Effects of Landau damping on ion-acoustic solitary waves in a semiclassical plasma

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We study the nonlinear propagation of ion-acoustic waves (IAWs) in an unmagnetized collisionless plasma with the effects of electron and ion Landau damping in the weak quantum (semiclassical) regime, i.e., when the typical ion-acoustic (IA) length scale is larger than the thermal de Broglie wavelength. Starting from a set of classical and semiclassical Vlasov equations for ions and electrons, coupled to the Poisson equation, we derive a modified (by the particle dispersion) Korteweg-de Vries (KdV) equation which governs the evolution of IAWs with the effects of wave-particle resonance. It is found that in contrast to the classical results, the nonlinear IAW speed ($\lambda$) and the linear Landau damping rate ($\gamma$) are no longer constants, but can vary with the wave number ($k$) due to the quantum particle dispersion. The effects of the quantum parameter $H$ (the ratio of the plasmon energy to the thermal energy) and the electron to ion temperature ratio ($T_e/T_i$) on the profiles of $\lambda$, $\gamma$ and the solitary wave amplitude are also studied. It is shown that the decay rate of the wave amplitude is reduced by the effects of $H$.

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I. INTRODUCTION

Since its theoretical prediction by Landau [1] and later its experimental verification by Malmberg and Wharton [2], the Landau damping has been a longstanding problem of wave-particle interactions in plasmas. The effects of such collisionless damping on solitary waves was first theoretically studied by Ott and Sudan [3]. However, their theory was limited to the effects of electron Landau damping only, and thereby cannot be applied to treat the ion resonance effects. The theory was later developed to include ion Landau damping adequately and consistently by Vandam et al. [4] by using a multi-scale asymptotic expansion scheme. It has been shown that the nonlinear resonance effects (such as trapping, reflection) can be significant along with the linear resonance of Landau damping. Though, there have been a number of similar works in recent times (see, e.g., Refs. [3,4]), however, the theory of Landau damping on nonlinear ion-acoustic waves (IAWs) is still undeveloped in the semiclassical or full quantum regimes.

The purpose of the present work is to extend and generalize the work of Vandam et al. [4] in the context of quantum plasmas. Our treatment is, however, limited to the weak quantum regime in which the typical ion-acoustic (IA) length scale is larger than the thermal de Broglie wavelength. We derive a modified Korteweg-de Vries (KdV) equation following the work of Vandam et al. [4], and show that even in the weak quantum (semiclassical) regime, the characteristics of IA solitary waves (IASWs) are significantly modified by the particle dispersion as well as the thermal contributions of electrons and ions.

II. BASIC EQUATIONS

We consider the nonlinear propagation of IAWs in an unmagnetized collisionless electron-ion semiclassical plasma. The Vlasov equations for electrons and ions, respectively, are [13,14]

\[
\frac{\partial f_e}{\partial t} + v \frac{\partial f_e}{\partial x} + \frac{e}{m_e} \frac{\partial \phi}{\partial x} \frac{\partial f_e}{\partial v} - \frac{e^2 h^2}{24 m_e^2 \omega_s^2} \frac{\partial^3 \phi}{\partial x^3} \frac{\partial^3 f_e}{\partial v^3} = 0,
\]

\[
\frac{\partial f_i}{\partial t} + v \frac{\partial f_i}{\partial x} - \frac{e}{m_i} \frac{\partial \phi}{\partial x} \frac{\partial f_i}{\partial v} = 0.
\]

Equations (1) and (2) are closed by the Poisson equation

\[
\frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum e_n f_n \, dv.
\]

The semiclassical Vlasov equation (1) is obtained from the Wigner-Moyal equation with the assumption that $h=m_e v_t c/L \equiv \lambda_B/L \ll 1$ with $\hbar \equiv h/2\pi$ denoting the reduced Planck’s constant, $v_t$ the electron thermal speed and $L$ the typical length scale of IAWs. From Eq. (1), the classical Vlasov equation can be recovered in the limit, $h \rightarrow 0$.

Next, we normalize the physical quantities according to $v \rightarrow v/c_s$, $\phi \rightarrow e\phi/k_B T_e$, $n_n \rightarrow n_n/n_0$, $f_n \rightarrow f_n c_s n_0$ where $c_s = \sqrt{k_B T_e/m_e} \equiv \omega_{pi}/\lambda_D$ is the IAW speed with $\omega_{pi} = \sqrt{4\pi n_0 e^2/m_e}$ and $\lambda_D = \sqrt{k_B T_e/4\pi m_e e^2}$ denoting, respectively, the ion plasma frequency and the plasma Debye length. Furthermore, $n_0$ is the equilibrium number density of electrons and ions, $T_i$ is the thermodynamic temperature of electrons ($j = e$) or ions ($j = i$), $k_B$ is the Boltzmann constant. The space and time variables are normalized by $\lambda_D$ and $\omega_{pi}^{-1}$ respectively.

Thus, from Eqs. (1)-(3), we obtain the following set of equations in dimensionless forms:

\[
\frac{\partial f_e}{\partial t} + v \frac{\partial f_e}{\partial x} + \frac{1}{m} \frac{\partial \phi}{\partial x} \frac{\partial f_e}{\partial v} - \frac{H^2}{24 m^2} \frac{\partial^3 \phi}{\partial x^3} \frac{\partial^3 f_e}{\partial v^3} = 0,
\]

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where $m = m_e/m_i$ is the electron to ion mass ratio, $H = h\omega_{pe}/k_B T_e$ is the dimensionless quantum energy to the thermal energy and $\theta_\alpha = \mp 1$ for electrons ($\alpha = e$) and ions ($\alpha = i$).

### III. DERIVATION OF KDV EQUATION

In this section, we derive the evolution equation for small amplitude IAWs following the same multi-scale asymptotic expansion technique as in Ref. 4 in which the physical quantities are perturbed from the equilibrium state as

$$
\phi = \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \cdots,
$$

$$
f_\alpha = f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + \cdots,
$$

where $\epsilon (\lesssim 1)$ is a small positive scaling parameter measuring the weakness of perturbations. The equilibrium distribution of electrons and ions, i.e., $f_\alpha^{(0)}$, for $\alpha = e, i$, are assumed to be the Maxwellian given by

$$
f_\alpha^{(0)} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{m_e T_e}{m_\alpha T_\alpha}} \exp \left( -\frac{m_\alpha T_e}{m_e T_\alpha} \frac{v^2}{2} \right).
$$

Such an assumption for electrons is valid when $T_e \gg T_F$, where $T_F$ is the electron Fermi temperature. Also, the expansion for $f_\alpha$ in (7) is valid only in the non-resonance region where $|v - \lambda| \gg o(\epsilon)$ with $\lambda$ denoting the phase velocity of the wave. However, in the resonance region, to be discussed later, where $v \approx \lambda$, a slightly different ordering of $\epsilon$ is to be considered. Furthermore, in order to properly include the contributions of resonant particles (trapped and/or free) (Note here that the Gardner-Morikawa transformation $\zeta = \epsilon^{1/2}(x - \lambda t), s = \epsilon^{3/2}x$, usually used in fluid models, can not be applied directly to the Vlasov equation), we introduce the stretched coordinates and the multi-scale Fourier-Laplace transforms for $f_\alpha - f_\alpha^{(0)}$ and $\phi$ as

$$
\xi = \epsilon^{1/2} x, \sigma = \epsilon^{1/2} t, s = \epsilon^{3/2} x,
$$

$$
f_\alpha^{(n)}(v, \xi, \sigma, s) = \frac{i}{(2\pi)^2} \int d\omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(k, \omega, s) f_\alpha^{(n)}(v, k, \omega, s) \times (\omega - k\lambda)^{-1} \exp[i(k\xi - \omega\sigma)],
$$

$$
\phi^{(n)}(\xi, \sigma, s) = \frac{i}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(k, \omega, s) \phi^{(n)}(k, \omega, s) \times (\omega - k\lambda)^{-1} \exp[i(k\xi - \omega\sigma)],
$$

where $\omega$ ($k$) is the wave frequency (number), $W$ is the usual Laplace transform contour lying above any poles of the integrals on the complex $\omega$-plane, and the transformed quantities $\hat{f}_\alpha^{(n)}$ and $\hat{\phi}^{(n)}$ are analytic for all $\omega$, i.e., the inverse transforms does not contain any factor $1/(\omega - \lambda\lambda)$ in their expressions. Next, we substitute Eqs. (11) and (12) into Eqs. (13)-16, and equate different powers of $\epsilon$. The results are obtained in Secs. III A and III B.

### A. First-order perturbations and nonlinear wave speed

Equating the coefficients of $\epsilon^{3/2}$ from Eqs. (1) and (5), and the coefficients of $\epsilon$ from Eq. (6), we, respectively, obtain

$$
\frac{\partial f_\alpha^{(1)}}{\partial t} + v \frac{\partial f_\alpha^{(1)}}{\partial x} - \theta_\alpha \left( \frac{m_i}{m_\alpha} \right) G_\alpha(v) \frac{\partial \phi^{(1)}}{\partial \xi} + R_\alpha \frac{\partial^2 G_\alpha(v)}{\partial v^2} \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = 0,
$$

$$
\sum_{\alpha = e, i} \theta_\alpha \int_{n.r.} f_\alpha^{(1)} dv = 0,
$$

where $n.r.$ stands for the non-resonance region in the velocity integral, $G_\alpha(v) = (\partial f_\alpha^{(0)}/\partial v)$, $R_\alpha = -H^2/24m_\alpha^2$ for quantum electrons and $R_i = 0$ for classical ions. Also, we assume that $R_\alpha(\partial^3 f_\alpha^{(0)}/\partial v^3) \sim O(\epsilon^{-1})$ in the non-resonance region in order to include the contribution from the particle dispersion in the linear and nonlinear regimes. Note that, the quantum effect of electrons gives rise a new term proportional to $R_e$, and Eq. (12) gives the quasineutrality condition, i.e., the electron and ion number densities do not differ until $O(\epsilon^2)$. The resonance effects such as the Landau damping will be shown to be at least of $O(\epsilon^2)$.

Transforming Eq. (11) according to the formula (10) and using the fact that $f_\alpha(x, v, t) = f_\alpha^{(0)}(v)$ and $\phi(x, t) = 0$ for $t < 0$, we obtain

$$
\hat{f}_\alpha^{(1)} = -\frac{k}{\omega - k\lambda} \left( \theta_\alpha \frac{m_i}{m_\alpha} G_\alpha(v) + k^2 R_\alpha \frac{\partial^2 G_\alpha(v)}{\partial v^2} \right) \hat{\phi}^{(1)}.
$$

Substituting the expression for $\hat{f}_\alpha^{(1)}$, obtained from the Fourier-Laplace inversion of Eq. (13), into Eq. (12), we obtain the following dispersion relation for the nonlinear wave speed $\lambda \equiv \omega/k$

$$
\sum_{\alpha = e, i} \int_{n.r.} \frac{1}{v - \lambda} \left( \frac{m_i}{m_\alpha} G_\alpha(v) + \theta_\alpha k^2 R_\alpha \frac{\partial^2 G_\alpha(v)}{\partial v^2} \right) dv = 0.
$$

Note that the dispersion relation is modified by the quantum correction proportional to $R_\alpha$, in absence of which we can recover the classical results as in Ref. 4. Using the unperturbed velocity distribution functions for electrons and ions given by Eq. (5), and appropriate limiting
expressions for the Plasma dispersion function we obtain (for details, see Appendix A) the following dispersion relation for IAWs in semiclassical plasmas:

\[
\lambda^2 = \frac{1 + \sqrt{1 + \frac{12}{7} \left(1 + \frac{H^2 k^2}{12}\right)}}{2 \left(1 + \frac{H^2 k^2}{12}\right)}. \tag{15}
\]

In the limit of \( T \equiv T_e/T_i \gg 1 \) and \( H^2 k^2/12 \ll 1 \), the expression for \( \lambda \) in Eq. (15) can be approximated as

\[
\lambda \approx 1 + \frac{3}{2T} \frac{H^2 k^2}{24}. \tag{16}
\]

In terms of the original dimensional variables, Eq. (16) is rewritten as

\[
\lambda = c_s \left(1 + \frac{3 T_i}{2 T_e} \frac{H^2 k^2}{24} \lambda_D^2\right). \tag{17}
\]

The expression (16) of \( \lambda \) exactly agrees with Eq. (2.37) of Ref. [3] in the formal semiclassical limit of \( H = 0 \). Thus, in contrast to the quantum fluid theory [18] or classical kinetic theory [3], the phase velocity of the nonlinear IAWs, in presence of the particle’s dispersion, may no longer be a constant but can vary with the wave number \( k \) and gets modified by the quantum parameter \( H \). The profiles of \( \lambda \) for different values of the quantum parameter \( H \) and the temperature ratio \( T \) are shown in Fig. 1. It is seen that the wave speed \( \lambda \) decreases with the wave number, and this diminution is greatly enhanced with a small increase of the quantum parameter \( H \) [Fig. 1(a)]. Furthermore, a significant reduction in \( \lambda \) is also seen to occur with an enhancement of the temperature ratio \( T \) [Fig. 1(b)]. From Eq. (16), one can obtain a value of the phase velocity for a given set of plasma parameters. For example, for typical parameter values with \( T = 20, H = 0.05 \) (which, e.g., correspond to plasmas with \( T_e \approx 7 \times 10^6 \) K, \( n_0 \approx 6 \times 10^{23} \) cm\(^{-3}\), so that \( \lambda_D = 3 \times 10^{-9} \) cm) and given a dimensionless wave number \( k = 2 \) (i.e., in dimensional variable, \( k = 6.5 \times 10^8 \) cm\(^{-1}\)), one obtains the phase velocity \( \lambda = 1.06 \) or in dimensional variable, \( \lambda = 9.6 \times 10^7 \) cm/s. We mention that further increase of the values of \( H \) and \( k \) may not be admissible as we are interested in the propagation of IAWs in the weak quantum regime \( H < 1 \). However, the values of \( T \) may be considered in the range \( 20 \lesssim T \lesssim 100 \). Furthermore, in the range of values of \( k \) (as in Fig. 1), \( \lambda \) is greater than the unity, implying that the relation \( v_{ti} \ll \lambda \ll v_{te} \) holds good for IAWs with \( T \gg 1 \) and \( m \approx m_e/m_i \ll 1 \). Here, \( v_{ta} = \sqrt{k_B T_i/m_a} \) is the thermal velocity of \( \alpha \)-spices particle. As expected, in the limit of \( k \to 0 \) and/or in the semiclassical limit \( H \to 0 \), the phase velocity of IAWs approaches a constant value, i.e., close to the ion-acoustic speed \( c_a \).

**B. Second-order perturbations and KdV equation**

Equating the coefficients of \( \epsilon^{5/2} \) from Eqs. (4) and (5), and the coefficient of \( \epsilon^3 \) from Eq. (4), we respectively, obtain

\[
\frac{\partial f_{\alpha}^{(2)}}{\partial \sigma} + v \left( \frac{\partial f_{\alpha}^{(2)}}{\partial \xi} + \frac{\partial f_{\alpha}^{(1)}}{\partial s} \right) - \theta_{\alpha} \left( \frac{m_i}{m_a} \right) \times \left[ G_{\alpha}(v) \left( \frac{\partial \phi^{(2)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial s} + \frac{\partial \phi^{(1)}}{\partial v} \right) + R_{\alpha} \left( \frac{\partial^3 \phi^{(2)}}{\partial s^2 \partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \right) \frac{\partial^3 f_{\alpha}^{(0)}}{\partial v^3} \right] = 0, \tag{18}
\]

and

\[
\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = -\sum \theta_{\alpha} \int_{n.r} f_{\alpha}^{(2)} dv \left( \int_{r.s} f_{\alpha} dv \right)^{(2)}, \tag{19}
\]

where the term in the angular brackets appears due to the second order density perturbation in the resonance region. Transforming Eq. (18) according to the formula (10) and eliminating \( f_{\alpha}^{(1)} \) by using Eq. (13), we obtain

\[
\hat{f}_{\alpha}^{(2)} = -k \left( \theta_{\alpha} \left( \frac{m_i}{m_a} \right) G_{\alpha}(v) + R_{\alpha} k^2 \frac{\partial^2 G_{\alpha}(v)}{\partial v^2} \right) \phi^{(2)} + \left( \theta_{\alpha} \left( \frac{m_i}{m_a} \right) i \omega (\omega - kv)^{-1} G_{\alpha}(v) + 3i R_{\alpha} k^2 \frac{\partial G_{\alpha}(v)}{\partial v^2} \right) \phi^{(1)} + \omega \theta_{\alpha} \left( \frac{m_i}{m_a} \right) (\omega - k\lambda) \hat{B}_{\alpha} (\omega - kv)^{-1}, \tag{20}
\]

where

\[
\hat{B}_{\alpha} = -i \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\sigma \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{\alpha}^{(1)}}{\partial v} \exp[-i(k\xi - \omega \sigma)], \tag{21}
\]

is the transform of the nonlinear term \( (\partial \phi^{(1)}/\partial \xi)(\partial f_{\alpha}^{(1)}/\partial v) \). Next, we substitute the expressions of \( f_{\alpha}^{(1)} \) and \( \phi^{(1)} \) from Eq. (10) into Eq. (21). By virtue of Eq. (13) and using the Convolution theorem, Eq. (21) reduces to

\[
\hat{B}_{\alpha}(\omega, k) = -i \theta_{\alpha} \left( \frac{m_i}{m_a} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega' - \lambda k' \int_{-\infty}^{\infty} \phi^{(2)}(\xi, \omega') \int_{-\infty}^{\infty} (v - \lambda)^{-1} G_{\alpha}(v) dv \left[ \frac{G_{\alpha}(v)}{\omega' - v k'} \right] d\omega' dk', \tag{22}
\]

where \( \text{Im}(\omega - \omega') > 0 \). Taking the Fourier inversion of Eq. (20) and integrating both sides of this equation over the non-resonance region, we obtain

\[
\int_{n.r.} f_{\alpha}^{(2)} dv = \theta_{\alpha} \left( \frac{m_i}{m_a} \right) \left[ \phi^{(2)}(\zeta, s) \int_{n.r.} (v - \lambda)^{-1} G_{\alpha}(v) dv - A_{\alpha} - \theta_{\alpha} B_{\alpha} + R_{\alpha} k^2 \phi^{(2)} \int_{n.r.} (v - \lambda)^{-1} \right] \frac{\partial^2 G_{\alpha}(v)}{\partial v^2} dv, \tag{23}
\]
where \( \zeta = \xi - \lambda \sigma \) and \( A_n, B_n \) are given by

\[
A_n = (2\pi)^{-2} \int \int_W \frac{dkd\omega}{\omega - \lambda k} \frac{\partial \phi^{(1)}}{\partial \lambda} \exp[i(k\xi - \omega\sigma)] \\
\times \int_{n.r.} \left( \omega G_\alpha(v) + 3\theta_\alpha R_\alpha k^2 \frac{\partial^2 G_\alpha(v)}{\partial v^2} \right) \frac{dv}{(\omega - k\nu)^2},
\]

\( B_n = (2\pi)^{-4} \int \int_W \frac{dkd\omega}{\omega - \lambda k} \int \int_W dk'd\omega'k'(k - k') \\
\times (\omega' - \lambda k')^{-1}[(\omega - \omega') - \lambda(k - k')]^{-1} \\
\times \phi^{(1)}(\omega, k') \phi^{(1)}(\omega, k) \exp[i(k\xi - \omega\sigma)] \\
\times L_\alpha(\omega, k; \omega', k'),
\]

with

\[
L_\alpha = -k \int_{n.r.} (\omega' - \lambda k')^{-1} (\omega - \lambda k)^{-2} G_\alpha(v) dv,
\]

and to the fact that over the non-resonance region, where \( |v - \lambda| > > \alpha \), \( G_\alpha(v), (\omega - k\nu)^{-1}, (\omega' - k'\nu)^{-1} \rightarrow 0 \). Closing the \( W \) contour from below and differentiating Eq. (24) with respect to the transformation \( \zeta = \xi - \lambda \sigma \), we obtain

\[
\frac{\partial A_n}{\partial \zeta} = k \frac{\partial \phi^{(1)}}{\partial \lambda} \int_{n.r.} \left( \omega G_\alpha(v) + 3\theta_\alpha R_\alpha k^2 \frac{\partial^2 G_\alpha(v)}{\partial v^2} \right) \\
\times (\omega - k\nu)^{-2} dv \\
= [A_n(\lambda) + C_n(\lambda)] \frac{\partial \phi^{(1)}}{\partial \lambda},
\]

where, by virtue of Eq. (24), \( A_n \) and \( C_n \) are given as

\[
\sum_{\alpha} A_n(\lambda) = \lambda \sum_{\alpha} \left( \frac{m_i}{m_\alpha} \right) \int_{n.r.} \frac{dv}{v - \lambda} \frac{\partial G_\alpha(v)}{\partial v},
\]

\[
\sum_{\alpha} C_n(\lambda) = 18k \sum_{\alpha} \left( \frac{m_i}{m_\alpha} \right) \theta_\alpha R_\alpha \frac{G_\alpha(v)}{(v - \lambda)^4} dv,
\]

and we have used the fact that over the non-resonance region, \( (v - \lambda)^{-1}, (v - \lambda)^{-2}, (v - \lambda)^{-3} \rightarrow 0 \). In the expression of \( B_n \) [Eq. (25)], the \( \omega' \)-integration is performed with the fact that the pole \( \omega' = \omega - \lambda(k - k') \) lies on the \( W \) contour which is above the \( W' \) contour, and we close the \( W' \) contour in the lower half plane, so that only the residue at \( \omega' = \lambda(k' - k') \) contributes. However, the \( \omega \)-integration is performed by closing the \( W \) contour from below and assuming the pole is at \( \omega = \lambda k \). Thus, we obtain

\[
B_n = -B_{n0}(\lambda)(2\pi)^{-2} \int \int_W k^{-1}(k - k') \frac{\partial \phi^{(1)}}{\partial \lambda}(k - k') \\
\times \phi^{(1)}(k') \exp[i(k\xi - \omega\sigma)] dkd\nu,
\]

where we have used

\[
L_n(k, k') = -k \int_{n.r.} \frac{1}{k^2 k'} \left( \frac{\omega'}{k'} - 1 \right)^{-1} \left( \frac{\omega}{k'} - 1 \right)^{-2} G_\alpha dv,
\]

and so, \( B_n(\lambda) \) is given by

\[
B_{n0}(\lambda) = \int_{n.r.} (v - \lambda)^{-3} G_\alpha dv.
\]

Next, differentiating Eq. (30) with respect to \( \zeta \) and using the convolution theorem, we obtain

\[
\frac{\partial B_n}{\partial \zeta} = -2B_{n0}(\lambda) \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \lambda},
\]
Finally, differentiating Eq. (19) with respect to $\zeta$ and using Eq. (23), we note that the coefficient of $\partial \phi^{(2)}/\partial \zeta$ vanishes by the dispersion relation [14]. Thus, we obtain the following modified KdV equation

$$\frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + a \frac{\partial \phi^{(1)}}{\partial s} + b \frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + \frac{\partial}{\partial \zeta} \sum_{\alpha} \theta_{\alpha} \left( \int f_{\alpha} dv \right)^{(2)} = 0, \quad (34)$$

where the coefficients $a$ and $b$ are given by

$$a = -\sum_{\alpha} \left( \frac{m_\alpha}{m_\alpha} \right) [A_\alpha(\lambda) + C_\alpha(\lambda)], \quad (35)$$

and

$$b = \sum_{\alpha} \theta_{\alpha} \left( \frac{m_\alpha}{m_\alpha} \right) \int_{\text{a.r.}} (v - \lambda)^{-3} G_\alpha(v) dv. \quad (36)$$

In order to evaluate the integral in Eq. (34) over the resonance region where the particle velocity approaches the phase velocity of the wave, a different ordering for the distribution function is to be considered. Here, we assume that $v - \lambda \sim \epsilon^{1/2}(m_i/m_\alpha)^{1/2} u$, where $u \sim \alpha(1)$, and along the particle path $f_{\alpha}(v, x, t) = f_{\alpha}(v, 0, 0) = f_{\alpha}^{(0)}(v)$. So, in the resonance region, we expand the distribution function as [4]

$$f_{\alpha} = f_{\alpha}^{(0)} + \epsilon^{3/2} \left( \frac{m_\alpha}{m_i} \right)^{1/2} f_{\alpha}^{(1)} + \cdots, \quad (37)$$

together with the ordering for the derivatives

$$\frac{\partial f_{\alpha}^{(0)}}{\partial v} \left( \frac{m_i}{m_\alpha} \right) \sim \epsilon, \quad \frac{\partial f_{\alpha}^{(1)}}{\partial v} \left( \frac{m_i}{m_\alpha} \right)^{1/2} \sim \epsilon^{1/2}. \quad (38)$$

The expansion for $\phi$ remains the same as in Eq. [17]. Thus, from Eqs. 41 and 43, the coefficients of $\epsilon^{3/2}$ yield

$$\epsilon^{1/2} \left( \frac{m_\alpha}{m_i} \right)^{1/2} \left[ \frac{\partial f_{\alpha}^{(1)}}{\partial \sigma} + v \frac{\partial f_{\alpha}^{(1)}}{\partial \xi} \right] - \theta_{\alpha} \left[ \left( \frac{m_i}{m_\alpha} \right) \epsilon^{-1} \right. \left. \frac{\partial f_{\alpha}^{(1)}}{\partial v} \right] \left( \frac{m_i}{m_\alpha} \right)^{1/2} \frac{\partial \phi^{(1)}}{\partial \xi} \right. \left. - \theta_{\alpha} \left[ \left( \frac{m_i}{m_\alpha} \right)^{1/2} \frac{\partial f_{\alpha}^{(1)}}{\partial v} \right] \frac{\partial \phi^{(1)}}{\partial \xi} \right. \left. \right) = 0, \quad (39)$$

where $\partial / \partial \sigma + v(\partial / \partial \xi) \sim \epsilon(v - \lambda)$ and clearly, all the terms in the left hand side of Eq. (39) are of the same order of magnitude. We note that the second and the third terms in Eq. (39) under the square brackets appear due to the Landau damping (linear resonance) and particle trapping (nonlinear resonance) respectively. Furthermore, with the orderings as above, the quantum effect $\propto R_c$ does not contribute to Eq. (39) implying that the results for the trapping will remain qualitatively the same as in the classical theory [4]. So, we will focus mainly on the linear Landau damping. Thus, Eq. (39), after the third term being dropped, becomes

$$\epsilon^{-1/2} \left( \frac{m_\alpha}{m_i} \right)^{1/2} \left[ \frac{\partial f_{\alpha}^{(1)}}{\partial \sigma} + v \frac{\partial f_{\alpha}^{(1)}}{\partial \xi} \right] - \theta_{\alpha} \left( \frac{m_i}{m_\alpha} \right) \epsilon^{-1} \times G_\alpha(v) \frac{\partial \phi^{(1)}}{\partial \xi} = 0. \quad (40)$$

Following Ref. [4] we obtain

$$\frac{\partial}{\partial \zeta} \sum_{\alpha} \theta_{\alpha} \left( \int f_{\alpha} dv \right)^{(2)} = cP \int_{-\infty}^{\infty} \frac{d\zeta^{'}}{\zeta - \zeta^{'}} \frac{\partial \phi^{(1)}(\zeta^{'})}{\partial \zeta^{'}} = \epsilon^{-1} \sum_{\alpha} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(\lambda). \quad (41)$$

Substituting Eq. (41) into Eq. (34), we obtain the following modified KdV equation for IAWs:

$$\frac{\partial \phi}{\partial s} + \phi \frac{\partial \phi}{\partial \zeta} + \beta \frac{\partial^3 \phi}{\partial \zeta^3} + \gamma P \int_{-\infty}^{\infty} \frac{d\zeta^{'}}{\zeta - \zeta^{'}} \frac{\partial \phi(\zeta^{'})}{\partial \zeta^{'}} = \epsilon^{-1} \sum_{\alpha} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(\lambda), \quad (42)$$

where $\phi = \phi^{(1)}$ and the coefficients are given by $\delta = b/a$, $\beta = b/a$ and $\gamma = c/a$. If the equilibrium distribution of particles be the Maxwellian [Eq. (3)], one can obtain the coefficients $a$, $b$ and $c$ (for details see Appendices B-D) as

$$a = 2\lambda^{-2} (1 + 6\lambda^{-2}T^{-1}) + \lambda H^2 k, \quad (44)$$

$$b = 3\lambda^{-4} + 30T^{-1}\lambda^{-6} - 1, \quad (45)$$

$$c = \epsilon^{-1} \frac{\lambda}{\sqrt{2\pi}} \left[ m_i^{1/2} + T^{3/2} \exp \left( -\frac{T\lambda^2}{2} \right) \right]. \quad (46)$$

We note that each of the coefficients $a$, $b$ and $c$ are modified by the quantum parameter $H$, in absence of which one recovers the classical results of Vandam et al. [4]. Considering a small effect of the Landau damping ($\propto \gamma$) with $\gamma \ll \alpha (\geq \beta)$, which holds for $T \gtrsim 20$ and $H < 1$, we find the solitary wave solution [6] of Eq. (43) as

$$\phi = \Psi \text{sech}^2 \left[ \left( \zeta - \frac{\delta}{3} \int_0^s \Phi ds \right) / W \right] + o(\gamma), \quad (47)$$

where $\Psi = \Phi_0 (1 + s/s_0)^{-2}$ is the amplitude of the solitary wave solution of the modified KdV equation (43), and $\Phi = 3U_0/\delta$ is the corresponding amplitude, $W = (12\beta/\Phi \delta)^{1/2} \equiv 4\beta/\Phi_0$ is the width and $U_0 = \Phi \delta/3$ is the constant phase speed (normalized by $c_s$) of the solitary wave solution of the KdV equation in absence of the Landau damping (i.e., when $\gamma = 0, c = 0$). For details
about the solution, readers are referred to Ref. 6. Also, \( \Phi = \Phi_0 \) at \( s = 0 \) and \( s_0 \) is given by

\[
s_0^{-1} = \frac{\gamma}{4} \sqrt{\frac{\delta \Phi_0}{3\beta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sech}^2 z \frac{\partial}{\partial z'} (\text{sech}^2 z') \ dz d'.
\]

(48)

IV. RESULTS AND DISCUSSION

The profiles of the linear damping rate \( \gamma \) are shown in Fig. 2 for different values of \( H \) and \( T \). From the subplots (a) and (b) it is clear that whatever be the values of \( T \) and \( H \), the damping rate always decreases with increasing values of the wave number \( k \). The value of \( \gamma \) also decreases with a small increase of the quantum parameter \( H \), and such a decrement is significant at higher values of \( k \). However, subplot (b) shows that there exists a critical value of \( T \sim 24 \), below (above) which the value of \( \gamma \) decreases (increases) with increasing value of \( T \).

We have also obtained a solitary wave solution (47) of the KdV equation assuming that the Landau damping effect is small compared to those of the nonlinearity and dispersion. It is found that the wave amplitude decays as time goes on. The profiles are shown in Fig. 3 for different values of \( H \). It is found that the decay rate is lower (than the classical case) in the semiclassical regime with a value of \( H \). The decay rate remains almost unaltered by the effects of \( T \).

We note that the trapping time of an electron by a solitary pulse is \( T_{\text{trap}} = \omega_B^2 \), where \( \omega_B \) is the bouncing frequency, given by, \( \omega_B = \sqrt{e\phi_0/m_e} \) with \( W \) denoting the soliton width. Since \( \phi \sim \epsilon \), \( \omega_B \sim \omega_{\text{pe}} \sqrt{\epsilon} \sim \omega_{\text{pe}} \epsilon^{-1/2} \) (for \( \epsilon \sim \sqrt{m_e/m_i} \)). However, from Fig. 2 it is clear that the Landau damping rate \( \gamma \) is of the order of \( 0.18 \omega_{\text{pi}} \) (depending on the values of \( T \), \( H \) and \( k \)), i.e., \( \gamma < \omega_B \), the usual criterion for electrons to be trapped. However, in case of ions, since \( \omega_B \sim \omega_{\text{pl}} \sqrt{\epsilon} \), we can have \( \gamma > \Omega_B \), i.e., ion trapping may be neglected. Furthermore, one can compare the magnitudes of the effects of trapping (nonlinear resonance) and the Landau damping (linear resonance). For the nonlinear resonance we find that

\[
\int_{\text{res}} f_\alpha dv \approx -\frac{\lambda}{\sqrt{2\pi}} \left[ \exp \left( \frac{-1}{2} m \lambda^2 \right) + \epsilon T^{3/2} \exp \left( \frac{1}{2} T \lambda^2 \right) \right].
\]

(49)

From Eqs. (49) and (49), it is clear that the effects of the linear resonance is relatively higher than that of the nonlinear one. On the other hand, ions may be reflected by a solitary pulse and propagate as a precursor, which is also not of interest in the present study.

In semiclassical plasmas, the thermodynamic temperature of electrons \( (T_e) \) is assumed to be larger than the Fermi temperature \( (T_F) \) in which case the Pauli blocking is reduced, and the particles’ collisional effects can have some role on the dynamics of IAWs. Nevertheless, the inclusion of a collisional term in the semiclassical Vlasov equation is not so straightforward. However, if a small collisional effect (e.g., Coulomb collision) is introduced, the effective electron-electron collision frequency scales as \( \nu_{\text{ef}} \sim \epsilon \omega_{\text{pe}} (n_0 \lambda_D^3)^{-1} \sim \epsilon^{-2} \omega_{\text{pe}} (n_0 \lambda_D^3)^{-1} \). For moderate density plasmas with \( n_0 \sim 6 \times 10^{23} \text{ cm}^{-3} \) and \( T_e \sim 7 \times 10^6 \) K, one can have \( (n_0 \lambda_D^3)^{-1} \sim (0.13) > \epsilon (\sim 0.02) \) and \( H \sim 0.05 \). Thus, \( \nu_{\text{ef}} (\gtrsim \epsilon^{-1} \omega_{\text{pi}}) > \omega_{\text{Be}} \sim \sqrt{\epsilon} \omega_{\text{pe}} \sim \omega_{\text{pe}} \epsilon^{-1/2} \), and consequently, the trapping of electrons will be destroyed. Furthermore, depending on the values of

FIG. 2. The Landau damping rate \( \gamma \) (normalized by \( \omega_{\text{pi}} \)) is plotted against the wave number \( k \) (normalized by \( \lambda_D^{-1} \)) in three different cases: (a) when \( T \) is fixed and \( H \) varies, (b) when \( H \) is fixed and \( T \) varies and (c) when the value of \( T \) is relatively lower than that in the plots (a) and (b).
$T$, $H$ and $k$, the Landau damping contribution $c$ can be larger than the damping due to the collisional effects.

V. CONCLUSION

We have studied the effects of Landau damping associated with the resonance of electrons and ions with ion-acoustic solitary waves in a semiclassical electron-ion plasma. The latter includes the quantum particle dispersion (QPD) in the weak sense, i.e., when the typical ion-acoustic length scale is larger than the thermal de Broglie wavelength. A modified KdV equation, which governs the evolution of IAWs, is derived by a multiscale asymptotic expansion method. It is found that in contrast to classical results [3], the IAW speed ($\lambda$) and the Landau damping rate ($\gamma$) are no longer constants by the effects of QPD, but can vary with the wave number $k$. Both $\lambda$ and $\gamma$ are seen to be significantly modified by the quantum parameter $H$ and the temperature ratio $T$. It is also found that the decay of the solitary wave amplitude can be suppressed by increasing the value of $H$.

To conclude, the nonlinear resonance effects such as trapping, reflection of electrons and ions should be properly considered especially in the semiclassical and quantum kinetic models in order to describe correctly the behaviors of IAWs in plasmas. The dynamics of solitary waves in the (strong) quantum regime, i.e., starting from a Wigner or Wigner-Moyal equation can also be a problem of interest but beyond the scope of the present study. This work is underway and will be communicated elsewhere.

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Appendix A: Expressions for the dispersion relation

Here, we give the details of the derivation of Eq. (A3). First of all, we calculate the classical part.

\[
\sum_{\alpha} \int_{\text{n.r.}} (v - \lambda)^{-1} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(v) dv 
= -\frac{1}{\sqrt{2\pi}} \sum_{\alpha} \left( \frac{m_i}{m_\alpha} \right)^{1/2} \left( \frac{T_e}{T_\alpha} \right)^{3/2} \int_{-\infty}^{\infty} v(v - \lambda) 
\times \exp \left[ -\frac{m_\alpha T_e v^2}{m_i T_\alpha} \right] dv. \quad (A2)
\]

Using the transformation $(v^2/2)(m_\alpha T_e/m_i T_\alpha) \rightarrow v^2$, Eq. (A2) yields

\[
\sum_{\alpha} \int_{\text{n.r.}} (v - \lambda)^{-1} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(v) dv 
= -\frac{1}{\sqrt{2\pi}} \sum_{\alpha} T_e \int_{-\infty}^{\infty} v \exp(-v^2) 
\times \left( v - \frac{\lambda}{\sqrt{2 \frac{m_i v^2}{m_\alpha}}} \right)^{-1} dv. \quad (A3)
\]

Differentiating the Plasma dispersion function once with respect to $\xi$, we obtain

\[
Z'(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-t^2) \frac{dt}{(t - \xi)^2}. \quad (A4)
\]

Equation (A4), after integrating by parts, gives

\[
Z'(\xi) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2t \exp(-t^2)}{(t - \xi)^2} dt = -2[1 + \xi Z(\xi)]. \quad (A5)
\]

Next, using the relation (A5), we obtain from Eq. (A3) as

\[
\sum_{\alpha} \int_{\text{n.r.}} (v - \lambda)^{-1} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(v) dv 
= -\sum_{\alpha} \frac{T_e}{T_\alpha} \left[ 1 + \sqrt{\frac{m_\alpha T_e}{m_i T_\alpha}} \frac{\lambda}{\sqrt{2}} \sqrt{\frac{m_i}{m_\alpha}} Z \left( \frac{\lambda}{\sqrt{2 \sqrt{\frac{m_i}{m_\alpha}}}} \right) \right]. \quad (A6)
\]

Assuming $\lambda \sqrt{m_e/m_i} \ll 1$ and $\lambda \sqrt{T_e/T_i} \ll 1$, we expand the Plasma dispersion function for small as well as large argument [19], and obtain from Eq. (A6) as

\[
\sum_{\alpha} \int_{\text{n.r.}} (v - \lambda)^{-1} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(v) dv 
= -(1 + T) - \frac{\lambda}{\sqrt{2}} \left[ \sqrt{m_i} (i\sqrt{\pi} - \lambda \sqrt{2m_i} - \frac{\sqrt{\pi}}{2} m_\alpha^2 \sqrt{m_i} \chi^3) + T^{3/2}(i\sqrt{\pi} - \frac{\sqrt{\pi}}{\lambda} T^{-1/2}) \right. 
\left. + \frac{\sqrt{\pi}}{2} m_\alpha^2 \chi^3 + T^{3/2}(i\sqrt{\pi} - \frac{\sqrt{\pi}}{\lambda} T^{-1/2}) \right] \left[ \sqrt{m_i} (i\sqrt{\pi} - \lambda \sqrt{2m_i} - \frac{\sqrt{\pi}}{2} m_\alpha^2 \sqrt{m_i} \chi^3) + T^{3/2}(i\sqrt{\pi} - \frac{\sqrt{\pi}}{\lambda} T^{-1/2}) \right], \quad (A7)
\]
where, $m = m_e/m_i$ and $T = T_e/T_i$. Neglecting the imaginary terms in the right hand side of Eq. (A7), we obtain

$$\sum_{\alpha} \int_{n.r.} (v - \lambda)^{-1} \left( \frac{m_i}{m_\alpha} \right) G_\alpha(v) dv = -1 + m\lambda^2 \left( 1 - \frac{m\lambda^2}{3} \right) + \frac{1}{\lambda^2} \left( 1 + \frac{3}{T\lambda^2} \right). \quad (A8)$$

Next, we calculate the contribution from the quantum particle dispersion.

$$-R_e k^2 \int_{n.r.} (v - \lambda)^{-1} \frac{\partial^2 G_e(v)}{\partial v^2} dv = \frac{H^2 k^2}{24} \left( \frac{m_i}{m_e} \right)^2 \left[ (v - \lambda)^{-1} \frac{\partial G_e(v)}{\partial v} + \int (v - \lambda)^{-2} \times \frac{\partial G_e(v)}{\partial v} dv \right]_{n.r.}$$

$$= \frac{H^2 k^2}{12} \frac{1}{\sqrt{2\pi}} \left( \frac{m_i}{m_e} \right)^{1/2} \int_{-\infty}^{\infty} v(v - \lambda)^{-3} \times \exp \left[ -\frac{m_e v^2}{m_i} \right] dv. \quad (A9)$$

Using the transformation $(v^2/2)(m_e/m_i) \to v^2$, Eq. (A9) yields

$$-R_e k^2 \int_{n.r.} (v - \lambda)^{-1} \frac{\partial^2 G_e(v)}{\partial v^2} dv = \frac{H^2 k^2}{24} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v \exp(-v^2)$$

$$\times \left[ v - \frac{\lambda}{\sqrt{2m_i/m_e}} \right]^{-3} \, dv. \quad (A10)$$

Next, differentiating the Plasma dispersion function thrice with respect to $\xi$, we obtain

$$Z'''(\xi) = -4[(2\xi^2 - 3)\xi Z(\xi) + 2\xi^2 - 2]. \quad (A11)$$

Also,

$$Z'''(\xi) = -\frac{4}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \exp(-t^2) \frac{1}{(t - \xi)^3} dt. \quad (A12)$$

From Eqs. (A11) and (A12), we have

$$(2\xi^2 - 3)\xi Z(\xi) + 2\xi^2 - 2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \exp(-t^2) \frac{1}{(t - \xi)^3} dt. \quad (A13)$$

Next, using the relation (A13), we obtain from Eq. (A10) as

$$-R_e k^2 \int_{n.r.} (v - \lambda)^{-1} \frac{\partial^2 G_e(v)}{\partial v^2} dv = \frac{H^2 k^2}{24} \left[ \left( \frac{\lambda^2 m_e}{m_i} - 3 \right) \frac{\lambda}{\sqrt{2}} \sqrt{\frac{m_e}{m_i}} \right.$$

$$\left. Z \left( \frac{\lambda}{\sqrt{2}} \sqrt{\frac{m_e}{m_i}} \right) + \frac{m_e \lambda^2}{m_i} \right] \frac{m_e \lambda^2}{m_i}. \quad (A14)$$

Assuming $\lambda \sqrt{m_e/m_i}/\sqrt{2} \ll 1$, and expanding the Plasma dispersion function for small argument [19] up to $O(\lambda^3)$, we obtain from Eq. (A14) as

$$-R e k^2 \int_{n.r.} (v - \lambda)^{-1} \frac{\partial^2 G_e(v)}{\partial v^2} dv = \frac{H^2 k^2}{24} \left[ (m\lambda^2 - 3) \frac{\lambda}{\sqrt{2}} \sqrt{m(i\sqrt{\pi} - \lambda\sqrt{2m} - i\sqrt{3}/2) m\lambda^2} \right.$$

$$+ \frac{\sqrt{3}}{3} m^{3/2} \lambda^3 \right] + m\lambda^2 - 2 \right]. \quad (A15)
Neglecting the imaginary terms in the right hand side of Eq. (A15), we obtain

\[
-R_e k^2 \int_{\text{n.r.}} (v - \lambda)^{-1} \frac{\partial^2 G_\alpha(v)}{\partial v^2} dv \\
= \frac{H^2 k^2}{24} m \left[ \frac{1}{3} m^2 \lambda^6 + 4 \lambda^2 - \frac{2}{m} \right] \\
= \frac{H^2 k^2}{24} \left[ \frac{1}{3} m^3 \lambda^6 + 4 m \lambda^2 - 2 \right]. \quad \text{(A16)}
\]

Next, substituting the values from Eqs. (A8) and (A16) into Eq. (A11), we obtain

\[
-1 + m \lambda^2 - \frac{1}{3} m^2 \lambda^4 + \frac{1}{2} T^2 + \frac{1}{2} \lambda^2 + \frac{H^2 k^2}{12} \left( \frac{1}{3} m^3 \lambda^6 + 4 m \lambda^2 - 2 \right) = 0,
\quad \text{(A17)}
\]

Neglecting the smaller terms compared to the larger ones in Eq. (A17), we obtain

\[
\left( 1 + \frac{H^2 k^2}{12} \right) \lambda^4 - \lambda^2 - \frac{3}{T} = 0. \quad \text{(A18)}
\]

Solving Eq. (A18), we obtain

\[
\lambda^2 = \frac{1 \pm \sqrt{1 + \frac{12}{R_e} \left( 1 + \frac{H^2 k^2}{12} \right)}}{2 \left( 1 + \frac{H^2 k^2}{12} \right)}. \quad \text{(A19)}
\]

Neglecting the smaller terms, we obtain the expression for nonlinear wave speed from Eq. (A19) as

\[
\lambda = 1 + \frac{3}{2T} - \frac{H^2 k^2}{24}. \quad \text{(A20)}
\]

**Appendix B: The coefficient \( a \)**

We simplify the expression for \( a \) [Eq. (B1)] as

\[
a = -\left[ \lambda \sum_{\alpha} \frac{m_i}{m_{\alpha}} \int_{\text{n.r.}} (v - \lambda)^{-1} \frac{\partial G_\alpha(v)}{\partial v} dv \\
+ 18 k \sum_{\alpha} \theta_\alpha R_\alpha \left( \frac{m_i}{m_{\alpha}} \right) \int_{\text{n.r.}} (v - \lambda)^{-4} G_\alpha(v) dv \right]. \quad \text{(B1)}
\]

The first term on the right hand side of Eq. (B1) is

\[
-\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \int_{\text{n.r.}} (v - \lambda)^{-1} \frac{\partial G_\alpha(v)}{\partial v} dv \\
= -\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \left[ (v - \lambda)^{-1} G_\alpha(v) + \int_{\text{n.r.}} \frac{G_\alpha(v)}{(v - \lambda)^2} dv \right] \\
= -\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \int_{\text{n.r.}} (v - \lambda)^{-2} G_\alpha(v) dv,
\]

\[
\lambda \sum_{\alpha} \left( \frac{T_e}{T_\alpha} \right) \int_{-\infty}^{\infty} v (v - \lambda)^{-2} f_\alpha^{(0)}(v) dv
\]

\[
= \lambda \sum_{\alpha} \left( \frac{T_e}{T_\alpha} \right) \int_{-\infty}^{\infty} v (v - \lambda)^{-2} \exp \left( -\frac{v}{\lambda^2} \right) dv.
\]

Differentiating the Plasma dispersion function twice with respect to \( \xi \), we obtain

\[
Z''(\xi) = -2(1 - 2\xi^2) Z(\xi) - 2\xi. \quad \text{(B3)}
\]

Also,

\[
Z''(\xi) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \exp(-t^2) dt, \quad \text{(B4)}
\]

Equations (B3) and (B4) yield

\[
(1 - 2\xi^2) Z(\xi) - 2\xi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \exp(-t^2) dt. \quad \text{(B5)}
\]

Using the transformation \((v^2/2)(m_\alpha T_e/m_i T_\alpha) \rightarrow v^2\), we see that Eq. (B2) can be rewritten as

\[
-\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \int_{\text{n.r.}} (v - \lambda)^{-1} \frac{\partial G_\alpha(v)}{\partial v} dv \\
= \frac{\lambda}{\sqrt{2\pi}} \sum_{\alpha} \frac{T_e}{T_\alpha} \sqrt{\frac{m_{\alpha} T_e}{m_i T_\alpha}} \int_{-\infty}^{\infty} v \exp(-v^2) \\
\times \left( v - \frac{\lambda}{\sqrt{2m_{\alpha} T_e/m_i T_\alpha}} \right) dv.
\]

Using the relation (B5), Eq. (B6) reduces to

\[
-\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \int_{\text{n.r.}} (v - \lambda)^{-1} \frac{\partial G_\alpha(v)}{\partial v} dv \\
= \frac{\lambda}{\sqrt{2}} \sum_{\alpha} \frac{T_e}{T_\alpha} \sqrt{\frac{m_{\alpha} T_e}{m_i T_\alpha}} \left( 1 - \lambda^2 \frac{m_{\alpha} T_e}{m_i T_\alpha} \right) Z \left( \frac{\lambda}{\sqrt{2}} \sqrt{\frac{m_{\alpha} T_e}{m_i T_\alpha}} \right)
\]

\[
- \lambda \sqrt{\frac{m_e/m_i}{\sqrt{2}}} \ll 1 \quad \text{and} \quad \lambda \sqrt{T_e/T_i/\sqrt{2}} \gg 1,
\]

and expanding the Plasma dispersion function for small
argument upto $o(\lambda^3)$ as well as for large argument upto $O(\lambda^{-7})$ [19], we obtain from Eq. (B7) the following

$$
-\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \int_{n.r.} (v - \lambda)^{-1} \frac{\partial G_\alpha(v)}{\partial v} dv
= \frac{\lambda}{\sqrt{2}} \left[ \sqrt{m_i}[(1 - m_i \lambda^2)(i\sqrt{\pi} - \lambda \sqrt{2} m_i^2 - \frac{i\sqrt{\pi}}{2} m_i^2 \lambda^2 + \frac{\sqrt{3}}{3} m_i^3 \lambda^3 + \frac{i\sqrt{\pi}}{8} m_i^2 \lambda^4 - \frac{\sqrt{3}}{15} m_i^5 \lambda^5)
- \frac{2\sqrt{2} m_i}{m_i^4 \lambda^5}
+ \frac{\sqrt{3}}{15} m_i^4 \lambda^7 + \frac{2\sqrt{3}}{\lambda^3} \frac{12\sqrt{3} T}{T} + \frac{90\sqrt{3}}{\lambda} \frac{1}{T^2}
- \frac{105\sqrt{3}}{\lambda^9} \frac{1}{T^3} \right].
$$

(B8)

Neglecting the imaginary part, Eq. (B8) reduces to

$$
-\lambda \sum_{\alpha} \left( \frac{m_i}{m_{\alpha}} \right) \int_{n.r.} (v - \lambda)^{-1} \frac{\partial G_\alpha(v)}{\partial v} dv
= \frac{\lambda}{\sqrt{2}} \left[ -2\sqrt{2} m_i^3 \lambda + \frac{4\sqrt{2}}{3} m_i^2 \lambda^3 - \frac{2\sqrt{2}}{5} m_i^3 \lambda^5
- \frac{\sqrt{3}}{15} m_i^4 \lambda^7 + \frac{2\sqrt{3}}{\lambda^3} \frac{12\sqrt{3} T}{T} + \frac{90\sqrt{3}}{\lambda} \frac{1}{T^2}
- \frac{105\sqrt{3}}{\lambda^9} \frac{1}{T^3} \right].
$$

(B9)

In a similar procedure, we obtain the second term on the right hand side of Eq. (B11) as

$$
-18k \sum_{\alpha} \theta_\alpha R_\alpha \left( \frac{m_i}{m_{\alpha}} \right) \int_{n.r.} (v - \lambda)^{-4} G_\alpha(v) dv
= \frac{H^2 k}{8} \left[ 8\lambda - 8\lambda^3 m + \frac{48}{15} \lambda^5 m^2 - \frac{11}{15} \lambda^7 m^3 + \frac{1}{15} \lambda^9 m^4 \right]
$$

(B10)

Next, adding Eqs. (B9) and (B10), and neglecting the smaller terms in comparison with the larger ones, we obtain

$$
a = \lambda^{-2}(2 + 12\lambda^{-2} T^{-1} + 90\lambda^{-4} T^{-2} - 105\lambda^{-6} T^{-3}) + \lambda H^2 k.
$$

(B11)

**Appendix C: The coefficient $b$**

We simplify the expression for $b$ [Eq. (30)] as

$$
b = \sum_{\alpha} \theta_\alpha \left( \frac{m_i}{m_{\alpha}} \right) \int_{n.r.} (v - \lambda)^{-3} G_\alpha(v) dv
= - \sum_{\alpha} \theta_\alpha \left( \frac{T_e}{T_{\alpha}} \right) \int_{-\infty}^{\infty} v(v - \lambda)^{-3} f_\alpha^{(0)}(v) dv
= - \frac{1}{\sqrt{2\pi}} \sum_{\alpha} \theta_\alpha \frac{T_e}{T_{\alpha}} \sqrt{\frac{m_i T_e}{m_i T_{\alpha}}} \int_{-\infty}^{\infty} v \exp(-v^2) d\nu
\times \left[ \frac{m_i T_e}{m_i T_{\alpha}} v^2 \right].
$$

(C1)

Differentiating the Plasma dispersion function thrice with respect to $\xi$, and noting that

$$Z'''(\xi) = -\frac{4}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \exp(-t^2) dt,
$$

we obtain

$$
(2\xi^2 - 3)Z(\xi) + 2\xi^2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \exp(-t^2) dt.
$$

(C3)

Using the transformation $(v^2/2)(m_i T_e/m_i T_{\alpha}) \rightarrow v^2$, we see that Eq. (C1) can be rewritten as

$$
b = -\frac{1}{\sqrt{2\pi}} \sum_{\alpha} \theta_\alpha \frac{T_e^2 m_{\alpha}}{m_i T_e T_{\alpha}} \int_{-\infty}^{\infty} v \exp(-v^2) d\nu
\times \left( v - \frac{\lambda}{\sqrt{2\pi}} \frac{m_i T_e}{m_i T_{\alpha}} \right)^3.
$$

(C4)

Using the relation (C3), Eq. (C4) becomes

$$
b = -\frac{1}{2} \sum_{\alpha} \theta_\alpha \frac{T_e^2 m_{\alpha}}{m_i T_e T_{\alpha}} \left( \frac{\lambda}{\sqrt{2\pi}} \frac{m_i T_e}{m_i T_{\alpha}} - 3 \right) \frac{\lambda}{\sqrt{2\pi}} \frac{m_i T_e}{m_i T_{\alpha}}
\left[ \frac{m_i T_e}{m_i T_{\alpha}} + \frac{m_i T_e}{m_i T_{\alpha}} \right] - 2 \frac{\lambda^2}{\lambda^2}.
$$

(C5)

Assuming $\lambda \sqrt{m_i/m_e}/\sqrt{2} \ll 1$ and $\lambda \sqrt{T_e/T_i}/\sqrt{2} \gg 1$ and expanding the Plasma dispersion function for small argument upto $o(\lambda^3)$ as well as for large argument upto $o(\lambda^{-7})$ [19], we obtain from Eq. (C6) the following

$$
b = \frac{1}{2} \left[ \frac{m_i T_e}{m_i T_i} (3 - m_i \lambda^2) \frac{\lambda}{\sqrt{2\pi}} \frac{m_i T_e}{m_i T_{\alpha}}
\left[ - \frac{i\sqrt{\pi}}{2} m_i \lambda^2 + \frac{\sqrt{3}}{3} m_i^3 \lambda^3 + \frac{i\sqrt{\pi}}{8} m_i^2 \lambda^4 - \frac{\sqrt{3}}{15} m_i^5 \lambda^5 \right)
- \frac{m_i^2 + 2}{T_e^2} T^2 [(3 - T_e^2) \frac{\lambda}{\sqrt{2\pi}} \frac{m_i T_e}{m_i T_{\alpha}}
\left[ \frac{\sqrt{3}}{2} T^3 \left( i\sqrt{\pi} - \frac{\sqrt{3}}{\lambda} T^{-1/2} \right.
- \frac{\sqrt{3}}{\lambda^3} T^{-3/2} - \frac{3}{\lambda^5} T^{-5/2} - \frac{105\sqrt{3}}{\lambda^9} T^{-9/2}
- \frac{15\sqrt{3}}{\lambda^7} T^{-7/2} - T^2 \lambda^2 + 2\lambda \right].
$$

(C6)
Next, considering the real parts of $b$, Eq. (C7) gives

$$b = \frac{1}{30}[mT(60m\lambda^2 - 30m^2\lambda^4 + 6m^3\lambda^6 - m^4\lambda^8 - 30) + 90T^{-1}\lambda^{-6} - 4725T^{-2}\lambda^{-8} + 90\lambda^{-4}].$$

Neglecting the small terms and also using the fact that, $(T_e/T_i)(m_e/m_i) \sim o(1)$, we have from Eq. (C7)

$$b \approx \frac{1}{30}[-30 + 900T^{-1}\lambda^{-6} + 90\lambda^{-4}] = 3\lambda^{-4} - 1 + 30T^{-1}\lambda^{-6}.$$ (C8)

### Appendix D: The coefficient $c$

We simplify the expression for $c$ [Eq. (42)] as

$$c = -\epsilon^{-1} \sum_\alpha \left( \frac{m_i}{m_\alpha} \right) G_\alpha(\lambda)$$

$$= -\epsilon^{-1} \frac{\lambda}{\sqrt{2\pi}} \left[ m^{1/2} \exp\left( -\frac{m\lambda^2}{2} \right) + T^{3/2} \exp\left( -\frac{T\lambda^2}{2} \right) \right],$$ (D1)

where, we have used the expression for $G_\alpha(\lambda)$ as

$$G_\alpha(\lambda) = -\frac{\lambda}{\sqrt{2\pi}} \left( \frac{m_\alpha T_e}{m_i T_\alpha} \right)^{3/2} \exp\left( -\frac{m_\alpha T_e \lambda^2}{m_i T_\alpha} \right),$$ (D2)

Assuming $\exp(-m\lambda^2/2) \approx 1$, we obtain from Eq. (D1) the following expression for $c$

$$c = \epsilon^{-1} \frac{\lambda}{\sqrt{2\pi}} \left[ m^{1/2} + T^{3/2} \exp\left( -\frac{T\lambda^2}{2} \right) \right].$$ (D3)