Dual Projection and Selfduality in Three Dimensions.

Rabin Banerjee\textsuperscript{b} and Clovis Wotzasek\textsuperscript{a,b}

\textsuperscript{a}Instituto de Física
Universidade Federal do Rio de Janeiro
21945, Rio de Janeiro, Brazil

\textsuperscript{b}S. N. Bose National Centre for Basic Sciences,
Block JD, Sector III, Salt Lake, Calcutta 700091, India.

Abstract

We discuss the notion of duality and selfduality in the context of the dual projection operation that creates an internal space of potentials. Contrary to the prevailing algebraic or group theoretical methods, this technique is applicable to both even and odd dimensions. The role of parity in the kernel of the Gauss law to determine the dimensional dependence is clarified. We derive the appropriate invariant actions, discuss the symmetry groups and their proper generators. In particular, the novel concept of duality symmetry and selfduality in Maxwell theory in (2+1) dimensions is analysed in details. The corresponding action is a 3D version of the familiar duality symmetric electromagnetic theory in 4D. Finally, the duality symmetric actions in the different dimensions constructed here manifest both the $SO(2)$ and $Z_2$ symmetries, contrary to conventional results.
1 Introduction

It is often claimed that the duality operation is only defined in even dimensional spacetimes and that selfduality is further restricted to twice-odd spacetime dimensional theories\cite{1}. The purpose of this paper is to extend the notion of both duality symmetry as well as selfduality to the odd (2+1) dimensional spacetime in the electromagnetic context. Naturally, the conventional duality symmetry in even dimensions is also contained in our approach. To achieve this we introduce an alternative definition for the duality operation which is valid in any dimensions. Specifically we analyse and explore the impact of the Gauss law kernel’s parity and the dual projection procedure over the duality operation of the Maxwell theory in different spacetime dimensions. The crucial issue concerns the spacetime dependence of the parity property of a generalized curl involved in the resolution of the Gauss law. We show that this property is decisive in determining the proper actions and the corresponding group of symmetry for each specific dimension.

The role of the duality operation in the investigation of concrete physical systems in different areas is by now well recognized\cite{2}. This is a symmetry transformation that is fundamental for investigations in arenas as distinct as quantum field theory, statistical mechanics and string theory. Duality is a general concept relating physical quantities in different regions of the parameter space. It relates a model in a strong coupling regime to its dual version working in a weak coupling regime, providing valuable information in the study of strongly interacting models. The selfduality present in D=4k-2 dimensions has attracted much attention because it seems to play an important role in many theoretical models\cite{3}.

The electromagnetic duality transformation is defined by the Hodge-star operation that involves multiplication by the appropriate $\epsilon$ symbol. Consider a general (n-1)-form and its field strength

$$F_{k_1 \ldots k_n} \equiv \partial_{[k_n} A_{k_1 \ldots k_{n-1}]}.$$

The dual field is then defined as,

$$\ast F^{k_1 \ldots k_n} = \frac{1}{n!} \epsilon^{k_1 \ldots k_{2n}} F_{k_{n+1} \ldots k_{2n}}.$$

Note that only in 2n dimensions will the n-form field be of the same rank as its dual. The action, field equation and Bianchi identity for a source free field are

$$S = -c_n \int d^{2n}x F_{k_1 \ldots k_n} F^{k_1 \ldots k_n}.$$
\[ 0 = \partial_k, F^{k_1 \ldots k_n} \]
\[ 0 = \partial_k * F^{k_1 \ldots k_n} \]  \hspace{1cm} (3)

where \( c_1 = 1/2, c_2 = 1/4, \) etc. The field equation and the Bianchi identity are of the same form so that the duality transformation \( F \leftrightarrow *F \) is a symmetry at the level of these equations but is not present at the level of the action. The dependence with dimensionality appears to be crucial. The duality symmetry is characterized by a one-parameter continuous SO(2) group for \( D=4k \) \( (k \in \mathbb{Z}_+) \) while for the \( D=4k-2 \) case it is described by a discrete group with just two elements. Most of the analysis of this dimensional dependence take the algebraic point of view where the distinction among different dimensions is manifest by the double dualization operation following from the identities,

\[ **F = \begin{cases} F, & \text{if } D = 4k - 2 \\ -F, & \text{if } D = 4k. \end{cases} \]  \hspace{1cm} (4)

It was then shown that the duality groups \( G \) preserving the form of the action were subgroups of those preserving equations of motion and Bianchi identities, obtained by taking the intersection of the former with the group \( \text{O}(2) \), the symmetry group for the energy-momentum tensor[4].

Most of the discussion about duality transformations as a symmetry for the actions and the existence of self-duality are based on these concepts. We feel that this algebraic viewpoint is rather restrictive. It is only defined for even dimensional spacetimes leading to separate consequences regarding the duality groups and actions. The usual lore is that only the 4D Maxwell theory and its 4k extensions possess duality as a symmetry but self-duality would only be definable in \( D=4k-2 \). On the other hand for the 2D scalar field theory and its \( 4k-2 \) extensions duality is not even a well defined concept. A solution for these problems came with the recognition of an internal structure in the space of potentials[5, 6]. The internal space effectively unifies the self-duality concept in the different \( 4k - 2 \) and \( 4k \) dimensions. The dual field is now defined to include an internal index \((\alpha, \beta)\) and an extended dualization is defined as,

\[ \tilde{F}^{\alpha} = \epsilon^{\alpha\beta*} F^{\beta}; \quad D = 4k \]
\[ \tilde{F}^{\alpha} = \sigma_1^{\alpha\beta*} F^{\beta}; \quad D = 4k - 2 \]  \hspace{1cm} (5)

where \( \sigma_k \) are the usual Pauli matrices and \( \epsilon_{\alpha\beta} \) is the fully antisymmetric \( 2 \times 2 \) matrix with \( \epsilon_{12} = 1 \).

Now, irrespective of the dimensionality, the double dual operation yields,

\[ \tilde{F} = F \]  \hspace{1cm} (6)

which generalises (4). Self and anti-self dual solutions are now well defined in all even \( D = 2k \) dimensions.
With the above background, it is useful to put our work in a proper perspective by observing the following:

• The procedure of dual projection developed here is basically analogous to a canonical transformation, except that the former is performed at the level of the actions while the latter, as is well known, is at the Hamiltonian level. Since throughout the paper only first order actions will be considered the equivalence between the dual projection and the canonical transformations becomes manifest.

• The Maxwell action in any even dimension is decomposed, by the dual projection method, into two pieces, one of which carries the $SO(2)$ symmetry while the other has the $Z_2$ symmetry. By specialising to $4k$ or $4k - 2$ dimensions, we find that one of the terms in the action becomes a total derivative which can be ignored. In this way the conventional results of duality symmetry characterising the $SO(2)$ group for $D = 4k$ and the $Z_2$ group for $D = 4k - 2$ dimensions are reproduced.

• The dual projection in $D = 4k - 2$ dimensions leads to a diagonal form of the actions with two pieces manifesting the opposite chiralities. It is then possible to impose a chiral constraint to eliminate either of the pieces. What remains is the action for a chiral boson. This generalises the usual construction of chiral bosons to higher dimensions.

• The case $D = 2$ seems to be a special point. Although it qualifies as a member of the $D = 4k - 2$ group, it will be shown that it allows for the realization of both $D = 4k - 2$ and $D = 4k$ constructions.

• By passing to the momentum space, it is possible to derive results analogous to the $D = 2$ case for higher dimensions. In other words there is a duality transformation among the Fourier modes showing the complementary nature of the symmetries: the $SO(2)$ symmetry for $D = 4k - 2$ dimensions and the $Z_2$ symmetry for the $D = 4k$ dimensions. It might be mentioned that this nature of duality symmetry was earlier shown by us using different methods.

• The algebraic analysis leaves out the possibility of obtaining duality symmetric electromagnetic Maxwell theory in odd dimensions. For the special case of three dimensions this will be achieved here.

In the next section we shall show that the use of the two key concepts; namely, the dimensional dependence of the parity property of the generalized rotation operator involved in the resolution
of the Gauss law and the dual projection method, reproduce the known results of the algebraic analysis, clarifying their physical origins. In the third section we discuss some special instances like the two-dimensional case that possess both $SO(2)$ and $Z_2$ representations. In the fourth section, which contains the central result of this paper, attention is given to the D=(2+1) Maxwell theory that is studied in full details. We disclose the presence of an internal space of potentials where duality is realized as a $SO(2)$ rotation and also as a discrete $Z_2$ symmetry. The special equivalence between Maxwell theory and the scalar field via Hodge-dualization is discussed from our approach. The last section is reserved to a discussion of our conclusions and perspectives.

2 Parity and Dual Projection in Even Dimensional Spacetimes

The main argument of this report is the dimensional dependence of the parity property of the generalized rotation operator and a canonical transformation that we call dual projection. A systematic derivation of self-dual actions for two and four dimensional cases was proposed in [6] using the dual projection procedure. Here we generalise the method to include all even dimensions. Subsequently these ideas will be exploited to discuss the consequences in a three dimensional theory. The generalized rotation operator, which is basically a functional curl, is defined as,

\[(\epsilon \partial) \equiv \epsilon_{k_1k_2\cdots k_{D-1}} \partial_{k_{D-1}}\]

Clearly the dimensional dependence of this operator’s parity is given by,

\[\mathcal{P}(\epsilon \partial) = \begin{cases} +1, & \text{if } D = 4k \\ -1, & \text{if } D = 4k - 2, \end{cases} \]

where parity is defined as

\[\int \Phi(\epsilon \partial \Psi) = \mathcal{P}(\epsilon \partial) \int \Psi(\epsilon \partial \Phi)\]

The consequence of this property is best appreciated after a dual projection of the action. First, the theory is reduced to its first-order form as,

\[S = \int d^D x \left[ \pi \cdot \dot{A} - \frac{1}{2} \pi \cdot \pi - \frac{1}{2} B \cdot B + A_0 (\partial \cdot \pi) \right]\]
where we used the notation (anti symmetrisation is implied by the brackets)
\[ \Phi \cdot \Psi \equiv \Phi_{[k_1 k_2 \cdots k_{D-1}]} \Psi_{[k_1 k_2 \cdots k_{D-1}]} \] (11)
and defined the magnetic field as
\[ B = (\epsilon \partial) \cdot A \] (12)

In the four dimensional case, \( B_k = \epsilon_{k m n} \partial_m A_k \) is a three-vector while in three dimensions, the magnetic field is a scalar, \( B = \epsilon_{k m} \partial_m A_k \). Clearly the two-dimensional instance represents a special situation due to the absence of a Gauss law and will be treated separately in the next section. There does not seem to exist any difficulty in dimensions \( D > 4 \). Note that \( A_0 \) in (10) generically denotes the multiplier in any even dimension, that enforces the Gauss constraint. For example, it is just \( A_0 \) in four dimensions while it is \( A_{0i} \) in six dimensions and so on.

The important point to observe is that the Gauss constraint is trivially solved, in any dimension, using the generalized curl (7),
\[ \pi = (\epsilon \partial) \cdot \phi \] (13)
where \( \phi \) is a \( (\frac{D}{2} - 1) \)-form potential. For instance, in \( D=4 \) and \( D=6 \) which are generic for \( D=4k \) and \( D=4k-2 \), this solution reads
\[ \pi_k = \epsilon_{k m n} \partial_m \phi_n \]
\[ \pi_{km} = \epsilon_{kmnpq} \partial_n \phi_{pq} \] (14)

The next step is to perform the canonical transformations,
\[ A = \Phi^{(+)} + \Phi^{(-)} \]
\[ \pi = \eta (\epsilon \partial) \cdot (\Phi^{(+)} - \Phi^{(-)}) \] (15)
with \( \eta = \pm 1 \) defining the signature of the operation. The effect of the dual projection procedure into the first-order Maxwell action is the creation of an internal space of potentials in which the duality symmetry is local and manifest. In terms of the internal space potentials \( \Phi^{(+)} \) and \( \Phi^{(-)} \) the action now reads,
\[ S = \int d^D x \left\{ \eta \left[ \hat{\Phi}^{(\alpha)} \sigma_3^{\alpha \beta} B^{(\beta)} + \hat{\Phi}^{(\alpha)} \epsilon^{\alpha \beta} B^{(\beta)} \right] - B^{(\beta)} B^{(\beta)} \right\} \]  

(16)

where \( B^{(\beta)} = (\epsilon \partial \cdot \Phi^{(\beta)}) \) and \( \sigma_3^{(\alpha \beta)} \) and \( \sigma_2^{(\alpha \beta)} = i \epsilon^{(\alpha \beta)} \) are the 2 \times 2 Pauli matrices. Notice that while the hamiltonian sector of the first-order action is unique, the symplectic sector is composed by two distinct parts with separate consequences. We can now appreciate the impact of the dimensionality over the symplectic structure of (16) and the role of the parity in selecting the proper action and the corresponding duality group. Parity (or dimensionality) has no influence over the hamiltonian since it only involves quadratic forms. For twice odd dimensions the second term of the symplectic sector is a total derivative and may be discarded. The remaining piece diagonalizes the action providing a generalization of the two-dimensional action describing chiral bosons\(^8\). The \( Z_2 \) property is manifest by the interchange between the internal space potentials \( \Phi^{(\pm)} \leftrightarrow \Phi^{(\mp)} \) mapping one chirality into the other.

For twice even dimensions, on the other hand, it is the first term that becomes a total derivative. The action does not diagonalize but presents an explicit one-parameter continuous SO(2) symmetry. In the D=4 case this action corresponds to duality symmetric Maxwell theory quoted in the literature\(^5, 6\). The important point to stress is that the derivative operator involved in the dual projection has been determined by the solution of the Gauss constraint. This automatically fixes the dependence of parity with dimensionality and explains its effect over the electromagnetic actions and duality groups. Incidentally, observe that due to its intrinsic diagonal form, the phase space of the chiral boson solution (D=4k-2) may be reduced if we impose a chiral constraint as,

\[ \pi = \pm(\epsilon \partial) \cdot A \]  

(17)

Each of these constraints eliminates one of the (internal) chiral potentials thereby leading to an action for chiral bosons. These are the generalisations of the usual actions for chiral bosons in two dimensions\(^8\). However, the same situation cannot be reached in the twice-even instance due to the special form of the symplectic sector (the hamiltonian poses no obstruction to reduction in either case). In the reduced phase-space the remaining chiral boson carries a representation for half the number of degrees of freedom of the original system. On the other hand, the duality symmetric system maintains the phase space structure intact. Therefore, this system should not be considered as the 4D analog of the 2D chiral boson. Although, due to the possibility of two distinct signatures in the dual projection, there exist either a self-dual or anti self-dual decomposition (but not simultaneously), we believe that this situation should not be confused with the distinct chiralities
that appear (simultaneously) in the dual projection of $D=4k-2$ dimensional systems.

We have therefore reproduced completely the results known from the algebraic approach plus presented a derivation of the appropriate actions displaying the internal potentials for each case. Also worth of mention is the fact that there are two and not one self-dual action, labelled by the signature $\eta$ of the dual projection, describing opposite aspect of the self-duality symmetry. The most useful and striking feature in this dual projection procedure is that it is not based on evidently even dimensional concepts and may be extended to the odd dimensional situation.

3 Special Examples

In this section we discuss some special dimensions and situations. The 4D Maxwell example deserves detailed analysis since it is the paradigm of the duality symmetry. The other example is the $(1+1)$ dimensional scalar theory. The absence of a gauss constraint leads to a crucial change from the Maxwell case.

3.1 The Electromagnetic Duality

Exploiting the ideas elaborated in the previous sections, it is straightforward to implement the selfduality projection in the electromagnetic theory. Let us start with the usual Maxwell action,

$$ S = -\frac{1}{4} \int d^4 x \, F_{\mu \nu} F^{\mu \nu} $$

which is expressed in terms of the electric and magnetic fields as,

$$ S = \frac{1}{2} \int d^4 x \left( E_i^2 - B_k^2 \right) $$

where,

$$ E_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0 $$

$$ B_i = \epsilon_{ijk} \partial_j A_k $$

The following duality transformation,

$$ E_k \rightarrow \mp B_k \; ; \; B_k \rightarrow \pm E_k $$

is known to preserve the invariance of the full set comprising Maxwell’s equations and the Bianchi identities although the action changes its signature. The Maxwell Lagrangean is next recast in
a symmetrised first order form that displays an $\text{Sp}(2,\mathbb{R})$ symmetry when we treat $(P_k, A_k)$ as a doublet,

$$\mathcal{L} = \frac{1}{2} \left( P_k \dot{A}_k - \dot{P}_k A_k \right) - \frac{1}{2} P_k^2 - \frac{1}{2} B_k^2 + A_0 \partial_k P_k$$

(22)

Next a canonical transformation is invoked. There are two possibilities (assigning different signatures for the dual projection) which translate from the old set $(P_k, A_k)$ to the new ones $(A_1^k, A_2^k)$. It is, however, important to recall that the Maxwell theory has a Gauss constraint that is implemented by the Lagrange multiplier $A_0$. The new variables are chosen in two different ways which solve this constraint and implement distinct signatures to the dual projection as,

$$P_k \rightarrow B_2^k; \quad A_k \rightarrow A_1^k$$

$$P_k \rightarrow B_1^k; \quad A_k \rightarrow A_2^k$$

(23)

It is now simple to show that, in terms of the new variables, the original Maxwell action takes the form,

$$S_\pm = \frac{1}{2} \int d^4x \left( \pm \dot{A}_k^\alpha \epsilon^{\alpha\beta} B_k^\beta - B_k^\alpha B_k^\alpha \right)$$

(24)

It is duality symmetric under the full $SO(2)$.

Let us next introduce the proper and improper $O(2)$ rotation matrices as $R^+(\theta)$ and $R^-(\phi)$ with determinant $+1$ and $-1$, respectively,

$$R^+ (\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

(25)

$$R^- (\phi) = \begin{pmatrix} \sin \phi & \cos \phi \\ \cos \phi & -\sin \phi \end{pmatrix}$$

(26)

Note that the matrix that corresponds to improper rotations, $R^-(\phi)$ switches the actions $S_+$ and $S_-$ into one another.

Using the basic brackets following from the canonical transformations or from the symplectic structure of the theory,

$$\left[ A_\alpha^i (x), \epsilon^{jkl} \partial^k A_\beta^l (y) \right] = \pm i \delta^{ij} \epsilon_{\alpha\beta} \delta (x - y)$$

(27)

we can verify that the generators of the $SO(2)$ rotations are given by a Chern-Simon like structure,

$$Q^{(\pm)} = \mp \frac{1}{2} \int d^3x \ A_\alpha^\alpha \ B_\alpha^\alpha$$

(28)
so that finite transformations are given by,

\[ A_{k}^{\alpha} \rightarrow A_{k}^{\prime \alpha} = e^{-iQ\theta} A_{k}^{\alpha} e^{iQ\theta} \]  

(29)

Let us stress on the fact that there are two distinct structures for the duality symmetric actions. These must correspond to the opposite aspects of some symmetry. By looking at the equations of motion obtained from (24),

\[ \dot{A}_{k}^{\alpha} = \pm \epsilon^{\alpha\beta} \nabla_{k} \times A_{k}^{\beta} \]  

(30)

it is possible to verify that these are just the self and anti-self dual solutions,

\[ F_{\mu\nu}^{\alpha} = \pm \epsilon^{\alpha\beta} \ast F_{\mu\nu}^{\beta}; \quad \ast F_{\mu\nu}^{\beta} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F_{\rho\lambda}^{\beta} \]  

(31)

obtained by setting \( A_{0}^{\alpha} = 0 \). It may be observed that the opposite aspects of the dual symmetry are contained in the internal space.

To close our arguments, let us now comment on another property, which is related to the existence of two distinct actions (24), by replacing (21) with a new set of transformations,

\[ E_{\alpha} \rightarrow R_{\alpha\beta}(\varphi) E_{\beta} \]

\[ B_{\alpha} \rightarrow R_{\alpha\beta}(\varphi) B_{\beta} \]  

(32)

Notice that these transformations preserve the invariance of the Hamiltonian following from either \( S_{+} \) or \( S_{-} \). The kinetic terms in the action change signatures so that \( S_{+} \) swaps to \( S_{-} \). The discretised version of (32) is obtained by setting \( \varphi = 0 \),

\[ E_{\alpha} \rightarrow \sigma_{1}^{\alpha\beta} E_{\beta} \]

\[ B_{\alpha} \rightarrow \sigma_{1}^{\alpha\beta} B_{\beta} \]  

(33)

It is precisely the \( \sigma_{1} \) matrix that reflects the proper into improper rotations,

\[ R^{+}(\theta)\sigma_{1} = R^{-}(\theta) \]  

(34)

which illuminates the reason behind the swapping of the actions in this example.

### 3.2 The Scalar Theory in 1+1 Dimensions

The ideas developed in the previous section are now implemented and elaborated in 1+1 dimensions. In particular we show that two distinct dual projections are possible in this case, leading to either
or SO(2) group of dualities. Notice first that in D=2 there is no photon and the Maxwell theory trivialises so that the electromagnetic field can be identified with a scalar field. Thus all the results presented here can be regarded as equally valid for the “photon” field. There is no Gauss constraint however so that we are free to choose any operator in the dual projection. Our computations will be presented in a very suggestive notation which illuminates the Maxwellian nature of the problem.

The action for the free massless scalar field is given by,

\[ S = \frac{1}{2} \int d^2x \left( \partial_\mu \phi \right)^2 \]  

(35)

and the equation of motion reads,

\[ \ddot{\phi} - \phi'' = 0 \]  

(36)

where the dot and the prime denote derivatives with respect to time and space components, respectively. Introduce a change of variables using electromagnetic symbols,

\[ E = \dot{\phi} \quad ; \quad B = \phi' \]  

(37)

Obviously, \( E \) and \( B \) are not independent but constrained by the identity,

\[ E' - \dot{B} = 0 \]  

(38)

In these variables the equation of motion and the action are expressed as,

\[ \dot{E} - B' = 0 \quad ; \quad S = \frac{1}{2} \int d^2x \left( E^2 - B^2 \right) \]  

(39)

so that the transformations,

\[ E \rightarrow \pm B \quad ; \quad B \rightarrow \pm E \]  

(40)

display a duality between the equation of motion and the ‘Bianchi’-like identity (38) but the action changes its signature. Note that there is a relative change in the signatures of the duality transformations (40) with respect to the true electromagnetic duality (21), arising basically from dimensional considerations. This symmetry corresponds to the improper group of rotations.

To illuminate the close connection with the Maxwell formulation, we introduce covariant and contravariant vectors with a Minkowskian metric \( g_{00} = -g_{11} = 1 \),

\[ F_\mu = \partial_\mu \phi \quad ; \quad F^\mu = \partial^\mu \phi \]  

(41)
whose components are just the ‘electric’ and ‘magnetic’ fields defined earlier,

\[ F_\mu = \left( E, B \right) ; \quad F^\mu = \left( E, -B \right) \]  

Likewise, with the convention \( \epsilon_{01} = 1 \), the dual field is defined,

\[ *F_\mu = \epsilon_{\mu\nu} \partial^\nu \phi = \epsilon_{\mu\nu} F^\nu \]

\[ = \left( -B, -E \right) \]  

(43)

The equation of motion and the ‘Bianchi’ identity are now expressed by typical electrodynamical relations,

\[ \partial_\mu F^\mu = 0 \]
\[ \partial_\mu *F^\mu = 0 \]  

(44)

To expose a duality symmetric action, the basic principle of our approach is adopted. We convert the original second order form (39) to its first order version displaying the Sp(2) symmetry and then invoke a canonical transformation to provide an internal index. An auxiliary field is therefore introduced at the first step,

\[ \mathcal{L} = P E - \frac{1}{2} P^2 - \frac{1}{2} B^2 \]  

(45)

where \( E \) and \( B \) have already been defined. The following canonical transformation,

\[ B \to \partial \left( \phi^+ + \phi^- \right) \]
\[ P \to \partial \left( \phi^+ - \phi^- \right) \]  

(46)

leads to an action with fields taking values in the internal space

\[ S = \int d^2 x \left[ \left( \partial \phi^+(\phi^+ - \phi^-) \right) + \left( \partial \phi^-(\phi^+ - \phi^-) \right) - \left( \partial \phi^+(\phi^- - \phi^+) \right) \right] \]  

(47)

As discussed previously, due to the absence of a true Gauss law in this case, we are free of any imposition regarding the choice of the operator \( \partial \) in the dual projection. To display this arbitrariness, we choose, for each group of symmetry transformation,

\[ \partial = \begin{cases} \\
\partial_x, & \text{leading to } Z_2 \\
\sqrt{-\partial_x}\partial_x, & \text{leading to } SO(2). \\
\end{cases} \]  

(48)
The first choice is traditional. The odd parity of the operator diagonalizes the action by eliminating the second term in the sympletic sector. The resulting actions,

\[ S = S_+ + S_- \]

\[ S_{\pm} = \int d^2 x \left( \dot{\Phi}^\pm \partial_x \Phi^\pm - \partial_x \Phi^\pm \partial_x \Phi^\pm \right) \]  

(49)
correspond to the well known right and left chiral boson theories[8].

To examine the symmetry content it is possible to recast (49) in a very suggestive form,

\[ S_{\pm} = \frac{1}{2} \int d^2 x \left[ \pm \partial_x \Psi^\alpha \sigma_1^{\alpha\beta} \Psi^\beta - \partial_x \Psi^\alpha \partial_x \Psi^\alpha \right] \]

\[ = \frac{1}{2} \int d^2 x \left[ \pm B_\alpha \sigma_1^{\alpha\beta} E_\beta - B_\alpha^2 \right] \]  

(50)

where \( \Psi^\pm = \Phi^+ \pm \Phi^- \). In the second line the action is expressed in terms of the electromagnetic variables. This action is duality symmetric under the transformations of the basic scalar fields,

\[ \Psi^\alpha \rightarrow \sigma_1^{\alpha\beta} \Psi^\beta \]  

(51)

which, in the notation of \( E \) and \( B \), is given by,

\[ B_\alpha \rightarrow \sigma_1^{\alpha\beta} B_\beta \]

\[ E_\alpha \rightarrow \sigma_1^{\alpha\beta} E_\beta \]  

(52)

It is quite interesting to observe that, contrary to the 4D electromagnetic theory, the transformation matrix in the \( O(2) \) internal space of potentials is not the epsilon, but rather a \( \sigma_1 \) Pauli matrix. This result is in agreement with that found from general algebraic arguments[4] which stated that for \( D = 4k - 2 \) dimensions there is a discrete \( \sigma_1 \) symmetry. Observe that (52) is a manifestation of the original duality[4] which was also effected by the same operation. It is important to stress that the above transformation is only implementable at the discrete level. Moreover, since it is not connected to the identity, there is no generator for it. In this sense it is observed that duality symmetry is not defined in these twice odd dimensions. To complete the picture, we also mention that the following rotation,

\[ \Psi^\alpha \rightarrow \epsilon_{\alpha\beta} \Psi^\beta \]  

(53)

interchanges the actions (50),

\[ S_+ \leftrightarrow S_- \]  

(54)
Thus, except for a rearrangement of the the matrices generating the various transformations, most features of the electromagnetic example are perfectly retained. The crucial point of departure is that now all these transformations are only discrete.

The second choice in (48) is new and unexpected since it does not fit into the known dimensional classification. It leads to a continuous \(SO(2)\) duality transformation, characteristic of the \(4k\) dimensional spacetimes, instead of the discrete \(Z_2\). The resulting action is,

\[
S = \int d^2x \left[ \left( \partial \phi^+ \dot{\phi}^- - \partial \phi^- \dot{\phi}^+ \right) - \left( \partial \phi^{\alpha} \partial \phi^{\alpha} \right) \right]
\]

which is appropriate for the even-parity dual projection. It is easy to obtain the basic symplectic brackets from here as,

\[
\{ \Phi^\alpha(x), \partial \Phi^\beta(y) \} = \mp \epsilon^{\alpha \beta} \delta(x-y)
\]

Now observe that the action (55) is manifestly invariant under the continuous duality transformations,

\[
\Phi^\alpha \rightarrow R^+_{\alpha \beta} \Phi^\beta
\]

where \(R^+_{\alpha \beta}\) is the usual \(SO(2)\) rotation matrix (25). The generator of the infinitesimal symmetry transformation is given by,

\[
Q^\pm = \mp \frac{1}{2} \int d^2x \Phi^\alpha \partial \Phi^\alpha
\]

while the transformation by a finite angle \(\theta\) is generated by,

\[
\Phi^\alpha \rightarrow \tilde{\Phi}^\alpha = e^{-i\theta Q^\alpha} e^{i\theta Q^-} = R^+_{\alpha \beta}(\theta) \Phi^\beta
\]

We therefore conclude that the scalar theory in two dimensions manifests all features of duality symmetry pertaining to either twice odd or twice even dimensions. It, therefore, goes beyond the results found by the algebraic approach.

### 4 Dual Projection in (2+1) dimensions

As we have stressed, conventional group theoretical arguments fail to discuss duality in odd dimensional theories. This is possible in our approach. As an example we consider the (2+1) dimensional Maxwell theory. We shall see that the solution of the Gauss constraint leads naturally to a differential operator to be used as the dual projector. In subsection 4.1 the canonical transformation in the
dual projection involves an even-parity operation. The resulting action displays a continuous \( \text{SO}(2) \) group of symmetry transformations. This is a new result that could not be disclosed by algebraic methods. However, since in three spacetime dimensions vector fields are duality related to scalars where there is no Gauss law restriction, an odd-parity kernel also exists. The complete diagonalization of the electromagnetic action into two distinct type actions, that would be a prototype of chiral bosons in three dimensions, cannot be done in the coordinate space. On the other hand, if momentum space approach of the dual-projection is adopted, such a structure is then shown to exist if a special combination of the Fourier modes is considered. This issue will be studied in subsection 4.2.

### 4.1 Even Parity Projection

To begin with we see that the solution of the Gauss constraint that takes proper care of the spatial indices must involve a canonical scalar field,

\[
\pi_k \sim \epsilon_{km} \partial_m \phi
\]

The problem here is that, in contrast to the even dimensional cases, parity is not a good property to look for in the generalized curl operator \((\epsilon \partial)\). However, exploiting the property,

\[
\nabla^2 = (\epsilon_{km} \partial_m) (\epsilon_{kn} \partial_n)
\]

we may find an even solution for the dual projection by the following canonical transformations,

\[
\pi_k = \eta \epsilon_{km} \partial_m (\phi^+ - \phi^-) \\
A_k = \frac{\epsilon_{km} \partial_m}{\sqrt{-\nabla^2}} (\phi^+ + \phi^-)
\]

where \(\eta\) gives the signature of the dual projection and \(\sqrt{-\nabla^2}\) is included for dimensional reasons. Substituting these into the Maxwell action (10) we obtain,

\[
S_{\eta} = \int d^3 x \left( \eta \phi^\alpha \epsilon^{\alpha\beta} B^\beta - B^\alpha B^\alpha \right)
\]

where \(B^\alpha\) is a shorthand for

\[
B^\alpha = \sqrt{-\nabla^2} \phi^\alpha \quad ; \quad \phi^\alpha = \phi^+ ; \phi^-
\]
Not surprisingly, the resulting action is explicitly duality invariant displaying a continuous $SO(2)$ symmetry.

Let us next examine the symmetry contents revealed in (63). These actions are manifestly invariant under the continuous $SO(2)$ transformations,

$$\phi_\alpha \to R^{+}_{\alpha\beta} \phi_\beta \tag{65}$$

where $R^{+}_{\alpha\beta}$ is the proper $SO(2)$ rotation matrix (25). The generator of the infinitesimal symmetry transformation is given by the Chern-Simons form,

$$Q_q = \frac{\eta}{2} \int d^3 x \phi_\alpha B_\alpha \tag{66}$$

and the finite transformations (65) are generated as,

$$\phi_\alpha \to \tilde{\phi}_\alpha = e^{-i\theta Q} \phi_\alpha e^{i\theta Q} = R^{+}_{\alpha\beta}(\theta) \phi_\beta \tag{67}$$

This result comes by using the basic symplectic brackets obtained from (63),

$$\{ \phi_\alpha(\vec{x}), B_\beta(\vec{y}) \} = \eta \epsilon_{\alpha\beta} \delta(\vec{x} - \vec{y}) \tag{68}$$

This is the parallel of the usual constructions done in the 4D Maxwell theory and its D=4k extensions to induce a duality symmetry in the action.

It is interesting to check the dual projection procedure when applied to a (2+1) dimensional scalar field theory. Recall that a vector field in 3D has only one degree of freedom and spin zero. The vector and the scalar fields are related by the dualization,

$$* F_{\mu\nu} = \epsilon_{\mu\nu\lambda} \partial^\lambda \phi \tag{69}$$

A simple analysis shows that under this transformation, $B \to \dot{\phi}$ and $E_k \to \epsilon_{km} \partial_m \phi$ meaning that there is an exchange of the potential and sympletic sectors between the models. Clearly the Maxwell Gauss law is automatically satisfied in the scalar representation,

$$\nabla.A = 0 \quad \rightarrow \quad \partial_k (\epsilon_{km} \partial_m \phi) = 0 \tag{70}$$

showing that in the dual point of view the gauge constraint has no dynamical consequences. It is now easy to apply the dual projection to the conventional scalar action, written in a first order form,
\[ S = \int dx \left[ \pi \dot{\phi} - \frac{1}{2} \pi^2 - \frac{1}{2} (\partial_t \phi)^2 \right] \] (71)

by performing the following canonical transformations,

\[ \begin{align*}
\phi &= \phi_+ + \phi_- \\
\pi &= \eta \sqrt{-\nabla^2} (\phi_+ - \phi_-)
\end{align*} \] (72)

The resulting action reproduces the result (63), as it should. This also shows the equivalence of these theories in the context of dual projection.

### 4.2 Odd Parity Projection

To disclose an odd parity projection we note the fundamental difference from the two dimensional case. In the latter, there is only one space dimension and hence it is straightforward to define the odd projection in the coordinate space. For any space dimension greater than one, this projection becomes ambiguous in the coordinate space. The standard way to bypass this problem is to go over to the momentum space. Let us therefore introduce a two-dimensional basis, \( \{ \hat{e}_a(k, x), a = 1, 2 \} \), with \((k, x)\) being conjugate variables and the orthonormalization condition given as,

\[ \int dx \hat{e}_a(k, x) \hat{e}_b(k', x) = \delta_{ab} \delta(k, k') \] (73)

We choose the vectors in the basis to be eigenvectors of the Laplacian, \( \nabla^2 = \partial \cdot \partial \),

\[ \nabla^2 \hat{e}_a(k, x) = -\omega^2(k) \hat{e}_a(k, x) \] (74)

The action of \( \partial \) over the \( \hat{e}_a(k, x) \) basis is

\[ \partial \hat{e}_a(k, x) = \omega(k) M_{ab} \hat{e}_b(k, x) \] (75)

that together with definition (74) gives \[ \tilde{M} \cdot M = -I \] (76)

\[ ^1 \text{Here we use the matricial notation where } (\tilde{M})_{ab} = M_{ba}. \]
Let us use this basis to represent the elementary fields. Since the Maxwell theory here is equivalent to a spin zero scalar, it suffices to analyze this last case. The canonical scalar and its conjugate momentum have the following expansion,

\[ \Phi(x) = \int dk \, q_a(k) \, \hat{e}_a(k, x) \]
\[ \Pi(x) = \int dk \, p_a(k) \, \hat{e}_a(k, x) \]  
(77)

with \( q_a \) and \( p_a \) being the expansion coefficients. It leads to a representation of the action as a two-dimensional oscillator. The phase-space is four-dimensional, representing two degrees of freedom per mode,

\[ S = \int dk \left\{ p_a \dot{q}_a - \frac{1}{2} p_a p_a - \frac{\omega^2}{2} q_a q_a \right\} \]  
(78)

Let us now consider the following canonical transformation,

\[ p_a(k) = \omega(k) \, \epsilon_{ab} \left( \varphi_b^{(+)} - \varphi_b^{(-)} \right) \]
\[ q_a(k) = \left( \varphi_b^{(+)} + \varphi_b^{(-)} \right) \]  
(79)

such that (78) gets diagonalized,

\[ S = S_+ + S_- \]  
(80)

where,

\[ S_\pm = \int dk \omega(k) \left( \pm \dot{q}_a \epsilon_{ab} q_b - \omega(k) q_a q_a \right) \]  
(81)

This action displays the \( Z_2 \) symmetry since, under the transformation \( \varphi_a \to \sigma_1^{a\beta} \varphi_a^{\beta} \), the two pieces \( S_+ \) and \( S_- \) are swapped. Hence the theory shows both \( SO(2) \) and \( Z_2 \) symmetries, depending upon the nature of the transformation.

It may be useful to point out that the actions \( S_\pm \) are the analogues actions for chiral bosons. Each of these actions characterises one degree of freedom per mode in phase space or half degree of freedom in configuration space that represent a chiral scalar. Since the Maxwell theory is equivalent to a scalar field theory in three dimensional spacetime, this is valid for the photon field as well. This shows that it is indeed possible to obtain a phase space reduced, diagonal selfdual solution for the Maxwell field in this odd dimensional spacetime.
5 Conclusions

In this paper we have developed a new technique for obtaining duality symmetric actions. Different aspects of duality symmetry were discussed. Our technique was based on an operation which was termed as a dual projection. Since the analysis was always carried out for first order systems an equivalence between the lagrangian and hamiltonian approaches was possible. Indeed the dual projection entailed a change of variables which was a canonical transformation in the phase space. The analysis was completely general, which required an appropriate definition of the functional curl used for solving the Gauss law constraint. The conventional results for the construction of duality symmetric actions in any even dimension was contained in a single expression. Zooming in on the particular even dimension, either $4k$ or $4k - 2$, immediately displayed the relevant $SO(2)$ or $Z_2$ symmetry, respectively. This sharp line of distinction was related to the parity of the functional curl. Our analysis, however, went beyond. Using the same techniques it was possible to discuss the property of duality symmetry in odd dimensions, which were not analysed earlier. Since the definition of parity of the functional curl was problematic in these dimensions, it was not straightforward to give a general discussion for arbitrary odd dimensions, as was the case for the even dimensions. Hence we elaborated our methods by concentrating on the three dimensional Maxwell theory.

Historically, the study of duality symmetry began by considering the symmetry among the electric and magnetic fields. This led to an invariance of the equations of motion but not of the actions. However it was felt that these were composites and one should study the symmetry properties in the context of potentials which were regarded as the basic entities. This was achieved by introducing an internal space. The new duality symmetry involving the potentials preserved the invariance of the corresponding actions. Two distinct classes of symmetries, pertaining to either $4k$ or $4k - 2$ dimensions were found. Now the potentials can also be regarded as composites that were defined in terms of their fourier modes. Pursuing this line of research and studying the symmetries in terms of these modes we were able to show that, suitably interpreted, the distinction among the duality groups could be obliterated. In other words, in all even dimensions, both $SO(2)$ and $Z_2$ symmetry groups could coexist.

The present work goes a step further. We show that it is possible to obtain duality symmetric actions for odd dimensions also. At the level of potentials, the duality symmetry in a three dimensional Maxwell theory was shown to posses the $SO(2)$ symmetry. Furthermore, by passing to the Fourier modes the $Z_2$ symmetry was also revealed.
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