Relativistic positioning in Schwarzschild space-time

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Abstract. In the Schwarzschild space-time created by an idealized static spherically symmetric Earth, two approaches –based on relativistic positioning– may be used to estimate the user position from the proper times broadcast by four satellites. In the first approach, satellites move in the Schwarzschild space-time and the photons emitted by the satellites follow null geodesics of the Minkowski space-time asymptotic to the Schwarzschild geometry. This assumption leads to positioning errors since the photon world lines are not geodesics of any Minkowski geometry. In the second approach –the most coherent one– satellites and photons move in the Schwarzschild space-time. This approach is a first order one in the dimensionless parameter GM/R (with the speed of light c=1). The two approaches give different inertial coordinates for a given user. The differences are estimated and appropriately represented for users located inside a great region surrounding Earth. The resulting values (errors) are small enough to justify the use of the first approach, which is the simplest and the most manageable one. The satellite evolution mimics that of the GALILEO global navigation satellite system.

1. Introduction
In any Relativistic Positioning System (RPS), which may be based on Minkowski Space-Time (MS-T), Schwarzschild Space-Time (SS-T), or any other space-time, the user to be located must receive -simultaneously- the proper times from four satellites. These proper times are called emission coordinates $\tau^A$, where $A$ runs from 1 to 4 to label the four satellites. The user space-time position is characterized by the inertial coordinates $x^\alpha$; here and hereafter any Greek index runs from 1 to 4. These inertial coordinates are associated to an almost inertial reference system. From emission to reception, the photons delivering the proper times follow null geodesics. This fact allow us to calculate the inertial coordinates from the emission ones and the satellite world lines (excepting bifurcation cases).

Any Latin index, excepting $A$, runs from 1 to 3. Quantities $G$, $M_\oplus$, $R_\oplus$, $t$ and $\tau$ stand for the gravitation constant, the Earth mass, the Earth radius, the coordinate time and the proper time, respectively. Units are chosen in such a way that the speed of light is $c = 1$.

The SS-T describes the external gravitational field of an idealized static spherically symmetric Earth. This space-time tends -asymptotically- to a MS-T, in which, the symmetry centre is at rest; hence, the SS-T may be considered as a perturbation of the asymptotic MS-T. The perturbing terms tend to zero as the distance to the symmetry centre tends to infinity.

Let us assume that the emission coordinates $\tau^A$ are received from four satellites. From these coordinates, the user position may be found by using two different approaches; in the first one, photons move in the MS-T asymptotic to the SS-T created by Earth, whereas in the second approach, photons move in the proper SS-T and calculations are performed up to first order
in $GM/R$. In both approaches, satellites move in SS-T. Thus, two different user positions are obtained, whose differences allow us to estimate the positioning E-errors, which are to be compared with the errors due to uncertainties in the satellite world lines (U-errors, see [4]). If the E-errors are small enough as compared with the U-errors, the user position may be calculated by using the first approach instead of the second one, which is more complicated from the numerical point of view.

2. First approach

If the satellite world lines are known in a certain inertial system of reference, the user position can be found, in the same inertial reference, using the following equations:

$$\eta_{\alpha\beta}[x^\alpha - x_A^\alpha(\tau^A)] [x^\beta - x_A^\beta(\tau^A)] = 0 \quad (1)$$

where $\eta_{\alpha\beta}$ is the MS-T metric tensor, whose non vanishing components are $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{44} = -1$, and the satellite world lines $x_A^\alpha = x_A^\alpha(\tau^A)$ are written in terms of the proper time parameter. These equations, which relate the emission coordinates $\tau^A$ and the inertial ones $x^\alpha$, are found taking into account that, in this first approach, the photons move along null-geodesics in MS-T from satellite emission to user reception.

An analytical solution of the system of equations (1) was found in [1]. This solution gives the inertial coordinates $(x^\mu_{I A})$ in terms of the emission ones, for photons moving in MS-T. A numerical code based on this solution -hereafter referred as the TX-code- was designed and tested in [2], [3] and [4]. The analytical solution holds for arbitrary satellite world lines.

Moreover, we can numerically calculate the emission coordinates (unknowns) from the inertial ones, by using the Newton-Raphson method [5] to solve the system (1). This is implemented in a numerical code, which is called XT-code.

We assume that the satellite trajectories are circumference concentric with Earth, which are travelled as it is predicted in SS-T. See [2] and [4], where the equations of the satellite world lines are given in terms of $\tau$. In order to compute E-errors, it may be assumed that satellites strictly follow nominal world lines of the GALILEO constellation.

3. Second approach

In this second approach, the photons move in SS-T. Each photon is emitted from the satellite A at point $P_{IA}$, whose coordinates $x_{IA}^\alpha(\tau^A)$ ($\vec{R}_{IA} \equiv x_{IA}^1$ and $t_{IA} = x_{IA}^4$) have been calculated from $\tau^A$. The user position is then the intersection of four null geodesics that pass through the points $P_{IA}$ and have appropriate propagation directions at these points. The coordinates of the intersection point $P_{S0}$ (user space-time position in SS-T) are denoted $x_{S0}^\mu$. Hereafter, $\vec{R}_{S0} = x_{S0}^1$ and $t_{S0} = x_{S0}^4$.

In the first approach, the photons emitted from the points $P_{IA}$ follow null geodesics in MS-T. Four of these geodesics intersect at user space-time position $P_{M0}$ (with coordinates $\vec{R}_{M0} = x_{M0}^1$ and $t_{M0} = x_{M0}^4$), which may be calculated as shown in the first approach (TX-code).

Moreover, SS-T may be seen as a perturbation of MS-T. Hence, the coordinates of each point on the SS-T geodesic may be written in the form $x_{S}^\mu = x_{M}^\mu + \Delta x_{M}^\mu$; so the SS-T null geodesic may be seen as a perturbation of the MS-T one. Quantities $\Delta x_{M}^\mu$ are the deviations between both geodesics.

The condition for the intersection of four null geodesics in SS-T is that, at point $P_{M0}$, the quantities $\Delta \vec{R}_{M0} = \Delta x_{M0}^1$ and $\Delta t_{M0} = \Delta x_{M0}^4$ are identical for the four satellites (see [6]). Lengthy but straightforward calculations (at first order in $GM/R$) allow us to impose this condition. So, we obtain the following system of equations:

$$\Delta t_{M0} - \frac{\vec{R}_{M0} - \vec{R}_{IA}}{|\vec{R}_{M0} - \vec{R}_{IA}|} \Delta \vec{R}_{M0} = 2GM_{\oplus} \ln \left[ \frac{||\vec{R}_{M0}|| + ||\vec{R}_{IA}|| + ||\vec{R}_{M0} - \vec{R}_{IA}||}{||\vec{R}_{M0}|| + ||\vec{R}_{IA}|| - ||\vec{R}_{M0} - \vec{R}_{IA}||} \right] \quad (2)$$
where \( r = |\vec{r}| \) is a radial isotropic coordinate. The relation between the radial coordinates \( R \) and \( r \), up to first order in \( GM_0/R \), is \( |\vec{r}| = |\vec{R}|(1 - GM_0/|\vec{R}|) \) (see [7]). The quantities \( \Delta \vec{R}_{M0} \) and \( \Delta t_{M0} \) are the unknowns. The system (2) is composed of four equations, one equation per satellite.

The determinant of this system of four equations is

\[
D = \begin{vmatrix}
\frac{x_1^1 - x_{M0}^1}{|\vec{R}_{M0} - \vec{R}_{i1}|} & \frac{x_1^2 - x_{M0}^2}{|\vec{R}_{M0} - \vec{R}_{i1}|} & \frac{x_1^3 - x_{M0}^3}{|\vec{R}_{M0} - \vec{R}_{i1}|} & 1 \\
\frac{x_2^1 - x_{M0}^1}{|\vec{R}_{M0} - \vec{R}_{i2}|} & \frac{x_2^2 - x_{M0}^2}{|\vec{R}_{M0} - \vec{R}_{i2}|} & \frac{x_2^3 - x_{M0}^3}{|\vec{R}_{M0} - \vec{R}_{i2}|} & 1 \\
\frac{x_3^1 - x_{M0}^1}{|\vec{R}_{M0} - \vec{R}_{i3}|} & \frac{x_3^2 - x_{M0}^2}{|\vec{R}_{M0} - \vec{R}_{i3}|} & \frac{x_3^3 - x_{M0}^3}{|\vec{R}_{M0} - \vec{R}_{i3}|} & 1 \\
\frac{x_4^1 - x_{M0}^1}{|\vec{R}_{M0} - \vec{R}_{i4}|} & \frac{x_4^2 - x_{M0}^2}{|\vec{R}_{M0} - \vec{R}_{i4}|} & \frac{x_4^3 - x_{M0}^3}{|\vec{R}_{M0} - \vec{R}_{i4}|} & 1 \\
\end{vmatrix}
\]

(3)

If this determinant vanishes for the emission coordinates \( \tau^A \), there are no solutions of the corresponding system of equations, which means that positioning at the MS-T point \( P_{M0} \) [with coordinates \( (x_{M0}^i, t_{M0}) \)] is not possible.

In [4], it was proved that, for low satellite velocities as those of any GNSS, the relation \( V_T \approx |J|/6 \) is satisfied, where \( J \) is the Jacobian of the transformation, \( \tau^A = \tau^A(x^a) \), calculated at point \( (\vec{R}_{M0}, t_{M0}) \), and \( V_T \) is the volume of the tetrahedron formed by the tips of the four user-satellite unit vectors having their common origin at point \( (\vec{R}_{M0}, t_{M0}) \) in MS-T. It is important to remark that, for low satellite velocities, the determinant \( D \) obtained here is exactly the same as the Jacobian \( J \). Therefore, \( D \) is proportional to the volume \( V_T \). See also [8], where this volume is related with the so-called dilution of precision in the framework of Global Positioning System (GPS).

Quantities \( D, J, \) and \( V_T \) vanish simultaneously. In the points where these three quantities are small, the U-errors and the E-errors become very large; by this reason, these errors are expected to be large in two cases: close to \( D = J = V_T = 0 \) points, and also at points located far from both \( D = 0 \) points and satellites. In this last case, the four satellites subtend a small solid angle and the tetrahedron volume \( V_T \) is expected to be small (large errors).

4. Numerical Results

Calculations will be performed inside a big sphere, hereafter E-sphere. The center of this sphere is the point \( E \), whose spherical inertial coordinates are \( r_E = R_\oplus, \theta_E = 60^\circ \) and \( \phi_E = 30^\circ \). Hence, \( E \) is on Earth surface. The radius of this sphere is \( 10^5 \, km \).

In this paper, as in [2] and [4], the HEALPix (hierarchical equal area isolatitude pixelization of the sphere) package ([9]) is used to depict appropriate maps. This pixelization was designed to construct and analyze maps of the cosmic microwave background. It is useful to display any scalar quantity depending on the observation direction (pixel). In our maps, there are 3072 pixels.

Given four satellites of the GALILEO constellation and an user position \( (x_{M0}^i, t_{M0}) \) in MS-T, the XT-code may be used to get the emission coordinates \( \tau^A \), which would be received by the chosen user in MS-T. From these emission coordinates and the satellite world lines (see [2] and [4]), the satellite positions at emission times \( (x_{iA}^i, t_{iA}) \) may be obtained and, then, the E-errors at the MS-T chosen point \( (x_{M0}^i, t_{M0}) \) may be calculated by solving the system (2) for the unknowns \( \Delta x_{M0}^i \) and \( \Delta t_{M0} \). Quantities \( \Delta R = |\vec{R}_{M0}| \) and \( \Delta t_{M0} \) are good estimators of the E-errors.
We have considered 3072 HEALPIX directions, and $10^3$ users uniformly distributed along each direction, from point E ($L = 0$) to $L = 10^5$ km. So the E-sphere is covered. The E-errors and the determinant D have been then calculated in the points of this coverage at coordinate time $t = 25$ h.

In Fig. 1, the E-errors corresponding to a HEALPIX particular direction without $D = 0$ points are shown. From this Figure, we can conclude that $\Delta_R$ and $|\Delta_t|$ increase with the distance $L$ to E. For the chosen direction, these quantities are smaller than 0.5 m, but they are close to this value nearby the E-sphere boundary.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Quantities $\Delta_R$ (left) and $c\Delta_t$ (right) are represented along a particular direction from $L = 0$ (point E) to $L = 10^2$ Mm. The determinant D does not vanish along the chosen segment.}
\end{figure}

In Fig. 2, the distance $L_D$, from $L = 0$ (point E) to the point where $D = 0$ for the first time is displayed. Grey pixels correspond to HEALPIX directions where the determinant D does not vanish from E to $L = 10^5$ km. Inside a sphere (centered at point E), whose radius is the minimum value of the colour bar (23500 km), the determinant D does not vanish.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{HEALPIX mollwide map. The quantity represented is the distance $L_D$, in kilometers, from E to the point where D vanishes for the first time.}
\end{figure}
\[ \rho = R_{\oplus} \]

\[ \rho = R_{\oplus} \]

\[ \rho = 10^4 \text{ km} \]

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\[ \rho = 1.5 \times 10^4 \text{ km} \]

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\[ \rho = 2 \times 10^4 \text{ km} \]

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Figure 3. HEALPix mollwide maps of $\Delta_R$ (left) and $c\Delta_t$ (right), in kilometers, on spherical surfaces concentric with Earth. The radius $\rho$ are given in the bottom of each panel. Users located on the grey pixels cannot see the four satellites at the same time (invisibility zones).

Finally, we have calculated the E-errors ($\Delta_R$ and $\Delta_t$) on spherical surfaces concentric with Earth for appropriate radius and $t = 25$ h. These radius are $\rho = R_{\oplus}, \rho = 10^4$ km,
\[ \rho = 1.5 \times 10^4 \text{ km} \] and \[ \rho = 2 \times 10^4 \text{ km} \]. Since the chosen radius are smaller than 23500 km, the determinant \( D \) does not vanish on the corresponding spherical surfaces (see Fig. 2).

In Fig. 3, the grey pixels cover invisibility zones, in which, positioning is not possible since the users do not receive, at time \( t = 25 \text{ h} \), the signals from the four satellites. It is due to the fact that one or more satellites are hidden by Earth.

All the spheres, whose E-errors are displayed in Fig. 3, are located inside the satellite orbits in a region where \( D \neq 0 \). On these spherical surfaces the order of these E-errors is between \( 10^{-1} \text{ cm} \) and \( 10 \text{ cm} \). The maximum values of the E-errors appear in pixels close to the boundary of the invisibility regions. It is due to the fact that, for these pixels, the photon passes close to Earth surface.

5. Conclusions

Our study has been restricted to regions without \( D = 0 \) points. One of these regions is surrounding Earth. From Fig. 3 it follows that, in this region, the E-errors are smaller than \( \sim 10 \text{ cm} \). According to Fig. 2, there are directions (grey pixels) without any \( D = 0 \) point (from E to the E-sphere boundary). The E-errors have been estimated along many of these directions. One of them has been considered in Fig. 1. From this study, we have concluded that the E-errors increase with the distance \( L \) to point E, reaching values of the order of 1 m for some directions (close to 0.5 m in Fig. 1); hence, positioning quality decreases as \( L \) grows.

In paper [4], U-errors due to uncertainties in the satellite world lines were calculated. Here, we have estimated the E-errors at the same user space-time positions considered in the U-errors calculations. It has been done to allow comparisons. The order of magnitude of the U-errors, inside the E-sphere, is between 10 m and 100 m. These errors were calculated under the assumption that the amplitude of the deviation in the satellite world lines -with respect to the nominal ones- is 10 m. Then, for the chosen deviation amplitude, we have verified that the E-errors are smaller than the U-errors inside the E-sphere. Taking into account that the U-error values are proportional to the deviation amplitude, if this amplitude would be as smaller as 1 m, in future, the U-errors would be of the same order of magnitude as the E-errors. In this paper, it has been verified that the conditions \( D = 0 \), \( J = 0 \) and \( V_T = 0 \) are equivalent. So, where the U-errors diverge \( (J = 0) \), the E-errors also diverge \( (D = 0) \).

Since E-errors are smaller than U-errors inside the E-sphere, we conclude that the approach based on the approximation that photons (satellites) are moving in MS-T(SS-T) can be used. This approach simplifies the calculations.

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