Hydrodynamics of a black brane in Gauss–Bonnet massive gravity

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Abstract
A black brane solution to Gauss–Bonnet massive gravity is introduced. In the context of AdS/CFT correspondence, the viscosity to entropy ratio is found by the Green–Kubo formula. The result indicates a violation of the well-known KSS bound, as expected in a higher derivative theory. Setting mass zero gives back the known viscosity to entropy ratio dependent on the Gauss–Bonnet coupling, while without the Gauss–Bonnet term, a nonzero mass parameter does not contribute to the ratio which saturates the bound of $\frac{1}{4\pi}$.

Keywords: Gauss–Bonnet gravity, massive gravity, shear viscosity
Green–Kubo formula

1. Introduction
For decades, various modifications of the Einstein gravity, like Lovelock gravity [1], brane world cosmology [2], scalar-tensor theories [3], and the so-called $F(R)$ gravity [4–6] have been proposed to address important problems in cosmology such as the cosmological constant, dark energy and dark matter. A common ingredient of these modified theories is the massless graviton, so it would be interesting to consider adding the massive graviton to these theories. The earliest attempt to study such theories goes back to Fierz and Pauli [7] with the linear ghost-free massive spin-two theory. However, its generalization to a non-linear massive gravity has been a great challenge for decades due to fatal instability problems [8]. Finally, a consistent covariant ghost-free non-linear massive gravity was introduced in [9] (see also [10, 11]).

To understand aspects of this new class of gravity it is important to find various spacetimes in different setups. One of the interesting setups includes the massive gravity in a higher derivative theory, such as the Gauss–Bonnet gravity, and looking for a black hole

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solution. The higher derivative theories are in turn important generalizations of the Einstein gravity which emerge from string theory as a deeper underlying theory of gravity. If one considers the second order curvature terms in the gravity action, the Gauss–Bonnet is the best combination; it does not alter the field equations in four-dimensions, but has nontrivial effects in higher dimensions. Indeed, for higher dimensions, it is the only curvature correction with no-ghost self-interacting gravitons [12].

Regarding the importance of both Gauss–Bonnet and massive gravity, we are motivated to combine them to find a black hole solution in this setup. This is the first part of our work given in section 2, where in the context of massive Gauss–Bonnet gravity with a cosmological constant, we find a class of solution which includes black holes with either spherical or hyperbolic horizons, and particularly a black brane solution with a flat horizon.

On the other hand, it is well known from the AdS/CFT duality [13–15] that quantum gravity and string theory on an AdS$_{d+1}$ background are dual to a $d$-dimensional CFT, which is a non-gravitating theory. The latter can be described by hydrodynamics as an effective theory of QFT at large distances and time-scales [16]. The AdS/CFT duality leads to fluid/geometry correspondence in this limit [17–19], so it is fascinating to understand field theory in its strong coupling limit by its gravity dual. The suitable dual solution for investigating the hydrodynamics of a field theory is the black brane in the bulk. Specifically, a black brane solution equipped with AdS/CFT duality can be used to find transport coefficients of the hydrodynamic description of a field theory [20].

The hydrodynamics equations are laws of conservation of energy and momentum [16, 21],

$$\nabla_{\mu} T^{\mu\nu} = 0,$$

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p g^{\mu\nu}.$$  

According to the dictionary of AdS/CFT, a black brane within the bulk is dual to a fluid on the boundary. It also implies that the Einstein equation in the bulk corresponds to the conservation of the energy-momentum equation on the boundary. The bulk metric is the dual field of the energy-momentum tensor of the boundary theory.

In the large wavelength limit, where the hydrodynamics regime is valid, the energy-momentum tensor can be expanded as follows,

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p g^{\mu\nu} - \sigma^{\mu\nu}$$

$$\sigma^{\mu\nu} = P^{\mu\nu} + g^{\mu\nu} - \frac{2}{3} \eta \nabla^{\mu}u^{\nu},$$

where $\eta$, $\zeta$, $\sigma^{\mu\nu}$ and $P^{\mu\nu}$ are the shear viscosity, bulk viscosity, shear tensor and projection operator, respectively [19, 22–25]. Here, we are interested in the shear viscosity which can be derived by the Green–Kubo formula [25, 26].

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int_0^\infty \sin x \frac{1}{x} \left[ T_s^\mu (x), T_s^\mu (0) \right] \left[ T_s^\nu (x), T_s^\nu (0) \right] - \lim_{\omega \to 0} \frac{1}{\omega} \left[ 3 G^{\mu\nu} (\omega, \vec{0}) \right].$$

Notice that the Green–Kubo prescription is independent of the details of the bulk theory. Furthermore, the Gauss–Bonnet massive gravity has the advantage of being second order in derivatives of metric perturbation, so the Green–Kubo procedure preserves for the Gauss–Bonnet massive gravity.
AdS/CFT duality is a fascinating tool for studying the hydrodynamical aspects of quantum field theory in strong coupling. From this prospective one of the important quantities to calculate is \( s_h \) in strongly coupled plasma. This quantity has a universal value equal to \( \frac{1}{4} \) for field theories dual to Einstein–Hilbert gravity regardless of the details of the theory \([27]\). KSS conjecture states that all relativistic quantum field theories at finite temperature and zero chemical potential have \( s_h \geq \frac{1}{4} \) \([20]\). Actually, this fact points to the Heisenberg uncertainty principle. In addition, the results of quark-gluon-plasma experiments are in good agreement with this bound. However, for higher derivative gravity, the value of \( \frac{2}{s_h} \) becomes less than \( \frac{1}{4} \) \([28]\).

In the following, after finding a black brane solution in the Gauss–Bonnet massive gravity, we calculate the shear viscosity to the entropy density ratio by applying the Green–Kubo formula. The result shows that this ratio depends on both the Gauss–Bonnet coupling and the graviton mass, and violates KSS conjecture as expected in a higher derivative theory.

2. Gravity setup and the black brane solution

The action of Gauss–Bonnet massive gravity is as follows \([10, 11]\),

\[
I = \frac{1}{16\pi G_S} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\lambda_{gb}}{2} L^2 \mathcal{L}_{gb} + m^2 \sum_{i=1}^{4} c_i \mathcal{U}_i(g,f) \right]
\]

where \( \mathcal{L}_{gb} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \) (6)

where \( R \) is the scalar curvature, \( f \) is a fixed rank-two symmetric tensor and \( m \) is the mass parameter. In equation (6), \( c_i \)'s are constants and \( \mathcal{U}_i \) are symmetric polynomials of the eigenvalues of the \( 5 \times 5 \) matrix \( \mathcal{K}_{\mu}^{\nu} = \sqrt{g_{\mu\nu}} f_{\mu\nu} \) given as \([10, 11]\)

\[
\mathcal{U}_1 = [\mathcal{K}]
\]

\[
\mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2]
\]

\[
\mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]
\]

\[
\mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^2][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]
\]

The square root in \( \mathcal{K} \) means \((\sqrt{A})_{\nu}^{\mu}(\sqrt{A})_{\rho}^{\sigma} = A_{\rho}^{\mu} \) and the rectangular brackets denote traces.

We consider the following metric ansatz for a five-dimensional planar AdS black brane,

\[
d^s = -\frac{r^2N(r)}{L^2}f(r)dt^2 + \frac{L^2dr^2}{rf(r)} + r^2h_{ij}dx^idx^j.
\]

A generalized version of \( f_{\mu\nu} \) was proposed in \([10, 11]\) with the form \( f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij}) \), where \( h_{ij} = \frac{1}{r^2} \).

The values of \( \mathcal{U}_i \) are calculated as below,

\[
\mathcal{U}_1 = \frac{3c_0}{r}, \quad \mathcal{U}_2 = \frac{6c_0^2}{r^2}, \quad \mathcal{U}_3 = \frac{6c_0^3}{r^3}, \quad \mathcal{U}_4 = 0.
\]

\(^2\) It was shown in \([29]\) that to preserve causality in the context of Gauss–Bonnet gravity, some corrections including massive higher spin fields are needed. However, this is beyond the scope of the present work. We thank Jose Edelstein for raising this point.
Inserting this ansatz into the action equation (6) yields,

\[ I = \frac{1}{16\pi G_s} \int d^4x \frac{3N(r)}{L^3} \]

\[ \times \left[ r^4 \left( 1 - f(r) + \lambda_{gh} f(r)^2 \right) + m^2 L^2 c_0 \left( \frac{c_1 r^3}{3} + c_0 c_1 r^2 + 2c_0^2 c_3 r \right) \right]' \]

in which \(^\prime\) denotes the derivative with respect to \(r\). The equation of motion is given by a variation of \(N(r)\) [30],

\[ r^4 \left( 1 - f(r) + \lambda_{gh} f(r)^2 \right) + m^2 L^2 c_0 \left( \frac{c_1 r^3}{3} + c_0 c_1 r^2 + 2c_0^2 c_3 r \right) = 0 \]

\(f(r)\) is determined by solving the following equation,

\[ r^4 \left( 1 - f(r) + \lambda_{gh} f(r)^2 \right) + m^2 L^2 c_0 \left( \frac{c_1 r^3}{3} + c_0 c_1 r^2 + 2c_0^2 c_3 r \right) = b^4 \]

in which \(b\) is an integration constant. This yields two solutions,

\[ f_\pm(r) = \frac{1}{2\lambda_{gh}} \left[ 1 \pm \sqrt{1 - 4\lambda_{gh} \left( 1 - \frac{b^4}{r^4} \right) - 4m^2 L^2 \lambda_{gh} \left( \frac{2c_0^2 c_2}{r^3} + \frac{c_0 c_1}{r^2} + \frac{c_1 c_0}{3r} \right)} \right] \]

Since the black brane should have an event horizon, one should set \(f(r) = 0\) to get the horizon location, and it is obvious that \(f_\pm(r)\) does not satisfy this. Thus, we choose \(f_-(r)\) of the above solution. Notice \(f_-(r)\) is zero at the event horizon, \(r_+\), so we have,

\[ b^4 = r_+^4 \left[ 1 + m^2 L^2 \left( \frac{2c_0^2 c_3}{r_+^3} + \frac{c_0^2 c_2}{r_+^2} + \frac{c_1 c_0}{3r_+} \right) \right] \]

It is easy to show that \(N(r)\) is constant by variation of \(f(r)\) from equation (8). We use the dimensionless variable \(z = \frac{r}{r_+}\)

\[ ds^2 = -\frac{2r_+^2}{L^2} N^2 f_-(z) dz^2 + \frac{L^2 dz^2}{z^2 f_-(z)} + \frac{r_+^2 z^2}{L^2} \sum_{i=1}^{3} dx_i^2 \]

substituting \(b^4\) in \(f_-(r)\),

\[ f_-(z) = \frac{1}{2\lambda_{gh}} \left[ 1 - \Gamma(z) \right] \]

\[ \Gamma(z) = \sqrt{1 + \frac{4\lambda_{gh}}{z^4} \left[ 1 - z^4 + m^2 L^2 \left( \frac{2c_0^2 c_3}{r_+^3} (1 - z) + \frac{c_0^2 c_2}{r_+^2} (1 - z^2) + \frac{c_1 c_0}{3r_+} (1 - z^3) \right) \right]} \]

In AdS/CFT correspondence, the speed of light in the boundary CFT is simply \(c = 1\). So, for the black brane solution in the asymptotic region we have \(\lim_{z \to \infty} N^2 f_-(z) = 1\) to recover a causal boundary. By applying this criterion we will have

\[ N^2 = \frac{1 + \sqrt{1 - 4\lambda_{gh}}}{2} \]
The temperature and the Bekenstein–Hawking entropy density follow,
\[ T = \frac{1}{2\pi} \left[ \frac{1}{\sqrt{-g_{rr}}} \frac{d}{dr} \sqrt{-g_{rr}} \right]_{b=r_c} = \frac{N r_c^2}{4\pi L^2} - f'_{\frac{1}{L}}(r_c) \]
\[ = \frac{N r_c}{\pi L^2} \left[ 1 + \frac{m^2 L^2}{4} \left( \frac{2c_0^2 c_3}{r_c^2} + \frac{2c_0 c_2}{r_c} + \frac{c_0 c_1}{r_c} \right) \right] = \frac{N r_c}{\pi L^2} [1 + \Delta] \] (16)
\[ s = \frac{4\pi}{V} \int d^3x \sqrt{-g} = 4\pi \left( \frac{r_c}{L} \right)^3, \] (17)
where in equation (16), the last equation defines parameter \( \Delta \) and in equation (17), we used \( \frac{1}{16\pi G} = 1 \) so \( \frac{1}{4\pi} = 4G \).

We also generalize the metric ansatz to include spherical and hyperbolic, as well as planar horizons:
\[ ds^2 = \left( k + \frac{r^2}{L^2} f(r) \right) N(r)^2 dt^2 + \frac{dr^2}{k + \frac{r^2}{L^2} f(r)} + r^2 h_{ij} dx^i dx^j, \] (18)
where \( h_{ij} dx^i dx^j \) is the following,
\[ k = +1: \quad d\Omega^2_3 \quad (\text{metric on } S^3), \]
\[ k = 0 : \quad \frac{1}{L^2} \sum_{i=1}^3 (dx^i)^2, \]
\[ k = -1: \quad d\Sigma^2_3 \quad (\text{metric on } H^3). \]

Note that for \( k = \pm 1 \), the above line element has unit curvature. The horizon is determined by \( f(r_c) = -k L^2 / r_c^2 \).

3. Shear viscosity for the black brane

To find the shear viscosity, let us start with a brief introduction to the Green–Kubo formula which explains shear viscosity in terms of the correlation function of the stress-energy tensor [25].

This can be achieved in the linear response theory. One can perturb the theory by a source \( J \) coupled to an appropriate operator \( \mathcal{O} \):
\[ S \rightarrow S + \int dx J(x) \mathcal{O}(x) \] (19)

Assuming \( J \) is small then the expectation value of \( \mathcal{O} \) can be written as
\[ \langle \mathcal{O} \rangle = -\int dx' G_R(x - x') J(x') \] (20)
where \( G_R \) is the retarded Green’s function of \( \mathcal{O} \).
\[ iG_R(x - x') = \theta(x^0 - x'^0) \left\{ \left[ \mathcal{O}(x), \mathcal{O}(x') \right] \right\} \] (21)

For our purposes, the operator \( \mathcal{O} \) is chosen to be the energy-momentum tensor \( T^{\mu\nu} \) on the boundary. From the AdS/CFT dictionary it is known that the source for the energy-
momentum tensor is the metric fluctuations. So, we consider small metric perturbations as
\[ g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu} \] (22)

Then
\[ \langle T^{\mu\nu}(x) \rangle \sim \int dx' \langle T^{\mu\nu}(x) T^{\alpha\beta}(x') \rangle h_{\alpha\beta} \] (23)

To be more specific, consider the black brane solution
\[ ds^2 = -\frac{r_s^2}{L^2}N^2f \, dt^2 + \frac{L^2}{f} \, dx^2 \]
\[ + \frac{r_s^2}{L^2} \sum_{i=1}^{3} dx_i^2, \] (24)
in which \( \tilde{f} = \frac{L^2}{r_s^2} f \). Then we perturb the background metric by \( h_{3}^{\gamma} \equiv \frac{r_s^2}{L} \phi(t, \bar{x}, z) \) with every other \( h_{\mu\nu} = 0 \),
\[ ds^2 = -\frac{r_s^2}{L^2}N^2\tilde{f} \, dt^2 + \frac{L^2}{f} \, dx^2 \]
\[ + \frac{r_s^2}{L^2} \sum_{i=1}^{3} dx_i^2 + 2\phi(t, \bar{x}, z) dx_3 \, dx_2 \] (25)

Since we are looking for the hydrodynamics limit of the fields, \( h_{3\gamma} \) needs to be varied slowly over the space-time. Moreover, it is a spin-two field, so it cannot excite velocity and temperature which are vector and scalar, respectively. Thus, we consider \( u^i = 0 \) in the expressions equations (3) and (4) and set \( \zeta = 0 \), then [25],
\[ T_{3\gamma} = P g_{3\gamma} - \eta \left( \nabla_i u_i + \nabla_i u_3 \right) = P h_{3\gamma} + \eta \partial_3 h_{3\gamma} \]
\[ = -\int dx' G_R(x - x') h_{3\gamma}(x') \] (26)

Then for the zero spatial momentum limit, one finds
\[ \langle T_{3\gamma}^\tau T_{3\gamma}^\tau \rangle(\omega, \vec{0}) = P - i\eta \omega \] (27)
This is equivalent to the Green–Kubo’s formula given in equation (5).

The retarded Green’s function can be read from the bulk as follows: plugging equation (25) into the action (6) and introducing the Fourier modes of \( \phi \), one finds
\[ \phi(t, \bar{x}, z) = \int \frac{d\omega dq}{(2\pi)^2} \phi(z, k) e^{-i\omega t - i\vec{q} \cdot \vec{x}}, \quad k = (\omega, 0, 0, q) \]
\[ I = -\frac{1}{2} \int \frac{d\omega dq}{(2\pi)^2} \left[ K \left( \partial_\phi \right)^2 - K_2 \phi^2 + \partial_3 \left( K_3 \phi^2 \right) \right], \] (28)

where \( \phi^2 = \phi(z, k) \phi(z, -k) \), \( \phi(z, -k) = \phi^*(z, k) \) and \( K \)’s functions will be introduced in the following. We expand the metric up to the second order of \( \phi \) and by using the general covariance symmetry\(^3\), the action can be written as follows,
\[ I = -\frac{1}{2} \int \frac{d\omega dq}{(2\pi)^2} \left[ g^{2\tau} \left( \partial_\tau \phi \right)^2 + g^{\alpha\beta} \left( \partial_\phi \phi \right)^2 + m^2 \frac{U(z)}{H} \phi^2 + \ldots \right], \] (29)

\(^3\) A general procedure is given in [31].
where,
\[ K = H g^c = \frac{H f}{L^2} \]
\[ \frac{K_2}{K} = \frac{g^a}{g^c} \left( -\omega^2 - \frac{m^2 U}{H g^a} \right) = \frac{L^4}{r^3_{+} N^2 f^2} \left( \omega^2 - \frac{m^2 r^2_{+} N^2 f^2}{L^2 H} \right) \]
\[ U = -\frac{N c_0 r^3_{+}}{4 L^3} \left[ 3c_1 r_{+} e^2 + 2c_0 c_2 z + 108c_0^3 c_4 z^{-1} \right] \]
\[ K = z^{2f} \left( z - \lambda_{gb} \partial_b f \right) \]
\[ H = \frac{N r^4}{L^3 \Gamma(z)} \left[ \left( 1 - 4\lambda_{gb} \right) z^3 - \lambda_{gb} L^2 m^2 \left( \frac{2c_0^3 c_3}{r^3_{+}} + \frac{2c_0^2 c_2 z}{r^2_{+}} + \frac{c_0 c_1 z^2}{r^2_{+}} \right) \right] \tag{30} \]
in which we consider zero momentum \( q = 0 \). Notice the surface term \( K_3 \) does not contribute to the viscosity. The mode equation can be derived from action (28) as follows,
\[ K \phi'' + K' \phi' + K_2 \phi = 0 \tag{31} \]
Firstly, we solve the mode equation near the horizon. In the near horizon limit we have,
\[ \frac{K_2}{K} \approx \frac{\omega^2}{B^2(z - 1)^2} \tag{32} \]
\[ \frac{K'}{K} \approx \frac{1}{z - 1} \tag{33} \]
By comparing \( K_2/K \) in equations (32) and (30), and ignoring the less singular term proportional to \( m^2 \), we obtain,
\[ B = \frac{N r_{+} \rho'}{L^2} (1) \tag{34} \]
in which \('\) denotes the derivative with respect to \( z \). This equation is singular at the horizon \( z = 1 \). The mode equation at the near horizon is,
\[ \phi''(z) - \frac{1}{1 - z} \phi'(z) + \frac{\omega^2}{B^2 (1 - z)} \phi(z) = 0, \tag{35} \]
where its solution is \( \phi(z) = (1 - z)^\beta \), with
\[ \beta = -\frac{2w}{B} \tag{36} \]
To solve the mode equation (31), we apply the following ansatz,
\[ \phi(z) = F(z)^\beta \left( 1 - \beta h + O(\beta^2) \right), \tag{37} \]
where \( F(z) = N^2 f(z) \). Inserting equation (37) into equation (31) to the first order of \( \beta \), we obtain,
\[ \left( \frac{K \phi'}{F} \right)' = K h'' - K' h' + m^2 Uh = 0. \tag{38} \]
It can easily be solved to find,

\[
\left( \frac{K F'}{F} - Kh' \right)' + m^2 U h = 0,
\]

(39)

\[
K F' - Kh' = A_1 - m^2 \int U(z') h(z') dz'
\]

(40)

\[
h = \log \frac{F}{A_2} - A_1 \int_1^z \frac{1}{K} dz + m^2 \int_1^z \frac{dz_2}{K(z_1)} \int_1^{z_2} U(z_2) h(z_2) dz_2
\]

(41)

where \( A_1 \) and \( A_2 \) are integration constants. From the above equation, \( h \) can be found perturbatively. However, we only need the near horizon and near boundary behavior of \( h \), as we perform in the appendix. For \( m^2 = 0 \) we have,

\[
h^{(0)} = \log \frac{F}{A_2} - A_1 \int_1^z \frac{1}{K} dz
\]

(42)

To be regular at the horizon it is sufficient to take \( A_1 = \frac{K'}{1} \), and this preserves all orders of \( m^2 \), as shown in the appendix.

We can read the retarded Green’s function by expanding the action up to the second order of the source or the value of field on the boundary,

\[
S = - \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} J_c(-k) F_a(k; z) J_a(k)|_{z=4},
\]

(43)

then the retarded Green’s function \( G_R(k) \) in momentum space for the boundary field dual is given by,

\[
G_R(k) = - \lim_{\epsilon \to 0} \mathcal{F}_a(k, z)|_{z=4},
\]

(44)

which can be calculated by the prescription given in [20, 32],

\[
G_R\left( \omega, 0 \right) \equiv K \delta^{\theta} \partial_{\theta} \phi + K_3 \phi^2 = K F^{-\beta} (1 + \beta \hbar \beta F') \left( \frac{F'}{F} - h' \right) = \beta \left( A_1 - E \int U(z') h(z') dz' \right) \bigg|_{z \to \infty} = - \omega \frac{A_1}{F}
\]

(45)

In the first line \( K_3 \phi^2 \) is real and does not contribute to viscosity, so we drop this term in the retarded Green’s function. In the second line, the integral term includes terms proportional to \( c_1, c_2 \) and \( c_4 \). A simple power counting indicates that the perturbative calculation of \( h \) diverges for \( z \to \infty \) unless \( c_1 = c_2 = 0 \) and \( c_4 \) term converges to zero as we show in the appendix.

Shear viscosity can now be found by using the Green–Kubo formula, as in the following,

\[
\eta = - \lim_{\omega \to 0} \frac{1}{\omega} \mathcal{J} G_R\left( \omega, 0 \right) = \left. \frac{A_1}{B} \right| = \left. \frac{L^2}{r_i N f'} \right| K'(1)
\]

(46)

We can calculate \( K'(1) \) as below,

\[
K'(1) = \left. \left( H g^{zz} \right) \right|_{z=1} = \left. \left( H' g^{zz} + H \left( g^{zz} \right)' \right) \right|_{z=1} = \left. H \left( g^{zz} \right)' \right|_{z=1} = \frac{H f'}{L^2} \left. \left|_{z=1} \right. \right.
\]

(47)
Then by substituting equation (47) in (46), we will have,
\[ \eta = \frac{A_1}{B} = \frac{H(z = 1)}{N r_c} \] (48)
and the ratio of shear viscosity to entropy density is,
\[ \frac{\eta}{s} = \frac{1}{4\pi} \frac{L^3}{N r_c^2} H(z = 1). \] (49)
For the Einstein–Hilbert massive gravity \( H = \sqrt{-g} = \frac{N r_c}{L^2} \), then the ratio of shear viscosity to entropy density is \( \frac{\eta}{s} = \frac{1}{4\pi} \) independent of the graviton mass.

We apply equation (49) to our black brane, with \( H \) given in equation (30)
\[ \eta \bigg/ s = \frac{1}{4\pi} \left[ 1 - \frac{4\pi L^2 T \lambda_{gb}}{N r_c} \right] \] (50)
\[ = \frac{1}{4\pi} \left[ 1 - \frac{4\pi L^2 T \lambda_{gb}}{N r_c} \right] \] (51)
where \( \Delta \) is in the following,
\[ \Delta = \frac{L^2 m^2}{4} \left( \frac{2c_0^3 c_3}{r_+^3} + \frac{2c_0^2 c_2}{r_+^2} + \frac{c_0 c_1}{r_+} \right) \] (52)
\[ = \frac{L^2 m^2 c_0^3 c_3}{2r_+^3} \] (53)
where in the last line we put \( c_1 = c_2 = 0 \). For \( m = 0 \) we have, \( \frac{\eta}{s} = \frac{1}{4\pi}[1 - 4\lambda_{gb}] \), which is the well-known result for the Gauss–Bonnet gravity.

4. Conclusion

The Gauss–Bonnet massive gravity theory has various important aspects. Firstly, the higher curvature theories of gravity are inevitable from a more fundamental theory like string theory. Among them, the Gauss–Bonnet is interesting since despite being of a higher order in curvature, has a second order differential equation of motion. The other ingredient of our model is the massive gravity, which in turn has the advantage of being a covariant stable generalization of massless gravity, and would be a candidate for a possible solution to important problems like the cosmological constant. As such, studying a black hole/brane solution in this setup is helpful in order to better understand the model.

In this paper, we introduced a new black hole/brane solution to this model. Then, we investigated its implications in an AdS/CFT context to describe the dual field theory on the boundary. In particular, we found viscosity and showed that the lower bound \( \eta/s = 1/4\pi \)—known as the KSS conjecture [33]—violates the black brane in this gravity. Although this is not surprising in the context of higher derivative theories [34–41], the ratio depends on the details of the model and there is no universal value in contrast to the Einstein gravity.

Results were exact in both the Gauss–Bonnet coupling and the graviton mass. It is interesting that for the zero Gauss–Bonnet coupling the graviton mass disappears in the viscosity to entropy ratio and the result saturates the KSS bound, \( 1/4\pi \). This means that the inclusion of the massive graviton does not affect the KSS bound. This point was not obvious \textit{a priori}. On the other hand, the inclusion of mass to the Gauss–Bonnet looks like rescaling
the coupling as \( \lambda_{gb} \rightarrow (1 + \Delta) \lambda_{gb} \) in the viscosity to entropy ratio, with \( \Delta \) being a mass-dependent parameter. Recalling the causal boundary criteria in equation (15), it indicates that \( 4\lambda_{gb} < 1 \). Now the rescaling of \( \lambda_{gb} \) either decreases or increases the bound on the \( \eta/s \) ratio depending on the sign of \( \Delta \). In any case, \( 1 + \Delta \) is a positive parameter proportional to temperature as given in equation (16).

It is worth mentioning that some calculations in section 2 may be simplified if one uses the results in \([42]\). For future developments, it would be interesting to apply this procedure to black strings, which are an extension of Schwarzschild geometry to one extra-dimension. Stability and the inclusion of these objects in an FRW cosmology on the brane are nontrivial, as done in \([43, 44]\).

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**Note added** As this article was being completed, we received the preprint \([45]\) which found the same solution as ours.

**Appendix**

Here we study the behavior of the \( h \) function at the horizon and boundary limits. Firstly, consider the following perturbative expansion of \( h \) in powers of \( m^2 \):

\[
 h(z) = h^0(z) + m^2h^{(1)}(z) + m^4h^{(2)}(z) + \cdots \tag{54}
\]

where \( h^0 \) is given in equation (42) and higher orders can be found by

\[
 h^{(n+1)}(z) = \int_1^z \frac{dz_1}{K(z_1)} \int_1^{z_1} U(z_2) h^{(n)}(z_2) dz_2 \tag{55}
\]

On the other hand, \( F \) and \( K \) are regular functions at \( z = 1 \),

\[
 F \approx F'(1)(z - 1) \\
 K \approx K'(1)(z - 1). \tag{56}
\]

Thus to have a non-singular \( h^0 \) at the horizon, \( A_1 \) is chosen to be \( A_1 = K'(1) \). This result preserves at higher orders. Notice that \( U \) is a constant at the horizon, so taking \( A_2 = 1 \) then \( h^l \approx C (z - 1)^l \) with \( C \) some constant and

\[
 h^{(1)}(z) \approx \int_1^z \frac{dz_1}{K(z_1)} \int_1^{z_1} U(z_2) [C(z_2 - 1)] dz_2 = \frac{U(1)C}{4K'(1)} (z - 1)^2 \tag{57}
\]

then \( h^{(n)}(z) = \frac{1}{(n+1)!} \left( \frac{U(1)C}{K'(1)} \right)^n (z - 1)^n + \cdots \). This guarantees that \( h \) vanishes smoothly near the horizon for any order of \( m^2 \).

In the same way, we analyze the near boundary behavior of \( h \). For \( z \rightarrow \infty \), we have \( K \sim z^2 \), \( F = N^2 f \sim 1 \), and with \( A_2 = 1 \), one finds \( h^0 \sim 1/z^2 \). Higher orders of \( h \) can be derived perturbatively from equation (55).

Recall that \( U(z) \) includes terms proportional to \( c_1 z^2 \), \( c_2 z^2 \) and \( c_4 z^{-1} \). It is easy to show that positive powers of \( z \) in \( U \) lead to an increasing power of \( z \) in each step of the perturbation for \( h \), and give a logarithmic divergence in a few steps. So, we have to take \( c_1 = c_2 = 0 \). For \( c_4 = 0 \) the integral converges at the boundary and indeed makes a zero contribution to the viscosity:
\[ h^{(1)}(z) \sim \int z^2 \frac{dz_1}{K(z_1)} \int z_2 \frac{dz_2}{z_2^2} \sim \frac{1}{z^4} \]  \hfill (58)

Therefore \( h^{(n)}(z) \sim 1/z^{2n+2} \) and the integral in equation (45) becomes

\[ \int z U(z') h^{(n)}(z') dz' \sim \int z \left( \frac{z'}{z^2} \right)^{2n+2} dz' \sim \frac{1}{z^{2n+2}} \rightarrow 0 \]  \hfill (59)

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