Abstract—This paper proposes a novel mathematical theorem that embeds the stability and accuracy assessment into a large-signal order reduction (LSOR) method of microgrids. Using the proposed method, the dynamic stability of full microgrid models can be assessed by only leveraging their derived reduced-order models and boundary layer models. In particular, when the reduced-order system is input-to-state stable and the boundary layer system is uniformly globally asymptotically stable, the original microgrids system is stable based on several common growth conditions. In addition, we develop the conditions to guarantee the accuracy of the reduced model. We show that the error between the solutions of reduced and original models is bounded and convergent under such conditions. Further, we provide the strict mathematical proof to illustrate that the proposed order reduction method is generic and can be applied to arbitrary dynamic systems. Simulation validation is conducted on microgrid systems to show the effectiveness of proposed method.

Index Terms—Microgrids, order reduction, singular perturbation, stability and accuracy assessment

I. INTRODUCTION

MICROGRIDS are localized small-scale power systems composed of interconnected loads and distributed energy resources (DERs) in low-voltage and medium-voltage distribution networks that can be operated in both grid-connected and islanded modes [1], [2]. The high penetration of low-inertia DERs makes the dynamic response of microgrids different from conventional networks dominated by synchronous machines. This low-inertia characteristic highlights the importance of dynamic modeling and stability studies of microgrids [3]–[7]. However, due to the high-order nature of microgrids caused by the large number of DERs and dynamic loads, it is intractable to analyze the total dynamic stability of microgrids [8]–[11]. Furthermore, the two-time-scale behavior of microgrids due to the variations of transient velocities of different state variables leads to a stiff differential equation problem [12], [13]. Solving this stiff problem requires extremely small time steps, which results in an unmanageable computational complexity [14], [15].

An effective approach to overcome the above challenges is to perform model order reduction. Considering the two-time-scale property of microgrids, the singular perturbation method (SPM) is suitable for order reduction. Different from conventional model reduction methods that simply neglect some state variables, SPM integrates fast dynamics into slow dynamics to remain the characteristics of fast ones [16]. The so-called “slow” and “fast” refer to the transient velocity of states. In addition, SPM converts the original stiff problem into a non-stiff problem, thus improving the computational efficiency. Consequently, SPM can develop a reduced-order model (ROM) that is represented by slow states and a boundary layer model (BLM) whose state variables are the errors between fast states and quasi-steady states. Hence, SPM is commonly used for order-reduction of microgrids.

The existing literatures on ROM of microgrids can be categorized into two classes. The first class focuses on developing the reduced model of microgrids, whereas the stability assessment is not included. [17] proposed a spatiotemporal model reduction method of microgrids using SPM and Kron reduction. In [18], a linear SPM was applied to small-signal models of microgrids. However, since it used the small-signal model, the result only holds in the neighborhood of a stable equilibrium point. The second class focuses on the stability assessment of microgrids. [19] simplified the stability assessment and applied it to an islanded microgrid with droop control by using inverter angles. Nevertheless, [20], [21] demonstrated that such simplification process could affect the accuracy of reduced models. [22] proposed a stability assessment criterion that used the input-to-state stability (ISS) of the ROM and global asymptotic stability (GAS) of the BLM to analyze the total stability of the original system. However, these works fall short of mathematical proof of error convergence between reduced and original models, which hinders the accuracy evaluation of reduced models.

To address the above shortcomings, we propose a novel theorem for assessing the total dynamic stability of microgrids on infinite time interval by only using the ROM and BLM in this paper. A key point is that we take the impact of external inputs into account. In particular, assuming the reduced system to be ISS and the BLM to be uniformly GAS, then the original system is totally ISS. Further, we develop the conditions that guarantee the accuracy of reduced models for both slow and fast dynamics. In addition, we provide a strict mathematical proof and illustrate that the proposed order reduction technique is generic for arbitrary dynamic systems. Utilizing our stability and accuracy assessment theorem, an improved large-signal order-reduction (LSOR) algorithm is proposed for inverter-dominated microgrids as demonstrated in Fig. 1. In the case studies, we show that the stability of original system can be properly assessed by analyzing its ROM and BLM. Numerical results show that our method gives an accurate ROM with much less computational time compared to the original model.

The main contributions can be summarized as follows:

- To our best knowledge, the existing studies on microgrid stability analysis do not consider the impact of external inputs such as power commands and voltage frequency references on the ISS. A typical way is considering the unforced system (neglecting the input). However, even though the unforced system is stable, a continuous input signal can make the system unstable. Thus, our theorem...
takes into account the ISS.

- We provide the conditions under which, the error between reduced and original models is bounded and converged as the perturbation coefficients decrease. These results are strictly proved. Using this theorem, we can evaluate the accuracy of reduced models.

- Our theorem improves the accuracy of the LSOR method by designing a feedback mechanism. When the derived ROM is identified to be inaccurate, the bounds of perturbation coefficients are calculated as an index to re-select slow/fast states to improve the accuracy of LSOR.

- The stability assessment in this paper is based on large-signal (nonlinear) models of microgrids without any linearization or simplification.

- The proposed theorem can be generalized to arbitrary dynamic systems in addition to microgrids.

II. IMPROVED LSOR BY EMBEDDING STABILITY AND ACCURACY ASSESSMENT THEOREM

The small-signal order reduction methods have been studied for a long time [18], [23]. However, due to the limitation of linearization, these methods are only valid locally around equilibrium points. This section will derive a globally effective ROM. Firstly, we first present the SPM-based LSOR approach. Then section II-B presents our main work as a stability and accuracy assessment theorem. Finally, we improve the LSOR algorithm by embedding the stability and accuracy assessment theorem, so that it can guarantee the accuracy of derived ROM and efficiently evaluate the stability of original models.

A. LSOR Approach using the SPM for microgrids

Due to the two-time-scale property, the dynamics of microgrids can be classified as slow and fast dynamics according to the transient velocity. The main idea of SPM is to freeze the fast dynamics and degenerate them to static equations. Thus, the ROM can be obtained by substituting the solutions of the static equations into the slow dynamic equations.

In real physical systems, one challenge of SPM is to identify the slow and fast dynamic states. A commonly-used approach is the knowledge discover-based method that relies on expert knowledge for specific domains. For example, in microgrids, some small parasitic parameters such as capacitances, inductances, and small time constants, can be selected as the perturbation coefficients $\varepsilon$. The states with respect to these small $\varepsilon$ are identified as fast states. This conventional empirical identification method fall short of efficiency and accuracy. Therefore, we propose a more efficient and accurate theorem to identify the slow/fast dynamics by finding the bound of $\varepsilon$. The detailed description is presented in the next subsection. Once we identify the slow and fast dynamics, we can rewrite a microgrid system in the following general singular perturbed form,

$$\dot{x}(t) = f(x(t), z(t), u(t), \varepsilon),$$  \hspace{1cm} (1a)

$$\varepsilon \dot{z}(t) = g(x(t), z(t), u(t), \varepsilon),$$  \hspace{1cm} (1b)

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ represent the state variables of microgrids such as voltages and currents. $u \in \mathbb{R}^p$ denotes the continuous microgrid inputs such as power commands, or voltage frequency commands. $\varepsilon \in [0, \varepsilon_0]$ are perturbation coefficients representing the small parameters in microgrids such as capacitances and inductances. $f$ and $g$ are locally Lipschitz functions on their arguments. For simplicity, we neglect the notation of time-dependency ($t$) in the rest of this paper.

Since $\varepsilon$ is small, the fast transient velocity $\dot{z} = g/\varepsilon$ can be much larger than the slow dynamics $\dot{x}$. To solve this two-time-scale problem, we can set $\varepsilon = 0$, then equation (1b) degenerates to the following algebraic equation,

$$0 = g(x, z, u, 0).$$  \hspace{1cm} (2)

If equation (2) has at least one isolated real root and satisfies the implicit function theory, then for each argument, we have the following closed-form solution,

$$z = h(x, u).$$  \hspace{1cm} (3)

Substitute equation (3) into equation (1a) and let $\varepsilon = 0$, we have a quasi-steady-state (QSS) model,

$$\dot{x} = f(x, h(x, u), u, 0).$$  \hspace{1cm} (4)

Note that the order of the QSS system (4) drops from $n+m$ to $n$. The inherent two-time-scale property can be described by introducing the BLM. Define a fast time scale variable $\tau = t/\varepsilon$, and a new coordinate $y = z - h(x, u)$. In this new coordinate, equation (1b) is rewritten as

$$\frac{dy}{d\tau} = g(x, y + h(x, u), u, \varepsilon) - \varepsilon \left[ \frac{\partial h}{\partial x} f(x, y + h(x, u), u, \varepsilon) + \frac{\partial h}{\partial u} \dot{u} \right].$$  \hspace{1cm} (5)

Let $\varepsilon = 0$, we obtain the BLM as follows,

$$\frac{dy}{d\tau} = g(x, y + h(x, u), u, 0).$$  \hspace{1cm} (6)

B. Stability and Accuracy Assessment Theorem

In this subsection, we propose a criterion to assess the stability of the original system and the accuracy of ROM and BLM. Consider the impact of external inputs on the stability of microgrids, we define the ISS as follows.

**Definition (ISS).** Consider such a nonlinear system

$$\dot{x} = f(x, v_1, v_2)$$  \hspace{1cm} (7)

where $x \in \mathbb{R}^n$ is the state vector, $v_1 \in \mathbb{R}^m$, $v_2 \in \mathbb{R}^p$ are input vectors, and $f$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$. The system (7) is ISS with Lyapunov gains $\alpha_{v_1}$ and $\alpha_{v_2}$ of class kappa (K), if there exists a class kappa-ell (KL) function $\beta$ such that...
for $x(0) \in \mathbb{R}^n$ and bounded inputs $v_1, v_2$, the solution of (7) exists and satisfies
\[ \|x(t)\| \leq \beta\left(\|x(0)\|, t + \alpha_{v_1}(\|v_1\|) + \alpha_{v_2}(\|v_2\|)\right). \] (8)

The above definition indicates that a microgrid system is ISS when all the trajectories are bounded by some functions of the input magnitudes. Two lemmas are introduced as follows according to [22] in order to prove the theorem.

**Lemma 1.** If (7) is ISS with Lyapunov gain $\alpha_{v_1}$ and $v_2(t) \equiv 0$, there exists a class $K_\infty$ function $\alpha_{v_2}$ and a function $p \times p$ non-singular matrix $M(x, v_1)$ of smooth functions for its arguments that is identity in a neighborhood of the origin, such that
\[ \dot{x} = f(x, v_1, M(x, v_1)v_2) \] (9)
is ISS with Lyapunov gains $\alpha_{v_1}$ and $\alpha_{v_2}$. Moreover, if $\alpha_{v_1}$ is of class $K_\infty$, then the matrix $M$ can be independent of $v_1$.

**Lemma 2.** If (7) is ISS with Lyapunov gain $\alpha_{v_1}$ and $v_2(t) \equiv 0$, there exist a class $K$ function $\alpha_{v_2}$, a class $KLD$ function $\beta$ and a continuous nonincreasing function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that for all $x(0) \in \mathbb{R}^n$ and bounded inputs $v_1, v_2$ that satisfies $\|v_2\| \leq \gamma(\max\{\|x(0)\|, \|v_1\|\})$, the solution of (7) exists and satisfies (8).

The proofs of these two lemmas are similar as in [22]. Then we give the following three assumptions which are the sufficient conditions for the theorem.

**Assumption 1.** (Growth conditions) The functions $f, g,$ and their first partial derivatives are continuous and bounded with respect to $(x, z, u, \varepsilon)$; $h$ and its first partial derivatives $\partial h/\partial x, \partial h/\partial u$ are locally Lipschitz; and the Jacobian $\partial g/\partial z$ has bounded first partial derivatives with respect to its arguments.

**Assumption 2.** (Stability of ROM) The ROM (4) is ISS with $\alpha_{x}$, and its unforced system has a globally exponentially stable equilibrium at the origin.

**Assumption 3.** (Stability of BLM) The origin of the BLM (6) is a GAS equilibrium, uniformly in $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$.

The conditions in Assumption 1 are commonly satisfied for most microgrids [20]. Assumptions 2 and 3 are the stability conditions on the ROM and BLM. We propose the stability and accuracy assessment of microgrids as the following theorem.

**Theorem.** If the microgrids system (1), its ROM (4) and the BLM (6) satisfy the Assumptions 1-3, then there exist positive constants $\varepsilon^*$, $\mu$, such that for all $t \in [0, \infty)$, $\max\{\|x(0)\|, \|y(0)\|, \|u\|, \|\hat{u}\|\} \leq \mu$, and $\varepsilon \in (0, \varepsilon^*)$ the errors between the solutions of the original microgrids system (1) and its ROM (4) and BLM (6) satisfy
\[ \|x(t, \varepsilon) - \hat{x}(t)\| = O(\varepsilon), \] (10)
\[ \|z(t, \varepsilon) - h(\hat{x}(t), u(t)) - \hat{y}(t/\varepsilon)\| = O(\varepsilon), \] (11)
where $\hat{x}(t)$ and $\hat{y}(\tau)$ are the solutions of ROM (4) and BLM (6), respectively. $\|x - \hat{x}\| = O(\varepsilon)$ means that $\|x - \hat{x}\| \leq \kappa \|\varepsilon\|$ for some positive constant $\kappa$. Furthermore, for any given $T > 0$, there exists a positive constant $\varepsilon^* \leq \varepsilon^*$ such that for $t \in [T, \infty)$ and $\varepsilon < \varepsilon^*$, it follows uniformly that
\[ \|x(t, \varepsilon) - h(\hat{x}(t), u(t))\| = O(\varepsilon). \] (12)

Moreover, there exist class $KL$ functions $\beta_x, \beta_y$, a Lyapunov gain $\alpha_x$ of class $K$ and positive constants $\xi$, such that the solutions of the original microgrids system (12) and (13) exist and satisfy
\[ \|x(t, \varepsilon)\| \leq \beta_x(\|x(0)\|, t + \alpha_x(\|u\|) + \xi, \] (13)
\[ \|y(t, \varepsilon)\| \leq \beta_y(\|y(0)\|, t/\varepsilon) + \xi. \] (14)

**Remark 1.** The errors between the solutions of reduced and original microgrids should be small and bounded to guarantee the accuracy. (10) and (17) show that for sufficiently small $\varepsilon$, these errors tend to be zero. Equation (12) means that for small enough $\varepsilon$, the solution $\hat{y}$ of the BLM decays to zero exponentially fast in time $T$, so that the fast solutions can be estimated by only QSS solutions $h(t, \hat{x}(t))$ after time $T$.

**Remark 2.** According to the theorem, if the ROM is ISS and BLM is GAS, then the original system is stable as shown in [13] and [14]. Moreover, note that these assumptions are the sufficient conditions that $\varepsilon^*$ exists. The detailed method to determine $\varepsilon^*$ and $\varepsilon^{**}$ is provided in the following proof.

**Proof:** The proof of the theorem is conducted in three steps. First, we prove the GAS of $y$ (14). This result will then be used in proving the accuracy of reduced model (10) (12). Finally, we provide the proof of ISS of $x$ (13).

Using the converse theorem and Assumption 3, there exist a smooth function $V_1(x, y, u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}_{\geq 0}$, and three class $K_{\infty}$ functions $\alpha_1, \alpha_2$ and $\alpha_3$, such that
\[ \alpha_1(\|y\|) \leq V_1(x, y, u) \leq \alpha_2(\|y\|), \] (15)
\[ \frac{\partial V_1}{\partial y}(x, y, h(x, u), u, 0) \leq -\alpha_3(\|y\|). \] (16)

Using Lemma 1, 2, (15) and (16), following the similar procedure in [22], it can be verified that there exist a class $K$ function $\alpha_y$, a class $KLD$ function $\beta_y$ and a continuous nonincreasing function $\gamma_y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that for essentially bounded inputs and $\varepsilon \leq \gamma_y(\max\{\|x\|, \|y(0)\|, \|u\|, \|\hat{u}\|\})$, the solution of (5) exists for all $t \geq 0$ and satisfies
\[ \|y(t, \varepsilon)\| \leq \beta_y(\|y(0)\|, t/\varepsilon) + \alpha_y(\varepsilon). \] (17)

Note that at this step we do not know the boundedness of $x$. To use the inequality (17), we apply the causality and signal truncations. Define a positive constant $\tilde{\mu}$ satisfying $\tilde{\mu} > \beta_x(\tilde{\mu}, 0) + \alpha_x(\mu) + \xi$. It can be verified that $\mu < \tilde{\mu}$. Considering the continuity for given initial condition, we can define $\tilde{T} > 0$ as the upper bound of $[0, T)$ within which $\|x\| \leq \tilde{\mu}$. Since $\gamma_y$ is nonincreasing, it follows that
\[ \gamma_y(\tilde{\mu}) < \gamma_y(\mu) \leq \gamma_y(\max\{\|x(0)\|, \|y(0)\|, \|u\|, \|\hat{u}\|\}), \] (18)
\[ \gamma_y(\tilde{\mu}) \leq \gamma_y(\|x\|). \] (19)

For $\varepsilon \leq \varepsilon_1 := \gamma_y(\tilde{\mu})$, (18) and (19) yield that $\varepsilon \leq \gamma_y(\max\{\|x\|, \|y(0)\|, \|u\|, \|\hat{u}\|\})$ holds for all $t \in [0, T)$. However, from the definition of $\tilde{\mu}$, there must exist a positive constant $\eta$, such that $\|x\| < \tilde{\mu}$ for all $t \in [0, T + \eta)$. This contradicts that $T$ is maximal, so $T = \infty$. Therefore, there exists an $\varepsilon_2$ satisfying $\alpha_y(\varepsilon_2) = \xi$, such that (14) holds for all $t \geq 0$, and $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$.

Then, we prove the second step. Define the error between solutions of reduced and original slow dynamics as $E_x = x - \hat{x}$. When $\varepsilon = 0$, $y = z - \tilde{h}(x, u) = 0$. Then, we have
\[ \dot{E}_x = f(E_x, 0, u, 0) + \Delta f, \] (20)
where \( \Delta f = [f(\hat{x} + E_x, 0, u, 0) - f(\hat{x}, 0, u, 0) - f(E_x, 0, u, 0)] + f(x, y, u, \varepsilon) - f(x, y, u, 0) \). According to Assumption 1, it follows that
\[
\|\Delta f\| \leq \ell_1 \|E_x\|^2 + \ell_2 \|E_x\| \|\hat{x}\|
+ \ell_3 \beta_y \left(\|y(0)\|, \frac{t}{\varepsilon}\right) + \ell_4 \xi + \ell_4 \varepsilon, \tag{21}
\]
for some positive constants \( \ell_1, \ell_2, \ell_3, \ell_4 \). The last term in system (20) can be viewed as a perturbation of
\[
\dot{E}_x = f(E_x, 0, u, 0). \tag{22}
\]
Since the origin of the system (22) is globally exponentially stable with \( u = 0 \), using the converse theorem, there exist a Lyapunov function \( V_2(E_x) \), and positive constants \( c_1, c_2, c_3, c_4 \), for which it follows that
\[
c_1 \|E_x\|^2 \leq V_2(E_x) \leq c_2 \|E_x\|^2, \tag{23}
\]
\[
\frac{\partial V_2}{\partial E_x} f(E_x, 0, u, 0) \leq -c_3 \|E_x\|^2, \tag{24}
\]
\[
\left\|\frac{\partial V_2}{\partial E_x}\right\| \leq c_4 \|E_x\|. \tag{25}
\]
Using (24), (21) and (23)-(25), the Lyapunov function of (22) along the trajectory of (20) satisfies
\[
V_2 = \frac{\partial V_2}{\partial E_x} f(E_x, 0, u, 0) + \frac{\partial V_2}{\partial E_x} \Delta f
\leq -c_3 \|E_x\|^2 + c_4 \|E_x\| \left[\ell_1 \|E_x\|^2 + \ell_2 \|E_x\| \|\hat{x}\|
+ \ell_3 \beta_y \left(\|y(0)\|, \frac{t}{\varepsilon}\right) + \ell_4 \xi + \ell_4 \varepsilon\right]. \tag{26}
\]
For \( \|E_x\| \leq c_3/(2c_1 \ell_1) \), using Assumption 2, it follows that
\[
V_2 \leq -2 \left\{c_3 - c_4 \ell_1 \left[\beta_x (\|\hat{x}(0)\|, t) + \alpha_x (\|u\|)\right]\right\} V_2
+ 2 \left[\ell_3 \varepsilon + \ell_4 \xi + \ell_4 \beta_y \left(\|y(0)\|, \frac{t}{\varepsilon}\right)\right] \sqrt{V_2}
\leq -2 \left\{\ell_\alpha - \ell_\alpha \beta_x (\|\hat{x}(0)\|, t)\right\} V_2
+ 2 \left[\ell_\alpha \varepsilon + \ell_\beta \beta_y (\|y(0)\|, \frac{t}{\varepsilon})\right] \sqrt{V_2}, \tag{27}
\]
where \( 0 < \ell_\alpha \leq c_3 - c_4 \ell_1 \alpha_x (\sup \|u\|), \ell_\alpha \geq \ell_\beta (1 + \xi/\varepsilon) > 0, \) and \( \ell_\alpha, \ell_\beta > 0 \). Using the comparison lemma, we have
\[
W_2(t) = \phi(t, 0) W_2(0)
+ \int_0^t \phi(t, s) \left[\ell_\alpha \varepsilon + \ell_\beta \beta_y (\|y(0)\|, \frac{t}{\varepsilon})\right] ds, \tag{28}
\]
where \( W_2 = \sqrt{V_2} \) and
\[
|\phi(t, s)| \leq \ell_\mu e^{-\ell_\mu t}, \quad \text{for } \ell_\mu, \ell_\beta > 0. \tag{29}
\]
Because
\[
\int_0^t e^{-\ell_\mu t} \beta_y (\|y(0)\|, \frac{t}{\varepsilon}) ds = O(\varepsilon), \tag{30}
\]
it can be verified that \( W_2(t) = O(\varepsilon) \). Then it follows that \( E_x(t, \varepsilon) = O(\varepsilon) \), and this means that (10) holds.

Since we have already verified that (14) holds in the first step, then by using (17) and Assumption 3, it follows that
\[
E_y(t, \varepsilon) = \|x(t, \varepsilon) - h(\hat{x}(t, \varepsilon), u(t)) - \hat{y}(t/\varepsilon)\|
\leq \beta_y (\|y(0)\|, t/\varepsilon) + \alpha_y(\varepsilon) + \beta_y (\|\hat{y}(0)\|, t/\varepsilon) = O(\varepsilon) \tag{31}
\]
for given initial points and all \( t > 0 \). This proves (11). According to Assumption 3, \( \hat{y}(t/\varepsilon) = \beta_y (\|y(0)\|, t/\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). Thus, the term \( \hat{y}(t/\varepsilon) = O(\varepsilon) \) for all \( t \geq T > 0 \) if \( \varepsilon \) is small enough to satisfy
\[
\beta_y (\|y(0)\|, t/\varepsilon) \leq k \varepsilon \tag{32}
\]
Let \( e^{**} \) and \( T \) denote a solution of (32) with equal sign. Subsequently, (32) holds for all \( \varepsilon \leq e^{**} \) uniformly on \( [T, \infty) \).

Finally, we prove the ISS of original slow dynamics. Since \( \|x(t, \varepsilon) - \|\hat{x}(t)\| \leq \|x(t, \varepsilon) - x(t)\| = O(\varepsilon) \), there exist some class \( \mathcal{KL} \) function \( \beta_x, \mathcal{K} \) function \( \alpha \) and a small positive constant \( \varepsilon_3 \), such that the solution of (14) exists for all \( t \geq 0 \) and \( \varepsilon \leq \varepsilon_3 \) satisfying
\[
\|x(t, \varepsilon)\| \leq \hat{\beta}_x (\|\hat{x}(0)\|, t) + \alpha_x (\|u\|) + O(\varepsilon) \leq \hat{\beta}_x (\|x(0)\|, t) + \alpha_x (\|u\|) + \xi. \tag{34}
\]
This completes the proof of (13). \( \blacksquare \)

C. Stability and Accuracy Assessment Embedded LSOR

This subsection develops a novel LSOR method by embedding the above theorem. The overall algorithm is proposed in Algorithm 1.

**Algorithm 1 Stability/Accuracy Assessment Embedded LSOR**

1. Choose the smaller parameters dominating the transient velocity as \( \varepsilon \). The states with respect to \( \varepsilon \) are identified as fast states, while the others as slow states.
2. **procedure** ROM AND BLM DERIVATION
   3. Let \( \varepsilon = 0 \), solve the algebraic equation (2) to obtain the isolated QSS solutions \( z = h(x, u) \).
   4. Substitute \( z \) into (12), obtaining the ROM (3).
   5. Derive the BLM using equation (6).
3. **end procedure**
4. **procedure** STABILITY ASSESSMENT
   5. **if** Assumption 2 and 3 are satisfied **then**
      6. Go to next procedure
   7. **else**
      8. Return to Step 1 to re-identify slow/fast dynamics.
   9. **end if**
10. **end procedure**
11. **procedure** CALCULATE THE BOUND OF \( \varepsilon \)
   12. Calculate \( \varepsilon^* = \min \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \) according to proof.
   13. Calculate \( e^{**} \) by solving equation (32) with equal sign.
14. **end procedure**
15. **procedure** ACCURACY ASSESSMENT
   16. **if** \( \varepsilon \leq e^{**} \) **then**
       17. **if** \( \varepsilon \leq \varepsilon^{**} \) **then**
          18. \( z = h(\hat{x}, u) \) is the solution of fast dynamics
       19. **else**
          20. Use \( z = h(\hat{x}, u) + \hat{y} \) by solving the BLM (6).
       21. **end if**
   22. **else**
      23. **Return** to Step 1 to re-identify slow/fast dynamics.
   24. **end if**
   25. **end procedure**

**Remark 3.** This algorithm is designed for microgrids with two-time-scale property, however, no basic assumptions of the microgrids are required. Therefore, the proposed method can be applied to arbitrary dynamic systems.
III. MICROGRID SYSTEM DESCRIPTION

Section II proposes our algorithm based on a general high-level expression of microgrid systems. This section introduces a detailed nonlinear mathematical model of microgrids, which will be used to demonstrate order reduction.

Depending on the research objectives, control strategies and operation modes, microgrids may have different models. According to [18], the transient response velocity of line dynamics is much faster than the slow ones in DERs due to the small line impedance. Moreover, the state equations are fully decoupled between DERs and lines. As a result, the line dynamics can be neglected. Therefore, this section focuses on the modeling of DERs, which are the main dynamic components in an inverter-dominated microgrid.

Fig. 2 shows a general control diagram of DERs. The model can switch between two subsystems according to the microgrid operation modes. In grid-tied mode, \( OM_{flag} \) switches to 1, then the voltage source inverter (VSI) is controlled by the power controller and current controller to follow the power command \( P^*, Q^* \). The microgrid bus voltage and system frequency are maintained by the main grid. Therefore, it must generate its own voltage and frequency references using droop controllers as follows,

\[
\omega^*_i = \omega_{ni} - m_i P_i, \quad V^*_{qi} = V_{oq,ni} - n_i Q_i. \quad \text{(40a, 40b)}
\]

These references will be used as the set points for voltage controllers. Two PI controllers are adopted for the voltage controllers as follows,

\[
\hat{\phi}_d = \omega P_{PLL,i} - \omega^*_i, \quad \text{(41a)}
\]

\[
I^*_{di} = K_{I,V,i}\hat{\phi}_d + K_{P,V,i}\dot{\phi}_d, \quad \text{(41b)}
\]

\[
\hat{\phi}_q = V^*_{qi} - V_{oq,ki}, \quad \text{(41c)}
\]

\[
I^*_{qi} = K_{I,V,qi}\hat{\phi}_q + K_{P,V,qi}\dot{\phi}_q. \quad \text{(41d)}
\]

E. Current Controllers

The PI controllers are adopted for current controllers. They generate the commanded voltage reference \( V'_{idqi} \) according to the error between the inductor currents reference \( I'_{idqi} \) and its feedback measurements \( I_{idqi} \):

\[
\hat{\dot{\Gamma}}_{id} = I'_{id} - I_{id}, \quad \text{(42a)}
\]

\[
V'_{i} = -\omega_{ni} L_{f1} I_{iq1} + K_{I,C_1}\dot{\Gamma}_{id} + K_{P,C_1}\Gamma_{id}, \quad \text{(42b)}
\]

\[
\hat{\dot{\Gamma}}_{qi} = I'_{iq} - I_{iq}, \quad \text{(42c)}
\]

\[
V'_{i} = -\omega_{ni} L_{f1} I_{idi} + K_{I,C_1}\dot{\Gamma}_{qi} + K_{P,C_1}\Gamma_{qi}. \quad \text{(42d)}
\]

F. LC Filters and Coupling Inductors

The dynamical models of LC filters and coupling inductors are as follows,

\[
\hat{I}_{odi} = (-R_{fi} I_{odi} + V_{odi} - V_{odi}) / L_{fi} - \omega_{ni} I_{odi}, \quad \text{(43a)}
\]

\[
\hat{\dot{\Gamma}}_{odi} = (-R_{fi} I_{odi} + V_{odi} - V_{odi}) / L_{fi} - \omega_{ni} I_{odi}, \quad \text{(43b)}
\]

\[
\hat{\dot{\Gamma}}_{odi} = (-R_{fi} I_{odi} + V_{odi} - V_{odi}) / L_{fi} - \omega_{ni} I_{odi}, \quad \text{(43c)}
\]

\[
\hat{I}_{odi} = (-R_{fi} I_{odi} + V_{odi} - V_{odi}) / L_{fi} - \omega_{ni} I_{odi}, \quad \text{(43d)}
\]

In islanded mode, when the microgrid system is operating in grid-tied mode, the mathematical model can be represented by equations (35)-(37), (39) and (42)-(43). In islanded mode, the microgrid model can be represented by equations (35)-(36), (38) and (40)-(43).

IV. CASE STUDY

A. Simulation Setup

The proposed method is tested on a modified IEEE-37 bus microgrid which can be operated in grid-tied or islanded modes. Fig. 3 shows the diagram of the modified IEEE-37 bus microgrid. According to [17], seven inverters are connected to buses 15, 18, 22, 24, 29, 33 and 34. When PCC is closed, the microgrid is operated in grid-tied mode. Otherwise, it is operated in islanded mode.

To verify the performance of the proposed method, we first let the microgrid be operated in grid-tied mode. In order to analyze the detailed dynamic properties of both slow and fast dynamics, a single bus of interest (bus 34) is chosen to show
its dynamic responses. The parameter setting of the DER on bus 34 is shown in Table I. Then, a simulation is conducted in islanded mode to show the dynamic responses of active and reactive powers of multiple buses with DERs, where the detailed parameter settings can be found in [17].

B. Simulation in Grid-tied Mode

In this subsection, we take the dynamic responses of bus 34 as an example to show how to use the proposed algorithm and evaluate its performance. According to the Algorithm 1 in section II-C, we first identify the slow and fast dynamics by finding the ε coefficients, which are utilized to classify the slow and fast states in this system:

$$ x_1 = [P, Q, \dot{x}_{PLL}, \delta, \dot{\phi}_P, \dot{\phi}_Q, \Gamma_d, \Gamma_q]^T, $$

$$ z_1 = [V_{odf}, I_{od}, I_{oq}, I_{od}, I_{oq}, V_{od}, V_{oq}]^T. $$  

Substituting the parameters in Table I into vector (44), it can be seen that the magnitudes of different parameters vary significantly, which is caused by the two-time-scale property of the system. The smaller parameters are selected as perturbation coefficients ε, which are utilized to classify the slow and fast states in this system:

$$ x_1 = [P, Q, \dot{x}_{PLL}, \delta, \dot{\phi}_P, \dot{\phi}_Q, \Gamma_d, \Gamma_q]^T, $$

$$ z_1 = [V_{odf}, I_{od}, I_{oq}, I_{od}, I_{oq}, V_{od}, V_{oq}]^T. $$  

We first set ε to 0 and calculate the QSS solution $z_1 = h(x_1, u_1)$ by solving the algebraic equation with respect to the fast dynamics (46). Then the ROM of the microgrid is obtained by substituting $z_1$ into the slow dynamic equations with respect to (45). Compare the numbers of state variables in equation (44) and (45), the order of original model is reduced to 53.33%. Then we derive the BLM using equation (6). Once we obtain the ROM and BLM, we use the conventional ISS and GAS judging theorems in [18] to evaluate the stability of them. It can be verified that the assumptions are satisfied. Based on this result, we are inclined to anticipate the stability of the original system. To ensure this, we still need to theoretically verify the accuracy of the ROM and BLM.

Following the technique in the proof, we can calculate the boundary of ε as $\varepsilon^* = \min \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = 7.92 \times 10^{-3}$. Note that $\max \{\varepsilon\} = 3.9 \times 10^{-3} < 7.92 \times 10^{-3} = \varepsilon^*$. Therefore, we can conclude that this microgrid system is stable and we can use the solutions of its ROM $\hat{x}$ and $z = h(x, u) + \hat{y}$ to accurately represent its real dynamic responses. Furthermore, given $T = 0.43 \text{ sec}$, we can find a $\varepsilon^{**}$ satisfying $\max \{\varepsilon\} < \varepsilon^{**} = 4.2 \times 10^{-3}$, which indicates that the term $\hat{y}$ will be $O(\varepsilon)$ after 0.43 sec. Here, a trade-off exists between the accuracy and efficiency. When the accuracy is prior, one can choose $z = h(x, u) + \hat{y}$ by computing an additional differential equation (BLM). When the efficiency dominates, use $z = h(x, u)$ suffering the inaccuracy only within $(0, T)$.

Then we conduct the simulation of the derived ROM using Matlab. The active power command changes to 1000 W at 2 sec and changes to 500 W at 4 sec. The reactive power command changes to 500 W at 2 sec and changes to 300 W at 4 sec. The simulation results are shown in the following
Fig. 4. Simulation results of slow dynamic responses of interest (a) Active power $P$, (b) Reactive power $Q$.

Fig. 5. Simulation results of fast dynamic responses of interest: (a) $d$-axis output voltage $V_{od}$, (b) $q$-axis output voltage $V_{oq}$.

Fig. 6. Simulation results of fast dynamic responses of interest: (a) $d$-axis output current $I_{od}$, (b) $q$-axis output current $I_{oq}$.

figures. Fig. 4 shows the simulation result of main slow dynamic responses (active/reactive power) of the microgrids system. The blue solid lines denote the dynamic response of the original full-order model, while the red dashed lines represent the ROM. The small differences between each pair of response curves show the effectiveness of the proposed method representing the slow dynamics.

We are interested in not only the active/reactive power, but also the output voltage and current of DER. For nominal parameter settings, these variables are usually identified as fast states. As discussed above, we can calculate the fast states using equations $z = h(\hat{x}, u)$ or $z = h + \hat{y}$ according to the users’ need. Fig. 5 and 6 show the simulation results of fast dynamic responses of $dq$-axes output voltage $V_{odq}$ and current $I_{odq}$, respectively. The blue solid lines denote the dynamic responses of the original model; the pink dotted lines show the fast responses only using the solution of static equation $h(\hat{x}, u)$; the red dashed lines are the fast responses with the addition of solution $\hat{y}$ of BLM (i.e., $z = h + \hat{y}$). The comparison shows that the introduction of $\hat{y}$ can significantly improve the accuracy of the fast dynamics of the reduced model. In contrast, if we use $z = h(\hat{x}, u)$, it doesn’t require to solve the BLM but only suffers the inaccuracy within 0.43 sec after transient period.

In order to evaluate the computational performance, two different ordinary differential equation (ODE) solvers are adopted: ode45 solver and ode15s solver. The ode45 solver uses 4th-order Runge-Kutta method with variable step sizes in order to solve the non-stiff ODE problems, whereas the ode15s solver is designed for stiff problems. The simulation time of reduced and full models using ode45 solver are 11.92 sec and 94.25 sec, respectively. The computational time is reduced by 87.4%. However, the simulation time of the two models using ode15s solver are 10.81 sec and 11.43 sec, respectively, only reduced by 5%. This comparison indicates that our LSOR method converts the original model from a stiff ODE problem to a non-stiff one. The adoption of the proposed method also improves the stability of ODE solving process by this conversion. In conclusion, the proposed method can reduce the computational time from two aspects: the order of system and the stiffness of the ODE problem.

Remark 4. Note that with the addition of the solution of BLM, we need to solve another set of differential equations. This seems that the proposed method has limited ability to reduce the computational burden. However, this is not the case. As discussed above, SPM reduces the computational burden not only by reducing the number of differential equations, but also by converting the stiff problem to a non-stiff one. Moreover, the adopted example is a possible worst case that the perturbation coefficients are not small enough. When $\varepsilon$ is sufficiently small, the converging time $T$ can be sufficiently small as well. Then we can directly use the algebraic equation to estimate the fast states.

Even though the stability can be verified mathematically using the proposed theorem, we can observe the simulation results from the stability’s point of view. The red lines in Fig. 4 - 6 show that, with bounded input power commands, both slow and fast dynamics of the reduced system converge rapidly after the change of inputs. This indicates that the ROM is ISS. Furthermore, the blue lines show that when the reduced system is stable, the original system is stable as well.

C. Simulation in Islanded Mode

In this subsection, we conduct the simulation in islanded mode to verify the performance of the proposed method by
showing the dynamic responses of the buses with DERs. Following the similar procedure in case 1, we can identify the slow and fast dynamics of this multi-bus system. Despite of the different parameter settings of inverters, the relative magnitudes of derivative terms’ coefficients still hold uniformly. That means we can obtain a uniform division of slow and fast dynamics. This fact is based on the nature of different component’s time-scale. The slow and fast states are divided as follows,

$$x_2 = [P_i, Q_i, \Phi_{PLL_i}, \delta, \Phi_{di}, \Phi_{qi}, \Gamma_{di}, \Gamma_{qi}]^T,$$

$$z_2 = [V_{odi}, I_{odi}, I_{odi}, I_{oqi}, V_{odi}, V_{oqi}]^T.$$  (47)

The ROM can be derived using the Algorithm 7. The order of original model is reduced from $10^5^{th}$ to $50^{th}$. A step change of power command is given to the system at $1$ sec. The simulation time of reduced and full models using ode45 solver are 11.25 sec and 140.25 sec, respectively. The computational time is reduced about 89.2%. The simulation time of reduced and full models using ode15s solver are 11.37 sec and 13.23 sec, respectively. Only about 14% of computational time is reduced. Fig. 7 shows the comparison between dynamic responses of active/reactive power of the original and reduced models of the seven buses with DERs connected. The comparison between the results of the original model and the reduced one shows the accuracy of the ROM and verifies the effectiveness of our method in islanded multi-bus systems.

V. CONCLUSION

This paper proposes a novel stability and accuracy assessment theorem for microgrids LSOR. The advantages of the proposed theorem can be summarized into two aspects: firstly, we can determine the stability of the original full-order system by only analyzing the stability of the ROM and BLM. This makes it easier and more feasible to determine the stability of a large-scale complex system. Secondly, it gives quantitative conditions to guarantee the accuracy of derived reduced model by finding the bound of error between the original and reduced model. Furthermore, we have strictly proved the theorem and illustrated that the proposed method is generic for arbitrary dynamic systems. Finally, we have conducted simulation on an IEEE standard microgrid system to verify the effectiveness of the proposed method.

REFERENCES

[1] M. Shahidehpour, “Role of smart microgrid in a perfect power system,” in Proc. IEEE PES General Meeting, Jul. 2010, p. 1.

[2] M. Shahidehpour and J. F. Clair, “A Functional Microgrid for Enhancing Reliability, Sustainability, and Energy Efficiency,” The Electricity Journal, vol. 25, no. 8, pp. 21 – 28, 2012. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1040619012002278

[3] Z. Wang, B. Chen, J. Wang, J. Kim, and M. M. Begovic, “Robust Optimization Based Optimal DG Placement in Microgrids,” IEEE Transactions on Smart Grid, vol. 5, no. 5, pp. 2173–2182, Sep. 2014.

[4] X. Lu, K. Sun, J. M. Guerrero, J. C. Vasquez, L. Huang, and J. Wang, “Impedance Enhancement for DC Microgrids Based on Virtual Impedance for DC Microgrids With Constant Power Loads,” IEEE Transactions on Smart Grid, vol. 6, no. 6, pp. 2770–2783, Nov. 2015.

[5] Y. Pan, L. Chen, X. Lu, J. Wang, F. Liu, and S. Mei, “Stability Region of Droop-Controlled Distributed Generation in Autonomous Microgrids,” IEEE Transactions on Smart Grid, vol. 10, no. 2, pp. 2288–2300, Mar. 2019.

[6] Y. Yan, D. Shi, D. Bian, B. Huang, Z. Yi, and Z. Wang, “Small-Signal Stability Analysis and Performance Evaluation of Microgrids Under Distributed Control,” IEEE Transactions on Smart Grid, vol. 10, no. 5, pp. 4848–4858, Sep. 2019.

[7] Y. Li, W. Gao, and J. Jiang, “Stability analysis of microgrids with multiple der units and variable loads based on nmt,” in 2014 IEEE PES General Meeting— Conference & Exposition, IEEE, 2014, pp. 1–5.

[8] Y. Li, P. Zhang, L. Zhang, and B. Wang, “Active Synchronous Detection of Deception Attacks in Microgrid Control Systems,” IEEE Transactions on Smart Grid, vol. 8, no. 3, pp. 373–375, Jan. 2017.

[9] Y. Li, P. Zhang, M. Althoff, and M. Yue, “Distributed Formal Analysis for Power Networks with Deep Integration of Distributed Energy Resources,” IEEE Transactions on Power Systems, p. 1. 2018.

[10] Y. Li, P. Zhang, and M. Yue, “Networked microgrid stability through distributed formal analysis,” Applied Energy, vol. 228, pp. 270 – 288, 2018. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0306261918300097

[11] Z. Li and M. Shahidehpour, “Small-Signal Modeling and Stability Analysis of Hybrid AC/DC Microgrids,” IEEE Transactions on Smart Grid, vol. 10, no. 2, pp. 2080–2095, Mar. 2019.

[12] J. Duan, Z. Yi, D. Shi, C. Lin, X. Lu, and Z. Wang, “Reinforcement-Learning-Based Optimal Control of Hybrid Energy Storage Systems in Hybrid AC/DC Microgrids,” in Proc. IEEE Power Energy Society General Meeting (PESGM), Aug. 2018, pp. 1–5.

[13] M. L. Crow and J. G. Chen, “The multivariate method for simulation of power system dynamics,” IEEE Transactions on Power Systems, vol. 9, no. 3, pp. 1684–1690, Aug. 1994.

[14] J. He and D. Del Vecchio, “Deterministic-like model reduction for a class of multiscale stochastic differential equations with application to biomolecular systems,” IEEE Transactions on Automatic Control, vol. 64, no. 1, pp. 351–358, 2018.

[15] H. K. Khalil, Nonlinear Systems, New Jersey: Prentice Hall, 2000.

[16] Luo, Ling and S. Dhople, “Spatiotemporal model reduction of inverter-based islanded microgrids,” IEEE Transactions on Energy Conversion, vol. 29, no. 4, pp. 823–832, 2014.

[17] M. Rashiduzzaman, J. A. Mueller, and J. W. Kimball, “Reduced-Order Small-Signal Model of Microgrid Systems,” IEEE Transactions on Sustainable Energy, vol. 6, no. 4, pp. 1292–1305, Oct. 2015.

[18] J. W. Simpson-Porco, F. Drfler, F. Bullo, Q. Shafiee, and J. M. Guerrero, “Stability; power sharing, distributed secondary control in droop-controlled microgrids,” in Proc. IEEE Int. Conf. Smart Grid Communications (SmartGridComm), Oct. 2013, pp. 672–677.

[19] V. Mariani, F. Vasca, J. C. Vásquez, and J. M. Guerrero, "Model order reductions for stability analysis of islanded microgrids with droop control," IEEE Transactions on Industrial Electronics, vol. 62, no. 7, pp. 4344–4354, Jul. 2015.

[20] I. P. Nikolakakos, H. H. Zeineldin, M. S. El-Moursi, and J. L. Kirtley, “Reduced-Order Model for Inter-Inverter Oscillations in Islanded Droop-Controlled Microgrids,” IEEE Transactions on Smart Grid, vol. 9, no. 5, pp. 4953–4963, Sep. 2018.

[21] P. D. Christofides and A. R. Teel, “Singular perturbations and input-to-state stability,” IEEE Transactions on Automatic Control, vol. 41, no. 11, pp. 1645–1650, Nov. 1996.

[22] P. Vorobev, P.-H. Huang, M. Al Hosani, J. L. Kirtley, and K. Turitsyn, “High-fidelity model order reduction for microgrids stability assessment,” IEEE Transactions on Power Systems, vol. 33, no. 1, pp. 874–887, 2017.