On a stronger reconstruction notion for monoids and clones

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Abstract

Motivated by reconstruction results by Rubin, we introduce a new reconstruction notion for permutation groups, transformation monoids and clones, called automatic action compatibility, which entails automatic homeomorphism. We further give a characterization of automatic homeomorphism for transformation monoids on arbitrary carriers with a dense group of invertibles having automatic homeomorphism. We then show how to lift automatic action compatibility from groups to monoids and from monoids to clones under fairly weak assumptions. We finally employ these theorems to get automatic action compatibility results for monoids and clones over several well-known countable structures, including the strictly ordered rationals, the directed and undirected version of the random graph, the random tournament and bipartite graph, the generic strictly ordered set, and the directed and undirected versions of the universal homogeneous Henson graphs.

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1 Introduction

Permutation groups, transformation monoids and clones carry a natural topology: the Tichonov topology (also known as the topology of pointwise convergence). Under this topology the corresponding group becomes a topological group since composition and taking inverses are continuous operations, similarly a transformation monoid becomes a topological monoid, and clones (see below for a definition) become topological clones; here again the composition becomes continuous with respect to the topology induced by the Tichonov topology.

When we pass from a (usually countable) structure $A$ to its automorphism group $\text{Aut}(A)$ seen as permutation group, then to $\text{Aut}(A)$ seen as a topological group, and finally to $\text{Aut}(A)$ as an abstract group, we keep losing information, and a principal problem is: how much of this information can be recovered?

Hence, some of the natural research questions in this field relate to ‘reconstruction’ issues. For automorphism groups of (usually countable) structures, the reconstruction problem can be roughly stated as the question of determining as much about a structure as possible given only its automorphism group (up to some notion of equivalence). If one restricts the class of structures one looks at, there are various famous instances in which a partial or complete answer can be found. For instance, in [31], [24] and [32], M. Rubin tackles reconstruction problems for chains, Boolean algebras, certain $\aleph_0$-categorical trees, a broad class of $\aleph_0$-categorical structures in general, and several other specific structures.

One popular method, which by no means applies in all cases, is to establish the ‘small index property’ (SIP) for a structure. SIP means in its original form that for a countable structure $A$ any subgroup of its automorphism group of countable index contains the pointwise stabilizer of a finite set. Reinterpreted via the natural topology on the automorphism group this is saying that any subgroup of countable index is open. Seeking a more robust notion of reconstruction leads to the definition that a countable structure $A$ is ‘reconstructible’ if any isomorphism from its automorphism group to a closed subgroup of $\text{Sym}(\omega)$ is an isomorphism of the topological groups, that is, a homeomorphism. While the SIP is mainly useful in the countable case, the latter concept also makes sense for carriers of arbitrary cardinality.

If we seek to apply the above notions to clones, or even to monoids naturally associated with $A$ such as the self-embedding or endomorphism monoid, there are immediate problems regarding the SIP. The notion of ‘index’ does not arise naturally in the semigroup context. The better way to view it therefore is in the formulation via ‘automatic homeomorphicity’ [13]. Automatic homeomorphicity means that every (abstract) isomorphism between two (closed) permutation groups, transformation semigroups or two clones on carriers of the same size is a homeomorphism with respect to the Tichonov topologies. That is, the isomorphism class of a particular (closed) permutation group, understood as an abstract group, coincides with its isomorphism class seen as a topological group, and likewise for monoids and clones. A naturally arising question is then, of course, which monoids or clones are ‘reconstructible’, or have automatic homeomorphicity.
One of Rubin’s central reconstruction results \cite{rubin} gives a criterion for the class of countable \( \aleph_0 \)-categorical relational structures without algebraicity. Whenever such a structure \( A \) has a so-called weak \( \forall \exists \)-interpretation and \( B \) is countable \( \aleph_0 \)-categorical without algebraicity, it is enough to know \( \text{Aut}(A) \cong \text{Aut}(B) \) in order to conclude that the permutation groups \( \langle \text{Aut}(A), A \rangle \) and \( \langle \text{Aut}(B), B \rangle \) are isomorphic. More precisely, if \( A \) and \( B \) are countable \( \aleph_0 \)-categorical structures without algebraicity, \( A \) has a weak \( \forall \exists \)-interpretation and \( \xi : \text{Aut}(A) \to \text{Aut}(B) \) is an isomorphism, then there exists a bijection \( \theta \) between the domain of \( A \) and the domain of \( B \) such that for all \( g \in \text{Aut}(A) \) we have \( \xi(g) = \theta \circ g \circ \theta^{-1} \).

We take this result as a motivation to define a stronger property analogous to automatic homeomorphicity: we ask that any abstract algebraic isomorphism from a permutation group, transformation monoid or clone to another such structure from a given class, having an equipotent carrier, is a concrete isomorphism, i.e., one that respects the action. So, for a permutation group, transformation monoid or clone with this ‘automatic action compatibility’ the algebraic isomorphism class coincides with its action isomorphism class. Such a property implies automatic homeomorphicity and gives rise to stronger reconstruction results for countably infinite first-order structures, see section 2.4.

For the example structures we consider, these immediate consequences concerning automatic homeomorphicity of the monoids of self-embeddings are special cases of results known from the literature (\cite{countable}, Corollary 22, p. 3726) for the countable universal homogeneous graph, directed graph, tournament, \( k \)-uniform hypergraph (\( k \geq 2 \), \cite{countable}, Theorem 2.6, p. 75) for \( (Q, \prec) \), \cite{rationals}, Theorems 3.7 and 4.1) for the rationals with the circular order or the separation relation, respectively; \cite{henson}, Theorem 2.2, p. 599) for the Henson graphs and digraphs). In fact, one can get these results in a uniform way by combining the arguments in Corollary 5.2 and Lemma 5.3 with Corollary 3.17 of \cite{polymorphism}, p. 11. However, with the exception of the random graph (see \cite{countable}, Theorem 5.2, p. 3737), the random directed graph, the \( k \)-uniform hypergraph (see \cite{henson}, Example 4.6, p. 19 et seq.) and the strictly ordered generic poset (\cite{polymorphism}, Theorem 4.7, p. 20) not much is known on the side of the polymorphism clones. Hence, all automatic homeomorphicity corollaries one can derive from our reconstruction results shed some fresh light on polymorphism clones related with the studied example structures. In particular, it follows from Lemma 5.3 that any polymorphism clone on \( Q \) having \( \text{End}(Q, \prec) \) as its unary part has automatic homeomorphicity w.r.t. the class of all polymorphism clones of countable \( \aleph_0 \)-categorical structures without algebraicity. As a special case we therefore obtain new theorems regarding automatic homeomorphicity of \( \text{Pol}(Q, \prec) \) itself and of the polymorphism clones of the reducts of the strictly ordered rationals, \( \text{Pol}(Q, \text{betw}) \), \( \text{Pol}(Q, \text{circ}) \) and \( \text{Pol}(Q, \text{sep}) \), see Lemma 5.3 and Corollaries 5.5 and 5.11. A better understanding of these structures has been one of our initial motivations for studying automatic action compatibility. However, the obtained results are conditional in that the scope of the automatic action compatibility / homeomorphicity condition is restricted to a proper subset of all closed clones on countable carrier sets. Proving an unconditional automatic homeomorphicity result for the polymorphism clone of \( (Q, \prec) \), as well as for the ones of its reducts, remains an open problem for the
time being.

The central contribution of this article is that we provide a machinery how to transfer automatic action compatibility from the group case to suitable transformation monoids and clones. In this respect, it turns out that in concrete instances the essential structural obstructions (i.e., additional assumptions that need to be made) come from the group case; our lifting techniques work fairly generally and have, for example, no restrictions on the cardinality of the carrier sets. The main technical condition that we have to impose to cross the border from groups to monoids is that the group lie dense in the monoid, but this does not present a severe restriction from the viewpoint of applications (cf. [13, section 3.7, p. 3716]). In this setting, as a by-product, we prove a new characterization of automatic homeomorphism for transformation monoids on arbitrary carrier sets. It involves a weakening of the technical assumption that the only injective monoid endomorphism fixing every group member be the identical one, which has featured in several earlier reconstruction results [13, 29, 35, 4]. We believe that our weakened version, so, by our characterization, automatic homeomorphism, would also have been sufficient in any of these previous cases.

Reconstruction questions have quite a long tradition in mathematics, for they can, in a broader sense, even be traced back to the problem of classifying (hence understanding) geometries from their symmetry groups in Felix Klein’s ‘Erlanger Programm’ [21, 20]. Recently, however, such questions have also gained importance in applied contexts, namely for studying the complexity of so-called ‘constraint satisfaction problems’ (CSPs) in theoretical computer science. These are decision problems regarding the satisfiability of primitive positive logical formulæ where the constraints are formulated using (usually relational) structures \( \mathbb{A} \) with finite signature, so-called ‘templates’, given as a defining parameter of the problem, \( \text{CSP}(\mathbb{A}) \). Besides finite structures, in particular countably infinite homogeneous structures obtained as Fraïssé limits play a key role here because they are \( \aleph_0 \)-categorical and can be treated from the point of view of the Galois connection of polymorphisms and invariant relations like their finite siblings [11, Theorem 5.1] Automatic homeomorphicity comes into play when comparing the complexity of CSPs given by two countable \( \aleph_0 \)-categorical template structures \( \mathbb{A} \) and \( \mathbb{B} \) of finite but possibly different relational signature. Namely, if there is a continuous homomorphism from the polymorphism clone of \( \mathbb{A} \) to that of \( \mathbb{B} \) whose image is dense (and in contrast to the finite case, continuity is a true requirement here), then there is a primitive positive interpretation of \( \mathbb{B} \) into \( \mathbb{A} \) [12, Theorem 1, p. 2529]. In this case one can find a log-space reduction from \( \text{CSP}(\mathbb{B}) \) to \( \text{CSP}(\mathbb{A}) \), meaning that \( \text{CSP}(\mathbb{B}) \) is not essentially harder to solve than \( \text{CSP}(\mathbb{A}) \). Thus the presence of a homomorphism between the clones that is also a homeomorphism indicates that \( \text{CSP}(\mathbb{A}) \) and \( \text{CSP}(\mathbb{B}) \) have the same decision complexity: the problems are interreducible. So from the viewpoint of CSPs, structures with automatic homeomorphicity are easier to study as they behave more like finite templates where continuity of homomorphisms is not an issue to establish reductions. As automatic action compatibility is a new and more powerful concept than automatic homeomorphicity, its full effect in the context of CSPs yet remains to be explored.
2 Preliminaries

2.1 Maps, groups, monoids, clones and their homomorphisms

Subsequently we write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the set of natural numbers, and for sets $A$ and $B$ we denote their inclusion by $A \subseteq B$ as opposed to proper set inclusion $A \subset B$. Further, we use the symbol $\mathfrak{P}(A)$ to denote the set of all subsets of $A$. The cardinality of a set $A$ will be written as $|A|$. Moreover, if $f: A \to B$ is a function, $U \subseteq A$ and $V \subseteq B$ are subsets, we write $f[U]$ for the image of $U$ under $f$ and $f^{-1}[V]$ for the preimage of $V$ under $f$. If $f[U] \subseteq V$, we write $f|_{V}$ for the function $g: U \to V$ that is given as restriction of $f$, i.e., $g(x) = f(x)$ for $x \in U$. If $V = B$, we use the abbreviation $f|_{U} := f|_{B}$. We also define the surjective function $f|_{U} := f|_{U}$ for any $U \subseteq A$. In this way, if $g: C \to D$ is another function and $U \subseteq A \cap C$, then $f|_{U} = g|_{U}$ is equivalent to $f(u) = g(u)$ for all $u \in U$, because for $|_{U}$ we do not have to care about the equality of the co-domain.

For a set $A$ and $n \in \mathbb{N}$ we denote by $O_{A}^{(n)} := \{ f \mid f: A^{n} \to A \} = A^{n}$ the set of all $n$-ary operations on $A$ and by $O_{A} = \bigcup_{n \in \mathbb{N}} O_{A}^{(n)}$ the set of all finitary operations on $A$. If $F \subseteq O_{A}$ is a set of finitary operations on $A$ and $n \in \mathbb{N}$, we usually write $F^{(n)}$ for $O_{A}^{(n)} \cap F$ and call it the $n$-ary part of $F$. This is consistent since $(O_{A})^{(n)} = O_{A}^{(n)}$. We compose maps from right to left, so if $g: A \to B$ and $f: B \to C$, then $f \circ g: A \to C$, mapping $a \in A$ to $f(g(a))$. A subset $S \subseteq O_{A}^{(1)}$ of unary operations on $A$ that is closed under composition is a transformation semigroup on $A$. If $S$ contains the identity $id_{A}$, it is a transformation monoid. Semigroup homomorphisms need to be compatible with the binary operation $\circ$, and monoid homomorphisms have to additionally preserve the neutral element. In contrast to this abstract notion a transformation semigroup homomorphism (or action homomorphism) between $S \subseteq O_{A}^{(1)}$ and $T \subseteq O_{B}^{(1)}$ consists of a pair $(\varphi, \theta)$ of maps, where $\varphi: S \to T$ is a homomorphism as above and $\theta: A \to B$ is a map such that $\varphi(f) \circ \theta = \theta \circ f$ for every $f \in S$. If $\theta$ is a bijection, this is the same as $\varphi(f) = \theta \circ f \circ \theta^{-1}$ and we say that the homomorphism $\varphi: S \to T$ is induced by conjugation by $\theta$. This happens in particular, if $(\varphi, \theta)$ is an action isomorphism, i.e., if it has an inverse action homomorphism $(\varphi^{-1}, \theta^{-1})$. Moreover, a semigroup homomorphism $\varphi: S \to T$ is induced by conjugation, if there is some bijection $\theta: A \to B$ such that $(\varphi, \theta)$ is an action homomorphism. In this case $\varphi$ is automatically injective and an isomorphism when restricted to its image $\varphi[S]$. So if $S$ is a transformation monoid (a permutation group), then $\varphi[S]$ will be a transformation monoid (a permutation group, respectively), and the restriction of $\varphi$ to $S$ and $\varphi[S]$ will be a monoid (group) isomorphism, and even gives an action isomorphism. Note also that any transformation semigroup, transformation monoid or permutation group on $A$ and any bijection $\theta: A \to B$ give rise to an action isomorphism $(\varphi, \theta)$ by defining $\varphi: S \to \varphi[S]$ by $\varphi(f) := \theta \circ f \circ \theta^{-1}$ for $f \in S$.

We shall also be concerned with composition structures of functions of higher...
arity (motivated by higher-ary symmetries of structures), that are perhaps less known: A set of finitary operations $F \subseteq O_A$ on a fixed set $A$ is a clone if it contains the projections and is closed under composition, i.e., $F$ is a clone if for all $n \in \mathbb{N} \setminus \{0\}$, the set $\{\xi_i^{(n)}: (x_1, \ldots, x_n) \mapsto x_i \mid i \in \{1, \ldots, n\}\}$ is contained in $F$ and for every $n$-ary operation $f \in F^{(n)}$ and $m$-ary operations $g_1, \ldots, g_n$ from $F$ the ($m$-ary) composition $f \circ (g_1, \ldots, g_n)$, given by
\[
f \circ (g_1, \ldots, g_n)(x) = f(g_1(x), \ldots, g_n(x))
\]
for every $x \in A^m$, also belongs to $F$.

An (abstract) clone homomorphism between clones $F \subseteq O_A$ and $F' \subseteq O_B$ is a map $\xi: F \to F'$ that respects arities and is compatible with projections and composition. It is a clone isomorphism if it has an inverse clone homomorphism $\xi^{-1}: F' \to F$ with which it composes to the identity on $F$ and $F'$, respectively. Generalizing the situation for transformation monoids, an action homomorphism (or concrete clone homomorphism) is a pair $(\xi, \theta)$ where $\xi: F \to F'$ is a clone homomorphism and $\theta: A \to B$ is a map such that for every $n \in \mathbb{N}$ and all $f \in F^{(n)}$ we have $\xi(f) \circ (\theta \times \cdots \times \theta) = \theta \circ f$, where the product is formed with exactly $n$ factors of $\theta$. Again, if $\theta: A \to B$ is a bijection, this means $\xi(f) = \theta \circ f \circ (\theta^{-1} \times \cdots \times \theta^{-1})$ for all $f \in F$, and in this case, by a slight abuse of terminology, we still say that $\xi$ is induced by conjugation by $\theta$. This situation clearly happens for action isomorphisms, having, by definition, an inverse action homomorphism $(\xi^{-1}, \theta^{-1})$. The same remarks as above are true: Any clone homomorphism induced by conjugation (i.e., by some bijection $\theta: A \to B$) is injective and a clone isomorphism (together with $\theta$ it even is an action isomorphism) when restricted to its image. Also any clone and any bijection to some other carrier set induces an action isomorphism in the natural way.

### 2.2 Uniformities and topologies on powers

Functions on a set $A$ of any fixed arity are members of a power $A^I$, namely where $I = A^n$ and $n$ is the arity of the function. Such powers of $A$ (and their subsets) carry a natural topological and even uniform structure induced on the subset by the power structure on $A^I$ when $A$ is initially understood as discrete topological or uniform space.

We only give a short introduction to uniform spaces. A uniformity on a set $A$ is a non-empty (lattice) filter $\mathcal{U} \subseteq \mathcal{P}(A \times A)$ of reflexive binary relations on $A$ that is closed under taking inverses (that is, with every $\alpha \in \mathcal{U}$ also $\alpha^{-1} = \{(y, x) \in A^2 \mid (x, y) \in \alpha\} \in \mathcal{U}$) and has the property that for every $\alpha \in \mathcal{U}$ there is some $\beta \in \mathcal{U}$ such that $\beta \circ \beta = \{(x, z) \in A^2 \mid \exists y \in A: (x, y), (y, z) \in \beta\} \subseteq \alpha$, which represents a property analogous to the triangle inequality for metric spaces. The members $\alpha$ of a uniformity $\mathcal{U}$ are called entourages, and the idea is that $(x, y) \in \alpha$ means that points $x$ and $y$ of $A$ are uniformly close to each other. A uniform space is a pair $(A, \mathcal{U})$ where $\mathcal{U}$ is a uniformity on $A$. A uniformity base $\mathcal{B}$ on $A$ is a filter base of binary relations on $A$ (called basic entourages) such that the filter generated by $\mathcal{B}$, that is, the set of all relations containing
some $\beta \in \mathcal{B}$, is a uniformity on $A$. This requires that $\mathcal{B} \subseteq \mathcal{U}$, so every $\beta \in \mathcal{B}$

must be reflexive. Thus a sufficient condition for $\mathcal{B} \subseteq \mathcal{U}(A)$ to be a uniformity base is that $\mathcal{B}$ is a non-empty downward directed collection of reflexive binary relations that is closed under taking inverses and for every $\beta \in \mathcal{B}$ there is a $\gamma \in \mathcal{B}$
such that $\gamma \circ \gamma \subseteq \beta$. It is well known that every uniformity $\mathcal{U}$ on $A$ induces a topology by saying that a set $U \subseteq A$ is open if for every $x \in U$ there is an

tourange $\alpha \in \mathcal{U}$ such that $[x]_\alpha = \{ y \in A \mid (x, y) \in \alpha \} \subseteq U$. This means the collection $\{ [x]_\alpha \mid \alpha \in \mathcal{U} \}$ forms a neighbourhood base of $x$ for each $x \in A$. A map $h: A \to B$ between uniform spaces $(A, \mathcal{U})$ and $(B, \mathcal{V})$ is said to be uniformly continuous if for every $\beta \in \mathcal{V}$ there is some $\alpha \in \mathcal{U}$ such that $(h \times h)[\alpha] \subseteq \beta$. It is clear that it is sufficient to require this condition to be satisfied for all $\beta \in \mathcal{B}$ where $\mathcal{B}$ is a uniformity base of $\mathcal{V}$. Moreover, we say that $h: A \to B$ is a uniform homeomorphism if it is a bijection and $h$ and $h^{-1}$ are both uniformly continuous. Of course uniform continuity implies continuity with respect to the topologies induced by the uniform spaces, so uniform homeomorphicity implies homemorphism.

For our purposes only two uniformities are relevant: the first is the discrete uniformity, which is generated by the base $\{ \Delta_{A}^{(2)} \}$ where $\Delta_{A}^{(2)} = \{ (x, x) \mid x \in A \}$. Thus the discrete uniformity has every reflexive binary relation as an entourage, and hence induces the discrete topology $\mathcal{U}(A)$. The second is the uniformity induced by the discrete one (on $A$) on sets $F \subseteq A^{I}$ where $I$ is any index set. The category of uniform spaces with uniformly continuous maps has products and they are given by equipping the Cartesian product with the least uniformity on the product such that all projections are uniformly continuous. Applied to our situation, this implies that a uniformity base of $A^{I}$ is given by all equivalence relations of the form $\alpha_{J} = \{ (f_{1}, f_{2}) \in (A^{I})^{2} \mid f_{1}|_{J} = f_{2}|_{J} \}$ where $J$ ranges over all finite subsets $J \subseteq I$. This induces a uniformity base on $F \subseteq A^{I}$ by restricting the basic entourages to $\alpha_{J} \cap F^{2} = \{ (f_{1}, f_{2}) \in F^{2} \mid f_{1}|_{J} = f_{2}|_{J} \}$. We note that the product uniformity on $A^{I}$ induces the standard product topology (Tichonov topology) on $A^{I}$.

Based on the product uniformity every permutation group / transformation monoid / transformation semigroup $F \subseteq A^{A}$ carries a natural uniform structure related to functions interpolating each other on finite subsets of their domain. Also every clone $F \subseteq O_{A}$ can be written as $F = \bigcup_{n \in \mathbb{N}} F^{(n)}$ and hence be equipped with the coproduct uniformity given by the uniform structures on each $F^{(n)}, n \in \mathbb{N}$. Whenever we will be using concepts like openness, closedness, topological closure, interior, continuity etc., we will implicitly be referring to the topology induced by the uniformity just described. In particular, it makes sense to ask whether homomorphisms between transformation semigroups or between clones are uniformly continuous. It follows from the definition of the coproduct and the uniformity induced on subspaces that a clone homomorphism $\xi: F \to F'$ is uniformly continuous if and only if for every $n \in \mathbb{N}$ the restriction $\xi|_{F^{(n)}}: F^{(n)} \to F'^{(n)}$ is uniformly continuous, and the uniformities involved in this condition are the ones previously described. For more detailed information on the general product, coproduct and subspace constructions occurring in the
above we refer the reader to the excellent and concise overview given in section 2 of [33]. A more thorough treatment of uniform spaces can also be found in chapter 9 of [36, p. 238].

It is worth noting that any clone (group, monoid, semigroup) isomorphism that is induced by conjugation automatically is a uniform homeomorphism. This underlines the importance of action isomorphisms in the context of automatic homeomorphism.

2.3 Relational structures

Our main source for permutation groups, transformation monoids/semigroups and for clones shall be sets of homomorphisms of relational structures. If \( A \) and \( B \) are relational structures of the same signature on \( A \) and \( B \), respectively, then a map \( h: A \rightarrow B \) is a homomorphism between \( A \) and \( B \) if for any \( m \in \mathbb{N} \) and any \( m \)-ary relational symbol of the common signature that is interpreted in \( A \) and \( B \) as \( R \subseteq A^m \) and \( S \subseteq B^m \), respectively, the following implication is true: for every \( x = (x_1, \ldots, x_m) \in R \) it is required that \( h \circ x = (h(x_1), \ldots, h(x_m)) \in S \).

We usually denote the truth of this fact by \( h: A \rightarrow B \) and collect all homomorphisms between \( A \) and \( B \) in the set \( \text{Hom}(A, B) \). If \( A = B \) then any homomorphism \( h: A \rightarrow B \) is an endomorphism of \( A \); the set of all endomorphisms of \( A \) is \( \text{End}(A) \), it forms a transformation monoid. A homomorphism \( h: A \rightarrow B \) having an inverse homomorphism \( h': B \rightarrow A \) is an isomorphism. Isomorphisms can be equivalently characterized as those bijections that not only preserve relations as described above, but also reflect relations, i.e., for isomorphisms the implication used to define the homomorphism property is a logical equivalence. An isomorphism which also is an endomorphism is an automorphism of a structure \( A \) and all automorphisms of \( A \) form the set \( \text{Aut}(A) \), which naturally carries a permutation group structure. An intermediate concept between homomorphism and isomorphism is that of an embedding that is an injective relation preserving and reflecting map, i.e., an isomorphism, when restricted to the image. The set of all self-embeddings of \( A \) is denoted as \( \text{Emb}(A) \) and gives us another source of transformation monoids that are closer to the automorphism group. Moreover, defining relations in the product structure component-wise, one can study homomorphisms between the \( n \)-th power \( A^n \) (\( n \in \mathbb{N} \)) and \( A \), which are called \( n \)-ary polymorphisms of \( A \). The set \( \text{Pol}(A) = \bigcup_{n \in \mathbb{N}} \text{Hom}(A^n, A) \) consists of all polymorphisms of \( A \). Such sets always form clones, and it is well-known that the polymorphism clones of structures on a given set \( A \) are exactly those clones that are closed in the Tichonov topology.

To obtain some good examples for our results we need to impose some model theoretic ‘niceness’ properties on infinite structures. One of these is countable categoricity. We say that a structure \( A \) is \( \aleph_0 \)-categorical if up to isomorphism there is exactly one model of cardinality \( \aleph_0 \) of the first-order theory of \( A \). Thus such \( A \) cannot be finite. If \( A \) itself is countably infinite, then \( \aleph_0 \)-categoricity implies that \( A \) is the only countable model of its first-order theory up to isomorphism. All examples occurring in this paper will be of the latter form, and, by the Ryll-Nardzewski Theorem [19, Theorem 7.3.1, p. 341], for such structures
countable categoricity can be equivalently formulated as a condition on $\text{Aut}(\mathcal{A})$ called oligomorphicity (cf. [19, p. 134]). Another property we shall need is homogeneity. We say that a first-order structure $\mathcal{A}$ is homogeneous (occasionally called ultra-homogeneous, cf. [19, p. 325]), if any isomorphism between any two finitely generated substructures of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$. For relational structures finitely generated substructures coincide with finite substructures, so homogeneity means that any isomorphism between finite substructures must be extendable to an automorphism. All example structures considered in this article have a purely relational signature. As a third type we shall meet structures $\mathcal{A}$ without algebraicity, that is, whose automorphism group $\text{Aut}(\mathcal{A})$ has no algebraicity. In this context, a permutation group $G$ on $\mathcal{A}$ has no algebraicity if the algebraic closure of any (finite) subset $B$ of $\mathcal{A}$ is trivial, i.e., equal to $B$ (see [19, pp. 134, 330]). This means for any (finite) set $B \subseteq \mathcal{A}$ the only points having finite orbit with respect to the pointwise stabilizer $G(B)$ of $B$ under $G$ are those in $B$.

In order to facilitate the study of concrete examples of countably infinite homogeneous relational structures, we need the following easy lemma collecting some basic properties. In this connection we denote by $\Delta^{(m)}_A$ the $m$-ary relation $\{(x, \ldots, x) \mid x \in A\}$ on a given carrier set $A$. The two final statements of the lemma even hold without the assumption of homogeneity.

**Lemma 2.1.** For any homogeneous relational structure $\mathcal{A} = (A, (R_i)_{i \in I})$, where $R_i \subseteq A^{m_i}$ is an $m_i$-ary relation for each $i \in I$, the following facts hold.

(a) If for every $i \in I$ we have $R_i \cap \Delta^{(m_i)}_A = \emptyset$ or $R_i \cap \Delta^{(m_i)}_A = \Delta^{(m_i)}_A$, then $\text{Aut}(\mathcal{A})$ is transitive.

(b) If $I$ is finite and $|A| = \aleph_0$, then $\mathcal{A}$ is $\aleph_0$-categorical.

(c) $\text{Aut}(\mathcal{A}) = \text{Emb}(\mathcal{A})$ and the invertible embeddings are precisely the automorphisms.

(d) The structure $\mathcal{A}^c = (A, (R_i)_{i \in I}, (A^{m_i} \setminus R_i)_{i \in I}, A^2 \setminus \Delta^{(2)}_A)$ has the property $\text{End}(\mathcal{A}^c) = \text{Emb}(\mathcal{A})$.

(e) If $\mathcal{A}$ has no algebraicity, then the centre of $\text{Aut}(\mathcal{A})$ contains only the identity.

**Proof.**

(a) By the assumed condition, for every $i \in I$ the $i$-th relation of the induced substructure of $\mathcal{A}$ on any singleton $\{a\}$ is either empty or contains the constant tuple $(a, \ldots, a)$. This means any two induced singleton substructures of $\mathcal{A}$ are isomorphic, and by homogeneity the unique isomorphism between $\{a\}$ and $\{b\}$ can be extended to an automorphism of $\mathcal{A}$, whatever $a, b \in A$.

(b) This statement is a consequence of the Ryll-Nardzewski Theorem, which can be found, for instance, in Corollary 3.1.3 of [23, p. 1607].
The inclusion $\text{Aut}(\mathcal{A}) \subseteq \text{Emb}(\mathcal{A})$ is generally true, the converse follows, because for any $f \in \text{Emb}(\mathcal{A})$ and any finite subset $B \subseteq A$ the restriction of $f$ to the substructures induced by $B$ and $f[B]$ is an isomorphism, which, by homogeneity, can be extended to an automorphism of $\mathcal{A}$. The additional statement characterizing the group of invertible elements of $\text{Emb}(\mathcal{A})$ is evident from the definitions.

A map $f : A \to A$ preserves the complement of a relation, if it reflects the relation. In particular, preserving the inequality relation is equivalent to injectivity. The statement follows from the fact that the embeddings are exactly the injective relation preserving and relation reflecting maps.

This argument does not require homogeneity of $\mathcal{A}$. If $f$ belongs to the centre of the automorphism group, its graph is invariant for any group member, and so the stabilizer of any point $a \in A$ must fix $f(a)$ since $\{f(a)\}$ is (primitive positively) definable from $\{a\}$ and the graph of $f$. Thus the algebraic closure of $\{a\}$ contains $f(a)$, however, since $\mathcal{A}$ has no algebraicity, it follows that $f(a) = a$ for every $a \in A$. Hence, the centre of $\text{Aut}(\mathcal{A})$ is the singleton $\{\text{id}_{\mathcal{A}}\}$.

Concerning the reconstruction of countable $\aleph_0$-categorical structures from their automorphism groups we rely on the notions of (strong) small index property and weak $\forall\exists$-interpretation. These are actually properties of the automorphism group considered as a permutation group, so let $G \subseteq \text{Sym}(A)$ be a permutation group on a countable set $A$. We say that $G$ has the small index property (SIP), see [19, p. 144], if every subgroup $U \leq G$ of countable index contains the pointwise stabilizer $G_{(B)}$ of a finite subset $B \subseteq A$. This is equivalent to asking that every subgroup $U \leq G$ with $|G/U| \leq \aleph_0$ is open in the Tikhonov topology on $A^A$. Strengthening this requirement, the permutation group $G$ has the strong small index property (SSIP), see [19, p. 146], if for every $U \leq G$ of countable index in $G$ there is a finite set $B \subseteq A$ such that $G_{(B)} \subseteq U \subseteq G_B$, that is, $U$ lies between the pointwise and the setwise stabilizer of $B$. The notion of weak $\forall\exists$-interpretation needs a lengthy and technical definition that is nowhere needed in this text, so we leave it as an undefined black box tool and refer the reader to [31] or [23, p. 1620] for more information. Only the consequences of this property in the context of countable $\aleph_0$-categorical structures without algebraicity (as stated in Theorem 2.3) are relevant for us. For brevity we say that a structure $\mathcal{A}$ has the SIP, SSIP or a weak $\forall\exists$-interpretation whenever $\text{Aut}(\mathcal{A})$ has it.

2.4 Reconstruction notions

For applications to such reconstruction questions that go beyond automorphism groups we need the notion of automatic homeomorphicity and, in close analogy to the latter, we introduce the concept of automatic action compatibility. Both can be defined for closed permutation groups, transformation monoids and clones, and in both definitions we allow some class $\mathcal{K}$ of permutation groups,
transformation monoids or clones to act as a parameter restricting the scope of
the condition. For transformation monoids and clones on countable sets the
concept of automatic homeomorphicity originated in [13, Definition 6, p. 3714];
to the best of our knowledge, the modification of automatic homeomorphicity
relative to a parameter class was first given in [27, Definition 4.1, p. 142]. Our
definition extends both as it allows for any carrier set and a scope parameter $K$.

**Definition 2.2.** Let $\{F\} \cup K$ be a class of permutation groups/transformation
monoids/clones and let $A$ be the carrier set of $F$. We say that $F$ has

(a) automatic homeomorphicity with respect to $K$, if for every $F' \in K$ on a
set $B$ of the same cardinality as $A$ any group/monoid/clone isomorphism
$\varphi: F \to F'$ is automatically a homeomorphism;

(b) automatic action compatibility with respect to $K$, if for every $F' \in K$ on a
set $B$ of the same cardinality as $A$ any group/monoid/clone isomorphism
$\varphi: F \to F'$ is automatically part of an action isomorphism, that is, if there
is some bijection $\theta: A \to B$ such that $\varphi$ is induced by conjugation by $\theta$.

It is customary to agree that $F$ having automatic homeomorphicity (action com-
patibility) without any restriction means $F$ having this property with respect the
class of all (closed, if $F$ is closed) permutation groups/transformation monoids/
clones on sets equipotent to $A$.

To shorten formulations we stipulate that if $\mathcal{C}$ is a class of structures, the
statement that $F$ has one of the above properties ‘with respect to $\mathcal{C}$’ means that
the permutation group/transformation monoid/clone $F$ has the property with
respect to the class $\mathcal{K}$ of all automorphism groups/endomorphism monoids/
polymorphism clones of structures in $\mathcal{C}$.

Because every isomorphism induced by conjugation is automatically an (even
uniform) homeomorphism, it is immediate that automatic action compatibility
with respect to $K$ implies automatic homeomorphicity with respect to $K$.

Often, automatic homeomorphicity is only considered for such $F$ that are
closed in the Tichonov topology. If in this case $K$ contains some $F' \cong F$ on
some set equipotent to the carrier of $F$, and $F'$ is not closed, then, trivially,
$F$ has neither automatic homeomorphicity nor automatic action compatibility
with respect to $K$ (because homeomorphisms preserve closedness). As such
parametrizations do not give very interesting notions, one often restricts the
definition to the case where $\{F\} \cup K$ consists only of closed groups/monoids/
clones. Definition 2.2 as given above, however, also allows for some possibly
non-closed $F$, where non-closed $F' \in K$ would possibly make sense. So as a rule
of thumb, if $F$ is closed, we normally only consider classes $K$ of closed sets of
functions; if it is not, we may use any $K$ in Definition 2.2.

It is clear from the definition that whether some $F \in K$ has automatic homeo-
orphicity/action compatibility with respect to $K$ only depends on those mem-
bers of $K$ that live on carrier sets equipotent to the carrier of $F$, and thus one
could in principle restrict the definition to the case where all members of $\{F\} \cup K$
have equipotent carriers. In fact, if $F' \in K$ has carrier $B$ and $\theta: B \to A$ is a
bijection with the carrier $A$ of $F$, then $F$ has automatic homeomorphicity / action compatibility with respect to $K$ if and only if this holds with respect to $(K \setminus \{F'\}) \cup \{F''\}$, where $F''$ on $A$ is (uniform homeomorphically) action isomorphic to $F'$ by conjugation with $\theta$. Similarly, all other members of $K$ that do not live on $A$ could be replaced by an action isomorphic copy on $A$, and one could do with classes $\{F\} \cup K$ all of whose members live on the same set (and then ‘on a set $B$ of the same cardinality as $A$’ could be dropped from the definition). We prefer the version above because it is more convenient to say ‘with respect to all self-embeddings monoids of relational structures without algebraicity’ than ‘with respect to all self-embeddings monoids of relational structures without algebraicity on carriers of size $\aleph_{42}$’.

One should note that the concepts given in Definition 2.2 give rise to reconstruction results in the $\aleph_0$-categorical setting. We explain this in the case of automorphism groups of relational structures; for polymorphism clones the situation is essentially the same—effectively in the following results ‘first-order’ has to be replaced by ‘primitive positive’—but the details are a bit more technical, see [13, section 2.1, p. 3708 et seq.] for references and further information and [13, p. 3714] for remarks on the monoid case. Given countable $\aleph_0$-categorical structures $\mathcal{A}$ and $\mathcal{B}$ where $\text{Aut}(\mathcal{A})$ has automatic homeomorphicity with respect to the class of (automorphism groups of) countable $\aleph_0$-categorical structures, the condition $\text{Aut}(\mathcal{A}) \cong \text{Aut}(\mathcal{B})$ implies that these permutation groups are isomorphic as topological groups, and thus by a theorem of Coquand (presented in [1, Corollary 1.4(ii), p. 67]), the structures $\mathcal{A}$ and $\mathcal{B}$ are first-order bi-interpretable. If on the other hand $\text{Aut}(\mathcal{A})$ has automatic action compatibility with respect to countable $\aleph_0$-categorical structures, then the same condition on the automorphism groups implies that these are even isomorphic as permutation groups, whence as a consequence of Ryll-Nardzewski’s Theorem, $\mathcal{A}$ and $\mathcal{B}$ are even first-order interdefinable (see also [31, Proposition 1.3, p. 226]).

Thus, automatic action compatibility entails a stronger reconstruction notion (up to first-order bidefinability from automorphism groups, up to primitive positive bidefinition from polymorphism clones [11, cf. Theorem 5.1, p. 365]) than automatic homeomorphicity. It is hence an important question, whether there are any good examples of countable $\aleph_0$-categorical relational structures having automatic action compatibility. Fortunately, there are two theorems that connect our concept to other well-studied properties, namely to Rubin’s weak $\forall\exists$-interpretations and to the strong small index property, which have been established for many known structures. The relevant results are the following:

**Theorem 2.3** ([31, Theorem 2.2, p. 228]). If $G = \text{Aut}(\mathcal{A})$ and $G' = \text{Aut}(\mathcal{B})$ are automorphism groups of countable $\aleph_0$-categorical structures without algebraicity and (the automorphism group of) $\mathcal{A}$ has a weak $\forall\exists$-interpretation, then every group isomorphism $\varphi: G \to G'$ is induced by conjugation.

**Theorem 2.4** ([26, Corollary 2]). If $G = \text{Aut}(\mathcal{A})$ and $G' = \text{Aut}(\mathcal{B})$ are automorphism groups of countable $\aleph_0$-categorical structures without algebraicity such that $G$ and $G'$ have the strong small index property, then every group isomorphism $\varphi: G \to G'$ is induced by conjugation.
Reformulated in terms of Definition 2.2, Rubin’s Theorem 2.3 says that the automorphism group of a countable \( \aleph_0 \)-categorical structure with a weak \( \forall \exists \)-interpretation and no algebraicity has automatic action compatibility with respect to all countable \( \aleph_0 \)-categorical structures without algebraicity. The more recent Theorem 2.4 by Paolini and Shelah states that the automorphism group of a countable \( \aleph_0 \)-categorical structure having SSIP and no algebraicity has automatic action compatibility with respect to the class of exactly these structures.

It is worth noting that the assumption of absence of algebraicity cannot easily be abandoned in Theorems 2.3 and 2.4. Otherwise one could take some countable \( \aleph_0 \)-categorical \( A \) with countable signature, SSIP and transitive automorphism group, e.g. \( A = (\mathbb{Q},<) \), and construct an extension \( B \) by adding a new element \( b \) to the carrier and \( \{b\} \) as a unary relation so that \( b \) becomes a fixed point of \( \text{Aut}(B) \). This construction preserves \( \aleph_0 \)-categoricity and SSIP, and although \( \text{Aut}(A) \) and \( \text{Aut}(B) \) would be isomorphic, they would not be action isomorphic for \( \text{Aut}(A) \) is transitive, but \( \text{Aut}(B) \) fixes \( b \).

The two theorems above will be our starting point to lift automatic action compatibility from permutation groups to monoids and finally to polymorphism clones of structures with certain additional properties in section 4. Concrete examples where our approach applies are given in section 5.

### 3 Characterizing automatic homeomorphicity

In the following we are going to give a characterization of automatic homeomorphicity for transformation monoids in the situation when the group of invertible elements lies dense. The importance of this particular case for applications has been outlined in [13, section 3.7, p. 3716]. For this we develop an improved version of [13, Proposition 11, p. 3720], which does not require the assumption of a countable carrier set. We thereby also eliminate any metric reasoning used to obtain it, which ultimately seems to be an artifice going back to an idea proposed by Lascar in [22, p. 31]. Even though the product topology (Tichonov topology) on \( A^A \) over a discrete space \( A \) is only metrizable if \( A \) is countable, a uniform argument can achieve the same conclusion without restriction to the countable case. In particular such a result can be obtained just applying interpolation on finite sets without mentioning completions of metric spaces via equivalence classes of Cauchy sequences.

The first step is a non-metric variant of [27, Lemma 4.2, p. 142].

**Lemma 3.1.** Let \( M \subseteq O^{(1)}_A \) and \( M' \subseteq O^{(1)}_B \) be transformation monoids such that \( M \) has a dense subset \( G \) of invertibles and let \( \xi: M \to M' \) be a continuous monoid homomorphism. Then \( \xi \) is uniformly continuous.

**Proof.** Consider any finite subset \( D \subseteq B \) determining the basic entourage \( \{ (h_1, h_2) \in M'^2 \mid h_1|_D = h_2|_D \} \). The set \( \{ h \in M' \mid h|_D = \text{id}_B|_D \} \) is a basic open neighbourhood of \( \text{id}_B \). For \( \xi \) is continuous, the preimage is an open neighbourhood of \( \text{id}_A \), and thus contains a basic open neighbourhood of \( \text{id}_A \) given by

\[ \{ h \in M' \mid h|_D = \text{id}_B|_D \} \]
some finite subset \( C \subseteq A \):

\[
\{ g \in M \mid g|_C = \text{id}_A|_C \} \subseteq \{ g \in M \mid \xi(g)|_D = \text{id}_B|_D \}.
\]

We claim

\[
(\xi \times \xi)[\{ (f_1, f_2) \in M^2 \mid f_1|_C = f_2|_C \}] \subseteq \{ (h_1, h_2) \in M^2 \mid h_1|_D = h_2|_D \},
\]

proving uniform continuity. Namely, if \( f_1, f_2 \in M \) satisfy \( f_1|_C = f_2|_C \), then by density of \( G \) the basic open neighbourhood \( \{ f \in M \mid f|_C = f|_C \} \) of \( f \) contains some invertible \( g \in G \). Hence \( g|_C = f_1|_C = f_2|_C \), wherefore we have \((g^{-1} \circ f_i)|_C = \text{id}_A|_C\) for both \( i \in \{1, 2\} \). By the above, we infer

\[
(\xi(g)^{-1} \circ \xi(f_i))|_D = \xi(g^{-1} \circ f_i)|_D = \text{id}_B|_D,
\]

or \( \xi(g)|_D = \xi(f_i)|_D \) for \( i \in \{1, 2\} \). Hence \( \xi(f_1)|_D = \xi(f_2)|_D \). \( \qed \)

Now we give the previously advertised strengthened version of [13, Proposition 11] having no restrictions on the carrier sets.

**Proposition 3.2.** Let \( M \subseteq \mathcal{O}^{(1)}_A \) and \( M' \subseteq \mathcal{O}^{(1)}_B \) be transformation monoids having subsets \( G \subseteq M \) and \( G' \subseteq M' \) of invertibles such that \( M \subseteq \mathcal{C} \) and \( M' \) is closed. For any continuous monoid homomorphism \( \xi : G \to G' \) there is a uniformly continuous monoid homomorphism \( \overline{\xi} : M \to M' \) extending \( \xi \). If \( M \) and \( M' \) are closed, \( G \) and \( G' \) are dense and \( \xi \) is a homeomorphic isomorphism, then \( \overline{\xi} \) is a uniform homeomorphism and a monoid isomorphism.

**Proof.** To define \( \overline{\xi} : M \to M' \) let \( f \in M \) and \( b \in B \) be given. Let \( D \subseteq B \) be any finite set such that \( b \in D \). For convenience, it can always be chosen as \( \{b\} \), but we are not going to assume this. The set \( \beta_D := \{ (h_1, h_2) \in G^2 \mid h_1|_D = h_2|_D \} \) is a basic entourage, given by \( D \). By Lemma 3.1 we know that \( \xi \) is uniformly continuous, hence there is an entourage \( \alpha \) such that \( (\xi \times \xi)[\alpha] \subseteq \beta_D \). Since every entourage contains a basic entourage there is some finite set \( C \subseteq A \) and a basic entourage \( \alpha_C = \{ (g_1, g_2) \in G^2 \mid g_1|_C = g_2|_C \} \subseteq \alpha \), which is hence mapped to \( \beta_D \). Let \( \tilde{C} \supseteq C \) be any possibly larger finite subset of \( A \). As \( G \) is dense in \( M \), there is some group element \( g \in G \) such that \( g|_{\tilde{C}} = f|_{\tilde{C}} \). We put \( \overline{\xi}(f)|_D := \xi(g)|_D \). Since \( b \in D \), we have thus defined \( \overline{\xi}(f)(b) \). Careful inspection shows that this is indeed well defined: namely, if for \( i \in \{1, 2\} \) we have finite subsets \( D_i \subseteq B \) with \( b \in D_i \), entourages \( \alpha_i \) with \( (\xi \times \xi)[\alpha_i] \subseteq \beta_{D_i} \), and finite subsets \( C_i \subseteq \tilde{C}_i \) of \( A \) with \( \alpha_{C_i} \subseteq \alpha_i \) and invertibles \( g_i \in G \) with \( f|_{C_i} = g_i|_{C_i} \), then for \( D = D_1 \cap D_2 \) we need to verify that \( \xi(g_1)|_D = \xi(g_2)|_D \). We put \( C := C_1 \cup C_2 \) and, by density of \( G \) in \( M \), we pick some \( g \in G \) such that \( f|_C = g|_C \). Since \( C \subseteq \tilde{C}_i \), it follows that \( g_i|_{C_i} = f|_{C_i} \). Likewise, we get \( g|_{C_i} = f|_{C_i} \) because \( C_i \subseteq C \). Consequently, \( g|_{C_i} = g|_{C_i} \), so \( (g_i, g) \in \alpha_{C_i} \subseteq \alpha_i \) and thus, \( (\xi(g_i), \xi(g)) \in (\xi \times \xi)[\alpha_i] \subseteq \beta_{D_i} \subseteq \beta_D \). This means \( \xi(g_1)|_D = \xi(g_2)|_D \), which is independent of the index \( i \). Hence we have \( \xi(g_1)|_D = \xi(g_2)|_D \), so that \( \xi(g_1) \) and \( \xi(g_2) \) really give non-contradictory values for \( \overline{\xi}(f) \) on \( D \). Note also
that, by definition, on any finite $D \subseteq B$, the function $\xi(f)$ coincides with $\xi(g)$ for some $g \in G$, so with some member of $G'$. Therefore, $\xi(f) \in \overline{G} \subseteq M' = M'$.

If $g \in G$ and $b \in B$, then taking $D = \{b\}$ and some finite subset $C \subseteq A$ with $(\xi \times \xi)[\alpha_C] \subseteq \beta(b)$ we are free to choose $g \in G$ as its own interpolant on $C$, to define $\overline{\xi(g)} \mid \{b\}$ via $\xi(g) \mid \{b\}$. This means $\overline{\xi(g)} = \xi(g)$ for any $g \in G$, so $\overline{\xi}$ extends $\xi$.

To prove that $\overline{\xi} : M \rightarrow M'$ is a homomorphism we consider $f_1, f_2 \in M$ and any $b \in B$. We want to show that $\overline{\xi}(f_2 \circ f_1)(b) = \overline{\xi}(f_2)(\overline{\xi}(f_1)(b))$ where $b' := \overline{\xi}(f_1)(b)$. According to the definition, we choose finite subsets $C, C' \subseteq A$ such that $(\xi \times \xi)[\alpha_C] \subseteq \beta(b)$ and $(\xi \times \xi)[\alpha_{C'}] \subseteq \beta(b')$. Moreover, for $C'_v = C_v \cup f_1(C_b)$ we take any $g_1, g_2 \in G$ such that $g_1 \mid C_b = f_1 \mid C_v$ and $g_2 \mid C'_v = f_2 \mid C'_v$. It follows that

$$\overline{\xi}(f_2)(b') = \xi(g_2)(b') = \xi(g_2)(\overline{\xi}(f_1)(b)) = \xi(g_2) \circ \xi(g_1)(b) = \xi(g_2 \circ g_1)(b) = \overline{\xi}(f_2 \circ f_1)(b)$$

since for every $x \in C_b$ we have $f_2(f_1(x)) = g_2(f_1(x)) = g_2(g_1(x))$ by the construction of $C'_v$ as a superset of $f_1(C_b)$.

To see that $\overline{\xi}$ is uniformly continuous, consider any finite set $D \subseteq B$ and the basic entourage $\beta_D$ given by it. According to the definition of $\overline{\xi}$ we choose a finite subset $C \subseteq A$ such that $(\xi \times \xi)[\alpha_C] \subseteq \beta_D$. Now if $f_1, f_2 \in M$ fulfill $f_1 \mid C = f_2 \mid C$, then we can use any $g \in G$ satisfying $f_1 \mid C = g \mid C = f_2 \mid C$ to define $\overline{\xi}(f_1)$ and $\overline{\xi}(f_2)$ on $D$, vide licet $\overline{\xi}(f_1)(b) = \xi(g)(b) = \overline{\xi}(f_2)(b)$ for every $b \in D$. This proves $\overline{\xi}(f_1) \mid D = \overline{\xi}(f_2) \mid D$ and hence uniform continuity.

Finally, we discuss the situation where $\xi$ is a bijection with inverse $\xi^{-1}$. Putting $f = \xi^{-1}(f')$ we have to check that $\overline{\xi}(f) = f'$ for every $f' \in M'$. Consider any $b \in B$ and pick a finite subset $C_b \subseteq A$ such that $(\xi \times \xi)[\alpha_{C_b}] \subseteq \beta(b)$. The value $\overline{\xi}(f)(b)$ is determined by any $G$-interpolant for $f = \xi^{-1}(f')$ on $C_b$; we need to choose this interpolant in a special way. We find a finite subset $D \subseteq B$ such that for $g_1, g_2 \in G'$ with $g_1 \mid D = g_2 \mid D$ we have $\xi^{-1}(g_1) \mid C_b = \xi^{-1}(g_2) \mid C_b$. We put $D = D \cup \{b\}$ and we take any $g' \in G'$ satisfying $g' \mid D = f' \mid D$. By the definition of $\xi^{-1}$, we obtain $f \mid C_b = \xi^{-1}(f')(\xi^{-1}(g')) \mid C_b = \xi^{-1}(g')(\xi^{-1}(f')) \mid C_b$, so $\xi^{-1}(g') \in G$ is a suitable interpolant for $f$ on $C_b$. Using the definition of $\overline{\xi}(f)$, we can infer $\overline{\xi}(f)(b) = \xi(\xi^{-1}(g'))(b) = g'(b) = f'(b)$, where the last equality holds since $b \in D$. This shows $\overline{\xi}(\xi^{-1}(f'))(b) = f'(b)$, and as we have symmetric assumptions, we can obtain $\xi^{-1}(\overline{\xi}(f)) = f$ for $f \in M$ by a dual argument. □

Note that to understand how the extension $\overline{\xi}$ works, it is sufficient to fix for each $b \in B$ one finite set $C_b \subseteq A$ such that $(\xi \times \xi)[\alpha_{C_b}] \subseteq \beta(b)$ (using the continuity of $\xi$ at $\mathrm{id}_A$, see Lemma 3.1). This information can be 'precomputed'. To see what $\xi(f)$ for some $f \in M$ does at $b \in B$, one then simply has to find an interpolant $g \in G$ of $f$ on $C_b$ and to observe, how $\xi(g)$ acts on $b$.

Next we prove the mentioned characterization of automatic homeomorphicity, which is closely related to the sufficient condition given in [13, Lemma 12, p. 3720]. Our criterion is again independent of the size of the underlying set and we shall see afterwards how to derive the analogue of Lemma 12 of [13] from it.
Proposition 3.3. Let $\mathcal{K}$ be a class of closed monoids and $\mathcal{G} = \{G(M) \mid M \in \mathcal{K}\}$ be the corresponding class of groups $G(M)$ of invertibles of monoids $M \in \mathcal{K}$. Moreover, we assume that $\{G \mid G \in \mathcal{G}\} \subseteq \mathcal{K}$, where $G$ denotes the closure in the full transformation monoid. Supposing that $M \subseteq O^{(1)}_A$ is a closed monoid, its group $G = G(M)$ of invertibles is dense in $M$ and has automatic homeomorphicity w.r.t. $\mathcal{G}$, the following facts are equivalent.

(a) For any set $B$ of the same cardinality as $A$ and any monoid $M' \subseteq O^{(1)}_B$ satisfying $M' \in \mathcal{K}$ and every monoid isomorphism $\theta : M \to M'$ it follows that $\theta[G]$ is dense in $M'$ and $\theta$ is a uniform homeomorphism.

(b) $M$ has automatic homeomorphicity with respect to $\mathcal{K}$.

(c) For any set $B$ of the same cardinality as $A$ and any closed monoid $M' \subseteq O^{(1)}_B$ satisfying $M' \in \mathcal{K}$ and every monoid isomorphism $\theta : M \to M'$ it follows that $\theta$ is continuous.

(d) For any set $B$ of the same cardinality as $A$ and transformation monoids $M_1 \subseteq M_2 \subseteq O^{(1)}_B$ such that $M_1, M_2 \in \mathcal{K}$ and for any monoid isomorphisms $\varphi_i : M \to M_i$ for $i \in \{1, 2\}$ the following implication holds:

$$
\varphi_1|_G = \varphi_2|_G \implies M_1 = M_2 \land \varphi_1 = \varphi_2.
$$

Note that every $M \in \mathcal{K}$ can be understood as an endomorphism monoid of some relational structure $A$; $G(M)$ is then the automorphism group of that structure and thus a closed permutation group. So the requirement that $G$ have automatic homeomorphicity w.r.t. $\mathcal{G}$ is reasonable. Moreover, $G(M)$ is always a submonoid of the monoid of self-embeddings of $A$, and if $A$ is homogeneous, both monoids are actually equal (see Lemma 2.1(c)). Thus, if $\mathcal{K}$ is the collection of all endomorphism monoids of a given class of homogeneous relational structures, then the condition that $\{G \mid G \in \mathcal{G}\} \subseteq \mathcal{K}$ means that for each of these structures, $\mathcal{K}$ also contains the corresponding monoid of self-embeddings. Furthermore, if $\mathcal{K}$ consists of the endomorphism monoids of all possible relational structures on sets of a certain size, i.e., up to isomorphism, $\mathcal{K}$ is the collection of all closed transformation monoids on a fixed set, then the assumption $\{G \mid G \in \mathcal{G}\} \subseteq \mathcal{K}$ is always satisfied (cf. Lemma 2.1(d)).

Proof of Proposition 3.3. As '(a) $\implies$ (b) $\implies$ (c)' is evident we start with the implication 'c $\implies$ (d)'. Assume condition (c) and let $M_1 \subseteq M_2 \subseteq O^{(1)}_B$ be transformation monoids on a set $B$ of the same cardinality as $A$ such that $M_1, M_2 \in \mathcal{K}$, so $M_1, M_2$ are closed. Suppose we have monoid isomorphisms $\varphi_i : M \to M_i$ that coincide on $G$. It follows that $\varphi_i$ is continuous. Now consider any $f \in M$ and any $b \in B$. The set $\{h' \in M_i \mid h'(b) = \varphi_i(f)(b)\}$ is a basic open neighbourhood of $\varphi_i(f)$, so, as $\varphi_i$ is continuous, its preimage $\{h \in M \mid \varphi_i(h)(b) = \varphi_i(f)(b)\}$ is an open neighbourhood of $f$ and hence contains a basic open neighbourhood.

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of $f$. Thus, there is a finite set $X_i \subseteq A$ such that for any finite $Y \subseteq A$ with $X_i \subseteq Y$ we have

$$f \in V_Y = \{ h \in M \mid h|_Y = f|_Y \} \subseteq \{ h \in M \mid \varphi_i(h)(b) = \varphi_i(f)(b) \}.$$ 

Now letting $Y = X_1 \cup X_2$ we have this inclusion for both $i \in \{1,2\}$. By density of $G$ in $M$, the basic open neighbourhood $V_Y$ of $f$ has non-empty intersection with $G$, so there is some $g \in G$ such that $g|_Y = f|_Y$ and consequently $\varphi_1(f)(b) = \varphi_1(g)(b) = \varphi_2(g)(b) = \varphi_2(f)(b)$, where the middle equality follows from $\varphi_1|G = \varphi_2|G$. This is true for any $b \in B$, so $\varphi_1(f) = \varphi_2(f)$; however, also $f \in M$ was arbitrary and $\varphi_1$ and $\varphi_2$ were surjective, so $M_1 = M_2$ and $\varphi_1 = \varphi_2$.

For the converse ‘(d) \Rightarrow (a)’ we assume condition (d) and consider any monoid isomorphism $\theta: M \to M'$ where $M' \in \mathcal{K}$ is a closed transformation monoid on $B$, $|B| = |A|$. Let $G' \subseteq M'$ be the subset of invertibles of $M'$. Certainly, $G' \in \mathcal{G}$. Clearly, $G'$ is dense in its closure $\overline{G'}$, which belongs to $\mathcal{K}$ by the assumption on $\mathcal{K}$. Indeed, $G'$ is also the set of invertibles of $\overline{G'}$. Since $M'$ is closed, we have $\overline{G'} \subseteq \overline{M'} = M'$. Moreover, as $\theta$ is an isomorphism, $\theta[G'] \subseteq G'$, and likewise $\theta^{-1}[G''] \subseteq G$, so $\theta[G] = G'$. Therefore, $\theta|G_G: G \to G'$ is a well-defined monoid isomorphism; moreover, as $G$ and $G'$ are group reducts, $\theta|G_G$ actually is a group isomorphism onto a group in $\mathcal{G}$. For $G$ has automatic homeomorphism w.r.t. $\mathcal{G}$, the isomorphism $\theta|G_G$ is a homeomorphism. By Proposition 3.2, there is an extension $\xi: M \to \overline{G'}$ of $\theta|G_G$, which is a monoid isomorphism and a uniform homeomorphism. Since $\xi$ and $\theta$ coincide on $G$, assumption (d) entails $\theta[G] = \overline{G'} = M'$ and $\theta = \xi$. In particular, $\theta$ is a uniform homeomorphism. 

**Remark 3.4.** Later it will be useful to observe that the assumptions that $G$ have automatic homeomorphism w.r.t. $\mathcal{G}$ and that $\{ \overline{H} \mid H \in \mathcal{G} \} \subseteq \mathcal{K}$ were only needed to prove the implication ‘(d) \Rightarrow (a)’; The ‘forward’ implications ‘(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)’ hold even without these preconditions, so for them the class $\mathcal{G}$ does not play any role.

With the help of the following lemma, we can reobtain [13, Lemma 12].

**Lemma 3.5.** Suppose that $M \subseteq O_A^{(1)}$ and $M_1 \subseteq M_2 \subseteq O_B^{(1)}$ are transformation monoids on sets $A$ and $B$, respectively, and let $G \subseteq M$ be any subset such that $E_G := \{ \psi \in \text{End}(M) \mid \psi \text{ injective} \land \psi|_G = \text{id}_M|_G \} = \{ \text{id}_M \}$, that is, the only injective monoid endomorphism of $M$ fixing $G$ pointwise is the identity. If $\varphi: M \to M_2$ is a monoid isomorphism and $\xi: M \to M_1$ is an injective monoid homomorphism such that $\xi|G = \varphi|_G$, then $\xi[M] = M_1 = M_2$ and $\xi = \varphi$.

Moreover, for a monoid $M \subseteq O_A^{(1)}$, a set $G \subseteq M$ and any class $\mathcal{K}$ of monoids such that $\mathcal{K} \supseteq \{ \psi[M] \mid \psi \in E_G \}$, condition (d) of Proposition 3.3 is equivalent to $E_G = \{ \text{id}_M \}$.

**Proof.** Applying the identical monoid embedding $\iota: M_1 \to M_2$, it follows that $\psi := \varphi^{-1} \circ \iota \circ \xi$ is an injective monoid endomorphism fixing every $g \in G$ since $\psi(g) = \varphi^{-1}(\xi(g)) = \varphi^{-1}(\varphi(g)) = g$. Thus, $\psi = \text{id}_M$, and hence $\varphi = \varphi \circ \psi = \iota \circ \xi$. 

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This means $M_2 = \varphi[M] = \xi[M] \subseteq M_1 \subseteq M_2$, giving $\xi[M] = M_1 = M_2$ and $\xi = \varphi$.

The fact just shown clearly entails part (d) of Proposition 3.3 (for any class of monoids); therefore, the latter is necessary given $E_G = \{\text{id}_M\}$. Conversely, knowing that $K \supseteq \{\psi[M] \mid \psi \in E_G\}$, for any $\psi \in E_G$, we let $M_1 = \psi[M]$, $M_2 = M$, $\psi_1 = \psi|_M^M$, and $\psi_2 = \text{id}_M$ in Proposition 3.3(d). We then conclude that $\psi[M] = M$, so $\psi$ is an isomorphism, and $\psi = \psi_1 = \psi_2 = \text{id}_M$. \hfill $\Box$

Lemma 12 of [13] is the special case of the following result where $A$ is countably infinite.

**Corollary 3.6.** Supposing that $M \subseteq O_A^{(1)}$ is a closed monoid, its group $G \subseteq M$ of invertibles is dense in $M$ and has automatic homeomorphicity and every injective monoid endomorphism $\theta: M \to M$ with $\theta(g) = g$ for $g \in G$ is the identity, then $M$ has automatic homeomorphicity.

**Proof.** Choosing $K$ as the class of all closed transformation monoids on carriers of the same cardinality as $A$, Lemma 3.5 demonstrates condition (d) of Proposition 3.3, so the claim follows from statement (b) of the same result. \hfill $\Box$

**Remark 3.7.** Let $K$, $G$, $M$ and $G \subseteq M$ be as in Proposition 3.3 and assume $K \supseteq \{\psi[M] \mid \psi \in E_G\}$ for $E_G = \{\psi \in \text{End}(M) \mid \psi \text{ injective} \land \psi|_G = \text{id}_M|_G\}$. It follows from Proposition 3.3 and Lemma 3.5 that $E_G = \{\text{id}_M\}$ is actually equivalent to $M$ having automatic homeomorphicity with respect to $K$.

Note that under the assumptions of Corollary 3.6 it does not automatically follow that the implication of the corollary is an equivalence. This is because the endomorphisms $\psi \in E_G$ are not necessarily closed maps, and therefore their images do not necessarily belong to the class $K$ of closed transformation monoids over sets equipotent with the carrier of $M$. This is the same type of complication that has made additional arguments necessary in proving automatic homeomorphicity of $\text{End}(\mathbb{Q}, \leq)$, see [4, Lemma 4.1, p. 79] and the discussion next to it.

### 4 Stronger reconstruction for monoids and clones

In this section we will show how to lift automatic action compatibility from permutation groups first to endomorphism monoids and then to clones. At our point of departure, we recall that Rubin in [31, Theorem 2.2, p. 228] shows that any $\aleph_0$-categorical structure without algebraicity is *group categorical* with respect to the class of all such structures, provided it has weak $\forall \exists$-interpretations. The exact meaning of this is stated in Theorem 2.3, saying that for every isomorphism $\varphi$: $\text{Aut}(A) \to \text{Aut}(B)$ between the automorphism groups of countable $\aleph_0$-categorical structures without algebraicity where $A$ has a weak $\forall \exists$-interpretation there is a bijection $\theta$: $A \to B$ between the carrier sets $A$ and $B$ of $A$ and $B$, respectively, such that $\varphi(g) = \theta \circ g \circ \theta^{-1}$ for all $g \in \text{Aut}(A)$.

On the other hand from Proposition 3.2, we have that for transformation monoids $M \subseteq O_A^{(1)}$ and $M' \subseteq O_B^{(1)}$ with dense groups of invertibles $G$ and $G'$.
on sets $A$ and $B$, any topological isomorphism $\xi : G \to G'$ can be extended to a uniform homeomorphism and isomorphism $\overline{\xi} : M \to M'$. With the additional knowledge about $\xi$ obtained from results such as Theorem 2.3 we can describe even more precisely how $\overline{\xi}(f)$ for $f \in M$ will look like.

Lemma 4.1. Assume that $M \subseteq O_1^A$ and $M' \subseteq O_1^B$ are closed transformation monoids on carrier sets $A$ and $B$ with subsets $G \subseteq M$ and $G' \subseteq M'$ of invertible elements such that $G$ is dense in $M$. Moreover, let $\varphi : M \to M'$ be a monoid isomorphism. If

(i) for any isomorphism $\psi : M \to \overline{G'}$ onto the closure of $G'$ in $O_1^B$ the condition $\psi|_G = \varphi|_G$ implies $\psi(f) = \varphi(f)$ for all $f \in M$, and

(ii) the isomorphism $\xi = \varphi|_G' : G \to G'$ is induced by conjugation by some bijection $\theta : A \to B$,

then the latter fact extends to the monoid isomorphism $\varphi$; in fact, $\varphi$ is induced by conjugation by the same $\theta : A \to B$.

Observe that with the help of Lemma 3.5 the first condition of the preceding lemma can be replaced by the stronger assumption that every injective monoid endomorphism $\psi : M \to M$ fixing $G$ pointwise is the identity. Moreover, it is clear from the second condition that the sets $A$ and $B$ necessarily have to be equipotent.

Proof. Invertibility is preserved under monoid isomorphisms, so $\varphi[G] = G'$, and thus the restriction $\xi : G \to G'$ is a well-defined monoid isomorphism. By the second assumption it is a uniform homeomorphism. The set of invertibles of $\overline{G'}$ is again $G'$, and it is dense in the closed monoid $\overline{G'}$. Thus, by Proposition 3.2, the isomorphism $\xi$ extends to a monoid isomorphism $\overline{\xi} : M \to \overline{G'}$ that also is a uniform homeomorphism. Using the first assumption of the lemma, we infer that $\varphi = \overline{\xi}$ and $\overline{G'} = M'$ since $\varphi$ and $\overline{\xi}$ are surjective. Now, by our second assumption we can find some bijection $\theta : A \to B$ inducing $\xi$. It only remains to lift this condition from $\xi$ to $\overline{\xi}$ and thus to $\varphi$. This can be done by only relying on the continuity of $\overline{\xi}$, but it is shorter to use the explicit description of $\overline{\xi}$ from the proof of Proposition 3.2.

Let $f \in M$ and $b \in B$. According to the construction given by Proposition 3.2, we take some finite set $C_b \subseteq A$ such that $g_1|_{C_b} = g_2|_{C_b}$ implies $\xi(g_1)(b) = \xi(g_2)(b)$ for any $g_1, g_2 \in G$. We put $\tilde{C} = C_b \cup \{\theta^{-1}(b)\}$ and consider any $g \in G$ such that $f|_{\tilde{C}} = g|_{\tilde{C}}$. Then we observe

$$\varphi(f)(b) = \overline{\xi}(f)(b) = \xi(g)(b) = (\theta \circ g \circ \theta^{-1})(b) = \theta(g(\theta^{-1}(b))) = \theta(f(\theta^{-1}(b))),$$

where the second equality holds by definition of $\overline{\xi}$, the third one by the assumption on $\xi$ and the last one since $\theta^{-1}(b) \in \tilde{C}$. This implies $\varphi(f) = \theta \circ f \circ \theta^{-1}$. □

With the preceding lifting lemma we can transfer automatic action compatibility from dense groups to monoids.
Corollary 4.2. Let $\mathcal{K}$ be a class of closed monoids and $\mathcal{G} = \{ G(M) \mid M \in \mathcal{K} \}$ be the corresponding class of groups $G(M)$ of invertibles of monoids $M \in \mathcal{K}$. Moreover, we assume that $\{ G \mid G \in \mathcal{G} \} \subseteq \mathcal{K}$, where $G$ denotes the closure in the full transformation monoid. Supposing that $M \subseteq O_A^{(1)}$ is a closed monoid with automatic homeomorphicity w.r.t. $\mathcal{K}$ and that its group $G = G(M)$ of invertibles is dense in $M$, then, provided $G$ has automatic action compatibility w.r.t. $\mathcal{G}$, also $M$ has automatic action compatibility w.r.t. $\mathcal{K}$.

It is possible to marginally weaken the assumption that $M$ have automatic homeomorphicity w.r.t. $\mathcal{K}$ because, in the proof, condition (i) of Lemma 4.1 will instantiate the universally quantified implication in statement (d) from Proposition 3.3 only for specific pairs $M_1 \subseteq M_2$ from $\mathcal{K}$. However the modification would be rather technical and probably have only very few applications.

Proof of Corollary 4.2. Given any $M' \in \mathcal{K}$ on some set $B$ that is equipotent with $A$ and any monoid isomorphism $\varphi: M \to M'$ we shall use Lemma 4.1 to prove that $\varphi$ is induced by conjugation. By the choice of $M$ and $\mathcal{K}$ both monoids are closed and $G \subseteq M$ is dense. Also, since $G' = G(M') \in \mathcal{G}$ and $G$ has automatic action compatibility w.r.t. $\mathcal{G}$, the isomorphism $\varphi[G']$ is induced by conjugation; so condition (ii) of Lemma 4.1 is satisfied. Condition (i) now follows from Proposition 3.3(d) since $\{ G \mid G \in \mathcal{G} \} \subseteq \mathcal{K}$, $M$ has automatic homeomorphicity w.r.t. $\mathcal{K}$ and $G$ has automatic homeomorphicity w.r.t. $\mathcal{G}$ for it even has automatic action compatibility w.r.t. $\mathcal{G}$. \hfill $\square$ 

Proving, for certain clones, automatic action compatibility w.r.t. all $\aleph_0$-categorical structures without algebraicity will be based on the next theorem. The technique to prove it is inspired by [32] and uses the assumption of being weakly directed (this notion has appeared in [32, 10.1, p. 60] as semi-transitive, but the latter term has been introduced in [30] to designate a different semigroup property recurring in a number of articles, e.g. [5, 6, 7, 8, 17, 18]). Hence, we say that an action of a semigroup $S$ on a set $A$ is weakly directed if for all $a, b \in A$ there are $f, g \in S$ and $c \in A$ such that $(f, c) \mapsto a$ and $(g, c) \mapsto b$. We call a transformation semigroup weakly directed if its action by evaluation at points of the underlying set has this property. Certainly every transitive action is weakly directed. Moreover, a straightforward inductive argument shows that a weakly directed action of $S$ on a non-empty set $A$ for every $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$ allows for finding $f_1, \ldots, f_n \in S$ and $a_0 \in A$ such that $(f_1, a_0) \mapsto a_i$ holds for all $1 \leq i \leq n$.

Theorem 4.3. Let $F \subseteq O_A$ and $F' \subseteq O_B$ be clones on carrier sets $A$, $B$ such that $F^{(1)}$ is weakly directed and let $\xi: F \to F'$ be a surjective clone-homomorphism such that the restriction $\xi^{(n)}_{F^{(1)}}: F^{(1)} \to F'^{(n)}$ is given by conjugation with some bijection $\theta: A \to B$, i.e.,

$$ \forall f \in F^{(1)}: \quad \xi(f) = \theta \circ f \circ \theta^{-1}, $$

then $\xi$ is induced by conjugation by the same $\theta$, so

$$ \forall n \in \mathbb{N} \forall h \in F'^{(n)}: \quad \xi(h) = \theta \circ h \circ (\theta^{-1} \times \cdots \times \theta^{-1}). $$
In particular, $\xi$ is a clone isomorphism and a uniform homeomorphism.

Note that the following proof also works for nullary operations, which only exist on non-empty carrier sets. Moreover, there is absolutely no restriction on the cardinality of the carriers $A$ and $B$ here.

**Proof.** If $A = \emptyset$, the assumed bijection $\theta : A \to B$ ensures that $B = \emptyset$ and thus $F = F' = O_{\emptyset}$, which contains only projections. So the claim is trivially true.

Now let $A \neq \emptyset$, let $n \in \mathbb{N}$, $h \in F^{(n)}$, and consider any $y_{1}, y_{2}, \ldots, y_{n} \in B$. We put $a_{i} = \theta^{-1}(y_{i})$ for all $1 \leq i \leq n$. Since $F^{(1)}$ is weakly directed, there is some $a \in A$ and $g_{1}, \ldots, g_{n} \in F^{(1)}$ such that $g_{i}(a) = a_{i}$ for all $1 \leq i \leq n$.

Consider $f := h \circ (g_{1}, \ldots, g_{n}) \in F^{(1)}$. By the assumption on $\xi|_{F^{(1)}}$, we have

$$\xi(f) = \theta \circ f \circ \theta^{-1} = \theta \circ h \circ (g_{1}, \ldots, g_{n}) \circ \theta^{-1},$$

on the other hand, since $\xi$ is compatible with $\circ$, we have

$$\xi(f) = \xi(h \circ (g_{1}, \ldots, g_{n})) = \xi(h) \circ (\xi(g_{1}), \ldots, \xi(g_{n})),
$$

and again because of the assumption on $\xi|_{F^{(1)}}$, we have

$$\xi(f) = \xi(h) \circ (\xi(g_{1}), \ldots, \xi(g_{n})) = \xi(h) \circ (\theta \circ g_{1} \circ \theta^{-1}, \ldots, \theta \circ g_{n} \circ \theta^{-1}).$$

Finally, we evaluate $\xi(f)$ at $y := \theta(a)$:

$$\xi(f)(y) = \theta(h(g_{1}(\theta^{-1}(y)), \ldots, g_{n}(\theta^{-1}(y)))) = \theta(h(g_{1}(a), \ldots, g_{n}(a))) = \theta(h(a_{1}, \ldots, a_{n})) = \theta(h(\theta^{-1}(y_{1}), \ldots, \theta^{-1}(y_{n})))
$$

and

$$\xi(f)(y) = \xi(h)(\theta(g_{1}(\theta^{-1}(y))), \ldots, \theta(g_{n}(\theta^{-1}(y)))) = \xi(h)(\theta(g_{1}(a)), \ldots, \theta(g_{n}(a))) = \xi(h)(\theta(a_{1}), \ldots, \theta(a_{n})) = \xi(h)(y_{1}, \ldots, y_{n}).$$

Because $\xi$ is surjective, it is an isomorphism and a uniform homeomorphism. \qed

**Remark 4.4.** If $F \subseteq O_{A}$ is a closed clone where $F^{(1)}$ is weakly directed and $F' \subseteq O_{B}$ is a clone that fails to be closed, then there is no surjective clone homomorphism $\xi : F \to F'$ whose restriction $\xi|_{F^{(1)}}$ to the monoid parts is induced by some bijection $\theta : A \to B$. Otherwise, Theorem 4.3 would imply that $\xi$ is a homeomorphism and hence, $F' = \xi[F]$ would have to be closed.

**Theorem 4.5.** Let $K$ and $C$ be classes of transformation monoids and clones, respectively, such that $\{ F^{(1)} \mid F' \in C \} \subseteq K$. Moreover, let $F \subseteq O_{A}$ be a clone with a weakly directed monoid $F^{(1)}$ having automatic action compatibility w.r.t. $K$. Then $F$ has automatic action compatibility (and thus automatic homeomorphicity) w.r.t. the class $C$.
Note that this theorem only makes, albeit strong, assumptions on the monoid part $F^{(1)}$ of the clone under consideration. Furthermore, no closedness requirements are made. They may, however, be necessary to provide the preconditions of the theorem for concrete instances.

**Proof.** Let $\xi: F \to F'$ be a clone isomorphism where $F' \in \mathcal{C}$ is a clone on a set $B$ of the same size as $A$. By assumption, $\xi|_{F^{(1)}}: F^{(1)} \to F'^{(1)}$ is a monoid isomorphism onto $F'^{(1)} \in \mathcal{K}$. Thus, it is given by conjugation by some bijection $\theta: A \to B$, which of course may depend on $\xi$ and $F'^{(1)}$. Applying Theorem 4.3, $\xi$ is induced by conjugation by $\theta$, as well, and $\xi$ is a uniform homeomorphism. □

## 5 Automatic action compatibility for concrete structures

We begin with a convenience result summarizing a set of assumptions that allows to combine all the previous results in a smooth manner. Other ways to put Theorem 4.3 to work (with different assumptions) are certainly conceivable.

**Corollary 5.1.** Let $\mathcal{K}$ be the class of all endomorphism monoids of countable $\aleph_0$-categorical structures without algebraicity, and let $\mathcal{K}$ be such a structure. Let $M$ be a closed transformation monoid on the carrier set of $\mathfrak{A}$, for instance, $M = \text{End}(\mathfrak{A})$ or $M = \text{Emb}(\mathfrak{A})$. If

1. $\mathfrak{A}$ has a weak $\forall \exists$-interpretation,
2. $\text{Aut}(\mathfrak{A})$ is dense in $M$ and coincides with the group of invertible elements $\{g \in M \mid \exists f \in M: f \circ g = g \circ f = \text{id}_A\}$,
3. $M$ is weakly directed, e.g. transitive, and
4. every injective monoid endomorphism of $M$ that fixes $\text{Aut}(\mathfrak{A})$ pointwise is the identity, or

   $M$ has automatic homeomorphicity w.r.t. a class $\mathcal{L} \supseteq \mathcal{K}$ of closed transformation monoids such that $G' \in \mathcal{L}$ for the set $G'$ of invertibles of any monoid $M' \in \mathcal{K}$,

then any closed clone $F$ on the carrier set of $\mathfrak{A}$ satisfying $F^{(1)} = M$ has automatic action compatibility (and thus automatic homeomorphicity) with respect to the class $\mathcal{C}$ of polymorphism clones of countable $\aleph_0$-categorical structures without algebraicity. Moreover, $M$ has automatic action compatibility with respect to $\mathcal{K}$.

**Proof.** Let us abbreviate $G = \text{Aut}(\mathfrak{A})$. By assumption, the unary part of our clone, $F^{(1)} = M$, is weakly directed. Let $\mathcal{K}$ and $\mathcal{C}$ be as described. Clearly, $\{F'^{(1)} \mid F' \in \mathcal{C}\} \subseteq \mathcal{K}$. Moreover, let $\xi: M \to M'$ be any monoid isomorphism onto any transformation monoid $M' \in \mathcal{K}$ on some set equipotent to $A$. As soon as we verify that $\xi$ is induced by conjugation, Theorem 4.5 will yield the desired conclusion.

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To achieve this, we shall employ Lemma 4.1. Certainly, $M'$ is closed since $M' \in \mathcal{K}$. Likewise, $M$ is a closed monoid with dense group of invertibles $G$. Furthermore, let $\psi : M \to G'$ be a monoid isomorphism onto the closure of the invertibles $G'$ inside $M'$ that coincides with $\xi$ on members of $G$. If every injective monoid endomorphism of $M$ that fixes $G$ pointwise is the identity, then Lemma 3.5 states that $\xi$ equals $\psi$. Otherwise, if $M$ has automatic homeomorphism w.r.t. $\mathcal{L}$, then Proposition 3.3(b) is satisfied (cf. Remark 3.4). So Proposition 3.3(d) yields that $\xi = \psi$ since the closed monoids $G' \subseteq M'$ belong to $\mathcal{L}$.

Finally, as $\mathcal{A}$ is $\aleph_0$-categorical without algebraicity and has a weak $\forall \exists$-interpretation, Theorem 2.3 yields that the restriction $\xi|_G : G \to G'$ of $\xi$ to $G$, is induced by conjugation. This follows for, by the construction of $\mathcal{K}$, $G'$ is an automorphism group of a countable $\aleph_0$-categorical structure without algebraicity and thus covered by Theorem 2.3.

It is interesting to observe that the main structural restrictions (countable categoricity, absence of algebraicity) for the preceding result come from the group case, that is, Theorem 2.3. This means a stronger result regarding the automorphism groups would allow for a wider ranging reconstruction result regarding the clones. Next we state a few less technical assumptions, allowing us to use Corollary 5.1.

**Corollary 5.2.** Let $\mathcal{A}$ be a countable $\aleph_0$-categorical homogeneous relational structure without algebraicity and $M = \text{Emb}(\mathcal{A})$. If $\text{Aut}(\mathcal{A})$ is transitive and supports a weak $\forall \exists$-interpretation, then Corollary 5.1 applies to $\mathcal{A}$ and $\text{Emb}(\mathcal{A})$, which in this case coincides with the monoid of elementary self-embeddings of $\mathcal{A}$.

We note that our argument for condition (4) is somewhat similar to the strategy employed in the proof of [28, Theorem 4.7, p. 21].

**Proof.** Since $\mathcal{A}$ is countable and $\aleph_0$-categorical, it is saturated (see [19, Example 5, p. 485]). For this reason, the automorphism group is dense in the monoid of elementary self-embeddings (cf. [29, p. 598]) and, by homogeneity, its closure coincides with $M = \text{Emb}(\mathcal{A})$, whose invertibles are exactly $\text{Aut}(\mathcal{A})$ (see Lemma 2.1(c)(d)). Transitivity of $\text{Aut}(\mathcal{A})$ is inherited by $M$. For $\mathcal{A}$ avoids any algebraicity, Lemma 2.1(c) gives that $\text{Aut}(\mathcal{A})$ has a trivial centre. As $\mathcal{A}$ is countable and saturated, and $M$ equals the monoid of elementary self-embeddings, Proposition 2.5 of [29, p. 600] provides the monoid condition (4) of Corollary 5.1 for $M$.

Subsequently, we consider a list of example structures. With two exceptions they satisfy all assumptions of Corollary 5.1; for the two particular cases only one condition will remain open.

As our first examples, we consider the non-trivial first-order reducts of the rationals $(\mathbb{Q}, <)$ studied in [14]. Each of them is given by a single relation: $(\mathbb{Q}, \text{betw})$, $(\mathbb{Q}, \text{circ})$, and $(\mathbb{Q}, \text{sep})$, where for any elements $x, y, z, t \in \mathbb{Q}$ we have

$$\text{betw}(x, y, z) \iff x < y < z \vee z < y < x,$$

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\[
\text{circ}(x, y, z) \iff x < y < z \vee y < z < x \vee z < x < y,
\]
\[
\text{sep}(x, y, z, t) \iff (\text{circ}(x, y, z) \land \text{circ}(x, t, y)) \lor (\text{circ}(x, z, y) \land \text{circ}(x, y, t)).
\]

These reducts featured prominently in the complexity classification of temporal constraint languages [10]. Moreover, a selection of further notorious candidates from the zoo of countable universal homogeneous structures will play a role.

**Lemma 5.3.** Let \( A \) be one of the following structures:

(a) a reduct \((\mathbb{Q}, \rho)\) of the strictly ordered rationals \((\mathbb{Q}, <)\), where \(\rho\) is one of the relations in \{<, betw, circ, sep\};

(b) the countable universal homogeneous poset \(\mathcal{P}\) (under the strict order);

(c) the Rado (also random) graph \(\mathcal{G}\);

(d) the random directed graph \(\mathcal{D}\);

(e) the universal homogeneous version of the random bipartite graph \(\mathcal{B}\) (with an additional relation for the bipartition, cf. [23, p. 1603]);

(f) the countable universal homogeneous tournament \(\mathcal{T}\);

(g) the countable dense local order \(\mathbb{S}_2\), see [15] or e.g. [23, p. 1604 (ii)];

(h) Cherlin’s myopic local order \(\mathbb{S}_3\), see [16] or [23, Example 2.3.1 2., p. 1605];

(i) the countable universal homogeneous \(k\)-uniform hypergraph \(\mathbb{H}_k\) for \(k \geq 2\);

(j) the countable universal homogeneous \(\mathbb{K}_n\)-free graph \(\mathcal{G}_{-\mathbb{K}_n}\) for \(n \geq 3\);

(k) any of the countably universal homogeneous Henson digraphs \(\mathcal{D}_X\), forbidding a certain set \(X\) of finite tournaments, see e.g. [23, p. 1604 (iii)].

Then \( A \) satisfies all assumptions of Corollary 5.1 other than (1) with respect to the closed monoid \( M = \text{Emb}(A) \), which coincides with the monoid of elementary self-embeddings of \( A \).

If \( A \notin \{(\mathbb{Q}, \text{betw}), (\mathbb{Q}, \text{circ}), (\mathbb{Q}, \text{sep})\} \), then Corollary 5.1 is indeed applicable and yields that \( M \) and every closed clone \( F \) on the carrier of \( A \) having \( M \) as its unary part, both have automatic action compatibility with respect to countable \( \aleph_0 \)-categorical structures without algebraicity. In particular, this holds for the clone \( F = \text{Pol}(A) = \text{Pol}(\mathbb{Q}, <) \).

Note that in the case of the reducts \((\mathbb{Q}, \rho)\) one can also use the endomorphism monoid, because \( \text{End}(\mathbb{Q}, \rho) = \text{Emb}(\mathbb{Q}, \rho) \). The proof of the latter fact is completely elementary, but (at least in the case of sep) a long and tedious case distinction. So it is perfectly suited to be left to an automated theorem prover.
Proof. Most of these structures $\mathbb{A}$ are, by definition, limits of Fraïssé classes, so they are countably infinite universal homogeneous structures. The reducts $(\mathbb{Q}, \rho)$ and $\mathbb{S}_2$, $\mathbb{S}_3$ were not defined in this way, but nonetheless are homogeneous, see e.g. [23, Example 2.3.1, p. 1605]. Since $\mathbb{A}$ has a finite relational signature, it is $\aleph_0$-categorical (Lemma 2.1(b)). By part (a) of the same lemma, $\text{Aut}(\mathbb{A})$ is transitive because all given structures are ‘loopless’ in the sense that the intersection of each of their fundamental relations with the appropriate $\Delta^m_A$ is empty. Moreover, all the listed structures have no algebraicity. To give some literature references, for $\mathbb{H}_k$ this information can be obtained from the proof of Corollary 22 of [13, p. 3726]. Moreover, every structure listed by Rubin in [31] as examples for his Theorem 2.2 has this property, see p. 234 et seq. for $\mathbb{D}_X$, $\mathbb{D} = \mathbb{D}_0$, $\mathbb{G}_m$, $\mathbb{G}_K$, $\mathbb{G}$, $\mathbb{B}$, $\mathbb{T}$ and p. 243 for $\mathbb{P}$, $\mathbb{S}_2$, $\mathbb{S}_3$ and $(\mathbb{Q}, <)$. For $(\mathbb{Q}, <)$ has no algebraicity, the same is true for any of its reducts as $\text{Aut}(\mathbb{Q}, <) \subseteq \text{Aut}(\mathbb{Q}, \rho)$ wherefore the reducts have even bigger orbits than those given by stabilizers of $\text{Aut}(\mathbb{Q}, <)$.

Except for $\forall\exists$-interpretations we are now ready to invoke Corollary 5.2 to have Corollary 5.1 deliver the conclusion. For this we assume that $\mathbb{A} \neq (\mathbb{Q}, \rho)$ for any $\rho \in \{\text{betw}, \text{circ}, \text{sep}\}$. Concerning the remaining condition, Rubin proves in [31, Theorem 3.2, p. 235] that every ‘simple’ structure has a weak $\forall\exists$-interpretation. The list of ‘simple’ structures given in [31, Examples(1)–(3), p. 234 et seq.] covers $\mathbb{G}$, $\mathbb{D}$, $\mathbb{B}$, $\mathbb{T}$, $\mathbb{D}_X$ and $\mathbb{G}_m$. The sporadic $\forall\exists$-interpretations given in [31, p. 243] cover $(\mathbb{Q}, <)$, $\mathbb{P}$, and $\mathbb{S}_2$ and $\mathbb{S}_3$. Finally, $\mathbb{H}_k$ is treated in [3], see section 1 and the discussion after Theorem 4.1 ibid.

Note that for $\mathbb{A} = (\mathbb{Q}, <)$, we have $\text{Emb}(\mathbb{Q}, <) = \text{End}(\mathbb{Q}, <) = \text{Pol}^{(1)}(\mathbb{Q}, <)$, so that one can avoid forming the structure $\mathbb{A}^6$.

\begin{remark}
For each of $\mathbb{G}$, $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{H}_k$ ($k \geq 2$), the monoid condition (4) of Corollary 5.1 also is a consequence of [13, Lemma 20, p. 3726], which is applicable since these structures have the joint extension property, cf. the proof of Corollary 22 (p. 3726) and the discussion after Definition 18 (p. 3724) in [13].

Regarding the reducts $(\mathbb{Q}, \rho)$, for $\rho = <$, assumption (4) was more explicitly verified in [4, Corollary 2.5]. For $\rho = \text{betw}$, this condition is shown in the proof of Theorem 2.3 of [35], for $\rho = \text{circ}$, it is stated in Corollary 3.5 of [35], and finally, for $\rho = \text{sep}$, this fact is discussed at the beginning of section 4 of [35], before Theorem 4.1.

It can be seen that for the reducts of $(\mathbb{Q}, <)$ the first condition of Corollary 5.1 has been left open so far. Christian Pech kindly pointed out to us that Silvia Barbina gave a weak $\forall\exists$-interpretation for $(\mathbb{Q}, \text{betw})$ in her PhD-thesis, see [2, Example 1.5.3, p. 38].

\begin{corollary}
Corollary 5.1 is applicable to $\mathbb{A} = (\mathbb{Q}, \text{betw})$ and $M = \text{End}(\mathbb{A})$. Hence, $M$ and any closed clone $F \subseteq \text{O}_2$ with $F^{(1)} = M$, e.g., $F = \text{Pol}(\mathbb{Q}, \text{betw})$, has automatic action compatibility with respect to countable $\aleph_0$-categorical structures without algebraicity.

\begin{proof}
Combine Lemma 5.3 with [2, Example 1.5.3], and recall that in this case $\text{Emb}(\mathbb{A}) = M = \text{Pol}^{(1)}(\mathbb{A})$.
\end{proof}
\end{corollary}
Barbina’s construction uses that $\text{Aut}(\mathbb{Q}, <)$ is a closed normal oligomorphic transitive subgroup of $\text{Aut}(\mathbb{Q}, \text{betw})$ and that this subgroup is existentially definable in $\text{Aut}(\mathbb{Q}, \text{betw})$ to transfer Rubin’s $\forall \exists$-interpretation for $(\mathbb{Q}, <)$, see [31, p. 243], to $(\mathbb{Q}, \text{betw})$. The detailed requirements on the interpretation for when such a transfer is possible are stated in [2, Proposition 1.5.1, p. 36]. Subsequently we are going to show that such a transfer is also possible between $(\mathbb{Q}, \text{circ})$ and $(\mathbb{Q}, \text{sep})$. However, we currently are not aware of $\forall \exists$-interpretations for $(\mathbb{Q}, \text{circ})$, even though [2, Proposition 1.2.9, p. 23] might possibly provide a route to them. Let us record this problem explicitly:

**Problem 1.** Which of the reducts $(\mathbb{Q}, \text{circ})$ and $(\mathbb{Q}, \text{sep})$ have weak $\forall \exists$-interpretations? Does $(\mathbb{Q}, \text{circ})$ have a weak $\forall \exists$-interpretation satisfying the additional assumptions 1.–4. mentioned in [2, Proposition 1.5.1]? 

For Barbina’s transfer result the existential definability of the smaller group in the bigger one can be particularly tricky. For this we present the following lemma, generalizing [2, Lemma 1.5.2, p. 37].

**Lemma 5.6.** Let $G$ be a Polish group and $H \trianglelefteq G$ a closed normal subgroup that has the following sort of locally generic (cf. [34, p. 122]) elements $h \in H$: there is a non-empty open subset $X \subseteq H$ such that every $k \in H$ is a product of two members of $X$ and there is $h \in X$ (said to be locally generic), the $H$-conjugacy class $C := C_h = \{ghg^{-1} \mid g \in H\}$ of which is comeagre in $X$, that is, $X \cap C$ is a comeagre subset of $X$. Under these conditions, $H$ is existentially definable (with parameter $h$) in $G$.

Note that $X \cap C \subseteq X$ being comeagre means that $X \setminus C = X \setminus (X \cap C)$ is a meagre subset of $X$. Here the notion of being meagre does not differ whether it is understood with respect to the topology of $H$ or the topology of the subspace $X$, since $X$ is open in $H$. Moreover, $H$ is again Polish since it is closed in $G$. In the proof we shall use the following observation, which is readily verified.

**Fact 5.7.** If $G$ is a group, $g \in G$ and $S \subseteq G$ any subset, then $g \in S \cdot S$ (i.e., $g$ is a product of two possibly equal elements from $S$) if and only if $S \cdot gS^{-1} \neq \emptyset$.

**Proof of Lemma 5.6.** Let $h \in X$, $X \subseteq H$ and $C$ be as described above. We begin by showing that $H \subseteq C \cdot C$ and thus consider an arbitrary element $k \in H$. We work in the Polish group $H$; since it is a topological group, taking inverses or left or right multiplications by some element of $H$ are homeomorphisms of $H$. We know that $X \setminus C = X \setminus (X \cap C)$ is meagre in $H$, so $X^{-1} \setminus C^{-1}$, which can be written as $X^{-1} \setminus ((C^{-1} \cap X^{-1}) = X^{-1} \setminus (C \cap X)^{-1}$ is meagre in $H$, too, and likewise $kX^{-1} \cap kC^{-1} = kX^{-1} \setminus (kC^{-1} \cap kX^{-1}) = kX^{-1} \setminus k(C \cap X)^{-1}$. Subsets of meagre sets are again meagre, so $(X \cap kX^{-1}) \setminus C \subseteq X \setminus C$ and $(X \cap kX^{-1}) \setminus kC^{-1} \subseteq kX^{-1} \setminus kC^{-1}$ are both meagre, and so is their union $(X \cap kX^{-1}) \setminus (C \cap kC^{-1})$. Exploiting the homeomorphisms, $X^{-1}$ and $kX^{-1}$ are open, and so $X \cap kX^{-1} \subseteq H$ is open. By the assumption on $X$ and Fact 5.7, $X \cap kX^{-1} \neq \emptyset$; since $H$ is Polish, this open and non-empty set must be non-
meagre. If \( C \cap kC^{-1} \) were empty, then \( X \cap kX^{-1} = (X \cap kX^{-1}) \setminus (C \cap kC^{-1}) \) would be meagre, thus we conclude that \( C \cap kC^{-1} \neq \emptyset \). By Fact 5.7, \( k \in C \cdot C \).

From \( H \subseteq C \cdot C \) we continue as in the proof of [2, Lemma 1.5.2]. We define \( C_G := \{ ghg^{-1} \mid g \in G \} \supseteq C \). Since \( H \subseteq G \), we have \( C_G \subseteq H \), so \( H \subseteq C \cdot C \subseteq C_G \cdot C_G \subseteq H \cdot H \subseteq H \), leading to

\[
H = C_G \cdot C_G = \{ x \in G \mid \exists g_1, g_2 \in G : x = g_1 hg_1^{-1} \cdot g_2 hg_2^{-1} \},
\]

which is the evaluation of an existential \( G \)-formula with parameter \( h \).

While Barbina employed generic automorphisms to obtain a weak \( \forall \exists \)-interpretation for \( (\mathbb{Q}, \text{betw}) \), we can now use locally generic automorphisms.

**Lemma 5.8.** \( H := \text{Aut}(\mathbb{Q}, \text{circ}) \) is a closed oligomorphic normal subgroup of index 2 of \( G := \text{Aut}(\mathbb{Q}, \text{sep}) \), which acts transitively on \( \mathbb{Q} \) and is existentially definable with a single parameter in \( G \).

**Proof.** By the definition of \( \text{sep} \) from \( \text{circ} \), \( H \subseteq G \) is a subgroup; it is closed and oligomorphic since it is the automorphism group of a countable \( \kappa_0 \)-categorical structure. \( H \) acts transitively on \( \mathbb{Q} \) by Lemma 2.1(a), and it is normal since it has index 2 in \( G \). The latter holds since \( G \) contains bijections preserving the circular order, i.e., members of \( H \), and bijections \( f \) reversing the circular order (in the sense that any \((x, y, z) \in \text{circ}) \) is mapped to \((f(x), f(z), f(y)) \in \text{circ}) \), very much analogously like \( \text{Aut}(\mathbb{Q}, \text{betw}) \) consists of order preserving and order reversing bijections with respect to \((\mathbb{Q}, <) \). Using orbit-stabilizer techniques and the observation that the \( G \)-stabilizer of a single point \( a \) of \( \mathbb{Q} \) is \( \text{Aut}(\mathbb{Q}, \text{betw}) \) with respect to a suitably defined betweenness relation on \( \mathbb{Q} \setminus \{a\} \) (some more details on this can be found in [9, section 11.3.4, p. 110 et seq.]), one can show that \( G \) actually consists of none other than two disjoint cosets of \( H \), namely \( H \) and \( fH \) where \( f \) is some bijection reversing the circular order, e.g., given by \( f(x) = -x \) for \( x \in \mathbb{Q} \).

To get that \( H \) is existentially definable in \( G \), we make use of Lemma 5.6. It is well known that the full symmetric group on a countable set, such as \( \mathbb{Q} \), is a Polish group (see, e.g., [25, section 2.6, p. 97]). Hence, any closed subgroup of it, that is, any automorphism group of a countable structure, e.g., \( H \), is Polish, too. Now, according to Example 5.6 of [34, p. 134], there is some \( \text{circ} \)-preserving permutation \( h \in X = \{ k \in H \mid \exists q \in \mathbb{Q} : k(q) = q \} \) which is a locally generic automorphism on the open subset \( X \subseteq H \). Using again orbit-stabilizer methods to get a more explicit description of the functions in \( H \), it can also be verified that every \( k \in H \) is a product of two elements from \( X \), i.e., of two automorphisms in \( H \) each having some fixed point. \( \Box \)

**Corollary 5.9.** If \( (\mathbb{Q}, \text{circ}) \) has a weak \( \forall \exists \)-interpretation satisfying the conditions in Proposition 1.5.1. 1.–4. from [2, p. 36], then \( (\mathbb{Q}, \text{circ}) \) and \( (\mathbb{Q}, \text{sep}) \) each will have one, and will satisfy the assumptions of Corollary 5.1, which will entail automatic action compatibility with respect to countable \( \kappa_0 \)-categorical structures of the embeddings monoid and any closed clone having this monoid as its unary part.

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Proof. The assumed weak \( \forall \exists \)-interpretation for \((\mathbb{Q}, \text{circ})\) and Lemma 5.8 provide the preconditions for [2, Proposition 1.5.1]. This result now yields that \((\mathbb{Q}, \text{sep})\) has a weak \( \forall \exists \)-interpretation, too, and this, together with the facts shown in Lemma 5.3, will make Corollary 5.1 applicable.

We mentioned earlier that there may be different ways to combine our theorems to obtain reconstruction results. In this way Theorem 2.4, recently proved by Paolini and Shelah (see [26]), can be used to provide reconstruction of the action for the reducts \((\mathbb{Q}, \text{circ})\) and \((\mathbb{Q}, \text{sep})\), with respect to a slightly smaller class than the one we would get from a solution to Problem 1.

Corollary 5.10. Let \( K' \) be the class of all endomorphism monoids of countable \( \aleph_0 \)-categorical structures with the strong small index property and no algebraicity, and let \( \mathbb{K} \) be such a structure. Let \( M \) be a closed transformation monoid on the carrier set of \( \mathbb{K} \), e.g. \( M = \text{End}(\mathbb{K}) \) or \( M = \text{Emb}(\mathbb{K}) \). If

1. \( \text{Aut}(\mathbb{K}) \) is dense in \( M \) and coincides with the group of invertible elements \( \{ g \in M \mid \exists f \in M : f \circ g = g \circ f = \text{id}_A \} \),
2. \( M \) is weakly directed, e.g. transitive, and
3. every injective monoid endomorphism of \( M \) that fixes \( \text{Aut}(\mathbb{K}) \) pointwise is the identity, or

\( M \) has automatic homeomorphism w.r.t. a class \( \mathcal{L} \supseteq K' \) of closed transformation monoids such that \( G' \in \mathcal{L} \) for the set \( G' \) of invertibles of any monoid \( M' \in K' \),

then any closed clone \( F \) on the carrier set of \( \mathbb{K} \) satisfying \( F^{(1)} = M \) has automatic action compatibility (and thus automatic homeomorphicity) with respect to the class \( C' \) of all polymorphism clones of countable \( \aleph_0 \)-categorical structures with the strong small index property and no algebraicity. Moreover, \( M \) has automatic action compatibility with respect to \( K' \).

Proof. Except for replacing \( K \) and \( C \) by \( K' \) and \( C' \), respectively, the proof is literally identical to the one of Corollary 5.1 with only small changes occurring in the last paragraph: \( \forall \exists \)-interpretations for \( \mathbb{K} \) are not needed any more since Theorem 2.3 is replaced by Theorem 2.4, and \( G' \) is now the automorphism group of a countable \( \aleph_0 \)-categorical structure without algebraicity whose group, that is \( G' \), has the strong small index property by the choice of \( K' \).

Corollary 5.11. If \( \mathbb{K} = (\mathbb{Q}, \rho) \), where \( \rho \in \{ \text{circ}, \text{sep} \} \), then \( \text{End}(\mathbb{K}) \) and every closed clone \( F \) on the carrier of \( \mathbb{K} \) with \( F^{(1)} = \text{End}(\mathbb{K}) \) have automatic action compatibility with respect to countable \( \aleph_0 \)-categorical structures with the strong small index property and no algebraicity. In particular, this holds for the polymorphism clones \( F = \text{Pol}(\mathbb{K}) \).

Proof. By Lemma 5.3 all assumptions of Corollary 5.10 have already been verified except for the strong small index property. For \((\mathbb{Q}, \text{circ})\) this is a direct consequence (cf. [19, Theorem 4.2.9, p. 146]) of the intersection condition proved in
Lemma 3.8 of [35], and is discussed in close proximity to this lemma in the mentioned article. For \((\mathbb{Q}, \text{sep})\) the strong small index property has been observed in [35] directly after Theorem 4.1.

For \((\mathbb{Q}, \text{betw})\) the previously presented approach is not applicable since this structure fails to have the strong small index property. To see this take for instance \(H = \text{Aut}(\mathbb{Q}, <)\), which has index two in \(G = \text{Aut}(\mathbb{Q}, \text{betw})\). If \(H\) were a subset of the setwise stabilizer of a finite \(B \subseteq \mathbb{Q}\) under \(G\), then every order preserving permutation of \(\mathbb{Q}\) would have to preserve \(B\), but unless \(B = \emptyset\), this is impossible. However, for \(B = \emptyset\), the pointwise (and setwise) stabilizer of \(B\) is the whole of \(G\), which is not contained in \(H\).

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