ADMISSIBILITY AND REALIZABILITY OVER NUMBER FIELDS

DANIEL NEFTIN

Abstract. Let $K$ be a number field. A finite group $G$ is $K$-admissible if there is a $K$-division algebra with a (maximal) subfield $L$ for which $\text{Gal}(L/K) \cong G$. The method that was used in most proofs of $K$-admissibility was to satisfy the local conditions in Schacher’s criterion and then find a global realization satisfying these local conditions. We shall see that this approach works in the cases of tame admissibility (in particular when $L$ is tamely ramified over $K$) of solvable groups, admissibility of most of the abelian groups and admissibility of some larger classes of groups. Many conjectures regarding $K$-admissibility are based on the guess that the $K$-admissible groups are those that satisfy the local conditions. We shall construct an example of a special case in which there is an abelian 2-group $A$ and a number field $K$ for which $A$ satisfies the local conditions but $A$ is not $K$-admissible.

1. Introduction

Admissibility was first introduced in 1967 by Schacher [19] in his study of maximal subfields of central division algebras. Schacher introduced two notions, the first known as $K$-adequacy:

Definition 1.1. Let $L/K$ be a finite extension of fields. The field $L$ is $K$-adequate if there is a division algebra $D$, with center $K$ and a maximal subfield $L$.

The second notion refers to Galois groups of $K$-adequate extensions and is known as $K$-admissibility:

Definition 1.2. Let $K$ be a field and let $G$ be a finite group. The group $G$ is $K$-admissible if there exist a $K$-adequate Galois $G$-extension $L/K$ ($\text{Gal}(L/K) \cong G$).

Remark 1.3. In some cases, including the case of a number field $K$, a subfield of a $K$-division algebra is a maximal subfield of some $K$-division algebra. Thus, over a number field $K$ a group $G$ is $K$-admissible (resp. an extension $L/K$ is $K$-adequate) if and only if there is a $G$-extension $M/K$ and a $K$-division algebra $D$ that contains $M$ (resp. $D$ contains $L$). Therefore the study of $K$-admissibility (resp. $K$-adequacy) leads to observations concerning both subfields of division algebras and to the study of crossed product division algebra central over $K$.

The following theorem revealed an important connection between $K$-adequacy and a problem of realization with prescribed local conditions.

Theorem 1.4. Let $K$ be a number field and let $L/K$ be a finite Galois extension. For every $p$ let $p^r$ be the maximal $p$-power that divides $[L : K]$. Then $L$ is $K$-adequate if and only if for every $p|[L : K]$ there are two primes $v_1, v_2$ of $K$ for which $p^r|[Lv_i : Kv_i]$, $i = 1, 2$.

In the second condition $Lv_i$ denotes a completion of $L$ at a prime divisor of $v_i$ (since $L/K$ is Galois, the local degree $[Lv_i : Kv_i]$ does not depend on the choice of the divisor).
We will use this notation throughout the text. As to admissibility Schacher deduces the following:

**Theorem 1.5.** (Schacher, [19]) Let $K$ be a number field and let $G$ be a finite group. Then $G$ is $K$-admissible if and only if there exists a Galois extension $L/K$ that satisfies:

1. $\text{Gal}(L/K) \cong G$,
2. For every rational prime $p||G|$, there are two primes $v_1, v_2$ of $K$ such that $\text{Gal}(L_{v_i}/K_{v_i})$ contains a $p$-Sylow subgroup of $G$.

Note that the property of containing the $p$-Sylow subgroup does not depend on the choice of the divisor of $v_i$.

Since Theorem (1.5) was proved by Schacher, many efforts were devoted to the classification of finite groups admissible over a given number field (mostly over $\mathbb{Q}$). However, the problem of determining the set of $K$-admissible groups for a number field $K$ remains far from being solved. Schacher’s criterion also supplies necessary conditions for the $K$-admissibility of a group which are rather easy to verify. Let us call these necessary conditions, $K$-preadmissibility:

**Definition 1.6.** Let $K$ be a number field. A finite group $G$ is $K$-preadmissible if there is a set $T = \{v_i(p)||G|, i = 1, 2\}$ of primes of $K$ and corresponding subgroups $G_{v_i} \leq G$ for every $v \in T$, such that for every $p||G|$:

1. $v_1(p) \neq v_2(p)$,
2. $G_{v_i(p)}$ is realizable over $K_{v_i(p)}$ for $i = 1, 2$,
3. $G_{v_i(p)} (i = 1, 2)$ contains a $p$-Sylow subgroup of $G$.

It is clear, by Schacher’s criterion, that a $K$-admissible group is also $K$-preadmissible. We shall often consider $K$-preadmissibility in order to find necessary conditions for $K$-admissibility. In many cases these necessary conditions will also be sufficient. For example, when considering $\mathbb{Q}$-preadmissibility, we have the following simple observation:

**Definition 1.7.** 1. A group $D$ is metacyclic if it contains a normal cyclic subgroup $C \triangleleft D$ for which $D/C$ is also cyclic.
2. A group $G$ is Sylow metacyclic if all the Sylow subgroups of $G$ are metacyclic.

**Proposition 1.8.** Let $G$ be a $\mathbb{Q}$-admissible group. Then $G$ is Sylow metacyclic.

This can be viewed as a direct conclusion from the fact that a $\mathbb{Q}$-preadmissible group is Sylow metacyclic. Moreover, as we shall see in Section 3 that a group $G$ is $\mathbb{Q}$-preadmissible if and only if $G$ is Sylow metacyclic. A natural question to ask is for which groups $K$-preadmissibility is equivalent to $K$-admissibility. For instance, if $K = \mathbb{Q}$, this question is also known as the conjecture of Schacher (19).

**Conjecture 1.9.** (Schacher) A finite group $G$ is $\mathbb{Q}$-admissible if and only if $G$ is Sylow metacyclic.

As not all Sylow metacyclic groups are known to be realizable over $\mathbb{Q}$, it is not known whether these groups are $\mathbb{Q}$-admissible. This conjecture was proved for solvable groups by Sonn ([25], [26], [27]).

**Theorem 1.10.** (Sonn) Let $G$ be a solvable group. Then $G$ is $\mathbb{Q}$-admissible if and only if $G$ is Sylow metacyclic.
In section 3 we shall follow the proof of Theorem 1.10 and prove the following generalization.

Let $\mu_n$ denote the set of $n$-th roots of unity and $\sigma_{t,n}$ the automorphism of $\mathbb{Q}(\mu_n)/\mathbb{Q}$ for which $\sigma_{t,n}(\zeta) = \zeta^t$ for $\zeta \in \mu_n$.

**Theorem 1.11.** Let $K$ be a number field and let $G$ be a solvable Sylow metacyclic group. For every $p|\lvert G\rvert$, let $G(p)$ denote a $p$-Sylow subgroup of $G$. Assume that for every $p|\lvert G\rvert$, $G(p)$ has a presentation:

$$G(p) \cong \langle x, y \mid x^n = y^t, y^n = 1, x^{-1}yx = y^t \rangle,$$

for which $\sigma_{t,n} \in \text{Gal}(\mathbb{Q}(\mu_n)/(\mathbb{Q}(\mu_n) \cap K))$. Then $G$ is $K$-admissible.

We shall then deduce the following:

**Corollary 1.12.** Let $K$ be a number field. Let $G$ be a solvable group such that for every $p|\lvert G\rvert$, there is a unique prime divisor of $p$ in $K$. Then $G$ is $K$-admissible if and only if for every $p|\lvert G\rvert$, any $p$-Sylow subgroup $G(p)$ is metacyclic and has a presentation for which $\sigma_{t,n} \in \text{Gal}(\mathbb{Q}(\mu_n)/(\mathbb{Q}(\mu_n) \cap K))$.

The kind of admissibility that appears in Theorem 1.11 shall be described better by the notion of tame $K$-admissibility defined in Section 3. The latter condition on $\sigma_{t,n}$ is called Liedahl’s condition. In [10], Liedahl proves Theorem 1.11 for metacyclic groups and first uses this type of condition. Note that if $G$ is a group (not necessarily solvable) and for every prime $p|\lvert G\rvert$ there is a unique prime divisor of $p$ in $K$, the Sylow subgroups of $G$ have the above presentation (Presentation 1.1) that satisfies Liedahl’s condition if and only if $G$ is $K$-preadmissible. Thus, $K$-preadmissibility can also be viewed as a generalization of Liedahl’s condition to outside the context of Corollary 1.12. As to $G$ and $K$ as in the assumption of Corollary 1.12, the conclusion is that $G$ is $K$-admissible if and only if $G$ is $K$-preadmissible.

The proof of Theorem 1.11 heavily relies on the main theorem in [13]. Let us describe the setup of the embedding problems in which this theorem applies. Let $G_K$ be the absolute Galois group of $K$ and let $\pi : G \rightarrow \Gamma$ and $\phi : G_K \rightarrow G$ be epimorphisms ($\phi$ corresponds to a realization of $G$ over $K$):

$$G_K \xrightarrow{\phi} G \xrightarrow{\pi} \Gamma \xrightarrow{0} 0.$$ 

Two homomorphisms $\psi_1, \psi_2 : G_K \rightarrow G$ are called equivalent if there is an $a \in \ker(\pi)$ such that $a^{-1}\psi_1(g)a = \psi_2(g)$ for all $g \in G_K$. A solution is an equivalence class of homomorphisms $\psi : G_K \rightarrow G$ that makes Diagram 1.2 commutative, i.e. $\phi = \pi\psi$. The set of solutions is denoted by $\text{Hom}_\Gamma(G_K, G)$ and the set of surjective solutions is denoted by $\text{Hom}_\Gamma(G_K, G)_{\text{sur}}$. By restriction every $\phi : G_K \rightarrow \Gamma$ induces $\phi_v : G_{K_v} \rightarrow \Gamma$ for a prime $v$ of $K$. This induces a local embedding problem. Every solution to the global embedding problem induces a solution to the local embedding problem.

**Theorem 1.13.** (Neukirch) Let $K$ be a number field, $L/K$ a $\Gamma$-extension and $m(L)$ the number of roots of unity in $L$. Let $\pi : G \rightarrow \Gamma$ be an epimorphism with a kernel of order that is prime to $m(L)$ (and therefore solvable). If

$$\prod_v \text{Hom}_\Gamma(G_{K_v}, G) \neq \emptyset$$
where the product is over all primes of $K$, then for every finite set $S$ of primes of $K$ the natural restriction map
\[ \theta^T_G : \text{Hom}_T(G_K, G)_{\text{sur}} \rightarrow \prod_{v \in S} \text{Hom}_T(G_{K_v}, G) \]
is surjective.

The following theorem is a corollary to Theorem 1.13:

**Theorem 1.14.** (Neukirch) Let $K$ be a number field with $m(K)$ roots of unity. Let $S$ be a finite set of primes of $K$. Let $G$ be a finite group with order prime to $m(K)$ (as $G$ is of odd order it must be solvable). For every $v \in S$, let $L^{(v)}/K_v$ be a Galois extension whose Galois group is a subgroup of $G$. Then there exist a Galois extension $L/K$ with $\text{Gal}(L/K) \cong G$ for which $L_v = L^{(v)}$ for all $v \in S$.

Theorem 1.14 supplies a Grunwald-Wang type of assertion and will be used repeatedly later.

In Corollary 1.12, the assumption that the prime $p$ has a unique prime divisor restricts the $K$-admissible $p$-groups to be metacyclic of a certain form and in some sense small. If a prime $p || |G|$ has more than one prime divisor in $K$, the set of $K$-admissible groups (and therefore also the set of $K$-preammissible groups) is usually much larger than the set of Sylow metacyclic groups. The first example for this appears in Section 2, where we determine $K$-admissibility of abelian groups by using the Grunwald-Wang Theorem. The original theorem of Grunwald holds, roughly speaking, for sets of odd primes (primes that do not divide 2):

**Theorem 1.15.** (Grunwald, [6]) Let $K$ be a number field and $A$ a finite abelian group. Let $S$ be a finite set of primes of $K$ which do not divide 2. For each $v \in S$ let $L^{(v)}/K_v$ be a Galois extension with $\text{Gal}(L^{(v)}/K_v) \cong A$. Then, there exists a finite Galois extension $L/K$ for which $\text{Gal}(L/K) \cong A$ and $L_v = L^{(v)}$ for all $v \in S$.

Wang has showed this does not necessarily hold if $A$ is of even order and the set $S$ contains even primes (primes that divide 2). In Section 2 we follow Wang’s survey ([31]) on Grunwald’s Theorem and rely on the set of special cases supplied by Wang, in order to construct the following example:

**Example 1.16.** There exist an abelian group $A$ and a number field $K$ for which $A$ is $K$-preadmissible but $A$ is not $K$-admissible.

In [3], Charbit and Sonn studied admissibility of abelian groups that do not fall into a special case (special case as defined in [12]). We shall make use of Wang’s study in order to determine admissibility of abelian groups that do fall into a special case (see Theorem 2.11) as well as understanding the non-special case by giving similar conditions to the conditions in [3].

In both discussions, on abelian groups and on Sylow metacyclic groups, our strategy is to verify $K$-preadmissibility (by that also choose local conditions) and solve the corresponding realization with prescribed local conditions problem.

At first, let us understand better the local conditions. The absolute Galois group $G_k$, of a $p$-adic field $k$, was studied throughout several extensive researches. Studies of Shafarevich ([16]), Demushkin ([1]), Serre ([22]) and Labute ([8]) resulted in a useful presentation of the Galois group of the maximal $p$-extension, $\text{Gal}(k(p)/k)$. Later on,
Jannsen and Wingberg ([7]) determined an explicit presentation of the absolute Galois group of a \( p \)-adic field for \( p \neq 2 \). These results can be applied in order to reduce the local realizability conditions (in \( K \)-preadmissibility) on a group \( G \) to group theoretical conditions. In order to show that \( G \) is \( K \)-preadmissible, one should find for every prime \( p\||G| \), two completions \( K_v \) for which there is an epimorphic image of \( G_{K_v} \) isomorphic to a subgroup of \( G \) that contains a \( p \)-Sylow subgroup.

We will be more concerned with showing the second step, namely that a group which is \( K \)-preadmissible is also \( K \)-admissible. To secure this, one desired property is the GN-property:

**Definition 1.17.** Let \( G \) be a finite group. We say \( G \) has the Grunwald-Neukirch property over \( K \) (GN-property), if for every finite set \( S \) of the form:

\[
S = \{(v_i, G^{(v_i)}) | v_i \in \text{spec}(K), G^{(v_i)} \leq G, i = 1, \ldots, r\},
\]

in which \( G^{(v)} \) is realizable over \( K_v \) for every \( (v, G^{(v)}) \in S \), there is a Galois \( G \)-extension \( L/K \) so that for every pair \( (v, G^{(v)}) \in S \),

\[
\text{Gal}(L_v/K_v) \cong G^{(v)}.
\]

In Section 4, we shall see examples of classes of groups that satisfy the GN-property or similar properties which guarantee equivalence between \( K \)-admissibility and \( K \)-preadmissibility. We shall see the strategy of showing a group is both \( K \)-preadmissible and has the GN-property (or the similar properties) will work for large classes of groups. However, there are \( K \)-admissible groups that do not satisfy the GN-property (see Remark 2.5). The relation between the classes of groups is described by the following diagram:

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2. Admissibility of Abelian Groups

In this section we shall focus on comparing \( K \)-preadmissibility and \( K \)-admissibility of abelian groups. The theorem of Grunwald (Theorem 1.15), implies roughly speaking that every abelian group of odd order is \( K \)-preadmissible if and only if it is \( K \)-admissible. A more general version is stated in [12]. There, the Grunwald-Wang Theorem is stated for all abelian groups that do not fall into a special case (the notion of a special case as defined in [12]). In [3], Charbit and Sonn used Grunwald’s Theorem in the version of [12] to provide criteria for admissibility of abelian groups that do not fall into such a
special case. Note, these criteria on a group $G$ are equivalent to the $K$-preadmissibility of $G$ and therefore gives a statement of the form $G$ is $K$-admissible if and only if $G$ is $K$-preadmissible. We shall use the theorem of Wang ([31], see also [2]) to list all the abelian $K$-admissible groups and more important produce examples of $K$-preadmissible groups that are not $K$-admissible. In order to do so, we should first understand Wang’s theorem for abelian 2-groups. Let us introduce the language Wang has used. Let $\mu_{2^K}$ denote the set of all 2-power roots of unity.

**Definition 2.1.** Let $K$ be a number field for which $K \cap \mathbb{Q}(\mu_{2^K})$ is totally real. A prime $\nu$ of $K$ is called an even prime if it divides 2. We call $\nu$ oddly even if $[K, \nu(\mu_{2^K}) : K] = [K(\mu_{2^K}, \nu) : K]$ for all $s$, otherwise $\nu$ is called evenly even.

Let $\eta_t$ denote the number $\cos(2\pi/2^t)$. The field $K \cap \mathbb{Q}(\mu_{2^K})$ is totally real if and only if there is a $t \in \mathbb{N}$ for which $\eta_t \in K$ while

$$i, \eta_{t+1}, i\eta_{t+1} \notin K.$$ Then $\nu$ is oddly even if and only if $i, \eta_{t+1}, i\eta_{t+1} \notin K$. Note that if we do not assume $K \cap \mathbb{Q}(\mu_{2^K})$ is totally real then we are not in a special case and Grunwald’s theorem holds. In [31], Wang develops a theory of embedding problems for cyclic groups. As to realization with prescribed local conditions we extract the following corollary to Wang’s Theorem:

**Theorem 2.2.** (Wang, [31]) Let $C$ be a cyclic group of order $2^s$. Let $K$ be a number field and $t \in \mathbb{N}$ for which $\eta_t \in K$ while $i, \eta_{t+1}, i\eta_{t+1} \notin K$. Let $S$ be a finite set of primes of $K$. For every $\nu \in S$, let $L(\nu)$ be a given cyclic extension of $K_\nu$ with Galois group $C^* \leq C$. Then there is cyclic Galois extension $L/K$ for which

1. $\text{Gal}(L/K) \cong C$,
2. $L_\nu = L(\nu)$ for every $\nu \in S$.

if and only if $S$ does not contain all oddly even primes of $K$ or:

(*) the number of oddly even primes of $K$ for which $\eta_{t+1}^{2^s}$ (here Wang originally used $(\sec(2\pi/2^{t+1}))^{2^s}$ instead of $\eta_{t+1}^{2^s}$) is not a norm of $L(\nu)/K_\nu$ is even.

**Remark 2.3.** (Wang, [31]) In the context of Theorem 2.2 if $[L(\nu) : K_\nu] < 2^s$ then $\eta_{t+1}^{2^s}$ is a norm from $L(\nu)/K_\nu$.

**Remark 2.4.** (Wang, [31],[32]) Consider $K = \mathbb{Q}$. Let $k/\mathbb{Q}_2$ denote the unramified $2^s$-extension of $\mathbb{Q}_2$, for $s \geq 3$. We have: $\mathbb{Q} \cap \mathbb{Q}(\mu_{2^s}) = \mathbb{Q}$ is totally real, $t = 2$ and thus the rational prime 2 is oddly even. Moreover, as a result of the discussion in [32], we have that $\eta_{t+1}^{2^s} = \sqrt{2^{2^s}} = 2^{2^{s-1}}$ is not a norm from $k/\mathbb{Q}_2$. Thus, there is no cyclic extension $F/\mathbb{Q}$ of degree $2^s$ in which 2 has full inertial degree, i.e in which 2 neither decomposes nor ramifies in $F$.

**Remark 2.5.** Remark 2.4 provides a simple example of a group that is $K$-admissible but does not have the GN-property. By Remark 2.4 $G = C_{2^s}$, $s \geq 3$, does not have the GN-property over $\mathbb{Q}$ but it is of course $\mathbb{Q}$-admissible.

The theorem also implies a condition on realization with abelian groups. Let us focus on abelian 2-groups. Note that from now on we shall say that the field extensions $F_i/K, \ldots, F_k/K$ are disjoint if for every $i \in \{1, \ldots, k\}$, $F_i \cap (F_1 \ldots \hat{F}_i \ldots F_k) = K$. 

Corollary 2.6. Let $A$ be an abelian 2-group that decomposes into cyclic groups as $A = C_1 \times \ldots \times C_r$. Let $K$ be a number field and let $S$ be a finite set of primes. Then there is a Galois $A$-extension $L/K$ for which $\text{Gal}(L_v/K_v) = A$ for every $v \in S$ if and only if one of the following conditions holds:

(i) $K \cap \mathbb{Q}(\mu_{2^\infty})$ is not totally real,
(ii) $S$ does not contain all oddly even primes of $K$,
(iii) for every $v \in S$ there are $r$ disjoint (over $K_v$) extensions $L_j(v)/K_v$, $j \in \{1,\ldots,r\}$ with Galois groups $\text{Gal}(L_j(v)/K_v) = C_j$ for every $j \in \{1,2,\ldots,r\}$ that satisfy the following condition:

(*) the number of oddly even primes $v$ for which $\eta_{t+1}^{C_j}$ is not a norm of $(L_j)_v/K_v$ is even for every $j \in \{1,2,\ldots,r\}$.

Proof. If there is a Galois $A$-extension $L/K$ for which

$$\text{Gal}(L_v/K_v) = A,$$

then $L$ can be decomposed into $L = L_1L_2\ldots L_r$ so that $\text{Gal}(L_i/K) = C_i$ for $i = 1,\ldots,r$. Hence for every $v \in S$,

$$[L : K] = \prod_{j=1}^{r}[L_j : K] \geq \prod_{j=1}^{r}[(L_j)_v : K_v] = [L_v : K_v] = [L : K].$$

As equality holds, we deduce that for every $v \in S$ and $j = 1,\ldots,r$, $[(L_j)_v : K_v] = |C_j|$ and locally $((L_j)_v)_{j=1}^{r}$ are disjoint. We may now apply Theorem 2.2 for $L_j/K$ and deduce that the following can not all hold together:

(1) there is a $t \in \mathbb{N}$ for which $\eta_t \in K$ while $i, \eta_{t+1}, i\eta_{t+1} \notin K$,
(2) $S$ contains all oddly even primes,
(3) the number of oddly even primes $v$ for which $\eta_{t+1}^{C_j}$ is not a norm of $(L_j)_v/K_v$ is even for every $j \in \{1,2,\ldots,r\}$.

To show the converse we assume that $K \cap \mathbb{Q}(\mu_{2^\infty})$ is totally real, that $S$ contains all oddly even primes and that for every $v \in S$ there are $r$ disjoint extensions $L_j(v)$, $j = 1,\ldots,r$, that satisfies condition (iii). For every $j = 1,\ldots,r$, the extensions $L_j(v)$, $v \in S$ satisfy the criteria of Theorem 2.2 and therefore there is an extension $L_j/K$ with Galois group $C_j$ such that

$$(L_j)_v = L_j(v), \text{ for every } v \in S \text{ and } j = 1,\ldots,r.$$ 

By assumption for $v \in S$ the extensions $L_j(v)$, $j = 1,\ldots,r$, are disjoint and therefore the extensions $L_1,L_2,\ldots,L_r$ are also disjoint. Thus $L = L_1L_2\ldots L_r$ has Galois group $\text{Gal}(L/K) = A$ and

$$\text{Gal}(L_v/K_v) = A \text{ for every } v \in S.$$ 

Remark 2.7. Let $A$ be a finite abelian group. It is easily verified that $A$ is $K$-admissible (resp. $K$-preadmissible) if and only if every $p$-primary component of $A$ is $K$-admissible (resp. $K$-preadmissible). Therefore the problem of determining the $K$-admissible abelian groups (resp. $K$-preadmissible) reduces to a problem of determining the $K$-admissible (resp. $K$-preadmissible) abelian $p$-groups for every prime $p$. 

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Remark 2.8. Let $A = C_m \times C_n$ be an abelian metacyclic group. Then $A$ is realizable over $K_v$ for every prime $v$ of $K$ for which $q_v \equiv 1 \pmod{n}$, where $q_v$ is the size of the residue field of $K_v$. By the Chebotarev density theorem (applied on $\overline{K}(\mu_n)$) there are infinitely many such primes $v$. Choose two such non-even primes $v_1, v_2$. As $v_1, v_2$ are non-even we may apply Corollary 2.6 to produce a Galois $A$-extension $L/K$ for which $Gal(L_{v_1}/K_v) \cong A$. Therefore any metacyclic abelian group is $K$-admissible.

Remark 2.9. Note that if $A$ is a $p$-group which is realizable over $K_v$ for a prime $v$ of $K$ that is not a divisor of $p$ then $A$ is metacyclic (see [33], Chapter 3). Thus if $K$ has only one prime divisor of $p$, all $K$-admissible $p$-groups are metacyclic.

Remark 2.10. Let $A$ be an abelian $p$-group of rank $n$ for which the smallest cyclic factor is of size $q$. Let $v$ be a prime of $K$ which divides $p$. Let us describe (in terms of simple invariants) the conditions on $A$ to be realizable over $K_v$. A similar description was given in [3]. Let $q_v = p^n$ be the highest power of $p$ for which $\mu_{q_v} \subseteq K_v$ and $n_v = [K_v : \mathbb{Q}_p] + 2$.

Let $U_{K_v}$ denote the group of units in $K_v$. By local class field theory (see [24], Chapter 14, Section 6) the maximal abelian extension $K_v^{ab}$ of $K_v$ has Galois group $Gal(K_v^{ab}/K_v) \cong \hat{\mathbb{Z}} \times U_{K_v}$ and therefore the maximal abelian pro-$p$ group realizable over $K_v$ is:

$$A_v = \mathbb{Z}/q_v \mathbb{Z} \times \mathbb{Z}_p^{n_v-1}.$$ 

Let $\leq$ be the lexicographical order, i.e. $(a,b) \leq (c,d)$ if $a < c$ or $a = c, b \leq d$. Then $A$ is realizable over $K_v$ if and only if $A$ is an epimorphic image of $A_v$, which happens if and only if $(n,q) \leq (n_v,q_v)$.

Now assume $A$ is a non-metacyclic $p$-group. Then $A$ is not realizable over any $K_v$ for any $v$ that is not a divisor of $p$. Let $v_1, \ldots, v_r$ denote the prime divisors of $p$ in $K$ sorted so that for $i < j$, $(n_{v_i}, q_{v_i}) \leq (n_{v_j}, q_{v_j})$. Then $A$ is $K$-preadmissible if and only if $r > 1$ and $(n,q) \leq (n_{v_2}, q_{v_2})$ ($A$ is realizable over $K_{v_1}, K_{v_2}$).

This allows an explicit determination of the odd order abelian groups that are $K$-admissible. To obtain such a description for 2-groups a more delicate analysis is required:

Theorem 2.11. Let $K$ be a number field and let $T$ be the set of oddly even primes in $K$. Let $A$ be an abelian non-metacyclic $K$-preadmissible 2-group which decomposes into a product of cyclic groups as: $A = C_1 \times \ldots \times C_r$. Then $A$ is not $K$-admissible if and only if all the following conditions hold:

1. $K \cap \mathbb{Q}(\mu_{2^{\infty}})$ is totally real,
2. $A$ is realizable over $K_v$ for exactly two (even) primes $v_1, v_2$ of $K$,
3. $T \subseteq \{v_1, v_2\}$ and $|T| \geq 1$,
4. for each $i = 1, 2$ and every $r$ disjoint Galois extensions $L_{i,j}/K_{v_i}$, $j \in \{1, \ldots, r\}$, for which $Gal(L_{i,j}/K_{v_i}) = C_j$, there is a $j \in \{1, \ldots, r\}$ for which $\eta_{k+1}^{[C_j]}$ is not a norm of $L_{i,j}/K_{v_i}$ for exactly one $v_i \in T$.

If Conditions (1)-(4) are satisfied we shall say $A$ falls into a special case of $K$-admissibility.

Proof. Let $v_1, \ldots, v_k$ be the even primes in $K$ sorted so that $(n_k, q_k) \leq \ldots \leq (n_1, q_1)$. As $A$ is $K$-preadmissible, $A$ is realizable over $K_{v_1}, K_{v_2}$. Let us assume at first that $A$ is not $K$-admissible. As there is no $A$-extension $L/K$ for which $Gal(L_v/K_v) = A$ for $v \in S = \{v_1, v_2\}$ we deduce from Corollary 2.6 that:

1. $K \cap \mathbb{Q}(\mu_{2^{\infty}})$ is totally real,
2. the set of oddly even primes $T$ is contained in $S$ (and hence $2 \geq |T|$),
LARGEST PRIME

LARGEST EVEN PRIME (WITH

|≥ |

This is a contradiction to the assumption that $A$ is not $K$-admissible and hence Condition (3) must hold.

For the converse let us assume Conditions (1-4) hold. We claim $A$ is not $K$-admissible. Indeed any $K$-adequate $A$-extension $L/K$ must have $\text{Gal}(L_v/K_v) = A$ for $i = 1, 2$ but by Wang’s Theorem applied to $S$ with the assumption of conditions (1-4), we fall into Wang’s special case and hence such an extension $L/K$ does not exist. 

\[ \Box \]

Remark 2.12. Let $A$ be a 2-group of rank $n$ and smallest factor $\mathbb{Z}/q\mathbb{Z}$ and let $K$ a number field. In a special case of $K$-admissibility we have: $(n_{v_3}, q_{v_3}) < (n, q) \leq (n_{v_2}, q_{v_2})$ where $v_3$ (resp. $v_2$) is the third (resp. second) largest prime with respect to $\leq$.

We can now summarize the above discussion on admissibility of abelian groups by:

Theorem 2.13. Let $K$ be a number field and $A$ a finite abelian group. Let $A_p$ denote the $p$-primary component of $A$. Let $n_p$ denote the rank of $A_p$ and let $\mathbb{Z}/q\mathbb{Z}$ be the smallest cyclic factor of $A_p$. Let $S_p$ be the set of prime divisors of $p$ in $K$. Then:

1. $A$ is $K$-admissible if and only if $A_p$ is $K$-admissible for every $p || A$.
2. If $|S_p| = 1$ then $A_p$ is $K$-admissible if and only if $A_p$ is of rank 2 (metacyclic).
3. If $A_p$ is metacyclic then $A_p$ is $K$-admissible.
4. If $|S_p| > 1$, $p$ is odd and $A_p$ is not metacyclic, let $v$ be the second largest prime divisor of $p$ (by the relation $\leq$ defined above). Then $A_p$ is $K$-admissible if and only if $(n_p, q_p) \leq (n_v, q_v)$.
5. If $|S_2| > 1$ and $A_2$ is not metacyclic, let $v$ be the second largest even prime (with respect to $\leq$). Then $A_2$ is $K$-admissible if and only if $(n_2, q_2) \leq (n_v, q_v)$ ($A_2$ is $K$-preamissible) and $A_2$ does not fall into a special case of $K$-admissibility.

Clearly, Theorem 2.11 implies a gap between $K$-admissibility and $K$-preamissibility. Let us construct an explicit example of a $K$-preamissible group that is not $K$-admissible by extending an example of Wang (31). Example after Lemma 1.

Example 2.14. Let $\theta$ be a root of the polynomial $f(x) = x^3 + x + 8$. Let $K$ be the number field $\mathbb{Q}(\theta)$ and let $A$ be the abelian group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z}$ (for $s \geq 3$). Then $A$ is $K$-preamissible but $A$ is not $K$-admissible.

Proof. First we note that $\sqrt{2}, i, \sqrt{-2} \notin K$, thus $t = 2$. Using Newton’s polygon one can observe that the polynomial $f$ has a root over $\mathbb{Q}_2$ of the form $-8\varepsilon$ where $\varepsilon$ is a unit in $\mathbb{Q}_2$. Dividing $f$ by $x + 8\varepsilon$ one has:

\[ x^3 + x + 8 = (x + 8\varepsilon)(x^2 - 8\varepsilon x + 1 + 64\varepsilon^2). \]
The roots of the quadratic factor in (2.1) are $4\varepsilon \pm i\sqrt{1 + 48\varepsilon^2}$. As $1 + 48\varepsilon^2$ is a square in $Q_2$, the splitting field over $Q_2$ of the quadratic factor is $Q_2(i)$. Thus $K$ has two even primes, one of which of ramification index and inertial degree 1 and therefore oddly even while the second prime is even as the corresponding completion contains $i$. Let $v$ be the oddly even prime, $w$ the evenly even prime and $S = \{v, w\}$ the set of even primes in $K$. Since $v$ is of inertia degree and ramification index 1, $Q_2 = K_v$. By Remark 2.4, $\eta_3^{2s}$ is not a norm from the unramified $2^s$-extension of $K_v$, for $s \geq 3 = t + 1$.

Let $\tilde{A}$ (resp. $\tilde{A}_{Q_2(i)}$) be the maximal abelian $2$-extension of $Q_2$ (resp. $Q_2(i)$). By local class field theory (see [24]) we have:

$$\tilde{A} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$\tilde{A}_{Q_2(i)} \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}_2)^3.$$

As $A$ is a quotient of both $\tilde{A}$ and $\tilde{A}_{Q_2(i)}$, $A$ is realizable over $K_v, K_w$ and hence $K$-admissible. Note that $A \cong \tilde{A}/2s\tilde{A}$ and hence $A$ is the maximal abelian group of exponent $2s$ that is realizable over $Q_2$.

We now turn to show that $A$ is not $K$-admissible. Assume on the contrary $A$ is $K$-admissible and $L/K$ is a $K$-adequate $A$-extension. As $A$ is not metacyclic, $A$ is realizable only over even primes and hence:

$$\text{Gal}(L_v/K_v) \cong A.$$  

As $A$ is the maximal abelian group of exponent $2s$ realizable over $Q_2$, the unique unramified cyclic $2s$-extension of $Q_2$ corresponds to an extension $\Omega/K$ contained in $L$ for which

$$\text{Gal}(\Omega/K) \cong \text{Gal}(\Omega_v/K_v) \cong C_{2s}.$$  

As $v$ is the only oddly even prime, by Theorem 2.11, $\eta_3^{2s}$ is a norm from $\Omega_v/K_v$ but by Remark 2.4, $\eta_3^{2s}$ is not a norm from the unramified $2^s$-extension of $K_v = Q_2$. This is a contradiction. We conclude $A$ is not $K$-admissible.  

3. TAME ADMISSIBILITY AND METACYCLIC GROUPS

Let us introduce a stronger notion of admissibility that will, roughly speaking, describe better the kind of admissibility we meet over $Q$. Let $K$ be a number field and $L/K$ a Galois extension with Galois group $G$. The Brauer group $Br(K)$ has the following well known characterization in terms of Hasse invariants,

$$0 \rightarrow Br(K) \rightarrow \bigoplus_{\pi \in \Pi_1} Q/\mathbb{Z} \bigoplus_{\pi \in \Pi_2} \frac{1}{2}\mathbb{Z}/\mathbb{Z} \rightarrow Q/\mathbb{Z} \rightarrow 0,$$

where $\Pi_1$ is the set of finite primes of $K$, $\Pi_2$ is the set of real primes of $K$, the first map is $\bigoplus_{\pi \in \Pi_1 \cup \Pi_2} inv_\pi$ and the second is summation. Denote $\Pi = \Pi_1 \cup \Pi_2$. Let $\alpha \in Br(K)$ and for $\pi \in \Pi$ denote $inv_\pi(\alpha) = a_\pi/b_\pi$ where $(a_\pi, b_\pi) = 1$. Then $L$ splits $\alpha$ if and only if $b_\pi|[L_\pi : K_\pi]$ for all $\pi \in \Pi$. Thus we have the following isomorphism of groups:

$$Br(L/K) \cong \bigoplus_{\pi \in \Pi} \left( \frac{1}{[L_\pi : K_\pi]} \mathbb{Z}/\mathbb{Z} \right)_0,$$

where ($\cdot)_0$ denotes that the sum of invariants is zero.

Remark 3.1. (Schacher, [19]) $L$ is $K$-adequate if and only if $Br(L/K)$ has an element of order $[L : K]$.  


Our definition of tame adequacy is motivated by Remark 3.1. But first let us denote by $M_{tr}$, for any local field $M$, the maximal tamely ramified extension of $M$.

**Definition 3.2.** Let the tamely ramified subgroup $Br(L/K)_{tr}$ of $Br(L/K)$ be the subgroup that corresponds to

$$ (3.1) \bigoplus_{\pi \in \Pi} \left[ L_{\pi} \cap (K_{\pi})_{tr} : K_{\pi} \right] \mathbb{Z}/\mathbb{Z}, $$

i.e. the subgroup that is split by the tamely ramified part of every completion of $L/K$.

This group contains the subgroup $Br(L/K)_{un}$ that is similarly defined in [20]. To be more precise, one can show:

$$ Br(L/K)/Br(L/K)_{tr} \cong \bigoplus_{p \mid |G|} \bigoplus_{\pi \mid p} e_{L_{\pi}/K_{\pi}}(p) \mathbb{Z}/\mathbb{Z}, $$

where $\pi \in \Pi$ and $e_{L_{\pi}/K_{\pi}}(p)$ denotes the maximal $p$-power that divides the ramification index of $L_{\pi}/K_{\pi}$ (this may be regarded as the wild ramification index).

We can now define tame adequacy and tame admissibility:

**Definition 3.3.** Let $L/K$ be finite Galois extension of number fields. We say $L$ is tamely $K$-adequate if there is an element of order $[L : K]$ in $Br(L/K)_{tr}$.

**Example 3.4.** Let $p = 2$, $K = \mathbb{Q}$ and $L = K(\sqrt{3})$. So that $L/K$ is a $C_2$-extension in which 2 is wildly ramified. Let us show $L$ is tamely $K$-adequate. Let $v_1 = (5)$ and $v_2 = (7)$ be two primes of $K$. Both $v_1$ and $v_2$ are inert in $L$. Let $D$ be the $K$-division algebra with invariants

$$ inv_{v_1}(D) = 1/2, \ inv_{v_2}(D) = -1/2, \ inv_{u}(D) = 0 \text{ for any } u \neq v_1, v_2. $$

Then $D$ is an element of order 2 in $Br(L/K)_{tr}$ and hence $L$ is tamely $K$-adequate.

**Definition 3.5.** Let $K$ be a number field and let $G$ be a finite group. We say $G$ is tamely $K$-admissible if there is a tamely $K$-adequate field $L$, Galois over $K$ with Galois group $Gal(L/K) \cong G$.

The following Lemma supplies an alternative definition to tame $K$-adequacy.

**Lemma 3.6.** A $G$-extension $L/K$ is tamely $K$-adequate if and only if there is a set $T = \{ v_i(p) | i = 1, 2, p\mid |G| \}$ of primes of $K$ so that for every $i = 1, 2$ and $p\mid |G|$

(1) $v_i(p)$ is not a divisor of $p$,

(2) $Gal(L_{v_{i}(p)}/K_{v_{i}(p)}) \supseteq G(p)$.

**Proof.** First, note that if $L/K$ is tamely $K$-adequate then there is a $[D] \in Br(L/K)_{tr}$ of exponent $|G|$. Thus, as in the proof of Schacher’s criterion, for every $p\mid |G|$ there are two primes $v_1, v_2$ of $K$ which are not divisors of $p$ for which $|G(p)| \mid \exp(D \otimes_K K_{v_i}), i = 1, 2$. As $L$ splits $D$, $\exp(D \otimes_K K_{v_i}) \mid [L_{v_i} : K_{v_i}]$ and hence the primes $v_1, v_2$ must satisfy Condition 2 as well. Therefore the set $T$ consisting of all such primes $v_1, v_2$ running over all $p$ is the required set.

For the converse, let us assume there is a set $T$ whose primes satisfy both Conditions (1),(2). Let $D_p$ be the $K$-division algebra whose invariants are:

$$ inv_{v_1(p)}(D_p) = \frac{1}{|G(p)|}, \ inv_{v_2(p)}(D_p) = -\frac{1}{|G(p)|} $$
and \( inv_v(D_p) = 0 \) for any other prime. Let \( D = \otimes_{p\mid |G|} D_p \pmod{D_p} \) (tensor over \( K \)). Then \( D_p \) and hence \( D \) are split everywhere by the tamely ramified parts of \( L/K \). Thus \( D \in Br(L/K)_{tr}, \exp(D) = |G| \) and hence \( L/K \) is tamely \( K \)-admissible.

Note that Lemma 3.6 uses only the language of fields and shall be useful later. We conjecture that the following definition will describe admissibility which is not tame.

**Definition 3.7.** A \( G \)-extension \( L/K \) is wildly \( K \)-admissible if \( L/K \) is \( K \)-admissible and for every set \( T = \{ v_i(p) | i = 1, 2, p|||G|\} \) of primes of \( K \) for which \( \text{Gal}(L_{v_i(p)}/K_{v_i(p)}) \supseteq G(p) \) there is a prime \( q|||G| \) for which \( v_1(q), v_2(q) \mid q \).

A \( K \)-admissible group \( G \) will be called wildly \( K \)-admissible if every \( K \)-admissible \( G \)-extension is wildly \( K \)-admissible. We shall say that an extension \( L/K \) (resp. a group \( G \)) is non-wildly \( K \)-admissible (resp. \( K \)-admissible) if \( L/K \) (resp. \( G \)) is \( K \)-admissible (resp. \( K \)-admissible) but not wildly \( K \)-admissible (resp. \( K \)-admissible).

Clearly if \( L/K \) is tamely \( K \)-admissible then \( L/K \) is non-wildly \( K \)-admissible. Note that Definition 3.7 and the alternative definition for tame \( K \)-admissibility in Lemma 3.6 are not negations of each other. Hence, it is not clear whether a non-wildly \( K \)-admissible group is tamely \( K \)-admissible. We shall see that this is the case for solvable groups.

Let us now focus on tame \( K \)-admissibility. Since a tamely ramified extension of a completion \( K_v \) has a metacyclic Galois group (see [33], Chapter 3), a tamely \( K \)-admissible group must be Sylow metacyclic. One can prove more, the following proposition is a light modification of Liedahl’s observation in [10] (we require only \( K \)-p-admissibility). The proof remains identical. Let \( \mu_n \) denote the set of \( n \)-th roots of unity and \( \sigma_{t,n} \) the automorphism of \( \mathbb{Q}(\mu_n)/\mathbb{Q} \) for which \( \sigma_{t,n}(\zeta) = \zeta^t \) for \( \zeta \in \mu_n \).

**Definition 3.8.** Let \( M \) be a metacyclic group. Then there are \( m, n, i, t \in \mathbb{Z} \) for which

\[
M \cong \langle x, y | x^m = y^i, y^n = 1, x^{-1}yx = y^t \rangle,
\]

and for which \( t^n \equiv 1 \pmod{n} \), \( n|t-1,i \). Denote such a presentation by \( M = \mathcal{M}(m, n, i, t) \).

**Proposition 3.9.** (Liedahl) Let \( K \) a number field and \( G \) a \( K \)-p-admissible group. Let \( G_1 \) be a subgroup of \( G \) for which \( P \leq G_1 \leq G \), where \( P \) is a \( p \)-Sylow subgroup of \( G \). Assume there is a prime \( v \) of \( K \) that does not divide \( p \) and for which \( G_1 \) is realizable over \( K_v \). Then \( P \) is a metacyclic \( p \)-group with a presentation \( \mathcal{M}(m, n, i, t) \) for which

\[
\sigma_{t,n} \in \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}(\mu_n) \cap K)).
\]

**Corollary 3.10.** A tamely (or even non-wildly) \( K \)-admissible group \( G \) is Sylow metacyclic and for every \( p|||G| \) a \( p \)-Sylow subgroup of \( G \) has a presentation \( \mathcal{M}(m, n, i, t) \) for which Condition 3.3 is satisfied.

Condition 3.3 forms a relation between the parameters of the presentation. We can also deduce the following congruence relation:

**Remark 3.11.** Let \( K \) be a number field. Let \( G \) be a group such that a \( p \)-Sylow subgroup \( G(p) \) admits a presentation \( \mathcal{M}(m_p, n_p, i_p, t_p) \) for which Condition 3.3 holds. Let \( d_p \) be defined by \( \mu_{n_p} \cap K = \mu_{d_p} \). Then \( t_p \equiv 1 \pmod{d_p} \).

**Proof.** The inclusion \( \mu_{d_p} \subseteq K \) implies \( \mathbb{Q}(\mu_{d_p}) \subseteq K \cap \mathbb{Q}(\mu_{n_p}) \). Thus,

\[
\text{Gal}(\mathbb{Q}(\mu_{n_p})/(K \cap \mathbb{Q}(\mu_{n_p}))) \subseteq \text{Gal}(\mathbb{Q}(\mu_{n_p})/\mathbb{Q}(\mu_{d_p}))
\]

and \( \sigma_{t_p,n_p} \in \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}(\mu_{d_p})) \). The last statement holds if and only if
In [11], Liedahl showed that if $G$ is a metacyclic group then the conditions of Proposition 3.9 are sufficient for $K$-admissibility. It turns out these conditions are sufficient for tame $K$-admissibility of solvable groups. In other words, we will show the converse of Corollary 3.10 for solvable groups.

We start by describing a type of $\mathbb{Q}$-preadmissibility for such groups. Let us define a tame supporting set of primes for a group and a compatible extension.

**Definition 3.12.** Let $G$ be a Sylow metacyclic group. A set $T$ of rational primes will be called a tame supporting set of primes for $G$ if for every $p|G|$ there are two distinct primes $v_1(p), v_2(p)$ in $T$ so that:

1. $v_i(p) \equiv t_p \pmod{n_p}$, $i = 1, 2$, for some presentation $G(p) = M(m_p, n_p, i_p, t_p)$,
2. $v_i(p) \neq v_j(q)$, for $p \neq q$ and $i, j = 1, 2$ (and also for $p = q$ and $i \neq j$),
3. $v_i(p) > 2$ for all $p||G|$, $i = 1, 2$.

**Definition 3.13.** Let $E/\mathbb{Q}$ be a Galois extension and let $T$ be a tame supporting set of primes for $G$. We say $E$ is compatible with the set $T$ if for every $p||G|$ and $v_i(p) \in T$ the following holds

$$\text{Gal}(E_{v_i(p)}/\mathbb{Q}) \supseteq G(p)$$

for some $p$-Sylow subgroup $G(p)$ of $G$.

As all decomposition groups of divisors of $v_i(p)$ are conjugate, Condition (3.5) is well defined.

We shall first consider a solvable Sylow metacyclic $\{2, 3\}$-group, by this we mean a group of order $2^33^4$. Let us explain briefly the classification of such groups, as it was done in [27].

Let $G$ be a solvable Sylow metacyclic $\{2, 3\}$-group. Let $G(p)$ denote a $p$-Sylow subgroup of $G$ and let $G(p) = M(m_p, n_p, i_p, t_p)$ be a presentations of $G(p)$ for $p \in \{2, 3\}$. We first consider the case in which $G(3)$ is not a normal subgroup of $G$. Let $F = F(G)$ denote the Fitting subgroup of $G$ (the maximal normal nilpotent subgroup of $G$), $F(2)$ and $F(3)$ a 2-Sylow and a 3-Sylow subgroups of $F$, respectively. Then $G/F$ is isomorphic either to $S_3$ or to $A_3$ and $F(2)$ is either the quaternion group $Q_8$ or a homocyclic group, i.e. of the form $C_2^u \times C_2^v$. The following cases cover all possibilities:

Case 1: $G/F \cong A_3$ and $F(2) \cong C_2^u \times C_2^v$. In such case $G/(F(3))$ is the unique extension of $C_2^u \times C_2^v$ by a non trivial automorphism of order 3.

Case 2: $G/F \cong A_3$ and $F(2) \cong Q_8$. In such case $G/(F(3)) \cong SL_2(3)$ (the unique extension of $Q_8$ by a non trivial automorphism of order 3).

Case 3: $G/F \cong S_3$ and $F(2)$ is homocyclic. Then $F(2) \cong C_2 \times C_2$ and $G/(F(3)) \cong S_4$.

Case 4: $G/F \cong S_3$, $F(2) \cong Q_8$. Then $G/(F(3))$ is one of the two central extensions of $S_4$ with kernel $C_2$, denote these by $S_4^*$ and $S_4^{**}$. The groups $S_4^*$ and $S_4^{**}$ have 2-Sylow metacyclic subgroups that are isomorphic to

$$Q_{16} = \langle x, y | x^2 = y^4, y^8 = 1, x^{-1}yx = y^7 \rangle$$

and

$$D_{16}^* = \langle x, y | x^2 = y^8 = 1, x^{-1}yx = y^3 \rangle,$$

respectively.
The above mentioned 2-Sylow subgroups all have unique parameters \( m, n, t \). Furthermore, the parameter \( i \) is also unique up to multiplication by an odd number. We shall need:

**Lemma 3.14.** Let \( G \cong \mathcal{M}(m, n, i, t) \).

1. If \( G \cong C_{2^n} \times C_{2^n} \) then \( m = 2^n, n = 2^n, t = 1 \).
2. If \( G \cong Q_8 \) then \( m = 2, n = 4, t = 7 \).
3. If \( G \cong D_{16} \) then \( m = 2, n = 8, t = 3 \).
4. If \( G \cong Q_16 \) then \( m = 2, n = 8, t = 7 \).

**Proof.** (1) is immediate. (2-4) are conclusions from Theorem 22, Case 3 in [10]. In Theorem 22, Liedahl gives a necessary and sufficient condition on a presentation \( \mathcal{M}(m, n, i, t) \) for a group as in one of the cases (2-4) to have an equivalent presentation with other parameters, this condition requires \( m \geq 4 \) which fails in all presentations (2-4). \( \square \)

Let us first consider the admissibility of metacyclic 2-groups. We shall need a refinement of Theorem 1 in [25].

**Lemma 3.15.** Let \( G \) be a metacyclic 2-group, \( S \) a set of odd rational primes and \( T = \{ v_1(2), v_2(2) \} \) a tame supporting for \( G \) for which \( S \cap T = \emptyset \). Then there is a \( \mathbb{Q} \)-adequate \( G \)-extension \( M \) that is compatible with \( T \) and in which all primes of \( S \) split completely.

**Proof.** We follow the proof in [25]. First note it is enough to prove the lemma for metacyclic 2-groups of the form \( G = \langle x, y | x^m = y^n = 1, x^{-1}yx = y^t \rangle \) as every metacyclic 2-group \( H \) is a quotient of such \( G \) that has the same parameters \( t, n \) as \( H \). Thus the same tame supporting set of primes for \( H \) is a tame supporting set for \( G \) as well. So we may fix a presentation \( G = \mathcal{M}(m, n, 0, t) \) for which \( v_i(2) \equiv t \pmod{n} \), \( i = 1, 2 \). Let \( x, y \) be the generators in the presentation \( \mathcal{M}(m, n, 0, t) \). The primes of \( S \cup T \) are not divisors of 2 and by Remark 2.3 we may apply Theorem 22. Thus there is a cyclic \( C_{m} \)-extension \( k/\mathbb{Q} \) for which

1. the primes of \( S \) split completely in \( k \),
2. for every \( q \in T \), \( k_q/\mathbb{Q}_q \) is the unramified \( C_{m} \)-extension of \( \mathbb{Q}_q \).

This guarantees \( k/\mathbb{Q} \) is compatible with \( T \).

As \( v_1(2) \equiv t \pmod{n} \) we have \( v_1(2)^m \equiv 1 \pmod{n} \). Therefore \( \mu_n \subseteq k_{v_1(2)} \) and \( k_{v_2(2)}(v_2(2)^{\frac{1}{2}}) \) is Galois over \( \mathbb{Q}_{v_2(2)} \) with Galois group \( G \).

We consider the embedding problem \( G \to Gal(k/\mathbb{Q}) \) where the following local conditions on a solution \( L \) are prescribed at the primes of \( S' = S \cup \{ v_1(2), v_2(2) \} \):

1. for every prime \( v \in S' \), \( L_v = \mathbb{Q}_v \), i.e \( v \) splits completely,
2. \( L_{v_2(2)} = k_{v_2(2)}(v_2(2)^{\frac{1}{2}}) \) for \( i = 1, 2 \).

We shall apply Theorems 6.4(b) and 2.5 of [13] in order to solve this embedding problem with its prescribed local conditions. The subgroup \( \langle y \rangle \) is a \( G_{\mathbb{Q}} \)-module via the map \( G_{\mathbb{Q}} \to \langle x \rangle = Gal(k/\mathbb{Q}) \). By a Theorem of Scholz (24) there is a solution for the global embedding problem. Thus by Theorem 2.5 in [13], if the map

\[
H^1(G_{\mathbb{Q}}, \langle y \rangle) \to \prod_{v \in S'} H^1(G_{\mathbb{Q}_v}, \langle y \rangle)
\]

is surjective then there is a solution \( L \) that satisfies the local conditions. To show the map (3.8) is surjective, we use Theorem 6.4(b) in [13]. Let us recall the notations. Let \( Y' = Hom(\langle y \rangle, \mu_n) \) be a \( G_{\mathbb{Q}} \)-module induced from the action of \( G_{\mathbb{Q}} \) on both \( \langle y \rangle \) and \( \mu_n \) and let \( H \leq G_{\mathbb{Q}} \) be the stabilizer of this action. Now let \( k' = \mathbb{Q}(Y') \) be the fixed field.
of \( H, G' = \text{Gal}(k'/\mathbb{Q}) \) and for a rational prime \( v \) let \( G'_v = \text{Gal}(k'_v/\mathbb{Q}_v) \). Now, one has \( k' \subseteq k(\mu_n) \) and thus for \( v \in T \) one has \( k'_v \leq (k(\mu_n))_v = k_v(\mu_n) = k_v \) which is cyclic over \( \mathbb{Q}_v \). On the other hand for \( v \in S \) one has \( k'_v \leq (k(\mu_n))_v = k_v(\mu_n) = \mathbb{Q}_v(\mu_n) \) which is also cyclic over \( \mathbb{Q}_v \). Thus \( G'_v \) is cyclic for every \( v \in S' \) and in which every prime in \( S \) splits completely. This completes the proof of Lemma 3.15.\( \Box \)

We shall now aim to embed these \( K \)-admissible metacyclic 2-groups into Sylow metacyclic groups. For this we shall use the notion of strong \( K \)-admissibility introduced by Sonn in [25].

**Definition 3.16.** Let \( K \) be a number field. A group \( G \) is strongly \( K \)-admissible if for every \( n \in \mathbb{N} \) there is a \( K \)-adequate \( G \)-extension \( L/K \) for which \( L \cap K(\mu_n) = K \).

We note that the following condition implies strong \( K \)-admissibility:

**Condition 3.17.** There is a finite set \( W \) of primes of \( K \) so that for every finite set \( S \) for which \( S \cap W = \emptyset \), there is a \( K \)-adequate \( G \)-extension \( L/K \) in which every prime in \( S \) splits completely.

Moreover:

**Lemma 3.18.** Let \( G \) be a group, \( K \) a number field and \( M/K \) a finite extension. If \( G \) satisfies Condition 3.17 over \( K \) then there is a \( K \)-adequate \( G \)-extension \( L/K \) for which \( L \cap M = K \). In particular \( G \) is strongly \( K \)-admissible.

**Proof.** Let \( W \) be the finite set as in Condition 3.17. Let \( \overline{M} \) be the Galois closure of \( M/K \). Let \( \mathcal{A} \) be the collection of all cyclic subgroups of \( \Gamma = \text{Gal}(\overline{M}/K) \). Then by Chebotarev density Theorem for every \( C \in \mathcal{A} \) there are infinitely many primes \( v \) of \( K \) for which \( \text{Gal}(\overline{M}_v/K_v) = C \), fix such a prime \( v_C \) that is not in \( W \). Let \( S \) be the set of all primes \( v_C \), running over all cyclic subgroups \( C \leq \Gamma \). Then Condition 3.17 guarantees the existence of a \( K \)-adequate \( G \)-extension \( L/K \) in which every prime in \( S \) splits completely. We claim \( \overline{M} \cap L = K \). Assume on the contrary there is a \( \sigma \in \Gamma \setminus \text{Gal}(\overline{M}/(\overline{M} \cap L)) \). Then for \( v = v_{(\sigma)} \), we have \( \text{Gal}((\overline{M} \cap L)_v/K_v) \neq \{e\} \) which contradicts the fact that \( v \) splits completely in \( L \).\( \square \)

We shall now prove a refinement of Theorem 3 in [27]. The proof is an adaptation of Sonn’s proof in [27].

**Proposition 3.19.** Let \( G \) be a Sylow metacyclic group of order \( 2^a 3^b \). Let \( S \) be finite set of odd rational primes and let \( T \) be a tame supporting set for \( G \) so that \( S \cap T = \emptyset \). Then there is a Galois \( G \)-extension \( L/\mathbb{Q} \) compatible with \( T \) for which every prime in \( S \) splits completely in \( L \).

**Proof.** Let \( n = |G| \). First, if \( G \) has a normal 3-Sylow subgroup we show such an extension can be constructed. Our supporting set is \( T = \{v_1(2), v_2(2), v_1(3), v_2(3)\} \). By Lemma 3.15 there is a \( \mathbb{Q} \)-adequate Galois extension \( M \) with Galois group \( \text{Gal}(\mathbb{M}/\mathbb{Q}) \cong G(2) \) for which:

1. every prime in \( S \cup \{v_1(3), v_2(3)\} \) splits completely in \( M \),
2. \( M \) is compatible with the supporting set \( \{v_1(2), v_2(2)\} \),
3. \( M \cap \mathbb{Q}(\mu_n) = \mathbb{Q} \).
Remark 3.20. Condition (3) can be obtained by Lemma 3.18. We may apply Theorem 1.13 to embed $M$ into a larger field $E$ Galois over $\mathbb{Q}$ with $\text{Gal}(E/\mathbb{Q}) = G$ for which:

\begin{equation}
\text{Gal}(E_{v_1(3)}/\mathbb{Q}_{v_1(3)}) \cong G(3),
\end{equation}

in which the primes of $S$ split completely and $\mathbb{Q}(\mu_n) \cap E = \mathbb{Q}$. The compatibility of $M$ with $\{v_1(2), v_2(2)\}$ and Condition 3.9 shows that $E$ is compatible with $T$ and therefore $\mathbb{Q}$-adequate.

We now return to the 4 cases in which $G(3)$ is not a normal subgroup of $G$. We shall need the following Lemma which is a conclusion from Proposition 2.5 in [29].

Lemma 3.21. Let $K$ be a number field Galois over $\mathbb{Q}$ and let $L = K(\sqrt{m}), \eta \in K$, be a Galois $G$-extension of $\mathbb{Q}$. Let $S$ be a finite set of primes of $\mathbb{Q}$ that split completely in $K$ and let $W$ be a finite set of rational primes for which $S \cap W = \emptyset$. Then there is a rational integer $m$ for which $L' = K(\sqrt{m\eta})$ is Galois over $\mathbb{Q}$ and satisfies

1. $\text{Gal}(L'/\mathbb{Q}) \cong G$,
2. every prime in $S$ splits completely in $L'$,
3. for every prime $p \in W$, $L'_p \cong L_p$ (at prime divisors of $p$).

Proof. Let $t \neq 2$ be a prime that is not in $S \cup W$ and let $\varepsilon$ be a non-square unit in $\mathbb{Q}_t$. For every $p \in P = S \cup W \cup \{t\}$ define $m_p$ by:

$$m_p = \begin{cases} 
1 & \text{if } p \in W \\
\eta & \text{if } p \in S \\
\varepsilon \eta & \text{if } p = t
\end{cases}$$

For every $p \in P$ let $u_p$ denote the standard valuation on $\mathbb{Q}_p$. By the approximation Theorem ([33] 3.1-4) there is a rational integer $m$ for which $u_p(m_{v}/m_p - 1)$ is large enough to insure $m \equiv m_p \pmod{K^*_v}$ for every $p \in P$ and any divisor $v$ of $p$. Now let $L' = K(\sqrt{m\eta})$. We deduce:

$$L'_v = \begin{cases} 
K_v(\sqrt{m\eta}) = K_v(\sqrt{\eta}) \cong L_v & \text{if } p \in W \\
K_v(\sqrt{m\eta}) = K_v(\sqrt{\eta^2}) = K_v(\mathbb{Q}_p) & \text{if } p \in S \\
K_v(\sqrt{m\eta}) = K_v(\sqrt{\eta}) \neq K_v & \text{if } p = t
\end{cases}$$

for every prime divisor $v$ of $p \in P$.

Since at $p = t$, $K_v$ is a non-trivial extension we have $L' \neq K$ and by Proposition 2.5 in [29], $L'$ has the same Galois group i.e $\text{Gal}(L'/\mathbb{Q}) \cong \text{Gal}(L/\mathbb{Q})$. Thus $L'$ is the required extension.

\[ \square \]

We proceed with the proof of Proposition 3.19.

Case 1: By Theorem 1.14, there is a Galois $\mathbb{Q}$-adequate $G(3)$-extension $E$ of $\mathbb{Q}$ such that $E$ is compatible with the supporting set $\{v_1(3), v_2(3)\}$ and all primes of $S \cup \{v_1(2), v_2(2)\}$ split completely in $E$. Let $k$ be the fixed subfield of $F(3)$, such $k$ is a cubic extension of $\mathbb{Q}$. Note, all the primes in $S \cup \{v_1(2), v_2(2)\}$ split completely in $k$ as well. We aim to embed $k/\mathbb{Q}$ into a $\mathbb{Q}$-adequate Galois $G/(F(3))$-extension $L/Q$ compatible with $T$ (T is also a tame supporting set of primes for $G/(F(3))$) in which all primes of $S$ split completely. For such $L$, $EL/\mathbb{Q}$ will be a Galois $\mathbb{Q}$-adequate $G$-extension compatible with $T$ in which all primes of $S$ split completely. To construct such an $L$, one has to solve
the embedding problem: \( G/(F(3)) \to Gal(k/Q) \). By Lemma 3.14, \( v_i(2) \equiv 1 \pmod{2^n} \) while by definition of \( k \), \( v_i(2) \) splits completely in \( k \). To solve the above embedding problem with the corresponding local conditions we again make use of Theorems 6.4(b) and 2.5 in [13]. Let \( P' = Hom(C_{2^n} \times C_{2^n}, \mu_n) \) and \( G' = Gal(Q(P')/Q) \). In this case \( G'_v(2) \leq Gal(k_v(2)(\mu_{2^\nu})/Q_v(2)) = 1 \) and for every \( v \) in \( S \): \( G'_v \) is cyclic and we may apply Theorems 6.4(b) and 2.5 of [13]. We deduce there is an \( L \supseteq k \) for which:

1. \( Gal(L/Q) \cong G/(F(3)) \),
2. \( Gal(L_{v_i(2)}/Q_v(2)) \cong F/(F(3)) \) (the 2-Sylow subgroup of \( G/(F(3)) \)),
3. all primes of \( S \) have trivial decomposition groups.

Thus all primes in \( S \) split completely in \( L \), \( L \) is compatible with \( T \) and hence \( Q \)-adequate.

**Remark 3.22.** We may also obtain such an \( L \) for which \( L \cap Q(\mu_n) = Q \).

**Case 2:** Let \( E \) and \( k \) be defined as in case 1 \( (Gal(E/Q) = G(3), Gal(k/Q) = C_3) \). We construct an \( L \) with Galois group \( G/(F(3)) \cong SL_2(3) \). Let \( \sigma \) be the generator of \( Gal(k/Q) \). By choice of \( k \), the primes \( v_i(2) \) split completely in \( k \), \( v_i(2) = p_ip_i^2p_i^2 \), \( i = 1, 2 \).

Let \( m \) be the modulus of \( k \) consisting of \( (8) \), the infinite primes, the ramified primes of \( k/Q, p_i^2 \), and the prime divisors of primes in \( S, i = 1, 2 \). Let \( \gamma \) be totally positive, congruent to 1 mod \( 8 \), the ramified primes of \( k/Q \), the prime divisors of primes in \( S, p_i^2 \) and let \( \gamma \) be congruent to a non-square unit mod \( p_i^2 \), \( i = 1, 2 \). By the generalized Dirichlet theorem the ray class mod \( m \) of \( p_1^{-1}p_2^{-1}\gamma \) contains a prime ideal \( q \) of degree 1.

Then there is an element \( \delta \in k \) for which \( \delta \equiv 1 \pmod{m} \) and \( (\gamma \delta) = p_1p_2q \). The element \( \beta = \gamma \delta \) satisfies the conditions imposed on \( \gamma \). Then \( \alpha = \beta^i \beta^{x_i} \) is a non-square at \( p_i \), \( \alpha^x \) is prime at \( p_i \) and \( \alpha^{x_i} \equiv a\alpha^x \pmod{k^{x_i}} \), for \( i = 1, 2 \). Thus, \( K = k(\sqrt{\alpha}, \sqrt{\alpha^x}) \) is Galois over \( Q \) with Galois group \( A_4 \), \( Gal(K_{v_i(2)}/Q_v(2)) \cong C_2 \times C_2 \) and all primes in \( S \) split completely in \( K \).

Consider the embedding problem \( SL_2(3) \to Gal(K/Q) \). By Lemma 3.14, \( v_i(2) \equiv -1 \pmod{4} \), \( i = 1, 2 \), thus \( K_{v_i(2)} \) is embedded uniquely into a \( Q_8 \)-extension. Therefore the embedding problem has a local solution at all primes except perhaps at \( q \) that lies below \( q \). By Lemma 2 in [25], there is a global solution \( L = K(\sqrt{\eta}) \) and the structure of \( SL_2(3) \) forces:

\[ Gal(L/Q) \cong SL_2(3), Gal(L_{v_i(2)}/Q_v(2)) \cong Q_8. \]

By applying Lemma 3.21 with \( W = \{ v_1(2), v_2(2) \} \) and \( S \) we obtain a \( Q \)-adequate Galois \( SL_2(3) \)-extension \( L'/Q \) that is compatible with \( T \) and in which every prime in \( S \) splits completely in \( L' \).

**Case 4** (including Case 3): In this case \( G/(F(3)) \cong S_4^* \) or \( S_4^{**} \). By Lemma 3.14 the 2-Sylow presentations translate into conditions on \( v_i(2) \):

1. \( v_i(2) \equiv -1 \pmod{8} \) if \( G/(F(3)) \cong S_4^* \), where \( S_4^*(2) = Q_{16} = \langle x, y \mid x^2 = y^4, y^8 = 1, x^{-1}yx = y^7 \rangle \),
2. \( v_i(2) \equiv 3 \pmod{8} \) if \( G/(F(3)) \cong S_4^{*} \), where \( S_4^{*}(2) = D_{16}^* = \langle x, y \mid x^2 = y^8 = 1, x^{-1}yx = y^3 \rangle \),

and by assumption we have
(3) \( v_1(2) \not\in S \).
Let \( t \) be a prime for which
(1) \( v_1(2)v_2(2)t \equiv 1 \) (mod \( u \)) for every odd \( u \) in \( S \),
(2) \( t \not\in S \cup T \) and
(3) \( t \equiv 1 \) (mod 8).

Then all primes of \( S \) (and 2) split in \( k = \mathbb{Q}(\sqrt{v_1(2)v_2(2)t}) \). By Theorem 1.13, \( k/Q \) can be embedded into a \( Q \)-adequate Galois \( G/F(2) \)-extension \( E/Q \) compatible with \( T \) for which all primes in \( S \) split completely in \( E/Q \) and the ramified prime \( t \) of \( k \) that lies above \( t \) also splits completely in \( E \). Set \( K = E^{F/(F(2))} \), the fixed field of \( F/(F(2)) \).

We aim to embed \( K/Q \) into a \( Q \)-adequate Galois \( G/(F(3)) \)-extension \( L/Q \) compatible with \( T \) in which the primes of \( S \) split completely. In such case \( EL/Q \) will constitute a \( Q \)-adequate Galois \( G \)-extension compatible with \( T \) in which the primes of \( S \) split completely. Fix a presentation \( \text{Gal}(K/Q) \cong S_2 \cong (\sigma, \tau) \sigma^3 = \tau^2 = 1, \tau^{-1} \sigma \tau = \sigma^{-1} \). Let \( F \) be the fixed subfield of \( \tau \). Then the primes \( v_1(2), v_2(2) \) decompose in the following way:

\[
(v_1(2))_F = p_1p_2^2, \quad (p_1)_K = p^2, \quad (p_2)_K = p^\sigma p^{\sigma^2}, \quad (v_2(2))_F = q_1q_2^2, \quad (q_1)_K = q^2, \quad (q_2)_K = q^{\sigma}q^{\sigma^2}.
\]

Let \( R \) be the set of primes of \( F \) whose prime divisors in \( K \) ramify in \( K/k \). Construct a modulus \( m \) of \( F \) consisting of \( (8) \), the infinite primes, \( p_1, q_1 \), the prime divisors of \( t \), the primes in \( R \) and the prime divisors of primes in \( S \). Choose \( \gamma \in F \) so that \( \gamma \) is congruent to 1 mod 8, the primes of \( R \), the prime divisors of primes in \( S \cup \{ t \} \) and congruent to a non-square unit at \( p_1, q_1 \). By the generalized Dirichlet Theorem, the ray class mod \( m \) of the ideal \( p_2^{-1}q_2^{-1}\gamma \) contains a prime ideal \( \mathfrak{r} \) and hence there exists a \( \delta \in F \) so that \( \delta \equiv 1 \) (mod \( m \)) and \( (\gamma \delta) = p_2q_2\mathfrak{r} \). The element \( \beta = \gamma \delta \in F \) satisfies the same conditions imposed above on \( \gamma \). Thus the field \( K_p(\sqrt{3^\alpha}, \sqrt{3^{\sigma^2}}) \) (resp. \( K_q(\sqrt{3^\alpha}, \sqrt{3^{\sigma^2}}) \)) is the maximal tamely ramified extension of \( K_p \) (resp. \( K_q \)) of exponent 2. In [27], Theorem 3, case 2, it is proved that \( \beta^{\sigma^2} \equiv \beta \sigma^2 \) (mod \( K_p^2 \)) (resp. mod \( K_q^2 \)). Setting \( \alpha = \beta^\sigma \) we have \( K_p(\sqrt{\alpha^2}, \sqrt{\alpha^{\sigma^2}}) = K_p(\sqrt{\alpha^\sigma}, \sqrt{\alpha^{\sigma^2}}) \) (resp. \( K_q(\sqrt{\alpha^2}, \sqrt{\alpha^{\sigma^2}}) = K_q(\sqrt{\alpha^\sigma}, \sqrt{\alpha^{\sigma^2}}) \)) and:

\[
\alpha^{\sigma^2} \equiv \alpha \alpha^\sigma \text{ (mod } K_p^2), \quad \alpha^\sigma = \alpha, \quad \alpha^{\sigma^2} = \alpha^\sigma \text{ and } \alpha^{\sigma^2} = \alpha^\sigma.
\]

Therefore \( M = K(\sqrt{\alpha}, \sqrt{\alpha^\sigma}) \) is a Galois \( S_4 \)-extension of \( Q \) for which all primes of \( S \) split completely in \( M \) and \( t \) has local Galois group \( \text{Gal}(M_t/Q_t) \cong \mathbb{Z}/2\mathbb{Z} \). The local Galois groups at \( v_1(2), v_2(2) \) are \( \text{Gal}(M_{v_1(2)}/Q_{v_1(2)}) \cong D_8 \) for \( i = 1, 2 \). Therefore \( M \) is compatible with \( T \) and hence \( Q \)-adequate. This proves case 3.

Consider the embedding problem \( G/(F(3)) \to \text{Gal}(M/Q) \). At the rational primes whose divisors are ramified in \( K/k \) the decomposition group is \( C_3 \) or \( S_3 \) (with odd ramification index). As the index of the embedding problem is 2, it is solvable at primes ramified in \( K \). Since \( t \equiv 1 \) (mod 8) and \( t \) splits completely at \( M \) the local embedding problem at \( t \) is solvable. For \( G/(F(3)) = S_4 \) (resp. \( S_4^* \)), as \( v_1(2) \equiv -1 \) (mod 8) (resp. \( v_1(2) \equiv 3 \) (mod 8)) the local Galois group \( (D_8) \) can be embedded in a \( Q_{16} \) (resp. \( D_{16}^* \)) extension. As the induced local embedding problems are solvable everywhere except perhaps at \( r \), by [28] Lemma 2, it is solvable at \( r \) and the global embedding problem has a solution \( N = M(\sqrt{\eta}) \) (the structure of \( G/(F(3)) \) forces \( N \) to be compatible with \( T \)).

By applying Lemma 3.21 with \( W = \{v_1(2), v_2(2)\} \) there is a \( Q \)-adequate Galois \( G/(F(3)) \)-extension \( N' \) compatible with \( T \) in which all primes of \( S \) split completely.
We are now ready to prove an adaptation of Theorem 1.10 which will later serve the discussion on tame $K$-admissibility over a general number field $K$.

**Proposition 3.23.** Let $G$ be a solvable Sylow metacyclic group and let $S$ be a finite set of odd rational primes. Let $T$ be a tame supporting set of primes for $G$ so that $S \cap T = \emptyset$. Then there is a Galois $G$-extension $L/\mathbb{Q}$ compatible with $T$ (and hence $\mathbb{Q}$-adequate) in which every prime in $S$ splits completely.

**Proof.** Let $n = |G|$. By Lemma 1.4 in [26], $G$ has a $\{2, 3\}$-normal complement. In other words, there is a normal subgroup $N \triangleleft G$ of order prime to 6 (to 2 and 3) and a $\{2, 3\}$-subgroup $A$ for which $G = NA$. Denote by $U$ the set $\{v_i(p)|i = 1, 2, p || N\}$.

By Proposition 3.19 there is a $\mathbb{Q}$-adequate Galois $A$-extension $K/\mathbb{Q}$ compatible with the supporting set $\{v_1(2), v_2(2), v_1(3), v_2(3)\}$ in which all primes of $U \cup S$ split completely. By Lemma 3.18 $K$ can be chosen so that $K \cap \mathbb{Q}(\mu_n) = \mathbb{Q}$. As $|N|, |A| = 1$ the embedding problem $G \rightarrow Gal(K/\mathbb{Q})$ is split and we may apply Theorem 1.13 (here, $\mu_n \not\in K$ is required). Theorem 1.13 guarantees a solution $L$ that satisfies the following conditions:

1. $Gal(L/\mathbb{Q}) = G$,
2. if $p \in S$ then $p$ splits completely in $L$,
3. if $p || |N|$ then $Gal(L_{v_i(p)}/\mathbb{Q}_{v_i(p)}) = N(p)$.

Thus $L$ is a Galois $G$-extension compatible with $T$ in which every prime in $S$ splits completely. \hfill \qed

This result will also be useful later to construct division algebras with infinitely many non isomorphic maximal $G$-subfields. We shall now lift the above construction to a general number field $K$.

**Theorem 3.24.** Let $K$ be a number field and let $G$ be a solvable Sylow metacyclic group. For every $p || |G|$, let $G(p)$ denote a $p$-Sylow subgroup of $G$. Then $G$ is tamely $K$-admissible if and only if for every $p || |G|$, $G(p)$ has a presentation $\mathcal{M}(m_p, n_p, i_p, t_p)$ for which Liedahl’s condition holds, i.e:

\[(3.11) \quad \sigma_{t_p, n_p} \in Gal(\mathbb{Q}(\mu_{n_p})/(\mathbb{Q}(\mu_{n_p}) \cap K)).\]

Moreover, if $S$ is a finite set of primes of $K$ which are not divisors of 2 and for every $p || |G|$ there is a presentation of $G(p)$ that satisfies Condition 3.11 then there is a $K$-adequate Galois $G$-extension $N/K$ for which every prime in $S$ splits completely in $N$.

**Proof.** First if $G$ is tamely $K$-admissible then by Proposition 3.9 the $p$-Sylow subgroups have such a presentation. We shall prove the converse statement.

Let $S_G$ be the set of rational primes that lie below the primes in $S$. Let $\overline{K}$ denote the $\mathbb{Q}$-Galois closure of $K$. Fix a prime $p || |G|$ and denote $M_p = \overline{K}(\mu_{n_p})$. As $\sigma_{t_p, n_p} \in Gal(\mathbb{Q}(\mu_{n_p})/(\mathbb{Q}(\mu_{n_p}) \cap K))$, $\sigma_{t_p, n_p}$ is also in $Gal(\mathbb{Q}(\mu_{n_p})/(\mathbb{Q}(\mu_{n_p}) \cap \overline{K}))$. Therefore there is an automorphism $\tau_p \in Gal(M_p/\overline{K})$ that fixes $\overline{K}$ and restricts to $\sigma_{t_p, n_p}$. By the Chebotarev density Theorem there are infinitely many rational primes $(v_i(p))_{i \in \mathbb{N}}$ whose Frobenius automorphism is $\tau_p$. Since $\tau_p$ is the Frobenius automorphism of $v_i(p)$ it follows that $v_i(p) \equiv t_p \mod n_p$, $i = 1, 2$, $p || |G|$.

Since for every $p || |G|$ there are infinitely many such primes $v_i(p)$, there can be fixed $w_i(p)$ for all $i \in \{1, 2\}$ and $p || |G|$ such that:
Proof. If every \( p\) \( (i \in \{1, 2\}, p||G) \) are distinct,
(2) \( w_i(p) \not\in S_Q \), for all \( i = 1, 2 \) and \( p||G \),
(3) \( w_i(p) > 2 \) for all \( p||G, i = 1, 2 \),
(4) \( \tau_p \) is the Frobenius automorphism of \( M_{w_i(p)}/Q_{w_i(p)} \),
(5) \( w_i(p) \neq p \) for all \( p||G, i = 1, 2 \).

The conditions imply that \( T = \{ w_i(p) | i = 1, 2, p||G \} \) is the Frobenius supporting set for \( G \) and every prime in \( T \) splits completely in \( K \). By Proposition 3.24 there is a Galois \( G \)-extension \( L/Q \) compatible with \( T \) for which every prime in \( S_Q \) splits completely in \( L \).

Let \( N := LK \). As \( w_i(p) \) split completely in \( K, \{ K_v : Q_{w_i(p)} \} = 1 \) for every prime divisor \( v|w_i(p) \) in \( K, i = 1, 2, p||G \). Then for every such prime divisor \( v \),

\[
(N_v : K_v) = \left[ \frac{N_v : Q_{w_i(p)}}{K_v : Q_{w_i(p)}} \right] = [N_v : Q_{w_i(p)}] = [L_{w_i(p)} : Q_{w_i(p)}] \equiv [N_v : L_{w_i(p)}].
\]

But, as \( [N_v : K_v][L_{w_i(p)} : Q_{w_i(p)}] \) we deduce \( [N_v : L_{w_i(p)}] = 1 \) and \( [N_v : K_v] = [L_{w_i(p)} : Q_{w_i(p)}] \). Thus, \( Gal(N_v/K_v) \subseteq G(p) \) for all \( w_i(p), i \in \{1, 2\}, p||G \). Note this also implies \( [N : K] = |G| = [L : Q] \) and hence \( K \cap L = Q \).

Let \( v' \in S \) and \( q \) its restriction in \( S_Q \). Then

\[
[N_{v'} : K_{v'}][L_q : Q_q] = 1,
\]

and therefore \( v' \) splits completely in \( N \). The Galois extension \( N/K \) is therefore \( K \)-adequate with Galois group \( G \) and every prime in \( S \) splits completely in \( N \). Moreover as \( w_i(p) \) and its divisors in \( K \) are not divisors of \( p \) for all \( i = 1, 2, p||G \), by Lemma 3.6 \( N/K \) is also tamely \( K \)-adequate.

Remark 3.25. In the proof of Theorem 3.24 we have constructed a \( Q \)-division algebra \( D \) that has a maximal subfield \( L \) so that

(1) \( L/Q \) is a Galois \( G \)-extension,
(2) \( N = LK \) is a maximal subfield of \( D \otimes_Q K \),
(3) \( Gal(N/K) = G \).

Thus, not only \( G \) is \( K \)-admissible but there is also a \( G \)-crossed product division algebra in the image of the restriction map from \( Q \), i.e \( [D] \in Im(res^K_Q) \).

Theorem 3.24 also supplies in a sense a converse statement to Proposition 3.9 For a solvable \( G \) and a number field \( K \), for which every \( p||G \) has a unique prime divisor in \( K \), we can determine precisely when \( G \) is \( K \)-admissible:

Corollary 3.26. Let \( K \) be a number field. Let \( G \) be a solvable group such that for every \( p||G \), there is a unique prime divisor of \( p \) in \( K \). Then \( G \) is \( K \)-admissible if and only if for every \( p||G \), any \( p \)-Sylow subgroup \( G(p) \) is metacyclic and has a presentation \( M(m_p, n_p, i_p, t_p) \) that satisfies Liedahl’s condition.

Proof. If \( G \) is \( K \)-admissible then by Proposition 3.9 for every \( p||G \), \( G(p) \) has such a presentation. The converse follows from Theorem 3.24.

Corollary 3.27. Let \( K \) be a number field and let \( G \) be a solvable group. Assume that every \( p||G \) has a unique prime divisor in \( K \). Then the following conditions are equivalent:

(1) \( G \) is tamely \( K \)-admissible,
(2) $G$ is $K$-admissible,
(3) $G$ is $K$-preadmissible,
(4) for every $p|\lvert G\rvert$, $G(p)$ has a presentation $\mathcal{M}(m_p, n_p, i_p, t_p)$ that satisfies Liedahl’s condition. If moreover $G$ is of odd order the conditions above are equivalent to:
(5) There is a $K$-adequate Galois $G$-extension $L/K$ which is everywhere tamely ramified.

**Proof.** The implications (5) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear from the definitions and Propositions 3.6 and 3.9. The implication (4) $\Rightarrow$ (1) follows from Theorem 3.24. Let us show that for an odd order group $G$, (1) $\Rightarrow$ (5). Let $S$ be the set of primes of $K$ lying above a rational $p$ that divides $\lvert G\rvert$. By Theorem 3.24 there is a $K$-adequate Galois $G$-extension $L/K$ in which all primes in $S$ split completely. Therefore $L$ is tamely ramified. □

The difficulty in proving Corollary 3.27 for any Sylow metacyclic group $G$ arises in the step (4) $\Rightarrow$ (5). When (4) is given, for every $p|\lvert G\rvert$, there are infinitely many primes over which there is a local realization for the $p$-Sylow subgroup. This infinite support and the large amount of realizations raises a belief that the following conjecture holds:

**Conjecture 3.28.** Let $K$ be a number field and $G$ a finite group (not necessarily solvable). Then there is a $K$-adequate Galois $G$-extension that is tamely ramified if and only if for every $p|\lvert G\rvert$, $G(p)$ is metacyclic and satisfies Condition (3.3) (Liedahl’s condition).

**Remark 3.29.** If Conjecture 3.28 holds then a finite group is tamely $K$-admissible if and only if for every $p|\lvert G\rvert$, $G(p)$ is metacyclic and satisfies Condition (3.3) (Liedahl’s condition).

Another use for Theorem 3.24 is to the notion of infinitely often $K$-admissibility. It is defined in [1] as:

**Definition 3.30.** Let $K$ be a field and let $G$ be a finite group. Then $G$ is *infinitely often $K$-admissible* if there are infinitely many disjoint adequate Galois $G$-extensions of $K$. In other words there is a sequence $(L_i)_{i \in \mathbb{N}}$ for which $L_{r} \cap L_{1} \ldots L_{r-1} = K$ and $L_{r}/K$ is a $K$-adequate $G$-extension for every $r \in \mathbb{N}$.

**Corollary 3.31.** Let $K$ be a number field and $G$ a solvable Sylow metacyclic group for which Liedahl’s condition holds. Then $G$ is $K$-admissible infinitely often. Moreover there is a division algebra $D$ with infinitely many disjoint (and non-isomorphic) maximal subfields $L/K$ for which $\text{Gal}(L/K) = G$.

**Proof.** By Remark 3.25 there is a tame supporting set $T = \{v_i(p)\}$ of rational primes that split completely in $K$ and a Galois $G$-extension $L/\mathbb{Q}$ compatible with $T$. Let $|G|(p)$ denote the maximal $p$-power dividing $G$. Let $D_p$ be the $\mathbb{Q}$-division algebra defined by the invariants:

$$\text{inv}_{v_1(p)}(D_p) = -\text{inv}_{v_2(p)}(D_p) = \frac{1}{|G|(p)}$$

and $\text{inv}_u(D) = 0$ if $u \neq v_i(p)$, $i \in \{1, 2\}$. Let $D = \otimes_{p|\lvert G\rvert} D_p$.

Then $L$ is a maximal subfield of $D$ and by Remark 3.25 for every such $L$, $LK$ is a maximal $G$-subfield of $D \otimes_{\mathbb{Q}} K$. We shall prove that $D \otimes_{\mathbb{Q}} K$ has infinitely many disjoint maximal subfields Galois over $K$ with Galois group $G$.

Let $\bar{K}$ be the normal closure of $K$ over $\mathbb{Q}$. Let $L_1, \ldots, L_r$ be a list of disjoint Galois $G$-extensions which are maximal subfields of $D$ and $M = L_1 \cdots L_r \bar{K}$. By Theorem 3.24...
and Lemma 3.18 there is a Galois $G$-extension $L_{r+1}/Q$ compatible with $T$ in which all primes of $S$ split completely. Thus, $L_{r+1} K$ is a maximal $G$ subfield of $D \otimes Q K$. As every prime divisor of $u_C$ in $K$ splits completely in $M$, $L_{r+1} K \cap M = K$. We conclude $D \otimes Q K$ has infinitely many disjoint (non-isomorphic) maximal subfields $L/K$ with Galois group $Gal(L/K) = G$.

**Remark 3.32.** Theorem 3.24 guarantees that a solvable group which is non-wildly $K$-admissible is tamely $K$-admissible. Similarly a proof of Conjecture 3.28 will imply: a finite group which is non-wildly $K$-admissible is tamely $K$-admissible.

**Remark 3.33.** Let us say $G$ is tamely $K$-preadmissible if the local conditions of tame $K$-admissibility are satisfied. Namely if there is a $K$-division algebra $D$ and for every $p || G$ there is a set $T = \{ v_i(p) || p || G, i = 1, 2 \}$ of primes of $K$ and corresponding Galois extensions $L^v/K_v$ for $v \in T$ so that

1) $v_1(p) \neq v_2(p),$
2) $G(p) \leq Gal(L^{v_i(p)}/K_{v_i(p)}) \leq G$ ($i = 1, 2$),
3) $L^{v_i(p)} \cap (K_{v_i(p)})_{tr}$ splits $D \otimes K_{v_i(p)}$.

Thus a group $G$ is tamely $K$-preadmissible if and only if for every $p || G$, $G(p)$ has a metacyclic representation that satisfies Liedahl’s Condition. Therefore a proof to Conjecture 3.28 will imply: If $G$ is tamely $K$-preadmissible then $G$ is tamely $K$-admissible. By Theorem 3.24 this holds for any solvable group $G$.

**Remark 3.34.** The main reason for choosing the strategy of lifting $Q$-adequate extensions to obtain $K$-adequate extensions was that Theorem 1.13 is valid over $Q$ for all odd order groups. In general this powerful tool does not remain valid.

**Remark 3.35.** Let $p || G$ be a prime that has more than one prime divisor in $K$. In such case as we shall see, the set of $K$-admissible groups (and therefore also the set of $K$-preadmissible groups) is usually considerably “larger” than the set of tamely $K$-admissible groups.

4. **Examples of $K$-preadmissible groups that are $K$-admissible**

There are special cases, as constructed in Section 2, of $K$-preadmissible groups that are not $K$-admissible. We shall see there is a large collection of groups for which $K$-preadmissibility does imply $K$-admissibility. One of the properties that guarantees equivalence between these notions is the GN-property. Theorem 2.13 provides an example in which a collection of groups has the GN-property over any field (for example odd order abelian groups). By that, Theorem 2.13 allows to determine explicit necessary and sufficient conditions on $K$-admissibility of abelian groups (in terms of local invariants of $K$). Let us mark some more advantages that groups with the GN-property have:

**Remark 4.1.** Let $G$ be a group that has the GN-property over a number field $K$. If $G$ is $K$-preadmissible then there is a $K$-division algebra $D$ which has infinitely many disjoint subfields $L/K$ Galois over $K$ for which $Gal(L/K) = G$. We deduce that for a group $G$ that has the GN-property over a number field $K$, the following conditions are equivalent:

1) $G$ is $K$-preadmissible,
2) $G$ is $K$-admissible,
3) $G$ is infinitely often $K$-admissible,
(4) there is a $K$-division algebra with infinitely many maximal (non-isomorphic) subfields Galois over $K$ with Galois group $G$.

**Proof.** Clearly (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), we shall show (1) $\Rightarrow$ (4). As $G$ is $K$-preadmissible there is a set $T = \{v_i(p)|p||G|, i = 1, 2\}$ of primes of $K$ ($v_1(p) \neq v_2(p)$) and corresponding subgroups $G_v$ of $G$ to every prime $v \in T$ so that for every $p||G|$ and $i \in \{1, 2\}$, $G_{v_i(p)}$ is realizable over $K_{v_i(p)}$ and $G_{v_i(p)}$ contains a $p$-Sylow subgroup.

For $p||G|$, let $D_p$ be the division algebra defined by the invariants:

$$\text{inv}_{v_1(p)}(D_p) = -\text{inv}_{v_2(p)}(D_p) = \frac{1}{|G|(p)}$$

and $\text{inv}_u(D) = 0$ if $u \neq v_i(p), i \in \{1, 2\}$. Let $D = \otimes_{p||G|} D_p$.

Let $L_1, \ldots, L_r$ be a list of maximal subfields of $D$ with Galois group $\text{Gal}(L_i/K) = G$. Since $G$ has the GN-property over $K$, there is a Galois $G$-extension $L_{r+1}/\mathbb{Q}$ such that for every $u \in T$, $\text{Gal}((L_{r+1})_u/K_u) = G_u$ and in which every prime in a finite set $S$ splits completely. By Lemma 3.18 $L_{r+1}$ can be chosen to be disjoint to the field $L_1L_2\ldots L_r$. Thus, $L_{r+1}$ is a maximal $G$-subfield of $D$ that is disjoint from $L_1\ldots L_r$ (and not isomorphic to $L_i$ for any $i \leq r$). \hfill $\square$

**Remark 4.2.** Let $G$ be a group with the GN-property over a number field $K$ and let $D$ be a $K$-division algebra (whose invariants we know) with $\text{inv}_u(D) = \frac{m_u}{n_u}$, for $m_u \neq 0$, $(m_u, n_u) = 1$. Given such $G$, $K$ and $D$, to determine whether $D$ has a maximal subfield $L$ which is Galois over $K$ with Galois group $\text{Gal}(L/K) = G$ again reduces to a group theoretical problem. Indeed, let $U$ be the finite set of primes of $K$ in which $m_u \neq 0$ (the support of $D$). Then $D$ is a $G$-crossed product if and only if $D$ is of exponent $|G|$ and for every $u \in U$ there is a subgroup $G_u \leq G$ that is realizable over $K_u$ and $n_u||G_u|$.

Theorem 1.13 supplies a large class of groups with the GN-property. It can also be used to say more on the $K$-admissibility of group extensions:

**Proposition 4.3.** Let $K$ be a number field and let $m(K)$ denote the number of roots of unity in $K$. Let $G = H \rtimes \Gamma$ for groups $\Gamma, H$ that satisfy:

(1) both $H$ and $\Gamma$ are $K$-preadmissible,
(2) $\Gamma$ has the GN-property,
(3) $(|H|, m(K)||\Gamma||) = 1$.

Then $G$ is $K$-admissible.

Condition (2) can be also replaced by the condition:

(2') $\Gamma$ is a solvable Sylow metacyclic group that satisfies Liedahl’s condition (Condition 3.3).

**Proof.** Fix a rational prime $p||G|$. Let $S_p$ be the set of primes $v$ of $K$ for which there is a subgroup $G_v \leq G$ that contains a $p$-Sylow subgroup of $G$ which is realizable over $K_v$. As $\Gamma$ and $H$ are $K$-preadmissible we have $|S_p| \geq 2$ for all $p||G|$.

Let $D$ be the set of rational primes $p||G|$ for which all primes in $S_p$ are divisors of $p$. If a prime $v \in S_p$ is not a divisor of $p$ then by Proposition 3.9 any $p$-Sylow subgroup $G(p)$ is metacyclic and it satisfies Liedahl’s condition. Such a $G(p)$ is necessarily realizable over infinitely many primes of $K$ (by Theorem 3.24).

So for $p \notin D$, we may fix two primes of $K$, $v_1(p), v_2(p)$ for which:

(G1) the restriction of $v_i(p)$ to $\mathbb{Q}$ does not divide $|G|$,
(G2) $G(p)$ is realizable over $K_{v_i(p)}$. 


(G3) all \( v_i(p), i = 1, 2, p \notin D \).

For a prime \( p \notin D \) for which \( p|\Gamma \) (resp. \( p||H| \)) denote \( \Gamma_{v_i(p)} = \Gamma(p) \) (resp. \( H_{v_i(p)} = H(p) \)) for \( i = 1, 2 \).

For \( p \in D \) for which \( p|\Gamma \) (resp. \( p||H| \)), let \( v_1(p), v_2(p) \) denote two primes of \( K \) for which there is a subgroup \( \Gamma_{v_i(p)} \leq \Gamma \) (resp. \( H_{v_i(p)} \leq H \)) that contains a \( p \)-Sylow subgroup of \( \Gamma \) (resp. \( H \)) so that \( \Gamma_{v_i(p)} \) (resp. \( H_{v_i(p)} \)) is realizable over \( K_{v_i(p)} \). There is such since \( \Gamma \) (resp. \( H \)) is \( K \)-preadmissible.

Note that by the choices above the primes \( v_i(p), i = 1, 2, p||G| \) are distinct. Since \( \Gamma \) has the Grunwald-Neukirch property (or if it is Sylow metacyclic and satisfies Liedahl’s Condition) there is a field \( L \) for which:

- \( \text{Gal}(L/K) \cong \Gamma \),
- \( \text{Gal}(L_{v_i(p)}/K_{v_i(p)}) \cong \Gamma_{v_i(p)} \) for all \( p|\Gamma \) and for all \( i \in \{1, 2\} \),
- \( \text{Gal}(L_{v_i(p)}/K_{v_i(p)}) \cong \{1\} \) for all \( p|H| \),
- \( L \cap K(\mu|H|) = K \) (there is such by Lemma 3.18).

Now since \( G \to \Gamma \) splits we have

\[
\prod_{v \in \text{spec}(K)} \text{Hom}_\Gamma(G_{K_v}, G) \neq \emptyset
\]

and by Theorem 1.13 (note the theorem can be applied since \( (m(L), |H|) = 1 \) the map

\[
\theta^G_v: \text{Hom}_\Gamma(G_K, G) \to \prod_{v \in S} \text{Hom}_\Gamma(G_{K_v}, G)
\]

is surjective for every finite set \( S \). Let \( S = \{v_i(p)|i = 1, 2, p||G|\} \). For every \( p||H| \), let \( \phi_{v_i(p)} \) be an epimorphism \( G_{K_{v_i(p)}} \to H_{v_i(p)} \). Since \( v_i(p) \) splits completely in \( L \), we have

\[
\phi_{v_i(p)} \in \text{Hom}_\Gamma(G_{K_{v_i(p)}}, G).
\]

Let \( \phi: G_K \to \Gamma \) be an epimorphism with kernel \( \text{Gal}(\tilde{K}/L) \) where \( \tilde{K} \) is the algebraic closure of \( K \). For every \( p||\Gamma| \), let \( \phi_{v_i(p)} \in \text{Hom}_\Gamma(G_{K_{v_i(p)}}, G) \) be the entry of \( \theta^G_v(\phi) \) that corresponds to the prime \( v_i(p) \in S \). Since \( \theta^G_v \) is surjective there is an element \( \psi \in \text{Hom}_\Gamma(G_K, G) \) that restricts to \( \phi_{v_i(p)} \) for every \( p||G|, i = 1, 2 \). Let \( M \) be the fixed field of the kernel of \( \psi \). Then \( M \) satisfies:

- (M1) \( \text{Gal}(M/K) = G \)
- (M2) for every \( p|\Gamma|, \text{Gal}(M_{v_i(p)}/K_{v_i(p)}) = \Gamma_{v_i(p)} \)
- (M3) for every \( p||H|, \text{Gal}(M_{v_i(p)}/K_{v_i(p)}) = H_{v_i(p)} \)

and therefore \( M/K \) is a \( K \)-adequate Galois \( G \)-extension. \( \square \)

Remark 4.4. It is important to note that the requirements on \( \Gamma \) posed in Condition (2) of Proposition 4.3 can be eased. The actual requirement that was used in the proof is: given a finite set \( S \) of primes of \( K \) which are not divisors of any prime \( p \) that divides \( |\Gamma| \), there is a \( K \)-adequate \( \Gamma \)-extension \( L/K \) in which every prime of \( S \) splits completely.

Remark 4.5. Proposition 4.3 implies \( G, H \) and \( \Gamma \) are all \( K \)-admissible. In general (omitting \( (|H|, |\Gamma|) = 1 \) if \( H, \Gamma \) are \( K \)-admissible, \( G = H \rtimes \Gamma \) need not be \( K \)-admissible. For example, \( H = C_2, \Gamma = C_2^2, G = H \times \Gamma \). Then \( H, \Gamma \) are \( \mathbb{Q} \)-admissible but \( G \) is not \( \mathbb{Q} \)-preadmissible.

Remark 4.6. In general the \( K \)-preadmissibility of \( G = H \rtimes \Gamma \) does not imply that of \( H \). For example, let \( p \geq 5 \) be an odd rational prime, \( G = C_p \wr C_p \) so that \( H = C_p^p, \Gamma = C_p \) and \( G = H \rtimes \Gamma \) where \( \Gamma \) acts on \( H \) by permuting the \( p \) copies of \( C_p \). Let
Let us give some examples. By Theorem 1.14 an odd order group has the GN-property over \( \mathbb{Q} \). Generally every group of order prime to \( m(K) \) has the GN-property over \( K \). By Theorem 1.15, every abelian group of odd order has the GN-property over any \( K \) and if further \( K \cap \mathbb{Q}(\mu_{2\infty}) \) is not totally real then by Corollary 2.6 any abelian group \( A \) has the GN-property over \( K \).

In [17], Saltman proves that a group that has a generic extension over \( K \) satisfies the GN-property. We may therefore substitute Condition (2) in Proposition 4.3 by:

\[(2^*) \Gamma \text{ has a generic extension over } K.\]

By [18], if \( \mu_p \subseteq K \) then any group of order \( p^3 \) which is not the cyclic group of order 8 has a generic extension over \( K \). Combining Theorem 1.14 we deduce that every group of order \( p^3 \) (which is not the cyclic group of order 8) can be chosen as the cokernel in Proposition 4.3. In [17], a collection of groups are proved to have a generic extension over a number field, in particular, any abelian group that does not have \( C_8 \) as a subgroup and the symmetric group \( S_n \). In [17] it is also proved that the class of groups with a generic extension is closed under wreath products and split epimorphic images, i.e:

\[(1) \text{ if } H, G \text{ have generic extensions then } H \wr G \text{ has a generic extension},\]
\[(2) \text{ if } G \text{ has a generic extension and if the epimorphism } G \to G/N \text{ splits then } G/N \text{ has a generic extension.}\]

Note this proves that the group \( G \) in Remark 4.6 is actually \( K \)-admissible (\( G = C_p \wr C_p \)). This class provides a large class of examples of groups for which \( K \)-preadmissibility is equivalent to \( K \)-admissibility.

Let us consider the following two classes:

**Definition 4.8.** Let \( SD \) be the minimal class of finite groups satisfying the following properties:

\[(1) \{e\} \in SD,\]
\[(2) \text{ if } H \in SD \text{ and } C \text{ is a finite cyclic group, then every semidirect product } C \rtimes H \in SD.\]

**Definition 4.9.** Let \( p \) be a prime. Let \( SC \) (resp. \( SC_p \)) be the minimal class of finite groups satisfying the following properties:

\[(1) \{e\} \in SC \text{ (resp. } \{e\} \in SC_p),\]
\[(2) \text{ if } H \in SC \text{ (resp. } H \in SC_p) \text{ and } C \text{ a finite cyclic group (resp. finite cyclic } p\text{-group), then every semidirect product } C \rtimes H \in SC \text{ (resp. } C \rtimes H \in SC_p),\]
\[(3) \text{ if } G \in SC \text{ (resp. } G \in SC_p) \text{ and } H < G, \text{ then } G/H \in SC \text{ (resp. } G/H \in SC_p).\]

We call such groups *semicyclic*. 

\[K = \mathbb{Q}(\sqrt{p+1}) \text{ (} p + 1 \text{ is a square mod } p \text{). Then } K \text{ has two prime divisors } v_1, v_2 \text{ of } p \text{ for which } K_{v_i} \cong \mathbb{Q}_p. \text{ By [4], the maximal pro-} p \text{ group } G_{\mathbb{Q}_p} \text{ that is realizable over } \mathbb{Q}_p \text{ is topologically generated by 3 elements } x, y, z \text{ that satisfy the relation } x^p[y, z] = 1. \text{ One notices that } G \text{ is an epimorphic image of } G_{\mathbb{Q}_p}. \text{ Thus, } G \text{ is realizable over } K_{v_1}, K_{v_2} \text{ and hence } K\text{-preadmissible. We shall soon see that } G \text{ is also } K\text{-admissible. On the other hand the maximal rank of an abelian group that is realizable over } K_{v_i} \text{ is 4 and hence } H \text{ is not } K\text{-preadmissible.}

**Remark 4.7.** Note that the GN-property is defined in the terminology of decomposition groups. It is a weaker requirement than the surjectivity of the map \( \theta_G \) for every finite set \( S \) as in Theorem 1.14.

Let us give some examples. By Theorem 1.14, an odd order group has the GN-property over \( \mathbb{Q} \). Generally every group of order prime to \( m(K) \) has the GN-property over \( K \). By Theorem 1.15, every abelian group of odd order has the GN-property over any \( K \) and if further \( K \cap \mathbb{Q}(\mu_{2\infty}) \) is not totally real then by Corollary 2.6 any abelian group \( A \) has the GN-property over \( K \).
The minimality of $SC$ (resp. $SD$) implies that every group in $SC$ (resp. $SD$) can be derived as a sequence of operations of type (1-3) (resp. (1-2)).

As we have seen (Example 2.14) for some number fields $K$, $K$-preadmissibility of a group in $SC$ or $SD$ does not guarantee $K$-admissibility. Let us focus on odd groups in $SC$ and $SD$. An odd order group of class $SD$ can be achieved by using only semidirect products with odd kernels. To understand the class of semicyclic groups better, note the following property which is similar to a property proved by Dentzer (in $[5]$) for semiabelian groups.

**Proposition 4.10.** Let $G$ be a finite group (resp. $p$-group). Then $G$ is semicyclic (resp. in $SC_p$) if and only if there exist a cyclic normal subgroup $C$ (resp. cyclic $p$-group) and a proper semicyclic subgroup $H$ (resp. in $SC_p$) for which $G = CH$.

**Proof.** First, if there exist such subgroups $C$ and $H$ then $H$ acts on $C$ by conjugation ($C$ is normal). Therefore there is a surjective homomorphism $\beta : C \rtimes H \to G$. Thus, $G \cong (C \rtimes H)/\ker \beta$ and $G$ is semicyclic (resp. $G \in SC_p$).

For the other direction, let $G$ be semicyclic (resp. $G \in SC_p$) and $G \neq \{1\}$. There is a sequence $(H_i)_{i=1}^r$ such that $H_{i+1} = (C_i \rtimes H_i)/K_i$, $H_1 = \{1\}$, $H_r = G$. Assume $(H_i)_{i=1}^r$ is a sequence of shortest length satisfying the above properties. Then $G$ is not a quotient of $H_{r-1}$, otherwise there would be a shorter sequence for $G$. The subgroups $C' = C_{r-1}K_{r-1}/K_{r-1}, H' = H_{r-1}K_{r-1}/K_{r-1}$ satisfy $C' H' = G$. Now, $H' \cong H_{r-1}/(H_{r-1} \cap K_{r-1})$ but as $G$ is not a quotient of $H_{r-1}$, the group $H'$ is a proper subgroup of $G$. Note $C'$ is normal in $G$ (resp. normal of a $p$-power order) and $H'$ is semicyclic, as a quotient of $H_{r-1}$. Therefore $G = C' H'$ is the required decomposition. \hfill $\square$

**Remark 4.11.** Let $SC_o$ be the minimal class of groups for which:

1. $\{e\} \in SC_o,
2. \text{if } H \in SC_o \text{ and } C \text{ a finite cyclic group of odd order then every semidirect product } C \rtimes H \in SC_o,
3. \text{if } G \in SC_o \text{ and } H \vartriangleleft G, \text{ then } G/H \in SC_o.$

Replacing $SC_p$ by $SC_o$ throughout the proof of Proposition 4.10 we have the following conclusion: An odd order group $G$ is in $SC_o$ if and only if there is a normal cyclic subgroup $C \vartriangleleft G$ (of odd order) and a proper subgroup $G \geq H \in SC_o$ (also of odd order) for which $G = CH$.

**Remark 4.12.** From the Proposition it is clear that the class $SC_p$ (resp. $SC_o$) consists exactly of the semicyclic $p$-groups (resp. semicyclic odd order groups).

**Remark 4.13.** The classes $SC, SD, SC_p, SC_o$ and the class of odd groups in $SD$ are closed to direct products.

Let $q \in \mathbb{N}$ be odd. In $[17]$ (Theorem 3.7) it is proved that if $G = C_q \rtimes H$, $H$ has a generic extension over $K$ and $\mu_q \subseteq K$ then $G$ has a generic extension over $K$. Though this supplies information on $SD$ we shall adopt a more classical approach. The following Theorem is a direct conclusion from Theorems 6.4(b) and 2.5 in $[13]$:

**Theorem 4.14.** (Neukirch) Let $\pi : G \to \Gamma$ be an epimorphism with cyclic odd order kernel. Let $M/K$ be a Galois $\Gamma$-extension and $\phi : G_K \to \text{Gal}(M/K)$ an epimorphism. Let $S$ be a finite set of primes of $K$. For every $v \in S$, let $\Gamma_v = \text{Gal}(M_v/K_v)$, $G_v =$
$\pi^{-1}(\Gamma_v)$. For every $v \in S$, let $\psi_v : G_{K_v} \to G$ be a solution to:

\[
\begin{CD}
G_{K_v} @>\psi_v>> G_v \\
@VV\pi V @VV\phi V \\
\Gamma_v @>>> 0.
\end{CD}
\]

If there is a global solution to the embedding problem:

\[
\begin{CD}
G_K @>>\phi>> \Gamma @>>> 0,
\end{CD}
\]

then there is a surjective solution $\psi : G_K \to G$ that restricts to $\psi_v$ for any $v \in S$.

Note that in case of a semidirect product $G = H \rtimes \Gamma$ the map $\pi : G \to \Gamma$ splits and the global embedding problem has a solution. Therefore the map

\[
\theta^\Gamma_G : \text{Hom}_\Gamma(G_K, G)_{\text{sur}} \to \prod_{v \in S} \text{Hom}_\Gamma(G_{K_v}, G)
\]

is surjective. By iteration of the process we shall have:

**Corollary 4.15.** Let $K$ be a number field. An odd order group in $\mathcal{SD}$ has the GN-property over $K$.

**Proof.** Let $K$ be a number field and $S$ a finite set of primes of $K$. Let us prove by induction on $|G|$ that the map

\[
\theta_G : \text{Hom}(G_K, G)_{\text{sur}} \to \prod_{v \in S} \text{Hom}(G_{K_v}, G),
\]

is surjective. The statement is trivial for $G = \{e\}$. Let $G = C \rtimes H$ for $H \in \mathcal{SD}$ and $C \neq \{e\}$ a cyclic group of odd order.

Fix an element $(f_v)_{v \in S} \in \prod_{v \in S} \text{Hom}(G_{K_v}, G)$. Denote the projection $G \to H$ by $\pi$. By induction the map

\[
\theta_H : \text{Hom}(G_K, H)_{\text{sur}} \to \prod_{v \in S} \text{Hom}(G_{K_v}, H),
\]

is surjective and therefore the element

\[
(4.1) \quad (h_v)_{v \in S} = (\pi \circ f_v)_{v \in S} \in \prod_{v \in S} \text{Hom}(G_{K_v}, H),
\]

has a source $\psi$ (under $\theta_H$) that defines a Galois $H$-extension $M/K$. Since $f_v$ is a solution to the embedding problem:

\[
\begin{CD}
G_{K_v} @>h_v>> G_v \\
@VV\pi V @VVH V \\
\Gamma_v @>>> 0,
\end{CD}
\]

$(f_v)_{v \in S}$ is an element of $\prod_{v \in S} \text{Hom}_H(G_{K_v}, G)$. Thus, by Theorem 4.14, $(f_v)_{v \in S}$ is in the image of $\theta^H_G$ and hence of $\theta_G$. \qed
One can actually prove more. For this, the following refinement of a splitting sequence can be useful:

**Definition 4.16.** The group extension

\[(4.2) \quad 1 \to H \to G \to \Gamma \to 1\]

with an epimorphism \(\pi : G \to \Gamma\) *meta-splits* if for every metacyclic subgroup \(D \leq \Gamma\) the group extension

\[1 \to H \to \pi^{-1}(D) \to D \to 1\]

splits.

For an example of a sequence that meta-splits but not splits see Remark 4.24. The following Remark discusses the compatibility between the approximation property of groups in \(SD\) and Theorem 1.13.

**Remark 4.17.** Let \(K\) be a number field and assume the extension in 4.2 meta-splits. If \(\Gamma \in SD\) is of odd order and \((|H|, m(K)) = 1\) then \(G\) has the GN-property over \(K\).

In particular, if \(G\) is \(K\)-preadmissible then \(G\) is also \(K\)-admissible.

**Proof.** Let \(S\) be a finite set of primes of \(K\), we may assume \(S\) contains all rational primes \(p\) for which \(p|\)\(|G|\). As a direct consequence of the proof of Corollary 4.15 the map

\[(4.3) \quad \theta_\Gamma : \text{Hom}(G_K, \Gamma)_{\text{sur}} \to \prod_{v \in S} \text{Hom}(G_{K_v}, \Gamma),\]

is surjective. For any \(\Gamma\)-extension \(L/K\) and \(v \notin S\), \(\text{Gal}(L_v/K_v)\) is metacyclic and since the exact sequence 4.12 is assumed to meta-split we have

\[(4.4) \quad \prod_{v \notin S} \text{Hom}_\Gamma(G_{K_v}, G) \neq \emptyset.\]

Let \((\phi^v)_{v \in S} \in \prod_{v \in S} \text{Hom}(G_{K_v}, G)\). Then by Proposition 4.3 there is an epimorphism \(\psi : G_K \to \Gamma\) so that \(\theta_\Gamma(\psi) = (\pi \circ \phi^v)_{v \in S}\). For the \(\Gamma\)-extension \(L/K\) defined by \(\psi\) we have

\[
\prod_{v \in S} \text{Hom}_\Gamma(G_{K_v}, G) \neq \emptyset.\]

Together with Assertion 4.4 the conditions of Theorem 1.13 are satisfied and for the \(\Gamma\)-extension \(L/K\) we have \(\theta_G\) is surjective. Thus, there is an epimorphism \(\phi : G_K \to G\) for which \(\theta_G(\phi) = (\phi^v)_{v \in S}\). By that we have showed \(\theta_G\) is surjective and \(G\) has the GN-property over \(K\). \(\square\)

Let us construct an example of a collection of \(K\)-preadmissible semicyclic groups. Let \(p\) be an odd prime, \(v\) a prime of \(K\) that divides \(p\) and \(k = [K_v : \mathbb{Q}_p] + 2\). Then the maximal pro-\(p\) group \(\text{Gal}(K(p)/K)\) is either a free pro-\(p\) group on \(k - 1\) generators or a pro-\(p\) group generated by \(x_1, ..., x_k\) with one relation (see \[4\]):

\[x_1^q[x_1, x_2]...[x_{k-1}, x_k].\]

In the latter case \(k\) is even and one can construct an epimorphism \(G_{K_v} \to F_p(\frac{k}{2})\), where \(F_p(\frac{k}{2})\) is the free pro-\(p\) group on \(\frac{k}{2}\) generators, and hence any \(p\)-group of rank \(\frac{k}{2}\) is realizable over \(K_v\).
More precisely, let $N_v$ denote the maximal rank of a free pro-$p$ group realizable over $K_v$. If $K_v$ does not contain the $p$-th roots of unity then $N_v = k - 1$ and otherwise, by [30], $N_v = \frac{k}{2}$. Let

$$N_p(K) = \max_{v_i, v_j \mid p} \{\min\{N_{v_i}, N_{v_j}\}\}.$$  

where $v_i, v_j$ are distinct primes of $K$ that divide $p$. In case $p$ has a unique prime divisor in $K$ define $N_p(K)$ to be 1. Let us also denote the set of homomorphisms $\{\pi : G_{K_v} \to G\}$ that split through a free pro-$p$ group (and hence through a free pro-$p$ group of rank $\leq N_v$) by $S_v(G)$.

Any $p$-group of rank $\leq N_p(K)$ is $K$-preadmissible. More generally, any group $G$ for which a $p$-Sylow subgroup $G(p)$ is of rank $\leq N_p(K)$ for every $p||G|$ is $K$-preadmissible. We shall prove any such semicyclic group of odd order is also $K$-admissible:

**Corollary 4.18.** Let $\Gamma$ be a semicyclic group of odd order that has $p$-Sylow subgroups $\Gamma(p)$ generated by at most $N_p(K)$ generators for every $p||G|$. Then $\Gamma$ is $K$-admissible.

Corollary 4.18 will be essentially deduced from the following Lemma:

**Lemma 4.19.** For a prime $v$ of $K$, let $\theta_v, N_v, S_v(G)$ be as above. Then for every $G \in \mathcal{SC}_o$ and every finite set $S$ of primes of $K$,

$$\prod_{v \in S} S_v(G) \subseteq \text{Im}(\theta_v).$$  

**Remark 4.20.** In the proof we shall use the fact that the free pro-$p$ group $F$ is projective in the category of profinite groups (not only in the category of pro-$p$ groups). Indeed given a $p$-group $P$ and two epimorphisms $\phi : F \to P, \psi : G \to P$, there is a $p$-Sylow subgroup $G(p) \leq G$ that maps onto $P$ (via $\psi$) and hence one can lift $\phi$ to a homomorphism $\hat{\phi} : F \to G(p)$ so that $\hat{\phi} \circ \psi = \phi$.

**Proof.** (Lemma 4.19) Fix a finite set $S$ of primes of $K$. For $G = \{e\}$ the claim is clear. The class of odd order semicyclic groups is generated by semidirect products with odd order cyclic groups and by quotients. We shall therefore show that if

1. $\Gamma \in \mathcal{SC}_o$,
2. $\prod_{v \in S} S_v(\Gamma) \subseteq \text{Im}(\theta_v)$ and
3. either $G = C \rtimes \Gamma$ for a cyclic odd order group $C$ or $G = \Gamma/K$ for $\Gamma$ of odd order, then $\prod_{v \in S} S_v(G) \subseteq \text{Im}(\theta_v)$. This will prove the assertion.

**Case A:** $G = C \rtimes \Gamma$ for some cyclic group $C$ of odd order. Let $\pi : G \to \Gamma$ be the projection. Fix an element $(g_v)_{v \in S} \in \prod_{v \in S} S_v(G)$. Then $f_v = \pi \circ g_v$ also splits through a free pro-$p$ group. Thus, $(f_v)_{v \in S} \in \prod_{v \in S} S_v(\Gamma)$. By the hypothesis there is an epimorphism $f : G_K \to \Gamma$ such that $\theta(\Gamma)(f) = (f_v)_{v \in S}$. Note that the diagram:

$$G \xrightarrow{\pi} \Gamma \xrightarrow{f_v} 0,$$

is commutative and hence $(g_v)_{v \in S} \in \prod_{v \in S} \text{Hom}_\Gamma(G_{K_v}, G)$. By Theorem 4.14 the map

$$\theta_G : \text{Hom}_\Gamma(G_K, G)_{\text{sur}} \to \prod_{v \in S} \text{Hom}_\Gamma(G_{K_v}, G)$$
is surjective. Thus \((\theta^*_G)^{-1}((g_v)_{v\in S}) \neq \emptyset\) and \(\theta_G^{-1}((g_v)_{v\in S}) \neq \emptyset\).

**Case B:** \(G = \Gamma/K\). Let \(\pi : \Gamma \to G\) be the projection. Fix an element \((g_v)_{v\in S} \in \prod_{v\in S} S_v(G)\). We shall make use of the following maps:

\[
\begin{align*}
\pi_* &: \text{Hom}(G_K, \Gamma) \to \text{Hom}(G_K, G), \\
\pi_*(v) &: \text{Hom}(G_{K_v}, \Gamma) \to \text{Hom}(G_{K_v}, G)
\end{align*}
\]

and the induced map

\[
\overline{\pi}_*(v) : S_v(\Gamma) \to S_v(G)
\]

for \(v \in S\). The map \(\overline{\pi}_*(v)\) is surjective since every \(f_v : G_{K_v} \to G\) that splits through \(F_p(N_v)\) can be lifted to \(f_v : G_{K_v} \to \Gamma\):

\[
\begin{array}{c}
G_{K_v} \\
\downarrow \\
F_p(N_v) \\
\downarrow \\
\Gamma \\
\downarrow \pi \\
G \\
\downarrow 0
\end{array}
\]

By Remark 4.20, the map \(\prod_{v\in S} \overline{\pi}_*(v)\) is surjective and hence there is an element \((\tilde{g}_v)_{v\in S} \in \prod_{v\in S} S_v(\Gamma)\) for which \(\big(\prod_{v\in S} \overline{\pi}_*(v)\big)((\tilde{g}_v)_{v\in S}) = (g_v)_{v\in S}\). By hypothesis \(\prod_{v\in S} S_v(\Gamma) \subseteq Im(\theta_\Gamma)\). Thus, there is an element \(\tilde{g} : G_K \to \Gamma\) for which \(\big(\prod_{v\in S} \overline{\pi}_*(v)\big) \circ \theta_\Gamma(\tilde{g}) = (g_v)_{v\in S}\).

As the following diagram

\[
\begin{array}{c}
\text{Hom}(G_K, \Gamma)_{\text{sur}} \\
\downarrow \pi_* \\
\prod_{v\in S} \text{Hom}(G_{K_v}, \Gamma) \\
\downarrow \theta_G \\
\prod_{v\in S} \text{Hom}(G_{K_v}, G)
\end{array}
\]

is commutative, we have \((\theta_G \circ \pi_*)(\tilde{g}) = (g_v)_{v\in S}\) and \((g_v)_{v\in S} \in Im(\theta_G)\).

Notice that this language is compatible with Theorem 1.13 and hence one can actually prove more than Corollary 4.18. Let us apply Lemma 4.19 to prove the following corollary (Corollary 4.18 will follow by choosing \(H = \{1\}\)).

**Corollary 4.21.** Let

\[
(4.7) \quad 1 \to H \to G \to \Gamma \to 1
\]

be an extension that meta-splits. If

1. \(\Gamma \in SC_o\),
2. \(|H|, m(K)\) = 1 and
3. \(G(p)\) is generated by at most \(N_p(K)\) generators for every \(p\parallel |G|\),

then \(G\) is \(K\)-admissible.

**Proof.** Let \(\pi\) be the epimorphism \(G \to \Gamma\). Let \(S_G\) be the set of rational primes \(p\) that divide \(|G|\) and let \(S^1_G\) be the subset of \(S_G\) of primes \(p\) for which \(N_p(K) > 1\).

For every \(p \in S^1_G\), fix two primes \(v_1(p), v_2(p)\) of \(K\) that divide \(p\) and for which \(F_p(N_p(K))\) is an epimorphic image of \(G_{K_{v_i(p)}}\), \(i = 1, 2\). Then \(G(p)\) is an epimorphic image of \(G_{K_{v_i(p)}}\) via a homomorphism \(f_{v_i(p)}\) that factors through a free pro-\(p\) group.
Let $T$ be the set $\{v_1(p), v_2(p) | p \in S_G^1\}$. Let $S$ be the rest of the primes $v$ of $K$ whose restriction to $\mathcal{O}$ is in $S_G$.

By Lemma 4.19 applied with $S \cup T$ there is an epimorphism $\psi : G_K \to \Gamma$ that induces a $\Gamma$-extension $L/K$ for which

1) $\theta_T(\psi) = (\pi \circ f_v)_{v \in S}$ for $T$,
2) every prime in $S$ splits completely in $L/K$,
3) $\Lambda \supset K(\mu_{2}) = K$.

Condition (2) can be guaranteed by Lemma 4.19 since a trivial homomorphism always splits through a free pro-$q$ group for every rational $q$. Condition (3) can be guaranteed by Lemma 3.18. Now as the sequence 4.7 meta-splits we have

\begin{equation}
\Hom_{\Gamma}(G_{K_v}, G) \neq \emptyset
\end{equation}

for every $v \notin S \cup T$. Condition 4.8 holds trivially for every $v \in S$. For $v \in T$, $f_v \in \Hom_{\Gamma}(G_{K_v}, G)$ and hence Condition 4.8 is satisfied for all $v$. By Theorem 1.13 there is an epimorphism $\phi \in \Hom_{\Gamma}(G_K, G)$ for which $\theta_G(\phi) = (f_v)_{v \in T}$. Then the $G$-extension $M/K$ induced by $\phi$ satisfies $\Gal(M_{v(p)}/K_{v(p)}) \supseteq G(p)$ for every $p \in S_G^1$, its chosen $v_1(p), v_2(p)$ and for some $p$-Sylow subgroup $G(p)$.

To complete the proof it is left to take care of the primes $p||G|$ with $N_p(K) = 1$. For such a $p$, the $p$-Sylow subgroup $G(p)$ is cyclic and by Chebotarev’s density theorem there are infinitely many primes $v$ of $K$ for which $\Gal(M_v/K_v) \cong G(p)$. We conclude $M/K$ is $K$-admissible and $G$ is $K$-admissible. \hfill $\Box$

Remark 4.22. The following condition:

(2)’ $\Gamma$ is semicyclic of odd order and for every $p||\Gamma|$ any $p$-Sylow subgroup $\Gamma(p)$ is generated by $N_p(K)$ generators,

can replace condition (2) of Proposition 4.3 (in such case $\Gamma$ is also automatically $K$-preadmissible). In particular we deduce that in this context $\Gamma$ is $K$-admissible. Condition (2)’ clearly allows to construct a $\Gamma$-extension $M/K$ for which:

1) for every $p||\Gamma|$, there are two primes $v$ of $K$ for which $\Gal(M_v/K_v) = \Gamma(p)$, denote this set by $T$,

2) given a set $S$ of primes of $K$ for which $S \cap T = \emptyset$, $M$ can be chosen so that every $v \in S$ splits completely in $M$. By Remark 4.3, this is exactly the required property of $\Gamma$ in the proof of Proposition 4.3.

Remark 4.23. Let $p$ be an odd prime. A short calculation shows that the smallest example of a $p$-group that is not semicyclic is the Hiesenberg group:

\[N = \langle x, y, z | x^p = y^p = z^p = 1, [x, y] = z, [x, z] = [y, z] = 1 \rangle.\]

It is a nilpotent group of order $p^3$. As we mentioned earlier for groups of order $p^3$, $N$ is $K$-preadmissible if and only if $N$ is $K$-admissible.

Remark 4.24. Let us give an example of a meta-split group extension that does not split. Let $G$ be given by:

\[G = \langle x, y, z, w, u | x^p = y^p = z^p = w^p = u^p = 1, [x, u] = [y, u] = [z, u] = [w, u] = [x, y] = [z, w] = [x, z] = 1, [x, w] = [y, z] = [y, w] = u \rangle\]

and let $\Gamma$ be the group $C_3^3$. There is a unique homomorphism $\pi : G \to \Gamma$ that satisfies:

\[
\pi(x) = (1, 0, 0), \pi(y) = (0, 1, 0), \pi(z) = (0, 1, 0), \pi(w) = (0, 0, 1), \pi(u) = (0, 0, 0).
\]
The kernel of this map is \( K = \langle y^{-1}z, u \rangle \). Straightforward calculations show that the group extension

\[
1 \to K \to G \to \Gamma \to 1
\]

meta-splits but does not split.

We now have a wider collection of examples in which \( K \)-preadmissibility is equivalent to \( K \)-admissibility. In fact the only groups for which we have found this equivalence does not hold are of even order. One can ask how wide is this class and whether every odd order \( p \)-group has the GN-property. We do not know of an example of an odd order group which is \( K \)-preadmissible but not \( K \)-admissible.

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Technion

E-mail address: neftind@tx.technion.ac.il