Stability-Preserving, Incentive-Compatible, Time-Efficient Mechanisms for Increasing School Capacity*

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Abstract

We address the following dynamic version of the school choice question: a city, named City, admits students in two temporally-separated rounds, denoted $R_1$ and $R_2$. In round $R_1$, the capacity of each school is fixed but in round $R_2$, the City is happy to allocate extra seats to specific schools per the recommendation of the mechanism; in turn, the latter has to meet specified requirements. We study three natural settings of this model, with the requirements getting increasingly stringent. For Settings I and II, we give pairs of polynomial time mechanisms $(M_1, M_2)$ which, besides addressing the specific requirements, find stable matchings, are dominant strategy incentive compatible (DSIC) w.r.t. reporting preference lists of students, and never break, in round $R_2$, a match created in round $R_1$.

In Setting III, the mechanism needs to deal with residents of the City who try to game the system by not appearing in round $R_1$ and only showing up in round $R_2$, in addition to gaming by misreporting preference lists. We note that the mechanisms described above were all oblivious in that they needed to know only the preference lists of students being considered for admission and not those who were not participating. After proving that no oblivious mechanism can satisfy the rather stringent requirements of Situation III in a stability-preserving, DISC manner, we turn to non-oblivious mechanisms. Moreover, since we were unable to achieve DSIC, we relax this notion to weak incentive compatible (WIC) and give such a pair of mechanisms.

Finally, we also give a procedure that outputs all possible stability-preserving extensions of a given stable matching (which may be exponentially many) with polynomial delay.

1 Introduction

School choice is among the most consequential events in a child’s upbringing, whether it is admission to elementary, middle or high school, and hence has been accorded its due importance not only in the education literature but also in game theory and economics. In order to deal with the flaws in the practices of the day, the seminal paper of Abdulkadiroglu and Sonmez [AS03] formulated this as a mechanism design problem. This approach has been enormously successful,

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especially in large cities involving the admission of hundreds of thousands of students into hundreds of schools, e.g., see [APR09, ACP+17, AS13, Pat11], and today occupies a key place in the area of market design in economics, e.g., see [RS12, Rot08, Rot16, ftToC19].

Once the basic issues in school choice were adequately addressed, researchers turned attention to the next level of questions. In this vein, in a very recent paper, Feigenbaum et. al. [FKLS18] remarked, “However, most models considered in this literature are essentially static. Incorporating dynamic considerations in designing assignment mechanisms ... is an important aspect that has only recently started to be addressed.”

Our paper deals with precisely this. We study three natural settings in which students need to be assigned to schools in two temporally-separated rounds, denoted $R_1$ and $R_2$, for which we give mechanisms $M_1$ and $M_2$, respectively. $M_1$ finds a stable matching $M$ in round $R_1$. In round $R_2$, additional students need to be assigned to the schools. We want that $M_2$ should also yield a stable matching, besides having other nice properties. Clearly, the task of $M_2$ would be a lot simpler if it were allowed to reassign the schools of a small number of students who were matched in round $R_1$. However, one of our central tenets is to disallow this altogether. Indeed, switching the school of a student midway, unsynchronized with her classmates, is well-known to cause traumatic effects, e.g., see [GDE12]. Hence, $M_2$ must extend $M$ to a stable matching $M'$.

The use of Gale-Shapley student-proposing deferred acceptance algorithm, which finds a student-optimal stable matching of students to schools, has emerged as a method of choice in the literature. Two main advantages of this method are:

1. As a consequence of stability, once the matching is done, no student and school, who are not matched to each other, will have the incentive to go outside the mechanism to strike a deal. Another advantage of stability is that it eliminates justified envy, i.e., the following situation cannot arise: there is a student $s_i$ who prefers another student $s_j$’s school assignment while being fully aware that $s_j$’s school preferred her to $s_i$.

2. This mechanism is dominant strategy incentive compatible (DSIC), for students. This entails that regardless of the preferences reported by other students, a student can do no better than report her true preference list, i.e., truth-telling is a dominant strategy for all students. This immediately simplifies the task of students and their parents, since they don’t need to waste any effort trying to game the system.

In all three settings, we give a pair of mechanisms $(M_1, M_2)$ which run in polynomial time and $M_2$ extends, in a stable manner, the stable matching found by $M_1$. Additionally, for the first two settings, $(M_1, M_2)$ are DSIC for students w.r.t. reporting their preference lists. The additional requirements are much more stringent in Setting III due to which we were unable to find DSIC mechanisms; see the requirements below. Instead, we define the weaker notion of weak incentive compatible (WIC) mechanism, under which a student cannot gain by misreporting her choices, if all other students are truthful, and we provide such a pair of mechanisms.

The three settings involve the admission of students of a city, named City, into schools; the preference lists of both are provided to the mechanisms. In round $R_1$, the capacity of each school is fixed but in round $R_2$, the City is happy to allocate extra seats to specific schools per the recommendation of mechanism $M_2$, which in turn has to meet specified requirements imposed by the City. In this round, in Settings I and II, $M_1$ finds a student-optimal stable matching $M$. 
Let $L$ be the set of *left-over students*, those who could not be admitted in this round.

In round $R_2$ of Setting I, the problem is to maximize the number of students admitted from $L$, by extending $M$ in a stability-preserving, DSIC manner. In Setting II, a set $N$ of *new students* also arrive from other cities and their preference lists are revealed to $M_2$. The requirement now is to admit as few students as possible from $N$ and subject to that, as many as possible from $L$, again in a stability-preserving, DSIC manner.

In Setting III, some students, who are residents of the City, try to game the system by not appearing in round $R_1$ but only showing up in round $R_2$, thereby getting admission to a better school. Therefore, $M_2$ needs to be incentive compatible not only w.r.t. preference lists but also late arrival. We note that the previous mechanisms were all *oblivious* in that they needed to know only the preference lists of students being considered for admission and not those who were not participating. We first prove there is no pair of oblivious, stability-preserving, DISC mechanisms $(M_1, M_2)$ that ensure that students can’t gain by deliberately not showing up in round $R_1$. We then give a pair of *non-oblivious* mechanisms, that know the preference lists of all students who reside in the City. As stated previously, these are WIC, though not DSIC mechanisms.

Finally, we also give a procedure that outputs all possible stability-preserving extensions of a given stable matching (which may be exponentially many) with polynomial delay.

### 1.1 Related work

Besides the references pointed out above on school choice, in this section, we will concentrate on recent work on dynamic matching markets, especially those pertaining to school choice. Feigenbaum et. al. [FKLS18] study the following issue that arises in NYC public high schools, which admits over 80,000 students annually: after the initial centralized allocation, about 10% of the students choose not attend the school allocated to them, instead going to private or charter schools. To deal with this, [FKLS18] give a two-round solution which maintains truthfulness and efficiency and minimizes the movement of students between schools.

An interesting phenomena that has been observed in matching markets is *unraveling*, under which matches are made early to beat the competition, even though it leads to inefficiencies due to unavailability of full information. A classic case, indeed one that motivated the formation of centralized clearing houses, is that of the market for medical interns in which contracts for interns were signed two years before the future interns would even graduate [Rot84]. A theoretical explanation of this phenomena was recently provided by [EP16].

We note that the phenomena we are studying in Setting III can be viewed as anti-unraveling: some students are able to game the system by making the match *late*. Clearly, this aspect deserves more work. We also note that this phenomenon is by no means rare, e.g., it occurred in the Pasadena School District and the authorities were made specific recommendations by economists from Caltech to counter it [Ech19].

[KK18] point out that stable pairings may not necessarily last forever, e.g., a student may switch from private to public school or a married couple may divorce. They study dynamic, multi-period, bilateral matching markets and they define and identify sufficient conditions for the existence of a dynamically stable matching.
1.2 Overview of structural and algorithmic ideas

The main idea for obtaining a stability-preserving mechanism in round $R_2$ for Settings I and II lies in the notion of a barrier which ensures that students admitted in $R_2$ do not form blocking pairs. At the same time, we need to place our barriers optimally to ensure that the number of students admitted is optimized (minimized or maximized) appropriately. The main idea in these settings for achieving DSIC mechanisms is to ensure that the best school that a student can be matched to is independent of her preference list. If so, her best outcome results from truthfully revealing her preference list.

Perhaps our most interesting result is the pair of mechanisms for Setting III. As already pointed out, no oblivious mechanism exists for this problem. The idea behind our non-oblivious mechanism $M_1$ is to prepare the round $R_1$ matching $M$ in such a way that the defectors, i.e., residents of the City who opt not to participate in round $R_1$, will not be able to get admission to a better school by arriving only in round $R_2$. For this, $M_1$ needs to know the preference lists of not only students participating in round $R_1$, but also the defectors, hence making it non-oblivious.

The algorithm for enumerating stable extensions of a stable matching, given in Section 5, relies heavily on the fundamental structural property of stable matchings given in Lemma 2. Enumerated matchings are extended by only one student in an iteration. At each step, the algorithm finds all such feasible extensions by one student in a way such that there must be at least one feasible assignment, for any student, at each step. This assurance is crucial in guaranteeing that the delay between any two enumerated matchings is polynomial.

2 Preliminaries

2.1 The stable matching problem for school choice

The stable matching problem takes as input a set $H = \{h_1, h_2, \ldots, h_m\}$ of $m$ public schools and a set $S = \{s_1, s_2, \ldots, s_n\}$ of $n$ students who are seeking admission to the schools. Each school $h_j \in H$ has an integer-valued capacity, $c(j)$, stating the maximum number of students that can be assigned to it. If $h_j$ is assigned $c(j)$ students, we will say that $h_j$ is filled, and otherwise it is under-filled.

Each student $s_i \in S$ has a strict and complete preference list, $l(s_i)$, over $H \cup \{\emptyset\}$. If $s_i$ prefers $\emptyset$ to $h_j$, then she prefers remaining unassigned rather being assigned to school $h_j$. We will assume that the list $l(s_i)$ is ordered by decreasing preferences. Therefore, if $s_i$ prefers $h_j$ to $h_k$, we can equivalently say that $h_j$ appears before $h_k$ or $h_k$ appears after $h_j$ on $s_i$’s preference list. Clearly, the order among the schools occurring after $\emptyset$ on $s_i$’s list is immaterial, since $s_i$ prefers remaining unassigned rather than being assigned to any one of them. Similarly, each school $h_j \in H$ has a strict and complete preference list, $l(h_j)$, over $S \cup \{\emptyset\}$. Once again, for each student $s_i$ occurring after $\emptyset$, $h_j$ prefers remaining under-filled rather than admitting $s_i$, and the order among these students is of no consequence.

Given a set of schools, $H' \subseteq H$, by the best school for $s_i$ in $H'$ we mean the school that $s_i$ prefers the most among the schools in $H'$. Similarly, given a set of students, $S' \subseteq S$, by the best student
for \( h_j \) in \( S' \) we mean the student whom \( h_j \) prefers the most among the students in \( S' \).

A matching \( M \) is a function, \( M : S \rightarrow H \cup \{\emptyset\} \), such that if \( M(s_i) = h_j \) then it must be the case that \( s_i \) prefers \( h_j \) to \( \emptyset \) and \( h_j \) prefers \( s_i \) to \( \emptyset \); if so, we say that student \( s_i \) is assigned to school \( h_j \). If \( M(s_i) = \emptyset \), then \( s_i \) is not assigned to any school. The matching \( M \) also has to ensure that the number of students assigned to each school \( h_j \) is at most \( c(j) \).

For a matching \( M \), a student-school pair \((s_i, h_j)\) is said to be a blocking pair if \( s_i \) is not assigned to \( h_j \), \( s_i \) prefers \( h_j \) to \( M(s_i) \) and one of the following conditions holds:

1. \( h_j \) prefers \( s_i \) to one of the students assigned to \( h_j \), or
2. \( h_j \) is under-filled and \( h_j \) prefers \( s_i \) to \( \emptyset \).

The blocking pair is said to be type 1 (type 2) if the first (second) condition holds. A matching \( M \) is said to be stable if there is no blocking pair for it.

Most of the mechanisms presented in this paper are dominant-strategy incentive-compatible, for the students, i.e., truth-telling is a weakly-dominant strategy for students: they cannot gain by being untruthful, regardless of what the others do. We will often shorten this term to DSIC mechanism.

### 3 Problem Definition and the Three Settings

As stated above, in this paper, we will study assignment of students to schools in two rounds, \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), which are temporally separated. In this section we state the three settings studied; for each, we will have two mechanisms, \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). In round \( \mathcal{R}_1 \), mechanism \( \mathcal{M}_1 \) finds a stable matching of students to schools, \( M \). In round \( \mathcal{R}_2 \), \( \mathcal{M}_2 \) extends \( M \) to \( M' \), which is also required to be stable; by extends we mean that additional student-school pairs are matched.

The students report their preference lists and the mechanisms operate on whatever is reported. We will assume that the schools’ preference lists are truthfully reported and we will show that in each of the three settings, \((\mathcal{M}_1, \mathcal{M}_2)\) are DSIC for students, hence showing that the students gain nothing by misreporting their preference lists.

We now state the common aspects of the first two settings before describing them completely; the third setting is quite different. In both, in round \( \mathcal{R}_1 \), the setup defined in Section 2.1 prevails and \( \mathcal{M}_1 \) simply computes the student-optimal stable matching denoted by \( M \). Let \( S_M \subseteq S \) be the set of students assigned to schools by \( M \) and \( L = S - S_M \) be the set of left-over students. As shown in [DF81], \( \mathcal{M}_1 \) is DSIC for students.

As described above, in round \( \mathcal{R}_1 \), the capacity of each school is fixed: to \( c(j) \) for \( h_j \). In round \( \mathcal{R}_2 \), the City has decided to extend matching \( M \) in a stable, DSIC manner to matching \( M' \) and will appropriately add extra seats to schools. Let \( c'(j) \) be the round \( \mathcal{R}_2 \) capacity of school \( h_j \), for \( h_j \in H \). Once we obtain the solution under this assumption, we will show how it can be modified in case the City can only add a fewer number of extra seats.

**Lemma 1.** For some student \( s_i \), let \( M(s_i) = h_j \). Then for any student \( s_k \in L \), \( s_k \) appears after \( s_i \) in \( l(h_j) \).
Proof. If $s_k$ were to appear before $s_i$ in $l(h_j)$, then $(s_k, h_j)$ will form a blocking pair for $M$, contradicting its stability. \hfill \square

3.1 Setting I

In this setting, the City wants to admit as many students from $L$ as possible in a stability-preserving, DSIC manner. We will call this problem $\text{Max}_L$. We will prove the following:

**Theorem 1.** There is a polynomial time mechanism $M_2$ that is DSIC for students and extends matching $M$ to $M'$ so that $M'$ is stable w.r.t. students $S$ and schools $H$. Furthermore, $M_2$ yields the largest matching that can be obtained by a mechanism satisfying the stated conditions.

Let $k$ be the maximum number of students that can be added from $L$, as per Theorem 1. Next, suppose that the City can only afford to add $k' < k$ extra seats. We show in Section 4.1 how this can be achieved while maintaining all the properties stated in Theorem 1.

3.2 Setting II

In this setting, a set $N$ of new students arrive from other cities in round $R_2$. In Setting II, the mechanisms are assumed to be oblivious in the following sense: they do not know the set of actual residents of the City, only what is reported to them. Thus, a resident of the City $s_i$ may opt to not report her preference list in round $R_1$ and the mechanisms will assume that she is not in $S$. In round $R_2$, $s_i$ may report her preference list, and if so, the mechanisms will assume she is in $N$ (in Setting III, the mechanisms will be non-oblivious). The preference lists of schools are also updated to include students in $N$ appropriately. In this setting, the City wants to give preference to students who were not matched in round $R_1$, i.e., $L$, over the new students, $N$. We will call this problem $\text{Min}_N \text{Max}_L$. We will prove the following:

**Theorem 2.** There is a polynomial time mechanism $M_2$ that is DSIC for students and accomplishes the following:

1. It finds smallest subset $N' \subseteq N$ with which the current matching can be extended in a stability-preserving manner.

2. Subject to the previous extension, it finds the largest subset $L' \subseteq L$ with which the current matching can be extended further in a stability-preserving manner.

3.3 Setting III

In Theorem 5, we will prove that no pair of mechanisms can ensure that revealing preference lists in round $R_1$ is a dominant strategy for the residents of the City. For this reason, in Setting III, we turn to mechanisms that are non-oblivious, i.e., they know the entire set $S$ of residents of the City and all residents are required to report their preference lists in round $R_1$. In this framework, we give a pair of mechanisms achieving a weaker incentive-compatibility guarantee (Theorem 3). In
round $R_1$, a subset $S_1 \subseteq S$ of students declare that they are seeking admission to public schools. Mechanism $M_1$ finds a student-optimal stable matching, $M$, on set $S_M \subseteq S_1$ of students. Let $L = S_1 - S_M$ denote the set of left-over students.

In round $R_2$, a subset of $(S - S_1)$ of the residents of the City decide to seek admission to public schools. In addition, new students arrive from other cities as well and provide their preference lists over schools and the schools also update their preference lists to include the new students. Let $N$ denote the set of all students who are seeking admission to public schools in round $R_2$; $N$ includes new students as well as residents.

As before, the City will not alter the matching of any students matched in $M$ and would like to maximize the number of students admitted from $L$, since they were left out in round $R_1$ and minimize the number of students matched from $N$; the extended matching, $M'$ again needs to be stable.

Finally, we require that $(M_1, M_2)$ be incentive compatible for students in the following sense: a student cannot gain by misreporting his preference list, participating only in $R_2$ even though he is in $S$, or both, assuming all other students are truthful. We say that $(M_1, M_2)$ is weak incentive-compatible (WIC) if the above condition is satisfied. We establish the following:

**Theorem 3.** There is a pair of polynomial time, non-oblivious mechanisms $(M_1, M_2)$ such that:

1. $M_1$ computes a stable matching, $M$, in round $R_1$ and $M_2$ extends it to $M'$ in a stability-preserving manner.

2. The combined mechanism is WIC for students, w.r.t. revealing true preference lists as well as not gaming their arrival time.

3. Subject to the above requirements, $M_2$ maximizes $L' \subseteq L$ among all mechanisms that minimize $N' \subseteq N$.

4 The Mechanisms

4.1 Setting I

We will first characterize situations under which a matching is not stable, i.e., admits a blocking pair. This characterization will be used for proving stability of matchings constructed in round $R_2$. For this purpose, assume that $M$ is an arbitrary matching, not necessarily stable nor related to the matching computed in round $R_1$. For each school $h_j \in H$, define the lowest preferred student assigned to $h_j$, denoted LPS-Assigned($h_j$), to be the student that $h_j$ prefers the least among the students that are assigned to $h_j$. F

Next, for each student $s_i \in S_M$, define the set of schools preferred by $s_i$, denoted Preferred-Schools($s_i$) by \{ $h_j$ | $s_i$ prefers $h_j$ to $M(s_i)$ \}; note that $m(s_i) = \emptyset$ is allowed in this definition. Further, for each school $h_j \in H$, define the set of students that prefer $h_j$ over the school they are assigned to, denoted Preferring-Students($h_j$) to be \{ $s_i$ | $h_j \in$ Preferred-Schools($s_i$) \}. Finally, define the best student preferring $h_j$, denoted BS-Preferring($h_j$), to be the student whom $h_j$ prefers the best in the set
Preferring-Students($h_j$). If Preferring-Students($h_j$) = $\emptyset$ then we will define BS-Preferring($h_j$) = $\emptyset$; in particular, this happens if $h_j$ is under-filled.

Lemma 2. W.r.t. matching $M$, there exists a blocking pair:

1. of type 1 iff there is a school $h_j$ s.t. $h_j$ prefers BS-Preferring($h_j$) to LPS-Assigned($h_j$).

2. of type 2 iff there is a school $h_j$ that is under-filled and a student $s_i$ such that $s_i$ prefers $h_j$ to $M(s_i)$ and $h_j$ prefers $s_i$ to $\emptyset$.

Proof. 1. Suppose for some school $h_j$, BS-Preferring($h_j$) = $s_i$ and LPS-Assigned($h_j$) = $s_k$, and $h_j$ prefers $s_i$ to $s_k$. Then, $s_i$ prefers $h_j$ to $M(s_i)$ and $h_j$ prefers $s_i$ to $s_k$. Therefore, $(s_i, h_j)$ is a blocking pair of type 1. Next, assume that $(s_i, h_j)$ is a blocking pair of type 1. Then, it must be the case that $s_i$ prefers $h_j$ to $M(s_i)$ and $h_j$ prefers $s_i$ to $s_k$, for some student $s_k$ that is assigned to $h_j$. Clearly, $h_j$ weakly prefers $s_k$ to LPS-Assigned($h_j$). Therefore $h_j$ prefers BS-Preferring($h_j$) to LPS-Assigned($h_j$).

2. Both directions follow from the definition of blocking pair of type 2.

The mechanism, $M_2$, for round $R_2$ for MaxL in Situation I is given in Figure 1. Step 1 simply ensures that the matching found by $M_2$ extends the round $R_1$ matching. Step 2 defines the Barrier for each school to be BStP($h_j$); observe that this could be $\emptyset$. Step 3 computes the subset of $L$ that needs to be assigned schools in a stability-preserving manner and Step 5 computes the school for each student in this subset.

Proof of Theorem 1: Suppose Barrier($h_j$) = $s_i$ (or, $\emptyset$). Since all students assigned to $h_j$ from $L$ appear before $s_i$ (respectively, $\emptyset$) in $l(h_j)$, therefore by Lemma 2, there is no type 1 (respectively, type 2) blocking pair. This establishes the stability of matching $M'$. Next, consider a student $s_k \in (L - L')$ and suppose she is assigned to school $h_j$. By the definition of $L'$, $h_j$ prefers Barrier($h_j$) to $s_k$, therefore, (Barrier($h_j$), $h_j$) form a blocking pair, which is of type 2 if Barrier($h_j$) = $\emptyset$ and type 1 otherwise. Hence the matching found in round $R_2$ is the largest stable extension of $M$.

For any student $s_i \in L'$, the Barriers are defined independent of her preference list and she is assigned to the best school in which she will not create a blocking pair. Hence, $M_2$ is DSIC for students.

For the problem of admitting fewer students, stated in Section 3.1, we give the following:

Proposition 1. Let $k$ be the total number of students added from $L$ in round $R_2$ in the previous theorem and let $k' < k$. There is a polynomial time mechanism $M_2$ that is stability-preserving, DSIC for students and extends matching $M$ to $M'$ so that $|M'| - |M| = k'$.

Proof. Let $c'$ denote the capacities of schools after round $R_2$ as per Theorem 1. Note that the total difference in capacities $c' - c$ over all schools is $k$, where $c$ is the capacity function in round $R_1$. Starting with $c'$, arbitrarily decrease the capacities of schools to obtain capacity function $c''$ so that for any school $h_j$, $c'(h_j) - c(h_j) \geq c''(h_j) - c(h_j) \geq 0$ and the total of $c'' - c$ over all schools is
**Max**$_L$($M, L$):

**Input:** Stable matching $M$ and set $L$.

**Output:** Stable, IC, $Max_L$ extension of $M$.

1. $\forall s_i \in S_M : M'(s_i) \leftarrow M(s_i)$
2. $\forall h_j \in H : \text{Barrier}(h_j) \leftarrow \text{BS-Preferring}(h_j)$.
3. $L' \leftarrow \{s_i \in L \mid \exists h_j \text{ s.t. } s_i \text{ appears before } \text{Barrier}(h_j) \text{ in } l(h_j),$
   and $h_j \text{ appears before } \emptyset \text{ in } l(s_i)\}$.
4. $\forall s_i \in L' : \text{Feasible-Schools}(s_i) \leftarrow \{h_j \mid s_i \text{ appears before } \text{Barrier}(h_j) \text{ in } l(h_j)\}$.
5. $\forall s_i \in L' : M'(s_i) \leftarrow \text{Best school for } s_i \text{ in } \text{Feasible-Schools}(s_i)$.
6. $\forall s_i \in (L - L') : M'(s_i) \leftarrow \emptyset$.
7. Return $M'$.

**Figure 1:** Mechanism for round $R_2$ for problem $Max_L$ in Setting I

$k'$. Starting with $M$, the round $R_1$ matching, run the Gale-Shapley algorithm with students from $L$ proposing and with current capacities fixed at $c''$.

We claim that when this algorithm terminates, the matching found will be student-optimal, stable and each school $h_j$ will be allocated $c''(h_j)$ students from $L$. To see the last claim, observe that the proposals received by any school $h_j$ will be weakly better than the $c'(h_j) - c(h_j)$ students of $L$ who were allocated to $h_j$ under matching $M'$.

### 4.2 Setting II

The mechanism for round $R_2$ for $Min_N Max_L$ in Situation II is given in Figure 2. Suppose there is a school $h_j$, student $s_k \in S_M$ is assigned to it and there is a student $s_i \in N$ such that $h_j$ prefers $s_i$ to $s_k$. Now, if $s_i$ is kept unmatched, $(s_i, h_j)$ will form a blocking pair of type 1 by Lemma 2. Next suppose $h_j$ is under-filled and there is a student $s_i \in N$ such that $h_j$ and $s_i$ prefer each other to $\emptyset$. This time, if $s_i$ is kept unmatched, $(s_i, h_j)$ will form a blocking pair of type 2 by Lemma 2. Therefore, all students in $N'$, computed in Step 3, need to be matched. Our mechanism keeps all students in $N - N'$ unmatched, thereby minimizing the number of students matched from $N$.

We next describe the various barriers that need to be defined. The first one, defined in Step 2, plays the same role as that in Figure 1. As before, if $h_j$ is under-filled, $\text{Barrier1}(h_j) = \emptyset$. If a student $s_i \in (N' \cup L')$ is assigned to $h_j$ so that $s_i$ appears after $\text{Barrier1}(h_j)$ in $l(h_j)$, then $(\text{Barrier1}(h_j), h_j)$ will form a blocking pair. The second one, $\text{Barrier2}(h_j)$ in $(N - N')$ defined in Step 4. Again, if $s_i \in (N' \cup L')$ is assigned to $h_j$ and $s_i$ appears after $\text{Barrier2}(h_j)$ in $l(h_j)$,
MinNMax\textsubscript{L}(M, N, L):

**Input:** Stable matching \(M\), and sets \(N\) and \(L\).

**Output:** Stable, IC, MinNMax\textsubscript{L} extension of \(M\).

1. \(\forall s_i \in S_M : M'(s_i) \leftarrow M(s_i)\)

2. \(\forall h_j \in H : \text{Barrier1}(h_j) \leftarrow \text{BS-Preferring}(h_j)\).

3. \(N' \leftarrow \{s_i \in N | \exists h_j \exists s_k \text{ s.t. } M(s_k) = h_j, h_j \text{ prefers } s_i \text{ to } s_k, \text{ and } s_i \text{ prefers } h_j \text{ to } \emptyset\} \cup \{s_i \in N | \exists h_j \text{ s.t. } h_j \text{ is under-filled and } h_j \text{ and } s_i \text{ prefer each other to } \emptyset\} \).

4. \(\forall h_j \in H : \text{Barrier2}(h_j) \leftarrow \text{Best student for } h_j \text{ in } (N - N')\).

5. \(\forall h_j \in H : \text{Barrier}(h_j) \leftarrow \text{Best student for } h_j \text{ in } \{\text{Barrier1}(h_j), \text{Barrier2}(h_j)\}\).

6. \(L' \leftarrow \{s_i \in L | \exists h_j \text{ s.t. } s_i \text{ appears before } \text{Barrier}(h_j) \text{ in } l(h_j), \text{ and } h_j \text{ appears before } \emptyset \text{ in } l(s_i)\}\).

7. \(\forall s_i \in N' \cup L' : \text{Feasible-Schools}(s_i) \leftarrow \{h_j | s_i \text{ appears before } \text{Barrier}(h_j) \text{ in } l(h_j)\}\).

8. \(\forall s_i \in N' \cup L' : M'(s_i) \leftarrow \text{Best school for } s_i \text{ in Feasible-Schools}(s_i)\).

9. \(\forall s_i \in ((L - L') \cup (N - N')) : M'(s_i) \leftarrow \emptyset\).

10. Return \(M'\).

Figure 2: Mechanism for round \(R_2\) for MinNMax\textsubscript{L} in Setting II

then \((\text{Barrier1}(h_j), h_j)\) will form a blocking pair. In step 5, \(\text{Barrier}(H_j)\) is defined to be the more stringent of these two barriers.

The final question is which school should \(s_i \in N'\) be matched to? One possibility is to compute for each student \(s_i\) the set

\[T(s_i) = \{h_j \in H | \exists s_k \text{ s.t. } M(s_k) = h_j \text{ and } h_j \text{ prefers } s_i \text{ to } s_k\},\]

and match \(s_i\) to her best school in \(T(s_i)\).

Assume that \(s_i\) is matched to \(h_j\) under this scheme. The problem now is that \(s_i\) may prefer school \(h_k\) (of course, \(h_k \notin T(s_i)\)) and moreover \(h_k\) prefers \(s_i\) to some student \(s_l \in L'\) who has been assigned to \(h_k\). If so, \((s_i, h_k)\) will form a blocking pair. One remedy is to redefine the barrier so \(s_l\) is not assigned to \(h_k\). However, this will make the barrier even more stringent and the resulting mechanism will, in general, match fewer students from \(L\) than our mechanism. The latter is as follows: simply match \(s_i\) to the best school which prefers her to the Barrier of that school.

**Proof. of Theorem 2:** The arguments given above already establish stability of matching \(M'\) com-
puted. Next, let us argue that the mechanism is DSIC for students in $N$ and $L$. The matching $M$ is not affected by the preference lists of $N$. Therefore the choice of $N'$ and hence $(N - N')$ is independent of the preference lists of $N$. Barrier1 is influenced only by preference lists of $S_M$ and Barrier2 by those of $(N - N')$. Hence Barrier is independent of the preference lists of $N'$ and $L'$. Hence, the matching of students in these two sets is also done in a DSIC manner.

As argued above, each student in $N'$ needs to be matched simply to preserve stability. Since our mechanism does not match any more students from $N$ it achieves $Min_N$. As argued above, not imposing the more stringent of the two barriers computed may result in a blocking pair. Therefore our mechanism imposes the minimum needed restrictions while matching students from $L$. Hence it achieves $Min_L Max_N$.

Next, we turn to a slightly different problem within Setting 2, namely find the largest subset of $(N \cup L)$ that can be matched in a stability-preserving and DSIC manner. We call this problem $Max_{N \cup L}$. As shown below, this mechanism also solves the problems $Max_N Max_L$ and $Max_L Max_N$, namely maximizing the number of students matched from $L$ after having maximized the number of students matched from $N$ and vice versa.

**Theorem 4.** There is a polynomial time mechanism $M_2$ that is DSIC for students and finds the largest subset of $(N \cup L)$ that can be matched to schools and added to the current matching while maintaining stability. This mechanism also solves $Max_N Max_L$ and $Max_L Max_N$.

**Proof.** We will show that the mechanism presented in Figure 1, with $(N \cup L)$ playing the role of $L$, suffices. Barriers for schools are computed as before in Step 2. Denote the subset of $(N \cup L)$ that is matched in round $R_2$ by $(N \cup L)'$; it consists of students $s_i \in (N \cup L)$ such that some school $h_j$ prefers $s_i$ to $\text{Barrier}(h_j)$ and $s_i$ prefers $h_j$ to $\emptyset$. If so, $s_i$ is assigned to the best such school.

The argument given in Theorem 1 suffices to show stability of the matching produced. Observe that Barrier, computed in Step 2, is independent of the preference lists of $N$ and $L$ and hence the mechanism is DSIC for $N$ and $L$. As before, matching any student from the rest of $(N \cup L)$ will lead to a blocking pair, and hence the mechanism maximizes the number of students matched in round $R_2$.

Finally, since this mechanism acts on $N$ and $L$ independently of each other, it also solves $Max_N Max_L$ and $Max_L Max_N$. 

**4.3 Setting III**

In the mechanisms given in Theorems 1, 2 and 4, no student can gain by falsifying her preference list. In Setting II, a different way for a student, who should legitimately be in the set $S$, to cheat is to arrive in $N$ instead in round $R_2$ and thereby get assigned to a better school. This is illustrated in Example 1.

**Example 1.** Assume there are 3 students $A, B, C$ and two schools 1, 2. The preference lists of the students are $l(A) = (2, 1), l(B) = (1, 2), l(C) = (1, 2)$. The preference lists of the schools are $l(1) = (C, A, B), l(2) = (B, A, C)$. In round $R_3$, each school has a capacity of 1 seat. $A$ and $C$ are both in set $S$
$\mathcal{M}_1(S, S_1, H, c)$:

**Input:** Sets $S$ of students, $H$ of schools, and $S_1 \subseteq S$ of students participating in round $\mathcal{R}_1$.

**Output:** Stable matching $M$ from $S_1$ to $H \cup \{\emptyset\}$.

1. $M \leftarrow$ Student-optimal stable matching from $S$ to $H \cup \{\emptyset\}$ under school capacities $c$.

2. Remove all students of $(S - S_1)$ from $M$.

3. While $\exists$ school $h_j$ which participates in a type 2 blocking pair:
   
   (a) Let $s_i = \text{BS-Preferring}(h_j)$.
   
   (b) $M \leftarrow M \cup \{s_i h_j\}$.

4. Return $M$.

**Figure 3:** Mechanism for round $\mathcal{R}_1$ in Setting III

In round $\mathcal{R}_1$. If B is also in set $S$ in round $\mathcal{R}_1$, she will be matched to school 2. On the other hand, if she is in set $N$ in round $\mathcal{R}_2$, she will be matched to school 1, which she prefers.

In fact, no mechanism can be DSIC w.r.t. application time in setting II, as shown in Example 2.

**Example 2.** Assume there are 4 students $A, B, C, D$ and 3 schools 1, 2, 3. The preferences for $A, B, C, D$ are $(1, 2, 3), (2, 1, 3), (2, 3, 1), (2, 3, 1)$ respectively. The preferences for 1, 2, 3 are $(B, A, C, D), (A, C, B, D), (C, B, A, D)$. In $\mathcal{R}_1$, each school has 1 seat. The only stable matching when all students come in $\mathcal{R}_1$ is $1B, 2A, 3C$. If C does not come in $\mathcal{R}_1$, there are two possible stable matchings $1B, 2A, 3D$ and $1A, 2B, 3D$. Consider two cases:

1. $1A, 2B, 3D$ is chosen. In this case, C will have incentive to come in $\mathcal{R}_2$. The reason is that, B is assigned to school 2 in $\mathcal{R}_1$, and hence, to maintain stability, school 2 must accept C in $\mathcal{R}_2$.

2. $1B, 2A, 3D$ is chosen. Now B will have incentive to come later in the instance with only B, A and D. Without B in $\mathcal{R}_1$, there is only one stable matching $1A, 2D$. When B comes later, she will be matched to 2 so that stability is preserved.

Hence, choosing neither of the two possible matchings results in a DSIC mechanism.

As a consequence of Example 2 we get:

**Theorem 5.** There is no pair of oblivious, stability-preserving, DISC mechanisms $(M_1, M_2)$ that ensure that residents of the City can’t gain by opting to participate only in round $\mathcal{R}_2$.

To get around this impossibility result, we move to the more stringent, non-oblivious Setting III described in Section 3.3 and prove Theorem 3.
\[ \mathcal{M}_2(M, N, L) : \]
**Input:** Stable matching \( M \) and sets \( N, L \).
**Output:** Stable, IC, \( \min_N \max_L \) extension of \( M \).

1. \( \forall s_i \in N : M'(s_i) \leftarrow M(s_i) \).
2. \( \forall s_i \in N : \text{Preferring-Schools}(s_i) \leftarrow \{ h_j \in H \mid h_j \text{ prefers } s_i \text{ to LPS-Assigned}(h_j), \text{ and } s_i \text{ prefers } h_j \text{ to } \emptyset \} \cup \{ h_j \in H \mid h_j \text{ is under-filled and } h_j \text{ and } s_i \text{ prefer each other to } \emptyset \} \).
3. \( N' \leftarrow \{ s_i \in N \mid \text{Preferring-Schools}(s_i) \neq \emptyset \} \).
4. \( \forall s_i \in N' : M'(s_i) \leftarrow \text{Best school for } s_i \text{ in Preferring-Schools}(s_i) \).
5. \( \forall s_i \in (N - N') : M'(s_i) \leftarrow \emptyset \).
6. \( \forall h_j \in H : \text{Barrier}(h_j) \leftarrow \text{BS-Preferring}(h_j) \text{ w.r.t. } M' \).
7. \( L' \leftarrow \{ s_i \in L \mid \exists h_j \text{ s.t. } s_i \text{ appears before } \text{Barrier}(h_j) \text{ in } l(h_j), \text{ and } h_j \text{ appears before } \emptyset \text{ in } l(s_i) \} \).
8. \( \forall s_i \in L' : \text{Feasible-Schools}(s_i) \leftarrow \{ h_j \mid s_i \text{ appears before } \text{Barrier}(h_j) \text{ in } l(h_j) \} \).
9. \( \forall s_i \in L' : M'(s_i) \leftarrow \text{Best school for } s_i \text{ in Feasible-Schools}(s_i) \).
10. \( \forall s_i \in (L - L') : M'(s_i) \leftarrow \emptyset \).
11. Return \( M' \).

Figure 4: Mechanism for round \( R_2 \) in Setting III
The mechanisms $M_1$ and $M_2$ are given in Figures 3 and 4 respectively. $M_1$ first computes the student-optimal matching for all students in $S$. Then all students in $(S - S_1)$ are removed from the matching. Clearly, there will be empty seats, and hence, type 2 blocking pairs may be formed. While there exists a blocking pair of type 2 including a school $h_j$, student $s_i = \text{BS-Preferring}(h_j)$ is reassigned to $h_j$. This cuts down an empty seat in $h_j$, but creates another empty seat in the school that $s_i$ was previously matched to. As shown in Lemma 3, this process terminates in polynomial time.

**Lemma 3.** Step 3 in mechanism $M_1$, given in Figure 3, runs in polynomial time.

**Proof.** When a reassignment happens, the school-assignment of the student improves, i.e., she prefers the new school to the previous match. Therefore, at most $nm$ reassignments can take place.

Reassigning $s_i$ to $h_j$ cannot create type 1 blocking pairs, as proven later in Lemma 4, but can potentially fix type 2 blocking pairs containing $h_j$. The mechanism returns the final matching when no blocking pairs remain.

**Lemma 4.** The matching $M$ returned by $M_1$ is stable.

**Proof.** Clearly, the matching is stable after Step 1. Moreover, Step 2 introduces only blocking pairs of type 2. Hence, at the beginning of Step 3, where the reassignments are made, only blocking pairs of type 2 exist. A reassignment for school $h_j$ matches $h_j$ to $s_i = \text{BS-Preferring}(h_j)$. Hence, after the reassignment, $s_i$ becomes LPS-Assigned($h_j$), and the new BS-Preferred($h_j$) if exists, must appear after $s_i$ in the preference list of $h_j$. By Lemma 2, no blocking pairs of type 1 are created. When Step 3 finishes, all blocking pairs of type 2 are also fixed. Since Step 3 must finish in polynomial time according to Lemma 3, the matching $M$ returned is stable.

The second mechanism $M_2$ is similar to the one given in Figure 2. The main difference is that, instead of allocating student $s_i \in N'$ in the best school among her possible matches (Feasible-Schools($s_i$)), we accept her in the best among the schools that prefer her to their least preferred student, or are under-filled (Preferring-Schools($s_i$)). As a consequence, Barrier($h_j$), for each $j$, also needs to be updated in a suitable way. By a similar argument to the one given in Section 4.2, we have the following lemma:

**Lemma 5.** The matching $M'$ returned by $M_2$ is stable.

Next we show that the pair of mechanisms ($M_1, M_2$) is WIC, i.e., no student $s_i$ can gain by not being truthful, assuming all other students are truthful. Throughout the argument, we consider two scenarios: the scenario where $s_i$ is truthful, denoted by superscript $t$, and the scenario where $s_i$ is cheating, denoted by superscript $c$. For example, $M^t$ is the matching returned by $M_1$ if $s_i$ is truthful, and $M^c$ is the matching returned by $M_1$ if $s_i$ is cheating. The analysis is divided into two cases: $s_i \in S$ and $s_i \notin S$.

**First case:** Suppose $s_i$ is a student in $S$. Then $s_i$ can cheat by misreporting her preferences, or arriving only in $R_2$, or both.
Lemma 6. If $s_i$ arrives in $\mathcal{R}_1$, she cannot gain by misreporting her preferences.

Proof. Since $s_i$ comes in $\mathcal{R}_1$, the first mechanism $M_1$ becomes simply computing the student-optimal matching on $S$.

If $s_i \in L^1$, i.e., $s_i$ is not matched in the first round even if she report her true preferences, then $s_i$ cannot be matched in the first round regardless of the preferences she reports. This dues to the incentive-compatibility of the student-proposing Gale-Shapley algorithm [DF81]. In $\mathcal{R}_2$, $s_i$ is assigned to the best possible school, according to her reported list, subject to the Barriers. Since the Barriers are independent of her preference list, she cannot gain by misreporting.

If $s_i \in S^t_M$, the school which she is assigned to in $M'$ will be her final school. Again, by the incentive-compatibility of Gale-Shapley, she cannot be matched to a better school.

Lemma 7. If $s_i$ only comes in $\mathcal{R}_2$, she is assigned to the same school that she would be assigned to had she come in $\mathcal{R}_1$ also, regardless of the preferences she reports.

Proof. Since $s_i$ comes in $\mathcal{R}_2$, $M'$ can be obtained from $M_0$, the student-optimal matching (w.r.t. the preference list $s_i$ report), by the following procedure:

- Let $h_j = M_0(s_i)$. Remove $s_i$ from $M_0$.
- If there is no student preferring $h_j$ than her current school, stop. Otherwise, let $s_{i_1}$ be BS-Preferring($h_j$) and $h_{j_1} = M_0(s_{i_1})$. Match $s_{i_1}$ to $h_{j_1}$.
- If there is no student preferring $h_{j_1}$ than her current school, stop. Otherwise, let $s_{i_2}$ be BS-Preferring($h_{j_1}$) and $h_{j_2} = M_0(s_{i_2})$. Match $s_{i_2}$ to $h_{j_1}$.

... The procedure ends when there is no student preferring $h_{j_k}$ for some $k$. We know that the procedure is finite according to Lemma 3. Construct a directed graph $G$ whose vertices are in $H \cup \{\emptyset\}$ and an edge for each reassignment:

$$E(G) = \{v \xleftarrow{x} u \mid \text{student } x \text{ is reassigned from school } u \text{ to school } v \text{ in the above procedure.}\}$$

Claim 1. $G$ is acyclic.

Proof. Consider building $G$ in the order of the procedure. Notice that each reassignment sends a student to the school where the previously reassigned student left. Hence, each new edge that appears will point towards the tail of the previous edge. Let $C$ be the first cycle that appears, for the sake of contradiction. Suppose

$$C = u_1 \xleftarrow{x_1} u_2 \xleftarrow{x_2} u_3 \ldots u_t \xleftarrow{x_t} u_1.$$ 

In particular, the reassignment of $x_1$ appears first and the reassignment of $x_t$ appears last among all assignments in the cycle. Consider $G$ at the point where $C$ appears. Since $C$ is the first cycle appearing, each school in $C$ has exactly one incoming edge. In other words, there are no students reassigned to any schools in $C$ before the reassignment of $x_1$. Therefore, $x_i = \text{BS-Preferring}(u_i)$ for all $1 \leq i \leq t$ w.r.t. $M_0$. By Lemma 2, applying the reassignments in $C$ to $M_0$ creates no
blocking pairs. Hence, the result matching \( M_1 \) is stable. Since all students are reassigned to better schools, \( M_0 \) is not student-optimal, which is a contradiction. 

Hence, \( E(G) \) forms a path from some school (possibly \( \emptyset \)) to \( h_j \). Now suppose that \( s_i \) is assigned to school \( h_k \) by \( M_2 \). For the sake of contradiction, assume that \( s_i \) prefers \( h_k \) to \( h_j \). Since \( s_i \) is assigned to school \( h_k \) by \( M_2 \), \( h_k \) is in Preferring-Schools(\( s_i \)). There are two possibilities:

1. \( h_k \) is under-filled in \( M^c \). Then \( h_k \) would have been assigned to \( s_i \) if \( s_i \) had come in \( R_1 \).
2. \( h_k \) prefers \( s_i \) to LPS-Assigned(\( h_k \)) at some point. In this case, \( h_k \) must be in the path formed by \( E(G) \). Otherwise, \( s_i h_k \) is a blocking pair of type 1 at \( M_0 \). Let \( P \) be the path from \( h_k \) to \( h_j = M_0(s_i) \) in \( G \):

\[
P = h_j \leftarrow s_{i_1} h_{j_1} \leftarrow s_{i_2} \ldots h_k.
\]

Let \( M_1 \) be the matching obtained by reassigning \( s_i \) from \( h_j \) to \( h_k \) and applying all reassignments in \( P \). Then all students reassigned in the process are BS-Preferring of the schools they are reassigned to w.r.t. \( M_0 \). By Lemma 2, no blocking pairs are created and \( M_1 \) is stable. Moreover, the reassigned students get better schools in \( M_1 \). This contradicts the student-optimality of \( M_0 \).

By Lemmas 6 and 7, a student in \( S \) cannot gain by misreporting her preferences, or arriving only in \( R_2 \), or both. This completes the analysis for the first case.

**Second case:** If \( s_i \) is a not student in \( S \), she comes in \( R_2 \). Hence, she can only cheat by misreporting her preference list. Since Preferring-Schools(\( s_i \)) is independent of the list she reports, and she is matched to the best school in Preferring-Schools(\( s_i \)) w.r.t the list, she cannot gain by misreporting.

From the argument in the above two cases, we have:

**Lemma 8.** \((M_1, M_2)\) is WIC.

**Proof of Theorem 3.** By Lemmas 4 and 5, both matchings returned by \( M_1 \) and \( M_2 \) are stable. Moreover, the pair \((M_1, M_2)\) is WIC according to Lemma 8. Notice that, all students in \( N' \) have to be matched to maintain stability. \( M_2 \) minimizes the set of matched students in \( N \). Moreover, in order for WIC to hold, a student \( s_i \) in \( N \) must be matched to the best school in Preferring-Schools(\( s_i \)). By the assignments of students in \( N' \), Barriers are created such that: if a student in \( L \) appears after the Barriers of all schools, assigning her will create a blocking pair. Hence, to preserve stability, those students cannot be matched. Since \( M_2 \) assigns all other students \( L' \) in \( L \), \( M_2 \) maximizes the set of matched students in \( L \), subject to \( N' \).

5 **Enumeration of Stable Extensions**

In this section we show how to enumerate all the possible stable extensions of a given stable matching with polynomial delay between any two enumerated matchings. Specifically, the al-
**StableExtension**(*M*, *c*, *N*):

**Input**: Stable matching *M*, capacity *c*, new students *N* = \{s₁, s₂ . . . sₖ\}.

**Output**: Stable extensions of *M*, with polynomial delay.

\[ M₀ ← M \]

\[ A₁ = \text{FeasibleAssignment}(M₀, c, s₁) \]

For *i₁* in *A₁*:

\[ M₁ ← \text{Starting from } M₀, \text{ match } s₁ \text{ to } i₁. \]

\[ A₂ = \text{FeasibleAssignment}(M₁, c, s₂) \]

For *i₂* in *A₂*:

\[ \vdots \]

\[ Aₖ = \text{FeasibleAssignment}(Mₖ₋₁, c, sₖ) \]

For *iₖ* in *Aₖ*:

\[ Mₖ ← \text{Starting from } Mₖ₋₁, \text{ match } sₖ \text{ to } iₖ. \]

Enumerate *Mₖ*.

---

**Figure 5**: Algorithm for enumerating stable extensions of *M*.

The algorithm takes as input a stable matching *M* from *S* to *H* satisfying capacity *c* and a set of new students *N* = \{s₁, s₂ . . . sₖ\} that can be added to the schools. Here the preference lists of all schools and students are also given. The algorithm enumerates all solutions *M'* from *S* ∪ *N* to *H* ∪ {∅} such that:

- all assignments in *M* are preserved in *M'*, and
- *M'* is stable with respect to capacity *c'* where

\[ c'(j) = \begin{cases} |M'⁻¹(h_j)| & \text{if } |M'⁻¹(h_j)| > c(j), \\
                 c(j) & \text{otherwise.} \end{cases} \]  

Note that *M'⁻¹(h_j)* is the set of students assigned to *h_j* under *M'*. We say that *M'* is a stable extension of *M* with respect to *N*.

The complete algorithm **StableExtension**(*M*, *c*, *N*) is given in Figure 5. At a high level, the algorithm maintains a stable extension *Mₐ* of *M* with respect to a subset *N'ₐ* of *N*. At each step,
**FeasibleAssignment**($M_c, c, s_i$):

**Input:** Stable matching $M_c$, capacity $c$, student $s_i$.

**Output:** Set $A_i$ of all possible assignments for $s_i$. Adding any assignment in $A_i$ to $M_c$ preserves stability.

1. Initialize $A_i$ to the empty set.

2. For each $h$ in $l(s_i)$, in decreasing order of preferences, do:
   
   (a) If $h = \emptyset$ then Return $A_i \cup \{\emptyset\}$.

   (b) Else $h = h_j$;
      
      i. If $|M_c^{-1}(h_j)| < c(j)$ then Return $A_i \cup \{h_j\}$.

      ii. If $s_i$ appears before LPS-Assigned$(h_j)$ then Return $A_i \cup \{h_j\}$.

      iii. If $s_i$ appears after LPS-Assigned$(h_j)$ and before BS-Preferring$(h_j)$ then $A_i \leftarrow A_i \cup \{h_j\}$.

Figure 6: Algorithm for finding feasible matches of $s_i$ w.r.t. current matching $M_c$.

A student $s_i$ is added to $N'$ and all possible assignments $A$ of $s_i$ that are compatible to $M_c$ are identified. In other words, adding each assignment in $A$ to $M_c$ gives a stable extension of $M$ with respect to $N' \cup \{s_i\}$. The algorithm branches to an assignment in $A$ and continues to the next student. When $N' = N$, the current matching is returned. The algorithm then backtracks to a previous branching point and continues.

Figure 6 gives the subroutine for finding compatible assignments. It takes on input the current matching $M_c$, capacity $c$, student $s_i$ and finds all possible assignments $A_i$ of $s_i$ to $H \cup \{\emptyset\}$ such that stability is preserved. Initially, $A_i$ is set to be an empty set. The subroutine then goes through the preference list of $s_i$ one by one in decreasing order. The considered school $h$ is added to $A_i$ and the subroutine terminates if at least one of the following happens:

- $h$ is $\emptyset$,
- $h$ is under-filled,
- $h$ prefers $s_i$ to LPS-Assigned$(h)$ with respect to $M_c$.

Notice that in the last two scenarios above, if $s_i$ was assigned to any school after $h$ in her preference list, $(s_i, h)$ would form a blocking pair. Assume none of the above scenarios happens. The subroutine adds $h$ to $A$ and continues if $h$ prefers $s_i$ to BS-Preferring$(h)$. Otherwise, $h$ prefers BS-Preferring$(h)$ to $s_i$. Hence, assigning $s_i$ to $h$ would create a blocking pair. The subroutine continues to the next school in this case. The following lemma says that **FeasibleAssignment** correctly finds all possible assignments of a student, given the current matching, at each step.
Lemma 9. Let \( N' \) be the set of students assigned (possibly to \( \emptyset \)) in \( M_e \), i.e., \( M_e \) is a stable extension of \( M \) with respect to \( N' \). FeasibleAssignment\((M_e, c, s_i)\) finds all possible assignments of \( s_i \) to \( H \cup \{ \emptyset \} \) such that adding each assignment to \( M_e \) gives a stable extension of \( M \) with respect to \( N' \cup \{ s_i \} \).

Proof. By the above argument, FeasibleAssignment considered all possible assignments. It suffices to prove that adding each to \( M_e \) gives a stable extension.

FeasibleAssignment only assigns student \( s_i \) to school \( h_j \) whenever \( s_i \) is before BS-Preferring\((h_j)\) in the preference list of \( h_j \). Moreover, whenever \( s_i \) is before LPS-Assigned\((h_j)\), no school after \( h_j \) in the list of \( s_i \) is considered. By Lemma 2, no blocking pair of type 1 is created.

Similarly, when \( h_j \) is under-filled, no school after \( h_j \) in the list of \( s_i \) is considered. Hence, no blocking pair of type 2 is created according to Lemma 2.

Finally, whenever \( h_j \) is filled, the capacity \( c(j) \) increases by 1 according to (1). Hence, adding the assignment of \( s_i \) to \( h_j \) gives a stable matching with respect to the new capacity.

Another important observation is that there is at least one possible assignment returned by FeasibleAssignment\((M_e, c, s_i)\) for any input of the subroutine. To see this, consider two cases:

1. if all schools \( h_j \) that appear before \( \emptyset \) in the preference list of \( s_i \) prefer BS-Preferring\((h_j)\) to \( s_i \), then \( \emptyset \) is a possible assignment for \( s_i \),
2. if at least one school \( h_j \) that appears before \( \emptyset \) in the preference list of \( s_i \) prefers \( s_i \) to BS-Preferring\((h_j)\), \( h_j \) is a possible assignment.

Hence, we have the following lemma:

Lemma 10. FeasibleAssignment\((M_e, c, s_i)\) returns at least one possible assignment.

From Lemmas 9 and 10, we can prove the main theorem of this section:

Theorem 6. StableExtension\((M_e, c, N)\) enumerates all possible stable extension of \( M \) with respect to \( N \). Moreover, the time between any two enumerations is \( O((k + n)m) \).

Proof. Let \( M' \) be a stable extension of \( M \) with respect to \( N \). Let \( i_1, i_2 \ldots i_k \) be the assignment of \( s_1, s_2 \ldots s_k \) respectively. For \( 1 \leq l \leq k \), denote by \( M_l \) the matching obtained by adding \( s_1i_1, s_2i_2, \ldots s_li_l \) to \( M \). By Lemma 9, \( i_{l+1} \) must be in the set of possible assignments returned by FeasibleAssignment\((M_l, c, s_i)\). Hence, StableExtension\((M_e, c, N)\) correctly enumerates \( M_k = M' \) at some point.

By Lemma 10, there are at most \( k \) calls of FeasibleAssignment between two matchings enumerated by StableExtension. Each call of FeasibleAssignment goes through a student preference list of at most \( m + 1 \) schools (including \( \emptyset \)). The time needed for initializing and updating LPS-Assigned\((h_j)\) and BS-Preferring\((h_j)\) for each school \( h_j \) at each step is \( O((k + n)m) \). Hence the total time is \( O((k + n)m) \).
6 Discussion

Designing a mechanism that actually gets used in practice involves consideration of numerous low-level details, which is not the focus of this paper. Indeed, this paper should be viewed as a work in algorithm design, motivated by natural questions from school choice, and adhering fairly closely to the ground rules of that discipline. At the same time, it is important to note that algorithmic insights obtained from work done in a mathematically clean, abstract setting can be immensely powerful and once put in the hands of practitioners, can lead to implementations that beat the ones obtained by simply working on the low-level details of the problem domain. Indeed, this has been observed time and again\(^1\). This precisely is the power of the “algorithmic way of thinking,” a key paradigm developed within Theoretical Computer Science.

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References

[ACP\(^{+}\)17] Atila Abdulkadiroglu, Yeon-Koo Che, Parag A Pathak, Alvin E Roth, and Olivier Tercieux. Minimizing justified envy in school choice: The design of new orleans’ oneapp. Technical report, National Bureau of Economic Research, 2017.

[APR09] Atila Abdulkadiroğlu, Parag A Pathak, and Alvin E Roth. Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match. American Economic Review, 99(5):1954–78, 2009.

[AS03] Atila Abdulkadiroğlu and Tayfun Sonmez. School choice: A mechanism design approach. American economic review, 93(3):729–747, 2003.

[AS13] Atila Abdulkadiroglu and Tayfun Sonmez. Matching markets: Theory and practice. Advances in Economics and Econometrics, 1:3–47, 2013.

[DF81] Lester E Dubins and David A Freedman. Machiavelli and the Gale-Shapley algorithm. The American Mathematical Monthly, 88(7):485–494, 1981.

[Ech19] Federico Echenique. Private communication, 2019.

[EP16] Federico Echenique and Juan Sebastián Pereyra. Strategic complementarities and unraveling in matching markets. Theoretical Economics, 11(1):1–39, 2016.

\(^1\)E.g., consider [MSVV05], which is also a matching market. A heuristic adaptation of the main idea of this algorithm, called bid scaling, is widely used by search engine companies today for implementing budgeted auctions.
[FKLS18] Itai Feigenbaum, Yash Kanoria, Irene Lo, and Jay Sethuraman. Dynamic matching in school choice: Efficient seat reallocation after late cancellations. 2018.

[ftToC19] Simons Institute for the Theory of Computing. Online and matching-based market design, 2019. https://simons.berkeley.edu/programs/market2019.

[GDE12] Joseph Gasper, Stefanie DeLuca, and Angela Estacion. Switching schools: Revisiting the relationship between school mobility and high school dropout. American Educational Research Journal, 49(3):487–519, 2012.

[KK18] Sangram V Kadam and Maciej H Kotowski. Multiperiod matching. International Economic Review, 59(4):1927–1947, 2018.

[MSVV05] A. Mehta, A. Saberi, U. Vazirani, and V. Vazirani. Adwords and generalized on-line matching. Journal of the ACM, 54(5), 2005.

[Pat11] Parag A Pathak. The mechanism design approach to student assignment. Annu. Rev. Econ., 3(1):513–536, 2011.

[Rot84] Alvin E. Roth. Stability and polarization of interests in job matching. Econometrica: Journal of the Econometric Society, pages 47–57, 1984.

[Rot08] Alvin E Roth. What have we learned from market design? Innovations: Technology, Governance, Globalization, 3(1):119–147, 2008.

[Rot16] Alvin E. Roth. Al Roth’s game theory, experimental economics, and market design page, 2016. http://stanford.edu/~alroth/alroth.html#MarketDesign.

[RS12] Alvin E. Roth and Lloyd S. Shapley. Nobel Memorial Prize in Economics, 2012. https://www.nobelprize.org/prizes/economic-sciences/2012/summary/.