Efficient Clustering for Stretched Mixtures: Landscape and Optimality

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Abstract

This paper considers a canonical clustering problem where one receives unlabeled samples drawn from a balanced mixture of two elliptical distributions and aims for a classifier to estimate the labels. Many popular methods including PCA and k-means require individual components of the mixture to be somewhat spherical, and perform poorly when they are stretched. To overcome this issue, we propose a non-convex program seeking for an affine transform to turn the data into a one-dimensional point cloud concentrating around $-1$ and $1$, after which clustering becomes easy. Our theoretical contributions are two-fold: (1) we show that the non-convex loss function exhibits desirable geometric properties when the sample size exceeds some constant multiple of the dimension, and (2) we leverage this to prove that an efficient first-order algorithm achieves near-optimal statistical precision without good initialization. We also propose a general methodology for clustering with flexible choices of feature transforms and loss objectives.

Keywords: clustering, dimensionality reduction, unsupervised learning, landscape, nonconvex optimization

1 Introduction

Clustering is a fundamental problem in data science, especially in the early stages of knowledge discovery. Its wide applications include genomics (Eisen et al., 1998; Remm et al., 2001), imaging (Filipovych et al., 2011), linguistics (Di Marco andNavigli, 2013), networks (Adamic and Glance, 2005), and finance (Arnott, 1980; Zhu et al., 2020), to name a few. They have motivated numerous characterizations for “clusters” and associated learning procedures.

In this paper, we consider a binary clustering problem where the data come from a mixture of two elliptical distributions. Suppose that we observe i.i.d. samples $\{X_i\}_{i=1}^n \subseteq \mathbb{R}^d$ from the latent variable model

$$X_i = \mu_0 + \mu Y_i + \Sigma^{1/2} Z_i, \quad i \in [n].$$

Here $\mu_0, \mu \in \mathbb{R}^d$ and $\Sigma > 0$ are deterministic; $Y_i \in \{\pm 1\}$ and $Z_i \in \mathbb{R}^d$ are independent random quantities; $\mathbb{P}(Y_i = -1) = \mathbb{P}(Y_i = 1) = 1/2$, and $Z_i$ is an isotropic random vector whose distribution is spherically symmetric with respect to the origin. $X_i$ is elliptically distributed (Fang et al., 1990) given $Y_i$. The goal of clustering is to estimate $\{Y_i\}_{i=1}^n$ from $\{X_i\}_{i=1}^n$. Moreover, it is desirable to build a classifier with straightforward out-of-sample extension that predicts labels for future samples.

As a warm-up example, assume for simplicity that $Z_i$ has density and $\mu_0 = 0$. The Bayes-optimal classifier is

$$\varphi_{\beta^*}(x) = \text{sgn}(\beta^* \top x) = \begin{cases} 1 & \text{if } \beta^* \top x \geq 0 \\ -1 & \text{otherwise} \end{cases},$$

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A good approximate solution to (2) leads to
\[ \min_{\beta \in \mathbb{R}^d} \sum_{i=1}^n f(\beta^T X_i). \]
where \( f(x) = (x^2 - 1)^2 \). We name this method as “Clustering via
Uncoupled REgression”, or CURE for short. Here \( f \) penalizes the discrepancy between predictions \( \{\beta^T X_i\}_{i=1}^n \)
and labels \( \{Y_i\}_{i=1}^n \). In the unsupervised setting, we have no access to the one-to-one correspondence but can
still enforce proximity on the distribution level, i.e.
\[ \frac{1}{n} \sum_{i=1}^n \delta_{\beta^T X_i} \approx \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1. \]
A good approximate solution to (2) leads to \( |\beta^T X_i| \approx 1 \). That is, the transformed data form two clusters
around \( \pm 1 \). The symmetry of the mixture distribution automatically ensures balance between the clusters.
Thus (2) is an uncoupled regression problem based on (3). Above we focus on the centered case \( (\mu_0 = 0) \)
merely to illustrate main ideas. Our general methodology
\[ \min_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}^d} \left\{ \frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i) + \frac{1}{2} (\alpha + \beta^T \mu_0)^2 \right\}, \]
where \( \mu_0 = \frac{1}{n} \sum_{i=1}^n X_i \), deals with arbitrary \( \mu_0 \) by incorporating an intercept term \( \alpha \).

**Main contributions.** We propose a clustering method through (4) and study it under the model (1)
without requiring the clusters to be spherical. Under mild assumptions, we prove that an efficient algorithm
achieves near-optimal statistical precision even in the absence of a good initialization.

- **(Loss function design)** We construct an appropriate loss function \( f \) by clipping the growth of the quartic
function \( (x^2 - 1)^2/4 \) outside some interval centered at 0. As a result, \( f \) has two “valleys” at \( \pm 1 \) and does
not grow too fast, which is beneficial to statistical analysis and optimization.

- **(Landscape analysis)** We characterize the geometry of the empirical loss function when \( n/d \) exceeds
some constant. In particular, all second-order stationary points, where the smallest eigenvalues of Hessians
are not significantly negative, are nearly optimal in the statistical sense.
• (Efficient algorithm with near-optimal statistical property) We show that with high probability, a perturbed version of gradient descent algorithm starting from 0 yields a solution with near-optimal statistical property after $O(n/d + d^2/n)$ iterations (up to polylogarithmic factors).

The formulation (4) is uncoupled linear regression for binary clustering. Beyond that, we introduce a unified framework which learns feature transforms to identify clusters with possibly non-convex shapes. That provides a principled way of designing flexible unsupervised learning algorithms.

We introduce the model and methodology in Section 2 conduct theoretical analysis in Section 3 present numerical results in Section 4 and finally conclude the paper with a discussion in Section 5.

Related work. Methodologies for clustering can be roughly categorized as generative and discriminative ones. Generative approaches fit mixture models for the joint distribution of features $X$ and label $Y$ to make predictions (Moitra and Valiant, 2010; Kannan et al., 2005; Anandkumar et al., 2014). Their success usually hinges on well-specified models and precise estimation of parameters. Since clustering is based on the conditional distribution of $Y$ given $X$, it only involves certain functional of parameters. Generative approaches often have high overhead in terms of sample size and running time. On the other hand, discriminative approaches directly aim for predictive classifiers. A common strategy is to learn a transform to turn the data into a low-dimensional point cloud that facilitates clustering. Statistical analysis of mixture models lead to information-based methods (Bridle et al., 1992; Krause et al., 2010), analogous to the logistic regression for supervised classification. Geometry-based methods uncover latent structures in an intuitive way, similar to the support vector machine. Our method CURE belongs to this family. Other examples include projection pursuit (Friedman and Tukey, 1974; Peña and Prieto, 2001a), margin maximization (Ben-Hur et al., 2001; Xu et al., 2005), discriminative $k$-means (Ye et al., 2008; Bach and Harchaoui, 2008), graph cut optimization by spectral methods (Shi and Malik, 2000; Ng et al., 2002) and semidefinite programming (Weinberger and Saul, 2006), correlation clustering (Bunea et al., 2020; Jarrow et al., 2020). Discriminative methods are easily integrated with modern tools such as deep neural networks (Springenberg, 2015; Xie et al., 2016). The list above is far from exhaustive.

The formulation (4) is invariant under invertible affine transforms of data and thus tackles stretched mixtures which are catastrophic for many existing approaches. A recent paper (Kushnir et al., 2019) uses random projections to tackle such problem but requires the separation between two clusters to grow at the order of $\sqrt{d}$, where $d$ is the dimension. There have been provable algorithms dealing with general models with multiple classes and minimal separation conditions (Brubaker and Vempala, 2008; Kalai et al., 2010; Belkin and Sinha, 2015). However, their running time and sample complexity are large polynomials in the dimension and desired precision. In the class of two-component mixtures we consider, CURE has near-optimal (linear) sample complexity and runs fast in practice. Another relevant area of study is clustering under sparse mixture models (Azizyan et al., 2013; Verzelen and Arias-Castro, 2017), where additional structures help handle non-spherical clusters efficiently.

The vanilla version of CURE in (2) is closely related to the Projection Pursuit (PP) (Friedman and Tukey, 1974) and Independent Component Analysis (ICA) (Hyvärinen and Oja, 2000). PP and ICA find the most nontrivial direction by maximizing the deviation of the projected data from some null distribution (e.g., Gaussian). Their objective functions are designed using key features of that. Notably, Peña and Prieto (2001a) propose clustering algorithms based on extreme projections that maximize and minimize the kurtosis; Verzelen and Arias-Castro (2017) use the first absolute moment and skewness to construct objective functions in pursuit of projections for clustering. On the contrary, CURE stems from uncoupled regression and minimizes the discrepancy between the projected data and some target distribution. This makes it generalizable beyond linear feature transforms with flexible choices of objective functions. Moreover, CURE has nice computational guarantees while only a few algorithms for PP and ICA do. The formulation (2) with double-well loss $f$ also appears in the real version of Phase Retrieval (PR) (Candes et al., 2015) for recovering a signal $\beta$ from noisy quadratic measurements $Y_i \approx (X_i^\top \beta)^2$. In both CURE and PR, one observes the magnitudes of labels/outputs without sign information. However, algorithmic study of PR usually require $\{X_i\}_{i=1}^n$ to be isotropic Gaussian; most efficient algorithms need good initializations by spectral methods. Those cannot be easily adapted to clustering. Our analysis of CURE could provide a new way of studying PR under more general conditions.
Here we want to achieve low excess risk. What they can handle. While multiplication by \( \Sigma \) is undesirable technicalities, and is commonly adopted in the study of parameter estimation in mixture models. There exists an easy fix for that. The last assumption in Model 1 makes the loss landscape regular, helps avoid undesired technicalities, and it is not clear whether there exists an easy fix for that. The last assumption in Model 1 makes the loss landscape regular, helps avoid undesired technicalities, and it is not clear whether there exists an easy fix for that.

2 Problem setup

2.1 Elliptical mixture model

Model 1. Let \( X \in \mathbb{R}^d \) be a random vector with the decomposition

\[
X = \mu_0 + \mu Y + \Sigma^{1/2} Z.
\]

Here \( \mu_0, \mu \in \mathbb{R}^d \) and \( \Sigma > 0 \) are deterministic; \( Y \in \{ \pm 1 \} \) and \( Z \in \mathbb{R}^d \) are random and independent. Let \( Z = e_i^T Z \), \( \rho \) be the distribution of \( X \) and \( \{ X_i \}_{i=1}^n \) be i.i.d. samples from \( \rho \).

- (Balanced classes) \( \mathbb{P}(Y = -1) = \mathbb{P}(Y = 1) = 1/2 \);
- (Elliptical sub-Gaussian noise) \( Z \) is sub-Gaussian with \( \|Z\|_{\psi_2} \) bounded by some constant \( M \), \( \mathbb{E} Z = 0 \) and \( \mathbb{E}(ZZ^\top) = I_d \); its distribution is spherically symmetric with respect to \( 0 \);
- (Leptokurtic distribution) \( \mathbb{E} Z^4 - 3 > \kappa_0 \) holds for some constant \( \kappa_0 > 0 \);
- (Regularity) \( \|\mu_0\|_2, \|\mu\|_2, \lambda_{\text{max}}(\Sigma) \) and \( \lambda_{\text{min}}(\Sigma) \) are bounded away from 0 and \( \infty \) by constants.

We aim to build a classifier \( \mathbb{R}^d \rightarrow \{ \pm 1 \} \) based solely on the samples \( \{ X_i \}_{i=1}^n \) from a mixture of two elliptical distributions. For simplicity, we assume that the two classes are balanced and focus on the well-conditioned case where the signal strength and the noise level are of constant order. This is already general enough to include stretched clusters incapacitating many popular methods including PCA, \( k \)-means and semi-definite relaxations (Brubaker and Vempala [2008]). One may wonder whether it is possible to transform the data into what they can handle. While multiplication by \( \Sigma^{-1/2} \) yields spherical clusters, precise estimation of \( \Sigma^{-1/2} \) or \( \Sigma \) is no easy task under the mixture model. Dealing with those \( d \times d \) matrices causes overhead expenses in computation and storage. The assumption on positive excess kurtosis prevents the loss function from having undesirable degenerate saddle points and facilitates the proof of algorithmic convergence. It rules out distributions whose kurtoses do not exceed that of the normal distribution, and it is not clear whether there exists an easy fix for that. The last assumption in Model 1 makes the loss landscape regular, helps avoid undesirable technicalities, and is commonly adopted in the study of parameter estimation in mixture models. The Bayes optimal classification error is of constant order, and we want to achieve low excess risk.

2.2 Clustering via Uncoupled Regression

Under Model 1 the Bayes optimal classifier for predicting \( Y \) given \( X \) is

\[
\hat{Y}^{\text{Bayes}}(X) = \text{sgn} (\alpha^{\text{Bayes}} + \beta^{\text{Bayes}}^T X),
\]

where \( \alpha^{\text{Bayes}}, \beta^{\text{Bayes}} = (-\mu_0^\top \Sigma^{-1} \mu, \Sigma^{-1} \mu) \). On the other hand, it is easily seen that the following (population-level) least squares problem \( \mathbb{E}[(\alpha + \beta^T X - Y)^2] \) has a unique solution \( \alpha^{LR}, \beta^{LR} = (-c_0 \Sigma^{-1} \mu, c \Sigma^{-1} \mu) \).
for some $c > 0$. For the supervised classification problem where we observe $\{(X_i, Y_i)\}_{i=1}^n$, the optimal feature transform can be estimated via linear regression

$$\frac{1}{n} \sum_{i=1}^n [(\alpha + \beta^T X_i) - Y_i]^2.$$  \hfill (5)

This is closely related to Fisher’s Linear Discriminant Analysis [Friedman et al. [2001]].

In the unsupervised clustering problem, we no longer observe individual labels $\{Y_i\}_{i=1}^n$ but have population statistics of labels, as the classes are balanced. While (6) directly forces $\alpha + \beta^T X_i \approx Y_i$ thanks to supervision, here we relax such proximity to the population level:

$$\frac{1}{n} \sum_{i=1}^n \delta_{\alpha + \beta^T X_i} \approx \frac{1}{2} \delta_1 + \frac{1}{2} \delta_1.$$  \hfill (6)

Thus the regression should be conducted in an uncoupled manner using marginal information about $X$ and $Y$. We seek for an affine transformation $x \mapsto \alpha + \beta^T x$ to turn the samples $\{X_i\}_{i=1}^n$ into two balanced clusters around $\pm 1$, after which $\text{sgn}(\alpha + \beta^T X)$ predicts $Y$ up to a global sign flip. It is also supported by the geometric intuition in Section 1 based on projections of the mixture distribution.

Clustering via Uncoupled REgression (CURE) is formulated as an optimization problem:

$$\min_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}^d} \left\{ \frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i) + \frac{1}{2} (\alpha + \beta^T \mu_0)^2 \right\},$$  \hfill (7)

where $\mu_0 = \frac{1}{n} \sum_{i=1}^n X_i$. $f$ attains its minimum at $\pm 1$. Minimizing $\frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i)$ makes the transformed data $\{\alpha + \beta^T X_i\}_{i=1}^n$ concentrate around $\{\pm 1\}$. However, there are always two trivial minimizers $(\alpha, \beta) = (\pm 1, 0)$, each of which maps the entire dataset to a single point. What we want are two balanced clusters around $-1$ and $1$. The centered case ($\mu_0 = 0$) discussed in Section 1 does not have such trouble as $\alpha$ is set to be $0$ and the symmetry of the mixture automatically balance the two clusters. For the general case, we introduce a penalty term $(\alpha + \beta^T \mu_0)^2 / 2$ in (7) to drive the center of the transformed data towards $0$. The idea comes from moment-matching and is similar to that in [Flammarion et al. [2017]]. If $\frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i)$ is small, then $|\alpha + \beta^T X_i| \approx 1$ and

$$\frac{1}{n} \sum_{i=1}^n \delta_{\alpha + \beta^T X_i} \approx \frac{|\{i : \alpha + \beta^T X_i \geq 0\}|}{n} \delta_1 + \frac{|\{i : \alpha + \beta^T X_i < 0\}|}{n} \delta_1.$$  \hfill (8)

Then, in order to get (8), we simply match the expectations on both sides. This gives rise to the quadratic penalty term in (7). The same idea generalizes beyond the balanced case. When the two classes $1$ and $-1$ have probabilities $p$ and $(1-p)$, we can match the mean of $\{\alpha + \beta^T X_i\}_{i=1}^n$ with that of a new target distribution $p\delta_1 + (1-p)\delta_{-1}$, and change the quadratic penalty to $[(\alpha + \beta^T \mu_0) - (2p - 1)]^2$. When $p$ is unknown, (7) can always be a surrogate as it seeks for two clusters around $\pm 1$ and uses the quadratic penalty to prevent any of them from being vanishingly small.

The function $f$ in (7) requires careful design. To facilitate statistical and algorithmic analysis, we want $f$ to be twice continuously differentiable and grow slowly. That makes the empirical loss smooth and concentrate well around its population counterpart. In addition, the coercivity of $f$, i.e. $\lim_{|x| \to \infty} f(x) = +\infty$, confines all minimizers within some ball of moderate size. Similar to the construction of Huber loss [Huber [1964]], we start from $h(x) = (x^2 - 1)^2 / 4$, keep its two valleys around $\pm 1$, clip its growth using linear functions and interpolate in between using cubic splines:

$$f(x) = \begin{cases} h(x), & |x| \leq a \\ h(a) + h'(a)(|x| - a) + \frac{h''(a)}{2}(|x| - a)^2 - \frac{h'''(a)}{6(a - b)}(|x| - a)^3, & a < |x| \leq b \\ f(b) + [h'(a) + \frac{b - a}{2}h''(a)](|x| - b), & |x| > b \end{cases}.$$  \hfill (8)

Here $b > a > 1$ are constants to be determined later. $f$ is clearly not convex, and neither is the loss function in (7). Yet we can find a good approximate solution efficiently by taking advantage of statistical assumptions and recent advancements in non-convex optimization [Jin et al. [2017]].
Algorithm 1 Clustering via Uncoupled REgression (meta-algorithm)

Input: Data \( \{X_i\}_{i=1}^n \) in a feature space \( \mathcal{X} \), embedding space \( \mathcal{Y} \), target distribution \( \nu \) over \( \mathcal{Y} \), discrepancy measure \( D \), function class \( \mathcal{F} \), classification rule \( g \).

Embedding: find an approximation solution \( \hat{\varphi} \) to \( \min_{\varphi \in \mathcal{X}} D(\varphi \# \hat{\rho}_n, \nu) \).

Output: \( \hat{Y}_i = g[\hat{\varphi}(X_i)] \) for \( i \in [n] \).

Algorithm 2 Perturbed gradient descent

Initialize \( \gamma^0 = 0 \).

For \( t = 0, 1, \ldots \) do

- If perturbation condition holds: Perturb \( \gamma^t \leftarrow \gamma^t + \xi_t^t \) with \( \xi_t^t \sim U(B(0, r)) \)
- If termination condition holds: Return \( \gamma^t \)

Update \( \gamma^{t+1} \leftarrow \gamma^t - \eta \nabla L_1(\gamma^t) \).

2.3 Generalization

The aforementioned procedure seeks for a one-dimensional embedding of the data that facilitates clustering. It searches for the best affine function such that the transformed data look like a two-point distribution. The idea of uncoupled linear regression can be easily generalized to any suitable target probability distribution \( \nu \) over a space \( \mathcal{Y} \), class of feature transforms \( \mathcal{F} \) from the original space \( \mathcal{X} \) to \( \mathcal{Y} \), discrepancy measure \( D \) that quantifies the difference between the transformed data distribution and \( \nu \), and classification rule \( g: \mathcal{Y} \rightarrow [K] \).

CURE for Model [1] above uses \( \mathcal{X} = \mathbb{R}^d, \mathcal{Y} = \mathbb{R}, \nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1, \mathcal{F} = \{ x \mapsto \alpha + \beta^T x : \alpha \in \mathbb{R}, \beta \in \mathbb{R}^d \}, g(y) = \text{sgn}(y) \) and

\[
D(\mu, \nu) = |E_{X \sim \mu} f(X) - E_{X \sim \nu} f(X)| + \frac{1}{2} |E_{X \sim \mu} X - E_{X \sim \nu} X|^2
\]

for any probability distribution \( \mu \) over \( \mathbb{R} \). Here we briefly show why (9) is true. Fix any \( f: x \mapsto \alpha + \beta^T x \) in \( \mathcal{F} \) and let \( \mu = \frac{1}{n} \sum_{i=1}^n \delta_{\alpha + \beta^T X_i} \) be the transformed data distribution. From \( f(-1) = f(1) = 0 \) and \( E_{X \sim \nu} X = 0 \) we see

\[
|E_{X \sim \mu} f(X) - E_{X \sim \nu} f(X)| = E_{X \sim \mu} f(X) = \frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i),
\]

\[
|E_{X \sim \mu} X - E_{X \sim \nu} X|^2 = \left( \frac{1}{n} \sum_{i=1}^n (\alpha + \beta^T X_i) \right)^2 = (\alpha + \beta^T \mu_0),
\]

\[
D(\mu, \nu) = \frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i) + \frac{1}{2} (\alpha + \beta^T \mu_0).
\]

On top of that, we propose a general framework for clustering (also named as CURE) and describe it at a high level of abstraction in Algorithm [2]. Here \( \hat{\rho}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \) is the empirical distribution of data and \( \varphi \# \hat{\rho}_n = \frac{1}{n} \sum_{i=1}^n \delta_{\varphi(X_i)} \) is the push-forward distribution. The general version of CURE is a flexible framework for clustering based on uncoupled regression (Rigollet and Weed [2019]). For instance, we may set \( \mathcal{Y} = \mathbb{R}^K \) and \( \nu = \frac{1}{K} \sum_{k=1}^K \delta_{c_k} \) when there are \( K \) clusters; choose \( \mathcal{F} \) to be the family of convolutional neural networks for image clustering; let \( D \) be the Wasserstein distance or some divergence. CURE is easily integrated with other tools, see Section A.2 in the supplementary material.

3 Theoretical analysis

3.1 Main results

Let \( L_1(\alpha, \beta) \) denote the objective function of CURE in (7). Our main result (Theorem [1]) shows that with high probability, a perturbed version of gradient descent (Algorithm 2) applied to \( L_1 \) returns an approximate minimizer that is nearly optimal in the statistical sense, within a reasonable number of iterations. Here
\( U(B(\mathbf{0}, r)) \) refers to the uniform distribution over \( B(\mathbf{0}, r) \). We omit technical details of the algorithm and defer them to Appendix B.4, see Algorithm 3 and Theorem 4 therein. For notational simplicity, we write \( \gamma = (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^d \) and \( \gamma_{\text{Bayes}} = (\alpha_{\text{Bayes}}, \beta_{\text{Bayes}}) = (-\mu^T \Sigma^{-1} \mu_0, \Sigma^{-1} \mu). \) \( \gamma_{\text{Bayes}} \) defines the Bayes-optimal classifier \( x \mapsto \text{sgn}(\alpha_{\text{Bayes}} + \beta_{\text{Bayes}}^T x) \) for Model 1.

**Theorem 1** (Main result). Let \( \gamma_0, \gamma_1, \cdots \) be the iterates of Algorithm 3 starting from 0. Under Model 1 there exist constants \( c, C_0, C_1, C_2 > 0 \) independent of \( n, d \) and \( s \) such that if \( n \geq Cd \) and \( b \geq 2a \geq C_0 \), then with probability at least 1 - \( C_1[(d/n)^{C_2d} + e^{-C_2n^{1/3}} + n^{-10}] \), Algorithm 3 terminates within \( \tilde{O}(n/d + d^2/n) \) iterations and the output \( \gamma \) satisfies

\[
\min_{s = \pm 1} \| s \gamma - c \gamma_{\text{Bayes}} \|_2 \lesssim \frac{d}{n} \log(n^2/d).
\]

Up to a \( \sqrt{\log(n/d)} \) factor, this matches the optimal rate of convergence \( O(\sqrt{d/n}) \) for the supervised problem with \( \{Y_i\}_{i=1}^n \) observed, which is even easier than the current one. Theorem 1 asserts that we can achieve a near-optimal rate efficiently without good initialization, although the loss function is non-convex. The two terms \( n/d \) and \( d^2/n \) in the iteration complexity have nice interpretations. When \( n \) is large, we want a small computational error in order to achieve statistical optimality. The term \( n/d \) reflects the cost for this. When \( n \) is small, the empirical loss function does not concentrate well and is not smooth enough either. Hence we choose a conservative step-size and pay the corresponding price \( d^2/n \).

**Corollary 1** (Misclassification rate). Consider the settings in Theorem 1 and suppose that \( Z = e_1^T Z \) has density \( p \in C^1(\mathbb{R}) \) satisfying \( \|p\|_\infty \leq C_3 \) and \( \|p'\|_\infty \leq C_3 \) for some constant \( C_3 > 0 \). For \( \gamma = (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^d \), define its misclassification rate (up to a global sign flip) as

\[
R(\gamma) = \min_{s = \pm 1} \mathbb{P}\{s \text{sgn} (\alpha + \beta^T X) \neq Y\}.
\]

There exists a constant \( C_4 \) such that

\[
\mathbb{P}(R(\hat{\gamma}) \leq R(\gamma_{\text{Bayes}}) + \frac{C_4d \log(n/d)}{n}) \geq 1 - C_1[(d/n)^{C_2d} + e^{-C_2n^{1/3}} + n^{-10}].
\]

### 3.2 Sketch of proof

The loss function \( \hat{L}_1 \) is non-convex in general. To find an approximate minimizer efficiently without good initialization, we need \( \hat{L}_1 \) to exhibit benign geometric properties that can be exploited by a simple algorithm. Our choice is the perturbed gradient descent algorithm in Jin et al. (2017), see Algorithm 3 in Appendix B.4 for more details. Provided that the function is smooth enough, it provably converges to an approximate second-order stationary point where the norm of gradient is small and the Hessian matrix does not have any significantly negative eigenvalue. Then it boils down to landscape analysis of \( \hat{L}_1 \) with precise characterizations of approximate stationary points. To begin with, define the population version of \( \hat{L}_1 \) as

\[
L_1(\alpha, \beta) = \mathbb{E}_{X \sim p} f(\alpha + \beta^T X) + \frac{1}{2}(\alpha + \beta^T \mu_0)^2.
\]

**Proposition 1.** There exist positive constants \( c, \varepsilon, \delta, \eta \) and a set \( S \subseteq \mathbb{R} \times \mathbb{R}^d \) such that

1. The only two local minima of \( L_1 \) are \( \pm \gamma^* \) with \( \gamma^* = -c \gamma_{\text{Bayes}} \);
2. All the other first-order critical points (i.e. with zero gradient) are within \( \delta \) distance to \( S \);
3. \( \| \nabla L_1(\gamma) \|_2 \geq \varepsilon \) if \( \text{dist}(\gamma, \{\pm \gamma^*\} \cup S) \geq \delta \);
4. \( \nabla^2 L_1(\gamma) \geq \eta I \) if \( \text{dist}(\gamma, \{\pm \gamma^*\}) \leq \delta \), and \( \lambda_{\min}[\nabla^2 L_1(\gamma)] \leq -\eta \) if \( \text{dist}(\gamma, S) \leq \delta \).
Figure 1: Visualization of the dataset via PCA. The left plot shows the transformed data via PCA. The right plot is a 2-dimensional visualization of the dataset using PCA.

Figure 2: In sample and out-of-sample misclassification rate (with error bar quantifying one standard deviation) vs. iteration count for CURE over 50 independent trials. The four plots corresponds to $N_2 = 6000, 3000, 2000$ and $1500$ respectively, while $N_1$ is always fixed to be 6000.

Proposition 1 shows that all of the approximate second-order critical points of $L_1$ are close to that corresponding to the Bayes-optimal classifier. Then we will prove similar results for the empirical loss $\hat{L}_1$ using concentration inequalities, which leads to the following proposition translating approximate second-order stationarity to estimation error.

**Proposition 2.** There exists a constant $C$ such that the followings happen with high probability: for any $\gamma \in \mathbb{R} \times \mathbb{R}^d$ satisfying $\|\nabla \hat{L}_1(\gamma)\|_2 \leq \varepsilon/2$ and $\lambda_{\min}[\nabla^2 \hat{L}_1(\gamma)] > -\eta/2$,

$$\min_{s = \pm 1} \|s\gamma - \gamma^*\|_2 \leq C \left( \|\nabla \hat{L}_1(\gamma)\|_2 + \sqrt{\frac{d}{n} \log \left( \frac{n}{d} \right)} \right).$$

To achieve near-optimal statistical error (up to a $\sqrt{\log(n/d)}$ factor), Proposition 2 asserts that it suffices to find any $\tilde{\gamma}$ such that $\|\nabla \tilde{L}_1(\tilde{\gamma})\|_2 \leq \sqrt{d/n}$ and $\lambda_{\min}[\nabla^2 \tilde{L}_1(\tilde{\gamma})] > -\eta/2$. Here the perturbed gradient descent algorithm comes into play, and we see the light at the end of the tunnel. It remains to estimate the Lipschitz smoothness of $\nabla \tilde{L}_1$ and $\nabla^2 \tilde{L}_1$ with respect to the Euclidean norm. Once this is done, we can directly apply the convergence theorem in Jin et al. (2017) for the perturbed gradient descent. A more comprehensive outline of the proof and all the details are deferred to the Appendix.

4 Numerical experiments

In this section, we conduct numerical experiments on a real dataset. We randomly select $N_1$ (resp. $M_1$) T-shirts/tops and $N_2$ (resp. $M_2$) pullovers from the Fashion-MNIST (Xiao et al., 2017) training (resp. testing) dataset, each of which is a $28 \times 28$ grayscale image represented by a vector in $[0, 1]^{28 \times 28}$. The goal is clustering, i.e., learning from those $N = N_1 + N_2$ unlabeled images to predict the class labels of both $N$ training samples and $M = M_1 + M_2$ testing samples. The inputs for CURE and other methods are raw images and their pixel-wise centered versions, respectively. To get a sense why this problem is difficult, we set $N_1 = N_2 = 6000$.
Figure 3: Histograms of transformed out-of-sample data for CURE. The red bins correspond to T-shirts/tops, and the blue bins correspond to pullovers.

| Method                      | $N_1 : N_2$ | 1 : 1 | 2 : 1 | 3 : 1 | 4 : 1 |
|-----------------------------|-------------|-------|-------|-------|-------|
| CURE                        |             | 5.2 ± 0.2% | 7.1 ± 0.4% | 9.3 ± 0.7% | 11.3 ± 1.1% |
| K-means                     |             | 45.1% | 49.7% | 46.8% | 45.1% |
| Spectral method (vanilla)   |             | 42.2% | 46.9% | 49.7% | 49.0% |
| Spectral method (Gaussian kernel) |       | 49.9% | 33.4% | 25.0% | 20.0% |

Table 1: Misclassification rate of CURE and other methods.

and plot the transformed data via PCA in the left panel of Figure 1, the transformation does not give meaningful clustering information, and the misclassification rate is 42.225%. A 2-dimensional visualization of the dataset using PCA (right panel of Figure 1) shows two stretched clusters, which cause the PCA to fail. In this dataset, the bulk of a image corresponds to the belly part of clothing with different grayscales, logos and hence contributes to the most of variability. However, T-shirts and Pullovers are distinguished by sleeves. Hence the two classes can be separated by a linear function that is not related to the leading principle component of data. CURE aims for such direction onto which the projected data exhibit cluster structures.

To show that CURE works beyond our theory, we set $N_1$ to be 6000 and choose $N_2$ from $\{6000, 3000, 2000, 1500\}$ to include unbalanced cases. We set $M_1$ to be 1000 and choose $M_2$ from $\{1000, 500, 333, 250\}$. We use gradient descent with random initialization from the unit sphere and learning rate $10^{-3}$ (instead of perturbed gradient descent) to solve (7) as that requires less tuning. Figure 2 shows the learning curves of CURE over 50 independent trials. Even when the classes are unbalanced, CURE still reliably achieves low misclassification rates. Figure 3 presents histograms of testing data under the feature transform learned by the last (50th) trial of CURE, showing two separated clusters around ±1 corresponding to the two classes. To demonstrate the efficacy of CURE, we compare its misclassification rates with those of K-means and spectral methods on the training sets. We include the standard deviation over 50 independent trials for CURE due to its random initializations; other methods use the default settings (in Python) and thus are regarded as deterministic algorithms. As is shown in Table 1, CURE has the best performance under all settings.

5 Discussion

Motivated by the elliptical mixture model (Model 1), we propose a discriminative clustering method CURE and establish near-optimal statistical guarantees for an efficient algorithm. It is worth pointing out that CURE learns a classification rule that readily predicts labels for any new data. This is an advantage over many existing approaches for clustering and embedding whose out-of-sample extensions are not so straightforward. We impose several technical assumptions (spherical symmetry, constant condition number, positive excess kurtosis, etc.) to simplify the analysis, which we believe can be relaxed. Achieving Bayes optimality in multi-class clustering is indeed very challenging. Under parametric models such as Gaussian mixtures, one may construct suitable loss functions for CURE based on likelihood functions and obtain statistical guarantees. Other directions that are worth exploring include the optimal choice of the target distribution and the discrepancy measure, high-dimensional clustering with additional structures, estimation of the number of
clusters, to name a few. We also hope to further extend our methodology and theory to other tasks in unsupervised learning and semi-supervised learning.

The general CURE (Algorithm 1) provides versatile tools for clustering problems. In fact, it is related to several methods in the deep learning literature [Springenberg, 2015; Xie et al., 2016; Yang et al., 2017]. When we were finishing the paper, we noticed that Genevay et al. (2019) develop a deep clustering algorithm based on k-means and use optimal transport to incorporate prior knowledge of class proportions. Those methods are built upon certain network architectures (function classes) or loss functions while CURE offers more choices. In addition to the preliminary numerical results, it would be nice to see how CURE tackles more challenging real data problems.

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A Additional numerical experiments

A.1 Two classes

In this section, we provide additional numerical experiments to compare CURE in (7) with other clustering methods on the same real dataset as Section 4. We focus on six methods: (i) discriminative K-means (DisKmeans) in Ye et al. (2008); (ii) a discriminative clustering formulation described in Bach and Harchaoui (2008); Flammarion et al. (2017); (iii) Model-based clustering (Mclust) in Fraley and Raftery (1999); (iv) Projection Pursuit (PP) in Peña and Prieto (2001b); (v) Adaptive LDA-guided K-means Clustering in Ding and Li (2007); and (vi) Minimum Density Hyperplane (MDH) in Pavlidis et al. (2016).

As suggested by Ye et al. (2008), the regularization parameter λ therein has a significant impact on the performance of DisKmeans. To resolve this issue, they provide an automatic tuning framework. Here we provide a comparison between CURE and DisKmeans. For the DisKmeans, we consider pre-chosen λ ∈ {0, 1, 10, 100} as well as λ from the automatic tuning procedure suggested by Ye et al. (2008), initialized from 1. Due to high computational cost of DisKmeans with automatic tuning (which includes eigendecomposition of (N1 + N2) × (N1 + N2) matrix in each iteration), we conduct the experiment on smaller dataset: we fix N1 = 1000 and choose N2 from {1000, 500, 333, 250}. As is shown in Table 2, CURE has lower misclassification rate under all settings. It is also worth mentioning that the automatic tuning procedure sends λ → ∞, in which case DisKmeans is equivalent to classical K-means.

Table 2: Misclassification rate of CURE and discriminative K-means.

| Method          | N1 : N2     | 1 : 1  | 2 : 1  | 3 : 1  | 4 : 1  |
|-----------------|-------------|--------|--------|--------|--------|
| CURE            | /           | 5.2 ± 0.3% | 6.7 ± 0.6% | 9.1 ± 0.9% | 11.2 ± 1.2% |
| Discriminative K-means [Ye et al., 2008] | λ = 0       | 49.9%  | 49.5%  | 49.5%  | 47.7%  |
|                 | λ = 1       | 48.8%  | 46.6%  | 49.4%  | 48.3%  |
|                 | λ = 10      | 46.5%  | 44.2%  | 47.4%  | 41.8%  |
|                 | λ = 100     | 6.6%   | 49.4%  | 46.5%  | 27.2%  |
|                 | automatic tuning | 43.3%  | 49.4%  | 47.5%  | 45.8%  |

For experiments comparing CURE with other five methods, we still adopt the usual setting of sample size: we fix N1 = 6000 and choose N2 from {6000, 3000, 2000, 1500}. Model-based clustering (Mclust) in Fraley and Raftery (1999), Projection Pursuit (PP) in Peña and Prieto (2001b) and Minimum Density Hyperplane (MDH) in Pavlidis et al. (2016) are implemented using open-source R packages with default settings. In addition:
1. The discriminative clustering method appeared in Bach and Harchaoui (2008); Flammarion et al. (2017) stems from the optimization problem
\[
\min_{v \in \mathbb{R}^d, y \in \{\pm 1\}^d} \| y - Xv \|_2^2,
\]
where \( X \) is the centered data matrix. We adopt the alternating minimization scheme: given \( v \), the optimal \( y \) is obtained by \( \text{sgn}(Xv) \) (or by running K-means on \( Xv \), which has similar empirical performance) while given \( y \), the optimal \( v \) is obtained from solving a least squares problem. In the first step, \( v \) is initialized from a uniform distribution over the unit sphere. The iterative algorithm is terminated when \( y \), the predicted label, no longer changes.

2. Following the instructions in Ding and Li (2007), we implement the adaptive LDA-guided K-means clustering algorithm (Algorithm 1 therein) by alternating between linear discriminant analysis and K-means until convergence.

Table 3 shows the misclassification rate and the standard deviation of CURE and the other five methods over 50 independent trials. It is clear that CURE is more accurate and stable than these five methods under all settings.

| Method            | \( N_1 : N_2 \) | 1 : 1   | 2 : 1   | 3 : 1   | 4 : 1   |
|-------------------|-----------------|---------|---------|---------|---------|
| CURE              | 5.2 ± 0.2%      | 7.1 ± 0.4% | 9.3 ± 0.7% | 11.3 ± 1.1% |
| Method (10)       | 31.1 ± 13.8%    | 32.9 ± 13.3% | 34.7 ± 12.7% | 36.8 ± 11.2% |
| Melust            | 48.7 ± 1.3%     | 39.1 ± 4.8% | 34.1 ± 8.0% | 28.2 ± 7.8% |
| Projection Pursuit| 36.9 ± 9.8%     | 37.4 ± 9.6% | 39.7 ± 6.9% | 40.6 ± 7.3% |
| LDA-guided K-means| 45.9%           | 49.0%     | 45.6%    | 44.3%    |
| MDH               | 48.6%           | 43.1%     | 38.3%    | 35.2%    |

A.2 Multiple classes

To illustrate how the general CURE in Section 2.3 works, we consider the clustering problem with the first 4 classes in Fashion-MNIST (T-shirt/top, Trouser, Pullover, Dress), each of which has 6000 training samples and 1000 testing samples. Our training process only uses features of training samples and does not touch any labels.

We let the number of classes \( K \) be 4, the embedding space \( \mathcal{Y} \) be \( \mathbb{R}^K \), the target distribution \( \nu \) be \( \frac{1}{K} \sum_{j=1}^{K} \delta_{e_j} \), the discrepancy measure \( D \) be the Wasserstein-1 distance, and define the classification rule \( g(y) = \argmin_{j \in [K]} \| y - e_j \|_2 \). We compare two classes \( \mathcal{F} \) of feature mappings: linear functions and fully-connected neural networks with one hidden layer that has 100 nodes. Initial values All of the weight parameters are initialized using i.i.d. samples from \( N(0, 0.05^2) \).

Let \( f_\theta \) be a feature transform in \( \mathcal{F} \), parametrized by \( \theta \). Denote by \( \{ x_i \}_{i=1}^n \) the samples, where \( n = 4 \times 6000 = 24000 \). The loss function is
\[
L(\theta) = W_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{f_\theta(x_i)}, \nu \right) = P_{\text{10d}} \min_{P_{1K} = 1_{n/K}} \sum_{i=1}^n \sum_{j=1}^K p_{ij} | f_\theta(x_i) - e_j |.
\]

It is natural to optimize with respect to \( P \) and \( \theta \) in an alternating manner. We apply random sampling techniques to speedup computation. In the \( t \)-th iteration,

1. Draw \( B = 200 \) samples \( \{ x_{i1} \}_{i=1}^B \) uniform at random (with replacement) from the dataset;
2. Use the Python function `ot.sinkhorn2` in library POT (Flamary and Courty, 2017) with `reg = 0.1` to obtain the solution $P_t$ to an entropy-regularized version of

$$
\min_{P \in [0,1]^B \times K} P = \frac{1}{B} \sum_{i=1}^{B} \sum_{j=1}^{K} p_{ij} |f_{\theta}(x_{ti}) - e_j|;
$$

3. Update model parameters by $\theta_{t+1} = \theta_t - \eta \partial L_t(\theta_t)$, where $\partial$ is the sub-differential operator, $\eta = 10^{-3}$ and

$$
L_t(\theta) = \sum_{j=1}^{K} \hat{p}_{ij} |f_{\theta}(x_{ti}) - e_j|, \quad \forall \theta.
$$

An epoch refers to $n/B = 12$ consecutive iterations. The learning curves in Figure 4 shows the advantage of neural network and demonstrates the flexibility of CURE with nonlinear function classes.

B Proof sketch of Theorem 1

B.1 Step 1: properties of the test function $f$

We now investigate the function $f$ defined in (8) and relate it to $h(x) = (x^2 - 1)^2/4$. As Lemma 1 suggests, $|f'|$, $|f''|$ and $|f'''|$ are all bounded by constants determined by $a$ and $b$; $|f' - h'|$ and $|f'' - h''|$ are bounded by polynomials that are independent of $a$ and $b$. See Appendix D for a proof.

**Lemma 1.** When $a$ is sufficiently large and $b \geq 2a$, $f$ has the following properties:

1. $f'$ is continuous with $F_1 \triangleq \sup_{x \in \mathbb{R}} |f'(x)| \leq 2a^2b$ and $|f'(x) - h'(x)| \leq 7|x|^{1}(1_{\{x \geq a\}})$;
2. $f''$ is continuous with $F_2 \triangleq \sup_{x \in \mathbb{R}} |f''(x)| \leq 3a^2$ and $|f''(x) - h''(x)| \leq 9x^21_{\{x \geq a\}}$;
3. $f'''$ exists in $\mathbb{R} \setminus \{\pm a, \pm b\}$ with $F_3 \triangleq \sup_{x \in \mathbb{R} \setminus \{\pm a, \pm b\}} |f'''(x)| \leq 6a$.

B.2 Step 2: landscape analysis of the population loss

To kick off the landscape analysis we investigate the population version of $\hat{L}_1$, namely

$$
L_1(\alpha, \beta) = E_{X \sim p} f(\alpha + \beta^T X) + \frac{1}{2}(\alpha + \beta^T \mu_0)^2.
$$

(11)
One of the main obstacles is the complicated piecewise definition of $f$, which prevent us from obtaining closed form formulae. We bypass this problem by relating the population loss with $f$ to that with the quartic function $h$. See Appendix [F] for a proof.

**Theorem 2** (Landscape of the population loss). Consider Model [7] and assume that $b \geq 2a$. There exist positive constants $A, \varepsilon, \delta$ and $\epsilon$ determined by $M$, $\mathbb{E} Z^4$, $\|\mu\|_2$, $\lambda_{\max}(\Sigma)$ and $\lambda_{\min}(\Sigma)$ but independent of $d$ and $n$, such that when $a > A,$

1. The only two global minima of $L_1$ are $\pm \gamma^*$, where $\gamma^* = (-c\beta^\top h \mu_0, c\beta^h)$ for some $c \in (1/2, 2)$ and

   $$\beta^h = \left( \frac{1 + 1/\|\mu\|_2^2 \Sigma^{-1} + 6\|\mu\|_2^2 \Sigma^{-1} + M_2}{\|\mu\|_2^2 \Sigma^{-1} + M_2} \right)^{1/2} \Sigma^{-1} \mu;$$

2. $\|\nabla L_1(\gamma)\|_2 \geq \varepsilon$ if $\text{dist}(\gamma, \{-\gamma^*\} \cup S) \geq \delta$, where $S = \{0\} \cup \{(-\beta^\top h \mu_0, \beta) : \mu^\top \beta = 0, \beta^\top \Sigma \beta = 1/M_2\};$

3. $\nabla^2 L_1(\gamma) \succeq \eta I$ if $\text{dist}(\gamma, \{-\gamma^*\}) \leq \delta$, and $u^\top \nabla^2 L_1(\gamma) u \leq -\eta$ if $\text{dist}(\gamma, S) \leq \delta$ with $u = (0, \Sigma^{-1}\mu/\Sigma^{-1} \mu_2).$

Theorem [2] precisely characterizes the landscape of $L_1$. In particular, all of its critical points make up the set $\{-\gamma^*\} \cup S$, where $\pm \gamma^*$ are global minima and $S$ consists of strict saddles. The local geometry around critical points is also desirable.

**B.3 Step 3: landscape analysis of the empirical loss**

Based on geometric properties of the population loss $L_1$, we establish similar results for the empirical loss $\hat{L}_1$ through concentration analysis. See Appendix [F] for a proof.

**Theorem 3** (Landscape of the empirical loss). Consider Model [7] and assume that $b \geq 2a \geq 4$. Let $\gamma^*$ and $S$ be defined as in Theorem [2]. There exist positive constants $A, C_0, C_1, C_2, M_1, \varepsilon, \delta$ and $\eta$ determined by $M, M_2, \|\mu\|_2, \lambda_{\max}(\Sigma)$ and $\lambda_{\min}(\Sigma)$ but independent of $d$ and $n$, such that when $a > A$ and $n \geq C_0 d$, the followings hold with probability exceeding $1 - C_1(d/n)^{2/3} - C_1 \exp(-C_2 n^{1/3})$:

1. $\|\nabla \hat{L}_1(\gamma)\|_2 \geq \varepsilon$ if $\text{dist}(\gamma, \{-\gamma^*\} \cup S) \geq \delta;$

2. $u^\top \nabla^2 \hat{L}_1(\gamma) u \leq -\eta$ if $\text{dist}(\gamma, S) \leq \delta$, with $u = (0, \Sigma^{-1} \mu/\Sigma^{-1} \mu_2);$

3. $\|\nabla \hat{L}_1(\gamma_1) - \nabla \hat{L}_1(\gamma_2)\|_2 \leq M_1 \|\gamma_1 - \gamma_2\|_2$ and $\|\nabla^2 \hat{L}_1(\gamma_1) - \nabla^2 \hat{L}_1(\gamma_2)\|_2 \leq M_1 \sqrt{d \log(n/d)/\sqrt{n}} \|\gamma_1 - \gamma_2\|_2$ hold for all $\gamma_1, \gamma_2 \in \mathbb{R} \times \mathbb{R}^d$.

Theorem [3] shows that a sample of size $n \gtrsim d$ suffices for the empirical loss to inherit nice geometric properties from its population counterpart. The corollary below illustrates that as long as we can find an approximate second-order stationary point, then the statistical estimation error can be well controlled by the gradient descent. We defer the proof of this to Appendix [G]

**Corollary 2.** Under the settings in Theorem [3], there exist constants $C, C_1, C_2$ such that the followings happen with probability exceeding $1 - C_1'(d/n)^{2/3} - C_1' \exp(-C_2' n^{1/3})$: for any $\gamma \in \mathbb{R} \times \mathbb{R}^d$ satisfying $\|\nabla \hat{L}_1(\gamma)\|_2 \leq \varepsilon$ and $\lambda_{\min}(\nabla^2 \hat{L}_1(\gamma)) > -\eta$,

$$\min_{s = \pm 1} \|s \gamma - \gamma^*\|_2 \leq C \left( \|\nabla \hat{L}_1(\gamma)\|_2 + \sqrt{\frac{d \log(n/d)}{n}} \right).$$

As a result, when the event above happens, any local minimizer $\hat{\gamma}$ of $\hat{L}_1$ satisfies

$$\min_{s = \pm 1} \|s \hat{\gamma} - \gamma^*\|_2 \leq C \sqrt{\frac{d \log(n/d)}{n}}.$$
Algorithm 3 Perturbed gradient descent

\[
\text{PerturbedGD}(\gamma_{\text{pgd}}, \ell, \rho, \varepsilon_{\text{pgd}}, \epsilon_{\text{pgd}}, \Delta_{\text{pgd}})
\]

\[
\chi \leftarrow 3 \max \{ \log(d \Delta_{\text{pgd}} / (c_{\text{pgd}} \varepsilon_{\text{pgd}}^2 \delta_{\text{pgd}})), 4 \}, \quad \eta_{\text{pgd}} \leftarrow \epsilon_{\text{pgd}} / \ell, \quad r \leftarrow \sqrt{c_{\text{pgd}} \varepsilon_{\text{pgd}} / (\chi^2 \ell)}, \quad g_{\text{thres}} \leftarrow \sqrt{c_{\text{pgd}} \varepsilon_{\text{pgd}} / \chi^2},
\]

\[
f_{\text{thres}} \leftarrow c_{\text{pgd}} \varepsilon_{\text{pgd}} / (\chi^2 \rho), \quad t_{\text{thres}} \leftarrow \chi / (c_{\text{pgd}} \varepsilon_{\text{pgd}} R_{\text{pgd}}), \quad t_{\text{noise}} \leftarrow -t_{\text{thres}} - 1.
\]

Initialize \( \gamma^0 = \gamma_{\text{pgd}} \).

For \( t = 0, 1, \ldots \) do

\[
\text{If } \| \nabla \hat{L}_1(\gamma^t) \|_2 \leq g_{\text{thres}} \text{ and } t - t_{\text{noise}} > t_{\text{thres}}:
\]

Update \( t_{\text{noise}} \leftarrow t \),

\[
\text{Perturb } \gamma^t \leftarrow \gamma^t + \xi^t \text{ with } \xi^t \sim \mathcal{N}(\mathbf{0}, r)
\]

\[
\text{If } t - t_{\text{noise}} = t_{\text{thres}} \text{ and } \hat{L}_1(\gamma^t) - \hat{L}_1(\gamma_{\text{noise}}^t) > -f_{\text{thres}}:
\]

Return \( \gamma_{\text{noise}}^{t_{\text{thres}}} \)

Update \( \gamma^{t+1} \leftarrow \gamma^t - \eta_{\text{pgd}} \nabla \hat{L}_1(\gamma^t) \).

B.4 Step 4: convergence guarantees for perturbed gradient descent

The landscape analysis above shows that all local minimizers of \( \hat{L}_1 \) are statistically optimal (up to logarithmic factors), and all saddle points are non-degenerate. Then it boils down to finding any \( \gamma \) whose gradient size is sufficiently small and Hessian has no significantly negative eigenvalue. Thanks to the Lipschitz smoothness of \( \nabla \hat{L}_1 \) and \( \nabla^2 \hat{L}_1 \), this can be efficiently achieved by the perturbed gradient descent algorithm (see Algorithm 3) proposed by Jin et al. (2017). Small perturbation is occasionally added to the iterates, helping escape from saddle points efficiently and thus converge towards local minimizers. Theorem 4 provides algorithmic guarantees for CURE on top of that. We defer the proof to Appendix H.

Implementation of the algorithm requires specification of hyperparameters \( a, b, M_1, \varepsilon \) and \( \eta \). Under the regularity assumptions in Model 1, many structural parameters are well-behaved constants and that helps choose hyperparameters at least in a conservative way. In theory, we can let \( b = 2a \); \( a \) and \( M_1 \) be sufficiently large; \( \varepsilon \) and \( \eta \) be sufficiently small. In our numerical experiments, the algorithm does not appear to be sensitive to choices of hyperparameters. We do not go into much details to avoid distractions.

**Theorem 4 (Algorithmic guarantees).** Consider the settings in Theorem 3 and adopt the constants \( M_1, \varepsilon \) and \( \eta \) therein. With probability exceeding \( 1 - C_1[[d/n]^{C_2} d + \varepsilon^{-C_2} n^{1/3} + \varepsilon^{-10}] \), Algorithm 3 with parameters \( \gamma_{\text{pgd}} = 0, \ell = M_1, \delta_{\text{pgd}} = n^{-11}, \rho = M_1 \max \{1, d \log(n/d)/\sqrt{n} \}, \varepsilon_{\text{pgd}} = \min \{ \sqrt{d \log(n/d)/n}, \ell^2 / \rho, \varepsilon^2 / \rho, \varepsilon \} \) and \( \Delta_{\text{pgd}} = 1/4 \) terminates within \( O(n/d + d^2/n) \) iterations and the output \( \hat{\gamma} \) satisfies

\[
\| \nabla \hat{L}_1(\hat{\gamma}) \|_2 \leq \sqrt{\frac{d}{n} \log \left( \frac{n}{d} \right)} \leq \varepsilon \quad \text{and} \quad \lambda_{\text{min}}(\nabla^2 \hat{L}_1(\hat{\gamma})) \geq -\eta.
\]

Theorem H and Corollary 2 immediately lead to

\[
\min_{s \neq 2} \| s \hat{\gamma} - \gamma^* \|_2 \leq \| \nabla \hat{L}_1(\gamma^*) \|_2 + \sqrt{\frac{d}{n} \log \left( \frac{n}{d} \right)} \leq \sqrt{\frac{d}{n} \log \left( \frac{n}{d} \right)},
\]

which finishes the proof of Theorem H.

C Preliminaries

Before we start the proof, let us introduce some notations. Recall the definition of the random vector \( \mathbf{X} = \mu_0 + \mu Y + \sum_{1}^{1/2} \mathbf{Z} \) and the i.i.d. samples \( \mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{R}^d \). Let \( \mathbf{X} = (1, \mathbf{X}), \mathbf{X}_i = (1, \mathbf{X}_i) \) and \( \mu_0 = (1, \mu_0) \). For any \( \gamma = (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^d \), define

\[
L_{\alpha}(\gamma) = L(\gamma) + \lambda R(\gamma) \quad \text{and} \quad \hat{L}_{\alpha}(\gamma) = \hat{L}(\gamma) + \lambda \hat{R}(\gamma),
\]

where

\[
L(\gamma) = \mathbb{E} f(\gamma^\top \mathbf{X}) = \mathbb{E} f(\alpha + \beta^\top \mathbf{X}), \quad \hat{L}(\gamma) = \frac{1}{n} \sum_{i=1}^{n} f(\gamma^\top \mathbf{X}_i) = \frac{1}{n} \sum_{i=1}^{n} f(\alpha + \beta^\top \mathbf{X}_i),
\]

\[
R(\gamma) = \frac{1}{2} (\alpha + \beta^\top \mu_0)^2 = \frac{1}{2} (\gamma^\top \mu_0)^2, \quad \hat{R}(\gamma) = \frac{1}{2} (\alpha + \beta^\top n^{-1} \sum_{i=1}^{n} \mathbf{X}_i)^2 = \frac{1}{2} (\gamma^\top n^{-1} \sum_{i=1}^{n} \mathbf{X}_i)^2.
\]
Note that the results stated in Section III and IV focus on the special case when $\lambda = 1$. The proof in the appendices allows for general choices of $\lambda \geq 1.$

**D  Proof of Lemma [1]**

By direct calculation, one has

\[
\begin{align*}
   f'(x) &= \begin{cases} 
   h'(x), & |x| \leq a \\
   [h'(a) + h''(a)(|x| - a) - \frac{h'''(a)}{2(b-a)}(|x| - a)^2] \operatorname{sgn}(x), & a < |x| \leq b, \\
   [h'(a) + \frac{b-a}{2}h''(a)] \operatorname{sgn}(x), & |x| > b.
   \end{cases} \\
   f''(x) &= \begin{cases} 
   h''(x), & |x| \leq a \\
   h''(a)(1 - \frac{|x-a|}{b-a}), & a < |x| \leq b, \\
   0, & |x| > b.
   \end{cases} \\
   f'''(x) &= \begin{cases} 
   h'''(x), & |x| < a \\
   \frac{h'''(a)}{b-a} \operatorname{sgn}(x), & a < |x| < b. \\
   0, & |x| > b.
   \end{cases}
\]

When $a$ is sufficiently large and $b \geq 2a$, we have $F_1 \triangleq \sup_{x \in \mathbb{R}} |f'(x)| = h'(a) + \frac{b-a}{2}h''(a) \leq 2a^2b$, $F_2 \triangleq \sup_{x \in \mathbb{R}} |f''(x)| = h''(a) \leq 3a^2$, and $F_3 \triangleq \sup_{|x| \neq a,b} |f'''(x)| = h'''(a) \vee \frac{h'''(a)}{b-a} \leq 6a$.

In addition, one can also check that when $a < |x| \leq b$, we have $|h'(a)| \leq |x|^3$ and $|h''(a)| \leq |x|^2$, thus

\[
|f'(x) - h'(x)| \leq |f'(x)| + |h'(x)| \leq |h'(a)| + |h''(a)(|x| - a) + h'''(a)(|x| - a)^2/(2a)| + |x^3 - x|
\]

\[
\leq |x|^3 + 3|x|^2 + \frac{3}{2}|x|^2 + |x|^3 \leq 7|x|^3
\]

provided that $b \geq 2a \geq 2$. When $|x| \geq b$, we have

\[
|f'(x) - h'(x)| \leq |f'(x)| + |h'(x)| \leq |h'(a)| + |(b-a)h''(a)/2| + |x^3 - x|
\]

\[
\leq |x|^3 + \frac{3}{2}|x|^2 + |x|^3 \leq 4|x|^3.
\]

This combined with $f'(x) = h'(x)$ when $|x| \leq a$ gives $|f'(x) - h'(x)| \leq 1_{\{|x| \geq a\}} 7|x|^3$. Similarly we have $|f''(x) - h''(x)| \leq 1_{\{|x| \geq a\}} 9x^2$.

**E  Proof of Theorem [2]**

It suffices to focus on the special case $\mu_0 = 0$ and $\Sigma = I_d$. We first give a theorem that characterizes the landscape of an auxiliary population loss, which serves as a nice starting point of the study of the actual loss functions that we use.

**Theorem 5** (Landscape of the auxiliary population loss). Consider model [1] with $\mu_0 = 0$ and $\Sigma = I_d$. Suppose that $M_Z > 3$. Let $h(x) = (x^2 - 1)^2/4$ and $\lambda \geq 1$. The stationary points of the population loss

\[
L_h^b(\alpha, \beta) = \mathbb{E} h(\alpha + \beta^\top X) + \frac{\lambda}{2} \alpha^2
\]

are $\{(\alpha, \beta) : \nabla L_h^b(\alpha, \beta) = 0\} = S_1^b \cup S_2^b$, where

1. $S_1^b = \{(0, \pm \beta^b)\}$ consists of global minima, with

\[
\beta^b = \left(\frac{1 + 1/\|\mu\|^2}{\|\mu\|^4 + 6\|\mu\|^2 + M_Z}\right)^{1/2} \mu;
\]
2. \( S^h_2 = \{(0, \beta): \mu^T \beta = 0, \|\beta\|^2_2 = 1/M_Z\} \cup \{0\} \) consists of saddle points whose Hessians have negative eigenvalues.

We also have the following quantitative results: there exist positive constants \( \varepsilon^h, \delta^h \) and \( \eta^h \) determined by \( M_Z, \|\mu\|_2 \) and \( \lambda \) such that

1. \( \|\nabla L^h_\lambda(\gamma)\|_2 \geq \varepsilon^h \) if \( \text{dist}(\gamma, S^h_1 \cup S^h_2) \geq \delta^h \);

2. \( \|\nabla^2 L^h_\lambda(\gamma)\|_2 \geq \eta^h I \) if \( \text{dist}(\gamma, S^h_1) \leq 3\delta^h \), and \( u^T \nabla^2 L^h_\lambda(\gamma)u \leq -\eta^h \) if \( \text{dist}(\gamma, S^h_2) \leq 3\delta^h \) where \( u = (0, \mu/\|\mu\|_2) \).

**Proof.** See Appendix E.1.

The following Lemma 2 controls the difference between the landscape of \( L_\lambda \) and \( L^h_\lambda \) within a compact ball.

**Lemma 2.** Let \( X \) be a random vector in \( \mathbb{R}^{d+1} \) with \( \|X\|_{\psi_2} \leq M \), \( f \) be defined in (8) with \( b \geq 2a \geq 4 \), \( h(x) = (x^2 - 1)^2/4 \) for \( x \in \mathbb{R} \), \( L_\lambda(\gamma) = E f(\gamma^T X) + \lambda a^2/2 \) and \( L^h_\lambda(\gamma) = Eh(\gamma^T X) + \lambda a^2/2 \) for \( \gamma \in \mathbb{R}^{d+1} \). There exist constants \( C_1, C_2 > 0 \) such that for any \( R > 0 \),

\[
\sup_{\|\gamma\|_2 \leq R} \|\nabla L_\lambda(\gamma) - \nabla L^h_\lambda(\gamma)\|_2 \leq C_2 R^2 M^4 \exp \left( - \frac{C_1 a^2}{R^2 M^2} \right),
\]

\[
\sup_{\|\gamma\|_2 \leq R} \|\nabla^2 L_\lambda(\gamma) - \nabla^2 L^h_\lambda(\gamma)\|_2 \leq C_2 R^2 M^4 \exp \left( - \frac{C_1 a^2}{R^2 M^2} \right).
\]

In addition, when \( E(XX^T) \geq \sigma^2 I \) holds for some \( \sigma > 0 \), there exists \( m > 0 \) determined by \( M \) and \( \sigma \) such that \( \inf_{\|\gamma\|_2 \geq m} \|\nabla L_\lambda(\gamma)\|_2 \geq m \) and \( \inf_{\|\gamma\|_2 \geq m} \|\nabla L^h_\lambda(\gamma)\|_2 \geq m \).

**Proof.** See Appendix E.2.

On the one hand, Lemma 2 implies that \( \inf_{\|\gamma\|_2 \geq m} \|\nabla L_\lambda(\gamma)\|_2 \geq m \) for some constant \( m > 0 \). Suppose that

\[
\varepsilon^h < m
\]

and define \( r = 3/\varepsilon^h \). Then

\[
\|\nabla L_1(\gamma)\|_2 \geq \varepsilon^h \quad \text{if} \quad \|\gamma\|_2 \geq r.
\]

Moreover, we can take \( a \) to be sufficiently large such that

\[
\sup_{\|\gamma\|_2 \leq r} \|\nabla L_1(\gamma) - \nabla L^h_1(\gamma)\|_2 \leq \varepsilon^h / 2.
\]

On the other hand, from Theorem 5 we know that

\[
\|\nabla L^h_\lambda(\gamma)\|_2 \geq \varepsilon^h \quad \text{if} \quad \text{dist}(\gamma, S^h_1 \cup S^h_2) \geq \delta^h.
\]

Taking (13), (14) and (15) collectively gives

\[
\|\nabla L_\lambda(\gamma)\|_2 \geq \varepsilon^h / 2 \quad \text{if} \quad \text{dist}(\gamma, S^h_1 \cup S^h_2) \geq \delta^h.
\]

Hence \( \{\gamma: \nabla L_\lambda(\gamma) = 0\} \subseteq \{\gamma: \text{dist}(\gamma, S^h_1 \cup S^h_2) \leq \delta^h\} \) and it yields a decomposition \( \{\gamma: \nabla L_\lambda(\gamma) = 0\} = S_1 \cup S_2 \), where

\[
S_j \subseteq \{\gamma: \text{dist}(\gamma, S^h_j) \leq \delta^h\}, \quad \forall j = 1, 2.
\]

Consequently, for \( j = 1, 2 \) we have

\[
\{\gamma: \text{dist}(\gamma, S_j) \leq 2\delta^h\} \subseteq \{\gamma: \text{dist}(\gamma, S^h_j) \leq 3\delta^h\} \subseteq \{\gamma: \|\gamma\|_2 \leq 3\delta^h + \max_{\gamma' \in S^h_1 \cup S^h_2} \|\gamma'\|_2\}.
\]

Now we work on the first proposition in Theorem 2 by characterizing \( S_1 \).
Lemma 3. Consider the model in (7) with $\mu_0 = 0$ and $\Sigma = I_d$. Suppose that $f \in C^2(\mathbb{R})$ is even, $\lim_{x \to +\infty} f'(x) = +\infty$ and $f''(0) < 0$. Define

$$L_\lambda(\alpha, \beta) = Ef(\alpha + \beta^T X) + \frac{\lambda}{2} \alpha^2, \quad \forall \alpha \in \mathbb{R}, \beta \in \mathbb{R}^d.$$ 

1. There exists some $c > 0$ determined by $\|\mu\|_2$, the function $f$, and the distribution of $Z$, such that $(0, \pm c \mu)$ are critical points of $L_\lambda$.

2. In addition, if $f''$ is piecewise differentiable and $|f'''(x)| \leq F_3 < \infty$ almost everywhere, we can find $c_0 > 0$ determined by $\|\mu\|_2$, $f'''(0)$, $F_3$ and $M$ such that $c > c_0$.

Proof. See Appendix E.3.

Lemma 3 asserts the existence of two critical points $\pm \gamma^* = (0, \pm c \beta^h)$ of $L_1$, for some $c$ bounded from below by a constant $c_0 > 0$. If

$$\delta^h < c_0 \|\beta^h\|_2 / 4,$$

then the property of $S^h_2$ forces

$$\text{dist}(\pm \gamma^*, S^h_2) \geq \|\gamma^*\|_2 = c \|\beta^h\|_2 \geq c_0 \|\beta^h\|_2 > 4 \delta^h > 3 \delta^h.$$ 

It is easily seen from (18) with $j = 2$ that $\text{dist}(\pm \gamma^*, S_2) > 2 \delta^h$ and $\pm \gamma^* \notin S_2$. Then $\{\gamma : \nabla L_1(\gamma) = 0\} = S_1 \cup S_2$ forces

$$\{\gamma^*, -\gamma^*\} \subseteq S_1.$$ 

Let us investigate the curvature near $S_1$. Lemma 2 and (18) with $j = 1$ allow us to take $a$ to be sufficiently large such that

$$\sup_{\text{dist}(\gamma, S_1) \leq 2 \delta^h} \|\nabla^2 L_\lambda(\gamma) - \nabla^2 L^h_\lambda(\gamma)\|_2 \leq \eta^h / 2.$$ 

Theorem 5 asserts that $\nabla^2 L^h_\lambda(\gamma) \succeq \eta^h I$ if $\text{dist}(\gamma, S^h_2) \leq 3 \delta^h$. By this, (18) with $j = 1$ and (22),

$$\nabla^2 L_\lambda(\gamma) \succeq (\eta^h / 2) I \quad \text{if} \quad \text{dist}(\gamma, S_1) \leq 2 \delta^h.$$ 

Hence $L_1$ is strongly convex in $\{\gamma : \text{dist}(\gamma, S_1) \leq 2 \delta^h\}$. Combined with (21), it leads to $S_1 = \{\pm \gamma^*\}$, and both points therein are local minima.

Let $\gamma^h = (0, \beta^h)$. The fact $S^1_1 = \{\pm \gamma^h\}$ and (17) yields

$$|c - 1| \cdot \|\beta^h\|_2 = \|\gamma^* - \gamma^h\|_2 = \text{dist}(\gamma^*, S^1_1) \leq \delta^h.$$ 

When

$$\delta^h < \|\beta^h\|_2 / 2,$$

we have $1 / 2 < c < 3 / 2$ as claimed. The global optimality of $\pm \gamma^*$ is obvious. Without loss of generality, in Theorem 5 we can always take $\delta^h < \|\beta^h\|_2 \min\{c_0 / 3, 1 / 2\}$ and then find $\varepsilon^h < m$. In that case, (17), (18) and (25) imply the first proposition in Theorem 2.

Next, we study the second proposition in Theorem 2. Let $S = S^h_2$. Given $S_1 = \{\pm \gamma^h\}$ and $S_1 = \{\pm \gamma^*\}$, from (24) we know that $\text{dist}(\gamma, \{\pm \gamma^*\} \cup S) \geq 2 \delta^h$ implies $\text{dist}(\gamma, S^1_1 \cup S^h_2) \geq 2 \delta^h$. This combined with (16) immediately gives

$$\|\nabla L_\lambda(\gamma)\|_2 \geq \varepsilon^h / 2 \quad \text{if} \quad \text{dist}(\gamma, \{\pm \gamma^*\} \cup S) \geq 2 \delta^h.$$ 

Hence the second proposition in Theorem 2 holds if

$$\varepsilon = \varepsilon^h / 2 \quad \text{and} \quad \delta = 2 \delta^h.$$ 

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Finally, we study the third proposition in Theorem 2. By (23), the first part of that proposition holds when
\[ \eta = \eta^h / 2 \quad \text{and} \quad \delta = 2\delta^h. \] (27)
It remains to prove the second part. Lemma 2 and (18) with \( j = 2 \) allow us to take \( a \) to be sufficiently large such that
\[ \sup_{\dist(\gamma, S) \leq 3\delta^h} \| \nabla^2 L_{\lambda}(\gamma) - \nabla^2 L_{\lambda}^h(\gamma) \|_2 \leq \eta^h / 2. \] (28)
Theorem 5 asserts that \( u^\top \nabla^2 L_{\lambda}^h(\gamma) u \leq -\eta^h \) for \( u = (0, \mu/\|\mu\|_2) \) if \( \dist(\gamma, S) \leq 3\delta^h \). By this, (18) with \( j = 2 \) and (28).
\[ \nabla^2 L_{\lambda}(\gamma) \leq -\eta^h / 2 \quad \text{if} \quad \dist(\gamma, S) \leq 3\delta^h. \] (29)
Hence (26) suffice for the second part of the third proposition to hold.

According to (26) and (27), Theorem 2 holds with \( \varepsilon = \varepsilon^h / 2, \delta = 2\delta^h \) and \( \eta = \eta^h / 2 \).

E.1 Proof of Theorem 5
E.1.1 Part 1: Characterization of stationary points

Note that
\[ \nabla L_{\lambda}^h(\alpha, \beta) = \mathbb{E} \left[ \left( \frac{1}{X} h'(\alpha + \beta^\top X) \right) + \left( \lambda \right) \right] \]
\[ = \left( \mathbb{E} h'(\alpha + \beta^\top X) + \lambda \right) + \left( \mathbb{E} |Y| h'(\alpha + \beta^\top X) |\mu\right) + \left( \mathbb{E} |Z| h'(\alpha + \beta^\top X) |\mu\right). \]

Now we will expand individual expected values in this sum. For the first term,
\[ \mathbb{E} h'(\alpha + \beta^T X) = \mathbb{E}(\alpha + \beta^T \mu Y + \beta^T Z)^3 - \mathbb{E}(\alpha + \beta^T \mu Y + \beta^T Z) \]
\[ = \alpha^3 + 3\alpha \mathbb{E}(\beta^T \mu Y)^2 + 3\alpha \mathbb{E}(\beta^T Z)^2 + \mathbb{E}(\beta^T \mu Y + \beta^T Z)^3 - \alpha \]
\[ = \alpha(\alpha^2 + 3(\beta^T \mu)^2 + 3(\beta^T Z)^2 - 1), \]
where the first line follows since \( h'(x) = x^3 - x \), the other two follows from \( \mathbb{E}(ZZ^\top) = I \) plus the fact that \( Y \) and \( Z \) are independent, with zero odd moments due to their symmetry.

Using similar arguments,
\[ \mathbb{E} |Y| h'(\alpha + \beta^T X) |\mu| = \mathbb{E} |Y| (\alpha + \beta^T \mu Y + \beta^T Z)^3 - \mathbb{E} |Y| (\alpha + \beta^T \mu Y + \beta^T Z) \]
\[ = 3\alpha^2 \mathbb{E} |Y| (\beta^T \mu Y + \beta^T Z)^3 + 3\mathbb{E} |Y| (\beta^T \mu Y)^2 (\beta^T Z) - \beta^T \mu \]
\[ = [3\alpha^2 + (\beta^T \mu)^2] \mathbb{E} |Y|^4 + 3(\beta^T Z)^2 |\beta| - 1] \beta^T \mu. \]

To work on \( \mathbb{E} |Z| h'(\alpha + \beta^T X) |\mu| = \mathbb{E} |Z| (\alpha + \beta^T \mu Y + \beta^T Z) |\mu| \), we define \( \beta = |\beta|/\|\beta\|_2 \) for \( \beta \neq 0 \) and \( \beta = 0 \) otherwise. Observe that \( (Y, \beta \beta^T X, (I - \beta \beta^T) Z) \) and \( (Y, \beta \beta^T Z, -(I - \beta \beta^T) Z) \) have exactly the same joint distribution. As a result,
\[ \mathbb{E} ([I - \beta \beta^T] Z) h'(\alpha + \beta^T X) = \mathbb{E} ([I - \beta \beta^T] Z) h'(\alpha + \beta^T \mu Y + \beta^T Z) = 0. \]
Hence,
\[ \mathbb{E} [Z h'(\beta^T X)] = \mathbb{E} [\beta \beta^T Z h'(\alpha + \beta^T X)] = \mathbb{E} [\beta \beta^T Z h'(\alpha + \beta^T \mu Y + \beta^T Z)] \beta \]
\[ = \mathbb{E} [\beta \beta^T Z (\alpha + \beta^T \mu Y + \beta^T Z)^3] \beta - \mathbb{E} [\beta \beta^T Z (\alpha + \beta^T \mu Y + \beta^T Z)] \beta \]
\[ = 3\alpha^2 \mathbb{E} [\beta \beta^T Z (\beta^T \mu Y + \beta^T Z)^3] \beta - \beta \]
\[ = (3\alpha^2 - 1) \beta + 3 \mathbb{E} (\beta^T \mu Y)^2 \beta + \mathbb{E} [\beta \beta^T Z (\beta^T Z)^3] \beta \]
\[ = [3\alpha^2 + 3(\beta^T \mu)^2 + M_2 |\beta|_2^2 - 1] \beta, \]
where besides the arguments we have been using we also employed identities $\|\beta\|_2 \bar{\beta} = \beta$ and $E(\gamma^T Z)^4 = M_Z$ for any unit-norm $\gamma$. Combining all these together, we get

$$
\nabla_\alpha L^k_\lambda(\alpha, \beta) = \alpha(\alpha^2 + 3(\beta^T \mu)^2 + 3\|\beta\|_2^2 + \lambda - 1),
$$

$$
\nabla_\beta L^k_\lambda(\alpha, \beta) = [3\alpha^2 + (\beta^T \mu)^2 + 3\|\beta\|_2^2 - 1](\mu^T \beta)\mu + [3\alpha^2 + 3(\mu^T \beta)^2] + M_Z\|\beta\|_2^2 - 1]\beta.
$$

Taking second derivatives,

$$
\nabla^2_{\alpha\alpha} L^k_\lambda(\alpha, \beta) = 3\alpha^2 + 3(\beta^T \mu)^2 + 3\|\beta\|_2^2 + \lambda - 1,
$$

$$
\nabla^2_{\beta\beta} L^k_\lambda(\alpha, \beta) = 3(\beta^T \mu)^2 + (3\alpha^2 + 3\|\beta\|_2^2 - 1)\mu^T + 6\mu^T \beta \beta^T + [3\alpha^2 + 3(\mu^T \beta)^2 + M_Z\|\beta\|_2^2 - 1]I + \beta(6(\mu^T \beta)\mu^T + 2M_Z\beta^T)
$$

$$
= [3\alpha^2 + 3(\mu^T \beta)^2 + M_Z\|\beta\|_2^2 - 1]I + [3\alpha^2 + 3(\beta^T \mu)^2 + (3\|\beta\|_2^2 - 1)]\mu^T + 6(\mu^T \beta)(\mu^T + \beta \beta^T) + 2M_Z\beta \beta^T.
$$

Now that we have derived the gradient and Hessian in closed form, we will characterize the landscape. Let $(\alpha, \beta)$ be an arbitrary stationary point, we start by proving that it must satisfy $\alpha = 0$.

**Claim 1.** If $\lambda \geq 1$ then $\alpha = 0$ holds for any critical point $(\alpha, \beta)$.

**Proof.** Seeking a contradiction assume that $\alpha \neq 0$. We start by assuming $\beta = c\mu$ for some $c \in \mathbb{R}$, then the optimality condition $\nabla_\alpha L^k_\lambda(\alpha, \beta) = 0$ gives $0 < \alpha^2 + 3\alpha^2\|\mu\|_2^2 (\|\mu\|_2^2 + 1) = 1 - \lambda \leq 0$, yielding a contraction.

Now, let us assume that $\mu$ and $\beta$ are linearly independent, this assumption together with $(30)$ and $(31)$ imply that

$$
\alpha^2 + 3(\beta^T \mu)^2 + 3\|\beta\|_2^2 + \lambda - 1 = 0,
$$

$$
[3\alpha^2 + (\beta^T \mu)^2 + 3\|\beta\|_2^2 - 1] \mu^T \beta = 0,
$$

$$
3\alpha^2 + 3(\mu^T \beta)^2 + M_Z\|\beta\|_2^2 - 1 = 0.
$$

There are only two possible cases:

**Case 1.** If $\beta^T \mu = 0$, then the optimality condition for $\alpha$ gives $\alpha^2 + 3\|\beta\|_2^2 = 1 - \lambda \leq 0$, which is a contradiction.

**Case 2.** If $\beta^T \mu \neq 0$, then $3\alpha^2 + (\beta^T \mu)^2 + 3\|\beta\|_2^2 - 1 = 0$ and by substracting it from $3\alpha^2 + 3(\beta^T \mu)^2 + (3\|\beta\|_2^2 - 1)\mu^T \beta = 0$, we get $0 < 2(\beta^T \mu)^2 + (M_Z - 3)\|\beta\|_2^2 = 0$, yielding a contradiction again.

This completes the proof of the claim.

This claim directly implies that the Hessian $\nabla^2 L^k_\lambda$, evaluated at any critical point, is a block diagonal matrix with $\nabla^2_{\beta\beta} L^k_\lambda(\alpha, \beta) = 0$. Furthermore its first block is positive if $\beta \neq 0$, as

$$
\nabla^2_{\alpha\alpha} L^k_\lambda(\alpha, \beta) = 3(\beta^T \mu)^2 + 3\|\beta\|_2^2 + \lambda - 1 > \lambda - 1 \geq 0.
$$

To prove the results regarding second order information at the critical points, it suffices to look at $\nabla_{\beta\beta} L^k_\lambda(\alpha, \beta)$.

Following a similar strategy to the one we used for the claim, let us start by assuming that $\beta$ and $\mu$ are linearly independent. Then, $(31)$ yields

$$
[(\beta^T \mu)^2 + 3\|\beta\|_2^2 - 1](\mu^T \beta) = 0,
$$

$$
3(\mu^T \beta)^2 + M_Z\|\beta\|_2^2 - 1 = 0.
$$

Consider two cases:

**Case 1.** If $\mu^T \beta = 0$, then $(37)$ yields $\|\beta\|_2^2 = 1/M_Z$ and $(0, \beta) \in S^k_2$.

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Case 2. If $\mu^T \beta \neq 0$, then (36) forces $(\beta^T \mu)^2 + 3|\beta|^2 = 1 = 0$. Since $M_Z > 3$, this equation and (37) force $\beta = 0$ and $\mu^T \beta = 0$, which leads to contradiction.

Therefore, $S^h_1 \setminus \{0\}$ is the collection of all critical points that are linearly independent of $(0, \mu)$. For any $(0, \mu) \in S^h_2 \setminus \{0\}$, we have

$$
\nabla^2_{\beta \beta} L^h_0(0, \beta) = (3|\beta|^2 - 1)\mu \mu^T + 2M_Z \beta \beta^T,
$$

$$
\mu^T \nabla^2_{\beta \beta} L^h_0(0, \beta) \mu = (3|\beta|^2 - 1)|\mu|^2 = -(1 - 3/M_Z)|\mu|^4,
$$

$$
u^T \nabla^2 L^h_0(0, \beta) \nu \leq -(1 - 3/M_Z)|\mu|^2 < 0,
$$

where $u = (0, \mu/|\mu|^2)$. Hence the points in $S^h_2 \setminus \{0\}$ are strict saddles.

Now, suppose that $\beta = c \mu$ and $\nabla L^h_0(0, \beta) = 0$. By (31),

$$
\nabla L^h_0(0, \beta) = [(c|\mu|^2)^a + (3c^2|\mu|^2 - 1)c|\mu|^2] \mu + [3(c|\mu|^2)^a + M_Z c^2 |\mu|^2 - 1]c \mu
$$

$$
= [(c|\mu|^2)^a + 6|\mu|^2 + M_Z)|\mu|^2 c^2 - (|\mu|^2 + 1)c \mu.
$$

It is easily seen that $\nabla L^h_0(0, \beta) = 0$. If $c \neq 0$, then

$$
(|\mu|^2 + 6|\mu|^2 + M_Z)|\mu|^2 c^2 = |\mu|^2 + 1.
$$

Hence $S^h_1 \cup \{0\}$ is the collection of critical points that live in span$\{(0, \mu)\}$, and $S^h_1 \cup S^h_2$ contains all critical points of $L^h_0$.

We first investigate $\{0\}$. On the one hand,

$$
\nabla^2_{\beta \beta} L^h_0(0, \beta) = -(I + \mu \mu^T) < 0.
$$

On the other hand,

$$
L^h_0(\alpha, 0) = h(\alpha) + \frac{\lambda}{2} \alpha^2 = \frac{1}{4}(\alpha^2 - 1)^2 + \frac{\lambda}{2} \alpha^2,
$$

$$
\nabla_{\alpha} L^h_0(\alpha, 0) = \alpha^3 + (\lambda - 1) \alpha = \alpha(\alpha^2 + \lambda - 1).
$$

It follows from $\lambda \geq 1$ that $0$ is a local minimum of $L^h_0(\cdot, 0)$. Thus $0$ is a saddle point of $L^h_0$ whose Hessian has negative eigenvalues.

Next, for $(0, \beta) \in S_1$, we derive from (34) that

$$
\nabla^2_{\beta \beta} L^h_0(0, \beta) = [3(c|\mu|^2)^a + M_Z c^2 |\mu|^2 - 1]I + [3(c|\mu|^2)^a + 3c^2 |\mu|^2 - 1] \mu \mu^T
$$

$$
+ 6c|\mu|^2 \cdot 2c \mu \mu^T + 2M_Z c^2 \mu \mu^T
$$

$$
= [(3|\mu|^2 + M_Z) c^2 |\mu|^2 - 1]I + [(3|\mu|^2 + 15|\mu|^2 + 2M_Z) c^2 - 1] \mu \mu^T.
$$

From (39) we see that

$$
(3|\mu|^2 + M_Z) c^2 |\mu|^2 - 1 = \frac{(3|\mu|^2 + M_Z) (|\mu|^2 + 1)}{|\mu|^2 + 6|\mu|^2 + M_Z} - 1 = \frac{2|\mu|^2 + (M_Z - 3)|\mu|^2}{|\mu|^2 + 6|\mu|^2 + M_Z} > 0,
$$

$$
(3|\mu|^2 + 15|\mu|^2 + 2M_Z) c^2 - 1 \geq 2(6|\mu|^2 + 6|\mu|^2 + M_Z) c^2 - 1 = \frac{2(|\mu|^2 + 1)}{|\mu|^2 + 6} - 1 > 0.
$$

Hence both points in $S_1$ are local minima because

$$
\nabla^2_{\beta \beta} L^h_0(0, \beta) \geq \frac{2|\mu|^2 + (M_Z - 3)|\mu|^2}{|\mu|^2 + 6|\mu|^2 + M_Z} I > 0, \quad \forall (0, \beta) \in S_1,
$$

which immediately implies global optimality and finishes the proof.
E.1.2 Part 2: Quantitative properties of the landscape

1. Lemma 2 implies that we can choose a sufficiently small constant $\epsilon^h > 0$ and a constant $R > 0$ correspondingly such that $|\nabla L^h_\lambda(\gamma)|_2 \geq \epsilon^h$ when $|\gamma|_2 \geq R$. Without loss of generality, we can always take $\delta^h \leq 1$ and $R > 1 + \max_{\gamma \in S^h_1 \cup S^h_2} |\gamma|_2$. In doing so, we have

$$S = \{ \gamma : |\gamma|_2 \leq R, \text{ dist}(\gamma, S^h_1 \cup S^h_2) \geq \delta^h \} \neq \emptyset.$$ 

We now establish a lower bound for $\inf_{\gamma \in S} |\nabla L^h_\lambda(\gamma)|_2$. Define

$$S_\beta = \text{span} \{ (0, \mu), (0, \beta), (1, 0) \} \cap S, \forall \beta \perp \mu,$$

$$\epsilon_\beta = \inf_{\gamma \in S_\beta} |\nabla L^h_\lambda(\gamma)|_2.$$

By symmetry, $\epsilon_\beta$ is the same for all $\beta \perp \mu$. Denote this quantity by $\epsilon^h_2$. Since $S = \cup_{\beta \perp \mu} S_\beta$,

$$\inf_{\gamma \in S} |\nabla L^h_\lambda(\gamma)|_2 = \inf_{\beta \perp \mu} \inf_{\gamma \in S_\beta} |\nabla L^h_\lambda(\gamma)|_2 = \inf_{\beta \perp \mu} \epsilon_\beta = \epsilon^h_2.$$

Take any $\beta \perp \mu$. On the one hand, the nonnegative function $|\nabla L^h_\lambda(\cdot)|_2$ is continuous and its zeros are all in $S^h_1 \cup S^h_2$. On the other hand, $S_\beta$ is compact and non-empty. Hence $\epsilon^h_2 = \epsilon_\beta > 0$ and it only depends on the function $L^h_\lambda$ restricted to a three-dimensional subspace, i.e. span $\{ (0, \mu), (0, \beta), (1, 0) \}$. It is then straightforward to check using the quartic expression of $L^h_\lambda$ and symmetry that $\epsilon^h_2$ is completely determined by $|\mu|_2$, $M_\lambda$, $\lambda$ and $\delta^h$. From now on we write $\epsilon^h_2(\delta^h)$ to emphasize its dependence on $\delta^h$, whose value remains to be determined.

To sum up, when $\delta^h \leq 1$ and $\epsilon^h \leq \min\{ \epsilon^h_1, \epsilon^h_2(\delta^h) \}$, we have the desired result in the first claim.

2. Given properties (38), (40) and (41) of Hessians at all critical points, it suffices to show that

$$|\nabla^2 L^h_\lambda(\gamma_1) - \nabla^2 L^h_\lambda(\gamma_2)|_2 \leq C|\gamma_1 - \gamma_2|_2, \forall \gamma_1, \gamma_2 \in B(0, R)$$

holds for some constant $C'$ determined by $|\mu|_2$ and $R$. In that case, we can take sufficiently small $\delta^h$ and $\eta^h$ to finish the proof.

Based on (32), (33) and (34), we first decompose $|\nabla^2 L^h_\lambda(\gamma)|$ into the sum of two matrices $I(\gamma)$ and $J(\gamma)$:

$$|\nabla^2 L^h_\lambda(\gamma)| = \begin{pmatrix} 3\alpha^2 + 3(\beta^T \mu)^2 + 3|\beta|_2^2 + \lambda - 1 & 6\alpha \left[ (\beta^T \mu) \mu + \beta \right]^T \\ 6\alpha \left[ (\beta^T \mu) \mu + \beta \right] & 3\alpha^2 (I + \mu \mu^T) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \nabla^2_{\beta \beta} L^h_\lambda(\gamma) - 3\alpha^2 (I + \mu \mu^T) \end{pmatrix} = I(\gamma) + J(\gamma).$$

For any $\gamma_1 = (\alpha_1, \beta_1), \gamma_2 = (\alpha_2, \beta_2) \in B(0, R)$, we have

$$|I(\gamma_1) - I(\gamma_2)|_2 \leq 3|\alpha_1^2 + 3(\beta_1^T \mu)^2 + 3|\beta_1|_2^2 - 3|\alpha_2^2 + 3(\beta_2^T \mu)^2 - 3|\beta_2|_2^2| + 2 |6\alpha_1 [(\beta_1^T \mu) \mu + \beta_1] - 6\alpha_2 [(\beta_2^T \mu) \mu + \beta_2]|_2 + |3(\alpha_1^2 - \alpha_2^2)(I + \mu \mu^T)|_2.$$ 

Let $\Delta = |\gamma_1 - \gamma_2|_2$ and note that $|\alpha_1^2 - \alpha_2^2| \leq 2R \Delta$, $|\beta_1^2 - \beta_2^2| \leq 2R \Delta$, $|\beta_1^2 \mu - (\beta_2^T \mu)| \leq 2R |\mu|_2 \Delta$, $|\alpha_1(\beta_1^T \mu) - \alpha_2(\beta_2^T \mu)| \leq 2R |\mu|_2 \Delta$, we immediately have

$$|I(\gamma_1) - I(\gamma_2)|_2 \lesssim (1 + |\mu|_2 + |\mu|_2^2)R|\gamma_1 - \gamma_2|_2.$$
According to Lemma 1, $J(\gamma)$ depends on $\beta$ but not $\alpha$. Moreover, we have the following decomposition for its bottom right block:

$$
\begin{align*}
&\left[3 (\mu^T \beta)^2 + M_Z \|\beta\|^2 - 1 \right] I + \frac{\left[3 (\beta^T \mu)^2 + (3 \|\beta\|^2 - 1) \right] \mu \mu^T}{\lambda_1(\beta)} \\
&+ \frac{6 (\mu^T \beta) (\mu \beta^T + \beta \mu) + 2 M_Z \beta \beta^T}{\lambda_1(\beta)}.
\end{align*}
$$

Similar argument gives $\|J_1(\beta_1) - J_1(\beta_2)\| \lesssim \|(\mu \beta^2 + M_Z) R \Delta\|$, $\|J_2(\beta_1) - J_2(\beta_2)\| \lesssim \|(\mu \beta^2 + M_Z) R \Delta\|$ and $\|J_3(\beta_1) - J_3(\beta_2)\| \lesssim M_Z R \Delta$. As a result, we have

$$
\|J(\gamma_1) - J(\gamma_2)\| \lesssim \|(\mu \beta^2 + M_Z) R \|\gamma_1 - \gamma_2\|_2.
$$

Hence we finally get (42).

### E.2 Proof of Lemma 2

By definition, $\nabla L_\lambda(\gamma) - \nabla L_\lambda^h(\gamma) = \mathbb{E} \left( X \left[ f' (\gamma^T X) - h' (\gamma^T X) \right] \right)$. From Lemma 1, we obtain that $|f'(x) - h'(x)| \lesssim |x|^3 1_{\{|x| \geq a\}}$ when $b \geq 2a$ and $a$ is sufficiently large. When $\|\gamma\|_2 \leq R$, we have

$$
\|\nabla L_\lambda(\gamma) - \nabla L_\lambda^h(\gamma)\|_2 = \sup_{u \in \mathbb{S}^d} \mathbb{E} \left( u^T X \left[ f'(\gamma^T X) - h'(\gamma^T X) \right] \right)
$$

(i) $\leq \sup_{u \in \mathbb{S}^d} \mathbb{E} (u^T X)^3 \mathbb{E} |\gamma^T X|^{9/3} \mathbb{P} (|\gamma^T X| \geq a)

(ii) $\leq \sup_{u \in \mathbb{S}^d} \|u^T X\|_{\psi_2} \|\gamma^T X\|_{\psi_2}^{3} \exp \left(- \frac{C_1a^2}{\|\gamma^T X\|_{\psi_2}^2} \right)

(iii) $\leq R^3 M^4 \exp \left(- \frac{C_1a^2}{R^2 M^2} \right)

for some constant $C_1 > 0$. Here (i) uses H"older’s inequality, (ii) comes from sub-Gaussian property [Vershynin, 2010], and (iii) uses $\|u^T X\|_{\psi_2} \leq \|u\|_2 \|X\|_{\psi_2} = \|u\|_2 M$, $\forall u \in \mathbb{R}^{d+1}$.

To study the Hessian, we start from $\nabla^2 L_\lambda(\gamma) - \nabla^2 L_\lambda^h(\gamma) = \mathbb{E} \left( XX^T \left[ f'' (\gamma^T X) - h'' (\gamma^T X) \right] \right)$. Again from Lemma 1, we know that $|f''(x) - h''(x)| \lesssim x^2 1_{\{|x| \geq a\}}$. When $\|\gamma\|_2 \leq R$, we have

$$
\|\nabla^2 L_\lambda(\gamma) - \nabla^2 L_\lambda^h(\gamma)\|_2 = \sup_{u \in \mathbb{S}^d} u^T \mathbb{E} \left( XX^T \left[ f'' (\gamma^T X) - h'' (\gamma^T X) \right] \right) u
$$

(i) $\leq \sup_{u \in \mathbb{S}^d} \mathbb{E} \left( |u^T X|^2 \|\gamma^T X\|^2 1_{\{|\gamma^T X| \geq a\}} \right)

(ii) $\leq \sup_{u \in \mathbb{S}^d} \mathbb{E} |u^T X|^6 \mathbb{E} |\gamma^T X|^{6/3} \mathbb{P} (|\gamma^T X| \geq a)

(iii) $\leq \sup_{u \in \mathbb{S}^d} \|u^T X\|_{\psi_2}^2 \|\gamma^T X\|_{\psi_2}^2 \exp \left(- \frac{C_1a^2}{\|\gamma^T X\|_{\psi_2}^2} \right)

\leq R^2 M^4 \exp \left(- \frac{C_1a^2}{R^2 M^2} \right)

for some constant $C_1 > 0$.

We finally work on the lower bound for $\|\nabla L_\lambda(\gamma)\|_2$. From $b \geq 2a \geq 4$ we get $f(x) = h(x)$ for $|x| \leq a$; $f'(x) \geq 0$ and $f''(x) \geq 0$ for all $x \geq 1$. Since $f'$ is odd,

$$
\inf_{x \in \mathbb{R}} x f'(x) = \inf_{|x| \leq 1} x f'(x) = \inf_{|x| \leq 1} x h'(x) = \inf_{|x| \leq 1} \{x^4 - x^2\} \geq -1,
$$

$$
\inf_{x \geq 2} x f'(x) \geq f'(2) = h'(2) = 2^3 - 2 = 6.
$$
Taking $a = 2$, $b = 1$ and $c = 6$ in Lemma\footnote{\ref{lem:11}} we get
\[
\|L_\alpha(\gamma)\|_2 \geq 6 \inf_{u \in \mathbb{S}^d} \mathbb{E}[u^T X] - \frac{12 + 1}{\|\gamma\|_2} \geq 6 \varphi(\|X\|_{\psi_2}, \lambda_{\min}(\mathbb{E}(XX^T))) - \frac{13}{\|\gamma\|_2} \geq 6 \varphi(M, \sigma^2) - \frac{13}{\|\gamma\|_2}
\]
for $\gamma \neq 0$. Here $\varphi$ is the function in Lemma\footnote{\ref{lem:12}}. If we let $m = \varphi(M, \sigma^2)$, then $\inf_{\|\gamma\|_2 \geq 3/m} \|L_\alpha(\gamma)\|_2 \geq m$. Follow a similar argument, we can show that $\inf_{\|\gamma\|_2 \geq 3/m} \|L_\alpha(\gamma)\|_2 \geq m$ also holds for the same $m$.

### E.3 Proof of Lemma\footnote{\ref{lem:12}}

To prove the first part, we define $\hat{\mu} = \mu/\|\mu\|_2$ and seek for $c > 0$ determined by $\|\mu\|_2$, the function $f$, and the distribution of $Z$ such that $\nabla L_1(0, \pm c\hat{\mu}) = 0$.

By the chain rule, for any $(\alpha, \beta, t) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ we have
\[
\nabla L_\alpha(\alpha, \beta, t) = \left( \frac{\mathbb{E} f'(\alpha + \beta X) + \lambda \alpha}{\mathbb{E} X f'(\alpha + \beta X)} \right) \quad \text{and} \quad \nabla L_1(0, t \hat{\mu}) = \left( \frac{\mathbb{E} f'(t \hat{\mu}^T X)}{\mathbb{E} X f'(t \hat{\mu}^T X)} \right).
\]

Since $f$ is even, $f'$ is odd and $t \hat{\mu}^T X$ has symmetric distribution with respect to 0, we have $\mathbb{E} f'(t \hat{\mu}^T X) = 0$. It follows from $(I - \hat{\mu} \hat{\mu}^T)X = (I - \hat{\mu} \hat{\mu}^T)Z$ that
\[
(I - \hat{\mu} \hat{\mu}^T)\mathbb{E} X f'(t \hat{\mu}^T X) = \mathbb{E}((I - \hat{\mu} \hat{\mu}^T)Z f'(t \hat{\mu}^T X)) = \mathbb{E}((I - \hat{\mu} \hat{\mu}^T)Z f'(t \|\mu\|_2 Y + t \hat{\mu}^T Z)).
\]

Thanks to the independence between $Y$ and $Z$ as well as the spherical symmetry of $Z$, $(Y, \hat{\mu}^T Z, (I - \hat{\mu} \hat{\mu}^T)Z)$ and $(Y, \hat{\mu}^T Z, - (I - \hat{\mu} \hat{\mu}^T)Z)'$ share the same distribution. Then
\[
(I - \hat{\mu} \hat{\mu}^T)\mathbb{E} X f'(t \hat{\mu}^T X) = 0 \quad \text{and} \quad \mathbb{E} X f'(t \hat{\mu}^T X) = \hat{\mu} \hat{\mu}^T \mathbb{E} X f'(t \hat{\mu}^T X).
\]

As a result,
\[
\nabla L_1(0, t \hat{\mu}) = \mathbb{E} [\mu^T X f'(t \hat{\mu}^T X)] \left( \frac{0}{\hat{\mu}} \right).
\]

Define $W = \hat{\mu}^T X = \|\mu\|_2 Y + \hat{\mu}^T Z$ and $\varphi(t) = \mathbb{E} [W f'(tW)]$ for $t \in \mathbb{R}$. The fact that $f$ is even yields $f'(0) = 0$ and $\varphi(0) = \mathbb{E} [W f'(0)] = 0$. On the one hand, $f''(0) < 0$ forces
\[
\varphi'(0) = \mathbb{E} [W^2 f''(tW)]|_{t=0} = f''(0) \mathbb{E} W^2 = f''(0)(\|\mu\|_2^2 + 1) < 0.
\]

Hence there exists $t_1 > 0$ such that $\varphi(t_1) < 0$. On the other hand, $\lim_{x \to +\infty} x f'(x) = +\infty$ leads to $\lim_{x \to +\infty} x \varphi(x) = \mathbb{E} [W f'(tW)] = +\infty$. Then there exists $t_2 > 0$ such that $\varphi(t_2) > 0$. By the continuity of $\varphi$, we can find some $c > 0$ such that $\varphi(c) = 0$. Consequently,
\[
\nabla L_1(0, c\hat{\mu}) = \varphi(c) \left( \frac{0}{\hat{\mu}} \right) = 0.
\]

In addition, from
\[
\varphi(-c) = \mathbb{E} [W f'(-cW)] = -\mathbb{E} [W f'(cW)] = -\varphi(c) = 0
\]
we get $\nabla L(0, -c\hat{\mu}) = 0$. It is easily seen that $t_1$, $t_2$ and $c$ are purely determined by properties of $f$ and $W$, where the latter only depends on $\|\mu\|_2$ and the distribution of $Z$. This finishes the first part.

To prove the second part, we first observe that
\[
|\varphi''(t)| = |\mathbb{E} [W^3 f'''(tW)]| \leq F_3 \mathbb{E} |W|^3 = F_3 (3^{-1/2} \mathbb{E}^{1/3} |W|)^3 \cdot 3^{3/2} \leq 3^{3/2} F_3 M, \quad \forall t \in \mathbb{R}.
\]

Let $c_0 = -f''(0)(\|\mu\|_2^2 + 1)/(3^{3/2} F_3 M)$. In view of (43),
\[
\varphi'(t) \leq \varphi'(0) + t \sup_{s \in \mathbb{R}} |\varphi''(s)| \leq f''(0)(\|\mu\|_2^2 + 1) + 3^{3/2} F_3 M t < 0, \quad \forall t \in [0, c_0).
\]

Thus $\varphi(t) < \varphi(0) = 0$ in the same interval, forcing $c > c_0$. 

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F Proof of Theorem 3

It suffices to prove the bound on the exceptional probability for each claim.

1. Claim 1 can be derived from Lemma 4, Theorem 2, and concentration of gradients within a ball (cf. Lemma 6).

Lemma 4. Let \( \{X_i\}_{i=1}^n \) be i.i.d. random vectors in \( \mathbb{R}^{d+1} \) with \( \|X_i\|_{\psi_2} \leq 1 \) and \( E(X_iX_i^T) \succeq \sigma^2 I \) for some \( \sigma > 0 \), \( f \) be defined in (8) with \( b \geq 2a \geq 4 \), and

\[
\hat{L}_\lambda(\gamma) = \frac{1}{n} \sum_{i=1}^n f(\gamma^T X_i) + \frac{\lambda}{2} (\gamma^T \hat{\mu})^2
\]

with \( \hat{\mu} = \frac{1}{n} \sum_{i=1}^n X_i \) and \( \lambda \geq 0 \). There exist positive constants \( C, C_1, C_2, R \) and \( \epsilon_1 \) determined by \( \sigma \) such that when \( n/d \geq C \),

\[
P \left( \inf_{\|\gamma\|_2 \geq R} \|\nabla \hat{L}_\lambda(\gamma)\|_2 > \epsilon_1 \right) > 1 - C_1(d/n)^{C_2d}.
\]

Proof. See Appendix F.1

Let \( R \) and \( \epsilon \) be the constants stated in Lemma 4 and Theorem 2 respectively. Lemma 6 asserts that

\[
P \left( \sup_{\gamma \in B(0,R)} \|\nabla \hat{L}_\lambda(\gamma) - \nabla L_\lambda(\gamma)\|_2 < \frac{\epsilon}{2} \right) > 1 - C_1(d/n)^{C_2d}
\]

for some constant \( C_1, C_2 > 0 \), provided that \( n/d \) is large enough. From Theorem 2 we know that \( \|\nabla L_\lambda(\gamma)\|_2 \geq \epsilon \) if \( \text{dist}(\gamma, \{\pm \gamma^*\} \cup S) \geq \delta \). The triangle inequality immediately gives

\[
P \left( \inf_{\gamma : \text{dist}(\gamma, \{\pm \gamma^*\} \cup S) \geq \delta} \|\nabla \hat{L}_\lambda(\gamma)\|_2 > \epsilon/2 \right) < 1 - C_1'(d/n)^{C_2'd},
\]

for some constants \( C_1' \) and \( C_2' \).

2. We invoke the following Lemma 5 to prove Claim 2.

Lemma 5. Let \( \{X_i\}_{i=1}^n \) be i.i.d. random vectors in \( \mathbb{R}^{d+1} \) with \( \|X_i\|_{\psi_2} \leq 1 \); \( u \in \mathbb{S}^d \) be deterministic; \( R > 0 \) be a constant. Let \( f \) be defined in (8) with constants \( b \geq 2a \geq 4 \), and

\[
\hat{L}_\lambda(\gamma) = \frac{1}{n} \sum_{i=1}^n f(\gamma^T X_i) + \frac{\lambda}{2} (\gamma^T \hat{\mu})^2
\]

with \( \hat{\mu} = \frac{1}{n} \sum_{i=1}^n X_i \) and \( \lambda \geq 0 \). Suppose that \( n/d \geq e \). There exist positive constants \( C, C_2, C \) and \( N \) such that when \( n > N \),

\[
P \left( \sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla \hat{L}_\lambda(\gamma_1) - \nabla \hat{L}_\lambda(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} < C \right) > 1 - C_1 e^{-C_2 n},
\]

\[
P \left( \sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla^2 \hat{L}_\lambda(\gamma_1) - \nabla^2 \hat{L}_\lambda(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} < C \max\{1, d \log(n/d)/\sqrt{n}\} \right) > 1 - C_1(d/n)^{C_2d},
\]

\[
P \left( \sup_{\|\gamma\|_2 \leq R} \|u^T [\nabla^2 \hat{L}_\lambda(\gamma) - \nabla^2 L_\lambda(\gamma)] u\| < C \sqrt{d \log(n/d)/n} \right) > 1 - C_1(d/n)^{C_2d} - C_1 e^{-C_2 n^{1/3}}.
\]

Proof. See Appendix F.2

From Theorem 2 we know that \( u^T \nabla^2 L_\lambda(\gamma) u \leq -\eta \) if \( \text{dist}(\gamma, S) \leq \delta \). Lemma 5 (after proper rescaling) asserts that

\[
P \left( \sup_{\|\gamma\|_2 \leq R} \|u^T [\nabla^2 \hat{L}_\lambda(\gamma) - \nabla^2 L_\lambda(\gamma)] u\| < \frac{\eta}{2} \right) > 1 - C_1(d/n)^{C_2d} - C_1 e^{-C_2 n^{1/3}}
\]

provided that \( n/d \) is sufficiently large. Then Claim 2 follows from the triangle’s inequality.

3. Claim 3 follows from Lemma 6 with proper rescaling.
F.1 Proof of Lemma 4

It is shown in Lemma 2 that when \( b \geq 2a \geq 4 \), we have \( \inf_{x \in \mathbb{R}} x f'(x) \geq -1 \) and \( \inf_{|x| \geq 2} f'(x) \text{sgn}(x) \geq 6 \). Using an empirical version of Lemma 5

\[
\nabla \hat{L}_\lambda(\gamma) \geq \inf_{\gamma \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} [u^\top X_i] - \frac{13}{\|\gamma\|_2}, \quad \forall \gamma \in \mathbb{R}^d.
\]

Define \( S_n(u) = \frac{1}{n} \sum_{i=1}^{n} (|u^\top X_i| - \mathbb{E}|u^\top X_i|) \) for \( u \in \mathbb{S}^d \). By the triangle inequality,

\[
\hat{L}_\lambda(\gamma) \geq \inf_{\gamma \in \mathbb{R}^d} \mathbb{E}|u^\top X_1| - \sup_{\gamma \in \mathbb{R}^d} |S_n(u)| - \frac{13}{\|\gamma\|_2}, \quad \forall \gamma \in \mathbb{R}^d.
\]

According to Lemma 9, \( \inf_{\gamma \in \mathbb{R}^d} \mathbb{E}|u^\top X_1| > \varphi \) for some constant \( \varphi > 0 \) determined by \( \sigma \). Then it suffices to prove

\[
\sup_{\gamma \in \mathbb{R}^d} |S_n(u)| = O_P(\sqrt{d \log(n/d)/n}; \; d \log(n/d)). \tag{44}
\]

We will use Theorem 1 in [Wang 2019] to get there.

1. Since \( \|X_i\|_{\psi_2} \leq 1 \), the Hoeffding-type inequality in Proposition 5.10 of [Vershynin 2010] asserts the existence of a constant \( c > 0 \) such that

\[
\mathbb{P}(|S_n(u)| \geq t) \leq e^{-cnt^2}, \quad \forall t \geq 0.
\]

Then \( \{S_n(u)\}_{u \in \mathbb{S}^d} = O_P(\sqrt{d \log(n/d)/n}; \; d \log(n/d)) \).

2. Let \( \varepsilon_n = \sqrt{d/n} \). According to Lemma 5.2 in [Vershynin 2010], there exists an \( \varepsilon_n \)-net \( \mathcal{N}_n \) of \( \mathbb{S}^d \) with cardinality at most \( (1 + 2R/\varepsilon_n)^d \). When \( n/d \) is large, \( \log |\mathcal{N}_n| = d \log(1 + \sqrt{n/d}) \lesssim d \log(n/d) \).

3. Define \( M_n = \sup_{u \in \mathbb{S}^d, v \in \mathbb{S}^d, u \neq v} \{ |S_n(u) - S_n(v)|/\|u - v\|_2 \} \). By Cauchy-Schwarz inequality,

\[
\left| \frac{1}{n} \sum_{i=1}^{n} [u^\top X_i] - \frac{1}{n} \sum_{i=1}^{n} [v^\top X_i] \right| \leq \frac{1}{n} \sum_{i=1}^{n} |(u - v)^\top X_i| \leq \left( \frac{1}{n} \sum_{i=1}^{n} |(u - v)^\top X_i|^2 \right)^{1/2} \leq \|u - v\|_2 \sup_{w \in \mathbb{S}^d} \left( \frac{1}{n} \sum_{i=1}^{n} |w^\top X_i|^2 \right)^{1/2} = \|u - v\|_2 \cdot \mathbb{E}(\|X_1 X_1^\top\|_2 \leq \|u - v\|_2).
\]

Hence \( M_n = O_P(1; \; n) \).

Then Theorem 1 in [Wang 2019] yields (44).

F.2 Proof of Lemma 5

It follows from Example 6 in [Wang 2019] that \( \|n^{-1} \sum_{i=1}^{n} X_i - \mu_0\|_2 = O_P(1; \; n) \). As a result \( \|n^{-1} \sum_{i=1}^{n} X_i\|_2 = O_P(1; \; n) \). This combined with Lemma 8 and Lemma 11 gives

\[
\sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla \hat{L}_\lambda(\gamma_1) - \nabla \hat{L}_\lambda(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} = O_P(1; \; n),
\]

\[
\sup_{\gamma_1 \neq \gamma_2} \frac{|u^\top [\nabla^2 \hat{L}_\lambda(\gamma_1) - \nabla^2 \hat{L}_\lambda(\gamma_2)] u|}{\|\gamma_1 - \gamma_2\|_2} = O_P(1; \; n^{1/3}),
\]

\[
\sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla^2 \hat{L}_\lambda(\gamma_1) - \nabla^2 \hat{L}_\lambda(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} = O_P(\max\{1, d \log(n/d)/\sqrt{n}; \; d \log(n/d)\})
\]
given $F_2 \leq 3a^2 \lesssim 1$ and $F_3 \leq 6a \lesssim 1$, provided that $n/d$ is sufficiently large. It is easily seen that there exist universal constants $(c_1, c_2, N) \in (0, +\infty)^3$ and a non-decreasing function $f : [c_2, +\infty) \to (0, +\infty)$ with $\lim_{t \to +\infty} f(x) = +\infty$, such that

\[
P \left( \sup_{\gamma \neq \gamma_1} \frac{\|\nabla L(\gamma_1) - \nabla L(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} \geq t \right) \leq c_1 e^{-n f(t)},
\]

\[
P \left( \sup_{\gamma \neq \gamma_1} \frac{|u^T (\nabla^2 L(\gamma_1) - \nabla^2 L(\gamma_2)) u|}{\|\gamma_1 - \gamma_2\|_2} \geq t \right) \leq c_1 e^{-n^{1/3} f(t)},
\]

\[
P \left( \sup_{\gamma \neq \gamma} \frac{\|\nabla^2 L(\gamma_1) - \nabla^2 L(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} \geq t \right) \leq t \max\{1, d \log(n/d) / \sqrt{n}\},
\]

as long as $n \geq N_1$ and $t \geq c_2$. We prove the first two inequalities in Lemma 5 by (43), (47) and choosing proper constants.

Let

\[X_n(\gamma) = u^T (\nabla^2 L(\gamma) - \nabla^2 L(\gamma_1)) u = u^T (\nabla^2 L(\gamma) - \nabla^2 L(\gamma_1)) u,
\]

$S_n = B(0, R)$ and $m = \log(n/d)$. We will invoke Theorem 1 in Wang (2019) to control $\sup_{\gamma \in S_n} |X_n(\gamma)|$ and prove the remaining claim.

1. By definition, $X_n(\gamma) = \frac{1}{n} \sum_{i=1}^{n} ((u^T X_i)^{2f''(\gamma) X_i}) - \mathbb{E}[(u^T X_i)^{2f''(\gamma^T X_i)}]$ and

\[\|u^T X_i\|_2^2 \leq F_2 \|u^T X_i\|_2^2 \leq F_2 \|u^T X_i\|_2^2 \leq 1.
\]

By the Bernstein-type inequality in Proposition 5.16 of Vershynin (2010), there is a constant $c'$ such that

\[P(|X_n(\gamma)| \geq t) \leq 2 e^{-c' n |t|^2}, \quad \forall t \geq 0, \gamma \in \mathbb{R}^d.
\]

When $t = s \sqrt{md/n}$ for $s \geq 1$, we have $nt^2 = s^2 md \geq smd$. Since $n/d \geq e$, we have

\[m = \log(n/d) = \log(1 + (n/d - 1)) \leq n/d - 1 \leq n/d,
\]

\[n \geq md \text{ and } nt = s \sqrt{md/n} \geq smd. \text{ This gives}
\]

\[P(|X_n(\gamma)| \geq s \sqrt{md/n}) \leq 2 e^{-c' n s^2 d}, \quad \forall s \geq 1, \gamma \in \mathbb{R}^d.
\]

Hence $\{X_n(\gamma)\}_{\gamma \in S_n} = O_P(\sqrt{md/n} \cdot md)$.

2. Let $\varepsilon_n = 2R/\sqrt{d/n}$. According to Lemma 5.2 in Vershynin (2010), there exists an $\varepsilon_n$-net $N_0$ of $S_n$ with cardinality at most $(1 + 2R/\varepsilon_n)^d$. Since $n/d \geq e$, $\log |N_0| = d \log(1 + \sqrt{n/d}) \leq d \log(n/d) = md$.

3. Define $M_n = \sup_{\gamma_1 \neq \gamma_2} \{ |X_n(\gamma_1) - X_n(\gamma_2)| / \|\gamma_1 - \gamma_2\|_2 \}$. Observe that by Lemma 8 and $\|X\|_{\psi_2} \leq 1$,

\[\sup_{\gamma_1 \neq \gamma_2} \frac{|u^T (\nabla^2 L(\gamma_1) - \nabla^2 L(\gamma_2)) u|}{\|\gamma_1 - \gamma_2\|_2} \leq \sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla^2 L(\gamma_1) - \nabla^2 L(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} \leq F_3 \sup_{u \in S^d} E[|u^T X|^3] \leq (\sqrt{3})^3 F_3 \lesssim 1.
\]

From this and (46) we obtain that $M_n = O_P(\sqrt{md/n} \cdot n^{1/3})$.

Based on these, Theorem 1 Wang (2019) implies that

\[\sup_{\gamma \in S_n} |X_n(\gamma)| = O_P(\sqrt{md/n + \varepsilon_n}; \quad md \wedge n^{1/3}) = O_P(\log(n/d) d/n; \quad d \log(n/d) \wedge n^{1/3}).
\]

As a result, there exist absolute constants $(c_1, c_2, N_1) \in (0, +\infty)^3$ and a non-decreasing function $g : [c_2, +\infty) \to (0, +\infty)$ such that

\[P \left( \sup_{\gamma \in S_n} |X_n(\gamma)| \geq t \sqrt{\log(n/d) d/n} \right) \leq c_1 e^{-n^{1/3} g(t)} \leq c_1 (e^{-md g(t)} + e^{-n^{1/3} g(t)}) \leq c_1 (d/n)^{g(t)} + c_1 e^{-n^{1/3} g(t)}, \quad \forall n \geq N_1, t \geq c_2.
\]

The proof is finished by taking $t = c_2$ and re-naming some constants above.
G Proof of Corollary 2

From Claim 1 in the second item of Theorem 3, we know that $\|\nabla \hat{L}_1(\gamma)\|_2 \leq \varepsilon$ implies $\text{dist}(\gamma, \{\pm \gamma^*\} \cup S) < \delta$. On the other side, since $\lambda_{\min}(\nabla^2 \hat{L}_1(\gamma)) = \varepsilon$, we have $\nabla^2 \hat{L}_1(\gamma) v \geq -\varepsilon$ for any unit vector $v$. Then in view of Claim 2 of Theorem 3, we know that $\text{dist}(\gamma, S) > \delta$. Therefore we arrive at $\text{dist}(\gamma, \{\pm \gamma^*\}) < \delta$. According to Theorem 2 $\nabla^2 \hat{L}_1(\gamma^* \geq \varepsilon I$ so long as $\text{dist}(\gamma^*, S_1) \leq \delta$. This and $\nabla L_1(\gamma^*) = 0$ lead to

$$
\min_{s = \pm 1} \|s\gamma - \gamma^*\|_2 \leq \frac{1}{\eta} \|\nabla L_1(\gamma) - \nabla L_1(\gamma^*)\|_2 = \frac{1}{\eta} \|\nabla \hat{L}_1(\gamma)\|_2 \\
\leq \frac{1}{\eta} \|\nabla \hat{L}_1(\gamma)\|_2 + \frac{1}{\eta} \|\nabla L_1(\gamma) - \nabla L_1(\gamma^*)\|_2.
$$

(48)

All of these hold with probability exceeding $1 - C_1(d/n)^{C_2d} - C_1 \exp(-C_2 n^{1/3})$.

The desired result is a product of (48) and Lemma 6 below.

Lemma 6. For any constant $R > 0$, there exists a constant $C > 0$ such that when $n \geq Cd$ for all $n$,

$$
\sup_{\|\gamma\|_2 \leq R} \|\nabla \hat{L}_1(\gamma) - \nabla \hat{L}_1(\gamma)\|_2 = O_p \left( \sqrt{\frac{d}{n}} \log \left( \frac{n}{d} \right) \right)
$$

(49)

Proof. See Appendix G.1

G.1 Proof of Lemma 6

Let $\gamma = (\alpha, \beta), \hat{L}(\gamma) = \frac{1}{n} \sum_{i=1}^n f(\alpha + \beta^T X_i), L(\gamma) = \mathbb{E} f(\alpha + \beta^T X), \hat{R}(\gamma) = \frac{1}{n} (\alpha + \beta^T \mu_0)^2$ and $R(\gamma) = \frac{1}{2} (\alpha + \beta^T \mu_0)^2$. Since $|f(0)| = 0, \sup_{x \in B} |f''(x)| = h'(a) + (b-a)h''(a) \leq 3a^2b \leq 1$ and $\|X_i\|_{\psi_2} \leq M \leq 1$, from Theorem 2 in Wang (2019) we get

$$
\sup_{\|\gamma\|_2 \leq R} \|\nabla \hat{L}(\gamma) - \nabla L(\gamma)\|_2 = O_p \left( \sqrt{\frac{d}{n}} \log \left( \frac{n}{d} \right) \right).
$$

Then it boils down to proving uniform convergence of $\|\nabla \hat{R}(\gamma) - \nabla R(\gamma)\|_2$. Let $\bar{X}_1 = (1, X_i), \bar{\mu}_0 = (1, \frac{1}{n} \sum_{i=1}^n X_i)$ and $\bar{\mu}_0 = (1, \mu_0)$. By definition,

$$
\nabla \hat{R}(\gamma) = (\gamma^T \bar{\mu}_0 - \bar{\mu}_0) \bar{\mu}_0 \quad \text{and} \quad \nabla R(\gamma) = (\gamma^T \mu_0) \mu_0.
$$

Since $\|\bar{X}_1 - \bar{\mu}_0\|_{\psi_2} \lesssim \|\bar{X}_1\|_{\psi_2} \lesssim 1$, we know that $\|\bar{\mu}_0 - \bar{\mu}_0\|_{\psi_2} \lesssim 1/\sqrt{n}$. In view of Example 6 in Wang (2019) and $\|\mu_0\|_2 \lesssim 1$, we know that $\|\bar{\mu}_0 - \mu_0\|_2 = O_p (\sqrt{d/n} \log(n/d); \log(n/d))$ and $\|\hat{\mu}_0\|_2 = O_p (1; d \log(n/d))$. As a result,

$$
\sup_{\|\gamma\|_2 \leq R} \|\nabla \hat{R}(\gamma) - \nabla R(\gamma)\|_2 \leq \sup_{\|\gamma\|_2 \leq R} \left\{ |\gamma^T (\bar{\mu}_0 - \mu_0)| \|\bar{\mu}_0\|_2 + |\gamma^T \mu_0| \|\bar{\mu}_0 - \mu_0\|_2 \right\}
$$

$$
\leq R \|\bar{\mu}_0 - \mu_0\|_2 (\|\bar{\mu}_0\|_2 + \|\mu_0\|_2) = O_p \left( \sqrt{\frac{d}{n}} \log \left( \frac{n}{d} \right) \right).
$$

H Proof of Theorem 4

To prove Theorem 4, we invoke the convergence guarantees for perturbed gradient descent in Jin et al. (2017).

Theorem 6 (Theorem 3 of Jin et al. (2017)). Assume that $F(\cdot)$ is $\ell$-smooth and $\rho$-Hessian Lipschitz. Then there exists an absolute constant $c_{\text{max}}$ such that, for any $\delta_{\text{pgd}} > 0$, $\varepsilon_{\text{pgd}} \leq \ell^2/\rho$, $\Delta_{\text{pgd}} \geq F(\gamma_{\text{pgd}}) - \inf_{\gamma \in \mathbb{R}^{d+1}} F(\gamma)$ and constant $c_{\text{pgd}} \leq c_{\text{max}}$, with probability exceeding $1 - \delta_{\text{pgd}}$, Algorithm terminates within

$$
T \lesssim \left( \frac{\ell (F(\gamma_{\text{pgd}}) - \inf_{\gamma \in \mathbb{R}^{d+1}} F(\gamma))}{\varepsilon_{\text{pgd}}^2} \right) \log^4 \left( \frac{d \ell \Delta_{\text{pgd}}}{\varepsilon_{\text{pgd}}^2 \delta_{\text{pgd}}^4} \right).
$$
iterations and the output $\gamma^T$ satisfies

$$\|\nabla F (\gamma^T)\|_2 \leq \varepsilon_{pgd} \quad \text{and} \quad \lambda_{\min}(\nabla^2 F(\gamma)) \geq -\sqrt{\rho c_{pgd}}.$$ 

Let $\mathcal{A}$ denote this event where all of the geometric properties in Theorem 3 holds. When $\mathcal{A}$ happens, $\hat{L}_1$ is $\ell$-smooth and $\rho$-Hessian Lipschitz with

$$\ell = M_1 \quad \text{and} \quad \rho = M_1 \left( 1 \vee \frac{d \log(n/d)}{\sqrt{n}} \right).$$

Let $\gamma_{pgd} = 0$ and $\Delta_{pgd} = 1/4$. Since $\inf_{\gamma \in \mathbb{R} \times \mathbb{R}^d} \hat{L}_1(\gamma) \geq 0$, we have

$$\Delta_{pgd} = \hat{L}_1(\gamma_{pgd}) \geq \hat{L}_1(\gamma_{pgd}) - \inf_{\gamma \in \mathbb{R} \times \mathbb{R}^d} \hat{L}_1(\gamma).$$

In addition, we take $\delta_{pgd} = n^{-11}$ and let

$$\varepsilon_{pgd} = \sqrt{\frac{d \log(n/d)}{n}} \wedge \frac{\ell^2}{\rho} \wedge \frac{\eta^2}{\rho} \wedge \varepsilon.$$ 

Here $\varepsilon$ and $\eta$ are the constants defined in Theorem 3.

Recall that $M_1, \xi, \varepsilon \approx 1$. Conditioned on the event $\mathcal{A}$, Theorem 3 asserts that with probability exceeding $1 - n^{-10}$, Algorithm 3 with parameters $\gamma_{pgd}$, $\ell$, $\rho$, $\varepsilon_{pgd}$, $c_{pgd}$, $\delta_{pgd}$, and $\Delta_{pgd}$ terminates within

$$T \lesssim \left( \frac{n}{d \log(n/d)} + \frac{d^2}{n} \log^2 \left( \frac{n}{d} \right) \right) \log^4 (nd) = O \left( \frac{n}{d} + \frac{d^2}{n} \right)$$

iterations, and the output $\hat{\gamma}$ satisfies

$$\|\nabla \hat{L}_1(\hat{\gamma})\|_2 \leq \varepsilon_{pgd} \leq \sqrt{\frac{d \log(n/d)}{n}} \quad \text{and} \quad \lambda_{\min}(\nabla^2 \hat{L}_1(\hat{\gamma})) \geq -\sqrt{\rho c_{pgd}} \geq -\eta.$$ 

Then the desired result follows directly from $\mathbb{P}(\mathcal{A}) \geq 1 - C_1(d/n)^C_{2d} - C_1 \exp(-C_2 n^{1/3})$ in Theorem 3.

I Proof of Corollary 1

Throughout the proof we suppose that the high-probability event

$$\min_{s = \pm 1} \|s \hat{\gamma} - c \gamma^{\text{Bayes}}\|_2 \lesssim \sqrt{\frac{d \log(n/d)}{n}}$$

in Theorem 1 happens. Write $\hat{\gamma} = (\hat{\alpha}, \hat{\beta})$ and $\gamma^* = (\alpha^*, \beta^*) = c \gamma^{\text{Bayes}}$. Without loss of generality, assume that $\mu_0 = 0$, $\Sigma = I_d$, $\arg\min_{s = \pm 1} \|s \hat{\gamma} - \gamma^*\|_2 = 1$ and $\beta^T \mu > 0$. Let $F$ be the cumulative distribution function of $Z = e_1^T Z$.

For any $\gamma = (\alpha, \beta)$ with $\beta^T \mu > 0$, we use $X = \mu Y + Z$ and the symmetry of $Z$ to derive that

$$\mathcal{R}(\gamma) = \frac{1}{2} \mathbb{P}(\alpha + \beta^T (\mu + Z) < 0) + \frac{1}{2} \mathbb{P}(\alpha + \beta^T (-\mu + Z) > 0)$$

$$= \frac{1}{2} \mathbb{P}(\beta^T Z < -\alpha - \beta^T \mu) + \frac{1}{2} \mathbb{P}(\beta^T Z > -\alpha + \beta^T \mu)$$

$$= \frac{1}{2} F \left( -\alpha \|\beta\|_2 - (\beta / \|\beta\|_2)^T \mu \right) + \frac{1}{2} F \left( \alpha \|\beta\|_2 - (\beta / \|\beta\|_2)^T \mu \right).$$

Define $\gamma_0 = (\alpha_0, \beta_0)$ with $\alpha_0 = \hat{\alpha} / \|\hat{\beta}\|_2$ and $\beta_0 = \hat{\beta} / \|\hat{\beta}\|_2$: $\gamma_1 = (\alpha_1, \beta_1)$ with $\alpha_1 = 0$ and $\beta_1 = \mu / \|\mu\|_2$. Recall that $\gamma^{\text{Bayes}} = c(0, \mu)$ for some constant $c > 0$. We have

$$\mathcal{R}(\hat{\gamma}) - \mathcal{R}(\gamma^{\text{Bayes}}) = \frac{1}{2} F(-\alpha_0 - \beta_0^T \mu) - \frac{1}{2} F(-\alpha_1 - \beta_1^T \mu) + \frac{1}{2} F(\alpha_0 - \beta_0^T \mu) - \frac{1}{2} F(\alpha_1 - \beta_1^T \mu).$$
Using Taylor's Theorem, $\|p'\|_\infty \lesssim 1$ and $\|\mu\|_2 \lesssim 1$, one can arrive at
\[
\left| E_1 - p(-\alpha_1 - \beta_1^\top \mu) \left( (\alpha_1 - \alpha_0 + (\beta_1 - \beta_0)^\top \mu) \right) \right| \lesssim \|\gamma_0 - \gamma_1\|_2^2,
\]
\[
\left| E_2 - p(\alpha_1 - \beta_1^\top \mu) \left( (\alpha_0 - \alpha_1 + (\beta_1 - \beta_0)^\top \mu) \right) \right| \lesssim \|\gamma_0 - \gamma_1\|_2^2.
\]

From $\alpha_1 = 0$, $\beta_1 = \mu/\|\mu\|_2$ and $\|p\|_\infty \lesssim 1$ we obtain that
\[
\mathcal{R}(\hat{\gamma}) - \mathcal{R}(\gamma^{Bayes}) \lesssim \|p(-\beta_1^\top \mu)[-\alpha_0 + (\beta_1 - \beta_0)^\top \mu] + p(-\beta_1^\top \mu)[\alpha_0 + (\beta_1 - \beta_0)^\top \mu]\| + \|\gamma_0 - \gamma_1\|_2^2
\leq \left| (\beta_1 - \beta_0)^\top \beta_1 \right| + \|\gamma_0 - \gamma_1\|_2^2.
\]

Since $\beta_0$ and $\beta_1$ are unit vectors,
\[
\|\beta_1 - \beta_0\|_2^2 = \|\beta_0\|_2^2 - 2\beta_0^\top \beta_1 + \|\beta_1\|_2^2 = 2(1 - \beta_0^\top \beta_1) = 2(\beta_1 - \beta_0)^\top \beta_1,
\]
\[
\mathcal{R}(\hat{\gamma}) - \mathcal{R}(\gamma^{Bayes}) \lesssim \|\beta_1 - \beta_0\|_2^2 + \|\gamma_0 - \gamma_1\|_2^2 \lesssim \|\gamma_0 - \gamma_1\|_2^2.
\]

Note that $\|\hat{\beta} - \beta^*\|_2 \leq \|\hat{\gamma} - \gamma^*\|_2 \lesssim \sqrt{d/n\log(n/d)}$ and $\|\beta^*\|_2 \simeq 1$. When $n/d$ is sufficiently large, we have $\|\hat{\beta}\|_2 \simeq 1$ and
\[
\|\beta_1 - \beta_0\|_2 \leq \|\hat{\beta}/\|\hat{\beta}\|_2 - \beta^*/\|\beta^*\|_2\|_2 \leq \|\beta^*\|_2 \|\hat{\beta} - \beta^*\|_2 \leq \|\beta - \beta^*\|_2.
\]

In addition, we also have $\|\alpha_0 - \alpha_1\| = \|\alpha_0\| = \|\alpha\|/\|\beta\|_2 \lesssim |\alpha| = |\alpha - \alpha^*|$. As a result, $\|\gamma_0 - \gamma_1\|_2 \leq |\alpha - \alpha^*| + \|\beta_1 - \beta_0\|_2 \lesssim \|\gamma - \gamma^*\|_2$. Plugging these bounds into (50), we get
\[
\mathcal{R}(\hat{\gamma}) - \mathcal{R}(\gamma^*) \lesssim \|\hat{\gamma} - \gamma^*\|_2 \lesssim \frac{d}{n} \log \left( \frac{n}{d} \right).
\]

### J Technical lemmas

**Lemma 7.** Let $X$ be a random vector in $\mathbb{R}^{d+1}$ with $\mathbb{E}\|X\|_2^3 < \infty$. Then
\[
\sup_{u, v \in \mathbb{S}^d} \mathbb{E}(u^\top X | v^\top X) = \sup_{u \in \mathbb{S}^d} \mathbb{E}u^\top X^3.
\]

**Proof.** It is easily seen that $\sup_{u, v \in \mathbb{S}^d} \mathbb{E}(u^\top X | v^\top X) \geq \sup_{u \in \mathbb{S}^d} \mathbb{E}u^\top X^3$. To prove the other direction, we first use Cauchy-Schwarz inequality to get
\[
\mathbb{E}(u^\top X^2 | v^\top X) = \mathbb{E}(u^\top X)^3/2(\mathbb{E}(u^\top X^2 | v^\top X)) \leq \mathbb{E}^{1/2}(u^\top X)^3 \cdot \mathbb{E}^{1/2}(u^\top X) \cdot (v^\top X^2).
\]

By taking suprema we prove the claim. \qed

**Lemma 8.** Let $X$ be a random vector in $\mathbb{R}^{d+1}$ and $f \in C^2(\mathbb{R})$. Suppose that $\mathbb{E}\|X\|_2^3 < \infty$, $\sup_{x \in \mathbb{R}} |f''(x)| = F_2 < \infty$ and $f''$ is $F_3$-Lipschitz. Define $\tilde{\mu} = \mathbb{E}X$. Then
\[
L_\lambda(\gamma) = \mathbb{E}f(\gamma^\top X) = \lambda(\gamma^\top \tilde{\mu})^2/2
\]
exists for all $\gamma \in \mathbb{R}^{d+1}$ and $\lambda \geq 0$, and
\[
\sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla L_\lambda(\gamma_1) - \nabla L_\lambda(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} \leq F_2 \sup_{u \in \mathbb{S}^d} \mathbb{E}u^\top X^2 + \lambda\|\tilde{\mu}\|_2^2,
\]
\[
\sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla^2 L_\lambda(\gamma_1) - \nabla^2 L_\lambda(\gamma_2)\|}{\|\gamma_1 - \gamma_2\|_2} \leq F_3 \sup_{u \in \mathbb{S}^d} \mathbb{E}(u^\top X)^3, \quad \forall u \in \mathbb{S}^{-1},
\]
\[
\sup_{\gamma_1 \neq \gamma_2} \frac{\|\nabla^2 L_\lambda(\gamma_1) - \nabla^2 L_\lambda(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} \leq F_3 \sup_{u \in \mathbb{S}^d} \mathbb{E}u^\top X^3.
\]
In addition, if there exist nonnegative numbers \(a, b\) and \(c\) such that \(\inf_{x \in \mathbb{R}} x f'(x) \geq -b\) and \(\inf_{|x| \geq a} f'(x) \operatorname{sgn}(x) \geq c\), then

\[
\|\nabla L_\gamma\|_2 \geq c \inf_{u \in \mathbb{R}^d} E[u^T X] - \frac{ac + b}{\|\gamma\|_2}, \quad \forall \gamma \neq 0.
\]

Proof. Let \(L(\gamma) = E f(\gamma^T X)\) and \(R(\gamma) = (\gamma^T \tilde{\mu})^2/2\). Since \(L_\gamma = L + \lambda R,\) \(\nabla^2 L(\gamma) = E[X X^T f''(\gamma^T X)]\) and \(\nabla^2 R(\gamma) = \mu \mu^T,\)

\[
\sup_{\gamma \neq \gamma_2} \frac{\|\nabla L_\gamma(\gamma_1) - \nabla L_\gamma(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} \leq F_2 \sup_{u \in \mathbb{R}^d} E(u^T X)^2 + \lambda \|\mu\|_2^2.
\]

For any \(u \in \mathbb{R}^d,\)

\[
|u^T [\nabla^2 L_\gamma(\gamma_1) - \nabla^2 L_\gamma(\gamma_2)] u| = |E[u^T X] f''(\gamma_1^T X) - E[(u^T X)^2 f''(\gamma_2^T X)]| \leq E[(u^T X)^2 | f''(\gamma_1^T X) - f''(\gamma_2^T X)]| \leq F_3 E[(u^T X)^2 | (\gamma_1 - \gamma_2)^T X] | \leq F_4 |\gamma_1 - \gamma_2| \sup_{u \in \mathbb{R}^d} E|(u^T X)^2 | v^T X| |
\]

As a result,

\[
\sup_{\gamma \neq \gamma_2} \frac{\|\nabla^2 L_\gamma(\gamma_1) - \nabla^2 L_\gamma(\gamma_2)\|_2}{\|\gamma_1 - \gamma_2\|_2} = \sup_{u \in \mathbb{R}^d} \frac{|u^T \nabla^2 L_\gamma(\gamma_1) - \nabla^2 L_\gamma(\gamma_2)] u|}{\|\gamma_1 - \gamma_2\|_2} \leq \frac{F_3}{F_2} \sup_{u \in \mathbb{R}^d} E|(u^T X)^2 | v^T X| |
\]

where the last equality follows from Lemma 9.

We finally come to the lower bound on \(\|\nabla L_\gamma\|_2\). Note that \(\|\nabla L_\gamma\|_2 \geq \langle \gamma, \nabla L_\gamma \rangle,\) \(\nabla L_\gamma = E[X f'(X^T \gamma)]\) and \(\nabla R(\gamma) = (\gamma^T \tilde{\mu})^2.\) The condition \(\inf_{|x| \geq a} f'(x) \operatorname{sgn}(x) \geq c\) implies that \(f'(x) \geq c|x|\) when \(|x| \geq a.\) By this and \(\inf_{x \in \mathbb{R}} x f'(x) \geq -b,\)

\[
\langle \gamma, \nabla L_\gamma \rangle = E[X f'(X^T \gamma)] = E[X^T \gamma f'(X^T \gamma)] = E[X^T \gamma f'(X^T \gamma) 1_{|X^T \gamma| \geq a} + E[X^T \gamma f'(X^T \gamma) 1_{|X^T \gamma| < a}] \geq c E[X^T \gamma 1_{|X^T \gamma| \geq a}] - b \geq c E[X^T \gamma 1_{|X^T \gamma| < a}] - b \geq c E[X^T \gamma] - (ac + b) \geq \|\gamma\|_2^c \inf_{u \in \mathbb{R}^d} E[u^T X] - (ac + b).
\]

In addition, we also have \(\langle \gamma, \nabla R(\gamma) \rangle = (\gamma^T \tilde{\mu})^2 \geq 0.\) Then the lower bound directly follows.

\begin{lemma}
There exists a continuous function \(\varphi : (0, +\infty)^2 \to (0, +\infty)\) that is non-increasing in the first argument and non-decreasing in the second argument, such that for any nonzero sub-Gaussian random variable \(X, E|X| \geq \varphi(\|X\|_{\psi_2}, E X^2)\).
\end{lemma}

Proof. For any \(t > 0,\)

\[
E|X| \geq E(|X| 1_{|X| \leq t}) \leq t^{-1}E(X^2 1_{|X| \leq t}) = t^{-1}[E X^2 - E(X^2 1_{|X| > t})].
\]

By Cauchy-Schwarz inequality and the sub-Gaussian property (\textit{Vershynin} 2010), there exist constants \(C_1, C_2 > 0\) such that

\[
E(X^2 1_{|X| > t}) \leq E^{1/2} X^4 \cdot P^{1/2}(|X| > t) \leq C_1 \|X\|_{\psi_2}^2 e^{-C_2 t^2/\|X\|_{\psi_2}^2}.
\]

By taking \(\varphi(\|X\|_{\psi_2}, E X^2) = \sup_{t > 0} t^{-1}(EX^2 - C_1 \|X\|_{\psi_2}^2 e^{-C_2 t^2/\|X\|_{\psi_2}^2})\) we finish the proof, as the required monotonicity is obvious.
Lemma 10. Let \( \{X_{ni}\}_{n \geq 1, i \in [n]} \) be an array of random variables where for any \( n \), \( \{X_{ni}\}_{i=1}^n \) are i.i.d. sub-Gaussian random variables with \( \|X_{ni}\|_{\psi_2} \leq 1 \). Fix some constant \( a \geq 2 \), define \( S_n = \frac{1}{n} \sum_{i=1}^n |X_{ni}|^a \) and let \( \{r_n\}_{n=1}^{\infty} \) be a deterministic sequence satisfying \( \log n \leq r_n \leq n \). We have

\[
S_n - \mathbb{E}|X_{ni}|^a = O_P(r_n^{(a-1)/2}/\sqrt{n}; r_n),
\]

\[
S_n = O_P(\max(1, r_n^{(a-1)/2}/\sqrt{n}); r_n).
\]

Proof. Define \( R_{nt} = t\sqrt{r_n} \) and \( S_{nt} = \frac{1}{n} \sum_{i=1}^n |X_{ni}|^a 1_{\{|X_{ni}| \leq R_{nt}\}} \) for \( n, t \geq 1 \). For any \( p \geq 1 \), we have

\[
2p \geq 2 > 1 \quad \text{and} \quad (2p)^{-1/2} \mathbb{E}[E(2p)|X_{ni}|^2] \leq \|X_{ni}\|_{\psi_2} \leq 1.
\]

Hence

\[
\mathbb{E}(|X_{ni}|^a 1_{\{|X_{ni}| \leq R_{nt}\}})^p = \mathbb{E}(|X_{ni}|^{ap} 1_{\{|X_{ni}| \leq R_{nt}\}}) = \mathbb{E}(|X_{ni}|^{2p} |X_{ni}|^{a-2}p 1_{\{|X_{ni}| \leq R_{nt}\}})
\]

\[
\leq \mathbb{E}|X_{ni}|^{2p} R_{nt}^{(a-2)p} \leq (2p)^{1/2} \|X_{ni}\|_{\psi_2}^{2p} R_{nt}^{(a-2)p} \leq (2p R_{nt}^{a-2})^p
\]

and \( \|X_{ni}|^a 1_{\{|X_{ni}| \leq R_{nt}\}}\|_{\psi_1} \leq 2R_{nt}^{a-2} \). By the Bernstein-type inequality in Proposition 5.16 of Vershynin (2010), there exists a constant \( c \) such that

\[
\mathbb{P}(|S_{nt} - \mathbb{E}S_{nt}| \geq s) \leq 2 \exp \left[ -c n \left( \frac{s^2}{R_{nt}^{2(a-2)}} \wedge \frac{s^8}{R_{nt}^{a-2}} \right) \right], \quad \forall t \geq 0, s \geq 0.
\]

(51)

Take \( t \geq 1 \) and \( s = r_n^{(a-1)/2}/\sqrt{n} \). We have

\[
\frac{s}{R_{nt}^{a-2}} = \frac{r_n^{(a-1)/2}/\sqrt{n}}{r_n^{a-2}/\sqrt{n}} = t\sqrt{r_n}/n,
\]

\[
\frac{s^2}{R_{nt}^{2(a-2)}} = \frac{r_n^{(a-2)/2}/\sqrt{n}}{r_n^{a-2}/\sqrt{n}} = \frac{t^2 r_n}{n} \wedge \frac{t \sqrt{r_n}}{n} \geq \frac{t r_n}{n},
\]

where the last inequality is due to \( r_n/n \leq 1 \leq t \). By (51),

\[
\mathbb{P}(|S_{nt} - \mathbb{E}S_{nt}| \geq t^{a-1} r_n^{(a-1)/2}/\sqrt{n}) \leq 2e^{-ct r_n}, \quad \forall t \geq 1.
\]

(52)

By Cauchy-Schwarz inequality and \( \|X_{ni}\|_{\psi_2} \leq 1 \), there exist \( C_1, C_2 > 0 \) such that

\[
0 \leq \mathbb{E}S_n - \mathbb{E}S_{nt} = \mathbb{E}(|X_{ni}|^a 1_{\{|X_{ni}| > t\sqrt{r_n}\}}) \leq \mathbb{E}^{1/2} |X_{ni}|^{2a} \mathbb{P}^{1/2}(|X_{ni}| > t\sqrt{r_n}) \leq C_1 e^{-C_2 t^2 r_n}
\]

holds for all \( t \geq 0 \). Since \( r_n \geq \log n \), there exists a constant \( C > 0 \) such that \( C_1 e^{-C_2 t^2 r_n} \leq t^{a-1} r_n^{(a-1)/2}/\sqrt{n} \) as long as \( t \geq C \). Hence (52) forces

\[
\mathbb{P}(|S_{nt} - \mathbb{E}S_{nt}| \geq t^{a-1} r_n^{(a-1)/2}/\sqrt{n}) \leq \mathbb{P}(|S_{nt} - \mathbb{E}S_{nt}| + |\mathbb{E}S_{nt} - \mathbb{E}S_n| \geq 2 t^{a-1} r_n^{(a-1)/2}/\sqrt{n})
\]

\[
\leq \mathbb{P}(|S_{nt} - \mathbb{E}S_{nt}| \geq t^{a-1} r_n^{(a-1)/2}/\sqrt{n}) \leq 2e^{-ct r_n}, \quad \forall t \geq C.
\]

Note that

\[
\mathbb{P}(|S_n - \mathbb{E}S_n| \geq 2 t^{a-1} r_n^{(a-1)/2}/\sqrt{n}) \leq \mathbb{P}(|S_n - \mathbb{E}S_n| \geq 2 t^{a-1} r_n^{(a-1)/2}/\sqrt{n}, S_n = S_{nt}) + \mathbb{P}(S_n \neq S_{nt})
\]

\[
\leq \mathbb{P}(|S_{nt} - \mathbb{E}S_{nt}| \geq 2 t^{a-1} r_n^{(a-1)/2}/\sqrt{n}) + \mathbb{P}(S_n \neq S_{nt})
\]

\[
\leq 2e^{-ct r_n} + \mathbb{P}(\max_{i \in [n]} |X_{ni}| > t\sqrt{r_n}), \quad \forall t \geq C.
\]

(54)

Since \( \|X_{ni}\|_{\psi_2} \leq 1 \), there exist constants \( C_1', C_2' > 0 \) such that

\[
\mathbb{P}(|X_{ni}| \geq t) \leq C_1' e^{-C_2' t^2}, \quad \forall n \geq 1, i \in [n], t \geq 0.
\]
By union bounds,
\[
P \left( \max_{i \in [n]} |X_{n,i}| > t\sqrt{n} \right) \leq nC_1' e^{-C_2't^2r_n} = C_1 e^{\log n - C_3't^2r_n}, \quad \forall t \geq 0.
\]
When \( t \geq \sqrt{2/C_2'} \), we have \( C_2't^2r_n \geq 2r_n \geq 2\log n \) and thus \( \log n - C_2't^2r_n \leq -C_3't^2r_n/2 \). Then (54) leads to
\[
P(\|S_n - ES_n\| \geq 2e^{a-1}r_n(1-1/\sqrt{n}) \leq e^{-c_2'r_n} + C_1'e^{-C_2't^2r_n}/2, \quad \forall t \geq C \sqrt{2/C_2'}.
\]
This shows \( S_n - ES_n \leq O_p(r_n^{a-1}/\sqrt{n}r_n) \). The proof is finished by \( E|X_{n,i}|^a \leq 1 \). \( \square \)

**Lemma 11.** Suppose that \( \{X_i\}_{i=1}^n \subseteq \mathbb{R}^{d+1} \) are independent random vectors, \( \max_{i \in [n]} \|X_i\|_{\psi_2} \leq 1 \) and \( n \geq md \geq \log n \) for some \( m \geq 1 \). We have
\[
\sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n \|u^\top X_i\|^2 = O_P(1; n), \quad \forall u \in S^d.
\]
\[
\sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n \|v^\top X_i\|^2 |u^\top X_i| = O_P(1; n^{1/3}), \quad \forall v \in S^d.
\]
\[
\sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n |u^\top X_i|^3 = O_P(\max\{1, md/\sqrt{n}; md\}).
\]

**Proof.** From \( 2^{-1/2}E^{1/2}(u^\top X)^2 \leq \|u^\top X\|_{\psi_2} \leq 1, \forall u \in S^d \) we get \( E(XX^\top) \preceq 2I \). Since \( n \geq d + 1 \), Remark 5.40 in [Vershynin 2010] asserts that
\[
\sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n \|u^\top X_i\|^2 = \left\| \frac{1}{n} \sum_{i=1}^n X_iX_i^\top - E(XX^\top) \right\|_2 + \|E(XX^\top)\|_2 = O_P(1; n).
\]
For any \( u, v \in S^d \), the Cauchy-Schwarz inequality forces
\[
\frac{1}{n} \sum_{i=1}^n (v^\top X_i)^2 |u^\top X_i| \leq \left( \frac{1}{n} \sum_{i=1}^n (v^\top X_i)^4 \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^n (u^\top X_i)^2 \right)^{1/2},
\]
\[
\sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n (v^\top X_i)^2 |u^\top X_i| \leq \left( \frac{1}{n} \sum_{i=1}^n (v^\top X_i)^4 \right)^{1/2} O_P(1; n).
\]
Since \( \{v^\top X_i\}_{i=1}^n \) are i.i.d. sub-Gaussian random variables and \( \|v^\top X_i\|_{\psi_2} \leq 1 \), Lemma 10 with \( a = 4 \) and \( r_n = n^{1/3} \) yields \( \frac{1}{n} \sum_{i=1}^n (v^\top X_i)^4 = O_P(1; n^{1/3}) \). Hence \( \sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n (v^\top X_i)^2 |u^\top X_i| = O_P(1; n^{1/3}) \).

To prove the last equation in Lemma 11 define \( Z_i = X_i - EX_i \). From \( \|Z_i\|_{\psi_2} = \|X_i - EX_i\|_{\psi_2} \leq 2\|X_i\|_{\psi_2} \leq 2 \) we get \( \sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n |u^\top Z_i|^2 = O_P(1; n) \). For \( u \in S^d \),
\[
|u^\top X_i|^3 = |u^\top Z_i|^3 + (|u^\top X_i| - |u^\top Z_i|) (|u^\top X_i|^2 + |u^\top Z_i|^2) \leq |u^\top Z_i|^3 + |u^\top (X_i - Z_i)| (|u^\top X_i|^2 + |u^\top Z_i|^2) \leq |u^\top Z_i|^3 + |u^\top EX_i| \leq 3\|u^\top X_i\|^2 + \|u^\top Z_i\|^2 \leq |u^\top Z_i|^3 + 3\|u^\top X_i\|^2 + \|u^\top Z_i\|^2,
\]
where the last inequality is due to \( |u^\top EX_i| \leq \|EX_i\|_2 \leq \|X_i\|_{\psi_2} \leq 1 \). Hence
\[
\sup_{u \in S^d} \frac{1}{n} \sum_{i=1}^n |u^\top X_i|^3 \leq \|u^\top EX_i\|_2 \leq \sup_{u \in S^d} |u^\top Z_i|^2 + O_P(1; n).
\]

Define \( S(u) = \frac{1}{n} \sum_{i=1}^n |u^\top Z_i|^3 \) for \( u \in S^d \). We will invoke Theorem 1 in [Wang 2019] to control \( \sup_{u \in S^d} S(u) \).
1. For any \( u \in \mathbb{S}^d \), \( \{ u^\top Z_i \}_{i=1}^n \) are i.i.d. and \( \| u^\top Z_i \|_2 \leq 1 \). Lemma 10 with \( a = 3 \) and \( r_n = md \) yields

\[
\{ S(u) \}_{u \in \mathbb{S}^d} = O_{\mathbb{P}}(\max\{1, md/\sqrt{n}; md\}).
\]

2. According to Lemma 5.2 in Vershynin (2010), for \( \varepsilon = 1/6 \) there exists an \( \varepsilon \)-net \( \mathcal{N} \) of \( \mathbb{S}^d \) with cardinality at most \( (1 + 2/\varepsilon)^d = 13^d \). Hence \( \log |\mathcal{N}| \lesssim md \).

3. For any \( x, y \in \mathbb{R} \), we have \( |x| - |y| \leq |x - y|, 2|xy| \leq x^2 + y^2 \) and

\[
| |x|^3 - |y|^3| \leq ||x| - |y|| (x^2 + |xy| + y^2) \leq \frac{3}{2} |x - y|(x^2 + y^2).
\]

Hence for any \( u, v \in \mathbb{S}^d \),

\[
|S(u) - S(v)| \leq \frac{1}{n} \sum_{i=1}^{n} |u_i^\top Z_i|^3 - |v_i^\top Z_i|^3| \leq \frac{3}{2} \frac{1}{n} \sum_{i=1}^{n} |(u - v)^\top Z_i||u_i^\top Z_i|^2 + |v_i^\top Z_i|^2
\]

\[
\leq 3\|u - v\|_2 \sup_{w_1, w_2 \in \mathbb{S}^d} \frac{1}{n} \sum_{i=1}^{n} |w_1_i^\top Z_i| \cdot |w_2_i^\top Z_i|^2 = \frac{1}{2\varepsilon} \|u - v\|_2 \sup_{w \in \mathbb{S}^d} S(w).
\]

where the last inequality follows from \( \varepsilon = 1/6 \) and Lemma 7.

Theorem 1 in Wang (2019) then asserts that \( \sup_{u \in \mathbb{S}^d} S(u) = O_{\mathbb{P}}(\max\{1, md/\sqrt{n}; md\}) \). We finish the proof using (55). \( \square \)

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