Higher elastica: geodesics in jet space

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Abstract
Carnot groups are subRiemannian manifolds. As such, they admit geodesic flows, which are left-invariant Hamiltonian flows on their cotangent bundles. Some of these flows are integrable; some are not. The \( k \)-jets space of real-valued functions on the real line forms a Carnot group of dimension \( k + 2 \). In this study, it is shown that its geodesic flow is integrable and that its geodesics generalize Euler’s elastica, with the case \( k = 2 \) corresponding to the elastica.

Keywords Hamiltonian dynamics · Integrable system · Carnot groups · Goursat distribution · SubRiemannian geometry

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1 Introduction
The \( k \)-jets space of real functions of a single real variable, denoted here by \( J^k \), is a \((k + 2)\)-dimensional manifold endowed with a canonical rank 2 distribution, i.e., a linear sub-bundle of its tangent bundle. This distribution is framed by two vector fields, below denoted by \( X_1, X_2 \), whose iterated Lie brackets endow \( J^k \) with the structure of a stratified group. Declaring \( X_1 \) and \( X_2 \) to be orthonormal endows \( J^k \) with the structure of a subRiemannian manifold, which is (left-) invariant under the group multiplication. Like any subRiemannian structure, the cotangent bundle \( T^* J^k \) is endowed with a Hamiltonian system whose underlying Hamiltonian \( H \) is that whose solution curves project to the subRiemannian geodesics on \( J^k \). We call this Hamiltonian system the geodesic flow on \( J^k \).
This paper has two main goals, the following theorem is the first one.

**Theorem 1.1** The geodesic flow for the subRiemannian structure on $\mathcal{J}^k$ is integrable.

The space $\mathcal{J}^1$ is isometric to the Heisenberg group, this result is well known, see [10, 14]. The space $\mathcal{J}^2$ is isometric to Engel’s group, and Ardentov and Sachkov showed that its subRiemannian geodesics correspond to Euler’s elastica. Their result inspired our Theorem 1.2.

The space $\mathcal{J}^k$ comes with a projection $\Pi: \mathcal{J}^k \rightarrow \mathbb{R}^2 = \mathbb{R}^2_{x, u_k}$ onto the Euclidean plane which projects the frame $X_1, X_2$ onto the standard coordinate frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial u_k}$ of $\mathbb{R}^2$. (See Sect. 2 for the meaning of the coordinates.) As a consequence, a horizontal curve $\gamma$ in $\mathcal{J}^k$ is parameterized by (subRiemannian) arc-length if and only if its planar projection $\Pi \circ \gamma$ to $\mathbb{R}^2$ is parameterized by arc-length. We will characterize geodesics on $\mathcal{J}^k$ in terms of their planar projections. As already mentioned, Ardentov and Sachkov [3] proved that when $k = 2$, the planar projections of geodesics are Euler’s elastica. These elastica have “directrix” the $u_2$-line, the line orthogonal to the $x$-axis. There are several ways to characterize Euler’s elastica, see, e.g., [7, 8, 11, 12]. The one we will use is as follows. Take a planar curve $c(s) = (x(s), y(s))$ and consider its curvature $\kappa = \kappa(s)$, where $s$ is the arc-length. Then the curve $c$ is an Euler’s elastica with a line directrix parallel to the $y$-axis if and only if $\kappa(s) = P(x(s))$ for some degree $(k-1)$-polynomial $P(x)$ in the coordinate $x$. Conversely, any plane curve $c(s)$ in the $(x, u_k)$ plane, which is parameterized by the arc-length $s$ and whose curvature $\kappa(s)$ equals $P(x(s))$ for some polynomial $P(x)$ of degree at most $k-1$ in $x$, is the projection of such a subRiemannian geodesic.

**Example 1.3** For the case $k = 1$ of the Heisenberg group, the theorem asserts that $\kappa = P(x)$, where $P$ is a degree 0 polynomial, i.e., a constant function. The only curves having constant curvature are lines and circles, and these are well known to be the projections of the Heisenberg geodesics.
A geodesic is called globally minimizing if each of its compact sub-arcs realizes the distance between its endpoints. The geodesic on $J^k$ will be classified and used to present Conjecture 6.2, which attempts to make a complete classification of global minimizing geodesics on $J^k$.

The outline of the paper is as follows. Section 2 introduces $J^k$, the $k$-th jet space as a subRiemannian manifold, and the notation that will be used throughout the work. Also, the Hamiltonian subRiemannian geodesic flow is defined and some details are given about the Carnot structure of $J^k$. In Sect. 3, the Poisson–Lie reduction is used to prove Theorem 1.1 and an explicit expression of the Casimir functions is presented. In Sect. 4, we use the Lie–Poisson bracket and the geodesic equations to prove Theorem 1.2. In Sect. 5, it is shown that the geodesics are generically periodic on $x$-coordinates and the geodesics on the $J^k$ are classified. Finally, in Sect. 6, Conjecture 6.2 is stated about a complete classification of globally minimizing geodesics on $J^k$.

2 Set-up

The $k$-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ is its $k$th order Taylor expansion at $x_0$. We will encode this $k$-jet as a $(k+2)$-tuple of real numbers as follows:

$$(j^k f)(x_0) = (x_0, f^k(x_0), f^{k-1}(x_0), \ldots, f'(x), f(x_0)) \in \mathbb{R}^{k+2}.$$  

As $f$ varies over smooth functions and $x_0$ varies over $\mathbb{R}$, these $k$-jets sweep out the $k$-jet space, denoted by $J^k$. One can see that $J^k$ is diffeomorphic to $\mathbb{R}^{k+2}$ and its points are coordinatized according to

$$(x, u_k, u_{k-1}, \ldots, u_1, y) \in \mathbb{R}^{k+2} := J^k.$$  

Recall that if $y = f(x)$, then $u_1 = dy/dx$, while $u_{j+1} = du_j/dx$, $j \geq 1$. Rearranging these equations into $dy = u_1 dx$, $du_j = u_{j+1} dx$, we see that $J^k$ is endowed with a natural rank 2 distribution $\mathcal{D} \subset TJ^k$ characterized by the $k$ Pfaffian equations

$$u_1 dx - dy = 0,$$

$$u_2 dx - du_1 = 0,$$

$$\ldots$$

$$u_k dx - du_{k-1} = 0.$$  

The typical integral curves of $\mathcal{D}$ are the $k$-jet curves $x \mapsto (j^k f)(x)$. In addition to these integral curves we have a distinguished family of curves which arise by varying only the highest derivative $u_k$, and which are the integral curves of the vector field $X_2$ below equation (2.1). These latter curves are $C^1$-rigid in the sense of Bryant–Hsu, see [9], and they exhaust the supply of $C^1$-rigid curves.

A subRiemannian structure on a manifold consists of a non-integrable distribution together with a smoothly varying family of inner products on the distribution. We have our distribution $\mathcal{D}$ on $J^k$. We arrive at our subRiemannian structure by observing that
\(\mathcal{D}\) is globally framed by the two vector fields
\[
X_1 = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} + \sum_{i=2}^{k} u_i \frac{\partial}{\partial u_{i-1}} \quad \text{and} \quad X_2 = \frac{\partial}{\partial u_k}
\] (2.1)
and then declaring these two vector fields to be orthonormal. Now the restrictions of the one-forms \(dx, du_k\) to \(\mathcal{D}\) form a global co-frame for \(\mathcal{D}^*\) which is dual to our frame, equation (2.1). Therefore an equivalent way to describe our subRiemannian structure is to say that its metric is \(dx^2 + du_k^2\) restricted to \(\mathcal{D}\).

For the purposes of Theorem 1.2 the following alternative characterization of the subRiemannian metric is crucial. Consider the projection
\[
\Pi: J^k \to \mathbb{R}^2_{x, u_k}; \quad \Pi(x, u_k, u_{k-1}, \ldots, u_1, y) = (x, u_k).
\]
Its fibers are transverse to \(\mathcal{D}\), and since \(\Pi_*X_1 = \frac{\partial}{\partial x}, \Pi_*X_2 = \frac{\partial}{\partial u_k}\), the frame pushes down to the standard frame for \(\mathbb{R}^2\). The metric on each two-plane \(\mathcal{D}_p, p \in J^k\), is characterized by the condition that \(d\Pi_p\) (which is just \(\Pi\) since \(\Pi\) is linear), restricted to \(\mathcal{D}_p\) is a linear isometry onto \(\mathbb{R}^2\), where \(\mathbb{R}^2\) is endowed with the standard metric \(dx^2 + du_k^2\). It follows immediately that the length of any horizontal path equals the length of its planar projection, that \(\Pi\) is a “submetry”, i.e., \(\Pi(B_r(p)) = B_r(\Pi(p))\), where \(B_r(p)\) denotes the metric ball of radius \(r\) about \(p\), and the horizontal lift of a Euclidean line in \(\mathbb{R}^2\) is a geodesic in \(J^k\).

### 2.1 Hamiltonian

Let \(P_1, P_2: T^*J^k \to \mathbb{R}\) be the ‘power functions’ of the vector fields \(X_1, X_2\) above (see [14, p. 8]). In terms of traditional cotangent coordinates \((x, u_k, u_{k-1}, \ldots, u_1, y, p_x, p_k, p_{k-1}, \ldots, p_1, p_y)\) for \(T^*J^k\), with \(p_i\) short for \(p_{u_i}\), we have
\[
P_1 = p_x + u_1 p_y + u_2 p_1 + \ldots + u_k p_{k-1}; \quad P_2 = p_k.
\]

Then the Hamiltonian governing the subRiemannian geodesic flow on \(J^k\) is
\[
H = \frac{1}{2} (P_1^2 + P_2^2)
\]
(see [14, p. 8]), where the condition \(H = 1/2\) implies that the geodesics are parameterized by arc-length, and this we will do in what follows.

**Remark 2.1** (\(C^1\)-rigidity) A curve tangent to \(\mathcal{D}\) is called \(C^1\)-rigid if it is a critical point of the endpoint map (see [14, Chapter 3]). The \(u_k\) curves are \(C^1\)-rigid for \(\mathcal{D}\), and form what Liu–Sussmann christened as the “regular-singular” curves for \(\mathcal{D}\). As such, they are geodesics for any subRiemannian metric \(Edx^2 + 2Fdxdu_k + Gdu_k^2\), restricted to \(\mathcal{D}\) such that \(ds^2\) is positive definite and for \(E, F, G\) functions of the jet coordinates \((x, u_k, u_{k-1}, \ldots, y)\), regardless on whether or not they satisfy the
corresponding (normal) geodesic equations. For the present metric each $u_k$-curve is indeed the projection to $\beta^k$ of a solution to our $H$, so we do not go to extra effort to account for these abnormal geodesics (REF [14, Chapter 3]).

2.2 Carnot group structure

Under iterated bracket, the frame $\{X_1, X_2\}$ generates a $(k + 2)$-dimensional nilpotent Lie algebra which can be identified pointwise with the tangent space to $\beta^k$. Specifically, if we write

$X_3 = [X_2, X_1], \quad X_4 = [X_3, X_1], \quad \ldots, \quad X_{k+2} = [X_{k+1}, X_1], \quad 0 = [X_{k+2}, X_1],$

then we compute that

$$X_{k+2} = \frac{\partial}{\partial y}, \quad X_{k+1} = \frac{\partial}{\partial u_1}, \quad X_k = \frac{\partial}{\partial u_2}, \quad \ldots, \quad X_3 = \frac{\partial}{\partial u_{k-1}}$$

while all other Lie brackets $[X_i, X_j]$ are zero. The span of the $X_i$ thus forms a $(k + 2)$-dimensional graded nilpotent Lie algebra $\mathfrak{g}_k = V_1 \oplus V_2 \oplus \cdots \oplus V_{k+1}$, where $V_1 = \text{span}\{X_1, X_2\}$, $V_i = \text{span}\{X_{i+1}\}$, $1 < i \leq k + 1$.

Like any graded nilpotent Lie algebra, this algebra has an associated Lie group which is a Carnot group $G$ w.r.t. the subRiemannian structure, and by using the flows of the $X_i$, we can identify $G$ with $\beta^k$, and the $X^i$ with left-invariant vector fields on $G \cong \beta^k$.

3 Integrability: Proof of Theorem 1.1

Our Hamiltonian $H$ is a left-invariant Hamiltonian on the cotangent bundle of a Lie group $G$. Let us recall the general ‘Lie–Poisson’ structure for such Hamiltonian flows, see [6, Appendix] or [13, Chapter 4]:

$$T^*G \xrightarrow{J_L} \mathfrak{g}^* \xleftarrow{J_R} \mathfrak{g}^*_+ \xrightarrow{J_L} \mathfrak{g}^*_+ \xleftarrow{J_R} \mathfrak{g}^*$$

The arrows $J_R, J_L$ are the momentum maps for the actions of $G$ on itself by right and left translation, lifted to $T^*G$. The subscripts ± are for a plus or minus sign in front of the Lie–Poisson (also known as Kostant–Kirillov–Souriau) bracket on $\mathfrak{g}^*$. $J_R$ corresponds to the left translation back to the identity and realizes the quotient of $T^*G$ by the left action. $J_L$ corresponds to the right translation of a covector back to the identity and forms the components of the momentum map for the left translation, lifted to the cotangent bundle. In this case, $\mathfrak{g}^* = \mathbb{R}^{k+2}$ and
\[ J_R = (P_1, P_2, P_3, \ldots, P_{k+2}) \]

with \( P_i \) as the power function associated to \( X_i \), so that

\[ P_3 = p_{k-1}, \quad P_4 = p_{k-2}, \quad \ldots, \quad P_{k+2} = p_y \]

in terms of standard canonical coordinates as above.

When the Hamiltonian \( H : T^*G \to \mathbb{R} \) is left-invariant, it can be expressed as a function of the components of \( J_R \), that is, \( H = h \circ J_R \) for some \( h : \mathfrak{g}^* \to \mathbb{R} \), and \( H \) Poisson commutes with every component of the left momentum map \( J_L \), so that these left-components are invariants. \( J_L \) and \( J_R \) are related by \( J_L(g, p) = (\text{Ad}_g)^* J_R(g, p) \), where we have written \( p \in T^*_g G \), and where \( \text{Ad}_g \) is the adjoint action of \( g \).

The reason underlying the integrability of this system is a simple dimension count.

**Proposition 3.1** If the generic co-adjoint orbit of \( \mathfrak{g}^* \) is 2-dimensional, then the left-invariant Hamiltonian flow on \( T^*G \) is integrable.

Let us recall that the symplectic reduced spaces for the left translation action are the co-adjoint orbits, for \( \mathfrak{g}^* + \), and that \( J_R \) realizes this symplectic reduction procedure, mapping each \( J_L^{-1}(\mu) \) onto the co-adjoint orbit through \( \mu \). The hypothesis of the proposition asserts that the symplectic reduced spaces associated to the \( G \)-action are zero or 2-dimensional, so, morally speaking, the system is automatically integrable by reasons of dimension count.

**Proof** We must produce \( n \) commuting integrals in involution, where \( n = \dim(G) \). The hypothesis asserts that there are \( n - 2 \) Casimirs \( C_1, \ldots, C_{n-2} \) for \( \mathfrak{g}^* \), these being the functions whose common level sets at a generic value define a generic co-adjoint orbit. These Casimirs are a functional basis for the \( \text{Ad}_G^* \)-invariant polynomials on \( \mathfrak{g}^* \).

When viewed as functions on \( T^*G \) via \( C_i \circ J_R \), the Casimirs Poisson commute with any left-invariant function on \( T^*G \), and in particular with \( H \) and with each other. Thus, \( H, C_1, C_2, \ldots, C_{n-2} \) yield \( n - 1 \) integrals. To get the last commuting integral, take any component of \( J_L \).

**Proof of Theorem 1.1** In order to use the proposition, we need to verify that the co-adjoint orbits are generically 2-dimensional. We have the Poisson brackets

\[ \{ P_i, P_i \} = P_{i+1}, \quad 1 < i < k + 1, \quad \text{and} \quad \{ P_i, P_{k+2} \} = 0, \]

with all other Poisson brackets \( \{ P_i, P_j \}, 1 < i < j \leq k + 2 \), being zero. Thus the Poisson tensor \( B \) at a point \( Z = (P_1, P_2, P_3, \ldots, P_{k+2}) \in \mathfrak{g}^*_+ \) is

\[ B := \begin{pmatrix} 0 & Z_k & 0 \\ Z_k^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad Z_k = (P_3, \ldots, P_{k+2}), \]

which generically has rank 2, and rank 0 if and only if \( Z_k = 0 \), i.e., if and only if \( P_i = 0 \) for \( 2 < i \).
Note When this paper was completed, we learned about closely related work by Anzaldo-Meneses and Monroy-Pérez, see [1, 2], who obtained many of the same results in a different way.

Thanks to Theorem 1.1, we know that the system has \( k \) Casimir functions.

**Theorem 3.2** Let \( C_i \) be the functions defined by

\[
C_i = P_{k+2}^{i-1} P_{k+2-i} + \sum_{j=1}^{i-2} (-1)^j P_{k+2}^{i-(j+1)} P_{k+2-j} \frac{P_{k+1}^j}{(j)!} + (-1)^{i-1} \frac{P_{k+1}^i}{(i-2)!}.
\]

for \( i > 1 \). If \( P_{k+2} \neq 0 \), then \( C_i \)s are constants of motion for the geodesic equations on \( \mathfrak{g}^k \), in others words they are Casimir functions\(^1\).

### 4 Integration and curvature: Proof of Theorem 1.2

Hamilton’s equations read as \( \frac{df}{dt} = \{ f, H \} \). With the Hamiltonian of this system, they expand to \( \frac{df}{dt} = \{ f, P_1 \} P_1 + \{ f, P_2 \} P_2 \). Returning to our coordinates \( x, u_k \), we compute \( \{ x, P_1 \} = 1, \{ x, P_2 \} = 0, \{ u_k, P_1 \} = 0, \{ u_k, P_2 \} = 1 \), so that

\[
\frac{dx}{dt} = P_1 \quad \text{and} \quad \frac{du_k}{dt} = P_2.
\]

(4.1)

Thus \( (P_1, P_2) \) are the components of the tangent vector to the plane curve \((x(s), u_k(s))\) obtained by projecting a geodesic to the plane. If \( H = 1/2 \), then this vector is a unit vector, the parameter \( t \) of the flow is the arc-length \( s \), and we can write

\[
(P_1, P_2, \dot{\theta}) = (\cos(\theta), \sin(\theta), \kappa).
\]

Using \( \{ P_1, P_2 \} = P_3 \), we see that \( P_1 \) evolves according to the equations

\[
\dot{P}_1 = P_3 P_2, \\
\dot{P}_2 = -P_3 P_1.
\]

(4.2)

We also have that \( \dot{P}_1 = -\dot{\theta} \sin(\theta) = -\dot{\theta} P_2 \) and \( \dot{P}_2 = +\dot{\theta} \cos(\theta) = +\dot{\theta} P_1 \), from which we see that

\[
- P_3 = \kappa.
\]

\(^1\) In the case \( i = 2 \) the sum is empty.
Now, for $k + 2 > i > 2$ we have that $\{P_i, P_1\} = -P_{i+1}$, $\{P_i, P_2\} = 0$, and $\{P_j, P_{k+2}\} = 0$ for all $j$, so that

$$
\begin{align*}
\dot{P}_3 &= -P_1 P_4, \\
\dot{P}_4 &= -P_1 P_5, \\
&\quad \ldots \\
\dot{P}_{k+1} &= -P_1 P_{k+2}, \\
\dot{P}_{k+2} &= 0.
\end{align*}
$$

(4.3)

**Proof of Theorem 1.2** Consider a geodesic $\gamma$ and an arc of the geodesic for which $\dot{x} \neq 0$. Instead of the arc-length $t = s$, we use $x$ to parameterize this arc. From equation (4.1) along this arc

$$\frac{d}{dx} = \frac{1}{P_1} \frac{d}{ds},$$

so that the equations for the evolution of $P_3$, $P_4$, $\ldots$, $P_{k+2}$ along the curve become

$$
\begin{align*}
\frac{dP_3}{dx} &= -P_4, \\
\frac{dP_4}{dx} &= -P_5, \\
&\quad \ldots \\
\frac{dP_{k+1}}{dx} &= -P_{k+2}, \\
\frac{dP_{k+2}}{dx} &= 0,
\end{align*}
$$

these equations can be summarized by

$$
\frac{d^k P_3}{dx^k} = 0,
$$

which asserts that the curvature $P_3$, of the projected curve $c = \pi \circ \gamma$, is a polynomial $p(x)$ of degree $k - 1$ in $x$, at least along this arc-length. Finally, since $\gamma$ is an analytic function of $s$, so are $c(s)$ and $\kappa(s)$, so that if $\kappa(s)$ enjoys a relation $\kappa(s) = p(x(s))$ along some subarc of $c(s)$, it enjoys this same relation everywhere along $c$.

To prove the converse, first consider a general smooth curve $c(s)$ in the $(x, u)$ plane along which $dx/ds > 0$. We can parameterize the curve either by the arc-length $c(s) = (x(s), u(s))$ or as a graph $u = u(x)$. Define the function $F(x)$, with $-1 \leq F(x) \leq 1$, by way of relating the two parameterizations

$$
(\dot{x}, \dot{u}) := \left(\frac{dx}{ds}, \frac{du}{ds}\right) = \left(\sqrt{1 - F(x)^2}, F(x)\right)
$$

(4.4)
so that \( dx = \sqrt{1 - F(x)^2} \, ds \) and

\[
u'(x) := \frac{du}{dx} = \frac{F(x)}{\sqrt{1 - F(x)^2}}.
\]

Therefore,

\[
u'' = \frac{F'}{(1 - F(x)^2)^{3/2}}.
\]

The curvature of \( c(s) \), when viewed as a graph, is well known to be

\[
\kappa(x) = \frac{u''(x)}{(1 + u'(x)^2)^{3/2}}
\]

and so

\[
(1 + u'(x)^2) = \frac{1}{1 - F(x)^2},
\]

from which we conclude that

\[
\kappa = F'(x).
\]  \hspace{1cm} (4.5)

To end this proof, suppose that we are given a curve \( c(s) \) in the plane \((x, u)\), with \( u = u_k \), whose curvature \( \kappa \) is a degree \( k - 1 \) polynomial in \( x \). Define \( F(x) \) by equation (4.4) along an arc of \( c(s) \) for which \( dx/ds > 0 \). From equation (4.5), we know that \( F \) is an anti-derivative of \( \kappa \) and so a polynomial of degree \( k \) in \( x \). The constant term in the integration \( F(x) = \int x \kappa \, dx \) is fixed by choosing any point \((x_*, u_*) = (x(s_*), u(s_*))\) along \( c(s) \) for which \( dx/ds > 0 \), so that \(-1 < du/ds|_{s=s_*} < 1\) and setting \( F(x_*) = du/ds|_{s=s_*} \). By the preceding analysis, \( c(s) \) has curvature \( \kappa(x(s)) \) along the entire arc \( dx/ds > 0 \) of \( c(s) \) which contains \((x_*, u_*)\). Moreover, \( u'(x) = F(x)/\sqrt{1 - F(x)^2} \). Set

\[
(P_1(x), P_2(x)) := \left( \sqrt{1 - (F(x))^2}, F(x) \right), \hspace{1cm} (4.6)
\]

and

\[
P_3(x) := F'(x), \quad \text{and} \quad P_{i+2} := (-1)^i \frac{d^i F}{d^i x}(x), \hspace{0.5cm} i > 1. \hspace{1cm} (4.7)
\]

We look at \( P_i \) as momentum functions. Reparameterize the momentum functions by \( s \) using \( dx/ds = P_1(x) \). Then we verify that \( P_i \) satisfies (4.2) and (4.3), so that the horizontal curve \( \gamma(x(s)) \) in \( J^k \) with these momenta satisfies the geodesic equations and projects on our given curve \( c(s) \).
Corollary 4.1 Suppose that the momentum functions $P_i$ are related to the degree $k$ polynomial $F(x)$ as per equations (4.6), (4.7), and that $H = 1/2$, then a critical point $x_0$ of $F(x)$ corresponds to a relative equilibrium for the reduced equations (4.2) and (4.3) if and only if $F(x_0) = \pm 1$.

Proof The equilibria of equations (4.2) and (4.3) are the points with $P_1 = 0$ and $P_3 = 0$, as long as $H \neq 0$. If $H = 1/2$, the condition $P_1 = 0$ forces $P_2 = \pm 1$, but $P_2 = F(x)$. Finally, $P_3 = F'(x)$. □

5 Structure of the higher elastica

From the proof of Theorem 1.2, we have some freedom selecting a primitive function for $p(x)$, then, given $F_k(x) = \int p(x) \, dx$, the dynamic is trivial if $F_k(x(s))$ is constant for all $s$, that is, when $F_k^{-1}([−1, 1])$ is empty or a set of isolated points. Then we take $F_k^{-1}([−1, 1])$ as follows:

$$F_k^{-1}([−1, 1]) := \bigcup_I [x_0^i, x_1^i], \quad \text{where } F_k^2(x_0^i) = 1, \quad F_k^2(x_1^i) = \pm 1,$$

$x_0^i < x_1^i < x_0^{i+1} < x_1^{i+1}$, and the condition that $|F_k(x)| < 1$ if $x \in (x_0^i, x_1^i)$; note that we allow $F_k(x_0^i) F_k(x_1^i) = 1$ or $F_k(x_0^i) F_k(x_1^i) = −1$, this dichotomy will help us to classify the geodesic in Sect. 5.1. Choose $[x_0^i, x_1^i] = [x_0, x_1]$.

Theorem 5.1 The curve $c(s)$ in $(x, u)$ with curvature $k((x(s)) = p(x)$ is bounded in the $x$-direction; generically the curve is periodic in $x$, and the period $L$ is given by

$$L := \int_{x_0}^{x_1} \frac{2 \, dx}{\sqrt{1 - F_k^2(x)}}, \quad \text{we also define } \Delta u := \int_{x_0}^{x_1} \frac{2F_k(x)}{\sqrt{1 - F_k^2(x)}}.$$

Finally, we have that $u(s + L) = u(s) + \Delta u$.

Due to Corollary 4.1, we know that the points $x_0$ and $x_1$ are equilibrium points if and only if they are critical points of the function $F_k(x)$, so it takes infinite time to arrive to them.

Let $x_0$ be a regular point, we notice that if $c(s)$ is reparameterized as a graph $u = u(x)$, as in the proof of Theorem 1.2, $c(s)$ stops to be a graph at the point $x_0$ (see Fig. 1). We will answer the question of how to extend the curve $c(s)$ to a “multiple value function” on $x$ at $x = x_0$ so that its lift to $Ω^k$ is a smooth solution to the geodesic equations. Set $(P_1, P_2) = (\cos(\theta), \sin(\theta))$ and $\dot{\theta} = p(x)$, since $F_k(x_0) = \pm 1$ define $\theta(x_0) = \pm \pi/2$ and $\dot{\theta}(x_0) \neq 0$, then $P_1$ changes sign, while $P_2$ does not change. Therefore, if $x(s_0) = x_0$ we define

$$(\dot{x}, \dot{u}) = \left\{ \begin{array}{ll} \left( \pm \sqrt{1 - F_k^2(x)}, F_k(x) \right) & \text{if } s_0 - L/2 \leq s \leq s_0, \\ (1 + \sqrt{1 - F_k^2(x)}, F_k(x)) & \text{if } s_0 \leq s \leq s_0 + L/2. \end{array} \right.$$
Hence, the curve stays in the interval \([x_0, x_1]\), we extend the curve \(c(s)\) to a “multiple value function” on \(x\) at \(x = x_1\) in the same way. If both are regular points, we can read the equation \(P_1 = \sqrt{1 - F_k(x)}\) as the restriction \(P_1(x)|_{H=1/2, C_1, \ldots, C_k}\) and consider action function \(I\) given by the area under the graph \(\sqrt{1 - F_k(x)}\) going from \(x_0\) to \(x_1\) and the area of \(-\sqrt{1 - F_k(x)}\) going from \(x_1\) to \(x_0\), i.e.,

\[
J(H = 1/2, C_1, \ldots, C_k) := 2 \int_{x_0}^{x_1} \sqrt{1 - F_k(x)} \, dx.
\]

Finally, the period is given by \(\frac{\partial J}{\partial H}|_{H=1/2, C_1, \ldots, C_k} = L\) (see [6, Chapter 10]). The period goes to infinity when \(x_0\) or \(x_1\) are critical points, much like the well-known homoclinic connection of a pendulum.

Let us consider \((x_0, u_*)\) the initial point of the curve, \(x(s) \in [x_0, x_1]\) and \(2s \leq L\), then

\[
u(s) + \tau = \int_{x_0}^{x(s)} \frac{F_k(x) \, dx}{\sqrt{1 - F_k^2(x)}} + \int_{x_0}^{x_1} \frac{2F_k(x) \, dx}{\sqrt{1 - F_k^2(x)}} + u_*
= \left( \int_{x_0}^{x(s)} + \int_{x(s)}^{x_1} + \int_{x_0}^{x_1} + \int_{x_0}^{x(s+L)} \right) \frac{F_k(x) \, dx}{\sqrt{1 - F_k^2(x)}} + u_* = u(s + L).
\]

Once again, we use the fact that \(\int_{x_0}^{x_1} \frac{F_k(x) \, dx}{\sqrt{1 - F_k^2(x)}} = \int_{x_0}^{x_1} \frac{F_k(x) \, dx}{-\sqrt{1 - F_k^2(x)}}\).

### 5.1 Geodesic classification

There are three cases:

- \((x\text{-periodic})\) geodesics whose \(x\)-coordinate is periodic, \(p(x_0) \neq 0\) and \(p(x_1) \neq 0\).
- \((\text{homoclinic})\) geodesics whose plane curve is asymptotic to a one line in both directions: \(p(x_0) = 0\) and \(p(x_1) \neq 0\), or \(p(x_0) \neq 0\) and \(p(x_1) = 0\).
- \((\text{heteroclinic})\) geodesics whose plane curve is asymptotic to two lines: \(p(x_0) = 0\) and \(p(x_1) = 0\).

In the heteroclinic case, we add one more dichotomy to the mix.

**Definition 5.2** A heteroclinic geodesic is said to be of turn-back type if \(F_k(x_0) F_k(x_1) = -1\). Otherwise, it is said that the heteroclinic is of direct type, in which case \(F_k(x_0) F_k(x_1) = -1\).

To have a better understanding of Definition 5.2 see Figs. 3 and 4.

### 5.2 General Euler soliton

The elastica equation has a distinguished solution called the *Euler Kink*. Other names for it are Euler’s soliton or Convict’s curve (see Figs. 1 and 2), see [3, 8, 11, 12].
We define the *Euler soliton* at the level $k$ in the sense that the curvature of the curve $(x, u_k(x))$ is always proportional to $x^{k-1}$, see Fig. 2.

**Theorem 5.3** If $1 < k$ then the level $k$ has a soliton curve.

Consider the polynomial $F_k(x) = x^k/a^k - \alpha$. Set $x = a \sqrt[2k]{\alpha + \cos(t)}$, we find the following expressions:

$$u(t) = \int_{t_0}^{t_1} \frac{\cos(t) \, dt}{(\alpha + \cos(t))^{(k-1)/k}},$$

$$t(x) = \frac{a}{k} \int_{t_0}^{t_1} \frac{dt}{(\alpha + \cos(t))^{(k-1)/k}}.$$

The case $k = 2$ is the classical solution for the elastica equation, see [11, p. 436]. If $\alpha = 1$, then we have the explicit expression

$$u(x) = \int_x^{\sqrt{2}} \frac{x^{k/2} \, dx}{\sqrt{2} - x^k} - \frac{2}{k\sqrt{2}} \ln \left( \frac{\sqrt{2} - \sqrt{2 - x^k}}{x^{k/2}} \right),$$

$$t(x) = \frac{1}{\frac{3}{2} \sqrt{2}} \ln \left( \frac{\sqrt{2} - \sqrt{2 - x^k}}{x^{k/2}} \right).$$

We can find an explicit second order ODE for $\dot{\theta}$,

$$p_{u_{k-1}} = \frac{\partial P}{\partial x}(x) = kx^{k-1} \quad \text{and} \quad \frac{\dot{\theta}^2}{k^2 a^{2k}} = (\cos \theta + \alpha)^{2(k-1)/k}. \quad (5.1)$$

If $k > 1$ and $\alpha = 1$, the ODE defined by the right-side of equation (5.1) has a unique equilibrium point $\theta = \pi$ and a homoclinic connection. In the case $k = 2$, this ODE is the same as the pendulum equation defined in [3].
5.3 Heteroclinic geodesics

The case $k = 1, 2$ does not admit heteroclinic geodesics, since a polynomial $p(x)$ of degree 1 has at most one root. In the case $2 < k$ we have the opposite.

Proposition 5.4 If $k > 2$, then $\partial^k$ admits heteroclinic geodesics.

Proof We split in the even and odd cases, in both cases the Hill interval is $[-1, 1]$:

- **Turn-back type:** Consider $F_{2k+1}(x) = -\frac{x^{2k+1}}{2k+1}$. The points $x = \pm 1$ are equilibrium points, since $F_{2k+1}(\pm 1) = \pm 1$ and $\frac{\partial F}{\partial x}(\pm 1) = 0$.
- **Direct type:** Consider $F_{2k}(x) = -\frac{x^{2k}}{1-k}$. The points $x = \pm 1$ are equilibrium points, since $F_{2k}(\pm 1) = -1$ and $\frac{\partial F}{\partial x}(\pm 1) = 0$. \qed
6 Future work

Definition 6.1 A geodesic $\gamma(t)$ is globally minimizing if each of its compact sub-arcs realizes the distance between its endpoints, in other words, $\gamma(t)$ is an isometric embedding of the real line.

In [3–5], Andertov and Sachkov proved that the Euler soliton for $k = 2$ is globally minimizing geodesic and, in [10], Hakavuori and Donne showed that neither periodic nor turn-back geodesics are globally minimizing geodesic. These results suggest the following conjecture.

Conjecture 6.2 The global minimizers geodesics on $\mathcal{J}^k$ are homoclinic and heteroclinic of direct type.

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