SYNTHETIC FOUNDATIONS OF CEVIAN GEOMETRY, II: THE CENTER OF THE CEVIAN CONIC

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Abstract. We study the conic $C_P$ on the five points $ABCPQ$, where $ABC$ is a given ordinary triangle and $Q$ is the isotomcomplement of $P$, defined as the complement of the isotomic conjugate $P'$ of $P$ with respect to triangle $ABC$. The properties of $C_P$ are shown to be related to the affine mapping $\lambda = T_P \circ T_P'$, where $T_P$ and $T_P'$ are the unique affine maps taking $ABC$ to the cevian triangles of $P$ and $P'$, respectively. We characterize the center $Z$ of $C_P$ as the unique fixed point of $\lambda$ in the extended plane, when $C_P$ is a parabola or an ellipse; and as the unique ordinary fixed point of $\lambda$, when $C_P$ is a hyperbola. When $P$ is the Gergonne point of $ABC$, this gives a new characterization of the Feuerbach point $Z$. All of our arguments are purely synthetic.

1. Introduction.

In this paper we continue the investigation begun in Part I [9], and study the conic $C_P = ABCPQ$ on the five points $A, B, C, P, Q$, where $Q = K \circ \iota(P)$ is the isotomcomplement of $P$, defined to be the complement of the isotomic conjugate of $P$ with respect to $ABC$. Here, as in Part I, $P$ is any point not on the extended sides of $ABC$ or its anticomplementary triangle. This conic is defined whenever $P$ does not lie on a median of triangle $ABC$. We show first that the symmetrically defined points $P' = \iota(P)$ and $Q' = K \circ \iota(P') = K(P)$ also lie on this conic, as well as 6 other points that can be given explicitly (see Theorem 2.1 and Figure 1).

Recall from Part I that the map $T_P$ is the unique affine map which takes triangle $ABC$ to the cevian triangle $DEF$ of $P$ with respect to $ABC$. The map

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$T_P$ is defined in the same way for the point $P'$. In Theorem 2.4 we show that if $P$ and $P'$ are ordinary points and do not lie on a median of $ABC$, then

$$
\eta T_p = T_{P'} \eta,
$$

where $\eta$ is the harmonic homology (affine reflection, see [4]) whose center is the infinite point $V_\infty$ on the line $PP'$ and whose line of fixed points is the line $GV$, where $G$ is the centroid of $ABC$ and $V = PQ \cdot P'Q'$. Thus, $T_P$ and $T_{P'}$ are conjugate maps in the affine group. We prove this formula synthetically by proving an interesting relationship between the centroids $G_1$ and $G_2$ of the cevian triangles $DEF$ and $D_3E_3F_3$ of the points $P$ and $P'$, respectively. Lemma 2.5 shows that $G$ is the midpoint of the segment $G_1G_2$ and $\eta(G_1) = G_2$.

After we introduce the affine map

$$
\lambda = T_{P'} \circ T_P^{-1}
$$

in Section 3, we show that the 6 points on $C_P$ mentioned above are the images of the vertices $A, B, C$ under $\lambda$ and $\lambda^{-1}$ (Theorem 3.4). The mapping $\lambda$ leaves the conic $C_P$ invariant as a set (Theorem 3.2), and is the main affine map considered in this paper. The relation (1) allows us to write the map $\lambda$ as

$$
\lambda = \eta \circ (T_P \circ \eta \circ T_P^{-1}) = \eta_1 \circ \eta_2,
$$

where both maps $\eta_1$ and $\eta_2$ are harmonic homologies. Using this representation we prove in Theorem 4.1 that the center $Z = Z_P$ of the conic $C_P$ is the intersection

$$
Z = GV \cdot T_P(GV),
$$

and that when $C_P$ is a parabola or an ellipse, $Z$ is the unique fixed point of $\lambda$ in the extended plane. When $C_P$ is a hyperbola, $Z$ is the unique ordinary fixed point of $\lambda$. In the latter case, $\lambda$ also fixes the two points at infinity on the asymptotes of $C_P$.

At the end of the paper we interpret the mapping $\lambda$ as an isometry on the model of hyperbolic geometry whose points are the interior points of the conic $C_P$ and whose lines are Euclidean chords.

In Part III [10] of this series of papers we will prove that the point $Z$ is a generalized Feuerbach point. In particular, this will show that $Z$ is the Feuerbach point of triangle $ABC$ when $P$ is the Gergonne point and $P'$ is the Nagel point of $ABC$ (see [1] and [8]). Our Theorem 4.1 therefore gives a representation of the Feuerbach point as the intersection of two lines. Corollary 4.2 shows that these two lines are $GV$ and $T_P(GV) = G_1J$, where $J$ is the midpoint of $PQ$. A third line through $Z$ is the line $G_2J'$, where $G_2 = T_P(G)$ and $J'$ is the midpoint of $P'Q'$. (See Figure 5 in Section 4.) In this case $C_P$ is a hyperbola, and the Feuerbach point $Z$ is the unique ordinary fixed point of $\lambda = T_{P'} \circ T_P^{-1}$.

We mention one more fact that we prove along the way. The mapping

$$
S' = T_{P'}T_PT_{P'}^{-1}T_P^{-1}
$$
is always a translation, which implies (Corollary 2.8 and Figure 4) that the
triangles $A_3B_3C_3 = T_P(D_3E_3F_3)$ and $A'_3B'_3C'_3 = T_{P'}(DEF)$ are always
congruent triangles.

We refer to Part I [9] for the notation that we use throughout this series
of papers; to [1], [11], [12], or [13] for definitions in triangle geometry; and
to [2] and [3] for classical results from projective geometry.

2. The conic $ABCPQ$ and the affine mapping $T_P$.

We start by proving

**Theorem 2.1.** If $P$ does not lie on the sides of triangle $ABC$ or its anticomplemen-
tary triangle, and not on a median of $ABC$, then there is a conic $C_P$ on the points
$A, B, C, P, Q, P', Q'$. This conic also passes through the points

$$ A_0P \cdot D_0Q', \quad B_0P \cdot E_0Q', \quad C_0P \cdot F_0Q', $$

and

$$ A'_0P' \cdot D_0Q, \quad B'_0P' \cdot E_0Q, \quad C'_0P' \cdot F_0Q, $$

where $D_0, E_0, F_0$ are the midpoints of the sides $BC, CA, AB$, and $A_0B_0C_0 = T_P(D_0E_0F_0)$ and $A'_0B'_0C'_0 = T_{P'}(D_0E_0F_0)$ are the medial triangles of $DEF$ and
$D_2E_2F_2$, respectively.

**Proof.** The condition that $P$ is not on a median of $ABC$ ensures that the points
$P$ and $Q$ are not collinear with one of the vertices. If $P, Q, A$ and $A$ collinear,
for example, then the points $D = AP \cdot BC$ and $D_2 = AQ \cdot BC$ coincide, so
$A_1 = T_P(D)$ and $A_2 = T_P(D_2)$ coincide, meaning that $Q'$ is collinear with $A$
and $G$ (by I, Theorem 3.5). But $P$ is collinear with $K(P) = Q'$ and $G$, so $P$
would lie on $AG$. By the same reasoning, $P, Q'$, and $A$ are not collinear and
neither are $P, P'$ and $A$.

Now the mapping $T_P$ is a projective mapping which takes the pencil
of lines $x$ on $Q'$ to the pencil $y = T_P(x)$ on $P$, since $T_P(Q') = P$ (I, Theorem
3.7). For the lines $x = AQ'$ and $y = AP$ we have $x \cdot y = A$; while
$x = BQ'$ and $y = EP = BP$ give $x \cdot y = B$; and $x = CQ'$ and $y = FP = CP$ give
$x \cdot y = C$. Thus the pencil $x$ is not perspective to the pencil $y$, so Steiner’s
theorem [3], p. 80 implies that the locus of points $x \cdot y$ is the conic $ABCPQ'$.
If $x = QQ'$ then $y = QP$, so $x \cdot y = Q$ is also on this conic. Hence the conic
$ABCPQ' = ABCQ'$. Arguing the same with the mapping $T_{P'}$ shows that
there is a conic $ABCPQ = ABCQQ'$. It follows that the conic $C_P = ABCQP'$
lies on $P'$ and $Q$, so that $C_P = ABCQP$. This proves the first assertion.
Letting $x = D_0Q'$ gives $y = A_0P$, so $A_0P \cdot D_0Q'$ is on $C_P$, as are all the
other listed intersections. \hfill \Box

**Corollary 2.2.** (a) If $Y$ is any point on the conic $C_P$ other than $P, Q'$ (respectively
$Q, P'$), then $T_P(Q'Y) = PY$ (resp. $T_P(QY) = P'Y$).
(b) The conic $C_P$ is the locus of points $Y$ for which $P, Y,$ and $T_P(Y)$ are collinear.
(c) In particular, $P, P', T_P(P'),$ and $T_{P'}(P)$ are collinear (whether $P$ lies on a median
or not).
Proof. Part (a) follows from the definition of the conic \( C_P \) as the locus of intersections \( x \cdot y \). For part (b), if \( Y \) lies on \( C_P \), then by part (a), \( P, Y, \) and \( T_P(Y) \) are collinear. This also clearly holds for \( Y = P \) and \( Y = Q' \). Conversely, suppose \( P, Y, \) and \( T_P(Y) \) are collinear. If \( Y \notin \{P, Q'\} \), then \( T_P \) maps the line \( x = Q'Y \) to the line \( y = PT_P(Y) = PY \), so \( x \cdot y = Y \) lies on \( C_P \). Part (c) follows from part (b) with \( Y = P' \) and from the analogous statement obtained by switching \( P \) and \( P' \), as long as \( P \) does not lie on a median of \( ABC \). Now assume that \( P \neq G \) lies on the median \( AG \). Then \( P, P', Q, \) and \( Q' \) are all on \( AG \). From the Collinearity Theorem (I, Theorem 3.5) the points \( D_i \) for \( 0 \leq i \leq 4 \) are all the same point, so \( A_0 = A_2 = A_3 \), the midpoint \( EF \cdot AG \) of \( EF \); and similarly, the points \( A'_i \) equal \( A'_0 = A'_3 \), the midpoint \( E_3F_3 \cdot AG \) of \( E_3F_3 \). From \( P' \) on \( AD_3 \) it follows that \( T_P(P') \) is on \( DA_3 = D_0A_2 = AG \). Similarly, \( T_P(P) \) lies on \( AG = PP' \). □

Remarks. 1. We know from I, Theorems 3.13 and 2.4 that the lines \( D_0Q', E_0Q', F_0Q' \) in (1) are the cevians for the point \( Q' \) with respect to the anticevian triangle \( A'B'C' \) of \( Q, \) since \( A'Q', B'Q', \) and \( C'Q' \) pass through the midpoints...
of the sides of $ABC$. See Theorem 3.4 below.

2. Part (b) of the corollary allows for an easy computation of an equation for the conic $C_P$.

We will assume the hypothesis of Theorem 2.1 anytime we make use of the conic $C_P = ABCPQ$. Alternatively, we could define $C_P$ to be a degenerate conic, the union of a median and a side of $ABC$, when $P$ does lie on a median. In that case, $P, Q, P', Q'$ are all on the median.

Assume now that the point $P$ is ordinary and does not lie on a median of triangle $ABC$ or on $(l_\infty)$. Then the points $P'$ and $Q$ are also ordinary. We shall use the conic $C_P$ to prove an interesting relationship between the maps $T_P$ and $T_{P'}$. First note that the lines $PP'$ and $QQ'$ are parallel, since $K(PQP') = Q'GQ$. The midpoint of $QQ'$ is clearly the complement of the midpoint of $PP'$, so the line joining them passes through the centroid $G$. Since the quadrangle $PP'QQ'$ is inscribed in $C_P$, its diagonal triangle is a self-polar triangle for $C_P$ ([3], p. 75). The vertices of this self-polar triangle are $PQ' \cdot P'Q = G, PQ \cdot P'Q' = V$, and $PP' \cdot QQ' = V_\infty$, a point on the line at infinity. Hence, the polar of $G$ is the line $VV_\infty$, which is the line through $V$ parallel to $PP'$. The polar of $V_\infty$ is $GV$. Since the segment $QQ'$ is parallel to $PP'$ and half its length, $Q$ is the midpoint of segment $PV$, so $V$ is the reflection of $P$ in $Q$, and the reflection of $P'$ in $Q'$. Considering the quadrangle $VQQGQ'$, it is not hard to see that $GV$ passes through the midpoints of segments $PP'$ and $QQ'$, since these midpoints are harmonic conjugates of $V_\infty$ with respect to the point pairs $(QQ')$ and $(PP')$.

Let $Z = Z_P$ be the center of the conic $C_P = ABCPQ$, so $Z$ is the pole of the line at infinity. Since $V_\infty$ lies on the polar of $Z$, $Z$ must lie on $GV$. Since $PP'$ and $QQ'$ are parallel chords on the conic $C_P$, the line through their midpoints passes through $Z$ ([2], p. 111). Hence we have:

**Proposition 2.3.** Assume that the ordinary point $P$ does not lie on a median of triangle $ABC$ or on $(l_\infty)$.

(a) The points $G, V = PQ \cdot P'Q'$, and $V_\infty = PP' \cdot QQ'$ form a self-polar triangle with respect to the conic $C_P = ABCPQ$.

(b) The center $Z$ of the conic $C_P$ lies on the line $GV = G(PQ \cdot P'Q')$, which is the polar of the point $V_\infty$.

(c) The line $GV = GZ$ passes through the midpoints of the parallel chords $PP'$ and $QQ'$.

(d) The harmonic homology $\mu_G$ with center $G$ and axis $VV_\infty$ maps the conic $C_P$ to itself. In other words, if a line $GY$ intersects the conic in points $X_1$ and $X_2$ and $GY \cdot VV_\infty = X_3$, then the cross-ratio $(X_1X_2G X_3) = -1$.

(e) The line $VV_\infty$ is the same as the line joining the anti-complements of $P$ and $P'$.

Thus, the polar of $G$ is the line $K^{-1}(PP')$.

(f) The point $V$ is the midpoint of the segment joining $K^{-1}(P)$ and $K^{-1}(P')$.

**Proof.** We have already proven parts (a)-(c). Part (d) is immediate from the fact that a conic is mapped into itself by any homology whose center is the pole of its axis ([3], p. 76, ex. 4). For part (e), note that $\mu_G(P) = Q'$, since $PQ'$ is a chord of the conic containing $G$. Hence $(PQ', GX_3) = -1$, where $GP \cdot VV_\infty = X_3$; this implies $PX_3 = -2X_3Q'$, which means that $Q'$
is the midpoint of \( PX_3 \). But \( Q' = K(P) \), so \( K(X_3) = P \). Thus \( X_3 \) is the anti-complement of \( P \). Similarly, the anti-complement of \( P' \) lies on \( VV_{\infty} \), and this proves part (e). Part (f) follows from the fact that the midpoint \( M \) of \( PP' \) lies on the line \( GV \), so that

\[
K^{-1}(M) = K^{-1}(PP') \cdot K^{-1}(GV) = K^{-1}(PP') \cdot GV = V,
\]

since the line \( GV \) is an invariant line of the complement map. \( \square \)

Let \( \eta = \eta_P \) be the harmonic homology whose center is \( V_{\infty} \) and whose axis is its polar \( GV \). The map \( \eta \) is an affine reflection ([4], p. 203), since it fixes the line \( GV \) and maps a point \( Y \) to the point \( Y' \) with the property that \( YY' \) is parallel to \( VV_{\infty} \) (or \( PP' \)) and \( YY' \cdot GV \) is the midpoint of \( YY' \). The map \( \eta \) is an involution on the extended plane. It takes the conic \( C_P \) to itself and interchanges the point pairs \( (PP') \) and \( (QQ') \), since the line \( GV \) passes through the midpoints of the chords \( PP' \) and \( QQ' \) and both lines \( PP' \) and \( QQ' \) lie on \( V_{\infty} \). Hence this homology induces an involution of points on \( C_P \).

**Remark.** It is not hard to show that the map \( \eta \) commutes with the complement map: \( K\eta = \eta K \).

We shall now prove

**Theorem 2.4.** Assume that the ordinary point \( P \) does not lie on a median of triangle \( ABC \) or on \( i(\omega) \). Then the maps \( T_P \) and \( T_{P'} \) satisfy the equation \( \eta T_P = T_{P'} \eta \), and so are conjugate to each other in the affine group.

To prove this theorem we need a lemma, which is of interest in its own right.

**Lemma 2.5.** Let \( G_1 = T_P(G) \) and \( G_2 = T_{P'}(G) \) be the centroids of the cevian triangles \( DEF \) and \( D_3E_3F_3 \) of \( P \) and \( P' \). Then the centroid \( G \) of \( ABC \) is the midpoint of the segment \( G_1G_2 \), which is parallel to \( PP' \). In other words, \( \eta(G_1) = G_2 \) (when \( P \) and \( P' \) are ordinary).

**Proof.** We first show that the line \( G_1G_2 \) lies on \( G \) and is parallel to \( PP' \). To begin with, assume \( P \) and \( P' \) are ordinary. Using I, Corollary 3.3, we know that the points \( Q, G_1, T_P(P') \) are collinear, with \( G_1 \) one-third of the way from \( Q \) to \( T_P(P') \). Since \( G \) is collinear with \( Q \) and \( P' \) and one-third of the way along \( QP' \), the triangles \( G_1QG \) and \( T_P(P')QP' \) are similar (SAS). Hence the line \( G_1G \) is parallel to the line \( T_P(P')P' = PP' \) (Corollary 2.2). Switching the roles of \( P \) and \( P' \) gives that \( G_2G \) is also parallel to \( PP' \), which implies that \( G_1G = G_2G \), proving the claim.

If \( P' = Q \) is infinite, then \( P \) and \( P' \) are ordinary, so we still get \( G_1G = G_2G \parallel PP' \). Applying the map \( T_P \) and using I, Theorem 3.14 gives that \( T_P(G_2G) \parallel T_P(PP') \), i.e., \( GG_1 \parallel PP' \), because \( T_P(G_2) = T_P(G) = K^{-1}(G) = G \) and \( T_P(PP') = T_P(PQ) = T_P(P)Q \parallel PQ = PP' \). Hence we get \( G_1G = G_2G \), as before. A similar argument works if \( P = Q' \) is infinite.

We now show that \( G \) is the midpoint of \( G_1G_2 \). (Cf. Figure 3.) Consider the sequence of triangles \( DEF, D_0E_F, D_3E_F \) with centroids \( G_1, G_0, G_{11} \) lying on the respective lines \( A_0D, A_0D_0, A_0D_3 \) (since \( A_0 \) is the midpoint of \( EF \)). By the properties of the centroid we know that the segment \( G_1G_{11} \) is the image of the segment \( DD_3 \) under a dilatation with center \( A_0 \) and ratio \( 1/3 \). Since
$D_0$ is the midpoint of $DD_3$ it follows that the vector $\overrightarrow{G_1G_{01}}$ is $\frac{1}{2}$ the vector $\overrightarrow{G_1G_{11}}$.

Next consider passing from triangle $D_3EF$ with centroid $G_{11}$ to the triangle $D_3E_3F$ with centroid $G_{12}$. Considering the dilatation with ratio $1/3$ from the midpoint of $D_3F$ shows that the vector $\overrightarrow{G_{11}G_{12}}$ is $\frac{1}{3}$ the vector $\overrightarrow{EE_3}$. In the same way, if we move from triangle $D_0EF$ with centroid $G_{01}$ to triangle $D_0E_0F$ with centroid $G_{02}$, the vector $\overrightarrow{G_{01}G_{02}}$ is $\frac{1}{3}$ the vector $\overrightarrow{EE_0}$, and therefore $\frac{1}{2}$ the vector $\overrightarrow{G_{11}G_{12}}$.

Finally, in passing from triangle $D_3E_3F$ with centroid $G_{12}$ to triangle $D_3E_3F_3$ with centroid $G_2$, the vector $\overrightarrow{G_{12}G_2}$ is $\frac{1}{3}$ the vector $\overrightarrow{FF_3}$. Similarly, in passing from $D_0E_0F$ with centroid $G_02$ to triangle $D_0E_0F_0$ with centroid $G$, the vector $\overrightarrow{G_{02}G}$ is $\frac{1}{3}$ of $\overrightarrow{FF_0}$. Hence, the vector $\overrightarrow{G_{02}G}$ is $\frac{1}{2}$ the vector $\overrightarrow{G_{12}G_2}$.

It follows that in passing from triangle $DEF$ to $D_0E_0F_0$, the centroid experiences a displacement represented by the vector

$$\overrightarrow{G_1G_{01}} + \overrightarrow{G_{01}G_{02}} + \overrightarrow{G_{02}G} = \frac{1}{2}(\overrightarrow{G_1G_{11}} + \overrightarrow{G_{11}G_{12}} + \overrightarrow{G_{12}G_2}),$$

which is $\frac{1}{2}$ the vector displacement from $G_1$ to $G_2$. Hence, $G_1G = \frac{1}{2}G_1G_2$, which proves that $G$ is the midpoint of $G_1G_2$. □

Proof of Theorem 2.4. We check that $\eta T_P(Y) = T_P \eta(Y)$ for three non-collinear ordinary points $Y$. This holds for $Y = G$ by the lemma. It also holds for the points $Q$ and $Q'$, since I, Theorems 3.2 and 3.7 imply that

$$\eta T_P(Q) = \eta(Q) = Q' = T_P(Q') = T_P \eta(Q)$$
and

$$\eta T_P(Q') = \eta(P) = P' = T_{P'}(Q) = T_{P'} \eta(Q').$$

The points $Q, Q'$, and $G$ are clearly not collinear, since $G$ is the intersection of the diagonals in the trapezoid $PP'Q'Q$ (and no three of these points lie on a line because they all lie on the conic $C_P$). This implies the theorem, since $\eta T_P$ and $T_{P'} \eta$ are affine maps.

**Remark.** The map $\eta$ is the unique involution $\psi$ in the affine group satisfying $\psi T_P = T_{P'} \psi$.

For the corollary, recall that the point $X = AA_3 \cdot BB_3$ is the fixed point of the map $S_1 = T_P T_{P'}$, and $X' = AA'_3 \cdot BB'_3$ is the fixed point of $S_2 = T_{P'} T_P$.

**Corollary 2.6.** Assume that the ordinary point $P$ does not lie on a median of triangle $ABC$ or on $\ell(\infty)$. If $X$ is an ordinary point, then $\eta(X) = X'$ and $XX'$ is parallel to $PP'$. Thus the line joining the $P$-ceva conjugate of $Q$ and the $P'$-ceva conjugate of $Q'$ is parallel to $PP'$.

**Proof.** We have $T_{P'} T_P(\eta(X)) = T_{P'} \eta(T_P(X)) = \eta T_{P'} T_P(X) = \eta(X)$, which shows that $\eta(X)$ is an ordinary fixed point of $S_2 = T_P T_{P'}$. Hence, $\eta(X) = X'$. This implies the assertion, by I, Theorems 3.8 and 3.10.

**Theorem 2.7.** The commutator $S' = T_{P'} T_P T_{P'}^{-1} T_P^{-1}$ is always a translation, in the direction $PP'$ by the distance $T_P(P') P' = PT_P^{-1}(P')$, if $P$ and $P'$ are ordinary; and by the distance $3|G_1 G_l|$, if $P$ or $P'$ is infinite.

**Proof.** For notational convenience write $T_1$ for $T_P$ and $T_2$ for $T_{P'}$. We first note that

$$S'(T_1(P')) = T_2 T_1 T_2^{-1}(P') = T_2 T_1(Q) = T_2(Q) = P'$$
and similarly

\[ S'(P) = T_2T_1T_2^{-1}(Q') = T_2T_1(Q') = T_2(P). \]

From this computation and Corollary 2.2, the mapping \( S' \) fixes the line \( PP' \).

By I, Theorem 3.8 and Corollary 3.11(c) we may assume that both \( S_1 = T_1T_2 \) and \( S_2 = T_2T_1 \) are homotheties, since otherwise the assertion is trivial. If \( P \) and \( P' \) are ordinary points, then \( S_2(Q) = P' \) implies that \( X', P', \) and \( Q \) are collinear, so part (b) of the same corollary implies that

\[
\frac{X'P'}{X'Q} = \frac{T_1(X'P')}{T_1(X'Q)} = \frac{XP}{XQ} = \frac{XP}{XQ'};
\]

the last equality being a consequence of \( S_1(Q) = T_1(P') \) and \( S_1(Q') = P \). This equation shows that the similarity ratios of \( S_1 \) and \( S_2 \) are equal and the mapping \( S' = S_2S_1^{-1} \) is an isometry which fixes \( l_\infty \) pointwise. It follows that \( S' \) is either a half-turn or a translation.

First assume that \( P \) does not lie on a median of \( ABC \). Then \( \eta(T_1(P')) = T_2(\eta(P')) = T_2(P) \), so that \( T_1(P') \) and \( T_2(P) \) are both inside or both outside the segment \( PP' \), and \( T_1(P')P' = PT_2(P) \) since \( \eta \) preserves lengths along \( PP' \). If the midpoints of segments \( T_1(P')P' \) and \( PT_2(P) \) are \( M_1 \) and \( M_2 \), then \( \eta(M_1) = M_2 \). If \( M_1 = M_2 \) then \( M_1 \) is the midpoint of \( PP' \), impossible since \( T_1(P') \neq P = T_1(Q') \). Thus, \( M_1 \neq M_2 \), and \( S' \) fixes the line \( PP' \) but is not a half-turn. Hence, \( S' \) is a translation. This proves the assertion in the case that \( P \) is not on a median.

Now assume that \( P \neq G \) lies on the median \( AG \). Then \( P, P', Q, \) and \( Q' \) are distinct points on \( AG \). (This follows easily from the fact that \( B, Q \) are on the opposite side of \( G \) on line \( AG \) from \( P', Q' \) and \( T_P(Q) = Q \) and \( T_P(Q') = P \). See I, Theorems 3.2 and 3.7.) From I, Theorem 3.5 the points \( A_i \) for \( 0 \leq i \leq 4 \) are all the same point \( A_0 = A_3 \), the midpoint \( EF \cdot AG \) of \( EF \); and similarly, the points \( A'_i = A'_0 = A'_3 \), the midpoint \( E_3F_3 \cdot AG \) of \( E_3F_3 \). Also, \( T_1(P') \) and \( T_2(P) \) are on \( PP' = AG \), so \( T_1, T_2 \) and \( S' \) all fix the line \( AG = AD = PP' \).

If \( P \) lies in the interior of triangle \( ABC \), then it is easy to see that \( E = T_1(B) \) and \( E_3 = T_2(B) \) are on the same side of the line \( AD \) as point \( C \), and hence that \( T_1 \) and \( T_2 \) interchange the sides of this line. Therefore, \( B_3 = T_1(E_3) \) and \( B'_3 = T_2(E) \) are on the same side of line \( AD = AG \), which implies that segments \( B_3B'_3 \) and \( A_3A'_3 \) do not intersect. Since

\[ S'(A_3B_3C_3) = T_2T_1T_2^{-1}(D_3E_3F_3) = T_2T_1(ABC) = A'_3B'_3C'_3, \]

we see that \( S' \) is not a half-turn, and is therefore a translation. On the other hand, if \( P \) is exterior to \( ABC \), points \( E \) and \( E_3 \) are on opposite sides of the line \( AD \), so one of \( T_1 \) and \( T_2 \) interchanges the sides of \( AD \) and the other leaves both sides of \( AD \) invariant. Hence \( B_3 = T_1(E_3) \) and \( B'_3 = T_2(E) \) are on the same side of the line \( AD \) and we get the same conclusion as before.

Finally, assume that \( P' = Q \) is infinite. Using I, Theorem 3.14, we have as in the proof of Lemma 2.5 that \( T_1(G) = G_3, T_1(G_2) = T_1T_2(G) = K^{-1}(G) = G \) and \( T_1(Q) = Q \). Since \( G \) is the midpoint of the segment \( G_1G_2 \), the fundamental theorem of projective geometry implies that \( T_1 \) acts as translation along the line \( GG_1 \). In this situation the map \( S_1 = K^{-1} \) fixes \( G \), so \( X = G \), while
\[ S_2 = T_2T_1 \text{ fixes } G_2 = X'. \] (See I, Theorem 3.10 and Corollary 3.11.) Furthermore, \( S_2(G) = T_2T_1(G) = T_1^{-1}K^{-1}(G_1) = T_1^{-1}(G_3) \), where \( G_3 = K^{-1}(G_1) = T_1^{-1}(G_2) \). Hence, \( S_2(G) = T_1^{-2}(G_2) \), and the similarity ratio of the homothety \( S_2 \) is \(-2\), which is equal to the similarity ratio of \( S_1 = K^{-1} \). Once again, \( S'_1 = S_2^{-1}S_1^{-1} = S_2K \) is an isometry. It is now straightforward to verify that the map \( S'_1 \), which is the commutator of \( T_1^{-1} \) and \( K^{-1} \), is equal to the translation \( T_1^{-3} \) on the line \( GG_1 \). For example, 

\[
S'_1(G) = S_2K(G) = T_1^{-2}(G_2) = T_1^{-3}(G),
\]

while 

\[
S'(G_3) = S_2(G_1) = T_1^{-1}K^{-1}T_1(G_1) = T_1^{-4}(G_2) = T_1^{-3}(G_3).
\]

Hence, the points on the line \( GG_1 \) experience a translation and not a half-turn. If instead, \( P = Q' \) is infinite, we apply the same argument to the inverse of \( S'_1 \).

This completes the proof. \( \square \)

**Corollary 2.8.** (a) The triangles \( A_3B_3C_3 \) and \( A'_3B'_3C'_3 \) are always congruent to each other. (See Figure 4.)

(b) If \( P \) and \( P' \) are ordinary, the length \( |T_P(P')P| \) of the segment \( T_P(P')P \) is equal to \( |QQ'||XP||XQ'| \).

**Proof.** Part (a) follows from \( S'(A_3B_3C_3) = A'_3B'_3C'_3 \). Part (b) follows from \( S_1(Q) = T_1T_2(Q) = T_1(P') \) and \( S_1(Q') = P \), since \( S_1 \) is a homothety with fixed point \( X \). \( \square \)

### 3. The affine map \( \lambda \).

We now set \( \lambda = T_P^{-1}T_P \), the unique affine map taking the cevian triangle \( DEF \) for \( P \) to the cevian triangle \( D_3E_3F_3 \) for \( P' \). We will show that \( \lambda \) maps the conic \( C_P \) to itself.

We use the fact that the diagonal triangle of any quadrangle inscribed in \( C_P \) is self-polar ([2], p. 73). Since \( A, B, C \) are on \( C_P \), if \( Y \) is any other point
on $C_P$, the cevian triangle whose vertices are the traces of $Y$ on the sides of $ABC$ is a self-polar triangle. We apply this to the point $Y = P$, obtaining that $DEF$ is a self-polar triangle.

Associated to the vertex $D$ of the self-polar triangle $DEF$ is the harmonic homology $\eta_D$ whose center is $D$ and whose axis is $EF$. We know $P$ and $A$ are collinear with $D$ and $A_Q$ (on $EF$) by the Collinearity Theorem. Furthermore, $\eta_D$ maps the conic to itself. Since $P$ and $A$ are on the conic $C_P$, the definition of $\eta_D$ implies that $\eta_D(A) = P$ and hence that $(AP, DA_4) = -1$. This proves

**Proposition 3.1.** The point sets $AA_4PD_1, BB_3PE_1, CC_4PF_1$ are harmonic sets.

This is the key fact we use to prove the following.

**Theorem 3.2.** If $P$ does not lie on a median of triangle $ABC$, then the map $\lambda = T_P^{-1}T_P$ takes the conic $C_P$ to itself: $\lambda(C_P) = C_P$.

**Proof.** Consider the anticevian triangle of $Q$, which, by I, Corollary 3.11, is $A'B'C' = T_P^{-1}(ABC)$. The cevian triangle for $Q$ with respect to this triangle is $ABC$, so $Q$ plays the role of $P$ in the above discussion and $ABC$ plays the role of $DEF$. We have that $AQ \cdot BC = D_2$, so Proposition 3.1 implies that $A'D_2QA$ is a harmonic set. Now we map this set by $T_P$. We have that $T_P(A'D_2QA) = T_P(A'A_2QD)$, so $T_P(A')$ is the harmonic conjugate of $Q$ with respect to $D$ and $A_2$. But $A_2$ is on $EF$, so the above discussion implies that $\eta_D(Q) = T_P(A')$. Since $Q$ is on $C_P$, so is $T_P(A') = T_P^{-1}(A) = \lambda^{-1}(A)$. Similar arguments apply to the other vertices, so $\lambda^{-1}$ maps $A, B, C$ to points on $C_P$. Now it is easy to see that $\lambda^{-1}(P') = Q$ and $\lambda^{-1}(Q') = P$, using I, Theorems 3.2 and 3.7. Hence $\lambda^{-1}$ maps 5 points on $C_P$ to 5 other points on $C_P$, so we must have $\lambda^{-1}(C_P) = C_P$. This implies the assertion.

If $P$ is ordinary and not on $l(\infty)$, then the map $\lambda = T_P^{-1}T_P = \eta T_P^{-1}T_P$ is the product of $\eta$ and $\eta\lambda = T_P\eta T_P^{-1}$, both of which are involutions (harmonic homologies) on the extended plane, and both of which fix the conic $C_P$. The involution $T_P\eta T_P^{-1}$ interchanges $P$ and $Q$. Its axis of fixed points is the line $T_P(GV)$ and its center $T_P(V_\infty) = T_P(P'Q'\cdot QQ')$ lies on $T_P(QQ') = PQ$, so that the midpoint of segment $PQ$ lies on $T_P(GV)$.

The map $T_P\eta T_P^{-1}$ is the map corresponding to $\eta$ for the conic $T_P(C_P) = T_P(ABCQ'Q) = DEFPQ$, which also lies on the points $T_P(P)$ and $T_P(P')$. Since $Q$ is the complement of $T_P(P')$ with respect to the triangle $DEF$ (I, Corollary 3.3), the conic $DEFPQ$ equals $C_R$ for triangle $DEF$ and the point $R = T_P(P)$. This is because the cevian triangle for $T_P(P)$ in $DEF$ is $A_1B_1C_1$, the cevian triangle for $T_P(P')$ in $DEF$ is $A_3B_3C_3$, and the points $T_P(P)$ and $T_P(P')$ are isotomic conjugates with respect to $DEF$.

**Remark.** It is easy to show that $\eta\lambda^{-1} = \lambda^{-1}$. Moreover, the map $\lambda$ is never a projective homology. (Hint: show $T_P(V_\infty)$ is never on the line $GV$ and use [3], p. 56, ex. 4.)

The last theorem has several interesting consequences.
**Theorem 3.3.** If $P$ does not lie on a median of triangle $ABC$, then the 6 vertices of the anticevian triangles for $Q$ and $Q'$ (with respect to $ABC$) lie on the conic $T_p^{-1}(C_P) = T_p^{-1}(C_P)$, along with the points $Q$ and $Q'$. This conic is $C_Q = A'B'C'QQ'$ for the anticevian triangle of $Q$, which is the same as $C_{Q'}$ for the anticevian triangle of $Q'$. Moreover, the vertices of the anticevian triangle of $R$ with respect to any self-polar triangle for the conic, also lie on this conic follows from $T_p^{-1}(Q) = Q$ and $T_p^{-1}(P) = Q'$. Theorem 3.12 of Part I implies the second assertion. Since the triangle $DEF$ is self-polar for $C_P$, it follows that $ABC$ is a self-polar triangle for $T_p^{-1}(C_P)$. The last assertion of the theorem follows from the general projective fact, that for any point $R$ on a conic $C$, the vertices of the anticevian triangle of $R$, with respect to any self-polar triangle for the conic, also lie on $C$. (See [3], p. 91, ex. 3.)

Note that the map $\eta$ fixes the conic $T_p^{-1}(C_P)$, since $\eta T_p^{-1}(C_P) = T_p^{-1}\eta(C_P) = T_p^{-1}(C_P)$.

**Remark.** If $P$ is the orthocenter of $ABC$, then $Q'$ is the circumcenter, $Q$ is the symmedian point ([7], Thm. 7), and $P'$ is the point $X(69)$, the symmedian point of the anticomplementary triangle (see [8]). In this case, the anticevian triangle of $Q$ is the tangential triangle [11], so Theorem 3.3 implies that $T_p^{-1}(C_P)$ is the Stammler hyperbola ([13], p. 21).

**Theorem 3.4.** Assume that the ordinary point $P$ does not lie on a median of triangle $ABC$ or on $l(\infty)$.

(a) The last-named points of Theorem 2.1 are

\[ A_0P \cdot D_0Q' = \lambda^{-1}(A), \quad B_0P \cdot E_0Q' = \lambda^{-1}(B), \quad C_0P \cdot F_0Q' = \lambda^{-1}(C); \]

\[ A_0'P' \cdot D_0Q = \lambda(A), \quad B_0'P' \cdot E_0Q = \lambda(B), \quad C_0'P' \cdot F_0Q = \lambda(C). \]

(b) Moreover, the lines $A_0P, D_0Q', DQ, A_3'P'$ are concurrent at the point $\lambda^{-1}(A)$, and $A_0'P', D_0Q, D_3Q', A_3P$ are concurrent at the point $\lambda(A)$, with similar statements holding for the other points in (a).

**Proof.** From the proof of Theorem 3.2 we have that $\lambda^{-1}(A)$ is on the intersection of the line $DQ$ with the conic $C_P$ and is distinct from $Q$. The proof of Theorem 2.1 together with $T_p(DQ) = T_p(D_3Q) = A_3'P'$ shows that $DQ \cdot A_3'P'$ is also on $C_P$. Now $DQ \cdot A_3'P'$ is not the point $Q$; otherwise $Q$ would lie on $A_3'P'$, which would imply the point $X'$ is on $A_3'P'$, since $X'$ is collinear with $Q$ and $P'$ (I, Theorem 3.8). However, the point $A$ is on $X'A_3'$ by the definition of $X'$ (I, Theorem 3.5), so $Q$ and $P'$ would be collinear with $A$. This contradicts the assumption that $P$ is not on $AG$. Hence, $DQ \cdot A_3'P' = \lambda^{-1}(A)$, since $DQ$ intersects the conic in exactly two points.
On the other hand, I, Corollary 3.11 and Theorems 3.12 and 2.4 give that the point \( A' = T_p^{-1}(A) \) is on both lines \( AQ \) and \( D_0Q' \), so we have that
\[
\lambda^{-1}(A) = T_pT_p^{-1}(A) = T_p(AQ \cdot D_0Q') = DQ \cdot A_0P.
\]
Note that \( DQ \cdot A_0P \neq P \) since \( Q \) does not lie on \( PD = AP \) (see the proof of Theorem 2.1). Finally, \( A_0P \cdot D_0Q' \) is on \( C_P \), by Theorem 2.1, and this intersection is not \( P \), since \( P \) and \( Q' \) are collinear with \( G \) and \( G \) is not on \( D_0Q' \). Hence, \( A_0P \cdot D_0Q' = \lambda^{-1}(A) \). Therefore, the lines \( A_0P, D_0Q', DQ \), and \( A_2P' \) meet at the point \( \lambda^{-1}(A) \). This proves the first assertion in (b). The second assertion follows immediately upon reversing the roles of \( P \) and \( P' \).

This gives two of the equalities in part (a), and the others follow by the same reasoning applied to the other vertices.

\[\square\]

**Corollary 3.5.** Under the assumptions of the theorem, the two quadrangles \( PQQP' \) and \( A_0DDA_2' \) are perspective, as are quadrangles \( PQP'Q' \) and \( A_3D_0D_3A_0' \).

(See Part I, Figure 5, where \( Y = \lambda(A) \).)

**Theorem 3.6.** The translation \( S' = T_pT_p^{-1}T_p^{-1} \) of Theorem 2.7 maps the conic \( DEFPQ \) to the conic \( D_3E_3F_3P'Q' \). In other words, these two conics are congruent.

**Proof.** From Theorem 3.3 we have, again with \( T_1 = T_p \) and \( T_2 = T_{p'} \), that
\[
S'(DEFPQ) = T_2T_1T_2^{-1}(ABCQ'Q) = T_2T_1(T_2^{-1}(C_P))
\]
\[
= T_2T_1(T_1^{-1}(C_P)) = T_2(C_P) = D_3E_3F_3P'Q'.
\]

\[\square\]

4. The center \( Z \) of \( C_P \).

In this section we study the center \( Z = Z_P \) of the conic \( C_P \), which we will recognize as a generalized Feuerbach point in Part III.

**Theorem 4.1.** Assume that the ordinary point \( P \) does not lie on a median of \( ABC \) or on \( l(\infty) \). Then the center \( Z \) of the conic \( C_P \) is given by \( Z = GV \cdot T_p(GV) \). If \( C_P \) is a parabola or an ellipse, \( Z \) is the unique fixed point in the extended plane of the affine mapping \( \lambda = T_pT_p^{-1} \). If \( C_P \) is a hyperbola, the infinite points on the asymptotes are also fixed, and these are the only other invariant points of \( \lambda \).

**Proof.** Since the map \( \lambda \) leaves invariant the conic and the line at infinity, it fixes the pole of this line, which is \( Z \).

To prove uniqueness write the map \( \lambda = \eta_1\eta_2 \) as the product of the harmonic homologies \( \eta_1 = \eta \) and \( \eta_2 = \eta \lambda = T_p\eta T_p^{-1} \) (see the discussion following Theorem 3.2). The center of \( \eta_1 \) is \( V_\infty \), lying on the line \( PP' \), and the center of \( \eta_2 \) is \( T_p(V_\infty) \), lying on the line \( T_p(QQ') = QP \). If \( R \) is any ordinary fixed point of \( \lambda \), then \( \eta_1(R) = \eta_2(R) = R' \). If \( R \) is distinct from \( R' \), this implies that \( RR' \) is parallel to both lines \( PP' \) and \( QP \), which is impossible. If \( R = R' \), then \( R \) is fixed by both \( \eta_1 \) and \( \eta_2 \), so \( R \) must be the intersection \( GV \cdot T_p(GV) \) of the axes of the two maps. This proves that \( Z = GV \cdot T_p(GV) \) if \( Z \) is ordinary.

Suppose now that the point \( Z = GV \cdot l(\infty) \) is an infinite point (so \( C_P \) is a parabola) and \( R \) is another infinite fixed point. Then \( \eta_1(R) = \eta_2(R) = R' \) is
also an infinite fixed point of $\lambda$, since $\lambda \eta(R) = \eta \lambda^{-1}(R) = \eta(R)$. If $R \neq Z, R'$, then $\lambda$ fixes three points on $I_\infty$ and is therefore the identity on $I_\infty$. But $\lambda(PQ) = Q'P'$, and $PQ \cdot P'Q'$ is the ordinary point $V$ (Proposition 2.3f), implying that $PQ$ cannot be parallel to $Q'P'$ and the line at infinity cannot be a range of fixed points. Thus, since $Z$ on $GV$ is fixed by $\eta_1$, we have $R = R'$, implying that $R$ is fixed by $\eta_1$ and $\eta_2$. We know that $V_\infty \neq T_p(V_\infty)$ (because $T_p(QQ') = QP$). Since $R$ is fixed by $\eta_1$ but different from $Z$, then $R = V_\infty$, which must lie on the axis $T_p(GV)$ in order to be fixed by $\eta_2$. Now $T_p(GV)$ lies on the point $G_1 = T_p(G)$ and on the midpoint $J$ of segment $PQ$ (see the comments following Theorem 3.2). The point $G_1$ lies on the line $GV_\infty$ through $G$ parallel to $PP'$ (Lemma 2.5), but $J$ is not on $GV_\infty$, since this line divides the segment $PQ$ in the ratio $2:1$. Hence, $T_p(GV) = G_1J$ cannot possibly lie on the point $V_\infty$. Thus $R = Z$ and $Z$ is the only infinite invariant point of $\lambda$.

We have already shown that the only possible ordinary fixed point of $\lambda$ is the point $Y = GV \cdot T_p(GV)$. We now show this point is infinite when $Z$ is infinite. Assume $Y$ is an ordinary point. Then $Y$ is on the line $GV$ with $Z$, so $GV = YZ$ is an invariant line; hence its pole $V_\infty$ is also invariant. Now use the fact that $\lambda(P) = Q'$ and $\lambda(Q) = P'$. Since $V_\infty$ is invariant the line $PP'$ must be mapped to the line $QQ'$. Since the conic is also invariant, $P'$ must map to the second point of intersection of $QQ'$ with the conic, which is $Q$. Hence, $\lambda$ interchanges $P'$ and $Q$, implying that the line $P'Q'$ is invariant; hence the infinite point on this line is invariant. This point is clearly not $V_\infty$; it is also not $Z$ because $P'Q \cdot GV = G$, so that if $Z$ were on $P'Q$ then $P'Q = GV$, which is not the case since $P'$ and $Q$ are not fixed by the map $\eta$. This shows that $\lambda$ has three invariant points on the line at infinity, which contradicts the fact that $\lambda$ is not the identity on $I_\infty$. Therefore, $\lambda$ has no ordinary fixed point in this case, and we conclude that $Z = GV \cdot T_p(GV)$ if $Z$ is infinite. Hence, $Z = GV \cdot T_p(GV)$ in all cases.

Suppose next that $Z$ is ordinary and $C_P$ is a hyperbola. Let $R_1$ and $R_2$ be the infinite points on the two asymptotes. Since the point $V_\infty$ is the pole of the axis $GV$ of the map $\eta_1$, the point $I_1 = GV \cdot I_\infty$ is conjugate to $V_\infty$ on $I_\infty$ and $GV = ZI_1$ and $ZV_\infty$ are conjugate diameters. By a theorem in Coxeter ([2], p. 111, 8.82), these two conjugate diameters are harmonic conjugates with respect to the asymptotes. (Note that $GV$ is never an asymptote: since $GVV_\infty$ is the diagonal triangle of the quadrangle $PQQ'P'$, the pole $V_\infty$ of $GV$ never lies on $GV$.) Hence we have the harmonic relation $H(R_1R_2, I_1V_\infty)$. By the definition of $\eta_1$ this implies that $\eta_1(R_1) = R_2$. In the same way, $T_p(V_\infty)$ is the center and $T_p(GV)$ is the axis of the harmonic homology $\eta_2$. Since $\eta_2$ fixes the conic, it fixes the pole of the line $T_p(GV)$ with respect to $C_P$; this pole is the center $T_p(V_\infty)$ since $T_p(V_\infty)$ is the only fixed point of $\eta_2$ on the line $T_p(GV)$. Hence we have the harmonic relation $H(R_1R_2, I_2T_p(V_\infty))$, where $I_2 = T_p(GV) \cdot I_\infty$, and this implies that $\eta_2(R_1) = R_2$. Therefore, $\lambda(R_1) = \eta_1\eta_2(R_1) = \eta_1(R_2) = R_1$ and $\lambda(R_2) = R_2$. There cannot be any other fixed points since $\lambda$ induces a non-trivial map on the line $I_\infty$.

The only case left to consider is the case when $C_P$ is an ellipse. In this case $Z$ is an interior point of the conic and any line through $Z$ intersects the conic in two points. Suppose that the infinite point on the line $ZR_1$ is fixed
by \( \lambda \), where \( R_1 \) lies on the conic. As before we have \( \eta_1(ZR_1) = \eta_2(ZR_1) \), so \( R_2 = \eta_1(R_1) \) lies on the conic, along with \( R_3 = \eta_2(R_1) \), and \( Z, R_2, \) and \( R_3 \) are collinear. The points \( R_2 \) and \( R_3 \) are distinct because \( Z \) is the only ordinary fixed point of \( \lambda \). Now line \( R_1R_2 \) is on the point \( V_\infty \) and \( R_1R_3 \) is on the point \( T_P(V_\infty) \), so \( R_1R_2R_3 \) is a triangle. Furthermore, \( T_P(GV) \) is on the midpoint of segment \( R_1R_3 \). Since \( Z \) is the center, it is the midpoint of the segment \( R_2R_3 \), and therefore the line \( T_P(GV) \), which lies on \( Z \) and the midpoint of \( R_1R_3 \), is parallel to \( R_1R_2 \). This implies that \( T_P(GV) \) lies on \( V_\infty \), which we showed above is never the case. Therefore, \( Z \) is the only fixed point of \( \lambda \) in this case. This completes the proof of the theorem. \( \square \)

**Corollary 4.2.**

(a) The point \( Z = GV \cdot G_1J \), where \( G_1 \) is the centroid of the cevian triangle of \( P \) and \( J \) is the midpoint of segment \( PQ \).

(b) The lines \( GV, T_P(GV) = G_1J, \) and \( T_P(GV) = G_2J' = \eta(G_1J) \) are concurrent at \( Z \). (See Figure 5.)

**Theorem 4.3.** Assume that the ordinary point \( P \) does not lie on a median of \( ABC \) but does lie on the Steiner circumellipse \( \iota(l_\infty) \). Then the point \( Z \) is one-third of the distance from the point \( G \) to the point \( G_1 = T_P(G) \) on the line \( GG_1 \) and is the only ordinary fixed point of the mapping \( \lambda \). In this case \( C_P \) is a hyperbola and the line \( l = GG_1 \) is an asymptote.

**Proof.** This will follow from the discussion in the last paragraph of the proof of Theorem 2.7, according to which \( T_P \) acts as translation on \( l = GG_1 \) and \( \lambda = T_P^{-1} = T_P^{-1}K^{-1}T_P^{-1} \). First of all, it follows easily from this representation of \( \lambda \) that \( \lambda \) fixes the point \( Z_1 \) on \( l \) for which \( GZ_1 : GG_1 = 1 : 3 \), and that this \( Z_1 \) is the only fixed point of \( \lambda \) on this line.
Furthermore, the mapping $T_p$ interchanges the sides of the line $GG_1$. This is because the point $D_2$ is on the line $AQ = AP = AD_2$, which is parallel to the fixed line $l$, and $A_2 = T_p(D_2)$ is on the opposite side of $l$, because $A_2$ is a vertex of the anticomplementary triangle (I, Corollary 3.14) and the point $G$ lies on segment $AA_2$. Since $K$ interchanges the sides of the line $l$, it follows that the mapping $\lambda$ has the same property. Thus, $\lambda$ has no ordinary fixed points off of $l$, and $Z_1$ is its only ordinary fixed point.

On the other hand, $\lambda$ fixes the center $Z$ of the conic $C_p$. If $Z \neq Z_1$, then $Z$ is infinite and $C_p$ is a parabola. Now the infinite point $Q$ lies on $l$ and $C_p$, so $Z = Q$, and the line $l$ must intersect the parabola in a second point, which would have to be a fixed point of $\lambda$, since $l$ and $C_p$ are both invariant under $\lambda$. This second point on $l \cap C_p$ is therefore the point $Z_1$. Now all the points (other than $Z_1$) of the conic $C_p$ lie on one side of the tangent line $l_1$ of $Z_1$. Note that $l \neq l_1$; otherwise $Q$ on $l_1$ would imply $l_\infty$ is on $Z_1$ by the polarity. If $U$ and $V$ are points on $C_p$ on either side of $Z_1$, then they lie on opposite sides of $l$ and the intersection $UV \cdot l$ is mapped to another point on $l$ by $\lambda$, which must lie on the same side of $l_1$. This would say that $\lambda$ which fixes the tangent $l_1$, leaves invariant both sides of this line. However, $\lambda(G) = T_p^{-1}K^{-1}T_p^{-1}(G) = T_p^{-1}K^{-1}(G_2) = T_p^{-1}T_p^2(G) = G_1,$ yet $G$ and $G_1$ are on opposite sides of $l_1$. This contradiction shows that $Z = Z_1$, proving the first assertion. This implies that $C_p$ is a hyperbola and the line $l = GG_1$ is an asymptote, since $l$ is on $Z$ and cannot intersect $C_p$ at a point other than $Q$ for the same reason as before - such an intersection would have to be a second ordinary fixed point of $\lambda$. This completes the proof. \qed

To view Theorem 4.1 from a different perspective, consider the model of hyperbolic geometry whose points are the interior points of the conic $C_p$ and whose lines are the intersections of Euclidean chords on the conic with the interior of $C_p$. Consider the case when $C_p$ is an ellipse. The involutions $\eta_1 = \eta$ and $\eta_2 = T_p\eta T_p^{-1}$ fix the conic $C_p$ and map the interior of $C_p$ to itself. Furthermore, the points $R$ and $\eta(R) = R'$ lie on a line “perpendicular” to $GV$ in this model, since $RR'$ lies on the pole of $GV$. (See Greenberg [6], p. 309.) Note that $GV$ contains points that are interior to $C_p$, because $Z$ is an interior point in this case. Since $\eta$ preserves cross-ratios it is a hyperbolic isometry. Thus, $\eta$ represents reflection across the diameter $GZ$ in this model ([6], pp. 341-343). The mapping $\eta_2 = \eta\lambda = T_p\eta T_p^{-1}$ is also a hyperbolic isometry fixing at least two points and thus a reflection across the diameter $T_p(GV) = G_1J = G_1Z$ ([6], p. 412). Thus, the map $\lambda = \eta_1\eta_2$ represents a hyperbolic rotation about the point $Z$.

When $C_p$ is a hyperbola, then $Z$ is an exterior point. If the lines $GV$ and $T_p(GV)$ are secant lines, they both pass through the pole of the line at infinity (also a secant line) and are therefore perpendicular to $l_\infty$ in the hyperbolic model. In this case $\lambda$ represents a translation along $l_\infty$. If the axis $GV$ does not intersect the conic, then the homology $\eta_1$ interchanges the sides of the line $l_\infty$ (in the model). Since the asymptotes separate the lines $GV$ and $ZV_\infty$ (see the proof of Theorem 4.1), the point $V_\infty$ is a fixed
point of $\eta_1$ in the hyperbolic plane, and all lines through $V_\infty$ are invariant. Hence, $\eta_1$ is a half-turn about $V_\infty$. If the axis $T_P(GV)$ is a secant line, then $\lambda$ is an indirect isometry with no fixed points and an invariant line, and is therefore a glide ([6], p. 430). If neither axis is a secant line, $\lambda$ is the product of two half-turns and is therefore a translation. When $C_P$ is a parabola, $Z$ is a point on the conic and therefore an ideal point in the hyperbolic model. In this case every line through $Z$ intersects the conic, so $GV$ and $T_P(GV)$ are secants and $\lambda$ is a parallel displacement ([6], p. 424).

References

[1] N. Altshiller-Court, College Geometry, An Introduction to the Modern Geometry of the Triangle and the Circle, Barnes and Noble, New York, 1952. Reprint published by Dover.
[2] H.S.M. Coxeter, The Real Projective Plane, McGraw-Hill Book Co., New York, 1949.
[3] H.S.M. Coxeter, Projective Geometry, 2nd edition, Springer, 1987.
[4] H.S.M. Coxeter, Introduction to Geometry, John Wiley & Sons, Inc., New York, 1969.
[5] H. Eves, College Geometry, Jones and Bartlett Publishers, Boston, 1995.
[6] M.J. Greenberg, Euclidean and Non-Euclidean Geometries, Development and History, 4th edition, W.H. Freeman and Co., New York, 2008.
[7] D. Grinberg, Hyacinthos Message #6423, http://tech.groups.yahoo.com/group/Hyacinthos.
[8] C. Kimberling, Encyclopedia of Triangle Centers, at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.
[9] I. Minevich and P. Morton, Synthetic foundations of cevian geometry, I: Fixed points of affine maps, http://arXiv.org/abs/1504.00210, to appear in J. of Geometry.
[10] I. Minevich and P. Morton, Synthetic foundations of cevian geometry, III: The generalized orthocenter, http://arXiv.org/abs/1506.06253, 2015.
[11] S. Wolfram, MathWorld, http://mathworld.wolfram.com.
[12] P. Yiu, Introduction to the Geometry of the Triangle, 2002, at http://www.math.fau.edu/Yiu/GeometryNotes020402.ps.
[13] P. Yiu, A Tour of Triangle Geometry, at http://www.math.fau.edu/yiu/TourOfTriangleGeometry/MAAFlorida37040428.pdf.