On the Path-Integral Derivation of the Anomaly for the Hermitian Equivalent of the Complex $PT$-Symmetric Quartic Hamiltonian

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Abstract

It can be shown using operator techniques that the non-Hermitian $PT$-symmetric quantum mechanical Hamiltonian with a “wrong-sign” quartic potential $-gx^4$ is equivalent to a Hermitian Hamiltonian with a positive quartic potential together with a linear term. A naïve derivation of the same result in the path-integral approach misses this linear term. In a recent paper by Bender et al. it was pointed out that this term was in the nature of a parity anomaly and a more careful, discretized treatment of the path integral appeared to reproduce it successfully. However, on re-examination of this derivation we find that a yet more careful treatment is necessary, keeping terms that were ignored in that paper. An alternative, much simpler derivation is given using the additional potential that has been shown to appear whenever a change of variables to curvilinear coordinates is made in a functional integral.

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1 Introduction

Hamiltonians of the form

$$H = \frac{1}{2}p^2 - g(ix)^N,$$

(1)
which are are $PT$-symmetric, but not Hermitian, have been shown\cite{1,2} to have a real, positive spectrum for $N \geq 2$. For $N < 4$ the energy eigenvalue problem can be posed on the real $x$-axis, but for $N \geq 4$ it must instead be imposed in the complex $x$-plane, along a contour which asymptotically remains within the Stokes wedges of the Hamiltonian. Such is indeed the case for $N = 4$, when the potential on the real axis is an “upside down” quartic. To that end we rewrite Eq. (1) for $N = 4$ in terms of the complex variable $z$:

$$H = \frac{1}{2}p_z^2 - gz^4.$$  

(2)

A general framework encompassing $PT$-symmetric Hamiltonians was given by Mostafazadeh\cite{3}, who showed that such Hamiltonians were related by a similarity transformation to an equivalent Hermitian Hamiltonian $h$ possessing the same spectrum. In only a few cases is it possible to construct $h$ explicitly, but this was done recently for Eq. (2) in Ref. [4], rediscovering an earlier result of Buslaev and Grecchi\cite{5}. The calculation was performed using the particular real parametrization

$$z = -2i\sqrt{1 + ix}$$

(3)

of the contour in the complex plane on which Eq. (2) was defined, and using operator techniques to perform the similarity transformation to $h$, whose form is ($\alpha \equiv 16g$)

$$h = \frac{1}{2}p_y^2 + \frac{1}{4}\alpha y^4 - \sqrt{\frac{\alpha}{8}}y.$$  

(4)

We should emphasize that this latter form is obtained only after a Fourier transform, which means that $y$ is really a momentum rather than a position, as will become apparent in the path-integral treatment. When a proper dimensional analysis is made\cite{6}, the linear term turns out to be proportional to $\hbar$, which means that it is in the nature of an anomaly.

In fact we will present various path-integral treatments, of various levels of sophistication. In the next section we give the simplest of these, a continuum version, which fails to reproduce the anomaly. Realizing that for a general change of variables in a functional integral there is an additional term, $\Delta V$, to be included over and above the functional determinant, we show, in Section 3 how $\Delta V$ is traded for the anomaly in an intermediate functional integration. In Section 4 we give the discretized version of this calculation, in order to provide a benchmark for comparison with the results obtained in Section 5 by two variants of the discretized calculation of Ref. [6]. In effect this
latter section gives an alternative derivation of $\Delta V$, as a consequence of the mismatch between the functional determinant and the coefficient of the kinetic term when our particular change of variable is made.

## 2 Naïve Continuum Treatment

The original path integral is the single Euclidean functional integral

$$Z = \int_C [D\psi] \exp \left\{ - \int dt \left[ \frac{1}{2} \dot{\psi}^2 - g\psi^4 \right] \right\}, \quad (5)$$

where the subscript $C$ is to remind us that the integrals are to be performed on an appropriate curve in the complex $\psi$ plane. Making the change of variable

$$\psi = -2i\sqrt{(1 + i\varphi)}, \quad (6)$$
in analogy with Eq. (3), we obtain

$$Z = \int \frac{[D\varphi]}{\text{Det}\sqrt{(1 + i\varphi)}} \exp \left\{ - \int dt \left[ \frac{1}{2} \frac{\dot{\varphi}^2}{1 + i\varphi} - \alpha(1 + i\varphi)^2 \right] \right\}, \quad (7)$$

where the $\varphi$ integrals are now along the real axis. We now rewrite this as a double functional integral in $\varphi$ and $\pi$ by means of the identity

$$\frac{1}{\text{Det}\sqrt{(1 + i\varphi)}} = \frac{1}{N} \int [D\pi] \exp \left\{ - \int dt \frac{1}{2} (1 + i\varphi) \left( \pi - i\frac{\dot{\varphi}}{1 + i\varphi} \right)^2 \right\}, \quad (8)$$

where $N$ is an appropriate normalization constant. This leads to the expression

$$Z = \frac{1}{N} \int [D\varphi][D\pi] \exp \left\{ - \int dt \left[ \frac{1}{2} (1 + i\varphi)^2 - i\pi\dot{\varphi} - \alpha(1 + i\varphi)^2 \right] \right\}. \quad (9)$$

Replacing $\pi\dot{\varphi}$ by $-\dot{\pi}\varphi$ under the $t$ integration, we have an exponent that is quadratic in $\varphi$, so the $\varphi$ functional integration can be performed. The result, after applying the compensating rescalings $\varphi \rightarrow \varphi/\sqrt{2\alpha}$ and $\pi \rightarrow \pi\sqrt{2\alpha}$, is

$$Z = \int [D\pi] \exp \left\{ - \int dt \left[ \frac{1}{2}\pi^2 - \sqrt{2\alpha}\pi \left( 1 - \frac{1}{2}\pi^2 \right) + \frac{\alpha}{4}\pi^4 \right] \right\}. \quad (10)$$

The middle term in the integrand is a perfect derivative, and so may be discarded, leaving the equivalent Lagrangian $\ell$ in the exponent, written in terms of $\pi$:

$$Z = \int [D\pi] \exp \left\{ - \int dt \left[ \frac{1}{2}\pi^2 + \frac{\alpha}{4}\pi^4 \right] \right\}, \quad (11)$$
except that the linear term is missing. The simple classical calculation we have just performed is unable to obtain this term. Note that the Fourier transform of the Schrödinger treatment occurs here naturally, since $Z$ is expressed in terms of the momentum variable $\pi$.

3 Correct Continuum Treatment

It is shown in various standard books on functional integration, for example \cite{7,8}, that when a general change of variables, such as that of Eq. (6), is made, an additional potential term $\Delta V$ must be included in the Lagrangian. Ultimately this term, which is actually of order $\hbar^2$, is derived from the discretized form of the functional integral when the particular form of discretization

\begin{equation}
\dot{\varphi}_n \equiv \frac{1}{a}(\varphi_{n+1} - \varphi_n) \tag{12}
\end{equation}

\begin{equation}
\bar{\varphi}_n \equiv \frac{1}{2}(\varphi_{n+1} + \varphi_n) \tag{13}
\end{equation}

is adopted, where $a$ is the lattice spacing, corresponding to Weyl ordering in the operator treatment. The importance of this prescription was indeed emphasized in Ref. \cite{6}.

The general form of $\Delta V$ for a change of variable from $\psi$ to $\varphi$ is \cite{7}

\begin{equation}
\Delta V = \frac{1}{8} \left[ \frac{d}{d\varphi} \left( \frac{d\varphi}{d\psi} \right) \right]^2, \tag{14}
\end{equation}

which for the particular transformation of Eq. (6) turns out to be

\begin{equation}
\Delta V = -\frac{1}{32} \frac{1}{1 + i\varphi}. \tag{15}
\end{equation}

The correct version of Eq. (7) is thus

\begin{equation}
Z = \int \frac{[D\varphi]}{\text{Det} \sqrt{(1 + i\varphi)}} \exp \left\{ -\int dt \left[ \frac{1}{2} \frac{\dot{\varphi}^2}{1 + i\varphi} - \frac{1}{32} \frac{1}{1 + i\varphi} - \alpha(1 + i\varphi)^2 \right] \right\}. \tag{16}
\end{equation}

It is now possible to write down a variant of the Gaussian identity \cite{8},

\begin{equation}
\frac{1}{\text{Det} \sqrt{(1 + i\varphi)}} = \frac{1}{N} \int [D\pi] \exp \left\{ -\int dt \frac{1}{2}(1 + i\varphi) \left( \pi - \frac{i\dot{\varphi} + \frac{1}{4}}{1 + i\varphi} \right)^2 \right\}, \tag{17}
\end{equation}

which serves to cancel the $\Delta V$ term as well as the kinetic term. In turn it introduces two additional terms in the exponent: (i) a term in $i\dot{\varphi}/(1 + i\varphi)$,
which is a perfect derivative and so can be discarded under the $t$ integration, and (ii) the anomaly $\int dt \, \pi/4$.

Finally, after rescaling as before, the corrected version of Eq. (11) is

$$Z = \int [D\pi] \exp \left\{ -\int dt \left[ \frac{1}{2} \pi^2 - \sqrt{\frac{\alpha}{8}} \pi + \frac{\alpha}{4} \pi^4 \right] \right\},$$  

(18)

### 4 Discretized Version

In this section we will go through the discretized version of the previous calculation, in order to provide a standard discretized formula with which we can compare the results of the (corrected) calculation of Ref. [6] and another calculation whereby the kinetic term is expanded in powers of the lattice spacing $a$.

In place of Eq. (16) we have

$$Z = \prod_n \int \frac{d\varphi_n}{\sqrt{1 + i\bar{\varphi}_n}} \exp \left\{ -a \left[ \frac{1}{2} \varphi_n^2 - \frac{1}{32} \frac{1}{1 + i\varphi_n} - \frac{1}{4} \varphi_n^2 \right] - \frac{1}{2} \frac{1}{1 + i\varphi_n} \right\}. \tag{19}$$

The Gaussian identity we will use is

$$\frac{1}{\sqrt{1 + i\bar{\varphi}_n}} = \frac{1}{N} \int d\bar{\pi}_n e^{-\frac{1}{2} \lambda(\bar{\pi}_n - B)^2}, \tag{20}$$

where $\lambda = a(1 + i\bar{\varphi}_n)$ and $B = (i\varphi_n + \frac{1}{4})/(1 + i\varphi_n)$.

Written out in full this is

$$\frac{1}{\sqrt{1 + i\varphi_n}} = \frac{1}{N} \int d\pi_n \exp \left\{ -a \left[ \frac{1}{2} \pi_n^2 (1 + i\varphi_n) - i\pi_n\varphi_n - \frac{1}{4} \pi_n^2 \right] - \frac{1}{2} \frac{1}{1 + i\varphi_n} \right\}.$$

Neglecting the term in $i\varphi_n/(1 + i\varphi_n)$ because the identity

$$\log \left( \frac{1 + i\varphi_{n+1}}{1 + i\varphi_n} \right) = \frac{i a \varphi_n}{1 + i\varphi_n} + O(a^3) \tag{21}$$

shows it to be a perfect difference up to a correction of order $a^3$, we obtain

$$Z = \frac{1}{N} \prod_n \int d\pi_n d\varphi_n \exp \left\{ -a \left[ \frac{1}{2} (1 + i\varphi_n)\pi_n^2 - i\pi_n\varphi_n - \frac{1}{4} \pi_n - \alpha(1 + i\varphi_n)^2 \right] \right\} \tag{22}$$
Thus the \( \Delta V \) term has been cancelled, and we are left with the anomaly \( \frac{1}{4} a \pi_n \) in the exponent. Now \( \pi_n \dot{\varphi}_n + \varphi_n \dot{\pi}_n \) is a perfect difference:

\[
 a(\pi_n \dot{\varphi}_n + \varphi_n \dot{\pi}_n) = \pi_n+1 \varphi_n+1 - \pi_n \varphi_n. \tag{23}
\]

So now we can “integrate by parts”, in the form \( \pi_n \dot{\varphi}_n \rightarrow -\varphi_n \dot{\pi}_n \).

Changing the integration measure from \( \int d\bar{\pi}_n d\varphi_n \) to \( \int d\pi_n d\bar{\varphi}_n \), which does not introduce any additional factors, Eq. (22) becomes

\[
 Z = \frac{1}{N} \prod_n \int d\pi_n \exp \left\{ -a \left[ \frac{\alpha}{4} \pi_n^4 + \sqrt{\frac{\alpha}{2}} \pi_n \dot{\pi}_n + \frac{1}{2} \pi_n^2 - \sqrt{\frac{\alpha}{8}} \pi_n \right] \right\},
\]

having dropped a perfect difference proportional to \( \dot{\pi}_n \).

Now we rescale: \( \bar{\varphi}_n \rightarrow \varphi_n/\sqrt{2\alpha} \) and \( \pi_n \rightarrow \pi_n \sqrt{2\alpha} \) and perform the \( \bar{\varphi}_n \) integration, with the result

\[
 Z = \frac{1}{N} \prod_n \int d\pi_n \exp \left\{ -a \left[ \frac{\alpha}{4} \pi_n^4 + \sqrt{\frac{\alpha}{2}} \pi_n \dot{\pi}_n + \frac{1}{2} \pi_n^2 - \sqrt{\frac{\alpha}{8}} \pi_n \right] \right\},
\]

This is the desired result, provided that we can neglect the term \( \pi_n^2 \dot{\pi}_n \). The identity

\[
 3a \pi_n^2 \dot{\pi}_n = \pi_n^3 + \pi_n - (\pi_n+1 - \pi_n)^3 / 4 \tag{24}
\]

shows that is a perfect difference up to a correction of order \( a^3 \), so that it can indeed be neglected. The resulting expression for \( Z \) is

\[
 Z = \prod_n \int d\pi_n \exp \left\{ -a \left[ \frac{\alpha}{4} \pi_n^4 + \sqrt{\frac{\alpha}{8}} \pi_n + \frac{\alpha}{4} \pi_n^4 \right] \right\}, \tag{25}
\]

the discrete version of Eq. (16). Equation (22), from which this was derived, will be a reference point for the calculations of the following two sections, which arise out of the treatment of Ref. [6].

The basis of that treatment was an exact, discretized, treatment of the kinetic term. In the original functional integral written in terms of \( \psi \), the time derivative is defined as \( \dot{\psi}_n = (\psi_{n+1} - \psi_n) / a \). In terms of \( \psi_n = -2i\sqrt{(1 + i\varphi_n)} \), this becomes \( \dot{\psi}_n = \varphi_n / A_n \), where

\[
 A_n = \frac{1}{2} \left( \sqrt{1 + i\varphi_n+1} + \sqrt{1 + i\varphi_n} \right) \tag{26}
\]

The relation between \( A_n \) and \( \sqrt{1 + i\varphi_n} \) can be written exactly as

\[
 \frac{1}{\sqrt{1 + i\varphi_n}} = \frac{1}{A_n} \left( 1 + \frac{ia\varphi_n}{4A_n\sqrt{1 + i\varphi_n}} \right) \tag{27}
\]
In this formulation the anomaly arises because the denominator \( A_n \) in the expression for \( \dot{\varphi}_n \) is not quite the same as that in the determinant of Eq. (27). That is, we have the expression \( \exp[-a\dot{\varphi}_n^2/(2A_n^2)]/\sqrt{1+i\varphi_n} \), and we need to expand one of these in terms of the other.

5 Expanding the Kinetic Term

In this case we take \( \lambda = a(1 + i\varphi_n) \), \( x = \bar{\pi}_n \) and \( B = i\dot{\varphi}_n/(1 + i\varphi_n) \) in the Gaussian identity

\[
\frac{1}{\sqrt{1+i\varphi_n}} = \frac{1}{N} \int dx \exp \left\{ -\frac{1}{2} \lambda (x-B)^2 \right\} \tag{28}
\]

\[
= \frac{1}{N} \int d\bar{\pi}_n \exp \left\{ -\frac{1}{2} a(1+i\varphi_n)\bar{\pi}_n^2 + ia\dot{\varphi}_n\bar{\pi}_n + \frac{1}{2} a\dot{\varphi}_n^2 \frac{1}{21 + i\varphi_n} \right\}
\]

Now we need to expand the kinetic term \(-\frac{1}{2}a\dot{\varphi}_n^2/A_n^2\) in terms of \( \frac{1}{2}\lambda B^2 \), namely \( \frac{1}{2}\lambda B^2 = -\frac{1}{2}a\dot{\varphi}_n^2/(1 + i\varphi_n) \). First write Eq. (27) in terms of \( B \):

\[
1 \sqrt{1+i\varphi_n} = \frac{1}{A_n} \left( 1 + \frac{aB}{4A_n} \sqrt{1+i\varphi_n} \right)
\]

In terms of \( R \equiv \sqrt{1+i\varphi_n}/A_n \), this reads

\[
\frac{1}{R} = 1 + \frac{1}{4}aBR \tag{29}
\]

which is a quadratic equation for \( R \), with solution

\[
R = \frac{2}{1 + \sqrt{1+4aB}}
\]

So the kinetic term is \(-\frac{1}{2}a\dot{\varphi}_n^2/A_n^2 = \frac{1}{2}\lambda B^2 R^2 = \frac{1}{2}\lambda B^2 + \frac{1}{2}\lambda B^2 (R^2 - 1)\). Hence

\[
e^{-\frac{1}{2}a\dot{\varphi}_n^2/A_n^2} = e^{\frac{1}{2}\lambda B^2} \left[ 1 - \frac{1}{4} a\lambda B^3 + \frac{a^2}{32} (5\lambda B^4 + \lambda^2 B^6) + \ldots \right] \tag{30}
\]

Now we need the Gaussian identities, under \( \int dt \exp\{ -\frac{1}{2} \lambda (x-B)^2 \} \):

\[
B^3 \equiv x^3 - \frac{3x}{\lambda},
\]

\[
B^4 \equiv x^4 - \frac{6x^2}{\lambda} + \frac{3}{\lambda^2}, \tag{31}
\]

\[
B^6 \equiv x^6 - \frac{15x^4}{\lambda} + \frac{45x^2}{\lambda^2} - \frac{15}{\lambda^3}.
\]
\[ e^{-\frac{1}{2}a\phi_n^2/A_n^2} = e^{\frac{1}{2}\lambda B^2} \left( 1 + \frac{3}{4}a\bar{\pi}_n + O(a^2) \right). \] (32)

Note that it is crucial to keep terms in Eq. (30) which are nominally of order \( a^2 \) (and possibly higher). The point is that the Gaussian identities bring in terms of order \( 1/\lambda, 1/\lambda^2 \) etc., which means that such terms may actually be of order \( a \). The net result so far is that we appear to have produced the anomaly, but with the wrong coefficient, \( 3/4 \) versus \( 1/4 \). However, it is important to realize that after these transformations we are left with

\[ -\frac{1}{2}a(1+i\bar{\phi}_n)\bar{\pi}_n^2 \]

of Eq. (22). The difference between them is

\[ -\frac{1}{2}a(1+i\bar{\phi}_n)\bar{\pi}_n^2 = -\frac{1}{2}a(1+i\bar{\phi}_n)\bar{\pi}_n^2 + \frac{1}{4}ia^2\dot{\phi}_n\bar{\pi}_n^2 = -\frac{1}{2}a(1+i\bar{\phi}_n)\bar{\pi}_n^2 + \frac{1}{4}\lambda aB\bar{\pi}_n^2 \]

In order to implement Gaussian identities with the new kinetic term we need to write the correction in terms of a new \( \tilde{\lambda} \) and \( \tilde{B} \), namely

\[ \tilde{\lambda} = a(1+i\bar{\phi}_n) \]

and \( \tilde{B} = i\dot{\phi}_n/(1+i\bar{\phi}_n) \), with the same property that \( \lambda \tilde{B} = \lambda B = ia\dot{\phi}_n \). The exponential of the additional term \( \frac{1}{4}\lambda aBx^2 \) expands to

\[ \exp\left(\frac{1}{4}\lambda aBx^2\right) = 1 + \frac{1}{4}\lambda aBx^2 + \frac{1}{32}\lambda^2 B^2 x^4 + \ldots \equiv 1 - \frac{1}{2}ax + O(a^2), \]

using the further Gaussian equivalences

\[ \tilde{B}x^2 \equiv x^3 - \frac{2x}{\lambda}, \]

\[ \tilde{B}^2x^4 \equiv x^6 - \frac{9x^4}{\lambda} + \frac{12x^2}{\lambda^2}. \] (33)

Thus, the additional term precisely corrects the coefficient of the anomaly from \( 3/4 \) in Eq. (32) to \( 1/4 \). A necessary ingredient for this to work is the lack of a term in \( 1/\lambda^3 \) in the Gaussian equivalence for \( \tilde{B}^2x^4 \).

### 6 Expanding the Determinant

In this case we take \( x = \bar{\pi}_n \), \( \lambda = aA_n^2 \) and \( B = i\dot{\phi}_n/A_n^2 \) in the Gaussian identity

\[ \frac{1}{A_n} = \frac{1}{N} \int dx \exp\left\{ -\frac{1}{2}\lambda(x - B)^2 \right\} \] (34)

\[ = \frac{1}{N} \int dx \exp\left\{ -\frac{1}{2}aA_n^2\bar{\pi}_n^2 + ia\dot{\phi}_n\bar{\pi}_n + \frac{a\dot{\phi}_n^2}{2A_n^2} \right\} \] (35)
In fact we can solve Eq. (27) for $\sqrt{1 + i\varphi_n}$ in terms of $A_n$:

$$\frac{1}{A_n} = \frac{1}{\sqrt{1 + i\varphi_n}} \left( 1 - \frac{ia\dot{\varphi}_n}{4A_n^2} \right) = \frac{1}{\sqrt{1 + i\varphi_n}} \left( 1 - \frac{1}{4} aB \right)$$

Thus, in terms of the new $B, R \equiv \sqrt{1 + i\varphi_n}/A_n = 1 - aB/4$.

So the determinant can be written as

$$\frac{1}{\sqrt{1 + i\varphi_n}} = \frac{1}{A_n} \cdot \frac{1}{R} = \frac{1}{A_n} \left( 1 + \frac{1}{4} aB + \frac{1}{16} a^2 B^2 + \ldots \right)$$

Now we use two further identities under the Gaussian integration of Eq. (34):

$$B \equiv x$$
$$B^2 \equiv x^2 - \frac{1}{\lambda}$$

to obtain

$$\frac{1}{\sqrt{1 + i\varphi_n}} \equiv \frac{1}{A_n} \left( 1 + \frac{1}{4} ax - \frac{a}{16 A_n^2} + O(a^2) \right)$$

The second term in ( ) correctly gives the anomaly when elevated to the exponent, but the third term, which was missed in Ref. [6], appears to be an unwanted addition. The final result is

$$Z = \frac{1}{N} \prod_n \int d\varphi_n d\bar{\pi}_n \exp \left\{ -a \left[ \frac{1}{2} A_n^2 \bar{\pi}_n^2 - i\bar{\pi}_n \dot{\varphi}_n - \frac{1}{4} \bar{\pi}_n + \frac{1}{16 A_n^2} - \alpha(1 + i\varphi_n)^2 \right] \right\}$$

This differs from Eq. (22) in two ways: we have a term resembling $\Delta V$, and the coefficient of $\bar{\pi}_n^2$ is $\frac{1}{2} A_n^2$ rather than $\frac{1}{2}(1 + i\varphi_n)$. The difference between $\varphi_n$ and $\varphi_n$ in the interaction term is not important.

Now

$$a A_n^2 \bar{\pi}_n^2 = a(1 + i\varphi_n) \bar{\pi}_n^2 + \lambda \bar{\pi}_n^2 \left( 1 - R^2 \right).$$

So, since $ia\varphi_n = ia(\varphi_n - \frac{1}{2}a\dot{\varphi}_n) = i\alpha \varphi_n - \frac{1}{2} \lambda aB$,

$$-\frac{1}{2} a A_n^2 \bar{\pi}_n^2 = -\frac{1}{2} a(1 + i\varphi_n) \bar{\pi}_n^2 + \frac{1}{4} \lambda (aB) \bar{\pi}_n^2 - \frac{1}{2} \lambda \bar{\pi}_n^2 \left( 1 - R^2 \right).$$

The additional terms expand to

$$\frac{1}{4} \lambda \bar{\pi}_n^2 (aB) - \frac{1}{2} \lambda \bar{\pi}_n^2 \left[ \frac{1}{2} (aB) - \frac{1}{16} (aB)^2 \right] = \frac{1}{32} \lambda \bar{\pi}_n^2 (aB)^2$$
and their exponential to \(1 + (aB)^2 \lambda \bar{n}^2/32 + \ldots\).

Again, because we have changed the coefficient of the kinetic term to \(\frac{1}{2}(1 + i \bar{\phi}_n)\) we strictly need to rewrite \(\lambda B^2\) in terms of \(\bar{\lambda}\) and \(\bar{B}\) according to \(\lambda B^2 = (\bar{\lambda} \bar{B}^2) \times (\bar{\lambda}/\lambda)\), where the correction factor is

\[
\frac{\bar{\lambda}}{\lambda} = \frac{1 + i \bar{\phi}_n}{A_n^2} = 1 + O(a \bar{B})^2
\]

The additional term does not in fact contribute to order \(a\). Thus we need the final equivalence

\[
\bar{B}^2 x^2 \equiv x^4 - \frac{5x^2}{\lambda} + \frac{2}{\lambda^2}
\]

for \(\bar{B}^2 x^2\) under Gaussian integration, which gives \(1 + a^2/(16 \bar{\lambda}) + O(a^3)\). When exponentiated this term precisely cancels the \(\Delta V\)-like term \(a/(16A_n^2)\) in Eq. (40) up to \(O(a^2)\).

### 7 Conclusions

The naïve functional integral formulation of the operator calculation can be regarded as the classical result, in that it takes no account of operator ordering, or equivalently of any particular discretization, and so fails to produce the linear, anomalous term, which is of order \(\hbar\). We have shown an elegant method of producing this term in the continuous formalism, provided we take account of the additional \(\Delta V\) term that has been shown (by careful discretization) to occur whenever a general change of variables is made.

The calculation of Ref. [6] indeed attempted a careful discretization, but missed terms that were nominally of higher order in the lattice spacing \(a\), but were in fact of the same order as the terms kept because of the particular nature of the Gaussian equivalences, which bring in factors of \(O(1/\lambda) = O(1/a)\). Starting from the recognition that the coefficient of the kinetic term does not exactly match the argument of the functional determinant we proceeded in two ways, expanding either the kinetic term or the functional determinant. In both cases we used the discretized version of the corrected continuum calculation as a canonical form with which to compare our results. The expansion of the kinetic term turned out to be the easier, needing only a change from the point variable \(\phi_n\) to the averaged variable \(\bar{\phi}_n\) to obtain agreement. The method of expanding the determinant used in Ref. [6] eventually produced the same result after more lengthy manipulations, which incidentally showed that the simple passage from Eq. (57) to Eq. (58) in Ref. [6] was not correct.
One can ask whether the same phenomenon of trading a $\Delta V$ term for an anomaly by means of some analogue of Eq. (17) is possible for other changes of variable. The answer seems to be in the negative: the parametrization of Eq. (3) seems to be very special, as indeed it is in other respects.

In general, the coefficient of the kinetic term in $\varphi$ starting from a standard kinetic term in $\psi$ is $\frac{1}{2} M$, where

$$M = \left( \frac{d\psi}{d\varphi} \right)^2,$$

(42)
to be compared with the expression for $\Delta V$ in Eq. (14). For the two to be proportional we require

$$\frac{d\psi}{d\varphi} \propto \frac{d}{d\varphi} \left( \frac{d\varphi}{d\psi} \right),$$

(43)
or equivalently $d^2\varphi/d\psi^2 = \text{const}$. This leads back to a relation of the general form of Eq. (4) up to a shift in $\psi$.

The main motivation for reformulating the quantum mechanical problem in path-integral terms in Ref. [6] was in order to attempt a generalization to higher dimensions, where there are indications from the non-Hermitian formulation that the theory is asymptotically free and naturally possesses a non-vanishing vacuum expectation value, making it an attractive alternative to the standard Higgs model. It is therefore natural to ask whether additional potentials of the nature of $\Delta V$, and the consequent production of an anomaly, occur in dimensions greater than 1. The answer appears to be negative, at least in the context of dimensional regularization. This is because algebraic field transformations lead to $O(h^2)$ terms proportional to $(\delta^{(n)}(0))^2$ in $n$ spatial dimensions, which vanish in such a regularization scheme. This is consistent with a diagrammatic expansion, for which no additional terms are necessary [9]. The reason for their presence in quantum mechanics versus their absence in field theory has been analyzed in some detail by Salomonson [10].

Even though it appears that this particular problem does not occur in field theory, there remain severe difficulties in carrying out the programme. The most promising approach (3) put forward in Ref. [6] still contains un-cancelled Jacobian factors, which can only be represented in the Lagrangian by the introduction of several auxiliary fields.

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