Electrons in a strong magnetic field on a disk

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Abstract

The problem of interacting electrons moving under the influence of a strong magnetic field in two dimensions on a finite disk is reconsidered. First, the results of exact diagonalizations for up to \( N = 9 \) electrons for Coulomb as well as for a short-range interaction are used in the search for a peculiar ground state corresponding to filling factor \( 1/3 \). Not for the Coulomb, but only for the short-range interaction, can the \( 1/3 \)-state be safely identified amongst the spectra of various filling factors close to \( 1/3 \). Second, the propositions of the concept of quasiparticles, as used in the hierarchical theory, are examined in view of the exact results for the disk geometry. Whereas the theory for the quasiholes is in complete accordance with the spectra, for the quasielectrons, finite size corrections make an analysis difficult. For the quasielectron energy, an extrapolation to \( N \to \infty \) is given and compared with the corresponding extrapolations of three different proposals for trial wave functions. While the limiting value for the best trial wave function is very close to the limit of the exact results, the behavior of the finite size corrections of the exact energies and of the trial wave functions, respectively, is qualitatively rather different.

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1 Introduction

The peculiar transport properties of a two-dimensional electronic system in a strong magnetic field, as seen in the integral quantum Hall effect (IQHE) \[1\] and the fractional quantum Hall effect (FQHE) \[2\], are just a part of the rich structure of the phase diagram in the electron density vs. temperature plane \[3, 4\]. Much work was devoted in the last years to a study of the Wigner crystal phase at low densities \[3\] and, recently, to the interesting properties at filling factor $\nu = 1/2$ \[5, 6, 7\]. Here, we wish to reconsider the FQHE reporting work done on the few-particle problem.

In both quantum Hall effects, there is a gap above the ground state at the magic filling factors in the ordered system. If a gap remains in the presence of disorder, a plateau in the Hall conductivity $\sigma_{xy}$ is to be expected in the experiments (around filling factors $\nu = p/q$ in the FQHE; $p, q$ – relative primes, $q$ – odd); the longitudinal conductivity $\sigma_{xx}$ vanishes wherever $\sigma_{xy}$ is in a plateau region. Despite the experimental similarity in the transport properties, the microscopic models considered for the IQHE and FQHE are quite different. The IQHE is understood as a localization–delocalization transition, as shown by non-interacting electrons moving in a strong magnetic field under the additional influence of a moderate random potential \[8\]. There, the energy gap is due to the cyclotron energy. In contrast, in the FQHE, even the starting point of a theoretical analysis, namely interacting spin-polarized electrons in the lowest Landau level without any disorder, is a non-tractable system itself because of the strong correlation. Here, even the existence of an energy gap in the many-particle spectra at the magic filling factors has to be proven.

An important step towards an understanding was taken by Laughlin \[9, 10, 11\] who showed the special character of the ground state at filling factors $1/q$ ($q$ – odd) by presenting a trial wave function. This success was partially based on conclusions drawn from few particle calculations. Laughlin’s notion of quasi-particles in connection with the hierarchical theory \[12, 13\] opened a way to explain the occurrence of filling factors with numerators different from one, e.g. $2/5$. However, the hierarchical theory is rather qualitative and allows all filling factors with an odd denominator; thus, a further theoretical analysis becomes necessary. Moreover, a microscopic derivation of an effective quasiparticle Hamiltonian which would start from the original electronic Hamiltonian is still missing.

A new and different point of view emerged in the work of Jain \[14, 15, 16\]. He circumvented the rather artificial construction of many particle states at $\nu = p/q$ via quasiparticles which is done in the hierarchical scheme. Instead, he suggested explicit electronic wave functions for all observed filling factors, $\nu = n/(2np + 1)$ ($n, p$ – integers) and particle–hole transformed states. Jain’s approach starts from $n$ completely filled Landau levels, then multiplies the wave function with a symmetric Jastrow factor of power $2p$ (“adds $2p$ flux quanta per electron”) in order to simulate the influence of the interaction and, finally, projects to the lowest Landau level. Numerical calculations for a small
number of particles \[17\], and also the Fermi–liquid like behavior seen in the experiments \[3, 4\] near \(\nu = 1/2\) corroborate Jain’s approach. Nevertheless, the reason for its success remains unclear if one considers the quite different energy scales of non-interacting and interacting electrons, respectively \[18\].

Other approaches attack the problem with field theoretical methods \[19, 20, 21\]. These attempts describe already known features of the theory such as the Laughlin ground state, the nature of the quasiparticles and the collective excitation spectrum \[22\].

Both, the trial wave function approach and the field theories consider only points of the phase diagram of interacting electrons in the lowest Landau level, because the analysis focuses on definite filling factors. It should be emphasized that the approaches considered so far have another thing in common: they do not depend very much on the specific form of the electron–electron interaction. On the one hand, this can be seen as expressing a universality of the FQHE. On the other hand, one would like to check such an assertion. A nice concept for a discussion of the interaction is that of the pseudopotential coefficients, i. e. the eigenenergies of the two-particle problem, introduced by Haldane \[23\]. In the lowest Landau level, the interaction is completely determined by these coefficients. The picture emerged, that the largest of these determines the FQHE ground state, not to be reached from the non–interacting ground state by perturbation theory, while the other coefficients can be included perturbatively and do not lead to a big difference. Is this true? Below, we will show the numerical spectra for two different interactions (i) the full Coulomb interaction and (ii) a pseudopotential interaction, where only the first coefficient is non–zero.

At first glance, the choice of an appropriate geometry for studying such a system seems to be only a matter of convenience. Three geometries were studied in the past: the disk geometry with a symmetric gauge and open boundary conditions \[10\], the torus geometry with periodic boundary conditions based on the Landau gauge, cf. \[24\], and the spherical geometry where the electrons move in the constant magnetic field of a magnetic monopole in the centre \[13\].

The analytical theory of Laughlin \[11\] was formulated on the disk and also numerical work was done for this geometry \[10, 25, 26, 27\]. Soon, the advantages of the spherical geometry with an additional symmetry and without boundary attracted a lot of interest. It became the favorable choice allowing exact diagonalizations for up to ten electrons at filling factor 1/3 \[28\] and the study of the hierarchical theory, because the lower energy levels are well separated from the rest of the spectrum.

In this paper, we come back to the disk geometry for several good reasons:
– From the experimental point of view, one should study a planar geometry, particularly, if one wants to account for boundaries and contacts.
– The topological differences between the different geometries can very well change the results. E. g. , the ground state at \(\nu = 1/q\) is nondegenerate in the disk and spherical geometry, whereby in the torus geometry it is \(q\)-fold degenerate \[29, 30, 31\].
– The Laughlin theory was originally formulated for the plane. Then, in the current understanding of the FQHE, the plateaus of the Hall conductivity in the vicinity of the magic filling factors are a result of the disorder. Up to now, calculations including disorder study the energy gap between the ground state and the excited states, see e.g. [32, 33, 34]; the calculation of the transport properties of a disordered and interacting electronic system in a magnetic field is an outstanding problem. Still, current theories of disorder use a planar geometry, see [35].

– If one wants to verify theories stressing the importance of edge excitations in the FQHE [36], one needs a geometry with an edge.

In this work, we present the results of a numerical study of the low–lying eigenstates of the FQHE Hamiltonian at and in the vicinity of filling factor 1/3 for the disk geometry. We compare the Coulomb with a pseudopotential interaction. While on the sphere, the differences between both interactions are not so marked, on the disk, the pseudopotential interaction plays a special role and this will be shown below. With our results, we evaluate properties of quasiparticles and thus check the foundations of the hierarchical theory, by comparing the exact spectra for small numbers of quasiparticles with energy expectation values of trial wave functions.

The paper is organized in the following way: in Section 2, the model with the interactions is introduced. Section 3 describes shortly the numerical method used for the diagonalization. In Section 4, we set out to identify the stable state in a system with a finite number of electrons, and we compare the result with Laughlin’s ground state wave function. The notion of quasiparticles is discussed in general in Section 5. Then, the results for the quasi-hole are discussed in Section 6, those for the quasielectron in Section 7. Here, we have extended our recent analysis [37] to a larger number of electrons. Section 8 gives the summary.

2 Model

We consider electrons of charge \(-e\) \((e > 0)\) moving in the x–y–plane under the influence of a perpendicular, constant magnetic field \(\vec{B} = -B\vec{e}_z \) \((B > 0)\). They interact via an isotropic, translationally invariant interaction \(V(|\vec{r} - \vec{r}'|)\).

We study the disk geometry and then, the appropriate gauge is the symmetric gauge \(\vec{A} = \frac{B}{2}(y, -x, 0)\).

The one–particle problem is equivalent to an isotropic two-dimensional harmonic oscillator, whose Hamiltonian \(H_0\) and angular momentum \(L\) can be expressed by two pairs of commuting Bose operators \(a, a^\dagger\) and \(b, b^\dagger\), so that \(H_0 = \hbar\omega_c(b^\dagger b + \frac{1}{2})\) with \(\omega_c = \frac{|eB|}{m}\) and \(L = \hbar(a^\dagger a - b^\dagger b)\). Then, the one–particle basis is

\[
|n, m > = [n!(n + m)!]\frac{1}{2^n} (b^\dagger)^m (a^\dagger)^{n+m} |0 >
\]

with \(n = 0, 1, \ldots; m = -n, \ldots, 0, \ldots\). The energies \(E_n = \hbar\omega_c(n + \frac{1}{2})\) are degenerate in each of the Landau levels \(n\) with respect to the angular momentum
Because the gap between adjacent Landau levels grows linearly with $B$, whereas the coupling constant for a $1/r$ interaction increases only with the square root of $B$, in the strong field limit (FQHE), the Landau level mixing can be neglected and the Hilbert space of the one–particle basis (1) is restricted to $n = 0$. Analogously, if the g–factor in the Zeeman term is not too small, the electrons are spin polarized (about the necessity to include the spin, see [38]).

The basis (1) becomes in real–space representation for the lowest Landau level

$$\varphi_m(z) = [2\pi 2^m m!]^{-\frac{1}{2}} z^m e^{-\frac{1}{4}|z|^2},$$

where $z = x + iy$ and all lengths are given in units of $l_c = \sqrt{\hbar/|eB|}$.

We want to calculate the energy spectrum for a finite number $N$ of electrons at a given filling factor. The ”filling factor” $\nu'$, naively is the ratio of the number of electrons to the number of available one–particle states. In the disk geometry with background potential, this latter number is given by the number $N_\Phi$ of flux quanta $\hbar/e$ through the area of the background. For the neutralizing background potential, we take a superposition of all one–particle states with an angular momentum from $m = 0$ to $m_{\text{max}}$; this leads in the thermodynamic limit to a homogeneous background charge density. Then, outside the disk with the area $2\pi N_\Phi = 2\pi(m_{\text{max}} + 1)$, the background charge density drops quickly to zero. Thus, there are just $m_{\text{max}} + 1$ one–particle states with a maximum expectation value of the area not exceeding the area of the background. Therefore, $\nu' = N / N_\Phi = \frac{N}{m_{\text{max}} + 1}$. For the numerical diagonalization, we restrict the single electron Hilbert space to only $h + 1$ one–particle states with $m = 0, 1, \ldots, h$. The Hamiltonian to be diagonalized numerically then reads:

$$H = \frac{1}{2} \sum_{m_1,m_2,m_3,m_4=0}^{h} W_{m_1,m_2,m_3,m_4} c_{m_1}^{\dagger} c_{m_2} c_{m_3} c_{m_4} - \nu' \sum_{m=0}^{m_{\text{max}}} \sum_{m'=0}^{m_{\text{max}}} W_{m,m',m',m} c_m^\dagger c_m + \frac{\nu'^2}{2} \sum_{m,m'=0}^{m_{\text{max}}} W_{m,m',m',m},$$

where the kinetic energy term has been already dropped since it is a constant for fixed $N$. The $c_m, c_m^\dagger$ are Fermi annihilation and creation operators and the two–particle matrix element is

$$W_{m_1,m_2,m_3,m_4} = \int \int \varphi_{m_1}^*(z) \varphi_{m_2}^*(z') V(|z-z'|) \varphi_{m_3}(z') \varphi_{m_4}(z) d^2zd^2z'.$$
will use a slightly different definition for the filling factor, \( \nu = \frac{N-1}{N_{\Phi} - 1} = \frac{N-1}{m_{\text{max}}} \), because of reasons which will become clear below. The difference between \( \nu \) and \( \nu' \) disappears in the thermodynamic limit. Then, in the case of the SRM, the Hamiltonian reduces to the first term in (3) with \( h = m_{\text{max}} \).

In first quantization, the first term of \( H \) can be decomposed as follows

\[
H^{1st} = \sum_{i>j=1}^{N} \sum_{m=0}^{\infty} V_{m} \hat{P}^{(m)}_{ij}.
\]

\( \hat{P}^{(m)}_{ij} \) is the projector which annihilates in a two-particle wave function all components except the one with relative angular momentum \( m \). This decomposition in angular momenta was originally introduced by Haldane for the sphere. The \( V_{m} \) are the eigenvalues of the two-particle problem and are called pseudopotential coefficients. Because of the fermionic nature of the problem, only odd \( m \) (\( m = 1, 3, \ldots \)) contribute to the sum in (5). The \( W_{m_1,m_2,m_3,m_4} \) are expressed by the \( V_{m} \) as follows:

\[
W_{m_1,m_2,m_3,m_4} = \delta_{m_1+m_2,m_3+m_4} \prod_{i=1}^{4} (2^{m_i} m_i!)^{-\frac{1}{2}} \sum_{m=0}^{m_1+m_2} (m_1 + m_2)! m! V_{m} \times \\
\times \sum_{\lambda=-\infty}^{\infty} \left( \begin{array}{c} m_1 \\ \lambda \end{array} \right) \left( \begin{array}{c} m_2 \\ m - \lambda \end{array} \right) (-1)^{(m-\lambda)} \sum_{\lambda'=\infty}^{\infty} \left( \begin{array}{c} m_3 \\ \lambda' \end{array} \right) \left( \begin{array}{c} m_4 \\ m - \lambda' \end{array} \right) (-1)^{\lambda'}.
\]

The Coulomb model is specified by the pseudopotential coefficients

\[
V_{m}^{CM} = \frac{\sqrt{\pi} (2m)!}{2^{2m+1} (m!)^2}.
\]

Here and below, all energies are measured in units of the coupling constant \( \frac{e^2}{4\pi \varepsilon \ell_c} \) of a Coulomb potential. The SRM is specified by the coefficients

\[
V_{1}^{SRM} = \frac{\sqrt{\pi}}{4}, \quad V_{m>1}^{SRM} = 0.
\]

Here, only electrons with relative angular momentum one interact and repel each other. An interaction in real space, which yields \( V_{m}^{SRM} \) as pseudopotential coefficients in the lowest Landau level, was introduced by Pokrovsky and Talapov [39]:

\[
H_{SRM}^{1st} = V_{1}^{SRM} \sum_{i>j=1}^{N} \nabla_i^2 \nabla_j^2 (r_i - r_j),
\]

hence the name short range model. Why should one study a SRM? The ground state of (3) at the magic filling factors \( \nu \), where the experiment shows the FQHE, cannot be reached by a simple perturbation theory from non-interacting electrons. If, on the other hand, the non-perturbative ground state of a SRM is already very close to the true ground state of (3), then the higher coefficients (\( V_{m>1} \) in our case) can be switched on perturbatively and do not
change the essence of the results. In this case, a SRM should be considered as a generic model for the FQHE. Indeed, the Laughlin wave function is the exact ground state for $\nu = 1/q$ for a SRM with $V_{m\geq q} = 0$ [13, 39, 40]. Particularly on the sphere, this reasoning is very successful. In addition, Halldane showed for $\nu = 1/3$ that if the $V_{m\geq 3}$ become too large compared with $V_1$, then the preference of the Laughlin wave function as the ground state wave function is destroyed [23].

For future reference, we give the translation operator. The generator of the magnetic translations is $\vec{t} = \vec{p} - e\vec{A}$ and commutes with the Hamiltonian for an infinite system. For $N$ electrons, the magnetic translation is a product of unitary one–particle operators $\hat{T}_\xi^{(i)}$

$$\hat{T}_\xi = \prod_{i=1}^{N} \hat{T}_\xi^{(i)} = \exp(i \sum_{i=1}^{N} \xi \hat{t}_i) = \exp\left(-\frac{1}{\sqrt{2}} \sum_{i=1}^{N} (a_\xi^+ \xi^* - a_\xi \xi)\right),$$

where we have used again the bosonic operators $a, a^+$.

The action of the operator $\hat{T}_\xi^{(1)}$ on the basis (2) yields

$$\hat{T}_\xi^{(1)} \varphi_m(z_1) = [2\pi 2^m m!]^{-\frac{1}{2}} (z_1 - \xi)^m e^{-\frac{1}{4} |z_1|^2 + \frac{1}{2} z_1 \xi^* - \frac{1}{4} |\xi|^2}. \quad (11)$$

Next, the factors $(z_1 - \xi)^m$ and $e^{\frac{1}{2} z_1 \xi^*}$ on the r. h. s. of (11) can be expanded with respect to $\xi, \xi^*$. Thus, the transformed function (a “ring” of radius $\sqrt{2(m+1)}$ around $\xi$) is a superposition of all the degenerate eigenfunctions of the lowest Landau level with angular momenta reaching up to infinity. For an electronic system of restricted size, one needs to truncate the angular momentum at $m_{\text{max}}$.

### 3 Numerical treatment

We want to study the energy spectrum, particularly the ground state and the lowest excited states, of $H$ at a fixed filling factor. Then, the electron number $N$ determines the area of the system expressed by the degeneracy $N_\Phi$ or by $m_{\text{max}}$. The dimension of the fermionic many–particle Hilbert space $\binom{N_{\Phi} + m_{\text{max}} + 1}{N}$ grows dramatically with increasing $N$. However, also for a finite system, the total angular momentum $M = \sum_{m=0}^{\infty} m c_m^\dagger c_m$ is conserved. Thus, the diagonalization can be performed in a Hilbert sub–space with fixed $M$, i.e., in the "block $M". The allowed values of $M$ are $M_{\text{min}} = N(N-1)/2 \leq M \leq M_{\text{max}} = N(2m_{\text{max}} - N + 1)/2$.

For the diagonalization of (3) in a block $M$, we construct the many–particle basis in terms of the Slater determinants of the one–particle states. Then, the sum of the $m_i$ ($1 \leq i \leq N, 0 \leq m_i \leq m_{\text{max}}$) is $M$ and all one–particle angular momenta have to be different (fermionic case). The dimensions of the blocks $M$ are symmetric between $M_{\text{min}}$ and $M_{\text{max}}$. The matrix with $M$ being in the middle has the largest dimension and its size limits the feasibility of the diagonalization. The number of these many–particle basis states with fixed $M$,
$g_F(N_\Phi, N, M)$, can be determined either numerically by a recursive formula or from the following generating function [11]

$$x^N (N-1)/2 \prod_{l=1}^{N} \frac{1 - x^l (N_\Phi - (l-1))}{1 - x^l} = \sum_{M=M_{\text{min}}}^{M_{\text{max}}} g_F(N_\Phi, N, M)x^M. \quad (12)$$

As an example, the largest Hilbert sub–space for $\nu = 1/3$ and 10 electrons has $M = 135$ and the dimension 246 448. The maximum dimension is the same as that in the spherical geometry for the same number of electrons at the same filling factor, cf. [28].

The $W_{m_1, m_2, m_3, m_4}$ in (8), determined by the pseudopotential coefficients via (6), are used to calculate numerically the matrix elements between the many–particle states; the diagonal elements are sums of the difference between the direct and the exchange terms, the off–diagonal elements are only single differences. With increasing $N$, more and more of the matrix elements are zero because of the angular momentum conservation and, therefore, the matrix becomes sparse. At filling factor $\nu = 1/3$, for $N = 6$, the matrix $M = 45$ (dimension 338) contains 6108 non–zero off–diagonal matrix elements out of 56953, whereas for $N = 9$ and $M = 108$ (matrix dimension 45207), there are 3016844 non–zero matrix elements of 1021813821, i. e. only 0.3% off–diagonal elements are non–zero. The number of non–zero off–diagonal elements grows approximately as the matrix dimension to the power 1.25. We have checked that there is no further (accidental) decomposition of a block $M$ into subblocks as long as we use the above basis of Slater determinants.

For the diagonalization, we apply two methods. For $N \leq 6$, we calculate all energy levels by combining the Householder and the QR method. For electron numbers $N \geq 7$ when the matrices become quite large and sparsely occupied, we use a Lanczos method in which we restrict the search for energy levels to the low–energy region.

In this work, we concentrate on the filling factor 1/3 and calculate the exact energy levels. We do not study overlaps between eigenfunctions and trial wave functions because our conclusions can be drawn already on the basis of quantum numbers and expectation values. In the spherical geometry, in contrast, one needs also the eigenfunctions for a determination of the angular momentum $\vec{L}^2$.

4 The stable state

In the following, we describe the results of our numerical calculations for a filling factor of precisely 1/3, where the experiment shows the FQHE. We study first the CM, then the SRM.

How can one identify in a series of spectra for various $m_{\text{max}}$ at fixed $N$ the one at filling factor $1/q$? An important hint is given by the trial wave function
for filling factor $1/q$, the Laughlin wave function $\Psi$:

$$
\Psi_{1/q}(z_1, \ldots, z_N) = \prod_{i>j=1}^{N} (z_i - z_j)^q \exp \left[ -\frac{1}{4} \sum_{i=1}^{N} |z_i|^2 \right].
$$

(13)

Its total angular momentum is $M = qN(N-1)/2 \equiv M_N(3)$; we will abbreviate $M_N(3)$ in the following as $M_N$. (13) exhibits a large overlap with the true, numerically determined ground states for small $N$ in the CM. This was shown only for up to four electrons on the disk and for the corresponding wave function on the sphere for up to ten electrons. There are many arguments why (13) is so successful. In particular, if $q$ is not too large ($q \leq 70$), it describes an incompressible, homogeneous liquid. For a finite number of interacting electrons on the sphere, the requirement of a homogeneous and nondegenerate state makes the identification of special spectra corresponding to filling factor $1/q$ simple, because the ground state must have zero total angular momentum.

We want to identify – in the disk geometry – in a series of spectra for the two interactions, CM and SRM, the ground state of the filling factor at $1/3$ as a special state which we call the stable state and which we compare with the Laughlin wave function. There will be problems in the case of the CM, but the identification will be straightforward for the SRM.

We start with the CM (with background potential included). For an overview, the complete spectrum of a system with 6 electrons on the disk calculated by the exact diagonalization procedure is shown in Fig. 1a ordered with respect to $M$. We have chosen $m_{\text{max}} = 17$ so that the naive filling factor is $\nu' = 1/3$. We use $h = m_{\text{max}}$ in (3); numerical tests show that larger $h$ have very small influence on the energy values but just increase the dimension of the Hilbert space. The ground state is nondegenerate and has $M_6 = 45$ as predicted by the Laughlin function. Still, it is not a priori clear that the ground state for this choice of $m_{\text{max}}$ is a finite $N$ approximation of the $1/3$–state. We will investigate this now by changing $m_{\text{max}}$; generally, as we increase $m_{\text{max}}$ from its minimum value $N-1$ (at $\nu' = 1$), the spectrum moves to larger $M$ and the ground state angular momentum increases. In order to show this in detail, we compare for 7 electrons the low–energy levels in the vicinity of $\nu' = 1/3$ where $m_{\text{max}}$ varies between 16 and 21 (Fig. 2a - f); the spectrum at $\nu' = 1/3$ is given in Fig. 2e. At first glance, the most promising candidate for a particularly stable ground state seems to be the one in Fig. 2a, $m_{\text{max}} = 16$, where the gap between the ground state energy level and the first excited state is the largest. But this $m_{\text{max}}$ does not correspond to $\nu' = 1/3$. Further, the state with one flux quantum more ($m_{\text{max}} = 17$) does not show any signature of a degeneracy in the ground state in contradistinction to the quasiparticle picture which is discussed in detail below in Section 5. Finally, the ground state spectrum with the formally correct $\nu' = 1/3$ given in Fig. 2e does not exhibit the expected total angular momentum for seven electrons $M_7 = 63$, but instead, $M = 57$. Table 1 summarizes the dilemma. It gives, from 5 to 7 electrons, the total angular momentum $M_g$ of the exact ground state for a range of filling.
factors around 1/3 expressed by \( N \Phi = m_{\text{max}} + 1 \). This range is chosen such that within it, the ground state angular momentum \( M_g \) increases through its Laughlin value \( M_N \). Also, \( m_{\text{max}} \) and the angular momentum of (13) are given. The table shows that for the CM and \( N = 5, 6, 7 \), an identification of the 1/3–ground state cannot be brought into agreement with what the Laughlin theory predicts. Either the maximum single particle angular momentum \( m_{\text{max}} \) or the total angular momentun \( M_g \) does not agree with (13). Certainly, finite size corrections could in principle, in the limit \( N \to \infty \), remedy this problem. But the tendency seems to be in the opposite direction. The situation does not improve at all for eight electrons where the ground state at \( \nu' = 1/3 \) has an angular momentum 77 unequal to \( M_8 = 84 \).

Disregarding this difficulty, we extrapolate the ground state energy per particle at \( \nu' = 1/3 \) for the block with Laughlin’s value \( M = M_N \) from data for up to 9 electrons to the thermodynamic limit. In the case \( \nu' = 1 \), the finite size corrections scale as \( \frac{1}{\sqrt{N}} \). Assuming this behavior also at \( \nu' = 1/3 \) yields

\[
\frac{E_g(1/3)}{N} = -0.409510 + 0.036220 \frac{1}{\sqrt{N}} + O(\frac{1}{N}).
\] (14)

Fig. 3 shows the exact data and the extrapolation. Eq. (14) is in good agreement with various other exact diagonalizations and with energy expectation value calculations using Laughlin’s trial wave function on the disk [27] and on the sphere [28, 44, 45].

We now change the model in two steps to arrive at the SRM and do a similar study there. At first, starting from the spectrum Fig. 1a, we omit the background. This changes the shape of the spectrum dramatically, see Fig. 1b. Without background, the repelling electrons tend to occupy the largest available angular momenta. In such a case, the Laughlin wave function cannot describe the ground state correctly. In the second step, changing the interaction to the SRM, we calculate the spectrum without background still using the same \( m_{\text{max}} = 17 \). Then, the ground state of this spectrum is highly degenerate, see Fig. 1c. This shows that a change from the CM with background to the SRM using the same external parameters \((N, m_{\text{max}})\) does not necessarily yield a nondegenerate ground state, as we expect at the stable state.

In order to find the nondegenerate ground state, we decrease \( m_{\text{max}} \) successively and find for \( N = 6 \) at \( m_{\text{max}} = 15 \) really a nondegenerate state of zero energy (Fig. 1e, for comparison see also the Figures 4a–f for 7 electrons including the stable state in Fig. 4c). For completeness, we show in Fig. 1d also the spectrum for the case of the SRM with a background where only \( V_1 \)–parts contribute to both electron–background and background–background interaction in (3).

The nondegenerate ground state as seen in the spectra of Fig. 1e and 4c for 6 and 7 electrons, respectively, can be immediately identified as the Laughlin wave function (13) for \( N = 6 \) and \( N = 7 \). It is well known that (13) is the exact and only solution of zero energy in the SRM at \( \nu = 1/3 \) because it is the only \( N \)–electron fermionic wave function not containing a component with relative angular momentum one for any two electrons [13, 33, 40]. The spectra
display the expected ground state angular momentum $M = M_N$ (in Fig. 1e, $45 = M_6$ and in Fig. 4c, $63 = M_7$). The situation is similar in the case of the sphere where the stable state has $\tilde{L}^2 = 0$ and is thus nondegenerate and homogeneous. Eventually, this finding justifies our use of $\nu$ which differs from the naive definition $\nu'$ by a finite size correction, since $\nu = 1/3$ for the parameters of Fig. 1e and 4c.

After having compared the spectra of the CM and the SRM at filling factor $1/3$ we conclude that the SRM seems to be – on the disk – the most promising model for a discussion of the concept of a stable $1/q$–state and of quasiparticles. This is true although thermodynamic instabilities are expected in this model [46]. In contrast to the case of the spherical geometry, where the results for the CM do not differ qualitatively from those for the SRM [23, 47], the SRM suggests itself for a study in the disk geometry. Thus, from now on, we will exclusively consider the SRM.

As a side remark, we supply an argument supporting the nondegeneracy of (13) in the disk geometry. We try to construct a state degenerate with (13) by applying a magnetic translation (commuting with $H$ for the infinite system)

$$\hat{\mathbf{T}}_\xi \Psi_{1/\eta} (z_1, \ldots, z_N) = \Psi_{1/\eta} (z_1, \ldots, z_N) e^{N/4 |\xi|^2} \prod_{i=1}^N e^{\frac{1}{2} z_i \xi^*}$$

An expansion of the product w.r.t. $\xi^*$ yields eigenfunctions with total angular momentum $M_N + l$

$$\sum_{l=0}^\infty \sum_{k=0}^l \sum_j N_{l,k,j} (\xi^*) \ P_{k,j}^{(l)} (z_1, \ldots, z_N) \Psi_{1/\eta} (z_1, \ldots, z_N).$$

(16)

The $P_{k,j}^{(l)}$ are symmetric polynomials of degree $l$, $k$ counts the maximum power of any variable $z_i$ ($0 \leq k \leq l$) occurring in $P_{l,j}^{(k)}$, and $j$ enumerates the individual polynomial for fixed $l$ and $k$. If $\Psi_{1/\eta}$ is an exact eigenstate, as it is e. g. for the SRM, then for an infinite large disk all the functions $P_{k,j}^{(l)} \Psi_{1/\eta}$ are again degenerate eigenfunctions, but with $\nu = \frac{N-1}{q(N-1)+k}$. However, we search for eigenfunctions with $\nu = 1/q$. Thus, we have to put $k = 0$ which leaves in (16) only the term $l = 0$, the Laughlin state (13). This shows how a translation of (13) followed by a restriction to the original area of the system (filling factor) leaves (13) invariant (see for the case of other topologies [31]). A similar procedure will prove useful in the case of the quasiparticles see below.

Finally, we comment on the degeneracy in the spectra of systems without background potential for small $M$. $M$–blocks between $M_{\text{min}}$ and $M_{\text{min}} - (N - 1) + m_{\text{max}} < M_N (q)$ are entirely unaffected by the confinement (this is only true without background). Energy levels with different $M$ are degenerate as the result of the center of mass motion (in Fig. 1b and 1c the part of the spectra between $M = 15$ and $M = 27$, and in Fig. 1e the part between $M = 15$ and $M = 25$). In some sense, even the Laughlin wave function is also such a state
because enlargement of the system does not influence the energy of the state with zero energy.

It should be mentioned that we have refrained from calculating the overlap between Laughlin’s wave function and the numerically calculated ground states because the agreement in \( M \), in connection with the energy of the state being zero, suffices for our conclusion. Summarizing our attempts to identify the \( 1/q \)-Laughlin state in numerically calculated spectra we see that for the SRM, the ground state can be identified uniquely already for a finite system. In the CM case, this is not possible. While on the sphere, the spectra of the two interactions for finite \( N \) are supposed to be related by perturbation theory, on the disk, they look completely different.

5 Quasiparticles

After having identified the stable state at the filling factor 1/3 in the SRM, we turn to the ground state properties at filling factors nearby. Laughlin [10] was the first who introduced the notion of quasiparticles at \( 1/q \) in the disk geometry. Here, we look for a definition applicable in our finite system. Starting from the stable state, we create the two kinds of quasiparticles by increasing and decreasing, respectively, the number of flux quanta through the area. The quasi-hole is then defined as the bulk ground state for a system with one additional flux quantum through the area covered by the electrons, i.e. \( m_{\text{max}} = q(N - 1) + 1 \), and a quasi-electron is the bulk ground state for a system with one flux quantum less, i.e., \( m_{\text{max}} = q(N - 1) - 1 \). This leads to a smaller and larger filling factor, respectively, with a deviation of the order \( 1/N \) from \( 1/q \).

In general, the quasiparticles are fundamentally different from ordinary quasiparticles of a Fermi liquid in that they cannot be constructed from the original fermions by a process of switching on the interaction adiabatically. They have various descriptions as anyons [12], bosons [13] or fermions [11]. They are not elementary particles as the electrons, but charge deviations from the homogeneous density of the Laughlin state. There are two applications of this notion of quasiparticles. The first is the description of excited states at an unchanged filling factor \( 1/q \). In this picture, quasihole and quasielectron, separated by an infinite distance, form a quasiexciton [18] which is a short-wavelength excitation \((k \to \infty)\) [22, 30, 19]. The energy of this excitation is the sum of the two quasiparticle energies; this is the gap seen in the activation measurements of \( \rho_{xx} \) [21]. Otherwise, for small \( k \), a collective theory yields the low-lying excitations [22].

Here, we are interested in the other use of the quasiparticles in which they describe the ground state properties near \( 1/q \) and in which they are the elements of the hierarchical theory [13]. This theory sets out to understand the occurrence of filling factors \( p/q \) (\( p \neq 1, q - \text{odd} \)). Starting with \( N \) free electrons in \( N_{\Phi} \) energetically degenerate states in the lowest Landau level and switching on an interaction (e.g. the SRM–interaction), one gets a stable state for the spe-
cial filling factors $\nu = 1/q$, i. e. for $N_\Phi - 1 = q(N - 1)$. The hierarchical theory now draws an analogy from the electrons to the quasiparticles. These can occupy of the order of $N$ degenerate states. Creating a macroscopic number $N'$ of quasiparticles one gets in the presence of a quasiparticle–quasiparticle interaction a new stable state, the "daughter state on the first level of the hierarchy", for the special filling factor $1/p$ of the quasiparticles, i. e. for $N = p(N' - 1)$ ($p$ – even for bosonic quasiparticles). Then, the electronic filling factor is given by

$$\nu = \frac{N - 1}{(N - 1)q + N'} \xrightarrow{N \to \infty} \frac{1}{q \pm \frac{1}{p}}. \quad (17)$$

For $q = 3$ and $p = 2$ this gives filling factors of $2/5$ and $2/7$. Iterating the scheme, one can find successively all filling factors between 0 and 1 with odd denominators in an unique manner on different levels of the hierarchy.

We want to emphasize that there are several propositions which the quasiparticles have to fulfill for a justification of this theory.

– The lowest one–quasiparticle levels must be energetically degenerate at least in the thermodynamic limit, and there must be an energy gap to the other states of higher energy.

– For a two–quasiparticle system (two flux quanta more or less), it must be possible to identify the low energy states as two free quasiparticles plus an interaction contribution. I. e. , these states must be separated from the energetically higher states which means that the interaction must be small enough to leave a gap between the two–quasiparticle states and the neglected upper states.

– The low–energy levels for more than two quasiparticles, particularly for a macroscopic number of them, must be describable by an effective Hamiltonian with up to two–particle interactions as determined from the two–quasiparticle spectra. Higher interactions are not allowed. This is the strongest condition and the test for the validity of the hierarchical theory itself.

In summary, the low–energy behavior of the effective quasiparticle Hamiltonian has to reproduce the low–energy behavior of the original fermionic problem. For the simplest non–trivial case of quasiholes at $\nu = 1$, this has been successfully checked on the disk with a special interaction [51].

In the past, there were a lot of efforts to justify quantitatively the quasiparticle picture. Almost all authors used the spherical geometry and thus neglected the effect of boundaries. The work was mainly concentrated on the determination of the quasiparticle energies at various filling factors [11, 27] and then, conclusions were drawn for the hierarchical theory. Béran and Morf [38] extracted the effective interaction of quasielectrons on the sphere and estimated the ground state energy and the gap of the $2/5$–stable state. Endesfelder and Terzidis [52] extrapolated the interaction of quasiholes and compared the spectra of a small number of bosons interacting via this interaction with electronic spectra of the same number of quasiholes. There is a large asymmetry between quasiholes and quasielectrons [53]. Their energies are of a different order of magnitude, and the subspaces of low lying excitations have different dimensions for the
quasiholes and the quasielectrons. Thus, it was tried to explain these different dimensions with a difference in the quasiparticle-interaction: both quasiparticles are bosons, but quasielectrons have a hard-core interaction.\[54\].

In the two following sections, we want to study whether a degeneracy of the quasiparticles is found in the disk geometry. In the spherical geometry, this degeneracy holds trivially because the angular momentum of the quasiparticles is $\vec{L}^2 = \frac{N}{2}(\frac{N}{2} + 1)$ and, therefore, there are $N + 1$ degenerate states ($L^z$-degeneracy)\[20\]. In contrast to the authors cited above, we investigate the quasiparticles in the disk geometry, where the situation is more difficult as a result of the boundary.

6 Quasiholes

A quasihole is created by adding one flux quantum to the system at $\nu = 1/q$ by increasing the area of the stable state system by one flux quantum, i.e., $m_{\text{max}} = (N - 1)q + 1$. For the short-range model, we know already such a state, an exact eigenfunction of zero energy, i.e. put $k = 1$ in \(\text{(16)}\). Expanding the exponential in \(\text{(15)}\) up to first order in $z_i$ (and substituting $\xi^* \rightarrow -2/\xi$) one finds a function of the form

$$\prod_{i=1}^{N}(z_i - \xi)\Psi_\frac{1}{q} = \sum_{l=0}^{N}(-\xi)^{N-l}\Psi_{\frac{1}{q}(-M_N(q)+l)}$$

i.e., the quasihole wave function earlier proposed by Laughlin\[10\] as a good trial wave function for the CM. The r. h. s. of \(\text{(18)}\) defines the expansion in components with angular momenta $M = M_N(q) + l$. Thus, there are $N + 1$ degenerate states with zero energy as on the sphere.

The inclusion of the Laughlin wave function \(\text{(13)}\) in \(\text{(18)}\) as the quasihole component $l = 0$ is the consequence of a somewhat ambiguous situation. The quasihole has a local charge deviation from the homogeneous value $\frac{\nu}{2\pi}$ of the charge distribution of the electrons on the disk. E.g. for $\nu = 1$, the one-particle density of a quasihole at $\xi$ is\[53\]

$$n(-\xi)(z) = \frac{1}{2\pi}(1 - e^{-\frac{1}{2}|z-\xi|^2}).$$

Around $\xi$ there is a charge depletion of magnitude $e^* = \nu e$. For a finite system, the charge missing at $\xi$ accumulates at the finite edge reflecting the enlargement of the disk area by one flux quantum. The component of \(\text{(18)}\) with the largest possible total angular momentum $M = M_N(q) + N$ is identical to the quasihole at $\xi = 0$. If the parameter $\xi$ is considered as a particle coordinate for a quasihole in the lowest Landau level, smaller angular momenta of the electrons correspond to larger radii for the quasiholes\[11\]. Then, the component of \(\text{(18)}\) with the smallest $M = M_N(q)$ corresponds to the quasihole on the border of the disk and cannot be distinguished from the Laughlin state. In this sense, the Laughlin state \(\text{(13)}\) is a quasihole too and therefore, for the
creation of a quasihole at a location $\xi \neq 0$, this is also needed.

The angular momentum component corresponding to a quasihole at $\xi = 0$
can be chosen to generate the other quasihole components with the help of
the magnetic translation as follows. Starting with $\prod_{i=1}^{N} z_i \Psi$, applying $\hat{T}_\xi$,
and annihilating all total angular momentum components with a filling factor
corresponding to $m_{\text{max}} > (N - 1)q + 1$ one recovers the quasihole function
(15). Physically, this means a shift of the quasihole at $\xi = 0$ against the
homogeneous background of the Laughlin wave function followed by a cut off
at the border of the original system.

Now let us turn to the spectra for one quasihole. For the Coulomb case, none
of the spectra for 7 electrons in Fig. 2 shows any evidence of a degeneracy
around $\nu' = 1/3$ in the low energy region. This observation expresses again
the difficulty with the interpretation of the spectra with Coulomb interaction
for a finite number of electrons. Therefore, coming back to the SRM, we discuss
Fig. 4d (7 electrons). Actually, the ground state is $(N+1)$–fold degenerate and
has zero energy and angular momenta reaching from $M = M_N$ to $M_N+N$ (here
$M = 63$ to $70$). This is obviously in agreement with the fact that the state
(15) is a zero energy eigenstate, which follows because the multiplication with
powers of $z_i$ can not change the minimum relative angular momentum from
3 in Laughlin’s wave function. The low energy states, appearing in branches
above the zero ground states with excitation energies of an order of magnitude
smaller than the energy scale given in the SRM by $V_{1}^{\text{SRM}} \approx 0.44$, are connected
with the edge excitations [56].

We want to determine the quasihole energy in the SRM. Since there are three
external parameters in our model, three mechanisms exist for the creation
of quasiparticles. Correspondingly, the three energies (the indices ± mean
quasihole and quasielectron, respectively) are the gross quasiparticle energy $\varepsilon_\pm$
(change of the electron number), the proper quasiparticle energy $\bar{\varepsilon}_\pm$ (change
of the magnetic field) and the neutral quasiparticle energy $\varepsilon^n_\pm$ (change of the
area) [27, 55]. There are relations between these energies involving the ground
state energy per particle of the stable state. Thus, it is sufficient to know one
of the energies. The energy taken from our spectra is the neutral quasiparticle
energy because we change the disk area for the creation of the quasiparticle.

In the SRM model with zero ground state energy, all these energies are equal,
i. e., $\varepsilon = \bar{\varepsilon} = \varepsilon^n (\nu < 1)$. The neutral quasihole energy for the $\nu = 1/q$–state
is defined as the difference of the ground state energy of the one–quasihole
spectrum ($m_{\text{max}} = q(N-1)+1$) and that of the stable state ($m_{\text{max}} = q(N-1)$).
We find from the exact spectra in agreement with the analytical arguments
given above $\varepsilon^n_{(1/3)} = \bar{\varepsilon}_{(1/3)} = \varepsilon_{(1/3)} = 0$.

Two–quasihole spectra are pictured in Fig. 1c for 6 electrons and in Fig. 4e
for 7 electrons ($m_{\text{max}} = 20$), where we have again a degenerate ground state.
But in contrast to the one–quasihole spectrum, where the zero energy levels
in one block (a line) were non–degenerate, the levels in this case are partially
degenerate; e. g., in Fig. 4e we find for the $M$ reaching from 63 to 77 ($M = M_7$
to $M = M_7 + 2 \cdot 7 = 77$) energy degeneracies of 1, 1, 2, 2, 3, 3, 4, 4, 4, 3, 3,
2, 2, 1, 1. This subspace of the lowest energy eigenstates (here of zero energy) can be considered as being made up of bosonic quasiholes. The quasihole can occupy $N + 1$ one-particle states, thus $d$ bosons (quasiholes) form a Hilbert space of dimension $\binom{(N+1)+d-1}{d} = \binom{N+d}{d}$. In particular, the total degeneracy for two quasiholes is just $\binom{N+2}{2}$, i.e., for $N = 7$, there are 36 states, and the exact spectrum actually shows this. In general, the degeneracy $g_B(N, d, \tilde{M})$ of a level with $M = M_N(q) + \tilde{M}$ in a $d$-particle Bose system with $N + 1$ one-particle states can be derived from the following generating function which is analogous to the fermionic formula (12):

$$\prod_{l=1}^{d} \frac{1 - x^{(N+1)+(l-1)}}{1 - x^l} = \sum_{\tilde{M}=0}^{Nd} g_B(N, d, \tilde{M}) x^{\tilde{M}}.$$  

(20)

Additionally, we confirm the validity of this description by inspection of the degeneracy of the zero energy levels of, e.g., the three-quasihole spectrum in Fig. 4f ($d = 3, N + 1 = 8, \tilde{M} = 0, \ldots, 21$). On the other hand, it is immediately clear that the total dimension of the Hilbert space is actually

$$\sum_{\tilde{M}=0}^{Nd} g_B(N, d, \tilde{M}) = \binom{N+d}{d}.$$  

We summarize that the $1/3$-quasihole spectra in the SRM corroborate strongly the quasihole picture already for finite $N$. Unfortunately, the interaction between the quasiholes is zero. A bosonic description of the quasiholes appears quite natural considering the degeneracies in our spectra. This conclusion is in agreement with investigations on the sphere where the Hilbert–space dimension can be understood in the boson–picture [13, 54].

7 Quasielectrons

In the hierarchical construction, the quasiparticles enter symmetrically, but in their energies and spectra, they are not very symmetric [53]. For the energies, this is quite inevitable, since a $1/3$-quasielectron is created by decreasing the maximum angular momentum to $m_{\text{max}} = q(N - 1) - 1$ in order to reach a larger $\nu$. Confining the electrons to a smaller disk gives rise to a stronger interaction contribution compared to the case of an enlargement of the disk. Thus, $\varepsilon_{n+}(1/3)$ must be positive.

Since the SRM plays the role of a canonical model, particularly with respect to the quasiholes, we hope that it reflects the essential physics in the quasielectron case too. Still, the situation for the quasielectrons is different from that for the quasiholes because we can not expect zero energy eigenvalues and we do not know an exact eigenfunction for the quasielectron up to now.

Let us look at the one–quasielectron spectrum for the SRM for 7 electrons in Fig. 4b. The ground state at $M_7 = 63$ has a non–zero energy. This is to be expected because it is not possible to avoid relative angular momentum one in any wave function with $M \leq M_N$ and $m_{\text{max}} = 3(N - 1) - 1$. Further, there are low lying energy levels with $M \geq M_N$ (see Fig. 4b) which are, as in the case of
the quasihole, again attributed to the edge of the system. The quasielectrons are the lowest energy levels with $M = M_N - N, \ldots, M_N - 1$ (here from $M=56$ to 62) whose energies are almost degenerate.

While this seems to be in contradiction to the usual definition of the quasielectron as a ground–state of the spectrum, we justify our focus on these angular momentum blocks by considering the quasielectron trial wave functions. That there are many proposals for such trial wave functions \cite{10, 27, 55, 57, 58} reflects the fact that one has for the quasielectron a more complicated situation than for the quasihole. Still, all proposals have angular momentum components in the range of $M = M_N - N, \ldots, M_N$. This becomes particularly clear, if one considers Haldane’s quasielectron on the sphere \cite{13}. This $N + 1$–fold degenerate wave function is a good trial wave function for the description of the quasielectron which is well exhibited in the spectra on the sphere \cite{13, 21}. After a projection onto the plane according to the rules given in \cite{28} the resulting wave function has $N + 1$ angular momenta components reaching from $M = M_N - N$ to $M_N$ and this is our trial function introduced in \cite{57}. The component with $M = M_N$ has much smaller energy than the other components and is therefore excluded.

After having identified the angular momenta of the quasielectron components we want to evaluate the energies. The exact energies for up to 8 electrons have been compared with those of Laughlin’s, Jain’s, MacDonald’s and our quasielectron trial wave functions in our previous work \cite{37}. We extend here this comparison to 9 electrons. The exact energies for $N = 9$ are pictured in Fig. 5a together with the trial function data from Table 1b in \cite{37}; the data from Table 1c in \cite{37} for $N = 10$ (without results from exact diagonalizations) are shown in Fig. 5b. For both particle numbers, the pictures are very similar with respect to the $M$–dependence of all trial wavefunctions; the exact eigenenergies are almost independent of $M$. Jain’s quasielectron wave function has, in comparison with the other trial functions, the best energy expectation values, but for higher $M$, the deviations from the exact values are still quite large.

The exact data in the one–quasielectron spectrum exhibit already for a finite electron number almost degeneracy. This degeneracy is not so well expressed in Fig. 4b as in Fig. 5a, because of the different scale; the relative deviation from the average value is for $N = 9$ only 5.6 %.

Next we try to find the neutral quasielectron energy $\varepsilon_n^0 (1/3)$ from the exact numerical data for up $N = 9$ electrons in the SRM. Unfortunately, an extrapolation with respect to $N$ for a single component of the quasielectron, e. g. for the one with the smallest angular momentum $M = M_N - N$, does not work because the energy fluctuates with $N$. Therefore, we calculated the arithmetic mean of the quasielectron energies of all $N$ components forming the new quantity $\langle \varepsilon_n^0 (1/3) \rangle$. Then, these data are extrapolated to the thermodynamic limit by minimizing the weighted quadratic deviation from an extrapolation curve with a finite size correction proportional to a negative rational power of $N$ to be determined. This way, we find in the thermodynamic limit

$$\langle \varepsilon_n^0 (1/3) \rangle = 0.1865 + 1.8790 \, N^{-2.6320} \quad (21)$$
This curve is shown in Fig. 6a. The finite size corrections vanish surprisingly fast. Arguments given by Morf et al. [27, 59] for a $N^{-\frac{3}{2}}$ correction are not applicable here because there is no background in the SRM. The result (21) can be compared with the numerical data for the discontinuity in the chemical potential in the case of short–range interacting electrons on the surface of the sphere. We extrapolate from Table 2 in [46] the value for the jump of the chemical potential and find that the neutral quasielectron energy at $\nu = 1/3$ is 0.1905 showing quite good agreement between our extrapolation and the result of the other geometry.

We now extrapolate the energy expectation values of the trial wave functions, see Fig. 6b, again using the average of the available components at a given $N$. Here, the leading finite size correction is unknown too. The extrapolation of Jain’s quasielectron is closest to the exact data. On the other hand, the finite size correction is qualitatively different from that of the exact data, see Fig. 6a, particularly the approximate $1/\sqrt{N}$–behavior of Jain’s trial wave function is not found in the exact data. Still Jain’s proposal comes closest to the extrapolated value of 0.1865. The proposal of [57], on the other hand, yields a higher limit, but the behavior of its finite size correction is much more similar to the exact result.

One of the most interesting questions for a test of the relevance of the notion of a quasielectron is the understanding of the many–quasiparticle behavior. The simplest case is the two–quasielectron spectrum shown for 7 electrons for the SRM (Fig. 4a). We realize that different from the case of the two–quasihole spectrum there is a complicated interaction superposed by finite size effects. It seems difficult to identify the two–particle states arising from the subspace generated by the quasielectrons. One of the reasons should be a strong interaction between the quasielectrons leading to a mixing with other energy levels of the many–electron system. Thus, a detailed investigation of the quasielectron interaction for two quasielectrons and further implications for more quasielectrons in the framework of the hierarchical scheme remain a matter of future research. Now, there is an intensive discussion on this point [17, 54, 60] and a satisfying explanation of the success of Jain’s wave function and its connection with the ideas of the hierarchical scheme are still missing.

We do not believe that a proposal reconciling the two pictures [60] is justified up to now.

8 Summary

The numerical diagonalization of the FQHE Hamiltonian for up to nine electrons on the disk shows that whether a unique identification of a particular, stable state at filling factor $1/q$ is possible in the spectra depends on the electronic interaction. This is in contrast to the spherical geometry. Whereas for the Coulomb interaction, the ground states of spectra around $1/3$ can not be related to the Laughlin wave function for small $N$, the latter turns out to be the exact ground state solution of the model with short–range interaction.
The picture that the spectra for Coulomb and for the short-range interaction are very similar and that the states of both interactions are related to each other by only a small perturbation could not be quantitatively confirmed for the present geometry of a finite disk.

For the identification of the stable state and the quasiholes, the SRM serves as a canonical model. Then, \( N + 1 \) zero–energy levels can be exactly identified in the one–quasihole spectra as quasihole components of the Laughlin type. The degeneracy of these levels in the spectra for two and more quasiholes justifies to treat the quasiholes as bosons.

In the quasielectron case, \( N \) components of the quasielectron are identified which show almost degeneracy with small deviations from the average value in our finite size calculations. The expectations values of trial wave functions of various authors are compared with the exact data. They deviate with increasing angular momentum more and more from the exact values leaving space for improvement. An extrapolation of the average quasielectron energy to the thermodynamic limit is in agreement with the results of other geometries.

It should be interesting to use the quasiparticle identification on the disk in order to extend the present treatment to the quasielectron–quasielectron interaction. Creation energies and interaction parameters could then be used as input parameters in an attempt to formulate an effective theory for the quasielectron.

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Table 1

| N  | $N_\Phi = m_{\text{max}} + 1$ | $M_a$ | Remarks                                      |
|----|-------------------------------|-------|----------------------------------------------|
| 5  |                               |       |                                              |
|    | 13                            | 25    | $\nu = 1/3$                                 |
|    | 14                            | 25    | $\nu = 1/3 + \text{one quasihole}$           |
|    | 15                            | 30    | $\nu = 1/3 + \text{two quasiholes, } \nu' = 1/3$ |
|    | 16                            | 30    | ...                                          |
|    | 17                            | 30    | ...                                          |
|    | 18                            | 30    | ...                                          |
|    | 19                            | 35    | ...                                          |
|    | 13                            | 30    | Laughlin state                               |
| 6  |                               |       |                                              |
|    | 16                            | 35    | $\nu = 1/3$                                 |
|    | 17                            | 39    | $\nu = 1/3 + \text{one quasihole}$           |
|    | 18                            | 45    | $\nu = 1/3 + \text{two quasiholes, } \nu' = 1/3$ |
|    | 19                            | 45    | ...                                          |
|    | 20                            | 45    | ...                                          |
|    | 21                            | 45    | ...                                          |
|    | 22                            | 51    | ...                                          |
|    | 16                            | 45    | Laughlin state                               |
| 7  |                               |       |                                              |
|    | 19                            | 51    | $\nu = 1/3$                                 |
|    | 20                            | 57    | $\nu = 1/3 + \text{one quasihole}$           |
|    | 21                            | 57    | $\nu = 1/3 + \text{two quasiholes, } \nu' = 1/3$ |
|    | 22                            | 63    | ...                                          |
|    | 23                            | 63    | ...                                          |
|    | 24                            | 63    | ...                                          |
|    | 25                            | 69    | ...                                          |
|    | 19                            | 63    | Laughlin state                               |

“Search for the stable state” in the Coulomb model (CM)
Figure Captions

Fig. 1: Complete many–particle energy spectrum for 6 electrons and various interactions. The levels are ordered with respect to the total angular momentum $M$.

a: Coulomb interaction with background, degeneracy 18 ($m_{\text{max}} = 17$), i. e. $\nu' = 1/3$.
b: as a, but without background.
c: as a, but for the short–range interaction (8) without background.
d: as c, but with background.
e: short–range interaction without background, $m_{\text{max}} = 15$, i. e. $\nu = 1/3$.

Fig. 2: Series of spectra showing for decreasing filling factor the low lying states for 7 electrons with Coulomb interaction; $m_{\text{max}} = 16, \ldots, 21$ (Fig. 2a – f).

Fig. 3: Energy per particle for the model with Coulomb interaction and background at $\nu' = 1/3$ in dependence on the number of electrons $N$ up to $N = 9$. The line extrapolates to the thermodynamic limit.

Fig. 4: Series of spectra showing for decreasing filling factor the low lying states for 7 electrons with the SRM interaction (8); $m_{\text{max}} = 16, \ldots, 21$ (Fig. 4a – f). Fig. 4c shows the stable state, Fig. 4d and 4b the one–quasihole state and the one–quasielectron state, respectively.

Fig. 5: Numerically calculated expectation values for Laughlin’s, our [57], and Jain’s quasielectron trial wave functions (from top to bottom) arranged with increasing $M$;

a: $N = 9$ electrons, the lowest levels (dotted line) are the exact energies.
b: $N = 10$ electrons.

Fig. 6: Averaged quasielectron energy in the SRM (8) extrapolated to the thermodynamic limit.
a: exact energies for up to $N = 9$.
b: Jain’s ($\times$), Laughlin’s (+) and our (∗) [57] quasielectron trial wave functions for up to $N = 10$. 
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