EQUIVALENCES OF DERIVED CATEGORIES OF SHEAVES ON SMOOTH STACKS

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Abstract. We extend Orlov’s representability theorem on the equivalence of derived categories of sheaves to the case of smooth stacks associated to normal projective varieties with only quotient singularities.

1. Introduction. Orlov’s representability theorem [12] is of fundamental importance in the study of derived categories of sheaves on smooth projective varieties. It says that any equivalence of derived categories for such varieties is representable as a Fourier-Mukai functor. The purpose of this paper is to extend this theorem to the case of projective orbifolds.

To any normal projective variety $X$ which has only quotient singularities, we can attach naturally a smooth stack (or an orbifold) $\mathcal{X}$. Many results on smooth varieties are expected to extend to such stacks. For example, any coherent sheaf on a smooth stack has a finite locally free resolution provided that the stack has a trivial stabilizer group at the generic point and the coarse moduli space is a separated scheme [14]. On the other hand, if we allow quotient singularities on varieties, then there are a lot more possibility of interesting examples such as flops than the case of smooth varieties. Note that there are more sheaves on the stack $\mathcal{X}$ than the underlying variety $X$, and as was shown in [10], the correct category which should be considered is that of the stacky sheaves, not the ordinary sheaves.

For a smooth stack $\mathcal{X}$, we denote by $D^b(\text{Coh}(\mathcal{X}))$, or sometimes simply $D^b(\mathcal{X})$, the derived category of bounded complexes of coherent sheaves. If $e \in D^b(\text{Coh}(\mathcal{X} \times \mathcal{Y}))$ is an object on the product stack, the integral functor

$$\Phi^e_{\mathcal{X} \to \mathcal{Y}} : D^b(\text{Coh}(\mathcal{X})) \to D^b(\text{Coh}(\mathcal{Y}))$$

between such categories is defined by

$$\Phi^e_{\mathcal{X} \to \mathcal{Y}}(a) = p_{2*}(e \otimes p_1^*a)$$
for $a \in D^b(\text{Coh}(\mathcal{X}))$, where $p_1^*$ and $\otimes$ are the left derived functors and $p_2^*$ is the right derived functor. An integral functor is called a Fourier-Mukai functor if it is an equivalence.

The main result of this paper is the following extension of [12] Theorem 2.2:

**Theorem 1.1.** Let $X$ and $Y$ be normal projective varieties with only quotient singularities and let $\mathcal{X}$ and $\mathcal{Y}$ be smooth stacks naturally associated to them. Let

$$F : D^b(\text{Coh}(\mathcal{X})) \to D^b(\text{Coh}(\mathcal{Y}))$$

be an exact functor which is fully faithful and has a left adjoint functor. Then there exist an object $e \in D^b(\text{Coh}(\mathcal{X} \times \mathcal{Y}))$ and an isomorphism of functors

$$F \cong \Phi^{\mathcal{X}}_{\mathcal{X} \times \mathcal{Y}}$$

Moreover, the object $e$ is uniquely determined up to isomorphism.

In §2 we review the way to attach a single object to a complex of objects. There are two different ways of construction, the left and right convolutions, which are both used in the proof of Theorem 1.1.

In §3 we construct an infinite left resolution of the structure sheaf of the diagonal of the self product for any projective variety (Theorem 3.2). The construction is based on a theorem of Backelin [1] which says that the Veronese subring of the homogeneous coordinate ring of sufficiently high degree becomes a Koszul algebra.

In §4 we extend the construction of §3 to smooth stacks by taking the invariant part under a group action (Theorem 4.1). We also prove that for any coherent sheaf on our stack, there exists a surjective morphism from a locally free sheaf of special type (Theorem 4.2). This gives an alternative proof of Totaro’s theorem [14] for our stack. As an example, we describe the resolution of the diagonal explicitly in the case of weighted projective spaces using Canonaco’s formula [6] in §5.

We prove the main theorem in §6. The proof is partly parallel to that of [12], but we also need different arguments because there are more sheaves on the stacks. Indeed, we need both left and right convolutions, an infinite resolution of the diagonal and a special locally free sheaf, which are explained in previous sections.

Section 7 is devoted to applications similar to those appearing in [9]. We consider the consequences of “$D$-equivalence” in Theorem 7.1, and for the group of autoequivalences in Theorem 7.2.

We work over $\mathbb{C}$, but most of our results hold for arbitrary fields.

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2. Convolution. We sometimes have to deal with a complex of objects in a derived category. But we cannot associate a single object to it in a similar way as a single complex associated to a double complex, because quasi-isomorphisms are inverted in the derived category.

We recall the theory of convolutions from [12]. Let $\mathcal{D}$ be a triangulated category, and let

\[(2.1) \quad 0 \longrightarrow a_m \xrightarrow{d_m} a_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} a_1 \xrightarrow{d_0} a_0 \longrightarrow 0\]

be a bounded complex of objects in $\mathcal{D}$, that is, we assume that $d_{i+1}d_i = 0$ for $1 \leq i < m$. Assume the condition

\[(2.2) \quad \text{Hom}_\mathcal{D}(a_p[r], a_q) = 0 \text{ for any } p > q, r > 0.\]

Then the right convolution is defined to be a morphism $d_0 : a_0 \rightarrow a$ to an object in $\mathcal{D}$ which is uniquely constructed inductively on the length $m$ in the following way. (We would like to call this “right” convolution instead of “left” as in [12] because the direction of the convolution is to the right.)

If $m = 0$, then $a = a_0$ and $d_0 = \text{Id}$. If $m \geq 1$, then let $a'_m$ be the cone of the morphism $a_m \rightarrow a_{m-1}$:

\[a_m \longrightarrow a_{m-1} \xrightarrow{j_{m-1}} a'_m \longrightarrow a_m[1]\]

is a distinguished triangle. We have an exact sequence

\[\text{Hom}_\mathcal{D}(a_m[1], a_{m-2}) \rightarrow \text{Hom}_\mathcal{D}(a'_m, a_{m-2}) \rightarrow \text{Hom}_\mathcal{D}(a_m, a_{m-2}).\]

Since the first term as well as the image of $d_{m-1}$ in the last term vanishes, there exists a unique morphism $d'_{m-1} : a'_m \rightarrow a_{m-2}$ such that $d'_{m-1}j_{m-1} = d_{m-1}$. Moreover, we deduce that $d'_{m-2}d'_{m-1} = 0$ by a similar diagram chasing.

Thus we obtain a new complex

\[(2.3) \quad 0 \longrightarrow a'_{m-1} \xrightarrow{d'_{m-1}} a_{m-2} \xrightarrow{d_{m-2}} \cdots \xrightarrow{d_1} a_1 \xrightarrow{d_0} a_0 \longrightarrow 0\]

for which we can check condition (2.2).

The right convolution $d_0 : a_0 \rightarrow a$ is obtained as that of the complex (2.3). Note that the object $a_0$ is unchanged during the above process. Thus we obtained:

**Lemma 2.1.** ([12] Lemma 1.5) Let (2.1) be a complex of objects in a triangulated category $\mathcal{D}$ which satisfies condition (2.2). Then there exists a right convolution $d_0 : a_0 \rightarrow a$ which is uniquely determined up to isomorphism.
Example 2.2. (1) If all the \(a_p\) are sheaves, then the right convolution is nothing but the complex itself with an obvious morphism \(a_0 \to a\). Indeed, we can take the intermediate objects \(a'_p\) to be the complex

\[a'_p = \{a_m \to a_{m-1} \to \cdots \to a_p\}\]

where \(a_p\) is at degree 0.

(2) If \(d_0 : a_0 \to a\) is the right convolution of the complex (2.1) and \(F\) is a fully faithful exact functor, then \(F(d_0) : F(a_0) \to F(a)\) is the right convolution of the complex

\[0 \to F(a_m) \to F(a_{m-1}) \to \cdots \to F(a_1) \to F(a_0) \to 0.\]

Indeed, the assumption is preserved by \(F\) and the objects of intermediate steps are given by \(F(a'_p)\).

**Lemma 2.3.** (cf. [12] Lemma 1.6) Let

\[
\begin{array}{ccccccccc}
0 & \to & a_m & \xrightarrow{d_m} & a_{m-1} & \xrightarrow{d_{m-1}} & \cdots & \xrightarrow{d_1} & a_0 & \to & 0 \\
\downarrow f_m & & \downarrow f_{m-1} & & \downarrow f_1 & & \downarrow f_0 & & \\
0 & \to & b_m & \xrightarrow{e_m} & b_{m-1} & \xrightarrow{e_{m-1}} & \cdots & \xrightarrow{e_1} & b_0 & \to & 0
\end{array}
\]

be a morphism between complexes which satisfy condition (2.2). Let \(d_0 : a_0 \to a\) and \(e_0 : b_0 \to b\) be the right convolutions of these complexes, and \(\epsilon : b \to b'\) a morphism. Assume in addition that

\[\text{Hom}_D(a_p[r], b_q) = 0 \text{ for any } p > q, r > 0.\]

Then there exists a morphism \(f : a \to b'\) in \(D\) satisfying the commutativity \(fd_0 = \epsilon e_0 f_0\):

\[
\begin{array}{ccccccccc}
a_0 & \xrightarrow{d_0} & a & \xrightarrow{f_0} & b_0 & \xrightarrow{e_0} & b & \xrightarrow{\epsilon} & b' \\
\downarrow f_0 & & \downarrow f & & & & & & \\
= & & a & & & & & &
\end{array}
\]

Moreover, this morphism is unique with this property if

\[\text{Hom}_D(a_p[r], b') = 0 \text{ for any } p, r > 0.\]

**Proof.** In order to find \(f\), we proceed by induction on the length \(m\). If \(m = 0\), then \(f = e_0 f_0\). If \(m \geq 1\), then we construct \(a'_{m-1}\) and \(b'_{m-1}\) as before. The
morphisms \( f_m \) and \( f_{m-1} \) induce a morphism \( f_{m-1}' : d_{m-1}' \to b_{m-1}' \) between the cones. We note that the existence of the morphism \( f_{m-1}' \) is guaranteed by the axiom of triangulated categories such that the following diagram

\[
\begin{array}{ccc}
    a_m & \xrightarrow{j_{m-1}} & a_{m-1} \\
f_m & \downarrow & \downarrow f_{m-1} \\
b_m & \xrightarrow{k_{m-1}} & b_{m-1}
\end{array}
\begin{array}{ccc}
da_{m-1}' & \xrightarrow{f_{m-1}'} & a_m' [1] \\
f_{m-1}' & \downarrow & \downarrow f_{m-1}' \\
b_{m-1}' & \xrightarrow{k_{m-1}'} & b_m'[1]
\end{array}
\]

is commutative, but it is not uniquely determined in general.

We have

\[
ed_{m-1}' j_{m-1} = e_{m-1}' k_{m-1} f_{m-1} = e_{m-1} f_{m-1} = f_m d_{m-1} = f_m d_{m-1}' j_{m-1}.
\]

Hence \( e_{m-1}' f_{m-1}' = f_m d_{m-1}' \) by condition (2.5). By diagram chasing, we can check that condition (2.5) is satisfied by the new morphism of complexes

\[
(2.8) \quad 0 \longrightarrow d_{m-1}' \longrightarrow a_{m-2} \longrightarrow \cdots \longrightarrow a_1 \longrightarrow a_0 \longrightarrow 0
\]

from which we obtain the morphism \( f : a \to b' \) by the induction hypothesis.

We note that \( f_0 : a_0 \to b_0 \) remains unchanged in the above process. By condition (2.7), \( f \) is uniquely determined by the commutativity of (2.6).

In the dual way, we define the left convolutions of complexes of objects. We consider a bounded complex of objects (2.1) which satisfies condition (2.2). The left convolution is a morphism \( d : a \to a_m \) from an object in \( \mathcal{D} \) which is uniquely constructed inductively on the length \( m \) in the following way.

If \( m = 0 \), then \( a = a_0 \) and \( d = \text{Id} \). If \( m \geq 1 \), then let \( a_1'[1] \) be the cone of the morphism \( a_1 \to a_0' \):

\[
a_1' \xrightarrow{j_1} a_1 \xrightarrow{d_1} a_0 \longrightarrow a_1'[1]
\]

is a distinguished triangle. There exists a unique morphism \( d_2' : a_2 \to a_1' \) such that \( j_1 d_2' = d_2' \) and \( d_2' d_3 = 0 \) as before.

Thus we obtain a new complex

\[
(2.9) \quad 0 \longrightarrow a_m \longrightarrow a_{m-1} \longrightarrow \cdots \longrightarrow a_2 \longrightarrow a_1' \longrightarrow 0
\]
which satisfies condition (2.2). The left convolution \( d : a \to a_m \) of the complex (2.1) is defined to be that of the new complex (2.9).

**Lemma 2.4.** Let (2.1) be a complex of objects in a triangulated category \( \mathcal{D} \) which satisfies condition (2.2). Then there exists a left convolution \( d : a \to a_m \) which is uniquely determined up to isomorphism.

**Example 2.5.** (1) If all the \( a_p \) are sheaves, then the left convolution is nothing but the complex itself with an obvious morphism \( a_*[-m] \to a_m \), where the term \( a_m \) of \( a_*[-m] \) is at degree 0. Indeed, we can take inductively

\[
a'_p = \{ a_p \to a_{p-1} \to \cdots \to a_1 \to a_0 \}
\]

where \( a_p \) is at degree 0.

(2) If \( d : a \to a_m \) is the left convolution of the complex (2.1) and \( F \) is a fully faithful exact functor, then \( F(d) : F(a) \to F(a_m) \) is the left convolution of the complex

\[
0 \to F(a_m) \to F(a_{m-1}) \to \cdots \to F(a_1) \to F(a_0) \to 0.
\]

**Lemma 2.6.** Let (2.4) be a morphism between complexes which satisfy condition (2.2). Let \( d : a \to a_m \) and \( e : b \to b_m \) be the left convolutions of these complexes, and \( \epsilon : a' \to a \) a morphism. Assume in addition that

\[
\text{Hom}_{\mathcal{D}}(a'[r], b[q]) = 0 \text{ for any } p > q, r > 0.
\]

Then there exists a morphism \( f : a' \to b \) in \( \mathcal{D} \) satisfying the commutativity \( ef = f_m d \epsilon \):

\[
\begin{array}{ccc}
a' & \xrightarrow{e} & a \xrightarrow{d} a_m \\
& \xrightarrow{f} & \downarrow f_m \\
b & \xrightarrow{=} & b \xrightarrow{e} b_m.
\end{array}
\]

Moreover, this morphism is unique with this property if

\[
\text{Hom}_{\mathcal{D}}(a'[r], b[q]) = 0 \text{ for any } q, r > 0.
\]

**Proof.** The proof is similar to that of Lemma 2.3.

**3. Resolution of the diagonal.** For a projective space \( \mathbb{P} \), there is a standard resolution of the structure sheaf of the diagonal subvariety \( O_{\Delta_{\mathbb{P}}} \) in the self-product \( \Delta_{\mathbb{P}} \subset \mathbb{P} \times \mathbb{P} \) by locally free sheaves of Künneth type ([2], see also (5.1)). We extend this in a weaker sense for general projective varieties.
Let $X$ be a projective algebraic variety, $L$ an ample line bundle, and let

$$A = \bigoplus_{m=0}^{\infty} A_m = \bigoplus_{m=0}^{\infty} H^0(X, mL)$$

be the homogeneous coordinate ring. We define vector spaces $B_m$ ($m \geq 0$) by

$$B_m = \text{Ker}(B_{m-1} \otimes \mathbb{C} A_1 \rightarrow B_{m-2} \otimes \mathbb{C} A_2)$$

for $m \geq 2$, where the homomorphism is induced from the multiplication in $A$. We set $A_m = B_m = 0$ for $m < 0$.

The graded $\mathbb{C}$-algebra $A$ is said to be a Koszul algebra if the sequence of natural homomorphisms

$$(3.1) \quad \cdots \rightarrow B_m \otimes \mathbb{C} A(-m) \rightarrow B_{m-1} \otimes \mathbb{C} A(-m+1) \rightarrow \cdots \rightarrow B_1 \otimes \mathbb{C} A(-1) \rightarrow A \rightarrow \mathbb{C} \rightarrow 0$$

is exact, where the shifted module $A(j)$ is defined by $A(j)_k = A_{j+k}$. In other words, $A$ is Koszul if the minimal $A$-free resolution of its residue field consists of homomorphisms of degree 1.

By [1] Theorem 2, the subring $A^{(d)} = \bigoplus_{m=0}^{\infty} A_{dm}$ of $A$ is a Koszul algebra for a sufficiently large integer $d$, i.e., $A$ becomes Koszul if we replace $L$ by its tensor power $L^d$. We assume that $L$ is already replaced so that $A$ is Koszul in the following.

We define sheaves $R_m$ ($m \geq 0$) on $X$ by $R_0 = \mathcal{O}_X$ and

$$R_m = \text{Ker}(B_m \otimes \mathbb{C} \mathcal{O}_X \rightarrow B_{m-1} \otimes \mathbb{C} L)$$

for $m \geq 1$, where the homomorphism is induced from the natural homomorphisms $B_m \rightarrow B_{m-1} \otimes \mathbb{C} A_1$ and $A_1 \otimes \mathbb{C} \mathcal{O}_X \rightarrow L$. We set $R_m = 0$ for $m < 0$.

If we take the sequence of associated sheaves to the tensor product of (3.1) with $A(m)$, we obtain an exact sequence

$$(3.2) \quad 0 \rightarrow R_m \rightarrow B_m \otimes \mathbb{C} \mathcal{O}_X \rightarrow B_{m-1} \otimes \mathbb{C} L \rightarrow \cdots \rightarrow B_1 \otimes \mathbb{C} L^{m-1} \rightarrow L^m \rightarrow 0.$$
Proof. We consider a double complex of sheaves

$$C^{i,j} = \begin{cases} A_i \otimes B_{m-i-j} \otimes L_j & \text{for } i, j, m - i - j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where the differentials

$$d^{i,j}_1 : A_i \otimes B_{m-i-j} \otimes L_j \to A_{i+1} \otimes B_{m-i-j-1} \otimes L_j$$
$$d^{i,j}_2 : A_i \otimes B_{m-i-j} \otimes L_j \to A_i \otimes B_{m-i-j-1} \otimes L^{j+1}$$

are induced from the homomorphisms in (3.1) and (3.2). The condition $$d^{i,j+1}_1 d^{i,j}_2 = \pm d^{i,j+1}_2 d^{i,j}_1$$ is satisfied, because both are obtained as a composition of natural homomorphisms

$$A_i \otimes B_{m-i-j} \otimes L_j \to A_i \otimes B_{m-i-j-2} \otimes A_1 \otimes A_1 \otimes L_j$$
$$\to A_{i+1} \otimes B_{m-i-j-2} \otimes L^{j+1}.$$

If we take the cohomologies of $$C^{i,j}$$ with respect to the differential $$d_1$$ first, then we obtain 0's except $$L_m$$ at $$(i, j) = (0, m)$$. On the other hand, if we take the cohomologies with respect to $$d_2$$ first, then we obtain $$A_i \otimes R_{m-i}$$ at $$j = 0$$ and 0 elsewhere by (3.2). Therefore, we obtain our exact sequence.

Now we consider sheaves on the product $$X \times X$$. By Lemma 3.1, we obtain a homomorphism $$L^{-m} \boxtimes R_m \to L^{-m+1} \boxtimes R_{m-1}$$ as a composition of the following homomorphisms

(3.3) \[ L^{-m} \boxtimes R_m \to A_1 \otimes_C (L^{-m} \boxtimes R_{m-1}) \to L^{-m+1} \boxtimes R_{m-1}. \]

Theorem 3.2. Let $$X$$ be a projective algebraic variety, $$L$$ an ample line bundle, and $$\Delta X \subset X \times X$$ the diagonal subvariety of the direct product. If $$L$$ is replaced by a sufficiently high power $$L^k$$, then the complex of sheaves on $$X \times X$$

$$\cdots \to L^{-m} \boxtimes R_m \to L^{-m+1} \boxtimes R_{m-1} \to \cdots$$
$$\to L^{-1} \boxtimes R_1 \to \mathcal{O}_X \boxtimes \mathcal{O}_X \to \mathcal{O}_{\Delta X} \to 0$$

is exact.

Proof. Let $$D^\bullet$$ denote the given complex, where we set $$D^m = L^{m+1} \boxtimes R_{-m-1}$$ for $$m \leq -1$$ and $$D^0 = \mathcal{O}_{\Delta X}$$. Assume that $$H^{m_0}(D^\bullet) \neq 0$$ for some non-positive integer $$m_0$$. We take a sufficiently large integer $$k$$ such that

$$R^p p_{2*}(H^q(D^\bullet) \otimes p_1^* L^k) = 0$$ for $$p > 0, q \geq m_0 - \dim X$$
\[ H^p(X, L^{k+q}) = 0 \text{ for } p > 0, q \geq m_0 - \dim X \]
\[ R^0 p_{2*}(H^{m_0}(D^\bullet) \otimes p_1^*L^k) \neq 0. \]

We consider a spectral sequence
\[ E_2^{p,q} = R^p p_{2*}(H^q(D^\bullet) \otimes p_1^*L^k) \Rightarrow R^{p+q} p_{2*}(D^\bullet \otimes p_1^*L^k). \]

By the assumption, we have \( E_2^{p,q} = 0 \) for \( p > 0 \) and \( q \geq m_0 - \dim X \), hence \( R^m p_{2*}(D^\bullet \otimes p_1^*L^k) \neq 0. \)

On the other hand, we consider another spectral sequence
\[ E_1^{p,q} = R^q p_{2*}(D^p \otimes p_1^*L^k) \Rightarrow R^{p+q} p_{2*}(D^\bullet \otimes p_1^*L^k). \]

We have
\[ E_1^{p,q} = \begin{cases} L^k & \text{for } p = 0 \text{ and } q = 0 \\ H^q(X, L^{k+p+1}) \otimes R_{-p-1} & \text{for } p \leq -1 \\ 0 & \text{otherwise.} \end{cases} \]

Thus \( E_1^{0,0} = A_{k+p+1} \otimes R_{-p-1} \) for \( p \leq -1 \) and \( E_1^{p,q} = 0 \) for \( p+1 \geq m_0 - \dim X \) and \( q > 0 \). By Lemma 3.1, we obtain that \( E_2^{p,q} = 0 \) for \( p+q = m_0 \), a contradiction. \( \square \)

**Corollary 3.3.** Assume the conditions of Theorem 3.2. Define an object \( e \in D^-(X \times X) \) by
\[
e = \{ \cdots \to L^{-m} \otimes R_m \to L^{-m+1} \otimes R_{m-1} \to \cdots \to L^{-1} \otimes R_1 \to \mathcal{O}_X \otimes \mathcal{O}_X \}.
\]

Then for any object \( a \in D^-(X) \), there are isomorphisms in \( D^-(X) \)
\[ a \cong p_{1*}(e \otimes p_2^*a) \cong p_{2*}(e \otimes p_1^*a) \]

where \( p_{1*} \) and \( p_{2*} \) are right derived functors and \( \otimes \), \( p_1^* \) and \( p_2^* \) are left derived functors.

**Proof.** We have \( p_{1*}(\mathcal{O}_X \otimes p_2^*a) \cong p_{2*}(\mathcal{O}_X \otimes p_1^*a) \cong a. \) \( \square \)

**4. Smooth stack.** We refer to [11] and [8] for readable account on Deligne-Mumford stacks.

Let \( X \) be a normal projective variety with only quotient singularities. Then there exists an etale covering \( \{ U_i \} \) of \( X \) with finite Galois coverings \( \sigma_i : U_i \to U_i \) from smooth varieties \( U_i \) which are etale in codimension 1. The data \( \{ U_i, \sigma_i \} \) defines a smooth stack \( \mathcal{X} \). Let \( \sigma : \mathcal{X} \to X \) denote the natural morphism.
Let $X^#$ be the normalization of $X$ in a common Galois extension of the fields $\mathbb{C}(U_i)$. Then we have a morphism $\pi : X^# \to \mathcal{X}$. Denote $G = \text{Gal}(X^#/X)$ and $G_i = \text{Gal}(U_i/U_i)$. We have the following diagram

$$
\begin{array}{ccc}
U_i^# & \xrightarrow{\pi_i} & U_i \\
\text{open} & \downarrow & \text{etale} \\
X^# & \xrightarrow{\pi} & \mathcal{X}
\end{array}
$$

where $U_i^#$ is an open subset of $X^#$ such that the induced morphism $\pi_i : U_i^# \to U_i$ is finite.

Let $a$ be a sheaf on $X^#$. Then the direct image sheaf $\pi_*a$ on $\mathcal{X}$ is given by the following collection of $G_i$-equivariant sheaves $(\pi_*a)_i$ on the $U_i$. Let $X^#_i = (X^# \times_X U_i)^\nu$ be the normalization of the fiber product with natural morphisms $p_1 : X^#_i \to X^#$ and $p_2 : X^#_i \to U_i$. Then

$$(\pi_*a)_i = p_2^*p_1^*a.$$ 

We note that $p_1$ is etale. Indeed, the normalization of the fiber product $U_{ij} = (U_i \times_X U_j)^\nu$ is etale over $U_j$, and $X_i^# \cap p_1^{-1}(U_j^#) = U_j^# \times_{U_j} U_{ij}$.

If $b$ is a sheaf on $\mathcal{X}$ given by the collection of $G_i$-equivariant sheaves $b_i$ on the $U_i$, then the inverse image sheaf $\pi^*b$ on $X^#$ is obtained as the etale descent of the collection of sheaves $p_2^*b_i$ on the $X_i^#$. Since

$$(\pi_*\pi^*b)_i = p_2^*p_2^*b_i$$

the Galois group $G$ acts on the sheaf $\pi_*\pi^*b$, and we have

$$(\pi_*\pi^*b)^G \cong b.$$ 

Let $G = \text{Gal}(X^#/X) \cong \Delta G \subset G \times G$ act diagonally on the product $X^# \times X^#$.

**Theorem 4.1.** Let $X$ be a normal projective variety with only quotient singularities, $\sigma : X \to X$ the natural morphism from the associated smooth stack, $L$ an ample line bundle, $\mathcal{L} = \sigma^*L$, and $\pi : X^# \to \mathcal{X}$ a finite Galois morphism from a projective variety as above. Define the sheaves $R_m$ on $X^#$ from $\pi^*\mathcal{L}$ as before. Define an object $e^- \in D^-(\text{Coh}(\mathcal{X} \times \mathcal{X}))$ by

$$(4.1) \quad e^- = \{ \cdots \to (\pi_*\pi^*\mathcal{L}^{-m} \boxtimes \pi_*R_m)^{\Delta G} \to (\pi_*\pi^*\mathcal{L}^{-m+1} \boxtimes \pi_*R_{m-1})^{\Delta G} \to \cdots \to (\pi_*\pi^*\mathcal{L}^{-1} \boxtimes \pi_*R_1)^{\Delta G} \to (\pi_*\pi^*\mathcal{O}_X \boxtimes \pi_*\pi^*\mathcal{O}_X)^{\Delta G}\}.$$
If $L$ is replaced by a sufficiently high power $L^d$, then there are isomorphisms in $D^-(\text{Coh}(\mathcal{X}))$

$$a \cong p_1^*(e^\# \otimes p_2^s a) \cong p_2^*(e^\# \otimes p_1^s a)$$

for any object $a \in D^-(\text{Coh}(\mathcal{X}))$.

**Proof.** Let

$$e^\# = \{ \cdots \to \pi^* \mathcal{L}^m \boxtimes R_m \to \pi^* \mathcal{L}^{m+1} \boxtimes R_{m-1} \to \cdots \to \pi^* \mathcal{L}^1 \boxtimes R_1 \to \pi^* \mathcal{O}_X \boxtimes \pi^* \mathcal{O}_X \} \in D^-(X^\# \times X^\#).$$

The homomorphisms are equivariant with respect to the action of $G$. Indeed, we have a commutative diagram of injective homomorphisms

$$\begin{array}{ccc}
R_m & \longrightarrow & B_m \otimes \mathcal{O}_{X^g} \\
\downarrow & & \downarrow \\
A_1 \otimes R_{m-1} & \longrightarrow & A_1 \otimes B_{m-1} \otimes \mathcal{O}_{X^g}.
\end{array}$$

Since the multiplication $A \otimes_C A \to A$ is $G$-equivariant with respect to the diagonal action, so is the right vertical arrow, hence the left. The homomorphism $A_1 \otimes \pi^* \mathcal{L}^m \to \pi^* \mathcal{L}^{m+1}$ is similarly $G$-equivariant. Therefore, the homomorphisms in (3.3) are equivariant with respect to the diagonal $G$-actions.

We have $G$-equivariant isomorphisms

$$\pi^* a \cong p_1^*(e^\# \otimes p_2^s \pi^* a) \cong p_2^*(e^\# \otimes p_1^s \pi^* a)$$

by Corollary 3.3. Thus

$$a \cong (\pi^* \pi^* a)^G \cong (\pi^* p_1^*(e^\# \otimes p_2^s \pi^* a))^G \cong (p_1^*(\pi \times \pi)_*(e^\# \otimes p_2^s \pi^* a))^G$$

$$\cong (p_1^*((\pi \times \pi)_* e^\# \otimes p_2^s a))^G \cong p_1^*((\pi \times \pi)_* e^\# \otimes p_2^s a) \cong p_1^*(e^- \otimes p_2^s a).$$

The second isomorphism is obtained similarly.

Since there are more sheaves on the stack $\mathcal{X}$ than the underlying variety $X$, the existence of a surjective homomorphism from a locally free sheaf as in the following is nontrivial. This gives an alternative proof of a theorem by Totaro [14] which says that there exists a finite locally free resolution for any coherent sheaf on $\mathcal{X}$.

**Theorem 4.2.** Let $X$ be a normal quasi-projective variety with only quotient singularities, $L$ a very ample invertible sheaf, $\sigma : \mathcal{X} \to X$ the morphism from the associated smooth stack, and $\mathcal{L} = \sigma^* L$. Then there exists a locally free sheaf $\mathcal{A}_0$ on
such that, for any coherent sheaf \( C \) on \( \mathcal{X} \), there exist positive integers \( k, l \) and a surjective homomorphism \((\mathcal{A}_0 \otimes \mathcal{L}^{-k})^\oplus l \to C\).

**Proof.** Let \( T \) be the tangent sheaf of \( \mathcal{X} \). There exists a positive integer \( j_0 \) such that any irreducible representation of the stabilizer group \( G_x \) of any point \( x \in \mathcal{X} \) appears in the representation \((\bigoplus_{j=0}^{j_0} T^\otimes j) \otimes \mathcal{O}_x\) of \( G_x \) ([7] Problem 2.37).

We set \( \mathcal{A}_0 = (\bigoplus_{j=0}^{j_0} T^\otimes j) \). Let \( C \) be any nonzero coherent sheaf on \( \mathcal{X} \), and \( x \in \mathcal{X} \) a point in its support. Then there is a nontrivial homomorphism \( \mathcal{A}_0 \otimes \mathcal{O}_x \to \mathcal{C} \otimes \mathcal{O}_x \). Hence the sheaf \( \sigma_\ast \text{Hom}(\mathcal{A}_0, C) \) on \( \mathcal{X} \) is nontrivial, and there exists a positive integer \( k \) such that \( \text{Hom}(\mathcal{A}_0 \otimes \mathcal{L}^{-k}, C) \neq 0 \). Let \( C' = \text{Im}(\text{Hom}(\mathcal{A}_0 \otimes \mathcal{L}^{-k}, C) \otimes \mathcal{A}_0 \otimes \mathcal{L}^{-k}, C) \).

If \( C/C' \neq 0 \), then there exists an integer \( k' \geq k \) such that \( \text{Hom}(\mathcal{A}_0 \otimes \mathcal{L}^{-k'}, C/C') \neq 0 \) and \( \text{Hom}(\mathcal{A}_0 \otimes \mathcal{L}^{-k'}, C'[1]) = 0 \). By Noetherian induction, we obtain our result.

\( \square \)

5. An example: weighted projective space. In this section, we consider the special case where \( X \) is a weighted projective space as an example of the general construction. The results will not be used in later sections.

Let \((a_0, \ldots, a_n)\) be a sequence of positive integers which is well formed, that is, any subset consisting of \( n \) integers from these \( n + 1 \) is coprime. Let a finite group \( G = \mu_{a_0} \times \cdots \times \mu_{a_n} \) act on \( \mathbb{P}^n = \mathbb{P}^n \) with homogeneous coordinates \((x_0^a, \ldots, x_n^a)\) diagonally so that \( \mathcal{P} = \mathbb{P}^n / G \cong \mathbb{P}(a_0, \ldots, a_n) \). Let \( \pi' : \mathbb{P}^n \to \mathcal{P} \) be the projection.

The weighted projective space \( \mathcal{P} \) has only quotient singularities. Let \( \mathcal{P} \) be the smooth stack associated to \( \mathcal{P} \), and \( \pi : \mathbb{P}^n \to \mathcal{P} \) the morphism induced by \( \pi' \). The stack \( \mathcal{P} \) is described as follows. Let \( D_i = \text{div}(x_i^a) \), \( U_i^\# = \mathbb{P}^n \setminus D_i^\# \), and \( U_i = U_i^\# / G \subset \mathbb{P} \). Then we have a factorization of \( \pi' = \pi'|_{U_i^\#}^{\sigma_i} \):

\[
\pi_i' : U_i^\# \xrightarrow{\pi_i} U_i \xrightarrow{\sigma_i} U_i = U_i / \mu_{a_i}
\]

where \( U_i \) is smooth and \( \sigma_i \) is etale in codimension 1. Thus the set of coverings \( \{\sigma_i\} \) define the stack \( \mathcal{P} \).

For \( i \neq j \), if we set \( U_{ij}^\# = U_i^\# \cap U_j^\# \) and \( U_{ij} = U_{ij}^\# / G \subset \mathbb{P} \), then \( \pi_{ij} = \pi'|_{U_{ij}^\#} \) is factorized as

\[
\pi_{ij} : U_{ij}^\# \xrightarrow{\pi_{ij}} U_{ij} \xrightarrow{\sigma_{ij}} U_{ij} = U_{ij} / (\mu_{a_i} \times \mu_{a_j})
\]

where \( U_{ij} = U_i \times \mathcal{P} U_j \).

We have

\[
\pi^* \mathcal{O}_\mathcal{P}(1) \cong \mathcal{O}_{\mathbb{P}^n}(1).
\]
The images $\mathcal{D}_i = \pi(D_i^\#)$ are prime divisors on $\mathcal{P}$ such that $\pi^*\mathcal{D}_i = a_i\mathcal{D}_i^\#$ and $\mathcal{O}_{\mathcal{P}^\#}(\mathcal{D}_i^\#) \cong \mathcal{O}_{\mathcal{P}^\#}(1), \quad \mathcal{O}_{\mathcal{P}}(\mathcal{D}_i) \cong \mathcal{O}_{\mathcal{P}}(a_i)$. The dualizing invertible sheaf is given by

$$\omega_{\mathcal{P}} \cong \mathcal{O}_{\mathcal{P}}(\sum_i a_i).$$

We have the following resolution of the structure sheaf of the diagonal in $\mathbb{P}^\# \times \mathbb{P}^\#$ due to Beilinson [2]. From a tautological exact sequence on $\mathbb{P}^\#

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^\#}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^\#}^{n+1} \rightarrow \mathcal{T}_{\mathbb{P}^\#}(-1) \rightarrow 0$$

we obtain a homomorphism of sheaves on $\mathbb{P}^\# \times \mathbb{P}^\#

$$s : \mathcal{O}_{\mathbb{P}^\#}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^\#} \rightarrow \mathcal{O}_{\mathbb{P}^\#} \boxtimes \mathcal{T}_{\mathbb{P}^\#}(-1)$$

given by

$$s(P, Q)(v) = v \mod Cw \text{ if } [v] = P \text{ and } [w] = Q$$

where $v, w \in V$ for $\mathbb{P}^\# = \mathbb{P}(V^*)$. We can also write

$$s = \sum_{i=0}^{n} x_i^\# \boxtimes \frac{\partial}{\partial y_i^\#}.$$ 

The cokernel of the homomorphism

$$s' : \mathcal{O}_{\mathbb{P}^\#}(-1) \boxtimes \Omega_{\mathbb{P}^\#}^1(1) \rightarrow \mathcal{O}_{\mathbb{P}^\#} \boxtimes \mathcal{O}_{\mathbb{P}^\#}$$

induced by $s$ is the structure sheaf of the diagonal $\Delta \mathbb{P}^\# \subset \mathbb{P}^\# \times \mathbb{P}^\#$, We note that $s'$ is equivariant under the action of the diagonal subgroup $\Delta G \subset G \times G$.

Therefore, we obtain an exact sequence called Beilinson resolution of the diagonal as a Koszul complex

$$(5.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^\#}(-n) \boxtimes \Omega_{\mathbb{P}^\#}^n(n) \rightarrow \mathcal{O}_{\mathbb{P}^\#}(-n+1) \boxtimes \Omega_{\mathbb{P}^\#}^{n-1}(n-1) \rightarrow \cdots \rightarrow \mathcal{O}_{\mathbb{P}^\#}(-1) \boxtimes \Omega_{\mathbb{P}^\#}^1(1) \rightarrow \mathcal{O}_{\mathbb{P}^\#} \boxtimes \mathcal{O}_{\mathbb{P}^\#} \rightarrow \mathcal{O}_{\Delta \mathbb{P}^\#} \rightarrow 0.$$ 

We note that the homomorphisms are $\Delta G$-equivariant. We extend the Beilinson resolution for the stacky sheaves on weighted projective spaces according to the description by Canonaco [6], where he considered the sheaves on $\mathbb{P}$ instead of $\mathbb{P}$. 


The group of characters $G^*$ of $G$ is isomorphic to $\mathbb{Z}_{a_0} \times \cdots \times \mathbb{Z}_{a_n}$. For a character $\chi = (\chi_0, \ldots, \chi_n) \in G^*$, we take representatives of the $\chi_i$ such that $0 \leq \chi_i < a_i$, and define $|\chi| = \sum_{i=0}^{n} \chi_i \in \mathbb{Z}$.

Since $\pi$ is ramified along the $D_i$ with ramification indices $a_i$, we have a decomposition of the direct image sheaf into eigenspaces according to the $G$-action:

$$\pi_* \mathcal{O}_{\mathbb{P}}^* \cong \bigoplus_{\chi} (\pi_* \mathcal{O}_{\mathbb{P}}^*)^\chi, \quad (\pi_* \mathcal{O}_{\mathbb{P}}^*)^\chi \cong \mathcal{O}_{\mathbb{P}}(-|\chi|).$$

By [6], we have also

$$\pi_* \Omega_{\mathbb{P}}^\chi \cong \bigoplus_{\chi} (\pi_* \Omega_{\mathbb{P}}^\chi)^\chi, \quad (\pi_* \Omega_{\mathbb{P}}^\chi)^\chi \cong \Omega_{\mathbb{P}}^\chi(\log D_{-\chi})(-|\chi|)$$

where $D_{-\chi} = \sum_{\chi_i \neq 0} D_i$.

Let $a$ be a coherent sheaf on $\mathbb{P}$, and consider the Beilinson resolution of the pull-back $a^\# = \pi^* a$, where we note that $\pi$ is a flat morphism because we consider the stack $\mathbb{P}$ instead of the variety $\mathbb{P}$. We have a quasi-isomorphism

(5.2) $\mathcal{O}_{\Delta \mathbb{P}}^* \otimes p_2^* a^\# \cong \{ 0 \to \mathcal{O}_{\mathbb{P}}(-n) \boxtimes (\Omega_{\mathbb{P}}^n(n) \otimes a^\#) \to \cdots \to \mathcal{O}_{\mathbb{P}}(1) \boxtimes (\Omega_{\mathbb{P}}^1(1) \otimes a^\#) \to \mathcal{O}_{\mathbb{P}} \boxtimes a^\# \to 0 \}$.

Since

$$p_1_*(\mathcal{O}_{\Delta \mathbb{P}}^* \otimes p_2^* a^\#) \cong a^\#$$

we have

$$a^\# \cong \text{sg}\{ 0 \to \mathcal{O}_{\mathbb{P}}(-n) \boxtimes \mathcal{R}\Gamma(\mathbb{P}, \Omega_{\mathbb{P}}^n(n) \otimes a^\#) \to \cdots \to \mathcal{O}_{\mathbb{P}}(1) \boxtimes \mathcal{R}\Gamma(\mathbb{P}, \Omega_{\mathbb{P}}^1(1) \otimes a^\#) \to \mathcal{O}_{\mathbb{P}} \boxtimes a^\# \to 0 \}$$

where $\text{sg}$ stands for the associated single complex and

$$\mathcal{R}\Gamma(\mathbb{P}, \Omega_{\mathbb{P}}^n(p) \otimes a^\#) = \sum_q H^q(\mathbb{P}, \Omega_{\mathbb{P}}^n(p) \otimes a^\#)[-q].$$

Since the group $G \cong \Delta G$ acts equivariantly on (5.2), we have

$$a \cong \text{sg}\{ 0 \to \bigoplus_{\chi} \mathcal{O}_{\mathbb{P}}(-n - |\chi|) \boxtimes \mathcal{R}\Gamma(\mathbb{P}, \Omega_{\mathbb{P}}^n(\log D_{-\chi})(n - |\chi|) \otimes a) \to \cdots \to \bigoplus_{\chi} \mathcal{O}_{\mathbb{P}}(1 - |\chi|) \boxtimes \mathcal{R}\Gamma(\mathbb{P}, \Omega_{\mathbb{P}}^1(1 - |\chi|) \otimes a) \to \bigoplus_{\chi} \mathcal{O}_{\mathbb{P}}(1 - |\chi|) \boxtimes \mathcal{R}\Gamma(\mathbb{P}, a(-|\chi|)) \to 0 \}.$$
In particular, we have a spectral sequence of Beilinson type

\[ E_1^{p,q} = \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(p - |\chi|) \otimes H^q(\mathcal{P}, \Omega_{\mathcal{P}}^{-p} \log \mathcal{D}_{-\chi})(-p - |\chi|) \otimes a \Rightarrow a \]

where the right hand side is at degree 0.

If \( H^q(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(p - |\chi|) \otimes a \) = 0 for \( q > 0 \) and all \( p \) and \( \chi \), then we obtain an exact sequence

\[ 0 \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(n - |\chi|) \otimes H^0(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(n - |\chi|) \otimes a \rightarrow \]

\[ \cdots \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(1 - |\chi|) \otimes H^0(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(1 - |\chi|) \otimes a \]

\[ \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(-|\chi|) \otimes H^0(\mathcal{P}, a(-|\chi|)) \rightarrow a \rightarrow 0. \]

We call this the left resolution of \( a \), because the direction of the resolution is to the left.

On the other hand, if \( H^q(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(p - |\chi|) \otimes a \) = 0 for \( q < n \) and all \( p \) and \( \chi \), then we obtain an exact sequence called the right resolution of \( a \):

\[ 0 \rightarrow a \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(n - |\chi|) \otimes H^0(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(n - |\chi|) \otimes a \rightarrow \]

\[ \cdots \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(1 - |\chi|) \otimes H^0(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(1 - |\chi|) \otimes a \]

\[ \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(-|\chi|) \otimes H^0(\mathcal{P}, a(-|\chi|)) \rightarrow 0. \]

We note that the left and right resolutions are related by the shift.

If we define an object \( e \in D^b(\text{Coh}(\mathcal{P} \times \mathcal{P})) \) by

\[ e = \begin{cases} 
0 \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(n - |\chi|) \boxtimes H^0(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(n - |\chi|) \rightarrow \\
\cdots \rightarrow \bigoplus_{\chi} \mathcal{O}_\mathcal{P}(1 - |\chi|) \boxtimes H^0(\mathcal{P}, \Omega_{\mathcal{P}}^{p} \log \mathcal{D}_{-\chi})(1 - |\chi|) \rightarrow \\
\bigoplus_{\chi} \mathcal{O}_\mathcal{P}(-|\chi|) \otimes \mathcal{O}_\mathcal{P}(-|\chi|) \rightarrow 0 
\end{cases} \]

then we have

\[ a \cong p_{1*}(e \otimes p_{2*}a) \cong p_{2*}(e \otimes p_{1*}a). \]
6. Proof of Theorem 1.1. The proof of the theorem is partly parallel to the original proof in [12]. But we cannot use an embedding in a projective space because $\mathcal{X}$ has more sheaves than $X$.

We denote by $\sigma : \mathcal{X} \to X$ the natural morphism, and $\pi : X^\# \to \mathcal{X}$ the covering considered in the last section. We fix a very ample line bundle $L$ on $X$ such that the pull-back $\pi^*L$ for $L = \sigma^*L$ on the covering $X^\#$ has the property described in Theorem 3.2.

6.1. Step 1. We prove the boundedness of the functor $F$ ([12] Lemma 2.4), that is, there exists a fixed interval of integers such that $H^k(F(A)) = 0$ for any integer $k$ outside this interval and for any coherent sheaf $A$ on $\mathcal{X}$.

Let $F^*$ be the left adjoint functor of $F$. Fix an embedding $Y \to \mathbb{P}^M$ into some projective space. By pulling back the Beilinson resolution to $Y$, we obtain right resolutions

$$O_Y(-j) \cong \{ V_M^j \otimes C O_Y \to V_{M-1}^j \otimes C O_Y(1) \to \cdots \to V_0^j \otimes C O_Y(M) \}$$

for integers $j > 0$, where $V_p^j = H^p(M, \Omega^p(p-j-M)) (0 \leq p \leq M)$.

Let $B_0$ be a locally free sheaf on $Y$ obtained in Theorem 4.2. We choose integers $k_1 < k_2$ such that

$$H^k(F^*(B_0(j))) = 0$$

for $k \notin [k_1, k_2]$ and $0 \leq j \leq M$. Then

$$H^k(F^*(B_0( -j))) = 0$$

for $k \notin [k_1, k_2 + M]$ and any $j > 0$.

For any coherent sheaf $A$ on $\mathcal{X}$, we have

$$\text{Hom}^k(B_0( -j), F(A)) \cong \text{Hom}^k(F^*(B_0( -j)), A) \cong 0$$

for $k \notin [-k_2 - M, -k_1 + \dim X]$ and for any $j > 0$. If we take $j$ sufficiently large, then we have

$$\text{Hom}^p(B_0( -j), H^q(F(A))) \cong 0$$

for $p > 0$ and any $q$. Hence $H^k(F(A)) = 0$ for $k \notin [-k_2 - M, -k_1 + \dim X]$.

By replacing $F$ by its shift, we assume from now on that there exists an integer $k_0$ such that $H^k(F(A)) = 0$ for $k \notin [-k_0, 0]$ and any sheaf $A$ on $\mathcal{X}$.

6.2. Step 2. We shall define an object $e \in D^b(\text{Coh}(\mathcal{X} \times Y))$. 

We fix an integer $m$ such that $m > \dim X + \dim Y + k_0$, where the length $k_0$ is given in Step 1, and let $c_\bullet$ be the complex defined by

$$c_p = \begin{cases} (\pi_+\pi^*\mathcal{L}^{-p} \boxtimes F(\pi_*\mathcal{R}^p))^{\oplus G} & \text{for } 0 \leq p \leq m \\ 0 & \text{otherwise} \end{cases}$$

with the morphisms $\delta_p : c_p \to c_{p-1}$ induced from (3.3).

We have

$$\text{Hom}(\pi_+\pi^*\mathcal{L}^{-p} \boxtimes F(\pi_*\mathcal{R}^p)[r], \pi_+\pi^*\mathcal{L}^{-q} \boxtimes F(\pi_*\mathcal{R}^q))$$

$$\cong \bigoplus_{r_1 + r_2 = r} \text{Hom}(\pi_+\pi^*\mathcal{L}^{-p}[r_1], \pi_+\pi^*\mathcal{L}^{-q}) \otimes \text{Hom}(F(\pi_*\mathcal{R}^p)[r_2], F(\pi_*\mathcal{R}^q))$$

$$\cong \bigoplus_{r_1 + r_2 = r} \text{Hom}(\pi_+\pi^*\mathcal{L}^{-p}[r_1], \pi_+\pi^*\mathcal{L}^{-q}) \otimes \text{Hom}(\pi_*\mathcal{R}^p[r_2], \pi_*\mathcal{R}^q)$$

$$\cong 0$$

if $r > 0$. Thus there exists a right convolution $\delta_0 : c_0 \to c' \in D^b(\text{Coh}(\mathcal{X} \times \mathcal{Y}))$.

**Lemma 6.1.** $H^p(c') = 0$ unless $p \in [-m - k_0, -m] \cup [-k_0, 0]$.

**Proof.** Assume that there exists $p_0 \not\in [-m - k_0, -m] \cup [-k_0, 0]$ such that $H^{p_0}(c') \neq 0$. For a locally free sheaf $\mathcal{A}$ on $\mathcal{X}$, we have a spectral sequence

$$E_2^{p,q} = R^p p_2_*(H^q(e') \otimes p_1^* \mathcal{A}) \Rightarrow H^{p+q}(p_2_*(e' \otimes p_1^* \mathcal{A})).$$

Hence there exists $\mathcal{A}$ such that $H^{p_0}(p_2_*(e' \otimes p_1^* \mathcal{A})) \neq 0$ by Theorem 4.2. We may also assume that

$$(6.2) \quad H^p(\mathcal{X}, \mathcal{A} \otimes \pi_*\pi^*\mathcal{L}^{-q}) = 0 \text{ for } p > 0, 0 \leq q \leq m + \dim X.$$

We consider a complex of sheaves $a_\bullet$ given by

$$a_p = \begin{cases} (H^0(\mathcal{X}, \mathcal{A} \otimes \pi_*\pi^*\mathcal{L}^{-p}) \otimes \pi_*\mathcal{R}^p)^{\oplus G} & \text{for } 0 \leq p \leq m \\ 0 & \text{otherwise} \end{cases}$$

This coincides with $p_2_*(e^- \otimes p_1^* \mathcal{A})$ up to degree $-m$ for $e^-$ given in Theorem 4.1.

If we denote $U(\mathcal{A}) = \text{Ker}(a_m \to a_{m-1})$, then there is a distinguished triangle

$$U(\mathcal{A})[m] \to a_\bullet \to \mathcal{A} \to U(\mathcal{A})[m + 1].$$

Since $\text{Hom}_{p_2(\mathcal{X})}(\mathcal{A}, U(\mathcal{A})[m + 1]) = 0$, we have a right convolution $d_0 : a_0 \to U(\mathcal{A})[m] \oplus \mathcal{A}$ of $a_\bullet$. Hence $F(d_0) : F(a_0) \to F(U(\mathcal{A})[m] \oplus \mathcal{A})$ is also a right convolution of the complex of objects $F(a_\bullet)$. 
On the other hand, since we have
\[ p_{2*}(c_p \otimes p_1^*A) = F(a_p) \]
\( \delta_0 \) induces a right convolution \( \delta_{0*} : F(a_0) \to p_{2*}(e' \otimes p_1^*A) \) of the complex \( F(a_0) \).

Since both \( F(U(A)[m] \oplus A) \) and \( p_{2*}(e' \otimes p_1^*A) \) are right convolutions of the same complex, there exists an isomorphism \( \tilde{f}(A) \), which is not uniquely determined, making the following diagram commutative
\[
\begin{array}{ccc}
F(a_0) & \xrightarrow{\delta_{0*}} & p_{2*}(e' \otimes p_1^*A) \\
\downarrow & & \downarrow \tilde{f}(A) \\
F(a_0) & \xrightarrow{F(d_0)} & F(U(A)[m] \oplus A)
\end{array}
\]

Therefore, we obtain a contradiction by Step 1.

It follows that there exist objects \( e, e'' \in D^b(\text{Coh}(X \times Y)) \) and a distinguished triangle
\[ e'' \to e' \to e \to e''[1] \]
such that \( H^p(e) = 0 \) (resp. \( H^p(e'') = 0 \)) unless \( p \in [-k_0, 0] \) (resp. \( [-m - k_0, -m] \)). Since \( \text{Hom}_{D^b(\text{Coh}(X \times Y))}(e, e''[1]) = 0 \), we have \( e' \cong e \oplus e'' \).

6.3. Step 3. We construct a functorial isomorphism from \( \Phi^e_{X \to Y} \) to \( F \) for a certain set of locally free sheaves.

**Lemma 6.2.** Let \( A \) be any locally free coherent sheaf on \( X \) such that
\[ H^p(X, A \otimes \pi_+ \pi^- L^{-q}) = 0 \]
for any \( p > 0 \) and \( 0 \leq q \leq m + \text{dim} X \). Then there exists an isomorphism
\[ f(A) : \Phi^e_{X \to Y}(A) \to F(A) \]
which is functorial in the sense that, for any homomorphism \( \alpha : A \to B \) to another locally free coherent sheaf \( B \) which satisfies the same condition, the following diagram is commutative
\[
\begin{array}{ccc}
\Phi^e_{X \to Y}(A) & \xrightarrow{f(A)} & F(A) \\
\Phi^e_{X \to Y}(\alpha) \downarrow & & \downarrow F(\alpha) \\
\Phi^e_{X \to Y}(B) & \xrightarrow{f(B)} & F(B)
\end{array}
\]
Proof. We define $a_\bullet$, $U(A)$, $d_0 : a_0 \to U(A)[m] \oplus A$, $\delta_0 : F(a_0) \to p_{2*}(e' \otimes p_1^* A) = \Phi_{X,Y}^*(A)$ and $\tilde{f}(A)$ as in Lemma 6.1.

The isomorphism $\tilde{f}(A)$ induces an isomorphism $p_{2*}(e \otimes p_1^* A) \to F(A)$ by Steps 1 and 2. But we note that it is not sufficient, because $\tilde{f}(A)$ is not uniquely determined.

Let $\epsilon(A) : F(U(A)[m] \oplus A) \to F(A)$ be the projection. Since

$$\text{Hom}(F(a_0)[r], F(A)) \cong \text{Hom}(a_0[r], A) \cong 0$$

for any $p$ and $r > 0$, $\epsilon(A)$ is the only morphism which makes the following diagram commutative by Lemma 2.3:

$$\begin{array}{ccc}
F(a_0) & \xrightarrow{F(d_0)} & F(U(A)[m] \oplus A) \\
\downarrow & & \downarrow \epsilon(A) \\
F(a_0) & \xrightarrow{\epsilon(A)F(d_0)} & F(A).
\end{array}$$

By the same reason as above, the projection $\epsilon'(A) : p_{2*}(e' \otimes p_1^* A) \to p_{2*}(e \otimes p_1^* A)$ is the only morphism making the following diagram commutative

$$\begin{array}{ccc}
F(a_0) & \xrightarrow{\epsilon'(A)} & p_{2*}(e' \otimes p_1^* A) \\
\downarrow & & \downarrow \epsilon'(A) \\
F(a_0) & \xrightarrow{\epsilon'(A)F(\delta_0)} & p_{2*}(e \otimes p_1^* A).
\end{array}$$

Therefore, there exists a uniquely determined isomorphism $f(A) : p_{2*}(e \otimes p_1^* A) \to F(A)$ such that the following diagram is commutative

$$\begin{array}{ccc}
F(a_0) & \xrightarrow{\epsilon'(A)F(\delta_0)} & p_{2*}(e \otimes p_1^* A) \\
\downarrow & & \downarrow f(A) \\
F(a_0) & \xrightarrow{\epsilon'(A)F(d_0)} & F(A).
\end{array}$$

Next, we consider a complex of sheaves $b_\bullet$ given by

$$b_p = \begin{cases} 
(H^0(X, B \otimes \pi_\star \pi^* L^{-p}) \otimes \pi_\star R_p)^{D_G} & \text{for } 0 \leq p \leq m \\
0 & \text{otherwise}.
\end{cases}$$

The homomorphism $\alpha : A \to B$ induces a morphism of complexes $\alpha_\bullet : a_\bullet \to b_\bullet$. Let $e_0 : b_0 \to U(B)[m] \oplus B$ be the right convolution.
There exists a uniquely determined morphism $g$ to make the following diagram commutative

\[
\begin{array}{ccc}
F(a_0) & \xrightarrow{\delta_{a_0}} & p_{2*}(e' \otimes p_1^* A) \\
\downarrow F(\alpha_0) & & \downarrow g \\
F(b_0) & \xrightarrow{\epsilon(BF(\alpha))} & F(B).
\end{array}
\]

It follows that

\[
g = f(B)\Phi^e_{X \to Y}(\alpha)e'(A) = F(\alpha)f(A)e'(A)
\]
hence $f(B)\Phi^e_{X \to Y}(\alpha) = F(\alpha)f(A)$. \qed

6.4. Step 4. We extend the functorial isomorphism from $F$ to $\Phi^e_{X \to Y}$ to an ample set of objects.

A sequence of objects $\{P_k\}_{k \in \mathbb{Z}}$ in an abelian category $\mathcal{C}$ is said to be ample if for any object $C \in \mathcal{C}$, there exists an integer $k_0(C)$ such that the following conditions are satisfied for any $k < k_0(C)$ ([12] Definition 2.12).

1. The canonical morphism $\text{Hom}(P_k, C) \otimes P_k \to C$ is surjective.
2. $\text{Hom}(P_k, C[r]) = 0$ if $r \neq 0$.
3. $\text{Hom}(C, P_k) = 0$.

We note that the set $\{P_k\}_{k \in \mathbb{Z}}$ becomes a spanning class of the derived category $D^b(\mathcal{C})$ if it has a Serre functor (cf. [4]).

By Theorem 4.2, we obtain the following:

**Lemma 6.3.** There exists a locally free sheaf $A_0$ on $X$ such that the sequence $\{A_0 \otimes L^k\}$ is ample in the category of coherent sheaves $\text{Coh}(\mathcal{X})$.

**Lemma 6.4.** There exist isomorphisms $f_k : F(A_0 \otimes L^k) \to \Phi^e_{X \to Y}(A_0 \otimes L^k)$ for any integers $k$ which are functorial in the sense that, for any homomorphism $\alpha : A_0 \otimes L^k \to A_0 \otimes L^{k'}$, the following diagram is commutative

\[
\begin{array}{ccc}
F(A_0 \otimes L^k) & \xrightarrow{f_k} & \Phi^e_{X \to Y}(A_0 \otimes L^k) \\
\downarrow F(\alpha) & & \downarrow \Phi^e_{X \to Y}(\alpha) \\
F(A_0 \otimes L^{k'}) & \xrightarrow{f_{k'}} & \Phi^e_{X \to Y}(A_0 \otimes L^{k'}).\n\end{array}
\]
Proof. We denote $\Phi = \Phi^e_{X \to Y}$. By Lemma 6.2, there exists an integer $k_1$ such that our assertion holds if $k, k' \geq k_1$. We proceed by the descending induction on such $k_1$.

Let us fix an embedding $X \to \mathbb{P}^N$ such that $L = \mathcal{O}_X(1)$. We have an exact sequence

\begin{equation}
0 \to \mathcal{O}_X \to V_N \otimes \mathcal{L} \to V_{N-1} \otimes \mathcal{L}^2 \to \cdots \to V_0 \otimes \mathcal{L}^{N+1} \to 0
\end{equation}

for $V_p = H^N(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}(p - N - 1))$.

For any integer $k$, we consider a complex of objects $a^{(k)}_\bullet$ and $b^{(k)}_\bullet$ defined by

\begin{align*}
a^{(k)}_p &= \begin{cases}
F(V_p \otimes A_0 \otimes \mathcal{L}^{k+N+1-p}) & \text{for } 0 \leq p \leq N \\
0 & \text{otherwise}
\end{cases} \\
b^{(k)}_p &= \begin{cases}
\Phi(V_p \otimes A_0 \otimes \mathcal{L}^{k+N+1-p}) & \text{for } 0 \leq p \leq N \\
0 & \text{otherwise}
\end{cases}
\end{align*}

If $k \geq k_1 - 1$, then there is an isomorphism of complexes $f^{(k)} : a^{(k)}_\bullet \to b^{(k)}_\bullet$ by the induction hypothesis, where

\[ f^{(k)}_p = \text{Id}_{V_p} \otimes f_{k+N+1-p}. \]

We have the left convolutions $d^{(k)} : F(A_0 \otimes \mathcal{L}^k) \to a^{(k)}_N$ and $e^{(k)} : \Phi(A_0 \otimes \mathcal{L}^k) \to b^{(k)}_N$, and there is a uniquely determined isomorphism $f_k : F(A_0 \otimes \mathcal{L}^k) \to \Phi(A_0 \otimes \mathcal{L}^k)$ such that $e^{(k)} f_k = f^{(k)}_N d^{(k)}$. Note that if $k \geq k_1$, then the morphism $f_k$ which we already have satisfies this condition thanks to the functoriality of the $f_k$.

Let $k, k' \geq k_1 - 1$. Then the homomorphism $\alpha$ induces a homomorphism of complexes $\alpha_{k'} : a^{(k)}_\bullet \to b^{(k')}_\bullet$ given by

\[ \alpha_{k'} = \text{Id}_{V_p} \otimes f_{k'+N+1-p} F(\alpha \otimes \text{Id}_{\mathcal{L}^{N+1-p}}). \]

Therefore, we have a uniquely determined homomorphism $g : F(A_0 \otimes \mathcal{L}^k) \to \Phi(A_0 \otimes \mathcal{L}^{k'})$ by the condition that $e^{(k')} g = \alpha_{k'} d^{(k)}$. Since both $\Phi(\alpha) f_k$ and $f_k F(\alpha)$ satisfy this condition, they coincide.

\[ \square \]

6.5. Step 5. By [12] Proposition 2.16, we obtain an isomorphism of functors $f : F \to \Phi^e_{X \to Y}$ by extending the $f_k$: 

**Lemma 6.5.** Let $C$ be an abelian category, $D^b(C)$ its derived category, and $D$ another triangulated category. Let $F : D^b(C) \to D$ be a fully faithful exact functor which has a left adjoint, and $G : D^b(C) \to D$ another exact functor which has a left adjoint. Assume that $D^b(C)$ and $D$ have Serre functors and that there is an ample
sequence $\Omega = \{P_k\}$ in $C$ with an isomorphism of functors $f_\Omega : F|_\Omega \to G|_\Omega$ when restricted to the full subcategory $\Omega \subset D^b(C)$. Then there exists an isomorphism of functors $f : F \to G$ which is an extension of $f_\Omega$.

**Proof.** Let $F^*$ and $F^l$ (resp. $G^*$ and $G^l$) be the left and right adjoint functors of $F$ (resp. $G$). Since $F$ is fully faithful, the natural morphism of functors $\text{Id}_{D^b(C)} \to F^lF$ is an isomorphism, because

$$\text{Hom}(a, b) \to \text{Hom}(a, F^lFb)$$

is an isomorphism for any objects $a$ and $b$.

Since $G$ is isomorphic to $F$ when restricted to a spanning class $\Omega$, it is also fully faithful, hence the natural morphism of functors $G^*G \to \text{Id}_{D^b(C)}$ is an isomorphism.

$G^*F$ is the left adjoint functor of $F^lG$. Since they are isomorphic to the identity functor when restricted to $\Omega$, it follows that they are fully faithful, hence they are quasi-inverses of each other. Thus $F^lG$ is an autoequivalence of $D^b(C)$, and there exists an extended isomorphism of functors $f^l : \text{Id}_{D^b(C)} \to F^lG$ by [12] Proposition 2.16. By adjunction, we obtain a morphism of functors $f : F \to G$.

We prove that $f$ is an isomorphism. For any object $a$, let

$$F(a) \xrightarrow{f(a)} G(a) \to c \to F(a)[1]$$

be a distinguished triangle. Since $F^l(f(a))$ is an isomorphism, we have $F^l(c) \cong 0$. From

$$\text{Hom}(\omega, G^l(c)) \cong \text{Hom}(G(\omega), c)$$

$$\cong \text{Hom}(F(\omega), c) \cong \text{Hom}(\omega, F^l(c)) \cong 0$$

for any $\omega \in \Omega$, we deduce that $G^l(c) \cong 0$. Hence $\text{Hom}(G(a), c) \cong 0$, thus $F(a) \cong G(a) \oplus c[-1]$. But

$$\text{Hom}(F(a), c[-1]) \cong \text{Hom}(a, F^l(c)[-1]) \cong 0$$

hence $c = 0$. □

**6.6. Step 6.** Let $e_1 \in D^b(\text{Coh}(X \times Y))$ be any object such that there is an isomorphism of functors $F \to \Phi_{X \to Y}^e$. We shall prove that $e_1 \cong e$. 

Let \((c_\bullet, \delta_\bullet)\) be the complex of objects considered in Step 2. We consider a commutative diagram

\[
\begin{array}{ccccccc}
\cdots & \rightarrow & c_1 & \rightarrow & c_{10} & \rightarrow & c_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bar{c}_1 & \rightarrow & \bar{c}_{10} & \rightarrow & \bar{c}_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
p_1^*\mathcal{L}^{-1} \otimes e_1 \otimes p_1^*\pi_*R_1 & \rightarrow & A \otimes_C (p_1^*\mathcal{L}^{-1} \otimes e_1) & \rightarrow & e_1
\end{array}
\]

where

\[
\begin{align*}
c_1 &= (p_1^*\pi_*\pi^*\mathcal{L}^{-1} \otimes p_2^*p_2_*(e_1 \otimes p_1^*\pi_*R_1))^\Delta G \\
c_{10} &= A_1 \otimes_C (p_1^*\pi_*\pi^*\mathcal{L}^{-1} \otimes p_2^*p_2_*(e_1 \otimes p_1^*\pi_*\pi^*\mathcal{O}_X))^\Delta G \\
c_0 &= (p_1^*\pi_*\pi^*\mathcal{O}_X \otimes p_2^*p_2_*(e_1 \otimes p_1^*\pi_*\pi^*\mathcal{O}_X))^\Delta G \\
\bar{c}_1 &= p_1^*\pi_*\pi^*\mathcal{L}^{-1} \otimes e_1 \otimes p_1^*\pi_*R_1 \\
\bar{c}_{10} &= A_1 \otimes_C (p_1^*\pi_*\pi^*\mathcal{L}^{-1} \otimes e_1 \otimes p_1^*\pi_*\pi^*\mathcal{O}_X) \\
\bar{c}_0 &= p_1^*\pi_*\pi^*\mathcal{O}_X \otimes e_1 \otimes p_1^*\pi_*\pi^*\mathcal{O}_X.
\end{align*}
\]

We define a morphism \(\bar{\delta}_0 : c_0 \rightarrow e_1\) as the composition of the arrows in the last column. Since the composition of the last row is zero, we have \(\bar{\delta}_0 \delta_0 = 0\).

We have

\[
\begin{align*}
\text{Hom}(\pi_*\pi^*\mathcal{L}^{-k} \boxtimes F(\pi_*R_k)[r], e_1) & \cong \text{Hom}(p_2^*\Phi^{e_1}(\pi_*R_k)[r], e_1 \otimes p_1^*\pi_*\pi^*\mathcal{L}^k) \\
& \cong \text{Hom}(\Phi^{e_1}(\pi_*R_k)[r], \Phi^{e_1}(\pi_*\pi^*\mathcal{L}^k)) \cong \text{Hom}(\pi_*R_k[r], \pi_*\pi^*\mathcal{L}^k) \cong 0
\end{align*}
\]

for any \(r > 0\). If we apply the argument of the proof of Lemma 2.1 to a longer complex

\[
0 \rightarrow c_m \rightarrow \cdots \rightarrow c_0 \rightarrow e_1 \rightarrow 0
\]

we deduce that there exists a uniquely determined morphism \(e : e' \rightarrow e_1\) such that \(e \delta_0 = \bar{\delta}_0\). Composing with the natural morphism \(e \rightarrow e'\), we obtain a morphism \(e \rightarrow e_1\).

Let \(c\) be the cone of this morphism and assume that \(c \not\cong 0\):

\[
e \rightarrow e_1 \rightarrow c \rightarrow e[1].
\]

We use the notation of Lemma 6.1, where we assume additionally that \(p_2_*(c \otimes p_1^*\mathcal{A}) \neq 0\). If we apply the functor \(p_2_*(\bullet \otimes p_1^*\mathcal{A})\) to the complex \(c_\bullet\) with the morphism \(\bar{\delta}_0 : c_0 \rightarrow e_1\), then we obtain a complex \(F(a_\bullet)\) with a morphism
δ_{0*}: F(a_0) \rightarrow \Phi c^i(A), and a commutative diagram

\[
\begin{array}{ccc}
F(a_0) & \xrightarrow{\delta_{0*}} & F(U(A)[m] \oplus A) \cong \Phi^e(A) \\
\downarrow & & \downarrow e(A) \\
F(a_0) & \xrightarrow{\delta_{0*}} & F(A) \cong \Phi c^i(A).
\end{array}
\]

Therefore, the induced morphism \( p_2^*(e \otimes p_1^* A) \rightarrow p_2^*(c \otimes p_1^* A) \) is an isomorphism, hence \( p_2^*(c \otimes p_1^* A) \cong 0 \), a contradiction.

This is the end of the proof of Theorem 1.1.

7. Applications. The following theorem is a generalization of [9] Theorem 2.3 to the case of varieties with quotient singularities.

**Theorem 7.1.** Let \( X \) and \( Y \) be normal projective varieties with only quotient singularities, and \( X \) and \( Y \) the associated smooth stacks. Assume that the bounded derived categories of coherent sheaves are equivalent as triangulated categories: \( D^b(Coh(X)) \cong D^b(Coh(Y)) \). Then the following hold:

1. \( \dim X = \dim Y \).
2. If \( K_X \) (resp. \( -K_X \)) is nef, then \( K_Y \) (resp. \( -K_Y \)) is also nef, and an equality on the numerical Kodaira dimension \( \nu(X) = \nu(Y) \) (resp. \( \nu(X, -K_X) = \nu(Y, -K_Y) \)) holds.
3. If \( \kappa(X) = \dim X \), i.e., \( X \) is of general type, or if \( \kappa(X, -K_X) = \dim X \), then \( X \) and \( Y \) are birationally equivalent. Moreover, there exist birational morphisms \( f: Z \rightarrow X \) and \( g: Z \rightarrow Y \) from a smooth projective variety \( Z \) such that the canonical divisors are \( \mathbb{Q} \)-linearly equivalent: \( f^* K_X \sim \mathbb{Q} g^* K_Y \).

**Proof.** The proof is parallel to that of [9] Theorem 2.3, and we only explain the outline.

By Theorem 1.1, there exists an object \( e \in D^b(Coh(X \times Y)) \) such that \( \Phi = \Phi c^e_{X \rightarrow Y} : D^b(Coh(X)) \rightarrow D^b(Coh(Y)) \) is an equivalence. Considering the right and left adjoint functors, we obtain an isomorphism of objects

\[
e c^e \otimes p_1^* \omega_X\{ \dim X \} \cong (\dim Y) \]  

Part (0) follows immediately.

Let \( \Gamma \) be the union of the supports of the cohomology sheaves \( H^i(e c^e) \) for all \( i \), and let \( \tilde{\nu}_1 : \tilde{Z}_1 \rightarrow \mathcal{Z}_1 \) be the normalization of an irreducible component of \( \Gamma \) which is at the same time an irreducible component of the support of some \( H^i(e c^e) \). Since \( \Phi c^e_{X \rightarrow Y} \) is an equivalence, there exists \( \mathcal{Z}_1 \) such that the projection \( \mathcal{Z}_1 \rightarrow \mathcal{Y} \) is surjective and that

\[
\tilde{\nu}_1^* p_1^* \omega_{\tilde{X}}^{\otimes m_1} \cong \tilde{\nu}_1^* p_2^* \omega_{\tilde{Y}}^{\otimes m_1}
\]
where $m_1$ is the rank of $\check{\nu}_1^*H^i(e^\vee)$. Let $Z_1$ be the image of $Z_1$ on $X \times Y$ and $\nu_1 : \tilde{Z}_1 \to Z_1$ the normalization.

If $K_X$ is nef or anti-nef, then $\nu_1^*p_1^*K_X \sim_\mathbb{Q} \nu_1^*p_2^*K_Y$ is also nef or anti-nef, hence so is $K_Y$. We have also $\nu(X, \pm K_X) \geq \nu(\tilde{Z}_1, \pm \nu_1^*p_2^*K_Y) = \nu(Y, \pm K_Y)$, thus $\nu(X, \pm K_X) = \nu(Y, \pm K_Y)$ by symmetry.

If $\kappa(X, \pm K_X) = \dim X$, then there exist an ample $\mathbb{Q}$-Cartier divisor $A$ and an effective $\mathbb{Q}$-Cartier divisor $B$ on $X$ such that $\pm K_X \sim A + B$ by Kodaira's lemma. We take $Z_1$ which dominates $X$ instead of $Y$. Then the projection $p_2|_{Z_1} : Z_1 \to Y$ is quasi-finite on $Z_1 \setminus p_1^{-1}(\text{Supp}(B))$. It follows that $\dim Z_1 = \dim X$ and $Z_1$ also dominates $Y$. The set $\Gamma \cap p_1^{-1}(x)$ consists of a single point for a general point $x \in X$, and $Z_1$ becomes a graph of a birational map. If we take $Z$ to be its resolution of singularities, then the conclusion holds.

We consider a generalization of a result in [3] on the autoequivalence group:

**Theorem 7.2.** Let $X$ be a normal projective variety with only quotient singularities, and $\mathcal{X}$ the associated smooth stack. Assume that $K_X$ or $-K_X$ is ample and that $K_X$ generates the local divisor class group at each point $x \in X$. Then the group of autoequivalences $\text{Autoeq}(D^b(\text{Coh}(\mathcal{X})))$ is isomorphic to the semidirect product of the automorphism group $\text{Aut}(X)$ and the trivial factor $\text{Pic}(\mathcal{X}) \times \mathbb{Z}$.

**Proof.** The theorem can be proved in the same way as in [3]. But we present here an alternative proof using Theorem 1.1 as in [9] Remark 2.4 (1).

We use the notation of the proof of Theorem 7.1. Since we can take $B = 0$, $Z_1$ becomes a graph of an automorphism of $X$, which induces an automorphism $h$ of $\mathcal{X}$. We have that

\[(7.1) \quad \Phi(a(jK_\mathcal{X})) \cong \Phi(a)(jK_\mathcal{X})\]

for any object $a$ and integer $j$.

We claim that the support of $e$ coincides with the graph of $h$. Indeed, if not, then there exists a point $x \in \mathcal{X}$ such that the support of $e \otimes p_1^*O_x$ is not connected. Since $K_X$ generates the local class groups, it follows that the support of $\Phi(O_\sigma)$ is not connected by (7.1), a contradiction.

Let $x \in \mathcal{X}$ be any point, $r_x$ the index of $K_X$ at $\sigma(x)$, and $j$ an integer such that $0 \leq j < r_x$. Then $\Phi(O_\sigma)$ is a shift of a sheaf, because we have otherwise $\text{Hom}_{D^b(\mathcal{X})}(\Phi(O_\sigma), \Phi(O_\sigma(jK_X))) \neq 0$ for some $p < 0$ and $j$. Moreover, since

\[
\text{Hom}_{D^b(\mathcal{X})}(\Phi(O_\sigma), \Phi(O_\sigma(jK_X))) \cong \begin{cases} 
\mathbb{C} & \text{if } j = 0 \\
0 & \text{if } 0 < j < r_x 
\end{cases}
\]

$\Phi(O_\sigma)$ is a shift of a skyscraper sheaf of length 1.
Assume that $e$ is not a shift of a sheaf, and let $i_1$ be the maximum of the integers $i$ such that $H^i(e) \neq 0$. We consider a spectral sequence

$$E_2^{p,q} = \text{Tor}_{-p}(H^q(e), p_1^* O_x) \Rightarrow e \otimes p_1^* O_x.$$ 

Take a general point $x$ of the support of $p_1^* H^i_1(e)$ which is also contained in the support of $p_1^* H^i_2(e)$ for some $i_2 < i_1$. We assume that $i_2$ is the largest such integer. If the support of $p_1^* H^i_1(e)$ is a proper subvariety of $X$, then both $E_2^{0,i_1}$ and $E_2^{-1,i_1}$ do not vanish and they survive at $E_\infty$. Otherwise, both $E_2^{0,i_1}$ and $E_2^{-1,i_1}$ do not vanish and survive at $E_\infty$. In any case, we conclude that $\Phi(O_x)$ is not a shift of a skyscraper sheaf of length 1, a contradiction. Hence $e[-i_1]$ is a sheaf. Moreover, there exists an invertible sheaf $M$ on $X$ such that $\Phi(e(a)) \cong h_*(a) \otimes M[i_1]$ for any $a \in D^b(X)$. 

**Remark 7.3.** Let $Y$ be a minimal algebraic surface of general type, and $X$ its canonical model. 

1. By [5], the derived categories $D^b(\text{Coh}(X))$ and $D^b(\text{Coh}(Y))$ are equivalent. Though $K_X$ is ample, $X$ and $Y$ may not be isomorphic. This is a difference from the case of smooth varieties [3]; if $X$ is smooth and $\pm K_X$ is ample, then any variety $Y$ with an equivalent derived category is isomorphic to $X$. The reason is that, there is no crepant birational map, which is not an isomorphism, from a smooth projective variety whose canonical or anti-canonical divisor is ample. 

2. The condition on the local class groups in Theorem 7.2 cannot be removed. For example, assume that $X$ has an ordinary double point $x_0 \in X$. Let $O_{x_0}(-1)$ be a skyscraper stacky sheaf over $x_0$ which has a non-trivial action of the stabilizer group, and $O_{\Delta X}$ the object on the product $X \times X$ corresponding to the identity functor. Then the object

$$e = \text{cone}\{O_{x_0}(-1)^{\vee} \otimes O_{x_0}(-1) \rightarrow O_{\Delta X}\}$$

corresponds to the twisting $T \in \text{Autoeq}(D^b(\text{Coh}(X)))$ defined by

$$T : a \mapsto \text{cone}\{\text{Hom}(O_{x_0}(-1), a) \otimes O_{x_0}(-1) \rightarrow a\}$$

for $a \in D^b(\text{Coh}(X))$ ([13]). We note that

$$O_{x_0}(-1)^{\vee} \cong O_{x_0}(-1)[-2].$$

**Remark 7.4.** The main results of this paper hold without change for more general situations where the stabilizer groups of the smooth stacks have fixed
loci of codimension 1. This extended version will be used in a forthcoming paper.

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