SINGULAR EIGENVALUE PROBLEMS ON THE CIRCLE

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Abstract. The eigenvalue problem on the circle for the non-self-adjoint operators \( L_m(V) = (-1)^m \frac{d^{2m}}{dx^{2m}} + V, \ m \in \mathbb{N} \) with singular complex-valued 2-periodic distributions \( V \in H^{m}_{per}[-1,1] \) is studied. Asymptotic formulae for the eigenvalues uniformly in \( V \) in the space \( H^m_{per}[-1,1] \) and local uniformly in \( V \) in the space \( H^-m_{per}[-1,1] \) are found.

1. Introduction

Let introduce the complex Sobolev spaces \( H^s_{per}[-1,1] \), \( s \in \mathbb{R} \), of 2-periodic functions or distributions. They are defined by means their Fourier coefficients:

\[
H^s_{per}[-1,1] := \left\{ f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ik\pi x} \mid \| f \|_s < \infty \right\},
\]

where

\[
\| f \|_s := \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} | \hat{f}(k) |^2 \right)^{1/2}, \quad \langle k \rangle := 1 + |k|,
\]

\[
\hat{f}(k) := \langle f, e^{ik\pi x} \rangle, \quad k \in \mathbb{Z}.
\]

The brackets denote the sesquilinear pairing between dual spaces \( H^s_{per}[-1,1] \) and \( H^{-s}_{per}[-1,1] \) extending the \( L^2_{per}[-1,1] \)-inner product

\[
\langle f, g \rangle := \frac{1}{2} \int_{-1}^{1} f(x) \overline{g(x)} \, dx, \quad f, g \in L^2_{per}[-1,1].
\]

In the paper we study the eigenvalue problem for the non-self-adjoint differential operators

\[
(1.1) \quad L \equiv L_m(V) := (-1)^m \frac{d^{2m}}{dx^{2m}} + V, \quad m \in \mathbb{N}
\]

with the singular complex-valued distributions \( V(x) \in H^{m}_{per}[-1,1] \). The operators \( L \) are well defined as unbounded linear operators in the Hilbert space \( H^{-m}_{per}[-1,1] \) with the inner product

\[
\langle f, g \rangle_{-m} := \sum_{k \in \mathbb{Z}} \langle k \rangle^{-2m} \hat{f}(k) \overline{\hat{g}(k)}
\]

and the domain

\[
D(L) = H^{m}_{per}[-1,1].
\]

Similarly as for the functions \( V(x) \in H^0_{per}[-1,1] \), it turns out that the spectrum \( \text{spec}(L) \) of the operator \( L \) when considered on the interval \([-1,1]\) and with periodic boundary conditions is discrete and consists of a sequence of the eigenvalues

\[
\lambda_k = \lambda_k(m, V), \quad k \in \mathbb{Z}_+
\]

with the property that

\[
(1.2) \quad \text{Re}\lambda_k \to +\infty \quad \text{as} \quad k \to +\infty.
\]

Here the eigenvalues \( \lambda_k \) are enumerated with their algebraic multiplicities and ordered lexicographically, so that

\[
\text{Re}(\lambda_k) < \text{Re}(\lambda_{k+1}), \quad \text{or} \quad \text{Re}(\lambda_k) = \text{Re}(\lambda_{k+1}) \text{ and } \text{Im}(\lambda_k) \leq \text{Im}(\lambda_{k+1}), \quad k \in \mathbb{Z}_+.
\]
One can prove that operator \( L \) is self-adjoint in the Hilbert space \( H_{per}^{-m}[-1,1] \) if and only if the 2-periodic distribution \( V(x) \) is real-valued, i.e.

\[
\hat{V}(n) = \overline{V(-n)}, \quad n \in \mathbb{Z}.
\]

The aim of the paper is to find asymptotic formulae for the eigenvalues \((\lambda_k)_{k \geq 0}\) which are uniform in \( V \) on appropriate sets of distributions. The case \( m = 1 \) was studied in [2, 6] using the same approach.

The following two theorems are the main results of the paper.

**Theorem 1.1.** For any \( V \in H_{per}^{-m}[-1,1] \), there exist \( \varepsilon > 0 \), \( M = M(\varepsilon) \geq 1 \) and \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) such that for any \( W \in H_{per}^{-m}[-1,1] \) with

\[
\|W - V\|_{-m} \leq \varepsilon
\]

the spectrum of the operator \( L_m(W) \) satisfies the estimates:

(a) There are precisely \( 2n_0 - 1 \) eigenvalues inside the bounded cone

\[
T_{M,n_0} = \{ \lambda \in \mathbb{C} \mid |Im \lambda| - M \leq Re \lambda \leq (n_0^{2m} - n_0^m)\pi^{2m}\}.
\]

(b) For any \( n \geq n_0 \) the pairs of eigenvalues \( \lambda_{2n-1}(W), \lambda_{2n}(W) \) are inside a disc around \( n^{2m} \pi^{2m} \) of the radius \( n^m \):

\[
|\lambda_{2n-1}(W) - n^{2m} \pi^{2m}| < n^m,
\]

\[
|\lambda_{2n}(W) - n^{2m} \pi^{2m}| < n^m.
\]

In the case \( m = 1 \) theorem 1.1 has been proved in [6] (see also [2]).

**Theorem 1.2.** Let \( R \geq 0, V \in H_{per}^{-m}[-1,1] \). For any \( W \in H_{per}^{-m}[-1,1] \) with

\[
\|W - V\|_{-m} \leq R
\]

the eigenvalues \((\lambda_k(m,W))_{k \geq 0}\) satisfy the asymptotic formula

\[
(1.3) \quad \lambda_{2n-1}(m,W), \lambda_{2n}(m,W) = n^{2m} \pi^{2m} + o(n^m), \quad n \to \infty
\]

uniformly in \( W \).

The estimates (1.3) is novel in the case \( m = 1 \) as well. In the case \( m = 1 \) and \( V \in H_{per}^{k}[-1,1], k \in \mathbb{Z}_+ \) an asymptotic behavior of the eigenvalues \( \lambda_k(1,V) \) has been investigated in monograph [4]. The case \( V \in H_{per}^{-ma}[-1,1], \alpha \in [0,1) \) was earlier studied in [5].

### 2. Preliminary Results

A purpose of this section to prove some qualitative results concerning the operator \( L_m(V) \). More precisely we are going to prove the following statement.

**Theorem 2.1.** The operator \( L_m(V) \), \( m \in \mathbb{R} \), \( V(x) \in H_{per}^{-m}[-1,1] \) is well defined as an unbounded linear operator in the Hilbert space \( H_{per}^{-m}[-1,1] \) with the domain

\[
D(L_m) = H_{per}^{-m}[-1,1].
\]

Moreover:

(a) The operator \( L_m(V) \) is closed.

(b) A resolvent set of the operator \( L_m(V) \) is not empty and the resolvent \( R(\lambda, L_m(V)) \) is a compact operator.

To prove the theorem 2.1 we need two preliminary lemmas.

As well known the space \( L_{per}^2[-1,1] \) can be isometric identified with the sequence space \( l^2(\mathbb{Z}) \) by means of Fourier coefficients of a function \( f(x) \in H_{per}^{0}[-1,1] \). Similarly by the Fourier transform the spaces \( H_{per}^{s}[-1,1] \) identifies with sequence spaces. For any \( n \in \mathbb{Z} \), and \( s \in \mathbb{R} \) we can define the weighted \( l^2 \)-spaces by

\[
h^{s,n} = h^{s,n}(\mathbb{Z}; \mathbb{C}).
\]

This space is the Hilbert space sequences \((a(k))_{k \in \mathbb{Z}} \) in \( \mathbb{C} \) with norm

\[
\|a\|_{h^{s,n}} := \left( \sum_{k \in \mathbb{Z}} |k+n|^{2s} |\hat{a}(k)|^2 \right)^{1/2}.
\]

For \( n = 0 \) we will simply write \( h^s \) instead of \( h^{s,0} \).
Further, the map
\[ f \mapsto (\hat{f}(k))_{k \in \mathbb{Z}} \]
is an isometric isomorphism of the space \( H^s_{per}[-1,1] \) onto \( h^s \), \( s \in \mathbb{R} \). For this isomorphism, multiplication of functions corresponds to convolution of sequences, where the convolution product of two sequences \( a = (a(k))_{k \in \mathbb{Z}} \) and \( b = (b(k))_{k \in \mathbb{Z}} \) (formally) defined as the sequence given by
\[ (a * b)(k) := \sum_{j \in \mathbb{Z}} a(k-j)b(j). \]
(2.1)

So, given two functions \( u, v \) formally,
\[ (u \cdot v)(k) = \sum_{j \in \mathbb{Z}} \hat{u}(k-j) \hat{v}(j). \]
(2.2)

The following Convolution Lemma is a starting point of our method.

**Lemma 2.2** (Convolution Lemma, [2]). Let \( n \in \mathbb{Z}, s, r \geq 0, \) and \( t \in \mathbb{R} \) with \( t \leq \min(s, r) \). If \( s + r - t > 1/2 \), then the convolution map is continuous (uniformly in \( n \)), when viewed as a map
\[
\begin{align*}
(a) & \quad h^{r-n} \times h^{n-r} \longrightarrow h^t, \\
(b) & \quad h^{-t} \times h^{r,n} \longrightarrow h^{-r,n}, \\
(c) & \quad h^{t} \times h^{-r,n} \longrightarrow h^{-r,n}.
\end{align*}
\]

So, the map
\[
H^{-m}_{per}[-1,1] \times H^{m}_{per}[-1,1] \longrightarrow H^{-m}_{per}[-1,1], \quad (V, f) \mapsto V \cdot f
\]
is continuous, when \( V \cdot f \) is given by formula (2).

Now, we can define the operator \( L_m(V) \), which is given by the differential expression
\[
l[n] := (-1)^m \frac{d^{2m}}{dx^{2m}} + V, \quad m \in \mathbb{N}
\]
with singular complex-valued potentials \( V \) in \( H^{-m}_{per}[-1,1] \). The operator \( L_m(V) \) is well defined as an unbounded linear operator in \( H^{-m}_{per}[-1,1] \) with the dense domain
\[
H^{m}_{per}[-1,1].
\]

Really, the derivative operator
\[
(-1)^m \frac{d^{2m}}{dx^{2m}} : H^{m}_{per}[-1,1] \rightarrow H^{-m}_{per}[-1,1]
\]
and the multiplication operator \( f \rightarrow Vf \) maps the space \( H^{m}_{per}[-1,1] \) into \( H^{-m}_{per}[-1,1] \) by (2.1).

**Lemma 2.3.** The multiplication operator \( V \) is \( L_m(0) \)-bounded and its relative-bound is 0, i.e. \( V \ll L_m(0) \).

**Proof.** According to the Convolution Lemma there exists the constant \( C_m > 0 \) such that
\[
\| Vu \|_{-m} \leq C_m \| V \| \| u \|_{-m}, \quad V \in H^{m}_{per}[-1,1], u \in H^{-m}_{per}[-1,1].
\]
Further, for any fixed \( \delta > 0 \) there exists a decomposition
\[
V = V_0 + V_\delta
\]
with
\[
V_0 \in H^{m}_{per}[-1,1], V_\delta \in H^{-m}_{per}[-1,1], \| V_\delta \|_{-m} \leq \frac{\delta}{C_m}.
\]
Taking to account that
\[
\| u \|_{-m} \leq \| u \|_{-m} + \| L_m(0) u \|_{-m}, \quad u \in H^{m}_{per}[-1,1]
\]
then we have the following estimates:
\[
\| Vu \|_{-m} \leq \| V_0 u \|_{-m} + \| V_\delta u \|_{-m} \leq C_m \| V_0 \| \| u \|_{-m} + \| V_\delta \| \| u \|_{-m} \leq \delta \| L_m(0) u \|_{-m} + (C_m \| V_0 \| + \delta) \| u \|_{-m}.
\]
Hence \( V \ll L_m(0) \).
Now we can prove Theorem 2.1. Statement (a) is a consequence of Lemma 2.3 and Theorem 1.11 ([3], Ch. IV) since the operator $L_m(0)$ is self-adjoint. Similarly statement (c) is a consequence of Lemma 2.3 and Theorem 3.17 ([3], Ch. IV) since a resolvent $R(\lambda, L_m(0))$ is a compact operator.

Remark that using the perturbation results ([1], Ch. V, §11) and [3, 7] one can prove the following statement

**Theorem 2.4.** For any $\varepsilon > 0$ the spectrum of the operator $L_m(V)$ belongs to the cone

$$S_\varepsilon := \{ \lambda \in \mathbb{C} \mid |\arg \lambda| < \varepsilon \}$$

except a finite number of the eigenvalues. The asymptotic formula

$$\lambda_n(m, V) \sim \lambda_n(m, 0), \quad n \to \infty$$

is valid.

Obviously that assertions of Theorems 1.1 and 1.2 are much stronger. Therefore the proof of Theorem 2.4 is omitted.

3. **Proofs of the Main Theorems**

To prove the theorems 1.1 and 1.2 it is useful to deal with the eigenvalue problem for the operator $\hat{L}_m(v)$ in the sequence Hilbert space $h^{-m}(\mathbb{Z})$. This operator has the same spectrum and is of the form

$$\hat{L}_m = D_m + B(v),$$

where $D_m$ and $B(v)$ are infinite matrices,

$$D_m(k, j) := k^{2m} \pi^{2m} \delta_{kj},$$

$$B(k, j) := v(k - j), \quad k, j \in \mathbb{Z}$$

with $v(k) := \hat{V}(k)$ in the sequence space $h^{-m}$. By the Convolution Lemma

$$B(v) \ast a = v \ast a$$

is well defined for $a \in h^m$ and hence the operator $\hat{L}_m(v) = D_m + B(v)$ is well defined as an unbounded linear operator in the Hilbert space $h^{-m}$ with the domain

$$D(\hat{L}_m) = h^m.$$

The eigenvalue problem

$$\hat{L}_m(v)f = \lambda f, \quad f \in h^m$$

is studied in this section. For this purpose we will compare the spectrum $\text{spec}(D_m + B(v))$ of the operator $\hat{L}_m(v) = D_m + B(v)$ to the spectrum of the operator $D_m$. It is clearly that

$$\text{spec}(D_m) = \{ k^{2m} \pi^{2m} | k \in \mathbb{Z}_+ \},$$

where the eigenvalue 0 is simple and other eigenvalues are double.

For given $M \geq 1$, $n \geq 1$, and $0 < r_n < n^m \pi^{2m}$ the following regions $\text{Ext}_M$ and $\text{Vert}_n^m(r)$, $m \in \mathbb{N}$ of complex plane will be used:

$$\text{Ext}_M := \{ \lambda \in \mathbb{C} \mid \Re \lambda \leq |\Im \lambda| - M \},$$

$$\text{Vert}_n^m(r_n) := \{ \lambda = n^m \pi^{2m} + z \in \mathbb{C} \mid |\Re z| \leq n^m \pi^{2m}, \; |z| \geq r_n \}.$$  

Let formulate the result which we will use bellow.

**Proposition 3.1** ([5]). Let $R > 0$. There exist $M \geq 1$ and $n_0 \in \mathbb{N}$ so that for any $v \in h^0$ with

$$\|v\|_{h^0} \leq R$$

the spectrum $\text{spec}(D_m + B(v))$ of the operator $\hat{L}_m(v) = D_m + B(v)$ with $B(v) = v \ast \cdot$ consists of a sequence $(\lambda_k(v))_{k \geq 0}$ such that:

(a) There are precisely $2n_0 - 1$ eigenvalues inside the bounded cone

$$T_{M, n_0} = \{ \lambda \in \mathbb{C} \mid |\Im \lambda| - M \leq |\Re \lambda| \leq (n_0^m - n_0^m) \pi^{2m} \}.$$

(b) For $n \geq n_0$ the pairs of eigenvalues $\lambda_{2n-1}(m, v)$, $\lambda_{2n}(m, v)$ are inside a disc around $n^m \pi^{2m}$,

$$|\lambda_{2n-1}(v) - n^{2m} \pi^{2m}| < (3^m \sqrt{2} + 1)R,$$

$$|\lambda_{2n}(v) - n^{2m} \pi^{2m}| < (3^m \sqrt{2} + 1)R.$$
In a straightforward way, one can prove the following two auxiliary lemmas.

**Lemma 3.2.** For any $s, t \in \mathbb{R}$ with $s - t \leq 2$ and any $\lambda \in \mathbb{C} \setminus \text{spec}(D_m)$, $m \in \mathbb{N}$ we have $(\lambda - D_m)^{-1} \in \mathcal{L}(h^m, h^m)$ with norm
\[
\| (\lambda - D_m)^{-1} \|_{\mathcal{L}(h^m, h^m)} = \sup_{k \in \mathbb{N}} \frac{\lambda - k^2\pi^2}{|k^2|} < \infty.
\]

**Lemma 3.3.** Uniformly for $n \in \mathbb{Z} \setminus \{0\}$ and $\lambda \in \text{Vert}_n^m(r_n)$
\[
\begin{align*}
(a) \quad & (\lambda - D_m)^{-1} \mathcal{L}(h^{-n}) = \frac{1}{r_n} O(1), \\
(b) \quad & (\lambda - D_m)^{-1} \mathcal{L}(h^{-n+1}) = \frac{1}{r_n} O(1), \\
(c) \quad & (\lambda - D_m)^{-1} \mathcal{L}(h^{-n}, h^{-n+1}) = \frac{1}{r_n} O(1), \\
(d) \quad & (\lambda - D_m)^{-1} \mathcal{L}(h^{-n}, h^{-n+1}) = \frac{n^2}{r_n} O(1), \\
(e) \quad & (\lambda - D_m)^{-1} \mathcal{L}(h^{-n}, h^{-n+1}) = O(1).
\end{align*}
\]

**Proposition 3.4.** Let $v \in h^{-m}$. There exist $\varepsilon > 0$, $M \geq 1$ and $n_0 \in \mathbb{N}$ (both depending on $\varepsilon$) so that for any $w \in h^{-m}$ with
\[
\| w - v \|_{h^{-m}} \leq \varepsilon
\]
the spectrum $\text{spec}(D_m + B(w))$ of the operator
\[
\hat{L}_m(w) = D_m + B(w)
\]
with $B(w) = w * \cdot$ consists of a sequence $(\lambda_k(m, w))_{k \geq 0}$ such that:
\[
\begin{align*}
(a) \quad & \text{There are precisely } 2n_0 - 1 \text{ eigenvalues inside the bounded cone} \\
& T_{M, n_0} = \{ \lambda \in \mathbb{C} \mid |Im\lambda| - M \leq Re\lambda \leq (n_0^2 - n_0^m)\pi^2m \}, \\
(b) \quad & \text{For } n \geq n_0 \text{ the pairs of eigenvalues } \lambda_{2n-1}(m, w), \lambda_{2n}(m, v) \text{ are inside a disc around } n^2 \pi^2 m, \\
& |\lambda_{2n-1}(m, w) - n^2 \pi^2 m| < n^m, \quad i = 0, 1.
\end{align*}
\]

**Proof.** Let $v \in h^{-m}$. Since the set $h^m$ is dense in the space $h^{-m}$, we can represent $v$ in the form
\[
v = v_0 + v_1,
\]
with $v_0 \in h^m$ and $\| v_1 \|_{h^{-m}} \leq \varepsilon$, where $\varepsilon > 0$ will be found below. We will show that for some $M \geq 1$ and $n_0 \in \mathbb{N}$, which both depending on $\| v_0 \|_{h^m}$, so that for any $w = v + \tilde{w} \in h^{-m}$ with $\| \tilde{w} \|_{h^{-m}} \leq \varepsilon$,
\[
\text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}_n^m(n^{2m}) \subseteq \text{Resol}(\hat{L}_m(w)),
\]
where $\text{Resol}(\hat{L}_m(w))$ denotes the resolvent set of the operator $\hat{L}_m(w) = D_m + B_0 + B_1$, and $B_0 = v_0 * \cdot$, and $B_1 = (v_1 + \tilde{w}) * \cdot$.

At first let consider $\lambda \in \text{Ext}_M$ for $M \geq 1$. Using the Convolution Lemma and the Lemma 3.2 one gets
\[
\| B_0(\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m})} \leq C_m \| v_0 \|_{h^m} \| (\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m})} = \| v_0 \|_{h^m} O(M^{-1}).
\]
Hence, for $M \geq 1$ large enough and $\lambda \in \text{Ext}_M$,
\[
L_\lambda := \lambda - D_m - B_0 = (Id - B_0(\lambda - D_m)^{-1})(\lambda - D_m)
\]
is invertible in $\mathcal{L}(h^{-m})$ with inverse
\[
L_\lambda^{-1} = (\lambda - D_m)^{-1}(Id - B_0(\lambda - D_m)^{-1})^{-1}.
\]
So, using the Convolution Lemma and the estimate
\[
\| (\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m}, h^m)} = O(1),
\]
we obtain
\[
\| B_0 L_\lambda^{-1} \|_{\mathcal{L}(h^{-m})} = O(\| (v_1 + \tilde{w}) \|_{h^{-m}} = O(\varepsilon)).
\]
Therefore, if $\varepsilon > 0$ is small enough, the resolvent of the operator
\[
\hat{L}_m(w) = D_m + B_0 + B_1
\]
exists in the space \( \mathcal{L}(h^{-m}) \) for \( \lambda \in \text{Ext}_M \) and is given by the formula
\[
(\lambda - D_m - B_0 - B_1)^{-1} = (L^{-1}_\lambda - B_1)^{-1} = L^{-1}_\lambda \Sigma_{k \geq 0} (B_1 L^{-1}_\lambda)^k.
\]
Consequently, for \( M \) large enough, \( \text{Ext}_M \subseteq \text{Resol}(\hat{L}_m(w)) \).

To treat \( \lambda \in \text{Vert}^m_n(n^m) \), first note that, unfortunately,
\[
\| (\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m}, h^m)} = O(n^m),
\]
and so we can not argue as above. However, we have (see the Lemma 3.3 (e'))
\[
\| (\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m, n}, h^m, n)} = O(1).
\]
Now, for \( \lambda \in \text{Vert}^m_n(n^m) \) with \( n \) large enough, we find that the following decomposition of the resolvent of \( \hat{L}_m(w) = D_m + B_0 + B_1 \) converges in the space \( \mathcal{L}(h^{-m}) \),
\[
(\lambda - D_m - B_0 - B_1)^{-1} = L^{-1}_\lambda + L^{-1}_\lambda T_\lambda (B_1 L^{-1}_\lambda) + L^{-1}_\lambda T_\lambda (B_1 L^{-1}_\lambda)^2,
\]
where \( L_\lambda = \lambda - D_m - B_0 \), and \( T_\lambda := \Sigma_{l \geq 0} (B_1 L^{-1}_\lambda)^{2l} \) is considered as an element in \( \mathcal{L}(h^{-m, n}) \). Using the Convolution Lemma (c) and the Lemma 3.3 (a'), (b') we can find \( n_0 \in \mathbb{N} \) such that, for any \( n \geq n_0 \) and \( \lambda \in \text{Vert}^m_n(n^m) \), the operator \( L_\lambda \) is invertible in the spaces \( \mathcal{L}(h^{-m}) \) and \( \mathcal{L}(h^{-m, n}) \) in the form (3.2). Using the Convolution Lemma (a) and the Lemma 3.3 (c'), one can obtain
\[
\| B_1 L^{-1}_\lambda \|_{\mathcal{L}(h^{-m, n}, h^m, n)} \leq C_m \| (v_1 + \hat{w}) \|_{h^{-m}} \| L^{-1}_\lambda \|_{\mathcal{L}(h^{-m, n}, h^m, n)} = O(\varepsilon).
\]
Therefore, if \( \varepsilon > 0 \) is small enough, the sum
\[
T_\lambda = \Sigma_{l \geq 0} (B_1 L^{-1}_\lambda)^{2l}
\]
converges in \( \mathcal{L}(h^{-m, n}) \). Then the representation (3.5) follows because \( B_1 L^{-1}_\lambda \in \mathcal{L}(h^{-m, h^m}) \) by the Convolution Lemma (a) and the Lemma 3.3 (d'), and \( L^{-1}_\lambda \in \mathcal{L}(h^{-m, n}, h^m) \) by the Lemma 3.3 (c').

Hence, for \( \varepsilon \geq 0 \), \( M \geq 1 \), and \( n_0 \in \mathbb{N} \) as above, the inclusion (3.1) holds. Let remark, that in fact, we have proved the inclusion
\[
\text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}^m_n(n^m) \subseteq \text{Resol}(\hat{L}_m(w(s))),
\]
where \( \text{Resol}(\hat{L}_m(w)) \) denotes the resolvent set of the operator \( \hat{L}_m(w) = D_m + B_0 + sB_1 \) for \( 0 \leq s \leq 1 \). Further, denoting the Riesz projector for \( 0 \leq s \leq 1 \) and any contour \( \Gamma \) in \( \text{Ext}_M \cup \bigcup_{n \geq n_0} \text{Vert}^m_n(n^m) \),
\[
P(s) := \frac{1}{2\pi i} \int_{\Gamma} (\lambda - D_m - B_0 - sB_1)^{-1} d\lambda \in \mathcal{L}(h^{-m}),
\]
one concludes that the operators \( D_m + B_0 \) and \( D_m + B_0 + B_1 \) have the same number of eigenvalues (counted with their algebraic multiplicity) inside \( \Gamma \). To complete the proof of Proposition 3.4 it is sufficient to apply Proposition 3.1 to the operator \( D_m + B_0 \) with \( B_0 = v_0 + \cdot \) and \( v_0 \in h^m \subseteq h^0 \).

**Proposition 3.5.** Consider the eigenvalue problem \( \hat{L}_m(v)f = \lambda f \) with \( v \) in \( h^{-m} \), and let \( R \geq 0 \). For any \( w \in h^{-m} \) with
\[
\| w - v \|_{h^m} \leq R
\]
the spectrum \( \text{spec}(D_m + B(w)) \) of the operator \( \hat{L}_m(w) = D_m + B(w) \) with \( B(w) := w \cdot \) consists of a sequence \( (\lambda_k(w))_{k \geq 0} \) ordered lexicographically of complex-valued eigenvalues counted with their algebraic multiplicity satisfies the asymptotic formula
\[
\lambda_{2n-1}(m, w) = n^{2m} \pi^{2m} + o(n^m), \quad i = 0, 1, \quad n \to \infty
\]
holds.

**Proof.** Let \( v \in h^{-m} \). Since the set \( h^m \) is dense in the space \( h^{-m} \), one decomposes
\[
v = v_0 + v_1,
\]
with \( v_0 \in h^m \) and \( \| v_1 \|_{h^{-m}} \leq \varepsilon \), where \( \varepsilon > 0 \) will be chosen below. We are going to show as above that there exists \( n_0 \in \mathbb{N} \) depending on \( \| v_0 \|_{h^m} \) and \( R \geq 0 \) such that, for any \( w = v + w_0 \in h^{-m} \) with \( \| w_0 \|_{h^{-m}} \leq R \),
\[
\bigcup_{n \geq n_0} \text{Vert}^m_n(n^m) \subseteq \text{Resol}(\hat{L}_m(w)),
\]
where \( \text{Resol}(\hat{L}_m(w)) \) denotes the resolvent set of the operator \( \hat{L}(w) = D_m + B_0 + B_1 \), and \( B_0 = (v_0 + w_0) \cdot \), and \( B_1 = v_1 \cdot \). Notice, that now we consider the strips \( \text{Vert}^m_n(r_n) \) with \( r_n = \delta n^m \) for some \( \delta \in (0, 1] \).
So, let $\lambda \in \text{Vert}_n^m(r_n)$. Using the Convolution Lemma and the Lemma 3.3 (b) one gets

\[
\| B_0(\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m,n})} \leq C_m \| (v_0 + w_0) \|_{h^m} \| (\lambda - D_m)^{-1} \|_{\mathcal{L}(h^{-m,n})} = \frac{\| (v_0 + w_0) \|_{h^m}}{r_n} O(1).
\]

Hence, for $n$ large enough and $\lambda \in \text{Vert}_n^m(r_n)$,

\[
L_\lambda := \lambda - D_m - B_0 = (Id - B_0(\lambda - D_m)^{-1})(\lambda - D_m)
\]

is invertible in $\mathcal{L}(h^{-m})$ with inverse

\[
L_\lambda^{-1} = (\lambda - D_m)^{-1}(Id - B_0(\lambda - D_m)^{-1})^{-1}.
\]

Further, for $n$ large enough, we can show that the following representation of resolvent of the operator $\hat{L}_m(w) = D_m + B_0 + B_1$ converges in $\mathcal{L}(h^{-m})$,

\[
(\lambda - D_m - B_0 - B_1)^{-1} = L_\lambda^{-1} + L_\lambda^{-1}T_\lambda(B_1L_\lambda^{-1}) + L_\lambda^{-1}T_\lambda(B_1L_\lambda^{-1})^2,
\]

where $L_\lambda = \lambda - D_m - B_0$, and $T_\lambda := \Sigma_{n \geq 0}(B_1L_\lambda^{-1})^{2n}$ is considered as an element in $\mathcal{L}(h^{-m,n})$. Using the Convolution Lemma and the Lemma 3.3 (e), we get

\[
\| B_1L_\lambda^{-1} \|_{\mathcal{L}(h^{-m,n}, h^{-m,n})} \leq C_m \| v_1 \|_{h^{-m}} \| L_\lambda^{-1} \|_{\mathcal{L}(h^{-m,n}, h^{-m,n})} = O(\varepsilon).
\]

Hence, if $\varepsilon > 0$ is small enough, the sum $T_\lambda = \Sigma_{n \geq 0}(B_1L_\lambda^{-1})^{2n}$ converges in the space $\mathcal{L}(h^{-m,n})$ and the representation (3.5) then follows because $B_1L_\lambda^{-1} \in \mathcal{L}(h^{-m,n}, h^{-m,n})$ by the Convolution Lemma and the Lemma 3.3 (d), and $L_\lambda^{-1} \in \mathcal{L}(h^{-m,n}, h^{-m})$ by the Lemma 3.3 (c).

Consequently, for some $\varepsilon \geq 0$ and $n_0 \in \mathbb{N}$ the inclusion (3.1) holds for $r_n = \delta n^m$, $\delta \in (0, 1]$. Defining the Riesz projector for $0 \leq s \leq 1$ and any contour $\Gamma$ in $\bigcup_{n \geq n_0} \text{Vert}_n^m(n^m)$,

\[
P(s) := \frac{1}{2\pi i} \int_{\Gamma} (\lambda - D_m - B_0 - sB_1)^{-1} d\lambda \in \mathcal{L}(h^{-m}),
\]

one concludes that the operators $D_m + B_0$ and $D_m + B_0 + B_1$ have the same number of eigenvalues (counted with their algebraic multiplicity) inside $\Gamma$. Applying Proposition 3.4 to the operator $D_m + B_0$ with $B_0 = (v_0 + w_0) \ast$ and $v_0 + w_0 \in h^m \subseteq h^0$ one gets:

the spectrum $\text{spec}(D_m + B(w))$ of the operator $\hat{L}_m(w) = D_m + B(w)$ with $B(w) = w \ast \cdot$ consists of a sequence $(\lambda_k(m, w))_{k \geq 0}$ of complex-valued eigenvalues, and for any $\delta \in (0, 1]$ there exists $n_0 \in \mathbb{N}$ such that the pairs of eigenvalues $\lambda_{2n-1}(m, w), \lambda_{2n}(m, w)$ there are inside a disc around $n^{2m}\pi^{2m}$,

\[
|\lambda_{2n-1}(w) - n^{2m}\pi^{2m}| < \delta n^m, \quad i = 0, 1.
\]

So, we conclude that the sequence $(\lambda_k(m, w))_{k \geq 0}$ of eigenvalues satisfies the asymptotic formula (1.3). 

\[\square\]

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