ON THE DIAMETER OF ODD-DIMENSIONAL SPHERES

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Abstract. We show that certain Riemannian metrics on each odd-dimensional sphere have a fixed point free involutory isometry such that the diameter is not attained between antipodal points. This result partially responds to a question by Yurii Nikonorov.

1. Introduction

Given \((M, \rho)\) a compact length metric space, its diameter is given by the maximum distance between any two points. Nikonorov \([Ni01]\) proved the following result.

**Theorem 1.** Let \((M, \rho)\) be a length metric space that is homeomorphic to the two-dimensional sphere \(S^2\) and let \(I : M \to M\) be an involutory isometry without fixed points. Then, there is \(x \in M\) such that \(\text{diam}(M, \rho) = \rho(x, I(x))\).

He raised the following natural question.

**Question 2.** Is there an analogue of Theorem 1 for length metric spaces homeomorphic to the sphere \(S^n\) for some \(n \geq 3\)?

Podobryaev \([Po18b]\) found counterexamples for \(n = 3\) as a consequence of the explicit expression for the diameter of every 3-dimensional Berger sphere established by himself in \([Po18a]\) (see also \([PS16]\)).

The goal of this short note is to provide counterexamples for Question 2 for every odd dimension \(n \geq 5\) (see Theorem 3). The proof extends the one for \(n = 3\), using an argument from spectral geometry instead of an explicit expression for the diameter of any \(n\)-dimensional Berger sphere, which is in fact unknown (see Problem 3).

The author has not been able to find this result in the mathematical literature. However, he would not be surprised if it is somewhere, probably using geometric arguments.

2. Proof

We set

\[
\begin{align*}
G &= U(n + 1) = \{ A \in \mathbb{G}L(n + 1, \mathbb{C}) : A^*A = I \}, \\
K &= \left\{ \begin{pmatrix} A & 0 \\ 0 & z \end{pmatrix} : A \in U(n), z \in U(1) \right\} \simeq U(n) \times U(1), \\
H &= \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in U(n) \right\} \simeq U(n).
\end{align*}
\]

These compact connected Lie groups satisfy \(H \subset K \subset G\). At the Lie algebra level, we write \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}\) and \(\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}\) for the orthogonal decompositions with respect to the Killing form of \(\mathfrak{g}\).

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The group $G$ acts transitively on the sphere $S^{2n+1} = \{ x \in \mathbb{C}^{n+1} : |x|^2 = 1 \}$ by left multiplication, that is, $A \cdot x = Ax$. The stabilizer subgroup of $e_{n+1} = (0, \ldots, 0, 1)^t \in S^{2n+1}$ is $H$, thus we have the diffeomorphism $G/H \simeq S^{2n+1}$. Similarly, $G/K \simeq \mathbb{C}P^n$.

The inner product $\langle X, Y \rangle := -\frac{1}{2} \text{Re}(\text{Tr}(XY))$ on $g$ is $\text{Ad}(G)$-invariant. For any $t > 0$, let $g_t$ be the $G$-invariant metric on $G/H = S^{2n+1}$ induced by the inner product

\begin{equation}
\langle \cdot, \cdot \rangle_t := \frac{1}{t^2} \langle \cdot, \cdot \rangle_p \oplus \langle \cdot, \cdot \rangle_m.
\end{equation}

We denote by $\rho_t$ the corresponding distance metric on $S^{2n+1}$.

It turns out that $(G/H, g_t) \to (G/K, g_{FS})$ is a Riemannian submersion with totally geodesic fibers (one of the Hopf fibrations), and $(S^{2n+1}, g_{\frac{1}{2}})$ has constant sectional curvature 1. The Riemannian manifolds $(S^{2n+1}, g_t)$ for $t > 0$ are called the \textit{generalized Berger spheres}, or just \textit{Berger spheres}.

The element $-\text{Id} \in G$ corresponds to the fixed point free involution $I : S^{2n+1} \to S^{2n+1}$ given by $I(x) = -x$, which is consequently an isometry of $g_t$ for every $t > 0$. Consequently, due to the homogeneity of the metric $g_t$, $\rho_t(x, -x) = \rho_t(e_{n+1}, -e_{n+1})$ for every $x \in S^{2n+1}$.

**Theorem 3.** One has that $\text{diam}(S^{2n+1}, g_t) > \rho_t(e_{n+1}, -e_{n+1})$ for every $t > \sqrt{8(n+1)}$.

**Proof.** We define the continuous curve $\gamma : [0, \pi] \to G/H \simeq S^{2n+1}$ given by

\begin{equation}
\gamma(s) = \exp \left( \begin{pmatrix} 0 & 0 \\ 0 & i s \end{pmatrix} \right) H = \left( \begin{pmatrix} 0 & 0 \\ 0 & e^{is} \end{pmatrix} \right) H \equiv \left( \begin{pmatrix} 0 \\ : \end{pmatrix} \\ 0 \right) e^{is},
\end{equation}

which connects $e_{n+1}$ and $-e_{n+1}$. Note that $\gamma'(s) = (0 \ 0)^t \in p$ for every $s$. It follows that

\begin{equation}
\rho_t(e_{n+1}, -e_{n+1}) \leq \int_0^\pi \langle \gamma'(s), \gamma'(s) \rangle_t^\frac{1}{2} ds = \frac{1}{t} \int_0^\pi \left\langle \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right\rangle^\frac{1}{2} ds = \frac{\pi}{\sqrt{2} t},
\end{equation}

and consequently, $\rho_t(e_{n+1}, -e_{n+1}) \to 0$ when $t \to +\infty$.

On the other hand, the smallest positive eigenvalue of the Laplace-Beltrami operator on $(S^{2n+1}, g_t)$ is given by (see [BP13 Prop. 5.3])

\begin{equation}
\lambda_1(S^{2n+1}, g_t) = \min \left\{ 4(n+1), 2(n+t^2) \right\} = \begin{cases} 4(n+1) & \text{if } t \geq \sqrt{n+2}, \\ 2(n+t^2) & \text{if } t < \sqrt{n+2}, \end{cases}
\end{equation}

and Peter Li’s eigenvalue estimate [Li80] valid for homogeneous Riemannian manifolds ensures that

\begin{equation}
\lambda_1(S^{2n+1}, g_t) \text{diam}(S^{2n+1}, g_t)^2 \geq \frac{\pi^2}{4} \quad \text{for every } t > 0.
\end{equation}

Combining these facts, we get

\begin{equation}
\text{diam}(S^{2n+1}, g_t) \geq \frac{\pi}{4\sqrt{n+1}} \quad \text{for every } t \geq \sqrt{n+2}.
\end{equation}

Applying (2.4) we conclude that

\begin{equation}
\text{diam}(S^{2n+1}, g_t) > \rho_t(e_{n+1}, -e_{n+1}) \quad \text{for every } t > \sqrt{8(n+1)},
\end{equation}

as claimed. \qed
3. Comments

We end the article with some comments and open problems.

Remark 4. The same argument cannot be applied to the even-dimensional case since their only homogeneous metrics have constant sectional curvature. Counterexample might be constructed with cohomogeneity-one metrics, though the current spectral geometry argument is not available any more.

Remark 5. The same argument as in Theorem 3 can be used with the canonical variation of the remaining Hopf fibrations:

\[
\begin{align*}
\text{Sp}(n+1)/\text{Sp}(n) &\simeq S^{4n+3} \longrightarrow \text{Sp}(n+1)/\text{Sp}(n) \times \text{Sp}(1) \simeq \mathbb{H}P^n, \\
\text{Spin}(9)/\text{Spin}(7) &\simeq S^{15} \longrightarrow \text{Spin}(9)/\text{Spin}(8) \simeq S^8.
\end{align*}
\]

Problem 6. Obtain an explicit expression for \(\text{diam}(S^{2n+1}, g_t)\) for every \(t > 0\), as well as for the canonical variation of the other Hopf fibrations in (3.1).

One of its consequences would be to have new examples of explicit (local) extremes of the functional

\[
(3.2) \quad g \mapsto \lambda_1(M, g) \text{diam}(M, g)^2
\]

restricted to certain family of compact connected homogeneous Riemannian manifolds \((M, g)\). This was done for \((S^3, g)\) in [La19, (1.8)]. For a higher dimensional sphere, the first Laplace eigenvalue of those metrics were computed in [BP13] (see also [BLP22a] and [BLP22b]).

The importance of having explicit expressions for these extremes is to give some light for the following open problems:

Problem 7. Give a sharp lower bound for (3.2) among arbitrary compact connected homogeneous Riemannian manifolds.

Problem 8. Prove or disprove the existence of an upper bound for (3.2) among compact connected homogeneous Riemannian manifolds of a fixed dimension.

For instance, Eldredge, Gordina and Saloff-Coste [EGS18] conjectured that such upper bound exist among left-invariant metric on a fixed compact Lie group \(G\) (see [La20], [La21] for partial answers). In [La22] is studied this question among \(G\)-invariant metrics on a fixed homogeneous space \(G/H\).

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