Starting from the coupling of a relativistic quantum particle to the curved Schwarzschild space-time, we show that the Dirac–Schwarzschild problem has bound states and calculate their energies including relativistic corrections. Relativistic effects are shown to be suppressed by the gravitational fine-structure constant \( \alpha_G = G m_1 m_2/(\hbar c) \), where \( G \) is Newton’s gravitational constant, \( c \) is the speed of light and \( m_1 \) and \( m_2 \) are the masses of the two particles. The kinetic corrections due to space-time curvature are shown to lift the familiar \((n,j)\) degeneracy of the energy levels of the hydrogen atom. We supplement the discussion by a consideration of an attractive scalar potential, which, in the fully relativistic Dirac formalism, modifies the mass of the particle according to the replacement \( m \to m(1 - \lambda/r) \), where \( r \) is the radial coordinate. We conclude with a few comments regarding the \((n,j)\) degeneracy of the energy levels, where \( n \) is the principal quantum number, and \( j \) is the total angular momentum, and illustrate the calculations by way of a numerical example.

PACS numbers: 11.10.-z, 03.70.+k, 03.65.Pm, 95.85.Ry, 04.25.dg, 95.36.+x, 98.80.-k

I. INTRODUCTION

As one combines relativistic quantum mechanics \[1, 2\] with general relativity \[3, 4\], one has to formulate the Dirac equation on a curved space-time \[6, 12\]. One of the most paradigmatic calculations concerns the analogue of the Dirac–Coulomb problem \[9, 13, 14\] which is obtained for a Dirac particle in the static Schwarzschild metric. The Dirac–Schwarzschild problem constitutes the analogue of the Dirac–Coulomb problem \[13, 17\], which is otherwise relevant for the Dirac particle bound to a central Coulomb potential, as opposed to a central gravitational field. The main problem is that, unlike for the Dirac–Coulomb problem, the gravitational central-field Dirac–Schwarzschild problem cannot be treated based on the correspondence principle alone.

Namely, the gravitational potential \(-G m_1 m_2/r\) cannot simply be inserted into the Dirac–Schwarzschild Hamiltonian. One first has to couple \[3, 8\] the Dirac particle to the curved space-time, using a fully covariant formalism, and then, identify the translation operator for the time coordinate with the Dirac Hamiltonian. This identification becomes unique in the Dirac–Schwarzschild problem when we demand that the time coordinate have a smooth limit to the flat-space-time in the regime of large separation \[10, 12, 13\].

We recall that for the Dirac–Coulomb problem, one simply adds the Coulomb potential \(-Ze^2/(4\pi\epsilon_0 r)\) to the free Dirac Hamiltonian, in the sense of a minimal coupling of the bound electron to the central electrostatic field of the nucleus \[15, 17\]. Here, \( Z \) is the nuclear charge number, \( e \) is the elementary charge, \( \epsilon_0 \) is the vacuum permittivity, and \( r \) is the distance from the center of the potential. Both the Dirac–Schwarzschild as well as the Dirac–Coulomb Hamiltonians take into account the gauge boson exchange (graviton exchange and Coulomb photon exchange, respectively) to all orders, but only in the classical approximation. This is sufficient to calculate the corrections of order \( \alpha_G^4 \) and \( \alpha_{\text{QED}}^4 \), where \( \alpha_G^4 \) and \( \alpha_{\text{QED}}^4 \) denote the gravitational and electrodynamical fine-structure constants, respectively.

We anticipate that the familiar \((n,j)\) degeneracy of the energy levels of the Dirac–Coulomb problem will be lifted for gravitational coupling, which implies that for example, the gravitationally coupled 2S and 2P\(_{1/2}\) levels are not degenerate. For the electromagnetically coupled hydrogen atom, the corresponding degeneracy is lifted only by the Lamb shift; the theoretical explanation involves a manifestly quantized electromagnetic field \[18\]. The reason for the lifted degeneracy, in the case of gravitational coupled, is different: Namely, we observe that it is due to the space-time curvature corrections to the kinetic term in the Dirac–Schwarzschild Hamiltonian. This finding is illustrated by a comparison to the energy levels of an attractive scalar potential, which are also calculated here, including relativistic corrections.

This paper is organized as follows. In Sec. \[II\] we consider the fine structure of the energy levels of the Dirac–Schwarzschild Hamiltonian and express the result in terms of the gravitational fine-structure constant \( \alpha_G \), and of the quantum numbers of the bound state. In passing, we clarify that the quantum mechanical gravitational central-field problem has bound states. For clarity, but without loss of generality, we consider a gravitationally coupled “atom” consisting of electron and proton. In Sec. \[III\] we compare to the energy levels of an attractive scalar potential. Having clarified the physical origin of the correction terms which lift the \((n,j)\) degeneracy, we continue in Sec. \[IV\] with the identification of a set of physical parameters for a gravitationally coupled system, where the calculations reported here might be phenomenologically relevant. These concern an electron gravitationally coupled, in a Rydberg state, to a black hole of mass \(10^{-11} M_E\), where \( M_E \) is the mass of the Earth. In the derivations, we use the electron mass \( m_e \) and the proton mass \( m_p \), Newton’s gravitational constant \( G \), Planck’s reduced quantum unit of action \( \hbar \), and the speed of light \( c \). Units with \( \hbar = c = \epsilon_0 = 1 \) are used in...
II. DIRAC–SCHWARZSCHILD FINE STRUCTURE

We start from the Dirac–Schwarzschild Hamiltonian $H$ for particle of mass $m_e$ in the central gravitational field of a particle (or planet) of mass $m_p \gg m_e$ (see Ref. [1]),

$$H = \frac{1}{2} \left\{ \vec{\alpha} \cdot \vec{p}, \left( 1 - \frac{G m_p}{2 r} \right) \right\} + \beta m_e \left( 1 - \frac{G m_p}{r} \right), \tag{1}$$

The mass parameters $m_e$ and $m_p$ are canonically associated with the electron and proton masses. However, the considerations reported in the following remain valid, without loss of generality, for any small mass $m_1 = m_e$ in the gravitational field of a larger, central mass $m_2 = m_p$. The nonrecoil approximation is employed. The vector of the Dirac $\vec{\alpha}$ matrices and the Dirac $\beta$ matrix are used in the standard representation [10,11,14,17].

After a Foldy–Wouthuysen transformation [19], one obtains the Dirac–Schwarzschild Hamiltonian $H_{DS}$, as is characterized by an overall prefactor matrix $\beta$, which expresses the particle–antiparticle symmetry inherent to the gravitationally coupled Dirac theory [see Eq. (28) of Ref. [9] and Eq. (21) of Ref. [11] for a manifestly Hermitian form]. In order to obtain the leading relativistic corrections, one may restrict the wave function to the “upper” two-component spinor, and the Dirac–Schwarzschild Hamiltonian $H_{DS}$ to its upper $(2 \times 2)$-submatrix,

$$H_{DS} = \frac{\vec{p}^2}{2 m_e} - \frac{G m_e m_p}{r} - \frac{\vec{p}^4}{8 m_e^2} - \frac{3 G m_p}{4 m_e} \left\{ \vec{p}^2, \frac{1}{r} \right\} + \frac{3 \pi G m_p}{2 m_e} \delta^{(3)}(\vec{r}) + \frac{3 G m_p}{4 m_e} \vec{\rho} \cdot \vec{L}. \tag{2}$$

The vector of $(2 \times 2)$–Pauli matrices is denoted as $\vec{\sigma}$. The momentum operator in Eq. (2) is given as $\vec{p} = -i \hbar \vec{\nabla}_r$, where we temporarily restore SI mkSA units for absolute clarity. We employ the following scaling to dimensionless quantities $\rho$,

$$r = \frac{\hbar^2}{G m_e^2 m_p}, \quad \vec{\nabla}_r = \frac{G m_e m_p}{\hbar^2} \vec{\nabla}_\rho, \tag{3a}$$

$$\vec{p} = -i \frac{G m_e^2 m_p}{\hbar} \vec{\nabla}_\rho. \tag{3b}$$

Here, $\vec{\nabla}_\rho$ is the dimensionless gradient operator, with respect to the dimensionless coordinate $\rho$. The scaled leading-order term has the Schrödinger–like structure

$$H_S = \frac{\vec{p}^2}{2 m_e} - \frac{G m_e m_p}{r} = \frac{\alpha_G^2 m_e c^2}{\hbar^2} \left( -\frac{1}{2} \nabla^2_\rho - \frac{1}{\rho} \right). \tag{4}$$

For the electron–proton system, employing the CODATA [20] value of $G = 6.67384(80) \times 10^{-11} \text{N m}^2/\text{kg}^2$, one obtains

$$\alpha_G = \frac{G m_e m_p}{\hbar c} = 3.21637(39) \times 10^{-42}. \tag{5}$$

Today, Newton’s gravitational constant $G$ remains one of the least well known physical constants to date, with a relative uncertainty of $1.2 \times 10^{-4}$. We should note that the numerically small value of the gravitational fine-structure constant $\alpha_G$ given in Eq. (5) is tied to the physical system under consideration, namely, the electron–proton system. The gravitational Bohr radius of the electron–proton system is

$$a_{0,G} = \frac{\hbar^2}{G m_e^2 m_p} \approx 1.20 \times 10^{29} \text{m}, \tag{6}$$

which is very large but depends on the masses employed. For other systems composed of elementary particles or black holes of various masses, the value of the gravitational fine-structure constant is different. One may remark that Eddington [21] observes that the electromagnetic fine-structure constant $\alpha_{\text{QED}} \approx 1/137.036$ and the gravitational fine-structure constant $\alpha_{\text{G(ee)}}$ for two gravitationally interaction electrons fulfill the approximate numerical relationship

$$\frac{\alpha_{\text{QED}}}{\alpha_{\text{G(ee)}}} = \frac{e^2}{4 \pi \epsilon_0 G m_e^2} \approx 4.2 \times 10^{42} \approx \sqrt{N_C}, \tag{7}$$

where $N_C$ is the number of charged particles in the Universe. We shall not comment on this numerical coincidence here except for reemphasizing that the gravitational interactions of elementary particles are much weaker than electromagnetic and “weak” interactions, as well as strong interactions. Still, to fix ideas, it is instructive to consider that bound electron-proton system, the Schrödinger eigenenergies of the eigenproblem $H_S|\phi \rangle = E_n|\phi \rangle$ are given as follows,

$$E_n = -\frac{\alpha_G^2 m_e c^2}{2 n^2}. \tag{8}$$

For the relativistic correction term given in Eq. (2), it is instructive to consider the scaling of the various relativistic correction terms separately, with full reference to the SI mkSA unit system,

$$- \frac{\vec{p}^4}{8 m_e^2 c^2} = \frac{\hbar^4 \nabla^4_\rho}{8 m_e^2 c^2} = -\frac{1}{8} \alpha_G^2 m_e c^2 \nabla^4_\rho, \tag{9a}$$

$$- \frac{\hbar^2}{c^2} \frac{3 G m_p}{4 m_e} \delta^{(3)}(\vec{r}) = \frac{3}{4} \alpha_G^2 m_e c^2 \left\{ \nabla^2_{\rho}, \frac{1}{\rho} \right\}, \tag{9b}$$

$$\frac{\hbar^2}{c^2} \frac{3 G m_p}{4 m_e} \vec{\Sigma} \cdot \vec{L} = \frac{3}{4} \alpha_G^2 m_e c^2 \frac{3 \vec{\Sigma} \cdot \vec{L}}{4 \rho^2}. \tag{9c}$$

These considerations manifestly identify the relativistic correction terms to be of order $\alpha_G^2$. The scaled Dirac–Schwarzschild Hamiltonian with relativistic corrections
Thus is given as follows,
\[
H_{DS} = \alpha_G^2 m_e c^2 \left( -\frac{1}{2} \frac{\nabla^2}{\rho} - \frac{1}{\rho} \right) + \alpha_G^4 m_e c^2 
\]
\[
\times \left( -\frac{1}{8} \frac{\nabla^2}{\rho} + \frac{3}{4} \left( \frac{\nabla^2}{\rho} \right) \frac{1}{\rho} + \frac{3\pi}{2} \delta^{(3)}(\rho) + \frac{3 \mathbf{\Sigma} \cdot \mathbf{E}}{4 \rho^3} \right).
\]

Using formulas given on p. 17 of Ref. [22], we may evaluate the relativistic corrections as a function of the bound-state quantum numbers \((n, j)\) and the orbital angular momentum quantum number, \(j\), being the total angular momentum quantum number. The calculation proceeds via first-order perturbation theory, starting from the Schrödinger–Pauli wave function \(\psi_{n\ell j}(\rho) = R_{nf}(\rho) \chi_{\ell j}(\rho)\), where
\[
\nu = 2(\ell - j)(j + 1/2) = (-1)^{j+1/2} \left( j + \frac{1}{2} \right) \quad \text{(11)}
\]
is the Dirac angular quantum number [15, 23]. Some exemplary radial parts \(R_{nf}(\rho)\) of the Schrödinger–Pauli wave functions are given in p. 15 of Ref. [22]. Knowing \(j\) and \(\ell\), one may calculate \(\nu\) using Eq. (11). Conversely, one may calculate \(\ell\) with the help of the formula \(\ell = |\nu + 1/2| - 1/2\). The relativistic corrections amount to
\[
E_{n\ell j} = -\frac{\alpha_G^2 m_e c^2}{2n^2} + \alpha_G^4 m_e c^2 \left( \frac{15}{8n^4} - \frac{7j + 5}{2} \frac{(j + 1)(2j + 1)}{n^3} \delta_{\ell j + 1/2} \right. \\
- \left. \frac{7j + 2}{j(2j + 1)n^3} \delta_{\ell j - 1/2} \right).
\]
\]
The \(S\) state energy can be obtained from Eq. (12) with the help of the term \(\ell = 0\) and \(j = 1/2\); \(S\) states are the only ones for which the expectation value of the Dirac-\(\delta\) term in Eq. (10) is nonvanishing; the result reads as
\[
E_{nS_{1/2}} = -\frac{\alpha_G^2 m_e c^2}{2n^2} + \alpha_G^4 m_e c^2 \left( \frac{15}{8n^4} - \frac{1}{2n^3} \right).
\]
\]
The \(2S_{1/2}, 2P_{1/2}\) and \(2P_{3/2}\) levels are given as follows,
\[
E_{2S_{1/2}} = -\frac{1}{8} \alpha_G^2 m_e c^2 - \frac{73}{128} \alpha_G^4 m_e c^2,
\]
\[
E_{2P_{1/2}} = -\frac{1}{8} \alpha_G^2 m_e c^2 - \frac{91}{384} \alpha_G^4 m_e c^2,
\]
\[
E_{2P_{3/2}} = -\frac{1}{8} \alpha_G^2 m_e c^2 - \frac{55}{384} \alpha_G^4 m_e c^2.
\]
\]
While there is no degeneracy, the hierarchy \(E_{2S_{1/2}} < E_{2P_{1/2}} < E_{2P_{3/2}}\), follows a somewhat general paradigm of bound-state theory [24], namely, that states with higher angular momentum quantum numbers have higher energy.

### III. Fine Structure for a Scalar Potential

The Dirac Hamiltonian with a \((r/1)\)-scalar potential [12] reads as follows (in natural units),
\[
H = \vec{\alpha} \cdot \vec{p} + \beta \left( m - \frac{\lambda}{r} \right),
\]
\]
After the Foldy–Wouthuysen transformation, we have
\[
H_{SP} = \beta \left( m + \frac{\vec{p}^2}{2m} - \frac{\lambda}{r} \right)
\]
\[
- \frac{\vec{p}^4}{8m^3} + \frac{\lambda}{4m^2} \left( \frac{\vec{p}^2}{m^2} \right) - \frac{\pi \lambda}{2m^2} \delta(\vec{r}) - \frac{\lambda \mathbf{\Sigma} \cdot \mathbf{E}}{4m^2 r^3}.
\]
\]
The scaling to dimensionless variables is analogous to Eqs. (13a),
\[
r = \frac{\lambda}{m} \rho, \quad \nabla_r = m \lambda \nabla_{\rho},
\]
\[
\vec{p} = -i m \lambda \vec{\nabla}_{\rho}, \quad \alpha_S \equiv \lambda.
\]
\]
The role of the “scalar fine-structure constant” is taken by the variable \(\alpha_S = \lambda\), and the scaled Hamiltonian reads as follows,
\[
H_{SP} = \frac{\alpha_S^2}{8} m \left( -\frac{1}{2} \frac{\nabla^2}{\rho} - \frac{1}{\rho} \right) + \alpha_S^4 m
\]
\[
\times \left( -\frac{1}{8} \frac{\nabla^2}{\rho} - \frac{1}{4} \left( \frac{\nabla^2}{\rho} \right) \frac{1}{\rho} - \frac{\pi}{2} \delta^{(3)}(\rho) - \frac{\mathbf{\Sigma} \cdot \mathbf{E}}{4 \rho^3} \right).
\]
The energy levels are given as
\[
E_{n\ell j} = -\frac{\alpha_S^2 m_e c^2}{2n^2} + \alpha_S^4 m_e c^2 \left( \frac{1}{8n^4} + \frac{1}{n^3(j + 1)} \right).
\]
\]
Here, an important observation can be made: In contrast to Eq. (12), the result for the relativistic corrections of order \(\alpha_S^2\) in the case of the scalar potential has a compact functional form, and the \((n, j)\) degeneracy familiar from the Dirac–Coulomb problem (see Appendix A) is restored. We also note that the Dirac–Schwarzschild Hamiltonian [2] and the scalar Dirac Hamiltonian [16] both entail “(1/r)-modifications of the mass term”, namely, the terms
\[
\beta m_e \left( 1 - \frac{G m_p}{r} \right) \Leftrightarrow \beta m \left( 1 - \frac{\lambda}{r} \right)
\]
\]
However, in addition to this modification, the Dirac–Schwarzschild Hamiltonian [2] contains a modification of the kinetic term \(\vec{\alpha} \cdot \vec{p}\) which is responsible for the lifting of the \((n, j)\) degeneracy, as a comparison of Eqs. (12) and (20) shows.
IV. NUMERICAL EXAMPLE

Let us consider a “tiny black hole” of mass $m_{\text{BH}}$ to be $10^{-11}$ times the mass $M_E$ of the Earth,

$$M_E \approx 5.9742 \times 10^{24} \text{ kg}, \quad m_{\text{BH}} = 5.9742 \times 10^{13} \text{ kg}, \quad (22)$$

and assume that the electric dipole polarizability of the very dense black hole is vanishing. The Schwarzschild radius $r_{S, \text{BH}}$ is given as follows,

$$r_{S, \text{BH}} = \frac{2 G m_{\text{BH}}}{c^2} = 8.8731 \times 10^{-14} \text{ m}. \quad (23)$$

The gravitational fine-structure constant for an electron gravitationally bound to the black hole is given as

$$\alpha_{G, \text{BH}} = \frac{G m_e m_{\text{BH}}}{\hbar c} = 0.1148. \quad (24)$$

The gravitational Bohr radius is

$$a_{0, \text{BH}} = \frac{\hbar^2}{G m_e m_{\text{BH}}} = 3.3612 \times 10^{-12} \text{ m}. \quad (25)$$

In accordance with Eq. (25), we define the Cartesian components of the scaled dimensionless coordinate $\vec{\rho}$ as follows,

$$\rho_x = \frac{x}{a_{0, \text{BH}}}, \quad \rho_y = \frac{y}{a_{0, \text{BH}}}, \quad \rho_z = \frac{z}{a_{0, \text{BH}}}. \quad (26)$$

In Fig. 1 we present a “scatter plot” of the bound state with quantum numbers $n = 10$, $\ell = 9$, and magnetic orbital angular momentum projection $m = |\ell| = 9$ (“circular Rydberg state”), where the points representing the wave function are distributed according to the probability density given by the absolute square of the wave function $|\psi|^2$. The probability density of the Rydberg state inside the Schwarzschild radius is negligible and the expectation value of the zitterbewegung term in the Dirac–Schwarzschild Hamiltonian vanishes. The non-relativistic Schrödinger–type approximation is justified because the gravitational fine-structure constant $\alpha_{G, \text{BH}}$ is small against unity. According to p. 17 of Ref. [22], the radial expectation value in the Schrödinger state is $\langle |\rho| \rangle = 105$ gravitational Bohr radii. A classical circular trajectory circling the black hole is indicated in red for comparison.

According to Eq. (12), the bound-state energies for the two states with $j = 9 \pm 1/2$ are given as follows,

$$E_{n=10, \ell=9, j=19/2} = \left( -\frac{\alpha_{G, \text{BH}}^2}{200} - \frac{263 \alpha_{G, \text{BH}}^4}{1520000} \right) m_e c^2,$$

$$= -33.7397 \text{ eV}, \quad (27a)$$

$$E_{n=10, \ell=9, j=17/2} = \left( -\frac{\alpha_{G, \text{BH}}^2}{200} - \frac{173 \alpha_{G, \text{BH}}^4}{912000} \right) m_e c^2,$$

$$= -33.7412 \text{ eV}, \quad (27b)$$

The higher value of the total angular momentum $j$ moves the state with $j = 19/2$ energetically upward. Both energies (27a) and (27b) are numerically close to the nonrelativistic approximation, which reads as $-\alpha_{G, \text{BH}}^2 m_e c^2/200 = -33.7243 \text{ eV}$. The probability density is indicated in red for comparison, and the black hole at the center is indicated as a black dot.

V. CONCLUSIONS

We have divided the current paper into three parts, the first of which (see Sec. III) deals with the leading-order relativistic corrections to the energies of bound states of the Dirac–Schwarzschild Hamiltonian, while the second part (Sec. IV) investigates the bound states of a Dirac Hamiltonian with a scalar $(1/r)$-potential. The latter potential modifies the mass term of the Dirac particle; it is commonly referred to as a scalar potential because of its properties under Lorentz transformations. Having clarified the origin of the terms that lift the $(n, j)$ degeneracy otherwise observed for scalar Dirac bound states and for the Dirac–Coulomb problem (see Sec. III and Appendix A, respectively), we then turn our attention back to the Dirac–Schwarzschild problem in Sec. IV and consider a numerical example for bound states of a “small” black hole of mass $10^{-11}$ times the Earth mass (third part of our investigation). This parameter combination leads to gravitational electronic bound states (the coupling constant $\alpha_{G, \text{BH}}$ given in Eq. (24) is small.
against unity). It is thus possible to compare to a classical treatment for circular Rydberg states, in terms of the trajectory shown in Fig. 4. In the nonrelativistic approximation, the circular symmetry (Schrödinger approximation) is restored, while the relativistic corrections, including the Fokker precession term (spin-orbit coupling term) enter the relativistic energies given in Eqs. (A5) and (27a).

In our investigations, we clarify, in particular, that the quantum mechanical gravitational central-field problem has quantum mechanical bound states. This result holds in the framework of curved space-times (general relativity, see Ref. [5]) and takes into account the fact that it is impossible, in contrast to the Dirac–Coulomb problem, to simply insert the gravitational potential (−Gm1m2/r) into the Dirac Hamiltonian by the corresponding principle. We evaluate the fine-structure formula for the Dirac–Schwarzschild Hamiltonian [see Eq. (12)], and calculate the αGκ corrections to the energy. The bound-state energies are obtained as a function of “good” quantum numbers.

Let us briefly comment on the appropriate quantum numbers for the Dirac–Schwarzschild problem. Because of the symmetries of the problem [10, 11], the principal quantum number n, the total angular momentum quantum number j, and the Dirac angular momentum quantum number κ constitute a set of “good” quantum numbers. The familiar spin-angular \( \chi_{\kappa\mu}(\vec{r}) \) is assembled from the fundamental spinors and the spherical harmonics as follows [13, 23, 25]. It has the property

\[
(\vec{\sigma} \cdot \vec{L} + 1) \chi_{\kappa\mu}(\vec{r}) = -\kappa \chi_{\kappa\mu}(\vec{r}).
\] (28)

In the conventions of Refs. [13, 22], we have \( \kappa = (-1)^{j+1/2} \left(j + \frac{1}{2}\right) \). Knowing j and κ, one may calculate the orbital angular momentum quantum number \( \ell = |\kappa + 1/2| - 1/2 \) even if the orbital angular momentum operator \( \vec{L} \) itself does not commute with the Dirac–Schwarzschild Hamiltonian [2]. Because κ can be mapped onto the orbital angular momentum quantum number \( \ell \) (i.e., onto the “spin orientation with respect to the orbital angular momentum”), the main result (12) is consistent.

For both the scalar Dirac Hamiltonian [10] as well as the Dirac–Coulomb Hamiltonian [A2], the explicit \( \ell \) dependence of the spin-orbit coupling accidentally cancels out against the “implicit” \( \ell \) dependence of the matrix elements of the momentum, and the position operator [see Ref. [22] and Eq. (A5)].

**Acknowledgments**

The research has been supported by the National Science Foundation (Grant PHY–1068547).

---

**Appendix A: Dirac–Coulomb Hamiltonian**

For comparison, we briefly recall the Dirac–Coulomb Hamiltonian [13, 26]

\[
H = \vec{\alpha} \cdot \vec{p} + \beta m_e - \frac{Z\alpha_{\text{QED}}}{r}, \quad (A1)
\]

where \( Z \) is the nuclear charge number, and \( \alpha_{\text{QED}} \approx 1/137.036 \) is the QED fine-structure constant. The nonrecoil approximation is employed. After a Foldy–Wouthuysen transformation, the Hamiltonian takes the form

\[
H_{\text{DC}} = \frac{\vec{p}^2}{2m_e} - \frac{Z\alpha_{\text{QED}}}{r} - \frac{\vec{p}^4}{8m_e^2} + \frac{\pi Z\alpha_{\text{QED}}}{2m_e^2} \delta^{(3)}(\vec{r}) + \frac{Z\alpha}{4m_e^2 r^3} \vec{\Sigma} \cdot \vec{L}. \quad (A2)
\]

The scaling corresponding to Eqs. (3a) and (17) reads as follows,

\[
r = \frac{\hbar}{m_e c} \rho, \quad \vec{\nabla}_r = \frac{m_e c}{\hbar} \vec{\nabla}_\rho, \quad \vec{p} = -i \frac{m_e c}{\hbar} \vec{\nabla}_\rho. \quad (A3)
\]

The familiar [13, 17] scaled Dirac–Coulomb Hamiltonian is obtained as

\[
H_{\text{DS}} = \alpha_{\text{QED}}^2 m_e c^2 \left( -\frac{1}{2} \nabla_\rho^2 - \frac{1}{\rho} + \frac{\vec{\nabla}_r^4}{8} + \frac{\pi}{2} \delta^{(3)}(\vec{p}) + \frac{\vec{\sigma} \cdot \vec{L}}{4 \rho^3} \right). \quad (A4)
\]

The energy levels are given as follows [17, 26]

\[
E_{n\ell j} = -\frac{\alpha_{\text{QED}}^2 m_e c^2}{2n^2} + \alpha_{\text{QED}}^4 m_e c^2 \left( \frac{3}{8n^4} - \frac{1}{n^3 (2j + 1)} \right). \quad (A5)
\]

When evaluating the matrix elements according to formulas given on pp. 15–17 of Ref. [22], one first obtains a functionally different formulas for \( j = \ell + 1/2 \) as opposed to \( j = \ell - 1/2 \) but they coincide for given \( j \). This is analogous to Eq. (20).
[1] P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 117, 610 (1928).
[2] P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 118, 351 (1928).
[3] A. Einstein, Königlich Preussische Akademie der Wissenschaften (Berlin), Sitzungsberichte, 799–801 (1915).
[4] D. Hilbert, Nachrichten der Königlichen Gesellschaft der Wissenschaften (Göttingen), Math.-phys. Klasse, 395–407 (1915).
[5] A. Einstein, Ann. Phys. (Leipzig) 49, 796 (1916).
[6] D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957).
[7] D. G. Boulware, Phys. Rev. D 12, 350 (1975).
[8] M. Soffel, B. Müller, and W. Greiner, J. Phys. A 10, 551 (1977).
[9] A. J. Silenko and O. V. Teryaev, Phys. Rev. D 71, 064016 (2005).
[10] U. D. Jentschura, Phys. Rev. A 87, 032101 (2013) [Erratum Phys. Rev. A 87, 069903(E) (2013)].
[11] U. D. Jentschura and J. H. Noble, Phys. Rev. A 88, 022121 (2013).
[12] U. D. Jentschura and J. H. Noble, J. Phys. A 47, 045402 (2014).
[13] K. Schwarzschild, Sitzungsberichte d. K. Preuss. Akad. d. Wiss. 7, 189 (1916).
[14] A. S. Eddington, The Mathematical Theory of Relativity (Cambridge University Press, Cambridge, England, 1924).
[15] R. A. Swainson and G. W. F. Drake, J. Phys. A 24, 79 (1991).
[16] R. A. Swainson and G. W. F. Drake, J. Phys. A 24, 95 (1991).
[17] R. A. Swainson and G. W. F. Drake, J. Phys. A 24, 1801 (1991).
[18] H. A. Bethe, Phys. Rev. 72, 339 (1947).
[19] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).
[20] P. J. Mohr, B. N. Taylor, and D. B. Newell, Rev. Mod. Phys. 84, 1527 (2012).
[21] A. S. Eddington, Proc. Camb. Phil. Soc. 27, 15 (1931).
[22] H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms (Springer, Berlin, 1957).
[23] M. E. Rose, Relativistic Electron Theory (J. Wiley & Sons, New York, NY, 1961).
[24] H. A. Bethe and R. Jackiw, Intermediate Quantum Mechanics (Perseus, New York, 1986).
[25] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, Quantum Theory of Angular Momentum (World Scientific, Singapore, 1988).
[26] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).