Toric 3-folds defined by quadratic binomials

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Abstract
Let $(X, A)$ be a polarized nonsingular toric 3-fold with $\Gamma(X, A + K_X) = 0$. Then for any ample line bundle $L$ on $X$ the image of the embedding $\Phi_L : X \to \mathbb{P}(\Gamma(X, L))$ is an intersection of quadrics.

Keywords Toric varieties · Lattice polytopes

Mathematics Subject Classification 14M25 · 52B20

1 Introduction

Sturmfels (1997, Conjecture 2.9) asked whether a nonsingular projective toric variety should be defined by quadrics if it is embedded by global sections of a normally generated ample line bundle. An evidence has been obtained by Koelman (1993b) before Sturmfels asked the question. Koelman showed that projective toric surfaces are defined by binomials (differences of two monomials) of degree at most three in Koelman (1993a, Theorem 2.7) and obtained a criterion when the surface needs defining equations of degree three (Koelman 1993b, Theorem 1). He used combinatorics of plane polygons.

Let $X$ be a projective algebraic variety and let $L$ an ample line bundle on it. If the natural homomorphism

$$\phi : S := \text{Sym} \, \Gamma(X, L) \longrightarrow R := \bigoplus_{k \geq 0} \Gamma(X, L^\otimes k)$$

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is surjective, then we call $L$ normally generated. This notation comes from Mumford (1969). A normally generated ample line bundle is always very ample, but not conversely. For example, we know from Lange and Martens (1985) that on a hyperelliptic curve of genus $g$ there exists no normally generated line bundle of degree $d \leq 2g$. We define the ideal $I(X, L)$ of the graded ring $S$ as

$$I(X, L) := \text{Ker } \phi \subset S = \bigoplus_{k \geq 0} S_k.$$  

We call $L$ normally presented if $I(X, L)$ is generated by elements of degree two. This notion also comes from Mumford (1969).

Let $X$ be a toric variety of dimension $n$ and $L$ an ample line bundle on it. In general, $L$ is not very ample if $n \geq 3$. On the other hand, $L^\otimes k$ is normally generated for $k \geq n - 1$ (see Ewald and Wessels 1991, Theorem 1). And the ideal $I(X, L^\otimes k)$ is generated by quadrics

1. for $k \geq n$ (see Bruns et al. 1997, Theorem 1.4.1; Nakagawa and Ogata 2002, Theorem 0.1), or
2. for $k \geq n - 1$ and $n \geq 3$ (see Ogata 2003, Theorem 1).

We know that there exists a polarized toric variety $(X, L)$ of dimension $n \geq 3$ such that $L$ is very ample but $L^\otimes(n-2)$ is not normally generated (see Bruns and Gubeladze 2009, Exercise 2.23; Ogata 2013, Remark 5 in case $n = 3$). We also know that any ample line bundle on a nonsingular toric variety is always very ample (Demazure 1970; see also Oda 1988, Corollary 2.15). Ogata (2012) showed that an ample line bundle $L$ on a nonsingular toric 3-fold $X$ is normally generated if the adjoint bundle $L + K_X$ is not big.

In this paper we give a partial answer to Sturmfels’ question.

**Theorem 1** Assume that $(X, A)$ is a polarized nonsingular toric variety of dimension three with $\Gamma(X, A + K_X) = 0$. Let $L$ be an ample line bundle on $X$ and $\Phi_L : X \to \mathbb{P}(\Gamma(X, L))$ the associated embedding. Then the image $\Phi_L(X)$ is the common zero of quadratic binomials.

The proof is deduced from two propositions: Proposition 3 in Sect. 4 and Proposition 5 in Sect. 6.

In Sect. 2 we recall the basic fact about toric varieties and ample line bundles on them, and corresponding lattice polytopes. In Sect. 3 we give an algebro-geometric proof of the result of Koelman and explain the classification of $(X, A)$ satisfying the condition $\Gamma(X, A + K_X) = 0$ in Theorem 1. In Sect. 4 we discuss the binomials defining affine charts of $\Phi_L(X)$ and give a strategy to prove Theorem 1. In Sect. 5 we point out some property of nonsingular lattice polygons (Proposition 4). In Sect. 6 we give a proof of the main part of Theorem 1 as Proposition 5.
2 Polarized toric varieties

In this section we recall the fact about toric varieties and ample line bundles on them and corresponding lattice polytopes [see, for example, Oda’s book (Oda 1988) or Fulton’s book (Fulton 1993)].

Let \( M \) be a free abelian group of rank \( n \) and \( M_\mathbb{R} := M \otimes \mathbb{Z} \cong \mathbb{R}^n \) the extension of coefficients. Set \( \mathbb{C}[M] \) the group algebra of \( M \) and \( T := \text{Spec} \mathbb{C}[M] \cong (\mathbb{C}^\times)^n \) the algebraic torus of dimension \( n \). Then the group of characters \( \text{Hom}_{\text{gr}}(T, \mathbb{C}^\times) \) is isomorphic to \( M \). For an element \( m \in M \) we denote by \( \chi^m : T \to \mathbb{C}^\times \) the character corresponding to \( m \).

A toric variety \( X \) is a normal algebraic variety with an algebraic action \( T \times X \to X \) of the algebraic torus \( T \) such that \( X \) contains an open orbit \( O \) isomorphic to \( T \) and that the action is compatible with the inclusion \( T \cong O \to X \) and the multiplication \( T \times T \to T \).

We define a lattice polytope as the convex hull \( P := \text{Conv}\{m_1, \ldots, m_r\} \) of a finite subset \( \{m_1, \ldots, m_r\} \) of \( M \) in \( M_\mathbb{R} \). We define the dimension of a lattice polytope \( P \) as that of the smallest affine subspace \( \mathbb{R}(P) \) containing \( P \).

Let \( X \) be a projective toric variety of dimension \( n \) and \( L \) an ample line bundle on \( X \). Then there exists a lattice polytope \( P \) of dimension \( n \) such that the space of global sections of \( L \) is described by

\[
\Gamma(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\chi^m,
\]

where \( \chi^m \) is considered as a rational function on \( X \) since \( T \) is identified with the dense open subset (see Oda 1988, Section 2.2 or Fulton 1993, Section 3.5). We also have

\[
\Gamma(X, L \otimes \omega_X) \cong \bigoplus_{m \in \text{int}(P) \cap M} \mathbb{C}\chi^m,
\]

where \( \omega_X \) is the dualizing sheaf of \( X \).

Conversely, for a lattice polytope \( P \) in \( M_\mathbb{R} \) of dimension \( n \) set \( V(P) \) the set of vertices of \( P \). For each vertex \( v \in V(P) \) define the convex cone \( C_v(P) := \mathbb{R}_{\geq 0}(P - v) \) and the dual cone \( \sigma = C_v(P)^\vee \) in \( N_\mathbb{R} \), where \( N \) is the dual lattice to \( M \). Let \( \Delta \) be the fan in \( N \) consisting of all faces of \( \sigma(v) \) for all vertices \( v \in V(P) \). Set \( X \) the toric variety defined by \( \Delta \). For a vertex \( v \in V(P) \), the affine toric variety \( U_v := \text{Spec} \mathbb{C}[C_v(P) \cap M] \) is an open subset of \( X \). We obtain an affine open covering:

\[
X = \bigcup_{v \in V(P)} U_v.
\]

We can define a line bundle \( L \) satisfying

\[
\Gamma(U_v, L) \cong \chi^v \mathbb{C}[C_v(P) \cap M].
\]
Then $L$ is ample and satisfies the equality (2) (see Oda 1988, Theorem 2.22 or Fulton 1993, Section 1.5).

Let $A$ and $B$ be two ample line bundles on $X$, and $P_A$ and $P_B$ the corresponding lattice polytopes. Then $A \otimes B$ corresponds to the Minkowski sum

$$P_A + P_B := \{x + y \in M_R : x \in P_A \text{ and } y \in P_B\}$$

(see Fulton 1993, Section 1.5).

If $X$ is nonsingular, then all $U_v$ are isomorphic to $\mathbb{C}^n$. This implies that there exists a $\mathbb{Z}$-basis $\{m_1, \ldots, m_n\}$ of $M$ such that

$$C_v(P) = \mathbb{R}_{\geq 0}m_1 + \cdots + \mathbb{R}_{\geq 0}m_n$$

(see Oda 1988, Theorem 1.10).

### 3 Algebro-geometric approach

We recall the results of Koelman. He treated the case of dimension two.

**Theorem 2** (Koelman 1993a, b) Any ample line bundle $L$ on a projective toric surface $X$ is normally generated and the ideal $I(X, L)$ is generated by elements of degree at most three. Moreover, it is generated by quadrics unless $\Gamma(X, L \otimes \omega_X) \neq 0$ and $\dim \Gamma(X, L) - \dim \Gamma(X, L \otimes \omega_X) = 3$.

In his proof Koelman uses combinatorics of lattice polygons. Let $P$ be the lattice polygon corresponding to a polarized toric surface $(X, L)$. The condition $\Gamma(X, L \otimes \omega_X) \neq 0$ is equivalent to $\text{int}(P) \cap M \neq \emptyset$ and the difference of dimensions of cohomologies is the number of lattice points in the boundary $\partial P$ of $P$. Thus the ideal is generated by quadrics except that $P$ is a triangle containing lattice points in its interior and only three lattice points on the boundary. Such triangle $P$ corresponds to a singular toric surface $X$ isomorphic to $\mathbb{P}^2/G$. In fact, set $M = \mathbb{Z}^2$. We may assume that the vertices of $P$ are $0, (1, 0), (a, b)$ with $b > a \geq 2$ and $\gcd(a, b) = \gcd(a - 1, b) = 1$. Then $(1, 1) \in \text{int}(P)$. Set $n_1 = (b, -a), n_2 = (0, 1), n_0 = (-b, a - 1)$ in $N = \mathbb{Z}^2$. Then $n_0 + n_1 + n_2 = 0$. Set $N' = \mathbb{Z}n_0 + \mathbb{Z}n_1 + \mathbb{Z}n_2 \subset N$. Let $\Delta$ be the fan in $N'$ whose cones are generated by any proper subsets of vectors $n_0, n_1, n_2$. $\Delta$ in $N'$ defines the toric surface isomorphic to $\mathbb{P}^2$. Thus $\Delta$ in $N$ defines the toric surface $X \cong \mathbb{P}^2/G$ with $G = N/N'$.

Here we give a proof of Theorem 2 by using a method from projective geometry. Let $C \in |L|$ be a general member of the linear system of $L$. Then $C$ is a nonsingular curve of genus $g = \sharp(\text{int}(P) \cap M)$. Let $L_C$ denote the restriction to $C$. Then we have

$$\deg L_C = \sharp(\partial P \cap M) + 2g - 2.$$ 

Since $P$ is a convex polygon, $\sharp(\partial P \cap M) \geq 3$. The theorems of Fujita (1977, Corollaries 1.11 and 1.14) say that $L_C$ is normally generated if $\deg L_C \geq 2g + 1$ and that $I(C, L_C)$ is generated by quadrics if $\deg L_C \geq 2g + 2$. By regular ladder theorem (Fujita
1977, Theorem 4.1), we see that \( L \) is always normally generated, and that \( I(\mathbf{X}, L) \) is generated by quadrics if \( \sharp(\partial P \cap M) \geq 4 \).

Next, we consider the case of dimension three. Ogata (2012) classified the polarized toric 3-folds satisfying the condition in Theorem 1.

**Proposition 1** (Cf. Ogata 2012) Let \((X, A)\) be a nonsingular polarized toric variety of dimension three with \( \Gamma(X, A + K_X) = 0 \). Then \( X \) is one of the followings.

1. a blow up \( \mathbb{P}^3 \) along at most 4 invariant points,
2. a blow up \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}^1 \) along at most 2 invariant points,
3. a \( \mathbb{P}^1 \)-bundle over a nonsingular toric surface.

Let \( M = \mathbb{Z}^3 \) with a basis \( \{e_1, e_2, e_3\} \). Let \( Q \) be the lattice polytope of dimension three corresponding \((X, A)\) in Proposition 1. The condition \( \Gamma(X, A + K_X) = 0 \) implies that \( \text{int}(\mathbf{Q}) \cap M = \emptyset \). Set \( \Delta_3 := \text{Conv}\{0, e_1, e_2, e_3\} \) the basic 3-simplex. In order to describe the shape of \( Q \) we need one more word. For a lattice polytope \( R \) and a vertex \( v \) of \( R \), we call \( \text{Conv}(\mathbf{R} \cap M \setminus \{v\}) \) the polytope removed \( v \) from \( R \).

In case (1), \( Q \) is \( k\Delta_3 \) for \( 1 \leq k \leq 3 \), a polytope removed one vertex from \( 2\Delta_3 \) or a polytope removed at most 4 vertices from \( 3\Delta_3 \). See Fig. 1.

In case (2), \( Q \) is a prism \( P_{a,b,c}^{(1)} \) with the base \( \Delta_2 = \text{Conv}\{0, e_1, e_2\} \) and three edges of length \( a, b, c \geq 1 \), a prism \( P_{d,e,f}^{(2)} \) with the base \( 2\Delta_2 \) and three edges of length \( d, e, f \geq 1 \) such that \( e - f \) and \( e - d \) are both even, or a polytope removed one vertex from the base or one vertex from the roof of a prism \( P_{d,e,f}^{(2)} \). See Fig. 2.

In case (3), \( Q \) has parallel two facets \( F_0 \) and \( F_1 \) such that \( F_i \) is a lattice polygon corresponding to a polarized nonsingular toric surface \((Y, L_i)\). \( F_0 \) and \( F_1 \) have the same number of edges and corresponding edges are parallel. From \( \text{int}(\mathbf{Q}) \cap M = \emptyset \), we can choose a coordinate \((x, y, z)\) of \( M_\mathbb{R} \) such that \( F_i \) contains in the plane \( z = i \) for \( i = 0, 1 \).

**Proposition 2** Let \((X, A)\) be a polarized nonsingular toric 3-fold in Proposition 1. If \( X \) is (1) or (2), then the ideal \( I(\mathbf{X}, A) \) is generated by quadrics.
Proof Let $S_1, S_2 \in |A|$ be two general members of the linear system of $A$. Set $C = S_1 \cap S_2$. Then $C$ is a nonsingular curve. From easy calculation, we see that

$$h^0(C, A_C) = h^0(X, A) - 2 = \#(Q \cap M) - 2,$$

$$h^1(C, A_C) = h^3(X, A^{-1}) = h^0(X, A + K_X) = \#(\text{int}Q \cap M).$$

Thus we have

$$\chi(A_C) = \#(Q \cap M) - 2 - \#(\text{int}(Q) \cap M) = \#(Q \cap M) - 2.$$

Set $g(C)$ the genus of $C$. Write $S = S_1$. Let $K_S$ and $K_C$ be the canonical divisors of $S$ and $C$, respectively. From the adjunction formula, we see that

$$h^0(S, K_S) = h^0(X, A + K_X) \quad \text{and} \quad h^1(S, K_S) = 0,$$

$$h^0(S, A_S + K_S) = h^0(X, 2A + K_X).$$

From the exact sequence

$$0 \rightarrow \mathcal{O}_S(K_S) \rightarrow A_S \otimes \mathcal{O}_S(K_S) \rightarrow \mathcal{O}_C(K_C) \rightarrow 0$$

we have

$$h^0(C, K_C) = h^0(S, A_S + K_S) - h^0(S, K_S) = h^0(X, 2A + K_X) - 2h^0(X, A + K_X).$$

Finally we have

$$\chi(A_C) = \#(Q \cap M) - 2 \quad \text{and} \quad g(C) = \#(\text{int}(2Q) \cap M).$$

From the Riemann–Roch formula we have

$$\deg A_C = g(C) - 1 + \chi(A_C).$$
If $\chi(A_C) \geq g(C) + 3$, that is, $\sharp(Q \cap M) - \sharp(\text{int}(2Q) \cap M) \geq 5$, then Fujita’s theorem (Fujita 1977, Corollary 1.14) says that $I(C, A_C)$ is generated by quadrics. By calculation of the numbers $\sharp(Q \cap M)$ and $\sharp(\text{int}(2Q) \cap M)$, we see $\deg A_C \geq 2g(C) + 2$. By regular ladder theorem (Fujita 1977, Theorem 4.1) of Fujita, we obtain a proof. □

4 Ideals of affine charts

Let a projective variety $X$ be embedded by the complete linear system of a very ample line bundle $L$, $\Phi_L : X \to \mathbb{P}^r$. If you want to find polynomials defining $\Phi_L(X)$, it is an effective way to investigate polynomials defining its affine charts $\Phi_L(X) \cap (\mathbb{P}^r \setminus H_i)$. We know that toric varieties are defined by binomials (Sturmfels 1996, Lemma 4.1).

Let $P \subset M_{\mathbb{R}}$ be a lattice polytope corresponding to a nonsingular polarized toric variety $(X, L)$ of dimension three. Since $X$ is a union of affine toric varieties $U_v (v \in V(P))$, the embedding $\Phi_L : X \to \mathbb{P}(\Gamma(X, L))$ is defined by that of affine charts $U_v \subset X$:

$$\Phi_L|U_v : U_v = \text{Spec } \mathbb{C}[C_v(P) \cap M] \to \text{Spec}(\text{Sym}(\langle P - v \rangle \cap M) \mathbb{C})$$

for all $v \in V(P)$.

Set $P \cap M = \{m_0, m_1, \ldots, m_r\}$. We assume that $m_0$ is a vertex of $P$ and that $m_1, m_2, m_3$ are the lattice points nearest to $m_0$ on three edges meeting each other at $m_0$. Then the convex cone $C_{m_0}(P)$ is

$$C_{m_0}(P) = \mathbb{R}_{\geq 0}(m_1 - m_0) + \mathbb{R}_{\geq 0}(m_2 - m_0) + \mathbb{R}_{\geq 0}(m_3 - m_0).$$

Since $\{m_1 - m_0, m_2 - m_0, m_3 - m_0\}$ is a $\mathbb{Z}$-basis of $M \cong \mathbb{Z}^3$,

$$U_{m_0} = \text{Spec } \mathbb{C}[C_{m_0}(P) \cap M] \cong \mathbb{C}^3.$$

Let $Z_0, Z_1, \ldots, Z_r$ be the homogeneous coordinates of $\mathbb{P}(\Gamma(X, L)) \cong \mathbb{P}^r$ corresponding to $P \cap M$. We consider the affine chart $\Phi_L(X) \cap (Z_0 \neq 0)$. Set $x_i = Z_i / Z_0$. Then $(Z_0 \neq 0) = \text{Spec } \mathbb{C}[x_1, \ldots, x_r] \cong \mathbb{C}^r$. Since $\{m_1 - m_0, m_2 - m_0, m_3 - m_0\}$ is a $\mathbb{Z}$-basis of $M \cong \mathbb{Z}^3$, for $i \geq 4$ we can write

$$m_i - m_0 = \sum_{j=1}^{3} a_{ij} (m_j - m_0) \quad (a_{ij} \geq 0)$$

in a unique way. From this expression we define binomials as

$$f_i = x_i - \prod_{j=1}^{3} x_j^{a_{ij}}.$$
Then we have

\[ \mathbb{C}[C_{m_0}(P) \cap M] \cong \mathbb{C}[x_1, \ldots, x_r]/(f_4, \ldots, f_r). \]

Here we define a property “2-D(m_0)”:

**Definition 1** Let \( P \) be a nonsingular lattice polytope in \( M_{\mathbb{R}} \) of dimension three. Let \( m_0 \) be a vertex of \( P \) and let \( m_1, m_2, m_3 \) be the lattice points nearest to \( m_0 \) on three edges meeting each other at \( m_0 \). Then we say that \( P \) satisfies the property 2-D(m_0) if for all \( m_i \in P \cap M \setminus \{m_0, m_1, m_2, m_3\} \) there exist \( m_k, m_l \in P \cap M \setminus \{m_0\} \) such that

\[ m_i + m_0 = m_k + m_l, \]

that is, \( m_i - m_0 = (m_k - m_0) + (m_l - m_0) \).

If \( P \) satisfies the property 2-D(m_0), then we define new binomials as

\[ g_i = x_i - x_k x_l, \]

and we have equality of ideals \((f_4, \ldots, f_r) = (g_4, \ldots, g_r)\). From \( g_i \) we obtain homogeneous binomials \( G_i := (g_4, \ldots, g_r) \) and we see that the affine chart \( \Phi_{\mathbb{L}}(X) \cap (Z_0 \neq 0) \) is the common zero set of \( G_4, \ldots, G_r \).

**Proposition 3** Let \( P \subset M_{\mathbb{R}} \) be a lattice polytope corresponding to a nonsingular polarized toric variety \((X, L)\) of dimension three. Assume that \( X \) is one of (1) or (2) in Proposition 1. Then for each vertex \( v \in V(P) \), the polytope \( P \) satisfies the property 2-D(v).

**Proof** In case that \( X \) is (1). We note that \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))\) corresponds to the polytope \( k\Delta_3 \). If \( X \) is a blow up \( \mathbb{P}^3 \) along one invariant point, then \((X, L)\) corresponds to a polytope of the form

\[ P^{(1)} = (k\Delta_3) \cap (0 \leq z \leq k - l_1) \]

with \( 1 \leq l_1 < k \). From this consideration we see that \( P \) is

\[ (k\Delta_3) \cap (0 \leq z \leq k - l_1, 0 \leq x \leq k - l_2, 0 \leq y \leq k - l_3, l_4 \leq x + y + z \leq k) \]

with \( 0 \leq l_i + l_j < k \) for \( i \neq j \) up to unimodular transformation of \( M \).

If \( m_0 \in V(P) \) is a vertex of \( k\Delta_3 \), then we take a coordinates of \( M \) as

\[ m_0 = 0, m_1 = (1, 0, 0), m_2 = (0, 1, 0), m_3 = (0, 0, 1). \]

This is the case \( l_4 = 0 \) in the expression (4). Set \( m_i = (a, b, c) \). Then \( a, b, c \geq 0 \). When \( c = 0 \), we have \( a \geq 1 \) or \( b \geq 1 \), hence, \((a - 1, b, 0) \in P \) or \((a, b - 1, 0) \in P \) and \((a, b, 0) = (a - 1, b, 0) + (1, 0, 0) \) or \((a, b, 0) = (a, b - 1, 0) + (0, 1, 0) \). When \( c \geq 1 \), we have \((a, b, c - 1) \in P \) and \((a, b, c) = (a, b, c - 1) + (0, 0, 1) \).
If \( m_0 \) is not a vertex of \( k \Delta_3 \), then after a unimodular transformation of \( M \) the vertex \( m_0 \) is one of three vertices on the plane \( x + y + z = l_4 \) with \( l_4 \geq 1 \) in the expression (4). Then we take a coordinates of \( M \) as

\[
m_0 = 0, \quad m_1 = (1, 0, -1), \quad m_2 = (0, 1, -1), \quad m_3 = (0, 0, 1).
\] (6)

Set \( m_i = (a, b, c) \). Then \( a, b \geq 0 \). When \( c \geq 0 \), we can do the same procedure as above. When \( c < 0 \), we have \((a - 1, b, c + 1) \in P\) or \((a, b - 1, c + 1) \in P\).

When \( X \) is (2), set \( P_{d,e,f}^{(k)} \) a prism with the base \( k \Delta_2 \) and three edges of length \( d, e, f \geq 1 \) such that \( e - f \) and \( e - d \) are in \( k \mathbb{Z} \) and \( k \geq 1 \). \( P \) is obtained by removing at most two parts isomorphic to \( l \Delta_3 \) from \( P_{d,e,f}^{(k)} \). If \( m_0 \in P \) is a vertex of \( P_{d,e,f}^{(k)} \), then we take a coordinates as in (5). If \( m_0 \) is not a vertex of \( P_{d,e,f}^{(k)} \), then it is one of three vertices on the plane \( x + y + z = l \) of \( P_{d,e,f}^{(k)} \cap (l \leq x + y + z) \) and we take a coordinates as in (6). In any case, we can apply the same process as in (1). \( \square \)

5 Nonsingular lattice polygons

In order to prove Theorem 1, we have to treat the case that \( X \) is a \( \mathbb{P}^1 \)-bundle over a nonsingular toric surface \( Y \).

The lattice polytope \( Q \) corresponding to (3) in Proposition 1 has two parallel facets \( F_0 \) and \( F_1 \). In order to compare lattice points on \( F_0 \) and \( F_1 \) we need to know some information near opposite vertices. Let \( M' = \mathbb{Z}^2 \). We call a lattice parallelogram \( S \subset M'_\mathbb{R} \) to be a basic diamond if \( \#(S \cap M') = 4 \).

Let \( \hat{F} \subset M'_\mathbb{R} \) be a nonsingular lattice polygon with \( s + 1 \) edges. Let \( u_0, u_1, \ldots, u_s \) be the vertices of \( F \) numbered as counter-clockwise. By an affine transform of \( M' \), we may set as

\[
u_0 = 0, \quad u_1 = (a, 0), \quad u_s = (0, b)
\]

with \( a, b \geq 1 \). Set \( E_0 = [u_0, u_1], E_s = [u_0, u_s] \) two edges of \( F \) meeting at \( u_0 \). If \( \text{int}(F) \cap M' \neq \emptyset \), then the point \((1, 1)\) is contained in the interior of \( F \).

Proposition 4 Let \( F \subset M'_\mathbb{R} \) be a nonsingular lattice polygon with \( s + 1 \) vertices \( u_0, u_1, \ldots, u_s \) as above. Assume \( \text{int}(F) \cap M' \neq \emptyset \).

1. If \( F \) has an edge \([u_{t-1}, u_t] \) parallel to \( E_s \), there exists a basic diamond \( S \) contained in \( F \) such that \( u_t \) is a vertex of \( S \), \([u_{t-1}, u_t] \) contains one edge of \( S \) and that \( S \) stays in \( F \) after the vertex \( m' \) of \( S \) opposite to \( u_t \) is translated to the origin, that is, \( S - m' \subset F \).

2. When \( F \) has no edges parallel to \( E_0 \) nor \( E_s \), set \( u_t \) the farthest vertex of \( F \) from \( u_0 \). Let \( S \subset F \) be a basic diamond such that \( u_t \) is a vertex of \( S \) and \( S \) has two edges contained in \([u_{t-1}, u_t] \) and \([u_t, u_{t+1}] \), respectively. Set \( m' \in S \) the vertex opposite to \( u_t \). Then \( S - m' \subset F \).

Proof First, consider the case (2). Set \( u_t = (p, q) \). Then \( p, q \geq 1 \). Set \( u_{t-1} = (p - k\alpha, q - k\beta), u_{t+1} = (p - l\gamma, q - l\delta) \) with \( k, l \geq 1 \). Since \( u_t \) is the farthest
from $u_0$ and $F$ has no edges parallel to $E_0$ nor $E_5$, we have $\alpha, \beta, \gamma, \delta \geq 1$. Since $F$ is nonsingular, $\beta \gamma - \alpha \delta = 1$. Since $\text{int}(F) \cap M' \neq \emptyset$, $(p - \alpha - \gamma, q - \beta - \delta) \in \text{int}(F)$. Set
\[
S := \text{Conv} \{(p - \alpha, q - \beta), (p, q), (p - \gamma, q - \delta), (p - \alpha - \gamma, q - \beta - \delta)\}
\]
and $m' = (p - \alpha - \gamma, q - \beta - \delta) \in \text{int}(F)$. Then $S$ is a basic diamond and $S \subset F$. From the convexity of $F$ we see $S - m' \subset F$.

When the case (1), since the edge $[u_{i-1}, u_i]$ is parallel to $E_5$, we see $\alpha = 0, \beta = \gamma = 1$. If $\delta \geq 0$, then set
\[
S := \text{Conv} \{(p, q - 1), (p, q), (p - 1, q - \delta), (p - 1, q - \delta - 1)\}
\]
and $m' = (p - 1, q - \delta - 1) \in \text{int}(F)$.

If $\delta < 0$, then set $S := \text{Conv} \{(p, q - 1), (p, q), (p - 1, q), (p - 1, q - 1)\}$ and $m' = (p - 1, q - 1)$. Then $m' \in \text{int}(F)$. In both cases, $S - m' \subset F$ from the convexity of $F$. \hfill \Box

6 Proof of Theorem 1

From the argument in Sect. 4, it is enough to prove the following proposition in order to obtain a proof of Theorem 1.

Proposition 5 Let $P \subset M_\mathbb{R}$ be a lattice polytope corresponding to a nonsingular polarized toric variety $(X, L)$ of dimension three. Assume that $X$ satisfies (3) in Proposition 1. Then for each vertex $v \in V(P)$, $P$ satisfies the property $2-D(v)$.

Proof First, we consider the lattice polytope $Q$ corresponding to $(X, A)$. As we explain below Proposition 1, $Q$ has two parallel facets $F_0$ and $F_1$. We may assume that a vertex $m_0$ of $Q$ is a vertex of $F_0$. From an affine transform of $M$, we may set as $m_0$ is the origin and choose a basis $\{e_1, e_2, e_3\}$ of $M$ so that $e_1$ and $e_2$ are contained in edges of $F_0$ and $e_3$ is a vertex of $F_1$. Set $M' = \mathbb{Z}e_1 + \mathbb{Z}e_2$. Then $M = M' \oplus \mathbb{Z}e_3$. We may consider as $F_0$, $F_1 \subset M'_\mathbb{R}$ and $Q = \text{Conv}\{F_0 \times 0, F_1 \times e_1\}$. Both of $F_0$ and $F_1$ have $s + 1$ edges with $s \geq 2$ and contain $e_1$ and $e_2$ in their edges.

Take $m_i \in Q \cap M \setminus \{m_0, e_1, e_2, e_3\}$. Set $m_i = (a, b, c)$. Then $a, b \geq 0$ and $a \geq 1, b \geq 1, c = 0, \text{or } c = 1$.

When $s = 2$, $(a - 1, b, c) \in Q$ or $(a, b - 1, c) \in Q$. Hence $(a, b, c) = (a - 1, b, c) + e_1$ or $(a, b, c) = (a, b - 1, c); e_2$.

When $s = 3$, both of $F_0$ and $F_1$ have at least one pair of parallel edges. Assume that they have edges parallel to $[0, e_2]$. Then $(a, b - 1, c) \in Q$ if $b \geq 1$.

Set $s \geq 4$. Since $F_0$ and $F_1$ are nonsingular, they contain lattice points in their interiors. We apply Proposition 4 to $F_0$. We have the basic diamond $S \subset F_0$ and the lattice point $m' \in S$. Set $\tilde{S} = S - m'$. Then $S = \tilde{S} + m'$. Since each edge of $F_1$ is parallel to corresponding edge of $F_0$, the basic diamond of $F_1$ is a parallel transform of $\tilde{S}$, that is, $\tilde{S} + m''$. Set $\tilde{S} = \{0, u_1', u_2', u_1' + u_2'\}$.\hfill \$\checkmark$ Springer
Consider the case $m_i \in F_0 \times 0$. Set $R_0 := \text{Conv}(\tilde{S}, \tilde{S} + m')$. If $m_i \in R_0 \setminus \{u'_1, u'_2\}$, then there exists an $m_j \in R_0 \cap M' \setminus \{0\}$ satisfying $m_i = m_j + u'_1$ or $m_i = m_j + u'_2$. When $m_i \notin R_0$, if it is contained in the side of $e_2$, then $m_i - e_2 \in F_0$, if it is contained in the side of $e_1$, then $m_i - e_1 \in F_0$. After several steps, it moves in $R_0$.

When $m_i \in F_1 \times e_3$, set $R_1 := \text{Conv}(\tilde{S}, \tilde{S} + m'')$ and $m_i = \tilde{m}_i \times e_3$. If $\tilde{m}_i = e_1$, then $m_i = e_1 + e_3$. We may set $\tilde{m}_i \neq e_1, e_2$. If $\tilde{m}_i \in R_1 \setminus \{u'_1, u'_2\}$, then there exists an $\tilde{m}_j \in R_1 \cap M' \setminus \{0\}$ satisfying $\tilde{m}_i = \tilde{m}_j + u'_1$ or $\tilde{m}_i = \tilde{m}_j + u'_2$. Then we have

$$m_i = \tilde{m}_i \times e_3 = \tilde{m}_j \times e_3 + u'_1 \times 0, \quad \text{or} \quad m_i = \tilde{m}_i \times e_3 + u'_2 \times 0.$$  

The same method holds even if $\tilde{m}_i \notin R_1$.

Next, for general $P$, since $P$ has the same number of facets as those of $Q$ and corresponding facets are parallel, we know that $P$ has also two parallel facets $F_0$ and $F_l$ with $l \geq 1$ such that facet $F_l$ lies on the plane $z = i$ for $i = 0, l$. Now, for $1 \leq k \leq l$ set $F_k$ the cross section $P \cap \{z = k\}$. Since $P$ is a nonsingular lattice polytope, each section $F_k$ is also a nonsingular lattice polygon with the same number of edges parallel to corresponding edges of $F_0$. Since $m_i \in P \cap M \setminus \{m_0, m_1, m_2, m_3\}$ is contained one $F_k$, we can apply the same process by replacing $F_1$ with $F_k$. \hfill \Box

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