Neutrinoless Double Beta Decay and Heavy Sterile Neutrinos

Manimala Mitra\textsuperscript{a} \textsuperscript{*}, Goran Senjanovi\textsuperscript{b} \textsuperscript{†}, Francesco Vissani\textsuperscript{a} \textsuperscript{‡}

\textsuperscript{a} INFN, Laboratori Nazionali del Gran Sasso, Assergi (AQ), Italy  
\textsuperscript{b} ICTP, Trieste, Italy

January 31, 2013

Abstract

The experimental rate of neutrinoless double beta decay can be saturated by the exchange of virtual sterile neutrinos, that mix with the ordinary neutrinos and are heavier than 200 MeV. Interestingly, this hypothesis is subject only to marginal experimental constraints, because of the new nuclear matrix elements. This possibility is analyzed in the context of the Type I seesaw model, performing also exploratory investigations of the implications for heavy neutrino mass spectra, rare decays of mesons as well as neutrino-decay search, LHC, and lepton flavor violation. The heavy sterile neutrinos can saturate the rate only when their masses are below some 10 TeV, but in this case, the suppression of the light-neutrino masses has to be more than the ratio of the electroweak scale and the heavy-neutrino scale; i.e., more suppressed than the naive seesaw expectation. We classify the cases when this condition holds true in the minimal version of the seesaw model, showing its compatibility (1) with neutrinoless double beta rate being dominated by heavy neutrinos and (2) with any light neutrino mass spectra. The absence of excessive fine-tunings and the radiative stability of light neutrino mass matrices, together with a saturating sterile neutrino contribution, imply an upper bound on the heavy neutrino masses of about 10 GeV. We extend our analysis to the Extended seesaw scenario, where the light and the heavy sterile neutrino contributions are completely decoupled, allowing the sterile neutrinos to saturate the present experimental bound on neutrinoless double beta decay. In the models analyzed, the rate of this process is not strictly connected with the values of the light neutrino masses, and a fast transition rate is compatible with neutrinos lighter than 100 meV.
1 Introduction

The study of neutrinoless double beta decay (0ν2β) transition (A, Z) → (A, Z + 2) + 2e− has a special relevance for testing the physics beyond the standard model. On the experimental side, there is a very lively situation [1, 2, 3, 4, 5, 6] and even an experimental claim, that the 0ν2β transition has been already measured at present [7]; but even postponing a judgment on these findings, it is remarkable that there are realistic prospects for order-of-magnitude improvements in the search for the 0ν2β lifetime [5, 6, 8, 9, 10, 11, 12, 13, 14, 15]. On the theoretical side, accepting that neutrinos have masses, see [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43], the investigation of 0ν2β becomes the most natural option; moreover, the observation of lepton-number violating processes would be a cogent manifestation of incompleteness of the standard model, and could be even considered as a step toward the understanding of the origin of the matter.

However, once we enter the theoretical discussion, one has to stress immediately an evident but essential point, that the meaning of 0ν2β depends on the model. To naive eyes, the fact that no neutrino is emitted in this transition leads one to wonder what is the link of 0ν2β with neutrinos. The most popular theoretical answer is that, the hypothesis that neutrinos have Majorana mass [44] suggests that the exchange of virtual, light neutrinos is a plausible mechanism for the occurrence of 0ν2β [45]. An additional (and more recent) theoretical argument is that, listing the effective operators that obey the standard model gauge symmetry, Majorana neutrino masses arise already as dimension-five operators [46]; thus any further contribution to 0ν2β (or, say, to proton decay) will be due to the higher-dimensional operators, and, as such, are expected to be suppressed. However, there is an implicit assumption underlying this approach: namely, that the new physics is at very high scale. This assumption may or may not hold. In fact, the possibility that 0ν2β is mostly due to mechanisms different from the conventional one (light neutrino exchange) has been proposed since long [47] and it is actively discussed, see e.g., [48, 49, 50, 51, 52, 53, 54, 55, 56, 57].

Among the simplest renormalizable extensions of the standard model, that are able to account for neutrino masses, the addition of heavy sterile neutrinos–i.e., so called Type I seesaw [58, 59, 60, 61]–is largely considered

1In fact, the 0ν2β process can correctly be described as a nuclear transition in which some basic constituent of the ordinary matter–i.e., two electrons–are created. Furthermore, once recognized the existence of transitions among different leptonic flavors (again, neutrino oscillations) one should conclude that any global symmetry of the standard model excepting at most B − L is broken; thus, the observation of a lepton number violating processes would imply that also the baryon number is at some level violated.
as the minimal option. These new particles, if lighter than a some 10 of TeV (see below) can act as new sources of lepton number violation, or in other terms, as potential additional contributors to $0\nu2\beta$. The main goal of the present paper is a systematic study of this possibility, considered occasionally in the past [56]. Note incidentally that the possibility that the heavy neutrinos are not ultra-heavy, and thus can be potentially tested experimentally, is the first one that has been considered in [55, 59].

The outline of the paper is the following: In Sect. 2, we review the basics of light as well as heavy neutrino exchange in $0\nu2\beta$ process. The most important result of this section is given in Sect. 2.2 using the updated nuclear matrix element of reference [62], the bound on the active-sterile mixing coming from $0\nu2\beta$ transition is re-examined. The improvement in the uncertainty of the nuclear matrix element leads the bound to be one order of magnitude tighter than the existing one [63]. On the face of this analysis, the bounds coming from other potentially relevant experiments, see [64] for a review, have become relatively less significant. Also see [65] for a specific realization, where the bound on sterile neutrino mass and mixing has been obtained from $0\nu2\beta$, astrophysical and cosmological informations.

Following this, we provide the detailed analysis on the nature of $0\nu2\beta$ transition for the simplest extensions of the standard model with heavy sterile neutrinos. In Sect. 3, we first concentrate on the usual Type I seesaw, and later in Sect. 4, we extend the discussion to the other seesaw scenarios as well, namely Extended seesaw [66, 67].

In Type I seesaw, the generic naive expectation (as precisely defined in Sect. 3.2) leads us to believe that the heavy sterile neutrino contribution in $0\nu2\beta$ process is much smaller than the light neutrino contribution. For one generation of light and heavy sterile neutrino state, this naive expectation is established by the very basic seesaw structure (see Sect. 3.2.1). Going beyond one generation, it is however possible to reach the opposite extreme; i.e., one can obtain a dominant and even saturating $1$ sterile neutrino contribution, which is not inherently linked with the light neutrino contribution. The systematic study of this possibility requires, that the light neutrino contribution should be smaller than the naive expectation suggested by seesaw. This consideration is carried out in Sect. 3.3 where we analyze the vanishing seesaw condition and its perturbation, leading to small neutrino masses. We classify the different cases, where the light neutrino spectra is not necessarily degenerate, and even possibly hierarchical. All these cases can provide a dominant sterile neutrino contribution in $0\nu2\beta$ process, as discussed in Sect. 3.4. We derive an useful parameterization in Appendix A to study these cases analytically. For completeness we also provide explicit numerical example in Sect. 3.4.6.

Assuming a saturating contribution $1$ from heavy sterile neutrino exchange, in Sect. 3.2.2 we provide a naive estimation on the prospect of heavy Majorana neutrino search at LHC [65, 70, 71], as well as in lepton flavor violating process [72]; these prospects turn out to be weak due to the stringent constraints coming from $0\nu2\beta$ process. The possible issue, like radiative stability of the light neutrino mass matrices, below the naive seesaw expectation, is discussed in Sect. 3.3.3. This, along with the request of a dominant heavy sterile neutrino contribution in $0\nu2\beta$ process gives an upper bound of about 10 GeV on the heavy sterile neutrino mass scale (Sect 3.4.5) and also an upper bound on the necessary perturbation of the vanishing seesaw condition.

Following the analysis of Type I seesaw, in Sect. 4, we then consider a natural seesaw extension, namely Extended seesaw scenario [66, 67]. We describe the basics of Extended seesaw in Sect. 4.1 and after that quantify the different sterile neutrino contributions in $0\nu2\beta$ process (Sect. 4.2). As for the Type I seesaw, the sterile neutrino states in this case can also give significant contributions in $0\nu2\beta$ process. In this particular seesaw scenario, the light neutrino contribution depends on a small lepton number violating parameter, while to the leading order, the active-sterile neutrino mixing is independent of that parameter. Due to this particular feature, the sterile neutrino contribution to $0\nu2\beta$ process is totally independent of the light neutrino contribution. In the next section i.e., Sect. 4.3, we discuss the possibility of obtaining a saturating contribution $1$ from the sterile neutrino states, the scope of finding sterile neutrinos at LHC [69, 70, 71], and as well as the possibility of obtaining a rapid lepton flavor violation [72]. Possible issues, such as, the higher-dimensional correction to the active-sterile mixing angle and $0\nu2\beta$ transition amplitude, that will arise due to the small lepton number violating scale, has also been discussed. The details of

---

2 A theoretical objection to the hypothesis of Type I seesaw, defined introducing the heavy sterile neutrinos as pure gauge singlets as in [60], is that the mass of right-handed neutrinos is not related to any gauge symmetry, differently from the masses of all other known particles. But, as pointed out originally [68] and stressed recently [65], even if such a gauge symmetry is introduced (most plausibly through a SU(2)$_R$ group) similar considerations hold: the mechanism of $0\nu2\beta$ is not necessarily light neutrino exchange. The phenomenology of this type of models is however different and to some extent richer than the one we will describe.
higher-dimensional correction to the mass and mixing matrix has been evaluated in the Appendix \[B\]. Finally, in Sect. 3 we present the summary of our work.

The analysis presented in this paper clearly shows that the heavy sterile neutrino states can certainly dominate the $0\nu 2\beta$ transition; this possibility has the potential to overcome any conflict \[73\] between the cosmological bound and the experimental hint on $0\nu 2\beta$ obtained by Klapdor and collaborators \[74\], or more in general it permits to reconcile a fast (and potentially observable) rate of $0\nu 2\beta$ and small neutrino masses. In addition, for Type I seesaw, the demand of radiatively stable light neutrino mass matrices lowers the mass scale of the heavy sterile neutrino states below 10 GeV. Note that, based on the updated nuclear matrix elements \[62\], the bound from $0\nu 2\beta$ is now much more stringent than the previous consideration \[63, 64\]. Further improvement in meson as well as neutrino-decay experiments have a certain potential to provide us with more information on (and possibly a measurement of) the active-sterile mixings, which is similar to the conclusion obtained in the $\nu$MSM model by Shaposhnikov and collaborators \[74\].

## 2 Light and heavy neutrino exchange

The phenomenological possibility that some heavy neutrino state contributes to $0\nu 2\beta$ transition amplitude has been remarked since long: see \[75\] for a model that however contradicts the current understanding of neutrino masses and interactions. \[68\] for the first modern discussion within gauge theories, \[76\] for a further earlier contribution.

The relevant discussion is summarized in this section, emphasizing: 1) the large number of free parameters, Sect. 2.1; 2) the role of (and the remarkably large uncertainties in) nuclear matrix elements for heavy neutrino exchange, Sec. 2.2. Subsequently, we will apply the results of this discussion to the specific model of interest for heavy neutrinos, Type I seesaw, where the number of free parameters is smaller.

### 2.1 Parameters of the $0\nu 2\beta$ amplitude

As we recall, several experiments testify that the usual left neutrinos $\nu_\ell$ ($\ell = e, \mu, \tau$) are subjected to flavor transformations, as expected if they have mass. Considering only the minimal case these neutrinos have Majorana mass; this minimal ansatz amounts to postulate that the particle content of the SU(2)$_L \times U(1)_Y$ standard model theory remains the same, while the Lagrangian is endowed with a non-renormalizable term, which after spontaneous symmetry breaking reads

$$M_\nu = U^* \text{diag}(m_i) U^\dagger,$$

(1)

In the above, the unitary matrix $U$ is the leptonic mixing matrix, $\nu_\ell = U_{\ell i} \nu_i$, $\nu_i$ and $\nu_\ell$ are respectively the flavor and mass basis; the physical masses $m_i$ of the neutrinos are real and non-negative; the possible Dirac and Majorana phases are included into $U_{\ell i}$.

This hypothesis not only accounts for oscillations, but also has some predictive power for the lepton number violating neutrinoless double beta decay process. Indeed, the ee-element of the mass matrix:

$$|\langle M_\nu \rangle_{ee}| = | \sum_i U_{ei}^2 m_i |,$$

(2)

contributes to the neutrinoless double beta decay rate (see \[77\], \[78\], \[79\], \[80\] for recent reviews). Evidently this quantity cannot exceed $\sum_i |U_{ei}^2| m_i$, and since the differences of neutrino mass squared are measured and the relevant elements of the mixing matrix are sufficiently well-known, there is a upper bound on $|\langle M_\nu \rangle_{ee}|$ as a function of the lightest neutrino mass $m_{\text{min}}$ \[10\] \[81\] \[82\], or equivalently of the other mass scales, such as the mass probed in direct search for neutrino masses $m_3^2 = \sum_i |U_{e3}^2| m_i^2$, or the sum of the neutrino masses $m_{\text{sum}} = \sum_i m_i$ probed in cosmology (see \[83\] and e.g., \[84\] \[85\]). For the lightest neutrino mass scale $m_{\text{min}} > 0.1$ eV, relevant to the case of present experimental sensitivities, one can approximate the bound as,

$$m_\beta \approx m_{\text{sum}}/3 \approx m_{\text{min}} > |\langle M_\nu \rangle_{ee}|,$$

(3)

with an accuracy better than 10%, that moreover improves for normal mass hierarchy. It is interesting that the experimental hint on $0\nu 2\beta$ obtained by Klapdor and collaborators \[74\], according to \[73\], challenges the bound
obtained in cosmology, though this conclusion depends on which cosmological bound is considered and which nuclear matrix element is used, see below for a more quantitative statement.

Of course, in less minimal models new contributions to $0\nu 2\beta$ are expected and the conflict between the cosmological bound and the experimental hint on $0\nu 2\beta$ [7] can be overcome. This can happen when the usual left-handed neutrinos contain heavy neutrino components, too,

$$\nu_\ell = \sum_{i=1}^{3} U_{\ell i} \nu_i + \sum_{i=1}^{n_h} V_{\ell i} N_i,$$

(4)

where we have considered $n_h$ heavy neutrinos $N_i$ with masses $M_i$ and small mixing $|V_{\ell i}| \ll 1$ (this condition is discussed later). In fact, the amplitude of $0\nu 2\beta$ is proportional to

$$A = \frac{U^2_{ei} m_i}{p^2} - \frac{V^2_{ei}}{M_i},$$

(5)

which includes the contribution due to heavy neutrino exchange (also called direct contribution, or contact term). Here, $p^2$ is the virtuality of the exchanged neutrino. We have $p^2 < 0$, for the time component $p_0$ is of the order of the $Q$ value of the reaction, few MeV, whereas its space component is much larger and essentially determined by the separation between neutrons, $|\vec{p}| \sim \hbar c/fm \approx 200$ MeV. In the above expression for the amplitude, we considered the case of interest,

$$m_i \ll 200 \text{ MeV} \ll M_i,$$

(6)

Indeed, if we have a virtual Majorana neutrino with mass $\mu$ and with momentum $p$, its propagator implies that the amplitude is proportional to:

$$\mu/(p^2 - \mu^2),$$

(7)

which in the limits of $\mu \ll 200$ MeV or $\mu \gg 200$ MeV reduces to $\mu/p^2$ and $-1/\mu$ respectively.

### 2.2 Role of the nuclear matrix elements

A traditional expression for the $0\nu 2\beta$ half-life is:

$$\frac{1}{T_{1/2}} = G_{0\nu} |\mathcal{M}_\nu \eta_\nu + \mathcal{M}_N \eta_N|^2,$$

(8)

where the parameters of light and heavy neutrinos, as defined in Eq. [5] are presented through the complex dimensionless parameters $\eta_\nu = U^2_{ei} m_i/m_e$ and $\eta_N = V^2_{ei} m_p/M_i$, and, conventionally, the reference mass scales are chosen to be the electron and the proton masses $m_e$ and $m_p$. In the above, $\mathcal{M}_e$ and $\mathcal{M}_N$ are the nuclear matrix elements corresponding to the light and heavy neutrino exchange in $0\nu 2\beta$ process.

Whenever necessary, we will assume for the nuclear matrix elements the values given in [62], where we read that, in the case of $^{76}\text{Ge}$ $0\nu 2\beta$ transition, the ‘phase space’ factor is $G_{0\nu} = 7.93 \times 10^{-15}$ yr$^{-1}$ and the matrix elements are $\mathcal{M}_e = 5.24 \pm 0.52$ and $\mathcal{M}_N = 363 \pm 44$. These values imply, for instance, that the result of Klapdor, $T_{1/2}(^{76}\text{Ge}) = 2.23^{+0.44}_{-0.31} \times 10^{25}$ yr [62] would be consistent with $\eta_\nu = 4.5 \times 10^{-7}$ and $\eta_N = 0$; with $\eta_\nu = 0$ and $\eta_N = 6.5 \times 10^{-9}$; or with linear combinations of these limiting cases (possibly allowing for an overall sign). The first possibility, where only the neutrino mass is present, would imply

$$|\langle \mathcal{M}_\nu \rangle_{ee} | = 0.23 \pm 0.02 \pm 0.02 \text{ eV},$$

(9)

namely, a degenerate neutrino spectrum, partly testable with KATRIN experiment and of great interest for cosmological investigations, since it implies $m_{\text{cosem}} > 0.69$ eV (see Eq. [3]). In the following, we will be especially interested to explore the opposite limit, when the $0\nu 2\beta$ is dominated by the second term, and the neutrino mass spectrum is not necessarily degenerate, and even possibly hierarchical.

The traditional expression in Eq. [8] can be recast into the following equivalent form:

$$\frac{1}{T_{1/2}} = K_{0\nu} \left| \frac{U^2_{ei} m_i}{(p^2)^2} - \frac{V^2_{ei}}{M_i} \right|^2,$$

(10)
where we set \( K_{0\nu} = G_{0\nu} (\mathcal{M}_N m_p)^2 \) and following [55] we defined,
\[
\langle p^2 \rangle \equiv -m_e m_p \frac{\mathcal{M}_N}{\mathcal{M}_\nu}.
\]
(11)

Note the resemblance of Eq. 10 with the expression of the amplitude given in Eq. 5. With the values of [62], we get \( \langle p^2 \rangle = -(182 \text{ MeV})^2 \), in remarkable accordance with the rough expectations for \( p^2 \) described in Sec. 2.1.

An alternative presentation of the lifetime, valid for generic values of the neutrino masses, is
\[
\frac{1}{T_{1/2}} = G_{0\nu} |\Theta_{ei}^2 \mathcal{M}(\mu_i) \mu_i / m_e|^2.
\]
(12)

This agrees with Eq. 8 if one identifies the following sets of parameters,
\[
(\mu_i, \Theta_{ei}) = \begin{cases} (m_i, U_{ei}) & \text{when } \mu_i \to 0, \\ (M_i, V_{ei}) & \text{when } \mu_i \to \infty \end{cases}
\]
and, at the same time, the following limits hold:
\[
limit_{\mu \to 0} \mathcal{M}(\mu) = \mathcal{M}_\nu \text{ and } \lim_{\mu \to \infty} \mu^2 \mathcal{M}(\mu) = m_e m_p \mathcal{M}_N \equiv -\langle p^2 \rangle \mathcal{M}_\nu,
\]
(14)

the scale of comparison being \(-\langle p^2 \rangle \sim (200 \text{ MeV})^2\). A simple and useful analytical approximation of the general expression of Eq. 12 has been proposed in [86]:
\[
\frac{1}{T_{1/2}} = K_{0\nu} \left| \Theta_{ei}^2 \frac{\mu_i}{\langle p^2 \rangle - \mu_i^2} \right|^2.
\]
(15)

The advantage of this formula is that it emphasizes the role of the neutrino propagator given in Eq. 7 and allows one to switch easily from the regimes of light and heavy neutrino exchange (compare with Eq. 10 in the limit of light and heavy neutrinos), even being slightly inaccurate in the region where \(-\langle p^2 \rangle \sim \mu_i^2\) [57, 63].

Using the last formula and the present experimental bound on \(0\nu2\beta\) lifetime \(T_{1/2} > 1.9 \times 10^{25} \text{ yr}\) [1], we obtain the upper bound on the mixing \(|\Theta_{ei}|^2\), which is shown in Fig. 1. When this is compared with the other experimental constraints on the model, compiled by [64], it is quite evident that \(0\nu2\beta\) play the most important role. The details of the figure are as follows,

- The upper yellow region is disallowed from neutrinoless double beta decay consideration. Part of this region is as well constrained from different meson decays, neutrino decay-searches as well as other experiments, shown explicitly in the figure. The lower blue region is the allowed one from \(0\nu2\beta\) as well as the various above mentioned experiments.
- The middle grey band which has been obtained considering the exchange of a single heavy neutrino with mass \(\mu_i\) in \(0\nu2\beta\) process, corresponds to the uncertainty of the nuclear matrix elements \(\mathcal{M}_\nu\) and \(\mathcal{M}_N\). For the thick black line in this grey band, we have adopted the parameters of [62], \(\mathcal{M}_\nu = 5.24\) and \(\mathcal{M}_N = 363\), i.e., \(\langle p^2 \rangle = -(182 \text{ MeV})^2\); for the upper thin black line \(\mathcal{M}_\nu = 3\) and \(\mathcal{M}_N = 69\), namely, \(\langle p^2 \rangle = -(105 \text{ MeV})^2\); while for the lower thin line \(\mathcal{M}_\nu = 7\) and \(\mathcal{M}_N = 600\), namely, \(\langle p^2 \rangle = -(203 \text{ MeV})^2\). The upper line agrees numerically with the results of [63] (see also [86]); the large difference with [62] should be attributed to the new short range correlations and improved nucleon form factors. The lower line, instead, is meant to convey a conservative idea of the uncertainties; see [57, 68] for the most stringent upper bound on \(0\nu2\beta\) we could have at present, depending on the size of the nuclear matrix elements. For each of these black lines, the region above the line is disallowed from \(0\nu2\beta\) transition.

\footnote{We thank F. Simkovic for clarifying discussions on this issue.}
Figure 1: Bounds on the mixing between the electron neutrino and a (single) heavy neutrino as obtained from Eq. 15. The upper thin black line corresponds to the result of ref. [63], the thick black one to ref. [62], while the lower thin black line is an attempt to convey a conservative assessment on the residual uncertainty. For comparison, we also show other experimental constraints as compiled in [64]. See text for details.

- The span of values of $\mathcal{M}_\nu$ used in Fig. 1 is much more conservative than the one of [62], quoted above. It corresponds to the range given in the compilation [89], see their Fig. 1. By comparing with a similar compilation of about 10 years ago [82], see their Fig. 2, one understand that the new nuclear physics calculations obtained a reduction of the uncertainty of a factor of two for the regime of light neutrino exchange. This improvement is of enormous importance: the lifetime scales only as the square of the nuclear matrix elements, while in presence of background, the improvement of the bound on the lifetime scales as the square root of the exposure.

- The limits from $0\nu2\beta$ which has been derived using the result of [62] and presented in Fig. 1, are significantly tighter than the previous limits on mass and mixing given in [63] (the result of [63] has also been adopted in recent global analysis [64]). Conversely, the impact of other constraints, in particular those from meson decays, neutrino-decay searches and other experiments, becomes relatively less important: See again [64] (and in particular their Fig. 2) where full reference to the original literature is provided.

3 Type I seesaw and the nature of the $0\nu2\beta$ transition

Type I seesaw is in many regards the simplest extension of the standard model: only heavy sterile neutrino states are added to the spectrum of the SU(3)$_C \times$SU(2)$_L \times$U(1)$_Y$ theory [58, 59, 60, 61], with a primary purpose to account for light neutrino masses in a renormalizable gauge model. However, these heavy states might lead to measurable effects, in particular, for the neutrinoless double beta decay.

In this section we discuss the nature of $0\nu2\beta$ transition within the Type I seesaw [59, 67], emphasizing the possibility discussed occasionally in the literature that the heavy neutrino exchange contribution plays the main role for $0\nu2\beta$. In the present study, we analyze in greater detail the parameter space of Type I seesaw.
Let us describe in detail the outline and scope of this section. First, we recall the basic notations for the model (Sect. 3.1). In Sect. 3.2 we provide a precise formulation of a naive and widespread expectation: within Type I seesaw, the contribution of the heavy neutrino states to the $0\nu\beta\beta$ decay is smaller than the one due to light neutrino states. Actually, for one generation this naive estimation works perfectly well (see Sect. 3.2.1) but for more than one generation, it is possible to obtain a large and dominant contribution to $0\nu\beta\beta$ from the heavy neutrino states, which is not necessarily inherently linked with the light neutrino contribution. This will be discussed in detail, after the mathematical premise of Sect. 3.3 aimed at outlining the cases when the light neutrino masses are much smaller than suggested by the naive expectations from seesaw. Finally, we discuss in Sect. 3.4.6 the possible cases when heavy neutrino exchange provide us with a large effect in $0\nu\beta\beta$. We exhibit explicit examples when this happens. We prove that this possibility can be implemented within Type I seesaw, without occurring into limitations on the structure of the light neutrino mass matrix. As an extreme possibility, we show that it is possible to arrange a large contribution from the heavy Majorana neutrino exchange, even when the light neutrino contribution to $0\nu\beta\beta$ is negligible. We discuss the possible issues, like radiative stability in Sect. 3.3.3, the possibility of relatively less fine-tuning in Sect. 3.4.2 and derive bounds on heavy neutrino mass scale and fine-tuning parameter in Sect. 3.4.5. Finally, in Sect. 3.4.6 we present an explicit numerical example, where the heavy neutrino contribution is the dominant one and the light neutrino contribution is negligibly smaller than the heavy neutrino contribution.

### 3.1 Notation

In our subsequent discussion of $0\nu\beta\beta$ process and its relation with Type I seesaw, we denote the standard model flavor neutrino states by $\nu_L$ and the heavy Majorana neutrinos by $N_L$. The Lagrangian describing the mass terms is the following,

$$ L = -\frac{1}{2} (\nu_L \quad N_L) \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} + \text{h.c.} \tag{16} $$

For three generation of standard model neutrinos $\nu_L$ and $n_h$ generation of sterile neutrino state $N_L$, $M_D^T$ and $M_R$ will be of $3 \times n_h$ and $n_h \times n_h$ dimension. From the above Lagrangian one obtains this following neutral lepton mass matrix,

$$ M_n = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{17} $$

The neutrino flavor state $(\nu_L \quad N_L)^T$ is related to the neutrino mass state $(\nu_m \quad N_m)^T$ by the unitary mixing matrix $U$ where,

$$ \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} = U \begin{pmatrix} \nu_m \\ N_m \end{pmatrix}. \tag{18} $$

The mixing matrix $U$ which diagonalizes the above mentioned neutral lepton mass matrix satisfies the following relation $U^T M_n U = M_n^d$, where $M_n^d$ is the diagonal neutrino mass matrix. We denote $M_n^d$ as follows,

$$ M_n^d = \begin{pmatrix} \text{diag}(m_i) & 0 \\ 0 & \text{diag}(M_i) \end{pmatrix}, \tag{19} $$

where $m_i$ and $M_i$ represent the light and heavy neutrino masses respectively. It is convenient to introduce a couple of auxiliary matrices $U_{1,2}$ as $U = U_1 U_2$. The first matrix $U_1$ block-diagonalizes $M_n$, namely it satisfies $U_1^T M_n U_1 = M_{bd}$. Subsequently, $M_{bd}$ is further diagonalized by the matrix $U_2$, that satisfies the relation $U_2^T M_{bd} U_2 = M_n^d$. Let us denote the block-diagonalized matrix as,

$$ M_{bd} = \begin{pmatrix} M_\nu & 0 \\ 0 & M_N \end{pmatrix}. \tag{20} $$

It is possible to operate a systematic expansion of $M_{bd}$ and $U_1$ is powers of $M_R$ \cite{40}, thus enforcing the seesaw approximation, $M_R \gg M_D$. Up to leading order in powers of $M_R$, we have simply $M_N = M_R$ for the heavy
neutrino mass matrix, while the light neutrino mass matrix reads,

\[ M_\nu = -M_D^T M_R^{-1} M_D. \]  

(21)

Keeping terms up to 2nd order in \( M_R^{-1} \), the mixing matrix \( U_1 \) is,

\[ U_1 = \begin{pmatrix} 1 - \frac{1}{2} M_D^1 M_R^{-1*} M_R^{-1} M_D & M_D^1 M_R^{-1*} \\ -M_R^{-1} M_D & 1 - \frac{1}{2} M_R^1 M_D M_D^* M_R^{-1*} \end{pmatrix}. \]  

(22)

Next, we denote the mixing matrix \( U_2 \) as follows,

\[ U_2 = \begin{pmatrix} U \\ 0 \\ W \end{pmatrix}, \]  

(23)

where the mixing matrices \( U \) and \( W \) diagonalize the light and heavy neutrino mass matrices \( M_\nu \) and \( M_R \) respectively: \( U^T M_\nu U = \text{diag}(m_i) \) and \( W^T M_R W = \text{diag}(M_i) \). From Eq. 22 and Eq. 23, one immediately obtains,

\[ U = \begin{pmatrix} (1 - \frac{1}{2} M_D^1 M_R^{-1*} M_R^{-1} M_D)U & M_D^1 M_R^{-1*} W \\ -M_R^{-1} M_D U & (1 - \frac{1}{2} M_R^1 M_D M_D^* M_R^{-1*})W \end{pmatrix}. \]  

(24)

To the leading order, the mixing matrix \( U \) is simply,

\[ U = \begin{pmatrix} U \\ -M_R^{-1} M_D U \end{pmatrix} W. \]  

(25)

Finally and quite importantly, we note that the mixing between light and heavy neutrino states is \( M_D^1 M_R^{-1*} W \). According to convention of Eq. 4, this mixing matrix is denoted as \( V \), namely

\[ V = M_D^1 M_R^{-1*} W. \]  

(26)

In the basis where the heavy Majorana neutrino mass matrix is diagonal, \( M_R = W^* \text{diag}(M_i) W \), we rewrite the mixing matrix \( V \) as follows,

\[ V = \hat{M}_D^1 M_i^{-1}, \]  

(27)

and the Dirac mass matrix in this basis is simply,

\[ \hat{M}_D = W^T M_D. \]  

(28)

We note in passing, that Eq. 4 is actually valid only when \( |V_{ei}| \ll 1 \); the deviations that should be expected (due to the unitarity constraints) are formally evident from Eq. 24 and are usually small.

### 3.2 Naive expectations for 0ν2β in Type I seesaw model

In this section, we aim to define precisely which are the naive expectations from the Type I seesaw model for various interesting measurable quantities. Subsequently, we argue for the interest in exploring alternative possibilities.

#### 3.2.1 Heavy neutrino exchange in the single flavor case

Let us begin the analysis by showing that, a single flavor Type I seesaw implies that the light neutrino exchange in 0ν2β process is never sub-leading. In other words, it is not possible to attribute the 0ν2β transition to the heavy neutrino exchange with one light and one heavy neutrinos only.

Assume one light and one heavy Majorana neutrino \( \nu_L \) and \( N_L \) with masses \( m_1 \) and \( M_1 \) respectively. The flavor state \( \nu_e \) is mixed with the mass states as follows,

\[ \nu_e = U_{e1} \nu_1 + V_{e1} N_1. \]  

(29)
and thus the $0\nu2\beta$ transition amplitude receives a contribution proportional to,

$$U_{e1}^2 \frac{m_1}{p^2} + V_{e1}^2 \frac{M_1}{p^2 - M_1^2}. \quad (30)$$

The expression is valid whatever value the mass $M_1$ has. Now, since by hypothesis we are considering Type I seesaw, the left-left element of the mass matrix given in Eq. 17 is zero. Hence we have, $U_{e1}^2 m_1 + V_{e1}^2 M_1 = 0$. Thus we conclude that the amplitude of Eq. 30 can be rewritten as:

$$U_{e1}^2 \frac{m_1}{p^2} \times \frac{M_1^2}{M_1^2 - p^2}. \quad (31)$$

Since $p^2 < 0$, we see that the effect of the heavier state can only reduce the strength of the $0\nu2\beta$ transition; this becomes negligible, leaving only the contribution due the light neutrino exchange, in the limit when $M_1^2 \gg -p^2$.

It is also useful to note that, upon expanding Eq. 31 in powers of $p^2/M_1^2$ we get the following,

$$U_{e1}^2 \frac{m_1}{p^2} \times \left(1 + \frac{p^2}{M_1^2}\right). \quad (32)$$

As expected from Eq. 5, the new contribution $U_{e1}^2 m_1/M_1^2$ has the form of a contact term—i.e., it is a constant. It is clearly evident from above that for the limit $M_1^2 \gg -p^2$, the second term within the bracket is much smaller than unity.

With this, we conclude that for one generation case, the contribution of the heavy sterile Majorana neutrino state to $0\nu2\beta$ process is always much smaller than the one of the light Majorana state. This implies that, in order to have a large contribution to $0\nu2\beta$ within Type I seesaw, we need to consider the multi-flavor case. However, such a property is not generic of a multi-flavor case, as shown later.

### 3.2.2 Naive expectations: $0\nu2\beta$, colliders and lepton flavor violation

Next, we consider the naive expectations from Type I seesaw. Since this discussion is quite important for the following discussion, we begin by providing a precise definition of what is meant by ‘naive expectations’. Most of these expectations correspond to simple scaling laws, obtained replacing the Dirac mass matrix $M_D$ in Eq. 17 with a single mass scale $m$, and likewise the Majorana mass matrix $M_R$ with a single mass scale $M$. In other words, here we assume that all heavy neutrino masses are of the order of $M$, all light neutrino masses of order of $m^2/M$ (see Eq. 21) all mixing angles with heavy neutrinos $V_{\ell i}$ (with $\ell = e, \mu, \tau$ and $i = 1, 2, 3$) are of the order of $m/M$ (see Eqs. 26 and 27); and, when we speak of “seesaw”, we simply mean that we restrict to the case $M \gg m$.

Fig. 2 shows the relevant portion of the $(m - M)$-plane. The three gray bands correspond to the following boundaries: (1) $M > 200$ MeV, i.e., heavy sterile neutrinos are assumed to act as point-like interactions in the nucleus, as discussed in Sect. 2.1; see in particular Eq. 7; (2) $m < 174$ GeV, in order to ensure perturbativity of the Yukawa couplings; (3) $M > m$, namely, to the seesaw in a conventional sense. The $(m - M)$-plane is divided in various regions (half-planes) by the three oblique lines, that corresponds to power laws in log-log plot, and are defined as follows:

- The leftmost oblique line separating white and blue region corresponds to the condition that the naive formula for neutrino mass gives a small enough result: $M_\nu \sim m^2/M = 0.1$ eV. The inequality $m^2/M > 0.1$ eV, excluded experimentally, corresponds to the region that lies below the line. (We will discuss below the cases when this bound can be evaded).

- The condition that the naive contribution from the heavy neutrino exchange to $0\nu2\beta$ amplitude: $V_{\ell i}^2/M_i \sim m^2/M^3$ saturates the present experimental bound gives the line close to the diagonal and which separates the pink and blue region of the parameter space. The half-plane below this line defines the region where this contribution is larger than allowed experimentally.

- The area below the oblique line separating the pink and yellow region is the region in $m - M$ plane, where the production of the heavy Majorana neutrinos in colliders is not suppressed by a small coupling. The oblique
Figure 2: Naive expectations on Type I seesaw model are displayed on the \((m - M)\)-plane. The constraints from 0\(\nu\)2\(\beta\) transition heavy Majorana neutrino searches in colliders and lepton flavor violating decays are shown. See the text for detailed explanation.

Note that the last region is divided in two parts by the \(\mu \rightarrow e\gamma\) bound that applies to \(V_{ei}^*V_{\mu i}\phi(M_i/M_W)\), with \(\phi(x) = x/2(1 - 6x + 3x^2 + 2x^3 - 6x^2 \ln x)/(1 - x)^4\), and that translates into \(m^2/M^2\phi(M/M_W) < 10^{-4}\) [92]. This excludes the rightmost region; the almost vertical line corresponds to the fact that when \(M < M_W\), the limit is relaxed. For one generation of standard model neutrino \(\nu_L\) and one heavy sterile state \(N_L\), of course the bound would be absent.

If taken rather literally, the naive expectations on Type I seesaw (as defined here) would suggest that the existing bounds on neutrino masses imply that there is no room for large contributions to 0\(\nu\)2\(\beta\) from heavy neutrinos, or to produce heavy neutrinos at colliders, or to expect a rapid \(\mu \rightarrow e\gamma\) transition.

### 3.2.3 Alternative possibilities in the multi-flavor case

However, as it is known in the literature and as we discuss in great details later, the naive expectation on \(M_\nu\) discussed in the previous section, can be strongly relaxed in certain multi-flavor cases.

This offers interesting possibilities for the phenomenology of Type I seesaw and in particular for the investigations of the nature of 0\(\nu\)2\(\beta\) transition within this model. Indeed, the impression that one receives from Fig. 2 is that, even after evading the constraint from neutrino mass, the constraints from 0\(\nu\)2\(\beta\) are likely to be relevant in a large portion of the parameter space, which agrees with the conclusion of [70]. We will show in the following that this impression is confirmed by a detailed analysis in a large portion of the parameter space. When we depart from the condition that the mixing angles are all the same, \(V_{ei} \sim V_{\mu i} \sim m/M\) we can decouple the size of the various amplitudes; in fact, the amplitude of 0\(\nu\)2\(\beta\) depends on \(V_{ei}^2\), the one of same-charge di-muon signal depends instead on \(V_{\mu i}^2\), and finally \(\mu \rightarrow e\gamma\) depends on \(V_{ei}^*V_{\mu i}\). This fact, already, provides a large freedom to phenomenological investigations. However, in this work we prefer to proceed systematically, and will be mostly concerned to classify which cases (which matrices) evade the constraint from neutrino mass, showing that at the same time, the heavy neutrino exchange contribution can play a relevant role for the 0\(\nu\)2\(\beta\) decay process. This can be achieved if light neutrino mass is strongly suppressed than the naive expectation from seesaw.

With these phenomenological motivations in mind, we proceed to investigate in detail the different cases when the neutrino masses are much smaller than suggested by the naive expectations.

However, we would like to sketch out in passing an important theoretical consideration, that will be developed in the following. Consider the case when the tree-level neutrino mass-matrix is very suppressed or zero. We expect...
that the radiative corrections will provide us with non-zero mass-matrix; let us say, for definiteness, of the order of \( g^2/(4\pi)^2 \times m^2/M \), where \( g \) is some order-one gauge coupling. Thus, the tighter constraint depicted in Fig. 2, the one denoted ‘\( \nu_L \) mass too large’–can be relaxed, but only by a couple of orders of magnitude. This implies that the sterile (i.e., heavy) neutrino exchange contribution to \( 0\nu2\beta \) can have a dominant role only in a limited region of the parameter space, when the sterile neutrinos are not too heavy. Stated differently, the lighter the sterile neutrinos, the less problematic is to reconcile a dominant role of the sterile neutrinos for the \( 0\nu2\beta \) transition with the values of the masses of the ordinary neutrinos.

3.3 Departing from the naive expectations for neutrino masses

We are interested to the cases when the naive expectation for the light neutrino mass, \( M_D^T M_R^{-1} M_D \sim m^2/M \), typical of Type I seesaw, does not hold. Thus, we first proceed in Sect. 3.3.1 by a mathematical analysis of the vanishing seesaw condition \( M_D^T M_R^{-1} M_D = 0 \) and then we classify in Sect. 3.3.2 which are the perturbations of this condition that permit us to have neutrino masses much smaller than suggested by this naive expectation.

3.3.1 Solving the condition \( M_D^T M_R^{-1} M_D = 0 \): Light-neutrino masses = 0

We describe here a direct procedure to find the non-trivial solutions to the matricial condition \( M_D^T M_R^{-1} M_D = 0 \). Consider the three flavor scenario with two \( 3 \times 3 \) matrices \( M_D \) and \( M_R \), where \( M_R \) is assumed to be invertible. Using a suitable bi-unitary transformation, we go to the basis where the Dirac mass matrix is diagonal, i.e., \( M_D = \text{diag}(q, n, m) \). This basis is very convenient but only to solve the condition \( M_D^T M_R^{-1} M_D = 0 \); eventually, we have to return to the original flavor basis, where the weak interactions and the charged lepton mass matrix are diagonal. The condition \( M_D^T M_R^{-1} M_D = 0 \) is compatible with an invertible matrix \( M_R \) if \( q = n = m = 0 \), that is the trivial solution, but also if one diagonal element in \( M_D \) is non-zero; all other cases are excluded.\(^4\) Let the non-zero element be the third one, i.e., \( q = n = 0 \) and \( m \neq 0 \): We have to satisfy \( (M_R^{-1})_{33} = 0 \). This means that the \( 2 \times 2 \) block including \((M_R)_{11}, (M_R)_{12}\) and \((M_R)_{22}\) has zero determinant, i.e.,

\[
(M_R)_{11} (M_R)_{22} - (M_R)_{12}^2 = 0. \tag{33}
\]

Rotating this \( 2 \times 2 \) block of \( M_R \) into a diagonal form, it has one diagonal element equal to zero. We conclude that, in a given basis, we are dealing with the following Lagrangian including the Dirac and the Majorana masses:

\[
\mathcal{L} = \frac{1}{2} \left( \nu_L, \nu_{L2}, \nu_{L3}, N_{L1}, N_{L2}, N_{L3} \right) \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & M_1 & M_1 \\ 0 & 0 & 0 & M_2 & M_3 & M_3 \\ 0 & 0 & m & M_1 & M_3 & M_4 \\ \end{array} \right) \left( \begin{array}{c} \nu_{L1} \\ \nu_{L2} \\ \nu_{L3} \\ N_{L1} \\ N_{L2} \\ N_{L3} \end{array} \right) \tag{34}
\]

where \( M_1 \) and \( M_2 \) are non-zero since \( \det(M_R) = -M_2^2 M_2 \), while \( M_3 \) and \( M_4 \) are free parameters.

Evidently, the characteristic of this \( 6 \times 6 \) matrix is 3: there are three null eigenvalues. More in details, we see that operating a rotation in the \( \nu_{L3} \) and the \( N_{L1} \) fields, we can transform it into a mass matrix where the new ‘Dirac’ block is just zero. The physical meaning of these mathematical results is that the condition \( M_D^T M_R^{-1} M_D = 0 \) always implies that the mass matrix of the light neutrinos is zero to all order, and that this condition can be realized in a non-trivial manner only arranging for a ‘large’ mixing (i.e., order \( m/M \)) between the left and the right neutrinos.

(This conclusion has been derived previously using a systematic expansion of the neutrino mass matrix \( \text{90} \) \( \text{93} \).) From the above proof, it is easy to understand that, up to change of basis, the previous non-trivial solution of the condition \( M_D^T M_R^{-1} M_D = 0 \) is the most general one.

\(^4\)If \( q, n \) and \( m \) are all non-zero, we immediately find \( M_R^{-1} = 0 \). If, e.g., only \( q \) and \( n \) are non-zero, we have \((M_R^{-1})_{11} = (M_R^{-1})_{12} = (M_R^{-1})_{22} = 0\), which implies \( \det(M_R^{-1}) = 0 \), that is again incompatible with the existence of the inverse of \( M_R \). In mathematical terms, recalling that the characteristic of a matrix is basis invariant, we conclude that \( M_D \) has characteristic 0 or 1.
3.3.2  Perturbing the condition \( M_D^T M_R^{-1} M_D = 0 \): Light-neutrino masses \( \neq 0 \)

Here, we perturb both \( M_D \) and \( M_R^{-1} \), maintaining the first one diagonal. From the previous discussion, it is pretty evident that to satisfy the vanishing seesaw condition \( M_D^T M_R^{-1} M_D = 0 \) with an invertible \( M_R \), one should have at most one non-zero diagonal element in \( M_D \). Hence, we want to keep only one large diagonal element in the Dirac mass matrix; while the other two diagonal elements appear due to perturbation: In formulae, we write

\[
M_D = m \, \text{diag}(\epsilon_1, \epsilon_2, 1),
\]

with \( \epsilon_1 \) and \( \epsilon_2 \ll 1 \). In the following, we denote by \( \epsilon \) a small parameter, which we will use to explicitly tune the smallness of the light neutrino masses. We will show how it is possible to organize the elements of \( M_R^{-1} \) in powers of \( \epsilon \) in order to maintain a special suppression of the light neutrino mass matrix. To simplify the notation, we refrain from writing explicitly the coefficients of \( \mathcal{O}(1) \) of the mass matrices \( M_D, M_R^{-1} \) and \( M_\nu = -M_D^T M_R^{-1} M_D \), but we will use the symbol \( \mathcal{O}(1) \) to keep track of this simplification; i.e., to say, in each of the matrix elements of \( M_D, M_R^{-1} \) and \( M_\nu \), we show only the leading order. In short, in the following formulae we emphasize the necessary suppressions of the matrix elements in powers of the small parameter \( \epsilon \). We identified 3 main cases:

**Case A:** Consider the following Dirac and Majorana mass matrices:

\[
M_D \overset{\mathcal{O}(1)}{=} m \, \text{diag}(0, \epsilon, 1) \quad M_R^{-1} \overset{\mathcal{O}(1)}{=} M^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \epsilon \end{pmatrix} \Rightarrow M_\nu \overset{\mathcal{O}(1)}{=} \frac{m^2}{M} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 & \epsilon \\ 0 & \epsilon & \epsilon \end{pmatrix} .
\]

(The elements (1-2) and (1-3) of \( M_R^{-1} \) could be much smaller without affecting the argument.) The analysis of this case essentially reduces to analysis of the \( 2 \times 2 \) matrices. This case yields one massless and two massive light neutrinos and will be discussed in details later, being a prototypical case.

**Cases B:** A similar situation is realized for the following mass matrices:

\[
M_D \overset{\mathcal{O}(1)}{=} m \, \text{diag}(\epsilon, \epsilon, 1) \quad M_R^{-1} \overset{\mathcal{O}(1)}{=} M^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \epsilon \end{pmatrix} , \quad M^{-1} \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix} , \quad M^{-1} \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix} , \quad M^{-1} \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon \end{pmatrix} ,
\]

that correspond to the following light neutrino mass matrices:

\[
M_\nu \overset{\mathcal{O}(1)}{=} \frac{m^2}{M} \begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \end{pmatrix} , \quad \frac{m^2}{M} \begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \end{pmatrix} , \quad \frac{m^2}{M} \begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \end{pmatrix} .
\]

The analysis of these cases is pretty similar to the analysis of the previous one. It is easy to see that, for all of them:

1. The elements of the neutrino mass matrix are at most of the order of \( \epsilon \). Thus all neutrino masses are more suppressed than the what naive seesaw formula would suggest.

2. However, the determinant of the light neutrino mass matrix is \( \mathcal{O}(\epsilon^4) \). This essentially implies that the lightest neutrino mass is order \( \epsilon^2 \), i.e., very small.

Both features of Case B are in common with those of Case A, where the lightest neutrino mass is just zero.

**Case C:** Finally we consider an interesting case, which is in favor of non-suppressed lightest neutrino mass, i.e.,

\[
M_D \overset{\mathcal{O}(1)}{=} m \, \text{diag}(\epsilon^2, \epsilon, 1) \quad M_R^{-1} \overset{\mathcal{O}(1)}{=} M^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \epsilon \\ 1 & \epsilon & \epsilon^2 \end{pmatrix} \Rightarrow M_\nu \overset{\mathcal{O}(1)}{=} \frac{m^2}{M} \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & \epsilon \end{pmatrix} .
\]

Now the elements of the light neutrino mass matrix are at most of the order of \( \mathcal{O}(\epsilon^2) \) (that is the same as before up to the redefinition \( \epsilon^2 \rightarrow \epsilon \)) but the determinant of the mass matrix is \( \mathcal{O}(\epsilon^6) \). Hence, depending on \( \epsilon \), it is possible to have a lightest neutrino mass, which is not small a priori.
Remark Before passing to the discussion of the previous mass matrices, we note that the presence of a very light
(or just massless) neutrino, common to Cases A and B— but not necessarily to Case C— could be tested experimentally
by future cosmological measurements [84, 85]. In fact, in the case of normal (resp., inverted) hierarchy, we would
expect that the sum of neutrino mass is (resp., twice) the atmospheric mass scale \( \sqrt{\Delta m_{\text{atm}}^2} \approx 50 \text{ meV} \).

3.3.3 Quantification of the fine-tuning and lower bound on \( \epsilon \)

The common feature of the neutrino mass matrices classified in the above section is that they are smaller than
suggested by the naive seesaw formula; using the symbols as in Sects. 3.2.2 and 3.3.2,

\[
M_\nu \sim \epsilon \frac{m^2_M}{M},
\]

where \( \epsilon \) is a small parameter. The question arises, whether such a structure is stable under radiative corrections.

It has been remarked in [71] that, for non-supersymmetric Type I seesaw, the decoupling of two heavy neutrinos
with different masses \( M_1 \) and \( M_2 \) produces a correction to the neutrino mass matrix of the order of

\[
\delta M_\nu \sim \frac{g^2}{(4\pi)^2} \frac{m^2_M}{M} \log(M_1/M_2),
\]

where \( g \) is a gauge or Higgs coupling, since the renormalization group evolution of the effective operator differs
from the evolution of the Yukawa couplings (or Dirac mass), as shown in [97]. This can be seen as a minimum
natural size of the coefficient in Eq. 40,

\[
\epsilon > \frac{g^2}{(4\pi)^2} \sim 10^{-2},
\]

unless we want to accept very fine-tuned mass matrices, an unattractive possibility that we could however consider,
if the data should force us to do so: see Sect. 3.4.7 for a discussion.

Besides the possibility to enforce \( M_1 = M_2 \), by an (approximate) global symmetry [71, 98] to put to zero the
radiative correction of Eq. 41, there are also other cases when the resulting condition on \( \epsilon \) (Eq. 42) can be, if not
avoided, at least relaxed. First, it is known [99] that in supersymmetry, the effective operator and the neutrino
mass matrix receive the same radiative correction, so that the conclusion of Eq. 41 does not hold. Second, also
in non-supersymmetric model, heavy neutrinos with masses below the electroweak scale, \( M < M_{\text{ew}} \), will not give
logarithmic corrections, but smaller polynomial corrections only:

\[
\delta M_\nu \sim \frac{g^2}{(4\pi)^2} \frac{m^2_M}{M} \frac{M^2}{M_{\text{ew}}^2}
\]

\[\text{For Case C, this requires a redefinition of } \epsilon^2 \rightarrow \epsilon.\]
simply because the electroweak physics should decouple. These corrections can be attributed to finite diagrams, where the usual tree level operator for neutrino mass is dressed by $Z$ (or by higgs boson) exchanges (see Fig. 3). The condition of radiative stability of the tree level neutrino mass matrix, $M_\nu > \delta M_\nu$ bounds $\epsilon$ from below:

$$\epsilon > (M/1 \text{ TeV})^2.$$

The discussions in this section can be summarized as follows,

$$\epsilon > \begin{cases} 
(M/1 \text{ TeV})^2 & \text{if } M < M_{\text{ew}} \\
10^{-2} & \text{if } M > M_{\text{ew}}
\end{cases}$$

where $M_{\text{ew}} \approx 100 \text{ GeV}$; as we see in the following, it is the regime where the corrections are smaller, $M < M_{\text{ew}}$, the one in which we will be mostly interested.

### 3.4 A dominant role of heavy neutrino exchange in $0\nu2\beta$

In this section we discuss how it is possible that heavy neutrino exchange provides us with a dominant contribution to $0\nu2\beta$ within Type I seesaw models. First, we state precisely in Sect. 3.4.1 what is the role of the heavy neutrino exchange for the amplitude of $0\nu2\beta$, and we consider in Sect. 3.4.2 the case when this contribution is large building on the mass matrices discussed in Sect. 3.3. Then, we pursue a detailed investigation of this case for two flavors (Sect. 3.4.3) and as well as for three flavor mass matrices (Sect. 3.4.4). We discuss the compatibility with fine-tuning issues in Sect. 3.4.5 and conclude in Sect. 3.4.6 with numerical example.

#### 3.4.1 The amplitude of $0\nu2\beta$ and the role of heavy neutrino exchange

The mixing between the heavy sterile neutrino state $N_L$ and the standard model neutrino state $\nu_L$ is caused by the Dirac mass matrix $M_D$ (see Eq. 16). As clear from Fig. 4, the amplitude for the $0\nu2\beta$ process is proportional to the following factor stemming from the vertices and the propagator,

$$A = \left[ \frac{1}{\hat{p}} \hat{M}_D^\dagger \text{diag}\left( \frac{1}{\hat{p} - M_i} \right) \hat{M}_D^* \frac{1}{\hat{p}} \right]_{ee}$$

where we consider the expressions in leading (second) order in $M_D$ and $\hat{M}_D$ is the Dirac mass matrix in the basis where the heavy neutrinos are diagonal, see Eq. 28. Reminding that this expression in sandwiched between chiral projectors, its contribution to the $0\nu2\beta$ amplitude is just,

$$A^* = \left[ \frac{M_D^\dagger W^* \text{diag}\left( \frac{1}{M_i} \left( \frac{1}{p^2 - M_i^2} - \frac{1}{p^2} \right) \right) W^\dagger M_D^*}{p^2 - M_D^2 M_R^{-1} M_R^{-1*} M_R^{-1} M_D^* + \mathcal{O}(M_R^{-5})} \right]_{ee}$$

where we have used the diagonalizing relation $M_R^{-1} = W \text{diag}(M_i^{-1}) W^T$. The first term in brackets, evidently due to the exchange of light Majorana neutrinos, is the usual one; the second one is the effect of heavy neutrino exchange in which we are interested. For real $M_R$, it can be written simply (up to the sign) as,

$$\left( M_D^T M_R^{-3} M_D \right)_{ee}$$

a quantity with dimension of an inverse mass. In the following, we will refer often to this as a ‘contact term’, having in mind the nature of the operator that induces the $0\nu2\beta$ transition. Using Eqs. 21 and 26, it is easy to verify that

\footnote{When $M \to 0$ we enforce lepton number conservation in the model; this agrees with the fact that Eq. 43 vanishes in this limit.}
Eq. 48 coincides with Eq. 5 fully; but the new expression is more convenient for the subsequent theoretical analysis of Type I seesaw model.

In the following, we will speak of a ‘saturating contribution’ [1] when the heavy neutrino exchange dominates the transition. In formulas and using the notations of Sect. 2.2, this implies that the (absolute value) of the contact term in Eq. 49 is equal to \(1/\sqrt{K_{0\nu}T_{1/2}}\), or, in numerical terms

\[ |(M_D^T M_R^{-3} M_D)_{ee}| = 7.6 \times 10^{-9} \text{ GeV}^{-1} \times \left(\frac{363}{M_N}\right) \times \left(\frac{1.9 \times 10^{25} \text{ yr}}{T_{1/2}}\right)^{1/2} \]  

(50)

3.4.2 The case for a large contact term

Being ready to consider the special class of neutrino mass matrices classified in Sect. 3.3, it is possible to understand the cases when the naive expectations of Type I seesaw discussed in Sect. 3 (and in particular Sect. 3.2.2) do not work. These special cases are of great phenomenological interest, especially for neutrinoless double beta decay, but have also some theoretical interest, in view of the fact that they are based on the simplest renormalizable extension of the standard model, that includes massive neutrinos.

In formal terms, and using the symbols as in Sects. 3.2.2 and 3.3.2, we are considering the possibility that the neutrino mass matrix is smaller than suggested by the seesaw formula and at the same time the contribution of heavy Majorana neutrino states in \(0\nu2\beta\) process,

\[ (M_D^T M_R^{-3} M_D)_{ee} = \kappa \frac{m^2}{M^3} \]  

(51)

where \(\kappa\) is a coefficient which depends on the specific particle physics model, and will be determined later in this section for the cases of interest. We are particularly interested in heavy neutrino masses that saturate the \(0\nu2\beta\) experimental bound [1],

\[ M = 16 \text{ TeV} \times \left(\frac{T_{1/2}}{1.9 \times 10^{27} \text{ yr}}\right)^{1/6} \left(\frac{M_N \times \kappa}{363 \times 1}\right)^{1/3} \left(\frac{m}{174 \text{ GeV}}\right)^{2/3} \]  

(52)

where the nuclear matrix element and the half-life apply both to \(^{76}\text{Ge}\) (see Sect. 2). It is important to note that by reducing the mass scales, the need of fine-tuning (i.e., too small \(\epsilon\)) diminishes; indeed Eq. 40 and 51 are left unchanged by the scaling

\[
\begin{cases}
M \rightarrow \alpha \times M \\
m \rightarrow \alpha^{3/2} \times m \\
\epsilon \rightarrow \alpha^{-1} \times \epsilon
\end{cases}
\]  

(53)

where \(\alpha < 1\) implies that \(m\) will be smaller than 174 GeV, maintaining perturbativity of the Yukawa couplings. (Actually, the fact that decreasing \(m\) and \(M\) we need less fine-tuning can be understood also from an inspection of Fig. 2) Also note the very mild dependency of the heavy neutrino mass scale in Eq. 52 on the true value of the half-life; if the central value found by Klapdor and collaborator is confirmed, this would change by only 3\%, but even if the true lifetime should turn out to be 100 times larger, i.e., \(T_{1/2} = 1.9 \times 10^{27} \text{ yr}\), the mass \(M\) in Eq. 52 would just be doubled. The dependence on the matrix elements is also quite mild. Finally, even stretching the perturbativity condition on \(Y_D = m/(174 \text{ GeV})\), from \(Y_D < 1\) to \(Y_D^2/(4\pi) < 1\), the mass \(M\) would increase only by a factor of 2.3.

3.4.3 Dominating heavy-neutrino contribution with two flavors

Here we analyze a prototypical two flavor case. We start from rather specific mass matrices, given in the basis where the Dirac mass matrix is diagonal, and then switch to the flavor basis (where the charged current interactions and the charged lepton masses are diagonal). We show that, in this way, we can obtain information on the structure of the contact term.
Structure of the contact term in the basis where $M_D$ is diagonal

Suppose that we have the $2 \times 2$ matrices

$$M_R \overset{\mathcal{O}(1)}{=} M \begin{pmatrix} \epsilon & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_D \overset{\mathcal{O}(1)}{=} m \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (54)

(compare with Sect. 3.3.2, Case A, Eq. 36). We derive immediately the neutrino mass matrix and the contact term:

$$M_D^T M_R^{-1} M_D \overset{\mathcal{O}(1)}{=} \frac{e m^2}{M} \begin{pmatrix} \epsilon & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_D^T M_R^{-3} M_D \overset{\mathcal{O}(1)}{=} \frac{m^2}{M^3} \begin{pmatrix} \epsilon & \epsilon \\ \epsilon & 1 \end{pmatrix}$$  \hspace{1cm} (55)

Using the results of [90], the next higher order contribution to the neutrino mass is found to be

$$\delta M_\nu \overset{\mathcal{O}(1)}{=} \frac{m^4 \epsilon}{M^3} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^3)$$  \hspace{1cm} (56)

This contribution is suppressed as compared to the leading contribution by a factor $m^2/M^2 \ll 1$; for $m = 174$ GeV and $M \sim 16$ TeV, the suppression is in fact $10^{-4}$. Hence, we do not take into account the sub-leading contribution any further.

The limit $\epsilon \to 0$

Keeping only the leading order entries in $\epsilon$, the light neutrino mass matrix becomes:

$$M_\nu \overset{\mathcal{O}(1)}{=} \frac{e m^2}{M} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^2) \quad \text{and} \quad M_D M_R^{-3} M_D \overset{\mathcal{O}(1)}{=} \frac{m^2}{M^3} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon)$$  \hspace{1cm} (57)

The interpretation is the following:

\footnote{This bound can be obtained as follows: denoting the heavy sterile neutrino contribution to $0\nu\beta\beta$ as $m^2/M^3 = k$, we find $m^2/M^2 = kM$; thus, we maximize $m^2/M^2$ when (1) $k$ is as large as possible, (i.e., when we saturate the experimental upper bound) and when (2) $M$ is as large as possible, that happens when $m = 174$ GeV and $M = 16$ TeV, as seen in the previous section.}
1. The parameter \( \epsilon \) can be used to diminish the value of neutrino masses to the desired value. As an example, for \( m = 174 \text{ GeV} \) and \( M = 16 \text{ TeV} \), eV-neutrino masses can be generated for \( \epsilon \sim 10^{-9} \); for smaller values of \( m \) and \( M \), we need less fine tuning, i.e., the parameter \( \epsilon \) increases.

2. Generically, we expect two neutrino masses of similar order and with large mixing (note that the first statement is basis independent, while the second is not; we are still in the basis where the Dirac mass matrix is diagonal).

3. The size of the contact term has been made independent from the size of neutrino masses. In other words, the contact term can take whatever value.

By a suitable choice of phases and restoring the coefficients of the order of 1 from here on, we see that the first non-trivial terms in \( \epsilon \) of the mass matrix in Eq. 57 can be rewritten as:

\[
M_{\nu} \approx \left( \begin{array}{c} 0 \\ \sqrt{m_1 m_2} \\ m_2 - m_1 \end{array} \right)
\]

where \( 0 \leq m_1 \leq m_2 \) are the mass eigenstates; think e.g., to the solar doublet.

**Structure of the contact term in the flavor basis** Let us then go to the flavor basis for the neutrino mass and see what happens to the contact term (note that both neutrino mass matrix and the contact term rotate in the same manner when we change basis). First, we apply a rotation by an angle

\[
\tan \theta = \frac{m_1}{m_2}
\]

and, in this way, go to the basis where the neutrino mass matrix reduces to the diagonal form \( \text{diag}(-m_1, m_2) \). Then we perform a second rotation with a solar mixing angle \( \theta_\odot \) to go to the flavor basis, and include a Majorana phase \( \phi \) to consider the most general mass matrix. In this way, we recover the well-known expression for ee-element of the neutrino mass matrix, relevant to the 0\( \nu \)2\( \beta \):

\[
(M_{\nu})_{ee} = \sin^2 \theta_\odot m_2 - \cos^2 \theta_\odot m_1 e^{i2\phi}
\]

and in the same basis (flavor basis), the contact term is

\[
(M_{\nu}^T M_{\nu}^{-1} M_D)_{ee}^{(F1.)} = \xi \frac{m^2}{M^3} \left( \frac{\sin \theta_\odot \sqrt{m_2 + \cos \theta_\odot \sqrt{m_1}} e^{i\phi}}{m_1 + m_2} \right)^2
\]

where \( \xi \) is a factor of the order of 1 (further discussed below) and in the above we used the freedom to redefine the phase \( \phi \) as \( \phi \rightarrow \phi + \pi \). Thus, in general the latter is non zero and it is correctly estimated dimensionally as \( m^2/M^3 \). This concludes the proof of the principle in the case of two flavors.

Before passing to the three flavor case, let us consider a couple of particular noticeable two flavor cases:

**Additional suppression of light neutrino contribution to 0\( \nu \)2\( \beta \)** An interesting special case is when we arrange \((M_{\nu})_{ee} = -(M_D^T M_R^{-1} M_D)_{ee} = 0\), a condition that can be realized in normal mass hierarchy but not in inverted mass hierarchy, as noted in [81, 82]. From Eq. 59 we see that for \((M_{\nu})_{ee} = 0\), we need to have \( \phi = 0 \) or \( \pi \) and

\[
\begin{cases}
  m_2 = \sqrt{\Delta m_2^2 \cos^2 \theta_\odot \over \cos 2\theta_\odot} \\
  m_1 = \sqrt{\Delta m_2^2 \sin^2 \theta_\odot \over \cos 2\theta_\odot}
\end{cases}
\]

i.e., \( m_1 \approx 4.5 \text{ meV} \) as calculated with \( \Delta m_2^2 = 7.6 \times 10^{-5} \text{ eV}^2 \) and \( \theta_\odot = 34^\circ \). If \( \phi = \pi \) the contact term is also suppressed, if \( \phi = 0 \), instead, it is unsuppressed. The last possibility offers a very explicit example of a case when the 0\( \nu \)2\( \beta \) transition is *entirely* due to heavy neutrino exchange. In the language of Schechter-Valle 'theorem', we
can say that this is one case when the black box of [100], that apparently should connect the double beta decay transition to non-zero neutrino masses, actually does not yield any contribution to the ee-element of the Majorana neutrino mass matrix.\footnote{In mathematics, such a case should be called a counterexample, but it should be understood that the Schechter-Valle 'theorem' is just an illustration of typical expectations and has no quantitative aims. In last analysis, and despite the evident fact that an observation of the 0ν2β decay would imply that electronic lepton number is broken, a true understanding of the connection between neutrino masses and neutrinoless double beta decay rate is possible only within specific extensions of the standard model. See [54, 55, 101] for more relevant examples and discussion.}

**Suppression of heavy neutrino contribution to 0ν2β** We encountered in the previous discussion a case ($\phi = \pi$ and $m_1 \approx 4.5$ meV) where the heavy neutrino contribution to 0ν2β is suppressed. The other case when this happens, as we see from Eq. [60] is when $\xi = 0$. In order to discuss the viability of this case, we need to restore the $O(1)$ coefficients in Eq. [54]. First, we note that the expression for $M_D$ can be regarded as a definition of $m \neq 0$ and $\epsilon \neq 0$, and thus can be left unchanged. Second, we write $M_R$ as

$$M_R = M \begin{pmatrix} e & 1 \\ 0 & b \end{pmatrix}$$

where we introduced two free parameters and provided incidentally a precise definition of the parameter $M \neq 0$. It is easy to show that

$$M_D M_R^{-1} M_D = \frac{e m^2}{M} \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix} + O(\epsilon^2) \quad \text{and} \quad M_D M_R^{-3} M_D = \frac{m^2}{M^3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \end{pmatrix} + O(\epsilon)$$

which is a more precise statement than Eq. [67]. This calculation shows that not only the mass scales, but also the order-one coefficient of the contact term is completely independent from those entering the neutrino mass matrix. In other words, we have the freedom to take the limit $\xi \to 0$ in Eq. [60] simply letting $b \to 0$ in Eq. [62]. Evidently, this limit amounts to enforcing an approximate global symmetry in the heavy sterile neutrino mass matrix, which assumes a ‘quasi Dirac’ structure [56, 71].

### 3.4.4 Dominating heavy-neutrino contribution with three flavors

A large part of the discussion and conclusions for the two flavor example can be repeated for the three flavor case. It is easy to show that in all cases A, B, C of Sect. 3.3.2 the leading part in $\epsilon$ of the contact term has the form

$$M_D^T M_R^{-3} M_D = \frac{m^2}{M^3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis where Dirac masses are diagonal. Here, $\xi$ is a combination of the order-one coefficients of the heavy Majorana and Dirac mass matrices. It is easy to show by direct calculation that this combination can be made independent from the coefficients order-one that regulate the light neutrino mass matrix.

However, we need to determine the contact term in *the flavor basis*. We begin from the light neutrino mass matrix $M_\nu = -M_D^T M_R^{-1} M_D$ in the basis where the Dirac mass matrix is diagonal; then, we change to mass basis by the diagonalization $O^T M_\nu O = \text{diag}(m_i)$; finally, we reach the flavor basis simply including the leptonic mixing matrix $U$. The same changes of basis apply to the contact term as well; thus, the quantity that regulates the heavy neutrino exchange contribution to 0ν2β is given by,

$$(M_D^T M_R^{-3} M_D)_{ee}^{(F_1)} \equiv (U^* O^T M_D^T M_R^{-3} M_D O U^\dagger)_{ee}$$

By comparing with Eq. [51] we see that the order one coefficient that correct the naive estimation for the contact term is given by

$$\kappa = \xi \times \varphi^2, \quad \text{with} \quad \varphi = \sum_{i=1}^{3} U^*_{ei} O_{3i}$$

(65)
The quantity $\varphi$ has modulus between 0 and 1, since it can be thought of as a scalar product between two unit vectors. In practice, all we need is to calculate is the matrix $O$, since the moduli of the leptonic mixing matrix $U_{\ell i}$ are known experimentally precisely enough: $|U_{e2}/U_{e1}| = \tan \theta_{12}$ and $|U_{e3}| = \sin \theta_{13}$, with $\theta_{12} \approx 34^\circ$ and $\theta_{13} \approx 8^\circ$. In the rest of this section, we provide its evaluation for all relevant cases. In the cases A and B, discussed in Sect. 3.3.2 working in the leading order $\epsilon_j$, and upon suitable rotations that do not modify the contact term, the neutrino mass matrix has the same structure

$$- M_D^T M_R^{-1} M_D \approx O(\epsilon) \frac{m^2}{M} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right)$$  \hspace{1cm} (67)$$

Again, it is easy to verify that the higher order contribution in $m/M$ can be safely neglected. As already remarked, these cases produce a very suppressed lightest neutrino mass; if we assume $m_{\text{min}} = m_1$, we consider normal mass hierarchy; if instead $m_{\text{min}} = m_3$, we consider inverted mass hierarchy. The later case is in fact already treated; it corresponds in all details to the calculation of the previous section, and the result of Eq. (60) are almost (up effects due to $\theta_{13}$) unchanged. In the basis where Dirac couplings are diagonal, the first case instead requires to identify the following light neutrino mass matrix,

$$- M_D^T M_R^{-1} M_D = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \sqrt{m_2 m_3} \\ 0 & \sqrt{m_2 m_3} & m_3 - m_2 \end{array} \right)$$  \hspace{1cm} (68)$$

To get the contact term in the flavor basis, one can repeat the same steps. In summary, we get the simple and explicit result for the cases A and B (for case B, considering the neutrino mass matrix up to $O(\epsilon)$)

$$(M_D^T M_R^{-3} M_D)^{(\text{Fl})}_{ee} = \frac{\epsilon^2 m^2}{M^2} \times \left\{ \begin{array}{ll} \frac{U_{e2} \sqrt{m_2 + U_{e3} \sqrt{m_3}}}{m_2 + m_3} & \text{with normal hierarchy} \\
\frac{U_{e3} \sqrt{m_2 + U_{e2} \sqrt{m_3}}}{m_1 + m_2} & \text{with inverted hierarchy} \end{array} \right.$$  \hspace{1cm} (69)$$

where the superscript “Fl” represents the flavor basis. For $\Delta m^2_{12} = 7.7 \times 10^{-5}$ eV$^2$, $\Delta m^2_{23} = 2.4 \times 10^{-3}$ eV$^2$, $\theta_{12} = 34^\circ$, $\theta_{23} = 42^\circ$ and $\theta_{13} = 8^\circ$, the modular of the order one coefficient in the bracket (i.e., $\varphi^2$ in Eq. (60) goes from about 0.12 to 0.007 in the case of normal hierarchy, and from about 0.94 to about 0.03 in the case of inverted hierarchy, depending on the Majorana phases. Correspondingly, we have the following light neutrino contribution, $|(M_{\nu})_{ee}| = |m_3 U_{e3}^2 - m_2 U_{e2}^2|$, for normal (resp., inverted) mass hierarchy; note that for normal hierarchy we can have an almost complete cancellation, i.e., an insignificant contribution to $0\nu2\beta$ from light neutrino exchange, for certain choices of the Majorana phases.

The calculations of case C of Sect. 3.3.2 are slightly more complicated, since the neutrino mass matrix at leading order in $\epsilon_j$ and in the basis where the Dirac couplings are diagonal is now

$$M_{\nu} \approx O(1) \frac{m^2 \epsilon^2}{M} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) + O(\epsilon^3)$$  \hspace{1cm} (70)$$

| sub-case | 0 meV | 3 meV | 10 meV | 30 meV | 100 meV | hierarchy |
|----------|-------|-------|--------|--------|--------|-----------|
| $m_1 < 0$ | .00-.57 | .00-.90 | .00-.97 | .15-.98 | .17-.98 | normal |
| $m_2 < 0$ | .08-.57 | .00-.90 | .00-.97 | .00-.98 | .00-.98 | inverted |
| $m_3 < 0$ | .08-.34 | .00-.43 | .00-.54 | .00-.69 | .00-.78 | inverted |
| $m_1 > 0$ | .00-.99 | .01-.10 | .00-.96 | .00-.88 | .00-.82 | inverted |
| $m_2 > 0$ | .08-.57 | .00-.97 | .00-.98 | .00-.98 | .00-.98 | normal |

Table 1: Maximum and minimum value of the numerical coefficient $|\varphi|$ for matrices belonging to the case C and in all $2 \times 3$ sub-cases (notation as in Appendix [A]), Eq. (105), calculated for 5 values of the lightest neutrino mass, ranging from zero to 100 meV.
However, it is still possible to obtain analytical expression for $\varphi$ as defined in Eq. 66 applying the results given in Appendix A for the matrix $O$. The explicit expressions are not particularly illuminating and thus we will omit them here, providing some sample calculation in Table 1. From this table and from the calculations we see that

1. For any hierarchy and for any value of the lightest neutrino mass $m_{\text{min}}$, the numerical coefficient $\varphi$ of the contact term can reach values close to 1.

2. There is not necessarily a suppression of this coefficient both for normal hierarchy as well as for inverted hierarchy.

3. The special possibility realized for certain masses in normal mass hierarchy when the contribution from light neutrino exchange is very small or just zero, is compatible with a large coefficient $\varphi$, i.e., it does not contradict the hypothesis that $0\nu\beta\beta$ is completely due to heavy neutrino exchange.

### 3.4.5 Upper bound on heavy neutrino mass $M$ and $\epsilon$

Having confirmed the mathematical consistency between Type I seesaw and a large contribution from heavy neutrinos to $0\nu\beta\beta$ transition, for rather specific Dirac and Majorana mass matrices, we now want to discuss briefly whether these scenarios run necessarily into the objection of fine-tuning, as well as we now derive the possible upper bounds on the fine-tuning parameter $\epsilon$ and the heavy neutrino mass scale $M$. We focus on non-supersymmetric models, which as discussed in Sect. 3.3.3, are more likely to encounter such an objection.

Considering the heavy neutrino contribution to $0\nu\beta\beta$ as the dominant one, we can assume that the sterile neutrino contribution saturates the present bound (or value) on the lifetime $T_{1/2} = 1.9 \times 10^{25}$ yr [1]. In terms of $m$, $M$ and $\kappa$ this can be interpreted as $\kappa m^2/M^4 = 7.6 \times 10^{-9}$ GeV$^{-1}$, where we have considered the nuclear matrix elements $M_\nu = 5.24$ and $M_N = 363$ [62]. Combining with the neutrino mass constraint, $m_{\text{min}}^2 < 0.1$ eV, which automatically ensures the subdominant contribution from the light neutrino exchange as long as $|p^2| > (120)^2$MeV$^2$, the upper bound on $\epsilon$ can be obtained as,

$$\epsilon \lesssim \kappa \left( \frac{100 \text{ MeV}}{M} \right)^2. \quad (71)$$

This condition remains unchanged if both contributions to $0\nu\beta\beta$ transition amplitude are scaled down by the same factor; this means that we will not be forced to abandon the hypothesis that the heavy neutrino contribution is the dominating one, even if the true lifetime will turn out to be lower than the present bound (or value) on the lifetime.

To recollect, the lower bound on $\epsilon$ given in Eq. [15] ensures the stability of tree level neutrino mass matrix. The above condition on $\epsilon$, combined with the lower bound on $\epsilon$, given in Eq. [12] or Eq. [44] can also be used to derive upper bound on the heavy neutrino mass scale $M$. If we use naively as the minimum value of $\epsilon$ the one given in Eq. [42] we find that $M < \sqrt{\kappa} \times 1$ GeV. However, for $M < M_{\text{ew}}$, it is clear that the right bound is obtained using Eq. [44] rather than Eq. [42]. Hence, the previously mentioned tight bound on the heavy neutrino mass $M$ is relaxed to

$$M \lesssim \kappa^{1/4} \times 10 \text{ GeV}, \quad (72)$$

which is a small subset of region indicated by Eq. [52]. This is a pretty interesting result: this is the region which automatically satisfies constraints coming from small neutrino mass, as well as from radiative stability of the tree level neutrino mass matrix; the only stringent bound applicable in this region of parameter space comes just from $0\nu\beta\beta$ process.

Note that, low mass scales of the sterile neutrinos have been considered for a variety of reasons in the literature, for instance: [102] for heavy neutrinos and naturalness; [103] for a mechanism of baryogenesis; [104] for a model of

---

9Redefining $\epsilon \rightarrow \epsilon^2$ into the light neutrino mass matrix (relevant for Case C), i.e., $M_\nu \propto m^2\epsilon^2/M$, the lower and upper bound on $\epsilon$ should be interpreted as $(M/1\text{TeV}) < \epsilon < \sqrt{\kappa}(100\text{MeV})/M$. Of course, the upper bound on $M$ remains unchanged by this re-interpretation.
heavy neutrinos ($\nu_{\text{MSM}}$) to account for baryon asymmetry of the universe and dark matter; \cite{105} for low scale leptogenesis in supersymmetry; etc. See \cite{64} for a review with a compilation of the experimental constraints from direct search experiments and earlier bound coming from from $0\nu2\beta$, but recall that in view of the revised matrix elements presented in \cite{62}, the actual bound on $0\nu2\beta$, as shown in our Fig. 1, is significantly stronger.

### 3.4.6 Numerical Example

To illustrate the discussions of the previous sections with numerical analysis, in this section we offer a numerical example, considering the normal hierarchical light neutrino masses $|m_1| < |m_2| < |m_3|$. This example clearly shows that it is possible to achieve large contact term; i.e., even when light neutrino contribution is small, the factor $\varphi$ can be very large to produce saturating contribution from the sterile neutrino states.

To proceed further, we include the coefficients of $M_D$ and $M_{R}^{-1}$ written in Eq. 39

$$M_D = m \begin{pmatrix} f \epsilon^2 & 0 & 0 \\ 0 & g \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \quad M_{R}^{-1} = \begin{pmatrix} a & b & k \\ b & c & d \epsilon \\ k & d \epsilon & e \epsilon^2 \end{pmatrix}$$

The light neutrino masses in leading order is the following,

$$M_\nu = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & \delta & \beta \\ \alpha & \beta & \gamma \end{pmatrix},$$

where, $\alpha$, $\beta$, $\gamma$ and $\delta$ satisfy the following relations in terms of $m_i$ and the free parameter $m_0$,\n
$$\alpha = k f \frac{\epsilon^2 m^2}{M} = \sqrt{-\frac{m_1 m_2 m_3}{m_0}} \tag{75}$$

$$\beta = d g \frac{\epsilon^2 m^2}{M} = \sqrt{\frac{(m_1 - m_0)(m_2 - m_0)(m_3 - m_0)}{m_0}}$$

$$\gamma = \epsilon^{-2} \frac{m^2}{M} = m_1 + m_2 + m_3 - m_0$$

$$\delta = c g^2 \epsilon^2 \frac{m^2}{M} = m_0$$

The leading order expression of the contact term $M_D^T M_{R}^{-3} M_D$ is the following,

$$M_D^T M_{R}^{-3} M_D = \frac{m^2}{M^3} ak^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{76}$$

For this specific example, we present the different numerical values of the input parameters of $M_D$ and $M_{R}^{-1}$ in Table 2.

| $M$ (GeV) | $m$ (MeV) | $\epsilon$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---------|---------|---------|-----|-----|-----|-----|-----|-----|-----|
| 5.00    | 0.935   | 0.02    | 1.00| 1.35| 0.90| 1.4576| 0.7942| 0.2898| 0.0948| 0.485 |

Table 2: The input parameters of $M_D$ and $M_{R}^{-1}$.

The light neutrino mass matrix in the Dirac diagonal basis is the following,

$$M_\nu = \begin{pmatrix} 0.251 \times 10^{-5} & 0.579 \times 10^{-3} & 0.895 \times 10^{-1} \\ 0.579 \times 10^{-3} & 0.24 & 0.269 \\ 0.895 \times 10^{-1} & 0.269 & 0.203 \end{pmatrix} \times 0.1 \text{eV} \tag{77}$$

22
where, for completeness, we have shown the higher order terms as well. For the choice of input parameters given in Table 2, the light as well as heavy neutrino masses and the solar and atmospheric mass square differences has been presented in Table 3. To go to the flavor basis, we further choose the mixing angles as $\theta_{12} = 34^\circ$, $\theta_{23} = 42^\circ$ and $\theta_{13} = 8^\circ$. The numerical values of flavor basis contact term, as well as the light neutrino contribution and the enhancement factor $\varphi$ has been shown in Table 4.

| $m_1$ (meV) | $m_2$ (meV) | $m_3$ (meV) | $\Delta m^2_{12}$ eV$^2$ | $\Delta m^2_{13}$ eV$^2$ | $M_1$ (GeV) | $M_2$ (GeV) | $M_3$ (GeV) |
|------------|------------|------------|----------------|----------------|----------|----------|----------|
| 3.95       | −9.60      | 49.9       | $7.66 \times 10^{-5}$ | $2.40 \times 10^{-3}$ | 1.99     | −4.77    | 5.02     |

Table 3: The light, heavy neutrino masses and the solar as well as atmospheric mass square differences.

and $\theta_{13}$ are below 10 GeV or so. Indeed, the mass matrix in Eq. 62 is constructed just to obey the condition of Eq. 78.

$$\alpha = \frac{\sqrt{4 + b^2}}{M} \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$$

(79)

Indeed, the mass matrix in Eq. 62 is constructed just to obey the condition of Eq. 78.

As already recalled in Sect. 3.3.3, one should consider the radiative corrections to light neutrino masses, that are potentially large. This is true for the one-loop correction estimated in 71, assuming that both sterile neutrino masses $M_2 > M_1$ are above the electroweak scale. Using again the matrix in Eq. 62 these read

$$\delta M_\nu \approx \frac{\alpha_2 - \alpha_1}{(4\pi)^2} \times M_\nu^{(1)} \times 2 \sinh^{-1}(b/2)$$

(80)

which is the same as Eq. 41 after including all the order-one factors. The different running of $M_\nu^{(1)}$ and $M_\nu^{(2)}$ is described by the order-one factor $\alpha_2 - \alpha_1 = \lambda + \frac{3}{2}g^2 + \frac{3}{2}g'^2$, and the last factor in Eq. 80 is just a compact expression of $\log(M_2/M_1)$ as a function of $b$. As anticipated in Sect. 3.2, such a correction is just a couple of orders of magnitude smaller than the naive expectation $m^2/M$ and thus it is non-negligible.

\footnote{For $\theta_{13} = 0$ and the same set of other input parameters, the enhancement factor $\varphi^2$ diminishes to 0.42, while $(M_\nu)_{ee}$ in flavor basis becomes $(M_\nu)_{ee} = 0.29$ meV.}

3.4.7 Conditions of validity and meaning of Eq. 72

In the last part of this section, we examine in depth the hypotheses that led to Eq. 72 which says that, in the absence of excessive fine-tunings, large contributions to $0\nu2\beta$ of sterile neutrinos are possible, only if their masses are below 10 GeV or so.

Consider the case when the tree-level neutrino mass $M_{\nu}^{\text{tree}}$ is zero due to a cancellation. This can be regarded as an effect of the opposite (and possibly large) contributions, due to the exchange of virtual sterile neutrinos $N_1$ and $N_2$,

$$M_{\nu}^{\text{tree}} = M_{\nu}^{(1)} + M_{\nu}^{(2)} = 0$$

(78)

where, following 71, we consider a two-dimensional system to simplify the analysis. This can be made explicit considering the mass matrix in Eq. 62 that in the limit $\epsilon \to 0$ has

$$M_{\nu}^{(1)} = -M_{\nu}^{(2)} = \frac{m^2}{M} \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \frac{1}{\sqrt{4 + b^2}}$$

(79)

As already recalled in Sect. 3.3.3, one should consider the radiative corrections to light neutrino masses, that are potentially large. This is true for the one-loop correction estimated in 71, assuming that both sterile neutrino masses $M_2 > M_1$ are above the electroweak scale. Using again the matrix in Eq. 62 these read

$$\delta M_\nu \approx \frac{\alpha_2 - \alpha_1}{(4\pi)^2} \times M_\nu^{(1)} \times 2 \sinh^{-1}(b/2)$$

(80)

which is the same as Eq. 41 after including all the order-one factors. The different running of $M_\nu^{(1)}$ and $M_\nu^{(2)}$ is described by the order-one factor $\alpha_2 - \alpha_1 = \lambda + \frac{3}{2}g^2 + \frac{3}{2}g'^2$, and the last factor in Eq. 80 is just a compact expression of $\log(M_2/M_1)$ as a function of $b$. As anticipated in Sect. 3.2, such a correction is just a couple of orders of magnitude smaller than the naive expectation $m^2/M$ and thus it is non-negligible.
At this point, we can focus on the case of interest. When we assume that the sterile neutrinos saturate the 0ν2β bound, the combination of parameters \( \varphi^2 b m^2/M^3 \) is fixed, see Eq. (66). This permits us to rewrite Eq. (80) as

\[
\delta M_\nu \approx 100 \text{ eV} \left( \frac{M}{100 \text{ GeV}} \right)^2 \left( \frac{\varphi^2 b m^2/M^3}{7.6 \times 10^{-9} \text{ GeV}^{-1}} \right) \left( \frac{\lambda(b)/\varphi^2}{0.2} \right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\] (81)

where \( \varphi \) is discussed in Sect. 3.4.3 and

\[
\lambda(b) = \frac{\sinh^{-1}(b/2)}{b/\sqrt{1 + (b/2)^2}}
\] (82)

is a decreasing function, with \( \lambda(0) = 1/2 \).

Note that when \( b \) grows, the mass of the lighter sterile neutrino \( M_1 = M(\sqrt{1 + (b/2)^2} - b/2) \) decreases; but this has to be above the electroweak scale to ensure the validity of the radiative corrections estimated in [61]. The numerical result of Eq. (81) means that the one-loop correction \( \delta M_\nu \) is just proportional to the heavy neutrinos contribution to the 0ν2β amplitude.

Formally, it is possible to replace the fine-tuning condition operated at tree-level, Eq. (78), with a fine-tuning involving also the radiative contributions. For instance, we can include the logarithmic corrections of [71], imposing the condition

\[
M_\nu^{\text{loop}} = M_\nu^{(1)} \left( 1 + \frac{\alpha_1}{(4\pi)^2} \log \frac{M_2}{M_1} \right) + M_\nu^{(2)} \left( 1 + \frac{\alpha_2}{(4\pi)^2} \log \frac{M_2}{M_1} \right) = 0
\] (83)

One can rewrite it approximatively as

\[
M_\nu^{(1)} + M_\nu^{(2)} \approx -(M_\nu^{(2)} - M_\nu^{(1)})(\alpha_2 - \alpha_1) \log \frac{M_2}{M_1} \frac{32\pi^2}{\alpha_1}
\] (84)

which emphasizes that we have to concoct a small and ad hoc tree level term to cancel the one-loop term.

Eq. (83) permits us to retain large sources of lepton number violation in the theory without contradicting the observed neutrino masses; moreover, it can be extended to include two-loop terms or, possibly, even finite corrections. While technically acceptable, we believe that Eq. (83) should be considered as an excessive fine-tuning, since it is not justified in terms of symmetries and it appears to be even more artificial than the tree level condition Eq. (78).

In the present section, guided by the phenomenological motivations described in the introduction and following a vast literature on the subject, we studied the consequences of accepting a moderate amount of fine-tuning, corresponding to Eq. (78). We could even resort to the fine-tuning condition of Eq. (83) eventually, if the data should force us to do so; e.g., if we would have evidence of the existence of sterile neutrinos of 100 GeV or 1 TeV. But, at present, we miss any convincing motivation to analyze the implications of Eq. (83) any further, and tentatively, we bar it as unlikely. If instead one is convinced on a theoretical basis that this type of conditions (either Eq. (78) or Eq. (83) should not be accepted, the conclusions are those already illustrated in Fig. 2.

### 4 Going beyond Type I seesaw

In the previous sections we have restricted the discussion of 0ν2β for the simple prototype Type I seesaw scenario. As it is well known, smallness of neutrino masses can be well explained by other seesaw scenarios as well, e.g., Type II [62, 63, 109], Type III [107, 108, 109, 110], Inverse seesaw [111, 112, 113, 114, 115], Extended seesaw [66, 67].

In the case of the Type II seesaw, it is clear that the new contribution due to doubly charged scalars in the triplet is always smaller than the neutrino mass one. This is evident from the fact that the new contribution is

\[^{11}\text{This is relevant for the works based on the assumption that sterile neutrinos with large masses provide the main contribution to 0\nu2\beta, e.g., [59]. A quantitative evaluation of the light-neutrino masses was not attempted there, though the authors seem to be aware of the potential issue, since after Eq. (39) they state “higher order (one or two loop) contributions lead to } m_\nu \neq 0\text{".}\]
again proportional to \((M_\nu)_{ee}\) element of the neutrino mass matrix, but now suppressed by the large mass of the doubly charged scalar. The situation changes in the case of the LR symmetric theory, where the right-handed triplet may enter the game since its contribution is proportional to the right-handed analog of \((M_\nu)_{ee}\). However, the constraints from lepton flavor violation processes seem to render this contribution sub-leading compared to the right-handed gauge boson one.\[55\]

The case of Type III proceeds in the same manner as the Type I, since one simply interchanges the fermion singlets, i.e. the right-handed neutrinos by the neutral components of the fermionic triplets. The crucial difference lies in the collider aspect of the model, since the triplets can be produced through the gauge couplings. We will not discuss this issue any further here; for a recent review of see e.g. \[\text{[55]}\]. Also, the question lepton flavor violation becomes more intricate \[\text{[116, 117, 118]}\].

In the cases of the Inverse and Extended seesaws, the smallness of the neutrino masses is inherently linked with a small lepton number violating element of the neutral lepton mass matrix. The implementation of Extended seesaw in low scale leptogenesis has been studied in literature \[66, 67\]. Below, we pursue the question of obtaining a large and dominant sterile neutrino contribution for the Extended seesaw scenario \[66, 67\], where the sterile neutrino sector has been extended by additional degrees of freedom.

Our discussion on the Extended seesaw proceeds as follows: Below, we first describe the Extended seesaw mass matrix and the relevant mixing matrix. We present a brief comparison between the analytic and numerical result for one generation case in Sect. 4.1.2. After that, in Sect. 4.1.2, we discuss the dominant role of sterile neutrino states in 0\(\nu\)2\(\beta\) process. Finally in Sect. 4.3, we discuss the major constraints on the Dirac mixing \(m\) and the lightest sterile neutrino mass \(m_s\), coming from 0\(\nu\)2\(\beta\) transition, as well as the heavy Majorana neutrino searches at LHC and the lepton flavor violating process. The detail of the diagonalization procedure for Extended seesaw and the higher-dimensional considerations of the mass and mixing matrix has been given in Appendix B.

### 4.1 Extended seesaw

In Extended seesaw (or Extended double seesaw according to \[66\]) framework \[66, 67\], we have \(n\)-generation \((n = 3)\) of standard model neutrino \(\nu_L\), \(m\)-generation of sterile neutrino state \(S_L\) and \(p\)-generation of sterile neutrino state \(N_L\). The Lagrangian describing the mass terms of the neutral leptons has the following form,

\[
L = -\frac{1}{2} (\nu_L \quad S_L \quad N_L) \begin{pmatrix}
0 & 0 & M_T^T D \\
0 & \mu & M_T^T S \\
M_D & M_S & M_R
\end{pmatrix}
\begin{pmatrix}
\nu_L \\
S_L \\
N_L
\end{pmatrix}
+ \text{h.c.}
\]

We denote the neutral lepton mass matrix as \(M_n\)\[72\], where,

\[
M_n = \begin{pmatrix}
0 & 0 & M_T^T D \\
0 & \mu & M_T^T S \\
M_D & M_S & M_R
\end{pmatrix}
\]

To understand the GUT realization of this seesaw scenario, see \[67\]. In this specific example, we work in a basis where the Majorana mass matrix \(M_R\) is real and \(M_D\) represents the mixing between the standard model flavored neutrino state \(\nu_L\) and the heavy sterile neutrino state \(N_L\). Furthermore, being a Majorana mass matrix of the heavy neutrino state \(S_L\), the matrix \(\mu\) is complex symmetric. In addition, throughout our analysis we adopt the following few assumptions,

- The generation of \(N_L\) and \(S_L\) are identical, i.e., \(m = p\). As a result, the matrix \(M_S\) is a square matrix.
- The matrices \(M_R\) and \(M_S\) are invertible.
- The different sub-matrices of the neutral lepton mass matrix follow this hierarchy, \(M_R > M_S > M_D \gg \mu\) and \(\mu < M_T^T M_R^{-1} M_S\), i.e., \(\mu < \mathcal{O}(\frac{M_T^2}{M_R})\).

\[\text{Note that, from hereon in the discussion of Extended seesaw, } \mu \text{ implies the Majorana mass of the sterile neutrino state } S_L.\]
Note that, this Extended seesaw scenario is very different from the inverse seesaw scenario [111, 112, 113, 114, 115], due to the simultaneous presence of both the heavy and small lepton number violating scales $M_R$ and $\mu$ respectively. The later has been widely discussed in the literature for its large contribution in lepton flavor violating processes [111, 112, 113, 114, 115]. In inverse seesaw, there is only one small lepton number violating scale $\mu$ and the lepton number is conserved in $\mu = 0$ limit. Hence, the $0\nu 2\beta$ transition amplitude also vanishes in this limit. On the contrary, in Extended seesaw, the heavy Majorana neutrino contribution can be the dominant contribution, even when the small lepton number violating scale $\mu$ vanishes. However, the standard model neutrino masses strongly depend on the small lepton number violating scale $\mu$ and hence in the $\mu = 0$ limit, the standard model neutrinos become massless. As a result, the contributions of the standard model neutrinos and the heavy Majorana neutrinos in $0\nu 2\beta$ process are completely decoupled from each other. This is the essence of our work on Extended seesaw, which we discuss in detail in the subsequent sections.

4.1.1 Mass and Mixing

We start with evaluating the mixing of the standard model neutrinos with these extra sterile states $S_L$ and $N_L$. The diagonalization of this Extended seesaw mass matrix (Eq. 86) is carried out by the $(n + 2m) \times (n + 2m)$-dimensional matrix $U$ where,

$$U^T M_n U = M'_n. \quad (87)$$

We decompose the mixing matrix $U$ as $U = U_1 U_2$, where $U_1$ and $U_2$ satisfy the relations $U_1^T M_n U_1 = M_{bd}$ and $U_2^T M_{bd} U_2 = M'_n$. $M_{bd}$ and $M'_n$ are respectively the block diagonal and diagonal mass matrices and are denoted as,

$$M_{bd} = \begin{pmatrix} m_\nu & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_n \end{pmatrix}; \quad M'_n = \begin{pmatrix} m_\nu & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_n \end{pmatrix}. \quad (88)$$

We define the block-diagonal basis and the diagonal mass basis with $(\nu_b, S_b, N_b)^T$ and $(\nu_m, S_m, N_m)^T$ respectively. The flavor state $(\nu_L, S_L, N_L)^T$ is related with the mass state $(\nu_m, S_m, N_m)^T$ as follows,

$$\begin{pmatrix} \nu_L \\ S_L \\ N_L \end{pmatrix} = U \begin{pmatrix} \nu_m \\ S_m \\ N_m \end{pmatrix}. \quad (89)$$

It is clearly evident from Eq. 88 that $m_\nu$, $m_s$ and $m_n$ are the mass matrices corresponding to the intermediate states $\nu_b, S_b$ and $N_b$, while $m_\nu^d$, $m_s^d$ and $m_n^d$ are the diagonal matrices containing the physical masses and correspond to the states $\nu_m, S_m$ and $N_m$ respectively. For one generation of $S_L$ and $N_L$ evidently, $m_s = m_s^d$ and $m_n = m_n^d$.

Following the parameterization in [90], to the leading order the mixing matrix $U_1$ is,

$$U_1 \sim \begin{pmatrix} 1 - \frac{1}{2} M_D^T (M_S^{-1})^T M_S^{-1} M_D & M_D^T (M_S^{-1})^T M_\mu M_S^{-1} M_D & M_D^T M_R^{-1} \\ -M_S^{-1} M_D & 1 - \frac{1}{2} M_S^{-1} M_D M_D^T (M_S^{-1})^T & -\frac{1}{2} M_S^{-1} M_\mu M_S^{-1} M_D \\ M_S^{-1} M_\mu M_S^{-1} M_D & -\frac{1}{2} M_S^{-1} M_\mu M_S^{-1} M_D & 1 - \frac{1}{2} M_R^{-1} M_S M_S^{-1} M_R^{-1} \end{pmatrix}. \quad (90)$$

In the above, we have neglected the $O(M_R^2)$ terms as compared to $O(M_D^2)$ and $O(M_D^2)$ terms, as $M_D < M_S, M_R$. Also, note that, $(U_1)_{31}$ is of the order $O(M_{bd}/M_S^2)$, which comes from the expansion of $O(M_{bd}/M_S^2)$ elements. In addition, in the diagonal elements $(U_1)_{11,33}$, we have shown the dominant sub-leading corrections, and in $(U_1)_{22}$, we have shown the correction which does not involve $\mu$. The corrections involving $\mu$ in the other elements of $U_1$ are smaller than the leading order terms and hence we do not show them explicitly. To the leading order, the light neutrino mass matrix $m_\nu$, and the heavy neutrino mass matrices $m_s, m_n$ have these following form,

$$m_\nu \sim M_D^T (M_S^{-1})^T M_S^{-1} M_D,$$

$$m_s \sim -M_S^{-1} M_\mu M_S^{-1} M_D,$$

$$m_n \sim M_R. \quad (91)$$

26
Figure 5: Feynman diagram of the 0ν2β process for Extended seesaw. The intermediate neutrino states are in mass basis.

Note that, due to $M_S < M_R$, the mass matrices $m_s$ and $m_n$ of the sterile neutrino states $S_b$ and $N_b$ satisfy the following inequality $m_s < m_n$. From Eq. 90 and Eq. 91, one can see that the standard model neutrino mass matrix depends on the parameter $\mu$, whereas to the leading order, the mixing ((U112 and U113)) between the standard model neutrinos $\nu_L$ and the sterile neutrino states $S_m$, $N_m$ are independent of that parameter. Hence, one can choose $\mu$ to be small to generate eV neutrino masses, still having large active-sterile neutrino mixings. The corresponding eigenvalues of the matrices $m_\nu$, $m_s$ and $m_n$ can be extracted by further diagonalization with the mixing matrix $U_2$, where the mixing matrix $U_2$ is denoted as,

$$U_2 = \begin{pmatrix} U & 0 & 0 \\ 0 & W_S & 0 \\ 0 & 0 & W_N \end{pmatrix}.$$  \hspace{1cm} (92)

The three matrices $U$, $W_{S,N}$ diagonalize the light and heavy neutrino matrices $m_\nu$ and $m_{s,n}$ respectively,

$$U^T m_\nu U = m_\nu^d = \text{diag}(m_i),$$  \hspace{1cm} (93)

$$W_S^T m_s W_S = m_s^d = \text{diag}(M_{S_i}),$$

$$W_N^T m_n W_N = m_n^d = \text{diag}(M_{N_i}).$$

In the above, $m_i$, $M_{S_i}$ and $M_{N_i}$ are the physical masses of the neutrino states $\nu_{m_i}$, $S_{m_i}$ and $N_{m_i}$ respectively. From Eq. [90] and Eq. [92] one immediately gets the following form of the mixing matrix $U$,

$$
\begin{pmatrix}
(1 - \frac{1}{2} M_D^1 (M_S^{-1})^1 M_S^{-1} M_D) U \\
-M_S^{-1} M_D U \\
M_S^{-1} \mu M_S^{-1} M_D U
\end{pmatrix}
\begin{pmatrix}
M_D^1 (M_S^{-1})^1 W_S \\
-M_S^{-1} M_D^1 W_S (1 - \frac{1}{2} M_S^{-1} M_D (M_S^{-1})^1 - \frac{1}{2} M_S^{-1} M_D^{-1} M_S) W_S \\
-M_D^{-1} M_S W_S (1 - \frac{1}{2} M_D^{-1} M_S M_S^{-1} M_D^{-1}) W_N
\end{pmatrix}
\begin{pmatrix}
M_D^1 M_R^{-1} W_N \\
M_D^1 M_R^{-1} W_N \\
(1 - \frac{1}{2} M_D^{-1} M_S M_S^{-1} M_D^{-1}) W_N
\end{pmatrix}
$$  \hspace{1cm} (94)

We provide the technical details of the block diagonalization in Appendix [B]. Below, we present the comparison between the analytical expression of the mixing matrix with the numerical result, considering one generation of standard model neutrino $\nu_L$ and one extra generation of sterile neutrino states $S_L$ and $N_L$. 

27
4.1.2 One generation consideration

For one generation of standard model neutrino $\nu_L$ and one extra generation of $S_L$ and $N_L$ states, the mass and mixing matrix is a simple $3 \times 3$ matrix. We present here a numerical estimation of the mass and mixing in this simplest case, and compare the numerical result with the analytical approximations. The mixing matrix is just:

$$U = \begin{pmatrix}
1 - \frac{M_D^2}{2M_S^2} & \frac{M_D}{M_S} & \frac{M_D}{M_R} \\
\frac{M_D}{M_S} & 1 - \frac{M_D^2}{2M_S^2} - \frac{M_D^2}{2M_R^2} & \frac{M_D}{M_R} \\
\frac{M_D}{M_R} - \frac{\mu}{M_S/M_R} & \frac{M_D}{M_R} - \frac{\mu}{M_S/M_R} & 1 - \frac{1}{2} \frac{M_S^2}{M_R^2}
\end{pmatrix}$$ \hspace{1cm} (95)

Few comments are in order,

- The leading order estimation of the diagonal terms is $U_{11,22,33} \sim 1$.
- $U_{12,13,23}$ elements are of the following order $U_{12} \sim \frac{M_D}{M_S}$, $U_{13} \sim \frac{M_D}{M_R}$ and $U_{23} \sim \frac{M_S}{M_R}$.
- For $\mu = 0$, $O(\frac{1}{M_R})$ (as well as higher orders in $O(M_R^{-1})$ terms in $U_{31}$ suffers mutual cancellation, resulting $U_{31} = 0$. For non-zero $\mu$, the leading order contribution in $U_{31}$ becomes $U_{31} \sim \frac{M_D}{M_R} \frac{\mu}{M_S/M_R}$. See Appendix B.1 for the detailed discussion about this particular feature.

For this simple one generation case, the numerical analysis has been shown in Table 5. The different elements $M_D$, $M_S$, $M_R$ and $\mu$ are the inputs; $m_\nu$ is the light neutrino mass, whereas the other two heavy neutrino masses are denoted by $m_s$ and $m_n$ respectively. Considering the sample values of the input parameters given in Table 5, the estimation of $U$ is the following,

$$U = \begin{pmatrix}
1.000 & 0.995 \times 10^{-3} & 0.985 \times 10^{-4} \\
-0.001 & 0.995 & 0.985 \times 10^{-1} \\
0.999 \times 10^{-10} & -0.985 \times 10^{-1} & 0.995
\end{pmatrix}$$ \hspace{1cm} (96)

As expected from Eq. 91 and Eq. 95

- $m_\nu = (\frac{M_D}{M_S})^2 \mu \sim 0.1\text{eV}$ and the other two heavy masses $m_s$ and $m_n$ are 99.02 GeV and 10099 GeV respectively. Note that to the leading order $m_n \sim M_R$. The analytical result $m_n \sim M_R + \frac{M_S^2}{M_R}$ (see Appendix B) resembles very closely the numerical estimation.
- The naive estimation of $U_{12,13,23}$ as $U_{12} \sim \frac{M_D}{M_S} \sim 10^{-3}$, $U_{13} \sim \frac{M_D}{M_R} \sim 10^{-4}$, $U_{23} \sim \frac{M_S}{M_R} \sim 10^{-1}$ and $U_{31} \sim \frac{M_D}{M_R} \frac{\mu}{M_S/M_R} \sim 10^{-10}$ matches well the numerical result.

| $M_R$ | $M_S$ | $M_D$ | $\mu$ | $m_\nu$ | $m_s$ | $m_n$ |
|-------|-------|-------|-------|--------|------|------|
| $10^4$ | $10^3$ | 1.0   | $10^{-4}$ | $10^{-10}$ | -99.0195 | 10099.0 |

Table 5: The light and heavy neutrino masses in GeV, for one generation Extended seesaw scenario.

4.2 Extended seesaw and $0\nu2\beta$ transition

In this subsection, we discuss the contributions of heavy Majorana neutrino states in the $0\nu2\beta$ process (see Fig. 5). Note that, in this case both the heavy Majorana neutrino states $S_m$ and $N_m$ will participate in $0\nu2\beta$ process. Depending on the masses and their mixings with the standard model neutrinos, the contributions of the heavy states $S_m$ and $N_m$ will differ. As for the Type I seesaw, we work in the following mass regime $m_n > m_s > 200$
MeV. As evident from Eq. (94) the electron flavor neutrino state $\nu_e$ mixes with the different active and sterile neutrino mass states $\nu_m, S_m$ and $N_m$ as follows,

$$\nu_e \sim U_{ei}\nu_{m_i} + (M_D^T(M_S^{-1})^TW_S)_{ek}S_{mk} + (M_D^TM_R^{-1}W_N)_{el}N_{ml},$$

where, $i,k,l$ represents the generations of the standard model neutrino state $\nu_m$ and the heavy sterile states $S_m$ and $N_m$ respectively. For simplicity, in the above, we have neglected any non-unitary effect associated with $U$. Note that, the mixing between the electron neutrino $\nu_e$ and the sterile neutrino mass eigenstates $S_m$ and $N_m$ are not constrained by the smallness of the standard model light neutrino mass.

The discussion of sterile neutrino contribution proceeds analogously as for the Type I seesaw. Denoting the two mixings matrices between the active and sterile states $M_D^T(M_S^{-1})^TW_S$ and $M_D^TM_R^{-1}W_N$ by the notations $V_{eS}$ and $V_{eN}$ respectively, the half-life time period of $0\nu\beta\beta$ transition can be expressed as follows,

$$\frac{1}{T_{1/2}} = K_{0\nu} \left| \frac{U^2_{ei}m_i}{(p^2)} - \frac{V^2_{eS_i}}{M_{S_i}} - \frac{V^2_{eN_i}}{M_{N_i}} \right|^2,$$

where the definition $K_{0\nu}$ and $(p^2)$ follow the Type I seesaw considerations, i.e., $K_{0\nu} = G_{0\nu}(\mathcal{M}_N m_p)^2$ and $(p^2) \equiv -m_e m_p \frac{M_S}{M_e}$ (Eq. [11] and [55]). In the above, $\frac{V^2_{eS_i}}{M_{S_i}}$ and $\frac{V^2_{eN_i}}{M_{N_i}}$ are the contributions of the sterile neutrino states $S_m$ and $N_m$ in $0\nu\beta\beta$ process respectively, and $M_{S_i}$ and $M_{N_i}$ are the corresponding masses. Expressing $V_{eS}$ and $V_{eN}$ back in terms of $M_D$, $M_S$ and $M_R$, the total amplitude of $0\nu\beta\beta$ process is the following,

$$A^t_{\nu} = m_{ee} - \left( M_D^T M_S^{-1} m_s^{-1} M_D \right)_{ee} - \left( M_D^T M_R^{-3} M_D \right)_{ee}. \quad (98)$$

In the above, $m_{ee}$ is the standard model neutrino mass contribution and is given by,

$$m_{ee} = \left( M_D^T(M_S^{-1})^T \mu M_S^{-1} M_D \right)_{ee}. \quad (99)$$

The quantity $m_s$ represents the mass matrix of the sterile intermediate state $S_0$ and is given in Eq. [91]. The contribution of the sterile neutrino mass states $S_m$ and $N_m$ to the neutrinoless double beta decay amplitude, i.e., the contact terms are respectively given by,

$$A_S = \left( M_D^T M_S^{-1} m_s^{-1} M_D \right)_{ee}$$
$$A_N = \left( M_D^T M_R^{-3} M_D \right)_{ee}, \quad (100)$$

For one generation of standard model neutrino $\nu_e$ and one extra sterile states $S_m$ and $N_m$, the amplitudes simplify to,

$$A_S = \left( \frac{M_D}{M_S} \right)^2 \frac{1}{m_s} = \frac{M_D^2}{m_n m_s^2},$$
$$A_N = \left( \frac{M_D}{M_R} \right)^2 \frac{1}{m_n} = \frac{M_D^2}{m_n^3}, \quad (101)$$

where we have used $m_s \sim \frac{M_S^2}{M_R}$ and $m_n \sim M_R$. Note that, for one generation of $\nu_L$ and $S_L, N_L$ states $m_s$ and $m_n$ are the physical masses of the sterile states $S_m$ and $N_m$. Few comments are in order,

- The contributions coming from the extra sterile states $S_m$ and $N_m$ are decoupled from the light neutrino contribution. This can be clearly seen from Eq. [100] as well as from Eq. [101].

- In the $\mu \rightarrow 0$ limit, when the contribution of the light neutrino $m_{ee}$ is zero, one can even get a non-zero and significant contribution to the neutrinoless double beta decay, due to the additional heavy neutrino states.

- As an example, the choice $M_D = 10\sqrt{0.1}$ GeV, $M_S = 10^4$ GeV and $M_R = 10^6$ GeV generate $m_s = 10^2$ GeV, $m_n = 10^6$ GeV. The mixing of standard model neutrinos with the state $S_m$ is $10^{-3}\sqrt{0.1}$. This generates the contribution $A_S = 10^{-9}$ GeV$^{-1}$. 

29
Figure 6: Constrains on $m - m_s$ parameter plane from $0\nu2\beta$, heavy Majorana neutrino searches at LHC and the lepton flavor violation. $m$ and $m_s$ are in GeV. In this figure $M_R$ has been set to $M_R = 1.69 \times 10^4$ GeV.

- The light neutrino mass scale is fixed by the lepton number violating parameter $\mu$. For the above mentioned numerical values, eV neutrino mass is generated by fixing $\mu = 10^{-3}$ GeV.
- In the present scenario, the mixing between the heavy Majorana neutrino state $N_m$ and the standard model neutrino $\nu_L$ is given by $\frac{M_D}{M_R} = \frac{M_D}{M_S} \frac{M_S}{M_R} < \frac{M_D}{M_S}$, where $\frac{M_D}{M_S}$ represents the mixing of the heavy state $S_m$ with the standard model neutrino state $\nu_L$. Also, the mass of the heavy states $N_m$ and $S_m$ are related by the following inequality $m_m > m_s$. Hence, excepting fine-tuning or cancellation among the active-sterile ($\nu_L - S_m$) neutrino mixing, the contribution of the heavy states $N_m$ will most likely be much smaller. We do not address any such cancellations among the sterile neutrino states $S_m$ in this present study. Hence, in this case, the two contributions $A_S$ and $A_N$ in Eq. 100 and Eq. 101 are related by the following inequality,

$$A_N \ll A_S.$$  

In the next subsection, we present our analysis in detail. Like for the Type I seesaw, we analyze the constraints on the Dirac mass matrix $M_D$ and the sterile neutrino mass $m_s$, keeping the mass $M_R$ of the sterile neutrino $N_m$ fixed.

### 4.3 Constraining $m - m_s$ parameter plane

In this section we discuss the different constraints on the lightest sterile neutrino mass $m_s$ and Dirac mixing $m$ coming from $0\nu2\beta$ process, and we provide a naive estimation on heavy Majorana neutrino searches at LHC as well as searches for lepton flavor violating processes. Our discussion relies on the assumption that $\mu$ is smaller than the light sterile neutrino mass $m_s \sim M_S^2/M_R$. For simplicity, we adopt the following considerations (compare with Sect. 3.2.2):

- The scale of $M_D$ is referred as $m$, while the scale of $M_S$ is denoted as $M$. Among the two scales, the scale of $M_D$ is bounded from perturbativity, i.e., $m < 174$ GeV; while being the mixing between two sterile neutrino states, the scale of $M_S$ can take larger value.
- We denote the mass scale of the sterile neutrino states $N_m$ by $M_R$, while the mass scale of the sterile neutrino states $S_m$ is fixed by $m_s$. This notation perfectly fits the following scenarios,

\footnote{For more than one generation, the masses $\text{diag}(M_S_i)$ may have significant hierarchy, for which further precise definition of $m_s$ as a mass-scale is required. Instead, in this example, we consider the masses $\text{diag}(M_S_i)$ are not strongly hierarchical, and hence, can be represented by $m_s$.}
- when the sterile neutrino masses are not very hierarchical.
- when the model is extended by only two sterile neutrino states $S_L$ and $N_L$.

Fig. 6 illustrates the naive constraints on the Dirac mixing $m$ and the sterile neutrino mass $m_s$, for Extended seesaw scenario. The bound on $m_s$ should be interpreted as the bound on the absolute value of $m_s$. We present our result for the sample case, where the heavy Majorana neutrino mass $M_R = 1.69 \times 10^4$ GeV. The details of the figure are as follows,

- The three grey bands are disallowed by the following considerations,
  - The upper grey band is disallowed, since it violates the Extended seesaw condition $M_S < M_R$. In other words, in this region the mass of the sterile neutrino states $S_m$ and $N_m$ satisfy the following inequality relation $m_s > M_R$, which is not permitted by the Extended seesaw criterion.
  - The disallowed lower grey band corresponds to $m_s < 200$ MeV.
  - The side grey band is disallowed from perturbativity bound on the mixing between the standard model neutrino $\nu_L$ and the sterile neutrino state $N_L$, i.e., $m < 174$ GeV.

- The oblique line separating the white and pink region corresponds to the saturating $0\nu2\beta$ contribution from the sterile neutrino states $S_m$. The amplitude of the sterile neutrino states $S_m$ across this oblique line satisfies the relation,
  $$\left(\frac{m}{M}\right)^2 \frac{1}{m_s} = \frac{1}{\sqrt{\Gamma_{0\nu}/K_{0\nu}}},$$
  where we have considered the half-life of germanium 76, $T_{1/2} = 1.9 \times 10^{25}$ yr [1]. As already has been discussed in Sect. 2.2, the factor $K_{0\nu} = G_{0\nu}(M_N m_p)^2 = 9.2 \times 10^{-10}$ GeV$^2$/yr, using the nuclear matrix element of [62]. Expressing the amplitude in terms of $M_R$, $m_s$ and the Dirac mixing $m$, the sterile neutrino contribution across the oblique line satisfies $m_s^2 \frac{1}{M_R m_s^2} = 7.6 \times 10^{-9}$ GeV$^{-1}$. The area below this oblique line is disallowed, since the sterile neutrino contribution in this region is larger than the above mentioned saturating contribution. The area above this oblique line is however allowed from $0\nu2\beta$ consideration and the sterile neutrino contribution in this region is smaller than the saturating value.

- The oblique line separating the pink and yellow region corresponds to the active-sterile neutrino mixing (mixing between $\nu_L$ and $S_m$ state) $\theta \sim \frac{m_s}{M} = 0.01$. In terms of $m_s$, $M_R$ and the Dirac mixing $m$, this condition can be written as $m_s = \frac{m_s^2}{M_R} 10^4$. The area below this oblique line corresponds to the large mixing $\theta > 0.01$ and is the favorable region for heavy sterile neutrino searches at LHC [69, 70, 71]. Note that, the region is further subdivided into two subregions, where the light gray region is ruled out by the lepton flavor violating processes $\mu \rightarrow e\gamma$ [14]. Also note that, for this simplified scenario, the $0\nu2\beta$ puts severe constraint on the heavy Majorana neutrino searches at LHC.

- The rightmost light gray region, which is disallowed from lepton flavor violating constraint, and as well as from $0\nu2\beta$ consideration, is further subdivided by a oblique line. For our choice $M_R = 1.69 \times 10^4$ GeV, the smaller gray region under right most oblique line violates the Extended seesaw condition $M > m$, i.e., in other words $m_s > \frac{m_s^2}{M_R}$, hence also disallowed from Extended seesaw criterion.

In our previous discussions about the different constraints, we have considered the mass of the sterile neutrino state $m_s \sim \frac{M^2}{M_R}$, and the active-sterile mixing angle as $\theta \sim \frac{m_s}{M}$. As, $\mu$ is smaller than $m_s \sim \frac{M^2}{M_R}$, our consideration is perfectly justified. However, for completeness, we discuss the possible corrections to the previously discussed bounds, which appear because of the small lepton number violating parameter $\mu$. Due to the presence of $\mu$, the physical mass $m_s$ of the sterile state $S_m$ is changed to $m_s \sim \mu - \frac{M^2}{M_R}$ (see Appendix B), where $\mu < \frac{M^2}{M_R}$. Also note that, the correction to the active-sterile mixing angle $(\nu_L - S_m$ mixing angle $\theta$) due to a non-zero small $\mu$.

---

[14] For one generation standard model neutrino $\nu_L$ and one generation of extra sterile states $S_L$ and $N_L$, this lepton flavor bound will be absent.
which is allowed by the $0^\nu \beta \beta$ process is $\delta A_S \sim O(\frac{m^2 M^2}{M^2 R^2})$, where the mass of the state $S_m$ is $m_s \sim \mu - \frac{M^2}{M_R}$. In terms of the physical mass $m_s$, the correction is $\delta A_S \sim \frac{m^2}{M^2 m_s^2} \frac{\mu}{m_s}$, hence suppressed than the leading order contribution $\frac{m^2}{M^2 m_s^2}$ by a factor $O(\frac{\mu}{m_s})$, which is very small due to $\mu < m_s$. To give an estimation, for $\mu = 10^{-3}$ GeV, $m_s = 0.1$ GeV, the factor $\frac{\mu}{m_s} = 10^{-4}$. Hence, for all practical purposes, it is negligible.

- Considering the physical mass of the $S_m$ state as $m_s \sim \mu - \frac{M^2}{M_R}$, the Extended seesaw conditions $M < M_R$ and $m < M$ is now modified to $-m_s < M_R - \mu$ and $-m_s > \frac{m^2}{M_R} - \mu$ respectively.

- Similarly, the bound coming from heavy Majorana searches at LHC $\frac{m}{M} > 10^{-2}$, will be corrected by a very small factor $O(\frac{m^2}{M^2 m_s^2} \frac{\mu}{m_s})$.

The discussions of the previous as well as this section clearly shows that the leading order contribution coming from the sterile neutrino state $S_m$ is independent of the small lepton number violating parameter. It is also clearly evident from the above discussion that the possible correction to $0^\nu \beta \beta$ amplitude due to the lepton number violating parameter $\mu$ will be extremely small. Hence, one can practically consider the $0^\nu \beta \beta$ bound as independent of this parameter. However, as pointed out before in Sect. 4.1.1 and stressed in the subsequent sections, the light neutrino masses strongly depend on this parameter. Changing $\mu$ to a relatively larger value, the $0^\nu \beta \beta$ allowed region can further be restricted from small neutrino mass constraint.

We present the possible comparison between the $0^\nu \beta \beta$ bound and neutrino mass constraint in Fig. 7. For a very small $\mu$ (as shown in Fig. 7), the grey region which are excluded from the theoretical constraints $M < M_R$ and $m < M$ remains almost unchanged as compared to Fig. 6. The pink region is disallowed from $0^\nu \beta \beta$ consideration $\mu$. Across the oblique black line separating the pink and white region, the contribution of the sterile neutrino state $S_m$ in $0^\nu \beta \beta$ process saturates the upper bound of $\mu$, i.e., $\frac{m^2}{M_R m_s} = 7.6 \times 10^{-9}$ GeV$^{-1}$.

On the other hand, the standard model light neutrino mass is $m_\nu = (\frac{\mu}{2})^2 \mu$; the dependence on $\mu$ is clearly evident. Expressing $m_\nu$ in terms of the physical masses $m_s$, $M_R$ and the Dirac mixing $m$, the eV neutrino mass constraint can be expressed, $\frac{m^2}{m_\nu M_R} \mu = 0.1eV$. The oblique blue line (Fig. 7) separating the blue and white region corresponds to $\mu = 10^{-3}$ GeV, and represents the above neutrino mass constraint. The area below this line violates $m_\nu < 0.1$ eV and is strongly disallowed from neutrino mass constraint. It is evident from the figure, the blue region which is allowed by the $0^\nu \beta \beta$ consideration is ruled out by the neutrino mass constraint. However, considering smaller values of $\mu$, the neutrino mass constraint on the $m - m_s$ plane can be comparatively relaxed. We have given illustrative example for two other $\mu$ values, the red oblique line corresponds to $\mu = 10^{-5}$ GeV and the orange oblique line corresponds to $\mu = 10^{-7}$ GeV. For each of the oblique lines, the area below the line is disallowed from the neutrino mass constraint. As it is clearly evident, for the smallest of these three values, i.e., for $\mu = 10^{-7}$ GeV, the neutrino mass constraint does not restrict the $m - m_s$ parameter space any further than the $0^\nu \beta \beta$ consideration. Hence, we will conclude that the possible additional restriction on $m - m_s$ parameter space coming from small neutrino mass can be evaded with the choice of smaller $\mu$, whereas the bound coming from lepton number violating $0^\nu \beta \beta$ is process possibly the most stringent one.

5 Summary and discussion

Undoubtedly, the study of neutrinoless double beta decay is one of the main available probes of the lepton number violation. The existing results of the Heidelberg-Moscow collaboration 1, as well as results from the other experiments, e.g., Cuoricino 2, IGEX 3, Nemo 4 provide lower bound on the half-life of this process. In addition, there is already an existing experimental hint on $0^\nu \beta \beta$ obtained by Klapdor and collaborators 7, which however, according to 73, violates the bound obtained from cosmology. The currently running experiment Gerda 5, as well
Figure 7: Constrains on $m - m_s$ parameter plane from $0\nu 2\beta$ and neutrino mass. The blue, red and orange oblique line correspond to $\mu = 10^{-3}, 10^{-5}, 10^{-7}$ GeV. The value of $M_R$ has been set to $M_R = 1.69 \times 10^4$ GeV.

as the different future experiments Cuore [6], EXO [8], SuperNEMO [9], Majorana [10], Lucifer [11], SNO+ [12], KamLAND-Zen [13], Cobra [14] and NEXT [15] will provide much more useful information regarding this process.

The simplest extension of standard model includes the heavy sterile neutrino states, which are responsible of generating light neutrino masses via Type I seesaw mechanism. While the naive expectations would attribute the violation of lepton number in the $0\nu 2\beta$ process totally to the light Majorana neutrino states, with heavy sterile neutrinos playing a subdominant role, there is however the possibility to achieve the opposite extreme within the minimal Type-I seesaw model only. In this work, this question of how to achieve this possibility has been analyzed in detail.

The existing bounds on the active-sterile mixing angle have been re-evaluated, and the leading role of $0\nu 2\beta$ reassessed. Due to the improved result of the nuclear matrix elements given in [62], the bounds on active-sterile mixing coming from $0\nu 2\beta$ is very stringent: the previous constraint on the mixing angle [63] has become one order of magnitude tighter. On the face of this new result, the bounds from various meson decay as well as other experiments [64] has become relatively less important in all parameter space, and almost entirely superseded by the $0\nu 2\beta$ bound—see Fig. 1.

The question of having a dominant heavy sterile neutrino contribution in $0\nu 2\beta$ process in the minimal Type I seesaw model (i.e., with three heavy neutrino states) has been explored in detail. In the case of one generation standard model neutrino and one extra sterile neutrino state, the seesaw structure of the neutrino mass matrix automatically guarantees the smallness of the extra sterile neutrino contribution in the $0\nu 2\beta$ process. The opposite regime, when the $0\nu 2\beta$ transition is saturated by the heavy sterile neutrino states, can be possibly achieved, for more than one generation case. To obtain this, the light neutrino mass has to be smaller than the naive expectation from the ordinary seesaw formula. This condition has been implemented beginning from an exact cancellation and then arranging small perturbation in several possible ways. A classification of the interesting Type I seesaw models emerged, with light neutrino mass matrices of all types but also with sterile neutrino states dominating the $0\nu 2\beta$ transition. All the cases can be studied by mean of simple analytical formulae. Moreover, in order to obtain a viable scenario where the light neutrino masses do not suffer from radiative instabilities, the perturbations as well as the sterile neutrinos have to obey additional constraints, and in particular, their masses have to be approximatively lighter than 10 GeV. Several explicit examples illustrate how the sterile neutrino and light neutrino contribution in $0\nu 2\beta$ process can possibly decouple, and the sterile neutrino contribution become dominant, for the two flavor and the three flavor scenarios. The analytical results have been verified numerically. The dominant sterile neutrino contribution in $0\nu 2\beta$ process provide a way to overcome the conflict between cosmology and the experimental hint obtained by Klapdor and collaborators [7], or more in general to have a relatively fast $0\nu 2\beta$ transition, even with very small neutrino masses.
Similar investigations have been carried in more complex seesaw scenarios, and in particular, in the Extended seesaw \[66, 67\]. In this model, one has additional heavy neutrino states. The lepton number violation is introduced by two main hierarchical mass scales. The light neutrino masses depend on the small lepton number violating scale, and therefore, the sub-eV neutrino masses can be explained by the small lepton number violating scale of the theory. Since the active-sterile neutrino mixing in leading order is independent of this small lepton number violating scale, the standard model neutrino contribution and the extra sterile state contribution are completely decoupled in this seesaw scenario. As a result, the sterile neutrino contribution can saturate the present experimental bound on $0\nu2\beta$ transition \[1\], while the light neutrino contribution can even be subdominant. Possible issues, such as, the higher-dimensional corrections to the active-sterile mixing angle due to the small lepton number violating scale, as well as to the $0\nu2\beta$ transition amplitude have been discussed in some detail. The details of higher-dimensional correction to the mass and mixing matrix have been evaluated in Appendix A.

These results have direct implications for the phenomenology of the (minimal and extended) seesaw models. In particular, they are potentially relevant for collider physics and rare transitions (such as $\mu \rightarrow e\gamma$) though the exploratory investigations here presented suggest only a marginal impact; but more promisingly, these models have interesting implications for meson decays, neutrino-decay searches and cosmology. A systematic study of these issues will be matter of a future work.

**Acknowledgments**

The authors would like to thank Sandhya Choubey, Srubabati Goswami for reading the manuscript, and Anushree Ghosh for computational help. This work has been partially supported with the INFN-ICTP exchange program and by Centro Fisica Astroparticellare (CFA) supported by POR Abruzzo.

### A A special type of Majorana mass matrices

Here a special type of Majorana mass matrix is analyzed. This type of matrix occurs repeatedly in the present study: as sub-block of the $6 \times 6$ mass matrix for the non-trivial solution of the equation $M_D^T M_R^{-1} M_D = 0$, see Eq. 34 in the discussion of $M_R$ and $M_\nu$, see e.g., Eq. 39 in the simplest version of Extended seesaw mass matrix, the one involving just 3 states $\nu_L, S_L, N_L$. This matrix is a $3 \times 3$ lower triangular matrix, say

$$
M = \begin{pmatrix} 0 & 0 & \alpha \\
0 & \delta & \beta \\
\alpha & \beta & \gamma \end{pmatrix}
$$

(102)

since this matrix will be considered as a Majorana mass matrix, the phases of the neutrino fields (i.e., the basis in the complex space) can be conveniently chosen in order to make the elements $\alpha, \beta, \delta$ real non-negative. Assuming the condition of reality, $M_{ij} = M_{ij}^\ast$ (i.e., also $\gamma$ is real for the same choice of phases) the matrix can be diagonalized simply by an orthogonal change of basis: $O^T M O = \text{diag}(m_i)$. Here $m_i$ are the eigenvalues, that are real but not necessarily positive, and that can be arranged in an increasing order

$$
|m_1| \leq |m_2| \leq |m_3|
$$

(103)

The elements $\alpha, \beta, \gamma$ can be expressed in terms of the eigenvalues by considering the characteristic polynomial, $p(x) = \text{det}(M - xI) = \prod_{i=1}^3 (m_i - x)$ and identifying the terms of the same order in $x$

$$
\begin{cases}
\alpha &= \sqrt{-\frac{m_1 m_2 m_3}{m_0}} \\
\beta &= \sqrt{(m_1 - m_0)(m_2 - m_0)(m_3 - m_0)} \\
\gamma &= m_1 + m_2 + m_3 - m_0
\end{cases}
$$

(104)
where, of course, \( \delta = m_0 \) acts as a free parameter (we changed the notation just to emphasize that it has the dimension of a mass). The condition that these parameters are real require that one of the following mutually exclusive conditions holds true:

\[
\begin{align*}
  m_1 & \leq 0 \leq m_2 \leq m_0 \leq m_3 \quad \text{or} \\
  m_2 & \leq 0 \leq m_1 \leq m_0 \leq m_3 \quad \text{or} \\
  m_3 & \leq 0 \leq m_1 \leq m_0 \leq m_2
\end{align*}
\]

(105)

Note incidentally that the conditions of Eqs. (105) imply that in the limit \( m_0 \to 0 \) also one eigenvalue is forced to go to zero, thus this limit does not need to be singular.

A last interesting result is the simple expression for the normalized eigenvectors, namely the vectors satisfying \( M e_i = m_i e_i \), that are also the columns of orthogonal matrix \( O \) that diagonalizes \( M \). These are given by

\[
e_i = \frac{1}{N_i} \begin{pmatrix}
  \alpha \\
  \frac{m_i}{m_i - m_0} \\
  \beta
\end{pmatrix}
\]
with

\[
N_i = \sqrt{\frac{(m_i - m_j)(m_i - m_k)}{m_i(m_i - m_0)}}
\]

where \( \{i, j, k\} = \{1, 2, 3\} \). It is easy to show that any of the condition of Eqs. (105) implies that \( N_i \) is real.

### B Derivation of Eq. (90)

In this appendix, we provide the details of the block diagonalization of Extended seesaw mass matrix and evaluate the mixing matrix \( \mathcal{U}_1 \). The neutral fermion mass matrix (Eq. [86]) is,

\[
M_n = \begin{pmatrix}
  0 & 0 & M_D \mathbf{T} \\
  0 & \mu & M_S \\
  M_D & M_S & M_R
\end{pmatrix}.
\]

(107)

For simplicity we consider the Majorana mass matrix \( M_R \) to be real. Furthermore, being a Majorana mass matrix, \( \mu \) is complex symmetric. We assume \( M_R > M_S > M_D \gg \mu \) and also \( \mu < M_D^T M_R^{-1} M_S = O \left( \frac{M_S}{M_R} \right) \). The block diagonalizing matrix \( \mathcal{U}_1 \) is \((n + 2m) \times (n + 2m)\)-dimensional and satisfies \( \mathcal{U}_1^T M_n \mathcal{U}_1 = \hat{M}_{bd} \). The matrix \( \hat{M}_{bd} \) has been written in Eq. [88]. To evaluate \( \mathcal{U}_1 \), we further decompose \( \mathcal{U}_1 = \mathcal{U}_1' \mathcal{U}_1'' \), where \( \mathcal{U}_1' \) and \( \mathcal{U}_1'' \) satisfy \( \mathcal{U}_1'^T M_n \mathcal{U}_1'' = \hat{M}_{bd} \) and \( \mathcal{U}_1''^T \hat{M}_{bd} \mathcal{U}_1'' = \hat{M}_{bd} \), \( \hat{M}_{bd} \) is the intermediate block-diagonal matrix. We follow the parameterization of [90], i.e.,

\[
\mathcal{U}_1' = \begin{pmatrix}
  \sqrt{1 - BB^T} & B \\
  -B^\dagger & \sqrt{1 - B^\dagger B}
\end{pmatrix},
\]

(108)

where \( B = \Sigma_j B_j \) and \( B_j \sim \frac{1}{M_R} \). Up to 2nd order in \( M_R^{-1} \), the mixing matrix \( \mathcal{U}_1' \) has the following form,

\[
\mathcal{U}_1' \sim \begin{pmatrix}
  1 - \frac{1}{2} M_D^T M_R^{-2} M_D & -\frac{1}{2} M_D^T M_R^{-2} M_S & M_D^T M_R^{-1} \\
  -\frac{1}{2} M_S^T M_R^{-2} M_D & 1 - \frac{1}{2} M_S^T M_R^{-2} M_S & M_S^T M_R^{-1} + \mu^* M_S^T M_R^{-2} \\
  -M_R^{-1} M_D & -(M_R^{-1} M_S + M_R^{-2} M_S^* \mu) & 1 - \frac{1}{2} M_R^{-1} (M_D M_R^{-1} + M_S M_S^* \mu) M_R^{-1}
\end{pmatrix}.
\]

(109)

The intermediate block diagonalized matrix \( \hat{M}_{bd} \) is (up to order \( M_R^{-2} \)),

\[
\hat{M}_{bd} \sim \begin{pmatrix}
  -M_D^T M_R^{-1} M_D & -M_D^T M_R^{-1} M_S - \frac{1}{2} M_D^T M_R^{-2} M_S^* \mu & 0 \\
  -M_S^T M_R^{-1} M_D - \frac{1}{2} \mu M_D^T M_R^{-2} M_D & \mu - M_S^T M_R^{-1} M_S - \frac{1}{2} (M_S^T M_R^{-2} M_S^* \mu + \mu M_S^T M_R^{-2} M_S) & 0 \\
  0 & 0 & M_R^{-1}
\end{pmatrix},
\]

(110)
where up to 2nd order in $M_R^{-1}$,

$$M'_R = M_R + [(M_D M_D^\dagger + M_S M_S^\dagger)M_R^{-1} + \frac{1}{2} M_S \mu^* M_S^T M_R^{-2} + \text{Trans.}].$$  \hspace{1cm} (111)

Note that, further consideration of $M_R^{-3}$ terms in $U'_1$ will open up the relative comparison between the different terms of $O(\frac{M_D}{M_R})^2, O(\frac{M_S}{M_R})^2$ and $O(\frac{\mu}{M_R})$. As the leading order terms in $U_1$ is unaffected by the inclusion of sub-leading terms, therefore we do not discuss the detail dynamics of sub-leading terms here.

The intermediate block diagonal matrix $\hat{M}_{bd}$ can further be diagonalized by the 2nd mixing matrix $U''_1$. As evident from the above, the leading order terms in $\hat{M}_{bd}$ is,

$$\hat{M}_{bd} \sim \begin{pmatrix}
-M_D^T M_D^{-1} M_D & -M_D^T M_D^{-1} M_S & 0 \\
-M_S^T M_S^{-1} M_D & \mu - M_S^T M_S^{-1} M_S & 0 \\
0 & 0 & M_R
\end{pmatrix}. \hspace{1cm} (112)$$

For $M_R > M_S > M_D \gg \mu$, we have $M_D^T M_D^{-1} M_S > M_D^T M_S^{-1} M_S > M_D^T M_S^{-1} M_D$, hence one can again apply the seesaw approximation on $\hat{M}_{bd}$. Assuming further $\mu < M_S^T M_S^{-1} M_S$, the block diagonal matrix $M_{bd}$ has the following form,

$$M_{bd} \sim \begin{pmatrix}
M_D^T (M_S^T)^{-1} \mu M_S^{-1} M_D & 0 & 0 \\
0 & -M_S^T M_S^{-1} M_S & 0 \\
0 & 0 & M_R
\end{pmatrix}. \hspace{1cm} (113)$$

To the leading order, the mixing matrix $U''_1$ is,

$$U''_1 \sim \begin{pmatrix}
1 - \frac{1}{2} M_D^\dagger (M_S^T)^{-1} M_S^{-1} M_D & M_D^\dagger (M_S^T)^{-1} & 0 \\
-M_S^{-1} M_D & 1 - \frac{1}{2} M_S^{-1} M_D M_D^\dagger (M_S^T)^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix}. \hspace{1cm} (114)$$

From the expression of $U'_1$ and $U''_1$ and neglecting the relatively smaller $O(\frac{M_D}{M_R})^2$ terms as compared to $O(\frac{M_S}{M_R})^2, O(\frac{\mu}{M_R})^2$ one obtains the following expression of the mixing matrix $U_1$, given in Eq. 90.

$$U_1 \sim \begin{pmatrix}
1 - \frac{1}{2} M_D^\dagger (M_S^T)^{-1} M_S^{-1} M_D & M_D^\dagger (M_S^T)^{-1} & M_D^\dagger M_D^{-1} \\
-M_S^{-1} M_D & 1 - \frac{1}{2} M_S^{-1} M_D M_D^\dagger (M_S^T)^{-1} & -\frac{1}{2} M_S^{-1} M_S^2 M_S \\
(M_S^T)^{-1} \mu M_S^{-1} M_D & -M_S^{-1} M_S & 1 - \frac{1}{2} M_R^{-1} M_S^T M_S M_R^{-1}
\end{pmatrix}. \hspace{1cm} (115)$$

In the above, the $(U_1)_{31}$ term is of the order $O(\frac{M_D}{M_R} \frac{\mu}{M_S} \frac{\mu}{M_R})$. To the leading order, the light and heavy neutrino mass matrices $m_\nu$, and $m_s, m_n$ of Eq. 88 are respectively the following,

$$m_\nu \sim M_D^\dagger (M_S^T)^{-1} \mu M_S^{-1} M_D,$$

$$m_s \sim -M_S^T M_R^{-1} M_S,$$

$$m_n \sim M_R. \hspace{1cm} (116)$$

Note that, in the above $(U_1)_{31}$ strongly depends on $\mu$. This can be very easily seen for the one generation case, which we discuss in some detail below.

### B.1 Higher Order Consideration

We discuss the possible higher order correction to the block diagonalized matrix $\hat{M}_{bd}$ and as well as to the mixing matrices $U'_1, U''_1$, considering one generation $\nu_L, S_L$ and $N_L$. The higher order terms are important to understand the possible corrections to the mixing matrix $U_1$ and as well as to understand a nonzero $(U_1)_{31}$. It is straightforward to verify the results for multiple generation and hence we do not repeat the task anymore. We first discuss the higher order corrections for the case $\mu = 0$ and then simply extend the discussion for $\mu \neq 0$.  

36
To calculate the other mixing matrix section and as well as in section 4.1.1, due to mutual cancellation between the elements of

\[ \hat{M}_{bd} = \begin{pmatrix} M_{ll} & M_{lh} \\ M_{lh}^T & M_{hh} \end{pmatrix}, \quad (117) \]

where \( M_{ll}, M_{lh} \) and \( M_{hh} \) are:

\[ M_{ll} = -M_D^TM_R^{-1}M_D + \frac{1}{2} \left( M_D^TM_R^{-1}M_D M_D^TM_R^{-2}M_D + M_D^TM_R^{-1}M_S M_S^TM_S^R M_R^{-1}M_D + \text{Trans.} \right) \]

\[ M_{hh} = -M_D^TM_R^{-1}M_S + \frac{1}{2} \left( M_D^TM_R^{-1}M_S M_S^TM_S^R M_S + M_D^TM_R^{-1}M_D M_D^TM_R^{-1}M_S M_S^TM_S^R M_R^{-1}M_D + \text{Trans.} \right) \]

\[ M_{lh} = -M_D^TM_R^{-1}M_S + \frac{1}{2} \left( M_D^TM_R^{-1}M_D M_D^TM_R^{-2}M_S + M_D^TM_R^{-1}M_D M_D^TM_R^{-1}M_S M_S^TM_S^R M_R^{-1}M_D \right) \]

\[ + \frac{1}{2} \left( M_D^TM_R^{-1}M_D M_D^TM_R^{-1}M_S M_S^TM_S^R M_R^{-1}M_D \right) , \quad (118) \]

In the above, we have shown explicitly up to \( \mathcal{O}(M_R^3) \). Considering one generation of \( \nu_L, S_L \) and \( N_L \), the intermediate block-diagonal matrix \( \hat{M}_{bd} \) simplifies to,

\[ \hat{M}_{bd} = -\left( \begin{array}{cc} \frac{M_{ll} M_D}{M_R} & \frac{M_{lh} M_D}{M_R} \\ \frac{M_{lh} M_S}{M_R} & \frac{M_{hh} M_S}{M_R} \end{array} \right) \left( 1 - \frac{M_D^2}{M_R^2} - \frac{M_S^2}{M_R^2} \right) . \quad (119) \]

It is clearly evident, as the determinant of this matrix vanishes, the light neutrino mass is zero. The mixing matrix \( \mathcal{U}' \) in this case has the following simple form (up to \( \mathcal{O}(M_R^3) \)),

\[ \mathcal{U}' = \begin{pmatrix} 1 - \frac{M_D^2}{2M_R^2} & -\frac{1}{2} \frac{M_D M_S}{M_R} & \frac{M_R}{M_R} \left( 1 - \frac{3}{2} \frac{M_D^2}{M_R^2} - \frac{3}{2} \frac{M_S^2}{M_R^2} \right) \\ -\frac{1}{2} \frac{M_D M_S}{M_R} & 1 - \frac{1}{2} \frac{M_S^2}{M_R^2} & \frac{M_R}{M_R} \left( 1 - \frac{3}{2} \frac{M_D^2}{M_R^2} - \frac{3}{2} \frac{M_S^2}{M_R^2} \right) \\ -\frac{M_D}{M_R} \left( 1 - \frac{3}{2} \frac{M_D^2}{M_R^2} - \frac{3}{2} \frac{M_S^2}{M_R^2} \right) & -\frac{M_D}{M_R} \left( 1 - \frac{3}{2} \frac{M_D^2}{M_R^2} - \frac{3}{2} \frac{M_S^2}{M_R^2} \right) & 1 - \frac{1}{2} \frac{M_D^2}{2M_R^2} + \frac{M_S^2}{2M_R^2} \end{pmatrix} . \quad (120) \]

To calculate the other mixing matrix \( \mathcal{U}'' \), we again follow [33]. Up to \( \mathcal{O}(\frac{M_D^2}{M_S^3}) \) the parameter \( B = \frac{M_D}{M_S} - \frac{1}{2} \frac{M_D^2}{M_S^2} \), where \( B_1 = \frac{M_D}{M_S} \) and \( B_1 = -\frac{1}{2} (\frac{M_D}{M_S})^3 \). The mixing matrix \( \mathcal{U}'' \) is (up to \( \mathcal{O}(\frac{M_D^2}{M_S^3}) \),

\[ \mathcal{U}'' = \begin{pmatrix} 1 - \frac{1}{2} \frac{M_D^2}{M_S^2} & \frac{M_D}{M_S} \left( 1 - \frac{1}{2} \frac{M_D^2}{M_S^2} \right) & 0 \\ -\frac{M_D}{M_S} \left( 1 - \frac{1}{2} \frac{M_D^2}{M_S^2} \right) & 1 - \frac{1}{2} \frac{M_S^2}{M_S^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (121) \]

Given \( \mathcal{U}' \) and \( \mathcal{U}'' \), one can straightforwardly calculate \( \mathcal{U}_1 = \mathcal{U}' \mathcal{U}'' \). Also note that, as discussed in the previous section and as well as in section [4.1.1], due to mutual cancellation between the elements of \( \mathcal{U}' \) and \( \mathcal{U}'' \), the element \( \langle \mathcal{U}_1 \rangle_{31} = 0 \). With a \( \mu \neq 0 \), it is possible to obtain a non-zero \( \langle \mathcal{U}_1 \rangle_{31} \).

Case II

For non-zero \( \mu \), the intermediate block diagonal matrix \( \hat{M}_{bd} \) changes to the following, where we have written up to \( M_R^{-3} \).

\[ \hat{M}_{bd} = \begin{pmatrix} 0 & 0 & \left( \frac{M_D M_D}{M_R} \frac{M_D M_S}{M_R} \frac{M_D M_S}{M_R} \right) \left( 1 - \frac{M_D^2}{M_R^2} - \frac{M_S^2}{M_R^2} \right) \end{pmatrix} . \quad (122) \]

The mixing matrix \( \mathcal{U}' \) (up to \( \mathcal{O}(M_R^3) \)), described in Eq. [120] now changes to \( \mathcal{U}_1 = \mathcal{U}_1^0 + \delta \mathcal{U}_1^0 \), where \( \mathcal{U}_1^0 \) is the same as \( \mathcal{U}'_1 \) of Eq. [120] while \( \delta \mathcal{U}_1^0 \) is the following,

\[ \delta \mathcal{U}_1^0 = \begin{pmatrix} 0 & -\frac{1}{2} \frac{M_D M_S}{M_R} & 0 \\ -\frac{1}{2} \frac{M_D M_S}{M_R} & -\frac{M_S^2}{M_R} & \frac{\mu M_S}{M_R} \left( \frac{\mu^2}{M_R^2} + M_S^2 \right) \\ 0 & -\frac{\mu M_S}{M_R} & -\frac{\mu^2 M_S}{M_R} \end{pmatrix} . \quad (123) \]
The other mixing matrix mixing matrix $\mathcal{U}_1''$ can be evaluated again following the parameterization \cite{90},

$$
\mathcal{U}_1'' = \left( \begin{array}{cc}
\sqrt{1 - B B^\dagger} & \frac{B}{\sqrt{1 - B^\dagger B}} \\
-B^\dagger & \sqrt{1 - B B^\dagger}
\end{array} \right)
$$

(124)

where $B_j = \mathcal{O}((\frac{M_2}{M_5})^j)$. We conclude the section with the following few remarks,

- For $\mu < \frac{M_2^2}{M_R}$, the light neutrino mass would be $m_\nu \sim \frac{M_D}{M_S} \mu \frac{M_D}{M_S}$.

- For $\mu \neq 0, \mu < \frac{M_2^2}{M_R}$, the expansion parameters $B_1$ and $B_3$ changes by $\delta B_1 \propto \frac{M_D}{M_S} \frac{\mu}{M_5/M_R}$ and $\delta B_3 \propto -\frac{5}{2} \frac{M_3^2}{M_5^2} \frac{\mu}{M_5/M_R}$, where we have only shown the dominant sub-leading correction in $\mu$. For $\mu \neq 0$, one will obtain the leading order contribution in $(\mathcal{U}_1)_{31} \sim \mathcal{O}(\frac{M_D}{M_R} \frac{\mu}{M_5/M_R})$. Also, note that the dependency of the active-sterile mixing $(\mathcal{U}_1)_{12}$ on the small lepton number violating parameter $\mu$ is as follows $(\mathcal{U}_1)_{12} \sim \frac{M_D}{M_S} \frac{\mu}{M_5/M_R}$.

- Considering leading order $B \sim \frac{M_D}{M_S}$, the mixing matrix $\mathcal{U}_1''$ will have the form given in Eq. [114].
References

[1] H. V. Klapdor-Kleingrothaus, A. Dietz, L. Baudis, G. Heusser, I. V. Krivosheina, S. Kolb, B. Majorovits, H. Pas et al., Eur. Phys. J. A12 (2001) 147-154 [hep-ph/0103062].

[2] C. Arnaboldi et al. [CUORICINO Collaboration], Phys. Rev. C78, 035502 (2008) arXiv:0802.3439 [hep-ex].

[3] C. E. Aalseth et al. [ IGEX Collaboration ], Phys. Rev. D65 (2002) 092007 [hep-ex/0202026].

[4] X. Sarazin et al. [NEMO Collaboration] [hep-ex/0006031]; J. Argyriades et al. [NEMO Collaboration], Phys. Rev. C80, 032501 (2009) arXiv:0810.0248 [hep-ex].

[5] I. Abt, M. F. Altmann, A. Bakalyarov, I. Barabanov, C. Bauer, E. Bellotti, S. T. Belyaev, L. B. Bezrukov et al., [GERDA Collaboration], Nucl. Phys. Proc. Suppl. 145, 242-245 (2005).

[6] C. Arnaboldi et al. [CUORE Collaboration], Nucl. Instrum. Meth. A518, 775-798 (2004) hep-ex/0212053.

[7] H. V. Klapdor-Kleingrothaus, I. V. Krivosheina, A. Dietz, O. Chkvorets, Phys. Lett. B586, 198-212 (2004) hep-ph/0404088; H. V. Klapdor-Kleingrothaus, I. V. Krivosheina, Mod. Phys. Lett. A21, 1547-1566 (2006).

[8] E. Conti et al. [EXO Collaboration], Phys. Rev. B68, 054201 (2003). hep-ex/0303008.

[9] R. Arnold et al. [SuperNEMO Collaboration], Eur. Phys. J. C70, 927-943 (2010) arXiv:1005.1241 [hep-ex].

[10] V. E. Guiseppe et al. [Majorana Collaboration] arXiv:0811.2446 [nucl-ex].

[11] F. Ferroni, J. Phys. Conf. Ser. 293, 012005 (2011) and J.W. Beeman et al., arXiv:1106.6286.

[12] C. Kraus [SNO+ Collaboration], http://www.sno.phy.queensu.ca/~alex/SNOLab.pdf and Prog. Part. Nucl. Phys. 57 (2006) 150.

[13] See talk of K. Inoue at Neutrino Telescope 2011, http://neutrino.pd.infn.it/Neutel2011/Program.html.

[14] K. Zuber, Phys. Lett. B519 (2001) 1 [nucl-ex/0105018]; T. Bloxham et al. [ COBRA Collaboration ], Phys. Rev. C76 (2007) 025501 [arXiv:0707.2756 [nucl-ex]].

[15] F. Granena et al. [The NEXT Collaboration], arXiv:0907.4054 [hep-ex].

[16] B. T. Cleveland et al., Astrophys. J. 496, 505 (1998); J. N. Abdurashitov et al. [SAGE Collaboration], Zh. Eksp. Teor. Fiz. 122, 211 (2002) [J. Exp. Theor. Phys. 95, 181 (2002)]; W. Hampel et al. [GALLEX Collaboration], Phys. Lett. B 447, 127 (1999); S. Fukuda et al. [Super-Kamiokande Collaboration], Phys. Lett. B 539, 179 (2002); B. Aharmim et al. [SNO Collaboration], Phys. Rev. C 72, 055502 (2005); B. Collaboration, arXiv:0708.2251 [astro-ph]. B. Aharmim et al. [ SNO Collaboration ], Phys. Rev. C81, 055504 (2010) arXiv:0910.2984 [nucl-ex].

[17] K. Eguchi et al., [KamLAND Collaboration], Phys.Rev.Lett.90 (2003) 021802; T. Araki et al. [KamLAND Collaboration], Phys. Rev. Lett. 94, 081801 (2005); A. Gando et al. [ The KamLAND Collaboration ], Phys. Rev. D83, 052002 (2011) arXiv:1009.4771 [hep-ex].

[18] Y. Fukuda et al. [ Super-Kamiokande Collaboration ], Phys. Rev. Lett. 81, 1562-1567 (1998) hep-ex/9807003; Y. Ashie et al. [Super-Kamiokande Collaboration], Phys. Rev. D 71, 112005 (2005) arXiv:hep-ex/0501064; R. Wendell et al. [ KAMIOKANDE Collaboration ], Phys. Rev. D81, 092004 (2010) arXiv:1002.3471 [hep-ex].

[19] M. Ambrosio et al. [ MACRO Collaboration ], Phys. Lett. B434, 451-457 (1998) hep-ex/9807005; M. Ambrosio et al. [ MACRO Collaboration ], Eur. Phys. J. C36, 323-339 (2004).

[20] W. W. M. Allison et al. [ Soudan-2 Collaboration ], Phys. Lett. B449, 137-144 (1999) hep-ex/9901024.

[21] E. Aliu et al. [ K2K Collaboration ], Phys. Rev. Lett. 94, 081802 (2005) hep-ex/0411038; M. H. Ahn et al. [ K2K Collaboration ], Phys. Rev. D74, 072003 (2006) hep-ex/0606032.

[22] K. Abe et al. [ T2K Collaboration ], arXiv:1106.2822 [hep-ex].

[23] P. Adamson et al. [ The MINOS Collaboration ], Phys. Rev. Lett. 106, 181801 (2011) arXiv:1103.0340 [hep-ex].

[24] M. Apollonio et al., Eur. Phys. J. C 27, 331 (2003).

[25] J. N. Bahcall, Neutrino Astrophysics, Cambridge Univ. Press (1989) 567p.

[26] R.N. Mohapatra, P. B. Pal, Massive neutrinos in physics and astrophysics, World Scientific (1991) 318p.

[27] M. Fukugita, T. Yanagida, Physics of neutrinos and applications to astrophysics, Springer (2003) 593p.

[28] C. Giunti, C. W. Kim, Fundamentals of neutrino physics and astrophysics, Oxford University Press (2007) 710p.
Measurements of Neutrino Mass, Enrico Fermi School, Vol. CLXX, ed. C. Brofferio, F. Ferroni and F. Vissani. IOS Press, Amsterdam (2009).

Seventy years of double beta decay: From nuclear physics to beyond-standard-model particle physics, H. V. Klapdor-Kleingrothaus, Hackensack, USA. World Scientific (2010).

B. M. Pontecorvo, Sov. Phys. Usp. 26 (1983) 1087-1108.

S. M. Bilenky, S. T. Petcov, Rev. Mod. Phys. 59, 671 (1987).

S. P. Mikheyev, A. Y. Smirnov, Prog. Part. Nucl. Phys. 23 (1989) 41-136.

M. Koshiha, Phys. Rept. 220 (1992) 229-381.

G. Gelmini, E. Roulet, Rept. Prog. Phys. 58, 1207-1266 (1995) [hep-ph/9412278].

K. Zuber, Phys. Rept. 305 (1998) 295 [hep-ph/9811267].

M. C. Gonzalez-Garcia, Y. Nir, Rev. Mod. Phys. 59, 671 (1987).

S. M. Bilenky, S. T. Petcov, Rev. Mod. Phys. 59, 671 (1987).

S. P. Mikheyev, A. Y. Smirnov, Prog. Part. Nucl. Phys. 23 (1989) 41-136.

M. Koshiha, Phys. Rept. 220 (1992) 229-381.

G. Gelmini, E. Roulet, Rept. Prog. Phys. 58, 1207-1266 (1995) [hep-ph/9412278].

K. Zuber, Phys. Rept. 305 (1998) 295 [hep-ph/9811267].

M. C. Gonzalez-Garcia, Y. Nir, Rev. Mod. Phys. 75, 345-402 (2003) [hep-ph/0202058].

K. Zuber, Acta Phys. Polon. B37 (2006) 1905, nucl-ex/0511009.

R. N. Mohapatra, A. Y. Smirnov, Ann. Rev. Nucl. Part. Sci. 56 (2006) 569-628 [hep-ph/0603118].

A. Strumia and F. Vissani, arXiv:hep-ph/0606054.

G. Senjanović, Riv. Nuovo Cim. 034, 1-68 (2011).

G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo, A. M. Rotunno, [arXiv:1106.6028 [hep-ph]].

M. Maltoni, T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 6, 229-381 (1992) [hep-ph/9502315]; M. Hirsch, H. V. Klapdor-Kleingrothaus, S. G. Kovalenko, Phys. Lett. B352, 1-7 (1995) [hep-ph/9506354].

S. Weinberg, Phys. Rev. Lett. 43, 1566-1570 (1979); F. Wilczek, A. Zee, Phys. Rev. Lett. 43, 1571-1573 (1979).

G. Feinberg, M. Goldhaber, Proc. Nat. Ac. Sci. USA 45, 1301 (1959); B. Pontecorvo, Phys. Lett. B26, 630-632 (1968).

R. N. Mohapatra, Phys. Rev. D34, 3457-3461 (1986).

K. S. Babu, R. N. Mohapatra, Phys. Rev. Lett. 75, 2276-2279 (1995) [hep-ph/9506354].

J. D. Vergados, Phys. Rev. D 25, 914 917 (1982); S. Bergmann, H. V. Klapdor-Kleingrothaus, H. Pas, Phys. Rev. D62, 113002 (2000) [hep-ph/0004048]; A. Faessler, T. Gutsche, S. Kovalenko, F. Smilkovic, Phys. Rev. D77, 113012 (2008) [arXiv:0710.3199 [hep-ph]].

M. Hirsch, H. V. Klapdor-Kleingrothaus, S. G. Kovalenko, Phys. Lett. B352, 1-7 (1995) [hep-ph/9502315]; M. Hirsch, H. V. Klapdor-Kleingrothaus, S. G. Kovalenko, Phys. Rev. D53, 1329-1348 (1996) [hep-ph/9502385]; M. Hirsch, H. V. Klapdor-Kleingrothaus, S. G. Kovalenko, Phys. Rev. D54, 4207-4210 (1996) [hep-ph/9603213]; M. Hirsch, H. V. Klapdor-Kleingrothaus, S. G. Kovalenko, Phys. Rev. D57, 1947-1961 (1998) [hep-ph/9707207]; M. Hirsch, J. W. F. Valle, Nucl. Phys. B557, 60-78 (1999) [hep-ph/9812463].

B. C. Allanach, C. H. Kom, H. Pas, Phys. Rev. Lett. 103, 091801 (2009) [arXiv:0902.4697 [hep-ph]].

V. Cirigliano, A. Kurylov, M. J. Ramsey-Musolf, P. Vogel, Phys. Rev. D70, 075007 (2004) [hep-ph/0404233]; V. Cirigliano, A. Kurylov, M. J. Ramsey-Musolf, P. Vogel, Phys. Rev. Lett. 93, 231802 (2004) [hep-ph/0406199].

K. W. Choi, K. S. Jeong, W. Y. Song, Phys. Rev. D66, 093007 (2002) [hep-ph/0207180].

A. Ibarra, E. Molinaro, S. T. Petcov, JHEP 1009, 108 (2010) [arXiv:1007.2378 [hep-ph]]; A. Ibarra, E. Molinaro, S. T. Petcov, arXiv:1101.5778 [hep-ph].

M. Blennow, E. Fernandez-Martinez, J. Lopez-Pavon, J. Menendez, JHEP 1007 (2010) 096 [arXiv:1005.3240 [hep-ph]].

P. Minkowski, Phys. Lett. B67, 421 (1977).

R. N. Mohapatra, G. Senjanović, Phys. Rev. Lett. 44, 912 (1980).
