THE ZERO INERTIA LIMIT OF ERICKSEN-LESLIE’S MODEL FOR LIQUID CRYSTALS

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Abstract. In this paper we study the zero inertia limit that is from the hyperbolic to parabolic Ericksen-Leslie’s liquid crystal flow. By introducing an initial layer and constructing an energy norm and energy dissipation functional depending on the solutions of the limiting system, we derive a global in time uniform energy bound to the remainder system under the small size of the initial data.

Keywords. Zero inertia limit; Ericksen-Leslie model; Uniform energy bounds; Initial layer

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1. Introduction and main theorem

The hydrodynamic theory of incompressible liquid crystals was established by Ericksen [2, 3, 4] and Leslie [11, 12] in the 1960’s (see also Section 5.1 of [17]). In this paper, we study the following hyperbolic Ericksen-Leslie’s incompressible liquid crystal flow in \((t, x) \in \mathbb{R}^+ \times \mathbb{T}^3\)

\[
\begin{aligned}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \frac{1}{2\mu_4} \Delta u^\varepsilon + \nabla p^\varepsilon &= -\text{div}(\nabla d^\varepsilon \otimes \nabla d^\varepsilon) + \text{div}\sigma^\varepsilon, \\
\text{div} u^\varepsilon &= 0, \\
\varepsilon D_{u^\varepsilon}^2 d^\varepsilon &= \Delta d^\varepsilon + \gamma^\varepsilon d^\varepsilon + \lambda_1 (D_{u^\varepsilon} d^\varepsilon + B d^\varepsilon) + \lambda_2 A^\varepsilon d^\varepsilon, \\
|d^\varepsilon| &= 1,
\end{aligned}
\]

(1.1)

where the Lagrangian multiplier \(\gamma^\varepsilon\) (for the geometric constraint \(|d^\varepsilon| = 1\)) is

\[
\gamma^\varepsilon = -\varepsilon |D_{u^\varepsilon} d^\varepsilon|^2 + |\nabla d^\varepsilon|^2 - \lambda_2 A^\varepsilon : d^\varepsilon \otimes d^\varepsilon,
\]

(1.2)

and the extra stress \(\sigma^\varepsilon\) is

\[
\sigma^\varepsilon \equiv \sigma(u^\varepsilon, d^\varepsilon) = \mu_1 (A^\varepsilon : d^\varepsilon \otimes d^\varepsilon)d^\varepsilon \otimes d^\varepsilon + \mu_2 (D_{u^\varepsilon} d^\varepsilon + B d^\varepsilon) \otimes d^\varepsilon + \mu_3 d^\varepsilon \otimes (D_{u^\varepsilon} d^\varepsilon + B d^\varepsilon) + \mu_5 (A^\varepsilon d^\varepsilon) \otimes d^\varepsilon + \mu_6 d^\varepsilon \otimes (A^\varepsilon d^\varepsilon).
\]

(1.3)
Here \( u^\varepsilon(t, x) \in \mathbb{R}^3 \) is the bulk velocity, \( d^\varepsilon(t, x) \in \mathbb{S}^2 \) is the unit director fields of the liquid molecules, \( p^\varepsilon(t, x) \in \mathbb{R} \) is the pressure. The notation \( D_{u^\varepsilon}d^\varepsilon \) denotes the first order material derivative of \( d^\varepsilon \) with respect to the velocity \( u^\varepsilon \) by
\[
D_{u^\varepsilon}d^\varepsilon = \partial_t d^\varepsilon + u^\varepsilon \cdot \nabla d^\varepsilon,
\]
and \( D_{u^\varepsilon}^2d^\varepsilon \) represents the second order material derivative,
\[
D_{u^\varepsilon}^2d^\varepsilon = D_{u^\varepsilon}(D_{u^\varepsilon}d^\varepsilon).
\]
In this paper we work on the periodic spatial domain \( \mathbb{T}^3 = \mathbb{R}^3/\mathbb{L}^3 \), where \( \mathbb{L}^3 \subset \mathbb{R}^3 \) is any 3-dimensional lattice. The notations \( A^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon + \nabla u^\varepsilon^\top) \) and \( B^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon - \nabla u^\varepsilon^\top) \) represent the rate of strain tensor, skew-symmetric part of the strain rate of by fluid velocity, respectively. More precisely, the entries of \( A^\varepsilon \) and \( B^\varepsilon \) are given as
\[
A_{ij}^\varepsilon = \frac{1}{2}(\partial_j u_i^\varepsilon + \partial_i u_j^\varepsilon), \quad B_{ij}^\varepsilon = \frac{1}{2}(\partial_j u_i^\varepsilon - \partial_i u_j^\varepsilon)
\]
for \( 1 \leq i, j \leq 3 \). One notices \( D_{\varepsilon}^i = -B_{ij}^\varepsilon \). The components of the vector \( B^\varepsilon d^\varepsilon \) and \( A^\varepsilon d^\varepsilon \) are \((B^\varepsilon d^\varepsilon)_i = B^\varepsilon_k d_k^\varepsilon\) and \((A^\varepsilon d^\varepsilon)_i = A^\varepsilon_k d_k^\varepsilon\), respectively. The entries of the matrix \( \nabla d^\varepsilon \circ \nabla d^\varepsilon \) are \((\nabla d^\varepsilon \circ \nabla d^\varepsilon)_ij = \partial_j d_k^\varepsilon \partial_i d_k^\varepsilon\) and the symbol \( a \circ b \) means \((a \circ b)_{ij} = a_i b_j \) for \( 1 \leq i, j \leq 3 \). For any two matrix \( M, N \in \mathbb{R}^{3 \times 3} \), we denote by \( M : N = M_{ij}N_{ij} \). Furthermore, the symbol \( \text{div}M \) means a vector field in \( \mathbb{R}^3 \) with the components \( \text{div}M_i = \partial_j M_{ij} \) for \( 1 \leq i \leq 3 \). We emphasize that the Einstein summation convention is used throughout this paper.

The parameter \( \varepsilon > 0 \) is the inertia constant, which is usually small in the physical experiments. \( \mu_4 > 0 \) is the viscosity of the flow. The material coefficients \( \lambda_1 \leq 0 \) and \( \lambda_2 \in \mathbb{R} \) reflect the molecular shape and the slippery part between the fluid and the particles. The coefficients \( \mu_i (i = 1, 2, 3, 5, 6) \), which may depend on material and temperature, are usually called Leslie coefficients, and are related to certain local correlations in the fluid. Moreover, the previous coefficients have the relations \( \mu_4 > 0 \) and
\[
\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \quad \mu_2 + \mu_3 = \mu_6 - \mu_5 .
\]
The first two relations are necessary conditions in order to satisfy the equation of motion identically, while the third relation is called Parodi’s relation, which is derived from Onsager reciprocal relations expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium.

For the system (1.1) we take the initial data independent of \( \varepsilon \), i.e.,
\[
u^\varepsilon(0, x) = u^{in}(x) \in \mathbb{R}^3, \quad d^\varepsilon(0, x) = d^{in}(x) \in \mathbb{S}^2, \quad (D_{u^\varepsilon}d^\varepsilon)(0, x) = \tilde{d}^{in}(x) \in \mathbb{R}^3
\]
with the compatibilities
\[
\text{div} u^{in}(x) = 0, \quad d^{in}(x) \cdot \tilde{d}^{in}(x) = 0.
\]

1.1. Initial layer vs well-prepared initial data. Formally letting \( u^\varepsilon \rightarrow u_0 \) and \( d^\varepsilon \rightarrow d_0 \) as \( \varepsilon \rightarrow 0 \) in the hyperbolic liquid crystal system (1.1) deduces to the parabolic liquid crystal model
\[
\begin{align*}
\partial_t u_0 + u_0 \cdot \nabla u_0 - \frac{1}{2} \mu_4 \Delta u_0 + \nabla p_0 &= -\text{div}(\nabla d_0 \circ \nabla d_0) + \text{div} \sigma_0, \\
\text{div} u_0 &= 0, \\
-\lambda_1 (D_{u_0}d_0 + B_0d_0) &= \Delta d_0 + \gamma_0 d_0 + \lambda_2 A_0d_0, \\
|d_0| &= 1,
\end{align*}
\]
where the Lagrangian multiplier \( \gamma_0 \) is
\[
\gamma_0 = |\nabla d_0|^2 - \lambda_2 A_0 : d_0 \circ d_0,
\]
and the extra stress \( \sigma_0 \) is
\[
\sigma_0 = \sigma(u_0, d_0),
\]
i.e. replacing \((u^\varepsilon, d^\varepsilon)\) by \((u_0, d_0)\) in (1.3). Furthermore, the initial data of the limit system (1.10) will naturally be
\[
u_0(0, x) = u^{in}(x) \in \mathbb{R}^3, \quad d_0(0, x) = d^{in}(x) \in \mathbb{S}^2.
\]

The limit considered in this paper is a limit from a hyperbolic-type system for \(\varepsilon > 0\) to a parabolic system for \(\varepsilon = 0\). One notices that \(d^\varepsilon\)-equation in (1.1) is a system of wave equations with two initial conditions, while the \(d_0\)-equation in (1.10) is a parabolic system with only a single initial condition. In general, the solution \((u_0, d_0)\) to the limit system (1.10)-(1.13) does not satisfy the third initial condition in (1.8). To overcome this disparity, we will take two ways: 1) introduce an initial layer in times; 2) give a well-prepared initial condition.

**Initial layer.** This disparity between the initial conditions of the hyperbolic type system (1.1) and of the parabolic type system (1.10) indicates that one should expect an initial layer in time, appearing in the limit process \(\varepsilon \to 0\). Specifically, this disparity, denoted by \(D^{in}(x)\), between the initial conditions (1.8) and (1.13) is defined as
\[
D^{in} := \tilde{d}^{in} - D_{u^0} d_0|_{t=0} = \tilde{d}^{in} + B^{in}d^{in} + \frac{1}{\lambda_1} (\Delta d^{in} + \gamma_0^{in}d^{in} + \lambda_2 A^{in}d^{in}).
\]
Here \(A^{in} = \frac{1}{2}(\nabla u^{in} + (\nabla u^{in})^\top)\), \(B^{in} = \frac{1}{2}(\nabla u^{in} - (\nabla u^{in})^\top)\) and \(\gamma_0^{in} = |\nabla d^{in}|^2 - \lambda_2 A^{in} : d^{in} \otimes d^{in}\).

We will justify rigorously this limit by employing the Hilbert expansion method in which the leading term is given by solutions to the limit system (1.10). In this approach, the solution to the limit system (1.10) is known beforehand. Then a special class of solutions to the original system (1.1) can be constructed around the limit system. A key of this approach is to construct a correct ansatz of the solutions to the original system (1.1). Besides the limit system (1.10) and the remainder term \((u^\varepsilon_R, d^\varepsilon_R)\), an initial layer \(\varepsilon^\beta d_I(\frac{t}{\varepsilon^\gamma}, x)\) to absorb the disparity \(D^{in}(x)\) should be included in the ansatz. More precisely, we take the following ansatz of the solution \((u^\varepsilon, d^\varepsilon)\) to the system (1.1)
\[
u(t, x) = u_0(t, x) + \sqrt{\varepsilon} u^\varepsilon_R(t, x), \quad d(t, x) = d_0(t, x) + \varepsilon^\beta d_I(\frac{t}{\varepsilon^\gamma}, x) + \sqrt{\varepsilon} d^\varepsilon_R(t, x)
\]
for a fixed \(\beta > 0\) to be determined, where \((u_0, d_0)\) is the solution to (1.10) with initial data (1.13).

It is easy to derive by plugging the expansions (1.15) into the system (1.1) that the leading relation is
\[
\varepsilon^{1-\beta} \partial^2 \tau \partial_I - \lambda_1 \partial_I \partial_I = \varepsilon^\beta \Delta d_I,
\]
where \(\tau = \frac{t}{\varepsilon^{\gamma}}\). Then we can design the initial layer satisfying the following linear damped wave system on \((\tau, x) \in \mathbb{R}^+ \times \mathbb{T}^3\) (called initial layer system):
\[
\begin{aligned}
\partial^2 \tau \partial_I + \frac{\gamma_0^{in}}{\varepsilon^{\gamma}} \partial_I \partial_I &= \varepsilon^{2\beta-1} \Delta d_I, \\
d_I(\infty, x) &= \lim_{\tau \to \infty} d_I(\tau, x) = 0, \\
\partial_\tau d_I(0, x) &= D^{in}(x).
\end{aligned}
\]
We emphasize that if \(\beta > \frac{1}{2}\), the Laplacian term \(\varepsilon^{2\beta-1} \Delta d_I\) is also a higher order term as \(\varepsilon \to 0\), which means that it can be ignored. For instance, in Jiang-Luo-Tang-Zarnescu’s work [10] to justify this limit corresponding to the background velocity \(u^\varepsilon \equiv 0\), the value of \(\beta\) is taken as \(\beta = 1\), and ignored the term \(\varepsilon \Delta d_I\). For completeness and generality (for example, if we consider the case \(\lambda_1 = 0\) in a forthcoming separate paper, the term \(\varepsilon^{2\beta-1} \Delta d_I\) must kept, and \(\beta\) should be taken as \(1/2\)), we keep this term in the initial layer structure here. Then, we can easily solve (1.16):
\[
d_I(\tau, x) = 2\varepsilon^{-\beta} \left(\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon \Delta}\right)^{-1} \exp \left(\frac{\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon \Delta}}{2\varepsilon^{1-\beta}} \tau\right) D^{in}(x).
\]
Here the operator \(\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon \Delta}\right)^{-1} \exp \left(\frac{\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon \Delta}}{2\varepsilon^{1-\beta}} \tau\right)\) is understood in the sense of Fourier multiplier. As a consequence, the initial layer \(d_I\) designed in (1.17) can be explicitly
presented as
\[
\varepsilon^\beta d_f^{\beta}(\frac{t}{\varepsilon}, x) = \varepsilon D_f^\varepsilon(t, x) := 2\varepsilon \left(\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon\Delta}\right)^{-1} \exp \left(\frac{\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon\Delta}}{2\varepsilon}\right) D^{in}(x). \tag{1.18}
\]

One observes that \(\varepsilon^\beta d_f^{\beta}(\frac{t}{\varepsilon}, x)\) is independent of \(\beta > 0\). In other words, this initial layer is an intrinsic structure resulted from the disparity \(D^{in}(x)\) of the initial conditions between the original system (1.1) and the limit equations (1.10). Since the real part \(\Re c(\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon|k|^2}) < 0\) for all \(k \in \mathbb{Z}^3\), both the initial layer \(D_f^\varepsilon(t, x)\) and its time derivative \(\partial_t D_f^\varepsilon(t, x)\) exponentially decay to zero as \(\varepsilon \to 0\) for every \((t, x) \in \mathbb{R}^+ \times \mathbb{T}^3\). This means that the disparity \(D^{in}(x)\) just affects the whole evolution process in a very short beginning time.

We now write the ansatz (1.15) as the form
\[
u^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon}u_R^\varepsilon(t, x), \quad d^\varepsilon(t, x) = d_0(t, x) + \varepsilon D_f^\varepsilon(t, x) + \varepsilon d_R^\varepsilon(t, x). \tag{1.19}
\]

Then, plugging the ansatz (1.19) into the original system (1.1) implies that the remainder \((u_R^\varepsilon, d_R^\varepsilon)\) satisfies the following system
\[
\begin{cases}
\partial_t u_R^\varepsilon - \frac{1}{2} \mu_1 \Delta u_R^\varepsilon + \nabla p_R^\varepsilon = \mu_1 \div \left(\frac{\Lambda_R^\varepsilon}{\varepsilon} : d_0 \otimes d_0\right) + \kappa u + \div (C_u + T_u + \sqrt{\varepsilon} R_u) + \varepsilon \div Q_u(D_f), \\
\varepsilon \div u_R^\varepsilon = 0,
\end{cases}
\]
\[
D_u^2 u_R^\varepsilon + \sqrt{\varepsilon} u_R^\varepsilon d_R^\varepsilon = \varepsilon \Delta d_R^\varepsilon + \partial_t (u_R^\varepsilon \cdot \nabla d_0 + \sqrt{\varepsilon} u_R^\varepsilon \cdot \nabla D_f^\varepsilon)
\]
with the constraint
\[
2d_0 \cdot (d_f^\varepsilon + \varepsilon D_f^\varepsilon) + \varepsilon |d_f^\varepsilon|^2 = 0, \tag{1.21}
\]
where the tensor \(C_u\) is
\[
C_u = \mu_2 \left(D_u + \sqrt{\varepsilon} u_R^\varepsilon d_R^\varepsilon + B_R^\varepsilon d_0\right) \otimes d_0 + \mu_3 d_0 \otimes (D_u + \sqrt{\varepsilon} u_R^\varepsilon d_R^\varepsilon + B_R^\varepsilon d_0)
\]
\[
+ \mu_5 (A_R^\varepsilon d_0) \otimes d_0 + \mu_6 d_0 \otimes (A_R^\varepsilon d_0), \tag{1.22}
\]
the vector field \(C_d\) is
\[
C_d = \lambda_1 B_R d_0 + \lambda_2 A_R d_0, \tag{1.23}
\]
and the tedious terms \(T_u, \kappa, S_d, \rho_d, \rho_u, Q_u(D_f)\) and \(Q_d(D_f)\) are defined as the forms (A.2), (A.3), (A.4), (A.5), (A.6), (A.7), (A.12) and (A.21) in Appendix A respectively.

Next we consider the initial conditions of the remainder system (1.20). Since our goal is to seek a solution to (1.1) with the form (1.19), the initial data of \((u_R^\varepsilon, d_R^\varepsilon)\) should subject to
\[
\begin{cases}
\sqrt{\varepsilon} u_R^\varepsilon(0, x) = u^\varepsilon(0, x) - u_0(0, x), \\
\sqrt{\varepsilon} d_R^\varepsilon(0, x) = d^\varepsilon(0, x) - d_0(0, x) - \varepsilon D_f^\varepsilon(0, x), \\
\sqrt{\varepsilon} (D_u + \sqrt{\varepsilon} u_R^\varepsilon d_R^\varepsilon)(0, x) = (D_u + d^\varepsilon)(0, x) - (D_u + d_0)(0, x) - \sqrt{\varepsilon}(u^\varepsilon \cdot \nabla d_0)(0, x) - \varepsilon \partial_t D_f^\varepsilon(0, x) - \varepsilon (u^\varepsilon \cdot \nabla D_f^\varepsilon)(0, x) - \varepsilon^2 (u^\varepsilon \cdot \nabla D_f^\varepsilon)(0, x).
\end{cases}
\tag{1.24}
\]
Recalling that the initial conditions of the original system (1.1) and the limit system (1.10) satisfy
\[
u^\varepsilon(0, x) = u_0(0, x) = u^{in}(x), \quad d^\varepsilon(0, x) = d_0(0, x) = d^{in}(x), \quad (D_u + d^\varepsilon)(0, x) = D^{in}(x), \quad (D_u + d_0)(0, x) = D^{in}(x) \tag{1.25}
\]
and the initial data of the initial layer \(d_f\) in (1.16) is imposed on
\[
\varepsilon \partial_t D_f^\varepsilon(0, x) = \partial_t d_f(0, x) = D^{in}(x), \tag{1.26}
\]
where \(D^{in}(x)\) is the disparity defined in (1.14), we derive from the initial relations (1.24) that the initial data of the remainder system (1.20) should be
\[
\begin{cases}
u_R^\varepsilon(0, x) = 0, \\
\partial_t d_R^\varepsilon(0, x) = -\sqrt{\varepsilon} D_f^\varepsilon(0, x) = -\sqrt{\varepsilon} D^{in}_f(x), \\
(D_u + \sqrt{\varepsilon} u_R^\varepsilon d_R^\varepsilon)(0, x) = -\sqrt{\varepsilon} (u_0 \cdot \nabla D_f^\varepsilon)(0, x) = -\sqrt{\varepsilon} (u^{in} \cdot \nabla D^{in}_f)(x),
\end{cases}
\tag{1.27}
\]
where the vector field $\tilde{D}_\varepsilon^{in}(x)$ is defined as

$$\tilde{D}_\varepsilon^{in}(x) = 2\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon\Delta}\right)^{-1}D^{in}(x).$$

(1.28)

We emphasize that for any fixed $s \geq 0$, the norm bound

$$\|\tilde{D}_\varepsilon^{in}\|_{H^s}^2 \leq \frac{4}{\lambda_1^2}\|D^{in}\|_{H^s}^2$$

(1.29)

holds uniformly in $\varepsilon > 0$, since the uniform lower bound $|\lambda_1 - \sqrt{\lambda_1^2 - 4\varepsilon|x|^2}| \geq -\lambda_1 > 0$ for all $k \in \mathbb{Z}^3$ reduces to

$$\|\tilde{D}_\varepsilon^{in}\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s \frac{4}{\lambda_1^2}\|D^{in}\|_{H^s}^2$$

(1.30)

where the vector field $\tilde{D}_\varepsilon^{in}(x)$ is defined as

$$\tilde{D}_\varepsilon^{in}(x) = 2\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon\Delta}\right)^{-1}D^{in}(x).$$

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$$\|\tilde{D}_\varepsilon^{in}\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s \frac{4}{\lambda_1^2}\|D^{in}\|_{H^s}^2$$

Here the $H^s$-norm will be defined in the later.

**Well-prepared initial data.** As mentioned before, one of our goal in this paper is to deal with the disparity $D^{in}(x)$ given in (1.14) resulted from the initial conditions (1.8). Besides introducing a so-called initial layer in time to overcome this disparity, we can also impose the original system (1.1) on the so-called well-prepared initial data. To be more precise, we can skilfully select the initial values (1.8) of the system (1.1) such that the disparity $D^{in}(x)$ vanishes, hence

$$\tilde{d}^{in}(x) = (D_{u_0}d_0)(0, x) = -(B^{in}d^{in})(x) - \frac{1}{\lambda_1}(\Delta d^{in} + \gamma_0 d^{in} + \lambda_2 A^{in}d^{in})(x).$$

(1.31)

Actually, if the disparity $D^{in}(x) = 0$, the initial layer $\varepsilon D_0^{in}(t, x)$ defined in (1.18) will automatically be zero. Consequently, we shall take ansatz that the system (1.1) imposed on the well-prepared initial data (1.8)- (1.31) has a solution $(u_0^\varepsilon, d_0^\varepsilon)$ with the form

$$\begin{cases}
  u_0^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon}u_R^\varepsilon(t, x), \\
  d_0^\varepsilon(t, x) = d_0(t, x) + \sqrt{\varepsilon}d_R^\varepsilon(t, x).
\end{cases}$$

(1.32)

By substituting the ansatz (1.32) into the original hyperbolic type system (1.1), one easily derives that the remainder $(u_R^\varepsilon, d_R^\varepsilon)$ subjects to the system with the similar structure of (1.20), just removing the terms $\varepsilon\partial_t(u_R^\varepsilon \cdot \nabla D)$, $\varepsilon\text{div}Q_u(D_I)$ and $Q_d(D_I)$ appearing in the system (1.20). More precisely, the remainder $(u_R^\varepsilon, d_R^\varepsilon)$ in (1.32) satisfies

$$\begin{align*}
  \partial_t u_R^\varepsilon - \frac{1}{2}\mu_4 \Delta u_R^\varepsilon + \nabla p_R^\varepsilon &= \mu_1 \text{div}\left([A_2^\varepsilon : d_0 \otimes d_0]d_0 \otimes d_0\right) + \mathcal{K}_u + \text{div}(\mathcal{C}_u + \mathcal{T}_u + \sqrt{\varepsilon}\mathcal{R}_u), \\
  \text{div}u_R^\varepsilon &= 0, \\
  D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon}^2 d_R^\varepsilon + \varepsilon\partial_t D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon}^2 d_R^\varepsilon - \frac{1}{2}\Delta d_R^\varepsilon + \partial_t(u_R^\varepsilon \cdot \nabla d_0) &= \frac{1}{\varepsilon}\mathcal{C}_d + \frac{1}{\varepsilon}\mathcal{S}_d^1 + \frac{1}{\varepsilon}\mathcal{S}_d^2 + \mathcal{R}_d
\end{align*}$$

(1.33)

with the constraint

$$2d_0 \cdot d_R^\varepsilon + \sqrt{\varepsilon}|d_R^\varepsilon|^2 = 0,$$

(1.34)

where the tensor term $\mathcal{C}_u$ and the vector term $\mathcal{C}_d$ are defined in (1.22) and (1.23) respectively, and the accurate expressions of the tedious terms $\mathcal{T}_u$, $\mathcal{K}_u$, $\mathcal{S}_d^1$, $\mathcal{S}_d^2$, $\mathcal{R}_d$ and $\mathcal{R}_u$ are all given in Appendix A. Furthermore, based on the initial relations (1.24), we know that the initial conditions of the remainder system (1.33) should be imposed on

$$u_R^\varepsilon(0, x) = 0, \quad d_R^\varepsilon(0, x) = 0, \quad (D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon}^\varepsilon)(0, x) = 0.$$  

(1.35)
1.2. Main results. To state our main results, we collect here the notations we will use throughout this paper. We denote by $A \sim B$ if there are two constants $C_1, C_2 > 0$, independent of $\varepsilon > 0$, such that $C_1 A \leq B \leq C_2 A$. For convenience, we also denote by

$$L^p = L^p(\mathbb{T}^3)$$

for all $p \in [1, \infty]$, which endows with the norm $\|f\|_{L^p} = \left( \int_{\mathbb{T}^3} |f(x)|^p dx \right)^{\frac{1}{p}}$ for $p \in [1, \infty)$ and $\|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{T}^3} |f(x)|$. For $p = 2$, we use the notation $\langle \cdot, \cdot \rangle$ to represent the inner product on the Hilbert space $L^2$.

For any multi-index $m = (m_1, m_2, m_3)$ in $\mathbb{N}^3$, we denote the $m^{th}$ partial derivative by

$$\partial^m = \partial^{m_1}_{x_1} \partial^{m_2}_{x_2} \partial^{m_3}_{x_3}.$$

If each component of $m \in \mathbb{N}^3$ is not greater than that of $\tilde{m}$’s, we denote by $m \leq \tilde{m}$. The symbol $m < \tilde{m}$ means $m \leq \tilde{m}$ and $|m| < |\tilde{m}|$, where $|m| = m_1 + m_2 + m_3$. We define the Sobolev space $H^N = H^N(\mathbb{T}^3)$ by the norm

$$\|f\|_{H^N} = \left( \sum_{|m| \leq N} \|\partial^m f\|_{L^2}^2 \right)^{\frac{1}{2}} < \infty,$$

or the equivalent norm

$$\|f\|_{H^N} = \left( \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < +\infty,$$

(1.36)

where the symbol $\hat{f}(k)$ is the Fourier transform of $f(x)$ on $x \in \mathbb{T}^3$, hence,

$$\hat{f}(k) = \int_{\mathbb{T}^3} f(x)e^{ix \cdot k} dx$$

for all $k \in \mathbb{Z}^3$. For any integer $N \geq 2$, we define a number $S_N \in \mathbb{N}$ as

$$S_N = \min \{ k \in \mathbb{N}; 2k \geq N + 2 \}. \quad (1.37)$$

Actually, if $N$ is even, $S_N = \frac{1}{2} N + 1$ and if $N$ is odd, $S_N = \frac{1}{2} (N + 3)$.

Now we state our main theorem:

**Theorem 1.1.** Let $N \geq 2$ be an integer and $(u^{in}(x), d^{in}(x), \hat{d}^{in}(x)) \in \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}^3$ satisfy the compatibility conditions (1.9) and $u^{in}, \hat{d}^{in}, \nabla d^{in} \in H^{2S_N}$. If the Leslie’s coefficients satisfy

$$\mu_4 > 0, \ \lambda_1 < 0, \ \mu_1 \geq 0, \ \mu_5 + \mu_6 + \frac{\lambda_3^2}{\lambda_1} \geq 0$$

and there exist small $\varepsilon_0, \xi_0 \in (0, 1]$, depending on the Leslie’s coefficients and $N$, such that

$$E^{in} \triangleright \|u^{in}\|^2_{H^{2S_N}} + \|\hat{d}^{in}\|^2_{H^{2S_N}} + \|\nabla d^{in}\|^2_{H^{2S_N}} \leq \xi_0$$

(1.39)

for all $\varepsilon \in (0, \varepsilon_0]$, then the system (1.1) with the initial conditions (1.8) admits a unique solution $(u^\varepsilon, d^\varepsilon)$ satisfying

$$u^\varepsilon, \nabla d^\varepsilon, D_u d^\varepsilon \in L^\infty(\mathbb{R}^+; H^N), \ \nabla u^\varepsilon \in L^2(\mathbb{R}^+; H^N). \quad (1.40)$$

Moreover, the solution $(u^\varepsilon, d^\varepsilon)$ is of the form

$$\begin{aligned}
    u^\varepsilon(t, x) &= u_0(t, x) + \sqrt{\varepsilon} u_0^\varepsilon(t, x), \\
    d^\varepsilon(t, x) &= D U_f^\varepsilon(t, x) + \varepsilon D_d^\varepsilon(t, x) + \varepsilon D_{\tilde{d}}(t, x),
\end{aligned} \quad (1.41)$$

where $(u_0, d_0)$ is the solution to the incompressible parabolic Ericksen-Leslie’s liquid crystal model (1.10) with the initial data (1.13), the initial layer $\varepsilon D_f^\varepsilon(t, x)$ is defined in (1.18), and
obey the remainder system (1.20) with the initial condition (1.27). Furthermore, $(u_\varepsilon^R, d_\varepsilon^R)$ satisfies the uniform energy bound

$$\left(\|u_\varepsilon^R\|_{H^N}^2 + \|D_{u_\varepsilon + \sqrt{\varepsilon}d_\varepsilon^R}\|_{H^N}^2 + \frac{1}{\varepsilon}\|d_\varepsilon^R\|^{2}_{H^{N+1}}(t) + \frac{1}{\varepsilon}\int_0^t \|\nabla u_\varepsilon^R\|_{H^N}^2(\tau)d\tau \leq C\xi_0 \right) \tag{1.42}$$

for all $t \geq 0$, $\varepsilon \in (0, \varepsilon_0]$ and for some constant $C > 0$, independent of $\varepsilon$ and $t$.

**Remark 1.1.** The small number $\xi_0$ is firstly smaller than $\beta_{S_{N,0}}$ mentioned in Proposition 3.1 proved by Wang-Zhang-Zhang in [21], so that the limit system (1.10)-(1.13) admits a unique global in time classical solution $(u_0, d_0)$ under the constraint of small size $\xi_0$ to the initial data. Thus, the vectors $u_0$ and $d_0$ appeared in the remainder system (1.20) can be regarded as the known coefficients.

**Remark 1.2.** Under the assumptions of Theorem 1.1, if the well-prepared initial data is further assumed, i.e., (1.31) holds, then the hyperbolic Ericksen-Leslie’s liquid crystal system (1.1)-(1.8) has a unique global classical solution $(u^\varepsilon, d^\varepsilon)$ with the form

$$\begin{cases}
  u^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon} u_\varepsilon^R(t, x), \\
  d^\varepsilon(t, x) = d_0(t, x) + \sqrt{\varepsilon} d_\varepsilon^R(t, x),
\end{cases} \tag{1.43}$$

and the other conclusions are the same as stated in Theorem 1.1.

**Remark 1.3.** The uniform bound (1.42) implies that

$$\|D_{u_\varepsilon + \sqrt{\varepsilon}u_\varepsilon^R}\|_{L^\infty(\mathbb{R}^+;H^N)} + \frac{1}{\varepsilon}\|d_\varepsilon^R\|^{2}_{L^\infty(\mathbb{R}^+;H^{N+1})} \leq C\xi_0, \tag{1.44}$$

which shows us the convergence rate $\sqrt{\varepsilon}$ on the orientation $d^\varepsilon$ we obtain is optimal. This optimality can be also seen in Remark 1.1 of Jiang-Luo-Zarnescu’s work [10]. We also derive from the uniform bound (1.42) that

$$\frac{1}{\varepsilon}\|u_\varepsilon^R\|^{2}_{L^\infty(\mathbb{R}^+;H^N)} + \frac{1}{\varepsilon}\|\nabla u_\varepsilon^R\|^{2}_{L^2(\mathbb{R}^+;H^N)} \leq C\xi_0. \tag{1.45}$$

Let $\bar{u}_\varepsilon^R = \sqrt{\varepsilon} u_\varepsilon^R$. Then the expansion of $u^\varepsilon$ can be rewritten as $u^\varepsilon(t, x) = u_0(t, x) + \varepsilon \bar{u}_\varepsilon^R(t, x)$ and the uniform bound $\|\bar{u}_\varepsilon^R\|^{2}_{L^\infty(\mathbb{R}^+;H^N)} + \|\nabla \bar{u}_\varepsilon^R\|^{2}_{L^2(\mathbb{R}^+;H^N)} \leq C\xi_0$ holds, which means that the convergence rate of the velocity $u^\varepsilon$ is $\varepsilon$.

### 1.3. Ideals and novelties

Generally speaking, there are three aspects of the disparity between the original system and the limit equations on the limit problem. First, the form of the limit equations is obviously different from that of the original system. Second, the initial conditions of original system and limit equations are not the same type, which will result to the initial layers. Finally, the boundary conditions will also lead to the disparity, which can be covered by the boundary layers.

In the current paper, we study the limit problem on $\mathbb{T}^3$ in the regime of classical solutions, which does not involve the boundary conditions. It is a very difficult problem to derive the energy bounds of $(u^\varepsilon, d^\varepsilon)$ uniform in small $\varepsilon > 0$ to the original hyperbolic Ericksen-Leslie’s liquid crystal model (1.1). So, we take the Hilbert expansion method to rigorously justify the limit from hyperbolic Ericksen-Leslie’s liquid crystal model to the parabolic case, in which the remainder term $(u_\varepsilon^R, d_\varepsilon^R)$ is utilized to deal with the difference between the forms of two systems. Because the $d^\varepsilon$-equation in (1.1) is a wave type equation and the corresponding limit $d_0$-equation in (1.10) is a parabolic type equation, the $d^\varepsilon$-equation of original system need impose on two initial conditions but the $d_0$-equation of limit system only need impose on one initial data. One of the methods to overcome this disparity is to introduce an initial layer $\varepsilon D_1^\varepsilon(t, x)$ (defined in (1.18)), so that the disparity $D^\varepsilon(x)$ is absorbed, for details see the analysis before. Thus we give the formal expansion (1.19), namely

$$u^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon} u_\varepsilon^R(t, x), \quad d^\varepsilon(t, x) = d_0(t, x) + \varepsilon D_1^\varepsilon(t, x) + \sqrt{\varepsilon} d_\varepsilon^R(t, x).$$
Another method to overcome the disparity of the initial conditions is to give the well-prepared initial data such that $D^m(t, x) = 0$, which immediately implies that the initial layer $D^l(t, x)$ vanishes. In this case, the expansion form will be (1.32), hence

$$u^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon} u^\varepsilon_R(t, x), \quad d^\varepsilon(t, x) = d_0(t, x) + \sqrt{\varepsilon} d^\varepsilon_R(t, x).$$

The main goal of this work is to derive the global energy bound of the remainder $(u^\varepsilon_R, d^\varepsilon_R)$ uniform in small $\varepsilon > 0$ under the small size of the initial data $(u^m, d^m, \tilde{d}^m)$. Since the initial layer $D^l(t, x)$ and its time derivative $\partial_t D^l(t, x)$ are infinitely small as $\varepsilon \ll 1$, we can merely consider the remainder $(u^\varepsilon_R, d^\varepsilon_R)$ in the expansion (1.32) with respect to the case of well-prepared initial data, which satisfying the system (1.33). The major structure of (1.33) reads

$$\begin{aligned}
\partial_t u^\varepsilon_R - \frac{1}{\varepsilon^2} \mu_4 \Delta u^\varepsilon_R + \nabla p^\varepsilon_R - \text{div} C_u = \text{some other terms}, \\
\text{div} u^\varepsilon_R = 0, \\
D^2_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R - \frac{\lambda_1}{\varepsilon} D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R - \frac{1}{\varepsilon^2} \Delta d^\varepsilon_R + \partial_t (u^\varepsilon_R \cdot \nabla d_0) - \frac{1}{\varepsilon} C_d = \text{some other terms},
\end{aligned}$$

where $\lambda_1 < 0$. The key point in this work is how to control the term $\partial_t (u^\varepsilon_R \cdot \nabla d_0)$ in the energy estimates to the remainder system (1.33). We will design a energy functional, which sensitively depends on the limit vector field $(u_0, d_0)$, to deal with this term. More precisely, we multiply by $u^\varepsilon_R$ and $D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R$ in the first and third equations of (1.46) respectively and integrate by parts over $x \in \mathbb{T}^3$. Then, combining the cancellation (2.1) of the case $m = 0$ in Lemma 2.1, hence

$$-\langle \text{div} C_u, u^\varepsilon_R \rangle - \langle C_d, D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \rangle = - \lambda_1 \| D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R + B^\varepsilon_R d_0 + \frac{\lambda_2}{\lambda_1} A^\varepsilon_R d_0 \|_{L^2}^2 \\
+ \lambda_1 \| D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \|_{L^2}^2 + (\mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1}) \| A^\varepsilon_R d_0 \|_{L^2}^2,$$

we obtain the main part of the $L^2$-energy equality

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \frac{1}{\varepsilon} \| u^\varepsilon_R \|_{L^2}^2 + \| D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \|_{L^2}^2 + \frac{1}{\varepsilon} \| \nabla d^\varepsilon_R \|_{L^2}^2 \right) + \langle \partial_t (u^\varepsilon_R \cdot \nabla d_0), D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \rangle \\
+ \frac{1}{2} \mu_4 \| \nabla u^\varepsilon_R \|_{L^2}^2 - \frac{\lambda_1}{\varepsilon} \| D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R + B^\varepsilon_R d_0 + \frac{\lambda_2}{\lambda_1} A^\varepsilon_R d_0 \|_{L^2}^2 + \frac{1}{\varepsilon} (\mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1}) \| A^\varepsilon_R d_0 \|_{L^2}^2 \\
= \ldots \ldots.
\end{aligned}$$

Under the coefficient conditions (1.38), the energy dissipative rate is positive. If we regard the term $\partial_t (u^\varepsilon_R \cdot \nabla d_0)$ as a source term, the quantity $\langle \partial_t (u^\varepsilon_R \cdot \nabla d_0), D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \rangle$ should be dominated by the energy and the energy dissipative rate defined in the above equality. However, it is impossible, since the regularity of $\partial_t (u^\varepsilon_R \cdot \nabla d_0)$ is equivalent to $\frac{1}{\varepsilon^2} \mu_4 \Delta u^\varepsilon_R$ by using the first $u^\varepsilon_R$-equation of (1.46) and the highest order regularity of the energy dissipative rate is $\nabla u^\varepsilon_R$. In order to overcome this difficulty, we try to design this quantity as a part of
the energy. More precisely, we take the following important deformation
\[
\langle \partial_t (u_\varepsilon \cdot \nabla d_0), D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle
\]
\[
= \frac{1}{\varepsilon} \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle - \langle u_\varepsilon \cdot \nabla d_0, \partial_t D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle
\]
\[
= \frac{1}{\varepsilon} \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle - \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle
\]
\[
+ \langle u_\varepsilon \cdot \nabla d_0, (u_0 + \sqrt{\varepsilon} u_\varepsilon) \cdot \nabla D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle
\]
\[
= \frac{1}{2\varepsilon} \left( \| u_\varepsilon \cdot \nabla d_0 \|^2_{L^2} + 2 \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle \right)
\]
\[
- \langle (u_0 + \sqrt{\varepsilon} u_\varepsilon) \cdot \nabla (u_\varepsilon \cdot \nabla d_0), D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle + \frac{1}{\varepsilon} \langle u_\varepsilon \cdot \nabla d_0, C \rangle
\]
\[
- \frac{1}{\varepsilon} \langle \nabla (u_\varepsilon \cdot \nabla d_0), \nabla d_\varepsilon \rangle + \frac{\Lambda_1}{\varepsilon} \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle
\]
\[
+ \ldots
\]
where the last equality is derived from the $d_\varepsilon$-equation of (1.46). Then, we obtain the relation
\[
\frac{1}{2\varepsilon} \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle + \frac{1}{\varepsilon} \| \nabla d_\varepsilon \|^2_{L^2} + \| u_\varepsilon \cdot \nabla d_0 \|^2_{L^2} + 2 \langle u_\varepsilon \cdot \nabla d_0, D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle
\]
\[
+ \frac{1}{2\varepsilon} \mu_4 \| \nabla u_\varepsilon \|^2_{L^2} - \frac{\Lambda_1}{\varepsilon} \| D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \|^2_{L^2} + B R_\varepsilon^2 \| d_\varepsilon \|^2_{L^2} + \frac{\Lambda_1}{\varepsilon} \| \Lambda d_\varepsilon \|^2_{L^2} + \frac{1}{\varepsilon} (\mu_5 + \mu_6 + \frac{\Lambda_1^2}{\varepsilon}) \| A d_\varepsilon \|^2_{L^2}
\]
\[
= P_1 + P_2 + \ldots
\]
where the quantities $P_1$ and $P_2$ can be controlled by the energy and energy dissipative rate. Although the new energy is not positive for all $\varepsilon > 0$, it will be a definitely positive sign with sufficiently small $\varepsilon > 0$ under the fixed coefficient $\nabla d_0$. Consequently, we have designed a complicated energy functional, which sensitively depends on the solutions to the limit system, to deal with the trouble quantity $\langle \partial_t (u_\varepsilon \cdot \nabla d_0), D_{u_0 + \sqrt{\varepsilon} u_\varepsilon} d_\varepsilon \rangle$.

The advantage of the expansion (1.32) of the solutions $(u_\varepsilon, d_\varepsilon)$ to the system (1.1) is such that the remainder equations (1.33) of $(u_\varepsilon, d_\varepsilon)$ have weaker nonlinearities than the original system (1.1), despite the system (1.33) is still nonlinear and singular (with singular terms of the type $\frac{1}{\varepsilon}$). To be more precise, after utilizing the expansion (1.32), the nonlinearity and singularity are separated. For instance, the term $(-\varepsilon |D u_\varepsilon|^2 + |\nabla d_\varepsilon|^2 - \lambda_2 A_2 $ $d_\varepsilon \otimes d_\varepsilon$ $d_\varepsilon$ in the original system (1.1) is replaced by either linear terms (the unknown $d_\varepsilon$ and $u_\varepsilon$ are superseded by the known $u_0$ and $d_0$) or a nonlinear term with the same form but with some higher order power of $\varepsilon$ in front. So, it will be easier to get the energy bound, uniform in small $\varepsilon > 0$, of the remainder system (1.33)-(1.35).

1.4. Historical remarks. In this subsection, we review some history of the mathematical analytic works on the liquid crystals, in particular Ericksen-Leslie’s system. The static analogue of the parabolic Ericksen-Leslie’s system (1.10) is the so-called Oseen-Frank model, whose mathematical study was initiated from Hardt-Kinderlehrer-Lin [5]. Since then there have been many works in this direction. In particular, the existence and regularity or partial regularity of the approximation (usually Ginzburg-Landau approximation as in [14]) dynamical Ericksen-Leslie’s system was started by the work of Lin and Liu in [14], [15] and [16]. For the simplest system preserving the basic energy law which can be obtained by neglecting the Leslie stress and by specifying some elastic constants, in 2-D case, the existence of global weak solutions with at most a finite number of singular times were proved by Lin-Lin-Wang.
[13]. Recently, Lin and Wang proved global existence of weak solution for 3-D case with the initial director field \( \mathbf{d}^n(x) \) lying in the hemisphere in [19]. For the more general parabolic Ericksen-Leslie’s system, local well-posedness is proved by Wang-Zhang-Zhang in [21], and in [7] existence of global solutions and regularity in \( \mathbb{R}^2 \) was established by Huang-Lin-Wang. For more complete review of the works for the parabolic Ericksen-Leslie’s system, please see the reference listed above.

For the hyperbolic Ericksen-Leslie’s system (1.1), much less is known. For the most simplified model, i.e. taking the bulk velocity field \( u = 0 \), neglecting the Leslie’s coefficients, and the spatial dimension is 1, the system (1.1) can be reduced to a so-called nonlinear variational wave equation which is already highly nontrivial. Zhang and Zheng studied systematically the dissipative and energy conservative solutions in series work starting from late 90’s ([23, 24, 22]).

Recently, there started some works on the original hyperbolic Ericksen-Leslie’s system (1.1) for multi-dimensional case. The authors of the current paper studied in [8] the well-posedness in the context of classical solutions of (1.1). More precisely, in [8] under some natural constraints on the Leslie coefficients which ensure the basic energy law is dissipative, it was proved the local-in-time existence and uniqueness of the classical solution to the system (1.1) with finite initial energy. Furthermore, with an additional assumption on the coefficients which provides a damping effect, i.e. \( \lambda_1 < 0 \), and the smallness of the initial energy, the unique global classical solution was established. Here we remark that the assumption \( \lambda_1 < 0 \) plays a crucial role in the global-in-time well-posedness. Later, Cai-Wang [1] made progress for the simplified Ericksen-Leslie system, namely, the case with \( \mu_i = 0, i = 1, \ldots, 6, i \neq 4 \) in (1.3). They proved the global regularity of (1.1) near the constant equilibrium by employing the vector field method. More recently, in [6], the authors of the current papers with Huang and Zhao, considered the more general case: still \( \mu_2 = \mu_3 = 0 \), but \( 0 \neq \mu_5 = \mu_6 > -\mu_4 \), and \( 0 \neq \mu_1 > -2(\mu_4 + \mu_5) \), and proved results similar to [1].

Regarding to the inertia limit, i.e. \( \varepsilon \to 0 \), for a given bulk velocity and well-prepared initial data, together with Tang, we justified this limit in [9]. For the case without the bulk velocity and general initial data, by constructing an initial layer, we with Tang and Zarnescu, justified this limit in [10]. In this sense, the current paper, proved this inertia limit for the much more general case with bulk velocity field, under the assumption \( \lambda_1 < 0 \). The case \( \lambda_1 = \lambda_2 = 0 \) will be analytically more subtle, for which the limiting system is the harmonic map to \( S^2 \), and furthermore, the initial layer will be a wave equation which preserve the energy. This work is under preparation, together with Huang and Zhao.

The organization of this paper is as follows: in the next section, we first derive the canceled relations between \( C_u \) and \( C_d \) contained in the remainder system of \( (u^R_\varepsilon, d^R_\varepsilon) \), which will play an essential role in estimating the energy of the remainder \( (u^R_\varepsilon, d^R_\varepsilon) \). Then we shown that the constraints (1.21) and (1.34) will hold at any time provided they initially hold, see Lemma 2.2. In Section 3, we estimate the uniform energy bound on small \( \varepsilon > 0 \) of the remainder system of \( (u^R_\varepsilon, d^R_\varepsilon) \). Then, based on the uniform energy estimates in the previous section, Theorem 1.1 of the current paper is proved in Section 4. Finally, we accurately present the all tedious term of the remainder system (1.20) (also (1.33)) in Appendix A.

2. Some basic cancellations and constraints

In this section, we will first derive some basic cancellations on the remainder equations (1.20) (or (1.33)), which will play an essential role in deriving the global in time energy estimates uniformly in \( \varepsilon > 0 \) to the remainder system (1.20) (or (1.33)) with small initial data. We then prove that the constraints (1.21) or (1.34), which come from the geometric constraint \( |d^\varepsilon| = 1 \) in the original system (1.1), will hold for all time \( t \geq 0 \) provided they initially hold.

First, we will give the following lemma, which displays the cancellations between the terms \( C_u \) and \( C_d \).
Lemma 2.1. Under the relations (1.7), for all multi-indexes \( m \in \mathbb{N}^3 \), one has

\[
\langle \text{div} \partial^m \mathbf{C}_u, \partial^m \mathbf{u}_R^\varepsilon \rangle + \langle \partial^m \mathbf{C}_d, \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R^\varepsilon} d_R^\varepsilon \rangle = \lambda_1 \left\| \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R^\varepsilon} d_R^\varepsilon \right\|_{L^2}^2 - \lambda_1 \left\| \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R} d_R^\varepsilon \right\|_{L^2}^2 - (\mu_5 + \mu_6 + \frac{\mu_7}{\lambda_1}) \left\| \partial^m A_{R}^\varepsilon \right\|_{L^2}^2 + \mathcal{G}_m,
\]

where the quantity \( \mathcal{G}_m \) is defined as follows: if \( m = 0 \),

\[
\mathcal{G}_m = 0,
\]

and if \( m \neq 0 \),

\[
\mathcal{G}_m = \sum_{m' < m} C_{m'} \left[ \lambda_1 (\partial^m B_{R}^\varepsilon) \partial^{m-m'} d_0 + \lambda_2 (\partial^m A_{R}^\varepsilon) \partial^{m-m'} d_0, \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R^\varepsilon} d_R^\varepsilon \right] \]

\[
- \langle \mu_2 \partial^{m-m'} (d_0, d_0, k) \partial^m \mathbf{B}_{R,ki}^\varepsilon + \mu_3 (d_0, id_0, k) \partial^m \mathbf{B}_{R,kj}^\varepsilon, \partial^m \partial_j \mathbf{u}_R^\varepsilon \rangle \]

\[
- \langle \mu_5 \partial^{m-m'} (d_0, j, d_0, k) \partial^m \mathbf{A}_{R,ki}^\varepsilon + \mu_6 (d_0, i, d_0, k) \partial^m \mathbf{A}_{R,kj}^\varepsilon, \partial^m \partial_j \mathbf{u}_R^\varepsilon \rangle \]

\[
- \langle \mu_2 \partial^{m-m'} (d_0, j, d_0, k) \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R} d_R^\varepsilon \rangle \]

\[
+ \mu_3 \partial^{m-m'} (d_0, i, d_0, k) \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R} d_R^\varepsilon \rangle + \mu_3 \partial^{m-m'} (d_0, i, d_0, k) \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R} d_R^\varepsilon \rangle \]

\]

(2.3)

Proof of Lemma 2.1. Recalling the definition of \( \mathbf{C}_u \) in (1.22), one deduces

\[
\langle \text{div} \partial^m \mathbf{C}_u, \partial^m \mathbf{u}_R^\varepsilon \rangle = \left\langle \partial_j \partial^m (\mu_2 d_0, j, d_0, k) + \mu_3 d_0, j, \partial^m \mathbf{B}_{R,ki}^\varepsilon \right\rangle \]

\[
+ \left\langle \partial_j \partial^m (\mu_5 d_0, j, d_0, k) \partial^m \mathbf{A}_{R,ki}^\varepsilon + \mu_6 d_0, j, \partial^m \mathbf{A}_{R,kj}^\varepsilon \right\rangle \]

\[
+ \left\langle \partial_j \partial^m \left[ \mu_2 d_0, j, (D_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R} d_R^\varepsilon) \right] + \mu_3 d_0, j \partial^m \mathbf{D}_{u_0 + \sqrt{\varepsilon} \mathbf{u}_R} d_R^\varepsilon \right\rangle \]

\]

(2.4)

Then we will directly calculate the terms \( U_1, U_2 \) and \( U_3 \) for the case \( m \neq 0 \). The case \( m = 0 \) can be similarly justified. For the term \( U_1 \), we derive from the integration by parts over \( x \in \mathbb{T}^3 \) that

\[
U_1 = - \langle \mu_2 d_0, j \partial^m \mathbf{B}_{R,ki}^\varepsilon d_0, k + \mu_3 d_0, j \partial^m \mathbf{B}_{R,kj}^\varepsilon d_0, k, \partial^m \partial_j \mathbf{u}_R^\varepsilon \rangle \]

\[
- \sum_{m' < m} C_{m'} \left\langle \mu_2 \partial^{m-m'} (d_0, j, d_0, k) \partial^m \mathbf{B}_{R,ki}^\varepsilon + \mu_3 \partial^{m-m'} (d_0, i, d_0, k) \partial^m \mathbf{B}_{R,kj}^\varepsilon, \partial^m \partial_j \mathbf{u}_R^\varepsilon \right\rangle \]

\]

(2.5)

Since \( \lambda_1 = \mu_2 - \mu_3, \lambda_2 = \mu_5 - \mu_6 \) and \( \mu_6 - \mu_5 = \mu_2 + \mu_3 \), we have

\[
U_{11} = - \langle \mu_2 - \mu_3 \rangle d_0, j \partial^m \mathbf{B}_{R,ki}^\varepsilon d_0, k \]

\[
+ \mu_3 (d_0, j) \partial^m \mathbf{B}_{R,ki}^\varepsilon d_0, k + d_0, i \partial^m \mathbf{B}_{R,kj}^\varepsilon d_0, k, \partial^m \mathbf{A}_{R,ij}^\varepsilon + \partial^m \mathbf{B}_{R,ij}^\varepsilon \rangle \]

\[
= - \lambda_1 \langle \mu_2 - \mu_3 \rangle d_0, j \partial^m \mathbf{B}_{R,ki}^\varepsilon d_0, k + \partial^m \mathbf{A}_{R,ij}^\varepsilon + \partial^m \mathbf{B}_{R,ij}^\varepsilon \rangle \]

\[
- \mu_3 \langle d_0, j \partial^m \mathbf{B}_{R,ki}^\varepsilon d_0, k + d_0, i \partial^m \mathbf{B}_{R,kj}^\varepsilon d_0, k, \partial^m \mathbf{A}_{R,ij}^\varepsilon \rangle \]

\[
= \lambda_1 \left\| (\partial^m \mathbf{B}_{R}) d_0 \right\|_{L^2}^2 - (\lambda_1 + 2 \mu_3) \langle \partial^m \mathbf{B}_{R,ki}^\varepsilon d_0, k, (\partial^m \mathbf{A}_{R,ij}^\varepsilon) d_0, j \rangle \]

\[
= \lambda_1 \left\| (\partial^m \mathbf{B}_{R}) d_0 \right\|_{L^2}^2 + \lambda_2 \langle (\partial^m \mathbf{B}_{R}) d_0, (\partial^m \mathbf{A}_{R}^\varepsilon) d_0 \rangle ,
\]

where we make use of the relation \( \mathbf{B}_{R,ij}^\varepsilon = -\mathbf{B}_{R,ji}^\varepsilon \).
For the term $U_2$, integrating by parts over $x \in \mathbb{T}^3$ implies

$$U_2 = - \langle \mu_5 d_{0,j} d_{0,k} (\partial^m A_{R,ki}^\varepsilon) + \mu_6 d_{0,l} d_{0,k} (\partial^m A_{R,kj}^\varepsilon), \partial^m \partial_j u_{R,i}^\varepsilon \rangle$$

$$- \sum_{m' < m} C_{m'} \left\langle \mu_5 \partial^{m-m'} (d_{0,j} d_{0,k}) \partial^{m'} A_{R,ki}^\varepsilon + \mu_6 \partial^{m-m'} (d_{0,l} d_{0,k}) \partial^{m'} A_{R,kj}^\varepsilon, \partial^m \partial_j u_{R,i}^\varepsilon \right\rangle. \quad (2.7)$$

By the similar calculations on $U_{11}$, we compute the term $U_{21}$

$$U_{21} = - \langle (\mu_5 - \mu_6) d_{0,j} d_{0,k} (\partial^m A_{R,ki}^\varepsilon) + \mu_6 (d_{0,j} d_{0,k} (\partial^m A_{R,ki}^\varepsilon) + d_{0,l} d_{0,k} (\partial^m A_{R,kj}^\varepsilon)), \partial^m A_{ij} + \partial^m B_{ij}^\varepsilon \rangle$$

$$- \lambda_2 \| (\partial^m A_{R}^\varepsilon) d_0 \|^2_L + \lambda_2 \langle (\partial^m A_{R}^\varepsilon) d_0, (\partial^m B_{R}^\varepsilon) d_0 \rangle - 2 \mu_6 \| (\partial^m A_{R}^\varepsilon) d_0 \|^2_L$$

$$- (\mu_5 + \mu_6) \| (\partial^m A_{R}^\varepsilon) d_0 \|^2_L + \lambda_2 \langle (\partial^m A_{R}^\varepsilon) d_0, (\partial^m B_{R}^\varepsilon) d_0 \rangle. \quad (2.8)$$

For the term $U_3$, we deduce from integrating by parts over $x \in \mathbb{T}^3$ that

$$U_3 = - \langle \mu_2 d_{0,j} \partial^m (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) i + \mu_3 d_{0,i} \partial^m (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) j, \partial^m \partial_j u_{R,i}^\varepsilon \rangle$$

$$- \sum_{m' < m} C_{m'} \left\langle \mu_2 \partial^{m-m'} d_{0,j} \partial^{m'} (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) i + \mu_3 \partial^{m-m'} d_{0,i} \partial^{m'} (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) j, \partial^m \partial_j u_{R,i}^\varepsilon \right\rangle. \quad (2.9)$$

It is easily yielded that by the analogous arguments in computations of $U_{11}$ or $U_{21}$

$$U_{31} = - \langle (\mu_2 - \mu_3) d_{0,j} \partial^m (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) i + \mu_3 (d_{0,j} \partial^m (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) i + d_{0,i} \partial^m (D_{u_0 + \sqrt{u_R} d_R^\varepsilon}) j), \partial^m A_{R,ij}^\varepsilon + \partial^m B_{R,ij}^\varepsilon \rangle$$

$$- \lambda_1 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m A_{R}^\varepsilon d_0, (\partial^m B_{R}^\varepsilon) d_0 \rangle - 2 \mu_3 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m A_{R}^\varepsilon d_0 \rangle$$

$$+ \lambda_1 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m B_{R}^\varepsilon d_0 \rangle$$

$$= \lambda_1 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m A_{R}^\varepsilon d_0 \rangle + \lambda_2 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m B_{R}^\varepsilon d_0 \rangle. \quad (2.10)$$

We thereby obtain

$$\langle \text{div} \partial^m C_u, \partial^m u_R^\varepsilon \rangle = \lambda_1 \| (\partial^m B_{R}^\varepsilon) d_0 \|^2_L + \lambda_2 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m A_{R}^\varepsilon d_0, (\partial^m B_{R}^\varepsilon) d_0 \rangle$$

$$- (\mu_5 + \mu_6) \| (\partial^m A_{R}^\varepsilon) d_0 \|^2_L + \lambda_1 \langle \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \partial^m B_{R}^\varepsilon d_0 \rangle$$

$$+ U_{12} + U_{22} + U_{32}. \quad (2.11)$$

We finally compute the quantity $\langle \partial^m C_d, \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \rangle$. Recalling the definition of $C_d$ in (1.23), we deduce

$$\langle \partial^m C_d, \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \rangle$$

$$= \lambda_1 \langle (\partial^m B_{R}^\varepsilon) d_0, \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \rangle + \lambda_2 \langle (\partial^m A_{R}^\varepsilon) d_0, \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \rangle$$

$$+ \sum_{m' < m} C_{m'} \left\langle \lambda_1 (\partial^m B_{R}^\varepsilon) \partial^{m-m'} d_0 + \lambda_2 (\partial^m A_{R}^\varepsilon) \partial^{m-m'} d_0, \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \right\rangle. \quad (2.12)$$
Adding the equality (2.11) to (2.12) tells us
\[
\langle \text{div}(\partial^m C_u, \partial^m u^\varepsilon_R) + \langle \partial^m C_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \rangle
\]
\[
= \lambda_1 \|(\partial^m D^\varepsilon_R) d_0\|^2_{L^2} + 2\lambda_2 \langle \partial^m D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R, (\partial^m A^\varepsilon_R) d_0 \rangle
\]
\[
- (\mu_5 + \mu_6) \|(\partial^m A^\varepsilon_R) d_0\|^2_{L^2} + 2\lambda_1 \langle \partial^m D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R, (\partial^m B^\varepsilon_R) d_0 \rangle
\]
\[
+ U_{12} + U_{22} + U_{32} + U_4
\]
\[
= \lambda_1 \left\| \partial^m D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R + (\partial^m B^\varepsilon_R) d_0 + \frac{\lambda_1}{\lambda_2} (\partial^m A^\varepsilon_R) d_0 \right\|_{L^2}^2 - \lambda_1 \left\| \partial^m D_{u_0 + \sqrt{\varepsilon} u^\varepsilon_R} d^\varepsilon_R \right\|_{L^2}^2
\]
\[
- (\mu_5 + \mu_6 + \frac{\lambda_2}{\lambda_1}) \|(\partial^m A^\varepsilon_R) d_0\|^2_{L^2} + U_{12} + U_{22} + U_{32} + U_4,
\]
and then the proof of Lemma 2.1 is completed. \(\square\)

Then, inspired by Lemma 4.1 of [8], the work of the same two authors of this paper, we derive the following lemma to show how the constraints (1.21) or (1.34) hold under the corresponding initial constraints.

**Lemma 2.2.** Assume \((u_0, d_0)\) and \((u^\varepsilon_R, d^\varepsilon_R)\) are two sufficiently smooth solutions to the limit system (1.10)-(1.13) and the remainder system (1.20)-(1.27), respectively. Let \(D^\varepsilon_f(t, x)\) is the initial layer defined in (1.18). If the initial constraint
\[
2(d_0 \cdot (d^\varepsilon_R + \sqrt{\varepsilon} D^\varepsilon_f))(0, x) + \sqrt{\varepsilon}|d^\varepsilon_R + \sqrt{\varepsilon} D^\varepsilon_f|^2(0, x) = 0
\]
holds, then the constraint (1.21), i.e.,
\[
2d_0 \cdot (d^\varepsilon_R + \sqrt{\varepsilon} D^\varepsilon_f) + \sqrt{\varepsilon}|d^\varepsilon_R + \sqrt{\varepsilon} D^\varepsilon_f|^2 = 0
\]
holds for all \(t \geq 0\).

**Remark 2.1.** For the remainder system (1.33)-(1.35) corresponding to the well-prepared initial data, the initial layer \(D^\varepsilon_f(t, x) \equiv 0\). If the initial constraint \(2(d_0 \cdot d^\varepsilon_R)(0, x) + \sqrt{\varepsilon}|d^\varepsilon_R|^2(0, x) = 0\) holds, then the constraint (1.34), hence \(2d_0 \cdot d^\varepsilon_R + \sqrt{\varepsilon}|d^\varepsilon_R|^2 = 0\) still holds for all \(t \geq 0\).

Before proving Lemma 2.2, for convenience to readers, we list Lemma 4.1 in [8] as follows:

**Lemma 2.3** (Lemma 4.1 in [8]). Assume \((u^\varepsilon, d^\varepsilon)\) is a sufficiently smooth solution to the system (1.1)-(1.8). If the constraint \(|d^\varepsilon| = 1\) is further assumed, then the Lagrangian multiplier \(\gamma^\varepsilon\) is (1.2), i.e.,
\[
\gamma^\varepsilon = -\varepsilon|D_{u^\varepsilon}d^\varepsilon|^2 + |\nabla d^\varepsilon|^2 - \lambda_2 A^\varepsilon : d^\varepsilon \otimes d^\varepsilon.
\]
Conversely, if we give the form of \(\gamma^\varepsilon\) as (1.2) and \(d^\varepsilon\) satisfies the initial compatibility \(|d^\varepsilon|_{t=0} = 1\) and \((d^\varepsilon \cdot D_{u^\varepsilon}d^\varepsilon)|_{t=0} = 0\), then \(|d^\varepsilon| = 1\) holds at any time.

**Proof of Lemma 2.2.** According to the formal analysis given in Section Introduction, we know that
\[
\begin{cases}
    u^\varepsilon(t, x) = u_0(t, x) + \sqrt{\varepsilon} u^\varepsilon_R(t, x), \\
    d^\varepsilon(t, x) = d_0(t, x) + \sqrt{\varepsilon} D^\varepsilon_f(t, x) + \sqrt{\varepsilon} d^\varepsilon_R(t, x)
\end{cases}
\]
satisfy the equations of (1.1) but without the geometric constraint \(|d^\varepsilon| = 1\). Via the initial relations (1.24), one easily derives from (2.14) and the compatibilities (1.9) that
\[
|d^\varepsilon|(0, x) = 1, \quad (d^\varepsilon \cdot D_{u^\varepsilon}d^\varepsilon)(0, x) = 0.
\]
Therefore, Lemma 2.3 implies that \(|d^\varepsilon| = 1\) holds for all \(t \geq 0\). Noticing that \(|d_0(t, x)| \equiv 1\), we easily derive from the expression of \(d^\varepsilon(t, x)\) and the geometric constraint \(d^\varepsilon(t, x) \in S^2\) that the constraint (1.21) holds for all \(t \geq 0\). Then the proof of Lemma 2.2 is finished. \(\square\)

The same arguments in justifying Lemma 2.2 will also imply the conclusions of Remark 2.1, just letting \(D^\varepsilon_f(t, x) \equiv 0\).
3. A priori uniform energy estimates for the remainder system

In this section, we will derive the a priori uniform (in small $\varepsilon > 0$) energy estimates for the remainder systems (1.20) or (1.33) globally in times. Notice that there are two differences between the equations (1.20) and (1.33):

1) the system (1.20) involves the terms $\sqrt{\varepsilon}\partial_t(u_R^e \cdot \nabla D_{ij}^e)$, $\varepsilon \text{div} Q_u(D_I)$ and $Q_d(D_I)$;

2) The initial conditions of (1.20) are nontrivial (see (1.27)) but that of (1.33) are all imposed on zero (see (1.35)).

However, the initial data (1.27) of (1.20) are infinitely small quantities and the initial layer $D_I(t, x)$ defined in (1.18) is also infinitely small as $\varepsilon \ll 1$. The terms $\varepsilon \text{div} Q_u(D_I)$ and $Q_d(D_I)$ involved in (1.20)-(1.27) will not result to any more difficulty in the energy estimates comparing to the energy estimates of the system (1.33)-(1.35). For term $\sqrt{\varepsilon}\partial_t(u_R^e \cdot \nabla D_{ij}^e)$, it will be dealt with the same way as the term $\partial_t(u_R^e \cdot \nabla d_0)$, the details of which will be given later. To avoid the tedious calculations in controlling the tedious terms $\varepsilon \text{div} Q_u$ and $Q_d$, we only derive the a priori estimates of the remainder system (1.33) with the initial conditions (1.35), for simplicity. Actually, its calculations remain very annoying and complicated.

Next, we aim at deriving the a priori estimates of the remainder equations (1.33) with the initial data (1.35). The key points are:

- The $H^N$-norm of term $\partial_t(u_R^e \cdot \nabla d_0)$ in the $d_R^e$-equation of (1.33) has the same regularity as the $H^N$-norm of $\Delta u_R^e$ and the energy dissipative rate will only supply a regularity of $\|\nabla u_R^e\|_{H^N}$, so that we can not crudely view it as a source term to be controlled in the right-hand side of the energy inequality. To overcome this, we deal with this term as an energy term, just like $\partial_t u_R^e$ or $\partial_t D_{uv} + \sqrt{\varepsilon} u_R^e \cdot D_{ij}^e$. Consequently, the energy functional of the remainder system (1.33) is depended on the vector field $d_0$, which is a solution to the limit equations (1.10).

- The relations (2.1) is essential to deal the the terms $C_u$ and $C_d$, which are all linearly dependent on the $u_R^e$ and $d_R^e$ and with the coefficient $d_0$. We derive some useful dissipative structures from these two terms. So, we will derive an energy dissipative rate of the remainder system depending also on $d_0$.

Before deriving the a priori uniform energy estimates on the remainder system (1.33), we state the global existence, which has been proved by Wang-Zhang-Zhang in [21], to the incompressible parabolic Ericksen-Leslie’s liquid crystal model (1.10) with small initial data (1.13). For convenience to readers, we restate this result here. We first define the following energy functionals $\mathcal{E}_{s, 0}$ and $\mathcal{D}_{s, 0}$ for any integer $s \geq 2$ (see Section 5 of [21]):

\[
\mathcal{E}_{s, 0}(t) = \|\nabla d_0\|_{L^2}^2 + \|\nabla \Delta^s d_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2,
\]

\[
\mathcal{D}_{s, 0}(t) = \frac{1}{\lambda_1} \|\Delta d_0\|_{L^2}^2 + \frac{1}{\lambda_1} \|\Delta^{s+1} d_0\|_{L^2}^2 + \frac{\mu_4}{2} \|\nabla u_0\|_{L^2}^2 + \frac{\mu_4}{2} \|\nabla \Delta^s u_0\|_{L^2}^2.
\]

By interpolation, one easily know that

\[
\mathcal{E}_{s, 0}(t) \sim \|u_0\|_{H^{2s}}^2 + \|\nabla d_0\|_{H^{2s}}^2,
\]

\[
\mathcal{D}_{s, 0}(t) \sim \|\nabla u_0\|_{H^{2s}}^2 + \|\Delta d_0\|_{H^{2s}}^2.
\]

Then, the following result holds:

**Proposition 3.1** (Wang-Zhang-Zhang in [21]). Let $s \geq 2$ be an integer. Assume that the Leslie coefficients satisfy (1.7) and (1.38), and the initial data $\nabla d^{in} \in H^{2s}$, $u^{in} \in H^{2s}$. Then if there exists a $\beta_{s, 0} > 0$ such that

\[
\|\nabla d^{in}\|_{H^{2s}}^2 + \|u^{in}\|_{H^{2s}}^2 \leq \beta_{s, 0},
\]

the incompressible parabolic Ericksen-Leslie’s liquid crystal model (1.10) with initial conditions (1.13) admits a unique global classical solution

\[
u_0, \nabla d_0 \in C(\mathbb{R}^+; H^{2s}), \quad \nabla u_0 \in L^2(\mathbb{R}^+; H^{2s})
\]
satisfying the uniform bound
\[ \|u_0\|_{H^2}^2 + \|\nabla d_0\|_{H^2}^2 \leq c_0^{-1} \mathcal{E}_{s,0} \leq c_0^{-1} C_0 \mathcal{E}_{s,0} \] (3.3)
holds for all \( t \geq 0 \). Moreover, the following energy inequality holds:
\[ \frac{d}{dt} \mathcal{E}_{s,0} + \mathcal{D}_{s,0} \leq 0, \quad \forall t \geq 0. \] (3.4)

We now introduce the following energy functional \( \mathcal{E}_{N,\varepsilon}(t) \)
\[ \mathcal{E}_{N,\varepsilon}(t) = \frac{1}{\varepsilon} \|u_R^\varepsilon\|_{H^2}^2 + (1 - \delta) \|D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon\|_{H^2}^2 + \frac{1}{\varepsilon} \|\nabla d_R^\varepsilon\|_{H^2}^2 \]
\[ + \left( - \frac{\delta \lambda_1}{\varepsilon} - \frac{5}{4} \delta \right) \|d_R^\varepsilon\|_{H^2}^2 + \|u_R^\varepsilon \cdot \nabla d_0 + \frac{\delta}{2} d_R^\varepsilon\|_{H^2}^2 \]
\[ + \delta \|D_{u_0 + \varepsilon u_R^\varepsilon} d^\varepsilon_R + d^\varepsilon_R\|_{H^2}^2 + 2 \sum_{|m| \leq N} \langle \partial^m (u_R^\varepsilon \cdot \nabla d_0), \partial^m D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon \rangle \] (3.5)
and the energy dissipative rate \( \mathcal{D}_{N,\varepsilon}(t) \)
\[ \mathcal{D}_{N,\varepsilon}(t) = \frac{3 \delta \lambda_1}{8 \varepsilon} \|\nabla u_R^\varepsilon\|_{H^2}^2 + \frac{3}{2 \varepsilon} \|\nabla d_R^\varepsilon\|_{H^2}^2 - \delta \|D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon\|_{H^2}^2 \]
\[ + \frac{5 \lambda_1}{\varepsilon} \sum_{|m| \leq N} \|\partial^m A_R^\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon} \|\nabla \partial^m (u_R^\varepsilon \cdot \nabla d_0)\|_{L^2}^2 \] (3.6)
\[ + \frac{\delta \lambda_1}{\varepsilon} \sum_{|m| \leq N} \|\partial^m D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon + (\partial^m B_R^\varepsilon)d_0 \| + \frac{\lambda_1}{\varepsilon} (\partial^m A_R^\varepsilon)d_0 \|_{L^2}^2, \]
where \( \delta \in (0, \frac{1}{4}) \) is a fixed constant, depending only on \( \lambda_1, \lambda_2 \) and \( N \).

One notices that the energy \( \mathcal{E}_{N,\varepsilon}(t) \) and the energy dissipative rate \( \mathcal{D}_{N,\varepsilon}(t) \) may not be nonnegative, since there is an indefinitely signed term \( \sum_{|m| \leq N} \langle \partial^m u_R^\varepsilon \cdot \nabla d_0, \partial^m D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon \rangle \) appearing in \( \mathcal{E}_{N,\varepsilon}(t) \) and the functional \( \mathcal{D}_{N,\varepsilon}(t) \) includes a negative term \( - \delta \|D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon\|_{H^2}^2 \). However, if the inertia constant \( \varepsilon > 0 \) is sufficiently small, the functionals \( \mathcal{E}_{N,\varepsilon}(t) \) and \( \mathcal{D}_{N,\varepsilon}(t) \) will be both nonnegative. More precisely, we derive the following lemma.

**Lemma 3.1.** If the integer \( N \geq 2 \) and the Leslie’s coefficients satisfy relations (1.7) and (1.38), then there is a small \( \varepsilon_0 > 0 \), depending only on \( N, \beta_{S_{N,0}}, \) and the all Leslie’s coefficients, such that the energy \( \mathcal{E}_{N,\varepsilon}(t) \) and the energy dissipative rate \( \mathcal{D}_{N,\varepsilon}(t) \) are both nonnegative for any \( \varepsilon \in (0, \varepsilon_0) \). Moreover, for all \( \varepsilon \in (0, \varepsilon_0) \), we have
\[ \mathcal{E}_{N,\varepsilon}(t) \sim \frac{1}{\varepsilon} \|u_R^\varepsilon\|_{H^2}^2 + \|D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon\|_{H^2}^2 + \frac{1}{\varepsilon} \|\nabla d_R^\varepsilon\|_{H^2}^2, \] (3.7)
and
\[ \mathcal{D}_{N,\varepsilon}(t) \sim \frac{1}{\varepsilon} \|\nabla u_R^\varepsilon\|_{H^2}^2 + \frac{1}{\varepsilon} \|\nabla d_R^\varepsilon\|_{H^2}^2 + \|D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon\|_{H^2}^2 \]
\[ + \frac{1}{\varepsilon} \sum_{|m| \leq N} \|\partial^m D_{u_0 + \varepsilon u_R^\varepsilon} d_R^\varepsilon + (\partial^m B_R^\varepsilon)d_0 \| + (\partial^m A_R^\varepsilon)d_0 \|_{L^2}^2. \] (3.8)

Here the small positive constant \( \beta_{S_{N,0}} \) is given in Proposition 3.1.

**Proof of Lemma 3.1.** We now find a constant \( \varepsilon_0 > 0 \) such that
\[ \mathcal{E}_{N,\varepsilon} \geq 0, \quad \text{and} \quad \mathcal{D}_{N,\varepsilon}(t) \geq 0 \] (3.9)
for all \( \varepsilon \in (0, \varepsilon_0) \). First, we require the coefficient \( - \frac{\delta \lambda_1}{\varepsilon} - \frac{5}{4} \delta \) in \( \mathcal{E}_{N,\varepsilon}(t) \) satisfies
\[ - \frac{\delta \lambda_1}{\varepsilon} - \frac{5}{4} \delta \geq - \frac{\delta \lambda_1}{2 \varepsilon} > 0, \] (3.10)
Noticing that $0 < \delta \leq \frac{1}{2}$, we have

$$
\| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \mathbf{d}_R^\varepsilon \|^2_{H^N} - \delta \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \\
\geq \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \\
\geq \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \geq 0 .
$$

Then by utilizing the Hölder inequality, the Sobolev embedding theory, the Young's inequality and the bound (3.3) in Proposition 3.1, we estimate

$$
\left| \sum_{|m| \leq N} \left( \partial^m (u^R_{\varepsilon} \cdot \nabla d_0), \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \right) \right| \\
\leq \| u^R_{\varepsilon} \cdot \nabla d_0 \|_{H^N} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \\
\leq C \| \nabla d_0 \|_{L^\infty(R^+;H^N)} \| u^R_{\varepsilon} \|^2_{H^N} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \\
\leq C \| \nabla d_0 \|_{L^\infty(R^+;H^N)} \| u^R_{\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \\
\leq C \beta S_{N,0} \| u^R_{\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N}
$$

for some constant $C = C(N) > 0$. Consequently, we deduce

$$
\frac{1}{\varepsilon} \| u^R_{\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \sum_{|m| \leq N} \left( \partial^m (u^R_{\varepsilon} \cdot \nabla d_0), \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \right) \\
\geq \left( \frac{1}{\varepsilon} - C \beta S_{N,0} \right) \| u^R_{\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} .
$$

If we require $\frac{1}{\varepsilon} - C \beta S_{N,0} \geq \frac{1}{2\varepsilon}$, hence $0 < \varepsilon \leq \frac{1}{2C\beta S_{N,0}}$, then

$$
\frac{1}{\varepsilon} \| u^R_{\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \sum_{|m| \leq N} \left( \partial^m (u^R_{\varepsilon} \cdot \nabla d_0), \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \right) \\
\geq \frac{1}{2\varepsilon} \| u^R_{\varepsilon} \|^2_{H^N} + \frac{1}{2} \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \geq 0 .
$$

As a result, if $0 < \varepsilon \leq \min \left\{ \frac{-2\lambda_1}{\delta}, \frac{1}{2C\beta S_{N,0}} \right\}$, we have $\mathcal{G}_{N,\varepsilon}(t) \geq 0$.

Next we consider the energy dissipative rate $\mathcal{G}_{N,\varepsilon}(t)$. One observes that there is only a negative term $-\delta \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N}$ in $\mathcal{G}_{N,\varepsilon}(t)$ under assumption (1.38). Via the following elementary estimates

$$
\| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \leq 2 \sum_{|m| \leq N} \left( \| \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \right) \\
\leq 2 \sum_{|m| \leq N} \left( \| \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \right) \\
+ 4(1 + \frac{\lambda_2}{\lambda_1}) \| \nabla u^R_{\varepsilon} \|^2_{H^N} ,
$$

we know

$$
\frac{3\mu_4}{8e} \| \nabla u^R_{\varepsilon} \|^2_{H^N} - 2\delta \| \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \frac{-2\lambda_1}{\varepsilon} \sum_{|m| \leq N} \| \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \| \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \frac{\lambda_2}{\lambda_1} (\partial^m \mathbf{A}_{R_0} \mathbf{d}_0) \|^2_{L^2} \\
\geq \left( \frac{3\mu_4}{8e} - 8\delta - \frac{8\delta\lambda_2}{\lambda_1} \right) \| \nabla u^R_{\varepsilon} \|^2_{H^N} \\
+ \left( \frac{-2\lambda_1}{\varepsilon} - 4\delta \right) \sum_{|m| \leq N} \| \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} + \| \partial^m \mathbf{D}_{u_0 + \sqrt{u_R} \mathbf{d}_R^\varepsilon} \|^2_{H^N} \frac{\lambda_2}{\lambda_1} (\partial^m \mathbf{A}_{R_0} \mathbf{d}_0) \|^2_{L^2} .
$$
If we require \(\frac{3\mu^4}{8\varepsilon^2} - 8\delta - \frac{8\lambda_1^2}{\lambda_2^2} \geq \frac{4\mu^4}{48\varepsilon^2}\) and \(-\frac{\lambda_1}{8\varepsilon} - 4\delta \geq \frac{\lambda_1^2}{2\varepsilon^2}\), namely, \(0 < \varepsilon \leq \min\left\{\frac{-\lambda_1}{8\varepsilon}, \frac{\mu^4\lambda_1^2}{64\delta(\lambda_2^2 + \lambda_3^2)}\right\}\), then the quantity in the right-hand side of the inequality (3.16) has a lower bound

\[
\frac{\mu^4}{48\varepsilon^2} \| \nabla u_R^\varepsilon \|_{L^2}^2 + \frac{-\lambda_1}{8\varepsilon} \sum_{|m| \leq N} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon + (\partial^m A_R^\varepsilon) d_0 + \frac{\lambda_1}{\lambda_2} (\partial^m A_R^\varepsilon) d_0 \|_{L^2}^2 \geq 0.
\]

In summary, we can take \(\varepsilon_0 = \min\left\{1, \frac{-2\lambda_1}{5}, \frac{-\lambda_1}{8\varepsilon}, \frac{\mu^4\lambda_1^2}{64\delta(\lambda_2^2 + \lambda_3^2)}, \frac{1}{2\varepsilon^2} \right\} > 0\), so that \(\mathcal{S}_{N,\varepsilon}(t) \geq 0\) and \(\mathcal{D}_{N,\varepsilon}(t) \geq 0\) hold for all \(\varepsilon \in (0, \varepsilon_0]\). Moreover, following the previous estimates, one can easily derive the lower bounds of the inequalities (3.7) and (3.8), and the upper bounds are obviously holds. Then the proof of Lemma 3.1 is finished. 

Next we derive the following key energy inequality, which will reduce to the uniform energy bound under the assumption of small size of the initial data. More precisely, we will give the following proposition.

**Proposition 3.2.** Let \(N \geq 2\) be an integer and assume that \((u_R^\varepsilon, d_R^\varepsilon)\) is a sufficiently smooth solution to the remainder system (1.20) on \([0, T]\). Then there are constants \(C > 0\) and \(\theta_0 > 1\), depending only on the Leslie coefficients and \(\beta_{S_N,0}\) given in Proposition 3.1, such that

\[
\frac{4}{\varepsilon} \left[ \mathcal{S}_{N,\varepsilon}(t) + \theta_0 \mathcal{S}_{S_N,0}(t) \right] + \mathcal{D}_{N,\varepsilon}(t) + \frac{\theta_0}{2} \mathcal{S}_{S_N,0}(t)
\]

\[
\leq C \left[ \mathcal{S}_{N,\varepsilon}^2(t) + \mathcal{S}_{S_N,0}^2(t) \right] \left[ \mathcal{D}_{N,\varepsilon}(t) + \frac{\theta_0}{2} \mathcal{S}_{S_N,0}(t) \right]
\]

holds for all \(t \in [0, T]\) and \(\varepsilon \in (0, \varepsilon_0]\), where the small positive constant \(\varepsilon_0\) is mentioned in Lemma 3.1, and the integer \(S_N\) is defined in (1.37).

**Proof of Proposition 3.2.** For all multi-indexes \(m \in \mathbb{N}^3\) with \(|m| \leq N\) \((N \geq 2)\), we take the derivative operator \(\partial^m\) on the first equation of the remainder system (1.20) and take \(L^2\)-inner product by multiplying \(\partial^m u_R^\varepsilon\) and integrating by parts over \(x \in \mathbb{T}^3\). We hence obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \partial^m u_R^\varepsilon \|_{L^2}^2 + \frac{\mu_1}{4} \| \nabla \partial^m u_R^\varepsilon \|_{L^2} + \mu_1 \| \partial^m A_R^\varepsilon : d_0 \otimes d_0 \|_{L^2} \right] - \mu_1 \sum_{0 \neq m' \leq m} \sum_{m'' \leq m'} \left\langle \left( \partial^{m-m'} A_R^\varepsilon : d_0 \otimes d_0 \right) \partial^{m''} (d_0 \otimes d_0), \nabla \partial^m u_R^\varepsilon \right\rangle
\]

\[
+ \langle \text{div} \partial^m C_u, \partial^m u_R^\varepsilon \rangle + \langle \partial^m K_u, \partial^m u_R^\varepsilon \rangle + \langle \partial^m \text{div}(T_u + \sqrt{\varepsilon} R_u), \partial^m u_R^\varepsilon \rangle,
\]

where we make use of the divergence-free property of \(u_R^\varepsilon\), the relation \(\nabla \partial^m u_R^\varepsilon = \partial^m A_R^\varepsilon + \partial^m B_R^\varepsilon\) and the skew-symmetry of \(\partial^m B_R^\varepsilon\).

Acting the derivative operator \(\partial^m\) on the third equation of (1.20), taking \(L^2\)-inner product by dot with \(\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\) and integrating by parts over \(x \in \mathbb{T}^3\), we know

\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \|_{L^2}^2 + \frac{1}{\varepsilon} \| \nabla \partial^m d_R^\varepsilon \|_{L^2}^2 \right)
\]

\[
+ \left\langle \partial \partial^m (u_R^\varepsilon \cdot \text{grad} d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle + \frac{\lambda_1}{\varepsilon} \left\| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\|_{L^2}^2
\]

\[
- \left\langle \partial^m (\sqrt{\varepsilon} u_R^\varepsilon \cdot \nabla D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle - \frac{1}{\varepsilon} \left\langle \nabla \partial^m d_R^\varepsilon, \nabla \partial^m (\sqrt{\varepsilon} u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle
\]

\[
+ \left( \frac{1}{\varepsilon^3} d_0 \right) + \frac{1}{\varepsilon} \partial^m \mathcal{S}_d + \frac{\lambda_1}{\varepsilon} \partial^m \mathcal{S}_d^2 + \partial^m \mathcal{R}_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle.
\]

(3.19)
Accounting for the cancellation (2.1) in Lemma 2.1, we add the \( \frac{1}{\varepsilon} \) times of (3.18) to (3.19), and then deduce that for \(|m| \leq N\)

\[
\begin{align*}
&\frac{1}{2\varepsilon} \left\| \partial^m u_R^\varepsilon \right\|_{L^2}^2 + \left\| \partial^m D_{u_0} + \nabla u_R^\varepsilon \partial^m \right\|_{L^2}^2 + \frac{1}{\varepsilon} \left\| \nabla \partial^m \partial^m \right\|_{L^2}^2 \\
+ \varepsilon \left( \partial^m \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) \left( \partial^m \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) + \left( \partial^m \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) \left( \partial^m \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) - \frac{1}{\varepsilon} \left( \nabla \partial^m \partial^m (\varepsilon u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) \\
+ \frac{1}{\varepsilon^2} \sum_{m' \leq m} C_{m''}^m \left( \epsilon \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) \left( \epsilon \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) + \frac{1}{\varepsilon^2} \epsilon \partial^m \partial^m (\nabla d_R^\varepsilon) \left( \epsilon \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right).
\end{align*}
\]

We next deal with the term \( \left( \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right) \). Straightforward calculation reduces to

\[
\begin{align*}
&\left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle \\
= \frac{d}{d\varepsilon} \left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \right\rangle \\
- \left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (\sqrt{\varepsilon} u_R^\varepsilon + D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon}) \right\rangle. \\
\end{align*}
\]

Then, from utilizing the third \( d_R^\varepsilon \)-equation of (1.20), we derive that

\[
\begin{align*}
&\left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \right\rangle \\
= \frac{d}{d\varepsilon} \left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m \partial^m (u_0 \cdot \nabla d_R^\varepsilon) \right\rangle \\
+ \frac{1}{\varepsilon} \left\langle \nabla \partial^m \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle - \frac{1}{\varepsilon^2} \left\langle \nabla \partial^m \partial^m (\varepsilon u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle.
\end{align*}
\]

In summary, we obtain the key relation to deal with the term \( \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \) that

\[
\begin{align*}
&\left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \right\rangle \\
= \frac{d}{d\varepsilon} \left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_0 \cdot \nabla d_R^\varepsilon) \right\rangle \\
- \left\langle \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_0 \cdot \nabla d_R^\varepsilon) \right\rangle \\
- \frac{1}{\varepsilon^2} \left\langle \nabla \partial^m \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle.
\end{align*}
\]

Consequently, it is derived from the equalities (3.20), (3.23) and summing up for \(|m| \leq N\) that

\[
\begin{align*}
&\frac{1}{2\varepsilon} \left\| u_R^\varepsilon \right\|_{H^N}^2 + \left\| D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} \partial^m \right\|_{H^N}^2 + \frac{1}{\varepsilon} \left\| \nabla d_R^\varepsilon \right\|_{H^N}^2 \\
+ \left\| u_R^\varepsilon \cdot \nabla d_R^\varepsilon \right\|_{H^N}^2 + 2 \sum_{|m| \leq N} \left\langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m (u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \right\rangle.
\end{align*}
\]
precisely, we act the derivative operator $\partial^m$ on (1.20), take $L^2$-inner product by dot with $\partial^m d_R$ and integrate by parts over $x \in \mathbb{T}^3$. Then we have for $|m| \leq N$

$$\frac{\lambda_1}{\varepsilon} \frac{d}{dt} \|\partial^m d_R\|_{L^2}^2 + \frac{\lambda_2}{\varepsilon} \|\partial^m d_R\|_{L^2}^2$$

$$+ \frac{\lambda_3}{\varepsilon} \sum_{|m| \leq N} \langle \partial^m D_{u_0 + \sqrt{\varepsilon} u_R} d_R, \partial^m d_R \rangle + \langle \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle$$

$$= \lambda_1 \left\langle \partial^m ((u_0 + \sqrt{\varepsilon} u_R) : \nabla d_R), \partial^m d_R \right\rangle$$

$$= \frac{\lambda_1}{\varepsilon} \left\langle \partial^m (u_0 + \sqrt{\varepsilon} u_R) : \nabla d_R, \partial^m d_R \right\rangle + \frac{\lambda_2}{\varepsilon} \sum_{|m| \leq N} \langle \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle$$

$$+ \frac{\lambda_3}{\varepsilon} \sum_{|m| \leq N} \langle \partial^m D_{u_0 + \sqrt{\varepsilon} u_R} d_R, \partial^m d_R \rangle + \langle \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle$$

It is easily calculated that

$$\langle \partial_t \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle$$

$$= \frac{d}{dt} \langle \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle - \langle \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle$$

$$= \frac{d}{dt} \langle \partial^m (u_R \cdot \nabla d_R), \partial^m d_R \rangle - \langle \partial^m (u_R \cdot \nabla d_R), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R} d_R \rangle$$

$$+ \langle \partial^m (u_R \cdot \nabla d_R), \partial^m ((u_0 + \sqrt{\varepsilon} u_R) \cdot \nabla d_R) \rangle.$$
\[ + \left< \partial^m (\sqrt{\varepsilon} u_R \cdot \nabla D_{u_0+\sqrt{\varepsilon} u_R} d_R^c), \partial^m d_R^c \right> - \left< \partial^m (\sqrt{\varepsilon} u_R \otimes D_{u_0+\sqrt{\varepsilon} u_R} d_R^c), \nabla \partial^m d_R^c \right> \\
= \frac{1}{2} \frac{d}{dt} \left( \|D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{L^2} - \|D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{H^N} \right) + \frac{1}{\varepsilon} \|\nabla d_R^c \|^2_{H^N} - \|D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{H^N} \\
- \|\partial^m D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{L^2} + \sum_{0 \neq m' \leq m} C_m^{m'} \left< \partial^m D_{u_0+\sqrt{\varepsilon} u_R} d_R^c, \nabla \partial^m d_R^c \right> \\
- \sum_{0 \neq m' \leq m} C_m^{m'} \left< \sqrt{\varepsilon} \partial^m u_R \otimes \partial^m m' D_{u_0+\sqrt{\varepsilon} u_R} d_R^c, \nabla \partial^m d_R^c \right> \tag{3.27} \]

Substituting the relations (3.26) and (3.27) into (3.25) and summing up for \(|m| \leq N\) reduce to

\[ \frac{1}{2} \frac{d}{dt} \left( \|D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{L^2} - \|D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{H^N} \right) + \frac{1}{\varepsilon} \|\nabla d_R^c \|^2_{H^N} - \|D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \|^2_{H^N} \]

\[ = - \sum_{|m| \leq N} \left< \partial^m ((u_0 + \sqrt{\varepsilon} u_R) \cdot \nabla d_0), \partial^m (u_0 + \sqrt{\varepsilon} u_R) \cdot \nabla d_R^c \right> + \sum_{|m| \leq N} \left< \partial^m (u_0 + \sqrt{\varepsilon} u_R) \cdot \nabla d_0, \partial^m D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \right> \]

\[ + \frac{1}{\varepsilon} \sum_{|m| \leq N} \left< \partial^m (u_0 + \sqrt{\varepsilon} u_R) \cdot \nabla d_0, \partial^m D_{u_0+\sqrt{\varepsilon} u_R} d_R^c \right> + \frac{1}{\varepsilon} \sum_{|m| \leq N} \left< \partial^m \mathcal{C}_d, \partial^m d_R^c \right> \]

\[ + \sum_{|m| \leq N} \left< \frac{1}{\varepsilon} \partial^m S_d^2 + \frac{1}{\varepsilon} \partial^m \mathcal{C}_d, \partial^m d_R^c \right> + \sum_{|m| \leq N} \sum_{0 \neq m' \leq m} C_m^{m'} \left< \sqrt{\varepsilon} \partial^m m' u_R \otimes \partial^m m' D_{u_0+\sqrt{\varepsilon} u_R} d_R^c, \nabla \partial^m d_R^c \right> \\
- \sum_{|m| \leq N} \sum_{0 \neq m' \leq m} C_m^{m'} \left< \partial^m D_{u_0+\sqrt{\varepsilon} u_R} d_R^c, \nabla \partial^m m' d_R^c \right> \tag{3.28} \]

for all \(\varepsilon > 0\). Recalling the definition of \(\mathcal{C}_d\) in (1.23), we calculate that

\[ \frac{1}{2} \sum_{1 \leq |m| \leq N} \left< \partial^m \mathcal{C}_d, \partial^m d_R^c \right> \]

\[ = \frac{1}{2} \sum_{1 \leq |m| \leq N} \left< \lambda_1 (\partial^m B_R) d_0 + \lambda_2 (\partial^m A_R^c) d_0, \partial^m d_R^c \right> \]

\[ + \frac{1}{\varepsilon} \sum_{1 \leq |m| \leq N} \sum_{0 \neq m' \leq m} C_m^{m'} \left< \lambda_1 (\partial^m \mathcal{C}_d) d_0 + \lambda_2 (\partial^m \mathcal{C}_d) d_0, \partial^m d_R^c \right>, \tag{3.29} \]

where the first term in the right-hand side of the previous equality can be bounded by

\[ \frac{1}{2} \sum_{1 \leq |m| \leq N} (|\lambda_1| \|\partial^m B_R\|_{L^2} + |\lambda_2| \|\partial^m A_R^c\|_{L^2}) \|\partial^m d_R^c\|_{L^2} \]

\[ \leq \frac{1}{2} \sqrt{c_0'(\lambda_1, \lambda_2, N)} \|\nabla u_R^c\|_{H^N} \|\nabla d_R^c\|_{H^N} \]

\[ \leq \frac{1}{4} c_0'(\lambda_1, \lambda_2, N) \|\nabla u_R^c\|^2_{H^N} + \frac{1}{4} \|\nabla d_R^c\|^2_{H^N} \tag{3.30} \]

for some constant \(c_0'(\lambda_1, \lambda_2, N) > 0\). Furthermore, thanks to \(\int_{\mathbb{T}^3} dx = |\mathbb{T}^3|^3 < \infty\), we derive from the Gagliardo-Nirenberg interpolation inequality \(\|f\|_{L^6} \leq C \|\nabla f\|_{L^2}\) and the Young’s
inequality that
\[ \frac{1}{\varepsilon} \langle \lambda_1 B_R \partial u_0 + \lambda_2 A_R \partial^m d_0, d_R^\varepsilon \rangle \leq \frac{1}{\varepsilon} (|\lambda_1| + |\lambda_2|) \left( \int_{\mathbb{T}^3} |d_0|^3 dx \right)^{\frac{2}{3}} \left\| \nabla u_R \right\|_{L^2} \left\| d_R^\varepsilon \right\|_{L^6} \]
\[ \leq \frac{1}{\varepsilon} (|\lambda_1| + |\lambda_2|) \left\| \nabla u_R \right\|_{H^N} \left\| \nabla d_R^\varepsilon \right\|_{H^N} \]
\[ \leq \frac{1}{\varepsilon} c_0^0 (|\lambda_1, \lambda_2|) \left\| \nabla u_R \right\|_{H^N}^2 + \frac{1}{\varepsilon} \left\| \nabla d_R^\varepsilon \right\|_{H^N}^2 \]  \quad (3.31)
for some constant \( c_0^0 (|\lambda_1, \lambda_2|) > 0 \). Consequently, we know that
\[ \frac{1}{\varepsilon} \sum_{|m|\leq N} \langle \partial^m C_d, \partial^m d_R^\varepsilon \rangle \leq \frac{1}{\varepsilon} c_0 (|\lambda_1, \lambda_2, N|) \left\| \nabla u_R \right\|_{H^N}^2 + \frac{1}{\varepsilon} \left\| \nabla d_R^\varepsilon \right\|_{H^N}^2 \]
\[ + \frac{1}{\varepsilon} \sum_{1 \leq |m| \leq N 0 \neq |m| \leq N} C_m^\varepsilon \left\langle \lambda_1 (\partial^{|m| - m'} B_R) \partial^m d_0 + \lambda_2 (\partial^{|m| - m'} A_R) \partial^m d_0, d_R^\varepsilon \right\rangle \]  \quad (3.32)
holds for \( N \geq 2 \), where \( c_0 (|\lambda_1, \lambda_2, N|) = c_0' (|\lambda_1, \lambda_2, N|) + c_0^0 (|\lambda_1, \lambda_2|) > 0 \).

We now take a small constant
\[ \delta = \min \left\{ \frac{1}{2}, \frac{\mu}{8 c_0 (|\lambda_1, \lambda_2, N|)} \right\} \in (0, \frac{1}{2}] \]  \quad (3.33)
such that \( \delta c_0 (|\lambda_1, \lambda_2, N|) \leq \frac{\mu}{8} \). Combining the relation (3.31), we multiply (3.28) by \( \delta \) and then add it to the (3.24), which gives us
\[ \frac{1}{2 \varepsilon} \frac{d}{dt} \mathcal{E}_{N, \varepsilon} (t) + \mathcal{P}_{N, \varepsilon} (t) \leq \mathcal{I}_{N}^{(1)} + \mathcal{I}_{N}^{(2)} + \mathcal{I}_{N}^{(3)} + \mathcal{I}_{N}^{(4)} \]  \quad (3.34)
for \( N \geq 2 \), where the symbols \( \mathcal{I}_{N}^{(i)} \) \( (1 \leq i \leq 4) \) are defined as
\[ \mathcal{I}_{N}^{(1)} = \frac{1}{\varepsilon} \sum_{|m|\leq N} \langle \partial^m \mathcal{K}_u, \partial^m u_R^\varepsilon \rangle + \frac{1}{\varepsilon} \sum_{|m|\leq N} \langle \nabla \partial^m d_R^\varepsilon, \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_0) \rangle \]
\[ + \frac{\delta_1}{\varepsilon} \sum_{|m|\leq N} \langle \partial^m ((u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \cdot \nabla d_R^\varepsilon), \partial^m d_R^\varepsilon \rangle \]
\[ - \sum_{|m|\leq N} \langle \partial^m (\sqrt{\varepsilon} u_R^\varepsilon \cdot \nabla D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \rangle \]
\[ - \frac{1}{\varepsilon} \sum_{|m|\leq N} \langle \nabla \partial^m d_R^\varepsilon, \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_R) \rangle \]
\[ - \sum_{|m|\leq N} \langle \sqrt{\varepsilon} \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon, u_R^\varepsilon \cdot \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_0) \rangle \]
\[ + \frac{\delta_1}{\varepsilon} \sum_{|m|\leq N} \langle \partial^m (u_R^\varepsilon \cdot \nabla d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \rangle \]
\[ - \frac{\mu_1}{\varepsilon} \sum_{|m|\leq N 0 \neq |m| \leq N} \sum_{m' \leq m} C_m^m C_m''' \left\langle (\partial^{m-m'} A_R \partial^{m'-m''} (d_0 \otimes d_0)) \partial^{m''} (d_0 \otimes d_0), \nabla \partial^m u_R^\varepsilon \right\rangle \]
\[ - \delta \sum_{|m|\leq N} \langle \partial^m (u_R^\varepsilon \cdot \nabla d_0), \partial^m ((u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \cdot \nabla d_R^\varepsilon) \rangle \]
\[ + \delta \sum_{1 \leq |m| \leq N 0 \neq |m| \leq m} C_m^m \left\langle \sqrt{\varepsilon} \partial^m u_R^\varepsilon \otimes \partial^{m-m'} D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon, \nabla \partial^m d_R^\varepsilon \right\rangle \]
\[ - \delta \sum_{1 \leq |m| \leq N 0 \neq |m| \leq m} C_m^m \left\langle \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon, \sqrt{\varepsilon} \partial^m u_R^\varepsilon \cdot \nabla \partial^{m-m'} d_R^\varepsilon \right\rangle \]  \quad (3.35)
for the functional \( \mathcal{K}_u \) defined in (A.3), and
\[ \mathcal{I}_{N}^{(2)} = \frac{1}{\varepsilon} \sum_{|m|\leq N} \langle \partial^m T_u, \nabla \partial^m u_R^\varepsilon \rangle - \frac{1}{\varepsilon} \sum_{|m|\leq N} \langle \partial^m \mathcal{R}_u, \nabla \partial^m u_R^\varepsilon \rangle \]  \quad (3.36)
for the expressions $\mathcal{T}_u, \mathcal{R}_u$ given in (A.2), (A.7), respectively, and

$$I_N^{(3)} = \frac{1}{\varepsilon} \sum_{|m| \leq N} G_m$$

(3.37)

with the quantity $G_m$ mentioned in Lemma 2.1, and

$$I_N^{(4)} = -\frac{1}{\varepsilon} \sum_{|m| \leq N} \langle \partial^m (u_R^e \cdot \nabla d_0), \partial^m C_d \rangle$$

$$+ \sum_{|m| \leq N} \langle \frac{1}{\varepsilon} \partial^m S_1^d + \frac{1}{\varepsilon^2} \partial^m S_2^d + \partial^m \mathcal{R}_d, \partial^m \mathcal{D}_u + \sqrt{\varepsilon} \mathcal{D}_u^d \mathcal{D}_R^e - \partial^m (u_R^e \cdot \nabla d_0) \rangle$$

$$+ \delta \sum_{|m| \leq N} \langle \frac{1}{\varepsilon^2} \partial^m S_1^d + \frac{1}{\varepsilon^3} \partial^m S_2^d + \partial^m \mathcal{R}_d, \partial^m d_R^e \rangle$$

$$+ \frac{\delta}{\varepsilon} \sum_{1 \leq |m| \leq N} \sum_{0 \neq m' \leq m} C_m^i \langle \lambda_1 (\partial^m \mathcal{M}^m \mathcal{R}^e) \partial^m d_0 + \lambda_2 (\partial^m \mathcal{M}^m \mathcal{R}^e) \partial^m d_0, \partial^m d_R^e \rangle.$$  

(3.38)

Here the vectors $C_d, S_1^d, S_2^d$ and $R_d$ are determined in (1.23), (A.4), (A.5) and (A.6), respectively.

It remains to control the terms $I_N^{(i)} (1 \leq i \leq 4)$ for $N \geq 2$. We emphasize that, in the following estimates, we will frequently use the Sobolev interpolation inequality $\|f\|_{L^p(T^3)} \leq C \|\nabla f\|_{L^q(T^3)}$, Sobolev embeddings $H^1(T^3) \hookrightarrow L^4(T^3)$ (or $L^8(T^3)$), $H^2(T^3) \hookrightarrow L^\infty(T^3)$ and the inequalities (3.7), (3.8) with the constraint $\varepsilon \in (0, \varepsilon_0)$ in Lemma 3.1. Furthermore, for $|m| \leq N$ ($N \geq 2$), the calculus inequalities (see [20], for instance)

$$\|\partial^m (f g)\|_{L^2} \leq C \|f\|_{H^N} \|g\|_{H^N}$$

(3.39)

will also be frequently utilized. The geometric constraint $|d_0| = 1$ is also considered in the following energy estimates.

**Step 1. Control the term $I_N^{(1)}$.**

Via the divergence-free property of $u_0$, we have

$$-\frac{1}{\varepsilon} \langle \partial^m (u^e_0 \cdot \nabla u^e_R), \partial^m u^e_R \rangle = -\frac{1}{\varepsilon} \sum_{0 \neq m' \leq m} C_m^i \langle \partial^m u^e_0 \cdot \nabla \partial^m \mathcal{M}^m u^e_R, \partial^m u^e_R \rangle$$

$$\leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^m u^e_0\|_{L^4} \|\nabla \partial^m \mathcal{M}^m u^e_R\|_{L^4} \|\partial^m u^e_R\|_{L^2}$$

$$\leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^m u^e_0\|_{H^1} \|\nabla \partial^m \mathcal{M}^m u^e_R\|_{H^1} \|\partial^m u^e_R\|_{L^2}$$

$$\leq \frac{C}{\varepsilon} \|\nabla u^e_0\|_{H^N} \|\nabla u^e_R\|_{H^N} \|u^e_R\|_{H^N} \leq C \varepsilon^{\frac{1}{2}}(t) \varepsilon^{\frac{1}{2}}(t) \|\nabla u^e_0\|_{H^N}$$

holds for all $|m| \leq N$. If $1 \leq |m| \leq N$, we estimate

$$-\frac{1}{\varepsilon} \langle \partial^m (u^e_R \cdot \nabla u^e_0), \partial^m u^e_R \rangle = -\frac{1}{\varepsilon} \sum_{m' \leq m} C_m^i \langle \partial^m u^e_R \cdot \nabla \partial^m \mathcal{M}^m u^e_0, \partial^m u^e_R \rangle$$

$$\leq \frac{C}{\varepsilon} \|u^e_R\|_{L^8} \|\nabla \partial^m u^e_0\|_{L^2} \|\partial^m u^e_R\|_{L^2}$$

$$+ \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^m u^e_0\|_{L^4} \|\nabla \partial^m \mathcal{M}^m u^e_0\|_{L^4} \|\partial^m u^e_R\|_{L^2}$$

$$\leq \frac{C}{\varepsilon} \|u^e_R\|_{H^N} \|\nabla u^e_R\|_{H^N} \|\nabla u^e_0\|_{H^N} \leq C \varepsilon^{\frac{1}{2}}(t) \varepsilon^{\frac{1}{2}}(t) \|\nabla u^e_0\|_{H^N}.$$
Moreover, for \( m = 0 \), we have
\[
-\frac{1}{\varepsilon} \langle \partial^m (u_R^\varepsilon \cdot \nabla u_0^\varepsilon), \partial^m u_R^\varepsilon \rangle \leq \frac{1}{\varepsilon} \| u_R^\varepsilon \|_{L^4}^2 \| \nabla u_0^\varepsilon \|_{L^2} \leq C \| \nabla u_R^\varepsilon \|_{L^2}^2 \| u_R^\varepsilon \|_{L^2}^2 \leq C \frac{1}{\varepsilon} \| u_R^\varepsilon \|_{H^1} \| \nabla u_R^\varepsilon \|_{L^2} \| \nabla u_0^\varepsilon \|_{L^2} \leq C \varepsilon \frac{1}{N_{\varepsilon}}(t) \partial_{N_{\varepsilon}}^2(t) \| \nabla u_R^\varepsilon \|_{H^N},
\]
where we utilize the Sobolev interpolation inequality \( \| f \|_{L^4} \leq C \| f \|_{L^2}^{\frac{1}{2}} \| \nabla f \|_{L^2}^{\frac{1}{2}} \). Then, we obtain
\[
-\frac{1}{\varepsilon} \langle \partial^m (u_R^\varepsilon \cdot \nabla u_0^\varepsilon), \partial^m u_R^\varepsilon \rangle \leq -\frac{1}{\varepsilon} \langle \partial^m (u_R^\varepsilon \cdot \nabla u_0^\varepsilon), \partial^m u_R^\varepsilon \rangle
\]
for all \( |m| \leq N \) \((N \geq 2)\). It is easy to derive from the divergence-free property of \( u_R^\varepsilon \) that
\[
-\frac{1}{\varepsilon} \langle \partial^m (\sqrt{\varepsilon} u_R^\varepsilon \cdot \nabla u_R^\varepsilon), \partial^m u_R^\varepsilon \rangle = -\frac{1}{\varepsilon} \sum_{0 \neq m' \leq m} C_{m'} \langle \partial^m' u_R^\varepsilon \cdot \nabla (\partial^m - m') u_R^\varepsilon, \partial^m u_R^\varepsilon \rangle
\]
\[
\leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \| \partial^m' u_R^\varepsilon \|_{L^4} \| \nabla (\partial^m - m') u_R^\varepsilon \|_{L^4} \| \partial^m u_R^\varepsilon \|_{L^2} \leq \frac{C}{\varepsilon} \| u_R^\varepsilon \|_{H^N} \| \nabla u_R^\varepsilon \|_{H^N}^2
\]
(3.44)
Moreover, we can deduce
\[
-\frac{1}{\varepsilon} \langle \partial^m \text{div}(\nabla d_0^\varepsilon \nabla d_R^\varepsilon, \nabla d_0^\varepsilon \nabla d_R^\varepsilon), \partial^m u_R^\varepsilon \rangle
\]
\[
= \frac{1}{\varepsilon} \langle \partial^m (\nabla d_0^\varepsilon \nabla d_R^\varepsilon + \nabla d_0^\varepsilon \nabla d_R^\varepsilon + \sqrt{\varepsilon} \nabla d_R^\varepsilon \nabla d_R^\varepsilon), \partial^m u_R^\varepsilon \rangle
\]
\[
\leq \frac{C}{\varepsilon} \| \partial^m (\nabla d_0^\varepsilon \nabla d_R^\varepsilon + \nabla d_0^\varepsilon \nabla d_R^\varepsilon + \sqrt{\varepsilon} \nabla d_R^\varepsilon \nabla d_R^\varepsilon) \|_{L^2} \| \partial^m u_R^\varepsilon \|_{L^2}
\]
(3.45)
for all \( |m| \leq N \). Recalling the definition of \( K_u \) in (A.3), we derive from the bounds (3.40), (3.43), (3.44) and (3.45) that for \( |m| \leq N \)
\[
\frac{1}{\varepsilon} \langle \partial^m K_u, \partial^m u_R^\varepsilon \rangle \leq C \varepsilon \frac{1}{N_{\varepsilon}}(t) \partial_{N_{\varepsilon}}^2(t) \| \nabla u_0^\varepsilon \|_{H^N} + C (\| \nabla d_0^\varepsilon \|_{H^N} + \varepsilon \frac{1}{N_{\varepsilon}}(t) \partial_{N_{\varepsilon}}^2(t) \partial_{N_{\varepsilon}}(t)) \partial_{N_{\varepsilon}}(t).
\]
(3.46)
For \( |m| \leq N \), we calculate
\[
\frac{1}{\varepsilon} \langle \nabla \partial^m d_R^\varepsilon \nabla (u_R^\varepsilon \nabla d_0^\varepsilon) \rangle = \frac{1}{\varepsilon} \langle \nabla \partial^m u_R^\varepsilon \nabla (u_R^\varepsilon \nabla d_0^\varepsilon) \rangle = \frac{1}{\varepsilon} \sum_{m' \leq m} C_{m'} \langle \nabla \partial^m u_R^\varepsilon \nabla (u_R^\varepsilon \nabla d_0^\varepsilon) \rangle
\]
\[
\leq \frac{C}{\varepsilon} \| \partial^m u_R^\varepsilon \|_{L^6} \| \nabla \partial^m u_R^\varepsilon \|_{L^6} \| \nabla u_R^\varepsilon \|_{L^2} \| \nabla \partial^m d_0^\varepsilon \|_{L^6} \| \nabla \partial^m u_R^\varepsilon \|_{L^2}
\]
(3.47)
Via the divergence-free property of \( u_R^\varepsilon \), we yield that
\[
\frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon), \partial^m d_R^\varepsilon \rangle = \frac{\delta \lambda}{\sqrt{\varepsilon}} \sum_{0 \neq m' \leq m} C_{m'} \langle \partial^m u_R^\varepsilon \cdot \nabla (\partial^m - m') d_R^\varepsilon, \partial^m d_R^\varepsilon \rangle
\]
\[
\leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \| \partial^m u_R^\varepsilon \|_{L^4} \| \nabla (\partial^m - m') d_R^\varepsilon \|_{L^4} \| \partial^m d_R^\varepsilon \|_{L^2}
\]
\[
\leq \frac{C}{\varepsilon} \| \nabla d_R^\varepsilon \|_{H^N} \| \nabla u_R^\varepsilon \|_{H^N} \| \nabla u_R^\varepsilon \|_{H^N} \leq C \varepsilon \frac{1}{N_{\varepsilon}}(t) \partial_{N_{\varepsilon}}(t)
\]
(3.48)
holds for $|m| \leq N$. Similarly, we deduce from the fact $\text{div} \hat{u}_R^\varepsilon = 0$ that
\begin{align*}
- \left\langle \partial^m (\sqrt{\varepsilon} u_R^\varepsilon \cdot \nabla D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon), \partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle
\leq C \varepsilon \sum_{|m'|=1} ||\partial^m u_R^\varepsilon||_{L^\infty} ||\nabla \partial^m u_R^\varepsilon\nabla D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2}
\leq C \varepsilon \sum_{2 \leq |m'| \leq |m|} ||\partial^m u_R^\varepsilon||_{L^4} ||\nabla \partial^m u_R^\varepsilon\nabla D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^4} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2}
\leq C \varepsilon ||\nabla u_R^\varepsilon||_{H^N} ||\nabla d_R^\varepsilon||_{L^2} \leq C \varepsilon \delta_{N,\varepsilon}^\frac{1}{2} (t) \mathcal{D}_{N,\varepsilon} (t)
\end{align*}
for all multi-indexes $m \in \mathbb{N}^3$ with $|m| \leq N$. For the term $- \frac{1}{\sqrt{\varepsilon}} \left\langle \nabla \partial^m d_R^\varepsilon, \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle$, one estimates that for $|m| \leq N$
\begin{align*}
- \frac{1}{\sqrt{\varepsilon}} \sum_{0 \neq m' \leq m} C_{m'} \left\langle \nabla \partial^m d_R^\varepsilon, \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle
\leq \frac{C}{\sqrt{\varepsilon}} \sum_{|m'|=1} ||\partial^m u_R^\varepsilon||_{L^\infty} ||\nabla \partial^m u_R^\varepsilon\nabla d_R^\varepsilon||_{L^2} ||\partial^m d_R^\varepsilon||_{L^2}
\leq \frac{C}{\sqrt{\varepsilon}} \sum_{2 \leq |m'| \leq |m|} ||\partial^m u_R^\varepsilon||_{L^4} ||\nabla \partial^m u_R^\varepsilon\nabla d_R^\varepsilon||_{L^4} ||\partial^m d_R^\varepsilon||_{L^2}
\leq \frac{C}{\sqrt{\varepsilon}} ||\nabla u_R^\varepsilon||_{H^N} ||\nabla d_R^\varepsilon||_{L^2} \leq C \varepsilon \delta_{N,\varepsilon}^\frac{1}{2} (t) \mathcal{D}_{N,\varepsilon} (t)
\end{align*}
where the fact $\text{div} \hat{u}_R^\varepsilon = 0$ is also utilized.

We derive the bound of the term $- \left\langle \sqrt{\varepsilon} \partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon, u_R^\varepsilon \cdot \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle$ as follows:
\begin{align*}
- \left\langle \sqrt{\varepsilon} \partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon, u_R^\varepsilon \cdot \nabla \partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon) \right\rangle
\leq \sqrt{\varepsilon} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2} ||u_R^\varepsilon||_{L^\infty} ||\partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon)||_{L^2}
\leq \sqrt{\varepsilon} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2} ||u_R^\varepsilon||_{L^\infty} ||\partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon)||_{L^2}
\leq \sqrt{\varepsilon} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2} ||u_R^\varepsilon||_{L^\infty} ||\partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon)||_{L^2}
\leq \sqrt{\varepsilon} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2} ||u_R^\varepsilon||_{L^\infty} ||\partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon)||_{L^2}
\leq \sqrt{\varepsilon} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2} ||u_R^\varepsilon||_{L^\infty} ||\partial^m (u_R^\varepsilon \cdot \nabla d_R^\varepsilon)||_{L^2}
\leq \sqrt{\varepsilon} ||\nabla d_R^\varepsilon||_{H^N} ||\nabla u_R^\varepsilon||_{H^N} ||\nabla d_R^\varepsilon||_{H^N} \leq C \varepsilon \delta_{N,\varepsilon}^\frac{1}{2} (t) \mathcal{D}_{N,\varepsilon} (t)
\end{align*}
For all $|m| \leq N$, it is derived that
\begin{align*}
- \frac{\lambda}{\varepsilon} \left\langle \partial^m (u_R^\varepsilon \cdot \nabla \partial^m d_R^\varepsilon), \partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle
\leq \frac{\lambda}{\varepsilon} ||u_R^\varepsilon||_{L^\infty} ||\nabla \partial^m d_R^\varepsilon||_{L^2} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2}
\leq \frac{\lambda}{\varepsilon} ||u_R^\varepsilon||_{L^\infty} ||\nabla \partial^m d_R^\varepsilon||_{L^2} ||\partial^m D_{u_0+\sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon||_{L^2}
\end{align*}
\begin{equation}
\frac{-C\lambda}{\varepsilon} \sum_{0 \neq m' \leq m} C'_{m'} \varepsilon^{m''} \left( \langle \partial^{m-m'} \nabla \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \nabla \partial^{m} \phi \rangle \right)
\leq C \sum_{0 \neq m' \leq m} \varepsilon^{m''} \left( \sum_{0 \neq m' \leq m} \varepsilon^{m''} \partial^{m-m'} \nabla \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \nabla \partial^{m} \phi \right)
\leq C \varepsilon \sum_{0 \neq m' \leq m} \varepsilon^{m''} \left( \sum_{0 \neq m' \leq m} \varepsilon^{m''} \partial^{m-m'} \nabla \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \nabla \partial^{m} \phi \right)
\leq C \varepsilon \sum_{0 \neq m' \leq m} \varepsilon^{m''} \left( \sum_{0 \neq m' \leq m} \varepsilon^{m''} \partial^{m-m'} \nabla \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \nabla \partial^{m} \phi \right)
\end{equation}

holds for all \( |m| \leq N \). For the term
\(- \langle \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \partial^{m} \phi \rangle \),
we have
\begin{equation}
\delta \left( \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \partial^{m} \phi \right)
\leq C \varepsilon \sum_{0 \neq m' \leq m} \varepsilon^{m''} \left( \sum_{0 \neq m' \leq m} \varepsilon^{m''} \partial^{m-m'} \nabla \partial^{m^0} \phi \cdot (d_0 \otimes d_0), \nabla \partial^{m} \phi \right)
\end{equation}
Via the similar arguments in \((3.56)\), we yield
\[
- \sum_{0 \neq m' \leq m} C_m' \left< \partial^m D_{u_0 + \sqrt{\varepsilon} u_R} \partial_{u_R}^{m'} \frac{d_R}{\sqrt{\varepsilon}}, \sqrt{\varepsilon} \partial^m \partial^{m'} u_\varepsilon \cdot \nabla \partial^{m-m'} d_R \right>
\leq C \sqrt{\varepsilon} \| \nabla u_\varepsilon \|_{H^N} \| \nabla d_R \|_{H^N} \| D_{u_0 + \sqrt{\varepsilon} u_R} d_R \|_{H^N}
\leq C \varepsilon \| \nabla d_R \|_{H^N} \| \nabla u_\varepsilon \|_{H^N} \Phi_{N, \varepsilon}(t).
\]
\[
(3.57)
\]

Collecting the bounds \((3.46)-(3.57)\) reduces to
\[
\mathcal{T}_N^{(1)} \leq C \varepsilon \Phi_{N, \varepsilon}^2(t) \| \nabla d_0 \|_{H^N} + C \| \nabla d_0 \|_{H^{N+1}} \Phi_{N, \varepsilon}(t)
+ C \varepsilon (1 + \| \nabla d_0 \|_{H^{N+2}}) \Phi_{N, \varepsilon}^2(t) \Phi_{N, \varepsilon}(t)
\]
\[
(3.58)
\]
for all \(\varepsilon \in (0, \varepsilon_0)\).

**Step 2. Control the term** \(\mathcal{T}_N^{(2)}\).

We need to estimate the terms
\[-\frac{\mu_\varepsilon}{\varepsilon} \left< \partial^m \left[ (A_0 : (d_{R}^\varepsilon \otimes d_0 + d_0 \otimes d_{R}^\varepsilon))\partial_{d_R} u_\varepsilon \right], \nabla \partial^{m} u_\varepsilon \right>
\leq C \varepsilon \sum_{m' \leq m} \| \partial^{m} u_\varepsilon \|_{L^6} \| \partial^{m-m'} (A_0 d_0 \otimes d_0 \otimes d_0) \|_{L^3} \| \nabla \partial^{m} u_\varepsilon \|_{L^2}
\leq C \varepsilon \| \nabla d_{R}^\varepsilon \|_{H^N} \| \nabla u_\varepsilon \|_{H^N} \Phi_{N, \varepsilon}(t)
+ C \| \nabla d_0 \|_{H^{N+2}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \Phi_{N, \varepsilon}(t)
\]
\[
(3.59)
\]
holds for all \(|m| \leq N\). We derive from the similar arguments in \((3.59)\) that for \(|m| \leq N\)
\[
-\frac{\mu_\varepsilon}{\varepsilon} \left< \partial^m \left[ (A_0 : d_0 \otimes d_0) \partial_{d_R} u_\varepsilon \right], \nabla \partial^{m} u_\varepsilon \right>
\leq C \| \nabla d_0 \|_{H^{N+2}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \Phi_{N, \varepsilon}(t).
\]
\[
(3.60)
\]
For the term
\[
-\frac{\mu_\varepsilon}{\varepsilon} \left< \partial^m \left[ (D_{u_0} d_0 + B_0 d_0) \partial_{d_R} u_\varepsilon \right], \nabla \partial^{m} u_\varepsilon \right>
\leq C \varepsilon \sum_{m' \leq m} \| \partial^{m} d_{R}^\varepsilon \|_{L^6} \| \partial^{m} (D_{u_0} d_0 + B_0 d_0) \|_{L^3} \| \nabla \partial^{m} u_\varepsilon \|_{L^2}
\leq C \| \nabla d_{R}^\varepsilon \|_{H^N} \| \nabla u_\varepsilon \|_{H^N} \Phi_{N, \varepsilon}(t)
+ C \| \nabla d_0 \|_{H^{N+2}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \Phi_{N, \varepsilon}(t).
\]
\[
(3.61)
\]
Recall that \(d_0\) obeys the third equation of \((1.10)\), hence
\[
-\lambda_1 (D_{u_0} d_0 + B_0 d_0) = \Delta d_0 + \gamma_0 d_0 + \lambda_2 A_0 d_0,
\]
where the Lagrangian multiplier \(\gamma_0\) is
\[
\gamma_0 = | \nabla d_0 |^2 - \lambda_2 A_0 : d_0 \otimes d_0.
\]
\[
(3.62)
\]
Then one can easily yield that if the integer \(N \geq 2\),
\[
\| D_{u_0} d_0 \|_{H^N} \leq C (\| \nabla u_0 \|_{H^N} + \| \Delta d_0 \|_{H^N})(1 + \| \nabla d_0 \|_{H^N}^3).
\]
\[
(3.64)
\]
and
\[
\| D_{u_0}^2 d_0 \|_{H^N} \leq C (\| \nabla u_0 \|_{H^{N+2}} + \| \Delta d_0 \|_{H^{N+2}})(1 + \| \nabla d_0 \|_{H^{N+2}}^6).
\]
\[
(3.65)
\]
Consequently, we obtain the bound
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [(D_{u_0} d_0 + B_0 d_0) \otimes d_R^e], \nabla \partial^m u_R^e \rangle \\
\leq C(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}) (1 + \|\nabla d_0\|_{H^{N+2}}^3) \mathcal{O}_{N, \varepsilon}(t). \tag{3.66}
\end{align}

Next, we deduce
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [(B_0 d_R^e + u_0 \cdot \nabla d_R^e + u_R^e \cdot \nabla d_0) \otimes d_0], \nabla \partial^m u_R^e \rangle \\
\leq \frac{C}{\varepsilon} \|\nabla \partial^m u_R^e\|_{L^2} \sum_{m' \leq m} \left( \|\partial^m d_R^e\|_{L^6} \|\partial^{m-m'} (B_0 \otimes d_0)\|_{L^3} \\
+ \|\partial^m d_R^e\|_{L^6} \|\partial^{m-m'} (\nabla d_0 \otimes d_0)\|_{L^3} \right) \\
+ \frac{C}{\varepsilon} \|u_0 \otimes d_0\|_{L^\infty} \|\nabla \partial^m d_R^e\|_{L^2} \|\nabla \partial^m u_R^e\|_{L^2} \\
+ \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^m (A_0 \otimes d_0)\|_{L^4} \|\nabla \partial^m d_R^e\|_{L^4} \|\nabla \partial^m u_R^e\|_{L^4} \tag{3.67}
\end{align}

Analogous estimates in (3.66) tell us
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [d_R^e \otimes (D_{u_0} d_0 + B_0 d_0)], \nabla \partial^m u_R^e \rangle \\
\leq C(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}) (1 + \|\nabla d_0\|_{H^{N+2}}^3) \mathcal{O}_{N, \varepsilon}(t), \tag{3.68}
\end{align}

and the similar calculations in (3.67) reduce to
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [(B_0 d_R^e + u_0 \cdot \nabla d_R^e + u_R^e \cdot \nabla d_0) \otimes d_0], \nabla \partial^m u_R^e \rangle \\
\leq C(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}) (1 + \|\nabla d_0\|_{H^{N+2}}) \mathcal{O}_{N, \varepsilon}(t). \tag{3.69}
\end{align}

It is easy to derive
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [(A_0 d_0) \otimes d_R^e + (A_0 d_R^e) \otimes d_0], \nabla \partial^m u_R^e \rangle \\
\leq \frac{C}{\varepsilon} \sum_{m' \leq m} \|\partial^m d_R^e\|_{L^6} \|\partial^{m-m'} (A_0 d_0)\|_{L^3} \|\nabla \partial^m d_R^e\|_{L^2} \tag{3.70}
\end{align}

\begin{align}
\leq \frac{C}{\varepsilon} \|\nabla d_R^e\|_{H^N} \|\nabla u_R^e\|_{H^N} \|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|_{H^N}) \\
\leq C(\|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|_{H^N}) \mathcal{O}_{N, \varepsilon}(t)
\end{align}

for all \(|m| \leq N\). Furthermore, via the analogous estimates in (3.70), we imply that
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [(d_R^e \otimes (A_0 d_0) + d_0 \otimes (A_0 d_R^e))], \nabla \partial^m u_R^e \rangle \\
\leq C(\|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|_{H^N}) \mathcal{O}_{N, \varepsilon}(t). \tag{3.71}
\end{align}

As a consequence, collecting the estimates (3.59), (3.60), (3.66), (3.67), (3.68), (3.69), (3.70) and (3.71), we deduce from the definition of \(\mathcal{T}_u\) in (A.2) that
\begin{align}
-\frac{1}{\varepsilon} \langle \partial^m \mathcal{T}_u, \nabla \partial^m u_R^e \rangle \leq C(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}) (1 + \|\nabla d_0\|_{H^{N+2}}^3) \mathcal{O}_{N, \varepsilon}(t). \tag{3.72}
\end{align}

We next estimate the quantity \(-\frac{1}{\varepsilon} \langle \partial^m \mathcal{R}_{u_i}, \nabla \partial^m u_R^e \rangle\) for \(1 \leq i \leq 4\). First, we estimate
\begin{align}
-\frac{\mu}{\varepsilon} \langle \partial^m [(A_0 : d_R^e \otimes d_R^e) d_0 \otimes d_0], \nabla \partial^m u_R^e \rangle \\
\leq \frac{C}{\varepsilon} \|\partial^m (A_0 : d_0 \otimes d_0)\|_{L^6} \|d_R^e\|_{L^2} \|\nabla \partial^m u_R^e\|_{L^2} \\
+ \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^{m-m'} (A_0 : d_0 \otimes d_0)\|_{L^4} \|\partial^m (d_R^e \otimes d_R^e)\|_{L^4} \|\nabla \partial^m u_R^e\|_{L^2}
\end{align}
\[
\leq \frac{C}{\sqrt{\varepsilon}} \|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|_{H^{N+2}}^2) \|\nabla d_{R}^{\varepsilon}\|_{H^{N}}^2 \|\nabla u^{\varepsilon}_{R}\|_{H^{N}}
\]
\[
\leq C\varepsilon \|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|_{H^{N+2}}^2) \frac{1}{2} N \varepsilon (t) \mathcal{D}_{N, \varepsilon}(t).
\]

The term \(-\frac{\mu}{\sqrt{\varepsilon}} \langle \partial^m \left[ 2(A^\varepsilon_R : d_0 \otimes d_R^\varepsilon) d_0 \otimes d_0 \right], \nabla \partial^m u^{\varepsilon}_{R} \rangle\) can be controlled by
\[
-\frac{\mu}{\sqrt{\varepsilon}} \langle \partial^m \left[ 2(A^\varepsilon_R : d_0 \otimes d_R^\varepsilon) d_0 \otimes d_0 \right], \nabla \partial^m u^{\varepsilon}_{R} \rangle
\leq \frac{C}{\sqrt{\varepsilon}} \|\nabla \partial^m u^{\varepsilon}_{R}\|_{L^2} (d_{R}^{\varepsilon} \otimes d_0) \|\nabla \partial^m u^{\varepsilon}_{R}\|_{L^2}^2
\leq \frac{C}{\sqrt{\varepsilon}} \|d_{R}^{\varepsilon}\|_{H^{N}}^2 \|u^{\varepsilon}_{R}\|_{H^{N}}^2 (\|d_{R}^{\varepsilon}\|_{H^{N}}^2 + \|u^{\varepsilon}_{R}\|_{H^{N}}^2) (1 + \|\nabla d_0\|_{H^{N+2}}^3)
\leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}^3) \frac{1}{2} N \varepsilon (t) \mathcal{D}_{N, \varepsilon}(t).
\]

Via the similar arguments in (3.73) and (3.74), we imply that for \(|m| \leq N\)
\[
-\frac{\mu}{\sqrt{\varepsilon}} \langle \partial^m \left[ A_0 : d_0 \otimes d_{R}^{\varepsilon} \right], \nabla \partial^m u^{\varepsilon}_{R} \rangle
-\frac{\mu}{\sqrt{\varepsilon}} \langle \partial^m \left[ 2A_0 : d_0 \otimes d_{R}^{\varepsilon} + A_{R} : d_0 \otimes d_0 \right], \nabla \partial^m u^{\varepsilon}_{R} \rangle
\leq C\varepsilon \|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|_{H^{N+2}}^2) \frac{1}{2} N \varepsilon (t) \mathcal{D}_{N, \varepsilon}(t)
\]
and similarly we have
\[ -\frac{\mu_k}{\varepsilon} \langle \partial^m [d_0 \otimes (B_R d_R^e)], \nabla \partial^m u_R^e \rangle \leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2}. \] (3.77)
We employ the similar arguments of (3.76) to obtain
\[ -\frac{\mu_k}{\varepsilon} \langle \partial^m [d_R \otimes (D_{u_0} + \sqrt{\varepsilon} u_R^e) d_R^e + u_0 \cdot \nabla d_R^e + u_R^e \cdot \nabla d_0 + B_0^e d_R^e + B_R^e d_0^e), \nabla \partial^m u_R^e \rangle \leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2}, \] (3.78)
and employ the analogous estimates of (3.77) to imply
\[ -\frac{\mu_k}{\varepsilon} \langle \partial^m [(A_R^e d_R^e) \otimes d_0 + (A_R^e d_R^e + A_0 d_R^e) \otimes d_R^e], \nabla \partial^m u_R^e \rangle \leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2} \] (3.79)
We now estimate that for all \(|m| \leq N\)
\[ -\frac{\mu_k}{\varepsilon} \langle \partial^m [(A_R^e d_R^e) \otimes d_0 + (A_R^e d_R^e + A_0 d_R^e) \otimes d_R^e], \nabla \partial^m u_R^e \rangle \leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2} \] (3.80)
and similarly we have
\[ -\frac{\mu_k}{\varepsilon} \langle \partial^m [(A_R^e d_R^e) \otimes d_0 + (A_R^e d_R^e + A_0 d_R^e) \otimes d_R^e], \nabla \partial^m u_R^e \rangle \leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2} \] (3.81)
Recalling the definition of \(M_1\) in (A.8), we derive from collecting the bounds (3.73), (3.74), (3.75), (3.76), (3.77), (3.78), (3.79), (3.80) and (3.81) that for all \(|m| \leq N\)
\[ -\frac{1}{\varepsilon} \langle \partial^m \mathcal{M}_i, \nabla \partial^m u_R^e \rangle \leq C\varepsilon (1 + \|\nabla d_0\|_{H^{N+2}}) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2} \] (3.82)
Next we estimate the term \(-\frac{1}{\varepsilon} \langle \sqrt{\varepsilon} \partial^m \mathcal{M}_2, \nabla \partial^m u_R^e \rangle\) for \(|m| \leq N\). Firstly, we have
\[ -\frac{\mu_k}{\varepsilon} \langle \sqrt{\varepsilon} \partial^m [(A_R^e : d_0 \otimes d_0 + 2A_0 : d_0 \otimes d_R^e) d_R^e \otimes d_R^e], \nabla \partial^m u_R^e \rangle \leq \mu_k \|\partial^m A_R^e \|_{L^2} \|d_R^e \otimes d_0 \|_{L^2} \|\nabla \partial^m u_R^e \|_{L^2} \] (3.83)
\[ \leq C(\|\nabla u_R^e \|_{H^N} \|d_R^e \|_{H^N}^2 + C(\|\nabla u_R^e \|_{H^N} \|d_R^e \|_{H^N}^2 + \|\nabla d_R^e \|_{H^N}^2)(1 + \|\nabla d_0\|_{H^N}^2) \] (3.84)
\[ + C\|\nabla u_R^e \|_{H^N} \|\nabla d_R^e \|_{H^N} \|\nabla d_R^e \|_{H^N}^2 \|\nabla d_0\|_{H^N} + \|\nabla d_0\|_{H^N}^2)(1 + \|\nabla d_0\|_{H^N}^2 + \|\nabla d_0\|_{H^N}^2) \] (3.85)
\[ \leq C(1 + \|\nabla d_0\|_{H^N}^2 + \|\nabla d_0\|_{H^N}^2) \frac{\varepsilon^3}{N,\varepsilon}(t) \frac{\partial N,\varepsilon(t)}{2} \] (3.86)
Via employing the analogous estimates in (3.83), we have

$$- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ (A^\varepsilon_R : d^\varepsilon_R \otimes d^\varepsilon_R) d_0 \otimes d_0 \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \varepsilon^2 (1 + \|d_0\|_{H^{N+2}}^2) \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) \quad (3.84)$$

and

$$- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ (A^\varepsilon_R : d_0 \otimes d^\varepsilon_R + A_0 : d^\varepsilon_R \otimes d^\varepsilon_R) (d_0 \otimes d^\varepsilon_R + d^\varepsilon_R \otimes d_0) \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \varepsilon^2 (1 + \|u_0\|_{H^{N+2}}^2 + \|d_0\|_{H^{N+2}}^2) \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) \quad (3.85)$$

For the term $- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ (B^\varepsilon_R d^\varepsilon_R) \otimes d^\varepsilon_R \right], \nabla \partial^m u^\varepsilon_R \right)$, we deduce that

$$- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ (B^\varepsilon_R d^\varepsilon_R) \otimes d^\varepsilon_R \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \|\partial^m B^\varepsilon_R\|_{L^2} \|d^\varepsilon_R \otimes d^\varepsilon_R\|_{L^\infty} \|\nabla \partial^m u^\varepsilon_R\|_{L^2} + C \sum_{0 \neq m' \leq m} \|\partial^{m-m'} B^\varepsilon_R\|_{L^4} \|\partial^{m'} (d^\varepsilon_R \otimes d^\varepsilon_R)\|_{L^4} \|\nabla \partial^m u^\varepsilon_R\|_{L^2} \quad (3.86)$$

$$\leq C \|\nabla d_0\|_{H^N}^2 + C \|\nabla d_0\|_{H^N}^2 (\|d^\varepsilon_R\|_{H^N}^2 + \|\nabla d^\varepsilon_R\|_{H^N}^2) \leq C \varepsilon^2 \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t).$$

By utilizing the similar calculations in (3.86), we yield that

$$- \frac{1}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ 2 \mu_3 d^\varepsilon_R \otimes (B^\varepsilon_R d^\varepsilon_R) + \mu_5 (A^\varepsilon_R d^\varepsilon_R) d^\varepsilon_R + \mu_6 d^\varepsilon_R \otimes (A^\varepsilon_R d^\varepsilon_R) \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \varepsilon^2 \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) \quad (3.87)$$

Recalling the definition of $\mathcal{M}_2$ in (A.9), one deduces from collecting the estimates (3.83), (3.84), (3.85), (3.86) and (3.87) that

$$- \frac{1}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \mathcal{M}_2, \nabla \partial^m u^\varepsilon_R \right) \leq C \varepsilon^2 (1 + \|u_0\|_{H^{N+2}}^2 + \|d_0\|_{H^{N+2}}^2) \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) \quad (3.88)$$

for all multi-indexes $m \in \mathbb{N}^3$ with $|m| \leq N$.

We next estimate the term $- \frac{1}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \mathcal{M}_3, \nabla \partial^m u^\varepsilon_R \right)$ for all $|m| \leq N$. First, we compute that

$$- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ 2 A^\varepsilon_R : d_0 \otimes d^\varepsilon_R \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \sqrt{\varepsilon} \|\partial^m A^\varepsilon_R\|_{L^2} \|d_0 \otimes d^\varepsilon_R\|_{L^\infty} \|\nabla \partial^m u^\varepsilon_R\|_{L^2} + C \sqrt{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^{m-m'} A^\varepsilon_R\|_{L^4} \|\partial^{m'} (d_0 \otimes d^\varepsilon_R)\|_{L^4} \|\nabla \partial^m u^\varepsilon_R\|_{L^2} \quad (3.89)$$

$$\leq C \sqrt{\varepsilon} \|\nabla u^\varepsilon_R\|_{H^N}^2 + C \sqrt{\varepsilon} \|\nabla d^\varepsilon_R\|_{H^N}^2 (1 + \|d_0\|_{H^{N+1}}^2) \leq C \varepsilon^2 (1 + \|d_0\|_{H^{N+1}}) \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t).$$

Similarly as in (3.89), we have

$$- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ (A^\varepsilon_R : d^\varepsilon_R \otimes d^\varepsilon_R) (d_0 \otimes d_0 + d_0 \otimes d^\varepsilon_R) \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \varepsilon^2 (1 + \|d_0\|_{H^{N+1}}) \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) \quad (3.90)$$

For all $|m| \leq N$, we deduce that

$$- \frac{m}{\sqrt{\varepsilon}} \left( \sqrt{\varepsilon} \partial^m \left[ (A^\varepsilon_R : d^\varepsilon_R \otimes d^\varepsilon_R) d_0 \otimes d_0 \right], \nabla \partial^m u^\varepsilon_R \right) \leq C \sqrt{\varepsilon} \|\partial^m A^\varepsilon_R\|_{L^2} \|d_0 \otimes d^\varepsilon_R\|_{L^\infty} \|\nabla \partial^m u^\varepsilon_R\|_{L^2} + C \sqrt{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^{m-m'} A^\varepsilon_R\|_{L^4} \|\partial^{m'} (d_0 \otimes d^\varepsilon_R)\|_{L^4} \|\nabla \partial^m u^\varepsilon_R\|_{L^2} \leq C \sqrt{\varepsilon} \|u_0\|_{H^{N+2}} \|\nabla u^\varepsilon_R\|_{H^N} \|d^\varepsilon_R\|_{H^N}^2 \leq C \varepsilon^2 \partial_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t).$$
Therefore, from the definition of $\mathcal{M}_3$ in (A.10), we collect the estimates (3.89), (3.90) and (3.91), and then obtain

$$-\frac{1}{\varepsilon^2} \left( \sqrt{\varepsilon^2 \partial^m \mathcal{M}_3, \nabla \partial^m u^\varepsilon_R} \right) \leq C \varepsilon^3 (1 + \| u_0 \|_{H^{N+2}} + \| \nabla d_0 \|_{H^{N+2}}) \mathcal{D}_{N, \varepsilon}(t).$$

(3.92)

Recalling the definition of $\mathcal{M}_4$ in (A.11), we calculate that

$$-\frac{1}{\varepsilon} \left( \sqrt{\varepsilon \partial^m \mathcal{M}_4, \nabla \partial^m u^\varepsilon_R} \right) = -\mu_1 \varepsilon \| \partial^m \mathcal{A}^\varepsilon_R : d^\varepsilon_R \otimes d^\varepsilon_R \|_{L^2}$$

$$- \mu_1 \varepsilon \sum_{0 \neq m' \leq m} \left( \partial^{m-m'} \mathcal{A}^\varepsilon_R : \partial^m \nabla u^\varepsilon_R \right) \left( \partial^{m-m'} \nabla u^\varepsilon_R \right) \left( \nabla \partial^m u^\varepsilon_R \right) \left( \nabla \partial^m u^\varepsilon_R \right) \| \partial^{m-m'} \nabla u^\varepsilon_R \|_{L^2}$$

$$\leq C \varepsilon \sum_{0 \neq m' \leq m} \| \partial^{m-m'} \mathcal{A}^\varepsilon_R \|_{L^4} \| \partial^m \nabla u^\varepsilon_R \|_{L^4} \| \partial^{m-m'} \nabla u^\varepsilon_R \|_{L^4} \| \partial^{m-m'} \nabla u^\varepsilon_R \|_{L^4}$$

(3.93)

Therefore, from the definition of $\mathcal{R}_u$ in (A.7) and the bounds (3.82), (3.88), (3.92) and (3.93), we derive that

$$-\frac{1}{\varepsilon^2} \left( \partial^m \mathcal{R}_u, \nabla \partial^m u^\varepsilon_R \right) = -\frac{1}{\varepsilon^2} \left( \partial^m (\mathcal{M}_1 + \sqrt{\varepsilon} \mathcal{M}_2 + \sqrt{\varepsilon^2} \mathcal{M}_3 + \sqrt{\varepsilon^3} \mathcal{M}_4), \nabla \partial^m u^\varepsilon_R \right)$$

$$\leq C \varepsilon^2 (1 + \| u_0 \|_{H^{N+2}}^3 + \| \nabla d_0 \|_{H^{N+2}}^3 \mathcal{D}_{N, \varepsilon}(t)$$

$$+ C \varepsilon^3 (1 + \| u_0 \|_{H^{N+2}}^3 + \| \nabla d_0 \|_{H^{N+2}}^3 \mathcal{D}_{N, \varepsilon}(t)$$

$$+ C \varepsilon^4 (1 + \| u_0 \|_{H^{N+2}}^3 + \| \nabla d_0 \|_{H^{N+2}}^3 \mathcal{D}_{N, \varepsilon}(t)$$

(3.94)

for $0 < \varepsilon \leq 1$. Consequently, via the estimates (3.72) and (3.94), we obtain

$$\mathcal{T}_N^{(2)} = -\frac{1}{\varepsilon} \left( \partial^m \mathcal{T}_u, \nabla \partial^m u^\varepsilon_R \right) - \frac{1}{\varepsilon^2} \left( \partial^m \mathcal{R}_u, \nabla \partial^m u^\varepsilon_R \right)$$

$$\leq C (\| u_0 \|_{H^{N+2}} + \| \nabla d_0 \|_{H^{N+2}}) \mathcal{D}_{N, \varepsilon}(t)$$

$$+ C \varepsilon (1 + \| u_0 \|_{H^{N+2}} + \| \nabla d_0 \|_{H^{N+2}}) \mathcal{D}_{N, \varepsilon}(t)$$

(3.95)

for all $|m| \leq N$ ($N \geq 2$) and for all $0 < \varepsilon \leq \varepsilon_0 \leq 1$.

Step 3. Control the quantity $\mathcal{T}_N^{(3)}$.

Now we carefully estimate the term $\mathcal{T}_N^{(3)}$ for $N \geq 2$. First, we have

$$\frac{1}{\varepsilon^2} \sum_{|m| \leq N} \sum_{m' \neq m} C_{m, m'} \left( \lambda_1 (\partial^{m-m'} \mathcal{B}^\varepsilon_R, \partial^m d_0, \partial^{m-m'} \nabla u^\varepsilon_R \right)$$

$$\leq C \varepsilon \sum_{|m| \leq N} \sum_{m' \neq m} \| \partial^{m-m'} \mathcal{B}^\varepsilon_R \|_{L^4} \| \partial^m \nabla u^\varepsilon_R \|_{L^4} \| \partial^{m-m'} \nabla u^\varepsilon_R \|_{L^4} \| \partial^{m-m'} \nabla u^\varepsilon_R \|_{L^4}$$

$$\leq C \varepsilon \| \nabla u^\varepsilon_R \|_{H^{N+2}} \| \nabla d_0 \|_{H^{N+2}} \| \partial^m \mathcal{D}_u + \sqrt{\varepsilon} u^\varepsilon_R \|_{L^2} \| \partial^m \mathcal{D}_u + \sqrt{\varepsilon} u^\varepsilon_R \|_{L^2}$$

$$\leq C \varepsilon \| \nabla d_0 \|_{H^{N+2}} \| \nabla u^\varepsilon_R \|_{H^{N+2}} \| \partial^m \mathcal{D}_u + \sqrt{\varepsilon} u^\varepsilon_R \|_{L^2} \| \partial^m \mathcal{D}_u + \sqrt{\varepsilon} u^\varepsilon_R \|_{L^2}$$
Furthermore, via the analogous calculations in (3.96), we imply
\[
\frac{1}{\varepsilon} \sum_{|m| \leq N} \sum_{m' < m} C_m^{m'} \left\langle \lambda_2 (\partial^{m'} A^\varepsilon_R) \partial^{m-m'} d_0, \partial^{m} D_{u_0 + \sqrt{\varepsilon} u^R_R} d^\varepsilon_R \right\rangle \leq C \|\nabla d_0\|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t), \tag{3.97}
\]
and
\[
-\frac{1}{\varepsilon} \sum_{|m| \leq N} \sum_{m' < m} \left\langle \mu_2 \partial^{m-m'} d_{0,j} \partial^{m'} (D_{u_0 + \sqrt{\varepsilon} u^R_R} d^\varepsilon_R)_{j}, \mu_3 \partial^{m-m'} d_{0,i} \partial^{m'} (D_{u_0 + \sqrt{\varepsilon} u^R_R} d^\varepsilon_R)_{i}, \partial^{m} \partial_j u^R_R \right\rangle \leq C \|\nabla d_0\|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t). \tag{3.98}
\]

It is easy to estimate that
\[
-\frac{1}{\varepsilon} \sum_{|m| \leq N} \sum_{m' < m} C_m^{m'} \left\langle \mu_2 \partial^{m-m'} (d_{0,j} d_{0,k}) \partial^{m'} B^\varepsilon_{R,ki}, \partial^{m} \partial_j u^R_R \right\rangle \leq C \sum_{|m| \leq N} \sum_{m' < m} \|\partial^{m-m'} (d_{0,j} d_{0,k})\|_{L^4} \|\partial^{m'} B^\varepsilon_{R,ki}\|_{L^4} \|\partial^{m} \partial_j u^R_R\|_{L^2} \leq C \|\nabla d_0\|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t). \tag{3.99}
\]
Furthermore, via the analogous calculations in (3.99), we imply
\[
-\frac{1}{\varepsilon} \sum_{|m| \leq N} \sum_{m' < m} C_m^{m'} \left\langle \mu_3 \partial^{m-m'} (d_{0,i} d_{0,k}) \partial^{m'} B^\varepsilon_{R,ki}, \partial^{m} \partial_j u^R_R \right\rangle \leq C \|\nabla d_0\|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t). \tag{3.100}
\]
Recalling the definition of \(G_m\) in (2.3) and noticing that \(\mathcal{I}_N^{(3)} = \frac{1}{\varepsilon} \sum_{|m| \leq N} \mathcal{G}_m\), we derive from the inequalities (3.96), (3.97), (3.98), (3.99) and (3.100) that
\[
\mathcal{I}_N^{(3)} \leq C \|\nabla d_0\|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t) \tag{3.101}
\]
holds for all \(N \geq 2\) and \(0 < \varepsilon \leq \varepsilon_0\).

**Step 4. Control the term \(\mathcal{I}_N^{(4)}\).**

We first rewrite the expression of \(\mathcal{I}_N^{(4)}\) in (3.38) as
\[
\mathcal{I}_N^{(4)} = -\frac{1}{\varepsilon} \sum_{|m| \leq N} \left\langle \partial^m (u^R_R \cdot \nabla d_0), \partial^m C_d \right\rangle
\]
\[
+ \frac{1}{\varepsilon} \sum_{|m| \leq N} \left\langle \partial^m S^1_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u^R_R} d^\varepsilon_R - \partial^m (u^R_R \cdot \nabla d_0) + \delta \partial^m d^\varepsilon_R \right\rangle
\]
\[
\mathcal{I}_N^{(4)}(S^1_d)
\]
\[ + \frac{1}{\varepsilon} \sum_{|m| \leq N} \left\langle \partial^m S^2_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \right\rangle \]

\[ + \sum_{|m| \leq N} \left\langle \partial^m R_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \right\rangle \]

\[ + \frac{\delta}{\varepsilon} \sum_{1 \leq |m| \leq N} \sum_{0 \neq m' \leq m} C_m^m \left\langle \lambda_1 (\partial^{m-m'}B_R^\varepsilon) \partial^{m'} d_0 + \lambda_2 (\partial^{m-m'}A_R^\varepsilon) \partial^{m'} d_0, \partial^m d_R^\varepsilon \right\rangle. \] (3.102)

We next estimate the previous terms one by one. Before doing this, we derive the following inequality, which will be frequently used. More precisely, for all $|m| \leq N$,

\[ \| \partial^m (u_R^\varepsilon \cdot \nabla d_0) \|_{L^2} \leq C \| u_R^\varepsilon \cdot \nabla \partial^m d_0 \|_{L^2} + C \sum_{0 \neq m' \leq m} \| \partial^{m'} u_R^\varepsilon \cdot \nabla \partial^{m-m'} d_0 \|_{L^4} \]

\[ \leq C \| u_R^\varepsilon \|_{L^6} \| \nabla \partial^m d_0 \|_{L^3} + C \sum_{0 \neq m' \leq m} \| \partial^{m'} u_R^\varepsilon \|_{L^4} \| \nabla \partial^{m-m'} d_0 \|_{L^4} \] (3.103)

\[ \leq C \| \nabla u_R^\varepsilon \|_{H^N} \| \nabla d_0 \|_{H^{N+1}}. \]

We initially estimate the quantity $I_N^{(4)}(C_d)$. It is easy to deduce that

\[ - \frac{1}{\varepsilon} \left\langle \lambda_1 \partial^m (B_R^\varepsilon d_0), \partial^m (u_R^\varepsilon \cdot \nabla d_0) \right\rangle \]

\[ \leq C \varepsilon \left[ \| \partial^m B_R^\varepsilon \|_{L^2} + \sum_{0 \neq m' \leq m} \| \partial^{m-m'} B_R^\varepsilon \|_{L^4} \| \partial^{m'} d_0 \|_{L^4} \right] \| \partial^m (u_R^\varepsilon \cdot \nabla d_0) \|_{L^2} \]

\[ \leq C \varepsilon \| \nabla u_R^\varepsilon \|^2_{H^N} \| \nabla d_0 \|_{H^{N+1}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \]

\[ \leq C \| \nabla d_0 \|_{H^{N+1}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \| \partial^m d_0 \|_{H^{N+1}} \] (3.104)

where we also make use of the inequality (3.103). Similarly as in (3.104), we yield that

\[ - \frac{1}{\varepsilon} \left\langle \lambda_2 \partial^m (A_R^\varepsilon d_0), \partial^m (u_R^\varepsilon \cdot \nabla d_0) \right\rangle \leq C \| \nabla d_0 \|_{H^{N+1}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \| \partial^m d_0 \|_{H^{N+1}} \] (3.105)

Based on the definition of $C_d$ in (1.23), the inequalities (3.104) and (3.105) reduces to

\[ I_N^{(4)}(C_d) \leq C \| \nabla d_0 \|_{H^{N+1}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \| \partial^m d_0 \|_{H^{N+1}} \| \partial^m d_0 \|_{H^{N+1}} \| \partial^m d_0 \|_{H^{N+1}} \] (3.106)

For the term $I_N^{(4)}(AB)$, we estimate that

\[ I_N^{(4)}(AB) \leq C \varepsilon \sum_{1 \leq |m| \leq N} \sum_{0 \neq m' \leq m} \left( \| \partial^{m-m'} B_R^\varepsilon \|_{L^4} + \| \partial^{m-m'} A_R^\varepsilon \|_{L^4} \right) \| \partial^{m'} d_0 \|_{L^4} \| \partial^m d_R^\varepsilon \|_{L^2} \]

\[ \leq C \varepsilon \| \nabla u_R^\varepsilon \|_{H^N} \| \nabla d_0 \|_{H^{N+1}} \| \nabla d_0 \|_{H^{N+1}} \leq C \| \nabla d_0 \|_{H^{N+1}} \| \partial^m d_0 \|_{H^{N+1}} \] (3.107)

We now estimate the quantity $I_N^{(4)}(S^1_d)$. First, we have

\[ \frac{1}{\varepsilon} \left\langle 2 \partial^m (\nabla d_0 \cdot \nabla d_R^\varepsilon) d_0, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \right\rangle \]

\[ \leq C \varepsilon \| \nabla \partial^m d_R^\varepsilon \|_{L^2} \| d_0 \|_{L^\infty} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]

\[ + \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \| \nabla \partial^{m-m'} d_R^\varepsilon \|_{L^4} \| \partial^{m'} (d_0 \otimes d_0) \|_{L^4} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]

\[ \leq C \varepsilon \| \nabla d_0 \|_{H^N} \| \nabla d_R^\varepsilon \|_{H^N} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]

\[ + \frac{C}{\varepsilon} \| \nabla d_0 \|_{H^N} (1 + \| \nabla d_0 \|_{H^N}) \| \nabla d_R^\varepsilon \|_{H^N} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]

\[ \leq C \varepsilon \| \nabla d_0 \|_{H^N} (1 + \| \nabla d_0 \|_{H^N}) \| \nabla d_R^\varepsilon \|_{H^N} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]

\[ + \frac{C}{\varepsilon} \| \nabla d_0 \|_{H^N} (1 + \| \nabla d_0 \|_{H^N}) \| \nabla d_R^\varepsilon \|_{H^N} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]

\[ + \frac{C}{\varepsilon} \| \nabla d_0 \|_{H^N} (1 + \| \nabla d_0 \|_{H^N}) \| \nabla d_R^\varepsilon \|_{H^N} \| \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^d} d_R^\varepsilon \|_{L^2} \]
It is easily estimated that

\[
\lambda C \|\nabla u_R\|_{H^N} (1 + \|\nabla d_0\|_{H^N}) \leq C \|\nabla d_0\|_{H^N} \mathcal{D}_{N, \varepsilon}(t) .
\]  

(3.108)

For the term \( \frac{1}{\varepsilon} \left( \partial^n (|\nabla d_0|^2 \partial_R^\varepsilon), \partial^n D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} \partial_R^\varepsilon \right) \), one can easily estimate that

\[
\lambda C \|\nabla d_0\|_{H^N} (1 + \|\nabla d_0\|_{H^N}) \leq C \|\nabla d_0\|_{H^N} \mathcal{D}_{N, \varepsilon}(t) .
\]  

(3.109)

We then derive the following bound

\[
\lambda C \|\nabla d_0\|_{H^N+1} \leq C \|\nabla d_0\|_{H^N} \leq C \|\nabla d_0\|_{H^N} \mathcal{D}_{N, \varepsilon}(t) .
\]  

(3.110)

Via the analogous arguments in (3.110), we imply that

\[
\lambda C \|\nabla d_0\|_{H^N+1} \leq C \|\nabla d_0\|_{H^N} \mathcal{D}_{N, \varepsilon}(t) .
\]  

(3.111)

It is easily estimated that

\[
\lambda C \|\nabla d_0\|_{H^N+2} (1 + \|\nabla d_0\|_{H^N+2}) \leq C \|\nabla d_0\|_{H^N} \mathcal{D}_{N, \varepsilon}(t) .
\]  

(3.112)
Similarly as in (3.112), we also have

$$-\frac{2\lambda}{\varepsilon} \left\langle \partial^m [(A_0 : d_0 \otimes d_R^\varepsilon) d_0], \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \leq C \|u_0\|_{H^{N+2}} (1 + \|\nabla d_0\|^2) \mathcal{D}_{N, c}(t).$$

(3.113)

Next, we estimate the quantity $-\frac{2\lambda}{\varepsilon} \left\langle \partial^m [(A^\varepsilon_R : d_0 \otimes d_0) d_0], \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle$ for all $|m| \leq N$. Straightforward calculations imply that

$$-\frac{2\lambda}{\varepsilon} \left\langle \partial^m [(A^\varepsilon_R : d_0 \otimes d_0) d_0], \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle$$
$$= -\frac{2\lambda}{\varepsilon} \left\langle \partial^m A^\varepsilon_R : d_0 \otimes d_0, \partial_t (d_0 \cdot \partial^m d^\varepsilon_R) - \partial_t (d_0 \cdot \partial^m d^\varepsilon_R) \right\rangle$$
$$+ \frac{\lambda^2}{\varepsilon} \sum_{0 \neq m' \leq m} C_m^m \left\langle \partial^{m-m'} A^\varepsilon_R : \partial^{m'} (d_0 \otimes d_0 \otimes d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle.$$  (3.114)

Recalling the constraint (1.34), hence $d_0 \cdot d^\varepsilon_R = -\frac{\sqrt{\varepsilon}}{2} |d_R^\varepsilon|^2$, we easily derive that

$$d_0 \cdot \partial^m d^\varepsilon_R = \partial^m (d_0 \cdot d^\varepsilon_R) - \sum_{0 \neq m' \leq m} C_m^m \partial^{m'} d_0 \cdot \partial^{m-m'} d^\varepsilon_R$$
$$= -\frac{\sqrt{\varepsilon}}{2} \partial^m |d_R^\varepsilon|^2 - \sum_{0 \neq m' \leq m} C_m^m \partial^{m'} d_0 \cdot \partial^{m-m'} d^\varepsilon_R,$$  (3.115)

which immediately reduces to

$$\partial_t (d_0 \cdot \partial^m d^\varepsilon_R) = -\sqrt{\varepsilon} \partial^m (d_R^\varepsilon \cdot \partial_t d^\varepsilon_R)$$
$$- \sum_{0 \neq m' \leq m} C_m^m \left( \partial^{m'} d_0 \cdot \partial^m d^\varepsilon_R + \partial^{m'} d_0 \cdot \partial^{m-m'} d^\varepsilon_R \right)$$
$$= -\sqrt{\varepsilon} \partial^m \left[ d_R^\varepsilon \cdot D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon - (u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \cdot \nabla d_R^\varepsilon \right]$$
$$- \sum_{0 \neq m' \leq m} C_m^m \left[ \partial^{m'} d_0 \cdot \partial^{m-m'} d_R^\varepsilon + \partial^{m'} d_0 \cdot \partial^{m-m'} D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right]$$
$$- \sum_{0 \neq m' \leq m} C_m^m \left[ \partial^{m'} d_0 \cdot \partial^{m-m'} ((u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \cdot \nabla d_R^\varepsilon) \right].$$  (3.116)

Then, plugging the relation (3.116) into the equality (3.114) implies that

$$\begin{align*}
\frac{\lambda}{\varepsilon} \left\langle \partial^m [(A^\varepsilon_R : d_0 \otimes d_0) d_0], \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle &\leq \frac{\lambda}{\varepsilon} \left\langle \partial^m A^\varepsilon_R : d_0 \otimes d_0, \partial^m [u_0 + \sqrt{\varepsilon} u_R^\varepsilon] \cdot \nabla d_R^\varepsilon \right\rangle \\
&+ \frac{\lambda}{\varepsilon} \left\langle \partial^m A^\varepsilon_R : d_0 \otimes d_0, \partial_t d_0 \cdot \partial^m d^\varepsilon_R \right\rangle \\
&+ \frac{\lambda}{\varepsilon} \left\langle \partial^m A^\varepsilon_R : d_0 \otimes d_0, \sqrt{\varepsilon} \partial^m (d_R^\varepsilon \cdot D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon) \right\rangle \\
&+ \frac{\lambda}{\varepsilon} \sum_{0 \neq m' \leq m} C_m^m \left\langle \partial^{m-m'} A^\varepsilon_R : \partial^{m'} (d_0 \otimes d_0 \otimes d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \\
&= \frac{\lambda}{\varepsilon} \sum_{0 \neq m' \leq m} C_m^m \left\langle \partial^{m-m'} A^\varepsilon_R : \partial^{m'} (d_0 \otimes d_0 \otimes d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle.
\end{align*}$$
+ \frac{\lambda_2}{\varepsilon} \sum_{0 \neq m' \leq m} C_{m'} \left\langle \partial^m A_{R}^{\xi} : d_0 \otimes d_0, \partial^m d_0 \cdot \partial^{m-m'} d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\rangle. \\

\text{For the term } I_1, \text{ we estimate}
\begin{align*}
I_1 & \leq \frac{C}{\varepsilon} \frac{\left\| \partial^m A_{R}^{\xi} \right\| L^2}{\left\| \partial^m (u_0 + \sqrt{\varepsilon} u_R) \cdot \nabla d_{R}^{\xi} \right\| L^2} \\
& \leq \frac{C}{\varepsilon} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left\| \left( \parallel u_0 \parallel_{H^N} + \sqrt{\varepsilon} \parallel u_R \parallel_{H^N} \right) \right\|_{H^N}} \\
& \leq C \left( \parallel u_0 \parallel_{H^N} + \varepsilon \sqrt{\varepsilon} \right) \mathcal{D}_{N, \varepsilon} (t). \tag{3.118}
\end{align*}

\text{For the term } I_2, \text{ it is easily controlled that}
\begin{align*}
I_2 & \leq \frac{C}{\varepsilon} \frac{\left\| A_{R}^{\xi} \right\|_{L^4} \left\| \partial^m (d_0 \otimes d_0 \otimes d_0) \right\|_{L^6} \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2}} \\
& + \frac{C}{\varepsilon} \sum_{0 \neq m' < m} \left\| \partial^{m-m'} A_{R}^{\xi} \right\|_{L^2} \left\| \partial^{m'} (d_0 \otimes d_0 \otimes d_0) \right\|_{L^6} \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2} \\
& \leq \frac{C}{\varepsilon} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left( 1 + \left\| \nabla d_{0} \right\|_{H^N}^2 \right) \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2}} \\
& \leq \frac{C}{\varepsilon} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left( 1 + \left\| \nabla d_{0} \right\|_{H^N}^2 \right) \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2}} \\
& \leq C \left( 1 + \left\| \nabla d_{0} \right\|_{H^N}^2 \right) \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \mathcal{D}_{N, \varepsilon} (t). \tag{3.119}
\end{align*}

The quantity $I_3$ can be bounded by
\begin{align*}
I_3 & \leq \frac{\lambda_2}{\varepsilon} \left\| \partial^m A_{R}^{\xi} \right\|_{L^2} \left\| \partial d_{0} \right\|_{L^3} \left\| \partial^m d_{R}^{\xi} \right\|_{L^6} \\
& \leq \frac{C}{\varepsilon} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left\| \partial d_{0} \right\|_{H^1}} \\
& \leq \frac{C}{\varepsilon} \left( 1 + \left\| \nabla d_{0} \right\|_{H^N}^3 \right) \left( \left\| u_0 \right\|_{H^N} + \left\| \nabla d_{0} \right\|_{H^N} \right) \left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left\| \partial d_{0} \right\|_{H^N} \\
& \leq C \left( 1 + \left\| \nabla d_{0} \right\|_{H^N}^3 \right) \left( \left\| u_0 \right\|_{H^N} + \left\| \nabla d_{0} \right\|_{H^N} \right) \mathcal{D}_{N, \varepsilon} (t). \tag{3.120}
\end{align*}

Here we make use of the bound
\begin{equation}
\left\| \partial d_{0} \right\|_{H^1} \leq C \left( 1 + \left\| \nabla d_{0} \right\|_{H^N}^3 \right) \left( \left\| u_0 \right\|_{H^N} + \left\| \nabla d_{0} \right\|_{H^N} \right) \tag{3.121}
\end{equation}
for $N \geq 2$, which is derived from the $d_0$-equation of (1.10). The term $I_4$ can be estimated as
\begin{align*}
I_4 & \leq \frac{\lambda_2}{\sqrt{\varepsilon}} \left\| \partial^m A_{R}^{\xi} \right\|_{L^2} \left\| \partial^m (d_{R}^{\xi} \cdot D_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi}) \right\|_{L^2} \\
& \leq \frac{C}{\sqrt{\varepsilon}} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left\| D_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{H^N}} \\
& \leq \frac{C}{\sqrt{\varepsilon}} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| d_{R}^{\xi} \right\|_{H^N} \left\| \partial d_{0} \right\|_{H^N}} \\
& \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{H^N} \left\| (\partial^m B_{R}^{\xi}) d_{0} + \frac{\lambda_2}{\sqrt{\varepsilon}} \left\| \partial^m A_{R}^{\xi} \right\|_{L^2} \right\|_{L^2} \\
& \leq C \varepsilon \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N} \left\| \partial d_{0} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N}} \\
& \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{H^N} \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2} \\
& \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2} \left\| \partial^m d_{u_0 + \sqrt{\varepsilon} u_R} d_{R}^{\xi} \right\|_{L^2} \tag{3.122}
\end{align*}

For the term $I_5$, we calculate that
\begin{align*}
I_5 & \leq \frac{C}{\varepsilon} \left\| \partial^m A_{R}^{\xi} \right\|_{L^2} \sum_{0 \neq m' \leq m} \left\| \partial^{m'} d_{0} \right\|_{L^3} \left\| \partial^{m-m'} d_{R}^{\xi} \right\|_{L^6} \\
& \leq \frac{C}{\varepsilon} \frac{\left\| \nabla u_{R}^{\xi} \right\|_{H^N} \left\| \partial d_{0} \right\|_{H^{N+1}} \left\| \nabla d_{R}^{\xi} \right\|_{H^N}} \\
& \leq \frac{C}{\varepsilon} \left( 1 + \left\| \nabla d_{0} \right\|_{H^{N+2}} \right) \left( \left\| u_0 \right\|_{H^{N+2}} + \left\| \nabla d_{0} \right\|_{H^{N+2}} \right) \left\| \nabla u_{R} \right\|_{H^N} \left\| \nabla d_{R}^{\xi} \right\|_{H^N}
\end{align*}
\[ \leq C(1 + \|\nabla d_0\|_{H^{N+2}}^3)(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\mathcal{D}_{N, \varepsilon}(t), \] (3.123)

where the bound \(\|\partial_t d_0\|_{H^{N+1}} \leq C(1 + \|\nabla d_0\|_{H^{N+2}})(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\) is utilized. We compute the quantity \(I_6\) that

\[ I_6 \leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^m A_R^\varepsilon\|_{L^2} \|\partial^{m'} d_0\|_{L^4} \|\partial^{m-m'} D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R\|_{L^4} \]

\[ \leq \frac{C}{\varepsilon} \|\nabla u_R^\varepsilon\|_{H^N} \|\nabla d_0\|_{H^N} \|D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R\|_{H^N} \]

\[ \leq \frac{C}{\varepsilon} \|\nabla d_0\|_{H^N} \|\nabla d_R^\varepsilon\|_{H^N} \times \left(\|\nabla u_R^\varepsilon\|_{H^N} + \sum_{|m| \leq N} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon + (\partial^m B_{R}^\varepsilon) d_0 + \frac{\sqrt{\varepsilon}}{A_t}(\partial^m A_R^\varepsilon) d_0\|_{L^4}\right) \]

\[ \leq C\|\nabla d_0\|_{H^N} \mathcal{D}_{N, \varepsilon}(t). \] (3.124)

It is easy to be estimated that

\[ I_7 \leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \|\partial^m A_R^\varepsilon\|_{L^2} \|\partial^{m'} d_0\|_{L^4} \|\partial^{m-m'} (u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \cdot \nabla d_R^\varepsilon\|_{L^4} \]

\[ \leq \frac{C}{\varepsilon} \|\nabla u_R^\varepsilon\|_{H^N} \|\nabla d_0\|_{H^N} \|(u_0 + \sqrt{\varepsilon} u_R^\varepsilon) \cdot \nabla d_R^\varepsilon\|_{H^N} \]

\[ \leq \frac{C}{\varepsilon} \|\nabla d_0\|_{H^N} \|\nabla u_R^\varepsilon\|_{H^N} \|\nabla d_R^\varepsilon\|_{H^N} (\|u_0\|_{H^N} + \sqrt{\varepsilon}\|u_R^\varepsilon\|_{H^N}) \]

\[ \leq C\|\nabla d_0\|_{H^N} (\|u_0\|_{H^N} + \varepsilon\mathcal{D}_{N, \varepsilon}^\frac{1}{2}(t)) \mathcal{D}_{N, \varepsilon}(t). \] (3.125)

Collecting the bounds of \(I_i\) (1 \(\leq i \leq 7\)) above, we obtain

\[ -\frac{\varepsilon}{\varepsilon} \left\langle \partial^m[(A_R^\varepsilon : d_0 \otimes d_0) d_0], \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \leq C\varepsilon(1 + \|\nabla d_0\|_{H^{N+2}})\mathcal{D}_{N, \varepsilon}^\frac{1}{2}(t) \mathcal{D}_{N, \varepsilon}(t) \]

\[ + C(1 + \|\nabla d_0\|_{H^{N+2}})^2(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\mathcal{D}_{N, \varepsilon}(t). \] (3.126)

Recalling that the expression of \(S^1_d\) in \(A.4\), we derive from the bounds (3.108), (3.109), (3.110), (3.111), (3.112), (3.113) and (3.126) that

\[ \frac{1}{\varepsilon} \sum_{|m| \leq N} \left\langle \partial^m S^1_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \leq C\varepsilon(1 + \|\nabla d_0\|_{H^{N+2}})^2 \mathcal{D}_{N, \varepsilon}^\frac{1}{2}(t) \mathcal{D}_{N, \varepsilon}(t) \]

\[ + C(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})(1 + \|\nabla d_0\|_{H^{N+2}})^2 \mathcal{D}_{N, \varepsilon}(t). \] (3.127)

Combining the similar arguments of the inequality (3.127) and the bound (3.103), we know that

\[ \frac{1}{\varepsilon} \sum_{|m| \leq N} \left\langle \partial^m S^1_d, \partial^m (u_R^\varepsilon \cdot \nabla d_0) \right\rangle \]

\[ \leq \frac{C}{\varepsilon^2} (\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})(1 + \|\nabla d_0\|_{H^{N+2}})^2 \mathcal{D}_{N, \varepsilon}^\frac{1}{2}(t) \|\partial^m (u_R^\varepsilon \cdot \nabla d_0)\|_{L^2} \]

\[ \leq \frac{C}{\varepsilon^2} (\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})(1 + \|\nabla d_0\|_{H^{N+2}})^2 \mathcal{D}_{N, \varepsilon}(t). \]

We next estimate the quantity \(\frac{d}{\varepsilon} \sum_{|m| \leq N} \left\langle \partial^m S^1_d, \partial^m d_R^\varepsilon \right\rangle\) for \(N \geq 2\). First, we have

\[ \frac{d}{\varepsilon} \left\langle \partial^m [(\nabla d_0 \cdot \nabla d_R^\varepsilon) d_0], \partial^m d_R^\varepsilon \right\rangle \]

\[ \leq \frac{C}{\varepsilon} \sum_{m' \leq m} \|\partial^{m-m'} (d_0 \otimes \nabla d_0)\|_{L^3} \|\nabla \partial^{m'} d_R^\varepsilon\|_{L^2} \|\partial^m d_R^\varepsilon\|_{L^6} \]

\[ \leq \frac{C}{\varepsilon} \|\nabla d_0\|_{H^{N+1}}(1 + \|\nabla d_0\|_{H^{N+1}}) \|\nabla d_R^\varepsilon\|_{H^N}^2 \]

\[ \leq C\|\nabla d_0\|_{H^{N+1}}(1 + \|\nabla d_0\|_{H^{N+1}}) \mathcal{D}_{N, \varepsilon}(t). \] (3.129)
We then estimate
\[ \frac{\delta m}{\varepsilon} \langle \partial^m ((\nabla d_0) \partial d_R^\varepsilon), \partial^m d_R^\varepsilon \rangle \]
\[ \leq \frac{C}{\varepsilon} \sum_{m' \leq m} \sum_{m'' \leq m'} \| \nabla \partial^{m''} d_0 \|_{L^5} \| \nabla \partial^{m''} d_0 \|_{L^3} \| \partial^{m-m'} d_R^\varepsilon \|_{L^6} \| \partial^m d_R^\varepsilon \|_{L^6} \]
\[ \leq \frac{C}{\varepsilon} \| \nabla d_0 \|_{H^{N+1}}^2 \| \nabla d_R^\varepsilon \|_{H^N}^2 \leq C \| \nabla d_0 \|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t) . \]

(3.130)

It is easy to calculate that
\[ \frac{\delta \lambda}{\varepsilon} \langle \partial^m (u_0 \cdot \nabla d_R^\varepsilon + u_R^\varepsilon \cdot \nabla d_0 + B_0 d_R^\varepsilon), \partial^m d_R^\varepsilon \rangle \]
\[ \leq \frac{C}{\varepsilon} \sum_{m' \leq m} \| \partial^{m'} u_0 \|_{L^3} \| \nabla \partial^{m-m'} d_R^\varepsilon \|_{L^2} \| \partial^m d_R^\varepsilon \|_{L^6} \]
\[ \leq \frac{C}{\varepsilon} \sum_{m' \leq m} \| \partial^{m'} u_R^\varepsilon \|_{L^6} \| \nabla \partial^{m-m'} d_0 \|_{L^2} + \| \partial^{m'} d_R^\varepsilon \|_{L^6} \| \partial^m d_R^\varepsilon \|_{L^6} \| \nabla d_0 \|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t) , \]
\[ \leq C(\| u_0 \|_{H^{N+1}} + \| \nabla d_0 \|_{H^{N+1}})(\| \nabla d_R^\varepsilon \|_{H^N} + \| \nabla u_R^\varepsilon \|_{H^N}) \| \nabla d_R^\varepsilon \|_{H^N} \]
\[ \leq C(\| u_0 \|_{H^{N+1}} + \| \nabla d_0 \|_{H^{N+1}}) \mathcal{D}_{N,\varepsilon}(t) , \]

where we intrinsically utilize the fact that the volume of $T^3$ is finite. Similarly as in (3.131), we obtain
\[ \frac{\delta \lambda}{\varepsilon} \langle \partial^m (A_0 d_R^\varepsilon), \partial^m d_R^\varepsilon \rangle \leq C \| u_0 \|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t) , \]

(3.132)

and
\[ -\frac{\delta \lambda}{\varepsilon} \langle \partial^m [(A_0 : d_0 \otimes d_0), \partial^m d_R^\varepsilon] - \frac{2\delta \lambda}{\varepsilon} \langle \partial^m [(A_0 : d_0 \otimes d_0) d_0], \partial^m d_R^\varepsilon \rangle \]
\[ \leq C(1 + \| \nabla d_0 \|_{H^{N+1}}) \mathcal{D}_{N,\varepsilon}(t) . \]

(3.133)

It remains to estimate the quantity $-\frac{\delta \lambda}{\varepsilon} \langle \partial^m ((A_R^\varepsilon : d_0 \otimes d_0) d_0), \partial^m d_R^\varepsilon \rangle$ for all $|m| \leq N$. From the relation (3.115), we easily derive that
\[ -\frac{\delta \lambda}{\varepsilon} \langle \partial^m ((A_R^\varepsilon : d_0 \otimes d_0) d_0), \partial^m d_R^\varepsilon \rangle \]
\[ = \frac{\delta \lambda}{\varepsilon} \sum_{m' \leq m} C_m^m \langle \partial^m A_R^\varepsilon : d_0 \otimes d_0, \partial^m d_R^\varepsilon \rangle \]
\[ + \frac{\delta \lambda}{\varepsilon} \sum_{0 \neq m' \leq m} C_m^m \langle \partial^m A_R^\varepsilon : d_0 \otimes d_0, \partial^m \partial^{m-m'} d_R^\varepsilon \rangle \]
\[ + \frac{\delta \lambda}{\varepsilon} \sum_{0 \neq m' \leq m} C_m^m \langle \partial^m (A_R^\varepsilon : d_0 \otimes d_0), \partial^m d_R^\varepsilon \rangle . \]

(3.134)

For the term $I_1$, we estimate
\[ I_1 \leq \frac{C}{\varepsilon} \sum_{m' \leq m} \| \partial^m A_R^\varepsilon \|_{L^2} \| \partial^m d_R^\varepsilon \|_{L^6} \| \partial^m d_R^\varepsilon \|_{L^6} \]
\[ \leq \frac{C}{\varepsilon} \| \nabla u_R^\varepsilon \|_{H^N} \| \nabla d_R^\varepsilon \|_{H^N} (\| d_R^\varepsilon \|_{H^N} + \| \nabla d_R^\varepsilon \|_{H^N}) \]
\[ \leq C \varepsilon \mathcal{D}_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) . \]

(3.135)

For the term $I_2$, we have
\[ I_2 \leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \| \partial^m A_R^\varepsilon \|_{L^2} \| \partial^m d_0 \|_{L^3} \| \partial^m d_R^\varepsilon \|_{L^6} \]
\[ \leq \frac{C}{\varepsilon} \| \nabla u_R^\varepsilon \|_{H^N} \| \nabla d_0 \|_{H^N} \| \nabla d_R^\varepsilon \|_{H^N} \leq C \| \nabla d_0 \|_{H^N} \mathcal{D}_{N,\varepsilon}(t) . \]

(3.136)
For the term $I_3$, we calculate that
\[
I_3 \leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \| \partial^m \partial^m A_R \|_{L^2} \| \partial^m (d_0 \otimes d_0) \|_{L^2} \| \partial^m d_R \|_{L^2} \\
\leq \frac{C}{\varepsilon} \| \nabla u_R \|_{H^N} \| \nabla d_0 \|_{H^N} (1 + \| \nabla d_0 \|^2_{H^N}) \| \nabla d_R \|_{H^N} \\
\leq C(1 + \| \nabla d_0 \|^2_{H^N}) \| \nabla d_0 \|_{H^N} \mathcal{D}_{N,\varepsilon}(t). 
\]
(3.137)

Summarizing the estimates $I_1$, $I_2$ and $I_3$, we obtain
\[
- \frac{\varepsilon}{\varepsilon} \mathcal{S}_4 \leq \frac{\varepsilon}{\varepsilon} \sum_{|m| \leq N} (\partial^m S_d, \partial^m d_R) \leq C(1 + \| \nabla d_0 \|^2_{H^N}) \| \nabla d_0 \|_{H^N} \mathcal{D}_{N,\varepsilon}(t) + C\varepsilon \mathcal{E}_{N,\varepsilon}^2(t) \mathcal{D}_{N,\varepsilon}(t). 
\]
(3.138)

Recalling the expression of $S_d$ in (A.4), we derive from the inequalities (3.129), (3.130), (3.131), (3.132), (3.133) and (3.138) that
\[
\mathcal{D}_{N,\varepsilon} \leq C(1 + \| \nabla d_0 \|^2_{H^N}) \| \nabla d_0 \|_{H^N} \mathcal{D}_{N,\varepsilon}(t). 
\]
(3.139)

Consequently, plugging the inequalities (3.127), (3.128) and (3.139) into the expression of $\mathcal{T}_N^4(S_d)$ in (3.102) reduces to
\[
\mathcal{T}_N^4(S_d) \leq C(\| u_0 \|_{H^{N+2}} + \| \nabla d_0 \|_{H^{N+2}})(1 + \| \nabla d_0 \|^3_{H^{N+2}}) \mathcal{D}_{N,\varepsilon}(t) + C\varepsilon \mathcal{E}_{N,\varepsilon}^2(t) \mathcal{D}_{N,\varepsilon}(t). 
\]
(3.140)

Next we estimate the quantity $\mathcal{T}_N^4(S_d)$ for any integer $N \geq 2$. First, we have
\[
\frac{1}{\varepsilon} \left( \partial^m (\| \nabla d_R \|^2 d_0), \partial^m D_{u_0 + \epsilon u_R} d_R \right) \\
\leq \frac{C}{\varepsilon} \sum_{m' \leq m} \| \partial^m \partial^m \|_{L^2} \| \partial^m d_0 \|_{L^\infty} \| \partial^m D_{u_0 + \epsilon u_R} d_R \|_{L^2} \leq \frac{C}{\varepsilon} \| \nabla d_R \|^2_{H^N} \| D_{u_0 + \epsilon u_R} d_R \|_{H^N} (1 + \| \nabla d_0 \|_{H^{N+2}}) \\
\leq C\sqrt{\varepsilon}(1 + \| \nabla d_0 \|_{H^{N+2}}) \mathcal{E}_{N,\varepsilon}^2(t) \mathcal{D}_{N,\varepsilon}(t) 
\]
and
\[
\frac{2}{\varepsilon} \left( \partial^m ((\nabla d_0 \cdot \nabla d_R) \partial^m d_R), \partial^m D_{u_0 + \epsilon u_R} d_R \right) \\
\leq \frac{C}{\varepsilon} \sum_{m' \leq m} \| \nabla \partial^m \|_{L^\infty} \| \partial^m \partial^m \|_{L^2} \| \partial^m \partial^m \|_{L^2} \| \partial^m \partial^m \|_{L^2} \leq \frac{C}{\varepsilon} \| \nabla d_R \|^2_{H^N} \| \nabla d_R \|^2_{H^N} \| D_{u_0 + \epsilon u_R} d_R \|_{H^N} (1 + \| \nabla d_0 \|_{H^{N+2}}) \\
\leq C\sqrt{\varepsilon}(1 + \| \nabla d_0 \|_{H^{N+2}}) \mathcal{E}_{N,\varepsilon}^2(t) \mathcal{D}_{N,\varepsilon}(t) 
\]
for all $|m| \leq N$. It is easy to deduce that
\[
- \frac{1}{\varepsilon} \left( \partial^m D_{u_0} d_0, \partial^m D_{u_0 + \epsilon u_R} d_R \right) \leq \frac{1}{\varepsilon} \| \partial^m D_{u_0} d_0 \|_{L^2} \| \partial^m D_{u_0 + \epsilon u_R} d_R \|_{L^2} \leq \frac{C}{\varepsilon} \| D_{u_0} \|^2_{H^N} (\| \partial^m D_{u_0 + \epsilon u_R} d_R \|_{L^2} + \| \Delta d_R \|_{H^N}) \\
\leq C(1 + \| u_0 \|^6_{H^{N+2}} + \| \nabla d_0 \|^6_{H^{N+2}})(\| \nabla u_0 \|^6_{H^{N+2}} + \| \Delta d_0 \|_{H^{N+2}}) \mathcal{E}_{N,\varepsilon}^2(t), 
\]
where we make use of the inequality (3.65).
For the term \(- \frac{1}{\sqrt{\varepsilon}} \left< \partial^m (|D u_0 d_0|^2 d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2 \right>\), we deduce that
\[
\begin{align*}
&\leq \frac{1}{\sqrt{\varepsilon}} ||\partial^m (|D u_0 d_0|^2 d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2||_L^2 ||\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2||_L^2 \\
&+ \frac{C}{\sqrt{\varepsilon}} \sum_{0 \neq m' \leq m} ||\partial^m m' (|D u_0 d_0|^2 L^2 ||\partial^m m' d_0||_L^2 ||\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2||_L^2 \\
&\leq \frac{C}{\sqrt{\varepsilon}} (1 + \|\nabla d_0\|_{H^{N+1}})^2 ||D u_0 d_0||_H^2 \\
&\times (||\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2 + (\partial^m B_R^2) d_0 + \frac{1}{\sqrt{\varepsilon}} (\partial^m A_R^2) d_0||_L^2 + ||\nabla u_R^2||_{H^{N}}) \\
&\leq C (1 + ||\nabla d_0||_{H^{N+1}}^2 + ||\Delta d_0||_{H^{N}}^2) \mathcal{D}_{N, \varepsilon} (t) \\
&\leq C (1 + ||u_0||_{H^{N+1}}^2 + ||\nabla d_0||_{H^{N+1}}^2) (||\nabla u_0||_{H^{N}} + ||\Delta d_0||_{H^{N}}) \mathcal{D}_{N, \varepsilon} (t),
\end{align*}
\]
where the inequality (3.64) is utilized. We next estimate that
\[
\begin{align*}
&\leq \frac{C}{\sqrt{\varepsilon}} (||\partial^m B_R^2||_L^2 ||\partial^m d_R^2||_L^2 + \sum_{0 \neq m' \leq m} ||\partial^m m' B_R^2||_L^2 ||\partial^m m' d_R^2||_L^2) ||\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2||_L^2 \\
&\leq \frac{C}{\sqrt{\varepsilon}} ||\nabla u_R^2||_{H^{N}} (||d_R^2||_{H^{N}} + ||\nabla d_R^2||_{H^{N}}) ||D u_0 + \sqrt{\varepsilon} u_R^2 d_R^2||_{H^{N}} \\
&\leq C \sqrt{\varepsilon} \mathcal{D}_{N, \varepsilon} (t) \mathcal{D}_{N, \varepsilon} (t).
\end{align*}
\]
Similarly as in (3.145), we have
\[
\begin{align*}
&\leq \frac{C}{\sqrt{\varepsilon}} (\partial^m (A_R^2 d_R^2), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2) \leq C \sqrt{\varepsilon} \mathcal{D}_{N, \varepsilon} (t) \mathcal{D}_{N, \varepsilon} (t).
\end{align*}
\]
We deduce the following bound
\[
\begin{align*}
&\leq \frac{C}{\sqrt{\varepsilon}} (\partial^m (A_R^2 (d_0 \otimes d_R^2), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2) \\
&\leq \frac{C}{\sqrt{\varepsilon}} \sum_{0 \neq m' \leq m} \sum_{m' \leq m} \sum_{m' \leq m'} ||\partial^m m' A_R^2 d_0 d_0||_L^2 ||\partial^m m' d_R^2||_L^2 ||\partial^m m' d_R^2||_L^2 ||\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2||_L^2 \\
&\leq \frac{C}{\sqrt{\varepsilon}} ||u_0||_{H^{N+2}}^2 + ||\nabla d_0||_{H^{N+2}}^2 2 ||\nabla u_R^2||_{H^{N}} ||D u_0 + \sqrt{\varepsilon} u_R^2 d_R^2||_{H^{N}} \\
&\leq C \sqrt{\varepsilon} ||u_0||_{H^{N+2}}^2 + ||\nabla d_0||_{H^{N+2}}^2 \mathcal{D}_{N, \varepsilon} (t) \mathcal{D}_{N, \varepsilon} (t).
\end{align*}
\]
We calculate that
\[
\begin{align*}
&\leq \frac{C}{\sqrt{\varepsilon}} ||\partial^m (A_R^2 (d_0 \otimes d_R^2), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2) \\
&\leq \frac{C}{\sqrt{\varepsilon}} ||\partial^m A_R^2||_L^2 ||d_0 \otimes d_R^2||_L^2 ||\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2||_L^2 \\
&+ \frac{C}{\sqrt{\varepsilon}} \sum_{0 \neq m' \leq m} \sum_{m' \leq m} \sum_{m' \leq m'} ||\partial^m m' A_R^2 d_0 d_0||_L^2 ||\partial^m m' d_R^2||_L^2 ||\partial^m m' d_R^2||_L^2 \leq \frac{C}{\sqrt{\varepsilon}} ||\nabla u_R^2||_{H^{N}} ||D u_0 + \sqrt{\varepsilon} u_R^2 d_R^2||_{H^{N}} (||d_R^2||_{H^{N}} + ||\nabla d_R^2||_{H^{N}}) (1 + ||\nabla d_0||_{H^{N+1}}^2) \\
&\leq C \sqrt{\varepsilon} (1 + ||\nabla d_0||_{H^{N+1}}^2) \mathcal{D}_{N, \varepsilon} (t) \mathcal{D}_{N, \varepsilon} (t).
\end{align*}
\]
It is derived from the analogous arguments in (3.148) that
\[
\begin{align*}
&\leq \frac{C}{\sqrt{\varepsilon}} ||\partial^m (A_R^2 (d_0 \otimes d_R^2 + d_R^2 \otimes d_0) d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^2} d_R^2) \\
&\leq C \sqrt{\varepsilon} (1 + ||\nabla d_0||_{H^{N+1}}^2) \mathcal{D}_{N, \varepsilon} (t) \mathcal{D}_{N, \varepsilon} (t).
\end{align*}
\]
Moreover, from the similar calculations in (3.147), we derive that

\[
- \frac{\varepsilon}{\sqrt{\varepsilon}} \left\langle \partial^m [(A_0 : d_R^\varepsilon \otimes \dot{d}_R^\varepsilon) d_0], \partial^m D_{u_0 + \sqrt{\varepsilon} \sigma u_0} d_R^\varepsilon \right\rangle \\
\leq C \sqrt{\varepsilon} \| u_0 \|_{H^{N+2}} (1 + \| \nabla d_0 \|_{H^{N+2}}) \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t) .
\]  

(3.150)

Recalling the definition of $\mathcal{S}_d^2$ in (A.5), we collect the inequalities (3.141), (3.142), (3.143), (3.144), (3.145), (3.146), (3.147), (3.148), (3.149) and (3.150), and then know that

\[
\frac{1}{\sqrt{\varepsilon}} \left\langle \partial^m \mathcal{S}_d^2, \partial^m (u_R^\varepsilon : \nabla d_0) \right\rangle \\
\leq C (1 + \| u_0 \|_{H^{N+2}}^2 + \| \nabla d_0 \|_{H^{N+2}}^2) \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t) \| u_R : \nabla d_0 \|_{H^{N}} \\
+ C (1 + \| u_0 \|_{H^{N+1}}^8 + \| \nabla d_0 \|_{H^{N+1}}^8) (\| \nabla u_0 \|_{H^{N}} + \| \Delta d_0 \|_{H^{N}}) \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t). 
\]  

(3.151)

Furthermore, the inequality (3.103) and similar estimates on (3.151) reduce to

\[
\frac{1}{\sqrt{\varepsilon}} \left\langle \partial^m \mathcal{S}_d^2, \partial^m (u_R^\varepsilon : \nabla d_0) \right\rangle \\
\leq C (1 + \| u_0 \|_{H^{N+2}}^3 + \| \nabla d_0 \|_{H^{N+2}}^3) \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t) \| u_R^\varepsilon : \nabla d_0 \|_{H^{N}} \\
+ C (1 + \| u_0 \|_{H^{N+1}}^9 + \| \nabla d_0 \|_{H^{N+1}}^9) (\| \nabla u_0 \|_{H^{N}} + \| \Delta d_0 \|_{H^{N}}) \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t). 
\]  

(3.152)

We next estimate the quantity $\frac{\delta}{\sqrt{\varepsilon}} \left\langle \partial^m \mathcal{S}_d^2, \partial^m d_R^\varepsilon \right\rangle$ for all $|m| \leq N$. First, we have

\[
\frac{\delta}{\sqrt{\varepsilon}} \left\langle \partial^m ([\nabla d_0 : \nabla d_0^\varepsilon] d_R^\varepsilon), \partial^m d_R^\varepsilon \right\rangle \\
\leq \varepsilon^{\frac{1}{2}} \| \partial^m |\nabla d_R^\varepsilon|^2 \|_{L^2} \| \partial^m d_R^\varepsilon \|_{L^2} \\
+ \varepsilon^{\frac{1}{2}} \sum_{0 \neq m' \leq m} \| \partial^m - m' \|_{L^4} \| \partial^m d_{R_0} \|_{L^4} \| \partial^m d_{R_0} \|_{L^2} \\
\leq \varepsilon^{\frac{1}{2}} \| \nabla d_R^\varepsilon \|_{H^{N}} \| \nabla d_0 \|_{H^{N+1}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \\
+ C \varepsilon (1 + \| \nabla d_0 \|_{H^{N+1}}) \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t). 
\]  

(3.153)

It is estimated that

\[
\frac{2\delta}{\sqrt{\varepsilon}} \left\langle \partial^m ([\nabla d_0 : \nabla d_0^\varepsilon] d_R^\varepsilon), \partial^m d_R^\varepsilon \right\rangle \\
\leq \varepsilon^{\frac{1}{2}} \sum_{m' \leq m} \sum_{m'' \leq m'} \| \nabla R^{m'-m'} d_R^\varepsilon \|_{L^2} \| \partial^m d_R^\varepsilon \|_{L^2} \| \nabla R^{m'-m''} d_{R_0} \|_{L^5} \| \partial^m d_{R_0} \|_{L^5} \| \partial^m d_{R_0} \|_{L^6} \\
\leq \varepsilon^{\frac{1}{2}} \| \nabla d_R^\varepsilon \|_{H^{N}} \| \nabla d_0 \|_{H^{N+1}} \leq C \| \nabla d_0 \|_{H^{N+1}} \varepsilon^{\frac{1}{2}} (t) \mathcal{D}_{N, \varepsilon} (t). 
\]  

(3.154)

The following bound holds

\[
\frac{\delta}{\sqrt{\varepsilon}} \left\langle \partial^m D_{u_0} d_0, \partial^m d_R^\varepsilon \right\rangle \leq \frac{\delta}{\sqrt{\varepsilon}} \| \partial^m D_{u_0} d_0 \|_{L^2} \| \partial^m d_R^\varepsilon \|_{L^5} \| T^3 \|_{T^3}^\frac{1}{2} \\
\leq \varepsilon^{\frac{1}{2}} \| D_{u_0} d_0 \|_{H^{N}} \| \nabla d_R^\varepsilon \|_{H^{N}} \\
\leq \varepsilon^{\frac{1}{2}} (1 + \| u_0 \|_{H^{N+2}}^6 + \| \nabla d_0 \|_{H^{N+2}}^6) (\| \nabla u_0 \|_{H^{N+2}} + \| \Delta d_0 \|_{H^{N+2}}) \| \nabla d_R^\varepsilon \|_{H^{N}} \\
\leq C (1 + \| u_0 \|_{H^{N+2}}^6 + \| \nabla d_0 \|_{H^{N+2}}^6) (\| \nabla u_0 \|_{H^{N+2}} + \| \Delta d_0 \|_{H^{N+2}}) \varepsilon^{\frac{1}{2}} (t) .
\]  

(3.155)
where the inequality (3.65) and the fact that the volume of $\mathbb{T}^3$ is finite are utilized. Moreover, from the bound (3.64) and the finiteness of the volume $\mathbb{T}^3$, we deduce that

$$\begin{align*}
& - \frac{C}{\sqrt{\varepsilon}} \langle \partial^m [\partial u_0 d_0], \partial^m d^\varepsilon_R \rangle \\
& \leq \frac{C}{\sqrt{\varepsilon}} \| \partial^{m'} [\partial u_0 d_0] \|_{L^2} \| \partial^{m'} d^\varepsilon_R \|_{L^6} \| \mathbb{T}^3 \|^{\frac{1}{3}} \\
& + \frac{C}{\sqrt{\varepsilon}} \sum_{0 \neq m'} \| \partial^{m-m'} [\partial u_0 d_0] \|_{L^2} \| \partial^{m'} d^\varepsilon_R \|_{L^4} \| \partial^m d^\varepsilon_R \|_{L^6} \| \mathbb{T}^3 \|^{\frac{1}{3}}
\end{align*}$$

(3.156)

For the term $\frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m (B^\varepsilon d^\varepsilon_R), \partial^m d^\varepsilon_R \rangle$, we estimate that

$$\begin{align*}
& \frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m (B^\varepsilon d^\varepsilon_R), \partial^m d^\varepsilon_R \rangle \\
& \leq \frac{C}{\sqrt{\varepsilon}} \sum_{m'} \| \partial^{m-m'} B^\varepsilon \|_{L^2} \| \partial^{m'} d^\varepsilon_R \|_{L^4} \| \partial^m d^\varepsilon_R \|_{L^6}
\end{align*}$$

(3.157)

Similarly as in (3.157), we have

$$\begin{align*}
& \frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m (A^\varepsilon R d^\varepsilon_R), \partial^m d^\varepsilon_R \rangle \leq C \varepsilon \mathcal{E}_{N,\varepsilon}^\frac{1}{3} (t) \mathcal{D}_{N,\varepsilon}^\frac{1}{3} (t).
\end{align*}$$

(3.158)

We next calculate that

$$\begin{align*}
& - \frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m [A_0 : (d_0 \otimes d^\varepsilon_R + d^\varepsilon_R \otimes d_0) d^\varepsilon_R], \partial^m d^\varepsilon_R \rangle \\
& \leq \frac{C}{\sqrt{\varepsilon}} \sum_{m' \leq m} \| \partial^{m-m'} [A_0 d_0] \|_{L^2} \| \partial^{m'} d^\varepsilon_R \|_{L^4} \| \partial^m d^\varepsilon_R \|_{L^6}
\end{align*}$$

(3.159)

One easily derives the following bound

$$\begin{align*}
& - \frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m [(A^\varepsilon_R : d_0 \otimes d_0) d^\varepsilon_R], \partial^m d^\varepsilon_R \rangle \\
& \leq \frac{C}{\sqrt{\varepsilon}} \sum_{m' \leq m} \| \partial^{m-m'} [A^\varepsilon_R d_0] \|_{L^2} \| \partial^{m'} \left((d_0 \otimes d_0) d^\varepsilon_R\right) \|_{L^4} \| \partial^m d^\varepsilon_R \|_{L^6}
\end{align*}$$

(3.160)

Via the analogous calculations in (3.160), we imply that

$$\begin{align*}
& - \frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m [A^\varepsilon_R : (d_0 \otimes d^\varepsilon_R + d^\varepsilon_R \otimes d_0) d_0], \partial^m d^\varepsilon_R \rangle \leq C (1 + \| \nabla d_0 \|_{H^N}^2 ) \mathcal{E}_{N,\varepsilon}^\frac{1}{3} (t) \mathcal{D}_{N,\varepsilon}^\frac{1}{3} (t).
\end{align*}$$

(3.161)

Furthermore, the similar calculations in (3.159) tell us

$$\begin{align*}
& - \frac{\delta \lambda}{\sqrt{\varepsilon}} \langle \partial^m [(A_0 : d^\varepsilon_R \otimes d^\varepsilon_R) d_0], \partial^m d^\varepsilon_R \rangle \leq C \varepsilon \| u_0 \|_{H^{N+1}} (1 + \| \nabla d_0 \|_{H^{N+1}}) \mathcal{E}_{N,\varepsilon}^\frac{1}{3} (t) \mathcal{D}_{N,\varepsilon}^\frac{1}{3} (t).
\end{align*}$$

(3.162)
Recalling that the definition of $S_d^2$ in (A.5), we imply by collecting the bounds (3.153), (3.154), (3.155), (3.156), (3.157), (3.158), (3.159), (3.160), (3.161) and (3.162) that

$$\frac{\delta}{\varepsilon} \left\langle \partial^m S_d^2, \partial^m d_R^\varepsilon \right\rangle \leq C\varepsilon (1 + \|u_0\|^2_{H^{N+2}} + \|\nabla d_0\|^2_{H^{N+2}}) \delta^2_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t)$$

(3.163)

$$+ C(1 + \|u_0\|^2_{H^{N+2}} + \|\nabla d_0\|^2_{H^{N+2}})(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}) \delta^2_{N,\varepsilon}(t) .$$

From the inequalities (3.151), (3.152) and (3.163), we deduce that

$$\mathcal{I}_N^{(4)}(S_d^2) = \frac{1}{\sqrt{\varepsilon}} \left\langle \partial^m S_d^2, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon - \partial^m (u_R^\varepsilon \cdot \nabla d_0) + \delta \partial^m d_R^\varepsilon \right\rangle$$

$$\leq C\varepsilon (1 + \|u_0\|^2_{H^{N+2}} + \|\nabla d_0\|^2_{H^{N+2}}) \delta^2_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t)$$

(3.164)

$$+ C(1 + \|u_0\|^2_{H^{N+2}} + \|\nabla d_0\|^2_{H^{N+2}})(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}) \delta^2_{N,\varepsilon}(t) .$$

We next estimate the quantity $\mathcal{I}_N^{(4)}(R_d)$ for the integer $N \geq 2$. We start with the term

$$\left\langle \partial^m R_d, \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle$$

for $|m| \leq N$. First, we have

$$\left\langle \partial^m (D_{u_0} d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \leq \|\partial^m (|\nabla d_R^\varepsilon|^2 d_R^\varepsilon)\|_{L^2} \|\nabla d_R^\varepsilon\|_{L^\infty} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{L^2}$$

$$+ C \sum_{0 \neq m' \leq m} \|\partial^{m-m'} d_R^\varepsilon\|_{L^4} \|\partial^{m'} d_R^\varepsilon\|_{L^4} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{L^2}$$

(3.165)

$$\leq C\varepsilon \|\nabla d_R^\varepsilon\|_{H^N} \|D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{H^N} (\|d_R^\varepsilon\|_{H^N} + \|\nabla d_R^\varepsilon\|_{H^N})$$

$$\leq C\varepsilon \delta_{N,\varepsilon}(t) \mathcal{D}_{N,\varepsilon}(t) .$$

Via the inequality (3.64), we deduce that

$$- \left\langle \partial^m (|D_{u_0} d_0|^2 d_R^\varepsilon), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \leq C \sum_{m' \leq m} \|\partial^{m-m'} |D_{u_0} d_0|^2\|_{L^6} \|\partial^{m'} d_R^\varepsilon\|_{L^6} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{L^2}$$

(3.166)

$$\leq C \|D_{u_0} d_0\|^2_{H^{N+1}} \|\nabla d_R^\varepsilon\|_{H^N} \|D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{H^N}$$

$$\leq C\varepsilon (1 + \|\nabla d_0\|^2_{H^{N+1}})(\|u_0\|^2_{H^{N+1}} + \|\Delta d_0\|_{H^{N+1}}) \mathcal{D}_{N,\varepsilon}(t) ,$$

$$\leq C\varepsilon (1 + \|\nabla d_0\|^2_{H^{N+2}})(\|u_0\|^2_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}) \mathcal{D}_{N,\varepsilon}(t) .$$

It is easily derived that

$$- 2 \left\langle \partial^m (D_{u_0} d_0 \cdot (D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon + u_R^\varepsilon \cdot \nabla d_0) d_0), \partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon \right\rangle \leq C \|D_{u_0} d_0 \otimes d_0\|_{L^\infty} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{L^2}$$

$$+ C \sum_{0 \neq m' \leq m} \|\partial^{m-m'} (D_{u_0} d_0 \otimes d_0)\|_{L^4} \|\partial^{m'} d_R^\varepsilon\|_{L^4} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{L^2}$$

(3.167)

$$+ C \sum_{m' \leq m} \|\partial^{m-m'} (D_{u_0} d_0 \otimes \nabla d_0)\|_{L^4} \|\partial^{m'} d_R^\varepsilon\|_{L^4} \|\partial^m D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{L^2}$$

$$\leq C \|D_{u_0} d_0\|_{H^N} (1 + \|\nabla d_0\|_{H^N}) \|D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{H^N}$$

$$+ C \|D_{u_0} d_0\|_{H^{N+1}} \|u_0\|^2_{H^{N+1}} (1 + \|\nabla d_0\|_{H^{N+1}}) \|\nabla u_R^\varepsilon\|_{H^N} \|D_{u_0 + \sqrt{\varepsilon} u_R^\varepsilon} d_R^\varepsilon\|_{H^N}$$

$$\leq C\varepsilon \|D_{u_0} d_0\|_{H^{N+1}} (1 + \|u_0\|^2_{H^{N+1}} + \|\nabla d_0\|^2_{H^{N+1}}) \mathcal{D}_{N,\varepsilon}(t) ,$$

$$\leq C\varepsilon (1 + \|u_0\|^2_{H^{N+2}} + \|\nabla d_0\|^2_{H^{N+2}})(\|u_0\|^2_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}) \mathcal{D}_{N,\varepsilon}(t)$$

for all $|m| \leq N$ and $0 < \varepsilon \leq 1$, where the bounds (3.64) is also utilized.
We now calculate that

\[- \lambda_2 \left< \partial^m [(A_R : d^e_R \otimes d^e_R) d_0], \partial^m D_{u_0 + \sqrt{u^e_R} d^e_R} \right> \]

and further, by the similar arguments as in (3.168), we immediately have

\[- \lambda_1 \left< \partial^m [A^e_R : (d^e_R \otimes d^e_R) d_0], \partial^m D_{u_0 + \sqrt{u^e_R} d^e_R} \right> \]

Furthermore, by the similar arguments as in (3.168), we immediately have

\[- \lambda_2 \left< \partial^m [(A_R : d^e_R \otimes d^e_R) d_0], \partial^m D_{u_0 + \sqrt{u^e_R} d^e_R} \right> \]

We now calculate that

\[- \lambda_2 \left< \partial^m [(A_R : d^e_R \otimes d^e_R) d_0], \partial^m D_{u_0 + \sqrt{u^e_R} d^e_R} \right> \]

The following bound holds for all \(|m| \leq N\) and \(0 < \varepsilon \leq 1: \)

\[- \sqrt{\varepsilon} \left< \partial^m [(D_{u_0 + \sqrt{u^e_R} d^e_R} d^e_R + u^e_R \cdot \nabla d_0)^2 d_0], \partial^m D_{u_0 + \sqrt{u^e_R} d^e_R} \right> \]

where the inequality (3.103) is also used. We then estimate that

\[- 2 \varepsilon \left< \partial^m [(D_{u_0 + \sqrt{u^e_R} d^e_R} d^e_R + u^e_R \cdot \nabla d_0)^2 d_0], \partial^m D_{u_0 + \sqrt{u^e_R} d^e_R} \right> \]
\[
\leq C \varepsilon (1 + \| \nabla d_0 \|^4_{H^{N+1}} + \| u_0 \|^4_{H^{N+1}}) (\| u_0 \|^2_{H^{N+1}} + \| \nabla d_0 \|^2_{H^{N+1}}) \phi_{N, \varepsilon}^2(t) \mathcal{D}_{N, \varepsilon}(t)
\]
\[
\leq C \varepsilon (1 + \| u_0 \|^5_{H^{N+1}} + \| \nabla d_0 \|^5_{H^{N+1}}) \phi_{N, \varepsilon}^3(t) \mathcal{D}_{N, \varepsilon}(t)
\]
for all \(|m| \leq N\) and \(0 < \varepsilon \leq 1\), where the last second inequality is implied the bound (3.103). Via the same calculations as in the bound (3.172), we have
\[
\leq C \varepsilon (1 + \| u_0 \|^2_{H^{N+1}} + \| \nabla d_0 \|^2_{H^{N+1}}) \phi_{N, \varepsilon}^2(t) \mathcal{D}_{N, \varepsilon}(t).
\]
It is easy to deduce that
\[
\leq \| \partial^m [u_R^t \cdot \nabla (D_{u_0} d_0)] \| L^2 \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \| L^2
\]
\[
\leq C \| u_R^t \|_{H^N} \| \nabla (D_{u_0} d_0) \|_{H^N} \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{H^N}
\]
\[
\leq C(1 + \| \nabla d_0 \|^3_{H^{N+1}})(\| \nabla u_0 \|_{H^{N+1}} + \| \Delta d_0 \|_{H^{N+1}}) \| u_R^t \|_{H^N} \| D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{H^N}
\]
\[
\leq C \varepsilon (1 + \| \nabla d_0 \|^3_{H^{N+1}})(\| \nabla u_0 \|_{H^{N+1}} + \| \Delta d_0 \|_{H^{N+1}}) \phi_{N, \varepsilon}^2(t) \mathcal{D}_{N, \varepsilon}(t),
\]
where the last second inequality is derived from the bound (3.64). We next estimate that
\[
\leq -\left\langle \partial^m [u_R^t \cdot \nabla (D_{u_0} d_0)], \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \right\rangle
\]
\[
= -\left\langle \partial^m (u_0 \cdot \nabla u_R^t \cdot \nabla d_0 + u_0 \cdot u_R^t \cdot \nabla \nabla d_0), \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \right\rangle
\]
\[
\leq C \| \nabla \partial^m u_R^t \|_{L^2} \| u_0 \cdot \nabla \nabla d_0 \|_{L^\infty} \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{L^2}
\]
\[
+ \sum_{0 \neq m' \leq m} \| \nabla \partial^m u_R^t \|_{L^2} \| u_0 \cdot \nabla \nabla d_0 \|_{L^4} \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{L^2}
\]
\[
+ C \sum_{m' \leq m} \| \partial^m (u_0 \cdot \nabla u_0 \cdot \nabla \nabla d_0) \|_{L^2} \| \partial^m u_R^t \|_{L^6} \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{L^2}
\]
\[
\leq C \| \nabla u_R^t \|_{H^N} \| D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{H^N} \| u_0 \|_{H^{N+2}} \| \nabla d_0 \|_{H^{N+2}} (1 + \| u_0 \|_{H^{N+2}})
\]
\[
\leq C \varepsilon \| u_0 \|_{H^{N+2}} \| \nabla d_0 \|_{H^{N+2}} (1 + \| u_0 \|_{H^{N+2}}) \mathcal{D}_{N, \varepsilon}(t).
\]
Furthermore, we have
\[
\leq -\sqrt{\varepsilon} \left\langle \partial^m [u_R^t \cdot \nabla (u_R^t \cdot \nabla d_0)], \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \right\rangle
\]
\[
\leq C \sqrt{\varepsilon} \| u_R^t \|_{L^2} \| \nabla \partial^m d_0 \|_{L^2} \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{L^2}
\]
\[
+ C \sqrt{\varepsilon} \sum_{0 \neq m' \leq m} \| \partial^m (u_R^t \otimes u_R^t) \|_{L^2} \| \partial^m \nabla \partial^m \nabla d_0 \|_{L^2} \| \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{L^2}
\]
\[
\leq C \sqrt{\varepsilon} \| \nabla d_0 \|_{H^{N+2}} \| u_R^t \|_{H^N} \| \nabla u_R^t \|_{H^N} \| D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \|_{H^N}
\]
\[
\leq C \sqrt{\varepsilon} \| \nabla d_0 \|_{H^{N+2}} \mathcal{D}_{N, \varepsilon}(t).
\]
Recalling the definition of \(R_d\) in (A.6) and collecting the previous bounds (3.165), (3.166), (3.167), (3.168), (3.169), (3.170), (3.171), (3.172), (3.173), (3.174), (3.175), (3.176) and (3.177), we obtain
\[
\left\langle \partial^m R_d, \partial^m D_{u_0 + \sqrt{u_R} d_R^\varepsilon} \right\rangle
\]
\[
\leq C \sqrt{\varepsilon} (1 + \| u_0 \|_{H^{N+2}}^5 + \| \nabla d_0 \|_{H^{N+2}}^5) \phi_{N, \varepsilon}^4(t) + \mathcal{D}_{N, \varepsilon}(t)
\]
\[
+ C \sqrt{\varepsilon} (1 + \| u_0 \|_{H^{N+2}}^5 + \| \nabla d_0 \|_{H^{N+2}}^5) (\| u_0 \|_{H^{N+2}}^2 + \| \nabla d_0 \|_{H^{N+2}}^2) \mathcal{D}_{N, \varepsilon}(t)
\]
\[
+ C \sqrt{\varepsilon} (1 + \| \nabla d_0 \|_{H^{N+2}}^2) (\| \nabla u_0 \|_{H^{N+2}}^2 + \| \Delta d_0 \|_{H^{N+2}}^2) \phi_{N, \varepsilon}^2(t) \mathcal{D}_{N, \varepsilon}(t).
\]
for all $|m| \leq N$ and $\varepsilon \in (0, \varepsilon_0]$.

By the similar estimates on the quantity $\left\langle \partial^m R_d, \partial^m D_{u_0^m + \sqrt{\varepsilon} \delta_R} d^m R \right\rangle$ in (3.178), we obtain

$$\left\langle \partial^m R_d, \partial^m (u^m_R \cdot \nabla d_0) \right\rangle$$

$$\leq C(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left[\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) + \varepsilon_{N,\varepsilon}^{\frac{3}{2}}(t)\right] \|u^m_R \cdot \nabla d_0\|_{H^N}$$

$$+ C(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \|u^m_R \cdot \nabla d_0\|_{H^N}$$

$$+ C(1 + \|\nabla d_0\|_{H^{N+2}})\left(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \|u^m_R \cdot \nabla d_0\|_{H^N}$$

(3.179)

$$\leq C\varepsilon(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left[\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) + \varepsilon_{N,\varepsilon}^{\frac{3}{2}}(t)\right] \mathcal{D}_{N,\varepsilon}(t)$$

$$+ C\varepsilon(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}\right)\mathcal{D}_{N,\varepsilon}(t)$$

$$+ C\varepsilon(1 + \|\nabla d_0\|_{H^{N+2}})\left(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \mathcal{D}_{N,\varepsilon}(t),$$

where we make use of the inequality (3.103), hence for $N \geq 2$

$$\|u^m_R \cdot \nabla d_0\|_{H^N} \leq C\|\nabla u^m_R\|_{H^N} \|\nabla d_0\|_{H^{N+1}} \leq C\varepsilon \|\nabla d_0\|_{H^{N+1}} \mathcal{D}_{N,\varepsilon}(t).$$

Notice that the bound

$$\|\partial^m d^m_R\|_{L^2} \leq \|\partial^m d^m_R\|_{L^6} |T^4|^{\frac{1}{6}} \leq C\|\nabla d^m_R\|_{H^N} \leq C\varepsilon \mathcal{D}_{N,\varepsilon}(t)$$

(3.180)

holds for all $|m| \leq N$. Then, from the similar arguments in (3.178) and the previous bound, we deduce that

$$\delta \left\langle \partial^m R_d, \partial^m d^m_R \right\rangle$$

$$\leq C(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left[\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) + \varepsilon_{N,\varepsilon}^{\frac{3}{2}}(t)\right] \|\partial^m d^m_R\|_{L^2}$$

$$+ C(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \|\partial^m d^m_R\|_{L^2}$$

$$+ C(1 + \|\nabla d_0\|_{H^{N+2}})\left(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \|\partial^m d^m_R\|_{L^2}$$

(3.181)

$$\leq C\varepsilon(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left[\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) + \varepsilon_{N,\varepsilon}^{\frac{3}{2}}(t)\right] \mathcal{D}_{N,\varepsilon}(t)$$

$$+ C\varepsilon(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}\right)\mathcal{D}_{N,\varepsilon}(t)$$

$$+ C\varepsilon(1 + \|\nabla d_0\|_{H^{N+2}})\left(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \mathcal{D}_{N,\varepsilon}(t).$$

Finally, from the inequalities (3.178), (3.179) and (3.181), we derive that

$$\mathcal{I}^{(t)}(R_d) = \sum_{|m| \leq N} \left\langle \partial^m R_d, \partial^m D_{u_0 + \sqrt{\varepsilon} \delta_R} d^m R - \partial^m (u^m_R \cdot \nabla d_0) + \delta \partial^m d^m R \right\rangle$$

$$\leq C\varepsilon(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left[\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) + \varepsilon_{N,\varepsilon}^{\frac{3}{2}}(t)\right] \mathcal{D}_{N,\varepsilon}(t)$$

(3.182)

$$+ C\varepsilon(1 + \|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})\left(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}}\right)\mathcal{D}_{N,\varepsilon}(t)$$

$$+ C\varepsilon(1 + \|\nabla d_0\|_{H^{N+2}})\left(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}}\right)\varepsilon_{N,\varepsilon}^{\frac{1}{2}}(t) \mathcal{D}_{N,\varepsilon}(t)$$

holds for all $N \geq 2$ and $0 < \varepsilon \leq \varepsilon_0$. 


As a consequence, substituting the inequalities (3.106), (3.107), (3.140), (3.164) and (3.182) into (3.102) reduces to

\[ I_N^{(4)} \leq C(1 + \|u_0\|_{H^{N+2}}^9 + \|\nabla d_0\|_{H^{N+2}}^9)(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}})D_{N,\varepsilon}^{\frac{1}{2}}(t) \]

\[ + C\sqrt{\varepsilon}(1 + \|u_0\|_{H^{N+2}}^6 + \|\nabla d_0\|_{H^{N+2}}^6)[\varepsilon_{N,\varepsilon}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^2(t)]D_{N,\varepsilon}(t) \]

\[ + C\sqrt{\varepsilon}(1 + \|u_0\|_{H^{N+2}}^8 + \|\nabla d_0\|_{H^{N+2}}^8)(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})D_{N,\varepsilon}(t) \]

\[ + C\sqrt{\varepsilon}(1 + \|\nabla d_0\|_{H^{N+2}}^3)(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}})\varepsilon_{N,\varepsilon}^\frac{1}{2}(t)D_{N,\varepsilon}^{\frac{1}{2}}(t) \]  

for all \( N \geq 2 \) and \( 0 < \varepsilon \leq \varepsilon_0 \).

**Step 5. Close the a priori uniform energy estimates.**

Via plugging the bounds (3.58), (3.95), (3.101) and (3.183) into the relation (3.34), we obtain

\[ \frac{1}{2} \frac{d}{dt}\varepsilon_{N,\varepsilon}(t) + D_{N,\varepsilon}(t) \]

\[ \leq C(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}})D_{N,\varepsilon}^{\frac{1}{2}}(t) \]

\[ + C(\|u_0\|_{H^{N+2}}^9 + \|\nabla d_0\|_{H^{N+2}}^9)(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}})D_{N,\varepsilon}^{\frac{1}{2}}(t) \]

\[ + C\sqrt{\varepsilon}(1 + \|u_0\|_{H^{N+2}}^6 + \|\nabla d_0\|_{H^{N+2}}^6)[\varepsilon_{N,\varepsilon}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^2(t)]D_{N,\varepsilon}(t) \]

\[ + C\sqrt{\varepsilon}(1 + \|u_0\|_{H^{N+2}}^8 + \|\nabla d_0\|_{H^{N+2}}^8)(\|u_0\|_{H^{N+2}} + \|\nabla d_0\|_{H^{N+2}})D_{N,\varepsilon}(t) \]

\[ + C\sqrt{\varepsilon}(1 + \|\nabla d_0\|_{H^{N+2}}^3)(\|\nabla u_0\|_{H^{N+2}} + \|\Delta d_0\|_{H^{N+2}})\varepsilon_{N,\varepsilon}^\frac{1}{2}(t)D_{N,\varepsilon}^{\frac{1}{2}}(t) \]  

for all \( 0 < \varepsilon \leq \varepsilon_0 \leq 1 \). Furthermore, by the relations (3.2) and the bound (3.3) we know that

\[ \|\nabla d_0\|_{H^{N+2}}^2 + \|u_0\|_{H^{N+2}}^2 \leq c_0^{-1}\varepsilon_{S_0,0}(t) \leq C(\beta_{S_0}(t)) \]

\[ \|\nabla u_0\|_{H^{N+2}}^2 + \|\Delta d_0\|_{H^{N+2}}^2 \leq c_0^{-1}\varepsilon_{S_0,0}(t) \]  

As a result, we deduce that for all \( N \geq 2 \) and \( 0 < \varepsilon \leq \varepsilon_0 \)

\[ \frac{1}{2} \frac{d}{dt}\varepsilon_{N,\varepsilon}(t) + D_{N,\varepsilon}(t) \]

\[ \leq C\varepsilon_{S_0,0}(t)D_{N,\varepsilon}^{\frac{1}{2}}(t) + C(\varepsilon_{S_0,0}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^\frac{1}{2}(t))D_{N,\varepsilon}^{\frac{1}{2}}(t) \]

\[ + C\varepsilon_{S_0,0}^\frac{1}{2}(t) + \varepsilon_{S_0,0}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^\frac{1}{2}(t)D_{N,\varepsilon}^{\frac{1}{2}}(t) \]  

which immediately implies by the Young’s inequality that

\[ \frac{d}{dt}\varepsilon_{N,\varepsilon}(t) + D_{N,\varepsilon}(t) \leq C\varepsilon_{S_0,0}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^\frac{1}{2}(t)D_{N,\varepsilon}^{\frac{1}{2}}(t) \]

\[ + C\varepsilon_{S_0,0}(t) + C(\varepsilon_{S_0,0}^\frac{1}{2}(t) + \varepsilon_{N,\varepsilon}^\frac{1}{2}(t))D_{N,\varepsilon}^{\frac{1}{2}}(t) \]  

for all \( 0 < \varepsilon \leq \varepsilon_0 \). Taking a large constant \( \theta_0 \gg 1 \) and adding the \( \theta_0 \) times of the differential inequality (3.4) to the previous inequality gives us

\[ \frac{d}{dt}\left[\varepsilon_{N,\varepsilon}(t) + \theta_0\varepsilon_{S_0,0}(t)\right] + D_{N,\varepsilon}(t) + \frac{\theta_0}{2}\varepsilon_{S_0,0}(t) \]

\[ \leq C[\varepsilon_{N,\varepsilon}^\frac{1}{2}(t) + \varepsilon_{S_0,0}^\frac{1}{2}(t) + \varepsilon_{S_0,0}^\frac{1}{2}(t)]D_{N,\varepsilon}(t) + \frac{\theta_0}{2}\varepsilon_{S_0,0}(t) \]  

for all \( N \geq 2 \) and \( 0 < \varepsilon \leq \varepsilon_0 \). Then the proof of Proposition 3.2 is finished. \( \Box \)
4. Global well-posedness of the remainder system: proof of Theorem 1.1

In this section, we aim at completing the proof of Theorem 1.1. Without loss of generality, based on the a priori energy estimate (3.17) derived in Proposition 3.2, we prove the global existence of the remainder system (1.33) with the initial data (1.8). Then, combining the solution \((u_0, d_0)\) to the limit equations (1.10) with initial conditions (1.13) given in Proposition 3.1, we know that \((u^\varepsilon, d^\varepsilon) = (u_0 + \sqrt{\varepsilon}u_R^\varepsilon, d_0 + \sqrt{\varepsilon}d_R^\varepsilon)\) obeys the first three equations of the system (1.1) with the initial data (1.8). We remark that the similar arguments will also justify the global existence to the remainder system (1.20)-(1.27), which is the remainder system with respect to the ill-posedness initial data. Thus, we obtain a global classical solution \((u^\varepsilon, d^\varepsilon) = (u_0 + \sqrt{\varepsilon}u_R^\varepsilon, d_0 + \varepsilon D^\varepsilon_f + \sqrt{\varepsilon}d_R^\varepsilon)\) to the original system (1.1)-(1.8). We omit the details of this case here.

We now introduce a mollifier over the periodic space variable. Recall that \(T^3 = \mathbb{R}^3/\mathbb{L}^3\), where \(\mathbb{L}^3 \subset \mathbb{R}^3\) is some 3-dimensional lattice. Let \(\varphi \in C^\infty(\mathbb{R}^3)\) be such that \(\varphi \geq 0\), \(\int_{\mathbb{R}^3} \varphi(x)dx = 1\), and \(\varphi(x) = 0\) for \(|x| > 1\). Then we define \(\varphi^\varepsilon(x) \in C^\infty(T^3)\) by

\[
\varphi^\varepsilon(x) = \frac{1}{\varepsilon} \sum_{l \in \mathbb{L}^3} \varphi\left(\frac{x-l}{\varepsilon}\right)
\]

for any \(\zeta > 0\). Then we define a mollifier \(J_\zeta\) as

\[
J_\zeta f(x) = \varphi^\varepsilon \ast f(x) = \int_{T^3} \varphi^\varepsilon(x-y)f(y)dy.
\]

Next we prove the main results of this paper.

**Proof of Theorem 1.1.** We first construct the following approximate system of the remainder equations (1.20)-(1.27)

\[
\begin{aligned}
\partial_t u_{R, \zeta}^\varepsilon - \frac{1}{2}H_4 J_\zeta \Delta J_\zeta u_{R, \zeta}^\varepsilon + \nabla P_{R, \zeta}^\varepsilon = \mu_1 J_\zeta \text{div}
\left((J_\zeta A_{R, \zeta}^\varepsilon : d_0 \otimes d_0) \otimes d_0\right)
+ \mathcal{J}_\zeta \mathcal{K}_{u, \zeta} + \mathcal{J}_\zeta \text{div}(C_{u, \zeta} + T_{u, \zeta} + \sqrt{\varepsilon}R_{u, \zeta}) ,
\end{aligned}
\]

\[
\text{div} u_{R, \zeta}^\varepsilon = 0 ,
\]

\[
\begin{aligned}
\partial_t \left(D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon} d_{R, \zeta}^\varepsilon\right) + \mathcal{J}_\zeta \left[(u_0 + \sqrt{\varepsilon}J_u u_{R, \zeta}^\varepsilon) \cdot \nabla \left(D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon} d_{R, \zeta}^\varepsilon\right)\right]
+ \frac{1}{\varepsilon} J_\zeta \Delta J_\zeta d_{R, \zeta}^\varepsilon + \partial_t \left(J_\zeta u_{R, \zeta}^\varepsilon \cdot \nabla d_0\right)
= \frac{1}{\varepsilon} J_\zeta \mathcal{C}_d, + \frac{1}{\varepsilon} J_\zeta S_{d, \zeta}^1 + \frac{1}{\varepsilon} S_{d, \zeta}^2 + \mathcal{J}_\zeta R_{d, \zeta},
\end{aligned}
\]

\[
\begin{aligned}
\partial_t d_{R, \zeta}^\varepsilon = \left(D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon} d_{R, \zeta}^\varepsilon\right) - \mathcal{J}_\zeta \left([u_0 + \sqrt{\varepsilon}J_u u_{R, \zeta}^\varepsilon] \cdot \nabla J_\zeta d_{R, \zeta}^\varepsilon\right)
\end{aligned}
\]

with the initial conditions

\[
(u_{R, \zeta}^\varepsilon, d_{R, \zeta}^\varepsilon, (D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon} d_{R, \zeta}^\varepsilon))|_{t=0} = (J_\zeta u_{R, \zeta}^{\varepsilon, \text{in}}, J_\zeta d_{R, \zeta}^{\varepsilon, \text{in}}, J_\zeta d_{R, \zeta}^{\varepsilon, \text{in}}),
\]

where

\[
A_{R, \zeta}^\varepsilon = \frac{1}{2}\left(\nabla u_{R, \zeta}^\varepsilon + (\nabla u_{R, \zeta}^\varepsilon)^T\right), \quad B_{R, \zeta}^\varepsilon = \frac{1}{2}\left(\nabla u_{R, \zeta}^\varepsilon - (\nabla u_{R, \zeta}^\varepsilon)^T\right),
\]

and the symbols \(\mathcal{K}_{u, \zeta}, \mathcal{C}_{u, \zeta}, \mathcal{T}_{u, \zeta}, \mathcal{C}_{R_{u, \zeta}}, \mathcal{C}_{d, \zeta}, \mathcal{S}_{d, \zeta}^1, \mathcal{S}_{d, \zeta}^2, \mathcal{R}_{d, \zeta}\) and \(\mathcal{R}_{d, \zeta}\) are the same form as \(\mathcal{K}_{u}, \mathcal{C}_{u}, \mathcal{T}_{u}, \mathcal{C}_{R_{u}}, \mathcal{C}_{d}, \mathcal{S}_{d}^1, \mathcal{S}_{d}^2\) and \(\mathcal{R}_{d}\) respectively (just replacing the symbols \(u_R^\varepsilon, d_R^\varepsilon\) and \(D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon} d_{R, \zeta}^\varepsilon\) with the corresponding symbols \(u_{R, \zeta}^\varepsilon, d_{R, \zeta}^\varepsilon\) and \(D_{u_0 + \sqrt{\varepsilon}u_R^\varepsilon} d_{R, \zeta}^\varepsilon\)). The previous approximate system can be regarded as an ordinary differential equation in \(H^N\) for any \(\varepsilon > 0\) by verifying the conditions of Cauchy-Lipschitz theorem. Thus it admits a unique solution \((u_{R, \zeta}^\varepsilon, d_{R, \zeta}^\varepsilon)\) in \(C([0, T_{\varepsilon, \zeta}); H^N]\) with the maximal time interval \([0, T_{\varepsilon, \zeta})\).
We define the following approximate energy \( \mathcal{E}^\xi_{N,\varepsilon}(t) \) and the approximate energy dissipative rate \( \mathcal{D}^\xi_{N,\varepsilon}(t) \):

\[
\mathcal{E}^\xi_{N,\varepsilon}(t) = \frac{1}{\varepsilon} \| u^\varepsilon_{R,\xi} \|_{H^N}^2 + (1 - \delta) \| (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) \|_{H^N}^2 + \frac{1}{\varepsilon} \| \nabla J^\xi_d\|_{H^N}^2 \\
+ \left( \frac{-\delta}{\varepsilon} \right) \| d^\xi_{R,\xi} \|_{H^N}^2 + \| J^\xi_d \cdot \nabla d^\xi_{R,\xi} \|_{H^N}^2 + \frac{\delta}{2} \| d^\xi_{R,\xi} \|_{H^N}^2 \\
+ \frac{1}{\varepsilon} \sum_{|m| \leq N} \left( \delta^m (J^\xi_d u^\varepsilon_{R,\xi} \cdot \nabla d^\varepsilon_{R,\xi}) , \delta^m (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) \right),
\]

and

\[
\mathcal{D}^\xi_{N,\varepsilon}(t) = \frac{\mu_4}{\varepsilon} \| \nabla J^\xi_d u^\varepsilon_{R,\xi} \|_{H^N}^2 + \frac{\mu_5}{\varepsilon} \| \nabla J^\xi_d d^\xi_{R,\xi} \|_{H^N}^2 + \delta \| (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) \|_{H^N}^2 \\
+ \frac{1}{\varepsilon} \sum_{|m| \leq N} \| \delta^m J^\xi_d A^\varepsilon_{R,\xi} \|_{L^2}^2 \\
+ \frac{1}{\varepsilon} (\mu_5 + \mu_6 + \lambda_3^2) \sum_{|m| \leq N} \| \delta^m J^\xi_d A^\varepsilon_{R,\xi} \|_{L^2}^2 \\
+ \frac{-\lambda_3}{\varepsilon} \sum_{|m| \leq N} \| \delta^m (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) + \delta^m J^\xi_d B^\varepsilon_{R,\xi} \|_{L^2}^2 ,
\]

where the constant \( \delta \in (0, \frac{1}{2}] \) is the same as that mentioned in \( \mathcal{E}^\xi_{N,\varepsilon}(t) \) and \( \mathcal{D}^\xi_{N,\varepsilon}(t) \). As shown in Lemma 3.1, when \( N \geq 2 \) and \( 0 < \varepsilon \leq \varepsilon_0 \), the approximate energy \( \mathcal{E}^\xi_{N,\varepsilon}(t) \) and the approximate energy dissipative rate \( \mathcal{D}^\xi_{N,\varepsilon}(t) \) are nonnegative. Moreover,

\[
\mathcal{E}^\xi_{N,\varepsilon}(t) \sim \frac{1}{\varepsilon} \| u^\varepsilon_{R,\xi} \|_{H^N}^2 + \| (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) \|_{H^N}^2 + \frac{1}{\varepsilon} \| \nabla J^\xi_d \|_{H^N}^2 + \frac{1}{\varepsilon} \| d^\xi_{R,\xi} \|_{H^N}^2 ,
\]

and

\[
\mathcal{D}^\xi_{N,\varepsilon}(t) \sim \frac{1}{\varepsilon} \| \nabla J^\xi_d u^\varepsilon_{R,\xi} \|_{H^N}^2 + \frac{1}{\varepsilon} \| \nabla J^\xi_d d^\xi_{R,\xi} \|_{H^N}^2 + \| (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) \|_{H^N}^2 \\
+ \frac{1}{\varepsilon} \sum_{|m| \leq N} \| \delta^m (D_{u_0} + \nabla u^\xi_{R,\xi} \cdot d^\xi_{R,\xi} \cdot \nabla) + \delta^m J^\xi_d B^\varepsilon_{R,\xi} \|_{L^2}^2 + \frac{\lambda_3}{\varepsilon} \| \delta^m J^\xi_d A^\varepsilon_{R,\xi} \|_{L^2}^2 .
\]

Via the similar arguments in Proposition 3.2, we can derive that for all \( 0 < \varepsilon \leq \varepsilon_0 \), \( \zeta > 0 \) and \( t \in [0, T_{\varepsilon,\zeta}] \)

\[
\frac{d}{dt} \left[ \mathcal{E}^\xi_{N,\varepsilon}(t) + \frac{\theta_0}{2} \mathcal{D}^\xi_{N,\varepsilon}(t) \right] + \mathcal{D}^\xi_{N,\varepsilon}(t) + \frac{\theta_0}{2} \mathcal{D}^\xi_{N,\varepsilon}(t) \leq C \left[ (\mathcal{E}^\xi_{N,\varepsilon}(t))^2 + (\mathcal{D}^\xi_{N,\varepsilon}(t))^2 + \frac{\theta_0}{2} \mathcal{D}^\xi_{N,\varepsilon}(t) \right] \left[ \mathcal{D}^\xi_{N,\varepsilon}(t) + \frac{\theta_0}{2} \mathcal{D}^\xi_{N,\varepsilon}(t) \right],
\]

where the constant \( C > 0 \), \( \varepsilon_0 \in (0, 1] \) and \( \theta_0 \gg 1 \) are mentioned in Proposition 3.2.

Next we prove that the maximal time \( T_{\varepsilon,\zeta} = +\infty \) under the small size constraint of the initial energy \( E^m \), defined in (1.39). More precisely, there is a small \( \zeta_0 > 0 \), independent of \( \varepsilon \) and \( \zeta \), such that if \( E^m \leq \zeta_0 \), we have \( T_{\varepsilon,\zeta} = +\infty \) and

\[
\mathcal{E}^\xi_{N,\varepsilon}(t) + \theta_0 \mathcal{D}^\xi_{N,\varepsilon}(t) + \frac{\theta_0}{2} \mathcal{D}^\xi_{N,\varepsilon}(t) \leq (1 + \theta_0)\zeta_0
\]

holds for all \( t \geq 0 \), \( \zeta > 0 \) and \( 0 < \varepsilon \leq \varepsilon_0 \).

Assume that \( T_{\varepsilon,\zeta} < +\infty \), We define a time number \( T^*_{\varepsilon,\zeta} \) as

\[
T^*_{\varepsilon,\zeta} = \sup \left\{ \tau \in [0, T_{\varepsilon,\zeta}) ; \sup_{t \in [0, \tau]} C \left[ (\mathcal{E}^\xi_{N,\varepsilon}(t))^2 + (\mathcal{D}^\xi_{N,\varepsilon}(t))^2 \right] \leq \frac{1}{2} \right\} \in [0, T_{\varepsilon,\zeta}) \cdot
\]
By Lemma 3.1 and the definition of the initial energy $E_{in}$ in (1.39), we have
\[
C[(\mathcal{E}_{N,\varepsilon}^\xi(0))^2 + (\mathcal{E}_{N,\varepsilon}^\xi(0))^\frac{1}{2} + \mathcal{E}_{S_N,0}^\frac{1}{2}(0)] \\
\leq C[(C_2 E_{in})^2 + 2\sqrt{C_2 E_{in}}] \\
\leq C[(C_2 \xi_0)^2 + 2\sqrt{C_2 \xi_0}]
\]
(4.10)
holds for all $0 < \varepsilon \leq \varepsilon_0$, where the constants $C_2$ and $\xi_0$ are given in Lemma 3.1. As listed in Proposition 3.1 of this paper, we first require the number $\xi_0 \leq \beta_{S_N,0}$ such that the initial energy bound $E_{in} \leq \xi_0$ guarantees the global existence results of Wang-Zhang-Zhang [21] to the limit system (1.10) with the initial data (1.13). We thus choose the small positive constant $\xi_0 \in (0, \beta_{S_N,0})$ such that
\[
C[(C_2 \xi_0)^2 + 2\sqrt{C_2 \xi_0}] \leq \frac{1}{4}.
\]
Specifically, one can take any $\xi_0 \in (0, \min\{1, \beta_{S_N,0}, \frac{1}{C_2(1+\theta_0)}\}) \subset (0,1]$. As a result, we know that if $E_{in} \leq \xi_0$, then for all $T_{\varepsilon,\xi} > 0$
\[
C[(\mathcal{E}_{N,\varepsilon}^\xi(0))^2 + (\mathcal{E}_{N,\varepsilon}^\xi(0))^\frac{1}{2} + \mathcal{E}_{S_N,0}^\frac{1}{2}(0)] \leq \frac{1}{4}.
\]
(4.11)
By the initial energy bound (4.11) and the continuity of the energy functional $\mathcal{E}_{N,\varepsilon}^\xi(t)$ in $t \in [0,T_{\varepsilon,\xi})$, we imply that $T_{\varepsilon,\xi} > 0$

We now claim that there is a small $\xi_0 \in (0, \min\{1, \beta_{S_N,0}, \frac{1}{C_2(1+\theta_0)}\})$ such that if $E_{in} \leq \xi_0$, then $T_{\varepsilon,\xi} = T_{\varepsilon,\xi}$ holds for all $0 < \varepsilon \leq \varepsilon_0$ and $\xi > 0$. Indeed, if $T_{\varepsilon,\xi} < T_{\varepsilon,\xi}$ for all $\xi_0 \in (0, \min\{1, \beta_{S_N,0}, \frac{1}{C_2(1+\theta_0)}\})$, the inequality (4.7) tells us that
\[
\frac{d}{dt}[\mathcal{E}_{N,\varepsilon}^\xi(t) + \theta_0 \mathcal{E}_{S_N,0}(t)] + \frac{1}{2}[\mathcal{D}_{N,\varepsilon}^\xi(t) + \frac{\theta_0}{2} \mathcal{D}_{S_N,0}(t)] \leq 0
\]
(4.12)
holds for all $t \in [0,T_{\varepsilon,\xi}^\ast]$ and for all $\xi > 0, \varepsilon \in (0,\varepsilon_0]$. Hence, integrating the previous differential inequality over $[0,t]$ reduces to
\[
\mathcal{E}_{N,\varepsilon}^\xi(t) + \theta_0 \mathcal{E}_{S_N,0}(t) + \frac{1}{2} \int_0^t \left[\mathcal{D}_{N,\varepsilon}^\xi(\tau) + \frac{\theta_0}{2} \mathcal{D}_{S_N,0}(\tau)\right] d\tau
\]
\[
\leq \mathcal{E}_{N,\varepsilon}^\xi(0) + \theta_0 \mathcal{E}_{S_N,0}(0)
\]
\[
\leq (1 + \theta_0)[\mathcal{E}_{N,\varepsilon}^\xi(0) + \mathcal{E}_{S_N,0}(0)]
\]
\[
\leq (1 + \theta_0)C_2 \xi_0
\]
(4.13)
holds for all $t \in [0,T_{\varepsilon,\xi}^\ast]$ and for all $\xi > 0, \varepsilon \in (0,\varepsilon_0]$, which means that
\[
\mathcal{E}_{N,\varepsilon}^\xi(t) + \theta_0 \mathcal{E}_{S_N,0}(t) \leq (1 + \theta_0)C_2 \xi_0.
\]
(4.14)
If we choose a fixed $\xi_0 = \min\{1, \beta_{S_N,0}, \frac{1}{C_2(1+\theta_0)}\} \in (0, \min\{1, \beta_{S_N,0}, \frac{1}{C_2(1+\theta_0)}\})$, then the previous energy bound implies that under the constraint $E_{in} \leq \xi_0$
\[
C[(\mathcal{E}_{N,\varepsilon}^\xi(t))^2 + (\mathcal{E}_{N,\varepsilon}^\xi(t))^\frac{1}{2} + \mathcal{E}_{S_N,0}^\frac{1}{2}(t)] \\
\leq C[(C_2(1 + \theta_0)\xi_0)^2 + 2\sqrt{C_2(1 + \theta_0)\xi_0}] \\
\leq \frac{1}{4} < \frac{1}{2}
\]
(4.15)
holds for all $t \in [0,T_{\varepsilon,\xi}^\ast]$. Thus the continuity of $\mathcal{E}_{N,\varepsilon}^\xi(t)$ yields that there exists a $t^* > T_{\varepsilon,\xi}^\ast$ such that
\[
\sup_{t \in [0,t^*]} C[(\mathcal{E}_{N,\varepsilon}^\xi(t))^2 + (\mathcal{E}_{N,\varepsilon}^\xi(t))^\frac{1}{2} + \mathcal{E}_{S_N,0}^\frac{1}{2}(t)] \leq \frac{1}{2},
\]
which contradicts to the definition of $T_{\varepsilon,\xi}^\ast$. Consequently, we have $T_{\varepsilon,\xi}^* = T_{\varepsilon,\xi} < +\infty$.

Therefore, it must hold that at time $t = T_{\varepsilon,\xi}$
\[
\mathcal{E}_{N,\varepsilon}^\xi(t) + \mathcal{E}_{S_N,0}^\frac{1}{2}(t) < +\infty.
\]
(4.16)
We then can extend the solution \((u^{\varepsilon}_{R,\xi}, d^{\varepsilon}_{R,\xi})\) of the approximate system \((4.1)\) to a larger interval \([0, T_{\varepsilon,\xi} + \kappa]\) for some \(\kappa > 0\). This contradicts to the maximality of \(T_{\varepsilon,\xi}\). As a consequence, there is a small \(\xi_0 > 0\), independent of \(\varepsilon\) and \(\xi\), such that if \(E^{in} \leq \xi_0\) for all \(\varepsilon \in (0, \varepsilon_0]\), we have \(T_{\varepsilon,\xi} = +\infty\) and

\[
\sup_{t \geq 0} \left[ \mathcal{E}_{N,\varepsilon}^\xi (t) + \mathcal{E}_{S_{N,0}}^\varepsilon (t) \right] + \frac{1}{2} \int_0^\infty \left[ \mathcal{D}_{N,\varepsilon}^\xi (\tau) + \mathcal{D}_{S_{N,0}}^\varepsilon (\tau) \right] d\tau \leq (1 + \theta_0) \xi_0
\]

holds for all \(\xi > 0\) and \(0 < \varepsilon \leq \varepsilon_0\).

Then, by compactness arguments (let \(\xi \to 0\)), we get vector field \((u^R_{\varepsilon}, d^R_{\varepsilon}) \in \mathbb{R}^3 \times \mathbb{R}^3\) satisfying

\[
u^R_{\varepsilon}, D_{u^\varepsilon} + \sqrt{\varepsilon u^\varepsilon_{R,\xi}}, \nabla d^R_{\varepsilon}, d^R_{\varepsilon} \in L^\infty(\mathbb{R}^+; H^N), \ \nabla u^R_{\varepsilon} \in L^2(\mathbb{R}^+; H^N)
\]

for all \(0 < \varepsilon \leq \varepsilon_0\), which solves the remainder system \((1.20)-(1.27)\). Moreover, \((u^R_{\varepsilon}, d^R_{\varepsilon})\) obeys the energy bound

\[
\sup_{t \geq 0} \left( \frac{1}{2} \left\| u^\varepsilon_R \right\|_{H^N}^2 + \left\| D_{u^\varepsilon} + \sqrt{\varepsilon u^\varepsilon_{R,\xi}} \right\|_{H^N}^2 + \frac{1}{2} \left\| d^\varepsilon_R \right\|_{H^{N+1}}^2 \right) (t) + \frac{1}{2} \int_0^\infty \left\| \nabla u^\varepsilon_R \right\|_{H^N}^2 dt \leq C \xi_0
\]

for some \(C > 0\), which is uniform in \(\varepsilon \in (0, \varepsilon_0]\). It is easy to know that

\[
2(d_0 \cdot d^R_{\varepsilon})(0, x) + \sqrt{\varepsilon} |d^R_{\varepsilon}|^2 (0, x) = 0.
\]

As a consequence, Lemma 2.2 (or Remark 2.1) implies the constraint \((1.34)\) holds at any time. Then the proof of Theorem 1.1 is finished. \(\square\)

**Appendix A. Remainder systems**

In this section, we will present the tedious terms of the remainder system \((1.20)\). After submitting the ansatz \((1.15)\) into the hyperbolic Ericksen-Leslie’s liquid crystal model \((1.1)\), one obtain the reminder system \((1.20)\), namely,

\[
\begin{aligned}
\partial_t u^R_{\varepsilon} - \frac{1}{2} \mu_4 \Delta u^R_{\varepsilon} + \nabla p^R_{\varepsilon} &= \mu_1 \text{div} \left( (A_{\varepsilon}^R : d_0 \otimes d_0) d_0 \otimes d_0 \right) \\
+ \mathcal{K}_u + \text{div} \left( C_u + T_u + \sqrt{\varepsilon} R_{\varepsilon} \right) + \varepsilon \text{div} Q_u(D_I), \\
- \text{div} u^R_{\varepsilon} &= 0, \\
\frac{D^2}{\varepsilon^2} u^\varepsilon_{R,\xi} + \frac{\lambda_1}{\varepsilon} D_{u^\varepsilon} u^\varepsilon_{R,\xi} + \frac{\lambda_2}{\varepsilon} \Delta d^\varepsilon_R + \partial_t (u^R_{\varepsilon} \cdot \nabla d_0 + \sqrt{\varepsilon} u^\varepsilon_{R,\xi} \cdot \nabla D_I) &= \frac{1}{\varepsilon} C_d + \frac{1}{\varepsilon^2} S_d^2 + \frac{1}{\varepsilon^2} \nabla d^\varepsilon_R + R_d + Q_d(D_I)
\end{aligned}
\]

with the constraint

\[
2(d_0 \cdot (d^R_{\varepsilon} + \sqrt{\varepsilon} D_I)) + \sqrt{\varepsilon} |d^R_{\varepsilon} + \sqrt{\varepsilon} D_I|^2 = 0, \quad (A.1)
\]

where the tensor \(C_u\) is defined in \((1.22)\) and the vector field \(C_d\) is given in \((1.23)\), the linear tensor term \(T_u\) is

\[
T_u = \mu_1 \left( (A_0 : (d^R_{\varepsilon} \otimes d_0 + d_0 \otimes d^R_{\varepsilon}) d_0 \otimes d_0 + (A_0 : d_0 \otimes d_0) (d_0 \otimes d^R_{\varepsilon} + d^R_{\varepsilon} \otimes d_0) \right)
+ \mu_2 \left( (D_{u^\varepsilon} d_0 + B_0 d_0) \otimes d^R_{\varepsilon} + (B_0 d^R_{\varepsilon} + u^R_{\varepsilon} \cdot \nabla d_0) \otimes d_0 \right)
+ \mu_3 \left( d^R_{\varepsilon} \otimes (D_{u^\varepsilon} d_0 + B_0 d_0) + d_0 \otimes (B_0 d^R_{\varepsilon} + u^R_{\varepsilon} \cdot \nabla d_0) \right)
+ \mu_5 \left( (A_0 d_0) \otimes d^R_{\varepsilon} + (A_0 d^R_{\varepsilon}) \otimes d_0 \right) + \mu_6 \left( d^R_{\varepsilon} \otimes (A_0 d_0) + d_0 \otimes (A_0 d^R_{\varepsilon}) \right), \quad (A.2)
\]

the linear vector field \(K_u\) is

\[
K_u = - u_0 \cdot \nabla u^R_{\varepsilon} - u^R_{\varepsilon} \cdot \nabla u_0 - \sqrt{\varepsilon} u^\varepsilon_{R,\xi} \cdot \nabla u_{\varepsilon},
- \text{div} (\nabla d_0 \otimes \nabla u^R_{\varepsilon} + \nabla d^R_{\varepsilon} \otimes \nabla d_0 + \sqrt{\varepsilon} \nabla d^R_{\varepsilon} \otimes \nabla d^R_{\varepsilon}), \quad (A.3)
\]

the singular linear term \(S_d^1\) is of the form

\[
S_d^1 = 2(\nabla d_0 \otimes \nabla d_0) d_0 + |\nabla d_0|^2 d^R_{\varepsilon} + \lambda_1 (u^R_{\varepsilon} \cdot \nabla d_0 + B_0 d^R_{\varepsilon})
+ \lambda_2 \left( A_0 d^R_{\varepsilon} - (A^R_{\varepsilon} : d_0 \otimes d_0) d_0 - (A_0 : d_0 \otimes d_0) d^R_{\varepsilon} - 2(A_0 : d_0 \otimes d^R_{\varepsilon}) d_0 \right), \quad (A.4)
\]
the nonsingular nonlinear term $S_d^2$ is defined as
\[ S_d^2 = \| \nabla \tilde{d}_R^2 \|_2^2 d_0 + 2(\nabla d_0 \cdot \nabla \tilde{d}_R^2) d_0^2 - D_{n u} d_0 - |D_{n u} d_0|^2 d_0 + \lambda_1 B_R^2 d_R^2 \]
\[ + \lambda_2 [A_R^2 d_R^2 - A_0 : (d_0 \otimes d_0 + d_R^2 \otimes d_0) d_R^2 - (A_R^2 : d_0 \otimes d_0) d_R^2] \]  
\[ - \lambda_2 [A_R^2 : (d_R^2 \otimes d_0 + d_0 \otimes d_R^2) d_R^2 - (A_0 : d_R^2 \otimes d_R^2) d_R^2] \quad \text{(A.5)} \]
the nonsingular nonlinear term $\mathcal{R}_d$ is
\[ \mathcal{R}_d = (\| \nabla \tilde{d}_R^2 \|_2^2 - |D_{n u} d_0|^2 d_0^2) d_R^2 - 2[D_{n u} d_0 \cdot (D_{n u} + \sqrt{\epsilon_0} \tilde{d}_R^2) d_R^2 + \tilde{u}_R^2 \cdot \nabla d_0] d_0 \]
\[ - \lambda_2 [A_R^2 : d_R^2 \otimes d_0 + A_0 : d_0 \otimes d_R^2] + (d_R^2 \otimes d_0 + d_0 \otimes d_R^2) d_R^2 + (A_0 : d_R^2 \otimes d_R^2) d_R^2 \]
\[ - \sqrt{\epsilon} [D_{n u} + \sqrt{\epsilon_0} \tilde{d}_R^2 d_R^2 + \tilde{u}_R^2 \cdot \nabla d_0]^2 d_0 + 2\lambda_2 (A_R^2 : d_R^2 \otimes d_R^2) d_R^2 \]
\[ + 2D_{n u} d_0 \cdot (D_{n u} + \sqrt{\epsilon_0} \tilde{d}_R^2 d_R^2 + \tilde{u}_R^2 \cdot \nabla d_0) d_R^2 \quad \text{(A.6)} \]
and the nonsingular nonlinear tensor $\mathcal{R}_u$ is
\[ \mathcal{R}_u = M_1 + \sqrt{\epsilon} M_2 + \sqrt{\epsilon^2} M_3 + \sqrt{\epsilon^3} M_4 . \quad \text{(A.7)} \]
Here the term $M_1$ is
\[ M_1 = M_1^2 [(A_0 : d_R^2 \otimes d_R^2 + 2A_R^2 : d_0 \otimes d_R^2) d_0 \otimes d_0 + (A_0 : d_0 \otimes d_0) d_R^2 \otimes d_R^2 \]
\[ + (2A_0 : d_0 \otimes d_R^2 + A_R^2 : d_0 \otimes d_0) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \]
\[ + \lambda_2 [B_R^2 : d_R^2 \otimes d_R^2 + (2A_0 : d_R^2 \otimes d_R^2 + A_R^2 : d_0 \otimes d_0) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \]
\[ + \lambda_2 [B_R^2 : d_R^2 \otimes d_R^2 + (2A_0 : d_R^2 \otimes d_R^2 + A_R^2 : d_0 \otimes d_0) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \quad \text{(A.8)} \]
the term $M_2$ is
\[ M_2 = M_2^2 [(A_0^2 : d_0 \otimes d_0 + 2A_R^2 : d_0 \otimes d_R^2) d_R^2 \otimes d_R^2 + (A_0 : d_R^2 \otimes d_R^2) d_0 \otimes d_0 \]
\[ + (2A_R^2 : d_0 \otimes d_R^2 + A_0 : d_R^2 \otimes d_0) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \]
\[ + \lambda_2 [B_R^2 : d_R^2 \otimes d_R^2 + (2A_0^2 : d_0 \otimes d_0 + A_R^2 : d_0 \otimes d_0) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \quad \text{(A.9)} \]
the term $M_3$ is
\[ M_3 = M_3^2 [(2A_R^2 : d_0 \otimes d_R^2) d_R^2 \otimes d_R^2 + (A_0 : d_R^2 \otimes d_R^2) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \]
\[ + (A_0 : d_R^2 \otimes d_R^2) (d_R^2 \otimes d_0 + d_0 \otimes d_R^2)] \quad \text{(A.10)} \]
the term $M_4$ is
\[ M_4 = M_4^2 (A_R^2 : d_R^2 \otimes d_R^2) d_R^2 \otimes d_R^2 \quad \text{(A.11)} \]
Moreover, the tensor term $Q_u(D_I)$ involving initial layer in the $u_R^2$-equation of (1.20) reads
\[ Q_u(D_I) = \frac{1}{\epsilon^2} Q_u^1 + \sqrt{\epsilon} Q_u^2 + \sqrt{\epsilon^2} Q_u^3 + \sqrt{\epsilon^3} Q_u^4 + \sqrt{\epsilon^4} Q_u^5 + \sqrt{\epsilon^5} Q_u^6 + \sqrt{\epsilon^6} Q_u^7 + \sqrt{\epsilon^7} Q_u^8 \quad \text{(A.12)} \]
where the term $Q_u^1$ is
\[ Q_u^1 = - \nabla d_0 \otimes \nabla D_R^2 + \nabla D_R^2 \otimes \nabla d_0 \]
\[ + \lambda_2 [2(A_0 : d_0 \otimes D_I^2) d_0 \otimes d_0 + (A_0 : d_0 \otimes d_0) (D_R^2 \otimes d_0 + d_0 \otimes D_I^2)] \]
\[ + \lambda_2 [2(D_{n u} d_0 + B_0 d_0) \otimes D_I^2 + (D_{n u} D_I^2 + B_0 D_I^2) \otimes d_0] \]
\[ + \lambda_2 [2(D_I^2 \otimes (D_{n u} d_0 + B_0 d_0) + d_0 \otimes (D_{n u} D_I^2 + B_0 D_I^2)] \]
\[ + \lambda_2 [(A_0 d_0) \otimes D_I^2 + (A_0 D_I^2) \otimes d_0] + \mu_0 [D_I^2 \otimes (A_0 d_0) + d_0 \otimes (A_0 D_I^2)] \quad \text{(A.13)} \]
the term $Q_u^2$ is
\[
Q_u^2 = -\nabla D_I^e \otimes \nabla d_R^e - \nabla d_R^e \otimes \nabla D_I^e + 2\mu_1(A_0 : D_I^e \otimes D_I^e)(d_R^e \otimes d_0 + d_0 \otimes d_R^e) \\
+ 2\mu_1(A_R^e : D_I^e \otimes d_0 + A_0 : D_I^e \otimes d_R^e)d_0 \otimes d_0 \\
+ \mu_1(A_R^e : d_0 \otimes d_0 + 2A_0 : d_0 \otimes d_R^e)(D_I^e \otimes d_0 + d_0 \otimes D_I^e) \\
+ \mu_2(u_R^e \cdot \nabla d_0 + D_{u_0} + \sqrt{\tau_{u_0}} d_R^e + B_R^e d_0 + B_0 d_R^e) \otimes D_I^e \\
+ \mu_2[(u_R^e \cdot \nabla D_I^e + B_R^e D_I^e) \otimes d_0 + (D_{u_0} D_I^e + B_0 D_I^e) \otimes d_R^e] \\
+ \mu_3 D_I^e \otimes (u_R^e \cdot \nabla D_I^e + d_R^e (u_R^e \cdot \nabla D_I^e + B_R^e D_I^e) + d_R^e (D_{u_0} D_I^e + B_0 D_I^e)] \\
+ \mu_5 [(A_0 d_R^e + A_R^e d_0) \otimes D_I^e + (A_0 D_I^e) \otimes d_R^e + (A_R^e D_I^e) \otimes d_0] \\
+ \mu_6 [D_I^e \otimes (A_0 d_R^e + A_R^e d_0) + d_R^e \otimes (A_0 D_I^e) + d_0 \otimes (A_R^e D_I^e)], \\
\] (A.14)

the term $Q_u^3$ is
\[
Q_u^3 = -\nabla D_I^e \otimes \nabla D_I^e + \mu_1[(A_0 : D_I^e \otimes D_I^e)d_0 \otimes d_0 + (A_0 : d_0 \otimes d_0)D_I^e \otimes D_I^e] \\
+ \mu_1[2(A_0 : D_I^e \otimes d_0)(D_I^e \otimes d_0 + d_0 \otimes D_I^e) + 2(A_R^e : D_I^e \otimes d_R^e)d_0 \otimes d_0] \\
+ \mu_1[2(A_0 : D_I^e \otimes d_0)d_R^e \otimes d_R^e + 2(A_R^e : D_I^e \otimes d_0)(d_0 \otimes d_R^e + d_R^e \otimes d_0)] \\
+ \mu_1(2A_0 : D_I^e \otimes d_R^e + A_R^e : D_I^e \otimes d_R^e)(d_0 \otimes d_R^e + d_R^e \otimes d_0) \\
+ \mu_1(A_0 : d_R^e \otimes d_R^e + 2A_R^e : d_R^e \otimes d_0)(d_0 \otimes D_I^e + D_I^e \otimes d_0) \\
+ \mu_1(A_R^e : d_0 \otimes d_0 + 2A_0 : d_0 \otimes d_R^e)(d_R^e \otimes D_I^e + D_I^e \otimes d_R^e) \\
+ \mu_2[(D_{u_0} D_I^e + B_0 D_I^e) \otimes D_I^e + (B_R^e d_R^e) \otimes D_I^e + (u_R^e \cdot \nabla D_I^e + B_R^e D_I^e) \otimes d_R^e] \\
+ \mu_3(D_I^e (d_0 \otimes d_R^e + d_R^e \otimes d_0) + d_R^e \otimes (u_R^e \cdot \nabla D_I^e + B_R^e D_I^e)) \\
+ \mu_5[(A_0 D_I^e) \otimes D_I^e + (A_R^e d_R^e) \otimes D_I^e + (A_R^e D_I^e) \otimes d_R^e] \\
+ \mu_6[D_I^e \otimes (A_0 D_I^e) + D_I^e \otimes (A_R^e d_R^e) + d_R^e \otimes (A_R^e D_I^e) \otimes d_R^e], \\
\] (A.15)

the term $Q_u^4$ is
\[
Q_u^4 = \mu_1[(A_R^e : D_I^e \otimes D_I^e)d_0 \otimes d_0 + (A_R^e : d_0 \otimes d_0 + 2A_0 : d_0 \otimes d_R^e)D_I^e \otimes D_I^e] \\
+ \mu_1[(A_0 : D_I^e \otimes D_I^e)(d_R^e \otimes d_0 + d_0 \otimes d_R^e) + 2(A_0 : D_I^e \otimes d_0)(d_R^e \otimes D_I^e + D_I^e \otimes d_0)] \\
+ 2\mu_1(A_R^e : D_I^e \otimes d_0 + A_0 : D_I^e \otimes d_R^e)(D_I^e \otimes d_0 + d_0 \otimes D_I^e) \\
+ 2\mu_1(A_0 : D_I^e \otimes d_R^e + A_R^e : D_I^e \otimes d_R^e)d_R^e \otimes d_R^e \\
+ \mu_1(2A_R^e : d_R^e \otimes d_0 + 2A_0 : d_R^e \otimes d_0)(D_I^e \otimes d_0 + d_0 \otimes D_I^e) \\
+ \mu_1(A_R^e : d_R^e \otimes d_R^e + 2A_0 : d_R^e \otimes d_0)(D_I^e \otimes d_R^e + d_R^e \otimes D_I^e) \\
+ \mu_1(A_R^e : d_R^e \otimes d_R^e)(d_R^e \otimes d_0 + d_0 \otimes d_R^e) + (A_R^e : d_R^e \otimes d_R^e)(D_I^e \otimes d_R^e + d_R^e \otimes D_I^e)] \\
+ \mu_2(u_R^e \cdot \nabla D_I^e + B_R^e D_I^e) \otimes D_I^e + \mu_3 D_I^e \otimes (u_R^e \cdot \nabla D_I^e + B_R^e D_I^e) \\
+ \mu_5(A_R^e D_I^e) \otimes D_I^e + \mu_6 D_I^e \otimes (A_R^e D_I^e), \\
\] (A.16)

the term $Q_u^5$ is
\[
Q_u^5 = \mu_1[2(A_0 : D_I^e \otimes d_0)D_I^e \otimes D_I^e + (A_0 : D_I^e \otimes D_I^e)(D_I^e \otimes d_0 + d_0 \otimes D_I^e)] \\
+ \mu_1[(A_0 : D_I^e \otimes D_I^e)(d_R^e \otimes d_0 + 2(A_R^e : D_I^e \otimes d_R^e)(d_0 \otimes D_I^e + D_I^e \otimes d_0)] \\
+ \mu_1(2A_R^e : d_R^e \otimes d_0 + A_0 : d_R^e \otimes d_R^e)D_I^e \otimes D_I^e \\
+ 2\mu_1(A_R^e : D_I^e \otimes d_0 + A_0 : D_I^e \otimes d_R^e)(D_I^e \otimes d_R^e + d_R^e \otimes D_I^e) \\
+ \mu_1[2(A_R^e : D_I^e \otimes d_R^e)d_R^e \otimes d_R^e + (A_R^e : d_R^e \otimes d_R^e)(D_I^e \otimes d_R^e + d_R^e \otimes D_I^e)], \\
\] (A.17)
the term $Q_0^\alpha$ is
\[ Q_0^\alpha = \mu_1 \left[ 2(A_R : D_f^\alpha \otimes d_0)D_f^\alpha \otimes D_f^\alpha + (A_R : D_f^\alpha \otimes D_f^\alpha)(d_0 \otimes D_f^\alpha + D_f^\alpha \otimes d_0) \right] \\
+ \mu_1 \left[ 2(A_0 : D_f^\alpha \otimes d_0)D_f^\alpha \otimes D_f^\alpha + (A_0 : D_f^\alpha \otimes D_f^\alpha)(d_0 \otimes D_f^\alpha + D_f^\alpha \otimes d_0) \right] \\
+ \mu_1 \left[ (A_R : D_f^\alpha \otimes D_f^\alpha) d_0 \otimes d_R + (A_0 : d_0 \otimes D_f^\alpha)D_f^\alpha \otimes D_f^\alpha \right] \\
+ 2\mu_1(A_R : D_f^\alpha \otimes d_R)(d_R \otimes D_f^\alpha + D_f^\alpha \otimes d_R), \]
(A.18)
and the term $Q_0^\alpha$ is
\[ Q_0^\alpha = \mu_1(A_R : D_f^\alpha \otimes D_f^\alpha)D_f^\alpha \otimes D_f^\alpha. \]
(A.19)
Finally, the vector field term $Q_d(D_f)$ involving the initial layer structure in the $d_R$-equation of \((A.20)\) is defined as
\[ Q_d(D_f) = \frac{1}{\sqrt{\varepsilon}} Q_d^1 + \sqrt{\varepsilon} Q_d^2 + \sqrt{\varepsilon^2} Q_d^3 + \sqrt{\varepsilon^4} Q_d^4 + \sqrt{\varepsilon^6} Q_d^5 + \sqrt{\varepsilon^8} Q_d^5, \]
where the term $Q_d^1$ is
\[ Q_d^1 = \lambda_1(u_0 \cdot \nabla D_f^\lambda + B_0 D_f^\lambda) + \lambda_2 A_0 D_f^\lambda + \gamma_0 D_f^\lambda + 2(\nabla d_0 \cdot \nabla D_f^\lambda - \lambda_2 A_0 : D_f^\alpha \otimes d_0)D_f^\alpha, \]
(A.22)
the term $Q_d^2$ is
\[ Q_d^2 = \lambda_1(u_R^\alpha : D_f^\alpha + B_R^\alpha D_f^\alpha) + \lambda_2 A_R^\alpha D_f^\alpha + 2(\nabla d_0 \cdot \nabla D_f^\alpha - \lambda_2 A_0 : D_f^\alpha \otimes d_0)d_R^\alpha \\
+ (2\nabla D_f^\alpha \cdot \nabla d_R^\alpha - \lambda_2 A_0 : D_f^\alpha \otimes d_R^\alpha - 2\lambda_2 A_R^\alpha : D_f^\alpha \otimes d_0)d_0 \\
+ (2\nabla d_0 \cdot \nabla d_R^\alpha - \lambda_2 A_0 : d_0 \otimes d_R^\alpha - \lambda_2 A_R^\alpha : d_0 \otimes d_0)D_f^\alpha, \]
(A.23)
the term $Q_d^3$ is
\[ Q_d^3 = -2u_0 \cdot \nabla \delta d_f^\lambda - \partial T u_0 \cdot \nabla D_f^\lambda - u_0 \cdot \nabla(u_0 \cdot \nabla D_f^\lambda) \\
+ ((\nabla D_f^\alpha)^2 - \lambda_2 A_0 : D_f^\alpha \otimes D_f^\alpha)D_f^\alpha \\
+ 2(\nabla d_0 \cdot \nabla D_f^\alpha - \lambda_2 A_0 : D_f^\alpha \otimes d_0 - 2\lambda_2 A_R^\alpha : D_f^\alpha \otimes d_0)D_f^\alpha \\
- \partial u_0 \cdot \nabla D_f^\alpha + 2(\nabla d_0 \cdot D_f^\alpha - \lambda_2 A_0 : d_0 \otimes d_0)D_f^\alpha \\
- 2(\delta D_f^\alpha \cdot (u_0^\alpha \cdot \nabla D_f^\alpha) + (u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) \cdot \nabla u_0^\alpha D_f^\alpha) \\
+ \partial u_0 \cdot \nabla D_f^\alpha + 2(\nabla d_0 \cdot (u_0^\alpha \cdot D_f^\alpha) + (u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) \cdot D_0 D_f^\alpha) \\
- \partial D_0 \cdot (u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) D_f^\alpha - 2(\delta D_0 \cdot D_0 D_f^\alpha) D_f^\alpha, \]
(A.24)
the term $Q_d^4$ is
\[ Q_d^4 = (\nabla D_f^\alpha)^2 - \lambda_2 A_0 : D_f^\alpha \otimes D_f^\alpha - \lambda_2 A_R^\alpha : D_f^\alpha \otimes D_f^\alpha \\
+ 2(\nabla d_0 \cdot \nabla D_f^\alpha - \lambda_2 A_0 : D_f^\alpha \otimes d_0 - 2\lambda_2 A_R^\alpha : D_f^\alpha \otimes d_0 - 2\lambda_2 A_R^\alpha : D_f^\alpha \otimes d_0)D_f^\alpha \\
- \partial u_0 \cdot \nabla D_f^\alpha + 2(\nabla d_0 \cdot D_0 - \lambda_2 A_0 : d_0 \otimes d_0)D_f^\alpha \\
- 2(\delta D_0 \cdot (u_0^\alpha \cdot \nabla D_f^\alpha) + (u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) \cdot D_0 D_f^\alpha) \\
+ \partial u_0 \cdot \nabla D_f^\alpha + 2(\nabla d_0 \cdot (u_0^\alpha \cdot D_0) + (u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) \cdot D_0 D_f^\alpha) \\
- \partial D_0 \cdot (u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) D_f^\alpha - 2(\delta D_0 \cdot D_0 D_f^\alpha) D_f^\alpha, \]
(A.25)
the term $Q_d^5$ is
\[ Q_d^5 = (\nabla D_f^\alpha)^2 - \lambda_2 A_0 : D_f^\alpha \otimes D_f^\alpha - |D_0 D_f^\alpha|^2 d_0 - 2(\delta D_f^\alpha \cdot D_0 D_f^\alpha)D_f^\alpha \\
- 2(\delta D_0 \cdot D_0 D_f^\alpha) d_0 + 2(u_0^\alpha \cdot \nabla d_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) \cdot (u_0^\alpha \cdot \nabla D_f^\alpha) \\
- 2(\delta D_0 \cdot (u_0^\alpha \cdot \nabla D_f^\alpha) + (u_0^\alpha \cdot \nabla d_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha) \cdot D_0 D_f^\alpha) \\
- \partial u_0^\alpha \cdot \nabla d_0 + D_0 + \sqrt{\varepsilon} u_R^\alpha d_R^\alpha |D_0 D_f^\alpha|^2 D_f^\alpha, \]
(A.26)
the term $Q_4^d$ is

$$
Q_4^d = - \lambda_2 (A_R^2 : D_I^f \otimes D_I^f) - |D_{uu}D_I^f|^2 d_R^f - 2D_{uu}D_I^f \cdot (u_R^f \cdot \nabla D_I^f) d_0
- 2D_{uu}d_0 \cdot (u_R^f \cdot \nabla D_I^f) D_I^f - 2D_{uu}D_I^f \cdot (u_R^f \cdot \nabla d_0 + D_{uu} + \sqrt{\sigma_{uu}} d_R^f) D_I^f
- 2|D_{uu}d_0 \cdot (u_R^f \cdot \nabla D_I^f) + D_{uu}D_I^f \cdot (u_R^f \cdot \nabla d_0 + D_{uu} + \sqrt{\sigma_{uu}} d_R^f) D_I^f|
- 2D_{uu}D_I^f \cdot (u_R^f \cdot \nabla D_I^f) d_0 - 2(u_R^f \cdot \nabla d_0) \cdot (u_R^f \cdot \nabla D_I^f) d_I^f - 2D_{uu} + \sqrt{\sigma_{uu}} D_{uu}^f \cdot (u_R^f \cdot \nabla D_I^f) d_I^f,
$$

(A.27)

the term $Q_4^s$ is

$$
Q_4^s = - |D_{uu}D_I^s|^2 d_I^s - |u_R^s \cdot \nabla D_I^s|^2 d_0 - 2D_{uu}D_I^s \cdot (u_R^s \cdot \nabla D_I^s) d_I^s
- 2(u_R^s \cdot \nabla D_I^s) \cdot (u_R^s \cdot \nabla d_0 + D_{uu} + \sqrt{\sigma_{uu}} d_R^s) D_I^s
$$

(A.28)

and the term $Q_4^a$ is

$$
Q_4^a = - 2D_{uu}D_I^a \cdot (u_R^a \cdot \nabla D_I^a) D_I^a - |u_R^a \cdot \nabla D_I^a|^2 (d_R^a + \sqrt{\sigma_{uu}} d_I^a).
$$

(A.29)

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