Solving the Goddard problem
by an influence diagram

Jiří Vomlel and Václav Kratochvíl

Institute of Information Theory and Automation,
Czech Academy of Sciences,
Pod vodárenskou věží 4, Prague 8, 182 08, Czechia
vomlel@utia.cas.cz,
http://www.utia.cas.cz/vomlel/

Abstract

Influence diagrams are a decision-theoretic extension of probabilistic graphical models. In this paper we show how they can be used to solve the Goddard problem. We present results of numerical experiments with this problem and compare the solutions provided by influence diagrams with the optimal solution.

1 Introduction

Formulated by Robert H. Goddard, see (Goddard, 1919), the problem is to establish the optimal thrust profile for a rocket ascending vertically from the Earth’s surface to achieve a given altitude with a given speed and pay load and with the minimum fuel expenditure. The aerodynamic drag and the gravitation varying with the altitude are considered. We study a problem version that assumes a bounded thrust. The problem has become a benchmark in the optimal control theory due to a characteristic singular arc behavior in connection with a relatively simple model structure, which makes the Goddard problem an ideal object of study, c.f. (Graichen and Petit, 2008).

2 The ODE model

The equation of motion of the rocket, subject to the forces of gravity, drag, and thrust are:

\[
\frac{cn(t)}{dt} + m \frac{dv(t)}{dt} + d(v, h) + mg(h) = 0 ,
\]

(1)

*This work was supported by the Czech Science Foundation (project 16-12010S).
where

\[
\begin{align*}
d(v, h) &= \frac{1}{2} \cdot s \cdot c_D \cdot \rho(h) \cdot v^2 \\
\rho(h) &= \rho_0 \cdot \exp \left( \beta \cdot \left(1 - \frac{h}{R}\right) \right) \\
g(h) &= g_0 \cdot \frac{R^2}{h^2}
\end{align*}
\]

using the following notation:

- \(c\) is the exhaust velocity (the jet speed),
- \(m\) is the rocket mass composed from the pay load \(m_p\), which is a constant, and the fuel \(m_f\), which is burnt during the rocket ascent,
- \(m_0\) is the initial rocket mass,
- \(t\) is the time,
- \(v\) is the speed of the rocket,
- \(h\) is the altitude measured as the distance from the Earth’s center,
- \(R = 6371\ \text{km}\) is the radius of the Earth,
- \(s\) is the cross-section area of the rocket,
- \(d(v, h)\) is the drag at speed \(v\) and altitude \(h\),
- \(c_D\) is the dimensionless drag constant,
- \(\rho(h)\) is the density of the air at altitude \(h\),
- \(\rho_0\) is the density of the air at the Earth’s surface,
- \(\beta\) is a dimensionless constant,
- \(g(h)\) is the acceleration of gravity at altitude \(h\), and
- \(g_0 = 9.81\ m/s^2\) is the gravitational acceleration at the Earth’s surface.

If we describe the system dynamics with respect to the altitude the formula transforms to:

\[
c \cdot v \cdot \frac{dm}{dh} + m \cdot v \cdot \frac{dv}{dh} + d(v, h) + m \cdot g(h) = 0 .
\]

3 Normalized Goddard Problem

In literature the problem is often presented in its nondimensional form. Let \(G = g_0 \cdot R^2\). We use tilde to denote the new nondimensionalized variables:

\[
\begin{align*}
\tilde{m} &= \frac{m}{m_0} \\
\tilde{h} &= h \cdot \frac{1}{R} \\
\tilde{t} &= t \cdot \sqrt{\frac{G}{R^3}} .
\end{align*}
\]
This leads to

\[ \tilde{v} = v \cdot \sqrt{\frac{R}{G}} \]
\[ \tilde{a} = a \cdot \frac{R^2}{G}. \]

The nondimensionalized model transformed from formulas (5) and (2)–(4) is the following:

\[ \tilde{c} \cdot \sqrt{\frac{G}{R}} \cdot m_0 \cdot \sqrt{\frac{G}{R^2}} \cdot \frac{d\tilde{m}}{dh} \cdot \tilde{v} + \tilde{m} \cdot m_0 \cdot \sqrt{\frac{G}{R}} \cdot \sqrt{\frac{G}{R^2}} \cdot \frac{d\tilde{v}}{dh} \cdot \tilde{v} + d(\tilde{v}, \tilde{h}) + \tilde{m} \cdot m_0 \cdot g(\tilde{h}) = 0, \quad (6) \]

where

\[ d(\tilde{v}, \tilde{h}) = \frac{1}{2} \cdot s \cdot R^2 \cdot c_D \cdot \rho(\tilde{h}) \cdot \left( \tilde{v} \cdot \sqrt{\frac{G}{R}} \right)^2 \quad (7) \]
\[ \rho(\tilde{h}) = \tilde{\rho}_0 \cdot m_0 \cdot \frac{G}{R^2} \cdot \exp \left( \beta \cdot \left( 1 - \tilde{h} \right) \right) \quad (8) \]
\[ g(\tilde{h}) = \frac{G}{R^2} \cdot \frac{R^2}{\tilde{h}^2} \cdot \frac{1}{R^2} \cdot \frac{1}{\tilde{h}^2} \quad (9) \]

By substituting (8) to (7) we get:

\[ d(\tilde{v}, \tilde{h}) = \frac{1}{2} \cdot s \cdot c_D \cdot \tilde{\rho}_0 \cdot \frac{m_0 \cdot G}{R^2} \cdot \exp \left( \beta \cdot \left( 1 - \tilde{h} \right) \right) \cdot \tilde{v}^2 \quad (10) \]

and by substituting (10) and (9) to (6) and dividing both sides of the equation by \( \frac{m_0 \cdot G}{R^2} \) we get:

\[ \tilde{c} \cdot \frac{d\tilde{m}}{dh} \cdot \tilde{v} + \tilde{m} \cdot \frac{d\tilde{v}}{dh} \cdot \tilde{v} + \frac{1}{2} \cdot s \cdot c_D \cdot \tilde{\rho}_0 \cdot \exp \left( \beta \cdot \left( 1 - \tilde{h} \right) \right) \cdot \tilde{v}^2 + \frac{\tilde{m}}{\tilde{h}^2} = 0, \quad (11) \]

**Remark 1.** In the sequel we will use the normalized Goddard Problem. For simplicity, we will omit tildes.

### 4 Optimal control problem formulation

The state variables are the rocket mass \( m \) (of pay load and fuel) and the rocket speed \( v \) at altitude \( h \). The control variable \( u \) controls the engine thrust, which is the derivative of mass \( m \) with respect to time \( t \) multiplied by the jet speed \( c \), i.e.

\[ u = c \cdot \frac{dm}{dt} = c \cdot \frac{dm}{dh} \cdot \frac{dh}{dt} = \frac{dm}{dh} \cdot c \cdot v. \quad (12) \]

which implies that the mass \( m \) at the altitude \( h \) is

\[ m(h) = m_0 + \int_{h'=0}^{h} c \cdot v(h') \cdot u(h') \, dh'. \quad (13) \]
where $m_0$ is the initial mass at the rocket launch. Please note that $u \leq 0$ due to the fact that the mass of the rocket can only decrease (by burning the fuel). The control will be restricted to $u \in [-3.5, 0]$.

The task is to find a control function $u(h)$ so that we get from the initial state $(m_0, v_0)$ to a terminal state $(m_T, v_T)$, where $m_T$ is the terminal mass and $v_T$ is the terminal speed at a given terminal altitude $h_T$, $v_0$ is the initial speed, and $m_0 > m_T$ is the initial rocket mass (including fuel) so that the with a minimal fuel consumption (i.e., with a maximal final mass).

Formula (11) can be rewritten as:

$$ u + m \cdot v \cdot \frac{dv}{dh} + m \cdot \frac{1}{h^2} + \frac{1}{2} \cdot s \cdot c_D \cdot \rho_0 \cdot \exp \left( \beta \cdot (1 - h) \right) \cdot v^2 = 0 . \tag{14}$$

The formula (12) can be written using a newly defined function $g(h, v)$ and formula (14) using a newly defined function $f(h, m, u, v)$ as:

$$ \frac{dm}{dh} = g(u, v) = \frac{u}{c \cdot v} \tag{15}$$

$$ \frac{dv}{dh} = f(h, m, u, v) = -\frac{u}{m \cdot v} - \frac{1}{2} \cdot \frac{s \cdot c_D \cdot \rho_0 \cdot \exp \left( \beta \cdot (1 - h) \right) \cdot v}{v \cdot h^2} . \tag{16}$$

This equation we will use for the derivation of approximate methods in the next section.

The control will be restricted to $u \in [-3.5, 0]$. It is assumed that the rocket is initially at rest at the surface of the Earth and that its fuel mass is 40% of the rocket total mass. The initial and terminal values will be

$$ h_0 = 1, \quad h_T = 1.01, \quad m_0 = 1, \quad v_0 = 0, \quad m_T \geq 0.6 \cdot m_0 = 0.6 .$$

### 4.1 Model parameters

The model parameters we consider correspond to those presented in (Tsiotras and Kelley, 1991) and (Seywald and Cliff, 1992). The aerodynamic data and the vehicle’s parameters originate from (Zlotskiy and Kiforenko, 1983) and correspond roughly to the Soviet SA-2 surface-to-air missile, NATO code-named Guideline. The nondimensionalized values of these constants are:

$$ \beta = 500, \quad s \cdot \rho_0 = 12400, \quad c_D = 0.05, \quad c = 0.5 . $$
5 ODE approximate solution methods

Now, consider a trajectory segment of length $\Delta h$ with the control being constant and equal to $u'$. Let $h'$ be the altitude, $m'$ the mass value, and $v'$ the speed – all at the beginning of the segment.

The Euler method

In the Euler method the following approximation is used:

$$\frac{m(h' + \Delta h)}{v(h' + \Delta h)} \approx \left( \frac{m' + \Delta h \cdot g(u', v')}{v' + \Delta h \cdot f(h', m', u', v')} \right)^{1}. \quad (17)$$

This method is not very accurate. Note that in the mass estimation a constant rocket speed is assumed for the whole segment, which means that during speed-ups the mass is underestimated and during slow-down the mass is overestimated. Also, note that in the speed estimation a constant rocket mass and a constant drag is assumed for the whole segment, which means the speed is underestimated. On the other hand the method is extremely fast.

The general approximation method

The general approximation of order $s$ takes the following form:

$$\frac{m(h' + \Delta h)}{v(h' + \Delta h)} \approx \left( \frac{m(h') + \Delta h \cdot \sum_{i=1}^{s} w_i \cdot \ell_i}{v(h') + \Delta h \cdot \sum_{i=1}^{s} w_i \cdot k_i} \right), \quad (18)$$

where for $i = 1, \ldots, s$

$$\ell_i = g \left( \frac{u'}{v' + \Delta h \cdot \sum_{j=1}^{s} a_{i,j} \cdot k_j} \right) \quad (19)$$

$$k_i = f \left( \frac{h' + z_i \cdot \Delta h, \quad m' + \Delta h \cdot \sum_{j=1}^{s} a_{i,j} \cdot \ell_j, \quad u'}{v' + \Delta h \cdot \sum_{j=1}^{s} a_{i,j} \cdot k_j} \right), \quad (20)$$

which reduces to one equation for each $i = 1, \ldots, s$:

$$k_i = f \left( \frac{h' + z_i \cdot \Delta h, \quad m' + \Delta h \cdot \sum_{j=1}^{s} a_{i,j} \cdot \ell_j, \quad u'}{v' + \Delta h \cdot \sum_{j=1}^{s} a_{i,j} \cdot k_j} \right). \quad (21)$$

The classical Runge–Kutta method

In the the classical Runge–Kutta method of order $s = 4$ (RK4) the coefficients’ values are given by the following Butcher tableau:

\[\begin{array}{c|ccc|c|c}
\end{array}\]
The computational advantage of RK4 is that (due to the zeroes in its Butcher tableau) the values of $\ell_i$ and $k_i$ for $i = 1, \ldots, s$ are specified explicitly. Unfortunately, for some problems the RK4 method can be numerically unstable unless the step size is extremely small. This may lead to wild oscillations of the control.

**Gauss–Legendre method**

The Butcher tableau of this method for $s = 2$ is

| $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|-----------|-----------|-----------|-----------|
| $a_{1,1}$ | $a_{1,2}$ | $a_{1,2}$ | $a_{1,4}$ |
| $a_{2,1}$ | $a_{2,2}$ | $a_{2,3}$ | $a_{2,4}$ |
| $a_{3,1}$ | $a_{3,2}$ | $a_{3,3}$ | $a_{3,4}$ |
| $a_{4,1}$ | $a_{4,2}$ | $a_{4,3}$ | $a_{4,4}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $0$ | $0$ | $1$ |
| $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ |

Note that the values of $k_i$ for $i = 1, \ldots, s$ are specified only implicitly and the non-linear system specified by (21) must be solved. When their values are found they can be substituted to formula (19) and (18). Contrary to RK4 the Gauss–Legendre method is A-stable (Dahlquist, 1963).

**Control constraints**

The rocket jets cannot produce an infinite force, which implies that the absolute value\(^1\) of control $u$ is restricted from above. The upper bound is assumed to be constant during the whole flight:

$$ |u| \leq |u_{\text{max}}| . $$

To avoid situations the rocket is not moving or falling down we require its speed $v > 0$. Also, the rocket cannot have its mass lower than is its payload, i.e., $m \geq m_p$. We realize these constraints by means of control restrictions. We allow only control values $u$ for which, when they are substituted to formula (18), it holds that

$$ m(h' + \Delta h) \geq m_p \quad \text{and} \quad v(h' + \Delta h) > 0 $$

except the terminal altitude $h_T$ where $v(h_T) \geq 0$. Note that the lower bound of $|u|$ is thus a function of the altitude $h$, the current speed $v$, and the current rocket mass $m$.

\(^1\)Note that $u \leq 0$. 

6
6 The influence diagram

In each segment $i$ ($i = 0, 1, \ldots, N$) of the influence diagram for the Goddard Problem, there are two state variables:

- a speed variable $V_i$ and
- a mass variable $M_i$.

In each segment $i$ there is also one decision variable:

- the control of the thrust of the rocket engine $U_i$.

Finally, in each segment $i = 1, \ldots, N$ one utility node is present:

- the fuel consumption in the segment $f_i$.

The structure of one segment of the influence diagram for the discrete version of the Goddard Problem is presented in Figure 1.

![Figure 1: A Segment of the Influence Diagram for the Goddard Problem](image)

In each segment a solution of a system of two ordinary differential equations is found by an approximate method as it is discussed in Section 5. Typically, the computed mass and speed values at the end of the segment will not lay in the discrete set of values of the mass and speed variables. Therefore we will approximate the state transformations by non-deterministic CPTs $P(V_{i+1}|V_i, M_i)$ and $P(M_{i+1}|V_i, M_i)$ as it is described in (Kratochvíl and Vomlel, 2016, Section 5.2).

7 Experimental results

In Figure 2 we compare the control, speed, and mass profiles of the optimal solution found by Bocop (Team Commands, 2016) with solutions found by influence
Figure 2: Comparisons of the optimal solution with influence diagram solutions.
diagrams with different discretizations and different approximation methods. It is known (Miele, 1963) that the optimal solution consists of three subarcs: (a) a maximum-thrust subarc, (b) a variable-thrust subarc, and (c) a coasting subarc, i.e., a subarc with the zero thrust.

We denoted the solutions found by influence diagrams using a name schema \texttt{v.u.m.M.h} composed from the parameters used in the experiments:

- \texttt{v} ... the number of states of the speed variables,
- \texttt{u} ... the number of states of the control variables,
- \texttt{m} ... the number of states of the mass variables,
- \texttt{M} ... the discretization method for solving ODEs (E stands for the Euler method and G for the Gauss–Legendre method), and
- \texttt{h} ... the length of the trajectory segment.

From Figure 2 we can conclude that the Euler method best approximates optimal control and speed profiles despite it suffers from small oscillations of the control. The Gauss–Legendre method solutions do not suffer much from oscillations but their approximation error is substantially larger for the control and speed profiles while they are better for the mass profile estimation.

8 Conclusions

We have shown how influence diagrams can be used to solve a control theory benchmark problem – the Goddard Problem. The numerical experiments reveal that the solution found by influence diagrams approximates well the optimal solution and quality of approximation improves with finer discretizations. From the tested ODE approximation methods the best results were achieved by the simplest one – the Euler method.

References

Dahlquist, G. G. (1963). A special stability problem for linear multistep methods. \textit{BIT Numerical Mathematics}, 3(1):27–43.

Goddard, R. H. (1919). A method for reaching extreme altitudes. \textit{Smithsonian Miscellaneous Collections}, 71(2).

Graichen, K. and Petit, N. (2008). Solving the goddard problem with thrust and dynamic pressure constraints using saturation functions. In \textit{Proceedings of the 17th World Congress of the International Federation of Automatic Control Seoul, Korea}, pages 14301–14306.

Kratochvíl, V. and Vomlel, J. (2016). Influence diagrams for speed profile optimization. \textit{International Journal of Approximate Reasoning}. (in press), \url{http://dx.doi.org/10.1016/j.ijar.2016.11.018}

Miele, A. (1963). A survey of the problem of optimizing flight paths of aircraft and missiles. In Bellman, R., editor, \textit{Mathematical Optimization Techniques}, pages 3–32. University of California Press.
Seywald, H. and Cliff, E. M. (1992). Goddard problem in presence of a dynamic pressure limit. *Journal of Guidance, Control, and Dynamics*, 16(4):776–781.

Team Commands, I. S. (2016). Bocop: an open source toolbox for optimal control. [http://bocop.org](http://bocop.org).

Tsiotras, P. and Kelley, H. J. (1991). Drag-law effects in the Goddard problem. *Automatica*, 27(3):481–490.

Zlatskiy, V. T. and Kiforenko, B. N. (1983). Computation of optimal trajectories with singular-control sections. *Vychislitel’naia i Prikladnaia Matematika*, 49:101–108.