Rigidity for piece-wise smooth circle maps and certain GIETs

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Abstract

The goal of this article is to show a rigidity property of conjugacies of generalized interval exchange transformations (GIETs). More precisely, we show that if two piecewise $C^3$ GIETs $f$ and $g$ of generic rotation number with mean-non-linearity 0 are homeomorphic, boundary-equivalent and their renormalizations approach in an appropriate way the set of affine interval exchange transformations, then their respective renormalizations converge to each other and the conjugating map is $C^1$. Moreover, if $f$ and $g$ are GIETs with rotation type combinatorial data, generic rotation number and they are break-equivalent as piecewise circle diffeomorphisms, they are actually $C^1$-conjugated as circle diffeomorphisms. These results generalize the work of K. Cunha and D. Smania [4] in the case of piecewise $C^3$ circle maps, where the authors prove an analogous result for GIETs with rotation type combinatorial data and a bounded rotation number.

1 Introduction

Generalized interval exchange transformations (GIETs) are piecewise smooth bijective maps of a compact interval, with a finite number of discontinuities and non-negative derivative. They appear naturally as first return maps to Poincaré sections of smooth flows on surfaces. Similar to the classical Poincaré Classification Theorem for circle maps, a typical GIET with no periodic points is semi-conjugated to a standard interval exchange transformation (IET), that is to a GIET which is a piecewise translation. Moreover, this semi-conjugacy is actually a conjugacy if and only if the GIET is minimal.

In this work, we are investigating the rigidity properties of GIETs. Namely, given two GIETs $f$ and $g$ which are topologically conjugated, we study the smoothness of this conjugacy. Let us point that throughout this work we will only consider GIETs with no periodic points.

Several rigidity questions have been previously studied in closely related settings. By a classical result of A. Denjoy, sufficiently smooth circle homeomorphisms with irrational rotation number are topologically conjugated to rotations. The regularity of the conjugacy map in this setting has been extensively studied by several authors [13], [27], [14], [22] and show an important interplay between the regularity of the transformation being considered and the arithmetic properties of its rotation number.

Smooth circle homeomorphisms with singularities, i.e. smooth circle maps whose derivative either vanishes at a finite number of points (critical circle diffeomorphisms) or is discontinuous at a finite number of points but admits non-vanishing left and right derivatives (circle diffeomorphisms with breaks), are, under quite general assumptions, topologically conjugated to circle rotations. Indeed, it follows from a result by J.C. Yoccoz [28] that critical circle maps with irrational rotation number and non-flat critical points are topologically conjugated to irrational rotations while Denjoy’s result remains valid in the case of singularities of break-type (see [13] Chapter VI). However, the rigidity question is slightly different in this setting. In fact, we cannot expect a circle map with singularities to be smoothly conjugated to a rotation,
but it is natural to ask about the regularity of the conjugacy between two circle maps with singularities having the same rotation number and the same type of singularities.

Rigidity for critical circle maps has been shown for sufficiently regular circle maps of irrational rotation number having exactly one singularity of odd criticality. See e.g. [5], [6], [15], [11], [12]. Similar rigidity results hold for circle diffeomorphisms with exactly one break. See e.g. [15], [23], [16], [19], [17], [1]. However, in this case rigidity holds only for almost every rotation number and cannot be extended to every irrational rotation number [15].

The setting of critical circle maps with multiple singularities has yet to be explored more thoroughly, but some first results [7], [8] point to the existence of rigidity phenomena in this setting. Rigidity for $C^3$ critical circle maps with irrational rotation number and a finite number of singularities has been formally conjectured by G. Estevez and E. de Faria in [7].

Concerning circle diffeomorphisms with several breaks, rigidity results have been recently obtained by K. Cunha and D. Smania [4] for an exceptional class of maps. One of the aims of this article is to generalize their result to typical circle diffeomorphisms with breaks of class $C^3$. As in the definition of the class of maps considered in [4], our notion of typical map will make use of the renormalization theory for generalized interval exchange transformations. Let us point out that our rigidity result holds for a more general class of maps, namely, for generalized interval exchange transformations with zero mean non-linearity which satisfy the EC Condition (see Definition [1.5]), which describes the way the GIET linearizes under the renormalization.

Let us mention that the EC Condition is known to hold for typical $C^3$ GIETs of rotation type (see [8] for a precise definition) or typical minimal $C^3$ GIETs of 4 or 5 intervals having zero mean non-linearity. In fact, this follows from a recent work by S. Ghazouani and C. Ulcigrai [10], where the authors introduce the so-called a priori bounds for GIETs (which is verified by the aforementioned classes of GIETs) and study the behavior of the renormalizations for maps in this class. In particular, it follows from [10] that GIETs with zero mean non-linearity and having a priori bounds, verify the EC Condition.

The smooth conjugacy class of IETs in the space of GIETs was first studied by S. Marmi, P. Moussa and J.C. Yoccoz in [21], where the authors show that, for a typical IET $T$, the set of GIETs smoothly conjugated to $T$ defines a smooth submanifold of positive finite codimension (in the space of $C^r$ deformations of $T$ which are tangent to $T$ at the singularities). A generalization of this result in the case of IETs of periodic type has been obtained by S. Ghazouani in [9]. More recently, S. Ghazouani and C. Ulcigrai [10] studied the rigidity phenomena for GIETs.

The main result of this article is Theorem 1.7 which states that for typical topologically conjugated GIETs $f$ and $g$ of class $C^3$ with zero mean non-linearity, having the same boundary (see Section 1.2) and satisfying the EC Condition, the conjugacy is actually of class $C^1$. Let us point out that our main result is stated in a more general fashion since the assumption of $C^0$ conjugacy can be dropped. Indeed, as a consequence of Theorem 1.9 and Lemmas 1.10 [5.2] for two typical GIETs of class $C^3$ satisfying the EC Condition to be topologically conjugated, it is sufficient to assume that they have the same boundary and the same generalized rotation number, which can be seen as an analogous notion for rotation number of circle homeomorphisms in the GIETs setting.

To prove our main result, we show in Theorem 1.9 that for a typical $C^3$ GIET with zero mean non-linearity, there exists an affine interval exchange transformation (AIET), that is, a piecewise affine GIET, such that they are $C^1$ conjugated to each other. It is worth mentioning that this piecewise affine transformation is not uniquely determined. There are uncountably many possible affine models for a typical GIET. In fact, the logarithms of slopes of these affine models form an affine vector space whose dimension is equal to that of the stable space of the
Zorich height cocycle (see Subsection 1.1 for precise definitions).

The above results apply to the case of piecewise smooth circle homeomorphisms, which can be naturally identified with GIETs with combinatorial data of rotation type. We prove that, typically, two sufficiently regular break-equivalent circle homeomorphisms in this class are $C^2$ conjugated to each other. Let us point out that the notion of typical in this result involves not only restrictions in the rotation number of these transformations but conditions in their generalized rotation number.

We now describe the structure of the paper. In the remainder of this section we present the notation used in this article. Some of it, which we deemed as standard, is exhibited in Subsection 1.1 in shortened form. Other objects are defined in the subsections that follow. In particular, in Subsection 1.4 we present the acceleration of the Zorich cocycle, which is crucial in proving majority of the results of this article. Finally, in Subsection 1.5 we state precisely the main results, while in Subsection 1.6 we formulate some of the auxiliary results which are used to prove the theorems stated in Subsection 1.5. In Subsection 1.5 we present also how to deduce Theorem 1.7 from Theorems 1.8 and 1.9.

In Section 2 we prove Theorem 1.8, which yields a natural candidate for the affine interval exchange transformation to consider in Theorem 1.9. This section is divided into subsections devoted to prove the auxiliary propositions which considered together imply Theorem 1.8. In Section 3 we show first the proofs of Propositions 1.16 and 1.17, which together imply Theorem 1.9. Finally, in Subsection 3.3 we explain how to obtain Theorem 1.11 using the previous results of the article in the setup of piecewise smooth circle homeomorphisms.

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1.1 Notations

An interval exchange transformation (IET) is a bijection of an interval $I$ (often $I = [0, 1)$) which is a right-continuous piecewise translation with finite number of discontinuities. It is often parametrized by the length vector of exchanged intervals and a permutation describing the order of exchange. We will adopt the following notations.

| Finite Alphabet | $\mathcal{A}$ of $d \geq 2$ elements |
|-----------------|----------------------------------|
| Length vector   | $\lambda \in \mathbb{R}^d_+$ or $\lambda \in \Delta_A := \{ \lambda \in \mathbb{R}^d_+ \mid |\lambda| = \sum_{a \in A} \lambda_a = 1 \}$ |
| Permutation     | $\pi \in \mathfrak{S}_A := \{(\pi_0, \pi_1) \mid \pi_0, \pi_1 : \mathcal{A} \to \{1, \ldots, d\} \text{ bijections} \}$ |
| Irreducible Permutations | $\mathfrak{S}^0_A := \{ \pi \in \mathfrak{S}_A \mid \pi_1 \circ \pi_0^{-1}([1, \ldots, k]) = \{1, \ldots, k\} \Rightarrow k = d \}$ |
| Space of IETs   | $\mathbb{R}^d_+ \times \mathfrak{S}^0_A$ |
| Space of Normalized IETs | $\Delta_A \times \mathfrak{S}^0_A$ |

Elements in $\mathbb{R}^d_+ \times \mathfrak{S}^0_A$ are in one to one correspondence with IETs not having non-trivial invariant subintervals containing the left end-point of the interval. Given $(\lambda, \pi) \in \mathbb{R}^d_+ \times \mathfrak{S}^0_A$, which encodes an IET $T : [0, |\lambda|) \to [0, |\lambda|)$, we associate the following objects.

| Partition of $[0, |\lambda|)$ | $\{I_a(T)\}_{a \in A}$, where $|I_a(T)| = \lambda_a$ |
|-------------------------------|--------------------------------------------------|
| Endpoints of the Partition    | $0 = u_0(T) < u_1(T) < \ldots < u_{d-1}(T) < u_d(T) = |\lambda|$ |
| Translation Matrix            | $\Omega_\pi : \mathbb{R}^d_+ \to \mathbb{R}^d_+$, where $\Omega_\pi + 1$ if $\pi(\alpha) > \pi(\beta)$ and $\pi(\alpha) < \pi(\beta)$, $\Omega_\pi - 1$ if $\pi(\alpha) < \pi(\beta)$ and $\pi(\alpha) > \pi(\beta)$, $0$ in other cases. |

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For typical IETs, there is a classical induction procedure called *Rauzy-Veech induction* which, for a given IET $T$, produces an IET with the same number of intervals by inducing $T$ in some appropriate subinterval. This induction can be repeated infinitely many times for IETs verifying the so-called *Keane’s condition*. Moreover, this procedure naturally induces a directed graph structure on $\mathcal{O}_0^0$ called the *Rauzy graph*. Each connected component in this graph is called a *Rauzy class*. The infinite path in the Rauzy graph defined by an IET is called *combinatorial rotation number*.

| IETs satisfying Keane’s condition | $X_A^+ \subset \mathbb{R}_+^d \times \mathcal{O}_A^0$ |
|----------------------------------|-----------------------------------------------|
| Winner and loser symbols         | $\alpha_e = \pi_e^{-1}(d)$, for $e \in \{0, 1\}$ |
| Type of $(\lambda, \pi) \in X_A^+$ | $\epsilon(\lambda, \pi) = \begin{cases} 0 & \text{if } \lambda_{00} > \lambda_{11}, \\ 1 & \text{if } \lambda_{00} < \lambda_{11}. \end{cases}$ |
| Rauzy-Veech induction            | $\mathcal{R}V : X_A^+ \to X_A^+ \quad T \mapsto T|_{[0,|\lambda|-\lambda_{0e}(T)]}$ |
| Combinatorial rotation number    | $\gamma(T)$ |
| Cocycle notation                | $F : X \to X$, $\phi : X \to GL(d, \mathbb{Z})$ and $n \geq 0$, $\phi^{(n)}(x) = \phi(F^{(n-1)}(x)) \ldots \phi(x)$ |
| Rauzy-Veech matrix               | $A : X_A^+ \to SL(d, \mathbb{Z}) \quad T \mapsto Id + E_{\alpha_1(T),\alpha_0(T)}$ |

The Rauzy-Veech matrices are very useful in describing several aspects of this induction procedure. In fact, given $n \geq 0$, $T = (\lambda, \pi) \in X_A^+$ and denoting $T = \mathcal{R}V^n(T)$, the cocycles $A^{-1}TA : X_A^+ \to SL(d, \mathbb{Z})$ associated to $A$ verify the following.

- $(|A(T)|)_{\alpha \in A} = (A^{-1})^{(n)}(T)$
- If $(T|_{\alpha_T(T)})_{\alpha \in A} = (T^{h_\alpha}|_{\alpha_T(T)})_{\alpha \in A}$ then $(h_\alpha)_{\alpha \in A} = (T^{\alpha})^{(n)}(T)$, where $T \in \mathbb{R}_+^d$ is the vector whose coordinates are all equal to 1.

Sometimes it will be useful to consider normalized variants of this procedure. For this purpose we introduce the following notation.

| Normalized vector            | $\hat{\xi} := \xi |_{\tilde{\xi}}$, where $\xi \in \mathbb{R}_+^d$ |
|------------------------------|---------------------------------------------------------------|
| Normalized operator          | $D(\xi) := D(\hat{\xi})$, where $D$ is a positive matrix |
| Normalized IETs satisfying Keane’s condition | $X_A \subset \Delta_A \times \mathcal{O}_A^0$ |
| Normalized RV-induction      | $\tilde{\mathcal{R}}V : X_A \to X_A$ |

The normalized RV-induction $\tilde{\mathcal{R}}V$ admits an infinite but no finite invariant measure equivalent to the Lebesgue measure. However, a proper acceleration of this procedure, which we denote by $\hat{Z}$, possesses a unique ergodic invariant probability measure $\mu_{\hat{Z}}$ with this property.

| Zorich acceleration time      | $z : X_A^+ \to \mathbb{N}$ |
|------------------------------|-------------------------------|
| Zorich induction             | $Z : X_A^+ \to X_A^+ \quad (\lambda, \pi) \mapsto \mathcal{R}V^{\lambda}(\lambda, \pi)$ |
| Zorich renormalization       | $\hat{Z} : X_A \to X_A \quad (\lambda, \pi) \mapsto \tilde{\mathcal{R}}V^{\lambda}(\lambda, \pi)$ |
| Zorich matrix                | $B : X_A^+ \to SL(d, \mathbb{Z}) \quad (\lambda, \pi) \mapsto \prod_{\alpha=0}^{\lambda,\pi} A(\mathcal{R}V^{\lambda}(\lambda, \pi))$ |
| Zorich length cocycle        | $B^{-1} : X_A^+ \to SL(d, \mathbb{Z})$ |
| Zorich height cocycle        | $T_B : X_A^+ \to SL(d, \mathbb{Z})$. |
As before, the cocycles described above describe several aspects of this accelerated induction. Moreover, they are integrable with respect to the measure $\mu_{\tilde{Z}}$ and thus admit an Oseledec's filtration.

**Oseledec's filtration for the height cocycle**

\[ E^s(\lambda, \pi) \subset E^{cs}(\lambda, \pi) \subset \mathbb{R}^A \]

**Oseledec's filtration for the length cocycle**

\[ F^s(\lambda, \pi) \subset F^{cs}(\lambda, \pi) \subset \mathbb{R}^A \]

However, it will be more useful for us to have an Oseledecs splitting associated to these cocycles. For this purpose, we consider a natural extension of the Zorich renormalization.

**Extended space of parameters**

\[ \mathcal{X}_A = \left\{ (\tau, \lambda, \pi) \in \mathbb{R}_+^d \times X_A^+ \mid \sum_{\sigma_0(a) \leq k} \tau_0 > 0; \sum_{\sigma_1(a) \leq k} \tau_0 < 0; \right\} \]

**Extended Zorich induction**

\[ Z_{ext} : \mathcal{X}_A^+ \rightarrow \mathcal{X}_A^+ \]

\[ (\tau, \lambda, \pi) \mapsto (B^{-1}(\lambda, \pi) \tau, Z(\lambda, \pi)) \]

**Normalized extended space of parameters**

\[ \tilde{Z}_{ext} : \mathcal{X}_A \rightarrow \mathcal{X}_A \]

\[ (\tau, \lambda, \pi) \mapsto (|B^{-1}(\lambda, \pi) \lambda| B^{-1}(\lambda, \pi) \tau, \tilde{Z}(\lambda, \pi)) \]

Let us point out that for a.e. $(\tau, \lambda, \pi) \in \mathcal{X}_A$ we have $E^s(\tau, \lambda, \pi) = E^s(\lambda, \pi)$ and $E^c(\tau, \lambda, \pi) \oplus E^u(\tau, \lambda, \pi) = \mathbb{R}^A$.

**Oseledec's splitting for the height cocycle**

\[ B : \mathcal{X}_A \rightarrow SL(d, \mathbb{Z}) \]

\[ E^s(\tau, \lambda, \pi) \oplus E^c(\tau, \lambda, \pi) \oplus E^u(\tau, \lambda, \pi) = \mathbb{R}^A \]

**Oseledec's splitting for the length cocycle**

\[ B^{-1} : \mathcal{X}_A \rightarrow SL(d, \mathbb{Z}) \]

\[ F^s(\tau, \lambda, \pi) \oplus F^c(\tau, \lambda, \pi) \oplus F^u(\tau, \lambda, \pi) = \mathbb{R}^A \]

Let us point out that for a.e. $(\tau, \lambda, \pi) \in \mathcal{X}_A$ we have $E^s(\tau, \lambda, \pi) = E^s(\lambda, \pi)$ and $E^c(\tau, \lambda, \pi) \oplus E^u(\tau, \lambda, \pi) = E^c(\lambda, \pi)$.

The operators defined above, as well as Keane’s condition, extend naturally to generalized interval exchange transformations (GIETs), that is, to piece-wise smooth right continuous bijections of an interval with a finite number of discontinuities. We consider as well affine interval exchange transformations (AIETs) which are piecewise linear GIETs. For a given GIET (and AIET), we denote the associated partition, the endpoints of the intervals in this partition, its combinatorial rotation number and the associated inductions/renormalizations as in the case of IETs. A GIET $f$ is parametrized by the permutation $\pi$ describing the order on which the intervals are exchanged, the lengths of the exchanged intervals $\{l_\alpha(f)\}_{\alpha \in A}$ before and after applying the function, and a rescaling of the different branches $f|_{l_\alpha} : l_\alpha \rightarrow f(l_\alpha)$. 
By \cite{proposition} Proposition 7, for a typical IET $T$, define the translation surface. Moreover, let $\sigma$ be the endpoints of the partition that are identified with the singularities in the associated translation surface. Then the relations between the endpoints of the partition and the singularities of the translation surface can be encoded as follows. Let $p$ and $g$ be continuous surjective maps such that, for each $\alpha \in \mathcal{A}$ and $f \in \mathcal{X}_A$, the GIET $\hat{f}$, conjugated to $T$, is $C^0$ semi-conjugated to $T$ via non-decreasing conjugating map. Let us recall that given two GIETs $f, g : I \to I$, we say that $f$ is semi conjugated to $g$ if there exists a non-decreasing continuous surjective map $h : I \to I$ such that $f \circ h = h \circ g$ and $h(I\alpha(f)) = I\alpha(g)$, for all $\alpha \in \mathcal{A}$.

It follows from our definition of GIET, that for any $f \in \mathcal{X}_A$, there exists $c > 0$ such that $Df, Dg > c$, wherever those derivatives are defined. In other words, the GIETs we consider, do not possess critical points.

Notice that, although the height and length cocycles also extend trivially to the GIET setting (in fact they depend only on the combinatorial rotation number of a GICT) and the height cocycle still describes the return times for the induced transformation, the length cocycle no longer describes the lengths of the intervals in the partition of the induced transformation.

### 1.2 Preliminaries

Given a GIET $f : I \to I$ such that $D(\log(Df)) \in L^1(I)$ we denote its mean-non-linearity by

$$N(f) = \int_I D\log(Df(x)) dx.$$ 

Given a partition $\{I_{\alpha}\}_{\alpha \in \mathcal{A}}$ of an interval $I$, we denote by $C^r(\bigcup_{\alpha \in \mathcal{A}} I_{\alpha})$ the set of real functions on $I$ such that, for each $\alpha \in \mathcal{A}$, the restriction to $I_{\alpha}$ extends to a $C^r$-function on $\overline{I_{\alpha}}$.

Recall that for any IET $T = (\lambda, \pi) \in \Delta \times \Phi^0_d$ the endpoints of the associated partition are identified with the singular points $\{s_1, \ldots, s_\kappa\}$ in the associated translation surface, where

$$\kappa = \dim(\text{Ker}(\Omega_\pi)) + 1.$$ 

The relations between the endpoints of the partition and the singularities of the translation surface can be encoded as follows. Let $p = \pi_1 \circ \pi_0^{-1}$ denote the monodromy invariant of $\pi$ and define $\sigma : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d\}$ by

$$\sigma(j) = \begin{cases} p^{-1}(1) - 1 & \text{if } j = 0, \\ p^{-1}(j) - 1 & \text{if } p(j) = d, \\ p^{-1}(p(j) + 1) - 1 & \text{otherwise}. \end{cases} \tag{1}$$

Then $\sigma$ is a well-defined bijection, having exactly $\kappa$ different orbits, and each orbit corresponds to endpoints of the partition that are identified with the same singularity in the associated translation surface. Moreover, $\sigma$ allows to define a base for $\text{Ker}(\Omega_\pi)$ given by the vectors

$$\{\lambda(O) \in \mathbb{R}^d \mid O \text{ is an orbit of } \sigma \text{ not containing } 0\},$$

where

$$\lambda(O)_{\sigma_0(j)} = \begin{cases} 1 & \text{if } j \in O \text{ and } j - 1 \notin O, \\ -1 & \text{if } j \notin O \text{ and } j - 1 \in O, \\ 0 & \text{otherwise.} \end{cases}$$
For details, we refer the interested reader to [25]. Following [21], given an irreducible permutation $\pi \in \Phi_d^0$, a partition $\{I_\alpha\}_{\alpha \in A}$ of $I$, and denoting by $s(u_i)$ the singularity associated to the endpoints $u_i$, for $i = 0, \ldots, d$ of the underlying IET, we define the boundary operator associated to $\pi$ and $\{I_\alpha\}_{\alpha \in A}$ as

$$B : \bigcup_{\alpha \in A} I_\alpha \to \mathbb{R}_\kappa,$$

$$\varphi \mapsto (B_s(\varphi))_{1 \leq s \leq \kappa},$$

with

$$B_s(\varphi) = \sum_{s(u_i) = s, \ 0 \leq i \leq d} \partial\varphi(u_i),$$

where $\partial\varphi(u_i)$ denotes the difference between the right and left limits of $\varphi$ at $u_i$ (assuming $\varphi$ is defined as 0 outside of $I$), namely

$$\partial\varphi(u_i) = \begin{cases} \varphi(u_0) & \text{if } i = 0, \\ \lim_{x \to u_i^-} \varphi(x) - \lim_{x \to u_i^+} \varphi(x) & \text{if } 0 < i < d, \\ -\lim_{x \to u_d^-} \varphi(x) & \text{if } i = d. \end{cases}$$

Finally, we define the boundary $B(f)$ of a GIET over an irreducible permutation $\pi \in \Phi_d^0$, as

$$B(f) = B(\log Df),$$

where $B$ is the boundary operator associated to $\pi$ and $\{I_\alpha(f)\}_{\alpha \in A}$. We say that two GIETs $f$ and $g$ with the same combinatorial data are boundary-equivalent iff $B(f) = B(g)$.

Let us point out that for AIETs, the boundary operator is closely related to the projection of the log-slope vector to the space $\text{Ker}(\Omega_x)$, where $\pi$ is the subjacent permutation. Indeed, if $S$ is an AIET with log-slope $\omega$ over $\pi \in \Phi_d^0$ and $O$ is the orbit of $\sigma_x$ associated to the singularity $s \in \{s_1, \ldots, s_n\}$, then

$$B_s(S) = \langle \omega, \lambda(O) \rangle.$$

In particular, the log-slope vectors of two AIETs over $\pi \in \Phi_d^0$ having the same boundary project to the same vector in $\text{Ker}(\Omega_x)$.

The boundary operator for GIETs verifies the following.

**Proposition 1.1.** Let $f$ be a $C^1$ GIET with genus $\kappa$ satisfying Keane’s condition. Then the following holds.

1. $B(\text{RV}(f)) = B(f)$.
2. $\mathcal{N}(f) = 0$ if and only if $\sum_{1 \leq i \leq \kappa} b_i = 0$, where $(b_i)_{1 \leq i \leq \kappa} = B(f)$.
3. Let $g$ be a $C^1$ GIET over $\pi$. If $f$ and $g$ are $C^1$ conjugated, then $B(f) = B(g)$.

For a proof of the properties above see [10]. By the first assertion in the above proposition and the previous discussion on boundary operator for AIETs, we have the following.

**Proposition 1.2.** Given two AIETs $f, g$ over $\pi \in \Phi_d^0$, with zero mean non-linearity and log-slope vectors $\omega_f, \omega_g$, $B(f) = B(g) \iff \pi_{\text{Ker}(\Omega_x)}(\omega_f) = \pi_{\text{Ker}(\Omega_x)}(\omega_g)$.

We define now an equivalent of a length cocycle for GIETs. Let $f$ be a GIET and let $T$ be such that $\gamma(f) = \gamma(T)$. Let $A^n := A(\lambda, \pi) \cdots A(\text{RV}^{n-1}(\lambda, \pi))$. For any $n \in \mathbb{N}$ consider a $d \times d$ matrix $A^n(f)$ defined as follows

$$A^n_{\alpha, \beta}(f) := \sum_{i=1}^{A^n} \frac{|f_{\alpha, \beta}(I_\alpha(R\text{V}^n(f)))|}{|I_\alpha(R\text{V}^n(f))|}.$$

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where \( m_i(\alpha, \beta) \) is the \( i \)-th return time of the interval \( I_\alpha(\mathcal{R}^n f) \) to the interval \( I_\beta(f) \) via \( f \). Note that by Mean Value Theorem we have

\[
A^\alpha_{\beta}(f) := A^\alpha_{\beta}(f) := \sum_{i=1}^{A^\alpha_{\beta}} Df^{m_i(\alpha, \beta)}(x^n_i(\alpha, \beta))
\]

for some points \( x^n_i(\alpha, \beta) \in I_\alpha(\mathcal{R}^n f) \).

The importance of the matrices defined above follows from the fact that

\[
\left( |I_\alpha(f)| \right)_{\alpha, \beta} = A^n(f) \left( |I_\alpha(\mathcal{R}^n f)| \right)_{\alpha, \beta}
\]

Whenever we are considering accelerations of \( \mathcal{R} \)-induction, the notation for the corresponding matrices will be switched to the ones we used for said acceleration.

### 1.3 Piece-wise smooth circle homeomorphisms

Our results apply to the setting of piece-wise smooth circle homeomorphisms with a finite number of branches, but as their formulations and hypotheses will be slightly different we introduce the following notations.

A **piece-wise smooth circle homeomorphism** \( f : \mathbb{T} \to \mathbb{T} \) is a smooth orientation preserving homeomorphism, differentiable away from countable many points, so called break-points, at which left and right derivatives, denoted by \( Df_- \), \( Df_+ \) respectively, exist but do not coincide. For any \( x \in \mathbb{T} \), we define the **jump ratio** of \( f \) at \( x \) by

\[
\sigma_f(x) = \frac{Df_-(x)}{Df_+(x)}.
\]

We denote the **set of break points** of \( f \) by

\[
BP(f) = \{ x \in \mathbb{T} \mid \sigma_f(x) \neq 1 \}.
\]

Notice that if the map has a finite number of break points, then the product of all jump ratios at the break points is equal to one, namely,

\[
\prod_{x \in BP(f)} \sigma_f(x) = 1.
\]  

Two piece-wise smooth circle homeomorphisms \( f \) and \( g \) are said to be **break-equivalent** if they are topologically conjugated by a map \( h \in \text{Hom}(\mathbb{T}) \), verifying \( h \circ f = g \circ h \), such that \( h(BP(f)) = BP(g) \) and \( \sigma_f(x) = \sigma_g(h(x)) \), for any \( x \in BP(f) \).

We denote by \( P^d_0(\mathbb{T}) \) the set of \( C^d \) piece-wise smooth circle homeomorphisms with exactly \( d \) breaks, having a break at \( \varphi(0) \) and such that the orbits of the breaks are 2 by 2 disjoint.

For any \( d \geq 3 \), we identify maps in \( P^d_{d-1}(\mathbb{T}) \) with GIETs on \( d \) intervals as follows. Let

\[
\varphi : \ [0,1) \to \mathbb{T} \quad x \mapsto e^{2\pi ix}.
\]

Then, for any \( f \in P^d_{d-1} \), the map \( F = \varphi^{-1} \circ f \circ \varphi \) is an infinitely renormalizable GIET of class \( C^d \) on \( d \) intervals. Notice that this map cannot be seen as a GIET on a smaller number of intervals since by assumption \( f \) has exactly \( d \) break points and the orbits of this breaks are 2 by 2 disjoint.

In this case, the boundary \( \mathcal{B}(F) \) defined in (2) is nothing more than the vector given by the logarithm of the jump ratios of \( f \) at the break points. Indeed, by definition of \( F \), it is easy to see that the combinatorial datum \( \pi = (\pi_0, \pi_1) \) of \( F \) satisfies

\[
\pi_1 \circ \pi_0^{-1}(i) - 1 = i + k \ (\text{mod} \ d), \quad \text{for} \ i = 1, \ldots, d.
\]
for some \( k \in \{0, \ldots, d - 2\} \). In this case, an explicit computation shows that the permutation \( \sigma \) defined in (1) is given by

\[
\sigma(j) = \begin{cases} 
  d - k - 1 & \text{if } j = 0, \\
  d & \text{if } j = d - k - 1, \\
  0 & \text{if } j = d, \\
  j & \text{otherwise}.
\end{cases}
\] 

(9)

Denote by \( 0 = u_0 < u_1 < \cdots < u_d = 1 \) the endpoints of the partition associated to \( F \). Notice that \( BP(f) = \{ \varphi(u_j) | 0 \leq j < d; j \neq d - k - 1 \} \).

By (10), \( u_0, u_{d-k-1} \) and \( u_d \) correspond to the same singularity (in the associated translation surface), which we denote by \( s_1 \). In addition, since \( \varphi(u_{d-k-1}) \not\in BP(f) \),

\[
B_{s_1}(\log DF) = \log D_+ F(u_0) + \log D_+ F(u_{d-k-1}) - \log D_- F(u_{d-k-1}) - \log D_- F(u_1)
= \log D_+ f(\varphi(0)) + \log D_+ f(\varphi(u_{d-k-1})) - \log D_- f(\varphi(u_{d-k-1})) - \log D_- f(\varphi(0))
= \log \sigma_f(\varphi(0)).
\]

Furthermore, the remaining \( d - 2 \) singularities, which we denote by \( s_2, \ldots, s_{d-1} \), are in one to one correspondence with the remaining endpoints. By associating \( s_j \) to \( u_{j-1} \) if \( 2 \leq j \leq d-k-1 \), and to \( u_j \) if \( d-k-1 < j < d \), we can express the boundary of \( f \) as

\[
B(F)_j = \begin{cases} 
  \log \sigma_f(\varphi(u_{j-1})) & \text{if } j \leq d - k - 1, \\
  \log \sigma_f(\varphi(u_j)) & \text{if } j > d - k - 1.
\end{cases}
\] 

(10)

The argument above shows a slightly more general fact, namely, if \( T \) is a GIET on \( d \) intervals with combinatorial datum verifying (8) and denoting by \( 0 = u_0 < u_1 < \cdots < u_{d+1} = 1 \), then we can express the boundary of \( T \) as

\[
B(T)_j = \begin{cases} 
  \partial \log DT(u_0) + \partial \log DT(u_{d-k-1}) + \partial \log DT(u_1) & \text{if } j = 1, \\
  \partial \log DT(u_{j-1}) & \text{if } 1 < j \leq d - k - 1, \\
  \partial \log DT(u_j) & \text{if } d - k - 1 < j < d.
\end{cases}
\] 

(11)

Finally, let us mention that given a GIET \( T \) on \( d \geq 3 \) intervals with a rotation combinatorial datum, the circle homeomorphism induced by \( T \) and given by \( \varphi \circ T \circ \varphi^{-1} \) is a piece-wise smooth circle homeomorphism with \( at \) most \( d \) breaks and thus it is not necessarily a map in \( P_{d-1}(\mathbb{T}) \). In fact, if the combinatorial datum of \( T \) verifies (8), for some \( k \in \{0, \ldots, d - 2\} \), and \( 0 = u_0 < u_1 < \cdots < u_d = 1 \) are the endpoints of the associated partition of \( [0, 1] \), then the circle homeomorphism \( \varphi \circ T \circ \varphi^{-1} \) belongs to \( P_{d-1}(\mathbb{T}) \) if and only if \( \partial \log DT(u_{d-k-1}) = 0 \) and \( B(T)_j \neq 0 \), for all \( j = 1, \ldots, d-1 \).

Moreover, it follows from (11) that for any two boundary-equivalent GIETs with combinatorial data of rotation type, the associated piece-wise smooth homeomorphisms (having at most \( d \) break points) have the same jump ratio in at least \( d - 2 \) break points. However, notice that it is still possible for two GIETs to be topologically conjugated and boundary-equivalent, while the associated piece-wise smooth homeomorphisms are topologically conjugated but not break-equivalent (in this case the associated circle maps would necessarily have \( d \) breaks).

### 1.4 An appropriate subsequence of Zorich times

We will consider an acceleration \( \mathcal{R} : X \to X \) of the Zorich extension \( Z_{ext} \), defined over a \( \mu_{Z_{ext}} \)-full-measure subset \( X \subset X_A \), and of the form

\[
\mathcal{R}(\tau, \lambda, \pi) = Z_{ext}^{N(\tau, \lambda, \pi)}(\tau, \lambda, \pi),
\]
where \(N : X \rightarrow N\) is a measurable function. We denote by
\[
Q : X \rightarrow SL(d, \mathbb{Z}),
\]
the associated accelerated Zorich cocycle, which is given by
\[
Q(\tau, \lambda, \pi) = \prod_{i=0}^{N(\tau, \lambda, \pi)-1} B(Z_{ext}^i(\tau, \lambda, \pi)),
\]
and, as before, we denote for any \(n > m,\)
\[
Q_{m,n}(\tau, \lambda, \pi) = \prod_{i=m}^{n-1} Q \left( R^i(\tau, \lambda, \pi) \right),
\]
\[
Q_{-n,-m}(\tau, \lambda, \pi) = (Q_{0,-n-m}(R^{-n}(\tau, \lambda, \pi)))^{-1}.
\]
Note that transformation \(Q\), defines an induced length and height cocycle given by \(Q^{-1}\) and \(T_{Q}\) respectively, similar to the case of non-accelerated Zorich cocycle. As in the case of \(Z_{ext}\), we denote by \(\tilde{R}\) the normalized variant of \(R\).

Given \((\tau, \lambda, \pi) \in X\), we denote its iterates with respect to \(R\) by \((\tau^n, \lambda^n, \pi^n)\), the induction intervals by \((I^n(\tau, \lambda, \pi))_{a \in A}\) and the associated heights by \((q^n(\tau, \lambda, \pi))_{a \in A}\). If there is no risk of confusion, we will sometimes omit the base point \((\tau, \lambda, \pi)\) in all of the previous notations.

The function \(N\) will be picked so that \(R\) will satisfy the following.

**Proposition 1.3.** For any \((\tau, \lambda, \pi) \in X\) the following holds:

1. The sequence \(m_n = \sum_{k=0}^{n-1} N(\tau^k, \lambda^k, \pi^k)\) is sublinear, that is,
   \[
   \sup_{n \geq 1} \frac{m_n}{n} < +\infty.
   \]
2. For any \(n \geq 0,\)
   \[
   \min_{a \in A} \frac{\lambda^n_a}{\lambda^n_b} > c_0,
   \]
   where \(c_0 > 0\) is a constant independent of \(n\).

   More precisely, \(\lambda^n \in A^\gamma(\mathbb{R}_+^d)\), where \(\gamma\) is a finite path in the Rauzy-Veech algorithm, independent of \(n\) and \((\tau, \lambda, \pi)\), such that the associated matrix \(A^\gamma\) is positive.
3. For any \(n \geq 1,\)
   \[
   C_0^{-1} \leq \| Q_{0,n} \|_{E^\gamma} \leq C_0,
   \]
   where \(C_0 > 0\) is a constant independent of \(n\).
4. There exists \(C > 0\) such that \(\sup_{a \in A} \frac{\log \| Q_{0,a} \|}{n} < C,\)
5. For any \(\tau > 0,\) there exists \(C > 0\) such that
   \[
   \| Q_{n-1,a} \| \leq C \| Q_{0,a} \|^{\tau},
   \]
   for every \(n \geq 1,\) In particular, by condition 4 there exists \(C' > 0\) such that
   \[
   \| Q_{0,a} \| \leq C' e^{n\tau}.
   \]

**Proof.** Conditions 2 and 3 are given by recurrence to an appropriate set of positive \(\mu_E\) measure in \(X_A\) (for Condition 2 we refer the reader to [25], while for Condition 3 to [24]). Condition 4 follows from the ergodicity of Zorich cocycle, Conditions 4 and 5 follow from its integrability (which is inherited by the induced cocycle). For the proof of this fact we refer the reader to [26].

\[\square\]
In view of Condition 1, the invariant Oseledet’s subspaces $E^c(\tau, \lambda, \pi)$, where $c \in \{s, c, u\}$, of the height cocycle $T_B$ form the Oseledet’s splitting of the accelerated height cocycle $T_Q$ as well, hence notation remains unchanged.

In [3] the authors define a metric $d_{C^r}$ on $C^r$ piecewise smooth diffeos with $d$ branches with a fixed combinatorial data (this metric is closely related to the one introduced in [10]), given by the formula

$$d_{C^r}(f, g) := \max_{\alpha \in A} \left\{ \| \Xi(f | I_{\alpha}(f)) - \Xi(g | I_{\alpha}(g)) \|_{C^r} + \| I_{\alpha}(f) - I_{\alpha}(g) \| + \| f(I_{\alpha}(f)) - g(I_{\alpha}(g)) \| \right\},$$

where $\Xi$ is the zoom operator. This operator, which is defined in both of the articles mentioned above, associates to any homeomorphism between two bounded intervals its rescaling by affine transformations to a homeomorphism of the unit interval. More precisely, if $h : I \to J$ is an homeomorphism between two closed bounded intervals, then

$$\Xi(h) = A_1 \circ h \circ A_2,$$

where $A_1 : J \to [0, 1]$, $A_2 : [0, 1] \to I$ are bijective orientation preserving homeomorphisms.

The following is a direct consequence of [21] Corollary 3.6.

**Proposition 1.4** (Cohomological equation). Let $f$ be a $C^0$ GIET satisfying Keane’s condition. Then, for any $\varphi \in C^0(\bigcup_{\alpha \in A} I_{\alpha}(f))$ verifying

$$\sup_{n \geq 0} \| S_n^f \varphi \| < +\infty,$$

where $S_n^f \varphi$ denotes $n$-th Birkhoff sum of $\varphi$ w.r.t. $f$, there exists $\psi \in C^0([0, 1])$ such that

$$\varphi = \psi \circ f - \psi.$$

### 1.5 Results

We will make use of the following assumptions on a GIET.

**Definition 1.5** (EC Condition). Let $f$ be a GIET with no connection, semi-conjugated to an IET $T = (\lambda, \pi)$. We say that $f$ satisfies the EC condition if there exists $\tau \in \mathbb{R}^A$ such that $(\tau, \lambda, \pi)$ is Oseledet’s generic w.r.t. to the Zorich cocycle, verifying

$$\max_{\alpha \in A} \| \Xi(Z^n f | I_{\alpha}(Z^n f) - Id) \|_{C^1([0, 1])} = O(c^n),$$

$$|L^n(Z^n f) - L^c(Z^n f)| = O(c^n),$$

for some $0 < c < 1$, where we decompose $L(Z^n f) = L^c(Z^n f) + L^e(Z^n f) + L^u(Z^n f)$ with respect to the Oseledet’s splitting at $Z^n_{osc}(\tau, \lambda, \pi)$.

This property is known to hold for $C^3$ GIETs of rotation type or minimal $C^3$ GIETs of $4$ or $5$ intervals. As discussed in the introduction, this is a consequence of a recent work by S. Ghazouani and C. Ulcigrai [10], which implies that a typical GIET with zero mean-nonlinearity having a priori bounds (see [10] for a precise definition) verifies the EC Condition. The authors in [10] conjectured that the a priori bounds should hold for a.e. minimal $C^3$ GIET with $\mathcal{N}(f) = 0$ without any assumption on the underlying combinatorics. Moreover, they proved that this conjecture is true if a generalization (to AIETs of any combinatorial type) of a result by S. Marmi-P. Moussa and J.C. Yoccoz [21] concerning Birkhoff sums for AIETs holds. One of the main consequences is that if $f$ satisfies EC Condition, then its renormalizations verify the following.
Lemma 1.10. Let equivalent GIETs of class $\omega$ and $S$ of the Rauzy-Veech induction. Let us point out that such an affine model is not necessarily unique. More precisely, we will show the following.

Theorem 1.8 (Affine shadow). For a.e. $T = (\lambda, \pi) \in \Delta_A \times \Theta^0_d$, any two boundary-equivalent GIETs $f, g$ of class $C^3$ with $\mathcal{N}(f) = \mathcal{N}(g) = 0$, $\gamma(f) = \gamma(T) = \gamma(g)$ and verifying the EC Condition are $C^1$ conjugated.

We will prove the theorem above by showing that any two maps $f, g$ as in the statement are $C^1$ conjugated to the same AIET. For any IET $T$ and $\omega \in \mathbb{R}^d$ we denote by Aff$(T, \omega)$ the set of AIETs whose log-slope vector is $\omega$ and combinatorial rotation number $\gamma(T)$.

In order to construct an affine model of a given mean non-linearity zero GIET $f$, we look for an AIET $S$ that shadows the orbit of the $f$ with respect to an appropriate acceleration of the Rauzy-Veech induction. Let us point out that such an affine model is not necessarily unique. More precisely, we will show the following.

Theorem 1.7 (Rigidity). For a.e. $T = (\lambda, \pi) \in \Delta_A \times \Theta^0_d$, any two boundary-equivalent GIETs $f, g$ of class $C^2$ with $\mathcal{N}(f) = 0$ and $\gamma(f) = \gamma(T)$ which verifies the EC Condition, there exists $\omega_1 \in \text{Ker}(\Omega_\pi)$, such that, for any $S \in \text{Aff}(T, \omega)$ with $\omega \in E^\infty(\lambda, \pi)$ verifying $\pi_{\text{Ker}(\Omega_\pi)}(\omega) = \omega_f$, we have $B(S) = B(f)$ and

$$d_{C^1}(\tilde{R}^n f, \tilde{R}^n S) = O(c^n),$$

for some $0 < c < 1$.

This will imply the following linearisation result for GIETs.

Theorem 1.9 (Linearisation). Let $T = (\lambda, \pi) \in \Delta_A \times \Theta^0_d$ verifying Theorem 1.7 and let $f$ and $S$ as in Theorem 1.8. Then $f$ and $S$ are $C^1$ conjugated (as GIETs).

Let us deduce Theorem 1.7 from Theorems 1.8, 1.9. For this, we need to show that the affine models of $f$ and $g$ given by Theorem 1.9 coincide. In fact, we have the following.

Lemma 1.10. Let $T = (\lambda, \pi) \in \Delta_A \times \Theta^0_d$ verifying Theorem 1.3 and let $f, g$ be two boundary-equivalent GIETs of class $C^2$ with $\mathcal{N}(f) = \mathcal{N}(g) = 0$ and $\gamma(f) = \gamma(T) = \gamma(g)$ verifying the EC Condition. Then $\omega_f = \omega_g$, where $\omega_f$ and $\omega_g$ are the vectors given by Theorem 1.8 when applied to $f$ and $g$, respectively.

Proof. Let $S_f, S_g$ be AIETs with $\gamma(S_f) = \gamma(T) = \gamma(S_g)$ and log-slopes $\omega_f$ and $\omega_g$, respectively. By Theorem 1.9 $f$ (resp. $g$) is $C^1$ conjugated, as GIET, to $S_f$ (resp. $S_g$). Hence

$$B(S_f) = B(f) = B(g) = B(S_g).$$

Therefore, $\omega_f = \omega_g$ by Proposition 1.2.

Theorem 1.7 is now a direct consequence of Theorems 1.8, 1.9 and the previous lemma.

Restricted to the piece-wise smooth circle homeomorphisms setting, we will prove the following.

Theorem 1.11. Let $d \geq 3$. For a.e. $T = (\lambda, \pi) \in \Delta_A \times \Theta^0_d$, with $\pi$ of rotation type, any two boundary-equivalent piece-wise smooth circle homeomorphisms $f, g \in P^d_{\lambda - 1}(T)$ with $\mathcal{N}(f) = \mathcal{N}(g) = 0$ and $\gamma(f) = \gamma(T) = \gamma(g)$ are $C^1$ conjugated as circle maps.
The proof of this theorem makes use of the above results but requires special considerations on the conjugating map. This is due to the fact that boundary-equivalence, which is a necessary condition for $C^1$ conjugacy of GIETs, does not imply break-equivalence, which is a necessary condition for $C^1$ conjugacy of circle homeomorphisms.

Thus, although any function $f$ in the statement of Theorem 1.11 will be $C^1$-conjugated, as GIET, to a 1 dimensional family of AIETs parametrized by the log-slope vector (notice that for an Oseledet’s generic IET $(\lambda, \pi)$ of rotation type the space $E^s(\lambda, \pi)$ has dimension 1), it will be $C^1$ conjugated, as a circle homeomorphism, only to AIETs having a precise log-slope vector. A proof of this result is given at the end of Section 3.

1.6 Outline

In the following we fix $(\tau, \lambda, \pi) \in X$ as in Proposition 1.3. For $f$ as in Theorem 1.8 we will prove the following.

Proposition 1.12. There exists $\omega \in E^c(\tau, \lambda, \pi)$ such that

$$|\omega^n - L^n| = O(c^n),$$

for some $0 < c < 1$, where

$$\omega^n = TQ_{0,n}(\tau, \lambda, \pi)\omega,$$

and $L^n = (L^n_\alpha)_{\alpha \in A}$ is given by

$$L^n_\alpha = \ln \left( \frac{1}{|I^n_\alpha(f)|} \int_{I^n_\alpha(f)} Df^n_\alpha(s) ds \right). \tag{12}$$

Proposition 1.13. Let $\omega$ as in Proposition 1.12 and $S \in \text{Aff}(T, \omega)$. Then

$$\max_{\alpha \in A} \frac{|I^n_\alpha(f)|}{|I^n(S)|} = O(c^n),$$

for some $0 < c < 1$.

From Proposition 1.12 and Proposition 1.13 we can conclude the following.

Proposition 1.14. Let $\omega$ as in Proposition 1.12 and $S \in \text{Aff}(T, \omega)$.

Then

$$\max_{\alpha \in A} \left| \frac{R^n f(I^n_\alpha(f))}{|I^n(f)|} - \frac{R^n S(I^n_\alpha(S))}{|I^n(S)|} \right| = O(c^n),$$

for some $0 < c < 1$.

Theorem 1.8 will follow from Proposition 1.12, 1.13, 1.14 and the EC Condition. A proof of this fact will be given at the beginning of Section 2.1.

Recall that for a typical IET $T$, any GIET $f$ with $\gamma(f) = \gamma(T)$ is semi-conjugated to $T$. Hence, for $f$ and $S$ as in Theorem 1.8 these two maps are semi-conjugated by a non-decreasing continuous map. In particular, $S \circ h = f \circ h$ for some continuous non-decreasing surjective function $h : [0, 1) \rightarrow [0, 1)$. To prove Theorem 1.9 we will show that this semi-conjugacy is actually a $C^1$ diffeomorphism.

We start by showing that $h$ is a conjugacy between $f$ and $S$. In order to do this, we will use the following result due to C. Ulcigrai and the second author [24], which states that in the AIET setting this semi-conjugacy is actually a conjugacy as long as the log-slope vector belongs to the central-stable space of $T$ for the Zorich cocycle. Let us point out that this was already known for log-slope vectors in the stable space (see [2, Theorem 1]).

Theorem 1.15 (Theorem 1 in [24]). For almost every IET $T = (\lambda, \pi)$, if $\omega \in E^c(\lambda, \pi)$ then any $S \in \text{Aff}(T, \omega)$ is $C^0$-conjugated to $T$. 13
The previous result implies that for \( f \) and \( S \) as in Theorem 1.8 the AIET \( S \) is topologically conjugated to an IET \( T \), in particular, \( S \) has no wandering intervals. Combining this with Theorem 1.8 we shall see in Lemma 3.2 that \( f \) cannot have wandering intervals, and thus it is also conjugated to \( T \). Hence, \( f \) and \( S \) are topologically conjugated, which means that the semi-conjugacy \( h \) is actually a conjugacy between these two maps.

Notice that if \( h \) was of class \( C^1 \), by taking derivatives and logarithm in \( S \circ h = f \circ h \), we obtain
\[
(\log DS) \circ h - \ln Df = (\log Dh) \circ f - \log Dh.
\]
This motivates us to consider the following cohomological equation
\[
(\log DS) \circ h - \ln Df = \psi \circ f - \psi. \tag{13}
\]

Using an appropriate Diophantine condition and Gottschalk-Hedlund Theorem we prove the following.

**Proposition 1.16.** Let \( f, S \) as in Theorem 1.8 and assume that \( f \) is semi-conjugated to \( S \) via \( h \). Then, there exist a continuous solution \( \psi : [0, 1] \to \mathbb{R} \) of (13).

Moreover, if \( f \) and \( S \) are break-equivalent circle homeomorphisms, \( \psi \) can be taken as a continuous function on the circle.

**Proposition 1.17.** Let \( f, S \) as in Theorem 1.8 and assume that \( f \) is semi-conjugated to \( S \) via \( h \). Suppose \( \psi : [0, 1] \to \mathbb{R} \) is a continuous solution of (13). Then, \( h \) is of class \( C^1 \) and there exists a constant \( C \) such that
\[
Dh = Ce^\psi.
\]

Clearly, Propositions 1.16 and 1.17 imply Theorem 1.9.

### 2 Existence of an affine shadow

Let us prove Theorem 1.8 assuming Propositions 1.12, 1.13, 1.14, whose proof we defer to the following subsections.

**Proof of Theorem 1.8.** Fix \( \tau \) such that \((\tau, \lambda, \pi) \in X\) verifies Proposition 1.3. Let \( \omega^* \in E^c(\tau, \lambda, \pi) \) be the vector given by Proposition 1.12 when applied to \( f \) and define \( \omega_f = \pi_{\ker(\Omega_\pi)}(\omega^*) \).

In the following, we fix \( \omega \in E^c(\tau, \lambda, \pi) \) such that \( \pi_{\ker(\Omega_\pi)}(\omega) = \omega_f \) and \( S \in \text{Aff}(T, \omega) \).

Notice that \( \omega - \omega^* \in E^c(\tau, \lambda, \pi) \), since
\[
\pi_{\ker(\Omega_\pi)} |_{E^c(\tau, \lambda, \pi)} : E^c(\tau, \lambda, \pi) \to \ker(\Omega_\pi)
\]
is an isomorphism. Therefore, \( \omega \) verifies the conclusions of Proposition 1.12 and \( S \) verifies Propositions 1.13 and 1.14.

By Proposition 1.12 1.13 1.14 and the EC Condition,
\[
d_{C^2}(\tilde{R}^n f, \tilde{R}^n S) = o(c^n),
\]
for some \( 0 < c < 1 \). Finally, it follows from Proposition 1.1 the EC Condition and the previous equation that
\[
B(f) = B(\tilde{R}^n f) = B(\tilde{R}^n S) + o(c^n) = B(S) + o(c^n),
\]
for some \( 0 < c < 1 \). Therefore \( B(f) = B(S) \).
2.1 Proof of Proposition 1.12

Let us start by showing that the sequence \( (L_n)_{n \in \mathbb{N}} \) behaves as a pseudo-orbit with respect to the heights cocycle.

**Lemma 2.1.** Then \( |L^{n+1} - TQ_{n,n+1}L^n| = O(c^n) \), for some \( 0 < c < 1 \).

**Proof.** For any \( \alpha \in \mathcal{A} \) and \( n \in \mathbb{N} \), let \( x^n_\alpha \in I^n_\alpha(f) \) such that

\[
L^n_\alpha = \ln Df^n_\alpha(x^n_\alpha).
\]

Let us fix \( n \in \mathbb{N} \). Given \( \alpha \in \mathcal{A} \), let

\[
b^n_\alpha = \sum_{\beta \in \mathcal{A}} \left( TQ_{n,n+1} \right)_{\alpha \beta} L^n_\beta.
\]

For any \( \alpha \in \mathcal{A} \), we can express \( q^{n+1}_\alpha \) uniquely as

\[
q^{n+1}_\alpha = \sum_{i=1}^{b^n_\alpha} q^n_{\delta_i(\alpha)},
\]

for some \( \delta_i(\alpha) \in \mathcal{A} \), such that

\[
f^{h_i}(I^{n+1}_\alpha(f)) \subset I^n_{\delta_i(\alpha)}(f), \quad \text{where} \quad h_i = \sum_{j=1}^{i-1} q^n_{\delta_j(\alpha)}.
\]

for \( i = 1, \ldots, b^n_\alpha \). Notice that

\[
\left( TQ_{n,n+1}L^n \right)_{\alpha \beta} = \sum_{\beta \in \mathcal{A}} \left( TQ_{n,n+1} \right)_{\alpha \beta} L^n_\beta,
\]

for any \( \alpha \in \mathcal{A} \), and

\[
\# \{ 1 \leq i \leq b^n_\alpha | \delta_i(\alpha) = \beta \} = \left( TQ_{n,n+1} \right)_{\alpha \beta},
\]

for any \( \alpha, \beta \in \mathcal{A} \). A simple calculation shows that

\[
(L^{n+1} - TQ_{n,n+1}L^n)_{\alpha \beta} = \sum_{\beta \in \mathcal{A}} \sum_{\delta_i(\alpha) = \beta} \ln Df^n_\beta \left( f^{h_i}(x^{n+1}_\alpha) \right) Df^n_\beta \left( x^n_\beta \right),
\]

for any \( \alpha \in \mathcal{A} \). By (14) and Corollary 1.6, the previous equation yields to

\[
\frac{|L^{n+1} - TQ_{n,n+1}L^n|}{\|TQ_{n,n+1}\|} = O(c^n),
\]

for some \( 0 < c < 1 \). The claim now follows from Condition 5 and the fact that \( \|TQ_{0,n}\| \) grows exponentially.

We shall see that this pseudo-orbit can be shadowed by the iterates of a vector with respect to the heights cocycle. For any \( n \geq 0 \), let us decompose \( L^n \) with respect to the Oseledet’s splitting at \( (\tau^n, \lambda^n, \pi^n) \) as

\[
L^n = L^n_s + L^n_c + L^n_u,
\]

and define

\[
v_n = TQ^{-1}_0 L^n_c.
\]

**Lemma 2.2.** There exists \( \omega \in E^c(\tau, \lambda, \pi) \) such that \( \lim_{n \to \infty} v_n = \omega \). Moreover

\[
|\omega - v_n| = O(c^n),
\]

for some \( 0 < c < 1 \).
Proof. We have
\[
|v_{n+1} - v_n| = \left| T Q_{0,n+1}^{-1} L^{n+1} - T Q_0^{-1} L^n \right|
\]
\[
= \left| T Q_{0,n+1}^{-1} \left( L^{n+1} - T Q_{n,n+1}^{-1} L^n \right) \right|
\]
\[
\leq \left| T Q_{0,n+1}^{-1} \right| \left| L^{n+1} - T Q_{n,n+1}^{-1} L^n \right|
\]
\[
= O(c^n),
\]
where the last equality follows from Condition 3 and Lemma 2.1.

Proof of Proposition 1.13. By the EC Condition we have
\[
|L^{n+1}|, |L^n| = O(c^n),
\]  
(15)
for some 0 < c < 1. Moreover, by Condition 3 and the previous lemmas,
\[
|\omega^n - L^{n,k}| = \left| T Q_{0,n} \omega - L^n \right|
\]
\[
= \left| T Q_{0,n} (\omega - T Q_0^{-1} L^n) \right|
\]
\[
\leq \left| T Q_{0,n} \right| |\omega - v_n|
\]
\[
= O(c^n),
\]
for some 0 < c < 1 which, together with (15) finishes the proof of Proposition 1.12.

2.2 Proof of Proposition 1.13

Before we prove the main result of this section let us state a few auxiliarly results as well as the notion of a matrix associated with the action of Zorich algorithm on the slope vector.

For \( k \in \mathbb{N} \) define
\[
U = U(\lambda, \pi, w) = (U)_{\alpha, \beta} =
\begin{cases}
1, & \text{if } \alpha = \beta \text{ and } \beta \neq \pi^{-1}(d); \\
\exp(\epsilon \cdot w_{\pi^{-1}(d)}) & \text{if } \alpha = \beta = \pi^{-1}(d); \\
\exp((1 - \epsilon) \cdot w_{\pi^{-1}(d)}) & \text{if } \alpha = \pi^{-1}(d) \text{ and } \beta = \pi^{-1}(d); \\
0 & \text{otherwise},
\end{cases}
\]
where \( \epsilon = \epsilon(\lambda, \pi) \) is the type. Let \( \omega \) be as in Proposition 1.12. Define a matrix
\[
V_n := \prod_{i=k(n)}^{k(n+1)-1} U(RY^i(\lambda, \pi), A^i(\lambda, \pi, \omega)).
\]
where \( k(n) \) is such that \( RY^k(\lambda, \pi, \omega) = R^n(\lambda, \pi, \omega) \). The matrix \( V_n \) gives a way of describing the change of lengths for AIETs which have the same log-slope \( \omega \) and follow the same finite path in the Rauzy graph as \( (\lambda, \pi) \). Then it holds that
\[
(|I^n(S)|)_{\alpha \in A} = V_n \left( |I^{n+1}(S)| \right)_{\alpha \in A},
\]
(16)
In particular, for the normalized length vector we have
\[
(|\tilde{I}^n(S)|)_{\alpha \in A} = V_n \left( |\tilde{I}^{n+1}(S)| \right)_{\alpha \in A}.
\]
For the sake of simplicity we will denote the vectors \( (|I^n(S)|)_{\alpha \in A} \) and \( (|I^n(f)|)_{\alpha \in A} \) as \( I^n(S) \) and \( I^n(f) \) respectively. The analogous abuse of notation applies to the normalized vectors as well.
Lemma 2.3. There exists a compact set $K \subset \Delta_A$ such that $I^n(f), I^n(S) \in K$, for any $n \geq 1$. In particular, there exists a constant $C_K > 0$, depending only on $K$, such that

$$\sup_{n \geq 1} \max_{\alpha, \beta \in A} \left\{ \frac{|I^n_\alpha(f)|}{|I^n_\beta(f)|}, \frac{|I^n_\alpha(S)|}{|I^n_\beta(S)|} \right\} < C_K. \quad (17)$$

Proof. Let $D > 0$ and define

$$K = \bigcup_{D_1 < A_{\alpha, \beta} < D} \hat{A}(\Delta_A),$$

where the union is taken over all $d \times d$ positive matrices such that their coefficients are uniformly bounded from above and below. It is not difficult to check that $K$ is a compact subset of $\Delta_A$ (see [25]) and that there exists a constant $C_K > 0$ such that

$$\max_{\alpha, \beta \in A} \frac{w_\alpha}{w_\beta} < C_K,$$

for any $w \in K$. Thus it is sufficient to show that $\tilde{I}^n(f), \tilde{I}^n(S) \in K$, for any $n \geq 1$, if $D$ is sufficiently big. By [5] we have

$$\tilde{I}^n(f) \in A^{\gamma}(\mathbb{R}^n f)(\Delta_A), \quad \tilde{I}^n(S) \in A^{\gamma}(\mathbb{R}^n S)(\Delta_A),$$

for any $n \geq 0$. Since the log-slope of $S$ belongs to $E^c(\tau, \lambda, \pi)$, by Condition [3] there exists a constant $C > 0$ such that

$$C^{-1} < \|D \mathbb{R}^n S \|_{I^n_\alpha(S)} < C \quad \text{for every } n \in \mathbb{N} \text{ and } \alpha \in A$$

Thus, by Proposition [1.12] and Corollary [1.6] by enlarging $C$ if necessary, we get

$$C^{-1} < \|D \mathbb{R}^n f \|_{I^n_\alpha(f)} < C \quad \text{for every } n \in \mathbb{N} \text{ and } \alpha \in A \quad (18)$$

Hence there exists a constant $\tilde{C} > 0$ such that

$$\tilde{C}^{-1} < A^{\gamma}(\mathbb{R}^n f), A^{\gamma}(\mathbb{R}^n S) < \tilde{C} \quad \text{for every } n \in \mathbb{N} \text{ and } \alpha, \beta \in A$$

Therefore $\tilde{I}^n(f), \tilde{I}^n(S) \in K$, for any $n \geq 1$, if $D$ is sufficiently big. \qed

Lemma 2.4. For every $n \in \mathbb{N}$ we have

$$\frac{1}{|I^n_\alpha(f)|} |I^n(f) - V_n I^{n+1}(f)| = O(c^n)$$

for some $0 < c < 1$.

Proof. Fix $n \geq 1$. We use the notations from the proof of Lemma [2.1]. Namely, for any $\alpha \in A$, we denote

$$b^n_\alpha = \sum_{\beta \in A} \left( T_q a_{n+1} \right)_{\alpha, \beta},$$

and express $q_{n+1}$ uniquely as

$$q^n_{n+1} = \sum_{i=1}^{b^n_\alpha} q^n_{h_i(\alpha)}$$

for some $\delta_i(\alpha) \in A$, such that

$$f_{h_i(\alpha)}(I^{n+1}_\alpha(f)) \subset I^n_{h_i(\alpha)}(f), \quad \text{where } h_i(\alpha) = \sum_{j=0}^{i-1} q^n_{h_j(\alpha)},$$

for $i = 1, \ldots, b^n_\alpha$. Note that $f_{n+1} I^n_{a_{n+1}} = f_{h_i(\alpha)} |I^{n+1}_\alpha|_{h_i(\alpha)}$. We define $\delta_0(\alpha) = 0 = h_0(\alpha)$, for any $\alpha \in A$. Notice that

$$\# \{ 1 \leq i \leq b^n_\alpha \mid \delta_i(\alpha) = \beta \} = (Q_{n,n+1})_{\alpha, \beta}.$$
for any $\alpha, \beta \in \mathcal{A}$. For the sake of simplicity, we will denote $(Q_{n,n+1})_{\alpha,\beta}$ simply by $Q_{\alpha,\beta}$.

By (5) we obtain

$$|I^n_\alpha(f)| = \sum_{\alpha \in \mathcal{A}} |I^{n+1}_\alpha(f)| \sum_{i=1}^{Q_{\alpha,\beta}} D f_{h_{m_i(\alpha,\beta)}(\alpha)}(x^n_i(\alpha,\beta)).$$  \hspace{1cm} (19)

Replacing $f$ by $S$ in the previous formula yields

$$|I^n_\alpha(S)| = \sum_{\alpha \in \mathcal{A}} |I^{n+1}_\alpha(S)| \sum_{i=1}^{Q_{\alpha,\beta}} \exp \left( \sum_{j=0}^{m_i(\alpha,\beta)-1} \omega^n_j(\alpha) \right),$$  \hspace{1cm} (20)

which is nothing more than the $\beta$-th coordinate of (16), since

$$(V_{n,n+1})_{\alpha,\beta} = \sum_{i=1}^{Q_{\alpha,\beta}} \exp \left( \sum_{j=0}^{m_i(\alpha,\beta)-1} \omega^n_j(\alpha) \right).$$  \hspace{1cm} (21)

By (19) and (21),

$$\frac{1}{|I^n_\alpha(f)|} \left| I^n_\alpha(f) - V_{n,n+1} I^{n+1}_\alpha(f) \right|_{\beta} =$$

$$\sum_{\alpha \in \mathcal{A}} \sum_{i=1}^{Q_{\alpha,\beta}} \left| I^{n+1}_\alpha(f) \right| \left( D f_{h_{m_i(\alpha,\beta)}(\alpha)}(x^n_i(\alpha,\beta)) - \exp \left( \sum_{j=0}^{m_i(\alpha,\beta)-1} \omega^n_j(\alpha) \right) \right).$$  \hspace{1cm} (22)

**Lemma 2.5.** For any $\alpha, \beta \in \mathcal{A}$ and any $1 \leq i \leq Q_{\alpha,\beta}$,

$$\left| \frac{I^{n+1}_\alpha(f)}{I^n_\alpha(f)} \right| \left| D f_{h_{m_i(\alpha,\beta)}(\alpha)}(x^n_i(\alpha,\beta)) - \exp \left( \sum_{j=0}^{m_i(\alpha,\beta)-1} \omega^n_j(\alpha) \right) \right| = O(c^n),$$

for some $0 < c < 1$.

Assuming Lemma 2.5, the result now follows from equations (17), (22) as well as Condition 5. \hfill \Box

**Proof of Lemma 2.5.** For any $\alpha, \beta \in \mathcal{A}$ and any $1 \leq i \leq Q_{\alpha,\beta}$, we have

$$\left| \ln D f_{h_{m_i(\alpha,\beta)}(\alpha)}(x^n_i(\alpha,\beta)) - \sum_{j=0}^{m_i(\alpha,\beta)-1} \omega^n_j(\alpha) \right| \leq \sum_{j=0}^{m_i(\alpha,\beta)-1} \left| \ln D f_{h^n_j(\alpha)}(h^{n+1}_j(\alpha), x^n_i(\alpha,\beta)) - \omega^n_j(\alpha) \right|$$

$$\leq \sum_{j=0}^{m_i(\alpha,\beta)-1} \left( |L^n_j(\alpha) - \omega^n_j(\alpha)| + O(c^n) \right)$$

$$= O(c^n),$$

for some $0 < c < 1$, where the third line in the above expression follows from the Corollary 1.6 and the fact that $m_i(\alpha,\beta) \leq \|Q_{n,n+1}\|$, while the last line follows from Proposition 1.12 as well as Condition 5.

By the previous equation,

$$\left| \frac{D f_{h_{m_i(\alpha,\beta)}(\alpha)}(x^n_i(\alpha,\beta))}{\exp \left( \sum_{j=0}^{m_i(\alpha,\beta)-1} \omega^n_j(\alpha) \right)} - 1 \right| = O(c^n).$$  \hspace{1cm} (23)

The following observation will be useful in the proof.
Claim. There exists $C > 0$ such that
\[
\max_{\alpha \in A} \frac{|I_{n+1}^\alpha(f)|}{|I_n(f)|} \leq C |Q_{n,n+1}|,
\]
for any $n \geq 1$.

Proof. Fix $\beta \in A$. By Lemma 2.4,
\[
\frac{|I_{n+1}^\beta(f)|}{|I_n(f)|} \leq dC_r \frac{|I_{n+1}^\beta(f)|}{|I_n^\beta(f)|} \leq C \frac{|I_{n+1}^\beta(f)|}{|I_n^\beta(f)|}.
\]
By (17), (19), (20) and (23)
\[
\frac{|I_{n+1}^\beta(f)|}{|I_n^\beta(f)|} \leq C_0 \sum_{\alpha \in A} \sum_{i=1}^{Q_\alpha} \frac{|I_{n+1}^\beta(f)|}{|I_n^\beta(f)|} \exp \left( \sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha) \right)
\]
\[
\leq C_0^2 \sum_{\alpha \in A} \sum_{i=1}^{Q_\alpha} \frac{|I_{n+1}^\beta(f)|}{|I_n^\beta(f)|} \exp \left( \sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha) \right)
\]
\[
\leq C_0^2 \sum_{\alpha \in A} \sum_{i=1}^{Q_\alpha} (1 + O(\epsilon^n))
\]
\[
\leq C \|Q_{n,n+1}\|,
\]
for some $C > 0$.

Finally, by (20), Condition 5 and the previous claim, there exists $C > 0$ such that
\[
\left| \frac{|I_{n+1}^\alpha(f)|}{|I_n(f)|} \right| \left| \frac{Df^h_{m_{\alpha}(\alpha, \beta)}(x_i^n(\alpha, \beta))}{\exp(\sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha))} - 1 \right| \left| \frac{|I_{n+1}^\alpha(f)|}{|I_n^\alpha(f)|} \exp \left( \sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha) \right) \right|
\]
\[
= \left| \frac{Df^h_{m_{\alpha}(\alpha, \beta)}(x_i^n(\alpha, \beta))}{\exp(\sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha))} - 1 \right| \left| \frac{|I_{n+1}^\alpha(f)|}{|I_n^\alpha(f)|} \exp \left( \sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha) \right) \right|
\]
\[
\leq C \left| \frac{Df^h_{m_{\alpha}(\alpha, \beta)}(x_i^n(\alpha, \beta))}{\exp(\sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha))} - 1 \right| \left| \frac{|I_{n+1}^\alpha(f)|}{|I_n^\alpha(f)|} \right| \left| \frac{|I_{n+1}^\alpha(f)|}{|I_n^\alpha(f)|} \exp \left( \sum_{j=0}^{m_{\alpha}(\alpha, \beta) - 1} \omega_{j_0}(\alpha) \right) \right|
\]
\[
= O(\epsilon^n)
\]
for some $0 < \epsilon < 1$.

As a corollary we have the following.

Corollary 2.6. For every $n \in \mathbb{N}$ we have
\[
\bar{I}_n(f) = \bar{V}_n I_{n+1}(f) + O(\epsilon^n),
\]
for some $0 < \epsilon < 1$.

Proof. By Lemma 2.4 we have
\[
|V_n I_{n+1}(f)| = (1 + O(\epsilon^n))|I_n(f)|.
\]
Thus, again by Lemma 2.4, we obtain
\[
\tilde{V}_n I^{n+1}(f) = \frac{V_n I^{n+1}(f)}{|V_n I^{n+1}(f)|} = \frac{V_n I^{n+1}(f)}{|I^n(f)|} \cdot \frac{|I^n(f)|}{|V_n I^{n+1}(f)|} = \tilde{I}^n(f) + O(c^n).
\]

For the purpose of formulating the next result let us recall a classical notion of a Hilbert projective metric \( d_p \) on \( \Delta_A \). It is defined as follows
\[
d_p(v, w) = \log \sup \left\{ \frac{v_\alpha}{w_\alpha} \cdot \frac{w_\beta}{v_\beta} : \alpha, \beta \in A \right\}.
\]
It is equivalent to the Euclidean metric in any compact subset of \( \Delta_A \).

**Lemma 2.7.** There exists \( 0 < \kappa < 1 \) such that for every \( n \in \mathbb{N} \) we have that \( V_n \) is a contraction in the projective metric with \( \kappa \) as a uniform contraction constant.

**Proof.** Note that \( V_n = A^{|\alpha|}(\mathbb{R}^n_S) \cdot B(\mathbb{R}^n_S) \), where \( B(\mathbb{R}^n_S) \) is some non-negative matrix. As a classical fact (see e.g. (92) in [25]), multiplication by non-negative matrix is non-expanding in the projective metric. It is thus enough to show that multiplication by \( A^{|\alpha|}(\mathbb{R}^n_S) \) is a contraction with a constant which does not depend on \( n \).

First note that \( A^{|\alpha|}(\mathbb{R}^n_S) \) is positive, since \( A^{|\alpha|} \) is positive. By Proposition 26.3 in [25] it is enough to show that the entries of \( V_n \) are uniformly bounded from below and above. However in view of Condition 3 (and the fact that \( \omega \in E^\infty(\tau, \lambda, \pi) \)) we have that
\[
\max_{\alpha \in A} |\omega_\alpha^n| < K,
\]
for some \( 1 \leq K < \infty \) uniformly in \( n \in \mathbb{N} \). In particular we get that for every \( n \in \mathbb{N} \) we have
\[
\max_{\alpha, \beta \in A} A^{|\alpha|, |\beta|}(\mathbb{R}^n_S) \leq (\#A)^{|\alpha|} K \cdot e^{2K} \cdot e^{4K} \cdot \ldots \cdot e^{2^{\alpha} K}
\]
and
\[
\min_{\alpha, \beta \in A} A^{|\alpha|, |\beta|}(\mathbb{R}^n_S) \geq e^{-K} \cdot e^{-2K} \cdot e^{-4K} \cdot \ldots \cdot e^{-2^{\alpha} K}.
\]
Since both bounds do not depend on \( n \), this concludes the proof of the lemma.

The ideas used for the following proof come from [4].

**Proof of Proposition 7.7.** For \( n < m \) let us denote
\[
\hat{V}_n^m := V_n \circ \ldots \circ V_m.
\]
By Corollary 2.6 and Lemma 2.7 with Lemma 2.3 we have
\[
d_p(\hat{I}^n(f), \hat{V}_n^{2n-1} \hat{I}^{2n}(f)) \\
\leq \sum_{i=0}^{n-1} d_p(\hat{V}_n^{2n-1-i} \hat{I}^{2n-i}(f), \hat{V}_n^{2n-(i+1)-1} \hat{I}^{2n-i-1}(f)) \\
\leq \sum_{i=0}^{n-1} \kappa^{n-i-1} O(e^{2n-(i+1)}) \leq (1 - \kappa)^{-1} O(c^n) = O(e^n).
\]
Thus, by (19),
\[
d_p(\hat{I}^n(f), \hat{I}^n(S)) \leq d_p(\hat{I}^n(f), \hat{V}_n^{2n-1} \hat{I}^{2n}(f)) \\
+ d_p(\hat{V}_n^{2n-1} \hat{I}^{2n}(f), \hat{V}_n^{2n-1} \hat{I}^{2n}(S)) \leq O(c^n) + \kappa^n \cdot D,
\]

(24)
where $D$ is the diameter in $d_p$ of a compact set given by Lemma 2.3 which contains both $\tilde{I}^{2n}(f)$ and $\tilde{I}^{2n}(S)$. Thus we get
\[
d_p(\tilde{I}^n(f), \tilde{I}^n(S)) \leq O(e^n) + O(\kappa^n),
\]
which yields
\[
d_p(\tilde{I}^n(f), \tilde{I}^n(S)) = O(\max\{e, \kappa\}^n).
\]

2.3 Proof of Proposition 1.14

We use the results of previous sections, in particular Proposition 1.12 and Proposition 1.13 to prove Proposition 1.14.

Proof of Proposition 1.14 Note first that
\[
\frac{|R^n(S(I^n(S))|}{|I^n(S)|^2} = e^{\omega n} \frac{|I^n(S)|^2}{|I^n(S)|^2}.
\]
Moreover,
\[
\frac{|R^n f(I^n(S)|}{|I^n(f)|} = e^{\omega n} \frac{|I^n(S)|^2}{|I^n(f)|}.
\]
We thus obtain
\[
\frac{|R^n f(I^n(S)|}{|I^n(f)|} - |R^n S(I^n(S)|}{|I^n(S)|} = \frac{e^{\omega n} |I^n(S)|^2}{|I^n(f)|} - \frac{e^{\omega n} |I^n(S)|^2}{|I^n(S)|^2}.
\]
By Proposition 1.13 we have
\[
\frac{e^{\omega n} |I^n(S)|^2}{|I^n(S)|} - \frac{e^{\omega n} |I^n(S)|^2}{|I^n(S)|} = e^{\omega n} O(e^n).
\]
On the other hand by 1.12 we also get
\[
\frac{e^{\omega n} |I^n(S)|^2}{|I^n(S)|} - \frac{e^{\omega n} |I^n(S)|^2}{|I^n(S)|} = e^{\omega n} O(e^n)
\]
The result now follows from 1.15. □

3 Smoothness of the conjugacy map

The following consequences of Theorem 1.8 will be useful.

Lemma 3.1. Let $f, S$, as in Theorem 1.8 Then
1. For any $\alpha \in A$, \[
\frac{|I^n(f)|}{|I^n(S)|} = 1 + O(e^n), \text{ for some } 0 < c < 1.
\]
2. \[
|I^n f^{-1}(S)| \leq 1 + O(e^n), \text{ for some } 0 < c < 1.
\]
3. For any $\alpha \in A$, \[
\frac{|R^n f(I^n(S)|}{|I^n(f)|} \to C_0 \text{ for some } C_0 > 0.
\]
Proof. The first assertion follows from Lemma 2.3 and Proposition 1.13. For any \( n \geq 1 \) we have

\[
\frac{|I^n(f)|}{|I^n(S)|} = \frac{Q_{n-1,n}I^n(S)}{|I^n(f)|} = 1 + \frac{|Q_{n-1,n}I^n(S)|}{|I^n(f)|} = 1 + O(c^n),
\]

for some \( 0 < c < 1 \), by Condition 5 as well as Theorem 1.8. This proves the second assertion.

Notice that by Theorem 1.8 and the first assertion, to prove the last assertion it is sufficient to show that \( \frac{|I^n(f)|}{|I^n(S)|} \) converges as \( n \to \infty \).

By the second assertion,

\[
\frac{|I^n(f)|}{|I^n(S)|} = \frac{|I^0(f)|}{|I^0(S)|} \prod_{i=1}^{n} \frac{|I^i(f)|}{|I^{i-1}(S)|} = \frac{|I^0(f)|}{|I^0(S)|} \prod_{i=1}^{n} (1 + O(c^n)),
\]

for some \( 0 < c < 1 \). Therefore, \( \frac{|I^n(f)|}{|I^n(S)|} \) converges as \( n \to \infty \).

\[\square\]

### 3.1 Proof of Proposition 1.16

Proof of Proposition 1.16. For any bounded \( \varphi : [0, 1) \to \mathbb{R} \), and any \( m \geq 0 \), denote by \( S_m^{\varphi} \) the \( m \)-th Birkhoff sum of \( \varphi \) w.r.t. \( f \). By decomposing Birkhoff sums of length \( m \) into Birkhoff sums along towers obtained via \( R \) (special Birkhoff sums) we get

\[
\|S_m^{\varphi}\|_{[0,1)} \leq \sum_{n \geq 0} \|T^n Q_{n,n+1}\| \max_{a \in A} \|S_m^{\varphi}\|_{I_\alpha^n(f)}.
\]

For details on this decomposition, we refer the interested reader to [20 Section 2.2.3]. Notice that for any \( m \geq 0 \),

\[
S_m^{f} (\log DS \circ h - \log Df) = \sum_{n \geq 0} \|T^n Q_{n,n+1}\| \max_{a \in A} \|S_m^{f}\|_{I_\alpha^n(f)}.
\]

Pick \( \varphi = \log DS \circ h - \log Df \). Notice that \( \varphi \in C^0\left( \bigcup_{a \in A} I_\alpha^n \right) \). By (20) together with Corollary 1.6 and Proposition 1.12 we get

\[
\max_{a \in A} \|S_m^{f}\|_{I_\alpha^n(f)} = O(c^n),
\]

for some \( 0 < c < 1 \). By the previous equation, (20) and Condition 5 for any \( \tau > 0 \) there exists \( C > 0 \) such that

\[
\|S_m^{\varphi}\|_{[0,1)} \leq C \sum_{n \geq 0} e^{n(\tau - |\log c|)},
\]

for any \( m \geq 0 \). Picking \( \tau < |\log c| \), it follows that

\[
\sup_{m \geq 0} \|S_m^{f}\|_{[0,1)} < +\infty.
\]

Hence, by Proposition 1.14 there exists \( \psi : [0, 1) \to \mathbb{R} \) continuous verifying (13).

Furthermore, if \( f \) and \( S \) are break-equivalent circle homeomorphisms, the function \( \varphi \) above is a well-defined continuous function on the circle. Then, considering \( \varphi \) and \( f \) as continuous functions on \( T \), by Gottschalk-Hedlund’s Theorem there exists \( \psi : T \to \mathbb{R} \) continuous verifying (13). \[\square\]
3.2 Proof of Proposition 1.17

Lemma 3.2. Let $f$, $S$ be as in Theorem 1.18 and assume that $f$ is semi-conjugated to $S$ via $h$. Then $h$ is a Lipschitz homeomorphism.

Proof. Let $\alpha \in \mathcal{A}$, $n \geq 0$ and $0 \leq j < q^n_\alpha$. Since $h(f^j(I^n_\alpha(S))) = S^j(I^n_\alpha(S))$, by the intermediate value theorem,

$$\frac{|h(f^j(I^n_\alpha(f)))|}{|f^j(I^n_\alpha(f))|} = \frac{|R^n_s(I^n_\alpha(S))|}{|R^n_s(f(I^n_\alpha(f)))|} \frac{DF^n_{\alpha}^{j-1}(x)}{DS^{\alpha}_{\alpha}^{j-1}(y)}$$

for some $x \in f^j(I^n_\alpha(f))$ and $y \in S^j(I^n_\alpha(S))$. Using Proposition 1.16 we can rewrite the previous equation as

$$\frac{|h(f^j(I^n_\alpha(f)))|}{|f^j(I^n_\alpha(f))|} = \frac{|R^n_s(I^n_\alpha(S))|}{|R^n_s(f(I^n_\alpha(f)))|} \frac{DS^{\alpha}_{\alpha}^{j-1}(h(x))}{DS^{\alpha}_{\alpha}^{j-1}(h(y))} \frac{DF^n_{\alpha}^{j-1}(x)}{DS^{\alpha}_{\alpha}^{j-1}(y)}$$

$$= \frac{|R^n_s(I^n_\alpha(S))|}{|R^n_s(f(I^n_\alpha(f)))|} \frac{DS^{\alpha}_{\alpha}^{j-1}(h(x))}{DS^{\alpha}_{\alpha}^{j-1}(h(y))} e^{\psi(f^{j-1}(x))} e^{\psi(x)}.$$ 

Since $S$ is an AIET, $y, h(x) \in I^n_\alpha(S)$ and $0 \leq q^n_\alpha - j < q^n_\alpha$, then $\frac{DS^{\alpha}_{\alpha}^{j-1}(h(x))}{DS^{\alpha}_{\alpha}^{j-1}(h(y))} = 1$. By Lemma 3.1 and Proposition 1.16 and the previous equation,

$$\sup_{n \geq 0} \max_{p \in \mathcal{P}_n(f)} \max_{0 \leq j < q^n_\alpha} \left\{ \frac{|h(j)|}{|j|}, \frac{|j|}{|h(j)|} \right\} < +\infty,$$

(27)

where $\mathcal{P}_n(f) = \{f^j(I^n_\alpha(f)); \alpha \in \mathcal{A}, 0 \leq j < q^n_\alpha\}$. Since any interval $(a, b) \subset [0, 1]$ can be expressed as a countable union of intervals in $\bigcup_{n \geq 0} \mathcal{P}_n(f)$, it follows that $h$ is a Lipschitz function.

Since by Theorem 1.15 $S$ is topologically conjugated to a minimal IET, we have that

$$\lim_{n \to \infty} \max_{0 \leq j < q^n_\alpha} |S^j(I^n_\alpha(S))| = 0.$$

Hence by (27) we obtain

$$\lim_{n \to \infty} \max_{0 \leq j < q^n_\alpha} |f^j(I^n_\alpha(f))| = 0.$$ 

Since for every $n \geq 0$ the wandering intervals are always included in the elements of the partition $\mathcal{P}_n(f)$, the above expression implies that $f$ does not have wandering intervals, which in turn implies that $h$ is a conjugacy.

Proof of Proposition 1.17. By Lemma 3.2 the map $h$ is a Lipschitz homeomorphism. It follows that it is differentiable almost everywhere. Let $x_0 \in [0, 1]$ such that $h$ is differentiable. WLOG, we may suppose that $x_0$ is not the endpoint of any interval in $\bigcup_{n \geq 0} \mathcal{P}_n(f)$. Then, there exist sequences $(\alpha_n)_{n \geq 1} \subset \mathcal{A}$ and $(j_n)_{n \geq 1} \subset \mathbb{N}$ such that

$$0 \leq j_n < q^n_\alpha, \quad \{x_0\} = \bigcap_{n \geq 1} f^{j_n}(I^n_{\alpha_n}(f)).$$

Therefore

$$Dh(x_0) = \lim_{n \to \infty} \frac{|h(f^{j_n}(I^n_{\alpha_n}(f)))|}{|f^{j_n}(I^n_{\alpha_n}(f))|}.$$ 

By the intermediate value theorem, there exist sequences $x_n \in f^{j_n}(I^n_{\alpha_n}(f))$ and $y_n \in S^{j_n}(I^n_{\alpha_n}(S))$ such that

$$\frac{|h(f^{j_n}(I^n_{\alpha_n}(f)))|}{|f^{j_n}(I^n_{\alpha_n}(f))|} = \frac{|S^{j_n}(I^n_{\alpha_n}(S))|}{|f^{j_n}(I^n_{\alpha_n}(f))|} = \frac{|R^n_s(I^n_{\alpha_n}(S))|}{|R^n_s(f(I^n_{\alpha_n}(f)))|} \frac{DF^n_{\alpha}^{j_n}(x_n)}{DS^{\alpha}_{\alpha}^{j_n}(y_n)}.$$
Using Proposition \[1.16\] we can rewrite the previous equation as

\[
\frac{|h(f^{n_0}(f^{n_0}(f)))|}{|f^{n_0}(f^{n_0}(f))|} = \frac{|\mathcal{R}^nS(f^{n_0}(f^{n_0}(S)))|}{|\mathcal{R}^nS(f^{n_0}(f^{n_0}(f)))|} \frac{DS^{n_0-jn}(h(x_n))}{DS^{n_0-jn}(y_n)} \frac{DF^{n_0}(x_n)}{DF^{n_0}(y_n)}
\]

\[
= \frac{|\mathcal{R}^nS(f^{n_0}(f^{n_0}(S)))|}{|\mathcal{R}^nS(f^{n_0}(f^{n_0}(f)))|} \frac{DS^{n_0-jn}(h(x_n))}{DS^{n_0-jn}(y_n)} e^{-\psi f^{n_0}-jn(x_n)} e^{\psi(x_n)}.
\]

We now analyse each term in the previous equation separately.

By Lemma \[5.1\], \( |\mathcal{R}^nS(f^{n_0}(f^{n_0}(f)))| \rightarrow C_6 \), for some constant \( C_6 \).

Since \( S \) is an AET, \( y_n, h(x_n) \in I^{n_0}(S) \) and \( 0 \leq q^{n_0}_{a_2-jn} < q^{n_0}_{a_2} \), then \( DS^{n_0-jn}(S^{n_0-jn}(h(x_n))) = 1 \).

Since \( f^{n_0-jn}(x_n) = 0 \), we have \( e^{\psi f^{n_0-jn}(x_n)} \rightarrow e^{\psi(0)} \).

Therefore, since \( x_n \rightarrow x_0 \), we have \( Dh(x_0) = C_6 e^{\psi(0)} e^{\psi(x_0)} \). Moreover, since \( h \) is Lipschitz and the derivative of \( h \) at almost every point coincides with the continuous function \( C_6 e^{\psi(0)} e^{\psi(0)} \), it follows that \( h \) is of class \( C^1 \)

\( \square \)

### 3.3 Proof of Theorem 1.11

**Proof of Theorem 1.11**

Let \((\lambda, \pi)\) as in Theorem 1.1 with \( \pi \) of rotation type, and let \( f, g \) as in the statement of Theorem 1.11. Denote by \( F \) and \( G \) the canonical identification of \( f \) and \( g \) as GIETs introduced in Section 1.3 namely \( F = \varphi \circ f \circ \varphi^{-1} \) and \( G = \varphi \circ g \circ \varphi^{-1} \), with \( \varphi : [0, 1] \rightarrow T \) given by \( \{T\} \).

By Theorems 1.8, 1.9 and Lemma 1.10 there exists \( \omega \in E^\circ(\tau, \lambda, \pi) \) and \( L \in \text{Aff}(T, \omega) \) such that \( F \) and \( G \) are \( C^1 \)-conjugated to \( L \) as GIETs. Denote by \( l \) the circle homeomorphism induced by \( L \) with respect to the parametrization \( \varphi \), namely, \( l = \varphi^{-1} \circ L \circ \varphi \).

Notice that, although \( F \) and \( G \) are \( C^1 \)-conjugated to \( L \) (in particular boundary-equivalent to \( L \)) the maps \( f \) and \( g \) are not necessarily break-equivalent to \( l \). In fact, assuming WLOG that \( \pi \) verifies \( \{S\} \) for some \( k \in \{0, \ldots, d - 2\} \), we have

\[
BP(l) \subseteq \{\varphi(u_j(L)) \mid 0 \leq j < d\},
\]

\[
BP(f) = \{\varphi(u_j(F)) \mid 0 \leq j < d; j \neq d - k - 1\},
\]

\[
BP(g) = \{\varphi(u_j(G)) \mid 0 \leq j < d; j \neq d - k - 1\}.
\]

Note that the conjugacy between \( f \) (resp. \( g \)) and \( l \), which we obtain as the induced circle homeomorphism associated to the conjugacy between \( F \) (resp. \( G \)) and \( L \), identifies the points \( \varphi(u_j(F)) \) (resp. \( \varphi(u_j(G)) \)) with \( \varphi(u_j(L)) \), for all \( 0 \leq j \leq d \). Since \( \varphi(u_{d-k-1}(L)) \) could be a break point for \( l \), the map \( l \) might not be break-equivalent to \( f \) and \( g \).

However, it follows from the discussion in Section 1.3 that

\[
\sigma_f(\varphi(u_j(F))) = \sigma_g(\varphi(u_j(G))) = \sigma_f(\varphi(u_j(L))),
\]

which are mapped by the conjugacy with the map \( f \) (resp. \( g \)) to break points of \( f \) (resp. \( g \)) with the same jump ratio. Thus, if \( \varphi(u_{d-k-1}(L)) \) is not a break point for \( l \) then the map \( l \) is break-equivalent to \( f \) and \( g \).

Indeed, if \( \varphi(u_{d-k-1}(L)) \) is not a break point for \( l \), then \( l \) would have at most \( d - 1 \) break points. Since \( f \) (resp. \( g \)) has exactly \( d - 1 \) break points and \( BP(l) \subseteq \{\varphi(u_j(L)) \mid 0 \leq j < d; j \neq d - k - 1\} \), it follows from \( \{9\} \) and \( \{25\} \) that \( l \) has a break point at \( \varphi(u_0(L)) \) with jump ratio \( \sigma_f(\varphi(u_0(F))) \) (resp. \( \sigma_g(\varphi(u_0(G))) \)). Since the conjugacy between \( f \) (resp. \( g \)) and \( l \) is sending break points to break points, these maps would be break-equivalent.

In general \( \varphi(u_{d-k-1}(L)) \) will be a break point for \( l \). However, we can show the following.
Hence, there exists a unique $a$ such that for any $L^* \in \text{Aff}(T, \omega + v)$, we have $\varphi(u_{d-k-1}(L^*)) \notin \text{BP}(t^*)$, where $t^* = \varphi^{-1} \circ L^* \circ \varphi$ denotes the circle homeomorphism induced by $L^*$.

**Proof of the Claim.** Since $\pi$ is of rotation type, it follows from the duality of the heights and lengths cocycle (see [30, Lemma 3.3] and [2, pages 384-385]) that

$$E^*(\lambda, \pi) = \lambda^+ \cap \text{Ker}(\Omega_\pi)^-.$$ (29)

Moreover, since $\pi$ verifies (5), a simple calculation shows that

$$E^*(\lambda, \pi) = \left\{ v \in \mathbb{R}^d \left| \begin{array}{l}
\langle v, \lambda \rangle = 0; \\
v_\alpha = v_\beta \text{ if } \pi_0(\alpha), \pi_0(\beta) \leq d - k - 1; \\
v_\alpha = v_\beta \text{ if } \pi_0(\alpha), \pi_0(\beta) > d - k - 1.
\end{array} \right. \right\}.$$ 

Let $v \in E^*(\lambda, \pi)$ and denote $a = v_{\pi_0^{-1}(1)}$ or $b = v_{\pi_0^{-1}(d)}$. Notice that any coordinate of $v$ is either equal to $a$ or $b$. Since $\langle \lambda, v \rangle = 0$, we have

$$b = ta, \quad \text{where} \quad t = -\frac{\sum_{\pi_0(\alpha) \leq d - k - 1} \lambda_\alpha}{\sum_{\pi_0(\alpha) > d - k - 1} \lambda_\alpha}.$$ 

With these notations it is easy to show that, for any $L^* \in \text{Aff}(T, \omega + v)$,

$$\sigma(\varphi(u_j(L^*))) = \left\{ \begin{array}{ll}
\sigma_1(\varphi(u_0(L))) \exp^{a(t-1)} & \text{if } j = 0, \\
\sigma_1(\varphi(u_{d-k-1}(L))) \exp^{a(1-t)} & \text{if } j = d - k - 1, \\
\sigma_1(\varphi(u_j(L))) & \text{otherwise.}
\end{array} \right.$$ 

Hence, there exists a unique $a \in \mathbb{R}$ (and thus a unique $v \in E^*(\lambda, \pi)$) such that $\sigma_L(\varphi(u_{d-k-1}(L^*) \}) = 1$, for any $L^* \in \text{Aff}(T, \omega + v)$. In particular, $\varphi(u_{d-k-1}(L^*))$ is not a break point of $t^* = \varphi^{-1} \circ L^* \circ \varphi$.

Recall that $E^*(\lambda, \pi)$ is orthogonal to $\text{Ker}(\Omega_\pi)$ by (29). Hence, by definition of $\omega$, any AIET $L^*$ as in the previous claim fulfills the conditions in Theorem 1.8 and therefore, by Theorem 1.9 is $C^1$ conjugated to $F$ and $G$ as GIETs. As before, the induced circle homeomorphism $t^*$ is topologically conjugated to $f$ and $g$ and $d - 2$ of the respective jump ratios at the break points coincide. Since $\varphi(u_{d-k-1}(L^*))$ is not a break point for $t^*$, it follows from the previous discussion that $t^*$ is break-equivalent to $f$ and $g$. Finally, by Propositions 1.16 and 1.17 $t^*$ is $C^1$-conjugated to $f$ and $g$ as circle homeomorphisms.

\[ \blacksquare \]

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