CURVE SHORTENING FLOW IN A RIEMANNIAN MANIFOLD

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Abstract. In this paper, we systematically study the long time behavior of the curve shortening flow in a closed or non-compact complete locally Riemannian symmetric manifold. Assume that we have a global flow. Then we can exhibit a limit for the global behavior of the flow. In particular, we show the following results. 1). Let $M$ be a compact locally symmetric space. If the curve shortening flow exists for infinite time, and

$$\lim_{t \to \infty} L(\gamma_t) > 0,$$

then for every $n > 0$,

$$\lim_{t \to \infty} \sup \{ |D^n T| \} = 0.$$

In particular, the limiting curve exists and is a closed geodesic in $M$. 2). For $\gamma_0$ is a ramp, we have a global flow and the flow converges to a geodesic in $C^\infty$ norm.

1. Introduction

In this paper, we systematically study the limiting behavior at infinity of the curve shortening flow in a locally symmetric Riemannian manifold. Curve shortening flows in the plane, surfaces, and the 3-dimensional Euclidean space were studied respectively by M.Gage and R.Hamilton [8], M.Grayson [10], and S.Altschuler and M.Grayson[2]. Very recently the curve shortening flow in a closed Riemannian manifold has been used by G.Perelman [15] to study the Ricci-Hamilton flow. All these papers can be considered as models for our study. It is clear that studying the curve shortening flow in a general Riemannian manifolds is not an easy work, but it is still quite interesting.

By definition, our curve shortening flow in the compact or complete Riemannian manifold $(M, g)$ is evolving the initial curve $\gamma_0$ along the flow

$$\frac{\partial \gamma}{\partial t} = \frac{DT}{\partial s},$$

where $T$ is the unit tangent vector of $\gamma_t$, $\frac{DT}{\partial s}$ is the covariant derivative of $T$ in the direction $T$ in the space $(M, g)$, and $s$ is the arc-length parameter of $\gamma_t$. It is shown that there always exists a short time flow for the curve shortening problem in any compact Riemannian manifold (see § 2 in [8]). We remark that the existence of a short time flow for the curve shortening problem in a non-compact complete Riemannian manifold can also be obtained as in [16].

In this work, we mainly prove the following

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**Theorem 1** (Main Theorem). Let $M$ be a compact locally symmetric space. If the curve shortening flow exists for infinite time, and
\[ \lim_{t \to \infty} L(\gamma_t) > 0, \]
then for every $n > 0$,
\[ \lim_{t \to \infty} \sup(|D^n T|) = 0. \]
In particular, the limiting curve exists and is a closed geodesic in $M$.

2) For $\gamma_0$ is a ramp, we have a global flow and the flow converges to a geodesic in $C^\infty$ norm.

Roughly speaking, a *ramp* is a curve with non-trivial height with respect to a conformal Killing or just Killing vector field. One may see section 6 below for concrete definition. This result is a special case of our Theorem 10 and Theorem 19 below. We also obtain many other results for the curve shortening problem in varied cases, in particular, when $M$ is the space forms.

The results are exhibited in the sequent sections. We remark that in many cases, we do not assume $M$ being compact. We only assume that $(M, g)$ is a complete Riemannian manifold. We give a remark on our assumption of the global flow. In fact, one can easily see from our estimate in section 3 that in a nontrivial path homotopy class, we can always have a global flow. This is also true for ramps. We will study the finite time blow up of the flow in a separate paper.

The paper is organized as follows:

1. In § 2, we include some fundamental formulae for the flow.
2. In § 3, we deduce some precise estimates for the flow in the locally symmetric Riemannian space which also reprove the standard long time existence result in $R^3$.
3. In § 4, we investigate the limiting behavior of the curve shortening flow with some necessary assumptions on the initial curves.
4. In § 5, we compute in details the evolutions of the curve shortening flows in space forms $H^3$ and $S^3$ respectively.
5. In § 6, we study the flow for ramps and use them to find closed geodesics.
6. In § 7, we analyze the curve shortening flow on manifolds with time-dependent metrics.

### 2. Preliminaries

On an $n$-dimensional Riemannian manifold $(M, g)$, let
\[ \gamma : S^1 \times (a, b) \to M \]
be an evolving immersed curve. Denote by $\gamma_t$ the associated trajectory, i.e.,
\[ \gamma_t(\cdot) =: \gamma(\cdot, t). \]

Then the length of $\gamma_t$ is
\[ L(\gamma_t) = \int_{S^1} \left| \frac{d}{du} \gamma_t \right| du = \int_{S^1} \left| \frac{\partial \gamma}{\partial u} \right| du = \int_{S^1} v du, \]
where
\[ v = \frac{\partial \gamma}{\partial u} \]
is the speed. We define the arc-length parameter \( s \) by
\[ \frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u} , \]
which implies
\[ ds = v du . \]
As usual, we denote by \( T \) the associated unit tangent vector, i.e.,
\[ T = \frac{\partial \gamma}{\partial s} = \frac{1}{v} \frac{\partial \gamma}{\partial u} . \]
Then the time derivative of length is
\[ \frac{d}{dt} L(\gamma_t) = \int_{S^1} \frac{\partial v}{\partial t} du \]
\[ = \int_{S^1} \left\langle \nabla_t \frac{\partial \gamma}{\partial u} , T \right\rangle du \]
\[ = \int_{S^1} \left\langle \nabla_u \frac{\partial \gamma}{\partial t} , T \right\rangle du \]
\[ = \int_{S^1} \left\{ \frac{\partial}{\partial u} \left\langle \frac{\partial \gamma}{\partial t} , T \right\rangle - \left\langle \frac{\partial \gamma}{\partial t} , \nabla_u T \right\rangle \right\} du \]
\[ = - \int_{S^1} \left\langle \frac{\partial \gamma}{\partial t} , \nabla_u T \right\rangle du \]
\[ = - \int_{S^1} \left\langle \frac{\partial \gamma}{\partial t} , DT \right\rangle ds \times . \]
If \( \gamma \) evolves according to the equation
\[ \frac{\partial \gamma}{\partial t} = DT \frac{\partial}{\partial s} , \]
then we find that
\[ \frac{d}{dt} L(\gamma_t) = - \int_{S^1} k^2 ds \leq 0 , \]
where
\[ k^2 = \left| DT \right|^2 \]
is the curvature squared. This leads us to give the following

**Definition 2.** For a curve shortening flow, we mean an evolving immersed curve \( \gamma(\cdot, t) \) satisfying the evolution equation
\[ \frac{\partial \gamma}{\partial t} = DT \frac{\partial}{\partial s} . \]
Remark 3. Obviously, we can regard $\gamma_t(S^1)$ as a 1-dimensional sub-manifold of $M$. With the induced metric from $M$, its mean curvature vector field is $H = (\nabla_T T)^\perp$.

Noting that $<T, T> \equiv 1$,

we have $<\nabla_T T, T> \equiv 0$.

This implies $\nabla_T T \perp T_\gamma$.

So $H = \nabla_T T$.

This shows that a curve shortening flow is a mean curvature flow.

Next we will give some fundamental computations. These formulae have already appeared in many papers, for example [10].

**Lemma 4.** The evolution of $v$ is

$$\frac{\partial v}{\partial t} = -k^2 v.$$  

**Proof.** By definition,

$$v^2 = <\frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}>.$$  

Differentiating it with respect to $t$, we get

$$2v \frac{\partial v}{\partial t} = 2 <\nabla_t \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}>$$

$$= 2 <\nabla_u \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial u}>$$

$$= 2v^2 <\nabla_T \frac{DT}{\partial s}, T>$$

$$= -2v^2 <\frac{DT}{\partial s}, \frac{DT}{\partial s}>$$

$$= -2k^2 v^2.$$  

□

**Lemma 5.** Covariant differentiation with respect to $s$ and $t$ are related by the equation

$$\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s + R(T, \frac{DT}{\partial s}),$$

where $R$ is the curvature operator on $M$. 
Proof. We have (see [4])
\[ \nabla_t \nabla_u = \nabla_u \nabla_t + R(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}) \]
and
\[ \nabla_s = \frac{1}{v} \nabla_u. \]
Using Lemma 3, we get
\[ \nabla_t \nabla_s = \frac{\partial}{\partial t} \left( \frac{1}{v} \right) \nabla_u + \frac{1}{v} \nabla_t \nabla_u \]
\[ = k^2 \frac{1}{v} \nabla_u + \frac{1}{v} \nabla_u \nabla_t + \frac{1}{v} R(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}) \]
\[ = \nabla_s \nabla_t + k^2 \nabla_s + R(T, \frac{D^2 T}{\partial s^2}). \]

\[ \square \]

Lemma 6. The covariant differentiation of \( T \) with respect to time \( t \) is
\[ \nabla_t T = k^2 T + \frac{D^2 T}{\partial s^2}. \]

Proof. The proof is a straightforward calculation.
\[ \nabla_t T = \nabla_t \left( \frac{1}{v} \frac{\partial \gamma}{\partial u} \right) \]
\[ = k^2 \frac{1}{v} \frac{\partial \gamma}{\partial u} + \frac{1}{v} \nabla_u \frac{\partial \gamma}{\partial t} \]
\[ = k^2 T + \frac{D^2 T}{\partial s^2}. \]

\[ \square \]

3. Bernstein type estimates

In this section and the next one, we shall assume that \( M \) is a locally symmetric space, i.e.,
\[ \nabla R = 0. \]
For a locally symmetric space, we have
\[ \nabla R(X, Y, Z, W) = (\nabla_X R)(Y, Z, W) \]
\[ = \nabla_X (R(Y, Z, W)) - R(\nabla_X Y, Z, W) - R(Y, \nabla_X Z, W) \]
\[ - R(Y, Z, \nabla_X W) \]
\[ = 0 \]
\[ \nabla_X(R(Y, Z, W)) = R(\nabla_X Y, Z, W) + R(Y, \nabla_X Z, W) + R(Y, Z, \nabla_X W), \]
for all \( X, Y, Z, W \in TM \). We shall also assume that \( M \) satisfies Condition (\( \Lambda \)), i.e., there exists a positive constant \( \Lambda \), such that
\[ R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \leq \Lambda, \]
for all unit vectors \( \bar{X}, \bar{Y}, \bar{Z}, \bar{W} \).

With these assumptions, we can give some precise estimates which will bound the evolution of \( |\frac{D^n T}{\partial s^n}|^2 \). This type of estimate also appears in \([2]\) for the curve shortening flow in \( \mathbb{R}^3 \). In that case, these estimates are dilation-invariant, and play an important role in singularity analysis.

First, let us compute the time derivative of \( |\frac{D^n T}{\partial s^n}|^2 \) as follows:
\[ \frac{\partial}{\partial t}(|\frac{D^n T}{\partial s^n}|^2) \]
\[ = 2 < \frac{D^n T}{\partial t}, \frac{D^n T}{\partial s^n} > \]
\[ = 2 < \frac{D D^n T}{\partial s \partial t}, \frac{D^n T}{\partial s^n} + k^2 \frac{D^n T}{\partial s^n} > + 2R(T, \frac{DT}{\partial s}, \frac{D^n - 1 T}{\partial s^n - 1}, \frac{D^n T}{\partial s^n}) \]
\[ = 2 < \frac{D}{\partial s} \left( \frac{D D^n - 2 T}{\partial s \partial t}, \frac{D^n T}{\partial s^n - 2} + k^2 \frac{D^n - 1 T}{\partial s^n - 1} + R(T, \frac{DT}{\partial s}, \frac{D^n - 2 T}{\partial s^n - 2}), \frac{D^n T}{\partial s^n} \right) \]
\[ + 2k^2 |\frac{D^n T}{\partial s^n}|^2 + 2R(T, \frac{DT}{\partial s}, \frac{D^n - 1 T}{\partial s^n - 1}, \frac{D^n T}{\partial s^n}) \]
\[ = 2 < \frac{D^2 D^n - 2 T}{\partial s \partial t}, \frac{D^n T}{\partial s^n - 2} > + 2 < \frac{D}{\partial s} (k^2 \frac{D^n - 1 T}{\partial s^n - 1}), \frac{D^n T}{\partial s^n} > + 2k^2 |\frac{D^n T}{\partial s^n}|^2 \]
\[ + 2 < \frac{D}{\partial s} \left( R(T, \frac{DT}{\partial s}) \frac{D^n - 2 T}{\partial s^n - 2}, \frac{D^n T}{\partial s^n} \right) > + 2R(T, \frac{DT}{\partial s}, \frac{D^n - 1 T}{\partial s^n - 1}, \frac{D^n T}{\partial s^n}) \]
\[ = \cdots = 2 < \frac{D^n D}{\partial s \partial t}, \frac{D^n T}{\partial s^n} > + 2 \sum_{i=0}^{n-1} < \frac{D^i}{\partial s^i} (k^2 \frac{D^n - i T}{\partial s^n - i}), \frac{D^n T}{\partial s^n} > \]
\[ + 2 \sum_{i=0}^{n-1} < \frac{D^i}{\partial s^i} \left( R(T, \frac{DT}{\partial s}) \frac{D^n - 1 - i T}{\partial s^n - 1 - i}, \frac{D^n T}{\partial s^n} \right) > . \]

Using Lemma 5, we get
\[ 2 < \frac{D^n}{\partial s^n \partial t}, \frac{D^n T}{\partial s^n} > \]
\[\begin{align*}
&= 2 < \frac{D^{n+2}T}{\partial s^{n+2}}, \frac{D^n T}{\partial s^n} > + 2 < \frac{D^n T}{\partial s^n} (k^2 T), \frac{D^n T}{\partial s^n} > \\
&= \frac{\partial^2}{\partial s^2} \left( \frac{D^n T}{\partial s^n} \right)^2 - 2 \left| \frac{D^{n+1} T}{\partial s^{n+1}} \right|^2 + 2 < \frac{D^n T}{\partial s^n} (k^2 T), \frac{D^n T}{\partial s^n} > .
\end{align*}\]

So
\[
\frac{\partial}{\partial t} \left( \left| \frac{D^n T}{\partial s^n} \right|^2 \right) = \frac{\partial^2}{\partial s^2} \left( \left| \frac{D^n T}{\partial s^n} \right|^2 \right) - 2 \left| \frac{D^{n+1} T}{\partial s^{n+1}} \right|^2 + 2 \sum_{i=0}^{n-1} < \frac{D^i}{\partial s^i} (k^2 \frac{D^{n-i} T}{\partial s^{n-i}}), \frac{D^n T}{\partial s^n} > .
\]

It is easy to see that
\[
(2) \quad \frac{D^i}{\partial s^i} (k^2 \frac{D^{n-i} T}{\partial s^{n-i}}) = \sum_{j+k\leq i} C_{ijk} < \frac{D^{j+1} T}{\partial s^{j+1}}, \frac{D^{k+1} T}{\partial s^{k+1}} > \frac{D^{n-j-k} T}{\partial s^{n-j-k}},
\]

and
\[
(3) \quad \frac{D^i}{\partial s^i} (R(T, \frac{DT}{\partial s}) \frac{D^{n-1-i} T}{\partial s^{n-1-i}}) = \sum_{j+k\leq i} C_{ijk} \frac{D^j T}{\partial s^j} \frac{D^{k+1} T}{\partial s^{k+1}} \frac{D^{n-1-j-k} T}{\partial s^{n-1-j-k}},
\]

where the coefficients \( C_{ijk} \) are constants. To obtain (3), we have repeatedly used (1).

Noting that \( M \) satisfies Condition (\( \Lambda \)), and then putting above equations together, we obtain
\[
\frac{\partial}{\partial t} (\left| \frac{D^n T}{\partial s^n} \right|^2) \leq \frac{\partial^2}{\partial s^2} (\left| \frac{D^n T}{\partial s^n} \right|^2) - \left| \frac{D^{n+1} T}{\partial s^{n+1}} \right|^2 + C_1 \left| \frac{D^n T}{\partial s^n} \right|^2 + C_2 \left| \frac{D^n T}{\partial s^n} \right|^2 \left| \frac{D^n T}{\partial s^n} \right|^2 + C_3 \frac{\partial^2 T}{\partial s^2} \left| \frac{D^n T}{\partial s^n} \right|^2 + C_4 \sum_{0\leq i,j,k<n} \left| \frac{D^i T}{\partial s^i} \right| \left| \frac{D^j T}{\partial s^j} \right| \left| \frac{D^k T}{\partial s^k} \right| \left| \frac{D^n T}{\partial s^n} \right| ,
\]

where \( C_i \) are positive constants depending on \( n \) and \( \Lambda \). In the last term, the range of indices satisfies in addition either
\[i + j + k = n + 2\]
or
\[i + j + k = n.\]

**Theorem 7.** Fix \( t_0 \in [0, +\infty) \). Let
\[M_{t_0} =: \max k^2(\cdot, t_0).\]
Assume
\[M_{t_0} < +\infty.\]
Then there exist constants $\tilde{c}_l < +\infty$ independent of $t_0$ such that for $t \in (t_0, t_0 + \frac{1}{2\Lambda} \log(1 + \frac{\Lambda}{4M_{t_0} + \Lambda + 1}))$, we have

$$|D_l^t T| \leq \frac{\tilde{c}_l M_{t_0}}{(t - t_0)^{l-1}}.$$ 

Proof. Without loss of generality, we may assume that $t_0 = 0$, and then translate the estimates.

(1) For $l = 1$, we have

$$\frac{\partial}{\partial t}(D_l^t T) = 2 < \frac{D}{D} D_l^t T + k^2 D_l^t T + R(T, D_l^t T) T, D_l^t T >$$

$$= 2 < \frac{D^3 T}{D} \frac{D}{D} T > + 4k^4 + 2k^2 R(T, N, T, N)$$

$$\leq \frac{\partial^2}{\partial s^2}(D_l^t T)^2 - 2|D_l^t T| + 4k^4 + 2\Lambda k^2$$

It follows from the maximum principle that $M_t$ satisfies

$$\log \frac{M_t}{\Lambda M_t + 1} - \log \frac{M_0}{\Lambda M_0 + 1} \leq 2\Lambda t.$$ 

If

$$t \leq \frac{1}{2\Lambda} \log(1 + \frac{\Lambda}{4M_{t_0} + \Lambda + 1}),$$

then

$$M_t \leq 2M_0.$$ 

So we may choose $\tilde{c}_1 = 2$.

(2) For $l = 2$, we have

$$\frac{\partial}{\partial t}(D^2_l T) = 2 < \frac{D}{D} D_l^t T + k^2 D_l^t T + R(T, D_l^t T) T, D_l^t T >$$

$$= 2 < \frac{D^3 T}{D} \frac{D}{D} T > + 2 < \frac{D}{D}(k^2 D_l^t T), \frac{D^2}{D} T >$$

$$+ 2 < \frac{D}{D}(R(T, D_l^t T) T), \frac{D^2}{D} T > + 2k^2 |D^2_l^t T|^2$$

$$+ 2R(T, \frac{D}{D} T, \frac{D^2}{D} T)$$

$$\leq \frac{\partial^2}{\partial s^2}(D^2_l^t T)^2 - |D^2_l^t T|^2 + 18 |\frac{D}{D} T|^2 |\frac{D^2}{D} T|^2$$

$$+ 2\Lambda |\frac{D^2}{D} T|^2 + 2\Lambda |D_l^t T|^4.$$ 

So

$$\frac{\partial}{\partial t}(t |D^2_l^t T|^2 + 3|D_l^t T|^2)$$
\[ \leq \frac{\partial^2}{\partial s^2} (t |D^2T|_2^2 + 3 |DT|_2^2) - t |D^3T|_2^2 + t(18 |DT|_2^2 + 2\Lambda - 5) |D^2T|_2^2 \\
+ (2\Lambda t + 12) |DT|_2^2 |\frac{\partial T}{\partial s}|^4 + 6\Lambda |\frac{\partial T}{\partial s}|^2. \]

Since
\[ t \leq \frac{1}{2\Lambda} \log (1 + \frac{\Lambda}{4M_0 + \Lambda + 1}) \leq \frac{1}{2(4M_0 + \Lambda + 1)}, \]
we have
\[ \frac{\partial}{\partial t} (t |D^2T|_2^2 + 3 |DT|_2^2) \leq \frac{\partial^2}{\partial s^2} (t |D^2T|_2^2 + 3 |DT|_2^2) + 52M_0^2 + 12\Lambda M_0. \]

Thus it follows that
\[ t |\frac{D^2T}{\partial s^2}|^2 + 3 |\frac{DT}{\partial s}|^2 \leq 16M_0, \]
and we may conclude on this time interval that
\[ |\frac{D^2T}{\partial s^2}|^2 \leq \frac{16M_0}{t}. \]

So we may choose \( \tilde{c}_2 = 16. \)

The induction hypothesis and repeated usage of the Peter-Paul inequality, i.e.,
\[ ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \]
allow us to find constants \( a_i \) and \( A, B \) on our time interval such that
\[ \frac{\partial}{\partial t} \left( \sum_{i=1}^{m} a_i t^{i-1} \left| \frac{D^i T}{\partial s^i} \right|^2 \right) \leq AM_0^2 + BM_0. \]

Thus we obtain \( \tilde{c}_m \) as before. \( \square \)

Note that these estimates prove the long time existence result. That is, as long as the curvature remains bounded on time interval \([0, \alpha]\), one can define a smooth limit for the tangent vector \( T \) at time \( \alpha \). Thus, by integrating the tangent vector \( T \), one can obtain a smooth limit curve.

4. CONVERGENCE AT INFINITY

In this section, we want to prove some convergent results which will be applied in § 6. Similar results can be found in [10] where the author dealt with Riemannian surfaces. Those results may be regarded as a generalization of ours at 2-dimensional case.

Let \( \omega \) be the maximal existence time of the curve shortening flow. Throughout this section, we shall assume that
\[ \omega = +\infty, \]
and
\[ \lim_{t \to +\infty} L(\gamma_t) > 0. \]
Then we have

**Lemma 8.** The $L^2$-norm of curvature converges to zero as $t \to +\infty$.

**Proof.** It is obvious that the time derivative of length,

$$- \int k^2 ds,$$

must be approaching zero at an $\epsilon$-dense set of sufficiently large times. So what we need to do is to bound its time derivative.

$$\frac{\partial}{\partial t} \int k^2 ds \leq \int \frac{\partial^2}{\partial s^2} (|\frac{DT}{\partial s}|^2 - 2|\frac{D^2 T}{\partial s^2}|^2 + 4k^4 + 2\Lambda k^2 ds - \int k^4 ds$$

$$\leq -2 \int \frac{D^2 T}{\partial s^2} |^2 ds + (3 \sup k^2 + 2\Lambda) \int k^2 ds.$$

But

$$\sup k^2 \leq (\inf |k| + \int |\frac{D^2 T}{\partial s^2}|^2 ds)^2 \leq \frac{2}{l_\infty} \int k^2 ds + 2l_0 \int |\frac{D^2 T}{\partial s^2}|^2 ds$$

$$\Rightarrow -2 \int \frac{D^2 T}{\partial s^2} |^2 ds \leq -\frac{1}{l_0} \sup k^2 + \frac{2}{l_\infty l_0} \int k^2 ds.$$

So

$$\frac{\partial}{\partial t} \int k^2 ds \leq (2\Lambda + \frac{2}{l_\infty l_0}) \int k^2 ds + \sup k^2 (3 \int k^2 ds - \frac{1}{l_0}).$$

Therefore $\int k^2 ds$ has at most exponential growth when it is sufficiently small. This implies that it must converge to zero at $t \to \infty$. \qed

**Lemma 9.**

$$\lim_{t \to \infty} \int |\frac{D^2 T}{\partial s^2}|^2 ds = 0.$$

**Proof.** Suppose not. We need only consider those times when $\int |\frac{D^2 T}{\partial s^2}|^2 ds$ is sufficiently greater than $\int k^2 ds$. Look at the time derivative of $\int |\frac{D^2 T}{\partial s^2}|^2 ds$.

Our rules for differentiating yields:

$$\frac{\partial}{\partial t} \int |\frac{D^2 T}{\partial s^2}|^2 ds \leq \int -|\frac{D^3 T}{\partial s^3}|^2 + 18\frac{DT}{\partial s} |^2 |\frac{D^2 T}{\partial s^2}|^2 + 2\Lambda |\frac{D^2 T}{\partial s^2}|^2 + \Lambda^2 |\frac{DT}{\partial s}|^2.$$

We will bound the last three terms in this integral by a fraction of the first.

Note that

$$\frac{\partial}{\partial s} < \frac{DT}{\partial s} \cdot \frac{DT}{\partial s} >= \frac{DT}{\partial s} \cdot \frac{D^3 T}{\partial s^3} > + |\frac{D^2 T}{\partial s^2}|^2$$

$$\Rightarrow 0 = \int < \frac{DT}{\partial s} \cdot \frac{D^3 T}{\partial s^3} > ds + \int |\frac{D^2 T}{\partial s^2}|^2 ds$$

$$\Rightarrow \int |\frac{D^2 T}{\partial s^2}|^2 ds \leq (\int |\frac{DT}{\partial s}|^2 ds \cdot \int |\frac{D^3 T}{\partial s^3}|^2 ds)^{\frac{1}{2}}.$$
If we assume that
$$\int |D^2 T|_s^2 ds > \alpha \cdot \int |DT|_s^2 ds,$$
then we get
$$\int |D^2 T|_s^2 ds \leq \alpha^{-1} \cdot \int |D^3 T|_s^3 ds.$$

Assume that
$$\int |DT|_s^2 ds \leq \epsilon$$
for some small $\epsilon > 0$. We estimate the second term:
$$\int |D^2 T|_s^2 |D^2 T|_s^2 ds \leq \epsilon \cdot \sup |D^2 T|_s^2.$$
But
$$\sup |D^2 T|_s^2 \leq \{ \inf |D^2 T|_s^2 + \int |\frac{\partial}{\partial s} (D^2 T)|_s^2 |ds\}^2$$
$$\leq (\inf |D^2 T|_s^2 + \int |D^3 T|_s^3 ds)^2$$
$$\leq 2 \int |D^2 T|_s^2 ds + 2l_0 \int |D^3 T|_s^3 ds$$
$$\leq (\frac{2}{\alpha l_\infty} + 2l_0) \int |D^3 T|_s^3 ds.$$

Hence
$$\frac{\partial}{\partial t} \int |D^2 T|_s^2 ds \leq [-1 + \epsilon \cdot (\frac{2}{\alpha l_\infty} + 2l_0) + \frac{2\Lambda}{\alpha} + \frac{\Lambda^2}{\alpha^2}] \int |D^3 T|_s^3 ds$$
$$\leq -\frac{1}{2} \int |D^3 T|_s^3 ds$$
$$\leq -\frac{1}{2} \int |D^2 T|_s^2 ds.$$

So, either $\int |D^2 T|_s^2 ds$ decays exponentially, or it is comparable to $\int |DT|_s^2 ds$. In either event, it decreases to zero. □

The following Sobolev inequality is useful, and will be used repeatedly.

**Lemma 10.** If $\|f\|_2 \leq C$ and $\|f'\|_2 \leq C$, then
$$\|f\|_\infty \leq (\frac{1}{\sqrt{2\pi}} + \sqrt{2\pi})C,$$
where $\| \cdot \|_2$ is the $L_2$ norm and $\| \cdot \|_\infty$ is the sup norm for functions on $S^1$. 
Notice
\[ \left| \frac{\partial}{\partial s} \left( \frac{D T}{\partial s} \right) \right|^2 \leq \left| \frac{D^2 T}{\partial s^2} \right|^2. \]

So, from Lemma 8, we have
\[ \lim_{t \to +\infty} \int \left| \frac{\partial}{\partial s} \left( \frac{D T}{\partial s} \right) \right|^2 ds = 0. \]

Then it follows from the Sobolev inequality that \( \sup(\frac{D T}{\partial s}) \) decreases to zero.

We deal with the higher derivatives in the same fashion. Integration and the Holder inequality yield:
\[ \left( \int \left| \frac{D^n T}{\partial s^n} \right|^2 ds \right)^2 \leq \int \left| \frac{D^{n-1} T}{\partial s^{n-1}} \right|^2 ds \cdot \int \left| \frac{D^{n+1} T}{\partial s^{n+1}} \right|^2 ds. \]

We start, knowing that
\[ \int \left| \frac{D^{n-1} T}{\partial s^{n-1}} \right|^2 ds \to 0, \]

and that
\[ \sup(\frac{D^m T}{\partial s^m}) \to 0 \]

for all \( m < n - 1 \). Then, as before, we can show that \( \int \left| \frac{D^n T}{\partial s^n} \right|^2 ds \) decreases exponentially when it is much bigger than some linear combination of \( \int \left| \frac{D^{n-1} T}{\partial s^{n-1}} \right|^2 ds \) and terms involving the lower order derivatives. Therefore
\[ \int \left| \frac{D^n T}{\partial s^n} \right|^2 ds \to 0. \]

And then it easily follows from the inequality
\[ \left| \frac{\partial}{\partial s} \left( \frac{D^{n-1} T}{\partial s^{n-1}} \right) \right|^2 \leq \left| \frac{D^n T}{\partial s^n} \right|^2, \]

and the Sobolev inequality that
\[ \sup(\left| \frac{D^{n-1} T}{\partial s^{n-1}} \right|) \to 0. \]

We leave the details as an exercise. Or one can refer to [10].

By now, we have proved

**Theorem 11.** \( M \) is a compact locally symmetric space satisfying Condition (\( \Lambda \)). If the curve shortening flow exists for infinite time, and
\[ \lim_{t \to \infty} L(\gamma_t) > 0, \]

then for every \( n > 0 \),
\[ \lim_{t \to \infty} \sup(\left| \frac{D^n T}{\partial s^n} \right|) = 0. \]

Moreover, the limiting curve exists and is a geodesic.
5. Curve shortening in space forms

In this section, we shall assume that $M$ is a 3-dimensional Riemannian manifold with constant sectional curvature $K$. In this case, the curvature operator $R$ has the following simple expression, i.e.,

$$R(X_1, X_2)X_3 = K(<X_1, X_3> X_2 - <X_2, X_3> X_1)$$

for all $X_i \in TM$. In particular,

$$R(T, N)T = KN$$

and

$$R(T, N)N = -KT.$$

For $n = 3$, we have the well-known Frenet matrix for a curve $\gamma$, with the arc-length parameter $s$,

$$\frac{D}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$ 

With these relations, we can make Lemma 5 more precise.

**Lemma 12.**

$$\nabla_t T = \frac{\partial k}{\partial s} N + \tau k B.$$

**Proof.** Note

$$\frac{D^2 T}{\partial s^2} = \frac{D}{\partial s} (kN) = \frac{\partial k}{\partial s} N + k(-kT + \tau B),$$

so

$$k^2 T + \frac{D^2 T}{\partial s^2} = \frac{\partial k}{\partial s} N + \tau k B.$$ 

Then, by Lemma 5, we get

$$\nabla_t T = k^2 T + \frac{D^2 T}{\partial s^2} = \frac{\partial k}{\partial s} N + \tau k B.$$

\[\square\]

Now, we can compute the evolution of curvature $k$.

**Lemma 13.**

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 - \tau^2 k + Kk.$$

**Proof.** Note

$$\nabla_t \nabla_s T = \nabla_s \nabla_t T + k^2 \nabla_s T + kR(T, N)T$$

$$= \nabla_s (\frac{\partial k}{\partial s} N + \tau k B) + k^3 N + Kk N$$

$$= \frac{\partial^2 k}{\partial s^2} N + \frac{\partial k}{\partial s} (-kT + \tau B) + \frac{\partial}{\partial s} (\tau k) B$$
\[ + (\tau k)(-\tau N) + k^3 N + KkN \]
\[ = -k\frac{\partial k}{\partial s} T + \left(\frac{\partial^2 k}{\partial s^2} + k^3 - \tau^2 k + Kk\right)N \]
\[ + (\tau \frac{\partial k}{\partial s} + \frac{\partial}{\partial s}(\tau k))B, \]
so
\[ \frac{\partial k}{\partial t} = \frac{\partial}{\partial t} \langle \nabla_s T, N \rangle \]
\[ = \langle \nabla_t \nabla_s T, N \rangle + \langle \nabla_s T, \nabla_t N \rangle \]
\[ = \frac{\partial^2 k}{\partial s^2} + k^3 - \tau^2 k + Kk. \]

We also need to know the rate at which the unit normal vector to the curve rotates. This can be got directly from the proof of Lemma 12.

**Corollary 14.**
\[ \nabla_t N = -\frac{\partial k}{\partial s} T + \left(\frac{\partial \tau}{\partial s} + 2\frac{\tau}{k} \frac{\partial k}{\partial s}\right)B. \]

**Proof.** Note
\[ \nabla_s T = kN, \]
so
\[ \nabla_t \nabla_s T = \nabla_t(kN) = \frac{\partial k}{\partial t} N + k \nabla_t N. \]
With this equation, and noticing that
\[ \langle \nabla_t N, N \rangle = 0, \]
we get the following relation from the proof of Lemma 12:
\[ k \nabla_t N = -k \frac{\partial k}{\partial s} T + \left(k \frac{\partial \tau}{\partial s} + 2\tau \frac{\partial k}{\partial s}\right)B. \]
Multiplying both sides \( \frac{1}{k} \), we obtain what we want. \( \square \)

Now, we can compute the evolution of torsion \( \tau \).

**Lemma 15.**
\[ \frac{\partial \tau}{\partial t} = \frac{\partial^2 \tau}{\partial s^2} + 2\frac{1}{k} \frac{\partial k}{\partial s} \frac{\partial \tau}{\partial s} + 2\frac{\tau}{k} \left(\frac{\partial^2 k}{\partial s^2} - \frac{1}{k} \left(\frac{\partial k}{\partial s}\right)^2 + k^3\right). \]
Proof. We have
\[ \nabla_s N = -kT + \tau B \]
\[ \Rightarrow \]
\[ (4) \quad \nabla_t \nabla_s N = -\frac{\partial k}{\partial t} T - k \nabla_t T + \tau \frac{\partial}{\partial t} B + \tau \nabla_t B. \]

The left hand side of (4) equals
\[ \nabla_s \nabla_t N + k^2 \nabla_s N + kR(T, N) N \]
\[ = \nabla_s (-\frac{\partial k}{\partial s} T + (\frac{\partial \tau}{\partial s} + 2 \frac{\tau}{k} \frac{\partial k}{\partial s}) B) + k^2 (-kT + \tau B) - KkT \]
\[ = -\frac{\partial^2 k}{\partial s^2} T - k \frac{\partial k}{\partial s} N + (\frac{\partial^2 \tau}{\partial s^2} + 2 \frac{\partial}{\partial s} (\tau \frac{\partial k}{\partial s})) B \]
\[ + (\frac{\partial \tau}{\partial s} + 2 \frac{\tau}{k} \frac{\partial k}{\partial s})(-\tau N) - k^3 T + \tau k^2 B - KkT. \]

The coefficient of \( B \) is
\[ \frac{\partial^2 \tau}{\partial s^2} + 2 \frac{\partial}{\partial s} (\tau \frac{\partial k}{\partial s}) + \tau k^2. \]

The right hand side of (4) equals
\[ -\frac{\partial k}{\partial t} T - k (\frac{\partial k}{\partial s} N + \tau k B) + \tau \frac{\partial}{\partial t} B + \tau \nabla_t B. \]

Note that
\[ < \nabla_t B, B > = 0. \]

So the coefficient of \( B \) is
\[ -\tau k^2 + \frac{\partial \tau}{\partial t}. \]

Hence we get
\[ \frac{\partial^2 \tau}{\partial s^2} + 2 \frac{\partial}{\partial s} (\tau \frac{\partial k}{\partial s}) + \tau k^2 = -\tau k^2 + \frac{\partial \tau}{\partial t} \]
\[ \Rightarrow \]
\[ \frac{\partial \tau}{\partial t} = \frac{\partial^2 \tau}{\partial s^2} + 2 \frac{\partial k}{\partial s} \frac{\partial \tau}{\partial s} + 2 \tau \frac{\partial^2 k}{\partial s^2} - \frac{1}{k} (\frac{\partial k}{\partial s})^2 + k^3. \]

\[ \Box \]

If both \( k \) and \( \tau \) only depend on time, then their evolutions become
\[ \begin{align*}
\frac{dk}{dt} &= k^3 - \tau^2 k + Kk \\
\frac{d\tau}{dt} &= 2\tau k^2
\end{align*} \tag{5} \]

In the following, we study the system in space forms. Since the flat case \( R^3 \) was treated by S.J. Altschuler and M.A. Grayson [2], we consider two remaining cases.

Case (1): \( M = H^3 \)
In this case, $K = -1$. The system (5) becomes

\[
\begin{align*}
\frac{dk}{dt} &= k^3 - k - \tau^2 k \\
\frac{d\tau}{dt} &= 2\tau k^2 \tag{6}
\end{align*}
\]

We solve (6) as follows.

For the sake of simplicity, we introduce the following notations. Let

\[
\begin{align*}
u &= k^2, \\
v &= \tau^2.
\end{align*}
\]

Then the system (6) is equivalent to

\[
\begin{align*}
\frac{du}{dt} &= 2u^2 - 2u - 2uv \cdots (\star) \\
\frac{dv}{dt} &= 4uv \cdots (\star\star)
\end{align*}
\]

Without loss of generality, we may assume the initial condition

\[v(0) > 0.\]

From (\star\star), we know that $v$ is non-decreasing. So

\[v(t) > 0\]

for all $t \geq 0$. Then we can divide (\star) by (\star\star) to get

\[\frac{du}{dv} = \frac{u - 1}{2v} - \frac{1}{2}.\]

Define

\[w = u - 1,\]

then

\[\frac{dw}{dv} = \frac{1}{2}(\frac{w}{v} - 1).\]

Also define

\[z = \frac{w}{v},\]

then

\[w = zv.\]

Substituting it into (8), we get

\[\frac{dz}{dv} = \frac{1}{2}(z - 1) - z = -\frac{z + 1}{2v}.\]

Integrating from time $0$ to $t$, we have

\[\frac{z(v(t)) + 1}{z(v(0)) + 1} = \left(\frac{v(t)}{v(0)}\right)^{-\frac{1}{2}}.\]

Note

\[z = \frac{u - 1}{v}.
\]

So

\[\frac{u(v(t)) - 1}{v(t)} + 1 = \left(\frac{v(t)}{v(0)}\right)^{-\frac{1}{2}}.\]
\[ u(v(t)) = 1 + v(t)\left( \frac{v(t)}{v(0)} \right)^{\frac{1}{2}} \left( \frac{u(v(0)) - 1}{v(0)} + 1 \right) - 1 \]

\[ = 1 + (v(t)v(0))^{\frac{1}{2}} \left( \frac{u(v(0)) - 1}{v(0)} + 1 \right) - v(t). \]

Still, without loss of generality, we may assume another initial condition

\[ u(v(0)) = 1. \]

Then

\[ u(v(t)) = 1 + (v(t)v(0))^{\frac{1}{2}} - v(t) \]

Substituting (9) into (**), we have

\[ \frac{dv}{dt} = 4v(1 + (vv_0)^{\frac{1}{2}} - v), \]

where \( v_0 =: v(0). \)

Let

\[ \tilde{\tau} =: \sqrt{v}, \]

i.e.,

\[ v = \tilde{\tau}^2, \]

then

\[ \frac{d\tilde{\tau}}{dt} = 2\tilde{\tau}(1 + \tilde{\tau}_0\tilde{\tau} - \tilde{\tau}^2), \]

where

\[ \tilde{\tau}_0 =: \tilde{\tau}(0). \]

Solving (10), we obtain

\[ \tilde{\tau}^a(-\tilde{\tau} + \frac{\tilde{\tau}_0 + \sqrt{\tilde{\tau}_0^2 + 4}}{2})^b(\tilde{\tau} - \frac{\tilde{\tau}_0 - \sqrt{\tilde{\tau}_0^2 + 4}}{2})^c \]

\[ = \tilde{\tau}_0^a(\frac{-\tilde{\tau}_0 + \sqrt{\tilde{\tau}_0^2 + 4}}{2})^b(\frac{\tilde{\tau}_0 + \sqrt{\tilde{\tau}_0^2 + 4}}{2})^c \cdot \exp(-2t), \ldots \ldots (\diamond) \]

where

\[ a = -1, \]

\[ b = \frac{1}{2} - \frac{\tilde{\tau}_0}{2\sqrt{\tilde{\tau}_0^2 + 4}}, \]

\[ c = \frac{1}{2} + \frac{\tilde{\tau}_0}{2\sqrt{\tilde{\tau}_0^2 + 4}}. \]

Notice that both \( b \) and \( c \) are positive. Let \( t \rightarrow +\infty \), then the right hand side of (\diamond) tends to 0. So it must be

\[ \tilde{\tau} \rightarrow \frac{\tilde{\tau}_0 + \sqrt{\tilde{\tau}_0^2 + 4}}{2}. \]
Together with (9), we see that
\[ u \to 0^+ . \]

**Case (2):** \( M = S^3 \)

In this case, \( K = 1 \). Using the same method as in (1), we know that, as \( t \to +\infty \),
\[ \sqrt{v} \to \frac{m + \sqrt{m^2 + 4}}{2} , \]
where
\[ m =: \sqrt{v(0)} \cdot \left( \frac{2}{v(0)} + 1 \right) , \]
and
\[ u \to 0^+ . \]

**Remark 16.** In both cases, the limiting curves, if they exist, are geodesics. Moreover, the non-zero torsion reflects the fact that the frames are twisting along the geodesics.

6. Ramps in the Flow

In this section, we deal with product Riemannian manifolds \( (M \times S^1, g + d\sigma^2) \). As before, define
\[ \gamma_t(\cdot) =: \gamma(\cdot, t) : S^1 \to M \times S^1 \]
is an evolving immersed curve along the curve shortening flow. Let
\[ \pi_{S^1} : M \times S^1 \to S^1 \]
be projection. It naturally induces a linear mapping
\[ (\pi_{S^1})_* : T(M \times S^1) \to T_{\pi_{S^1}(\cdot)}S^1 . \]

**Definition 17** (definition). We shall call \( \gamma_t \) a ramp if there exists a unit tangent vector field \( U \) to \( S^1 \) such that
\[ < (\pi_{S^1})_*(T), U >_{S^1} > 0 \]
along \( \gamma_t \).

From this definition, it is easy to deduce the following

**Proposition 18.** An immersed curve is a ramp iff the \( TS^1 \)-component of its tangent vector is non-zero everywhere.

**Remark 19.** Ramp is not a new concept. In fact, many authors have studied it before (see [3], [5], [15]). As we will see, ramps have very good properties.

**Claim:** For a curve shortening flow, if \( \gamma_0 \) is a ramp, then for all \( t > 0 \), \( \gamma_t \) is a ramp, too.
Proof. By definition, there exists a unit tangent vector field $U \in T S^1$, such that
\[ u := \langle (\pi_{S^1}), T \rangle, U > S^1 > 0 \]
at $t = 0$. The time derivative of $u$ is
\[ \frac{\partial u}{\partial t} = u'' + k^2 u. \]
Here $t$ denotes differential with respect to $s$. If we define
\[ \mu_t =: \min_{S^1} u(\cdot, t), \]
then $\mu_0 > 0$. (11) tells us that $\mu_t$ is non-decreasing. So we obtain the Claim. □

**Proposition 20.** Assume the sectional curvature of $M \times S^1$ has an upper bound $\Xi > 0$, and $\gamma_0$ is a ramp.

(1) Let
\[ \kappa_t =: \min_{S^1} k(\cdot, t). \]
If $\kappa_t < 0$ for all $t \geq 0$, then
\[ k(\cdot, t) \geq C_1 \exp(\Xi t) \]
for all $t \geq 0$, where $C_1$ is negative and only depends on $\gamma_0$.

(2) Let
\[ \lambda_t =: \max_{S^1} k(\cdot, t). \]
If $\lambda_t > 0$ for all $t \geq 0$, then
\[ k(\cdot, t) \leq C_2 \exp(\Xi t) \]
for all $t \geq 0$, where $C_2$ is positive and only depends on $\gamma_0$.

Proof. Since $\gamma_0$ is a ramp, our Claim guarantees that $\gamma_t$ is always a ramp. So we may divide $k$ by $u$. The time derivative of $k/u$ is
\[ \frac{\partial}{\partial t} \left( \frac{k}{u} \right) = \frac{\partial^2}{\partial s^2} \left( \frac{k}{u} \right) + 2u' \frac{\partial}{\partial s} \left( \frac{k}{u} \right) + \frac{k}{u} \left( k^2 - |DN| \right)^2 + \frac{k}{u} R(T, N, T, N). \]
Notice the third term of the right hand side of (12) is non-positive, and the sectional curvature $R(T, N, T, N)$ is bounded from above by $\Xi$.

(1) If we define
\[ \Phi_t =: \min_{S^1} \frac{k}{u}(\cdot, t), \]
then, by the assumption,
\[ \Phi_t < 0 \]
for all $t \geq 0$. Formula (12) tells us that $\Phi_t$ satisfies
\[ \frac{\partial}{\partial t} \Phi_t \geq \Phi_t (k^2 - |DN|^2) + \Phi_t R(T, N, T, N) \]
\[ \Rightarrow \]
\[ \Phi_t^{-1} \frac{\partial}{\partial t} \Phi_t \leq \Xi \]
\( \Rightarrow \)
\[ \Phi_t \geq \Phi_0 \exp(\Xi t). \]
Note \( \Phi_0 < 0 \) and \( u \leq 1 \). Then we can obtain (1) easily.

(2) If we define

\[ \Psi_t =: \max_{s^i} \frac{k_i}{u}(\cdot, t), \]
then, by the assumption,

\[ \Psi_t > 0 \]

for all \( t \geq 0 \). Formula (12) tells us that \( \Psi_t \) satisfies

\[ \frac{\partial}{\partial t} \Psi_t \leq \Psi_t(k^2 - |\frac{DN}{\partial s}|^2) + \Psi_t R(T, N, T, N) \]
\( \Rightarrow \)

\[ \Psi_t^{-1} \frac{\partial}{\partial t} \Psi_t \leq \Xi \]
\( \Rightarrow \)

\[ \Psi_t \leq \Psi_0 \exp(\Xi t). \]

Note \( \Psi_0 > 0 \) and \( u \leq 1 \). Then we can obtain (2) easily.

By now, Proposition 18 is proved.

The following theorem is a direct consequence of Theorem 10 and Proposition 18.

**Theorem 21.** \( M \times S^1 \) is a compact locally symmetric space. If \( \gamma_0 \) is a ramp, then the curve shortening flow will converge to a geodesic in the \( C^\infty \) norm.

**Proof.** For \( \gamma_0 \) is a ramp, Proposition 18 guarantees that the curve shortening flow will not blow-up in finite time. This means that the flow will exists for infinite time. Moreover, from the proof of Claim, we know that \( \mu_t \) is non-decreasing. This will guarantee that

\[ \lim_{t \to \infty} L(\gamma_t) > 0. \]

Then Theorem 10 tells us that the limiting curve exists and is a geodesic. \( \square \)

**Remark 22.** As we know, closed geodesics theory is a fundamental part of Riemannian geometry. There are a lot of nice works in this theory (\cite{12}). In 1929, L.Lusternik and L.Schnirelmann (\cite{13}) outlined that any Riemannian 2-sphere has at least three simple closed geodesics. Unfortunately, there was a shortcoming in their proof. Later, M.Grayson used the curve shortening flow to prove the three geodesics theorem beautifully (\cite{10}). But, for higher dimensional case, there are only few results. The importance of Theorem 19 is that it points out a possible way to find closed geodesics on a compact locally symmetric space \( M \times S^1 \), i.e., finding closed ramps representing non-trivial homology classes in the path space \( (\Sigma, \Sigma_0) \) of closed curves relative to the point curve, and then evolving them along the curve shortening flow.
7. SHORTENING CURVES IN EVOLVING METRIC

In this section, we want to study how to evolve the metric to make a specific curve shortening flow exists as long as the metric is non-singular. One motivation for this problem is G. Perelman’s work (15) where he considers the curve shortening flow during the Ricci flow.

First, we consider a simple case.

Let \( M \) be an oriented differentiable manifold. Let \( g_0 \) is a metric on \( M \) at \( t = 0 \). For an initial curve \( \gamma_0 \), we shall evolve it according to the curve shortening flow. Meanwhile, we shall change the metric as time goes on according to the conformal flow. More precisely, let

\[
g_t = \exp(f) g_0
\]

be the time-dependent metric on \( M \) at time \( t \geq 0 \), where \( f : M \times [0, +\infty) \to \mathbb{R} \) is a \( C^\infty \) function, satisfying

\[
f(\cdot, 0) \equiv 0.
\]

Theorem 23. Define

\[
\Lambda_t = \{ p \in S^1 | k^2(p, t) = M_t \}.
\]

If \( f \) satisfies

\[
\left. \frac{\partial f}{\partial t} \right|_{\Lambda_t} = 2k^2 + 2R(T, N, T, N),
\]

then the curve shortening flow will exist as long as \( g_t \) keeps non-singular.

Proof. The different from before is that the metric on \( M \) depends on time. So we need to consider its time derivative. For convenience, we will use \( < \cdot, \cdot >_t \) representing \( g_t \).

Let

\[
v_t = \left( \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right)_t^{\frac{1}{2}}.
\]

Then the time derivative of \( v_t \) is

\[
\frac{\partial v_t}{\partial t} = -k^2 v_t + \frac{1}{2} \frac{\partial f}{\partial t} v_t.
\]

The second term comes from the change of metric (compare this equation with Lemma 3).

Next we want to show that covariant differentiation with respect to \( s \) and \( t \) are related by the equation:

\[
\nabla_t \nabla_s = \nabla_s \nabla_t + (k^2 - \frac{1}{2} \frac{\partial f}{\partial t}) \nabla_s + R(T, \frac{DT}{\partial s}).
\]

In fact,

\[
\nabla_t \nabla_s = \frac{\partial}{\partial t} \left( \frac{1}{v_t} \right) \nabla_u + \frac{1}{v_t} \left( \nabla_u \nabla_t + R\left( \frac{\partial}{\partial u}, \frac{\partial}{\partial t} \right) \right).
\]
\[
= (k^2 - \frac{1}{2} \frac{\partial f}{\partial t}) \nabla_s + \nabla_s \nabla_t + R(T, \frac{DT}{\partial s})
\]

Last, the covariant differentiation of \(T\) with respect to time \(t\) is
\[
\nabla_t T = \frac{\partial}{\partial t} \left( \frac{1}{v_t} \frac{\partial \gamma}{\partial u} + \frac{1}{v_t} \nabla_u \frac{\partial \gamma}{\partial t} \right) = (k^2 - \frac{1}{2} \frac{\partial f}{\partial t}) T + \frac{D^2 T}{\partial s^2}.
\]

Now, with these preparations, we can compute the evolution of \(k^2\). The desired result is
\[
\frac{\partial}{\partial t} (k^2) = (k^2)'' - 2 \left( \frac{D^2 T}{\partial s^2} \right)^2 - k^4 + k^2 [2k^2 + 2R(T, N, T, N) - \frac{\partial f}{\partial t}].
\]
Note that the second term on right hand side is non-positive. So if \(f\) satisfy the evolution equation
\[
\frac{\partial f}{\partial t} |_{\Lambda_t} = 2k^2 + 2R(T, N, T, N),
\]
then \(M_t\) is non-increasing. Therefore, it is bounded by \(M_0\) for all \(t > 0\). This implies that, at time \(t\), the curve can go on flowing if \(g_t\) is non-singular. \(\square\)

**Example 24.** Consider on \(\mathbb{R}^2\),
\[
g_0 = dx^2 + dy^2.
\]
Let
\[
\gamma_0(\theta) = (\cos \theta, \sin \theta), \theta \in S^1,
\]
i.e., the unit circle. For this initial curve, we have
\[
k^2(\cdot, 0) \equiv 1.
\]
If we change the metric according to the conformal equation
\[
g_t = \exp(2t) g_0,
\]
i.e., let
\[
f \equiv 2t,
\]
then the evolving curve satisfies
\[
\gamma_t = \exp(-t) \gamma_0.
\]
It is easy to see that
\[
k^2(\cdot, t) \equiv 1
\]
for all \(t > 0\).

Observe that the circle collapses to a point in infinite time rather than finite time. Similar phenomenon also appears in [1] where the authors modified the usual curve shortening flow and introduced new time parameter. We will not go further in that direction. Readers who are interested in this topic can read their excellent paper.
Finally, we will investigate a little more complicated case.

\((M \times S^1, g + \exp(f) du^2)\) is a warped Riemannian manifold (for a warped Riemannian manifold, see, for example, [13]). In this case, we will fix \(g\), and only change \(f\) as time goes on. We want to decide how to change \(f\) to make a specific curve shortening flow exist as long as the metric keeps non-singular.

As before, let

\(\pi_{S^1} : M \times S^1 \to S^1\)

be projection, with induced mapping

\((\pi_{S^1})_* : TM \times S^1 \to T\pi_{S^1}(\cdot)S^1\).

Still, for convenience, we will use \(<\cdot, \cdot>_t\) and \(<\cdot, \cdot>_t^{S^1}\) representing metrics on \(M \times S^1\) and \(S^1\) respectively. Let

\[ v_t := (<\frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}>_t)^{\frac{1}{2}}. \]

Then the time derivative of \(v_t\) is

\[ \frac{\partial v_t}{\partial t} = -k^2 v_t + \frac{1}{2} \frac{\partial f}{\partial t} \left( (\pi_{S^1})_* (T) \right)^{S^1}_t v_t. \]

A straightforward calculation shows

\[ \nabla_t \nabla_s = \nabla_s \nabla_t + (k^2 - \frac{1}{2} \frac{\partial f}{\partial t} \left( (\pi_{S^1})_* (T) \right)^{S^1}_t) \nabla_s + R(T, \frac{DT}{\partial s}), \]

and

\[ \nabla_t T = (k^2 - \frac{1}{2} \frac{\partial f}{\partial t} \left( (\pi_{S^1})_* (T) \right)^{S^1}_t) T + \frac{D^2 T}{\partial s^2}. \]

Now we compute the evolution of \(k^2\):

\[ \frac{\partial}{\partial t} (k^2) = (k^2)'' - 2 \left( \frac{D^2 T}{\partial s^2} \right)^2 - k^4 + 2k^2 \left( \frac{2k^2 + 2R(T, N, T, N)}{2((\pi_{S^1})_* (T))^{S^1}} \right) - \frac{\partial f}{\partial t} \left[ 2((\pi_{S^1})_* (T))^{S^1} - ((\pi_{S^1})_* (N))^{S^1} \right]. \]

If

\[ 2((\pi_{S^1})_* (T))^{S^1} - ((\pi_{S^1})_* (N))^{S^1} \neq 0 \]

on \(\Lambda_t\) at time \(t\), then we can require \(f\) satisfy

\[ \left. \frac{\partial f}{\partial t} \right|_{\Lambda_t} = \frac{2k^2 + 2R(T, N, T, N)}{2((\pi_{S^1})_* (T))^{S^1} - ((\pi_{S^1})_* (N))^{S^1}}. \]

So \(M_t\) is non-increasing. By now, we have eventually proved

**Theorem 25.** Assume

\[ 2((\pi_{S^1})_* (T))^{S^1} - ((\pi_{S^1})_* (N))^{S^1} \neq 0 \]

on \(\Lambda_t\) for all \(t \geq 0\), and \(f\) satisfies

\[ \left. \frac{\partial f}{\partial t} \right|_{\Lambda_t} = \frac{2k^2 + 2R(T, N, T, N)}{2((\pi_{S^1})_* (T))^{S^1} - ((\pi_{S^1})_* (N))^{S^1}}. \]
Then the curve shortening flow will exist as long as \( g \) keeps non-singular.

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