Spectrum from the warped compactifications with the de Sitter universe

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Abstract: We discuss the spectrum of the tensor metric perturbations and the stability of warped compactifications with the de Sitter spacetime in the higher-dimensional gravity. The spacetime structure is given in terms of the warped product of the non-compact direction, the spherical internal dimensions and the four-dimensional de Sitter spacetime. To realize a finite bulk volume, we construct the brane world model, using the cut-copy-paste method. Then, we compactify the spherical directions on the brane. In any case, we show the existence of the massless zero mode and the mass gap of it with massive Kaluza-Klein modes. Although the brane involves the spherical dimensions, no light massive mode is excited. We also investigate the scalar perturbations, and show that the model is unstable due to the existence of a tachyonic bound state, which seems to have the universal negative mass square, irrespective of the number of spacetime dimensions.

Keywords: Higher-dimensional gravity.
1. Introduction

The realization of inflation and dark energy is the challenging issue in cosmology and particle physics. Since string theory is higher-dimensional, one has expected that such an accelerating universe may be realized via dynamics of the extra-dimensional space. Realization of an accelerating universe depends on ansatz of the spacetime metric and fields. Whether such a solution can serve as a realistic model will require more studies.
The traditional expectation has been that one obtains the four-dimensional de Sitter universe as the exact solution of higher-dimensional gravity, in particular in superstring- or M-theory. One of the most successful constructions of the de Sitter universe in higher dimensions is the warped compactification in five dimensions discussed in Refs. [1, 2]. This is the cosmological generalization of the famous Randall-Sundrum model [3]. Recently, higher dimensional warped compactifications with the de Sitter universe in the pure gravity have been obtained in Ref. [4]. New solutions of warped compactifications including the bulk matter have also been derived in Ref. [5]. In these solutions, the spacetime is given by the warped product of the non-compact extra dimension, the internal spherical direction and the four-dimensional de Sitter spacetime. As the bulk volume diverges, we construct a codimension-one braneworld, by cutting the spacetime at a certain place of it, and then gluing the remaining piece to its copy at the same position. By construction, there is the \( \mathbb{Z}_2 \)-symmetry with respect to the brane. This is the ordinary cut-copy-paste method. The difference from the five-dimensional case is that the braneworld involves the spherical dimensions as well as the de Sitter spacetime. We then compactify the spherical dimensions to obtain the four-dimensional cosmology. Such a way of construction of the braneworld model from a higher-dimensional theory is known as the Kaluza-Klein braneworld in the literature [6]. The junction condition requires the positive brane tension. The insertion of the braneworld is hence equivalent to adding a positive potential energy to the effective theory. Though these solutions cannot be counterexamples of the NO-GO theorem [7], they give us an interesting class of the cosmological braneworld models in spacetimes of higher than six dimensions. In this paper, we study the spectrum of the gravitational waves and the stability in our model.

In order to investigate whether these models are realistic, we have to see the localizability of the massless zero mode which could reproduce the four-dimensional physics after compactifying the spherical dimensions. The inflationary four-dimensional universe after the compactification also should not suffer excitations of any light massive mode. Here, we define the light mode as follows: The effective four-dimensional metric after the compactification on the brane is given by the de Sitter spacetime

\[
ds_4^2 = -dt^2 + c_0^2 e^{2Ht} \delta_{ij} dx^i dx^j,
\]

where \( H \) is the Hubble expansion rate and \( c_0 \) is the size of the universe at \( t = 0 \). In the effective four-dimensional point of view, the time evolution equation of each four-dimensional mode of mass \( m \) and comoving momentum \( k \) whose mode function is given by \( \varphi_{m,k} \), is written as

\[
\left( \frac{d^2}{dt^2} + 3H \frac{d}{dt} + \frac{k^2}{c_0^2 e^{2Ht}} + m^2 \right) \varphi_{m,k}(t) = 0.
\]
The solution to Eq. (1.2) is given by

\[ \varphi_{m,k} \propto e^{-\frac{3}{2}Ht}Z_{\nu_m}\left(\frac{k}{c_0}e^{-Ht}\right), \quad \nu_m := \frac{m^2}{H^2} - \frac{9}{4} = \frac{1}{H^2}(m^2 - m_c^2), \quad (1.3) \]

where \( Z_\nu \) denotes the Bessel functions of order \( \nu \). Let us review the behavior in the homogeneous limit \( k \to 0 \). The late time behavior of Eq. (1.3) depends on the mass. Heavy modes of \( m > m_c = \frac{3}{2}H \) decay rapidly as \( |\varphi_{m,k}| \propto a^{-\frac{3}{2}} \), while light modes of \( m < m_c \) decay more slowly and in particular for \( m \ll m_c, \ |\varphi_{m,k}| \propto a^{-\frac{3}{4}m_c^2} \). Thus the contribution of heavy KK modes is diluted rapidly during inflation, while that of light ones may survive and affect the late time cosmology. We will show that in any model the mass gap between the zero and massive modes is always greater than \( \frac{3}{2}H \), and hence warped compactifications with the de Sitter universe are free from massive excitations. The five-dimensional model has been investigated in e.g., Ref. [8], which showed that there is always mass gap given by \( \frac{3}{2}H \).

This paper is constructed as follows. In Sec. II, we review warped compactification solutions with the de Sitter universe discussed in [4, 5]. In Sec. III, we investigate the tensor perturbations with respect to the three-dimensional space and mass spectrum. In Sec. IV, we discuss the stability of the solutions with a spherical internal space. The last Sec. V is devoted to give summary and conclusion.

2. Warped compactifications with the de Sitter spacetime

In this section, we review solutions of warped compactifications with the de Sitter spacetime discussed in [4] as well as [5].

2.1 The five-dimensional model

We consider the Einstein gravity with a negative cosmological constant in a five-dimensional spacetime including the braneworld

\[ S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left( R - 2\Lambda_5 \right) - \int d^4x \sqrt{-q}\sigma_5, \quad (2.1) \]

where \( \Lambda_5 < 0 \) is the cosmological constant, \( \sigma_5 \) is the brane tension and \( q_{\mu\nu} \) is the induced metric on the brane. There is the warped compactification solution with the de Sitter universe

\[ ds^2 = A(y)^2 \left(-dt^2 + c_0^2e^{2Ht}\delta_{ij}dx^i dx^j + dy^2\right), \quad (2.2) \]

where

\[ A(y)^{-1} = e^{H|y|} - \frac{|\Lambda_5|e^{-H|y|}}{24H^2}. \quad (2.3) \]
The bulk solution is the five-dimensional anti de Sitter (AdS) spacetime \([1, 2, 8]\). By defining the curvature radius of the AdS spacetime \(\ell := \sqrt{\frac{6}{|\Lambda_5|}}\), the warp factor \(A\) is rewritten into the more familiar form

\[
A(y)^{-1} = \frac{1}{H\ell} \sinh \left[ H(|y| + y_0) \right], \quad e^{H y_0} := 2(H\ell).
\] (2.4)

To realize the finite volume of the extra space, the brane boundary is put at \(y = y_b > 0\) and the \(Z_2\)-symmetry across it is also imposed. The brane position \(y = y_b\) is given by

\[
e^{H y_b} = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{|\Lambda_5|}{6H^2}} \right). \tag{2.5}
\]

The induced metric on the brane is that of the four-dimensional de Sitter spacetime Eq. (1.1). The junction condition gives \(\kappa_5^2 \sigma_5 = \sqrt{6(6H^2 + |\Lambda_5|)} > 0\).

2.2 The \(D(>6)\)-dimensional models

In this subsection, we consider warped compactification solutions of \(D(>6)\) dimensions with a spherical internal space.

\(\textit{(1) In the case of the pure gravity}\)

First of all, we consider the \(D\)-dimensional Einstein gravity including the braneworld

\[
S = \frac{1}{2\kappa_D^2} \int d^Dx \sqrt{-g} R - \int d^{D-1}x \sqrt{-g} \sigma_D,
\] (2.6)

where \(\sigma_D\) is the brane tension and \(q_{\mu\nu}\) is the induced metric on the brane. The warped compactification of the de Sitter universe \([4]\) is given by

\[
ds^2 = A(y)^2 \left[ -dt^2 + c_0^2 e^{2Ht} \delta_{ij} dx^i dx^j + \frac{1}{H^2} \left( G(y)dy^2 + \frac{D-6}{3} d\Omega_{D-5}^2 \right) \right], \tag{2.7}
\]

with

\[
A(y) = \left( \frac{\cosh M(|y| + a)}{\cosh Ma} \right)^{-p}, \quad G(y) = \frac{D-2}{3} p^2 M^2 \tanh^2 M(|y| + a). \tag{2.8}
\]

We assume \(p > 0\) and redefine the coordinate of the noncompact direction \(dy = \frac{H}{G^{1/2}} dY\), which gives

\[
Y = \frac{p}{H} \sqrt{\frac{D-2}{3}} \ln \cosh[M(|y| + a)]. \tag{2.9}
\]

For \(p > 0\), the direction of the increasing \(y\) corresponds to that of the increasing \(Y\). Setting the coordinate \(Y\), the \(D\)-dimensional metric (2.7) is rewritten by

\[
ds^2 = A(Y)^2 \left[ -dt^2 + c_0^2 e^{2Ht} \delta_{ij} dx^i dx^j + dY^2 + \frac{D-6}{3H^2} d\Omega_{D-5}^2 \right]. \tag{2.10}
\]
where the warp factor $A$ is

$$A(Y) = e^{-H \sqrt{\frac{2}{3}} |Y - Y_0|}. \quad (2.11)$$

The spacetime structure is given in terms of the warped product of $dS_4$, $\mathbb{R}$ and $S^{D-5}$. We emphasize that in the new frame $Y$ there is no explicit dependence of the metric on $p$ and $M$, which indicates that these parameters are not physical and can be absorbed by the redefinition of coordinates. We assume that the $(D-1)$-dimensional braneworld is located at $Y = Y_0 = \frac{H}{\sqrt{3}} \ln \cosh(Ma)$ where $A(Y_0) = 1$. And we impose the $Z_2$-symmetry across it. The conformal metric in the square bracket of Eq. (2.10) is given by the product of the four-dimensional de Sitter spacetime, the noncompact $Y$ direction and the $(D-5)$-dimensional sphere. The internal space has a deficit solid angle at $Y \to \infty$, given by

$$\Delta \Omega^{(D)}_{D-5} = \Omega_{D-5} \left[ 1 - \left( \frac{D-6}{D-2} \right)^{\frac{D-5}{2}} \right]. \quad (2.12)$$

The induced metric on the brane is given by

$$ds^2_{\text{ind}} = -dt^2 + c_0^2 e^{2Ht} \delta_{ij} dx^i dx^j + \frac{D-6}{3H^2} d\Omega^2_{D-5}. \quad (2.13)$$

The junction condition gives $\kappa^2_D \sigma_D = 2 \sqrt{3(D-2)H}$.

We now generalize the solution Eq. (2.10) to the case with a cosmological constant

$$S = \frac{1}{2\kappa^2_D} \int d^Dx \sqrt{-g} \left( R - 2\Lambda_D \right) - \int d^{D-1}x \sqrt{-q} \sigma_D. \quad (2.14)$$

The solution discussed in [4] is given by the metric Eq. (2.7) with

$$A(y) = \left( \frac{\cosh\left[M(|y| + a)\right]}{\cosh(Ma)} \right)^{-1},$$

$$G(y) = \frac{(D-2)M^2 \sinh^2\left[M(|y| + a)\right]}{3 \cosh^2\left[M(|y| + a)\right] + \frac{2|\Lambda_D|}{(D-1)H^2} \cosh^2(Ma)}. \quad (2.15)$$

This solution is the generalization of that in the previous subsection with $p = 1$. As before, it is also useful to redefine the coordinate $dy = \frac{H}{G^2} dY$, which gives

$$Y = \frac{1}{H} \sqrt{\frac{D-2}{3}} \ln \left[ \frac{1}{2} \left( \cosh[M(|y| + a)] + \sqrt{\cosh^2[M(|y| + a)] + \frac{2|\Lambda_D| \cosh^2(Ma)}{3(D-1)H^2}} \right) \right]. \quad (2.16)$$
The spacetime metric can be rewritten as Eq. (2.10) with
\[
A(Y) = \left( \cosh \left[ \sqrt{\frac{3}{D - 2}} H |Y - Y_0| \right] + \sqrt{1 + \frac{2|\Lambda_D|}{3(D - 1)H^2}} \sinh \left[ \sqrt{\frac{3}{D - 2}} H |Y - Y_0| \right] \right)^{-1}.
\]
(2.17)
The internal space has a deficit solid angle at \( Y \to \infty \), given by
\[
\Delta \Omega^{(D)}_{D-5} = \Omega_{D-5} \left[ 1 - \left( \frac{D - 6}{D - 2} \right)^{\frac{D - 5}{2}} \right].
\]
(2.18)
The \((D - 1)\)-dimensional braneworld is located at
\[
Y = Y_0 := \frac{1}{H} \sqrt{\frac{D - 2}{3}} \ln \left[ \frac{\cosh \left( \sqrt{\frac{3}{2}} \frac{Ma}{2} \right)}{2} \left( 1 + \sqrt{1 + \frac{2|\Lambda_D|}{3(D - 1)H^2}} \right) \right].
\]
We impose the \( Z_2 \)-symmetry in the \( Y \)-direction across it. The induced metric on the brane is the same as Eq. (2.13). The junction condition gives \( \kappa_D^2 \sigma_D = 2H \sqrt{3(D - 2)} \left( 1 + \frac{2|\Lambda_D|}{3(D - 1)H^2} \right) \).

(2) In the case of the scalar-tensor theory

We consider the scalar-tensor theory with a negative potential including the braneworld
\[
S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - 2e^{\beta \phi} \Lambda_s \right) - \int d^{D-1} x \sqrt{-q_s} e^{\gamma_s \phi},
\]
(2.19)
where \( \beta \) represents the coupling parameter, \( \Lambda_s < 0 \) is constant, and \( \gamma_s \) denotes the brane coupling to the scalar field. The solution discussed in [5] is given by the metric Eq. (2.10) with
\[
A = e^{-H \sqrt{\frac{3\Lambda_s}{c(D - 2)}} |Y - Y_0|}.
\]
(2.20)
The scalar field configuration is given by
\[
\phi = 2H \sqrt{\frac{3}{c(D - 2)}} |Y - Y_0|, \quad c := \beta^2 - \frac{2}{D - 2}.
\]
(2.21)
The expansion rate of the de Sitter universe is given by
\[
H^2 = -\frac{1}{3} c \Lambda_s.
\]
(2.22)
According to \( \Lambda_s < 0 \), one has \( \beta^2 > \frac{2}{D - 2} \) so that \( H^2 > 0 \). The internal space has a deficit or surplus solid angle at \( Y \to \infty \) which is given by
\[
\Delta \Omega^{(s)}_{D-5} = \Omega_{D-5} \left[ 1 - \left\{ \frac{D - 6}{D - 2} \left( 1 + \frac{2}{c(D - 2)} \right) \right\}^{\frac{D - 5}{2}} \right].
\]
(2.23)
For \( c = c_* := \frac{D-6}{2(D-2)} \), the internal space becomes a flat space, where \( \beta_* = \frac{1}{\sqrt{2}} \). The metric (2.23) gives a deficit solid angle for \( c > c_* \) while the \( D \)-dimensional spacetime has a surplus solid angle for \( c < c_* \). We assume that the braneworld is located at \( Y = Y_0 \), and we impose the \( Z_2 \) symmetry. The induced metric on the brane is the same as Eq. (2.13). The junction conditions of the metric and the scalar field give

\[
\kappa_D^2 \sigma_s = 2\sqrt{\frac{3(D-2)c+2}{c}} H \quad \text{and} \quad \gamma_s = \frac{1}{(D-6)^3}.
\]

(3) In the case with a form field strength

Finally, we consider the theory with a form field strength including the braneworld

\[
S = \frac{1}{2\kappa_D^2} \int d^Dx\sqrt{-g} \left[R - 2e^{-\frac{\alpha\phi}{D-2}} \Lambda_f - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2(D-5)!} e^{\alpha\phi} F_{(D-5)}^2 \right] - \int d^{D-1}x\sqrt{-q} \sigma_f e^{\gamma_f \phi}.
\]

With the warp factor \( A(Y) \), the solution discussed in [5] is given by

\[
d s^2 = A(Y)^2 \left[ -dt^2 + c_0^2 e^{2Ht} \delta_{ij} dx^i dx^j + dY^2 + \frac{D-6}{3H^2 + \frac{f^2}{2}} d\Omega_{D-5}^2 \right],
\]

\[
\phi = \frac{2(D-6)}{\alpha} \ln A,
\]

\[
F = f \left( \frac{D-6}{3H^2 + \frac{f^2}{2}} \right)^{\frac{D-5}{2}} \sqrt{\gamma} dz^1 \wedge dz^2 \wedge \cdots \wedge dz^{D-5},
\]

where \( \gamma_{ab} \) is the metric of an unit \( S^{D-5} \), \( f \) represents the strength of form field, and the warp factor and the expansion rate are given by

\[
A(Y) = e^{-\frac{3}{D-2} \left( 1 + \frac{D-6}{D-2} \right) H|Y-Y_0|}, \quad H^2 = \frac{\zeta}{3} \left( -\frac{2\Lambda_f}{D-6} + \frac{f^2}{2} \right),
\]

respectively. Here \( \zeta \) is defined as \( \zeta := \frac{\alpha^2}{2(D-6)} - \frac{D-6}{D-2} \). In the case of \( f = 0 \), the solution in the scalar-tensor theory is reproduced with the following replacements, \( \alpha = -(D-6)\beta \) and \( \zeta = \frac{D-6}{2} c \). The deficit or surplus angle of the internal space at \( Y \rightarrow \infty \) is given by

\[
\Delta \Omega_{D-5}^{(f)} = \Omega_{D-5} \left[ 1 - \left( \frac{D-6}{D-2} \right)^2 \right],
\]

The induced metric thus turns out to be

\[
ds_{\text{ind}}^2 = -dt^2 + c_0^2 e^{2Ht} \delta_{ij} dx^i dx^j + \frac{D-6}{3H^2 + \frac{f^2}{2}} d\Omega_{D-5}^2.
\]

The junction conditions of the metric and the scalar field give

\[
\kappa_D^2 \sigma_f = 2H \sqrt{3(D-2 + \frac{D-6}{\zeta})}, \quad \gamma_f = -\frac{D-6}{(D-2)\alpha}.
\]
The form field strength is continuous across the brane and hence we do not need to introduce any coupling of it with the brane matter.

For all the solutions discussed in this subsection, there is a curvature singularity at \( Y \to \infty \) where the Kretschmann invariant \( R_{ABCD}R^{ABCD} \) diverges. The singularity is due to the presence of a deficit (or surplus) solid angle there \([5]\). In the presence of the matter fields, we can eliminate such a singularity for the particular coupling constant, for example, for \( c = c_* \) from Eq. (2.23) in the case of the scalar-tensor theory.

### 3. Spectrum of the tensor metric perturbations

In this section, we discuss the tensor metric perturbations with respect to the ordinary three-dimensional space which satisfy the transverse-traceless (TT) conditions. We focus on the existence of the zero mode and the possible excitations of Kaluza-Klein (KK) modes.

#### 3.1 The five-dimensional model

We briefly review the five-dimensional case. The tensor perturbation for the metric Eq. (2.2) is given by

\[
ds^2 = A^2 \left( -dt^2 + \epsilon_0^2 e^{2Ht} (\delta_{ij} + h_{ij}) dx^i dx^j + dy^2 \right),
\]

where \( h_{ij} \) satisfies the TT conditions \( h_{ij,j} = h_{ii} = 0 \).

Decomposing the tensor perturbations into the Fourier modes

\[
h_{ij} = \int dmd^3k f_m(y) \varphi_{m,k}(t)e^{ik\cdot x} \hat{e}_{ij},
\]

where \( \hat{e}_{ij} \) denotes two independent polarizations, we find the equation of motion for each mode

\[
f_m'' + 3\frac{A'}{A} f_m' = -m^2 f_m.
\]

The solution of \( \varphi_{m,k} \) is given by Eq. (1.3). It is convenient to rewrite the bulk equation into the form of the Schrödinger equation by introducing new variable \( f_m = A^{-\frac{3}{2}} X_m \),

\[
\left[ -\frac{d^2}{dy^2} + V(y) \right] X_m(y) = m^2 X_m(y),
\]

where

\[
V(y) = -3H \sqrt{1 + \frac{|\Lambda_5|}{6H^2}} \delta(y - y_b) + \frac{9H^2}{4} + \frac{360|\Lambda_5|H^4}{(24H^2e^{H|y|} - |\Lambda_5|e^{-H|y|})^2}.
\]
The attractive delta function term represents the contribution from the brane and ensures the existence of the massless zero mode. The zero and KK modes satisfy the normalization conditions

\[ 2 \int_{y_b}^{\infty} dy A^3 f_0 f_0 = 1, \quad 2 \int_{y_b}^{\infty} dy A^3 f_m f_{m'} = \delta(m - m'), \quad (3.5) \]

respectively, where \( m, m' \neq 0 \) in the second relation. The factor 2 in front of the integrals reflects the \( Z_2 \)-symmetry with respect to the brane.

The mode function for the zero mode is given by \( f_0 = C \), where

\[ C^{-2} = 2 \int_{y_b}^{\infty} dy A^3 \]

\[ = \frac{\sqrt{6}}{|A_5|^{3/2}} \left\{ |A_5|^{1/2} \sqrt{6H^2 + |A_5|} + 6H^2 \ln \left( \frac{\sqrt{6H^2}}{|A_5| + \sqrt{6H^2 + |A_5|}} \right) \right\}. \quad (3.6) \]

The mode function gives \( C^2 \simeq \sqrt{\frac{|A_5|}{6}} \) at the low energy scales \( H \ll |A_5|^{1/2} \), while one finds \( C^2 \simeq \frac{3H^2}{2} \) at the high energy scales \( H \gg |A_5|^{1/2} \). This was the well-known result obtained in Ref. [8]. The effective four-dimensional gravitational mass is given by

\[ M_{4,\text{eff}}^2 = \frac{1}{\kappa_5^2 C^2} \]

\[ = \frac{\sqrt{6}}{\kappa_5^3 |A_5|^{3/2}} \left\{ |A_5|^{1/2} \sqrt{6H^2 + |A_5|} + 6H^2 \ln \left( \frac{\sqrt{6H^2}}{|A_5| + \sqrt{6H^2 + |A_5|}} \right) \right\}. \quad (3.7) \]

We find that there is a well-behaved low energy limit of Eq. (3.7) with a fixed \( |A_5| \), as \( M_{4,\text{eff}}^2 = \frac{1}{\kappa_5^3 C^2} \mid_{H \to 0} = \frac{1}{\kappa_5^3} \sqrt{\frac{6}{|A_5|}} \). Eq. (3.4) shows that since the potential \( V > \frac{9H^2}{4} \), there is always the mass gap between the zero mode and the continuum of KK modes given by \( \frac{3}{2}H \). Thus there is no excitation of light KK modes of \( 0 < m < \frac{3}{2}H \).

### 3.2 The \( D(> 6) \)-dimensional models

In this subsection, we consider the tensor perturbations in higher-dimensional warped compactifications with the de Sitter universe. The metric including tensor perturbations is given by

\[ ds^2 = A(y)^2 \left[ -dt^2 + c_0^2 e^{2Ht} (\delta_{ij} + h_{ij}) dx^i dx^j + dY^2 + \omega_{ab} dz^a dz^b \right], \quad (3.8) \]

where \( \omega_{ab} \) denotes the metric of the \( (D - 5) \)-dimensional spherical dimensions with a given radius in each case.

Decomposing the tensor perturbations into the Fourier modes

\[ h_{ij} = \int dmd^3k \sum_{\{L\}} f_m(Y)Y_{(L)}\varphi_{m,k}(t)e^{ik^ij} \hat{e}_{ij}, \]
where the solution of $\varphi_{m,k}$ is given by Eq. (1.3), and $m, \{L\}$ denote the KK mass and a set of quantum numbers associated with the $(D-5)$ sphere, respectively. In particular, $L$ denotes the azimuthal quantum number, and $Y_{\{L\}}$ corresponds to the harmonic function on $S^{D-5}$, which satisfies $\Delta_{D-5} Y_{\{L\}} = -L(L+D-6)Y_{\{L\}}$, and is normalized as $\int d\Omega_{D-5} Y_{\{L\}} Y_{\{L'\}} = \delta_{\{L\},\{L'\}}$.

(1) In the case of the pure gravity

We take the perturbation in the metric (2.10) so that the perturbed line element is of the form (3.8), where

$$A(Y) = e^{-H\sqrt{\frac{D-2}{3}}|Y-Y_0|}, \quad \omega_{ab} dz^a dz^b = \frac{D-6}{3H^2} d\Omega_{D-5}. \quad (3.9)$$

We find that each mode satisfies the equation of motion

$$\frac{H^2}{A^{D-2}} \frac{d}{dY} \left( A^{D-2} \frac{d}{dY} f_m \right) = -\left( m^2 - \frac{3L(L+D-6)H^2}{D-6} \right) f_m. \quad (3.10)$$

The zero and KK modes satisfy the normalization conditions

$$2 \int_{Y_0}^\infty dY \Omega_{D-5} \left( \frac{D-6}{3} \right)^{\frac{D-5}{2}} \frac{1}{H^{D-4}} A^{D-2} f_0 f_0 = 1, \quad (3.11)$$

$$2 \int_{Y_0}^\infty dY \Omega_{D-5} \left( \frac{D-6}{3} \right)^{\frac{D-5}{2}} \frac{1}{H^{D-4}} A^{D-2} f_m f_m' = \delta(m-m'),$$

where $m, m' \neq 0$ in the second relation.

By the redefinition of $f_m = H^\frac{D-2}{2} A^{\frac{D-2}{2}} X_m$, Eq. (3.10) reduces to

$$\left[ -\frac{d^2}{dY^2} + \frac{3(D-2)}{4} H^2 - \frac{\sqrt{3H^2(D-2)}Y-Y_0}{D-2} \right] X_m = \left( m^2 - \frac{3H^2L(L+D-6)}{D-2} \right) X_m. \quad (3.12)$$

All modes of $m^2 = \frac{3L(L+D-6)}{D-6} H^2$ give the normalizable solutions of Eq. (3.12). Thus excitations of the massive bound states associated with the angular dimensions are allowed. However, the masses of these modes are always greater than the critical mass of the de Sitter spacetime. This is summarized in Fig. 1. On the other hand, for $L = 0$, the mass gap between the zero mode and the continuum of the KK modes associated with the non-compact $y$-direction is given by $\sqrt{\frac{3(D-2)}{2}} H$. Thus it does not depend on the parameters of $M$ and $p$. As the number of extra dimensions increases, the mass gap also increases. For the cases of $D = 10$ and 11, we obtain the mass gaps of $\sqrt{6H}$ and $\frac{3\sqrt{3}}{2} H$, respectively. Thus although the brane involves the internal angular dimensions, in the four-dimensional universe no light massive mode is excited. The zero mode solution is given by

$$f_0 = C_1 + C_2 A^{-(D-2)}. \quad (3.13)$$
Figure 1: The thick (red), dashed (green), dotted (blue) curves represent the mass gap between the zero mode and the continuum of KK modes associated with the $y$-direction, and masses of the first and second massive bound states associated with the angular directions, respectively, in the unit of $H$ and as the functions of $n := D - 4$ which denotes the dimensionality of the internal space. The solid line shows the critical mass of the de Sitter spacetime.

The normalizability forbids the second solution and we set $C_2 = 0$. Then, through Eq. (3.11), we obtain

$$C_1^2 = \frac{\sqrt{3(D - 2)}}{2\Omega_{D-5}} \left(\frac{3}{D - 6}\right) \frac{D-6}{2} H^{D-4}. \tag{3.14}$$

For $D = 7$, we get $C_1^2 = \frac{3\sqrt{7}h^3}{8\pi}$. Also, in the limit of $H \to 0$, the amplitude of the zero mode vanishes. The effective four-dimensional gravitational mass is given by

$$M_{2,\text{eff}}^2 = \frac{1}{\kappa^2 D C_1^2} = \frac{2\Omega_{D-5}}{3(3(D - 2))\kappa^2 D H^{D-4}} \left(\frac{D - 6}{3}\right) \frac{D-6}{2}, \tag{3.15}$$

which is ill-defined in the limit of $H \to 0$.

In order to see the difference from the analysis Ref. [4], it is better to write down the bulk equation Eq. (3.10) in terms of the original $y$ coordinate

$$\frac{H^2}{G^2 D^{D-2}} \frac{d}{dy} \left(\frac{A^{D-2}}{G^2} \frac{d}{dy} f_m\right) = -\left(m^2 - \frac{3H^2L(L + D - 6)}{D - 6}\right)f_m. \tag{3.16}$$

The equation Eq. (3.10) for $D = 7$ and $L = 0$ disagrees with Eq. (3.4) of Ref. [4]. The difference should be due to the fact that in Eq. (3.4) of Ref. [4], $m^2$ should be replaced with $m^2G(y)$ in the separation of variables, which is expected from Eq. (3.3). Therefore, the subsequent results and conclusions are different. For example, in Ref. [4] the potential (3.14) diverges at the brane position and the final expression for the mass gap becomes $\frac{25p^2M^2}{4}$ which explicitly depends on $p$ and $M$. As we have
mentioned in the previous section, both $p$ and $M$ are not physical parameters. This can be seen, for instance, from the fact that they do not appear in the metric after defining the new coordinate $Y$ as Eq. (2.10), which is the proper coordinate in the conformal frame. The physical coordinate $dZ = A(Y)dY$ also does not depend on $p$ and $M$, since $A(Y)$ does not depend on them. In the $Y$ coordinate system (hence also in the $Z$ coordinate system), except for the term of the delta function representing the contribution from the brane, the potential for the bulk eigen equation is smooth at the brane.

In the case with a cosmological constant, the metric including the tensor perturbations is given by Eq. (3.8) with Eq. (2.17). As in the previous case, decomposing into the Fourier modes we find that each mode satisfies Eq. (3.10) with Eq. (2.15). By the redefinition of $f_m = H^2 A^{-\frac{D-2}{2}} X_m$, the bulk mode function becomes

$$\left[ -\frac{d^2}{dY^2} + V(Y) - \sqrt{D-2} \sqrt{3H^2 + \frac{2|\Lambda_D|}{D-1} \delta(Y-Y_0)} \right] X_m = \left( m^2 - \frac{3H^2 L(L+D-6)}{D-6} \right) X_m. \quad (3.17)$$

The bulk potential $V(Y)$ is greater than $\frac{3(D-2)H^2}{4}$, monotonically decreases as increasing $Y$ and asymptotically approaches this value. The behaviors of the modes are the same as those in the previous subsection, which are summarized in Fig. 1. All discrete modes of $m^2 = \frac{3L(L+D-6)}{D-6} H^2$ are normalizable and satisfy the boundary conditions. But for any $D$, all the massive bound states are heavier than the critical mass of de Sitter $m_c$. On the other hand, for $L = 0$, the mass gap between the zero mode the continuum of KK modes associated with the non-compact $y$-direction is given by $\sqrt{\frac{3(D-2)}{2}} H$. Thus the mass gap does not depend on the parameters of $M$ and $p$. As the number of extra dimensions increases, the mass gap also increases.

With the normalizable zero mode solution $f_0 = C_1$, the effective four-dimensional gravitational mass is given by $M^2_{4,\text{eff}} = \frac{1}{\kappa_D C_1^2}$. The normalization constant for $D = 7$ is given by

$$C_1^{-2} = \frac{\pi}{H^2|\Lambda_7|^\frac{1}{2}} \sqrt{\frac{5}{3}} \left\{ \frac{243H^4}{2} \ln \left( \frac{\sqrt{|\Lambda_7|} + \sqrt{9H^2 + |\Lambda_7|}}{-\sqrt{|\Lambda_7|} + \sqrt{9H^2 + |\Lambda_7|}} \right) - (27H^2 - 2|\Lambda_7|) \sqrt{9H^2 + |\Lambda_7|} \right\}. \quad (3.18)$$

For $H \gg |\Lambda_7|^{\frac{1}{2}}$, the normalization constant becomes $C_1^2 \approx \frac{3\sqrt{15}H^3}{8\pi}$, while we find $C_1^2 \approx \frac{H^2}{2\pi} \sqrt{\frac{3|\Lambda_7|}{5}}$, for $H \ll |\Lambda_7|^{\frac{1}{2}}$. In the first case, we recovered the result in the
previous subsection. Similarly for $D = 11$, the normalization constant is given by

$$C_{11}^{-2} = \frac{25\sqrt{5}\pi^3}{216H^6|\Lambda_{11}|^2} \times \left\{ 2|\Lambda_{11}|^{\frac{3}{2}}\sqrt{15H^2 + |\Lambda_{11}|} \left( -118125H^6 + 5250H^4|\Lambda_{11}| - 280H^2|\Lambda_{11}|^2 + 16|\Lambda_{11}|^3 \right) + 1771875H^8\ln\left( \sqrt{15H^2 + |\Lambda_{11}|} + \sqrt{|\Lambda_{11}|} \right) \right\}. \quad (3.19)$$

For $H \gg |\Lambda_{11}|^{\frac{1}{2}}$, Eq. (3.19) gives $C_{11}^2 \simeq \frac{243\sqrt{3}H^7}{800\pi^3}$, while we obtain $C_{11}^2 \simeq \frac{27\sqrt{|\Lambda_{11}|H^6}}{100\sqrt{5}\pi^3}$ for $H \ll |\Lambda_{11}|^{\frac{1}{2}}$. The first limit coincides with Eq. (3.14) for $D = 11$. In the $H \to 0$ limit, the zero mode amplitude vanishes. Similar properties are obtained for any value of $D$. The four-dimensional gravitational mass is still ill-defined for $H \to 0$.

(2) In the case of the scalar-tensor theory

We then consider perturbations about the solution in the scalar-tensor theory with the negative potential Eq. (2.21). The bulk mode obeys the eigen equation

$$\left[ -\frac{d^2}{dY^2} + \left( \frac{3(2+c(D-2))H^2}{4c} - H\sqrt{\frac{3(2+c(D-2))}{c}}\delta(Y-Y_0) \right) X_m \right. = \left. \left( m^2 - \frac{3H^2L(L+D-6)}{D-6} \right) X_m. \right] \quad (3.20)$$

The solution for the zero mode is given by

$$X_0 \propto e^{-\frac{1}{2}H\sqrt{\frac{3(2+c(D-2))}{c}}|Y-Y_0|}. \quad (3.21)$$

Other than the massless state, there are massive bound states of $m^2 = \frac{3H^2L(L+D-6)}{D-6}$ which behave as Eq. (3.21). The effective four-dimensional gravitational mass is given by

$$M_{4,\text{eff}}^2 = \frac{1}{\kappa_D^2 C_1^2} = \frac{2\Omega_{D-5}}{\kappa_D^2 H^{D-4}} \left( \frac{D-6}{3} \right)^{\frac{D-5}{2}} \sqrt{\frac{c}{3((D-2)c + 2)}}. \quad (3.22)$$

The mass gap between the zero mode and KK modes associated with the warped dimensions is given by $\Delta m^2 = \frac{3(2+c(D-2))H^2}{4c}$, which is bigger than the gap in the case without a scalar field, $\frac{3(D-2)}{4}H^2$. Thus in the case of the scalar-tensor theory, it becomes more difficult to excite the KK modes, in particular for the limit $c \to 0$.

(3) In the case with a form field strength

Finally, we consider perturbations about the solution in the theory including scalar and gauge fields Eq. (2.25). The metric including perturbations is given by Eq. (3.8) with

$$\omega_{ab}dz^a dz^b = \frac{D-6}{3H^2 + \frac{f^2}{2}} d\Omega_{(D-5)}^2. \quad (3.23)$$
The bulk mode obeys the eigen equation

\[
\left[ -\frac{d^2}{dY^2} + \frac{3(D-2)H^2}{4} \left( 1 + \frac{D-6}{(D-2)\zeta} \right) \right. \\
- H \sqrt{3(D-2)} \left( 1 + \frac{D-6}{(D-2)\zeta} \right) \delta(Y - Y_0) \left. \right] X_m \\
= \left( m^2 - \frac{3H^2(1 + \frac{f^2}{6H^2})L(L + D - 6)}{D - 6} \right) X_m.
\] (3.24)

The normalizable solution for the zero mode is given by

\[X_0 \propto e^{-\frac{1}{2}H\sqrt{3(D-2 + \frac{D-6}{\zeta})}|Y - Y_0|}. \] (3.25)

Besides the zero mode, there are massive bound states with \(m^2 = \frac{3H^2L(L + D - 6)}{D - 6} \) which satisfy Eq. (3.25). The effective four-dimensional gravitational mass is given by

\[M_{4,\text{eff}}^2 = \frac{1}{\kappa_D^2 C_1^2} = \frac{2\Omega_{D-5}}{\kappa_D^2 H^{D-3}} \frac{D - 6}{3(1 + \frac{f^2}{6H^2})} \frac{D - 5}{\sqrt{3(D - 2 + \frac{D-6}{\zeta})}}. \] (3.26)

There are two \(H \to 0\) limits: One is to set \(\zeta = 0\). The other is to choose \(f^2 = \frac{4\Lambda_f}{D-6}\). In the latter case, the effective gravitational coupling diverges. In the former case, the effective gravitational mass in the limit of \(H \to 0\) is well-defined with

\[M_{4,\text{eff}}^2 = \frac{2\Omega_{D-5}}{\kappa_D^2} \left( \frac{2(D - 6)}{f^2} \right)^{D-5} \frac{1}{\sqrt{-2\Lambda_f + \frac{f^2}{2}(D - 6)}}. \] (3.27)

The mass gap between the zero mode and KK modes associated with the warped dimensions is given by

\[\Delta m^2 = \frac{3(D - 2)H^2}{4} \left( 1 + \frac{D-6}{\zeta(D-2)} \right) \]
\[= \frac{1}{4} \left( - \frac{2\Lambda_f}{D - 6} + \frac{f^2}{2} \right) (\zeta(D - 2) + D - 6). \] (3.28)

In the limit of \(\zeta \to 0\), the mass gap is still finite as

\[\Delta m^2 = \frac{3(D - 2)H^2}{4} \left( 1 + \frac{D - 6}{\zeta(D - 2)} \right) = \frac{1}{4} \left( - 2\Lambda_f + \frac{f^2}{2}(D - 6) \right). \] (3.29)

In the case of \(f^2 = \frac{4\Lambda_f}{D-6}\), there is no mass gap. For a finite \(H\), the mass gap is always greater than \(\frac{3(D-2)H^2}{4}\) in the case of the pure gravity.
4. Scalar metric perturbations and stability

Finally, we investigate stability against the scalar perturbations. Here the term “scalar” is with respect to the four-dimensional de Sitter spacetime. It has been argued that the de Sitter solutions become unstable against the scalar metric perturbations, for example, in the product spacetime of $dS_4 \times S^n$ with flux [9, 10] and in the five-dimensional model with two de Sitter branes [11, 12]. Thus, for these new de Sitter solutions [4, 5], it is important to investigate the stability.

The most important purpose of this section is to see whether the lowest mode becomes tachyonic. But then, it is also important to investigate how large the negative mass square of a tachyonic mode is, even if it exists. For an inflationary model, the existence of a tachyonic mode whose absolute value of the negative mass square is much smaller than the Hubble parameter $H$ may be required to terminate inflation successfully with an appropriate e-folding number of cosmic expansion. Otherwise, inflation is terminated within a few Hubble time scales and the model becomes unrealistic. Thus, we will focus on the value of the lowest mode and also how its value is affected by the existence of the matter fields in the higher-dimensional theory.

Taking a longitudinal-type gauge, the scalar perturbed metric is given by

$$\begin{align*}
\text{d}s^2 &= A(Y)^2 \left[ (1 + 2\phi_1)\gamma_{\mu\nu}dx^\mu dx^\nu + (1 + 2\phi_2)dy^2 + (1 + 2\phi_3)\omega_{ab}dz^a dz^b \right],
\end{align*}$$

(4.1)

where $\gamma_{\mu\nu}$ is the de Sitter metric and $\omega_{ab}$ is that for an $(D-5)$-sphere of the given radius which is dependent on the model. $\phi_i$ ($i = 1, 2, 3$) are the moduli of each direction. The components of the Christoffel symbol and the curvature tensors obtained from the metric Eq. (4.1) are summarized in the Appendix A. We derive the equation which determines the stability, by combining the Einstein equations

$$\delta G^A_B = \kappa_D^2 \delta T^A_B,$$

(4.2)

where $\delta T^A_B$ is perturbation of the energy-momentum tensor of the bulk matter.

4.1 In the case of the pure gravity

Firstly, we consider the pure gravity solution Eq. (2.7). Assuming the vacuum bulk and combining the Einstein equations, we find the equation

$$\begin{align*}
(\Box_4 + 6H^2)(\phi_2 - \phi_3) - \sqrt{3(D-2)}H(\phi_2 - \phi_3)' + (\phi_2 - \phi_3)'' &= 0.
\end{align*}$$

(4.3)

The detailed derivation of this relation is outlined in the Appendix B. 1. This equation is the same as that for the tensor perturbations, except for the replacement of $\Box_4$ with $(\Box_4 + 6H^2)$. Decomposing into the effective four-dimensional modes, $\phi_2 - \phi_3 = \int dm \psi_m(x^\mu)g_m(Y)$, the bulk eigen equation is the same as that in the case of the tensor perturbations Eq. (3.10). Thus the zero mode of $m = 0$ gives $g_0(Y) = \text{const}$ and the effective four-dimensional equation

$$\begin{align*}
(\Box_4 + 6H^2)\psi_0 &= 0,
\end{align*}$$

(4.4)
and hence the lowest mode becomes tachyonic. There is also the mass gap between the zero mode and higher modes \( \Delta m^2 = \frac{3(D-2)}{4} H^2 \) and the lowest KK mass is given by \( m^2 = \frac{3(D-10)}{4} H^2 \). For \( D \geq 10 \), all the KK modes are not tachyonic. The absolute value of the tachyonic mass is too large for the model [4] to be a realistic model.

We should also mention the normalizability of the scalar perturbations. The normalization condition for the scalar perturbations essentially remains the same as that for the tensor modes: \( \int_0^\infty dY A^{D-2} g_0(Y)^2 < \infty \) for a discrete mode, and also \( \int_0^\infty dY A^{D-2} g_m(Y) g_{m'}(Y) \propto \delta(m - m') \) for continuous modes \( m \neq 0 \) and \( m' \neq 0 \). The boundary condition for \( g_m(Y) \) on the brane which is obtained by integrating over it is also the same as that for the tensor perturbations. Hence the normalizable solution for the tachyonic zero mode in the scalar perturbations is the same as that for the zero mode in the tensor perturbations, and hence the corresponding mode is also normalizable and physical.

### 4.2 In the case of the scalar-tensor theory

We then discuss the stability of the solution in the scalar-tensor theory Eq. (2.21), outlined in the Appendix B. 2. We obtain the equation

\[
\left( \Box + 6H^2 \right) (\phi_2 - \phi_3) - \sqrt{3(D-2)} \left( 1 + \frac{2}{c(D-2)} \right) H (\phi_2 - \phi_3)' + (\phi_2 - \phi_3)'' = 0. \tag{4.5}
\]

The differential operator has the same structure as that in the pure gravity model. Decomposing into the effective four-dimensional modes, the lowest mode becomes tachyonic with the same mass square \(-6H^2\). Thus, this model is unstable against the scalar perturbations. Since the normalization and boundary conditions for a bulk mode remain the same as those in the pure gravity model, the tachyonic zero mode is also normalizable. The lowest mass for the KK continuum is given by \( \frac{3(2 + (D-10)c)}{4c} H^2 \). The absolute value of the tachyonic mass is too large for the model in the scalar-tensor theory [5] to be a realistic model.

### 4.3 In the case with a form field strength

We finally consider the solution with a form field strength, outlined in the Appendix B. 3. For simplicity, we have ignored the perturbations of the form field strength. The solution Eq. (2.25) leads to the equation

\[
\left( \Box + 6H^2 \right) (\phi_2 - \phi_3) - \sqrt{3(D-2)} \left( 1 + \frac{D-6}{\zeta(D-2)} \right) H (\phi_2 - \phi_3)' + (\phi_2 - \phi_3)'' = f^2 \left( 2\phi_3 + \alpha \delta \phi \right). \tag{4.6}
\]
Although the differential operator in the left-hand side is very similar to the previous cases, the perturbation equation becomes inhomogeneous due to the existence of the source term. Such a source term is induced for the nonvanishing flux. In general the solution can be written in terms of the linear combination of the particular and homogeneous solutions. From the homogeneous part, the lowest mode becomes tachyonic with the same mass square $-6H^2$. Inclusion of the perturbations of the form field modifies the form of the source term, but does not affect the homogeneous, hence geometrical part of the equation Eq. (B.41). Thus, this model is also unstable against the scalar perturbations. Since the normalization and boundary conditions for a bulk mode remain the same as those in the previous two models, the tachyonic zero mode is normalizable. The lowest mass of the KK continuum is given by

$$
\frac{3}{4}H^2 \left( D - 10 + \frac{D - 6}{\zeta} \right).
$$

(4.7)

The absolute value of the tachyonic mass is too large for the model in the gravitational theory coupled to the form field [5] to be a realistic model.

### 4.4 Interpretation of the tachyonic mode

Although we find that adding the matter degrees of freedom does not contribute to alleviate the instability, our result suggests that the value of the tachyonic mass in our model relies on the geometrical (warped) structure of the spacetime. If we can construct a more general class of solutions of the warped de Sitter compactification where the spacetime structure and the warp factor can be determined by several different mass scales, it could provide a sufficiently large parameter space where a suppressed value of the tachyonic mass is obtained, even if a tachyonic mode exists. Having such a class of solutions, we could realize inflation with a sufficient e-folding number.

It should be mentioned that similar situations have been analyzed in the past works. Let us review these models and then clarify the difference of our model from them. In Refs. [13, 14], Garriga and Vilenkin analyzed the fluctuations of a thin domain wall in the $(N + 1)$-dimensional Minkowski spacetime. They showed that the wall fluctuation mode is represented by a scalar field living on the $N$-dimensional de Sitter space which describes the internal metric on the domain wall, and the scalar field has the negative mass squared ($-NH^2$). It turned out, however, that the wall fluctuation cannot be seen by an observer living on the wall because the fluctuation does not change the intrinsic curvature of the domain wall. In addition, in Refs. [11, 12] it was shown that in the system of a single de Sitter brane in the five-dimensional spacetime the fluctuations of the brane position cannot be seen by an observer on the brane, while in the system of two de Sitter branes the fluctuation of the relative displacement of branes induces physically non-intrinsic effects on the
brane, since this is purely geometrical effect which can exist without any source on the branes. In other words, only the relative displacement of branes can be physical.

In contrast to the case of a domain wall and that of a single de Sitter brane in five dimensions, as we have mentioned in the previous subsections, in our model the scalar mode is normalizable and physical even in the single brane system without any source on the brane. Note that the important difference from the single-brane model in five dimensions is the existence of the internal \((D-5)\)-sphere. The scalar mode \(\phi_{2} - \phi_{3}\) is clearly interpreted as the fluctuation of the size of the noncompact direction \(Y\) relative to the \((D-5)\)-sphere, and this is in a situation similar to that of the relative displacement of two branes in five dimensions. Thus an observer on the brane would observe non-intrinsic effects from extra dimensions, which must be revealed in the future studies.

Before closing this section, we should also comment on the moduli instability in the lower-dimensional effective theory. In Ref. [5], we have analyzed the lower-dimensional effective theory obtained via integrating over the spherical internal directions. In the effective theory two moduli fields appear, which are associated with the size of the sphere and the overall rescaling of the warp factor, respectively. It was found that the first modulus can be stabilized by the contribution of the field strength. On the other hand, the second modulus associated with the warp factor cannot be stabilized by the classical ingredients in the original theory. We expect that the instability of the scalar metric perturbations found in this paper corresponds to the modulus instability in the lower-dimensional effective theory. Thus the stabilization should be achieved by some other mechanism. A possible mechanism for the stabilization is via the quantum corrections of the bulk matter fields, which is under active study [15].

5. Conclusion

In this paper, we have investigated the spectrum of the tensor metric perturbations from the warped solutions with the four-dimensional de Sitter spacetime in the pure gravity, in the scalar-tensor theory and also in the theory with a form field strength. The solutions have more than seven spacetime dimensions. In these solutions, the metric is given by the warped product of the non-compact extra dimension, spherical extra dimensions and the four-dimensional de Sitter spacetime. To make the volume of extra dimensions finite, we construct a braneworld using the so-called cut-copy-paste method. The difference from the five-dimensional case is that the braneworld involves the spherical dimensions as well as the de Sitter spacetime. Thus we compactify the spherical dimensions to obtain the four-dimensional cosmology. The junction condition gives the positive brane tension.

In all these models, the tensor spectrum contains the massless zero mode. In general, in the low energy limit the amplitude of the zero mode vanishes, which
leads to the divergence of the four-dimensional gravitational mass. Adding a bulk cosmological constant does not provide a resolution to this problem. It implies that in such models one cannot smoothly connect the de Sitter inflation phase to the Friedmann universe. Only the exceptional case is the case with a form field strength, where the de Sitter expansion rate can be zero for a particular value of the coupling parameter to the field strength.

In the known five-dimensional case, there is the mass gap between the zero and the continuum of KK modes, which is equal to the critical mass of de Sitter spacetime. The mass gap between the zero mode and the continuum of the KK modes associated with the warped dimension is greater than the critical mass of the de Sitter spacetime. In addition, there exist the massive bound states associated with the excitations along the spherical dimensions. These modes are heavier than the critical mass in de Sitter spacetime. Therefore, although the braneworld involves the internal angular dimensions, no light KK mode is excited.

We then have argued the stability of the solutions against the scalar perturbations with respect to the four-dimensional symmetry. We have shown the existence of the tachyonic zero mode in all models. The differential operator for the scalar perturbations is the same as that for the tensor perturbations, except that the four-dimensional operator is shifted by the tachyonic mass. Irrespective of the presence of matter fields, the mass of the lowest mode always takes $-6H^2$ which seems to be universal and irrespective of the number of dimensions.

The absolute value of the tachyonic mass is too large for the de Sitter compactifications discussed in Refs. [4, 5] to be realistic inflationary models. But if we can construct a more general class of solutions of the warped de Sitter compactification where the spacetime structure and the warp factor depend on several different mass scales, we expect that it could give a sufficiently large parameter space where a suppressed value of the tachyonic mass suitable for inflation with a sufficient e-folding number is realized. This subject is worth being investigated. Finally, we also mention that the existence of the unstable mode is consistent with the analysis of the lower-dimensional effective theory. In the effective potential obtained after integrating over the spherical internal spaces, the warp factor $A$ cannot be fixed, which was originally found in Ref. [5]. We expect that the instability of the scalar metric perturbations corresponds to such a modulus instability in the effective theory. The stabilization via the quantum effects of the bulk fields is currently studied in [15]. We hope to report these results in the future publication.

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A. The components of the Christoffel symbol and the curvature tensors

In this Appendix, we present the components of the Christoffel symbol and the curvature tensors obtained from the metric Eq. (4.1) which include the linear order scalar metric perturbations.

The components of the Christoffel symbol obtained from the perturbed metric Eq. (4.1) are given by

\[
\Gamma^Y_{\mu\nu} = -\frac{A'}{A} \gamma_{\mu\nu} - \frac{A'}{A} (2\phi_2 + 2\phi_1) \gamma_{\mu\nu} - \phi_1' \gamma_{\mu\nu}, \quad (A.1)
\]

\[
\Gamma^Y_{\nu\mu} = \phi_{2,\mu}, \quad (A.2)
\]

\[
\Gamma^Y_{\nu\nu} = \frac{A'}{A} + \phi_2', \quad (A.3)
\]

\[
\Gamma^Y_{ab} = -\left(\frac{A'}{A} + \frac{A'}{A} (2\phi_3 - 2\phi_2) + \phi_3'\right) \omega_{ab}, \quad (A.4)
\]

\[
\Gamma^Y_{\mu a} = \phi_{2,a}, \quad (A.5)
\]

\[
\Gamma^Y_{\mu Y} = -\gamma^{\mu\nu} \phi_{2,\nu}, \quad (A.6)
\]

\[
\Gamma^Y_{\nu Y} = -\phi_{2,\nu} \omega_{ab}, \quad (A.7)
\]

\[
\Gamma^Y_{\mu Y} = \delta^\mu_\nu \left(\frac{A'}{A} + \phi_1'\right), \quad (A.8)
\]

\[
\Gamma^Y_{Y b} = \delta^a_b \left(\frac{A'}{A} + \phi_3'\right), \quad (A.9)
\]

\[
\Gamma^\mu_{\alpha\beta} = \tilde{\Gamma}^\mu_{\alpha\beta} + \left(\phi_{1,\beta} \delta^\mu_\alpha + \phi_{1,\alpha} \delta^\mu_\beta - \phi_{1,\nu} \gamma^{\mu\nu} \gamma_{\alpha\beta}\right), \quad (A.10)
\]

\[
\Gamma^a_{bc} = \tilde{\Gamma}^a_{bc} + \left(\phi_{1,b} \delta^a_c + \phi_{1,c} \delta^a_b - \phi_{1,d} \omega^{ad} \omega_{bc}\right), \quad (A.11)
\]

\[
\Gamma^b_{\mu\nu} = -\gamma_{\mu\nu} \phi_{1,a} \omega_{ab}, \quad (A.12)
\]

\[
\Gamma^a_{\alpha\nu} = -\omega_{ab} \phi_{3,\beta} \gamma^{\beta\alpha}, \quad (A.13)
\]

\[
\Gamma^\mu_{a\nu} = \delta^\mu_\nu \phi_{1,\alpha}, \quad (A.14)
\]

\[
\Gamma^a_{\mu b} = \delta^a_b \phi_{3,\mu}, \quad (A.15)
\]

where \(\tilde{\Gamma}^\mu_{\alpha\beta}\) and \(\tilde{\Gamma}^a_{bc}\) are background Christoffel symbols computed from \(\gamma_{\mu\nu}\) and \(\omega_{ab}\), respectively.
The components of the Riemann tensor are given by

\[
R^Y_{\mu Y \nu} = - \left( \frac{A'}{A} \right) ' \gamma_{\mu \nu} - 2 \left( \frac{A'}{A} \right) ' (\phi_2 + \phi_1) \gamma_{\mu \nu} - \frac{A'}{A} (\phi_2' + \phi_1') \gamma_{\mu \nu} \\
- \phi_1'' \gamma_{\mu \nu} - D_\mu D_\nu \phi_2, \quad \tag{A.16}
\]

\[
R^Y_{a Y b} = - \left( \frac{A'}{A} \right) ' \omega_{ab} - 2 \left( \frac{A'}{A} \right) ' (\phi_2 + \phi_3) \omega_{ab} - \frac{A'}{A} (\phi_2' + \phi_3') \omega_{ab} \\
- \phi_3'' \omega_{ab} - D_a D_b \phi_2, \quad \tag{A.17}
\]

\[
R^\mu_{Y Y \alpha} = \phi_{1, \nu} \delta^\mu_\alpha - \phi_{1, \rho} \gamma^{\rho \mu} \gamma^\nu_\alpha + \frac{A'}{A} \phi_{2, \rho} \gamma^{\mu \rho} \gamma^\nu_\nu - \delta^\mu_\alpha \frac{A'}{A} \phi_{2, \nu}, \tag{A.20}
\]

\[
R^a_{b Y c} = \phi_{3, b} \delta^a_c - \phi_{3, d} \gamma^{da} \omega_{bc} + \frac{A'}{A} \phi_{2, d} \omega^{da} \omega_{bc} - \delta^a_c \frac{A'}{A} \phi_{2, b}, \tag{A.21}
\]
\[ R^\alpha_{\beta\mu\nu} = \tilde{R}^\alpha_{\beta\mu\nu} + (\delta^\alpha_{\mu} \gamma_{\beta\nu} - \delta^\alpha_{\nu} \gamma_{\beta\mu}) \left(\frac{A'}{A}\right)^2 \]
\[ + D_\mu D_\beta \phi_1 \delta^\alpha_{\nu} - D_\mu D_\beta \phi_1 \gamma_{\nu\beta} - D_\nu D_\beta \phi_1 \delta^\alpha_{\mu} + D_\nu D_\beta \phi_1 \gamma_{\mu\beta} \]
\[ - (\delta^\alpha_{\mu} \gamma_{\beta\nu} - \delta^\alpha_{\nu} \gamma_{\beta\mu}) \left[ \left(\frac{A'}{A}\right)^2 (-2\phi_2 + 2\phi_1) + 2\phi_1' A' A \right], \quad (A.22) \]
\[ R^a_{\beta c d} = \tilde{R}^a_{\beta c d} + (\delta^a_{\alpha} \omega_{b\delta} - \delta^a_{\beta} \omega_{\delta c}) \left(\frac{A'}{A}\right)^2 \]
\[ + D_c D_b \phi_3 \delta^a_{\alpha d} - D_c D_b \phi_3 \omega_{b\delta d} - D_d D_b \phi_3 \delta^a_{\beta c} + D_d D_b \phi_3 \omega_{\delta c} \]
\[ - (\delta^a_{\alpha} \omega_{b\delta d} - \delta^a_{\beta} \omega_{\delta c}) \left[ \left(\frac{A'}{A}\right)^2 (-2\phi_2 + 2\phi_3) + 2\phi_3' A' A \right], \quad (A.23) \]
\[ R^Y_{\mu Y \alpha} = R^Y_{a Y \mu} = - \phi_{2, \alpha\mu}, \quad (A.24) \]
\[ R^\alpha_{\mu b c} = - \left(\frac{A'}{A}\right)^2 \delta^\alpha_{b \gamma_{\mu\nu}} \]
\[ - 2 \left(\frac{A'}{A}\right)^2 (\phi_1 - \phi_2) \delta^\alpha_{b \gamma_{\mu\nu}} - \frac{A'}{A} (\phi_1' + \phi_3') \delta^\alpha_{b \gamma_{\mu\nu}} - \gamma_{\mu\nu} D^a D_b \phi_1 \]
\[ - \delta^\alpha_{b} D_{\mu} D_{\nu} \phi_3, \quad (A.25) \]
\[ R^\mu_{\alpha b} = - \left(\frac{A'}{A}\right)^2 \delta^\mu_{\nu} \omega_{ab} \]
\[ - 2 \left(\frac{A'}{A}\right)^2 (\phi_3 - \phi_2) \delta^\mu_{\nu} \omega_{ab} - \frac{A'}{A} (\phi_1' + \phi_3') \delta^\mu_{\nu} \omega_{ab} - \omega_{ab} D^\mu D_\nu \phi_3 \]
\[ - \delta^\mu_{\nu} D_{\alpha} D_{b} \phi_1, \quad (A.26) \]
\[ R^\alpha_{\mu \beta \alpha} = - \phi_{1, \mu\beta} \delta^\alpha_{\alpha} + \phi_{1, \nu \alpha} \gamma^\nu_{\alpha} \gamma_{\mu\beta} \quad (A.27) \]
\[ R^a_{\alpha b \mu} = - \phi_{3, c \mu} \delta^a_{b \gamma_{\nu c}} \quad (A.28) \]
\[ R^a_{\alpha \beta \mu} = \delta^a_{\mu} \phi_{1, a \beta} - \delta^a_{\beta} \phi_{1, a \mu} \quad (A.29) \]
\[ R^a_{\mu b c} = \delta^a_{c} \phi_{3, \mu b} - \delta^a_{b} \phi_{3, \mu c} \quad (A.30) \]
\[ R^a_{Y b \mu} = - \delta^a_{b} \phi_{3, ' \mu} + \delta^a_{b} \phi_{2, \mu} \frac{A'}{A} \quad (A.31) \]
\[ R^a_{Y \beta a} = - \delta^a_{b} \phi_{3, ' a} + \delta^a_{b} \phi_{2, a} A' A \quad (A.32) \]
\[ R^a_{Y \beta \gamma} = \delta^a_{c} \phi_{3, ' b} - \delta^a_{b} \phi_{3, c} \]
\[ + \delta^a_{b} A' A \phi_{2, c} - \delta^a_{c} A' A \phi_{2, b}, \quad (A.33) \]
\[ R^a_{Y b \gamma} = \delta^a_{c} \phi_{3, ' b} - \delta^a_{b} \phi_{3, c} \]
\[ + \delta^a_{b} A' A \phi_{2, c} - \delta^a_{c} A' A \phi_{2, b} \quad (A.34) \]

where \( \tilde{R}^a_{\beta\mu\nu} \) and \( \tilde{R}^a_{\beta c d} \) are the background Riemann tensors with respect to \( \gamma_{\mu\nu} \) and \( \omega_{ab} \), respectively.

The components of the background Ricci tensor are given by
\[ R_{Y Y}^{(0)} = -(D - 1) \left(\frac{A'}{A}\right)' \quad (A.35) \]
\[ R_{\mu\nu}^{(0)} = - \left(\frac{A'}{A}\right)' \gamma_{\mu\nu} - (D - 2) \left(\frac{A'}{A}\right)^2 \gamma_{\mu\nu} + \tilde{R}_{\mu\nu}, \quad (A.36) \]
\[ R_{ab}^{(0)} = - \left(\frac{A'}{A}\right)' \omega_{ab} - (D - 2) \left(\frac{A'}{A}\right)^2 \omega_{ab} + \tilde{R}_{ab}, \quad (A.37) \]
where $\tilde{R}_{\mu\nu}$ and $\tilde{R}_{ab}$ are the background Ricci tensors with respect to $\gamma_{\mu\nu}$ and $\omega_{ab}$, respectively. Those of the perturbed Ricci tensor are given by

$$
\delta R_{YY} = -\left(\Box + \Delta_{D-5}\right)\phi_2 - 4\phi_1'' - (D - 5)\phi_3'' + (D - 1)\frac{A'}{A}\phi_2' - 4\frac{A'}{A}\phi_1',
$$

$$
\delta R_{\mu\nu} = -2D\mu D\nu\phi_1 - \gamma_{\mu\nu} \left(\Box + \Delta_{D-5}\right)\phi_1 - (D - 5)D\mu D\nu\phi_3 - D\mu D\nu\phi_2
$$

$$
+ 2(D - 2)\left(\frac{A'}{A}\right)'(\phi_1 - \phi_2)\gamma_{\mu\nu} - (D + 2)\frac{A'}{A}\phi_1'\gamma_{\mu\nu} - (D - 5)\frac{A'}{A}\phi_3'\gamma_{\mu\nu}
$$

$$
+ \frac{A'}{A}\phi_2'\gamma_{\mu\nu} - \phi_1''\gamma_{\mu\nu} - 2\left(\frac{A'}{A}' - \phi_2 + \phi_1\right)\gamma_{\mu\nu},
$$

$$
\delta R_{ab} = -2D_a D_b\phi_3 - \omega_{ab} \left(\Box + \Delta_{D-5}\right)\phi_3 - 4D_a D_b\phi_1 - D_a D_b\phi_2
$$

$$
- 2(D - 2)\left(\frac{A'}{A}\right)'(\phi_3 - \phi_2)\omega_{ab} - (2D - 7)\frac{A'}{A}\phi_3'\omega_{ab} - 4\frac{A'}{A}\phi_1'\omega_{ab}
$$

$$
+ \frac{A'}{A}\phi_2'\omega_{ab} - \phi_3''\omega_{ab} - 2\left(\frac{A'}{A}' - \phi_2 + \phi_3\right)\omega_{ab},
$$

$$
\delta R_{\mu\alpha} = -\left(\phi_2 + 3\phi_1 + (D - 6)\phi_3\right)\gamma_{\mu\alpha},
$$

$$
\delta R_{Y\mu} = -3\phi_1' + (D - 2)\frac{A'}{A}\phi_2, -(D - 5)\phi_3',
$$

$$
\delta R_{Ya} = -(D - 6)\phi_3, + (D - 2)\frac{A'}{A}\phi_2, - 4\phi_1'.
$$

### B. The derivation of the equations of motion for the scalar perturbations

In this Appendix, we briefly summarize the equations of motion for the scalar perturbations. In what follows, we use the results shown in the Appendix A.

#### B.1 In the case of the pure gravity

In the case of the pure gravity model, assuming the metric form Eq. (4.1), it is straightforward to confirm that

$$
A = e^{-\sqrt{D-2}H|y-y_0|}, \quad \omega_{ab} dz^a dz^b = \frac{D - 6}{3H^2}d\Omega_{D-5}^2,
$$

$$
\gamma_{\mu\nu} dx^\mu dx^\nu = -dt^2 + e_0^2 e^{2Ht} \delta_{ij} dx^i dx^j,
$$

is the solution to the background equations $R_{AB}^{(0)} = 0$, where we have employed $\tilde{R}_{\mu\nu} = 3H^2\gamma_{\mu\nu}$ and $\tilde{R}_{ab} = 3H^2\omega_{ab}$.

Taking the property of the background equation $\frac{A'}{A} = -\sqrt{\frac{D-2}{D-3}}H$ and $(\frac{A'}{A})' = 0$, the components of the perturbed Einstein tensor associated with the metric (4.1) are
given by

\[ A^2 \delta G^\mu = -D^\mu D_\nu (2\phi_1 + (D - 5)\phi_3 + \phi_2) + \left[ \Box_4 (2\phi_1 + (D - 5)\phi_3 + \phi_2) + \Delta_{D-5} (3\phi_1 + \phi_2 + (D - 6)\phi_3) + 6H^2 (\phi_1 - \phi_2) + 3(D - 5)H^2 (\phi_3 - \phi_2) - \sqrt{3(D - 2)}H (3\phi_1' + (D - 5)\phi_3' - \phi_2') + 3\phi_1'' + (D - 5)\phi_3'' \right] \delta^\mu_\nu (B.2) \]

\[ A^2 \delta G^a_b = -D^a D_b (4\phi_1 + (D - 7)\phi_3 + \phi_2) + \left[ \Delta_{D-5} (4\phi_1 + (D - 7)\phi_3 + \phi_2) + \Box_4 (3\phi_1 + \phi_2 + (D - 6)\phi_3) + 12H^2 (\phi_1 - \phi_2) + 3(D - 7)H^2 (\phi_3 - \phi_2) + \sqrt{3(D - 2)}H (4\phi_1' + (D - 6)\phi_3' - \phi_2') + 3\phi_1'' + (D - 6)\phi_3'' \right] \delta^a_b (B.3) \]

\[ A^2 \delta G^\nu = \Box_4 (3\phi_1 + (D - 5)\phi_3) + \Delta_{D-5} (4\phi_1 + (D - 6)\phi_3) - \sqrt{3(D - 2)}H (4\phi_1' + (D - 5)\phi_3') + 3\phi_1'' + (D - 1)\phi_2', \]

\[ \delta G^\mu_a = -\left( \Box_4 (3\phi_1 + (D - 6)\phi_3) - \Delta_{D-5} (4\phi_1 + (D - 6)\phi_3) \right) \delta^\mu_a, \]

\[ \delta G^\nu_a = -\left( \Box_4 (3\phi_1 + (D - 5)\phi_3) - \Delta_{D-5} (4\phi_1 + (D - 5)\phi_3) \right) \delta^\nu_a, \]

The components of the perturbed Einstein equations are given by

\[ 0 = -D^\mu D_\nu (2\phi_1 + (D - 5)\phi_3 + \phi_2) + \left[ \Box_4 (2\phi_1 + (D - 5)\phi_3 + \phi_2) + \Delta_{D-5} (3\phi_1 + \phi_2 + (D - 6)\phi_3) + 6H^2 (\phi_1 - \phi_2) + 3(D - 5)H^2 (\phi_3 - \phi_2) - \sqrt{3(D - 2)}H (3\phi_1' + (D - 5)\phi_3' - \phi_2') + 3\phi_1'' + (D - 5)\phi_3'' \right] \delta^\mu_\nu, \]

\[ 0 = -D^a D_b (4\phi_1 + (D - 7)\phi_3 + \phi_2) + \left[ \Delta_{D-5} (4\phi_1 + (D - 7)\phi_3 + \phi_2) + \Box_4 (3\phi_1 + \phi_2 + (D - 6)\phi_3) + 12H^2 (\phi_1 - \phi_2) + 3(D - 7)H^2 (\phi_3 - \phi_2) + \sqrt{3(D - 2)}H (4\phi_1' + (D - 6)\phi_3' - \phi_2') + 3\phi_1'' + (D - 6)\phi_3'' \right] \delta^a_b, \]

\[ 0 = \Box_4 (3\phi_1 + (D - 5)\phi_3) + \Delta_{D-5} (4\phi_1 + (D - 6)\phi_3) - \sqrt{3(D - 2)}H (4\phi_1' + (D - 5)\phi_3') + 3\phi_1'' + (D - 1)\phi_2', \]

\[ 0 = \Box_4 (3\phi_1 + (D - 5)\phi_3) + \Delta_{D-5} (4\phi_1 + (D - 6)\phi_3) - \sqrt{3(D - 2)}H (4\phi_1' + (D - 5)\phi_3') + 3\phi_1'' + (D - 1)\phi_2', \]
0 = -\left( \phi_2 + 3\phi_1 + (D - 6)\phi_3 \right)_{,\mu}^{,\alpha}, \quad (B.10)
0 = -3\phi_{1,\mu}' - \sqrt{3(D - 2)}H\phi_{2,\mu} - (D - 5)\phi_{3,\mu}', \quad (B.11)
0 = -(D - 6)\phi_{3,a}' - \sqrt{3(D - 2)}H\phi_{2,a} - 4\phi_{1,a}'. \quad (B.12)

Here, we ignore the massive excitations in the $S^{D-5}$ direction and set the terms with $\partial_a$ to be zero. Then, the perturbed Einstein equations reduce simply to

\begin{align*}
0 &= -D^\mu D_\nu(2\phi_1 + (D - 5)\phi_3 + \phi_2) + \left[ \Box_4(2\phi_1 + (D - 5)\phi_3 + \phi_2) \\
&+ \quad 6H^2(\phi_1 - \phi_2) + 3(D - 5)H^2(\phi_3 - \phi_2) \\
&- \quad H\sqrt{3(D - 2)}\left(3\phi_1' + (D - 5)\phi_3' - \phi_2'\right) + 3\phi_1'' + (D - 5)\phi_3'' \right] \delta^{\mu}_{\nu}, \quad (B.13)
0 &= \left[ \Box_4(3\phi_1 + \phi_2 + (D - 6)\phi_3) \\
&+ \quad 12H^2(\phi_1 - \phi_2) + 3(D - 7)H^2(\phi_3 - \phi_2) \\
&- \quad \sqrt{3(D - 2)}H\left(4\phi_1' + (D - 6)\phi_3' - \phi_2'\right) + 4\phi_1'' + (D - 6)\phi_3'' \right] \delta^{a}_{b}, \quad (B.14)
0 &= \Box_4(3\phi_1 + (D - 5)\phi_3) - H\sqrt{3(D - 2)}\left(4\phi_1' + (D - 5)\phi_3'\right) \\
&+ \quad 3H^2\left(4\phi_1 + (D - 5)\phi_3 - (D - 1)\phi_2\right), \quad (B.15)
0 &= -3\phi_{1,\mu}' - \sqrt{3(D - 2)}H\phi_{2,\mu} - (D - 5)\phi_{3,\mu}', \quad (B.16)
\end{align*}

where ($\mu, a$) and ($Y, a$) components become trivial.

We assume that there is no anisotropic stress in the four dimensions and $\delta T_{Y \mu} = 0$. Eq. (B.16) gives

\begin{equation}
3\phi_1' + (D - 5)\phi_3' + \sqrt{3(D - 2)}H\phi_2 = 0. \quad (B.17)
\end{equation}

We choose $\phi_1$ so that $\phi_1$ obeys the equation

\begin{equation}
2\phi_1 = -(D - 5)\phi_3 - \phi_2. \quad (B.18)
\end{equation}

Combined Eq. (B.17) with Eq. (B.18), we obtain

\begin{equation}
3\phi_2' + (D - 5)\phi_3' - 2\sqrt{3(D - 2)}H\phi_2 = 0. \quad (B.19)
\end{equation}

Taking the difference between the trace of Eq. (B.14) and Eq. (B.15), we get

\begin{align*}
0 &= \Box_4(\phi_2 - \phi_3) + 6H^2(\phi_2 - \phi_3) + \sqrt{3(D - 2)}H\left(\phi_2' + \phi_3'\right) + 4\phi_1'' + (D - 6)\phi_3'' \\
&= \Box_4(\phi_2 - \phi_3) + 6H^2(\phi_2 - \phi_3) + \sqrt{3(D - 2)}H\left(\phi_2' + \phi_3'\right) \\
&- \left(2\phi_2'' + (D - 4)\phi_3''\right), \quad (B.20)
\end{align*}

where we have used Eq. (B.18). Furthermore, we find

\begin{align*}
2\phi_2'' + (D - 4)\phi_3'' &= -(\phi_2'' - \phi_3'') + 3\phi_2'' + (D - 5)\phi_3'' \\
&= -(\phi_2 - \phi_3)' + 2\sqrt{3(D - 2)}H\phi_2', \quad (B.21)
\end{align*}
where we have used Eq. (B.19). Combining with Eq. (B.20), we find

\[ 0 = \left( \Box + 6H^2 \right) (\phi_2 - \phi_3) - \sqrt{3(D-2)}H (\phi_2 - \phi_3)' + (\phi_2 - \phi_3)'' \]  

(B.22)

### B.2 In the case of the scalar-tensor theory

We then consider the perturbations about the solution in the scalar-tensor theory Eq. (2.21). We consider the perturbations of the scalar field as well as the metric. For the metric form (4.1), the background solution of Einstein equations is obtained as

\[ A = e^{-\sqrt{\frac{3}{(D-2)}}}H|Y - Y_0|, \quad \omega_{ab}d\zeta^a d\zeta^b = \frac{D-6}{3H^2}d\Omega^2_{(D-5)}, \]

\[ \gamma_{\mu\nu} dx^\mu dx^\nu = -dt^2 + e^{2Ht} \delta_{ij} dx^i dx^j. \]  

(B.23)

Also, \( \tilde{R}_{\mu\nu} = 3H^2 \gamma_{\mu\nu} \) and \( \bar{R}_{ab} = 3H^2 \omega_{ab} \).

The energy-momentum tensor is given by

\[ \kappa_D^2 T_{AB} = \frac{1}{2} \partial_A \phi \partial_B \phi - \frac{1}{2} g_{AB} \left( \frac{1}{2} g^{CD} \partial_C \phi \partial_D \phi - 2 e^{\beta \phi} \Lambda_s \right). \]  

(B.24)

The components of the background energy-momentum tensor are given by

\[ \kappa_D^2 T^{(0)Y} = \frac{1}{4A^2} \phi' \delta \phi' - \phi_2 \phi'^2 + e^{\beta \phi} \Lambda_s, \]

\[ \kappa_D^2 T^{(0)\mu}_{\nu} = \left[ - \frac{1}{4A^2} \phi'^2 + e^{\beta \phi} \Lambda_s \right] \delta^\mu_{\nu}, \]

\[ \kappa_D^2 T^{(0)a}_{b} = \left[ - \frac{1}{4A^2} \phi'^2 + e^{\beta \phi} \Lambda_s \right] \delta^a_b. \]  

(B.25)

The components of the perturbed energy-momentum tensor are given by

\[ \kappa_D^2 \delta T^{Y} = \frac{1}{2A^2} \left( \phi' \delta \phi' - \phi_2 \phi'^2 \right) + \beta e^{\beta \phi} \Lambda_s \delta \phi, \]

\[ \kappa_D^2 \delta T^{\mu}_{\nu} = \left[ - \frac{1}{2A^2} \left( \phi' \delta \phi' - \phi_2 \phi'^2 \right) + \beta e^{\beta \phi} \Lambda_s \delta \phi \right] \delta^\mu_{\nu}, \]

\[ \kappa_D^2 \delta T^{a}_{b} = \left[ - \frac{1}{2A^2} \left( \phi' \delta \phi' - \phi_2 \phi'^2 \right) + \beta e^{\beta \phi} \Lambda_s \delta \phi \right] \delta^a_b, \]

\[ \kappa_D^2 \delta T^{Y}_{\mu} = \frac{1}{2} \phi' \delta \phi'_{, \mu}, \]

(B.26)

where we have ignored the excitations along \( S^{D-5} \) directions and set the terms with \( \partial_a \) to be zero. Hence,

\[ \kappa_D^2 \delta \left( \frac{1}{D-5} T^{a}_{a} - T^{Y} \right) = \frac{1}{A^2} \left( \phi_2 \phi'^2 - \phi' \delta \phi' \right). \]  

(B.27)
Now we derive the equation for the perturbations. We consider the following combination

\[
\delta \left( \frac{1}{D-5} G^a_a - G^Y_Y \right) = \frac{1}{D-5} g^{(0)ac} \delta R_{ac} - g^{YY} \delta R_{YY} + \frac{1}{D-5} \delta g^{ac} R^{(0)}_{ac}
\]

\[
= A^{-2} \left[ \Box_4 (\phi_2 - \phi_3) - (D - 2) \frac{A'}{A} (\phi_2' + \phi_3') + 4 \phi_1'' + (D - 6) \phi_3'' \right.
\]

\[
+ 2(D - 2) \left( \frac{A'}{A} \right)^2 (\phi_2 - \phi_3) \left. - 2 \phi_3 \frac{R^{(0)}a_a}{D-5} \right] + A^{-2} \left[ \Box_4 (\phi_2 - \phi_3) + (D - 2) \frac{A'}{A} (\phi_2' - \phi_3') + \phi_2'' - \phi_3'' \right.
\]

\[
+ 2(D - 2) \left( \frac{A'}{A} \right)^2 (\phi_2 - \phi_3) - \phi' \delta \phi' \left] - 2 \phi_3 \frac{R^{(0)}a_a}{D-5}, \right. \quad \text{(B.28)}
\]

where we have used the \((T, \mu)\) and the off-diagonal part of \((\mu, \nu)\) components of the perturbed Einstein equation

\[
\frac{1}{2} \phi' \delta \phi_{,\mu} = \left( - 3 \phi_1' + (D - 2) \frac{A'}{A} \phi_2 - (D - 5) \phi_3 \right), \quad 2 \phi_1 + (D - 5) \phi_3 + \phi_2 = 0. \quad \text{(B.29)}
\]

For the case of the solution Eq. (2.21), we obtain

\[
2 \phi_3 \frac{R^{(0)}a_a}{D-5} + A^{-2} \phi^2 \phi_2 = \frac{12 H^2}{c(D-2)} \frac{1}{A^2} (\phi_2 - \phi_3). \quad \text{(B.30)}
\]

From Eqs. (B.27) and (B.28) with Eq. (B.30), we find

\[
\left( \Box_4 + 6 H^2 \right) (\phi_2 - \phi_3) - \sqrt{3(D - 2) \left( 1 + \frac{2}{c(D-2)} \right) H (\phi_2 - \phi_3)'}
\]

\[
+ (\phi_2 - \phi_3)'' = 0. \quad \text{(B.31)}
\]

**B.3 In the case with a form field strength**

We finally consider the perturbations of the theory including the form field strength Eq. (2.24). Here, we simply consider the perturbations of the scalar field \(\phi \rightarrow \phi + \delta \phi\) as well as the metric perturbations. For the metric form (4.1), the background solution of the Einstein equations is obtained as

\[
A = e^{-\sqrt{\frac{2}{D-2} \left( 1 + \frac{D-6}{D-2} \right) H |Y - Y_0|}}, \quad \omega_{ab} dz^a dz^b = \frac{D - 6}{3 H^2 + \frac{2}{2}} d\Omega^2_{(D-5)},
\]

\[
\gamma_{\mu \nu} dx^\mu dx^\nu = -dt^2 + \gamma_0^2 e^{2Ht} \delta_{ij} dx^i dx^j. \quad \text{(B.32)}
\]

Also, \(\tilde{R}_{\mu \nu} = 3 H^2 \gamma_{\mu \nu}\) and \(\tilde{R}_{ab} = \left( 3 H^2 + \frac{2}{2} \right) \omega_{ab}\). For simplicity, we set the perturbations of the form field strength to be zero, but expect that this would not change the
structure of the evolution equation for the radionic mode. The energy-momentum tensor obtained from Eq. (2.24) is given by
\[
\kappa_D^2 T_{AB} = -e^{-\frac{\alpha_\phi}{D-6}}\Lambda_f g_{AB} + \frac{1}{2} \partial_A \phi \partial_B \phi - \frac{1}{4} g_{AB} g^{CD} \partial_C \phi \partial_D \phi \\
+ \frac{1}{2(D-5)!} e^{\phi} \left[(D-5)F_{A M_1 \cdots M_{D-5}} F_{B}^{M_1 \cdots M_{D-5}} - \frac{1}{2} g_{AB} F_{M_1 \cdots M_{D-5}} F^{M_1 \cdots M_{D-5}} \right].
\] (B.33)

The background components of the energy-momentum tensor are given by
\[
\kappa_D^2 A^2 T^{(0)y} = \frac{3H^2}{2(D-1)} \left( \frac{D-6}{\zeta(D-2)} \right) - \frac{D-5}{4} f^2,
\]
\[
\kappa_D^2 A^2 T^{(0)a} = \left[ \frac{3H^2}{2(D-3)} \left( \frac{D-6}{\zeta(D-2)} \right) - \frac{D-7}{4} f^2 \right] \delta^a_a,
\]
\[
\kappa_D^2 A^2 T^{(0)\mu} = \left[ \frac{3H^2}{2(D-3)} \left( \frac{D-6}{\zeta(D-2)} \right) - \frac{D-5}{4} f^2 \right] \delta^\mu^\mu,
\] (B.34)

where we have used
\[
e^\phi f_a_{1 \cdots a_{D-5}} f^{a_1 \cdots a_{D-5}} = \frac{(D-5)! f^2}{A^2}, \quad \phi'^2 = \frac{3H^2 D - 6}{2\zeta(D-2)},
\]
\[
\Lambda_f = - \frac{3(D-6)H^2}{2\zeta} + \frac{(D-6)f^2}{4}.
\] (B.35)

Ignoring the excitations along the \(S^{D-5}\) direction, the components of the perturbed energy-momentum tensor reduce to
\[
\kappa_D^2 A^2 \delta T^y_y = \frac{1}{2} \left( \phi' \delta \phi' - \phi_2 \phi'^2 \right) - \frac{f^2}{4} \alpha \delta \phi + \frac{\alpha \Lambda_f}{D-2} \delta \phi,
\]
\[
\kappa_D^2 A^2 \delta T^a_a = \left[ - \frac{1}{2} \left( \phi' \delta \phi' - \phi_2 \phi'^2 \right) + \frac{f^2}{4} \alpha \delta \phi + \frac{\alpha \Lambda_f}{D-2} \delta \phi \right] \delta^a_a,
\]
\[
\kappa_D^2 A^2 \delta T^\mu^\mu = \left[ - \frac{1}{2} \left( \phi' \delta \phi' - \phi_2 \phi'^2 \right) - \frac{f^2}{4} \alpha \delta \phi + \frac{\alpha \Lambda_f}{D-2} \delta \phi \right] \delta^\mu^\mu,
\]
\[
\kappa_D^2 \delta Y_{\mu} = \frac{1}{2} \phi' \delta \phi_{\mu},
\] (B.36)

where others are zero and we have ignored the perturbations of the form field. Hence, Eq. (B.36) gives
\[
\kappa_D^2 \delta \left( \frac{1}{D-5} T^a_a - T^y_y \right) = \frac{1}{A^2} \left( \phi_2 \phi'^2 - \phi' \delta \phi' \right) + \frac{f^2}{2} \frac{\alpha}{A^2} \delta \phi.
\] (B.37)

We now derive the evolution equation for the perturbations. As for the previous case, we compute the following combination
\[
\delta \left( \frac{1}{D-5} G^a_a - G^y_y \right) = A^{-2} \left[ \square_4 (\phi_2 - \phi_3) + (D-2) \frac{A'}{A} (\phi_2' - \phi_3') + \phi_2'' - \phi_3'' \right. \\
+ \left. 2(D-2) \left( \frac{A'}{A} \right)^2 (\phi_2 - \phi_3) - \phi' \delta \phi' \right] - 2 \phi_3 \frac{R^{(0)\mu}}{D-5} a^a.
\] (B.38)
where we have used the \((T, \mu)\) and the off-diagonal part of \((\mu, \nu)\) components of the perturbed Einstein equation. For the case of the solution Eq. (2.25), we can write
\[
2\phi_3 \frac{A^2R(0)^a}{D-5} + \phi'^2 \phi_2 = -\left[\frac{6(D-6)}{(D-2)\zeta}H^2 - f^2\right]\phi_3 + \frac{6(D-6)}{(D-2)\zeta}H^2\phi_2
= f^2\phi_3 + \frac{6(D-6)H^2}{(D-2)\zeta}(\phi_2 - \phi_3),
\]
where we have used
\[
A^2R(0)^a = -\frac{3(D-6)}{(D-2)\zeta}H^2 + \frac{f^2}{2}.
\]
From Eqs. (B.37) and (B.38) with Eq. (B.39), we find
\[
\left(\Box_4 + 6H^2\right)(\phi_2 - \phi_3) - \sqrt{3(D-2)}\left(1 + \frac{D-6}{\zeta(D-2)}\right)H(\phi_2 - \phi_3)' + (\phi_2 - \phi_3)'' = \frac{f^2}{2}(2\phi_3 + \alpha \delta \phi).
\]
Thus, in contrast to the previous cases of the pure gravity and the scalar-tensor theory, there is the source term in the right-hand side. Since this source term is proportional to \(f^2\), it is induced if there is the non-zero flux.

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