The Poisson equations in the nonholonomic Suslov problem: integrability, meromorphic and hypergeometric solutions

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Abstract
We consider the long standing problem of integrability of the Poisson equations describing spatial motion of a rigid body in the classical nonholonomic Suslov problem. We obtain necessary conditions for their solutions to be meromorphic and show that under some further restrictions these conditions are also sufficient. This leads to a family of explicit meromorphic solutions, which correspond to rather special motions of the body in space. We also give explicit extra polynomial integrals in this case.

In the more general case (but still under one restriction), the Poisson equations are transformed into a generalized third order hypergeometric equation. A study of its monodromy group allows us also to calculate the ‘scattering’ angle: the angle between the axes of limit permanent rotations of the body in space.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In some cases of the rigid body dynamics, in particular, in the problem of motion of a solid about a fixed point, the Euler equations for the angular velocity vector $\omega \in \mathbb{R}^3$ separate and can be integrated. Then, given a generic solution $\omega(t)$, to determine the motion of the solid
in space it is necessary to solve the reconstruction problem, that is, to find three independent
solutions of the linear Poisson equations
\[ \dot{\gamma} = \gamma \times \omega(t), \]  
(1.1)
\( \gamma \in \mathbb{R}^3 \) being a unit vector fixed in space.

The most known example of solvable Poisson equations appears in the Euler top problem,
when generic \( \omega(t) \) are elliptic (i.e. doubly periodic) functions and one particular solution \( \gamma(t) \)
is also elliptic, whereas the other two are quasiperiodic (see, e.g. [7, 18]).

Similar, but formally more complicated solutions \( \gamma(t) \) appear in the case of the
Zhukovsky–Volterra gyrostat (see [17, 21]).

A nontrivial integrable generalization of the Euler–Poisson equations was considered
in [3], where the Euler equations have the standard form and (1.1) are replaced by the equations
\[ \dot{\gamma} = \chi \gamma \times \omega(t), \]
\( \chi \) being an arbitrary odd integer number. It was shown that, as in the Euler top problem, the
latter equations possess an extra algebraic integral; however, a general solution for \( \gamma \) is still
unknown.

In this paper, following Suslov [15], we consider the motion of the rigid body about a
fixed point in the presence of the constraint \( \langle \omega, a \rangle = 0 \), \( a \) being a fixed vector in the body
frame. Let \( \mathbb{I} : \mathbb{R}^3 \mapsto \mathbb{R}^3 \) be the symmetric inertia tensor of the body. The Euler equations
for the angular velocity vector \( \omega \) separate and take the following simple form
\[ \frac{d}{dt}(\mathbb{I} \omega) = \mathbb{I} \omega \times \omega + \lambda a, \]  
(1.2)
where \( \times \) denotes the vector product in \( \mathbb{R}^3 \) and \( \lambda \) is the Lagrange multiplier. Differentiating the
constraint, we find \( \lambda = -\langle \mathbb{I} \omega \times \omega, \mathbb{I}^{-1} a \rangle / \langle a, \mathbb{I}^{-1} a \rangle \). Therefore, (1.2) can be represented as
\[ \frac{d}{dt}(\mathbb{I} \omega) = \frac{1}{\langle a, \mathbb{I}^{-1} a \rangle} \mathbb{I}^{-1} a \times (((\mathbb{I} \omega) \times \omega) \times a), \]
which, in view of \( \langle \omega, a \rangle = 0 \), is equivalent to
\[ \frac{d}{dt}(\mathbb{I} \omega) = \frac{\langle \mathbb{I} \omega, a \rangle}{\langle a, \mathbb{I}^{-1} a \rangle} \mathbb{I}^{-1} a \times \omega. \]  
(1.3)
In the following, without loss of generality, we assume that \( a = (0, 0, 1)^T \), which, in view of
the constraint, implies \( \omega_3 \equiv 0 \). This simplifies the Poisson equations to the form
\[ \gamma' = -\omega_2(t)\gamma_3, \quad \gamma' = \omega_1(t)\gamma_3, \quad \gamma' = \omega_2(t)\gamma_1 - \omega_1(t)\gamma_2. \]  
(1.4)
We also assume that the tensor \( \mathbb{I} \) is disbalanced, i.e. is not diagonal in the chosen frame. (If \( a \)
is an eigenvector of \( \mathbb{I} \), then all the solutions of (1.3) are equilibria).

It is known that under these assumptions system (1.3) restricted to the plane \( \omega_3 = 0 \) has
a line of equilibria points \( I_{13} \omega_1 + I_{23} \omega_2 = 0 \) and that the trajectories \( \omega(t) \) are elliptic arcs that
form the heteroclinic connection between the asymptotically unstable and stable equilibria
(see figure 1).

Whereas the reduced system (1.3) is elementary integrable in terms of hyperbolic
functions, it is believed that the corresponding Poisson equations (1.4) are not, although we
did not find a proof of that in the literature. A study of some properties of their complex
solutions was made in [11]. A qualitative analysis of the behaviour of \( \gamma(t) \) in the classical
Suslov problem, as well as in its multi-dimensional generalization, was made in [20], whereas
some other interesting generalizations of the problem were studied in [8].
In section 2 we present generic solutions of the Euler equations for the Suslov problem and formulate the problem of integrability of system (1.3) and (1.4) by the Euler–Jacobi theorem.

In section 3 necessary and sufficient conditions of meromorphicity of solutions are obtained, they both require that one of the components \( I_{13}, I_{23} \) of the inertia tensor must be zero.

Section 4 discusses general properties of asymptotic solutions of the Poisson equations in the specific case \( I_{13} = 0 \), which are compared with generic solutions of another famous nonholonomic system, the Chaplygin sleigh. It is also shown that the meromorphicity conditions on \( I \) are compatible with the restrictions on the inertia tensor of a physical rigid body.

In section 5 we show that in the case \( I_{13} = 0 \) the Poisson equations can be transformed to a generalized third order hypergeometric equation. Using its monodromy group, we solve the classical problem of calculating the angle between the axes of limit permanent rotations of the body in space.

When the sufficient conditions of meromorphicity are satisfied, we observe that the corresponding hypergeometric series solutions reduce to products of polynomials and exponents, which are explicitly calculated.

Next, we apply the differential Galois analysis to the Poisson equations and prove that when the parameters of the problem satisfy the necessary conditions of meromorphicity, but not the sufficient ones, these equations and, therefore, the whole Suslov system are not solvable in the class of Liouvillian functions.

In section 6 we present all the meromorphic solutions of the problem and the corresponding extra polynomial integrals in the explicit form.

In conclusion some relevant open problems are briefly discussed.

2. Generic solutions of the Euler and the Poisson equations

In the general case the components \( I_{13}, I_{12} \) of the inertia tensor \( \mathbf{I} \) are not zero, but, by an appropriate choice of the frame that preserves the constraint one can always make \( I_{12} = 0 \). The condition that \( \mathbf{I} \) is positively defined means that all main minors are greater than zero and we obtain \( I_{11} > 0 \) and \( I_{11} I_{22} > 0 \) that gives \( I_{22} > 0 \).

Then the Euler–Poisson equations (1.3) and (1.4) have the form

\[
\begin{align*}
\dot{\omega}_1 &= -\frac{1}{I_{11}} (I_{13}\omega_1 + I_{23}\omega_2)\omega_2, \\
\dot{\omega}_2 &= \frac{1}{I_{22}} (I_{13}\omega_1 + I_{23}\omega_2)\omega_1
\end{align*}
\]
and

\[ \dot{y}_1 = -\omega_2(t)y_3, \quad \dot{y}_2 = \omega_1(t)y_3, \quad \dot{y}_3 = \omega_2(t)y_1 - \omega_1(t)y_2. \]  \hspace{1cm} (2.2)

They have two first integrals, the energy and the trivial geometric one:

\[ F_1 = I_{11}\omega_1^2 + I_{22}\omega_2^2, \quad F_2 = \langle \gamma, \gamma \rangle = \gamma_1^2 + \gamma_2^2 + \gamma_3^2. \]  \hspace{1cm} (2.3)

The main question of this paper is: for which values of \( I_{ij} \) is system (2.1) and (2.2) integrable? Here the integrability will be understood in the context of the classical Euler–Jacobi theorem, which relies upon the existence of an invariant measure and a sufficient number of independent integrals.

Note that although system (2.1) and (2.2) does not have an invariant measure in the strict sense (the density of the volume form tends to 0 as \( t \to \pm \infty \)), we can consider its restriction on the energy level \( F_1 = E > 0 \), which consists of two open components. On each of them we choose time \( t \) and \( y \) as coordinates, and then the restricted system in the autonomous form reads as

\[ i = 1, \quad \dot{y}_1 = -\omega_2(t)y_3, \quad \dot{y}_2 = \omega_1(t)y_3, \quad \dot{y}_3 = \omega_2(t)y_1 - \omega_1(t)y_2. \]  \hspace{1cm} (2.4)

It possesses the trivial integral \( F_2 \), as well as the invariant volume form

\[ \mu = dr \wedge dy_1 \wedge dy_2 \wedge dy_3. \]

Thus, for the integrability in the Jacobi sense only one additional first integral \( F(t, \gamma) \) is required.

It appears that the presence of such an integral is closely related to the existence of single-valued solutions of the Poisson equations. Namely, let \( P(t) = (P_1, P_2, P_3) \) be a single-valued vector-function satisfying \( \dot{P} = P \times \omega \). Then system (2.4) possesses the single-valued integral \( F = \langle P(t), \gamma \rangle \). Indeed,

\[ \dot{F} = \langle P(t), \gamma \rangle + \langle P(t), \dot{\gamma} \rangle = \langle \dot{P}(t), \gamma \rangle + \langle P(t), \gamma \times \omega \rangle = \langle P(t), \gamma \times P(t), \gamma \rangle = 0. \]

If, moreover, the components of \( P(t) \) can be expressed as single-valued functions of the solutions \( \omega_1(t), \omega_2(t) \), then system (2.1) and (2.2) admits an additional integral

\[ F_3(\omega_1, \omega_2, y_1, y_2, y_3) = \langle P(\omega), \gamma \rangle \]

functionally independent with \( F_1 \) and \( F_2 \).

In the following we assume that \( I_{13}^2 + I_{23}^2 \neq 0 \), otherwise all solutions of the dynamical equations (2.1) are equilibria. Then their generic solutions have the form

\[ \omega_1(t) = \frac{a(e^{At} - e^{-At}) + c_1}{e^{At} + e^{-At}}, \quad \omega_2(t) = \frac{b(e^{At} - e^{-At}) + c_2}{e^{At} + e^{-At}}, \quad b = -a \frac{I_{13}}{I_{23}}, \]  \hspace{1cm} (2.5)

where

\[ a = -\frac{AI_{11}I_{22}I_{23}}{I_{13}^2I_{22} + I_{23}^2I_{11}}, \quad c_1 = \pm \frac{2AI_{13}I_{22}\sqrt{I_{11}I_{22}}}{I_{13}^2I_{22} + I_{23}^2I_{11}}, \quad c_2 = \pm \frac{2AI_{23}I_{11}\sqrt{I_{11}I_{22}}}{I_{13}^2I_{22} + I_{23}^2I_{11}}. \]

\( A \) being an arbitrary positive constant related to the energy integral. Note that for \( t \to \pm \infty \) these expressions give points on the equilibria line \( I_{13}\omega_1 + I_{23}\omega_2 = 0 \), as required.

**Remark 2.1.** For each \( A \) fixed, the Poisson equations (2.2) and the above functions \( \omega(t) \) are invariant with respect to time rescaling \( t \to t/A \), which reduces the solution of (2.1) to the form

\[ \omega_1(t) = \frac{a(e^t - e^{-t}) + c_1}{e^t + e^{-t}}, \quad \omega_2(t) = \frac{b(e^t - e^{-t}) + c_2}{e^t + e^{-t}}, \quad b = -a \frac{I_{13}}{I_{23}}, \]  \hspace{1cm} (2.6)

\[ a = -\frac{I_{11}I_{22}I_{23}}{I_{13}^2I_{22} + I_{23}^2I_{11}}, \quad c_1 = \pm \frac{2I_{13}I_{22}\sqrt{I_{11}I_{22}}}{I_{13}^2I_{22} + I_{23}^2I_{11}}, \quad c_2 = \pm \frac{2I_{23}I_{11}\sqrt{I_{11}I_{22}}}{I_{13}^2I_{22} + I_{23}^2I_{11}}. \]  \hspace{1cm} (2.7)
This implies that, without loss of generality, one can study solutions of (2.2) with the coefficients $\omega_1, \omega_2$ having the form (2.6), (2.7). This will be assumed in the following.

Note that choosing $\omega(t)$ in the form (2.6) implies that the energy integral is fixed to be

$$F_1 = \frac{I_{11}^2 I_{22}}{I_{13} I_{22} + I_{23} I_{11}}.$$  \hfill (2.8)

### 3. Painlevé property

Since generic solution of equations (2.2) with the coefficients (2.6) is not known, it is natural at least to ask under which conditions on the parameters $I_{ij}$ the complex functions $\gamma_i(t)$ are meromorphic or single valued. The answer is given by the following theorem.

**Theorem 3.1.** All solutions of (2.2) are meromorphic solutions if and only if either

$$I_{13} = 0 \quad \text{and} \quad \frac{I_{22}(I_{11} - I_{22})}{I_{23}^2} = p^2, \quad (3.1)$$

or

$$I_{23} = 0 \quad \text{and} \quad \frac{I_{11}(I_{11} - I_{22})}{I_{13}^2} = -p^2, \quad (3.2)$$

where $p$ is a nonzero integer.

Note that the second case (3.2) can be obtain from (3.1) by the linear transformation

$$\omega_1 \rightarrow -\omega_2, \quad \omega_2 \rightarrow -\omega_1, \quad \gamma_1 \rightarrow \gamma_2, \quad \gamma_2 \rightarrow \gamma_1, \quad \gamma_3 \rightarrow \gamma_3.$$  \hfill (3.3)

Below we also show that conditions (3.1) expressed in terms of the parameters $(a, b, c_1, c_2)$ take the following form

$$c_1 = b = 0 \quad \text{and} \quad c_2^2 - 4a^2 = 4p^2, \quad p \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}. \quad (3.4)$$

The proof of theorem 3.1 is given in next two subsections.

#### 3.1. Necessary conditions—Kovalevskaya analysis

Let us show that if all solutions of system (2.2) are single valued, then one of conditions (3.1) and (3.2) is satisfied.

Let $v$ denote the right-hand sides of the Euler–Poisson equations (2.1) and (2.2). We will analyse whether all solutions of this system are single valued in a neighborhood of a particular solution of the form $x(t) = d/t$, where $d \in \mathbb{C}^5 \setminus \{0\}$. As it is easy to check, such a solution exists if $d$ is a nonzero solution of the algebraic equations

$$v(d) = -d.$$  

They have two solutions. The first of them is given by

$$\tilde{\omega}_1 = -\frac{\sqrt{I_{11} I_{22}}}{\sqrt{I_{11} I_{23} + i I_{13} \sqrt{I_{22}}}}, \quad \tilde{\omega}_2 = \frac{I_{13} \sqrt{I_{22}}}{I_{13} \sqrt{I_{22}} - i \sqrt{I_{11} I_{23}}}, \quad \tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}_3 = 0,$$  \hfill (3.5)
and the second one is its complex conjugate. According to the Lyapunov theorem, see, e.g. [10],
if all solutions of (2.1) and (2.2) are single valued, then

1. all eigenvalues of the Kovalevskaya matrix
   \[ K(d) = \frac{\partial v}{\partial x}(d) + \text{Id} \]
   are integers, and
2. the Kovalevskaya matrix is semi-simple.

For the solution \( d \) given by (3.5) the characteristic polynomial of \( K(d) \) is

\[ \det(K(d) - \lambda \text{Id}) = \frac{(\lambda - 2)(\lambda - 1)(\lambda + 1)W(\lambda)}{(I_{13}\sqrt{I_{22}} - iI_{23}\sqrt{I_{11}})^2}, \tag{3.6} \]

where

\[ W(\lambda) = I_{11}I_{22}(I_{12} - I_{11}) - (I_{13}\sqrt{I_{22}} - iI_{23}\sqrt{I_{11}})^2(\lambda - 1)^2. \]

The roots of \( W(\lambda) \) have the form

\[ \lambda_{1,2} = 1 \pm \Lambda, \quad \Lambda = \sqrt{(I_{11} - I_{22})I_{11}I_{22}(\sqrt{I_{11}I_{22} + iI_{13}\sqrt{I_{22}}})^2}{(\sqrt{I_{11}I_{22} + iI_{13}\sqrt{I_{22}}})^2}. \]

If \( \lambda_1 \) and \( \lambda_2 \) are integer, then, in particular, they are real, which implies

\[ I_{13}I_{23}(I_{11} - I_{22}) = 0. \tag{3.7} \]

Moreover, if \( \lambda_1, \lambda_2 \in \mathbb{Z} \), then \( \Lambda^2 = p^2 \), where \( p \in \mathbb{Z} \). This gives

\[ p^2 = \frac{(I_{11} - I_{22})I_{11}I_{22}(I_{11}I_{23}^2 - I_{22}I_{13}^2)}{(I_{11}I_{23} + I_{22}I_{13})^2}. \tag{3.8} \]

Condition (3.7) gives three possibilities. Either \( I_{13} = 0 \), and in this case we have (3.1), or \( I_{23} = 0 \), which gives (3.2), or, finally, \( I_{11} = I_{22} \). In the last case, the Kovalevskaya matrix has the multiple eigenvalue \( \lambda = 1 \) and is not semi-simple. If \( I_{13} = 0 \), then \( p = 0 \) if \( I_{11} = I_{22} \), but then the Kovalevskaya matrix is not semi-simple. Similarly, if \( I_{23} = 0 \), then \( p = 0 \) if \( I_{11} = I_{22} \), but then again the Kovalevskaya matrix is not semi-simple. As a result, we proved the ‘if’ part of theorem 3.1.

In another, but in an almost equivalent form, condition (3.8) was obtained previously in [11].

**Remark 3.2.** In terms of the parameters \( a, b, c_1, c_2 \) conditions (3.7) and (3.8) have the form

\[ ac_1 + bc_2 = 0 \quad \text{and} \quad c_1^2 + c_2^2 = 4(a^2 + b^2) = 4p^2, \quad p \in \mathbb{Z}^*, \tag{3.9} \]

which also implies (3.4).

### 3.2. Sufficient conditions—analysis of the monodromy group

The general solution (2.6) of the Euler equations is single valued. Thus, all solutions of the Euler–Poisson equations (2.1) and (2.2) are single valued if and only if all the solutions of the linear Poisson equations (2.2) with \( \omega_1(t) \) and \( \omega_2(t) \) as in (2.6) are single valued.

Note that the only singular points of (2.2) are simple poles located at \( t_0 = \pm \frac{\pi}{2} i \mod \pi i \). Then, a necessary condition for the single valuedness is that the monodromy matrices at all the singular points are identities.
To show this, rewrite system (2.2) in the form
\[ \dot{\gamma} = A(t)\gamma, \quad A(t) = \begin{bmatrix} 0 & 0 & -\omega_2(t) \\ 0 & 0 & -\omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} \] (3.10)
and observe that
\[ A(t) = \frac{1}{t-t_0} A_0 + O(t-t_0), \quad A_0 = \begin{bmatrix} 0 & 0 & -b \pm \frac{1}{2}ic_2 \\ 0 & 0 & -a \pm \frac{1}{2}ic_1 \\ b \mp \frac{1}{2}ic_2 & -a \pm \frac{1}{2}ic_1 & 0 \end{bmatrix} \] (3.11)

\(a, b, c_1, c_2\) being specified in (2.7).

The monodromy matrix of the canonical loop around \(t_0\) is \(M_{t_0} = \exp 2\pi i A_0\). A direct calculation shows that the eigenvalues of \(A_0\) are
\[ \rho_1 = 0 \quad \text{and} \quad \rho_{2,3} = \pm \frac{1}{2}\sqrt{c_1^2 + c_2^2 - 4(a^2 + b^2) + 4i(ac_1 + bc_2)}. \] (3.12)

Hence, if conditions (3.9) are satisfied, then \(A_0\) has eigenvalues \(\rho_1 = 0, \rho_2 = p\) and \(\rho_3 = -p\) with \(p \in \mathbb{Z}^\star\). In this case \(M_{t_0} = \text{Id}\) provided that there are no local logarithmic solutions in a neighbourhood of \(t_0\). In order to check this, we use the following theorem (see, e.g. [1] for the details).

**Theorem 3.3.** Assume that in the linear system
\[ \dot{Y} = B(t)Y, \quad B(t) = \sum_{i=0}^{\infty} B_i t^i, \] (3.13)
the matrix coefficient \(B_0\) does not have a pair of eigenvalues such that their difference is a nonzero integer. Then, there exists a matrix \(T\) given by a convergent power series
\[ T = \sum_{i=0}^{\infty} T_i t^i, \quad T_0 = \text{Id}, \]
such that the linear map \(Y = TZ\) transforms (3.13) into the following form
\[ \dot{Z} = \frac{1}{t} B_0 Z. \] (3.14)

The form of the fundamental matrix \(Z_{B_0}\) of (3.14) is well known. Namely, let \(P\) be the similarity transformation reducing \(B_0\) into its Jordan form \(J = P^{-1} B_0 P\), where \(P\) is an invertible matrix with constant coefficients, and let \(Z_J\) denote the new fundamental matrix. Then \(Z_{B_0} = P Z_J P^{-1}\) and \(Z_J\) has a block-diagonal structure such that the Jordan block (of dimension \(v\)) corresponding to an eigenvalue \(\lambda\) has the form
\[ t^\lambda \begin{bmatrix} 1 & \ln t & \cdots & \frac{\ln^{v-1} t}{(v-1)!} \\ 0 & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ln t \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \]

Now assume that conditions (3.1) or (3.2) are satisfied. By the existence of the linear transformation (3.3), without loss of generality we can assume that conditions (3.1) are satisfied.
Expressed in terms of parameters \((a, b, c_1, c_2)\), the latter take the form (3.4), or, setting 
\[
(c^2 - 1)a^2 = p^2.
\] (3.15)

We also denote \(d^2 = c^2 - 1\).

Now we check the presence of logarithmic terms in the solutions.

**Theorem 3.4.** The Poisson equations (2.2) with \(\omega_i\) given by (2.6) with \(c_1 = b = 0\), \(c_2 = -2ca = -2\sqrt{d^2 + 1}p/d\), \(p \in \mathbb{Z}\), do not have logarithmic terms in local solutions around singular points \(t_0\).

**Proof.** For \(c_1 = b = 0\), \(c_2 = -2ca = -2\sqrt{d^2 + 1}p/d\), the matrix \(A_0\) reads

\[
A_0 = \begin{pmatrix}
0 & 0 & -ip\sqrt{d^2 + 1} & \frac{p}{d} \\
0 & 0 & \frac{p}{d} & 0 \\
-\frac{ip\sqrt{d^2 + 1}}{d} & -\frac{p}{d} & 0 & 0 \\
-\frac{p}{d} & 0 & 0 & 0
\end{pmatrix}.
\] (3.16)

Using the similarity transformation \(S\), one obtains its diagonal form \(\tilde{A}_0 = S^{-1}A_0S\) with

\[
\tilde{A}_0 = \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -p & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
\frac{-i\sqrt{d^2 + 1}}{d} & \frac{i\sqrt{d^2 + 1}}{d} & 0 & 0 \\
\frac{i}{\sqrt{d^2 + 1}} & \frac{-1}{d} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Thus we see that all the eigenvalues of \(A_0\) are integer and their differences are also nonzero integers. Under this transformation the matrix \(A\) of the linear Poisson equations takes the form

\[
\tilde{A} = \begin{pmatrix}
\frac{ip\text{sech}(t)(d^2 + 1 + i \sinh(t))}{2d} & \frac{ip(\text{sech}(t) + itanh(t))}{d^3} & 0 & 0 \\
\frac{(d^2 + 1)p(-i \text{sech}(t) + \tanh(t))}{d^3} & \frac{0}{d^3} & (d^2 + 1)p(-i \text{sech}(t) + \tanh(t)) & 0 \\
0 & \frac{ip(\text{sech}(t) + itanh(t))}{2d} & \frac{p[-i(d^2 + 1) \text{sech}(t) + \tanh(t)]}{d^2} & 0
\end{pmatrix}.
\]

Transformation \(\gamma \rightarrow \Gamma\) defined by \(\gamma = U\Gamma\) with

\[
U = \begin{pmatrix}
(t - \frac{i\pi}{2})^{2p} & 0 & 0 \\
0 & (t - \frac{i\pi}{2})^{p} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

gives the equivalent linear system \(\dot{\Gamma} = B(t)\Gamma\) with \(B(t) = U^{-1}AU - U^{-1}\dot{U}\).

This matrix has residual part diagonal

\[
B_0 = \begin{pmatrix}
-p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{pmatrix}.
\]
which satisfies the assumptions of theorem 3.3. Since all the Jordan blocks of $B_0$ are one-dimensional, the logarithmic terms do not appear in the solutions of the Poisson equations.

4. General properties of solutions $\gamma(t)$ in the special case $I_{13} = 0$

Assume now that, apart from $I_{12} = 0$, the first meromorphicity condition in (3.1) is also satisfied, that is, $I_{13} = 0$, which imposes an essential restriction on $\gamma$.

Then the reparametrized solutions (2.6) and (2.7) are reduced to

$$\omega_1(t) = \frac{a(e^t - e^{-t})}{e^t + e^{-t}}, \quad \omega_2(t) = -\frac{2ac}{e^t + e^{-t}},$$

(4.1)

$$a = -\frac{I_{22}}{I_{23}}, \quad c = \sqrt{\frac{I_{11}}{I_{22}}}.$$  

It is seen that they have equilibria along the line $\omega_2 = 0$.

Using the parameter $p$ introduced in (3.1), and setting $d = p/a = \sqrt{(I_{11} - I_{22})/I_{22}}$, we can rewrite the whole system (2.1), (2.2) in the form

$$\dot{\omega}_1 = \pm \frac{d}{p(d^2 + 1)} \omega_2^2, \quad \dot{\omega}_2 = -\frac{d}{p} \omega_1 \omega_2,$$

(4.2)

$$\dot{\gamma}_1 = -\omega_2 \gamma_3, \quad \dot{\gamma}_2 = \omega_1 \gamma_3, \quad \dot{\gamma}_3 = \omega_2 \gamma_1 - \omega_1 \gamma_2.$$  

Since $d = \sqrt{c^2 - 1}$ and $p$ can take positive as well as negative values, we choose sign $+$ in the first equation in (4.2).

As it follows from theorem 3.1, the solution of this system is meromorphic if and only if $p = \sqrt{I_{22}(I_{11} - I_{22})/I_{23}}$ is a nonzero integer.

A natural question is whether this condition can be satisfied for the components of the tensor of inertia of a ‘physical’ rigid body.

**Proposition 4.1.** For any $p \geq 1$ there exists tensor $\mathbb{I}$ with

$$I_{13} = I_{12} = 0, \quad I_{11} > I_{22} > I_{33} > 0 \quad \text{and} \quad I_{33} + I_{22} > I_{11},$$

(4.3)

such that its eigenvalues are positive, satisfy the triangular inequalities, and $I_{ij}$ satisfy the meromorphicity condition (3.1).

**Proof.** Let (4.3) hold. Since $I_{13} = I_{12} = 0$, the eigenvalues of $\mathbb{I}$ are $J_1 = I_{11} > 0$ and $J_2, J_3$ such that

$$J_2 + J_3 = I_{33} + I_{22}, \quad J_2 J_3 = I_{22} I_{33} - I_{23}^2.$$  

(4.4)

Then, if (3.1) holds

$$J_2 J_3 = I_{22} I_{33} - \frac{I_{22}(I_{11} - I_{22})}{p^2}.$$  

(4.5)

If $p = 1$, then $J_2 J_3 = I_{22} (I_{33} + I_{22} - I_{11})$, which is positive due to the second inequality in (4.3). If $p > 1$, then the right-hand side of (4.5) is even bigger. Hence, in any of these cases, $J_2 J_3 > 0$ and, therefore, $J_2, J_3$ are positive.

Next, since $J_2 + J_3 = I_{33} + I_{22}$, the second inequality in (4.3) implies the triangular inequality $J_3 + J_2 > J_1$. To prove the other two inequalities, $J_1 + J_2 > J_3, J_1 + J_3 > J_2$, we note that they are equivalent to

$$J_1^2 > (J_2 - J_3)^2.$$  

(4.6)
In view of (4.4) and (3.1),
\[
(J_2 - J_3)^2 = (J_2 + J_3)^2 - 4J_2J_3 = (I_{33} + I_{22})^2 - 4(I_{22}I_{33} - I_{23}^2) = (I_{33} - I_{22})^2 + 4\frac{I_{22}(I_{11} - I_{22})}{p^2}.
\] (4.7)

Then, if \( p = 1 \),
\[
(J_2 - J_3)^2 - J_1^2 = -I_{22}^2 - 2I_{22}(I_{33} + I_{22}) - I_{11}^2 + I_{33}^2 + 4I_{11}I_{22}.
\]
Setting here \( I_{33} + I_{22} = I_{11} + \Delta, \Delta = I_{33} + I_{22} - I_{11} > 0 \), we get
\[
(J_2 - J_3)^2 - J_1^2 = -(I_{22} - I_{11})^2 - 2I_{22}\Delta + I_{33}^2 = -(I_{33} - \Delta)^2 + I_{33}^2 - 2I_{22}\Delta
\]
\[= 2\Delta(I_{33} - I_{22}) - \Delta^2,
\]
which is negative, since \( I_{22} > I_{11} \) and \( \Delta > 0 \). Hence, (4.6) holds.

Now, if \( p > 1 \), the right-hand side of (4.7) is even less, and (4.6) holds as well. As a result, under conditions (4.3) all the three triangular inequalities hold.

**Steady-state rotations of the body in space.** Solutions of the Poisson equations (1.4) with the components of \( \omega(t) \) given by (2.6) describe the evolution of the rigid body in space, which, as was shown in [15], is an asymptotic evolution from one steady-state rotation to another one. The equilibria of \( \omega(t) \) correspond to the steady-state rotations themselves.

In the particular case \( I_{13} = 0 \), as \( t \to \pm \infty \), the motion in space tends to rotations with the angular velocities \( \mp a \) about the axis \((1, 0, 0)\) fixed in the body. Note that the spatial orientations of the axes of the above steady-state rotations are generally not the same.

This can be illustrated by solutions of (1.4) with the limit condition
\[
\lim_{t \to -\infty} \gamma(t) = (-1, 0, 0)^T.
\] (4.8)
As it follows from the structure of (1.4) and (4.1), in this case, as \( t \to +\infty \), \( \gamma(t) \) tends to the periodic trajectory
\[
(\cos(\Delta \psi), \sin(\Delta \psi) \cos(at + \delta), \sin(\Delta \psi) \sin(at + \delta))^T,
\]
where \( \Delta \psi \) is the angle between the above axes of the steady-state rotations in space and \( \delta \) is a phase. That is, the motion of \( \gamma(t) \) in the body frame tends to a uniform rotation about the axis \((1, 0, 0)\), as shown in figure 2(a), when \( p = 2.5 \). Figure 2(b) illustrates the corresponding asymptotic motion of the body axis \((1, 0, 0)\) in space frame.

**Comparison with the Chaplygin sleigh.** A similar behaviour occurs in the ‘noncompact’ version of the Suslov problem: the Chaplygin sleigh, a rigid body moving on a horizontal plane supported at three points, two of which slide freely without friction, while the third is a knife edge (a blade), which allows no motion orthogonal to its direction (see for the details, among others, [4, 12]). The configuration space of this dynamical system is \( \text{SE}(2) \), the group of Euclidean motions of the two-dimensional plane \( \mathbb{R}^2 \), which can be parametrized by the angular orientation \( \theta \) of the blade, and the position \((x, y)\) of the contact point of the blade. The presence of the blade determines a nonholonomic distribution on the phase space \( T \text{SE}(2) \).

As in the Suslov problem, the equations for the two components of the linear momentum of the sleigh and its angular momentum in the body frame separate and can be solved in terms of hyperbolic functions.

In contrast to what is known about the solvability of the Poisson equations (1.4), the kinematic equations for the Chaplygin sleigh are integrable for any initial conditions [12].
In particular, as \( t \to \pm \infty \), the motion of the sleigh tends to a straight line uniform motion, however, along different directions. (This can be compared with the steady-state rotations of the body about different axes in the Suslov problem.) A typical trajectory of the contact point of the blade is shown in figure 3.

The solution of the kinematic equations found in [12] permits calculation of the angle between the limit straight line motions of the sleigh, which does not depend on the initial conditions, but only on dynamical parameters of the body.

Going back to the Suslov problem, due to the invariance of the Poisson equations (2.2) and the components of \( \omega(t) \) with respect to time rescaling \( t \to t/A \) (remark 2.1), one concludes that the angle \( \Delta \psi \) between the axes of the limit steady-state rotations depends only on the components of the inertia tensor \( I \) (that is, it does not depend on the energy of the motion).

Then, the following natural question arises: is it possible to calculate the scattering angle \( \Delta \psi \) as a function of \( I_{ij} \) explicitly, without solving the Poisson equations? It appears that at least in the case \( I_{13} = 0 \) the answer is positive and is given by theorem 5.4.

5. Solvability analysis

5.1. Solvability in the class of generalized hypergeometric functions

In this subsection we again assume that \( I_{13} = 0 \) and fix the energy level at the value determined by (2.8). Hence, \( \omega_1(t) \) and \( \omega_2(t) \) are given by (4.1), with

\[
a = \frac{p}{d}, \quad d^2 = c^2 - 1.
\]

Note that here we do not assume that \( p \in \mathbb{Z}^* \).

The Poisson equations (2.2) can be rewritten as the following third order equation:

\[
\dot{a}_0 \dot{\gamma}_1 + a_1 \ddot{\gamma}_1 + a_2 \dot{\gamma}_1 + a_3 \gamma_1 = 0,
\]

where

\[
a_0(t) = (d^2 + 1) p^2 \omega_1, \quad a_1(t) = dp(2(d^2 + 1) \omega_1^2 - \omega_2^2),
\]

\[
a_2(t) = (d^2 + 1) \omega_1[(d^2 + p^2) \omega_1^3 + p^2 \omega_2^2], \quad a_3(t) = -dp \omega_2^2[(d^2 + 1) \omega_1^2 + \omega_2^2]
\]
Now introduce new independent variable
\[
z(t) = \frac{d^2}{(d^2 + 1)p^2} \frac{\omega^2}{(e^t + e^{-t})^2},
\]
(5.4)
Then
\[
\dot{z} = -\frac{2d \omega}{p} \frac{\omega}{z}, \quad \ddot{z} = \frac{2z(2p^2 + 2d^2 p^2 - 3(1 + d^2) p^2 z)}{(d^2 + 1)p^2},
\]
and (5.2), (5.3) transform to
\[
\gamma'''' + b_1 \gamma''' + b_2 \gamma'' + b_3 \gamma = 0,
\]
(5.5)
where the prime denotes the differentiation with respect to \(z\), and
\[
b_1 = \frac{2}{z} + \frac{1}{z - 1}, \quad b_2 = \frac{(d^2 + 1)p^2}{8d^2(z - 1)^2 z^2}, \quad b_3 = -\frac{p^2(z - 1) + d^2(1 + z(-6 - p^2(z - 1) + 4z))}{4d^2(z - 1)^2 z^2}.
\]
(5.6)
Now, setting \(\gamma_1 = \sqrt{z - 1} u(z)\), we obtain the following equation for the function \(u(z)\):
\[
z^2(1 - z)u''' + z\left[1 + \frac{1}{2}(4 - 9z)z u'' + \left[\frac{1}{4}\left(1 + \frac{p^2}{d^2}\right) + \frac{1}{4}(p^2 - 13)z\right] u' + \frac{1}{8}(p^2 - 1)u\right] = 0.
\]
(5.7)
This is a generalized third order hypergeometric equation, whose canonical form is (see, e.g. [13])
\[
z^2(1 - z)u''' + z(1 + \beta_1 + \beta_2 - (3 + \alpha_1 + \alpha_2 + \alpha_3)z)u'' + [\beta_1 \beta_2 - (1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1)]z u' - \chi_1 \alpha_2 \chi_3 u = 0.
\]
(5.8)
One can identify (5.8) and (5.7) by setting
\[
\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1 + p}{2}, \quad \alpha_3 = \frac{1 - p}{2}, \quad \beta_1 = \frac{d - ip}{2d}, \quad \beta_2 = \frac{d + ip}{2d}.
\]
(5.9)
Equation (5.8) belongs to the class of generalized hypergeometric equations of order \(n\) for \(n = 3\). If we denote by \(\theta = zd/dz\), then this class takes the form
\[
[(\theta + \beta_1 - 1) \cdots (\theta + \beta_{n-1} - 1)\theta - z(\theta + \alpha_1) \cdots (\theta + \alpha_n)]u = 0,
\]
(5.10)
see, e.g. [2, 14]. Recall that one of its solutions is the generalized hypergeometric function introduced by Thomae [16]

\[ _nF_{n-1}\left(\begin{array}{c}
\alpha_1, \ldots, \alpha_n \\
\beta_1, \ldots, \beta_{n-1}
\end{array}; z\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k z^k}{(\beta_1)_k \cdots (\beta_{n-1})_k k!}. \]

(5.11)

Here \((a)_k = a(a + 1) \cdots (a + k)\) is the Pochhammer symbol and \((a)_0 = 1\).

It is known (see, e.g. [13]) that equation (5.8) has three regular singularities over \(\mathbb{C}P^1\) with the exponents

\[ 0, 1 - \beta_1, 1 - \beta_2, \quad \text{at } z = 0, \]
\[ 0, 1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3, \quad \text{at } z = 1, \]
\[ \alpha_1, \alpha_2, \alpha_3, \quad \text{at } z = \infty, \]

and if \(\beta_1, \beta_2, \beta_1 - \beta_2 \not\in \mathbb{Z}\), then the general solution of (5.7) around the origin is given by the linear combination

\[ u = C_1 F_1(z) + C_2 F_2(z) + C_3 F_3(z), \]

(5.12)

\[ F_1(z) = _3F_2\left(\begin{array}{c}
\alpha_1, \alpha_2, \alpha_3 \\
\beta_1, \beta_2
\end{array}; z\right), \]
\[ F_2(z) = z^{1-\beta_3} _3F_2\left(\begin{array}{c}
\alpha_1 - \beta_1 + 1, \alpha_2 - \beta_1 + 1, \alpha_3 - \beta_1 + 1 \\
\beta_2 - \beta_1 + 1, 2 - \beta_1
\end{array}; z\right), \]
\[ F_3(z) = z^{1-\beta_3} _3F_2\left(\begin{array}{c}
\alpha_1 - \beta_2 + 1, \alpha_2 - \beta_2 + 1, \alpha_3 - \beta_2 + 1 \\
\beta_1 - \beta_2 + 1, 2 - \beta_2
\end{array}; z\right), \]

(5.13)

\(C_1, C_2\) and \(C_3\) being constants of integration.

An immediate observation is that if one of \(\alpha_i\) in (5.11) is a negative integer or zero, then the series truncates. In view of (5.4), this happens precisely when \(p\) is an odd integer, and then the particular solution \(F_1(z)\) takes the form

\[ F_1(z) = 1 + \sum_{j=1}^{(p-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{d_j}{(d_j + p_j)(d_j + p_j - 1)} z^{(p_j + 1)}/z^{(p_j + 1)}/. \]

(5.14)

Since \(\gamma_1 = \sqrt{1 - z^2}(z)\) and \(z = 4/(e^t + e^{-t})\), this corresponds to a solution \(\gamma_1(t)\), rational in \(e^t\) and meromorphic in \(t\), having poles of order \(p\) at \(t = \pi i/2 \text{ (mod } \pi i)\), as predicted by theorem 3.1.

**Factorization of \(F_3(z)\), \(F_3(z)\).** Due to the symmetry in definition (5.9) of the parameters \(\alpha, \beta\), the above series \(F_2(z), F_3(z)\) can be expressed in terms of customary hypergeometric functions

\[ F(\alpha, \beta; \gamma; y) = 1 + \frac{\alpha \beta}{\gamma} y + \frac{\alpha (\alpha + 1) \beta (\beta + 1)}{\gamma (\gamma + 1) 2!} y^2 + \ldots \]

of the variable \(y = e^t\).

**Proposition 5.1.** Under condition (5.9) the following factorization holds

\[ F_2, F_3 = \frac{y}{(1 + y)^{\alpha - 1}(\alpha - 1)!} \cdot F(\alpha_1, b_1; c_1; -y) \cdot F(\alpha_2, b_2; c_2; -y), \]

(5.15)
functions \(F(\alpha, \beta; y)\) and \(\gamma\). The first hypergeometric function \(F(\alpha, \beta; y)\) can be written as a series in \(y\) by using the following quadratic transformation:

\[
F\left(\frac{a}{2}; \frac{a}{2} + \frac{1}{2} - b, 1 + a - b; \frac{4y}{(1 + y)^2}\right) = (1 + y)^{a-b} F(a, b, 1 + a - b, -y),
\]

where \(\gamma = \alpha + \beta + 1, \gamma_2 = \alpha_\pm + \beta_\pm + \frac{2}{3} \pm \frac{ip}{2d}, \quad x = \frac{4y}{(y + 1)^2}\).

Next, due to the special form of \(\gamma\), the first hypergeometric function \(F(\alpha, \beta; y; x)\) can be written as a series in \(y\) by using the following quadratic transformation:

\[
F\left(\frac{a}{2}; \frac{a}{2} + \frac{1}{2} - b, 1 + a - b; \frac{4y}{(1 + y)^2}\right) = (1 + y)^{a-b} F(a, b, 1 + a - b, -y),
\]

see, e.g. page 64 in [6]. Now identifying \(a_\pm = a_\pm/2\) and \(\beta_\pm = \alpha_\pm/2 + 1/2 - b_\pm\), we obtain

\[
a_\pm = \frac{2 - p}{2} \pm \frac{ip}{2d}, \quad b_\pm = \frac{1 - p}{2}, \quad c_\pm = 1 + a_\pm - b_\pm = \frac{3}{2} \pm \frac{ip}{2d},
\]

which coincide with (5.16). Inserting these parameters into (5.19), we get

\[
F\left(\alpha_\pm; \gamma_\pm; y; \frac{4y}{(y + 1)^2}\right) = (y + 1)^{\gamma_\pm - 1}\gamma_\pm F(a_\pm; b_\pm; c_\pm, -y).
\]

It remains to apply a similar quadratic transformation to the second factor \(F(\alpha_\pm, \beta_\pm; \gamma_\pm - 1; 4y/(y + 1)^2)\) in (5.18). To do this, we first use the relation

\[
(\gamma - n)_n z^{\gamma - 1 - n} F(\alpha, \beta; \gamma - n; z) = \frac{d^n}{dz^n} [z^{\gamma - 1} F(\alpha, \beta; \gamma; z)],
\]

(\(\gamma - n)_n\) being the corresponding Pochhammer symbol (see, e.g. equation (22) on page 102 in [6]). Setting here \(n = 1\), we obtain

\[
F(\alpha_\pm, \beta_\pm; \gamma_\pm - 1; z) = \frac{1}{\gamma_\pm} F(\alpha_\pm, \beta_\pm; \gamma_\pm; z) + \frac{z}{\gamma_\pm (\gamma_\pm - 1)} \frac{d}{dz} F(\alpha_\pm, \beta_\pm; \gamma_\pm; z),
\]

\[
z = \frac{4y}{(y + 1)^2}.
\]
In view of the chain rule
\[
\frac{d}{dz} = \frac{(y + 1)^3}{4(y - 1)} \frac{d}{dy},
\]
and formula (5.17), expression (5.21) takes the form
\[
F\left(\alpha_\pm, \beta_\pm; \gamma_\pm - 1; \frac{4y}{(y + 1)^2}\right) = \frac{2d(y + 1)^{a_i}}{(d \pm ip)(3d \pm ip)(y - 1)}
\cdot \left\{ [d(p - 1)y - d \mp ip]F(a_\pm, b_\pm; c_\pm, -y) - 2dy(y + 1) \frac{d}{dy} F(a_\pm, b_\pm; c_\pm, -y) \right\}.
\]

Substituting this, as well as (5.20), into (5.18) and simplifying, we finally get expression (5.15).

Note that when \( p \) is an odd positive integer, the parameters \( b_\pm = (1 - p)/2 \) become negative integer, and the hypergeometric series \( F(a_\pm, b_\pm; c_\pm, -y) \) in (5.15) convert into polynomials of degree \((p - 1)/2\),
\[
F(a_\pm, b_\pm; c_\pm, -y) = \sum_{k=0}^{(p-1)/2} \frac{(a_\pm)_k (b_\pm)_k}{(c_\pm)_k k!} (-y)^k. \quad (5.22)
\]
Next, after a simplification, the series \( \hat{F}(a_\pm, b_\pm; c_\pm, -y) \) in (5.15) become the following polynomials of degree \((p - 1)/2 + 1\) annihilate:
\[
\hat{F}(a_\pm, b_\pm; c_\pm, -y) = \sum_{k=0}^{(p-1)/2} \frac{(a_\pm)_k (b_\pm)_k}{(c_\pm)_k k!} \frac{(2d k + d + pi)(4d k - dp \pm ip)}{dp - 2d k \mp ip} (-y)^k. \quad (5.23)
\]
As a result, we conclude that in this case,
\[
F_{2,3}(z(y)) = \frac{\kappa_\pm}{(1 + y)^{p-1} (y - 1)} y^{\frac{d_{\text{top}}}{d_{\text{top}}}} P_\pm(y), \quad \kappa_\pm = \frac{2d}{(dp \pm ip)(3d \pm ip)}, \quad (5.24)
\]
where \( P_\pm(y) = F(a_\pm, b_\pm; c_\pm, -y) \hat{F}(a_\pm, b_\pm; c_\pm, -y) \) are polynomials of degree \( p - 1 \) with complex conjugated coefficients.

The fact that in this case the series \( \mathcal{F}_1 \) is a rational function of \( y \) follows from formula (5.14).

The explicit form of the corresponding meromorphic solutions of the Poisson equations (2.2) for odd \( p \) is given in section 6.

**Monodromy for \( \mathcal{F}_1(z) \) and the angle \( \Delta \psi \).** Now we show how to use the monodromy group of equation (5.10), which was analysed in [2, 13, 14], one can calculate the angle \( \Delta \psi \) between the axes of the limit steady-state rotations of the body in space for generic \( p \in \mathbb{R} \). As it follows from (5.4), the values \( \tau = \pm \infty, \tau = 0 \) and \( \tau = \pi i/2 \) (mod \( \pi i \)) correspond to the singular points \( z = 0, z = 1, \) and \( z = \infty \), respectively. As time \( \tau \) evolves along the real axis from \(-\infty\) to \( \infty \) passing by \( \tau = 0 \), the variable \( z \) makes a loop in \( \mathbb{C} \) embracing 1 in the positive or negative direction.

Next, we need the following lemma.

**Lemma 5.2.** Under the substitution (5.4), the component \( \gamma_1 \) of the special solution \( \gamma(t) \in S^2 \) satisfying the boundary condition (4.8) has the following form around the origin \( z = 0 \):
\[
\gamma_1(t) = -\sqrt{1 - z(t)} \mathcal{F}_1(z(t)) \equiv -\sqrt{1 - z^3} F_2 \left( \begin{array}{c} 1/2 \\ 1/2 - ip/(2d) \end{array} \begin{array}{c} (1 + p)/2 \\ (1 - p)/2 \end{array} ; z \right). \quad (5.25)
\]
Proof. As it follows from (5.12), the most general possible expression for the component \( \gamma_1 \), which ensures \( \lim_{\omega_1 \to -\infty} \gamma_1 = -1 \), is

\[
\hat{\gamma}_1(t) = -\sqrt{1 - \tilde{z}(F_1(z) + c_2 F_2(z) + c_3 F_3(z))},
\]

\( c_2, c_3 \) being arbitrary constants. On the other hand, from the Poisson equations and solutions (4.1) one has

\[
\gamma_3 = -\frac{\gamma_1}{\omega_1} = -4 \frac{e^t - e^{-t}}{ac(e^t + e^{-t})^2} \frac{dy_1}{dz} = -\frac{e^t - e^{-t}}{ac} \frac{d\gamma_1}{dz}.
\]

The above special solution requires that \( \lim_{\omega_1 \to -\infty} \gamma_3 = 0 \). Then, setting here \( \gamma_1 = \hat{\gamma}_1 \), differentiating \( F_{1,2,3}(z) \) and taking into account the values of \( \beta_1, \beta_2 \), we see that this condition holds if and only if \( c_2, c_3 \) above are zero.

When \( z \) makes a loop around 1, the root \( \sqrt{1 - \tilde{z}} \) changes sign, whereas the hypergeometric series \( F_1(z) \) transforms according to the following proposition, which is a corollary of theorem 4.1 in [13].

**Proposition 5.3.** When \( z \in \mathbb{C} \) makes a loop around the singular point \( z = 1 \) in the positive direction, the solution \( F_1(z) \) around the origin undergoes the monodromy

\[
F_1 \to F_1 - 2i \exp(\pi i (\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)) (\sigma_1 F_1 + \sigma_2 F_2 + \sigma_3 F_3),
\]

\( \sigma_1 = \frac{\sin(\pi \alpha_1) \sin(\pi \alpha_2) \sin(\pi \alpha_3)}{\sin(\pi \beta_1) \sin(\pi \beta_2)} \),

\( \sigma_2 = -\frac{\sin(\pi (\beta_1 - \alpha_1)) \sin(\pi (\beta_1 - \alpha_2)) \sin(\pi (\beta_1 - \alpha_3))}{\sin(\pi \beta_1) \sin(\pi (\beta_1 - \beta_2))} \),

\( \sigma_3 = -\frac{\sin(\pi (\beta_2 - \alpha_1)) \sin(\pi (\beta_2 - \alpha_2)) \sin(\pi (\beta_2 - \alpha_3))}{\sin(\pi (\beta_2 - \beta_1)) \sin(\pi \beta_2)} \).

Note that in our case, when (5.9) holds,

\[
\hat{\beta}_1 + \hat{\beta}_2 - \hat{\alpha}_1 - \hat{\alpha}_2 - \hat{\alpha}_3 = \frac{1}{2}, \quad \sigma_1 = \frac{\cos^2(\pi p/2)}{\cos^2((\pi i p)/(2d))}.
\]

Then, combining formula (5.25) with the above proposition and observing that at \( z = 0 \): \( F_1 = 1, F_2 = F_3 = 0 \), one concludes that

\[
\lim_{t \to -\infty} \gamma_1 = 1 - 2i \exp\left(\frac{\pi i}{2}\right) \sigma_1 F_1(0) = 1 - 2\frac{\cos^2(\pi p/2)}{\cos^2((\pi i p)/(2d))}.
\]

Since \( \lim_{t \to +\infty} \gamma_1 = -1 \) and \( \cos(\Delta \psi) = -\lim_{t \to +\infty} \gamma_1 \), we obtain the following result.

**Theorem 5.4.** The angle \( \Delta \psi \) between the axes of the limit steady-state rotations of the body does not depend on the energy and, when \( I_{13} = 0 \), is uniquely defined by relation

\[
\cos \frac{\Delta \psi}{2} = \frac{\cos(\pi p/2)}{\cos((\pi i p)/(2d))},
\]

where, as above,

\[
d = \sqrt{\frac{I_{11} - I_{22}}{I_{22}}}, \quad p = \sqrt{\frac{I_{22}(I_{11} - I_{22})}{I_{23}^2}}.
\]

Note that when the parameter \( p \) is an odd integer, \( \Delta \psi \) is always \( \pi \), regardless of the value of \( q \).

We also add that the formula of theorem 5.4 stands in perfect correspondence with numerical integration tests.

It remains to study the case of even integer \( p \), when, according to theorem 3.1, all the solutions of the Poisson equations (2.2) are meromorphic, but are not given by truncated generalized hypergeometric series. This will be one of the subjects of the next subsection.
5.2. **Differential Galois analysis**

Here we recall one result of G. Darboux which was formulated in Chapter II of his *Théorie générale des surfaces* [5].

**Lemma 5.5.** Assume that $\gamma'(1) = \gamma'(1)(t)$ is a real solution of the Poisson equation

$$\dot{\gamma} = \gamma \times \omega(t),$$

(5.27)

where $\omega(t)$ is a real vector, and $\gamma'(1)$ satisfies $\langle \gamma(1), \gamma(1) \rangle = 1$. Then the remaining two solutions of (5.27) linearly independent of $\gamma(1)$ can be found by a single quadrature.

**Proof.** Let us restrict equation (5.27) to the unit sphere $\langle \gamma, \gamma \rangle = 1$. As coordinates on it we choose

$$u_1 = \frac{\gamma_3 + 1}{\gamma_1 - i\gamma_2}, \quad u_2 = -\frac{\gamma_1 + i\gamma_2}{\gamma_3 + 1},$$

(5.28)

so

$$\gamma_1 = \frac{1 - u_1 u_2}{u_1 - u_2}, \quad \gamma_2 = \frac{1 + u_1 u_2}{u_1 - u_2}, \quad \gamma_3 = \frac{u_1 + u_2}{u_1 - u_2}.$$  

(5.29)

Now, it is easy to check that $u_1$ and $u_2$ satisfy the following Riccati equation:

$$\dot{u} = A + Bu + Cu^2,$$

(5.30)

where

$$A = \frac{i}{2}(\omega_2(t) - i\omega_1(t)), \quad B = -i\omega_3(t), \quad C = \frac{i}{2}(\omega_2(t) + i\omega_1(t)).$$

(5.31)

Formulae (5.29) show that knowing two different solutions of (5.30) we determine one solution of (5.27). On the other hand, having one real solution $\gamma'(1)$ of (5.27) we have two different solutions of Riccati equation (5.30) which are given by formulae (5.28).

Let $u_0$ be a solution of (5.30). Then, as it is easy to check, $u_1 = -1/u_0^*$, where $z^*$ denotes the complex conjugate of $z$, is also a solution of this equation. It is well known that the general solution $u$ of a Riccati equation is determined by its three different particular solutions $u_0$, $u_1$ and $u_2$. Namely, we have

$$\frac{(u - u_0)(u_2 - u_1)}{(u - u_1)(u_2 - u_0)} = C_0,$$

(5.32)

where $C_0$ is an arbitrary constant. From this formula we deduce that knowing only two different solutions $u_0$ and $u_1$, the general solution can be obtain by a single quadrature

$$\frac{u - u_0}{u - u_1} = C_0 \exp \left[ \int C(s)(u_0(s) - u_1(s)) \, ds \right].$$

(5.33)

□

Putting $u = -v/C$, we transform equation (5.30) to the form

$$\dot{v} = A_1 + B_1 v - v^2,$$

where $A_1 = -AC$, $B_1 = B + \dot{C}/C$.  

(5.34)

Then $v = \dot{w}/w$ where

$$\dot{w} - B_1 \dot{w} - A_1 w = 0.$$  

(5.35)

The general question is whether we can find an explicit form of solutions of Poisson equation (5.27) for given functions $\omega_i(t)$. A proper setting to this question is given by the differential algebra. Namely, we assume that $\omega_i(t)$ are elements of a certain differential field $L$ with $\mathbb{C}$ as a subfield of constants. In the considered Suslov problem $L = \mathbb{C}(e^t)$.
Let $K \supset L$ be the Picard–Vessiot extension for equation (5.27). We say that the equation is solvable iff its extension $K \supset L$ is a Liouvillian extension. In this case all solutions of the equation are Liouvillian. By the known Kolchin theorem, the extension $K \supset L$ is a Liouvillian extension iff the identity component of the differential Galois group $G(K/L)$ is solvable, see e.g. [9].

**Proposition 5.6.** Assume that equation (5.27) has a solution Liouvillian over $L = \mathbb{C}(e^t)$. Then all its solutions are Liouvillian.

**Proof.** Let $\gamma^{(1)}$ be a Liouvillian solution of (5.27). Assume that $\langle \gamma^{(1)}, \gamma^{(1)} \rangle = 1$. From the proof of lemma 5.5 we know that to find other solutions of (5.27) it is enough to find a general solution of the Ricatti equation (5.30). But $\gamma^{(1)}$ gives us one Liouvillian solution $u_0$ of equation (5.30). The general solution of (5.30) is given by

$$u = u_0 + \frac{1}{y}, \quad (5.36)$$

where $y$ satisfies linear equation

$$\dot{y} = -(B + 2Cu_0)y - C. \quad (5.37)$$

Hence $y$ is Liouvillian and all solutions of (5.30) are Liouvillian, and thus all solutions of (5.27) are Liouvillian.

If $\gamma^{(1)}$ is a Liouvillian solution of (5.27) and $\langle \gamma^{(1)}, \gamma^{(1)} \rangle = 0$, then, as $\omega_i(t)$ are real, $\text{Re} \gamma^{(1)}$ and $\text{Im} \gamma^{(1)}$ are linearly independent real solutions of (5.27). Moreover,

$$\gamma = \alpha_1 \text{Re} \gamma^{(1)} + \alpha_2 \text{Im} \gamma^{(1)} + \alpha_3 (\text{Re} \gamma^{(1)}) \times (\text{Im} \gamma^{(1)}) \quad (5.38)$$

is a solution of (5.27) for arbitrary $\alpha_1$, $\alpha_2$ and $\alpha_3$. In effect all the solutions of (5.27) are Liouvillian. \hfill \Box

From the above proved fact we obtain the following.

**Proposition 5.7.** Poisson equation (5.27) has a Liouvillian solution iff all solutions of linear equation (5.35) are Liouvillian.

**Proof.** If equation (5.27) has a Liouvillian solution, then, by proposition 5.6, all its solutions are Liouvillian and formulae (5.28) show that all solutions of the Riccati equation (5.30), as well as the transformed Riccati equation (5.34), are Liouvillian. As $v = w/w$, $w$ is Liouvillian, and thus all solutions of (5.35) are Liouvillian.

On the other hand, if equation (5.35) has a nonzero Liouvillian solution, then all solutions of the Riccati equation (5.34), as well as (5.30) are Liouvillian. Thus, by formulae (5.29), Poisson equation (5.27) has a Liouvillian solution, so, by proposition 5.6, all its solutions are Liouvillian. \hfill \Box

For the considered problem $\dot{y} = \gamma \times \omega(t)$, equation (5.35) reads as

$$w'' + p(z)w' + q(z)w = 0, \quad z = e^t, \quad \prime \equiv \frac{d}{dz}, \quad (5.39)$$

where

$$p(z) := \frac{z^2(z^2 + 4icz - 4) - 1}{z(z^2 + 1)(z^2 + 2icz - 1)},$$

$$q(z) := \frac{p^2}{4(c^2 - 1)} \frac{1}{z^2} + \frac{p^2}{(z^2 + 1)^2}. \quad (5.40)$$
This equation is derived under the assumption that relation (3.15) holds true. In what follows, we will work with the reduced form of the above equation, i.e. with

\[ y'' = r(z)y \]

where \( r(z) = \frac{P}{Q} = \frac{1}{5}p'(z) + \frac{1}{4}p(z)^2 - q(z) \), (5.41)

where \( P \) is a polynomial of eight degree

\[ P = \sum_{i=0}^{8} p_i z^i, \] (5.42)

with the following coefficients:

\[ p_0 = p_8 = d^2 + p^2, \quad p_1 = p_7 = 4ic(2d^2 + p^2), \quad p_2 = p_6 = -4(4d^2 + p^2), \] (5.43)

\[ p_4^* = p_5 = 4ic((4p^2 - 2)d^2 + p^2), \] (5.44)

\[ p_4 = -2((-8d^2 + 8(d^2 + 2)p^2 + 1)d^2 + 5p^2), \quad d^2 = c^2 - 1 \] (5.45)

and

\[ Q = 4(c^2 - 1)z^2(z^2 + 1)^2(2cz - i(z^2 - 1))^2. \] (5.46)

Note that we assumed that \( c \neq \pm 1 \), i.e. taking into account relation (3.15), \( p \neq 0 \). This case we will consider later.

From (5.46) it follows that equation (5.41) has six regular singularities \( s_i, i = 0, \ldots, 5 \), and \( s_0 = 0, s_1 = s_2^* = i, s_3 = -i(c + d), s_4 = -i(c - d) \) and \( s_5 = \infty \). The respective differences of exponents \( \Delta_i \) at these points are as follows:

\[ \Delta_0 = \Delta_3 = \frac{p}{d}, \quad \Delta_1 = \Delta_2 = p, \quad \Delta_3 = \Delta_4 = 2. \] (5.47)

We observe immediately that at four singularities \( s_1, s_2, s_3 \) and \( s_4 \) differences of exponents are integer, thus at local expansions of solutions around these points logarithmic terms can appear. Here we only sketch how this happens, for more details see, e.g. [19]. For a linear equation of the second order local solutions \( y_1 \) and \( y_2 \) around a singular point \( s_i \) are postulated in the form of infinite series with leading terms that are equal to the exponents at this point.

\[ \rho_i^{(1/2)} = \frac{1}{2}(1 \pm \Delta_i). \]

In the case when the difference of exponents \( \Delta_i = \rho_i^{(1)} - \rho_i^{(2)} \) is integer such two expansions are in general not functionally independent and then \( y_2 \) is constructed in a different way from \( y_1 \) by a quadrature that usually leads to logarithms.

By direct calculations one can check that solutions around \( s_3 \) and \( s_4 \) do not have such terms independently of the value of \( p \). For singular points \( s_1 \) and \( s_2 \) direct calculations for small \( p \) show that the following conjecture is true.

**Conjecture 1.** Singular points \( s_1 \) and \( s_2 \) are nonlogarithmic for any odd integer and logarithmic for any even \( p \in \mathbb{Z}^* \).

For singular points \( s_1 \) and \( s_2 \) the difference of exponents depends on \( p \) and when \( p \) grows we have to calculate the expansions of \( u_1 \) with more and more terms. For this reason we cannot check the presence of logarithmic terms for an arbitrary \( p \) but we can make such calculations effectively for any chosen value of \( p \) and we did this up to \( p = 10 \).

The presence of logarithmic terms restricts very strongly the possible forms of elements from the differential Galois group, thus at first up to the end of this section we will consider the case of \( p \) even and we prove the following statement.

**Proposition 5.8.** The differential Galois group of equation (5.41) for even \( p \in \mathbb{Z}^* \) is \( SL(2, \mathbb{C}) \).
Proof. The presence of logarithms means that the differential Galois group of the considered equation cannot be neither a subgroup of the infinite dihedral group (because it contains a nondiagonalizable element) nor a finite group. If it is contained in triangular group, then by the same reason it cannot be its proper subgroup, i.e. diagonal subgroup. Thus we have two possibilities that it
1. is contained in the whole triangular group,
2. is SL$(2, \mathbb{C})$.

If the first possibility occurs, then equation (5.41) has an exponential solution of the form
\[ y = R(z) \prod_{i=0}^{4} (z - s_i)^{\rho_i}, \]
where \( R(z) \in \mathbb{C}[z], \) (5.48)

and \( \rho_i \in \{\rho_1^{(1)}, \rho_2^{(2)}\} \) are exponents at points \( s_i \), for \( i = 0, \ldots, 4 \). Expanding this solution at the infinity we find that
\[ n = - \sum_{i=0}^{5} \rho_i \geq 0, \quad \text{where } n = \deg R. \]
(5.49)

We have
\[ \rho_i = \frac{1}{2}(1 + \Delta_i) \quad \text{for } i = 0, \ldots, 4 \]
(5.50)

and
\[ \rho_5 = -\frac{1}{2}(1 + \Delta_s). \]
(5.51)

The above implies that we have to choose \( \rho_0 \) and \( \rho_5 \) such that \( \rho_0 + \rho_5 = 0 \). As points \( s_1 \) and \( s_2 \) are logarithmic, then the only choice is \( \rho_1^{(1)} = \rho_2^{(1)} = (1 + p)/2 \) for \( p > 0 \), or \( \rho_1^{(2)} = \rho_2^{(2)} = (1 - p)/2 \) for \( p < 0 \). At \( s_3 \) and \( s_4 \) both exponents \( \rho_1^{(1)} = 3/2 \) and \( \rho_2^{(2)} = -1/2 \) for \( i = 3, 4 \) are possible. Thus for \( p > 0 \), \( n \in \{-p, -p - 2, -p - 4\} \) and for \( p < 0 \) we have \( n \in \{p, p - 2, p - 4\} \) but all these admissible values are negative. This shows that the equation does not have an exponential solution (5.48). Thus, the differential Galois group of the equation is SL$(2, \mathbb{C})$. □

By the Kolchin theorem this means that in this case the Poisson equations are not solvable or, more precisely, the following theorem holds.

Theorem 5.9. Euler–Poisson equations in the meromorphic case defined by (3.4) for even \( p \in \mathbb{Z}^* \) are not solvable in the class of Liouvillian functions.

6. Explicit meromorphic solutions and first integrals

6.1. Meromorphic solutions

As was observed in section 5.1, for odd \( p \in \mathbb{N} \) the third order hypergeometric equation (5.7) for the variable \( u = y/\sqrt{z^2 - 1} \) has three independent quasi-polynomial solutions \( \mathcal{F}_{1,2,3}(z) \) given by (5.14) and (5.24), that is,
\[ \mathcal{F}_1(z) = Q(z), \quad \mathcal{F}_{2,3}(z(y)) = \frac{x_{\pm}}{(1 + y)^{p-1} (y - 1)^{p-1}} y^{\frac{4p}{y^{2p}} P_{\pm}(y)}, \]
\[ z = \frac{4}{(e^y + e^{-y})^2}, \quad y = e^{2y}, \]

\( Q(z), P_{\pm}(y) \) being polynomials of degree \((p - 1)/2\) and \( p - 1 \), respectively.

Now, taking into account the relation between the solutions \( w(z) \) of this hypergeometric equation and those of the Poisson equation (5.2), we arrive at
Theorem 6.1. For odd \( p \in \mathbb{N} \), the third order equation (5.2) for \( \gamma_1(t) \) has independent meromorphic solutions

\[
\gamma_1^{(1)}(t) = \frac{e' - e^{-t}}{(e^t + e^{-t})^p} \sum_{k=0}^{(p-1)/2} a_k (e^t + e^{-t})^{p-2k},
\]

\[
a_k = 4^k \frac{(2k-1)!!}{(2k)!!} \frac{d(2k)(p^2 - 1) \cdots (p^2 - (2k - 1)^2)}{(d^2 + p^2) \cdots ((2k-1)d^2 + p^2)}, \quad a_0 = 1,
\]

\[
\gamma_1^{(2)}(t) = \frac{e^{1 - \frac{2i}{p}}}{(1 + e^{2i})^p} \sum_{k=0}^{(p-1)/2} b_k e^{2kt} \equiv e^{-i \frac{2t}{p}} b_0 e^{-(p-1)i} + \cdots + b_{p-1} e^{(p-1)i} \frac{e^{i \frac{2t}{p}}}{(e^t + e^{-t})^p},
\]

\[
\gamma_1^{(3)}(t) = \frac{e^{1 + \frac{2i}{p}}}{(1 + e^{2i})^p} \sum_{k=0}^{(p-1)/2} b_k^* e^{2kt} \equiv e^{i \frac{2t}{p}} b_0^* e^{-(p-1)i} + \cdots + b_{p-1}^* e^{(p-1)i} \frac{e^{i \frac{2t}{p}}}{(e^t + e^{-t})^p},
\]

where \((\cdot)^*\) denotes the complex conjugation, and the coefficients \( b_k \) are uniquely determined from the polynomial product

\[
\sum_{k=0}^{p-1} b_k y^k = \sum_{k=0}^{(p-1)/2} (a_k b_k) \frac{(p-1)/2)!}{(c_k k)!} (-y)^k \sum_{k=0}^{(p-1)/2} (a_k b_k) \frac{(2dk + d + pi)(4dk - dp + ip)}{dp - 2dk - ip} (-y)^k,
\]

\[
a = \frac{2 - p + ip}{2d}, \quad b = \frac{1 - p}{2}, \quad c = \frac{3 + ip}{2d}.
\]

The proof is straightforward: substitution of expression (5.14) into

\[
\gamma = \sqrt{z - 1} F_1(z) = \frac{e' - e^{-t}}{e^t + e^{-t}} \frac{4}{(e^t + e^{-t})'},
\]

yields (6.1). Next, in view of (5.24), the product

\[
\gamma = \frac{e' - e^{-t}}{e^t + e^{-t}} F_2(z(y))
\]

gives (6.2) and (6.3). \( \square \)

Note that all these solutions satisfy

\[
\lim_{t \to \pm \infty} \gamma_1^{(1)}(t) = \pm 1, \quad \lim_{t \to \pm \infty} \gamma_1^{(2,3)} = 0.
\]

Now let \( \gamma^{(i)} = (\gamma_1^{(i)}, \gamma_2^{(i)}, \gamma_3^{(i)})^T, i = 1, 2, 3 \) be the corresponding independent vector solutions of the Poisson equations (1.4). Given \( \gamma_1^{(i)} \) as in theorem 6.1, the two remaining components of \( \gamma^{(i)} \) can be calculated by differentiations, using system (1.4).

Namely, let us write the above meromorphic solutions in the form

\[
\gamma_1^{(1)} = \frac{P_1(x)}{(1 + x^2)^p}, \quad \gamma_1^{(2)} = e^{-x} \frac{x P_2(x)}{(1 + x^2)^p}, \quad x = e^t,
\]

\( P_1(x), P_2(x) \) being polynomials of degrees \( 2p \) and \( 2(p - 1) \), respectively. Then, in view of (1.4), one has

\[
\gamma_3 = -\frac{\gamma_1^{(1)}}{\omega_2}, \quad \text{and then} \quad \gamma_3 = \frac{\omega_2 \gamma_1^{(1)} - \gamma_3^{(1)}}{\omega_1}, \quad (\gamma' = \frac{d}{dx}).
\]

Applying these formulae and using expression (4.1) for \( \omega_1, \omega_2 \), we get

\[
\gamma_3^{(1)} = \frac{d((1 + x^2)^2 P_1(x) - 2px P_1(x))}{2p \sqrt{d^2 + 1(1 + x^2)^p}},
\]

\[
\gamma_2^{(1)} = \frac{d(2x^2 + 1)((x^2 + 1) P_1(x) + (2 - 4p)x P_1'(x) + 2p(2p + d^2(2p + 1)(x^2 + 1)) P_1(x))}{2p \sqrt{d^2 + 1(1 + x^2)^p}}.
\]
and
\[ \gamma_{3}^{(2)} = \frac{x^{-\frac{p}{2}}[dx(x^2 + 1)P'_2 + (d - ip + (d - ip - 2dp)x^2)P_2]}{2p\sqrt{d^2 + 1(1 + x^2)^p}}, \]

\[ \gamma_{2}^{(2)} = \frac{x^{-\frac{p}{2}}W}{2p^2\sqrt{d^2 + 1(1 + x^2)^p(x^2 - 1)}}, \]

\[ W = -dx(x^2 + 1)(dx(x^2 + 1)P''_2(x) + 2(d - 2d(p - 1)x^2 - ip(x^2 + 1))P'_2(x)) \]
\[ + (p^2(x^2 - 1)^2 - 2d^2(1 + p(2p - 3))x^2(x^2 + 1) \]
\[ - idp(x^2 + 1)(-1 + (4p - 3)x^2))P_2(x). \]

Expressions for \( \gamma_{i}^{(i)} \) are complex conjugations of those for \( \gamma_{j}^{(2)}. \)

It remains to mention that all the components of \( \gamma^{(i)} \) have poles of order \( p \) at \( t = \frac{\pi}{2}i + \pi iN, N \in \mathbb{Z}. \)

**Remark 6.2.** Note that for real moments of inertia \( I_{ij} \) and, therefore, a real constant \( d, \) solution (6.1) is real. To get real solutions for the other components one just takes real and imaginary parts of (6.2) and (6.3), as well as \( \gamma_{2}^{(1)}, \gamma_{3}^{(1)}, \) using the formula \( e^{it} = \cos(\alpha t) + i \sin \alpha t. \) These parts should be normalized to obtain unit vector solutions \( \gamma_{2}^{(2)}, \gamma_{3}^{(2)}. \)

In view of (6.4), as \( t \to \pm \infty \) the real vector solution \( \gamma^{(1)} \) starts and ends along the axis \((1, 0, 0), \) although for finite \( t \) its evolution can be complicated (see figures 4 and 5).

### 6.2. Examples of meromorphic solutions

Below we show two examples of independent meromorphic solutions \( \{\gamma_{1}, \gamma_{2}, \gamma_{3}\} \) of the Poisson equations for \( p = 1 \) and \( p = 3, \) which are real for \( t \in \mathbb{R}. \)

First, solutions (6.1) and (6.2) were taken, their real and imaginary parts were extracted, then formulae (6.5) were applied. We write the obtained real solutions in terms of hyperbolic and trigonometric functions.

The simple solutions for \( p = 1 \) are
\[ \gamma_{1}^{(1)} = \left( \tanh(t), -\frac{\sech(t)}{\sqrt{1 + d^2}}, \frac{d\sech(t)}{\sqrt{1 + d^2}} \right), \]
\[ \gamma_{2}^{(2)} = \left( \cos\left(\frac{t}{d}\right) \frac{\tanh(t) - d \sin\left(\frac{t}{d}\right)}{\sqrt{d^2 + 1}}, -\frac{d \cos\left(\frac{t}{d}\right) \tanh(t) + \sin\left(\frac{t}{d}\right)}{\sqrt{d^2 + 1}} \right), \]
\[ \gamma_{3}^{(3)} = \left( \sin\left(\frac{t}{d}\right) \frac{\tanh(t) + d \cos\left(\frac{t}{d}\right)}{\sqrt{d^2 + 1}}, -\frac{d \sin\left(\frac{t}{d}\right) \tanh(t) - \cos\left(\frac{t}{d}\right)}{\sqrt{d^2 + 1}} \right), \]

where \( \sech(t) = 2/(e^t + e^{-t}). \)

As one can check, these vectors form an orthonormal basis, \( \langle \gamma^{(i)}, \gamma^{(j)} \rangle = \delta_{ij}, \) and, as \( t \to \pm \infty, \) we have \( \gamma_{1}^{(1)} = (\pm 1, 0, 0) \) and, respectively,
\[ \gamma_{2}^{(2)} = \left( \frac{\cos(t/d) - d \sin(t/d)}{\sqrt{d^2 + 1}}, -\frac{d \cos(t/d) + \sin(t/d)}{\sqrt{d^2 + 1}} \right), \]
\[ \gamma_{3}^{(3)} = \left( 0, \frac{\cos(t/d) + d \sin(t/d)}{\sqrt{d^2 + 1}}, -\frac{d \cos(t/d) - \sin(t/d)}{\sqrt{d^2 + 1}} \right). \]
Since $d > 0$, as $t \to -\infty$ (respectively $t \to +\infty$), in the body frame the vectors $\gamma^{(2)}, \gamma^{(3)}$ perform uniform rotations in the plane $(0, 1, 1)$ in the counterclockwise (respectively clockwise) direction.

This implies the limit spatial motions of the body are rotations about the same axis, with the same angular velocity $1/d$ and in the same direction; however, the rotation angle undergoes the phase shift, which is computed to be \(\arccos \left( \frac{1 - d^2}{1 + d^2} \right)\).

An example of the above solutions is illustrated in figure 4. The corresponding trajectories of the axis $(1, 0, 0)$ of the body in space and of the angular velocity vector in space are shown in figures 5(a) and (b).

For $p = 3$ the corresponding solutions are more complicated. Setting

\[
\alpha = \frac{1}{\sqrt{d^2 + 1(d^2 + 9)}},
\]
we have
\[\gamma^{(1)} = \alpha(\sqrt{d^2 + 1}(9 + d^2 - 4d^2 sech(t)^2) \tanh(t)),\]
\[\gamma^{(2)} = -\frac{2\alpha \sqrt{d^2 + 1}e^t}{(e^{2t} + 1)^3} \left[ \left( d^2(3 - 10e^{2t} + 3e^{4t}) - 9(1 + e^{2t})^2 \right) \cos \left( \frac{3t}{d} \right) + 12d(e^{4t} - 1) \sin \left( \frac{3t}{d} \right) \right],\]
\[\gamma^{(2)} = \frac{4\alpha e^{3t}}{(e^{2t} + 1)^3} \left[ \left( d^2(9 - 7d^2) \cosh(2t) - 9 - 17d^2 \right) \cosh(t) \sin \left( \frac{3t}{d} \right) + \left( 9 - 7d^2 \right) \cosh(2t) + 9 + d^2 \right] \cosh(t) \sin \left( \frac{3t}{d} \right) \right],\]
\[\gamma^{(3)} = \frac{4\alpha e^{3t}}{(e^{2t} + 1)^3} \left[ \left( 9 - 7d^2 \right) \cosh(2t) + 9 + 17d^2 \right) \cosh(t) \cos \left( \frac{3t}{d} \right) \sinh(t) \cos \left( \frac{3t}{d} \right) \right].\]

These vector solutions also form an orthonormal basis and the corresponding spatial motion of the body is similar to that in the previous case \( p = 1 \). An example of these solutions is illustrated in figure 6.

6.3. Extra first integrals for odd \( p \)

As mentioned at the beginning of section 2, if \( P(t) = (P_1, P_2, P_3) \) is a single-valued vector solution of the Poisson equations and, moreover, the components of \( P(t) \) are single-valued functions of \( \omega_1(t), \omega_2(t) \), then system (2.1) and (2.2) admits an additional first integral
\[ F_3 = P_1(\omega_1, \omega_2)\gamma_1 + P_2(\omega_1, \omega_2)\gamma_2 + P_3(\omega_1, \omega_2)\gamma_3. \] (6.6)

Since this system is homogeneous, then \( P_i \) are homogeneous polynomials of a certain degree \( k \).
Now we use the meromorphic solutions $\gamma^{(1)}$ obtained in section 6.1 for odd $p$ to construct the corresponding extra integrals. In the simplest case $p = 1$, comparing the components of $\gamma^{(1)}$ with solutions (4.1) for $\omega_1, \omega_2$ and recalling the definition of $p, d$, we easily obtain

$$
P_1(t) = \frac{\omega_1(t)}{a} = d\omega_1,
$$
$$
P_2(t) = \frac{4a\omega_1(t)}{b^2} = \frac{d}{d^2 + 1}\omega_2,
$$
$$
P_3(t) = \frac{-4\omega_1(t)}{b^2} = -\frac{d^2}{d^2 + 1}\omega_2,
$$

which yields the integral

$$
F_3 = d\omega_1\gamma_1 + \frac{d}{d^2 + 1}\omega_2\gamma_2 - \frac{d^2}{d^2 + 1}\omega_2\gamma_3.
$$

In the case of generic odd $p$ the extra integral can be written by means of the special polynomial solution (5.14) of the generalized hypergeometric equation (5.7) with the parameters defined in (5.9). To do this, it is convenient to define the following homogeneous function:

$$
Q = Q(\omega_1, \omega_2) := F_1^{(p-1)/2}F_1(z),
$$

where

$$
F_1 := (d^2 + 1)\omega_1^2 + \omega_2^2,
$$
$$
z := \frac{\omega_2^2}{F_1},
$$

and $F_1(z)$ is the special polynomial solution (5.14) of (5.7), that is,

$$
F_1(z) = 1 + \sum_{j=1}^{(p-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{d^j(p^2 - 1) \cdots (p^2 - (2j - 1)^2)}{(d^2 + p^2) \cdots ((2j - 1)d^2 + p^2)} z^j.
$$

The above definitions imply that if $p$ is an odd natural number, then $Q_1 \in \mathbb{R}[\omega_1, \omega_2]$ is a homogeneous polynomial of degree $(p - 1)$. Moreover, $Q_1(\omega_1, \omega_2)$ is an even function of $\omega_1$, as well as an even function of $\omega_2$. 

Figure 6. Vector solutions $\gamma^{(1)}$ and $\gamma^{(2)}$ for $p = 3$ and $d = 1/2$. 

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Theorem 6.3. If $p$ is odd a natural number, then there exist homogeneous polynomials $P_1, P_2, P_3 \in \mathbb{R}[\omega_1, \omega_2]$ of the same degree $p$ such that (6.6) is a polynomial first integral of the Poisson equations. Moreover,

$$P_1 = \omega_1 Q$$

and

$$P_3 := \frac{d}{p} \left( \frac{1}{d^2 + 1} \frac{\partial P_1}{\partial \omega_2} - \frac{\partial P_1}{\partial \omega_1} \right),$$

$$P_2 := \frac{d}{p} \left( \frac{1}{d^2 + 1} \frac{\partial P_3}{\partial \omega_2} - \frac{\partial P_3}{\partial \omega_1} \right) + \frac{\omega_2}{\omega_1} P_1.$$  \hfill (6.10)

**Proof.** As follows from the definition of $P_1$, expressions (6.10) for $P_1$ and $P_2$ are polynomials. This is clear for $P_3$. Let us also show that $P_2 \in \mathbb{R}[\omega_1, \omega_2]$. To do this we observe that

$$\frac{\partial P_3}{\partial \omega_1} = -2pd\omega_2 \frac{\partial Q}{\partial \omega_1} - p\omega_2 \frac{\partial^2 Q}{\partial \omega_1^2} + p\omega_1 (1 + d^2) \frac{\partial Q}{\partial \omega_2} + \frac{2 \partial Q}{\partial \omega_1} + \omega_1 \frac{\partial^2 Q}{\partial \omega_1 \partial \omega_2}.$$  \hfill (6.11)

Since $Q$ is an even function of $\omega_1$, we have

$$\frac{\partial Q}{\partial \omega_1} = \omega_1 \tilde{Q}, \quad \text{where } \tilde{Q} \in \mathbb{R}[\omega_1, \omega_2],$$

and thus, from (6.11), we also have

$$\frac{\partial P_3}{\partial \omega_1} = \omega_1 \tilde{P}_3,$$ \quad \text{where } \tilde{P}_3 \in \mathbb{R}[\omega_1, \omega_2].$$

Now from (6.10) it easily follows that $P_2 \in \mathbb{R}[\omega_1, \omega_2]$.

It remains to show that with $P_1$, $P_2$ and $P_3$ given by (6.9) and (6.10), function (6.6) is an integral of the system. Indeed, if it is a first integral, then its time derivative vanishes, i.e.

$$0 = \dot{F}_3 = \frac{d}{p} \sum_{i=1}^{3} \left( \frac{1}{d^2 + 1} \frac{\partial P_i}{\partial \omega_2} - \frac{\partial P_i}{\partial \omega_1} \right) \omega_2 \gamma_i + P_3 \omega_1 \gamma_1 - P_3 \omega_1 \gamma_2 + (P_2 \omega_1 - P_1 \omega_2) \gamma_3.$$

The right-hand side of the above equation is a linear form in $\gamma_1$, hence we have the following system of three partial differential equations:

$$\frac{d}{p} \left( \frac{1}{d^2 + 1} \frac{\partial P_1}{\partial \omega_2} - \frac{\partial P_1}{\partial \omega_1} \right) + P_3 = 0,$$

$$\frac{d}{p} \left( \frac{1}{d^2 + 1} \frac{\partial P_2}{\partial \omega_2} - \frac{\partial P_2}{\partial \omega_1} \right) \omega_2 - P_3 \omega_1 = 0,$$

$$\frac{d}{p} \left( \frac{1}{d^2 + 1} \frac{\partial P_3}{\partial \omega_2} - \frac{\partial P_3}{\partial \omega_1} \right) \omega_2 + P_2 \omega_1 - P_1 \omega_2 = 0.$$ \hfill (6.12)

The form of this system shows that $P_2$ and $P_3$ can be expressed in terms of $P_1$ and its partial derivatives. The explicit forms of these expressions are given by (6.10). Using them we eliminate $P_2$ and $P_3$ in (6.12) and obtain one partial differential equation for $P_1$. Next, taking into account the fact that $P_1$ is assumed to be a homogeneous polynomial, we obtain an ordinary linear equation for $P_1$. It remains to show that one of its solutions is given by formula (6.9).

Introduce a new variable $\omega = \omega_2/\omega_1$ and define

$$P_i(\omega_1, \omega_2) = \omega_i^p P_i(1, \omega_2/\omega_1) =: \omega_i^p p_i(\omega), \quad i = 1, 2, 3.$$  

Then, we have

$$\frac{\partial P_i}{\partial \omega_1} = \omega_i^{p-1} [p p_i - \omega p_i'], \quad \frac{\partial P_i}{\partial \omega_2} = \omega_i^{p-1} p_i', \quad i = 1, 2, 3.$$
where the prime denotes the derivative with respect to $\omega$. This implies that our system of partial differential equations (6.12) on $P_1$ can be written as the system of ordinary differential equations on $p_i$

\[-d(\omega^2 + d^2 + 1)p_1' + d p \omega p_1 + p(d^2 + 1)p_3 = 0,\]
\[d(\omega^2 + d^2 + 1)p_2' - d p \omega^2 p_2 + p(d^2 + 1)p_3 = 0,\]
\[d(\omega^2 + d^2 + 1)p_3' - d p \omega^2 p_3 + p(d^2 + 1)p_3 - p(d^2 + 1)p_2 = 0.\]  

(6.13)

Resolving this system with respect to $p_2$, $p_3$, we find

\[p_3 = \frac{-d(d^2 + 1 + \omega^2)}{p(d^2 + 1)} p_1' - \frac{d}{(d^2 + 1)} \omega p_1,\]
\[p_2 = \frac{d^2(1 + d^2 + \omega^2)^2 \omega - 2d^2(p - 1) \omega^2 (1 + d^2 + \omega^2)}{(1 + d^2)^2 p^2} p_1' + \left(1 - \frac{d^2 p (1 + d^2 - (p - 1) \omega^2)}{(1 + d^2)^2 p^2}\right) \omega p_1.\]  

(6.14)

Substituting the above expressions into the second equation in (6.13), we obtain the following third order linear equation:

\[p_3'' + q_1 p_1' + q_2 p_1' + q_3 p_1 = 0,\]  

(6.15)

\[q_1 = \frac{1}{\omega} - \frac{3(p - 2) \omega}{1 + d^2 + \omega^2};\]
\[q_2 = \frac{d^2 \omega^2 (-1 + d^2)(5p - 4) + (p - 1)(3p - 8) \omega^2) + (1 + d^2)^2 (1 + \omega^2)p^2}{d^2 \omega^2 (1 + d^2 + \omega^2)^2};\]
\[q_3 = -\frac{1}{d^2 \omega (1 + d^2 + \omega^2)^2} \left[ d^2 p((1 + d^2)^2 - 4(1 + d^2)(p - 1) \omega^2 + (3 + (p - 4)p) \omega^4) \right.\]
\[\left. - (1 + d^2)^2 (d^2 - (p - 1)(1 + \omega^2)p^2).\right]\]

Now, making change of independent variable to

\[z = \frac{\omega^2}{\omega^2 + d^2 + 1} \equiv \frac{4}{(e^z + e^{-z})^2},\]

and using

\[\frac{dz}{d\omega} = \frac{2\omega(z - 1)^2}{d^2 + 1}, \quad \frac{d^2 z}{d\omega^2} = -2(z - 1)^2 (4z - 1), \quad \frac{d^3 z}{d\omega^3} = -\frac{24 \omega(z - 1)^3 (2z - 1)}{(d^2 + 1)} ,\]

we transform equation (6.15) into

\[p_3'' + b_1 p_1' + b_2 p_1' + b_3 p_1 = 0, \quad \equiv \frac{d}{dz}\]

\[b_1 = \frac{2}{z} + \frac{2 + 3p}{2(z - 1)},\]
\[b_2 = \frac{d^2 (1 + (p - 6 - 8p)z + (4 + 3p(2 + p) - p^2)z^2) - p^2 (z - 1)}{4d^2 (z - 1)^2 z^2},\]
\[b_3 = \frac{p^2 (-1 + p + z - pz) + d^2 (p^2 (z - 1) - 4p^2 z + p^3 z^2 + p(1 + (2 - p^2 (z - 1)))z))}{8d^2 (z - 1)^2 z^2} .\]
Now under the change of the dependent variable
\[ p_1(z) = (z - 1)^{1/2} v(z), \tag{6.16} \]
the equation for \( v(z) \) becomes exactly the generalized hypergeometric equation (5.7) defining the function \( _3F_2 \) with the parameters (5.9). On the other hand, it is easy to note that, up to a multiplicative constant, the dehomogenized \( P_1 \) given by (6.9) and expressed in terms of the variable \( z \) coincides with (6.16).

\[ \square \]

6.4. Conclusion

As follows from the results of sections 3 and 5, when \( I_{13} = 0 \) and the parameter \( p \) is an even integer, an interesting situation takes place: all the solutions of the Poisson equations are meromorphic, but the equations itself are not solvable in the class of Liouvillian functions. In particular, they do not possess extra meromorphic first integrals. In this case it is natural to expect that the Poisson equations are reducible to one of Painlevé equations.

It also should be emphasized that we managed to reduce the Poisson equations to the higher hypergeometric equation under the restriction \( I_{13} = 0 \). The study of solutions of the third order equation (5.2) in the general case, especially of its monodromy group, is an interesting open problem.

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