A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels

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Abstract

We deal with the problem of approximating a scalar conservation law by a conservation law with nonlocal flux. As convolution kernel in the nonlocal flux, we consider an exponential-type approximation of the Dirac distribution. This enables us to obtain a total variation bound on the nonlocal term. By using this, we prove that the (unique) weak solution of the nonlocal problem converges strongly in $C(L^1_{\text{loc}})$ to the entropy solution of the local conservation law. We conclude with several numerical illustrations which underline the main results and, in particular, the difference between the solution and the nonlocal term.

Keywords: Nonlocal conservation laws, nonlocal flux, balance laws, singular limits, approximation of local conservation laws, entropy solution.

2010 MSC: 35L65

1. Introduction

Nonlocal conservation laws have been studied and analyzed quite intensively over the last decade from an application point of view with a particular focus on traffic flow \cite{5, 20, 30, 45, 53, 37}, supply chains \cite{39, 55, 52}, pedestrian flow/crowd dynamics \cite{21}, opinion formation \cite{2, 51}, chemical engineering \cite{50, 57}, sedimentation \cite{6}, conveyor belts \cite{54} and more. For the underlying dynamics existence and uniqueness \cite{35, 40, 45, 44, 46, 12, 25}, (optimal) control problems \cite{33, 41, 19, 38, 24}, and suitable numerical schemes \cite{11, 11, 13, 28, 52} have been analyzed.

In this work, “nonlocal” refers to the fact that the velocity $V$ of the corresponding flux, i.e., $f(s) = V(s)s$, $s \in \mathbb{R}$, does not depend on the solution locally at a given space point but on an integral of the solution on a neighborhood.

First in \cite{3}, it has been observed that, at least numerically, there is some hope that the solution of the nonlocal conservation law converges to the local entropy solution when the nonlocal term approaches a Dirac delta. Positive results in this direction were obtained in \cite{59}, provided that the limit entropy solution is smooth and the convolution kernel is even, and in \cite{42} for a large class of nonlocal conservation laws under the assumption of having monotone initial data. Under the assumption that the initial datum...
has bounded total variation, is bounded away from zero and satisfies a one-sided Lipschitz condition, a positive result was obtained in [17]. In [9], for an exponential weight in the nonlocal term, it was shown – provided that the initial datum is bounded away from zero and has bounded total variation (but without monotonicity assumptions) – that, the nonlocal solutions converge (up to subsequences) to weak solutions of the corresponding local conservation law; they also showed that the limit is the unique entropy solution under the additional assumption that \( V \) is an affine function. More recently, in [8], the result was extended to more general fluxes.

A viscous nonlocal conservation law with kernel of exponential type was considered in [15]: as the nonlocal term together with the viscosity approaches zero, the sequence of solutions converges to the local entropy solution. The positive effect of viscosity in the nonlocal-to-local approximation process was previously studied in [18] [16] for more general compactly supported kernels (see also [10] in the case of more regular initial data and linear velocity). In contrast to the proof in [18], which was based on a priori estimates obtained by extensively using energy estimates for the heat kernel and the Duhamel representation formula, in [15], the authors established an energy estimate on the nonlocal term by relying just on the special structure provided by the exponential kernel (as in Remark 3.1 below) and used it to apply Tatar’s compensated compactness theory (see [58]).

In conclusion, although some progress has been made under quite restrictive assumptions, a general theory is missing. Even more, [14] demonstrates via a counterexample that a total variation blow-up of the solution of the nonlocal conservation law can occur if the data is not bounded away from zero so that the standard methods via compactness in \( L^1 \) seemed to be out of reach.

This is why, in this work, we focus on the nonlocal term: it turns out that the nonlocal term itself satisfies a local transport equation with nonlocal source (see Lemma 3.1) and we can use this to show a uniform total variation bound (see Theorem 3.2). Thanks to the specific structure of the nonlocal term this directly implies that also the solution of the conservation law converges strongly in \( L^1 \) (see Theorem 4.1 and Corollary 4.1).

More precisely, in the present paper, we consider the following setting. For a nonlocal parameter \( \eta \in \mathbb{R}_{>0} \), let \( q_\eta \) be the unique weak solution (weak solutions are unique in the nonlocal setup) of the nonlocal conservation law on \( \mathbb{R} \)

\[
\partial_t q_\eta(t,x) + \partial_x \left( V(W_\eta[q_\eta](t,x))q_\eta(t,x) \right) = 0 \quad (t,x) \in \Omega_T
\]

\[
q_\eta(0,x) = q_0(x) \quad x \in \mathbb{R}
\]

with \( \Omega_T := (0,T) \times \mathbb{R} \), supplemented by the nonlocal term \( W_\eta \) with exponential weight

\[
W_\eta[q](t,x) := \frac{1}{\eta} \int_x^\infty \exp\left( \frac{y-x}{\eta} \right) q(t,y) \, dy \quad (t,x) \in \Omega_T
\]

and let \( q \) be the entropy solution of the corresponding local conservation law on \( \mathbb{R} \):

\[
\partial_t q(t,x) + \partial_x \left( V(q(t,x))q(t,x) \right) = 0 \quad (t,x) \in \Omega_T \quad (1)
\]

\[
q(0,x) = q_0(x) \quad x \in \mathbb{R}. \quad (2)
\]

For the “local theory” and corresponding Entropy solutions, we refer to [7] [31] [26] [36]. Finally, in Theorem 4.2, we prove the convergence of the nonlocal solution to the local entropy solution when \( \eta \) approaches zero, i.e., the nonlocal term approaches a Dirac distribution

\[
q_\eta \xrightarrow{\eta \to 0} q \text{ in } C([0,T]; L^1_{\text{loc}}(\mathbb{R})).
\]

We do this by first analyzing the nonlocal term \( W_\eta[q_\eta] \) which – as can be shown – satisfies its own transport equation with a nonlocal source and possess a uniform total variation bound. Thanks to the relation \( \eta \partial_x W_\eta[q_\eta] = W_\eta[q_\eta] - q_\eta \) it immediately follows also the strong convergence of \( q \) to a weak solution of the local conservation law. Then, we can use [8] to obtain that the solution is indeed also entropic. Even more, we also obtain that the nonlocal term \( W_\eta[q_\eta] \) also converges to the local entropy solution.
Our “nonlocal-to-local convergence” result closes the gap between local and nonlocal modelling of phenomena governed by conservation laws; moreover, it provides a way of defining the entropy admissible solutions of local conservation laws as limits of weak solutions to nonlocal conservation laws, which usually do not require an entropy condition for uniqueness (see [23, 41, 44, 15]). This kind of singular limit would be an alternative to the classical vanishing viscosity approach (see [36, 7, 26] and references therein). In the case of a nonlocal approximation without artificial viscosity, no smoothing phenomena happen and the character of the approximating equation remains somewhat “hyperbolic” (finite propagation of mass, but infinite propagation of information).

Such a convergence result would also give additional insights into questions related to control theory (see [4]), in the spirit of [22, 34, 29, 47]. Showing control results for nonlocal conservation laws might be easier as the problem with adjoint equations and shocks of the local equations prohibiting differentiability in a certain state to the initial datum. Optimal control problems might also become mathematically more approachable as the problem with adjoint equations and shocks of the local equations prohibiting differentiability in a certain local framework might be resolvable in the nonlocal theory and one might then just consider the limit controls when the nonlocal term approaches a Dirac.

2. Preliminary results on nonlocal conservation laws

**Definition 2.1** (The nonlocal conservation law and the weak solution). Let $T \in \mathbb{R}_{>0}$ be given. We consider for $\eta \in \mathbb{R}_{>0}$ the following nonlocal conservation law in the “density” $q_\eta : \Omega_T \rightarrow \mathbb{R}$, $\Omega_T := (0, T) \times \mathbb{R}$

$$\partial_t q_\eta(t, x) + \partial_x \left( V(W_\eta[q_\eta](t, x))q_\eta(t, x) \right) = 0 \quad (t, x) \in \Omega_T$$

$$q_\eta(0, x) = q_0(x) \quad x \in \mathbb{R}$$

supplemented by the nonlocal term $W_\eta$

$$W_\eta[q_\eta](t, x) := \frac{1}{\eta} \int_x^\infty \exp \left( \frac{t-y}{\eta} \right) q_\eta(t, y) \, dy \quad (t, x) \in \Omega_T.$$  

We call $q_\eta : \mathbb{R} \rightarrow \mathbb{R}$ an initial datum and $W_\eta[q_\eta] : \Omega_T \rightarrow \mathbb{R}$ the nonlocal impact affecting the velocity function $V : \mathbb{R} \rightarrow \mathbb{R}$ of the nonlocal conservation law. We say that $q_\eta \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ is a weak solution for $q_\eta \in L^1_{\text{loc}}(\mathbb{R})$ and $\eta \in \mathbb{R}_{>0}$ iff $\forall \varphi \in C^1_0((-42, T) \times \mathbb{R})$ it holds that

$$\int_{\Omega_T} \partial_t \varphi(t, x)q_\eta(t, x) + \partial_x \varphi(t, x)V(W_\eta[q_\eta](t, x))q_\eta(t, x) \, dx \, dt + \int_{\mathbb{R}} \varphi(0, x)q_0(x) \, dx = 0.$$  

For the analysis and well-posedness, we require the following not restrictive assumptions:

**Assumption 2.1** (Assumptions on input data). The involved functions in **Definition 2.1** satisfy

- $q_0 \in L^\infty(\mathbb{R}; \mathbb{R}_{>0}) \cap TV(\mathbb{R})$
- $V \in W^{1, \infty}_{\text{loc}}(\mathbb{R})$ : $V'(s) \leq 0 \forall s \in \{\text{ess- inf } q_0(x), \|q_0\|_{L^\infty(\mathbb{R})}\}$.

**Theorem 2.1** (Existence and uniqueness of weak solutions and maximum principle). Given **Assumption 2.1** there exists a unique weak solution $q \in C([0, T]; L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0, T); L^\infty(\mathbb{R})) \cap L^\infty((0, T); TV(\mathbb{R}))$ of the nonlocal conservation law in **Definition 2.1** and the following maximum principle is satisfied

$$\text{ess- inf}_{x \in \mathbb{R}} q_0(x) \leq q(t, x) \leq \|q_0\|_{L^\infty(\mathbb{R})} \ a.e. \ (t, x) \in \Omega_T.$$  

**Proof.** See [40] Theorem 2.20 & Theorem 3.2 & Corollary 4.3].

**Remark 2.1** (Generalization of the assumptions on the velocity function $V$). The assumption on $V$ being monotonically decreasing (see **Assumption 2.1**) can be changed to $V$ monotonically increasing as long as one also changes the nonlocal range for $q \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ as

$$W_\eta[q](t, x) := \int_{-\infty}^x \exp \left( \frac{t-x}{\eta} \right) q(t, y) \, dy, \quad (t, x) \in \Omega_T.$$  


Analogously, the results can be extended to hold also for non-positive initial datum when changing the nonlocal term accordingly. We do not go into details.

Even more when assuming that $V'(s)$ has a sign for all $s \in \mathbb{R}$, one does not need even a maximum principle to be satisfied and thus the initial datum can be chosen arbitrarily in $L^\infty(\mathbb{R}) \cap TV(\mathbb{R})$ (no sign restrictions). However, then one does not obtain convergence of $q_\eta$ but of $W_\eta$ which remains essentially bounded and for which the total variation bound derived in Theorem 3.2 still holds. However, Theorem 4.2 is not directly applicable and we are left with that the limit is a weak solution. Compare also Remark 3.2.

3. Total variation bound on the nonlocal term

As we will tackle the convergence first in the nonlocal term, $W_\eta[q_\eta]$, we deduce a transport equation with a nonlocal source which will enable us to study $W_\eta[q_\eta]$ without $q_\eta$ itself.

**Lemma 3.1** (The transport equation with nonlocal source satisfied by the nonlocal term). Given the dynamics in Definition 2.1 the nonlocal term $W_\eta[q_\eta]$ as in Eq. (8) is Lipschitz-continuous and satisfies the following transport equation with nonlocal source in the strong sense

$$
\partial_t W_\eta(t, x) + V(W_\eta(t, x)) \partial_x W_\eta(t, x) = -\frac{1}{\eta} \int_x^\infty \exp(\frac{x-y}{\eta}) V'(W_\eta(t, y)) \partial_y W_\eta(t, y) W_\eta(t, y) dy \quad (t, x) \in \Omega_T
$$

(8)

$$
W_\eta(0, x) = \frac{1}{\eta} \int_x^\infty \exp(\frac{x-y}{\eta}) q_0(y) dy \quad x \in \mathbb{R}.
$$

(9)

In particular, for $\eta \in \mathbb{R}_{>0}$, we have $W_\eta \in W^{1, \infty}(\Omega_T)$.

**Proof.** We first show that $W_\eta[q_\eta]$ is Lipschitz-continuous. To this end, recall the definition Eq. (8) and compute for $(t, x) \in \Omega_T$

$$
\partial_x W_\eta[q_\eta](t, x) = \partial_x \frac{1}{\eta} \int_x^\infty \exp(\frac{x-y}{\eta}) q_\eta(t, y) dy = \frac{1}{\eta} W_\eta[q_\eta](t, x) - \frac{1}{\eta} q_\eta(t, x).
$$

(10)

However, as $\eta \in \mathbb{R}_{>0}$, $W_\eta[q_\eta] \in L^\infty(\Omega_T)$ and $q_\eta \in L^\infty(\Omega_T)$ thanks to Theorem 2.1 we obtain the uniform boundedness on the spatial derivative. The time derivative is slightly more tricky. Due to the lack of regularity we use the method of characteristics analyzed in [40, Lemma 2.6] to write down the solution $q_\eta$ and have on $(t, x) \in \Omega_T$

$$
\partial_t W_\eta[q_\eta](t, x) = \partial_t \frac{1}{\eta} \int_x^\infty \exp(\frac{x-y}{\eta}) q_\eta(t, y) dy
$$

(11)

$$
= \partial_t \frac{1}{\eta} \int_x^\infty \exp(\frac{x-y}{\eta}) q_0(\xi(t, y; 0)) \partial_2 \xi(t, y; 0) dy
$$

(12)

$$
= \partial_t \frac{1}{\eta} \int_0^\infty \exp\left(\frac{-x - \xi(t, x, 0)}{\eta}\right) q_0(z) dz
$$

(13)

$$
= -\frac{1}{\eta} \int_0^\infty \exp\left(\frac{-x - \xi(t, x, 0)}{\eta}\right) q_0(z) \partial_2 \xi(0, z; t) dz - \frac{1}{\eta} q_0(\xi(t, x, 0)) \partial_1 \xi(t, x; 0).
$$

(14)

Recalling some nice properties of the characteristics [40, Lemma 2.6] and in particular

$$
\partial_2 \xi(0, \xi(t, y; 0); t) = V(W_\eta[q_\eta](t, y)) \quad \forall (t, y) \in \Omega_T
$$

$$
\partial_1 \xi(t, y; 0) = -\partial_2 \xi(t, y; 0) V(W_\eta[q_\eta](t, y)) \quad \forall (t, y) \in \Omega_T
$$

we obtain by continuing Eq. (14)

$$
\partial_t W[q_\eta](t, x) = -\frac{1}{\eta} \int_0^\infty \exp\left(\frac{-x - \xi(t, x, 0)}{\eta}\right) q_0(z) \partial_2 \xi(0, z; t) dz - \frac{1}{\eta} q_0(\xi(t, x, 0)) \partial_1 \xi(t, x; 0)
$$

(14)
which is a direct consequence of the definition of \( \tilde{\eta} \). The nonlocal term can also be transformed into a fully local equation (as in [15]):

\[
\partial_t W_\eta[q_\eta](t, x) + V(W_\eta[q_\eta](t, x)) \partial_x W_\eta[q_\eta](t, x) = \frac{1}{\eta} \int_x^\infty \exp \left( \frac{x-u}{\eta} \right) q_\eta(t, y) V(W_\eta[q_\eta](t, y)) \, dy
\]

This expression is essentially bounded for \( \eta \in \mathbb{R}_{>0} \) so that we obtain the differentiability. Next, we show that the nonlocal operator indeed satisfies the Cauchy problem in Eqs. (8) to (9). Using the identity compute for \( W_t \) above we have for the left hand side of Eq. (8) and \((t, x) \in \Omega_T\):

\[
\begin{align*}
\partial_t W_\eta[q_\eta](t, x) + V(W_\eta[q_\eta](t, x)) \partial_x W_\eta[q_\eta](t, x) \\
= \frac{1}{\eta} q_\eta(t, x) V(W_\eta[q_\eta](t, x)) - \frac{1}{\eta^2} \int_x^\infty \exp \left( \frac{x-u}{\eta} \right) q_\eta(t, y) V(W_\eta[q_\eta](t, y)) \, dy \\
+ V(W_\eta[q_\eta](t, x)) \left( \frac{1}{\eta} W_\eta[q_\eta](t, x) - \frac{1}{\eta} q_\eta(t, x) \right) \\
= V(W_\eta[q_\eta](t, x)) \frac{1}{\eta} W_\eta[q_\eta](t, x) \\
- \frac{1}{\eta} \int_x^\infty \exp \left( \frac{x-u}{\eta} \right) W_\eta[q_\eta](t, y) - \eta \partial_y W_\eta[q_\eta](t, y) V(W_\eta[q_\eta](t, y)) \, dy \\
= V(W_\eta[q_\eta](t, x)) \frac{1}{\eta} W_\eta[q_\eta](t, x) \\
- \frac{1}{\eta} \int_x^\infty \exp \left( \frac{x-u}{\eta} \right) W_\eta[q_\eta](t, y) V(W_\eta[q_\eta](t, y)) \, dy \\
+ \frac{1}{\eta} \int_x^\infty \exp \left( \frac{x-u}{\eta} \right) \partial_y W_\eta[q_\eta](t, y) V(W_\eta[q_\eta](t, y)) \, dy \\
- \frac{1}{\eta} \int_x^\infty \exp \left( \frac{x-u}{\eta} \right) V(W_\eta[q_\eta](t, x)) \partial_y W_\eta[q_\eta](t, y) W_\eta[q_\eta](t, y) \, dy
\end{align*}
\]

where we have used two times the identity in Eq. (10) and integration by parts. However, the last term indeed satisfies the Cauchy problem in Eqs. (8) to (9). Using the identity compute for \( W_t \) above we have for the left hand side of Eq. (8) and \((t, x) \in \Omega_T\):

Let Assumption 2.1 hold, and let \( C_1, C_2 \in \mathbb{R}_{>0} \) be given such that

\[
Q(C_1, C_2) := \{ u \in TV_{\text{loc}}(\mathbb{R}) : \| u \|_{L^\infty(\mathbb{R})} \leq C_1 \wedge |u|_{TV(\mathbb{R})} \leq C_2 \}.
\]

Let \( q_0 \in Q(C_1, C_2) \) be given and denote by \( q \) the solutions to the corresponding nonlocal conservation law.

Then, the solutions to the corresponding nonlocal conservation laws (denoted by \( q \)) satisfy the following \( C([0, T]; L^1(\mathbb{R})) \) stability estimate, i.e.

\[
\forall \epsilon \in \mathbb{R}_{>0} \exists \delta \in \mathbb{R}_{>0} : \forall q_0 \in Q(C_1, C_2) \text{ with } \| q_0 - \tilde{q}_0 \|_{L^1(\mathbb{R})} \leq \delta \implies \| q - \tilde{q} \|_{C([0, T]; L^1(\mathbb{R}))} \leq \epsilon
\]

where \( \tilde{q} \) is the solution to the corresponding nonlocal conservation law with initial datum \( \tilde{q}_0 \).
Proof. Almost the required result can be found in [42, Theorem 4.17] with the difference that the kernel of the nonlocal operator is supposed to be finite while here we have the exponential kernel (Eq. (5)). However, the changes for this result are minor, we do not go into details.

Theorem 3.2 (Total variation bound in the spatial component of $W$ – uniformly in $\eta$). The nonlocal term $W_\eta$ defined in [Eq. (5)] but which also satisfies the identity demonstrated in [Lemma 3.1] satisfies – uniformly in $\eta$ a total variation bound, i.e.

$$|W_\eta(t,\cdot)|_{TV(\mathbb{R})} \leq |W_\eta(0,\cdot)|_{TV(\mathbb{R})} \leq |q_0|_{TV(\mathbb{R})} \forall \eta \in \mathbb{R}_>0.$$ 

Proof. We take advantage of the stability result in [Theorem 3.1] which tells us that when smoothing $q_0$ by $q_0^\varepsilon \equiv q_0 \ast \varphi_\varepsilon$ with $\varphi_\varepsilon$ being a standard mollifier [48, C.4 Mollifiers] with smoothing parameter $\varepsilon \in \mathbb{R}_>0$, the corresponding solution $q_0^\varepsilon$ will be close in the $C([0,T]; L^1(\mathbb{R}))$ topology. Additionally, as the initial datum is smooth, so is the corresponding solution (see [40, Corollary 5.3]) which we will denote by $q_0^\varepsilon$. We now prove the total variation bound. As the solution is smooth, the total variation coincides with the $L^1$-norm of the derivative and we obtain for $t \in [0,T]$, which can be estimated as follows.

$$\frac{d}{dt} \int_{\mathbb{R}} |\partial_x W^\varepsilon_\eta(t, x)| \, dx = \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) \partial^2_{xx} W^\varepsilon_\eta(t, x) \, dx$$

$$= - \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) V(W^\varepsilon_\eta(t, x)) \partial^2_{xx} W^\varepsilon_\eta(t, x) \, dx$$

$$- \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) V'(W^\varepsilon_\eta(t, x))(\partial_x W^\varepsilon_\eta(t, x))^2 \, dx$$

$$+ \frac{1}{\eta^2} \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) V'(W^\varepsilon_\eta(t, x)) W^\varepsilon_\eta(t, x) \partial_\eta W^\varepsilon_\eta(t, x) \, dx$$

$$- \frac{1}{\eta^2} \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) \int_x^\infty \exp(-\frac{y}{\eta}) V'(W^\varepsilon_\eta(t, y)) \partial_\eta W^\varepsilon_\eta(t, y) W^\varepsilon_\eta(t, y) \, dy \, dx$$

$$= \int_{\mathbb{R}} 2\delta_0(\partial_x W^\varepsilon_\eta(t, x)) V(W^\varepsilon_\eta(t, x)) \partial_x W^\varepsilon_\eta(t, x) \partial^2_{xx} W^\varepsilon_\eta(t, x) \, dx$$

$$+ \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) V'(W^\varepsilon_\eta(t, x))(\partial_x W^\varepsilon_\eta(t, x))^2 \, dx$$

$$- \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) V'(W^\varepsilon_\eta(t, x))(\partial_x W^\varepsilon_\eta(t, x))^2 \, dx$$

$$+ \frac{1}{\eta^2} \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) V'(W^\varepsilon_\eta(t, x)) W^\varepsilon_\eta(t, x) \partial_\eta W^\varepsilon_\eta(t, x) \, dx$$

$$- \frac{1}{\eta^2} \int_{\mathbb{R}} \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) \int_x^\infty \exp(-\frac{y}{\eta}) V'(W^\varepsilon_\eta(t, y)) \partial_\eta W^\varepsilon_\eta(t, y) W^\varepsilon_\eta(t, y) \, dy \, dx$$

$$+ \frac{1}{\eta^2} \int_{\mathbb{R}} |\partial_x W^\varepsilon_\eta(t, x)| V'(W^\varepsilon_\eta(t, x)) W^\varepsilon_\eta(t, x) \, dx$$

$$- \frac{1}{\eta} \int_{\mathbb{R}} V'(W^\varepsilon_\eta(t, y)) \partial_\eta W^\varepsilon_\eta(t, y) W^\varepsilon_\eta(t, y) \int_y^\infty \text{sgn} (\partial_x W^\varepsilon_\eta(t, x)) \exp(-\frac{y}{\eta}) \, dy \, dx$$

$$\leq \frac{1}{\eta} \int_{\mathbb{R}} |\partial_x W^\varepsilon_\eta(t, x)| V'(W^\varepsilon_\eta(t, x)) W^\varepsilon_\eta(t, x) \, dx$$

$$- \frac{1}{\eta} \int_{\mathbb{R}} V'(W^\varepsilon_\eta(t, y)) |\partial_\eta W^\varepsilon_\eta(t, y)| W^\varepsilon_\eta(t, y) \int_y^\infty \exp(-\frac{y}{\eta}) \, dy \, dy$$

$$= \frac{1}{\eta} \int_{\mathbb{R}} |\partial_x W^\varepsilon_\eta(t, x)| V'(W^\varepsilon_\eta(t, x)) W^\varepsilon_\eta(t, x) \, dx$$
We thus obtain

$$|W^s_\eta(t, \cdot)|_{TV(\mathbb{R})} \leq |W^s_\eta(0, \cdot)|_{TV(\mathbb{R})} \leq |\phi_0|_{TV(\mathbb{R})},$$

where the last inequality follows from the assumption on $q_0 \in TV(\mathbb{R})$ as stated in Assumption 2.1 and the definition of the initial value for $W_\eta$ as in Eq. (9).

\[ \|W^s_\eta(0, \cdot)\|_{TV(\mathbb{R})} = \sup_{\psi \in C^1_c(\mathbb{R}) : \|\psi\|_{L^\infty(\mathbb{R})} \leq 1} \int_\mathbb{R} \psi'(x)W^s_\eta[\phi_0](x)dx = \sup_{\psi \in C^1_c(\mathbb{R}) : \|\psi\|_{L^\infty(\mathbb{R})} \leq 1} \int_\mathbb{R} \psi'(x)\frac{1}{\eta} \int_{\mathbb{R}, \eta} \exp\left(\frac{x-y}{\eta}\right)q^s_\eta(y)dydx \]

\[ = \sup_{\psi \in C^1_c(\mathbb{R}) : \|\psi\|_{L^\infty(\mathbb{R})} \leq 1} \int_\mathbb{R} \psi'(x)\int_{\mathbb{R}, \eta} \exp\left(\frac{x-y}{\eta}\right)q^s_\eta(y)dydxdy = \sup_{\psi \in C^1_c(\mathbb{R}) : \|\psi\|_{L^\infty(\mathbb{R})} \leq 1} \int_\mathbb{R} \psi'(x)\int_{\mathbb{R}, \eta} \varphi_\eta(x-y)q^s_\eta(0)dzdxdy \]

\[ \leq |\phi_0|_{TV(\mathbb{R})}. \]

\[ \square \]

**Remark 3.2** (Total variation bound and the required assumptions on the velocity $V$). The key step in the proof of the total variation bound stated in Theorem 3.3 can be located in the estimate around Eq. (15). Reconnecting to Remark 2.1 it is enough to assume the velocity to satisfy $V'(s) \leq 0 \ \forall s \in \mathbb{R}$ to obtain the uniform total variation bound without any sign restriction on the initial datum.

### 4. Compactness argument and proof of the convergence result

**Theorem 4.1** (Compactness of $W_\eta$ in $C([0, T]; L^1_{loc}(\mathbb{R}))$). The set \( \{W_\eta\}_{\eta \in \mathbb{R}, \eta > 0} \subseteq C([0, T]; L^1_{loc}(\mathbb{R})) \) of solutions to Eqs. (8) to (9) is compactly embedded into $C([0, T]; L^1_{loc}(\mathbb{R}))$, i.e.

\[ \{W_\eta \in C([0, T]; L^1_{loc}(\mathbb{R})) : W_\eta \text{ satisfies Eqs. (8) to (9)} \} \xrightarrow{\ast} C([0, T]; L^1_{loc}(\mathbb{R})). \]

**Proof.** The proof consists basically of applying the Ascoli theorem in [56, Lemma 1]. We state the details in the following. Let $B$ be a Banach space.

Then, a set $F \subset C([0, T]; B)$ is relatively compact in $C([0, T]; B)$ iff

- $F(t) := \{f(t) \in B : f \in F\}$ is relatively compact in $B \ \forall t \in [0, T].$
- $F$ is uniformly equi-continuous, i.e.

\[ \forall \sigma \in \mathbb{R}, \exists \delta \in \mathbb{R}, \forall f \in F, \forall (t_1, t_2) \in [0, T]^2 \text{ with } |t_1 - t_2| \leq \delta : \|f(t_1) - f(t_2)\|_B \leq \sigma. \]

We start with setting $B = L^1_{loc}(\mathbb{R})$ and $F(t) := \{W_\eta(t, \cdot) \in L^1_{loc}(\mathbb{R}) : \eta \in \mathbb{R}, \eta > 0\}$. Thanks to Theorem 3.2 we know that $W_\eta(t, \cdot)$ has a uniform total variation bound and by [45, Theorem 13.35], the set $F(t)$ is compact in $L^1_{loc}(\mathbb{R})$, i.e.

\[ F(t) \subseteq L^1_{loc}(\mathbb{R}), \ \forall t \in [0, T]. \]
It remains to show the second point, the uniformly equi-continuity. To this end, we again smooth the initial datum $q_0$ by a $q_\eta$ for $\eta \in \mathbb{R}_{>0}$ as in the proof of Theorem 3.2 and call the corresponding smooth nonlocal term $W_\eta$ for an $\eta \in \mathbb{R}_{>0}$. Then, we can estimate

$$\|W_\eta(t_1, \cdot) - W_\eta(t_2, \cdot)\|_{L^1(\mathbb{R})} = \left\| \int_{t_2}^{t_1} \partial_t W_\eta(s, \cdot) \, ds \right\|_{L^1(\mathbb{R})}$$

plugging Eq. (8) in

$$\leq \left\| \int_{t_2}^{t_1} V(W_\eta(s, \cdot)) \partial_y W_\eta(s, \cdot) \, ds \right\|_{L^1(\mathbb{R})} + \left\| \int_{t_2}^{t_1} \frac{1}{\eta} \exp \left( \frac{-|W_\eta(s, y)|}{\eta} \right) \partial_y W_\eta(s, y) W_\eta(s, y) \, dy \, ds \right\|_{L^1(\mathbb{R})}$$

integrating by parts

$$\leq \| V \|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})})]} \| W_\eta \|_{L^\infty((0,T);TV(\mathbb{R}))} |t_1 - t_2| + \| V' \|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})})]} \| W_\eta \|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \| W_\eta \|_{L^\infty((0,T);TV(\mathbb{R}))} |t_1 - t_2|$$

applying Theorem 3.2 and Eq. (7)

$$\leq \left( \| V \|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})})]} + \| V' \|_{L^\infty((0,\|q_0\|_{L^\infty(\mathbb{R})})]} \| q_0 \|_{L^\infty(\mathbb{R})} \right) |q_0|_{TV(\mathbb{R})} |t_1 - t_2|.$$

As this is a uniform bound in $\eta \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$ we have the uniform equi-continuity so that we obtain by applying Ascoli’s theorem indeed the claimed compactness.

As a direct result, from the strong convergence of $W_\eta$ we have also the strong convergence of $q_\eta$ to a weak solution of the local conservation law as the following corollary states:

**Corollary 4.1** (Limit of $q_\eta$ and $W_\eta$ are weak solution to the local equation). For every sequence $(\eta_k)_{k \in \mathbb{N}_{>1}} \subset \mathbb{R}_{>0}$ with $\lim_{k \to \infty} \eta_k = 0$ there exists a subsequence (for reasons of convenience again denoted by $\eta_k$) and a function $q^* \in C([0,T];L^1_{loc}(\mathbb{R}))$ so that the solution $q_\eta \in C([0,T];L^1_{loc}(\mathbb{R}))$ of the nonlocal conservation law as given in Definition 2.1 converges in $C([0,T];L^1_{loc}(\mathbb{R}))$ to the limit point $q^*$ and so does the nonlocal term $W_\eta$ as given in Eq. (5). Additionally, $q^*$ is a weak solution of the local conservation law Eqs. (1) to (2) in equations,

$$\lim_{\eta \to 0} \| q_\eta - q^* \|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0 \land \lim_{\eta \to 0} \| W_\eta - q^* \|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0,$$

when $q^*$ satisfies $\forall \varphi \in C^1((-42, T) \times \mathbb{R})$

$$\int_{\Omega_T} \partial_t \varphi(t, x) q^*(t, x) + \partial_x \varphi(t, x) V(q^*(t, x)) q^*(t, x) \, dx \, dt + \int_{\mathbb{R}} \varphi(0, x) q_0(x) \, dx = 0.$$

**Proof.** Thanks to Theorem 4.1, $W := \{ W_{\eta_k}; k \in \mathbb{N}_{>1} \} \subset C([0,T];L^1_{loc}(\mathbb{R}))$, i.e., the set $W$ is compact in $C([0,T];L^1_{loc}(\mathbb{R}))$ and there exists a limit point $q^* \in C([0,T];L^1_{loc}(\mathbb{R}))$ so that we obtain

$$\lim_{k \to \infty} \| W_{\eta_k} - q^* \|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0.$$

The identity in Eq. (10) directly implies

$$\| W_{\eta_k}(t, \cdot) - q_\eta(t, \cdot) \|_{L^1(\mathbb{R})} = \eta_k \| W_{\eta_k}(t, \cdot) \|_{TV(\mathbb{R})} \leq \eta_k |q_0|_{TV(\mathbb{R})}$$

and thus we also obtain

$$\lim_{k \to \infty} \| q_\eta - q^* \|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0.$$

It remains to be shown that $q^*$ is indeed a weak solution. This directly follows from the strong convergence of $q_{\eta_k}$ to $q^*$ in $C([0,T];L^1_{loc}(\mathbb{R}))$ and due to the essential and uniform bound on $q_\eta$ as given in Theorem 2.1 in Eq. (7).
Figure 1: Solution of the nonlocal balance law with exponential kernel (top, Eq. (5)) and constant kernel (bottom, Eq. (17)) supplemented by the piecewise constant initial datum stated in Eq. (18) plotted in the space-time domain. From left to right η is decreasing, η ∈ {10^{-1}, 10^{-2}, 10^{-3}}. The rightmost figure is “by eye” not distinguishable from the corresponding local solution. Colorbar: 0 1

However, the previous result can actually be strengthened and indeed we obtain that the limit q^∗ is unique (in particular, every subsequence converges) and that this limit is the weak entropy solution of the corresponding local conservation law.

**Theorem 4.2** (Convergence to the Entropy solution). Given Assumption 2.1 the nonlocal term W_η[q_η] and the corresponding nonlocal solution q_η ∈ C([0,T];L^1_{loc}(R)) of the nonlocal conservation law Definition 2.1 converges in C([0,T];L^1_{loc}(R)) to the Entropy solution of the corresponding local conservation law (see Eqs. (1) to (2)).

**Proof.** This is a direct consequence of the convergence of W_η,q_η to a weak solution of the local conservation laws in C([0,T];L^1_{loc}(R)), Corollary 4.1 and of [8]. Therein, by taking advantage of the minimal entropy condition in [27, 49], it is shown that a solution q_η of the nonlocal conservation law in Definition 2.1 with uniform TV bound converges to the entropy solution of the local problem. However, when checking the proof carefully, it turns out that it suffices to assume that the solution q_η converges strongly to a weak solution q^∗.

5. Numerical illustrations

Some numerical results concerning the convergence can already be found in [42]. We rely on a solver based on characteristics [43] which is non dissipative. On the basis of a simple example we want to shed more light on the difference between the total variation of q_η and the nonlocal counterpart W_η[q_η]. We
Figure 2: Left: Solution of the nonlocal balance law with exponential kernel (top, Eq. (5)) and constant kernel (bottom, Eq. (17)) supplemented by the piecewise constant initial datum stated in Eq. (18) and its corresponding nonlocal term plotted for $t = 0.5$ and $\eta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. Right: Evolution of the corresponding total variations showing a monotone decreasing nature in terms of the nonlocal term (dotted lines) which is also the case for the local counterpart. In terms of the total variation of the solution itself (dashed dotted lines), the total variation approaches 3. This is due to that the zero in the initial datum $(x \in (\frac{1}{3}, \frac{2}{3}))$ moves and shrinks but does not vanish for all $\eta \in \mathbb{R}_{>0}$ and $t \in (0, T]$, resulting in an additional total variation of 2 compared to the total variation of the solution to the local equation being 1 for all $t \in (1, T)$.
further demonstrate that the result should still hold for general nonlocal kernels by using as “worst case” a constant kernel, i.e. for $q \in C([0,T]; L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T); L^\infty(\mathbb{R}))$

$$W_\eta[q](t,x) := \frac{1}{\eta} \int_x^{x+\eta} q(t,y) \, dy, \quad x \in \mathbb{R}. \tag{17}$$

It seems to be true that a total variation bound on the nonlocal term holds and that also the solution still converges to the local entropy solution. The following examples rely on the following initial datum:

$$q_0 \equiv \frac{1}{2} \chi_{(0,\frac{1}{3})} + \chi_{\mathbb{R} \setminus \left[0,\frac{1}{3}\right]} \tag{18}$$

The crucial point of the chosen initial datum are the roots for $x \in (\frac{1}{3}, \frac{2}{3})$. These roots are moving but kept in the nonlocal solution $q_\eta$ for all times. This results in an increase of the total variation. In the nonlocal term $W$ there are by construction of the initial datum as well as the exponential kernel no roots and the solution is smoothed resulting in a – as proven – non-increasing total variation.

6. Future work

What remains an open question is whether it is possible to obtain the same results for different kernels still satisfying the required monotonicity assumption for the solution to satisfy a maximum principle (see for this particularly Section 5 and Fig. 1 bottom). The considered exponential kernel clearly provides a nice structure which seems to be crucial in our analysis for showing the stated results.

Another interesting problem consists of what happens in the case of a fully symmetric nonlocal kernel which is sensitive to both propagating directions. However, such a kernel immediately implies that the solutions cannot satisfy a maximum principle (for an illustration see for instance [12, Example 7.3, Fig. 9]).

Then, recalling [17] it is also apparent that one cannot expect the solution to converge in a strong or weak sense to the entropy solution, but there is hope – compare particularly the numerics in [12, Example 7.3] – for convergence in a measure valued sense.

Acknowledgments

G. M. Coclite is a member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). He has been partially supported by the Research Project of National Relevance “Multiscale Innovative Materials and Structures” granted by the Italian Ministry of Education, University and Research (MIUR Prin 2017, project code 2017J4EAYB and the Italian Ministry of Education, University and Research under the Programme Department of Excellence Legge 232/2016 (Grant No. CUP - D94118000260001).

J.-M. Coron acknowledges funding from the Miller Institute and from the Agence Nationale de La Recherche (ANR), grant ANR Finite4SoS (ANR-15-CE23-0007). He also thanks the Miller Institute and UC Berkeley for their hospitality.

N. De Nitti has been partially supported by the Alexander von Humboldt Foundation and by the TRR-154 project of the DFG.

L. Pflug has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 416229255 – SFB 1411.

References

[1] A. Aggarwal, R. Colombo, and P. Goatin. Nonlocal systems of conservation laws in several space dimensions. *SIAM Journal on Numerical Analysis*, 53(2):963–983, 2015.

[2] G. Aletti, G. Naldi, and G. Toscani. First-order continuous models of opinion formation. *SIAM Journal on Applied Mathematics*, 67(3):837–853, 2007.

[3] P. Amorim, R. M. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. *ESAIM Math. Model. Numer. Anal.*, 49(1):19–37, 2015.
[4] A. Bayen, J.-M. Coron, N. De Nitti, A. Keimer, and L. Pflug. Boundary controllability and asymptotic stabilization of a nonlocal traffic flow model. Preprint, 2020.

[5] A. Bayen, A. Keimer, L. Pflug, and T. Veeravalli. Modeling multi-lane traffic with moving obstacles by nonlocal balance laws. Preprint (09 2020), doi, 10, 2020.

[6] F. Betancourt, R. Bürger, K. H. Karlsen, and E. M. Tory. On nonlocal conservation laws modelling sedimentation. Nonlinearity, 24(3):855–885, 2011.

[7] A. Bressan. Hyperbolic systems of conservation laws, volume 20 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[8] A. Bressan and W. Shen. Entropy admissibility of the limit solution for a nonlocal model of traffic flow. preprint, 2020.

[9] A. Bressan and W. Shen. On traffic flow with nonlocal flux: a relaxation representation. Arch. Ration. Mech. Anal., 237(3):1213–1236, 2020.

[10] P. Calderoni and M. Pulvirenti. Propagation of chaos for Burgers’ equation. Ann. Inst. H. Poincaré Sect. A (N.S.), 39(1):85–97, 1983.

[11] C. Chalons, P. Goatin, and L. M. Villada. High-order numerical schemes for one-dimensional nonlocal conservation laws. SIAM Journal on Scientific Computing, 40(1):A288–A305, 2018.

[12] F. A. Chiarello, P. Goatin, and E. Rossi. Stability estimates for non-local scalar conservation laws. Nonlinear Analysis: Real World Applications, 45:668 – 687, 2019.

[13] F. A. Chiarello, P. Goatin, and L. M. Villada. Lagrangian-antidiffusive remap schemes for non-local multi-class traffic flow models. Computational and Applied Mathematics, 39(2):1–22, 2020.

[14] J. Chu, P. Shang, and Z. Wang. Controllability and stabilization of a conservation law modeling a highly re-entrant manufacturing system. Nonlinear Anal., 189:111577, 19, 2019.

[15] G. M. Coclite, N. De Nitti, A. Keimer, and L. Pflug. Singular limits with vanishing viscosity for nonlocal conservation laws. Preprint, 2020.

[16] M. Colombo, G. Crippa, M. Graff, and L. V. Spinolo. Recent results on the singular local limit for nonlocal conservation laws, 2019.

[17] M. Colombo, G. Crippa, E. Marconi, and L. V. Spinolo. Local limit of nonlocal traffic models: convergence results and total variation blow-up, 2018.

[18] M. Colombo, G. Crippa, and L. V. Spinolo. On the singular local limit for conservation laws with nonlocal fluxes. Arch. Ration. Mech. Anal., 235(3):1131–1167, 2019.

[19] R. Colombo, M. Herty, and M. Mercier. Control of the continuity equation with a non local flow. ESAIM Control Optim. Calc. Var., 17(2):353–379, 2011.

[20] R. M. Colombo, M. Garavello, and M. Lécureux-Mercier. A class of nonlocal models for pedestrian traffic. Math. Models Methods Appl. Sci., 22(4):1150023, 34, 2012.

[21] R. M. Colombo, M. Lécureux-Mercier, and M. Garavello. Crowd dynamics through conservation laws. In Crowd Dynamics, Volume 2, pages 83–110, Springer, 2020.

[22] J.-M. Coron and S. Guerrero. Singular optimal control: a linear 1-D parabolic-hyperbolic example. Asymptot. Anal., 44(3-4):237–257, 2005.

[23] J.-M. Coron, M. Kawski, and Z. Wang. Analysis of a conservation law modeling a highly re-entrant manufacturing system. Discrete Contin. Dyn. Syst. Ser. B, 14(4):1337–1359, 2010.

[24] J.-M. Coron and Z. Wang. Controllability for a scalar conservation law with nonlocal velocity. Journal of Differential Equations, 252(1):181–201, 2012.

[25] G. Crippa and M. Lécureux-Mercier. Existence and uniqueness of measure solutions for a system of continuity equations with non-local flux. Nonlinear Differential Equations and Applications NoDEA, 20(3):523–537, 2013.

[26] C. M. Dafermos. Hyperbolic conservation laws in continuum physics, volume 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, fourth edition, 2016.

[27] C. De Lellis, F. Otto, and M. Westdickenberg. Minimal entropy conditions for burgers equation. Quarterly of applied mathematics, 62(4):687–700, 2004.

[28] J. Friedrich, O. Kolb, and S. Göttlich. A godunov type scheme for a class of lwr traffic flow models with non-local flux. Networks & Heterogeneous Media, 13:531, 2018.

[29] O. Glass and S. Guerrero. On the uniform controllability of the Burgers equation. SIAM J. Control Optim., 46(4):1211–1238, 2007.

[30] P. Goatin and S. Scialanga. Well-posedness and finite volume approximations of the lwr traffic flow model with non-local velocity. Networks and Heterogeneous Media, 11(1):107–121, 2016.

[31] E. Godlewski and P.-A. Raviart. Hyperbolic systems of conservation laws. Ellipses, 1991.

[32] X. Gong and M. Kawski. Weak measure-valued solutions to a nonlinear hyperbolic conservation law modeling a highly re-entrant manufacturing system. arXiv preprint arXiv:1903.00797, 2019.

[33] M. Gröschel, A. Keimer, G. Leugering, and Z. Wang. Regularity Theory and Adjoint Based Optimality Conditions for a Nonlinear Transport Equation with Nonlocal Velocity. SIAM Journal on Control and Optimization, 52(4):2141–2163, 2014.

[34] S. Guerrero and G. Lebeau. Singular optimal control for a transport-diffusion equation. Comm. Partial Differential Equations, 32(10-12):1813–1836, 2007.

[35] M. Gugat, A. Keimer, G. Leugering, and Z. Wang. Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks. Networks & Heter. Media, 10(4):749–785, 2015.

[36] H. Holden and N. H. Risebro. Front tracking for hyperbolic conservation laws, volume 152 of Applied Mathematical Sciences. Springer, Heidelberg, second edition, 2015.

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