Nonlinear Stability of the Periodic Traveling Wave Solution for a Class of Coupled KdV Equations

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In this paper, by applying the Jacobian ellipse function method, we obtain a group of periodic traveling wave solution of coupled KdV equations. Furthermore, by the implicit function theorem, the relation between some wave velocity and the solution’s period is researched. Lastly, we show that this type of solution is orbitally stable by periodic perturbations of the same wavelength as the underlying wave.

1. Introduction

In fluid mechanics of the density stratification, the mechanism of propagation of nonlinear long wave is being researched by physicists and mathematicians. This wave usually appears in the lakes, fjords, and temperature jump layer in the coastal waters. Generally, the KdV equation is often used to describe the wave of this type when the depth of fluid is much shorter than the length of it. Moreover, weak interactions occur in the internal of the nonlinear long waves when wave phase speeds are unequal. In this case, each wave satisfies a KdV equation, and the interaction among waves could be depicted by the phase shift. However, strong interactions occur when these wave phase speeds are nearly equal although the waves belong to different modes. This case is described by a coupled Korteweg-de Vries equations, which has the following form:

\[
\begin{align*}
&u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x = 0, \\
&b_1 v_t + rv_x + vv_x + b_2 u_x + v_{xxx} + b_2 a_1 (uv)_x = 0,
\end{align*}
\]

(1)

where \(a_1, a_2, a_3, b_1, b_2, \) and \(r\) are real constants with positive \(b_1\) and \(b_2\). Here, \(u = u(x,t)\) and \(v = v(x,t)\) are real-valued functions, \(x \in \mathbb{R}, t \in \mathbb{R}^+\).

System (1) was derived from Gear and Grimshaw [1]. Due to its significant physical meaning, many scholars at home and abroad carried out relevant research about these equations and obtained some results. As in [2], Marshall et al. have proved that (1) is globally well-posed when the initial data is rough. Moreover, Bona et al. also considered the similar question (for details, please see [3]).

Recently, a number of researchers are interested in the theory of stability of the Korteweg-de Vries equation. For example, in [4], Linares and Pazotob considered the exponential stabilization with an initial boundary value problem for following the Korteweg-de Vries equation, which has the following form:

\[
\begin{align*}
u_t + u_x + uu_x + u_{xxx} + a(x)u &= 0, x, t \in \mathbb{R}^+, \\
u(0, t) &= 0, t > 0, \\
u(x, 0) &= u_0(x), x > 0.
\end{align*}
\]

(2)

When \(a(x) > 0\), system (2) was shown to be exponentially stable. In [5], Russell and Zhang studied the KdV equation with the periodic boundary conditions, as follows:

\[
\begin{align*}
u_t + u_x + uu_x + u_{xxx} &= 0, u(x, 0) = \varphi(x), \\
u(0, t) &= u(\pi, t), u_{xx}(0, t) = u_{xx}(\pi, t), u_x(\pi, t) = au_x(0, t).
\end{align*}
\]

(3)
Here, $a \in (-1, 1)$. It was shown that if $a \neq -1/2$, then (3) was locally exponentially stable.

Furthermore, in [6], Bona et al. researched this type KdV equation with an initial boundary problem which posed in a quarter plane and appended a damping term, as follows:

$$\begin{align*}
\begin{cases}
\psi_t + u\psi_x + uu_x + u_{xxx} + au = 0, x, t \geq 0, \\
\psi(x, 0) = \varphi(x), \psi(0, t) = h(t).
\end{cases}
\end{align*}$$

They obtain such a result that the time-periodic solution $u^t(x, t)$ of (4) is either locally or globally exponentially stable in $H^s(\mathbb{R}^n)$ when $s \in (3/4, 1]$ or $s > 1$, respectively.

In recent decades, an extensive development of stability theory about solitary waves solutions has been obtained by Benjamin, Bona, Weinstein, Grillakis, and Strauss [7–12].

In [13], Pava considered a modified Korteweg-de Vries equation, which has the following form:

$$u_t + 3u^2u_x + u_{xxx} = 0,$$  

(5)

where $x \in \mathbb{R}$, $t > 0$. By applying the Lyapunov stability method ([10]), he gave an important conclusion that the periodic traveling solution $\eta_1 dn((\eta_1 \xi / \sqrt{2}) , k)$ of (5) is orbitally stable in energy space $H^1_{per}([0, L])$.

In this paper, I am interested in the existence of a smooth periodic solution and the orbital stability of the solution of a coupled Korteweg-de Vries (equation (1)).

Our paper is organized as follows. In Section 2, we applied the Jacobian ellipse function method to obtain a class of smooth cnoidal wave solution for system (1). Section 3 is devoted to studying the relation in a neighborhood of the relevant wave velocity between the period of above cnoidal wave solution and wave velocity. In Section 4, we present the orbital stability theory for the cnoidal wave solutions of system (1).

2. The Periodic Solution for the Coupled Korteweg-de Vries Equations

In this section, by implying the Jacobian ellipse function method, we will show the existence of a class of smooth cnoidal wave solution for system (1).

Firstly, we suppose that system (1) possesses solitary wave solution of the following form:

$$\begin{align*}
\begin{cases}
u(x, t) = \varphi(\xi), \\
\xi = x - ct,
\end{cases}
\end{align*}$$

(6)

where $c$ is a traveling wave speed.

By substituting (6) into (1), we get

$$\begin{align*}
\begin{cases}
c \varphi' + \varphi \varphi' \varphi'' + a_2 \varphi'' + a_1 \varphi \varphi' + a_2 (\varphi \varphi')' = 0, \\
c b_1 \psi' + c \psi' + \psi \psi' + b_2 a_2 \varphi \varphi' + \psi \varphi' + b_2 a_1 (\varphi \varphi')' = 0.
\end{cases}
\end{align*}$$

(7)

Next, we study the form the solution $(u(x, t), v(x, t)) = (\varphi(\xi), 0)$ of system (1).

Let $b_2 = 0$, so that we have

$$c \varphi' + \varphi \varphi' \varphi'' = 0.$$  

(8)

Integrating (8),

$$c \varphi + \frac{\varphi^2}{2} \varphi'' = h_1,$$  

(9)

where $h_1$ is a positive integration constant.

Multiply (9) by $\varphi'$ and integrate its equation, and we obtain

$$\frac{c \varphi^2}{2} + \frac{\varphi^3}{6} + \frac{(\varphi')^2}{2} = h_1 \varphi + h_2.$$  

(10)

Here, $h_2$ is an integration constant.

By further calculating, (10) converts

$$\left( \varphi' \right)^2 = \frac{1}{3} (\varphi - \alpha_1) (\varphi - \alpha_2) (\alpha_3 - \varphi).$$

(11)

Besides, $\alpha_1$, $\alpha_2$, $\alpha_3$, and $c$ satisfy the following relation:

$$\begin{align*}
\alpha_1 \alpha_2 \alpha_3 + 6h_2, \\
\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_2 = -6h_2, \\
\alpha_1 + \alpha_2 + \alpha_3 = -3c, \\
\alpha_2 > c.
\end{align*}$$

(12)

Let $\zeta = \varphi / \alpha_3$, $\eta_1 = \alpha_1 / \alpha_3$, and $\eta_2 = \alpha_2 / \alpha_3$; hence, (11) can be written as

$$\left( \zeta' \right)^2 = \frac{\alpha_3}{3} (\zeta - \eta_1) (\zeta - \eta_2) (1 - \zeta),$$  

(13)

where $\eta_1 = \alpha_1 / \alpha_3$ and $\eta_2 = \alpha_2 / \alpha_3$.

Furthermore, we define a new function $\chi = \chi(\xi)$, satisfying the relation

$$\zeta = 1 + (\eta_2 - 1) \sin^2 \chi.$$  

(14)

Substituting (14) into (13), through a tedious calculation, we get that

$$\left( \chi' \right)^2 = \frac{\alpha_3}{12} (1 - \eta_1) (1 - k^2 \sin^2 \chi).$$  

(15)

Here, $k^2 = 1 - \eta_2 / 1 - \eta_1$, $0 < k^2 < 1$.

By (15) and the definition of [14], we obtain that

$$\zeta = 1 + (1 - \eta_2) \chi_n \left( \sqrt{\frac{\alpha_3 (1 - \eta_2)}{12} \xi, k} \right).$$  

(16)
So that

\[
\varphi = a_3 + (a_3 - a_2)\text{cn}^2\left(\frac{\sqrt{a_3 - a_2}}{12}, k\right).
\]  

(17)

Furthermore, \(\text{cn}\) has fundamental period \(2K(k)\), i.e., \(\text{cn}(x + 2K(k)) = \text{cn}(x)\) and \(K(k)\) is the complete elliptic integral of the first kind, so \(\text{cn}^2(x + 2K(k)) = \text{cn}^2(x)\) and this traveling wave solution \(\varphi\) have a fundamental period \(T\), as follows:

\[
T = T(a_2, a_3) = \frac{4\sqrt{3}K(k)}{\sqrt{a_3 - a_2}}.
\]

(18)

System (1) has the following exact periodic solution:

\[
\begin{align*}
u(x, t) &= \varphi(\xi) = a_3 + (a_3 - a_2)\text{cn}^2\left(\frac{\sqrt{a_3 - a_2}}{12}, k\right), \\
\psi(x, t) &= \psi(\xi) = 0.
\end{align*}
\]

(19)

3. The Periodic Property of the Exact Periodic Solution of the Coupled KdV Equations

In this section, by the implicit function theorem, the function relation between the period \(T\) of the periodic solution of system (1) and wave velocity is obtained. \(H_1, H_2, a_1 + a_2 + a_3 = -3c, k^2\) can be rewritten as

\[
k^2 = \frac{a_3 - a_2}{2a_3 + a_2 - 3c}.
\]

(20)

Therefore,

\[
T(a_2, a_3) = \frac{4\sqrt{3}K(k)}{\sqrt{a_3 - a_2}}.
\]

(21)

From (12), we can obtain that

\[
a_3 = \frac{-(a_3 + 2c) + \sqrt{(a_3 + 2c)^2 - 4(a_3 + 3a_2c - 6h_1)}}{2} = g(a_3, c).
\]

(22)

so that \(T = T(a_2, a_3) = T(g(a_3, c), c)\).

If it exists a fixed \(L > 0, c_0 > \sqrt{2h_1 + 1/12}, a_{3,0} > 4c_0 + 2\sqrt{3(a_1^2 - 2h_1)}\) and satisfies \(T(c_0, a_{3,0}) = L\), there is a function relation \(T = T(g(a_3, c), c) = T(c)\) in neighborhoods \(U(c_0)\) of \(c_0\) and \(U(a_{3,0})\) of \(a_{3,0}\).

Theorem 1. If a fixed \(L > 0, c_0 > \sqrt{2h_1 + 1/12}, a_{3,0} > 4c_0 + 2\sqrt{3(a_1^2 - 2h_1)}\) and satisfies \(T(c_0, a_{3,0}) = L\), there is a neighborhood \(U(c_0)\) of \(c_0\), a neighborhood \(U(a_{3,0})\) of \(a_{3,0}\) and a unique smooth function \(T : U(c_0) \rightarrow U(a_{3,0}), a_3 = T(c), T = T(a_3, c) = T(c), \) where \(c \in U(c_0), a_3 \in U(a_{3,0}).\)

Proof. The idea and method are derived from [13]. Let

\[
\Omega = \left\{ (c, a_3) \mid c > \sqrt{2h_1 + 1/12}, a_3 > 4c + 2\sqrt{3(c^2 - 2h_1)} \right\}.
\]

(23)

Now we define \(\Phi : \Omega \rightarrow R\) by

\[
\Phi = \Phi(a_3, c) = \frac{4\sqrt{3}K(k)}{a_3 - g(a_3, c)} - L.
\]

(24)

Moreover, \(\Phi(a_{3,0}, c_0) = 0, \Phi, \partial \Phi/\partial a_3, \partial \Phi/\partial c\) are continuous in \(\Omega\). Then, we will prove that \(\partial \Phi/\partial a_3 < 0\). Obviously,

\[
\frac{\partial \Phi}{\partial a_3} = 2\sqrt{3}[2(\partial K/\partial k)(\partial k/\partial a_3)(a_3 - g(a_3, c)) + (1 - \partial g/\partial a_3)]
\]

(25)

By [15], (12), and (20), we can obtain that

\[
\begin{align*}
\frac{\partial K}{\partial k} &= \frac{3}{2k(2a_3 + a_2 - 3c)^2} > 0, \\
\frac{\partial g}{\partial a_3} &= \frac{-1 - [a_3 - g(a_3, c)]^{-1/2}(a_3 - 4c - 1)}{2} < 0.
\end{align*}
\]

(26)

Hence, \(\partial \Phi/\partial a_3 < 0\) in \(\Omega\). By the implicit function theorem, we can obtain that neighborhood \(U(c_0)\) of \(c_0\), neighborhood \(U(a_{3,0})\) of \(a_{3,0}\), and smooth function \(T : U(c_0) \rightarrow U(a_{3,0}), \) s.t. \(a_3 = T(c)\), so that \(T = T(a_3, c) = T(c)\).

This completes the proof.

4. The Stability of the Periodic Traveling Wave Solution of the Coupled KdV Equations

In this section, we devote to researching the nonlinear stability of the periodic traveling wave solution \(U\) of system (1), which is as follows:

\[
U = (u, v)' = \left(a_3 + (a_3 - a_2)\text{cn}^2\left(\frac{\sqrt{a_3 - a_2}}{12}, k\right), 0\right)'.
\]

(27)

4.1. Definition (Orbital Stability). A periodic traveling wave solution \(U\) of system (1) is orbitally stable. If every \(\varepsilon > 0\) and there exists a \(\delta > 0\), then the following assumption holds: If \(V(t, x) \in C([0, T]; H^2)\) is a solution of system (1) with \(\|V(0, x) - U(0, x)\|_{H^2} < \delta\), every \(t \in [0, T]\) satisfies

\[
\inf_{y \in \mathbb{R}}\|V(x, t) - U(\cdot, -y)\| \leq \varepsilon.
\]

(28)

Otherwise, the solution is unstable.
We will apply the method which was established and developed by Angulo and Natali in [16] to show the orbital stability of the above periodic traveling wave solution for the coupled KdV equation (1).

Initially, system (1) can be expressed as an abstract Hamiltonian system

\[
\frac{dU}{dt} = JE'(U),
\]

with

\[
U = (u, v), J = \begin{pmatrix}
\partial_x & 0 \\
0 & -\partial_x
\end{pmatrix},
\]

and

\[
E(U) = \int_0^T \left( -\frac{u_x^2}{2} - \alpha_3 uv_{xx} - \alpha_4 u^2 - \alpha_5 v^2 - \frac{1}{b_1} v_{xx} + \frac{b_2}{b_1} v_{axx} + \frac{b_3}{b_1} x v_{xx} + \frac{b_4}{b_1} u_{axx} + \frac{b_5}{b_1} x u_{xx} \right) dx
\]

is conserved quantity.

Moreover, the coupled KdV equation (1) possesses the conserved quantity

\[
F(U) = \frac{1}{4} \int_0^T (u^2 + v^2) dx.
\]

To be more precise, we have the following set of conditions which guarantees the stability of the periodic traveling solution $U$ of system (1):

(i) It exists a nontrivial smooth curve of periodic traveling solution $U$ for system (1). Moreover, there are functionals $E(U)$ and $F(U)$ which are conserved for system (1)

(ii) $E'(U) + cF'(U) = 0$

(iii) The operator $L(u, 0)$ has a unique negative eigenvalue, which is simple. Furthermore, The remainder of the spectrum consists of a discrete and positive set of eigenvalues.

(iv) $d''(c)$ is positive in a neighborhood of $c$.

Next, we focus on verifying these conditions hold. Firstly, in Section 2, we have obtained that system (1) exists a periodic traveling solution (19) and conserves quantities $E(U)$ and $F(U)$. Therefore, condition (i) holds. Moreover, substituting (19) into $E'(U) + cF'(U)$, we can easily get that

\[
E'(U) + cF'(U) = 0.
\]

Hence, condition (ii) holds.

Next, we study the spectral properties associated to the linear operator $L(u, 0) = E''(u, 0) + cF''(u, 0)$ determined by the periodic solution $U$. We will show that the spectrum of $L(u, 0)$ has simple negative eigenvalue, zero is simple eigenvalue, and the rest of the spectrum are away from zero.

In the first place, we calculate the quadratic form of the operator $L(u, 0)$, for $(u, 0) \in H^1(0, L) \times H^1(0, L)$, by

\[
Q_{L, 2} = (L(u, 0), (u, 0)) = \int_0^L \left(-\partial_x^2 u - u^2 + \frac{u^2 c}{2}\right) dx.
\]

It represents the quadratic form of the operator

\[
L_1 = -\partial_x^2 - \frac{u^2}{2} \left(1 - \frac{c}{2}\right).
\]

From (8), we can get that $L_1 \varphi = 0$, so that zero is an eigenvalue whose eigenfunction is $\varphi$. We will give the behavior of the eigenvalues related to operator $L_1$ in the following theorem.

**Theorem 2.** Let $\varphi$ be the cnoidal periodic traveling wave solution given by (17), then operator $L_1$ defined in $L^2_{\text{per}}(0, L)$ with domain $H^2_{\text{per}}(0, L)$ has one simple negative eigenvalue, zero is simple eigenvalue, and the rest of spectrum are away from zero.

**Proof.** The idea and method are derived from [13, 17].

In fact, $L_1$ is a self-adjoint operator. Since $H^2_{\text{per}}(0, L)$ is compactly embedded in $L^2_{\text{per}}(0, L)$, operator $L_1$ has a compact resolvent, so that the essential spectrum of $L_1$ is an empty set, and the discrete spectrum of operator $L_1$ consists of isolated
eigenvalues with finite algebraic multiplicities. Consequently, operator $L_1$ has only point spectrum. Because $L_1\varphi' = 0$, by the oscillation theorem in [18], the eigenvalue zero is the second or third eigenvalue of $L_1$. Next, we will prove that zero is the second one.

The periodic eigenvalue problem associated with (37) is given by

$$
\begin{aligned}
L_1 v &= \lambda v, \\
v(0) &= v(L), \\
v'(0) &= v'(L).
\end{aligned}
$$

(38)

Let $\Psi(x) = v, u^2 = 2 - 4k^2\sin^2(x; k)2 - c$. We get the equivalent of problem (38):

$$
\begin{aligned}
\frac{d^2\Psi}{dx^2} + [(1 - 2k^2)\sin^2(x, k)]\Psi &= \tilde{\lambda}\Psi, \\
\Psi(0) &= \Psi(L), \\
\Psi'(0) &= \Psi'(L).
\end{aligned}
$$

(39)

Here, $\tilde{\lambda} = -\lambda$. Moreover, (39) with the Jacobian form is the Lamé’s differential equation. According to the Floquet theory, we can get that (39) has two intervals of instability, so that the third eigenvalue, $\theta_3$ is 1 - $k^2$ is a positive eigenvalue.

This completes the proof.

Furthermore, let us consider the eigenvalue equation:

$$
L(u, 0) \begin{pmatrix} u \\ 0 \end{pmatrix} = \theta_i \begin{pmatrix} u \\ 0 \end{pmatrix},
$$

(40)

where $i = 1, 2, 3$. It is clear that the smallest eigenvalue $\theta_1$ associated with operator $L(u, 0)$ is negative. Now, we will show that the next eigenvalue of $L(x, 0)$ is $\theta_3(\theta_2 = 0)$, which is simple. The third eigenvalue, $\theta_3$, is strictly positive. By min–max characterization in [19], we have that

$$
\theta_2 = \max_{(\psi_1, \psi_2)\in Z} \min_{(g, h)\in Z} \frac{Q_{L2}}{\|g\|^2 + \|h\|^2},
$$

(41)

where $Z = H^2_{per}(0, L) \times H^2_{per}(0, L)$, $(g, \psi_1) + (h, \psi_2) = 0$. If we consider $\psi_1 = v, \psi_2 = 0$, we can get that

$$
\theta_2 \geq \min_{(g, h)\in Z} \frac{Q_{L2}}{\|g\|^2 + \|h\|^2} \geq 0.
$$

(42)

so that $\theta_2 = 0$.

Next, we prove that $\theta_3 > 0$. Through taking the two-dimensional subspace spanned by $(v, 0)$ and $(\varphi', 0)$, since $g \perp v, g \perp \varphi'$, therefore,

$$
\theta_3 \geq \min_{(v, \varphi')} \frac{Q_{L2}}{\|v\|^2 + \|\varphi'\|^2} > 0,
$$

(43)

where $\theta_3$ the third eigenvalue related to $L(u, 0)$ is obviously positive.

So, we deduce that operator $L(u, 0)$ has one negative eigenvalue and one zero eigenvalue, and the rest of the spectrum is positive and bounded away from zero, which proves the condition (iii) holds.

In the following, we will prove $d''(c) > 0$. In fact,

$$
d'(c) = \frac{dU}{dc} [E'(U) + cF'(U)] + F(U) = F(U).
$$

(44)

Furthermore,

$$
d''(c) = F'(U) \frac{dU}{dc} = \frac{1}{4} \int_0^T u d\xi \cdot \frac{du}{dc}
$$

(45)

where

$$
\frac{du}{dc} = -\frac{32 K^2 (1 + k^2) T dT}{kT} - \frac{96 K dT}{T^3} \cdot \cos \left(\sqrt{\frac{\alpha_3 - \alpha_2 \xi}{12}} k\right)
$$

$$
- 2(\alpha_3 - \alpha_2) \cdot \cos \left(\sqrt{\frac{\alpha_3 - \alpha_2 \xi}{12}} k\right) \cdot \sin \left(\sqrt{\frac{\alpha_3 - \alpha_2 \xi}{12}} k\right)
$$

$$
\cdot \sin \left(\sqrt{\frac{\alpha_3 - \alpha_2 \xi}{12}} k\right) \cdot \frac{2 T dT}{dc} - \frac{96 K}{T^3} \cdot \cos \left(\sqrt{\frac{\alpha_3 - \alpha_2 \xi}{12}} k\right) + 1.
$$

(46)

In Section 3, we have calculated $dT/dc < 0$; obviously, $du/dc < 0$, so that condition (iv) holds.

$$
d''(c) > 0.
$$

(47)

So, an application of the orbital stability theorem in [16] gives the following theorem:

Theorem 3. If conditions (i), (ii), (iii), and (iv) hold, the periodic solution (19) of system (1) is orbitally stable.

Data Availability

No data were used to support this study. In my manuscript, any reader can access the data supporting the conclusions of the study and this clearly outlines the reasons why unavailable data cannot be released.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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