Multiple contact problem for bodies with nonuniform coatings of variable thickness and regular systems of punches with complex base shape

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Abstract. Plane problem of multiple contact interaction for a viscoelastic foundation with nonuniform coating of variable thickness and regular system of rigid punches with complex base shape is considered. We obtain basic system of integral equations and additional conditions for this problem. The analytic solutions for one version of the problem is presented.

Introduction

Multiple contact problems for coated layers have previously been considered for following cases: 1) contact interaction between system of punches and surface nonuniform foundations (surface nonuniformity is can be described by rapidly changing or discontinuous function) [1]; 2) conformal contact between system of punches and layered foundation (punch base shapes and coating height conformal and can be described by rapidly changing function) [2, 3]; 3) contact interaction between system of irregularly shaped punches and layered foundation (punch base forms can be described by rapidly changing functions) [4, 5]. This paper deals with the case where shape and elastic properties of coating and punch base shapes described by different rapidly changing functions.

1. Statement of the plane contact problem

Viscoelastic aging foundation lies on a rigid undeformable basis (figure 1). This foundation consist of two layers: the lower homogeneous layer of constant thickness $h_{\text{lower}}$ made of a viscoelastic aging material at time $\tau_{\text{lower}}$ and the upper layer (coating) of variable thickness $h(x)$ made of elastic material ($x$ is longitudinal coordinate). Coating have nonuniformity, i.e. its Young’s modulus $E(x, z)$ and Poisson’s ratio $\nu(x, z)$ depends on longitudinal coordinate and depth. Functions $h(x)$, $E(x, z)$, and $\nu(x, z)$ has a period $\Delta a$ over $x$, i.e. $h(x) = h(x + \Delta a)$, $E(x, z) = E(x + \Delta a, z)$, and $\nu(x, z) = \nu(x + \Delta a, z)$. We assume that the coating rigidity is less than the rigidity of the lower layer or they are of the same order of magnitude, i.e., Young’s modulus $E(x, z)$ of the upper layer is less than Young’s modulus $E_{\text{lower}}(t)$ of the lower layer. We consider the plain strain problem.

At instant $\tau_0 \geq \tau_{\text{lower}}$, the forces $P_i(t)$ with eccentricities $e_i(t)$ starts to indent rigid punches into the surface of such a foundation ($i = 1, 2, \ldots, n$, $n$ is the number of the punches). The
system of punches is regular, i.e. the distances between neighbor punches are the same, punch lengths $\bar{a}$ are equal, and punch base shapes are similar. The contact area lengths $\bar{a} = (b_i - a_i) \ll h(x)$ for all $x$, where $a_i$ and $b_i$ are left and right coordinates of $i$th punch. Moreover the period $\Delta a$ is equal to $a_{i+1} - a_i$ for all $i = 1, 2, \ldots, n - 1$. The backlash between $i$th punch and the coating in undeformable state is described by a function $g_i(x) \equiv g(x) \left(g_i(x) \geq 0, \exists x_0 \in [a_i, b_i]: g_i(x_0) = 0\right)$. As a result of the interaction, the punches are immersed into the layered foundation to the depths $\delta_i(t)$ and rotated through the angles $\alpha_i(t)$.

To derive the mathematical model of the problem, we replace the system of punches by some normally distributed load acting on regions $x \in [a_i, b_i], i = 1, 2, \ldots, n$ (under the punches), and equal to zero outside these regions:

$$ q(x, t) = \begin{cases} 
q_1(x, t), & x \in [a_1, b_1], \\
q_2(x, t), & x \in [a_2, b_2], \\
\ldots \ldots \ldots \ldots, \\
q_n(x, t), & x \in [a_n, b_n], \\
0, & x \notin [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_n, b_n]. 
\end{cases} $$

Then the vertical displacement of the upper face of the foundation described above on each region (under each punch) can be written as (we use known solutions for thick and thin elastic and viscoelastic layers [6–8])

$$ u_{iz}(x, t) = - \int_{\tau_0}^{h(x)} \frac{dz}{R(x, z)} q_i(x, t) - \frac{2(1 - \nu^2)}{\pi} \left[ \sum_{j=1}^{n} \int_{a_j}^{b_j} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right) \frac{q_j(\xi, t)}{E_{lower}(t - \tau_{lower})} d\xi \right] \\
- \int_{\tau_0}^{t} K(t - \tau_{lower}, \tau - \tau_{lower}) \int_{a_j}^{b_j} k_{pl} \left( \frac{x - \xi}{h_{lower}} \right) \frac{q_j(\xi, \tau)}{E_{lower}(\tau - \tau_{lower})} d\xi d\tau, \quad (1) 
$$

where $\nu_{lower}$ is the Poisson’s ratio of lower layer; $R(x, z)$ is contact rigidity of the coating depend on its Young modulus and Poisson’s ratio (see, for example, [9]): in the case of a smooth coating-lower layer contact we have $R(x, z) = E(x, z)/[1 - \nu^2(x, z)]$, and in the case of an perfect contact
\[ R(x, z) = E(x, z)\left[1 - \nu(x, z)\right]/\left[1 - \nu(x, z) - 2\nu^2(x, z)\right]; \]
\[ k_{pl}(s) \text{ is known kernel of the plane contact problem, which has the form \[10\] } \]
\[ k_{pl}(s) = \int_{0}^{\infty} L(u) \frac{u}{u} \cos(su) du, \quad L(u) = \left\{ \begin{array}{ll}
\frac{\cosh(2u) - 1}{2\kappa \sinh(2u) - 4u}, & \text{smooth layer-basis contact,} \\
\frac{2\kappa \cosh(2u) + 4u^2 + 1 + x^2}{4}, & \text{perfect layer-basis contact;} \end{array} \right. \]
\[ K(t, \tau) \text{ is creep kernel which has a form (see \[7,11,12\]) } \]
\[ K(t, \tau) = E_{lower}(\tau) \frac{\partial}{\partial \tau} \left[ \frac{1}{E_{lower}(\tau)} + C_{lower}(t, \tau) \right], \]
\[ C_{lower}(t, \tau) \text{ is the tensile creep functions.} \]

But the vertical displacement of the upper face of the foundation connect with displacement of the rigid punches and backlash function by the formula (figure 2)

\[ u_{xz}(x, t) = -[\delta_i(t) + \alpha_i(t)(x - \eta_i) - g_i(x)], \quad x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n, \quad (2) \]

where \( \eta_i = \frac{1}{2}(a_i + b_i) \) is midpoint of \( i \)th punch, \( \delta_i(t) \) is its settlement, and \( \alpha_i(t) \) is its tilt angle.

By equating right sides of (1) and (2) we obtain the system of integral equations of our problem in the form

\[ \int_{0}^{b(x)} \frac{dz}{R(x, z)} q_i(x, t) + \frac{2(1 - \nu^2_{lower})}{\pi} \left[ \sum_{j=1}^{n} \int_{a_j}^{b_j} k_{pl}\left(\frac{x - \xi}{h_{lower}}\right) \frac{q_j(\xi, t)}{E_{lower}(t - \tau_{lower})} d\xi \right] - \int_{\tau_0}^{t} K(t - \tau_{lower}, \tau - \tau_{lower}) \int_{a_j}^{b_j} k_{pl}\left(\frac{x - \xi}{h_{lower}}\right) \frac{q_j(\xi, \tau)}{E_{lower}(\tau - \tau_{lower})} d\xi d\tau \]
\[ = \delta_i(t) + \alpha_i(t)(x - \eta_i) - g_i(x), \quad x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n. \quad (3) \]

The resulting system of equations should be supplemented with equilibrium conditions for punches under the action of distributed load \( q_i(x, t) \) and a concentrated force \( P_i(t) \):

\[ \int_{a_i}^{b_i} q_i(\xi, t) d\xi = P_i(t), \quad (4) \]
\[ \int_{a_i}^{b_i} q_i(\xi, t)(\xi - \eta_i) d\xi = M_i(t) \equiv P_i(t)e_i(t), \quad x \in [a_i, b_i], \quad t \geq \tau_0, \quad i = 1, 2, \ldots, n. \]
Make a change of variables in (3) and (4) according to the formulas

\begin{align}
\delta^*(t^*) &= \frac{2\delta(t)}{a}, \\
\alpha^*(t^*) &= \frac{2\alpha(t)}{a}, \\
\tau^*_k &= \frac{\tau_k}{\tau_0}, \\
\xi^* &= \frac{2(\xi - \eta)}{a}, \\
x^* &= \frac{2(x - \eta)}{a}, \\
t^* &= \frac{t}{\tau_0}, \\
\xi^* &= \frac{2(\xi - \eta)}{a}, \\
\tau^*_k &= \frac{\tau_k}{\tau_0}, \\
\lambda &= \frac{2h_{\text{lower}}}{a}, \\
m^*(x^*) &= \frac{E_0}{a(1 - \nu^2_{\text{lower}})} \int_0^{b(x)} \frac{dz}{R(x, z)}, \\
g^*(x^*, t^*) &= \frac{2g_i(x)}{a}, \\
c^*(t^*) &= \frac{\lambda}{E_0}, \\
P^*(t^*) &= \frac{4P_i(t)(1 - \nu^2_{\text{lower}})}{E_{\text{lower}}(t - \tau_{\text{lower}})}, \\
M^*(t^*) &= \frac{8M_i(t)(1 - \nu^2_{\text{lower}})}{E_{\text{lower}}(t - \tau_{\text{lower}})} \\
F^{ij^*}(x^*) &= \int_{-1}^1 k^{ij}(x^*, \xi^*) f(\xi^*) d\xi^*, \\
V^{*}(t^*) &= \int_{-1}^1 K^*(t^*, \tau^*) f(\tau^*) d\tau^*, \\
k^{ij}(x^*, \xi^*) &= \frac{1}{\pi} k_{\text{pl}} \left( \frac{x - \xi}{\eta_{\text{lower}}} \right), \\
\eta^{ij*} &= \frac{2\eta_i}{a}, \\
K^*(t^*, \tau^*) &= K(t - \tau_{\text{lower}}, \tau - \tau_{\text{lower}}) \tau_{\text{lower}}, \\
i, j &= 1, 2, \ldots, n.
\end{align}

Here \( E_0 \) some dimensional modulus, for example, \( E_0 = \max_{x \in (-\infty, +\infty)} E(x, z) \).

Then we obtain following system of mixed integral equations and additional conditions:

\begin{align}
c^*(t^*)m^*(x^*)q^{ij^*}(x^*, t^*) + (I - V^*) \sum_{j=1}^n F^{ij^*} q^{ij^*}(x^*, t^*) &= \delta^*(t^*) + \alpha^*(t^*) x^* - g^*(x^*), \\
\int_{-1}^1 q^{ij^*}(\xi, t^*) d\xi &= P^{ij^*}(t^*), \\
\int_{-1}^1 q^{ij^*}(\xi, t^*) \xi d\xi &= M^{ij^*}(t^*), \\
i &= 1, 2, \ldots, n, \\
x^* &\in [-1, 1], \\
t^* \geq 1.
\end{align}

Figure 3. Condition types on each punch.

There are exist four different condition types on each punch (figure 3): 1) the settlement and tilt angle of the punch are given; 2) the punch settlement and the moment of the load application are given; 3) the tilt angle of the punch and the force of the load application are given; 4) the force and the moment of the load application are given. Thus, there are fifteen different versions of for the multiple plane contact problem: four versions if all conditions are equal; 2) six versions if exists two punch groups with different condition types; 3) four versions if exists three punch groups with different conditions; and 4) one version if exists four groups of punches with different condition types.

2. Dimensionless form

Make a change of variables in (3) and (4) according to the formulas

\begin{align}
x^* &= \frac{2(x - \eta_i)}{a}, \\
\xi^* &= \frac{2(\xi - \eta_i)}{a}, \\
t^* &= \frac{t}{\tau_0}, \\
\tau^*_k &= \frac{\tau_k}{\tau_0}, \\
\lambda &= \frac{2h_{\text{lower}}}{a}, \\
m^*(x^*) &= \frac{E_0}{(1 - \nu^2_{\text{lower}})} \int_0^{b(x)} \frac{dz}{R(x, z)}, \\
g^*(x^*, t^*) &= \frac{2g_i(x)}{a}, \\
c^*(t^*) &= \frac{\lambda}{E_0}, \\
P^*(t^*) &= \frac{4P_i(t)(1 - \nu^2_{\text{lower}})}{E_{\text{lower}}(t - \tau_{\text{lower}})}, \\
M^*(t^*) &= \frac{8M_i(t)(1 - \nu^2_{\text{lower}})}{E_{\text{lower}}(t - \tau_{\text{lower}})} \\
F^{ij^*}(x^*) &= \int_{-1}^1 k^{ij}(x^*, \xi^*) f(\xi^*) d\xi^*, \\
V^{*}(t^*) &= \int_{-1}^1 K^*(t^*, \tau^*) f(\tau^*) d\tau^*, \\
k^{ij}(x^*, \xi^*) &= \frac{1}{\pi} k_{\text{pl}} \left( \frac{x - \xi}{\eta_{\text{lower}}} \right), \\
\eta^{ij*} &= \frac{2\eta_i}{a}, \\
K^*(t^*, \tau^*) &= K(t - \tau_{\text{lower}}, \tau - \tau_{\text{lower}}) \tau_{\text{lower}}, \\
i, j &= 1, 2, \ldots, n.
\end{align}

Here \( E_0 \) some dimensional modulus, for example, \( E_0 = \max_{x \in (-\infty, +\infty)} E(x, z) \).

Then we obtain following system of mixed integral equations and additional conditions:

\begin{align}
c^*(t^*)m^*(x^*)q^{ij^*}(x^*, t^*) + (I - V^*) \sum_{j=1}^n F^{ij^*} q^{ij^*}(x^*, t^*) &= \delta^*(t^*) + \alpha^*(t^*) x^* - g^*(x^*), \\
\int_{-1}^1 q^{ij^*}(\xi, t^*) d\xi &= P^{ij^*}(t^*), \\
\int_{-1}^1 q^{ij^*}(\xi, t^*) \xi d\xi &= M^{ij^*}(t^*), \\
i &= 1, 2, \ldots, n, \\
x^* &\in [-1, 1], \\
t^* \geq 1.
\end{align}
Here $I$ is identity operator. These equations can be represented as
\[
c^*(t^*)m^*(x^*)q^*(x^*, t^*) + (I - V^*)F^*q^*(x^*, t^*) = \delta^*(t^*) + \alpha^*(t^*)x^* - g^*(x^*),
\]
\[
\int_{-1}^{1} q^*(\xi, t^*) \, d\xi = P^*(t^*), \quad \int_{-1}^{1} q^*(\xi, t^*)\xi \, d\xi = M^*(t^*),
\]
where $x^* \in [-1, 1], \ t^* \geq 1,$

where
\[
q^*(x^*, t^*) = \sum_{i=1}^{n} q^{is}(x^*, t^*)i^i, \quad P^*(t^*) = \sum_{i=1}^{n} M^{is}(t^*)i^i, \quad M^*(t^*) = \sum_{i=1}^{n} M^{is}(t^*)i^i,
\]
\[
\delta^*(t^*) = \sum_{i=1}^{n} \delta^{is}(t^*)i^i, \quad \alpha^*(t^*) = \sum_{i=1}^{n} \alpha^{is}(t^*)i^i, \quad k^*(x, \xi) = \sum_{i,j=1}^{n} k^{ij}\bar{s}(x, \xi)i^i j^j,
\]
\[
F^*f(x) = \int_{-1}^{1} k^*(x, \xi) \cdot f(\xi) \, d\xi,
\]
where
\[
k^*(x, \xi) \cdot f(\xi) = k^{ij}\bar{s}(x, \xi)f^j(\xi)i^i, \quad j \text{th column}
\]
\[
i^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \leftarrow \text{ith row}, \quad i^i j^j = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \leftarrow \text{ith row}.
\]

Further, we will omit asterisks and it will be the summation over repeated upper indices $i, i_0,$ and $j$ from 1 to $n$ if the left side of the formula is independent of the index.

Note that functions $m(x)$ and $g(x)$ connect with coating rigidity, coating width, and punch base forms. These parameters can be described by a rapidly changing function.

3. Analytical Solution
We should see the solution for all versions of the problem in the form
\[
q(x, t) = \tilde{q}(x, t) - \frac{g(x)}{c(t)m(x)},
\]
where $\tilde{q}(x, t)$ is new function to be determined. In this case main equation and additional conditions (7) take a form
\[
c(t)m(x)\tilde{q}(x, t) + (I - V)\tilde{q}(x, t) = \delta(t) + \alpha(t)x - \tilde{c}(t)\tilde{g}(x),
\]
\[
\int_{-1}^{1} \tilde{q}(\xi, t) \, d\xi = \tilde{P}(t), \quad \int_{-1}^{1} \tilde{q}(\xi, t)\xi \, d\xi = \tilde{M}(t), \quad x \in [-1, 1], \ t \geq 1,
\]
where
\[
\tilde{g}(x) = \int_{-1}^{1} \frac{k(x, \xi) \cdot g(\xi)}{m(\xi)} \, d\xi, \quad \tilde{c}(t) = - (I - V) \frac{1}{c(t)},
\]
\[
\tilde{P}(t) = P(t) + \frac{1}{c(t)} \int_{-1}^{1} \frac{g(\xi)}{m(\xi)} \, d\xi, \quad \tilde{M}(t) = M(t) + \frac{1}{c(t)} \int_{-1}^{1} \frac{g(\xi)\xi}{m(\xi)} \, d\xi.
\]
We obtain operator equation with different integral operators and one rapidly changing function \( m(x) \) supplemented by two vector conditions. Last term in right-hand side of operator equation (9) is “good”: its smoothness defined by kernel \( k(x, \xi) \). Obtained operator equation with additional conditions has the same form as the main equation and additional conditions in [1]. (Only the last known term in the right-hand side contain \( t \)-dependent factor.) Hence the solution method will be similar.

In this paper we will show how to construct the solution if we have two punch groups: one group \( G_1 \) with force conditions and another group \( G_2 \) with kinematic conditions (figure 4). So we have following version of problem:

\[
\begin{aligned}
P_i(t) \text{ and } M_i(t) \text{ are known, } \alpha_i(t) \text{ and } \delta_i(t) \text{ are unknown} \\
\text{if } i \in G_1, \\
\alpha_i(t) \text{ and } \delta_i(t) \text{ are known, } P_i(t) \text{ and } M_i(t) \text{ are unknown} \\
\text{if } i \in G_2.
\end{aligned}
\]

Note that \( G_1 \cup G_2 = \{1, 2, \ldots, n\} \). Solution constructing will be similar for other versions of the problem.

3.1. Solution form and special basis

By introducing notations in (9)

\[
Q(x, t) = \sqrt{m(x)} \tilde{q}(x, t), \quad K(x, \xi) = \frac{k(x, \xi)}{\sqrt{m(x)m(\xi)}}, \quad Gf(x) = \int_{-1}^{1} K(x, \xi) \cdot f(\xi) \, d\xi,
\]

we obtain new operator equation and additional conditions

\[
\begin{aligned}
c(t)Q(x, t) + (I - V)GQ(x, t) &= \frac{\delta(t) + \alpha(t)x - \tilde{c}(t)\tilde{g}(x)}{\sqrt{m(x)}}, \\
\int_{-1}^{1} \frac{Q(\xi, t)}{\sqrt{m(\xi)}} \, d\xi &= \tilde{P}(t), \quad \int_{-1}^{1} \frac{Q(\xi, t)}{\sqrt{m(\xi)}} \, d\xi = \tilde{M}(t), \quad x \in [-1, 1], \quad t \geq 1.
\end{aligned}
\]

We will find solution of operator equation with additional conditions (12) in the class of vector–functions continuous in time \( t \) in Hilbert space \( L_2([-1, 1], V) \) (see [13]). First of all we should construct special basis. To this end we will orthonormal on \([-1, 1]\) the linearly independent system of vector-functions

\[
\left\{ \frac{i^1}{\sqrt{m(x)}}, \frac{x i^1}{\sqrt{m(x)}}, \frac{x^2 i^1}{\sqrt{m(x)}}, \ldots, \frac{i^2}{\sqrt{m(x)}}, \frac{x i^2}{\sqrt{m(x)}}, \frac{x^2 i^2}{\sqrt{m(x)}}, \ldots, \frac{i^n}{\sqrt{m(x)}}, \frac{x i^n}{\sqrt{m(x)}}, \frac{x^2 i^n}{\sqrt{m(x)}}, \ldots \right\}
\]
by the formulas [14]:
\[
P^i_k(x) = P^i_k(x) = \frac{P^i_k(x)}{\sqrt{m(x)}}, \quad d_{-1} = 1, \quad J_k = \int_{-1}^{1} \frac{\xi_k d\xi}{m(\xi)}.
\]

Following [13], we introduce the operators of orthogonal projection
\[
d_k = \begin{bmatrix}
J_0 & \cdots & J_k \\
\vdots & \ddots & \vdots \\
J_k & \cdots & J_{2k}
\end{bmatrix}, \quad p^i_k(x) = \frac{1}{\sqrt{d_{-1}d_k}} \begin{bmatrix}
J_0 \\
J_1 \\
\vdots \\
J_{k-1} \\
1
\end{bmatrix}
\begin{bmatrix}
J_0 & J_1 & \cdots & J_k \\
\vdots & \ddots & \vdots & \vdots \\
J_{k-1} & J_k & \cdots & J_{2k-1} \\
1 & x & \cdots & x^k
\end{bmatrix}
\]

\[i = 1, 2, \ldots, n, \quad k = 0, 1, 2, \ldots, \quad x \in [-1, 1].\]

We will use following scalar product in \(L_2([-1, 1], V)\)
\[
(f_1(x), f_2(x)) = \int_{-1}^{1} f_1(x) \cdot f_2(x) \, dx = \int_{-1}^{1} f_1(x) f_2(x) \, dx,
\]
where \(f_1(x)\) and \(f_2(x)\) are components of vector-functions \(f_1(x)\) and \(f_2(x)\), respectively.

### 3.2. Orthoprojectors, spectral problem, and solution

Divide the space \(L_2([-1, 1], V)\) into two parts (two subspaces): \(L_2^{(1)}([-1, 1], V)\) and \(L_2^{(2)}([-1, 1], V)\) with following bases:
\[
\{p^{i_1}_{0}(x), p^{i_2}_{0}(x)\}_{i_1 \in G_1} \text{ in } L_2^{(1)}([-1, 1], V),
\{p^{i_2}_{0}(x), p^{i_2}_{1}(x), p^{i_2}_{2}(x)\}_{i_2 \in G_2; i=1,2,\ldots; k=2,3,4,\ldots} \text{ in } L_2^{(2)}([-1, 1], V).
\]

Following [13], we introduce the operators of orthogonal projection \(P_1 : L_2([-1, 1], V) \to L_2^{(1)}([-1, 1], V)\) and \(P_2 : L_2([-1, 1], V) \to L_2^{(2)}([-1, 1], V)\):
\[
P_1 f(x) = (p^{i_1}_{0}(x), f(x)) p^{i_1}_{0}(x) + (p^{i_2}_{0}(x), f(x)) p^{i_2}_{0}(x),
\]
\[
P_2 f(x) = (p^{i_2}_{0}(x), f(x)) p^{i_2}_{0}(x) + (p^{i_2}_{2}(x), f(x)) p^{i_2}_{2}(x) + \sum_{k=2}^{\infty} (p^{i_k}_{k}(x), f(x)) p^{i_k}_{k}(x) = (I - P_1) f(x).
\]

Further, it will be the summation over repeated upper indices \(i_1\) (or \(i_2\)) and \(i_2\) (or \(i_2\)) over numbers from sets \(G_1\) and \(G_2\), respectively, if the left side of the formula is independent of the indices. Hence integrand and right-hand side of operator equation (12) can be represented as algebraic sum of terms from \(L_2^{(1)}([-1, 1], V)\) and \(L_2^{(2)}([-1, 1], V)\), i.e.
\[
Q(x, t) = Q_1(x, t) + Q_2(x, t), \quad \frac{\delta(t) + \alpha(t) x - \tilde{c}(t) \tilde{g}(x)}{\sqrt{m(x)}} = \Delta_1(x, t) + \Delta_2(x, t),
\]
where \(Q_1(x, t), \Delta_1(x, t) \in L_2^{(1)}([-1, 1], V), Q_2(x, t), \Delta_2(x, t) \in L_2^{(2)}([-1, 1], V),\) and
\[
\Delta_1(x, t) = P_1 \left[ \frac{\delta(t) + \alpha(t) x - \tilde{c}(t) \tilde{g}(x)}{\sqrt{m(x)}} \right] = \Delta_1^0(t) p^{i_1}_{0}(x) + \Delta_1^1(t) p^{i_1}_{1}(x),
\]
\[
\Delta_2(x, t) = P_2 \left[ \frac{\delta(t) + \alpha(t) x - \tilde{c}(t) \tilde{g}(x)}{\sqrt{m(x)}} \right] = \Delta_2^0(t) p^{i_2}_{0}(x) + \Delta_2^1(t) p^{i_2}_{1}(x) + \sum_{m=2}^{\infty} \Delta_2^m(t) p^{i_2}_{m}(x),
\]
\[
\Delta_1^0(t) = \sqrt{J_0} \delta(t) + \frac{J_1}{J_0} \alpha^0(t) - \tilde{g}_0 \tilde{c}(t), \quad \Delta_1^1(t) = \sqrt{J_0 J_2 - J_1^2} \alpha^1(t) - \tilde{g}_1 \tilde{c}(t),
\]
\[
\Delta_2^i(t) = -\tilde{g}_i \tilde{c}(t), \quad i = 1, 2, \ldots, n, \quad m = 2, 3, 4, \ldots
\]
To calculate coefficients $\tilde{g}_m^i$, we should represent kernel $K(x, \xi)$ as

$$K(x, \xi) = \sum_{m, l=0}^{\infty} K_{ml}^{ij} P_m(x) P_l^i(\xi),$$

and use relations (10), (11), and (13). Then:

$$\tilde{g}_m^i = \left( \frac{\tilde{g}(x)}{\sqrt{m(x)}}, P_m(x) \right) = \sum_{l=0}^{\infty} K_{ml}^{ij} \int_{-1}^{1} \frac{p_l^j(x) p_l^i(x) dx}{m(x)}, \quad m = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, n. \quad (18)$$

Function $Q_1(x, t)$ can be represented as

$$Q_1(x, t) = z_0^1(t) P_0^1(x) + z_1^1(t) P_1^1(x), \quad (19)$$

where $z_0^1(t)$ and $z_1^1(t)$ are unknown functional coefficients. They can be determined from the additional conditions (12). To this end substitute the representation (15) for $Q(x, t)$ into left-hand side of the additional conditions (12) in components with known forces and moments and use (13) and (19):

$$\int_{-1}^{1} \frac{Q_1^1(\xi, t)}{\sqrt{m(\xi)}} d\xi = \int_{-1}^{1} \frac{Q_1^1(\xi, t) + Q_2^1(\xi, t) - \frac{1}{\sqrt{m(\xi)}}}{m(\xi)} d\xi = \int_{-1}^{1} [z_0^1(t) p_0(\xi) + z_1^1(t) p_1(\xi) + Q_1^1(\xi, t)] \sqrt{J_0} p_0(\xi) d\xi = z_0^1(t) \sqrt{J_0},$$

$$\int_{-1}^{1} \frac{Q_1^1(\xi, t)}{\sqrt{m(\xi)}} \xi d\xi = \int_{-1}^{1} \frac{Q_1^1(\xi, t) + Q_2^1(\xi, t) \xi}{\sqrt{m(\xi)}} d\xi,$$

$$= \int_{-1}^{1} [z_0^1(t) p_0(\xi) + z_1^1(t) p_1(\xi) + Q_1^1(\xi, t)] \left[ \frac{J_1}{\sqrt{J_0}} p_0(\xi) + \sqrt{\frac{J_0 J_2 - J_1^2}{J_0}} p_1(\xi) \right] d\xi = z_0^1(t) \frac{J_1}{\sqrt{J_0}} + z_1^1(t) \sqrt{\frac{J_0 J_2 - J_1^2}{J_0}}, \quad i_1 \in G_1.$$

Equating derived expressions to the right-hand sides of additional conditions we obtain the expressions for $z_0^1(t)$ and $z_1^1(t)$:

$$z_0^1(t) = \frac{\tilde{p}_0^{i_1}(t)}{\sqrt{J_0}}, \quad z_1^1(t) = \frac{J_0 \tilde{p}_1^{i_1}(t) + J_1 \tilde{m}_0^{i_1}(t)}{\sqrt{J_0(J_0 J_2 - J_1^2)}}. \quad (20)$$

The formula (12) for $Q(x, t)$ contains known term $Q_1(x, t) \in L^2([-1, 1], V)$ and the term $Q_2(x, t) \in L^2([-1, 1], V)$ must be found. For the right-hand side, one should find $\Delta_1(x, t) \in L^2([-1, 1], V)$, while $\Delta_2(x, t) \in L^2([-1, 1], V)$ is known. These peculiarities permit one to class the resulting problem as a specific case of the generalized projection problem stated in [13, 15].

We apply the orthogonal projection operator $P_2$ to operator equation (12). As a result, we obtain the equation for determining $Q_2(x, t)$ with a known right-hand side

$$c(t) Q_2(x, t) + (I - V) P_2 G Q_2(x, t) = -(I - V) P_2 G Q_1(x, t) + \Delta_2(x, t). \quad (21)$$
It is necessary to construct its solution in the form of a series in the eigenfunctions of the operator \( \mathbf{P}_2 \mathbf{G} \), which, as one can show, is a compact strongly positive self-adjoint operator \( L_2^{(2)}([-1, 1], V) \rightarrow L_2^{(2)}([-1, 1], V) \). The system of eigenfunctions of such an operator is a basis in the space \( L_2^{(2)}([-1, 1], V) \). The spectral problem for the operator \( \mathbf{P}_2 \mathbf{G} \) can be written in the form

\[
\mathbf{P}_2 \mathbf{G} \varphi_k(x) = \gamma_k \varphi_k(x),
\]

where \( \varphi_k(x) \) is linearly combination of basis functions (13) in \( L_2([-1, 1], V) \), i.e.

\[
\varphi_k(x) = \psi_{i_k}^{j_k} P_{i_k}^{j_k}(x) + \psi_{k_1}^{j_1} P_{k_1}^{j_1}(x) + \sum_{m=2}^{\infty} \psi_{k_m}^{j_m} P_{k_m}^{j_m}(x), \quad k = 2, 3, 4, \ldots \quad (22)
\]

Using representations (11) and (14) for operators \( \mathbf{G} \) and \( \mathbf{P}_2 \) and (17) for kernel decomposition we can obtain spectral problem about coefficients \( \gamma_k \) and \( \psi_{km}^j \):

\[
\mathbf{P}_2 \left\{ \int_{-1}^{1} \sum_{m,l=0}^{\infty} K_{ml}^{ij} P_{m}^{i}(x) P_{l}^{j}(\xi) \left[ \psi_{i_k}^{j_k} P_{i_k}^{j_k}(\xi) + \psi_{k_1}^{j_1} P_{k_1}^{j_1}(\xi) + \sum_{m_2=2}^{\infty} \psi_{k_m}^{j_m} P_{m_2}^{j_m}(\xi) \right] d\xi \right\} = \mathbf{P}_2 \left\{ K_{m0}^{ij} \psi_{i_k}^{j_k} P_{m}^{i}(x) + K_{m1}^{ij} \psi_{k_1}^{j_1} P_{m}^{i}(x) + \sum_{m=0}^{\infty} \left( \sum_{l=2}^{\infty} K_{ml}^{ij} \psi_{kl}^{j} \right) P_{m}^{i}(x) \right\}
\]

\[
= \mathbf{P}_2 \left\{ \gamma_k \left[ \psi_{i_k}^{j_k} P_{i_k}^{j_k}(x) + \psi_{k_1}^{j_1} P_{k_1}^{j_1}(x) + \sum_{m=2}^{\infty} \psi_{k_m}^{j_m} P_{m}^{i}(x) \right] \right\},
\]

\[
K_{00}^{ij} \psi_{i_k}^{j_k} + K_{01}^{ij} \psi_{k_1}^{j_1} + \sum_{l=2}^{\infty} K_{l0}^{ij} \psi_{kl}^{j} = \gamma_k \psi_{i_k}^{j_k}, \quad i = 1, 2, \ldots, n, \quad i_2 \in G_2, \quad k, m = 2, 3, 4, \ldots
\]

We expand the functions \( \mathbf{Q}_2(x,t), \mathbf{P}_2 \mathbf{GQ}_1(x,t), \) and \( \Delta_2(x,t) \) with respect to the new basis functions \( \varphi_k(x) \) \((k = 2, 3, 4, \ldots)\) in \( L_2^{(2)}([-1, 1], V) \), i.e.,

\[
\mathbf{Q}_2(x,t) = \sum_{k=2}^{\infty} \gamma_k(t) \varphi_k(x), \quad \mathbf{P}_2 \mathbf{GQ}_1(x,t) = \sum_{k=2}^{\infty} \sigma_k(t) \varphi_k(x), \quad \Delta_2(x,t) = \sum_{k=2}^{\infty} \Delta_k(t) \varphi_k(x), \quad (23)
\]

where coefficients \( \sigma_k(t) \) and \( \Delta_k(t) \) defined by (we use formulas (11), (14), (16), (17), (19),
and (22))

\[ \sigma_k(t) = (P_2GQ_1(x, t), \varphi_k(x)) = \left( P_2 \left\{ \int_{-1}^{1} \left\{ \sum_{m=0}^{\infty} K_m^{i_0} p_m^i(x) [\xi \cdot (z_m^{i_0}(t) p_m^{i_0}(\xi) + z_m^i(t) p_m^i(\xi))] \right\} d\xi \right\} \right), \]

\[ \psi_k^{i_0} p_0^i(x) + \psi_k^{i_3} p_1^i(x) + \sum_{m=2}^{\infty} \psi_{km}^i p_m^i(x) = \left( P_2 \left\{ \sum_{m=0}^{\infty} \left[ K_m^{i_0} z_m^{i_0}(t) p_m^i(x) + K_m^{i_1} z_m^{i_1}(t) p_m^i(x) \right] \right\} \right), \]

\[ \psi_k^{i_0} p_0^i(x) + \psi_k^{i_3} p_1^i(x) + \sum_{m=2}^{\infty} \psi_{km}^i p_m^i(x) = \left( K_m^{i_0} p_0^i(x) + K_m^{i_1} p_1^i(x) + \sum_{m=2}^{\infty} K_m^{i_2} p_m^i(x) \right) z_m^i(t) + \left( K_m^{i_0} p_0^i(x) + K_m^{i_1} p_1^i(x) + \sum_{m=2}^{\infty} K_m^{i_2} p_m^i(x) \right) z_m^i(t), \]

\[ \Delta_k(t) = (\Delta_2(x, t), \varphi_k(x)) = \left( \Delta_k^{i_0}(t) p_0^i(x) + \Delta_k^{i_1}(t) p_1^i(x) + \sum_{m=2}^{\infty} \Delta_k^{i_2}(t) p_m^i(x), \right) \]

\[ \psi_k^{i_0} p_0^i(x) + \psi_k^{i_3} p_1^i(x) + \sum_{m=2}^{\infty} \psi_{km}^i p_m^i(x) = \Delta_k^{i_0}(t) p_0^i(x) + \Delta_k^{i_1}(t) p_1^i(x) + \sum_{m=2}^{\infty} \Delta_k^{i_2}(t) p_m^i(x). \]

Considering the above, the equation (21) is represented as

\[ \sum_{k=2}^{\infty} \left[ c(t) z_k(t) + (I - V) \gamma_k z_k(t) \right] \varphi_k(x) = \sum_{k=2}^{\infty} \left[ -(I - V) \sigma_k(t) + \Delta_k(t) \right] \varphi_k(x). \]

We can find functions \( z_k(t) \) from this equation:

\[ z_k(t) = (I + W_k) \frac{-(I - V) \sigma_k(t) + \Delta_k(t)}{c(t) + \gamma_k}, \quad W_k f(t) = \int_1^t R_k(t, \tau) f(\tau) d\tau, \quad (24) \]

where \( R_k(t, \tau) \) is is the resolvent of the kernel

\[ K_k^*(t, \tau) = \frac{\gamma_k K(t, \tau)}{c(t) + \gamma_k}. \]

Note that the final expression (according to formulas (8), (11), (15), (19), and (23)) for contact pressure vector-function has the following structure

\[ q(x, t) = \frac{1}{m(x)} \left[ z_m^i(t) p_m^{i_0}(x) + z_m^i(t) p_m^{i_1}(x) + \sum_{k=2}^{\infty} z_k(t) \varphi_k^i(x) - \frac{g(x)}{c(t) m(x)} \right], \quad (25) \]

where \( \varphi_k^i(x) = \sqrt{m(x)} \varphi_k^i(x) \) are polynomials. It means that there is a “good” function in square brackets and one can explicitly write out the weight functions \( m(x) \) and \( g(x) \) in the solution. Note that the coating thickness and its nonuniformity are related to \( m(x) \) and backlash function is related to \( g(x) \) (see (5)). This formula permit obtaining efficient analytic solutions for the cases when coating nonuniformities and forms of contact surfaces described by complicated and rapidly changing functions. Such a result can hardly be done by other known methods.
Knowing the stresses we can find forces and moments under the punches from group $G_2$ from additional conditions (7) and formulas (13), (22), and (25):

$$P^{i2}(t) = \int_{-1}^{1} q^{i2}(\xi, t) \, d\xi = \sum_{k=2}^{\infty} \sqrt{J_0} \psi_{k0}^{i2} \int_{-1}^{1} g(\xi) \, d\xi,$$
$$M^{i2}(t) = \int_{-1}^{1} q^{i2}(\xi, t) \xi \, d\xi = \sum_{k=2}^{\infty} \left[ \frac{J_1}{\sqrt{J_0}} \psi_{k0}^{i2} + \sqrt{\frac{J_0 J_2 - J_1^2}{J_0}} \psi_{k1}^{i2} \right] \psi_{k0}^{i2} \int_{-1}^{1} \xi g(\xi) \, d\xi.$$

Knowing the stresses we can find settlements and tilt angles under the punches from group $G_1$ from equation (12). To this end we apply the orthogonal projection operator $P_1$ to operator equation (12). As a result, we obtain the equation for determining $\delta^{i1}(t)$ and $\alpha^{i1}(t)$:

$$c(t)Q_1(x, t) + (I - V)P_1 GQ(x, t) = \Delta_1(t).$$

Using representations (11), (14)–(17), (19), (22), (23) is converted to the following form:

$$c(t)[z^{i1}_0(t)p_0^{i1}(x) + z^{i1}_1(t)p_1^{i1}(x)] + (I - V)P_1 \int_{-1}^{1} \sum_{m, l=0}^{\infty} K_{m1}^{i1} p_m^{ij}(x) \cdot \left\{ z_0^{i1}(t) p_0^{i1}(x) + z_1^{i1}(t) p_1^{i1}(x) \right\} \, d\xi$$
$$+ \sum_{k=2}^{\infty} \left[ \psi_{k0}^{i2} p_0^{i2}(\xi) + \psi_{k1}^{i2} p_1^{i2}(\xi) + \sum_{m_2=2}^{\infty} \psi_{km_2}^{i2} p_0^{i2}(\xi) \right] \, d\xi$$
$$= \Delta_0^{i1}(t) p_0^{i1}(x) + \Delta_1^{i1}(t) p_1^{i1}(x).$$

Hence

$$c(t)[z^{i1}_0(t)p_0^{i1}(x) + z^{i1}_1(t)p_1^{i1}(x)] + (I - V)P_1 \sum_{m=0}^{\infty} \left\{ K_{m0}^{i1} z_0^{i1}(t) + K_{m1}^{i1} z_1^{i1}(t) \right\}$$
$$+ \sum_{k=2}^{\infty} \left[ K_{m0}^{i2} \psi_{k0}^{i2} + K_{m1}^{i2} \psi_{k1}^{i2} + \sum_{l=2}^{\infty} K_{ml}^{i2} \psi_{kl}^{i2} \right] \, d\xi$$
$$= \left\{ \sqrt{J_0} \delta^{i1}(t) + \frac{J_1}{\sqrt{J_0}} \alpha^{i1}(t) - g_0^{i1} c(t) \right\} p_0^{i1}(x) + \left\{ \sqrt{\frac{J_0 J_2 - J_1^2}{J_0}} \alpha^{i1}(t) - g_1^{i1} c(t) \right\} p_1^{i1}(x).$$

Final formulas for settlements and tilt angles has a form

$$\alpha^{i1}(t) = \sqrt{\frac{J_0}{J_0 J_2 - J_1^2}} \left\{ g_0^{i1} c(t) + c(t) z_0^{i1}(t) \right\}$$
$$+ (I - V) \left\{ K_{00}^{i1} z_0^{i1}(t) + K_{01}^{i1} z_1^{i1}(t) + \sum_{k=2}^{\infty} \left[ K_{k0}^{i1} \psi_{k0}^{i2} + K_{k1}^{i1} \psi_{k1}^{i2} + \sum_{l=2}^{\infty} K_{kl}^{i1} \psi_{kl}^{i2} \right] \right\},$$
$$\delta^{i1}(t) = - \frac{J_1}{J_0} \alpha^{i1}(t) + \frac{1}{\sqrt{J_0}} \left\{ g_0^{i1} c(t) + c(t) z_0^{i1}(t) \right\}$$
$$+ (I - V) \left\{ K_{00}^{i1} z_0^{i1}(t) + K_{11}^{i1} z_1^{i1}(t) + \sum_{k=2}^{\infty} \left[ K_{k0}^{i1} \psi_{k0}^{i2} + K_{k11}^{i1} \psi_{k1}^{i2} + \sum_{l=2}^{\infty} K_{kl}^{i1} \psi_{kl}^{i2} \right] \right\}. $$
Main Results and Conclusions

- Plane problem of multiple contact interaction for base with coating and regular system of punches is posed. The corresponding mathematical model is given and analyzed. Possible variants of the problem statement are formulated.
- The analytical solution for one version of this problem is obtained using Manzhirov generalized projection method. In relation of contact stresses contact stiffness, coating thickness, and backlash function are represented by separate terms and factors. It is allows one to perform effective computations for actual nonuniformities and shapes.
- Analytical representation of the solution allow one to analyze carefully the behavior of the punches on the layer, taking into account the complex properties of the contacting surfaces and the mutual influence of the punches.

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