Minimal Model for Noise-Induced Intermittent Dynamics in a Many-body System

Guram Gogia and Justin C. Burton
Department of Physics, Emory University, Atlanta, GA, 30322
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In various ecological, biochemical, and physical systems, intermittent oscillations between two or more distinct states commonly emerge due to a coupling between external random fluctuations and internal spatial structure. Despite this ubiquity, a universal mechanism for noise-induced switching dynamics does not yet exist, especially in spatially-extended systems with structural heterogeneities. Using experiments combined with numerical simulations, here we show how these dynamics arise in an electrostatically-levitated layer of charged microspheres. Upon noisy forcing in the vertical direction perpendicular to the layer, the system experiences intermittent switching between crystalline and gas-like phases. Using normal mode analysis, we show how particle polydispersity leads to localized vibrational modes that serve to nucleate an energy cascade and an eventual transition to the gas-like phase. Finally, we provide a minimal model that quantitatively captures the intermittent dynamics. The vertical and horizontal mechanical energies of the system represent two competing features contributing to an emergent phenomenon in seemingly different systems. This can be done by employing a minimal model explanation [13] [15]. Such a model would undoubtedly accelerate the imminent paradigmatic shift in the understanding of “noise as a creator” [16] and allow greater insight into noise-induced intermittency in complex systems.

Here we show how a simple, many-body system manifests intermittent switching between two distinct phases due to nonlinear coupling between external noise and structural heterogeneities. A spatially-extended, horizontal layer of charged particles, forced by white noise in the vertical direction, continuously switches between crystalline and gas-like states. A small amount of quenched disorder arising from particle polydispersity is necessary for the switching to occur. Despite the nonequilibrium nature of the phenomenon, an analysis of the harmonic vibrations in the system shows that the dynamics are governed by spatially-localized modes. These modes exist at high frequencies and are predominantly polarized in the vertical direction. Upon excitation, they nonlinearly couple to the horizontal degrees of freedom. Furthermore, we introduce a minimal model to describe the evolution of the total mechanical energy in the vertical and horizontal directions. The energies are nonlinearly coupled, experience damping, and the vertical energy is driven by noise. The model quantitatively captures the quasi-cycle oscillations of horizontal kinetic energy, and the ensuing dynamics closely resemble the ecological dynamics of a predator-prey system at the brink of extinction.
Dusty Plasma Experiments

Our previous study introduced an experimental discovery of emergent bistability in a quasi-two-dimensional dusty plasma crystal consisting of colloidal particles immersed in a weakly ionized gas \[18\]. Such dusty plasmas are commonly used as model systems to study many-body dynamics such as phase transitions \[19\] and phase separations \[20\]. In the experiment, electrostatically-levitated microspheres (Fig. 1A) experience cyclic switching between crystalline (Fig. 1B) and gas-like states (Fig. 1C). Usually the gaseous state is preceded by a transient, liquid-like melted state. The system is inherently nonequilibrium; energy is sourced from the fluctuating plasma environment, leading to large-amplitude oscillations of each particle in the vertical \(z\)-direction \[21\ [23\]. Accompanying numerical simulations (described below) reproduced the experimental results. Fig. 1D-E shows the average horizontal kinetic energy per particle in the experiment and the numerical simulation, respectively. Both indicate that bistable switching could emerge only if the particles in the crystalline layer have some finite polydispersity in their size. Beyond the qualitative observations, we did not quantify the mechanism of coupling between the external forcing and the spatial organization of the system.

Numerical Simulations

We used a custom molecular dynamics code to model the many-body dynamics. For the sake of simplicity and generality, we ignored any plasma-specific properties, such as ion drag force and thermophoresis. The spherical particles interact via a pairwise Yukawa potential of the form:

\[
U(\mathbf{r}_{ij}) = \frac{q_i q_j e^{-r_{ij}/\lambda_D}}{4\pi \varepsilon_0 r_{ij}},
\]

where \(q = 4\pi \varepsilon_0 a\) is the charge on each particle of radius \(a\), \(r_{ij}\) is the distance between the centers of particle \(i\) and \(j\), and \(\lambda_D = 1\) mm is the Debye screening length. Gravity acts in the vertical direction, \(\mathbf{F}_g = -mg\hat{z}\), where \(m\) is the particle mass, and the particles are confined vertically and horizontally with harmonic, electrostatic potentials: \(U_v = \chi_v z^2/2\) and \(U_h = \chi_h (x^2 + y^2)/2\), where \(\chi_v = 4.13 \times 10^9\) V/m^2 and \(\chi_h = 2200\) V/m^2. These values were chosen according to the experimentally measured oscillation frequencies of particles (i.e. \(\approx 20\) Hz for vertical oscillations and \(\approx 0.5\) Hz for horizontal oscillations). The large discrepancy between horizontal and vertical confinement leads to a natural separation of the corresponding vibrational frequency bands in the quasi-two-dimensional layer of charged particles.

In addition, we introduce two non-conservative forces: hydrodynamic drag due to momentum transfer to the neutral gas atoms, \(\mathbf{F}_d = -\gamma m\mathbf{v}\), where \(\gamma = 0.2\) s\(^{-1}\) is the damping coefficient, and a spatially-uniform Langevin force in the \(z\)-direction to simulate the vertical oscillations, \(\mathbf{F}_s = \eta(\beta, t) m\sqrt{\Delta V_0/\Delta z}\), where \(\eta(\beta, t)\) is a Wiener process with zero mean and standard deviation...
Energy Cascades and Early Warning Signals

Prior to the transition to the gas-like state, most of mechanical energy in the condensed phase exists in the high frequency, vertical oscillations (Δ_{xy} ≪ 1). The transition occurs through a nucleation event and subsequent energy cascade that redistributes the mechanical energy into the low frequency, horizontal oscillations. The nucleation event comes in the form of a nonlinear scattering of two particles when their separation is too small. Fig. 2A and the accompanying inset illustrate scattering in a one-dimensional chain of 6 particles. The particles behave as underdamped harmonic oscillators driven by noise. As such, they each respond sharply at their fundamental frequency, which varies with particle size. Neighboring particles can eventually oscillate out-of-phase, increasing the chances of a scattering event (Fig. 2A, II-III). If all the particles are driven sinusoidally, they oscillate at the same driving frequency and nearly the same phase, suppressing the scattering of neighboring particles. Additionally, if the particles are individually driven with uncorrelated noise, the system transitions to the gas-like state quickly since energy can no longer be pumped into the center-of-mass motion of the entire system.

For our system, the scattering event represents a tipping point leading to a “regime shift” from the crystalline to gaseous state. In dynamical systems, tipping points are usually foreshadowed by a critical slow-down, as measured by increase of variance and/or auto-correlation of state variables [24, 25]. Recent studies showed that for spatially extended systems, the early warning signals for a critical transition can be extracted from the emerging spatial, rather than temporal, patterns [26, 27]. For our system, one spatial parameter that characterizes the approach to the critical transition is the minimum horizontal distance, δ_{min}, between any two particles in the layer. Two particles may collide and scatter due to mutual electrostatic repulsion when their separation is small.

As mentioned previously, for the switching to occur, the system should have some finite amount of quenched disorder. One possibility is to make the sizes polydisperse by choosing the radii (a) from a Gaussian distribution with small coefficient of variation (c_v). Fig. 1E shows the evolution of horizontal kinetic energy over 700 s for a system comprised of 500 particles with average radius (a) = 5 μm and c_v = 1.25%, being forced with β = 0.5 m/s². We also introduced disorder by making the particles bidisperse, with N_α particles with radius αa and the rest with radius a.

For each system’s evolution we calculated fractional horizontal kinetic energy, Δ_{xy} = KE_{xy} / (KE + KE_{xy}). Δ_{xy} is close to zero when energy is mostly confined to the vertical oscillations in the crystalline state, and approaches ~ 2/3 for the thermalized, gas-like state. We characterized the switching intensity by integrating the area under the Fourier transform of Δ_{xy}, for frequencies smaller than 0.02 Hz, which corresponds to periods of switching longer than 50 s (Fig. S1). In contrast to systems with intermittent dynamics, the switching intensity was small for systems that remained in persistent crystalline or gas-like states. Video S1 shows a cycle of sublimation and recrystallization of a bidisperse sample (N_α = 80, α = 1.08) for β = 0.52 m/s².
Localized Vibrational Modes

The observed intermittent dynamics in our system arise from its inherent nonequilibrium and nonlinear nature. Nevertheless, an analysis of the harmonic vibrational modes in the system can shed light on how quenched disorder facilitates switching between states. We used dynamical matrix formalism to calculate the normal modes of the system [30–32]. Systems of N particles are initially placed at random locations in the xy-plane and subsequently quenched to the nearest local equilibrium in three dimensions using the FIRE algorithm [33]. The weighted Hessian matrix was computed about this equilibrium position. The 3N eigenvalues of this matrix correspond to squares of mode frequencies, whereas the normalized eigenvectors are the polarizations of the particle displacements.

In order to characterize the spatial extent of each mode, we calculated the participation ratio, \( P_r \), for each eigenvector. \( P_r \) characterizes the fraction of particles participating in a given vibrational mode:

\[
P_r = \frac{\left( \sum_{i,j} |\hat{e}_{m,i}|^2 \right)^2}{N \sum_{i,j} |\hat{e}_{m,i}|^4},
\]

where \( \hat{e}_{m,i} \) is the polarization vector of the \( i \)-th particle in the \( m \)-th unit eigenvector. The modes with \( P_r \) close to unity represent coherent motion of large fraction of particles, whereas the modes with \( P_r \ll 1 \) correspond to the motion of only a few particles. Fig. 2A–C shows \( P_r \) versus mode frequency for three different systems comprised of 500 particles: monodisperse (A), polydisperse (B), and bidisperse (C). There are two distinct frequency bands. High frequency modes near \( f \approx 20 \) Hz correspond to vertical motion, and the low frequency modes with \( f < 10 \) Hz correspond to horizontal motion. Polydisperse and bidisperse quenched disorder leads to localized modes with \( P_r \ll 1 \) in the vertical frequency band, whereas the horizontal frequency band remains nearly the same. Upon excitation with spatially-uniform (wavevector \( k = 0 \)), stochastic noise, these low-\( P_r \) modes are excited since they contain low-\( k \) Fourier components. Additionally, since there are fewer particles participating in these modes, their amplitude of motion is larger, and thus more susceptible to nonlinearities and coupling to other modes in the system. In this way they serve as the progenitors of the energy cascade from vertical to horizontal motion.
In order to quantify the relationship between the degree of quenched disorder, the number of low \( P \) modes, and the switching dynamics, we used bidisperse samples and varied both the size ratio \( \alpha \) and the number of particles \( N_\alpha \). We defined a threshold corresponding to the lowest value of \( P \) for a completely monodisperse sample (dotted line in Fig. 3A-C), and calculated the fraction of vertical modes below the threshold, (red circle in Fig. 3B-C). The resulting heatmap is presented on Fig. 3D. Both small and large values of \( N_\alpha \) mean the system is rather monodisperse, so there are very few localized modes. When there is a large discrepancy in size between the two species of particles (\( \alpha \) far from unity), there are two distinct, monodisperse layers of particles due to the vertical balance of gravity and electrostatic confinement, and the modes are no longer localized due to disorder because the layers are decoupled.

We simulated each bidisperse sample with a fixed stochastic forcing (\( \beta = 0.62 \text{ m/s}^2 \)) and calculated the switching intensity from the evolution of \( \Delta_{xy} \) (Fig. S1). Fig. 3C shows the heatmap of switching intensity. By comparing to Fig. 3D, we see that the switching intensity is maximized (dark red) for an intermediate number of localized modes. There must be some disorder (finite \( \alpha \) and \( N_\alpha \)), yet too many localized modes result in a persistently gaseous system. We observe similar heatmaps of switching intensity for smaller values of \( \beta \) (Fig. S2), although the peaks in the switching intensity requires slightly less disorder. The surprising result from this analysis is that the equilibrium, linear properties of the system (Fig. 3D) can predict the location of the nonequilibrium, intermittent dynamics with a reasonable degree of accuracy.

**Minimal Model for Intermittent Switching**

An ecosystem of two or more competing species is an archetypal model for studying oscillations in complex systems. The simplest case is the two-species Lotka-Volterra predator-prey model \[34, 35\]. Recently, a predator-prey framework was used to explain intermittent dynamics in various systems, such as the oscillation of the light emission from dust-forming plasmas \[36\], intermittent precipitation in aerosol-cloud-rain climate models \[37\], and the recurrence of turbulent “puffs” in transitional pipe flow \[17\].

Despite the many-body nature of our system, we can derive a similar framework by considering the total horizontal mechanical (\( A \)) as a predator, and the total vertical mechanical energy (\( B \)) as a prey to be consumed:

\[
\frac{dA}{dt} = -\gamma A + cAB(B - A),
\]

\[
\frac{dB}{dt} = -\gamma B - cAB(B - A) + \sqrt{B} F_n.
\]

Here we have assumed that the kinetic and potential energy are equipartitioned in each direction, so that \( A = 2 \langle KE_{xy} \rangle \) and \( B = 2 \langle KE_z \rangle \). The first term on the right hand side in the both equations corresponds to the total power dissipated through hydrodynamic damping, \( \sum F_d \cdot \dot{v} = -\sum \gamma m(v_x^2 + v_y^2 + v_z^2) = -\gamma (A + B) \), where the sum runs over each particle. The second term on the right hand side in both equations characterizes energy conservation and “predation”. The constant \( c \) controls the coupling between vertical and horizontal energies, and parameterizes the polydispersity in the many-body system. The cubic form of the coupling is the simplest that satisfies the symmetries of the system: \( A \) and \( B \) must by non-zero for some finite coupling, and energy will flow from the larger to the smaller quantity. Finally, the third term in Eq. 4 characterizes the noisy forcing in the vertical direction, \( F_n = \eta(\phi, t) \sqrt{\Delta t_0 / \Delta t} \), where \( \eta(\phi, t) \) is a Wiener process with zero mean and standard deviation \( \phi \). \( \Delta t = 0.25 \text{ ms} \) is the time step in the simulation and \( \Delta t_0 = 1 \text{ ms} \). The multiplication by \( \sqrt{B} \) is necessary since we are interested in the rate of energy input (force \( \times \) velocity). Similar noise terms are commonly used in predator-prey equations to model demographic stochasticity, which remarkably has been found to induce oscillations in the predator-prey populations \[38\].

Interestingly, these equations recreate the three dynamical regimes observed in experiment \[18\] and numerical simulations. For small values of coupling and forcing, \( A \ll B \) and energy is mostly confined to the vertical direction. For large values of \( c \) or \( \phi \), \( A \approx B \) and the system is “excited”. Finally, for intermediate values of \( c \) or \( \phi \), the system exhibits intermittent dynamics characterized by recurrent cascades of energy from \( B \) to \( A \). Fig. 4B shows trajectories of \( A \) and \( B \) for two separate cycles of excitation and relaxation. Energy cascades are represented by counterclockwise paths that emanate from the vertical axis when \( B \) is large.

To directly compare with the numerical simulations, we define \( R \equiv A/(A + B) \) as the fractional horizontal mechanical energy, in analogy with \( \Delta_{xy} \) (Fig. 2B). Fig. 4C illustrates the intermittent behavior of \( R \) for intermediate values of \( c \) and \( \phi \). Increasing \( \phi \) by a factor of 3 results in the excited state, where \( A \approx B \) (Fig. 4D). The dependence of the switching intensity (Fig. S1) on both parameters can be seen in Fig. 4E. The heatmap shows that the coupling (\( c \)) and forcing (\( \phi \)) follow an inverse relationship, and there is a distinct region where intermittent dynamics can be observed. The extent of this region can be found through a non-dimensionalization of the coupled equations, yielding a single dimensionless number describing the dynamics, \( G^2 = c(\phi \sqrt{\Delta t_0 / \Delta t}^4 / \gamma B^2) \), which represents the ratio of the energy input and coupling to dissipation. For \( G \lesssim 70 \), the dynamics are quiescent (\( A \ll B \)). For \( 70 \lesssim G \lesssim 180 \), the system exhibits intermittent switching, and for \( G \gtrsim 180 \), the system is excited. Recurrent energy cascades are observed for a
relatively narrow range of $G$, which bears a striking resemblance to transitional pipe flow, where intermittent turbulence is observed for intermediate Reynolds numbers ($1700 \lesssim \text{Re} \lesssim 2300$) [39].

As a final point of comparison, we examine the distributions of time spent in the quiescent, crystalline phase ($\tau_q$), and the excited gas-like phase ($\tau_e$). We defined a threshold for the excited state as $A \geq 0.05$. Fig. 5 shows that all distributions are peaked at a characteristic time, typically 20-200 s. However, for both the numerical simulations and the minimal model, the distribution of $\tau_q$ is much broader. This is likely due to the stochastic nature of nucleating an energy cascade from $B$ to $A$, whereas $\tau_e$ is set mostly by the amount of energy released and the damping constant. Additionally, Fig. S3 shows that the distributions for both $\tau_q$ and $\tau_e$ have exponential tails so that it is rare to spend a long time in either state. Similar distributions of quiescent and excited states are commonly observed in excitable systems, where the excitation process is stochastic and the relaxation path is deterministic [40].

**Summary**

Intermittent dynamics are a ubiquitous feature among a diverse range of complex systems. Often these dynamics result from environmental noise coupled to the system’s inherent spatial heterogeneities. Inspired by our recent experimental work showing bistable switching in a spatially-extended system of charged microparticles [18], here we have shown how noise and quenched disorder directly interact to give rise to intermittency using numerical simulations, and provided a minimal model that quantitatively captures the salient features of the dynamics. In the quasi-two-dimensional crystalline layer of particles, external noise acts as an engine that provides energy to the system in one direction, and structural heterogeneities act as a rudder that allows energy to leak into many other degrees of freedom in the form of recurrent energy cascades. The linear properties of the system are able to reasonably predict the ensuing dynamics. Localized, harmonic vibrational modes act as nucleation sites for the energy cascades since they are the most anharmonic, and can easily couple to other modes upon excitation.

Our well-mixed, minimal model ignores the spatial variation of mechanical energy in the system and instead considers the evolution of the total vertical and horizontal mechanical energy. The model accurately captures dissipation and stochastic forcing, and parameterizes the transfer of energy using a simple analytic form. The resulting equations strongly resemble modified predator-
prey equations, where the vertical energy acts as prey that experiences stochastic “death” and “birth”, and the horizontal energy consumes the growing prey population. The model displays the same intermittent dynamics as the numerical simulations for a well-defined regime of coupling, stochastic forcing, and dissipation. Despite the particulate nature of our many-body system, it is worth noting that this phenomenon bears striking resemblance to classical examples of intermittent dynamics such as transitional turbulence in pipe flow. In this case, energy is also supplied to the system at the global scale, and irregularities such as wall roughness seed turbulent eddies that facilitate a cascade of energy to smaller and smaller length scales. In the same vein, we expect our results to provide a framework for understanding similar dynamical behavior in other spatially-heterogeneous, complex, and noisy systems.

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