The problem concerning the minimum time for an initial state to evolve up to a target state plays an important role in the Classic Optimal Control theory. In the quantum context, as quantum states are so sensitive to environmental influences, the problem is more complex, but its formulation and solution are decisive to implement quantum information processing systems. As is well known, the decoherence phenomenon is unavoidable and the time-energy uncertainty relation must be used to study quantum dynamics. Here, the time-energy uncertainty relations are revisited, being fundamental to propose performance measures based on minimum time evolution. A minimum time performance measure is defined for quantum control problems. Then, some practical examples are considered and the minimum time performance measure is applied providing results that are supposed to be useful for researchers to pursue strategies to optimize the control of the states of a quantum system.

Keywords: Minimum Time; Optimal Control; Quantum Systems; Time-Energy Uncertainty.

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1 Introduction

Along the 20th century, Physics and Cybernetics experienced a strong development decisively contributing to modern science in spite of the weak interactions between them. This, may be is due to the different approaches with Physics being a descriptive science and Cybernetics, a prescriptive one [1].

However, it can not be denied that automatic systems play an important role concerning physical experiments, but in most cases, the control theory is considered to be secondary with no effective contributions to explaining physical phenomena [1].

In the late 1980s, with the development of ultra fast lasers, methods based on optimal control were developed to control molecular systems [2]. The interest in this kind of problem increased in the early 1990s and the concepts of classical and quantum approaches were developed [3, 4].
Additionally, researches on quantum computation hardware can be implemented by manipulating the quantum state of trapped ions via laser or electrical fields [5]. Consequently, it seems to be important to formulate optimal control strategies for quantum particles allowing, for example, to minimize times, average distances and/or energy costs of the processes.

Luo [6] developed a theoretical method to calculate the minimum time for a state evolution that inspired the work conducted here, dedicated to go deeper into showing how the time-energy uncertainty affects the expression of the minimum time for state evolution and proposing performance measures for this kind of process.

This paper starts with a section explaining the derivation of time-energy uncertainty relation, following the seminal work by Mandelstam and Tamm [7], resulting an expression to the minimum time between states, called Bhattacharyya limit [8], which is valid considering constant Hamiltonian evolution.

In the next section, the Bhattacharyya limit is used to define a performance measure, essential to design quantum control devices, allowing a criterion to evaluate control systems in the minimum time optimization point of view [9], with quantum fidelity playing an important role in defining the target state reachability.

In order to show the practical application of the defined performance measure, three different problems are explored in a new section. The first one is about the transition between two orthogonal states, the formulation of which is useful for quantum information processing. The second is related to the Fahri and Gutmann formulation [10] for digital quantum computation and the third considers a general maximum fidelity [11] state transition. A conclusion section finishes the work.

2 Mandelstam-Tamm uncertainty relations and Bhattacharyya limit

The first theoretically satisfactory formulation of the time-energy uncertainty was proposed by Leonid Mandelstam and Igor Tamm in 1945 [7]. In this section, the time-energy uncertainty is derived in a slightly different form, in order to simplify the expressions of the performance measure to be defined.

Considering the quantum observables $\hat{R}$ and $\hat{S}$, the following relations can be written [7]:

$$\Delta \hat{S}.\Delta \hat{R} \geq \frac{1}{2} \left| < \hat{R}\hat{S} - \hat{S}\hat{R} > \right|;$$ (1)

and

$$\frac{d < \hat{R} >}{dt} = \frac{1}{i\hbar} < [\hat{R}, \hat{H}] > .$$ (2)

In expression (1), $\Delta \hat{S}$ and $\Delta \hat{R}$ are the dispersions of operators $\hat{S}$ and $\hat{R}$, respectively. Equation (2) is the dynamical evolution for the mean value of operator $\hat{R}$.

Making $\hat{S} = \hat{H}$ and considering the relations (1) and (2), one can write:

$$\Delta \hat{H}.\Delta \hat{R} \geq \frac{\hbar}{2} \left| \frac{d < \hat{R} >}{dt} \right|;$$ (3)

giving the relation between the total energy dispersion $\Delta \hat{H}$, or energy uncertainty, and the uncertainty of another dynamical quantity, relating then with the mean value of this quantity.
As the absolute value of the integral of a function is lower than or equal to the integral of the value absolute of the function, (3) can be integrated between times $t$ and $t + \Delta t$, with $H$ constant during this interval, resulting:

$$\Delta \hat{H} \Delta t \geq \frac{\hbar}{2} \left| < R_{t+\Delta t} > - < R_t > \right|,$$

with $< \Delta \hat{R} >$ being the mean value of $\Delta \hat{R}$ in the time interval $\Delta t$.

From now on, symbol $\Delta t$ represents the minimum time for the mean value of a physical quantity to be varied by its standard deviation. Consequently, $\Delta t$ is called uncertainty in time and (4) becomes:

$$\Delta \hat{H} \Delta t \geq \frac{\hbar}{2}, \quad (5)$$

Considering the projection operator $\hat{\Lambda} = |\psi_0 > < \psi_0 |$, with $\hat{\Lambda}^2 = \hat{\Lambda} \hat{\Lambda} = \hat{\Lambda}$, its mean value can be seen as the probability of the system to be in a quantum state $| \psi >$. This fact can be justified as:

$$\hat{\Lambda}_\Psi = < \psi | \hat{\Lambda} | \psi > = < \psi | \psi_0 > < \psi_0 | \psi > = | < \psi | \psi_0 > |^2 = P_\Psi.$$

(6)

It can be noticed that $0 \leq < \hat{\Lambda} > \leq 1$ and, considering the definition of the standard deviation:

$$\Delta \hat{\Lambda} = \sqrt{< \hat{\Lambda}^2 > - < \hat{\Lambda} >^2} = \sqrt{< \hat{\Lambda} > - < \hat{\Lambda} >^2}.$$

(7)

Replacing $\hat{R}$ by $\hat{\Lambda}$ in (3) and considering expression (7), results:

$$\Delta \hat{H} \sqrt{< \hat{\Lambda} > - < \hat{\Lambda} >^2} \geq \frac{\hbar}{2} \frac{d < \hat{\Lambda} >}{dt}.$$

(8)

Expression (8) contains only the quantities $\hat{\Lambda} = \hat{\Lambda}(t)$ and its derivative depending on time. Consequently, it can be integrated in time. If, for instance, $\hat{\Lambda}(0) = 1$, indicating that for $t = 0$ it is certain that the state is $\psi_0$, for $t \geq 0$, the integration gives:

$$\frac{\pi}{2} - \arcsin \sqrt{< \hat{\Lambda}(t) >} \leq \frac{\Delta \hat{H} \cdot t}{\hbar}.$$

(9)

By using some trigonometric identities and algebraic manipulation, (9) is transformed into:

$$< \hat{\Lambda}(t) > \geq \cos^2 \left( \frac{\Delta \hat{H} \cdot t}{\hbar} \right).$$

(10)

As the mean value of the operator $\hat{\Lambda}$ corresponds to the probability of finding the system in the state $< \psi_t >$ in time $t$, starting from state $< \psi_0 >$ in $t = 0$, that is denoted by $P_t$, expression (10) is rewritten as:

$$P_t \geq \cos^2 \left( \frac{\Delta \hat{H} \cdot t}{\hbar} \right).$$

(11)

Observing (11), the time physically possible for a state transition, considering a time independent Hamiltonian, is given by:
Performance measure in quantum control

\[ t \geq \frac{\hbar}{\Delta \hat{H}} \arccos \sqrt{P_t}, \tag{12} \]

considering the dispersion \( \Delta \hat{H} \), \( t \) the transition time and \( P_t = |<\Psi_t|\Psi_0>|^2 \).

Expression (12) is normally called “Bhattacharyya limit” due to the important work presented in [8].

3 Minimum Time Performance Measure

Concerning practical quantum control systems, it is important to define a quantity for measuring how far from the optimal conditions the operational conditions are. Here, it is proposed a minimum time performance measure, \( \eta_t \) defined as:

\[ \eta_t = \frac{t_{\text{min}}}{t_{CQS}}, \tag{13} \]

with \( t_{\text{min}} \) representing the inferior limit of expression (12) and \( t_{CQS} \), the effective time to the target transition state.

The minimum time performance measure \( \eta_t \) is a real number belonging to the interval \([0, 1]\). If the state transition has not occurred or the control algorithm has not converged, it is considered that \( t_{CQS} \to \infty \), i.e., \( \eta_t = 0 \). For the ideal state transition \( (t_{QCS} = t_{\text{min}}) \), \( \eta_t = 1 \).

Taking expression (12) into the minimum time performance measure, its complete expression becomes:

\[ \eta_t = \frac{\hbar \arccos \sqrt{P_t}}{\Delta H t_{CQS}}, \tag{14} \]

If a specific state transition between an initial state \( |\Psi_I> \) and a target state \( |\Psi_G> \) is considered, expression (13) is modified to:

\[ \eta_t = \frac{\hbar \arccos |<\Psi_G|\Psi_I>|}{\Delta H t_{CQS}}. \tag{15} \]

Frequently, it is not possible for practical applications or numerical simulations of optimal control algorithms to obtain the exact transition to the target state. In these cases, it is important to find optimal controls \( u^*(t) \) to maximize de quantum fidelity \( F \) [11] between the final state \( |\Psi_F> \) and the target state \( <\Psi_G> \), defined as:

\[ F = |<\Psi_G|\Psi_F>|^2. \tag{16} \]

4 Performance measure for particular state transitions

In this section, three different state transitions are considered and, in each case, the minimum time performance measure is calculated.

4.1 Transition between two orthogonal states

Considering the transition between an initial state \( \Psi \) and its orthogonal state \( \Psi^\perp \), occurring in a time \( t \), as \( P_t = |<\Psi^\perp|\Psi>|^2 \), equation (11) gives:
\[ |<\Psi^\perp|\Psi>|^2 \geq \cos^2 \left( \frac{\Delta \hat{H}t\bar{\hbar}}{\hbar} \right). \]  
(17)

The dynamical evolution between these states is subjected to a temporal evolution operator \( \hat{U}(t, t_0) \), with \( \hat{U}(t, t_0) = e^{-i(\hat{H}(t-t_0))} \) [12], for a time independent Hamiltonian \( \hat{H} \).

Consequently:

\[ |<\Psi|\hat{U}(t, t_0)|\Psi>|^2 \geq \cos^2 \left( \frac{\Delta \hat{H}t\bar{\hbar}}{\hbar} \right), \]  
(18)

with \( |<\Psi|\hat{U}(t, t_0)|\Psi>|^2 \) considered to be the surviving probability of state \( \Psi \), while \( \hat{U}(t, t_0) \) acts over the system.

If the system evolves from state \( \Psi \) to its orthogonal state \( \Psi^\perp \), this probability vanishes, i.e., the transition time obeys:

\[ t_{\Psi \rightarrow \Psi^\perp} = \inf \{ t \geq 0 : |<\Psi|\hat{U}(t, t_0)|\Psi>| = 0 \}, \]  
(19)

and, therefore:

\[ t_{\Psi \rightarrow \Psi^\perp} \geq \frac{\pi \hbar}{2\Delta \hat{H}}. \]  
(20)

Hence, the minimum time performance measure defined by (13) for transitions between orthogonal states is given by:

\[ \eta^\hat{H}_{\Psi \rightarrow \Psi^\perp} = \frac{\pi \hbar}{2\Delta \hat{H}_{t\text{CQS}}}. \]  
(21)

### 4.2 Digital quantum computation model

Here an example related to digital quantum computation [10], which can be formulated as a quantum searching algorithm [13], is studied, in order to derive its minimum time performance measure.

The initial idea is to consider \( |a> \) and \( |b> \) as initial and target states, respectively with the system Hamiltonian given by:

\[ \hat{H} = E_a|a><a| + E_b|b><b|, \]  
(22)

with \( E_a \) and \( E_b \) positive constants.

As the exact formulation for the evolution from state \( |a> \) to \( |b> \) is almost impossible, an alternative formulation that considers fidelity \( F \) is proposed. The reasoning is to obtain the minimum possible time, in order to maximize \( F \), given by:

\[ F = P_1 = |<b|\hat{U}(t, t_0)|a>|^2. \]  
(23)

In order to perform the calculations, it is necessary to choose a normalized orthogonal basis in the space generated by \( |a> \) and \( |b> \), composed of the kets \( |b> \) and \( |\tilde{b}> = \frac{1}{\sqrt{1-s^2}}(|a> - s|b>) \), with \( s = <a|b> \).

On this basis, \( |a> \) and \( |b> \) are expressed as:
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\[ |a> = \begin{bmatrix} \frac{s}{\sqrt{1-s^2}} \end{bmatrix}, \quad |b> = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \] (24)

Considering these expressions of the kets \(|a>\) and \(|b>\) on the new basis, the Hamiltonian becomes:

\[
\hat{H} = \begin{bmatrix} 1 + s^2 - \frac{E}{\mu}(1-s^2) & (1 + \frac{E}{\mu})s\sqrt{1-s^2} \\ (1 + \frac{E}{\mu})s\sqrt{1-s^2} & 1 - s^2 + \frac{E}{\mu}(1-s^2) \end{bmatrix} = \frac{E}{2} \begin{bmatrix} 1 + \lambda & \sqrt{\mu^2 - \lambda^2} \\ \sqrt{\mu^2 - \lambda^2} & 1 - \lambda \end{bmatrix}, \quad (25)
\]

with \(E = E_a + E_b, \quad x = E_a - E_b, \quad \mu = \sqrt{s^2 + \left(\frac{E}{\mu}\right)^2(1-s^2)}, \) and \(\lambda = s^2 - \left(\frac{E}{\mu}\right)(1-s^2).\)

From these considerations, it is possible to diagonalize the Hamiltonian operator as follows:

\[
\hat{H} = U \begin{bmatrix} \frac{E}{2}(1 + \mu) & 0 \\ 0 & \frac{E}{2}(1 - \mu) \end{bmatrix} U^{-1}, \quad (26)
\]

with \(U\) given by:

\[
U = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1 + \frac{\lambda}{\mu}} & \sqrt{1 - \frac{\lambda}{\mu}} \\ \sqrt{1 - \frac{\lambda}{\mu}} & -\sqrt{1 + \frac{\lambda}{\mu}} \end{bmatrix} U^{-1}, \quad (27)
\]

corresponding to the unitary operator constructed with the eigenvalues of \(\hat{H}\).

Taking into account equations (22) to (27), it is possible to obtain the fidelity, expressed by the probability of, starting with state \(|a>\), following the dynamical operator \(\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}\), to reach the target state \(|b>\), i.e.:

\[
P_t = \langle b | e^{-i\hat{H}t/\hbar} | a \rangle = s^2[(\frac{1}{\mu^2} - 1) \sin^2(\frac{\mu Et}{2\hbar}) + 1], \quad (28)
\]

considering \(t_0 = 0.\)

It can be noticed that \(s \leq \mu \leq 1\) and, therefore, maximum value of the probability \(P_t\), imposing \(\sin^2(\frac{\mu Et}{2\hbar}) = 1\), is:

\[
P_{\text{max}} = \max P_t(t \geq 0) = \left(\frac{s}{\mu}\right)^2. \quad (29)
\]

Consequently, it is natural to derive the expression for \(t_{\text{min}}\), the minimum time to obtain maximum fidelity, by simply applying the same conditions above. Therefore:

\[
t_{\text{min}} = \inf \{t \geq 0 : P_t = P_{\text{max}}\} = \frac{\pi\hbar}{E\mu}. \quad (30)
\]

Replacing equation (30) in definition (15), the expression for measuring minimum time optimal control performance in a Fahri-Gutmann system [10], with Hamiltonian independent of time, is:

\[
\eta_{FG}^H = \frac{\pi\hbar}{E\mu t_{\text{CQS}}}. \quad (31)
\]
4.3 General state transition

Here, the minimum time quantum control performance measure will be derived for a transition between two general states. In order to obtain this expression, it is necessary to start with a transition between a state and the state orthogonal to the other, deriving an intermediate expression that is used to complete the task.

4.3.1 Transition between a state and a state orthogonal to an other

Considering the states |a> and |c>, with |<c|a>|^2 = cos φ, and φ ∈ [0, π/2], if the dynamical evolution of the system is subjected to a temporal evolution operator \( \hat{U}(t, t_0) = e^{-i \hat{H}(t-t_0) / \hbar} \), for a time independent Hamiltonian \( \hat{H} \), it is possible to study the state transition between |a> and |c^⊥>, orthogonal to |c>, with temporal transition probability:

\[
F = P_t = |<c|\hat{U}(t, t_0)|a>|^2. \tag{32}
\]

Defining \( \hat{R} = |c><c| \), \( \Delta \hat{R} = \sqrt{P_t - P_t^2} \), relation (3) can be modified as:

\[
\frac{dP_t}{dt} \leq \frac{2\Delta \hat{H}}{\hbar} \sqrt{P_t(1 - P_t)}. \tag{33}
\]

Integrating (33) for \( P(0) = \cos^2 \phi \), it is possible to write:

\[
P_t \geq \cos^2(\Delta \hat{H} t / \hbar + \phi). \tag{34}
\]

In order to find the minimum time for the transition from the initial state |a> to the target state |c^⊥>, orthogonal to |c>, the probability given by (32) must be zero. Consequently,

\[
t_{a \rightarrow c^⊥}^{\text{min}} = \inf \{ t \geq 0 : P_t = 0 \}, \tag{35}
\]

resulting:

\[
t_{a \rightarrow c^⊥} \geq \frac{\hbar(\pi - 2\phi)}{2\Delta \hat{H}}. \tag{36}
\]

Therefore, for this case, the minimum time quantum control performance measure results:

\[
\eta_{a \rightarrow c^⊥}^H = \frac{\hbar(\pi - 2\phi)}{2\Delta \hat{H}_{CQS}}. \tag{37}
\]

4.3.2 Transition between two general states

Considering the states |a> and |b>, with |<b|a>|^2 = cos φ, and φ ∈ [0, π/2], if the dynamical evolution of the system is subjected to a temporal evolution operator \( \hat{U}(t, t_0) \), with \( \hat{U}(t, t_0) = e^{-i \hat{H}(t-t_0) / \hbar} \), for a time independent Hamiltonian \( \hat{H} \), it is possible to study the state transition between |a> and |b> and calculate the minimum time transition:

\[
t_{a \rightarrow b}^{\text{min}} = \inf \{ t \geq 0 : P_t = |<b|\hat{U}(t, t_0)|a>|^2 \} = 1. \tag{38}
\]
This calculation can be done following the steps proposed in [6], considering that the evolution from $|a\rangle$ to $|b\rangle$ is equivalent to the evolution from $|a\rangle$ to $|c^\perp\rangle$, orthogonal to $|c\rangle$, for any $|c\rangle$ orthogonal to $|b\rangle$.

As $|<b|a\rangle|^2 = \cos^2 \phi$ with $\phi \in [0, \frac{\pi}{2}]$, the maximum value of $|<c|a\rangle|^2$, when $|c\rangle$ assumes all the states orthogonal to $|b\rangle$, is $\cos^2(\frac{\pi}{2} - \phi)$. Considering the sub-space of the kets $c = b^\perp$, orthogonal to $|b\rangle$, expression (36) can be applied. Finally, for the whole transition:

$$t_{a \rightarrow b} \geq \sup \{t_{a \rightarrow c^\perp}\} = \frac{\hbar \phi}{\Delta \hat{H}}.$$  \hspace{1cm} (39)

Therefore, for this case, the minimum time quantum control performance measure results:

$$\eta^H_{a \rightarrow b} = \frac{\hbar \phi}{\Delta \hat{H} t_{CQS}}.$$ \hspace{1cm} (40)

5 Conclusions

By using time-energy uncertainty relations, expressions for minimum time to quantum transitions were developed for three cases: state to orthogonal state transition; digital quantum computation, and general state transition.

These cases are useful to build quantum control and computation systems that need to have their efficiency evaluated. In order to do this evaluation, a performance measure was defined and expressed for the three cases studied, considering minimum time quantum control systems.

It was considered that the time evolution of the systems obeys transitions with time independent Hamiltonian. This hypothesis is compatible with quantum computation and control processes that are supposed to occur in a very short time to avoid decoherence [11].

1. A.L. Fradkov, Cybernetical Physics (Springer, Berlin, 2007).
2. S. Krempl, T. Eisenhammer, A. Hubler, G. Mayer-Kress and P.W. Milonni, “Optimal Stimulation of a Conservative Nonlinear Oscillator - Classical and Quantum-Mechanical Calculations” Phys. Rev. Lett. 69(3) (1992) 430.
3. M. Dahleh, A. Pierce, H. Rabitz and V. Ramakrishna, “Numerical analysis of complex dynamics in atomic force microscopes” Proc. IEEE 84(1), (1996) 7.
4. P.W. Brumer and M. Shapiro, Principles of Quantum Control of Molecular Processes, (Willey-Interscience, Hoboken NJ, 2003).
5. C. Hangan, A. M. Bloch, C. Monroe and P.H. Bucksbaum, “Control of trapped-ion quantum states with optical pulses” Phys. Rev. Lett. 92, (2004) 113004.
6. S.L. Luo, “How fast can a quantum state evolve into a target state?” Physica D 189(1), (2004) pp. 1-7.
7. R. Bonifacio, “Time-energy uncertainty relation and irreversibility in Quantum Mechanics” 83/41 - Internal Report - International Centre for Theoretical Physics, (1983); http://streaming.ictp.trieste.it/preprints/P/83/041.pdf.
8. K. Bhattacharyya, “Quantum decay and the Mandelstam-Tamm time-energy inequality”, Journal of Physics A: Mathematical and General 16, (1983) pp. 2993-2996.
9. D.E. Kirkov, Optimal Control Theory: An Introduction (Prentice Hall, Englewood Cliffs, N.J., 1970).
10. E. Fahri and S. Gutmann, “Analog analogue of a digital quantum computation”, Physical Review A 57(4) (1998), pp. 2403-2406.
11. V. Vedral, Introduction to Quantum Information Science (Oxford University Press: Oxford, 2006).
12. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge, UK: Cambridge University Press, 2000).

13. L. K. Grover, “Quantum Mechanics helps in searching for a needle in a haystack”, *Physical Review Letters* 79(2), (1997), pp. 325-328.