = \|U^*TU\| = \|T\| = \|(P - P_R)(Q - P_R)\| \leq 1,
\]

hence \( \|(S^*S)^n\| = \|S^*S\|^n = \|S\|^{2^n} \to 0 \) as \( n \to \infty \). For each \( n \in \mathbb{N} \), it is clear that

\[
(S^*S)^n = (\pi(W))^n - P_{N\pi},
\]

so \( \{(\pi(W))^n\}_{n=1}^{\infty} \) is norm-convergent and thus is a Cauchy sequence, and so does \( \{W^n\}_{n=1}^{\infty} \) by the faithfulness of \( \pi \). Due to the completeness of \( \mathcal{L}(H) \), \( \{W^n\}_{n=1}^{\infty} \) is norm-convergent. The assertion then follows from Lemma 4.4.

\(^4\) Alternatively, is it possible to find a matched triple \( (P, Q, H) \) such that \( (I - Q, I - P, H) \) is not a matched triple?
Remark 4.1 Our next example shows that there exists a matched triple \((P, Q, H)\) such that the sequence \(\{(PQP)^n\}_{n=1}^{\infty}\) does not converge strongly to \(P_R\).

Example 4.1 Let \(A, H, P\) and \(Q\) be as in Example 2.1. According to (2.8), we have \(P_R = 0\). From (2.7) we obtain
\[
\|(PQP)^n\|(0) = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)
\]
for every \(n \geq 1\). It follows that
\[
\|(PQP)^nP\| \geq \|(PQP)^nP\|(0) = 1, \quad \forall n \geq 1.
\]
Thus, \(\{(PQP)^n\}_{n=1}^{\infty}\) does not converge strongly to zero.

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