POINT CONFIGURATIONS AND TRANSLATIONS

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Abstract. The spaces of point configurations on the projective line up to the action of $\text{SL}(2, \mathbb{K})$ and its maximal torus are canonically compactified by the Grothendieck-Knudsen and Losev-Manin moduli spaces $\overline{M}_{0,n}$ and $\overline{L}_n$, respectively. We examine the configuration space up to the action of the maximal unipotent group $G_a \subseteq \text{SL}(2, \mathbb{K})$ and define an analogous compactification. For this we first assign a canonical quotient to the action of a unipotent group on a projective variety. Moreover, we show that similar to $\overline{M}_{0,n}$ and $\overline{L}_n$ this quotient arises in a sequence of blow-ups from a product of projective spaces.

1. Introduction

In the present paper we examine point configurations on the projective line up to translations. In general, let us consider $n$ distinct points on $\mathbb{P}_1$. Then the open subset $U \subseteq \mathbb{P}_1^n$ consisting of pairwise different coordinates is the space of possible configurations. For an algebraic group $G$ acting on $\mathbb{P}_1$ the question arises what the resulting equivalence classes of configurations are, i.e. we ask for a quotient $U/G$ of the diagonal action and a possible canonical compactification. In the case of the full automorphism group $G = \text{SL}(2, \mathbb{K})$ this problem has been thoroughly studied. The space of configuration classes is canonically compactified by the famous Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$, i.e. we have

$$\overline{M}_{0,n} = U / \text{SL}(2, \mathbb{K}) \subseteq \overline{M}_{0,n}.$$  

Originally introduced as moduli space of certain marked curves Kapranov shows in [16] that $\overline{M}_{0,n}$ has (among others) the following two equivalent descriptions. Firstly it arises as the GIT-limit of $\mathbb{P}_1^n$ with respect to the $G$-action, i.e. the limit of the inverse system of Mumford quotients. Secondly, it can be viewed as the blow-up of $\mathbb{P}_{n-3}$ in $n-1$ general points and all the linear subspaces of dimension at most $n-5$ spanned by them.

Later this setting has been studied in the case where the full automorphism group $G = \text{SL}(2, \mathbb{K})$ this problem has been thoroughly studied. The space of configuration classes is canonically compactified by the famous Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$, i.e. we have

$$\overline{M}_{0,n} = U / \text{SL}(2, \mathbb{K}) \subseteq \overline{M}_{0,n}.$$  

In this paper we treat point configurations on $\mathbb{P}_1$ up to the action of the maximal connected unipotent subgroup $G_a \subseteq \text{SL}(2, \mathbb{K})$. It consists of upper triangular matrices with diagonal elements equal to $1_\mathbb{K}$ and can be thought of as group of translations. Since $G_a$ is not reductive, we are faced with the additional problem of first finding to suitable replacement for the GIT-limit, i.e. assigning a canonical quotient to this action. This will be overcome in the following manner.

Doran and Kirwan introduce in [9] the notion of finitely generated semistable points admitting so-called enveloped quotients. Moreover, in [2] Arzhantsev, Hausen and

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Celik propose a Gelfand-MacPherson type construction which allows to apply methods from reductive GIT to obtain these enveloped quotients. Building on this work we obtain again an inverse system and the corresponding GIT-limit. In general the enveloped quotients are not projective, hence one cannot expect the GIT-limit to be so.

We then show that (up to normalisation) the limit quotient, i.e. a canonical component of the GIT-limit, is canonically compactified by an iterated blow-up of $\mathbb{P}^{n-1}_1$. To make this a little more precise consider a subset $A \subseteq \{2, \ldots, n\}$. Denoting by $T_2, S_2, \ldots, T_n, S_n$ the homogeneous coordinates on $\mathbb{P}^{n-1}_1$ we associate to $A$ a subscheme $X_A$ on $\mathbb{P}^{n-1}_1$ given by the ideal

$$\langle T_i^2, T_j S_k - T_k S_j; i, j, k \in A, j < k \rangle.$$

The scheme-theoretic inclusions give rise to a partial order of these subschemes. Let $\text{Bl}(\mathbb{P}^{n-1}_1)$ denote the blow-up of $\mathbb{P}^{n-1}_1$ in all these subschemes in non-descending order.

**Theorem.** If $\mathbb{P}^{n-1}_1 \setminus G_a$ and $\text{Bl}(\mathbb{P}^{n-1}_1)$ denote the normalisations of the limit quotient and the above blow-up of $\mathbb{P}^{n-1}_1$ respectively, then we have open embeddings

$$U/G_a \subseteq \mathbb{P}^{n-1}_1 \setminus G_a \subseteq \text{Bl}(\mathbb{P}^{n-1}_1).$$

In the case of two distinct points, i.e. $n = 2$, the latter space is simply $\mathbb{P}_1$. If $n = 3$ holds, then the compatification $\text{Bl}(\mathbb{P}_1 \times \mathbb{P}_1)$ is the unique non-toric, Gorenstein, log del Pezzo $K^*$-surface of Picard number 3 with a singularity of type $A_1$. Similar to $\overline{M}_{0,5}$ which arises as a single Mumford quotient of the cone over the Grassmannian $\text{Gr}(2,5)$, this surface is the Mumford quotient of the cone over the Grassmannian $\text{Gr}(2,4)$. For higher $n$ an analogous Mumford quotient needs to be blown up as will be described in Section 5.

The paper is organized as follows. In Section 2 we recall the results of [2] and introduce the non-reductive GIT-limit and limit quotient. In the following Section 3 we apply these constructions to the action of $G_a$ on $\mathbb{P}^n_1$. We discuss explicitly the GIT-fan which contains the combinatorial data needed to make the limit quotient accessible. The blow-ups of $\mathbb{P}^{n-1}_1$ will be dealt with in a mostly combinatorial way, i.e. as proper transforms with respect to toric blow-ups. For this we prove a result on combinatorial blow-ups in the spirit of Feichtner and Kozlov, see [10]. This will be carried out in Section 4. The final Section 5 then is dedicated to the proof of the main theorems.

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2. The non-reductive GIT-limit

In this section we deal with the problem of assigning a canonical quotient to the action of a unipotent group $G$ on a Mori Dream Space $X$, i.e. a $\mathbb{Q}$-factorial, projective variety with finitely generated Cox ring $\mathcal{R}(X)$. For reductive groups an answer to this problem is the GIT-limit, i.e. the limit of the inverse system consisting of the Mumford quotients $X_{ss}^\infty(D)/G$. However, this method relies on Hilbert's Finiteness Theorem which guarantees, that for a linear action of a reductive group $G$ on any affine algebra the invariant algebra is affine again. So we make a further finiteness assumption on certain $G$-invariants which for example holds when $G = G_a$.

In [9, Definition 4.2.6] Doran and Kirwan introduce the notion of finitely generated semistable sets for the action of a unipotent group, namely the sets $X_{ss}^{fg}(D) := \bigcup X_f$ where $D$ is some ample divisor, $f \in \mathcal{O}_{nd}(X)^G$ is an invariant section for some $n > 0$ and $\mathcal{O}(X)^G$ is finitely generated. These sets possess enveloped quotients

$$r: X_{ss}^\infty(D) \rightarrow r(X_{ss}^\infty(D)) \subseteq X/DG$$

where the enveloping quotient $X/DG$ can be obtained by gluing together the affine varieties $\text{Spec}(\mathcal{O}(X)^G)$. Using a Gelfand-MacPherson type correspondence described in [2] we now turn this collection of enveloped quotients into an inverse system.

Consider the action of an affine-algebraic, simply connected group $G$ with trivial character group $X(G)$ on the normal, projective variety $X$. Let $K \subseteq \text{WDiv}(X)$ be a free and finitely generated group of Weil divisors mapping isomorphically onto the divisor class group $\text{Cl}(X)$. We then associate to $X$ a sheaf of graded algebras

$$\mathcal{R} := \bigoplus_{D \in K} \mathcal{O}_D.$$ 

We suppose that the algebra of global sections $\mathcal{R}(X)$, i.e. the Cox ring of $X$, is finitely generated. The $K$-grading yields an action of the torus $H := \text{Spec}(K[K])$ on the relative spectrum $\hat{X} := \text{Spec}_X(\mathcal{R})$ and the canonical morphism $p: \hat{X} \rightarrow X$ is a good quotient for this action. By linearisation the $G$-action on $X$ lifts to a unique action of $G$ on the total coordinate space $\overline{X} := \text{Spec}(\mathcal{R}(X))$ which commutes with the $H$-action and turns $p$ into an equivariant morphism, see [12, Section 1].

Now suppose that the algebra of invariants $\mathcal{R}(X)^G$ is finitely generated as well and let $\overline{Y}$ be its spectrum. The inclusion of the invariants gives rise to a morphism $\kappa: \overline{X} \rightarrow \overline{Y}$. Since $\kappa$ is not necessarily surjective, it need not have the universal property of quotients. However, passing to the category of constructible spaces we obtain a categorical quotient $\kappa: \overline{X} \rightarrow \overline{Y} := \kappa(\overline{X})$, see [2] for details.

For every ample $D \in K$ standard geometric invariant theory provides us with a set of semistable points

$$\overline{Y}^s(D) := \bigcup \overline{Y}_f \quad \text{where} \quad f \in \mathcal{R}(X)^G_{nd} \quad \text{and} \quad n > 0.$$ 

These sets admit good quotients for the $H$-action which are isomorphic to the enveloping quotient $X_{ss}^{\infty}(D)$ in the sense of Doran and Kirwan. The set of finitely generated semistable points $X_{ss}^\infty(D)$ can be retrieved from $\overline{Y}^s(D)$ by

$$X_{ss}^\infty(D) = p(\hat{U}) \quad \text{where} \quad \hat{U} := \kappa^{-1}(\overline{Y}^s(D)).$$
The situation fits into the following commutative diagram:

\[ X \xrightarrow{\kappa} \hat{X} \xrightarrow{\kappa} \bar{Y} \xrightarrow{\zeta} Y \]

In this setting [2, Corollary 5.3] answers the question whether the morphisms \( q \) and \( r \) are categorical quotients.

**Proposition 2.1.** If for every \( v \in V \) the closed \( H \)-orbit lying in \( q^{-1}(v) \) is contained in \( Y \) (e.g. \( q \) is geometric), then \( q \) and \( r \) are categorical quotients for the \( H \) - and \( G \)-actions respectively.

In order to define a canonical quotient for the action of \( G \) on \( X \) we first recall the respective methods in reductive geometric invariant theory. For the affine variety \( Y \) let \( Y_1, \ldots, Y_r \) be the sets of semistable points arising from ample divisors. Whenever we have \( Y_i \subseteq Y_j \) for two of these set we obtain a commutative diagram.

\[ Y_i \rightarrow Y_j \]

The morphisms \( \varphi_{ij} : Y_i/H \rightarrow Y_j/H \) turn the collection of quotients into an inverse system, the \( GIT \)-system. Its inverse limit \( Y_{\text{GIT}}/H \) is called \( GIT \)-limit. There exists a canonical morphism

\[ \bigcap Y_i \rightarrow Y_{\text{GIT}}/H \]

and the closure of its image is the limit quotient \( Y_{\text{GIT}}/H \) of \( Y \) with respect to \( H \). Note that in the literature this space is also called 'canonical component' or 'GIT-limit'. In general, the limit quotient need not be normal; its normalisation is the normalised limit quotient \( Y_{\text{GIT}}/H \).

We now turn to the non-reductive case. As constructible subsets of \( Y_i/H \) the corresponding enveloped quotients \( V_i \) inherit the above morphisms \( \varphi_{ij} \), and again form an inverse system.

**Definition 2.2.** The (non-reductive) \( GIT \)-limit \( X_{\text{GIT}}/G \) of \( X \) with respect to the \( G \)-action is the limit of the inverse system of enveloped quotients.

The non-reductive \( GIT \)-limit \( X_{\text{GIT}}/G \) is a constructible subset of the reductive \( GIT \)-limit \( Y_{\text{GIT}}/G \). Analogously, we obtain a canonical morphism into the (non-reductive) \( GIT \)-limit \( X_{\text{GIT}}/G \)

\[ \bigcap (Y' \cap V_i) \rightarrow X_{\text{GIT}}/G. \]
Definition 2.3. The (non-reductive) limit quotient $X \overset{LQ}{\to} G$ of $X$ with respect to the $G$-action is the closure of the image of the above morphism. Its normalisation is the normalised limit quotient $X \overset{LQ}{\to} G$.

The limit quotient in general appears to be relatively hard to access. However, if $Y$ is factorial we can realise it up to normalisation as a certain closed subset of a toric variety as follows. For this consider homogeneous generators $f_1, \ldots, f_r$ of the $K$-graded algebra $\mathcal{O}(Y)$. With $\deg(T_i) := \deg(f_i)$ we obtain a graded epimorphism

$$\mathbb{K}[T_1, \ldots, T_r] \to \mathcal{O}(Y); \quad T_i \mapsto f_i.$$ 

This gives rise to an equivariant closed embedding of $Y$ into $\mathbb{K}^r$. We denote by $Q$ the matrix recording the weights $\deg(f_i)$ as columns and fix a Gale dual matrix $P$, i.e. a matrix with $PQ^t = 0$. The Gelfand-Kapranov-Zelevinsky-decomposition (GKZ-decomposition) of $P$ is the fan

$$\Sigma := \{ \sigma(v); \, v \in \mathbb{Q}^{r - \text{rk}(K)} \}, \quad \sigma(v) := \bigcap_{v \in \tau} \tau$$

where $\tau$ is a cone generated by some of the columns of $P$. It is known that the normalised limit quotient $\mathbb{K}^r \overset{LQ}{\to} H$ is a toric variety with corresponding fan $\Sigma$. Now suppose that $Y$ is factorial. Then every set of semistable points of $Y$ arises as intersection of $\Sigma$ with a set of semistable points on $\mathbb{K}^r$. In this situation we obtain a closed embedding of the GIT-limits $Y \overset{\nu}{\to} \mathbb{K}^r \overset{\nu}{\to} H$ and hence of the respective limit quotients. The inverse image of $Y \overset{\nu}{\to} \mathbb{K}^r \overset{\nu}{\to} H$ under the normalisation map $\nu: \mathbb{K}^r \overset{\nu}{\to} \mathbb{K}^r \overset{\nu}{\to} H$ is in general not normal. However, its normalisation coincides with the normalised limit quotient $Y \overset{\nu}{\to} H$. The situation fits into the following commutative diagram.

$$Y \overset{\nu}{\to} \mathbb{K}^r \overset{\nu}{\to} H \quad \text{and} \quad Y \overset{\nu}{\to} \mathbb{K}^r \overset{\nu}{\to} H$$

Finally, if $T$ is the dense torus in $\mathbb{K}^r$, then $\nu^{-1}(Y \overset{\nu}{\to} H)$ coincides with the closure of $(Y \cap T)/H$ in $\mathbb{K}^r \overset{\nu}{\to} H$. Hence we obtain a normalisation map

$$\nu: Y \overset{\nu}{\to} H \to (Y \cap T)/H.$$ 

3. Point configurations on $\mathbb{P}^1$ and translations

In this section we examine point configurations on $\mathbb{P}^1_\mathbb{K}$ up to translations. For this we consider the diagonal action of $G_a$ on $\mathbb{P}^n_\mathbb{K}$ and explicitly perform the Gelfand-MacPherson type construction introduced in the preceding section. We determine the GIT-fan describing the variation of quotients and show that it is closely related to the well known GIT-fan stemming from the action of the full automorphism group $\text{SL}(2, \mathbb{K})$ on $\mathbb{P}^n_\mathbb{K}$.

For this we consider the unipotent group

$$G_a = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}; \, k \in \mathbb{K} \right\} \subseteq \text{SL}(2, \mathbb{K}),$$

and its action on $X := (\mathbb{K}^n)^2$ given by

$$A \cdot \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} := \begin{bmatrix} A(x_1) & \cdots & A(x_n) \\ A(y_1) & \cdots & A(y_n) \end{bmatrix}.$$
Viewing \([x_i, y_i]\) as homogeneous coordinates of the factors in \(\mathbb{P}^n_1\) this gives rise to an induced action on \(X := \mathbb{P}^n_1\). Note that the Cox ring of \(X\) is
\[
\mathcal{R}(X) = \mathcal{O}(X) = \mathbb{K}[T_1, \ldots, T_n, S_1, \ldots, S_n]
\]
together with a \(\text{Cl}(X)\)-grading defined by \(\deg(T_i) = \deg(S_i) = e_i \in \mathbb{Z}^n = \text{Cl}(X)\).

A first Proposition concerns the algebra of invariants in \(\mathcal{R}(X)\) and its spectrum.

**Proposition 3.1.** Consider the above \(G_a\)-action on \(X\).

(i) The subalgebra \(\mathcal{O}(X)^{G_a} \subseteq \mathcal{O}(X)\) is generated by \(S_1, \ldots, S_n, T_{jk} S_k - T_k S_j\), with \(1 \leq j < k \leq n\).

(ii) The canonical morphism \(\kappa' : X \to Y\) where \(Y := \text{Spec}(\mathcal{O}(X)^{G_a})\) fits into a commutative diagram

\[
\begin{array}{ccc}
X & \overset{\kappa}{\longrightarrow} & \mathbb{K}^{n+1} \\
\kappa' \downarrow & & \downarrow \\
Y & \overset{\iota}{\longrightarrow} & \text{closed embedding}
\end{array}
\]

where \(\iota\) is a closed embedding and its image \(\iota(Y)\) is the affine cone over the Grassmannian \(\text{Gr}(2, n+1)\). Its vanishing ideal is generated by the Plücker relations
\[
T_{ij} T_{kl} - T_{ik} T_{jl} + T_{il} T_{jk}, \quad 0 \leq i < j < k < l \leq n,
\]
where \(T_{ij} = (e_i \wedge e_j)^*\) are the dual basis vectors of the standard basis.

**Proof.** The invariants have been described by Shmelkin, see [18, Theorem 1.1]. For (ii) we define \(\iota\) by its comorphism
\[
\iota^*: T_{0i} \mapsto S_i, \quad T_{jk} \mapsto T_{jk} S_k - T_k S_j \quad \text{where} \quad 1 \leq i \leq n, \quad 1 \leq j < k \leq n.
\]
Clearly, \(\iota^*\) is surjective, hence \(\iota\) is an embedding. Moreover, the pullback of the Plücker relations with \(\iota^*\) gives the zero ideal. Thus \(Y\) lies in the affine cone \(C(\text{Gr}(2, n+1))\). It now suffices to show that \(\text{Im}(\kappa')\) has dimension \(2n - 1\).

For this consider two points \((x, y), (x', y')\) with only non-zero coefficients. If they have distinct orbits, then the orbits are separated by the invariants: If \(y \neq y'\) holds, then there exists a separating \(S_i\). Otherwise we can choose a separating \(T_i S_j - T_j S_i\). Hence, over an open set the fibres of \(\kappa'\) are one-dimensional and thus the image of \(\kappa'\) is \((2n - 1)\)-dimensional. \(\square\)

While for reductive groups the quotient morphism \(\kappa'\) is surjective, this fails in general. We provide a description of the image of
\[
\kappa: X = (\mathbb{K}^n)^2 \to \bigwedge^2 \mathbb{K}^{n+1}; \quad (x, y) \mapsto (1, x) \wedge (0, y).
\]

Via the embedding of the preceding proposition we view \(\bar{Y}\) as subset of \(\bigwedge^2 \mathbb{K}^{n+1}\). Observe that \(\bar{Y}\) contains the affine cone \(\bar{Y}'\) of the smaller Grassmannian \(\text{Gr}(2, n)\) in the following canonical manner:
\[
\bar{Y}' = \{(0, x) \wedge (0, y); \ x, y \in \mathbb{K}^n\} \subseteq \bar{Y}.
\]

**Proposition 3.2.** The image of \(\kappa\) is \(\kappa(X) = (\bar{Y} \setminus \bar{Y}') \cup \{0\}\).
Observe that we have inclusions \( \Im(\kappa) \subseteq \mathbb{K}^{n} \); note that \( y \neq 0 \) holds. With the identification \( \mathbb{K}^{n} = \{0\} \times \mathbb{K}^{n} \subseteq \mathbb{K}^{n+1} \) we obtain an affine subspace \( W_{y} \) by

\[
W_{y} := e_{0} \wedge y + \bigwedge^{2} \mathbb{K}^{n} \subseteq \left( \mathbb{K}e_{0} \wedge \mathbb{K}^{n} \right) \oplus \bigwedge^{2} \mathbb{K}^{n} = \bigwedge^{2} \mathbb{K}^{n+1}.
\]

Since \( z \) lies in \( W_{y} \cap \mathbb{Y} \), it suffices to show that \( \kappa(\cdot, y) \) maps \( \mathbb{K}^{n} \) onto \( W_{y} \cap \mathbb{Y} \). Clearly, by definition of \( \kappa \), the image of \( \kappa(\cdot, y) \) lies in \( W_{y} \cap \mathbb{Y} \). To show surjectivity we regard \( W_{y} \) as a vector space with origin \( e_{0} \wedge y \). Then there is a linear map

\[
\varphi : W_{y} \rightarrow \bigwedge^{3} \mathbb{K}^{n}; \quad e_{0} \wedge y + u \wedge v \mapsto u \wedge v \wedge y.
\]

Observe that we have inclusions \( \Im(\kappa(\cdot, y)) \subseteq Z_{y} \subseteq \ker(\varphi) \). We claim that equality holds in both cases. Since \( \kappa(\cdot, y) \) is linear of rank \( n - 1 \), the claim follows from

\[
\dim(\ker(\varphi)) = \dim(W_{y}) - \text{rank}(\varphi) = \binom{n}{2} - \binom{n - 1}{2} = n - 1.
\]

\[\square\]

We recall from \cite[Section 2]{6} the definition of the GIT-fan. Let the algebraic torus \( H := (\mathbb{K}^{*})^{n} \) act diagonally on \( \mathbb{K}^{r} \) via the characters \( \chi^{w_{1}}, \ldots, \chi^{w_{r}} \), \( w_{i} \in \mathbb{Z}^{n} \), i.e.

\[
h \cdot z := (\chi^{w_{1}}(h) z_{1}, \ldots, \chi^{w_{r}}(h) z_{r})
\]

and suppose that \( Y \subseteq \mathbb{K}^{r} \) is invariant under this action. Then the GIT-fan is defined as the collection of cones

\[
\Lambda_{H}(Y) := \{ \lambda(w); w \in \mathbb{Q}^{n} \}; \quad \lambda(w) := \bigcap_{w \in \omega_{I}} \omega_{I} \subseteq \mathbb{Q}^{n},
\]

where \( \omega_{I} := \cone(w_{i}; i \in I) \) is the cone associated to a \( Y \)-set \( I \), i.e. a subset \( I \subseteq \{1, \ldots, r\} \) for which the corresponding stratum \( \{ y \in Y; y_{i} \neq 0 \iff i \in I \} \) is non-empty.

We turn back to our setting. The \( \Cl(X) \)-grading of the Cox ring \( \mathcal{R}(X) = \mathcal{O}(\mathbb{X}) \) yields a diagonal action of the algebraic torus \( H := (\mathbb{K}^{*})^{n} = \Spec(\mathbb{K}[\Cl(X)]) \) on \( \mathbb{X} = (\mathbb{K}^{n})^{2} \) where

\[
h \cdot (x, y) = (h_{1}x_{1}, \ldots, h_{n}x_{n}, h_{1}y_{1}, \ldots, h_{n}y_{n}).
\]

Since the subalgebra \( \mathcal{O}(\mathbb{X})^{G_{a}} \) inherits the \( \Cl(X) \)-grading, the \( H \)-action descends to its spectrum \( \mathbb{Y} \subseteq \bigwedge^{2} \mathbb{K}^{n+1} \), turning \( \kappa \) into an equivariant morphism. Here the action is explicitly described by

\[
h \cdot e_{0} \wedge e_{j} = h_{j} e_{0} \wedge e_{j}, \quad h \cdot e_{i} \wedge e_{j} = h_{i} h_{j} e_{i} \wedge e_{j}.
\]

Note that this action differs from the well known maximal torus action. It rather is a submaximal action, with some connection to the maximal one, see Proposition \ref{3.6}.

In order to obtain the GIT-fan \( \Lambda_{H}(\mathbb{Y}) \) we consider the two-block partitions of \( N := \{1, \ldots, n\} \), i.e. partitions where \( N \) is a union of two disjoint subsets \( A, A^{c} \). To each such partition \( R = \{A, A^{c}\} \) we associate the hyperplane

\[
\mathcal{H}_{R} := \left\{ x \in \mathbb{Q}^{n}; \sum_{i \in A} x_{i} = \sum_{i \in A^{c}} x_{i} \right\}.
\]
Recall that the cones over the Grassmannians lie in the wedge products
\[ \Lambda^2 \mathbb{K}^n, \] 
then claim that
\[ y \] 
Moreover, we define
\[ k \] 
assertion follows from the induction hypothesis. We turn to the case where there
\[ n \] 
being of the Plücker relations that a
\[ E \] 
two graph graph structures on
\[ N \] 
for which
\[ N \subseteq \mathbb{N} \] 
where the embedding of the surrounding wedge products is reflected by the inclusion
\[ N := \{ 1, \ldots, n \} \] 
N := \{ \{ i, j \}; 1 \leq i < j \leq n \} 
N := \{ \{ 0, \ldots, n \} \} 
N := \{ \{ i \}; 0 \leq i < j \leq n \} 
Recall that the cones over the Grassmannians lie in the wedge products
\[ \mathbb{Y} \subseteq \Lambda^2 \mathbb{K}^n, \mathbb{Y} \subseteq \Lambda^2 \mathbb{K}^{n+1}. \] 
We use the above index sets \( N \) and \( N_0 \) to refer to the coordinate indices where \( \{ i, j \} \) labels \( e_i \wedge e_j \).

**Proposition 3.4.** A subset \( I \subseteq N_0 \) is a \( \mathbb{Y} \)-set if and only if \( I \) satisfies the following condition
\[ (*): \{ i, j \}, \{ k, l \} \subseteq I \implies \{ j, l \}, \{ i, k \} \subseteq I \quad \text{or} \quad \{ j, k \}, \{ i, l \} \subseteq I. \]

**Proof.** It follows from the nature of the Plücker relations that a \( \mathbb{Y} \)-set \( I \) has in fact the property \( (*) \). We prove that a subset of \( N \) satisfying \( (*) \) is a \( \mathbb{Y} \)-set by induction on \( n \). For this recall that we have commutative diagram of closed embeddings
\[
\begin{array}{ccc}
C(\text{Gr}(2, n+1)) & \xrightarrow{\mathbb{Y}} & (\mathbb{K}e_0 \wedge \mathbb{K}^n) \oplus \Lambda^2 \mathbb{K}^{n+1} \\
& & \xrightarrow{\mathbb{I}} \\
C(\text{Gr}(2, n)) & \xrightarrow{\mathbb{Y}^*} & \Lambda^2 \mathbb{K}^n \\
\end{array}
\]
where the embedding of the surrounding wedge products is reflected by the inclusion
\[ N \subseteq N_0. \] 
Let \( I \subseteq N_0 \) be a set with the property \( (*) \). If \( I \subseteq N \) holds, then the assertion follows from the induction hypothesis. We turn to the case where there exists \( k \in N \) such that \( \{ 0, k \} \) lies in \( I \). We will explicitly construct an element \( z \in \mathbb{Y} \) for which \( z_{ij} \) vanishes if and only if \( \{ i, j \} \) does not lie in \( I \). For this we introduce two graph graph structures on \( N \) by \( G_{12} := (N, E_1 \cup E_2) \) and \( G_2 := (N, E_2) \), where \( E_1, E_2 \) are sets of edges on \( N \) defined by
\[ E_1 := \{ \{ i, j \} \subseteq I; \{ 0, i \} \cap I \text{ or } \{ 0, j \} \cap I \}, \]
\[ E_2 := \{ \{ 0, j \} \in N \setminus I; \{ 0, i \} \cap I \}. \]

From the definition of the edge sets of the respective graphs we know that if \( \{ i \} \) is a connected component of \( G_{12} \), then it also is a connected component of \( G_2 \). Let \( F_1, \ldots, F_p \) be the connected components of \( G_2 \) different from a component \( \{ i \} \) of \( G_{12} \). We define a vector \( x \in \mathbb{K}^n \) by
\[ x_i := \begin{cases} 0 & \text{if } \{ i \} \text{ is a component of } G_{12}, \\ p & \text{if } \{ i \} \subseteq F_p \text{ holds.} \end{cases} \]
Moreover, we define \( y \in \mathbb{K}^n \) by \( y_j := 1 \) if \( \{ 0, j \} \in I \) and \( y_j := 0 \) if \( \{ 0, j \} \notin I \). We then claim that
\[ z_{ij} \neq 0 \iff \{ i, j \} \subseteq I. \]
Since \( z_{0j} = y_j \) holds, it is clear that the claim is true for the components of this type. For \( 0 \neq i < j \) the components of \( z \) can be written as
\[
    z_{ij} = x_i y_j - x_j y_i = \begin{cases} 
        0 & \text{if } \{0, i\}, \{0, j\} \notin I, \\
        \pm x_i & \text{if } \{0, i\} \notin I, \{0, j\} \in I, \\
        x_i - x_j & \text{if } \{0, i\}, \{0, j\} \in I.
    \end{cases}
\]

We now go through these three cases and verify for each that \( \{i, j\} \) lies in \( I \) if and only if \( z_{ij} \neq 0 \) holds.

Assume that \( \{0, i\}, \{0, j\} \notin I \) holds and recall that there exists a \( k \in N \) with \( \{0, k\} \in I \). It follows from \((*)\) applied to \( \{0, k\}, \{i, j\} \) that \( \{i, j\} \) does not lie in \( I \).

For the second case suppose that \( \{0, i\} \notin I \) and \( \{0, j\} \in I \) hold. We then have
\[
    x_i \neq 0 \iff \text{there exists } l \in N \text{ such that } \{i, l\} \in \mathcal{E}_1 \text{ or } \{i, l\} \in \mathcal{E}_2
\]
\[
    \iff \text{there exists } l \in N \text{ such that } \{i, l\} \in \mathcal{E}_1
\]
\[
    \iff \{i, j\} \in I
\]

For the second equivalence note that \( \{0, i\} \notin I \) holds which implies \( \{i, l\} \notin \mathcal{E}_2 \). The third equivalence is due to an application of \((*)\) to \( \{0, j\} \{i, l\} \).

In the last case where \( \{0, i\}, \{0, j\} \in I \) holds we obtain
\[
    x_i = x_j \iff \text{\( i, j \) lie in the same connected component of } \mathcal{G}_2
\]
\[
    \text{or } \{i\}, \{j\} \text{ are connected components of } \mathcal{G}_{12}
\]
\[
    \iff \{i, j\} \in \mathcal{E}_2 \text{ or } \{i\}, \{j\} \text{ are connected components of } \mathcal{G}_{12}
\]
\[
    \iff \{i, j\} \notin I
\]

For the second equivalence we use that each connected component of \( \mathcal{G}_2 \) is a complete graph, which follows \((*)\). \( \square \)

**Remark 3.5.** The affine cone \( \mathcal{Y}^* \) over the smaller Grassmannian \( \text{Gr}(2, n) \) is invariant under the \( H \)-action. The corresponding GIT-fan \( \Lambda_H(\mathcal{Y}^*) \) of this restricted action is well known, it was described in terms of walls in [5 Example 3.3.2] and [3 Example 8.5] as follows: Set
\[
    \Omega^* := \text{cone}(e_i + e_j; \ 1 \leq i < j \leq n) \subseteq \mathbb{Q}^n_{\geq 0}.
\]

Then the GIT fan \( \Lambda_H(\mathcal{Y}^*) \) is the fan supported on \( \Omega^* \) with walls given by the intersections of \( \Omega^* \) with the above hyperplanes \( \mathcal{H}_R \).

**Proposition 3.6.** The GIT-fan \( \Lambda_H(\mathcal{Y}^*) \) is a subfan of \( \Lambda_H(\mathcal{Y}) \).

**Remark 3.7.** Proposition 3.6 the universal property of the limit quotient (see [5 Remark 2.3]) and the inclusion \( \mathcal{Y}^* \subseteq \mathcal{Y} \) show that there is a closed embedding of the respective limit quotients \( \mathcal{M}_{0,n} = \mathcal{Y}^*\downarrow H \subseteq \mathcal{Y}\downarrow H \).

Moreover, for any \( \lambda \in \Lambda_H(\mathcal{Y}^*) \) the preimage \( p^{-1}(\mathcal{Y}^{ss}(\lambda)) \) lies in \( \mathcal{Y}^{ss}(\lambda) \) where
\[
    p: \bigwedge^2 \mathbb{K}^{n+1} = \mathbb{K}e_0 \bigwedge \mathbb{K}^n \oplus \bigwedge^2 \mathbb{K}^n \to \bigwedge^2 \mathbb{K}^n
\]
is the projection. This gives rise to an open subset \( U \subseteq \mathcal{Y}\downarrow H \) of the limit quotient and a surjective morphism \( \mathfrak{p}: U \to \mathcal{M}_{0,n} \).

**Example 3.8.** Consider the weights of the coordinates of the \( H \)-action on \( \bigwedge^2 \mathbb{K}^{n+1} \)
\[
    w_{01} := e_1, \quad \ldots, \quad w_{0n} := e_n, \quad w_{jk} := e_j + e_k, \ 1 \leq j < k \leq n.
\]

The following pictures of polytopal complexes arise from intersecting the GIT-fan \( \Lambda_H(\mathcal{Y}) \) with the hyperplane given by \( 1 = x_1 + \ldots + x_n \) in the cases \( n = 3, 4 \). The shaded area indicates the support \( \Omega^* \) of \( \Lambda_H(\mathcal{Y}^*) \).
In the case $n = 3$ the three walls of the GIT-fan are generated by two of the vectors $w_{12}, w_{13}, w_{23}$ and correspond to the two-block partitions

$$\{\{1\}, \{2, 3\}\}, \quad \{\{2\}, \{1, 3\}\} \quad \text{and} \quad \{\{3\}, \{1, 2\}\}.$$

In the case $n = 4$ again the hyperplanes separating $\Omega^*$ from the remaining 4 cones correspond to the partitions of the type $\{\{i\}, \{j, k, l\}\}$. The dotted lines in the right picture indicate the fan structure inside $\Lambda_H(\mathcal{Y})$. There are eight maximal cones arising from 3 hyperplanes of the form $\{\{i,j\}, \{k,l\}\}$.

**Proof of Proposition 3.6.** Recall that the weights of the coordinates of the $H$-action are

$$w_{01} := e_1, \ldots, w_{0n} := e_n, \quad w_{jk} := e_j + e_k, \quad 1 \leq j < k \leq n.$$

The GIT-fans $\Lambda_H(\mathcal{Y})$ and $\Lambda_H(\mathcal{Y}^*)$ are the collections of cones which arise as intersections of cones $\omega_I = \text{cone}(w_{ij}; \{i,j\} \in I)$ associated to $\mathcal{Y}$- or $\mathcal{Y}^*$-sets respectively. From Proposition 3.4 we know that every $\mathcal{Y}$-set is also a $\mathcal{Y}^*$-set. This means we only have to show that for every $\mathcal{Y}$-set $I \subseteq \mathbb{N}_0$ there exists a $\mathcal{Y}^*$-set $J \subseteq \mathbb{N}$ such that $\omega_I \cap \Omega^* = \omega_J$ holds. For a $\mathcal{Y}$-set $I \subseteq \mathbb{N}_0$ we set

$$J := J_1 \cup J_2, \quad J_1 := I \cap \mathbb{N}, \quad J_2 := \{\{i, j\}; \{0, i\}, \{0, j\} \in I\}$$

and prove that $J$ has the required properties. We first claim that $J$ is an $\mathcal{Y}^*$-set. For this we check that the condition of Proposition 3.4 applies to any two elements of $J$. If these two elements lie either both in $J_1$ or $J_2$ then the claim follows from $I$ being a $\mathcal{Y}$-set or the construction of $J_2$ respectively. For the remaining case consider $\{j, k\} \in J_1$ and $\{i_1, i_2\} \in J_2$. Since both $\{0, i_1\}$ and $\{j, k\}$ lie in $I$, we can without loss of generality assume that also $\{i_1, j\}$ and $\{0, k\}$ lie in $I$. Finally with $\{0, i_2\} \in I$ we conclude that $\{i_2, j\}, \{i_1, k\}$ are elements of $J$. This shows that $J$ is a $\mathcal{Y}^*$-set.

We now prove $\omega_I \cap \Omega^* = \omega_J$. It is easy to see that $\omega_J$ is in fact contained in $\omega_I \cap \Omega^*$; we turn to the reverse inclusion. With non-negative $a_i, a_{jk}$ let

$$x := \sum_{i \in \mathbb{N}} a_i w_{0i} + \sum_{j \in \mathbb{N}} a_{jk} w_{jk}$$

lie in $\omega_I \cap \Omega^*$. We show that $x$ is a non-negative linear combination of elements $w_{\eta}, \eta \in J$. Let $a_{i_1}$ be minimal among all $a_i$ with $\{0, i\} \in I$. For an arbitrary $\{0, i_2\} \in I$ we then replace in the above sum

$$a_{i_1} w_{0i_1} + a_{i_2} w_{0i_2} \quad \text{by} \quad (a_{i_2} - a_{i_1}) w_{0i_2} + a_{i_1} w_{i_1 i_2}.$$

Note that now $\{i_1, i_2\}$ lies in $J_2$. Iterating this process we see that there exists some $\{0, i\} \in I$ such that $x$ has the form

$$x = b_{i_1} w_{0i_1} + \sum_{j \in J_1 \cup J_2} b_{jk} w_{jk}.$$
Without loss of generality we assume that $i = 1$ holds. The condition $x \in \Omega^*$ implies $x_1 \leq x_2 + \ldots + x_n$, hence we have

\[ b_1 \leq 2 \sum_{(j,k) \in J, j,k \neq 1} b_{jk} \quad \text{and} \quad b_1 = 2 \sum_{(j,k) \in J, j,k \neq 1} b_{jk} \]

for certain $0 \leq b_{jk} \leq b_{jk}$. Plugging $w_{01} = \frac{1}{2}(w_{1j} + w_{1k} - w_{jk})$ into (**) we obtain a non-negative linear combination

\[ x = \sum_{(j,k) \in J, j,k \neq 1} ((b_{1j} + b_{jk})w_{1j} + (b_{1k} + b_{jk})w_{1k} + (b_{jk} - b_{jk})w_{jk}) + \sum_{(i,j) \in J} b_{jk}w_{jk}. \]

The last step to show is that for $(j,k) \in J$ both $\{1,j\}$ and $\{1,k\}$ lie in $J$. Recall that we have $\{0,1\} \in I$. If $(j,k)$ lies in $J_2$, then this follows directly from construction of $J_2$. Otherwise we can without loss of generality assume that $\{0,j\}, \{1,k\}$ lie in $I$. The claim again follows from the construction of $J_2$. \( \square \)

**Proof of Theorem 5.8** As before we denote the weights of coordinates with respect to the $H$-action by $w_{ij} = e_i$, $w_{jk} = e_j + e_k$. From Proposition 3.6 we know that $\Lambda_H(\bar{\Omega})$ has the asserted form on $\Omega^*$. Note that the remaining support $\Omega \setminus \text{relint}(\Omega^*)$ is the union of the cones

\[ \sigma_i := \text{cone}(w_{ij}, j \in N \setminus \{i\}), \quad i = 1, \ldots, n. \]

None of the hyperplanes $H_i$ intersect $\sigma_i$ in its relative interior. This means that we have to prove that $\sigma_i$ is a cone in the GIT-fan $\Lambda_H(\bar{\Omega})$, i.e. the intersection of cones $\omega_I$ associated to $\Omega$-sets. Note that $\sigma_i$ itself is a cone associated to a $\Omega$-set. Hence, it suffices to show that for any $\Omega$-set $I \subseteq N_0$ the intersection $\omega_I \cap \sigma_i$ is a face of $\sigma_i$. Without loss of generality we assume that $i$ equals 1 and set $\sigma := \sigma_1$. We now claim that $\omega_I \cap \sigma = \omega_J$ holds where

\[ J := J_1 \cup J_2; \quad J_1 := I \cap \{\{1,j\}; j \in N_0 \setminus \{1\}\}, \quad J_2 := \{\{1, j\}; \{0, j\} \in I\}. \]

To prove $\omega_J \subseteq \omega_I \cap \sigma$ note that any $w_{1j}$ with $\{1,j\} \in J_1$ clearly lies in $\omega_I \cap \sigma$. Hence, it suffices to show that for $w_{1j}$ with $\{0,j\} \in I$ the same holds. In case $\{0,1\} \in I$ this follows from $w_{1j} = w_{01} + w_{0j} \in \omega_I \cap \sigma$. Otherwise there must exist $\{1,l\} \in I$ and from Proposition 5.4 we know $\{0, l\}, \{1, j\} \in I$. This implies $w_{1j} \in \omega_I \cap \sigma$.

For the reverse inclusion $\omega_I \cap \sigma \subseteq \omega_J$ consider the non-negative linear combination

\[ x := a_{01}w_{01} + \sum_{(i,j) \in I, j \neq 0} a_{ij}w_{1j} + \sum_{(0,j) \in I} a_{0j}w_{0j} + \sum_{(j,k) \in I} a_{jk}w_{jk} \in \omega_I \]

Since $x$ lies in $\sigma$, we have $x_1 \geq x_2 + \ldots + x_n$ and this amounts to

\[ a_{01} \geq \sum_{(0,j) \in I} a_{0j} + 2 \sum_{(j,k) \in I} a_{jk}. \]

If $\{0,1\} \notin I$ holds, i.e. $a_{01} = 0$, then $x$ lies in the cone generated by the $w_{1j}$, $\{1,j\} \in J_1$. Otherwise with $w_{0j} = w_{1j} - w_{01}$ and $w_{jk} = w_{1j} + w_{1k} - 2w_{01}$ we get
a non-negative linear combination
\[
x = \sum_{i \in I} a_1 w_i + \sum_{j \neq 0} a_0, w_j + \sum_{j \neq 1} a_{jk} (w_{1j} + w_{1k}) + \sum_{j \neq 1} a_01 - \sum_{j \neq 0} a_0j - 2 \sum_{j \neq 0, 1} a_{jk} \right) w_{01}.
\]

The last thing to check is that all the above \( w_i \) lie in \( \omega J \). For this suppose that \( \{ j, k \} \in I \) holds. Since \( \{ 0, 1 \} \) is contained in \( I \), it follows from the construction of \( J \) that both \( \{ 1, j \} \) and \( \{ 1, k \} \) lie in \( J \). \qed

4. Combinatorial blow-ups

In this section we will provide a criterion whether a given cone lies in the iterated stellar subdivision of a simplicial fan. In [10] Feichtner and Kozlov deal with this problem in the more general setting of semilattices and give a nice characterisation in the case where the collection of subdivided cones forms a building set. We approach the issue of blowing up non-building sets, see Theorem 4.3. For details on stellar subdivisions see e.g. [13, Definition 5.1].

Let \( \mathcal{V} \) be a family of rays in a vector space and consider a \( \mathcal{V} \)-fan \( \Sigma_0 \), i.e. a fan with rays given by \( \mathcal{V} \). We then choose additional rays \( \nu_i, i = 1, \ldots, r \) lying in the relative interiors \( \sigma_i^\circ \) of pairwise different cones \( \sigma_i \in \Sigma_0 \). Moreover, we assume that \( \sigma_i \not\subseteq \sigma_j \) implies \( j < i \), which means that the larger the cone the earlier it will be subdivided. Now the questions comes up what the cones of the fan \( \Sigma_r \) are which arises from \( \Sigma_0 \) by the subsequent stellar subdivisions in the rays \( \nu_i \).

We call a subset \( S' \) of \( S := \{ \sigma_1, \ldots, \sigma_r \} \) conjunct, if the union \( \bigcup_{\sigma \in S'} (\sigma \setminus \{ 0 \}) \) is a connected subset in the usual sense and we set
\[
\langle S \rangle := \left\{ \sum_{\sigma \in S'} \sigma : S' \subseteq S \text{ conjunct} \right\}.
\]

A collection \( \mathcal{C} \subseteq \mathcal{V} \cup S \) is called geometrically nested, if for any subset \( \mathcal{H} \subseteq \mathcal{C} \) of pairwise incomparable cones with \( |\mathcal{H}| \geq 2 \) the following holds:
\[
\sum_{\tau \in \mathcal{H}} \tau \in \Sigma_0 \setminus \langle S \rangle.
\]

Proposition 4.1. Let \( \Sigma_0 \) be a simplicial fan and \( \nu_i, i = 1, \ldots, r \) rays in the relative interiors of pairwise different cones \( \sigma_i \in \Sigma_0 \). Assume that \( \sigma_i \not\subseteq \sigma_j \) implies \( j < i \) and let \( \Sigma_r \) be the iterated stellar subdivision of \( \Sigma_0 \) in the rays \( \nu_1, \ldots, \nu_r \) in order of ascending indices. If in the above notation \( \mathcal{C} \subseteq \mathcal{V} \cup S \) is geometrically nested, then \( \text{cone}(v, \nu_i; v \in \mathcal{C} \cap \mathcal{V}, \sigma_i \in \mathcal{C} \cap S) \) lies in \( \Sigma_r \).

We will prove this using the technique of combinatorially blowing up elements in a semilattice developed by Feichtner and Kozlov in [10]. For this we introduce some notation. Let \( (\mathcal{L}, \leq) \) be a finite (meet)-semilattice, i.e. a finite partially ordered set such that any non-empty subset \( \mathcal{X} \subseteq \mathcal{L} \) possesses a greatest lower bound \( \bigwedge \mathcal{X} \) called meet. Any meet-semilattice has a unique minimal element 0. Moreover, for a subset \( \mathcal{X} \subseteq \mathcal{L} \) the set \( \{ z \in \mathcal{L} : z \geq x \text{ for all } x \in \mathcal{X} \} \) is either empty or has a unique minimal element \( \bigvee \mathcal{X} \) called join. For \( y \in \mathcal{L} \) we denote \( \mathcal{X} \leq y := \{ x \in \mathcal{X} : x \leq y \} \), and finally, the semilattice \( \mathcal{L} \) is called distributive if the equation \( x \land (y \lor z) = (x \land y) \lor (x \land z) \) holds for any \( x, y, z \in \mathcal{L} \).
We now turn to blow-ups of semilattices in the sense of [10, Definition 3.1]. The blow-up of \((L,\geq)\) in an element \(\xi \in L\) is the semilattice \(\text{Bl}_{\{\xi\}}(L)\) consisting of
\[ x \in L \text{ with } x \not\geq \xi \] and \((\xi, x)\) where \(L \ni x \not\geq \xi\) and \(x \lor \xi\) exists.

The order relation \(\triangleright_{\text{Bl}}\) of the blow-up is given by
\[ x \triangleright_{\text{Bl}} y \text{ if } x > y, \quad (\xi, x) >_{\text{Bl}} (\xi, y) \text{ if } x > y, \quad (\xi, x) >_{\text{Bl}} y \text{ if } x \geq y, \]
in all three cases \(x, y \not\geq \xi\) holds.

We now want to iterate this process. Let \(G = (\xi_1, \ldots, \xi_r)\) be a family of elements \(\xi_i \in L\). The blow-up of \(L\) in \(G\) is simply the subsequent blow-up of \(L\) in the elements \(\xi_i\) in order of ascending indices. When we speak of a subfamily \((\xi_{i_1}, \ldots, \xi_{i_s})\) of \(G\) we always tacitly assume, that the order is preserved, i.e. that \(j < k\) implies \(i_j < i_k\).

We call \(G\) sorted if \(\xi_i > \xi_j\) implies \(i < j\). Moreover, we denote the underlying set of the family \(G\) by \(S_G\).

The subset \(S \subseteq L \setminus \{0\}\) is a building set for \(L\), if for every \(x \in L \setminus \{0\}\) the interval \(\{y \in L; 0 \leq y \leq x\}\) as a poset decomposes into a product of certain smaller intervals given by elements of \(S\). For a precise definition see [10, Definition 2.2].

For two subsets \(C \subseteq S \subseteq L\) we call \(C\) nested (in \(S\)) if for any subset \(\mathcal{H} \subseteq C\) of pairwise incomparable elements with \(|\mathcal{H}| \geq 2\) the join \(\bigvee \mathcal{H}\) exists and does not lie in \(S\). Note that the collection of nested sets forms an abstract simplicial complex \(\mathcal{C}(S)\) with vertex set \(S\).

**Theorem 4.2 ([10, Theorem 3.4]).** Assume that \(G\) is a sorted family in the semilattice \(L\) such that the underlying set \(S_G\) is a building set. Then we have an isomorphism of posets
\[ \mathcal{C}(S_G) \rightarrow \text{Bl}_G(L); \quad C \mapsto \bigvee_{\xi \in C}(\xi, 0). \]

We now describe a sufficient criterion to test whether an element lies in the blow-up \(\text{Bl}_F(L)\) in the case where \(S_F\) is not a building set.

**Theorem 4.3.** Let \(F\) be a sorted family in \(L\) and consider a subset \(C\) of the underlying set \(S_F\). If there exists a building set \(S\) of \(L\) with \(S_F \subseteq S\) such that \(C\) is nested in \(S\), then \(\bigvee_{\xi \in C}(\xi, 0)\) exists in the blow-up \(\text{Bl}_F(L)\).

Before we enter the proof of the Theorem we consider an example. Furthermore, for distributive \(L\) we provide an explicit construction of such a building set in the case where \(S_F\) generates \(L\) by \(\lor\), see Construction [10] Lemma 4.7.

**Example 4.4.** The face poset of a polyhedral fan is a semilattice in which the stellar subdivision in a ray \(\nu \in \sigma\) corresponds to the blow-up of the element \(\sigma\), see [10, Proposition 4.9]. Viewing the positive orthant \(\Sigma := \mathbb{Q}_+^2\) as a fan, we ask for the combinatoric structure of its stellar subdivisions \(\Sigma_1\) and \(\Sigma_2\) in the sorted families
\[ G_1 := (\nu_1, \nu_2, e_1, e_2, e_3), \quad G_2 := (\nu_2, \nu_1, e_1, e_2, e_3), \]
where \(\nu_1 := (1, 1, 0), \quad \nu_2 := (0, 1, 1)\).

If \(G_1\) and \(G_2\) were building sets, then Theorem 4.2 would imply that the fans \(\Sigma_1\) and \(\Sigma_2\) coincide. Clearly, this is not the case.
We now add to $G_1$ and $G_2$ a ray lying in the relative interior of the join of the faces $\text{cone}(e_1, e_2)$ and $\text{cone}(e_2, e_3)$, e.g. $\nu_0 = (1, 1, 1)$. This yields two building sets

$$G_{1a} := (\nu_0, \nu_1, \nu_2, e_1, e_2, e_3), \quad G_{2a} := (\nu_0, \nu_2, \nu_1, e_1, e_2, e_3).$$

Both families give rise to the same subdivided fan. Note that the faces of $\Sigma_{1a} = \Sigma_{2a}$ not having $\nu_0$ as a ray lie in both $\Sigma_1$ and $\Sigma_2$. This is essentially the idea of the proof of Proposition 4.3.

![Diagram](image)

**Definition 4.5.** Let $S = \{\xi_1, \ldots, \xi_r\}$ be a subset of the semilattice $L$. We call a (non-ordered) pair $\{\xi, \xi_j\}$ harmonious (with respect to $S$) if at least one of the following conditions is satisfied:

$$\xi_i \land \xi_j = 0 \quad \text{or} \quad \xi_i \lor \xi_j \quad \text{does not exist} \quad \text{or} \quad \xi_i \lor \xi_j \in S.$$

**Construction 4.6.** Let $S = \{\xi_1, \ldots, \xi_r\}$ be a subset of $L$. For all pairs of non-harmonious elements $\{\xi_i, \xi_j\}$ we add to $S$ the element $\xi_i \lor \xi_j$:

$$S' := S \cup \{\xi_i \lor \xi_j: \{\xi_i, \xi_j\} \text{ non-harmonious with respect to } S\}.$$

We continue this process with the new set $S'$ instead of $S$ until all pairs are harmonious and denote the final set by $\langle S \rangle$. Since $L$ is finite, clearly this process terminates after finitely many steps.

**Lemma 4.7.** Assume that $L$ is distributive and a subset $S \subseteq L \setminus \{0\}$ generates it by $\lor$. Then the following assertions hold.

1. If for any $x \in L$ and distinct $\xi_i, \xi_j \in \text{max}(S_{\leq x})$ their meet $\xi_i \land \xi_j$ equals 0, then $S$ is a building set for $L$.
2. The set $\langle S \rangle$ is a building set for $L$.

**Proof.** For the proof of (i) we check the two conditions of [10, Proposition 2.3 (4)]. Fix an $x \in L$ and a subset $\{y_1, y_2, \ldots, y_t\} \subseteq \text{max}(S_{\leq x})$. By assumption we have

$$0 = (y \land y_1) \lor \ldots \lor (y \land y_t) = y \land (y_1 \lor \ldots \lor y_t).$$

Since $0 \notin S$ holds, this implies $S_{\leq y} \cap S_{\leq y_1 \lor \ldots \lor y_t} = \emptyset$. For the second condition let $z < y$. Clearly $z \lor y_1 \lor \ldots \lor y_t \leq y \lor y_1 \lor \ldots \lor y_t$ holds. If they were equal, so would be the respective meets with $y$ and this would imply $z = y$.

We now prove the second assertion (ii). By construction of $\langle S \rangle$, for any $x \in L$ and $\xi_i, \xi_j \in \text{max}(\langle S \rangle_{\leq x})$ the pair $\{\xi_i, \xi_j\}$ is harmonious (with respect to $\langle S \rangle$). Its join exists but - by maximality of $\xi_i$ and $\xi_j$ - does not lie in $\langle S \rangle$. This implies that $\xi_i \land \xi_j = 0$ holds and the assertion follows from (i).

**Proof of Theorem 4.3** Before we enter the proof let us recall the join rules of blow-ups from [10, Lemma 3.2]. Let $x, y, \xi$ lie in the semilattice $L$ and consider the blow-up $L'$ of $L$ in $\xi$. Then the join $(\xi, x) \lor_{L'} y$ exists if and only if $x \lor_{L} y \in L'$ and $x \lor_{L} y \notin \xi$ holds. The join $x \lor_{L'} y$ exist if and only if $x \lor_{L} y \in L'$. In case the joins exist the following formulae hold

$$(\xi, x) \lor_{L'} y = (\xi, x \lor_{L} y), \quad x \lor_{L'} y = x \lor_{L} y.$$

We turn back to our case and fix some notation. We write $F = (\xi_1, \ldots, \xi_r)$ and denote the elements lying in $C$ by $\xi_{ij}$, $j = 1, \ldots, s$ where we assume that the order is preserved, i.e. $j < j'$ is equivalent to $i_j < i_{j'}$. Moreover, for $k = 1, \ldots, r$ let $L_k$
be the blow-up of \( \mathcal{L} \) in \( (\xi_1, \ldots, \xi_k) \) and for consistency we set \( \mathcal{L}_0 := \mathcal{L} \). In \( \mathcal{L}_k \) we consider the following (a priori non-existent) join
\[
\bigvee_{j=1}^{j(k)} (\xi_{j(k)}, 0) \vee \bigvee_{j=j(k)+1}^{s} \xi_{j(k)}, \text{ where } j(k) := \max\{0 \cup \{j; i_j \leq k\}\}.
\]
In case this join does exist, we denote it by \( z_k \). Note that from the definition of \( j(k) \) it follows that \( i_{j(k)} \) is the largest index, such that \( \xi_{i_1}, \ldots, \xi_{i_{j(k)}} \) are among the \( \xi_1, \ldots, \xi_k \). We prove the existence of \( z_r = \bigvee_{\xi \in \mathcal{C}} (\xi, 0) \) by induction on \( k \). Since \( \mathcal{C} \) is nested, it is clear that \( z_0 = \bigvee \mathcal{C} \) does exist in \( \mathcal{L}_0 \). Now assume that \( z_k \in \mathcal{L}_k \) exists. We discriminate two possible cases: In the first case \( \xi_{k+1} \) does not lie in \( \mathcal{C} \) in the second case it does.

Assume that \( \xi_{k+1} \notin \mathcal{C} \) holds and note that this is equivalent to \( j(k) = j(k + 1) \). Hence, as elements in \( \mathcal{L}_k \) we have \( z_{k+1} = z_k \) and the only thing to check is that \( z_k \geq \xi_{k+1} \) holds. For this note that the iterated application of the above join rules shows that
\[
z_k = (\xi_{i_{j(k)}}, 0) \vee \left( \bigvee_{j=1}^{j(k)-1} (\xi_{j(k)}, 0) \vee \xi_k \right) \vspace{2mm} = \ldots = (\xi_{i_{j(k)}}, (\ldots(\xi_{i_1}, \xi_k)\ldots)) \quad \text{where } \xi_k := \bigvee_{j=j(k)+1}^{s} \xi_{j(k)}.
\]
If we had \( z_k \geq \xi_{k+1} \), then this would mean \( (\xi_{i_{j(k)}}, (\ldots(\xi_{i_1}, \xi_k)\ldots)) \geq \xi_{k+1} \). Iterating this argument we would get \( \xi_k \geq \xi_{k+1} \) in \( \mathcal{L}_0 \) which would imply \( \xi_{k+1} \in \mathcal{S}_{\leq \xi_k} \). Since \( \mathcal{S} \) is a building set, by [10, Proposition 2.8 (2)]
\[
\max(\mathcal{S}_{\leq \xi_k}) = \max(\xi_{i_1}, \ j = j(k) + 1, \ldots, s)
\]
holds. Hence there must exist \( j_0 \geq j(k) + 1 \) with \( \xi_{k+1} \leq \xi_{i_{j_0}} \). Since \( \xi_{k+1} \notin \mathcal{C} \) holds, we have \( \xi_{k+1} \neq \xi_{i_{j_0}} \). In particular, this implies \( k > i_{j_0} - i_{j(k) + 1} - 1 \). However, from the definition of \( j(k) \) we easily see that \( k \leq i_{j(k) + 1} - 1 \) holds, a contradiction.

We turn to the second case where \( \xi_{k+1} \in \mathcal{C} \) holds which is equivalent to \( j(k) + 1 = j(k+1) \). In \( \mathcal{L}_k \) we consider the element
\[
y_k := \bigvee_{j=1}^{j(k)} (\xi_{i_1}, 0) \vee \bigvee_{j=j(k)+2}^{s} \xi_{i_j}.
\]
Since \( z_k \) exists, it follows that also \( y_k \) and the join \( \xi_{k+1} \vee y_k \) exist. Then the last thing to show is that \( y_k \geq \xi_{k+1} \) holds. This follows from the same argument as above with \( y_k \) instead of \( z_k \).

\[
\square
\]

\textbf{Proof of Proposition 4.1} First note that since \( \Sigma_0 \) is simplicial so is the iterated stellar subdivision \( \Sigma_r \). In particular, the further application of stellar subdivisions in the original rays \( V \) leaves \( \Sigma_r \) unchanged. From [10, Proposition 4.9] we know that a stellar subdivision in a ray \( \nu \in \sigma^0 \) corresponds to the blow-up of the face poset of the original fan in \( \sigma \). More precisely, as posets \( \Sigma_r \) and \( \text{Bl}(\mathcal{S}_0) \) are isomorphic, where
\[
F := (\sigma_1, \ldots, \sigma_r, v_1, \ldots, v_t), \quad V = \{v_1, \ldots, v_t\}.
\]
For the proof of the Proposition we now check the assumptions of Theorem 4.3

First note that \( \Sigma_0 \) is simplicial, hence it is distributive as a semilattice. Its joins and meets can be computed by taking convex geometric sums and intersections respectively. Also, with \( \mathcal{S} = \{\sigma_1, \ldots, \sigma_r\} \) it is clear that \( V \cup \mathcal{S} \), the underlying set
of \( \mathcal{F} \), generates \( \Sigma_0 \setminus \{0\} \) by +. In particular, from Lemma 4.7 we infer that \( \langle V \cup S \rangle \) is a building set for \( \Sigma_0 \).

Now note that \( \langle V \cup S \rangle \setminus V \) equals \( \langle S \rangle \) and from the respective constructions it follows that \( \langle S \rangle \subseteq \langle \langle V \cup S \rangle \rangle \). Together this means

\[
\Sigma_0 \setminus \langle S \rangle \subseteq \Sigma_0 \setminus \langle \langle V \cup S \rangle \rangle = (\Sigma_0 \setminus \langle V \cup S \rangle) \cup V.
\]

Since \( C \subseteq V \cup S \) is geometrically nested, it follows that it is also nested in \( \langle V \cup S \rangle \) in the sense of semilattices. From Theorem 4.3 we now know that \( \bigvee_{c \in C} (c, 0) \) lies in \( \text{Bl}_F(\Sigma_0) \). Under the above isomorphy \( \Sigma_r \cong \text{Bl}_F(\Sigma_0) \) this means

\[
\text{cone}(v, \nu_i; \nu \in C \cap V, \sigma_i \in C \cap S) \in \Sigma_r.
\]

\[\square\]

5. The limit quotient as blow-up

This section is devoted to the main result and its proof. As before, let \( Y \subseteq \bigwedge^2 \mathbb{A}^{n+1} \) be the affine cone over the Grassmannian \( \text{Gr}(2, n+1) \) and consider the torus \( H = (\mathbb{A}^\ast)^n \) acting on \( Y \) by

\[
h \cdot e_i = h_i e_i \quad \text{and} \quad h \cdot e_i \wedge e_j = h_i h_j e_i \wedge e_j.
\]

We assert that the normalised limit quotient \( Y \bigwedge H \) normalises the following iterated blow-up of \( \mathbb{P}^{n-1}_1 \). We set \( N_2 := \{2, \ldots, n\} \) and consider a subset \( A \subseteq N_2 \) with at least two elements. Labeling by \( T_2, S_2, \ldots, T_n, S_n \) the homogeneous coordinates of \( \mathbb{P}^{n-1}_1 \) we associate to \( A \) the subscheme of \( \mathbb{P}^{n-1}_1 \) given by the ideal

\[
\langle T_i^2, T_j S_k = T_k S_j; \ i, j, k \in A, j < k \rangle.
\]

The collection \( X \) of corresponding subschemes \( X_A \) comes with a partial order given by the scheme-theoretic inclusions with \( X_{N_2} \) being the minimal element. A linear extension of this partial order is a total order on \( X \) which is compatible with the partial order.

**Theorem 5.1.** Fix a linear extension of the partial order on \( X \). Then the normalised limit quotient \( Y \bigwedge H \) normalises the blow-up of \( \mathbb{P}^{n-1}_1 \) in all the subschemes \( X_A \) (i.e. their respective proper transforms) in ascending order.

Recall that the above action stems from the action of \( \mathbb{G}_a \) on \( X = \mathbb{P}^n_1 \) as shown in Sections 2 and 3. Moreover, keep in mind that the enveloped quotients \( V_i \) of \( X \) are only subsets of the Mumford quotients of \( Y \). Hence the non-reductive limit quotient \( X \bigwedge \mathbb{G}_a \) in general only is a subset of the reductive limit quotient. This is reflected in the second step of the following procedure to obtain \( X \bigwedge \mathbb{G}_a \).

**Theorem 5.2.** The normalised limit quotient \( X \bigwedge \mathbb{G}_a \) can be obtained by the following procedure.

\begin{itemize}
  
  (i) Let \( X_1 \) be the blow-up of \( \mathbb{P}^{n-1}_1 \) in the subscheme \( X_{N_2} \).
  
  (ii) Let \( X'_1 := X_1 \setminus E \) be the quasiprojective subvariety of \( X_1 \) where \( E \) is the intersection of the proper transform of \( V(T_2, \ldots, T_n) \subseteq \mathbb{P}^{n-1}_1 \) with the expectional divisor in \( X_1 \).
  
  (iii) Fix a linear extension of the partial order on \( X \) and blow up \( X'_1 \) in the respective proper transforms of the remaining subschemes \( X_A, A \subseteq N_2 \) in ascending order.
  
  (iv) Normalise the resulting space.
\end{itemize}
We briefly outline the structure of our proof. For this consider $E_n$ the identity matrix and

$$Q := (E_n, D_n), \quad \text{where} \quad D_n := (e_j + e_k)_{1 \leq j < k \leq n}.$$ 

Note that $Q$ is the matrix recording the weights of the coordinates of the above $H$-action. We denote the first $n$ columns of $Q$ by $w_{0i}$ and the remaining ones by $w_{jk}$. Furthermore, we fix a Gale dual matrix $P$ of $Q$, i.e. a matrix with $PQ^t = 0$, and analogously write $v_{0i}, v_{jk}$ for its columns. Denoting by $T$ the dense algebraic torus of $\mathbb{A}^n_{\mathbb{K}}$ we recall from Section 2.1 that there is a normalisation map

$$\mathbb{Y} \sslash H \to \left((\mathbb{Y} \cap T) / H\right)^*,$$

where the latter is the closure in the toric variety associated to the fan $\Sigma := \text{GKZ}(P)$. With this the proof of Theorem 5.1 will be split into two parts. As a first step we will prove that the blow-up of $\mathbb{P}^{n-1}_1$ in the subscheme $X_{N_1}$ yields one of the Mumford quotients $X_1$ of $\mathbb{Y}$. This quotient comes with a canonical embedding into a simplicial toric variety $Z_1$, which arises from a simplicial fan $\Sigma_1$ with rays generated by the columns of $P$. Finally we show that the iterated stellar subdivision of $\Sigma_1$ and the fan $\Sigma$ share a sufficiently large subfan. This implies that the proper transform of $X_1$ under the corresponding toric blow-ups and the limit quotient $\mathbb{Y} \sslash H$ share a common normalisation.

In the case $n = 2$ the normalised limit quotient is the projective line. If we consider three distinct points the resulting normalised limit quotient is the unique non-toric, Gorenstein, log del Pezzo $\mathbb{K}^*$-surface of Picard number 3 and a singularity of type $A_1$, see [14, Theorem 5.27]. The standard construction of this surface is the blow-up of three points on $\mathbb{P}_2$ followed by the contraction of a $(-2)$-curve. However, we realise it as a single (weighted) blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in the subscheme associated to $(T_2^2, T_3^2, T_2S_3 - T_3S_2)$ where $T_2, S_2, T_3, S_3$ are the homogeneous coordinates on $\mathbb{P}_1 \times \mathbb{P}_1$. Similar to $\overline{M}_{0,5}$ which is isomorphic to a single Mumford quotient of the cone over the Grassmannian $\text{Gr}(2, 5)$, this surface arises as Mumford quotient of the cone over the Grassmannian $\text{Gr}(2, 4)$. For higher $n$ an analogous Mumford quotient needs to be blown up as described above to obtain the limit quotient.

**Step 1.** Recall that each chamber in the GIT-fan $\Lambda_H(\mathbb{Y})$ gives rise to a set of semistable points admitting a Mumford quotient. We define two particular chambers and look at their respective quotients. For this consider the following linear forms on $\mathbb{Q}^n$:

$$f_1 := e_1^* - \sum_{i \neq 1} e_i^*; \quad f_{ij} := e_1^* + e_j^* - \sum_{i \neq 1, j} e_i^*.$$

The zero sets of these linear forms are precisely the walls arising from the partitions $\{\{1\}, N \setminus \{1\}\}$ and $\{\{1, j\}, N \setminus \{1, j\}\}$ of $N = \{1, \ldots, n\}$ in the sense of Section 3. We define the following two full dimensional cones in the GIT-fan

$$\lambda_0 := \Omega \cap \{w \in \mathbb{Q}^n; \ f_1(w) \geq 0\}; \quad \lambda_1 := \Omega \cap \{w \in \mathbb{Q}^n; \ f_1(w) \leq 0, \ f_{1j}(w) \geq 0 \text{ for } j = 2, \ldots, n\},$$

where $\Omega$ is the support of $\Lambda_H(\mathbb{Y})$. While $\lambda_1$ lies inside $\Omega^* = \text{supp}(\Lambda_H(\mathbb{Y}))$ the cone $\lambda_0$ does not. The two cones are adjacent in the sense that they share a common facet, namely $\Omega \cap \ker(f_1)$. Now consider the corresponding Mumford quotients $X_i := \mathbb{Y}^{\#}(\lambda_i)/H$ with $i = 0, 1$.

**Proposition 5.3.** In the above notation $X_0$ is isomorphic to $\mathbb{P}^{n-1}_1$. Moreover, $X_1$ is isomorphic to the blow-up of $X_0$ in the subscheme $X_{N_1}$. 
Recall that \( \lambda_1 \in \Lambda_H(Y) \) gives rise to the enveloped quotient \( V_1 \) which is the image of the restricted Mumford quotient \( Y^\bullet(\lambda_1) \cap Y \to X_1 \).

**Proposition 5.4.** Let \( E \) denote the intersection of the exceptional divisor of \( X_1 \to X_0 \) with the proper transform of \( V(T_2, \ldots, T_n) \). Then the enveloped quotient \( V_1 \) is given by \( X_1 \setminus E \). In particular, it is quasiprojective.

**Proposition 5.5.** Let \( A \subseteq N_2 \) be a subset with at least two elements. Then the cone \( \text{cone}(\nu; \eta \subseteq A \cup \{0\}) \) lies in \( \Sigma_0 \). Moreover, consider the ray

\[
\nu := \text{cone} \left( \sum_{i \in A} v_i + 2 \sum_{\eta \subseteq A} v_\eta \right)
\]

in the relative interior of the above cone. Let \( X' \) be the proper transform of \( X_0 \) under the blow-up corresponding to the stellar subdivision of \( \Sigma_0 \) in \( \nu \). Then \( X' \) is isomorphic to the blow-up of \( X_0 \) in the subscheme \( X_A \subseteq X_0 \) given by

\[
\langle T_2^j, T_jS_k - T_kS_j; \; i, j, k \in A, \; j < k \rangle.
\]

Before we look at our case we recall the connection between blow-ups and stellar subdivisions in general. For this let \( \Sigma_1 \to \Sigma_0 \) be the stellar subdivision of a simplicial fan in \( \mathbb{Z}^n \) in the ray \( \nu \). To any homogeneous ideal in the Cox ring of the corresponding toric variety \( Z_0 \) we can associate a subscheme of \( Z_0 \) in the sense of Cox, see [7 Section 3] for details. We now ask for an ideal and the associated subscheme \( Z_\nu \) such that the blow-up of \( Z_0 \) in \( Z_\nu \) is isomorphic to the toric variety \( Z_1 \) corresponding to \( \Sigma_1 \).

For this short reminder set \( P \) as the matrix mapping the standard basis vectors \( f_i, \; i = 1, \ldots, r \) of \( F := \mathbb{Z}^r \) to the primitive lattice vectors \( v_i \in \mathbb{Z}^n \) in the rays of \( \Sigma_0 \). Then the Cox ring of \( Z_0 \) is \( \mathbb{K}[E \cap \gamma] \) where \( E := F^* \) and \( \gamma \) is the positive orthant in \( E \otimes \mathbb{Q} \). If the ray \( \nu \) lies in the support of \( \Sigma_0 \), then there exists a subset \( I \subseteq \{1, \ldots, r\} \) and minimal positive integers \( \alpha_i \in \mathbb{Z}_{\geq 1} \) such that

\[
\nu = \text{cone} \left( \sum_{i \in I} \alpha_i v_i \right)
\]

holds. Denoting by \( \langle e_1, \ldots, e_r \rangle \) the dual basis of \( \langle f_1, \ldots, f_r \rangle \) we set

\[
E_I := \text{cone}(e_i; \; i \in I), \quad f := \sum_{i \in I} \alpha_i f_i \in F, \quad c := \text{lcm}(\alpha_i; \; i \in I).
\]

We obtain a homogeneous ideal in the Cox ring of \( Z_0 \) and from it a subscheme \( Z_\nu \) of \( Z_0 \) by

\[
\langle c e; \; e \in E_I, \; \langle e, f \rangle = c \rangle \subseteq \mathbb{K}[E \cap \gamma].
\]

**Proposition 5.6.** Let \( \Sigma_1 \to \Sigma_0 \) be the above stellar subdivision of the simplicial fan \( \Sigma_0 \) in \( \nu \). If \( Z_0, Z_1 \) are the toric varieties arising from \( \Sigma_0, \Sigma_1 \) respectively, then \( Z_1 \) is isomorphic to the blow-up of \( Z_0 \) in the subscheme \( Z_\nu \).

We turn back to our setting. We prove Propositions 5.3, 5.4 and 5.5 using the method of ambient modifications, see [13 Proposition 6.7]. For this note that \( X_0 \) and \( X_1 \) come with canonical embeddings into simplicial toric varieties. We provide an explicit construction, for the general case see [11 Chapter III, Section 2.5]. For the index sets we use the same notation as in Section 3:

\[
N = \{\{i, j\}; \; 1 \leq i < j \leq n\}, \quad N_0 = \{\{i, j\}; \; 0 \leq i < j \leq n\}.
\]
Viewing $\Lambda^2 \mathbb{K}^{n+1}$ as the toric variety arising from the positive orthant $\delta$ in $\Lambda^2 \mathbb{Q}^{n+1}$ we define a subset as follows. We set
\[ \text{envs}(\lambda_i) = \{ I \subseteq \mathbb{N}_0; J \subseteq I, \lambda_0^i \subseteq \omega_2^j \subseteq \omega_1^j \text{ for some } \mathbb{F}\text{-set } J \} \]
as the collection of enveloping sets. Denoting by $f_\eta$ with $\eta \in \mathbb{N}_0$ the standard basis vector in $\Lambda^2 \mathbb{Q}^{n+1}$ we consider the subfan of $\delta$
\[ \hat{\Sigma} \_i := \{ \text{cone}(f_\eta; \eta \in J); J \subseteq \mathbb{N}_0 \setminus I \text{ for some } I \in \text{envs}(\lambda_i) \} \]
and the corresponding toric variety $\hat{Z}_i \subseteq \Lambda^2 \mathbb{K}^{n+1}$. Then $\hat{Z}_i$ admits a good quotient $\hat{Z}_i \to Z_i$; the quotient space is toric again and the quotient morphism corresponds to the lattice homomorphism $P: \mathbb{Z}^{(n_2^i)} \to \mathbb{Z}^{(\hat{\Sigma}^i)}$. The fan of $Z_i$ is given by
\[ \Sigma_i = \{ \text{cone}(v_\eta; \eta \in \mathbb{N}_0 \setminus I); I \in \text{envs}(\lambda_i) \} \]
We now turn to the embedded spaces. Starting with the embedding $\mathcal{Y} \subseteq \Lambda^2 \mathbb{K}^{n+1}$ we have $\mathcal{Y} \cap \hat{Z}_i = \mathcal{Y}^{\text{envs}}(\lambda_i)$ and the quotient $\hat{Z}_i \to Z_i$ restricts to the good quotient $\mathcal{Y}^{\text{envs}}(\lambda_i) \to X_i$. The situation fits into the following commutative diagram where the vertical arrows are closed embeddings.
\[ \begin{array}{ccc}
\Lambda^2 \mathbb{K}^{n+1} & \xrightarrow{\mathcal{Y}} & Z_i \\
\mathcal{Y}^{\text{envs}}(\lambda_i) & \xrightarrow{\hat{Z}_i} & X_i \\
\end{array} \]

Proofs of Proposition 5.3, 5.4 and 5.5. We prove the first part of Proposition 5.3. For this we set $J := \mathbb{N}_0 \setminus \{ \eta; \eta \subseteq A \cup \{0\} \}$. With Proposition 4.4 it is easy to see, that $J$ is a $\mathbb{F}$-set. Moreover, $\lambda_0^i \subseteq \omega_2^j$ holds. By definition of $\Sigma_0$ it is now clear that it contains $\text{cone}(v_\eta; \eta \subseteq A \cup \{0\})$.

We now perform the ambient modification. For this note that the weight $w_0$ is extremal in $\Lambda_H(\mathcal{Y})$, hence we can contract $v_0$. It can be written as a non-negative linear combination
\[ v_0 = \sum_{\eta \in \mathbb{N}_0} \alpha_\eta v_\eta, \text{ where } \alpha_\eta = \begin{cases} 1 & \text{if } \eta = \{0\}, \\
0 & \text{if } \eta \neq \{0\}. \end{cases} \]
In particular, it lies in the above cone $\text{cone}(v_\eta; \eta \subseteq \{0\} \cup \mathbb{N}_2)$. The total coordinate spaces of the embedding toric varieties $Z_0$ and $\hat{Z}_1$ are affine spaces, they are given by
\[ Z_0 = \mathbb{K}^{\mathbb{N}_0 \setminus \{0,1\}} \quad \text{and} \quad Z_1 = \Lambda^2 \mathbb{K}^{n+1} = \mathbb{K}^{\mathbb{N}_0}. \]
Furthermore, the ambient modification $\Sigma_1 \to \Sigma_0$ gives rise to a morphism of the total coordinate spaces of the respective toric varieties
\[ c: \widehat{Z}_1 \to Z_0; \quad (x_\eta)_{\eta \in \mathbb{N}_0} \mapsto (x_0^\alpha \ x_\eta)_{\eta \in \mathbb{N}_0 \setminus \{0,1\}}. \]
We label the variables of the total coordinate space $\mathcal{Z}_0$ by $S_\eta$ where $\eta$ runs through $\mathbb{N}_0 \setminus \{0,1\}$. Recall that we have a closed embedding $\mathcal{Y} \subseteq \mathcal{Z}_1$. The vanishing ideal of the image $\mathcal{X}_0 := \mathcal{c}(\mathcal{Y})$ in the Cox ring is given as
\[ (S_{ij} - S_{0i}S_{1j} + S_{0j}S_{1i}; \ 2 \leq i < j \leq n) \subseteq \mathcal{R}(Z_0). \]
It turns out that $\mathcal{X}_0$ is in fact isomorphic to the affine space via
\[ \iota: \mathbb{K}^{n-1} \times \mathbb{K}^{n-1} \to Z_0 \quad (x, y) \mapsto (x, y, (x_iy_j - x_jy_i);_{i < j}). \]
The original $H$-action on $\mathcal{Y}$ descends via $\iota^{-1} \circ c$ to $\mathbb{K}^{n-1} \times \mathbb{K}^{n-1}$ and is explicitly given by the weight matrix $Q_0 = [E_{n-1}, E_{n-1}]$ where $E_{n-1}$ is the identity matrix.
This shows that \( X_0 \) is isomorphic to \( \mathbb{P}^{n-1}_1 \). For convenience we summarise the situation in the following commutative diagram.

\[
\begin{array}{c}
\mathbb{P}^{n-1}_1 \\
\downarrow e \\
Y \\
\downarrow e \\
X_0 \\
\end{array} \quad \begin{array}{c}
\mathcal{O}(\mathbb{Z}_0) = \mathcal{R}(Z_0) \\
\downarrow \iota \\
\mathbb{P}^{2(n-1)} \end{array}
\]

The next step of the proof is the second half of Proposition 5.5. From Proposition 5.6 we infer that the ideal in \( \mathcal{O}(\mathbb{Z}_0) = \mathcal{R}(Z_0) \) yielding the center of the blow-up is given by

\[
(S_{0}, S_{1}, \ldots, S_{n-1}) : \ i, \in A, \ \eta \subseteq A.
\]

If we pullback this ideal via \( \iota^* \) (see [4, Lemma 2.1]), then in homogeneous coordinates over \( \mathbb{P}^{n-1}_1 \) we obtain

\[
(T_i, T_jS_k - T_kS_j; \ i, j, k \in A, \ j < k).
\]

In the case of the ambient modification of Proposition 5.3 we set \( A = N_2 \) to obtain the assertion. Finally, we turn to Proposition 5.4 and determine the enveloped quotient. For this recall that the image of the categorical quotient in Section 3 was given by \( Y = (Y \setminus \mathcal{Y}) \cup \{0\} \), see Proposition 3.2. This means that the enveloped quotient \( V_1 \subseteq X_1 \) is given as the image of

\[
\pi : Y^{\ssss}_{\mathcal{Y}}(\lambda_1) \setminus D \rightarrow X_1,
\]

where \( D := V(S_{0}; \ i = 1, \ldots, n) \subseteq \mathbb{A}^1 \mathbb{K}^{n+1} \). The quotient is geometric, hence the enveloped quotient is \( V_1 = X_1 \setminus \pi(D) \). Now consider the subvariety \( V(T_2, \ldots, T_n) \subseteq \mathbb{P}^{n-1}_1 \). Transferring it via \( \iota \) and then taking the proper transform we obtain the subvariety of \( X_1 \) given by \( (S_{02}, \ldots, S_{0n}) \) in the Cox ring \( \mathcal{R}(Z_1) \). The intersection with the exceptional divisor is precisely the set \( E = \pi(D) \).

\[
\square
\]

**Step 2.** In this step we show that the remaining blow-ups lead to the limit quotient \( Y_{\mathcal{Y}} / H \). As before, \( Q = (E_n, D_n) \) is the matrix recording the weights of the coordinates of the \( H \)-action and we label its columns by \( w_0 \) with \( \eta \in N_0 \) and \( N_0 = \{(i, j); \ 0 \leq i < j \leq n\} \). We then have the Gale dual matrix \( P \) with columns denoted by \( v_0 \). Moreover, \( \Sigma_1 \) is the simplicial fan in \( \mathbb{Z}^{(2)} \) from the preceeding step and we recall that \( X_1 = Y^{\ssss}_{\mathcal{Y}}(\lambda_1) / H \) is embedded into the corresponding toric variety \( Z_1 \).

Now let \( R = \{A_1, A_2\} \) be a true two-block partition of \( N \), i.e. a partition with \( |A_1|, |A_2| \geq 2 \). To every such partition we associate a ray

\[
\nu_R := \text{cone} \left( \sum_{v \in A_1} v_{0i} + 2 \sum_{j < k \in A_2} v_{jk} \right) = \text{cone} \left( \sum_{v \in A_1} v_{0i} + 2 \sum_{j < k \in A_2} v_{jk} \right),
\]

Clearly, there exists \( A_R \in \{A_1, A_2\} \) with \( 1 \not\in A_R \). From Proposition 5.5 we now infer that the cone \( \sigma_R := \text{cone}(v_0; \ \eta \subseteq \{0\} \cup A_R) \) containing \( \nu_R \) in its relative interior lies in \( \Sigma_1 \).

Note that no two rays lie in the relative interior of the same cone of \( \Sigma_1 \). The above defined collection of rays hence comes with a natural partial order inherited from the fan \( \Sigma_1 \):

\[
\nu_R \leq \nu_S : \iff \sigma_R \subseteq \sigma_S \iff A_R \subseteq A_S.
\]
We choose a linear extension of this partial order. Beginning with the maximal ray we then consider the iterated stellar subdivision of $\Sigma_1$ in all the rays in descending order. The resulting fan we denote by $\Sigma_r$.

While it is not true that $\Sigma_r$ coincides with the GKZ-decomposition $\Sigma = \text{GKZ}(P)$, both fans share a sufficiently large subfan. To make this precise let $T$ be the dense torus of $\mathbb{A}^2 \mathbb{K}^{n+1}$. To $\mathcal{Y} \cap T$ we can associate its tropical variety $\text{Trop}(\mathcal{Y} \cap T)$, which is the support of a quasifan in $\mathbb{A}^2 \mathbb{Q}^{n+1}$. For a detailed description of this space see [19]. For our purposes it suffices to know that the image $\Delta := P(\text{Trop}(\mathcal{Y} \cap T))$ intersects the relative interior cone($v_\eta; \eta \in J$) of a cone if and only if $N_0 \setminus J$ is a $\mathcal{Y}$-set, see [20 Proposition 2.3]. We now define the $\Delta$-reduction of $\Sigma$ as the fan

$$\Sigma^\Delta := \{\sigma; \sigma \preceq \tau \in \Sigma \text{ for some } \tau \text{ with } \tau^\circ \cap \Delta \neq \emptyset\}.$$ 

Note that the relative interiors of all maximal cones of $\Sigma^\Delta$ intersect $\Delta$. Moreover, by [20] Proposition 2.3 the closure of $(\mathcal{Y} \cap T)/H$ in the toric variety corresponding to $\Sigma$ is already contained in the toric subvariety defined by $\Sigma^\Delta \subseteq \Sigma$.

**Proposition 5.7.** The $\Delta$-reduction $\Sigma^\Delta$ is a subfan of $\Sigma_r$.

**Corollary 5.8.** The proper transform of the Mumford quotient $X_1 \subseteq Z_1$ under the toric morphism arising from $\Sigma_r \rightarrow \Sigma_1$ and the limit quotient $\mathcal{Y} \cap H$ share a common normalisation.

**Proof.** For this just note that the following closures coincide and the first morphism is the normalisation map.

$$\mathcal{Y} \cap H \rightarrow (\mathcal{Y} \cap T)/H = (\mathcal{Y} \cap T)/H^{\Sigma^\Delta} = (\mathcal{Y} \cap T)/H^{\Sigma_r}. \square$$

**Remark 5.9.** In fact, with only minor modifications the Step 2 works for every Mumford quotient of $\mathcal{Y}$ which arises from a fulldimensional chamber $\lambda$ lying in $\Omega^*$. The idea of the proof of Proposition 5.7 is to give a combinatorial description of the cones in $\Sigma^\Delta$ and to show that these are geometrically nested in the sense of Section 4.

For the moment let $Q \in \text{Mat}(k, r; \mathbb{Z})$ and $P \in \text{Mat}(n, r; \mathbb{Z})$ be arbitrary Gale dual matrices. We set $R := \{1, \ldots, r\}$. For a subset $I \subseteq R$ we denote by $\gamma_I \subseteq \mathbb{Q}^n$ the cone generated by the $e_i$, $i \in I$ and by $\omega_I := Q(\gamma_I)$ its image under $Q$. Moreover, if $v_i, i \in R$ are the columns of $P$ we set $\sigma_J := \text{cone}(v_j; j \in J)$. A system $\mathcal{B}$ of subsets of $R$ is a separated $R$-collection if any two $I_1, I_2 \in \mathcal{B}$ admit an invariant separating linear form $f$, in the sense that

$$P^*(\mathbb{Q}^n) \subseteq \ker(f), \quad f|_{\gamma_{I_1}} \geq 0, \quad f|_{\gamma_{I_2}} \leq 0, \quad \ker(f) \cap \gamma_{I_1} = \gamma_{I_1} \cap \gamma_{I_2}.$$ 

The separated $R$-collections come with a partial order; for two $R$-collections $\mathcal{B}_1, \mathcal{B}_2$ we write $\mathcal{B}_1 \leq \mathcal{B}_2$ if for every $I_1 \in \mathcal{B}_1$ there exists $I_2 \in \mathcal{B}_2$ such that $I_1 \subseteq I_2$ holds. A separated $R$-collection $\mathcal{B}$ will be called normal if it cannot be enlarged as an $R$-collection and the cones $\omega_I, I \in \mathcal{B}$ form the normal fan of a polyhedron. With respect to the above partial order there exists a unique maximal normal $R$-collection, namely $(R)$ which consists of all subsets which are invariantly separable from $R$. By $\mathcal{M}$ we denote the submaximal normal $R$-collections in the sense, that $(R)$ is the only dominating normal $R$-collection. Finally, for a fixed normal $R$-collection $\mathcal{B}$ let $\mathcal{M}(\mathcal{B})$ consist of those collections of $\mathcal{M}$ lying above $\mathcal{B}$. 
If \( P \) consists of pairwise linearly independent columns, then by \([1]\) Section II.2 there is an order reversing bijection

\[
\{ \text{normal } R\text{-collections} \} \rightarrow \Sigma; \quad B \mapsto \bigcap_{I \in B} \sigma_{R \setminus I}.
\]

where again \( \Sigma = \text{GKZ}(P) \) is the GKZ-decomposition. It is clear that each maximal \( R \)-collection \( A \in \mathcal{M} \) gives rise to a ray \( \nu_A = \bigcap_B \sigma_{R \setminus I} \) of \( \Sigma \).

**Proposition 5.10.** Let \( B \) be a normal \( R \)-collection. Then the cone corresponding to \( B \) can be written as

\[
\bigcap_{I \in B} \sigma_{R \setminus I} = \text{cone}(\nu_B; \ A \in \mathcal{M}(B)).
\]

**Proof.** From the order reversing property of the above bijection it is clear that every ray \( \nu_B \) with \( A \in \mathcal{M}(B) \) lies in \( \sigma := \bigcap_B \sigma_{R \setminus I} \). Moreover, there must exist a set of maximal \( \gamma \)-collections \( \mathcal{N} \subseteq \mathcal{M} \) such that the extremal rays of \( \sigma \) are precisely the \( \nu_B \) with \( A \in \mathcal{N} \). Again from the above bijection we know that this means \( A \geq B \). The assertion then follows from the maximality of \( A \). \( \square \)

We now return to our special case where \( Q = (E_n, D_n) \) holds and the index set \( R \) equals \( N_0 \). We are interested in a description of the submaximal collections \( \mathcal{M}(B) \) where \( B \) consists of \( \mathcal{Y} \)-sets. The reason is the following Proposition.

**Proposition 5.11.** Let \( B \) be a normal \( N_0 \)-collection and suppose that its associated cone \( \bigcap_{I \in B} \sigma_{N_0 \setminus I} \) is a maximal cone in \( \Sigma^\Delta \). Then \( B \) is a collection of \( \mathcal{Y} \)-sets.

**Proof.** Since \( (\bigcap_B \sigma_{N_0 \setminus I})^\circ \cap \Delta \neq \emptyset \) holds the same is true for every \( \sigma_{N_0 \setminus I} \) with \( I \in B \). By \([20]\) Proposition 2.3 this implies that \( B \) is a collection of \( \mathcal{Y} \)-sets. \( \square \)

**Proposition 5.12.** Suppose that \( B \) is a normal \( N_0 \)-collection of \( \mathcal{Y} \)-sets and \( A \in \mathcal{M}(B) \) is a submaximal collection dominating it. Then \( A \) is of one of the following types.

(i) The collections \((I)\) where \( I := N_0 \setminus \{\eta\} \) for some \( \eta \in N \).

(ii) The collections \((I_1, I_2)\) where \( I_i := \{\eta; \ \eta \cap A_i \neq \emptyset\} \) for a two-block partition \( R = \{A_1, A_2\} \) of \( N \).

Moreover, if a collection of the second type lies over \( B \), then \( B \) contains the set \( J_0 := \{\eta; A_i \cap \eta \neq \emptyset \text{ for } i = 1, 2\} \).

Since every collection of the first type is uniquely determined by the element \( \eta \), we write it as \( A_\eta \). The ray \( \vartheta_\eta \) of \( \Sigma \) corresponding to this submaximal collection is generated by \( \nu_\eta \).

If a submaximal collection is of the second type, then it is characterised by the partition \( R \) of \( N \); for it we write \( A_R \). Moreover, the associated ray arises as intersection of \( \sigma_{N_0 \setminus I_1} \) and \( \sigma_{N_0 \setminus I_2} \). We now have to discriminate two cases. If the partition \( R \) is of the form \([i] := \{i\}, N \setminus \{i\}\), then the corresponding ray \( \vartheta_{[i]} = \vartheta_\eta \) is generated by \( \nu_\eta \). Otherwise, if \( R \) is a true two-block partition, by \([5]\) Proposition 4.1 we know that this ray is precisely \( \nu_R \), which was defined at the beginning of Step 2.

**Proof of Proposition 5.12** Consider an \( I \in A \) such that \( \omega_I \) is full dimensional. We now discriminate two cases. For the first case assume that \( \omega_I = \Omega \) holds. Since \( A \) is submaximal, \( I = N_0 \setminus \{\eta\} \) for some \( \eta \in N_0 \). If we had \( 0 \in \eta \), then \( \omega_I \) would be a proper subset of \( \Omega \).
We turn to the second case where \( \omega_J \subseteq \Omega \) holds. Then there exists an \( I' \in \mathfrak{A} \) such that \( \omega_{J'} \) is a facet of \( \omega_I \) and \( \omega_{J'} \cap \Omega' \) is non-empty. Since \( \mathfrak{B} \) cannot be enlarged as \( \mathbb{N}_0 \)-collection, there moreover exist \( J, J' \in \mathfrak{B} \) such that

\[
\omega_{J'} \subseteq \omega_J, \quad \omega_{J'} \text{ is a facet of } \omega_J, \quad \omega_{J'} \cap \omega_{J'} \neq \emptyset.
\]

Now \( \omega_{J'} \) is a subset of one of the walls of \( \Lambda_H(\mathfrak{Y}) \). Thus, from Theorem 5.3, we know that there exists some partition \( \{A_1, A_2\} \) of \( N \) such that \( J' \) is a subset of \( J_0 := \{\eta; A_i \cap \eta \neq \emptyset \text{ for } i = 1, 2\} \). We now claim that \( J' \) equals \( J_0 \).

For this let \( i_1 \in A_1, i_2 \in A_2 \) be two indices. Since \( \omega_{J'} \) is of dimension \( n - 1 \), there exist \( i_1' \in A_1, i_2' \in A_2 \) such that \( \{i_1, i_1'\} \) and \( \{i_2, i_2'\} \) lie in \( J' \). From the inclusion \( J' \subseteq J_0 \) we know that \( \{i_1, i_1'\} \) does not lie in \( J' \), hence from the characterisation of \( \mathfrak{Y} \)-sets in Proposition 5.4 it follows that \( \{i_1, i_2\} \) lies in \( J' \). This proves our claim.

Now let \( \mathfrak{A}' \) be the normal \( \mathfrak{R} \)-collection consisting of all faces which are invariantly separable from

\[ \{\eta; \eta \cap A_1 \neq \emptyset\} \text{ and } \{\eta; \eta \cap A_2 \neq \emptyset\}. \]

Then \( \mathfrak{A}' \) is submaximal and the assertion follows if we show that \( \mathfrak{A} \subseteq \mathfrak{A}' \) holds. For this note that \( \omega_{J'} \) is the intersection of \( \Omega \) with the zero set of

\[
l := \sum_{i \in A_1} e_i^* - \sum_{i \in A_2} e_i^*.
\]

Since the collection \( \{\omega_K; K \in \mathfrak{A}\} \) forms a fan with support \( \Omega \), for every cone \( \omega_K, K \in \mathfrak{A} \) we have \( l_{|\omega_K} \geq 0 \) or \( l_{|\omega_K} \leq 0 \). This implies that \( \mathfrak{A} \subseteq \mathfrak{A}' \) holds.

\[ \square \]

Recall that we want show that the (maximal) cones of \( \Sigma^\mathfrak{A} \) are geometrically nested in the sense of Section 4 and hence lie in \( \Sigma_r \). The relevant property of the corresponding \( \mathbb{N}_0 \)-collections shall be discusses in the sequel.

Let \( R = \{A_1, A_2\} \) and \( S = \{B_1, B_2\} \) be two-block partitions of \( N \) and \( \eta \in \mathbb{N}_0 \). We then call the pair \( \{\eta, R\} \) compatible if \( \eta \) lies in \( A_1 \) or in \( A_2 \). Moreover, we call \( \{R, S\} \) compatible, if there exist \( i, j \in \{1, 2\} \) such that \( A_i \subseteq B_j \) holds. The pairs of submaximal collections \( \{\mathfrak{A}_\eta, \mathfrak{A}_R\} \) and \( \{\mathfrak{A}_R, \mathfrak{A}_S\} \) are compatible, if the corresponding pairs \( \{\eta, R\} \) and \( \{R, S\} \) are compatible.

**Proposition 5.13.** Let \( \mathfrak{B} \) be a normal \( \mathbb{N}_0 \)-collection of \( \mathfrak{Y} \)-sets. Then the submaximal collections in \( \mathcal{M}(\mathfrak{B}) \) are pairwise compatible.

**Proof.** Let \( \mathfrak{A}_\eta, \mathfrak{A}_R \geq \mathfrak{B} \) be two submaximal collections with \( R = \{A_1, A_2\} \). Then \( \{i, j\} := \eta \) is contained in no \( I \in \mathfrak{B} \). However, the cones \( \omega_I, I \in \mathfrak{B} \) cover \( \Omega \). Since \( w_{ij} = w_{0i} + w_{0j} \) is the only positive linear combination of \( w_{ij} \), the sets \( \{0, i\}, \{0, j\} \) must lie in a common \( I \in \mathfrak{B} \). From the characterisation in Proposition 5.12(ii) we can now infer that without loss of generality \( i, j \in A_1 \) holds and this implies compatibility of \( \eta \) with \( R \).

Suppose we have \( \mathfrak{A}_R, \mathfrak{A}_S \geq \mathfrak{B} \) with \( R = \{A_1, A_2\} \) and \( S = \{B_1, B_2\} \). From Proposition 5.12 we infer that the set \( J_0 = \{\eta; \eta \cap A_i \neq \emptyset \text{ for } i = 1, 2\} \) lies in \( \mathfrak{B} \). This means that \( J_0 \) lies in one of the maximal sets of \( \mathfrak{A}_S \). In other words, there exists \( j \) such that

\[ \eta \cap A_1 \neq \emptyset \text{ and } \eta \cap A_2 \neq \emptyset \implies \eta \cap B_j \neq \emptyset. \]

This implies that there exists \( i \) such that \( A_i \subseteq B_j \) holds and hence \( \{R, S\} \) is compatible. \( \square \)
The final thing we show is that the cones defined by compatible submaximal collections are geometrically nested in the sense of Section 4. For this we define $S$ as the collection of two-block partitions of $N$ and set $S_{\geq 2}$ as the subcollection of true two-block partitions; i.e. the partitions $\{A_1, A_2\}$ with $|A_1|, |A_2| \geq 2$.

We set $V = \{v_\eta; \eta \in N_0\}$ as the set of rays of $\Sigma_1$. Keep in mind that the rays $v_\eta$ stem from the partitions $[i] = \{(i), N \setminus \{i\}\}$, hence we have $v_0 = v_{[i]}$.

Moreover, we define $S := \{\sigma_R; R \in S_{\geq 2}\}$ as the collection of cones in $\Sigma_1$ associated to true two-block partitions. This is precisely the collection of cones containing the rays $v_R$ in their relative interiors.

**Lemma 5.14.** Consider the collection of cones

$C := \{v_\eta, \sigma_R; A_0, A_R \in N\} \text{ for some } N \subseteq \{A_0, A_R; \eta \in N, R \in S\}$.

If any pair in $N$ is compatible, then $C$ is geometrically nested in $V \cup S$.

**Proof.** Consider a subset $H \subseteq C$ of incomparable elements with $|H| \geq 2$. Moreover, take $S' \subseteq S$ to be a non-empty conjunct subset. Assuming that

$$\sigma := \sum_{\tau \in S'} \tau = \sum_{\tau \in H} \tau \in \Sigma_1$$

holds we have to show that there exist an incompatible pair in $N$.

Recall that the cones of $S = \{\sigma_R; R \in S_{\geq 2}\}$ have the form

$$\sigma_R = \text{cone}(v_\eta; \eta \subseteq \{0\} \cup A_R) \text{ where } 1 \notin A_R \in R.$$  

Consider two cones $\sigma_1, \sigma_2 \in \Sigma_1$ such that their sum lies in $\Sigma_1$ as well. Since $\Sigma_1$ is simplicial, the rays of $\sigma_1 + \sigma_2$ are precisely given by the union of the rays of $\sigma_1$ and $\sigma_2$. In particular, if $\varrho$ is a ray of some $\tau \in H$, then there exists $\tau' \in S'$ such that $\varrho$ is a ray of $\tau'$. Clearly, the same is true with $H$ and $S'$ exchanged.

If $|H \cap S| = 0$ holds, i.e. $H$ is a subset of $V$, then one easily sees that there exist $v_{ij_1}, v_{ij_2} \in H$. Clearly, $[i]$ and $\{i, j\}$ are incompatible, hence $A_{[i]}, A_{\{i, j\}} \in N$ are the incompatible partitions.

We consider the case $|H \cap S| = 1$ and denote the single cone in $H \cap S$ by $\sigma_R$. Since $|H| \geq 2$ holds there exists an element $\varrho \in H \cap V$. We distinguish two subcases.

In the first case let this ray be of the form $\varrho = v_{ij}$. Then we find $\sigma_S \in S'$ with $\varrho \preceq \sigma_S$. From the special form of the cone $\sigma_R$ we know that there also exists $j \in N$ with $v_{ij_j} \preceq \sigma_S$. By the assumption made on $H$ we have $v_{ij_j} \not\preceq \sigma_R$; and the special form of $\sigma_R$ then means that also $v_{ij_j} \not\preceq \sigma_R$ holds. Hence $v_{ij_j}$ lies in $H$ and $A_{[i]}, A_{\{i, j\}} \in N$ are the incompatible collections.

In the second case where $\varrho = v_{ij}$ holds we again find $\sigma_S \in S'$ with $v_{ij} \preceq \sigma_S$. From the special form of the cone $\sigma_S$ we know that both $v_{ij} \subseteq \sigma_S$. Since $v_{ij} \not\preceq \sigma_R$ holds, at least one of the rays $v_{ij_1}, v_{ij_2}$ is not a ray of $\sigma_R$. Without loss of generality this implies that again $v_{ij_1}$ lies in $H$ and $A_{[i]}, A_{\{i, j\}} \in N$ are the incompatible collections.

Now we assume that $|H \cap S| \geq 2$ holds. Then there exist $\sigma_R_1, \sigma_S \in H \cap S$. For $\eta, \zeta \in N$ let $v_\eta \preceq \sigma_R$ and $v_\zeta \preceq \sigma_S$ be rays such that $v_\eta \not\preceq \sigma_S$ and $v_\zeta \not\preceq \sigma_R$ hold.

Since $S'$ is conjunct, we find $\xi_1, \ldots, \xi_r \in N$ with

$$\xi_1 = \eta, \quad \xi_r = \zeta, \quad v_{\xi_1} \preceq \sigma \quad \text{and} \quad \xi_i \cap \xi_{i+1} \neq \emptyset.$$  

Let $i'$ be the smallest index, for which $v_{\xi_i'}$ is not a ray of $\sigma_R$. If $v_{\xi_i'}$ lies in $H$, then we know $v_{\xi_i'} \not\preceq \sigma_R$ holds. This means that $\xi_i'$ and $R$ are incompatible. If $v_{\xi_i'}$ does not lie in $H$, then there exists an $\sigma_R \in H$ such that $v_{\xi_i'}$ is a ray of $\sigma_R$. This implies that $R$ and $R'$ are incompatible.  \[ \square \]
Proof of Proposition 5.7. Consider the cone $\sigma \in \Sigma^\Delta$. In order to show show that $\sigma$ lies in $\Sigma_r$ we can without loss of generality assume that $\sigma$ is maximal. Let $\mathcal{B}$ be the associated normal $\mathbb{N}_0$-collection with

$$\sigma = \bigcap_{I \in \mathcal{B}} \sigma_{\mathbb{N}_0 \setminus I}.$$ 

By Propositions 5.11, 5.13 we know that $\mathcal{M}(\mathcal{B})$ is a set of compatible normal $\mathbb{N}_0$-collections. Furthermore, by Proposition 5.10

$$\sigma = \text{cone}(\nu_\mathcal{A}; \mathcal{A} \in \mathcal{M}(\mathcal{B})) = \text{cone}(\rho, \nu_R; \rho \in \mathcal{V} \cap \mathcal{C}, \sigma_R \in S \cap \mathcal{C})$$

holds. Lemma 5.14 shows that $C = \{\rho_\eta, \sigma_R; \mathcal{A}_\eta, \mathcal{A}_R \in \mathcal{M}(\mathcal{B})\}$ is geometrically nested in $\mathcal{V} \cup \mathcal{S}$. And finally, from Proposition 4.1 we infer that $\sigma$ lies in $\Sigma_r$. □

Proof of Theorem 5.1. The Theorem now follows directly from Proposition 5.3 and Corollary 5.8. □

Proof of Theorem 5.2. As in the reductive case we performed the first blow-up in Proposition 5.3. In Proposition 5.4 determined the subset of $X_1$ that has to be removed due to the fact that the morphism $\kappa: \mathbb{K}^{2n} \to \bigwedge^2 \mathbb{K}^{n+1}$ is not surjective. Finally the remaining blow-ups are performed as in the reductive case, see Corollary 5.8. □

References

[1] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface. Cox rings. 2011. arXiv:1003.4229v2 [math.AG].
[2] I. V. Arzhantsev, D. Celik, and J. Hausen. Factorial algebraic group actions and categorical quotients. J. Algebra, 387:87–98, 2013.
[3] I. V. Arzhantsev and J. Hausen. Geometric invariant theory via cox rings. Journal of Pure and Applied Algebra, 213:154–172, 2009.
[4] H. Bäker. On the Cox ring of blowing up the diagonal. preprint: arXiv:1402.5509.
[5] H. Bäker, J. Hausen, and S. Keicher. On Chow quotients of torus actions. preprint: arXiv:1203.3759.
[6] F. Berchtold and J. Hausen. GIT-Equivalence beyond the ample cone. Michigan Mathematical Journal, 54(3):483–515, 2006.
[7] D. Cox. The homogeneous coordinate ring of a toric variety. J. Algebraic Geom, 4(1):17–50, 1995.
[8] I. V. Dolgachev and Y. Hu. Variation of geometric invariant theory quotients. Publications Mathematiques De L Ihes, 85:7–51, 1998.
[9] B. Doran and F. Kirwan. Towards non-reductive geometric invariant theory. Pure Appl. Math. Q., 3(1, part 3):61–105, 2007.
[10] E.-M. Feichtner and D. N. Kozlov. Incidence combinatorics of resolutions. Selecta Math. (N.S.), 10(1):37–60, 2004.
[11] I. M. Gel’fand, M. M. Kapranov, and A. V. Zelevinsky. Discriminants, resultants and multidimensional determinants. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2008. Reprint of the 1994 edition.
[12] J. Hausen. Geometric invariant theory based on Weil divisors. Compos. Math., 140(6):1518–1536, 2004. MR2098480.
[13] J. Hausen. Cox rings and combinatorics II. Moscow Mathematical Journal, 8(4):711–757, 2008.
[14] E. Huggenberger. Fano Varieties with Torus Action of Complexity One. Doctoral Thesis http://urn:nbn:de:bsz:21-opus-69570.
[15] M. Kapranov, B. Sturmfels, and A. Zelevinsky. Quotients of toric varieties. Mathematische Annalen, 290(4):643–655, 1991.
[16] M. M. Kapranov. Chow quotients of Grassmannians. I. In I. M. Gel’fand Seminar, volume 16 of Adv. Soviet Math., pages 29–110. Amer. Math. Soc., Providence, RI, 1993. MR1237834.
[17] A. Losev and Y. Manin. New moduli spaces of pointed curves and pencils of flat connections. Michigan Math. J., 48:443–472, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.
[18] D. A. Shmelkin. First fundamental theorem for covariants of classical groups. *Adv. Math.*, 167(2):175–194, 2002.
[19] D. Speyer and B. Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.
[20] J. Tevelev. Compactifications of subvarieties of tori. *Amer. J. Math.*, 129(4):1087–1104, 2007.

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