ON THE EVRARD’S FIBRANT REPLACEMENT
OF A FUNCTOR

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ABSTRACT. We give a new proof of an important result from M.Evrard’s Thesis [Ev1,2], giving an explicit “fibrant replacement” of a functor between two small categories. More precisely, we prove that the Evrard’s “fibrant replacement” of a functor fulfills the assumptions of Quillen Theorem B [Q]. We also prove a refined version of this result, which seems to be new.

INTRODUCTION

The problem we deal here with is how to replace a functor $f : C \to D$ between small categories by a “fibration of categories”. By the latter we mean a functor fulfilling the assumptions of Quillen Theorem B [Q, Section 1].

There are apparently many different explicit constructions for that. We consider here the “most economic construction”, due to Marcel Evrard [Ev1,2], which follows very closely the way of how a map of topological spaces is replaced by a Serre fibration, using the path space (we recall it below in the Introduction).

Evrard claims in his Thesis [Ev1] (Chapter D, Example III 2c(iii)) that the corresponding functor $f_h : \mathcal{H}(f) \to D$ (the “fibrant replacement” of $f$) is a fibration in his sense (what in particular means that is fulfills the assumptions of Quillen Theorem B). In this paper we provide a new direct proof of this result in Theorem 2.6. The author uses this result in his paper in progress [Sh3], thus he thought it might make sense to write down a concise proof of this claim, for future references. As well, we prove a new refined version of it in Theorem 4.2.

In fact, Evrard uses in his Thesis [Ev1] an intrinsic concept of homotopy equivalence of two functors $f, g : \mathcal{C} \to \mathcal{D}$, different from the common one used in [Q]. Whence Quillen refers to the classifying spaces $B\mathcal{C}$ and $B\mathcal{D}$ and says that the functors $f, g$ are homotopy equivalent iff the corresponding maps $Bf,Bg : B\mathcal{C} \to B\mathcal{D}$ of topological spaces are homotopy equivalent, Evrard gives his definition in terms of categories. Namely, he constructs a “free path” category $\Lambda \mathcal{D}$, with two projections $p_0, p_1 : \Lambda \mathcal{D} \to \mathcal{D}$ (the start and the end points of the path), and says that $f$ and $g$ are homotopy equivalent iff there is a functor $h : \mathcal{C} \to \Lambda \mathcal{D}$ such that $p_0 \circ h = f$ and $p_1 \circ h = g$. In a sense, this definition uses the “right cylinder homotopy relation” instead of the “left cylinder homotopy relation” used by Quillen. In fact, the two homotopy relations coincide, see Proposition 2.2.
Then the category $\mathcal{H}(f)$, for $f: \mathcal{C} \to \mathcal{D}$, is defined as the cartesian product

\[
\begin{array}{c}
\mathcal{H}(f) \\
\downarrow \\
\mathcal{C}
\end{array} \quad \Lambda \mathcal{D} \\
\downarrow \\
\mathcal{D}
\begin{array}{c}
f \\
\downarrow \\
f
\end{array}
\]

where the functor $p_1$ gives the functor $f_h: \mathcal{H}(f) \to \mathcal{D}$, the “fibrant replacement” of $f$.

Note that it follows literally the line of the construction of a fibrant replacement for a map $f: X \to Y$ of topological spaces. For that, one considers the free path space $P(Y)$, with two projections $p_0, p_1: P(Y) \to Y$, and defines $X_h$ as the cartesian product

\[
\begin{array}{c}
X_h \\
\downarrow \\
X
\end{array} \quad P(Y) \\
\downarrow \\
\begin{array}{c}
p_0 \\
p_1
\end{array}
\begin{array}{c}
f \\
f
\end{array}
\begin{array}{c}
Y
\end{array}
\]

where $p_1$ defines a map $f_h: X_h \to Y$ known to be a Serre fibration.

The paper is organized as follows. The first two Sections do not contain anything new. In Section 1 we recall the main definitions and facts about the homotopy equivalence of categories, following closely [Q], and recall as well the Grothendieck construction we use in Section 2. In Section 2 we recall the Evrard’s constructions of the free path category $\Lambda(\mathcal{D})$ and of the functor $f_h: \mathcal{H}(f) \to \mathcal{D}$. In Section 3 we give a proof of the main claim that the functor $f_h: \mathcal{H}(f) \to \mathcal{D}$ fulfills the assumptions of Quillen Theorem B. In Section 4 we provide a refined version of this Evrard’s result, which seems to be new.

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1 Basic facts on homotopy theory of small categories

1.1 The basic principles of the homotopy theory of categories

Here we recall some very elementary facts about the homotopy theory of categories, following the first few pages of [Q, Section 1].

The theory starts from the classifying space functor $B: \mathcal{C}at \to \mathcal{Top}$ from small categories to topological spaces, introduced by G.Segal in [Seg1].

Firstly one defines the nerve $N\mathcal{C}$ of a small category $\mathcal{C}$, which is the simplicial set whose $n$-simplices are chains of $n$ composable morphisms:

\[
X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \ldots X_{n-1} \xrightarrow{f_n} X_n
\]
The \( i \)-th face map \( \delta_i: \mathcal{N}C_n \to \mathcal{N}C_{n-1} \) is obtained by deleting of \( X_i \) in (1.1), and if \( i \neq 0, n \), by replacing the maps \( f_i \) and \( f_{i+1} \) by their composition. The \( i \)-th degeneracy map \( \varepsilon_i: \mathcal{N}C_n \to \mathcal{N}C_{n+1} \) is obtained by inserting of another copy of \( X_i \) at the \( i \)-th position, and inserting the identity map between the two copies \( X_i \).

It is a simplicial set, functorially depending on \( C \). The geometric realization of \( \mathcal{N}C \) is a tological space, called the classifying space of \( C \). It is denoted by \( BC \):  

\[
BC = |\mathcal{N}C| \quad (1.2)
\]

Any functor \( f: \mathcal{C}_1 \to \mathcal{C}_2 \) defines a map of topological spaces \( B(f): BC_1 \to BC_2 \). For three categories \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \) and functors \( f: \mathcal{C}_1 \to \mathcal{C}_2, g: \mathcal{C}_2 \to \mathcal{C}_3 \) one has:

\[
B(g \circ f) = B(g) \circ B(f) \quad (1.3)
\]

as the nerve enjoys this property, and the geometrical realization is a functor.

**Definition 1.1.** Two functors \( f, g: \mathcal{C}_1 \to \mathcal{C}_2 \) are said to be homotopic (in the sense of Quillen) if the corresponding maps \( Bf, Bg: BC_1 \to BC_2 \) are homotopic maps of topological spaces. A functor \( f: \mathcal{C}_1 \to \mathcal{C}_2 \) is a homotopy equivalence if there is a functor \( f': \mathcal{C}_2 \to \mathcal{C}_1 \) such that the compositions \( f \circ f' \) and \( f' \circ f \) are homotopy equivalent to the identity maps.

The data consisting of two functors \( f, g: \mathcal{C}_1 \to \mathcal{C}_2 \) and a natural transformation \( h: f \to g \) can be interpreted in the following way. Denote by \( I \) the category with two objects \( 0 \) and \( 1 \) and the only non-identity morphism \( i: 0 \to 1 \). Then the above data is the same that a single functor \( F_{f,g,h}: \mathcal{C}_1 \times I \to \mathcal{C}_2 \). It results in a map:

\[
BC_1 \times I \sim BC_1 \times BI \to B(\mathcal{C}_1 \times I) \to BC_2 \quad (1.4)
\]

where \( I \) is the closed interval, \( I = BI \).

We have:

**Proposition 1.2.** (i) Let \( \mathcal{C}_1, \mathcal{C}_2 \) be two small categories, \( f, g: \mathcal{C}_1 \to \mathcal{C}_2 \) two functors, and \( h: f \to g \) a map of functors. Then \( h \) defines a homotopy between \( B(f), B(g): BC_1 \to BC_2 \);

(ii) let \( \mathcal{C} \) and \( \mathcal{D} \) be small categories, and

\[
F: \mathcal{C} \rightleftarrows \mathcal{D}: G
\]

a pair of adjoint functors. Then \( BC \) and \( BD \) are homotopy equivalent topological spaces;

(iii) suppose a small category \( \mathcal{C} \) has an initial (resp., a final object). Then \( BC \) is contractible.

**Proof.** We have just shown (i). The claim (ii) follows immediately from (i) and from (1.3), as there are adjunction maps of functors \( F \circ G \to \text{id} \) and \( G \circ F \to \text{id} \). For (iii), if \( \mathcal{C} \) has an initial (resp., a final) object, the projection functor of \( \mathcal{C} \) to the category \( \ast \) with a single object and with the only (identity) morphism, admits a left (resp. a right) adjoint. \( \square \)
**Definition 1.3.** Let $\mathcal{C}_1, \mathcal{C}_2$ are two small categories. We say that they are *homotopy equivalent* if there are functors $f: \mathcal{C}_1 \to \mathcal{C}_2$ and $g: \mathcal{C}_2 \to \mathcal{C}_1$ such that the two compositions $B(g) \circ B(f)$ and $B(f) \circ B(g)$ are homotopy to the identity maps of $B\mathcal{C}_1$ and of $B\mathcal{C}_2$, correspondingly.

As follows from Proposition 1.2(i), to prove that $\mathcal{C}_1$ and $\mathcal{C}_2$ are homotopy equivalent, it is enough to construct maps of functors $g \circ f \to \text{id}_{\mathcal{C}_1}$ and $f \circ g \to \text{id}_{\mathcal{C}_2}$.

### 1.2 Pre(co-)fibred categories, and Quillen Theorem B

Here we recall the definitions of (pre-)fibred and of (pre-)cofibred categories (due to Grothendieck [SGA1, Exposé VI]), and formulate the Quillen’s Theorem B [Q, Section 1].

As a motivation, any functor $f: \mathcal{C} \to \mathcal{D}$ defines, after passing to the classifying spaces, a map of topological spaces $f_{\text{top}}: BCE \to BD$. We can ask the question whether it is possible to formulate intrinsically in categorical terms some conditions on $F$ which guarantee that $f_{\text{top}}$ is a Serre fibration. It is done by Quillen in his Theorem B and in the Corollary of it [Q, Section 1], using the concepts of (pre-)fibred and (pre-)cofibred categories of Grothendieck [SGA1, Exposé VI]. Note that both fibred and cofibred categories are corresponded to the case when $f_{\text{top}}$ is a Serre fibration, that is the terminology is a bit confusing. A better terminology might be a right (pre-)fibred and a left (pre-)fibred category.

Let $f: \mathcal{C} \to \mathcal{D}$ be a functor. Let $Y$ be a fixed object of $\mathcal{D}$.

Denote by $Y \setminus f$ the category whose objects are pairs $(X, v)$ where $X \in \mathcal{C}$, and $v: Y \to f(X)$ is a morphism in $\mathcal{D}$. A morphism $(X, v) \to (X', v')$ is a morphism $w: X \to X'$ in $\mathcal{C}$ such that $f(w) \circ v = v'$.

As well, denote by $f \setminus Y$ the category whose objects are pairs $(X, v)$ where $X \in \mathcal{C}$, and $v: f(X) \to Y$ is a morphism in $\mathcal{D}$. A morphism $(X, v) \to (X', v')$ is a morphism $w: X \to X'$ in $\mathcal{C}$ such that $v' \circ f(w) = v$.

The Quillen’s Theorem A [Q], Section 1 says that if, for a functor $f: \mathcal{C} \to \mathcal{D}$ the category $Y \setminus f$ (correspondingly, $f \setminus Y$) is contractible for each $Y \in \mathcal{D}$ then $f$ is a homotopy equivalence (that is, $f_{\text{top}}: BCE \to BD$ is a homotopy equivalence).

Along with the category $Y \setminus f$ and $f \setminus Y$, one considers the “set-theoretical fiber” $f^{-1}Y$. It is the subcategory of $\mathcal{C}$ of objects $X$ such that $f(X) = Y$ and of morphisms $w: X \to X'$ such that $f(w) = \text{id}_Y$.

The main advantage of the categories $Y \setminus f$ and $f \setminus Y$, comparably with $f^{-1}(Y)$, is their functorial behaviour. Let $v: Y \to Y'$ be a morphism in $\mathcal{D}$. Then one has the natural functors

$$[v^*]: Y' \setminus f \to Y \setminus f \text{ and } [v_*]: f \setminus Y \to f \setminus Y'$$

In the same time, for the existence of functors $v^*: f^{-1}(Y') \to f^{-1}(Y)$ and $v_*: f^{-1}(Y) \to f^{-1}(Y')$ one should impose some extra assumptions (see below).

We use the notations $Y \setminus \mathcal{D}$ and $\mathcal{D} \setminus Y$, for $Y \in \mathcal{D}$. The category $Y \setminus \mathcal{D}$ has as its objects the pairs $(Y', v)$ where $v: Y \to Y'$, and a morphism $t: (Y', v) \to (Y'', u)$ is a morphism $t: Y' \to Y''$ such that $t \circ v = u$, and the category $\mathcal{D} \setminus Y$ is defined accordingly.
Note that the category $Y \setminus \mathcal{D}$ has $Y \xrightarrow{id} Y$ as its initial object, and the category $\mathcal{D} \setminus Y$ has $Y \xrightarrow{id} Y$ as its final object. Therefore, both categories $Y \setminus \mathcal{D}$ and $\mathcal{D} \setminus Y$ are contractible, by Proposition 1.2(iii).

**Theorem 1.4 (Quillen Theorem B).** Let $f: \mathcal{C} \to \mathcal{D}$ be a functor such that for any arrow $v: Y \to Y'$ in $\mathcal{D}$ the induced functor $[v^*]: Y' \setminus f \to Y \setminus f$ is a homotopy equivalence. Then for any $Y \in \mathcal{D}$ the cartesian square of categories

\[ \begin{array}{ccc}
Y \setminus f & \xrightarrow{j} & \mathcal{C} \\
[f] & \downarrow & \downarrow f \\
Y \setminus \mathcal{D} & \xrightarrow{[j]} & \mathcal{D}
\end{array} \]  

is homotopy cartesian, where

\[ j(X, v) = X, \quad [f](X, v) = (f(X), v), \quad [j](Y', v) = Y' \]  

As well, there is a “dual” version, where with the assumption that for any $v: Y \to Y'$ the induced functor $[v_*]$ is a homotopy equivalence, the cartesian diagram

\[ \begin{array}{ccc}
f \setminus Y & \xrightarrow{j'} & \mathcal{C} \\
(f) & \downarrow & \downarrow f \\
\mathcal{D} \setminus Y & \xrightarrow{[j']} & \mathcal{D}
\end{array} \]  

is homotopy cartesian.

As we have shown just before the Theorem that the categories $Y \setminus \mathcal{D}$ and $\mathcal{D} \setminus Y$ are contractible, the meaning of this Theorem is that $Bf: B\mathcal{C} \to B\mathcal{D}$ is a quasi Serre fibration, with $B(Y \setminus f)$ (corresp., $B(f \setminus Y)$) the fiber over the contractible space $B(Y \setminus \mathcal{D})$ (corresp., $B(\mathcal{D} \setminus Y)$). That is, the fibers of $Bf$ are homotopy equivalent to $B(Y \setminus f)$ (corresp., $B(f \setminus Y)$).

In some extra assumptions it is possible to formulate a more precise statement, describing a fiber $(Bf)^{-1}(Y)$ as $B(f^{-1}(Y))$ (that is, giving a description of the fibers over points rather then the fibers over contractible sets containing points). It uses the pre-(co)fibred functors.

There are the natural imbeddings functor

\[ i_Y: f^{-1}Y \to Y \setminus f, \quad X \mapsto (X, \text{id}_Y) \]

and

\[ j_Y: f^{-1}Y \to f \setminus Y, \quad X \mapsto (X, \text{id}_Y) \]
The functor \( f : \mathcal{C} \to \mathcal{D} \) is called \textit{pre-fibration}, and the category \( \mathcal{C} \) is called \textit{pre-fibred} over \( \mathcal{D} \), if for any \( Y \in \mathcal{D} \) the functor \( i_Y \) has a right adjoint.

As well, the functor \( f : \mathcal{C} \to \mathcal{D} \) is called \textit{pre-cofibration}, and the category \( \mathcal{C} \) is called \textit{pre-cofibred} over \( \mathcal{D} \), if for any \( Y \in \mathcal{D} \) the functor \( j_Y \) has a left adjoint.

The right adjoint functor \( R_Y \) to \( i_Y \) (if it exists) assigns to each \((X,v)\) an object of \( f^{-1}Y \), which is denoted by \( v^*X \).

The left adjoint functor \( L_Y \) to \( j_Y \) (if it exists) assigns to each \((X,v)\) an object of \( f^{-1}Y \), denoted by \( v_*X \).

Let \( v : Y \to Y' \) be a morphism in \( \mathcal{D} \). Then the composition

\[
v^* := R_Y \circ [v^*] \circ i_Y : f^{-1}Y' \to f^{-1}Y
\]

is called the \textit{base-change functor}.

As well, for the same \( v : Y \to Y' \), the composition

\[
v_* := L_Y \circ [v_*] \circ j_Y : f^{-1}Y \to f^{-1}Y'
\]

is called the \textit{cobase-change} functor.

The Corollary of Quillen’s Theorem B says the following:

**Theorem 1.5** (Corollary to Quillen Theorem B). Let \( f : \mathcal{C} \to \mathcal{D} \) be pre-fibred (corresp., pre-cofibred), and that for any morphism \( u : Y \to Y' \) in \( \mathcal{D} \) the base change functor \( u^* : f^{-1}Y' \to f^{-1}Y \) (corresp., the cobase change functor \( u_* : f^{-1}Y \to f^{-1}Y' \)) is a homotopy equivalence. Then for any \( Y \in \mathcal{D} \) the category \( f^{-1}Y \) is the homotopy fiber of \( f \) over \( Y \). More precisely, the cartesian diagram

\[
f^{-1}Y \quad \xrightarrow{f} \quad \mathcal{C} \quad \xleftarrow{f} \quad \mathcal{D}
\]

is homotopy cartesian, for any object \( Y \in \mathcal{D} \).

This property on the level of topological spaces just means that \( f_{\text{top}} : B\mathcal{C} \to B\mathcal{D} \) is a quasi Serre fibration, with the fibers \( B(f^{-1}Y) \).

It is clear that the fulfillment of the assumptions of Theorem 1.5 implies the fulfillment of the assumptions of Theorem 1.4. Indeed, the functors \( i_Y : f^{-1}Y \to Y \setminus f \) and \( j_Y : f^{-1}Y \to f \setminus Y \) are homotopy equivalences, as they have adjoints, by Proposition 1.2(ii). Then to say that \( v^* \) (corresp., \( v_* \)) is a homotopy equivalence is the same that to say that \([v^*]\) (corresp., \([v_*]\)) is a homotopy equivalence.
1.3 The Grothendieck construction

The Evrard’s construction of Section 2 fits well in the framework of Grothendieck construction in category theory [SGA1, ExposéVI.8]. Here we recall what it is, restricting ourselves only for its properties necessary for the sequel. We refer the reader to loc.cit. and to [Th] for more detail.

Let $F : \mathcal{K} \to \mathbf{Cat}$ be a (strict) functor. The objects of $\mathcal{K} \int F$ are pairs $(K, X)$ were $K \in \mathcal{K}$ and $X \in F(K)$. A morphism $(k, x) : (K_1, X_1) \to (K_0, X_0)$ is given by a morphism $k : K_1 \to K_0$ in $\mathcal{K}$ and a morphism $x : F(k)(X_1) \to X_0$ in $F(K_0)$.

The composition is defined as $(k, x) \circ (k', x') = (kk', x \cdot F(k)(x'))$.

**Lemma 1.6.** A natural transformation of functors $f : F \to F'$ of functors $\mathcal{K} \to \mathbf{Cat}$ induces a functor $[f] : \mathcal{K} \int F \to \mathcal{K} \int F'$

This correspondence is functorial, making the Grothendieck construction a functor from the category whose objects are functors $\mathcal{K} \to \mathbf{Cat}$ (for fixed $\mathcal{K}$), and morphisms are the $\mathcal{K}$-morphisms of functors, to the category $\mathbf{Cat}$.

The following Proposition (see e.g. [Th, Prop. 1.3.1]) characterizes the Grothendieck construction category $\mathcal{K} \int F$ as the “lax colimit” of the functor $F : \mathcal{K} \to \mathbf{Cat}$.

**Proposition 1.7.** Let $\mathcal{K}$ be a small category, $F : \mathcal{K} \to \mathbf{Cat}$ a strict functor, $\mathcal{E}$ a category. Then there is a bijection between the set of functors $g : \mathcal{K} \int F \to \mathcal{E}$, and the set of data consisting of

1. for each object $K \in \mathcal{K}$, a functor $g(K) : F(K) \to \mathcal{E}$,
2. for each morphism $k : K \to K'$ in $\mathcal{K}$, a natural transformation
   \[ g(k) : g(K) \to g(K') \circ F(k) \]
   such that $g(\text{id}_K) = \text{id} : g(K) \to g(K)$, and for $K' \xrightarrow{k} K' \xleftarrow{k'} K$ one has
   \[ g(k'k) = g(k') \circ g(k) \quad (1.13) \]

See e.g. [Th, Prop. 1.3.1] for a short and direct proof.

We will be using this Proposition successively in the next Sections, omitting the explicit reference to it.
2 The Evrard’s construction

There is a nice construction due to M.Evrard [Ev1,2] of replacing a functor \( f: \mathcal{C} \to \mathcal{D} \) between small categories by a functor \( f_h: \mathcal{H}(f) \to \mathcal{D} \), where \( \mathcal{E}_h \) is another category, such that:

(i) there is a functor \( i: \mathcal{C} \to \mathcal{H}(f) \) which is a homotopy equivalence,

(ii) the functor \( f_h \) fulfills the assumptions of Quillen Theorem B (Theorem 1.4 above),

(iii) the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i} & \mathcal{H}(f) \\
\downarrow f & & \downarrow f_h \\
\mathcal{D} & & \\
\end{array}
\]

commutes.

2.1 The free paths category

In the categorical setting, one defines firstly the category \( \Lambda \mathcal{D} \) of a category \( \mathcal{D} \) which is the analogue of the “free paths space” \( PY \).

Define the category \( \Lambda_n \mathcal{D} \) for \( n \geq 1 \). An object of \( \Lambda_n \mathcal{D} \) is a zig-zag

\[
\bar{Y}_0 \to Y_1 \leftarrow \bar{Y}_1 \to Y_2 \leftarrow \bar{Y}_2 \to Y_3 \leftarrow \cdots \to Y_n \leftarrow \bar{Y}_n
\]

(2.2)

which we denote by \( \underline{Y}(n) \), and a morphism \( \underline{Y}(n) \to \underline{Z}(n) \) is the set of maps \( t_i: Y_i \to Z_i \) and \( \bar{t}_i: \bar{Y}_i \to \bar{Z}_i \) making all squares commutative. Such a morphism is denoted by \( \underline{t}: \underline{Y}(n) \to \underline{Z}(n) \).

Consider the category \( \Delta_{\text{str}} \) whose objects are \{[1],[2],\ldots\} and whose morphisms \([m] \to [n]\) are the strictly monotonous maps from \( 1 < \cdots < m \) to \( 1 < \cdots < n \) (that is, such maps \( \phi \) that \( \phi(i) < \phi(j) \) for \( i < j \); in particular a morphism exists only when \( m \leq n \)).

Consider the functor, for a fixed category \( \mathcal{D} \),

\[
\Lambda: \Delta_{\text{str}} \to \text{Cat}
\]

such that \( \Lambda([n]) = \Lambda_n \mathcal{D} \) on the objects, and for a morphism \( \phi: [m] \to [n] \), and for a \( \underline{Y} \in \Lambda_m \mathcal{D} \), one sets

\[
\Lambda(\phi)(\underline{Y}) =
\bar{Y}_0 \xrightarrow{\text{id}} \bar{Y}_0 \xleftarrow{\text{id}} \bar{Y}_0 \xrightarrow{\text{id}} \ldots \xrightarrow{\text{id}} \bar{Y}_0 \xrightarrow{\text{id}} Y_1 \xleftarrow{\text{id}} \bar{Y}_1 \xrightarrow{\text{id}} \ldots \xrightarrow{\text{id}} \bar{Y}_1 \xleftarrow{\text{id}} Y_2 \xrightarrow{\text{id}} \bar{Y}_2 \xleftarrow{\text{id}} \bar{Y}_2 \xrightarrow{\text{id}} \ldots
\]

(2.3)
Finally, we define $\Lambda D$ as the Grothendieck construction

$$\Lambda D = \Delta_{\text{str}} \int \Lambda$$

(2.4)

Define two maps $p_0, p_1: \Lambda D \to D$ setting

$$p_0([n], Y) = \bar{Y}_0, \quad p_1([n], Y) = \bar{Y}_n$$

(2.5)

Evrard gives [Ev1] the following definition:

**Definition 2.1.** Let $f, g: \mathcal{C}_1 \to \mathcal{C}_2$ be functors. They are called *homotopic in the sense of Evrard* if there is a functor $H: \mathcal{C}_1 \to \Lambda_n \mathcal{C}_2$, for some $n \geq 1$, such that $p_0 \circ H = f$ and $p_1 \circ H = g$. A functor $f: \mathcal{C}_1 \to \mathcal{C}_2$ is called a *homotopy equivalence in the sense of Evrard* if there exists a functor $f': \mathcal{C}_2 \to \mathcal{C}_1$ such that both compositions $f \circ f'$ and $f' \circ f$ are homotopy equivalent to the identity functors in the sense of Evrard.

One easily shows that if there is a natural transformation $f \to g$ between two functors $f, g: \mathcal{C}_1 \to \mathcal{C}_2$, they are homotopic in the sense of Evrard. That is, all statements of Proposition 1.2 hold if the homotopy is understood in the sense of Definition 2.1. The Evrard homotopy relation is a priori a weaker homotopy relation than the Quillen’s one, see Definition 1.1.

We have:

**Proposition 2.2.** Let $f, g: \mathcal{C}_1 \to \mathcal{C}_2$ be functors homotopic in the sense of Evrard. Then the corresponding maps $Bf, Bg: B\mathcal{C}_1 \to B\mathcal{C}_2$ are homotopic. That is, the Evrard homotopy relation coincides with the Quillen homotopy relation.

**Proof.** Let $f, g: \mathcal{C}_1 \to \mathcal{C}_2$ be homotopic in the sense of Evrard. Then the corresponding functor $H: \mathcal{C}_1 \to \Lambda_n \mathcal{C}_2$ gives rise to $2n + 1$ functors

$$\bar{Y}_0, \bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_n, \tilde{Y}_n: \mathcal{C}_1 \to \mathcal{C}_2$$

and natural transformations $\gamma_i: \bar{Y}_{i-1} \to \bar{Y}_i$ and $\delta_i: \tilde{Y}_i \to \tilde{Y}_i$, $i = 1 \ldots n$. Moreover, $\bar{Y}_0 = f$ and $\tilde{Y}_n = g$. Now the result follows from Proposition 1.2(1).

In the same time, the Evrard homotopy relation leaves us more flexibility for constructions of homotopies, as we will now see.

Let $Y \in \Lambda_n D$ be a free path. Consider the path $p_0(Y)$ which is, in the notations of (2.2), the following path in $\Lambda_n D$:

$$\bar{Y}_0 \xrightarrow{id} \bar{Y}_0 \xleftarrow{id} \bar{Y}_0 \xrightarrow{id} \bar{Y}_0 \xleftarrow{id} \bar{Y}_0 \ldots$$

(2.6)

**Proposition 2.3.** The functor $p_0: \Lambda_n D \to \Lambda_n D$ is Evrard homotopic to the identity functor of $\Lambda_n D$. 
Proof. For \( k \leq n \), define the objects \( \bar{Y}^{(k)} \) and \( Y^{(k)} \) in \( \Lambda_n \mathcal{D} \), as the paths

\[
\bar{Y}_0 \to Y_1 \leftarrow \bar{Y}_1 \to \cdots \leftarrow \bar{Y}_{k-1} \to Y_k \leftarrow Y_k \leftrightarrow Y_k \to \cdots \quad (2.7)
\]

and

\[
\bar{Y}_0 \to Y_1 \leftarrow \bar{Y}_1 \to \cdots \leftarrow \bar{Y}_{k-1} \to Y_k \leftarrow \bar{Y}_k \leftarrow Y_k \leftarrow \bar{Y}_k \to \cdots \quad (2.8)
\]
correspondingly.

In these notations, \( p_0(\bar{Y}) = Y_{(0)} \), and \( \bar{Y} = Y_{(n)} \).

We claim that there is a homotopy in the sense of Evrard from \( p_0 \) to \( \text{id} \), given by the functor

\[
H : \Lambda_n \mathcal{D} \to \Lambda(\Lambda_n \mathcal{D})
\]
sending \( \bar{Y} \) to the path

\[
p_0(\bar{Y}) = Y_{(0)} \xrightarrow{\alpha_1} Y_{(1)} \xrightarrow{\beta_1} Y_{(2)} \xrightarrow{\alpha_2} Y_{(3)} \xrightarrow{\beta_2} \cdots \xrightarrow{\alpha_n} Y_{(n)} \xrightarrow{\beta_n} Y_{(n)} = Y \quad (2.9)
\]
where the maps \( Y_{(i-1)} \xrightarrow{\alpha_i} Y_{(i)} \) and \( Y_{(i)} \xrightarrow{\beta_i} Y_{(i)} \) are as follows:

\[
\begin{align*}
Y_{(k)} : & \quad \bar{Y}_0 \longrightarrow Y_1 \longleftarrow \cdots \leftarrow \bar{Y}_{k-1} \leftarrow Y_k \leftarrow Y_k \leftarrow Y_k \leftarrow Y_k \leftarrow \cdots \\
\beta_k & \downarrow \quad \text{id} & \downarrow \quad \text{id} & \downarrow \quad \text{id} & \downarrow b & \downarrow b \\
Y^{(k)} : & \quad \bar{Y}_0 \longrightarrow Y_1 \longleftarrow \cdots \leftarrow Y_{k-1} \leftarrow \bar{Y}_{k-1} \leftarrow Y_k \leftarrow Y_k \leftarrow Y_k \leftarrow Y_k \leftarrow \cdots \\
\alpha_k & \downarrow \quad \text{id} & \downarrow \quad \text{id} & \downarrow a & \downarrow a & \downarrow a \\
Y_{(k-1)} : & \quad \bar{Y}_0 \longrightarrow Y_1 \longleftarrow \cdots \leftarrow Y_{k-1} \leftarrow \bar{Y}_{k-1} \leftarrow Y_{k-1} \leftarrow Y_{k-1} \leftarrow Y_{k-1} \leftarrow \cdots \\
\end{align*}
\]  

(2.10)

We are done. \( \square \)

Corollary 2.4. The categories \( \Lambda_n \mathcal{D} \), \( n \geq 0 \), and \( \Lambda \mathcal{D} \) are homotopy equivalent to the category \( \mathcal{D} \). \( \square \)

2.2 The Evrard’s “fibrant replacement” of a functor

For a functor \( f : \mathcal{C} \to \mathcal{D} \), define the category \( \mathcal{H}(f) \) as the pull-back limit

\[
\begin{array}{ccc}
\mathcal{H}(f) & \longrightarrow & \Lambda \mathcal{D} \\
\downarrow & & \downarrow p_0 \\
\mathcal{C} & \underset{f}{\longrightarrow} & \mathcal{D}
\end{array}
\]  

(2.11)
The functor \( p_1 : \Lambda \mathcal{D} \to \mathcal{D} \) defines the composition \( \mathcal{H}(f) \to \Lambda \mathcal{D} \xrightarrow{p_1} \mathcal{D} \) which is denoted by \( f_h \).

Explicitly, an object of \( \mathcal{H}(f) \) is a triple \((X, [n], Y)\) with \( X \in \mathcal{C}, Y \in \Lambda [n] \mathcal{D} \), and with \( \bar{Y}_0 = f(X) \). Then \( p_1 \) assigns to \((X, [n], Y)\) the rightmost object \( \bar{Y}_n \) in the string \( Y \).

There are functors \( q : \mathcal{H}(f) \to \mathcal{C} \), assigning to \((X, Y)\) the object \( X \), and \( i : \mathcal{C} \to \mathcal{H}(f) \), assigning to \( X \) the pair \((X, f(X))\), where \( f(X) \) is the constant string whose all terms are \( f(X) \) and all maps are identities.

**Proposition 2.5.** Let \( f : \mathcal{C} \to \mathcal{D} \) be a functor. Then the functors \( i \) and \( q \) are homotopy equivalences, homotopy inverse to each other. Therefore, the categories \( \mathcal{C} \) and \( \mathcal{H}(f) \) are homotopy equivalent.

**Proof.** We have \( q \circ i = \text{id} \), and the composition \( i \circ q \) is the functor assigning to \((X, [n], Y)\) the object \((X, [n], f(X))\). We prove that it is a homotopy equivalence repeating the construction used in the proof of Proposition 2.3.

The main result of the paper is a new proof of the following result of Evrard [Ev1] (along with its refinement in Section 4):

**Theorem 2.6.** Let \( f : \mathcal{C} \to \mathcal{D} \) be a functor. Then the functor \( f_h : \mathcal{H}(f) \to \mathcal{D} \) fulfills the assumptions of the Quillen’s Theorem B (Theorem 1.4 above).

We prove Theorem 2.6 in the next Section.

**Remark 2.7.** It is not true that the functor \( f_h : \mathcal{H}(f) \to \mathcal{D} \) fulfills the assumptions of Corollary to Quillen Theorem B (Theorem 1.5 above), which are stronger than the assumptions of Quillen Theorem B. In fact, the inclusion functor \( f_h^{-1}(Y) \to f_h \setminus Y \) (corresp., \( f_h^{-1}(Y) \to Y \setminus f_h \)) does not admit a left (corresp., a right) adjoint. That is, the functor \( f_h \) is neither pre-cofibred nor pre-fibred.

### 3 Proof of Theorem 2.6

The proof relies on several Lemmas.

#### 3.1 The Shift Functor

Define the functor \( T_n : \Lambda_n(\mathcal{D}) \to \Lambda_{n+1}(\mathcal{D}) \) as

\[
T_n(Y) = \cdots \leftarrow Y_{n-1} \to \bar{Y}_{n-1} \leftarrow Y_n \to \bar{Y}_n \leftarrow \id \bar{Y}_n \to \bar{Y}_n
\]

where \( \bar{Y} \in \Lambda_n(\mathcal{D}) \).

The functors \( T_n \) define a functor \( T : \Lambda(\mathcal{D}) \to \Lambda(\mathcal{D}) \), and for a functor \( f : \mathcal{C} \to \mathcal{D} \) the functors \( T_n \) define a functor \( T_f : \mathcal{H}(f) \to \mathcal{H}(f) \), by

\[
T_f(X, [n], Y) = (X, [n+1], T_n(Y))
\]
One has

**Lemma 3.1.** Let \( f : \mathcal{C} \to \mathcal{D} \) be a functor. Then \( T_f : \mathcal{H}(f) \to \mathcal{H}(f) \) is a homotopy equivalence, and is homotopy equivalent to the identity functor. The functor \( T_f \) preserves the category \( f^{-1}(Y) \) for any \( Y \in \mathcal{D} \). The corresponding functor \( T_f : f^{-1}(Y) \to f^{-1}(Y) \) is also a homotopy equivalence, homotopy equivalent to the identity functor.

**Proof.** We construct a natural transformation \( \text{id}_{\mathcal{H}(f)} \to T_f \), then the result follows from Proposition 1.2(i). That is, we need to define for any \( (X, [n], Y) \) a morphism in \( \mathcal{H}(f) \)

\[
\theta : (X, [n], Y) \to (X, [n + 1], T_n(Y))
\]

such that for any morphism \( g \) in \( \mathcal{H}(f) \) the diagram

\[
\begin{array}{ccc}
(X, [n], Y) & \xrightarrow{\theta} & (X, [n + 1], T_n(Y)) \\
\downarrow g & & \downarrow g \\
(X', [m], Y') & \xrightarrow{\theta} & (X', [m + 1], T_m(Y'))
\end{array}
\]

commutes.

We define the component \( \theta([n]) \) as the map \( \theta_n : [n] \to [n + 1] \) with \( \theta_n(i) = i, 1 \leq i \leq n \).

Then the corresponding map \( \Lambda(\theta_n)(Y) \to T_n(Y) \) is defined as the identity map (note that \( \Lambda(\theta_n)(Y) = T_n(Y) \)). The commutativity of (3.3) for any \( g \) is clear.

\[\square\]

### 3.2 The Functors \( u_\dagger \) and \( u^\dagger \)

Let \( u : Y \to Y' \) be a morphism in \( \mathcal{D} \). We introduce the functors \( u_\dagger : f^{-1}_h(Y) \to f^{-1}_h(Y') \) and \( u^\dagger : f^{-1}_h(Y') \to f^{-1}_h(Y) \), and prove that they are homotopy equivalences.

The functor \( u_\dagger \) assigns to \( Y \in \Lambda_n(\mathcal{D}) \) of the form

\[
\cdots \to \bar{Y}_{n-1} \leftarrow Y_n \xrightarrow{\alpha} \bar{Y}_n =_Y
\]

the element

\[
\cdots \to \bar{Y}_{n-1} \leftarrow Y_n \xrightarrow{u \circ \alpha} Y'
\]

in \( \Lambda_n(\mathcal{D}) \). It defines naturally a functor \( u_\dagger : f^{-1}_h(Y) \to f^{-1}_h(Y') \). The functor \( u^\dagger \) assigns to \( Z \in \Lambda_n(\mathcal{D}) \) of the form

\[
\cdots \to \bar{Z}_{n-1} \leftarrow Z_n \xrightarrow{\beta} \bar{Z}_n =_{Y'}
\]

the following element in \( \Lambda_{n+1}(\mathcal{D}) \):

\[
\cdots \to \bar{Z}_{n-1} \leftarrow Z_n \xrightarrow{\beta} \bar{Z}_n \xleftarrow{u \circ \beta} Y \xrightarrow{\text{id}} Y
\]

It defines naturally a functor \( u^\dagger : f^{-1}_h(Y') \to f^{-1}_h(Y) \).
**Lemma 3.2.** There are natural transformations \( \theta_1 : \text{id}_{f_h^{-1}(Y)} \to u^! \circ u^! \) and \( \theta_2 : u^! \circ u^! \to T_f \circ \text{id}_{f_h^{-1}(Y')} \).

**Proof.** For \( Y \in \Lambda_n(Y) \) and \( Z \in \Lambda_n(Y') \) as in (3.4), (3.6), define \( \theta_1(X, [n], Y) \) and \( \theta_2(X, [n], Z) \) as follows. The value \( \theta_1(X, [n], Y) \) is a morphism in \( f_h^{-1}(Y) \) from \( (X, [n], Y) \) to \( (X, [n + 1], u^! \circ u^!(Y)) \). Define it as the identity on the component \( X \), as the map \( \phi : [n] \to [n + 1] \) in \( \Delta_{str} \) such that \( \phi(i) = i \) for all \( i = 1 \ldots n \), and the component \( \Lambda(\phi)(Y) \to u^! \circ u^!(Y) \) is given as the following morphism in \( \Lambda_{n+1}(Y') \):

\[
\begin{array}{cccccccccc}
\ldots & \longrightarrow & Y_{n-1} & \leftarrow & Y_n & \alpha \longrightarrow & \text{id} & Y & \longrightarrow & Y \\
\text{id} & & \text{id} & & u & & \text{id} & & \text{id} \\
\ldots & \longrightarrow & Y_{n-1} & \leftarrow & Y_n & \longrightarrow & Y & \longrightarrow & Y \\
\end{array}
\] (3.8)

The value \( \theta_2(X, [n], Z) \) is a morphism in \( f_h^{-1}(Y') \) from \( (X, [n + 1], u^! \circ u^!(Z)) \) to \( (X, [n + 1], T_f \circ \text{id}(Z)) \). It is defined as the identity on \( X \), as the identity on \( [n + 1] \), and the corresponding morphism \( u^! \circ u^!(Z) \to T_f(Z) \) in \( \Lambda_{n+1}(Y') \) is given as

\[
\begin{array}{cccccccccc}
\ldots & \longrightarrow & Z_{n-1} & \leftarrow & Z_n & \beta \longrightarrow & \text{id} & Y' & \longrightarrow & Y' \\
\text{id} & & \text{id} & & u & & \text{id} & & \text{id} \\
\ldots & \longrightarrow & Z_{n-1} & \leftarrow & Z_n & \longrightarrow & Y' & \longrightarrow & Y' \\
\end{array}
\] (3.9)

In both cases, these definitions are compatible with the morphisms in \( f_h^{-1}(Y) \) (corresp., in \( f_h^{-1}(Y') \)), and thus define natural transformations.

**Corollary 3.3.** For any \( u : Y \to Y' \) in \( \mathcal{D} \), the functors \( u^! : f_h^{-1}(Y) \to f_h^{-1}(Y') \) and \( u^! : f_h^{-1}(Y') \to f_h^{-1}(Y) \) are homotopy equivalences, homotopy mutually inverse to each other.

It follows from Lemma 3.1 and Lemma 3.2.

### 3.3

Let \( u : Y \to Y' \) be a morphism in \( \mathcal{D} \). Consider the following diagram:

\[
\begin{array}{ccc}
f_h^{-1}(Y) & \overset{i_Y}{\longrightarrow} & f_h \setminus Y \\
\downarrow u^! & & \downarrow u^! \\
f_h^{-1}(Y') & \overset{i_{Y'}}{\longrightarrow} & f_h \setminus Y' \\
\end{array}
\] (3.10)
One sees immediately that it commutes.

We know that the left-hand side vertical arrow $u$ is a homotopy equivalence, see Corollary 3.3. Therefore, to prove that $u_*$ is a homotopy equivalence, it is enough to prove that the horizontal maps $i_Y$ and $i_{Y'}$ also are.

For this end, define a functor $\ell_Y : f_h \downarrow Y \to f_h^{-1}(Y)$ as follows. It sends $(X, [n], Y, s)$ where $s : \bar{Y}_n \to Y$ a morphism in $\mathcal{D}$, to $(X, [n + 1], Z)$, where $Z$ is given as

$$\cdots \leftarrow Y_n \to \bar{Y}_{n-1} \leftarrow Y_n \overset{\alpha}{\to} \bar{Y}_n \overset{id}{\leftarrow} \bar{Y}_n \overset{s}{\to} Y$$

(3.11)

and it is defined on the morphisms accordingly.

**Lemma 3.4.** Let $Y \in \mathcal{D}$. Then the composition $\ell_Y \circ i_Y = T_f$, where $T_f$ is the functor homotopy equivalent to the identity, see Lemma 3.1, and there is a natural transformation $\omega : id_{f_h \downarrow Y} \to i_Y \circ \ell_Y$.

**Proof.** The first claim is straightforward. For the second, consider the commutative diagram

$$\cdots \leftarrow Y_n \overset{\alpha}{\to} \bar{Y}_n \overset{id}{\leftarrow} \bar{Y}_n \overset{id}{\leftarrow} \bar{Y}_n \overset{s}{\to} Y$$

(3.12)

**Corollary 3.5.** The functors $i_Y$ and $\ell_Y$ are homotopy equivalences, homotopy inverse to each other.

**Theorem 2.6** is proven.

---

4 A variation on the Evrard’s result

Here we prove a version of Theorem 2.6 using slightly different construction of the free paths category of a category $\mathcal{D}$, which we denote here $\Lambda' \mathcal{D}$. The idea is that the category $\Delta_{str}$ (see Section 2.1) contains too much morphisms. The observation is that the morphisms we really used in the course of proof of Theorem 2.6 are those which form the following subcategory of the category $\Delta_{str}$.
Define the category $\Delta \leq$ having the objects $\{[1], [2], [3], \ldots\}$, and the only morphism from $[m]$ to $[n]$ if $m \leq n$, and the empty set of morphisms otherwise. We interpret this morphism as a morphism $f$ in $\Delta \str$ from $1 < 2 < \cdots < m$ to $1 < 2 < \cdots < n$ such that $f(i) = i$ ($m \leq n$). It gives an imbedding of categories $\Delta \leq \rightarrow \Delta \str$.

Recall the categories $\Lambda_n \mathcal{D}$, defined for a small category $\mathcal{D}$ and for $n \geq 1$. The composition of the functor $\Delta \leq \rightarrow \Delta \str$ with the functor $\Lambda: \Delta \str \rightarrow \mathcal{C}at$, sending $[n]$ to $\Lambda_n \mathcal{D}$, and a morphism $\phi: [m] \rightarrow [n]$ to the functor $\Lambda(\phi)$ as in (2.3), gives a functor $\Lambda': \Delta \leq \rightarrow \mathcal{C}at$

We set $\Lambda' \mathcal{D} = \Delta \leq \int \Lambda'$

This category $\Lambda' \mathcal{D}$ is another candidate for the “free paths category” of $\mathcal{D}$. There are two functors $p_0, p_1: \Lambda' \mathcal{D} \rightarrow \mathcal{D}$, the start point and the end point of the path functors, as in the case of $\Lambda \mathcal{D}$, see Section 2.1.

Then we define, for a functor $f: \mathcal{C} \rightarrow \mathcal{D}$, the category $\mathcal{H}'(f)$ from the cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{H}'(f) & \longrightarrow & \Lambda' \mathcal{D} \\
\downarrow & & \downarrow p_0 \\
\mathcal{C} & \rightarrow & \mathcal{D}
\end{array}
$$

There is a functor $f'_h: \mathcal{H}'(f) \rightarrow \mathcal{D}$ given by $(X, Y) \mapsto p_1(Y)$.

We have a direct analogue of Proposition 2.5:

**Proposition 4.1.** The functor $q: \mathcal{H}'(f) \rightarrow \mathcal{C}$, assigning $X \in \mathcal{C}$ to $(X, Y)$, is a homotopy equivalence.

The proof is literally the same.

The main result in this Section is:

**Theorem 4.2.** Let $f: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between small categories, and let $f'_h: \mathcal{H}'(f) \rightarrow \mathcal{D}$ be the functor constructed above. Then the functor $f'_h: \mathcal{H}'(f) \rightarrow \mathcal{D}$ fulfills the assumptions of Quillen Theorem B.

**Proof.** The only thing we should mention in addition to the proof of Theorem 2.6 is that our main “tool” in the proof of Theorem 2.6 is the shift functors $T_n: \Lambda_n \mathcal{D} \rightarrow \Lambda_{n+1} \mathcal{D}$, introduced in Section 3.1, and the corresponding functor $T_f: \mathcal{H}(f) \rightarrow \mathcal{H}(f)$ constructed out of them. That is, we can replace the category $\Delta \str$ by any subcategory (containing all its objects), where the the functors $T_n$ can be defined. It is clear that the category $\Delta \leq$ is the minimal among such categories. Then the proof can be repeated straightforwardly.
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