PAIR-OF-PANTS DECOMPOSITIONS OF 4-MANIFOLDS DIFFEOMORPHIC TO GENERAL TYPE HYPERSURFACES

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Abstract. In this paper, we show that a smooth 4-manifold diffeomorphic to a complex hypersurface in \( \mathbb{CP}^3 \) of degree \( d \geq 5 \) can be decomposed as the union of \( d(d-4)^2 \) copies of 4-dimensional pair-of-pants and certain subsets of K3 surfaces.

1. Introduction

A compact Riemann surface \( \Sigma \) admits Riemannian metrics of constant Gaussian curvature, while a 3-dimensional manifold may not admit any constant curvature metric. Instead the Perelman’s theorem on the Thurston’s geometrisation conjecture asserts that a 3-manifold can be canonically decomposed into domains, and some of them carry complete constant curvature Riemannian metrics of finite volume (cf. [46],[30]). A difference is that unlike the 3-dimensional case, if the genus of a Riemann surface is bigger than one, the hyperbolic metrics are not unique, and form a moduli space. However \( \Sigma \) admits so called pair-of-pants decompositions which restore certain aspects of the uniqueness (It is a standard topic in topology. See ‘Pair of pants (mathematics)’ in www.wikipedia.org).

A pair-of-pants \( \mathcal{P}^1 \) of real dimension two, or complex dimension one, is defined as the complement set of three generic points in \( \mathbb{CP}^1 \), i.e. \( \mathcal{P}^1 = \mathbb{CP}^1 \setminus \{0, 1, \infty\} \). There is a unique complete Riemannian metric \( g \) on \( \mathcal{P}^1 \) with Gaussian curvature \(-1\) and finite volume \( \text{Vol}_g(\mathcal{P}^1) = 2\pi \). There is a fibration structure \( \mathcal{P}^1 \to Y \) from \( \mathcal{P}^1 \) to a graph of \( Y \)-shape with generic fibres \( S^1 \), and one singular fibre of shape \( \ominus \) over the vertex.

If \( \Sigma \) is a compact Riemann surface of genus \( g \geq 2 \), then there is an open subset \( \Sigma^o \subset \Sigma \) consisting exactly of \( 2g - 2 \) copies of pair-of-pants, and the complement \( \Sigma \setminus \Sigma^o \) is the disjoint union of \( 3g - 3 \) circles. Each circle is not homotopic to a constant curve in \( \Sigma \). Therefore \( \Sigma \) is decomposed into canonical local pieces, the pair-of-pants, glued along circles. The Gauss-Bonnet formula expresses the Euler number via the volume of \( g \)

\[
\chi(\Sigma) = \frac{1}{2\pi} \sum_{1}^{2g-2} \text{Vol}_g(\mathcal{P}^1) = \frac{1-g}{\pi} \text{Vol}_g(\mathcal{P}^1).
\]
Figure 1.1. Fibration $\mathcal{P}^1 \rightarrow Y$, and two graphs associated with pair-of-pants decompositions of Riemann surfaces of genus 2.

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The combinatorial structure of the decomposition $\Sigma = \Sigma^o \cup (\Sigma \setminus \Sigma^o)$ is represented by a cubic graph, i.e. any vertex is of shape $Y$, where each pair-of-pants corresponds to one vertex, and any edge associates with a circle in $\Sigma \setminus \Sigma^o$.

In [40], Mikhalkin has generalised the notion of pair-of-pants to the case of any even dimension, and proved that a smooth complex hypersurface in $\mathbb{CP}^{n+1}$, and more general toric manifolds, admits pair-of-pants decompositions. [24] studied pair-of-pants decompositions for real 4-dimensional manifolds from the topology perspective. It is shown in [24] that every finitely presented group is the fundamental group of a 4-manifold admitting pair-of-pants decompositions. In this paper, we study some differential/algebraic geometry aspects of pair-of-pants decompositions for general type hypersurfaces in $\mathbb{CP}^3$.

A pair-of-pants $\mathcal{P}^2$ of real dimension 4, equivalently complex dimension 2, is defined as the complement of 4 general positioned lines $D$ in $\mathbb{CP}^2$, i.e. $\mathcal{P}^2 = \mathbb{CP}^2 \setminus D$, where $D$ can be chosen as

\[ D = \{ [z_0, z_1, z_2] \in \mathbb{CP}^2 | (z_0 + z_1 + z_2)z_0z_1z_2 = 0 \}. \]

Equivalently,

\[ \mathcal{P}^2 = \{ (w_1, w_2) \in (\mathbb{C}^*)^2 | 1 + w_1 + w_2 \neq 0 \}. \]

If the compact pair-of-pants is defined as $\mathcal{P}^2 = \mathbb{CP}^2 \setminus \tilde{D}$, where $\tilde{D}$ denotes the union of small tubular open neighbourhoods of the 4 generic lines, then $\mathcal{P}^2 \subset \mathcal{P}^2$, and the interior $\text{int}(\mathcal{P}^2)$ is diffeomorphic to $\mathcal{P}^2$. The boundary $\partial \mathcal{P}^2$ consists of 6 copies of 2-torus $T^2$ and 4 copies of the total space $F$ of the trivial $S^1$-bundle over $\mathcal{P}^1$. By composing with the fibration $\mathcal{P}^1 \rightarrow Y$, $F \rightarrow Y$ is a fibration with generic fibres $T^2$, and one singular fibre of shape $\Theta \times \bigcirc$ over the vertex of $Y$.

As in the case of Riemann surfaces, $\mathcal{P}^2$ carries a natural complete Einstein metric of finite volume. Many works were devoted to generalise Yau and Aubin’s theorem ([69], [70], [8]) on the Calabi conjecture of projective manifolds with negative first Chern class to the quasi-projective case under various conditions (cf. [16], [82], [83], [62] etc.). In the current case, $\mathcal{P}^2$ is a quasi-projective manifold, and the log-canonical divisor $K_{\mathbb{CP}^2} + D = H$ is ample, where $H$ denotes the hyperplane class. Since $D$ has only simple normal
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**Figure 1.2.** 4–dimensional pair-of-pants $\mathcal{P}^2 = \mathbb{CP}^2 \setminus D$

![Diagram of 4-dimensional pair-of-pants $\mathcal{P}^2 = \mathbb{CP}^2 \setminus D$]

crossing singularities, Theorem 1 in [32] shows that there is a complete Kähler-Einstein metric $\omega$ on $\mathcal{P}^2$ with Ricci curvature $-1$ and finite volume $2\pi^2(K_{\mathbb{CP}^2} + D)^2 = 2\pi^2$.

If $X$ is a smooth complex hypersurface in $\mathbb{CP}^3$ of degree $d \geq 5$, then the canonical divisor $K_X$ of $X$ is very ample, and more precisely the canonical bundle $\mathcal{O}_X(K_X) \cong \mathcal{O}_{\mathbb{CP}^3}(d-4)|_X$ by the adjunction formula. $X$ is a minimal algebraic surface of general type. The Yau-Aubin-Calabi theorem asserts that there is a unique Kähler-Einstein metric with Ricci curvature $-1$ and volume $2\pi^2 K^2_X = 2\pi^2 d(d-4)^2$. When we deform the complex structure of $X$, the Kähler-Einstein metric varies along the deformation. Therefore, if $M$ is a smooth 4-manifold diffeomorphic to $X$, Kähler-Einstein metrics on $M$ are not unique, and form a moduli space in a certain sense, which is a situation analogous to the case of Riemann surfaces of positive genus.

The Mikhalkin’s theorem in [40] asserts that $X$ admits a pair-of-pants decomposition consisting of $d^3$ copies of pair-of-pants, i.e. there is an open dense subset $X^0 \subset X$ diffeomorphic to the disjoint union of $d^3$ copies of pair-of-pants $\mathcal{P}^2$. The number of pair-of-pants is bigger than that we expect as shown in the following example.

We consider a family of hypersurfaces $X_t$ in $\mathbb{CP}^3$ of degree 5 defined by

$$\sum_{i=0}^{3} z_i^5 + t(z_0 + z_1 + z_2 + z_3) \prod_{j=0}^{3} z_j = 0, \quad [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3,$$

where $t \in [0, \infty)$. When $t \to \infty$, $X_t$ tends to the singular variety $X_0$ given by

$$(z_0 + z_1 + z_2 + z_3)z_0z_1z_2z_3 = 0,$$

which is the union of 5 generic hyperplanes in $\mathbb{CP}^3$, and the regular locus $X_0^0$ consists of 5 copies of pair-of-pants $\mathcal{P}^2$. We can diffeomorphically embed $X_0^0 \hookrightarrow X_t$ for $t \gg 1$, and obtain a decomposition of $X_t$ with 5 copies of pair-of-pants. And $5 < 5^3 = 125$. Furthermore, the number 5 is the intersection number $K^2_{X_t}$.

The following theorem shows that the number of pair-of-pants can be reduced to the expected one, if we allow components of other types appearing in the decomposition.
Theorem 1.1. Let $M$ be a 4-manifold diffeomorphic to a hypersurface $X$ in $\mathbb{C}P^3$ of degree $d \geq 5$, and $p_g$ be the geometric genus of $X$.

i) There is an open subset $M^o \subset M$ such that $M^o$ is diffeomorphic to the disjoint union of $d(d-4)^2$ copies of pair-of-pants $\mathcal{P}^2$, i.e.
$$M^o = \coprod d(d-4)^2 \mathcal{P}^2.$$ 

ii) There is a fibration $\lambda: \overline{M^o} \setminus M^o \to B$ onto a graph $B$ with generic fibres 2-torus $T^2$ and finite singular fibres of shape $\ominus \times \ominus$, where $\overline{M^o}$ denotes the closure of $M^o$ in $M$.

iii) $M \setminus \overline{M^o}$ is the union of subsets of the $K3$ surface $Y$ including $(\mathbb{C}^*)^2$ and $\mathbb{C} \times \mathbb{C}^*$. Furthermore, there is an open subset $M' \subset M \setminus \overline{M^o}$ admitting a smooth embedding $\iota: M' \to Y$ such that the closure of the image is the $K3$ surface, i.e. $\iota(M') = Y$.

iv) Any generic fibre $T^2$ of $\lambda$ represents a non-trivial homological class of $M$, i.e.
$$0 \neq \left[T^2\right] \in H_2(M, \mathbb{R}).$$
Moreover, if $\omega$ is a symplectic form representing $c_1(\mathcal{O}_{\mathbb{C}P^3}(1))|_X$, then
$$\int_{T^2} \omega = 0.$$ 

v) If $\chi(M)$ is the Euler number of $M$, and $\tau(M)$ denotes the signature of $M$, then
$$2\chi(M) + 3\tau(M) = \frac{d(d-4)^2}{2\pi^2} \text{Vol}_\omega(\mathcal{P}^2),$$
where $\omega$ is the complete Kähler-Einstein metric with Ricci curvature $-1$ on $\mathcal{P}^2$.

vi) There is a family $X_t$, $t \in [0, \infty)$, of degree $d$ hypersurfaces in $\mathbb{C}P^3$, and there are embeddings $\Psi_t: X_t \to \mathbb{C}P^{p_g-1}$ with $\mathcal{O}_{X_t}(K_{X_t}) = \Psi_t^* \mathcal{O}_{\mathbb{C}P^{p_g-1}}(1)$, such that $\Psi_t(X_t)$ converges to a singular variety $X_0$ in $\mathbb{C}P^{p_g-1}$ as analytic subsets, when $t \to \infty$, and the regular locus of $X_0$ is diffeomorphic to $M^o$.

In this theorem, we also regard certain subsets of the complex torus $(\mathbb{C}^*)^2$, $\mathbb{C} \times \mathbb{C}^*$, and the $K3$ surface as basic building blocks besides the pair-of-pants. It has been explored to decompose general type hypersurfaces into Calabi-Yau components in [39]. The pair-of-pants part $M^o$ is an analogue of hyperbolic pieces, and the complement $M \setminus \overline{M^o}$ plays a similar role as graph manifolds in the decompositions of 3-manifolds.

The conclusions ii) and iv) resemble the facts that hyperbolic pieces are glued with other parts along incompressible tori in the 3-dimensional case, and the circles in the pair-of-pants decompositions of Riemann surfaces represent non-trivial classes in the fundamental group. For example, $\mathbb{C}P^2$ admits a decomposition $\mathbb{C}P^2 = \mathcal{P}^2 \cup \overline{D}$, and the generic fibre $T^2$ in $\partial \mathcal{P}^2$ represents the zero class in $H_2(\mathbb{C}P^2, \mathbb{R})$. At least in this case, iv) plays a similar
role as those incompressible tori in the 3-dimensional decomposition. Unfortunately, iv) is much weaker than either cases as the homological group is used.

The assertion v) expresses the Hitchin-Thorpe inequality via the volume of the Kähler-Einstein metric on the pair-of-pants, and generalises (\ref{thm:main}). vi) shows that the expected pair-of-pants appear in a classical algebro-geometric way. What happens is that the $d(d-4)^2$ copies of pair-of-pants obtained in i) converge nicely to the regular locus of $X_0$, while those K3 components are crushed into the singularities of $X_0$.

We are motivated by a paper \[14\] of Cheeger and Tian about the collapsing of Einstein 4-manifolds, and we recall the relevant results in \[14\]. Let $g_i$ be a sequence of Einstein metrics on a 4-manifold $M$ with Ricci curvature $\text{Ric}(g_i) = -g_i$ and volume $\text{Vol}_{g_i}(M) \equiv \text{const.}$, which could be obtained by applying Yau and Aubin’s theorem on the Calabi conjecture to hypersurfaces $X$ of degree $d \geq 5$. Theorem 10.5 of \[14\] shows that by passing to a subsequence and choosing certain base points, $(M, g_i)$ converges to $\prod_{\nu=1}^{K} (M_{\nu}, g_\infty)$, when $i \to \infty$, in the pointed Gromov-Hausdorff sense, and

$$\text{Vol}_{g_i}(M) = \sum_{\nu=1}^{K} \text{Vol}_{g_\infty}(M_{\nu}),$$

where each $M_\nu$ admits at most finite isolated orbifold points as singularities, and $g_\infty$ are complete negative Einstein metrics in the orbifold sense. Consequently, for $i \gg 1$, $g_i$ induces a thick-thin decomposition $M = M_0 \cup (M \setminus M_0)$, where the thick part $(M_0, g_i)$ is close to the pointed Gromov-Hausdorff limit $\prod_{\nu=1}^{K} (M_{\nu}, g_\infty)$, and the thin part $M \setminus M_0$ supports metrics with bounded Ricci curvature and volume collapsing. Moreover, $g_i$ has bounded sectional curvature on a subset $M' \subset M \setminus M_0$ by Theorem 0.8 of \[14\], and therefore $M'$ admits an $F$-structure of positive rank in the Cheeger-Gromov sense (cf. \[12\] \[13\]).

In Theorem 1.1, $M^0$ is expected to be the thick part, the complement $M \setminus M^0$ should be the thin part, and the fibration $M^0 \setminus M^0 \to B$ is an example of $F$-structure of positive rank. The limiting behaviour of negative Kähler-Einstein metrics along degenerations of algebraic manifolds with negative first Chern class has been studied under various hypotheses (cf. \[58\] \[48\] \[49\] \[50\] \[71\] \[53\] and the references in them), which can certainly be applied to the current situation. Especially, \[50\] has explored the connection between pair-of-pants degenerations and the convergence of Kähler-Einstein metrics.

We outline the paper briefly. Section 2 presents a variant of Theorem 1.1 which shows a correlation between the appearance of pair-of-pants and the negativity of scalar curvature of metrics allowed on 4-manifolds in the current case. In Section 3, we give some further remarks about the geometry of pair-of-pants. We prove the assertions i), ii), iii), and v) of Theorem 1.1.
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in Section 4, which depends on Mikhalkin’s work \[40\] and tropical geometry. Finally, Section 5 proves the conclusions iv) and vi) of Theorem 1.1.

In this paper, we use the Riemannian geometric convention of scalar curvature instead of the Kähler geometry one, i.e. the scalar curvature of a metric on a Riemann surface equals to the twice of the Gaussian curvature.

2. Yamabe invariant

In the case of Riemann surfaces, (1.1) shows a strong correlation between the appearance of pair-of-pants and the negativity of scalar curvature of metrics allowed on the Riemann surfaces. However this information is lost in v) of Theorem 1.1. The goal of this section is to present a variant of v) in Theorem 1.1 which restores this correlation.

First, we recall the definition of the Yamabe invariant. Let \( M \) be a smooth compact real manifold of dimension \( n \). For any conformal class \( c \) on \( M \), the Yamabe constant of \( c \) is defined as

\[
σ(M, c) = \inf_{g ∈ c} \left( \text{Vol}_g \frac{2-n}{n} (M) \int_M R_g dv_g \right),
\]

where \( R_g \) denotes the scalar curvature of \( g \). The supremum

\[
σ(M) = \sup_{c ∈ C} σ(M, c)
\]

is a diffeomorphism invariant, and is called the Yamabe invariant of \( M \) (cf. \[51, 31\]), where \( C \) denotes the set of conformal classes on \( M \). If \( n = 2 \), the Gauss-Bonnet formula implies that the Yamabe invariant equals to \( 2π\chi(M) \).

When \( n = 3 \), a Perelman’s theorem (Section 8.2 in \[46\], and see also Proposition 93.9 in \[30\]) answers a conjecture of Anderson (cf. \[4\]), and says that if \( σ(M) < 0 \), then there is an open subset \( M_0 ⊂ M \) admitting a complete hyperbolic metric \( g \) with sectional curvature \( -1 \), and the Yamabe invariant is realised by the volume of \( g \), i.e.

\[
σ(M) = -6\text{Vol}_g^\frac{2}{3} (M_0).
\]

Actually, Perelman used the \( \bar{λ} \)-invariant introduced in \[45\], which was proved to equal to the Yamabe invariant when \( σ(M) ≤ 0 \) by \[1\].

In the case of dimension 4, LeBrun proved an analogous equality in \[36, 37\] for manifolds diffeomorphic to minimal Kähler surfaces of non-negative Kodaira dimension. We take this opportunity to give an alternative proof of the case of general type surfaces.

**Theorem 2.1** (Theorem 7 in \[36\]). Let \( M \) be a smooth 4-manifold diffeomorphic to a minimal projective surface \( X \) of general type, i.e. the canonical divisor \( K_X \) of \( X \) is nef and big \( K_X^2_X > 0 \). Then the Yamabe invariant

\[
σ(M) = -\sqrt{32π^2K_X^2_X} = -4\sqrt{\text{Vol}_ω(X_0)},
\]

where \( ω \) is a Kähler-Einstein metric with Ricci curvature \( -1 \) on an open subset \( X_0 ⊂ X \).
An alternative proof. We estimate the upper bound and the lower bound of the Yamabe invariant $\sigma(M)$, and show that they match. The upper bound follows LeBrun’s argument via the Seiberg-Witten equation with a minor twist, and we present it here for the completeness. The difference is to prove the lower bound where we use the Kähler-Ricci flow instead of the original approach in [36].

First, we recall the reinterpretation of the Yamabe invariant due to Lott and Kleiner [30]. For any 4-manifold $M$, (93.6) in [30] asserts that if $\sigma(M) \leq 0$, then

$$\sigma(M) = \sup_{g \in M} \hat{R}(g) \sqrt{\text{Vol}_g(M)},$$

where $\hat{R}(g)$ denotes the minimal of the scalar curvature of $g$, i.e.

$$\hat{R}(g) = \inf_{x \in M} R(g)(x).$$

If $c$ is a Spin$^c$ structure on $M$, and $S^+_c$ denotes the Spin$^c$ bundle, then the Seiberg-Witten equation is introduced in [67], and reads

$$D_A \phi = 0, \quad F^+_A = q(\phi),$$

for an unknown positive spinor $\phi$ and an unknown $U(1)$-connection $A$ on the determinant bundle $\mathcal{L}$ of $c$, where $D_A$ denotes the Dirac operator, $F_A$ is the curvature of $A$, and $q(\phi)$ is a quadratic form of $\phi$ satisfying $|q(\phi)|^2 = \frac{1}{8} |\phi|^4$.

The Seiberg-Witten invariant $SW_M(c)$ is a diffeomorphism invariant defined via the moduli space of the solutions $(\phi, A)$ module $U(1)$-gauge changes (cf. [67, 42]). For example, if $M$ is diffeomorphic to a minimal Kähler surface $X$ of general type, and $L$ is the anti-canonical bundle $L = \mathcal{O}_X(K_X^{-1})$, then $SW_M(c) \neq 0$ by [35], which implies that for any Riemannian metric $g$, the Seiberg-Witten equation has a solution $(\phi, A)$. Since $c_1^2(\mathcal{L}) = K_X^2 > 0$, $F^+_A$ and $\phi$ are not identical to zero.

The Weitzenböck formula says

$$4\nabla^*_A \nabla_A \phi + R(g)\phi + |\phi|^2 \phi = 0.$$

By producing with $\phi$ and integration, we have

$$\hat{R}(g) \int_M |\phi|^2 dv_g \leq \int_M (4|\nabla_A \phi|^2 + R(g)|\phi|^2) dv_g = -\int_M |\phi|^4 dv_g,$$

and by the Schwarz inequality,

$$\hat{R}(g) \sqrt{\text{Vol}_g(M)} \leq -\sqrt{\int_M |\phi|^4 dv_g} = -\sqrt{8 \int_M |F^+_A|^2 dv_g} \leq -\sqrt{32 \pi^2 c_1^2(\mathcal{L})}.$$

Therefore, $M$ does not admit any Riemannian metric of positive scalar curvature, and the Yamabe invariant $\sigma(M) \leq 0$. We obtain the upper bound

$$\sigma(M) \leq -\sqrt{32 \pi^2 K_X^2},$$
by taking the supremum and (2.1) (See [19] for another proof via the Perelman’s $\bar{\lambda}$-functional).

Now we prove the lower bound by considering the Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) - \omega_t, \quad t \in [0, T),$$

on $X$ with an initial Kähler metric $\omega_0$. This version of Kähler-Ricci flow was intensely studied in recent years for the differential geometric understanding of the minimal model programme (cf. [65, 63, 54, 64] and the references in them). See also [20, 21] for some interactions between the real Ricci flow and the Seiberg-Witten equation.

Let $X_{\text{can}}$ be the canonical model of $X$, and $\pi : X \to X_{\text{can}}$ be the contraction map. Note that $X_{\text{can}}$ is a 2-dimensional normal variety with only finite A-D-E singularities, and $-c_1(X) = c_1(\mathcal{O}_X(K_X)) = \pi^*\alpha$ for an ample class $\alpha$ on $X_{\text{can}}$. In [65, 63], it is proved that the solution $\omega_t$ exists for a long time $t \in [0, \infty)$, i.e. $T = \infty$, and $\omega_t$ converges to a semi-positive current $\omega$ presenting $-2\pi c_1(X)$ with bounded local potential functions when $t \to \infty$.

Furthermore, $\omega$ is a smooth Kähler-Einstein metric with Ricci curvature $-1$ on $\pi^{-1}(X_{\text{can}}^o)$, where $X_{\text{can}}^o$ is the regular locus of $X_{\text{can}}$, and

$$2\pi^2 K_X^2 = 2\pi^2 c_1^2(X) = \text{Vol}_\omega(X_{\text{can}}^o).$$

The evolution equation of scalar curvature is

$$\frac{\partial R_t}{\partial t} = \Delta_t R_t + 2|\text{Ric}_t|^2 + R_t = \Delta_t R_t + 2|\text{Ric}_t^0|^2 - (R_t + 4)$$

(cf. Lemma 2.38 in [15]), where $R_t = R(\omega_t)$, $\text{Ric}_t = \text{Ric}(\omega_t)$ and $\text{Ric}_t^0 = \text{Ric}_t + \omega_t$, which satisfies $|\text{Ric}_t^0|^2 = |\text{Ric}_t|^2 + R_t + 2$ by $R_t = 2\text{tr}_{\omega_t}\text{Ric}_t$. By the maximal principle, the minimal $\bar{R}_t$ of $R_t$ satisfies

$$\frac{\partial \bar{R}_t}{\partial t} \geq - (\bar{R}_t + 4),$$

and therefore we obtain

$$\bar{R}_t \geq -4 - Ce^{-t} \to -4,$$

for a constant $C > 0$ independent of $t$.

Since the Chern-Weil theory shows that the cohomological classes evolve as

$$\frac{\partial [\omega_t]}{\partial t} = -2\pi c_1(X) - [\omega_t],$$

we solve the ordinary differential equation and obtain

$$[\omega_t] = -2\pi c_1(X) + e^{-t}(2\pi c_1(X) + [\omega_0]).$$

Thus the volumes

$$\text{Vol}_{\omega_t}(X) = \frac{1}{2} \int_X \omega_t^n = 2\pi^2 K_X^2(X) = 2\pi^2 K_X^2,$$
when \( t \to \infty \). We obtain the lower bound of the Yamabe invariant

\[
\sigma(M) \geq \lim_{t \to \infty} \tilde{R}_t \sqrt{\text{Vol}_{\omega}(X)} = -\sqrt{32\pi^2 K_X^2},
\]

which is the same as the upper bound. We obtain the conclusion by letting

\[X_0 = \pi^{-1}(X^\text{can}).\]

\(\square\)

The Riemann-Roch theorem asserts

\[
K_X^2 = 2\chi(X) + 3\tau(X)
\]

for any compact complex surface \( X \). The canonical divisor \( K_X \) of a hypersurface \( X \) in \( \mathbb{CP}^3 \) of degree \( d \geq 5 \) is ample, and thus \( X \) is of minimal general type. We reach a variant of Theorem 1.1 by using Theorem 2.1, which shows a correlation between the negativity of the Yamabe invariant and the appearance of pair-of-pants.

**Theorem 2.2.** Under the setup of Theorem 1.1, the Yamabe invariant of \( M \) is

\[
\sigma(M) = -4\sqrt{d(d-4)}\text{Vol}_{\omega}(P^2),
\]

where \( \omega \) is the Kähler-Einstein metric with Ricci curvature \(-1\) on the pair-of-pants \( P^2 \). Equivalently, the number of pair-of-pants in \( M^0 \) equals to

\[
\frac{1}{32\pi^2}(\max\{0,-\sigma(M)\})^2.
\]

Finally, we remark that the assertions ii) and iv) in Theorem 1.1 provide a certain constraint on the Yamabe invariant. The proof can be applied to more broader scenarios, and therefore we present a general result, which might have some independent interests.

**Theorem 2.3.** Let \( M \) be a compact 4-dimensional symplectic manifold, and \( \varpi \) be a symplectic form on \( M \). If there is a Lagrangian submanifold \( \Sigma \) in \( M \) such that \( \Sigma \) is a Riemann surface of genus \( g \geq 1 \), and represents a non-trivial homological class in \( M \), i.e.

\[0 \neq [\Sigma] \in H_2(M, \mathbb{R}),\]

then \( M \) does not admit any Riemannian metric of positive scalar curvature. Furthermore, if \( M \) is minimal, then \( 2\chi(M) + 3\tau(M) \geq 0 \), and the Yamabe invariant

\[
\sigma(M) \leq -\sqrt{32\pi^2(2\chi(M) + 3\tau(M))}.
\]

If \( M \) is a finite covering of an oriented manifold \( \bar{M} \), then 2.3 also holds for \( \bar{M} \).

**Proof.** The Weinstein neighbourhood theorem (cf. [68]) says that there is a tubular neighbourhood of \( \Sigma \) in \( M \) diffeomorphic to a neighbourhood of the zero section in the cotangent bundle \( T^*\Sigma \). Since the first Chern number of \( T^*\Sigma \) is the minus of the Euler number of \( \Sigma \), i.e. \( \int_{\Sigma} c_1(T^*\Sigma) = -\chi(\Sigma) \), the self-intersection number \( \Sigma \cdot \Sigma = 2g - 2 \geq 0 \).
Let $g$ be a Riemannian metric compatible with $\varpi$, i.e. $g(\cdot, \cdot) = \varpi(\cdot, J\cdot)$ for an almost complex structure $J$ compatible with $\varpi$. Then $\varpi$ is a self-dual 2-form with respect to $g$, i.e. $*\varpi = \varpi$, where $*$ denotes the Hodge star operator of $g$. We identify $H^2(M, \mathbb{R})$ with the sum of spaces of self-dual and anti-self-dual harmonic 2-forms via the Hodge theory, i.e. $H^2(M, \mathbb{R}) = \mathcal{H}_+(M) \oplus \mathcal{H}_-(M)$. If $A \in H^2(M, \mathbb{R})$ is the Poincaré dual of $[\Sigma]$, then $A = \alpha + \beta$ such that $d\alpha = d\beta = 0$, $*\alpha = \alpha$, $*\beta = -\beta$, and

$$0 \leq \Sigma \cdot \Sigma = \int_M (\alpha + \beta)^2 = \int_M \alpha^2 + \int_M \beta^2.$$ 

Thus

$$\int_M \alpha^2 \geq -\int_M \beta^2 = \int_M \beta \wedge *\beta \geq 0.$$ 

Note that if $\alpha = 0$, then $\beta = 0$ and $A = 0$, which is a contradiction. Thus $\alpha \neq 0$, and

$$\int_{\Sigma} \alpha = \int_M \alpha \wedge (\alpha + \beta) = \int_M \alpha^2 \neq 0.$$ 

Since

$$\int_{\Sigma} \varpi = 0,$$

$\varpi$ and $\alpha$ are linearly independent in the space of self-dual harmonic 2-forms $\mathcal{H}_+(M)$. Therefore, $b^+_2 \geq 2$.

Taubes’s theorem (cf. [55, 56, 34]) asserts that the Seiberg-Witten invariant is non-zero, i.e. $SW_M(c) \neq 0$, for a certain Spin$^c$ structure $c$. The same argument as in the proof of Theorem 2.1 proves that there does not exist any Riemannian metric of positive curvature on $M$, and $\sigma(M) \leq 0$. Furthermore, if $M$ is minimal, then $\sigma(M) \leq -\sqrt{32\pi^2 c_1^2(\mathfrak{L})}$, since $c_1^2(\mathfrak{L}) = 2\chi(M) + 3\tau(M) \geq 0$ by Theorem 4.11 and Corollary 4.9 in [34] (cf. [57]), where $\mathfrak{L}$ is the determinant bundle of the Spin$^c$ structure $c$.

Finally, if $M \to \overline{M}$ is a finite $\nu$-sheets covering, then $\sqrt{\nu}\sigma(\overline{M}) \leq \sigma(M)$, $\nu\chi(\overline{M}) = \chi(M)$, and $\nu\tau(\overline{M}) = \tau(M)$. We obtain the conclusion. \hfill \Box

Note that if we replace the Lagrangian condition in this theorem by the existence of a local $T^2$-fibration satisfying iv) in Theorem 1.1 then the self-intersection number $T^2 \cdot T^2 = 0$, and the same argument proves the same conclusion. Certain submanifolds provide obstructions for the existence of Riemannian metrics of positive scalar curvature by Schoen-Yau [52] and Gromov-Lawson [25]. Unlike these earlier pioneer works, the obstruction in Theorem 2.3 is the constraint provided by the Seiberg-Witten theory. Of course, we should ideally use the techniques as in [52, 25] to prove that the existence of pair-of-pants under some topological hypotheses implies the negativity of the Yamabe invariant.
3. Remarks on pair-of-pants

In this section, we give some remarks on pair-of-pants, which are not directly used in the proof of the main theorem, but provide a better understanding of the geometry of pair-of-pants.

First, we recall the existence of complete Kähler-Einstein metrics with negative Ricci curvature on quasi-projective manifolds, which was studied by many authors under various hypotheses (cf. [16, 32, 33, 62] etc.). Let $(X, D)$ be a pair of log Kähler surface such that the divisor $D$ has only simple normal crossing singularities and the log canonical divisor $K_X + D$ is ample. Then there is a complete Kähler-Einstein metric $\omega$ on $X\setminus D$ with finite volume

$$\text{Vol}_\omega(X\setminus D) = 2\pi^2(K_X + D)^2, \quad \text{and} \quad \text{Ric}(\omega) = -\omega,$$

by Theorem 1 in [32]. Since $(K_X + D)^2 \in \mathbb{Z}$, the smallest possible value of the volumes of complete Kähler-Einstein metrics constructed by this method is $2\pi^2$, i.e. $(K_X + D)^2 = 1$.

We apply this result to $(\mathbb{C}P^2, D)$ with $D$ being the sum of 4 general positioned lines, i.e. $\mathbb{C}P^2\setminus D = \mathcal{P}^2$, and obtain a complete Kähler-Einstein metric $\omega$ on the 4-dimensional pair-of-pants $\mathcal{P}$ with

$$(3.1) \quad \text{Ric}(\omega) = -\omega, \quad \text{and} \quad \text{Vol}_\omega(\mathcal{P}^2) = 2\pi^2,$$

since $K_{\mathbb{C}P^2} = -3H$ and $D = 4H$, where $H$ denotes the hyperplane class. Note that complete Kähler-Einstein metrics with the numerical data (3.1) are certainly not unique. For example, if $D'$ is a smooth curve of degree 4, then there is a complete Kähler-Einstein metric on $\mathbb{C}P^2\setminus D'$ satisfying (3.1), and $\mathbb{C}P^2\setminus D'$ is not diffeomorphic to $\mathcal{P}^2$. However, the pair-of-pants is still special as the sum of 4 generic lines is the most degenerated reduced curve of degree 4.

We provide a criterion for 4-dimensional pair-of-pants via the numerical data (3.1) in the following proposition.

**Proposition 3.1.** Let $X$ be a smooth minimal projective surface with vanishing first Betti number, i.e. $b_1(X) = 0$, and $D$ be a reduced effective divisor on $X$ with only simple normal crossing singularities. Assume that log canonical divisor $K_X + D$ is ample, and $D$ is a most degenerated divisor in the sense that the number of irreducible components of $D$ is bigger or equal to the number of irreducible components of any reduced divisor $D'$ linearly equivalent to $D$. Then

$$(K_X + D)^2 = 1,$$

if and only if $X = \mathbb{C}P^2$ and $X\setminus D$ is the pair-of-pants, i.e. $X\setminus D = \mathcal{P}^2$.

**Proof.** We only need to prove that $(K_X + D)^2 = 1$ implies $X = \mathbb{C}P^2$ and $X\setminus D = \mathcal{P}^2$.

Note that $(K_X + D) \cdot D \in \mathbb{Z}$, $(K_X + D) \cdot K_X \in \mathbb{Z}$ and

$$(K_X + D)^2 = (K_X + D) \cdot D + (K_X + D) \cdot K_X = 1.$$
By the adjunction formula for singular curves in surfaces, \((K_X + D) \cdot D = 2(g(D) - 1)\), where \(g(D)\) is the virtual genus of \(D\). Since \(K_X + D\) is ample, \((K_X + D) \cdot D > 0\), which implies \((K_X + D) \cdot D \geq 2\), and
\[(K_X + D) \cdot K_X < 0.
\]
Hence \(K_X\) is not nef, and \(X\) is either \(\mathbb{CP}^2\) or a minimal ruled surface over a curve by the classification of minimal surfaces (cf. Theorem 4 in Chapter 10 of \[22\]).

Assume that \(X\) is a ruled surface over a curve \(C\). Since the first Betti number of \(X\) is zero, \(C = \mathbb{CP}^1\) (cf. Lemma 13 in Chapter 5 of \[22\]), and \(X\) is a Hirzebruch surface, i.e. \(X = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-\mu))\). Let \(s\) be the 0-section with \(s^2 = -\mu\), \(\mu \geq 0\), and \(l\) be the fibre class which satisfies \(l^2 = 0\) and \(s \cdot l = 1\). If \(K_X + D = as + bl\) for \(a \in \mathbb{Z}\) and \(b \in \mathbb{Z}\), then
\[(K_X + D) \cdot l = a > 0, \quad (K_X + D) \cdot s = -\mu a + b > 0,
\]
by the ampleness. Thus
\[1 = (K_X + D)^2 = -\mu a^2 + 2ab > \mu a^2 \geq 0,
\]
and \(\mu = 0\). We obtain a contradiction \(1 = 2ab > 0\).

Therefore, \(X = \mathbb{CP}^2\), which implies \(K_X = -3H\) and \(D = 4H\), where \(H\) is the hyperplane class. We obtain the conclusion since the sum of 4 generic lines is the most degenerated curve of degree 4.

Orbifold singularities, especially the A-D-E singularities, appear naturally in the study of minimal models for algebraic surfaces, and limit spaces of Einstein 4-manifolds in the Gromov-Hausdorff convergence (cf. \[3\] \[14\]). If singularities present, we certainly have examples of complete Kähler-Einstein metrics in the orbifold sense with volume less than \(2\pi^2\).

**Example 3.2.** We consider the \(\mathbb{Z}_3\)-action on \(\mathbb{CP}^2\) by permuting the homogeneous coordinates, i.e. \(\gamma \cdot [z_0, z_1, z_2] = [z_2, z_0, z_1]\) for the generator \(\gamma\) of \(\mathbb{Z}_3\). There are three fixed points \([1, 1, 1], [1, \varepsilon^2, \varepsilon], [1, \varepsilon, \varepsilon^2]\) where \(\varepsilon = \exp\frac{2\pi \sqrt{-1}}{3}\), and the volume form \(\frac{du_1 \wedge dw_2}{w_1 w_2}\) is preserved where \(w_1 = z_1 z_0^{-1}\) and \(w_2 = z_2 z_0^{-1}\). Note that \(D = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 | z_0 z_1 z_2 (z_0 + z_1 + z_2) = 0\}\) is \(\mathbb{Z}_3\)-invariant, and \([1, \varepsilon^2, \varepsilon], [1, \varepsilon, \varepsilon^2] \in D\) by \(1 + \varepsilon + \varepsilon^2 = 0\). Thus \(\mathbb{Z}_3\) acts on the pair-of-pants \(\mathcal{P}^2\) with only one fixed point \([1, 1, 1]\), and the quotient \(\mathcal{P}^2/\mathbb{Z}_3\) is an orbifold with one isolated singularity. Note that the class \(c_1(\mathcal{O}_{\mathbb{CP}^2}(K_{\mathbb{CP}^2} + D))\) is also invariant, and the complete Kähler-Einstein metric \(\omega\) on \(\mathcal{P}^2\) descends to a complete orbifold metric on \(\mathcal{P}^2/\mathbb{Z}_3\), denoted still as \(\omega\), which satisfies
\[
\text{Ric}(\omega) = -\omega, \quad \text{and} \quad \text{Vol}_\omega(\mathcal{P}^2/\mathbb{Z}_3) = \frac{2\pi^2}{3}.
\]

We remark that the Kähler-Einstein metric \(\omega\) on \(\mathcal{P}^2\) is not complex hyperbolic as follows. Note that the Euler number of \(\mathcal{P}^2\) is one (cf. Proposition 3.3 in \[3\]) and...
2.5 in [24]), and the Gauss-Bonnet-Chern formula still holds in the current case (cf. Theorem 4.5 in [14]), i.e.

\[ 1 = \chi(P^2) = \frac{1}{8\pi^2} \int_{P^2} (R^2 \frac{24}{24} + |W^+|^2 + |W^-|^2) \, dv_\omega, \]

where \( W^\pm \) denotes the self-dual/anti-self-dual Weyl curvature of \( \omega \) (cf. [9]). Since \( \omega \) is Kähler, \( 24|W^+|^2 = R^2 \), and we obtain

\[ \int_{P^2} |W^-|^2 \, dv_\omega = \frac{16}{3} \pi^2. \]

Hence \( W^- \) is not identical to zero, and \( \omega \) is not a complex hyperbolic metric. Furthermore, (3.2) indicates that we might not expect that complex hyperbolic manifolds admit sensible pair-of-pants decompositions, but we would rather view them as another basic negative pieces because of the Mostow rigidity.

Many works have been done towards certain decompositions of smooth 4-manifolds and canonical metrics from various perspectives, for example, metric geometry, topology, algebraic geometry and symplectic geometry etc. See [7, 5, 6, 59, 60, 61, 17, 18, 11, 38] and the references in them, and especially [6, 60, 17, 18] for some expository discussions. In this paper, we explore decompositions for 4-manifolds of a very specific type, i.e. projective hypersurfaces of general type, while keeping in mind the general questions whether there is a geometrisation theory for 4-manifolds, and if it exists at all, what the zoo of fundamental 4-manifolds might consists of.

Next, we remark that the product of two 2-dimensional pair-of-pants, \( P^1 \times P^1 \), deforms to a singular variety containing two copies of pair-of-pants \( P^2 \). Therefore, we may not regard \( P^1 \times P^1 \) as a fundamental building block.

**Example 3.3.** There is a degeneration \( f: X \rightarrow \mathbb{C} \) satisfying that \( X \) is smooth,

\[ (f^{-1}(0))^o = P^2 \coprod P^2, \quad \text{and} \quad f^{-1}(1) = P^1 \times P^1, \]

where \( (f^{-1}(0))^o \) denotes the regular locus of \( f^{-1}(0) \).

**Proof.** Let

\[ \tilde{X} = \{(w_0, w_1, w_2, w_3) \in (\mathbb{C})^3 \times \mathbb{C}^*| w_0 w_3 = w_1 w_2 \}, \]

and \( f: \tilde{X} \rightarrow \mathbb{C} \) be given by \( (w_0, w_1, w_2, w_3) \mapsto w_0 \). Note that \( \tilde{X} \) is smooth since the defining equation defines one ordinary double point that is not in \( (\mathbb{C})^3 \times \mathbb{C}^* \). If \( X_{w_0} = f^{-1}(w_0) \), then \( X_0 \) is a singular variety given by \( w_1 w_2 = 0 \), i.e.

\[ X_0 = \{(w_1, w_3) \in \mathbb{C} \times \mathbb{C}^* \} \cup \{(w_2, w_3) \in \mathbb{C} \times \mathbb{C}^* \} \subset \mathbb{C}^2 \times \mathbb{C}^*, \quad \text{and} \]

\[ X_1 = \{(w_1, w_2, w_3) \in \mathbb{C}^2 \times \mathbb{C}^*| w_3 = w_1 w_2 \neq 0 \} = \{(w_1, w_2) \in (\mathbb{C}^*)^2 \}. \]

If we let

\[ D = \{(w_0, w_1, w_2, w_3) \in \tilde{X}| 1 + w_1 + w_2 + w_3 = 0 \}, \]

then...
then the regular locus of $X_0 \setminus D$ is the Zariski open subset
\[
\{(w_1, w_3) \in (\mathbb{C}^*)^2 | 1 + w_1 + w_3 \neq 0\} \cup \{(w_2, w_3) \in (\mathbb{C}^*)^2 | 1 + w_2 + w_3 \neq 0\},
\]
i.e. the regular locus of $X_0 \setminus D$ consists of two copies of pair-of-pants $\mathcal{P}^2$. By
\[
1 + w_1 + w_2 + w_1 w_2 = (1 + w_1)(1 + w_2),
\]
we obtain the conclusion by setting $\mathcal{X} = \tilde{\mathcal{X}} \setminus D$.

If $D$ is a divisor in $\mathbb{CP}^2$ consisting of $d \geq 5$ generic positioned lines, then $(K_{\mathbb{CP}^2} + D)^2 = (d - 3)^2$, and we expect that $\mathbb{CP}^2 \setminus D$ deforms to $(d - 3)^2$-copies of pair-of-pants $\mathcal{P}^2$. Thus we might not obtain more sensible building blocks by removing lines from the pair-of-pants. $D$ is an example of arrangements of lines, and one technique to construct general type surfaces is to use branched coverings of $\mathbb{CP}^2$ along arrangements of lines (cf. [29]). We work out the concrete case of $d = 5$, and the proof would become more transparent by using the techniques from toric geometry.

**Example 3.4.** We assume that
\[
D = \{(z_0, z_1, z_2) \in \mathbb{CP}^2 | z_0 z_1 z_2 (z_0 + z_1 + z_2)(a_0 z_0 + a_1 z_1 + a_2 z_2) = 0\}
\]
for certain generic chosen $a_0, a_1, a_2 \in \mathbb{C}$. Then there is a degeneration $f : \mathcal{X} \to \mathbb{C}$ such that
\[
(f^{-1}(0))^o = \prod_{i=1}^4 \mathcal{P}^2, \quad \text{and} \quad f^{-1}(1) = \mathbb{CP}^2 \setminus D,
\]
where $(f^{-1}(0))^o$ denotes the regular locus of $f^{-1}(0)$.

**Proof.** We consider a family of Veronese embeddings $\Psi_w : \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^5$ given by
\[
Z_0 = w z_0^2, \quad Z_1 = z_0 z_1, \quad Z_2 = z_0 z_2, \quad Z_3 = z_1 z_2, \quad Z_4 = w z_1^2, \quad Z_5 = w z_2^2,
\]
where $w \in \mathbb{C}^*$, and $[Z_0, Z_1, Z_2, Z_3, Z_4, Z_5]$ are homogeneous coordinates of $\mathbb{CP}^5$. The image is given by the equations
\[
Z_0 Z_4 = w^2 Z_1^2, \quad Z_0 Z_5 = w^2 Z_2^2, \quad Z_4 Z_5 = w^2 Z_3^2,
\]
\[
w Z_1 Z_2 = Z_0 Z_3, \quad w Z_2 Z_3 = Z_1 Z_5, \quad w Z_1 Z_3 = Z_2 Z_4.
\]
When $w \to 0$, $\Psi_w(\mathbb{CP}^2)$ converges to a singular variety $X_0$ with 4 irreducible components, i.e. $X_0 = X_0^1 \cup X_0^2 \cup X_0^3 \cup X_0^4$, where
\[
X_0^1 = \{(Z_0, Z_1, Z_2, 0, 0, 0) \in \mathbb{CP}^5\}, \quad X_0^2 = \{(0, Z_1, Z_2, Z_3, 0, 0) \in \mathbb{CP}^5\},
\]
\[
X_0^3 = \{(0, Z_1, 0, Z_3, Z_4, 0) \in \mathbb{CP}^5\}, \quad X_0^4 = \{(0, 0, Z_2, Z_3, 0, Z_5) \in \mathbb{CP}^5\}.
\]
We let $D^0 = \{(0, 0, 0, Z_3, Z_4, Z_5) \in \mathbb{CP}^5\}$, $D^1 = \{(Z_0, 0, Z_2, 0, 0, Z_5) \in \mathbb{CP}^5\}$, $D^2 = \{(Z_0, Z_1, 0, 0, Z_4, 0) \in \mathbb{CP}^5\}$, and define $D' \subset \mathbb{CP}^3$ by
\[
a_0 Z_0 + (a_0 + a_1) Z_1 + (a_0 + a_2) Z_2 + (a_1 + a_2) Z_3 + a_1 Z_4 + a_2 Z_5 = 0.
\]
Figure 3.1. Intersection complex of $X_0$ in Example 3.4.

Note that $D' \cap X_0 \setminus Z_i$, $i = 1, 2, 3, 4$, is a generic line in $\mathbb{CP}^2$. If we regard $\mathbb{CP}^2$ as a toric manifold and $(\mathbb{C}^*)^2 \subset \mathbb{CP}^2$, then $(\mathbb{C}^*)^2 \setminus (D' \cap X_0)$ is a pair-of-pants $P^2$, and does not intersect with any $D'_j$, $j = 0, 1, 2$. Furthermore, $\Psi_1(D) = \Psi_1(\mathbb{CP}^2) \cap (D' \cup D^1 \cup D^3 \cup D^4)$.

We obtain the conclusion by letting

$$\mathcal{X} = (X_0 \bigcup_{w \in \mathbb{C}^*} \Psi_w(\mathbb{CP}^2)) \setminus (D' \cup D^1 \cup D^3 \cup D^4) \subset \mathbb{CP}^5 \times \mathbb{C},$$

and $f$ be the projection to $w$. □

We finish this section by showing that there is a pair-of-pants in the original Godeaux surface.

Example 3.5. We consider the Fermat quintic

$$X = \left\{ [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 | z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0 \right\},$$

which is a smooth minimal surface of general type with $K^2_X = 5$. If $\mathbb{Z}_5$ acts on $\mathbb{CP}^3$ by $\epsilon \cdot [z_0, z_1, z_2, z_3] = [z_0, \epsilon^2 z_1, \epsilon z_2, \epsilon^3 z_3]$ where $\epsilon = \exp \frac{2\pi\sqrt{-1}}{5}$, then $\mathbb{Z}_5$ acts on $X$ freely. The quotient $\tilde{X} = X/\mathbb{Z}_5$ is a smooth surface of general type with $K^2_{\tilde{X}} = 1$, called a Godeaux surface (cf. [17]). The geometric genus $p_g(\tilde{X}) = 0$ and the fundamental group $\pi_1(\tilde{X}) = \mathbb{Z}_5$.

Theorem 3.6. If $M$ is diffeomorphic to the Godeaux surface $\tilde{X}$, then there is an open subset $M^0 \subset M$ diffeomorphic to the pair-of-pants $P^2$, and the Yamabe invariant

$$\sigma(M) = -4\sqrt{\text{Vol}_\omega(P^2)} = -\sqrt{32\pi^2}.$$

Furthermore, if $v : M' \to M$ is the universal covering of $M$, then there is a 2-torus $T^2$ in $M'$ such that

i) $v(T^2) \subset M^0$,

ii) $T^2$ represents a non-trivial homological class in $M'$, i.e.

$$0 \neq [T^2] \in H^2(M', \mathbb{R}),$$

iii) the self-intersection number

$$T^2 \cdot T^2 = 0, \quad \text{and} \quad \int_{T^2} \omega = 0,$$
for a symplectic form $\omega$ on $M'$.

Proof. If $H$ is the hyperplane given by $h_0 = z_0 + z_1 + z_2 + z_3 = 0$, then $\varepsilon^i H$ is defined by $h_i = z_0 + \varepsilon^i z_1 + \varepsilon^{2i} z_2 + \varepsilon^{3i} z_3 = 0$ for $i = 0, \ldots, 4$. Let $X_0 = H \cup \varepsilon H \cup \varepsilon^2 H \cup \varepsilon^3 H \cup \varepsilon^4 H$, which is defined by $h_0 h_1 h_2 h_3 h_4 = 0$.

We claim that the regular locus of $X_0$ consists of 5 copies of pair-of-pants, i.e. for any $i$, $\varepsilon^i H \setminus \bigcup_{j \neq i} \varepsilon^j H$ is a pair-of-pants $P^2$. Since $\mathbb{Z}_5$ acts on $X_0$ and switches the irreducible components, we only need to prove that $H \setminus (\varepsilon H \cup \varepsilon^2 H \cup \varepsilon^3 H \cup \varepsilon^4 H)$ is a pair-of-pants. Note that $\varepsilon^2 (H \cap \varepsilon H \cap \varepsilon^2 H \cap \varepsilon^3 H) = H \cap \varepsilon^2 H \cap \varepsilon^3 H \cap \varepsilon^4 H$, and $\varepsilon^4 (H \cap \varepsilon H \cap \varepsilon^2 H \cap \varepsilon^3 H) = H \cap \varepsilon H \cap \varepsilon^2 H \cap \varepsilon^4 H$. Thus we only need to prove that $H \cap \varepsilon H \cap \varepsilon^2 H \cap \varepsilon^3 H$ is empty. The coefficient matrix of the equations $h_0 = h_1 = h_2 = h_3 = 0$ is the Vandermonde matrix, and the determinant is equal to $\varepsilon^4 (\varepsilon - 1)^6 (\varepsilon + 1)^2 (\varepsilon^2 + \varepsilon + 1) \neq 0$. We obtain the claim.

If we let $X_t = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}P^3 | z_0^5 + z_1^5 + z_2^5 + z_3^5 + t h_0 h_1 h_2 h_3 h_4 = 0 \}$, $t \in (1, \infty)$, then $X_t$ is invariant under the $\mathbb{Z}_5$-action, and $X_t$ converges to $X_0$ in $\mathbb{C}P^3$ as varieties when $t \to \infty$. Note that $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$ are the fixed points when $\mathbb{Z}_5$ acts on $\mathbb{C}P^3$, and $1 + t \varepsilon^j \neq 0$. Thus the $\mathbb{Z}_5$-action on $X_1$ is free. For $t \gg 1$, there is an open subset $X_t^0 \subset X_t$ diffeomorphic to the regular locus of $X_0$, which consists of 5 copies of pair-of-pants, and $X_t^0$ is invariant under the $\mathbb{Z}_5$-action. Therefore the quotient $X_t^0 / \mathbb{Z}_5$ is a pair-of-pants in $X_t / \mathbb{Z}_5$.

The rest of the assertion seems a corollary of Theorem 1.1. However we give a direct calculation proof since the degeneration used here is different from the family of varieties in the proof of Theorem 1.1.

We choose new homogeneous coordinates $Z_0, Z_1, Z_2, Z_3$ on $\mathbb{C}P^3$ such that $\varepsilon^i H$ is defined by $Z_i = 0$, $i = 0, 1, 2, 3$. Equivalently, $Z_i = z_0 + \varepsilon^i z_1 + \varepsilon^{2i} z_2 + \varepsilon^{3i} z_3$. Then $\varepsilon^4 H$ is given by $a_0 Z_0 + a_1 Z_1 + a_2 Z_2 + a_3 Z_3 = 0$, and $X_t$ is defined by

$$f_t = Z_0 Z_1 Z_2 Z_3 (a_0 Z_0 + a_1 Z_1 + a_2 Z_2 + a_3 Z_3) + t^{-1} P_5 = 0,$$

where $a_0 Z_0 + a_1 Z_1 + a_2 Z_2 + a_3 Z_3 = z_0 + \varepsilon^4 z_1 + \varepsilon^{8} z_2 + \varepsilon^{12} z_3$, and $P_5$ is a homogeneous polynomial of degree 5. If $w_j = Z_j / Z_0$, $j = 1, 2, 3$, then $\varepsilon^j H \cap \{Z_0 \neq 0\}$ is given by $w_j = 0$, $j = 1, 2, 3$, and $\varepsilon^4 H \cap \{Z_0 \neq 0\}$ is given by $a_0 + a_1 w_1 + a_2 w_2 + a_3 w_3 = 0$. We let

$$f_t = f_t / Z_0^5 = w_1 w_2 w_3 (a_0 + a_1 w_1 + a_2 w_2 + a_3 w_3) + t^{-1} P_5',$$

where $P_5' = P_5 / Z_0$. Note that $a_0 \neq 0$ since, otherwise, $(0, 0, 0)$ solves $w_i = 0$, $i = 1, 2, 3$, and $a_0 + a_1 w_1 + a_2 w_2 + a_3 w_3 = 0$, which is a contradiction.

We consider the hyperplane $\varepsilon^3 H \cap \{Z_0 \neq 0\} \subset \mathbb{C}^3$, and $\varepsilon^3 H \cap \{Z_0 \neq 0\} \setminus (\varepsilon^1 H \cup \varepsilon^2 H \cup \varepsilon^4 H)$ is the pair-of-pants $P^2$ belonging to the irreducible
component $\varepsilon^3H$. We choose a 2-torus

$$T^2 = \{(w_1, w_2) \in \mathcal{P}^2 \mid |w_1| = r_1, |w_2| = r_2\} \subset \varepsilon^3H,$$

for certain $r_1, r_2 \in \mathbb{R}^1 \setminus \{0\}$. If $\theta_i$, $i = 1, 2$, is the angle of $w_i$, i.e. $w_i = r_i \exp \sqrt{-1}\theta_i$, then $\theta_1$ and $\theta_2$ are angular coordinates on $T^2$. If $V$ is a small neighbourhood of $T^2$ in $\mathbb{P}^3$, then $X_t \cap V$ is given by $f_t = 0$ and $\varepsilon^3H \cap V$ is given by $w_3 = 0$. We choose $V$ such that $V$ does not intersect with $\varepsilon H$, $\varepsilon^2H$, and $\varepsilon^4H$.

Since the meromorphic form

$$\Omega = \frac{Z_0^2Z_1Z_2Z_3dw_1 \wedge dw_2 \wedge dw_3}{f_tw_1w_2w_3}$$

has only a simple pole along $X_t$, the Poincaré residue formula gives a holomorphic 2-form $\Omega_t = \text{res}_{X_t}(\Omega)$ on $X_t$, which defines a non-trivial cohomological class, i.e.

$$0 \neq [\Omega_t] \in H^2(X_t, \mathbb{C}).$$

Note that

$$\Omega_t = \frac{dw_1 \wedge dw_2}{\frac{\partial f_t}{\partial w_3}} = \frac{w_1w_2d\log(w_1) \wedge d\log(w_2)}{\frac{\partial f_t}{\partial w_3}},$$

on $X_t \cap V$. We calculate

$$\frac{\partial f_t}{\partial w_3} = w_1w_2(a_0 + a_1w_1 + a_2w_2) + 2a_3w_1w_2w_3 + t^{-1}\frac{\partial P_5'}{\partial w_3},$$

and thus

$$\Omega_t = \frac{d\log(w_1) \wedge d\log(w_2)}{a_0 + a_1w_1 + a_2w_2 + 2a_3w_3 + t^{-1}w_1^{-1}w_2^{-1}\frac{\partial P'}{\partial w_3}}.$$

We choose $r_1$ and $r_2$ such that $|a_1|r_1 + |a_2|r_2 \ll \frac{1}{2}|a_0|$ since $a_0 \neq 0$, and $r_1r_2 > \epsilon > 0$ for a small constant $\epsilon > 0$. Since $|a_0 + a_1w_1 + a_2w_2 + a_3w_3| > \epsilon' > 0$ for any $w \in V$ and a constant $\epsilon' > 0$, the equation $f_t = 0$ shows that

$$\epsilon' |w_3|_{X_t \cap V} < |P_5'|^{-1} \to 0,$$

when $t \to 0$. For any isotropic embedding $\phi_t : \mathcal{P}^2 \hookrightarrow X_t$ with $\phi_\infty = \text{id}$,

$$\int_{\phi_t(T^2)} \Omega_t \to \int_{T^2} \frac{d\theta_1 \wedge d\theta_2}{a_0 + a_1w_1 + a_2w_2} = \int_{T^2} \frac{(a_0 + a_1r_1e^{-\sqrt{-1}\theta_1} + a_2r_2e^{-\sqrt{-1}\theta_2})d\theta_1 \wedge d\theta_2}{|a_0 + a_1r_1e^{\sqrt{-1}\theta_1} + a_2r_2e^{\sqrt{-1}\theta_2}|^2} \neq 0,$$

when $t \to \infty$. Thus for $t \gg 1$, $\phi_t(T^2) \subset \phi_t(\mathcal{P}^2) \subset X_t^\circ$, and

$$0 \neq [\phi_t(T^2)] \in H_2(X_t, \mathbb{R}).$$

By varying $r_1$ and $r_2$, we obtain that the self-intersection number $\phi_t(T^2) \cdot \phi_t(T^2) = 0$. If $\varpi$ is a toric symplectic form on $\mathbb{CP}^3$, then $\varpi$ is preserved by
the \( \mathbb{Z}_5 \)-action, and \( \varpi|_{T^2} \equiv 0 \). Therefore
\[
\int_{\phi_t(T^2)} \varpi = \int_{T^2} \varpi = 0.
\]
Note that if \( \nu : X_t \to X_t/\mathbb{Z}_5 \) denotes the quotient map, then \( \nu : \phi_t(P^2) \to M' \) is a diffeomorphism. We obtain the conclusion since \( M \) is diffeomorphic to \( X_t/\mathbb{Z}_5 \), and \( M' = X_t \). \( \square \)

The difference between the assertion ii) in this theorem and iv) in Theorem 1.1 is that the torus \( \nu(T^2) \) is homological zero since \( \Omega_t \) is not preserved by the \( \mathbb{Z}_5 \)-action, and the geometric genus \( p_g(\tilde{X}) = 0 \). However \( \nu^{-1}(\nu(T^2)) \) consists of 5 copies of \( T^2 \) in the universal covering, and each connected component represents a non-trivial class in \( M' \). The assertion ii) is also different from the case of \( P^2 \subset \mathbb{C}P^2 \), where \( \mathbb{C}P^2 \) is simply connected, and numerical Lagrangian 2-tori in \( \mathbb{C}P^2 \) are homological zero. Moreover, Theorem 2.3 can be applied to the current case, and shows that the assertions ii) and iii) of Theorem 3.6 imply directly the Yamabe invariant \( \sigma(M) \leq 0 \). Therefore we would like to think that Theorem 3.6 provides a sensible decomposition of the Godeaux surface.

4. Pair-of-pants vs K3 surfaces

The goal of this section is to prove the assertions i), ii), iii), and v) of Theorem 1.1 which depends heavily on Mikhalkin’s paper [40] and tropical geometry (cf. Chapter 1 of [26] and [41]). We review the relevant facts first.

4.1. Amoebas. \((\mathbb{C}^*)^3\) denotes the complex torus of dimension 3, and for any \( m = (m_1, m_2, m_3) \in \mathbb{Z}^3 \), \( w^m = w_1^{m_1}w_2^{m_2}w_3^{m_3} \) where \( w_1, w_2, w_3 \) are coordinates on \((\mathbb{C}^*)^3\). Let \( X^o \) be the hypersurface in \((\mathbb{C}^*)^3\) defined by a Laurent polynomial
\[
(4.1) \quad f(z) = \sum_{m \in S} a_m w^m = 0,
\]
where \( a_m \in \mathbb{C} \), and \( S \subset \mathbb{Z}^3 \) is finite. The Newton polytope \( \Delta \subset \mathbb{R}^3 \) of \( X^o \) is defined as the convex hull of \( m \in S \) such that \( a_m \neq 0 \). We consider the log-map
\[
\log : (\mathbb{C}^*)^3 \to \mathbb{R}^3, \quad \log(w) = (\log |w_1|, \log |w_2|, \log |w_3|).
\]
The amoeba of \( X^o \) is defined in [23] (see also [41, 44] for analytic treatments) as the image
\[
\mathcal{A} = \log(X^o) \subset \mathbb{R}^3.
\]
Let \( \nu : \Delta(\mathbb{Z}) \to \mathbb{R} \) be a function where \( \Delta(\mathbb{Z}) = \Delta \cap \mathbb{Z}^3 \) is the set of lattice points of \( \Delta \). \( \nu \) induces a rational polyhedral subdivision \( \mathcal{T}_\nu \) of \( \Delta \) as follows. If \( \tilde{\Delta} \) is the upper convex hull of \( \{(m, \nu(m))|m \in \Delta(\mathbb{Z})\} \) in \( \Delta \times \mathbb{R} \), i.e.
\[
\tilde{\Delta} = \{(a, b) \in \Delta \times \mathbb{R} | \exists b' \in \text{Cov}\{(m, \nu(m))|m \in \Delta(\mathbb{Z})\}, \quad b \geq b'\},
\]

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\[
\tilde{\Delta} = \{(a, b) \in \Delta \times \mathbb{R} | \exists b' \in \text{Cov}\{(m, \nu(m))|m \in \Delta(\mathbb{Z})\}, \quad b \geq b'\},
\]
then $\mathcal{T}_v$ is the set of the images of proper faces under the projection $\tilde{\Delta} \to \Delta$ (cf. Chapter 1 in [26]). The discrete Legendre transform $L_v : \mathbb{R}^3 \to \mathbb{R}$ of $v$ is defined as

$$L_v(x) = \max_{m \in \Delta(\mathbb{Z})} \{ l_m(x) \}, \quad l_m(x) = \langle m, x \rangle - v(m),$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric. Then $L_v$ is a convex piecewise linear function.

The non-smooth locus $\Pi_v$ of $L_v$ is called the tropical hypersurface defined by $v$, or the non-archimedean amoeba, which is a balanced polyhedral complex dual to the subdivision $\mathcal{T}_v$ of $\Delta$ (cf. Proposition 2.1 in [40]). $\mathcal{T}_v$ is a strong deformational retract of the amoeba $\mathcal{A}$ (cf. [41, 44]). Furthermore, there is a one-to-one corresponding between the set of $k$-dimensional cells $\check{\rho} \subset \Pi_v$, $k = 0, 1, 2$, and the set of $(3 - k)$-dimensional cell $\rho$ in $\mathcal{T}_v$, i.e.

$$\{ k \text{- cells } \check{\rho} \subset \Pi_v \} = \{ (3 - k) \text{- cells } \rho \in \mathcal{T}_v \}, \text{ via } \check{\rho} \mapsto \rho,$$

and the cell $\check{\rho}$ is unbounded if and only if $\rho \subset \partial \Delta$. The vertices of $\mathcal{T}_v$ are one-to-one corresponding to the connected components of the complement of $\Pi_v$. Two cells $\rho_1 \subset \rho_2 \in \mathcal{T}_v$ if and only if $\check{\rho}_1 \supset \check{\rho}_2$ in $\Pi_v$.

We deform $X^o$ via the Viro’s patchworking polynomials (cf. [66]). The patchworking family $X_t^o$, $t \in (1, \infty)$, is defined by

$$f_t(w) = \sum_{m \in S} a_m t^{-v(m)} w^m = 0.$$  

If the deformed log map $\text{Log}_t : (\mathbb{C}^*)^3 \to \mathbb{R}^3$ is defined by

$$\text{Log}_t(w) = \left( \frac{\log |w_1|}{\log t}, \frac{\log |w_2|}{\log t}, \frac{\log |w_3|}{\log t} \right),$$

then we have a family of the amoebas

$$\mathcal{A}_t = \text{Log}_t(X_t^o) \subset \mathbb{R}^3.$$  

Corollary 6.4 in [40] asserts that

$$\mathcal{A}_t \to \Pi_v, \text{ when } t \to \infty,$$

in the Hausdorff sense. Therefore, $\Pi_v$ is a good approximation of $\mathcal{A}_t$ when $t \gg 1$.

If

$$H^o = \{ (w_1, w_2, w_3) \in (\mathbb{C}^*)^3 | 1 + w_1 + w_2 + w_3 = 0 \}$$

$$= \{ (w_1, w_2) \in (\mathbb{C}^*)^2 | 1 + w_1 + w_2 \neq 0 \},$$

then $H^o$ is the 4-dimensional pair-of-pants, i.e. $H^o = \mathcal{P}^2$. The Newton polytope $\Delta$ is the standard simplex in $\mathbb{R}^3$, i.e.

$$\Delta = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 \leq 1, x_i \geq 0, i = 1, 2, 3 \}.$$  

If $v \equiv 0$, then the corresponding tropical hypersurface $\Pi_v$ is the non-linear locus of $L_v(x) = \max \{ 0, x_1, x_2, x_3 \}$. 

4.2. Decompositions. Let $M$ be a smooth manifold diffeomorphic to a hypersurface $X$ in $\mathbb{CP}^3$ of degree $d \geq 5$. We regard $\mathbb{CP}^3$ as a toric manifold by $(\mathbb{C}^*)^3 \subset \mathbb{CP}^3$. For any positive integer $d$, the corresponding polytope $\Delta_d$ of $(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(d))$ is the standard simplex in $\mathbb{R}^3$, i.e.

$$\Delta_d = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 \leq d, x_i \geq 0, i = 1, 2, 3\}. \tag{4.5}$$

If $\Delta_d(\mathbb{Z}) = \Delta_d \cap \mathbb{Z}^3$, then for any $m \in \Delta_d(\mathbb{Z})$, $w^m = w_1^{m_1}w_2^{m_2}w_3^{m_3}$ is viewed as a section of $\mathcal{O}_{\mathbb{CP}^3}(d)$.

Let $\Delta_d^\circ$ be the convex hull of $\text{int}(\Delta_d) \cap \mathbb{Z}^3$, i.e. $\Delta_d^\circ = \text{Cov}(\text{int} \Delta_d \cap \mathbb{Z}^3)$, and $\Delta_d^\circ(\mathbb{Z}) = \Delta_d^\circ \cap \mathbb{Z}^3$. Note that $\Delta_d^\circ$ is a translation of the simplex $\Delta_{d-4}$, i.e.

$$\Delta_d^\circ = (1, 1, 1) + \Delta_{d-4}. \tag{4.6}$$

If $p_g(d)$ denotes the geometric genus of $X$ in $\mathbb{CP}^3$, then

$$p_g(d) = \frac{(d-1)(d-2)(d-3)}{6} = \# \Delta_d^\circ(\mathbb{Z}),$$

by the Noether’s formula $\chi(X) = 12(1 - p_g(d)) - K_X^2$, since $X$ is simply connected, $K_X^2 = d(d-4)^2$ and the Euler number $\chi(X) = d^3 - 4d^2 + 6d$.

Let $v : \Delta_d(\mathbb{Z}) \to \mathbb{R}$ be a function such that the induced subdivision $T_v$ is a uni-modular lattice triangulation, i.e. any 3-cell in $T_v$ is a simplex with Euclidean volume $\frac{1}{6}$. Since the Euclidean volume of $\Delta_d$ is $\frac{d^3}{6}$, $T_v$ has $d^3$ 3-dimensional cells, and $\Pi_v$ has $d^3$ vertices. $v$ defines a patchworking family
Figure 4.3. A decomposition of projective curve of degree 4 in \( \mathbb{C}P^2 \), which consists of 4 copies of pair-of-pants \( P^1 \), and 3 copies of \((0,1) \times S^1 \subset \mathbb{C}^*\).

\[
X_t, \ t \in (1, \infty), \text{ given by}
\]

\[
f_t(w) = \sum_{m \in \Delta_d(\mathbb{Z})} t^{-v(m)}w^m = 0.
\]

Since any smooth hypersurface of degree \( d \) is diffeomorphic to the same differential manifold, a smooth \( X_t \) is diffeomorphic to \( M \). We let \( X_t^o = X_t \cap (\mathbb{C}^*)^3 \).

In [40], Mikhalkin proved that \( M \) admits a pair-of-pants decomposition.

**Theorem 4.1 ([40]).** There is a fibration \( \lambda : X_t^o \rightarrow \Pi_v \) satisfying:

i) For any vertex \( \tilde{q} \in \Pi_v \), let \( U_{\tilde{q}} \subset \Pi_v \) be the interior of the star of the barycentre triangulation of \( \Pi_v \), called a primitive piece associated with \( \tilde{q} \). Then \( U_{\tilde{q}} \cap U_{\tilde{q}'} \) is empty if \( \tilde{q} \neq \tilde{q}' \), and \( \lambda^{-1}(U_{\tilde{q}}) \) is diffeomorphic to the pair-of-pants \( P^2 \).

ii) \[
\bigcup_{\tilde{q} \in \{ \text{vertices in } \Pi_v \}} \lambda^{-1}(U_{\tilde{q}}) \subset X_t^o
\]
is open dense. Therefore \( X_t \) admits a pair-of-pants decomposition consisting of \( d^3 \) copies of \( P^2 \).

iii) If a point \( x \) belongs to the interior of a 2-cell in \( \Pi_v \), then the fibre \( f^{-1}(x) \) is a torus \( T^2 \), and if \( x \) is in the interior of a 1-cell of \( \Pi_v \), then \( f^{-1}(x) \) has the shape of \( \ominus \times \ominus \).

The goal of Theorem 1.1 is to reduce the number of pair-of-pants from \( d^3 \) to \( d(d - 4)^2 \) by showing that those extra pair-of-pants form certain subsets of K3 surfaces. To decompose \( X_t \) into a union of subsets of Calabi-Yau manifolds was previously studied by Leung and Wan in [39].

We remark that there is an alternative way to view the pair-of-pants decomposition in the current case. Assume that any vertex \( \tilde{q} \in \Pi_v \) is an integral vector, i.e. \( \tilde{q} \in \mathbb{Z}^3 \), and for any \( m \in \Delta_d(\mathbb{Z}) \), \( v(m) \in \mathbb{Z} \). The pair
(Δ_d, v) induces a toric degeneration \( \mathcal{X} \to \mathbb{C} \) with a relative ample line bundle \( \mathcal{O}_{\mathcal{X}/\mathbb{C}}(d) \) (cf. Chapter 1 of [26] and [27]) satisfying:

i) Any generic fibre \( \mathcal{X}_{w_0}, w_0 \neq 0 \), is \( \mathbb{CP}^3 \), and the restriction of \( \mathcal{O}_{\mathcal{X}/\mathbb{C}}(d) \) on \( \mathcal{X}_{w_0} \) is \( \mathcal{O}_{\mathbb{CP}^3}(d) \).

ii) The central fibre \( \mathcal{X}_0 \) is a singular variety consisting of \( d^3 \) copies of \( \mathbb{CP}^3 \) as irreducible components. The singular set of \( \mathcal{X}_0 \) belongs to the union of toric boundary divisors of \( \mathbb{CP}^3 \).

iii) There is a one-to-one corresponding between 3-cells \( \rho \) of \( \mathcal{T}_v \) and irreducible components \( \mathcal{X}_{0, \rho} \) of \( \mathcal{X}_0 \). Furthermore, the pair \( (\Delta_d, \mathcal{T}_v) \) can be regarded as the intersection complex of \( \mathcal{X}_0 \).

iv) For any \( m \in \Delta_d(\mathbb{Z}) \), \( Z_m = w_0^{v(m)} w^m \) defines a section of \( \mathcal{O}_{\mathcal{X}/\mathbb{C}}(d) \) such that the restriction of \( Z_m \) on the irreducible component \( \mathcal{X}_{0, \rho} \) is identical to zero if \( m \) does not belong to \( \rho \). Furthermore, if \( \{ m^0, m^1, m^2, m^3 \} = \rho \cap \Delta_d(\mathbb{Z}) \), then \( Z_{m^i}, i = 0, 1, 2, 3 \), are homogeneous coordinates on \( \mathbb{CP}^3 \). \( Z_m \) are generalised theta functions in the Gross-Siebert sense (cf. [28]).

v) There is a divisor \( D \subset \mathcal{X} \) such that the intersection of \( D \) with any generic fibre \( \mathcal{X}_{w_0} \) is the toric boundary divisor of \( \mathcal{X}_{w_0} \). Moreover, if \( \mathcal{X}_0^o \) denotes the regular locus of \( \mathcal{X}_0 \), then \( \mathcal{X}_0^o \setminus D \) is the disjoint union of \( d^3 \) copies of \( (\mathbb{C}^*)^3 \).

The patchworking polynomial (4.7) reads

\[
\sum_{m \in \Delta_d(\mathbb{Z})} Z_m = 0,
\]

which defines a subvariety of \( \mathcal{X} \) over \( \mathbb{C} \), denoted as \( \tilde{\mathcal{X}} \to \mathbb{C} \). When \( w_0 = t^{-1} \), the fibre \( \tilde{\mathcal{X}}_{w_0} = \mathcal{X}_t \), and therefore \( \tilde{\mathcal{X}} \) can be regarded as an extension of the patchworking family. For any irreducible component \( \mathcal{X}_{0, \rho} \) in the central fibre, \( \tilde{\mathcal{X}}_0 \cap \mathcal{X}_{0, \rho} \) is given by

\[
Z_{m^0} + Z_{m^1} + Z_{m^2} + Z_{m^3} = 0,
\]

where \( \{ m^0, m^1, m^2, m^3 \} = \rho \cap \Delta_d(\mathbb{Z}) \), and \( \tilde{\mathcal{X}}_0 \cap \mathcal{X}_{0, \rho} \cap (\mathcal{X}_0^o \setminus D) \) is the pair-of-pants \( P^2 \). Thus \( \tilde{\mathcal{X}}_0 \cap (\mathcal{X}_0^o \setminus D) \) consists of \( d^3 \) copies of pair-of-pants \( P^2 \), and can be diffeomorphically embedded into \( \mathcal{X}_t \) for \( t \gg 1 \).

4.3. Proofs of i), ii), iii), and v) in Theorem (1.1). We fix a function \( v : \mathbb{Z}^3 \to \mathbb{R} \) by

(4.8) \[
v(m) = 4 \sum_{i=1}^{3} m_i^2 + (2m_1 + 2m_2 + 3m_3)^2,
\]

for any \( m = (m_1, m_2, m_3) \in \mathbb{Z}^3 \), which is the restriction of a positive defined quadratic form \( \tilde{v} \) on \( \mathbb{R}^3 = \mathbb{Z}^3 \otimes \mathbb{R} \). \( v \) induces a Delaunay polyhedral decomposition Del_v of \( \mathbb{R}^3 \) (cf. [2]). Since \( v \) has a \( \mathbb{Z}_2 \)-symmetry by switching \( x_1 \) and \( x_2 \), Del_v is invariant under the \( \mathbb{Z}_2 \)-action. By Lemma 1.8 of [2], Del_v is obtained by projecting the facets of the convex hull of countable points in
the paraboloid \( \{(m, v(m)) \in \mathbb{Z}^2 \times \mathbb{R} | m \in \mathbb{Z}^3 \} \subset \{(x, \tilde{v}(x)) \in \mathbb{R}^3 \times \mathbb{R} | x \in \mathbb{R}^3 \} \). If \( \rho \in \text{Del}_v \) is a 3-cell, then there is a linear affine function \( \ell_\rho : \mathbb{R}^3 \to \mathbb{R} \), called the supporting function of \( \rho \), such that \( \ell_\rho(m) = v(m) \) for \( m \in \rho \cap \mathbb{Z}^3 \), and \( \ell_\rho(m) < v(m) \) for \( m \in (\mathbb{R}^3 \setminus \rho) \cap \mathbb{Z}^3 \).

It is well-known that \( \text{Del}_v \) is invariant under the translation by \( m \in \mathbb{Z}^3 \) as follows. Since, for any \( m = (m_1, m_2, m_3) \in \mathbb{Z}^3 \),

\[
v(m' + m) = v(m') + \sum_{i=1}^{3} (8m_i + 2(2m_1 + 2m_2 + 3m_3))m_i' + v(m),
\]

\( v(m' + m) - v(m') \) is a linear affine function of \( m' \), and \( v(m' + m) \) induces the same decomposition \( \text{Del}_v \), i.e. \( \{ m + \rho \} \mid \rho \in \text{Del}_v \} = \text{Del}_v \).

**Lemma 4.2.** For any integer \( \tilde{d} \geq 1 \), the restriction of \( \text{Del}_v \) on \( \Delta_{\tilde{d}} \) is a unimodular subdivision \( T_v \) of \( \Delta_{\tilde{d}} \), which has \( \tilde{d}^3 \) simplices of dimension 3.

**Proof.** Table 1 shows the restriction of the decomposition \( \text{Del}_v \) on the unit cube \([0,1]^3\), which consists of 6 simplices of dimension 3. We let \( p_0 = (0,0,0) \), \( p_1 = (1,0,0) \), \( p_2 = (0,1,0) \), \( p_3 = (0,0,1) \), \( p_{12} = (1,1,0) \), \( p_{13} = (1,0,1) \), \( p_{23} = (0,1,1) \) and \( p_{123} = (1,1,1) \), and \( \overline{p_ip_jp_kp_l} \) be the convex hull of points \( p_i, p_j, p_k, p_l \), i.e. \( \overline{p_ip_jp_kp_l} = \text{Cov}(p_i, p_j, p_k, p_l) \).

| 3-cells          | functions | \( p_0 \) | \( p_1 \) | \( p_2 \) | \( p_3 \) | \( p_{12} \) | \( p_{13} \) | \( p_{23} \) | \( p_{123} \) |
|------------------|-----------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( p_0p_1p_2p_3 \) | \( 8x_1 + 8x_2 + 13x_3 \) | 0       | 8       | 8       | 13      | 24      | 33      | 33      | 61      |
| \( p_0p_1p_2p_3 \) | \( 16x_1 + 16x_2 + 21x_3 - 8 \) | -8      | 8       | 8       | 13      | 24      | 29      | 29      | 45      |
| \( p_{23}p_0p_1p_2 \) | \( 16x_1 + 20x_2 + 25x_3 - 12 \) | -12     | 4       | 8       | 13      | 24      | 29      | 33      | 49      |
| \( p_{12}p_0p_1p_3 \) | \( 20x_1 + 16x_2 + 25x_3 - 12 \) | -12     | 4       | 8       | 13      | 24      | 33      | 29      | 49      |
| \( p_{123}p_0p_1p_2 \) | \( 20x_1 + 20x_2 + 29x_3 - 16 \) | -16     | 4       | 4       | 13      | 24      | 33      | 33      | 53      |
| \( p_{123}p_0p_1p_2 \) | \( 28x_1 + 28x_2 + 37x_3 - 32 \) | -32     | 4       | -4      | 5       | 24      | 33      | 33      | 61      |

We need to verify that the supporting functions do not support any lattice points outside the cube. Because of the \( \mathbb{Z}^3 \)-translation symmetry and \( \mathbb{Z}^2 \)-reflection symmetry of \( \text{Del}_v \), we check the nearby points \( p'_1 = (0, -1, 0) \), \( p'_2 = (1, -1, 0) \), \( p'_3 = (1, -1, 1) \), \( p'_4 = (0, -1, 1) \), \( p'_5 = (0, 0, -1) \), \( p'_6 = (1, 0, -1) \), \( p'_7 = (1, 1, -1) \), and \( p'_8 = (0, 1, -1) \). Table 2 presents the data.

The \( \mathbb{Z}^3 \)-translation invariance of \( \text{Del}_v \) implies that all 3-cells of \( \text{Del}_v \) are simplices with Euclidean volume \( \frac{1}{6} \). The restriction of \( \text{Del}_v \) on the hyperplanes given by \( x_1 + x_2 + x_3 = \tilde{d} \), for any \( \tilde{d} \geq 1 \), is a subdivision on \( \Delta_{\tilde{d}} \). Since the Euclidean volume of \( \Delta_{\tilde{d}} \) is \( \frac{\tilde{d}^3}{6} \), \( \Delta_{\tilde{d}} \) contains \( \tilde{d}^3 \) simplices.

By using this lemma, we obtain the triangulation \( T_v \) on \( \Delta_{\tilde{d}} \), and the restriction of \( T_v \) on \( \Delta_4 \) is the restriction of \( \text{Del}_v \) on \( \Delta_4 \). The \( \mathbb{Z}^3 \)-translation
Table 2. Values of the functions on nearby lattice points.

| 3-cells   | functions   | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ |
|-----------|-------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $p_0p_1p_2p_3$ | $8x_1 + 8x_2 + 13x_3$ | -8    | 0     | 13    | 5     | -13   | -5    | 3     | -5    |
| $p_1p_2p_3p_4$ | $16x_1 + 16x_2 + 21x_3 - 8$ | -24   | -8    | 13    | -3    | -29   | -13   | 3     | -13   |
| $p_2p_3p_4p_5$ | $16x_1 + 20x_2 + 25x_3 - 12$ | -32   | -16   | 9     | -7    | -37   | 21    | -1    | 17    |
| $p_1p_2p_3p_5$ | $20x_1 + 16x_2 + 25x_3 - 12$ | -28   | -8    | 17    | -5    | -37   | -17   | 1     | -21   |
| $p_2p_3p_4p_5$ | $20x_1 + 20x_2 + 29x_3 - 16$ | -36   | -16   | 13    | -7    | -45   | -25   | 5     | -25   |
| $p_3p_4p_5p_6p_7$ | $28x_1 + 28x_2 + 37x_3 - 32$ | -60   | -32   | 5     | -23   | -69   | -41   | -13   | -41   |

Figure 4.4. Decomposition Del$_v$ on $[0,1]^3$.

The invariance of Del$_v$ implies that for any $m \in \Delta^o_d(Z)$, the restriction of $\mathcal{T}_v$ on $m - (1,1,1) + \Delta_4$ is the translation of the restriction of $\mathcal{T}_v$ on $\Delta_4$.

Again $\Pi_v$ denotes the tropical variety of the restriction of $v$ on $\Delta_d$, i.e. the non-linear locus of the discrete Legendre transform (4.2) of $v|_{\Delta_d}$, which is dual to $\mathcal{T}_v$. The restriction of $v$ on any $m - (1,1,1) + \Delta_4$ induces the same tropical variety $\Pi_{K_3}$ dual to the restriction of $\mathcal{T}_v$ on $\Delta_4$ by the $\mathbb{Z}^3$-translation.

**Lemma 4.3.** Let $\mathcal{T}^o$ be the set of 3-cells $\rho \in \mathcal{T}_v$ such that either $\rho$ belongs to $\Delta^o_d$, i.e. $\rho \subset \Delta^o_d$, or $\rho$ shares a 2-face $\rho'$ with the boundary $\partial \Delta^o_d$, i.e. $\rho' \in \mathcal{T}_v$, $\rho' = \rho \cap \partial \Delta^o_d$, and $\dim \rho' = 2$. Then $\mathcal{T}^o$ consists of $d(d - 4)^2$ simplices, i.e.

$$\# \mathcal{T}^o = d(d - 4)^2.$$  

**Proof.** Note that $\Delta^o_d = (1,1,1) + \Delta_{d-4}$, and the boundary $\partial \Delta^o_d = (1,1,1) + \partial \Delta_{d-4}$ is the union of some 2-cells in $\mathcal{T}_v$. The restriction of $\mathcal{T}_v$ on $(1,1,1) + \Delta_{d-4}$ consists of $(d - 4)^3$ simplices of dimension 3, and $\partial \Delta_{d-4}$ has $4(d - 4)^2$ 2-cells of $\mathcal{T}_v$. For a 2-cell $\rho' \subset \partial \Delta^o_d$, there is a unique 3-cell $\rho$ in $\mathcal{T}_v$ such that $\rho \supset \rho'$ and $\rho$ does not belong to $\Delta^o_d$. Therefore

$$\# \mathcal{T}^o = (d - 4)^3 + 4(d - 4)^2 = d(d - 4)^2.$$
**Figure 4.5.** Illustration of $\Pi_v$ vs $\Pi_{K_3}$ via cases of curves, where $\Pi_1$ is dual to a triangulation $\mathcal{T}$ of $\Delta_4 \subset \mathbb{R}^2$, and $\Pi_2$ is dual to the restriction of $\mathcal{T}$ on $\Delta_3$.

\[\begin{array}{c}
\text{\Pi}_1 \\
\text{\equiv} \\
\text{\Pi}_2
\end{array}\]

**Lemma 4.4.** For any $m \in \Delta_3^o(\mathbb{Z})$, let $T^m$ be the set of 3-cells $\rho \in \mathcal{T}_v$ such that $\rho \subset m - (1, 1, 1) + \Delta_4$. Then

\[
\bigcup_{m \in \Delta_3^o(\mathbb{Z})} \{(1, 1, 1) - m + \rho \mid \rho \in T^m \setminus T^o\} = \{3\text{-cells } \rho \in \mathcal{T}_v \mid \rho \subset \Delta_4\}.
\]

**Proof.** Note that $\Delta_4 = \Delta_3 \cup ((1, 0, 0) + \Delta_3) \cup ((0, 1, 0) + \Delta_3) \cup ((0, 0, 1) + \Delta_3)$, and if a 3-cell $\rho \in \mathcal{T}_v$ belongs to $\Delta_3$, then $\rho$ shares at most one point with $\Delta_3^o(\mathbb{Z})$. For example, if $\rho \subset \Delta_3$, then for any $x \in \rho$, $x_1 + x_2 + x_3 \leq 3$, while for an $x \in \Delta_3^o$, $x_1 + x_2 + x_3 > 3$ except $x = (1, 1, 1)$. Therefore, $\rho$ does not belong to $T^o$.

It is clear that $(1, 1, 1) - m + \rho \subset \Delta_4$ for any $\rho \in T^m$. If $\rho$ is a 3-cell of $\mathcal{T}_v$ and $\rho \subset \Delta_4$, then $\rho$ belongs to one of $\Delta_3$, $(1, 0, 0) + \Delta_3$, $(0, 1, 0) + \Delta_3$ or $(0, 0, 1) + \Delta_3$. For example, if $\rho \subset (1, 0, 0) + \Delta_3$ without loss of generality, then $(d - 4, 0, 0) + \rho \subset (d - 3, 0, 0) + \Delta_3$. We obtain the conclusion by the fact that $\Delta_3 \subset \Delta_4$, $(d - 3, 0, 0) + \Delta_3 \subset (d - 4, 0, 0) + \Delta_4$, $(0, d - 3, 0) + \Delta_3 \subset (0, d - 4, 0) + \Delta_4$ and $(0, 0, d - 3) + \Delta_3 \subset (0, 0, d - 4) + \Delta_4$. \(\square\)

**Proofs of i), ii), iii), and v) in Theorem 1.1.** First, we recall the construction in Section 4 of [40]. Let $\overline{P}^2 = \mathbb{CP}^2 \setminus (\tilde{H}_1 \cup \tilde{H}_2 \cup \tilde{H}_3 \cup \tilde{H}_4)$, where $\tilde{H}_i$ denotes a tubular neighbourhood of the hyperplane $H_i$. The interior $\text{int}(\overline{P}^2)$ is diffeomorphic to $P^2$. The boundary $\partial \overline{P}^2$ admits a stratification $\partial \overline{P}^2 = \partial_0 P^2 \cup \partial_1 P^2$ by Proposition 2.24 of [40], where $\partial_0 P^2$ is the disjoint union of 6 copies of $T^2$, and $\partial_1 P^2$ consists of 4 connected components where each component is diffeomorphic to the total space of a trivial $S^1$-bundle on...
the 2-dimensional pair-of-pants $\mathcal{P}^1$, i.e. $S^1 \times \mathcal{P}^1$ under a certain trivialisation. Moreover $\partial_0 \mathcal{P}^2 \subset \partial_1 \mathcal{P}^2$. The boundary $\partial \mathcal{P}^2$ is obtained by gluing the closures of components of $\partial_1 \mathcal{P}^2$ along $T^2$ components of $\partial_0 \mathcal{P}^2$.

Proposition 4.6 in [30] shows that there exists a proper submanifold $Q^n \subset (\mathbb{C}^*)^{n+1}$, $n = 0, 1, 2$, such that

i) $Q^0$ is a point.

ii) $Q^n$ is isotopic to the hyperplane $H^n = \{(w_1, \cdots, w_{n+1}) \in (\mathbb{C}^*)^{n+1}|w_1 + \cdots + w_{n+1} = 0\} \subset (\mathbb{C}^*)^{n+1}$. Note that the pair-of-pants $\mathcal{P}^n$ is bi-holomorphic to $H^n$.

iii) $Q^n$ is invariant under the symmetric group $\mathbb{S}_{n+2}$-action on $(\mathbb{C}^*)^{n+1}$, which interchanges the homogeneous coordinates on $\mathbb{C}P^{n+1} \supset (\mathbb{C}^*)^{n+1}$.

iv) For a $\varrho \ll -1$,

$$Q^n \cap (\mathbb{C}^*)^n = Q^{n-1} \times \mathbb{C}^*,$$

where $(\mathbb{C}^*)^n = \{(w_1, \cdots, w_{n+1}) \in (\mathbb{C}^*)^{n+1}|\log |w_{n+1}| < \varrho\}$ and $\mathbb{C}^* = \{w \in \mathbb{C}^*|\log |w| < \varrho\}$. Furthermore, $Q^n \cap (\mathbb{C}^*)^n$ is invariant under the translation $w_{n+1} \mapsto cw_{n+1}$ for $0 < c < 1$.

v) If $\tilde{Q}^n = Q^n \cap \log^{-1}(R\Delta)$ where

$$R\Delta = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}|x_i \geq -R, \ i = 1, \cdots, n+1, \ \sum_{i=1}^{n+1} x_i \leq R\},$$

for $R \gg 1$, then $\tilde{Q}^n$ is diffeomorphic to $\overline{\mathcal{P}}^n$ as manifolds with corners.

For each 3-cell $q \in \mathcal{T}_v$, we take a copy $\tilde{Q}^2_q$ of $\overline{\mathcal{P}}^2$, which is identified with $\tilde{Q}^2 \subset (\mathbb{C}^*)^3$. $Q^2_q$ denotes the interior of $\tilde{Q}^2_q$, and $\bar{q}$ is the vertex of $\Pi_v$ corresponding to $q$. Each connected component of $\partial_0 \tilde{Q}^2_q = \partial_1 \mathcal{P}^2$ corresponds to a 2-cell of $\mathcal{T}_v$, which is a face of $q$. If $\rho$ is a 2-cell of $\mathcal{T}_v$ such that $\rho = q \cap q'$, we consider the corresponding components $F_q$ and $F_{q'}$ of $\partial_1 \tilde{Q}^2_q$ and $\partial_1 \tilde{Q}^2_{q'}$ respectively. We glue $\tilde{Q}^2_q$ and $\tilde{Q}^2_{q'}$ along the closures of $F_q$ and $F_{q'}$. More precisely, if we assume that both $F_q$ and $F_{q'}$ are given by $\log |w_3| = -R$ via certain transforms of the symmetry group $\mathbb{S}_4$, then we attach $F_q$ and
Figure 4.7. $\bar{Q}^1$, and $\mathcal{P}^1 = Q^1 \to Y$.

$F_q$ by the map $(w_1, w_2, w_3) \mapsto (w_1, w_2, \bar{w}_3)$, i.e. gluing the 2-dimensional pair-of-pants canonically, and reversing the orientation of the $S^1$-fibre. In such way, we obtain a manifold $Q$ with boundary, i.e.

$$Q = \bigcup_{\text{all 3-cells } q \in T_v} \bar{Q}^2_q.$$  

The boundary $\partial Q$ is obtained by gluing the closures of connected components $F$ of $\partial_1 \bar{Q}^2_q$ corresponding to 2-cells of $T_v$ in $\partial \Delta_d$. Note that any two $F$ and $F'$ are glued along a certain $T^2$ component. Thus we have stratified structure $\partial Q = \partial_0 Q \cup \partial_1 Q$ where $\partial_0 Q$ consists of finite $T^2$, and each connected component of $\partial_1 Q$ is a $S^1$-bundle.

Let $W^o = Q \setminus \partial Q$, and let $W$ be the space obtained by collapsing the $S^1$ fibres of $\partial_1 Q$ (cf. Section 4 in [40]). Note that $W$ is a differential manifold since the collapsing locally coincides with collapsing the boundary of $P^2$ in $\mathbb{C} P^2$. Theorem 4 of [40] proves that $W^o$ is diffeomorphic to $X^o \subset X_t$, and $W$ is diffeomorphic to $X_t$. Thus $X_t$ is decomposed into $d^3$ copies of pair-of-pants.

Let

$$M^o = \coprod_{q \in T^o} Q^2_q,$$

which is the disjoint union of $d(d - 4)^2$ copies of pair-of-pants $P^2$ by Lemma 4.3. Note that $q$ intersects with the boundary $\partial \Delta_d$ at most one point, i.e. a vertex of $q$, which corresponds to a connected component of the complement of $\Pi_v$ in $\mathbb{R}^3$. Therefore any non-zero dimensional face $\rho \subset q$ does not belong to the boundary $\partial \Delta_d$, and the closure $\overline{M^o} \subset W^o$. Furthermore,

$$\overline{M^o} \setminus M^o = \bigcup_{q \in T^o} \partial \bar{Q}^2_q.$$  

Note that there is a fibration $P^1 \to Y$ from $P^1$ to a graph of $Y$-shape with generic fibres $S^1$ and one singular fibre of shape $\Theta$. For each $\bar{Q}^2_q$, any component $F$ of $\partial_1 \bar{Q}^2_q$ admits a fibration $F \to Y$ to a $Y$-shape graph $Y$ with generic fibres $T^2$ and one singular fibre of shape $\Theta \times \bigcirc$. Since $\partial_0 \bar{Q}^2_q$ consists of 6 copies of $T^2$, we have a fibration $\partial \bar{Q}^2_q \to B_q$ where $B_q$
is a graph consisting of 4 vertices and 6 edges with generic fibres \( T^2 \) and 6 singular fibres \( \ominus \times \odot \). If we have glued \( \bar{Q}^2_q \) and \( \bar{Q}^2_q' \) along the components \( F_q \) and \( F_q' \) of \( \partial_1 \bar{Q}_q \) and \( \partial_1 \bar{Q}_q' \) respectively, then we glue the closures of the corresponding graphs \( Y_q \) and \( Y_q' \), and obtain a graph \( B \). Moreover, there is a fibration \( M^o \setminus M^o \to B \) with generic fibres \( T^2 \) and finite singular fibres of shape \( \ominus \times \odot \).

We have proved i) and ii) of Theorem 1.1, and now we prove the statement iii).

We apply the above construction to the restriction of \( T_v \) on \( \Delta_4 \) and \( \Pi_{K3} \), and obtain a manifold \( Y \) diffeomorphic to the K3 surface. More precisely, for any \( m \in \Delta_4^o(Z) \), we consider \( m - (1, 1, 1) + \Delta_4 \). If we let \( Q_m = \bigcup_{q \in T^m} \bar{Q}_q^2 \subset Q \), where \( T^m \) is defined in Lemma 4.4, then as above, \( Q_m \) is a manifold with boundary \( \partial Q_m \), which is obtained by gluing the closures of connected components \( F \) of \( \partial_1 \bar{Q}_q^2 \) corresponding to 2-cells in \( m - (1, 1, 1) + \Delta_4 \). Since the restriction of \( T_v \) on \( m - (1, 1, 1) + \Delta_4 \) is the translated to the restriction of \( T_v \) on \( \Delta_4^o \), \( Q_m \) is independent of the choice of \( m \), and we obtain \( Y \) by collapsing the boundary \( \partial Q_m \) as the above construction. And \( Y \) is a K3 surface.

If \( Q'_m = \bigcup_{q \in T^m \setminus T^o} \bar{Q}_q^2 \subset Q_m \), then we collapse the closures of the components of \( \partial_1 \bar{Q}_q^2 \) corresponding to 2-cells in \( \partial \Delta_d \cap \partial (m - (1, 1, 1) + \Delta_4) \). Note that if \( q \) shares only one 2-cell (or two 2-cells) with \( \partial \Delta_d \), then the interior of the collapsed \( \bar{Q}_q \) is \((\mathbb{C}^*)^2\) \((\mathbb{C} \times \mathbb{C}^*)\) respectively. We obtain a manifold \( Y^m \) with boundary, and denote the interior as \( Y^o_m = Y_m \setminus \partial Y_m \), which satisfies

\[ Y^o_m \supset \coprod_{q \in T^m \setminus T^o} Q^2_q. \]

Then \( Y^o_m \) can be regarded as a subset both in \( Y \) and \( W \), i.e. \( Y^o_m \subset Y \) and \( Y^o_m \subset W \), and can have many connected components including \((\mathbb{C}^*)^2\) and \(\mathbb{C} \times \mathbb{C}^*\).

Since \[ \bigcup_{m \in \Delta_4^o(Z)} (T^m \setminus T^o) \cup T^o \] consists of all 3-cells of \( T_v \),

\[ W = \overline{M^o} \bigcup \bigcup_{m \in \Delta_4^o(Z)} \overline{Y^o_m}, \quad \text{and} \quad Y = \bigcup_{m \in \Delta_4^o(Z)} \overline{Y^o_m}, \]

by regarding \( Y^o_m \subset Y \) and Lemma 4.4. Note that \( Y^o_m \) does not intersect with \( Q^2_q \) for any \( m \in \Delta_4^o(Z) \) and \( q \in T^o \), when they are regarded as subsets in \( W \), and thus \( Y^o_m \cap M^o \) is empty. We obtain iii) by choosing an open dense subset \( M' \subset Y \) such that any connected component of \( M' \) belongs to a certain \( Y^o_m \).
Finally the Riemann-Roch formula shows that
\[ 2\chi(X) + 3\tau(X) = K_X^2 = d(d - 4)^2, \]
and we obtain v) by (3.1). \qedhere

We remark that many arguments in this section may be carried out in
the more general frame work, the Gross-Siebert programme (cf. [27, 28]).
We expect to generalise Theorem 1.1 to more general cases, and leave it to
the future research.

5. Degenerations

The final section finishes the proof of Theorem 1.1. We are continuing to
use the notations and the conventions in Section 4.

Recall that \( X_t \) is the patchworking family given by the polynomial
\begin{equation}
(5.1) \quad f_t(w) = \sum_{m \in \Delta_d(Z)} t^{-v(m)} w^m, \quad t \in (1, \infty),
\end{equation}
where \( v \) is defined by (4.8), which is a family of degree \( d \) hypersurfaces in
\( \mathbb{C}P^3 \). If \( [z_0, z_1, z_2, z_3] \) are the homogeneous coordinates on \( \mathbb{C}P^3 \), then
\( w_i = z_i/z_0, \quad i = 1, 2, 3 \). Note that the canonical bundle \( \mathcal{O}_{X_t}(K_{X_t}) = \mathcal{O}_{\mathbb{C}P^3}(d-4)|_{X_t} \)
by the adjunction formula, and is very ample.

For any \( m = (m_1, m_2, m_3) \in \Delta^3(Z) \), since \( m_1 + m_2 + m_3 \leq d - 1 \), and
\( m_i \geq 1, \quad i = 1, 2, 3 \),
\begin{equation}
- v(m) \frac{w^m dw_1 \wedge dw_2 \wedge dw_3}{w^{(1,1,1)} f_t} = - v(m) \frac{w_1^{m_1}w_2^{m_2-1}w_3^{m_3-1} dw_1 \wedge dw_2 \wedge dw_3}{f_t}
\end{equation}
is a meromorphic 3-form on \( \mathbb{C}P^3 \), and has a simple pole along \( X_t \). The
Poincaré residue formula gives a holomorphic 2-form
\begin{equation}
\Omega_m = - v(m) \text{Res}_{X_t} \frac{w^m dw_1 \wedge dw_2 \wedge dw_3}{w^{(1,1,1)} f_t}
\end{equation}
on \( X_t \), i.e.
\begin{equation}
\Omega_m = - v(m) \frac{w^m dw_2 \wedge dw_3}{w^{(1,1,1)} \frac{\partial f_t}{\partial w_1}} = - v(m) \frac{w^m dw_1 \wedge dw_3}{w^{(1,1,1)} \frac{\partial f_t}{\partial w_2}} = - v(m) \frac{w^m dw_1 \wedge dw_2}{w^{(1,1,1)} \frac{\partial f_t}{\partial w_3}}
\end{equation}
on \( X_t \). Note that \( \Omega_m \) represents a non-trivial cohomological class, i.e.
\begin{equation}
0 \neq [\Omega_m] \in H^2(X_t, \mathbb{C}).
\end{equation}

5.1. Proof of iv) in Theorem 1.1. Let \( \rho \) be a 3-cell in \( T_v \) such that
\( \rho \cap \Delta^3_d(Z) \) is not empty, and let \( m \in \rho \cap \Delta^3_d(Z) \). If \( m' \in \rho \cap \Delta^3_d(Z) \) is a
different vertex of \( \rho \), i.e. \( m \neq m' \), then the 1-cell \( m, m' \subseteq \rho \)
connecting \( m \) and \( m' \) corresponds a 2-cell \( \Pi_{m,m'} \) in \( T_v \). For any \( x \in \text{int}(\Pi_{m,m'}) \),
\begin{equation}
l_m(x) = l_{m'}(x) > l_{m''}(x),
\end{equation}
for any \( m'' \in \Delta_d(Z) \setminus \{m, m'\} \), where \( l_m(x) = \langle m, x \rangle - v(m) \). We assume that
\( m_3 - m'_3 \neq 0 \) without loss of generality.
By Corollary 6.4 in [10], the amoebas $A_t$ converges to $\Pi_c$ when $t \to \infty$, and furthermore, Theorem 5 of [10] asserts that a certain normalisation of $X_t^o$ converges to a limit in $\mathbb{C}^*$.$^3$. We take a close look at the limit in the current case.

If $V$ is a small open neighborhood of a point in $\text{int}(\Pi_{m,m'})$ such that $V \cap \Pi_c \subset \text{int}(\Pi_{m,m'})$, then $V \cap \Pi_c$ belongs to the hyperplane given by the equation

$$l_m(x) - l_{m'}(x) = \langle m - m', x \rangle - v(m) + v(m') = 0.$$ Let $\theta_i$ be the angle of $w_i$, i.e.

$$w_i = \exp(x_i \log(t) + \sqrt{-1} \theta_i), \quad i = 1, 2, 3.$$ We regard $(\theta_1, \theta_2, \theta_3)$ as angular coordinates on $T^3$, and identify $(\mathbb{C}^*)^3 = \mathbb{R}^3 \times T^3$ by $w \mapsto (x, \theta)$, where $x = (x_1, x_2, x_3)$ and $\theta = (\theta_1, \theta_2, \theta_3)$. Equivalently, we regards $(\mathbb{C}^*)^3$ as the quotient of $\mathbb{C}^3$ by $\sqrt{-1} \mathbb{Z}^3$.

**Lemma 5.1.** If $H_t : (\mathbb{C}^*)^3 \to \mathbb{R}^3 \times T^3$ is defined by $w \mapsto (x, \theta)$, then $H_t(X_t^o) \cap (V \times T^3)$ converges to $W_V$ in the Hausdorff sense, when $t \to \infty$, where

$$W_V = \{(x, \theta) \in V \times T^3 | l_m(x) = l_{m'}(x), \quad \langle m - m', \theta \rangle = \pi + 2\pi \mathbb{Z}\},$$

and $\langle m, \theta \rangle = m_1 \theta_1 + m_2 \theta_2 + m_3 \theta_3$.

**Proof.** Note that the log map $\text{Log}_t$ is the composition of $H_t$ with the projection $\mathbb{R}^3 \times T^3 \to \mathbb{R}^3$. For any $x \in V \cap A_t$ and $w \in \text{Log}_t^{-1}(x) \cap X_t^o$, $t^{-v(m)}u^m = l_m(x)$, and the patchworking polynomial (5.1) says that

$$0 = e^{\sqrt{-1} \langle m, \theta \rangle} + t^{1}_{m'}(x) l_m(x) e^{\sqrt{-1} \langle m', \theta \rangle} + \sum_{m'' \in \Delta_d(Z) \setminus \{m, m''\}} t^{1}_{m''}(x) - l_m(x) e^{\sqrt{-1} \langle m'', \theta \rangle},$$

which is the defining equation of $X_t^o \cap (V \times T^3)$. When $t \to \infty$, since $t^{1}_{m''}(x) - l_m(x) \to 0$, we obtain

$$e^{\sqrt{-1} \langle m, \theta \rangle} + t^{1}_{m'}(x) - l_m(x) e^{\sqrt{-1} \langle m', \theta \rangle} \to 0, \quad \text{and} \quad t^{1}_{m'}(x) - l_m(x) \to 1.$$

The limiting equations are

$$\langle m - m', \theta \rangle = \pi + 2\pi \mathbb{Z}, \quad \text{and} \quad l_m = l_{m'},$$

which define $W_V$. If we define a neighbourhood $\Xi_t$ of $W_V$ in $V \times T^3$ by the inequality

$$\left| e^{\sqrt{-1} \langle m, \theta \rangle} + t^{1}_{m'}(x) - l_m(x) e^{\sqrt{-1} \langle m', \theta \rangle} \right| < 2 \sup_{x' \in V} \left( \sum_{m'' \in \Delta_d(Z) \setminus \{m, m''\}} t^{1}_{m''}(x') - l_m(x') \right),$$

then $X_t^o \cap (V \times T^3) \subset \Xi_t$, and converges to $W_V$ in the Hausdorff sense. $\square$

We have an isotopic embedding $\phi_t : W_V \to X_t^o$ with $\phi_\infty = \text{id}$, which is certainly not unique. One way to obtain $\phi_t$ is to integrate a vector field $(u, \vartheta)$ satisfying

$$\frac{\partial f_t}{\partial m} + \langle \frac{\partial f_t}{\partial x_2}, u \rangle + \langle \frac{\partial f_t}{\partial \vartheta}, \vartheta \rangle = 0,$$

where $f_t = t^{-l_m(x)}f_t$, since a direct
calculation shows that $|\frac{\partial f_t}{\partial x_3}| > \frac{1}{2}|m'_3 - m_3| \log(t)$, and $|\frac{\partial f_t}{\partial \varphi}| > \frac{1}{2}|m'_3 - m_3|$, for $t \gg 1$.

If

$$T_{m,m'} = \{ \theta \in T^3 | (m - m', \theta) = \pi + 2\pi \mathbb{Z} \},$$

then

$$W_V = (V \cap \Pi_v) \times T_{m,m'}.$$  

**Lemma 5.2.** For any isotopic embedding $\phi_t : W_V \to X_t^0$ with $\phi_\infty = \text{id}$ and $x \in V \cap \Pi_{m,m'}$,

$$\int_{\phi_t \{ x \} \times T_{m,m'}} \Omega_m \neq 0,$$

and consequently, $\phi_t \{ x \} \times T_{m,m'}$ represents a non-zero class in $H_2(X_t, \mathbb{R})$ for $t \gg 1$, i.e.

$$0 \neq [\phi_t \{ x \} \times T_{m,m'}] \in H_2(X_t, \mathbb{R}).$$

Furthermore, if $\omega$ is a toric symplectic form on $\mathbb{CP}^3$, then

$$\int_{\phi_t \{ x \} \times T_{m,m'}} \omega = 0.$$

**Proof.** A direct calculation shows that

$$w_3 \frac{\partial f_t}{\partial w_3} = m_3 t^{-v(m)} w^m + m'_3 t^{-v(m')} w^{m'} + \sum_{m'' \in \Delta_s(\mathbb{Z}) \setminus m,m'} m''_3 t^{-v(m'')} w^{m''}$$

$$= (m_3 - m'_3) t^{-v(m)} w^m + m'_3 f_t + \sum_{m'' \in \Delta_s(\mathbb{Z}) \setminus m,m'} (m''_3 - m'_3) t^{-v(m'')} w^{m''}$$

$$= (m_3 - m'_3) t^{-v(m)} w^m + m'_3 f_t + o(t).$$

Note that on $V$,

$$\left| \frac{t^{-v(m'')} |w^{m''}|}{t^{-v(m)} |w^m|} \right| = t(l_{m''} - l_m)(\log_t(w)) \to 0,$$

and thus

$$\left| \frac{o(t)}{t^{-v(m)} |w^m|} \right| \to 0, \text{ when } t \to \infty.$$  

Therefore

$$\Omega_m \mid_{\log_t^{-1}(V) \cap X_t} = \frac{d \log(w_1) \wedge d \log(w_2)}{m_3 - m'_3 + t^{v(m)} w^m o(t)},$$

and

$$\int_{\phi_t \{ x \} \times T_{m,m'}} \Omega_m \to \int_{T_{m,m'}} \frac{d\theta_1 \wedge d\theta_2}{m'_3 - m_3} \neq 0, \text{ when } t \to \infty.$$  

We obtain the first conclusion.

Since $(\mathbb{C}^*)^3 = \mathbb{R}^3 \times T^3 \to \mathbb{R}^3$ is a Lagrangian fibration with respect to $\omega$, i.e. $\omega \{ x \} \times T^3 \equiv 0$, we have

$$\int_{\phi_t \{ x \} \times T_{m,m'}} \omega = \int_{\{ x \} \times T_{m,m'}} \omega = 0.$$
We continue to prove Theorem 1.1.

Proof of iv) in Theorem 1.1. Let $Q^2 \subset (\mathbb{C}^*)^3$ be the submanifold constructed in Proposition 4.6 of [10]. Recall that the last assertion of Proposition 4.6 of [10] says that for a certain $\varrho \ll -1$, $Q^n \cap (\mathbb{C}^*)^n_\varrho = Q^{n-1} \times C^n_\varrho$, where $n = 1, 2$, and $(\mathbb{C}^*)^2_\varrho = \{(w_1, \cdots, w_{n+1}) \in (\mathbb{C}^*)^{n+1} | \log |w_{n+1}| < \varrho \}$, and $Q^n \cap (\mathbb{C}^*)^{n+1}$ is invariant under the translation $w_{n+1} \mapsto cw_{n+1}$ for $0 < c < 1$. Hence

\[ \tilde{Q}_1 = Q^2 \cap \{(w_1, w_2, w_3) \in (\mathbb{C}^*)^3 | \log |w_2| < \varrho, \log |w_3| < \varrho \} \]

\[ = \{(w_1, w_2, w_3) \in (\mathbb{C}^*)^3 | \log |w_2| < \varrho, \log |w_3| < \varrho, w_1 \equiv -1 \}, \]

where we choose $w_1 \equiv -1$ that defines $Q^0 \subset \mathbb{C}^*$. If we consider the identification $(\mathbb{C}^*)^3 = \mathbb{R}^3 \times T^3$, via $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\theta = (\theta_1, \theta_2, \theta_3) \in T^3$, and $w_i = \exp(x_i + \sqrt{-1} \theta_i)$, $i = 1, 2, 3$, then

\[ \tilde{Q}_1 = \{x \in \mathbb{R}^3 | x_1 = 0, x_2 < \varrho, x_3 < \varrho \} \times \{\theta \in T^3 | \theta_1 = \pi \}. \]

If $\Pi$ denotes the tropical hyperplane of the pair-of-pants $H^0$ defined by $1 + w_1 + w_2 + w_3 = 0$, i.e. the non-smooth locus of $\max \{0, x_1, x_2, x_3 \}$, then $\Pi_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0, x_2 < 0, x_3 \leq 0 \}$ is the 2-cell in $\Pi$ corresponding to the 1-cell $(0, 0, 0), (1, 0, 0)$ in the standard simplex $\Delta$. Since the Log map is the projection of $\mathbb{R}^3 \times T^3 \rightarrow \mathbb{R}^3$, $\Log(\tilde{Q}_1) \subset \Pi_1$,

\[ \Log_t(\tilde{Q}_1) = \{x \in \mathbb{R}^3 | x_1 = 0, x_2 < \frac{\varrho}{\log(t)}, x_3 < \frac{\varrho}{\log(t)} \} \subset \Pi_1, \]

and $\bigcup_{t > 1} \Log_t(\tilde{Q}_1) = \text{int}(\Pi_1)$. Furthermore, if $V$ is an open subset such that $V \cap \Pi_1 \subset \text{int}(\Pi_1)$, then the same argument as in the proof of Lemma 5.4 shows that $H_t(H^0 \cap (V \times T^3))$ converges to $(\Pi_1 \cap V) \times \{\theta \in T^3 | \theta_1 = \pi \} \subset \tilde{Q}_1$ in the Hausdorff sense.

By the $\mathbb{S}_3$-symmetries of $\Pi$ and $Q^2$, there is an open subset $\tilde{Q}^2 \subset Q^2$ consisting of 6 connected components, and each component is a copy of $\tilde{Q}_1$.
by passing to the $S_4$-action on $(\mathbb{C}^*)^3$. Moreover, $\bigcup_{t>1} \log_t(\hat{Q}^2)$ is the union of the interior of 2-cells of $\Pi$.

Now, we recall the proof of Theorem 4 in [40], which is in Subsection 6.6 of [40]. Note that $\text{ASL}(3, \mathbb{Z})$ acts on $\mathbb{Z}^3$ by $A(m)_j = \sum a_j m_i + b_j$ for any $A \in \text{ASL}(3, \mathbb{Z})$, and acts on $(\mathbb{C}^*)^3$ by $w^m \mapsto w^A(m)$. For any 3-cell $q$ in $\mathcal{T}_v$, there is an open neighbourhood $V_q \subset \mathbb{R}^3$ of the vertex $o \in \Pi$, and an $A_q \in \text{ASL}(3, \mathbb{Z})$ such that $\bar{q} = A_q(o)$ where $\bar{q} \in \Pi_v$ is corresponding vertex of $q$, and

$$\Pi_v = \bigcup_{\text{all 3-cells } q \in \mathcal{T}_v} A_q(V_q \cap \Pi).$$

Then $Q_t = \bigcup_{\text{all 3-cells } q \in \mathcal{T}_v} A_q((V_q \times T^3) \cap \Pi_v(Q^2))$ is diffeomorphic to $W^o$, and is isotopic to $X_t^o$ for $t \gg 1$ by Subsection 6.6 of [40].

Under the identification $W^o = Q_t$, we have $M^o \subset Q_t^o$, and $Q_t^2 \subset \text{ASL}(V_q \times T^3) \cap \Pi_v(Q^2)$ for any $q \in \mathcal{T}_v$. If $T^2$ is a generic fibre of $\lambda : \text{ASL}(M^o \setminus M^o) \to B$, then $T^2$ is a generic fibre of $\lambda_q : \partial Q_t^2 \to B_q$ for a certain $q \in \mathcal{T}_v$. Recall that $Q_t^2 = \text{ASL}((R \Delta \times T^3) \cap \Pi_v(Q^2))$ where $R\Delta = \{x \in \mathbb{R}^3|x_i \geq -R, i = 1, 2, 3, x_1 + x_2 + x_3 \leq \mathcal{R}\}$ for a certain $R > 0$, and $R\Delta \subset V_q$. The boundary $\partial Q_t^2 = \text{ASL}((\partial(R\Delta) \times T^3) \cap \Pi_v(Q^2))$ consists of 4 copies of $S^1$-bundles over $\mathcal{P}^1$ gluing along 6 copies of $T^2$. Note that for any 2-face $\rho_2 \subset \partial(R\Delta)$, for example given by $x_1 = -R$, the intersection

$$(\rho_2 \times T^3) \cap \Pi_v(Q^2) \subset Q^1 \times \{w_1 \in \mathbb{C}^*|\log|w_1| = -R\} = Q^1 \times S^1,$$

and is diffeomorphic to $\mathcal{P}^1 \times S^1$. The fibration $\lambda_q$ is obtained by gluing $(\rho_2 \times T^3) \cap \Pi_v(Q^2) \to \rho_2 \cap \Pi$ along boundaries, and $\rho_2 \cap \Pi$ is the Y-shape graph.

Let $T^2$ be a generic fibre of $\lambda : \text{ASL}(M^o \setminus M^o) \to B$. By letting $t \gg 1$, we can take $A_{-1}(T^2)$ as a fibre of $\text{ASL}(\rho_2 \times T^3) \cap \Pi_v(Q^2) \to \rho_2 \cap \log_t(\hat{Q}^2)$, which is the restriction of the $T^2$-fibration $\hat{Q}^2 \to \log_t(\hat{Q}^2)$. By passing to the $S_4$-action, we can assume that

$$A_{-1}(T^2) = \{(x, \theta) \in \mathbb{R}^3 \times T^3|\theta_3 = \pi, x_1 = -R, x_2 = -\frac{R}{2}, x_3 = 0\}.$$

We apply Lemma 5.2 to $T^2 \subset A_q((V_q \times T^3) \cap \Pi_v(Q^2))$, and obtain the conclusion iv).

### 5.2. Proof of vi) in Theorem 1.1

Note that $\{\Omega_m|m \in \Delta^o(\mathbb{Z})\}$ is a basis of $H^0(X_t, K_{X_t})$, and defines an embedding

$$\Psi_t : X_t \to \mathbb{CP}^N, \quad N = \#\Delta^o(\mathbb{Z}) - 1 = p_g - 1,$$

such that $\mathcal{O}_{X_t}(K_{X_t}) = \Psi_t^* \mathcal{O}_{\mathbb{CP}^N}(1)$. If

$$Z_m = t^{-v(m)}w^m, \quad m \in \Delta^o(\mathbb{Z}),$$
then we embed $X^o_\rho$ in $(\mathbb{C}^*)^N \subset \mathbb{CP}^N$ via $Z_m/Z_{(1,1,1)}$, $m \in \Delta_o(Z)$, and the closure of $X^o_\rho$ in $\mathbb{CP}^N$ is the image $W_t(X_\rho)$. We regard $Z_m$, $m \in \Delta_o(Z)$, as the homogeneous coordinates of $\mathbb{CP}^N$. Furthermore, $Z_m$ are generalised theta functions in the sense of [28].

For any 2-cell $\rho_2$ in $\partial \Delta^0_\rho$ of the subdivision $\mathcal{T}_\nu$, let

$$X_{\rho_2} = \{Z_m = 0, \ m \in \Delta^0_\rho(Z) \setminus \rho_2\} \subset \mathbb{CP}^N,$$

and for any 3-cell $\rho_3 \subset \Delta^0_\rho$, let

$$X_{\rho_3} = \{Z_{m_0} + Z_{m_1} + Z_{m_2} + Z_{m_3} = 0, \ \{m_0, m_1, m_2, m_3\} = \Delta^0_d(Z) \cap \rho_3, \quad \text{and} \quad Z_m = 0, \ m \in \Delta^0_\rho(Z) \setminus \rho_3\} \subset \mathbb{CP}^N.$$ 

We define a singular variety

$$X_0 = \left( \bigcup_{\text{all \ 3-cells } \rho_3 \subset \Delta^0_\rho} X_{\rho_3} \right) \bigcup \left( \bigcup_{\text{all \ 2-cells } \rho_2 \subset \partial \Delta^0_\rho} X_{\rho_2} \right),$$

which has $d(d - 4)^2$ irreducible components by Lemma 5.3 and each component is the complex projective plane, i.e. $X_\rho = \mathbb{CP}^2$.

**Lemma 5.3.** The regular locus $X^o_0$ of $X_0$ consists of $d(d - 4)^2$ copies of pair-of-pants, i.e. $X^o_0 = \prod_{d(d - 4)^2} \mathbb{P}^2$.

**Proof.** If $X_\rho$ is the corresponding component of a 3-cell $\rho \subset \Delta^0_\rho$, then the intersection with another component $X_{\rho'}$ is a hyperplane $\{Z_{m_0} = 0\} \cap X_{\rho'}$ without loss of generality, where $\{m_0, m_1, m_2, m_3\} = \Delta^0_d(Z) \cap \rho$ and $\{m_1, m_2, m_3\} \subset \rho \cap \rho'$. Thus the regular locus

$$X^o_\rho = \{[Z_{m_0}, Z_{m_1}, Z_{m_2}, Z_{m_3}] \in X_\rho | Z_{m_0} Z_{m_1} Z_{m_2} Z_{m_3} \neq 0\}$$

is a pair-of-pants, i.e. $X^o_\rho = \mathbb{P}^2$. More precisely, if we regards $X_\rho$ as the projective plane $\{[Z_{m_1}, Z_{m_2}, Z_{m_3}] \in \mathbb{CP}^2\}$, then $Z_{m_i} = 0, \ i = 1, 2, 3$, give two coordinates axis, and the infinite line, and $-Z_{m_0} = Z_{m_1} + Z_{m_2} + Z_{m_3} = 0$ defines the fourth line.

If $X_\rho$ corresponds to a 2-cell $\rho \in \partial \Delta^0_\rho$, then the intersection with other three components $X_{\rho'}$ corresponding three different 2-cells $\rho' \in \partial \Delta^0_\rho$ is given by $Z_{m_1} Z_{m_2} Z_{m_3} = 0$ where $\{m_1, m_2, m_3\} = \Delta^0_d(Z) \cap \rho$, which is the union of coordinates axis and the infinite line. Since there is a 3-cell $\hat{\rho} \subset \Delta^0_\rho$ such that $\rho \subset \partial \hat{\rho}$, $X_\rho \cap X_{\hat{\rho}}$ is given by $-Z_{m_0} = Z_{m_1} + Z_{m_2} + Z_{m_3} = 0$ where $m_0$ is the vertex of $(\hat{\rho}) \cap \Delta^0_d(Z)$. Hence the regular locus $X^o_\rho$ of $X_\rho$ is a pair-of-pants.

**Lemma 5.4.** Let $\rho$ be a 3-cell of $\mathcal{T}_\nu$ such that $\rho \subset \Delta^0_\rho$, $\{m_0, m_1, m_2, m_3\} = \rho \cap \Delta^0_d(Z)$, and $\ell_\rho$ be the supporting function of $\rho$.

1) For any $m \in \Delta_d(Z)$,

$$w^m = t^{\ell_\rho(m)} Z_{m_0} Z_{m_1} Z_{m_2} Z_{m_3},$$

where $m = a_0 m_0 + a_1 m_1 + a_2 m_2 + a_3 m_3$, and $a_0 + a_1 + a_2 + a_3 = 1$. 


ii) If \( m \in \Delta_d^3(\mathbb{Z}) \), then
\[
Z_m^0 Z_m^1 Z_m^2 Z_m^3 = t^{\ell_\rho(m)} - v(m) Z_m^0 Z_m^1 Z_m^2 Z_m^3 \,
\]
where \( b_i = \max\{0, -a_i\} \), \( i = 0, 1, 2, 3 \).

iii) If \( \rho' \) is a 3-cell sharing a 2-face with \( \rho \), i.e. \( \rho \cap \rho' \) is a 2-cell in \( T_v \), and \( \{m^1, m^2, m^3, m^4\} = \rho \cap \Delta_d(\mathbb{Z}) \), then
\[
w^{m^1 + m^0} = w^{m^1 + m^2 + m^3} \quad \text{and} \quad w^{m^4} Z_m^0 = t^{\ell_{\rho'}(m^4)} Z_m^0 Z_m^2 Z_m^3 \,
\]
where two elements of \( \{\epsilon_1, \epsilon_2, \epsilon_3\} \) equal to 1, and one element is 0, e.g. \( \epsilon_1 = \epsilon_2 = 1 \) and \( \epsilon_3 = 0 \).

**Proof.** Note that \( \rho \) is a standard simplex generated by a \( \mathbb{Z} \)-basis \( e_1, e_2, e_3 \in \mathbb{Z}^3 \), i.e. \( \rho = \{x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathbb{R}^3 | x_i \geq 0, i = 1, 2, 3, x_1 + x_2 + x_3 \leq 1\} \) and \( \mathbb{Z}^3 = Z \cdot e_1 + Z \cdot e_2 + Z \cdot e_3 \). For example, \( \epsilon_1 = (1, -1, 0) \), \( \epsilon_2 = (1, 0, 0) \) and \( \epsilon_3 = (0, -1, 1) \) generate a simplex in \( T_v \).

If we regard \( m^0 \) as the origin, then \( \epsilon_i = m^i - m^0 \), \( i = 1, 2, 3 \), is the \( \mathbb{Z} \)-basis. Therefore, for any \( m \in \Delta_d(\mathbb{Z}) \), \( m = m^0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \) where \( a_i \in \mathbb{Z} \). Since \( w^{m^j} = w^{m^j} w^{e_j} \), \( j = 0, 1, 2, 3 \), we have
\[
w^m = w^{m^0} \prod_{i=1}^{3} (w^{e_i})^{a_i} = t^{v(m^0)} Z_m^0 \prod_{i=1}^{3} \left(t^{v(m^i) - v(m^0)} Z_m^i / Z_m^0\right)^{a_i}.
\]

Let \( \ell_\rho(x) = \langle n_\rho, x \rangle + b_\rho \) be the supporting function of the cell \( \rho \), i.e. \( \ell_\rho(m^j) = v(m^j) \), \( j = 0, 1, 2, 3 \), and \( v(m) > \ell_\rho(m) \) for any \( m \in \Delta_d(\mathbb{Z}) \setminus \rho \). Then
\[
v(m^i) - v(m^0) = \langle n_\rho, e_i \rangle, \quad \text{and} \quad \ell_\rho(m) = v(m^0) + \langle n_\rho, \sum_{i=1}^{3} a_i e_i \rangle.
\]

Thus
\[
w^m = t^{\ell_\rho(m)} Z_m^0 Z_m^1 Z_m^2 Z_m^3,
\]
where \( a_0 = 1 - \sum_{i=1}^{3} a_i \).

If we write \( m^i = \epsilon_0 m^0 + \epsilon_1 m^1 + \epsilon_2 m^2 + \epsilon_3 m^3 \), then \( m^4 - m^0 = \sum_{i=1}^{3} \epsilon_i (m^i - m^0) \). Note that \( \rho' \) belongs to the cube generated by \( m^i - m^0 \), i.e. \( 0 \leq \epsilon_i \leq 1 \), for \( i = 1, 2, 3 \), and \( \rho \cap \rho' \) belongs to the hyperplane \( \{x_1 (m^i - m^0) | x_1 + x_2 + x_3 = 1\} \). Thus \( \epsilon_0 = 1 - \epsilon_1 - \epsilon_2 - \epsilon_3 < 0 \), and \( \sum_{i=1}^{3} (m^i - m^0) \) belongs to a 3-cell sharing no faces with \( \rho \). We assume that \( \epsilon_0 = -1, \epsilon_1 = \epsilon_2 = 1, \) and \( \epsilon_3 = 0 \) without loss of generality. Therefore
\[
w^{m^4 + m^0} = w^{m^1 + m^2 + m^3}.
\]
Figure 5.2. \( m^0 \in \rho \) vs \( m^4 \in \rho' \), and \( w^{m^4} Z_{m^0} = t^{\ell}(m^4) Z_{m^2} Z_{m^3} \).

and we obtain the conclusion. \( \square \)

Note that if \( d = 5 \), then the canonical bundle \( \mathcal{O}_{X_t}(K_{X_t}) = \mathcal{O}_{\mathbb{CP}^3}(1)|_{X_t} \), and \( X_t \subset \mathbb{CP}^3 \) is the canonical embedding. Thus vi) in Theorem 1.1 holds as shown in the introduction. Now we assume \( d > 5 \). Then \( \Delta_d \) contains \( (d - 4)^3 \) 3-cells of \( \mathcal{T}_v \).

Lemma 5.5. Let \( \rho \subset \Delta_d \) be a 3-cell in \( \mathcal{T}_v \), and \( \rho' \) be another 2-cell sharing a 2-face with \( \rho \), i.e. \( \rho \cap \rho' \) is a 2-cell in \( \mathcal{T}_v \). Denote \( \{m^0, m^1, m^2, m^3\} = \rho \cap \Delta_d(\mathbb{Z}) \) and \( \{m^1, m^2, m^3, m^4\} = \rho' \cap \Delta_d(\mathbb{Z}) \).

i) If \( \rho' \subset \Delta_d \), then there are integers \( c_1 \geq 0, c_2 \geq 0, c_3 \geq 0 \) such that

\[
Z_{m^1} Z_{m^2} Z_{m^3} \left( \sum_{m \in \Delta_d(\mathbb{Z})} Z_m \right) + o_1(t) = 0,
\]

where \( o_1(t) \) is a polynomial.

ii) Assume that \( m^4 \in \partial \Delta_d \). If \( \rho'' \subset \Delta_d \) is another 3-cell such that \( m^0 \) is not a vertex of \( \rho'' \), i.e. \( \rho'' \cap \Delta_d(\mathbb{Z}) = \{m^5, m^6, m^7, m^8\} \) and \( m^0 \neq m^j \), \( j = 5, 6, 7, 8 \), then there are integers \( c_5 \geq 0, c_6 \geq 0, c_7 \geq 0, c_8 \geq 0 \) such that

\[
Z_{m^5} Z_{m^6} Z_{m^7} Z_{m^8} Z_{m^0} \left( \sum_{m \in \Delta_d(\mathbb{Z})} Z_m \right) + o_2(t) = 0,
\]

for a polynomial \( o_2(t) \).

iii) Both \( o_1(t) \) and \( o_2(t) \) are polynomials of \( Z_m \) with the coefficients tending to zero when \( t \to \infty \), i.e.

\[
o_1(t) \to 0, \quad \text{and} \quad o_2(t) \to 0.
\]

Proof. The patchworking polynomial reads

\[
f_t = \sum_{m \in \Delta_d(\mathbb{Z})} t^{-\nu(m)} w^m = \sum_{m \in \Delta_d(\mathbb{Z})} Z_m + \sum_{m' \in \partial \Delta_d \cap \mathbb{Z}^3} t^{-\nu(m')} w^{m'} = 0,
\]

by \( \Delta_d(\mathbb{Z}) = \text{int}(\Delta_d) \cap \mathbb{Z}^3 \).

Assume that there is another 3-cell \( \rho' \subset \Delta_d \) such that \( \rho \cap \rho' \) is the 2-cell with vertices \( m^1, m^2, m^3 \). If \( m^4 \) denotes the other vertex of \( \rho' \), then by
Lemma 5.4. \( w^{m^4+m^0} = w^{\epsilon_1 m^1+\epsilon_2 m^2+\epsilon_3 m^3} \), or equivalently,

\[
Z_{m^4}Z_{m^0} = t^{\ell_\rho(m^4)-v(m^3)}Z_{m^1}^2Z_{m^2}Z_{m^3},
\]

for some \( 0 \leq \epsilon_i \leq 1, i = 1, 2, 3 \), and

\[
t^{-v(m')}w^{m'} = t^{\ell_{\rho}(m')-v(m')}Z_{m^0}a_{1}Z_{m^2}a_{2}Z_{m^3} = t^{\ell_{\rho'}(m')-v(m')}Z_{m^4}Z_{m^1}Z_{m^2}Z_{m^3},
\]

for certain \( a_i \) and \( b_i \), \( i = 0, 1, 2, 3 \), where \( m' = a_0 m^0 + \sum_{i=1}^{3} a_i m^i = b_0 m^4 + \sum_{i=1}^{3} b_i m^i \), and \( a_0 + b_0 = 0 \). We choose the monomial with the power of \( Z_{m^0} \) or \( Z_{m^4} \) being non-negative in the above expression, i.e., if \( a_0 \geq 0 \) without loss of generality,

\[
t^{-v(m')}w^{m'} = t^{\ell_{\rho}(m')-v(m')}Z_{m^0}a_{1}Z_{m^2}a_{2}Z_{m^3}.
\]

Equivalently, we choose either the cell \( \rho \) or \( \rho' \) such that the only possible poles of \( w^{m'} \) are along \( \{ Z_{m^i} = 0 \}, i = 1, 2, 3 \). Thus there are \( c_1 \geq 0, c_2 \geq 0, c_3 \geq 0 \) such that

\[
Z_{m^1}c_1 Z_{m^2}c_2 Z_{m^3}c_3 \left( \sum_{m \in \Delta^0_d(\mathbb{Z})} Z_{m} \right) + \sum_{m' \in \partial \Delta_d \cap \Delta^3} t^{\ell(m')-v(m')}M_{m'} = 0,
\]

where \( M_{m'} \) is a monomial, and either \( \ell = \ell_\rho \) or \( \ell = \ell_{\rho'} \). Let

\[
o_1(t) = \sum_{m' \in \partial \Delta_d \cap \Delta^3} t^{\ell(m')-v(m')} M_{m'},
\]

which satisfies \( o_1(t) \to 0 \) as polynomials when \( t \to \infty \), since \( t^{\ell_{\rho}(m')-v(m')} \) and \( t^{\ell_{\rho'}(m')-v(m')} \) have negative powers.

Now we assume that \( m^4 \in \partial \Delta_d \). By Lemma 5.4.

\[
Z_{m^4} \left( \sum_{m \in \Delta^0_d(\mathbb{Z})} Z_{m} \right) + \sum_{m' \in \partial \Delta_d \cap \Delta^3} t^{-v(m')} w^{m'} + t^{\ell_{\rho}(m)-v(m')} Z_{m^1}^4 Z_{m^2}^2 Z_{m^3} = 0.
\]

If \( \rho'' \subset \Delta^0_{d} \) is another 3-cell such that \( m^0 \) is not a vertex of \( \rho'' \), i.e., \( \rho'' \cap \Delta_d(\mathbb{Z}) = \{m^5, m^6, m^7, m^8\} \) and \( m^0 \neq m^j, j = 5, 6, 7, 8 \), then by Lemma 5.4.

\[
t^{-v(m')} w^{m'} = t^{\ell_{\rho''}(m')-v(m')} Z_{m^5}^4 Z_{m^6}^2 Z_{m^7} Z_{m^8}.
\]

The same argument as above shows that there are integers \( c_5 \geq 0, c_6 \geq 0, c_7 \geq 0, c_8 \geq 0 \) such that

\[
Z_{m^5}c_5 Z_{m^6}c_6 Z_{m^7}c_7 Z_{m^8}c_8 Z_{m^0} \left( \sum_{m \in \Delta^0_d(\mathbb{Z})} Z_{m} \right) + o_2(t) = 0,
\]

where

\[
o_2(t) = \sum_{m' \in \partial \Delta_d \cap \Delta^3} t^{\ell''(m')-v(m')} M_{m'}' \to 0,
\]

when \( t \to \infty \), \( M_{m'} \) are monomials of \( Z_{m^0}, Z_{m^1}, Z_{m^2}, Z_{m^3}, Z_{m^5}, Z_{m^6}, Z_{m^7}, Z_{m^8} \), and either \( \ell'' = \ell_\rho \) or \( \ell'' = \ell_{\rho''} \).
Proof of vi) in Theorem 1.1. When $t \to \infty$, $\Psi_t(X_t)$ converges to an analytic subset $X_\infty$ of dimension 2 in $\mathbb{CP}^n$ by passing to a subsequence if necessary (cf. [10]). The convergence is in the sense of analytic spaces. Recall that an analytic subset $Y$ is locally defined by finite holomorphic functions $u_1 = 0, \ldots , u_k = 0$, and the function sheaf $\mathcal{O}_Y$ on $Y$ is locally given by the quotient $\mathcal{O}_{\mathbb{CP}^n}/(u_1, \ldots , u_k)$. Note that nilpotent functions are allowed. An analytic subset $Y' \subset Y$ is a subset of $Y$ with surjections $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_{Y'}$. A family of subsets $Y_t$ converges to $Y$, if $Y_t$ is locally given by $u'_1 = 0, \ldots , u'_k = 0$, and $u'_j \to u_j$, $j = 1, \ldots , k$, when $t \to \infty$. For example, if $Y$ is defined by $x^2 = 0$ in $\mathbb{C}^2$, and $Y'$ is given by $x = 0$, then $Y' \subset Y$ via $\mathbb{C}[x,y]/(x^2) \to \mathbb{C}[x,y]/(x)$.

We could view $Y$ as two copies of $Y'$ stacking together, i.e. $Y = 2Y'$. A family $Y_t$, defined by $x(x + t^{-1}y) = 0$, converges to $Y$.

For any $m \in \Delta_0^0(\mathbb{Z})$, let $H_m$ be the hyperplane in $\mathbb{CP}^n$ defined by $Z_m = 0$, and for any 3-cell $\rho \subset \Delta_0^0$ of $\mathcal{T}_\rho$, let $X_\rho$ be the subset given by $Z_m' = 0$ for all $m' \in \Delta_0^0(\mathbb{Z}) \setminus \rho$, which is the projective space of dimension 3, i.e.

$$X_\rho = \{(Z_{m^0}, Z_{m^1}, Z_{m^2}, Z_{m^3}) \in \mathbb{CP}^3\} = \bigcap_{m' \in \Delta_0^0(\mathbb{Z}) \setminus \rho} H_{m'},$$

where \( \{m^0, m^1, m^2, m^3\} = \rho \cap \Delta_0^0(\mathbb{Z}) \). Furthermore, if $\rho' \subset \Delta_0^0$ is another 3-cell, then

$$\tilde{X}_\rho \cap \tilde{X}_{\rho'} = \tilde{X}_\rho \cap H_{m''},$$

where $m'' \in (\rho' \setminus \rho) \cap \Delta_0^0(\mathbb{Z})$.

The image $\Psi_t(X_t)$ satisfies the equations in Lemma 5.4 and Lemma 5.5. If $\rho \subset \Delta_0^0$ with vertices $\{m^0, m^1, m^2, m^3\}$, then ii) of Lemma 5.4 asserts that $X_\infty$ satisfies $Z_{m^0} B_{m^1} Z_{m^2} B_{m^3} Z_m = 0$ for all $m \in \Delta_0^0(\mathbb{Z}) \setminus \rho$, and thus, as analytic subset,

$$X_\infty \subset \tilde{X}_\rho \cup b_0 H_{m^0} \cup b_1 H_{m^1} \cup b_2 H_{m^2} \cup b_3 H_{m^3},$$

for certain $b'_i \geq 0$, $i = 0, 1, 2, 3$. Since $\tilde{X}_\rho \subset H_m$ for all $m \in \Delta_0^0(\mathbb{Z}) \setminus \rho$, we obtain

$$X_\infty \subset \bigcup_{\text{all } 3\text{-cells } \rho \subset \Delta_0^0} \tilde{X}_\rho = \tilde{X},$$

as analytic subsets. Note that $\tilde{X}$ is a subvariety, and consists of $(d-4)^3$ irreducible components. Each irreducible component of $\tilde{X}$ is a copy of $\mathbb{CP}^3$.

We consider two 3-cells $\rho$ and $\rho'$ in $\Delta_0^0$, which share a 2-face, i.e. $\rho \cap \rho'$ is a 2-cell in $\mathcal{T}_\rho$. Let $\{m^0, m^1, m^2, m^3\} = \rho \cap \Delta_0^0(\mathbb{Z})$ and $\{m^1, m^2, m^3, m^4\} = \rho' \cap \Delta_0^0(\mathbb{Z})$. By Lemma 5.5, $X_\infty$ satisfies

$$Z_{m^1}^c Z_{m^2}^c Z_{m^3}^c \left( \sum_{m \in \Delta_0^0(\mathbb{Z})} Z_m \right) = 0,$$

for certain $c_i \geq 0$, $i = 1, 2, 3$, and thus, $X_\infty \cap \tilde{X}_\rho$ is given by

$$Z_{m^1}^c Z_{m^2}^c Z_{m^3}^c \left( Z_{m^0} + Z_{m^1} + Z_{m^2} + Z_{m^3} \right) = 0.$$
Consequently,
\[ X_\infty \cap \tilde{X}_\rho \subset X_\rho \cup c_1(H_{m^1} \cap \tilde{X}_\rho) \cup c_2(H_{m^2} \cap \tilde{X}_\rho) \cup c_3(H_{m^3} \cap \tilde{X}_\rho), \]
and \( X_\infty \cap \tilde{X}_\rho \cap H_{m^0} \subset X_\rho \cap H_{m^0}, \) i.e. a line in the projective plane \( \tilde{X}_\rho \cap H_{m^0} \). Therefore, the plane \( \tilde{X}_\rho \cap H_{m^0} = \tilde{X}_\rho' \cap H_{m^1} \) corresponding to the 2-cell \( \rho \cap \rho' \) is not a irreducible component of \( X_\infty \), and only intersects with \( X_\infty \) at most a line.

Assume that \( \rho_2 = \rho \cap \partial \Delta_d^0 \) is a 2-cell in \( T_v \). If the vertex \( m^0 \subset \rho \) does not belong to \( \rho_2 \), then \( X_{\rho_2} \) is given by \( Z_{m^0} = 0 \), i.e. \( X_{\rho_2} = \tilde{X}_\rho \setminus H_{m^0} \). If \( \rho'' \subset \Delta_d^0 \) is another 3-cell such that \( m^0 \) is not a vertex of \( \rho'' \) with \( \rho'' \cap \Delta_d(Z) = \{ m^5, m^6, m^7, m^8 \} \), then \( X_\infty \cap \tilde{X}_\rho \) satisfies that
\[ Z_{m^5} Z_{m^6} Z_{m^7} Z_{m^8} Z_{m^9} \left( Z_{m^0} + Z_{m^1} + Z_{m^2} + Z_{m^3} \right) = 0, \]
for certain \( c_j \geq 0, j = 5, 6, 7, 8 \) by Lemma 5.5. Hence
\[ X_\infty \cap \tilde{X}_\rho \subset X_{\rho_2} \cup X_\rho \cup c'_1(H_{m^1} \cap \tilde{X}_\rho) \cup c'_2(H_{m^2} \cap \tilde{X}_\rho) \cup c'_3(H_{m^3} \cap \tilde{X}_\rho), \]
for certain \( c_j' \geq 0 \). Thus \( X_\infty \) is a closed subvariety of \( X_0 \), i.e.
\[ X_\infty \subset X_0, \]
where \( X_0 \) is defined by (5.4).

Since
\[ c_1^2(O_{\mathbb{CP}^N}(1)|_{X_\infty}) = c_1^2(O_{\mathbb{CP}^N}(1)|_{X_t}) = K^2_{X_\infty} = d(d - 4)^2 = c_1^2(O_{\mathbb{CP}^N}(1)|_{X_0}), \]
there is no more irreducible components in \( X_0 \) besides \( X_\infty \). We obtain the conclusion \( X_\infty = X_0 \).

\[ \square \]

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