On Nonlinear Parabolic Equation in Nondivergent Form with Implicit Degeneration and Embedding Theorems

Kamal N. Soltanov and Mahmud A. Ahmadov

Abstract. The mixed problem for the implicit degenerating nonlinear parabolic equation is considered, and the solvability and behavior of solutions of this problem are studied. Furthermore, some classes of function spaces and their relations with Sobolev spaces are investigated, embedding and compactness theorems for these spaces are proved.

1. Introduction

Consider the following problem

\[ \frac{\partial u}{\partial t} - |u|^\rho \Delta u + b_0 |u|^{\mu+1} = h(t,x), \quad (t,x) \in Q_T \equiv (0,T) \times \Omega, \]

\[ u(0,x) = 0, \quad x \in \Omega \subset \mathbb{R}^n, \quad n \geq 1, \]

\[ u(t,x) \mid _{\Gamma} = 0, \quad \Gamma \equiv [0,T] \times \partial \Omega, \quad T > 0, \]

Here \( \Omega \) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \) (for example, \( \partial \Omega \in C^1 \)), \( \Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) is a Laplacian, \( \rho > 0, \mu \geq 0, b_0 \in R^1 \) are some numbers, \( h(t,x) \) is a certain function.

The equation (1.1) describes the behavior of a flow on a boundary layer (see, [14, 23, 8]) and is also called Prandtl-von Mises type equation. The solvability of such type of equations and the behavior of their solutions are considered in many works (for example, [7, 12, 16, 17, 18, 22, 23, 24, 25] and references therein).

In one-dimensional case, the existence of solution of the considered equation, and functional spaces where the solution belongs to are obtained in [16, 20] (see also references in [18]).

The main point of this work is considering the problem (1.1) - (1.3) in \( n \)-dimensional case without additional conditions. Namely, the existence theorem is proved; spaces generated by the considered problem, their properties (particularly, some smoothness results of solutions are obtained as corrolaries of proved embedding theorems) and the behavior of solution are studied.

2000 Mathematics Subject Classification. Primary 35K55, 35K65, 35G30; Secondary 46E40, 46.20, 46T99.

Key words and phrases. Nonlinear parabolic equation, implicit degenerating, nonlinear functional spaces, embedding theorems.
Boundary value problems often lead to study of functional spaces related to the considered problems directly. More precisely, mentioned spaces are domains of operators generated by boundary value problems. For instance, we can say that the Sobolev spaces and their different generalizations appear while studying boundary operators generated by boundary value problems. For instance, we can say that the considered problems directly. More precisely, mentioned spaces are domains of the corresponding operators, roughly speaking, are subsets of linear spaces, but not possessing the linear structure. Therefore, in the beginning we would be concentrated on investigation of these infinity dimensional manifolds.

2. Existence Theorem

Define the following function space:

\[(2.1) \quad P_{1,p,q} (Q_T) \equiv W^1_q (0,T; L_q (\Omega)) \cap L_p \left(0, T; S^{0,\rho,2} \Delta (\Omega) \right),\]

where \(p, q \geq 1, \rho \geq 0\) are some numbers, \(W^1_q (0,T; L_q (\Omega))\) is a vector Sobolev space, and for functions \(u : \Omega \rightarrow R^1\)

\[(2.2) \quad S^{0,\rho,2} \Delta (\Omega) \equiv \left\{ u \in L_1 (\Omega) \left| [u]^{0,\rho,2}_{\Delta} \equiv \int_\Omega |u|^\rho |\Delta u|^2 dx < +\infty, \quad u (x) |_{\partial \Omega} = 0 \right. \right\}.

and for functions \(u : Q_T \rightarrow R^1\)

\[(2.3) \quad L_p \left(0, T; S^{0,\rho,2} \Delta (\Omega) \right) \equiv \left\{ u \in L_1 (Q_T) \left| [u]^p_{L_p} \equiv \int_0^T \left[ [u]^{0,\rho,2}_{\Delta} \right]^p dt < +\infty, \quad u (t,x) |_{[0,T] \times \partial \Omega} = 0 \right. \right\}.

Our main result on solvability of the problem \((1.1) - (1.3)\) is

**Theorem 1.** Let \(\rho > 0\), \(\min \{0, q - 1\} \leq \mu < \rho \leq 2\) or \(\frac{q}{2} - 1 \leq \mu < \rho\) and \(b_0 \in R^1\). Then, for any \(h \in L_2 \left(0, T; W^1_\rho (\Omega) \right)\) the problem \((1.1) - (1.3)\) is solvable in \(P (Q) \equiv P_{1,p,q} (Q_T) \cap \{ u (t,x) | u (0,x) = 0 \}\), where \(p = \rho + 2\), \(q = p' = \frac{p+2}{\rho-1}\).

The proof is based on a general result (Theorem 3) that is given below.

Let \(X\) and \(Y\) be Banach spaces with duals \(X^*\) and \(Y^*\) respectively, \(Y\) is a reflexive Banach space, \(M_0 \subseteq X\) be a weakly complete "reflexive" pn-space (see, Appendix A or [S3, S5]), \(X_0 \subseteq M_0 \cap Y\) is a separable vector topological space such that \(X_0^{\sim} M_0 \equiv M_0, \quad X_0^{\sim} Y \equiv Y\) and

\[i) \quad f : P_{1,p,q} (0,T; M_0, Y) \rightarrow L_q (0,T; Y)\text{ is a weakly compact (weakly continuous) mapping, where}\]

\[P_{1,p,q} (0,T; M_0, Y) \equiv L_p (0,T; M_0) \cap W^1_q (0,T; Y) \cap \{ x (t) | x (0) = 0 \},\]

\[1 < \max \{q, q'\} \leq p < \infty, \quad q' = \frac{q}{q-1};\]
(ii) there is a linear continuous operator $L : W^s_{p_0} (0,T;X_0) \to W^s_{p_0} (0,T;Y^*)$, $s \geq 0$, $p_0 \geq 1$ such that $L$ commutes with $\frac{\partial}{\partial t}$ and the conjugate operator $L^*$ has $ker(L^*) = \{0\}$; 

(iii) there exist a continuous function $\varphi : R_+^1 \cup \{0\} \to R_+^1 \cup \{0\}$ and numbers $\tau_0 \geq 0$ and $\tau_1 > 0$ such that $\varphi(r)$ is not decreasing for $\tau \geq \tau_0$, $\varphi(\tau_1) > 0$ and for any $x \in L_p (0,T;X_0)$ operators $f$ and $L$ satisfy the inequality
\[
\int_0^T (f(t,x(t)), Lx(t)) dt \geq \varphi ([x]_{L_p(M_0)}) [x]_{L_p(M_0)} ;
\]

(iv) there exist a linear bounded operator $L_0 : X_0 \to Y$ and constants $C_0 > 0$, $C_1, C_2 \geq 0$, $\nu > 1$ such that the inequalities
\[
\int_0^T \xi(t), L_0 \xi(t) dt \geq C_0 \|L_0 \xi\|_{L_q(0,T;Y)}^{\nu} - C_2, \\
\int_0^t \frac{dx}{d\tau} L_0 x(\tau) d\tau \geq C_1 \|L_0 x\|_Y(t) - C_2, \quad a.e. \ t \in [0,T]
\]
hold for arbitrary $x \in W^1_p (0,T;X_0)$ and $\xi \in L_p (0,T;X_0)$. 

**Theorem 2.** Assume that conditions (i) - (iv) are fulfilled. Then, for any $y \in G \subseteq L_q (0,T;Y)$, $G \equiv \bigcup_{r \geq \tau_1} G_r$:
\[
G_r = \left\{ y \in L_q (0,T;Y) \mid \int_0^T |(y(t), L_0 x(t))| dt \leq \int_0^T \langle f(t,x(t)), Lx(t) \rangle dt - c, \right. \\
\left. \text{ for all } x \in L_p (0,T;X_0), \ [x]_{L_p(0,T;M_0)} = r \right\}, \quad C_2 < c < \infty
\]
the Cauchy problem
\[
(2.4) \quad \frac{dx}{dt} + f(t,x(t)) = y(t), \quad y \in L_q (0,T;Y); \quad x(0) = 0
\]
is solvable in $P_{1,p,q} (0,T;M_0,Y)$ in the following sense
\[
\int_0^T \left( \frac{dx}{dt} + f(t,x(t)), y^* (t) \right) dt = \int_0^T \langle y(t) , y^* (t) \rangle dt, \quad \forall y^* \in L_{q'} (0,T;Y^*) .
\]

The proof of this result is presented in Appendix C (one can also refer to proofs of similar theorems in [18, 20]). The next proposition follows immediately from the last theorem.

**Corollary 1.** Under assumptions of Theorem 2 the problem (2.4) is solvable in $P_{1,p,q} (0,T;M_0,Y)$ for any $y \in L_q (0,T;Y)$ satisfying the condition: there is $r > 0$ such that the inequality
\[
\|y\|_{L_q(0,T;Y)} \leq \varphi ([x]_{L_p(0,T;M_0)})
\]
assumes that \( \beta > L \). These spaces are necessary in the application of Theorem 2 (and Corollary 1) to the function spaces \((p_n\text{-spaces})\) that are connected to the considered problem directly.

\[ (3.1) \]

We account for \( \eta \) where \( \alpha \), and obtain

\[ (3.5) \]

holds for any \( x \in L_p(0,T;X_0) \) with \( |x|_{L_p(M_0)} = r \). Furthermore, if \( \varphi(\tau) \to \infty \) as \( \tau \to \infty \), then the problem (2.4) is solvable in \( P_{1,p,q}(0,T;M_0,Y) \) for any \( y \in L_q(0,T;Y) \) satisfying the inequality

\[ \sup \left\{ \frac{1}{|x|_{L_p(0,T;M_0)}} \int_0^T \langle y(t), Lx(t) \rangle \, dt \mid x \in L_p(0,T;X_0) \right\} < \infty. \]

3. Embedding Theorems on \( pn \)-Spaces

In this section we introduce and investigate properties of a class of nonlinear function spaces \((pn\text{-spaces})\) that are connected to the considered problem directly. These spaces are necessary in the application of Theorem 2 (and Corollary 1) to the considered problem.

Consider the following function spaces (class of functions \( u : \Omega \to R \))

\[ (3.1) \quad S_{1,\alpha,\beta}(\Omega) = \left\{ u \in L_1(\Omega) \mid \| u \|_{S_1}^{\alpha+\beta} = \int_{\Omega} \left[ |u|^{\alpha+\beta} + |u|^\alpha |\nabla u|^\beta \right] \, dx \leq \infty, \right\}, \]

\[ (3.2) \quad S_{\Delta,\alpha,\beta}(\Omega) = \left\{ u \in L_1(\Omega) \mid \| u \|_{S_\Delta}^{\alpha+\beta} = \int_{\Omega} \left[ |u|^{\alpha+\beta} + \int_{\Omega} |u|^\alpha |\Delta u|^\beta \, dx \right] \leq \infty, \right\}, \]

where \( \alpha \geq 0, \frac{\alpha}{\beta} > -1, \beta, \beta_1 \geq 1 \) and \( \alpha_1 + \beta_1 = \alpha + \beta \). Here and hereafter we assume \( \beta > 1 \). Further, we consider the case \( \frac{\beta}{\beta} > -1, \beta > 1, \alpha > \beta - 1 \), as well.

Also, consider the following spaces of functions \( u : Q_T \to R^1 \)

\[ (3.3) \quad L_p(0,T;S_{1,\alpha,\beta}(\Omega)) = \left\{ u \in L_1(\Omega) \mid \| u \|_{L(S_1)}^p = \int_0^T \| u \|_{S_1}^p \, dt \leq \infty, \right\}, \]

\[ (3.4) \quad P_{p_0,p_1}(0,T;S_{\Delta,\alpha,\beta}(\Omega);X) = W_{p_0}^1(0,T;X) \cap L_{p_1}(0,T;S_{\Delta,\alpha,\beta}(\Omega)), \]

where \( p, p_0, p_1, \beta > 1, \alpha \geq 0 \) and \( X \) is a Banach space. Particularly, \( X \) can be chosen in such a way that \( L_{p_0}(\Omega) \subseteq X \) for some \( p_0 \geq 1 \).

The space \( L_{p_1}(0,T;S_{\Delta,\alpha,\beta}(\Omega)) \) is defined as \( L_p(0,T;S_{1,\alpha,\beta}(\Omega)) \) by using (3.2) instead of (3.1).

The equivalence

\[ \mathcal{M}_{\eta,W^1_\beta} = \left\{ u \in L_1(\Omega) \mid \eta(u) \in W^1_\beta(\Omega), \eta(u) \equiv |u|^{\frac{\beta}{\beta}} u \right\} \]

that express relations between \( W^1_\beta(\Omega) \) and \( S_{1,\alpha,\beta}(\Omega) \) follows immediately from (3.1). Indeed, it is enough to note that \( \eta(u) \equiv |u|^{\frac{\beta}{\beta}} u = v \iff u = |v|^{\frac{\beta}{\beta}} v \equiv \eta^{-1}(v) \).

Taking the last equivalency and definition (3.2) of the space \( S_{\Delta,\alpha,\beta}(\Omega) \) into account we get

\[ (3.5) \quad S_{\Delta,\alpha,\beta}(\Omega) = \mathcal{M}_{\eta,W^1_\beta} \cap \left\{ u \mid |u|^{\frac{\beta}{\beta}} \Delta u \in L_\beta(\Omega) \right\}. \]

In our next step we are going to express the relations between the second order Sobolev spaces and \( S_{\Delta,\alpha,\beta}(\Omega) \). To this end we use a few auxilary results.
Then \( \alpha(3.6) = \Delta \left( |u|^{\frac{\alpha}{\beta}} u \right) = \nabla \cdot \left( \frac{\alpha + \beta}{\beta} |u|^{\frac{\alpha}{\beta}} \nabla u \right) = \)

\[
\frac{\alpha + \beta}{\beta} |u|^{\frac{\alpha}{\beta}} \Delta u + \frac{\alpha (\alpha + \beta)}{\beta^2} |u|^{\frac{\alpha}{\beta} - 2} u |\nabla u|^2.
\]

**Proposition 1.** Let \( \alpha > -1, \beta \geq \beta_0 \geq 0, \beta \geq 1 \) be some numbers, \( \beta_0 + \beta \geq 2 \) and \( \Omega \subset \mathbb{R}^n, n \geq 1 \), be a bounded domain with sufficiently smooth boundary \( \partial \Omega \). Then the inequality

\[
(3.7) \int_{\Omega} |u|^\alpha |\nabla u|^{\beta_0 + \beta} dx \leq c(\varepsilon) \sum_{i=1}^{n} \int_{\Omega} |u|^{\alpha + \beta_0} |D_i^2 u|^\beta dx + \varepsilon_1 \kappa (\beta - \beta_0) \int_{\Omega} |u|^{\alpha + \beta_0 + \beta} dx
\]

holds for any \( u \in C^2(\Omega) \cap C^1(\Omega) \), where \( \varepsilon > 0, \varepsilon_1 = \varepsilon_1(\varepsilon) \) are some numbers, \( \kappa(s) = 1 \) if \( s > 0 \), and \( \kappa(s) = 0 \) if \( s = 0 \).

**Proof.** We have

\[
\int_{\Omega} |u|^\alpha |\nabla u|^{\beta_0 + \beta} dx \leq \sum_{i=1}^{n} \int_{\Omega} |u|^\alpha |D_i u|^{\beta_0 + \beta} dx
\]

Rewriting the expression under the integral in the following form

\[
\left( |u|^{\frac{\alpha + \beta_0}{\beta_0 + \beta}} u \right) \left( |u|^{\frac{\beta_0 + \beta}{\beta_0 + \beta}} |D_i u|^{\beta_0 + \beta} \right) \left( |u|^{\frac{\alpha + \beta_0}{\beta_0 + \beta}} D_i^2 u \right) \text{ if } \beta > \beta_0
\]

or

\[
\left( |u|^{\frac{\alpha}{\beta}} u |D_i u|^{2\beta - 2} \right) \left( |u|^{\frac{\beta}{\beta + \beta - 2}} D_i^2 u \right) \text{ if } \beta = \beta_0
\]

and applying Young’s inequality with exponents

\[
p_0 = \frac{\beta (\beta_0 + \beta)}{\beta - \beta_0}, p_1 = \frac{\beta_0 + \beta}{\beta_0 + \beta - 2}, p_2 = \beta \text{ if } \beta > \beta_0
\]

or

\[
p_0 = \beta', p_1 = \beta \text{ if } \beta = \beta_0, \frac{1}{\beta'} + \frac{1}{\beta} = 1
\]

we get

\[
\leq \varepsilon \sum_{i=1}^{n} \int_{\Omega} \left[ \kappa (\beta - \beta_0) |u|^\alpha |D_i u|^{\beta_0 + \beta} + |u|^\alpha |D_i u|^{\beta_0 + \beta} \right] dx + c(\varepsilon) \sum_{i=1}^{n} \int_{\Omega} |u|^{\alpha + \beta_0 + \beta} |D_i^2 u|^\beta dx \leq
\]

\[
\text{or } \varepsilon_1 \kappa (\beta - \beta_0) \int_{\Omega} |u|^{\alpha + \beta_0 + \beta} dx + \varepsilon_2 \int_{\Omega} |u|^\alpha |\nabla u|^{\beta_0 + \beta} dx
\]
The second term in (3.8) is obtained by using the equivalency

\[
\int_{\Omega} |u|^\alpha \sum_{i=1}^{n} |D_i u|^{\beta_0 + \beta} dx \leq \int_{\Omega} |u|^\alpha |\nabla u|^{\beta_0 + \beta} dx \leq n \int_{\Omega} |u|^\alpha \sum_{i=1}^{n} |D_i u|^{\beta_0 + \beta} dx.
\]

Note that the first term of (3.8) vanishes if \( \beta = \beta_0 \). Obtained inequalities prove the statement of the proposition. \( \square \)

**Remark 1.** It is not difficult to see that if \( \alpha + \beta_0 + \beta > 1, \beta_0 \geq 0, \beta_1 \geq 1 \) then

\[
\int_{\Omega} |u|^{\alpha + \beta_0 + \beta} dx \leq c \int_{\Omega} |u|^{\alpha + \beta_0} |\nabla u|^{\beta} dx \quad \text{or} \quad \int_{\Omega} |u|^{\alpha + \beta_0 + \beta} dx \leq c \int_{\Omega} |u|^{\alpha} |\nabla u|^{\beta_0 + \beta} dx
\]

and if \( 1 \leq \alpha_0 + \beta_0 \leq \alpha_1 + \beta_1, 1 \leq \beta_0 \leq \beta_1, \alpha_0 \beta_1 \geq \alpha_1 \beta_0 \) then

\[
\int_{\Omega} |u|^{\alpha_0} |\nabla u|^{\beta_0} dx \leq c \int_{\Omega} |u|^{\alpha_1} |\nabla u|^{\beta_1} dx + c_1
\]

hold for any \( u \in C^1_0 (\Omega) \), where

\[ c = c (\alpha, \beta_0, \beta, \text{mes} \; \Omega) > 0, \; c_1 = c_1 (\alpha_0, \beta_0, \alpha_1, \beta_1, \text{mes} \; \Omega) \geq 0, \]

Moreover, if \( \alpha_0 + \beta_0 = \alpha_1 + \beta_1 \) then \( c_1 = 0 \).

**Proposition 2.** Let \( \alpha > -1, \beta \geq 1 \) be some numbers, \( \alpha + \beta \geq 2 \) and \( \Omega \subset \mathbb{R}^n, n \geq 1 \), be a bounded domain with sufficiently smooth boundary \( \partial \Omega \). Then the inequality

\[
\int_{\Omega} |u|^{\alpha + \beta} dx \leq c \int_{\Omega} |u|^{\alpha} |\Delta u|^{\beta} dx.
\]

holds for any \( u \in C^2 (\Omega) \cap C^1_0 (\Omega) \), where \( c = c (\alpha, \beta, \text{mes} \; \Omega) > 0 \).

**Proof.** Rewriting \( \alpha + \beta \) as \( \alpha + \beta - 2 + 2 = \alpha + \beta_0 + \beta_1 \) with \( \beta_0 = \beta - 2 \) and \( \beta_1 = 2 \) and applying the first one of inequalities (2.10) we get

\[
\int_{\Omega} |u|^{\alpha + \beta} dx \equiv \int_{\Omega} |u|^{\alpha + \beta - 2 + 2} dx \leq c \int_{\Omega} |u|^{\alpha + \beta - 2} |\nabla u|^2 dx.
\]

The right hand side of the last inequality is estimating as

\[
- \frac{1}{\alpha + \beta - 1} \int_{\Omega} |u|^{\alpha + \beta - 2} u \Delta u dx \leq \int_{\Omega} |u|^{\alpha + \beta - 1} |\Delta u| dx = c \int_{\Omega} |u|^{\alpha + \beta - 1} |\Delta u| dx
\]

\[ c \int_{\Omega} |u|^{\alpha + \beta - 1} \varphi |\Delta u| dx \]
Now, applying the Young’s inequality with exponents \((\beta, \frac{\beta}{\beta-1})\) and arbitrary \(\varepsilon > 0\) gives
\[
(3.14) \quad \leq c(\varepsilon) \int_{\Omega} |u|^\alpha |\Delta u|^\beta \, dx + \varepsilon \int_{\Omega} |u|^\alpha + \beta \, dx.
\]

The inequality (3.12) follows from (3.13) taking (3.14) into consideration and making \(\varepsilon\) sufficiently small. \(\square\)

The following result is a special case of the main inequality (3.22)

**Lemma 1.** Let \(\alpha > -1, \beta > \frac{n}{n-1}\) be some numbers, \(\Omega \subset \mathbb{R}^n, n \geq 2,\) be a bounded domain with sufficiently smooth boundary \(\partial \Omega.\) Then the inequality
\[
(3.15) \quad \int_{\Omega} |u|^\alpha |\nabla u|^{2\beta} \, dx \leq c_1 \int_{\Omega} |u|^\alpha + \beta |\Delta u|^\beta \, dx + c_2 \int_{\Omega} |u|^\alpha + 2\beta \, dx.
\]
holds for any \(u \in C^2(\Omega) \cap C^1_0(\Omega),\) where \(c_1 = c(\alpha, \beta) > 0.\)

**Proof.** The proof of the inequality (3.15) is based on the boundedness in the Lebesque space \(L_p(\Omega)\) of the local Hardy-Littlwood maximal function
\[
M_{\Omega} w(x) = \sup_{0 < r < \text{dist}(x, \partial \Omega)} \frac{1}{|B_r(x)|} \int_{B_r(x)} w(y) \, dy;
\]
when \(1 < p < +\infty\) (see [15]), the local spherical maximal function
\[
(A_r w)(x) = \sup_{0 < r < \text{dist}(x, \partial \Omega)} \int_{S_r(x)} w(x + ry) \, dS(y); \quad S_r(x) = \partial B_r(x)
\]
when \(p > \frac{n}{n-1}, n \geq 2\) (see [15]), and on \(L_p(\Omega)-\text{convergency of averages of a}\) function to the function itself
\[
(3.16) \quad \lim_{r \searrow 0} \int_{\Omega} \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} w(y) \, dy - w(x) \right|^p \, dx = 0
\]
Let’s put \(w(x) \equiv |u(x)|^\rho |\nabla u(x)|^2\) for a function \(u \in C^2(\Omega) \cap C^1_0(\Omega).\) Then, under the conditions of Proposition 1 and boundedness of the local Hardy-Littlwood maximal function, for \(\rho = \frac{n}{2}\) we have
\[
(3.17) \quad \int_{\Omega} \left( \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \, dy \right)^\beta \, dx \leq c \int_{\Omega} \left( |u|^\rho |\nabla u|^2 \right)^\beta \, dx.
\]

1 For \(n = 1\) the similar results to results of this section was proved in the earlier works (see, for example, [S1, S5]). Therefore, it is enough to consider just dimension \(n \geq 2.\)

2 It should be noted that this approach of the proof is suggested by the second author.
Moreover, it is obvious that
\[
\int_{S_r(x)} \frac{1}{|B_r(x)|} \int_{S_r(x)} \rho^n \frac{\partial u}{\partial \nu} \, dS(y) =
\]
\[
\int_{S_1(0)} \frac{1}{|B_1(0)|} \rho^n \int_{S_1(0)} |u(x + r\eta)|^\rho u(x + r\eta) (\nabla u(x + r\eta) \cdot \nu) \, r^n \, dS(\eta),
\]
Therefore, from the boundedness of a local spherical maximal function, we have
\[
\int_{\Omega} \left| \int_{S_1(0)} \frac{1}{|S_1(0)|} \int_{S_1(0)} |u(x + r\eta)|^\rho u(x + r\eta) (\nabla u(x + r\eta) \cdot \nu) \, dS(\eta) \right|^\beta \, dx \leq \frac{c}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \, dy,
\]
where the positive constant \( c \) does not dependent on the function \( u(x) \). According to (3.6) we have
\[
\nabla \cdot (|u|^\rho \nabla u) = |u|^\rho \Delta u + (\rho + 1) |u|^\rho |\nabla u|^2
\]
Taking the integral of both sides of this equality on \( B_r(x) \), for \( x \in \Omega \), and \( 0 < r < \text{dist}(x, \partial \Omega) \) we recieve
\[
\frac{\rho + 1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \, dy =
\]
\[
\frac{1}{|B_r(x)|} \int_{B_r(x)} \nabla \cdot (|u|^\rho \nabla u) \, dy - \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho \Delta u \, dy
\]
or
\[
\frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \, dy =
\]
\[
(3.19) \quad \frac{1}{\rho + 1} \left\{ -\frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho \Delta u \, dy + \frac{1}{|B_r(x)|} \int_{S_r(x)} |u|^\rho \frac{\partial u}{\partial \nu} \, dS(y) \right\}
\]
Using (3.19), the left part of (3.15) is estimated in the following way
\[
\int_{\Omega} \left| |u|^\rho |\nabla u|^2 \right|^\beta \, dx \leq c \int_{\Omega} \left| |u|^\rho |\nabla u|^2 - \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \, dy \right|^\beta \, dx +
\]
\[
(3.20) \quad c \int_{\Omega} \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \, dy \right|^\beta \, dx = I_1(r) + I_2(r)
\]
According to (3.16) we have \( \lim_{r \to 0} I_1(r) = 0 \). Therefore, it is enough to show that \( I_2(r) \) is estimated uniformly with respect to the \( r \). Taking (3.17), (3.18) and (3.19) into consideration in \( I_2(r) \) we get

\[
I_2(r) = c \int_{\Omega} \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho |\nabla u|^2 \; dy \right|^\beta dx = \\
c \int_{\Omega} \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^\rho u \Delta u \; dy + \frac{1}{|B_r(x)|} \int_{S_r(x)} |u|^\rho u \frac{\partial u}{\partial \nu} \; dS(y) \right|^\beta dx \\
c_1 \int_{\Omega} \left| |u|^\rho u \Delta u \right|^\beta dx + c_1 \int_{\Omega} \left| \frac{1}{|B_r(x)|} \int_{S_r(x)} |u|^\rho u \frac{\partial u}{\partial \nu} \; dS(y) \right|^\beta dx \\
\leq \\
c_1 \int_{\Omega} \left| |u|^\rho u \Delta u \right|^\beta dx + \varepsilon \int_{\Omega} \left| |u|^\rho |\nabla u|^2 \right|^\beta dx + c_2(\varepsilon) \int_{\Omega} |u|^{(\rho + 2)\beta} dx.
\]

Consequently

(3.21) \( I_2(r) \leq c_1 \int_{\Omega} |u|^{\alpha + \beta} |\Delta u|^\beta dx + \varepsilon \int_{\Omega} |u|^{\alpha} |\nabla u|^{2\beta} dx + c_2(\varepsilon) \int_{\Omega} |u|^{\alpha + 2\beta} dx. \)

where \( c_1,c_2 \) are positive quantities not dependent on \( r \). Choosing sufficiently small \( \varepsilon > 0 \), such that \( \varepsilon < 1 \), then substituting the right side of (3.21) into (3.20) and passing to the limit by \( r \to 0 \) in the obtained inequality we get the desired inequality (3.15).

\[\square\]

Proposition 2 and Lemma 1 imply

**Corollary 2.** Under the conditions of Lemma 1 the inequality

(3.22) \( \int_{\Omega} |u|^{\alpha} |\nabla u|^{2\beta} dx \leq c \int_{\Omega} |u|^{\alpha + \beta} |\Delta u|^\beta dx \)

holds with \( c = c(\alpha, \beta) \) that is not dependent on \( u \).

Our next goal is considering relations between the spaces \( W^2_\beta(\Omega) \) and \( \mathcal{S}_{\Delta,\alpha,\beta}(\Omega) \).

We start with definition of the second order Sobolev space:

\[ W^2_\beta(\Omega) = \{ u \in L_1(\Omega) \mid u, D_i u, D_i D_j u \in L_\beta(\Omega), \quad i, j = 1, n \}. \]

It is well known (4) that

\[ W^2_\beta(\Omega) = \{ u \in L_1(\Omega) \mid u, D_i^2 u \in L_\beta(\Omega), \quad i = 1, n \}. \]

Moreover, for sufficiently smooth domains (14).

\[ W^2_\beta(\Omega) \cap W^{0,1}_\beta(\Omega) = \{ u \mid \Delta u \in L_\beta(\Omega), \quad u \mid_{\partial \Omega} = 0 \}. \]
We also define the following class of functions

\[ \mathcal{M}_{\Delta \eta, L_\beta(\Omega)} \equiv \left\{ u \mid \Delta \circ \eta (u) \in L_\beta(\Omega), \; \eta (u) \equiv |u|^\beta u \right\}. \]

Now, we are ready to compare spaces defined in (3.5) and (3.23)

**Lemma 2.** Let \( \alpha \geq 0, \alpha_1 > -1, \beta_1 \geq \beta \geq \frac{\beta_1}{2} \geq 1, \beta > \frac{n}{n-1} \) be some numbers, \( \alpha + \beta = \alpha_1 + \beta_1 \), (if \( \beta_1 = 2\beta \text{ then } \alpha > \beta - 1 \)) and \( \Omega \subset \mathbb{R}^n, n \geq 1 \), be a bounded domain with sufficiently smooth boundary \( \partial \Omega \). Then, the following inclusion

\[ 0^0 \mathcal{S}_{\Delta, \alpha, \beta} (\Omega) = \mathcal{S}_{\Delta, \alpha, \beta} (\Omega) \cap \{ u \mid u |_{\partial \Omega} = 0 \} \subseteq \mathcal{M}_{\Delta \eta, L_\beta(\Omega)} \cap \{ u \mid u |_{\partial \Omega} = 0 \} \]

takes place.

**Proof.** Let \( u \in 0^0 \mathcal{S}_{\Delta, \alpha, \beta} (\Omega) \) be an arbitrary function. Then, according to (3.23) \( \eta_1 (u) \equiv |u(x)|^\frac{\alpha}{\beta} u(x) \in W^{\frac{\beta}{\beta_1}}_\lambda (\Omega) \) as far as \( u |_{\partial \Omega} = 0 \iff \eta_1 (u) |_{\partial \Omega} = 0 \), and \( |u|^\frac{\alpha}{\beta} \Delta u \in L_\beta(\Omega) \). Moreover, if \( \alpha_1 > 0 \) then \( u |_{\partial \Omega} = 0 \implies \frac{\partial}{\partial n} \eta_1 (u) |_{\partial \Omega} = 0 \).

Under the conditions of Lemma, according to (3.22) and (3.11), the inequality

\[ \int_\Omega |u|^{\alpha_1} |\nabla u|^{\beta_1} \, dx \leq c \int_\Omega |u|^{\alpha} |\Delta u|^{\beta} \, dx \]

takes place with \( c = c (\alpha, \beta, \alpha_1, \beta_1) \) that is not dependent on \( u \).

Taking this and Corollary 2 into account we conclude

\[ (3.24) \; \mathcal{S}_{\Delta, \alpha, \beta} (\Omega) \equiv \left\{ u(x) \mid |u|^\frac{\alpha}{\beta} \Delta u \in L_\beta(\Omega) \right\} \cap \{ u \mid u |_{\partial \Omega} = 0 \}, \]

On the other hand, the definition (3.23) implies that \( u \in \mathcal{M}_{\Delta \eta, L_\beta(\Omega)} \) is equivalent to \( \Delta v \equiv \Delta \eta (u) \in L_\beta(\Omega) \). Indeed, using (3.6) and estimating \( L_\beta(\Omega) \) of \( \Delta v \) we get

\[ \| \Delta v \|^\beta_{L_\beta(\Omega)} = \int_\Omega |u|^{\frac{\alpha}{\beta}} \Delta u + \frac{\alpha}{\beta} |u|^\frac{\alpha-2}{\beta} \, |\nabla u|^2 \, dx \leq \]

\[ c \left\{ \int_\Omega |u|^{\frac{\alpha}{\beta}} \Delta u \, dx + \int_\Omega |u|^{\frac{\alpha-1}{\beta}} \, |\nabla u|^2 \, dx \right\}. \]

Taking the inequality (3.22) and the equivalence (3.24) into account we obtain

\[ 0^0 \mathcal{S}_{\Delta, \alpha, \beta} (\Omega) \subseteq \mathcal{M}_{\Delta \eta, L_\beta(\Omega)} \cap \{ u \mid u |_{\partial \Omega} = 0 \} \]

\[ \square \]

**Corollary 3.** Under the conditions of Lemma 2 the implication

\[ u \in 0^0 \mathcal{S}_{\Delta, \alpha, \beta} (\Omega) \implies v \equiv \eta (u) \in W^2_\beta (\Omega) \cap W^{\frac{1}{\beta}}_\lambda (\Omega) \]

holds.

**Proof.** If \( v(x) \equiv \eta (u) \equiv |u(x)|^\frac{\alpha}{\beta} u (x) \) then \( \nabla v (x) \equiv \left( \frac{\alpha}{\beta} + 1 \right) |u(x)|^\frac{\alpha}{\beta} \nabla u (x) \) and (3.6) takes place for \( \Delta v \).

According to mentioned above, the inclusion \( v \equiv \eta (u) \in W^2_\beta (\Omega) \cap W^{\frac{1}{\beta}}_\lambda (\Omega) \) is equivalent to \( \Delta v = \Delta \circ \eta (u) \in L_\beta (\Omega) \), as far as \( u |_{\partial \Omega} = 0 \iff \eta (u) |_{\partial \Omega} = 0 \) (moreover, if \( \alpha > 0 \) then \( u |_{\partial \Omega} = 0 \implies \frac{\partial}{\partial n} \eta (u) |_{\partial \Omega} = 0 \)). This implies that
\( \eta(u) \in W^2_\beta(\Omega) \cap W^1_\beta(\Omega) \) is equivalent to \( u \in M_{\Delta \alpha \eta, L_p(\Omega)} \cap \{ u \mid u|_{\partial \Omega} = 0 \} \). Therefore, taking the lemma into account we conclude the desired implication. \( \square \)

**Notation 1.** If parameters \( \alpha, \alpha_1 \geq 0, \beta, \beta_1, p, p_0, p_1 \geq 1 \) satisfy certain conditions some relations between spaces \( S_{\Delta, \alpha, \beta} (\Omega), S_{1, \alpha_1, \beta_1} (\Omega), L_p (\Omega) \), \( P_{p_0, p_1} (0, T; S_{\Delta, \alpha, \beta} (\Omega)) \), \( L_p (0, T; S_{1, \alpha_1, \beta_1} (\Omega)) \) can be obtained according to their definitions. More precise inclusion and compactness results for them can be proved on the way that is similar to our earlier works \([17, 18, 19, 20]\). Here, we are presenting some of such type of results.

**Theorem 3.** Let \( \alpha, \alpha_1 \geq 0, \beta > \frac{n}{n-\alpha}, \beta_1 \geq 1 \) be such numbers that \( \frac{\alpha + \beta}{\alpha_1 + \beta_1} \geq \beta \beta_1^{-1} \) and \( \alpha \beta \geq 1, \alpha > \beta - 1 \) then \( S_{\Delta, \alpha, \beta} (\Omega) \subseteq S_{1, \alpha_1, \beta_1} (\Omega) \).

The proof follows from the inequality
\[
\int_\Omega |u|^{\alpha_1} |\nabla u|^{\beta_1} \, dx \leq c(\varepsilon) \int_\Omega |u|^{\alpha} |\Delta u|^{\beta} \, dx + \varepsilon \left( \int_\Omega |u|^s \, dx \right)^{\frac{\alpha + \beta}{n}} ,
\]
where \( s = s(\alpha, \alpha_1, \beta, \beta_1) \leq \alpha + \beta, \) that can be derived by using the inequalities \(3.10, 3.11, 3.15 \) and \(3.22\) (for details refer to \(18, 19\)).

**Remark 2.** Note that it is not difficult to verify that if \( \frac{\alpha + \beta}{\alpha_1 + \beta_1} \geq 1, \beta \geq \beta_1 \) and \( \frac{\alpha (\alpha + \beta)}{n} \geq p, n > \beta_1 \) then the following inclusions
\[
S_{1, \alpha, \beta} (\Omega) \subseteq S_{1, \alpha_1, \beta_1} (\Omega) \subseteq L_p (\Omega), \quad S_{\Delta, \alpha, \beta} (\Omega) \subseteq S_{\Delta, \alpha_1, \beta_1} (\Omega)
\]
take place. Moreover, arguments similar to those that express relations between the considered and Sobolev spaces show that the inclusion \( S_{1, \alpha, \beta} (\Omega) \subseteq L_p (\Omega) \) and consequently, \( S_{\Delta, \alpha, \beta} (\Omega) \subseteq L_p (\Omega) \) are compact (for detail one can refer to \(18, 19, 20\)).

**Corollary 4.** Assume that the conditions of Theorem 3 are fulfilled. Then, the following inclusions
\[
P_{p_0, p_1} (0, T; S_{\Delta, \alpha, \beta} (\Omega) ; X) \subseteq P_{p_0, p_1} (0, T; S_{1, \alpha_1, \beta_1} (\Omega) ; \tilde{X}) ,
\]
\[
P_{p_0, p_1} (0, T; S_{\Delta, \alpha, \beta} (\Omega) ; X) \subseteq L_p (0, T; S_{1, \alpha_1, \beta_1} (\Omega)) ,
\]
hold if \( X \subseteq \tilde{X} \), and \( p_0 \geq \tilde{p}_0 \geq 1, p_1 \geq \tilde{p}_1 \geq 1, p_1 \geq p \geq 1 \).

**Remark 3.** If \( \alpha \geq 0, \frac{\alpha}{\alpha_1} > -1, \frac{1}{\beta_1} = \beta > \frac{n}{n-\alpha_1} \) such numbers that \( \alpha + \beta = \alpha_1 + \beta_1, \alpha > \beta - 1, \) then
\[
S_{\Delta, \alpha, \beta} (\Omega) \iff \left\{ u(x) \mid \eta(u) \equiv |u|^\rho u \in W^2_\beta (\Omega) \right\}
\]
i.e.
\[
u \equiv \eta^{-1} (v) \equiv |v|^{-\alpha/\beta} v \in S_{\Delta, \alpha, \beta} (\Omega)
\]
under the conditions (see, \([13]\) and also \([18, 19]\)) that all operations make a sense.
Furthermore, note that \( S_{1,\alpha,\beta}(\Omega) \) and \( S_{\Delta,\alpha,\beta}(\Omega) \) are metric spaces ([18] [19], [20]) with the corresponding metrics of the form:

\[
d_{S_{1,\alpha,\beta}(\Omega)}(u; v) \equiv \|\eta(u) - \eta(v)\|_{W_{\rho}^{\alpha,\beta}(\Omega)}^{(\rho+1)^{-1}}, \quad \eta(\tau) \equiv |\tau|^{\rho}, \rho = \frac{\alpha}{\beta}, \alpha \geq 0, \beta > 1,
\]

\[
d_{S_{\Delta,\alpha,\beta}(\Omega)}(u; v) \equiv \|\eta_1(u) - \eta_1(v)\|_{W_{\rho}^{\alpha,\beta}(\Omega)}^{(\rho+1)^{-1}} + \|u|^{\rho}\Delta u - |v|^{\rho}\Delta v\|_{L_{\rho}^{\alpha,\beta}(\Omega)}^{(\rho+1)^{-1}},
\]

where \( \rho_1 = \frac{\alpha}{\beta} \), \( \eta_1(\tau) \equiv |\tau|^{\rho_1} \), and \( \alpha_1 + \beta_1 = \alpha + \beta \) (see [27]).

Moreover, it is not difficult to see that the metrics of spaces \( S_{1,\alpha,\beta}(\Omega) \) and \( S_{\Delta,\alpha,\beta}(\Omega) \) have the form:

\[
d_{S_{1,\alpha,\beta}(\Omega)}(u; v) \equiv \|\|u\|^{\rho}\nabla u - |v|^{\rho}\nabla v\|_{L_{\rho}^{\alpha,\beta}(\Omega)}^{(\rho+1)^{-1}};
\]

\[
d_{S_{\Delta,\alpha,\beta}(\Omega)}(u; v) \equiv \|\|u\|^{\rho}\Delta u - |v|^{\rho}\Delta v\|_{L_{\rho}^{\alpha,\beta}(\Omega)}^{(\rho+1)^{-1}}
\]
correspondingly.

Based on Theorem 3, Corollary 4 and Embedding Theorems for the Sobolev spaces we prove the following:

**Theorem 4.** Let \( \alpha, \alpha_1 \geq 0, \beta > \frac{n}{n-1}, \beta_1 \geq 1 \) be such numbers that \( \beta_1 < \frac{n\beta}{n-\beta} \), \( \beta < n \), \( \alpha_1 + \beta_1 < \frac{(\alpha+\beta)}{n-\beta} \) and \( \alpha_1 \beta_1 \geq \alpha_1 \beta \), \( \alpha > \beta - 1 \). Then, the inclusion

\[
0
\]

\[
S_{\Delta,\alpha,\beta}(\Omega) \subset S_{1,\alpha_1,\beta_1}(\Omega)
\]
is compact.

**Proof.** Since \( u \in S_{\Delta,\alpha,\beta}(\Omega) \), we have \( \eta(u) \equiv v \in W_{\rho}^{\alpha,\beta}(\Omega) \cap W_{\rho_1}^{\alpha_1,\beta_1}(\Omega) \) and \( u \in S_{1,\alpha_1,\beta_1}(\Omega) \) if and only if \( \eta_1(u) \equiv v \in W_{\rho_1}^{\alpha_1,\beta_1}(\Omega) \) \( (\eta_1(u) \equiv |u|^{\rho_1} u, \rho_1 = \frac{\alpha}{\beta}) \). Moreover, as far as \( W_{\rho}^{\alpha,\beta}(\Omega) \subset W_{\rho_1}^{\alpha_1,\beta_1}(\Omega) \) is compact for \( \beta_1 < \frac{n\beta}{n-\beta} \), we get the compactness of the inclusion \( \eta(G) \subset W_{\rho_1}^{\alpha_1,\beta_1}(\Omega) \) for any bounded subset \( G \) from \( S_{\Delta,\alpha,\beta}(\Omega) \). This implies the desired statement.

**Corollary 5.** If \( 0 < \rho \leq 2 \), then \( 0 S_{\Delta,\rho,\beta}(\Omega) \subset W_{\rho}^{1}(\Omega) \), \( p = \rho + 2 \).

Mentioned above and known results ([10], [11], [18], [19]) allow us to prove the compact embeddings for the following vector spaces: \( L_{p}(0, T; S_{1,\alpha_1,\beta_1}(\Omega)) \), \( P_{1,\rho_0,\rho_1}(0, T; S_{\Delta,\alpha,\beta}(\Omega); X) \). We need the following

**Lemma 3.** Let \( \alpha, \alpha_1, \alpha_2 \geq 0, \beta, \beta_1, \beta_2 \geq 1, \beta \geq \beta_1 \geq 1 \) be such numbers that \( \alpha + \beta = \alpha_1 + \beta_1 = \alpha_2 + \beta_2, \beta_1 < \beta < \beta_2, 1 \leq \beta_1 < \beta < \beta_2 \). Then, for any \( \varepsilon > 0 \) there exists \( c(\varepsilon) > 0 \) such that the inequality

\[
[u]_{S_{\Delta,\alpha,\beta}} \leq \varepsilon[u]_{S_{\Delta,\alpha_2,\beta_2}} + c(\varepsilon)[u]_{S_{\Delta,\alpha_1,\beta_1}}, \forall u \in S_{\Delta,\alpha_2,\beta_2}(\Omega)
\]

holds.

The proof is obvious.
Lemma 4. Let \( \alpha, \alpha_1 \geq 0, \beta, \beta_0, p \geq 1, \frac{\beta}{2} \geq \beta > \frac{\alpha}{n-1} \) be such numbers that \( \beta_0 < \frac{n\beta}{n-\beta}, \beta < n, \alpha_0 + \beta_0 < \frac{n(\alpha+\beta)}{n-\beta} \) and \( \alpha \beta_0 \geq \alpha_0 \beta, \alpha > \beta - 1, p \leq \alpha + \beta \). Then, for any \( \varepsilon > 0 \) there exists \( c(\varepsilon) > 0 \) such that the inequality
\[
d_{S_{\alpha_0,\beta_0}}(u,v) \leq \varepsilon \left( [u]_{S_{\alpha,\beta}} + [v]_{S_{\alpha,\beta}} \right) + c(\varepsilon) \|u - v\|_{L_p}, \forall u,v \in S_{\Delta,\alpha_2,\beta_2}(\Omega)
\]
holds.

The proof is similar to the proof of the same type results from [6, 10, 18, 19] and is based on the compactness of the inclusion \( S_{\Delta,\alpha_2,\beta_2}(\Omega) \subset S_{\alpha_1,\beta_1}(\Omega) \subset L_p(\Omega) \).

These lemmas allow us to get the following compactness.

Theorem 5. Let \( S_{\alpha_1,\beta_1}(\Omega), S_{\Delta,\alpha,\beta}(\Omega) \) and \( X \) be spaces defined above and \( S_{\Delta,\alpha,\beta}(\Omega) \subset S_{\alpha_1,\beta_1}(\Omega) \) is compact. Let \( \alpha_1 \geq 0, \beta, \beta_1, p, p_0, p_1 \geq 1 \) be such numbers that \( \alpha + \beta = p = p_1, \alpha \beta_1 \geq \alpha_1 \beta, \beta > \frac{n}{n-1}, \alpha > \beta - 1 \). Then, the inclusion \( P_{1,p_0,p_1}(0, T; S_{\Delta,\alpha,\beta}(\Omega); X) \subset L_p(0, T; S_{1,\alpha_1,\beta_1}(\Omega)) \) is compact.

The proof is similar to the proof of the same type of results from [10, 17, 18, 19, 20]. Therefore, we are not providing it here. The other compactness theorems similar to Theorem 4 and Lemma 3 can also be proved, but we are not presenting them here, as well. However, if it would be necessary, we are going to use those theorems for the spaces \( P_{1,p_0,p_1}(0, T; S_{\alpha_2,\beta_2}(\Omega); L_q(\Omega)) \), \( L_p(0, T; S_{\alpha_1,\beta_1}(\Omega)) \) under the corresponding conditions on parameters \( \alpha, \alpha_1, \beta, \beta_1, p, p_0, p_1 \) and refer reader to our earlier works [17, 18, 19, 20] for further details.

4. The Proof of the Solvability Theorem

Now we can lead the proof by using Theorem 2 (Corollary 1), and in order to apply it we introduce the following spaces and mappings:

\[ \mathcal{M}_0 \equiv S_{\Delta,\rho,2}(\Omega), \quad X_0 \equiv W_p^2(\Omega) \cap W_p^1(\Omega), \quad Y \equiv L_q(\Omega), \quad X \equiv L_p(\Omega), \]

\[ f(u) \equiv -|u|^\rho \Delta u + b_0|u|^{\mu+1}, \quad L \equiv -\Delta, \quad L_0 \equiv \nabla, \quad Y^* \equiv L_p(Q), \quad p = \rho + 2 \]

\[ P_{0,1,p,q}(0, T; \mathcal{M}_0, Y) \equiv P_{0,1,p,q}(0, T; S_{\Delta,\rho,2}(\Omega); L_q(\Omega)) \cap L^\infty(0, T; W_{\rho,2}(\Omega)) \]

where

\[ P_{0,1,p,q}(0, T; S_{\Delta,\rho,2}(\Omega); L_q(\Omega)) \equiv L_p(0, T; S_{\Delta,\rho,2}(\Omega)) \cap W_{\rho,q}^1(0, T; L_q(\Omega)). \]

It not is difficult to see that
\[ \langle f(u), Lu \rangle \equiv \left< -|u|^\rho \Delta u + b_0|u|^{\mu+1}, -\Delta u \right> = \int_\Omega |u|^\rho (\Delta u)^2 dx + \int_\Omega b_0 |u|^{\mu+1} \Delta u dx \]
for any \( u \in W_p^2(\Omega) \cap W_p^1(\Omega) \) and \( u \in L_p(0, T; W_p^2(\Omega) \cap W_p^1(\Omega)) \).
Taking into account the embedding theorems from Section 3, the last equality implies that, if \( \min \{ 0, \frac{q}{\rho} - 1 \} \leq \mu < \rho \leq 2 \) or \( \frac{q}{\rho} - 1 \leq \mu < \rho \) and \( b_0 \in \mathbb{R}^1 \) then

\[
\int_0^T \langle f(u), Lu \rangle \, dt \geq (1 - \varepsilon) \int_0^T \int_\Omega |u|^p (\Delta u)^2 \, dx \, dt - c_1(\varepsilon) =
\]

\[
(1 - \varepsilon) \int_0^T [u]^{p+2}_{S_{\Delta, \rho, 2}(\Omega)} \, dt - c_1(\varepsilon) \equiv \varphi \left( [u]^{0}_{L_{p+2}(S_{\Delta, \rho, 2})} \right) \left[ u \right]^{0}_{L_{p+2}(S_{\Delta, \rho, 2})},
\]

where \( c_0 > 0 \), \( c_1, \varepsilon \geq 0 \), and \( \varepsilon \) is a sufficiently small nonnegative number.

Furthermore, it is obvious that \( \int_0^T \langle \frac{\partial u}{\partial t}, Lu \rangle \, dt \equiv \frac{1}{2} \| \nabla u(t) \|_{L_2}^2 \) for any \( u \in W^1_p(0, T; X_0) \) and almost any \( t \in (0, T) \). Moreover, \( \int_0^T \langle w, Lu \rangle \, dt \equiv \| \nabla w \|_{L_2(Q)}^2 \) for any \( w \in L_p(0, T; X_0) \), where \( \equiv \frac{\partial u}{\partial t} \in L_p(0, T; X_0) \).

Using the generalized coercivity of pair \( f \) and \( -\Delta \) on \( L_p \left( 0, T; W^2_p(\Omega) \cap \frac{\partial}{\partial t} W^{1}_p(\Omega) \right) \cap W^1_q(0, T; L_q(\Omega)) \) the following apriori estimations for a solution \( u(t, x) \) of considered problem are obtained in a common way:

\[
[u]^{0}_{L_p(S_{\Delta, \rho, 2})} \leq c \left( \| b_0 \|, \| h \|_{L_2(W^2_2(\Omega))}, \rho, \mu \right), \quad p = \rho + 2, \quad q = p',
\]

and

\[
\| u \|_{W^2_q(L_q) \cap L_\infty(W^2_2)} \leq c \left( \| b_0 \|, \| h \|_{L_2(W^2_2(\Omega))}, \rho, \mu \right).
\]

Thus, each possible solution \( u(t, x) \) of the considered problem belongs to a bounded subset of

\[
L_p \left( 0, T; S_{\Delta, \rho, 2}(\Omega) \right) \cap W^1_q(0, T; L_q(\Omega)) \cap L_\infty \left( 0, T; \frac{\partial}{\partial t} W^{1}_2(\Omega) \right),
\]

and, consequently, the solutions belong to a bounded subset of \( P_{1,p,q} \left( 0, T; S_{\Delta, \rho, 2}(\Omega); L_q(\Omega) \right) \) and \( L_\infty \left( 0, T; \frac{\partial}{\partial t} W^{1}_2(\Omega) \right) \).

To apply Theorem 2 (Corollary 1) it remains to show that \( f \) is a weakly compact (continuous) mapping from \( P_1(Q_T) \equiv P_{1,p,q} \left( 0, T; S_{\Delta, \rho, 2}(\Omega); L_q(\Omega) \right) \) into \( L_q(Q_T) \). To this end, it is enough to use the following expressions:

\[
|u|^\rho \quad u \quad |\nabla u|^2 = (|u|^{\rho} \quad \nabla u) \cdot (|u|^{(1-\gamma)\rho - 2} \quad u \quad \nabla u),
\]

\[
|u|^\rho \quad u \quad |\nabla u|^2 = \frac{1}{\rho \rho (\rho + 1) (1 - \theta)} \Delta (|u|^\rho \quad u) - \frac{1}{\rho \rho (\rho + 1) (1 - \theta)} |u|^{(1-\theta)\rho} \quad \Delta (|u|^{\rho} \quad u),
\]

because of

\[
|u|^\rho \quad \Delta u = \frac{1}{\rho + 1} \Delta (|u|^\rho \quad u) - \rho \quad |u|^{\rho - 2} \quad u \quad |\nabla u|^2,
\]
where $\gamma$ is a number from condition 4) if $\rho \geq 1$, and $\theta$ is such a number that $\frac{1}{\gamma} \leq \theta < 1$ if $0 < \rho \leq 2$. Particularly, if $\theta = \frac{2}{3}$ it is sufficient to use the expression:

$$|u|^\rho - 2 u \mid \nabla u \mid^2 = \frac{3}{\rho (\rho + 1)} (|u|^\rho u) - \frac{3}{\rho} |u|^{\rho - 1} |\nabla u|,$$

then $|u|^{\frac{2}{\gamma}} \Delta u \in L_{\frac{2(\rho + 2)}{\gamma}} (Q)$ and $|u|^{\frac{2}{\gamma}} \in L_{\frac{2(\rho + 2)}{\gamma}} (Q)$.

Thus, according to the embedding theorems mentioned above, the solution $u(t,x) \in P_{1,p,q} \left( 0, T; \tilde{S}_{\Delta,\alpha,\beta} (\Omega); L_q (\Omega) \right)$ and $u(t,x) \in L_p \left( 0, T; \tilde{S}_{\Delta,\alpha,\beta} (\Omega); L_q (\Omega) \right)$, if the parameters $\alpha_1 \geq 0$, $\beta, \beta_1, p, p_0, p_1 \geq 1$, $\alpha > \beta - 1$ satisfy one of the following conditions: 1) $\alpha_1 = (\rho - 1) q$, $\beta_1 = 2 q$, $p = \rho + 2$, $q = p' = \frac{p}{p - 1}$; 2) $\alpha = \rho$, $\beta = 2$, $p_0 = \rho + 2$, $q_0 = q$; 3) $\alpha = s \beta$, $\beta > 1$, $p_0 = (s + 1) \beta$, $q_0 = \beta$, $1 \leq s \leq \frac{2}{\gamma - 2}$, or 4) $\alpha = \gamma \rho \beta$, $\beta = \frac{\rho + 2}{\gamma + 1}$, $\frac{1}{\gamma} \leq \theta < 1$, $p_0 = \rho + 2$, $q_0 = q$.

Therefore, if a sequence $\{u_m\} \subset C_{1,p,q} \left( 0, T; \tilde{S}_{\Delta,\rho,2} (\Omega); L_q (\Omega) \right)$ converges weakly to $u \in C_{1,p,q} \left( 0, T; \tilde{S}_{\Delta,\rho,2} (\Omega); L_q (\Omega) \right)$ in $C_{1,p,q} \left( 0, T; \tilde{S}_{\Delta,\rho,2} (\Omega); L_q (\Omega) \right)$, then, according to the compactness theorem from Section 3, one of the factors in (4.1) and in the second term of (4.2) converges weakly and the other one converges strongly in the corresponding spaces. This implies that $f(u_m) \rightarrow f(u)$ in $L_q (Q)$.

Hence, all conditions of Corollary 1 are fulfilled. Applying it to the considered problem (1.1)-(1.3) we obtain the statement of Theorem 1.

Remark 4. The solvability theorem such as Theorem 1 for the problem (1.1)-(1.3), but with $u(0,x) = u_0 (x)$ for $u_0 \in \tilde{S}_{\Delta,\rho,2} (\Omega)$ is also valid and can be proved as in [19] (or [18]).

Remark 5. The problem (1.1)-(1.3) can also be considered with the term $b(t,x,u)$ instead of $b_0 |u|^\mu + 1$. In this case, it is enough to assume holding of the following conditions: The function $b(t,x,u)$ is the Carathéodory function on $Q \times R^1$, there exist functions $b_0(t,x), b_1(t,x) \geq 0$ and number $\mu \geq 0$ such that $\min \left\{ 0, \frac{2}{\mu} - 1 \right\} \leq \mu < \rho$ and

$$|b(t,x,u)| \leq b_0 (t,x) |u|^\mu + 1 + b_1 (t,x),$$

where

$$b_0 \in L^\infty (Q), \quad b_1 \in L_2 \left( 0, T; W^1_2 (\Omega) \right) \quad \text{if } \mu \geq \frac{\rho}{2} - 1;$$

$$b \in L^\infty \left( 0, T; W^{1,\infty} (\Omega); C^1 (R^1) \right) \text{ and }$$

$$\begin{align*}
|D_i b(t,x,u)| &\leq \tilde{b}_0 (t,x) |u|^\mu + 1 + b_1 (t,x), \quad i = 1, n, \\
|b_3 (t,x,\xi)| &\leq b_2 (t,x) |\xi|^\mu + b_3 (t,x),
\end{align*}$$

$$\tilde{b}_0, b_2 \in L^\infty (Q), \quad \tilde{b}_1 \in L_2 (Q), \quad b_3 \in L_2 \left( 0, T; W^1_2 (\Omega) \right), \quad q = p' = \frac{p}{p - 1}, \text{ and } p = \rho + 2 \text{ if } \mu < \frac{2}{\mu} - 1.$$
5. On a Behavior of the Solutions of Problem (1.1)-(1.3)

In this section we investigate the behavior of solutions for different $\mu \geq 0$:

min $\{0, \frac{\mu}{2} - 1\} \leq \mu < \rho$ and $u(0, x) = u_0(x)$, and in the case $\mu = \rho$.

Theorem 6. Let $\min \{0, \frac{\mu}{2} - 1\} \leq \mu < \rho$, $u_0 \in W^{1}_p(\Omega)$, $h \in L^\infty \left(R^{1\prime}_1; L_q(\Omega)\right)$ and $\|h\|_{L_q}(t) \leq C_0$. Then, the solution of the problem (1.1)-(1.3) with the initial condition $u(0, x) = u_0(x)$ satisfies the inequality

$$
\|u(t)\|_{L^2(\Omega)}^2 \leq \left(\frac{C + C_2 \|h\|_{L^\infty(\Omega)}^q}{C_1}\right)^{\frac{2}{p}} + \left(C_1 \frac{\rho t}{2}\right)^{\frac{2}{p}},
$$

i.e. the solution of the problem (1.1)-(1.3) remains bounded as $t \not\to \infty$, where $C_j = C_j \left(\rho, \mu, b_0, C_0, \|u_0\|_{W^1_p}, mes \Omega\right)$.

Proof. Consider the functional

$$
\Phi(t) \equiv \Phi(u(t)) \equiv \frac{1}{2} \int_{\Omega} |u(t)|^2 \, dx \equiv \frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2.
$$

If $u(t)$ is a solution of the problem (1.1)-(1.3) then, the function $\Phi(t)$ has the property

$$
\Phi'(t) = \langle u', u \rangle = \left\langle |u|^\mu \Delta u - b_0 |u|^\mu + h, u \right\rangle = - (\rho + 1) \left\langle |u|^\mu \nabla u, \nabla u \right\rangle - \langle b_0 |u|^\mu u + (h, u) \rangle \leq - (\rho + 1) \left\| |u|^\frac{\mu}{2} \nabla u \right\|_{L^2}^2 + \langle b_0 |u|^\mu + (h, u) \rangle.
$$

Applying the results of Section 3 we get

$$
\Phi'(t) \leq - (\rho + 1) \left\| |u|^\frac{\mu}{2} \nabla u \right\|_{L^2}^2 + \langle b_0 |u|^\mu + (h, u) \rangle \leq - \left(\rho + 1\right) \frac{(\rho + 2)^2}{4} \left\| \nabla \left(|u|^\frac{\mu}{2} u\right) \right\|_{L^2}^2 + 2\varepsilon \left\| u \right\|_{L^p}^p + C(\varepsilon) \left(1 + \|h\|_{L^q}^q\right) \leq 0,
$$

$$
\left\| \left| |u|^\frac{\mu}{2} u \right|_{L^2}^2 + C(\varepsilon) \left(1 + \|h\|_{L^q}^q\right) \right\| \leq - C_0 \left\| u \right\|_{L_p}^p + C(\varepsilon) \left(1 + \|h\|_{L^q}^q\right) \leq - C_1 \left\| u \right\|_{L_2} + C(\varepsilon) + k(\varepsilon) \|h\|_{L^\infty(\Omega)}^q
$$

because of $S_{1,\rho,2}(\Omega) \subset L_2(\Omega)$ with the corresponding inequalities. Hence we have

$$
\Phi'(t) + C_1 (\Phi(t))^\frac{2}{p} \leq C + C_2 \|h\|_{L^\infty(\Omega)}^q,
$$

where $C = C \left(\rho, \mu, b_0, C_0, \|u_0\|_{W^1_p}, mes \Omega\right)$, $C_1 = C_1 \left(\rho, \mu, b_0, C_0, \|u_0\|_{W^1_p}, mes \Omega\right)$ and $\Phi(0) = \|u_0\|_{L^2}^2$.

Then, applying the following form of Gronwall’s lemma (Lemma 5) to the inequality (5.2), which was proved by Ghidaglia, with $y(t) \equiv \Phi(t)$, $\theta = C_1$, $\eta = C + C_2 \|h\|_{L^q}^q$, $l = \frac{\rho}{2}$ we obtain the inequality (5.1). □
Lemma 5. Let \( y(t) \) be a positive absolutely continuous function on \( \mathbb{R}_+ \) which satisfies

\[
y' + \theta y^l \leq \eta, \quad l > 1, \theta > 0, \eta \geq 0.
\]

Then, for \( t \geq 0 \),

\[
y(t) \leq \left( \frac{\eta}{\theta} \right)^{\frac{1}{l-1}} + (\theta (l-1) t)^{-\frac{1}{l-1}}.
\]

Now, consider the following problem:

\[
\frac{\partial u}{\partial t} - |u|^\rho \Delta u - b(x)|u|^\rho + 1 = 0, \quad (t, x) \in Q,
\]

\[
u(0, x) = u_0(x) \geq 0, \quad u \mid_{\Gamma} = 0 , \quad \Gamma = [0, T] \times \partial \Omega,
\]

Let \( \lambda_1 \) be the first eigenvalue and \( v_1(x) \) be the corresponding eigenfunction of the problem

\[-\Delta v = \lambda v, \quad x \in \Omega \quad v \mid_{\partial \Omega} = 0 \, .
\]

Lemma 6. Let \( \rho, b(x) > 0, u_0(x) \geq 0 \) and \( \mu = \rho \), moreover \( u_0 \in L^2(\Omega), \)

\[|b|_{L^\infty(\Omega)} \leq c, \quad c = c(\Omega) > 0, \quad \Omega \subset \subset R^n \] be as above. Then, if \( M = \frac{c(\rho+2)^2}{4(\rho+1)} \) < \( \lambda_1 \) then a solution of the problem \([5.3]-[5.4]\) remains bounded as \( t \nearrow \infty \), i.e. the inequality of the type \([5.7]\) takes place also.

Proof. Let \( \rho, b(x) > 0 \) and \( \mu = \rho \). Then, using the previous reasoning we get

\[
\Phi'(t) = \langle u', u \rangle = - (\rho + 1) \langle |u|^\rho \nabla u, \nabla u^\rho \rangle + \langle b(x) |u|^\rho + 1, u \rangle \leq
\]

\[
- (\rho + 1) \frac{4}{(\rho + 2)^2} \left( \left| |u|^\rho \nabla u \right|_{L^2} \right)^2 + \left| |u|^\rho \nabla \right|_{L^\infty}^2 .
\]

Since \( M < \lambda_1 \) a solution remains bounded when \( t \nearrow \infty \) as in the previous case. \( \square \)

Remark 6. Suppose \( b(x) > \lambda_1 \) and \( u(t, x) > 0 \) for \( x \in \Omega \) or on a subdomain \( \overline{\Omega} \subset \subset \Omega \). Note that this case was studied under various conditions in \([7, 24, 22, 25]\). In our consideration we study the problem \([5.3]-[5.7]\) in the following way:

If \( u(t, x) > 0 \) for \( x \in \Omega \), then the equation \([5.3]\) can be represented as

\[
u^{-\rho} \frac{\partial u}{\partial t} - \Delta u - b(x) u = 0, \quad (t, x) \in Q.
\]

Hence, we have

\[
\left\langle u^{-\rho} \frac{\partial u}{\partial t}, v_1 \right\rangle = \langle \Delta u + b(x) u, v_1 \rangle \rightarrow \left\langle u^{-\rho} \frac{\partial u}{\partial t}, v_1 \right\rangle = -\lambda_1 \langle u, v_1 \rangle + \langle b(x) u, v_1 \rangle
\]

or

\[
\left\langle u^{-\rho} \frac{\partial u}{\partial t}, v_1 \right\rangle \geq \delta \langle u, v_1 \rangle, \quad (b(x) - \lambda_1) \geq \delta > 0 \rightarrow (1 - \rho)^{-1} \frac{\partial}{\partial t} \left\langle u^{1-\rho}, v_1 \right\rangle \geq \delta \langle u, v_1 \rangle .
\]

The blow-up result can be obtained from here as in \([25]\) (see \([7, 24, 22, 25]\) and references therein).
6. Appendixes

6.1. Appendix A. Let \( X, Y \) be a locally convex vector topological spaces, \( B \subseteq Y \) be a Banach space and \( g : D(g) \subseteq X \to Y \). Let’s introduce the following subset of \( X \)
\[
\mathcal{M}_{gB} = \{ x \in X \mid g(x) \in B, \quad \text{Im} g \cap B \neq \emptyset \}. 
\]

**Definition 1.** A subset \( \mathcal{M} \subseteq X \) is called a \( pn \)-space (i.e. pseudonormed space) if \( S \) is a topological space and there is a function \([\cdot]_\mathcal{M} : \mathcal{M} \to R_+^1 \equiv [0, \infty) \) (which is called \( p \)-norm of \( \mathcal{M} \)) such that
\(\nu_n [x]_\mathcal{M} \geq 0, \forall x \in \mathcal{M} \) and \( x = 0 \implies [x]_\mathcal{M} = 0; \)
\(\nu_n [x_1]_\mathcal{M} \neq [x_2]_\mathcal{M} \implies x_1 \neq x_2, \) for \( x_1, x_2 \in \mathcal{M}, \) and \([x]_\mathcal{M} = 0 \implies x = 0; \)

The following conditions are often fulfilled in the spaces \( \mathcal{M}_{gB} \).

N) There exist a convex function \( \nu : R^1 \to R_+^1 \) and number \( K \in (0, \infty] \) such that \([\lambda x]_\mathcal{M} \leq \nu(\lambda) [x]_\mathcal{M} \) for any \( x \in \mathcal{M} \) and \( \lambda \in R^1, \) \( |\lambda| < K \), moreover,
\[
\lim_{|\lambda| \to \lambda_j} \frac{\nu(\lambda)}{|\lambda|} = c_j, \quad j = 0, 1 \text{ where } \lambda_0 = 0, \lambda_1 = K \text{ and } c_0 = c_1 = 1 \text{ or } c_0 = 0, c_1 = \infty, \text{ i.e. if } K = \infty \text{ then } \lambda x \in \mathcal{M} \text{ for any } x \in \mathcal{M} \text{ and } \lambda \in R^1.
\]

Let \( g : D(g) \subseteq X \to Y \) be such a mapping that \( \mathcal{M}_{gB} \neq \emptyset \) and the following conditions are fulfilled

\( G_1 \) \( g : D(g) \hookrightarrow \text{Im} g \) is a bijection and \( g(0) = 0; \)

\( G_2 \) there is a function \( \nu : R^1 \to R_+^1 \) satisfying condition \( N \) such that
\[
\| g(\lambda x) \|_B \leq \nu(\lambda) \| g(x) \|_B, \quad \forall x \in \mathcal{M}_{gB}, \quad \forall \lambda \in R^1.
\]

If the mapping \( g \) satisfies conditions \( G_1 \) and \( G_2 \) then \( \mathcal{M}_{gB} \) is a \( pn \)-space with \( p \)-norm defined in the following way: there is a one-to-one function \( \psi : R_+^1 \to R_+^1, \)
\( \psi(0) = 0, \psi, \psi^{-1} \in C^0 \) such that \([x]_{\mathcal{M}_{gB}} \equiv \psi^{-1} (\| g(x) \|_B). \) In this case \( \mathcal{M}_{gB} \) is a metric space with a metric: \( d_{\mathcal{M}}(x_1; x_2) \equiv \| g(x_1) - g(x_2) \|_B. \) Further, we consider just such type of \( pn \)-spaces.

**Definition 2.** The \( pn \)-space \( \mathcal{M}_{gB} \) is called weakly complete if \( g(\mathcal{M}_{gB}) \) is weakly closed in \( B. \) The \( pn \)-space \( \mathcal{M}_{gB} \) is "reflexive" if each bounded weakly closed subset of \( \mathcal{M}_{gB} \) is weakly compact in \( \mathcal{M}_{gB}. \)

It is clear that if \( B \) is a reflexive Banach space and \( \mathcal{M}_{gB} \) is a weakly complete \( pn \)-space, then \( \mathcal{M}_{gB} \) is "reflexive". Moreover, if \( B \) is a separable Banach space, then \( \mathcal{M}_{gB} \) is separable, also.

6.2. Appendix B. In the beginning we consider an operator equation
\[
f(x) = y, \quad y \in Y,\]
where \( f : D(f) \subseteq X \to Y \) is a nonlinear bounded operator, and prove a general solvability theorem for it. It is clear that (6.1) is equivalent to the following functional equation:
\[
\langle f(x), y^* \rangle = \langle y, y^* \rangle, \quad \forall y^* \in Y^*.
\]

We consider the following conditions:

1) \( f : \mathcal{M}_0 \subseteq D(f) \to Y \) is a weakly compact (weakly "continuous") mapping, i.e. for any weakly convergence sequence \( \{x_m\}_{m=1}^\infty \subseteq \mathcal{M}_0 \) in \( \mathcal{M}_0 \) (i.e. \( x_m \overset{M_0}{\to} x_0 \in \mathcal{M}_0 \)) there is a subsequence \( \{x_{m_k}\}_{k=1}^\infty \subseteq \{x_m\}_{m=1}^\infty \) such that \( f(x_{m_k}) \overset{Y}{\to} f(x_0) \)
weakly in \( Y \) (or for a general sequence if \( \mathcal{M}_0 \) not is separable space) and \( \mathcal{M}_0 \) be a weakly complete \( pn \)-space;

2) there exists a mapping \( g : X_0 \subseteq X \rightarrow Y^* \) and a continuous function \( \varphi : R_+^1 \rightarrow R^1 \) nondecreasing for \( \tau \geq \tau_0 \geq 0 \) and \( \varphi (\tau_1) > 0 \) for a number \( \tau_1 > 0 \) such that it generates a "coercive" pair in a generalized sense with \( f \) on the topological space \( X_1 \subseteq X_0 \cap \mathcal{M}_0 \), i.e.

$$\langle f (x), g (x) \rangle \geq \varphi ([x]_{\mathcal{M}_0}) [x]_{\mathcal{M}_0}, \; \forall x \in X_1,$$

where \( X_1 \) is such a topological space that \( X_1 \cong X_0 \) and \( X_1 \cong \mathcal{M}_0 \), and \( \langle \cdot, \cdot \rangle \) is a dual form of the pair \((Y, Y^*)\), moreover, one of the following conditions \((a)\) or \((b)\) holds:

\((a)\) if \( g \equiv L \) is a linear continuous operator, then \( X_1 \) is a "reflexive" space (see [S3, S4]), \( X_0 \equiv X_1 \subseteq \mathcal{M}_0 \) is a separable topological vector space which is dense in \( \mathcal{M}_0 \) and \( \ker L^* = \{0\} \).

\((b)\) if \( g \) is a bounded operator (linear or nonlinear), then \( Y \) is a reflexive separable space, \( g(X_1) \) contains an everywhere dense linear manifold of \( Y^* \) and \( g^{-1} \) is weakly compact (weakly continuous) operator from \( Y^* \) to \( \mathcal{M}_0 \).

**Theorem 7.** Let conditions 1 and 2 hold. Then the equation \((6.1)\) (or \((6.2)\)) is solvable in \( \mathcal{M}_0 \) for any \( y \in Y \) satisfying the following inequality: there exists \( r > 0 \) such that

$$\varphi ([x]_{\mathcal{M}_0}) [x]_{\mathcal{M}_0} \geq \langle y, g (x) \rangle, \; \forall x \in X_1 \; \text{ with } \; [x]_{\mathcal{M}} \geq r.$$ 

**Proof.** Assume that the conditions 1 and 2 \((a)\) are fulfilled and \( y \in Y \) such that \((6.3)\) holds. We are going to use Galerkin’s approximation method. Let \( \{x^k\}_{k=1}^\infty \) be a complete system in the (separable) space \( X_1 \equiv X_0 \). Then, we are looking for approximate solutions in the form \( x_m = \sum_{k=1}^{m} c_{mk}x^k \), where \( c_{mk} \) are unknown coefficients, that might be determined from the system of algebraic equations

$$\Phi_k (c_m) := \langle f (x_m), g (x^k) \rangle - \langle y, g (x^k) \rangle = 0, \; \forall x \in X_1 \; \text{with } \; [x]_{\mathcal{M}} \geq r.$$ 

with \( c_m \equiv (c_{m1}, c_{m2}, ..., c_{mm}) \).

We observe that the mapping \( \Phi (c_m) := (\Phi_1 (c_m), \Phi_2 (c_m), ..., \Phi_m (c_m)) \) is continuous by virtue of condition 1. \((6.4)\) implies the existence of such \( r = r (\|y\|_Y) > 0 \) that the "acute angle" condition is fulfilled for all \( x_m \) with \( [x_m]_{\mathcal{M}_0} \geq r \), i.e. for any \( c_m \in S_{\|r_1\|} \cap R^m, \; r_1 \geq r \) the inequality

$$\sum_{k=1}^{m} \langle \Phi_k (c_m), c_{mk} \rangle \equiv \langle f (x_m), g (\sum_{k=1}^{m} c_{mk}x^k) \rangle - \langle y, g (\sum_{k=1}^{m} c_{mk}x^k) \rangle \geq 0,$$

$$\forall c_m \in \mathbb{R}^m, \|c_m\|_{\mathbb{R}^m} = r_1,$$

holds. The solvability of system \((6.4)\) for each \( m = 1, 2, \ldots \) follows from a well-known lemma on the "acute angle" \((10, 19)\), which is equivalent to the Brouwer’s fixed-point theorem. Thus, the sequence \( \{x_m | m \geq 1\} \) of the approximate solutions, that is contained in a bounded subset of the space \( \mathcal{M}_0 \). Further arguments are analogous to those from \(10, 19\) therefore we omit them. It remains to pass to the limit in \((6.4)\) by \( m \) and use a weak convergency of a subsequence of the sequence \( \{x_m | m \geq 1\} \), the weak compactness of the mapping \( f \), and finally, the completeness of the system \( \{x^k\}_{k=1}^\infty \) in the space \( X_1 \).
Hence, we get the limit element \( x_0 = w - \lim_{j \to \infty} x_{m_j} \in \mathcal{M}_0 \) that is a solution of the equation

\[
(f(x_0), g(x)) = \langle y, g(x) \rangle, \quad \forall x \in X_0,
\]

or

\[
(g^* \circ f(x_0), x) = \langle g^* \circ y, x \rangle, \quad \forall x \in X_0.
\]

In the second case, i.e. when the conditions 1 and 2 (\( \beta \)) are fulfilled and \( y \in Y \) such that (6.3) holds, the approximate solutions suppose to be looked for in the form

\[
x_m = g^{-1} \left( \sum_{k=1}^{m} c_{mk} y_k^* \right) \equiv g^{-1} \left( y_{(m)}^* \right), \quad \text{i.e.} \quad x_m = g^{-1} \left( y_{(m)}^* \right)
\]

where \( \{y_k^*\}_{k=1}^{\infty} \subset Y^* \) is a complete system in the (separable) space \( Y^* \) and belongs to \( g(X_1) \). The unknown coefficients \( c_{mk} \), might be determined from the system of algebraic equations

\[
\Phi_k(c_m) := \langle f(x_m), y_k^* \rangle - \langle y, y_k^* \rangle = 0, \quad k = 1, 2, \ldots, m
\]

with \( c_m \equiv (c_{m1}, c_{m2}, \ldots, c_{mm}) \). Taking this and our conditions into account we get

\[
f(x_m), y_k^* \rangle - \langle y, y_k^* \rangle = \langle f \left( g^{-1} \left( y_{(m)}^* \right) \right), y_k^* \rangle - \langle y, y_k^* \rangle = 0,
\]

for \( k = 1, 2, \ldots, m \).

As it was observed above the mapping

\[
\Phi(c_m) := \left( \Phi_1(c_m), \Phi_2(c_m), \ldots, \Phi_m(c_m) \right)
\]

is continuous by virtue of the conditions 1 and 2(\( \beta \)). Also, (6.3) implies the existence of such \( \bar{r} > 0 \) that the “acute angle” condition is fulfilled for all \( y_{(m)}^* \) with \( \|y_{(m)}^*\|_Y \geq \bar{r} \), i.e. for any \( c_m \in S_{\bar{r}}^m (0) \subset R^m \), \( \bar{r}_1 \geq \bar{r} \) the inequality

\[
\sum_{k=1}^{m} \langle \Phi_k(c_m), c_{mk} \rangle = \langle f(x_m), \sum_{k=1}^{m} c_{mk} y_k^* \rangle - \langle y, \sum_{k=1}^{m} c_{mk} y_k^* \rangle =
\]

\[
\langle f \left( g^{-1} \left( y_{(m)}^* \right) \right), y_{(m)}^* \rangle - \langle y, y_{(m)}^* \rangle = \langle f(x_m), g(x_m) \rangle - \langle y, g(x_m) \rangle \geq 0,
\]

\( \forall c_m \in R^m, \|c_m\|_{R^m} = \bar{r}_1 \)

holds by virtue of our conditions. Consequently, the solvability of system (6.8) (or (6.9)) for each \( m = 1, 2, \ldots \) follows from the “acute angle” lemma as above. Thus, we obtained a sequence \( \{y_{(m)}^* | m \geq 1\} \) of the approximate solutions, that is contained in a bounded subset of \( Y^* \). This implies an existence of a subsequence \( \{y_{(m)}^* | m \geq 1\} \) that convergences weakly in \( Y^* \). Consequently, the sequence

\[
\{x_m\}_{j=1}^{\infty} \equiv \left\{ g^{-1} \left( y_{(m_j)}^* \right) \right\}_{j=1}^{\infty}
\]

converges weakly in the space \( \mathcal{M}_0 \) by virtue of the condition 2(\( \beta \)) (may be after passing to the subsequence). It remains to pass to the limit in (6.9) by \( j \) and use a weak convergency of the subsequence of the sequence \( \{y_{(m)}^* | m \geq 1\} \), the weak compactness of mappings \( f \) and \( g^{-1} \), and the completeness of the system \( \{y_k^*\}_{k=1}^{\infty} \) in the space \( Y^* \).
Hence, we get a limit element $x_0 = w - \lim_{j \to \infty} x_{m_j} = w - \lim_{j \to \infty} g^{-1}\left(y_{m_j}\right) \in \mathcal{M}_0$

that is the solution of the equation

$$\langle f(x_0), y^* \rangle = \langle y, y^* \rangle, \quad \forall y^* \in Y^*.$$  

Q.e.d.

**Remark 7.** It is obvious, that if there exists a function $\psi : R^1_+ \to R^1_+$, $\psi \in C^0$ such that $\psi(\xi) = 0 \iff \xi = 0$ and the inequality $\|f(x_1) - f(x_2)\|_Y$ is fulfilled for all $x_1, x_2 \in \mathcal{M}_0$, then a solution of the equation (6.2) is unique.

**Corollary 6.** Assume that the conditions of Theorem 7 are fulfilled and there is a continuous function $\varphi_1 : R^1_+ \to R^1_+$ such that $\|g(x)\|_{Y^*} \leq \varphi_1([x]_{\mathcal{M}_0})$ for any $x \in X_0$ and $\varphi(\tau) \nearrow +\infty$ and $\frac{\varphi(\tau)}{\tau}$ $\nearrow +\infty$ as $\tau \nearrow +\infty$. Then, the equation (6.2) is solvable in $\mathcal{M}_0$ for any $y \in Y$.

### 6.3. Appendix C

Now, we are ready to provide the proof of Theorem 7. Let $\{x^k\}_{k=1}^\infty$ be a complete system in the (separable) space $X_0$ and $\{\theta^s(t)\}_{s=1}^\infty$ be a complete system in the (separable) space $L_p(0, T)$, then $\{\theta^s(t)x^k\}_{s,k=1}^\infty$ is a complete system in the separable space $L_p(0, T; X_0)$.

**Proof of the Theorem 7.** We are going to use the method of elliptic regularization (see, for example, [10]). Namely, first we prove the solvability of the following auxiliary elliptic problem with a small parameter $\varepsilon > 0$.

$$\begin{align*}
-\varepsilon \frac{d^2x_\varepsilon}{dt^2} + \frac{dx_\varepsilon}{dt} + f(t, x_\varepsilon(t)) &= y(t), \\
x_\varepsilon(0) &= 0, \quad \frac{dx_\varepsilon}{dt} \mid_{t=T} = 0, \quad \varepsilon > 0.
\end{align*}$$

A solution of the problem (6.11)-(6.12) would be understood as an element $x_\varepsilon(t) \in P_{1,p,q}(0, T; \mathcal{M}_0, Y)$ that satisfies the following functional equation

$$\begin{align*}
\varepsilon \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, \frac{dy^*}{dt}\right\rangle dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, y^* \right\rangle dt + \int_0^T \langle f(t, x_\varepsilon(t)), y^* \rangle dt = \int_0^T \langle y, y^* \rangle dt
\end{align*}$$

for any $y^* \in W^1_q(0, T; Y^*) \cap \{y^*(t) \mid y^*(0) = 0\}$.

**Lemma 7.** Under the conditions of Theorem 7, the equation (6.13) is solvable in the space $P_{1,p,q}(0, T; \mathcal{M}_0, Y)$ for any $y \in G$ where $G$ is defined in Theorem 7.

---

3see, also, Soltanov K. N., Sprekels J. - Nonlinear equations in nonreflexive Banach spaces and fully nonlinear equations, Advances in Mathematical Sciences and Applications, 1999, v. 9, no. 2, 939-972.
to our conditions, for each fixed
\( \varepsilon > (6.14) \)
expression in (6.14) is greater than zero under the conditions of Th eorem 2. These
\( y \) is defined properly. Thus, the function
result are fulfilled and, consequently, the equation (6.13) is solvable (see also 
Thus, for each
\( y \)
we obtain
\( 22 \) KAMAL N. SOLTANOV AND MAHMUD A. AHMADOV

The statement of this lemma follows from Theorem [7] of Appendix B (see, also [18, 20]). Indeed, the mapping generated by the considered problem (6.11)-(6.12) is weakly compact from \( P_{0,\varepsilon} \cap \{ y^* (t) \mid y^* (0) = 0 \} )^* 
by virtue of condition (ii) and because of first two terms are linear bounded op-
Moreover, inequalities

\[
\varepsilon \int_0^T \left\langle \frac{dx}{dt}, L \frac{dx}{dt} \right\rangle \, dt + \int_0^T \left\langle \frac{dx}{dt}, Lx_\varepsilon \right\rangle \, dt + \\
\int_0^T \langle f(t, x_\varepsilon (t)), Lx_\varepsilon \rangle \, dt - \int_0^T \langle y, Lx_\varepsilon \rangle \, dt \geq \varepsilon C_0 \left\| \frac{dx_\varepsilon}{dt} \right\|_{L_q(0,T;Y)} + \\
\varphi \left( \| x_\varepsilon \|_{L_p(M_0)} \right) - c_1 \left\| \| y \|_{L_q(0,T;Y)} \right\|_{L_p(M_0)} - (1 + \varepsilon) C_2 \geq
\]

(6.14)

\[
\left( \varphi \left( \| x_\varepsilon \|_{L_p(0,T;M_0)} \right) - c_1 \left\| \| y \|_{L_q(0,T;Y)} \right\|_{L_p(M_0)} - \left[ x_\varepsilon \right]_{L_p(0,T;M_0)} - c \right)
\]

are fulfilled for any \( x_\varepsilon \in W^1_p (0, T; X_0) \cap \{ x_\varepsilon (t) \mid x_\varepsilon (0) = 0 \} \). It is also clear that for a sufficiently large \( p \)-norm of \( x_\varepsilon (t) \) there is a subset of \( L_q (0, T; Y) \) such that the last expression in (6.14) is greater than zero under the conditions of Theorem [2]. These and conditions iii and iv show that the other conditions of the above mentioned result are fulfilled and, consequently, the equation (6.13) is solvable (see also [20]). Thus, for each \( y \in L_q (0; T; Y) \) there is a function \( x_\varepsilon \in P_{0,\varepsilon} \cap \{ y^* (t) \mid y^* (0) = 0 \} \) that satisfies the equation (6.13) for any \( \forall y^* \in W^1_q (0, T; Y^*) \) i.e.

\[
\varepsilon \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, \frac{dy^*}{dt} \right\rangle \, dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, y^* \right\rangle \, dt + \\
\int_0^T \langle f(t, x_\varepsilon (t)), y^* \rangle \, dt = \int_0^T \langle y, y^* \rangle \, dt, \quad \forall y^* \in W^1_q (0, T; Y^*).
\]

(6.15)

The equality (6.15) can be rewritten in the form

\[
\varepsilon \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, \frac{dy^*}{dt} \right\rangle \, dt = \int_0^T \left\langle y - \frac{dx_\varepsilon}{dt} - f(t, x_\varepsilon (t)), y^* \right\rangle \, dt
\]

where \( y - \frac{dx_\varepsilon}{dt} - f(t, x_\varepsilon (t)) \) belongs to \( L_q (0, T; Y) \) because of \( y \in L_q (0, T; Y) \) and \( \frac{dx_\varepsilon}{dt} \), \( f(t, x_\varepsilon (t)) \) \( L_q (0, T; Y) \) for any \( x_\varepsilon \in P_{0,\varepsilon} \cap \{ y^* (t) \mid y^* (0) = 0 \} \). Hence, according to our conditions, for each fixed \( \varepsilon > 0 \), and boundedness of the right part of (6.14) we obtain \( \frac{dx_\varepsilon}{dt} \in L_q (0, T; Y) \) and consequently, the boundary condition \( \frac{dx_\varepsilon}{dt} \mid _{t=T} \) is defined properly. Thus, the function \( x_\varepsilon (t) \) is a solution of the equation

\[
-\varepsilon \int_0^T \left\langle \frac{d^2x_\varepsilon}{dt^2}, y^* \right\rangle \, dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, y^* \right\rangle \, dt +
\]
On the other hand, considering this equation for any \( y^* \in W^1_{1,p} (0,T; Y^*) \) and comparing it with (6.15) we obtain \( \frac{dx_{\varepsilon}}{dt} \big|_{t=T} = 0 \) (by using argumentations similar to those from [10, 6]).

Hence, we proved that the problem (6.11)-(6.12) is solvable in the space \( P^{T, p,q}_{0,1} (0,T; M_0, Y) \cap W^2_{p} (0,T; Y) \cap \{ x_\varepsilon (t) \big| \frac{dx_{\varepsilon}}{dt} \big|_{t=T} = 0 \} \) for each fixed \( \varepsilon > 0 \).

Now, it is necessary to pass to the limit at \( \varepsilon \searrow 0 \). To this end, we need the uniformly on \( \varepsilon \) estimation of \( \frac{dx_{\varepsilon}}{dt} \).

Further, we consider the equation

\[
\begin{align*}
-\varepsilon \int_0^T \frac{d^2}{dt^2} \langle x_\varepsilon, L x_\varepsilon \rangle \theta^s (t) dt + \int_0^T \frac{dx_{\varepsilon}}{dt} \langle x_\varepsilon, L x_\varepsilon \rangle \theta^s (t) dt = \\
\langle y(t) - f(t, x_\varepsilon (t)), L x_\varepsilon \rangle \theta^s (t) dt
\end{align*}
\]

(6.17)

where \( \{ \theta^s (t) x^k \}_{k=1}^\infty \) is a complete system in \( W^1_p (0,T; X_0) \), and consequently, \( \frac{dx_{\varepsilon}}{dt} \) is a solution of the problem

\[
(6.18) \quad -\varepsilon \frac{d^2 x_{\varepsilon}}{dt^2} + \frac{dx_{\varepsilon}}{dt} = f_0 (t), \quad t \in (0,T)
\]

(6.19) \( x_\varepsilon (0) = 0, \quad \frac{dx_{\varepsilon}}{dt} \big|_{t=T} = 0 \),

as \( x_\varepsilon (t) \) belongs to a bounded subset of \( L_p (0,T; M_0) \) and \( f_0(t, x_\varepsilon (t)) \) belongs to a bounded subset of \( L_q (0,T; Y) \) at \( \varepsilon \searrow 0 \), consequently \( f_0 \in L_q (0,T) \) and belongs to a bounded subset of this space at \( \varepsilon \searrow 0 \) and under the conditions of \( y(t) \).

The solution of the problem (6.18)-(6.19) satisfies

\[
\begin{align*}
\frac{dx_{\varepsilon}}{dt} (t) = \frac{1}{\varepsilon} \int_0^T f_0 ( T - \tau ) \exp \left\{ -\frac{T - t - \tau}{\varepsilon} \right\} d\tau.
\end{align*}
\]

Applying the generalized Minkowski’s inequality and taking into account that \( \frac{1}{\varepsilon} \int_0^\infty \exp \left\{ -\frac{\tau}{\varepsilon} \right\} d\tau = 1 \) we get \( \| \frac{dx_{\varepsilon}}{dt} \|_{L_q (0,T; Y)} \leq C < \infty \) for some positive \( C \) that is independent on \( \varepsilon \).

Thus, for each \( y(t) \in L_q (0,T; Y) \) the function \( x_\varepsilon (t) \) belongs to a bounded subset of the space \( P^{T, p,q}_{1,1} (0,T; M_0, Y) \) uniformly on \( \varepsilon \). The "reflexivity" of \( M_0 \) and the reflexivity of \( Y \) allow us to pass to the limit for \( \varepsilon \searrow 0 \) in all terms of the (6.17) except for the first one. Therefore, it remains to estimate just the first term of (6.17). We have

\[
\begin{align*}
\left| -\varepsilon \int_0^T \left\langle \frac{d^2 x_{\varepsilon}}{dt^2}, y^* \right\rangle dt \right| \leq \varepsilon \int_0^T \left| \left\langle \frac{dx_{\varepsilon}}{dt}, \frac{dy^*}{dt} \right\rangle \right| dt \leq
\end{align*}
\]
for any $y^* \in W^1_q(0, T; Y^*) \cap \{ y^*(t) \mid y^*(0) = 0 \}$. Taking into account the estimation

$$
\varepsilon \left\| \frac{dx}{dt} \right\|_{L^q(0,T;Y)} + \left\| \frac{dy^*}{dt} \right\|_{L^q(0,T;Y^*)} 
$$

for any $y^* \in W^1_q(0, T; Y^*)$ \cap \{ $y^*(t)$ $\mid$ $y^*(0) = 0$ \}. Taking into account the estimation

$$
\varepsilon \left\| \frac{dx}{dt} \right\|_{L^q(0,T;Y)} \leq C < \infty
$$

for $\nu > 1$, that is valid by virtue of the a priori estimations, we get

$$
\left| -\varepsilon \int_0^T \left( \frac{d^2x}{dt^2}, y^* \right) dt \right| \leq \varepsilon^{\frac{\nu-1}{\nu}} C \left\| \frac{dy^*}{dt} \right\|_{L^q(0,T;Y^*)}.
$$

This means that the first term of (6.17) vanishes when $\varepsilon \searrow 0$. Thus, considering the equation (6.17) for any $\xi \in L^p(0, T; X_0)$, passing to the limit at $\varepsilon \searrow 0$ and taking into account that $\ker L^* = \{ 0 \}$ complete the proof of Theorem 2.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. Comm. Pure Appl. Math. 12, 1959, 623–727; Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II. Comm. Pure Appl. Math. 17, 1964, 35–92.

[2] S. Angenent, On the formation of singularities in the curve shortening flow, J. Differential Geom., 1991, 33, 3, 601-633.

[3] J. Bebernes, V. A. Galaktionov, On classification of blow-up patterns for a quasilinear heat equation, Differential Integral Equations 9 (4) (1996) 655-670.

[4] O. V. Besov, V. P. Il’in, S. M. Nikol’skii, Integrated submissions of functions and embedding theorem, Nauka, Moscow, 1996 (Russian).

[5] R. Courant, D. Hilbert, Methods of Mathematical Physics, v. I, II. (1953, 1962), Interscience, N.-Y.

[6] Ju. A. Dubinskii, Weakly convergence into nonlinear elliptic and parabolic equations, Matem. Sborn., (1965), 67, n. 4.

[7] A. Friedman, B. McLeod, Blow-up of solutions of nonlinear degenerate parabolic equations, Arch. Rational Mech. Anal., 1987, 96, 55-80.

[8] R. S. Hamilton, The Ricci flow on surfaces, Contemp. Math., 1988, 71, 237-262.

[9] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, (1983), Springer-Verlag.

[10] J.-L. Lions, Quelques methodes de resolution des problemes aux limites non lineares, (1969) DUNOD, Gauthier-Villars, Paris.

[11] J.-L. Lions, E. Magenes, Nonhomogeneous boundary value problems and applications, (1972), Springer-Verlag, N.-Y.

[12] S. Luckhaus, R. Dal Passo, A degenerate diffusion problem not in divergence form, J. Differential Equations, 1987, 69, 1-14.

[13] C. B. Jr. Morrey, Multiple integrals in the calculus of Variations, (1966), Springer-Grunl. Math. Viss., 130.

[14] O. A. Oleinik, The mathematical problems of the boundary layer theory, 1968, Math. Survey, 23, 3, 3-65 (Russian).

[15] E. M. Stein, HarmonicAnalysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, (1993), Princeton Univ. Press, Princeton, New Jersey.

[16] K. N. Soltanov, Periodic solutions of certain nonlinear parabolic equations with implicit degeneracy, Dokl. Akad. Nauk SSSR, 1975, 222, no. 1-2, 291–294 (Russian).

[17] K. N. Soltanov, Solvability nonlinear equations with operators the form of sum the pseudomonotone and weakly compact, Soviet Math. Dokl., 1992, v.324, no.5, 944-948.

[18] K. N. Soltanov, Some applications of nonlinear analysis to differential equations, 2002, ELM, Baku (Russian).

[19] K. N. Soltanov, Some embedding theorems and its applications to nonlinear equations. Differential'nie uravnenia, 20, 12, 1984.
[20] K. N. Soltanov, On some modification Navier-Stokes equations, Nonlinear Analysis, TMA, (2003), 52, 4.
[21] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, (1988), Springer-Verlag, N.-Y.
[22] T. Tsutsumi, M. Ishiwata, Regional blow-up of solutions to the initial boundary value problem for $u_t = u^\delta (\Delta u + u)$, Proc.Roy.Soc. Edinburgh Sect. A, 1997, 127, 4, 871-887.
[23] W. Walter, Existence and convergence theorems for the boundary layer equations based on the line method, Arch. Rational Mech. and Anal., 1970, 39, 3, 169-188.
[24] M. Wiegner, A degenerate diffusion equation with a nonlinear source term, Nonlinear Anal.TMA, 1997, 28, 1977-1995.
[25] M. Winkler, Blow-up of solutions to a degenerate parabolic equation not in divergence form, J. Diff. Eq. 2003, 192, 445 - 474.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, HACETTEPE UNIVERSITY
BEYTEPE, ANKARA, TR-06532, TURKEY
E-mail address: soltanov@hacettepe.edu.tr; sultan_kamal@hotmail.com

Current address: Department of Mathematics, Holyoke Community College, 303 Homestead Ave., Holyoke, MA, 01040, USA
E-mail address: mahmadov@hcc.edu