How can quantum field operators encode entanglement?

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Abstract

We present techniques to construct the Deutsch-Hayden representation for quantum field operators and apply them to an entangled state of identical nonrelativistic spin-1/2 fermions localized in well-separated spatial regions. Using these entangled field operators we construct operators measuring spin in localized spatial regions, and verify that matrix elements of the spin-measurement operators in the information-free Deutsch-Hayden state yield the expected correlations between pairs of both entangled and unentangled particles. The entangled Deutsch-Hayden-representation field operators furnish an explicitly separable description of the entangled system.

Key words: entanglement, quantum field theory, Deutsch-Hayden representation, identical particles, separability, locality.

1 Introduction

The phenomenon of entanglement is commonly regarded as demonstrating that quantum theory is nonlocal, with the example of Einstein-Podolsky-Rosen-Bohm (EPRB) correlations [1], [2, pp. 614–623] most often cited. In the EPRB scenario, perfect anticorrelation between spin measurements on pairs of spatially-separated spin-1/2 particles in the singlet state, an entangled state, occurs when the measurements are made along the same axis for each of the paired particles. From these anticorrelations and the principle of locality — “all physical effects are propagated with finite, subluminal velocities, so that no effects can be communicated between systems separated by a space-like interval” [3] — the conclusion is drawn that information determining the outcomes of spin measurements along any axis that might be measured must reside in each particle (see, e.g., [4–7]), information Mermin has termed “instruction sets” [8].
When, however, the results of spin measurements on different axes for each particle are also taken into account, Bell’s theorem [4] shows that the correlations observed between the results measured on members of singlet-state pairs are in fact inconsistent with the existence of such instruction sets.

From this contradiction between locality and the experimentally-verified (see, e.g., [9] and references therein to earlier experiments) predictions of quantum mechanics the conclusion is generally drawn that the principle of locality does not in fact hold in the physical world (see e.g., [5, 10]). Norsen, for example, states that “what should be concluded from the experimentally observed violations of Bell-type inequalities is [that we must] simply conclude that locality – that the prohibition on faster-than-light causation that seems somehow to be implied by relativity theory – is false. Relativistic local causality is wrong, is in conflict with experimental data. Faster-than-light causal influences really exist in Nature!” [7, p. 234].

Some authors argue that locality can be preserved if one relinquishes instead the principle of separability — “any two spatially separated systems possess their own separate real states” [3] — so that information in some sense resides holistically in both particles. Howard, for example, claims that “these [Bell] experiments should be interpreted as refuting the separability principle” [3]. Brown and Timpson argue that “how a non-separable theory can locally explain Bell-inequality violating correlations would be that the correlations are entailed by some suitable non-separable joint state” [12].

We are not compelled to accept these conclusions regarding the nonlocality or the nonseparability of quantum mechanics if we consider quantum mechanics in the Everett interpretation. Bell’s theorem does not apply to quantum mechanics in the Everett interpretation (see e.g., [13–27]). As Vaidman succinctly puts it, “Bell’s theorem . . . cannot get off the ground in the framework of the [Everett interpretation] because it requires a single outcome of a quantum experiment” [21].

Still, we would like to know: If not as instruction sets, then how is the information that leads to the correlations between entangled particles, both perfect correlations in the same-axis case and imperfect correlations in the different-axis case, encoded in the quantum formalism?

In the present paper we focus on the case of entangled identical particles in nonrelativistic quantum field theory. We consider a state containing three spin-1/2 fermions, two of which are entangled. Correlations between spin measurements on the two entangled particles will be different from those between either of the entangled particles and the third unentangled particle. What are the elements of the quantum-field-theoretic formalism responsible for this difference? How is this difference consistent with the identity of the particles, each of which is but an excitation of the underlying quantum field? Does the information that determines the spin correlations divide up into portions that can be thought of as inhering respectively in individual particles, thus satisfying the principle of separability?
The answers to these questions begin with the observation that to be able to specify that two of the three particles are entangled and the other is not, all the particles must have distinct spatial locations to at least some degree. One cannot speak of “particle 1” or “particle 2” or “particle 3,” but, if the particles are sufficiently well-localized in distinct spatial regions, one can speak of “this particle over here” or “that particle over there.” It is thus natural to look to the field operators as elements of the formalism in which information pertaining to each particle could be encoded, since the field operators are indexed with spatial location from the outset.

However, although the field operators have spatial locations, in the usual representation of quantum field theory they do not by themselves encode information about entanglement or anything else. “The operator corresponding to a given observable represents not the value of the observable, but rather all the values that the observable can assume under various conditions, the values themselves being the eigenvalues... The dynamical variables of a system, being operators, do no represent the system other than generically. That is, they represent not the system as it really is, but rather all the situations in which the system might conceivably find itself... Which situation a system is actually in is specified by the state vector. Reality is therefore described jointly by the dynamical variables and the state vector” [28, p. 182]. To have field operators that in and of themselves encode information, in particular but not limited to information about spin correlations, we must make use of the Deutsch-Hayden representation.

In a seminal paper, Deutsch and Hayden [19] show that quantum computational networks encode and transport information in a local manner. A key step in demonstrating this is to point out that one can perform a unitary transformation to a representation in which the state vector is mapped to a standard state containing no physical information, while the information which is usually encoded in the state vector is transferred to operators. We refer to this representation as the Deutsch-Hayden representation, and to the transformation from the usual representation to the Deutsch-Hayden representation as the Deutsch-Hayden transformation.

To see in detail how information regarding entanglement is encoded in quantum-field-theoretic operators, we will perform a Deutsch-Hayden transformation from an entangled

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1Deutsch and Hayden [19] and Deutsch [20] argue that this proof of locality extends from quantum computational networks to all quantum systems as a consequence of the universality of quantum computation [29] and the Beckenstein bound [30]. Marletto, Tibau Vidal and Vedral [31] question the applicability of this argument to fermions. See also Sec. 5 below.

2In a previous paper [25] I referred to the Deutsch-Hayden “picture.” I now feel that this is inappropriate terminology. The Schrödinger and Heisenberg pictures are different ways of describing time evolution of quantum systems. The Deutsch-Hayden transformation maps all information residing in the state vector at a given time to the operators. If time evolution is subsequently performed using the Heisenberg picture, the state vector will remain the information-free standard state, since the state vector in the Heisenberg picture is constant. So, it is more natural to perform time evolution in the Deutsch-Hayden representation using the Heisenberg picture.
quantum field state and obtain the field operators in the Deutsch-Hayden representation. Our approach to explicitly performing this Deutsch-Hayden transformation and verifying its properties utilizes four key ingredients:

1. Auxiliary fields (Sec. 2.1). These additional fermionic fields, one for each particle in the system, carry no physical information but allow for the effective locality of the Deutsch-Hayden transformation of the field operators (Sec. 3.3), as first pointed out in [25].

2. Widely-separated wavepackets (Sec. 2.2), so that we can distinguish otherwise-identical particles by their approximate spatial location.

3. Two-step procedure (Sec. 4.1) for obtaining the Deutsch-Hayden transformation appropriate to an entangled state using the simpler Deutsch-Hayden transformation appropriate to a “nearby” unentangled state.

4. Localized spin operators (Sec. 2.4) constructed using aperture functions matched to the widely-separated wavepackets, to measure the spin associated with each particle.

This paper is organized as follows. In Sec. 2 we define an unentangled state containing three identical nonrelativistic spin-1/2 fermions localized in three different well-separated regions, with the particle localized in region 1 spin-up and the particles respectively localized in regions 2 and 3 spin-down. We define operators corresponding to the measurement of spin in a specified localized region, verify that the unentangled three-particle state is an eigenstate of these operators, and verify that it has the expected pairwise spin correlations. In Sec. 3 we construct a Deutsch-Hayden transformation for this state, compute the Deutsch-Hayden-transformed field operators, and point out the sense in which, and the conditions under which, this transformation is effectively local. In Sec. 4 we present the two-step procedure for obtaining a Deutsch-Hayden transformation for an entangled state from the Deutsch-Hayden representation for an unentangled state from which it can be generated, present an operator that generates the entangled state, compute the entangled Deutsch-Hayden field operators and localized spin operators, and verify that the latter have the correct expectation values and correlations in the information-free Deutsch-Hayden state. We present and discuss our conclusions in Sec. 5. In Appendix A we calculate spin expectation values and correlations for the corresponding first-quantized system. Appendix B examines the role played by auxiliary fields in the formalism.
2 Unentangled state in the usual representation

2.1 Physical and auxiliary field operators

We will employ the term “usual representation” to refer to states and operators on which a Deutsch-Hayden transformation has not been performed.

We will be working at a single time that we will take to be the time $t = 0$ at which Heisenberg-picture states and operators are equal to their Schrödinger-picture counterparts. The particles in the system are nonrelativistic spin-1/2 fermions; we will associate an index 1 to a particle which is spin-up with respect to the $x_3$ axis, and an index 2 to a particle which is spin-down with respect to this axis. So, a particle with spin-up ($i = 1$) or spin-down ($i = 2$) is created at point $\vec{x}$ at time $t = 0$ by the creation operator $\hat{\phi}_i^\dagger(\vec{x})$ satisfying the anticommutation relations

$$\{\hat{\phi}_i(\vec{x}), \hat{\phi}_j^\dagger(\vec{y})\} = \delta^3(\vec{x} - \vec{y})\delta_{i,j}, \quad i, j = 1, 2, \quad (1)$$

$$\{\hat{\phi}_i(\vec{x}), \hat{\phi}_j(\vec{y})\} = \{\hat{\phi}_i^\dagger(\vec{x}), \hat{\phi}_j^\dagger(\vec{y})\} = 0, \quad i, j = 1, 2, \quad (2)$$

as well as

$$\hat{\phi}_i(\vec{x})|0\rangle = 0, \quad i = 1, 2, \quad (3)$$

where $|0\rangle$ is the usual vacuum state.

In addition to these familiar creation and annihilation field operators, which we will refer to as physical field operators, we introduce auxiliary field operators $\hat{\alpha}_{(i)}^\dagger(\vec{x})$, $i = 1, 2, 3$, satisfying

$$\{\hat{\alpha}_{(i)}(\vec{x}), \hat{\alpha}_{(j)}^\dagger(\vec{y})\} = \delta^3(\vec{x} - \vec{y})\delta_{i,j}, \quad i, j = 1, 2, 3, \quad (4)$$

$$\{\hat{\alpha}_{(i)}(\vec{x}), \hat{\alpha}_{(j)}(\vec{y})\} = \{\hat{\alpha}_{(i)}^\dagger(\vec{x}), \hat{\alpha}_{(j)}^\dagger(\vec{y})\} = 0, \quad i, j = 1, 2, 3, \quad (5)$$

as well as

$$\hat{\alpha}_{(i)}(\vec{x})|0\rangle = 0, \quad i = 1, 2, 3, \quad (6)$$

and anticomuting with the physical field operators:

$$\{\hat{\phi}_i(\vec{x}), \hat{\alpha}_{(j)}(\vec{y})\} = \{\hat{\phi}_i^\dagger(\vec{x}), \hat{\alpha}_{(j)}(\vec{y})\} = \{\hat{\phi}_i(\vec{x}), \hat{\alpha}_{(j)}^\dagger(\vec{y})\} = \{\hat{\phi}_i^\dagger(\vec{x}), \hat{\alpha}_{(j)}^\dagger(\vec{y})\} = 0,$$

$$i = 1, 2, \quad j = 1, 2, 3. \quad (7)$$

We will refer to (1)-(7), including (3) and (6), as the equal-time anticommutation relations (ETARs).
2.2 Widely-separated wavepackets

To construct the initial state we will make use of three complex c-number functions, \( \psi_{\{i\}}(\vec{x}) \), \( i = 1, 2, 3 \), which are normalized,

\[
\int d^3 \vec{x} |\psi_{\{i\}}(\vec{x})|^2 = 1, \quad i = 1, 2, 3,
\]

and which correspond to the three regions in which the particles are localized. We will refer to these as physical wavefunctions. The key property that these functions possess is that they are “widely-separated wavepackets,” i.e., their supports are approximately nonoverlapping:

\[
\psi_{\{i\}}(\vec{x}) \psi_{\{j\}}(\vec{x}) \approx 0, \quad i \neq j, \quad i = 1, 2, 3.
\]

We will refer to (9) as the widely-separated wavepacket (WSW) conditions, and in what follows we will treat them as exact equalities.

2.3 Unentangled state

For the unentangled three-particle state, in the usual representation, we take

\[
|\psi_{\{1,2,3\}}\rangle = \int d^3 \vec{x}_1 d^3 \vec{x}_2 d^3 \vec{x}_3 d^3 \vec{y}_1 d^3 \vec{y}_2 d^3 \vec{y}_3 \psi_{\{1\}}(\vec{x}_1) \psi_{\{2\}}(\vec{x}_2) \psi_{\{3\}}(\vec{x}_3) \psi_{\{1\}}(\vec{y}_1) \psi_{\{2\}}(\vec{y}_2) \psi_{\{3\}}(\vec{y}_3) \hat{\phi}_{\{1\}}^\dagger(\vec{x}_1) \hat{\phi}_{\{2\}}^\dagger(\vec{x}_2) \hat{\phi}_{\{3\}}^\dagger(\vec{x}_3) \hat{\alpha}_{\{1\}}^\dagger(\vec{y}_1) \hat{\alpha}_{\{2\}}^\dagger(\vec{y}_2) \hat{\alpha}_{\{3\}}^\dagger(\vec{y}_3) |0\rangle.
\]

The auxiliary-field wavefunctions, \( \psi_{\{i\}}(\vec{x}) \), \( i = 1, 2, 3 \), are complex c-number functions that are completely arbitrary except for normalization:

\[
\int d^3 \vec{x} |\psi_{\{i\}}(\vec{x})|^2 = 1, \quad i = 1, 2, 3.
\]

In the usual representation, the auxiliary fields do not appear in operators corresponding to measurement of any physical quantity. Since we are always free to work in the usual representation, expectation values in the state (10) of operators that are functions only of physical field operators will be independent of the arbitrary but (importantly) normalized auxiliary-field wavefunctions. This will be true at all times, since auxiliary fields do not appear in the Hamiltonian, and the Hamiltonian only contains even powers of fermionic fields, so they do not evolve in time. For the same reasons it will also be true of expectation values in the entangled state generated from the unentangled state using the operator \( \hat{H} \) introduced in Sec. 4.2, in whatever representation they are computed, as will be seen in the computations presented below.
The physical predictions of the theory thus depend only on the physical operators and wavefunctions and would be unchanged if the auxiliary operators and wavefunctions were removed. However, as was first pointed out in [25] and as will be discussed below in Sec. 3.3, Sec. 5, and Appendix B, the inclusion of the auxiliary fields allows for the construction of an effectively local Deutsch-Hayden transformation.

2.4 Localized spin operators

We now wish to that the state (10) in fact represents three unentangled particles localized in regions 1, 2, and 3, where region $i$ is that volume where the support of $\psi_{\{i\}}(\vec{x})$ is non-negligible, and where the particle in region 1 is spin-up and the particles in regions 2 and 3 are spin-down. To this end we define operators corresponding to the measurement of spin, in units of $\hbar/2$, in a localized region along a specified axis.

The operator measuring, at point $\vec{x}$, the density of spin in the direction of a unit vector $\vec{u}$ is

$$\hat{S}_{\vec{u}}(\vec{x}) = \hat{N}_{\vec{u},1}(\vec{x}) - \hat{N}_{\vec{u},2}(\vec{x}),$$

where

$$\hat{N}_{\vec{u},i}(\vec{x}) = \hat{\phi}^\dagger_{\vec{u},i}(\vec{x})\hat{\phi}_{\vec{u},i}(\vec{x}), \quad i = 1, 2$$

are the number densities for particles which are spin-up ($i = 1$) and spin-down ($i = 2$) along $\vec{u}$, and where, referring to [32, eqs. (A2a),(A2b)], the field operators for spin defined along $\vec{u}$ ($\hat{\phi}_{\vec{u},i}, i = 1, 2$) are related to those for spin along $x_3$ ($\hat{\phi}_{i}, i = 1, 2$) by the relations

$$\hat{\phi}^\dagger_{\vec{u},1}(\vec{x}) = e^{-i\phi/2} \cos(\theta/2)\hat{\phi}^\dagger_1(\vec{x}) + e^{i\phi/2} \sin(\theta/2)\hat{\phi}^\dagger_2(\vec{x}),$$

$$\hat{\phi}^\dagger_{\vec{u},2}(\vec{x}) = -e^{-i\phi/2} \sin(\theta/2)\hat{\phi}^\dagger_1(\vec{x}) + e^{i\phi/2} \cos(\theta/2)\hat{\phi}^\dagger_2(\vec{x})$$

and their adjoints, with

$$\vec{u} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

Next, define three “aperture functions,” each corresponding to one of the regions in which the particles are localized. These functions take on at each point $\vec{x}$ either the value 0 or the value 1 and have nonoverlapping support:

$$A_{\{i\}}(\vec{x})A_{\{j\}}(\vec{x}) = \delta_{i,j}A_{\{j\}}(\vec{x}), \quad i, j = 1, 2, 3.$$ (17)

Each aperture function is matched to a physical wavepacket, in that it has value 1 where the wavepacket has nonnegligible value and 0 elsewhere, so

$$A_{\{i\}}(\vec{x})\psi_{\{j\}}(\vec{x}) \approx \delta_{i,j}\psi_{\{j\}}(\vec{x}), \quad i, j = 1, 2, 3.$$ (18)

3So named to suggest the aperture of a physical device that measures the spin in a region, say by detecting the associated magnetic field of the particles.
We will treat (18) as an exact equality. Note that, with the normalization conditions (8), (18) implies
\[ \int d^3 \bar{x} A_i(\bar{x}) |\psi_j(\bar{x})|^2 = \delta_{i,j}, \quad i, j = 1, 2, 3. \] (19)

We will refer to (17)-(19) as the aperture conditions.

Using the aperture functions, define the operators measuring spin along \( \vec{u}_{(i)} \) in region \( i \):
\[ \hat{S}_{\vec{u}_{(i)},i} = \hat{N}_{\vec{u}_{(i)},1,i} - \hat{N}_{\vec{u}_{(i)},2,i}, \quad i = 1, 2, 3, \] (20)

where
\[ \vec{u}_{(i)} = (\sin \theta_{(i)} \cos \phi_{(i)}, \sin \theta_{(i)} \sin \phi_{(i)}, \cos \theta_{(i)}), \quad i = 1, 2, 3. \] (21)

\[ \hat{N}_{\vec{u}_{(i)},j,i} = \int d^3 \bar{x} A_i(\bar{x}) \hat{N}_{\vec{u},j}(\bar{x}), \quad j = 1, 2, \quad i = 1, 2, 3. \] (22)

Using (13)-(15) and (20)-(22),
\[ \hat{S}_{\vec{u}_{(i)},i} = \int d^3 \bar{x} A_i(\bar{x}) \left[ \cos \theta_{(i)} \left( \hat{\phi}_1(\bar{x}) \hat{\phi}_1(\bar{x}) - \hat{\phi}_2(\bar{x}) \hat{\phi}_2(\bar{x}) \right) \right. \]
\[ \left. + \sin \theta_{(i)} \left( e^{i\phi_{(i)}} \hat{\phi}_2(\bar{x}) \hat{\phi}_1(\bar{x}) + e^{-i\phi_{(i)}} \hat{\phi}_1(\bar{x}) \hat{\phi}_2(\bar{x}) \right) \right]. \] (23)

Using the ETARs (1)-(7), (10), and the aperture conditions (17)-(19) with (23),
\[ \hat{S}_{\vec{u}_{(1)},1} |\psi_{1,(1),2,(2);2,(3)}\rangle = \cos \theta_{(1)} |\psi_{1,(1),2,(2);2,(3)}\rangle \]
\[ + \sin \theta_{(1)} e^{i\phi_{(1)}} |\psi_{2,(1),2,(2);2,(3)}\rangle \] (24)
\[ \hat{S}_{\vec{u}_{(2)},2} |\psi_{1,(1),2,(2);2,(3)}\rangle = -\cos \theta_{(2)} |\psi_{1,(1),2,(2);2,(3)}\rangle \]
\[ + \sin \theta_{(2)} e^{-i\phi_{(2)}} |\psi_{1,(1),1,(2);2,(3)}\rangle \] (25)
\[ \hat{S}_{\vec{u}_{(3)},3} |\psi_{1,(1),2,(2);2,(3)}\rangle = -\cos \theta_{(3)} |\psi_{1,(1),2,(2);2,(3)}\rangle \]
\[ + \sin \theta_{(3)} e^{-i\phi_{(3)}} |\psi_{1,(1),2,1,(3)}\rangle \] (26)

where
\[ |\psi_{2,(1),2,(2);2,(3)}\rangle = \int d^3 \bar{x}_1 d^3 \bar{x}_2 d^3 \bar{x}_3 d^3 \bar{y}_1 d^3 \bar{y}_2 d^3 \bar{y}_3 \]
\[ \psi_{(1)}(\bar{x}_1) \psi_{(2)}(\bar{x}_2) \psi_{(3)}(\bar{x}_3) \psi_{(1)}(\bar{y}_1) \psi_{(2)}(\bar{y}_2) \psi_{(3)}(\bar{y}_3) \]
\[ \hat{\phi}_2(\bar{x}_1) \hat{\phi}_2(\bar{x}_2) \hat{\phi}_2(\bar{x}_3) \hat{\alpha}^+_{(1)}(\bar{y}_1) \hat{\alpha}^+_{(2)}(\bar{y}_2) \hat{\alpha}^+_{(3)}(\bar{y}_3) |0\rangle \] (27)
\[ |\psi_{1,(1),1,(2);2,(3)}\rangle = \int d^3 \bar{x}_1 d^3 \bar{x}_2 d^3 \bar{x}_3 d^3 \bar{y}_1 d^3 \bar{y}_2 d^3 \bar{y}_3 \]
\[ \psi_{(1)}(\bar{x}_1) \psi_{(2)}(\bar{x}_2) \psi_{(3)}(\bar{x}_3) \psi_{(1)}(\bar{y}_1) \psi_{(2)}(\bar{y}_2) \psi_{(3)}(\bar{y}_3) \]
\[ \hat{\phi}_1(\bar{x}_1) \hat{\phi}_1(\bar{x}_2) \hat{\phi}_1(\bar{x}_3) \hat{\alpha}^+_{(1)}(\bar{y}_1) \hat{\alpha}^+_{(2)}(\bar{y}_2) \hat{\alpha}^+_{(3)}(\bar{y}_3) |0\rangle \] (28)
\[ |\psi_{1,(1),2,(2),1,(3)}\rangle = \int d^3 \vec{x}_1 d^3 \vec{x}_2 d^3 \vec{x}_3 d^3 \vec{y}_1 d^3 \vec{y}_2 d^3 \vec{y}_3 \]
\[ \psi_1(\vec{x}_1) \psi_2(\vec{x}_2) \psi_3(\vec{x}_3) \psi(\vec{y}_1) \psi(\vec{y}_2) \psi(\vec{y}_3) \]
\[ \hat{\mathbf{y}}_1^\dagger(\vec{x}_1) \hat{\mathbf{y}}_2^\dagger(\vec{x}_2) \hat{\mathbf{y}}_3^\dagger(\vec{x}_3) \hat{\mathbf{y}}_1^\dagger(\vec{y}_1) \hat{\mathbf{y}}_2^\dagger(\vec{y}_2) \hat{\mathbf{y}}_3^\dagger(\vec{y}_3)|0\rangle \]  

(29)

### 2.5 Eigenvalue-eigenvector link

If in each region we utilize localized spin operators measuring spin along the \( x_3 \) axis, i.e., we take
\[ \theta_{(1)} = \theta_{(2)} = \theta_{(3)} = 0, \]
then, defining
\[ \vec{x}_3 = (0, 0, 1) \]
and using (21) and (24)- (26), we obtain
\[ \hat{S}_{x_3,(1)} |\psi_{1,(1),2,(2),2,(3)}\rangle = |\psi_{1,(1),2,(2),2,(3)}\rangle \]  
\[ \hat{S}_{x_3,(2)} |\psi_{1,(1),2,(2),2,(3)}\rangle = -|\psi_{1,(1),2,(2),2,(3)}\rangle \]  
\[ \hat{S}_{x_3,(3)} |\psi_{1,(1),2,(2),2,(3)}\rangle = -|\psi_{1,(1),2,(2),2,(3)}\rangle \]  

That is, the unentangled state \( |\psi_{1,(1),2,(2),2,(3)}\rangle \) is a spin-up eigenstate of the operator measuring spin along the \( x_3 \) axis in regions 1, and a spin-down eigenstate of the operators measuring spin along the \( x_3 \) axis in regions 2 and 3.

### 2.6 Pairwise spin correlations

For arbitrary directions \( \vec{u}_{(1)}, \vec{u}_{(2)}, \vec{u}_{(3)} \), the pairwise correlations between spins in different regions in the state \( |\psi_{1,(1),2,(2),2,(3)}\rangle \) can be computed using (21)– (26). From (23) we see that \( \hat{S}_{\vec{u}_{(i)}} \) is Hermitian, and from the ETARs (11)–(7), the WSW conditions (9) and the normalizations (8), (11) it follows that the four vectors \( |\psi_{1,(1),2,(2),2,(3)}\rangle, |\psi_{2,(1),2,(2),2,(3)}\rangle, |\psi_{1,(1),1,(2),2,(3)}\rangle \) and \( |\psi_{1,(1),2,(2),1,(3)}\rangle \) are normalized and mutually orthogonal. Using in addition the aperture conditions (17)–(19) we obtain
\[ \langle \psi_{1,(1),2,(2),2,(3)} | \hat{S}_{\vec{u}_{(1)},(1)} \hat{S}_{\vec{u}_{(2)},(2)} | \psi_{1,(1),2,(2),2,(3)} \rangle = -u_{(1),3}u_{(2),3} \]
\[ \langle \psi_{1,(1),2,(2),2,(3)} | \hat{S}_{\vec{u}_{(2)},(2)} \hat{S}_{\vec{u}_{(3)},(3)} | \psi_{1,(1),2,(2),2,(3)} \rangle = u_{(2),3}u_{(3),3} \]
\[ \langle \psi_{1,(1),2,(2),2,(3)} | \hat{S}_{\vec{u}_{(3)},(3)} \hat{S}_{\vec{u}_{(1)},(1)} | \psi_{1,(1),2,(2),2,(3)} \rangle = -u_{(3),3}u_{(1),3} \]  

These are the correlations we expect to see for three unentangled spins with one spin-up and two spin-down along the \( x_3 \) axis; see Appendix A (and set to zero the entangling coupling \( \lambda \) that appears there.)
3 Deutsch-Hayden representation for unentangled state

3.1 Deutsch-Hayden transformation

As mentioned in Sec. 1, the Deutsch-Hayden transformation is a unitary transformation that takes the state at \( t = 0 \) to a standard state containing no information about the physical state of the system. For quantum field theory a natural choice for such a standard state is the vacuum state, and that is the choice we make here. We therefore wish to construct a unitary operator \( \hat{V} \) with the property

\[
\hat{V} |\psi_{1,1};2,2,2\rangle = |0\rangle. \tag{38}
\]

We will construct \( \hat{V} \) sequentially as a product of three unitary operators each of which removes a particle in one region. Define

\[
\hat{W}_{1,\{1\}} = g_1 \int d^3\vec{x} d^3\vec{y} \left( \psi_{\{1\}}^\dagger (\vec{x}) \psi_{\{1\}} (\vec{y}) \hat{\alpha}_{\{1\}} (\vec{y}) \hat{\beta}_1 (\vec{x}) - \psi_{\{1\}} (\vec{x}) \psi_{\{1\}}^\dagger (\vec{y}) \hat{\beta}_1^\dagger (\vec{y}) \hat{\alpha}_{\{1\}}^\dagger (\vec{x}) \right) \tag{39}
\]

with the constant \( g_1 \) real,

\[
g_1^* = g_1, \tag{40}
\]

so \( \hat{W}_{1,\{1\}} \) is skew-Hermitian,

\[
\hat{W}_{1,\{1\}}^\dagger = -\hat{W}_{1,\{1\}}. \tag{41}
\]

Applying the ETARs (1)-(7), in particular the consequences of the fermionic nature of the field operators such as

\[
\int d^3\vec{x} d^3\vec{y} \psi_{\{1\}} (\vec{x}) \psi_{\{1\}}^\dagger (\vec{y}) \hat{\alpha}_{\{1\}} (\vec{x}) \hat{\beta}_1^\dagger (\vec{y}) = 0, \tag{42}
\]

as well as the normalization conditions (8), (11), we find the action of \( \hat{W}_{1,\{1\}} \) on the unentangled three-particle state to be

\[
\hat{W}_{1,\{1\}} |\psi_{1,\{1\}};2,2,2\rangle = g_1 |\psi_{2,2,2}\rangle, \tag{43}
\]

where

\[
|\psi_{2,2,2}\rangle = \int d^3\vec{x}_2 d^3\vec{x}_3 d^3\vec{y}_2 d^3\vec{y}_3 \psi_{\{2\}} (\vec{x}_2) \psi_{\{2\}} (\vec{x}_3) \psi_{\{3\}} (\vec{y}_2) \psi_{\{3\}} (\vec{y}_3) \hat{\alpha}_{\{2\}}^\dagger (\vec{x}_2) \hat{\alpha}_{\{3\}}^\dagger (\vec{x}_3) \hat{\beta}_2^\dagger (\vec{y}_2) \hat{\beta}_3^\dagger (\vec{y}_3) |0\rangle. \tag{44}
\]

Application of \( \hat{W}_{1,\{1\}} \) to \( |\psi_{2,2,2}\rangle \) gives

\[
\hat{W}_{1,\{1\}} |\psi_{2,2,2}\rangle = -g_1 |\psi_{1,\{1\}};2,2,2\rangle. \tag{45}
\]

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So, if we define, for $\theta_1$ a real number,

$$\hat{V}(\theta_1)_{1,\{1\}} = \exp \left( \theta_1 \hat{W}_{1,\{1\}} \right),$$  \hspace{1cm} (46)

we find, using (43), (45) and mathematical induction, that

$$\hat{V}(\theta_1)_{1,\{1\}} |\psi_{1,\{1\};2,\{2\};2,\{3\}\rangle = \cos (\theta_1 g_1) |\psi_{1,\{1\};2,\{2\};2,\{3\}\rangle + \sin (\theta_1 g_1) |\psi_{2,\{2\};2,\{3\}\rangle. \hspace{1cm} (47)$$

Choose $\theta_1$ and $g_1$ so that

$$\cos (\theta_1 g_1) = 0,$$  \hspace{1cm} (48)

and define

$$s_1 = \sin (\theta_1 g_1) = \pm 1.$$  \hspace{1cm} (49)

Defining

$$\hat{V}_{1,\{1\}} = \hat{V}(\theta_1)_{1,\{1\}} \hspace{1cm} (50)$$

where $\theta_1$ and $g_1$ satisfy (48), (49), we obtain

$$\hat{V}_{1,\{1\}} |\psi_{1,\{1\};2,\{2\};2,\{3\}\rangle = s_1 |\psi_{2,\{2\};2,\{3\}\rangle. \hspace{1cm} (51)$$

Similarly, define

$$\hat{W}_{2,\{2\}} = g_2 \int d^3x d^3\tilde{y} \left( \psi^*_{\{2\}}(\tilde{x}) \psi^*_{\{2\}}(\tilde{y}) \hat{\alpha}_{\{2\}}(\tilde{y}) \hat{\phi}_2(\tilde{x}) - \psi_{\{2\}}(\tilde{x}) \psi_{\{2\}}(\tilde{y}) \hat{\phi}_2^\dagger(\tilde{x}) \hat{\alpha}_{\{2\}}^\dagger(\tilde{y}) \right). \hspace{1cm} (52)$$

Because both regions 2 and 3 have particles of the same spin we must make use of the WSW conditions (9) in addition to the ETARs (1)-(7) and normalization conditions (8), (11) to obtain

$$\hat{W}_{2,\{2\}} |\psi_{2,\{2\};2,\{3\}\rangle = -g_2 |\psi_{2,\{3\}\rangle, \hspace{1cm} (53)$$

$$\hat{W}_{2,\{2\}} |\psi_{2,\{3\}\rangle = g_2 |\psi_{2,\{2\};2,\{3\}\rangle, \hspace{1cm} (54)$$

where

$$|\psi_{2,\{3\}\rangle = \int d^3\tilde{x} d^3\tilde{y} \psi_{\{3\}}(\tilde{x}_3) \psi_{\{3\}}(\tilde{y}_3) \hat{\phi}_{2,\{3\}}^\dagger(\tilde{x}_3) \hat{\alpha}_{\{3\}}^\dagger(\tilde{y}_3) |0\rangle. \hspace{1cm} (55)$$

Exponentiating $\hat{W}_{2,\{2\}}$, we have

$$\hat{V}(\theta_2)_{2,\{2\}} = \exp \left( \theta_2 \hat{W}_{2,\{2\}} \right), \hspace{1cm} (56)$$

whence

$$\hat{V}(\theta_2)_{2,\{2\}} |\psi_{2,\{2\};2,\{3\}\rangle = \cos (\theta_2 g_2) |\psi_{2,\{2\};2,\{3\}\rangle - \sin (\theta_1 g_1) |\psi_{2,\{3\}\rangle. \hspace{1cm} (57)$$

Choosing $\theta_2$ and $g_2$ so that

$$\cos (\theta_2 g_2) = 0 \hspace{1cm} (58)$$
and defining
\[ s_2 = \sin(\theta_2 g_2) = \pm 1, \tag{59} \]
it follows that
\[ \hat{V}_{2,\{2\}}|\psi_{2,\{2\},\{3\}}\rangle = -s_2|\psi_{2,\{3\}}\rangle, \tag{60} \]
where
\[ \hat{V}_{2,\{2\}} = \hat{V}(\theta_2)_{2,\{2\}} \tag{61} \]
with \( \theta_2 \) and \( g_2 \) satisfying (58), (59).

Finally, to remove the spin-down particle in region 3, define
\[ \hat{W}_{2,\{3\}} = g_3 \int d^3 x d^3 y \left( \psi^*_{\{3\}}(\vec{x})\psi^*_{\{3\}}(\vec{y})\hat{\alpha}_{\{3\}}(\vec{x})\hat{\phi}_2(\vec{x}) - \psi_{\{3\}}(\vec{x})\psi_{\{3\}}(\vec{y})\hat{\alpha}^\dagger_{\{3\}}(\vec{x})\hat{\phi}^\dagger_2(\vec{x}) \right), \tag{62} \]
satisfying
\[ \hat{W}_{2,\{3\}}|\psi_{2,\{3\}}\rangle = g_3|0\rangle \tag{63} \]
\[ \hat{W}_{2,\{3\}}|0\rangle = -g_3|\psi_{2,\{3\}}\rangle, \tag{64} \]
so
\[ \hat{V}(\theta_3)_{2,\{3\}} = \exp \left( \theta_3 \hat{W}_{2,\{3\}} \right) \tag{65} \]
satisfies
\[ \hat{V}(\theta_3)_{2,\{3\}}|\psi_{2,\{3\}}\rangle = \cos(\theta_3 g_3)|\psi_{2,\{3\}}\rangle + \sin(\theta_3 g_3)|0\rangle. \tag{66} \]
Choosing \( \theta_3 \) and \( g_3 \) so that
\[ \cos(\theta_3 g_3) = 0, \tag{67} \]
and defining
\[ s_3 = \sin(\theta_3 g_3) = \pm 1 \tag{68} \]
and
\[ \hat{V}_{2,\{3\}} = \hat{V}(\theta_3)_{2,\{3\}} \tag{69} \]
with \( \theta_3 \) and \( g_3 \) satisfying (67), (68), we have
\[ \hat{V}_{2,\{3\}}|\psi_{2,\{3\}}\rangle = s_3|0\rangle. \tag{70} \]

Defining
\[ \hat{V} = \hat{V}_{2,\{3\}}\hat{V}_{2,\{2\}}\hat{V}_{1,\{1\}} \tag{71} \]
it follows from (46), (56), (55) and the skew-Hermiticity of \( \hat{W}_{1,\{1\}}, \hat{W}_{2,\{2\}} \) and \( \hat{W}_{2,\{3\}} \) that \( \hat{V} \) is unitary, and from (51), (60) and (70) that
\[ \hat{V}|\psi_{1,\{1\},\{2\},\{2\},\{3\}}\rangle = -s_1 s_2 s_3|0\rangle. \tag{72} \]
So, provided the \( \theta \)'s and \( g \)'s are chosen so that
\[ s_1 s_2 s_3 = -1, \tag{73} \]
\( \hat{V} \) implements the Deutsch-Hayden transformation (38).
3.2 Unentangled operators in the Deutsch-Hayden representation

Since the state transforms according to (38), the physical operators are transformed to the Deutsch-Hayden representation by the relations

$$\hat{\phi}_{i,DH}(\vec{x}) = \hat{V}\hat{\phi}_{i}(\vec{x})\hat{V}^\dagger, \quad i = 1, 2.$$  \hspace{1cm} (74)

To explicitly relate the operators (74) in the Deutsch-Hayden representation to those in the usual representation, we make use of the formula [33, p. 222]

$$e^{-y\hat{F}}G e^{y\hat{F}} = G + y \left[\hat{G}, \hat{F}\right] + \frac{y^2}{2!} \left[\left[\hat{G}, \hat{F}\right], \hat{F}\right] + \frac{y^3}{3!} \left[\left[\left[\hat{G}, \hat{F}\right], \hat{F}\right], \hat{F}\right] + \ldots$$ \hspace{1cm} (75)

Using this with the ETARs (1), (2), (4), (5), and (7), the normalization conditions (8), (11), the unitary-operator definitions (46), (56), (65), (71), the parameter conditions (48), (58), (67), the definitions (49), (59), (68) and mathematical induction, (74) becomes, for $i = 1$,

$$\hat{\phi}_{1,DH}(\vec{x}) = \hat{\phi}_{1}(\vec{x}) + \psi_{\{1\}}(\vec{x}) \int d^3\vec{x}'\psi_{\{1\}}^*(\vec{x}')\hat{\phi}_{1}(\vec{x}') + s_1 \int d^3\vec{x}'\psi_{\{1\}}(\vec{x}')\hat{\alpha}_{\{1\}}^\dagger(\vec{x}') + s_2 \int d^3\vec{x}'\psi_{\{2\}}(\vec{x}')\hat{\alpha}_{\{2\}}^\dagger(\vec{x}') + s_3 \int d^3\vec{x}'\psi_{\{3\}}(\vec{x}')\hat{\alpha}_{\{3\}}^\dagger(\vec{x}') \hspace{1cm} (76)$$

Using in addition the WSW conditions (9), we obtain from (74) for $i = 2$

$$\hat{\phi}_{2,DH}(\vec{x}) = \hat{\phi}_{2}(\vec{x}) + \psi_{\{2\}}(\vec{x}) \int d^3\vec{x}'\psi_{\{2\}}^*(\vec{x}')\hat{\phi}_{2}(\vec{x}') + s_2 \int d^3\vec{x}'\psi_{\{2\}}(\vec{x}')\hat{\alpha}_{\{2\}}^\dagger(\vec{x}') + \psi_{\{3\}}(\vec{x}) \int d^3\vec{x}'\psi_{\{3\}}^*(\vec{x}')\hat{\phi}_{2}(\vec{x}') + s_3 \int d^3\vec{x}'\psi_{\{3\}}(\vec{x}')\hat{\alpha}_{\{3\}}^\dagger(\vec{x}') + s_3 \int d^3\vec{x}'\psi_{\{3\}}(\vec{x}')\hat{\alpha}_{\{3\}}^\dagger(\vec{x}') \hspace{1cm} (77)$$

It will be useful (see Sec. 4.4) to have the action of the Deutsch-Hayden physical operators on the vacuum state. From (76), (77) and the ETARs (1)-(7),

$$\hat{\phi}_{1,DH}(\vec{x})|0\rangle = s_1\psi_{\{1\}}(\vec{x}) \int d^3\vec{x}'\psi_{\{1\}}^*(\vec{x}')\hat{\alpha}_{\{1\}}^\dagger(\vec{x}')|0\rangle, \hspace{1cm} (78)$$

$$\hat{\phi}_{2,DH}(\vec{x})|0\rangle = s_2\psi_{\{2\}}(\vec{x}) \int d^3\vec{x}'\psi_{\{2\}}^*(\vec{x}')\hat{\alpha}_{\{2\}}^\dagger(\vec{x}')|0\rangle + s_3\psi_{\{3\}}(\vec{x}) \int d^3\vec{x}'\psi_{\{3\}}^*(\vec{x}')\hat{\alpha}_{\{3\}}^\dagger(\vec{x}')|0\rangle, \hspace{1cm} (79)$$

$$\hat{\phi}_{1,DH}^\dagger(\vec{x})|0\rangle = \hat{\phi}_{1}^\dagger(\vec{x})|0\rangle - s_1\psi_{\{1\}}^*(\vec{x}) \int d^3\vec{x}'\psi_{\{1\}}^*(\vec{x}')\hat{\phi}_{1}(\vec{x}')|0\rangle, \hspace{1cm} (80)$$

$$\hat{\phi}_{2,DH}^\dagger(\vec{x})|0\rangle = \hat{\phi}_{2}^\dagger(\vec{x})|0\rangle - s_1\psi_{\{2\}}^*(\vec{x}) \int d^3\vec{x}'\psi_{\{2\}}^*(\vec{x}')\hat{\phi}_{2}(\vec{x}')|0\rangle - s_2\psi_{\{3\}}^*(\vec{x}) \int d^3\vec{x}'\psi_{\{3\}}^*(\vec{x}')\hat{\phi}_{2}(\vec{x}')|0\rangle. \hspace{1cm} (81)$$
3.3 Effective locality of the unentangled Deutsch-Hayden transformation

From (76), (77) we see that Deutsch-Hayden-transformed physical operators differ significantly from their values in the usual representation only at locations where quanta corresponding to those operators are present. By a field operator \(\hat{\phi}_j(\vec{x})\) “corresponding” to particle \(i\) we mean that \(\vec{x}\) is in the effective support of the wavefunction \(\psi_{(i)}(\vec{x})\) and the spin of particle \(i\) is \(j\).

For example, examining eq. (77) and noting the factors of \(\psi_{|2}(\vec{x})\) and \(\psi_{|3}(\vec{x})\) in front of the square brackets, we see that \(\hat{\phi}_{2,DH}(\vec{x})\) can differ significantly from \(\hat{\phi}_2(\vec{x})\) only where \(\vec{x}\) is in the effective support of either \(\psi_{|2}\) or \(\psi_{|3}\), i.e., at \(\vec{x}\) where either \(\psi_{|2}(\vec{x})\) or \(\psi_{|3}(\vec{x})\) is non-negligible. (From the WSW conditions (9) we know that at most one of these functions can be nonnegligible at any particular \(\vec{x}\).)

Furthermore, for \(\vec{x}\) in a given region, the difference between \(\hat{\phi}_{2,DH}(\vec{x})\) and \(\hat{\phi}_2(\vec{x})\) only depends on the values of wavefunctions in that region. E.g., if \(\vec{x}\) is in the effective support of \(\psi_{|2}\), the difference between \(\hat{\phi}_{2,DH}(\vec{x})\) and \(\hat{\phi}_2(\vec{x})\) depends only on the wavefunction \(\psi_{|2}\), not on \(\psi_{|3}\), since \(\psi_{|2}\) but not \(\psi_{|3}\) appears in the integral multiplying \(\psi_{|2}(\vec{x})\).

However, for the Deutsch-Hayden transformation of the physical operators to be “effectively local” in the sense that the difference between an operator at \(\vec{x}\) and the Deutsch-Hayden-transformed version of that operator only depends on physical information near \(\vec{x}\), i.e., on the values of physical wavefunctions near \(\vec{x}\), we must impose, in addition to the WSW conditions, the requirement that the effective support of each physical wavefunction is concentrated in a single localized volume, e.g., a narrow Gaussian. Imposing this additional requirement, letting \(\vec{x}_{|i}\) be a point about which the effective support of \(\psi_{|i}\) is localized (i.e., the only points \(\vec{x}\) where \(\psi_{|i}(\vec{x})\) is of significant magnitude are those \(\vec{x}\) close to \(\vec{x}_{|i}\)), and returning to the the example of eq. (77), we see that if \(\vec{x}\) is within the effective support of \(\psi_{|2}(\vec{x})\), then the difference between \(\hat{\phi}_{2,DH}(\vec{x})\) and \(\hat{\phi}_2(\vec{x})\) only comes significantly from values of wavefunctions near \(\vec{x}_{|2}\): The physical wavefunction \(\psi_{|2}(\vec{x})\) is only significantly different from zero for \(\vec{x} \approx \vec{x}_{|2}\); and the integral \(\int d^3\vec{x}' \psi_{|2}^*(\vec{x}')\hat{\phi}_2(\vec{x}')\) that multiplies \(\psi_{|2}(\vec{x})\) only receives significant contribution from parts of the integrand within the effective support of \(\psi_{|2}(\vec{x})\), i.e., from the parts with \(\vec{x}' \approx \vec{x}_{|2}\). Without imposing the additional localized-volume requirement, the difference between \(\hat{\phi}_{2,DH}(\vec{x})\) and \(\hat{\phi}_2(\vec{x})\) would depend on \(\psi_{|2}^*(\vec{x}')\) at all \(\vec{x}'\) within the effective support of \(\psi_{|2}\), not just at those \(\vec{x}'\) for which \(\vec{x}' \approx \vec{x}\).

It is the desire to employ a Deutsch-Hayden transformation that is effectively local that leads us to include the auxiliary fields in the formalism, as discussed in Appendix B. The significance of having a Deutsch-Hayden transformation that is effectively local is discussed

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4In this section we distinguish between a function and its value for a particular value of its argument, e.g., \(\psi_{|1}\) and \(\psi_{|1}(\vec{x})\).
in Sec. 5.

4 Deutsch-Hayden representation for entangled state

4.1 Entangled operators in the Deutsch-Hayden representation from “time-evolved” operators in the usual representation

We wish to examine field operators in a Deutsch-Hayden representation for an entangled state. The process of obtaining these field operators is somewhat simplified if we can express the entangled state as the “time-evolved” version of an unentangled state for which we already know a Deutsch-Hayden transformation.

Denote the unentangled state by $|\psi_{un}\rangle$, and the Deutsch-Hayden transformation that maps $|\psi_{un}\rangle$ to the vacuum state by $\hat{V}_{un}$:

$$\hat{V}_{un}|\psi_{un}\rangle = |0\rangle.$$  \hfill (82)

Let $\hat{H}_{u-e}$ be a Hamiltonian that, in the Schrödinger picture, acting over a time interval $\Delta t$ takes the unentangled state $|\psi_{un}\rangle$ to the entangled state $|\psi_{en}\rangle$; i.e.,

$$|\psi_{en}\rangle = \exp\left(-\frac{i\hat{H}_{u-e}\Delta t}{\hbar}\right)|\psi_{un}\rangle.$$  \hfill (83)

Then a Deutsch-Hayden transformation for $|\psi_{en}\rangle$, i.e., a unitary transformation that maps $|\psi_{en}\rangle$ to the vacuum state, is

$$\hat{V}_{en} = \hat{V}_{un} \exp\left(\frac{i\hat{H}_{u-e}\Delta t}{\hbar}\right),$$  \hfill (84)

since, from (83) and (84),

$$\hat{V}_{en}|\psi_{en}\rangle = \hat{V}_{un} \exp\left(\frac{i\hat{H}_{u-e}\Delta t}{\hbar}\right) \exp\left(-\frac{i\hat{H}_{u-e}\Delta t}{\hbar}\right)|\psi_{un}\rangle$$

$$= \hat{V}_{un}|\psi_{un}\rangle$$

$$= |0\rangle$$  \hfill (85)

using (82).

Let $\tilde{\chi}$ denote an arbitrary Schrödinger-picture operator in the usual representation. Applying the entangled-state Deutsch-Hayden transformation (84), the Deutsch-Hayden representation of $\tilde{\chi}$ is

$$\tilde{\chi}_{DH,en} = \hat{V}_{en}\tilde{\chi}\hat{V}_{en}^\dagger$$

$$= \hat{V}_{un}\tilde{\chi}(\Delta t)\hat{V}_{un}^\dagger,$$  \hfill (86)
where
\[
\hat{\chi}(\Delta t) = \exp \left( \frac{i \hat{H}_{u-e} \Delta t}{\hbar} \right) \hat{\chi} \exp \left( -\frac{i \hat{H}_{u-e} \Delta t}{\hbar} \right)
\] (87)
is recognized as the usual-representation Heisenberg-picture operator, at time \( t = \Delta t \), in that Heisenberg picture in which states and operators are equal to their Schrödinger-picture counterparts at time \( t = 0 \).

Note, however, that, although we refer to \( \hat{H}_{u-e} \) as a Hamiltonian, this is not necessarily the physical Hamiltonian that effects the actual Schrödinger-picture time evolution of the state \( |\psi_{un}\rangle \). Indeed, below (Sec. 4.2) we will take \( \hat{H}_{u-e} \) to be explicitly nonlocal (eq. (94)). We will continue to describe the transformation relating \( |\psi_{un}\rangle \) to \( |\psi_{en}\rangle \) as “time evolution,” but keep in mind that the state \( |\psi_{en}\rangle \) and the operator \( \hat{\chi}_{DH,en} \) are, respectively a Schrödinger-picture state at time \( t = 0 \) and a Schrödinger-picture operator at time \( t = 0 \), equal of course to their Heisenberg-picture counterparts at time \( t = 0 \).

Further computational simplification can be anticipated if we focus on “slightly entangled” states, those which are close to the unentangled states from which they arise via (83). That is, we approximate
\[
\exp \left( -\frac{i \hat{H}_{u-e} \Delta t}{\hbar} \right) \approx 1 - \frac{i \hat{H}_{u-e} \Delta t}{\hbar},
\] (88)
thereby allowing us to replace (87) with
\[
\hat{\chi}(\Delta t) = \hat{\chi} + \frac{i \Delta t}{\hbar} [\hat{H}_{u-e}, \hat{\chi}].
\] (89)

Eq. (86) then becomes
\[
\hat{\chi}_{DH,en} = \hat{\chi}_{DH} + \frac{i \Delta t}{\hbar} [\hat{H}_{u-e,DH}, \hat{\chi}_{DH}],
\] (90)
where
\[
\hat{\chi}_{DH} = \hat{V}_{un} \hat{\chi} \hat{V}_{un}^\dagger,
\] (91)
\[
\hat{H}_{u-e,DH} = \hat{V}_{un} \hat{H}_{u-e} \hat{V}_{un}^\dagger.
\] (92)

### 4.2 Generation of entanglement

For a Hamiltonian which, in the Schrödinger-picture, would cause the unentangled state \( |\psi_{1,\{1\},2,\{2\},2,\{3\}}\rangle \) to evolve over a time interval \( \Delta t \) to an entangled state \( |\psi_{1,\{1\},2,\{2\},2,\{3\}}(\Delta t)\rangle \), i.e., \( \hat{H} \) such that
\[
|\psi_{1,\{1\},2,\{2\},2,\{3\}}(\Delta t)\rangle = e^{-\frac{i \hat{H} \Delta t}{\hbar}} |\psi_{1,\{1\},2,\{2\},2,\{3\}}\rangle,
\] (93)
we choose
\[ \hat{H} = -i\lambda \int d^3 \bar{x}_1 d^3 \bar{x}_2 d^3 \bar{z}_1 d^3 \bar{z}_2 \]
\[ (\psi_{(1)}(\bar{x}_1)\psi_{(2)}(\bar{x}_2)\psi_{(1)}^*(\bar{z}_1)\psi_{(2)}^*(\bar{z}_2)\hat{\phi}_1^\dagger(\bar{x}_1)\hat{\phi}_2^\dagger(\bar{x}_2)\hat{\phi}_1(\bar{z}_1)\hat{\phi}_2(\bar{z}_2) ) \]
\[ -\psi_{(1)}^*(\bar{x}_1)\psi_{(2)}^*(\bar{x}_2)\psi_{(1)}(\bar{z}_1)\psi_{(2)}(\bar{z}_2)\hat{\phi}_1^\dagger(\bar{z}_1)\hat{\phi}_2^\dagger(\bar{z}_2)\hat{\phi}_1(\bar{x}_1)\hat{\phi}_2(\bar{x}_2) \].

The action of \( \hat{H} \) on \( |\psi_{(1),2,2,2,3}\rangle \) is, from (1)-(3), (8)-(10) and (94),
\[ \hat{H} |\psi_{(1),2,2,2,2,3}\rangle = -i\lambda |\psi_{(2),1,2,2,2,3}\rangle, \]
where
\[ |\psi_{(2),1,2,2,2,3}\rangle = \int d^3 \bar{x}_1 d^3 \bar{x}_2 d^3 \bar{x}_3 d^3 \bar{y}_1 d^3 \bar{y}_2 d^3 \bar{y}_3 \]
\[ \psi_{(1)}(\bar{x}_1)\psi_{(2)}(\bar{x}_2)\psi_{(3)}(\bar{x}_3)\psi_{(1)}(\bar{y}_1)\psi_{(2)}(\bar{y}_2)\psi_{(3)}(\bar{y}_3) \]
\[ \hat{\phi}_1^\dagger(\bar{x}_1)\hat{\phi}_2^\dagger(\bar{x}_2)\hat{\phi}_3^\dagger(\bar{x}_3)\hat{\phi}_1(\bar{y}_1)\hat{\phi}_2(\bar{y}_2)\hat{\phi}_3(\bar{y}_3)|0\rangle. \]

From this point forward we will work to first order in \( \frac{\Delta t}{\hbar} \), so (93) becomes, using (95),
\[ |\psi_{(1),2,2,2,2,2,3}\rangle(\Delta t) = |\psi_{(1),2,2,2,2,3}\rangle - \frac{\lambda \Delta t}{\hbar} |\psi_{(2),1,2,2,2,3}\rangle. \]

To verify that the action of \( \hat{H} \) is what we desire, we work in the usual representation and use (1)-(3), (8)-(10), (17)-(19), (23), (96) and (97) to compute the spin expectation values and spin correlations in the state \( |\psi_{(1),2,2,2,2,3}\rangle(\Delta t) \), obtaining
\[ \langle \psi_{(1),2,2,2,2,3}\rangle(\Delta t) | S_{\bar{u}_{(1),1},1} | \psi_{(1),2,2,2,2,3}\rangle(\Delta t) \rangle = u_{(1),3}, \]
\[ \langle \psi_{(1),2,2,2,2,3}\rangle(\Delta t) | S_{\bar{u}_{(2),2}} | \psi_{(1),2,2,2,2,3}\rangle(\Delta t) \rangle = -u_{(2),3}, \]
\[ \langle \psi_{(1),2,2,2,2,3}\rangle(\Delta t) | S_{\bar{u}_{(3),3}} \psi_{(1),2,2,2,2,3}\rangle(\Delta t) \rangle = -u_{(3),3}, \]
\[ \langle \psi_{(1),2,2,2,2,3}\rangle(\Delta t) | \hat{S}_{\bar{u}_{(1),1}} | \psi_{(1),2,2,2,2,3}\rangle(\Delta t) \rangle = \]
\[ -\left( 1 - \frac{2\lambda \Delta t}{\hbar} \right) u_{(1),3}u_{(2),3} - \frac{2\lambda \Delta t}{\hbar} \bar{u}_{(1),1} \cdot \bar{u}_{(2),2}, \]
\[ \langle \psi_{(1),2,2,2,2,3}\rangle(\Delta t) | \hat{S}_{\bar{u}_{(2),2}} | \psi_{(1),2,2,2,2,3}\rangle(\Delta t) \rangle = u_{(2),3}u_{(3),3}, \]
\[ \langle \psi_{(1),2,2,2,2,3}\rangle(\Delta t) | \hat{S}_{\bar{u}_{(3),3}} \psi_{(1),2,2,2,2,3}\rangle(\Delta t) \rangle = -u_{(3),3}u_{(1),3}. \]

These expectation values and correlations match to \( O(\lambda) \), those calculated in the analogous first-quantized system to \( O(\lambda^2) \), as shown in Appendix A, eqs. (A-8)-(A-13).
One might have expected that the generation of entanglement between spins 1 and 2 would have decreased the correlations between those spins and spin 3, i.e., that the magnitudes of \((102), (103)\) would be smaller than those of \((36), (37)\), respectively. This expectation is correct, but the decrease only shows up at second order in the coupling \(\lambda\) generating the entanglement; see \((A-12), (A-13)\) in Appendix A.

### 4.3 Entangled Deutsch-Hayden operators

We now make use of the method of Sec. 4.1 to obtain the entangled Deutsch-Hayden field operators, i.e. the field operators transformed from the usual representation using the Deutsch-Hayden transformation that maps the entangled state to the vacuum. Taking the entangling Hamiltonian \(\hat{H}_{u-e}\) of Sec. 4.1 to be \(\hat{H}\) of eq. (91), the unentangled Deutsch-Hayden transformation \(\hat{V}_{un}\) of Sec. 4.1 to be \(\hat{V}\) of eq. (71), \(\chi\) of Sec. 4.1 to be \(\phi_i(\vec{x}), i = 1, 2,\) and applying (90)-(92), we find the entangled field operators to be

\[
\hat{\phi}_{1,DH,en}(\vec{x}) = \hat{\phi}_{1,DH}(\vec{x}) + \frac{\lambda \Delta t}{\hbar} \int d^3 \vec{x}_1 d^3 \vec{z}_1 d^3 \vec{z}_2 \\
\left( \psi_2(\vec{x}) \psi_{11}^* (\vec{z}_1) \psi_{21}^* (\vec{z}_2) \psi_{11} (\vec{x}_1) - \psi_{11}(\vec{x}) \psi_{11}^* (\vec{z}_2) \psi_{21} (\vec{z}_1) \psi_{21} (\vec{x}_1) \right) \\
\hat{\phi}_{2,DH}(\vec{x}_1) \hat{\phi}_{2,DH}(\vec{z}_2) \hat{\phi}_{1,DH}(\vec{z}_1),
\]

(104)

\[
\hat{\phi}_{2,DH,en}(\vec{x}) = \hat{\phi}_{2,DH}(\vec{x}) + \frac{\lambda \Delta t}{\hbar} \int d^3 \vec{x}_1 d^3 \vec{z}_1 d^3 \vec{z}_2 \\
\left( \psi_{11}(\vec{x}) \psi_{22}^* (\vec{z}_1) \psi_{11}^* (\vec{z}_2) \psi_{22} (\vec{x}_1) - \psi_{22}(\vec{x}) \psi_{11}^* (\vec{z}_2) \psi_{22} (\vec{z}_1) \psi_{11} (\vec{x}_1) \right) \\
\hat{\phi}_{1,DH}(\vec{x}_1) \hat{\phi}_{1,DH}(\vec{z}_2) \hat{\phi}_{2,DH}(\vec{z}_1),
\]

(105)

where the unentangled Deutsch-Hayden-representation field operators \(\hat{\phi}_{1,DH}(\vec{x}), \hat{\phi}_{2,DH}(\vec{x})\) are as in \((76), (77)\).

### 4.4 Expectation values and correlations of spins of entangled and unentangled particles calculated in the entangled Deutsch-Hayden representation

Eqs. (104) and (105) provide an answer to the question posed in the title of this paper. In this section we verify that working in the entangled Deutsch-Hayden representation, i.e., using the entangled Deutsch-Hayden operators (104) and (105) and the Deutsch-Hayden-representation state vector \(|0\rangle\), we obtain the correct results for spin expectation values and correlations.
Transforming (23) to the entangled Deutsch-Hayden representation,
\[
\hat{S}_{\tilde{u}(1)\{i\},DH,en} = \int d^3 \tilde{x} A_{\{i\}}(\tilde{x}) \left[ \cos \theta_{\{i\}} \left( \hat{\phi}_{1,DH,en}^\dagger(\tilde{x}) \hat{\phi}_{1,DH,en}(\tilde{x}) - \hat{\phi}_{2,DH,en}^\dagger(\tilde{x}) \hat{\phi}_{2,DH,en}(\tilde{x}) \right) + \sin \theta_{\{i\}} \left( e^{i\phi_{\{i\}}} \hat{\phi}_{2,DH,en}(\tilde{x}) \hat{\phi}_{1,DH,en}(\tilde{x}) + e^{-i\phi_{\{i\}}} \hat{\phi}_{1,DH,en}(\tilde{x}) \hat{\phi}_{2,DH,en}(\tilde{x}) \right) \right].
\] (106)

We require the action of \(\hat{S}_{\tilde{u}(1)\{i\},DH,en}\) on \(|0\rangle\), i.e.,
\[
\hat{S}_{\tilde{u}(1)\{i\},DH,en}|0\rangle = \int d^3 \tilde{x} A_{\{i\}}(\tilde{x}) \left[ \cos \theta_{\{i\}} \left( \hat{\phi}_{1,DH,en}^\dagger(\tilde{x}) \hat{\phi}_{1,DH,en}(\tilde{x}) - \hat{\phi}_{2,DH,en}^\dagger(\tilde{x}) \hat{\phi}_{2,DH,en}(\tilde{x}) \right) + \sin \theta_{\{i\}} \left( e^{i\phi_{\{i\}}} \hat{\phi}_{2,DH,en}(\tilde{x}) \hat{\phi}_{1,DH,en}(\tilde{x}) + e^{-i\phi_{\{i\}}} \hat{\phi}_{1,DH,en}(\tilde{x}) \hat{\phi}_{2,DH,en}(\tilde{x}) \right) \right]|0\rangle.
\] (107)

Using (78), (79), (104), (105), the ETARs (1)-(7) and the WSW conditions (9),
\[
\hat{\phi}_{1,DH,en}|0\rangle = s_1 \psi_{\{1\}}(\tilde{x}) \int d^3 x' \psi_{\{1\}}(x') \hat{\alpha}_{\{1\}}^\dagger(x')|0\rangle - s_1 s_2 \frac{\lambda \Delta t}{\hbar} \psi_{\{2\}}(\tilde{x}) \cdot \int d^3 x_1 d^3 x'' d^3 y'' \psi_{\{1\}}(x_1) \psi_{\{2\}}(y'') \hat{\phi}_{1,DH,en}^\dagger(x_1) \hat{\alpha}_{\{1\}}^\dagger(y'')|0\rangle,
\] (108)
\[
\hat{\phi}_{2,DH,en}|0\rangle = s_2 \psi_{\{2\}}(\tilde{x}) \int d^3 x' \psi_{\{2\}}(x') \hat{\alpha}_{\{2\}}^\dagger(x')|0\rangle + s_3 \psi_{\{3\}}(\tilde{x}) \int d^3 x' \psi_{\{3\}}(x') \hat{\alpha}_{\{3\}}^\dagger(x')|0\rangle + s_1 s_2 \frac{\lambda \Delta t}{\hbar} \psi_{\{1\}}(\tilde{x}) \cdot \int d^3 x_1 d^3 x'' d^3 y'' \psi_{\{2\}}(x_1) \psi_{\{3\}}(y'') \hat{\phi}_{2,DH,en}^\dagger(x_1) \hat{\alpha}_{\{2\}}^\dagger(y'')|0\rangle.
\] (109)

Using (76), (77), (104), (105), (108), (109), the ETARs (1)-(7), WSW conditions (9), normalizations (8), (11) and the aperture conditions (17)-(19) in (107), we find
\[
\hat{S}_{\tilde{u}(1)\{i\},DH,en}|0\rangle = \cos \theta_{\{1\}}|0\rangle + \sin \theta_{\{1\}} e^{i\phi_{\{i\}}} s_1 \int d^3 \tilde{x} d^3 x' \psi_{\{1\}}(\tilde{x}) \psi_{\{1\}}(x') \hat{\phi}_{2,DH,en}^\dagger(\tilde{x}) \hat{\alpha}_{\{1\}}^\dagger(x')|0\rangle
\]
\[
+ \frac{\lambda \Delta t}{\hbar} \left( 2 \cos \theta_{\{1\}} s_1 s_2 \int d^3 \tilde{z}_1 d^3 z_2 d^3 \tilde{x} d^3 y' \psi_{\{1\}}(\tilde{z}_1) \psi_{\{2\}}(z_2) \psi_{\{1\}}(\tilde{x}') \psi_{\{2\}}(y') \hat{\phi}_{1,DH,en}^\dagger(\tilde{z}_1) \hat{\phi}_{2,DH,en}^\dagger(z_2) \hat{\alpha}_{\{1\}}^\dagger(x') \hat{\alpha}_{\{2\}}^\dagger(y')|0\rangle.
\]
where \(\tilde{\mathcal{S}}_{\tilde{u}_{(1)},\{1\},DH,\text{en}}|0\rangle = -\cos \theta_{(2)}|0\rangle + \sin \theta_{(2)} e^{-i\phi_{(2)}} s_2 \int d^3\vec{x} d^3\vec{y} \psi_{(2)}(\vec{x}) \psi_{(2)}(\vec{y}) \phi_{(2)}^\dagger(\vec{x}) \tilde{\chi}_{(2)}^\dagger(\vec{y})|0\rangle \\
- \frac{\lambda \Delta t}{\hbar} \left( 2 \cos \theta_{(2)} s_1 s_2 \int d^3\vec{x} d^3\vec{x}' d^3\vec{y} d^3\vec{y}' \psi_{(1)}(\vec{x}') \psi_{(1)}(\vec{x}) \psi_{(2)}(\vec{y}') \psi_{(2)}(\vec{y}) \phi_{(1)}^\dagger(\vec{x}) \tilde{\chi}_{(1)} \tilde{\chi}_{(2)}^\dagger(\vec{y})|0\rangle + \sin \theta_{(2)} e^{i\phi_{(2)}} s_1 \int d^3\vec{x} d^3\vec{x}'' \psi_{(1)}(\vec{x}') \psi_{(1)}(\vec{x}) \phi_{(1)}(\vec{x}'') \phi_{(1)}^\dagger(\vec{x}) \tilde{\chi}_{(2)} \tilde{\chi}_{(3)}^\dagger(\vec{y})|0\rangle \right),
\end{align}

\[
\tilde{\mathcal{S}}_{\tilde{u}_{(3)},\{3\},DH,\text{en}}|0\rangle = -\cos \theta_{(3)}|0\rangle + \sin \theta_{(3)} e^{-i\phi_{(3)}} s_3 \int d^3\vec{x} d^3\vec{y} \psi_{(3)}(\vec{x}) \psi_{(3)}(\vec{y}) \phi_{(1)}^\dagger(\vec{x}) \tilde{\chi}_{(3)} \tilde{\chi}_{(3)}^\dagger(\vec{y})|0\rangle.
\]

Since creation operators acting to the left annihilate the vacuum, we obtain immediately from (110)-(112) the expectation values of spin in the three regions:

\[
\langle 0| \tilde{\mathcal{S}}_{\tilde{u}_{(1)},\{1\},DH,\text{en}}|0\rangle = u_{(1),3},
\]

\[
\langle 0| \tilde{\mathcal{S}}_{\tilde{u}_{(2)},\{2\},DH,\text{en}}|0\rangle = -u_{(2),3},
\]

\[
\langle 0| \tilde{\mathcal{S}}_{\tilde{u}_{(3)},\{3\},DH,\text{en}}|0\rangle = -u_{(3),3}.
\]

From (110)-(112), the Hermiticity of the localized spin operators, the ETARs (11)-(17), the WSW conditions (9) and the normalizations (8), (11) we compute the spin correlations:

\[
\langle 0| \tilde{\mathcal{S}}_{\tilde{u}_{(1)},\{1\},DH,\text{en}} \tilde{\mathcal{S}}_{\tilde{u}_{(2)},\{2\},DH,\text{en}}|0\rangle = - \left( 1 - \frac{2\lambda \Delta t}{\hbar} \right) u_{(1),3} u_{(2),3} - \frac{2\lambda \Delta t}{\hbar} \tilde{u}_{(1)} \cdot \tilde{u}_{(2)},
\]

\[
\langle 0| \tilde{\mathcal{S}}_{\tilde{u}_{(2)},\{2\},DH,\text{en}} \tilde{\mathcal{S}}_{\tilde{u}_{(3)},\{3\},DH,\text{en}}|0\rangle = u_{(2),3} u_{(3),3},
\]

\[
\langle 0| \tilde{\mathcal{S}}_{\tilde{u}_{(3)},\{3\},DH,\text{en}} \tilde{\mathcal{S}}_{\tilde{u}_{(1)},\{1\},DH,\text{en}}|0\rangle = -u_{(3),3} u_{(1),3}.
\]

These are in agreement with the expectation values and correlations calculated in the usual representation, (98)-(103).

5See Sec. 3.3 re: correspondence between operators and particles.

5Conclusions and discussion

A Deutsch-Hayden-representation quantum field operator can encode entanglement, in the case of a small degree of entanglement, via modification to the unentangled operator corresponding to a given particle consisting of an addition of a term containing unentangled field operators corresponding to the particle \(5\) with which the given particle is entangled, as in (102), (103).
For example, suppose the point $\vec{x}$ is in region 1, the effective support of $\psi_{1}(\vec{x})$. Then from (104) and the WSW conditions we see that the difference between the entangled operator $\hat{\phi}_{1,\text{DH,en}}(\vec{x})$ and the unentangled operator $\hat{\phi}_{1,\text{DH}}(\vec{x})$ involves unentangled operators in region 2, the effective support of $\psi_{2}(\vec{x})$, weighted by the region-2 wavefunction and its conjugate, specifically $\int d^{3}\vec{z}_{1}d^{3}\vec{x}_{1}\psi_{2}^{*}(\vec{z}_{1})\psi_{2}(\vec{x}_{1})\hat{\phi}_{2,\text{DH}}(\vec{x}_{1})\hat{\phi}_{1,\text{DH}}(\vec{z}_{1})$.

On the other hand, we see from (104), (105) that entangled field operators corresponding to the unentangled particle, i.e., those for which $\vec{x}$ are in region 3, are identical to their unentangled counterparts. The $\mathcal{O}(\lambda)$ terms in (104), (105), involving as they do factors of $\psi_{1}(\vec{x})$ and $\psi_{2}(\vec{x})$, vanish by virtue of the WSW conditions if $\vec{x}$ is in the effective support of $\psi_{3}(\vec{x})$.

The representation of physical properties in the entangled system provided by the Deutsch-Hayden field theory described above is separable. It is indeed the case that “spatially separated systems are characterized by separate real states of affairs” [3]. Expectation values of spin along arbitrary directions $\vec{u}_{i}$ in region $i$ are encoded in operators $\hat{S}_{\vec{u}_{i},\{i\},\text{DH,en}}$ (see (113)-(115)) which are functions of the operators $\hat{\phi}_{j,\text{DH,en}}(\vec{x})$ in region $i$ (see (105)). It is the operators $\hat{\phi}_{j,\text{DH,en}}(\vec{x})$ that are the “separate real states of affairs.”

Of course determining correlations between spins in two regions, as in (116)-(118), requires use of operators in both of those regions. But no more, since the state vector $|0\rangle$ carries no information. As befits a separable system, “fixing the states of the parts fixes the state of the whole . . . The whole is ‘just the sum of the parts’ ” [34, p. 202].

The unentangled field operator $\hat{\phi}_{i,\text{DH}}(\vec{x})$ with $\vec{x}$ in a given region only depends on physical information in the same given region, i.e., values of physical wavefunctions in that region (see Sec. 3.3). For the entangled field operators this is not the case. The entangled field operator $\hat{\phi}_{i,\text{DH,en}}(\vec{x})$ with $\vec{x}$ in region 1 is a function of information in (possibly very distant) region 2, i.e., it depends on $\psi_{2}(\vec{x})$; and $\hat{\phi}_{i,\text{DH,en}}(\vec{x})$ in region 2 similarly depends on $\psi_{1}(\vec{x})$ (see (104), (105)). This possible dependence on distant information is not surprising, given that the entangled operators are constructed from the unentangled ones using a generator $\hat{H}$ (eq. (94)) that is explicitly nonlocal. Were we to regard $\hat{H}$ as the actual Hamiltonian acting on the system for a time interval $\Delta t$, it would involve instantaneous action-at-a-distance.

However, in the actual physical world, entanglement is produced locally, in an action-by-contact fashion, and formalisms exist to model this process. In particular, a previous paper [35] by the present author examining entanglement in Deutsch-Hayden field theory employs local interactions to generate entanglement starting from unentangled operators.

\[\text{Pachos and Solano} \quad \text{compute the generation of entanglement between two relativistic spin-1/2 fermions in QED. Van Leent et al.} \quad \text{have analyzed, and performed, an experiment in which spatially-separated atoms are entangled using photons transmitted over 33 km of optical fiber. Note that the entanglement swapping employed in this experiment is in fact local, as demonstrated by Hewitt-Horsman and Vedral using the Deutsch-Hayden approach for qubits.}\]
So, by combining the techniques employed in the present paper to construct in an effectively local manner unentangled Deutsch-Hayden field operators with the formalism [39, pp. 79-80] used in [35] for local propagation of field-theoretic information and generation of entanglement, it should be possible to present a quantum-field-theoretic model of entangled systems that is explicitly both separable and effectively local. Effective locality, however, will not be present unless the Deutsch-Hayden transformation from the usual representation is effectively local — hence the attention to effective locality in the present paper (see Sec. 3.3).

Arntzenius, in agreement with Deutsch and Hayden and the present author regarding the demonstration by Deutsch and Hayden of locality and separability in quantum computational networks, comments: “Technicalities aside, Deutsch and Hayden could equally well have taken the Heisenberg picture in quantum field theory and used the field states at locations in spacetime in place of qubits” [40, p. 116]. The “technicalities” are, to say the least, important, as they specify how to map physical information present in the usual representation of quantum field theory into the Deutsch-Hayden representation. Pienaar, Myers and Ralph conclude, as we have, that “finding an explicit form for the unitary [operator effecting a field-theoretic Deutsch-Hayden transformation] is a nontrivial matter” [41]. A nontrivial matter, but not, however, an insuperable problem, as we have shown.

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7As pointed out in [25] and discussed in Appendix B of the present paper, the Deutsch-Hayden transformation employed in [35], lacking auxiliary fields, is not effectively local. The Deutsch-Hayden transformation employed in [25] makes use of auxiliary fields and is effectively local. However the starting point is a more complicated state with distinguishable spatially-coincident entangled particles as well as three observers. While a Deutsch-Hayden transformation that correctly maps the initial-time state to the vacuum state and yields effectively local transformations of the operators is obtained for this system, closed forms for all operators in the Deutsch-Hayden representation have not been obtained and the time-dependent calculations are therefore done in the usual representation. So, separability is not explicit. We note also that in both of these previous papers all quanta, particles as well as observers, are distinguishable, in contrast to those in the present paper.

8Rather than computing the Deutsch-Hayden transformation, Pienaar, Myers and Ralph [41] take an alternative approach to incorporating information into field theory operators, generalizing a model of photon generation by parametric amplification. Recently Tibau Vidal, Vedral and Marletto [42] have proposed a local model for fermionic quantum field theory utilizing results of Raymond-Robichaud [43,44] and Bédard [45].
Appendix A. Expectation values and correlations of first-quantized spins to second order

Consider a system of three spins or qubits, i.e., distinguishable spin-1/2 particles with no spatial degrees of freedom. The state space of the $i$th spin, $i = 1, 2, 3$, is spanned by the kets $|1\rangle_i$ and $|2\rangle_i$ which are, respectively, spin-up and spin-down with respect to the $x_3$ axis. An unentangled state with particle 1 spin-up and particles 2 and 3 spin-down is

$$|\psi_{1Q,un}\rangle = |1\rangle_1|2\rangle_2|2\rangle_3. \quad (A-1)$$

Define a Hamiltonian $\tilde{H}_{1Q}$ that acts nontrivially only on particles 1 and 2,

$$\tilde{H}_{1Q} = -i\lambda (|2\rangle_1|1\rangle_2\langle 1|_1\langle 1|_2 - |1\rangle_1|2\rangle_2\langle 2|_1\langle 2|_2), \quad (A-2)$$

and the entangled state $|\psi_{1Q,en}\rangle$ that results from this Hamiltonian acting on $|\psi_{1Q,un}\rangle$ for a time $\Delta t$,

$$|\psi_{1Q,en}\rangle = \exp\left(-\frac{i\tilde{H}_{1Q}\Delta t}{\hbar}\right)|\psi_{1Q,un}\rangle. \quad (A-3)$$

To second order in $\frac{\Delta t^2}{\hbar}$,

$$|\psi_{1Q,en}\rangle = \left(1 - \frac{\lambda^2\Delta t^2}{2\hbar^2}\right)|\psi_{1Q,un}\rangle - \frac{\lambda\Delta t}{\hbar}|\tilde{\psi}_{1Q,un}\rangle, \quad (A-4)$$

where

$$|\tilde{\psi}_{1Q,un}\rangle = |2\rangle_1|1\rangle_2|2\rangle_3. \quad (A-5)$$

To see that $|\psi_{1Q,en}\rangle$ is in fact entangled use (A-1) and (A-5) to write (A-4) as

$$|\psi_{1Q,en}\rangle = \left(1 - \frac{\lambda^2\Delta t^2}{\hbar^2}\right)|1\rangle_1|2\rangle_2 - \frac{\lambda\Delta t}{\hbar}|2\rangle_1|1\rangle_2\right) |2\rangle_3. \quad (A-6)$$

The first factor in (A-6) is in the form of eq. (A5) of [32], and the analysis that follows there shows that (A-6) is entangled provided $\lambda\Delta t \neq 0$ (and provided as well that $1 - \frac{\lambda^2\Delta t^2}{\hbar^2} \neq 0$; but the vanishing of the latter would indicate we were well outside the range of validity of perturbation theory.)

The operator that measures the spin (in units of $\hbar/2$) of the $i$th particle along the unit vector $\vec{u}_i$ is $\vec{\sigma}_i \cdot \vec{\sigma}_i$, where the Pauli operators for the $i$th particle satisfy

$$\begin{align*}
\sigma_{1,[i]}|1\rangle_i = |2\rangle_i, & \quad \sigma_{1,[i]}|2\rangle_i = |1\rangle_i, \\
\sigma_{2,[i]}|1\rangle_i = i|2\rangle_i, & \quad \sigma_{2,[i]}|2\rangle_i = -i|1\rangle_i, \\
\sigma_{3,[i]}|1\rangle_i = |1\rangle_i, & \quad \sigma_{3,[i]}|2\rangle_i = -|2\rangle_i. \quad (A-7)
\end{align*}$$

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Using (A-6) with (A-7) we obtain the spin expectation values

\[ \langle \psi_{1Q, en} | \vec{u}_1 \cdot \hat{\sigma}_1 | \psi_{1Q, en} \rangle = \left( 1 - \frac{2\lambda^2 \Delta t^2}{\hbar^2} \right) u_{[1,3]}, \]  
(A-8)

\[ \langle \psi_{1Q, en} | \vec{u}_2 \cdot \hat{\sigma}_2 | \psi_{1Q, en} \rangle = - \left( 1 - \frac{2\lambda^2 \Delta t^2}{\hbar^2} \right) u_{[2,3]}, \]  
(A-9)

\[ \langle \psi_{1Q, en} | \vec{u}_3 \cdot \hat{\sigma}_3 | \psi_{1Q, en} \rangle = -u_{[3,3]}, \]  
(A-10)

and the spin correlations

\[ \langle \psi_{1Q, en} | (\vec{u}_1 \cdot \hat{\sigma}_1) (\vec{u}_2 \cdot \hat{\sigma}_2) | \psi_{1Q, en} \rangle = - \left( 1 - \frac{2\lambda^2 \Delta t}{\hbar} \right) u_{[1,3]u_{[2,3]} - \frac{2\lambda \Delta t}{\hbar} \vec{u}_1 \cdot \vec{u}_2; \]  
(A-11)

\[ \langle \psi_{1Q, en} | (\vec{u}_2 \cdot \hat{\sigma}_2) (\vec{u}_3 \cdot \hat{\sigma}_3) | \psi_{1Q, en} \rangle = \left( 1 - \frac{2\lambda^2 \Delta t^2}{\hbar^2} \right) u_{[2,3]u_{[3,3]}, \]  
(A-12)

\[ \langle \psi_{1Q, en} | (\vec{u}_3 \cdot \hat{\sigma}_3) (\vec{u}_1 \cdot \hat{\sigma}_1) | \psi_{1Q, en} \rangle = - \left( 1 - \frac{2\lambda^2 \Delta t^2}{\hbar^2} \right) u_{[3,3]u_{[1,3]}. \]  
(A-13)

**Appendix B. Deutsch-Hayden transformation without auxiliary fields**

It is possible to construct Deutsch-Hayden transformations for fermionic fields without including the auxiliary fields we employ here. The difficulty with all such constructions we have examined to date, however, arises in attempting to obtain effectively local transformations of the field operators.

For example, consider a state with a single fermionic particle,

\[ |\psi_1 \rangle = \int d^3 \vec{x} \psi(\vec{x}) \hat{\phi}^\dagger(\vec{x}) |0\rangle. \]  
(B-1)

Defining

\[ \hat{W}_1 = \int d^3 \vec{x} \left( \psi^*(\vec{x}) \hat{\phi}(\vec{x}) - \psi(\vec{x}) \hat{\phi}^\dagger(\vec{x}) \right) \]  
(B-2)

and

\[ \hat{V}_1(\theta) = \exp(\theta \hat{W}_1) \]  
(B-3)

we find

\[ \hat{V}_1(\theta) |\psi_1 \rangle = \cos(\theta) |\psi_1 \rangle + \sin(\theta) |0\rangle. \]  
(B-4)
So
\[ \hat{V}_1 = \hat{V}_1 \left( \frac{\pi}{2} \right) \] (B-5)
is a Deutsch-Hayden transformation for \(|\psi_1\rangle\),
\[ \hat{V}_1|\psi_1\rangle = |0\rangle. \] (B-6)

The field operator in the Deutsch-Hayden representation is
\[
\hat{\phi}_{DH}(\vec{x}) = \hat{V}_1 \hat{\phi}(\vec{x}) \hat{V}_1^\dagger = \hat{\phi}(\vec{x}) - \frac{\pi}{2} [\hat{\phi}(\vec{x}), \hat{W}_1] + \frac{(\pi/2)^2}{2!} [[\hat{\phi}(\vec{x}), \hat{W}_1], \hat{W}_1] - \ldots. \] (B-7)

But
\[
[\hat{\phi}(\vec{x}), \hat{W}_1] = 2\hat{\phi}(\vec{x}) \int d^3 \vec{x}' (\psi^*(\vec{x}') \hat{\phi}(\vec{x}') - \psi(\vec{x}') \hat{\phi}^\dagger(\vec{x}')) + \psi(\vec{x}). \] (B-8)

Even if the effective support of \(\psi(\vec{x})\) is within a localized volume concentrated around say \(\vec{x} = \vec{x}''\), the integral in the first term on the right-hand side of (B-8) will depend on \(\psi(\vec{x}'')\) regardless of the distance between \(\vec{x}\) and \(\vec{x}''\). So the first term in the difference between \(\hat{\phi}_{DH}(\vec{x})\) and \(\hat{\phi}(\vec{x})\) (i.e., the second term on the right-hand side of (B-7)) will depend on the value of \(\psi(\vec{x}'')\) no matter how far apart \(\vec{x}\) and \(\vec{x}''\) are.

A similar issue arises in the case of the Deutsch-Hayden transformation without auxiliary fields employed in [35], as was pointed out in [25]. There the change in the field operator for one of two species of fermions can be a function of the wavefunction for the other species at a distant point (see [35, eq. (151)]).

On the other hand, using auxiliary fields as in [25] and the present paper, we find, e.g., that
\[
[\hat{\phi}_1(\vec{x}), \hat{W}_{1,1}] = -g_1 \psi_{(1)}(\vec{x}) \int d^3 \vec{x}' \psi_{(1)}(\vec{x}') \hat{\alpha}^\dagger_{(1)}(\vec{x}'), \] (B-9)
so the change in the field operator at \(\vec{x}\) due to this term only depends on the physical wavefunction near \(\vec{x}\), and ultimately the complete Deutsch-Hayden transformation for the field operator is effectively local as discussed in Sec. [33].

None of the above is meant to imply that there with certainty does not exist an effectively-local Deutsch-Hayden transformation for nonrelativistic field theory of spin-1/2 fermions that does not employ auxiliary fields. However, as of this writing we have not discovered one.

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