GLOBAL WELL-POSEDNESS AND SCATTERING
FOR THE FOCUSING
NONLINEAR SCHRÖDINGER EQUATION
IN THE NONRADIAL CASE

Pigong Han

Abstract. The energy-critical, focusing nonlinear Schrödinger equation in the nonradial case reads as follows:

\[ i \partial_t u = -\Delta u - |u|^{\frac{4}{N-2}} u, \quad u(x, 0) = u_0 \in H^1(\mathbb{R}^N), \quad N \geq 3. \]

Under a suitable assumption on the maximal strong solution, using a compactness argument and a virial identity, we establish the global well-posedness and scattering in the nonradial case, which gives a positive answer to one open problem proposed by Kenig and Merle [Invent. Math. 166 (2006), 645–675].

Keywords: critical energy, focusing Schrödinger equation, global well-posedness, scattering.

Mathematics Subject Classification: 35Q40, 35Q55.

1. INTRODUCTION AND THE MAIN RESULT

We consider the energy-critical nonlinear Schrödinger equation in \( \mathbb{R}^N (N \geq 3) \):

\[
\begin{cases}
  i \partial_t u = -\Delta u \pm |u|^{\frac{4}{N-2}} u & \text{in } \mathbb{R}^N \times \mathbb{R}, \\
  u(x, 0) = u_0 & \text{in } \mathbb{R}^N,
\end{cases}
\]

where \( u = u(x, t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C} \) denotes the complex-valued wave function, \( i = \sqrt{-1} \).

The sign "-" corresponds to the focusing problem, while the sign "+" corresponds to the defocusing problem. Cazenave-Weissler [6, 7] showed that if \( \| \nabla u_0 \|_2 \) is suitably small, then there exists a unique solution \( u \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \) of (1.1) satisfying \( \| u \|_{L^{2(N+2)}(\mathbb{R}; L^{2(N+2)}(\mathbb{R}^N))} < \infty \). In the defocusing case, if \( u_0 \in H^1(\mathbb{R}^N) \) is radial, Bourgain [1] proved the global well-posedness for (1.1) with \( N = 3, 4 \), and that for more regular \( u_0 \), the solution preserves the smoothness for all time. (Another
proof of this last fact is due to Grillakis [13] for $N = 3$.) Bourgain's result is then extended to $N \geq 5$ by Tao [29], still under the assumption that $u_0$ is radial. Subsequently, Colliander-Keel-Staffilani-Takaoka-Tao [8] obtained the result for general $u_0 \in H^1(\mathbb{R}^3)$. Ryckman-Visan [26] extended this result to $N = 4$ and finally to $N \geq 5$ by Visan [30]. In the focusing case, these results do not hold. In fact, the classical virial identity shows that if $E(u_0) < 0$ and $|x|u_0 \in L^2(\mathbb{R}^N)$, the corresponding solution breaks down in finite time.

Ginibre-Velo [11] considered a general case:

$$
\begin{aligned}
\left\{ 
\begin{array}{l}
  i\partial_t u = -\Delta u - |u|^{4N/(4N-2)} u \\
  u(x,0) = u_0
\end{array}
\right. \\
\text{in } \mathbb{R}^N \times \mathbb{R},
\end{aligned}
$$

(1.2)

and established the local well-posedness of the Cauchy problem (1.2) (focusing case) in the energy space $H^1(\mathbb{R}^N)$ with $1 < q < 1 + \frac{4}{N}$. Furthermore, they proved the global existence for both small and large initial data in the $L^2$-subcritical case: $1 < q < 1 + \frac{4}{N}$. In the $L^2$-supercritical case: $1 + \frac{4}{N} < q < 1 + \frac{4}{N-2}$, Glassey [12], Ogawa-Tsutsumi [24,25] showed that the strong solution of the Cauchy problem (1.2) blows up in finite time for a class of initial data, especially for negative energy initial data. Holmer-Roudenko [15] established sharp conditions on the existence of global solutions of (1.2) with $q = 3$. In the $L^2$-critical case: $q = 1 + \frac{4}{N}$, Weinstein [31] gave a crucial criterion in terms of $L^2$-mass initial data. Relevant work on the above topics of (1.2) is referred to [2,3,9,14,16,18,20,23,27] and the references therein.

Using the concentration compactness, which is obtained by Keraani [18], Kenig-Merle [19] considered problem (1.1) in the focusing case for $N = 3, 4, 5$, and discussed global well-posedness and blow-up for the energy-critical problem (1.1) in the radial case. Moreover, they expected their results could be extended to the case of radial data for $N \geq 6$, and believed that it remained an interesting problem to remove the radial symmetry assumption. Subsequently, Killip-Visan [22] considered the focusing problem (1.1) with dimensions $N \geq 5$, and proved that if a maximal-lifespan solution $u : I \times \mathbb{R}^N \to \mathbb{C}$ obeys $\sup_{t \in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2$, then it is global and scatters both forward and backward in time. Here $W$ denotes the ground state, which is a stationary solution of the equation of the focusing problem (1.1). In particular, if a local strong solution has both energy and kinetic energy less than those of the ground state $W$ at some point in time, then the local strong solution is global and scatters in higher dimensions $N \geq 5$. Further results are referred to [10,17].

In the present paper, under a suitable assumption on the local strong solution, we establish the global well-posedness and scattering for the focusing problem (1.1) in the nonradial case, which gives a positive answer to one open problem proposed by Kenig-Merle in [19].

In order to state our main result conveniently, we rewrite the focusing problem (1.1) as follows:

$$
\begin{aligned}
\left\{ 
\begin{array}{l}
  i\partial_t u = -\Delta u - |u|^{4N/(4N-2)} u \\
  u(x,0) = u_0 \in H^1(\mathbb{R}^N),
\end{array}
\right. \\
\text{in } \mathbb{R}^N \times \mathbb{R},
\end{aligned}
$$

(1.3)
Theorem 1.1. Assume that \( u \) satisfies (1.3) defined on the maximal interval \( (T_-(u_0), T_+(u_0)) \) obeys conservations of charge and energy:

\[
\int_{\mathbb{R}^N} |u(x,t)|^2 dx = \int_{\mathbb{R}^N} |u_0(x)|^2 dx, \quad \forall t \in (-T_-(u_0), T_+(u_0)),
\]

and

\[
E(u(t)) = E(u_0), \quad \forall t \in (-T_-(u_0), T_+(u_0)),
\]

where

\[
E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} |u(x,t)|^2 dx, \quad 2^* = \frac{2N}{N-2}.
\]

Talenti [28] proved that the function

\[
W(x) = \left( \frac{N(N-2)}{(1+|x|^2)^{\frac{N-2}{2}}} \right)
\]

satisfies \(|\nabla W| \in L^2(\mathbb{R}^N)\) and solves the elliptic equation

\[-\Delta W = |W|^{\frac{4}{4-N}} W \quad \text{in} \quad \mathbb{R}^N.\]

The main result of this paper reads as follows.

**Theorem 1.1.** Assume that \( u_0 \in H^1(\mathbb{R}^N), \ N = 3, 4, 5. \) Then there exists a unique solution \( u \) of (1.3) defined on the maximum existence of interval \( (-T_-(u_0), T_+(u_0)) \) with \( u \in C((-T_-(u_0), T_+(u_0)), H^1(\mathbb{R}^N)), \) where \( 0 < T_-(u_0), T_+(u_0) \leq +\infty. \) Let \( E(u_0) < E(W), \ ||u_0||_{L^2(\mathbb{R}^N)} < ||W||_{L^2(\mathbb{R}^N)}. \) Assume that there exists a non-negative real-valued function \( \varphi \in C^\infty(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} \varphi|u_0|^2 dx > 0 \quad \text{and} \quad \inf_{t \in (0, T_+(u_0))} f(t) \geq 0 \quad \left( \text{resp.} \sup_{t \in (-T_-(u_0), 0)} f(t) \leq 0 \right),
\]

where

\[
f(t) \triangleq \text{Im} \int_{\mathbb{R}^N} \bar{\varphi}(x,t) \nabla \varphi(x) \cdot \nabla u(x,t) dx.
\]

Then \( T_-(u_0) = T_+(u_0) = +\infty, \) the solution \( u \) belongs to \( C(\mathbb{R}^1, H^1(\mathbb{R}^N)), \) and there exists \( u_{0,+}, u_{0,-} \in H^1(\mathbb{R}^N) \) such that

\[
\lim_{t \to +\infty} \|u(t) - e^{it\Delta}u_{0,+}\|_{H^1(\mathbb{R}^N)} = 0, \quad \lim_{t \to -\infty} \|u(t) - e^{it\Delta}u_{0,-}\|_{H^1(\mathbb{R}^N)} = 0.
\]
Remark 1.2. (i) Let \( \varphi_R \in C_0^\infty(\mathbb{R}^N) \) be a cut-off function, which satisfies \( \varphi_R(x) \equiv 1 \) if \( |x| \leq R \); \( \varphi_R(x) \equiv 0 \) if \( |x| \geq 2R \); \( |\nabla \varphi_R(x)| \leq \frac{C}{R} \) for any \( x \in \mathbb{R}^N \). Then it follows from Lemma 2.2 below that

\[
\sup_{t \in (-\infty, T_\tau(u_0), T_\tau(u_0))} \left| \int_{\mathbb{R}^N} \nabla \varphi_R \cdot \nabla u(x, t) \, dx \right| \leq \frac{C}{R} \left\| u(t) \right\|_{L^2(R \leq |x| \leq 2R)} \left\| \nabla u(t) \right\|_{L^2(R \leq |x| \leq 2R)} \leq \frac{C}{R} \left\| u_0 \right\|_{L^2(\mathbb{R}^N)} \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty,
\]

which implies that for any \( \epsilon > 0 \), there exists a large number \( R > 0 \) such that

\[
\inf_{t \in (0, T_\tau(u_0))} \int_{\mathbb{R}^N} \nabla \varphi_R \cdot \nabla u(x, t) \, dx \geq -\epsilon.
\]

However, this estimate does not work in obtaining (2.26) below because we have to let \( t = t_j \longrightarrow +\infty \) in (2.26). That is why we need the additional assumption (1.6) in Theorem 1.1.

(ii) If the initial datum \( u_0 \in \dot{H}^1(\mathbb{R}^N) \) \((N = 3, 4, 5)\) is radial. The global existence of the strong solution of (1.3) and the scattering in \( \dot{H}^1(\mathbb{R}^N) \) are proved in [19] without assumption (1.6). Here we do not need the radial symmetry assumption on \( u_0 \), which is replaced by (1.6). Therefore, our conclusion (i.e., Theorem 1.1) improves the results in [19] in some sense.

(iii) It is well known that if \( E(u_0) < 0 \), \( u_0 \in H^1(\mathbb{R}^N) \) with \( |x|u_0 \in L^2(\mathbb{R}^N) \), then the solution \( u \) of (1.3) blows up at some finite time. But it does not contradict Theorem 1.1. In fact, under the assumptions in Theorem 1.1, the initial energy \( E(u_0) \geq 0 \). Indeed, using the assumption \( \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)} < \left\| \nabla W \right\|_{L^2(\mathbb{R}^N)} \) and the Sobolev inequality, we get

\[
E(u_0) = \frac{1}{2} \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2^*} \left\| u_0 \right\|_{L^{2^*}(\mathbb{R}^N)}^2 \geq \left( \frac{1}{2} - \frac{N - 2}{2N} C_N \frac{2}{N} \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)} \right) \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)} \geq \left( \frac{1}{2} - \frac{N - 2}{2N} C_N \frac{2}{N} \left\| \nabla W \right\|_{L^2(\mathbb{R}^N)} \right) \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)} = \frac{1}{N} \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^N)}^2,
\]

where \( C_N = \left\| \nabla W \right\|_{L^2(\mathbb{R}^N)}^{\frac{2}{N}} \) is the best Sobolev constant (see [28] for details).

Throughout this paper, we denote the norm of \( H^1(\mathbb{R}^N), \dot{H}^1(\mathbb{R}^N) \) by \( \| u \|_{H^1} = \left( \int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2) \, dx \right)^{\frac{1}{2}}, \| u \|_{\dot{H}^1} = \left( \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}}, \) respectively, and positive constants (possibly different line to line) by \( C \).
2. PROOF OF THE MAIN RESULT

**Lemma 2.1.** Let \( u \in C((-T_-(u_0), T_+(u_0)), H^1(\mathbb{R}^N)) \) be a solution of (1.3), and let \( \varphi \in C^4([0, \infty)) \) with \( \varphi(s) \equiv \text{const} \) if \( s > 0 \) is large. Then for any \( t \in (-T_-(u_0), T_+(u_0)) \)

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \varphi(|x|)|u(x, t)|^2 \, dx = 2 \text{Im} \int_{\mathbb{R}^N} \nabla \varphi(|x|) \cdot \nabla u(x, t) \overline{u}(x, t) \, dx
\]

and

\[
\frac{d^2}{dt^2} \int_{\mathbb{R}^N} \varphi(|x|)|u(x, t)|^2 \, dx = 4 \int_{\mathbb{R}^N} \varphi''(|x|)|\nabla u(x, t)|^2 \, dx - \frac{4}{N} \int_{\mathbb{R}^N} \Delta \varphi(|x|)|u(x, t)|^{2^*} \, dx -
\]

\[
- \int_{\mathbb{R}^N} \Delta^2 \varphi(|x|)|u(x, t)|^2 \, dx.
\]

**Proof.** Since the proof is similar to those of Lemma in [12] and Lemma 7.6.2 in [5], we omit the details here.

The following variational estimates are Theorem 3.9 and Corollary 3.13 in [19].

**Lemma 2.2 ([19]).** Suppose that

\[
\int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx < \int_{\mathbb{R}^N} |\nabla W|^2 \, dx \quad \text{and} \quad E(u_0) < (1 - \delta_0)E(W), \quad \text{where} \quad \delta_0 \in (0, 1).
\]

Let \( I \ni 0 \) be the maximal interval of existence of the solution \( u \in C(I, H^1(\mathbb{R}^N)) \) of (1.3). Then there exists \( \overline{\delta} = \overline{\delta}(\delta_0, N) > 0 \) such that for each \( t \in I \)

\[
\int_{\mathbb{R}^N} |\nabla u(x, t)|^2 \, dx < (1 - \overline{\delta}) \int_{\mathbb{R}^N} |\nabla W|^2 \, dx,
\]

\[
\overline{\delta} \int_{\mathbb{R}^N} |\nabla u(x, t)|^2 \, dx < \int_{\mathbb{R}^N} (|\nabla u(x, t)|^2 - |u(x, t)|^2) \, dx,
\]

\[
E(u(t)) \geq 0.
\]

Furthermore, \( E(u(t)) \simeq \int_{\mathbb{R}^N} |\nabla u(x, t)|^2 \, dx \simeq \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx \), for all \( t \in I \) with comparability constants which depend only on \( \delta_0 \).

The following rigidity theorem plays a fundamental role in the proof of Theorem 1.1.

**Theorem 2.3.** Assume that \( u_0 \in H^1(\mathbb{R}^N) \) satisfies

\[
\int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx < \int_{\mathbb{R}^N} |\nabla W|^2 \, dx \quad \text{and} \quad E(u_0) < E(W).
\]
Let \( u \) be the solution of (1.3) with the maximal interval of existence \((-T_-(u_0), T_+(u_0))\), and let the assumption (1.6) hold. Suppose that there exists \( \lambda(t) > 0 \), \( x(t) \in \mathbb{R}^N \) with the property that

\[
K = \left\{ v(x,t) = \frac{1}{\lambda(t)^{\frac{4}{N-2}}} u\left(\frac{x-x(t)}{\lambda(t)}\right) : t \in [0,T_+(u_0)) \right\}
\]

is such that \( K \) is compact in \( \dot{H}^1(\mathbb{R}^N) \). Then \( T_+(u_0) = +\infty \), \( u_0 \equiv 0 \) in \( \mathbb{R}^N \).

**Remark 2.4.** If \( x(t) \equiv 0 \) or \( \lambda(t) \geq A > 0 \) and \( |x(t)| \leq C_0 \), Theorem 2.3 is verified in [19] for \( u_0 \in \dot{H}^1(\mathbb{R}^N) \).

**Proof of Theorem 2.3.** Step 1. \( T_+(u_0) = +\infty \). If \( T_+(u_0) < +\infty \), then from Lemma 2.11 in [19], one has

\[
\|u\|_{S(0,T_+(u_0))} = +\infty, \quad \text{where} \quad \|u\|_{S(t)} = \left\| u \right\|_{L^{\frac{2(N+2)}{N-2}}(I; L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))}.
\]

Now we claim that

\[
\lambda(t) \to +\infty \quad \text{as} \quad t \to T_+(u_0).
\]

Indeed if there exists a sequence \( \{t_j\} \), \( t_j \to T_+(u_0) \) such that \( \lambda(t_j) \to A < +\infty \) as \( j \to +\infty \).

Set \( v_j(x) = u(x,t_j) = \frac{1}{\lambda(t_j)^{\frac{4}{N-2}}} u\left(\frac{x-x(t_j)}{\lambda(t_j)}\right) \). It follows from the compactness of \( K \) in \( \dot{H}^1(\mathbb{R}^N) \) that there is a subsequence (still denoted by \( \{v_j\} \)) and \( v_0 \in \dot{H}^1(\mathbb{R}^N) \) such that

\[
v_j \to v_0 \quad \text{in} \quad \dot{H}^1(\mathbb{R}^N).
\]

Then it holds

\[
u\left( y - \frac{x(t_j)}{\lambda(t_j)} \right) = \lambda(t_j)^{\frac{N-2}{2}} v_j(\lambda(t_j)y) \to A^{\frac{N-2}{2}} v_0(Ay) \quad \text{in} \quad \dot{H}^1(\mathbb{R}^N).
\]

If \( A = 0 \), it follows from (2.3) that \( u(y - \frac{x(t_j)}{\lambda(t_j)} \to 0 \) in \( \dot{H}^1(\mathbb{R}^N) \). So

\[
\|\nabla u(t_j)\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t_j \to T_+(u_0).
\]

Using the conservation of energy (1.5), one has

\[
E(u_0) = E(u(t_j)) \to 0 \quad \text{as} \quad t_j \to T_+(u_0).
\]

In addition, (iii) in Remark 1.2 and the assumption: \( \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2} \) yield

\[
\|\nabla u_0\|_{L^2}^2 \leq NE(u_0).
\]

Combining (2.5) and (2.6), we infer \( \|\nabla u_0\|_{L^2} = 0 \). So \( u_0 \equiv 0 \) in \( \mathbb{R}^N \). Using the conservation of charge (1.4), one has for \( t \in [0,T_+(u_0)) \)

\[
\int_{\mathbb{R}^N} |u(t,x)|^2 \, dx = \int_{\mathbb{R}^N} |u_0(x)|^2 \, dx = 0,
\]
which implies us that \( u \equiv 0 \) a.e. on \( \mathbb{R}^N \times [0, T_+(u_0)) \). This is a contradiction with (2.1).

If \( \lim_{j \to \infty} \lambda(t_j) = A \in (0, +\infty) \). Let \( h(x, t) \) be the solution of (1.3) (which is guaranteed by Remark 2.8 in [19]) on the interval \( I_{\eta} = (T_+(u_0) - \eta, T_+(u_0) + \eta) \), \( h(x, T_+(u_0)) = A^{\frac{2}{N-2}} v_0(Ax) \), \( \|h\|_{S(I_{\eta})} < +\infty \), where \( \eta = \eta(\|\nabla v_0\|_{L^2(\mathbb{R}^N)}) \).

Let \( h_j(x, t) \) be the solution of (1.3) with \( h_j(x, T_+(u_0)) = u(x - \frac{x(t_j)}{\lambda(t_j)}, t_j) \). Then the convergence in (2.3) and the continuous dependence on the initial data (see Remark 2.17 in [19]) imply that

\[
\|h_j - h\|_{S(I_{\eta}^2)} \to 0 \quad \text{as} \quad j \to +\infty.
\]

Then

\[
\sup_j\|h_j\|_{S(I_{\eta}^2)} < +\infty. \tag{2.7}
\]

In addition, the uniqueness theorem on the strong solution of (1.3) (see Definition 2.10 in [19]) yields

\[
h_j(x, t) = u\left(x - \frac{x(t_j)}{\lambda(t_j)}, t + t_j - T_+(u_0)\right) \quad \text{for every} \quad t \in I_{\eta}^2. \tag{2.8}
\]

Combining (2.7) and (2.8), we get

\[
+\infty > \sup_j\|h_j\|_{S(I_{\eta}^2)} \geq \lim_{j \to +\infty} \|u\|_{S(t_j - \frac{\eta}{2}, t_j + \frac{\eta}{2})} \geq \|u\|_{S(T_+(u_0) - \frac{\eta}{2}, T_+(u_0))} = +\infty,
\]

which contradicts (2.1).

From the above arguments, we know that (2.2) holds.

Let \( \psi \in C_0^\infty(\mathbb{R}^N), \psi(x) = \psi(|x|), \psi \equiv 1 \) for \( |x| \leq 1 \), \( \psi \equiv 0 \) for \( |x| \geq 2 \), \( |\nabla \psi| \leq 2 \).

Define \( \psi_R(x) = \psi\left(\frac{|x|}{R}\right) \) and

\[
y_R(t) = \int_{\mathbb{R}^N} |u(x, t)|^2 \psi_R(x) dx, \quad \forall t \in [0, T_+(u_0)].
\]

Then from Lemma 2.1 and the conservation of charge (1.4), one has

\[
|y_R'(t)| \leq 2\left|\text{Im} \int_{\mathbb{R}^N} \overline{u} \nabla u \cdot \nabla \psi_R(x) dx \right| \leq \frac{C}{R} \left( \int_{\mathbb{R}^N} |\nabla u(x, t)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u(x, t)|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{R} \left( \int_{\mathbb{R}^N} |\nabla W(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u_0(x)|^2 dx \right)^{\frac{1}{2}}. \tag{2.9}
\]
Note that \( u(x, t) = \lambda(t)^{N/2} v(\lambda(t) x + x(t), t) \), we deduce for any \( R > 0, \epsilon > 0 \)
\[
\int_{|x| < R} |u(x, t)|^2 dx = \lambda(t)^{-2} \int_{|y-x(t)| < R\lambda(t)} |v(y, t)|^2 dy = \\
= \lambda(t)^{-2} \int_{B(x(t), R\lambda(t)) \cap B(0, \epsilon R\lambda(t))} |v(y, t)|^2 dy + \\
+ \lambda(t)^{-2} \int_{B(x(t), R\lambda(t)) \setminus B(0, \epsilon R\lambda(t))} |v(y, t)|^2 dy. 
\] (2.10)

Using Hölder inequality and the compactness property of \( K \) in \( \dot{H}^1(\mathbb{R}^N) \), we conclude from (2.2) that
\[
\lambda(t)^{-2} \int_{B(x(t), R\lambda(t)) \cap B(0, \epsilon R\lambda(t))} |v(y, t)|^2 dy \leq CR^2 \epsilon^2 \int_{|y| < R\lambda(t)} |v(y, t)|^2^* dy \leq CR^2 \epsilon^2 \int_{\mathbb{R}^N} |\nabla W|^2 dx
\] (2.11) 

and
\[
\lambda(t)^{-2} \int_{B(x(t), R\lambda(t)) \setminus B(0, \epsilon R\lambda(t))} |v(y, t)|^2 dy \leq CR^2 \epsilon^2 \int_{|y| \geq R\lambda(t)} |v(y, t)|^2^* dy \rightarrow 0 \quad \text{as} \quad t \rightarrow T_+(u_0). 
\] (2.12)

Combining (2.10), (2.11) and (2.12), we derive for all \( R > 0 \)
\[
\int_{|x| < R} |u(x, t)|^2 dx \rightarrow 0 \quad \text{as} \quad t \rightarrow T_+(u_0),
\]
and so
\[
y_R(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow T_+(u_0). \] (2.13)

From (2.9), (2.13), we obtain for any \( t \in [0, T_+(u_0)) \) and \( R > 0 \)
\[
y_R(t) = |y_R(t) - y_R(T_+(u_0))| \leq \\
\leq \frac{C}{R}(T_+(u_0) - t) \left( \int_{\mathbb{R}^N} |\nabla W(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u_0(x)|^2 dx \right)^{\frac{1}{2}}. 
\] (2.14)

Let \( R \rightarrow +\infty \) in (2.14), we get
\[
\int_{\mathbb{R}^N} |u(t, x)|^2 dx = 0 \quad \text{for each} \quad t \in [0, T_+(u_0)),
\]
and then $u \equiv 0$ a.e. on $\mathbb{R}^N \times [0, T_+(u_0))$, which contradicts (2.1). Therefore, $T_+(u_0) = +\infty$.

Step 2. $u_0 \equiv 0$ in $\mathbb{R}^N$. If $u_0 \not\equiv 0$ in $\mathbb{R}^N$, it holds true that
\[ \sup_{t \in [0, +\infty)} |x(t)| < +\infty. \]  
(2.15)

In fact, assume that there exists an increasing sequence $\{t_j\}$, $t_j \to +\infty$ ($= T_+(u_0)$) as $j \to +\infty$ such that
\[ |x(t_j)| \to +\infty \quad \text{as} \quad j \to +\infty. \]  
(2.16)

It follows from the Hardy inequality and the compactness property of $K$ in $\dot{H}^1(\mathbb{R}^N)$ that for any $\epsilon > 0$, there exists a large number $M(\epsilon) > 0$ such that for any $M \geq M(\epsilon)$
\[ \sup_{t \in [0, +\infty)} \int_{|y| \geq M(\epsilon)} (|\nabla v(y, t)|^2 + |v(y, t)|^{2*}) dy < \epsilon. \]  
(2.17)

Note that for any $Q > R > 0$ and $t \in [0, +\infty)$
\[ \int_{R < |x| < Q} |\nabla u(x, t)|^2 dx = \int_{R(\lambda(t)) < |y - x(t)| < Q \lambda(t)} |\nabla v(y, t)|^2 dy. \]  
(2.18)

In the next discussion, we analyze the three possible cases of the limit of the sequence $\{\frac{\lambda(t_j)}{|x(t_j)|}\}$ (select a subsequence if necessary).

(1) If $\lim_{j \to +\infty} \frac{\lambda(t_j)}{|x(t_j)|} = 0$, then for any $Q > 0$
\[ \lim_{j \to +\infty} \left( |x(t_j)| - Q \lambda(t_j) \right) = \lim_{j \to +\infty} \left( |x(t_j)| \left( 1 - \frac{Q \lambda(t_j)}{|x(t_j)|} \right) \right) = +\infty > M(\epsilon). \]

From (2.17) and (2.18), one has for any $Q > 0$
\[ \lim_{j \to +\infty} \int_{|x| < Q} |\nabla u(x, t_j)|^2 dx \leq \lim_{j \to +\infty} \int_{|y| \geq |x(t_j)| - Q \lambda(t_j)} |\nabla v(y, t_j)|^2 dy \leq \sup_{t \in [0, +\infty)} \int_{|y| \geq M(\epsilon)} |\nabla v(y, t)|^2 dy \leq \epsilon. \]  
(2.19)

Similarly, using the Sobolev inequality, we infer that for any $Q > 0$
\[ \lim_{j \to +\infty} \int_{|x| < Q} |u(x, t_j)|^2 dx \leq \epsilon. \]  
(2.20)

Combination of (2.19), (2.20) yields that (selecting a subsequence if necessary) for any $Q > 0$
\[ u(x, t_j) \to 0 \quad \text{a.e. on} \quad \{ x \in \mathbb{R}^N; \ |x| < Q \} \quad \text{as} \quad j \to +\infty. \]  
(2.21)
On the other hand, it follows from the conservation of charge (1.4) and Lemma 2.2 that
\[ \sup_j \|u(t_j)\|_{H^1} < \infty. \]
Up to a subsequence if necessary,
\[ u(x, t_j) \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(\mathbb{R}^N) \quad \text{and} \quad L^2(\mathbb{R}^N) \quad \text{as } j \to +\infty; \quad (2.22) \]
and
\[ u(x, t_j) \to \tilde{u} \quad \text{a.e. on } \mathbb{R}^N \quad \text{as } j \to +\infty. \quad (2.23) \]
From (2.21) and (2.23), we infer that
\[ \tilde{u} = 0 \quad \text{a.e. on } \{ x \in \mathbb{R}^N : |x| < Q \} \quad \text{as } j \to +\infty; \quad (2.24) \]
and so
\[ \tilde{u} = 0 \quad \text{a.e. on } \mathbb{R}^N \quad \text{due to the arbitrariness of } Q. \]
From (2.21)–(2.24), up to a subsequence if necessary, we derive
\[ u(x, t_j) \to 0 \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N) \quad \text{as } j \to +\infty. \quad (2.25) \]
Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) be the given real-valued function in (1.6). Then it follows from assumption (1.6) and Lemma 2.1 that for any \( t > 0 \)
\[ \int_{\mathbb{R}^N} \varphi(x)|u(x, t)|^2 \, dx \geq \int_{\mathbb{R}^N} \varphi(x)|u_0(x)|^2 \, dx. \quad (2.26) \]
Letting \( t = t_j \to +\infty \) in (2.26), together with (2.25), we deduce that
\[ \int_{\mathbb{R}^N} \varphi(x)|u_0(x)|^2 \, dx \leq 0, \]
which is a contradiction because of the assumption: \( \int_{\mathbb{R}^N} \varphi(x)|u_0(x)|^2 \, dx > 0. \)

(2) If \( \lim_{j \to +\infty} \frac{\lambda(t_j)}{|x(t_j)|} \in (0, +\infty) \), there exist \( R > 0 \) (which is independent of \( j, \epsilon \)) and \( j_1 = j_1(\epsilon) > 0 \) such that \( R \frac{\lambda(t_j)}{|x(t_j)|} \geq 2 \) and \( |x(t_j)| \geq M(\epsilon) \) for any \( j \geq j_1 \). Then from (2.17) and (2.18), one gets for any \( j \geq j_1 \),
\[ \int_{|x| > R} |\nabla u(x, t_j)|^2 \, dx \leq \int_{|y| \geq (R \frac{\lambda(t_j)}{|x(t_j)|})^{-1}|x(t_j)|} |\nabla v(y, t_j)|^2 \, dy \leq \sup_{t \in [0, +\infty]} \int_{|y| \geq M(\epsilon)} |\nabla v(y, t)|^2 \, dy \leq \epsilon. \quad (2.27) \]
If \( \lim_{j \to +\infty} \frac{\lambda(t_j)}{|x(t_j)|} = +\infty \), there exists \( j_2 = j_2(\epsilon) > 0 \) such that \( (\frac{\lambda(t_j)}{|x(t_j)|} - 1)|x(t_j)| \geq M(\epsilon) \) for any \( j \geq j_2 \). Then from (2.17) and (2.18), we derive for any \( j \geq j_2 \),

\[
\int_{|x| > 1} |\nabla u(x, t_j)|^2 dx \leq \int_{|y| \geq \left(\frac{\lambda(t_j)}{|x(t_j)|} - 1\right)|x(t_j)|} |\nabla v(y, t_j)|^2 dy \leq \sup_{t \in [0, +\infty)} \int_{|y| \geq M(\epsilon)} |\nabla v(y, t)|^2 dy \leq \epsilon.
\]

(2.28)

Set \( J = \max\{j_1, j_2\} \). From (2.27) and (2.28), we conclude that there exists a positive number \( R \), which is independent of \( j, \epsilon \), such that for any \( j \geq J \)

\[
\int_{|x| > R} |\nabla u(x, t_j)|^2 dx \leq \epsilon.
\]

(2.29)

Using the Sobolev inequality and the Hardy inequality, after a similar argument, we conclude for any \( j \geq J \)

\[
\int_{|x| > R} |u(x, t_j)|^2 dx \leq C(\epsilon), \quad \text{where} \quad C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.
\]

(2.30)

Here we take the same symbols \( R, J \) in (2.29) and (2.30) for the sake of simplicity.

Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \), \( \varphi(x) = \varphi(|x|) \), \( \varphi \equiv |x|^2 \) for \( |x| \leq 1 \); \( \varphi \equiv 0 \) for \( |x| \geq 2 \). Define \( \varphi_R(x) = R^2 \varphi\left(\frac{x}{R}\right) \) and

\[
z_R(t) = \int_{\mathbb{R}^N} |u(x, t)|^2 \varphi_R(x) dx, \quad \forall t \in [0, +\infty).
\]

It follows from Lemmas 2.1, 2.2 and the Hardy inequality that for any \( t \in [0, +\infty) \)

\[
|z'_R(t)| \leq 2 \left| Im \int_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \varphi_R(x) dx \right| \leq CR^2 \left( \int_{\mathbb{R}^N} |\nabla u(x, t)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|u(x, t)|^2}{|x|^2} dx \right)^{\frac{1}{2}} CR^2 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx.
\]

(2.31)

From (2.29), (2.30) and Lemma 2.2, one has for any \( j \geq J \)

\[
8 \int_{|x| \leq R} (|\nabla u(x, t_j)|^2 - |u(x, t_j)|^2) dx \geq C(\delta_0) \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx,
\]

(2.32)

where \( R \) is independent of \( j \).
From (2.29), (2.30), (2.32) and Lemmas 2.1, 2.2, we obtain for any $j \geq J$

$$z''_R(t_j) = 4 \int_{\mathbb{R}^N} \phi_R''(|x|) |\nabla u(x, t_j)|^2 dx - \frac{4}{N} \int_{\mathbb{R}^N} \Delta \varphi_R(|x|) |u(x, t_j)|^2 dx -$$

$$- \int_{\mathbb{R}^N} \Delta^2 \varphi_R(|x|) |u(x, t_j)|^2 dx \geq$$

$$\geq 8 \int_{|x| \leq R} (|\nabla u(x, t_j)|^2 - |u(x, t_j)|^2) dx -$$

$$- C \int_{|x| > R} (|\nabla u(x, t_j)|^2 + |u(x, t_j)|^2) dx -$$

$$- C \int_{R \leq |x| \leq 2R} (|u(x, t_j)|^2 \phi_R) dx \geq C \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx,$$

(2.33)

where $R$ is given in (2.31), and independent of $j$.

Combining (2.31), (2.32) and (2.33), we conclude for any $j \geq J$

$$CR^2 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx \geq |z''_R(2t_j) - z''_R(t_j)| =$$

$$= t_j \int_0^1 z''_R(2st_j + (1-s)t_j) ds \geq C t_j \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx,$$

from which we get a contradiction if $j \geq J$ is sufficiently large, because $t_j \to +\infty$ as $j \to +\infty$, and $R$ is independent of $j$. Here we have used the fact: replacing $t_j$ by any $t$ with $t \geq t_j$, $j \geq J$, (2.33) still holds. This is not difficult to verify because the sequence $\{t_j\}$ is taken to be increasing on $j$.

Whence (2.15) holds. Now we claim that there exists a positive number $C_0$ (which is independent of $t$) such that

$$\lambda(t) \geq C_0 \quad \text{for any} \quad t \in [0, +\infty).$$

(2.34)

We present a proof by contradiction. Assume that there is a sequence $\{t_m\}$, $t_m \to +\infty$ as $m \to +\infty$ such that

$$\lambda(t_m) \to 0 \quad \text{as} \quad m \to +\infty.$$

Observe that $u(x, t) = \lambda(t)^{-\frac{N-2}{2}} v(\lambda(t)x + x(t), t)$. From the conservation of charge (1.4), one has

$$\int_{\mathbb{R}^N} |v(x, t_m)|^2 dx = \lambda(t_m)^2 \int_{\mathbb{R}^N} |u(x, t_m)|^2 dx = \lambda(t_m)^2 \int_{\mathbb{R}^N} |u_0(x)|^2 dx,$$
which implies that

\[ v(x, t_m) \rightarrow 0 \text{ a.e. on } \mathbb{R}^N \text{ as } m \rightarrow \infty. \]

Whence from the compactness property of the set \( K \) in \( \dot{H}^1(\mathbb{R}^N) \), we can find a subsequence of \( \{ v(x, t_m) \} \) (still denoted by \( \{ v(x, t_m) \} \)) such that

\[ v(x, t_m) \rightarrow 0 \text{ in } \dot{H}^1(\mathbb{R}^N) \text{ as } m \rightarrow \infty. \] (2.35)

However, one gets from Lemma 2.2

\[ \int_{\mathbb{R}^N} |\nabla v(x, t_m)|^2 dx = \int_{\mathbb{R}^N} |\nabla u(x, t_m)|^2 dx \simeq \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx > 0. \] (2.36)

This contradicts (2.35) by passing the limit \( m \rightarrow \infty \) in (2.36). Therefore (2.34) holds.

From (2.15) and (2.34), we conclude that for any \( t \in [0, +T_1(u_0)) \) and \( R > 0 \)

\[ \int_{|x| > R} |\nabla u(x, t)|^2 dx = \int_{|y-x(t)| > R\lambda(t)} |\nabla v(y, t)|^2 dy \leq \int_{|y| > R\lambda(t)-|x(t)|} |\nabla v(y, t)|^2 dy \leq \int_{|y| > C R-C} |\nabla v(y, t)|^2 dy. \]

Whence it follows from (2.34) that for \( \epsilon > 0 \), there exists a large number \( R(\epsilon) > 0 \) such that for any \( t \in [0, +\infty) \)

\[ \int_{|x| > R(\epsilon)} (|\nabla u(x, t)|^2 + |u(x, t)|^2^*) dx < \epsilon. \] (2.37)

In addition, Lemma 2.2 implies that

\[ 8 \int_{\mathbb{R}^N} (|\nabla u(x, t)|^2 - |u(x, t)|^2^*) dx \geq \tilde{C}_0 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \] (2.38)

It follows from (2.37) and (2.38) that there exists a sufficiently large number \( M_0 > 0 \) such that for all \( t \in [0, +\infty) \)

\[ 8 \int_{|x| \leq M_0} (|\nabla u(x, t)|^2 - |u(x, t)|^2^*) dx \geq C \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx, \] (2.39)

where we take \( \epsilon = \epsilon_0 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx \) in (2.37) with \( \epsilon_0 > 0 \) suitably small.

Let \( z_R(t) \) be defined as in the above. From Lemma 2.1, one has for any \( t \in [0, +\infty) \)

\[ |z'_R(t) - z'_R(0)| \leq C R^2 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 dx. \] (2.40)
From (2.40) and Lemmas 2.1, 2.2, we obtain for every $t \in [0, +\infty)$
\[
z_M''(t) = 4 \int_{\mathbb{R}^N} \varphi_M'(|x|) \nabla u(x,t)^2 \, dx - \frac{4}{N} \int_{\mathbb{R}^N} \Delta \varphi_M(|x|) |u(x,t)|^2 \, dx - \int_{\mathbb{R}^N} \Delta^2 \varphi_M(|x|) |u(x,t)|^2 \, dx \geq 8 \int_{\mathbb{R}^N} (|\nabla u(x,t)|^2 - |u(x,t)|^2^2) \, dx - C \int_{|x| > M_0} (|\nabla u(x,t)|^2 + |u(x,t)|^2^2) \, dx - C \int_{M_0 < |x| \leq 2M_0} (|u(x,t)|^2^2) \, dx \geq C \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 \, dx.
\]
Combining (2.40) and (2.41), we obtain for every $t \in [0, +\infty)$
\[
CM_0^2 \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 \, dx \geq |z_M'(t) - z_M'(0)| = \int_0^t z_M'(s) \, ds \geq Ct \int_{\mathbb{R}^N} |\nabla u_0(x)|^2 \, dx,
\]
from which we get a contradiction if $t > 0$ is large enough unless $\int_{\mathbb{R}^N} |\nabla u_0(x)|^2 \, dx = 0$.

From the above argument of Steps 1, 2, we complete the proof of Theorem 2.3. \( \square \)

**Proof of Theorem 1.1.** We first introduce notation (see [19]): $(SC)(u_0)$ holds if for the particular function $u_0$ with $\int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx < \int_{\mathbb{R}^N} |\nabla W|^2 \, dx$ and $E(u_0) < E(W)$. Let $u$ be the corresponding strong solution of problem (1.3) with maximal interval of existence $I$, then $I = (-\infty, +\infty)$ and $\|u\|_{\mathcal{S}((-\infty, +\infty))} < \infty$, where $\| \cdot \|_{\mathcal{S}(I)} = \| \cdot \|_{L^{\frac{2(N+2)}{N-2}}(J, L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))}$. Note that if $\|\nabla u_0\|_{L^2(\mathbb{R}^N)} \leq \delta$, $(SC)(u_0)$ holds. Whence there exists a number $E_C$ with $\delta \leq E_C \leq E(W)$ such that if $u_0$ is as in $(SC)(u_0)$ and $E(u_0) < E_C$, $(SC)(u_0)$ holds and $E_C$ is optimal with this property.

From Remark 2.8 in [19] and the uniqueness theory on strong solutions of (1.3) (see Definition 2.10 in [19]), we know that problem (1.3) admits a unique maximal strong solution $u \in ((-T_- (u_0), T_+ (u_0)), H^1(\mathbb{R}^N))$. If $T_+ (u_0) < +\infty$ then by Lemma 2.11 in [19], $\|u\|_{\mathcal{S}(I_+)} = +\infty$, where $I_+ = [0, T_+(u_0)]$. By the definition of $E_C$, we infer that $E(u_0) \geq E_C$. If $E(u_0) = E_C$, then by Proposition 4.2 in [19], there exists $x(t) \in \mathbb{R}^N$ and $\lambda(t) \in \mathbb{R}^+$ such that
\[
K = \left\{ v(x,t) = \frac{1}{\lambda(t)^{\frac{2}{N-2}}} u \left( \frac{x - x(t)}{\lambda(t)} \right) : t \in I_+ \right\}
\]
has the property that $\overline{K}$ is compact in $H^1(\mathbb{R}^N)$. Therefore it follows from Theorem 2.3 that $T_+(u_0) = +\infty$, $u_0 \equiv 0$ in $\mathbb{R}^N$, which is a contradiction (we may always
assume $u_0 \not= 0$ in $\mathbb{R}^N$. Otherwise, the uniqueness theory on strong solutions of (1.3) in Definition 2.10 in [19] implies that problem (1.3) has only a trivial (global) solution.

If $E(u_0) > E_C$. Note that $E(su_0) \to 0$ as $s \to 0$, there exists $s_0 \in (0, 1)$ such that $E(s_0u_0) = E_C$. Repeating the proof in the case $E(u_0) = E_C$, we also infer $u_0 \equiv 0$ in $\mathbb{R}^N$, which is a contradiction. Similarly, a contradiction appears if $T_-(u_0) < \infty$.

From the above arguments, we conclude that $(SC)$ holds. That is, $T_-(u_0) = +\infty$ and $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$, $u \in L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N)$. Moreover from Remark 2.8 in [19] and following the proof of Theorem 2.5 in [19], $\nabla u \in L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N)$.

Note that

$$u(t) = e^{it\Delta}u_0 + i \int_0^t e^{i(t-s)\Delta}|u(s)|^{\frac{4}{N-2}}u(s)ds.$$ 

Set $F(t) = e^{it\Delta}$. Then the solution $u$ can be rewritten as

$$u(t) = F(t)u_0 + i \int_0^t F(t-s)|u(s)|^{\frac{4}{N-2}}u(s)ds.$$ 

Let $v(t) = F(-t)u(t)$. It follows from the Strichartz estimates (see [4, 21]) that for any $0 < \tau < t$

$$\|v(t) - v(\tau)\|_{H^1} =$$

$$= \|F(t)(v(t) - v(\tau))\|_{H^1} = \|i \int_\tau^t F(t-s)|u(s)|^{\frac{4}{N-2}}u(s)ds\|_{H^1} \leq$$

$$\leq C \left( \|u\|_{L^\frac{2(N+2)}{N-2}(\mathbb{R})} \right)^4 \left( \|\nabla(u)|^{\frac{4}{N-2}}u\|_{L^\frac{4(N+2)}{N-2}(\mathbb{R})} \right) \leq$$

$$\leq C \|u\|_{S(\tau, t)} \left( \|u\|_{W(\tau, t)} + \|\nabla u\|_{W(\tau, t)} \right),$$

where $\|u\|_{S(\tau, t)} = \|u\|_{L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N)}$, $\|u\|_{W(\tau, t)} = \|u\|_{L^{\frac{2(N+2)}{N-2}(\mathbb{R})}}$, and the Sobolev inequality is used: $\|u\|_{S(\tau, t)} \leq C \|u\|_{W(\tau, t)} \forall \tau \in \mathbb{R}$.

Whence $\|v(t) - v(\tau)\|_{H^1} \to 0$ as $\tau, t \to +\infty$. Therefore, there exists $u_+ \in H^1(\mathbb{R}^N)$ such that $v(t) \to u_+$ in $H^1(\mathbb{R}^N)$ as $t \to +\infty$. So

$$\|u(t) - e^{it\Delta}u_+\|_{H^1(\mathbb{R}^N)} =$$

$$= \|F(t)(v(t) - u_+)\|_{H^1(\mathbb{R}^N)} = \|v(t) - u_+\|_{H^1(\mathbb{R}^N)} \to 0 \text{ as } t \to +\infty.$$ 

Similarly there exists $u_- \in H^1(\mathbb{R}^N)$ such that $\|u(t) - e^{it\Delta}u_-\|_{H^1(\mathbb{R}^N)} \to 0$ as $t \to -\infty$.
Acknowledgments
This work was completed with the support in part by the Key Laboratory of Random Complex Structures and Data Science, Chinese Academy of Sciences; National Natural Science Foundation of China under grant No. 11071239.

REFERENCES

[1] J. Bourgain, Global well-posedness of defocusing critical nonlinear Schrödinger equation in the radial case, J. Amer. Math. Soc. 12 (1999), 145–171.

[2] D. Cao, P. Han, Inhomogeneous critical nonlinear Schrödinger equations with a harmonic potential, J. Math. Phys. 51 (2010), 043505, 24 pp.

[3] R. Carles, S. Keraani, On the role of quadratic oscillations in nonlinear Schrödinger equations. II. The $L^2$-critical case, Trans. Amer. Math. Soc. 359 (2007), 33–6.

[4] T. Cazenave, Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics, vol. 10, New York UniversityCourant Institute of Mathematical Sciences, New York, 2003.

[5] T. Cazenave, A. Haraux, An introduction to semilinear evolution equations, Oxford Lecture Series in Mathematics and Its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998.

[6] T. Cazenave, F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$, Nonlinear Anal. 14 (1990), 807-836.

[7] T. Cazenave, F.B. Weissler, Some remarks on the nonlinear Schrödinger equation in the critical case, Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987), 18–29, Lecture Notes in Math., 1394, Springer, Berlin, (1989).

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $R^3$, Ann. Math. 167 (2008), 767–865.

[9] T. Duyckaerts, J. Holmer, S. Roudenko, Scattering for the non-radial 3D cubic nonlinear Schrödinger equation, Math. Res. Lett. 15 (2008), 1233–1250.

[10] T. Duyckaerts, F. Merle, Dynamic of threshold solutions for energy-critical NLS, Geom. Funct. Anal. 18 (2009), 1787–1840.

[11] J. Ginibre, G. Velo, On a class of nonlinear Schrödinger equations, J. Funct. Anal. 32 (1979), 1–71.

[12] R.T. Glassey, On the blowup of nonlinear Schrödinger equations, J. Math. Phys. 18 (1977), 1794–1797.

[13] M.G. Grillakis, On nonlinear Schrödinger equations, Commun. Partial Differ. Equations. 25 (2000), 1827–1844.

[14] T. Hmidi, S. Keraani, Remarks on the blowup for the $L^2$-critical nonlinear Schrödinger equations, SIAM J. Math. Anal. 38 (2006), 1035–1047.
Well-posedness and scattering for nonlinear Schrödinger equation

[15] J. Holmer, S. Roudenko, On blow-up solutions to the 3D cubic nonlinear Schrödinger equation, Appl. Math. Res. Express. AMRX 2007, No. 1, Art. ID abm004, 31 pp.

[16] J. Holmer, S. Roudenko, A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation, Comm. Math. Phys. 282 (2008), 435–467.

[17] D. Li, X. Zhang, Dynamics for the energy critical nonlinear Schrödinger equation in high dimensions, J. Funct. Anal. 256 (2009), 1928–1961.

[18] S. Keraani, On the defect of compactness for the Strichartz estimates of the Schrödinger equations, J. Differential Equations. 175 (2001), 353–392.

[19] C.E. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166 (2006), 645–675.

[20] J. Krieger, W. Schlag, Non-generic blow-up solutions for the critical focusing NLS in 1-D, J. Eur. Math. Soc. 11 (2009), 1–125.

[21] M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998), 955–980.

[22] R. Killip, M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, arXiv:0804.1018v1 [math.AP].

[23] F. Merle, Nonexistence of minimal blow-up solutions of equation $iu_t = -\Delta u - k(x)|u|^4 u$ in $\mathbb{R}^N$, Ann. Inst. Henri Poincaré Physique Thérique 64 (1996), 33–85.

[24] T. Ogawa, Y. Tsutsumi, Blow-up of $H^1$-solution for the nonlinear Schrödinger equation, J. Differ. Equations 92 (1991), 317–330.

[25] T. Ogawa, Y. Tsutsumi, Blow-up of $H^1$-solution for the nonlinear Schrödinger equation with critic power nonlinearity, Proc. Amer. Math. Soc. 111 (1991), 487–496.

[26] E. Ryckman, M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $R^{1+4}$, Amer. J. Math. 129 (2007), 1–60.

[27] W. Schlag, Stable manifolds for an orbitally unstable nonlinear Schrödinger equation, Ann. Math. 169 (2009), 139–227.

[28] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976), 353–372.

[29] T. Tao, Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data, New York J. Math. 11 (2005), 57–80.

[30] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, Duke Math. J. 138 (2007), 281–374.

[31] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1983), 567–576.
Pigong Han
pghan@amss.ac.cn

Chinese Academy of Sciences
Academy of Mathematics and Systems Science
Beijing 100190, China

Received: June 8, 2011.
Revised: August 4, 2011.
Accepted: August 4, 2011.