THE CENTRAL LIMIT PROBLEM FOR RANDOM VECTORS WITH SYMMETRIES

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Abstract. Motivated by the central limit problem for convex bodies, we study normal approximation of linear functionals of high-dimensional random vectors with various types of symmetries. In particular, we obtain results for distributions which are coordinatewise symmetric, uniform in a regular simplex, or spherically symmetric. Our proofs are based on Stein’s method of exchangeable pairs; as far as we know, this approach has not previously been used in convex geometry and we give a brief introduction to the classical method. The spherically symmetric case is treated by a variation of Stein’s method which is adapted for continuous symmetries.

1. Introduction

Given a random vector $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$, $n \geq 2$, and a fixed $\theta \in S^{n-1}$, consider the random variable

$$W_\theta = \langle X, \theta \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. A typical example of interest is when $X$ is distributed uniformly in a convex body. In this paper we are interested in determining sufficient conditions under which $W_\theta$ is approximately normal, and in obtaining specific error estimates, possibly depending on $\theta$. To do this, we apply Stein’s method of exchangeable pairs. This technique has not previously been used in studying problems from convex geometry, and we believe it will continue to be useful in that context.

To begin with, we will assume that $X$ is isotropic, that is, that $W_\theta$ has mean 0 and variance 1 for every $\theta \in S^{n-1}$. Equivalently, $X$ is isotropic if

$$\mathbb{E}X_i = 0 \quad \text{and} \quad \mathbb{E}X_iX_j = \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. This is no real restriction, since every random vector with finite second moment which is not supported on a proper affine subspace has an affine image which is isotropic.

In the case that the components of $X$ are independent, bounds on the distance of $W_\theta$ from normal follow from classical results. For example, the Berry-Esseen theorem for sums of independent, non-identically distributed random variables implies that if $X$ is isotropic with independent components, then

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}[W_\theta \leq t] - \Phi(t) \right| \leq 0.8 \left( \max_{1 \leq i \leq n} \mathbb{E}|X_i|^3 \right) \sum_{j=1}^{n} |\theta_j|^3,$$

where $\Phi$ denotes the standard normal distribution function. There is also a body of work going back to Sudakov [31] (see [6] for a recent contribution and further references) on
randomized central limit theorems, which show that under quite general assumptions on $X$, $W_\theta$ is close to some average distribution for most $\theta$. In these results the average distribution may or may not be normal, and “most” may refer to the rotation invariant probability measure on $S^{n-1}$ or to some other distribution on weights.

Our motivation in studying this problem comes in part from the so-called central limit problem for convex bodies, which is to show that the uniform measure on any high-dimensional convex body has some one-dimensional projection which is approximately normal, or that most one-dimensional projections are approximately normal. Most of the results on this problem [1, 9, 28] prove some form of the latter conjecture (under appropriate assumptions on the convex body), and thus fit into the framework of randomized central limit theorems; none of the results in these papers identify any specific $\theta$ for which $W_\theta$ is approximately normal. The paper [10] studies approximate normality of $W_\theta$ for specific $\theta$ when $X$ is uniformly distributed in a cube, Euclidean ball, crosspolytope, or simplex, but in the last two cases only for a very restricted set of $\theta$ and with rather limited quantitative information.

Of course, there is no hope to identify any specific $\theta$ for which $W_\theta$ is approximately normal without some additional assumptions on the distribution of $X$. Here the additional assumptions we consider are more geometric than probabilistic in nature. Specifically, we consider distributions which have a sufficiently rich class of symmetries, although we emphasize that our results do not require $X$ to be drawn from a convex body, or even to be continuous. Stein’s method, described in Section 3, allows us to take advantage of these symmetries in order to reduce normal approximation to estimation of certain low-order moments.

Our first main result treats distributions which are symmetric with respect to reflection in a suitable collection of hyperplanes. Our hypothesis encompasses both the class of distributions which are coordinatewise symmetric (Corollary 2) and those with the symmetries of a regular simplex (Corollary 6). The error bounds for the approximations are in many cases small enough to derive multivariate randomized versions which improve on existing results.

Our second main result, treating spherically symmetric distributions, is proved by a variation of the classical version of Stein’s method adapted to take advantage of continuous symmetries. This result has as corollaries several classical results as well as some new applications. We also make connections with Poincaré inequalities for probability measures on $\mathbb{R}^n$.

The layout of this paper is as follows. We first define notations which will be used throughout the paper. In Section 2 we state our results and several corollaries, and give comparisons to existing results. Section 3 gives a brief introduction to Stein’s method. Section 4 contains the proofs of the first main theorem and its corollaries. Section 5 contains the proof of the second main theorem using the variation of Stein’s method described above, and the proofs of its corollaries.

**Notation.** Let $\ell^p_n = (\mathbb{R}^n, \| \cdot \|_p)$, where $\| \cdot \|_p$ denotes the norm

$$
\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}
$$
for $1 \leq p < \infty$, and
\[ \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|. \]

For $v \in \mathbb{R}^n$, define $v \otimes v : \mathbb{R}^n \to \mathbb{R}^n$ by
\[ v \otimes v(x) = \langle x, v \rangle v; \]
if $v \in S^{n-1}$, $v \otimes v$ is the orthogonal projection onto the span of $v$. A set of vectors $u_1, \ldots, u_m \in S^{n-1}$ such that
\[ \sum_{i=1}^m u_i \otimes u_i = \frac{m}{n} I_n, \]
is known in the signal processing literature as a normalized tight frame ($I_n$ is the identity on $\mathbb{R}^n$). By taking the trace of both sides of (1), one can see that $\frac{m}{n}$ is the only possible constant that can appear.

The Grassmann manifold of $k$-dimensional subspaces of $\mathbb{R}^n$ is denoted $G_{n,k}$; it is equipped with a unique rotation-invariant probability measure $\lambda_{n,k}$. For a fixed subspace $E \subset \mathbb{R}^n$, let $P_E$ denote the orthogonal projection onto $E$, and $\gamma_E$ the standard Gaussian measure on $E$.

For a random variable or random vector $X$, let $\mathcal{L}(X)$ denote the distribution of $X$. Given two probability measures $\mu$ and $\nu$ on $E$, define the $T$-distance between them as
\[ T(\mu, \nu) = \sup \{ |\mu(H) - \nu(H)| : H \text{ is an affine half-space of } E \}. \]

In particular,
\[ T(\mathcal{L}(P_E(X)), \gamma_E) = \sup_{\theta \in E \cap S^{n-1}, t \in \mathbb{R}} |\mathbb{P}[W_\theta \leq t] - \Phi(t)|. \]
This is a geometrically natural notion of distance between measures since it is invariant under nonsingular affine transformations and is thus not tied to any particular coordinate system. In addition, the topology induced by $T$ on the space of probability measures on $E$ is stronger than the $w^*$ topology.

The total variation distance between two probability measures $\mu$ and $\nu$ is
\[ d_{TV}(\mu, \nu) = 2 \sup \{ |\mu(A) - \nu(A)| : A \text{ is measurable} \}. \]
Recall that if $\mu$ and $\nu$ both have densities, then their total variation distance is the $L_1$ distance between their densities.

Finally, note that symbols like $c$, $c_1$, etc. which represent absolute constants may have different values from one appearance to the next.

2. Statements of results

Theorem 1, the first main result of this paper, is based on existing normal approximation results proved via Stein’s method. Corollaries 2, 4, 5, and 6 are all applications of Theorem 1.

**Theorem 1.** Let $u_1, \ldots, u_m \in S^{n-1}$ be a normalized tight frame, and for any $x \in \mathbb{R}^n$ let $x_{(i)} = \langle x, u_i \rangle$. Suppose that $X$ is a random vector whose distribution is invariant under
reflections in each of the hyperplanes $u_i \perp \nu$. Let $\theta \in S^{n-1}$ be fixed. Then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}[W_\theta \leq t] - \Phi(t)| \leq 2 \sqrt{\frac{n^2}{m^2} \sum_{i,j=1}^m \theta_{(i)}^2 \theta_{(j)}^2 \mathbb{E}[X_{(i)}^2 X_{(j)}^2]} - 1 + \left(\frac{8}{\pi}\right)^{1/4} \sqrt{\frac{n}{m} \max_{1 \leq j \leq m} \mathbb{E}[|X_{(j)}|^3]} \sum_{i=1}^m |\theta_{(i)}|^3. $$

If in addition $\max_{1 \leq i \leq m} |X_{(i)}| \leq a$ almost surely, then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}[W_\theta \leq t] - \Phi(t)| \leq 24 \sqrt{\frac{n^2}{m^2} \sum_{i,j=1}^m \theta_{(i)}^2 \theta_{(j)}^2 \mathbb{E}[X_{(i)}^2 X_{(j)}^2]} - 1 + 172na^3 \max_{1 \leq i \leq m} |\theta_{(i)}|^3. $$

The constants that appear in these statements, and explicit constants which appear in any of the results that follow, are generally not the best possible, and are included only for concreteness.

It is not obvious from the statement that the error estimate in Theorem 1 is useful. However, the proofs of Corollaries 2 and 6 will show that Theorem 1 allows different cases of geometric interest to be treated easily in this unified framework.

Borrowing terminology from the geometry of Banach spaces, call $X$ unconditional if its distribution is invariant under reflections in the coordinate hyperplanes, or equivalently, if $X$ has the same distribution as $(\varepsilon_1 X_1, \ldots, \varepsilon_n X_n)$ for any choice of $\varepsilon \in \{-1, 1\}^n$.

**Corollary 2.** Let $X$ be unconditional and isotropic, and let $\theta \in S^{n-1}$ be fixed. Then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}[W_\theta \leq t] - \Phi(t)| \leq 2 \sqrt{\max_{1 \leq i \leq n} \mathbb{E}X_i^2} \|\theta\|_4^4 + \max_{i \neq j} \text{Cov}(X_i^2, X_j^2) + \left(\frac{8}{\pi}\right)^{1/4} \left(\max_{1 \leq i \leq n} \sqrt{\mathbb{E}|X_i|^3}\right) \|\theta\|_3^{3/2}. $$

If moreover $X \in [-a, a]^n$ almost surely, then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}[W_\theta \leq t] - \Phi(t)| \leq 24 \sqrt{\max_{1 \leq i \leq n} \mathbb{E}X_i^4} \|\theta\|_4^4 + \max_{i \neq j} \text{Cov}(X_i^2, X_j^2) + 172na^3 \|\theta\|_\infty^3. $$

The statement of Corollary 2 motivates the introduction of the following definition, taken from [23]. A random vector $X$ has the **square negative correlation property** if

$$\mathbb{E}X_i^2 X_j^2 \leq (\mathbb{E}X_i^2)(\mathbb{E}X_j^2) \quad \text{for } i \neq j,$$

i.e. if $\text{Cov}(X_i^2, X_j^2) \leq 0$ for $i \neq j$. 
A random vector $X$ is called \textit{log-concave} if it has a logarithmically concave density with respect to Lebesgue measure on $\mathbb{R}^n$. In particular, a random vector which is uniformly distributed on a convex body is log-concave.

It natural to conjecture (cf. Section 5 of [23]) that an isotropic unconditional log-concave random vector has the square negative correlation property. However, this is not the case, as shown by the example [5] of the density
\begin{equation}
 a_n e^{-b_n \|x\|_\infty}
\end{equation}
on $\mathbb{R}^n$, where $a_n$ and $b_n$ are appropriate normalizing constants. (Counterexamples also exist which are uniformly distributed in a convex body [5].) However, the weaker conjecture that under these conditions
\begin{equation}
 \text{Cov}(X_i^2, X_j^2) \leq \frac{c}{n} \quad \text{for } i \neq j,
\end{equation}
for some absolute constant $c$, is open (and is in particular satisfied by $X$ with the density (2)). This conjecture is related to the Kannan-Lovász-Simonovits conjecture on isoperimetric constants [17] (see [8] for a discussion of this issue, and cf. Corollary 11 below).

The error bounds in Theorem II are small enough in many cases to show that $W_\theta$ is uniformly close to normal for all unit vectors $\theta$ in a typical subspace $E \subset \mathbb{R}^n$ of relatively large dimension. This means that the projection of $X$ onto $E$ is close to normally distributed in the sense of $T$-distance. In order to quantify this phenomenon, for $\varepsilon > 0$, define
\begin{equation}
 A_{n,k}(\varepsilon) = \{ E \in G_{n,k} : T(\mathcal{L}(P_E(X)), \gamma_E) \leq \varepsilon \};
\end{equation}
that is, $A_{n,k}(\varepsilon)$ is the set of $k$-dimensional subspaces $E \subset \mathbb{R}^n$ such that the projection of $X$ onto $E$ is $\varepsilon$-close to normal in the sense of $T$-distance.

The following lemma, proved in [9], allows normal approximation in the sense of total variation to be deduced from approximation of distribution functions in the case of log-concave distributions.

**Lemma 3** (Brehm-Hinow-Vogt-Voigt). \textit{There is an increasing function $\beta : (0, \infty) \to (0, 2]$ satisfying}
\begin{equation}
 \beta(t) = O\left(\sqrt{-t \log t}\right) \quad \text{as } t \to 0
\end{equation}
such that for any log-concave random variable $W$,
\begin{equation}
 d_{TV}(\mathcal{L}(W), \gamma_{\mathbb{R}}) \leq \beta \left( \sup_{t \in \mathbb{R}} |\mathbb{P}[W \leq t] - \Phi(t)| \right).
\end{equation}

All references to $\beta$ in the next three results are to the function in Lemma 3.

**Corollary 4.** Let $X$ be unconditional and isotropic, with the square negative correlation property. Then there are constants $c_1, \ldots, c_{11}$, independent of $X$ and $n$, such that each of the following holds.

1. If $\mathbb{E}|X_i|^3 \leq a$ and $\mathbb{E}X_i^4 \leq b$ for all $i$, then
\begin{equation}
 \sup_{t \in \mathbb{R}} |\mathbb{P}[W_\theta \leq t] - \Phi(t)| \leq 2\left(\sqrt{b}\|\theta\|^2_3 + \sqrt{a}\|\theta\|^{3/2}_3\right)
\end{equation}
for all $\theta \in S^{n-1}$, and
$$\lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - c_1 \exp\left[-c_2 \min\{a^{-2/3} \varepsilon^{4/3}, b^{-1/2} \varepsilon\} n\right]$$
for $\varepsilon \geq c_3 \max\{\sqrt{an}^{-1/4}, \sqrt{bn}^{-1/2}\}$ and $k \leq c_4 \min\{a^{-2/3} \varepsilon^{4/3}, b^{-1/2} \varepsilon\} n$.
(2) If $X$ is log-concave, then
$$\sup_{t \in \mathbb{R}} \left|\mathbb{P}[W_{\theta} \leq t] - \Phi(t)\right| \leq c_5 \|\theta\|^3_{3/2}$$
for all $\theta \in S^{n-1}$, and
$$\lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - c_6 \exp\left[-c_7 \varepsilon^{4/3} n\right]$$
for $\varepsilon \geq c_8 \varepsilon^{4/3} n$. Furthermore,
$$d_{TV}(\mathcal{L}(W_{\theta}), \gamma_{\mathbb{R}}) \leq \beta\left(c_5 \|\theta\|^3_{3/2}\right)$$
for all $\theta \in S^{n-1}$.
(3) If $X \in [-a, a]^n$ almost surely, then
$$\sup_{t \in \mathbb{R}} \left|\mathbb{P}[W_{\theta} \leq t] - \Phi(t)\right| \leq 196 \varepsilon a^3 \|\theta\|^3_{\infty}$$
for all $\theta \in S^{n-1}$, and
$$\lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - c_9 a^{-2} \varepsilon^{2/3} n^{1/3}$$
for $\varepsilon \geq c_{10} a^3 (\log n)^{3/2} n^{-1/2}$ and $k \leq c_{11} a^{-2} \varepsilon^{2/3} n^{1/3}$. If moreover $X$ is log-concave, then
$$d_{TV}(\mathcal{L}(W_{\theta}), \gamma_{\mathbb{R}}) \leq \beta\left(196 \varepsilon a^3 \|\theta\|^3_{\infty}\right)$$
for all $\theta \in S^{n-1}$.

The square negative correlation property is included as a hypothesis of Corollary 4 only for convenience. Replacing it with the hypothesis (3) would result only in a weakening of the constants that appear, and in fact the even weaker hypothesis
$$\text{Cov}(X_i^2, X_j^2) \leq \frac{c}{\sqrt{n}} \quad \text{for } i \neq j$$
would suffice for the same conclusion in part 2. In particular, the conclusion of part 2 applies to $X$ distributed according to the density (2).

One could also deduce randomized total variation results for one-dimensional projections in parts 2 and 3 of Corollary 4, but the statements are more complicated.

Naor and Romik [23, Theorem 5] proved a result comparable to the randomized statement in part 1 of Corollary 4. Under similar hypotheses (but without unconditionality), they showed
$$\lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - \frac{c_1}{\varepsilon} \exp\left[-c_2 b^{-1} \varepsilon^4 n\right]$$
for $\varepsilon > 0$ and $k \leq c_3 b^{-1} \varepsilon^4 n$, so Corollary 4 improves on the dependence on both $b$ and $\varepsilon$ in the unconditional case. In the case that $X$ is uniform in a convex body (hence log-concave)
and has the square negative correlation property, Antilla, Ball, and Perissinaki [1, Theorem 2] showed
\[ \lambda_{n,1}(A_{n,1}(\varepsilon)) \geq 1 - n \exp \left[ -c_1 \varepsilon^2 n \right] \]
for \( \varepsilon \geq c_2 n^{-1/3} \). Part 2 of Corollary 1 improves (in the unconditional case) on their dependence on \( \varepsilon \), although for a slightly more restricted range of \( \varepsilon \), and does not require that \( X \) be chosen from a convex body. In the case of certain bounded distributions, part 3 improves further on the results of [1, 23], as will be illustrated by Corollary 5 below.

The next corollary treats a class of examples of particular interest in asymptotic convex geometry, namely, \( X \) chosen from various natural distributions on the unit balls of the spaces \( \ell^n_p \). In addition to the uniform measure on the interior, there are two geometrically natural measures on the boundary of a convex body \( K \) whose interior contains the origin. First there is \( (n-1) \)-dimensional Hausdorff measure, or surface measure. Second, there is cone measure \( \mu \), defined by
\[ \mu(A) = \text{vol} \left( \bigcup_{t \in [0,1]} tA \right) \quad \text{for } A \subset \partial K. \]
Cone measure is the measure on \( \partial K \) for which there is a straightforward extension of the familiar polar integration formula, with \( \partial K \) replacing \( S^{n-1} \).

**Corollary 5.** For \( 1 \leq p \leq \infty \), let \( X \) have one of the following distributions:
1. uniform measure on the ball of \( \ell^n_p \), scaled to be isotropic;
2. normalized cone measure on the sphere of \( \ell^n_p \), scaled to be isotropic; or
3. normalized surface measure on the sphere of \( \ell^n_p \), scaled such that the normalized cone measure is isotropic.

Then there are absolute constants \( c_1, \ldots, c_5 \) and constants \( d_{1,p}, \ldots, d_{4,p} \) depending only on \( p \) such that
\[ \sup_{t \in \mathbb{R}} \left| \mathbb{P}[W_\theta \leq t] - \Phi(t) \right| \leq \min \left\{ c_1 \|\theta\|_3^{3/2}, d_{1,p} n^{1+\frac{2}{p}} \|\theta\|_\infty \right\} \]
for all \( \theta \in S^{n-1} \),
\[ \lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - c_2 \exp \left[ -c_3 \varepsilon^{4/3} n \right] \]
for \( \varepsilon \geq c_4 n^{-1/4} \) and \( k \leq c_5 \varepsilon^{4/3} n \), and
\[ \lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - c_2 \exp \left[ -d_{2,p} \varepsilon^{2/3} n^{\frac{1}{p} - \frac{2}{p}} \right] \]
for \( \varepsilon \geq d_{3,p}(\log n)^{3/2} n^{\frac{3}{2p} - \frac{1}{2}} \) and \( k \leq d_{4,p} \varepsilon^{2/3} n^{\frac{1}{p} - \frac{2}{p}} \).

Furthermore, in the case that \( X \) is chosen uniformly from the rescaled \( \ell^n_p \) ball,
\[ d_{TV}(\mathcal{L}(W_\theta), \gamma_\mathbb{R}) \leq \beta \left( \min \left\{ c_1 \|\theta\|_3^{3/2}, d_{1,p} n^{1+\frac{2}{p}} \|\theta\|_\infty \right\} \right) \]
for all \( \theta \in S^{n-1} \).

To compare the two bounds in (5), note that since \( \|\theta\|_\infty \leq \|\theta\|_3 \leq \|\theta\|_2 = 1 \) and \( \|\theta\|_3 \geq n^{-1/6} \), the first error bound is better for all \( \theta \) when \( 1 \leq p < 4 \) (ignoring the constant factors). On the other hand, for the principal diagonal \( \theta = n^{-1/2} \sum e_i \) (which roughly captures typical
behavior for $\ell_p^n$ norms on $S^{n-1}$) the second bound is better for $p \geq 12$. In particular, for $p > 18$, Corollary 5 improves on the typical rate of convergence to normality of about $n^{-1/3}$ which follows from (14). Corollary 5 also improves on Theorems 7 and 8 of [23], which show

$$\lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - \frac{c_1}{\varepsilon} \exp \left[ -c_2 \varepsilon^4 n \right]$$

for $\varepsilon > 0$ and $k \leq c_3 \varepsilon^4 n$ in cases 2 and 3 of Corollary 5.

Brehm and Voigt [10] considered $X$ uniformly distributed in the rescaled $\ell_p^n$ ball for $p = 1, 2, \infty$. See the discussion of Corollary 9 below for the case $p = 2$. In the case $p = \infty$, they derive sharper error bounds for general $\theta$ than here. In the case $p = 1$ (the crosspolytope) however, they consider only the case $\theta = n^{-1/2} \sum e_i$ for $n \to \infty$, and do not obtain an explicit rate of convergence, so Corollary 5 provides a substantial generalization and strengthening.

As discussed earlier, the form of Theorem 1 is general enough to accommodate the symmetries both of unconditional distributions and of a regular simplex, which is treated in the next result.

**Corollary 6.** Let $\Delta_n = \sqrt{n(n+2)} \text{conv} \{v_1, \ldots, v_{n+1}\}$ be an isotropic regular simplex, where $v_i \in S^{n-1}$. Let $X$ be uniformly distributed in $\Delta_n$, and let $\theta \in S^{n-1}$ be fixed. Then there are constants $c_1, \ldots, c_5$, independent of $n$, such that

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}[W_{\theta} \leq t] - \Phi(t) \right| \leq c_1 \sqrt{\sum_{i=1}^{n+1} \left| \langle \theta, v_i \rangle \right|^3}$$

and

$$d_{TV}(\mathcal{L}(W_{\theta}), \gamma_{\mathbb{R}}) \leq \beta \left( c_1 \sqrt{\sum_{i=1}^{n+1} \left| \langle \theta, v_i \rangle \right|^3} \right)$$

for all $\theta \in S^{n-1}$, and

$$\lambda_{n,k}(A_{n,k}(\varepsilon)) \geq 1 - c_2 \exp \left[ -c_3 \varepsilon^4 n \right]$$

for $\varepsilon \geq c_4 n^{-1/4}$ and $k \leq c_5 \varepsilon^{4/3} n$.

The case in which $X$ is uniformly distributed in a regular simplex was also considered in [10]. However, the results there consider only a certain discrete set of $\theta$ (roughly those for which the behavior of $W_{\theta}$ is best) for $n \to \infty$, and do not derive any explicit rate of convergence to normality.

The remaining results are not based on existing normal approximation theorems; instead, the proofs use a variation of Stein’s method of exchangeable pairs, adapted to situations in which there are continuous symmetries. This variation was introduced by Stein in [30] and developed further by the first-named author in [21] in studying functions on the classical matrix groups.
Theorem 7. Let $X$ be an isotropic random vector, with finite third moment, whose distribution is spherically symmetric. Then for any $\theta \in S^{n-1}$,
\[
d_{TV}(\mathcal{L}(W_\theta), \gamma_{\mathbb{R}}) \leq 4\mathbb{E}\left(|1 - \mathbb{E}[X_2^2 | X_1]|\right)
\leq \frac{4}{n-1}\mathbb{E}\left[\|X\|_2^2 - n\right] + \frac{8}{n-1}
\leq \frac{4}{n-1}\sqrt{\text{Var}(\|X\|_2)} + \frac{8}{n-1}.
\]

The latter bounds in Theorem 7 reduce normal approximation of one-dimensional projections of spherically symmetric random vectors to the problem of estimating deviations of $\|X\|_2$ from its mean. Some kind of such concentration of $\|X\|_2$ is either explicitly a hypothesis, or closely related to the key hypothesis, in many of the existing results on the central limit problem for convex bodies, cf. [1, 8, 9, 28].

Before proceeding to some classical consequences of Theorem 7, we first state a corollary giving a new randomized central limit theorem, which in particular gives information about the central limit problem for convex bodies. Suppose $X$ is a (not necessarily spherically symmetric) isotropic random vector and $U$ is a random $n \times n$ orthogonal matrix, distributed according to Haar measure independently of $X$. Notice that the distribution of $W = \langle UX, e_1 \rangle = \langle X, U^{-1}e_1 \rangle$ is the average (with respect to the rotation invariant probability on $S^{n-1}$) of the distributions of $W_\theta$ over all $\theta \in S^{n-1}$. As mentioned in the introduction, the distribution of $W$ is the object of some of the work on randomized central limit theorems. Since $\tilde{X} = UX$ is a spherically symmetric isotropic random vector and $\|\tilde{X}\|_2 = \|X\|_2$, the following is an immediate consequence of Theorem 7.

Corollary 8. Let $X$ be an isotropic random vector with finite third moment, and let $W$ be as defined above. Then
\[
d_{TV}(\mathcal{L}(W), \gamma_{\mathbb{R}}) \leq \frac{4}{n-1}\mathbb{E}\left[\|X\|_2^2 - n\right] + \frac{8}{n-1}
\leq \frac{4}{n-1}\sqrt{\text{Var}(\|X\|_2)} + \frac{8}{n-1}.
\]

In the case in which $X$ is uniformly distributed in $K \subset \mathbb{R}^n$, the density of $W$ gives the average $(n-1)$-dimensional volume of a hyperplane section of $K$ at a given distance from the origin. For $K$ a convex body, Bobkov and Koldobsky [8] proved a pointwise bound on the difference between the density of $W$ and the standard normal density, with a bound which also explicitly involves the variance which appears in Corollary 8 (and which is also essentially of the order $n^{-1}$ as long as the variance is not too big). Corollary 8 gives instead an $L_1$ bound on the difference of these densities. See [18] for an earlier asymptotic result in the case that $K$ is a cube, and [4] for a generalization of the result of [8] to arbitrary distributions and a multivariate version for sections by $k$-codimensional affine subspaces.

The following easy corollary of Theorem 7 is well-known; versions for higher-dimensional projections are proved in [13] for the sphere and in [10] for the ball. Theorem 7 allows the cases of both the ball and the sphere to be presented simply as part of a unified framework.
Corollary 9. If $X$ has the uniform distribution on the Euclidean ball of radius $\sqrt{n+2}$ or the uniform distribution on the sphere of radius $\sqrt{n}$, then for any $\theta \in S^{n-1}$
\[ d_{TV}(\mathcal{L}(W_\theta), \gamma_{\mathbb{R}}) \leq \frac{a}{n-1}, \]
where $a = 16$ in the case of the ball and $a = 8$ in the case of the ball.

Corollary 9 gives the correct order of approximation in both cases, although the constants can be improved.

The first error estimate in Theorem 7 is also strong enough to recover, as an immediate consequence, a version of the characterization of the normal distribution as the unique spherically symmetric product measure on $\mathbb{R}^n$. (The two-dimensional version of this characterization is the classical Herschel-Maxwell theorem; see [11] for various other versions.)

**Corollary 10.** A random vector $X$ with finite third moment has the standard normal distribution if and only if $X$ is spherically symmetric and has independent components with variance 1.

Recall that a random vector $X$ is said to satisfy a Poincaré inequality with constant $\lambda_1$ (the spectral gap of $X$) if
\[ \lambda_1 \text{Var}(f(X)) \leq \mathbb{E}\|\nabla f(X)\|^2 \]
for every smooth $f : \mathbb{R}^n \to \mathbb{R}$. The last estimate in Theorem 7 provides a connection between normal approximation and spectral gap estimates. A similar connection has been observed in a different but related context by Bobkov and Koldobsky [8].

**Corollary 11.** Let $X$ be an isotropic spherically symmetric random vector with spectral gap $\lambda_1$. Then for any $\theta \in S^{n-1}$,
\[ d_{TV}(\mathcal{L}(W_\theta), \gamma_{\mathbb{R}}) \leq \frac{10}{\sqrt{n} \lambda_1}. \]

One concrete application of Corollary 11 is the following.

**Corollary 12.** Let $X$ have the isotropic spherically symmetric exponential density
\[ a_n e^{-b_n \|x\|^2}, \]
where $a_n$ and $b_n$ are appropriate normalization constants. Then
\[ d_{TV}(\mathcal{L}(W_\theta), \gamma_{\mathbb{R}}) \leq \frac{10\sqrt{13}}{n^{1/2}}. \]

Bobkov [7] showed that for this distribution, $\frac{1}{13} \leq \lambda_1 \leq 1$, and so Corollary 12 is immediate from Corollary 11. Since this distribution is given explicitly, one can obtain an error estimate of the same order by directly estimating the variance in Theorem 7; however, there is a large literature on spectral gap estimates in much less explicit contexts using only certain geometric assumptions, typically diameter and/or curvature bounds (see [19] for a survey and further references). Corollary 11 thus allows the treatment of distributions about which one has
geometric information resulting in spectral gap estimates, but for which direct computation of the variance term in Theorem 7 is not possible.

There is also the following complex analogue of Theorem 7. All of the previously defined notation of this paper used here should be reinterpreted for vectors in $\mathbb{C}^n$ in the most obvious way.

**Theorem 13.** Let $X \in \mathbb{C}^n$ be a random vector with finite third moment such that $EX_i = 0$ for each $i$, $EX_iX_j = EX_jX_i = 0$ if $i \neq j$, and $E(\text{Re}X_i)^2 = E(\text{Im}X_i)^2 = 1$. Suppose the distribution of $X$ is invariant under multiplication by a unitary matrix. Then for any $\theta \in S_n^{n-1}$,

$$d_{TV}(\mathcal{L}(\text{Re} W_\theta), \gamma_\mathbb{R}) \leq 4E \left|1 - \frac{n}{2(n+1)}E[|X_2|^2|X_1]| + \frac{1}{n-1}\right|
\leq \frac{2n}{n^2 - 1} \sqrt{\text{Var} (||X||^2)} + \frac{5}{n-1}.$$

Note that under the hypotheses of Theorem 13 $E||X_2|^2|X_1| = 2E[(\text{Re}X_2)^2|X_1]$, and so the appearance of the factor of 2 inside the first bound above is to be expected.

3. Background on Stein’s method

The essential idea of Stein’s method is the notion of a characterizing operator. Say that $T_\circ$ is a characterizing operator for a distribution $\mu$ on $\mathbb{R}$ if the following conditions hold:

1. $\int T_\circ f(t) \, d\mu(t) = 0$ for all $f$ such that $T_\circ f$ is $\mu$-integrable, and
2. if $\nu$ is a probability measure on $\mathbb{R}$ such that $\int T_\circ f(t) \, d\nu(t) = 0$ for all $f$ with $T_\circ f \nu$-integrable, then $\mu = \nu$.

A characterizing operator is a strong characterization of a distribution, in the sense that if $T_\circ$ is characterizing for $\mu$ and $\nu$ is a measure such that $\int T_\circ f(t) \, d\nu(t)$ is small for a large class of test functions $f$, then $\nu \approx \mu$ in some sense.

This idea is quantified in the method of exchangeable pairs as follows. Let $W = W(\omega)$ be a random variable defined on a probability space $(\Omega, \mathbb{P})$, and let $\mathcal{X}$ be a space of measurable functions on $\Omega$. Think of $\mathbb{E}$ as a linear map from $\mathcal{X}$ to $\mathbb{R}$, $\mathbb{E} f = \int f(\omega) \, d\mathbb{P}(\omega)$. Let $\mathcal{X}_\circ$ be a space of measurable functions on $\mathbb{R}$ and let $\mathbb{E}_\circ$ be the linear function on $\mathcal{X}_\circ$ defined by $\mathbb{E}_\circ f = \int f(t) \, d\mu(t)$ for some fixed measure $\mu$. In our applications, $\mu$ will be the standard normal distribution, but one of the advantages of Stein’s method is that the set-up is quite general and can be adapted to various other measures. The random variable $W$ induces a map $\beta : \mathcal{X}_\circ \to \mathcal{X}$ defined by

$$\beta f(\omega) = f(W(\omega)).$$

Now, construct a symmetric probability $Q$ on $\Omega \times \Omega$ with margins $\mathbb{P}$ (i.e., $Q(A \times B) = Q(B \times A)$ and $Q(A \times \Omega) = \mathbb{P}(A)$). Note that this is the same as constructing an exchangeable pair $(W,W') = (W(\omega),W(\omega'))$ from $W$. Let $\mathcal{F}$ be a space of measurable, antisymmetric functions on $\Omega \times \Omega$ and use $Q$ to define a map $T : \mathcal{F} \to \mathcal{X}$ by

$$T f(\omega) = \mathbb{E}_Q [f(\omega,\omega')|\omega].$$
note that by exchangeability and antisymmetry, \( ET \equiv 0 \) on \( \mathcal{F} \). Let \( \mathcal{F}_o \) be another space of measurable functions on \( \mathbb{R} \), possibly the same as \( \mathcal{X}_o \), and let \( T_o : \mathcal{F}_o \rightarrow \mathcal{X}_o \) be a characterizing operator of \( \mu \). Let \( U_o : \mathcal{X}_o \rightarrow \mathcal{F}_o \) be a pseudo-inverse to \( T_o \), in the sense that

\[
T_o U_o f(t) = f(t) - E_o f.
\]

Finally, let \( \alpha : \mathcal{F}_o \rightarrow \mathcal{F} \). All of these definitions are summarized in the following diagram:

The following easy lemma \[29\] is the quantitative version of the heuristic at the beginning of the section.

**Lemma 14** (Stein). Suppose that in the diagram of spaces and maps above, \( ET = 0 \) and \( T_o U_o = \text{Id} - E_o \). Then

\[
E\beta - E_o = E(\beta T_o - T\alpha) U_o.
\]

Note that for \( f \in \mathcal{X}_o \),

\[
E\beta f - E_o f = E f(W) - E f(Z),
\]

where \( Z \) is a random variable with distribution \( \mu \). The strategy is apply Stein’s lemma to bound this difference uniformly over a large class of test functions.

To apply the method of exchangeable pairs, one needs an approximating distribution and a characterizing operator. In this paper, the measure \( \mu \) will be standard Gaussian, with characterizing operator

\[
T_o f(t) = f'(t) - tf(t)
\]

and pseudo-inverse

\[
U_o f(t) = e^{\frac{1}{2} t^2} \int_{-\infty}^{t} \left[ f(s) - E_o f \right] e^{-\frac{1}{2} s^2} ds.
\]

That \( E_o T_o = 0 \) can be verified by integration by parts, and verifying \( T_o U_o = \text{Id} - E_o \) is just calculus.

The following estimates for \( U_o \) are proved in \[29\, p. 25\] and are useful in estimating the error term from Lemma 14.

\[
\|U_o f\|_{\infty} \leq \sqrt{\frac{\pi}{2}} \|f - E_o f\|_{\infty} \leq \sqrt{2\pi} \|f\|_{\infty},
\]

\[
\|(U_o f)'\|_{\infty} \leq 2 \|f - E_o f\|_{\infty} \leq 4 \|f\|_{\infty},
\]

\[
\|(U_o f)''\|_{\infty} \leq 2 \|f'\|_{\infty}.
\]

Using this general set-up one can prove the following abstract normal approximation theorem, which is the main tool used in the proof of Theorem 1. The first statement was
proved by Stein in [29, Lecture III]; the second statement was proved by Rinott and Rotar in [26].

**Proposition 15** (Stein; Rinott-Rotar). Let \((W, W')\) be an exchangeable pair of random variables such that

\[ E W = 0, \quad E W^2 = 1, \]

and

\[ E[W - W'|W] = \lambda W \]

for some \(\lambda \in (0, 1)\). Then

\[ \sup_{t \in \mathbb{R}} |P[W \leq t] - \Phi(t)| \leq \frac{1}{\lambda} \sqrt{\text{Var} E[(W - W')^2|W]} + (2\pi)^{-1/4} \sqrt{\frac{1}{\lambda} E|W - W'|^3}. \]

If moreover \(|W - W'|\) is almost surely bounded, then

\[ \sup_{t \in \mathbb{R}} |P[W \leq t] - \Phi(t)| \leq \frac{12}{\lambda} \sqrt{\text{Var} E[(W - W')^2|W]} + \frac{43}{\lambda} \|W - W'\|_\infty^3. \]

**Remarks**

(1) For Stein’s lemma to hold, no assumptions are needed on the map \(\alpha : \mathcal{F}_0 \to \mathcal{F}\) other than that it does in fact produce antisymmetric functions. For normal approximation, one usually uses the map

\[ \alpha f(\omega, \omega') = a(W(\omega') - W(\omega)) \left[ f(W(\omega')) + f(W(\omega)) \right], \]

for some suitable choice of \(a\).

(2) In the description of the method, we have been quite cavalier about exactly which function spaces should be used. Of course, the choice of the space of test functions \(X_0\) determines the type of convergence; in practice, one generally fixes \(X_0\) first and chooses the remaining spaces in some way which guarantees that all of the maps do fit into the diagram as shown. The diagram and Stein’s lemma are described here mainly as motivation for the approach taken in Section 5 and are often not used directly but as a guide for how to proceed.

(3) Proposition 15 has been applied in many different contexts. In particular, Holmes and Reinert [15] use it in analyzing statistics similar to our \(W_\theta\) while studying process approximation for the bootstrap.

(4) This paper is the first that we know of to apply the exchangeable pairs approach to Stein’s method in convex geometry. However, Reitzner [25] used a rather different approach to Stein’s method (based on dependency graphs) in proving central limit theorems for random polytopes.

(5) We have described here how to apply Stein’s lemma to normal approximation; application to Poisson approximation has also been extensively developed, see [12].
4. Proof of Theorem 1 and its consequences

Proof of Theorem 1. Let $X$ be a random vector invariant under reflections in the hyperplanes defined by a normalized tight frame $u_1, \ldots, u_m$. Define an exchangeable pair $(W, W')$ as in Proposition 15 as follows. Let $I$ be chosen uniformly from $\{1, \ldots, m\}$, independently of $X$, and define

$$X' = X - 2X(I)u_I,$$

i.e., $X'$ is obtained from $X$ by reflection in the hyperplane $u_I^\perp$. Then $(X, X')$ is an exchangeable pair of random vectors by assumption. Define $W = W_\theta = \langle X, \theta \rangle$ and $W' = \langle X', \theta \rangle$.

Now $E[W] = 0$ and $E[W^2] = 1$ since $X$ is isotropic, and

$$E[W - W'|W] = E\left[\frac{1}{m} \sum_{i=1}^{m} 2X(I)\theta(I) | W \right]$$

$$= \frac{2}{m} E \left[ \left\langle \sum_{i=1}^{m} u_i \otimes u_i(X), \theta \right\rangle \left| W \right. \right]$$

$$= \frac{2}{n} W$$

by equation (1).

To apply Proposition 15 (with $\lambda = \frac{2}{n}$), it remains to estimate the quantities

$$\text{Var}_E[(W - W')^2|W], \quad E|W - W'|^3, \quad \text{and} \quad \|W - W'\|_\infty$$

(the last in the case that $\max |X(I)| \leq a$ almost surely). First,

$$E(E[(W - W')^2|W]) = E(E[W^2 + (W')^2 - 2WW'|W]) = \frac{4}{n},$$

and by the conditional form of Jensen’s inequality,

$$E(E[(W - W')^2|W])^2 \leq E(E[(W - W')^2|X])^2$$

$$= E(E[(2X(I)\theta(I))^2|X])^2$$

$$= \frac{16}{m^2} \sum_{i,j=1}^{m} \theta_{i,j}^2 E[X_i X_j^2],$$

so

$$\text{Var}_E[(W - W')^2|W] = E(E[(W - W')^2|W])^2 - \frac{16}{n^2}$$

$$\leq \frac{16}{n^2} \left( \frac{n^2}{m^2} \sum_{i,j=1}^{m} \theta_{i,j}^2 \theta_{i,j} \right) E[X_i X_j^2] - 1\right).$$

(14)
Next,
\[
\mathbb{E}|W - W'|^3 = 8\mathbb{E}|X(I)\theta(I)|^3
\leq \frac{8}{m} \sum_{i=1}^{m} |\theta(I)|^3 \mathbb{E}|X(I)|^3 \leq \frac{8}{m} \left( \max_{1 \leq i \leq m} \mathbb{E}|X(i)|^3 \right) \sum_{i=1}^{m} |\theta(i)|^3.
\] (15)

Finally, if \( \max |X(i)| \leq a \) almost surely, then
\[
|W - W'| = 2|\theta(I)||X(I)| \leq 2a \max |\theta(i)|
\] almost surely. Inserting the estimates (14), (15), and (16) into Proposition 15 now proves Theorem 1. \( \square \)

**Remark:** One can also prove, by the same method, a version of Theorem 1 for random vectors which are invariant under reflections in subspaces of arbitrary dimensions. Rather than a normalized tight frame, one would consider a decomposition

\[
\sum_{i=1}^{m} P_{E_i} = \alpha I_n
\]

for subspaces \( E_1, \ldots, E_m \subset \mathbb{R}^n \). In applications this can lead to a nontrivial approximation result if the maximum dimension of the \( E_i \) remains bounded (or grows very slowly) as \( n \) grows. This context covers, for example, \( X \) uniformly distributed in the \( \ell_p \)-sum of \( m \) copies of a fixed symmetric convex body \( K \subset \mathbb{R}^k \).

**Proof of Corollary 2** The corollary follows easily from Theorem 1 by taking as the normalized tight frame the standard basis of \( \mathbb{R}^n \) (so that \( x(i) = x_i \)). The form of the second error term in both statements follows immediately from Theorem 1. To estimate the first error term, note that
\[
\sum_{i,j=1}^{m} \theta_i^2 \theta_j^2 \mathbb{E}[X_i^2 X_j^2] \leq \left( \max_{1 \leq j \leq m} \mathbb{E}X_j^4 \right) \sum_{i=1}^{m} \theta_i^4 + \left( \max_{i \neq j} \mathbb{E}X_i^2 X_j^2 \right) \sum_{i,j=1}^{m} \theta_i^2 \theta_j^2
\]

\[
= \left( \max_{1 \leq j \leq m} \mathbb{E}X_j^4 \right) \sum_{i=1}^{m} \theta_i^4 + \max_{i \neq j} \mathbb{E}X_i^2 X_j^2
\]
since \( \|\theta\|_2 = 1 \), and
\[
\mathbb{E}X_i^2 X_j^2 = \text{Cov}(X_i^2, X_j^2) + 1
\]
since \( X \) is isotropic. \( \square \)

The following lemma is used to prove the randomized statements in Corollaries 4, 5, and 6. It follows easily from a concentration inequality implicitly proved by Gordon [14] (see Theorem 6 in [23] for an explicit statement), together with the well-known asymptotic orders of the averages of \( \ell_p^n \) norms over \( S^{n-1} \).
Lemma 16. There are absolute constants $c_1, \ldots, c_4$ such that the following hold.

1. If $\delta \geq c_1 n^{-1/6}$ and $k \leq c_2 \delta^2 n$, then
   $$\lambda_{n,k}(\{E \in G_{n,k} : \|\theta\|_3 \leq \delta \forall \theta \in E \cap S^{n-1}\}) \geq 1 - c_3 e^{-c_4 \delta^2 n}.$$  

2. If $\delta \geq c_1 n^{-1/4}$ and $k \leq c_2 \delta^2 n$, then
   $$\lambda_{n,k}(\{E \in G_{n,k} : \|\theta\|_4 \leq \delta \forall \theta \in E \cap S^{n-1}\}) \geq 1 - c_3 e^{-c_4 \delta^2 n}.$$  

3. If $\delta \geq c_1 \sqrt{\log n}$ and $k \leq c_2 \delta^2 n$, then
   $$\lambda_{n,k}(\{E \in G_{n,k} : \|\theta\|_\infty \leq \delta \forall \theta \in E \cap S^{n-1}\}) \geq 1 - c_3 e^{-c_4 \delta^2 n}.$$  

Proof of Corollary 4.

1. The first statement is immediate from the first statement of Corollary 2; the second follows from parts 1 and 2 of Lemma 16.

2. It is a well-known consequence of Borell’s lemma (cf. [20, Section 2.2]) that there is an absolute constant $c$, independent of $X$, $n$, and $p$, such that
   \[
   \left(\mathbb{E}|\langle X, y \rangle|^p\right)^{1/p} \leq c p \left(\mathbb{E}|\langle X, y \rangle|^2\right)^{1/2} 
   \]
   for any log-concave random vector $X$, $p \geq 2$, and fixed vector $y$. Thus part 1 applies with some absolute constants $a$ and $b$. Furthermore, since $\|\theta\|_4 \leq \|\theta\|_3 \leq \|\theta\|_2 = 1$, the first term in the r.h.s. of part 1 is not of larger order than the second term. The total variation bound follows from Lemma 3 and the fact that any projection of a log-concave measure is again log-concave.

3. From the second statement of Corollary 2 and the trivial estimate $\mathbb{E}X_i^4 \leq a^4$ we obtain
   $$\sup_{t \in \mathbb{R}} |\mathbb{P}[W_{\theta} \leq t] - \Phi(t)| \leq 24 a^2 \|\theta\|_3^2 + 172 na^3 \|\theta\|_\infty^3.$$  

The estimates
   $$\|\theta\|_4 \leq n^{1/4} \|\theta\|_\infty$$  

and
   $$1 = \|\theta\|_2 \leq \sqrt{n} \|\theta\|_\infty$$  

are well-known, and $a \geq 1$ since $X$ is isotropic. Therefore
   $$a^2 \|\theta\|_4^2 \leq a^3 \sqrt{n} \|\theta\|_\infty^2 \leq a^3 n \|\theta\|_\infty^3,$$

which proves the first statement. The randomized statement follows from part 3 of Lemma 16 and the total variation bound follows from Lemma 3.

\[\square\]
Proof of Corollary 5. The square negative correlation property was proved for the uniform measure on the ball of $\ell_p^n$ in [3] (see also [1]) and for the cone measure in [23]. The uniform measure on the ball is log-concave by the Brunn-Minkowski theorem, and it is not hard to show (cf. [23]) that $E|X_i|^4 \leq c$ for some absolute constant in the case of cone measure. Finally, it is well-known that if the $\ell_p^n$ ball is scaled so that its uniform measure is isotropic, then it is contained in $[-a_p n^{1/p}, a_p n^{1/p}]^n$, where $a_p$ is a constant depending only on $p$; it is not hard to show the same is true of the normalized cone measure.

Using all these facts, the statements for uniform measure on the ball and cone measure on the sphere follow from Corollary 4; the total variation bound for the ball follows from Lemma 3.

The statements for the surface measure then follow from the fact, proved in [23, 22] that the total variation distance between the cone and surface measures is at most $c \sqrt{n}$ for some absolute constant $c$, and both of the error estimates are of at least this order (cf. the proofs of 2 and 3 of Corollary 4).

\[ \square \]

Proof of Corollary 6. First, if $\Delta_n = \sqrt{n(n+2)} \conv \{v_1, \ldots, v_{n+1}\}$ is a regular simplex, then the $v_i$ form a normalized tight frame and also satisfy

\[ \sum_{i=1}^{n+1} v_i = 0 \]

Both of these facts are well-known and can be seen as consequences of John’s theorem on contact points between a convex body and the minimal volume ellipsoid containing it [16].

To see Corollary 6 as a consequence of Theorem 1, consider the vectors $u_{ij} = \sqrt{\frac{n}{2(n+1)}} (v_i - v_j), \quad 1 \leq i, j \leq n+1, \quad i \neq j.$

It is not hard to show from (18) that $\|u_{ij}\|_2 = 1$ for each $i \neq j$, and that the $u_{ij}$ form a normalized tight frame because the $v_i$ do. Reflection in $u_{ij}^\perp$ is a reflection which interchanges the vertices in the directions $v_i$ and $v_j$ and leaves all other vertices of $\Delta_n$ fixed. Theorem 4 can thus be applied (with $m = n(n+1)$), provided that $X$ is indeed isotropic under this scaling. This is not obvious at this point, but follows easily from (19) below.

In order to compute the relevant expectations, one can embed $\Delta_n$ isometrically in $\mathbb{R}^{n+1}$ by the affine map with $\sqrt{n(n+2)}v_i \mapsto \sqrt{(n+1)(n+2)}e_i$; the image of $\Delta_n$ under this map is $\Delta_n' = \sqrt{(n+1)(n+2)} \conv \{e_1, \ldots, e_{n+1}\}$.

Let $Y$ be the image of $X$ under this isometry; $Y$ is uniformly distributed in $\Delta_n'$. Then

\[ \langle X, v_i \rangle = \left\langle \sqrt{\frac{n+1}{n}} \left( e_i - \frac{1}{n+1} \sum_{j=1}^{n+1} e_j \right) \right\rangle, \]
and so, adapting the notation of Theorem 1

\[ X_{ij} = \langle X, u_{ij} \rangle \]

\[ = \sqrt{\frac{n}{2(n+1)}} \left( \langle X, v_i \rangle - \langle X, v_j \rangle \right) \]

\[ = \sqrt{\frac{n}{2(n+1)}} \left( \langle Y, \sqrt{\frac{n+1}{n}} e_i \rangle - \langle Y, \sqrt{\frac{n+1}{n}} e_j \rangle \right) \]

\[ = \frac{1}{\sqrt{2}} (Y_i - Y_j). \]

The joint moments of the \( Y_i \) are given by

\[ \mathbb{E} \left[ \prod_{i=1}^{n+1} Y_i^{r_i} \right] = \left( \frac{(n+1)(n+2)}{(n+3)(n+4)} \right)^{(n+r)!} \prod_{i=1}^{n+1} r_i!, \]

where \( r = \sum r_i \). This formula follows easily from Lemma 1 of [27]. Using this one can show that \( \mathbb{E}X_{ij}^2 = 1 \) and thus that the stated normalization for \( \Delta_n \) is correct. In addition, if \( i \neq j \) and \( k \neq l \), then

\[ \mathbb{E}X_{ij}^2 X_{kl}^2 = \frac{(n+1)(n+2)}{(n+3)(n+4)} \begin{cases} 1 & \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \\ 3 & \text{if } |\{i, j\} \cap \{k, l\}| = 1, \\ 6 & \text{if } |\{i, j\} \cap \{k, l\}| = 2, \end{cases} \]

and

\[ \mathbb{E}|X_{ij}|^3 \leq 3\sqrt{2}\frac{(n+1)(n+2)}{n+3} < 3\sqrt{2}. \]

(What is really needed about this latter quantity is just that it is bounded by an absolute constant; this also follows immediately from Borell’s lemma [17].)

To estimate the first error term from Theorem 1 by (20),

\[ \sum_{(ij), (kl)} \theta_{ij}^2 \theta_{kl}^2 \mathbb{E}\left[ X_{ij}^2 X_{kl}^2 \right] \]

\[ = \frac{(n+1)(n+2)}{(n+3)(n+4)} \left[ \sum_{(ij), (kl)} \theta_{ij}^2 \theta_{kl}^2 + 2 \sum_{|\{i, j\} \cap \{k, l\}| = 1} \theta_{ij}^2 \theta_{kl}^2 + 10 \sum_{(ij)} \theta_{ij}^4 \right], \]

where in all of the above sums, indices run from 1 to \( n+1 \), \( i \neq j \), and \( k \neq l \). Since the \( u_{ij} \) form a normalized tight frame,

\[ \sum_{(ij)} \theta_{ij}^2 = \sum_{(ij)} \langle u_{ij} \otimes u_{ij}(\theta), \theta \rangle = n+1, \]

and by the Cauchy-Schwarz inequality,

\[ \sum_{|\{i, j\} \cap \{k, l\}| = 1} \theta_{ij}^2 \theta_{kl}^2 \leq \sum_{|\{i, j\} \cap \{k, l\}| = 1} \theta_{ij}^4 = 4n \sum_{(ij)} \theta_{ij}^4. \]
By (18) and the fact that the \( v_i \) form a normalized tight frame,
\[
\sum_{(ij)} \theta^4_{(ij)} = \left( \frac{n}{2(n+1)} \right)^2 \sum_{i,j=1}^{n+1} \left( \langle \theta, v_i \rangle - \langle \theta, v_j \rangle \right)^4
\]
\[
= \frac{n^2}{2(n+1)} \sum_{i=1}^{n+1} \langle \theta, v_i \rangle^4 + 6,
\]
and so
\[
(22) \quad \frac{1}{(n+1)^2} \sum_{(ij),(kl)} \theta^2_{(ij)} \theta^2_{(kl)} \mathbb{E}[X^2_{(ij)}X^2_{(kl)}] - 1 \leq 4 \sum_{i=1}^{n+1} \langle \theta, v_i \rangle^4 + O\left( \frac{1}{n} \right).
\]

To estimate the second error term from Theorem 11,
\[
\left( \sum_{(ij)} |\theta_{(ij)}|^3 \right)^{1/3} = \sqrt[3]{\frac{n}{2(n+1)}} \left( \sum_{i,j=1}^{n+1} | \langle \theta, v_i \rangle - \langle \theta, v_j \rangle |^3 \right)^{1/3}
\]
\[
\leq \sqrt[3]{\frac{n}{2}} \left( \sum_{i=1}^{n+1} | \langle \theta, v_i \rangle |^3 \right)^{3/3}
\]
by the triangle inequality for the \( \ell_3^{(n+1)^2} \) norm. The first statement of the Corollary now follows by inserting (21), (22), and (23) into Theorem 11, and noting as in the proof of part 2 of Corollary 4 that the first error term is of smaller order than the second error term. The total variation bound then follows from Lemma 3.

The randomized statement essentially follows from Lemma 16 as in the proof of Corollary 4. In this case one actually needs not part 11 of Lemma 16, but the same estimate for the norm
\[
||\theta|| = \left( \sum_{i=1}^{n+1} | \langle \theta, v_i \rangle |^3 \right)^{1/3}
\]
which can be proved in the same way.

**Remarks:** The same issue of estimating covariances of the squares of frame components of \( X \), which is explicit in the error bound of Corollary 2, also arises implicitly in the proof of Corollary 6. In the latter case \( \text{Cov}(X^2_{(ij)}, X^2_{(kl)}) \) is not always small; the key point is that it is at worst a positive constant, and this only happens for a negligible fraction of pairs \((ij), (kl)\). Also, it is clear that the proof of Corollary 6 can be adapted to treat other distributions which posses the same symmetries as a centered regular simplex.

### 5. Infinitesimal Rotations

This section is mainly devoted to the proof of Theorem 7. In the usual exchangeable pairs approach to Stein’s method described in Section 3, one starts with a random variable \( W \) and makes a small change to get a new random variable \( W' \) so that the pair \((W, W')\) is exchangeable. This pair is then used cleverly to estimate differences in expectations of
test functions with respect to $W$ and some standard distribution. Since the symmetries in Theorem 7 are continuous rather than discrete, it is possible to make an “infinitesimal” change in $W$ by making a small change, scaling appropriately, and then taking a limit.

Proof of Theorem 7. Suppose that $X$ is spherically symmetric and isotropic. By the spherical symmetry, we may assume $\theta = e_1$, so $W = X_1$.

To prove the theorem, it suffices to bound $|E f(W) - E f(Z)|$, where $Z$ is a standard normal random variable and $f : \mathbb{R} \to \mathbb{R}$ is smooth with compact support. (The proof does not require this much regularity of $f$, but it produces no loss in generality.) Recall that the standard normal distribution is characterized by the identity $E[g'(Z) - Zg(Z)] = 0$ for all sufficiently regular $g$, and that given a test function $f$,

$$g(t) = U_o f(t) = e^{\frac{t^2}{2}} \int_{-\infty}^{t} [f(s) - E f(Z)] e^{-\frac{1}{2} s^2} ds$$

satisfies the differential equation

$$g'(t) - tg(t) = f(t) - E f(Z).$$

In particular

$$(24) \quad E[g'(W) - W g(W)] = E f(W) - E f(Z).$$

To carry out the infinitesimal rotations idea described above, define a family of random variables $\{W_\varepsilon\}$, for $\varepsilon \in (0, \frac{1}{2})$ as follows. Let $A_\varepsilon$ be the $n \times n$ orthogonal matrix

$$A_\varepsilon = \begin{pmatrix} \sqrt{1-\varepsilon^2} & \varepsilon \\ -\varepsilon & \sqrt{1-\varepsilon^2} \end{pmatrix} \oplus I_{n-2}.$$ 

Now let $U$ be a random $n \times n$ orthogonal matrix, chosen independently of $X$ according to Haar measure; $U^T A_\varepsilon U$ is a rotation in a random two-dimensional subspace through an angle $\sin^{-1}(\varepsilon)$. Define

$$W_\varepsilon = \langle (U^T A_\varepsilon U) X, e_1 \rangle.$$ 

By the rotational invariance of $X$, $(W, W_\varepsilon)$ is an exchangeable pair for each $\varepsilon$.

The following facts about the joint distribution of $(W, W_\varepsilon)$ will be needed:

$$(25) \quad E[W - W_\varepsilon | W] = \left(1 + O(\varepsilon^2)\right) \frac{\varepsilon^2}{n} W,$$

$$(26) \quad E[(W - W_\varepsilon)^2 | W] = \left(1 + O(\varepsilon)\right) \frac{2\varepsilon^2}{n} E[X_2^2 | W] + O(\varepsilon^4) W^2,$$

$$(27) \quad E|W - W_\varepsilon|^3 = O(\varepsilon^3).$$

Here and throughout this proof the $O$ notation refers to asymptotic behavior as $\varepsilon \to 0$, with deterministic implied constants (that may depend on $n$, $f$, or the distribution of $X$). The proof of (25) is given below. The proofs of (26) and (27) are similar; analogous estimates are proved in detail in [21].

First observe that by exchangeability,

$$E((W - W_\varepsilon)[g(W) + g(W_\varepsilon)]) = 0,$$
because the expression is antisymmetric in $W$ and $W_\varepsilon$. (In the language of Section 3, this is essentially the observation that $E T \alpha = 0$, where $\alpha$ has been chosen as in the remark at the end of Section 3.) Now by Taylor’s theorem,
\[(W - W_\varepsilon)[g(W) + g(W_\varepsilon)] = 2(W - W_\varepsilon)g(W) + (W - W_\varepsilon)[g(W_\varepsilon) - g(W)]\]
\[= 2(W - W_\varepsilon)g(W) - (W - W_\varepsilon)^2 g'(W) + R,
\]
where
\[|R| \leq \frac{1}{2}|W - W_\varepsilon|^3 \|g''\|_\infty \leq |W - W_\varepsilon|^3 \|f''\|_\infty\]
by (13).

By (25), (26), (27), and (11),
\[0 = \frac{n}{2\varepsilon^2} E [2g(W)(W - W_\varepsilon) - 2(W - W_\varepsilon)^2 g'(W)] + O(\varepsilon)\]
\[= \frac{n}{2\varepsilon^2} E [E [2g(W)(W - W_\varepsilon) - 2(W - W_\varepsilon)^2 g'(W) | W]] + O(\varepsilon)\]
\[= E [Wg(W) - E[X_2^2 | W]g'(W)] + O(\varepsilon),\]
and so, letting $\varepsilon \to 0$,
\[E[Wg(W)] = E[E[X_2^2 | W]g'(W)].\]

Therefore
\[(28) \quad E[f(W) - f(Z)] = E[g'(W) - Wg(W)] = E [(1 - E[X_2^2 | W])g'(W)]\]
for any smooth, bounded $f$. In particular,
\[d_{TV}(\mathcal{L}(W), \gamma_{\mathbb{R}}) = \sup_f |E[f(W) - f(Z)]| \leq 4E|1 - E[X_2^2 | W]|,\]
where the supremum may be taken over smooth, compactly supported $f$ with $\|f\|_\infty \leq 1$, so that $\|g'\|_\infty \leq 4$ by (12). This proves the first estimate.

To prove the second and third estimates, observe that by spherical symmetry,
\[E[X_2^2 | X_1] = \frac{1}{n-1} \left( E[\|X\|_2^2 | X_1] - X_1^2 \right),\]
and therefore
\[E|1 - E[X_2^2 | X_1]| \leq \frac{1}{n-1} \left( E|n - E[\|X\|_2^2 | X_1]| + E|X_1^2 - 1| \right)\]
\[\leq \frac{1}{n-1} \left( E|\|X\|_2^2 - n| + 2 \right)\]
\[\leq \frac{1}{n-1} \left( \sqrt{\text{Var}(\|X\|_2^2)} + 2 \right),\]
by the Cauchy-Schwarz inequality and the isotropicity of $X$.

Finally, to prove (25), first observe that
\[A_\varepsilon = I_n + \left[ \varepsilon J - \left( 1 + O(\varepsilon^2) \right) \frac{\varepsilon^2}{2} I_2 \right] \oplus 0_{n-2},\]
where $0_{n-2}$ denotes the $(n - 2) \times (n - 2)$ zero matrix and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

Denote by $K$ the $2 \times n$ matrix consisting of the first two rows of the random orthogonal matrix $U$. Then

$$(U^T A_\varepsilon U)X = X + K^T \left[ \varepsilon J - \left(1 + O(\varepsilon^2)\right) \frac{\varepsilon^2}{2} I_2 \right] KX,$$

and so

$$W - W_\varepsilon = -\varepsilon \left\langle (K^T J K)X, e_1 \right\rangle + \left(1 + O(\varepsilon^2)\right) \frac{\varepsilon^2}{2} \left\langle (K^T K)X, e_1 \right\rangle.$$  

Now if $u_{ij}$ denote the entries of $U$, then by expanding in components,

$$\mathbb{E} \left[ \left\langle (K^T J K)X, e_1 \right\rangle \left| X \right. \right] = \sum_{i=2}^{n} X_i \mathbb{E}(u_{11}u_{2i} - u_{21}u_{1i})$$

and

$$\mathbb{E} \left[ \left\langle (K^T K)X, e_1 \right\rangle \left| X \right. \right] = \sum_{i=1}^{n} X_i \mathbb{E}(u_{11}u_{1i} + u_{21}u_{2i}).$$

Computing these expectations is not difficult because the distribution of $U$ is unchanged by multiplying any row or column by $-1$, and any row or column of $U$ is distributed uniformly on $S^{n-1}$. Therefore

$$\mathbb{E} u_{ij} u_{kl} = \delta_{ik}\delta_{jl} \frac{1}{n},$$

and so

$$\mathbb{E} \left[ \left\langle (K^T J K)X, e_1 \right\rangle \left| X \right. \right] = 0$$

and

$$\mathbb{E} \left[ \left\langle (K^T K)X, e_1 \right\rangle \left| X \right. \right] = \frac{2}{n} W.$$

Putting these together proves (25).

The proofs of (26) and (27) follow similarly from (29). One needs in addition that $\mathbb{E}|X_i|^3 < \infty$ and that $\mathbb{E}[X_i^2 | X_1] = 0$ and $\mathbb{E}[X_i^2 | X_1] = \mathbb{E}[X_i^2 | X_1]$ for $i \neq 1$. One also needs values of fourth order moments of the entries of $U$, which can be found, e.g., in [2].

The proof of Theorem 13 is essentially the same as the proof of Theorem 7; the only difference is that $A_\varepsilon$ is conjugated by a random unitary matrix instead of a random orthogonal matrix. The relevant mixed moments of entries of a random unitary matrix can also be found in [2].

Proof of Corollary 9. If $X$ is uniformly distributed on the sphere of radius $\sqrt{n}$, then

$$\text{Var} \left( \|X\|_2^2 \right) = 0,$$

and the result follows immediately from Theorem 7. If $X$ is uniformly distributed on the ball of radius $\sqrt{n + 2}$, then it is easy to show by integration in polar coordinates that
Var(∥X∥^2) < 4, from which the result follows. In both cases the constants, although not the order in n, can be improved by working directly from the first estimate in Theorem 7.

□

Proof of Corollary 11. By applying the Poincaré inequality (6) to the function f(x) = ∥x∥^2, one obtains

Var(∥X∥^2) ≤ \frac{4n}{\lambda_1},

and so by Theorem 7

d_{TV}(\mathcal{L}(W_\theta), \gamma_\mathbb{R}) \leq \frac{8}{n - 1} \left( \sqrt{n} \lambda_1 + 1 \right).

By testing (3) on a linear functional f, one obtains that \lambda_1 ≤ 1 when X is isotropic, and therefore

\[ d_{TV}(\mathcal{L}(W_\theta), \gamma_\mathbb{R}) \leq \frac{8}{\sqrt{n \lambda_1} (\sqrt{n} - 1)} \]

Since \[ d_{TV} \leq 2 \] always, this gives a result only for \[ n > 25 \], for which the stated estimate now follows.

□

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