AN EXAMPLE RELATED TO BRODY’S THEOREM

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ABSTRACT. We discuss an example related to the method of Brody.

1. INTRODUCTION

1.1. Bloch principle. In one-dimensional function theory there is a general philosophy which supposedly goes back to A. Bloch (see e.g. [12], [2]): If \( P \) is a sufficiently reasonable class of holomorphic maps or functions, then the following statements should be equivalent:

(1) Every map in class \( P \) defined on the complex line \( \mathbb{C} \) is constant.

(2) The set of all maps in class \( P \) defined on the unit disk \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) is a normal family.

(A family of maps is called a “normal family” if every sequence in it is either compactly divergent or contains a subsequence which converges uniformly on compact sets. A sequence of maps \( f_n : X \to Y \) between topological spaces is “compactly divergent”, if for every pair of compact subsets \( K \subset X, C \subset Y \) there are only finitely many \( f_n \) with \( f_n(K) \cap C \neq \{\} \).

For example, every bounded holomorphic function on \( \mathbb{C} \) is constant by Liouville’s theorem and due to Montel’s theorem the family of all bounded holomorphic functions on \( \Delta \) is a normal family. Thus the Bloch principle is valid for the family \( P \) of all bounded holomorphic functions with values in \( \mathbb{C} \).

1.2. Brody’s theorem. Let \( Y \) be a complex manifold. It is called “taut” if the family of all holomorphic maps \( f : \Delta \to Y \) is a normal family. Let us from now on assume that \( Y \) is compact. Then being “taut” is easily seen to be equivalent with hyperbolicity in the sense of Kobayashi. The theorem of Brody (see [3]) states that this is furthermore equivalent with the property that every holomorphic map from \( \mathbb{C} \) to \( Y \) is constant. In other words: Brody’s theorem states that the

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Bloch principle hold for the class of holomorphic maps with values in a (fixed) compact complex manifold $Y$.

Now we may raise the question: What about holomorphic maps to a compact complex manifold fixing some given base points? Given a compact complex manifold $Y$ and a point $y \in Y$, let us consider the following two statements:

- Every holomorphic map $f : \mathbb{C} \to Y$ with $f(0) = y$ is constant.
- The family of all holomorphic maps $f : \Delta \to Y$ with $f(0) = y$ is a normal family.

Are they equivalent?

Using the notion of the infinitesimal Kobayashi-Royden pseudometric as introduced in [10] this can be reformulated into the following question: "If the infinitesimal Kobayashi-Royden pseudometric on a compact complex manifold $Y$ degenerates for some point $y \in Y$, does this imply that there exists a holomorphic map $f : \mathbb{C} \to Y$ with $y \in f(\mathbb{C})$?"

Thanks to Brody’s theorem it is clear that there exists some non-constant holomorphic map $f : \mathbb{C} \to Y$ if the Kobayashi-Royden pseudometric is degenerate at some point $y$ of $Y$. But it is not clear that $f$ can be chosen in such a way that $y$ is in the image or at least in the closure of the image. Of course, at first it looks absurd that degeneracy of the Kobayashi-Royden pseudometric at one point $y$ should only imply the existence of a non-constant holomorphic map to some part of $Y$ far away of $y$ and should not imply the existence of a non-constant map $f : \mathbb{C} \to Y$ whose image comes close to $y$.

Thus one is led to postulate

**Conjecture.** Let $X$ be a compact complex manifold, $x \in X$. Assume that the infinitesimal Kobayashi-Royden pseudometric is degenerate on $T_x X$.

Then there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ with $f(0) = x$.

1.3. **Bounded derivatives.** Let $X$ be a complex manifold equipped with a hermitian metric $h$. For each holomorphic map $f : \mathbb{C} \to X$ and each point $z \in \mathbb{C}$ we may now calculate the norm of the derivative $Df$ at $z$ with respect to the euclidean metric on $\mathbb{C}$ and $h$ on $X$. Let $P$ be a class of holomorphic maps $f : \mathbb{C} \to (X, h)$ with bounded derivatives (i.e. for every $f \in P$ there is a number $C > 0$ such that the inequality $||Df_z|| < C$ holds for all $z \in \mathbb{C}$). Let $f : \mathbb{C} \to X$ be a non-constant map in this class $P$. Via $f_n(x) = f(nx)$ this map $f$ yields a non-normal family of maps $f_n : \Delta \to X$. 
Now let $P'$ denote the set of those maps in $P$ for which the derivative (calculated with respect to the euclidean metric on $\mathbb{C}$ resp. $\Delta$ and the hermitian metric on $X$) is bounded. For each of the $f_n$ defined above the derivative is clearly bounded, since $\Delta$ is relatively compact in $\mathbb{C}$, and $f_n : \mathbb{C} \to X$ extends through the boundary. Thus $f_n$ is a non-normal family in $P'$. If the Bloch principle holds for $P'$, this implies the existence of a non-constant holomorphic map $F : \mathbb{C} \to X$ in $P$.

Thus: If the Bloch principle holds for $P'$, the existence of a non-constant holomorphic map $f$ in $P$ implies the existence of a non-constant holomorphic map $F$ in $P$ with the additional property that $||DF||$ is bounded.

Brody’s theorem implies that this is indeed true if, given a compact complex manifold $X$, we consider the set $P$ of all holomorphic maps with values in $X$.

However, we will give an example of a compact complex manifold $X$, an open subset $\Omega$ and a point $x \in \Omega$ such that this property does not hold if $P$ is chosen as the family of all holomorphic maps $f$ with image contained in $\Omega$ and $f(0) = x$.

1.4. Reparametrization. The key method for proving a Bloch principle is the following: Let $f_n : \Delta \to Y$ be a non-normal family. Then we look for an increasing sequence of disk $\Delta_{r_n}$ which exhausts $\mathbb{C}$ (i.e. $\lim r_n = +\infty$) and a sequence of holomorphic maps $\alpha_n : \Delta_{r_n} \to \Delta$ such that a subsequence of $f_n \circ \alpha_n$ converges (locally uniformly) to a non-constant holomorphic map from $\mathbb{C}$ to $Y$.

For the proof of his theorem Brody used this idea, taking combinations of affine-linear maps with automorphisms of the disk for the $\alpha_n$.

Zalcman ([12]) investigated other reparametrizations where the $\alpha_n$ themselves are affin-linear maps, a concept which has the advantage that it can also be applied to harmonic maps.

1.5. Subvarieties of abelian varieties. Let $A$ be a complex abelian variety (i.e. a compact complex torus which is simultaneously a projective algebraic variety) and $X$ a subvariety. Let $E$ denote the union of all translates of complex subtori of $A$ which are contained in $X$. It is known that this union is either all of $X$ or a proper algebraic subvariety ([6]).

Since $A$ is a compact complex torus there is a flat hermitian metric on $A$ induced by the euclidean metric on $\mathbb{C}^g$ via $A \simeq \mathbb{C}^g/\Gamma$. A holomorphic map $f : \mathbb{C} \to A$ has bounded derivative with respect to this metric if and only if it is induced by an affine-linear map from $\mathbb{C}$ to $\mathbb{C}^g$. 
From this, one can deduce that \( f(\mathbb{C}) \subset E \) for every holomorphic map \( f : \mathbb{C} \rightarrow X \) with bounded derivative. Given the previous considerations about the Bloch principle, it is thus natural to conjecture:

**Conjecture.** For every non-constant holomorphic map \( f : \mathbb{C} \rightarrow X \) the image is contained in \( E \). The Kobayashi-pseudodistance on \( X \) is a distance outside \( E \).

For example, this statement is a consequence of the more general conjecture VIII.I.4 by S. Lang in [9]. In the context of classification theory the above statement has also be conjectured by F. Campana ([4], §9.3).

In the spirit of the analogue between diophantine geometry and entire holomorphic curves as pointed out by Vojta [11], the conjecture above is also supported by the famous result of Faltings ([5]) with which he solved the Mordell conjecture. This result states the following: If we assume that \( A \) und \( X \) are defined over a number field \( K \), then with only finitely many exceptions every \( K \)-rational point of \( X \) is contained in \( E \).

1.6. **Our example.** We construct an example of the following type: There is a compact complex manifold \( X \), equipped with some hermitian metric, an open subset \( \Omega \) and a point \( p \in \Omega \). There exists a non-constant holomorphic map \( f : \mathbb{C} \rightarrow \Omega \) with \( f(0) = p \). Via \( f_n(z) = f(nz) \) this yields a non-normal family of holomorphic maps \( f_n : \Delta \rightarrow \Omega \) with bounded derivatives such that \( f_n(0) = p \).

But there is no non-constant holomorphic map \( f : \mathbb{C} \rightarrow \Omega \) with \( f(0) = p \) and bounded derivative.

2. **The example**

2.1. **Statement of main results.** We construct an example which shows that Brody reparametrization sometimes necessarily changes the image of the curve.

**Theorem 1.** There exists a compact complex hermitian manifold \((T, h)\) and open subsets \( \Omega_2 \subset \Omega_1 \subset T \) such that:

1. \( \Omega_2 \) is not dense in \( \Omega_1 \) and neither is \( \Omega_1 \) in \( T \).
2. For every point \( p \in \Omega_1 \) there is a non-constant holomorphic map \( f : \mathbb{C} \rightarrow \Omega_1 \) with \( p = f(0) \).
3. If \( f : \mathbb{C} \rightarrow T \) is a non-constant holomorphic map with bounded derivative (with respect to the euclidean metric on \( \mathbb{C} \) and \( h \) on \( T \)) and \( f(\mathbb{C}) \subset \Omega_1 \), then \( f(\mathbb{C}) \subset \Omega_2 \).
Recall that Brody’s method, starting from any holomorphic map from $\mathbb{C}$ to $T$, yields a holomorphic map from $\mathbb{C}$ to $T$ with bounded derivative. Thus this examples provides a picture in which Brody’s method really changes the properties of $f : \mathbb{C} \to T$ fundamentally.

Responding to some additional questions which may be asked, we prove a little bit more.

**Theorem 2.** There exists a compact complex torus $T$, equipped with a flat hermitian metric $h$ and open subsets $\Omega_2 \subset \Omega_1 \subset T$ such that:

1. $\Omega_2$ is not dense in $\Omega_1$ and neither is $\Omega_1$ in $T$.
2. For every point $p \in \Omega_1$ and every $v \in T_p \Omega_1$ there is a non-constant holomorphic map $f : \mathbb{C} \to \Omega_1$ with $p = f(0)$, $v = f'(0)$ and $\Omega_1 = f(\mathbb{C})$.
3. If $f : \mathbb{C} \to T$ is a non-constant holomorphic map with bounded derivative (with respect to the euclidean metric on $\mathbb{C}$ and $h$ on $T$) and $f(\mathbb{C}) \subset \Omega_1$, then $f(\mathbb{C}) \subset \Omega_2$. Moreover $f$ is affine-linear and $\overline{f(\mathbb{C})}$ is a closed analytic subset of $T$.

We remark that this implies in particular that the infinitesimal Kobayashi-Royden pseudometric vanishes identically on $\Omega_1$.

Furthermore, it provides examples of holomorphic maps from $\mathbb{C}$ into a compact complex torus with a rather “bad” image: The closure of the image with respect to the euclidean topology is $\overline{\Omega_2}$ and thus a set with non-empty interior such that the complement has also non-empty interior. This is in strong contrast to the Zariski-analytic closure: By the theorem of Green-Bloch-Ochiai for every holomorphic map $f$ from $\mathbb{C}$ to a compact complex torus $T$ the closure of the image $f(\mathbb{C})$ with respect to the analytic Zariski topology (i.e. the smallest closed analytic subset of $T$ containing $f(\mathbb{C})$) is always a translated subtorus of $T$.

We will now describe our example.

We precede the construction with some elementary observations about tori: Let $T = \mathbb{C}^n / \Lambda$ be a torus, equipped with the flat euclidean metric and the corresponding distance function $d_T(\cdot, \cdot)$. Let

$$\rho = \frac{1}{2} \min_{\gamma \in \Lambda \setminus \{0\}} ||\gamma||.$$  

This is the *injectivity radius*, in other words $\rho$ is the largest real number such that the natural projection $\pi : \mathbb{C}^n \to T$ induces a homeomorphism between the ball

$$B_\epsilon(\mathbb{C}^n; 0) = \{ v \in \mathbb{C}^n : ||v|| < \epsilon \}$$

and

$$B_\epsilon(T; e) = \{ x \in T : d_T(x, e) < \epsilon \}$$
for all $\epsilon < \rho$. Evidently, the injectivity radius $\rho$ is a lower bound for the diameter

$$\rho \leq \text{diam} = \max_{x,y \in T} d_T(x, y)$$

If we pass from $T$ to a subtorus $S \subset T$, the injectivity radius can only increase, while the diameter can only decrease. As a consequence we obtain:

**Lemma 1.** Let $T$ be a compact (real or complex) torus with injectivity radius $\rho$. Then for every real positive-dimensional subtorus $S \subset T$ the diameter

$$\text{diam}(S) = \max_{x,y \in S} d_T(x, y)$$

is at least $\rho$.

Furthermore, if $0 < \epsilon < \rho$ and $x \in T$, then the ball $B_\epsilon(T; x)$ contains no translate of any positive-dimensional real subtorus of $T$.

Before giving the details of the construction of our example, let us try to express its idea in a drawing:

![Diagram](image)

Now let us start the precise construction of the example. Let $E' = \mathbb{C}/\Gamma'$ and $E'' = \mathbb{C}/\Gamma''$ be elliptic curves and $T = E' \times E''$. Let $\pi' : \mathbb{C} \to E'$, $\pi'' : \mathbb{C} \to E''$ and $\pi = (\pi', \pi'') : \mathbb{C}^2 \to T$ denote the natural projections. We assume that $E'$ is not isogenous to $E''$. (For example, we might choose $E' = \mathbb{C}/\mathbb{Z}[i]$ and $E'' = \mathbb{C}/\mathbb{Z}[\sqrt{2}i]$.) Then $E' \times \{0\}$ and $\{0\} \times E''$ are the only non-trivial complex subtori of $T$.

Now $T = \mathbb{C}^2/\Gamma$ with $\Gamma = \Gamma' \times \Gamma''$. The compact complex torus $T$ carries a hermitian metric $h$ induced by the euclidean metric on $\mathbb{C}^2$ (i.e. $h = dz_1 \otimes \overline{dz_1} + dz_2 \otimes \overline{dz_2}$). The associated distance function is called $d$, the injectivity radius $\rho$ is defined as explained above.

We choose numbers $0 < \rho' < \rho'' < \rho$ and define $W = B_{\rho'}(E', e)$. 
Furthermore we choose $0 < \delta < \frac{1}{2}\rho$ and we choose a holomorphic map $\sigma : \mathbb{C} \to E''$ such that there exist complex numbers $t, t' \in B'_\rho(\mathbb{C}, 0)$ (i.e. $|t|, |t'| < \rho'$) and

$$d_{E''}(\sigma(t), \sigma(t')) > 2\delta.$$  

(This is possible since $2\delta$ is smaller than the injectivity radius $\rho$ of $T$ which in turn is a lower bound for the diameter of $E''$).

We denote by $s : \mathbb{C} \to \mathbb{C}$ a holomorphic function such that $\sigma = \pi'' \circ s$.

Since $\pi' : \mathbb{C} \to E'$ restricts to an isomorphism between $B'_\rho(\mathbb{C}, 0)$ and $B_\rho(E', e)$, the holomorphic maps $s$ and $\sigma$ induce maps from $B_\rho(E', e)$ to $\mathbb{C}$ resp. $E''$. By abuse of notation these maps will also be denoted by $s$ resp. $\sigma$.

Now define $\Omega_2 = (E' \setminus \bar{W}) \times E''$ and $\Omega_1 = \Omega_2 \cup \Sigma$ with

$$\Sigma = \{(x, y) : x \in \bar{W}, y \in E'', d_{E''}(y - \sigma(x)) < \delta\}$$

Let us now fix some point $p \in \Omega_1$ and $v = (v_1, v_2) \in T_p(T) = \mathbb{C}^2$. We have to show that there exists a holomorphic map $f$ as stipulated in (2) of theorem 2.

Let $(p_1, p_2) \in \mathbb{C}^2$ be a point mapped on $p$ by $\pi : \mathbb{C}^2 \to T$. If $p \in \Sigma$, we require $|p_1| \leq \rho'$ and $|s(p_1) - p_2| < \delta$ and define $\delta' = \delta - |s(p_1) - p_2|$. If $p \notin \Sigma$, we require $|p_1| > \rho''$ and define $\delta' = \delta$.

As the next step, we will choose a pair of entire functions $(Q, H)$.

**Claim 1.** There is a pair of entire functions $(Q, H)$ with the following properties:

1. $Q$ is a non-constant polynomial,
2. $(Q(0), H(0)) = (p_1, p_2)$ and
3. $(Q'(0), H'(0))$ and $v$ are parallel.
4. If $p \in \Sigma$, we require furthermore that $(Q(z), H(z) + y) \in \pi^{-1}(\Sigma)$ for all $z$ and $y$ with $|Q(z)| \leq \rho'$ and $|y| \leq \frac{1}{2}\delta'$.

Let us first discuss the case where $p \notin \Sigma$.

Then it suffices to choose

$$Q(z) = z^2 + v_1 z + p_1$$

and

$$H(z) = v_2 z + p_2.$$  

If $p \in \Sigma$, we proceed as follows: First, for $r, t \in \mathbb{C}$ we define

$$Q_t(z) = (z + t)^2 + p_1 - t^2$$

and

$$H_{r,t}(z) = p_2 - s(p_1) + s(Q_t(z)) + rz.$$
We will set $Q = Q_t$ and $H = H_{r,t}$ for appropriately chosen parameters $r, t$.

Evidently $Q_t$ is a polynomial for any choice of $t$. Furthermore $(Q_t(0), H_{r,t}(0)) = (p_1, p_2)$ independent of the choice of $r, t$:

$$Q_t(0) = t^2 + p_1 - t^2 = p_1$$

and

$$H_{r,t}(0) = p_2 - s(p_1) + s(p_1) + 0 = p_2.$$ 

Let $\Phi_{r,t} = (Q_t, H_{r,t})$. We have

$$\Phi'_{r,t}(0) = \left( Q'_t(0), s'(Q_t(0))Q'_t(0) + r \right) = \left( 2t, 2s'(p_1)t + r \right)$$

Observe that $(r, t) \mapsto 2t$ defines a meromorphic function on $\mathbb{C}^2$ with a point of indeterminacy at $(0, 0)$. This is true regardless of the value of $s'(p_1)$.

Thus every neighborhood of $(0, 0)$ contains a point $(r, t) \neq (0, 0)$ such that $\Phi'_{r,t}(0)$ is a non-zero multiple of $v$.

Next we note that $(t, z) \mapsto Q_t(z)$ defines a proper map from $B_1(\mathbb{C}, 0) \times \mathbb{C}$ to $\mathbb{C}$. Therefore there is a constant $C > 0$ such that $|z| < C$, whenever there exists a parameter $t$ such that $|t| \leq 1$ and $|Q_t(z)| \leq \rho$.

It is therefore possible to choose two numbers $r, t$ in such a way that

1. $\Phi'_{r,t}(0)$ is a non-zero multiple of $v$,
2. $|t| < 1$ and
3. $|2rC| < \delta'$.

Now assume that $z, y \in \mathbb{C}$ with $|Q_t(z)| \leq \rho'$ and $|y| < \frac{1}{2}\delta'$. By the definition of the constant $C$, this implies $|z| < C$. Let $(w_1, w_2) = \Phi_{r,t}(z) + (0, y)$. Then

$$|w_2 - s(w_1)| = |p_2 - s(p_1) + rz + y| < |p_2 - s(p_1)| + |rC| + \frac{1}{2}\delta' <$$

$$< (\delta - \delta') + \frac{1}{2}\delta' + \frac{1}{2}\delta' = \delta.$$ 

Now $|w_2 - s(w_1)| < \delta$ in combination with $|w_1| = |Q_t(z)| \leq \rho'$ implies $\pi(w_1, w_2) \in \Sigma$. Hence $\Phi_{r,t}(z) + (0, y) \in \pi^{-1}(\Sigma)$ under this assumption. Thus the claim is proved. Q.E.D.

Our next step is to construct a closed subset $A$ of $\mathbb{C}$ to which we will apply Arakelyan approximation.

Let $A_0$ be the union of $\overline{B_{\rho'}(0)}$ and $\overline{B_{\rho'}(\gamma)}$ for all $\gamma \in \Gamma'$. If $p \notin \Sigma$, then $p_1 \notin A_0$. Hence in this case we can choose $\eta > 0$ such that $\overline{B_{\eta}(p_1)}$
is disjoint to $A_0$ and define $A_1$ as the union of $A_0$ with this closed ball $\overline{B_\eta(p_1)}$. If $p \in \Sigma$, we simply take $A_1 = A_0$.

Next we choose dense countable subsets $S_1 \subset \text{int}(\Sigma)$ (where $\text{int}(\Sigma)$ denotes the interior of $\Sigma$) and $S_2 \subset \Omega_2$. We observe that $\mathbb{C} \setminus A_1$ projects surjectively onto $E' \setminus \overline{W}$ and that the fibers of this projection are infinite discrete subsets of $\mathbb{C}$. For this reason we can find sequences $a_n, b_n$ in $\mathbb{C}$ such that

$$S_2 = \{\pi(a_n, b_n) : n \in \mathbb{N}\}$$

and all the $a_n$ are distinct elements of $\mathbb{C} \setminus A_1$ with $\lim_{n \to \infty} |a_n| = +\infty$.

It follows that

$$\Theta = \{a_n : n \in \mathbb{N}\}$$

is a discrete subset of $\mathbb{C}$ which has empty intersection with $A_1$. We define $A_2 = A_1 \cup \Theta$.

We fix a bijection $\xi : \Gamma' \setminus \{0\} \sim S_1$ and an enumeration $n \mapsto \gamma_n$ of $\Gamma' \setminus \{0\}$. Then we can choose sequences of complex numbers $c_n, d_n$ such that the following properties hold for all $n \in \mathbb{N}$

1. $\pi(c_n, d_n) = \xi(\gamma_n)$,
2. $|c_n - \gamma_n| < \rho'$ and
3. $|d_n - s(c_n)| < \delta$.

We define $A = Q^{-1}(A_2)$.

Claim 2. Arakelyan approximation is applicable to $A$, i.e. $\{\infty\} \cup (\mathbb{C} \setminus A)$ is connected and locally connected.

Observing that we can deform $B_{\rho'}(\mathbb{C}, 0)$ to $B_{\rho'}(\mathbb{C}, 0)$, we deduce from prop. 1 that $Q^{-1}(A_0)$ has the desired property. Now $A$ and $Q^{-1}(A_0)$ differ only by removing the preimage of a closed disc and by removing a discrete countable set (namely $Q^{-1}(\Theta)$). This can not destroy connectivity, hence not only $\{\infty\} \cup (\mathbb{C} \setminus Q^{-1}(A_0))$ but also $\{\infty\} \cup (\mathbb{C} \setminus A)$ is connected and locally connected. Thus the claim is proved.

We will now define a continuous function $h$ on $A$, which is holomorphic in its interior, and which we will then approximate by an entire function, using Arakelyan’s theorem.

If $p \notin \Sigma$, we take $h(z) = H(z)$ on $Q^{-1}(B_\eta(p_1))$ and $h = s$ on $Q^{-1}(B_{\rho'}(0))$.

If $p \in \Sigma$, we define $h$ on $Q^{-1}(B_{\rho'}(0))$ as $H(z)$.

Next, for every $n \in \mathbb{N}$, we define $h(z)$ as

$$h(z) = s(Q(z) - \gamma_n) + d_n - s(c_n)$$

whenever $|Q(z) - \gamma_n| \leq \rho'$.

Finally, we define $h$ on $Q^{-1}(\Theta)$ by stipulating that $h(z) = b_n$ whenever $Q(z) = a_n$ for a number $n \in \mathbb{N}$.
By the construction of \((Q, H)\) we know that \(\pi(Q(0), h(0)) = p\) and that \((Q'(0), h'(0))\) is a multiple of \(v\). The choice of \(h\) implies moreover that \(S_1 \cup S_2\) is contained in the image of \(z \mapsto \pi(Q(z), h(z))\).

Next we define a continuous positive function \(\epsilon : A \to \mathbb{R}^+\) as follows:

- \(\epsilon \equiv 1\) on \(Q^{-1}(B_{\eta}(p_1))\) if \(p \notin \Sigma\).
- \(\epsilon \equiv \frac{1}{2}\delta'\) on \(Q^{-1}(B_{\rho''}(0))\).
- \(\epsilon(z) = \frac{1}{n}\) if \(Q(z) = a_n\).
- \(\epsilon(z) = \min\{\frac{1}{n}, \frac{1}{2}(\delta - \lvert d_n - s(c_n)\rvert)\}\) whenever \(\lvert Q(z) - \gamma_n\rvert \leq \rho'\).

Using prop. 2, we deduce that there exists an entire function \(F : \mathbb{C} \to \mathbb{C}\) such that

(1) \(|F(z) - h(z)| < \epsilon(z)\) for all \(z \in A\).
(2) \(F(0) = h(0)\) and \(F'(0) = h'(0)\).

By the second condition we obtain that \(\pi(Q(0), F(0)) = p\) and that \((Q'(0), F'(0))\) is a multiple of \(v\). The first condition ensures that \(\pi(Q(z), F(z)) \in \Omega\) for all \(z \in \mathbb{C}\). It also ensure that the image is dense: Indeed, let \(w \in \Omega_2\). Then there is a sequence of points in \(S_2\) converging to \(w\). But \(S_2 = \{\pi(a_n, b_n) : n \in \mathbb{N}\}\) and the construction of \(F\) implies that for every \(n \in \mathbb{N}\) there exists a number \(z_n \in \mathbb{C}\) such that \(Q(z_n) = a_n\) and \(|F(z_n) - b_n| < \frac{1}{n}\). It follows that there is a subsequence \(z_{n_k}\) such that \(\lim_k \pi(Q(z_{n_k}), F(z_{n_k})) = w\). If \(w \in \Sigma\), we argue similarly, with \(S_1\) in the role of \(S_2\). Thus the whole set \(\Omega_1\) is in the closure of the image of the map \(z \mapsto \pi(Q(z), F(z))\) from \(\mathbb{C}\) to \(T\).

Finally, let \(\mu\) be a complex number such that \(\mu(Q'(0), F'(0)) = v\) and define

\[ f(z) = \pi(Q(\mu z), F(\mu z)) \]

Then \(f : \mathbb{C} \to \Omega_1\) is a holomorphic map with the desired properties.

### 3. Arakelyan Approximation with interpolation

We will need a slight improvement of Arakelyan’s theorem. We recall the theorem of Arakelyan (see [1]):

**Theorem 3.** Let \(A\) be a closed subset of \(\mathbb{C}\), \(U = \mathbb{P}_1(\mathbb{C}) \setminus A\), \(\epsilon : A \to \mathbb{R}^+\) a continuous function and \(f_0 : A \to \mathbb{C}\) a continuous function which is holomorphic in the interior of \(A\). Assume that \(U\) is connected and locally connected.

Then there exists a holomorphic function \(F : \mathbb{C} \to \mathbb{C}\) with \(|F(z) - f(z)| < \epsilon(z)\) for all \(z \in A\).

We want to verify that Arakelyan’s theorem is applicable in our situation.
Proposition 1. Let $\Gamma$ be a lattice in $\mathbb{C}$ and $\rho'$ a real number with

$$0 < \rho' < \rho = \frac{1}{2} \min_{\gamma \in \mathbb{C} \setminus \{0\}} |\gamma|$$

Let $A' = \{z \in \mathbb{C} : d(z, \Gamma) \leq \rho'\} = \bigcup_{\gamma \in \mathbb{C}} B_{\rho'}(\gamma)$, $P : \mathbb{C} \to \mathbb{C}$ a non-constant polynomial and $U = \{\infty\} \cup (\mathbb{C} \setminus P^{-1}(A'))$.

Then $U$ is connected and locally connected.

Proof. First we want to verify that $U$ contains no bounded connected component. Indeed, assume that there is such a connected component $C$. Its boundary $\partial C$ is a connected set mapped into $\bigcup_{\gamma \in \mathbb{C}} B_{\rho'}(\gamma)$ by $P$. This is a disjoint union due to the choice of $\rho'$. Hence continuity of $P$ implies that there is one element $\gamma \in \mathbb{C}$ such that $|P(z) - \gamma| \leq \rho'$ for all $z \in \partial C$.

But $C \subset U$ implies $|P(z) - \gamma| > \rho'$ for all $z \in C$, $\gamma \in \Gamma$. This is in contradiction with the maximum principle for the holomorphic function $P$. Hence there can not exist a bounded connected component $C \subset U$.

For each $n \in \mathbb{N}$ we choose a simple closed curve $R_n \subset \mathbb{C} \setminus A'$ such that the open bounded subset $V_n \subset \mathbb{C}$ which is enclosed by $R_n$ has the property that $B_n(\mathbb{C}, 0) \subset V_n$.

The ramification locus

$$Z = \{z \in \mathbb{C} : \exists w \in \mathbb{C} : P(w) = z, P'(w) = 0\}$$

is a finite set. Let $N_0 = \max\{|z| : z \in Z\}$. Now the restriction of $P$ to $P^{-1}(\mathbb{C} \setminus V_n) \to \mathbb{C} \setminus V_n$ is an unramified covering of degree $d = \deg(P)$.
for all \( n > N_0 \). As a polynomial map, \( P \) extends to a proper map \( P : \mathbb{C} \cup \{ \infty \} \to \mathbb{C} \cup \{ \infty \} \). For a suitably chosen local coordinate \( w \) at \( \infty \) the map \( P \) near \( \infty \) can be described as \( w \mapsto w^d \). Using this fact and the fact that by construction each curve \( R_n \) defines a generator for \( \pi_1(\mathbb{C}^*) \cong \pi_1(\mathbb{C} \setminus B_n(\mathbb{C},0)) \)

we can conclude that \( P^{-1}(R_n) \) is connected for all \( n > N_0 \). Then \( P^{-1}(\Omega) \) is connected for every open subset \( \Omega \subset \mathbb{C} \setminus B_n(\mathbb{C},0) \) with \( R_n \subset \Omega \).

In particular

\[
W_n = U \setminus P^{-1}(V_n)
\]

is connected for all \( n > N_0 \). The collection of all these open sets \( W_n \) constitutes an neighborhood basis of \( U \) at \( \infty \), implying that \( U \) is locally connected at infinity. Furthermore, the connectedness of the sets \( W_n \) implies that \( U \) is only one unbounded connected component. Since we have already seen that \( U \) is no bounded connected component, this completes the proof that \( U \) is connected and locally connected. \( \square \)

**Proposition 2.** Let \( A \) be a closed subset in \( \mathbb{C} \), \( A \neq \mathbb{C} \), and suppose that for every function \( f \) on \( A \) which is holomorphic in its interior and every continuous map \( \varepsilon : A \to \mathbb{R}^+ \) there is an entire function \( F : \mathbb{C} \to \mathbb{C} \) with \( |F(z) - f(z)| < \varepsilon(z) \) for all \( z \in A \).

Let \( q \) be a point in the interior of \( A \). Then we can find such an entire function \( F \) with the additional properties \( F(q) = f(q) \) and \( F'(q) = f'(q) \).

**Proof.** Let \( U = \{ \infty \} \cup (\mathbb{C} \setminus A) \). By assumption \( U \) is connected and locally connected at infinity. Let \( p \in \mathbb{C} \setminus A \) and let \( W \) be a bounded connected open subset of \( \mathbb{C} \) containing both \( p \) and \( q \). Choose \( \delta > 0 \) such that

\[
\delta < \min \{ d(q, \partial W), d(q, \partial A), d(p, q) \}
\]

and define

\[
\tilde{A} = \overline{B_\delta(q)} \cup A \setminus W
\]

and

\[
\tilde{U} = \{ \infty \} \cup \left( \mathbb{C} \setminus \tilde{A} \right) = U \cup \left( W \setminus \overline{B_\delta(q)} \right).
\]

Now both \( U \) and \( \left( W \setminus \overline{B_\delta(q)} \right) \) are connected, and their intersection is non-empty, since it contains \( p \). Therefore \( \tilde{U} \) is connected. Moreover, \( \tilde{U} \) is locally connected at infinity, because it coincides with \( U \) near \( \infty \). Thus we have Arakelyan approximation for \( \tilde{A} \).
We choose constants $\xi_0, \xi_1 \in \mathbb{C} \setminus \{0\}$ such that
\[ |\xi_0| < \frac{1}{16} \epsilon(z) \]
and
\[ |\xi_1(z - q)| < \frac{1}{16} \epsilon(z) \]
for all $z \in \overline{B_\delta(q)}$.

Then we define functions $g, h : \tilde{A} \to \mathbb{C}$ via
\[ g(z) = \begin{cases} 
\xi_0 & \text{if } z \in \overline{B_\delta(q)} \\
0 & \text{else}
\end{cases} \]
and
\[ h(z) = \begin{cases} 
\xi_1(z - q) & \text{if } z \in \overline{B_\delta(q)} \\
0 & \text{else}
\end{cases} \]

Clearly, $g$ and $h$ are continuous and holomorphic in the interior of $\tilde{A}$. The choice of $\xi_0, \xi_1$ implies that $|g(z)| < \frac{1}{10} \epsilon(z)$ and $|h(z)| < \frac{1}{10} \epsilon(z)$ for all $z \in A$.

By the Arakelyan property we find sequences of entire functions $g_n, h_n : \mathbb{C} \to \mathbb{C}$ such that
\[ |g_n(z) - g(z)| < \frac{1}{8n} \epsilon(z) \]
for all \( n \in \mathbb{N} \), \( z \in \hat{A} \). Locally uniform convergence on \( \hat{A} \) implies that inside the interior of \( \hat{A} \) the derivatives converge as well. Hence we obtain
\[
\lim_{n \to \infty} \begin{pmatrix} g_n(q) & h_n(q) \\ g'_n(q) & h'_n(q) \end{pmatrix} = \begin{pmatrix} g(q) & h(q) \\ g'(q) & h'(q) \end{pmatrix} = \begin{pmatrix} \xi_0 & 0 \\ 0 & \xi_1 \end{pmatrix}.
\]
Thus, for \( n \) sufficiently large the vectors \( (g_n(q), g'_n(q)) \) and \( (h_n(q), h'_n(q)) \) are linearly independent.

Next we observe that \( A \setminus \hat{A} \) is relatively compact in \( A \). Therefore, for sufficiently large numbers \( n, C \) the functions \( \alpha = \frac{1}{C} g_n \) and \( \beta = \frac{1}{C} h_n \) have the following properties:

1. \( \alpha, \beta \) are entire functions,
2. \( |\alpha(z)|, |\beta(z)| < \frac{1}{8} \epsilon(z) \) for all \( z \in A \), and
3. the vectors \( (\alpha(q), \alpha'(q)) \) and \( (\beta(q), \beta'(q)) \) are linearly independent.

By the approximation property for \( A \) there are sequences of entire functions \( \alpha_n, \beta_n, f_n : \mathbb{C} \to \mathbb{C} \) such that
\[
\max\{|\alpha_n(z) - \alpha(z)|, |\beta_n(z) - \beta(z)|, |f_n(z) - f(z)|\} < \frac{1}{n} \epsilon(z)
\]
for all \( n \in \mathbb{N} \), \( z \in A \). The locally uniform convergence of \( \lim \alpha_n = \alpha \), \( \lim \beta_n = \beta \) and \( \lim f_n = f \) on \( A \) implies that in the interior of \( A \) the respective derivatives converge as well. In particular, this happens at \( q \). Hence the matrix
\[
A_n = \begin{pmatrix} \alpha_n(q) & \beta_n(q) \\ \alpha'_n(q) & \beta'_n(q) \end{pmatrix}
\]
converges to
\[
\lim_{n \to \infty} A_n = A = \begin{pmatrix} \alpha(q) & \beta(q) \\ \alpha'(q) & \beta'(q) \end{pmatrix}.
\]
Since \( A \) is invertible, it follows that \( A_n \) is likewise invertible for all sufficiently large \( n \). Hence we can define (for sufficiently large \( n \)) sequences \( \lambda_n, \mu_n \) via
\[
\begin{pmatrix} \lambda_n \\ \mu_n \end{pmatrix} = A_n^{-1} \cdot \begin{pmatrix} f(q) - f_n(q) \\ f'(q) - f'_n(q) \end{pmatrix}.
\]
Now \( \lim f_n = f \), \( \lim f'_n = f' \) and \( \lim A_n^{-1} = A^{-1} \). Therefore \( \lim \lambda_n = 0 = \lim \mu_n \).

Thus we can choose a natural number \( N \in \mathbb{N} \) with the following properties:
(1) $A_N$ is invertible,
(2) $|\lambda_N|, |\mu_N| < 1$,
(3) and $N > 4$.

We define
\[ F(z) = f_N(z) + \lambda_N \alpha_N(z) + \mu_N \beta_N(z). \]

By the choice of $\lambda_n, \mu_n$ we have
\[
\begin{pmatrix}
F(q) \\
F'(q)
\end{pmatrix} = \begin{pmatrix}
f_N(q) + \lambda_N \alpha_N(q) + \mu_N \beta_N(q) \\
f'_N(q) + \lambda_N \alpha'_N(q) + \mu'_N \beta'(q)
\end{pmatrix} = \begin{pmatrix}
f_N(q) \\
f'_N(q)
\end{pmatrix} + A_N \cdot \begin{pmatrix}
\lambda_N \\
\mu_N
\end{pmatrix} = \begin{pmatrix}
f(q) \\
f'(q)
\end{pmatrix}.
\]

Furthermore
\[
|F(z) - f(z)| \leq |f_N(z) - f(z)| + |\lambda_N| \cdot |(\alpha_N(z) - \alpha(z)) + |\alpha(z))| +
+ |\mu_N| \cdot |(\beta_N(z) - \beta(z)) + |\beta(z))|
\leq \frac{1}{N} \epsilon(z) + \frac{1}{N} \epsilon(z) + \frac{1}{8} \epsilon(z) + \frac{1}{N} \epsilon(z) + \frac{1}{8} \epsilon(z) < \left( \frac{3}{4} + \frac{2}{8} \right) \epsilon(z) = \epsilon(z)
\]
for all $z \in A$. Thus $F$ is an entire function with the desired properties. \hfill \Box

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