Formation and construction of a multidimensional shock wave for the first order hyperbolic conservation law with smooth initial data

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Abstract

In this paper, the problem on formation and construction of a multidimensional shock wave is studied for the first order conservation law $\partial_t u + \partial_x F(u) + \partial_y G(u) = 0$ with smooth initial data $u_0(x, y)$. It is well-known that the smooth solution $u$ will blow up on the time $T^* = -\frac{1}{\min H(\xi, \eta)}$ when $\min H(\xi, \eta) < 0$ holds for $H(\xi, \eta) = \partial_\xi (F'(u_0(\xi, \eta))) + \partial_\eta (G'(u_0(\xi, \eta)))$, more precisely, only the first order derivatives $\nabla_{t,x,y} u$ blow up on $t = T^*$ meanwhile $u$ itself is still continuous until $t = T^*$. Under the generic nondegenerate condition of $H(\xi, \eta)$, we construct a local weak entropy solution $u$ for $t \geq T^*$ which is not uniformly Lipschitz continuous on two sides of a shock surface $\Sigma$. The strength of the constructed shock is zero on the initial blowup curve $\Gamma$ and then gradually increases for $t > T^*$. Additionally, in the neighbourhood of $\Gamma$, some detailed and precise descriptions on the singularities of solution $u$ are given.

Keywords: Hyperbolic conservation law, multidimensional shock wave, generic nondegenerate condition, entropy condition, Rankine-Hugoniot condition.

Mathematical Subject Classification 2000: 35L05, 35L72

1 Introduction

1.1 Setting of the problem and statement of the main result

In this paper, we shall study the problem of a multidimensional shock formation for the following first order 2D conservation law

$$\begin{cases}
\partial_t u + \partial_x F(u) + \partial_y G(u) = 0, \\
u(0, x, y) = u_0(x, y),
\end{cases}$$

(1.1)\textsuperscript{*}

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where \((t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2\), \(F(u)\) and \(G(u)\) are \(C^5\) smooth functions of \(u\), \(u_0(x, y) \in C^4(\mathbb{R}^2)\). Let \(f(u) = F'(u)\) and \(g(u) = G'(u)\). Define the characteristics \((x(t; \xi, \eta), y(t; \xi, \eta))\) of (1.1) starting from the initial point \((\xi, \eta)\) as follows
\[
\begin{cases}
\frac{dx}{dt}(t; \xi, \eta) = f(u(t, x; \xi, \eta), y(t; \xi, \eta)), \\
\frac{dy}{dt}(t; \xi, \eta) = g(u(t, x; \xi, \eta), y(t; \xi, \eta)), \\
x(0; \xi, \eta) = \xi, \quad y(0; \xi, \eta) = \eta.
\end{cases}
\] (1.2)

As long as the \(C^1\) solution \(u\) of (1.1) exists (actually \(u \in C^4\) due to \(u_0 \in C^4\) and \(F(u), G(u) \in C^5\)), it then follows from (1.1) and (1.2) that along the characteristics \((t, x(t; \xi, \eta), y(t; \xi, \eta))\),
\[
\frac{dt}{du} u(t, x(t; \xi, \eta), y(t; \xi, \eta)) = 0,
\] (1.3)

which derives \(u(t, x(t; \xi, \eta), y(t; \xi, \eta)) \equiv u_0(\xi, \eta)\). In this case, we have that from (1.2)
\[
\begin{cases}
x(t; \xi, \eta) = \xi + tf(u_0(\xi, \eta)), \\
y(t; \xi, \eta) = \eta + tg(u_0(\xi, \eta)).
\end{cases}
\] (1.4)

Obviously, if \(\xi = \xi(t, x, y) \in C^1\) and \(\eta = \eta(t, x, y) \in C^1\) are obtained from (1.4), then \(u(t, x, y) = u_0(\xi(t, x, y), \eta(t, x, y)) \in C^1\) will be the classical solution of (1.1). In fact, in terms of \(\det \frac{\partial(x, y)}{\partial(\xi, \eta)} = 1 + tH(\xi, \eta)\) with
\[
H(\xi, \eta) = \partial_\xi(f(u_0(\xi, \eta)) + \partial_\eta(g(u_0(\xi, \eta))),
\]

by the implicit function theorem \((\xi(t, x, y), \eta(t, x, y)) \in C^1\) can be achieved for all \(t \geq 0\) when \(\min H(\xi, \eta) \geq 0\); or for \(0 \leq t < T^*\) when \(\min H(\xi, \eta) < 0\) and \(T^* = -\frac{1}{\min H(\xi, \eta)}\) since \(\det \frac{\partial(x, y)}{\partial(\xi, \eta)} > 0\) holds in these two cases. For \(\min H(\xi, \eta) < 0\), it follows from Theorem 3.1 of [13] that the \(C^1\) solution \(u\) of (1.1) blows up on \(T^* = -\frac{1}{\min H(\xi, \eta)}\); more precisely, the first order derivatives \(\nabla_{t,x,y} u\) blow up on \(t = T^*\) meanwhile \(u\) itself is still continuous until \(t = T^*\). In the paper, we are concerned with the multidimensional shock formation problem of (1.1) for \(t \geq T^*\) when \(\min H(\xi, \eta) < 0\) happens.

For brevity, we denote \(\phi(\xi, \eta) = f(u_0(\xi, \eta))\) and \(\psi(\xi, \eta) = g(u_0(\xi, \eta))\). Then
\[
H(\xi, \eta) = \partial_\xi \phi(\xi, \eta) + \partial_\eta \psi(\xi, \eta).
\] (1.5)

In addition, we pose the following generic nondegenerate condition on \(H(\xi, \eta)\):

There exists a unique point \((\xi_0, \eta_0)\) such that \(H(\xi_0, \eta_0) = \min_{(\xi, \eta) \in \mathbb{R}^2} H(\xi, \eta), \) and \(\partial_\xi H(\xi_0, \eta_0) = \partial_\eta H(\xi_0, \eta_0) = 0, \) \((\nabla_{\xi, \eta}^2 H)(\xi_0, \eta_0)\) is symmetric positive. (GNC)

Here we point out that (GNC) has been used in [11] to show the blowup of smooth small data solution to the second order quasilinear wave equations when the corresponding null conditions are not fulfilled. For convenience and without loss of generality, we assume that in (GNC),
\[
(\xi_0, \eta_0) = (0, 0) \text{ and } \phi(0, 0) = \psi(0, 0) = 0,
\] (1.6)
\[
H(0, 0) = \min_{(\xi, \eta) \in \mathbb{R}^2} H(\xi, \eta) = -1,
\] (1.7)
\[
\left( \begin{array}{cc}
\partial_\xi^2 H & \partial_\eta^2 H \\
\partial_\eta^2 H & \partial_\eta^2 H
\end{array} \right)(0, 0) = \left( \begin{array}{cc}
6 & 0 \\
0 & 6
\end{array} \right)
\] (1.8)

and
\[
\partial_\eta \psi(0, 0) \geq \frac{1}{2} \geq \partial_\xi \phi(0, 0).
\] (1.9)
In this case, the unique blowup point \((1, 0, 0)\) of \((1.1)\) will appear firstly. Additionally, from \((1.6) - (1.9)\), then there exists a small \(\delta > 0\) such that for \((\xi, \eta) \in B_\delta = \{(\xi, \eta) : |\xi| + |\eta| \leq \delta\}, \)

\[
\partial_\eta \psi \geq -\frac{2}{3}, \quad H \leq -\frac{3}{4}, \quad \text{and} \quad \phi, \psi, \partial_\xi H, \partial_\eta H, \partial^2_{\xi\eta} H = O(\delta), \quad \partial^2_\xi H, \partial^2_\eta H = 6 + O(\delta). \tag{1.10}
\]

Under conditions \((1.6) - (1.9)\), we will prove that for \(t \geq T^*_\star = 1, \) \((1.1)\) admits a shock surface \(\Sigma: x = \varphi(t, y) \in C^2, \) which starts from the space-like blowup curve \(\Gamma: t = T^*_\star(y), \) \(x = x^*(y), \) \(y \in (-\delta, \delta)\) (\(\Gamma\) will be defined in Lemma \(2.1\) below). Denote \(u_-(t, x, y)\) and \(u_+(t, x, y)\) by the solution of \((1.1)\) on the left \((x < \varphi(t))\) and right \((x > \varphi(t))\) side of \(\Sigma\) respectively (see Figure 1). Then \(\varphi(t, y)\) satisfies the Rankine-Hugoniot condition on \(\Sigma:\)

\[
\partial_t \varphi(t, y)[u] - [F(u)] + \partial_\eta \varphi(t, y)[G(u)] = 0, \tag{1.11}
\]

where \([u] = u_+ - u_-\) with \(u_\pm = u_\pm(t, x, y)|_{\Sigma} = u_\pm(t, \varphi(t, y), y).\) Note that the formation of shock \(\Sigma\) is due to the compression of characteristics, then the geometric entropy condition on \(\Sigma\) is

\[
(1, f(u_+), g(u_+)) \cdot (-\partial_\eta \varphi(t, y), 1, -\partial_\eta \varphi(t, y)) < 0 < (1, f(u_-), g(u_-)) \cdot (-\partial_\eta \varphi(t, y), 1, -\partial_\eta \varphi(t, y)). \tag{1.12}
\]

Note that \((1, f(u_\pm), g(u_\pm))\) is just the tangent direction of characteristics \(\gamma_\pm,\) where \(\gamma_\pm\) stands for the right/left characteristics of \((1.1)\) starting from the point \((t, \varphi(t, y), y) \in \Sigma\) (see Figure 2).
The main results in this paper are

**Theorem 1.1.** Under conditions [1.6]-[1.9], for small constants $\varepsilon > 0$ and $\delta > 0$,

1. there exist a space-like blowup curve $\Gamma$ for $t \geq 1$: $t = T^*(y)$, $x = x^*(y)$ with $y \in (-\delta, \delta)$ and $(T^*(0), x^*(0)) = (1, 0)$, and a shock surface $\Sigma = x = \varphi(t, y)$ starting from $\Gamma$ in the domain $\Omega = \{(t, x, y) : 1 \leq t < T^*(y) + \varepsilon, |x| < \delta, |y| < \delta\}$ such that the Rankine-Hugoniot condition [1.11] and the entropy condition [1.12] hold.

2. $x = \varphi(t, y) \in C^2(\Omega)$

and

$u \in C^1(\Omega \setminus \Sigma)$.

3. near $\Gamma$ and $t \in (1 - \varepsilon, 1 + \varepsilon)$,

\[
|u(t, x, y) - u(T^*(y), x^*(y), y)| = O \left( |t - T^*(y)|^{1/2} + |x - x^*(y)| - (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi})(t - T^*(y)) \right),
\]

\[
|\nabla_{t,x,y} u(t, x, y)| = O \left( \left( |t - T^*(y)|^{1/2} + |x - x^*(y)| - (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi})(t - T^*(y)) \right)^{1/2} \right),
\]

\[
|\partial_t u(t, x, y) + (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi}) \partial_x u(t, x, y)| = O \left( \left( |t - T^*(y)|^{1/2} + |x - x^*(y)| - (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi})(t - T^*(y)) \right)^{1/2} \right),
\]

where $\partial_t + (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi}) \partial_x$ is the tangent derivative along the tangent direction $(1, \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi}, 0)$ of $\Sigma$ at the point $(T^*(y), x^*(y), y) \in \Gamma$ with the variable $y$ being fixed and $\phi^* = \phi(\xi(t, x, y), \eta(t, x, y))|_{t=T^*(y), x=x^*(y)}$ (the meanings of $\phi^*, \partial_y \phi^*$ and $\partial_\xi \phi^*$ are the same as $\phi^*$).

### 1.2 Remarks and sketch of proof

**Remark 1.1.** Theorem 1.1 can be extended into the more general multidimensional first order hyperbolic conservation law

\[
\begin{cases}
\partial_t u + \sum_{i=1}^n \partial_i (F_i(u)) = 0, \\
u(0, x) = u_0(x),
\end{cases}
\]

where $x = (x_1, \cdots, x_n)$, $u_0(x) \in C^4(\mathbb{R}^n)$ ($n \geq 2$), and $F_i(u)$ ($1 \leq i \leq n$) is $C^5$ smooth on its argument $u$. If follows from Theorem 3.1 of [18] that the solution $u$ of (1.16) blows up on $T^* = \frac{1}{\min H(\xi)}$ with $H(\xi) = \sum_{i=1}^n \partial_{\xi_i} (F_i(u_0(\xi)))$ as long as $\min H(\xi) < 0$. The corresponding generic nondegenerate condition on $H(\xi)$ is as follows

There exists a unique point $\xi_0 \in \mathbb{R}^n$ such that $H(\xi_0) = \min H(\xi)$, and $\nabla_\xi H(\xi_0) = 0$, $(\nabla_\xi^2 H)(\xi_0)$ is symmetric positive.

**Remark 1.2.** For the 1-D conservation law

\[
\begin{cases}
\partial_t v + \partial_x f(v) = 0, \\
v(0, x) = v_0(x),
\end{cases}
\]

4
where $f(v) \in C^2(\mathbb{R})$ and $v_0(x) \in C^1(\mathbb{R})$. It is well-known that the $C^1$ solution $v$ of (1.17) will blow up at the time $T^* = -\frac{1}{\min g'(x)}$ with $g(x) = f'(v_0(x))$ and $\min_{x \in \mathbb{R}} g'(x) < 0$. If we further assume $g(x) \in L^\infty(\mathbb{R}) \cap C^p(\mathbb{R})$ with $p \geq 4$, and pose the following generic nondegenerate condition:

There exists a unique point $x_0$ such that $g'(x_0) = \min g'(x) < 0, g''(x_0) = 0, g^{(3)}(x_0) > 0$.  

(1.18)

Then by Theorem 2 of [14], a local weak entropy solution $u$ of (1.17) together with the shock curve $x = \varphi(t)$ starting from the blowup point $(T^*, x^* = x_0 + g(x_0)T^*)$ can be locally obtained. Moreover, $\varphi(t) \in C^p(T^*, T^* + \varepsilon) \cap C^2(T^*, T^* + \varepsilon)$, and if $g(x_0) = 0$, then in some neighbourhood of $(T^*, x^*)$,

$$
\begin{align*}
&|v(t, x) - v(T^*, x^*)| \leq C((t - T^*)^3 + (x - x^*)^2)^{\frac{1}{2}}, \\
&|\partial_t v(t, x)| \leq C((t - T^*)^3 + (x - x^*)^2)^{-\frac{1}{2}}, \\
&|\partial_x v(t, x)| \leq C((t - T^*)^3 + (x - x^*)^2)^{-\frac{1}{2}}.
\end{align*}
$$

(1.19)

By comparing (1.19) with (1.13)-(1.14), the descriptions on the singularities of $\partial_x v$ and $\nabla_{x,y} u$, $\partial_t v$ and $\partial_t u + (\phi^* + \psi^* \frac{\partial \phi}{\partial \nu}) \partial_x u$ are analogous. Note that $\partial_x v$ is actually the tangent direction along the shock curve at the blowup point $(T^*, x^*)$, which corresponds to $\partial_x u + (\phi^* + \psi^* \frac{\partial \phi}{\partial \nu}) \partial_x u$ in (1.15). On the other hand, when condition (1.18) is removed, we also study the formation and construction of the shock solution to (1.17) in [23].

Remark 1.3. We point out that the shock formation problem in Theorem 1.1 is different from the usual Riemann problem with the discontinuous shock initial datum on $t = T^*$. For the latter, the initial data $u_{\pm}(T^*, x, y)$ are piecewise smooth on the left/right side of $\Gamma$ and are discontinuous across $\Gamma$ (see Figure 3 and [16, 17, 20]), then the shock solution $u_{\pm}(t, x, y)$ are also piecewise smooth on the left/right side of $\Sigma$ for $t \geq T^*$, where $\Sigma$ is the resulting shock surface starting from $\Gamma$.

![Figure 3. Riemann problem of (1.1) on $T^*$.](image)

However, in Theorem 1.1, the initial datum $u(T^*, x, y)$ is continuous but is not piecewise smooth (even not uniformly Lipschitz continuous on two sides of $\Gamma$). More precisely, the strength of the constructed shock solution $u$ is zero on $\Gamma$ and then gradually increases for $t > T^*$ (see (1.13)-(1.14) of Theorem 1.1).
Remark 1.4. Although the global existence of BV solution $u$ of (1.1) has been early obtained (see [12] or [23]), from the viewpoint of understanding the physical process of the appearance of singularities, it is also an interesting problem to give a clear picture on the generation of singularities from a blowup curve, in particular, that of the singularity of the shock type.

Remark 1.5. When $u_0(x, y) \in L^\infty$, under the entropy condition

$$\partial_t \Phi(u) + \partial_x F_1(u) + \partial_y G_1(u) \leq 0 \quad \text{in the sense of distribution,}$$

(1.20)

where $\Phi$ is any $C^1$ convex function, $F_1(u) = \Phi(u)F'(u)$ and $G_1(u) = \Phi(u)G'(u)$, the global existence of a unique weak solution $u$ of (1.1) has been proved (see Theorem 3.4.3 of [12]). Next we illustrate that (1.20) means the geometric entropy condition (1.12) for our shock formation problem. In fact, by the statements in Pages 44 of [12], near the blowup curve $\Gamma$, the entropy condition (1.20) can be described as follows: Let $\nu = (\nu_0, \nu_1, \nu_2)$ be the normal vector of $\Sigma$, then the function

$$[u_-, u_+] \ni \theta \mapsto \text{sgn}(u_+ - u_-) (F(\theta)\nu_1 + G(\theta)\nu_2)$$

(1.21)

lies above the linear interpolation between its values at $u_\pm$. Without loss of generality, we assume $u_- < u_+$. Then one knows that $\Psi(\theta) \equiv F(\theta)\nu_1 + G(\theta)\nu_2$ is concave in $[u_-, u_+]$. This yields that $\Psi'(\theta) = f(\theta)\nu_1 + g(\theta)\nu_2$ is decreasing in $[u_-, u_+]$. Therefore $f(u_+)\nu_1 + g(u_+)\nu_2 < f(u_-)\nu_1 + g(u_-)\nu_2$. Due to $\nu = (-\partial_\nu \varphi, 1, -\partial_y \varphi)$, then we arrive at

$$f(u_+) - \partial_y \varphi g(u_+) < f(u_-) - \partial_y \varphi g(u_-).$$

(1.22)

In addition, it follows from (1.11) that

$$\partial_\nu \varphi = \frac{\Psi(u_+) - \Psi(u_-)}{u_+ - u_-},$$

(1.23)

which derives that there exists a $\theta_0 \in (u_-, u_+)$ such that $\partial_\nu \varphi = \Psi'(\theta_0)$. Note that $\Psi'(\theta)$ is decreasing in $[u_-, u_+]$. Then

$$f(u_+) - \partial_y \varphi g(u_+) < \partial_y \varphi < f(u_-) - \partial_y \varphi g(u_-),$$

(1.24)

which is equivalent to (1.12).

Remark 1.6. Generally speaking, the descriptions on the singularities of $u$ in (1.13)-(1.15) are optimal. Indeed, if we consider the following problem

$$\begin{cases} \partial_t v + \partial_x (\frac{v^2}{2}) + \partial_y (\frac{v^2}{2}) = 0, \\ v(0, x, y) = -x + x^3, \end{cases}$$

(1.25)

then as in Sec.10 of [14] or Remark 3.1 of [23], the regularity of $\partial_\nu \varphi = O((t - T^*)^3 + (x - x^*)^3)$ with $T^* = 1$ and $x^* = 0$ is optimal.

Now we briefly mention some interesting works on the shock formation and construction for the hyperbolic conservation laws. Under the generic nondegenerate condition of the initial data, for the 1-D scalar conservation law or 1-D $2 \times 2$ system of polytropic gases, the authors in [13]-[14] and [6] obtain the formation and construction of a shock wave starting from the blowup point under some variant assumptions; for the 1-D $3 \times 3$ strictly hyperbolic conservation laws with the small initial data or the 3-D full compressible Euler equations with symmetric structure and small perturbation, the authors in [5], [24] and [9] also get the formation and construction of the resulting shock waves, respectively. From these works, we know that the formation of a shock is caused by the squeeze of characteristics. On the other hand, in recent years, the
study on the blowup and shock formation of smooth solutions to the multidimensional hyperbolic conservation laws or the second order potential equations of polytropic gases have made much progress (see [2]-[4], [7]-[11], [15], [19], [21] and [23]), which illustrate that the formation of the multidimensional shock is due to the compression of the characteristic surfaces. However, the related constructions of multidimensional shock wave after the blowup of smooth solutions are not obtained. In the present paper, we are concerned with the construction of a multidimensional shock wave for the scalar conservation law under the generic nondegenerate condition of the initial data.

In order to prove Theorem 1.1, our focus is to solve the singular and nonlinear first order partial differential equation (1.11) of \( \varphi(t, y) \). The equation (1.11) is actually equivalent to \( \partial_t \varphi + h_1(t, y, \varphi)\partial_y \varphi = h_2(t, y, \varphi) \), where \( h_1(t, y, \varphi) = \int_0^1 g(\theta u_\pm(t, \varphi, y) + (1-\theta)\psi(t, \varphi, y))d\theta \) and \( h_2(t, y, \varphi) = \int_0^1 f(\theta u_\pm(t, \varphi, y) + (1-\theta)\psi(t, \varphi, y))d\theta \). Note that the functions \( h_i(t, y, \varphi) \) \( (i = 1, 2) \) are not Lipschitzian with respect to the variables \( (t, y) \) and the unknown function \( \varphi \) since the first order derivatives of \( \nabla \varphi \) are not Lipschitz and the second order potential equations of polytropic gases have made much progress (see [2]-[4], [7]-[11], [15], [19], [21] and [23]), which illustrate that the formation of the multidimensional shock is due to the compression of the characteristic surfaces. However, the related constructions of multidimensional shock wave after the blowup of smooth solutions are not obtained. In the present paper, we are concerned with the construction of a multidimensional shock wave for the scalar conservation law under the generic nondegenerate condition of the initial data.

Our paper is organized as follows. In Section 2, we give some key analysis on the characteristic surface and determine the blowup curve \( \Gamma \) of equation (1.1) near the blowup point \( (1, 0, 0) \). Then, complete the construction of the shock surface \( \Sigma \). In Section 3, the behaviors of solution \( u \) to problem (1.1) around \( \Gamma \) are given in detail and then Theorem 1.1 is proved.

### 2 Construction of the shock surface \( \Sigma \)

From (1.10) and by the implicit function theorem, it follows from the second equation \( y = \eta + t\psi(\xi, \eta) \) of (1.4) that
\[
\eta = Y(t, \xi, y) \in C^4,
\]
where \( t \in [0, \frac{1}{2}) \), \( \xi \in (-\delta, \delta) \) and \( y \in (-\delta, \delta) \) for sufficiently small \( \delta > 0 \). Meanwhile, it is easy to check that
\[
(\partial_t Y, \partial_\xi Y, \partial_\eta Y) = -\frac{1}{1 + t \partial_\eta \psi} (\psi, t \partial_\xi \psi, -1),
\]
where \( 1 + t \partial_\eta \psi \geq \frac{1}{4} \) for \( t \in [0, \frac{1}{2}) \). Note that all the derivatives of \( Y \) in (2.2) are uniformly bounded and \( \partial_t Y = O(\delta) \) holds when \( t \in [0, \frac{1}{2}), \xi \in (-\delta, \delta) \) and \( y \in (-\delta, \delta) \).

At first, we study the property of the surface \( \Sigma_0 \) generated by
\[
D(t, \xi, Y(t, \xi, y)) \equiv 1 + tH(\xi, Y(t, \xi, y)) = 0,
\]
where \( t \geq T^* = 1 \). Here we point out that the variable \( \xi \) in (2.3) can be actually expressed a function of \( (t, x, y) \) from (1.4) when \( (t, x, y) \in \Sigma_0 \), but \( \xi \) has two different expressions for the left part and the right part of \( \Sigma_0 \) (see Figure 4).

**Lemma 2.1.** The surface \( \Sigma_0 \) is of cusp type, whose edge \( \Gamma \) (blowup curve) is space-like.
Proof. For each fixed $y \in (-\delta, \delta)$ and the function $Y(t, \xi, y)$ defined in (2.1), let
\[ D(t, \xi, Y(t, \xi, y)) = 0. \] (2.4)
Then one has
\[ \partial_t (D(t, \xi, Y(t, \xi, y))) = H(\xi, Y(t, \xi, y)) - t \partial_n H(\xi, Y(t, \xi, y)) \frac{\psi}{1 + \Omega_n \psi}. \] (2.5)
This together with (1.10) yields that for $t \in [0, \frac{7}{6}]$ and $\xi, y \in (-\delta, \delta),$
\[ \partial_t (D(t, \xi, Y(t, \xi, y))) \leq -\frac{3}{4} + \frac{7}{6} \cdot O(\delta) < 0. \] (2.6)
Hence, for $\xi \in (-\delta, \delta)$ and $y \in (-\delta, \delta),$ there exists a unique function $t = t(\xi, y) \in C^1$ (the equation of $\Sigma_0$) satisfying $D(t(\xi, y), \xi, y) \equiv 0$ by the implicit function theorem.

Next we discuss the blowup time $T^*(y) = \min_\xi t(\xi, y)$ of solution $u$ to (1.1) for fixed $y \in (-\delta, \delta).$ Note that
\[ \partial_\xi t H(\xi, Y) + t (\partial_\xi H + \partial_\eta H \cdot (\partial_\xi Y + \partial_\eta \partial_\xi t)) = 0. \] (2.7)
If taking $t_\xi = 0$ in (2.7), we then obtain that the related $\xi$ should satisfy at $t = T^*(y),$
\[ D'(t, \xi, Y(t, \xi, y)) \triangleq \partial_\xi H(\xi, Y(t, \xi, y)) + \partial_\eta H(\xi, Y(t, \xi, y)) \cdot \partial_\xi Y(t, \xi, y) = 0. \] (2.8)
By (1.8), (1.10) and direct computation, the following Jacobian determinant holds that for $\xi, y \in (-\delta, \delta)$ and small $\delta > 0,$
\[
\begin{vmatrix}
\partial_\xi (D(t, \xi, Y(t, \xi, y))) & \partial_\xi (D(t, \xi, Y(t, \xi, y))) \\
\partial_\eta (D(t, \xi, Y(t, \xi, y))) & \partial_\eta (D'(t, \xi, Y(t, \xi, y)))
\end{vmatrix}
\]
\[= \left| \begin{array}{c}
H + t \partial_\eta H \partial_\xi Y \\
\partial_\eta H \partial_\xi Y + \partial_\xi H \partial_\eta Y + \partial_\eta H \partial_\xi^2 Y + \partial_\xi^2 H + 2 \partial_\xi H \partial_\eta Y + \partial_\eta H \partial_\xi Y^2 + \partial_\eta H \partial_\xi^2 Y \\
\end{array} \right| \]
\[= (H + t \partial_\eta H \partial_\xi Y) (\partial_\xi^2 H + 2 \partial_\xi H \partial_\eta Y + \partial_\xi^2 H (\partial_\xi Y)^2 + \partial_\eta H \partial_\xi^2 Y) \]
where \( \theta_0 = \left( \frac{\partial y^*(0,0)}{\partial y^*(0,0)} \right) ^2 \). Thus by the implicit function theorem, we obtain the solution \( (\zeta, \eta) = (\Xi^*(y), Y^*(y)) \) from equations (2.4) and (2.8). Furthermore, we can get the blowup curve \( \Gamma \) with the parameter \( y \) as follows

\[
t = T^*(y) = - \frac{1}{H(\Xi^*(y), Y^*(y))} , \quad x = \Xi^*(y) + T^*(y) \phi(\Xi^*(y), Y^*(y)),
\]

which comes from the first equation of the characteristics (1.4) starting from the initial point \( (t, \xi, \eta) = (0, \Xi^*(y), Y^*(y)) \).

Below we study the properties of surface \( \Sigma_y \). For each \( y \in (-\delta, \delta) \), we have the Taylor's expansion of \( H(\xi, Y(t, \xi, y)) \) at the point \( (t, \xi) = (T^*(y), \Xi^*(y)) \):

\[
H(\xi, Y(t, \xi, y)) = H^* + \partial_{\eta} H^* \partial_Y Y^*(t - T^*(y)) + \frac{1}{2} \left[ \partial^2_{\xi \eta} H^* + 2 \partial^2_{\xi} H^* \partial_Y Y^* + \partial^2_{\eta} H^* (\partial_Y Y^*)^2 + \partial_{\eta} H^* \partial^2_{\xi} Y^* \right] (\xi - \Xi^*(y))^2 + \cdots
\]

and

\[
a^*(y) = \frac{1}{6} \left[ \partial^2_{\xi} H^* + 3 \partial^2_{\xi} \partial_{\eta} H^* \partial_Y Y^* + 3 \partial_{\eta} \partial^2_{\xi} H^* (\partial_Y Y^*)^2 + 2 \partial^2_{\eta} H^* \partial^2_{\xi} Y^* + \partial_{\eta} H^* \partial^3_{\xi} Y^* \right] + \cdots
\]

From now on, the notation \( F^* = F|_{(t, \xi, y) = (T^*(y), \Xi^*(y), Y^*(y))} \) stands for the value of function \( F \) at the point \( (t, \xi, y) = (T^*(y), \Xi^*(y), Y^*(y)) \). In addition, by \( T^*(y) = -\frac{1}{H^*} \), we have

\[
D(t, \xi, Y(t, \xi, y)) = 1 + T^*(y) H(\xi, Y(t, \xi, y)) + (t - T^*(y)) H(\xi, Y(t, \xi, y))
\]

\[
= -D_0(y)(t - T^*(y)) + D_1(y)(\xi - \Xi^*(y))^2 + D_2(y)(t - T^*(y))(\xi - \Xi^*(y)) + D_3(y)(\xi - \Xi^*(y))^3 + O((t - T^*(y))(\xi - \Xi^*(y))^2 + (t - T^*(y))^2 + |\xi - \Xi^*(y)|^3),
\]

where

\[
D_0(y) = - (H^* + T^*(y) \partial_{\eta} H^* \partial_Y Y^*),
\]

\[
D_1(y) = \frac{T^*(y)}{2} \left[ \partial^2_{\xi} H^* + 2 \partial^2_{\xi} \partial_{\eta} H^* \partial_Y Y^* + \partial^2_{\eta} H^* (\partial_Y Y^*)^2 + \partial_{\eta} H^* \partial^2_{\xi} Y^* \right],
\]

\[
D_2(y) = T^*(y) \left[ \partial^2_{\xi} H^* \partial_Y Y^* + \partial^2_{\xi} \partial_{\eta} H^* \partial_Y Y^* + \partial_{\eta} H^* \partial^2_{\xi} Y^* \right],
\]

\[
D_3(y) = a^*(y) T^*(y).
\]
Note that \( H^* = -1 + o(\delta), \) \( T^*(y) = -\frac{1}{\delta^4} = 1 + O(\delta), \) \( \partial_t^2 H^* = 6 + O(\delta), \) \( \partial^2_t H^* = 6 + O(\delta) \) and \( \partial_t H^*, \partial_t Y^*, \partial^2_t Y^* = O(\delta) \) for \( y \in (-\delta, \delta) \) and \( \delta > 0 \) sufficiently small.

To obtain the expansion of \( \xi - \Xi^*(y) \) in (2.14), we consider the equation

\[
h(s, x) = -sx^2 + bsx + cx^3 + O(s^2 + s|x|^2 + x^4) = 0,
\]

where \( a > 0, b, c \in \mathbb{R} \) and \( h \in C^4 \). Taking \( \omega = s^2 \) and \( \sigma = \frac{s}{\sqrt{a}} \), (2.16) is equivalent to

\[
\tilde{h}(\omega, \sigma) \triangleq \frac{h(\omega^2, \omega \sigma)}{\omega^2} = -1 + a\sigma^2 + b\omega + c\omega \sigma^3 + O(\omega^2) = 0,
\]

where \( \tilde{h}(\omega, \sigma) \in C^2 \). Since \( \tilde{h}(0, \pm \frac{1}{\sqrt{a}}) = 0 \) and \( \frac{\partial h}{\partial \sigma} = \pm 2\sqrt{a} \neq 0 \), thus by implicit function theorem we obtain

\[
\sigma_{\pm}(\omega) = \pm \frac{1}{\sqrt{a}} - \frac{ab + c}{2a^2} \omega + O(\omega^2) \in C^2
\]

satisfying \( \tilde{h}(\omega, \sigma_{\pm}(\omega)) = 0 \) for \( \omega \) near 0. Then (2.16) has the two real roots \( x_{\pm}(s) \in C^{1/2} \) for \( s \in [0, \varepsilon) \) satisfying

\[
x_{\pm}(s) = \pm \frac{1}{\sqrt{a}} s^2 - \frac{ab + c}{2a^2} s + O(s^{3/2}).
\]

This yields that for \( (t, \xi, y) \in \Sigma_0 \),

\[
\xi = \Xi_{\pm}(t, y) = \Xi^*(y) \pm A_1(y)(t - T^*(y))^\frac{1}{2} + A_2(y)(t - T^*(y)) + O((t - T^*(y))^{3/2}),
\]

where

\[
A_1(y) = \sqrt{\frac{D_0(y)}{D_1(y)}} = \frac{1}{\sqrt{3 + 3l_0}} + O(\delta), \quad A_2(y) = -\frac{D_1(y)D_2(y) + D_0(y)D_3(y)}{2(D_1(y))^2}.
\]

In addition, by recalling \( x^*(y) = \Xi^*(y) + T^*(y) \phi^* \) and \( T^*(y) (\partial_\xi \phi^* + \partial_\eta \phi^* \partial_\xi Y^*) = -1, \) for \( (t, \xi, y) \in \Sigma_0 \) with \( t \geq T^*(y) \) and \( y \in (-\delta, \delta) \), we can arrive at

\[
\begin{align*}
x_{\pm}(t, y) & \equiv \Xi_{\pm}(t, y) + T^*(y) \phi_{\Xi_{\pm}(t, y), Y(t, \Xi_{\pm}(t, y), y)} + (t - T^*(y)) \phi_{\Xi_{\pm}(t, y), Y(t, \Xi_{\pm}(t, y), y)} \\
& = \Xi_{\pm}(t, y) + T^*(y) \phi^* + T^*(y) \partial_\eta \phi^* \partial_\xi Y^* (t - T^*(y)) + T^*(y) (\partial_\xi \phi^* + \partial_\eta \phi^* \partial_\xi Y^*) (\Xi_{\pm}(t, y) - \Xi^*(y)) \\
& \quad + \frac{T^*(y)}{2} [\partial^2_\xi (\phi_\xi Y(t, \xi, y))]_{\xi = T^*(y), \xi = \Xi^*(y)} (\Xi_{\pm}(t, y) - \Xi^*(y))^2 \\
& \quad + T^*(y) [\partial^2_\xi (\phi_\xi Y(t, \xi, y))]_{\xi = T^*(y), \xi = \Xi^*(y)} (t - T^*(y))(\Xi_{\pm}(t, y) - \Xi^*(y)) \\
& \quad + \frac{T^*(y)}{6} [\partial^3_\xi (\phi_\xi Y(t, \xi, y))]_{\xi = T^*(y), \xi = \Xi^*(y)} (\Xi_{\pm}(t, y) - \Xi^*(y))^3 \\
& \quad + (t - T^*(y)) \phi^* + (t - T^*(y)) (\partial_\xi \phi^* + \partial_\eta \phi^* \partial_\xi Y^*) (\Xi_{\pm}(t, y) - \Xi^*(y)) \\
& \quad + O((t - T^*(y))^2) + (t - T^*(y)) (\Xi_{\pm}(t, y) - \Xi^*(y))^2 + |\Xi_{\pm}(t, y) - \Xi^*(y)|^4 \\
& = x^*(y) + (\phi^* + \psi^* \partial_\xi \phi^*) (t - T^*(y)) + b^*(y)(t - T^*(y))^\frac{1}{2} + O((t - T^*(y))^2).
\end{align*}
\]
Remark 2.1. We now explain the geometric meaning of the coefficient \( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \) of \( t - T^*(y) \) in (1.13)-(1.15). Note that the surface \( \Sigma_0 \) is determined by the parametric equations of \( (\xi, \eta) \)

\[
\begin{align*}
  t &= -\frac{1}{\eta}, \\
  x &= \xi - \frac{1}{\eta} \phi(\xi, \eta), \\
  y &= \eta - \frac{1}{\eta} \psi(\xi, \eta).
\end{align*}
\]

Therefore, we have that for small \( \delta > 0 \),

\[
b^*(y) = -\frac{2}{3 \partial_\xi \phi(0,0) \sqrt{3 + 3 \delta}} + O(\delta) > 0.
\]

This means that \( \Sigma_0 \) is of a cusp-typed surface. Additionally, the edge \( \Gamma \) of \( \Sigma_0 \) is space-like in terms of 2.10. \( \square \)

### Remark 2.1

We now explain the geometric meaning of the coefficient \( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \) of \( t - T^*(y) \) in (1.13)-(1.15). Note that the surface \( \Sigma_0 \) is determined by the parametric equations of \( (\xi, \eta) \):

\[
\begin{align*}
  t &= -\frac{1}{\eta}, \\
  x &= \xi - \frac{1}{\eta} \phi(\xi, \eta), \\
  y &= \eta - \frac{1}{\eta} \psi(\xi, \eta).
\end{align*}
\]
Thus its normal vector is

\[
(\partial_t t, \partial_t x, \partial_t y) \times (\partial_y t, \partial_y x, \partial_y y) = \frac{1}{H^2} (\partial_\xi H (\phi \partial_\xi \phi + \psi \partial_\psi \phi) + \partial_\eta H (\phi \partial_\xi \psi + \psi \partial_\psi \phi) - (\partial_\xi H \partial_\xi \phi + \partial_\eta H \partial_\xi \psi), -(\partial_\xi H \partial_\eta \phi + \partial_\eta H \partial_\xi \psi) ).
\]  

(2.32)

Thus for fixed \( y \in (-\delta, \delta) \), one has a tangent direct of \( \Sigma_0 \) as follows

\[
\frac{dx}{dt} = \frac{\partial_\xi H (\phi \partial_\xi \phi + \psi \partial_\psi \phi) + \partial_\eta H (\phi \partial_\xi \psi + \psi \partial_\psi \phi)}{\partial_\xi H \partial_\xi \phi + \partial_\eta H \partial_\xi \psi} = \phi + \frac{\psi (\partial_\xi H \partial_\eta \phi + \partial_\eta H \partial_\xi \psi)}{\partial_\xi H \partial_\xi \phi + \partial_\eta H \partial_\xi \psi}.
\]  

(2.33)

Due to \( \partial_\xi \phi < 0 \), then

\[
\frac{\partial_\xi H \partial_\eta \phi + \partial_\eta H \partial_\xi \psi}{\partial_\xi H \partial_\xi \phi + \partial_\eta H \partial_\xi \psi} = \frac{\partial_\xi (\partial_\xi H \partial_\eta \phi + \partial_\eta H \partial_\xi \psi)}{\partial_\xi H \partial_\xi \phi + \partial_\eta H \partial_\xi \psi} = \frac{\partial_\eta \phi}{\partial_\xi \phi}.
\]  

(2.34)

Substituting this into (2.33) yields

\[
\frac{dx}{dt} = \phi + \psi \frac{\partial_\eta \phi}{\partial_\xi \phi},
\]  

(2.35)

which means that \( \phi^* + \psi^* \frac{\partial_\eta \phi^*}{\partial_\xi \phi^*} \) is just the tangent direction of \( \Gamma \) at the point \( (T^*(y), x^*(y), y) \) for the fixed \( y \in (-\delta, \delta) \).

We rewrite the first equation of the characteristics \([1,4]\) starting from the point \( (T^*(y), \Xi^*(y), Y^*(y)) \) as

\[
x - x^* = (\xi - \Xi^*(y)) + t \phi(\xi, Y(t, \xi, y) - T^*(y)\phi(\Xi^*(y), Y^*(y)).
\]  

(2.36)

For the fixed \( y \in (-\delta, \delta) \) and \( t > T^*(y) \), we introduce the following transformation

\[
s = (t - T^*(y))^2, \quad \mu = \frac{\xi - \Xi^*(y)}{s}, \quad \lambda = \frac{x - x^*(y) - (\phi^* + \psi^* \frac{\partial_\eta \phi^*}{\partial_\xi \phi^*})(t - T^*(y))}{s^3}.
\]  

(2.37)

Lemma 2.2. For fixed \( y \in (-\delta, \delta) \), \( t > T^*(y) \), and small \( \varepsilon > 0 \), when \( s \in [0, \varepsilon) \) and \( |\lambda| < \varepsilon \), there exist two real roots for the characteristics equation \([1,4]\) with the following expansions

\[
\xi_+(t, x, y) = \Xi^*(y) + s \left( \frac{c_1(y)}{c_2(y)} + \frac{\lambda}{2c_1(y)} \right) + O(s^2 + s^3\lambda^2),
\]  

(2.38)

\[
\xi_-(t, x, y) = \Xi^*(y) + s \left( -\frac{c_1(y)}{c_2(y)} + \frac{\lambda}{2c_1(y)} \right) + O(s^2 + s\lambda^2),
\]  

(2.39)

where \( c_1(y) \) and \( c_2(y) \) are some suitable positive functions. Furthermore, \( s \to s^\alpha (\xi_\pm - \Xi^*(y)) \) is of \( C^{2+\alpha} \) for \( \alpha = -1, 0, 1 \).
Proof. At first, we have that for fixed $y \in (-\delta, \delta)$,

$$t\phi(\xi, Y(t, \xi, y)) - T^*(y)\phi(\Xi^*(y), Y^*(y))$$

$$= (\phi^* + T^*(y)\partial_\eta\phi^*\partial_Y^*)(t - T^*(y)) + T^*(y)(\partial_\eta\phi^* + \partial_\eta\phi^*\partial_\xi Y^*)(\xi - \Xi^*(y)) + t\phi(\xi, Y(t, \xi, y))$$

$$+ \frac{T^*(y)}{2} [\partial_\xi^2 (\phi(\xi, Y(t, \xi, y)))_{t=T^*(y), \xi=\Xi^*(y)} (\xi - \Xi^*(y))^2 + (t - T^*(y))(\xi - \Xi^*(y))$$

$$+ O ((t - T^*(y))^2 + (t - T^*(y)))\xi - \Xi^*(y)^2 + |\xi - \Xi^*(y)|^4) + \frac{T^*(y)}{6} [\partial_\xi^3 (\phi(\xi, Y(t, \xi, y)))_{t=T^*(y), \xi=\Xi^*(y)} (\xi - \Xi^*(y))^3 + O ((t - T^*(y))^2 + (t - T^*(y)))\xi - \Xi^*(y)^2 + |\xi - \Xi^*(y)|^4).$$

Similarly to (2.28) and (2.29), we can arrive at

$$c_1(y) \triangleq - \left( T^*(y) \left[ \partial_\xi^2 (\phi(\xi, Y(t, \xi, y)))_{t=T^*(y), \xi=\Xi^*(y)} + (\partial_\eta\phi^* + \partial_\eta\phi^*\partial_\xi Y^*) \right] - \frac{1}{\partial_\xi\phi(0, 0)} + O(\delta) > 0$$

(2.41)

and

$$c_2(y) \triangleq T^*(y) \partial_\xi^3 (\phi(\xi, Y(t, \xi, y)))_{t=T^*(y), \xi=\Xi^*(y)} = - \frac{1}{\partial_\xi\phi(0, 0)} + O(\delta) > 0.$$ 

(2.42)

In this case, (2.36) can be rewritten as

$$G(s, \lambda, \mu) \triangleq -c_1(y)\mu + c_2(y)\mu^3 + O(s^2 + s\mu^2) = \lambda = 0.$$ 

(2.43)

By the implicit function theorem near the point $(s, \mu, \lambda) = (0, \pm \sqrt{\frac{c_1(y)}{c_2(y)}}), \lambda = |\lambda| < \varepsilon$ and $y \in (-\delta, \delta)$, there exist two real roots $\mu_\pm(s, \lambda)$ of (2.43) to fulfill

$$\mu_\pm(s, \lambda) = \pm \sqrt{\frac{c_1(y)}{c_2(y)}} + \frac{\lambda}{2c_1(y)} + O(s + \lambda^2).$$

(2.44)

Then (2.38) and (2.39) are proved.

**Remark 2.2.** Although there are three real roots for the equation (2.36) of $\xi$, which are denoted by $\xi_+ > \xi_c > \xi_-$, where $\xi_\pm$ are related to $\mu_\pm|_{s=0} = \pm \sqrt{\frac{c_2(y)}{c_1(y)}}$ and $\xi_c$ is related to $\mu_c|_{s=0} = 0$, the root $\xi_c$ is not utilized in the paper.

We denote the domain $\Omega_1$ formed by the cusp surface $\Sigma_1$:

$$\left( x - x^*(y) - (\phi^* + \psi^*\partial_\xi\phi^*)(t - T^*(y)) \right)^2 = \varepsilon^2(t - T^*(y))^2,$$

(2.45)
where \( t \in [T^*(y), T^*(y) + \varepsilon] \) and \( y \in (-\delta, \delta) \). It is easy to check \( \Omega_1 \subset \Omega_0 \) with the cusp domain \( \Omega_0 \), where \( \Omega_0 \) is bounded by \( \Sigma_0 \). Next we construct the shock surface \( \Sigma \) in domain \( \Omega_1 \).

It follows from \((1.11)\) that
\[
\begin{aligned}
\partial_t \varphi(t, y) + \left[ \frac{G(u)}{u} \right] \partial_y \varphi(t, y) &= \left[ \frac{F(u)}{u} \right], \\
\varphi(T^*(y), y) = x^*(y) = \Xi^*(y) + T^*(y) \phi(\Xi^*(y), Y^*(y)).
\end{aligned}
\tag{2.46}
\]

**Lemma 2.3.** Under conditions \((1.6) - (1.9)\), for small \( \varepsilon > 0 \), the solution \( x = \varphi(t, y) \) of \((2.46)\) exists uniquely in the domain \( \{(t, y) : 0 \leq t - T^*(y) < \varepsilon, \ y \in (\frac{-\delta}{2}, \frac{\delta}{2})\} \) and satisfies that
1. \( \varphi(t, y) \) is a \( C^2 \) function on the variables \((s, y)\), where \( s \in [0, \varepsilon) \) and \( y \in (\frac{-\delta}{2}, \frac{\delta}{2})\);
2. \( \varphi(t, y) \in C^2 \left( \{(t, y) : 0 \leq t - T^*(y) < \varepsilon, \ y \in (\frac{-\delta}{2}, \frac{\delta}{2})\} \right) \) admits the expansion
   \[
   \varphi(t, y) = x^*(y) + (\phi^* + \psi^* \partial_y \psi^*) (t - T^*(y)) + O((t - T^*(y))^2);
   \tag{2.47}
   \]
3. the entropy condition \((1.12)\) holds.

**Proof.** At first, we derive the asymptotic properties of \( \frac{[F(u)]}{[u]} \) and \( \frac{[G(u)]}{[u]} \) near \( \Gamma \). It follows from direct computation that
\[
\frac{[F(u)]}{[u]} = \frac{1}{2} \left[ \phi(\xi_+, Y(t, \xi_+, y)) + \phi(\xi_-, Y(t, \xi_-, y)) \right] + O \left( (\xi_+ - \Xi^*(y))^2 + (\xi_- - \Xi^*(y))^2 \right)
= \phi^* + \frac{1}{2} (\partial_\xi \phi^* + \partial_y \psi^* \partial_\xi Y^*) \left( (\xi_+ - \Xi^*(y)) + (\xi_- - \Xi^*(y)) \right) + O \left( (\xi_+ - \Xi^*(y))^2 + (\xi_- - \Xi^*(y))^2 \right)
= \phi^* - \frac{1}{T^*(y)} \frac{s \lambda}{2 c_1(y)} + O(s^2 + s^2) \tag{2.48}
\]
and
\[
\frac{[G(u)]}{[u]} = \psi^* + \frac{1}{2} (\partial_\xi \psi^* + \partial_y \psi^* \partial_\xi Y^*) \left( (\xi_+ - \Xi^*(y)) + (\xi_- - \Xi^*(y)) \right) + O \left( (\xi_+ - \Xi^*(y))^2 + (\xi_- - \Xi^*(y))^2 \right)
= \psi^* - \frac{\partial_\xi \psi^*}{T^*(y) \partial_y \psi^*} \frac{s \lambda}{2 c_1(y)} + O(s^2 + s^2). \tag{2.49}
\]

In addition, by \((2.8)\) and \((2.9)\), we deduce
\[
\frac{d}{dy} T^*(y) = - \frac{T^*(y) \partial_t H^* \partial_y Y^*}{H^* + T^*(y) \partial_t H^* \partial_y Y^*}. \tag{2.50}
\]
Due to \( s = (t - T^*(y))^\frac{1}{2} \), we then have
\[
\partial_t s = \frac{1}{2s}, \quad \partial_y s = - \frac{1}{2} \frac{d}{dy} T^*(y) = \frac{T^*(y) \partial_t H^* \partial_y Y^*}{2 s (H^* + T^*(y) \partial_t H^* \partial_y Y^*)}. \tag{2.51}
\]
Thus the equation in \((2.46)\) becomes
\[
\frac{1}{2s} \left( 1 - \frac{G(u)}{[u]} \frac{d}{dy} T^*(y) \right) \partial_\nu \varphi + \frac{G(u)}{[u]} \partial_y \varphi = \frac{F(u)}{[u]} \tag{2.52}
\]
Set $\lambda = \frac{\varphi - x^*(y) - (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi}) (t - T^*(y))}{s^3}$. Then $\varphi = \Xi^*(y) + T^*(y) \phi^* + (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi}) s^2 + s^3 \lambda$, and we have that

$$\partial_s \varphi = s^3 \partial_s \lambda + 3s^2 \lambda + 2s \left( \phi^* + \frac{\partial \phi^*}{\partial \xi} \psi^* \right)$$

(2.53)

and

$$\partial_y \varphi = s^3 \partial_y \lambda + \frac{d}{dy} (\Xi^* + T^*(y) \phi^*) + s^2 \frac{d}{dy} \left( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \psi^* \right).$$

(2.54)

Thus (2.46) becomes

$$\begin{cases}
    sC_0(s, \lambda, y) \partial_s \lambda + s^2 C_1(s, \lambda, y) \partial_y \lambda = C_2(s, \lambda, y), \\
    \lambda(0, y) = 0,
\end{cases}$$

(2.55)

where

$$C_0 = \frac{1}{2} \left[ 1 - \frac{[G(u)]}{[u]} \frac{d}{dy} T^*(y) \right] = \frac{1}{2} + O(y),$$

$$C_1 = \frac{[G(u)]}{[u]} = O(y + s^2 + s \lambda),$$

$$C_2 = -\frac{3 \lambda}{2} \left[ 1 - \frac{[G(u)]}{[u]} \frac{d}{dy} T^*(y) \right] - s \frac{[G(u)]}{[u]} \frac{d}{dy} \left( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \psi^* \right) - \frac{1}{s} \left[ \frac{[F(u)]}{[u]} - \left( 1 - \frac{[G(u)]}{[u]} \frac{d}{dy} T^*(y) \right) \cdot \left( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \psi^* \right) - \frac{[G(u)]}{[u]} (\Xi^* + T^*(y) \phi^*) \right]$$

$$= \lambda \left\{ \frac{3}{2} \left( 1 - \psi^* \frac{d}{dy} T^*(y) \right) - \frac{1}{2 T^*(y) c_1(y)} - \frac{\partial \xi \psi^*}{2 T^*(y) \phi^* c_1(y)} - \frac{d}{dy} T^*(y) \left( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \psi^* \right) \right\}$$

$$+ \frac{1}{s} \left\{ \phi^* - \left( 1 - \psi^* \frac{d}{dy} T^*(y) \right) \cdot \left( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \psi^* \right) - \psi^* \frac{d}{dy} (\Xi^* + T^*(y) \phi^*) \right\}. \quad (2.56)$$

Due to $y \equiv Y^*(y) + T^*(y) \psi(\Xi^*(y), Y^*(y))$, direct computation yields

$$\phi^* - \left( 1 - \psi^* \frac{d}{dy} T^*(y) \right) \cdot \left( \phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi} \psi^* \right) - \psi^* \frac{d}{dy} (\Xi^* + T^*(y) \phi^*)$$

$$= \psi^* \left\{ -\frac{\partial \phi^*}{\partial \xi} + \psi^* \frac{\partial \phi^*}{\partial \xi} \frac{d}{dy} T^*(y) - \frac{d}{dy} \Xi^*(y) - T^*(y) \left[ \frac{\partial \xi \phi^*}{\partial \xi} \Xi^*(y) + \frac{\partial \phi^*}{\partial y} \right] \right\}$$

$$= \psi^* \left\{ -\frac{\partial \phi^*}{\partial \xi} + \psi^* \frac{\partial \phi^*}{\partial \xi} \frac{d}{dy} T^*(y) + T^*(y) \psi^* \frac{d}{dy} \Xi^*(y) - T^*(y) \partial \phi^* \frac{d}{dy} Y^* \right\}$$

$$= \frac{\partial \phi^*}{\partial \xi} \psi^* \left\{ -1 + \psi^* \frac{d}{dy} T^*(y) + T^* \xi \psi^* \frac{d}{dy} \Xi^*(y) + T^*(y) \partial \psi^* \frac{d}{dy} Y^* + \frac{d}{dy} Y^* \right\}.$$ 

$$= 0. \quad (2.57)$$

Consequently,

$$C_2 = \left( -\frac{3}{2} - \frac{1}{2 c_1} + O(y) \right) \lambda + O(s + \lambda^2), \quad (2.58)$$

where $c_1 = c_1(0) = -\frac{1}{\partial \lambda(0, 0)}$. 

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We define the characteristics of \(2.55\) starting from the point \((0, \beta)\) with \(\beta \in (-\delta, \delta)\) as follows

\[
\begin{aligned}
\frac{dx}{dt} = y(s; \beta) = s \frac{\partial s}{\partial \beta}(s, \lambda(s, y(s; \beta)), y(s; \beta)), \\
y(0; \beta) = \beta.
\end{aligned}
\]  

(2.59)

Along this characteristics, set \(\Lambda(s; \beta) = \lambda(s, y(s; \beta))\), we then have

\[
\begin{aligned}
\frac{d}{ds}\Lambda(s; \beta) &= \frac{1}{2} \frac{\partial s}{\partial \beta}(s, \Lambda(s; \beta), y(s; \beta)), \\
\Lambda(0; \beta) &= 0.
\end{aligned}
\]  

(2.60)

Collecting (2.59) and (2.60), we obtain a nonlinear ordinary differential equation system of \((y(s; \beta), \Lambda(s; \beta))\) as follows

\[
\begin{aligned}
\frac{d}{ds}y(s; \beta) &= s \frac{\partial s}{\partial \beta}(s, \Lambda(s; \beta), y(s; \beta)), \\
s \frac{d}{ds}\Lambda(s; \beta) &= \frac{\partial s}{\partial \beta}(s, \Lambda(s; \beta), y(s; \beta)), \\
y(0; \beta) &= \beta, \\
\Lambda(0; \beta) &= 0,
\end{aligned}
\]  

(2.61)

where \(\frac{\partial s}{\partial \beta}(s, \Lambda, y), \frac{\partial s}{\partial \beta}(s, \Lambda, y) \in C^2\). By Lemma 4.1 in Appendix, (2.61) has a unique solution \((y(s; \beta), \Lambda(s; \beta)) \in C^2([0, \epsilon) \times (-\delta, \delta), C^2)\), moreover, \(\frac{\partial s}{\partial \beta}(s, \beta) \leq \frac{\delta}{2}\) holds for \(s, \beta\in[0, \epsilon)\) and \(C^2([0, \epsilon) \times (-\delta, \delta))\) by the implicit function theorem and then \(\Lambda(s, \beta(s; y)) \in C^2([0, \epsilon) \times (-\frac{\delta}{2}, \frac{\delta}{2}))\). In addition, it follows from (2.60) that \(\Lambda(s, \beta(s; y)) = O(s)\) holds. Therefore,

\[
x = \varphi(t, y) = s^3 \Lambda(s, \beta(s; y)) + x^*(y) + s^2(\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi^*}) \in C^2\left([0, \epsilon) : 0 \leq t - T^*(y) < \epsilon, y \in (-\frac{\delta}{2}, \frac{\delta}{2})\right)
\]

and then the expansion (2.47) is proved.

At last we verify the entropy condition (1.12). At the point \(P = (t, \varphi(t, y), y)\), the normal vector of \(\Sigma\) is \(\langle -\partial_t \varphi, 1, -\partial_y \varphi \rangle\) and the related two characteristic vectors are \((1, \phi(\xi, Y(t, \xi, y)), \psi(\xi, Y(t, \xi, y)))\), where \(\xi = \xi_\pm(t, \varphi(t, y), y)\) (see Figure 5 below). Without loss of generality, \(\xi_- < \xi_+\) is assumed. Since the Rankine-Hugoniot condition on \(\Sigma\) is

\[
\begin{cases}
[F(u)]_{u_1} = 0, \\
[\frac{G(u)}{u_1}]_{u_1} = 0,
\end{cases}
\]

(2.62)

then for fixed \(t\) and \(y\), there is a \(\xi \in [\xi_-, \xi_+]\) such that

\[
\partial_t \varphi = \phi(\xi, Y(t, \xi, y)) - \psi(\xi, Y(t, \xi, y)) \cdot \partial_y \varphi.
\]  

(2.63)

Next we prove that \(F(\xi) \equiv \phi(\xi, Y(t, \xi, y)) - \psi(\xi, Y(t, \xi, y)) \cdot \partial_y \varphi\) is decreasing for \(\xi \in [\xi_-, \xi_+]\) when the fixed \(t \in (T^*(y), T^*(y) + \epsilon]\) and \(y \in (-\delta, \delta)\). Indeed, Note that

\[
F'(\xi) = \frac{\partial \xi \phi}{1 + t \partial_y \psi} - \frac{\partial \xi \psi}{1 + t \partial_y \psi} \cdot \partial_y \varphi.
\]  

(2.64)

In addition, by (2.32) we have

\[
\partial_y \varphi(t, y) = -\frac{\partial \xi \phi}{1 + T^*(y) \partial_y \psi} - \frac{\partial \xi \psi}{1 + T^*(y) \partial_y \psi} \cdot \frac{\partial \xi \phi}{\partial \xi \phi^*} + O(s) = -\frac{1}{1 + T^*(y) \partial_y \psi^*} + O(s)
\]  

(2.65)

Thus by \(H^* = \partial \xi \phi^* + \partial \xi \psi^* = -1 + O(y)\), we obtain that for \(t \in [T^*(y), T^*(y) + \epsilon]\) and \(y \in (-\delta, \delta)\),

\[
F'(\xi) = \frac{\partial \xi \phi}{1 + T^*(y) \partial_y \psi} + \frac{\partial \xi \psi}{1 + T^*(y) \partial_y \psi} \cdot \frac{\partial \xi \phi}{\partial \xi \phi^*} + O(s) = -\frac{1}{1 + T^*(y) \partial_y \psi^*} + O(s + |y|) < 0.
\]  

(2.66)
This means that $F(\xi)$ is decreasing on the variable $\xi \in [\xi_-, \xi_+]$ and then $F(\xi_+) < F(\xi) < F(\xi_-)$ holds. This together with (2.63) yields that on $\Sigma$

$$\phi(\xi_+, Y(t, \xi_+, y)) - \psi(\xi_+, Y(t, \xi_+, y)) \cdot \partial_y \varphi < \phi(\xi_-, Y(t, \xi_-, y)) - \psi(\xi_-, Y(t, \xi_-, y)) \cdot \partial_y \varphi,$$

which derives the entropy condition (1.12).

\[ \square \]

\textbf{Figure 5.}

3 Behavior of the solution $u$ near $\Gamma$ and proof of Theorem 1.1.

It is easy to see that the map $(\xi, \eta) \to (x(t, \xi, \eta), y(t, \xi, \eta))$ in (1.4) is invertible outside the cusp domain $\Omega_0$. Moreover, for $t \in [T^*(y), T^*(y) + \varepsilon]$ and $y \in (-\delta, \delta)$, there exists a positive constant $\alpha_0(\delta)$ depending on $\delta$ such that for $x > x^*(y) + (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi})(t - T^*(y)) + \left(\frac{2}{3\delta^0(0,0) \sqrt{3 + 3\delta_0^2}} + \alpha_0(\delta)\right)(t - T^*(y))^2$, there exists a $\xi_+(t, x, y)$ satisfying (1.4); and for $x < x^*(y) + (\phi^* + \psi^* \frac{\partial \phi^*}{\partial \xi})(t - T^*(y)) + \left(-\frac{2}{3\delta^0(0,0) \sqrt{3 + 3\delta_0^2}} + \alpha_0(\delta)\right)(t - T^*(y))^2$, there exists a $\xi_-(t, x, y)$ satisfying (1.4). From (2.36), we start to improve Lemma 2.2 so that the better asymptotic behaviors of $\xi_\pm(t, x, y)$ are obtained.

\textbf{Lemma 3.1.} For each number $c > -\frac{2}{3\delta^0(0,0) \sqrt{3 + 3\delta_0^2}} + \alpha_0(\delta)$, there exist a constant $\varepsilon(c) > 0$ such that for $s \in [0, \varepsilon(c))$ and $|\lambda - c| < \varepsilon(c)$, $(s, \lambda) \to \xi_+$ satisfies

$$\xi_+(t, x, y) = \Xi^*(y) + s \left(\mu_c(y) + \frac{\lambda - c}{-c_1(y) + 3c_2\mu^2(y)}\right) + O(s^2 + s|\lambda - c|^2),$$

and for $s \in [0, \varepsilon(c))$ and $|\lambda + c| < \varepsilon(c)$, $(s, \lambda) \to \xi_-$ satisfies

$$\xi_-(t, x, y) = \Xi^*(y) + s \left(-\mu_c(y) + \frac{\lambda + c}{-c_1(y) + 3c_2\mu^2(y)}\right) + O(s^2 + s|\lambda + c|^2),$$

(3.1) and (3.2)
where $\mu_c(y)$ is the solution of the following algebraic equation for $y \in (-\delta, \delta)$

$$G_0(\mu; y) = -c_1(y)\mu + c_2(y)\mu^3 = c.$$  \tag{3.3}

**Proof.** Note that for $c > -\frac{2\sqrt{\Delta_{y_0}(0,y)}\sqrt{c_3(y)}}{a_0(\delta)}$, there is a real root $\mu_c(y) > \frac{1}{\sqrt{c_3(y)}}$ for (3.3). On the other hand, by (2.43), one can arrive at

$$\partial_\mu G(0, \pm c, \pm \mu_c(y)) = -c_1(y) + 3c_2(y)\mu_c^2(y) > 0,$$  \tag{3.4}

$$\partial_\mu G(0, \pm c, \pm \mu_c(y)) = -1.$$  \tag{3.5}

Similarly to the proof of Lemma 2.2 by implicit function theorem, we can obtain (3.1) and (3.2).

**Remark 3.1.** We point out that if $c \geq 0$, then $\mu_c(y) \geq \sqrt{\frac{c_1(y)}{c_2(y)}}\frac{1}{\sqrt{1+a_0(\delta)}} + O(\delta) > 0$ and $-c_1(y) + 3c_2(y)\mu^2(y) \geq c_0 > 0$ hold for some constant $c_0 > 0$ when $\delta > 0$ is sufficiently small.

To study the asymptotic behaviors of $\xi_\pm$ near $x = x^*(y)$, we now take the following transformation

$$\zeta = \left(x - x^*(y) - (\phi^* + \psi^* \partial_y \phi^*) (t - T^*(y))\right)^\frac{1}{3}, \quad \eta = \frac{t - T^*(y)}{\zeta}, \quad \nu = \frac{\xi - \Xi^*(y)}{\zeta}.$$  \tag{3.6}

By dividing the factor $\zeta^3$ for (2.36), we have

$$H(\eta, \zeta, \nu) \triangleq -c_1(\eta)\nu + c_2(\nu)^3 + O(\eta^2\zeta + \zeta^2\eta + \zeta^4\nu^2) - 1 = 0.$$  \tag{3.7}

Note that for $|\eta|$ small enough, there is a unique real root $\nu$ for (3.7). Furthermore, we have

**Lemma 3.2.** There exists a constant $\varepsilon > 0$ such that for $(\eta, \zeta) \in \{|\eta| \leq \varepsilon, |\zeta| < \varepsilon\}$, the expansion of $\xi(t, x, y)$ on $(\eta, \zeta)$ admits

$$\xi(t, x, y) = \Xi^*(y) + \zeta \left(\frac{1}{\sqrt{c_2(y)}} + \frac{c_1(y)}{3(c_2(y))^{\frac{3}{2}}}\eta\right) + O(\zeta^2 + \zeta^2\eta^2),$$  \tag{3.8}

where $y \in (-\delta, \delta)$.

**Proof.** For $\eta = 0$ and $\zeta = 0$, we have $\nu = \frac{1}{\sqrt{c_2(y)}}$. Due to

$$\partial_\nu H(0, 0, \frac{1}{\sqrt{c_2(y)}}) = 3\sqrt{c_2(y)} > 0,$$  \tag{3.9}

$$\partial_\zeta H(0, 0, \frac{1}{\sqrt{c_2(y)}}) = \frac{c_1(y)}{\sqrt{c_2(y)}},$$  \tag{3.10}

we then have that for $|\eta| \leq \varepsilon$ and $|\zeta| < \varepsilon$ with $\varepsilon$ being small,

$$\nu = \frac{1}{\sqrt{c_2(y)}} + \frac{c_1(y)}{3(c_2(y))^{\frac{3}{2}}}\eta + O(\zeta + \eta^2),$$  \tag{3.11}

which derives (3.8). \qed
Next we consider the behavior of $\xi(t, x, y)$ near $\Gamma$ for $t < T^*(y)$. Without of confusion to 2.37, we still denote the transformation

$$s = (T^*(y) - t)^{\frac{1}{2}}, \quad \mu = \frac{x - T^*(y)}{s}, \quad \lambda = \frac{(\phi^* + T^*(y)\partial y\phi^*\partial Y^*)(t - T^*(y))}{s^3}. \quad (3.12)$$

Similarly to (2.43), (2.36) becomes

$$\tilde{G}(s, \lambda, \mu) \triangleq c_1(\mu) + c_2(y)\mu^2 + O(s^2 + s\mu^2) - \lambda = 0. \quad (3.13)$$

Then we have

**Lemma 3.3.** For each $c \in \mathbb{R}$, there exists a positive constant $\varepsilon(c)$ such that for $s \in [0, \varepsilon(c))$ and $|\lambda - c| < \varepsilon(c)$, $(s, \lambda) \to \xi$ satisfies

$$\xi(t, x, y) = \Xi^*(y) + s\left(\tilde{\mu}_c(y) + \frac{\lambda - c}{c_1(y) + 3c_2\tilde{\mu}_c^2(y)}\right) + O(s^2 + |s\lambda - c|^2), \quad (3.14)$$

where $y \in (-\delta, \delta)$, and $\tilde{\mu}_c(y)$ is the real root of

$$\tilde{G}_0(\mu, y) = c_1(\mu) + c_2(y)\mu^2 = c. \quad (3.15)$$

**Proof.** It is easy to check that for $c \in \mathbb{R}$, there is a real root $\tilde{\mu}_c(y)$ of (3.15). On the other hand, by (3.13), one has that

$$\partial_\mu \tilde{G}(0, \pm c, \tilde{\mu}_c(y)) = c_1(y) + 3c_2(y)\tilde{\mu}_c^2(y) > 0, \quad (3.16)$$

$$\partial_x \tilde{G}(0, \pm c, \tilde{\mu}_c(y)) = -1. \quad (3.17)$$

Similarly to the proof of Lemma 2.2 by implicit function theorem, we then obtain (3.14). \qed

**Remark 3.2.** It is obvious that for $y \in (-\delta, \delta)$, $c_1(y) + 3c_2(y)\tilde{\mu}_c^2(y) \geq -\frac{1}{O_0(0, 0)} + O(\delta) > 0$ holds.

**Proof of Theorem 1.1 (3).**

**Proof.** We now establish the behavior of the solution $u$ and its derivatives near $\Gamma$. Denote $B_{\rho} = \{(t, x, y) : |x - x^*(y)| < \rho, |t - T^*(y)| < \rho, |y| < \rho\}$ for some positive constant $\rho$ defined below. Let $\varepsilon > 0$ be the constant obtained in Lemma 3.2. Set

$$\Omega_{x} = B \cap \{(t, x, y) : |x - x^*(y)| < \varepsilon, |t - T^*(y)| < \varepsilon |x - x^*(y)|^\frac{1}{2}, y \in (-\delta, \delta)\}, \quad (3.18)$$

where $x^*(y) = x^*(y) + (\phi^* + \psi^* \frac{\partial \phi^*}{\partial y})t - T^*(y)$. In addition, let

$$\Omega_0 = B \cap \{(t, x, y) : t < T^*(y), |x - x^*(t, y)| < \frac{2}{\varepsilon^\frac{3}{2}}(T^*(y) - t)^{\frac{1}{2}}, y \in (-\delta, \delta)\}, \quad (3.19)$$

$$\Omega_{t,0} = B \cap \{(t, x, y) : 0 < t < T^*(y), |x - x^*(t, y)| < \varepsilon, |t - T^*(y)| < \varepsilon |x - x^*(t, y)|^\frac{1}{2}, y \in (-\delta, \delta)\}, \quad (3.20)$$

$$\Omega_{t,+} = B \cap \{(t, x, y) : t > T^*(y), \frac{\varepsilon}{2} < \frac{x - x^*(t, y)}{(T^*(y) - t)^{\frac{1}{2}}} < \frac{2}{\varepsilon^{\frac{3}{2}}}, y \in (-\delta, \delta)\}, \quad (3.21)$$

$$\Omega_{t,-} = B \cap \{(t, x, y) : t > T^*(y), -\frac{2}{\varepsilon^{\frac{3}{2}}} < \frac{x - x^*(t, y)}{(T^*(y) - t)^{\frac{1}{2}}} < -\frac{\varepsilon}{2}, y \in (-\delta, \delta)\}. \quad (3.22)$$

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where $\epsilon$ is the same as in $\text{(3.18)}$, and $\epsilon$ satisfies the requirements in Lemma 2.2. By Heine-Borel property of compactness, one can choose \( \{c_{j,\pm}, \epsilon_{j,\pm} = \epsilon_{j,\pm}(c_{j,\pm})\}_{j=1}^{n} \) and \( \{c_{j,0}, \epsilon_{j,0} = \epsilon_{j,0}(c_{j,0})\}_{j=1}^{n} \) such that

\[
\Omega_{t,+} \subset \bigcup_{j=1}^{n} \Omega_{t,+}^{j}, \quad \Omega_{t,-} \subset \bigcup_{j=1}^{n} \Omega_{t,-}^{j}, \quad \Omega_0 \subset \bigcup_{j=1}^{n} \Omega_0^{j},
\]

(3.23)

where

\[
\Omega_{t,+}^{j} = \{(t,x,y) : 0 < (t - T^*(y))^\frac{1}{2} < \epsilon_{j,+}, \quad c_{j,+} - \epsilon_{j,+} < \frac{x - x^{**}(t,y)}{(t - T^*(y))}\frac{1}{2} < c_{j,+} + \epsilon_{j,+}\},
\]

(3.24)

\[
\Omega_{t,-}^{j} = \{(t,x,y) : 0 < (t - T^*(y))^\frac{1}{2} < \epsilon_{j,-}, \quad c_{j,-} - \epsilon_{j,-} < \frac{x - x^{**}(t,y)}{(t - T^*(y))}\frac{1}{2} < c_{j,-} + \epsilon_{j,-}\},
\]

(3.25)

\[
\Omega_{0}^{j} = \{(t,x,y) : 0 < (T^*(y) - t)^{\frac{1}{2}} < \epsilon_{j,0}, \quad c_{j,0} - \epsilon_{j,0} < \frac{x - x^{**}(t,y)}{(T^*(y) - t)\frac{1}{2}} < c_{j,0} + \epsilon_{j,0}\},
\]

(3.26)

and these domains satisfy the corresponding properties in Lemma 3.1 and 3.3.

We take $\rho > 0$ sufficiently small such that

\[
B_{\rho} = \Omega_{x,+} \cup \Omega_{x,-} \cup \Omega_{t,+} \cup \Omega_{t,-} \cup \Omega_{0}.
\]

(3.27)

In order to derive the behaviors of $u$ and its derivatives near $\Gamma$, it suffices to only consider them in the domains $\Omega_{x,+}, \Omega_{t,+}^{j}$ and $\Omega_{0}^{j}$ since the other cases can be treated analogously.

It follows from direct computation that for fixed $y \in (-\delta, \delta)$,

\[
|u(t,x,y) - u(T^*(y), x^{*}(y), y)| = |u_0(\xi(t,x,y), Y(t,x,y)) - u_0(\Xi^{*}(y), Y(T^*(y), \Xi^{*}(y), y))| \lesssim |\xi(t,x,y) - \Xi^{*}(y)| + |t - T^*(y)| \lesssim |\zeta|^\frac{1}{2},
\]

(3.28)

here we have used the boundedness of the derivatives of $u_0$ and the variable $Y$. Thus, for $(t,x,y) \in \Omega_{x,+}$,

\[
|u(t,x,y) - u(T^*(y), x^{*}(y), y)| \lesssim |\xi_+(t,x,y) - \Xi^{*}(y)| + (t - T^*(y)) \lesssim (t - T^*(y))^{\frac{1}{2}};
\]

(3.29)

and below $\zeta = x - x^{**}(t,y)$; for $(t,x) \in \Omega_{t,+}^{j}$,

\[
|u(t,x,y) - u(T^*(y), x^{*}(y), y)| \lesssim |\xi_+(t,x,y) - \Xi^{*}(y)| + (t - T^*(y)) \lesssim (t - T^*(y))^{\frac{1}{2}};
\]

(3.30)

and for $(t,x) \in \Omega_{0}^{j}$,

\[
|u(t,x,y) - u(T^*(y), x^{*}(y), y)| = |u_0(y(t,x))| \lesssim |\xi(t,x,y) - \Xi^{*}(y)| + (T^*(y) - t) \lesssim (T^*(y) - t)^{\frac{1}{2}}.
\]

(3.31)

Collecting (3.29), (3.30) and (3.31), then (1.13) is obtained.

Next we consider the estimate (1.14) on the derivatives of $u$. By (1.4) and direct computation, it follows that

\[
\left(\begin{array}{ccc}
\partial_{\xi} & \partial_{\xi} & \partial_{\xi} \\
\partial_{\eta} & \partial_{\eta} & \partial_{\eta} \\
\partial_{x} & \partial_{x} & \partial_{x}
\end{array}\right) = -\frac{1}{1 + tH} \left(\begin{array}{ccc}
\phi + t(\phi \partial_{\eta} \psi - \partial_{\eta} \psi) & -(1 + t \partial_{\eta} \psi) & t \partial_{\eta} \\
\psi + t(\partial_{x} \phi \psi - \phi \partial_{x} \psi) & t \partial_{x} & -(1 + t \partial_{x} \phi) \\
\end{array}\right).
\]

(3.32)

Then

\[
\partial_{t} u = \partial_{\xi} u_0 \partial_{\xi} + \partial_{\eta} u_0 \partial_{\eta} = -\frac{1}{1 + tH} (\phi \partial_{\xi} u_0 + \psi \partial_{\eta} u_0),
\]

(3.33)

\[
\partial_{x} u = \frac{\partial_{x} u_0}{1 + tH},
\]

(3.34)

\[
\partial_{y} u = \frac{\partial_{y} u_0}{1 + tH}.
\]

(3.35)
Near \( \Gamma \), we have

\[
1 + tH(\xi, Y(t, \xi, y)) = \frac{1}{2} T^*(y) \partial_x^2 H(\xi, Y(t, \xi, y))|_{t=T^*(y), \ x=x^*(y)} (\xi - \Xi^*(y))^2 + O\left((t - T^*(y))^2 + |t - T^*(y)||\xi - \Xi^*(y)|\right)
\]

For \((t, x) \in \Omega^+_1\), by (3.1) in Lemma 3.1 we have

\[
1 + tH(\xi, Y(t, \xi, y)) \\
= (1 + (3 + 3\theta_0)\mu^2_{c_{\delta, \rho}} (y) + O(\delta)) (t - T^*(y)) + O((t - T^*(y))^2 + |t - T^*(y)||\xi - \Xi^*(y)|)
\]

\[
\geq (t - T^*(y)) + |\xi|^\frac{3}{2},
\]

\[
(3.36)
\]

where the fact of \(-1 + (3 + 3\theta_0)\mu^2_{c_{\delta, \rho}} (y) + O(\delta) > 0\) in Remark 3.1 has been used.

For \((t, x) \in \Omega_2\), by (3.8) in Lemma 3.2 we have

\[
1 + tH(\xi, Y(t, \xi, y)) \\
= (1 + (3 + 3\theta_0)\mu^2_{c_{\delta, \rho}} (y) + O(\delta)) (t - T^*(y)) + O((t - T^*(y))^2 + |t - T^*(y)||\xi - \Xi^*(y)|)
\]

\[
\geq (t - T^*(y)) + |\xi|^\frac{3}{2},
\]

\[
(3.38)
\]

where the fact of \(-1 + (3 + 3\theta_0)\mu^2_{c_{\delta, \rho}} > 0\) has been used.

Therefore, \(1 + tH \geq |t - T^*(y)| + |\xi|^\frac{3}{2}\) holds for \((t, x, y) \in B_\rho\). On the other hand for fixed \(y \in (-\delta, \delta)\), we denote the tangent derivative along the \(\Sigma\) on \(\Gamma\) by

\[
\partial_T u(t, x, y) = \partial_t u + (\phi^* + T^*(y))\partial_{\phi^*} \partial_t Y^* \partial_x u
\]

\[
= \frac{1}{1 + tH} \left[ \left( \phi \partial_x u_0 + \psi \partial_t u_0 \right) - \left( \phi^* + \psi^* \partial_{\phi^*} \partial_t u_0 \partial_{\phi^*} \partial_t Y^* \right) \partial_x u_0 \right]
\]

\[
= \frac{1}{1 + tH} \left[ \left( \phi - \phi^* \right) + \left( \psi \partial_{\phi^*} \partial_t u_0 - \psi^* \partial_{\phi^*} \partial_t u_0 \partial_{\phi^*} \partial_t Y^* \right) \partial_x u_0 \right]
\]

\[
(3.40)
\]

here we have used the fact \(\partial_x \phi \partial_t u_0 = \partial_t \phi \partial_x u_0\). Therefore similarly to the proof of (3.28), we have

\[
|\partial_T u(t, x, y)| \lesssim \left( |t - T^*(y)|^\frac{3}{2} + |\xi|^\frac{3}{2} \right)^{-1},
\]

\[
(3.41)
\]

and then (1.15) is proved.
Proof of Theorem 1.1.

**Proof.** Theorem 1.1 (1) and (2) come from Lemma 2.3 directly. Theorem 1.1 (3) has been obtained. □

4 Appendix

In this section, we study problem (2.61) and (4.1) and (4.2) come from Lemma 2.3 directly. Theorem 1.1 (3) has been obtained. In addition, where \(|\beta| \leq \delta\) with \(\delta > 0\) being small, \(P, Q \in \mathbb{C}^2\) satisfy

\[|P(s, \Lambda, y)| \leq M|y + s^2 + s\Lambda|, \quad Q(s, \Lambda, y) = -\alpha \Lambda + \bar{Q}(s, \Lambda, y)\]

with \(|\bar{Q}(s, \Lambda, y)| \leq M|s + y + \Lambda^2|\), and the constants \(M > 1\) and \(\alpha \geq 2\).

**Lemma 4.1.** For small \(\delta\) and \(\varepsilon > 0\), (4.1) with assumption (4.2) admits a unique local solution

\[(y(s; \beta), \Lambda(s; \beta)) \in C^2([0, \varepsilon] \times [-\delta, \delta]).\]

**Proof.** Taking the following iterative scheme

\[
\begin{aligned}
y_k(s; \beta) &= \beta + \int_0^s \theta P(\theta, \Lambda_{k-1}(\theta), y_{k-1}(\theta)) \, d\theta, \\
\Lambda_k(s; \beta) &= s^{-\alpha} \int_0^s \theta^{\alpha-1} Q(\theta, \Lambda_{k-1}(\theta), y_{k-1}(\theta)) \, d\theta, \\
y_0 &= \beta, \quad \Lambda_0 = 0,
\end{aligned}
\]

where \((y, \Lambda) \in S \triangleq \{(y, \Lambda) \in C([0, \varepsilon]) : |y| \leq 2\delta, |\Lambda| \leq M s\}.

If \((y_{k-1}, \Lambda_{k-1}) \in S\), we then have that for small \(\delta\) and \(\varepsilon\),

\[
\begin{aligned}
|y_k| &\leq \delta + \varepsilon \|P\|_{L^\infty([0, \varepsilon])} \leq \delta + \varepsilon M (2\delta + \varepsilon^2 + M \varepsilon^2) \leq 2\delta, \\
|\Lambda_k| &\leq s^{-\alpha} \int_0^s \theta^{\alpha} \|\bar{Q}\|_{L^\infty([0, \varepsilon])} \leq \frac{s}{\alpha + 1} \|\bar{Q}\|_{L^\infty([0, \varepsilon])} \leq \frac{M s}{\alpha + 1} (1 + 2M \delta + M^2 \varepsilon) \leq M s.
\end{aligned}
\]

In addition,

\[
\begin{aligned}
|y_k - y_{k-1}| &\leq \int_0^s |\theta (P(\theta, \Lambda_{k-1}(\theta), y_{k-1}(\theta)) - P(\theta, \Lambda_{k-2}(\theta), y_{k-2}(\theta)))| \, d\theta \\
&\leq C \int_0^s \left( |y_{k-1}(\theta) - y_{k-2}(\theta)| + |\Lambda_{k-1}(\theta) - \Lambda_{k-2}(\theta)| \right) \, d\theta \\
&\leq C \varepsilon \left( \|y_{k-1} - y_{k-2}\|_{L^\infty([0, \varepsilon])} + \|\Lambda_{k-1} - \Lambda_{k-2}\|_{L^\infty([0, \varepsilon])} \right),
\end{aligned}
\]

\[
\begin{aligned}
|\Lambda_k - \Lambda_{k-1}| &\leq s^{-\alpha} \int_0^s \theta^{\alpha-1} \left| Q(\theta, \Lambda_{k-1}(\theta), y_{k-1}(\theta)) - \bar{Q}(\theta, \Lambda_{k-2}(\theta), y_{k-2}(\theta)) \right| \, d\theta \\
&\leq C s^{-\alpha} \int_0^s \theta^{\alpha-1} \left( 2M \theta |y_{k-1}(\theta) - y_{k-2}(\theta)| + 4\delta |\Lambda_{k-1}(\theta) - \Lambda_{k-2}(\theta)| \right) \, d\theta \\
&\leq C \left( \|y_{k-1} - y_{k-2}\|_{L^\infty([0, \varepsilon])} + \|\Lambda_{k-1} - \Lambda_{k-2}\|_{L^\infty([0, \varepsilon])} + \delta \right) \|\Lambda_{k-1} - \Lambda_{k-2}\|_{L^\infty([0, \varepsilon])},
\end{aligned}
\]

where \(C\) is a positive constant independent of \(\delta\) and \(\varepsilon\). Thus for small \(\delta\) and \(\varepsilon\), \((y_k, \Lambda_k)\) converges uniformly to some functions \((y(s), \Lambda(s)) \in C^2([0, \varepsilon])\) satisfying (4.4), moreover \((y(s), \Lambda(s)) \in S\). Furthermore, by (4.2) and (4.4), \(y(s)\) and \(\Lambda(s)\) are in \(C^2([0, \varepsilon])\) hold.
References

[1] S. Alinhac, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions*. Ann. of Math. 149 (1999), no. 1, 97-127.

[2] T. Buckmaster, S. Shkoller, V. Vicol, *Formation of shocks for 2D isentropic compressible Euler*, arXiv: 1907.03784, 8 Jul 2019.

[3] T. Buckmaster, S. Shkoller, V. Vicol, *Formation of point shocks for 3D compressible Euler*, arXiv: 1912.04429, 22 Jun 2020.

[4] T. Buckmaster, S. Shkoller, V. Vicol, *Shock formation and vorticity creation for 3d Euler*, arXiv:2006.14789 (2020)

[5] Chen Shuxing, Xin Zhouping, Yin Huicheng, *Formation and construction of shock wave for quasilinear hyperbolic system and its application to inviscid compressible flow*. The Institute of Mathematical Sciences at CUHK, 2010, Research Reports: 2000-10(069)

[6] Chen Shuxing, Dong Liming, *Formation of shock for the p-system with general smooth initial data*. Sci. in China, Ser. A, 44 (2001), no. 9, 1139-1147.

[7] D. Christodoulou, *The formation of shocks in 3-dimensional fluids*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.

[8] D. Christodoulou, *The shock development problem*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2019. ix+920 pp.

[9] D. Christodoulou, A. Lisibach, *Shock development in spherical symmetry*. Ann. PDE 2 (2016), no. 1, Art. 3, 246 pp.

[10] D. Christodoulou, Miao Shuang, *Compressible flow and Euler’s equations*, Surveys of Modern Mathematics, 9. International Press, Somerville, MA; Higher Education Press, Beijing, 2014.

[11] G. Holzegel, S. Klainerman, J. Speck, Wong Willie Wai-Yeung, *Small-data shock formation in solutions to 3D quasilinear wave equations: an overview*. J. Hyperbolic Differ. Equ. 13 (2016), no. 1, 1-105.

[12] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathematics and Applications 26, Springer-Verlag, Berlin, 1997.

[13] Kong Dexing, *Formation and propagation of singularities for $2 \times 2$ quasilinear hyperbolic systems*. Trans. Amer. Math. Soc. 354 (2002), no. 8, 3155-3179.

[14] M.P. Lebaud, *Description de la formation d’un choc dans le p–système*. J. Math. Pures Appl. (9) 73 (1994), no. 6, 523-565.

[15] J. Luk, J. Speck, *Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity*. Invent. Math. 214 (2018), no. 1, 1-169.

[16] A. Majda, *The stability of multidimensional shock fronts*. Mem. Amer. Math. Soc. 41 (1983), no. 275, iv+95 pp.

[17] A. Majda, *The existence of multidimensional shock fronts*. Mem. Amer. Math. Soc. 43 (1983), no. 281, v+93 pp
[18] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*. Applied Mathematical Sciences, 53, Springer-Verlag, New York, 1984.

[19] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel, *On the implosion of a three dimensional compressible fluid*, arXiv: 1912.11009, 13 Jun 2020.

[20] G. Métivier, *Stability of multidimensional shocks. Advances in the theory of shock waves*, 25-103, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[21] Miao Shuang, Yu Pin, *On the formation of shocks for quasilinear wave equations*. Invent. Math. 207 (2017), no. 2, 697-831.

[22] J. A. Smoller, *Shock waves and reaction-diffusion equations*, Berlin-Heiderberg-New York, Springer-Verlag, New York, 1984.

[23] J. Speck, *Shock formation for 2D quasilinear wave systems featuring multiple speeds: blowup for the fastest wave, with non-trivial interactions up to the singularity*. Ann. PDE 4 (2018), no. 1, Art. 6, 131 pp.

[24] Yin Huicheng, *Formation and construction of a shock wave for 3-D compressible Euler equations with the spherical initial data*. Nagoya Math. J. 175 (2004), 125-164.

[25] Yin Huicheng, Zhu Lu, *The shock formation and optimal regularities of the resulting shock curves for 1-D scalar conservation laws*, arXiv:2103.07837, 13 March, 2021.