A boundary problem with integral gluing condition for parabolic-hyperbolic equation involving the Caputo fractional derivative

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Abstract. In the present work we investigate the Tricomi problem with integral gluing condition for parabolic-hyperbolic equation with the Caputo fractional order derivative. Using the method of energy integrals we prove the uniqueness of the solution for considered problem. The existence will be proved using methods of ordinary differential equations, Fredholm integral equations and solution will be represented in an explicit form.

1 Introduction and formulation of a problem

It is well-known that investigations of fractional analogs of main ODE and PDEs appear as a result of the finding mathematic models for real-life processes [1-3]. They are also interesting for mathematicians as natural generalization of integer order ODE and PDEs. The theory of boundary problems for PDEs is also developing on this direction. There are many works [4-8] devoted to the investigation of various boundary problems for PDEs.

Distinctive side of this work is the usage of gluing condition of the integral form. We note, for the first time integral gluing condition was used in the work [9]. Then some generalizations of this work were published in works [10-11]. Special gluing condition of the integral form for parabolic-hyperbolic equation with the Riemann-Liouville fractional differential operator was used in the work [12].

In the present work we use integral gluing condition with more general kernel then in the work [13]. When we prove the uniqueness of the solution we must put some restrictions to the kernel (see theorem 1), but for the existence of solution we don’t need these conditions (see theorem 2).

Consider an equation

\[ 0 = \begin{cases} u_{xx} - cD_0^\alpha u, & y > 0, \\ u_{xx} - u_{yy}, & y < 0 \end{cases} \]

in the domain \( \Omega = \Omega^+ \cup \Omega^- \cup AB \), where \( 0 < \alpha < 1 \), \( AB = \{(x, y) : 0 < x < 1, y = 0\}, \Omega^+ = \{(x, y) : 0 < x < 1, 0 < y < 1\}, \Omega^- = \{(x, y) : -y < x < y + 1, -1/2 < y < 0\}, \]

\[ cD_0^\alpha f = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} f'(t) \, dt \]

is Caputo fractional differential operator of the order \( \alpha \) (0 < \( \alpha \) < 1), \( \Gamma(\cdot) \) is Euler’s gamma-funtion [14].
Problem. Find a solution of the equation (1) from the following class of functions:

\[ W = \{ u (x, y) : u \in C \left( \overline{\Omega} \right) \cap C^2 \left( \Omega^- \right) , u_{xx} \in C \left( \Omega^+ \right) , c D_0^\alpha u \in C \left( \Omega^+ \right) \} \]

satisfying boundary conditions

\[
\begin{align*}
    u (0, y) &= \varphi_1 (y) , \quad 0 \leq y \leq 1 , \\
    u (1, y) &= \varphi_2 (y) , \quad 0 \leq y \leq 1 , \\
    u (x, -x) &= \psi (x) , \quad 0 \leq x \leq 1/2
\end{align*}
\]

and gluing condition

\[
\lim_{y \to +0} y^{1-\alpha} u_y (x, y) = \gamma_1 u_y (x, 0) + \gamma_2 \int_0^x u_y (t, -0) Q (x, t) \, dt , \quad 0 < x < 1 .
\]

Here \( \varphi_1, \varphi_2, \psi, Q (\cdot, \cdot) \) are given functions, such that \( \varphi_1 (0) = \psi (0), \gamma_1, \gamma_2 = \text{const}, \gamma_1^2 + \gamma_2^2 \neq 0. \)

2 The uniqueness of a solution

Introduce designations:

\[
\begin{align*}
    u (x, +0) &= \tau_1 (x) , \quad 0 \leq x \leq 1, \quad u (x, -0) = \tau_2 (x) , \quad 0 \leq x \leq 1 , \\
    u_y (x, -0) &= \nu_2 (x) , \quad 0 < x < 1 , \quad \lim_{y \to +0} y^{1-\alpha} u_y (x, y) = \nu_1 (x) , \quad 0 < x < 1, \\
    u_x (x, +0) &= \tau'_1 (x) , \quad 0 < x < 1, \quad u_x (x, -0) = \tau'_2 (x) , \quad 0 < x < 1.
\end{align*}
\]

Known that solution of the Cauchy problem for Eq.(1) in \( \Omega^- \) can be represented as follows:

\[
    u (x, y) = \frac{1}{2} \left[ \tau_1 (x + y) + \tau_2 (x - y) - \int_{x-y}^{x+y} \nu_2 (t) \, dt \right] . \tag{6}
\]

After using a condition (4) from (6) we find

\[
    \tau'_2 (x) - 2 \psi' (x/2) = \nu_2 (x) , \quad 0 < x < 1. \tag{7}
\]

From the Eq.(1) at \( y \to +0 \) we get \[14\]

\[
    \tau''_1 (x) - \Gamma (\alpha) \nu_1 (x) = 0 . \tag{8}
\]

Below we prove the uniqueness of the solution of the formulated problem. For this aim we use usual way, i.e. first we suppose that the problem has two solutions, then denoting difference of these solutions as \( u \) we will get appropriate homogeneous problem. If we prove that this homogeneous problem has only trivial solution, then we can state that the original problem has unique solution.
Equation (8) we multiply to a function \( \tau_1(x) \) and integrate from 0 to 1:

\[
\int_0^1 \tau''_1(x) \tau_1(x) \, dx - \Gamma(\alpha) \int_0^1 \tau_1(x) \nu_1(x) \, dx = 0. \tag{9}
\]

Investigate the integral \( I = \int_0^1 \tau_1(x) \nu_1(x) \, dx \). Considering designations and gluing condition (5) we have

\[
\nu_1(x) = \gamma_1 \nu_2(x) + \gamma_2 \int_0^x \nu_2(t) Q(x,t) \, dt, \quad 0 < x < 1. \tag{10}
\]

Taking (7) into account at \( \psi(x) = 0 \) from (10) we obtain

\[
\nu_1(x) = \gamma_1 \tau'_2(x) + \gamma_2 \int_0^x \tau'_2(t) Q(x,t) \, dt, \quad 0 < x < 1. \tag{11}
\]

(12) we substitute into the integral \( I \) and considering \( \tau_1(0) = 0, \tau_1(1) = 0 \) (which deduced from the conditions (2), (3) in homogeneous case), have

\[
I = \int_0^1 \tau_1(x) \nu_1(x) \, dx = \gamma_2 \int_0^1 \tau'^2_1(x) Q(x,x) \, dx - \gamma_2 \int_0^1 \tau_1(x) \, dx \int_0^x \tau_1(t) \frac{\partial}{\partial t}Q(x,t) \, dt. \tag{12}
\]

Let

\[
\frac{\partial}{\partial t}Q(x,t) = -Q_1(x)Q_1(t).
\]

Then

\[
I = \gamma_2 \int_0^1 \tau'^2_1(x) Q(x,x) \, dx + \frac{\gamma_2 \Phi^2(1)}{2}, \tag{13}
\]

where

\[
\Phi(x) = \int_0^x \tau_1(t) Q_1(t) \, dt, \quad Q(x,x) = Q(x,0) - \int_0^t Q_1(x)Q_1(z) \, dz.
\]

From (9) and (13) we get

\[
\int_0^1 \tau'^2_1(x) \, dx + \Gamma(\alpha) \gamma_2 \left[ \int_0^1 \tau'^2_1(x) Q(x,x) \, dx + \frac{\Phi^2(1)}{2} \right] = 0. \tag{14}
\]

Since \( \Gamma(\alpha) > 0 \) for \( 0 < \alpha < 1 \), then if \( \gamma_2 \geq 0, \ Q(x,x) > 0 \) from (14) we easily get \( \tau_1(x) = 0 \) for \( \forall x \in [0,1] \).
Based on the solution of the first boundary problem for the Eq. (1) in the domain \( \Omega^+ \) we obtain \( u(x, y) \equiv 0 \) in \( \overline{\Omega^+} \). Since \( u(x, y) \in C(\overline{\Omega}) \), we get that \( u(x, y) \equiv 0 \) in \( \overline{\Omega} \).

Hence, we proved the following

**Theorem 1.** Let \( \gamma_2 \geq 0 \), \( \frac{\partial}{\partial t} Q(x, t) = -Q_1(x)Q_1(t) \) and \( Q(x, x) > 0 \). Then if there exists a solution, then it is unique.

As an example of the existence of the function \( Q(x, t) \) we note a function

\[ Q(x, t) = e^{-x}(1 + e^{-t}). \]

### 3 The existence of a solution

From (7), (8) and (10) we have

\[ \tau''_1(x) - A\tau_1(x) = F_1(x), \]  

(15)

where \( A = \Gamma(\alpha)\gamma_1 \),

\[ F_1(x) = \gamma_2\Gamma(\alpha) \left[ \int_0^\infty \tau'_1(t)Q(x, t)dt - \Gamma(\alpha) \left[ \gamma_1\psi\left(\frac{x}{2}\right) + \gamma_2\int_0^\infty \psi'(\frac{t}{2})Q(x, t)dt \right] \right]. \]  

(16)

Solution of the equation (15) together with conditions

\[ \tau_1(0) = \psi(0), \quad \tau_1(1) = \varphi_2(0) \]  

(17)

has a form

\[ \tau_1(x) = \frac{1}{1 - e^A} \left[ \varphi_2(0) \left(1 - e^{Ax}\right) + \psi(0) \left(e^{Ax} - e^A\right) \right] + \int_0^1 G_0(x, \xi)F_1(\xi)d\xi, \]  

(18)

where

\[ G_0(x, \xi) = \frac{1}{A[e^{Ax} - e^{A(x-1)}]} \begin{cases} 
(1 - e^{A\xi})(1 - e^{A(x-1)}), & 0 \leq \xi \leq x, \\
(1 - e^{A(\xi-1)})(1 - e^{Ax}), & x \leq \xi \leq 1 
\end{cases} \]  

(19)

is the Green’s function of the problem (15), (17). Considering (16) and using formulae of integration by parts, after some evaluations we deduce

\[ \tau_1(x) - \int_0^1 \tau_1(\xi)K(x, \xi)d\xi = F_2(x). \]  

(20)

Here

\[ K(x, \xi) = \gamma_2\Gamma(\alpha) \left[ G_0(x, \xi)Q(\xi, \xi) + \int_\xi^1 G_0(\xi, t)\frac{\partial}{\partial \xi}Q(t, \xi)dt \right], \]  

(21)

\[ F_2(x) = \frac{1}{1 - e^A} \left[ \varphi_2(0) \left(1 - e^{Ax}\right) + \psi(0) \left(e^{Ax} - e^A\right) \right] - \]
\[ -\Gamma(\alpha) \int_0^1 G_0(x, \xi) \left[ \gamma_1 \psi \left( \frac{\xi}{2} \right) + \gamma_2 \int_0^\xi \psi' \left( \frac{t}{2} \right) Q(\xi, t) dt \right] d\xi. \] (22)

Since kernel \( K(x, \xi) \) is continuous and function in right-hand side \( F_2(x) \) is continuously differentiable, solution of integral equation (20) we can write via resolvent-kernel:

\[ \tau_1(x) = F_2(x) - \int_0^1 F_2(\xi) R(x, \xi) d\xi, \] (23)

where \( R(x, \xi) \) is the resolvent-kernel of \( K(x, \xi) \).

Unknown functions \( \nu_1(x) \) and \( \nu_2(x) \) will be found by the following formulas:

\[ \nu_1(x) = \frac{1}{\Gamma(\alpha)} \left[ F''_2(x) - \int_0^1 F(\xi) \frac{\partial^2}{\partial x^2} R(x, \xi) d\xi \right], \]

\[ \nu_2(x) = F'_2(x) - \int_0^1 F(\xi) \frac{\partial}{\partial x} R(x, \xi) d\xi - \psi' \left( \frac{x}{2} \right). \]

Solution of the problem in the domain \( \Omega^+ \) we write as follows

\[ u(x, y) = \int_0^y G(x, y, 0, \eta) \varphi_1(\eta) d\eta - \int_0^y G(x, y, 1, \eta) \varphi_2(\eta) d\eta + \int_0^1 \tilde{G}(x - \xi, y) \tau_1(\xi) d\xi. \] (24)

Here

\[ \tilde{G}(x - \xi, y) = \frac{1}{\Gamma(1 - \alpha)} \int_0^y \eta^{-\alpha} G(x, y, \xi, \eta) d\eta, \]

\[ G(x, y, \xi, \eta) = \frac{(y - \eta)^{\beta - 1}}{2} \sum_{n=-\infty}^{\infty} c_{1, \beta}^{1, \beta} \left( -\frac{|x - \xi + 2n|}{(y - \eta)^{\beta}} \right) - c_{1, \beta}^{1, \beta} \left( -\frac{|x + \xi + 2n|}{(y - \eta)^{\beta}} \right) \]

is the Green’s function of the first boundary problem for Eq.(1) in the domain \( \Omega^+ \) with the Riemann-Liouville fractional differential operator instead of the Caputo ones [15], \( \beta = \alpha/2 \),

\[ c_{1, \beta}^{1, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\beta - \beta n)} \]

is the Wright type function [16].

Solution of the problem in the domain \( \Omega^- \) will be found by the formulae (6). Hence, we proved the following

**Theorem 2.** If \( \varphi_i(y), \psi(x) \in C[0, 1] \cap C^1(0, 1) \), \( Q(x,t) \in C([0, 1] \times [0, 1]) \), then there exists a solution of the problem and it can be represented in the domain \( \Omega^+ \) by formulae (24) and in the domain \( \Omega^- \) by the formulae (6).

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