Transparency of PT-symmetric complex potentials for coherent injection

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Abstract

It is known that when two identical waves are injected from left and right on a complex PT-symmetric scattering potential the two-port s-matrix can have uni-modular eigenvalues. If this happens for all energies, there occurs a perfect emission of waves at both ends. We call this phenomenon transparency. Using the versatile PT-Symmetric complex Scarf II potential, we demonstrate analytically that the transparency occurs when the potential has real discrete spectrum i.e., when PT-symmetry is exact (unbroken). Next, we find that exactness of PT-symmetry is only sufficient but not necessary for the transparency. Two other PT-symmetric domains of Scarf II reveal transparency without the PT-symmetry being exact. In these two cases there exist only scattering states. In one case the real part of the potential is a well devoid of real discrete spectrum and in the other real part is a barrier. Other numerically solved models also support our findings.
The handedness (non-reciprocity) [1,7] of reflection probability for left and right injection of waves at a complex non-symmetric potential has given rise to very interesting phenomenon in one-dimensional scattering. These phenomena are spectral singularity [7,8], coherent perfect absorption without [9] and with [10] lasing, uni-modular eigenvalues of s-matrix [11], invisibility [12], pseudo-unitarity [13], anisotropic transmission resonances [13]. These novel phenomena[7,14] have been proposed mostly based on very strong intuitions. Therefore, they occur as a possibility rather than a necessity in scattering from complex non-Hermitain potentials/mediums. Also at the time of their proposals, these effects appeared to occur more generally however later they were found to have interesting limitations.

For instance the for complex non-Hermitian potentials spectral singularity means a discrete positive energy at which both the transmission and reflection probabilities become infinite. It now turns out that SS does not [15] occur in the case of complex PT-symmetric potentials with unbroken PT-symmetry. Coherent perfect absorption (CPA) without lasing was supposed to be the property of complex mediums and it has now been proved [15] that complex PT-symmetric potentials are exceptional in this regard. CPA with lasing [10] was claimed to be the property of PT-symmetric (equal gain/loss) mediums, later it has been found [11] to be a property of the domains of broken PT-symmetry.

For the injection of two identical waves in all respects at a potential from left and right the s-matrix connecting the out-going and incoming waves is given as [11,13]:

\[
S = \begin{pmatrix} r_{\text{left}} & t \\ t & r_{\text{right}} \end{pmatrix}, \tag{1}
\]

The complex eigenvalues of two-port S-matrix (6) are given as

\[
s_{\pm} = \frac{r_{\text{left}} + r_{\text{right}} \pm \sqrt{(r_{\text{left}} - r_{\text{right}})^2 + 4t^2}}{2}, \tag{2}
\]

which from theory of matrices follow \(s_+ s_- = \det S\). Then in view of the result that for PT-symmetric potentials we have \(|\det S| = 1\), two cases may arise. Firstly, when one of \(|s_-|\) and \(|s_+|\) is < 1 and other one > 1. Secondly, both the eigenvalues are uni-modular: \(|s_+| = 1 = |s_-|\), representing coherent perfect emission (CPE) from both sides of the potential. By defining an interesting quantity, \(B(k)\), the condition of uni-modularity of \(s_\pm\) is given as [11,13]

\[
B(k) = \left| \frac{r_{\text{left}}}{t} - \frac{r_{\text{right}}}{t} \right| \leq 2. \tag{3}
\]

For usual Hermitian potentials the scattering is always reciprocal so \(r_{\text{left}} = r_{\text{right}}\). Hence, real Hermitian potentials are transparent for coherent injection.

We would like to remark that even for complex PT-symmetric scattering potentials (s.t., \(V(\pm\infty) = 0\)), the question as to whether or not this condition will be met in various domains
requires computations and search. In this work, we report that the versatile exactly solvable Scarf II potential \cite{2,16} is a special potential which helps in sorting out various parametric domains for transparency explicitly and analytically. We also confirm that other numerically solved complex PT-symmetric potential display similar results.

We write the complex PT-symmetric potential as \cite{17}

$$V(x) = -V_1 \text{sech}^2 x + iV_2 \text{sech} x \tanh x, V_1, V_2 \in \mathbb{R}, V_1 > 0,$$  \hspace{1cm} (4)

and propose to discuss transparency in three domains: (1) when $V_1 > 0$ and $|V_2| < V_1 + 1/4$, (2) when $V_1 > 0$ but $|V_2| > V_1 + 1/4$, (3) $V_1 < 0$.

(1): $V_1 > 0$ and $|V_2| < V_1 + 1/4$: real discrete spectrum (PT-symmetry is exact)

PT-symmetry of a complex PT-symmetric potential is known to be exact (unbroken) \cite{18}, if it has real discrete spectrum and the energy-eigenstates are also (simultaneous) eigenstates of joint operator PT: [Parity $(x \rightarrow -x)$, Time-reversal $(i \rightarrow -i)$]. For (4), two branches of real discrete energy eigenvalues of (4) are given as \cite{17,19} (choosing $2m = 1 = \hbar^2$)

$$E_n = -(n - a)^2, 0 \leq n < a, \quad E_m = -(m - 1/2 - b)^2, 0 \leq m < b + 1/2,$$  \hspace{1cm} (5)

where

$$a = [\sqrt{V_1 + |V_2| + 1/4} + \sqrt{V_1 - |V_2| + 1/4} - 1]/2, \quad b = [\sqrt{V_1 + |V_2| + 1/4} - \sqrt{V_1 - |V_2| + 1/4}]/2, \tag{6}$$

are real if $|V_2| \leq V_0 = V_1 + 1/4$ \cite{17}. The value $V_0$ can now be called the exceptional point (EP \cite{20}) of the potential (4). So for $|V_2| > V_0$ the eigenvalues are complex conjugate pair, PT-symmetry is spontaneously broken and it follows that

$$\text{PT}[\psi_{E_n}(x)] = \psi_{E_n}(x).$$  \hspace{1cm} (7)

We find that exactness of PT-symmetry is only sufficient but not necessary for the transparency. We present cases when a complex PT-symmetric potential can sustain transparency even for $(|V_2| > V_0)$ -the exceptional point (EP\cite{20}). Also, when complex PT-symmetric potential has only scattering states, transparency is sustained for small dissipation(small imaginary part).

Using the elegant analytic amenability of the scarf II potential for reflection and transmission amplitudes \cite{2,16}, we have earlier found the simple forms of the transmission and reflection coefficients: $T(k)$ and $R(k)$ \cite{15}. Here we need to give the simplified expression of $B(k)$ for this case. We find

$$B(k) = 2|\cos \pi a \sin \pi b| \text{sech}\pi k \leq 2, \quad k = \sqrt{2mE/\hbar}, \tag{8}$$
for real values of $a$ and $b$ (5). So it follows that transparency exist in the domain where the PT-symmetric potential has real discrete spectrum and PT-symmetry is exact.

(2) $V_1 > 0$ but $|V_2| > V_1 + 1/4$: PT-symmetry broken:

We find that the parametrization

$$V_1(c) = 2[(c + 1/2)^2 - d^2] + 1/4, |V_2(c)| = 2[d^2 + (c + 1/2)^2], c > 0$$

(9)
gives rise to real part as a potential well when yet no real eigenvalues as $|V_2(c)| > V_1(c) + 1/4 = V_α(c)$. Using the available [2,16] analytic forms of $t(k)$ and $r(k)$, In this case we find

$$r_{\text{left, right}}/t = i \left[ \mp \frac{(\cos 2\pi c + \cosh 2\pi d)}{2\cosh \pi k} + \frac{(\cosh 2\pi d - \cos 2\pi c)}{2\sinh \pi k} \right],$$

(10)

and the transmission coefficient ($T = |t|^2$) as

$$T(k) = \frac{\sinh^2 \pi k \cosh^2 \pi k}{[\sin^2 \pi c + \sinh^2 \pi (d - k)] [\sin^2 \pi c + \sinh^2 \pi (d + k)]}$$

(11)

Inserting the expressions (10) in (3) we obtain

$$B(k) = [\cos 2\pi c + \cosh 2\pi d] \text{ sech} \pi k \leq 2.$$  

(12)

The expressions (10-12) are valid for real values of $c$ and $d$. $B(k)$ is a monotonically decreasing function of $k$ its maximum value is $B(0)$. Here $\cosh 2\pi d \geq 1$ and $-1 < \cos 2\pi c < 1$, therefore the condition of transparency will be met within the contour in $c - d$ plane which is determined by (12) as

$$d(c) \leq \frac{\cosh^{-1}[2 - \cos 2\pi c]}{2\pi}, \quad d_{\text{max}} = d(1/2) = 0.2805,$$

(13)

and is shown in Fig.1(a). Since all the results are periodic function of $c$ so without a loss of generality we can assume $c \in (0,1)$, for other real values of $c$ the contour will be invariant. The curve $V_β(c)$ is obtained by substituting $d(c)$ from Eq. (13) in $V_2(c)$ in Eq. (9). The potential (4) with (9) for any pair of $(c,d)$ as allowed by the condition (13) will enjoy transparency despite breaking of PT-symmetry. In other words in terms of the Scarf II (4), for a given value of $c$, $V_1(c)$ is determined by (9). Further, if $V_α(c) < |V_2| \leq V_β(c)$ transparency is observed without the existence of real discrete spectrum. When $|V_2| < V_α(c)$, transparency is observed along with the existence real discrete spectrum in $V(x)$.

We choose, $c = \pm 0.4, d = 0.3$ in (9) (see Fig. 2(a,b)) to get real part of Scarf II as a well(barrier). However as $d > d_{\text{max}}, s_±$ are not uni-modular up to $E \sim 0.06$. Hence Fig. 2(c) depicts the scenario of non-occurrence of transparency. In both the cases $V(x)$ supports only scattering states and $B(k) > 2$. 


(3) $V_1 < 0$: only scattering states:
In the case (2), notice that if $-1 < c < 0$ and $d(c)$ is again as per Eq. (13), we get

$$V_1 = 2[(c + 1/2)^2 - d^2] + 1/4, |V_2| = 2[d^2 + (c + 1/2)^2], c < 0$$

(14)

and $V_1$ becomes negative. Consequently the real part of the potential becomes a barrier and there are only scattering states. The results (10-13) are again valid here and hence the transparency occurs.

Another branch of Scarf II (4) can be invoked by parameterising

$$V_1 = 2c^2 - 2c + 1/4, V_2 = -(c - 1/2)^2, c \in \left(\frac{2 - \sqrt{2}}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2 + \sqrt{2}}{2}\right).$$

(15)

For this case the transmission coefficient is

$$T(k) = \frac{\sinh^2 \pi k \cosh^2 \pi k}{(\sinh^2 \pi k + \sin^2 \pi c)^2},$$

(16)

and

$$F_{left, right}(k) = i \left[\frac{\pm \cos^2 \pi c}{\cosh \pi k} + \frac{\sin^2 \pi c}{\sinh \pi k}\right].$$

(17)

The reflection coefficient is given as

$$R_{left, right}(k) = T(k)|F_{left, right}(k)|^2.$$  

(18)

Finally we get

$$B(k) = 2 \cos^2 \pi c \sech \pi k \leq 2,$$

(19)

ensuring transparency in this domain that is parametrized in Eq. (15).

Next we show that the well known rectangular complex PT-symmetric

$$V_R(x) = -V_1 \Theta_1(x) + iV_2 \Theta_2(x), \quad \Theta_1(x) = \begin{cases} 1, & |x| \leq L, \\ 0, & |x| > L \end{cases}, \quad \Theta_2(x) = \begin{cases} 0, & |x| \geq L \\ -1, & -L < x < 0 \\ 1, & 0 \leq x < L \end{cases}$$

(20)

and other potentials also displays these feature. We solve them by integrating the Schrödinger equation

$$\frac{d^2 \Psi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\Psi(x) = 0,$$

(21)

numerically. For the injection from left, we use

$$\psi(x < -L) = Ae^{ikx} + Be^{-ikx}$$

(22)

$$\psi(|x| \leq L) = f(x)$$

$$\psi(x \geq L) = Ce^{ikx}.$$
We start the integration of Eq. (21) from right $x = L$, using $\psi(L) = Ce^{ikL}$, $\psi'(L) = Cike^{ikL}$. Here $C$ is any arbitrary real or non-real number. We integrate upto $x = -L$ and save $f(-L)$ to find

$$r_{\text{left}}(E) = e^{-2ikL} \frac{ikf(-L) - f'(-L)}{ikf(-L) + f'(-L)}, \quad t_{\text{left}}(E) = e^{-ikL} \frac{2ik}{ikf(-L) + f'(-L)}.$$  \hspace{1cm} (23)

For the injection from right, we have

$$\psi(x > L) = Be^{ikx} + Ae^{-ikx}$$

$$\psi(|x| \leq L) = g(x)$$

$$\psi(x < -L) = Ce^{-ikx}.$$  \hspace{1cm} (24)

This time we integrate Eq. (21) from $x = -L$, using $\psi(-L) = Ce^{ikL}$, $\psi'(-L) = -Cike^{ikL}$ and save $g(L)$. So for the injection from right, we get:

$$r_{\text{left}}(E) = e^{-2ikL} \frac{ikg(L) + g'(L)}{ikg(L) - g'(L)}, \quad t_{\text{left}}(E) = e^{-ikL} \frac{2ik}{ikg(L) - g'(L)}.$$  \hspace{1cm} (25)

Using this numerical procedure, we study the complex PT-symmetric potential (20) by fixing $V_1 = 5, a = 2$ ($2m = 1 = \hbar^2$) and varying $V_2$. We find that for $|V_2| \leq 2.19$ the scattering coefficients $R_{\text{left}}, R_{\text{right}}$ and $T_{\text{left}} = T_{\text{right}}$ commonly show two (poles) spikes indicating the existence of two real discrete bound states. In Fig. 3, we present only one (left) $R(E)$. The PT-symmetry breaks down spontaneously when $|V_2| = 2.20$ and poles at real discrete energies disappear to give way to a single maximum. In this case the real eigenvalues coalesce to complex conjugate pairs of energy eigenvalues. Thus, $|V_2| = 2.20$ is an exceptional point [20] of this non-Hermitian potential.

In figure 4, notice that even at/above the value 2.20 the potential (20) sustains transparency for $|V_2| = 2.20, 2.30$ and $B(k) \leq 2$. However, for $|V_2| = 2.40$ transparency can not be observed as $s_\pm$ are not uni-modular up to $E \sim 0.35$, also we get $B(k) > 2$.

Not shown here are the cases when the real part of $V(x)$ is a barrier having only scattering states, yet displaying transparency. Once such case is when $V_1 = -0.70, |V_2| = 0.10$ in (20).

We find that in the cases when the complex PT symmetric potential is of finite support there exists an energy say $E_s$ at which and interesting transition takes place: if $s_+(E < E_s) > s_-(E < E_s)$ then $s_+(E > E_s) < s_-(E > E_s)$. See the vertical line in the loop in Fig. 4(c).

Apart from the potentials (4,20), we have also investigated the potentials of the type

$$V(x) = V_1 \phi_e(x) + iV_2 \phi_o(x), \quad \phi_{e,o}(\pm \infty) = 0,$$  \hspace{1cm} (26)

here $e(o)$ denote even(odd) parity functions.
The features presented by the rectangular potential (20) are also displayed by several
other numerically solved potentials wherein we use parabolic, triangular, Gaussian profiles
of finite support. However, in case of long ranged potentials like (un-truncated) Gaussian
when transparency does not occur we do not find the said cross-over of $|s_\pm|$. Instead, we get
the scenario (see Fig. 2(c)) like that of the long ranged Scarf II potential.

Based on the potentials (4, 20, 26), below we summarize our findings for the phenomenon
of transparency of complex PT-symmetric for coherent injection.

- Complex PT-symmetric potentials share yet another feature common with the Hermitian
  potentials. This feature is transparency of the potential for coherent injection.
- Complex PT-symmetric potentials with real part as a well, have two critical values of the
  strength parameter of the imaginary part of the potential, they are $V_\alpha$ and $V_\beta$. The former
  is the exceptional point of the non-Hermitian potential below which the potential has real
  discrete spectrum and PT-symmetry is unbroken. We find that above $V_\alpha$ there exists an-
  other exceptional value $V_\beta$ upto which transparency (uni-modularity of $s_\pm$ at any energy)
  is observed. This implies that unbroken PT-symmetry is only sufficient but not necessary
  for transparency. The special and versatile complex Scarf II demonstrates these features
  explicitly and analytically.
- The complex PT-symmetric scattering potentials having real part as a barrier display
  transparency for small values of dissipation (small imaginary part).
- The complex PT-symmetric potentials of compact support give rise to a critical value
  of energy, $E_s$, about which the values $|s_\pm| \neq 1$ display a cross-over transition. It will be
  interesting to carry-out more investigation in this regard.
- For complex PT-symmetric potentials these are small values of energy where the eigenval-
  ues of s-matrix may not be uni-modular whereas for higher energies they are so. Therefore,
  if $B(k) \leq 2$ the potential will be transparent for coherent injection.
- The parametric domain of transparency in a complex PT-symmetric scattering potential
  is devoid of spectral singularity. See transmission coefficients (11,16) and in Eq. 22 (of Ref.
  [15]) for the three cases presented above.
- The simple analytic expressions in Eqs. (8,11,12,16-19) presented here are new.

We have called, the uni-modularity of the eigenvalues of s-matrix at any energy of a
complex PT-symmetric potential, transparency and investigated it in details. We hope that
the present work will generate further interest and investigations in transparency.
References

[1] Z. Ahmed, Phys. Rev. A 64 (2001) 042716.
[2] G. Levai, F. Cannata and A. Ventura, J. Phys. A: Math. Gen. 34 (2001) 839.
[3] G.S. Agarwal and S. D. Gupta, Opt. Lett. 27 (2002) 1205.
[4] R. N. Deb, A. Khare, B.D. Roy, Phys. Lett A 307 (2003) 215.
[5] Z. Ahmed, Phys. Lett. A 324 (2004) 152.
[6] F. Cannata, J.-P. Dedonder and A. Ventura, Ann. Phys.(N.Y.) 322 (2007) 397.
[7] A. Mostafazadeh, Phys. Rev. Lett. 102 (2009) 220402.
[8] Z. Ahmed, J. Phys. A: Math. Theor. 42 (2009) 472005; 45 (2012) 032004.
[9] Y. D. Chong, Li Ge. Hui Cao and A. D. Stone, Phys. Rev. Lett. 105 (2010) 053901.
[10] S. Longhi, Phys. Rev. A 82 (2010) 031801 (R).
[11] Y.D. Chong, Li Ge, and A.D. stone, Phys. Rev. Lett. 106 (2011) 093902.
[12] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao and D.N. Chistodoulides, Phys. Rev.
     Lett. 106 (2011) 213901.
[13] Li Ge, Y.D. Chong and A.D. Stone, Phys. Rev. A 85 (2012) 032802.
[14] A. Mostafazadeh, J. Phys. A Math. Theor. 45 (2012) 444024.
[15] Z. Ahmed, J. Phys. A Math. Theor. 47 (2014) 385303.
[16] A. Khare and U.P. Sukhatme, J. Phys. A 21 (1988) L501.
[17] Z. Ahmed, Phys. Lett. A 282 (2001) 343; 287(2001) 295.
[18] C.M. Bender and S. Boettcher,Phys. Rev. Lett. 80(1998) 5243.
[19] B. Bagchi and C. Quesne, Phys. Lett. A 273 (2000) 285.
[20] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. 34 (2001) L391.
     M. Znojil, J. Phys. A: Math. Gen. 39 (2006) 441.
FIG. 1: (a): The plot of \( d(c) \) (13). This curve marks the boundary above which \( (d_{\text{max}} = 0.28, \) see the dashed line) the \( s_\pm \) will not be uni-modal for all energies of injection. (b): The solid line denotes \( V_1(c) \), the short-dashed curve denotes \( V_\alpha(c) = V_1(c) + 0.25 \) and the long-dashed curve denotes \( V_\beta \). These values are such that \( V_1(c) < V_\alpha(c) < V_\beta(c) \). Given \( c \), \( V_1(c) \) gets fixed then if \( V_\alpha(c) < |V_2| < V_\beta(c) \), transparency is observed without the existence of real discrete spectrum. When \( |V_2| < V_\alpha \), transparency is observed along with the existence real discrete spectrum in the \( V(x) \).

FIG. 2: The depiction of non-occurrence of transparency. Scarf II potentials Fig. 2(a): \( c = 0.4 \) and \( d = 0.3 \) and Fig. 2(b): \( c = -0.4, d = 0.3 \). Notice that \( d > d_{\text{max}} = 0.28 \) (see Eq. (13)). In Fig. 2(a), the real part is a well and in Fig. 2(b) it is a barrier. In both the cases commonly, the eigenvalues of \( s \)-matrix i.e., \( s_\pm \) are not uni-modal at all energies. In Fig. 2(c), see the loop formed by solid and short-dashed lines for energies up to \( \sim 0.06 \). For higher energies they are uni-modal. Notice that \( B(0) \) value goes above the critical value 2 at lower values of energy.
FIG. 3: For rectangular potential (20) with $V_1 = 5, a = 2$, $R(E)$ for left injection is plotted to show the occurrence of two bound levels (poles) at negative energies: Fig. 3(a) when $V_2 = 2.00$. In Fig. 3(b) when $V_2 = 2.19$ the real eigenvalues are very close by. In Fig. 3(c) for $V_2 = 2.20$ these two levels coalesce to become complex conjugate pairs and PT-symmetry is spontaneously broken, the poles disappear to give rise to a single maximum. $V_2 = 2.20$ is the exceptional point of this non-Hermitian potential.

FIG. 4: $s_{\pm}$ for the same potential as in Fig. 3. Fig. 4(a): $|V_2| = 2.20$ and Fig. 4(b): $V_2 = 2.30$), the transparency is sustained. However, in Fig. 4(c): $V_2 = 2.40$ the transparency is not observed as $s_{\pm}$ do not remain uni-modular at lower energies of injection (see a loop of short dashed lines and solid line in Fig. 4(c)). Also see the vertical line in the loop sowing the discontinuous cross-over of $|s_{\pm}|$ at $E \sim 0.12$. Also see that $\mathcal{B}(k)$ (long-dashed line) remain $\leq 2$ in Figs. 4(a,b) but in Fig. 4(c) it is $> 2$. 