Boundary value problem with transmission conditions

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Abstract: One important innovation here is that for the Sturm-Liouville considered equation together with eigenparameter dependent boundary conditions and two supplementary transmission conditions at one interior point. We develop Green’s function method for spectral analysis of the considered problem in modified Hilbert space.

Keywords: Sturm-Liouville problems, Green’s function.

1 Introduction

In physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. This derivation based on that in [6], however, is more thorough than that in most elementary physics texts; while most parameters such as density and other thermal properties are treated as constant in such treatments, the following allows fundamental properties of the bar to vary as a function of the bars length, which will lead to a Sturm-Liouville problem of a more general nature. In this study we shall investigate one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$\Gamma(y) := -y''(x, \lambda) + q(x)y(x, \lambda) = \lambda y(x, \lambda) \quad (1.1)$$

to hold in finite interval $(-\pi, \pi)$ except at one inner point $0 \in (-\pi, \pi)$, where discontinuity in $u$ and $u'$ are prescribed by transmission conditions

$$\Gamma_1(y) := a_1y'(0-, \lambda) + a_2y(0-, \lambda) + a_3y'(0+, \lambda) + a_4y(0+, \lambda) = 0, \quad (1.2)$$

$$\Gamma_2(y) := b_1y'(0-, \lambda) + b_2y(0-, \lambda) + b_3y'(0+, \lambda) + b_4y(0+, \lambda) = 0, \quad (1.3)$$

together with the boundary conditions

$$\Gamma_3(y) := \cos \alpha y(-\pi, \lambda) + \sin \alpha y'(-\pi, \lambda) = 0, \quad (1.4)$$
\[ \Gamma_4(y) := \cos \beta y(\pi, \lambda) + \sin \beta y'(\pi, \lambda) = 0, \quad (1.5) \]

where the potential \( q(x) \) is real-valued, continuous in each interval \([-1, 0) \) and \((0, 1]\) and has a finite limits \( q(\pm c) ; \alpha_0, \beta_0, \beta_1 \) are real numbers; \( \lambda \) is a complex eigenparameter. In this study by using an own technique we introduce a new equivalent inner product in the Hilbert space \( L_2(-1,0) \oplus L_2(0,1) \) and a linear operator in it such a way that the considered problem can be interpreted as eigenvalue problem for this operator.

2 Preliminary Results

Let \( T = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix} \). Denote the determinant of the i-th and j-th columns of the matrix \( T \) by \( \rho_{ij} \). Note that throughout this study we shall assume that \( \rho_{12} > 0 \) and \( \rho_{34} > 0 \).

In this section we shall define two basic solutions \( \phi(x, \lambda) \) and \( \chi(x, \lambda) \) by own technique as follows. At first, let us consider solutions of the equation (1.1) on the left-hand \([-\pi, 0) \) of the considered interval \([\pi, 0) \cup (0, \pi] \) satisfying the initial conditions

\[ y(-\pi, \lambda) = \sin \alpha, \quad \frac{\partial y(-\pi, \lambda)}{\partial x} = -\cos \alpha \quad (2.1) \]

By virtue of well-known existence and uniqueness theorem of ordinary differential equation theory this initial-value problem for each \( \lambda \) has a unique solution \( \phi_1(x, \lambda) \). Moreover \([12], \text{Theorem 7}\) this solution is an entire function of \( \lambda \) for each fixed \( x \in [-\pi, 0) \). Using this solutions we can prove that the equation (1.1) on the right-hand interval \( \in (0, \pi] \) of the considered interval \([-\pi, 0) \cup (0, \pi] \) has the solution \( u = \phi_2(x, \lambda) \) satisfying the initial conditions

\[ y(0, \lambda) = \frac{1}{\rho_{12}} (\rho_{23} \phi_1(0, \lambda) + \rho_{24} \frac{\partial \phi_1(0, \lambda)}{\partial x}) \quad (2.2) \]
\[ y'(0, \lambda) = -\frac{1}{\rho_{12}} (\rho_{13} \phi_1(0, \lambda) + \rho_{14} \frac{\partial \phi_1(0, \lambda)}{\partial x}). \quad (2.3) \]

By applying the method of \([5]\) we can prove that the equation (1.1) on \((0, \pi]\) has an unique solution \( \phi_2(x, \lambda) \) satisfying the conditions (2.2)-(2.3) which also is an entire function of the parameter \( \lambda \) for each fixed \( x \in (0, \pi] \). Consequently, the function \( \phi(x, \lambda) \) defined by

\[ \phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda) & \text{for } x \in [-\pi, 0) \\ \phi_2(x, \lambda) & \text{for } x \in (0, \pi]. \end{cases} \quad (2.4) \]
satisfies equation (1.1), the first boundary condition (1.4) and the both transmission conditions (1.2) and (1.3). Similarly, \( \chi_2(x, \lambda) \) be solutions of equation (1.1) on the left-right interval \((0, \pi]\) subject to initial conditions
\[
y(\pi, \lambda) = -\sin \beta, \quad \frac{\partial y(\pi, \lambda)}{\partial x} = \cos \beta.
\]
(2.5)

By virtue of [12], Theorem 7 each of these solutions are entire functions of \( \lambda \) for fixed \( x \). By applying the same technique we can prove there is an unique solution \( \chi_1(x, \lambda) \) of equation (1.1) the left-hand interval \([-\pi, 0)\) satisfying the initial condition
\[
y(0, \lambda) = \frac{-1}{\rho_{34}} (\rho_{14} \chi_2(0, \lambda) + \rho_{24} \frac{\partial \chi_2(0, \lambda)}{\partial x}),
\]
(2.6)
\[
y'(0, \lambda) = \frac{1}{\rho_{34}} (\rho_{13} \chi_2 + \rho_{23} \frac{\partial \chi_2(0, \lambda)}{\partial x}).
\]
(2.7)

By applying the similar technique as in [5] we can prove that the solutions \( \chi_1(x, \lambda) \) are also an entire functions of parameter \( \lambda \) for each fixed \( x \). Consequently, the function \( \chi(x, \lambda) \) defined as
\[
\chi(x, \lambda) = \begin{cases} 
\chi_1(x, \lambda), & x \in [-\pi, 0) \\
\chi_2(x, \lambda), & x \in (0, \pi] 
\end{cases}
\]
satisfies the equation (1.1) on whole \([-\pi, 0) \cup (0, \pi]\), the other boundary condition (1.5) and the both transmission conditions (1.2) and (1.3). In the Hilbert Space \( \mathcal{H} = L_2[-1,0] \oplus L_2(0,1] \) of two-component vectors we define an inner product by
\[
< y, z >_{\mathcal{H}} := \rho_{12} \int_{-\pi}^{0} y(x)\overline{z(x)}dx + \rho_{34} \int_{0}^{\pi} y(x)\overline{z(x)}dx
\]
for \( y = y(x) \), \( z = z(x) \in \mathcal{H} \). We introduce the linear operator \( A : \mathcal{H} \to \mathcal{H} \) with domain of definition satisfying the following conditions
i) \( y \) and \( y' \) are absolutely continuous in each of intervals \([-\pi, 0) \) and \((0, \pi]\) and has a finite limits \( y(c+) \) and \( y'(c+) \)
ii) \( \Gamma y(x) \in \mathcal{H}, \quad \Gamma_1 y(x) = \Gamma_2 y(x) = \Gamma_3 y(x) = \Gamma_4 y(x) = 0 \).

Obviously \( D(A) \) is a linear subset dense in \( \mathcal{H} \). We put
\[
(Ay)(x) = \Gamma y(x), \quad x \in \mathcal{H}
\]
for \( y \in D(A) \). Then the problem (1.1) – (1.5) is equivalent to the equation
\[
Ay = \lambda y
\]
in the Hilbert space $\mathcal{H}$. Taking in view that the Wronskians $W(\phi_i, \chi_i; x) := \phi_i(x, \lambda)\chi'_i(x, \lambda) - \phi'_i(x, \lambda)\chi_i(x, \lambda)$ are independent of variable $x$ we shall denote $w_i(\lambda) = W(\phi_i, \chi_i; x) \ (i = 1, 2)$. By using (1.2) and (1.3) we have $\rho_{12}w_1(\lambda) = \rho_{34}w_2(\lambda)$ for each $\lambda \in \mathbb{C}$. It is convenient to introduce the notation 

$$w(\lambda) := \rho_{34}w_1(\lambda) = \rho_{12}w_2(\lambda). \quad (2.8)$$

**Theorem 2.1.** For all $y, z \in D(A)$, the equality 

$$< Ay, z > = < y, Az > \quad (2.9)$$

holds.

**Proof.** Integrating by parts we have for all $y, z \in D(A)$, 

$$< Ay, z > = \rho_{12} \int_{-\pi}^{\pi} \Gamma y(x)\overline{z(x)}dx + \rho_{34} \int_{0}^{\pi} \Gamma y(x)\overline{z(x)}dx 
= \rho_{12} \int_{-\pi}^{\pi} y(x)\overline{\Gamma z(x)}dx + \rho_{34} \int_{0}^{\pi} y(x)\overline{\Gamma z(x)}dx + \rho_{12}W[y, \overline{z}; 0] 
- \rho_{12}W[y, \overline{z}; -\pi] + \rho_{34}W[y, \overline{z}; \pi] - \rho_{34}W[y, \overline{z}; 0] 
= < y, Az > + \rho_{12}W[y_0, \overline{z_0}; -\pi] - \rho_{12}W[y_0, \overline{z_0}; -\pi] 
+ \rho_{34}W[y_0, \overline{y_0}; \pi] - \rho_{34}W[y_0, \overline{y_0}; 0] \quad (2.10)$$

From the boundary conditions (1.2)-(1.3) it is follows obviously that 

$$W(y, \overline{z}; -\pi) = 0 \quad \text{and} \quad W(y, \overline{z}; \pi) = 0 \quad (2.11)$$

The direct calculation gives 

$$\rho_{12}W(y, \overline{z}; 0) = \rho_{34}W(y, \overline{z}; 0). \quad (2.12)$$

Substituting (2.11) and (2.12) in (2.10) we obtain the equality (2.9). \qed

Relation (2.9) shows that the operator $A$ is symmetric. Therefore all eigenvalues of the operator $A$ are real and two eigenfunctions corresponding to the distinct eigenvalues are orthogonal in the sense of the following equality 

$$\rho_{12} \int_{-\pi}^{0} y(x)\overline{z(x)}dx + \rho_{34} \int_{0}^{\pi} y(x)\overline{z(x)}dx = 0. \quad (2.13)$$

**Theorem 2.2.** $D(A)$ is densely in the Hilbert space $\mathcal{H}$.

**Proof.** \qed

**Theorem 2.3.** The operator $A$ is self-adjoint.

**Proof.** \qed
3 The Green’s function

Let us consider the inhomogeneous differential equation

\[ y'' + (\lambda - q(x))y = f(x), \quad x \in [-\pi, 0] \cup (0, \pi] \quad (3.1) \]

together with the boundary conditions (1.2)-(1.4) and the transmission conditions (1.5). Making use of the definitions of the functions \( \phi_i, \chi_i \) (i = 1, 2) we see that the general solution of the differential equation (3.1) can be represented in the form

\[
y(x, \lambda) = \begin{cases} 
\frac{\chi_1(x, \lambda)}{\omega_1(\lambda)} \int_{-\pi}^{\pi} \phi_1(\xi, \lambda) f(\xi) d\xi + \frac{\phi_1(x, \lambda)}{\omega_1(\lambda)} \int_0^\pi \chi_1(\xi, \lambda) f(\xi) d\xi + c_1 \phi_1(x, \lambda) + d_1 \chi_1(x, \lambda), & \text{for } x \in [-\pi, 0) \\
\frac{\chi_2(x, \lambda)}{\omega_2(\lambda)} \int_0^\pi \phi_2(\xi, \lambda) f(\xi) d\xi + \frac{\phi_2(x, \lambda)}{\omega_2(\lambda)} \int_\xi^\pi \chi_2(\xi, \lambda) f(\xi) d\xi + c_2 \phi_2(x, \lambda) + d_2 \chi_2(x, \lambda), & \text{for } x \in (0, \pi] 
\end{cases} \quad (3.2)
\]

where \( c_i, d_i \) (i = 1, 2) are arbitrary constants. By substitution into the boundary conditions (1.2)-(1.4) we have at once that \( d_1 = 0, \ c_2 = 0 \). \quad (3.3)

Further, substitution (3.2) into transmission conditions (1.2)-(1.3) we have the inhomogeneous linear system of equations for \( c_1 \) and \( d_1 \), the determinant of which is equal to \(-\omega(\lambda)\) and therefore is not vanish by assumption. Solving that system we find

\[
c_1 = \frac{1}{\omega_2(\lambda)} \int_0^\pi \chi_2(\xi, \lambda) f(\xi) d\xi, \quad d_2 = \frac{1}{\omega_1(\lambda)} \int_{-\pi}^0 \phi_1(\xi, \lambda) f(\xi) d\xi \quad (3.4)
\]

Putting this equations in (3.2) we deduce that the problem (3.1), (1.2)-(1.4) has an unique solution \( y := y_f(x, \lambda) \) in the form

\[
y_f(x, \lambda) = \begin{cases} 
\frac{\rho_{11} \chi_1(x, \lambda)}{\omega(\lambda)} \int_{-\pi}^{\pi} \phi_1(\xi, \lambda) f(\xi) d\xi + \frac{\rho_{11} \phi_1(x, \lambda)}{\omega(\lambda)} \int_0^\pi \chi_1(\xi, \lambda) f(\xi) d\xi + c_1 \phi_1(x, \lambda) + d_1 \chi_1(x, \lambda), & \text{for } x \in [-\pi, 0) \\
\frac{\rho_{12} \chi_2(x, \lambda)}{\omega(\lambda)} \int_0^\pi \phi_2(\xi, \lambda) f(\xi) d\xi + \frac{\rho_{12} \phi_2(x, \lambda)}{\omega(\lambda)} \int_\xi^\pi \chi_2(\xi, \lambda) f(\xi) d\xi + c_2 \phi_2(x, \lambda) + d_2 \chi_2(x, \lambda), & \text{for } x \in (0, \pi] 
\end{cases} \quad (3.5)
\]

From this formula we derive that the Green’s function of the problem (3.1), (1.2)-(1.4) can be represented as

\[
G(x, \xi; \lambda) = \begin{cases} 
\frac{\phi(\xi, \lambda) \chi(x, \lambda)}{\omega(\lambda)} & \text{for } -\pi \leq \xi \leq x \leq \pi, \ x, \xi \neq 0 \\
\frac{\phi(x, \lambda) \chi(\xi, \lambda)}{\omega(\lambda)} & \text{for } -\pi \leq x \leq \xi \leq \pi, \ x, \xi \neq 0
\end{cases} \quad (3.6)
\]
and the formula (3.5) can be rewritten in terms of this Green’s function as

\[ y_f(x, \lambda) = \rho_{12} \int_{-\pi}^{0} G(x, \xi; \lambda) f(\xi) d\xi + \rho_{34} \int_{0}^{\pi} G(x, \xi; \lambda) f(\xi) d\xi \] (3.7)

**Theorem 3.1.** The resolvent operator \( R(\lambda, A) \) is compact.

**Proof.**

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