Classifying \( k \)-Edge Colouring for \( H \)-free Graphs\(^*\)

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Abstract. A graph is \( H \)-free if it does not contain an induced subgraph isomorphic to \( H \). For every integer \( k \) and every graph \( H \), we determine the computational complexity of \( k \)-Edge Colouring for \( H \)-free graphs.

1 Introduction

A graph \( G = (V,E) \) is \( k \)-edge colourable for some integer \( k \) if there exists a mapping \( c : E \rightarrow \{1, \ldots, k\} \) such that \( c(e) \neq c(f) \) for any two edges \( e \) and \( f \) of \( G \) that have a common end-vertex. The chromatic index of \( G \) is the smallest integer \( k \) such that \( G \) is \( k \)-edge colourable. Vizing proved the following classical result.

Theorem 1 ([27]). The chromatic index of a graph \( G \) with maximum degree \( \Delta \) is either \( \Delta \) or \( \Delta + 1 \).

The Edge Colouring problem is to decide if a given graph \( G \) is \( k \)-edge colourable for some given integer \( k \). Note that \((G,k)\) is a yes-instance if \( G \) has maximum degree at most \( k-1 \) by Theorem 1 and that \((G,k)\) is a no-instance if \( G \) has maximum degree at least \( k+1 \). If \( k \) is fixed, that is, \( k \) is not part of the input, then we denote the problem by \( k \)-Edge Colouring. It is trivial to solve this problem for \( k = 2 \). However, the problem is NP-complete if \( k \geq 3 \), as shown by Holyer for \( k = 3 \) and by Leven and Galil for \( k \geq 4 \).

Theorem 2 ([14, 20]). For \( k \geq 3 \), \( k \)-Edge Colouring is NP-complete even for \( k \)-regular graphs.

Due to the above hardness results we may wish to restrict the input to some special graph class. A natural property of a graph class is to be closed under vertex deletion. Such graph classes are called hereditary and they form the focus of our paper. To give an example, bipartite graphs form a hereditary graph class, and it is well-known that they have chromatic index \( \Delta \). Hence, Edge Colouring is polynomial-time solvable for bipartite graphs, which are perfect and triangle-free. In contrast, Cai and Ellis [4] proved that for every \( k \geq 3 \), \( k \)-Edge Colouring is NP-complete for \( k \)-regular comparability graphs, which are also perfect. They also proved the following two results, the first one of which shows that Edge Colouring is NP-complete for triangle-free graphs (the graph \( C_s \) denotes the cycle on \( s \) vertices).

Theorem 3 ([4]). Let \( k \geq 3 \) and \( s \geq 3 \). Then \( k \)-Edge Colouring is NP-complete for \( k \)-regular \( C_s \)-free graphs.

Theorem 4 ([4]). Let \( k \geq 3 \) be an odd integer. Then \( k \)-Edge Colouring is NP-complete for \( k \)-regular line graphs of bipartite graphs.

It is also known that Edge Colouring is polynomial-time solvable for chordless graphs [22], series-parallel graphs [16], split-indifference graphs [26] and for graphs of treewidth at most \( k \) for any constant \( k \) [1].

It is not difficult to see that a graph class \( \mathcal{G} \) is hereditary if and only if it can be characterized by a set \( \mathcal{F}_G \) of forbidden induced subgraphs (see, for example, [17]). Malyshev determined the complexity of 3-Edge Colouring for every hereditary graph class \( \mathcal{G} \), for which \( \mathcal{F}_G \) consists of graphs that each have at most five vertices, except perhaps two graphs that may contain six vertices [23]. Malyshev performed a

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similar complexity study for Edge Colouring for graph classes defined by a family of forbidden (but not necessarily induced) graphs with at most seven vertices and at most six edges [24].

We focus on the case where \( F_G \) consists of a single graph \( H \). A graph \( G \) is \( H \)-free if \( G \) does not contain an induced subgraph isomorphic to \( H \). We obtain the following dichotomy for \( H \)-free graphs.

**Theorem 5.** Let \( k \geq 3 \) be an integer and \( H \) be a graph. If \( H \) is a linear forest, then \( k \)-Edge Colouring is polynomial-time solvable for \( H \)-free graphs. Otherwise \( k \)-Edge Colouring is \textsc{NP}-complete even for \( k \)-regular \( H \)-free graphs.

We obtain Theorem 5 by combining Theorems 3 and 4 with two new results. In particular, we will prove a hardness result for \( k \)-regular claw-free graphs for even integers \( k \) (as Theorem 4 is only valid when \( k \) is odd).

## 2 Preliminaries

The graphs \( C_n \), \( P_n \) and \( K_n \) denote the path, cycle and complete graph on \( n \) vertices, respectively. A set \( I \) is an independent set of a graph \( G \) if all vertices of \( I \) are pairwise nonadjacent in \( G \). A graph \( G \) is bipartite if its vertex set can be partitioned into two independent sets \( A \) and \( B \). If there exists an edge between every vertex of \( A \) and every vertex of \( B \), then \( G \) is complete bipartite. The claw \( K_{1,3} \) is the complete bipartite graph with \( |A| = 1 \) and \( |B| = 3 \).

Let \( G_1 \) and \( G_2 \) be two vertex-disjoint graphs. The join operation \( \times \) adds an edge between every vertex of \( G_1 \) and every vertex of \( G_2 \). The disjoint union operation \( + \) merges \( G_1 \) and \( G_2 \) into one graph without adding any new edges, that is, \( G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \). We write \( rG \) to denote the disjoint union of \( r \) copies of a graph \( G \).

A forest is a graph with no cycles. A linear forest is a forest of maximum degree at most 2, or equivalently, a disjoint union of one or more paths. A graph \( G \) is a cograph if \( G \) can be generated from \( K_1 \) by a sequence of join and disjoint union operations. A graph is a cograph if and only if it is \( P_4 \)-free (see, for example, [3]). The following well-known lemma follows from this equivalence and the definition of a cograph.

**Lemma 1.** Every connected \( P_4 \)-free graph on at least two vertices has a spanning complete bipartite subgraph.

Let \( G = (V,E) \) be a graph. For a subset \( S \subseteq V \), the graph \( G[S] = (S, \{uv \in E \mid u, v \in S\}) \) denotes the subgraph of \( G \) induced by \( S \). We say that \( S \) is connected if \( G[S] \) is connected. Recall that a graph \( G \) is \( H \)-free for some graph \( H \) if \( G \) does not contain \( H \) as an induced subgraph. A subset \( D \subseteq V(G) \) is dominating if every vertex of \( V(G) \setminus D \) is adjacent to at least one vertex of \( D \). We will need the following result of Camby and Schaudt.

**Theorem 6 ([5]).** Let \( t \geq 4 \) and \( G \) be a connected \( P_t \)-free graph. Let \( X \) be any minimum connected dominating set of \( G \). Then \( G[X] \) is either \( P_{t-2} \)-free or isomorphic to \( P_{t-2} \).

Let \( G = (V,E) \) be some graph. The degree of a vertex \( u \in V \) is equal to the size of its neighbourhood \( N(u) = \{v \mid uv \in E\} \). The graph \( G \) is \( r \)-regular if every vertex of \( G \) has degree \( r \). The line graph of \( G \) is the graph \( L(G) \), which has vertex set \( E \) and an edge between two distinct vertices \( e \) and \( f \) if and only if \( e \) and \( f \) have a common end-vertex in \( G \).

## 3 The Proof of Theorem 5

To prove our dichotomy, we first consider the case where the forbidden induced subgraph \( H \) is a claw. As line graphs are claw-free, we only need to deal with the case where the number of colours \( k \) is even due to Theorem 4. For proving this case we need another result of Cai and Ellis, which we will use as a lemma. Let \( c \) be a \( k \)-edge colouring of a graph \( G = (V,E) \). Then a vertex \( u \in V \) misses colour \( i \) if none of the edges incident to \( u \) is coloured \( i \).
**Lemma 2** ([4]). For even $k \geq 2$, the complete graph $K_k$ has a $k$-edge colouring with the property that $V(K_k)$ can be partitioned into sets $\{u_i, u'_i\}$ ($1 \leq i \leq \frac{k}{2}$), such that for $i = 1, \ldots, \frac{k}{2}$, vertices $u_i$ and $u'_i$ miss the same colour, which is not missed by any of the other vertices.

We use Lemma 2 to prove the following result, which solves the case where $k$ is even and $H = K_{1,3}$.

**Lemma 3.** Let $k \geq 4$ be an even integer. Then $k$-Edge Colouring is NP-complete for $k$-regular claw-free graphs.

**Proof.** Recall that $k$-Edge Colouring for $k$-regular graphs is NP-complete for every integer $k \geq 4$ due to Theorem 2. Consider an instance $(G,k)$ of $k$-Edge Colouring, where $G$ is $k$-regular for some even integer $k = 2\ell \geq 4$. From $G$ we construct a graph $G'$ as follows. First we replace every vertex $v$ in $G$ by the gadget $H(v)$ shown in Figure 1. Next we connect the different gadgets in the following way. Every gadget $H(v)$ has exactly $k$ pendant edges, which are incident with vertices $v_1, \ldots, v_{\ell}, v_{\ell+1}, \ldots, v_{2\ell}$, respectively. As $G$ is $k$-regular, every vertex has $k$ neighbours in $G$. Hence, we can identify each edge $uv$ of $G$ with a unique edge $u_hv_i$ in $G'$, which is a pendant edge of both $H(u)$ and $H(v)$. It is readily seen that $G'$ is $k$-regular and claw-free.

![Fig. 1. The gadget $H(v)$ where $K_i(v)$ is a complete graph of size $2\ell$ for $i = 1, 2$. Note that edges inside $K_1(v)$ and $K_2(v)$ are not drawn.](image)

First suppose that $G$ is $k$-edge colourable. Let $c$ be a $k$-edge colouring of $G$. Consider a vertex $v \in V(G)$. For every neighbour $u$ of $v$ in $G$, we colour the pendant edge in $H(v)$ corresponding to the edge $uv$ with colour $c(uv)$. As $c$ assigned different colours to the edges incident to $v$, the $2\ell$ pendant edges of $H(v)$ will receive pairwise distinct colours, which we denote by $x_1, \ldots, x_{\ell}, y_1, \ldots, y_{\ell}$. By Lemma 2, we can colour the edges of $K_1(v)$ in such a way that for $i = 1, \ldots, \ell$, $v_i$ and $v'_i$ miss colour $x_i$. For $i = 1, \ldots, \ell$, we can therefore assign colour $x_i$ to edge $v_iw$. Similarly, we may assume that for $i = 1, \ldots, \ell$, $v_{\ell+i}$ and $v'_{\ell+i}$ miss colour $y_i$. For $i = 1, \ldots, \ell$, we can therefore assign colour $y_i$ to edge $v'_{\ell+i}w$. Recall that the colours $x_1, \ldots, x_{\ell}, y_1, \ldots, y_{\ell}$ are all different. Hence, doing this procedure for each vertex of $G$ yields a $k$-edge colouring $c'$ of $G'$.

Now suppose that $G'$ is $k$-edge colourable. Let $c'$ be a $k$-edge colouring of $G'$. Consider some $v \in V(G)$. Denote the pendant edges of $H(v)$ by $e_i$ for $i = 1, \ldots, 2\ell$, where $e_i$ is incident to $v_i$ (and to some vertex $u_h$ in a gadget $H(u)$ for each neighbour $u$ of $v$ in $G$). Suppose that $c'$ gave colour $x$ to an edge $wv'_i$ for some $1 \leq i \leq \ell$, say to $wv'_1$, but not to any edge $e_i$ for $i = 1, \ldots, \ell$. Note that $wv'_2, \ldots, wv'_\ell$ cannot be coloured $x$. As every vertex of $G'$ has degree $k = 2\ell$, every $v_i$ with $1 \leq i \leq \ell$ and every $v'_j$ with $2 \leq j \leq \ell$ is incident to some edge coloured $x$. $x$ is neither the colour of $e_1, \ldots, e_{\ell}$ nor the colour of $wv'_2, \ldots, wv'_\ell$, the complete graph $K_1(v) - v'_1$ contains a perfect matching all of whose edges have colour $x$. However, $K_1(v) - v'_1$ has odd size $2\ell - 1$. Hence, this is not possible. We conclude that each of the (pairwise distinct) colours of $wv'_1, \ldots, wv'_\ell$, which we denote by $x_1, \ldots, x_{\ell}$, is the colour of an edge $e_i$ for some $1 \leq i \leq \ell$. 

3
Let $y_1, \ldots, y_{\ell}$ be the (pairwise distinct) colours of $uv_{t+1}', \ldots, uv_{2\ell}'$, respectively. By the same arguments as above, we find that each of those colours is also the colour of a pendant edge of $H(v)$ that is incident to a vertex $v_{t+i}$ for some $1 \leq i \leq \ell$. Note that $x_1, \ldots, x_{\ell}, y_1, \ldots, y_{\ell}$ are $2\ell$ pairwise distinct colours, as they are colours of edges incident to the same vertex, namely vertex $w$. Hence, we can define a $k$-colouring $c$ of $G$ by setting $c(uv) = c'(u_kv_i)$ for every edge $uv \in E(G)$ with corresponding edge $u_kv_i \in E(G')$.

We note that the graph $G'$ in the proof of Lemma 3 is not a line graph, as the gadget $H(v)$ is not a line graph: the vertices $v_1', v_2', v_1, w$ form a diamond and by adding the pendant edge incident to $v_1$ and the edge $uv_{t+1}'$ we obtain an induced subgraph of $H(v)$ that is not a line graph.

To handle the case where the forbidden induced subgraph $H$ is a path, we make the following observation.

**Observation 1** If a graph $G$ of maximum degree $k$ has a dominating set of size at most $p$, then $G$ has at most $p(k+1)$ vertices.

We use Observation 1 to prove the following lemma.

**Lemma 4.** Let $k \geq 0$ and $t \geq 1$. Every connected $P_t$-free graph of maximum degree $k$ has at most $f(k,t)$ vertices for some function $f$ that only depends on $k$ and $t$.

**Proof.** Let $G$ be a connected $P_t$-free graph of maximum degree at most $k$. We use induction on $t$.

First suppose $t = 4$ (and observe that if the claim holds for $t = 4$, it also holds for $t \leq 3$). As $G$ is connected, $G$ has a dominating set of size 2 due to Lemma 1. Hence, by Observation 1, we find that $G$ has at most $f(k,2) = 2(k+1)$ vertices.

Now suppose $t \geq 5$. Let $X$ be an arbitrary minimum connected dominating set of $G$. By Theorem 6, $G[X]$ is either $P_{t-2}$-free or isomorphic to $P_{t-2}$. In the first case we use the induction hypothesis to conclude that $G[X]$ has at most $f(k,t-2)$ vertices. Hence, $G$ has at most $f(k,t-2)(k+1)$ vertices by Observation 1.

In the second case, we find that $G$ has at most $(t-2)(k+1)$ vertices. We set $f(k,t) = \max\{f(k,t-2)(k+1), (t-2)(k+1)\}$. 

We use Lemma 4 to prove our next lemma.

**Lemma 5.** Let $k \geq 3$ and $t \geq 1$. Then $k$-Edge Colouring is linear-time solvable for $P_t$-free graphs.

**Proof.** Let $G$ be a $P_t$-free graph. We compute the set of connected components of $G$ in linear time. For each connected component $\Delta_D$ of $G$ we do as follows. We first compute in linear time the maximum degree $\Delta_D$ of $D$. If $\Delta_D \leq k-1$, then $D$ is $k$-edge colourable by Theorem 1. If $\Delta_D \geq k+1$, then $D$ is not $k$-edge colourable. Hence, we may assume that $\Delta_D = k$. By Lemma 4, $D$ has at most $f(k,t)$ vertices for some function $f$ that only depends on $k$ and $t$. As we assume that $k$ and $t$ are constants, this means that we can now check in constant time if $D$ is $k$-edge colourable. Note that $G$ is $k$-edge colourable if and only if every connected component of $G$ is $k$-edge colourable. Hence, by using the above procedure, deciding if $G$ is $k$-edge colourable takes linear time.

We are now ready to prove Theorem 5, which we restate below.

**Theorem 5. (restated)** Let $k \geq 3$ be an integer and $H$ be a graph. If $H$ is a linear forest, then $k$-Edge Colouring is linear-time solvable for $H$-free graphs. Otherwise $k$-Edge Colouring is NP-complete even for $k$-regular $H$-free graphs.

**Proof.** First suppose that $H$ contains a cycle $C_s$ for some $s \geq 3$. Then the class of $H$-free graphs is a superclass of the class of $C_s$-free graphs. This means that we can apply Theorem 3. From now on assume that $H$ contains no cycle, so $H$ is a forest. Suppose that $H$ contains a vertex of degree at least 3. Then the class of $H$-free graphs is a superclass of the class of $K_{1,3}$-free graphs, which in turn forms a superclass of the class of line graphs. Hence, if $k$ is odd, then we apply Theorem 4, and if $k$ is even, then we apply Lemma 3. From now on assume that $H$ contains no cycle and no vertex of degree at least 3. Then $H$ is a linear forest, say with $\ell$ connected components. Let $t = \ell|V(H)|$. Then the class of $H$-free graphs is contained in the class of $P_t$-free graphs. Hence we may apply Lemma 5. This completes the proof of Theorem 5. 

4
4 Conclusions

We gave a complete complexity classification of $k$-Edge COLOURING for $H$-free graphs, showing a dichotomy between linear-time solvable cases and NP-complete cases. We saw that this depends on $H$ being a linear forest or not. It would be interesting to prove a dichotomy result for Edge COLOURING restricted to $H$-free graphs. Note that due to Theorem 5 we only need to consider the case where $H$ is a linear forest. However, even determining the complexity for small linear forests $H$, such as the cases where $H = 2P_2$ and $H = P_4$, turns out to be a difficult problem. In fact, the computational complexity of Edge COLOURING for split graphs, or equivalently, $(2P_2, C_4, C_5)$-free graphs [10] and for $P_4$-free graphs has yet to be settled, despite the efforts towards solving the problem for these graph classes [6, 8, 21].

On a side note, a graph is $k$-edge colourable if and only if its line graph is $k$-vertex colourable. In contrast to the situation for Edge COLOURING, the computational complexity of Vertex COLOURING has been fully classified for $H$-free graphs [19]. However, the computational complexity for $k$-VERTEX COLOURING restricted to $H$-free graphs has not been fully classified. It is known that for every $k \geq 3$, $k$-VERTEX COLOURING on $H$-free graphs is NP-complete if $H$ contains a cycle [9] or an induced claw [14, 20], but the case where $H$ is a linear forest has not been settled yet. The complexity status of $k$-VERTEX COLOURING is even still open for $P_t$-free graphs. More precisely, it is known that the cases $k \leq 2, t \geq 1$ (trivial), $k \geq 3, t \leq 5$ [13], $k = 3, 6 \leq t \leq 7$ [2] and $k = 4, t = 6$ [7] are polynomial-time solvable and that the cases $k = 4, t \geq 7$ [15] and $k \geq 5, t \geq 6$ [15] are NP-complete. However, the remaining cases, that is, the cases where $k = 3$ and $t \geq 8$ are still open. We refer to the survey [11] or some recent papers [12, 18, 25] for further background information.

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