RANDOM MORSE FUNCTIONS AND SPECTRAL GEOMETRY

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ABSTRACT. We study random Morse functions on a Riemann manifold \((M^m, g)\) defined as a random Gaussian weighted superpositions of eigenfunctions of the Laplacian of the metric \(g\). The randomness is determined by a fixed Schwartz function \(w\) and a small parameter \(\varepsilon > 0\). We first prove that as \(\varepsilon \to 0\) the expected distribution of critical values of this random function approaches a universal measure on \(\mathbb{R}\), independent of \(g\), that can be explicitly described in terms the expected distribution of eigenvalues of the Gaussian Wigner ensemble of random \((m + 1) \times (m + 1)\) symmetric matrices. In contrast, we prove that the metric \(g\) and its curvature are determined by the statistics of the Hessians of the random function for small \(\varepsilon\).

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1. Overview

1.1. The setup. Suppose that \((M, g)\) is a smooth, compact, connected Riemann manifold of dimension \(m > 1\). We denote by \(|dV_g|\) the volume density on \(M\) induced by \(g\). We assume that the metric is normalized so that

\[
\text{vol}_g(M) = 1. \tag{*}
\]

For any \(u, v \in C^\infty(M)\) we denote by \((u, v)_g\) their \(L^2\) inner product defined by the metric \(g\). The \(L^2\)-norm of a smooth function \(u\) is denoted by \(\|u\|\).

Let \(\Delta_g : C^\infty(M) \to C^\infty(M)\) denote the scalar Laplacian defined by the metric \(g\). Fix an orthonormal Hilbert basis \((\Psi_k)_{k \geq 0}\) of \(L^2(M)\) consisting of eigenfunctions of \(\Delta_g\),

\[
\Delta_g \Psi_k = \lambda_k \Psi_k, \quad \|\Psi_k\| = 1, \quad k_0 < k_1 \Rightarrow \lambda_{k_0} \leq \lambda_{k_1}.
\]

Fix an even measurable functions function \(w : \mathbb{R} \to [0, \infty)\) such that

\[
\lim_{t \to \infty} t^n w(t) = 0, \quad \forall n \in \mathbb{Z}_{>0}.
\]

For \(\varepsilon > 0\) and \(k \geq 0\) we set

\[
w_\varepsilon(t) := w(\varepsilon t), \quad \forall t \in \mathbb{R}, \quad v_\varepsilon^k = w_\varepsilon \left(\sqrt{\lambda_k}\right). \tag{1.1}
\]

Consider random functions on \(M\) of the form

\[
u^\varepsilon = \sum_{k \geq 0} X_k \sqrt{v_k^\varepsilon} \Psi_k, \tag{1.2}
\]

where the coefficients \(X_k\) are independent standard Gaussian random variables. Note that

\[
\Delta^N u^\varepsilon = \sum_{k \geq 0} \lambda_k^N u_k \Psi_k, \quad \forall N > 0.
\]

The fast decay of \(w\), the Weyl asymptotic formula, [10, VI.4], coupled with the Borel-Cantelli lemma imply that for any \(N > 0\) the function \(\Delta^N u^\varepsilon\) is almost surely (a.s.) in \(L^2\). In particular, this shows that \(u^\varepsilon\) is a.s. smooth.

The covariance kernel of the Gaussian random function \(u^\varepsilon\) is given by the function

\[
\mathcal{E}^\varepsilon : M \times M \to \mathbb{R}, \quad \mathcal{E}^\varepsilon(p, q) = E\left(u^\varepsilon(p) u^\varepsilon(q)\right) = \sum_{k \geq 0} w_\varepsilon \left(\sqrt{\lambda_k}\right) \Psi_k(p) \Psi_k(q).
\]

The eigenfunctions \(\Psi_k\) satisfy the known pointwise estimates (see [22, Thm. 17.5.3] or [28, Thm 1.6.1]),

\[
\|\Psi_k\|_{C^{\nu}(M)} = O\left(\lambda_k^{\frac{m + \nu}{2}}\right) \quad \text{as} \quad k \to \infty, \quad \forall \nu \geq 0.
\]

Since \(w_\varepsilon\) is rapidly decresing the above estimates imply that \(\mathcal{E}^\varepsilon\) is a smooth function. More precisely, \(\mathcal{E}^\varepsilon\) is the Schwartz kernel of the smoothing operator

\[
w(\varepsilon \sqrt{\Delta}) : C^\infty(M) \to C^\infty(M).
\]

Let us observe that if \(w(0) = 1\), then as \(\varepsilon \searrow 0\) the function \(w_\varepsilon\) converges uniformly on compacts to the constant function \(w_0(t) \equiv 1\) and \(w_\varepsilon(\sqrt{\Delta})\) converges weakly to the identity operator. The Schwartz kernel of this limiting operator is the \(\delta\)-function on \(M \times M\) supported along the diagonal. It defines a generalized random function in the sense of [16] usually known as white noise. For this reason, we will refer to the \(\varepsilon \to 0\) limits as white noise limits.
In the papers [26, 27] we investigated the distribution of critical points and critical values of the random function $u_\varepsilon$ in special case

$$w(t) = I_{[-1,1]} := \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$

In this paper we consider we investigate the same problem assuming that $w$ is a Schwartz function. We will discuss later the similarities and the differences between these two situations.

The asymptotic estimates in Proposition 2.2 show that the random field $du^\varepsilon$ satisfies the hypotheses of [1, Cor. 11.2.2] for $\varepsilon \ll 1$. Invoking [1, Lemma 11.2.11] we obtain the following technical result.

**Proposition 1.1.** The random function $u^\varepsilon$ is almost surely Morse if $\varepsilon \ll 1$. \qed

For any $u \in C^1(M)$ we denote by $\text{Cr}(u) \subset M$ the set of critical points of $u$ and by $D(u)$ the set of critical values of $u$. To a Morse function $u$ on $M$ we associate a Borel measure $\mu_u$ on $M$ and a Borel measure $\sigma_u$ on $\mathbb{R}$ defined by the equalities

$$\mu_u := \sum_{p \in \text{Cr}(u)} \delta_p, \quad \sigma_u := u_\varepsilon(\mu_u) = \sum_{t \in \mathbb{R}} |u^{-1}(t) \cap \text{Cr}(u)| \delta_t.$$ 

Observe that

$$\text{supp } \mu_u = \text{Cr}(u), \quad \text{supp } \sigma_u = D(u).$$

When $u$ is not Morse, we set

$$\mu_u := |dV_g|, \quad \sigma_u = \delta_0$$

the Dirac measure on $\mathbb{R}$ concentrated at the origin.

Observe that for any Morse function $u$ and any Borel subset $B \subset \mathbb{R}$ the number $\sigma_u(B)$ is equal to the number of critical values of $u$ in $B$ counted with multiplicity. We will refer to $\sigma_u$ as the variational complexity of $u$.

To the random function $u^\varepsilon$ we associate the random (or empirical) measure $\sigma_{u_\varepsilon}$. Its expectation

$$\sigma^\varepsilon = E(\sigma_{u^\varepsilon})$$

is the measure on $\mathbb{R}$ uniquely determined by the equality

$$\int_{\mathbb{R}} f(t) \sigma^\varepsilon(dt) = E \left( \int_{\mathbb{R}} f(t) d\sigma_{u^\varepsilon}(dt) \right),$$

for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$. In \S 2.1 we show that the measure $\sigma^\varepsilon$ is well defined for $\varepsilon \ll 1$. We will refer to it as the expected variational complexity of the random function $u_\varepsilon$.

(i) Describe the white noise limit of $\sigma^\varepsilon$.

(ii) Recover the geometry of $(M, g)$ from white noise statistics of the random function $u_\varepsilon$.

**Remark 1.2.** Before we state precisely our main results we believe that it is instructive to discuss some elementary topologic and geometric features of the white noise behavior of $u_\varepsilon$. For simplicity we assume that $w(0) = 1$ so that $u^\varepsilon$ does converge to the white noise on $M$.

It is not hard to prove that for any given Morse function $f : M \to \mathbb{R}$ and any $h > 0$, the probability that $\|f - u^\varepsilon\|_{C^3} < h$ is positive for $\varepsilon$ sufficiently small. If $f$ happens to be a stable Morse function, i.e., it has at most one critical point per level set, then for $h$ sufficiently small, any $C^3$-function $g : M \to \mathbb{R}$ satisfying $\|f - g\|_{C^3} < h$ is topologically equivalent to $f$. Thus as $\varepsilon \to 0$ the random function $u_\varepsilon$ samples all the topological types of Morse functions.

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1The set $D(u)$ is sometime referred to as the discriminant set of $u$. 
The rescaling $w \to w_\epsilon$ can be alternatively implemented as follows. Consider the rescaled metric $g_\epsilon := \epsilon^{-2}g$. As $\epsilon \to 0$ the metric $g_\epsilon$ becomes flatter and flatter. The Laplacian of $g_\epsilon$ is $\Delta_{g_\epsilon} = \epsilon^2 \Delta_g$. Its eigenvalues are $\lambda_k^\epsilon = \epsilon^2 \lambda_k$ and the collection $\Psi_k^\epsilon = \epsilon \Psi_k$ is an orthonormal eigen-basis of $L^2(M, |dV_{g_\epsilon}|)$. For any $\epsilon > 0$ we define the random function

$$v^\epsilon = \sum_{k \geq 0} X_k w \left( \sqrt{\lambda_k^\epsilon} \right) \Psi_k^\epsilon = \sum_{k \geq 0} X_k \sqrt{\epsilon} \Psi_k^\epsilon,$$

where the coefficients $X_k$ are independent standard Gaussian random variables. Observe that $v^\epsilon = \epsilon^{\frac{m}{2}} u^\epsilon$. This shows that the expected distribution $\sigma^\epsilon(v)$ of critical values of $v^\epsilon$ is a rescaling of $\sigma^\epsilon$. $\square$

1.2. Statements of the main results. Observe that if $u : M \to \mathbb{R}$ is a fixed Morse function and $c$ is a constant, then

$$Cr(c + u) = Cr(u), \quad \mu_{c+u} = \mu_u,$$

but

$$D(u + c) = c + D(u), \quad \sigma_{u+c} = \delta_c \ast \sigma_u,$$

where $\ast$ denotes the convolution of two finite measures on $\mathbb{R}$. More generally, if $X$ is a scalar random variable with probability distribution $\nu_X$, then the expected variational complexity of the random function $X + u$ is the measure $E(\sigma_{X+u}) = \nu_X \ast \sigma_u$. If $u$ itself is a random function, and $X$ is independent of $u$, then the above equality can be rephrased as

$$E(\sigma_{X+u}) = \nu_X \ast E(\sigma_u).$$

In particular, if the distribution $\nu_X$ is a Gaussian, then the measure $E(\sigma_u)$ is uniquely determined by the measure $E(\sigma_{X+u})$ since the convolution with a Gaussian is an injective operation. It turns out that it is easier to understand the statistics of the variational complexity of a perturbation of $u_\epsilon$ with an independent Gaussian variable of cleverly chosen variance.

To explain this perturbation we need to introduce several quantities that will play a crucial role throughout this paper. We define

$$s_m := \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} w(|x|)dx, \quad d_m := \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} x_1^2 w(|x|)dx,$$

$$h_m := \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} x_1^2 x_2^2 w(|x|)dx. \quad \text{ (1.3)}$$

The statistical relevance of these quantities is explained in Proposition 2.2. If we set

$$I_k(w) := \int_0^\infty w(r)r^k dr, \quad \text{ (1.4)}$$

then we deduce from [25, Lemma 9.3.10]

$$(2\pi)^m s_m = \left( \int_{|x|=1} dA(x) \right) I_{m-1}(w) = \frac{2\pi^\frac{m}{2}}{\Gamma(\frac{m}{2})} I_{m-1}(w),$$

$$(2\pi)^m d_m = \left( \int_{|x|=1} x_1^2 dA(x) \right) I_{m+1}(w) = \frac{\pi^\frac{m}{2}}{\Gamma(1 + \frac{m}{2})} I_{m+1}(w) = \frac{2\pi^\frac{m}{2}}{m\Gamma(\frac{m}{2})} I_{m+1}(w),$$

$$(2\pi)^m h_m = \left( \int_{|x|=1} x_1^2 x_2^2 dA(x) \right) I_{m+1}(w) = \frac{\pi^\frac{m}{2}}{2\Gamma(2 + \frac{m}{2})} I_{m+1}(w) = \frac{2\pi^\frac{m}{2}}{m(m+2)\Gamma(\frac{m}{2})} I_{m+1}(w).$$

We set

$$q_m := \frac{s_m h_m}{d_m^2} = \frac{m}{m + 2} \frac{I_{m-1}(w)I_{m+3}(w)}{I_{m+1}(w)^2}. \quad \text{ (1.5)}$$
The Cauchy inequality implies that $I_{m+1}(w)^2 \leq I_{m-1}(w)I_{m+3}(w)$ so that

$$q_m \geq \frac{m}{m+2}.$$  (1.6)

The sequence $(q_m)_{m \geq 1}$ can be interpreted as a measure of the tail of $w$, the heavier the tail, the faster the growth of $q_m$ as $m \to \infty$; see Section 3 for more details. We set

$$r_n := \max(1, q_n),$$

and define $\omega_m \geq 0$ via the equality

$$r_n = \frac{(s_m + \omega_m)h_m}{d_m^2}.$$  (1.7)

Set $\tilde{s}_m := s_m + \omega_m$ so that

$$r_m = \frac{\tilde{s}_m h_m}{d_m^2}.$$  (1.8)

Observe that

$$\omega_m = 0 \iff q_m = r_m \geq 1 \iff \tilde{s}_m = s_m,$$  (1.9)

while the inequality (1.6) implies that

$$\lim_{m \to \infty} \frac{\omega_m}{s_m} = 0, \quad \lim_{m \to \infty} \frac{r_m}{q_m} = 1.$$  (1.10)

Choose a scalar Gaussian random variable $X_{\omega(\varepsilon)}$ with mean 0 and variance $\omega(\varepsilon) := \omega_m \varepsilon^{-m}$ independent of $u_{\varepsilon}$ and form the new random function

$$\tilde{u}_\varepsilon := X_{\omega(\varepsilon)} + u_{\varepsilon}.$$ We denote by $\tilde{\sigma}^\varepsilon$ the expected variational complexity of $\tilde{u}_\varepsilon$. We have the equality

$$\tilde{\sigma}^\varepsilon = \gamma_{\omega(\varepsilon)} \ast \sigma^\varepsilon, \quad \omega(\varepsilon) := \omega_m \varepsilon^{-m},$$  (1.11)

Note that

$$N^\varepsilon = \int_\mathbb{R} \tilde{\sigma}^\varepsilon(dt) = \int_\mathbb{R} \sigma^\varepsilon(dt)$$

is the expected number of critical points of the random function $u^\varepsilon$.

To formulate our main results we need to briefly recall some terminology from random matrix theory.

For $\varepsilon \in (0, \infty)$ and $N$ a positive integer we denote by $\text{GOE}^\varepsilon_N$ the space $\text{Sym}_N$ of real, symmetric $N \times N$ matrices $A$ equipped with a Gaussian measure such that the entries $a_{ij}$ are independent, zero-mean, normal random variables with variances

$$\text{var}(a_{ii}) = 2\varepsilon, \quad \text{var}(a_{ij}) = \varepsilon, \quad \forall 1 \leq i < j \leq N.$$ Let $\rho_{N,\varepsilon} : \mathbb{R} \to \mathbb{R}$ be the normalized correlation function of $\text{GOE}^\varepsilon_N$. It is uniquely determined by the equality

$$\int_\mathbb{R} f(\lambda) \rho_{N,\varepsilon}(\lambda) d\lambda = \frac{1}{N} \mathbb{E}_{\text{GOE}^\varepsilon_N} (\text{tr} f(A)), $$

for any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. The function $\rho_{N,\varepsilon}(\lambda)$ also has a probabilistic interpretation: for any Borel set $B \subset \mathbb{R}$ the expected number of eigenvalues in $B$ of a random $A \in \text{GOE}^\varepsilon_N$ is equal to

$$N \int_B \rho_{N,\varepsilon}(\lambda) d\lambda.$$
For any \( t > 0 \) we denote by \( \mathcal{R}_t : \mathbb{R} \rightarrow \mathbb{R} \) the rescaling map \( \mathbb{R} \ni x \mapsto tx \in \mathbb{R} \). If \( \mu \) is a Borel measure on \( \mathbb{R} \) we denote by \( (\mathcal{R}_t)_* \mu \) its pushforward via the rescaling map \( \mathcal{R}_t \). The celebrated Wigner semicircle theorem, \([3, 24]\), states that as \( N \rightarrow \infty \) the rescaled probability measures

\[
(\mathcal{R}_{\frac{1}{\sqrt{N}}})_* (\rho_{N,v}(\lambda) d\lambda)
\]

converge weakly to the semicircle measure given by the density

\[
\rho_{\infty,v}(\lambda) := \frac{1}{2\pi v} \times \begin{cases} 
\sqrt{4v - \lambda^2}, & |\lambda| \leq \sqrt{4v} \\
0, & |\lambda| > \sqrt{4v}.
\end{cases}
\]

We can now state the main results of this paper.

**Theorem 1.3.** For \( v > 0 \) and \( N \in \mathbb{Z}_{>0} \) we set

\[
\theta_{N,v}^{\pm}(x) := \rho_{N,v}(x)e^{\pm \frac{x^2}{4v}}.
\]

(a) There exists a constant \( C = C_m(w) \) that depends only on the dimension \( m \) and the weight \( w \) such that

\[
N^\varepsilon \sim C_m(w) \varepsilon^{-m} \left( 1 + O(\varepsilon) \right) \text{ as } \varepsilon \rightarrow 0. \quad (1.12)
\]

More precisely

\[
C_m(w) = 2^{m+4} r_m^{\frac{1}{2}} \left( \frac{h_m}{2 \pi d_m} \right)^{\frac{m}{2}} \Gamma \left( \frac{m+3}{2} \right) \int_{\mathbb{R}} (\gamma_{r_m-1} * \theta_{m+1,r_m}^{+})(y) \gamma_1(y) dy. \quad (1.13)
\]

(b) As \( \varepsilon \downarrow 0 \) the rescaled probability measures

\[
\frac{1}{N^\varepsilon} \left( \mathcal{R}_{\frac{1}{\sqrt{N^\varepsilon}}} \right)_* \sigma^{\varepsilon}
\]

converge weakly to a probability measure \( \sigma_m \) on \( \mathbb{R} \) uniquely determined by the proportionalties

\[
\sigma_m \propto \left( \gamma_{r_m-1} * \theta_{m+1,r_m}^{+} \right)(y) \gamma_1(y) dy \quad (1.14a)
\]

\[
\propto \theta_{m+1,r_m}^{-} * \gamma_{r_m-1}^{-}(y) dy. \quad (1.14b)
\]

One immediate consequence of Theorem 1.3 is the following universality result.

**Corollary 1.4 (Universality).** As \( \varepsilon \rightarrow 0 \) the rescaled probability measures

\[
\frac{1}{N^\varepsilon} \left( \mathcal{R}_{\frac{1}{\sqrt{N^\varepsilon}}} \right)_* \sigma^{\varepsilon}
\]

converge weakly to a probability measure \( \sigma_m \) uniquely determined by the convolution equation

\[
\gamma_{r_m} \ast \sigma_m = \sigma_m. \]

Wigner’s semicircle theorem \([3, \text{Thm. 2.1.1}]\) allows us extract a bit more about the measure \( \sigma_m \) for \( m \) large, provided that the behavior of \( w \) at \( \infty \) is not too chaotic.

**Theorem 1.5 (Central limit theorem).** Suppose that the weight \( w \) is regular, i.e., the sequence \( r_m \) has a limit \( r \in [1, \infty) \) as \( m \rightarrow \infty \). Then

\[
\lim_{m \rightarrow \infty} \sigma_m = \gamma_{r+1}. \]
The above regularity assumption on $w$ is a constraint on the behavior of its tail. In Section 3 we describe many classes of regular weights.

**Corollary 1.6.** As $m \to \infty$ we have

$$C_m(w) \sim \frac{8}{2\sqrt{\pi m}} m \left( \frac{m + 3}{2} \right) \left( \frac{h_m}{\pi d_m} \right)^m \left( \frac{2I_{m+3}(w)}{\pi(m + 2)I_{m+1}(w)} \right)^{\frac{m}{2}}.$$  

(1.15)

Following [1, §12.2] we define symmetric $(0, 2)$-tensor $h^\varepsilon$ on $M$ given by

$$h^\varepsilon(X, Y) = \frac{e^{m+2}}{d_m} E(X u^\varepsilon(p), Y u^\varepsilon(p)), \quad \forall p \in M, \; X, Y \in \text{Vect}(M),$$  

(1.16)

where $X u$ denotes the derivative of the smooth function $u$ along the vector field $X$.

**Theorem 1.7 (Probabilistic reconstruction of the geometry).** (a) For $\varepsilon > 0$ sufficiently small the tensor $h^\varepsilon$ defines a Riemann metric on $M$.

(b) For any vector fields $X, Y$ on $M$ the function $h^\varepsilon(X, Y)$ converges uniformly to $g(X, Y)$ as $\varepsilon \searrow 0$.

(c) The sectional curvatures on $h^\varepsilon$ converge to the corresponding sectional curvatures of $g$ as $\varepsilon \searrow 0$.

The $C^0$-convergence of $h^\varepsilon$ towards the original metric was observed earlier by S. Zelditch [34]. The main novelty of the above theorem is part (c) which, as detailed below, implies the $C^\infty$ convergence of $h^\varepsilon$ to $g$. However, the qualitative jump from $C^0$ to $C^\infty$-converges requires requires novel input.

The construction of the metrics $h^\varepsilon$ generalizes the construction in [6]. Note that for any $\varepsilon > 0$ we have a smooth map $\Xi_{\varepsilon} : M \to L^2(M, g)$

$$M \ni p \mapsto \Xi_{\varepsilon}(p) := \left( \frac{e^{m+2}}{d_m} \right)^{\frac{1}{2}} \sum_{k \geq 0} w_k \sqrt{\lambda_k} \Psi_k(p) \Psi_k \in L^2(M, g).$$  

(1.17)

Then $h^\varepsilon$ is the pullback by $\Xi_{\varepsilon}$ of the Euclidean metric on $L^2(M, g)$. Let us point out that [6, Thm.5] is a special case of Theorem 1.7 corresponding to the weight $w(t) = e^{-t^2}$.

Theorem 1.7 coupled with the results in [30] imply that the metrics $h^\varepsilon$ converge $C^{1,\alpha}$ to $g$ as $\varepsilon \searrow 0$. The convergence of sectional curvatures coupled with the technique of harmonic coordinates in [2, 30] can be used to bootstrap this convergence to a $C^\infty$ convergence.

**Remark 1.8.** Suppose that $w$ has compact support, say $\text{supp } w \subset [-1, 1]$. In this case the map $\Xi_{\varepsilon}$ is actually a map to the finite dimensional Euclidean space

$$U_{\varepsilon} := \text{span} \{ \Psi_k; \; \lambda_k \leq \varepsilon^{-2} \} \subset L^2(M, g).$$

Theorem 1.7 implies that for $\varepsilon > 0$ sufficiently small the map $\Xi_{\varepsilon}$ is a near-isometric embedding of $M$ in a finite dimensional space. It is conceivable that this near-isometric embedding could be deformed to an actual isometry by using the strategy of X. Wang and K. Zhu [33].

We should add a few words about the nontrivial analytic result hiding behind Theorem 1.7. Fix a point $p \in M$ and normal coordinates $(x^i)$ at $p$. The techniques pioneered by L. Hörmander [20] show that as $\varepsilon \searrow 0$ we have the 1-term asymptotic expansions

$$E\left( \partial_{x_i x_j}^2 u^\varepsilon(p) \cdot \partial_{x_i x_j}^2 u^\varepsilon(p) \right) = h_{ij} \varepsilon^{-(m+4)} \left( 1 + O(\varepsilon^2) \right),$$  

(1.18a)
These and several other similar 1-term asymptotic expansions involving the Schwartz kernel of \( w_\varepsilon(\sqrt{\Delta}) \) (see Proposition 2.2) are responsible for Theorem 1.3. All these 1-term expansions are independent of the background metric \( g \). Note that (1.18a) and (1.18b) imply the estimate

\[
E\left( \partial^2_{x^ix^j} u^\varepsilon(p) \cdot \partial^2_{x^ix^j} u^\varepsilon(p) \right) = h_m \varepsilon^{-(m+4)} \left( 1 + O(\varepsilon^2) \right).
\] (1.18b)

Theorem 1.7 is equivalent with the following sharp estimate

\[
E\left( \partial^2_{x^ix^j} u^\varepsilon(p) \cdot \partial^2_{x^ix^j} u^\varepsilon(p) \right) - E\left( \partial^2_{x^ix^j} u^\varepsilon(p) \cdot \partial^2_{x^ix^j} u^\varepsilon(p) \right) \sim d_m K^g_{ij}(p) \varepsilon^{-(m+2)},
\]

where \( K^g_{ij}(p) \) denotes the sectional curvature of \( g \) at \( p \) along the 2-plane spanned by \( \partial_{x^i}, \partial_{x^j} \) which is an obvious refinement of (1.19). In fact the proof of Theorem 1.7 is based on refinements of the 1-term expansions (1.18a) and (1.18b) to explicit 2-term expansions that we prove in Appendix B.

The convergence of the metrics \( h^\varepsilon \) leads to a cute probabilistic proof of the Gauss-Bonnet theorem for the original metric \( g \) (and thus for any metric on \( M \)). Here is the simple principle behind this proof.

Assume for simplicity that \( M \) is oriented and \( m = \dim M \) is even. To a Morse function \( f \) we associate the signed measure

\[
\nu_f = \sum_{df(p)=0} (-1)^{\text{ind}(f,p)} \delta_p,
\]

where \( \text{ind}(f,p) \) denotes the Morse index of the critical point of the Morse function \( f \). The Poincaré-Hopf theorem implies that

\[
\int_M \nu_f = \chi(M).
\] (1.20)

We can also think of \( \nu_f \) as a degree 0-current. The random function \( u^\varepsilon \) then determines a random 0-current \( \nu_{u^\varepsilon} \). It turns out (see Section 4) that the expectation of this current is a current represented by a rather canonical top degree form. More precisely, we prove that,

\[
E(\nu_{u^\varepsilon}) = e_{h^\varepsilon}(M),
\] (1.21)

where \( e_{h^\varepsilon}(M) \) is the Euler form defined by the metric \( h^\varepsilon \) which appears in the Gauss-Bonnet theorem. Using (1.20) we conclude that

\[
\int_M e_{h^\varepsilon}(M) = \int_M E(\nu_{u^\varepsilon}) = E\left( \int_M \nu_{u^\varepsilon} \right) = \chi(M),
\]

and as a bonus we the Gauss-Bonnet theorem for the metric \( h^\varepsilon \). Letting \( \varepsilon \to 0 \) we obtain the Gauss-Bonnet theorem for \( g \) since \( h^\varepsilon \to g \) and \( e_{h^\varepsilon}(M) \to e_g(M) \). In particular, this shows that \( E(\nu_{u^\varepsilon}) \) converges in the sense of currents to \( e_g(M) \), the Euler form determined by the metric \( g \).

1.3. Some perspective. In [27] we proved the counterparts of Theorem 1.3, Corollary 1.4 and Theorem 1.5 in the case of the singular weight \( w = I_{[-1,1]} \). In this case the random function \( u^\varepsilon \) could be loosely interpreted as a random polynomial of large degree because since this is the case when \( (M,g) \) is the round sphere.

The fact that the results in the singular case \( w = I_{[-1,1]} \) are very similar to the results in the smooth case when \( w \) is Schwartz function could be erroneously interpreted as an indication that there are no qualitative differences between these two situations. This is not the case.

There is one subtle and meaningful qualitative difference buried in the proofs of Theorem 1.3 and Theorem 1.5. It has to do with the size of the tail of \( w \) as encoded by the quality \( q_m = q_m(w) \) defined in (1.5). Loosely speaking, a large \( q_m \) is an indication of a heavy tail.
The proof of Theorem 1.3 requires different arguments depending on whether \( q_m > 1 \) or \( q_m \leq 1 \); see Case 1 and Case 2 in the proof of Theorem 1.3. Since \( q_m(w) \geq \frac{m}{m+2} \) for any \( w \) we see that for \( m \) large the situation \( q_m < 1 \) is rather atypical. The case of the singular weight \( w = I_{[0,1]} \) is atypical because \( q_m(w) = \frac{m+2}{m+4} < 1 \).

The size of the tail plays an even more fundamental role in the proof of the Central Limit Theorem 1.5. The large \( m \)-limit of \( \sigma_m \) exists because of two facts: Wigner’s semicircle theorem and the fact limit \( \lim_m q_m = r = r(w) \) exists. However, the proof depends heavily on the size of the tail and there are two dramatically different cases, \( r < \infty \) and \( r = \infty \). The fact that the central limit theorem has a similar statement in both cases is a bit surprising because different forces are at play in these two cases.

In Section 3 we show that the two behaviors, \( r < \infty \) and \( r = \infty \) are not just theoretically possible, they can actually happen for various choices of \( w \). For example, if

\[
w(t) \sim \frac{1}{t \log \log(t)} \quad \text{as} \quad t \to \infty,
\]

then \( r = \infty \). If \( w(t) \sim e^{-c(\log t)^2} \) as \( t \to \infty \) for some \( c > 0 \), then \( r = e^{8/c} \), while if \( w(t) = e^{-t^2} \), then \( r = 1 \).

The quantity \( r(w) \) also affects the size of the constant \( C_m(w) \) in (1.12) which states that the expected number of critical points of \( u^\varepsilon \) is asymptotic to \( C_m(w)\varepsilon^{-m} \) as \( \varepsilon \to 0 \). For example if

\[
w(t) \sim \frac{1}{t \log \log(t)} \quad \text{as} \quad t \to \infty,
\]

then

\[
\log C_m(w) \sim \frac{m}{2} e^{m+2}(e^2 - 1) \quad \text{as} \quad m \to \infty.
\]

If \( w(t) \sim e^{-c(\log t)^2} \) as \( t \to \infty \), then

\[
\log C_m(w) \sim \frac{2}{C} m^2 \quad \text{as} \quad m \to \infty.
\]

If \( w(t) = e^{-t^2} \), then

\[
\log C_m(w) \sim \frac{1}{2} m \log m \quad \text{as} \quad m \to \infty.
\]

It is well known that if \( w \) is a Schwartz function, then Schwartz kernel of \( w_\varepsilon(\sqrt{\Delta}) \) has a complete asymptotic expansion as \( \varepsilon \downarrow 0 \); see e.g. [31, Chap. XIII]. Very little was known about the precise nature of the expansion. In Theorem B.5 and we obtain an explicit description of the second term of this expansion in terms of geometric invariants of the Riemann manifold \((M, g)\).

Theorem B.5 is a new result and we have delegated it to an appendix not to diminish its importance, but to help the reader separate the two conceptually different facts responsible for Theorem 1.7.

The first fact is purely probabilistic and states that the Riemann tensor of the approximate metric \( h_\varepsilon \) is completely determined by the statistics of the Hessians of \( u^\varepsilon \); see (2.23) and (4.4).

The second fact is purely analytic: the asymptotics of the statistics of the Hessians as \( \varepsilon \to 0 \) are determined by the 2-term expansion proved in Theorem B.5.

1.4. The organization of the paper. The remainder of the paper is organized as follows. Section 2 contains the proofs of the main results. In Section 3 we describe many classes of regular weights \( w \). In particular, these examples show that the limit \( r = \lim_{m \to \infty} r_m \) that appears in the statement of Theorem 1.5 can have any value in \([1, \infty]\). Section 4 contains the details of the probabilistic proof of the Gauss-Bonnet theorem outlined above.
To smooth the flow of the presentation we gathered in Appendices various technical results used in
the proofs of the mains results. In Appendix A we describe the jets of order \( \leq 4 \) along the diagonal of
the square of the distance function \( \text{dist}_g : M \times M \to \mathbb{R} \) which are needed in the two-step asymptotics
of the correlation kernel. This feels like a classical problem, but since precise references are hard to
find we decided to include a complete proof. Our approach, based on the Hamilton-Jacobi equation
satisfied by the distance function is similar to the one sketched in [12, p.281-282].

In Appendix B we describe small \( \varepsilon \) asymptotics of the Schwartz kernel of \( w(\varepsilon \sqrt{\Delta}) \) using a strategy
pioneered L. Hörmander [20] based on a good understanding of the short time asymptotics for the
wave kernel. For the applications in this paper we need explicit, two-term asymptotics. The central
result in this appendix is Theorem B.5 which, to the best of our knowledge, was never published
before. It essentially states that the Riemann curvature tensor can be recovered from the second order
terms of the \( \varepsilon \to 0 \) asymptotics of the fourth order jets along the diagonal of the Schwartz kernel of
\( w(\varepsilon \sqrt{\Delta}) \).

In Appendix C we describe a few facts about Gaussian measures in a coordinate free form suitable
for our geometric purposes. Finally, in Appendix D we have collected some facts about a family of
Gaussian random symmetric matrices that appear in our investigation and are less familiar to differential
geometers.

1.5. Notations.

(i) For any set \( S \) we denote by \( |S| \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) its cardinality. For any subset \( A \) of a set \( S \) we
denote by \( I_A \) its characteristic function

\[
I_A : S \to \{0, 1\}, \quad I_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \in S \setminus A \end{cases}.
\]

(ii) For any point \( x \) in a smooth manifold \( X \) we denote by \( \delta_x \) the Dirac measure on \( X \) concen-
trated at \( x \).

(iii) For any smooth manifold \( M \) we denote by \( \text{Vect}(M) \) the vector space of smooth vector fields
on \( M \).

(iv) For any random variable \( \xi \) we denote by \( E(\xi) \) and respectively \( \text{var}(\xi) \) its expectation and
respectively its variance.

(v) For any finite dimensional real vector space \( V \) we denote by \( V^\vee \) its dual, \( V^\vee := \text{Hom}(V, \mathbb{R}) \).

(vi) For any Euclidean space \( V \) we denote by \( \text{Sym}(V) \) the space of symmetric linear operators
\( V \to V \). When \( V \) is the Euclidean space \( \mathbb{R}^m \) we set \( \text{Sym}_m := \text{Sym}(\mathbb{R}^m) \). We denote by
\( \mathbb{1}_m \) the identity map \( \mathbb{R}^m \to \mathbb{R}^m \).

(vii) We denote by \( S(\mathbb{R}^m) \) the space of Schwartz functions on \( \mathbb{R}^m \).

(viii) For \( \nu > 0 \) we denote by \( \gamma_\nu \) the centered Gaussian measure on \( \mathbb{R} \) with variance \( \nu \),

\[
\gamma_\nu(x)dx = \frac{1}{\sqrt{2\pi}\nu}e^{-\frac{x^2}{2\nu}}dx.
\]

Since \( \lim_{\nu \to 0} \gamma_\nu = \delta_0 \), we set \( \gamma_0 := \delta_0 \). For a real valued random variable \( X \) we write
\( X \in \mathcal{N}(0, \nu) \) if the probability distribution of \( X \) is \( \gamma_\nu \).

(ix) If \( \mu \) and \( \nu \) are two finite measures on a common space \( X \), then the notation \( \mu \propto \nu \) means that

\[
\frac{1}{\mu(X)} = \frac{1}{\nu(X)}.
\]
2. PROOFS

2.1. A Kac-Rice type formula. The key result behind Theorem 1.3 is a Kac-Rice type result which we intend to discuss in some detail in this section. This result gives an explicit, yet quite complicated description of the measure $\tilde{\sigma}^\varepsilon$. More precisely, for any Borel subset $B \subset \mathbb{R}$, the Kac-Rice formula provides an integral representation of $\tilde{\sigma}^\varepsilon(B)$ of the form

$$\tilde{\sigma}^\varepsilon(B) = \int_M f_{\varepsilon,B}(p) |dV_g(p)|,$$

for some integrable function $f_{\varepsilon,B} : M \to \mathbb{R}$. The core of the Kac-Rice formula is an explicit probabilistic description of the density $f_{\varepsilon,B}$.

Fix a point $p \in M$. This determines three Gaussian random variables

$$\tilde{u}_\varepsilon(p) \in \mathbb{R}, \quad d\tilde{u}_\varepsilon(p) \in T_p^*M, \quad \text{Hess}_p(\tilde{u}_\varepsilon) \in \text{Sym}(T_pM),$$

where $\text{Hess}_p(\tilde{u}_\varepsilon) : T_pM \times T_pM \to \mathbb{R}$ is the Hessian of $u_\omega$ at $p$ defined in terms of the Levi-Civita connection of $g$ and then identified with a symmetric endomorphism of $T_pM$ using again the metric $g$. More concretely, if $(x^i)_{1 \leq i \leq m}$ are $g$-normal coordinates at $p$, then

$$\text{Hess}_p(\tilde{u}_\varepsilon) \partial_{x^i} = \sum_{i=1}^m \partial_{x^i}^2 \tilde{u}_\varepsilon(p) \partial_{x^i}.$$

For $\varepsilon > 0$ sufficiently small the covariance form of the Gaussian random vector $d\tilde{u}_\varepsilon(p)$ is positive definite; see (2.3). We can identify it with a symmetric, positive definite linear operator

$$S(\ d\tilde{u}_\varepsilon(p)) : T_pM \to T_pM.$$

More concretely, if $(x^i)_{1 \leq i \leq m}$ are $g$-normal coordinates at $p$, then we identify $S(\ d\tilde{u}_\varepsilon(p))$ with a $m \times m$ real symmetric matrix whose $(i, j)$-entry is given by

$$S_{ij}(\ d\tilde{u}_\varepsilon(p)) = E(\partial_{x^i}\tilde{u}_\varepsilon(p) \cdot \partial_{x^j}\tilde{u}_\varepsilon(p)).$$

**Theorem 2.1.** Fix a Borel subset $B \subset \mathbb{R}$. For any $p \in M$ define

$$f_{\varepsilon,B}(p) := (\det(2\pi S(\tilde{u}_\varepsilon(p))))^{-\frac{1}{2}} E\left( |\det \text{Hess}_p(\tilde{u}_\varepsilon)| \cdot I_B(\tilde{u}_\varepsilon(p)) \mid d\tilde{u}_\varepsilon(p) = 0 \right),$$

where $E(\var\mid\text{cons})$ stands for the conditional expectation of the variable $\var$ given the constraint $\text{cons}$. Then

$$\tilde{\sigma}^\varepsilon(B) = \int_M f_{\varepsilon,B}(p) |dV_g(p)|.$$

This theorem is a special case of a general result of Adler-Taylor, [1, Cor. 11.2.2]. Proposition 2.2 below shows that the technical assumptions in [1, Cor. 11.2.2] are satisfied if $\varepsilon \ll 1$.

For the above theorem to be of any use we need to have some concrete information about the Gaussian random variables ($RV$). All the relevant statistical invariants of these variables can be extracted from the covariance kernel of the random function $\tilde{u}_\varepsilon$. 

\qed
2.2. Proof of Theorem 1.3. Fix $\varepsilon > 0$. For any $p \in M$, we have the centered Gaussian random vector

$$
(\bar{u}_\varepsilon(p), d\bar{u}_\varepsilon(p), \text{Hess}_p(\bar{u}_\varepsilon)) \in \mathbb{R} \oplus T^*_pM \oplus \text{Sym}(T_pM).
$$

We fix normal coordinates $(x^i)_{1\leq i \leq m}$ at $p$ and we can identify the above Gaussian vector with the centered Gaussian vector

$$
(\bar{u}_\varepsilon(p), (\partial_{x^i} \bar{u}_\varepsilon(p))^1_{1\leq i \leq m}, \partial^2_{x^i x^j} (\bar{u}_\varepsilon(p))^1_{1\leq i,j \leq m}) \in \mathbb{R} \oplus \mathbb{R}^m \oplus \text{Sym}_m.
$$

The next result is the key reason the Kac-Rice formula can be applied successfully to the problem at hand.

**Proposition 2.2.** For any $1 \leq i, j, k, \ell \leq m$ we have the uniform in $p$ asymptotic estimates as $\varepsilon \searrow 0$

$$
E(\bar{u}_\varepsilon(p)^2) = s_m \varepsilon^{-m}(1 + O(\varepsilon^2)),
$$

$$
E(\partial_{x^i} \bar{u}_\varepsilon(p) \partial_{x^j} \bar{u}_\varepsilon(p)) = d_m \varepsilon^{-(m+2)} \delta_{ij}(1 + O(\varepsilon^2)),
$$

$$
E(\partial^2_{x^i x^j} \bar{u}_\varepsilon(p) \partial^2_{x^k x^l} \bar{u}_\varepsilon(p)) = h_m \varepsilon^{-(m+4)}(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})(1 + O(\varepsilon^2)),
$$

$$
E(\bar{u}_\varepsilon(p) \partial^2_{x^i x^j} \bar{u}_\varepsilon(p)) = -d_m \varepsilon^{-(m+2)} \delta_{ij}(1 + O(\varepsilon^2)),
$$

$$
E(\bar{u}_\varepsilon(p) \partial_{x^i} \bar{u}_\varepsilon(p)) = O(\varepsilon^{-m}),
$$

where $s_m = s_m + \omega_m$ and the constants $s_m, d_m, h_m$ are defined by (1.3).

**Proof.** Denote by $\bar{\sigma}_\varepsilon$ the covariance kernel of the random function

$$
\bar{u}_\varepsilon = X\omega(\varepsilon) + u_\varepsilon.
$$

Note that

$$
\bar{\sigma}_\varepsilon(p, q) = \omega(\varepsilon) + \bar{\sigma}_\varepsilon(p, q) = \omega_m \varepsilon^{-m} + \bar{\sigma}_\varepsilon(p, q).
$$

Fix a point $p_0 \in M$ and normal coordinates at $p_0$ defined in an open neighborhood $O_0$ of $p_0$. The restriction of $\bar{\sigma}_\varepsilon$ to $O_0 \times O_0$ can be viewed as a function $\bar{\sigma}_\varepsilon(x, y)$ defined in an open neighborhood of $(0, 0)$ in $\mathbb{R}^m \times \mathbb{R}^m$. For any $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^m$ we have

$$
E(\partial^\alpha_{x^i} \bar{u}_\varepsilon(p_0) \partial^\beta_{x^j} \bar{u}_\varepsilon(p)) = \partial^\alpha_{x^i} \partial^\beta_{x^j} \varepsilon(\varepsilon)(x, y)_{x=y=0}.
$$

Proposition 2.2 is now a consequence of the spectral estimates (B.1) in Appendix B. □

From the estimate (2.2b) we deduce that

$$
S(d\bar{u}_\varepsilon(p)) = d_m \varepsilon^{-(m+2)}(1 + O(\varepsilon^2)),
$$

so that

$$
\sqrt{\det S(\bar{u}_\varepsilon(p))} = (d_m)^{\frac{m}{2}} \varepsilon^{-\frac{(m+2)}{2}(1 + O(\varepsilon^2))} \text{ as } \varepsilon \to 0.
$$

Consider the rescaled random vector

$$
(\tilde{s}_\varepsilon, v_\varepsilon^i, H^\varepsilon) := (\varepsilon^{\frac{m}{2}} \bar{u}_\varepsilon(p), \varepsilon^{-\frac{m+2}{2}} d\bar{u}_\varepsilon(p), \varepsilon^{-\frac{m+4}{2}} \nabla^2 \bar{u}_\varepsilon(p)).
$$

Form Proposition 2.2 we deduce the following (uniform in $p$) estimates as $\varepsilon \searrow 0$.

$$
E((\tilde{s}_\varepsilon^2)^2) = s_m(1 + O(\varepsilon^2)),
$$

$$
E(v_\varepsilon^i v_\varepsilon^j) = d_m \delta_{ij}(1 + O(\varepsilon^2)),
$$

$$
E(H^\varepsilon_{ij} H^\varepsilon_{k\ell}) = h_m(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})(1 + O(\varepsilon^2)),
$$

$$
E(\tilde{s}_\varepsilon H^\varepsilon_{ij}) = -d_m \delta_{ij}(1 + O(\varepsilon^2)),
$$

$$
E(\tilde{s}_\varepsilon v_\varepsilon^i) = O(\varepsilon),
$$

$$
E(v_\varepsilon^i H^\varepsilon_{jk}) = O(\varepsilon).
$$
The probability distribution of the variable \( s^\varepsilon \) is

\[
d\gamma_{\tilde{s}_m(\varepsilon)}(x) = \frac{1}{\sqrt{2\pi \tilde{s}_m(\varepsilon)}} e^{-\frac{x^2}{2\tilde{s}_m(\varepsilon)}} |dx|,
\]

where

\[
\tilde{s}_m(\varepsilon) = \tilde{s}_m + O(\varepsilon).
\]

Fix a Borel set \( B \subset \mathbb{R} \). We have

\[
E\left( |\det \nabla^2 \tilde{u}_\varepsilon(p) I_B(\tilde{u}_\varepsilon(p)) | \big| d\tilde{u}_\varepsilon(p) = 0 \right) = \varepsilon^{-\frac{m(m+4)}{2}} E\left( |\det H^\varepsilon I_{\varepsilon \tilde{s}_m} B(\tilde{s}_\varepsilon) | v^\varepsilon = 0 \right)
\]

\[
= \varepsilon^{-\frac{m(m+4)}{2}} \int_{\varepsilon \tilde{s}_m B} E\left( |\det H^\varepsilon| \big| \tilde{s}_\varepsilon = x, v^\varepsilon = 0 \right) e^{-\frac{\varepsilon^2}{2\tilde{s}_m(\varepsilon)}} |dx|.
\]

Using (2.4) and (2.6) we deduce from Theorem 2.1 that

\[
\sigma^\varepsilon(B) = \varepsilon^{-m} \left( \frac{1}{2\pi d_m} \right)^\frac{m}{2} \int_M q_{\varepsilon,p}(\varepsilon \tilde{s}_m B) \rho_L(p) |dV_g(p)|,
\]

where \( \rho_{\varepsilon} : M \to \mathbb{R} \) is a function that satisfies the uniform in \( p \) estimate

\[
\rho_{\varepsilon}(p) = 1 + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.
\]

Hence

\[
\varepsilon^m \left( \mathcal{R}_{\varepsilon \tilde{s}_m} \right)_* \sigma^\varepsilon(B) = \left( \frac{1}{2\pi d_m} \right)^\frac{m}{2} \int_M q_{\varepsilon,p}(B) \rho_{\varepsilon}(p) |dV_g(p)|.
\]

To continue the computation we need to investigate the behavior of \( q_{\varepsilon,p}(B) \) as \( \varepsilon \). More concretely, we need to elucidate the nature of the Gaussian vector

\[
( H^\varepsilon | \tilde{s}_\varepsilon = x, v^\varepsilon = 0 )
\]

We will achieve this via the regression formula (C.3). For simplicity we set

\[
Y^\varepsilon := (\tilde{s}^\varepsilon, v^\varepsilon) \in \mathbb{R} \oplus \mathbb{R}^m.
\]

The components of \( Y^\varepsilon \) are

\[
Y^\varepsilon_0 = \tilde{s}^\varepsilon, \quad Y^\varepsilon_i = v^\varepsilon_i, \quad 1 \leq i \leq m.
\]

Using (2.5a), (2.5b) and (2.5e) we deduce that for any \( 1 \leq i, j \leq m \) we have

\[
E(Y^\varepsilon_i Y^\varepsilon_j) = \tilde{s}_m \delta_{0i} + O(\varepsilon), \quad E(Y^\varepsilon_i Y^\varepsilon_j) = d_m \delta_{ij} + O(\varepsilon^2).
\]

If \( S(Y^\varepsilon) \) denotes the covariance operator of \( Y \), then we deduce that

\[
S(Y^\varepsilon)^{-1}_{0i} = \frac{1}{\tilde{s}_m} \delta_{0i} + O(\varepsilon), \quad S(Y^\varepsilon)^{-1}_{ij} = \frac{1}{d_m} \delta_{ij} + O(\varepsilon).
\]

We now need to compute the covariance operator \( \text{Cov}(H^\varepsilon, Y^\varepsilon) \). To do so we equip \( \text{Sym}_m \) with the inner product

\[
(A, B) = \text{tr}(AB), \quad A, B \in \text{Sym}_m
\]

The space \( \text{Sym}_m \) has a canonical orthonormal basis

\[
\tilde{E}_{ij}, \quad 1 \leq i \leq j \leq m,
\]
where

\[
\tilde{E}_{ij} = \begin{cases} 
E_{ij}, & i = j \\
\frac{1}{\sqrt{2}} E_{ij}, & i < j 
\end{cases}
\]

and \( E_{ij} \) denotes the symmetric matrix nonzero entries only at locations \((i, j)\) and \((j, i)\) and these entries are equal to 1. Thus a matrix \( A \in \text{Sym}_m \) can be written as

\[
A = \sum_{i \leq j} a_{ij} E_{ij} = \sum_{i \leq j} \hat{a}_{ij} \tilde{E}_{ij},
\]

where

\[
\hat{a}_{ij} = \begin{cases} 
a_{ij}, & i = j, \\
\sqrt{2} a_{ij}, & i < j. 
\end{cases}
\]

The covariance operator \( \text{Cov}(H^\varepsilon, Y^\varepsilon) \) is a linear map

\[
\text{Cov}(H^\varepsilon, Y^\varepsilon) : \mathbb{R} \oplus \mathbb{R}^m \to \text{Sym}_m
\]
given by

\[
\text{Cov}(H^\varepsilon, Y^\varepsilon) \left( \sum_{\alpha=0}^{m} y_\alpha e_\alpha \right) = \sum_{i < j, \alpha} E(\tilde{H}_{ij}^\varepsilon Y^{\varepsilon}_\alpha) y_\alpha \tilde{E}_{ij} = \sum_{i < j, \alpha} E(H^\varepsilon_{ij} Y^{\varepsilon}_\alpha) y_\alpha E_{ij},
\]

where \( e_0, e_1, \ldots, e_m \) denotes the canonical orthonormal basis in \( \mathbb{R} \oplus \mathbb{R}^m \). Using (2.5d) and (2.5e) we deduce that

\[
\text{Cov}(H^\varepsilon, Y^\varepsilon) \left( \sum_{\alpha=0}^{m} y_\alpha e_\alpha \right) = - y_0 d_m \mathbb{1}_m + O(\varepsilon). \tag{2.10}
\]

We deduce that the transpose \( \text{Cov}(H^\varepsilon, Y^\varepsilon)^\vee \) satisfies

\[
\text{Cov}(H^\varepsilon, Y^\varepsilon)^\vee \left( \sum_{i \leq j} \hat{a}_{ij} \tilde{E}_{ij} \right) = - d_m \text{tr}(A)e_0 + O(\varepsilon). \tag{2.11}
\]

Set

\[
Z^\varepsilon := (H^\varepsilon | \tilde{s}^\varepsilon = x, v^\varepsilon = 0) - E(H^\varepsilon | \tilde{s}^\varepsilon = x, v^\varepsilon = 0).
\]

Above, \( Z^\varepsilon \) is a centered Gaussian random matrix with covariance operator

\[
S(Z^\varepsilon) = S(H^\varepsilon) - \text{Cov}(H^\varepsilon, Y^\varepsilon) S(Y^\varepsilon)^{-1} \text{Cov}(H^\varepsilon, Y^\varepsilon)^\vee.
\]

This means that

\[
E(\tilde{z}_{ij}^\varepsilon \tilde{z}_{k\ell}^\varepsilon) = (\tilde{E}_{ij}, S(Z^\varepsilon) \tilde{E}_{k\ell}).
\]

Using (2.9), (2.10) and (2.11) we deduce that

\[
\text{Cov}(H^\varepsilon, Y^\varepsilon) S(Y^\varepsilon)^{-1} \text{Cov}(H^\varepsilon, Y^\varepsilon)^\vee \left( \sum_{i \leq j} \hat{a}_{ij} \tilde{E}_{ij} \right) = \frac{d^2_m}{s_m} \text{tr}(A) \mathbb{1}_m + O(\varepsilon)
\]

\[
E(\tilde{z}_{ij}^\varepsilon) = h_m + O(\varepsilon), \quad E(z_{ii}^\varepsilon z_{jj}^\varepsilon) = h_m - \frac{d^2_m}{s_m} + O(\varepsilon), \quad \forall i < j,
\]

\[
E(\tilde{z}_{ii}^\varepsilon) = 3h_m - \frac{d^2_m}{s_m} + O(\varepsilon), \quad \forall i
\]

and

\[
E(z_{ij}^\varepsilon z_{k\ell}^\varepsilon) = O(\varepsilon), \quad \forall i < j, \quad k \leq \ell, \quad (i,j) \neq (k,\ell).
\]
We can rewrite these equalities in the compact form

$$E(z_{ij}^\varepsilon z_{k\ell}^\varepsilon) = \left( h_m - \frac{d_m^2}{s_m} \right) \delta_{ij} \delta_{k\ell} + h_m (\delta_{ik} \delta_{j\ell} + \delta_{il} \delta_{jk}) + O(\varepsilon).$$

Note that

$$h_m - \frac{d_m^2}{s_m} = \frac{r_m - 1}{r_m} h_m.$$ 

We set

$$\kappa_m := \frac{(r_m - 1)}{2r_m},$$

so that

$$E(z_{ij}^\varepsilon z_{k\ell}^\varepsilon) = 2\kappa_m h_m \delta_{ij} \delta_{k\ell} + h_m (\delta_{ik} \delta_{j\ell} + \delta_{il} \delta_{jk}) + O(\varepsilon).$$

Using (C.4) we deduce that

$$E(H^\varepsilon|\hat{s}^\varepsilon = x, v^\varepsilon = 0) = \text{Cov}(H^\varepsilon, Y^\varepsilon) S(Y^\varepsilon)^{-1}(x e_0) = -\frac{x d_m}{s_m} \mathbb{1}_m + O(\varepsilon).$$

We deduce that the Gaussian random matrix $(H^\varepsilon|\hat{s}^\varepsilon = x, v^\varepsilon = 0)$ converges uniformly in $p$ as $\varepsilon \to 0$ to the random matrix $A - \frac{x}{r_m(m+1)} \mathbb{1}_m$, where $A$ belongs to the Gaussian ensemble $\text{Sym}_{m, h_m}^2$ described in Appendix D. Thus

$$\lim_{\varepsilon \to 0} q_{\varepsilon, p}(B) = q_{\text{Sc}}(B) := \int_B \mathcal{E}_{\text{Sym}_{m, h_m}^2} \left( \left| \det \left( A - \frac{x d_m}{s_m} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi s_m}} \, dx$$

$$= (h_m)^\frac{m}{2} \int_B \mathcal{E}_{\text{Sym}_{m, 1}^2} \left( \left| \det \left( A - \frac{x}{s_m \sqrt{h_m}} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi s_m}} \, dx$$

$$= (h_m)^\frac{m}{2} \int_{(s_m)^{-\frac{1}{2}}} B \mathcal{E}_{\text{Sym}_{m, 1}^2} \left( \left| \det \left( A - \frac{y}{\sqrt{h_m}} \mathbb{1}_m \right) \right| \right) e^{-\frac{y^2}{2}} \, dy,$$

where

$$\alpha_m = \frac{d_m}{\sqrt{s_m h_m}} = \frac{1}{\sqrt{r_m}}.$$ 

This proves that

$$\lim_{\varepsilon \searrow 0} \mathcal{R}_{(s_m e^{-m})^{-\frac{1}{2}}} q_{\varepsilon, p}(B) = (h_m)^\frac{m}{2} \int_{B} \mathcal{E}_{\text{Sym}_{m, 1}^2} \left( \left| \det \left( A - \frac{y}{\sqrt{r_m}} \mathbb{1}_m \right) \right| \right) e^{-\frac{y^2}{2}} \, dy.$$ 

Using the last equality, the normalization assumption (*) and the estimate (2.7) in (2.8) we conclude

$$(\mathcal{R}_{(s_m e^{-m})^{-\frac{1}{2}}}^* \sigma^\varepsilon)(B) = e^{-m} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \mu_m(B) + O(\varepsilon)$$

as $\varepsilon \to 0$. (2.13)

In particular

$$\mathcal{N}^\varepsilon = e^{-m} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \mu_m(\mathbb{R}) + O(\varepsilon)$$

as $\varepsilon \to 0$. (2.14)

Observe that the density of $\mu_m$ is

$$\frac{d\mu_m}{dy} = \mathcal{E}_{\text{Sym}_{m, 1}^2} \left( \left| \det \left( A - \frac{y}{\sqrt{r_m}} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

(2.15)
\( \vec{A} = \sqrt{r_m} A \)

\[
E_{\text{Sym}^2 m, r_m} \left( \left| \det \left( \vec{A} - \frac{y}{\sqrt{r_m}} \mathbb{1}_m \right) \right| \right) = r_m^\frac{m}{2} e^{-\frac{y^2}{2}} \sqrt{2\pi}
\]

\( (D.7b) \)

\[
2k_m r_m = r_m - 1
\]

\[
\frac{d\mu}{dy} = 2^\frac{m+3}{2} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\pi \sqrt{r_m}} \left( \frac{r_m^2 + 1 - 2r_m y^2}{4r_m} \right) \int_\mathbb{R} \rho_{m+1,1}(\lambda) e^{-\frac{y}{4\sqrt{r_m}}} \frac{(\lambda - (r_m^2 + 1)\frac{y}{\sqrt{r_m}})^2 + \frac{r_m y^2}{2(r_m + 1)}}{2} d\lambda.
\]

This proves part (a) and (1.14a) in Theorem 1.3. To prove (1.14b) we distinguish two cases.

Case 1. \( r_m > 1 \). From Lemma D.2 we deduce that

\[
E_{\text{Sym}^2 m, 1} \left( \left| \det \left( A - \frac{y}{\sqrt{r_m}} \mathbb{1}_m \right) \right| \right) = 2^\frac{m+3}{2} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\pi \sqrt{r_m}} \int_\mathbb{R} \rho_{m+1,1}(\lambda) e^{-\frac{y}{4\sqrt{r_m}}} \frac{(\lambda - (r_m^2 + 1)\frac{y}{\sqrt{r_m}})^2 + \frac{r_m y^2}{2(r_m + 1)}}{2} d\lambda,
\]

where

\[
\tau_m := \frac{r_m}{\pi} - 1 = \frac{r_m - 1}{r_m + 1}.
\]

Thus

\[
\frac{d\mu_m}{dy} = 2^\frac{m+3}{2} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\pi \sqrt{r_m}} \frac{(\tau_m^2 + 1 - 2\tau_m y^2)}{4r_m} \int_\mathbb{R} \rho_{m+1,1}(\lambda) e^{-\frac{y}{4\sqrt{r_m}}} \frac{(\lambda - (\tau_m^2 + 1)\frac{y}{\sqrt{r_m}})^2 + \frac{\tau_m y^2}{2(r_m + 1)}}{2} d\lambda.
\]

An elementary computation yields a pleasant surprise

\[
-\frac{1}{4\tau_m} \left( \lambda - (\tau_m^2 + 1)\frac{y}{\sqrt{r_m}} \right)^2 - \frac{r_m y^2}{2(r_m + 1)} = -\frac{1}{4} \lambda^2 - \left( \frac{1}{2(r_m - 1)} - y \frac{r_m}{2(r_m - 1)} \right)^2.
\]

Now set

\[
\beta_m := \frac{1}{r_m - 1}.
\]

We deduce

\[
\frac{d\mu_m}{dy} = 2^\frac{m+3}{2} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\pi \sqrt{r_m}} \int_\mathbb{R} \rho_{m+1,1}(\lambda) e^{-\frac{y}{4\sqrt{r_m}}} \frac{1}{2} e^{-\frac{\beta_m y}{2}\lambda} d\lambda.
\]

\( (\lambda := \sqrt{r_m}) \)

\[
= 2^\frac{m+3}{2} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\pi \sqrt{r_m}} \int_\mathbb{R} \rho_{m+1,1}(\sqrt{r_m} \lambda) e^{-\frac{y}{4\sqrt{r_m}}}(\lambda)^2 e^{-\frac{\beta_m y}{2}\lambda} d\lambda
\]

\( (D.6) \)

\[
= 2^\frac{m+3}{2} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\pi \sqrt{r_m}} \int_\mathbb{R} \rho_{m+1,1/r_m}(\lambda) e^{-\frac{r_m y^2}{4\sqrt{r_m}}} \frac{1}{\beta_m r_m} (y - \lambda) d\lambda.
\]

\( (\kappa_m r_m \beta_m = \frac{1}{2}) \)

\[
= 2^\frac{m+1}{2} \Gamma \left( \frac{m+3}{2} \right) \int_\mathbb{R} \rho_{m+1,1/r_m}(\lambda) e^{-\frac{r_m y^2}{4\sqrt{r_m}}} \frac{1}{\beta_m r_m} (y - \lambda) d\lambda
\]
we obtain the following counterpart of (2.13) we obtain the case \( r_m > 1 \) (1.14b) of Theorem 1.3.

**Case 2.** \( r_m = 1 \). The proof of Theorem 1.3 in this case follows a similar pattern. Note first that in this case \( \kappa_m = 0 \) so invoking Lemma D.1 we obtain the following counterpart of (2.16)

\[
E_{\text{GOE}_m}\left( \left| \det \left( A - y^\sharp m \right) \right| \right) = 2^{\frac{m+4}{2}} \Gamma \left( \frac{m+3}{2} \right) e^{\frac{\sigma^2}{4}} \rho_{m+1,1}(y).
\]

Using this in (2.15) we deduce

\[
\frac{d\mu_m}{dy} = 2^{\frac{m+4}{2}} \Gamma \left( \frac{m+3}{2} \right) e^{\frac{\sigma^2}{4}} \rho_{m+1,1}(y)
\]

which is (1.14b) in the case \( r_m = 1 \). This completes the proof of Theorem 1.3.

\[\square\]

### 2.3. Proof of Corollary 1.4.

According to (1.11) we have

\[
\gamma_{\omega_m, e^{-m}} \ast \sigma^\varepsilon = \tilde{\sigma}^\varepsilon.
\]

Thus

\[
\gamma_{\omega_m, e^{-m}} \ast \left( \Re \frac{1}{\sqrt{\omega_m, e^{-m}}} \right) \ast \sigma^\varepsilon = \left( \Re \frac{1}{\sqrt{\omega_m, e^{-m}}} \right) \ast \tilde{\sigma}^\varepsilon.
\]

Hence

\[
\lim_{\varepsilon \to 0} \frac{1}{N\varepsilon} \gamma_{\omega_m, e^{-m}} \ast \left( \Re \frac{1}{\sqrt{\omega_m, e^{-m}}} \right) \ast \sigma^\varepsilon = \sigma_m.
\]

We can now conclude by invoking Lévy’s continuity theorem [23, Thm. 15.23(ii)] or [32, Thm. 2.4].

\[\square\]

### 2.4. Proof of Theorem 1.5.

We have

\[
\tilde{\sigma}_m = \frac{1}{K_m} \theta_{m+1, \frac{1}{m}} \ast \gamma_{\omega_m, e^{-m}} dy
\]

where

\[
\theta_{m+1, \frac{1}{m}}(\lambda) = \rho_{m+1, \frac{1}{m}}(\lambda) e^{-\frac{m\lambda^2}{2}},
\]

and

\[
K_m = \int_{\mathbb{R}} \theta_{m+1, \frac{1}{m}} \ast \gamma_{\omega_m, e^{-m}}(y) dy = \int_{\mathbb{R}} \theta_{m+1, \frac{1}{m}}(\lambda) d\lambda = \int_{\mathbb{R}} \rho_{m+1, \frac{1}{m}}(\lambda) e^{-\frac{m\lambda^2}{2}} d\lambda.
\]

We set

\[
R_m(\lambda) := \rho_{m+1, \frac{1}{m}}(\lambda), \quad R_\infty(x) = \frac{1}{2\pi} I_{\{|x| \leq 2\}} \sqrt{4 - x^2}.
\]

Fix \( c \in (0, 2) \). In [26, §4.2] we proved that

\[
\lim_{m \to \infty} \sup_{|x| \leq c} |\bar{R}_m(x) - R_\infty(x)| = 0, \quad (2.18a)
\]

\[
\sup_{|x| \geq c} |R_m(x) - R_\infty(x)| = O(1) \quad \text{as} \quad m \to \infty. \quad (2.18b)
\]

Then

\[
\rho_{m+1, \frac{1}{m}}(\lambda) = \sqrt{\frac{r_m}{m}} R_m \left( \sqrt{\frac{r_m}{m}} \lambda \right), \quad \theta_{m+1, \frac{1}{m}}(\lambda) = \sqrt{\frac{r_m}{m}} R_m \left( \sqrt{\frac{r_m}{m}} \lambda \right) e^{-\frac{r_m\lambda^2}{2}}.
\]

We now distinguish two cases.
Case 1. \( r = \lim_{m \to \infty} r_m < \infty \). In particular, \( r \in [1, \infty) \). In this case we have

\[
K_m = \sqrt{\frac{\tau_m}{m}} \int_{\mathbb{R}} R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) e^{-\frac{\tau_m \lambda^2}{4}} d\lambda,
\]

and using (2.18a)-(2.18b) we deduce

\[
\lim_{m \to \infty} \int_{\mathbb{R}} R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) e^{-\frac{\tau_m \lambda^2}{4}} d\lambda = R_\infty(0) \int_{\mathbb{R}} e^{-\frac{\lambda^2}{4}} d\lambda = R_\infty(0) \sqrt{\frac{4\pi}{r}}.
\]

Hence

\[
K_m \sim K_m' = R_\infty(0) \sqrt{\frac{4\pi}{m}} \text{ as } m \to \infty.
\] (2.19)

Now observe that

\[
\frac{1}{K_m'} \theta_{m+1, r_m}^{-1} \lambda d\lambda = \frac{1}{R_\infty(0)} R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) \frac{\tau_m}{\sqrt{4\pi}} e^{-\frac{\tau_m \lambda^2}{4}} d\lambda = \frac{1}{R_\infty(0)} R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) \gamma_{\frac{r}{r_m}}(d\lambda)
\]

Using (2.18a) and (2.18b) we conclude that the sequence of measures

\[
\frac{1}{K_m'} \theta_{m+1, r_m}^{-1} \lambda d\lambda
\]

converges weakly to the Gaussian measure \( \gamma_{\frac{r}{r}} \). Using this and the asymptotic equality (2.19) in (2.17) we deduce

\[
\lim_{m \to \infty} \sigma_m = \gamma_{\frac{r}{r}} \ast \gamma_{\frac{r+1}{r}} = \gamma_{\frac{r+1}{r}}.
\]

This proves Theorem 1.5 in the case \( r < \infty \) since

\[
\gamma_{\frac{\omega_m}{s_m}} \ast \sigma_m = \sigma_m \quad \text{and} \quad \lim_{m \to \infty} \omega_m = 0.
\] (1.10)

Case 2. \( \lim_{m \to \infty} r_m = \infty \). In this case we write

\[
\theta_{m+1, r_m}^{-1} \lambda d\lambda = \sqrt{\frac{4\pi}{m}} R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) \gamma_{\frac{r}{r_m}}(\lambda)d\lambda.
\]

Lemma 2.3. The sequence of measures

\[
R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) \gamma_{\frac{r}{r_m}}(\lambda)d\lambda
\]

converges weakly to the measure \( R_\infty(0)\delta_0 \).

Proof. Fix a bounded continuous function \( f : \mathbb{R} \to \mathbb{R} \). Observe first that

\[
\lim_{m \to \infty} \int_{\mathbb{R}} \left( R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) - R_\infty \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{r}{r_m}}(\lambda)d\lambda = 0. \tag{2.20}
\]

Indeed, we have

\[
D_m = \int_{|\lambda| < \sqrt{\frac{r_m}{\tau_m}}} \left( R_m \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) - R_\infty \left( \sqrt{\frac{\tau_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{r}{r_m}}(\lambda)d\lambda
\]

\[
=: D'_m
\]
\[
+ \int_{|\lambda|>c} \frac{1}{\sqrt{m}} \left( R_m \left( \sqrt{\frac{r_m}{m}} \lambda \right) - R_\infty \left( \sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.
\]

Observe that
\[
D'_m \leq \sup_{|x| \leq c} |R_m(x) - R_\infty(x)| \int_{|\lambda|<c} \frac{1}{\sqrt{m}} \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.
\]
and invoking (2.18a) we deduce
\[
\lim_{m \to \infty} D'_m = 0.
\]
Using (2.18b) we deduce that there exists a constant \( S > 0 \) such that
\[
D'_m \leq S \int_{|\lambda|>c} \frac{1}{\sqrt{m}} \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.
\]
On the other hand, Chebyshev's inequality shows that
\[
\int_{|\lambda|>c} \frac{1}{\sqrt{m}} \gamma_{\frac{2}{r_m}}(\lambda) d\lambda \leq \frac{2}{c^2 m}.
\]
Hence
\[
\lim_{m \to \infty} D'_m = 0.
\]
This proves (2.20).

The sequence of measures \( \gamma_{\frac{2}{r_m}}(\lambda) d\lambda \) converges to \( \delta_0 \) so that
\[
R_\infty(0) f(0) = \lim_{m \to \infty} \int_{\mathbb{R}} R_\infty(0) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.
\]
Using (2.20) and the above equality we deduce that the conclusion of the lemma is equivalent to
\[
\lim_{m \to \infty} \int_{\mathbb{R}} \left( R_\infty(0) - R_\infty \left( \sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda = 0. \tag{2.21}
\]
To prove this we decompose \( F_m \) as follows.
\[
F_m = \int_{|\lambda|<m^{-\frac{1}{4}} \sqrt{r_m}} \left( R_\infty(0) - R_\infty \left( \sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda
\]
\[
+ \int_{|\lambda|>m^{-\frac{1}{4}} \sqrt{r_m}} \left( R_\infty(0) - R_\infty \left( \sqrt{\frac{r_m}{m}} \lambda \right) \right) f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda.
\]
Observe that
\[
F'_m \leq \sup_{|x| \leq m^{-\frac{1}{4}} \sqrt{r_m}} |R_\infty(0) - R_\infty(x)| \int_{|\lambda|<m^{-\frac{1}{4}} \sqrt{r_m}} f(\lambda) \gamma_{\frac{2}{r_m}}(\lambda) d\lambda
\]
and since \( R_\infty \) is continuous at 0 we deduce
\[
\lim_{m \to \infty} F'_m = 0.
\]
Since $R_\infty$ and $f$ are bounded we deduce that there exists a constant $S > 0$ such that

$$F''_m \leq S \int_{|\lambda| > m} \frac{1}{\sqrt{\tau_m}} \gamma_{\tau_m}^2(\lambda) d\lambda.$$  

On the other hand, Chebyshev’s inequality shows that

$$\int_{|\lambda| > m} \frac{1}{\sqrt{\tau_m}} \gamma_{\tau_m}^2(\lambda) d\lambda \leq \frac{2}{\sqrt{m}}.$$

Hence

$$\lim_{m \to \infty} F''_m = 0.$$  

This proves (2.21) and the lemma.

Lemma 2.3 shows that

$$K_m \sim K'_m = \sqrt{\frac{4\pi}{m}} R_\infty(0),$$
and

$$\lim_{m \to \infty} \frac{1}{K_m} \theta_{m+1, \tau_m}(\lambda) d\lambda = \delta_0.$$  

On the other hand

$$\lim_{m \to \infty} \gamma_{\tau_m}(\lambda) d\lambda = \gamma_1(\lambda) d\lambda,$$
so that

$$\lim_{m \to \infty} \sigma_m = \delta_0 \ast \gamma_1 = \gamma_1.$$  

This completes the proof of Theorem 1.5.

2.5. **Proof of Corollary 1.6.** Using (2.14) we deduce

$$\epsilon^m N^\epsilon = \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \mu_m(\mathbb{R}) + O(\epsilon)$$

$$= 2^{m+1} \frac{\Gamma\left( \frac{m+3}{2} \right)}{2 \Gamma\left( \frac{m+1}{2} \right)} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \int_{\mathbb{R}} \theta_{m+1, \tau_m}^{-} \ast \gamma_{\tau_m}^{-}(y) dy + O(\epsilon)$$

$$= 2^{m+1} \frac{\Gamma\left( \frac{m+3}{2} \right)}{2 \Gamma\left( \frac{m+1}{2} \right)} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \int_{\mathbb{R}} \theta_{m+1, \tau_m}^{-}(\lambda) d\lambda + O(\epsilon)$$

$$= 2^{m+1} \frac{\Gamma\left( \frac{m+3}{2} \right)}{2 \Gamma\left( \frac{m+1}{2} \right)} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} K_m + O(\epsilon).$$

Lemma 2.3 implies that as $m \to \infty$ we have

$$K_m \sim \sqrt{\frac{4\pi}{m}} R_\infty(0) = \frac{2}{\sqrt{\pi m}}.$$  

We deduce that

$$C_m(w) \sim \frac{2^{m+6}}{\sqrt{\pi m}} \left( \frac{m+3}{2} \right)^{\frac{m}{2}} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \right)$$

as $m \to \infty$.  

\[\square\]
2.6. **Proof of Theorem 1.7.** Fix a point \( p \in M \) and normal coordinates \((x^i)\) near \( p \). The equality (2.2b) shows that as \( \varepsilon \to 0 \) we have the following estimate, uniform in \( p \):

\[
E\left( \partial_{x^i} \bar{\bar{u}}_{\varepsilon}(p) \partial_{x^j} \bar{\bar{u}}_{\varepsilon}(p) \right) = d_m \varepsilon^{-(m+2)} \left( \delta_{ij} + O(\varepsilon^2) \right).
\]

Hence

\[
h^\varepsilon(\partial_{x^i}, \partial_{x^j}) = \delta_{ij} + O(\varepsilon^2) = g_p(\partial_{x^i}, \partial_{x^j}) + O(\varepsilon^2).
\]

(2.22)

This proves (a) and (b) of Theorem 1.3.

With \( p \) and \((x^i)\) as above we set

\[
E_{i_1, \ldots, i_n, j_1, \ldots, j_k} := \frac{\partial^{a+b} E_{\varepsilon}(x, y)}{\partial x^{i_1} \ldots \partial x^{i_n} \partial y^{j_1} \ldots \partial y^{j_k}} \bigg|_{x=y=0},
\]

\[
h^\varepsilon := h^\varepsilon_p(\partial_{x^i}, \partial_{x^j}), \quad 1 \leq i, j \leq m.
\]

We denote by \( K^\varepsilon_{ij} \) the sectional curvature of \( h^\varepsilon \) along the plane spanned by \( \partial_{x^i}, \partial_{x^j} \). Using [1, Lemma 12.2.1] and that the sectional curvatures of a metric are inverse proportional to the metric we deduce as in [26, §3.3] that

\[
K^\varepsilon_{ij} = \frac{d_m}{\varepsilon^{m+2}} \times \frac{E_{i_1, \ldots, i_n, j_1, \ldots, j_k} - E_{i_1, \ldots, i_n, j_1, \ldots, j_k}^\varepsilon}{E_{i_1, \ldots, i_n, j_1, \ldots, j_k}^\varepsilon E_{i_1, \ldots, i_n, j_1, \ldots, j_k}^\varepsilon - (\delta_{ij})^2}.
\]

Using Theorem B.5 we deduce that there exists a universal constant \( Z_m \) that depends only on \( m \) and \( w \) such that

\[
E_{i_1, \ldots, i_n, j_1, \ldots, j_k}^\varepsilon - E_{i_1, \ldots, i_n, j_1, \ldots, j_k} = \varepsilon^{-(m+2)} Z_m K_{ij}(p) \left( 1 + O(\varepsilon^2) \right),
\]

(2.23)

where \( K_{ij}(p) \) denotes the sectional curvature of \( g \) at \( p \). The estimate (2.2b) implies that

\[
E_{i_1, \ldots, i_n, j_1, \ldots, j_k}^\varepsilon - (\delta_{ij})^2 = d_m^2 \varepsilon^{-2(m+2)} \left( 1 + O(\varepsilon^2) \right).
\]

Thus

\[
K^\varepsilon_{ij} = \frac{Z_m}{d_m} K_{ij}(p) \left( 1 + O(\varepsilon^2) \right).
\]

To determine the constant \( \frac{Z_m}{d_m} \) it suffices to compute it on a special manifold. Assume that \( M \) is the unit sphere \( S^m \) equipped with the round metric. This is is a homogeneous space equipped with an invariant metric \( g \) with positive sectional curvatures. The metrics \( h^\varepsilon \) are also invariant so there exists a constant \( C_{\varepsilon} > 0 \) such that \( h^\varepsilon = C_{\varepsilon} g \). The estimate (2.22) implies that \( C_{\varepsilon} = 1 \) and thus \( K^\varepsilon_{ij} = K_{ij}(p) \) so that \( \frac{Z_m}{d_m} = 1 \). \( \square \)

### 3. Some Examples

We want to discuss several examples of weights \( w \) satisfying the assumptions of the central limit theorem, Theorem 1.5. Observe first that

\[
r_n(w) \sim R_m(w) = \frac{I_{m-1}(w) I_{m+3}(w)}{I_{m+1}(w)} \quad \text{as} \quad m \to \infty.
\]

Moreover

\[
R_n(w_\varepsilon) = R_n(w).
\]

**Example 3.1.** Suppose that \( w(t) = e^{-t^2} \). In this case \( E_{\varepsilon}^\varepsilon \) is the Schwartz kernel of the heat operator \( e^{-\varepsilon \Delta} \) whose asymptotics as \( \varepsilon \to 0 \) have been thoroughly investigated. The momenta (1.4) are

\[
I_k(w) = \int_0^\infty t^k e^{-t^2} dt = \frac{1}{2} \int_0^\infty s^{k-1/2} e^{-s} ds = \frac{1}{2} \Gamma \left( \frac{k+1}{2} \right).
\]
Hence
\[ R_m(w) = \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2} + 2\right)}{\Gamma\left(\frac{m}{2} + 1\right)^2} = \frac{m + 4}{m + 2} \geq 1, \quad q_m = \frac{m(m + 4)}{(m + 2)^2} < 1, \quad \forall m \]
so that \( r_m = 1 \) for all \( m \). Moreover, in this case we have
\[ \frac{I_{m+3}(w)}{I_{m+1}(w)} = m + 2, \]
so that
\[ C_m(w) \sim \frac{2^{m+6}}{\sqrt{m\pi} m^{m/2}} \Gamma\left(\frac{m+3}{2}\right) \]
as \( m \to \infty \), and Stirling’s formula implies
\[ \log C_m(w) \sim \frac{m}{2} \log m \quad \text{as} \quad m \to \infty. \quad (3.1) \]

Example 3.2. Suppose that
\[ w(t) = \exp\left(-\left(\log t\right) \log(\log t)\right), \quad \forall t \geq 1. \]
Observe that
\[ I_k(w) = \int_0^1 r^k w(r) dr + \int_1^{\infty} r^k \exp\left(-\left(\log r\right) \log(\log r)\right) dr. \]
This proves that
\[ I_k(w) \sim J_k := \int_1^{\infty} r^k \exp\left(-\left(\log r\right) \log(\log r)\right) dr \quad \text{as} \quad k \to \infty. \]
Using the substitution \( r = e^t \) we deduce
\[ J_k = \int_0^{\infty} e^{(k+1)t - t \log t} dt. \]
We want to investigate the large \( \lambda \) asymptotics of the integral
\[ T_\lambda = \int_0^{\infty} e^{-\phi_\lambda(t)} dt, \quad \phi_\lambda(t) = \lambda t - t \log t. \quad (3.2) \]
We will achieve this by relying on the Laplace method [9, Chap. 4]. Note that
\[ \phi'_\lambda(t) = \lambda - \log t - 1, \quad \phi''_\lambda(t) = -\frac{1}{t}. \]
Thus \( \phi_\lambda(t) \) has a unique critical point
\[ \tau = \tau(\lambda) := e^{\lambda-1}. \]
We make the change in variables \( t = \tau s \) in (3.2). Observe that
\[ \lambda e^{\lambda-1} s - e^{\lambda-1} s \log(e^{\lambda-1} s) = e^{\lambda-1} s - (\lambda - 1)e^{\lambda-1} s - e^{\lambda-1} \log s = e^{\lambda-1} s(1 - \log s) \]
and we deduce
\[ T_\lambda = \tau \int_0^{\infty} e^{-\tau h(s)} ds, \quad h(s) = s(\log s - 1). \]
The asymptotics of the last integral can be determined using the Laplace method and we have, [9, §4.1]
\[ T_\lambda \sim \tau e^{-\tau h(1)} \sqrt{\frac{2\pi}{\tau h''(1)}} = \sqrt{2\pi \tau e^\tau}. \]
Hence
\[ J_k = T_{k+1} \sim \sqrt{2\pi \tau(k+1)e^{\tau(k+1)}} = \sqrt{2\pi e^k e^k} \quad \text{as} \ k \to \infty. \]

In this case
\[ R_m(w) \to \infty \quad \text{as} \ m \to \infty. \]

Note that
\[ \frac{h_m}{d_m} = \frac{2I_{m+3}(w)}{(m+2)I_{m+1}(w)}. \]

We deduce that
\[ \log \left( \frac{h_m}{d_m} \right) \sim e^m - e^{m+2} \quad \text{as} \ m \to \infty. \]

Hence
\[ \log C_m(w) \sim \frac{m}{2} e^{m+2}(e^2 - 1) \quad \text{as} \ m \to \infty. \quad (3.3) \]

\[ \square \]

\textbf{Example 3.3.} Suppose that
\[ w(r) = \exp(-C(\log r)^\alpha), \quad C > 0, \ r > 1, \ \alpha > 1. \]

Arguing as in Example 3.2 we deduce that as \( k \to \infty \)
\[ I_k(w) \sim \int_1^\infty r^k \exp(-C(\log r)^\alpha) \, dr = \int_0^\infty e^{(k+1)t-Ct^\alpha} \, dt. \]

Again, set
\[ T_\lambda := \int_0^\infty e^{-\phi_\lambda(t)} \, dt, \quad \phi_\lambda(t) := C t^\alpha - \lambda t. \]

We determine the asymptotics of \( T_\lambda \) as \( \lambda \to \infty \) using the Laplace method. Note that
\[ \phi_\lambda'(t) = \alpha C t^\alpha - \lambda. \]

The function \( \phi_\lambda \) has a unique critical point
\[ \tau = \tau(\lambda) = \left( \frac{\lambda}{\alpha C} \right)^{\frac{1}{\alpha-1}}. \]

Observe that
\[ \phi_\lambda(\tau s) = a(s^\alpha - bs), \quad a := \left( \frac{\lambda}{C^{1/\alpha-1}} \right)^{\frac{\alpha}{\alpha-1}}, \quad b := \alpha^{-\frac{1}{\alpha-1}}, \]
\[ T_\lambda = \tau(\lambda) \int_0^\infty e^{-a(s^\alpha - bs)} \, ds. \]

We set \( g(s) := s^\alpha - bs \). Using the Laplace method [9, §4.2] we deduce
\[ T_\lambda \sim \tau(\lambda) e^{-ag(1)} \sqrt{\frac{2\pi}{ag''(1)}} = \sqrt{\frac{2\pi}{a\alpha(\alpha-1)}} e^{a(b-1)} \]
\[ \sqrt{\frac{2\pi}{a\alpha(\alpha-1)}} e^{a(b-1)}. \]

Hence
\[ \log T_\lambda \sim \left( \frac{\lambda^\alpha}{C} \right)^{\frac{1}{\alpha-1}} \frac{a^{\alpha-1} - 1}{\alpha^{\frac{1}{\alpha-1}}} =: Z(\alpha, C) \lambda^{\frac{\alpha}{\alpha-1}}. \]

Hence
\[ \log R_m(w) \sim \log T_m + \log T_{m+4} - 2 \log T_{m+2} \]
\[ \sim Z(\alpha, C) \left( m^{\frac{\alpha}{\alpha-1}} + (m+4)^{\frac{\alpha}{\alpha-1}} - 2(m+2)^{\frac{\alpha}{\alpha-1}} \right) \]
Then, for example, assume that on the positive semi-axis it is given by
\[ Z(\alpha, C)m^{\frac{\alpha}{\alpha - 1}} \left( 1 + \left( 1 + \frac{4}{m} \right)^{\frac{\alpha}{\alpha - 1}} - 2 \left( 1 + \frac{2}{m} \right)^{\frac{\alpha}{\alpha - 1}} \right) \]

which shows that \( r \) can have any value in \([1, \infty]\). Note that in this case
\[ \log I_{m+3}(w) - \log I_{m+1}(w) \sim Z(\alpha, C)m^{\frac{\alpha}{\alpha - 1}} \left( \left( 1 + \frac{4}{m} \right)^{\frac{\alpha}{\alpha - 1}} - \left( 1 + \frac{2}{m} \right)^{\frac{\alpha}{\alpha - 1}} \right) \]

so that
\[ \log C_m(w) \sim \frac{Z(\alpha, C)}{\alpha - 1} m^{\frac{\alpha}{\alpha - 1}}, \quad m \to \infty. \] (3.5)

**Example 3.4.** Suppose now that \( w \) is a weight with compact support disjoint from the origin. For example, assume that on the positive semi-axis it is given by
\[ w(x) = \begin{cases} \frac{1}{1 - (x-c)}, & |x-c| \leq 1, \\ 0, & |x-c| > 1, \end{cases} \quad c > 1. \]

Then
\[ I_k(w) = \int_{c-1}^{c+1} t^k e^{-\frac{1}{1-(t-c)^2}} dt = \int_{-1}^{1} (t+c)^k e^{-\frac{1}{1-t^2}} dt \]

\[ = \int_{-1}^{0} (t+c)^k e^{-\frac{1}{1-t^2}} dt + \int_{0}^{1} (t+c)^k e^{-\frac{1}{1-t^2}} dt. \]

Observe that
\[ \lim_{k \to \infty} c^{-k} I_k^+ = 0. \]

On the other hand
\[ I_k^+ = \int_{0}^{1} (c+1-t)^k e^{-\frac{1}{1-t^2}} dt, \]

and we deduce
\[ c^k \int_{0}^{1} e^{-\frac{1}{1-t^2}} dt \leq I_k^+ \leq (c+1)^k \int_{0}^{1} e^{-\frac{1}{1-t^2}} dt. \]

Hence the asymptotic behavior of \( I_k(w) \) is determined by \( I_k^+ \). We will determine the asymptotic behavior of \( I_k^+ \) by relying again on the Laplace method. Set \( a := (c+1) \) so that
\[ I_k^+ = \int_{0}^{1} (a-t)^k e^{-\frac{1}{1-t^2}} dt = a^k \int_{0}^{1} (1-s)^k e^{-\frac{1}{a^2 s^2}} ds = a^k \int_{a}^{\infty} (u-1)^k u^{-(k+2)} e^{-\frac{u^2}{a^2}} du. \]

Consider the phase
\[ \phi_h(s) = \frac{1}{h} \log(1-s) - \frac{1}{h^2 a^2}, \quad h \searrow 0, \]
and set
\[ P_h = a^\frac{1}{\pi} \int_0^\frac{1}{a} e^{\phi_h(s)} \]
so that
\[ I_k^+ = P_{1/k}. \]
We have
\[ \phi_h'(s) = -\frac{1}{h(1-s)} + \frac{2}{a^2 s^3}; \quad \phi_h''(t) = -\frac{1}{h(1-s)^2} - \frac{6}{a^2 s^4}. \]
The phase \( \phi_h \) as a unique critical point \( \tau = \tau(h) \in (0, 1/a) \) satisfying
\[ h = \frac{a^2 \tau^3}{2(1 - \tau)} = \frac{a^2 \tau^3}{2}(1 + O(\tau)), \]
so that
\[ \tau = \left( \frac{2h}{a^2} \right)^{\frac{1}{3}} \left( 1 + O\left( \frac{1}{h^{\frac{1}{3}}} \right) \right) \quad \text{as} \quad h \searrow 0. \quad (3.6) \]
Set
\[ v := v(h) := -\frac{1}{\phi_h''(\tau)} \sim \frac{a^2 \tau^4}{6} \sim \frac{(2h)^{\frac{1}{3}}}{6a^{\frac{2}{3}}} = \frac{1}{6} \left( \frac{2h^2}{a} \right)^{\frac{2}{3}}. \quad (3.7) \]
We make the change in variables \( s = \tau + \sqrt{v}x \) and we deduce
\[ P_h = e^{\phi_h(\tau)} a^\frac{1}{\pi} \sqrt{v} \int_{J(h)} e^{\phi_h(\tau + \sqrt{v}x) - \phi_h(\tau)} dx, \quad J(h) = \left[ -\frac{\tau}{\sqrt{v}}, \frac{1/a - \tau}{\sqrt{v}} \right]. \]
We claim that
\[ \lim_{h \to 0} \int_{J(h)} e^{\phi_h(\tau + \sqrt{v}x) - \phi_h(\tau)} dx = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}. \quad (3.8) \]
It is convenient to think of \( \tau \) as the small parameter and then redefine
\[ h = h(\tau) = \frac{a^2 \tau^3}{2(1 - \tau)} \]
and think of \( v \) as a function of \( \tau \). Finally set \( \sigma := \sqrt{v} \) and
\[ \varphi_\tau(x) := \phi_h(\tau + \sigma x) - \phi_h(\tau) = \frac{2(1 - \tau)}{a^2 \tau^3} \log(1 - s) - \frac{1}{a^2 s^2} \]
\[ = \frac{2(1 - \tau)}{a^2 \tau^3} \left( \log(1 - \tau - \sigma x) - \log(1 - \tau) \right) - \frac{1}{a^2 \tau^2} \left( \frac{1}{(1 + \tau x)^2} - \frac{1}{\tau^2} \right) \]
\[ = \frac{2(1 - \tau)}{a^2 \tau^3} \log \left( \frac{1 - \sigma}{1 - \tau} x \right) - \frac{1}{a^2 \tau^2} \left( \frac{1}{(1 + \frac{\tau}{\tau} x)^2} - 1 \right) \]
\[ = \frac{1}{a^2 \tau^2} \left( \frac{2(1 - \tau)}{\tau} \log \left( \frac{1 - \sigma}{1 - \tau} x \right) - \left( \frac{1}{(1 + \frac{\tau}{\tau} x)^2} - 1 \right) \right). \]
The equality \( (3.8) \) is equivalent to
\[ \lim_{\tau \to \infty} \int_{J(h)} e^{\varphi_\tau(x)} = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx. \quad (3.9) \]
By construction, we have
\[ \varphi_\tau(0) = \varphi'_\tau(0) = 0, \quad \varphi''_\tau(0) = -1, \quad \varphi_\tau(x) \leq 0, \quad \forall x \in J(h). \]
Let us observe that
\[
\lim_{\tau \to 0} \varphi_\tau(x) = \frac{1}{2} \varphi''_\tau(0)x^2 = \frac{-x^2}{2}, \quad \forall x \in \mathbb{R}.
\] (3.10)

Indeed, fix \(x \in \mathbb{R}\) and assume \(\tau\) is small enough so that
\[
\tau|x| < \frac{1}{2}.
\] (3.11)

Observe that
\[
\varphi^{(j)}_\tau(0) = \frac{1}{a^2\tau^2} \left( \frac{2(1 - \tau)}{\tau} \frac{d^j}{dx^j}|_{x=0} \log \left( 1 - \frac{\sigma}{1 - \tau}x \right) \right) - \frac{d^j}{dx^j}|_{x=0} \left( \frac{1}{(1 + \frac{2}{\tau}x)^2} - 1 \right)
\]
\[
= \frac{1}{a^2\tau^2} \left( - \frac{2(1 - \tau)}{\tau} \left( \frac{\sigma}{1 - \tau} \right)^j + (-1)^j + 1 \right) \left( \frac{\sigma}{\tau} \right)^j.
\]
Using the estimate \(\sigma = O(\tau^2)\) as \(\tau \to 0\) we deduce that there exists \(C > 0\) such that, for any \(j \geq 0\) we have
\[
|\varphi^{(j)}_\tau(0)| \leq C(j + 1)!\tau^{j-2}.
\]

Hence
\[
\frac{1}{j!} |\varphi^{(j)}_\tau(0)x^j| \leq C|\tau x|^j x^{-2} x^2, \quad \forall j \geq 2.
\]

Thus if \(\tau\) satisfies (3.11), we have
\[
\varphi_\tau(x) + \frac{x^2}{2} = \varphi_\tau(x) - \varphi'_\tau(0)x - \frac{1}{2} \varphi''_\tau(0)x^2 = \sum_{j \geq 3} \frac{1}{j!} \varphi^{(j)}_\tau(0)x^j,
\]
where the series in the right-hand side is absolutely convergent. Hence
\[
|\varphi_\tau(x) + \frac{x^2}{2}| \leq Cx^2|\tau x| \sum_{j \geq 3} j|\tau x|^{j-3} \leq C|\tau x|x^2 \sum_{j \geq 3} j2^{j-3}.
\]
This proves (3.10).

Next we want to prove that there exists a constant \(A > 0\) such that
\[
\varphi_\tau(x) \leq A(1 - |x|), \quad \forall x \in J(h), \quad \forall \tau \ll 1.
\] (3.12)

We will achieve this by relying on the concavity of \(\varphi_\tau\) over the interval \(J(h)\). The graph of \(\varphi_\tau\) is situated below either of the lines tangent to the graph at \(x = \pm 1\). Thus
\[
\varphi_\tau(x) \leq \varphi_\tau(1) + \varphi'_\tau(1)(x - 1) \leq -\varphi'_\tau(1) + \varphi'_\tau(1)x,
\]
\[
\varphi_\tau(x) \leq \varphi'_\tau(-1) + \varphi'_\tau(-1)(x + 1) \leq \varphi'_\tau(-1) + \varphi'_\tau(-1).
\]

Now observe that
\[
\frac{d}{dx} \varphi_\tau(x) = \frac{1}{a^2\tau^2} \left( - \frac{2\sigma}{\tau} \frac{1}{1 - \frac{\sigma}{1 - \tau}x} + \frac{2\sigma}{\tau} \frac{1}{(1 + \frac{2}{\tau}x)^3} \right) = \frac{2\sigma}{a^2\tau^3} \left( \frac{1}{1 + \frac{2}{\tau}x} - 1 \right).
\]
Using the fact that \(\sigma = O(\tau^2)\) we deduce from the above equality that
\[
|\varphi'_\tau(\pm 1)| = O(1), \quad \text{as } \tau \to 0.
\]
This proves (3.12). Using (3.10), (3.12) and the dominated convergence theorem we deduce
\[
\lim_{\tau \to \infty} \int_{J(h)} e^{\varphi_\tau(x)} dx = \int_{\mathbb{R}} e^{\frac{x^2}{2}} dx = \sqrt{2\pi}.
\]
We conclude that
\[
P_h \sim e^{\phi_h(\tau)} a^h \sqrt{2\pi} \quad \text{as } h \to 0
\] (3.13)
Now observe that
\[ \phi_n(\tau) = \frac{1}{h} \log(1 - \tau) - \frac{1}{a^2 \tau^2} = \frac{2(1 - \tau) \log(1 - \tau)}{a^2 \tau^2} - \frac{1}{a^2 \tau^2} \sim -\frac{3}{a^2 \tau^2}. \]

Using (3.6) we deduce
\[ \phi_n(\tau) \sim -\frac{3}{a^2} \left( \frac{1}{2h} \right)^{\frac{3}{2}} = -3 \left( \frac{k}{2a} \right)^{\frac{3}{2}}, \quad k = \frac{1}{h}. \]

Also
\[ e^{\phi_n(\tau)} = (1 - \tau) \frac{2(1 - \tau)}{a^2 \tau^2} e^{-\frac{1}{a^2 \tau^2}}. \]

In any case, using (3.6), (3.7) and (3.13) we deduce that
\[ \log I_k(w) \sim k \log a = k \log(c + 1) \quad \text{as} \quad k \to \infty. \] (3.14)

Thus
\[ \log r_m(w) = \log \left( \frac{I_{m-1}(w)I_{m+3}(w)}{I_{m+1}(w)} \right) = 0, \]
so that
\[ \lim_{m \to \infty} q_m = \lim_{m \to \infty} r_m = 1. \quad \square \]

**Example 3.5.** If we let \( c = 0 \) in the above example, then we deduce that
\[ I_k(w) = \int_0^1 t^k e^{-\frac{1}{1 - \tau^2}} dt \sim e^{\phi_n(\tau)} \sqrt{2\pi v(h)} \]
where
\[ \phi_n(\tau) \sim -3 \left( \frac{k}{2} \right)^{\frac{3}{2}}, \quad v(h) \sim \frac{1}{6} \left( \frac{2}{k^2} \right)^{\frac{3}{2}}. \]

Hence
\[ \log I_k(w) \sim -3 \left( \frac{k}{2} \right)^{\frac{3}{2}}, \]
\[ \log r_m(w) \sim -\frac{3}{2} \left( (m - 1)^{\frac{3}{2}} + (m + 3)^{\frac{3}{2}} - (m + 1)^{\frac{3}{2}} \right) \to 0, \]
so that
\[ \lim_{m \to \infty} q_m = \lim_{m \to \infty} r_m = 1. \quad \square \]

4. A PROBABILISTIC PROOF OF THE GAUSS-BONNET THEOREM

Suppose that \( M \) is a smooth, compact, connected orientated manifold of even dimension \( m \). For any Riemann metric \( g \) we can view the Riemann curvature tensor \( R_g \) as a symmetric bundle morphism \( R_g : \Lambda^2 TM \to \Lambda^2 TM \). Equivalently, using the metric identification \( T^* M \cong TM \) we can view \( R_g \) as a section of \( \Lambda^2 T^* M \otimes \Lambda^2 T^* M \).

We will denote by \( \Omega^{p,q}(M) \) the sections of \( \Lambda^p T^* M \otimes \Lambda^q T^* M \) and we will refer to them of double forms of type \( (p,q) \). Thus \( R_g \in \Omega^{2,2}(M) \). We have a natural product
\[ \cdot : \Omega^{p,q}(M) \times \Omega^{p',q'}(M) \to \Omega^{p+p',q+q'}(M) \]
defined in a natural way; see [1, Eq. (7.2.3)] for a precise definition.

Using the metric \( g \) we can identify a double-form in \( \Omega^{k,k}(M) \) with a section of \( \Lambda^k T^* M \otimes \Lambda^k T^* M \), i.e., with a bundle morphism \( \Lambda^k T^* M \to \Lambda^k T^* M \) and thus we have a linear map
\[ \text{tr} : \Omega^{k,k}(M) \to C^\infty(M). \]
For $1 \leq k \leq \frac{m}{2}$ we have a double form
\[
R^*_g = R_g \cdot \cdots \cdot R_g \in \Omega^{2k,2k}(M).
\]
We denote by $dV_g \in \Omega^m(M)$ the volume form on $M$ defined by the metric $g$ and the orientation on $M$. We set
\[
e_g(M) := \frac{1}{(2\pi)^{\frac{m}{2}} \frac{m}{2}!} \text{tr}\left( -R_g \right) dV_g \in \Omega^m(M).
\]
The form $e_g(M)$ is called the Euler form of the metric $g$ and the classical Gauss-Bonnet theorem states that
\[
\int_M e_g(M) = \chi(M) =: \text{the Euler characteristic of } M. \tag{4.1}
\]
In this section we will show that the Gauss-Bonnet theorem for any metric $g$ is an immediate consequence of the Kac-Rice formula coupled with the approximation theorem Thm. 1.7.

Fix a metric $g$. For simplicity we assume that $\text{vol}_g(M) = 1$. This does not affect the generality since $e_{cg}(M) = e_g(M)$ for any constant $c > 0$. Consider the random function $u^\varepsilon$ on $M$ defined by (1.2, 1.1). Set
\[
v^\varepsilon = \left( \frac{\varepsilon^{m+2}}{d_m} \right)^{\frac{1}{2}} u^\varepsilon.
\]
Observe that for $\varepsilon > 0$ sufficiently small, any $X, Y \in \text{Vect}(M)$ and any $p \in M$ we have
\[
h(\varepsilon X(p), Y(p)) = E\left( X v^\varepsilon(p), Y v^\varepsilon(p) \right)
\]
where $h$ is the metric on $M$ that appears in the approximation theorem, Theorem 1.7.

For any smooth function $f : M \to \mathbb{R}$ and any $p \in M$ we denote by $\text{Hess}_p^\varepsilon(f)$ the Hessian of $f$ at $p$ defined in terms of the metric $h$. More precisely
\[
\text{Hess}_p^\varepsilon(f) = XY f(p) - (\nabla^\varepsilon_X Y) f(p), \quad \forall X, Y \in \text{Vect}(M),
\]
where $\nabla^\varepsilon$ denotes the Levi-Civita connection of the metric $h$. Using the metric $h$ we can identify this Hessian with a symmetric linear operator
\[
\text{Hess}_p^\varepsilon(f) : (T_p M, h^\varepsilon) \to (T_p M, h^\varepsilon).
\]
For any $p \in M$ we have a random vector $d v^\varepsilon(p) \in T_p^* M$. Its covariance form $S(d v^\varepsilon(p))$ is precisely the metric $h^\varepsilon$, and if we use the metric $h^\varepsilon$ to identify this form with an operator we deduce that $S(d v^\varepsilon(p))$ is identified with the identity operator.

For every smooth Morse function $f$ on $M$ and any integer $0 \leq k \leq m$ we have a measure $\nu_{f,k}$ on $M$
\[
\nu_{f,k} = \sum_{df(p) = 0, \text{ind}(f,p) = k} \delta_p,
\]
where $\text{ind}(f,p)$ denotes the Morse index of the critical point $p$ of the Morse function $f$. We set
\[
\nu_f = \sum_{k=0}^{m} (-1)^k \nu_{f,k}
\]
The Poincaré-Hopf theorem implies that for any Morse function we have
\[
\int_M \nu_f(d p) = \chi(M). \tag{4.2}
\]
Using the random Morse function $\nu^\varepsilon$ we obtain the random measures $\nu_{\nu^\varepsilon, p}$ and $\nu_{\nu^\varepsilon}$. We denote by $\nu^\varepsilon_k$ and respectively $\nu^\varepsilon$ their expectations. The Kac-Rice formula implies that

$$\nu_k = \frac{1}{(2\pi)^{\frac{m}{2}}} \rho^\varepsilon_k(p) |dV_{h^\varepsilon}(p)|,$$

where

$$\rho^\varepsilon_k(p) = \frac{1}{\sqrt{\det S(v^\varepsilon(p))}} E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| |d\nu^\varepsilon(p) = 0, \text{ind Hess}^\varepsilon_p(v^\varepsilon) = k \right)$$

$$= (-1)^k E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| |d\nu^\varepsilon(p) = 0, \text{ind Hess}^\varepsilon_p(v^\varepsilon) = k \right).$$

As shown in [1, Eq. (12.2.11)], the Gaussian random variables $\text{Hess}^\varepsilon_p(v^\varepsilon)$ and $d\nu^\varepsilon(p)$ are independent so that

$$\rho^\varepsilon_k(p) = (-1)^k E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| |\text{ind Hess}^\varepsilon_p(v^\varepsilon) = k \right).$$

Thus

$$\nu^\varepsilon = \frac{1}{(2\pi)^{\frac{m}{2}}} \sum_{k=0}^{m} (-1)^k \rho^\varepsilon_k(p) |dV_{h^\varepsilon}(p)|,$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \sum_{k=0}^{m} E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| |\text{ind Hess}^\varepsilon_p(v^\varepsilon) = k \right) |dV_{h^\varepsilon}(p)|$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| \right) |dV_{h^\varepsilon}(p)|.$$

From the Poincaré-Hopf equality (4.2) we deduce

$$\chi(M) = \int_M \nu^\varepsilon(dp) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| \right) |dV_{h^\varepsilon}(p)|, \quad (4.3)$$

Observe that Hessian $\text{Hess}^\varepsilon(f)$ of a function $f$ can also be viewed as a double form

$$\text{Hess}^\varepsilon(f) \in \Omega^{1,1}(M).$$

In particular, $\text{Hess}^\varepsilon(v^\varepsilon)$ is a random $(1,1)$ double form and we have the following equality, [1, Lemma 12.2.1]

$$-2R_{h^\varepsilon} = E\left( \text{Hess}^\varepsilon(v^\varepsilon) \right)^2,$$

where $R_{h^\varepsilon}$ denotes the Riemann curvature tensor of the metric $h^\varepsilon$. On the other hand we have the equality [1, Eq. (12.3.1)]

$$\det \text{Hess}^\varepsilon(v^\varepsilon) = \frac{1}{m!} \text{tr} \text{Hess}^\varepsilon(v^\varepsilon)^m$$

(4.5)

Using (4.4), (4.5) and the algebraic identities in [1, Lemma 12.3.1] we conclude that

$$\frac{1}{(2\pi)^{\frac{m}{2}}} E\left( |\det \text{Hess}^\varepsilon_p(v^\varepsilon)| \right) = \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{\text{tr} \left(-R_{h^\varepsilon}^\varepsilon \right)}{m!}.$$ 

This proves (1.21). Using this equality in (4.3) we deduce

$$\chi(M) = \int_M e_{h^\varepsilon}(M),$$

i.e., we have proved the Gauss-Bonnet theorem for the metric $h^\varepsilon$. Now let $\varepsilon \to 0$. As we have mentioned, Theorem 1.7 implies that $h^\varepsilon \to g$ so in the limit, the above equality reduced to the Gauss-Bonnet theorem for the original metric $g$. 
APPENDIX A. JETS OF THE DISTANCE FUNCTION

Suppose that \((M, g)\) is a smooth, \(m\)-dimensional manifold, \(p_0 \in M, U\) is an open, geodesically convex neighborhood of \(p_0\) and \((x^1, \ldots, x^m)\) are normal coordinates on \(U\) centered at \(p_0\). We have a smooth function

\[ \eta : U \times U \to [0, \infty), \quad \eta(p, q) = \text{dist}_g(p, q)^2. \]

We want to investigate the partial derivatives of \(\eta\) at \((p_0, p_0)\). Using the above normal coordinates we regard \(\eta\) as a function \(\eta = \eta(x, y)\) defined in an open neighborhood of \((0, 0) \in \mathbb{R}^m \times \mathbb{R}^m\).

If \(f = f(t^1, \ldots, t^N)\) is a smooth function defined in a neighborhood of \(0 \in \mathbb{R}^N\) and \(k\) is a nonnegative integer, then we denote by \([f]_k\) the degree \(k\)-homogeneous part in the Taylor expansion of \(f\) at 0, i.e.,

\[ [f]_k = \frac{1}{k!} \sum_{|\alpha|=k} \partial^\alpha f |_{t=0} t^\alpha \in \mathbb{R}[t^1, \ldots, t^N]. \]

In the coordinates \((x^i)\) the metric \(g\) has the form (using Einstein’s summation convention throughout)

\[ g = g_{ij} dx^i dx^j, \]

where \(g_{ij}\) satisfy the estimates [18, Cor. 9.8]

\[ g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikj\ell}(0) x^i x^j + O(|x|^3). \] \hspace{1cm} (A.1)

We deduce that

\[ g^{ij} = \delta_{ij} + \frac{1}{3} R_{ikj\ell}(0) x^i x^j + O(|x|^3). \] \hspace{1cm} (A.2)

The function \(\eta\) satisfies a Hamilton-Jacobi equation, [29, p. 171],

\[ g^{k\ell} \partial_{x^k} \eta(x, y) \partial_{x^\ell} \eta(x, y) = 4\eta(x, y), \quad \forall x, y. \] \hspace{1cm} (A.3)

Moreover, \(\eta\) satisfies the obvious symmetry conditions

\[ \eta(x, y) = \eta(y, x), \quad \eta(0, x) = \eta(x, 0) = |x|^2 := \sum_{i=1}^m (x^i)^2. \] \hspace{1cm} (A.4)

As shown in [7, Lemma 2.2] we have

\[ [\eta]_2 = |x - y|^2 = \sum_{i=1}^m (x^i - y^i)^2. \] \hspace{1cm} (A.5)

The symmetries (A.4) suggest the introduction of new coordinates \((u, v)\) on \(U \times U\),

\[ u_i = x^i - y^i, \quad v_j = x^j + y^j. \]

Then

\[ x^i = \frac{1}{2}(u_i + v_i), \quad y^j = \frac{1}{2}(v_j - u_j), \quad \partial_{x^i} = \partial_{u_i} + \partial_{v_i}. \]

The equality (A.2) can be rewritten as

\[ g^{k\ell}(x) = \delta^{k\ell} + \frac{1}{12} \sum_{i,j} R_{ikj\ell}(u_i + v_i)(u_j + v_j) + O(3). \] \hspace{1cm} (A.6)

The symmetry relations (A.4) become

\[ \eta(u, v) = \eta(-u, v), \quad \eta(u, u) = |u|^2, \] \hspace{1cm} (A.7)

while (A.5) changes to

\[ [\eta]_1 = 0, \quad [\eta]_2 = |u|^2. \] \hspace{1cm} (A.8)
The equality (A.3) can be rewritten
\[
\sum_{k,l} g^{kl}(x) \left( \eta'_{uk} + \eta'_{vk} \right) \left( \eta'_{\ell u} + \eta'_{\ell v} \right) = 4\eta. \tag{A.9}
\]
Note that
\[
[A_k]_0 = [A_\ell]_0 = [g^{k,\ell}]_1 = 0, \tag{A.10}
\]
while (A.8) implies that
\[
[A_k]_1 = 2u_k. \tag{A.11}
\]
We deduce
\[
4[\eta]_3 = \sum_{k,\ell} [g^{kl}]_1 \left( [A_k]_1 [A_\ell]_2 + [A_k]_2 [B_\ell]_1 \right) = \sum_k 2[A_k]_2 [A_k]_1 = 4 \sum_k u_k [A_k]_2.
\]
We can rewrite this last equality as a differential equation for $[\eta]_3$ namely
\[
[\eta]_3 = \sum_k u_k (\partial u_k + \partial v_k) [\eta]_3.
\]
We set $P = [\eta]_3$ so that $P$ is a homogeneous polynomial of degree 3 in the variables $u, v$. Moreover, according to (A.7) the polynomial $P$ is even in $u$ and $P(u, u) = 0$. Thus $P$ has the form
\[
P = \sum_i C_i(u) v_i + P_0(v),
\]
where $C_i(u)$ is a homogeneous polynomial of degree 2 in the variables $u$, and $P_0(v)$ is homogeneous of degree 3 in the variables $v$.

We have
\[
\sum_k u_k \partial v_k P_2 = \sum_k C_k(u) u_k, \quad Q_1 := \sum_k u_k \partial v_k P_0, \quad \sum_k u_k \partial u_k P_0 = 0,
\]
and the classical Euler equations imply
\[
\sum_k u_k \partial u_k P_2 = 2P_2.
\]
We deduce
\[
P = 2P_2 + Q_3 + Q_1,
\]
where the polynomials $Q_3$ and $Q_1$ are odd in the variable $u$. Since $P$ is even in the variable $u$ we deduce
\[
Q_3 + Q_1 = 0,
\]
so that $P_2 + P_0 = P = 2P_2$. Hence $P_2 + P_0 = 0$ and thus
\[
[\eta]_3 = 0. \tag{A.11}
\]
In particular
\[
[A_k]_2 = 0, \quad \forall k. \tag{A.12}
\]
Going back to (A.9) and using (A.10) and (A.12) we deduce
\[ 4[\eta]_4 = \sum_{k,\ell} [g^{k\ell}]_2 [A_k]_1 [A_{\ell}]_1 + \sum_{k,\ell} [g^{k\ell}]_0 ([A_k]_1 [A_{\ell}]_3 + [A_k]_3 [A_{\ell}]_1) \]
\[ = 4 \sum_{k,\ell} [g^{k\ell}]_2 u_k u_{\ell} + 2 \sum_k u_k [A_k]_3. \]  
(A.13)

We set \( P = [\eta]_4 \). The polynomial \( P \) is homogeneous of degree 4 in the variables \( u, v \), and it is even in the variable \( u \). We can write \( P = P_0 + P_2 + P_4 \), where
\[ P_4 = \sum_k c_{ikj} u_i u_j u_k u_{\ell}, \quad P_2 = \sum_{i,j} Q_{ij} (u_i v_j), \]
and \( P_0 \) is homogeneous of degree 4 in the variables \( v \), \( Q_{ij} (u) \) is a homogeneous quadratic polynomial in the variables \( u \). We have
\[ \sum_k u_k [A_k]_3 = \sum_k u_k (\partial_{u_k} + \partial_{v_k}) P. \]

We have
\[ \sum_k u_k \partial_{u_k} P_{2\nu} = 2\nu P_{2\nu}, \quad \nu = 0, 1, 2. \]
\[ \sum_k u_k \partial_{v_k} P_4 = 0, \]
\[ \sum_k u_k \partial_{v_k} P_2 = \sum_{k,i,j} u_k Q_{ij} (\delta_{ki} v_j + \delta_{kj} v_i) = \sum_{k,j} (Q_{kj} u_k v_j + Q_{jk} v_j u_k) \]

Using these equalities in (A.13) we deduce
\[ 4P_4 + 4P_2 + 4P_0 = 4 \sum_{k,\ell} [g^{k\ell}]_2 u_k u_{\ell} + 4P_4 + 2P_2 + \sum_k u_k \partial_{v_k} P_0 \]
\[ + \sum_{k,j} (Q_{jk} + Q_{kj}) u_k v_j. \]

This implies \( P_0 = 0 \) so that \( P = P_4 + P_2 \), and we can then rewrite the above equality as
\[ P_2 = 2 \sum_{k,\ell} [g^{k\ell}]_2 u_k u_{\ell} + \sum_{k,j} (Q_{jk} + Q_{kj}) u_k v_j. \]  
(A.14)

Note that the equality \( r(u, u) = |u|^2 \) implies \( P(u, u) = 0 \) so that
\[ P_4(u) = P_4(u, u) = -P_2(u, u). \]

There fore it suffices to determine \( P_2 \). This can be achieved using the equality (A.6) in (A.14). We have
\[ 2 \sum_{k,\ell} [g^{k\ell}]_2 u_k u_{\ell} = \frac{1}{6} \sum_{i,j,k,\ell} R_{ikj\ell} (u_i + v_i) (u_j + v_j) u_k u_{\ell} \]
\[ = \frac{1}{6} \sum_{i,j} \left( \sum_{k,\ell} R_{ikj\ell} u_k u_{\ell} \right) v_i v_j + \sum_j S_j(u) v_j, \]
\[ \hat{Q}_{ij}(u) \]
where \( S_j(u) \) denotes a homogeneous polynomial of degree 3 in \( u \). The equality (A.14) can now be rewritten as

\[
\sum_{i,j} Q_{ij}(u)v_i v_j = \frac{1}{6} \sum_{i,j} \tilde{Q}_{ij}(u)v_i v_j + \sum_j S_j(u) v_j + \frac{1}{2} \sum_{k,j} \left( Q_{jk} + Q_{kj} \right) u_k v_j.
\]

From this we read easily

\[
Q_{ij}(u) = \frac{1}{6} \tilde{Q}_{ij}(u) = \frac{1}{6} \sum_{k,\ell} R_{ikj\ell} u_k u_\ell.
\]

This determines \( P_2 \).

\[
P_2(u, v) = \frac{1}{6} \sum_{i,j} \tilde{Q}_{ij}(u)v_i v_j.
\]

As we have indicated above \( P_2 \) determines \( P_4 \).

\[
P_4(u) = -P_2(u, u) = -\frac{1}{6} \sum_{i,j,k,\ell} R_{ikj\ell} u_i u_j u_k u_\ell.
\]

The skew symmetries of the Riemann tensor imply that \( P_4 = 0 \) so that

\[
[\eta]_4(u, v) = \frac{1}{6} \sum_{i,j} \tilde{Q}_{ij}(u)v_i v_j, \quad \tilde{Q}_{ij}(u) = \sum_{k,\ell} R_{ikj\ell} u_k u_\ell.
\]

**Example A.1.** Suppose that \( M \) is a surface, i.e., \( m = 2 \). Set

\[
K = R_{1212} = R_{2121} = -R_{1221}.
\]

Note that \( K \) is the Gaussian curvature of the surface. Then

\[
\tilde{Q}_{11} = \sum_{k,\ell} R_{1k1\ell} u_k u_\ell = Ku_2^2, \quad \tilde{Q}_{22} = \sum_{k,\ell} R_{2k2\ell} u_k u_\ell = Ku_1^2.
\]

\[
\tilde{Q}_{12} = \sum_{k,\ell} R_{1k2\ell} u_k u_\ell = -K u_1 u_2 = \tilde{Q}_{21}.
\]

Hence

\[
P_2(u, v) = \frac{K}{6} \left( u_2^2 v_1^2 + u_1^2 v_2^2 - 2u_1 u_2 v_1 v_2 \right) = \frac{K}{6} (u_1 v_2 - u_2 v_1)^2.
\]

\[\square\]

**APPENDIX B. SPECTRAL ESTIMATES**

As we have already mentioned, the correlation function

\[
\mathcal{E}_\varepsilon(p, q) = \sum_{k \geq 0} w_\varepsilon(\sqrt{k}) \Psi_k(p) \Psi_k(q)
\]

is the Schwartz kernel of the smoothing operator \( w_\varepsilon(\sqrt{\Delta}) \). In this appendix we present in some detail information about the behavior along the diagonal of this kernel as \( \varepsilon \to 0 \). We will achieve this by relying on the wave kernel technique pioneered by L. Hörmander, [20].

The fact that such asymptotics exist and can be obtained in this fashion is well known to experts; see e.g [11] or [31, Chap.XII]. However, we could not find any reference describing these asymptotics with the level of specificity needed for the considerations in this paper.
Theorem B.1. Suppose that \( w \in S(\mathbb{R}) \) is an even, nonnegative Schwartz function, and \((M, g)\) is a smooth, compact, connected \( m \)-dimensional Riemann manifold. We define
\[
\delta^\varepsilon: M \times M \to \mathbb{R}, \quad \delta^\varepsilon(p, q) = \sum_{k \geq 0} w(\varepsilon \sqrt{\lambda_k}) \Psi_k(p) \Psi_k(q),
\]
where \((\Psi_k)_{k \geq 1}\) is an orthonormal basis of \( L^2(M, g) \) consisting of eigenfunctions of \( \Delta_g \).

Fix a point \( p_0 \in M \) and normal coordinates at \( p_0 \) defined in an open neighborhood \( \Omega_0 \) of \( p_0 \). The restriction of \( \delta^\varepsilon \) to \( \delta^\varepsilon \) to \( \Omega_0 \times \Omega_0 \) can be viewed as a function \( \delta^\varepsilon(x, y) \) defined in an open neighborhood of \((0, 0)\) in \( \mathbb{R}^m \times \mathbb{R}^m \). Fix multi-indices \( \alpha, \beta \in (\mathbb{Z}_{\geq 0})^m \). Then
\[
\partial_x^\alpha \partial_y^\beta \delta^\varepsilon(x, y)|_{x=y=0} = \varepsilon^{-m+2|\alpha|+|\beta|} \int_{\mathbb{R}^m} w(|x|) x^{\alpha+\beta} dx + O(\varepsilon^2), \quad \varepsilon \to 0, \quad (B.1)
\]
where
\[
d(\alpha, \beta) := \left\lfloor \frac{\alpha + \beta}{2} \right\rfloor.
\]
Moreover, the constant implied by the symbol \( O(\varepsilon) \) in \((B.1)\) uniformly bounded with respect to \( p_0 \).

Proof. For the reader’s convenience and for later use, we go in some detail through the process of obtaining these asymptotics. We skip many analytical steps that are well covered in [22, Chap. 17] or [28].

Observe that for any smooth \( f : M \to \mathbb{R} \) we have
\[
w(\sqrt{\Delta})f = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{w}(t)e^{it\sqrt{\Delta}} f dt = \frac{1}{2\pi \varepsilon} \int_{\mathbb{R}} \hat{w}(t/\varepsilon) e^{it\sqrt{\Delta}} f dt.
\]
The Fourier transform \( \hat{w}(t) \) is a Schwartz function so \( \hat{w}(t/\varepsilon) \) is really small for \( t \) outside a small interval around 0 and \( \varepsilon \) sufficiently small. Thus a good understanding of the kernel of \( e^{it\sqrt{\Delta}} \) for \( t \) sufficiently small could potentially lead to a good understanding of the Schwartz kernel of \( w(\sqrt{\Delta}) \).

Fortunately, good short time asymptotics for the wave kernel are available. We will describe one such method going back to Hadamard, [19, 29]. Our presentation follows closely [22, §14.4] but we also refer to [28] where we have substantially expanded the often dense presentation in [22].

To describe these asymptotics we need to introduce some important families homogeneous generalized functions (or distributions) on \( \mathbb{R} \). We will denote by \( C^{-\infty}(\Omega) \) the space of generalized functions on the smooth manifold \( \Omega \), defined as the dual of the space compactly supported 1-densities, [17, Chap. VI].

For any \( a \in \mathbb{C} \), \( \Re a > 1 \) we define \( \chi_+^a : \mathbb{R} \to \mathbb{R} \) by
\[
\chi_+^a(x) = \frac{1}{\Gamma(a + 1)} x_+^a, \quad x_+ = \max(x, 0).
\]
Observe that we have the following equality in the sense of distributions
\[
\frac{d}{dx} \chi_+^{a+1} = \chi_+^a(x), \quad \Re a > 1.
\]
We can use this to define for any \( a \in \mathbb{C} \)
\[
\chi_+^a := \frac{d^k}{dx^k} \chi_+^{a+k} \in C^{-\infty}(\mathbb{R}), \quad k > 1 - \Re a.
\]
For \( \Re a > 0 \) we denote by \( |\chi|^a \) the generalized function defined by the locally integrable function
\[
|\chi|^a(x) = \frac{1}{\Gamma(\frac{a+1}{2})} |x|^a.
\]
The correspondence $a \mapsto |\chi|^a$ is a holomorphic map \( \{ \text{Re } z > 0 \} \to C^{-\infty}(\mathbb{R}) \) which admits a holomorphic extension to the whole complex plane, [15, Chap. 1, 28]. This is a temperate generalized function, and its Fourier transform is given by, [15, 28],

$$|\chi|^a(\xi) = \sqrt{\pi}2^{a+1}|\chi|^{-(a+1)}(\xi), \quad \forall a \in \mathbb{C}. \quad (B.3)$$

Denote by $K_t(x, y)$ the Schwartz kernel of $e^{it\sqrt{\Delta}}$. We then have the following result [22, §17.4] or [28].

**Theorem B.2.** Set $n := m + 1$, and let

$$\eta(x, y) = \text{dist}_g(x, y)^2, \quad x, y \in M.$$  

There exists a positive constant $c > 0$, smaller than the injectivity radius of $(M, g)$, such that for $\text{dist}_g(x, y) < c$ we have the following asymptotic expansion as $t \to 0$

$$K_t(p, q) \sim \sum_{k=1}^{\infty} U_k(p, q)d_m(2k)\mathcal{H}_k(t, p, q), \quad |t| < c, \quad (B.4)$$

where for $\text{Re } a > 0$ we have

$$\mathcal{H}_a(t, p, q) = \partial_t \left( \chi_+^{a-\frac{n}{2}}(t_+^2 - \eta(p, q)) - \chi_+^{a-\frac{n}{2}}(t_-^2 - \eta(p, q)) \right),$$

$$d_m(2a) = \frac{\Gamma\left(\frac{2a+1}{2}\right)}{\pi^{\frac{m}{2}}\Gamma(2a)}.$$  

Let us explain in more detail the meaning of the above result. The functions $U_k$ are smooth functions defined in the neighborhood $\text{dist}_g(p, q) < c$ of the diagonal in $M \times M$. For fixed $q$, the functions $p \mapsto V_k(p) := U_k(p, q)$ are determined as follows.

Fix normal coordinates $x$ at $q$, set $|g| := \det(g_{ij})$, and

$$h(x) := -\frac{1}{2}g(\nabla \log |g|, x) = -\frac{1}{2} \sum_{j,k} g^{jk}x^j\partial_k \log |g|.$$  

Then $V_k(x)$ are the unique solutions of the differential recurrences

$$\begin{aligned}
V_1(0) &= 1, \quad 2x \cdot \nabla V_1 = hV_1, \quad |x| < c, \\
\frac{1}{k}x \cdot \nabla V_{k+1} + \left(1 - \frac{1}{2k}h\right)V_{k+1} &= -\Delta g V_k, \quad V_{k+1}(0) = 0, \quad |x| < c, \quad k \geq 1.
\end{aligned} \quad (B.5)$$

We have the following important equality

$$\lim_{\text{dist}_g(p, q) \to 0} \mathcal{H}_a(t, p, q) = |\chi|^{2a-2-m}(t), \quad \forall a \in \mathbb{C}. \quad (B.7)$$

The asymptotic estimate (B.4) signifies that for any positive integer $\mu$ there exists a positive integer $N(\mu)$ so that for any $N \geq N(\mu)$ the tail

$$\mathcal{F}_N(t, p, q) := K_t(p, q) - \sum_{k=1}^{N} U_k(p, q)d_m(2k)\mathcal{H}_k(t, p, q)$$

belongs to $C^{\mu}( (-c, c) \times M \times M )$ and satisfies the estimates

$$\|\partial_t^j \mathcal{F}_N(t, -,-)\|_{C^{\mu-j}(M \times M)} \leq C|t|^{2N-n-1-\mu}, \quad |t| \leq c, \quad j \leq \mu, \quad N \geq N(\mu). \quad (B.8)$$

Fix a point $p_0 \in M$ and normal coordinates at $p_0$ defined in a neighborhood $\mathcal{O}_0$ of $p_0$. Then we can identify a point $(p, q) \in \mathcal{O}_0 \times \mathcal{O}_0$ with a point $(x, y)$ in a neighborhood of $(0, 0)$ in $\mathbb{R}^m \times \mathbb{R}^m$. 
Lemma B.3.

Using (B.2) we deduce

\[ \partial_x^\alpha \partial_y^\beta \sigma^\varepsilon(x, y)|_{x=y} = \frac{1}{\varepsilon} \left\langle \partial_x^\alpha \partial_y^\beta K_t(x, y)|_{x=y}, \hat{\omega} \left( \frac{t}{\varepsilon} \right) \right\rangle, \quad \text{(B.9)} \]

Choose an even, nonnegative cutoff function \( \rho \in C_0^\infty(\mathbb{R}) \) such that

\[ \rho(t) = \begin{cases} 1, & |t| \leq \frac{c}{2}, \\ 0, & |t| \geq \frac{c}{2}, \end{cases} \]

where \( c > 0 \) is the constant in Theorem B.2. Then

\[ \partial_x^\alpha \partial_y^\beta \sigma^\varepsilon(x, y)|_{x=y} = \frac{1}{\varepsilon} \left\langle K_t^{\alpha, \beta}, \rho(t) \hat{\omega} \left( \frac{t}{\varepsilon} \right) \right\rangle + \frac{1}{\varepsilon} \left\langle K_t^{\alpha, \beta}, (1 - \rho(t)) \hat{\omega} \left( \frac{t}{\varepsilon} \right) \right\rangle. \]

Let us observe that that for any \( N > 0 \)

\[ \frac{1}{\varepsilon} \left\langle K_t^{\alpha, \beta}, (1 - \rho(t)) \hat{\omega} \left( \frac{t}{\varepsilon} \right) \right\rangle = O(\varepsilon^N) \quad \text{as} \quad \varepsilon \to 0 \]

Thus

\[ \forall N > 0 \quad \partial_x^\alpha \partial_y^\beta \sigma^\varepsilon(x, y)|_{x=y} \sim \frac{1}{\varepsilon} \left\langle K_t^{\alpha, \beta}, \rho(t) \hat{\omega} \left( \frac{t}{\varepsilon} \right) \right\rangle + O(\varepsilon^N), \quad \varepsilon \to 0. \quad \text{(B.10)} \]

On the other hand

\[ \partial_x^\alpha \partial_y^\beta K_t(x, y) \sim \sum_{k=1}^{\infty} d_m(2k) \partial_x^\alpha \partial_y^\beta \{ U_k(x, y) \hat{\mathcal{H}}_k(t, x, y) \}. \quad \text{(B.11)} \]

Recall that

\[ d(\alpha, \beta) = \left\lfloor \frac{1}{2} |\alpha + \beta| \right\rfloor. \]

One can show (see [7, 28])

\[ \partial_x^\alpha \partial_y^\beta K_t(x, y)|_{x=y=0} \sim \sum_{k=0}^{\infty} A_{m, \alpha, \beta, k} |\chi|^{-m-2d(\alpha, \beta)+2k} t^k, \quad \text{(B.12)} \]

where \( A_{m, \alpha, \beta, 0} \) is a universal constant depending only on \( m, \alpha, \beta \), which is equal to 0 if \( |\alpha + \beta| \) is odd.

**Lemma B.3.** (a) For any \( r \in \mathbb{Z} \) and any \( N > 0 \) we have

\[ \frac{1}{\varepsilon} \langle |\chi|^r, \rho \hat{\omega}_{\varepsilon} \rangle = \varepsilon^r \left( \langle |\chi|^r, \hat{\omega} \rangle + O(\varepsilon^N) \right) \quad \text{as} \quad \varepsilon \to 0. \]

(b) For every positive integer \( r \) we have

\[ \langle |\chi|^{-r}, \hat{\omega} \rangle = \frac{\sqrt{\pi} 2^{1-r}}{\Gamma\left( \frac{r}{2} \right)} \int_{\mathbb{R}} |\tau|^{r-1} w(\tau) d\tau. \]

**Proof.** (a) For transparency we will use the integral notation for the pairing between a generalized function and a test function. We have

\[ \langle |\chi|^r, \eta \hat{\omega}_{\varepsilon} \rangle = \frac{1}{\varepsilon} \int_{\mathbb{R}} |\chi|^r(t) \rho(t) \hat{\omega}(t/\varepsilon) dt = \int_{\mathbb{R}} |\chi|^r(\varepsilon t) \rho(\varepsilon t) \hat{\omega}(t) dt \]

\[ = \varepsilon^r \int_{\mathbb{R}} |\chi|^r(t) \rho(\varepsilon t) \hat{\omega}(t) dt = \varepsilon^r \langle |\chi|^r, \rho_{\varepsilon} \hat{\omega} \rangle, \quad \rho_{\varepsilon}(t) = \rho(\varepsilon t). \]
Now observe that \( \rho_\varepsilon \tilde{w} - \tilde{w} = \tilde{w}(\rho_\varepsilon - 1) \to 0 \) in \( S(\mathbb{R}) \). More precisely for \( k \geq 0 \) we have
\[
\frac{\partial^k}{\partial \varepsilon^k}(\rho_\varepsilon - 1) = O(\varepsilon^N t^N) \quad \text{as} \; \varepsilon \to 0.
\]
This implies that
\[
\langle |x|^r, \tilde{w}(\rho_\varepsilon - 1) \rangle = O(\varepsilon^N) \quad \text{as} \; \varepsilon \to 0,
\]
so that
\[
\langle |x|^r, \rho_\varepsilon \tilde{w} \rangle = \langle |x|^r, \tilde{w} \rangle + \langle |x|^r, \tilde{w}(\rho_\varepsilon - 1) \rangle = \langle |x|^r, \tilde{w} \rangle + O(\varepsilon^N) \quad \text{as} \; \varepsilon \to 0.
\]
(b) We have
\[
\langle |x|^{-r}, \tilde{w} \rangle = \langle \left| \frac{|x|}{\varepsilon} \right|^{-r}, w \rangle \quad \text{(B.3)}
\]

\[
= \frac{\sqrt{\pi} 2^{1-r}}{\Gamma(\frac{r}{2})} \int_{\mathbb{R}} |\tau|^{r-1} w(\tau) d\tau.
\]

Using (B.10) and the above lemma we deduce
\[
\partial_x^\alpha \partial_y^\beta \varepsilon^\varepsilon(x, y)|_{x=y} = D_{m, \alpha, \beta} \varepsilon^{-m-2d(\alpha, \beta)} + O\left(\varepsilon^{-m-2d(\alpha, \beta)+2}\right) \quad \text{as} \; \varepsilon \to 0, 
\]
where \( D_{m, \alpha, \beta} \) is a universal constant that depends only on \( m, \alpha, \beta \) which is \( 0 \) if \( |\alpha + \beta| \) is odd,
\[
D_{m, \alpha, \beta} = A_{m, \alpha, \beta, 0} \frac{\sqrt{\pi} 2^{1-r}}{\Gamma(\frac{r}{2})} \int_{\mathbb{R}} |\tau|^{r-1} w(\tau) d\tau, \quad r = m + 2d(\alpha, \beta). 
\]

To determine the constant \( D_{m, \alpha, \beta} \) it suffices to compute it for one particular \( m \)-dimensional Riemann manifold. Assume that \( (M, g) \) is the torus \( T^m \) equipped with the flat metric
\[
g = \sum_{i=1}^{m} (d\theta^i)^2, \quad 0 \leq \theta^i \leq 2\pi.
\]
The eigenvalues of the corresponding Laplacian \( \Delta_m \) are
\[
|\vec{k}|^2, \quad \vec{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m.
\]
Denote by \( \prec \) the lexicographic order on \( \mathbb{Z}^m \). For \( \vec{\theta} = (\theta^1, \ldots, \theta^m) \in \mathbb{R} \) and \( \vec{k} \in \mathbb{Z}^m \) we set
\[
\langle \vec{k}, \vec{\theta} \rangle := \sum_{j=1}^{m} k_j \theta^j.
\]
A real orthonormal basis of \( L^2(T^m) \) is given by the functions
\[
\Psi_{\vec{k}}(\vec{\theta}) = \frac{1}{(2\pi)^{\frac{m}{2}}} \begin{cases} 
\frac{1}{2^\frac{m}{2}} \sin(\vec{k}, \vec{\theta}), & \vec{k} \succ \vec{\theta} \\
\frac{1}{2^\frac{m}{2}} \cos(\vec{k}, \vec{\theta}), & \vec{k} \prec \vec{\theta} \\
1, & \vec{k} = \vec{\theta} \end{cases}
\]

Then
\[
\varepsilon^\varepsilon(\vec{\theta}, \vec{\varphi}) = \frac{1}{(2\pi)^m} \sum_{\vec{k} \in \mathbb{Z}^m} w(\varepsilon|\vec{k}|) e^{i(\vec{k}, \vec{\theta} - \vec{x})},
\]
so that
\[
\frac{\partial^\alpha}{\partial_{\vec{\theta}}^\alpha} \frac{\partial^\beta}{\partial_{\vec{\varphi}}^\beta} \varepsilon^\varepsilon(\vec{\theta}, 0) = \frac{2^{\frac{m}{2} - |\alpha - \beta|}}{(2\pi)^{\frac{m}{2}}} \sum_{\vec{k} \in \mathbb{Z}^m} w(\varepsilon|\vec{k}|) \vec{k}^{\alpha + \beta} e^{i(\vec{k}, \vec{\theta})}.
\]
Define
\[ W_m, u_\varepsilon : \mathbb{R}^m \to \mathbb{R}, \quad W_m(x) = w(|x|), \quad u_\varepsilon(x) = W_m(\varepsilon x)x^{\alpha+\beta}. \]

Using the Poisson summation formula [21, §7.2] we deduce
\[
\partial_\theta^\alpha \partial_\beta^\beta \delta^\xi (0, 0) = \frac{i^{\lfloor \alpha \rfloor - \lfloor \beta \rfloor}}{(2\pi)^m} \sum_{\nu \in \mathbb{Z}^m} \tilde{u}_\varepsilon(2\pi \nu).
\]

Observe that
\[
\tilde{u}_\varepsilon(\xi) = \int_{\mathbb{R}^m} e^{-i(\xi, x)} w(\varepsilon|x|)x^{\alpha+\beta} \, dx = (i\partial_\xi)^{\alpha+\beta} \left( \int_{\mathbb{R}^m} e^{-i(\xi, x)} W_m(\varepsilon x) \, dx \right)
\]
\[= \varepsilon^{-m}(i\partial_\xi)^{\alpha+\beta} \left( \int_{\mathbb{R}^m} e^{-i(\frac{\xi}{\varepsilon}, y)} W_m(y) \, dy \right) = \varepsilon^{-m}(i\partial_\xi)^{\alpha+\beta} \tilde{W}_m(\frac{1}{\varepsilon} \xi). \]

Hence
\[
\partial_\theta^\alpha \partial_\beta^\beta \delta^\xi (\theta, 0) = \frac{i^{\lfloor \alpha \rfloor - \lfloor \beta \rfloor}}{(2\pi)^m} \sum_{\nu \in \mathbb{Z}^m} \left\{ (i\partial_\xi)^{\alpha+\beta} \tilde{W}_m(\frac{1}{\varepsilon} \xi) \right\}_{\xi=2\pi \nu}.
\]

As \( \varepsilon \to 0 \) we have
\[
\partial_\theta^\alpha \partial_\beta^\beta \delta^\xi (0, 0) = \varepsilon^{-m-|\alpha+\beta|} \frac{i^{\lfloor \alpha \rfloor - \lfloor \beta \rfloor}}{(2\pi)^m} \left( (i\partial_\xi)^{\alpha+\beta} \tilde{W}_m(0) + O(\varepsilon^N) \right), \quad \forall N.
\]

Now observe that
\[ (i\partial_\xi)^{\alpha+\beta} \tilde{W}_m(0) = \int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta} \, dx. \]

so that
\[
\partial_\theta^\alpha \partial_\beta^\beta \delta^\xi (0, 0) = \varepsilon^{-m-|\alpha+\beta|} \frac{i^{\lfloor \alpha \rfloor - \lfloor \beta \rfloor}}{(2\pi)^m} \left( \int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta} \, dx + O(\varepsilon^N) \right), \quad \forall N.
\]

This shows that
\[
D_{m, \alpha, \beta} = \frac{i^{\lfloor \alpha \rfloor - \lfloor \beta \rfloor}}{(2\pi)^m} \int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta} \, dx.
\]

This completes the proof of Theorem B.1.

\[ \square \]

**Remark B.4.** Note that
\[
\int_{\mathbb{R}^m} w(|x|)x^{\alpha+\beta} \, dx = \left( \int_{|x|=1} x^{\alpha+\beta} dA(x) \right) \left( \int_0^\infty w(r)r^{m+|\alpha+\beta|-1} \, dr \right).
\]

On the other hand, according to [25, Lemma 9.3.10] we have
\[
\int_{|x|=1} x^{\alpha+\beta} dA(x) = Z_{m, \alpha, \beta} := \left\{ \frac{2\prod_{i=1}^{k} \Gamma(\frac{\alpha_i+\beta_i+1}{2})}{\Gamma(\frac{m+|\alpha+\beta|}{2})} \right\}, \quad (\alpha, \beta) \in (2\mathbb{Z}_{\geq 0})^m,
\]
\[
\text{otherwise.}
\]

We can now rewrite (B.16) as
\[
D_{m, \alpha, \beta} = \varepsilon^{-m-|\alpha+\beta|} \frac{i^{\lfloor \alpha \rfloor - \lfloor \beta \rfloor}}{(2\pi)^m} Z_{m, \alpha, \beta} I_{m, \alpha, \beta}(w).
\]

\[ \square \]
Theorem B.5. Fix a point \( p \in M \) and normal coordinates \((x^i)\) near \( p \). For \( i \neq j \) we denote by \( K_{ij}(p) \) the sectional curvature of \( g \) at \( p \) along the plane spanned by \( \partial_{x^i}, \partial_{x^j} \). For any multi-indices \( \alpha, \beta \in (\mathbb{Z}_{>0})^m \) we set
\[
\varepsilon_{\alpha, \beta}^\varepsilon := \partial_x^\varepsilon \partial_y^\varepsilon (x, y)|_{x=y=0}.
\]
Then there exists a universal constant \( Z_m \) that depends only on the dimension of \( M \) and the weight \( w \) such that
\[
\varepsilon_{ii;jj} - \varepsilon_{ij;ij} = Z_m K_{ij}(p) \varepsilon^{-m-2}(1 + O(\varepsilon^2)) \quad \text{as } \varepsilon \to 0. \tag{B.19}
\]

Proof. Using (B.12) we deduce
\[
\varepsilon_{ii;jj} - \varepsilon_{ij;ij} \sim \frac{1}{\varepsilon} \left( K_{i}^{ii,jj} - K_{i}^{ij,ij}, \eta(t) \bar{w}(t) \right) + O(\varepsilon^N), \quad \varepsilon \to 0 \tag{B.20}
\]

On the other hand from (B.9) we conclude
\[
K_{i}^{ii,jj} - K_{i}^{ij,ij} \sim \sum_{k=1}^{\infty} d_{m}(2k) \left( \partial_x^{2} \partial_y^{2} - \partial_x^{2} \partial_y^{2} \right) \{ U_k(x, y) \mathcal{H}_k(t, x, y) \} |_{x=y=0} \tag{B.21}
\]

To investigate the above asymptotics we use the technology in [28].

Let us introduce some notations. For a positive integer \( k \) we denote by \( \partial^k \) a generic mixed-partial derivative of order \( k \) in the variables \( x^i, y^j \). We denote by \( \partial^k \eta \) the collection of \( k \)-th order derivatives of \( \eta(x, y) \). \( P_i(X) \) will denote a homogeneous polynomial of degree \( i \) in the variables \( X \), while \( P_k(X)P_{l}(Y) \) will denote a polynomial which is homogeneous of degree \( k \) in the variables \( X \) and of degree \( l \) in the variables \( Y \). We then have the equalities
\[
\mathcal{H}_a = P_1(\partial \eta) \mathcal{H}_{a-1}, \tag{B.22}
\]
\[
\partial^2 \mathcal{H}_a = P_2(\partial \eta) \mathcal{H}_{a-2} + P_1(\partial^2 \eta) \mathcal{H}_{a-1}, \tag{B.23}
\]
\[
\partial^3 \mathcal{H}_a = P_3(\partial \eta) \mathcal{H}_{a-3} + P_1(\partial \eta)^2 \mathcal{H}_{a-2} + P_1(\partial^3 \eta) \mathcal{H}_{a-1}, \tag{B.24}
\]
\[
\partial^4 \mathcal{H}_a = P_4(\partial \eta) \mathcal{H}_{a-4} + (P_2(\partial \eta)^2 \mathcal{H}_{a-3} + (P_2(\partial^2 \eta) \mathcal{H}_{a-2} + P_1(\partial^4 \eta) \mathcal{H}_{a-1}. \tag{B.25}
\]

To simplify the presentation we will assume that in (B.19) we have \( i = 1, j = 2 \). Also, we will denote by \( O(1) \) a function \( f(x, y) \) such that \( f(x, y)|_{x=y=0} = 0 \). The computations in Section A show that for \( x = 0 \) terms of the form \( P_2(\partial \eta) \) and \( P_{k}(\partial^3 \eta) \) are \( O(1) \). In particular, the above equalities show that the 1st and 3rd order derivatives of \( \mathcal{H}_a \) are \( O(1) \). We have
\[
\partial_{x^1}^2 \partial_{x^2}^2 (U_k \mathcal{H}_k) = \partial_{x^2}^2 \left( \left( \partial_{x^2}^2 U_k \right) \mathcal{H}_k + 2 \partial_{x^2} U_k \partial_{y^2} \mathcal{H}_k + U_k \partial_{y^2} \mathcal{H}_k \right) \tag{B.26}
\]
\[
\partial_{x^1}^2 \partial_{y^1}^2 (U_k \mathcal{H}_k) = \partial_{x^1}^2 \left( \left( \partial_{y^1}^2 U_k \right) \mathcal{H}_k + 2 \partial_{y^1} U_k \partial_{x^1} \mathcal{H}_k + U_k \partial_{x^1} \mathcal{H}_k \right) \tag{B.27}
\]
He deduce that
\[
\left( \partial^2_{x^2} \partial^2_{y^2} - \partial^2_{x^1 x^2} \partial^2_{y^1 y^2} \right) (U_k \mathcal{K}_k)_{x=y=0} = \sum_{j=0}^{4} T^j_k \mathcal{K}^{k-j} \big|_{x=y=0},
\]
where the coefficients \( T^j_k \) are polynomials in the derivatives of \( U_k \) and \( \eta \) at \((x, y) = (0, 0)\). Using (B.22)-(B.25) we deduce
\[
T^4_k = T^3_k = 0.
\]
Moreover, the terms that appear in \( T^2_k \) appear only when we take forth order derivatives of \( \mathcal{K}_k \).

Upon inspecting (B.26) and (B.27) we see that the 4th order derivatives of \( \mathcal{K}_k \) are multiplied by \( U_k \).

According to (B.6) the function \( U_k \) is \( O(1) \) if \( k > 1 \). Hence \( T^2_k = 0 \) for \( k > 1 \). We deduce
\[
K^{ij, jj}_k - K^{ij, jj}_t \sim \sum_{k=1}^{\infty} d_m(2k) \left( T^0_k \mathcal{K}_k + T^1_k \mathcal{K}_{k-1} + T^2_k \mathcal{K}_{k-2} \right) \big|_{x=y=0}
\]
\[
= B_{-1} \mathcal{K}_{-1} \big|_{x=y=0} + B_0 \mathcal{K}_0 \big|_{x=y=0} + B_1 \mathcal{K}_1 \big|_{x=y=0} + \cdots,
\]
where
\[
B_{-1} = d_m(2) T^2_1, \quad B_0 = d_m(2) T^1_1, \quad B_1 = d_m(2) T^0_1 + d_m(4) T^1_2, \ldots.
\]
The term \( B_{-1} \) can be alternatively described as
\[
B_{-1} = A_{m, ii; jj, 0} - A_{m, ij; ij, 0},
\]
where the coefficients \( A_{m, \alpha, \beta, 0} \) are defined as in (B.12). Using (B.14) and (B.16) we deduce
\[
B_{-1} = 0.
\]

To compute \( T^1_1 \) we observe first that
\[
\eta(x - y) = \sum_i (x^i - y^i)^2 + \text{higher order terms}.
\]
Using (B.23) we can simplify (B.26) and (B.27) in the case \( k = 1 \) as follows.
\[
\partial^2_{x^1} \partial^2_{y^2} (U_1 \mathcal{K}_1) = \left( \partial^2_{x^1} \partial^2_{y^2} U_1 \mathcal{K}_1 + U_1 \partial^2_{x^1} \partial^2_{y^2} \mathcal{K}_1 + O(1) \right),
\]
\[
\partial^2_{x^1 x^2} \partial^2_{y^1 y^2} (U_1 \mathcal{K}_1) = \left( \partial^2_{x^1 x^2} \partial^2_{y^1 y^2} U_1 \mathcal{K}_1 + \partial^2_{x^1 y^1} U_1 \partial^2_{x^2 y^2} \mathcal{K}_1 + O(1) \right) + 2 \left( \partial^2_{x^1 y^1} U_1 + \partial^2_{x^2 y^2} U_1 \right) \big|_{(0,0)}.
\]

Using (B.23), (B.25) and (B.28) we deduce that
\[
T^1_1 = \left( \partial^2_{x^1} \partial^2_{y^2} - \partial^2_{x^1 x^2} \partial^2_{y^1 y^2} \right) \eta \big|_{(0,0)} + 2 \left( \partial^2_{x^1 y^1} U_1 + \partial^2_{x^2 y^2} U_1 \right) \big|_{(0,0)}.
\]
Using the transport equation (B.5) we obtain as in [10, VI.3] that \( U_1 \) coincides with the function \( \varphi(x, y) \) in [10, VI.3 Eq.(33)] or the function \( u_0(x, y) \) in [6, p. 380]. For our purposes an explicit description of \( U_1 \) is not needed. All we care is that
\[
U_1(x, y) = U_1(y, x), \quad U_1(x, x) \equiv 1.
\]
These conditions imply that the Hessian of \( U_1(x, y) \) at \( (0, 0) \) is a quadratic form in the variables \( u_i = (x^i - y^i) \) so that
\[
\partial^2_{x^1} U_1(0, 0) + \partial^2_{x^1 y^1} U_1(0, 0) = \partial^2_{y^2} U_1(0, 0) + \partial^2_{x^2 y^2} U_1(0, 0) = 0.
\]
Hence
\[
T^1_1 = \left( \partial^2_{x^1} \partial^2_{y^2} - \partial^2_{x^1 x^2} \partial^2_{y^1 y^2} \right) \eta \big|_{(0,0)}.
\]
Using (A.17) we conclude that
\[ T_1^i = Z R_{1212} = Z K_{12}(p), \]
where \( Z \) is a universal constant, independent of \((M, g)\). Hence
\[ K_{iijj}^t - K_{ijij}^t \sim d_m(2) Z K_{12}(p) H_0^{(x=y=0)} + \sum_{k \geq 1} B_k H_{k|x=y=0}. \]
The equality (B.19) now follows from the above equality by using (B.20), (B.7) and Lemma B.3. \( \square \)

APPENDIX C. GAUSSIAN MEASURES AND GAUSSIAN VECTORS

For the reader’s convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [8]. A Gaussian measure on \( \mathbb{R} \) is a Borel measure \( \gamma_{\mu,v} \), \( v \geq 0, m \in \mathbb{R} \), of the form
\[ \gamma_{\mu,v}(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(x-\mu)^2}{2v}} dx. \]
The scalar \( \mu \) is called the mean, while \( v \) is called the variance. We allow \( v \) to be zero in which case \( \gamma_{\mu,0} = \delta_\mu \) is the Dirac measure on \( \mathbb{R} \) concentrated at \( \mu \).

For a real valued random variable \( X \) we write \( X \in \mathcal{N}(\mu, v) \) (C.1)
if the probability measure of \( X \) is \( \gamma_{\mu,v} \).

Suppose that \( V \) is a finite dimensional vector space. A Gaussian measure on \( V \) is a Borel measure \( \gamma \) on \( V \) such that, for any \( \xi \in V^\vee \), the pushforward \( \xi_*(\gamma) \) is a Gaussian measure on \( \mathbb{R} \),
\[ \xi_*(\gamma) = \gamma_{\mu(\xi),v(\xi)}. \]
One can show that the map \( V^\vee \ni \xi \mapsto \mu(\xi) \in \mathbb{R} \) is linear, and thus can be identified with a vector \( \mu_\gamma \in V \) called the barycenter or expectation of \( \gamma \) that can be alternatively defined by the equality
\[ \mu_\gamma = \int_V v d\gamma(v). \]
Moreover, there exists a nonnegative definite, symmetric bilinear map
\[ \Sigma : V^\vee \times V^\vee \to \mathbb{R} \] such that \( v(\xi) = \Sigma(\xi, \xi), \forall \xi \in V^\vee. \)
The form \( \Sigma \) is called the covariance form and can be identified with a linear operator \( S : V^\vee \to V \) such that
\[ \Sigma(\xi, \eta) = \langle \xi, S\eta \rangle, \forall \xi, \eta \in V^\vee, \]
where \( \langle -,- \rangle : V^\vee \times V \to \mathbb{R} \) denotes the natural bilinear pairing between a vector space and its dual. The operator \( S \) is called the covariance operator and it is explicitly described by the integral formula
\[ \langle \xi, S\eta \rangle = \Sigma(\xi, \eta) = \int_V \langle \xi, v - \mu_\gamma \rangle \langle \eta, v - \mu_\gamma \rangle d\gamma(v). \]
The Gaussian measure is said to be nondegenerate if \( \Sigma \) is nondegenerate, and it is called centered if \( \mu = 0 \). A Gaussian measure on \( V \) is uniquely determined by its covariance form and its expectation.

Example C.1. Suppose that \( U \) is an \( n \)-dimensional Euclidean space with inner product \( (-,-) \). We use the inner product to identify \( U \) with its dual \( U^\vee \). If \( A : U \to U \) is a symmetric, positive definite operator, then
\[ \gamma_A(du) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det A}} e^{-\frac{1}{2}(A^{-1}u,u)} |du| \] (C.2)
is a centered Gaussian measure on $U$ with covariance form described by the operator $A$. □

If $V$ is a finite dimensional vector space equipped with a Gaussian measure $\gamma$ and $L : V \to U$ is a linear map, then the pushforward $L_\ast \gamma$ is a Gaussian measure on $U$ with expectation $\mu_{L_\ast \gamma} = L(\mu_\gamma)$ and covariance form
\[
\Sigma_{L_\ast \gamma} : U^\vee \times U^\vee \to \mathbb{R}, \quad \Sigma_{L_\ast \gamma}(\eta, \eta) = \Sigma_\gamma(L^\vee \eta, L^\vee \eta), \quad \forall \eta \in U^\vee,
\]
where $L^\vee : U^\vee \to V^\vee$ is the dual (transpose) of the linear map $L$. Observe that if $\gamma$ is nondegenerate and $L$ is surjective, then $L_\ast \gamma$ is also nondegenerate.

Suppose $(\mathcal{S}, \mu)$ is a probability space. A Gaussian random vector on $(\mathcal{S}, \mu)$ is a (Borel) measurable map
\[
X : \mathcal{S} \to V, \quad V \text{ finite dimensional vector space}
\]
such that $X_\ast \mu$ is a Gaussian measure on $V$. We will refer to this measure as the associated Gaussian measure, we denote it by $\gamma_X$ and we denote by $\Sigma_X$ (respectively $S(X)$) its covariance form (respectively operator),
\[
\Sigma_X(\xi_1, \xi_2) = E\left( \langle \xi_1, X - E(X) \rangle \langle \xi_2, X - E(X) \rangle \right).
\]
Note that the expectation of $\gamma_X$ is precisely the expectation of $X$. The random vector is called nondegenerate, respectively centered, if the Gaussian measure $\gamma_X$ is such.

Let us point out that if $X : \mathcal{S} \to U$ is a Gaussian random vector and $L : U \to V$ is a linear map, then the random vector $LX : \mathcal{S} \to V$ is also Gaussian. Moreover
\[
E(LX) = LE(X), \quad \Sigma_{LX}(\xi, \xi) = \Sigma_X(L^\vee \xi, L^\vee \xi), \quad \forall \xi \in V^\vee,
\]
where $L^\vee : V^\vee \to U^\vee$ is the linear map dual to $L$. Equivalently, $S(LX) = LS(X)L^\vee$.

Suppose that $X_j : \mathcal{S} \to V_j, j = 1, 2$, are two centered Gaussian random vectors such that the direct sum $X_1 \oplus X_2 : \mathcal{S} \to V_1 \oplus V_2$ is also a centered Gaussian random vector with associated Gaussian measure
\[
\gamma_{X_1 \oplus X_2} = p_{X_1 \oplus X_2}(x_1, x_2)|dx_1|dx_2|.
\]
We obtain a bilinear form
\[
cov(X_1, X_2) : V_1^\vee \times V_2^\vee \to \mathbb{R}, \quad cov(X_1, X_2)(\xi_1, \xi_2) = \Sigma(\xi_1, \xi_2),
\]
called the covariance form. The random vectors $X_1$ and $X_2$ are independent if and only if they are uncorrelated, i.e.,
\[
cov(X_1, X_2) = 0.
\]
We can then identify $cov(X_1, X_2)$ with a linear operator $Cov(X_1, X_2) : V_2 \to V_1$, via the equality
\[
E\left( \langle \xi_1, X_1 \rangle \langle \xi_2, X_2 \rangle \right) = cov(X_1, X_2)(\xi_1, \xi_2)
\]
\[
= \langle \xi_1, Cov(X_1, X_2)\xi_2 \rangle, \quad \forall \xi_1 \in V_1, \quad \xi_2 \in V_2^\vee,
\]
where $\xi_2 \in V_2$ denotes the vector metric dual to $\xi_2$. The operator $Cov(X_1, X_2)$ is called the covariance operator of $X_1, X_2$.

The conditional random variable $(X_1 | X_2 = x_2)$ has probability density
\[
p_{(X_1 | X_2 = x_2)}(x_1) = \frac{p_{X_1 \oplus X_2}(x_1, x_2)}{\int_{V_1} p_{X_1 \oplus X_2}(x_1, x_2)|dx_1|}.
\]
For a measurable function $f : V_1 \to \mathbb{R}$ the conditional expectation $E(f(X_1) | X_2 = x_2)$ is the (deterministic) scalar
\[
E(f(X_1) | X_2 = x_2) = \int_{V_1} f(x_1)p_{(X_1 | X_2 = x_2)}(x_1)|dx_1|.
If \( X_2 \) is nondegenerate, the regression formula, \([5]\), implies that the random vector \((X_1|X_2 = x_2)\) is a Gaussian vector with covariance operator
\[
S(Y) = S(X_1) - \text{Cov}(X_1, X_2)S(X_2)^{-1}\text{Cov}(X_2, X_1),
\]
and mean
\[
E(X_1|X_2 = x_2) = Cx_2,
\]
where \( C \) is given by
\[
C = \text{Cov}(X_1, X_2)S(X_2)^{-1}.
\]

**APPENDIX D. A CLASS OF RANDOM SYMMETRIC MATRICES**

We denote by \( \text{Sym}_m \) the space of real symmetric \( m \times m \) matrices. This is an Euclidean space with respect to the inner product
\[
(A, B) := \text{tr}(AB).
\]
This inner product is invariant with respect to the action of \( \text{SO}(m) \) on \( \text{Sym}_m \). We set
\[
\hat{E}_{ij} := \begin{cases} E_{ij}, & i = j \\ \frac{1}{\sqrt{2}}E_{ij}, & i < j. \end{cases}
\]
The collection \((\hat{E}_{ij})_{i \leq j}\) is a basis of \( \text{Sym}_m \) orthonormal with respect to the above inner product. We set
\[
\hat{a}_{ij} := \begin{cases} a_{ij}, & i = j \\ \sqrt{2}a_{ij}, & i < j. \end{cases}
\]
The collection \((\hat{a}_{ij})_{i \leq j}\) the orthonormal basis of \( \text{Sym}_m \) dual to \((\hat{E}_{ij})\). The volume density induced by this metric is
\[
|dA| := \prod_{i \leq j} d\hat{a}_{ij} = 2^{\frac{m(m-1)}{2}} \prod_{i \leq j} da_{ij}.
\]
Throughout the paper we encountered a 2-parameter family of Gaussian probability measures on \( \text{Sym}_m \). More precisely for any real numbers \( u, v \) such that
\[
v > 0, mu + 2v > 0,
\]
we denote by \( \text{Sym}_m^{u,v} \) the space \( \text{Sym}_m \) equipped with the centered Gaussian measure \( d\Gamma_{u,v}(A) \) uniquely determined by the covariance equalities
\[
E(a_{ij}a_{k\ell}) = u\delta_{ij}\delta_{k\ell} + v(\delta_{ik}\delta_{j\ell} + \delta_{ij}\delta_{k\ell}), \quad \forall 1 \leq i, j, k, \ell \leq m.
\]
In particular we have
\[
E(a_{ii}^2) = u + 2v, \quad E(a_{ii}a_{jj}) = u, \quad E(a_{ij}^2) = v, \quad \forall 1 \leq i \neq j \leq m,
\]
while all other covariances are trivial. The ensemble \( \text{Sym}_m^{0,v} \) is a rescaled version of the Gaussian Orthogonal Ensemble (GOE) and we will refer to it as \( \text{GOE}_m^v \).
For \( u > 0 \) the ensemble \( \text{Sym}_m^{u,v} \) can be given an alternate description. More precisely a random \( A \in \text{Sym}_m^{u,v} \) can be described as a sum
\[
A = B + X 1_m, \quad B \in \text{GOE}_m^v, \quad X \in N(0, u), \quad B \text{ and } X \text{ independent}.
\]
We write this
\[
\text{Sym}_m^{u,v} = \text{GOE}_m^v \hat{+} N(0, u) 1_m, \quad (D.1)
\]
where \( \hat{+} \) indicates a sum of independent variables.
The Gaussian measure \( d\Gamma_{u,v} \) coincides with the Gaussian measure \( d\Gamma_{u+2v,u,v} \) defined in \([26, \text{App. B}]\). We recall a few facts from \([26, \text{App. B}]\).
The probability density \( d\Gamma_{u,v} \) has the explicit description
\[
d\Gamma_{u,v}(A) = \frac{1}{(2\pi)^{n(m+1)/4}} e^{-\frac{1}{4} \text{tr} A^2 - \frac{u'}{2} (\text{tr} A)^2} |dA|,
\]
where
\[
D(u, v) = (2v)^{(m-1)} \left( \frac{mu + 2v}{2} \right),
\]
and
\[
u' = \frac{1}{m} \left( \frac{1}{mu + 2v} - \frac{1}{2v} \right) = -\frac{1}{2v(mu + 2v)}.
\]
In the special case \( \text{GOE}_m \) we have \( u = u' = 0 \) and
\[
d\Gamma_{0,v}(A) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}} e^{-\frac{1}{4} \text{tr} A^2} |dA|.
\]
We have a Weyl integration formula [3] which states that if \( f : \text{Sym}_m \to \mathbb{R} \) is a measurable function which is invariant under conjugation, then the value \( f(A) \) at \( A \in \text{Sym}_m \) depends only on the eigenvalues \( \lambda_1(A) \leq \cdots \leq \lambda_n(A) \) of \( A \) and we have
\[
E_{\text{GOE}_m^v}(f(X)) = \frac{1}{Z_m(v)} \int_{\mathbb{R}^m} f(\lambda_1, \ldots, \lambda_m) \left( \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{2}} |d\lambda_1 \cdots d\lambda_m|, \right.
\]
where the normalization constant \( Z_m(v) \) is defined by
\[
Z_m(v) = \int_{\mathbb{R}^m} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{2}} |d\lambda_1 \cdots d\lambda_m| = (2v)^{\frac{m(m-1)}{4}} \int_{\mathbb{R}^m} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{2}} |d\lambda_1 \cdots d\lambda_m|.
\]
The precise value of \( Z_m \) can be computed via Selberg integrals, [3, Eq. (2.5.11)], and we have
\[
Z_m = (2\pi)^{\frac{m}{2}} m! \prod_{j=1}^m \Gamma \left( \frac{j}{2} \right) = 2^m \pi^m m! \prod_{j=1}^m \Gamma \left( \frac{j}{2} \right).
\]
For any positive integer \( n \) we define the normalized 1-point correlation function \( \rho_{n,v}(x) \) of \( \text{GOE}_n^v \) to be
\[
\rho_{n,v}(x) = \frac{1}{Z_n(v)} \int_{\mathbb{R}^{n-1}} Q_{n,v}(x, \lambda_2, \ldots, \lambda_n)d\lambda_1 \cdots d\lambda_n.
\]
For any Borel measurable function \( f : \mathbb{R} \to \mathbb{R} \) we have [11, §4.4]
\[
\frac{1}{n} E_{\text{GOE}_n^v}(\text{tr } f(X)) = \int_{\mathbb{R}} f(\lambda) \rho_{n,v}(\lambda)d\lambda.
\]
The equality (D.5) characterizes \( \rho_{n,v} \). Let us observe that for any constant \( c > 0 \), if
\[
A \in \text{GOE}_n^v \iff cA \in \text{GOE}_n^{cv^2}.
\]
Hence for any Borel set \( B \subset \mathbb{R} \) we have
\[
\int_{cB} \rho_{n,cv^2}(x)dx = \int_B \rho_{n,v}(y)dy.
\]
We conclude that
\[ c \rho_{n,c^2v}(cy) = \rho_{n,v}(y), \quad \forall n, c, y. \tag{D.6} \]
The behavior of the 1-point correlation function \( \rho_{n,v}(x) \) for \( n \) large is described by the Wigner semicircle theorem [3, Thm.2.1.1] which states that for any \( v > 0 \) the sequence of probability measures on \( \mathbb{R} \)
\[ \rho_{n,v}(x)dx = n^{\frac{1}{2}} \rho_{n,v}(n^{\frac{1}{2}}x)dx \]
converges weakly as \( n \to \infty \) to the semicircle distribution
\[ \rho_{sc,v}(x)dx = I_{\{|x| \leq 2v\}} \frac{1}{2\pi v} \sqrt{4v - x^2} |dx|. \]
The expected value of the absolute value of the determinant of a random \( A \in G_{m}^{\text{GOE}} \) can be expressed neatly in terms of the correlation function \( \rho_{m+1,v} \). More precisely, we have the following result first observed by Y.V. Fyodorov [14] in a context related to ours.

**Lemma D.1.** Suppose \( v > 0 \). Then for any \( c \in \mathbb{R} \) we have
\[ E_{\text{GOE}_{m}}( |\det(A - c1_{m})|) = 2^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} \Gamma \left( \frac{m+3}{2} \right) e^{\frac{e^2}{4v}} \rho_{m+1,v}(c). \]

**Proof.** Using the Weyl integration formula we deduce
\[ E_{\text{GOE}_{m}}( |\det(A - c1_{m})|) = \frac{1}{Z_{m}(v)} \int_{\mathbb{R}^{m}} \prod_{i=1}^{m} e^{-\frac{\lambda_{i}^{2}}{4v}} |c - \lambda_{i}| \prod_{i \leq j} |\lambda_{i} - \lambda_{j}| d\lambda_{1} \cdots d\lambda_{m} \]
\[ = \frac{e^{\frac{e^2}{4v}}}{Z_{m}(v)} \int_{\mathbb{R}^{m}} e^{-\frac{1}{4v} \sum_{i=1}^{m} \lambda_{i}^{2}} |c - \lambda_{i}| \prod_{i \leq j} |\lambda_{i} - \lambda_{j}| d\lambda_{1} \cdots d\lambda_{m} \]
\[ = \frac{e^{\frac{e^2}{4v}} Z_{m+1}(v)}{Z_{m}(v)} \frac{1}{Z_{m+1}(v)} \int_{\mathbb{R}^{m}} Q_{m+1,v}(c, \lambda_{1}, \ldots, \lambda_{m}) d\lambda_{1} \cdots d\lambda_{m} \]
\[ = \frac{e^{\frac{e^2}{4v}} Z_{m+1}(v)}{Z_{m}(v)} \rho_{m+1,v}(c) = v^{\frac{m+1}{2}} e^{\frac{e^2}{4v}} \frac{Z_{m+1}}{Z_{m}} \rho_{m+1,v}(c) \]
\[ = (m + 1)^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} e^{\frac{e^2}{4v}} \Gamma \left( \frac{m+3}{2} \right) \rho_{m+1,v}(c) = 2^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} \Gamma \left( \frac{m+3}{2} \right) e^{\frac{e^2}{4v}} \rho_{m+1,v}(c). \]

The above result admits the following generalization, [4, Lemma 3.2.3].

**Lemma D.2.** Let \( u, v > 0 \). Set
\[ \theta^{\pm}_{m,v}(x) := \rho_{m+1,v}(x) e^{\frac{x^2}{4v}}. \]

Then
\[ E_{\text{Sym}_{m}^{u,v}}( |\det(A - c1_{m})|) = 2^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{\frac{(c-x)^2}{4v}} \frac{x^2}{4v} dx \tag{D.7a} \]
\[ = 2^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} \Gamma \left( \frac{m+3}{2} \right) (\gamma_{u} * \theta^{+}_{m+1,v})(c). \tag{D.7b} \]

In particular, if \( u = 2kv, k < 1 \) we have
\[ E_{\text{Sym}_{m}^{2kv,v}}( |\det(A - c1_{m})|) = 2^{\frac{3}{2}} (2v)^{\frac{m}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\sqrt{2\pi k}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{-\frac{(c-x)^2}{4v} + \frac{(x+c)^2}{4v} + \frac{(x+c)^2}{4v}} dx, \]
(\lambda := c - x)
\begin{align*}
(\lambda) &= \frac{3}{2} v \frac{4}{2} \Gamma \left( \frac{m + 3}{2} \right) \frac{1}{\sqrt{2\pi k}} \int_{\mathbb{R}} \rho_{m+1,v}(\lambda) e^{-\frac{1}{4\pi k} \left( \lambda - (t_k^2 + 1)c + \frac{(t_k^2 - 1)c^2}{4} \right)} d\lambda \\
\text{where} \\
& \quad = \frac{1}{k - 1} = \frac{k}{1 - k}.
\end{align*}

\textbf{Proof.} Recall the equality (D.1)
\begin{align*}
\text{Sym}_{m}^{u,v} = \text{GOE}_{m}^{u,v} + \mathcal{N}(0, u) \mathbb{1}_m.
\end{align*}

We deduce that
\begin{align*}
E_{\text{Sym}_{m}^{u,v}} \left( | \det \left( A - c \mathbb{1}_m \right) \right) &= E \left( | \det \left( B + (X - c) \mathbb{1}_m \right) \right) \\
&= \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} E_{\text{GOE}_{m}^{u}} \left( | \det \left( B - (c - X) \mathbb{1}_m \right) | \right) e^{-\frac{X^2}{2u}} dx \\
&= \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} E_{\text{GOE}_{m}^{u}} \left( | \det \left( B - (c - x) \mathbb{1}_m \right) | \right) e^{-\frac{x^2}{2u}} dx \\
&= 2^{\frac{3}{2}} (2v)^{\frac{m+3}{2}} \Gamma \left( \frac{m + 3}{2} \right) \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \rho_{m+1,v}(x - c) e^{-\frac{(x - c)^2}{4u}} dx .
\end{align*}

Now observe that if \( u = 2kv \) then
\begin{align*}
\frac{(c - x)^2}{4v} - \frac{x^2}{2u} - \frac{x^2}{4kv} + \frac{1}{4v} (x^2 - 2cx + c^2) \\
&= \frac{1}{4v} \left( \frac{1}{t_k^2} x^2 - 2cx - c^2 t_k^2 \right) + \frac{c^2 (1 + t_k^2)}{4v} = -\frac{1}{4vt_k^2} (x + t_k c)^2 + \frac{c^2 (1 + t_k^2)}{4v}.
\end{align*}

\[ \square \]

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