Strongly Regular Graphs as Laplacian Extremal Graphs

Fan-Hsuan Lin∗ † Chih-wen Weng†

November 25, 2014

Abstract

The Laplacian spread of a graph is the difference between the largest eigenvalue and the second-smallest eigenvalue of the Laplacian matrix of the graph. We find that the class of strongly regular graphs attains the maximum of largest eigenvalues, the minimum of second-smallest eigenvalues of Laplacian matrices and hence the maximum of Laplacian spreads among all simple connected graphs of fixed order, minimum degree, maximum degree, minimum size of common neighbors of two adjacent vertices and minimum size of common neighbors of two nonadjacent vertices. Some other extremal graphs are also provided.

keyword Laplacian matrix, Laplacian spread, strongly regular graph.

1 Introduction

Let \( G = (V, E) \) be a simple connected graph of order \( n \) with vertex set \( V = \{1, 2, \cdots, n\} \) and edge set \( E \). Let \( A = A(G) \) be the adjacency matrix of \( G \), i.e. the binary matrix with \( ij \)-entry 1 iff \( i \) and \( j \) are distinct and adjacent. The degree \( d_i \) of vertex \( i \in V \) is the number \( |G_1(i)| \), where \( G_1(i) \) is the set of vertices which are adjacent to \( i \). Let \( D(G) = \text{diag}(d_1, d_2, \cdots, d_n) \) be the diagonal matrix with entries \( d_1, d_2, \cdots, d_n \) in the diagonal. Then the matrix

\[
L(G) = D(G) - A(G)
\]

is called the Laplacian matrix of \( G \). We call the eigenvalues of \( L(G) \) the Laplacian eigenvalues of \( G \). It is well-known that \( L(G) \) is symmetric, positive semidefinite, and every row-sum being zero [8], so we denote the Laplacian eigenvalues of \( G \) in nonincreasing order as \( \ell_1(G) \geq \ell_2(G) \geq \cdots \geq \ell_n(G) = 0 \). The eigenvalues \( \ell_1(G), \ell_{n-1}(G) \) and \( \ell_n(G) \) are called Laplacian index, algebraic connectivity and trivial eigenvalue, respectively. The Laplacian spread of \( G \) is defined as \( S_L(G) := \ell_1(G) - \ell_{n-1}(G) \).

A graph \( G \) is called \( k \)-regular if any vertex has degree \( k \). Moreover \( G \) is called strongly regular with parameters \((n, k, \lambda, \mu)\) if \( G \) is a \( k \)-regular graph with order \( n \) which has \( \lambda \) (resp. \( \mu \)) common neighbors of any pair of two adjacent (resp. nonadjacent) vertices. The Laplacian matrix of a \( k \)-regular graph is \( kI - A \), so its eigenvalues are easily obtained from those of adjacency matrix. It is well-known that a strongly regular graph with parameters \((n, k, \lambda, \mu)\) and \( n \neq k + 1 \) has two distinct nontrivial Laplacian eigenvalues \( \ell_1, \ell_{n-1} \), and indeed

\[
\ell_1(G), \ell_{n-1}(G) = \frac{2k - \lambda + \mu \pm \sqrt{\lambda - \mu}^2 + 4(k - \mu)}{2}.
\]

(1.1)

∗Corresponding author. E-mail address: fanhsuan.am03g@nctu.edu.tw (F.-H Lin).
†Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C..
See for example [11, Chapter 21]. Note that $G$ is strongly regular iff its complement $G^c$ is strongly regular [2 Theorem 1.3.1].

Let $\delta := \min_{i \in V} d_i$ and $\Delta := \max_{i \in V} d_i$. Motivated by the definition of strongly regular graphs, we define another two graph parameters $\lambda(G)$ and $\mu(G)$ for any graph $G$:

$$\lambda(G) := \min_{i,j \in E} |G_1(i) \cap G_1(j)|, \quad \mu(G) := \min_{i,j \in E} |G_1(i) \cap G_1(j)|.$$

In Section 2 we review some previously known bounds for the Laplacian index $\ell_1(G)$ and algebraic connectivity $\ell_{n-1}(G)$. Our main theorem, Theorem 3.1 in Section 3, is an inequality involving the graph parameters $n, d_i, m_i, \lambda(G), \mu(G)$ and a nontrivial eigenvalue $\ell$ with its corresponding eigenvector $(x_1, x_2, \ldots, x_n)^T$ of a graph $G$, where $m_i = (\sum_{j \in E} d_j)/d_i$ is called the average $2$-degree of a vertex $i$. We give two ways to eliminate the eigenvector parameters in the above inequality, and provide many lower bounds and upper bounds of the Laplacian eigenvalue $\ell$. Among them upper bounds for $\ell = \ell_1(G)$ and lower bounds of $\ell = \ell_{n-1}(G)$ are of most interest. See Corollary 3.2 and (3.9), (3.10) in Corollary 3.3. We find that the class of strongly regular graphs attains both bounds. In Section 4 we provide more extremal graphs, attaining bounds of some but not all of the inequalities in Section 3. In Section 5 the Laplacian index $\ell_1(G)$ and algebraic connectivity $\ell_{n-1}(G)$ for all connected $(n-3)$-regular graphs $G$ of order $n$ are determined. With a class of exceptions, they are extremal for all the inequalities obtained in this paper.

2 Some known bounds

We recall some basic properties of Laplacian matrices [15] and known bounds about Laplacian index $\ell_1(G)$ and algebraic connectivity $\ell_{n-1}(G)$ in this section. For the basic properties, one can find from [15].

The most important important property of $L(G)$ probably is

$$X^T L(G) X = \sum_{j < k, j \in E} (x_j - x_k)^2,$$

where $X$ is a column vector. As the property $L(G) + L(G^c) = nI - J$, where $J$ is the all one matrix, the Laplacian matrices $L(G)$ and $L(G^c)$ of $G$ and its complement $G^c$ respectively share the same set of eigenvectors. Hence if $\ell$ is a nontrivial eigenvalue of $L(G)$ with associated eigenvector $X$, then $n - \ell$ is an eigenvalue of $L(G^c)$ with the same associated eigenvector $X$. Thus an upper bound of $\ell_1(G^c)$ also gives a lower bound of $\ell_{n-1}(G)$.

In 1973 [8], Fiedler showed the following upper bounded about $\ell_{n-1}(G)$

$$\ell_{n-1}(G) \leq \kappa(G) \leq \delta,$$  \hspace{1cm} (2.2)

intriguing the study of algebraic connectivity, where $\kappa(G)$ is the vertex connectivity of $G$. Note that $\ell_{n-1}(G) = 0$ iff $G$ is disconnected.

The upper bounds of Laplacian index $\ell_1(G)$ were studied by many authors. In 1985 [11], Anderson and Morley showed that

$$\ell_1(G) \leq \max_{i,j \in E} \{d_i + d_j\}.$$  \hspace{1cm} (2.3)

Note that $d_i + m_i = d_i + (\sum_{j \in E} d_j)/d_i \leq d_i + \max_{j \in E} \{d_j\} \leq \max_{j \in E} \{d_i + d_j\}$. In 1998 [14], Merris improved the bound in (2.3) by showing

$$\ell_1(G) \leq \max_{i \in V} \{d_i + m_i\}.$$  \hspace{1cm} (2.4)
As another way to improve the bound in (2.3), in 2000 [16], Rojo et al. showed
\[ \ell_1(G) \leq \max_{i,j \in E} \{d_i + d_j - |G_1(i) \cap G_1(j)|\} . \] (2.5)

In 2001 [12], Li and Pan gave a bound, as follows
\[ \ell_1(G) \leq \max_{i \in V} \left\{ \sqrt{2d_i(d_i + m_i)} \right\} . \] (2.6)

In 2004 [19], Zhang showed the following result, which is always better than the bound (2.6).
\[ \ell_1(G) \leq \max_{i \in V} \left\{ d_i + \sqrt{d_i m_i} \right\} . \] (2.7)

One of our results in Corollary 3.2 is an extension of (2.7).

For the lower bound of \( \ell_1(G) \) in 1994 [9], Grone and Merris showed that
\[ \ell_1(G) \geq \Delta + 1. \] (2.8)

The studies of Laplacian spread \( \mathcal{S}_L(G) = \ell_1(G) - \ell_{n-1}(G) \) can be found in [3, 5, 6, 7, 18]. They are interested in the graphs with a few more edges than the number of edges in a tree. The results in this paper with different favor are about graphs of higher vertex connectivity.

### 3 New bounds

The following is our main theorem.

**Theorem 3.1.** Let \( G = (V, E) \) be a simple connected graph of order \( n \). Let \( \ell \) be a nontrivial Laplacian eigenvalue of \( G \) with associated eigenvector \( X = (x_1, x_2, \ldots, x_n) ^\top \). Let \( d_i \) and \( m_i \) be degree and average 2-degree respectively of vertex \( i \in V \), and let \( \lambda \leq \lambda(G) \) and \( \mu \leq \mu(G) \) be two given numbers. Then
\[ \sum_{i=1}^{n} [(d_i - \ell)^2 - d_i m_i + \lambda \ell + \mu(n - \ell)] x_i^2 \leq 0. \] (3.1)

Moreover, the equality in (3.1) holds if and only if for any distinct vertices \( i, j \in V \), the following two statements hold:
\[ \begin{align*}
  ij \in E \quad \text{and} \quad x_i \neq x_j & \quad \Rightarrow \quad |G_1(i) \cap G_1(j)| = \lambda(G), \quad (3.2) \\
  ij \notin E \quad \text{and} \quad x_i \neq x_j & \quad \Rightarrow \quad |G_1(i) \cap G_1(j)| = \mu(G). \quad (3.3)
\end{align*} \]

**Proof.** Because \( X \) is an eigenvector of \( L(G) \) corresponding to \( \ell \),
\[ \| (D(G) - \ell I)X \|^2 = \| (D(G) - L(G))X \|^2 = \| A(G)X \|^2. \] (3.4)

As the \( ij \)-entry of \( A(G)^2 \) is the number \( w_{ij} \) of walks of length 2 from \( i \) to \( j \) and noting that \( w_{ii} = d_i \), we have
\[ \| A(G)X \|^2 = X ^\top A(G)^2 X \] (3.5)
\[ = \sum_{i \in V} d_i x_i^2 + 2 \sum_{j<k} w_{jk} x_j x_k \]
\[ = \sum_{i \in V} d_i x_i^2 + \sum_{j<k} w_{jk} (x_j^2 + x_k^2 - (x_j - x_k)^2) \]
\[ = \sum_{i \in V} \left( \sum_{j \in E} d_j x_i^2 \right) + \sum_{j<k \in E} w_{jk} (x_j - x_k)^2 - \sum_{j<k \in E} w_{jk} (x_j - x_k)^2. \] (3.6)
As $\lambda \leq \lambda(G) = \min_{ij \in E} w_{ij}$ and $\mu \leq \mu(G) = \min_{ij \notin E} w_{ij}$ and by (3.4), (3.6), we have
\[
\sum_{i \in V} (d_i - \ell)^2 x_i^2 \leq \sum_{i \in V} d_i m_i x_i^2 - \lambda \sum_{j<k \in E} (x_j - x_k)^2 - \mu \sum_{j<k \notin E} (x_j - x_k)^2.
\] (3.7)

Applying (2.1) and that $n - \ell(G)$ is eigenvalue of $L(G^c)$ with the same eigenvector $X$ to (3.7), we have
\[
\sum_{i \in V} (d_i - \ell)^2 x_i^2 \leq \sum_{i \in V} d_i m_i x_i^2 - \lambda X^\top L(G)X - \mu X^\top L(G^c)X
\]
\[
= \sum_{i \in V} d_i m_i x_i^2 - \lambda \|X\|^2 - \mu \|n - \ell\|X\|^2
\]
\[
= \sum_{i \in V} d_i m_i x_i^2 - \lambda \ell \sum_{i \in V} x_i^2 - \mu \|n - \ell\| \sum_{i=1}^n x_i^2,
\]
and (3.1) immediately follows from this. Note that the equality holds in (3.1) if and only if the equality in (3.7) holds, and this equivalent to (3.2), (3.3).

The expression
\[
(d_i - \ell)^2 - d_i m_i + \lambda \ell + \mu (n - \ell) = \ell^2 - (2d_i - \lambda + \mu) \ell + (d_i^2 - d_i m_i + \mu n)
\] (3.8)
inside the summation in (3.1) is a quadratic polynomial in variable $\ell$ and is not positive for some $i$. Solving the quadratic polynomial, we have the following upper bound of Laplacian index $\ell_1(G)$, lower bound of the algebraic connectivity $\ell_{n-1}(G)$ and upper bound of Laplacian spread $\mathcal{S}_L(G)$ of $G$.

**Corollary 3.2.** Referring to the notations in Theorem 3.1, the following three inequalities hold:

\[
\ell_1(G) \leq \max_{i \in V} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\},
\]

\[
\ell_{n-1}(G) \geq \min_{i \in V} \left\{ \frac{2d_i - \lambda + \mu - \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\}
\]

and

\[
\mathcal{S}_L(G) \leq \max_{i \in V} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\}
\]
\[
- \min_{i \in V} \left\{ \frac{2d_i - \lambda + \mu - \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\},
\]

where the $\max$ and $\min$ are over vertices $i \in V$ and exclude the terms with a negative term in the square root.

Note that the above upper bound of $\ell_1(G)$ with $\lambda = 0$ and $\mu = 0$ is (2.7), and with $\lambda = \lambda(G)$ and $\mu = 0$ is previously given in [10, Theorem 3.2].

The following corollary is another application of (3.1).
Corollary 3.3. Referring to the notations in Theorem 3.1, the following three inequalities hold:

\[
\ell_1(G) \leq \frac{2\Delta - \lambda + \mu + \sqrt{(2\Delta - \lambda + \mu)^2 - 4\mu n}}{2},
\]

\[
\ell_{n-1}(G) \geq \frac{2\delta - \lambda + \mu - \sqrt{(2\delta - \lambda + \mu)^2 - 4\mu n - 4\delta^2 + 4\Delta^2}}{2},
\]

and

\[
\mathcal{S}_L(G) \leq \Delta - \delta + \frac{1}{2} \left[ \sqrt{(2\Delta - \lambda + \mu)^2 - 4\mu n} + \sqrt{(2\delta - \lambda + \mu)^2 - 4\mu n - 4\delta^2 + 4\Delta^2} \right],
\]

where \(\delta\) and \(\Delta\) are the maximum degree and the minimum degree in \(G\). Moreover if \(G\) is \(k\)-regular then

\[
\mathcal{S}_L(G) \leq \sqrt{(2k - \lambda + \mu)^2 - 4\mu n}.
\]

Proof. By (3.1) with \(\ell = \ell_1(G)\) there exists \(i \in V\) such that the term in (3.8) is not positive. Using \(\ell_1(G) \geq \Delta + 1 > d_i\), and \(\Delta \geq m_i\), we have

\[(\Delta - \ell_1(G))^2 - \Delta^2 + \lambda \ell_1(G) + \mu(n - \ell_1(G)) \leq (d_i - \ell_1(G))^2 - d_im_i + \lambda \ell_1(G) + \mu(n - \ell_1(G)) \leq 0.\]

Solving the above quadratic inequality on the left for \(\ell_1(G)\), we have (3.9). Similarly, by considering \(\ell = \ell_{n-1}(G)\) in (3.1), there exists \(j \in V\) such that (3.8) with \(i = j\) is not positive. Using \(\ell_{n-1}(G) \leq \delta \leq d_j\) and \(\Delta \geq m_j\), we have

\[(\delta - \ell_{n-1}(G))^2 - \Delta^2 + \lambda \ell_{n-1}(G) + \mu(n - \ell_{n-1}(G)) \leq (d_j - \ell_{n-1}(G))^2 - d_jm_j + \lambda \ell_{n-1}(G) + \mu(n - \ell_{n-1}(G)) \leq 0.\]

Solving the quadratic inequality on the left for \(\ell_{n-1}(G)\), we have (3.10). The line (3.11) is immediate from (3.9), (3.10), and (3.12) is from (3.11).

Next we prove that the strongly regular graphs satisfy all the above equalities.

Corollary 3.4. If \(G\) is a strongly regular graph with parameters \((n, k, \lambda(G), \mu(G))\), then \(k = \delta = \Delta\) and the equality in (3.1), the three equalities in Corollary 3.2 and the three equalities (3.9), (3.10), (3.11) all hold for \(\lambda = \lambda(G)\) and \(\mu = \mu(G)\).

Proof. This is clear since (3.2) and (3.3) hold in a strongly regular graph.

4 Other extremal graphs

A graph is extremal for an inequality holding for graphs if the graph attains the equality. In this section, we shall provide extremal graphs for inequalities mentioned in the previous section, excluding strongly regular graphs. Throughout this section we assume \(\lambda = \lambda(G)\) and \(\mu = \mu(G)\). Let

\[
\alpha_1(G) = \max_{i \in V} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{4d_im_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\}
\]

\[
\left( \text{resp. } \beta_1(G) = \min_{i \in V} \left\{ \frac{2d_i - \lambda + \mu - \sqrt{4d_im_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\} \right)
\]
denote the upper bound (resp. lower bound) of Laplacian index $\ell_1(G)$ (resp. algebraic connectivity $\ell_{n-1}(G)$) described in Corollary \ref{corollary} and let

$$\alpha_2(G) = \frac{2\Delta - \lambda + \mu + \sqrt{(2\Delta - \lambda + \mu)^2 - 4\mu n}}{2}$$

and

$$\beta_2(G) = \frac{2\delta - \lambda + \mu - \sqrt{(2\delta - \lambda + \mu)^2 - 4\mu n - 4\delta^2 + 4\delta^2}}{2}$$

denote the upper bound (resp. lower bound) of Laplacian index $\ell_1(G)$ (resp. algebraic connectivity $\ell_{n-1}(G)$) described in Corollary \ref{corollary}

**Example 4.1.** The graph $G = X_8$ depicted on the left of Figure 1 is 3-regular of order 8 with $\lambda(X_8) = 0$, $\mu(X_8) = 1$ and $\ell_1(X_8), \ell_2(X_8) = (7 \pm \sqrt{17})/2 = \ell, \mathcal{A}_L(G) = \sqrt{17}$, so is extremal for (3.1). Note that $\alpha_1(X_8) = \alpha_2(X_8) = (7 + \sqrt{17})/2$ and $\beta_1(X_8) = \beta_2(X_8) = (7 - \sqrt{17})/2$. Hence $X_8$ is extremal for the three inequalities in Corollary \ref{corollary}, \ref{inequality}, \ref{inequality} and \ref{inequality}. On the other hand, the complement graph $G^c = X_8^c$ of $X_8$ has $\lambda(X_8^c) = 1$, $\mu(X_8^c) = 2$ and $\ell_1(X_8^c), \ell_2(X_8^c) = (9 \pm \sqrt{17})/2 = \ell, \mathcal{A}_L(X_8^c) = \sqrt{17}$, so is also extremal for (3.1). Note that $\alpha_1(X_8^c) = \alpha_2(X_8^c) = (9 + \sqrt{17})/2$ and $\beta_1(X_8^c) = \beta_2(X_8^c) = (9 - \sqrt{17})/2$. Hence $X_8$ is extremal for the three inequalities in Corollary \ref{corollary}, \ref{inequality}, \ref{inequality} and \ref{inequality}.

![Figure 1](image1.png)

Figure 1: The graph $X_8$ on the left has $\lambda(X_8) = 0$, $\mu(X_8) = 1$ and its complement graph $X_8^c$ on the right has $\lambda(X_8^c) = 1$, $\mu(X_8^c) = 2$.

**Example 4.2.** The graph $G = Y_8$ depicted on the left of Figure 2 is obtained from the complete graph $K_8$ of order 8 by deleting two vertex disjoint cycles of order 4. It is 5-regular of order 8 with $\lambda(Y_8) = 2$, $\mu(Y_8) = 4$, $\ell_1(Y_8) = 8$ and $\ell_2(Y_8) = 4$, so is extremal for (3.1) with $\ell = 8$, 4 respectively. Note that $\alpha_1(Y_8) = \alpha_2(Y_8) = 8$, $\beta_1(Y_8) = \beta_2(Y_8) = 4$. Hence $G$ is extremal for for the three inequalities in Corollary \ref{corollary}, \ref{inequality}, \ref{inequality} and \ref{inequality}.

**Example 4.3.** The graph $G = Z_8$ depicted on the right of Figure 2 is obtained from $K_8$ by deleting two vertex disjoint cycles of order 3 and 5 respectively. It is 5-regular of order 8 with $\lambda(Z_8) = 2$, $\mu(Z_8) = 4$, $\ell_1(Z_8) = 8$ and so is extremal for (3.1) with $\ell = 8$. Note that $\alpha_1(Z_8) = \alpha_2(Z_8) = 8$, $\beta_1(Z_8) = \beta_2(Z_8) = 4$ and $\ell_2(Z_8) = (11 - \sqrt{5})/2$. Hence $Z_8$ is extremal for the first inequality in Corollary \ref{corollary} and \ref{inequality} and is not extremal for other inequalities.
Figure 2: The graph $Y_8$ on the left has $\lambda(Y_8) = 2$, $\mu(Y_8) = 4$, and the graph $Z_8$ on the right has $\lambda(Z_8) = 2$, $\mu(Z_8) = 4$.

Example 4.4. The graph $G = U_8$ depicted on the left of Figure 3 is 4-regular of order 8 with $\lambda(U_8) = 0$, $\mu(U_8) = 2$, $\ell_7(U_8) = 2$, so is extremal for $(3.1)$ with $\ell = 2$. Note that $\alpha_1(U_8) = \alpha_2(U_8) = 8$, $\beta_1(U_8) = \beta_2(U_8) = 2$ and $\ell_1(U_8) = 6$. Hence $U_8$ is extremal for the second inequality in Corollary 3.2 and $(3.10)$, and is not extremal for other inequalities. On the other hand, the complement graph $G = U_8^c$ of $U_8$ depicted on the right of Figure 3 is 3-regular of order 8 with $\lambda(U_8^c) = \mu(U_8^c) = 0$, $\ell_1(U_8^c) = 6$, so is extremal for $(3.1)$ with $\ell = 6$. Note that $\alpha_1(U_8^c) = \alpha_2(U_8^c) = 6$, $\beta_1(U_8^c) = \beta_2(U_8^c) = 0$ and $\ell_1(U_8^c) = 2$. Hence $U_8^c$ is extremal for the first inequality in Corollary 3.2 and $(3.9)$, and is not extremal for other inequalities.

Figure 3: The graph $U_8$ on the left has $\lambda(U_8) = 0$, $\mu(U_8) = 2$, and its complement graph $U_8^c$ on the right has $\lambda(U_8^c) = \mu(U_8^c) = 0$.

Example 4.5. Let $G = K_{a,b}$ be the complete bipartite graph of bipartition orders $a$ and $b$, respectively, where $a < b$ and $n = a+b$. Then $\lambda(K_{a,b}) = 0$, $\mu(K_{a,b}) = a$, $(d_i,m_i) = (a,b)$ or $(b,a)$, so the lower bound of $\ell_{n-1}(K_{a,b})$ in the second inequality of Corollary 3.2 is $\beta_1(K_{a,b}) = \min\{a,(2b + a - \sqrt{a(4b-3a)})/2\} = a$. Also $\ell_{n-1}(K_{a,b}) = a$ since $\ell_{n-1}(K_{a,b}) \leq \kappa(K_{a,b}) \leq \delta = a$. Hence $K_{a,b}$ is extremal for $(3.1)$ with $\ell = \ell_{n-1}(K_{a,b})$ and the second inequality in Corollary 3.2. Note that $\alpha_1(K_{a,b}) = (2b + a + \sqrt{a(4b-3a)})/2$, $\alpha_2(K_{a,b}) = (2b + a + \sqrt{4b^2-3a^2})/2$, $\beta_2(K_{a,b}) = 2a - b$ and $\ell_1(K_{a,b}) = a+b$. Hence $K_{a,b}$ is not extremal for other inequalities.

Next we provide a graph which is extremal only for $(3.1)$ about $\ell = \ell_1(G)$.

Example 4.6. Let $G = F_t$ ($t > 1$) be a fan of order $n = 2t+1$ as depicted in Figure 4. Then $\lambda(F_t) = 1$, $\mu(F_t) = 1$, $(d_i,m_i) = (2,t+1)$ or $(2t,2)$, and $X = (1_{2t},-2t)^\top$ is an eigenvector corresponding to
the eigenvalue \( \ell_1(F_t) = 2t + 1 \), where \( 1 \) is all one vector. One can check that \( F_t \) is extremal for (5.1) about \( \ell = \ell_1(F_t) \). On the other hand, \( \alpha_1(F_t) = (2t + \sqrt{2t^2 - 2t - 1}) \), \( \alpha_2(F_t) = 2t + \sqrt{4t^2 - 2t - 1} \), \( \beta_1(F_t) = \min\{2t - \sqrt{2t^2 - 2t - 1}, 1\} \leq \ell_{2t}(F_t) \leq \kappa(F_t) = 1 \), \( \beta_2(F_t) = 2 - \sqrt{4t^2 - 2t - 1} \), so \( F_t \) is also extremal for (3.1) with \( \ell = \ell_{2t}(F_t) \), the second inequality in Corollary 3.2, but is not extremal for any other inequalities.

![Diagram of the fan graph](image)

Figure 4: The fan graph \( F_t \) of order \( 2t + 1 \) with \( \lambda(F_t) = 1 \), \( \mu(F_t) = 1 \).

The following table summarizes the extremal graphs mentioned in this section which are not strongly regular.

| Graph | \( \ell_1(G) \) | \( \alpha_1(G) \) | \( \alpha_2(G) \) | \( \ell_{n-1}(G) \) | \( \beta_1(G) \) | \( \beta_2(G) \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( X_8 \) | \((7 + \sqrt{17})/2 \) | \((7 + \sqrt{17})/2 \) | \((7 + \sqrt{17})/2 \) | \((7 - \sqrt{17})/2 \) | \((7 - \sqrt{17})/2 \) | \((7 - \sqrt{17})/2 \) |
| \( X_8^c \) | \((9 + \sqrt{17})/2 \) | \((9 + \sqrt{17})/2 \) | \((9 + \sqrt{17})/2 \) | \((9 - \sqrt{17})/2 \) | \((9 - \sqrt{17})/2 \) | \((9 - \sqrt{17})/2 \) |
| \( Y_8 \) | 8 | 8 | 8 | 4 | 4 | 4 |
| \( Z_8 \) | 8 | 8 | 8 | \((11 - \sqrt{5})/2 \) | 4 | 4 |
| \( U_8 \) | 6 | 8 | 8 | 2 | 2 | 2 |
| \( U_8^c \) | 6 | 6 | 6 | 2 | 0 | 0 |
| \( K_{a,b} \) | \( a + b \) | \( x \) | \( x \) | \( a \) | \( a \) | \( 2a - b \) |
| \( F_t \) | \( 2t + 1 \) | \( y \) | \( z \) | 1 | 1 | \( w \) |

\( a < b, t > 1, x = \frac{2b + a + \sqrt{a(4b - 3a)}}{2}, y = 2t + \sqrt{2t - 1}, z = 2t + \sqrt{4t^2 - 2t - 1}, w = 2 - \sqrt{4t^2 - 2t - 1} \).

Table 1: Extremal graphs that are not strongly regular.

5 The \((n - 3)\)-regular graphs of order \( n \)

From now on let \( G \) denote an \((n - 3)\)-regular graph \( G \) of order \( n \). Note that \( G \) is obtained from the complete graph \( K_n \) by deleting some edges whose union forms vertex disjoint cycles of order \( n \). Denote \( G \) in notation \( G = K_n - (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t}) \), where \( n_1 \geq n_2 \geq \cdots \geq n_t \geq 3 \) is a nonincreasing integer sequence satisfying \( n = n_1 + n_2 + \cdots + n_t \). For example, \( Y_8 = K_8 - 2C_4 \) in Example 4.2 and \( Z_8 = K_8 - (C_5 \cup C_5) \) in Example 4.3. Note that \( G \) is connected iff \( n \geq 5 \). If \( n = 5 \) then \( G = K_5 - C_5 = C_5 \) is a cycle of order 5 and is a strongly regular graph with \( \ell_1(G), \ell_4(G) = (5 + \sqrt{5})/2 \). Hence we assume \( n \geq 6 \).
Proposition 5.1. The \((n - 3)\)-regular graph \(G = K_n - (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})\) has \(\lambda(G) = n - 6\) and
\[
\mu(G) = \begin{cases} 
    n - 3, & \text{if } n_i = 3 \text{ for all } i; \\
    n - 4, & \text{otherwise}.
\end{cases}
\]

Moreover \(G\) is strongly regular iff \(n_i = 3\) for all \(1 \leq i \leq t\), and in this case \(\ell_1(G) = n\) and \(\ell_{n-1}(G) = n - 3\).

Proof. As \(G\) is \((n - 3)\)-regular, \(|G_1(i) \cap G_1(j)| \geq n - 6\) for \(i, j \in V\). To prove \(\lambda(G) = n - 6\), we need to choose \(ij \in E\) such that in the deleting cycles that \(i, j\) belong, two neighbors of \(i\) do not overlap the two neighbors of \(j\). This can be done if \(n_1 \geq 6\) of course, and also can be done if \(n_1 < 6\) since then as \(n \geq 6\), there are two deleting cycles and we can choose \(i, j\) in different cycles.

If \(i'j' \notin E\) then they are in the same deleting cycle and are adjacent in the cycle. Hence \(|G_1(i') \cap G_1(j')| \geq n - 4\), where the excluding four vertices are \(i', j'\), the other neighbor \(a\) of \(i'\), the other neighbor \(b\) of \(j'\), and \(a = b\) iff \(i', j'\) are inside \(C_3\). This proves the line of \(\mu(G)\).

The only possible for a union of cycles to be strongly regular is when the cycles are all triangles. Hence \(G\) is strongly iff \(n_i = 3\) for all \(i\). The \(\ell_1(G) = n\) and \(\ell_{n-1}(G) = n - 3\) are determined from (1.1) by using \(\lambda = n - 6\) and \(\mu = n - 3\). Therefore, we complete the proof. \(\square\)

The Laplacian eigenvalues of a cycle is well-known \([4, \text{Section 1.4.3}]\), indeed
\[
\ell_1(C_s) = \begin{cases} 
    4, & s \text{ is even}; \\
    2 + 2 \cos(\pi/s), & s \text{ is odd},
\end{cases} \quad (5.1)
\]
and
\[
\ell_{s-1}(C_s) = 2 - \cos(2\pi/s). \quad (5.2)
\]

Proposition 5.2. The \((n - 3)\)-regular graph \(G = K_n -(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})\) with some \(m_i > 3\) has \(\alpha_1(G) = \alpha_2(G) = n, \beta_1(G) = \beta_2(G) = n - 4\),
\[
\ell_1(G) = \begin{cases} 
    n - \cos(2\pi/n), & t = 1; \\
    n, & t \geq 2
\end{cases}
\]
and
\[
\ell_{n-1}(G) = \begin{cases} 
    n - 2 - 2 \cos(\pi/n), & n_i \text{ is odd for all } i; \\
    n - 4, & \text{otherwise}.
\end{cases}
\]

Proof. Applying \(\lambda(G) = n - 6\) and \(\mu(G) = n - 4\) to the definitions, one finds \(\alpha_1(G) = \alpha_2(G) = n\) and \(\beta_1(G) = \beta_2(G) = n - 4\) immediately. From (5.2),
\[
\ell_1(G) = n - \ell_{n-1}(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t}) = \begin{cases} 
    n - \cos(2\pi/n), & t = 1; \\
    n, & t \geq 2
\end{cases}
\]
From (5.1),
\[
\ell_{n-1}(G) = n - \ell_1(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t}) = \begin{cases} 
    n - 2 - 2 \cos(\pi/n), & \text{if } n_i \text{ is odd for all } i; \\
    n - 4, & \text{otherwise}.
\end{cases}
\]

From Proposition 5.2 we find that if \(t \geq 2\) and \(n_i\) is even for some \(i\) then the regular \((n - 3)\)-regular graph \(G = K_n -(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})\) is extremal for (3.1) with \(\ell = \ell_1(G), \ell_{n-1}(G)\), the three inequalities in Corollary 3.2 (3.2), (3.11) and (3.11).
Acknowledgments

This research is supported by the Ministry of Science and Technology of Taiwan R.O.C. under the project NSC 102-2115-M-009-009-MY3.

References

References

[1] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, *Linear Multilinear Algebra*, 18 (1985), 141-145.

[2] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[3] Y.-H. Bao, Y.-Y. Tan, Y.-Z. Fan, The Laplacian spread of unicyclic graphs, *Applied Mathematics Letters*, 22(2009), 1011-1015.

[4] A.E. Brouwer and W. H. Haemers *Spectra of Graphs*, 2012: Springer-Verlag.

[5] Y.Q. Chen, L.G. Wang, The Laplacian spread of tricyclic graphs, *The Electronic Journal of Combinatorics*, 16 (2009), #R80.

[6] Y.-Z. Fan, S.-D. Li, Y.-Y. Tan, The Laplacian spread of bicyclic graphs, *Journal of Mathematical Research and Exposition*, Jan., vol.30 No. 1 (2010), pp. 17-28.

[7] Y.-Z. Fan, J. Xu, Y. Wang, D. Liang, The Laplacian spread of a tree, *Discrete Mathematics and Theoretical Computer Science*, vol.10:1(2008), 79-86.

[8] M. Fiedler, Praha, Algebraic connectivity of graphs, *Czechoslovak Mathematical Journal* 23 (98), (1973), Praha.

[9] R. Grone and R. Merris, The Laplacian spectrum of a graph II, *SIAM Journal Discrete Math*, 7:229-237, 1994.

[10] J.M. Guo, J. Li, W.C. Shiu, A note on the upper bounds for the Laplacian spectral radius of graphs, *Linear Algebra and its Applications*, 439 (2013), 1657-1661.

[11] J.H. van Lint and R.M. Wilson, *A course in combinatorics*, Cambridge University Press, 2001.

[12] J.S. Li, Y.L. Pan, De Caen’s inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra and its Applications*, 328 (2001), 153-160.

[13] R. Merris, Laplacian graph eigenvectors, *Linear Algebra and its Applications* 278 (1998), 221-236.

[14] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 285 (1998), 33-35.

[15] M. W. Newman, *The Laplacian Spectrum of Graphs*, University of Manitoba, Winnipeg, MB, Canada, 2000.
[16] O. Rojo, R. Soto, H. Rojo, An always nontrivial upper bound for Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 312 (2000), 155-159.

[17] L.S. Shi, Bounds on the (Laplacian) spectral radius of graphs, *Linear Algebra and its Applications*, 422 (2007), 755-770.

[18] Z. You, B. Liu, The minimal Laplacian spread of unicyclic graphs, *Linear Algebra and its Applications*, 432 (2010), 499-504.

[19] X.D. Zhang, Two sharp upper bounds for the Laplacian eigenvalues, *Linear Algebra and its Application* 376 (2004), 207-213.

[20] X.D. Zhang, R. Luo, The Laplacian eigenvalues of mixed graphs, *Linear Algebra and its Application* 362 (2003), 109-119.