A Complete Global Solution to the Pressure Gradient Equation

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Abstract

We study the domain of existence of a solution to a Riemann problem for the pressure gradient equation in two space dimensions. The Riemann problem is the expansion of a quadrant of gas of constant state into the other three vacuum quadrants. The global existence of a smooth solution was established in Dai and Zhang [Arch. Rational Mech. Anal., 155(2000), 277-298] up to the free boundary of vacuum. We prove that the vacuum boundary where the system is degenerate is the trivial coordinate axes.

Keyword: Regularity, vacuum boundary, two dimensional Riemann problem, characteristic decomposition.

AMS subject classification: Primary: 35L65, 35J70, 35R35; Secondary: 35J65.

1 Introduction

The pressure gradient system

\[
\begin{align*}
    u_t + p_x &= 0, \\
    v_t + p_y &= 0, \\
    E_t + (pu)_x + (pv)_y &= 0,
\end{align*}
\]

(1.1)

where \( E = (u^2 + v^2)/2 + p \), appeared first in the flux-splitting method of Li and Cao [5] and Agarwal and Halt [1] in numerical computation of the Euler system of a

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compressible gas. Later, an asymptotic derivation was given in Zheng [10, 13] from the two-dimensional full Euler system for an ideal fluid

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho U) &= 0, \\
(\rho U)_t + \nabla \cdot (\rho U \otimes U + pI) &= 0, \\
(\rho E)_t + \nabla \cdot (\rho EU + pU) &= 0,
\end{align*}
\]

where \( U = (u, v), E = (u^2 + v^2)/2 + p/((\gamma - 1)\rho) \), and \( \gamma > 1 \) is a gas constant. We refer the reader to the books of Zheng [11] and Li et. al. [7] for more background information, and the papers [10, 9, 2, 12, 6, 13] for recent studies. After being decoupled from the pressure gradient system (1.1), the pressure satisfies the following second order quasi-linear hyperbolic equation

\[
\left( \frac{p_1}{p} \right)_t - \Delta (x,y)p = 0. \tag{1.2}
\]

Dai and Zhang [2] studied a Riemann problem for system (1.1), see also Yang and Zhang [8] by the hodograph method. In the self similar variables \( \xi = x/t, \eta = y/t \), the value of the pressure variable of the Riemann data is

\[
\begin{align*}
p(\xi, \eta) &= \xi^2, & \text{for } 0 < \xi, \eta \leq \sqrt{p_1}, \\
p(\xi, \eta) &= \eta^2, & \text{for } 0 < \xi, \eta \leq \sqrt{p_1}, \quad \xi^2 + (\eta - \sqrt{p_1})^2 = p_1.
\end{align*}
\]  

Here \( p_1 \) is any positive number. They showed that the Goursat problem for system (1.2) admits a global solution in the self-similar plane, which is smooth with a possible vacuum near the origin (see Figure 1, where \( a = \sqrt{p_1} \)). We are interested in the size of the vacuum boundary \( \{(\xi, \eta) \mid p(\xi, \eta) = 0\} \) where the pressure gradient system (1.2) is degenerate. Somewhat surprisingly, our result shows that the vacuum bubble is trivial and the entire vacuum boundary is the trivial coordinate axes in the self-similar plane (see Figure 2, where \( a = \sqrt{p_1} \)), which is stated in our main theorem at the end of Section 3.

Further motivation for the study of the current problem is that the study of boundaries such as a sonic curve is important in establishing the global existence of a solution to a general two dimensional Riemann problem of the pressure gradient system. In addition, the solution of the current problem covers wave interaction problems in which only some fractions of the plane waves are involved. Wave interactions of these kinds are common in two dimensional Riemann problems. Finally, the study of the pressure gradient system has motivated work on two dimensional full Euler systems, see Li [4] and Zheng [14, 15].

2 Integration along characteristics

In the self-similar variables \( \xi = x/t, \eta = y/t \), the pressure gradient equation (1.2) takes the form

\[
\frac{(\xi \partial_\xi + \eta \partial_\eta)^2 p}{p} - \Delta (\xi, \eta)p + \frac{(\xi \partial_\xi + \eta \partial_\eta)p}{p} - \frac{((\xi \partial_\xi + \eta \partial_\eta)p)^2}{p^2} = 0. \tag{2.1}
\]
Figure 1. Solution with a presumed vacuum bubble.
Figure 2. The global solution.
In the polar coordinates
\[
\begin{align*}
  r &= \sqrt{\xi^2 + \eta^2}, \\
  \theta &= \arctan \frac{\eta}{\xi},
\end{align*}
\]
equation (2.1) can be decomposed along the characteristics into the following form (see [6, 3])
\[
\begin{align*}
  \partial_+ \partial_- p &= mp_\theta \partial_- p, \\
  \partial_- \partial_+ p &= -mp_\theta \partial_+ p,
\end{align*}
\]
(2.2)
provided that
\[ p < r^2, \]
where
\[ m = \frac{\lambda r^4}{2p^2}, \]
and
\[
\begin{align*}
  \partial_\pm &= \partial_\theta \pm \frac{1}{\lambda} \partial_r, \\
  \lambda &= \sqrt{\frac{r}{r^2 - p}}.
\end{align*}
\]
(2.3)

Note that the equation is invariant under the following scaling transformation
\[(\xi, \eta, p) \rightarrow (\frac{\xi}{\sqrt{p_1}}, \frac{\eta}{\sqrt{p_1}}, \frac{p}{p_1}). \quad (p_1 > 0)\]
Thus, without loss of generality, the corresponding boundary condition (1.3) in the polar coordinates can be set as the form
\[
\begin{align*}
  p &= \xi^2 = r^2 \cos^2 \theta = 4 \cos^4 \theta, \quad \text{on} \quad r = 2 \cos \theta, \quad \pi/4 \leq \theta \leq \pi/2; \\
  p &= \eta^2 = r^2 \sin^2 \theta = 4 \sin^4 \theta, \quad \text{on} \quad r = 2 \sin \theta, \quad 0 \leq \theta \leq \pi/4.
\end{align*}
\]
(2.4)
The solution exists in the interaction zone up to a possible vacuum bubble. See Figure 3.

The characteristic form (2.2) of the pressure gradient equation enjoys a number of useful properties. For example, the quantities \( \partial_\pm p \) keep their positivities/negativities along characteristics of a plus/minus family, and the sign-persevering quantities yield monotonicity of the primary variable \( p \) (see [6], also [3] where the authors propose to call them \textit{Riemann sign-persevering variables}), and the fact that a state adjacent to a constant state for the pressure gradient system must be a simple wave in which \( p \) is constant along the characteristics of a plus/minus family [6].

The characteristic decomposition (2.2) has played an important role and was a powerful tool for building the existence of smooth solutions in the work of Dai and Zhang [2]. We are interested in the size of the vacuum boundary \( \{(r, \theta) \mid p(r, \theta) = 0\} \)
Figure 3. Presumed vacuum in the polar coordinates.
where the pressure gradient equation is degenerate. For this purpose, we rewrite (2.2) as
\[
\begin{align*}
\partial_+ \partial_- p &= q \partial_+ p \partial_- p - q (\partial_- p)^2, \\
\partial_- \partial_+ p &= q \partial_+ p \partial_- p - q (\partial_+ p)^2,
\end{align*}
\] (2.5)
where
\[ q = \frac{r^2}{4p(r^2 - p)}. \] (2.6)
Define the characteristic curves \( r_a^- (\theta) \) and \( r_b^+ (\theta) \) as
\[
\left\{ \begin{array}{l}
dr_a^-(\theta) = \frac{1}{\lambda(r_a^- (\theta), \phi)}, \quad \text{and} \quad \\
\theta_a^-(\theta) = 2 \sin \theta_a,
\end{array} \right. \quad \left\{ \begin{array}{l}
dr_b^+(\theta) = \frac{1}{\lambda(r_b^+ (\theta), \phi)}, \quad \text{and} \quad \\
r_b^+(\theta) = 2 \cos \theta_b.
\end{array} \right.
\] (2.7)
We point out here that for convenience, we also use the notation \( r_a^- (r, \theta) \) (\( r_b^+(r, \theta) \), respectively) which represents the characteristic passing through the point \((r, \theta)\) and intersecting the lower (upper, respectively) boundary at point \(a\) (\(b\), respectively). See Figure 3.

Now let us rewrite the system (2.5) as the following form
\[
\begin{align*}
\partial_+ \left( \frac{1}{\partial_- p} \exp \int_{\theta_a}^{\theta} q \partial_+ p (r_b^+(\phi), \phi) d\phi \right) &= q \exp \int_{\theta_a}^{\theta} q \partial_+ p (r_b^+(\phi), \phi) d\phi, \\
\partial_- \left( \frac{1}{\partial_+ p} \exp \int_{\theta_a}^{\theta} q \partial_- p (r_a^- (\phi), \phi) d\phi \right) &= q \exp \int_{\theta_a}^{\theta} q \partial_- p (r_a^- (\phi), \phi) d\phi.
\end{align*}
\]
Integrate the above equations along the positive and negative characteristics \( r_b^+ (\theta) \) and \( r_a^- (\theta) \) from \( \theta_b \) and \( \theta_a \) to \( \theta \), respectively, with respect to \( \theta \), one can get the iterative expressions of \( \partial_+ p \) and \( \partial_- p \):
\[
\begin{align*}
\frac{1}{\partial_- p} \exp \int_{\theta_b}^{\theta} q \partial_+ p (r_b^+(\phi), \phi) d\phi &= \frac{1}{\partial_- p} (2 \cos \theta_b, \theta_b) \\
+ \int_{\theta_b}^{\theta} q (r_b^+(\psi), \psi) \exp \int_{\theta_b}^{\psi} q \partial_+ p (r_b^+(\phi), \phi) d\phi d\psi,
\end{align*}
\] (2.8)
\[
\frac{1}{\partial_+ p} \exp \int_{\theta_a}^{\theta} q \partial_- p (r_a^- (\phi), \phi) d\phi = \frac{1}{\partial_+ p} (2 \sin \theta_a, \theta_a) \\
+ \int_{\theta_a}^{\theta} q (r_a^- (\psi), \psi) \exp \int_{\theta_a}^{\psi} q \partial_- p (r_a^- (\phi), \phi) d\phi d\psi.
\]
On the other hand, noting the boundary condition (2.4), a straightforward calculation shows that
\[
\exp \int_{\theta_b}^{\theta} q \partial_+ p (r_b^+(\phi), \phi) d\phi \]
\[
= \exp \frac{1}{4} \int_{\theta_b}^{\theta} \left( \frac{1}{p} + \frac{1}{r^2 - p} \right) \partial_+ p (r_b^+(\phi), \phi) d\phi
\]
\[
= \frac{1}{\sqrt{2 \cos \theta_b}} \exp \left\{ \frac{1}{4} \int_{p(2 \cos \theta_b, \theta_b)}^{p(r_b^+(\theta), \theta)} \frac{1}{r^2 (r_b^+(\theta) - p)} dp \right\}.
\]
Similarly, one has

$$\exp \int_{\theta_a}^{\theta} q \partial_{-} p \left( r^a_{-} (\phi), \phi \right) d\phi$$

$$= \frac{p^+_1 \left( r^a_{-} (\theta), \theta \right)}{\sqrt{2} \sin \theta_a} \exp \left\{ \frac{1}{4} \int_{p(2 \sin \theta_a, \theta_a)}^{\theta} \frac{1}{r^2 (r^a_{-}) - p_1} dp_1 \right\}.$$  \hfill (2.10)

Then, we have

$$\int_{\theta_b}^{\theta} q \left( r^b_{+} (\psi), \psi \right) \left\{ \exp \int_{\theta_b}^{\theta} q \partial_{+} p \left( r^b_{+} (\phi), \phi \right) d\phi \right\} d\psi$$

$$= \frac{1}{4 \sqrt{2} \cos \theta_b} \int_{\theta_b}^{\theta} \frac{r^2 p - \frac{3}{4}}{r^2 - p} \left( r^b_{+} (\psi), \psi \right) \times \exp \left\{ \frac{1}{4} \int_{p(2 \cos \theta_b, \theta_b)}^{\theta} \frac{1}{r^2 (r^b_{+}) - p_1} dp_1 \right\} d\psi,$$  \hfill (2.11)

and

$$\int_{\theta_a}^{\theta} q \left( r^a_{-} (\psi), \psi \right) \exp \int_{\theta_a}^{\theta} q \partial_{-} p \left( r^a_{-} (\phi), \phi \right) d\phi d\psi$$

$$= \frac{1}{4 \sqrt{2} \sin \theta_a} \int_{\theta_a}^{\theta} \frac{r^2 p - \frac{3}{4}}{r^2 - p} \left( r^a_{-} (\psi), \psi \right) \times \exp \left\{ \frac{1}{4} \int_{p(2 \sin \theta_a, \theta_a)}^{\theta} \frac{1}{r^2 (r^a_{-}) - p_1} dp_1 \right\} d\psi.$$  \hfill (2.12)

Finally, by substituting (2.10) and (2.12) into the second equality of (2.8), we arrive at a new iterative expression for \( \partial_{+} p \):

$$\partial_{+} p (r, \theta)$$

$$= \frac{\exp \int_{\theta_a}^{\theta} q \partial_{-} p \left( r^a_{-} (\phi), \phi \right) d\phi}{\frac{1}{4 \sin \theta_a} \left( 2 \sin \theta_a, \theta_a \right) + \int_{\theta_a}^{\theta} q \left( r^a_{-} (\psi), \psi \right) \exp \int_{\theta_a}^{\psi} q \partial_{-} p \left( r^a_{-} (\phi), \phi \right) d\phi d\psi}$$

$$= \frac{4p^+_1 (r, \theta) \exp \left\{ \frac{1}{4} \int_{p(2 \sin \theta_a, \theta_a)}^{\theta} \frac{1}{r^2 (r^a_{-}) - p_1} dp_1 \right\}}{\frac{1}{2 \sqrt{2} \sin \theta_a \cos \theta_a} + \int_{\theta_a}^{\theta} \frac{r^2 p - \frac{3}{4}}{r^2 - p} \left( r^a_{-} (\psi), \psi \right) \exp \left\{ \frac{1}{4} \int_{p(2 \sin \theta_a, \theta_a)}^{\theta} \frac{1}{r^2 (r^a_{-}) - p_1} dp_1 \right\} d\psi}.$$  \hfill (2.13)

Similarly, by substituting (2.9) and (2.11) into the first equality of (2.8), one obtains
the new iterative formula for $\partial_- p$:

$$
\partial_- p(r, \theta) = -\exp \int_{\theta_0}^{\theta} q \partial_+ p(r^b_+(\phi), \phi) d\phi
= \frac{1}{\sigma_-(2 \cos \theta_0, \theta_0) + \int_{\theta_0}^{\theta} q(r^b_+(\psi), \psi) \exp \int_{\theta_0}^{\psi} q \partial_+ p(r^b_+(\phi), \phi) d\phi d\psi}
\left( \frac{4p_1}{2} \exp \left\{ \frac{1}{4} \int p(r^b_+(\psi), \psi) \frac{1}{r^2(r^b_+ - p_1)} dp_1 \right\} \right)
\left( \frac{1}{2} \int_{\theta_0}^{\theta} \frac{r^2 r^b_+}{r^2 - p} (r^b_+(\psi), \psi) \exp \left\{ \frac{1}{4} \int p(r^b_+(\psi), \psi) \frac{1}{r^2(r^b_+ - p_1)} dp_1 \right\} d\psi \right).
$$

3 The boundary of the vacuum bubble

In this section, we use the iterative formulas (2.13) and (2.14) to prove that the vacuum bubble $\{ (r, \theta) \mid p(r, \theta) = 0, \theta \in (0, \pi/2) \}$ is in fact the trivial origin $\{ (0, 0) \}$ in the self-similar plane. We use the method of contradiction. Assume to the contrary that there is a bubble with boundary $r_0(\theta) \geq 0$ for all $\theta \in (0, \pi/2)$, and $r_0(\theta) > 0$ for some $\theta \in (0, \pi/2)$. That means $p(r_0(\theta), \theta) = 0$ for $\theta \in (0, \pi/2)$, and the solution is smooth in the domain bounded by the bubble $r_0(\theta)$ and the upper and lower characteristic boundaries. We intend to deduce contradictions, which in turn proves that $r_0(\theta) = 0$ for all $\theta \in (0, \pi/2)$ and the vacuum bubble $\{ (r_0(\theta), \theta) \}$ is in fact the trivial origin $\{ (0, 0) \}$ in the self-similar plane.

Before we start the above procedure, we point out two observations which roughly imply the nonexistence of the vacuum bubble. For presenting the observations, we assume further that $r_0(\theta) > 0$ for all $\theta \in (0, \pi/2)$.

First, we can compute easily the characteristic slope

$$
d\theta/dr = -\lambda = -\frac{1}{2 \sin \theta} \sim -\frac{1}{2}
$$

along the upper characteristic boundary and consider $\theta \to \pi/2$. Similar result holds on the lower boundary. Now assuming there is a bubble, noting that along the plus characteristics, by definition of $\lambda$ in (2.13), we see easily that

$$
d\theta/dr = \lambda = \sqrt{\frac{p}{r^2(r^2 - p)}} \to 0
$$

as $p \to 0$ (near the vacuum bubble) except for $\theta = 0$ and $\theta = \frac{\pi}{2}$. The above computation reveals that there is some kind of inconsistency for the slopes of the characteristics at $(0, 0)$ and $(0, \pi/2)$ in the polar coordinate plane.

Next, let us calculate the decay rate of $p$ along the middle line $\theta = \pi/4$. By using the symmetry of system (2.1) (see Figure 3), we can obtain that

$$
\partial_{\pm} p \sim \pm M_0 p^{1/2}
$$
asymptotically. To show the details, we propose
\[ \partial_\pm p \sim \pm M_0 p^{4+\delta}, \]
as \( (r, \theta) \) tends to a point of the vacuum bubble. Then, by (2.13), we have
\[
\partial_+ p(r, \theta) = 4p^{\frac{4}{3}}(r, \theta) \exp\left\{ \frac{1}{4} \int_{p(2 \sin \theta, \theta_a)}^{p(r_a, \theta)} \frac{1}{r^2(r^2 - p_1)} dp_1 \right\} / \left\{ \frac{1}{2\sqrt{2} \sin^2 \theta_a \cos \theta_a} \right\}
\[ + \int_{\theta_a}^{\theta} \frac{r^2 p^{-\frac{4}{3}}}{(r^2 - p) \partial_- p(r_a(\psi), \psi)} \exp\left\{ \frac{1}{4} \int_{p(2 \sin \theta_a, \theta_a)}^{p(r_a(\psi), \psi)} \frac{1}{r^2(r^2 - p_1)} dp_1 \right\} dp \]
\[ = M_0 \frac{4}{\delta} p^{\frac{4}{3}+\delta} + \text{high order terms}, \]
which implies that \( \delta \) should be \( \frac{1}{4} \).

Thus, by using (2.3), we have
\[ \partial_r p \sim \frac{M_0}{r^2} p \]
asymptotically as \( (r, \theta) \) tends to a point of the vacuum bubble. Therefore,
\[ p \sim c \exp \left( -\frac{M_0}{r} \right) \] (3.1)
asymptotically as \( (r, \theta) \) tends to a point of the vacuum bubble on the line \( \theta = \pi/4 \), which implies that there is no interior vacuum at least at \( \theta = \frac{\pi}{4} \).

We point out incidentally that Zheng’s previous numerical computation of the bubble, referred to in Dai and Zhang [2], is probably caused by the fast exponential decay (3.1).

In what follows, we concentrate on establishing the above formal intuition rigorously. With the symmetry of system (2.5), let us restrict our arguments on \( \theta \in (0, \pi/4] \).

Fix a point \( (r, \theta) \) on the bubble boundary. Let us denote \( \mathbb{D}(r, \theta) \) the bounded domain determined by the positive and negative characteristic curves starting from \( (r, \theta) \) and \((\sqrt{2}, \pi/4)\). See Figure 4.

For \( 0 < \epsilon < 1 \), define a curve \( r_\epsilon(\theta), \theta \in (0, \pi/2) \), by
\[ p(r_\epsilon(\theta), \theta) = \epsilon. \] (3.2)
From (2.13) and (2.14), it is easy to see that
\[
\begin{cases}
\partial_+ p(r, \theta) > 0, \\
\partial_- p(r, \theta) < 0,
\end{cases}
\text{for } 0 < \theta \leq \frac{\pi}{4}. \] (3.3)

Thus, by (2.3) and (3.3), we have
\[ p_r(r, \theta) > 0, \] (3.4)
Figure 4. Domain and notation

Figure 4:
which implies that the curve \( r_\varepsilon(\theta), \theta \in (0, \frac{\pi}{2}) \), defined in (3.2), is smooth if \( \varepsilon \in (0, 1) \). Here we point out that sometimes we still denote the characteristics passing through \((r_\varepsilon(\theta), \theta)\) or any other point \((r, \theta)\) by \( r_\varepsilon^a(\theta) \) and \( r_\varepsilon^b(\theta) \) or \( r^a(r, \theta) \) and \( r^b(r, \theta) \), when in fact \( a = a_\varepsilon \) is dependent on \( \varepsilon \), etc. We caution the reader that the intersection is determined by the point \((r, \theta)\) and the characteristic, which is hidden for notational convenience and always well understood.

Now we fix \( \varepsilon_0 = 1 \) (\( \varepsilon_0 \) can be any fixed positive constant in \((0, 1]\), which do not change any of the following arguments). Define

\[
M_1 = \max_{over S} \left\{ p^{-\frac{1}{2}} \partial_+ p, -p^{-\frac{1}{2}} \partial_- p \right\},
\]

where

\[
S := \{ (r, \theta) \mid \theta \in (0, \pi/2), r_{\varepsilon_0}(\theta) \leq r \leq \sqrt{2} \}.
\]

Then we have

\[
\begin{align*}
\partial_+ p & \leq M_1 p^{\frac{1}{2}}, \\
-\partial_- p & \leq M_1 p^{\frac{1}{2}},
\end{align*}
\]

for all \((r, \theta)\) with \( r \geq r_{\varepsilon_0}(\theta) \). Note that \( M_1 \) depends only on \( \varepsilon_0 \) and does not depend on \((r, \theta)\).

Next, let

\[
\begin{align*}
A(r, \theta) &= \exp \left\{ -\frac{2p^{-\frac{1}{2}}}{2\sqrt{2} \sin^2 \theta_a \cos \theta_a} \int_{r_{\varepsilon_0}(\theta)}^{r} p^{1} dp_1 \right\}, \\
B(r, \theta) &= \exp \left\{ -\frac{2p^{-\frac{1}{2}}}{2\sqrt{2} \cos^2 \theta_b \sin \theta_b} \int_{r_{\varepsilon_0}(\theta)}^{r} p^{1} dp_1 \right\}.
\end{align*}
\]

Then, let

\[
M_2 = \max \left\{ \max_{(r, \theta) \in \mathbb{D}(\overline{\tau}, \overline{\theta})} \frac{4p^{-\frac{1}{2}}(2 \sin \theta_a, \theta_a)}{A(r, \theta)}, \max_{(r, \theta) \in \mathbb{D}(\overline{\tau}, \overline{\theta})} \frac{4p^{-\frac{1}{2}}(2 \cos \theta_b, \theta_b)}{B(r, \theta)} \right\},
\]

and

\[
M_3 = \max \left\{ M_1, M_2 + 1 \right\}.
\]

Just as what we had pointed out before, the intersections \( a \) and \( b \) in (3.7) and (3.8) vary with \((r, \theta)\) and characteristics, which are not expressed explicitly for notational convenience. We intend to prove that the inequalities (3.6) are still valid for all points \((r, \theta) \in \mathbb{D}(\overline{\tau}, \overline{\theta})\) with \( M_1 \) being replaced by \( M_3 \). Namely, there hold

\[
\begin{align*}
\partial_+ p & \leq M_3 p^{\frac{1}{2}}, \\
-\partial_- p & \leq M_3 p^{\frac{1}{2}},
\end{align*}
\]

\((r, \theta) \in \mathbb{D}(\overline{\tau}, \overline{\theta})\).

Note that the positive constant \( M_3 (\geq M_1) \) is independent of \((r, \theta) \in \mathbb{D}(\overline{\tau}, \overline{\theta})\), but depends on the fixed \((\overline{\tau}, \overline{\theta})\).
We start to prove \((3.10)\). Suppose that \((3.10)\) is correct up to a line segment \(r(\theta) \subseteq \mathbb{D}(\pi, \bar{\theta})\), then we improve \((3.10)\) to strict inequalities on the line segment \(r(\theta)\) in the same domain. In fact, for \((r, \theta) \in \mathbb{D}(\pi, \bar{\theta})\) with \(r \leq r_\alpha(\theta)\), let us compute

\[
\partial_+ p(r, \theta) = 4p^\frac{1}{4}(r, \theta) \left/ \left\{ A(r, \theta) + \int_{\theta_\alpha}^{\theta} \frac{r^2 p^{-\frac{3}{4}}}{r^2 - p} \left( r_\alpha^a(\psi), \psi \right) \right\} \right. \times \exp \left\{ \frac{1}{4} \int_p \left( r_\alpha^a(\psi), \theta \right) \frac{1}{r^2 - p} dp_1 \right\} d\psi \right\}
\]

\[
= 4p^\frac{1}{4}(r, \theta) \left/ \left\{ A(r, \theta) + \int_{\theta_\alpha}^{\theta} \frac{-\partial_r pr^2 p^{-\frac{3}{4}}}{(\partial_r p)(r^2 - p)} \left( r_\alpha^a(\psi), \psi \right) \right\} \right. \times \exp \left\{ \frac{1}{4} \int_p \left( r_\alpha^a(\psi), \theta \right) \frac{1}{r^2 - p} dp_1 \right\} d\psi \right\}
\]

\[
\leq M_3p^\frac{1}{4}(r, \theta) \left/ \left\{ \frac{M_3 A(r, \theta)}{4} + \int_p \left( r_\alpha^a(\theta), \theta \right) \frac{-1}{2} p^{\frac{1}{4}} r^2 \left( r_\alpha^a(\psi), \psi \right) \right\} \right. \times \exp \left\{ \frac{1}{4} \int_p \left( r_\alpha^a(\psi), \theta \right) \frac{1}{r^2 - p} dp_1 \right\} dp \right\}
\]

\[
\leq \frac{M_3p^\frac{1}{4}(r, \theta)}{\frac{M_3 A(r, \theta)}{4} + f(\theta^-) \int_p \left( r_\alpha^a(\theta), \theta \right) \frac{-1}{2} p^{\frac{1}{4}} r^2 \left( r_\alpha^a(\psi), \psi \right) \right\} dp} \times \exp \left\{ \frac{1}{4} \int_p \left( r_\alpha^a(\psi), \theta \right) \frac{1}{\sqrt{2}} dp_1 \right\}
\]

where

\[
f(\theta^-) = \frac{r^2}{r^2 - p} \left( r_\alpha^a(\theta^-), \theta^- \right) \exp \left\{ \frac{1}{4} \int_p \left( r_\alpha^a(\theta^-), \theta^- \right) \frac{1}{r^2 - p} dp_1 \right\}
\]

\[
> \exp \left\{ \frac{1}{4} \int_p \left( r_\alpha^a(\theta^-), \theta^- \right) \frac{1}{\sqrt{2}} dp_1 \right\}
\]

\[
= \exp \left\{ \frac{p \left( r_\alpha^a(\theta^-), \theta^- \right) - p \left( r_\alpha^a(\theta^-), \theta^- \right)}{8} \right\} > 1
\]

with some \(\theta_a < \theta^- < \theta\) and \(A(r, \theta)\) is defined in \((3.7)\). Thus, by \((3.8), (3.11)\) and \((3.12)\), we have

\[
\partial_+ p(r, \theta) < \frac{M_3p^\frac{1}{4}(r, \theta)}{\frac{M_3 A(r, \theta)}{4} + [p^{\frac{1}{4}}(r, \theta) - p^{\frac{1}{4}}(2 \sin \theta_a, \theta_a)]} \leq M_3p^\frac{1}{4}(r, \theta).
\]
Similarly, by (2.8), (2.9) and (2.11), we have

\[-\partial_\rho p(r, \theta) = 4p^{1/4}(r, \theta) \sqrt{B(r, \theta)} - \frac{1}{\sqrt{r^2 - p}} \left\{ \frac{1}{4} \int_{p(r_+^b(\psi), \psi)}^{p(r_+^b(\theta), \theta)} \frac{1}{r^2(r_+^b) - p_1} dp_1 \right\} \]

\[\leq \frac{M_3 p^{1/4}(r, \theta)}{4} + g(\theta^+) \left[ p^{-1/4}(r, \theta) - p^{-1/4}(2 \cos \theta, \theta) \right], \]

where

\[g(\theta^+) = \frac{r^2}{r^2 - p}(r_+^b(\theta^+), \theta^+) \exp \left\{ \frac{1}{4} \int_{p(r_+^b(\psi), \psi)}^{p(r_+^b(\theta), \theta)} \frac{1}{r^2(r_+^b) - p_1} dp_1 \right\} \]

\[> \exp \left\{ \frac{1}{4} \int_{p(r_+^b(\theta), \theta)}^{p(r_+^b(\theta^+), \theta^+)} \frac{1}{\sqrt{r^2 - p}} dp_1 \right\} \]

\[= \exp \left\{ \frac{p(r_+^b(\theta^+), \theta^+)}{8} - p(r_+^b(\theta), \theta) \right\} > 1 \]

with some \( \theta < \theta^+ < \theta_b \) and \( B(r, \theta) \) is given in (3.7). Thus, the similar estimate as (3.13) holds:

\[-\partial_\rho p(r, \theta) < M_3 p^{1/4}(r, \theta). \quad (3.15)\]

Since \( M_3 \) depends only on \((r, \theta)\), particularly, is independent of \((r, \theta) \in \mathbb{D}(\overline{r, \bar{\theta}})\), the proof of (3.10) follows by (3.13) and (3.15).

Now with the aid of (2.3), we add up (3.13) and (3.15) to yield

\[p_r \leq \frac{M_3}{\sqrt{r^2 - p}} p \leq \frac{2M_3}{r^2} p, \]

for all \((r, \theta)\) in a small neighborhood of \((\overline{r, \bar{\theta}})\) in \( \mathbb{D}(\overline{r, \bar{\theta}}) \). Thus, a simple integration of the above inequality with respect to \( r \) from \( \overline{r} \) to \( r \) yields

\[p(\overline{r, \bar{\theta}}) \geq p(r, \bar{\theta}) \exp \left\{ -\frac{M_3}{r^2} (r - \overline{r}) \right\} \quad \text{for} \quad \overline{r} < r. \quad (3.16)\]

On the other hand, by (3.4) and the fact that \( p(r_0(\theta), \theta) = 0 \), we have

\[p(r, \theta) > 0 \]

for all \((r, \theta)\) with \( r > r_0(\theta) \), which together with (3.16) result in that \( p(\overline{r, \bar{\theta}}) > 0 \). Since \((\overline{r, \bar{\theta}})\) is a point on the bubble, we arrive at a contradiction.

Summing up, we in fact proved the following theorem.

**Theorem 3.1.** The Riemann problem \((1.2), (2.4)\) for the pressure gradient equation admits a unique smooth solution. The pressure of the solution is strictly positive in \( \xi > 0, \eta > 0 \).
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