Exchanging role of the phase space and symmetry group of integrable Hamiltonian systems related to Lie bialgebras of bi-symplectic types

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Abstract

We construct integrable Hamiltonian systems with Lie bialgebras \((\mathfrak{g}, \widetilde{\mathfrak{g}})\) of the bi-symplectic type for which the Poisson-Lie groups \(G\) play the role of the phase spaces, and their dual Lie groups \(\widetilde{G}\) play the role of the symmetry groups of the systems. We give the new transformations to exchange the role of phase spaces and symmetry groups and obtain the relations between integrals of motions of these integrable systems. Finally, we give some examples of real four-dimensional Lie bialgebras of bi-symplectic type.

keywords: Integrable Hamiltonian systems, Lie bialgebra, Bi-symplectic structure.

1 Introduction

A Hamiltonian system with \(N\) degrees of freedom is integrable in the sense of Liouville theorem if it has \(N\) invariants (globally defined and functionally independent) in involution (for review see [1], [2], [3]). If an integrable Hamiltonian system is invariant under some transformations on the phase space variables (such that these transformations construct a group); then we have a symmetry group for this Hamiltonian system [3]. In most of the symmetric integrable Hamiltonian systems, the symmetry group is a Lie group. A Lie group with compatible Poisson structure on it; is called Poisson-Lie group [4]. Poisson-Lie groups and their algebraic forms i.e. Lie bialgebras play important roles in constructing of classical integrable Hamiltonian systems [5]. In [6], we have constructed integrable and superintegrable Hamiltonian systems for which the symmetry Lie groups are also the phase spaces of the systems. Also in [7], we classified all four-dimensional real Lie bialgebras \((\mathfrak{g}, \widetilde{\mathfrak{g}})\) of symplectic type and also give two examples as the physical applications, such that for these integrable systems the Poisson-Lie groups \(G\) play the role of the phase spaces, and their dual Lie groups \(\widetilde{G}\) play the role of the symmetry Lie groups of the systems. In this work, we give new transformations for exchanging the role of the phase space and the symmetry Lie group. We use Lie bialgebras of bi-symplectic types for constructing of the integrable Hamiltonian systems for which the Poisson-Lie groups \(G\) play the role of the phase spaces, and their dual Lie groups \(\widetilde{G}\) play the role of the symmetry groups of the systems; then using these of transformation, the role of \(G\) and \(\widetilde{G}\) will be exchanged. The outline of the paper is as follows. For self-containing of the paper, we review the construction of an integrable Hamiltonian system by using Lie bialgebras in section 2. Then, in section 3, we consider integrable Hamiltonian systems related to Lie bialgebras of bi-symplectic types and present new transformations that exchange the role of phase spaces and symmetry Lie groups. In section 4, we give some examples related to real four-dimensional Lie bialgebras of bi-symplectic types [7].

2 Review of the construction of integrable Hamiltonian system with Lie bialgebra

For introducing the notations, let us have a survey on some definitions about Lie bialgebras [4] and related integrable Hamiltonian systems that can be constructed from them (for a review see [5]).
Definition[3]: A Lie bialgebra is a Lie algebra \( g \) with a skew-symmetric linear map \( \delta : g \to g \otimes g \) such that:

a) \( \delta \) is a one-cocycle, i.e.:
\[
\delta([X,Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)], \quad \forall X, Y \in g.
\]

where 1 is the identity map on \( g \).

b) \( \delta^\mathbf{t} : \hat{g} \otimes \hat{g} \to \hat{g} \) is a Lie bracket on \( \hat{g} \) (\( \hat{g} \) is the dual space of the vector space \( g \)):
\[
(\xi \otimes \eta, \delta(X)) = (\delta^\mathbf{t}(\xi \otimes \eta), X) = ([\xi, \eta]_g, X), \quad \forall X \in g; \xi, \eta \in \hat{g},
\]

where \((\cdot, \cdot)\) is a standard inner product between \( g \) and \( \hat{g} \). The Lie bialgebra defined in this way will be denoted by \((g, \hat{g})\) or \((g, \delta)\). In terms of \( \{X_i\} \) and \( \{\hat{X}^i\} \) (the bases of the Lie algebras \( g \) and \( \hat{g} \) respectively), we have the following commutation relations [5]:
\[
[X_i, X_j] = f_{ij}^k X_k, \quad [\hat{X}^i, \hat{X}^j] = \hat{f}^{ij}_k \hat{X}^k,
\]
\[
[X_i, \hat{X}^j] = \hat{f}^{jk}_i X_k + f_{ki}^j \hat{X}^k,
\]

such that \( \delta(X_i) = \hat{f}^{jk}_i X_k \otimes X_k \) and on the vector space \( D = g \otimes \hat{g} \) we have a Lie algebra structure and ad-invariant isotropic bilinear form \( < \cdot, \cdot > \):
\[
<X_i, X_j> = <\hat{X}^i, \hat{X}^j> = 0, \quad <X_i, \hat{X}^j> = \delta^i_j.
\]

the triple \((D, g, \hat{g})\) is called Manin triple [5].

For the coboundary Lie bialgebra, there is an \( r \)-matrix \( (r \in g \otimes g) \) such that:
\[
\delta(X) = [1 \otimes X + X \otimes 1, r],
\]

if the \( r \)-matrix \( r = r^{ij} X_i \otimes X_j \) satisfy the classical Yang-Baxter equation (CYBE) [8]:
\[
[r, r] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0;
\]

(6)

(\( r_{12} = r^{ij} X_i \otimes X_j \otimes 1, r_{13} = r^{ij} X_i \otimes 1 \otimes X_j \) and \( r_{23} = r^{ij} 1 \otimes X_i \otimes X_j \)); then the Lie bialgebra \((g, \hat{g})\) is called triangular Lie bialgebra [5]. The CYBE (6) can be rewritten in the term of structure constants \( f_{ij}^k \) as:
\[
r^{ij} r^{kl} f_{ik}^m + r^{mi} r^{kl} f_{ik}^j + r^{mi} r^{jk} f_{ik}^l = 0.
\]

(7)

Now, let us consider the method of constructing classical dynamical system which have symmetry Lie group. Let us consider 2n dimensional manifold \( M \) with the symplectic structure \( \omega_{ij} \) and local coordinate \( \{x_i\} \) as a phase space. The Poisson bracket \( \{\cdot, \cdot\} \) (defined by symplectic structure \( \omega_{ij} \)) of arbitrary functions \( f, g \in C^\infty(M) \) is given by
\[
\{f, g\} = P^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}
\]

(8)

where \( P^{ij} \) is the inverse of the matrix \( \omega_{ij} \). For a dynamical system with symmetry Lie group \( G \) we have independent dynamical functions \( S_k = S_k(x_i), (k = 1, ..., dim(g)) \) which are constructed as functions on \( M \) and satisfy the following relations
\[
\{S_i, S_j\} = f_{ij}^k S_k,
\]

(9)

where \( f_{ij}^k \) are structure constants of the Lie algebra \( g \) of the symmetry Lie group \( G \). For specifying the dynamical system with symmetry group, there are two methods:

a) Consider an \( r \)-matrix related to the Lie bialgebra \((g, \hat{g})\). If one can choose a matrix representation for the Lie algebra \( g \); then the satisfaction of the \( g \)-valued functions:
\[
Q = S_i r^{ij} X_j,
\]

(10)

in the relation:
\[
\{Q \otimes Q\} + [Q \otimes I + I \otimes Q, r] = 0,
\]

(11)
is equivalent to the CYBE [7] for the $r$-matrix [9]. Then, one can see that the following functions are the constants of motion of a dynamical system [9]:

$$I_k = \text{trace}(Q^k), \quad k \in \mathbb{N}. \hspace{1cm} (12)$$

b) For the case that there is no relevant matrix representation of the Lie algebra $g$; one can use a less accurate method; so that after writing relations [9]; one can specify the maximal number of the dynamical Hamiltonian of the system [3].

We know that a dynamical system with $n$ degrees of freedom would be completely integrable (in the sense of Liouville theorem) if there are $n$ independent constants of motion. In the next section, we will construct integrable systems by using Lie bialgebra $(g, \tilde{g})$ of symplectic type such that for these systems the phase spaces are symplectic Poisson-Lie groups $G$ for which the related dual Lie groups $\tilde{G}$ are symmetry Lie groups and vice versa. In section four we will consider examples related to real four-dimensional Lie bialgebras of bi-symplectic type [7].

3 Integrable Hamiltonian systems related to Lie bialgebras of bi-symplectic type, exchanging the role of phase spaces and symmetry Lie groups

Let us first consider the Lie bialgebra of bi-symplectic type and as a first step we consider the definition of the symplectic structure on a Lie algebra [10].

Definition [10]: A two form $\omega \in (\tilde{g} \otimes \tilde{g})$ on a Lie algebra $g$ is called symplectic if $d\omega \in \wedge^3 \tilde{g}$

$$d\omega(X, Y, Z) = -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X), \quad \forall X, Y, Z \in g,$$ \hspace{1cm} (13)

is closed i.e. $d\omega = 0$ and $\omega$ is nondegenerate [1]. In terms of Lie algebra basis $\omega = \omega_{ij} \tilde{X}^i \wedge \tilde{X}^j$ we must have:

$$\omega_{ij} = \omega_{ik} \omega_{kj} + \omega_{ik} \omega_{kj} - \omega_{kj} \omega_{ik} = 0,$$ \hspace{1cm} (14)

the inverse of $\omega_{ij}$ can be considered by $P_{ij}$ as Poisson structure on a Lie algebra $g$. The Poisson structure on the corresponding Lie groups $G$ can be obtained by using of vielbeins $e^i_j$ [11]

$$P_{ij}(x) = e^i_k e^j_l P_{kl},$$ \hspace{1cm} (15)

such that

$$dgg^{-1} = e^i_j \omega X_j, \quad \forall g \in G,$$ \hspace{1cm} (16)

and

$$e^i_j e^j_k = \delta^i_k,$$ \hspace{1cm} (17)

where $e^i_k$ is inverse of $e^i_j$ and $\{x^i\}$ and $\{X_i\}$ are coordinates and generators of the Lie group $G$. In the same way, if one of the Lie algebras $g$ or $\tilde{g}$ from Lie bialgebra $(g, \tilde{g})$ is of the symplectic type, then it is called symplectic Lie bialgebra. In the case that both of $g$ and $\tilde{g}$ of $(g, \tilde{g})$ are of the symplectic type, then the Lie bialgebra $(g, \tilde{g})$ is called bi-symplectic type [7] Note that for these Lie bialgebras of bi-symplectic type the dimension of Lie algebra $g$ and $\tilde{g}$ must be even $(2n)$. In [10] all real four-dimensional Lie algebras of symplectic type are classified; also all real four-dimensional Lie bialgebras of symplectic and bi-symplectic types are classified in [7].

Now, we construct a dynamical system by using of a $2n$ dimensional real Lie bialgebra $(g, \tilde{g})$ of bi-symplectic type (for which Lie algebras $g$ and $\tilde{g}$ are isomorphic) such that for this dynamical system the Poisson-Lie group $G$ (with coordinates $(x^1, ..., x^{2n})$) plays the role of the phase space, and its dual Lie group $\tilde{G}$ (with coordinates $(y^1, ..., y^{2n})$) plays the role of symmetry Lie group, so we have

$$\{S^i(x^1, ..., x^{2n}), S^j(x^1, ..., x^{2n})\} = f^{ij}_{kl} S^k(x^1, ..., x^{2n}).$$ \hspace{1cm} (18)

Now, by using the assumption that $g$ and $\tilde{g}$ are isomorphic i.e., there is a matrix $C$ for which:

$$\tilde{X}^i = C^{il} X_l, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}_{kl} \tilde{X}^k.$$ \hspace{1cm} (19)

\footnote{Note that here $\tilde{g}$ is the dual space of $g$ and $(\cdot, \cdot)$ is the standard inner product between $g$ and $\tilde{g}$.}
then we have

\[ \tilde{f}_{ij}^k = C^{il}C^{jm}f_{lm}(C^{-1})_{sk}, \]  

(20)

where \( C^{ij} \) is an invertible isomorphism matrix; such that by substituting \( 20 \) in \( 18 \) and defining

\[ \tilde{S}_m = (C^{-1})_{ml}S^l, \]  

(21)

we have

\[ \{ \tilde{S}_i(x^1, ..., x^{2n}), \tilde{S}_m(x^1, ..., x^{2n}) \} = f_{lm} \tilde{k} \tilde{S}_k(x^1, ..., x^{2n}). \]  

(22)

On the other hand, one can consider the coordinates \( \{ x^i \} \) on a Poisson-Lie group \( \tilde{G} \) (as a phase space with Poisson structure \( P^{ij} \)) as functions of the coordinates \( \{ y^i \} \) of the Poisson-Lie group \( \tilde{G} \). So, on a symplectic Poisson-Lie group \( G \) we have:

\[ \{ x^i(y^1, ..., y^{2n}), x^j(y^1, ..., y^{2n}) \} = \tilde{P}^{ik} \frac{\partial x^i(y^1, ..., y^{2n})}{\partial y^j} \frac{\partial x^j(y^1, ..., y^{2n})}{\partial y^k} = P^{ij}, \]  

(23)

where the functions \( \tilde{S}_i(x^1(y^1, ..., y^{2n}), ..., x^{2n}(y^1, ..., y^{2n})) = \tilde{S}_i(y^1, ..., y^{2n}) \) can be considered as dynamical functions on the space \( G \). i.e. relation \( 22 \) tells us that under transformations \( 23 \) \( x^i = x^i(y^1, ..., y^{2n}) \) the role of phase space \( G \) and symmetry Lie group \( \tilde{G} \) is exchanged. Furthermore, using \( 10, 33 \) and definition of \( \tilde{S}_i \), \( 21 \), one can show that the \( g \)-valued functions \( Q \) \( 10 \) is transformed to \( g \)-valued functions \( \tilde{Q} \) as follows

\[ Q(x^1(y^1, ..., y^{2n}), ..., x^{2n}(y^1, ..., y^{2n})) = S^i(x^1(y^1, ..., y^{2n}), ..., x^{2n}(y^1, ..., y^{2n})) \tilde{r}_{ij} \tilde{X}^j \]

\[ = \tilde{Q}(y^1, ..., y^{2n}) = \tilde{S}_i(y^1, ..., y^{2n}) r^{ij} X_j, \]  

(24)

if

\[ \tilde{r}_{ij} = (C^{-1})_{kl} r^{kl} (C^{-1})_{ij}, \]  

(25)

where \( \tilde{r}_{ij} \) and \( r^{ij} \) are the solutions of CYBE for \( \tilde{g} \) and \( g \), respectively. Furthermore, one can show that (using \( 20 \) and \( 25 \)) CYBE for the Lie algebra \( g \) transform the CYBE for the Lie algebra \( \tilde{g} \) and vice versa. In this way, we have the following theorem for exchanging the role of the phase space \( G \) and symmetry Lie group \( \tilde{G} \) and vice versa.

**Theorem 1:** Let \((g, \tilde{g})\) is a bi-symplectic Lie bialgebra for which the Lie algebras \( g \) and \( \tilde{g} \) are isomorphic. The dynamical functions of the dynamical system with Lie group \( G \) as a phase space and its dual Lie group \( \tilde{G} \) as symmetry Lie group are related to the dynamical functions of the dynamical system with Lie group \( G \) as a phase space and Lie group \( \tilde{G} \) as symmetry Lie group, as follows:

\[ \tilde{S}_j(y^1, ..., y^{2n}) = (C^{-1})_{ji} S^l(x^1(y^1, ..., y^{2n}), ..., x^{2n}(y^1, ..., y^{2n})), \]  

(26)

for which the transformation \( x^i = x^i(y^1, ..., y^{2n}) \) is a solution of \( 25 \) and the matrix \( C^{ij} \) is the isomorphism matrix. Furthermore, the integrals of motion of these systems are related to each other by using of \( 27 \).

In this way, we find transformations \( 22 \) which change the role of phase space and symmetry Lie group. In the next section, we will consider Hamiltonian dynamical systems related to real four-dimensional Lie bialgebras of bi-symplectic type as some examples.

### 4 Some examples

Now, we consider some integrable systems obtained by using the real four-dimensional Lie bialgebras of bi-symplectic type \( 7 \). In these examples, we consider the Lie group \( G \) related to the Lie bialgebra \((g, \tilde{g})\) as a phase space, and its dual Lie group \( \tilde{G} \) as a symmetry group of the system; such that by using theorem 1 we exchange the role of phase space and symmetry group and obtain other integrable system for which the Lie

Note that this transformation is not (in general) canonical transformation.
group $\mathbf{G}$ plays the role of symmetry Lie group and its dual Lie group $\tilde{\mathbf{G}}$ plays the role of a phase space of the system, for this propose we use the formalisms mentioned in the previous section for calculation of integrable Hamiltonian systems with some symmetry groups $\tilde{\mathbf{G}}$. In example one, we use the method a) and in the other examples 2-5 we use the b) method.

**Example 1** Lie bialgebra $(A_{0,9,iv}^0, A_{4,9}^0)$ [7].

Consider the Lie group $A_{0,9,iv}^0$ as a phase space and $A_{4,9}^0$ as symmetry Lie group of a Hamiltonian system. For this example, the Darboux coordinates have the following forms [6]:

$$
\begin{align*}
A
z_1 &= e^{-x_1}, \\
A z_3 &= -\frac{e^{2x_1}(x_2 + 2x_4)}{2(-1 + e^{x_1})}, \\
A z_4 &= x_3 - \frac{e^{-x_1}}{2},
\end{align*}
$$

such that in this Darboux coordinates the symplectic structure on the Lie group $A_{0,9,iv}^0$ which has the following form [7]:

$$
\{x_1, x_4\} = 1 - e^{-x_1}, \quad \{x_2, x_3\} = 1 - e^{-x_1}, \quad \{x_2, x_4\} = x_2e^{-x_1}, \quad \{x_3, x_4\} = \frac{1 + e^{-2x_1} - 2e^{-x_1}}{2},
$$

which can be simplified as

$$
\{z_1, z_3\} = 1, \quad \{z_2, z_4\} = 1.
$$

In this way, we have the following forms for the dynamical functions $S^i$ according to [6]

$$
\begin{align*}
S^1 &= -z_3 = \frac{e^{2x_1}(x_2 + 2x_4)}{2(-1 + e^{x_1})}, \\
S^2 &= -z_4 = x_3 + \frac{e^{-x_1}}{2}, \\
S^3 &= -z_2z_3 = \frac{x_2e^{3x_1}(x_2 + 2x_4)}{2(-1 + e^{x_1})}, \\
S^4 &= -z_1z_3 - z_2z_4 = \frac{x_2(1 + e^{x_1}(1 - 2x_3)) + 2e^{x_1}x_4}{2(-1 + e^{x_1})},
\end{align*}
$$

such that these functions satisfy the following Poisson brackets

$$
\{S^1, S^2\} = S^1, \quad \{S^2, S^3\} = S^1, \quad \{S^2, S^4\} = S^2,
$$

i.e., a Poisson bracket $\{S^1, S^2\} = \tilde{j}^{ij}S^k$, where $\tilde{j}^{ij}$ are the structure constants of Lie algebra $A_{4,9}^0$. We can take a representation of four-dimensional triangular matrices for the basis of the Lie algebra $A_{4,9}^0$ as follows:

$$
\begin{align*}
\tilde{X}^1 &= \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{X}^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{X}^3 &= \begin{pmatrix} 0 & ab & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \quad \tilde{X}^4 &= \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{align*}
$$

where $a, b, c$ and $d$ are nonzero arbitrary real constants. Then from (10) and (12) for the $r$–matrix $\tilde{r} = -1/2\tilde{X}^1 \wedge \tilde{X}^2 - \tilde{X}^1 \wedge \tilde{X}^4 - \tilde{X}^2 \wedge \tilde{X}^3$, [7] the constants of motion are obtained as follows:

$$
\begin{align*}
I_1 &= dz_4 + 2z_3 = d(x_3 - \frac{e^{-x_1}}{2}) + \frac{e^{2x_1}(x_2 + 2x_4)}{1 - e^{x_1}}, \\
I_2 &= (dz_4)^2 + 2(z_3)^2 = (d(x_3 - \frac{e^{-x_1}}{2}))^2 + \frac{e^{4x_1}(x_2 + 2x_4)^2}{2(e^{x_1} - 1)^2}.
\end{align*}
$$

Note that in terms of the Darboux coordinates $z_i$ the constants of motion $I_1$ and $I_2$ can be related to the physical systems. Now, by using (23) and Poisson structures on the Lie groups $A_{0,9}^0$ and $A_{4,9,iv}^0$ [7] one can find $x_i$’s as functions of $y_i$ as follows:

$$
\begin{align*}
x_1 &= y_4, \quad x_2 = y_3, \quad x_3 = -y_2 + y_4, \quad x_4 = -y_1,
\end{align*}
$$

(34)
For the Lie group $A_{4,9}^0$ as phase space the Darboux coordinates have the following forms \[35\]:

\[
\begin{align*}
\tilde{z}_1 &= \frac{e^{-y_4}}{2} + y_2 - y_4, \\
\tilde{z}_2 &= e^{y_4} + \frac{e^{2y_4}(-2y_1 + y_3)}{2(e^{y_4} - 1)}, \\
\tilde{z}_3 &= \frac{e^{y_4}y_3}{e^{y_4} - 1}, \\
\tilde{z}_4 &= e^{-y_4},
\end{align*}
\]

such that in this Darboux coordinates, the symplectic structure on the Lie group $A_{4,9}^0$ has the following form \[7\]:

\[
\{y_1, y_2\} = 1/2(1 - e^{-2y_4}), \quad \{y_1, y_3\} = y_3e^{-y_4}, \quad \{y_1, y_4\} = 1 - e^{-y_4}, \quad \{y_2, y_3\} = 1 - e^{-y_4},
\]

which has the following form

\[
\{\tilde{z}_1, \tilde{z}_3\} = 1, \quad \{\tilde{z}_2, \tilde{z}_4\} = 1.
\]

Then by substituting \[34\] in \[30\] and using \[21\] with

\[
C = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]

we have

\[
\begin{align*}
\tilde{S}_1 &= -\tilde{z}_1\tilde{z}_3 + \tilde{z}_4(\tilde{z}_2 + \tilde{z}_4) = -\frac{2y_1e^{y_4} + y_3 + y_3e^{y_4} - 2e^{y_4}y_3(y_4 - y_2)}{2 - 2e^{y_4}}, \\
\tilde{S}_2 &= \tilde{z}_2 - \tilde{z}_4 + \tilde{z}_3(\tilde{z}_4 - \tilde{z}_2) = -\frac{e^{2y_4}(1 + e^{y_4}(-1 + y_3))(2y_1 + y_3)}{2(e^{y_4} - 1)^2}, \\
\tilde{S}_3 &= \tilde{z}_1 = \frac{e^{-y_4}}{2} + y_2 - y_4, \\
\tilde{S}_4 &= \tilde{z}_2 - \tilde{z}_4 = \frac{e^{2y_4}(-2y_1 + y_3)}{2e^{y_4} - 2},
\end{align*}
\]

such that, they satisfy in the following Poisson brackets

\[
\{\tilde{S}_1, \tilde{S}_2\} = \tilde{S}_4, \quad \{\tilde{S}_1, \tilde{S}_3\} = \tilde{S}_3, \quad \{\tilde{S}_1, \tilde{S}_4\} = \tilde{S}_4, \quad \{\tilde{S}_2, \tilde{S}_3\} = \tilde{S}_4,
\]

i.e., a Poisson bracket \(\{\tilde{S}_i, \tilde{S}_j\} = f_{ij}^k \tilde{S}_k\), where \(f_{ij}^k\) are the structure constants of \(A_{4,9}^0\) and constants of motion on the Lie group \(A_{4,9}^0\) as a phase space can be obtained as follows:

\[
\begin{align*}
\tilde{I}_1 &= 2(\tilde{z}_2 - \tilde{z}_4) + d\tilde{z}_1 = \frac{e^{2y_4}(2y_1 - y_3)}{e^{y_4} - 1} + d\left(\frac{e^{-y_4}}{2} + y_2 - y_4\right), \\
\tilde{I}_2 &= 2(\tilde{z}_2 - \tilde{z}_4)^2 + (d\tilde{z}_1)^2 = \frac{e^{4y_4}(-2y_1 + y_3)^2}{2(e^{y_4} - 1)^2} + d^2\left(\frac{e^{-y_4}}{2} + y_2 - y_4\right)^2.
\end{align*}
\]

We see that in terms of the Darboux coordinates, these constants of motion are related to the physical systems. Note that using relations \[33\], \[34\] and \[27\] one can find the following transformations between coordinates \(\{\tilde{z}_1\}\) and \(\{z_1\}\):

\[
\tilde{z}_1 = -z_4, \quad \tilde{z}_2 = \frac{1}{z_1} - z_3, \quad \tilde{z}_3 = z_2, \quad \tilde{z}_4 = z_1.
\]

These are transformations between the Darboux coordinates on the Lie groups \(A_{4,9}^0\) and \(A_{4,9}^0\) as phase spaces. One can see that these transformations are preserved the canonical Poisson structure \[29\] and \[37\] but not transform the constants of motions \(I_1\) and \(I_2\) \[41\] to \(I_1\) and \(I_2\) of \[33\]. In this sense these are not canonical transformations.
In this way, we have the following forms for the dynamical functions $S$

\[ S^1 = \frac{e^{2x_3}x_4 - e^{x_3}}{1 + e^{x_3}}, \quad S_2 = x_2, \]
\[ S_3 = e^{-x_3}, \quad S_4 = \frac{x_2^2 - x_4}{q x_2}, \]

such that in this Darboux coordinates, the symplectic structure on the Lie group $(\mathbb{A}_2 \oplus \mathbb{A}_2)$ has the following form [7]:

\[ \{x_1, x_2\} = q x_2, \quad \{x_1, x_4\} = 1 + e^{-x_3}, \]

which can be simplified as

\[ \{z_1, z_3\} = 1, \quad \{z_2, z_4\} = 1. \]

In this way, we have the following forms for the dynamical functions $S^i$ according to [6]

\[ S^1 = q z_2 z_4 = -x_1 + x_2^2, \]
\[ S^2 = -a z_4 = \frac{a x_1}{q x_2} - \frac{a x_2}{q}, \]
\[ S^3 = -b z_3 = -b e^{-x_3}, \]
\[ S^4 = -z_1 z_3 = 1 - e^{x_3} x_4, \]

where $a, b$ and $q$ are nonzero arbitrary constants. These functions satisfy in the following Poisson brackets

\[ \{S^1, S^2\} = q S^2, \quad \{S^3, S^4\} = S^3 \]

i.e., a Poisson bracket $\{S^i, S^j\} = \tilde{f}^{ij}_k S^k$, where, $\tilde{f}^{ij}_k$ are the structure constants of the symmetry Lie algebra $(\mathbb{A}_2 \oplus \mathbb{A}_2).vi$. The invariants of the above system are $(S^1, S^3), (S^2, S^4), (S^1, S^4)$ or $(S^2, S^3)$ such that one can consider one of these $S^i$ as Hamiltonian of the integrable systems; in terms of Darboux coordinates these systems are physical once. Now, by using [23] and Poisson structures on Lie group $(\mathbb{A}_2 \oplus \mathbb{A}_2)$ and $(\mathbb{A}_2 \oplus \mathbb{A}_2).vi$ one can find $x_i$’s as functions of $y_k$ as follows:

\[ x_1 = q (y_1 - y_2), \quad x_2 = -\frac{y_2}{q (y_2 - y_1)}, \quad x_3 = y_4, \quad x_4 = -y_3, \]

The Darboux coordinates on the Lie group $(\mathbb{A}_2 \oplus \mathbb{A}_2).vi$ have the following forms [6]:

\[ \tilde{z}_1 = -\frac{e^{y_4} (1 + e^{y_4} y_3)}{e^{y_4} - 1}, \quad \tilde{z}_2 = \frac{y_2}{q (y_1 - y_2)}, \]
\[ \tilde{z}_3 = e^{-y_4}, \quad \tilde{z}_4 = \frac{q^3 (y_1 - y_2)^3 - y_2^2}{q y_2 (-y_1 + y_2)}, \]

such that in this Darboux coordinates, the symplectic structure on the Lie group $(\mathbb{A}_2 \oplus \mathbb{A}_2).vi$ has the following form [7]:

\[ \{y_1, y_2\} = y_2, \quad \{y_3, y_4\} = 1 - e^{y_4}, \]

which can be simplified as

\[ \{\tilde{z}_1, \tilde{z}_3\} = 1, \quad \{\tilde{z}_2, \tilde{z}_4\} = 1. \]
we have

\[
\begin{align*}
\tilde{S}_1 &= \tilde{z}_2 \tilde{z}_4 = -\frac{q^3(y_1 - y_2)^3 + y_2^2}{q^3(y_1 - y_2)^2}, \\
\tilde{S}_2 &= -\tilde{z}_4 = -\frac{q^3(y_1 - y_2)^3 + y_2^2}{q^2(-y_1 + y_2)y_2}, \\
\tilde{S}_3 &= \tilde{z}_1 \tilde{z}_3 = -\frac{1 - e^{y_1 y_2}}{1 + e^{y_4}}, \\
\tilde{S}_4 &= -\tilde{z}_3 = -e^{-y_4},
\end{align*}
\]

such that, they satisfy the following Poisson brackets

\[
\{\tilde{S}_1, \tilde{S}_2\} = \tilde{S}_2, \quad \{\tilde{S}_3, \tilde{S}_4\} = \tilde{S}_4,
\]

(53)
i.e., a Poisson bracket \(\{\tilde{S}_i, \tilde{S}_j\} = f_{ij}^k \tilde{S}_k\), where, \(f_{ij}^k\) are the structure constants of the symmetry Lie algebra \(A_2 \oplus A_2\). The invariants of the above system are \((\tilde{S}_1, \tilde{S}_3), (\tilde{S}_2, \tilde{S}_4), (\tilde{S}_1, \tilde{S}_4)\) or \((\tilde{S}_2, \tilde{S}_3)\) such that one can consider one of these \(\tilde{S}\) as Hamiltonian of the integrable systems, which in terms of the Darboux coordinates \(\{\tilde{z}_i\}\) its related to a physical systems. For this example using \(49\), \(19\) and \(43\) one can find the following trivial transformations between coordinates \(\tilde{z}_i\) and \(z_i\):

\[
\tilde{z}_1 = z_1, \quad \tilde{z}_2 = z_2, \quad \tilde{z}_3 = z_3, \quad \tilde{z}_4 = z_4.
\]

(54)
In this case these transformations preserved the canonical Poisson structures and transform the constant of motions to each other \((S^1 \to q\tilde{S}_1; S^2 \to a\tilde{S}_2; S^3 \to b\tilde{S}_4; S^4 \to -\tilde{S}_3)\) \(S^1\) transform to \(\tilde{S}_1\) and \(S^2\) transform to \(\tilde{S}_2\) but \(S^3\) transform to \(\tilde{S}_4\), and \(S^4\) transform to \(\tilde{S}_3\); so these transformations are canonical transformation. Note that these are transformation between to phase space \((A_2 \oplus A_2)\) and \((A_2 \oplus A_2)\).

**Example 3** Lie bialgebra \((A_{0,9}^0, A_{0,9}.iv)\) [7]:

Consider the Lie group \(A_{0,9}^0\) as a phase space and \(A_{0,9}.iv\) as symmetry Lie group of a Hamiltonian system.

For this example, the Darboux coordinates have the following forms [6]:

\[
\begin{align*}
z_1 &= -x_3, \\
z_2 &= \frac{e^{x_4}(-2e^{x_4}x_1 + 2e^{2x_4}x_1 - x_3 - e^{x_4}x_3 + 2e^{x_4}x_2x_3)}{2(-1 + e^{x_4})}, \\
z_3 &= \frac{1 + 2e^{x_4}x_2}{2(-1 + e^{x_4})}, \\
z_4 &= e^{-x_4},
\end{align*}
\]

(55)
such that in this Darboux coordinates, the symplectic structure on the Lie group \(A_{0,9}^0\) which has the following form [7]:

\[
\{x_1, x_2\} = \frac{1 - e^{-2x_4}}{2}, \quad \{x_1, x_3\} = e^{-x_4}x_3, \quad \{x_1, x_4\} = 1 - e^{-x_4}, \quad \{x_2, x_3\} = 1 - e^{-x_4},
\]

(56)
can be simplified as

\[
\{z_1, z_3\} = 1, \quad \{z_2, z_4\} = 1.
\]

(57)
In this way, we have the following forms for the dynamical functions \(S^i\) according to [6]

\[
\begin{align*}
S^1 &= z_1z_3 + z_2z_4 = \frac{2x_4 + 2e^{x_4}(x_1 + 2x_2x_3) - e^{2x_4}(2x_1 + x_3 + 2x_2x_3)}{2(-1 + e^{x_4})^2}, \\
S^2 &= -z_3 + z_2z_3 \\
&= \frac{1 + 2e^{x_4}x_2)(2 + e^{x_4}(-4 - x_3 + e^{x_4}(2 + 2(-1 + e^{x_4})x_1 + (-1 + e^{x_4} - 2x_2)x_3))}}{4(-1 + e^{x_4})^3}, \\
S^3 &= -z_4 = -e^{-x_4}, \\
S^4 &= -az_3 = \frac{a + 2ae^{x_4}x_2}{2 - 2e^{x_4}},
\end{align*}
\]

(58)
where \(a \in \mathbb{R} - \{0\}\). These functions satisfy in the following Poisson brackets

\[
\{S^1, S^2\} = S^4, \quad \{S^1, S^3\} = S^3, \quad \{S^1, S^4\} = S^4, \quad \{S^2, S^3\} = S^4.
\]

(59)
i.e., a Poisson bracket \( \{S^i, S^j\} = \tilde{f}^i_k S^k \), with \( \tilde{f}^i_k \) are the structure constants of the symmetry Lie algebra \( A_{4,9}^0,iv \). The invariants of the above system are \((S^3, S^4)\) or \((S^2, S^4)\), such that one can consider one of these \( S^i \) as Hamiltonian of the integrable systems, for which in terms of the Darboux coordinates these are shown a physical system. Now, by using (23) and Poisson structures on Lie group \( A_{4,9}^0,iv \) one can find \( x_i \)'s as functions of \( y_i \) as follows:

\[
x_1 = -y_1, \quad x_2 = y_1 - y_3, \quad x_3 = y_2, \quad x_4 = y_1,
\]

(60)

For the Lie group \( A_{4,9}^0,iv \) as phase space the Darboux coordinates have the following forms \[6\] and \( \tilde{z}_i \) which has the following form \[7\]:

\[
\{y_1, y_4\} = 1 - e^{-y_1}, \quad \{y_2, y_3\} = 1 - e^{-y_1}, \quad \{y_2, y_4\} = y_2 e^{-y_1}, \quad \{y_3, y_4\} = \frac{1 + e^{-2y_1} - 2e^{-y_1}}{2}, \quad \]

(62)

so these transformations are

\[
\{\tilde{z}_1, \tilde{z}_3\} = 1, \quad \{\tilde{z}_2, \tilde{z}_4\} = 1.
\]

(63)

Now after using \(68\) and substituting \(60\) and using \(21\) with

\[
C = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\alpha & 0 & 0 & 0
\end{pmatrix},
\]

(64)

we have

\[
\begin{align*}
\tilde{S}_1 &= -\tilde{z}_3 - \frac{1 + 2e^{y_1}(y_1 - y_3)}{2 - 2e^{y_1}}, \\
\tilde{S}_2 &= -\tilde{z}_4 - e^{-y_1}, \\
\tilde{S}_3 &= -\tilde{z}_2 \tilde{z}_3 - \frac{e^{y_1}(1 + 2e^{y_1}(y_1 - y_3))(-y_2 + e^{2y_1}(y_2 - 2y_4) - e^{y_1}(y_2 - 2y_2(y_1 - y_3) - 2y_4))}{4(-1 + e^{y_1})^3}, \\
\tilde{S}_4 &= -\tilde{z}_1 \tilde{z}_3 - \tilde{z}_2 \tilde{z}_4 - \frac{2y_2 + 2e^{y_1}(2y_2(y_1 - y_3) - y_4) - e^{2y_1}(y_2 + 2y_2(y_1 - y_3) - 2y_4))}{2(-1 + e^{y_1})^2},
\end{align*}
\]

such that, they satisfy the following Poisson brackets

\[
\{\tilde{S}_1, \tilde{S}_2\} = \tilde{S}_1, \quad \{\tilde{S}_2, \tilde{S}_3\} = \tilde{S}_2, \quad \{\tilde{S}_3, \tilde{S}_4\} = \tilde{S}_3, \quad \{\tilde{S}_4, \tilde{S}_1\} = \tilde{S}_4,
\]

(65)

i.e., a Poisson bracket \( \{\tilde{S}_i, \tilde{S}_j\} = f_{ij}^k \tilde{S}_k \), with \( f_{ij}^k \) are the structure constants of the symmetry Lie algebra \( A_{4,9}^0,iv \). The invariants of the above system are \((\tilde{S}_1, \tilde{S}_3), (\tilde{S}_3, \tilde{S}_4)\) or \((\tilde{S}_1, \tilde{S}_2)\) such that one can consider one of these \( \tilde{S}_i \) as Hamiltonian of the integrable systems, where these are physical systems, in terms of Darboux coordinates. For this example using \(61, 60\) and \(58\) one can find the following transformations between coordinates \( \tilde{z}_i \) and \( z_i \):

\[
\begin{align*}
\tilde{z}_1 &= z_1, \quad \tilde{z}_2 = z_2, \quad \tilde{z}_3 = z_3, \quad \tilde{z}_4 = z_4.
\end{align*}
\]

(66)

In this case these transformations preserved the canonical Poisson structures and transform the constant of motions to each other (i.e., \( S^1 \to -\tilde{S}_4; S^3 \to \tilde{S}_1 - \tilde{S}_3; S^3 \to \tilde{S}_2; S^4 \to a \tilde{S}_1 \)); so these transformations are canonical transformation. Note that these are transformation between to phase space \( A_{4,9}^0,iv \) and \( A_{4,9}^0,iv \).
Example 4) Lie bialgebra \((A^1_{4,9}, A^1_{4,9}.i)\) \([7]\): Consider the Lie group \(A^1_{4,9}\) as a phase space and \(A^1_{4,9}.i\) as symmetry Lie group of a Hamiltonian system. For this example, the Darboux coordinates have the following forms \([6]\):

\[
\begin{align*}
  z_1 &= -\frac{2e^{2x_4x_2}}{e^{2x_4} - 1}, &
  z_2 &= \frac{e^{-4x_4}(-2x_1 + 2e^{2x_4}x_1 - 2x_2x_3 + e^{2x_4}x_2x_3)}{(-1 + e^{2x_4})^2}, \\
  z_3 &= x_3, &
  z_4 &= e^{-2x_4},
\end{align*}
\]

such that in this Darboux coordinates, the symplectic structure on the Lie group \(A^1_{4,9}\) which has the following form \([7]\):

\[
\{x_1, x_2\} = -\frac{x_2}{4}, \quad \{x_1, x_3\} = \frac{x_3 - 2e^{-2x_4}x_3}{4}, \quad \{x_1, x_4\} = -\frac{1}{4}(1 - e^{-2x_4}), \quad \{x_2, x_3\} = -\frac{1}{2}(1 - e^{-2x_4}),
\]

can be simplified as

\[
\{z_1, z_3\} = 1, \quad \{z_2, z_4\} = 1.
\]

In this way, we have the following forms for the dynamical functions \(S^i\) according to \([6]\)

\[
\begin{align*}
  S^1 &= -1/4(2z_1z_3 + z_2z_4), &
  S^2 &= -z_2z_3 = -\frac{e^{2x_4}x_3(2(e^{2x_4} - 1)x_1 + (2 - 3e^{2x_4})x_2x_3)}{(-1 + e^{2x_4})^2}, \\
  S^3 &= -z_4 = -e^{-2x_4}, &
  S^4 &= -z_3 = -x_3.
\end{align*}
\]

such that these functions satisfy in the following Poisson brackets

\[
\{S^1, S^2\} = -\frac{1}{4}S^2, \quad \{S^1, S^3\} = -\frac{1}{4}S^3, \quad \{S^1, S^4\} = -\frac{1}{4}S^4, \quad \{S^2, S^3\} = -S^4.
\]

i.e., a Poisson bracket \(\{S^i, S^j\} = \tilde{f}^i_j S^k\), with \(\tilde{f}^i_j\) are the structure constants of the symmetry Lie algebra \(A^1_{4,9}.i\). The invariants of the above system are \((S^2, S^4)\) or \((S^3, S^4)\), such that one can consider one of these \(S^i\) as Hamiltonian of the integrable systems; in terms of Darboux coordinates these are physical systems. Now, by using \([24]\) and Poisson structures on Lie group \(A^1_{4,9}\) and \(A^1_{4,9}.i\) one can find \(x_i\)'s as functions of \(y_i\) as follows:

\[
x_1 = \frac{1}{4}y_4, \quad x_2 = -y_3, \quad x_3 = \frac{1}{4}y_2, \quad x_4 = -\frac{1}{4}y_1,
\]

For the Lie group \(A^1_{4,9}.i\) as phase space the Darboux coordinates have the following forms \([6]\):

\[
\begin{align*}
  \hat{z}_1 &= \frac{2y_3}{1 - e^{-\frac{y_4}{4}}}, &
  \hat{z}_2 &= \frac{(2 - e^{-\frac{y_4}{4}})y_2y_3 + 2(e^{-\frac{y_4}{4}} - 1)y_4}{4(e^{-\frac{y_4}{4}} - 1)^2}, \\
  \hat{z}_3 &= \frac{y_2}{4}, &
  \hat{z}_4 &= e^{\frac{y_4}{4}},
\end{align*}
\]

such that in this Darboux coordinates, the symplectic structure on \(A^1_{4,9}.i\) which has the following form \([7]\):

\[
\{y_1, y_4\} = 4(-1 + e^{\frac{y_4}{4}}), \quad \{y_2, y_3\} = 2(-1 + e^{\frac{y_4}{4}}), \quad \{y_2, y_4\} = y_2(-1 + 2e^{\frac{y_4}{4}}), \quad \{y_3, y_4\} = y_3,
\]

can be simplified as

\[
\{\hat{z}_1, \hat{z}_3\} = 1, \quad \{\hat{z}_2, \hat{z}_4\} = 1.
\]

Now after using \([70]\) and substituting \([72]\) and using \([21]\) with

\[
C = \begin{pmatrix}
  0 & 0 & 0 & 1/4 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix},
\]
In this case these transformations preserved the canonical Poisson structures and transform the constant of 
example using (73), (72) and (67) one can find the following transformations between coordinates \( \tilde{\Sigma} \) Hamiltonian of the integrable systems, which are physical systems in terms of Darboux coordinates. For this
In this way, we have the following forms for the dynamical functions \( S \):

\[
\begin{align*}
\tilde{S}_1 &= -\tilde{z}_3 = -y_2/4, \\
\tilde{S}_2 &= -\tilde{z}_4 = -e^{-y_1/2}, \\
\tilde{S}_3 &= -\tilde{z}_2\tilde{z}_3 = -e^{-y_1/2}y_2((2e^{y_1/2} - 1)y_2y_3 - 2(-1 + e^{y_1/2})y_4), \\
\frac{16(-1 + e^{y_1/2})^2}{4(-1 + e^{y_1/2})^2}, \\
\tilde{S}_4 &= -2\tilde{z}_1\tilde{z}_3 - \tilde{z}_2\tilde{z}_4 = -\frac{(2e^{y_1/2} - 1)y_4 + (3 - 2e^{y_1/2})y_2y_3}{4(-1 + e^{y_1/2})^2},
\end{align*}
\]

such that, they satisfy the following Poisson brackets

\[
\{\tilde{S}_1, \tilde{S}_4\} = 2\tilde{S}_1, \quad \{\tilde{S}_2, \tilde{S}_3\} = \tilde{S}_1, \quad \{\tilde{S}_2, \tilde{S}_4\} = \tilde{S}_2, \quad \{\tilde{S}_3, \tilde{S}_4\} = \tilde{S}_3,
\]

i.e., a Poisson bracket \( \{\tilde{S}_i, \tilde{S}_j\} = f_{ij}^k \tilde{S}_k \), with \( f_{ij}^k \) are the structure constants of the symmetry Lie algebra \( A_{4,9} \). The invariants of the above system are \( (\tilde{S}_1, \tilde{S}_2) \) or \( (\tilde{S}_1, \tilde{S}_3) \), such that one can consider one of these \( \tilde{S}_i \) as Hamiltonian of the integrable systems, which are physical systems in terms of Darboux coordinates. For this example using (73), (72) and (67) one can find the following transformations between coordinates \( \tilde{z}_i \) and \( z_i \):

\[
\tilde{z}_1 = z_1, \quad \tilde{z}_2 = z_2, \quad \tilde{z}_3 = z_3, \quad \tilde{z}_4 = z_4.
\]

In this case these transformations preserved the canonical Poisson structures and transform the constant of 

**Example 5** Lie bialgebra \( (A_{4,7}.i, A_{4,7}) \) [7]:

Consider the Lie group \( A_{4,7}.i \) as a phase space and \( A_{4,7} \) as symmetry Lie group of a Hamiltonian system. For this example, the Darboux coordinates have the following forms [6]:

\[
\begin{align*}
z_1 &= e^{-x_1}, \\
z_3 &= \frac{e^{2x_1}(2x_4 + e^{x_1}(x_2^2 + 2x_2x_3 + 2x_4))}{4(e^{x_1} - 1)^2}, \\
z_2 &= x_3, \\
z_4 &= \frac{e^{x_1}x_2}{1 - e^{x_1}},
\end{align*}
\]

such that in this Darboux coordinates, the symplectic structure on \( A_{4,7}.i \) which has the following form [7]:

\[
\{x_1, x_4\} = 2 - 2e^{-x_1}, \quad \{x_2, x_3\} = 1 - e^{-x_1}, \quad \{x_2, x_4\} = -x_2(1 - 2e^{-x_1}), \quad \{x_3, x_4\} = x_2 + x_3,
\]

(80)

can be simplified as

\[
\{z_1, z_3\} = 1, \quad \{z_2, z_4\} = 1.
\]

(81)

In this way, we have the following forms for the dynamical functions \( S^i \) according to [6]

\[
\begin{align*}
S^1 &= -z_3 = \frac{e^{2x_1}(-2x_4 + e^{x_1}(x_2^2 + 2x_2x_3 + 2x_4))}{4(-1 + e^{x_1})^2}, \\
S^2 &= -z_2z_3 = \frac{e^{2x_1}x_2(-2x_4 + e^{x_1}(x_2^2 + 2x_2x_3 + 2x_4))}{4(-1 + e^{x_1})^2}, \\
S^3 &= z_4 = \frac{x_2e^{x_1}}{1 - e^{x_1}}, \\
S^4 &= -z_2z_4 = -\frac{e^{x_1}(8x_2x_3 + x_4) + e^{2x_1}x_2^2(x_2^2 + 2x_2x_3 + 2x_4) + 2e^{x_1}(2x_2(x_2 + 4x_3) + (4 + x_3^2)x_4))}{8(e^{x_1} - 1)^2}.
\end{align*}
\]

(82)

such that these functions satisfy in the following Poisson brackets

\[
\{S^1, S^4\} = 2S^1, \quad \{S^2, S^3\} = S^1, \quad \{S^2, S^4\} = S^2, \quad \{S^3, S^4\} = S^2 + S^3.
\]

(83)

i.e., a Poisson bracket \( \{S^i, S^j\} = f^j_{ik} S^k \), with \( f^j_{ik} \) are the structure constants of the symmetry Lie algebra \( A_{4,7} \). The invariants of the above system are \( (S^1, S^2) \) or \( (S^1, S^3) \), such that one can consider one of these \( S^i \).
as Hamiltonian of the integrable systems, where in terms of Darboux coordinates these are physical systems. Now, by using (20) and Poisson structures on Lie group $A_{4.7}$ and $A_{4.7.1}$ one can find $x_i$'s as functions of $y_i$ as follows:

$$x_1 = 2y_4, \quad x_2 = y_2, \quad x_3 = y_3, \quad x_4 = \frac{4y_1 - e^{2y_4}(4y_1 + y_2^2 + 4y_2y_3 - y_3^2) + 4y_2y_3}{2(e^{2y_4} - 1)},$$

(84)

For the Lie group $A_{4.7}$ as phase space the Darboux coordinates have the following forms:

$$\tilde{z}_1 = e^{-2y_4}, \quad \tilde{z}_3 = \frac{e^{4y_4}(4(1 + e^{2y_4})y_1 - y_3(-2(-2 + e^{2y_4})y_2 + e^{2y_4}y_3))}{4(e^{2y_4} - 1)^2}, \quad \tilde{z}_4 = \frac{e^{2y_4}y_2}{1 - e^{2y_4}},$$

(85)

such that in this Darboux coordinates, the symplectic structure on $A_{4.7}$ which has the following form:

$$\{y_1, y_2\} = \frac{y_2 - y_4}{2}, \quad \{y_1, y_3\} = y_3(e^{-2y_4} - \frac{1}{2}), \quad \{y_1, y_4\} = \frac{1}{2}(1 - e^{-2y_4}), \quad \{y_2, y_3\} = 1 - e^{-2y_4},$$

(86)

can be simplified as

$$\{\tilde{z}_1, \tilde{z}_3\} = 1, \quad \{\tilde{z}_2, \tilde{z}_4\} = 1.$$

(87)

Now after using (82) and substituting (83) and using (21) with

$$C = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix},$$

(88)

we have:

$$\tilde{S}_1 = \tilde{z}_1\tilde{z}_3 - 1/4\tilde{z}_2(\tilde{z}_2\tilde{z}_3 - 2\tilde{z}_4) = \frac{e^{2y_4}(4(e^{2y_4} - 1)y_1(-4 + e^{2y_4}y_2^2) + y_3(4e^{2y_4}y_3 - e^{4y_4}y_3^2 + 2y_2(4 - e^{2y_4}y_3^2 + e^{4y_4}y_3^2)))}{16(e^{2y_4} - 1)^2},$$

$$\tilde{S}_2 = \tilde{z}_4 = \frac{e^{2y_4}y_2}{1 - e^{2y_4}},$$

$$\tilde{S}_3 = \tilde{z}_2\tilde{z}_3 = -\frac{e^{4y_4}y_3(-4(1 + e^{2y_4})y_1 + y_3(-2(-2 + e^{2y_4})y_2 + e^{2y_4}y_3))}{4(e^{2y_4} - 1)^2},$$

$$\tilde{S}_4 = -\tilde{z}_3/2 = -\frac{e^{4y_4}(-4(1 + e^{2y_4})y_1 + y_3(-2(-2 + e^{2y_4})y_2 + e^{2y_4}y_3))}{8(e^{2y_4} - 1)^2},$$

such that, they satisfy the following Poisson brackets

$$\{\tilde{S}_1, \tilde{S}_2\} = \frac{1}{2}\tilde{S}_2 - \frac{1}{2}\tilde{S}_3, \quad \{\tilde{S}_1, \tilde{S}_3\} = \frac{1}{2}\tilde{S}_3, \quad \{\tilde{S}_1, \tilde{S}_4\} = \tilde{S}_4, \quad \{\tilde{S}_2, \tilde{S}_3\} = 2\tilde{S}_4,$$

(89)

i.e., a Poisson bracket \(\{\tilde{S}_i, \tilde{S}_j\} = f_{ij}^k \tilde{S}_k\), with \(f_{ij}^k\) are the structure constants of the symmetry Lie algebra $A_{4.7.1}$. The invariants of the above system are \((\tilde{S}_2, \tilde{S}_3)\) or \((\tilde{S}_3, \tilde{S}_4)\), such that one can consider one of these \(\tilde{S}_i\) as Hamiltonian of the integrable systems, which are physical systems in terms of Darboux coordinates. For this example using (55), (84) and (79) one can find the following transformations between coordinates \(\tilde{z}_i\) and \(z_i\):

$$\tilde{z}_1 = z_1, \quad \tilde{z}_2 = z_2, \quad \tilde{z}_3 = z_3, \quad \tilde{z}_4 = z_4.$$

(90)

In this case these transformations preserved the canonical Poisson structures and transform the constant of motions to each other (i.e., $S^1 \rightarrow 2\tilde{S}_4$; $S^2 \rightarrow -\tilde{S}_3$; $S^3 \rightarrow \tilde{S}_2$; $S^4 \rightarrow -2\tilde{S}_1$); so these transformations are canonical transformation. Note that these are transformation between to phase space $A_{4.7}$ and $A_{4.7.1}$.
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