The Cauchy integral, Bounded and Compact Commutators

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Abstract
We study the commutator of the well-known Cauchy integral with a locally integrable function $b$ on $\mathbb{R}$, and establish a characterisation of the BMO space on $\mathbb{R}$ via the $L^p$ boundedness of this commutator. Moreover, we also establish a characterisation of the VMO space on $\mathbb{R}$ via the compactness of this commutator.

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1 Introduction and Statement of Main Results

The commutator of a singular integral operator $T$ with a function $b$ is defined by

$$[b, T](f) := bT(f) - T(bf).$$

Commutators arise in various contexts. Here we focus on their use in characterising the BMO and VMO spaces of functions of bounded and vanishing mean oscillation, respectively. The first characterisation of BMO via boundedness of commutators is due to Coifman, Rochberg and Weiss [CRW76]. They showed that a function $b$ is in $\text{BMO}(\mathbb{R})$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$, where $T$ is a convolution singular integral operator (SIO). The first characterisation of VMO via compactness of commutators is due to Uchiyama [Uch78]. He showed that a function $b$ is in $\text{VMO}(\mathbb{R})$ if and only if the commutator $[b, T]$ is compact on $L^p(\mathbb{R}^n)$, where $T$ is a convolution SIO. Since then, many other proofs of these fundamental results have appeared, and they have been extended to various settings. Specifically, the commutators considered are with certain singular integral operators, including linear, nonlinear and multilinear operators acting on a variety of underlying spaces. See for example [Blo85, FL02, KL2, HLW17, LW17, LOR, BDMT15, BT13, CT15, TYY] and the references therein.

The purpose of this paper is to establish such characterisations when the operator $T$ is the well-known Cauchy integral $C_\Gamma$, which is a particular example of a non-convolution operator. We state our main results as follows, starting with the boundedness result.

**Theorem 1.1.** Suppose $b \in \bigcup_{1 < q < \infty} L^q_{\text{loc}}(\mathbb{R})$ and suppose $p \in (1, \infty)$. Then the following results hold.

1. If $b$ is in $\text{BMO}(\mathbb{R})$, then $[b, C_\Gamma]$ is bounded on $L^p(\mathbb{R})$ with
   $$\| [b, C_\Gamma] : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \| \leq C_1 \|b\|_{\text{BMO}(\mathbb{R})}.$$

2. If $[b, C_\Gamma]$ is bounded on $L^p(\mathbb{R})$, then $b$ is in $\text{BMO}(\mathbb{R})$ with
   $$\|b\|_{\text{BMO}(\mathbb{R})} \leq C_2 \| [b, C_\Gamma] : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \|.$$

We also establish the following compactness result.

**Theorem 1.2.** Suppose $b \in \text{BMO}(\mathbb{R})$ and suppose $p \in (1, \infty)$. Then the following results hold.

1. If $b$ is in $\text{VMO}(\mathbb{R})$, then $[b, C_\Gamma]$ is a compact operator on $L^p(\mathbb{R})$.

2. If $[b, C_\Gamma]$ is a compact operator on $L^p(\mathbb{R})$, then $b$ is in $\text{VMO}(\mathbb{R})$. 
In this paper, all functions considered are real-valued functions defined on \(\mathbb{R}\). For every \(x \in \mathbb{R}\), \(r \in \mathbb{R}_+\), we define the interval
\[
I(x, r) := (x - r, x + r).
\]

For \(\lambda > 0\) we define the dilate \(\lambda J\) of an interval \(J\) to be the interval with the same midpoint as \(J\) and length \(\lambda |J|\). In particular, \(\lambda I(x, r) = I(x, \lambda r)\) for all \(x \in \mathbb{R}\), \(r \in \mathbb{R}_+\) and \(\lambda > 0\). Given \(y \in \mathbb{R}\), we define the translate \(J + y := \{x + y : x \in J\}\) of an interval \(J\). We use the notation \(\int_J f(x) \, dx := \frac{1}{|I|} \int_I f(x) \, dx\). Throughout the paper, we denote by \(C\) and \(\tilde{C}\) positive constants that are independent of the main parameters, but that may vary from line to line. For \(p \in (1, \infty)\), \(p'\) means the conjugate of \(p\): \(1/p' + 1/p = 1\). If \(f \leq Cg\), we write \(f \preceq g\) or \(g \succeq f\); and if \(f \preceq g \preceq f\), we write \(f \sim g\).

This paper is organised as follows. In Section 2, we recall some definitions and theorems which will be used in the proofs of our results. In Section 3, we prove our first result, which is about the relationship between \(\text{BMO}\) functions and the boundedness of the commutator. In Section 4, we prove our second result, which is about the relationship between \(\text{VMO}\) functions and the compactness of the commutator.

## 2 Preliminaries

In this section we recall the space \(\text{BMO}\) of functions of bounded mean oscillation, the space \(\text{VMO}\) of functions of vanishing mean oscillation, singular integral operators, Calderón–Zygmund operators, the Frechét–Kolmogorov theorem, and the Cauchy integral.

### 2.1 BMO and VMO spaces

**Definition 2.1.** (BMO) A locally integrable real-valued function \(f : \mathbb{R} \to \mathbb{R}\) is said to be of bounded mean oscillation, written \(f \in \text{BMO}\) or \(f \in \text{BMO}(\mathbb{R})\), if
\[
\|f\|_{\text{BMO}} := \sup_{x \in \mathbb{R}, \ r > 0} M(f, I(x, r)) := \sup_I \frac{1}{|I|} \int_I |f(x) - f_I| \, dx < \infty,
\]
where
\[
f_I := \frac{1}{|I|} \int_I f(y) \, dy
\]
is the average of the function \(f\) over the interval \(I\). Here \(I\) denotes an interval in \(\mathbb{R}\).
We denote by \( \text{VMO}(\mathbb{R}) \) the space of functions of \textit{vanishing mean oscillation}, defined to be the \( \text{BMO}(\mathbb{R}) \)-closure of the set \( \mathcal{D} := C_c^\infty(\mathbb{R}) \) of \( C_c^\infty(\mathbb{R}) \) functions with compact support. We note that our definition of \( \text{VMO}(\mathbb{R}) \) is the same as that of \( \text{VMO}(\mathbb{R}) \) in [CW77], as well as that of \( \text{CMO}(\mathbb{R}) \) in [Uch78].

There are several characterisations of \( \text{VMO} \) in the literature. Here we use the characterisation appearing in [Daf02].

**Definition 2.2.** (VMO) [Daf02] A \( \text{BMO} \) function \( f : \mathbb{R} \to \mathbb{R} \) is said to be of \textit{vanishing mean oscillation}, written \( f \in \text{VMO} \) or \( f \in \text{VMO}(\mathbb{R}) \), if

\[
\begin{align*}
(1) \quad & \lim_{\delta \to 0} \sup_{I, |I| \leq \delta} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx = 0, \\
(2) \quad & \lim_{R \to \infty} \sup_{I, |I| \geq R} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx = 0, \text{ and} \\
(3) \quad & \lim_{R \to \infty} \sup_{I, I \cap (0, R) = \emptyset} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx = 0.
\end{align*}
\]

In [Uch78] Lemma, Section 3, p.166], Uchiyama characterises the space \( \text{VMO}(\mathbb{R}^n) \) (denoted there by \( \text{CMO}(\mathbb{R}^n) \)) in terms of three conditions similar to those in Definition 2.2, but which are expressed in terms of the quantity \( \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy \). As we note in Section 4 this infimum is attained when \( c \) is any median \( \alpha_I(f) \) of \( f \) on \( I \). It is straightforward to show directly that Uchiyama’s definition is equivalent to Definition 2.2.

We refer the reader to Bourdaud’s paper [Bou02] for a careful treatment of various \( \text{BMO} \) and \( \text{VMO} \) spaces, and in particular a clarification of the confusion of the \( \text{VMO} \) and \( \text{CMO} \) notation.

### 2.2 Singular Integral Operators

**Definition 2.3.** [Chr90b] (Standard kernel) A \textit{kernel} \( K \) on \( \mathbb{R} \) is a function \( K : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \). A kernel \( K \) is said to \textit{satisfy standard estimates} if there exist \( \delta > 0 \) and \( C < \infty \) such that for all distinct \( x, y \in \mathbb{R} \) and all \( y' \) with \( |y - y'| < |x - y|/2 \) we have:

\[
\begin{align*}
(\text{i}) \quad & |K(x, y)| \leq C|x - y|^{-1}, \\
(\text{ii}) \quad & |K(x, y) - K(x, y')| \leq C\left(\frac{|y-y'|}{|x-y|}\right)^\delta |x - y|^{-1}, \text{ and} \\
(\text{iii}) \quad & |K(y, x) - K(y', x)| \leq C\left(\frac{|y-y'|}{|x-y|}\right)^\delta |x - y|^{-1}.
\end{align*}
\]

The smallest constant \( C \) for which properties (i)–(iii) hold is denoted by \( |K|_{CZ} \).

**Definition 2.4.** [Chr90b] (Operators associated to a kernel) Let \( \mathcal{D}' \) denote the space of distributions dual to \( \mathcal{D} = C_c^\infty(\mathbb{R}) \). A continuous linear
operator $T : C^\infty_c(\mathbb{R}) \to \mathcal{D}'$ is said to be associated to a kernel $K$ if whenever $f, g \in C^\infty_c(\mathbb{R})$ have disjoint supports, we have

$$\langle Tf, g \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} K(x, y) f(y) g(x) \, dy \, dx.$$ 

Here the brackets denote the natural pairing of $\mathcal{D}'$ with $C^\infty_c(\mathbb{R})$. Since $Tf$ is in the dual $\mathcal{D}'$ of $C^\infty_c(\mathbb{R})$, it is a bounded linear functional that acts on functions $g$ in $C^\infty_c(\mathbb{R})$.

**Definition 2.5.** [Chr90b] (Singular integral operators on $\mathbb{R}$) A singular integral operator (SIO) on $\mathbb{R}$ is a continuous linear mapping from $C^\infty_c(\mathbb{R})$ to $\mathcal{D}'$ which is associated to a standard kernel.

**Definition 2.6.** [Chr90b] (Calderón–Zygmund operators on $\mathbb{R}$) Let $T$ be a SIO on $\mathbb{R}$. $T$ is a Calderón–Zygmund operator (CZO) on $\mathbb{R}$ if it extends to a bounded operator from $L^2(\mathbb{R})$ to itself.

A SIO $T$ is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, for $p \in (0, \infty)$, if there exists a constant $C$ such that $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p(\mathbb{R})$. A SIO $T$ is compact on $L^p(\mathbb{R})$ if for all bounded sets $E \subset L^p(\mathbb{R})$, $T(E)$ is precompact. A set $S$ is precompact if its closure is compact. A common way to check precompactness is to use the criteria established in the well known Fréchet–Kolmogorov theorem [Yos80, p.275], [Bre10, pp.111–114].

**Theorem 2.7.** (Fréchet–Kolmogorov theorem) For $1 < p < \infty$, a subset $E$ of $L^p(\mathbb{R})$ is totally bounded (or precompact) if and only if the following three statements hold:

(a) $E$ is uniformly bounded, i.e., $\sup_{f \in E} \|f\|_{L^p(\mathbb{R})} < \infty$;

(b) $E$ vanishes uniformly at infinity, i.e., for every $\varepsilon > 0$, there exists a compact region $K_\varepsilon$ such that for every $f \in E$, $\|f\|_{L^p(K_\varepsilon)} < \varepsilon$; and

(c) $E$ is uniformly equicontinuous, i.e., for every $f \in E$, $\lim_{|z| \to 0} \|f(\cdot + z) - f(\cdot)\|_{L^p(\mathbb{R})} = 0$.

### 2.3 Cauchy Integral

Suppose $\Gamma$ is a curve in the complex plane $\mathbb{C}$ and $f$ is a function defined on the curve $\Gamma$. The Cauchy integral of $f$ is the operator $\mathcal{C}_\Gamma$ given by

$$\mathcal{C}_\Gamma(f)(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$
A curve $\Gamma$ is said to be a **Lipschitz curve** if it can be written in the form $\Gamma = \{x + iA(x) : x \in \mathbb{R}\}$ where $A : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition
\begin{equation}
|A(x_1) - A(x_2)| \leq L|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}.
\end{equation}

The best constant $L$ in (2.2) is referred to as the **Lipschitz constant** of $\Gamma$ or of $A(x)$. One can show that $A$ satisfies a Lipschitz condition if and only if $A$ is differentiable almost everywhere on $\mathbb{R}$ and $A' \in L^\infty(\mathbb{R})$. The Lipschitz constant is $L = \|A'||_\infty$.

The Cauchy integral associated with the Lipschitz curve $\Gamma$ is the SIO $\tilde{C}_\Gamma$ given by
\begin{equation}
\tilde{C}_\Gamma(f)(x) := \text{p.v.} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{(1 + iA'(y))f(y)}{y - x + i(A(y) - A(x))} \, dy,
\end{equation}
where $f \in C_\infty^c(\mathbb{R})$. The kernel of $\tilde{C}_\Gamma$ is
\begin{equation}
\tilde{C}_\Gamma(x, y) = \frac{1}{\pi i} \frac{1 + iA'(y)}{y - x + i(A(y) - A(x))},
\end{equation}
which is not a standard kernel because the function $1 + iA'$ does not necessarily possess any smoothness. As noted in [Gra04, p.289], the $L^p$-boundedness of $\tilde{C}_\Gamma$ is equivalent to that of the related operator $C_\Gamma$ defined by
\begin{equation}
C_\Gamma(f)(x) := \text{p.v.} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(y)}{y - x + i(A(y) - A(x))} \, dy.
\end{equation}
Moreover, as we will see in Lemma 3.3, the kernel
\begin{equation}
C_\Gamma(x, y) = \frac{1}{\pi i} \frac{1}{y - x + i(A(y) - A(x))}
\end{equation}
of $C_\Gamma$ satisfies standard estimates, and while $C_\Gamma(f)$ is initially defined for $f \in C_\infty^c(\mathbb{R})$, the operator $C_\Gamma$ can be extended to all $f \in L^p(\mathbb{R})$, for each $p \in (1, \infty)$.

Note that the operators $C_\Gamma, \tilde{C}_\Gamma$ and $C_\Gamma$ defined in (2.1), (2.3) and (2.4) are all distinct. For the rest of this paper, we work with the operator $C_\Gamma$ given by equation (2.4); we call $C_\Gamma$ the **Cauchy integral**. Also, for convenience we omit the factor $1/((\pi i)$ from its kernel from here on.

### 3 Proof of Theorem 1.1: Boundedness of $[b, C_\Gamma]$}

In this section, we prove our first result, which is about the boundedness of the commutator $[b, C_\Gamma]$. The main ingredient in the proof of Theorem 1.1 is...
is the characterisation of the function space $\text{BMO}(\mathbb{R}^n)$ via commutators in a multilinear ($m$-linear) setting. The necessity of the BMO condition was proved in [CRW76] in the linear setting ($m = 1$) on $\mathbb{R}^n, n \geq 1$. For the $m$-linear setting on $\mathbb{R}^n, n \geq 1$, it was proved in [Cha16]. The sufficiency of the BMO condition in the $m$-linear setting on $\mathbb{R}^n, n \geq 1$ was shown in [LOPTT09, PT03, Tan08]. These results are also stated as Theorem 1.4 in [LW17]. See also the recent paper [LOR]. In this paper, we work in the linear setting ($m = 1$) with the real line ($n = 1$) being the underlying space. Below we state these results in the special case where $m = n = 1$.

**Theorem 3.1.** [CRW76] Suppose that $T$ is an $L^p$-bounded SIO for some $p$ with $1 < p < \infty$. If $b$ is in $\text{BMO}(\mathbb{R})$, then the commutator $[b, T]$ is a bounded map from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for all $p$ with $1 < p < \infty$, with

$$
\| [b, T] : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \| \leq C \| b \|_{\text{BMO}(\mathbb{R})}.
$$

**Theorem 3.2.** [LOPTT09, PT03, Tan08] Suppose that $b \in L^p_{\text{loc}}(\mathbb{R})$ and $T$ is 1-1-homogeneous. If $[b, T]$ is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for some $p$ with $1 < p < \infty$, then $b$ is in $\text{BMO}(\mathbb{R})$ with

$$
\| b \|_{\text{BMO}(\mathbb{R})} \leq C \| [b, T] : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \|.
$$

**Definition 3.3.** A SIO $T$ is called $m$-$n$-homogeneous if there exists a constant $C > 0$ such that for all $M > 10$, for all $r > 0$, and for all collections of $m+1$ pairwise disjoint balls $B_0(x_0, r), \ldots, B_m(x_m, r)$ in $\mathbb{R}^n$ satisfying the condition

$$
|y_0 - y_l| \sim Mr \quad \text{for all } y_0 \in B_0 \text{ and for all } y_l \in B_l, l = 1, 2, \ldots, m,
$$

we have

$$
|T(\chi_{B_1}, \ldots, \chi_{B_m})(x)| \geq \frac{C}{M_m} \quad \text{for all } x \in B_0(x_0, r).
$$

We note again that in this paper, $m = n = 1$. If we can show that the Cauchy integral $C_T$ satisfies the hypotheses of Theorems 3.1 and 3.2, then Theorem 1.1 is proved. Specifically, if we can show that $C_T$ is an $L^p$-bounded SIO for some $p \in (1, \infty)$, then the first part of Theorem 1.1 is proved. Similarly, if we can show that $C_T$ is 1-1-homogeneous, then the second part of Theorem 1.1 is proved. These results are presented in Lemma 3.4 and 3.5.

**Lemma 3.4.** The Cauchy integral $C_T$ is an $L^p$-bounded SIO, for every $p \in (1, \infty)$.
Proof. The Cauchy integral $C_\Gamma$ is a SIO if it is a continuous linear mapping
from $C^\infty_c(\mathbb{R})$ to $\mathcal{D}'$ that is associated to a standard kernel. Recall that the
kernel of $C_\Gamma$ is
$$C_\Gamma(x, y) = \frac{1}{y - x + i(A(y) - A(x))}.$$ 
In Example 4.1.6 in [Gra04], it is noted that $C_\Gamma(x, y)$ is a standard kernel.
In particular, $C_\Gamma(x, y)$ has the following properties, for all $x, y, y' \in \mathbb{R}$ such that $|y - y'| \leq \frac{1}{2}|y - x|$:

$$|C_\Gamma(x, y)| \leq \frac{1}{|y - x|}, \quad (3.1)$$

$$|C_\Gamma(x, y) - C_\Gamma(x, y')| \leq \frac{2(L + 1)|y' - y|}{|y - x|^2}, \quad (3.2)$$
where $L > 0$ is the Lipschitz constant of $A(x)$, and

$$|C_\Gamma(y, x) - C_\Gamma(y', x)| \leq \frac{2(L + 1)|y' - y|}{|y - x|^2}. \quad (3.3)$$

Therefore, the Cauchy integral $C_\Gamma$ is a SIO.

Coifman, McIntosh and Meyer [CMM82] showed that $C_\Gamma$ is bounded
on $L^2$. Additionally, Calderón and Zygmund showed that a SIO which is
bounded on $L^2$ is also bounded on $L^p$ for all $p \in (1, \infty)$. Thus the Cauchy
integral $C_\Gamma$ is bounded on $L^p$ for all $p \in (1, \infty)$. This completes the proof
of Lemma 3.4.

Lemma 3.5. The Cauchy integral $C_\Gamma$ is 1-1-homogeneous.

Proof. We need to show that there exists a constant $C > 0$ such that for
all $M > 10$, for all $r > 0$, and for all disjoint intervals $I_0 = I_0(x_0, r)$
and $I_1 = I_1(x_1, r)$ satisfying the condition

$$|y_0 - y_1| \sim Mr \quad \text{for all } y_0 \in I_0, y_1 \in I_1, \quad (3.4)$$

we have

$$|C_\Gamma(\chi_{I_1})(x)| \geq \frac{C}{M} \quad \text{for all } x \in I_0(x_0, r).$$

Fix an $M > 10$, $r > 0$, and disjoint intervals $I_0 = I_0(x_0, r)$ and $I_1 =
I_1(x_1, r)$ satisfying the condition (3.4). Note that by the choice of the intervals $I_0$ and $I_1$, for each fixed $y_0 \in I_0$ we have either

$$y_1 > y_0 \quad \text{for all } y_1 \in I_1, \quad \text{or} \quad y_1 < y_0 \quad \text{for all } y_1 \in I_1.$$
We will consider the case $y_1 > y_0$. The case $y_1 < y_0$ follows exactly the same reasoning. Now for each $x \in I_0, y \in I_1$ satisfying the condition $|x-y| \sim Mr$ and Lipschitz function $A$ we have

$$|C_{\Gamma}(\chi_{I_1})(x)| = \left| \mathrm{p.v.} \int_{\mathbb{R}} \frac{\chi_{I_1}(y)}{y - x + i(A(y) - A(x))} \, dy \right|$$

$$= \left| \int_{y \in I_1} \frac{y - x}{(y - x)^2 + (A(y) - A(x))^2} \, dy \right|$$

$$\geq \int_{y \in I_1} \frac{y - x}{(y - x)^2 + (A(y) - A(x))^2} \, dy$$

$$\geq \frac{1}{(L^2 + 1)} \frac{1}{Mr} |I_1|$$

$$= \frac{2}{(L^2 + 1)} M.$$ 

This estimate holds for all $M > 10$, for all $r > 0$, for all disjoint intervals $I_0 = I_0(x_0, r)$ and $I_1 = I_1(x_1, r)$ satisfying the condition (3.4), so $C_{\Gamma}$ is 1-1-homogeneous, as required.

As noted above, the results of Theorems 3.1 and 3.2 coupled with Lemmas 3.4 and 3.5 establish Theorem 1.1.

4 Proof of Theorem 1.2: Compactness of $[b, C_{\Gamma}]$

The idea of the proof of Theorem 1.2 is originally due to Uchiyama [Uch78]. The main ingredients of the proof are the VMO characterisation (Definition 2.2) and the Frechét-Kolmogorov theorem (Theorem 2.7). To prove the sufficiency in Theorem 1.2 that is if $[b, C_{\Gamma}]$ is a compact operator on $L^p(\mathbb{R})$, then $b \in \text{VMO}(\mathbb{R})$, we use contradiction argument via Definition 2.2. Specifically, we show that if $b$ fails to satisfy any one of the conditions (1)–(3) in Definition 2.2 then the commutator $[b, C_{\Gamma}]$ is not compact. To prove the necessity in Theorem 1.2 that is if $b \in \text{VMO}(\mathbb{R})$, then $[b, C_{\Gamma}]$ is a compact operator on $L^p(\mathbb{R})$, we first reduce to showing that $[b, C_{\Gamma}]$ is compact for $b \in C_c^\infty(\mathbb{R})$. Then we show that for all bounded subsets $E \subset L^p(\mathbb{R})$, $[b, C_{\Gamma}]E$ is precompact, using Theorem 2.7. This implies that $[b, C_{\Gamma}]$ is compact on $L^p(\mathbb{R})$.

The proof of Theorem 1.2 requires lower and upper bounds for integrals of $|[b, C_{\Gamma}]f_j|^p$ over certain intervals, where $\{f_j\}_j$ is a certain bounded subset of $L^p(\mathbb{R})$ and $b \in \text{BMO}(\mathbb{R})$. These bounds will be obtained in Lemma 4.1 below.
Lemma 4.1. Assume that $b \in \text{BMO}(\mathbb{R})$ with $\|b\|_{\text{BMO}(\mathbb{R})} = 1$ and there exist $\varepsilon > 0$ and a sequence $\{I_j\}_{j=1}^\infty := \{I(x_j, r_j)\}_j$ of intervals such that for each $j \in \mathbb{N},$

\begin{equation}
M(b, I_j) = \int_{I_j} |b(y) - b_{I_j}| \, dy > \varepsilon.
\end{equation}

For $j \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}$, let

$I_j := (x_j + 2^k r_j, x_j + 2^{k+1} r_j).

Fix $p \in (1, \infty)$. Then there exist functions $\{f_j\}_j \subset L^p(\mathbb{R})$ and positive constants $A_1 > 4$, $\tilde{C}_1$ and $\tilde{C}_2$ such that for all $j \in \mathbb{N}$ and $k \geq \lfloor \log_2 A_1 \rfloor,$ we have

\begin{equation}
\|f_j\|_{L^p(\mathbb{R})} \lesssim 1 \quad \text{with constant independent of } j,
\end{equation}

\begin{equation}
\int_{I_j} |[b, C\Gamma] f_j(y)|^p \, dy \geq \tilde{C}_1 \varepsilon^p \frac{1}{2^{(p-1)}}, \quad \text{and}
\end{equation}

\begin{equation}
\int_{2^{k+1} I_j \setminus 2^k I_j} |[b, C\Gamma] f_j(y)|^p \, dy \leq \tilde{C}_2 \frac{1}{2^{(p-1)}}.
\end{equation}

The functions $\{f_j\}$ and the constants $\tilde{C}_1$ and $\tilde{C}_2$ depend on $p$ but not on $j$ or $k$, while $A_1$ is independent of $p$, $j$, and $k$.

Before proving Lemma 4.1, we recall some results related to the median value $\alpha_I(f)$ of a function $f$ on an interval $I$. See [CSS12], [Ler11], [Gra04, pp.160–166] and [Jou83, p.30] for more details. When $f \in L^1_{\text{loc}}(\mathbb{R})$ and $I$ is any interval on $\mathbb{R}$, the constants $c = \alpha_I(f)$ for which $\inf_c \frac{1}{|I|} \int_I |f(x) - c| \, dx$ is attained are the ones that satisfy

\begin{equation}
|\{x \in I : f(x) > \alpha_I(f)\}| \leq \frac{1}{2} |I| \quad \text{and}
\end{equation}

\begin{equation}
|\{x \in I : f(x) < \alpha_I(f)\}| \leq \frac{1}{2} |I|.
\end{equation}

Note that given a function $f$ and an interval $I$, the median $\alpha_I(f)$ may not be uniquely determined. In each such case, we mean by $\alpha_I(f)$ a particular fixed value of the median. Additionally, using the John–Nirenberg inequality and Hölder’s inequality, we obtain that for all $p$ with $1 \leq p < \infty$ and for all $f \in L^1_{\text{loc}}(\mathbb{R}),$

\begin{equation}
\sup_I \left( \frac{1}{|I|} \int_I |f(x) - f_I|^p \, dx \right)^{1/p} \sim \|f\|_{\text{BMO}}, \quad \text{and}
\end{equation}
\[
\sup_I \left( \frac{1}{|I|} \int_I |f(x) - \alpha_I(f)|^p \, dx \right)^{1/p} \sim \|f\|_{\text{BMO}}.
\]

Also, as shown in [CSS12, Equation (2.2)], for each interval \(I \subset \mathbb{R}\),
\[
\frac{1}{|I|} \int_I |f(x) - f_I| \, dx \sim \frac{1}{|I|} \int_I |f(x) - \alpha_I(f)| \, dx.
\]

Now we will prove Lemma 4.1.

**Proof of Lemma 4.1.** For each \(j \in \mathbb{N}\), define
\[
f_j := |I_j|^{-1/p} (f_j^1 - f_j^2),
\]
where
\[
f_j^1 := \chi_{I_{j,1}} - \chi_{I_{j,2}} := \chi_{\{x \in I_j : b(x) > \alpha_{I_j}(b)\}} - \chi_{\{x \in I_j : b(x) < \alpha_{I_j}(b)\}},
\]
\[
f_j^2 := a_j \chi_{I_j},
\]
and \(a_j\) is a constant chosen so that
\[
\int_{\mathbb{R}} f_j(x) \, dx = 0.
\]

We claim that the following properties hold:

\[
|a_j| \leq 1/2,
\]
\[
\text{supp}(f_j) \subset I_j,
\]
\[
f_j(y) \left[ b(y) - \alpha_{I_j}(b) \right] \geq 0 \quad \text{for all } y \in I_j, \quad \text{and}
\]
\[
|f_j(y)| \sim |I_j|^{-1/p} \quad \text{for all } y \in (I_{j,1} \cup I_{j,2}).
\]

To see (4.11), we start with equation (4.10). By the definition of \(f_j^1\) and \(f_j^2\),
and using property (4.6) of the median, we see that
\[
0 = \int_{\mathbb{R}} f_j(x) \, dx = \int_{\mathbb{R}} |I_j|^{-1/p} (\chi_{I_{j,1}}(x) - \chi_{I_{j,2}}(x) - a_j \chi_{I_j}(x)) \, dx
\]
\[
= |I_j|^{-1/p} (|I_{j,1}| - |I_{j,2}| - a_j |I_j|)
\]
\[
\geq |I_j|^{-1/p} (|I_{j,1}| - \frac{|I_j|}{2} - a_j |I_j|)
\]
\[
\geq - \left( \frac{1}{2} + a_j \right) |I_j|^{-1/p} |I_j| = - \left( \frac{1}{2} + a_j \right) |I_j|^{1/p'}.\]

Hence \(a_j \geq -1/2\). Similarly, using (4.10) and property (4.5) of the median,
we see that \(a_j \leq 1/2\). Hence \(|a_j| \leq 1/2\), as required. Equation (4.12) is immediate from the definition of \(f_j\).
To see (4.13), we consider the three cases when \( y \in I_{j,1}, \ y \in I_{j,2}, \) and \( y \in I_j \setminus (I_{j,1} \cup I_{j,2}) \). If \( y \in I_{j,1} \), then by the definitions of \( f_j^1 \) and \( f_j^2 \) and equation (4.11) we have

\[
b(y) > \alpha_{I_j}(b), \quad f_j^1(y) = 1 > 0 \quad \text{and} \quad f_j^2 \leq \frac{1}{2}.
\]

These yield

\[
f_j(y) \left[ b(y) - \alpha_{I_j}(b) \right] > 0 \quad \text{for all} \ y \in I_{j,1}.
\]  

(4.15)

The case of \( y \in I_{j,2} \) is similar. Next, if \( y \in I_j \setminus (I_{j,1} \cup I_{j,2}) \), then \( b(y) = \alpha_{I_j}(b) \) and so \( f_j(y) \left[ b(y) - \alpha_{I_j}(b) \right] = 0 \). Thus inequality (4.13) holds for all \( y \in I_j \).

To see (4.14) we first note that

\[
|f_j(y)| = |I_j|^{-1/p} \left| f_j^1(y) - f_j^2(y) \right| \geq \frac{1}{2} |I_j|^{-1/p} \quad \text{for all} \ y \in I_{j,1} \cup I_{j,2}.
\]

Second, for all \( y \in I_{j,1} \cup I_{j,2} \) we also have

\[
|f_j^1(y) - f_j^2(y)| \leq |\chi_{I_{j,1}}(y)| + |\chi_{I_{j,2}}(y)| + |a_j \chi_{I_j}(y)| \leq \frac{5}{2}.
\]

Thus

\[
|f_j(y)| = |I_j|^{-1/p} \left| f_j^1(y) - f_j^2(y) \right| \leq \frac{5}{2} |I_j|^{-1/p} \quad \text{for all} \ y \in I_{j,1} \cup I_{j,2}.
\]

(4.17)

So from inequalities (4.16) and (4.17) we obtain the equivalence in (4.14).

Now, to see (4.2) in Lemma 4.1, using (4.11), (4.12) and (4.14) we compute

\[
\|f_j\|_{L^p(\mathbb{R})}^p = \int_{I_{j,1} \cup I_{j,2}} |f_j(x)|^p \, dx + \int_{\mathbb{R} \setminus (I_{j,1} \cup I_{j,2})} |f_j(x)|^p \, dx \lesssim 1 + \frac{1}{2^p} \lesssim 1,
\]

as required.

Next, fix a constant \( A_1 > 4 \). Then for any integer \( k \geq \lfloor \log_2 A_1 \rfloor \), we claim that

\[
2^{k+1} I_j \subset 8 I_j^k = \left( x_j - \frac{5}{2} \cdot 2^k r_j, x_j + \frac{11}{2} \cdot 2^k r_j \right) \subset 2^{k+3} I_j.
\]

(4.18)

To see the first inclusion, we recall that \( I_j = I(x_j, r_j) = (x_j - r_j, x_j + r_j) \).

Hence

\[
2^{k+1} I_j = I(x_j, 2^{k+1} r_j) = (x_j - 2^{k+1} r_j, x_j + 2^{k+1} r_j), \quad \text{and}
\]

\[
2^{k+3} I_j = I(x_j, 2^{k+3} r_j) = (x_j - 4 \cdot 2^{k+1} r_j, x_j + 4 \cdot 2^{k+1} r_j).
\]

(4.19)\( (4.20) \)
Also, as defined in Lemma 4.1
\[ I_j^k := (x_j + 2^kr_j, x_j + 2^{k+1}r_j) = I \left( x_j + 3 \cdot 2^{k-1}r_j, 2^{k-1}r_j \right), \]
and so
\[
8I_j^k = \left( x_j - 5 \cdot 2^{k-1}r_j, x_j + 11 \cdot 2^{k-1}r_j \right)
\]
(4.21)
\[
= \left( x_j - \frac{5}{4} \cdot 2^{k+1}r_j, x_j + \frac{11}{4} \cdot 2^{k+1}r_j \right).
\]
The inclusions in (4.18) follow from equations (4.19)–(4.21), since
\[ x_j - 4 \cdot 2^{k}r_j \leq x_j - \frac{5}{4} \cdot 2^{k+1}r_j \leq x_j - 2^{k+1}r_j, \]
and
\[ x_j + 2^{k+1}r_j \leq x_j + \frac{11}{4} \cdot 2^{k+1}r_j \leq x_j + 4 \cdot 2^{k+1}r_j. \]
We turn to inequality (4.3) in Lemma 4.1. Observe that
\[
(4.22) \quad \|[b, C_T]f_j\| = \left| C_T \left[ b - \alpha_{I_j}(b), f_j \right] \right|_{A(\cdot)} - \left| \frac{b - \alpha_{I_j}(b)}{\Gamma}\right|_{B(\cdot)}
\]
Using (4.22) and Minkowski’s inequality for the $L^p(I_j^k)$ norm, we have
\[
(4.23) \quad \|[b, C_T]f_j\|_{L^p(I_j^k)} = \|A(\cdot) - B(\cdot)\|_{L^p(I_j^k)} \geq \|A(\cdot)\|_{L^p(I_j^k)} - \|B(\cdot)\|_{L^p(I_j^k)}.
\]
We will estimate the $L^p$-norms of $A$ and $B$ in (4.23).
We start with $\|B(\cdot)\|_{L^p(I_j^k)}$. Note that $|z - x_j| < \frac{1}{2}|y - x_j|$ for any $z \in I_j$ and $y \in \mathbb{R} \setminus 2I_j$. Also, recall that the kernel $C_T(x, y)$ of the Cauchy integral is standard. Using the fact that supp$(f_j) \subset I_j$, equations (4.10), (4.2) and (4.14), we see that for all $y \in \mathbb{R} \setminus 2I_j$ and $z \in I_j$,
\[
|B(y)| = \left| \left[ b(y) - \alpha_{I_j}(b) \right] C_T(f_j)(y) \right| \\
\leq \left| b(y) - \alpha_{I_j}(b) \right| \int_{I_j} |C_T(y, z) - C_T(y, x_j)||f_j(z)| \, dz \\
\leq \left| b(y) - \alpha_{I_j}(b) \right| \int_{I_j} \frac{|x_j - z|}{|x_j - y|^2} |I_j|^{-1/p} \, dz \\
= \frac{|b(y) - \alpha_{I_j}(b)|}{|I_j|^{1/p} |x_j - y|^2} \int_{I_j} |x_j - z| \, dz \\
\leq \frac{|b(y) - \alpha_{I_j}(b)|}{|I_j|^{1/p} |x_j - y|^2} \int_{I_j} r_j \, dz
\]
we have
Thus we obtain
\begin{equation}
(4.24)
= r_j |I_j|^{1/p'} \frac{|b(y) - \alpha_{I_j}(b)|}{|x_j - y|^2} \\
= 2^{-1} |I_j|^{1+1/p'} \frac{|b(y) - \alpha_{I_j}(b)|}{|x_j - y|^2}.
\end{equation}
Note that \( I_j^k = (x_j + 2^kr_j, x_j + 2^{k+1}r_j) \subset (\mathbb{R} \setminus 2I_j) \) for all \( k \geq \lfloor \log_2 A_1 \rfloor \) and \( A_1 > 4 \). Also, for all \( y \in I_j^k \), we have
\[ |x_j - y| \geq 2^k r_j = 2^{k-1}|I_j|. \]
Thus by (4.24) we get
\begin{equation}
(4.25)
\|B(\cdot)\|_{L^p(I_j^k)} \lesssim \frac{2^{-1} |I_j|^{1+1/p'}}{2^{2(k-1)}|I_j|^2} \|b - \alpha_{I_j}(b)\|_{L^p(I_j^k)} = \frac{2 |I_j|^{-1/p}}{2^{2k}} \|b - \alpha_{I_j}(b)\|_{L^p(I_j^k)}.
\end{equation}
We consider \( \|b - \alpha_{I_j}(b)\|_{L^p(I_j^k)} \). Note that for all \( k \geq \lfloor \log_2 A_1 \rfloor \) and \( A_1 > 4 \) we have
\[ I_j^k = (x_j + 2^kr_j, x_j + 2^{k+1}r_j) \subset (x_j - 2^{k+1}r_j, x_j + 2^{k+1}r_j) = 2^{k+1}I_j. \]
Thus we obtain
\begin{equation}
(4.26)
\begin{aligned}
\|b - \alpha_{I_j}(b)\|_{L^p(I_j^k)} &\leq \|b - \alpha_{2^{k+1}I_j}(b) + \alpha_{2^{k+1}I_j}(b) - \alpha_{I_j}(b)\|_{L^p(2^{k+1}I_j)} \\
&\leq \|b - \alpha_{2^{k+1}I_j}(b)\|_{L^p(2^{k+1}I_j)} + \|\alpha_{2^{k+1}I_j}(b) - \alpha_{I_j}(b)\|_{L^p(2^{k+1}I_j)}.
\end{aligned}
\end{equation}
For the first term in the last line of (4.26), using equation (4.8), for every interval \( I \) we have
\[ \int_I |b(y) - \alpha_{I}(b)|^p dy \lesssim |I|\|b\|_{BMO}^p \lesssim |I|. \]
Thus the first term in the last line of (4.26) is controlled by \( 2^{(k+1)/p}|I_j|^{1/p}: \)
\begin{equation}
(4.27)
\|b - \alpha_{2^{k+1}I_j}(b)\|_{L^p(2^{k+1}I_j)} \lesssim 2^{(k+1)/p}|I_j|^{1/p}.
\end{equation}
For the second term in the last line of (4.26), using equation (4.8) we have
\[ |\alpha_{2^{k+1}I_j}(b) - \alpha_{I_j}(b)| = \int_{I_j} |\alpha_{2^{k+1}I_j}(b) - \alpha_{I_j}(b)| dy \\
\leq \int_{2^{k+1}I_j} |\alpha_{2^{k+1}I_j}(b) - \alpha_{I_j}(b)| dy + \int_{I_j} |b(y) - \alpha_{I_j}(b)| dy \\
\lesssim \|b\|_{BMO} = 1.
\]
As a result
\[
\|\alpha_{2k+1}I_j(b) - \alpha I_j(b)\|_{L^p(2k+1I_j)} = \left| 2^{k+1}I_j \right|^{1/p} \left| \alpha_{2k+1}I_j(b) - \alpha I_j(b) \right| \\
\lesssim 2^{(k+1)/p} |I_j|^{1/p}.
\]
(4.28)

Using (4.27) and (4.28) we can estimate the left-hand side of (4.26) by
\[
\|b - \alpha I_j(b)\|_{L^p(I_j)} \lesssim 2^{(k+1)/p} |I_j|^{1/p} + 2^{(k+1)/p} |I_j|^{1/p} \lesssim 2^{(k+1)/p} |I_j|^{1/p}.
\]

Consequently, we can now estimate (4.25):
\[
\|B(\cdot)\|_{L^p(I_j)} \lesssim \frac{2 |I_j|^{-1/p}}{2^{2k}} 2^{(k+1)/p} |I_j|^{1/p} = C_4 \frac{1}{2^{k} 2^{k(p-1)/p}},
\]
where $C_4 = C_4(p)$ is independent of $k$ and $j$.

Next, we will estimate $\|A(\cdot)\|_{L^p(I_j)}$. Observe that for all $y \in I_j$ and $z \in I_j$ we have $y > z$ and
\[
|y - z| \leq |x_j + 2k+1r_j - (x_j - r_j)| = (2^{k+1} + 1)r_j \leq 2^{k+2}r_j.
\]

Using (2.2), (4.13), (4.14), (4.9) and (4.11), for all $y \in I_j$ and $z \in I_j$, we deduce a lower bound for $|A(y)|$:
\[
|A(y)| = \left| \int_{(J_1 \cup J_2, z)} C_{\Gamma}(y, z) \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz \right|
\]
\[
= \int_{(J_1 \cup J_2, z)} \frac{1}{z-y + i(A(z) - A(y))} \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz
\]
\[
= \int_{(J_1 \cup J_2, z)} \frac{z-y}{(z-y)^2 + (A(z) - A(y))^2} \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz
\]
\[
- i \int_{(J_1 \cup J_2, z)} \frac{A(z) - A(y)}{(z-y)^2 + (A(z) - A(y))^2} \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz
\]
\[
\geq \int_{(J_1 \cup J_2, z)} \frac{z-y}{(z-y)^2 + (A(z) - A(y))^2} \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz
\]
\[
\geq \int_{(J_1 \cup J_2, z)} \frac{y-z}{(z-y)^2 + L^2(z-y)^2} \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz
\]
\[
\geq \int_{(J_1 \cup J_2, z)} \frac{1}{y-z} \left[ b(z) - \alpha I_j(b) \right] f_j(z) \, dz
\]
\[
= \int_{(J_1 \cup J_2, z)} \frac{1}{|y-z|} \left| b(z) - \alpha I_j(b) \right| |f_j(z)| \, dz.
\]
\[
\geq \int_{(J_1 \cup J_2, z)} \frac{1}{|y-z|} \left| b(z) - \alpha I_j(b) \right| |I_j|^{-1/p} \, dz
\]
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\[ \geq \frac{|I_j|^{-1/p}}{2^{k+2}r_j} \int_{I_j} |b(z) - \alpha_{I_j}(b)| \, dz \]

\[ \geq \frac{|I_j|^{-1/p}}{2^{k+2}r_j} M(b, I_j)|I_j| \]

\[ \geq \frac{\varepsilon |I_j|^{-1/p}}{2^{k+2}r_j} = \frac{\varepsilon}{2^{k+1}}|I_j|^{-1/p}. \]

Consequently,

\[ \|A(\cdot)\|_{L^p(I_j^k)} \gtrsim \frac{\varepsilon}{2^{k+1}}|I_j|^{-1/p} |I_j^k|^{1/p} = \varepsilon 2^{\frac{1-p}{p}} \frac{1}{2^{k(p-1)/p}} = C_5\varepsilon \frac{1}{2^{k(p-1)/p}}, \]

where \( C_5 = C_5(p) \) is independent of \( k \) and \( j \).

Therefore, returning to (4.23), we have

\[ \|b, C\Gamma f_j\|_{L^p(I_j^k)} \gtrsim \frac{C_5\varepsilon}{2^{k(p-1)/p}} - C_4 \frac{1}{2^k} \frac{1}{2^{k(p-1)/p}} \]

\[ = \left( C_5\varepsilon - C_4 \frac{1}{2^k} \right) \frac{1}{2^{k(p-1)/p}}. \]

Take \( A_1 \) large enough that for any integer \( k \geq \lfloor \log_2 A_1 \rfloor \),

\[ C_5\varepsilon - C_4 \frac{1}{2^k} \geq \frac{C_5 \varepsilon}{2}. \]

Then for all such \( k \) we have

\[ \|b, C\Gamma f_j\|_{L^p(I_j^k)} \gtrsim \frac{C_5 \varepsilon}{2^{k(p-1)/p}}. \]

It follows that

\[ \int_{I_j^k} |[b, C\Gamma f_j(y)|^p \, dy \gtrsim \frac{C_5^p \varepsilon^p}{2^p} \frac{1}{2^{k(p-1)}} = \tilde{C}_1 \varepsilon^p \frac{1}{2^{k(p-1)}}. \]

This shows the inequality (4.3).

Finally we show the inequality (4.4) in Lemma 4.1 Using equation (4.22) we have

\[ \|b, C\Gamma f_j\|_{L^p(2^{k+1}I_j \backslash 2^k I_j)} = \|A(\cdot) - B(\cdot)\|_{L^p(2^{k+1}I_j \backslash 2^k I_j)} \]

(4.29)

\[ \leq \|A(\cdot)\|_{L^p(2^{k+1}I_j \backslash 2^k I_j)} + \|B(\cdot)\|_{L^p(2^{k+1}I_j \backslash 2^k I_j)}. \]

Consider the term \( \|A(\cdot)\|_{L^p(2^{k+1}I_j \backslash 2^k I_j)} \) in (4.29). Note that for all \( z \in I_j \) and \( y \in \mathbb{R} \setminus 2^k I_j \), we have

\[ |y - z| \geq |x_j + 2^k r_j - x_j - r_j| = (2^k - 1)r_j. \]
Using the fact that $\text{supp}(f_j) \subset I_j$, together with (3.1), (4.14) and (4.8), we deduce that for all $z \in I_j$ and $y \in \mathbb{R} \setminus 2^k I_j$,

$$|A(y)| \leq \int_{I_j} |C_{\Gamma}(y, z)| |b(z) - \alpha_{I_j}(b)| |f_j(z)| \, dz$$

\[ \lesssim \int_{I_j} \frac{1}{|y - z|} |b(z) - \alpha_{I_j}(b)| |I_j|^{-1/p} \, dz \]

\[ \lesssim \frac{|I_j|^{-1/p}}{(2^k - 1) r_j} \int_{I_j} |b(z) - \alpha_{I_j}(b)| \, dz \]

\[ \lesssim \frac{|I_j|^{-1/p}}{(2^k - 1) r_j} \|b\|_{BMO(I_j)} \]

\[ \lesssim \frac{|I_j|^{-1/p}}{2^{k-2}}. \]  \hspace{1cm} (4.30)

The upper bound for $|A(y)|$ in (4.30) gives us

$$\|A(\cdot)\|_{L^p(2^{k+1} I_j \setminus 2^k I_j)} \lesssim \frac{|I_j|^{-1/p}}{2^{k-2}} |2^{k+1} I_j|^{1/p} \lesssim 2^{1/p} \frac{1}{2^{k(p-1)/p}}$$

\[ \quad = C_6 \frac{1}{2^{k(p-1)/p}}, \]  \hspace{1cm} (4.31)

where $C_6 = C_6(p)$ is independent of $k$ and $j$.

Consider now the term $\|B(\cdot)\|_{L^p(2^{k+1} I_j \setminus 2^k I_j)}$ in (4.29). Following the same argument for estimating $\|B(\cdot)\|_{L^p(I_j)}$ above, we obtain

$$\|B(\cdot)\|_{L^p(2^{k+1} I_j \setminus 2^k I_j)} \lesssim C_7 \frac{1}{2^{k} r_j} \frac{1}{2^{k(p-1)/p}},$$

where $C_7 = C_7(p)$ is independent of $k$ and $j$. Using (4.31) and (4.32), for all $k \geq \lceil \log_2 A_1 \rceil$, we have

$$\|[b, C_{\Gamma}] f_j\|_{L^p(2^{k+1} I_j \setminus 2^k I_j)} \lesssim \left(C_6 + C_7 \frac{1}{2^k}\right) \frac{1}{2^{k(p-1)/p}}.$$

It follows that

$$\int_{2^{k+1} I_j \setminus 2^k I_j} |[b, C_{\Gamma}] f_j(y)|^p \, dy \leq \tilde{C}_2 \frac{1}{2^{k(p-1)}},$$

which is (4.4), as required. This completes the proof of Lemma 4.1. \hspace{1cm} \[\square\]

With Lemma 4.1 in hand, we now return to the proof of our main result.

**Proof of Theorem 1.2.** Sufficiency: We first show that if $[b, C_{\Gamma}]$ is a compact operator on $L^p(\mathbb{R})$, then $b \in \text{VMO}$. Since $[b, C_{\Gamma}]$ is compact on $L^p(\mathbb{R})$, $[b, C_{\Gamma}]$ is bounded on $L^p(\mathbb{R})$. Without loss of generality, we may assume that $\|b\|_{BMO(\mathbb{R})} = 1$. To show $b \in \text{VMO}$, we use a contradiction argument
via Definition 2.2. Observe that if \( b \notin \text{VMO} \), \( b \) does not satisfy at least one of conditions (1)–(3) in Definition 2.2. We consider the three cases separately.

**Case 1:** Suppose \( b \) does not satisfy condition (1) in Definition 2.2 that is,

\[
\lim_{\delta \to 0} \sup_{|I| < \delta} \int_I |f(x) - f_I| \, dx \neq 0.
\]

Then there exist \( \varepsilon > 0 \) and a sequence \( \{I_j\}_{j=1}^{\infty} \) of intervals satisfying

\[
M(b, I_j) > \varepsilon \quad \text{for each } j
\]

and \( |I_j| \to 0 \) as \( j \to \infty \). Let \( f_j, \tilde{C}_1, \tilde{C}_2, A_1 \) be as in Lemma 4.1 and let \( A_2 > A_1 \) be a large number to be chosen later. Since \( |I_j| \to 0 \) as \( j \to \infty \), we may choose a subsequence \( \{I_{j_\ell}\} \) of \( \{I_j\} \) such that

\[
\frac{|I_{j_\ell}^{(1)}|}{|I_{j_\ell+1}^{(1)}|} < \frac{1}{A_2} \quad \text{for all } \ell \in \mathbb{N}.
\]

For fixed \( \ell, m \in \mathbb{N} \), denote

\[
\mathcal{J} := \left(x_{j_\ell}^{(1)} + A_1 r_{j_\ell}^{(1)}, x_{j_\ell}^{(1)} + A_2 r_{j_\ell}^{(1)}\right),
\]

\[
\mathcal{J}_1 := \mathcal{J} \setminus \left\{ y \in \mathbb{R} : y - x_{j_\ell}^{(1)} \leq A_2 r_{j_\ell+m}^{(1)} \right\},
\]

\[
\mathcal{J}_2 := \left\{ y \in \mathbb{R} : y - x_{j_\ell+m}^{(1)} > A_2 r_{j_\ell}^{(1)} \right\}.
\]

Note that

\[
\mathcal{J}_1 \subset \left\{ y \in \mathbb{R} : y - x_{j_\ell}^{(1)} \leq A_2 r_{j_\ell+m}^{(1)} \right\} \cap \mathcal{J}_2 \text{ and } \mathcal{J}_1 = \mathcal{J} \cap \mathcal{J}_2 = \mathcal{J} \setminus (\mathcal{J} \setminus \mathcal{J}_2).
\]

We then have

\[
\| [b, C_T] (f_{j_\ell}) - [b, C_T] (f_{j_\ell+m}) \|_{L^p(\mathbb{R})} \\
\geq \| [b, C_T] (f_{j_\ell}) - [b, C_T] (f_{j_\ell+m}) \|_{L^p(\mathcal{J}_1)} \\
\geq \| [b, C_T] (f_{j_\ell})\|_{L^p(\mathcal{J}_1)} - \| [b, C_T] (f_{j_\ell+m})\|_{L^p(\mathcal{J}_1)} \\
\geq \| [b, C_T] (f_{j_\ell})\|_{L^p(\mathcal{J}_1)} - \| [b, C_T] (f_{j_\ell+m})\|_{L^p(\mathcal{J}_2)} \\
= \left( \int_{\mathcal{J} \setminus (\mathcal{J} \setminus \mathcal{J}_2)} [b, C_T] (f_{j_\ell}) (y)^p \, dy \right)^{1/p} - \left( \int_{\mathcal{J}_2} [b, C_T] (f_{j_\ell+m}) (y)^p \, dy \right)^{1/p} \\
=: F_1 - F_2.
\]

We first consider the term \( F_1 \). To begin with, we estimate the measure of the set \( E_{j_\ell} := \mathcal{J} \setminus \mathcal{J}_2 \). Assume that \( E_{j_\ell} \neq \emptyset \). Then \( E_{j_\ell} \subset A_2 I_{j_\ell+m}^{(1)} \). Hence, we have

\[
|E_{j_\ell}| \leq |A_2 I_{j_\ell+m}^{(1)}| = A_2 |I_{j_\ell+m}^{(1)}| < |I_{j_\ell}^{(1)}|,
\]

(4.34)
where the last inequality follows from (4.33).

Now for each \( k \geq \lceil \log_2 A_1 \rceil \), as in Lemma 4.1 let
\[
I^k_{j\ell} := \left(x^{(1)}_{j\ell} + 2^k r_{j\ell}^{(1)}, x^{(1)}_{j\ell} + 2^{k+1} r_{j\ell}^{(1)}\right).
\]

Then
\[
(4.35) \quad |I^k_{j\ell}| = 2^k r_{j\ell}^{(1)} = 2^k \frac{|I^{(1)}_{j\ell}|}{2} = 2^{k-1} |I^{(1)}_{j\ell}| \geq |E_{j\ell}| \quad \text{for all} \quad k \geq \lceil \log_2 A_1 \rceil.
\]

Notice also that by definition,
\[
(4.36) \quad E_{j\ell} \subset J \subset \bigcup_{k = \lceil \log_2 A_1 \rceil}^{\infty} I^k_{j\ell}.
\]

Here the second inclusion holds because the left endpoint of \( I^{\lceil \log_2 A_1 \rceil}_{j\ell} \) is \( x^{(1)}_{j\ell} + 2^{\lceil \log_2 A_1 \rceil} r_{j\ell}^{(1)} \), which lies to the left of the left endpoint of \( J \).

From inequality (4.35) and the fact (4.36), it follows that \( E_{j\ell} \) is covered by the union of at most two (adjacent) intervals \( I^k_{j\ell} \). That is, there is some \( k_0 \geq \lceil \log_2 A_1 \rceil \) such that \( E_{j\ell} \subset (I^k_{j\ell} \cup I^{k_0+1}_{j\ell}) \). By inequality (4.3) in Lemma 4.1,
\[
F^p_1 \geq \sum_{k = \lceil \log_2 A_1 \rceil + 1, k \neq k_0, k_0+1}^{\lfloor \log_2 A_2 \rfloor} \int_{I^k_{j\ell}} \left|[b, C_T] (f_{j\ell})(y)\right|^p dy
\]
\[
\geq \tilde{C}_1 \varepsilon^p \sum_{k = \lceil \log_2 A_1 \rceil + 1, k \neq k_0, k_0+1}^{\lfloor \log_2 A_2 \rfloor} \frac{1}{2^{k(p-1)}}
\]
\[
\geq \tilde{C}_1 \varepsilon^p \sum_{k = \lfloor \log_2 A_1 \rfloor + 3}^{\lfloor \log_2 A_2 \rfloor} \frac{1}{2^{k(p-1)}}
\]
\[
\geq 8^{(1-p)} \tilde{C}_1 \varepsilon^p A_1^{1-p} =: A_3.
\]

If \( E_{j\ell} := J \setminus J_2 = \emptyset \), then inequality (4.37) still holds.

On the other hand, using (4.4) in Lemma 4.1 we deduce that
\[
F^p_2 \leq \sum_{k = \lceil \log_2 A_2 \rceil}^{\infty} \int_{I^k_{j\ell}} [b, C_{T_{j\ell+m}}] (f_{j\ell+m})(y) dy
\]
\[
\leq \tilde{C}_2 \varepsilon \sum_{k = \lceil \log_2 A_2 \rceil}^{\infty} \frac{1}{2^{k(p-1)}}
\]
\[
\leq \tilde{C}_2 \frac{1/2^{p-1}}{1 - 1/2^{p-1}}
\]
\[ \leq \frac{C_2}{1 - 2^{1-p} 2^{|\log_2 A_2|(p-1)}}. \]

If we choose \( A_2 > A_1 \) large enough such that
\[ (4.38) \quad A_3 := 8^{(1-p)} \tilde{C}_1 C_1^{p} A_1^{(1-p)} > \frac{2 \tilde{C}_2}{1 - 2^{1-p} 2^{|\log_2 A_2|(p-1)}}, \]
then we have
\[ (4.39) \quad F^p_2 \leq \frac{A_3}{2}. \]

By inequalities (4.37) and (4.39), we get
\[ \| [b, C_1] (f_{j_\ell}) - [b, C_1] (f_{j_\ell+m}) \|_{L^p(\mathbb{R})} \gtrsim A_3^{1/p} > 0. \]

Thus, \( \{[b, C_1] f_j\}_j \) is not relatively compact in \( L^p(\mathbb{R}) \), which implies that \( [b, C_1] \) is not compact on \( L^p(\mathbb{R}) \). This contradiction implies that, \( b \) satisfies condition (1) in Definition 2.2.

**Case 2:** Suppose \( b \) violates condition (2) in Definition 2.2, that is,
\[ \lim_{R \to \infty} \sup_{|I| > R} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx \neq 0. \]

In this case, there exist \( \varepsilon > 0 \) and a sequence \( \{I_j\} \) of intervals satisfying \( M(b, I_j) > \varepsilon \) and that \( |I_j| \to \infty \) as \( j \to \infty \). We take a subsequence \( \{I^{(2)}_{j_\ell}\} \) of \( \{I_j\} \) such that
\[ (4.40) \quad \frac{|I^{(2)}_{j_\ell}|}{|I^{(2)}_{j_\ell+1}|} < \frac{1}{A_2} \quad \text{for all } l \in \mathbb{N}, \]
where \( A_2 \) is chosen as in Case 1 above. We use a similar method to that in the previous case, but redefine our sets with the roles of \( j_\ell \) and \( j_{\ell+m} \) reversed. That is, for fixed \( \ell \) and \( m \), let
\[ \tilde{\mathcal{J}} := \left\{ x^{(2)}_{j_{\ell+m}} + A_1 r^{(2)}_{j_{\ell+m}}, x^{(2)}_{j_{\ell+m}} + A_2 r^{(2)}_{j_{\ell+m}} \right\}, \]
\[ \tilde{\mathcal{J}}_1 := \tilde{\mathcal{J}} \setminus \left\{ y \in \mathbb{R} : |y - x^{(2)}_{j_\ell}| \leq A_2 r^{(2)}_{j_\ell} \right\}, \quad \text{and} \]
\[ \tilde{\mathcal{J}}_2 := \left\{ y \in \mathbb{R} : |y - x^{(2)}_{j_\ell}| > A_2 r^{(2)}_{j_\ell} \right\}. \]

Then we have that
\[ \tilde{\mathcal{J}}_1 \subset \left\{ y \in \mathbb{R} : |y - x^{(2)}_{j_{\ell+m}}| \leq A_2 r^{(2)}_{j_{\ell+m}} \right\} \cap \tilde{\mathcal{J}}_2 \text{ and } \tilde{\mathcal{J}}_1 = \tilde{\mathcal{J}} \cap \tilde{\mathcal{J}}_2 = \tilde{\mathcal{J}} \setminus \left( \tilde{\mathcal{J}} \setminus \tilde{\mathcal{J}}_2 \right). \]
Consequently,

\[
\| [b, C_T] (f_{j+\ell}^{\ell + m}) - [b, C_T] (f_{j\ell}) \|_{L^p(\mathbb{R})} \\
\geq \left( \int_{\tilde{J} \setminus (\tilde{J} \setminus \tilde{J}_2)} |[b, C_T] (f_{j+\ell}^{\ell + m})(y)|^p \, dy \right)^{1/p} - \left( \int_{\tilde{J}_2} |[b, C_T] (f_{j\ell})(y)|^p \, dy \right)^{1/p} \\
=: \tilde{F}_1 - \tilde{F}_2.
\]

By inequalities (4.3) and (4.4) in Lemma 4.1 and the definition of \( A_3 \) in (4.38), we can deduce that \( \tilde{F}_1^p \geq A_3 \) and \( \tilde{F}_2^p \leq A_3/2 \), just as \( F_1^p \) and \( F_2^p \) in Case 1. As a consequence,

\[
\| [b, C_T] (f_{j+\ell}^{\ell + m}) - [b, C_T] (f_{j\ell}) \|_{L^p(\mathbb{R})} \gtrsim (A_3)^{1/p}.
\]

As in Case 1, by Lemma 4.1 and inequality (4.40), we see that \([b, C_T] \) is not compact on \( L^p(\mathbb{R}) \). This contradiction implies that \( b \) satisfies condition (2) of Definition 2.2.

**Case 3:** By Cases 1 and 2, we may assume that conditions (1) and (2) in Definition 2.2 hold for \( b \). Suppose condition (3) in Definition 2.2 fails, that is,

\[
\lim_{R \to \infty} \sup_{I \cap I(0, R) = \emptyset} \frac{1}{|I|} \int_I |f(x) - f_I| \, dx \neq 0.
\]

Then there exist \( \varepsilon > 0 \) such that for each \( R > 0 \), there exists an interval \( I \) such that \( I \cap (-R, R) = \emptyset \) with \( M(b, I) > \varepsilon \). We claim that for the \( \varepsilon \) above, there exists a sequence \( \{I_j\}_j \) of intervals such that for all \( j \in \mathbb{N} \),

\[
M(b, I_j) > \varepsilon,
\]

and that for all \( \ell \neq m \), and for the constant \( A_2 \) chosen in Case 1 above,

\[
A_2 I_{\ell} \cap A_2 I_m = \emptyset.
\]

To see this, first note that as \( b \) satisfies condition (2) in Definition 2.2 for the aforementioned \( \varepsilon \) there exists a constant \( \tilde{C}_\varepsilon \) such that

\[
M(b, I) < \varepsilon
\]

for every interval \( I \) satisfying \( |I| > \tilde{C}_\varepsilon \). Let \( C_\varepsilon := \tilde{C}_\varepsilon / 2 \). Then for \( R_1 > C_\varepsilon \), there exists an interval \( I_1 := I(x_1, r_1) \subset \mathbb{R} \setminus I(0, R_1) \) such that (4.41) holds. Similarly, for \( R_j := |x_{j-1}| + 4A_2C_\varepsilon, j = 2, 3, \ldots \), there exists \( I_j := I(x_j, r_j) \subset \mathbb{R} \setminus I(0, R_j) \) satisfying (4.41). Repeating this procedure, we obtain a collection \( \{I_j\}_j \) of intervals satisfying (4.41) for each \( j \).
By the choice of \( \{I_j\} \), namely \( M(b, I_j) > \varepsilon \), we have that \( |I_j| \leq \tilde{C}_\varepsilon \), and so \( r_j \leq \tilde{C}_\varepsilon / 2 = C_\varepsilon \) for all \( j \in \mathbb{N} \). Thus
\[
A_2 r_j < A_2 C_\varepsilon < 4 A_2 C_\varepsilon.
\]
Therefore for all \( \ell \neq m \) we have
\[
d(A_2 I_\ell, A_2 I_m) \geq R_j - (x_{j-1} + A_2 r_{j-1}) = x_{j-1} + 4 A_2 C_\delta - x_{j-1} - A_2 r_{j-1} \geq 4 A_2 C_\delta - A_2 C_\delta = 3 A_2 C_\delta.
\]
This establishes the claim.

Now we define
\[
\mathcal{J}_1 := (x_\ell + A_1 r_\ell, x_\ell + A_2 r_\ell), \quad \text{and} \quad \mathcal{J}_2 := \{ y \in \mathbb{R} : |y - x_{\ell+m}| > A_2 r_{\ell+m} \}.
\]
Note that \( \mathcal{J}_1 \subset \mathcal{J}_2 \). Thus, similarly to the estimates of \( F_1 \) and \( F_2 \) in Case 1, for all \( \ell, m \in \mathbb{N} \), we get
\[
\| [b, C_T] (f_\ell) - [b, C_T] (f_{\ell+m}) \|_{L^p(\mathbb{R})} \geq \left\{ \int_{\mathcal{J}_1} \| [b, C_T] (f_\ell)(y) - [b, C_T] (f_{\ell+m})(y) \|^p dy \right\}^{1/p} \\
\geq \left\{ \int_{\mathcal{J}_1} \| [b, C_T] (f_\ell)(y) \|^p dy \right\}^{1/p} - \left\{ \int_{\mathcal{J}_2} \| [b, C_T] (f_{\ell+m})(y) \|^p dy \right\}^{1/p} \\
=: \mathcal{F}_1 - \mathcal{F}_2.
\]
Again, by (4.3) and (4.4) in Lemma 4.1 and the definition of \( A_3 \) in (4.38), we deduce that \( \mathcal{F}_1 \geq A_3 \) and \( \mathcal{F}_2 \leq A_3 / 2 \), as for \( \mathcal{F}_1^p \) and \( \mathcal{F}_2^p \) in Case 1. As a result, we get
\[
\| [b, C_T] (f_\ell) - [b, C_T] (f_{\ell+m}) \|_{L^p(\mathbb{R})} \gtrsim (A_3)^{1/p}.
\]
Thus, \( \{[b, C_T] f_\ell\}_\ell \) is not relatively compact in \( L^p(\mathbb{R}) \), which implies that \( [b, C_T] \) is not compact on \( L^p(\mathbb{R}) \). This contradicts the compactness of \([b, C_T] \) on \( L^p(\mathbb{R}) \), so \( b \) satisfies condition (3) in Definition 2.2. This completes the proof of the sufficiency in Theorem 1.2.

**Necessity:** To see the converse, we must show that when \( b \in \text{VMO}(\mathbb{R}) \), the commutator \( [b, C_T] \) is compact on \( L^p(\mathbb{R}) \). By a density argument, it suffices to show that \( [b, C_T] \) is a compact operator for \( b \in C_c^\infty(\mathbb{R}) \).

Let \( b \in C_c^\infty(\mathbb{R}) \). To show \([b, C_T] \) is compact on \( L^p(\mathbb{R}) \), it suffices to show that for every bounded subset \( E \subset L^p(\mathbb{R}) \), the set \([b, C_T]E \) is precompact.
Thus, we only need to show that \([b, C_T]E\) satisfies the hypotheses (a)–(c) in
the Frechét–Kolmogorov theorem (Theorem 2.7). We first point out that by
Theorem 1.1 and the fact that \(b \in \text{BMO}(\mathbb{R})\), \([b, C_T]\) is bounded on \(L^p(\mathbb{R})\),
which implies that \([b, C_T]E\) satisfies hypothesis (a) in Theorem 2.7.

Next, we show that \([b, C_T]E\) satisfies hypothesis (b) in Theorem 2.7. We
may assume that \(b \in C_c^\infty(\mathbb{R})\) with \(\text{supp}\, b \subset I(0, R)\). For \(t > 2\), set \(K^c := \{x \in \mathbb{R} : |x| > tR\}\). Then

\[
\|[b, C_T]f(x)\|_{L^p(K^c)} = \|bC_T(f)(x) - C_T(bf)(x)\|_{L^p(K^c)} \\
\leq \|bC_T(f)(x)\|_{L^p(K^c)} + \|C_T(bf)(x)\|_{L^p(K^c)}.
\]

Since \(\text{supp}\, b \cap K^c = \emptyset\), we have

\[
\int_{|x| > tR} |bC_T(f)(x)|^p \, dx = 0,
\]

and so

\[
(4.43) \quad \|[b, C_T]f(x)\|_{L^p(K^c)} \leq \|C_T(bf)(x)\|_{L^p(K^c)}.
\]

Using equation (3.1) and the fact that \(\text{supp}\, b \subset I(0, R)\) we have

\[
|C_T(bf)(x)| \leq \int_{|y| < R} |C_T(x, y)|\|b(y)\|\|f(y)\| \, dy \\
\leq \int_{|y| < R} \frac{1}{|x - y|} |b(y)\|f(y)\| \, dy.
\]

Notice that for \(|x| > tR, t > 2\) and \(|y| < R\) we have \(|x - y| > |x|/2\). In view
of this fact and Hölder’s inequality, inequality (4.44) yields

\[
|C_T(bf)(x)| \leq \frac{2}{|x|} \int_{|y| < R} |b(y)\|f(y)\| \, dy \\
\leq \frac{2}{|x|} \left( \int_{|y| < R} |b(y)|^{p'} \, dy \right)^{1/p'} \left( \int_{|y| < R} |f(y)|^p \, dy \right)^{1/p} \\
\leq \frac{2}{|x|} \|b\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})} (2R)^{1/p'} \\
= 2^{1+1/p'} \|b\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})} R^{1/p'} \frac{1}{|x|},
\]

since \(b \in C_c^\infty(\mathbb{R})\). With this estimate of \(|C_T(bf)(x)|\), inequality (4.43) be-
comes

\[
\|[b, C_T]f(x)\|_{L^p(K^c)} \leq 2^{1+1/p'} \|b\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})} R^{1/p'} \left( \int_{|x| > tR} \frac{1}{|x|^p} \, dx \right)^{1/p}.
\]
Finally, given each $\varepsilon > 0$, we can choose $t$ large enough such that $C t^{-1/p'} < \varepsilon$. Here the constant $C$ depends on $b$ and on the bound on $\|f\|_{L^p(\mathbb{R})}$ for $f \in E$. Hence hypothesis (b) in Theorem 2.7 holds for $[b, C_T]E$.

It remains to prove that $[b, C_T]E$ also satisfies hypothesis (c) of Theorem 2.7. Let $\varepsilon$ be a fixed positive constant in $(0, 1/2)$. Since $b \in C^\infty_c(\mathbb{R})$, it is uniformly continuous. Choose $z_0 = z_0(b, \varepsilon)$ sufficiently small that for all $z \in (0, z_0)$, we have both $|z| < \varepsilon^2$ and for all $x \in \mathbb{R}$, $|b(x) - b(x+z)| < \varepsilon$. Fix $z \in (0, z_0)$. Then for all $x \in \mathbb{R}$,

$$[b, C_T]f(x) - [b, C_T]f(x + z)$$

$$= \int_{\mathbb{R}} C_T(x,y)[b(x) - b(y)]f(y) \, dy$$

$$- \int_{\mathbb{R}} C_T(x+z,y)[b(x+z) - b(y)]f(y) \, dy$$

$$= \int_{|x-y| < \varepsilon^{-1}|z|} C_T(x,y)[b(x) - b(x+z)]f(y) \, dy$$

$$+ \int_{|x-y| < \varepsilon^{-1}|z|} [C_T(x,y) - C_T(x+z,y)][b(x+z) - b(y)]f(y) \, dy$$

$$+ \int_{|x-y| \leq \varepsilon^{-1}|z|} C_T(x,y)[b(x) - b(y)]f(y) \, dy$$

$$- \int_{|x-y| \leq \varepsilon^{-1}|z|} C_T(x+z,y)[b(x+z) - b(y)]f(y) \, dy$$

$$=: \sum_{j=1}^4 L_i.$$  

We start with $L_2$. Since $\varepsilon \in (0, 1/2)$, it follows that

$$|x-y| > \varepsilon^{-1}|z| \Rightarrow |(x+z) - x| < \frac{|x-y|}{2}.$$ 

Thus we may apply the smoothness condition of the kernel $C_T(x,y)$, concluding that

$$|C_T(x,y) - C_T(x+z,y)| \leq \frac{2(L+1)|x+z-x|}{|y-x|^2} = \frac{2(L+1)|z|}{|y-x|^2}.$$ 

Using this inequality together with the fact that $b \in C^\infty_c(\mathbb{R})$, we get

$$|L_2| \lesssim |z| \int_{|x-y| > \varepsilon^{-1}|z|} \frac{|f(y)|}{|y-x|^2} \, dy.$$
From this and Hölder’s inequality, we have
\[ \int_\mathbb{R} |L_2|^p \, dx \lesssim |z|^p \int_\mathbb{R} \left[ \int_{|x-y| \geq \epsilon^{-1}|z|} \frac{1}{|y-x|^{2/p'}} |f(y)| \, dy \right]^p \, dx \]
\[ = |z|^p \int_\mathbb{R} \left\{ \int_{|x-y| \geq \epsilon^{-1}|z|} \frac{1}{|y-x|^2} \, dy \right\}^{p/p'} \int_{|x-y| \geq \epsilon^{-1}|z|} \frac{|f(y)|^p}{|x-y|^2} \, dx \, dy \]
\[ \lesssim |z|^p \int_\mathbb{R} (\epsilon |z|^{-1})^{p/p'} \int_{|x-y| \geq \epsilon^{-1}|z|} \frac{|f(y)|^p}{|x-y|^2} \, dx \, dy \]
\[ \lesssim |z|^p (\epsilon |z|^{-1})^{p/p'} \int_\mathbb{R} \epsilon |z|^{-1} |f(y)|^p \, dy \]
\[ = |z|^p (\epsilon |z|^{-1})^p \|f\|_{L^p(\mathbb{R})}^p \]
(4.45)
\[ = \epsilon^p \|f\|_{L^p(\mathbb{R})}^p. \]

Turning to \( L_3 \), by (3.1), the fact that \( b \in C^\infty_c(\mathbb{R}) \) and the Mean Value Theorem, we conclude that
\[ |L_3| \lesssim \int_{|x-y| \leq \epsilon^{-1}|z|} |f(y)| \, dy. \]

Then using Hölder’s inequality as for \( L_2 \) we see that
\[ \int_\mathbb{R} |L_3|^p \, dx \lesssim \int_\mathbb{R} \left[ \int_{|x-y| \leq \epsilon^{-1}|z|} |f(y)| \, dy \right]^p \, dx \]
\[ \lesssim \int_\mathbb{R} \left\{ \int_{|x-y| \leq \epsilon^{-1}|z|} dy \right\}^{p/p'} \int_{|x-y| \leq \epsilon^{-1}|z|} |f(y)|^p \, dy \right\} \, dx \]
\[ \lesssim (\epsilon^{-1}|z|)^p \|f\|_{L^p(\mathbb{R})}^p \]
(4.46)
\[ < \epsilon^p \|f\|_{L^p(\mathbb{R})}^p. \]
by our choice of \( z \). Similarly, we obtain the same estimate for \( L_4 \):

(4.47)
\[ \int_\mathbb{R} |L_4|^p \, dx \lesssim \epsilon^p \|f\|_{L^p(\mathbb{R})}^p. \]

Lastly, we consider \( L_1 \):
\[ |L_1| \leq |b(x) - b(x + z)| \sup_{t > 0} \left| \int_{|x-y| > t} C_T(x, y) f(y) \, dy \right| \]
\[ = |b(x) - b(x + z)| C_T f(x). \]

Here we will use the following standard result.

**Theorem 4.2.** ([Duo01], Theorem 5.14, p.102) If \( T \) is a Calderón–Zygmund operator, then \( T^* \) is weak (1,1) and strong (p,p) for all \( p \in (1, \infty) \). \( T^* \) is defined by
\[ T^* f(x) := \sup_{t > 0} \left| \int_{|x-y| > t} K(x, y) f(y) \, dy \right|. \]
Thus we see that $C^*_T$ is bounded on $L^p(\mathbb{R})$ for all $p \in (1, \infty)$. Recall that $|b(x) - b(x + z)| < \varepsilon$ by our choice of $z$. Hence

$$
\int_{\mathbb{R}} |L_1|^p \, dx \leq \int_{\mathbb{R}} |b(x) - b(x + z)|^p |C^*_T f(x)|^p \, dx
< \varepsilon^p \int_{\mathbb{R}} |C^*_T f(x)|^p \, dx
\lesssim \varepsilon^p \|f\|_{L^p(\mathbb{R})}^p.
$$

(4.48)

Combining the estimates (4.45)–(4.48) of $L_i$, $i \in \{1, 2, 3, 4\}$, we conclude that

$$
\left[ \int_{\mathbb{R}} \left| [b, C_T] f(x) - [b, C_T] f(x + z) \right|^p \, dx \right]^{1/p} \lesssim \sum_{i=1}^4 \left( \int_{\mathbb{R}} |L_i|^p \, dx \right)^{1/p}
\lesssim \varepsilon \|f\|_{L^p(\mathbb{R})}.
$$

This shows that $[b, C_T]E$ satisfies hypothesis (c) in Theorem 2.7. Hence, $[b, C_T]$ is a compact operator. This completes the proof of Theorem 1.2. □

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