Nonlinear quantum input-output analysis using Volterra series

Jing Zhang Member IEEE Yu-xi Liu Re-Bing Wu Member IEEE Kurt Jacobs Sahin Kaya Ozdemir Lan Yang Tzyh-Jong Tarn Life Fellow IEEE Franco Nori

Abstract—Quantum input-output theory plays a very important role for analyzing the dynamics of quantum systems, especially large-scale quantum networks. As an extension of the input-output formalism of Gardiner and Collet, we develop a new approach based on the quantum version of the Volterra series which can be used to analyze nonlinear quantum input-output dynamics. By this approach, we can ignore the internal dynamics of the quantum input-output system and represent the system dynamics by a series of kernel functions. This approach has the great advantage of modeling weak-nonlinear quantum networks. In our approach, the number of parameters, represented by the kernel functions, used to describe the input-output response of a weak-nonlinear quantum network, increases linearly with the scale of the quantum network, not exponentially as usual. Additionally, our approach can be used to formulate the quantum network with both nonlinear and nonconservative components, e.g., quantum amplifiers, which cannot be modeled by the existing methods, such as the Hudson-Parthasarathy model and the quantum transfer function model. We apply our general method to several examples, including Kerr cavities, optomechanical transducers, and a particular coherent feedback system with a nonlinear component and a quantum amplifier in the feedback loop. This approach provides a powerful way to the modeling and control of nonlinear quantum networks.

Index Terms—Nonlinear quantum systems, Volterra series, quantum input-output networks, quantum coherent feedback control, quantum control.

I. INTRODUCTION

There has been tremendous progress in the last few years in the fields of quantum communication networks and quantum internet [1]–[3], quantum biology [4], quantum chemistry, hybrid quantum circuits [5], quantum computing and quantum simulation [6], [7], and quantum control [8]–[26]. These progresses pave the way to the development of large-scale quantum networks. Although scalable quantum networks exhibit advantages in information processing and transmission, many problems are still left to be solved to model such complex quantum systems. Different approaches have been proposed to analyze quantum networks, among which the input-output formalism of Gardiner and Collet [27], [28] is a useful tool to describe the input-output dynamics of such systems. In fact, using the input-output response to analyze and control the system dynamics is a standard method in engineering. The quantum input-output theory [27], [28] has been extended to cascaded-connected quantum systems [29], [30] and even more complex Markovian feedforward and feedback quantum networks, including both dynamical and static components [31]–[45]. Two main different formulations are proposed in the literature to model such quantum input-output networks: (i) the time-domain Hudson-Parthasarathy formalism [46], which can be considered as the extension of the input-output theory developed by Gardiner and Collet; and (ii) the frequency-domain quantum transfer function formalism [47], [48].

In the existing literature, quantum input-output theory is mainly applied to optical systems, in which the “memory” effects of the environment are negligibly small and the non-linear effects of the systems are weak and thus sometimes omitted. These lead to the development of the Markovian and linear quantum input-output network theory [31], [47]. [49]. In more general cases, such as in mesoscopic solid-state systems, both the linear and Markovian assumptions may not be valid. Recently, the Markovian quantum input-output theory has been extended to the non-Markovian case for single quantum input-output components [50] or even quantum networks [51]. However, how to model and analyze nonlinear quantum input-output systems is still an open problem.

Recent experimental progresses [36], [39], [52]–[59], especially those in solid-state quantum circuits [40], [57]–[60], motivate us to find some ways to analyze nonlinear quantum input-output networks. It should be pointed out that the
Hudson-Parthasarathy model can in principle formulate particular nonlinear quantum input-output systems. However, it cannot be applied to more general cases, such as those with both nonlinear components and quantum amplifiers. An additional problem yet to be solved is the computational complexity for modelling large-scale quantum input-output networks. The Hudson-Parthasarathy model gives the input-output response in terms of the internal system dynamics, and this will lead to an exponential increase of the computational complexity when we apply it to large-scale quantum networks composed of many components. For most cases, this exponentially-increased large-scale model contains redundant information. Not all the internal degrees of freedom are necessary to be known. For example, let us consider a quantum feedback control system composed of the controlled system and the controller in the feedback loop. We may not be interested in the internal dynamics of the controller, but only concern how the controller modifies the signal fed into it. This can be obtained by an input-output response after averaging over the internal degrees of freedom of the controller, which may greatly reduce the computational complexity of the quantum network analysis. For linear quantum networks, such an input-output response can be obtained by the quantum transfer function model. However, it may not be applied to nonlinear quantum networks.

To solve all these problems, we establish a nonlinear quantum input-output formalism based on the so-called Volterra series. This formalism gives a simpler form to model weak-nonlinear quantum networks. This paper is organized as follows: a brief review of quantum input-output theory is first presented in Sec. II and then the general form of the Volterra series for m-port quantum input-output systems is introduced in Sec. III. The Volterra series approach is extended in Sec. IV to more general cases in the frequency domain to analyze more complex quantum input-output networks with multiple components connected in series products and concatenation products. Our general results are then applied to several examples in Sec. V. Conclusions and discussion of future work are given in Sec. VI.

II. BRIEF REVIEW OF QUANTUM INPUT-OUTPUT THEORY

A. Gardiner-Collet input-output formalism

The original model of a general quantum input-output system is a plant interacting with a bath. Under the Markovian approximation (in which the coupling strengths between the system and different modes of the bath are assumed to be constants for all frequencies), an arbitrary system operator \( Z(t) \) satisfies the following quantum stochastic differential equation (QSDE)

\[
\dot{Z} = -i[Z,H_S] + \frac{1}{2} \{L^\dagger[Z,L] + [L^\dagger,Z]L\} + \{b_{in}[L^\dagger,Z] + [Z,L]b_{in}^\dagger\},
\]

with \( L = (L_1, \ldots, L_m)^T \), where \( L_i \)'s are the system operators representing the dissipation channels of the system coupled to the input fields. Let \( b_{in}(t) = [b_{1,in}(t), \ldots, b_{m,in}(t)]^T \) and output field \( b_{out}(t) = [b_{1,out}(t), \ldots, b_{m,out}(t)]^T \) be the time-varying input fields that are fed into the system and the output fields being about to propagate away, one has the relation

\[
b_{out}(t) = b_{in}(t) + L(t).
\]

This is the standard Gardiner-Collet input-output relation.

B. Hudson-Parthasarathy model

The Gardiner-Collet input-output theory can be extended to more general cases to include the static components such as quantum beam splitters by the Hudson-Parthasarathy model. A more general multi-input and multi-output open quantum systems can be characterized by the following tuple of parameters

\[
G = (S, L, H),
\]

where \( H \) is the internal Hamiltonian of the system; \( S \) is a \( n \times n \) unitary scattering matrix induced by the static components. The notations given in Eq. (3) can be used to describe a wide range of dynamical and static systems. For example, the traditional quantum input-output systems represented by Eqs. (1) and (2) can be written as \( G_{LH} = (I, L, H) \), and the quantum beam splitter with scattering matrix \( S \) can be represented by \( G_{BS} = (S, 0, 0) \).

To obtain the dynamics of the input-output system given by Eq. (3), we first introduce the quantum Wiener process \( B(t) \) and the quantum Poisson process \( \Lambda(t) \) as

\[
B(t) = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}, \quad \Lambda(t) = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix},
\]

which are defined by

\[
B_i(t) = \int_0^t b_{i,in}(\tau) d\tau, \quad B_{ij}(t) = \int_0^t b_{i,in}(\tau) b_{j,in}(\tau) d\tau.
\]

In the Heisenberg picture, the system operator \( Z(t) \) satisfies the following quantum stochastic differential equation

\[
dZ = \{L_L(Z) - i[Z,H_S]\} dt + dB^T S^T \{Z,L\} + [L^\dagger,Z]S dB + \text{tr}\{S^T ZS - Z\} d\Lambda^T,
\]

where the Liouville superoperator \( L_L(\cdot) \) is defined by

\[
L_L(X) = \frac{1}{2} L^\dagger[X,L] + \frac{1}{2} [L^\dagger,X]L
\]

\[
= \sum_{j=1}^n \left\{ \frac{1}{2} L_j^\dagger[X,L_j] + \frac{1}{2} [L_j^\dagger,X]L_j \right\},
\]

which is of the standard Lindblad form. Similar to Eq. (2), we can obtain the following input-output relation

\[
dB_{out} = S dB + L dt,
\]

\[
d\Lambda_{out} = S^* d\Lambda S^T + S^* dB^T L^T + L^* dB^T S^T + L^* L^T dt,
\]

where \( dB_{out} \) and \( d\Lambda_{out} \) are the output fields corresponding to the quantum Wiener process \( dB \) and Poisson process \( d\Lambda \).
C. Quantum transfer function model

The Gardiner-Collet input-output theory, or the more general Hudson-Parthasarathy model introduced in subsections [11A] and [11B] can be used to represent a large class of quantum input-output systems. However, the Hudson-Parthasarathy model is complex if the interior degrees of freedom of the system are very high. The quantum transfer function model can be applied to some cases that the Hudson-Parthasarathy model is invalid or inefficient.

Different from the Hudson-Parthasarathy model, which is in the time domain, the quantum transfer function model is a frequency-domain approach [47], [48] and can only be applied to linear quantum input-output systems. The system we consider is composed of $r$ harmonic oscillators \( \{a_j : j = 1, \cdots, r\} \), which satisfy the following canonical commutation relations

\[
[a_j, a_k^\dagger] = \delta_{jk}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0.
\]

We are interested in a general linear quantum system, which, in the \((S, L, H)\) notation given by Eq. (3), satisfies the following conditions: (i) the dissipation operators \( L_j \)'s are linear combinations of \( a_k \), i.e., \( L_j = \sum_k c_{jk} a_k \); and (ii) the system Hamiltonian \( H \) is a quadratic function of \( a_k \), i.e., \( H = \sum_j \omega_j a_j^\dagger a_j \). Under these conditions, we can obtain the following equivalent expression of Eq. (3):

\[
G = (S, C, \Omega), \tag{9}
\]

where

\[
C = \begin{pmatrix} c_{11} & \cdots & c_{1r} \\ \vdots & \ddots & \vdots \\ c_{r1} & \cdots & c_{rr} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1r} \\ \vdots & \ddots & \vdots \\ \omega_{r1} & \cdots & \omega_{rr} \end{pmatrix}.
\]

Let us introduce an operator vector called the state vector of the system \( a = (a_1, \cdots, a_r)^T \), then from Eqs. (6) and (8), we can obtain the following Heisenberg-Langevin equation and input-output relation

\[
\dot{a}(t) = A a(t) - C^\dagger S b_{in}(t), \tag{10}
\]

\[
b_{out} = S b_{in} + C a(t), \tag{11}
\]

where \( A = -C^\dagger C/2 - i\Omega \). Such kind of linear equations can be solved in the frequency domain. To show this, let us introduce the Laplace transform which is defined for \( \Re(s) > 0 \) by

\[
R(s) = \int_0^\infty \exp(-st) R(t) \, dt. \tag{12}
\]

In the frequency domain, Eqs. (10) and (11) can be solved as

\[
a(s) = -(sI_r - A)^{-1} C^\dagger S b_{in}(s), \tag{13}
\]

\[
b_{out}(s) = S b_{in} + C a(s). \tag{14}
\]

Then, we can obtain the input-output relation of the whole system

\[
b_{out}(s) = \Xi(s) b_{in}(s), \tag{15}
\]

where \( \Xi(s) \) is the transfer function of the linear quantum system, which can be calculated by

\[
\Xi(s) = S - C (sI_r - A)^{-1} C^\dagger S. \tag{16}
\]
Eq. (10) is $L = \text{span}\{−iX_α\}$, where $−iX_α$’s are the basis elements of the Lie algebra $L$ which satisfy the following commutation relation

$$[X_α, X_β] = −i \sum_γ C^γ_{αβ} X_γ.$$  \hspace{1cm} (18)

$C^γ_{αβ}$’s are the structure constants of the Lie algebra $L$. Let us define an operator vector $X = (X_α)$, of which the entries come from the basis elements of the Lie algebra $L$. From Eq. (10), we can obtain the following dynamical equation for the operator vector $X$

$$X(t) = AX(t) + \left(B^* b_{in}(t) + B b_{in}^\dagger\right) X(t).$$  \hspace{1cm} (19)

From Eq. (19), we have the following formal series solution for the above equation

$$X(t) = e^{At} X + \int_0^t e^{A(t−τ)} [B^* b_{in}(τ_1)$$

$$+ B b_{in}^\dagger(τ_1)] X(τ_1) dτ_1.$$  \hspace{1cm} (20)

where $X$ is the operator vector in the Schrödinger picture. By solving $X(τ_1)$ in the integral of Eq. (20), we can obtain the iterative solution

$$X(t) = e^{At} X + \sum_{n=1}^\infty \int_0^t \int_0^{τ_1} \cdots \int_0^{τ_{n−1}} e^{A(t−τ)} [B^* b_{in}(τ_1)$$

$$+ B b_{in}^\dagger(τ_1)] e^{A(τ_{n−1}−τ_{n−2})} \cdots e^{Aτ_1} X dτ_1 \cdots dτ_n.$$  \hspace{1cm} (21)

Since $\{−iX_α\}$ is the basis of the dynamical Lie algebra of the quantum input-output system, the system operator $L$ in the output equation (11) can be written as the linear combination of $\{X_α\}$, i.e.,

$$L = \sum_α l_α X_α,$$  \hspace{1cm} (22)

where $l_α \in \mathbb{C}$. Let us then assume that the total input-output system composed of the internal degrees of freedom and the external input field is initially in a separable state $ρ_{tot}(0) = ρ_0 ⊗ ρ_ν$, where $ρ_0$ and $ρ_ν$ are, respectively, the initial states of the internal system and the external input field. If we average over the internal degrees of freedom of the quantum input-output system, the output equation (22) can be rewritten as

$$b_{out}(t) = b_{in}(t) + \sum_α l_α \langle X_α(t) \rangle_0,$$  \hspace{1cm} (23)

where $\langle X_α(t) \rangle_0 = \text{tr} [X_α(t) ρ_0]$. By substituting Eq. (21) into Eq. (23), we can obtain the Volterra series of a general nonlinear quantum input-output component given by Eq. (17).

As shown in Eq. (17), the system input-output response is fully determined by the set of parameters $\{k_{j_1, j_2, \cdots j_n} (τ_1, \cdots, τ_n)\}$ called Volterra kernels. It is also shown in the proof of theorem (11) that these kernel functions are just determined by the high-order quantum correlations of the interior dynamics of the quantum input-output system. Notice that the quantum Volterra series is different from the classical Volterra series because the terms like $b_{in}^{(1)}(τ_1), \cdots, b_{in}^{(n)}(τ_n)$ do not commute with each other and the vacuum fluctuations in the input field should be considered when we analyze quantum input-output response.

Different from the Volterra series used for for classical systems, the kernel functions of the quantum Volterra series method we present here should satisfy additional physically-realizable conditions [31], [48] constrained by the theory of quantum mechanics. For example, for a Markovian input-output system [32], the commutation relation should be preserved from the input field to the output field, i.e., $[b_{out}(t), b_{out}^{(n)}(t')] = [b_{in}(t), b_{in}^{(n)}(t')]$. This leads to additional equality constraints for the kernel functions $k_{j_1, j_2, \cdots j_n} (τ_1, \cdots, τ_n)$.

Although the right side of Eq. (17) is an infinite series, i.e., a series with infinitely many terms, we can use its finite truncations to represent the input-output response under particular conditions. In fact, for linear systems, there are only linear terms in Eq. (17). Motivated by this consideration, we then study a weak-nonlinear quantum system with $r$ internal modes given by the annihilation (creation) operators $a_{i=1, \cdots, r}$ and $a_{i=1, \cdots, r}^\dagger$. For such a weak nonlinear quantum system, the system Hamiltonian $H$ and the dissipation operator $L$ in Eq. (1) can be expressed as

$$H = H_l + μ H_{nl}, \hspace{1cm} L = L_l + μ L_{nl},$$  \hspace{1cm} (24)

where $H_l$ and $L_l$ are quadratic and linear functions of the annihilation and creation operators $a_i$ and $a_i^\dagger$; $H_{nl}$ and $L_{nl}$ are higher-order nonlinear terms of $a_i$ and $a_i^\dagger$; and $μ$ is a parameter introduced to determine the nonlinear degree of the system. For a weak-nonlinear system, we have $μ ≪ 1$. The following theorem shows that we can use the finite truncation up to low-order terms and omit higher-order nonlinear terms in the Volterra series for weak-nonlinear systems satisfying Eq. (24).

**Theorem 2:** (Volterra series for weak-nonlinear systems) For a weak-nonlinear quantum input-output system with Hamiltonian and dissipation operators given by Eq. (24), the Volterra series for this quantum input-output system can be written as

$$b_{j_2, j_3, \cdots j_n}^{(i)}(t) = \int_0^t k_{j_2, j_3, \cdots j_n}^{(i)}(t−τ) b_{j_1, in}^{(i)}(τ_1) dτ_1$$

$$+ μ \int_0^t dτ_1 dτ_2 k_{j_2, j_3, \cdots j_n}^{(i)}(τ_1, τ_2) b_{j_1, in}^{(i)}(τ_1) b_{j_2, in}^{(i)}(τ_2)$$

$$k_{j_2, j_3, \cdots j_n}^{(i)}(t−τ_1, τ_1−τ_2) + o(μ).$$  \hspace{1cm} (25)
The proof of the theorem is given in the appendix. It can be easily seen that Eq. (25) is just the traditional convolution representation, or equivalently the transfer function representation, when \( \mu = 0 \), which corresponds to linear quantum systems.

**IV. FREQUENCY ANALYSIS OF NONLINEAR QUANTUM INPUT-OUTPUT NETWORKS**

The Volterra series can be expressed as a simpler form in the frequency domain, especially for quantum networks with several components. In the frequency domain, the quantum input-output relation can be rewritten as

\[
b_{j, \text{out}}^\pm (\omega) = \sum_{n=1}^{\infty} \int \ldots \int \chi_{j, i_1 \cdots i_n}^\pm \omega_1 \cdots \omega_n d\omega_1 \cdots d\omega_n, \tag{26}
\]

where

\[
\chi_{j, i_1 \cdots i_n}^\pm (\omega_1, \cdots, \omega_n) = K_{j, i_1 \cdots i_n}^\pm (\omega_1 + \cdots + \omega_n)
\]

and \( K_{j, i_1 \cdots i_n}^\pm (\omega_1, \cdots, \omega_n) \) is the \( n \)-th order Fourier transform of the kernel function \( K_{j, i_1 \cdots i_n}^\pm (\tau_1, \cdots, \tau_n) \) defined by

\[
K_{j, i_1 \cdots i_n}^\pm (\omega_1, \cdots, \omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} k_{j, i_1 \cdots i_n}^\pm (t_1, \cdots, t_n) e^{-i\omega t_1 \cdots i\omega t_n} dt_1 \cdots dt_n. \tag{27}
\]

The coefficients \( \{\chi_{j, i_1 \cdots i_n}^\pm\} \) can be seen as the quantum version of the \( n \)-th order nonlinear susceptibility coefficients.

The main merit of the Volterra series approach is that it can greatly reduce the computational complexity of quantum input-output network analysis in the frequency domain. In fact, from Eq. (26), we can find that the input-output response of a multi-input nonlinear component is fully determined by the quantum susceptibility coefficients \( \{\chi_{j, i_1 \cdots i_n}^\pm (\omega_1, \cdots, \omega_n)\} \).

The following theorem shows that the quantum susceptibility coefficients of a large-scale quantum network with several components can be expressed as the polynomial functions of the lower-order quantum susceptibility coefficients of each component.

**Theorem 3:** (Susceptibility coefficients for networks)

The \( n \)-th order quantum susceptibility coefficients \( \{\chi_{j, i_1 \cdots i_n} (\omega_1, \cdots, \omega_n)\} \) of a multi-component quantum network can be expressed as polynomials of lower-order quantum susceptibility coefficients of each component.

**Proof:** To prove our main results, we can see that an arbitrary nonlinear quantum network can be decomposed into two basic types of connections between different components, i.e., the concatenation product and the series product. Thus, we only need to verify the main results for these two types of basic quantum networks.

The concatenation product describes two components that are simply assembled together without any connection between them [see Fig. 2(a)]. Let us assume that \( \{\chi_{j, i_1 \cdots i_n}^1 (\tau_1, \cdots, \tau_n)\} \) and \( \{\chi_{j, i_1 \cdots i_n}^2 (\tau_1, \cdots, \tau_n)\} \) are the quantum susceptibility coefficients of the two components and \( \{\chi_{j, i_1 \cdots i_n} (\tau_1, \cdots, \tau_n)\} \) are the quantum susceptibility coefficients of the total system, then it can be easily verified that

\[
\chi_{j, i_1 \cdots i_n}^1 (\omega_1, \cdots, \omega_n) = \begin{cases} 
\chi_{j, i_1 \cdots i_n}^1 (\omega_1, \cdots, \omega_n), & i_k, j_k \in \{1, \cdots, r\}; \\
\chi_{j, i_1 \cdots i_n}^2 (\omega_1, \cdots, \omega_n), & i_k, j_k \in \{r + 1, \cdots, m\}; \\
0, & \text{otherwise}.
\end{cases}
\tag{28}
\]

The series product can be used to describe two cascade-connected components [see Fig. 2(b)], i.e., the output of the first system is taken as the input of the second system. The \( n \)-th quantum susceptibility coefficients \( \chi_{j, i_1 \cdots i_n}^1 \) of the quantum network in the series product can be calculated by the following equation

\[
\chi_{j, i_1 \cdots i_n}^1 (\omega_1, \cdots, \omega_n) = \sum_{\alpha_1 + \cdots + \alpha_r = n} \chi_{j, k_1 \cdots k_\alpha_1}^1 \chi_{k_1, k_{\alpha_1} \cdots k_\alpha_2}^2 (\omega_1 + \cdots + \omega_{\alpha_1}, \cdots, \omega_n), \tag{29}
\]

where \( \chi_{j, i_1 \cdots i_n}^1 \) and \( \chi_{j, i_1 \cdots i_n}^2 \) are the quantum susceptibility coefficients of the two components.

From Eq. (28) and Eq. (29), we can see that the \( n \)-th quantum susceptibility coefficients of a quantum network in the concatenation product and series product can be expressed as the polynomials of lower-order quantum susceptibility co-
The quantum susceptibility coefficients of the series-product component is cascade-connected to a nonlinear component. (a) The input field is first fed into the linear component with quantum transfer function \(G^{(1)}(i\omega)\) and then transmits through a nonlinear component with quantum susceptibility coefficients \(\chi^{(1)}_{j_1\cdots j_n}(\omega_1, \ldots, \omega_n)\); (b) the two components are connected in the opposite way: the input field is first fed into a nonlinear component with quantum susceptibility coefficients \(\chi^{(1)}_{j_1\cdots j_n}(\omega_1, \ldots, \omega_n)\) and then a linear component with quantum transfer function \(G^{(2)}(i\omega)\).

Remark 1: Theorem 3 shows that the computational complexity to describe the input-output response of a multi-component nonlinear quantum network increases linearly with the number of the components in the quantum network, in comparison to the traditional exponentially increasing complexity for describing a complex quantum input-output network.

As examples of multi-component nonlinear quantum networks, let us consider a quantum network in which a linear component is cascade-connected to a nonlinear component. This can be divided into two different cases (see Fig. 3):

(i) The input field is first fed into a linear component with the quantum transfer function \(G^{(1)}(s), s \in \mathbb{C}\), and then transmits through a nonlinear component with quantum susceptibility coefficients \(\chi^{(2)}_{j_1\cdots j_n}(\omega_1, \ldots, \omega_n)\). The quantum susceptibility coefficients of the total system can be calculated by

\[
\chi^{i_1\cdots i_n}_{j_1\cdots j_n}(\omega_1, \ldots, \omega_n) = \chi^{(2)}_{i_1\cdots i_n}(\omega_1, \ldots, \omega_n) G^{(1)}(i\omega_i) . \quad (30)
\]

(ii) The input field is first fed into a nonlinear component with quantum susceptibility coefficients \(\chi^{(1)}_{j_1\cdots j_n}(\omega_1, \ldots, \omega_n)\), and then guided into a linear component with quantum transfer function \(G^{(2)}(s), s \in \mathbb{C}\). The quantum susceptibility coefficients of the series-product system can be expressed as

\[
\chi^{i_1\cdots i_n}_{j_1\cdots j_n}(\omega_1, \ldots, \omega_n) = G^{(2)}\left(\sum_{i=1}^{n} \omega_i\right) \chi^{(1)}_{i_1\cdots i_n}(\omega_1, \ldots, \omega_n) . \quad (31)
\]

Note that the above examples are quite useful for modelling a large class of important quantum input-output networks, such as the network with a nonlinear component cascaded connected to a quantum amplifier, that cannot be modelled appropriately by the existing approaches.

V. APPLICATIONS

The Volterra series approach we introduce here can be applied to various linear and nonlinear quantum input-output systems, especially those with weak nonlinearity. To show this, we study the input-output relation of some conventional nonlinear components, which can be taken as the basic elements of more complex quantum networks.

Example 1: (Kerr Cavity)

As a first example, we consider a Kerr cavity with free Hamiltonian \(H = \omega_0 a^\dagger a + \chi (a^\dagger a^2 a)\) and dissipation operator \(L = \gamma a\) coupled to the input field, where \(a\) and \(a^\dagger\) are the annihilation and creation operators of the cavity. Here \(\omega_0, \chi, \gamma\) are the frequency of the fundamental mode, the nonlinear Kerr coefficient of the cavity, and the coupling strength between the cavity and the input field (see Fig. 3). Let us consider the weak-nonlinear assumption such that \(\chi \ll \omega, \gamma\), then from theorem 2 we can expand the quantum Volterra series up to the third-order terms. If we further assume that the cavity is initially in the vacuum state, there is only one nonzero first-order Volterra kernel

\[
k_{-}\left(\tau\right) = -\gamma \exp\left[-\left(\frac{\gamma}{2} + i\omega_0\right)\tau\right] \quad (32)
\]

and four nonzero third-order Volterra kernels

\[
k_{\pm}\left(\tau_1, \tau_2, \tau_3\right) = \frac{4i\gamma^2\chi^2}{-\gamma + i\chi} e^{-\gamma/2(\tau_1 + \tau_3) - i\omega_0(\tau_1 - \tau_3) - \gamma\tau_2} \left(1 - e^{-\gamma\tau_1}\right), \quad (33)
\]

\[
k_{\pm}\left(\tau_1, \tau_2, \tau_3\right) = -\frac{4i\gamma^2\chi^2}{-\gamma + i\chi} e^{-\gamma/2(\tau_1 + \tau_3) - i\omega_0(\tau_1 - \tau_3) - \gamma\tau_2} \left(1 - e^{-\gamma\tau_1}\right).
\]

See the derivations in the appendix.

For this example, the Volterra series approach gives a more exact description of the quantum input-output response compared with other approximation approaches, such as the truncation approximation approach in the Fock space which is mainly used for low-excitation quantum systems and the semiclassical approximation, which is traditionally introduced to study highly-excited systems. To show this, let us see the
field with damping rate $\gamma_a$ and coupling strength. The cavity mode is coupled to the input mechanical mode. Let us assume that the optomechanical mode and the mechanical oscillator; frequencies of these two modes; and transducer. The Hamiltonian of this system can be written mode cavity parametrically coupled to a mechanical oscillator by the few-photon truncation in the Fock space and the well with the ideal trajectory compared with those obtained drive the Kerr cavity by an external field with strength $\chi/\omega_6$. In order to obtain the output spectrum, we drive the Kerr cavity by an external field with strength $\epsilon_2 = 0.6\omega_a$. Here $\tau = 2\pi/\omega_a$ is a normalized unit of time. The black triangle curve is the ideal trajectory. The black triangle curve, the green dashed curve, and the red curve with plus signs are the trajectories obtained by the Volterra series approach, semiclassical approximation, and the few-photon truncation with expansion up to five-photon Fock state. The trajectory obtained by the Volterra series approach coincides very well with the ideal one compared with the other two approaches.

Fig. 5. (color online) (a) Time evolution of the output field $x_{out} = (b_{out} + b_{out}^\dagger)/\sqrt{2}$ and (b) logarithmic output spectra for a Kerr cavity with $(\chi/\omega_a, \gamma/\omega_a) = (0.01, 0.2)$. In order to obtain the output spectrum, we drive the Kerr cavity by an external field with strength $\epsilon_2 = 0.6\omega_a$. Here $\tau = 2\pi/\omega_a$ is a normalized unit of time. The black triangle curve is the ideal trajectory. The black triangle curve, the green dashed curve, and the red curve with plus signs are the trajectories obtained by the Volterra series approach, semiclassical approximation, and the few-photon truncation with expansion up to five-photon Fock state. The trajectory obtained by the Volterra series approach coincides very well with the ideal one compared with the other two approaches.

Example 2: (Optomechanical transducer)

In the second example, let us concentrate on a single-mode cavity parametrically coupled to a mechanical oscillator (see Fig. 6(b)), which received a high degree of attention recently [68], [72]–[76]. These systems can be used as sensitive detectors to detect spin and mass, or a sensitive mechanical transducer. The Hamiltonian of this system can be written as $H = \omega_a a^\dagger a + \omega_b b^\dagger b + g a^\dagger a(b + b^\dagger)$, where $a$ ($a^\dagger$) and $b$ ($b^\dagger$) are the annihilation (creation) operators of the cavity mode and the mechanical oscillator; $\omega_a$ and $\omega_b$ are the angular frequencies of these two modes; and $g$ is the optomechanical coupling strength. The cavity mode is coupled to the input field with damping rate $\gamma_a$, and $\gamma_b$ is the damping rate of the mechanical mode. Let us assume that the optomechanical coupling is weak enough such that $g \ll \omega_a, \omega_b, \gamma_a, \gamma_b$. Note that $g$ determines the nonlinearity of the optomechanical systems, thus the above assumption means that the nonlinearity of the optomechanical system we consider is weak. From theorem [2], we can expand the quantum Volterra series to the third-order terms and omit higher-order terms. If we further assume that the cavity and the mechanical oscillator are both initially in the vacuum states and note that $\omega_a \gg \omega_b$ and $\gamma_a \gg \gamma_b$, there are only one non-zero first-order Volterra kernel $k_{\pm}(\tau) = -\gamma_a \exp \left[ -\left( \frac{i\omega_a}{\beta_a} + i\omega_b \right) \tau \right]$ and two non-zero third-order Volterra kernels

$$k_{\pm \pm}(\tau_1, \tau_2, \tau_3) = 
\gamma_a^2 g^2 e^{-\gamma_a \tau_3} \left( 
\frac{e^{-\gamma_a \tau_1} + e^{-\gamma_a \tau_1}}{-\gamma_a^2 + \gamma_a^2} \right) \left( 
\frac{e^{-\gamma_b \tau_2} - e^{-\gamma_b \tau_2}}{-\gamma_b^2 + \gamma_a} \right),
$$

where $\gamma_a^2 = 2\pi/\omega_a$ and $\gamma_b^2 = 2\pi/\omega_b$. The derivations of Eq. (34) are similar to those of Eqs. (32) and (33) given in the appendix, thus we omit those here.

Example 3: (Nonlinear coherent feedback network with weak Kerr nonlinearity)

The Volterra series approach gives a simpler way to analyze nonlinear quantum coherent feedback control systems [44], [68], [72]–[76]. To show this, let us consider a simple coherent feedback system in Fig. 7(a). In this system, the controlled system is a linear cavity $S_1$, and in the feedback loop there are a quantum amplifier $S_2$ and a Kerr nonlinear component $S_3$. Notice that this coherent feedback system cannot be modelled by the existing approaches such as the Hudson-Parthasarathy model and the quantum transfer function model, but we can describe it by our approach. The total system can be seen as a cascade-connected system $S = S_1 \otimes S_2 \otimes S_3$. The system dynamics can be obtained from Eqs. (30), (31), and example 1. This quantum coherent feedback loop induces an interesting phenomenon: the nonlinear component in the coherent feedback loop changes the dynamics of the linear cavity and make it a nonlinear cavity. This nonlinear effect is additionally amplified by the quantum amplifier in the feedback loop. Let $\omega_a$, $\chi_b$ be the effective frequency and damping rate of the controlled linear cavity, $\omega_b$, $\chi_b$, $\gamma_b$ are the effective frequency, nonlinear Kerr coefficient, and damping rate of the nonlinear Kerr cavity in the feedback loop. $G$ is
the power gain of the quantum amplifier in the feedback loop. Under the condition that $\gamma_a \gg \omega_a$, the controlled cavity can be seen as a nonlinear Kerr cavity with effective Kerr coefficient $\chi_a = G\chi_b$. This amplified nonlinear Kerr effect leads to nonlinear quantum phenomena in the controlled cavity. For example, if the initial state of the controlled cavity is a coherent state, this state will evolve into a non-Gaussian state, which is highly nonclassical. In Fig. 7(b), we use the measure

$$\delta [\rho] = \frac{\text{tr} [\rho - \sigma^2 / 2]}{\text{tr} [\rho^2]} \in [0, 1]$$

to evaluate the non-Gaussian degree of the quantum states generated in the controlled cavity [77], where $\sigma$ is a Gaussian state with the same first and second-order quadratures of the non-Gaussian state $\rho$. Simulation results in Fig. 7(b) show that higher-quality non-Gaussian states can be obtained if we increase the power gain $G$ of the quantum amplifier in the feedback loop. We should point out that we have predicted a similar quantum feedback nonlinearity phenomenon in Ref. [44]. But in that paper, the nonlinearity is induced by the nonlinear dissipation interaction between the controlled system and the mediated quantum field, and the feedback loop is linear. Here, we show that nonlinear coherent feedback loop can induce quantum nonlinearity [76], which can be further amplified by the quantum amplifier in the feedback loop.

VI. Conclusion

We have introduced a new formalism of quantum input-output networks using the so-called Volterra series. It gives a simpler way to describe large-scale nonlinear quantum input-output networks especially in the frequency domain, and can be also used to analyze more general quantum networks with both nonlinear components and quantum amplifiers that cannot be modeled by the existing methods such as the Hudson-Parthasarathy model and the quantum transfer function model. An application to quantum coherent feedback systems shows that it can be used to show the quantum feedback nonlinearity effects, in which the nonlinear components in the coherent feedback loop can change the dynamics of the controlled linear system and these quantum nonlinear effects can be amplified by a linear quantum amplifier. Our work opens up new perspectives in nonlinear quantum networks, especially quantum coherent feedback control systems.

Appendix

Proof of the theorem 2 Let us assume that $X_1$ is an operator vector of which the components are linear terms of the annihilation and creation operators $a_i$ and $a_i^\dagger$, $i = 1, \cdots, r$, and $X_2$ is another operator vector of which the components are higher-order nonlinear terms of $a_i$ and $a_i^\dagger$. The system dynamics can be fully determined by the vector $X = (X_1^T, X_2^T)^T$. From Eq. (1) and the special form of the system Hamiltonian $H$ and dissipation operator $L$ in Eq. (5), we can obtain the dynamical equations of $X_1$ and $X_2$ as follows

$$\dot{X}_1 = (A_{11} X_1 + \mu A_{12} X_2) + \left( B_{11}^* b_{in} + B_{12}^* b_{in}^\dagger \right) X_1 + \mu \left( B_{21}^* b_{in} + B_{22}^* b_{in}^\dagger \right) X_1,$$

$$\dot{X}_2 = (A_{21} X_1 + A_{22} X_2) + \left( B_{21}^* b_{in} + B_{22}^* b_{in}^\dagger \right) X_1 + \mu \left( B_{21}^* b_{in} + B_{22}^* b_{in}^\dagger \right) X_2,$$

where $A_{ij}, B_{ij}$ are constant matrices determined by $H_i, H_{nl}, L_i$, and $L_{nl}$. By solving Eqs. (35) and (36), we have

$$X_1(t) = e^{A_{11} t} X_1 + \mu \int_0^t e^{A_{11} (t - \tau)} A_{12} X_2(\tau) d\tau + \int_0^t e^{A_{11} (t - \tau)} \left[ B_{11}^* b_{in}(\tau) + B_{12}^* b_{in}^\dagger(\tau) \right] d\tau,$$

$$X_2(t) = e^{A_{22} t} X_2 + \mu \int_0^t e^{A_{22} (t - \tau)} A_{21} X_1(\tau) d\tau + \int_0^t \left[ B_{21}^* b_{in}(\tau) + B_{22}^* b_{in}^\dagger(\tau) \right] X_1(\tau) d\tau + \mu \int_0^t \left[ B_{21}^* b_{in}(\tau) + B_{22}^* b_{in}^\dagger(\tau) \right] X_2(\tau) d\tau.$$

Noticing that

$$b_{out}(t) = b_{in}(t) + L_1 \langle X_1(t) \rangle_0 + \mu L_2 \langle X_2(t) \rangle_0,$$

we can obtain Eq. (25) by iterating Eq. (38) into Eq. (37).

Derivations of Eqs. (52) and (53): Under the weak nonlinearity assumption $\chi \ll \omega_a, \gamma$, we can expand the system dynamics of the Kerr cavity up to the third-order terms of
by Eq. (21), we can calculate the Volterra kernels using

\[ k(\tau) = i T e^A T x_0, \]

and

\[ k_{\pm \pm} = (\tau_1, \tau_2, \tau_3) = i T e^{A T x_0}, \]

where

\[ l = (010 \cdots 0)^T \] and \[ x_0 = (X)_0 = (10 \cdots 0)^T. \]

\[ \langle \cdot \rangle_0 \] is the average taking over the initial vacuum state of the Kerr cavity.

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\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 - \left( \frac{i}{2} + i\omega_a \right) & 0 & 0 & 0 & 0 & 0 & -2i\chi \\
0 & 0 & 0 & 0 & 0 & 0 & 2\chi \\
0 & 0 - \left( \frac{i}{2} - i\omega_a \right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma & 0 & 0 & 0 \\
0 & 0 & -\gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
B_- = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\gamma} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\gamma} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\gamma} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
B_+ = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{\gamma} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\gamma} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\gamma} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(39)

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**Jing Zhang** received his B.S. degree from Department of Mathematical Science and Ph.D. degree from Department of Automation, Tsinghua University, Beijing, China, in 2001 and 2006, respectively.

From 2006 to 2008, he was a Postdoctoral Fellow at the Department of Computer Science and Technology, Tsinghua University, Beijing, China, and a Visiting Researcher from 2008 to 2009 at the Advanced Science Institute, the Institute of Physical and Chemical Research (RIKEN), Japan. In 2010, he worked as a Visiting Assistant Professor at Department of Physics and National Center for Theoretical Sciences, National Cheng Kung University, Taiwan. He is now an Associate Professor at the Department of Automation, Tsinghua University, Beijing, China. His research interests include quantum control and nano manipulation.

**Yu-xi Liu** received his B.S. degree, M.S. degree and Ph.D. degree from Department of Physics, Shaanxi Normal University, Jilin University and Peking University in 1989, 1995 and 1998, respectively.

From 1998 to 2000, he was a Post-doctor at the Institute of Theoretical Physics, the Chinese Academy of Sciences, China. From 2000 to 2002, he was a JSPS Postdoctoral fellow at the Graduate University for Advanced Studies (SOKENDAI), Japan. From 2002 to 2009, he was a research scientist in the Institute of Physical and Chemical Research (RIKEN), Japan.

Since 2009, he has been a Professor with Institute of Microelectronics, Tsinghua University. His research interests include solid state quantum devices, quantum information processing, quantum optics and quantum control theory.

**Re-Bing Wu** received his B.S. degree in Electrical Engineering and Ph.D. degree in Control Science and Engineering from Tsinghua University, Beijing, China, in 1998 and 2004, respectively.

From 2005 to 2008, he was a Research Associate Fellow at the Department of Chemistry, Princeton University, USA. Since 2009, he has been an Associate Professor at the Department of Automation, Tsinghua University, Beijing, China. His research interests include quantum mechanical control theory and nonlinear control theory.

**Kurt Jacobs** received a B.S. and M.S. degree in Physics from the University of Auckland, and a Ph.D. degree in Physics from Imperial College, London, which he completed in 1998. He then held postdoctoral positions at Los Alamos National Laboratory, Griffith University, and Louisiana State University, before joining the University of Massachusetts at Boston as an Assistant Professor. He has been an Associate Professor there since 2011. His research interests include quantum measurement theory, feedback control in mesoscopic systems, and quantum thermodynamics. He is the author of *Stochastic Processes for Physicists, and Quantum Measurement Theory and Its Applications*, from Cambridge University Press.

**Kurt Jacobs** received a B.S. and M.S. degree in Physics from the University of Auckland, and a Ph.D. degree in Physics from Imperial College, London, which he completed in 1998. He then held postdoctoral positions at Los Alamos National Laboratory, Griffith University, and Louisiana State University, before joining the University of Massachusetts at Boston as an Assistant Professor. He has been an Associate Professor there since 2011. His research interests include quantum measurement theory, feedback control in mesoscopic systems, and quantum thermodynamics. He is the author of *Stochastic Processes for Physicists, and Quantum Measurement Theory and Its Applications*, from Cambridge University Press.

**Lan Yang** received her B.S. degree in Materials Physics and M.S. degree in Solid State Physics from University of Science and Technology of China in 1997 and 1999, and received her M.S. degree in Materials Science and Ph.D. degree in Applied Physics from Caltech in 2000 and 2005.

From 2005 to 2006, she was a Post-doctoral Scholar/Research Associate at Department of Applied Physics, Caltech. From 2007 to 2012, she was an Das Family Distinguished Career Development Assistant Professor at the Preston M. Green Department of Electrical and Systems Engineering, Washington University in St. Louis. Since 2012, she has been an Associate Professor at the Preston M. Green Department of Electrical and Systems Engineering, Washington University in St. Louis. She won the NSF CAREER Award and the Presidential Early Career Award for Scientists and Engineers (PECASE) in 2010. Her research interests include micro/nano photonics, quantum optics, and quantum information processing.
Tzyh-Jong Tarn (M71-SM83-F85) received the D.Sc degree in control system engineering from Washington University at St. Louis, Missouri, USA.

He is currently a Senior Professor in the Department of Electrical and Systems Engineering at Washington University, St. Louis, USA. He also is the director of the Center for Quantum Information Science and Technology at Tsinghua University, Beijing, China.

An active member of the IEEE Robotics and Automation Society, Dr. Tarn served as the President of the IEEE Robotics and Automation Society, 1992-1993, the Director of the IEEE Division X (Systems and Control), 1995-1996, and a member of the IEEE Board of Directors, 1995-1996.

He is the first recipient of the Nakamura Prize (in recognition and appreciation of his contribution to the advancement of the technology on intelligent robots and systems over a decade) at the 10th Anniversary of IROS in Grenoble, France, 1997, the recipient of the prestigious Joseph F. Engelberger Award of the Robotic Industries Association in 1999 for contributing to the advancement of the science of robotics, the Auto Soft Lifetime Achievement Award in 2000 in recognition of his pioneering and outstanding contributions to the fields of Robotics and Automation, the Pioneer in Robotics and Automation Award in 2003 from the IEEE Robotics and Automation Society for his technical contribution in developing and implementing nonlinear feedback control concepts for robotics and automation, and the George Saridis Leadership Award from the IEEE Robotics and Automation Society in 2009. In 2010 he received the Einstein Chair Professorship Award from the Chinese Academy of Sciences and the John R. Ragazzini Award from the American Automatic Control Council. He was featured in the Special Report on Engineering of the 1998 Best Graduate School issue of US News and World Report and his research accomplishments were reported in the Washington Times, Washington D.C., the Financial Times, London, Le Monde, Paris, and the Chicago Sun-Times, Chicago, etc. Dr. Tarn is an IFAC Fellow.

Franco Nori received his M.S. and Ph.D. in Physics from the University of Illinois at Urbana-Champaign, USA, in 1982 and 1987. From 1987 to 1989, he was a Postdoctoral Research Fellow at the Institute for Theoretical Physics, University of California, Santa Barbara. Since 1990, he has been Assistant Professor, Associate Professor, Full Professor and Research Scientist at the Department of Physics, University of Michigan, Ann Arbor, USA. Also, since 2002, he has been a Team Leader at the Advanced Science Institute, RIKEN, Saitama, Japan. Since 2013, he is a RIKEN Chief Scientist, as well as a Group Director of the Quantum Condensed Matter Research Group, at CEMS, RIKEN.

In 1997 and 1998, he received the “Excellence in Education Award” and “Excellence in Research Award” from the Univ. of Michigan. In 2002, he was elected Fellow of the American Physical Society (APS), USA. In 2003, he was elected Fellow of the Institute of Physics (IoP), UK. In 2007, he was elected Fellow of the American Association for the Advancement of Science (AAAS), USA. In 2013 he received the Prize for Science and Technology, the Commendation for Science and Technology, by the Minister of Education, Culture, Sports, Science and Technology, Japan.

His research interests include nano-science, condensed matter physics, quantum circuitry, quantum information processing, the dynamics of complex systems, and the interface between mesoscopics, quantum optics, atomic physics, and nano-science.