NON-LEFT-ORDERABLE SURGERIES ON 1-BRIDGE BRAIDS

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ABSTRACT. Boyer, Gordon, and Watson [BGW13] have conjectured that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. Since Dehn surgeries on knots in $S^3$ can produce large families of L-spaces, it is natural to examine the conjecture on these 3-manifolds. Greene, Lewallen, and Vafaee [GLV16] have proved that all 1-bridge braids are L-space knots. In this paper, we consider three families of 1-bridge braids. First we calculate the knot groups and peripheral subgroups. We then verify the conjecture on the three cases by applying the criterion [CGHV16] developed by Christianson, Goluboff, Hamann, and Varadaraj, when they verified the same conjecture for certain twisted torus knots and generalized the criteria in [CW13] and [IT15].

1. Introduction

Let $Y$ be a rational homology sphere, and denote by $\hat{HF}(Y)$ the “hat” version of Heegaard Floer homology, as defined in [OS04b]. The following result is shown by Ozsváth and Szabó in [OS04a]: $\text{rk} \ \hat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$. As a space with minimal Heegaard Floer homology, an L-space is defined as follows:

**Definition 1.** A closed, connected, orientable 3-manifold $Y$ is an L-space if it is a rational homology sphere with the property

$$\text{rk} \ \hat{HF}(Y) = |H_1(Y;\mathbb{Z})|.$$ 

It is interesting that L-spaces might be characterized by properties of their fundamental groups, which seem to be unrelated to Heegaard Floer homology. Recall the following definition:

**Definition 2.** A non-trivial group $G$ is called left-orderable if there exists a strict total ordering $<$ on $G$ which is left-invariant, i.e.,

$$g < h \Rightarrow fg < fh, \forall f, g, h \in G.$$ 

The identity element is always denoted by symbol 1 in this paper, and the symbols $>, \leq$, and $\geq$ have the usual meaning.

In [BGW13], Boyer, Gordon, and Watson make the following conjecture that indicates a connection between L-spaces and left-orderability of their fundamental groups.

**Conjecture 3 (BGW13 Conjecture 3).** An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

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Since Dehn surgeries on knots in $S^3$ provide large families of 3-manifolds, it is natural to consider if the conjecture can be verified on them. The concept of an L-space knot is needed to simplify our discussion:

**Definition 4.** An knot $K$ in $S^3$ is an L-space knot if it admits some non-trivial Dehn surgery yielding an L-space.

Conjecture 3 has been verified for certain families of Dehn surgeries. For instance, it has been verified for sufficiently large surgery on some twisted torus knots in [CGHV16, Theorem 14]. Our goal of this paper is to verify the conjecture on another similar family of knots. The specific family of knots that will be worked on is 1-bridge braids, which are first studied by Berge and Gabai in [Ber] and [Gab90], and are a natural subset of a broad family of $(1, 1)$ knots that are L-space knots ([GLV16]). They are defined as follows:

**Definition 5 ([GLV16, Definition 1.3]).** A knot in the solid torus $D^2 \times S^1$ is a 1-bridge braid if it is isotopic to a union of two arcs $\rho \cup \tau$ such that:

- $\rho \subset \partial (D^2 \times S^1)$ is braided, i.e., transverse to each meridian $\partial D^2 \times pt$, and
- $\tau$ is a bridge, i.e., properly embedded in some meridional disk $D^2 \times pt$.

It is positive if $\rho$ is a positive braid in the usual sense. A knot in $S^3$ is a 1-bridge braid if it is isotopic to a 1-bridge braid supported in a solid torus coming from a genus-1 Heegaard splitting of $S^3$.

To present a 1-bridge braid, let

$$B_\omega = \langle \sigma_1, \sigma_2, \cdots, \sigma_{\omega-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq \omega - 2, \sigma_j \sigma_k = \sigma_k \sigma_j, 1 \leq j, k \leq \omega - 1, |j - k| \geq 2 \rangle$$

denote the braid group with $\omega$ strands, where $\sigma_i$ gives strands $i$, $i + 1$ a right-hand half twist, and the braids are composed from right to left. It follows from Gabai’s classification theorem of 1-bridge braids in the solid torus ([Gab90, Proposition 2.3]) that every 1-bridge braid in $S^3$ is of the form $B(\omega, t + m\omega, b) = \overline{B(\omega, t + m\omega, b)}$, i.e., the braid closure of

$$B(\omega, t + m\omega, b) = (\sigma_1 \sigma_2 \cdots \sigma_b)(\sigma_1 \sigma_2 \cdots \sigma_{\omega-1})^{t + m\omega} \in B_\omega,$$

$$1 \leq b \leq \omega - 2, 1 \leq t \leq \omega - 2, m \in \mathbb{Z}.$$
Theorem 6 ([GLV16, Theorem 1.4]). All 1-bridge braids in $S^3$ are L-space knots.

In particular, positive 1-bridge braids admit positive L-space surgeries. We will verify Conjecture 3 on three families of 1-bridge braids. We first claim that they are all knots, and the proof will be given in Section 2.

**Proposition 7.** $B(\omega, t + m\omega, b)$ is a knot if
\[ t = 1, b = 2k, 1 \leq k \leq [(\omega - 2)/2]. \]

**Proposition 8.** $B(\omega, t + m\omega, b)$ is a knot if
\[ \omega = 2n + 1, t = 2n - 1, b = 2k, 1 \leq k \leq n - 1. \]

**Proposition 9.** $B(\omega, t + m\omega, b)$ is a knot if
\[ \omega = 2n, t = 2n - 2, b = 2k - 1, 1 \leq k \leq n - 1. \]

The following result due to Ozsváth and Szabó completely characterizes which surgeries on an L-space knot yield L-spaces.

**Theorem 10 ([OS11]).** Let $K \subset S^3$, and suppose that there exists $\frac{p}{q} \in \mathbb{Q}_{\geq 0}$ such that $S^3_{p/q}(K)$ is an L-space. Then $S^3_{p/q}(K)$ is an L-space if and only if $\frac{p}{q} \geq 2g(K) - 1$.

In the theorem, $S^3_{p/q}(K)$ denotes $p/q$-surgery on knot $K$ in $S^3$. In order to provide a sufficient condition on the knot group of $K$ in $S^3$ to imply that $r$-surgery on $K$ yields a manifold with non-left-orderable fundamental group, the following equivalent condition for left-orderability is required.

**Theorem 11 ([Ghy01, Theorem 6.8]).** Let $G$ be a countable group. Then the following are equivalent:

- $G$ acts faithfully on the real line by order-preserving homeomorphisms.
- $G$ is left-orderable.

With this theorem, Christianson, Goluboff, Hamann, and Varadaraj generalized Ichihara and Temma’s result in [IT15], and got the following criterion for non-left-orderability:

**Theorem 12 ([CGHV16, Theorem 10]).** Let $K$ be a non-trivial knot in $S^3$. Let $G$ denote the knot group of $K$, and let $G(p/q)$ be the quotient of $G$ resulting from $p/q$-surgery. Let $\mu$ be a meridian of $K$ and $s$ be a $v$-framed longitude with $v > 0$. Suppose that $G$ has two generators, $x$ and $y$, such that $x = \mu$ and $s$ is a word which excludes $x^{-1}$ and $y^{-1}$ and contains at least one $x$. Suppose further that every homomorphism $\Phi : G(p/q) \to \text{Homeo}^+(\mathbb{R})$ satisfies $\Phi(x)t > t$ for all $t \in \mathbb{R}$ implies $\Phi(y)t \leq t$ for all $t \in \mathbb{R}$. If $p, q > 0$, then, for $p/q \geq v$, $G(p/q)$ is not left-orderable.

We will use similar notations to [CGHV16] and work with the usual order on $\mathbb{R}$. Denote by $M(\omega, t + m\omega, b, r)$ the 3-manifold produced by $r$-surgery on $B(\omega, t + m\omega, b)$, and let $G(\omega, t + m\omega, b) = \pi_1(S^3 \setminus B(\omega, t + m\omega, b))$. $G(\omega, t + m\omega, b, r) = \pi_1(M(\omega, t + m\omega, b, r))$. Throughout, we will assume $\omega \geq 3$, $1 \leq t \leq \omega - 2$, $1 \leq b \leq \omega - 2$, $m \geq 0$, $r \in \mathbb{Q}_{\geq 0}$.

In this paper, we apply Theorem 12 to prove the following result.

**Theorem 13.** For sufficiently large $r$, the L-spaces produced by $r$-surgery on $B(\omega, t + m\omega, b)$, where $\omega$, $t$, and $b$ satisfy the conditions in Propositions 7, 8, and $m \geq 0$.
all have non-left-orderable fundamental groups, i.e., \( G(\omega, 1 + m\omega, 2k, r) \), \( G(2n + 1, 2n-1+m(2n+1), 2k, r) \), and \( G(2n, 2n-2+2mn, 2k-1, r) \) are non-left-orderable if \( r \geq \omega + 2k-1 \), \( r \geq 2n[2n-1+m(2n+1)]+2k \), and \( r \geq (2n-1)(2n-2+2mn)+2k-1 \) respectively.

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2. Proving All the Three Families of 1-bridge Braids are Knots

First, we define a surjective group homomorphism \( h : B_\omega \rightarrow S_\omega \) by mapping every braid \( \sigma_i \) to \( s_i \) in symmetric group

\[
S_\omega = (s_1, s_2, \ldots, s_{\omega-1}|s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, 1 \leq i \leq \omega - 2),
\]

\[
s_j s_k = s_k s_j, s_j^2 = 1, 1 \leq j, k \leq \omega - 1, |j - k| \geq 2).
\]

For any braid \( \sigma \in B_\omega \), \( h(\sigma) \) is called the permutation induced by \( \sigma \), so \( \sigma \) acts on the \( \omega \) endpoints of the braid by permutation \( h(\sigma) \). The permutations are composed from right to left. Denote \( h(B(\omega, t + m\omega, b))(i) \) briefly by \( B(\omega, t + m\omega, b)(i) \).

In this section, we prove Propositions 7, 8, and 9.

**Proof of Proposition 7** We claim that \( B(\omega, 1 + m\omega, 2k) \) induces the action on the endpoints by the permutation

\[
p = (1, 3, 5, \ldots, 2k - 1, 2k + 1, 2k + 2, 2k + 3, \ldots, \omega, 2, 4, 6, \ldots, 2k),
\]

by forgetting how the strands twist and cross, i.e., if strand \( i \) is sent to be strand \( j \) by \( B(\omega, 1 + m\omega, 2k) \), then \( p(i) = j \) (see Figure 2).

In fact, \( B(\omega, 1 + m\omega, 2k) \) has the expression:

\[
B(\omega, 1 + m\omega, 2k) = (\sigma_1 \sigma_2 \cdots \sigma_{2k})(\sigma_1 \sigma_2 \cdots \sigma_{\omega-1})^{1 + m\omega}.
\]

If \( 1 \leq i \leq 2k-1 \) is odd, \((\sigma_1 \sigma_2 \cdots \sigma_{\omega-1})^{1 + m\omega}(i) = i + 1 \leq 2k \); and \((\sigma_1 \sigma_2 \cdots \sigma_{2k})(i + 1) = i + 2 = p(i) \).

Similarly, if \( 2k+1 \leq i \leq \omega - 1 \), \((\sigma_1 \sigma_2 \cdots \sigma_{\omega-1})^{1 + m\omega}(i) = i + 1 \geq 2k + 2 > 2k + 1 \), so \((\sigma_1 \sigma_2 \cdots \sigma_{2k})(i + 1) = i + 1 \). Thus, \( B(\omega, 1 + m\omega, 2k)(i) = i + 1 = p(i) \).

For strand \( \omega \), \((\sigma_1 \sigma_2 \cdots \sigma_{\omega-1})^{1 + m\omega}(\omega) = 1 \), and then \((\sigma_1 \sigma_2 \cdots \sigma_{2k})(1) = 2 \), so \( B(\omega, 1 + m\omega, 2k)(\omega) = 2 = p(\omega) \).
If $2 \leq i \leq 2k - 2$ is even, $(\sigma_1 \sigma_2 \cdots \sigma_{i-1})^{1+m\omega}(i) = i + 1 \leq 2k - 1$, and $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(i+1) = i + 2$, so $B(\omega, 1 + m\omega, 2k)(i) = i + 2 = p(i)$.

Finally, for strand $2k$, $(\sigma_1 \sigma_2 \cdots \sigma_{i-1})^{1+m\omega}(2k) = 2k+1$, and $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(2k+1) = 1$, so $B(\omega, 1 + m\omega, 2k)(2k) = 1 = p(2k)$.

Therefore, $B(\omega, 1 + m\omega, 2k)$ is a knot. \hfill \square

**Proof of Proposition** We claim that $B(2n + 1, 2n - 1 + m(2n+1), 2k)$ induces the action on the endpoints by the permutation

$$p = (1, 2n, 2n - 2, 2n - 4, \ldots, 2k + 2, 2k + 1, 2k, \cdots, 2, 2n + 1, 2n - 1, 2n - 3, \cdots, 2k + 3),$$

as shown in Figure 3.

In fact, $B(2n + 1, 2n - 1 + m(2n+1), 2k)$ has the expression:

$$B(2n + 1, 2n - 1 + m(2n+1), 2k) = (\sigma_1 \sigma_2 \cdots \sigma_{2n}) \cdot (\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}.$$

For strand $2k$, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(1) = 2n$. Then, since $n \geq k + 1$, $2n \geq 2k + 2 > 2k + 1$, $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(2n) = 2n$. Thus, $B(2n + 1, 2n - 1 + m(2n+1), 2k)(1) = 2n = p(1)$.

For strand $2k + 4 \leq i \leq 2n + 1$ an even number, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(i) \equiv i + 2n - 1 \mod 2n + 1).$ Since $1 < 2k + 2 \leq i + 2n - 1 - \omega = i - 2 \leq 2n - 2 < \omega$, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(i) = i - 2$. Then, since $i - 2 \geq 2k + 2 > 2k + 1$, $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(i - 2) = i - 2$. Therefore, $B(2n + 1, 2n - 1 + m(2n+1), 2k)(i) = i - 2 = p(i)$.

For strand $3 \leq i \leq 2k + 2$, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(i) \equiv i + 2n - 1 \mod 2n + 1).$ Since $1 \leq i + 2n - 1 - \omega = i - 2 \leq 2k < \omega$, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(i) = i - 2$. Then, since $1 \leq i - 2 \leq b$, $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(i - 2) = i - 1$. Therefore, $B(2n + 1, 2n - 1 + m(2n+1), 2k)(i) = i - 1 = p(i)$.

For strand $2k + 5 \leq i \leq 2n + 1$ an odd number, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(i) \equiv i + 2n - 1 \mod 2n + 1).$ Since $1 \leq 2k + 3 \leq i + 2n - 1 - \omega = i - 2 \leq 2n - 1 < \omega$, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(i) = i - 2$. Then, since $i - 2 \geq 2k + 3 > 2k + 1$, $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(i - 2) = i - 2$. Therefore, $B(2n + 1, 2n - 1 + m(2n+1), 2k)(i) = i - 2 = p(i)$.

Finally, for strand $2k + 3$, we have $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(2k + 3) \equiv 2k + 2n + 2 \mod 2n + 1).$ Then, since $1 \leq 2k + 2n + 2 - \omega = 2k + 1 \leq 2n - 1 < \omega$, $(\sigma_1 \sigma_2 \cdots \sigma_{2n})^{2n - 1 + m(2n+1)}(2k + 3) = 2k + 1$. Since $(\sigma_1 \sigma_2 \cdots \sigma_{2k})(2k + 1) = 1$, $B(2n + 1, 2n - 1 + m(2n+1), 2k)(2k + 3) = 1 = p(2k + 3)$.

Therefore, $B(2n + 1, 2n - 1 + m(2n+1), 2k)$ is a knot. \hfill \square
Proof of Proposition 9. We claim that \( B(2n, 2n-2+2mn, 2k-1) \) induces the action on the endpoints by the permutation

\[ p = (1, 2n-1, 2n-3, 2n-5, \ldots, 2k+1, 2k, 2k-1, \ldots, 2, 2n-2, 2n-4, \ldots, 2k+2), \]

as shown in Figure 4.

In fact, since \( B(2n, 2n-2+2mn, 2k-1) \) has the expression:

\[ B(2n, 2n-2+2mn, 2k-1) = (\sigma_1 \sigma_2 \cdots \sigma_{2k-1})(\sigma_1 \sigma_2 \cdots \sigma_{2n-1})^{2n-2+2mn}, \]

\( (\sigma_1 \sigma_2 \cdots \sigma_{2n-1})^{2n-2+2mn}(1) = 2n + 1 \). Then, since \( 2n-1 \geq 2k+1 > 2k \), \( (\sigma_1 \sigma_2 \cdots \sigma_{2k-1})(2n+1) = 2n + 1 \). Thus, \( B(2n, 2n-2+2mn, 2k-1)(1) = 2n + 1 = p(1) \).

For strand \( 2k + 3 \leq i \leq 2n-1 \) an odd number, \( (\sigma_1 \sigma_2 \cdots \sigma_{2n-1})^{2n-2+2mn}(i) \equiv i + 2n - 2 (\text{mod } 2n) \). Since \( 1 < 2k+1 \leq i + 2n - 2 - \omega = i - 2 \leq 2n - 3 < \omega \), \( (\sigma_1 \sigma_2 \cdots \sigma_{2n-1})^{2n-2}(i) = i - 2 \). Then, since \( i - 2 \geq 2k+1 > 2k \), \( (\sigma_1 \sigma_2 \cdots \sigma_{2k-1})(i - 2) = i - 2 \). Therefore, \( B(2n, 2n-2+2mn, 2k-1)(i) = i - 2 = p(i) \).

The remainder of the proof of the claim follows analogously to the proof of Proposition 8.

Therefore, \( B(2n, 2n-2+2mn, 2k-1) \) is a knot. \( \square \)

3. Computing Knot Groups and Peripheral Subgroups

It is a fact that (see [Lic97]), for a non-trivial knot, the fundamental group of the boundary of the knot complement injects into the knot group, and its image up to conjugation is the peripheral subgroup, so all peripheral subgroups are abelian. We first derive the knot groups of \( B(\omega, 1 + m\omega, 2k) \), \( B(2n+1, 2n-1 + m(2n+1), 2k) \), and \( B(2n, 2n-2+2mn, 2k-1) \). We will achieve this for all three cases by the same way both Clay and Watson in [CW13] and Christianson et al. in [CGHV16] did: by finding a genus-2 Heegaard splitting of \( S^3 \) with the knot embedded on the Heegaard surface, we can apply Seifert-van Kampen Theorem on the two handlebodies of genus 2 (see Figures 6, 10, and 14).

Proposition 14. For \( B(\omega, 1 + m\omega, 2k) \),

(a) The knot group is

\[ G(\omega, 1 + m\omega, 2k) = \langle \alpha, \beta | (\beta \alpha)^k \beta = (\beta \alpha)^k \beta \alpha^{-2k} (\beta \alpha)^k \beta^{-k} \rangle. \]

(b) The peripheral subgroup is generated by the meridian

\[ \mu = \alpha \]
Figure 5. Basepoint $x_0$ and generators $\alpha, \beta, \delta, \gamma$ for the fundamental group of the Heegaard surface $\Sigma$.

Figure 6. The red graph is homotopy equivalent to the complement of $B(\omega, 1 + m\omega, 2k)$ on the Heegaard surface $\Sigma$. Here, we show $B(5, 1, 2)$ as an example. The knot is shown in gray.
Figure 7. The first generator for the fundamental group of $\Sigma \setminus \nu(B(\omega, 1 + m\omega, 2k))$.

and the surface framing

$$s = \mu^{(\omega-1)(1+m\omega)+2k}\lambda = \alpha(\beta\alpha)^k\beta^{\omega-2k}(\alpha\beta)^k.$$ 

Proof. Let $S^3 = U \cup \Sigma V$ be the genus-2 Heegaard splitting of $S^3$ with the knot $B(\omega, 1 + m\omega, 2k)$ embedded on the Heegaard surface, as shown in Figure 6. Then $\pi_1(U)$ is the free group on the generators $\alpha$ and $\beta$, and $\pi_1(V)$ is the free group on the generators $\delta$ and $\gamma$ (see Figure 5). Using Seifert-van Kampen Theorem, we can then express $G(\omega, 1 + m\omega, 2k)$ as a free product with amalgamation of $\pi_1(U)$ and $\pi_1(V)$. In order to figure out the amalgamation, we need the images of the generators of $\pi_1(\Sigma \setminus \nu(B(\omega, 1 + m\omega, 2k))$ under the map induced by inclusion into $\pi_1(U)$ and $\pi_1(V)$.

Now, $\Sigma \setminus \nu(B(\omega, 1 + m\omega, 2k))$ is homotopy equivalent to a twice-punctured genus-1 surface whose fundamental group is generated by the green, purple, and olive loops in Figures 7, 8, and 9. The green loop has image $\alpha(\beta\alpha)^k\beta^{\omega-2k}\alpha^{-1}$ in $\pi_1(U)$ and image $\gamma^{m(\omega-k)+1}$ in $\pi_1(V)$, so we get the relation

$$\alpha(\beta\alpha)^k\beta^{\omega-2k}\alpha^{-1} = \gamma^{m(\omega-k)+1}.$$

Likewise, the purple loop gives

$$\alpha(\beta\alpha)^k\beta^{-1} = \gamma^{m(k+1)}\delta,$$

and the olive loop gives

$$\alpha(\alpha\beta)^k = \gamma^{mk}\delta.$$

The second and the third relation give us the expression for $\gamma^m$ in terms of $\alpha$ and $\beta$:

$$\gamma^m = \alpha(\beta\alpha)^k\beta^{\omega-2k}(\alpha\beta)^{-1}.$$
Then by the first relation, replacing $\gamma^m(\omega-k)$ of the right side, we can get an expression for $\gamma$ in terms of $\alpha$ and $\beta$:

$$\gamma = \alpha(\beta\alpha)^k\beta\omega^{-2k}\alpha^{-1}\gamma^{-m(\omega-k)} = \alpha(\beta\alpha)^k\beta\omega^{-2k}\alpha^{-1}[\alpha(\beta\alpha)^k\beta\alpha^{-1}(\alpha\beta)^{-k}\alpha^{-1}]^{-\omega+k} = \alpha(\beta\alpha)^k\beta\omega^{-2k}[(\beta\alpha)^k\beta\alpha^{-1}(\alpha\beta)^{-k}]^{-\omega+k}\alpha^{-1}.$$ 

Also, from the third relation, we can write $\delta$ in terms of $\alpha$ and $\beta$:

$$\delta = \gamma^{-mk}\alpha(\alpha\beta)^k = [\alpha(\beta\alpha)^k\beta\alpha^{-1}(\alpha\beta)^{-k}]^{-k}\alpha(\alpha\beta)^k = \alpha[(\beta\alpha)^k\beta\alpha^{-1}(\alpha\beta)^{-k}]^{-k}(\alpha\beta)^k.$$ 

We can substitute these three relations for the original ones. The last two relations provide us with expressions for $\gamma$ and $\delta$ in terms of $\alpha$ and $\beta$. By getting rid of $\gamma$ of the expression for $\gamma^m$, we are left with only one relation:

$$(\beta\alpha)^k\beta = [(\beta\alpha)^k\beta\omega^{-2k}[(\beta\alpha)^k\beta\alpha^{-1}(\alpha\beta)^{-k}]^{-\omega+k}]^m(\alpha\beta)^k\alpha.$$ 

Thus, we have

$$G(\omega, 1 + m\omega, 2k) = \langle \alpha, \beta | (\beta\alpha)^k\beta = [(\beta\alpha)^k\beta\omega^{-2k}[(\beta\alpha)^k\beta\alpha^{-1}(\alpha\beta)^{-k}]^{-\omega+k}]^m(\alpha\beta)^k\alpha \rangle.$$ 

For the peripheral subgroup, we will compute the meridian $\mu$ and the surface framing $s$ which is a push-off of the knot $B(\omega, 1 + m\omega, 2k)$ in $\Sigma$. It is clear that

$$s = \gamma^{1+m\omega} \delta = \alpha[(\beta\alpha)^k\beta\omega^{-2k}(\alpha\beta)^k].$$ 

Using the relation in the expression of $G(\omega, 1 + m\omega, 2k)$, $s$ can be simplified as follows

$$s = \gamma^{1+m\omega} \delta = \alpha(\beta\alpha)^k\beta\omega^{-2k}(\alpha\beta)^k.$$
Figure 9. The third generator for the fundamental group of $\Sigma \setminus \nu(B(\omega, 1 + m\omega, 2k))$.

In order to compute $\mu$, consider the meridian based at $x_0$. It can be isotoped to be sitting in the handlebody $U$, and then it goes around the right genus of the handlebody once, so is isotopic to the generator $\alpha$. Thus,

$$\mu = \alpha.$$

Finally, we note that the linking number between $B(\omega, 1 + m\omega, 2k)$ and a push-off in $\Sigma$, by the construction of the 1-bridge braid, is $(\omega - 1)(1 + m\omega) + 2k$, which gives us

$$s = \mu^{(\omega-1)(1+m\omega)+2k} \lambda.$$

Proposition 15. For $B(2n + 1, 2n - 1 + m(2n + 1), 2k)$,

(a) The knot group is

$$G(2n + 1, 2n - 1 + m(2n + 1), 2k) = \langle \alpha, \beta \mid (\alpha \beta^{n-k+1} \alpha \beta^{-n+k})^{n+k} (\alpha \beta^{-2k+1} \beta^{-n+k-1})^{m+1} = \alpha \beta^{n-k+1} \alpha \beta^{-n+k} \rangle.$$

(b) The peripheral subgroup is generated by the meridian

$$\mu = \alpha$$

and the surface framing

$$s = \mu^{2n[2n-1+m(2n+1)]+2k\lambda} = \alpha \beta^{2n-k+1} (\alpha \beta)^{2k-1} \alpha \beta^{n-k+1}.$$

Proof. Let $S^3 = U \cup_{\Sigma} V$ again be the genus-2 Heegaard splitting of $S^3$ specified in Figure 10. Then $\pi_1(U)$ is the free group on the generators $\alpha$ and $\beta$, and $\pi_1(V)$ is the free group on the generators $\delta$ and $\gamma$ (see Figure 11). Using Seifert-van Kampen Theorem, we can then write $G(2n + 1, 2n - 1 + m(2n + 1), 2k)$ as a free product
Figure 10. The red graph is homotopy equivalent to the complement of $B(2n + 1, 2n - 1 + m(2n + 1), 2k)$ on the Heegaard surface $\Sigma$. Here, we show $B(5, 3, 2)$ as an example. The knot is shown in gray.

with amalgamation of $\pi_1(U)$ and $\pi_1(V)$. To work out the amalgamation, we need the images of the generators of $\pi_1(\Sigma \setminus \nu(B(2n + 1, 2n - 1 + m(2n + 1), 2k)))$ under the maps induced by inclusion into $\pi_1(U)$ and $\pi_1(V)$.

Now, $\Sigma \setminus \nu(B(2n + 1, 2n - 1 + m(2n + 1), 2k))$ is homotopy equivalent to a twice-punctured genus-1 surface whose fundamental group is generated by the green, purple, and olive loops in Figures 11, 12, and 13. The image of the green loop is $\alpha\beta^{n-k+1} \alpha^{2k-1} \alpha^{-1}$ under the map induced by inclusion to $\pi_1(U)$, and is $\gamma^{n+k-1+m(n+k)}$ under the map induced by inclusion to $\pi_1(V)$, so we get the relation

$$\alpha\beta^{n-k+1} \alpha^{2k-1} \alpha^{-1} = \gamma^{(m+1)(n+k)-1}.$$  

Likewise, from the purple loop we get

$$\alpha\beta^{n-k} \alpha^{-1} = \gamma^{(m+1)(n-k)-1},$$

and from the olive loop we get

$$\alpha^2 \beta^{n-k+1} = \gamma^{(m+1)(n-k+1)-1}.$$

Using the second and the third relation, we have

$$\alpha^2 \beta^{n-k+1} = \gamma^{(m+1)(n-k+1)-1} \delta = \gamma^{m+1} \gamma^{(m+1)(n-k)-1} \delta = \gamma^{m+1} \alpha\beta^{n-k-1} \alpha^{-1}.$$  

$$\Leftrightarrow \gamma^{m+1} = \alpha^2 \beta^{n-k+1} \alpha^{-n+k} \alpha^{-1}.$$
Figure 11. The first generator for the fundamental group of $\Sigma \setminus \nu(B(2n + 1, 2n - 1 + m(2n + 1), 2k))$.

Figure 12. The second generator for the fundamental group of $\Sigma \setminus \nu(B(2n + 1, 2n - 1 + m(2n + 1), 2k))$. 
Figure 13. The third generator for the fundamental group of $\Sigma \setminus \nu(B(2n + 1, 2n - 1 + m(2n + 1), 2k))$.

This relation can replace the second relation. Moreover, it provides us with an expression for $\gamma^{m+1}$ in terms of $\alpha$ and $\beta$, so with the first relation, we have

$$
\gamma = \gamma^{(m+1)(n+k)}(\alpha\beta)^{-2k+1} \beta^{-n+k-1} \alpha^{-1} = (\alpha^2 \beta^{n-k+1} \alpha \beta^{-n+k} \alpha^{-1})^{n+k} \alpha(\alpha \beta)^{-2k+1} \beta^{-n+k-1} \alpha^{-1} = \alpha(\alpha \beta^{n-k+1} \alpha \beta^{-n+k})^{n+k} (\alpha \beta)^{-2k+1} \beta^{-n+k-1} \alpha^{-1},
$$

which can replace the first relation. We can also write $\delta$ in terms of $\alpha$ and $\beta$ by replacing $\gamma$ of the second relation with its expression in terms of $\alpha$ and $\beta$:

$$
\delta = \gamma^{-(m+1)(n-k)+1} \alpha \beta^{n-k} \alpha^{-1} = [\alpha(\alpha \beta^{n-k+1} \alpha \beta^{-n+k})^{n+k} (\alpha \beta)^{-2k+1} \beta^{-n+k-1} \alpha^{-1}]^{-(m+1)(n-k)+1} \alpha \beta^{n-k} \alpha^{-1} = \alpha(\alpha \beta^{n-k+1} \alpha \beta^{-n+k})^{n+k} (\alpha \beta)^{-2k+1} \beta^{-n+k-1} \alpha^{-1}.
$$

By substitution, we are left with only one relation:

$$
[(\alpha \beta^{n-k+1} \alpha \beta^{-n+k})^{n+k} (\alpha \beta)^{-2k+1} \beta^{-n+k-1}]^{m+1} = \alpha \beta^{n-k+1} \alpha \beta^{-n+k}.
$$

Thus, we have

$$
G(2n + 1, 2n - 1 + m(2n + 1), 2k) = \langle \alpha, \beta \mid (\alpha \beta^{n-k+1} \alpha \beta^{-n+k})^{n+k} (\alpha \beta)^{-2k+1} \beta^{-n+k-1}]^{m+1} = \alpha \beta^{n-k+1} \alpha \beta^{-n+k} \rangle.
$$

For the peripheral subgroup, the surface framing $s$ is a push-off of the knot $B(2n + 1, 2n - 1 + m(2n + 1), 2k)$ along $\Sigma$, so

$$
s = \gamma^{2n-1+m(2n+1)} \delta.
$$

We note that this is actually the product of the right sides of the first and the
Figure 14. The red graph is homotopy equivalent to the complement of $B(2n, 2n - 2 + 2mn, 2k - 1)$ on the Heegaard surface $\Sigma$. Here, we show $B(6, 4, 3)$ as an example. The knot is shown in gray.

third relations, so we get

$$s = \alpha^2 \beta^{n-k+1} (\alpha \beta)^{2k-1} \alpha \beta^{n-k+1}.$$

In order to compute $\mu$, for the same reason as in the proof of Proposition 14, since the meridian based at $x_0$ can be isotoped to be going around the right genus of the handlebody $U$ once as in Figure 10, it is isotopic to the generator $\alpha$. Thus,

$$\mu = \alpha.$$

Finally, we note that the linking number between $B(2n+1, 2n-1+m(2n+1), 2k)$ and a push-off along $\Sigma$, by the construction of the 1-bridge braid, is $2n[2n-1 + m(2n+1)] + 2k$, which gives us

$$s = \mu^{2n[2n-1 + m(2n+1)] + 2k} \lambda.$$

Proposition 16. For $B(2n, 2n - 2 + 2mn, 2k - 1)$,

(a) The knot group is

$$G(2n, 2n - 2 + 2mn, 2k - 1) = \langle \alpha, \beta | \alpha \beta^{n-k+1} (\alpha \beta)^{n-k+1} (\alpha \beta)^{2k+2} \beta^{-n+k-1} \rangle^{m+1} = \alpha \beta^{n-k+1} \alpha \beta^{-n+k}.$$

(b) The peripheral subgroup is generated by the meridian

$$\mu = \alpha$$

and the surface framing

$$s = \mu^{(2n-1)(2n-2+2mn)+2k-1} \lambda = \beta^{n-k+1} (\alpha \beta)^{2k-2} \alpha \beta^{n-k+1} \alpha.$$
Figure 15. The first generator for the fundamental group of $\Sigma \setminus \nu(B(2n, 2n - 2 + 2mn, 2k - 1))$.

Proof. We use the same method as in the proof of the Propositions 14 and 15. Let $S^3 = U \cup_\Sigma V$ be the genus-2 Heegaard splitting. Then $\pi_1(U)$ and $\pi_1(V)$ are the free group on the generators $\alpha$ and $\beta$ and the free group on the generators $\delta$ and $\gamma$ respectively (see Figure 5). Using Seifert-van Kampen Theorem, $G(2n, 2n - 2 + 2mn, 2k - 1)$ is a free product with amalgamation of $\pi_1(U)$ and $\pi_1(V)$. Now we will give the images of the generators of $\pi_1(\Sigma \setminus \nu(B(2n, 2n - 2 + 2mn, 2k - 1)))$ under the maps induced by inclusion into $\pi_1(U)$ and $\pi_1(V)$.

$\Sigma \setminus \nu(B(2n, 2n - 2 + 2mn, 2k - 1))$ is homotopy equivalent to a twice-punctured genus-1 surface whose fundamental group is generated by the green, purple, and olive loops in Figures 15, 16, and 17. The image of the green loop is $\alpha\beta^{n-k+1}(\alpha\beta)^{2k-2}\alpha^{-1}$ under the map induced by inclusion to $\pi_1(U)$, and is $\gamma^{(m+1)(n+k-1)-1}$ under the map induced by inclusion to $\pi_1(V)$, so we get the relation

$$\alpha\beta^{n-k+1}(\alpha\beta)^{2k-2}\alpha^{-1} = \gamma^{(m+1)(n+k-1)-1}.$$  

As for the purple loop, we have

$$\alpha\beta^{n-k}\alpha^{-1} = \gamma^{(m+1)(n-k)-1}\delta,$$

and for the olive loop we have

$$\alpha^2\beta^{n-k+1} = \gamma^{(m+1)(n-k+1)-1}\delta.$$  

Using the second and the third relation, we have

$$\alpha^2\beta^{n-k+1} = \gamma^{(m+1)(n-k+1)-1}\delta = \gamma^{m+1}\gamma^{(m+1)(n-k)-1}\delta = \gamma^{m+1}\alpha\beta^{n-k}\alpha^{-1}.$$  

$$\Leftrightarrow \gamma^{m+1} = \alpha^2\beta^{n-k+1}\alpha\beta^{-n+k}\alpha^{-1}.$$
Figure 16. The second generator for the fundamental group of $\Sigma \setminus \nu(B(2n, 2n - 2 + 2mn, 2k - 1))$.

This relation can replace the second relation. Moreover, it provides us with an expression for $\gamma^{m+1}$ in terms of $\alpha$ and $\beta$, so with the first relation, we have

$$
\gamma = \gamma^{(m+1)(n+k-1)} \alpha(\alpha \beta)^{-2k+2} \beta^{-n+k-1} \alpha^{-1}
= (\alpha^2 \beta^{-n-k+1} \alpha \beta^{-n+k-1})^{n+k-1} \alpha(\alpha \beta)^{-2k+2} \beta^{-n+k-1} \alpha^{-1}
= \alpha(\alpha \beta^{-n+k+1} \alpha \beta^{-n+k-1})^{n+k-1} (\alpha \beta)^{-2k+2} \beta^{-n+k-1} \alpha^{-1},
$$

which can replace the first relation. We can also write $\delta$ in terms of $\alpha$ and $\beta$ by replacing $\gamma$ of the second relation with its expression in terms of $\alpha$ and $\beta$:

$$
\delta = \gamma^{-(m+1)(n-k)+1} \alpha \beta^{-n-k} \alpha^{-1}
= \alpha(\alpha \beta^{-n+k+1} \alpha \beta^{-n+k-1})^{n+k-1} \alpha(\alpha \beta)^{-2k+2} \beta^{-n+k-1} \alpha^{-1}
= \alpha((\alpha \beta^{-n+k+1} \alpha \beta^{-n+k-1})^{n+k-1} (\alpha \beta)^{-2k+2} \beta^{-n+k-1})^{m+1}(n+k) \beta^{-n-k} \alpha^{-1}.
$$

By substitution, we are left with only one relation:

$$
[\alpha^2 \beta^{-n-k+1} \alpha \beta^{-n+k-1} (\alpha \beta)^{-2k+2} \beta^{-n+k-1}]^{m+1} = \alpha \beta^{-n-k+1} \alpha \beta^{-n+k}.
$$

Thus, we have

$$
G(2n, 2n - 2 + 2mn, 2k - 1) = \langle \alpha, \beta | (\alpha \beta^{-n+k-1} \alpha \beta^{-n+k-1} (\alpha \beta)^{-2k+2} \beta^{-n+k-1})^{m+1} = \alpha \beta^{-n-k+1} \alpha \beta^{-n+k} \rangle.
$$

For the peripheral subgroup,

$$
s = \gamma^{2n+2mn-2} \delta,
$$

which is actually the product of the right sides of the first and the third relations, so we get

$$
s = \alpha \beta^{-n-k+1} (\alpha \beta)^{-2k+2} \alpha \beta^{-n-k+1}.
$$
Finally, we note that the linking number between $B(2n, 2n - 2 + 2mn, 2k - 1)$ and a push-off along $\Sigma$ is $(2n - 1)(2n - 2 + 2mn) + 2k - 1$, i.e.,

$$s = \mu^{(2n-1)(2n-2+2mn)+2k-1}\lambda.$$ 

\[\Box\]

4. Non-Left-Orderability of Certain 1-bridge Braids

In this section, we prove Theorem 13 by applying Theorem 12 on the three families of 1-bridge braids specified in Propositions 14, 15, and 16.

First, from Propositions 14, 15, and 16 we note that in each of the three cases, the knot group $G(\omega, t + m\omega, b)$ can be generated by $\alpha, \beta$, satisfying $\mu = \alpha$, $s$ is a word that excludes $\alpha^{-1}$ and $\beta^{-1}$ and contains at least one $\alpha$. Furthermore, the framing of the longitude $s$ is always $(\omega - 1)(t + m\omega) + b > 0$. Therefore, it is sufficient to prove that for any homomorphism $\Phi : G(\omega, t + m\omega, b) \to \text{Homeo}^+(\mathbb{R})$, $\alpha t > t$ for all $t \in \mathbb{R}$ implies $\beta t \geq t$ for all $t \in \mathbb{R}$, where $\omega t$ denotes a shorthand for $\Phi(\omega)t$ for any word $w$.

It is a fact that if $a$ and $b$ are real numbers, $a < b$, and $w \in \text{Homeo}^+(\mathbb{R})$, then $wa < wb$. We give the following two lemmas about groups acting on the real line that are needed in our proof of Theorem 13.

**Lemma 17.** Let $\Phi : G \to \text{Homeo}^+(\mathbb{R})$ and $\alpha \in G$. If $\alpha t > t$ for all $t \in \mathbb{R}$, then $w_1w_2t > w_1\alpha^{-1}w_2t$ for all $t \in \mathbb{R}$ and any $w_1, w_2 \in G$. 

---

**Figure 17.** The third generator for the fundamental group of $\Sigma \setminus \nu(B(2n, 2n - 2 + 2mn, 2k - 1))$. 

$$\mu = \alpha.$$ 

Finally, we note that the linking number between $B(2n, 2n - 2 + 2mn, 2k - 1)$ and a push-off along $\Sigma$ is $(2n - 1)(2n - 2 + 2mn) + 2k - 1$, i.e.,

$$s = \mu^{(2n-1)(2n-2+2mn)+2k-1}\lambda.$$ 

\[\Box\]
Proof. For any words \( w_1, w_2 \in G, w_1t > \alpha^{-1}w_1t \), since \( \alpha' > t' \) for \( t' = \alpha^{-1}w_1t \). Then since \( \Phi(w_2) \in \text{Homeo}^+(\mathbb{R}) \), applying \( \Phi(w_2) \) on both sides, we have \( w_2w_1t > w_2\alpha^{-1}w_1t \).

Lemma 18. Let \( \Phi : G \to \text{Homeo}^+(\mathbb{R}) \) and \( w_i \in G, i = 1, 2, 3, 4 \). If \( w_1t > w_2t \) for all \( t \in \mathbb{R} \) and \( w_3t > w_4t \) for all \( t \in \mathbb{R} \), then \( w_1w_3t > w_2w_4t \) for all \( t \in \mathbb{R} \).

Proof. First we have \( w_1w_3t > w_2w_3t \), since \( w_1t' > w_2t' \) for \( t' = w_3t \). Since \( \Phi(w_2) \in \text{Homeo}^+(\mathbb{R}) \), we get \( w_2w_3t > w_2w_4t \). Thus \( w_1w_3t > w_2w_3t > w_2w_4t \) for all \( t \in \mathbb{R} \).

Lemma 19. For any homomorphism \( \Phi : G(\omega, 1 + m\omega, 2k) \to \text{Homeo}^+(\mathbb{R}), 1 \leq k \leq \lfloor (\omega - 2)/2 \rfloor, m \geq 0, \alpha > t \) for all \( t \in \mathbb{R} \) implies \( \beta t \geq t \) for all \( t \in \mathbb{R} \).

Proof. Recall the expression for the knot group:
\[
G(\omega, 1 + m\omega, 2k) = \langle \alpha, \beta \mid (\beta\alpha)^k \beta = \{ \{ \beta\alpha \}^k \beta \omega - 2k \{ \beta\omega \}^k \beta^{-1} \omega - k \{ \beta\alpha \}^k \omega \beta^{-1} \beta^{-1} \}^m \{ \beta\alpha \}^k \rangle.
\]
The relation can be rewritten slightly as:
\[
1 = \{ \{ \beta\alpha \}^k \beta \omega - 2k \{ \beta\omega \}^k \beta^{-1} \omega - k \{ \beta\alpha \}^k \omega \beta^{-1} \beta^{-1} \}^m \{ \beta\alpha \}^k \beta t.
\]
Applying both sides of the relation on \( t \), we have
\[
1t = \{ \{ \beta\alpha \}^k \beta \omega - 2k \{ \beta\omega \}^k \beta^{-1} \omega - k \{ \beta\alpha \}^k \omega \beta^{-1} \beta^{-1} \}^m \{ \beta\alpha \}^k \beta^{-1} t.
\]
where \( C_0 = \{ \{ \beta\alpha \}^k \alpha \beta^{-1} \}^m \{ \beta\alpha \}^k \).

Assume \( \alpha > t \) for all \( t \in \mathbb{R} \). By Lemmas 17 and 18, we can remove all \( \alpha \)'s from \( \{ \beta\alpha \}^k \) and the \( \alpha \) in the middle from \( C_0 \) and get \( \beta^k \) and \( \beta^{-1} \) at the expense of getting an appropriate strict inequality on any \( t \), so we have
\[
1t > \{ \beta^k \beta \omega - 2k \beta^{-1} \omega \}^m \beta^{-1} t = \beta^{-1} t.
\]

Since \( \beta \) is order-preserving, applying it on both sides, we get
\[
\beta t > t
\]
for all \( t \in \mathbb{R} \).

Remark. If \( m = 0 \), by applying Markov’s Theorem and the Type II Markov Move, \( B(\omega, 1, 2k) \) is actually the torus knot \( T(2, 2k + 1) \). The fact the conjecture is true for torus knots follows from a result of Moser [M87], which shows that surgery along a torus knot is always either a lens space, a connected sum of lens spaces, or Seifert fibered. A result of Bover, Gordon, and Watson [BGW13, Theorem 4] shows that the conjecture is true for Seifert fibered spaces.

Lemma 20. For any homomorphism \( \Phi : G(2n + 1, 2n - 1 + m(2n + 1), 2k, r) \to \text{Homeo}^+(\mathbb{R}), 1 \leq k \leq n - 1, m \geq 0, \alpha > t \) for all \( t \in \mathbb{R} \) implies \( \beta t \geq t \) for all \( t \in \mathbb{R} \).

For any homomorphism \( \Phi : G(2n, 2n - 2 + 2mn, 2k - 1, r) \to \text{Homeo}^+(\mathbb{R}), 1 \leq k \leq n - 1, m \geq 0, \alpha > t \) for all \( t \in \mathbb{R} \) implies \( \beta t \geq t \) for all \( t \in \mathbb{R} \).

Proof. We note that the two knot groups can be written in a unified form:
\[
G(\omega, t + m\omega, b) = \langle \alpha, \beta \mid \{ (C_1C_2)^{n-k+b} \{ \beta\alpha \}^{-b+1} \beta^{-n-k-1} \}^m \{ \beta\alpha \}^k \rangle = C_1C_2,
\]
where \( C_1 = \alpha \beta^{n-k+1} \) and \( C_2 = \alpha \beta^{n+k} \), and in the first case, \( \omega = 2n+1, t = 2n-1, b = 2k \), while \( \omega = 2n, t = 2n-2, b = 2k-1 \) in the second. We can rewrite the left side in a slightly different way:

\[
(C_1 C_2)^{n-k+b}[(\alpha \beta)^{-b+1} \beta^{-n+k-1} (C_1 C_2)^{n-k+b}]^m (\alpha \beta)^{-b+1} \beta^{-n+k-1} = C_1 C_2.
\]

Thus, we have

\[
(C_1 C_2)^{n-k+b-1}[(\alpha \beta)^{-b+1} \beta^{-n+k-1} (C_1 C_2)^{n-k+b}]^m = \beta^{n-k+1} (\alpha \beta)^{b-1}.
\]

For any \( t_0 \in \mathbb{R} \), applying both sides of the relation on \( t_0 \), we have

\[
\beta^{n-k+1} (\alpha \beta)^{b-1} t_0 = (C_1 C_2)^{n-k+b-1}[(\alpha \beta)^{-b+1} \beta^{-n+k-1} (C_1 C_2)^{n-k+b}]^m t_0.
\]

Assume \( \alpha t > t \) for all \( t \in \mathbb{R} \). By Lemmas 17 and 18, we can add \( \alpha^{-1} \) on one side of the equation to get a strict inequality on any \( t_0 \). Thus,

\[
(C_1 C_2)^{n-k+b-1} t_0 = (C_1 C_2)^{n-k} (C_1 C_2)^{b-1} t_0
\]

for all \( t_0 \in \mathbb{R} \). Similarly,

\[
(C_1 C_2)^{n-k+b} t_0 > \beta^{n-k+1} (\alpha \beta)^{b-1} t_0
\]

for all \( t_0 \in \mathbb{R} \). Thus,

\[
(C_1 C_2)^{n-k+b-1}[(\alpha \beta)^{-b+1} \beta^{-n+k-1} (C_1 C_2)^{n-k+b}]^m t_0
\]

for all \( t_0 \in \mathbb{R} \). Given any \( t \in \mathbb{R} \), let \( t_0 = (\alpha \beta)^{-b+1} \beta^{-n+k} t \). We get

\[
\beta t > t.
\]

Therefore, we have proved Theorem 13.

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