Nöther Charges, Brown–York Quasilocal Energy and Related Topics

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Abstract. The Lagrangian proposed by York et al. and the covariant first order Lagrangian for General Relativity are introduced to deal with the (vacuum) gravitational field on a reference background. The two Lagrangians are compared and we show that the first one can be obtained from the latter under suitable hypotheses. The induced variational principles are also compared and discussed. A conditioned correspondence among Nöther conserved quantities, quasilocal energy and the standard Hamiltonian obtained by 3 + 1 decomposition is also established. As a result, it turns out that the covariant first order Lagrangian is better suited whenever a reference background field has to be taken into account, as it is commonly accepted when dealing with conserved quantities in non–asymptotically flat spacetimes. As a further advantage of the use of a covariant first order Lagrangian, we show that all the quantities computed are manifestly covariant, as it is appropriate in General Relativity.

1. Introduction

Many approaches to variational principles, conserved quantities and related topics can be found in the current literature about General Relativity (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and references quoted therein). We shall hereafter compare two of them, both dealing with a dynamical metric $g$ and a reference background metric $\bar{g}$ over a spacetime.

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manifold $M$ of arbitrary dimension $n$, with $n \geq 3$. A large part of the paper will deal with the explicit case $n = 4$, in view of applications to standard General Relativity. We shall deal with the vacuum case; the generalization to the case of presence of matter fields is straightforward.

The first variational principle we shall deal with is based on the covariant first order action functional for General Relativity (see [12], [14], [15], [16]). The second one ([3], [7]) is based on the action functional due to York et al., which is defined with the aim of dealing with the fixing of the $(n - 1)$–metric induced on the boundary $\partial D$ of any region $D$ of spacetime $M$.

The covariant first order action was introduced to set General Relativity in a standard covariant first order variational framework. As is well known, in fact, the Hilbert action functional $L = (1/2\kappa) \sqrt{g} R ds$ is second order in the metric field, so that field equations are expected to be of the fourth order. Einstein field equations are second order equations instead, as if the action were first order only. This is due to the well known fact that locally second derivatives of the metric field appearing in the scalar curvature may be hidden under a divergence, thus not appearing in field equations (a fact which was clear to Einstein from the very beginning; see, e.g. [17]). Of course, however, this cannot be done in general in a global and covariant way; that is why General Relativity is usually considered as a second order field theory or, whenever it is treated on a first order basis, something is lost (e.g. covariance or boundary terms; see [4], [5]).

The covariant first order action functional is the following:

$$A_D[g, \bar{g}] = \frac{1}{2\kappa} \int_D \sqrt{\bar{g}} \bar{R} ds - \frac{1}{2\kappa} \int_{\partial D} \sqrt{g} g^\mu\nu u^\alpha_{\mu\nu} ds_\alpha - \frac{1}{2\kappa} \int_D \sqrt{\bar{g}} \bar{R} ds \quad (1.1)$$

where $\kappa$ is a constant ($\kappa = 8\pi G/c^4$ in General Relativity with $\dim(M) = 4$), $\sqrt{\bar{g}}$ is the square root of the absolute value of the determinant of the dynamical metric $g$, $\sqrt{\bar{g}} = \sqrt{|\det\bar{g}|}$ the analogous quantity for the background metric $\bar{g}$ and $ds$ and $ds_\alpha = \partial_\alpha ds$ are the standard local bases for $n$–forms and $(n - 1)$–forms over $M$, respectively. We systematically denote by a bar the quantities referred to the background, i.e. here and hereafter we shall use the following notation

- $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$: covariant metric
- $g^{\mu\nu}$, $\bar{g}^{\mu\nu}$: contravariant metric
- $\Gamma^\alpha_{\beta\nu}$, $\bar{\Gamma}^\alpha_{\beta\nu}$: Levi–Civita connection
- $R^\alpha_{\beta\mu\nu}$, $\bar{R}^\alpha_{\beta\mu\nu}$: Riemann tensor
- $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, $\bar{R}_{\mu\nu} = \bar{R}^\alpha_{\mu\alpha\nu}$: Ricci tensor
- $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu}$: scalar curvature
- $u^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \delta^\mu_{(\alpha} \Gamma^\epsilon_{\beta)\epsilon}$, $\bar{u}^\mu_{\alpha\beta} = \bar{\Gamma}^\mu_{\alpha\beta} - \delta^\mu_{(\alpha} \bar{\Gamma}^\epsilon_{\beta)\epsilon}$:
Let us also introduce the following relative quantities:

\[ q_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\mu - \bar{\Gamma}_{\alpha\beta}^\mu, \quad w_{\alpha\beta}^\mu = u_{\alpha\beta}^\mu - \bar{u}_{\alpha\beta}^\mu. \]

The action functional (1.1) is associated to the so–called covariant first order Lagrangian

\[ L = \frac{1}{2\kappa} \left( \sqrt{g} R - d_\alpha (\sqrt{g} g^{\mu\nu} w_{\mu\nu}^\alpha) - \sqrt{\bar{g}} \bar{R} \right) ds \]

or equivalently

\[ L = \frac{1}{2\kappa} \left( (\sqrt{g} - \sqrt{\bar{g}}) \bar{R} + \sqrt{g} g^{\alpha\beta} (q_{\alpha\sigma}^\rho q_{\rho\beta}^\sigma - q_{\sigma\rho}^\alpha q_{\rho\beta}^\sigma) \right) ds \]

From the first expression (1.2.a), it can be easily seen that the fields \( g \) and \( \bar{g} \) do not interact and they both obey vacuum Einstein field equations (provided that suitable boundary conditions are satisfied, as we shall discuss shortly after). From the second expression (1.2.b), the Lagrangian is however recognised to be first order in \( g \) and second order in \( \bar{g} \). Being both \( q_{\alpha\beta}^\mu \) and \( w_{\alpha\beta}^\mu \) tensors, \( L \) is a covariant Lagrangian. It is, of course, the truly covariant counterpart of the so–called Hilbert–Palatini first order Lagrangian (see [18]), to which it reduces by suitable non–covariant cancellations and background fixings. We stress that in the variational principle induced by (1.1) the dynamical metric \( g \) is endowed with a direct physical meaning, while the reference background metric \( \bar{g} \) is, at least for the moment, introduced to provide covariance and as a reference value for conserved quantities, as discussed below. Notice that if the action is computed for \( g = \bar{g} \), then it identically vanishes.

We remark that the Hilbert–Einstein Lagrangian \( L = (1/2\kappa) \sqrt{g} R \, ds \) is a second order Lagrangian and thus it would a priori define a variational principle in which the dynamical metric \( g \) together with its first derivatives are kept fixed on the boundary. In fact, in standard variational calculus, for a covariant field theory described by a Lagrangian of order \( k \), one keeps fixed the fields on the boundary together with their derivatives up to order \( k - 1 \) (see Appendix B). However, the degeneracy of the Hilbert–Einstein action functional has been globally removed by introducing a reference background metric \( \bar{g} \) which provides a covariant way to cancel out the second order derivatives of the dynamical metric \( g \), as shown in (1.2.b) above. As we already remarked, the Lagrangian (1.2.b) is first order in \( g \) and second order in \( \bar{g} \) so that it defines a standard variational principle in which the dynamical metric \( g \) is kept fixed on the boundary, while the reference background metric \( \bar{g} \) is kept fixed together with its first derivatives on the boundary. Of course, nothing more can be done without breaking down covariance, i.e. without choosing an ADM
foliation, or without requiring suitable additional matching conditions between
the two metrics.

On the other hand, York’s action was originally built (see [3], [7], [19])
to provide a variational principle suited to deal with the boundary conditions
which are specified by keeping the induced metric on \( \partial D \) fixed. Boundary
terms are added to the standard Hilbert–Einstein action \( \frac{1}{2\kappa} \int_D R\sqrt{g} \, ds \).
They are exactly needed so that the boundary contribution to the variation of
the action vanishes when the induced \((n - 1)\)--metric is kept fixed on \( \partial D \).

The York’s action functional is adapted to an ADM foliation induced by
dragging a spacelike hypersurface \( \Sigma \) along a timelike vector field \( \zeta \).

An ADM foliation of a spacetime region \( D \) is obtained and it is para-
metrized by the affine parameter \( t \) of \( \zeta \) (see Fig. 1). The region \( D \) is thence
topologically the product of \( \Sigma \) times a real line interval \([t_0, t_1]\). Let the
generic leaf be \( \Sigma_t \) and \( B_t \) its boundary, which is obviously \((n - 2)\)--dimen-
sional. Let us then denote by \( \bar{u} \) the future directed timelike unit normal to
the leaf \( \Sigma_t \). The boundary \( \partial D \) is formed by the union of all \((n - 2)\)--
boundaries \( B_t \), which will be denoted by \( B \), together with the initial and fi-
nal leaves \( \Sigma_{t_0} \) and \( \Sigma_{t_1} \) of the sandwich, which will be called lids. Let
us denote by \( \bar{n} \) the outward spacelike unit normal to \( B \), by \( \gamma_{ij} \) the metric
induced by \( g \) on \( B \), by \( h_{ab} \) the metric induced on \( \Sigma_t \) and by \( \sigma_{AB} \) the metric
induced on the \((n - 2)\)--boundary \( B_t \). Here and in the sequel indices run in
the following ranges: Greek indices from 0 to \( n - 1 \), lower case Roman letters
\( a, b, \ldots \) from 1 to \( n - 1 \), lower case middle Roman letters \( i, j, \ldots \) take the
values 0, 2, \ldots \( n - 1 \) and upper case Roman letters \( A, B \ldots \) range from 2 to
\( n - 1 \) (see also Appendix A for the notation).

We shall finally denote by \( \Theta_{ij} \) the extrinsic curvature of \( B \) in the space-
time \( M \), by \( K_{ab} \) the extrinsic curvature of \( \Sigma_t \) in \( M \) and by \( K_{AB} \) the extrinsic
curvature of \( B_t \) in \( \Sigma_t \) (see Appendix A). We shall denote by \( \Theta = \gamma^{ij} \Theta_{ij} \),
\( K = h^{ab} K_{ab} \) and \( K = \sigma^{AB} K_{AB} \) the traces of the extrinsic curvatures of \( B \),
\( \Sigma_t \) and \( B_t \), respectively. It is assumed that the dynamical metric \( g \) and the
reference background \( \bar{g} \) induce the same metric \( \gamma_{ij} \) on \( B \). Furthermore, let
us also assume that the hypersurfaces \( B \) and \( \Sigma_t \) intersect orthogonally (or
equivalently that \( \bar{n} \) and \( \bar{u} \) are orthogonal on any \( B_t \), i.e. \( u^\mu n_\mu|_B = 0 \)). We
stress that here we are considering a region $D$ in which the timelike vector field $\zeta$ has no fixed points, so that the hypersurfaces $\Sigma_t$ do not intersect each other and span the whole region $D$.

According to this notation, York's action in presence of a background $\bar{g}$ may be written as

$$I_D[g, \bar{g}] = I_D[g] - I_D[\bar{g}]$$  \hspace{1cm} (1.3)

where the functional $I_D[g]$ is defined by:

$$I_D[g] = \frac{1}{2\kappa} \int_D \sqrt{g} R \, ds + \frac{1}{\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} K \ u^\alpha \sqrt{g} \, ds_\alpha - \frac{1}{\kappa} \int_B \Theta n^\alpha \sqrt{g} \, ds_\alpha$$  \hspace{1cm} (1.4)

and $I_D[\bar{g}]$ is the same functional calculated for the background $\bar{g}$. Notice that $\sqrt{g} u^\alpha \, ds_\alpha = \sqrt{n} \, d^3 x$ and $\sqrt{g} n^\alpha \, ds_\alpha = \sqrt{\gamma} \, d^3 x$ are the volume elements on $\Sigma_t$ and $B$, respectively. In the functional (1.4) we also set the convenient notation

$$\int_{\Sigma_{t_0}}^{\Sigma_{t_1}} \equiv \int_{\Sigma_{t_1}} - \int_{\Sigma_{t_0}}.$$

In the current literature, there exists a whole family of action functionals similar to (1.3) each adapted to the particular problem under consideration. In general, one can add to the functional (1.4) an arbitrary functional depending on the data fixed on the boundary, i.e. depending on the boundary metric (see [3]). This arbitrariness does not affect the equations of motion since the boundary metric is kept fixed in the variational principle. The choice (1.3) is motivated by the requirement $I_D[g, \bar{g}] = 0$, i.e. the requirement that the action functional vanishes when computed with $g = \bar{g}$. The same property is satisfied by (1.1).

Moreover, under additional hypotheses, the integral on the lids $\Sigma_{t_0}$ and $\Sigma_{t_1}$ in (1.4) is usually discarded since the lids are either ignored or identified by periodic boundary conditions (see [20], [21]).

For these reasons we are forced to first review the literature and choose homogeneous notation to compare with the covariant first order Lagrangian.

We are not aware of a theoretical comparison between these two variational principles in the current literature. A definition of energy based on the action functional (1.3) has been proposed by Brown and York. It is called quasilocal energy and it is often quoted because it reproduces the ADM mass in the asymptotically flat case, though it has been recognised to be also suitable for more general boundary conditions (e.g. in asymptotically anti–de–Sitter (see [22]) or asymptotically locally flat cases; see [23]). The main advantage of Brown–York method is that it allows to define the energy within a finite region as well as the total one. We remark that the covariant first order Lagrangian (1.1) was originally proposed for exactly the same reasons (see [14], [15]). We also remark that in both cases it has been recognized that absolute
conserved quantities have a meaning just in particular cases, while for general boundary conditions just conserved quantities relative to a reference background are meaningful. Furthermore, even “absolute” conserved quantities should be interpreted as conserved quantities relative to some canonical background (e.g. flat Minkowski space in asymptotically flat spaces). We also remark that reference backgrounds are particularly relevant in General Relativity, as well as in other non–linear field theories, since whenever fields are endowed with a vector space structure then a canonical choice for the reference exists, namely the zero section. If the configuration bundle (see Appendix B for a short geometric insight into variational calculus and field theory) is not a vector bundle, as it happens in General Relativity, as well as e.g. in Yang–Mills theories, there is no canonical choice for the vacuum state. The vacuum state has then to be arbitrarily fixed. When doing that it sounds physically reasonable to require also the background to be a solution of field equations, so that the relative mass can be interpreted as the energy “spent to go” from the background solution to the dynamical one (and analogously for other currents). Furthermore, it is essential that the choice of the reference background does not effect the evolution of the dynamical fields, i.e. they have to be decoupled. In view of these considerations both action functionals (1.1) and (1.3) incorporate the background from the very beginning. Thus we believe that the relationship between the two methods is worth investigating.

In this paper we shall also discuss the very notion of conserved quantity. On one hand, in fact, Noether theorem provides currents $\mathcal{E}$ which are covariantly conserved, i.e. $d\mathcal{E} = 0$ identically or on shell (i.e., along solutions), meaning that their integral on the boundary $\partial D$ of any $n$–region $D$ in spacetime $M$ vanishes or, equivalently, that the conserved quantity obeys a continuity equation. On the other hand, physicists are often interested in quantities which are conserved in time, meaning that, once an ADM foliation of a region of spacetime has been chosen, such a quantity $Q$ may be computed by integration on each leaf and it turns out not to depend on the particular leaf labelled by the time $t$. Clearly, ADM foliations are far not unique and different foliations of the same region $D$ correspond to different ways of defining time. Furthermore, such a quantity $Q$ may be conserved in the time defined by an ADM foliation without being conserved with respect to other foliations (see Appendix C). From a theoretical General Relativity viewpoint, quantities conserved in time are not (manifestly) covariant in nature. They are, in fact, conserved with respect to a special parameter, while, at a fundamental level, the principle of general covariance forbids, at least in principle, the selection of a preferred time.

Nevertheless, quantities conserved in time may be interesting to be investigated. In our perspective, in fact, they can be obtained from covariantly conserved quantities. To be more precise, we can consider a variational principle, an infinitesimal generator $\xi$ of Lagrangian symmetries and a solution of
field equations. Then we can compute covariantly conserved currents $\mathcal{E}[\xi]$ by Nöther theorem. Let us then fix a spacelike $(n - 1)$–region $\Sigma$ and integrate the Nöther current on it to define a conserved quantity $Q[\xi]$. Any timelike vector field $\zeta$ allows then to evolve the region $\Sigma$ along its flow, parametrized by its affine parameter $t$. Under this viewpoint, the question arises whether there exists a vector field $\zeta$ (possibly depending on the region $\Sigma$) such that the covariantly conserved quantity generated by $\xi$ is also conserved in the “time” induced by $\zeta$. At a first glance, if we have $\partial D = \Sigma_{t_1} - \Sigma_{t_0} + \mathcal{B}$ (see Fig. 1), conservation in time is equivalent to require that the integral of the Nöther current $\mathcal{E}[\xi]$ on $\mathcal{B}$ vanishes (for any time interval $[t_0, t_1]$). In fact, we have the covariant conservation law $d\mathcal{E}[\xi] = 0$, so that

$$0 = \int_D d\mathcal{E}[\xi] = \int_{\partial D} \mathcal{E}[\xi] = \int_{\Sigma_{t_1}} \mathcal{E}[\xi] - \int_{\Sigma_{t_0}} \mathcal{E}[\xi] + \int_{\mathcal{B}} \mathcal{E}[\xi] \Rightarrow$$

$$\Rightarrow \int_{\Sigma_{t_1}} \mathcal{E}[\xi] - \int_{\Sigma_{t_0}} \mathcal{E}[\xi] = -\int_{\mathcal{B}} \mathcal{E}[\xi] \tag{1.5}$$

Thence the conserved quantity $\int_{\Sigma_t} \mathcal{E}[\xi]$ computed on a leaf does not depend on the particular leaf if and only if $\int_{\mathcal{B}} \mathcal{E}[\xi] = 0$ (for any time interval $[t_0, t_1]$). Physically speaking, this amount to require that the flow of the current $\mathcal{E}[\xi]$ through $\mathcal{B}$ is vanishing. Clearly, different ADM foliations may evolve $\Sigma$ in different ways. In general, just few of them will lead to time–conserved quantities. The vanishing of $\int_{\mathcal{B}} \mathcal{E}[\xi]$ has then to be guaranteed by additional hypotheses, possibly in many different ways. Under stronger hypotheses on $\xi$ (or on $\zeta$, or on the boundary conditions which $g$ and $\bar{g}$ have to satisfy) the set of ADM foliations leading to time–conserved quantities with respect to different times may be possibly enlarged. Different sets of conditions which guarantee time–conservation will be discussed below and we shall compare them to those found in the current literature; see also Appendix C for some examples.

In Section 2 we shall prove that the two action functionals coincide provided that the metric $g$ and the background $\bar{g}$ agree on the boundary $\partial D$ of the region $D$ under consideration.

In Section 3 the variational principle for York’s action functional will be reviewed and the definition of quasilocal energy recalled. We shall also briefly review the variational principle for the first order covariant Lagrangian. The two variational principles are relevant by their own and of course are found to be equivalent when $g$ and $\bar{g}$ are required to agree on the boundary $\partial D$, i.e. when the action functionals are equivalent.

In Section 4 a $(3+1)$ decomposition of York’s action functional will be reviewed. The obtained ADM Hamiltonian will be later compared with the
Norther conserved quantity of the first order covariant Lagrangian as introduced in Section 5 and 6.

In Section 5 we shall review the covariant approach to Norther conserved quantities and comment on the state of the art in the definition of the energy (mass, angular momentum, etc.) in General Relativity. Here most of the geometric techniques and bundle framework have been isolated and reviewed in Appendix B.

In Section 6 the Norther theorem will be specialized to define the covariantly conserved quantities of the first order covariant Lagrangian. Such quantities are involved in the definition of the covariant ADM Hamiltonians (one for each spacetime vector field $\xi$). They will in fact be compared with the ADM Hamiltonian introduced in Section 4 and found to agree (under the usual matching conditions required on $g$ and $\bar{g}$). This result will be obtained by comparing the two quantities with respect to the same ADM foliation of the region $D$ under consideration. Finally, the relation between covariant ADM Hamiltonian and quasilocal energy will also be analysed. The main result is that quasilocal energy is covariantly conserved and it is the Norther charge associated to the unit normal to the leaves of the ADM foliation.

In Section 7 we shall discuss different sets of conditions under which conservation in time follows from covariant conservation.

In Appendix A we collect formulae which are used during the paper to translate covariant objects to objects adapted to the ADM foliation and viceversa.

As we already said, Appendix B contains a quick review of the bundle framework and Norther theorem at bundle level.

Finally in Appendix C we present two worked examples which in various occasions are quoted throughout the paper. The first one is the computation of various Norther charges of the Schwarzschild solution relative to Minkowski metric matched on a finite sphere. Conservation in time of various foliations is analysed and quasilocal energy is obtained in agreement with previously known results (see [3]). The second example is a Kerr–Newman solution matched at spatial infinity with the Minkowski metric. The Norther conserved quantities are obtained. Such an example is interesting because it does not obey the same matching conditions required all over the rest of the paper. It nevertheless produces a current which is time–conserved, showing that all the conditions discussed along the paper are sufficient but not necessary.

2. Comparison of the Action Functionals

From now on we assume $\dim(M) = 4$, unless explicitly stated. We shall here decompose the first order covariant action functional (1.1) along an ADM
foliation of \( D \) in order to prove that the action functional (1.3) and (1.1) are equal if the 4–metrics \( g \) and \( \bar{g} \) are required to coincide on \( \partial D \).

Of course, the ADM splitting breaks down the explicit covariance in the action allowing a comparison with respect to the same ADM foliation. In particular, the boundary term in the covariant action (1.1) splits into a contribution on \( B \) and a contribution on the lids \( \Sigma_{t_0} \) and \( \Sigma_{t_1} \).

Let us consider local coordinates \((t, r, x^A)\) adapted both to \( B \) and the ADM splitting on \( D \). In this coordinate system \( B \) has the expression \( r = \text{constant} \) while the leaves \( \Sigma_t \) are the hypersurfaces of equation \( t = \text{constant} \). The metric tensor \( g \) can be split with respect to the ADM–foliation obtaining the expression (A.13) in the Appendix A. Similarly, one can consider the foliation of spacetime in the hypersurfaces \( r = \text{constant} \) and obtain the expression (A.19) in the Appendix A. Analogous expressions can be obtained for the reference background metric \( \bar{g} \).

Let us evaluate the boundary term of the covariant first order action functional (1.1) on \( B \) and on the lids \( \Sigma_{t_0} \) and \( \Sigma_{t_1} \), i.e.:

\[
A_B = -\frac{1}{2\kappa} \int_B \sqrt{g} g^{\mu\nu} w_{\mu\nu}^\alpha \, d\sigma^\alpha \quad \quad A_{\Sigma_{t_0}} = -\frac{1}{2\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} \sqrt{g} g^{\mu\nu} w_{\mu\nu}^\alpha \, d\sigma^\alpha \quad (2.1)
\]

By using results which are summarized in Appendix A (see equations (A.18) and (A.23)) one obtains:

\[
A_B = -\frac{1}{2\kappa} \int_B \left( g^{\mu\nu} w_{\mu\nu}^\alpha \, n_{\alpha} \sqrt{\gamma} \, d^3x \right) = -\frac{1}{2\kappa} \int_B \left\{ 2\Theta + \right. \\
- \Theta_{ij} \left( \frac{\bar{V}}{V} \tilde{\gamma}^{ij} + \frac{V}{\bar{V}} \gamma^{ij} \right) - \frac{\tilde{\Theta}_{ij}}{\bar{V}V} (V^i - \bar{V}^i) (V^j - \bar{V}^j) + + \frac{1}{\bar{V}V} \partial_i \bar{V} (V^i - \bar{V}^i) - \frac{1}{V} \bar{D}_i (V^i - \bar{V}^i) \left\} \right. \sqrt{\gamma} \, d^3x \quad (2.2)
\]

where \( \bar{D}_i \) is the covariant derivative with respect to the 3–metric \( \tilde{\gamma}_{ij} \) induced on \( B \) by the background metric \( \bar{g} \), while \( V \) and \( V^i \) are the radial lapse and the radial shift as defined by equation (A.19) in the Appendix A. We stress that no matching condition is required to obtain the above result. The behaviours of the metrics \( g \) and \( \bar{g} \) are completely unrelated till now.
Analogously, on the lids the ADM splitting of the boundary term (2.1) gives an extra contribution of the following form:

\[ A_{\Sigma_{t_1}}^{\Sigma_{t_0}} = \frac{1}{2\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} g^{\mu\nu} u^\alpha_{\mu\nu} \sqrt{h} \, d^3x = -\frac{1}{2\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} \left\{ -2K + \frac{\bar{K}_{ab}}{N\bar{N}} \left( \frac{\bar{N}}{N} \bar{h}^{ab} + \frac{N}{\bar{N}} h^{ab} \right) - \frac{\bar{K}_{ab}}{NN} (N^a - \bar{N}^a) (N^b - \bar{N}^b) + \frac{1}{N\bar{N}} \partial_a \bar{N} \left( N^a - \bar{N}^a \right) + \frac{1}{\bar{N}} \bar{D}_a (N^a - \bar{N}^a) \right\} \sqrt{h} \, d^3x \]

(2.3)

where \( \bar{D}_a \) is the covariant derivative with respect to the 3–metric \( \bar{h}_{ab} \) induced on \( \Sigma \) by the background metric \( \bar{g} \), while \( N \) and \( N^a \) are the lapse and the shift of the metric as defined by equation (A.13) in the Appendix A. Once again we stress that no matching condition is required to obtain the result. In general (i.e. if no physical requirement about the matching of the dynamical metric and the background on the boundary is imposed) the two action functionals (1.1) and (1.3) are fairly different.

However, let us assume that the dynamical metric \( g \) and the background \( \bar{g} \) coincide on the hypersurface \( B \) so that, in particular, they induce the same 3–metric on \( B \) (i.e. \( \gamma_{ij} \mid_B = \bar{\gamma}_{ij} \mid_B \)) and they have the same radial lapse function (i.e. \( V \mid_B = \bar{V} \mid_B \)) and radial shift vector (i.e. \( V^i \mid_B = \bar{V}^i \mid_B \)). Then, under these additional hypotheses, the contribution \( A_B \) reduces to:

\[ A_B = -\frac{1}{\kappa} \int_B \left( \sqrt{\gamma} \Theta - \sqrt{\bar{\gamma}} \bar{\Theta} \right) d^3x = -\frac{1}{\kappa} \int_B \left( \sqrt{\gamma} \Theta - \sqrt{\bar{\gamma}} \bar{\Theta} \right) n^\alpha \, ds_\alpha \]

(2.4)

Analogously, if the metric \( g \) and \( \bar{g} \) are required to agree on the lids (i.e. if \( h_{ij} = \bar{h}_{ij} \), \( N = \bar{N} \) and \( N^i = \bar{N}^i \) on \( \Sigma_{t_0} \) and \( \Sigma_{t_1} \), then the contribution (2.3) on the lids reduces to:

\[ A_{\Sigma_{t_1}}^{\Sigma_{t_0}} = \frac{1}{\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} \left( \sqrt{h} K - \sqrt{\bar{h}} \bar{K} \right) d^3x = \frac{1}{\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} \left( \sqrt{\gamma} K - \sqrt{\bar{\gamma}} \bar{K} \right) u^\alpha \, ds_\alpha \]

(2.5)

Then the boundary term in the covariant first order action (1.1) can be written as:

\[ A_B + A_{\Sigma_{t_1}}^{\Sigma_{t_0}} = -\frac{1}{\kappa} \int_B \left( \sqrt{\gamma} \Theta - \sqrt{\bar{\gamma}} \bar{\Theta} \right) n^\alpha \, ds_\alpha + \frac{1}{\kappa} \int_{\Sigma_{t_0}}^{\Sigma_{t_1}} \left( \sqrt{\gamma} K - \sqrt{\bar{\gamma}} \bar{K} \right) u^\alpha \, ds_\alpha \]

(2.6)

and the action functionals (1.1) and (1.3) clearly coincide.
We stress that this result has been obtained by requiring the aforementioned matching conditions between $g$ and $\bar{g}$ on the complete boundary $\partial D = B + \Sigma_{t_1} - \Sigma_{t_0}$ of the region $D$.

We should also remark that for time–independent solutions the matching on the lids cannot be required, because if the two metrics $g$ and $\bar{g}$ agree on a spacelike hypersurface $\Sigma_t$, they necessarily agree on the whole region $D$. Also for this reason, the contributions on the lids are never considered in applications; this is usually done by restricting to situations in which the lids are not present (e.g. by considering a non–compact region $D$ in which $t_0$ and $t_1$ are let to tend to $-\infty$ and $\infty$, respectively) or are identified. Identification is obtained, e.g., when the solution is time–periodic, as it may happen in the Euclidean sector (see [20], [21], [24]) and in approaches based on path–integrals for evaluating the grand–canonical partition function or the density of states for General Relativity, where the sum over periodic histories has to be considered (see [21]). In all those cases the boundary $\partial D$ is required to have the topology $\partial \Sigma \times S^1$, i.e. it is assumed to be a single boundary component $\partial D = B$.

The matching condition of the 4–metrics $g$ and $\bar{g}$ is a stronger requirement than the one introduced in [10]. There, only the induced 3–metrics $\gamma_{ij}$ and $\bar{\gamma}_{ij}$ are required to agree on $B$, where $B$ is let to tend to infinity. Although the matching of the 4–metrics at infinity is not too hard to be implemented in applications (see [15], [16], [24]), we are aware that this matching, when possible, may become hard to implement in a finite region. Here the matching is required to have a direct theoretical comparison between the action functionals (1.1) and (1.3) (and between their Hamiltonians and conserved charges). In Appendix C we shall explicitly discuss a simple but relevant example of the matching of the 4–metrics in a finite region.

3. Comparison of the Variational Principles

We shall here review the variational principles associated to the action functionals (1.1) and (1.3). They are found to agree when the matching conditions already discussed in Section 2 are again imposed. We shall also recall the definition of quasilocal energy which will be later compared with Nöther conserved quantities.

Let us analyse the variation of the action functional (1.3). It reads as (see [7]):

$$\delta I_D[g] = \frac{1}{2\kappa} \int_D G_{\mu\nu} \delta g^{\mu\nu} \, ds + \int_B (\Pi^{ij} \delta \gamma_{ij}) \, d^3 x + \int_{\Sigma_{t_1}} (P^{ab} \delta h_{ab}) \, d^3 x + \int_{\Sigma_{t_0}} (P^{ab} \delta h_{ab}) \, d^3 x \quad (3.1)$$
where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, while $\Pi^{ij}$ and $P^{ab}$ denote the gravitational momenta conjugated to the metrics $\gamma_{ij}$ and $h_{ab}$, respectively; they are given by:

$$
\left\{
\begin{array}{l}
\Pi^{ij} = -\frac{1}{2\kappa} \sqrt{\gamma} (\Theta \gamma^{ij} - \Theta^{ij}) \\
P^{ab} = \frac{1}{2\kappa} \sqrt{h} (K h^{ab} - K^{ab})
\end{array}
\right.
$$

(3.2)

Analogous terms come from the variation $\delta I_B[\bar{g}]$. Notice that, if boundary conditions which keep the 3–metric fixed on the boundary are imposed, namely $\delta \gamma|_B = 0$ and $\delta h|_{\Sigma_{t_0}} = \delta h|_{\Sigma_{t_1}} = 0$, then the action functional (1.3) is correctly extremized by the solutions of Einstein equations $G_{\mu\nu} = 0$. [We recall that to obtain the expression (3.1) the hypothesis of orthogonal boundaries has been assumed. Otherwise further terms would appear which have to be cancelled through the introduction of a further boundary term into the action (1.3); see [8], [9], [10].]

The contribution $\delta I_B = \int_B (\Pi^{ij} \delta \gamma_{ij}) d^3x$ in (3.1) is considered (see [3]) in order to define the quasilocal energy $E[g]$, i.e. the energy of a region of finite spatial extent:

$$
E_t[g] = -\int_{B_t} \frac{\delta I_B}{\delta N} d^2 x
$$

(3.3)

It represents the change of the action in time, where changes in time are governed by the lapse $N$ on the boundary $B$. We stress that the interpretation of this quantity as the energy of a gravitating system was suggested in [3] through a Hamilton–Jacobi analysis of the action functional.

Inserting into (3.1) the following decomposition (see [3] for the details):

$$
\delta \gamma_{ij} = -2 \frac{u_i u_j}{N} \delta N - 2 \frac{\sigma^k(i u_j)}{N} \delta N^k + \sigma^k j^k \delta \sigma_{kh}
$$

(3.4)

we obtain:

$$
E_t[g] = 2 \int_{B_t} \left( \frac{\Pi^{ij}}{N} u_i u_j \right) d^2 x = \frac{1}{\kappa} \int_{B_t} K \sqrt{\sigma} d^2 x
$$

(3.5)

(in the latter equality we have used the definition (3.2) together with the expression (A.11) and (A.12) in the Appendix A). A contribution analogous to (3.5) comes from the background metric $\bar{g}$ and from the variation $\delta I_B[\bar{g}]$. Assuming that the induced 3–metric $\gamma$ and $\bar{\gamma}$ are matched on the surface $B$ we finally have:

$$
E_t[g, \bar{g}] = \frac{1}{\kappa} \int_{B_t} (K - \bar{K}) \sqrt{\sigma} d^2 x
$$

(3.6)

This is the explicit expression of Brown and York quasilocal energy computed for the action functional (1.3).
Let us now consider the action functional (1.1). We can perform the covariant variation with respect to the metrics before choosing any ADM splitting. In this way we obtain:

\[ \delta A_D[g, \bar{g}] = \frac{1}{2\kappa} \int_D G_{\mu\nu} \delta g^{\mu\nu} \, ds - \frac{1}{2\kappa} \int_D \bar{G}_{\mu\nu} \delta \bar{g}^{\mu\nu} \, ds + \]

\[ + \frac{1}{2\kappa} \int_{\partial D} \left( \sqrt{g} g^{\mu\nu} \delta \bar{u}_\mu^\alpha - \sqrt{\bar{g}} \bar{g}^{\mu\nu} \delta u_\mu^\alpha - \delta(\sqrt{g} g^{\mu\nu} w_\mu^\alpha) \right) ds_\alpha = \]

\[ \frac{1}{2\kappa} \int_D G_{\mu\nu} \delta g^{\mu\nu} \, ds - \frac{1}{2\kappa} \int_D \bar{G}_{\mu\nu} \delta \bar{g}^{\mu\nu} \, ds + \]

\[ + \frac{1}{2\kappa} \int_{\partial D} \left( (\sqrt{g} g^{\mu\nu} - \sqrt{\bar{g}} \bar{g}^{\mu\nu}) \delta \bar{u}_\mu^\alpha - \delta(\sqrt{g} g^{\mu\nu} w_\mu^\alpha) \right) ds_\alpha \]

(3.7)

Then, if both \( g \) and \( \bar{g} \) are solutions of Einstein equations \( G_{\mu\nu} = 0 \) and \( \bar{G}_{\mu\nu} = 0 \), they extremize the action (1.1), provided that \( \delta g^{\mu\nu} = 0 \) and \( \delta \bar{u}_\mu^\alpha = 0 \) on the boundary \( \partial D \).

This is a stronger requirement than the fixing of the metric induced on \( \partial D \). In fact, it amounts to fix the whole 4-metric restricted on \( \partial D \) (as well as the first order derivatives of the reference metric \( \bar{g} \)). In any case it is what can be done in order to preserve covariance if no additional hypotheses on the background are required.

Nevertheless, we again stress that the action (1.1) is extremized by solutions of Einstein equations also if we require \( \delta g^{\mu\nu}|_{\partial D} = 0 \) together with the matching condition \( g^{\mu\nu}|_{\partial D} = \bar{g}^{\mu\nu}|_{\partial D} \). This is obvious from expression (3.7) but it also follows from the demonstration carried out in the previous Section: under the assumption \( g_{\mu\nu}|_{\partial D} = \bar{g}_{\mu\nu}|_{\partial D} \) the action functional (1.1) equals the action functional (1.3). Hence the variation (3.7) turns out to coincide with the variation (3.1).

4. ADM Hamiltonian

We shall here review the \((3 + 1)\) Hamiltonian formulation of the action functional (1.3). The ADM Hamiltonian will be later compared with Nöther charges.

As it is well known, one of the possible way to give a Hamiltonian description of General Relativity is the so-called ADM formulation (see [4], [6], [25]). We remark that in this framework the concept of (manifest) general covariance is lost from the beginning owing to the ADM splitting of spacetime into space + time. We shall see below how a covariant ADM formulation of the problem, deeply related to conserved quantities and Nöther theorem, can be
formulated. But, first of all, in order to compare known results with ours on the subject let us review the standard ADM Hamiltonian formulation of the action (1.3).

To obtain the Hamiltonian of the system we need to separate out the terms in the volume integral \( \frac{1}{2\kappa} \int_D \sqrt{g} R \, ds \) which are pure divergences and then become boundary integrals. Using the decomposition (A.13) of the metric we have (see [6]):

\[
\sqrt{g} R = \sqrt{g} \left( R + K_{\mu\nu} K^{\mu\nu} - K^2 \right) - 2\sqrt{g} \nabla_\mu \left( K u^\mu + a^\mu \right) \quad (4.1)
\]

where \( R \) is the scalar curvature of the 3–metric \( h_{\mu\nu} \) while \( a^\mu = u^\nu \nabla_\nu u^\mu \) is the covariant acceleration of the unit normal \( u^\mu \). Inserting this expression into (1.4) we obtain:

\[
I_D[g] = \frac{1}{2\kappa} \int_D \sqrt{g} \left( R + K_{\mu\nu} K^{\mu\nu} - K^2 \right) \, ds
\]

\[
+ \frac{1}{\kappa} \int_{\Sigma_{t_1}} u_\mu (K u^\mu + a^\mu) \sqrt{h} \, d^3 x - \frac{1}{\kappa} \int_{\partial B} n_\mu (K u^\mu + a^\mu) \sqrt{\gamma} \, d^3 x \quad (4.2)
\]

\[
+ \frac{1}{\kappa} \int_{\Sigma_{t_0}} K \sqrt{h} \, d^3 x - \frac{1}{\kappa} \int_{\partial B} \Theta \sqrt{\gamma} \, d^3 x
\]

Owing to the properties \( u_\mu u^\mu = -1 \) and \( u_\mu a^\mu = 0 \), the integral \( \int_{\Sigma_{t_1}} \) on the lids vanishes. Moreover, since we have imposed the condition \( n_\mu u^\mu |_{\partial B} = 0 \) of orthogonal boundaries, the action functional \( I_D[g] \) reduces to:

\[
I_D[g] = \frac{1}{2\kappa} \int_D \left( R + K_{\mu\nu} K^{\mu\nu} - K^2 \right) \sqrt{g} \, ds - \frac{1}{\kappa} \int_{\partial B} (\Theta + n_\mu a^\mu) \sqrt{\gamma} \, d^3 x \quad (4.3)
\]

By using formula (A.12) in the Appendix A we obtain:

\[
I_D[g] = \frac{1}{2\kappa} \int_D \left( R + K_{\mu\nu} K^{\mu\nu} - K^2 \right) \sqrt{g} \, ds - \frac{1}{\kappa} \int_{\partial B} K \sqrt{\gamma} \, d^3 x \quad (4.4)
\]

This action may be written in canonical form by inserting the equality:

\[
K_{ab} K^{ab} - K^2 = \frac{2\kappa}{\sqrt{g}} \left\{ P_{ab} P_t h_{ab} - 2P_{ab} D_a N_b - \frac{\kappa N}{\sqrt{h}} \left( 2P_{ab} P_{ab} - P^2 \right) \right\} \quad (4.5)
\]
(see definition (3.2)) and by removing the term involving the derivatives \( D_a N_b \)
by an integration by parts. We finally have:

\[
I_D[g] = \int dt \int \Sigma_t \left( P^{ab} \partial_t h_{ab} - N \mathcal{H} - N_a \mathcal{H}^a \right) d^3 x + \\
- \int dt \int_{B_t} \left( \frac{N}{\kappa} \mathcal{K} + N^a 2 \frac{P^{bc}}{\sqrt{h}} n_b \sigma_{ac} \right) \sqrt{\sigma} d^2 x
\]  

(4.6)

where

\[
\mathcal{H} = - \frac{1}{2\kappa} \sqrt{h} \mathcal{R} + \frac{\kappa}{\sqrt{h}} (2P_{ab} P^{ab} - P^2) \\
\mathcal{H}^a = -2D_b P^{ab}
\]

(4.7)

are the Hamiltonian and the momentum constraint, respectively. The Hamiltonian for vacuum General Relativity is thus identified with the term:

\[
H(g) = \int \Sigma_t (N\mathcal{H} + N_a \mathcal{H}^a) d^3 x + \frac{1}{\kappa} \int_{B_t} (N \mathcal{K} - N^a K^{bc} n_b \sigma_{ac}) \sqrt{\sigma} d^2 x
\]

(4.8)

(in the latter term we have taken into account that \( h^{bc} n_b \sigma_{ac} = 0 \)). The Hamiltonian \( H(g) \) is the sum of a constrained volume term, which is vanishing when computed on a solution, and a boundary term.

We may repeat the above analysis also for the background action functional \( I_D[\bar{g}] \) obtaining an Hamiltonian \( \bar{H}(\bar{g}) \) which agrees with (4.8) provided that we replace the terms there involved with the corresponding barred ones. The total Hamiltonian \( H(g, \bar{g}) \) is then given by the difference:

\[
H(g, \bar{g}) = H(g) - \bar{H}(\bar{g})
\]

(4.9)

When it is evaluated on solutions of field equations, the constrained volume terms vanish and (4.9) reduces to the boundary terms:

\[
H(g, \bar{g}) \simeq \frac{1}{\kappa} \int_{B_t} (N \mathcal{K} - N^a K^{bc} n_b \sigma_{ac}) \sqrt{\sigma} d^2 x \\
- \frac{1}{\kappa} \int_{B_t} (\bar{N} \bar{K} - \bar{N}^a \bar{K}^{bc} \bar{n}_b \bar{\sigma}_{ac}) \sqrt{\sigma} d^2 x
\]

(4.10)

where the symbol \( \simeq \) denotes equality on–shell.

Moreover, if the metric \( g \) and its relative background \( \bar{g} \) agree on the boundary the Hamiltonian simplifies as follows:

\[
H(g, \bar{g}) \simeq \frac{1}{\kappa} \int_{B_t} \left\{ N (\mathcal{K} - \bar{\mathcal{K}}) - N^a (K^{bc} - \bar{K}^{bc}) n_b \sigma_{ac} \right\} \sqrt{\sigma} d^2 x
\]

(4.11)
The mass associated with the time translation \( t^\mu = Nu^\mu + N^\mu \), relative to two solutions \( g \) and \( \bar{g} \), is simply defined to be the value of the Hamiltonian (4.10). Clearly the mass \( H(\bar{g}, \bar{g}) \) of the background is equal to zero.

Notice that, choosing a Gaussian gauge, i.e. setting \( N = 1 \) and \( N^a = 0 \) in (4.11), we obtain the quasilocal energy (3.6) (see [3]).

In [26] it was shown that the definition (4.10) agrees with the expressions of energy already defined in literature for spacetimes with different asymptotic behaviour.

5. Conserved Quantities

In this Section we shall analyse conserved quantities associated to the covariant first order Lagrangian (1.1) by the Nöther theorem. As we claimed above, the Lagrangian (1.1) was originally introduced because it provides a simple framework to determine the density of conserved quantities. We shall here apply the general framework (see e.g. [14], [15], [16], [27] and references quoted therein).

The Nöther theorem is a direct consequence of the general covariance of General Relativity (as well as of any other generally covariant field theory). It algorithmically defines a Nöther conserved current, i.e. a map \( \mathcal{E}[\xi, \sigma] \) which associates to any spacetime vector field \( \xi \) and any field configuration \( \sigma = (g, \bar{g}) \) an \((n-1)\)–form on spacetime \( M \) of dimension \( n \) which is closed on–shell, i.e. when the configuration \( \sigma \) is a solution of field equations. For the action functional (1.1) we obtain:

\[
\mathcal{E}[\xi, \sigma] = \frac{1}{2\kappa} \left[ \sqrt{g} \left( (g^{\lambda\alpha} g_{\mu\nu} - \delta^{\lambda}_{\mu} \delta^{\alpha}_{\nu}) \nabla_{\alpha} \mathcal{L}_{\xi} g^{\mu\nu} - \xi^\lambda \bar{R} \right) + \right. \\
- \left. \left( \mathcal{L}_{\xi} (\sqrt{g} g^{\mu\nu} w^\lambda_{\mu\nu}) - \xi^\lambda d_{\alpha} (\sqrt{g} g^{\mu\nu} w^\alpha_{\mu\nu}) \right) + \right. \\
- \sqrt{\bar{g}} \left( (\bar{g}^{\lambda\alpha} \bar{g}_{\mu\nu} - \delta^{\lambda}_{\mu} \delta^{\alpha}_{\nu}) \nabla_{\alpha} \mathcal{L}_{\xi} \bar{g}^{\mu\nu} - \xi^\lambda \bar{R} \right) \bigg] \, ds_{\lambda} 
\]

The differential of the \((n-1)\)–form \( \mathcal{E}[\xi, \sigma] \) satisfies the following property:

\[
d\mathcal{E}[\xi, \sigma] = \mathcal{W}[\xi, \sigma] 
\]

where \( \mathcal{W}[\xi, \sigma] = -(1/2\kappa) \left( G_{\mu\nu} \mathcal{L}_{\xi} g^{\mu\nu} - \bar{G}_{\mu\nu} \mathcal{L}_{\xi} \bar{g}^{\mu\nu} \right) \, ds \) is proportional to field equations. Consequently, the Nöther current \( \mathcal{E}[\xi, \sigma] \) is closed along solutions.

The equation (5.2) is called weak conservation law for the Nöther current (5.1). We stress that equation (5.2) is written on spacetime \( M \), where it does not single out a unique Nöther current \( \mathcal{E}[\xi, \sigma] \). In fact, if a closed form is added to \( \mathcal{E}[\xi, \sigma] \) we get another solution of (5.2). This is why Nöther’s theorem
should be regarded as a claim on (a suitable prolongation of) the configuration bundle (see Appendix B), i.e. \( \mathcal{E}[\xi, \sigma] \) has to be regarded as a map \( \mathcal{E}[\xi] \) which associates a form on (the jet prolongation of) the configuration bundle to any spacetime vector field \( \xi \). As shown in [27] and [28], the Nöther current \( \mathcal{E}[\xi] \) at bundle level is canonically associated to the Lagrangian. Then it can be computed along a configuration \( \sigma \) to give \( \mathcal{E}[\xi, \sigma] \), i.e. a \((n-1)\)–form on the spacetime \( M \).

Of course, the Nöther currents, as well as the conserved quantities associated to them, explicitly depend on boundary terms in the Lagrangian. This is a very well known feature in Physics, as it can be simply seen, e.g., in thermodynamics. It is in fact well known that boundary terms are related to boundary conditions. As we remarked in Section 4, different boundary terms need different boundary conditions to keep field equations satisfied by action extremals (see [7], [21], [29]). And it is well known that, e.g., for a correct definition of energy in thermodynamics, different boundary conditions correspond to different definitions of energy, such as internal energy, free energy etc. We stress that all these energies are true physical energies of thermodinamical systems. Which one is to be used in practice is determined by the particular system under consideration and the boundary conditions we decided to impose. As it is physically relevant to notice, we may decide to keep temperature fixed on the boundary of a gas box or we may impose adiabatic conditions; this different choice corresponds to a different apparatus which selects a different energy flow through the boundary so that the boundary conditions are satisfied. We stress that this corresponds to an external action on the system which turns out to change the physical energy of the system itself. In the covariant first order approach to General Relativity something fully analogous holds: the background \( \bar{g} \) canonically selects both the boundary conditions and the corresponding energy to be used and different choices of the background correspond to different physically meaningful definition of energy.

One can also define (see Appendix B) the superpotential \( \mathcal{U}[\xi, \sigma] \) and the reduced current \( \tilde{\mathcal{E}}[\xi, \sigma] \) as those currents such that:

\[
\mathcal{E}[\xi, \sigma] = \tilde{\mathcal{E}}[\xi, \sigma] + d\mathcal{U}[\xi, \sigma] \tag{5.3}
\]

where the reduced current is required to vanish on–shell.

Once again both \( \tilde{\mathcal{E}}[\xi, \sigma] \) and \( \mathcal{U}[\xi, \sigma] \) are not uniquely identified by equation (5.3). The superpotential is defined modulo forms which are closed on–shell, while the reduced current is defined modulo forms vanishing along solutions. The decomposition (5.3) of Nöther’s current is again well–defined only at bundle level, where a decomposition algorithm can be constructed (see [28], [27]). The bundle superpotential \( \mathcal{U}[\xi] \) and the reduced current \( \tilde{\mathcal{E}}[\xi] \) are canonically
and globally defined and then they are computed along the configuration \( \sigma \) to give \( U[\xi, \sigma] \) and \( \tilde{E}[\xi, \sigma] \), respectively.

The **conserved quantity in a region** \( \Omega \) is defined as:

\[
Q_\Omega[\xi, \sigma] = \int_\Omega E[\xi, \sigma] = \int_\Omega \tilde{E}[\xi, \sigma] + \int_{\partial \Omega} U[\xi, \sigma] = \int_{\partial \Omega} U[\xi, \sigma]
\]  

(5.4)

where, in the last equality, \( \sigma \) is assumed to be a solution so that \( \tilde{E}[\xi, \sigma] = 0 \).

In the case of the first order covariant action functional (1.1) we obtain in particular:

\[
U[\xi] = \frac{1}{2\kappa} \left[ \sqrt{g} \nabla^\beta \xi^\alpha + \sqrt{\bar{g}} g^{\mu\nu} u^\beta_{\mu\nu} \xi^\alpha - \sqrt{\bar{g}} \bar{\nabla}^\beta \xi^\alpha \right] \text{d}s_{\alpha\beta}
\]

\[
\tilde{E}[\xi] = \frac{1}{\kappa} \left[ \sqrt{g} g^{\mu\lambda} G_{\mu\nu} \xi^\nu - \sqrt{\bar{g}} \bar{g}^{\mu\lambda} \bar{G}_{\mu\nu} \xi^\nu \right] \text{d}s_{\lambda}
\]

(5.5)

where \( g^{\mu\nu} \) (as well as Christoffel’s symbols \( \Gamma^\lambda_{\sigma\mu} \)) has to be regarded as local coordinates on (the jet prolongation of) the configuration bundle \( \text{Lor}(M) \) of all Lorentzian metrics on \( M \). They become the metric components, i.e. functions of spacetime point \( x \in M \), only when they are calculated along a configuration \( \sigma(x) = (g(x), \bar{g}(x)) \).

The conserved quantity \( Q_\Omega[\xi, \sigma] \) depends on the dynamical metric \( g \) and on the reference background metric \( \bar{g} \). It has to be interpreted as the **relative conserved quantity of** \( g \) **with respect to** \( \bar{g} \). For example, if the energy is considered (by choosing the vector field \( \xi \) in a suitable way, see below), \( Q_\Omega[\xi, \sigma] \) represents the amount of energy which is necessary to pass from the spacetime \( (M, \bar{g}) \) to the spacetime \( (M, g) \). As it is physically reasonable, this energy can be infinite in principle. Of course, if the metrics are “near” the energy between them can be expected to be finite, though we cannot even be sure that we can “continuously” join two very different metrics. In general we can expect conserved quantities to classify accessibility classes of metrics, though we do not want here to enter the problem of providing the set of sections of the configuration bundle with a topology or a differentiable structure. We certainly expect that if \( \sigma = (\bar{g}, \bar{g}) \), i.e. if we set \( g = \bar{g} \), the conserved quantity \( Q_\Omega[\xi, \sigma] \) vanishes. This condition is satisfied by prescription (5.4) because of the form of the superpotential (5.5).

We remark that the conserved quantities defined by (5.4) are **covariantly conserved**, meaning that their flows through boundaries \( \partial D \) of 4-dimensional regions \( D \) in spacetime \( M \) identically vanish. If one chooses \( g \) to be a solution asymptotically flat according to one of the current definitions (e.g. the Kerr–Newman solution), \( \bar{g} \) to be the flat reference background (which matches the
dynamical metric at infinity where “infinity” is prescribed by the definition of asymptotic flatness) and \( \Omega \) to be a spacelike hypersurface in \( M \), then \( Q_\Omega[\xi, \sigma] \) reproduces the expected value for mass (by choosing \( \xi = \partial_t \), i.e. the vector field which corresponds to asymptotic time translation) and angular momentum (by choosing \( \xi = -\partial_\phi \), i.e. the vector field which corresponds to asymptotic rotation; see e.g. [11], [15], [27]). Furthermore, the same results are achieved for non–asymptotically flat solutions by choosing suitable reference backgrounds. Examples are the \((2+1)\) BTZ solution (which is asymptotically anti–de–Sitter, see [16]), the Euclidean Taub–Bolt solution (which is asymptotically locally flat, see [30]). In addition, the same techniques are used successfully in gauge–natural theories, i.e. when the field theory owns both covariance and gauge invariance (e.g. BCEO theory, Einstein–Maxwell theory; see [27], [31]).

We stress that the conserved quantities \( Q_\Omega[\xi, \sigma] \) associated to the covariant first order action principle (1.1) are not affected by the anomalous factor problems as the ones associated to the standard Hilbert–Einstein Lagrangian (see [14]). As is well known, in fact, the Komar superpotential (see [32]):

\[
U_K = \frac{1}{2\kappa} \sqrt{g} \nabla^\beta \xi^\alpha \, ds_{\alpha\beta}
\]

(5.6)

which is the superpotential associated to the Hilbert–Einstein Lagrangian, when computed, for example, on the Kerr–Newman solution, produces the correct angular momentum, but just one–half of the expected mass. Of course one could postulate the mass to be associated to the vector field \( \xi' = 2\partial_t \). However, the factor appears to depend on the particular solution under investigation, thus such a prescription seems to be incorrect as well as unmotivated. This is usually interpreted as a hint of a correction needed in the definition of conserved quantities. The correction can be obtained by the ADM techniques (see [4]) by restricting to asymptotically flat solutions and by choosing an ADM foliation of spacetime \( M \). As shown in [33], [34], [35] and Section 6, the ADM formalism appears as a particular case of the technique exposed above, which furthermore applies to much more general situations.

6. Covariant ADM Hamiltonian

We shall here specialize the general framework introduced in Section 5 to the first order action functional (1.3). We thence obtain a map \( Q[\xi] \) which associate to each spacetime vector field a covariantly conserved quantity, called the covariant ADM Hamiltonian. It will be compared with the standard ADM Hamiltonian introduced in Section 4 and with the quasilocal energy defined in Section 3.
Let us consider the covariant conserved quantity $Q_{\Sigma_t}[\xi, \sigma]$ for the first order action functional (1.1) in the domain $\Sigma_t$ and relative to a vector field $\xi$ and a section $\sigma = (g, \bar{g})$. As we have already outlined in the previous Section, the quantity $Q_{\Sigma_t}[\xi, \sigma]$ is defined to be the integral of the superpotential (5.5) on the 2–dimensional surfaces $B_t$ (see Fig. 1 and Appendix A for the notation), i.e.

$$Q_{\Sigma_t}^{tot}[\xi, \sigma] = Q_{\Sigma_t}[\xi, g] + Q_{\Sigma_t}[\xi, g, \bar{g}] + Q_{\Sigma_t}[\xi, \bar{g}]$$  \hspace{1cm} (6.1)

where:

$$Q_{\Sigma_t}[\xi, g] = \frac{1}{2\kappa} \int_{B_t} \sqrt{g} \nabla^\beta \xi^\alpha \sigma_{\alpha\beta}$$

$$Q_{\Sigma_t}[\xi, g, \bar{g}] = \frac{1}{2\kappa} \int_{B_t} \sqrt{g} g^{\mu\nu} u^\beta_{\mu\nu} \xi^\alpha \sigma_{\alpha\beta}$$  \hspace{1cm} (6.2)

$$Q_{\Sigma_t}[\xi, \bar{g}] = -\frac{1}{2\kappa} \int_{B_t} \sqrt{\bar{g}} \bar{\nabla}^\beta \xi^\alpha \sigma_{\alpha\beta}$$

In order to simplify the ADM decomposition of the expression (6.1) so to be able to compare the results obtained with the standard ones of Section 4 and [3] let us assume, as usual, that the metrics $g$ and $\bar{g}$ are matched on the hypersurface $\mathcal{B}$ and that the boundaries are orthogonal (i.e. $u^\mu n_\mu|_{\mathcal{B}} = 0$).

[We stress that under our viewpoint the matching condition between $g$ and $\bar{g}$ is unessential, since Nöther currents are covariantly conserved. One may consider the second example analysed in Appendix C where the Kerr solution is studied and its mass inside the finite $\mathcal{R}$–sphere is obtained with respect to a flat background matched at infinity. In this Section we require the matching on $\mathcal{B}$ in order to compare the Nöther charges expression with the aforementioned standard $(3+1)$ Hamiltonian and quasilocal energy.]

Let us also assume that the vector field $\xi$ is tangent to the hypersurface $\mathcal{B}$ (i.e. $\xi^\mu n_\mu|_{\mathcal{B}} = 0$). First of all let us consider the first contribution $Q_{\Sigma_t}[\xi, g]$ into (6.1), i.e. the integral of the Komar superpotential. It may be rewritten as:

$$Q_{\Sigma_t}[\xi, g] = \frac{1}{2\kappa} \int_{B_t} (u_\beta n_\alpha - u_\alpha n_\beta) g^{\beta\mu} \nabla_\mu \xi^\alpha \sqrt{\sigma} d^2 x$$  \hspace{1cm} (6.3)

where $\sqrt{\sigma} d^2 x$, is the volume element on $B_t$. We can manipulate algebraically the latter expression in the following way:

$$Q_{\Sigma_t}[\xi, g] = \frac{1}{2\kappa} \int_{B_t} \{g^{\beta\mu} \nabla_\mu \xi^\alpha (2u_\beta n_\alpha - u_\beta n_\alpha - u_\alpha n_\beta)\} \sqrt{\sigma} d^2 x =$$

$$= \frac{1}{2\kappa} \int_{B_t} \{2u^\mu \nabla_\mu \xi^\alpha n_\alpha - (u^\mu n^\alpha + u^\alpha n^\mu) \nabla_\mu \xi_\alpha\} \sqrt{\sigma} d^2 x =$$  \hspace{1cm} (6.4)

$$= \frac{1}{2\kappa} \int_{B_t} \{-2u^\mu \xi_\alpha \nabla_\mu n^\alpha - u^\mu n^\alpha \xi_\alpha g_{\mu\alpha}\} \sqrt{\sigma} d^2 x.$$
By means of the identity \( n^\alpha \mathcal{L}_g \mu = \mathcal{L}_\xi n^\mu_g - g_{\mu \alpha} \mathcal{L}_\xi n^\alpha \) we obtain:

\[
Q_{\Sigma_t}[\xi, g] = \frac{1}{2\kappa} \int_{B_t} \left\{ -2u^\mu \xi_\alpha \nabla_\mu n^\alpha - u^\mu (\xi^\alpha \nabla_\alpha n^\mu + \nabla_\mu \xi^\alpha n_\alpha) + u_\alpha \mathcal{L}_\xi n^\alpha \right\} \sqrt{\sigma} d^2 x
\]

Taking formula (A.9) repeatedly into account together with the condition of orthogonal boundaries \( u^\mu n_\mu |_B = 0 \) and the condition \( \xi^\mu n_\mu |_B = 0 \) we finally obtain

\[
Q_{\Sigma_t}[\xi, g] = \frac{1}{\kappa} \int_{B_t} \{ \Theta_{\mu \alpha} u^\mu \xi^\alpha \} \sqrt{\sigma} d^2 x + \frac{1}{2\kappa} \int_{B_t} u_\alpha \mathcal{L}_\xi n^\alpha \sqrt{\sigma} d^2 x \tag{6.5}
\]

A similar expression may be found for the third contribution \( Q_{\Sigma_t}[\xi, \bar{g}] \) into formula (6.2), i.e. the Komar contribution of the matched background \( \bar{g} \):

\[
Q_{\Sigma_t}[\xi, \bar{g}] = \frac{1}{\kappa} \int_{B_t} \{ \bar{\Theta}_{\mu \alpha} u^\mu \xi^\alpha \} \sqrt{\sigma} d^2 x - \frac{1}{2\kappa} \int_{B_t} u_\alpha \mathcal{L}_\xi \bar{n}^\alpha \sqrt{\sigma} d^2 x \tag{6.6}
\]

It now remains to calculate the second contribution \( Q_{\Sigma_t}[\xi, g, \bar{g}] \) into formula (6.2). We stress that this is the contribution arising from the boundary term into the action functional (1.1). It can be written as:

\[
Q_{\Sigma_t}[\xi, g, \bar{g}] = \frac{1}{2\kappa} \int_{B_t} (u_\beta n_\alpha - u_\alpha n_\beta) \xi^\alpha w^\beta_{\mu \nu} g^{\mu \nu} \sqrt{\sigma} d^2 x =
\]

\[
= -\frac{1}{2\kappa} \int_{B_t} u_\alpha \xi^{\alpha \beta} n_{\beta} w^\beta_{\mu \nu} g^{\mu \nu} \sqrt{\sigma} d^2 x \tag{6.7}
\]

We remind that in the radial ADM decomposition (A.19) of the metric we have \( n_\beta dx^\beta = V \, dr \) where \( V \) is the radial lapse. Hence, by making use of the expression (A.23) in the Appendix A we obtain:

\[
Q_{\Sigma_t}[\xi, g, \bar{g}] = -\frac{1}{\kappa} \int_{B_t} u_\alpha \xi^\alpha (\Theta - \bar{\Theta}) \sqrt{\sigma} d^2 x \tag{6.8}
\]

[Notice that only the projection \( u_\alpha \xi^\alpha \) of the vector field \( \xi^\alpha \) along the timelike normal \( u^\alpha \) gives a contribution to the term \( Q_{\Sigma_t}[\xi, g, \bar{g}] \). This is the reason why the boundary term into the action (1.1) allows to correct the anomalous factor of the Komar superpotential which, as we said above, appears in the computation of mass while it does not enter in the computation of angular momentum.]
The conserved quantity in the region $\Sigma_t$ relative to the infinitesimal generator of spacetime symmetries $\xi$ is given by the sum of (6.5), (6.6) and (6.8):

$$Q^{\text{Tot}}_{\Sigma_t}[\xi, \sigma] = \frac{1}{\kappa} \int_{B_t} \left\{ \Theta_{\mu\alpha} u^\mu \xi^\alpha - \Theta u^\mu \xi^\mu \right\} \sqrt{\sigma} d^2x +
\quad - \frac{1}{\kappa} \int_{B_t} \left\{ \bar{\Theta}_{\mu\alpha} u^\mu \xi^\alpha - \bar{\Theta} u^\mu \xi^\mu \right\} \sqrt{\sigma} d^2x +
\quad + \frac{1}{2\kappa} \int_{B_t} u_{\alpha} \left\{ \mathcal{L}_\xi n^\alpha - \mathcal{L}_\xi \tilde{n}^\alpha \right\} \sqrt{\sigma} d^2x$$

This latter formula may be recasted in a form which is better suited to be analysed. Because of $K_{\mu\nu} u^\mu = 0$ and $\sigma^\nu_{\mu} u^\mu = 0$, from formula (A.11) in the Appendix A it follows in fact:

$$\Theta_{\mu\alpha} u^\mu \xi^\alpha = -u_{\alpha} \xi^\alpha \ n_{\mu} - \sigma^\rho_{\alpha} \xi^\alpha \ K_{\rho\beta} \ n^\beta$$

Inserting this latter expression together with (A.12) into (6.9) we obtain:

$$Q^{\text{Tot}}_{\Sigma_t}[\xi, \sigma] = -\frac{1}{\kappa} \int_{B_t} \left\{ u_{\alpha} \xi^\alpha (K - \bar{K}) + \sigma^\rho_{\alpha} \xi^\alpha n^\beta (K_{\rho\beta} - \bar{K}_{\rho\beta}) \right\} \sqrt{\sigma} d^2x +
\quad + \frac{1}{2\kappa} \int_{B_t} u_{\alpha} \left\{ \mathcal{L}_\xi n^\alpha - \mathcal{L}_\xi \tilde{n}^\alpha \right\} \sqrt{\sigma} d^2x$$

Let us stress that, until now, no assumption has been made on the vector field $\xi$, apart from the requirement $\xi^\mu n_{\mu}|_B = 0$.

An easy computation shows that the difference $u_{\alpha} \left( \mathcal{L}_\xi n^\alpha - \mathcal{L}_\xi \tilde{n}^\alpha \right)$ is always zero if the metrics $g$ and $\bar{g}$ are matched on $B$.

On the contrary the terms $u_{\alpha} \mathcal{L}_\xi n^\alpha$ and $u_{\alpha} \mathcal{L}_\xi \tilde{n}^\alpha$ appearing in the last contribution to (6.11) separately disappear if we choose $\xi$ to be tangent to the 2-surfaces $B_t$, that is $\xi^\mu n_{\mu}|_{B_t} = 0$, or also if we choose $\xi$ to be the time-like vector field $\partial_t$, i.e. $\xi^\alpha = N u^\alpha + N^\alpha$. In both cases the flow of the vector field $\xi$ maps each hypersurface $\Sigma_t$ into itself or, respectively, into another surface $\Sigma_t'$. Since the vector field $n^\alpha$ is tangent to each $\Sigma_t$ it turns out that also $\mathcal{L}_\xi n^\alpha$ is tangent to $\Sigma_t$ and then $u_{\alpha} \mathcal{L}_\xi n^\alpha = 0$ in these cases.

Hence if we specialize formula (6.11) for the vector field $\xi = \partial_t$ we obtain the covariant conserved quantity which we call the Hamiltonian of the system. It is given by the expression:

$$Q^{\text{Tot}}_{\Sigma_t}[\partial_t, \sigma] = \frac{1}{\kappa} \int_{B_t} \left\{ N (K - \bar{K}) - N^\alpha (K^{bc} - \bar{K}^{bc}) n_b \sigma_{ac} \right\} \sqrt{\sigma} d^2x$$
and it coincides exactly with the expression of the \((3+1)\) Hamiltonian \((4.11)\).

Let us notice that this definition of Hamiltonian can be correctly considered as the definition of a covariant ADM formulation (see \([33],[34],[35]\)). In fact it does not require, \textit{a priori}, a \((3+1)\) decomposition of spacetime. We stress that in the covariant ADM approach, the Hamiltonian, or energy, contained in a \(3\)-dimensional region \(\Omega\) and relative to a solution \(\sigma\), is defined by \((5.4)\) as a Nöther conserved quantity:

\[
Q_{\Omega}^{\text{tot}}[\xi, \sigma] = \int_{\partial\Omega} U[\xi, \sigma]
\]  

(6.13)

This is a well-posed definition of Hamiltonian provided only that the non-vanishing vector field \(\xi\) be transverse to the hypersurface \(\Omega\). Hence, by considering the parameter of the flow of \(\xi\) as the “time” parameter and transporting \(\Omega\) along the flow of \(\xi\) we obtain a world tube foliated by hypersurfaces diffeomorphic to \(\Omega\). In this covariant context, rather then starting from a preferred local foliation into hypersurfaces, the starting point is a non-vanishing vector field the flow of which defines the local time, i.e. the flow of evolution. Then, by specializing the definition \((6.13)\) to the ADM foliation depicted in Fig. 1, under the additional assumptions of orthogonal boundaries and of the matching between the metric and its background, the \textit{covariant} Hamiltonian \(Q_{\Sigma_t}^{\text{tot}}[\partial_t, \sigma]\) exactly coincides with the standard Hamiltonian \(H(g, \bar{g})\) derived from a \((3+1)\) splitting of the York action functional (see equation \((4.11)\)).

Another relevant Nöther conserved quantity is obtained by specializing formula \((6.11)\) to a vector field \(\xi^\alpha = N^\alpha\) tangent to the \(2\)-surfaces \(B_t\), i.e. \(N^\alpha u_\alpha|_{B_t} = 0\) and \(\sigma^\rho N^\alpha = N^\rho\). Because of the vanishing of the first and third term in the right hand side of \((6.11)\) we obtain:

\[
Q_{\Sigma_t}^{\text{tot}}[N^\alpha, \sigma] = -\frac{1}{\kappa} \int_{B_t} N^\alpha n^\beta (K_{\alpha\beta} - \bar{K}_{\alpha\beta}) \sqrt{\sigma} d^2x
\]  

(6.14)

In asymptotically flat spacetimes when \(\xi\) corresponds to a rotation at spatial infinity, the Nöther charge \((6.14)\) may be taken as the definition of \textit{angular momentum}.

The last Nöther charge we consider is the one relative to the unit vector field \(\xi = u\) normal to the leaves of the ADM foliation. From \((6.11)\) we obtain:

\[
Q_{\Sigma_t}^{\text{tot}}[u, \sigma] = \frac{1}{\kappa} \int_{B_t} (\mathcal{K} - \bar{\mathcal{K}}) \sqrt{\sigma} d^2x
\]  

(6.15)

We observe that it agrees with the definition \((3.6)\) of quasilocal energy, i.e. it is the value of the Hamiltonian \((6.12)\) with \(N|_{B_t} = 1\) and \(N^\alpha|_{B_t} = 0\).
the aforementioned hypotheses, quasilocal energy may then be considered as a Nöther charge.

7. Time Conservation

We shall here discuss two different sets of sufficient conditions for time-conservation. The quantities (6.12), (6.14) and (6.15), are all covariantly conserved quantities independently on the hypothesis that $\xi$ is a Killing vector field or not. In fact they have been defined by means of Nöther theorem through a construction which relies only on the covariant nature of the Lagrangian. Hence, on a solution of field equations, the covariant conservation law $d_{\mu} E_{\mu}^{\xi, \sigma} = 0$ always holds for the Nöther current $E^{\xi, \sigma}$ and for all vector fields $\xi$. This property, together with the property of existence of superpotentials for any natural theory, has allowed us to define the covariantly conserved Nöther charges $Q_{\Sigma}^{\text{Tot}}[\xi, \sigma]$ (see (5.4)). On the contrary the charges $Q_{\Sigma}^{\text{Tot}}[\xi, \sigma]$ are conserved in “time” if they do not depend on the chosen hypersurface $\Sigma_t$, i.e. if $Q_{\Sigma_t}^{\text{Tot}}[\xi, \sigma] = Q_{\Sigma_t'}^{\text{Tot}}[\xi, \sigma]$. This is a stronger condition that has to be supported by additional requirements. If $B$ is the 3–dimensional region such that $\Sigma' - \Sigma + B$ is the boundary of a region $D$, from the conservation law $d_{\mu} E_{\mu}^{\xi, \sigma} = 0$ we obtain a time–conserved quantity if $\int_{B} E_{\mu}^{\xi, \sigma} ds_{\mu} = 0$, i.e. if the net flow of the Nöther current $E$ through the hypersurface $B$ vanishes. A stronger condition amounts to require the integrand to be equal to zero on $B$, i.e. $E_{\mu} n_{\mu}|_{B} = 0$. In this case $Q_{\Sigma}^{\text{Tot}}[\xi, \sigma]$ is conserved not only with respect to the given foliation in hypersurfaces $\Sigma_t$ but it is time–conserved with respect to the time of any foliation of the region $D$.

For the action functional (1.1) the Nöther current (5.1) may be rewritten as:

$$E^{\alpha}[\xi, \sigma] = \frac{1}{2\kappa} \left\{ (\sqrt{g} g^{\mu\nu} - \sqrt{\bar{g}} \bar{g}^{\mu\nu}) \mathcal{L}_{\xi} \bar{u}^{\alpha}_{\mu\nu} - \mathcal{L}_{\xi}(\sqrt{g} g^{\mu\nu}) w^{\alpha}_{\mu\nu} - \xi^{\alpha} \mathcal{L} \right\} \quad (7.1)$$

The corresponding quantity $E^{\mu} n_{\mu}|_{B}$ will be equal to zero if some condition is verified. We do not explicitly known a set of necessary requirements for the occurence of this situation; we can only provide two examples of sufficient conditions.

We may require, as a first example, that the following properties hold true:

A) the vector field $\xi$ is a Killing vector field for the metric, i.e. $\mathcal{L}_{\xi} g_{\mu\nu} = 0$;
B) the vector field $\xi$ is a symmetry for the background in the sense that $\mathcal{L}_{\xi} \bar{u}^{\alpha}_{\mu\nu} = 0$;
C) $\xi$ is tangent to the boundary $B$, i.e. $\xi^{\mu} n_{\mu}|_{B} = 0$. 

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These three requirements together ensure that $\mathcal{E}^\mu n_\mu|_B = 0$. These properties are clearly satisfied if we are dealing with the Killing vector fields of an asymptotically flat stationary solution and we choose the flat metric as a background. Nevertheless, we stress that the time-conserved quantities $Q^{\Sigma_t}_{\Sigma_t}[^{\xi, \sigma]} = \int_{B_t} \mathcal{U}[^{\xi, \sigma]}$ can be computed on a finite region, i.e. it is not necessary that $B_t$ is identified with spatial infinity. Moreover, we do not have explicitly required here the matching between the metrics on $\mathcal{B}$ (see the example of the Kerr metric in Appendix C).

Another set of sufficient conditions may be imposed to fulfill the condition $\mathcal{E}^\mu n_\mu|_B = 0$. They closely resemble the ones of [3]. We may require the matching of the metrics on the boundary $\mathcal{B}$ (so that the first term in (7.1) vanishes) and we again require $\xi$ to be tangent to the boundary $\mathcal{B}$: $\xi^\mu n_\mu|_B = 0$ (in order to make the third term in (7.1) vanishing when contracted with the normal $n_\alpha$). We are left with the term:

$$\int_{B} \mathcal{E}^\alpha d\sigma_\alpha = -\frac{1}{2\kappa} \int_{B} \mathcal{L}_\xi (\sqrt{g} g^{\mu\nu}) w^\alpha_{\mu\nu} \, ds_\alpha$$

Owing to the matching requirement $g_{\mu\nu}|_B = \bar{g}_{\mu\nu}|_B$ the latter expression may be recast into the equivalent form:

$$\int_{B} \mathcal{E}^\alpha d\sigma_\alpha = \int_{B} (\Pi^{ij} - \bar{\Pi}^{ij}) \mathcal{L}_\xi \gamma_{ij} \, d^3x$$

It vanishes if we require $\xi$ to be a Killing vector of the boundary 3–metric: $\mathcal{L}_\xi \gamma_{ij} = D_i \xi_j + D_j \xi_i = 0$. Hence, also in this latter situation, we obtain time–conserved quantities $Q^{\Sigma_t}_{\Sigma_t}[^{\xi, \sigma]}$, for the time $t$ of any foliation of $D$ in hypersurfaces $\Sigma_t$.

8. Conclusion and Perspectives

We proved that once suitable matching conditions are required (i.e. the 4–metrics $g$ and $\bar{g}$ are required to agree on the boundary of the region under consideration) the two action functionals (1.1) and (1.3) agree. Consequently the action functional (1.3) may be considered as the ADM counterpart of the covariant action functional (1.1).

The second important result achieved here is the characterization of the quasilocal energy as the Nöther charge associated to the (timelike) unit vector normal to the leaves of the ADM foliation.

These seem to be new results which should enable us to extend the analysis further ahead to the prescription for the entropy in General Relativity. In fact, the quasilocal energy as well as the action functional (1.3) appeared also as
the starting point of a statistically–oriented approach to black hole entropy (see [21], [22], [24], [26], [36], [37] and references quoted therein). A different approach to black hole entropy based on Nöther approach (see [16], [27], [30], [38] and references quoted therein) may be found in literature. In view of the present comparison between the covariant first order approach (which the Nöther approach is based on) and the York’s action functional (which quasilocal energy is based on) as well as between the Nöther charges and the quasilocal energy themselves, we believe that the two different approaches to entropy can be now successfully compared (see also [39], [40]).

Another interesting perspective is to extend the present comparison to the more general situation of non–orthogonal boundaries which sometimes appeared in the literature (see [8], [9], [10]). It would be of some interest to know whether non–orthogonality of the boundaries in the ADM decomposition of the covariant action functional (1.1) exactly produces the additional boundary terms which are derived for the (modified) York’s action (see [9]), as we guess to be true.

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9. Appendix A: Notation

In order to make this paper self–contained we here briefly summarize some of the formulae and expressions which are used throughout the paper. We follow the convention and notation adopted in [3] and we also refer the reader to Fig. 1 for notation.

We assume the hypersurfaces $\Sigma_t$ to be spacelike and we assume the hypersurface $\mathcal{B}$ to be timelike. The metric $h_{\mu\nu}$ induced on the hypersurfaces $\Sigma_t$ may be written as:

$$h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu}$$

(A.1)

while for the metric $\gamma_{\mu\nu}$ on the hypersurface $\mathcal{B}$ we have:

$$\gamma_{\mu\nu} = g_{\mu\nu} - n_{\mu} n_{\nu}$$

(A.2)

(in the sequel Greek indices are always raised and lowered with the 4–dimensional metric). The two vectors $\vec{u}$ and $\vec{n}$ denote the future directed unit normal to $\Sigma_t$ and the outward pointing unit normal to the hypersurfaces $\mathcal{B}$, respectively. They satisfy the normalization relations $u^\mu u_\mu = -1$ and $n^\mu n_\mu = 1$. 

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respectively. Any spacetime tensor may be projected onto the hypersurfaces \( \Sigma_t \) by means of the projection tensor:

\[
h_{\mu
u} = \delta_{\mu
u} + u^\mu u_\nu
\]  

(A.3)

Any tensorial object may be also projected onto \( \mathcal{B} \) with the projection tensor

\[
\gamma_{\mu\nu} = \delta_{\mu\nu} - n^\mu n_\nu
\]  

(A.4)

The 2–metric \( \sigma_{\mu\nu} \) on the boundaries \( B_t \) is given by:

\[
\sigma_{\mu\nu} = \gamma_{\mu\nu} + u_{\mu} u_{\nu} = h_{\mu\nu} = n_{\mu} n_{\nu}
\]  

(A.5)

and the respective projection tensor is \( \sigma^{\mu}_{\nu} = g^{\mu\rho} \sigma_{\rho\nu} \).

The extrinsic curvatures \( K_{\mu\nu} \) of \( \Sigma_t \) in \( M \), \( \Theta_{\mu\nu} \) of \( \mathcal{B} \) in \( M \) and \( K_{\mu\nu} \) of \( B_t \) embedded in \( \Sigma_t \) are defined, respectively, as follows:

\[
K_{\mu\nu} = -h_{\mu}^{\alpha} \nabla_{\alpha} u_{\nu}
\]

\[
\Theta_{\mu\nu} = -\gamma_{\mu}^{\alpha} \nabla_{\alpha} n_{\nu}
\]

\[
K_{\mu\nu} = -\sigma_{\mu}^{\alpha} D_{\alpha} n_{\nu}
\]  

(A.6)

where \( D_{\alpha} \) denotes the covariant derivative on \( \Sigma_t \) compatible with the metric \( h \).

The extrinsic curvature \( K_{\mu\nu} \) is a symmetric tensor on \( \Sigma_t \), i.e. it satisfies the conditions \( K_{\mu\nu} u^\mu = 0 \), \( K_{\mu\nu} h_{\rho\mu} = K_{\rho\nu} \). Instead the extrinsic curvature \( \Theta_{\mu\nu} \) is a symmetric tensor on \( \mathcal{B} \): \( \Theta_{\mu\nu} n^\mu = 0 \), \( \Theta_{\mu\nu} \gamma_{\rho}^{\mu} = \Theta_{\rho\nu} \), while the extrinsic curvature \( K_{\mu\nu} \) is a symmetric tensor on \( B_t \), i.e. \( K_{\mu\nu} u^\mu = K_{\mu\nu} n^\mu = 0 \).

We also denote by:

\[
a_{\nu} = u^\mu \nabla_{\mu} u_{\nu}
\]

\[
b_{\nu} = n^\mu \nabla_{\mu} n_{\nu}
\]  

(A.7)

the (covariant) accelerations of the two normals \( u^\mu \) and \( n^\mu \), respectively. They satisfy the orthogonality properties: \( u^\mu a_\mu = 0 \) and \( n^\mu b_\mu = 0 \).

By making use of the property (A.4), we obtain:

\[
\nabla_{\nu} n^\mu = \delta_{\nu}^\alpha \nabla_{\alpha} n^\mu = \gamma_{\nu}^{\alpha} \nabla_{\alpha} n^\mu + n^\alpha n_\nu \nabla_{\alpha} n^\mu
\]

(A.8)

Taking into account definitions (A.6) and (A.7) we have:

\[
\nabla_{\nu} n^\mu = -\Theta^{\mu}_{\nu} + n_{\nu} b^\mu
\]  

(A.9)

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Performing calculations in the same manner for the vector $u^\mu$, it is easy to check the analogous relation:

$$\nabla_\nu u^\mu = -K_\nu^\mu - u_\nu a^\mu$$  \hspace{1cm} (A.10)

Moreover, projecting the indices of the extrinsic curvature $\Theta_{\mu\nu}$ normally and tangentially to the hypersurfaces $\Sigma_t$ we obtain the useful formula (see [3] for detailed computations):

$$\Theta_{\mu\nu} = K_{\mu\nu} + u_\mu u_\nu n_\alpha a^\alpha + 2\sigma_{(\mu} u_{\nu)} n^\beta K_{\alpha\beta}$$  \hspace{1cm} (A.11)

[We remind that this relation is true only under the assumption of orthogonal boundaries, i.e. $u^\mu n_\mu|_B = 0$.] Contracting the latter expression with $g^{\mu\nu}$ we also easily obtain:

$$\Theta = K - n_\alpha a^\alpha$$  \hspace{1cm} (A.12)

Let us now consider the ADM decomposition of the metric:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = (h_{\mu\nu} - u_\mu u_\nu) dx^\mu \otimes dx^\nu =$$

$$= - N^2 dt^2 + h_{ab}(dx^a + N^a dt) \otimes (dx^b + N^b dt)$$  \hspace{1cm} (A.13)

The coordinate system $(x^\mu) = (t, x^a) = (t, r, x^A)$ is adapted both to $B$ and to the ADM foliation on $D$. The surfaces $\Sigma_t$ are surfaces of constant $t$ while $B$ is the hypersurface of constant $r$. In other words, indices $a, b, c, \cdots$ run from 1 to 3 and denote indices on the spacelike hypersurfaces $\Sigma_t$, while indices $A, B, C, \cdots$ run from 2 to 3 and they instead denote indices on the boundary $B_t$ of $\Sigma_t$. Hence, tensors on $\Sigma_t$ are labelled by early Roman letters $a, b, \cdots$. When they are considered as tensors on spacetime $M$ the same tensors are instead denoted by Greek letters. For example, the extrinsic curvature $K$ in (A.6) can be denoted as $K_{\mu\nu}$ or $K_{ab}$, according to notational convenience.

The unit normal is given by $(u_\mu) = (-N, 0, 0, 0)$ while the timelike coordinate vector field $\vec{\partial}_0 = \partial/\partial t$ reads as $\vec{\partial}_0 = N\vec{u} + \vec{N}$, being $N$ the lapse function and $\vec{N} = (N^\mu) = (0, N^a)$ the spatial shift vector: $\vec{N} \cdot \vec{u} = N^\mu u_\mu = 0$ (we remind that if we also assume the vector field $\vec{\partial}_0$ be tangent to the 3-dimensional hypersurface $B$, the orthogonal boundaries condition reads as $\vec{N} \cdot \vec{n}|_B = N^\mu n_\mu|_B = 0$. Nevertheless this latter hypothesis will be not relevant for the computations in the rest of this Appendix). The extrinsic curvature $K_{\mu\nu}$ and the acceleration $a^\mu$ defined in (A.6) and (A.7), respectively, read as:

$$K_{ab} = -N \Gamma^0_{ab} = \frac{1}{2N} \left[ -\partial_0 h_{ab} + D_a N_b + D_b N_a \right]$$

$$(a^\mu) = (0, a^b) = (0, \frac{\partial^b N}{N})$$  \hspace{1cm} (A.14)

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(Roman indices are here raised and lowered with the 3–metric $h_{ab}$).

The Levi–Civita connection coefficients are given by:

\[
\begin{align*}
\Gamma^0_{0a} &= \frac{1}{N}(\partial_a N - N^b K_{ba}) \\
\Gamma^0_{ab} &= -\frac{K_{ab}}{N} \\
\Gamma^0_{00} &= \frac{\partial_b N}{N} + \frac{N^b}{N}\partial_b N - \frac{N^a N^b}{N} K_{ab} \\
\Gamma^a_{b0} &= -NK^a_b + \frac{N^a N^c}{N} K_{cb} + D_b N^a - \frac{N^a}{N} \partial_b N \\
\Gamma^a_{bc} &= 3 \Gamma^a_{bc} + \frac{N^a}{N} K_{bc}
\end{align*}
\]

(where $3 \Gamma^a_{bc}$ denotes the Levi–Civita connection of the 3–metric $h_{ab}$). A similar splitting may be performed with the background metric $\bar{g}$; we obviously obtain the same relations by replacing the objects involved with the corresponding barred ones. By means of formulae (A.15) one may easily compute the quantities $u^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \delta^\mu_{(\alpha} \Gamma^\epsilon_{\beta)\epsilon}$. For example, one can verify the following:

\[
g^{\mu\nu} u^0_{\mu\nu} = -\frac{2}{N} K^a_a + \frac{1}{N^2} \partial_a N^a
\]

and

\[
g^{\mu\nu} \bar{u}^0_{\mu\nu} = \bar{K}_{ab} \left[ \frac{1}{N^2 N}(\bar{N}^a - N^a)(\bar{N}^b - N^b) - \frac{1}{N} \left( \bar{N} \bar{h}_{ab} + \frac{N}{N} h_{ab} \right) \right] + \left( \frac{1}{N} \bar{N} N^a \partial_a \bar{N} + \frac{1}{N^2} \partial_a N^a + \frac{\bar{N}^b}{N^2} 3 \Gamma^a_{cb} - \frac{N^b}{N^2} 3 \Gamma^a_{ab} \right)
\]

which together give the expression

\[
g^{\mu\nu} u^\alpha_{\mu\nu} u_\alpha = g^{\mu\nu} (u^\alpha_{\mu\nu} - \bar{u}^\alpha_{\mu\nu}) u_\alpha = -N g^{\mu\nu} (u^0_{\mu\nu} - \bar{u}^0_{\mu\nu})
\]

The latter term is involved in the contribution on the lids of the boundary term in the action functional (1.1). To evaluate instead the action functional contribution on the hypersurface $\mathcal{B}$ we have to make use of the adapted splitting of the metric:

\[
g = g_{\mu\nu} dx^\mu \otimes dx^\nu = (\gamma_{\mu\nu} + n_\mu n_\nu) dx^\mu \otimes dx^\nu = V^2 dr^2 + \gamma_{ij}(dx^i + V^i dr) \otimes (dx^j + V^j dr)
\]
where \((x^i) = (t, x^A)\). Middle Roman letters \(i, j, k, \ldots\) denote indices on the timelike hypersurface \(B\) while \(x^A (A = 2, 3)\) denote again the coordinates over the two dimensional surfaces \(B_t\). The function \(V\) is the radial lapse while \(V^i\) is the radial shift. Hence the unit, outward pointing, radial normal \(\vec{n}\) reads as \(\vec{n} = (1/V)[\vec{\partial}_r - V^i \vec{\partial}_i]\). The extrinsic curvature \(\Theta_{ij}\) of the “cylinder” \(B = \{r = \text{constant}\}\) is given by:

\[
\Theta_{ij} = V \Gamma^r_{ij} = \frac{1}{2V} [-\partial_r \gamma_{ij} + \mathcal{D}_i V_j + \mathcal{D}_j V_i]
\]

(A.20)

where \(\mathcal{D}_i\) denotes the covariant derivative on \(B\) induced by the Levi-Civita connection \(3\Gamma^j_{ik}\) of the 3–metric \(\gamma_{ij}\). The coefficients of the 4–dimensional Levi-Civita connection can now be decomposed as:

\[
\begin{align*}
\Gamma^r_{ri} &= \frac{1}{V} (\partial_i V + V^j \Theta_{ji}) \\
\Gamma^r_{ij} &= \frac{\Theta_{ij}}{V} \\
\Gamma^r_{rr} &= \frac{1}{V} \partial_r V + \frac{V^i}{V} \partial_i V + \frac{V^i V^j}{V} \Theta_{ij} \\
\Gamma^i_{jr} &= -V \Theta^i_j - \frac{V^i V^k}{V} \Theta_{kj} + \mathcal{D}_j V^i - \frac{V^i}{V} \partial_j V \\
\Gamma^i_{jk} &= 3 \Gamma^r_{jk} - \frac{V^i}{V} \Theta_{jk}
\end{align*}
\]

(A.21)

The latter expressions are not simply obtained from (A.15) by exchanging tensors on \(\Sigma_t\) with the corresponding tensors on \(B\). Because the metric \(h_{ab}\) and \(\gamma_{ij}\) have different signatures, a change of sign may appear in some terms of (A.21) if compared with the decomposition (A.15).

By means of (A.21), the following expression are then easily computed:

\[
\begin{align*}
g^{\mu\nu} u^r_{\mu\nu} &= \frac{2}{V} \Theta^i_i - \frac{1}{V^2} \partial_i V^i \\
g^{\mu\nu} \bar{u}^r_{\mu\nu} &= \bar{\Theta}_{ij} \left[ \frac{1}{V^2 V} (\bar{V}^i - V^i)(\bar{V}^j - V^j) + \frac{1}{V} \left( \frac{\bar{V}}{V} \bar{\gamma}^{ij} + \frac{V}{V} \gamma^{ij} \right) \right] + \\
&+ \frac{1}{V V^2} (\bar{V}^i - V^i) \partial_i \bar{V} - \frac{1}{V^2} \partial_i \bar{V}^i - \frac{\bar{V}^j}{V^2} 3 \bar{\Gamma}^i_{ij} + \frac{\bar{V}^j}{V^2} 3 \bar{\Gamma}^i_{ij}
\end{align*}
\]

(A.22)

Expressions (A.22) give the expression

\[
g^{\mu\nu} u^r_{\mu\nu} n_{\alpha} = g^{\mu\nu} (u^r_{\mu\nu} - \bar{u}^r_{\mu\nu}) n_{\alpha} = V g^{\mu\nu} (u^r_{\mu\nu} - \bar{u}^r_{\mu\nu})
\]

(A.23)
which is the contribution on \(B\) of the boundary term in the action functional (1.1).

We stress that in (A.18) and (A.23) computations are performed without any hypothesis of orthogonal boundaries and without requiring any matching conditions between the metric \(g\) and its background \(\bar{g}\).

10. Appendix B: Bundle Formalism for Variational Calculus

We hereafter briefly recall the bundle framework for variational calculus and summarize how one can algorithmically construct the Noether conserved currents and the superpotentials out of the Lagrangian.

Let \(\mathcal{C} = (\mathcal{C}, M, \pi; F)\) be a bundle and \((x^\mu; y^i)\) be fibered coordinates (relative to a trivialization chosen on \(\mathcal{C}\)). In fibered coordinates the projection reads as \(\pi : (x^\mu; y^i) \mapsto x^\mu\). A local section defined on \(U \subset M\) is a map \(\sigma : U \to C\) such that \(\pi \circ \sigma = \mathbb{1}_U\), i.e. it is locally given by \(\sigma : x^\mu \mapsto (x^\mu; y^i(x))\). A local section is thence associated to a map \(\sigma^i : x^\mu \mapsto y^i(x)\). When \(U = M\) the section is called a global section on \(\mathcal{C}\). In variational calculus \(y^i(x)\) are identified with the values of dynamical fields at the point \(x \in M\). The bundle \(\mathcal{C}\) is called the configuration bundle.

Starting from the bundle \(\pi : \mathcal{C} \to M\) we can define the bundle \(\pi^k : J^k\mathcal{C} \to M\) which is called the \(k\)-order jet prolongation of \(\mathcal{C}\). A point in \(J^k\mathcal{C}\) is denoted by \(j^k x \sigma\) and it is an equivalence class of local sections (defined in a neighbourhood of \(x \in M\)) having contact of order \(k\) at \(x\), i.e. having the same Taylor expansion at \(x\) up to order \(k\). Points in \(J^k\mathcal{C}\) are then parametrized by the value of a section \(\sigma\) and its partial derivatives at the point \(x \in M\) up to order \(k\) included. Consequently, if \((x^\mu; y^i)\) are fibered “coordinates” on \(\mathcal{C}\), \((x^\mu; y^i, y^i_\mu, \ldots, y^i_{\mu_1 \ldots \mu_k})\) are fibered coordinates on \(J^k\mathcal{C}\). We remark that the “coordinates” \(y^i_{\mu_1 \ldots \mu_h}\) \((h \leq k)\) are symmetric in the lower indices. If \(\sigma\) is a section of \(\mathcal{C}\), we can prolong it to a section \(j^k \sigma\) of \(J^k\mathcal{C}\) which is defined by \(j^k \sigma(x) = j^k_x \sigma\). If \(\sigma : x^\mu \mapsto (x^\mu; y^i(x))\), the \(k\)-order prolongation \(j^k \sigma\) is given by

\[
j^k \sigma : x^\mu \mapsto (x^\mu; y^i(x), \partial_\mu y^i(x), \ldots, \partial_{\mu_1 \ldots \mu_k} y^i(x)) \quad (B.1)
\]

If a section \(\sigma\) is identified with a field configuration, \(j^k \sigma\) describes fields and their derivatives up to order \(k\).

Let us denote by \(\Phi : \mathcal{C} \to \mathcal{C}'\) a bundle morphism projecting onto a diffeomorphism \(\phi : M \to M'\). By definition of bundle morphism it preserves the fibers (i.e. \(\pi' \circ \Phi = \phi \circ \pi\)) and it is locally given by

\[
\begin{align*}
x' &= \phi(x) \\
y' &= \Phi(x, y)
\end{align*} \quad (B.2)
\]
We can prolong it to a bundle morphism $j^k \Phi : J^k C \rightarrow J^k C'$ defined by the following (composition preserving) rule

$$j^k \Phi : j_x^k \sigma \mapsto j_{\phi(x)}^k \left[ \Phi \circ \sigma \circ \phi^{-1} \right] \quad (B.3)$$

(see [41], [42] for further details). Analogously, for any infinitesimal generator of automorphisms, i.e. any projectable vector field $\Xi = \xi^\mu (x) \partial_\mu + \xi^i (x, y) \partial_i$ on $\mathcal{C}$, we can define the prolongation $j^k \Xi = \xi^\mu \partial_\mu + \xi^i \partial_i + \xi^i \partial^i + \ldots + \xi_{\mu_1 \ldots \mu_k} \partial^k_{\mu_1 \ldots \mu_k}$

(here we set $\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial_i = \frac{\partial}{\partial y^i}, \ldots, \partial^i_{\mu_1 \ldots \mu_k} = \frac{\partial}{\partial y_{\mu_1 \ldots \mu_k}}$ for the natural basis induced in the tangent space $T J^k C$ by fibered coordinates). It is recursively given by

$$\begin{cases}
\xi^i_\mu = d_\mu \xi^i - y^i_\mu \partial_\mu \xi^\nu = d_\mu (\xi^i - y^i_\nu \xi^\nu) + y^i_{\mu \nu} \xi^\nu \\
\ldots \\
\xi^i_{\mu_1 \mu_2 \ldots \mu_k} = d_{\mu_1} \xi^i_{\mu_2 \ldots \mu_k} - y^i_{\mu_2 \ldots \mu_k} \partial_{\mu_1} \xi^\nu = d_{\mu_1 \ldots \mu_k} (\xi^i - y^i_{\mu} \xi^\nu) + y^i_{\mu_1 \ldots \mu_k} \xi^\nu
\end{cases}$$

where $d_\mu$ denotes the total derivative at any order and it is (locally) defined by

$$d_\mu = \partial_\mu + y^i_{\mu \nu} \partial_i + y^i_{\nu \mu} \partial^i + \ldots \quad (B.5)$$

We define a variational principle of order $k$ to be a pair $(\mathcal{C}, L)$ where $\mathcal{C} = (C, M, \pi; F)$ is a bundle called the configuration bundle and $L : J^k C \rightarrow A_n (M)$ is a (vertical) bundle morphism called the Lagrangian (of order $k$). Here and below $A_n (M)$ ($h \leq n$) is the bundle of $h$-forms on $M$ and $n = \dim (M)$. Locally a Lagrangian reads as $L = \mathcal{L}(x^\mu, y^i, y^i_{\mu_1 \ldots \mu_k}) \, ds$ (where $ds = dx^1 \wedge \ldots \wedge dx^n$); $\mathcal{L}$ is the Lagrangian density and it depends on the spacetime point $x^\mu$, on fields $y^i$ and on derivatives of fields $(y^i_{\mu_1 \ldots \mu_k})$ up to a finite order $k$. The Lagrangian $L$ associates to any configuration $\sigma$ an $n$-form $L \circ j^k \sigma = (\mathcal{L} \circ j^k \sigma) \, ds$ over the spacetime $M$.

In General Relativity, the configuration bundle is the bundle $\mathcal{C} = \text{Lor}(M)$ of Lorentzian metrics on the spacetime $M$. It has local fibered coordinates $(x^\mu; g_{\mu \nu})$. The Hilbert Lagrangian $L = (1/2\kappa) \sqrt{g} R \, ds$ is second order, thus it is interpreted as a map $L : J^2 \text{Lor}(M) \rightarrow A_n (M)$. The second order jet bundle $J^2 \text{Lor}(M)$ has natural fibered coordinates $(x^\mu, g_{\mu \nu}, g_{\mu \nu, \lambda}, g_{\mu \nu, \lambda \sigma})$ or, equivalently, it can be parametrized by the more convenient set of variables $(x^\mu, g_{\mu \nu}, \Gamma^\lambda_{\mu \nu}, d_\sigma \Gamma^\lambda_{\mu \nu})$ where $\Gamma^\lambda_{\mu \nu}$ denote the Christoffel symbols of the metric $g$. 

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We stress that in this framework, the Lagrangian is a map between finite-dimensional manifolds. It induces the action functional:

\[ A_D(\sigma) = \int_D L \circ j^k\sigma \]  \hspace{1cm} (B.6)

where \( D \subset M \) is a region in spacetime \( M \), i.e. a compact submanifold (of dimension \( n \)) with a boundary \( \partial D \) which is again a compact submanifold (of dimension \( n - 1 \)).

Let now \( C = (C, M, \pi; F) \) be a fiber bundle. We can define the sub-bundle \( \tau_V : V(C) \to C \) of the tangent bundle \( \tau : TC \to C \), which turns out to be a vector sub-bundle of \( TC \). A vector \( v \in T_pC \) belongs to \( V_pC \) if and only if \( T_p\pi(v) = 0 \); in this case we say that \( v \) is a vertical vector at \( p \in C \). Accordingly, \( V(C) \) is defined to be the kernel \( \ker(T\pi) \) of the fiberwise linear map \( T\pi : TC \to TM \). A vertical vector is “tangent to the fiber” \( F \) and its local expression (using fibered coordinates \((x^\mu; y^i)\) on \( C \)) is \( v = v^i \partial_i \) where \( \partial_i = \frac{\partial}{\partial y^i} \) is a local pointwise basis of vertical vectors. A section of the bundle \( V(C) \to C \) is called a vertical vector field. When \( C \) is the configuration bundle of a field theory, a vertical vector field \( X = (\delta y^i) \partial_i \) represents a deformation of the dynamical fields.

We stress that the dual vertical bundle can be defined for any bundle; we can consider the dual vertical bundle \( V^*(C) \) of the configuration bundle as well as the vector bundle \( V^*(J^kC) \) of its prolongations. The importance of these objects is that they allow to translate the variation of the action functional into the variation of the Lagrangian morphism. For any Lagrangian \( L : J^kC \to A_n(M) \) we can in fact define a bundle morphism \( \delta L : J^kC \to V^*(J^kC) \otimes A_n(M) \) such that for any vertical vector fields \( X : C \to V(C) \) we have:

\[ < \delta L | j^kX > = \left( \frac{d}{ds}(L \circ j^k\Phi_s) \right)_{s=0} \]  \hspace{1cm} (B.7)

where \( \Phi_s : C \to C \) is the (vertical) flow of \( X \) and \( < | > \) denotes the canonical duality between \( V^*(J^kC) \) and \( V(J^kC) \) given by evaluation. Locally, the morphism \( \delta L \) is given by

\[ \delta L = [p^i \partial y^i + p^\mu_i \partial y^i_\mu + \ldots + p^\mu_1^{\mu_2 \ldots \mu_k}_i \partial y^i_{\mu_1 \ldots \mu_k}] \otimes ds \]  \hspace{1cm} (B.8)
where we defined the conjugate momenta \((p_i = \partial_i L, p_i^\mu = \partial_i^\mu L, \ldots, p_i^{\mu_1 \ldots \mu_k} = \partial_i^{\mu_1 \ldots \mu_k} L)\) to be the derivatives of the Lagrangian density \(L\) with respect to the “coordinates” \((y_i, y_i^\mu, \ldots, y_i^{\mu_1 \ldots \mu_k})\), respectively.

The variation of the action along a deformation \(X \in V(C)\) is then given by

\[
\delta X A_D(\sigma) = \int_D < \delta L | j^k X > \circ j^k \sigma \quad (B.9)
\]

Field equations now follow by Hamilton principle, i.e. requiring the action to be stationary along any deformation around classical solutions. In other words, we say that a configuration \(\sigma : M \to C\) is a classical solution if, for all regions \(D\) and for all deformations \(X\) such that \(j^{k-1}X|_{\partial D} = 0\) (i.e. such that the deformation \(X\) vanishes together with its derivatives up to order \(k-1\) on the boundary \(\partial D\) of the region \(D\)) we have:

\[
\delta X A_D(\sigma) = 0 \quad (B.10)
\]

Specializing to the Lagrangian \((1.1)\), which is first order in \(g\) and second order in \(\bar{g}\), the deformation \(X = X_g + X_{\bar{g}} = (\delta g^{\mu\nu}) \partial_{\mu\nu} + (\delta \bar{g}^{\mu\nu}) \bar{\partial}_{\mu\nu}\) splits into a deformation \(X_g\) of the dynamical metric \(g^{\mu\nu}\) which is required to vanish on the boundary \(\partial D\) (i.e. \(X_g|_{\partial D} = 0\)) and a deformation \(X_{\bar{g}}\) of the background metric \(\bar{g}^{\mu\nu}\) which is required to vanish together with its first order derivatives (i.e. \(j^1X_{\bar{g}}|_{\partial D} = 0\)) on the boundary \(\partial D\).

One can prove that there exist a (unique) global bundle morphism \(E : J^{2k}C \to V^*(C) \otimes A_n(M)\) (called the Euler–Lagrange morphism) and (a family of) global bundle morphisms \(F : J^{2k-1}C \to V^*(J^{k-1}C) \otimes A_{n-1}(M)\) (called the Poincaré–Cartan morphisms) such that the so-called first variation formula holds true, i.e.:

\[
< \delta L | j^k X > \circ j^k \sigma = < E | X > \circ j^{2k} \sigma + d(< F | j^{k-1}X > \circ j^{2k-1} \sigma) \quad (B.11)
\]

Since in the Hamilton principle we consider deformations such that \(j^{k-1}X|_{\partial D} = 0\) the Poincaré–Cartan contribution vanishes when integrated in \((B.9)\) and we get field equations \(E \circ j^{2k} \sigma = 0\), i.e. Euler–Lagrange field equations.

If, for example, we consider \(k = 2\) we obtain

\[
\begin{align*}
E &= [p_i - d_\mu p_i^\mu + d_\mu p_i^{\mu\nu}] \tilde{dy}_i \otimes ds \\
F &= [(p_i^\mu - d_\nu p_i^{\mu\nu}) \tilde{dy}_i + p_i^{\mu\nu} \tilde{dy}_\nu] \otimes ds_\mu
\end{align*}
\quad (B.12)
\]

A bundle \(C = (C, M, \pi; F)\) is called natural if there exists a natural lift to \(C\) of spacetime diffeomorphisms, i.e. if one can associate to any spacetime
diffeomorphism \( \phi : M \to M \) a (unique) bundle automorphism \( \Phi : C \to C \) so to preserve compositions. As a consequence, on natural bundles one can canonically and naturally lift spacetime vector fields \( \xi = \xi^\mu(x) \partial_\mu \) to bundle vector fields \( \hat{\xi} = \xi^\mu(x) \partial_\mu + \xi^i(x,y) \partial_i \). A field theory is called a natural theory if its configuration bundle \( C \) is a natural bundle and, in addition, any spacetime diffeomorphism is a Lagrangian symmetry, i.e. we have

\[
< \delta L \mid j^k \mathcal{L}_\xi \sigma > \circ j^k \sigma = \mathcal{L}_\xi (L \circ j^k \sigma) \quad (B.13)
\]

where \( \mathcal{L}_\xi (L \circ j^k \sigma) = (d \circ i_\xi + i_\xi \circ d)(L \circ j^k \sigma) \) is the Lie derivative of the Lagrangian and \( \mathcal{L}_\xi \sigma = T \sigma \circ \xi - \hat{\xi} \circ \sigma \) is the Lie derivative of the section \( \sigma \) along a spacetime vector field \( \xi \). Locally we have

\[
\mathcal{L}_\xi \sigma = (\xi^\mu d_\mu y^i(x) - \xi^i(x,y(x))) \partial_i \quad (B.14)
\]

Notice that \( \mathcal{L}_\xi \sigma \) is a vertical vector field on the section \( \sigma \) so that the l.h.s. of formula (B.13) is meaningful. We also recall that equation (B.13) is equivalent to the requirement that the so–called Poincaré–Cartan form is invariant under the flow of \( \xi \) (see [2]).

The global Nöther theorem is now easily obtained. Let \( L \) be a natural Lagrangian; inserting (B.11) into equation (B.13) we obtain

\[
< \mathcal{E} \mid \mathcal{L}_\xi \sigma > + d < \mathcal{F} \mid j^{k-1} \mathcal{L}_\xi \sigma > = d (i_\xi L) \quad (B.15)
\]

i.e.

\[
d(< \mathcal{F} \mid j^{k-1} \mathcal{L}_\xi \sigma > - i_\xi L) = - < \mathcal{E} \mid \mathcal{L}_\xi \sigma > \quad (B.16)
\]

The Nöther current and the work density are thence defined by

\[
\mathcal{E}(L, \xi, \sigma) = < \mathcal{F} \mid j^{k-1} \mathcal{L}_\xi \sigma > - i_\xi L
\]

\[
\mathcal{W}(L, \xi, \sigma) = - < \mathcal{E} \mid \mathcal{L}_\xi \sigma >
\]

and they satisfy

\[
d\mathcal{E}(L, \xi, \sigma) = \mathcal{W}(L, \xi, \sigma) \quad (B.17)
\]

In any natural theory, both the Nöther current \( \mathcal{E}(L, \xi) \) and the work density \( \mathcal{W}(L, \xi) \) can be recasted, by suitable covariant integration by parts, as:

\[
\mathcal{E}(L, \xi, \sigma) = \tilde{\mathcal{E}}(L, \xi, \sigma) + d(\mathcal{U}(L, \xi, \sigma))
\]

\[
\mathcal{W}(L, \xi, \sigma) = B(L, \xi, \sigma) + d(\tilde{\mathcal{E}}(L, \xi, \sigma))
\]

(19)
One can easily prove that $B(L, \xi, \sigma) = 0$ off–shell (i.e. on any configuration not necessarily a solution of field equations), that $\hat{\mathcal{E}}(L, \xi, \sigma) = \tilde{\mathcal{E}}(L, \xi, \sigma)$ and that $\tilde{\mathcal{E}}(L, \xi, \sigma)$ vanishes on–shell (i.e. along solutions of field equations). The identity $B(L, \xi, \sigma) = 0$ is called the generalized Bianchi identity; the current $\hat{\mathcal{E}}(L, \xi, \sigma)$ is called the reduced current, while $\mathcal{U}(L, \xi, \sigma)$ is called the superpotential. We stress once again that all the objects introduced are algorithmically constructed out of the Lagrangian.

We remark (see (B.18)) that on shell the Noether current $\mathcal{E}(L, \xi, \sigma)$ obeys a continuity equation $d\mathcal{E}(L, \xi, \sigma) = 0$, i.e. it is covariantly conserved. The conservation laws of this kind are called weak conservation laws since they hold on solutions of field equations only, in opposition to the strong conservation laws which hold also off–shell. An example of strongly conserved quantity is $\mathcal{E}(L, \xi, \sigma) - \tilde{\mathcal{E}}(L, \xi, \sigma)$ which is a closed form along any configuration $\sigma$, even if it is not a solution of field equations (see (B.19)).

An example of second order natural theory is General Relativity. Some details about this case are found in Section 5. Further details can be found in [2], [28], [27]. The Noether currents, as long as the superpotentials, are used in Section 5 to define covariantly conserved quantities in the particular case of General Relativity.

11. Appendix C: Examples

We shall here discuss some simple examples to illustrate and clarify some of the topics introduced above in the paper.

Let us first consider the conserved quantities of the Schwarzschild solution given in its standard form:

$$g = - \left(1 - \frac{2M}{\rho}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{\rho}\right)} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (C.1)$$

For computational convenience we rewrite it in isotropic coordinates $(t, r, \theta, \phi)$:

$$g = -\left(\frac{2r - M}{2r + M}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right) \quad (C.2)$$

where, see [6]:

$$2r = -M + \rho + \sqrt{\rho(\rho - 2M)} \quad \iff \quad \rho = \frac{(M + 2r)^2}{4r} \quad (C.3)$$

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Let us choose as a background the Minkowski spacetime in isotropic coordinates

\[ \bar{g} = \frac{(2R - M)^2}{(2R + M)^2} \, dt^2 + \left(1 + \frac{M}{2R}\right)^4 \left(dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)\right) \]  

(C.4)

The background \( \bar{g} \) is in fact a flat metric as one can easily check by direct computation of the Riemann tensor. Furthermore, the two metrics \( g \) and \( \bar{g} \) are matched on the hypersurface \( B \) defined by the equation \( r = R \).

Let us then consider a family of foliations of the region \( r \leq R \) generated by the infinitesimal generator

\[ \zeta = (1 + \epsilon (\sin(\phi) + 1)) \, \partial_t \]  

(C.5)

where \( \epsilon \) is a (small) parameter. The flow parameter is denoted by \( s \) so that, if \( \Sigma \) is a spacelike hypersurface, it can be dragged along the flow \( \Phi_s \) of \( \zeta \) to obtain the foliation:

\[ \Sigma_s^\epsilon = \{ s \text{ constant} \} \quad s = \frac{t}{1 + \epsilon (\sin(\phi) + 1)} \]

Since \( s \) denotes the affine parameter along the flow of \( \zeta \), it can be interpreted as the time associated to the ADM foliation. We remark that we are in the hypothesis of orthogonal boundaries as required throughout the paper. Notice also that for \( \epsilon = 0 \) we recover the ordinary asymptotic time translation \( \zeta = \partial_t \) and the ordinary ADM foliation by the hypersurfaces \( t = \text{constant} \) (see Fig. 2).

Let us then choose the (3–parameter) vector field

\[ \xi = \alpha \, \partial_t + (\beta + \gamma \sin(\theta) \cos \phi) \partial_\phi \]  

(C.6)
(where $\alpha$, $\beta$ and $\gamma$ are three real constants) which is a generator of symmetries for the first order Lagrangian (1.1). We remark that $\xi$ is a well defined vector field on $B$; in particular it extends to $\theta = 0$ and $\theta = \pi$. For $\alpha = 1$ and $\beta = \gamma = 0$, $\xi$ reduces to the ordinary time translation $\xi = \partial_t$ so that we expect the corresponding conserved quantity $Q_{\Sigma_s}^{\text{Tot}}[\xi]$ to be interpreted as the mass of $g$ relative to $\bar{\gamma}$ on the leaf $\Sigma_s$ in the region $r \leq R$. We also stress that for $\gamma = 0$ the vector field $\xi$ is a Killing vector both for $g$ and $\bar{\gamma}$. On the contrary, for $\gamma \neq 0$, $\xi$ is not Killing both for $g$ and $\bar{\gamma}$. In the case $\gamma \neq 0$, $\xi$ has not a direct physical interpretation though it is a symmetry generator for the Lagrangian (1.1) and it algorithmically generates the covariant Nöther conserved quantity $Q_{\Sigma_s}^{\text{Tot}}[\xi]$. We shall use it to illustrate how time conservation may be related to covariant conservation along different foliations.

If we calculate the Nöther conserved quantity (according to eq. (6.1) with $2k = 16\pi$, i.e. in geometric units $G = c = 1$) we get the following result

\[
Q_{\Sigma_s}[\xi, g] = \frac{\alpha}{2} M - \frac{\gamma}{16} \frac{\pi}{R} \left(1 - \frac{M}{2R}\right)^2 s \epsilon \\
Q_{\Sigma_s}[\xi, g, \bar{\gamma}] = \left(\frac{\alpha}{2} M - \frac{\gamma}{16} \frac{\pi M}{R}\right) \left(1 - \frac{M}{R}\right) s \epsilon \\
Q_{\Sigma_s}[\xi, \bar{\gamma}] = \frac{\gamma}{16} \frac{\pi R}{16} \left(1 - \frac{M^2}{4R^2}\right) s \epsilon \\
Q_{\Sigma_s}^{\text{Tot}}[\xi] = \alpha \left(M - \frac{M^2}{2R}\right) + \gamma \left(\frac{\pi M^2}{32R} s \epsilon\right)
\]

Thus, when $\xi$ is the Killing vector $\partial_t$ ($\alpha = 1$, $\beta = \gamma = 0$) the relative mass

\[
Q_{\Sigma_s}^{\text{Tot}}[\partial_t] = M - \frac{M^2}{2R}
\]

is time–conserved along any foliation of the family generated by $\zeta$, since it does not depend on the affine parameter $s$. We also stress that the conserved quantity (i.e. letting $R$ tend to infinity) is always $Q_{\Sigma_s}^{\text{Tot}}[\partial_t] = M$, i.e. it reduces to the expected value for total mass.

If $\gamma \neq 0$ the vector $\xi$ is not Killing. Despite of the fact that the conserved quantity is not time–conserved in general, it is still time–conserved along a particular foliation (namely, the one corresponding to $\epsilon = 0$, i.e. $\zeta = \partial_t$; see Fig. 2 A). We remark that in this case the flow of the Nöther current $\mathcal{E}[\xi] = \mathcal{E}^\mu ds_\mu$ through $\partial \Sigma_s$ vanishes even if $\mathcal{E}^\mu n_\mu|_B \neq 0$.

On the contrary, when $\epsilon \neq 0$ (see Fig. 2 B) the Nöther charge $Q_{\Sigma_s}^{\text{Tot}}$ explicitly depends on the time $s$, i.e. on the particular leaf on which it is computed.
We finally remark that in any case the quantity $Q_{\Sigma_t}^{\text{Tot}}[\xi]$ given by equation (C.7) is covariantly conserved (as any Nöther conserved quantity is) and it consequently obeys a continuity equation.

To end up this first example let us now compute the quasilocal energy (6.15). We consider the case depicted in Fig. 2 A, i.e. the foliation induced by $\zeta = \partial_t$. We choose, as infinitesimal symmetry $\xi$, the unit timelike normal $u$:

$$\xi = u = \frac{2r + M}{2r - M} \partial_t \quad (C.9)$$

If we calculate the Nöther conserved quantity (6.1) we now obtain the following result:

$$Q_{\Sigma_t}[u, g] = \frac{1}{4} \frac{M(2R + M)}{2R - M}$$

$$Q_{\Sigma_t}[u, g, \bar{g}] = \frac{1}{2} \frac{(2R^2 - MR - M^2)M}{R(2R - M)}$$

$$Q_{\Sigma_t}[u] = \frac{1}{4} \frac{M(4R^2 - M^2)}{(2R - M)^2}$$

$$Q_{\Sigma_t}^{\text{Tot}}[u] = M + \frac{M^2}{2R} \quad (C.10)$$

In spherical coordinates $(t, \rho, \theta, \phi)$ the fourth expression of (C.10) may be rewritten as:

$$Q_{\Sigma_t}^{\text{Tot}}[u] = \rho_0 \left[ 1 - \sqrt{1 - \frac{2M}{\rho_0}} \right] \quad (C.11)$$

where $2R = -M + \rho_0 + \sqrt{\rho_0(\rho_0 - 2M)}$; see (C.3). As expected, expression (C.11) perfectly agrees with the value of the energy computed in [3], formula (6.14).

Let us now consider another example of conserved quantity in a finite region $D$. It is a completely different example since it does not require the match of the solutions on the boundary $B$ of the finite region under consideration. Neither the condition of orthogonal boundaries is here required.

Let us consider the Kerr spacetime in ingoing Kerr–Schild coordinates $(t, r, \theta, \phi)$, given by:

$$g = \bar{g} + 2Mr \rho^{-2} \left[ dt + dr - a \sin^2 \theta d\phi \right]^2 \quad (C.12)$$
where \( \rho^2 = r^2 + a^2 \cos^2 \theta \), \( M^2 \geq a^2 \). Let us choose the flat background \( \bar{g} \) as:

\[
\bar{g} = -dt^2 + \left[ dr - a \sin^2 \theta \, d\phi \right]^2 + \rho^2 \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right]
\]  

(C.13)

The metrics \( g \) and \( \bar{g} \) are matched at infinity. Let us consider the regions \( D \) inside the hypersurface \( B \) defined by \( r = R \) and the ADM foliation \( \Sigma_t = \{ t \text{constant} \} \) generated by the vector field \( \partial_t \). We stress that \( g \) and \( \bar{g} \) do not match on \( B \) unless \( R \) is let to tend to infinity.

Let us finally choose as symmetry generator the (2-parameter) vector

\[
\xi = \alpha \partial_t + \beta \partial_\phi
\]

(C.14)

which is a Killing vector for \( g \) and \( \bar{g} \) (\( \alpha \) and \( \beta \) are two real constants).

The Nöther conserved quantities one obtains are

\[
Q_{\Sigma}[\xi, g] = (\alpha \frac{M}{2} - \beta Ma)
\]

\[
Q_{\Sigma}[\xi, g, \bar{g}] = \alpha \frac{M}{2}
\]

\[
Q_{\Sigma}[\xi, \bar{g}] = 0
\]

\[
Q_{\Sigma}^{\text{Tot}}[\xi] = \alpha M - \beta Ma
\]

(C.15)

which reproduce the expected values of the relative mass and of the angular momentum in the region \( t = \text{constant} \) and \( r \leq R \). Notice that the result is independent on \( R \) meaning that all the energy and angular momentum is “buried in the singularity”. We remark that setting anywhere \( a = 0 \) the Schwarzschild solution is recovered. The relative mass \( M \) we obtain in this case (see equation (C.15) with \( \alpha = 1 \), \( \beta = 0 \)) does not agree with the value found above (see equation (C.8)) because of the two different matches selected.

We remark that the quantities (C.15) are also time-conserved (even if the metrics are not matched at \( B \)). In fact the flow integral \( \int_B \mathcal{E} \) vanishes since \( \xi \) is tangent to \( B \) and it is a Killing vector of both \( g \) and \( \bar{g} \) (see Section 7). Notice also that \( \mathcal{E}^\alpha n_\alpha = 0 \), i.e. the Nöther current has no flow through any part of \( B \). Consequently, the associated conserved quantity is time conserved along any ADM foliation of the region \( D \).

[The calculations in this Appendix have been carried out by using tensor package of MapleV, see [43]. They are the very direct application of formula (6.1), just computed on the configuration \((g, \bar{g})\).]
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