Perfect matchings in down-sets

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January 12, 2022

Abstract

In this paper, we show that, given two down-sets (simplicial complexes) there is a matching between them that matches disjoint sets and covers the smaller of the two down-sets. This result generalizes an unpublished result of Berge from circa 1980. The result has nice corollaries for cross-intersecting families and Chvátal’s conjecture. More concretely, we show that Chvátal’s conjecture is true for intersecting families with covering number 2.

A family \( \mathcal{F} \subset 2^{[n]} \) is intersection-union (IU) if for any \( A, B \in \mathcal{F} \) we have \( 1 \leq |A \cap B| \leq n - 1 \). Using the aforementioned result, we derive several exact product-and sum-type results for IU-families.

1 Introduction

For a positive integer \( n \) let \( [n] := \{1, \ldots, n\} \) be the standard \( n \)-element set. For or a set \( X \) let \( 2^X \) denote its power set. Subsets of \( 2^X \) are called families. A family \( \mathcal{F} \subset 2^X \) is intersecting if for any \( A, B \in \mathcal{F} \) we have \( A \cap B \neq \emptyset \). Similarly, \( \mathcal{F} \subset 2^X \) is union if for any \( A, B \in \mathcal{F} \) we have \( A \cup B \neq X \). One should note that \( \mathcal{F} \) is union iff the family of complements, \( \mathcal{F}^c := \{X \setminus A : A \in \mathcal{F}\} \) is intersecting.

Erdős, Ko and Rado were the first to investigate intersecting families.

Theorem 1 (Non-uniform Erdős–Ko–Rado Theorem [4]). Suppose that \( \mathcal{F} \subset 2^{[n]} \) is intersecting. Then

\[
|\mathcal{F}| \leq 2^{n-1}.
\]  

Moreover, there exists some intersecting family \( \mathcal{F}' \subset 2^{[n]} \) with \( \mathcal{F} \subset \mathcal{F}' \) and \( |\mathcal{F}'| = 2^{n-1} \).

Note that the above result implies the existence of myriads of non-isomorphic intersecting families attaining equality in (1).

Let us mention that [1] states that an intersecting family contains at most half of all subsets of \( [n] \). This statement is nearly trivial. Namely, out of every pair \( A, [n] \setminus A \) an intersecting family may contain at most one.

Definition 1. A family \( \mathcal{B} \subset 2^X \) is called a down-set (up-set) if for all \( B \in \mathcal{B} \) and \( A \subset X \) with \( A \subset B \) (\( B \subset A \)), respectively, \( A \in \mathcal{B} \) holds.

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There is an important correlation inequality concerning down-sets and up-sets.

**Theorem 2** (Harris-Kleitman Inequality \([8], [12]\)). Suppose that \(\mathcal{A} \subset 2^{[n]}\) is an up-set and \(\mathcal{B} \subset 2^{[n]}\) is a down-set. Then

\[
\frac{|\mathcal{A} \cap \mathcal{B}|}{2^n} \leq \frac{|\mathcal{A}|}{2^n} \frac{|\mathcal{B}|}{2^n}.
\] (2)

For a family \(\mathcal{F} \subset 2^X\) let \(\mathcal{F}^\uparrow\) and \(\mathcal{F}^\downarrow\) stand for the up-set and down-set generated by \(\mathcal{F}\), respectively. That is,

\[
\mathcal{F}^\uparrow := \{G : \exists F \in \mathcal{F}, F \subset G \subset X\}, \quad \mathcal{F}^\downarrow := \{G : \exists F \in \mathcal{F}, G \subset F\}.
\]

As one can see, \(\mathcal{F}^\downarrow\) is independent of \(X\).

The most natural examples of intersecting families are stars, that is, families for which there is an element common to all its members.

**Conjecture 1** (Chvátal \([2]\)). Suppose that \(\mathcal{D} \subset 2^X\) is a down-set, \(\mathcal{F} \subset \mathcal{D}\) is intersecting. Then

\[
|\mathcal{F}| \leq \max_{x \in X} \{|F \in \mathcal{D} : x \in F\}| =: \Delta(\mathcal{D}).
\] (3)

Unfortunately, after half a century, very little is known about Chvátal’s conjecture. The Harris-Kleitman Correlation Inequality implies the following weaker result.

**Theorem 3.** Suppose that \(\mathcal{B} \subset 2^{[n]}\) is a down-set and \(\mathcal{F} \subset \mathcal{B}\) is intersecting. Then

\[
|\mathcal{F}| \leq |\mathcal{B}|/2.
\] (4)

**Proof.** Set \(\mathcal{A} := \mathcal{F}^\uparrow\). Then \(\mathcal{F} \subset \mathcal{A} \cap \mathcal{B}, \mathcal{A}\) is an intersecting up-set. In view of (1), \(|\mathcal{A}| \leq 2^n - 1\). Now (4) follows from (2). \(\square\)

We refer to [7] for much more on Chvátal’s conjecture and correlation inequalities. Berge \([1]\) gave a different proof of (4) based on basic graph theory. To a family \(\mathcal{F}\) correspond its Kneser graph \(KG(\mathcal{F})\) with vertex set \(\mathcal{F}\) and edge set \(\{(F, G) : F \cap G = \emptyset\}\).

**Theorem 4** (Berge \([1]\)). Suppose that \(\mathcal{B}\) is a down-set. Then (i) or (ii) holds.

(i) \(|\mathcal{B}|\) is even and \(KG(\mathcal{B})\) contains a perfect matching;

(ii) \(|\mathcal{B}|\) is odd and \(KG(\mathcal{B} \setminus \{\emptyset\})\) contains a perfect matching.

It should be clear that Theorem 4 implies (4). We will present one proof of this theorem at the end of Section 3.

One of our main results is a two-families version of Theorem 4. To state it let us take two arbitrary families \(\mathcal{F}\) and \(\mathcal{G}\) and define the bipartite Kneser graph \(KG(\mathcal{F}, \mathcal{G})\) with partite sets \(\mathcal{F}\) and \(\mathcal{G}\), with edge set \(\{(F, G) : F \in \mathcal{F}, G \in \mathcal{G}, F \cap G = \emptyset\}\).

**Theorem 5.** Suppose that \(\mathcal{F}\) and \(\mathcal{G}\) are down-sets, \(|\mathcal{F}| \leq |\mathcal{G}|\). Then there is a perfect matching of \(\mathcal{F}\) in \(KG(\mathcal{F}, \mathcal{G})\).

The key in proving this theorem is to enlarge the class of objects from families to monotone functions. We give the general statement and the proof in Section 2.

One of the nice corollaries of this result is the following theorem.
Theorem 6. If \( \mathcal{F}, \mathcal{G} \subset 2^{[n]} \) are cross-intersecting, then
\[
|\mathcal{G}| + |\mathcal{F}| \leq \max\{|\mathcal{F}^\downarrow|, |\mathcal{G}^\downarrow|\}.
\]

We provide a modest contribution to Chvátal’s conjecture as well. For a family \( \mathcal{F} \) of non-empty subsets one defines the covering number \( \tau(\mathcal{F}) \) as the minimal integer \( t \) such that there exists a \( t \)-set \( T \) with \( T \cap F \neq \emptyset \) for all \( F \in \mathcal{F} \). Let us note that stars have covering number 1. In many problems involving intersecting families, families with covering number 2 are suboptimal.

Theorem 7. Suppose that \( \mathcal{F} \subset \mathcal{G} \subset 2^{[n]} \), \( \mathcal{G} \) is a down-set and \( \mathcal{F} \) is intersecting. If \( \tau(\mathcal{F}) \leq 2 \) then (3) holds.

Our remaining results concern so-called IU-families.

Definition 2. A family \( \mathcal{F} \subset 2^X \) is an intersecting-union family or IU-family for short if for all \( F, G \in \mathcal{F} \) both \( F \cap G \neq \emptyset \) and \( F \cup G \neq X \) hold.

Example. Let \([n] = X \cup Y\) be a partition, \( \mathcal{A} \subset 2^X \) an intersecting family and \( \mathcal{B} \subset 2^Y \) a union family. Define \( \mathcal{A} \times \mathcal{B} := \{ A \cup B : A \in \mathcal{A}, B \in \mathcal{B} \} \). Then \( |\mathcal{A} \times \mathcal{B}| = |\mathcal{A}||\mathcal{B}| \) and \( \mathcal{A} \times \mathcal{B} \) is an IU-family.

Using Theorem 1 one can construct many IU-families of size \( 2^n - 2 \). Apparently unaware of (2) several people, including Daykin–Lovász [3], Schönheim and Seymour (private communication, cf. [3]) proved that \( 2^n - 2 \) is the maximum.

Theorem 8. If \( \mathcal{F} \subset 2^{[n]} \) is IU then
\[
|\mathcal{F}| \leq 2^n - 2.
\]

Proof. Since \( \mathcal{F}^\uparrow \) is intersecting and \( \mathcal{F}^\downarrow \) is union, \( \mathcal{F} \subset \mathcal{F}^\uparrow \cap \mathcal{F}^\downarrow \) implies (3) via (2):
\[
\frac{|\mathcal{F}|}{2^n} \leq \frac{|\mathcal{F}^\uparrow|}{2^n} \frac{|\mathcal{F}^\downarrow|}{2^n} \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},
\]
yielding (6). \( \square \)

Note that Theorem 6 implies a stronger version of (6): if we assume in (5) that \( \mathcal{F}, \mathcal{G} \) are both union, then so are \( \mathcal{F}^\downarrow \) and \( \mathcal{G}^\downarrow \), and so the right hand side of (5) is at most \( 2^{n-1} \).

Putting \( \mathcal{F} = \mathcal{G} \), we recover (6).

Actually, (2) implies a two-families version as well. Given two families \( \mathcal{F}, \mathcal{G} \subset 2^{[n]} \), we say that they are cross-intersecting if \( A \cap B \neq \emptyset \) for any \( A \in \mathcal{F}, B \in \mathcal{G} \). Cross-union and cross-IU are defined analogously.

Theorem 9. Suppose that \( \mathcal{A}, \mathcal{B} \subset 2^{[n]} \) are cross-IU. Then
\[
|\mathcal{A}||\mathcal{B}| \leq 2^{2n-4}.
\]

Applying (7) to \( \mathcal{F} = \mathcal{A}, \mathcal{F} = \mathcal{B} \) obviously yields (6). Considering the sum \( |\mathcal{A}| + |\mathcal{B}| \) would only yield \( |\mathcal{A}| + |\mathcal{B}| \leq 2^n \) because of the trivial choice \( \mathcal{A} = 2^{[n]}, \mathcal{B} = \emptyset \). To circumvent this problem we consider several families that are pairwise IU.

Theorem 10. Let \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_d \subset 2^{[n]} \) be pairwise cross-IU. Then
\[
\sum_{i=1}^d |\mathcal{A}_i| \leq \max \{2^n, d \cdot 2^{n-2}\}.
\]

From the proof it is also clear that the equality holds only if \( d \leq 4 \) and \( \mathcal{A}_1 = 2^{[n]} \) for some \( i \) or \( d \geq 4 \) and \( \mathcal{A}_1 = \ldots = \mathcal{A}_d \). We should mention that Hilton proved an analogous result for pairwise intersecting families \( \mathcal{A}_i \subset \binom{[n]}{k} \).
2 Proof of Theorem 5

In what follows we assume that \( N \) includes 0. As we have mentioned, the key to the proof is to work with monotone functions \( f := 2^{[n]} \to N \). We say that a function \( f \) is monotone (decreasing), if for any set \( A \) and element \( i \) we have \( f(A) \leq f(A \setminus \{i\}) \). For a function \( f : 2^{[n]} \to N \), put \( |f| = \sum_{X \in 2^{[n]}} f(X) \). Similarly, for a function \( p : 2^{[n]} \times 2^{[n]} \to N \), put \( |p| = \sum_{X,Y \in 2^{[n]}} p(X,Y) \).

**Theorem 11.** Given two monotone functions \( f, g : 2^{[n]} \to N \), \( |f| \leq |g| \), there is a function \( p : 2^{[n]} \times 2^{[n]} \to N \), such that

(i) \( p(X,Y) \neq 0 \) only if \( X, Y \subset [n] \) are disjoint;

(ii) \( |p| = |f| \).

(iii) We have \( \sum_{X \in 2^{[n]}} p(X,Y) \leq g(Y) \) for every \( Y \subset [n] \) and \( \sum_{Y \subset 2^{[n]}} p(X,Y) = f(X) \) for every \( X \subset [n] \).

One should think of the function \( p \) as of a weighted matching between disjoint sets (condition (i)) that ‘respects’ the restrictions on the number of occurrences of each set, imposed by \( f, g \) (condition (iii)). Condition (ii) states that ‘\( p \) is a matching of \( f \)’. (Condition (ii) is actually superfluous since it follows from the second part of (iii) via a summation over \( X \subset [n] \).)

If we take \( f, g \) to be the characteristic functions of the down-sets \( \mathcal{F}, \mathcal{G} \) (i.e., \( f(X) = 1 \) if \( X \in \mathcal{F} \) and \( f(X) = 0 \) if \( X \not\in \mathcal{F} \)), then the statement above is equivalent to Theorem 5. Indeed, for each \( A \in \mathcal{F} \) we can put \( \phi(A) = B \), where \( B \) is the only set such that \( p(A,B) \neq 0 \). Condition (i) guarantees that \( A \cap B = \emptyset \), condition (iii) guarantees that \( \phi : \mathcal{F} \to \mathcal{G} \) is injective and covers all \( \mathcal{F} \), i.e., it is a matching of \( \mathcal{F} \).

**Proof.** To simplify the presentation, we assume that \( |f| = |g| \). Otherwise, reduce some of the values of \( g(F), F \in 2^{[n]} \), so that \( |g| = |f| \), while preserving monotonicity. Clearly, if we construct the desired function \( p \) for such \( g \), then the same \( p \) will work for the original \( g \). In this case, we have equality in the inequality from (iii) for any \( X \).

The proof is by induction on \( n \). The statement is easy to see for \( n = 1 \). Assume that it is true for monotone functions on \( [2, n] \) and let us prove it for \( [n] \). For a pair of functions as in the statement, define two functions \( f', g' : 2^{[2,n]} \to N \) as follows: for each \( X \subset [2,n] \), put \( f'(X) = f(X) + f(X \cup \{1\}) \) and \( g'(X) = g(X) + g(X \cup \{1\}) \).

By induction, there is a function \( p' \) for \( f', g' \) as in the statement of the theorem. Now form a bipartite multigraph \( G \) between two copies \( \mathcal{A}, \mathcal{B} \) of \( 2^{[2,n]} \), where for any \( X \in \mathcal{A}, Y \in \mathcal{B} \) we have \( p'(X,Y) \) edges between sets \( X \) and \( Y \). For a set \( F \in 2^{[2,n]} \), we denote by \( F_a, F_b \) its copies in \( \mathcal{A} \) and \( \mathcal{B} \), respectively. By the condition (iii) applied to \( p' \), the degree \( d_{F_a} \) of \( F_a \) in \( G \) is \( f'(F) \) and \( d_{F_b} = g'(F) \).

We shall need the following simple claim.

**Claim 1.** Let \( G \) be a bipartite multigraph and denote \( d_v \) the degree of \( v \in V(G) \). Assume that each \( v \) is assigned a number \( u_v \) such that \( 2u_v \leq d_v \). Then there exists a subset of edges \( W \subset E(G) \) and an orientation on the edges of \( W \), such that the outdegree of \( v \) is exactly \( u_v \) for each \( v \).

**Proof.** The proof is by induction on \( |E| \). This is trivial for \( |E| = 0 \). First, assume that there is a vertex \( v \) of degree 1 in the graph and \( w \) is the only neighbor of \( v \). Then \( u_v = 0 \).
If \( u_w = 0 \) then simply delete the edge \( uw \). If \( u_w > 0 \) then orient the edge \( vw \) towards \( w \), reducing \( u_w \) by 1. Repeating this, we may assume that any vertex in \( G \) has either degree 0 or at least 2. In particular, there is an (even) cycle in \( G \).

Next, take an even cycle \( C = \{v_1, \ldots, v_k\} \) in \( G \) (it may have length 2 there are two edges between \( v_1 \) and \( v_2 \)). If \( u_{v_i} > 0 \) then orient the edge \( v_iv_{i+1} \) (where indices are cyclic modulo \( k \)) from \( v_i \) to \( v_{i+1} \). Otherwise, leave this edge without an orientation. Include the oriented edges of \( C \) into \( W \), remove the edges of \( C \) from \( G \) and reduce every non-zero \( u_{v_i}, i = 1, \ldots, k \), by 1. We obtain a new graph \( G' \) and an assignment \( u'_v, v \in V(G') = V(G) \).

We apply induction to this graph and assignment, but first we want to make sure that the condition \( 2u'_v \leq d'_v \) for every \( v \in V(G') \), where \( d'_v \) is the degree of \( v \) in \( G' \). Indeed, the degree only changed for the vertices of \( C \), and, whenever \( u_{v_i} \) was non-zero, \( u'_{v_i} = u_{v_i} - 1 \) and \( d'_{v_i} = d_{v_i} - 2 \).

Next, apply Claim \( \square \) to \( G \) with \( u_{F_a} = f(F \cup \{1\}) \) and \( u_{F_b} = g(F \cup \{1\}) \). Note that this is a correct assignment of \( u \)'s since \( 2u_{F_a} = 2f(F \cup \{1\}) \leq f(F) + f(F \cup \{1\}) = f'(F) = d_{F_a}(G) \), and similarly for \( u_{F_b} \). We are ready to define the function \( p \).

For a pair of sets \( X, Y \subset 2^{[2,n]} \)

- we put \( p(X \cup \{1\}, Y) \) to be the number of edges in \( G \) between \( X_a, Y_b \) and that are oriented from \( X_a \) to \( Y_b \);
- we put \( p(X, Y \cup \{1\}) \) to be the number of edges in \( G \) between \( X_a, Y_b \) and that are oriented from \( Y_b \) to \( X_a \);
- we put \( p(X, Y) \) to be the number of non-oriented edges in \( G \) between \( X_a \) and \( Y_b \).

Let us verify that \( p \) has all the required properties. First, \( p \) satisfies condition (i) from the proposition (by the definition and since \( p' \) satisfied it). Second, \(|p| = |E(G)| = |p'| = |f'| = |f| = |g| \). Third, for any \( Y \subset [2,n] \) we have

\[
\sum_{X \subset 2^{[2,n]}} p(X, Y \cup \{1\}) = \sum_{X \subset 2^{[2,n]}} p(X, Y \cup \{1\}) = u_Y = g(Y \cup \{1\})
\]

\[
\sum_{X \subset 2^{[2]}} p(X, Y) = \sum_{X \subset 2^{[2,n]}} p(X, Y) + p(X, \{1\}, Y) = d_Y - u_Y = g(Y) - g(Y \cup \{1\}) = g(Y).
\]

The symmetric equalities (roles of \( X \) and \( Y \), as well as \( f \) and \( g \) being switched) are also valid and are checked analogously. This verifies (iii). \( \square \)

### 3 Other proofs

**Proof of Theorem \( \square \)** W.l.o.g., assume that \(|\mathcal{F}^1| \leq |\mathcal{G}^1|\). From Theorem \( \Box \) it follows that we can obtain a matching of \( \mathcal{F}^1 \) in \( KG(\mathcal{F}^1, \mathcal{G}^1) \). Clearly, out of each pair at most one can be included in the respective family (\( \mathcal{F} \) or \( \mathcal{G} \)), and thus \(|\mathcal{F}| + |\mathcal{G}| \leq |\mathcal{G}^1|\). \( \square \)

**Proof of Theorem \( \Box \)** Suppose that \( \tau(\mathcal{F}) = 2 \) and by symmetry that \( \{1, 2\} \) is a cover. For \( i \in [2] \) define

\[
\mathcal{F}_i := \{ F \setminus \{i\} : F \in \mathcal{F}, F \cap [2] = \{i\} \},
\]

\[
\mathcal{F}_{12} := \{ F \setminus \{1, 2\} : F \in \mathcal{F}, \{1, 2\} \subset F \}.
\]
We think of these families as subfamilies of $2^{[3,n]}$. Since $\{1, 2\}$ is a cover for $\mathcal{F}$,

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_{12}|.$$  

Note that, with the analogous notations $|\mathcal{G}(1)| = |\mathcal{G}_1| + |\mathcal{G}_{12}$, $|\mathcal{G}(2)| = |\mathcal{G}_2| + |\mathcal{G}_{12}|$ and obviously $\mathcal{F}_{12} \subset \mathcal{G}_{12}$. Thus (8) will follow from

$$|\mathcal{F}_1| + |\mathcal{F}_2| \leq \max\{|\mathcal{G}_1|, |\mathcal{G}_2|\}. \quad (8)$$

Since $\mathcal{F}$ is intersecting, $\mathcal{F}_1$ and $\mathcal{F}_2$ are cross-intersecting. Noting that $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[3,n]}$ are down-sets, Theorem 6 implies that

$$|\mathcal{F}_1| + |\mathcal{F}_2| \leq \max\{|\mathcal{F}_1^↓|, |\mathcal{F}_2^↓|\} \leq \max\{|\mathcal{G}_1|, |\mathcal{G}_2|\}.$$  

Proof of Theorem 9

Set $\alpha := |\mathcal{A}|/2^n, \alpha^↑ := |\mathcal{A}^↑|/2^n, \alpha^↓ := |\mathcal{A}^↓|/2^n$ and define $\beta, \beta^↑, \beta^↓$ analogously. Note that the Harris–Kleitman correlation inequality implies

$$\alpha \leq \alpha^↑ \alpha^↓, \quad \beta \leq \beta^↑ \beta^↓. \quad (9)$$

Since $\mathcal{A}^↑$ and $\mathcal{B}^↑$ are cross-intersecting, $\alpha^↑ + \beta^↑ \leq 1$, implying

$$\alpha^↑ \beta^↑ \leq \frac{1}{4}. \quad (10)$$

Similarly, $\mathcal{A}^↓$ and $\mathcal{B}^↓$ are cross-union, yielding

$$\alpha^↓ \beta^↓ \leq \frac{1}{4}. \quad (11)$$

Combining (9), (10), (11), we immediately obtain

$$\alpha \beta \leq (\alpha^↑ \beta^↑)(\alpha^↓ \beta^↓) \leq 2^{-4}.$$  

□

Theorem 10 immediately follows from the next two results: Corollary 1 and Theorem 12. But first we need a simple lemma.

Lemma 1. Suppose that $d \geq 1$ and $1 \leq x \leq d$. Then

$$x + \frac{d}{x} \leq 1 + d. \quad (12)$$

Proof. Rearranging yields $(x - 1)(x - d) \leq 0$, which is true for $1 \leq x \leq d$. \hfill \Box

Corollary 1. Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_d \subset 2^{[n]}$ are pairwise cross-IU families, $d \geq 5$ and $|\mathcal{A}_1| \geq |\mathcal{A}_2| \geq \ldots \geq |\mathcal{A}_d|$. Then

$$|\mathcal{A}_1| + \ldots + |\mathcal{A}_d| \leq d2^{n-2}. \quad (13)$$

Moreover, the inequality is strict unless $\mathcal{A}_1 = \ldots = \mathcal{A}_d$. 

Proof. Set \( a_i := |A_i|/2^{n-2} \). We have \( a_1 \geq a_1 \geq \ldots \geq a_d \). In proving (13) we may assume \( 1 \leq a_1 \leq 4 \). In view of (7), \( a_i \leq 1/a_1 \) for \( 2 \leq i \leq d \). Moreover, if there is \( F \in A_i \cap A_j \) for some \( i, j \geq 2 \), then \( A_1 \) and \( \{F\} \cup A_j \) are cross-IU, and we get that \( a_j < 1/a_1 \). Similarly, if there is a set \( F \in A_i \cap A_j \), then \( \{F\} \cup A_i \) and \( A_j \) are cross IU for any \( j \neq 1, i \), and again we get \( a_j < 1/a_1 \). Now (12) implies \( a_1 + \ldots + a_d \leq a_1 + \frac{d-1}{a_1} \leq d \), as desired, and the first inequality is strict unless \( A_1 \supseteq A_2 = \ldots = A_d \). The second inequality is strict unless \( a_1 = \ldots = a_d \), in which case the previous sentence implies \( A_1 = \ldots = A_d \), or if \( a_1 = d - 1 \). This is only possible for \( d = 5 \), but then \( A_2 = \ldots = A_d = \emptyset \), and the inequality (13) is strict. \( \square \)

**Theorem 12.** Let \( A, B \subset 2^{[n]} \) be cross-IU families, \(|A| \geq |B|\). then

\[
|A| + 3|B| \leq 2^n. \tag{14}
\]

**Proof.** Set \( x := |A|/2^n, y = |B|/2^n \). By (7), we know that \( xy \leq 1/16 \). If \( x \leq 3/4 \) then (14) follows from (12). For convenience, set \( z = 1 - x \), and note that w.l.o.g. \( 0 \leq z \leq \frac{1}{2} \).

Note that \((1 - q)q\) is a decreasing function of \( q \) for \( 0 \leq q \leq \frac{1}{2} \). Obviously, \(|A^\uparrow| \geq |A|\), implying \(|A^\uparrow||B^\uparrow| \leq 2^nz(1 - z) + 3z \) and the same for \(|A^\downarrow||B^\downarrow|\). Using the Kleitman-Harris correlation inequality, this leads to

\[
\frac{|A||B|}{2^{2n}} \leq z^2(1 - z)^2
\]

and thereby to \(|B|/2^n \leq z^2(1 - z) = z(z(1 - z)) < \frac{z}{4} \). Consequently,

\[
\frac{|A|}{2^n} + 3 \cdot \frac{|B|}{2^n} \leq 1 - z + \frac{3z}{4} = 1 - \frac{z}{4} \leq 1.
\]

\( \square \)

In the remainder of this section, we will present a proof of Theorem 4.

Let us say that a family \( F \) can be matched to itself if either (i) or (ii) from Theorem 4 holds \( F \), i.e., if either \( F \) or \( F \setminus \emptyset \) can be partitioned into pairs of disjoint sets. Let us restate Theorem 4 for convenience.

**Theorem 13.** If \( A \subset 2^{[n]} \) is a down-set, then it can be matched to itself.

We use the following simple observation.

**Observation 1.** Let \( G_1, G_2 \subset \binom{X}{2} \) be two matchings, then \( G_1 \cup G_2 \) is bipartite.

**Proof of theorem 13.** The proof is by induction on \( n \). Let \( A \subset 2^{[n]} \) be a down-set. Consider the two families \( A(n), A(\bar{n}) \subset 2^{[n-1]} \). They are both down-sets, and \( A(n) \subset A(\bar{n}) \).

Let \((A_i, B_i), 1 \leq i \leq |A(n)|/2 \) be a perfect matching of \( A(n) \) or \( A(n) \setminus \emptyset \). Denote this matching \( G_1 \). Let \((C_i, D_i), 1 \leq i \leq |A(\bar{n})|/2 \) be a perfect matching of \( A(\bar{n}) \) or \( A(\bar{n}) \setminus \emptyset \). The edges with both \( C_i \) and \( D_i \) form the matching \( G_2 \). By Observation 1, the graph \( G_1 \cup G_2 \) is bipartite. Consequently, we can reorder some pairs (replacing \((A_i, B_i)\) with \((B_i, A_i)\) when necessary) to make \( \mathcal{I} := \{A_i : 1 \leq i \leq |A(n)|/2\} \) an independent set in \( G_1 \cup G_2 \).

Next, we define the matching \((A_i, B_i \cup \{n\}), 1 \leq i \leq |A(n)|/2 \) and \((C_j', D_j'), \) where \( C_j' \in \{C_j, C_j \cup \{n\}\}, D_j' \in \{D_j, D_j \cup \{n\}\}, \) and we add \( \{n\} \) only if the corresponding set belongs to \( \mathcal{I} \). Since \( \mathcal{I} \) is an independent set in \( G_2 \), at most one set from \( C_j, D_j \) is enlarged. Thus, we get pairs of disjoint sets that cover almost all \( A \).
In case when both $|A(n)|$ and $|A(\bar{n})|$ are odd, the only two unmatched sets in $A$ are $\{n\}$ and $\emptyset$, so we add an extra pair $(\{n\}, \emptyset)$ to the matching. In case when only $|A(\bar{n})|$ is odd, then the only unmatched set in $A$ is $\emptyset$, and so the matching constructed a paragraph earlier is already the matching of $A$. In case when only $A(n)$ is odd, then replace $A(\bar{n})$ with $A'(\bar{n})$ in the previous argument, where $A'(\bar{n}) = A(\bar{n}) \setminus A$, and $A \in A(\bar{n}) \setminus A(n)$ is any inclusion-maximal set. Then both $A(n)$ and $A'(n)$ are odd, and the argument above gives a matching that covers all sets of $A$, except for $\{n\}, \emptyset$ and $A$. Since $A \in A(\bar{n})$, we can then add a pair $(\{n\}, A)$ to the matching, obtaining the matching of $A$. □

4 Concluding remarks and open problems

In the present paper we proved results related to IU-families. It is very natural to consider the following quantitative version.

Definition 3. Let $t$ and $s$ be positive integers and $F \subset 2^{[n]}$. If $|A \cap B| \geq t$ for any $A, B \in F$ then $F$ is called $t$-intersecting. If $|A \cup B| \leq n - s$ for any $A, B \in F$ then $F$ is called $s$-union. Finally, if $F$ satisfies both properties, then we call it an $(t, s)$-family.

Let $m(n, t) = \max\{|F| : F \subset 2^{[n]}$ is $t$–intersecting}. The exact value of $m(n, t)$ was determined by Katona [10]. Let $(m(n, t, s))$ denote the maximum of $|F|$ over all $(t, s)$-families $F \subset 2^{[n]}$.

Attaching an extra element to the ground set without any sets containing it shows that

$$m(n + 1, t, 1) \geq m(n, t).$$

(15)

Katona [11] conjectured that equality holds in (15). This was proved in [3]. Extending the example above in the natural way shows

$$m(n + n', t, s) \geq m(n, t)m(n', s).$$

(16)

Conjecture 2. Suppose that $n \geq t + s$. Then

$$m(n, t, s) = \max \{m(n', t)m(n - n', s) : t \leq n' \leq n - s\}.$$

(17)

Defining the cross-$(t, s)$ property in the obvious way we have

Conjecture 3. Let $n \geq t + s$ and suppose that $F, G \subset 2^{[n]}$ are cross-$(t, s)$ families. Then

$$|F||G| \leq m(n, t, s)^2.$$

(18)

Let us note that (18) was proved in [6, Chapter 13] for the special case $s = 1$.

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