Racah coefficients and extended HOMFLY polynomials for all 5-, 6- and 7-strand braids

A.Anokhina§, A.Mironov¶, A.Morozov∥, And.Morozov∗∗

Abstract

Basing on evaluation of the Racah coefficients for SU_q(3) (which supported the earlier conjecture of their universal form) we derive explicit formulas for all the 5-, 6- and 7-strand Wilson averages in the fundamental representation of arbitrary SU(N) group (the HOMFLY polynomials). As an application, we list the answers for all 5-strand knots with 9 crossings. In fact, the 7-strand formulas are sufficient to reproduce all the HOMFLY polynomials from the katas.org: they are all described at once by a simple explicit formula with a very transparent structure. Moreover, would the formulas for the relevant SU_q(3) Racah coefficients remain true for all other quantum groups, the paper provides a complete description of the fundamental HOMFLY polynomials for all braids with any number of strands.

1 Introduction

Knot theory now comes to the avant-scene of theoretical physics. This was anticipated long ago [1], because it actually studies the Wilson loop averages in a simplified (topological) version of Yang-Mills theory, and is supposed to play the same crucially important role for generic Yang-Mills studies as topological strings play for the full string theory. In the underlying 3d Chern-Simons theory [1, 2] many non-trivial effects are already present, including sophisticated perturbation theory, non-perturbative effects and various dualities: just because of the topological nature they can be sometime simplified and even eliminated by the gauge choices. What is new today: a tremendous amount of ”experimental” data is available at [3] about the Wilson loop averages (called HOMFLY polynomials [4] within the Chern-Simons context). Also effective theoretical methods are developed to calculate these quantities based on studies of last years in adjacent fields: group theory, matrix models, conformal theories and AGT relations. These advances led to a discovery of vast net of interrelations between different HOMFLY polynomials including: various difference equations [5], the AMM/EO topological recursion [6, 7] and even some traces of integrability [8], which now need to be systematized, understood and converted into more standard forms common for other branches of science. All this gives a new vim to the study of old, but partly abandoned subjects like Racah coefficients for quantum groups and character expansions.

Today, generic simple formulas are known for the HOMFLY polynomials for all the 2,3,4-strand knots and links in the fundamental representation [9]; to write them down it turned sufficient to know the representation theory of SU_q(2), where the Racah coefficients are well known [10] and were widely used in other physical applications. In this letter we report the extension of these results to the 5-, 6- and 7-strand cases, where SU_q(3) representation theory (and its simple evident generalization) is needed. Evaluation per se of these Racah coefficients will be described in a separate paper [11]; what is important, this calculation confirms the universal ansatz suggested in [9], which now seems relatively safe to use for m > 7 strands as well, this will be done elsewhere. Note that the single 6-strand formula (30), explicitly written in the present paper, is sufficient to describe all the content of the Rolfsen tables [3] concerning the HOMFLY polynomials in the fundamental representation. Moreover, it is sufficient to describe the HOMFLY polynomials of almost all knots in [3], since all the tables are restricted to no more than 12 crossings, and all such knots, except just 17, admit the 6-strand braid representation at most. The remaining 17 knots⁠¹ require the 7-strand calculation

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§MIPT, Dolgoprudny, Russia and ITEP, Moscow, Russia; anokhina@itep.ru
¶Lebedev Physics Institute and ITEP, Moscow, Russia; mironov@itep.ru; mironov@lpi.ru
∥ITEP, Moscow, Russia; morozov@itep.ru
∗∗Moscow State University and ITEP, Moscow, Russia; Andrey.Morozov@itep.ru

¹In accordance with [3], these are: 12a0125; 12a0126; 12a0181; 12a0183; 12a0197; 12a0448; 12a0471; 12a0477; 12a0482; 12a0690; 12a0691; 12a0693; 12a1124; 12a1127; 12a1166; 12a1202; 12a1287.
which was also done with an evident generalization of the Racah coefficients to the $SU_q(4)$ Young diagram [4,1,1,1] in (16) and (18).

According to [8], the HOMFLY polynomials (Wilson averages) acquire the simplest form, if calculated in the gauge $A_0 = 0$ [12] and expanded in the Schur functions $S_Q\{p\}$ (characters of the linear groups):

$$ H^B_R(q) = \sum_{Q^m | R|} C^B_{RQ}(q) S_Q\{p\} $$

where the expansion coefficients

$$ C^B_{RQ}(q) = \text{Tr}_{N_{RQ} \times N_{RQ}} \left( \prod_{i=1}^{m-1} \prod_{j=1}^{a_{ij}} \hat{R}^{a_{ij}}_j \right) $$

Here

- $B$ is an $m$-strand braid parameterized by a sequence of integers

$$ B = (a_{11}, a_{12}, \ldots, a_{1,m-1}) a_{21}, a_{22}, \ldots, a_{2,m-1} a_{31}, a_{32} \ldots) $$

These integers enter eq.(2) as powers of $\hat{R}_j$-matrices along the braid: moving along the braid one first meets $a_{1k}$ links of $k$-th and $k+1$-th strands with some $k$, then $a_{1l}$ links of $l$-th and $l+1$-th strands with some $l \neq k$ (the sign of link is also taken into account). When one meets the links of the $k$-th and $k+1$-th strands for the second time, one associates with them the number $a_{2k}$. The meaning of these integers can be understood from the figure for the 3-strand braid:

[Figure]

In this figure when moving from the left, one first meets 2 links of the second and the third strands, which gives (with account of sign) $a_{12} = -2$. Since the first and the second links do not cross at the beginning, one puts $a_{11} = 0$. Then, there are two links of the first and the second strands with the opposite sign: $a_{21} = 2$. The next link of the second and the third strands gives us $a_{22} = -1$, and finally there are 3 links of the first and the second strands again, i.e. $a_{31} = 3$. This is knot 810.

- $R$ is the representation parameterized by the Young diagram $R = [r_1 \geq r_2 \geq \ldots 0]$ with $|R| = \sum_k r_k$ boxes. For the fundamental representation $R = \square = [1]$. The product $R^{\otimes m}$ is expanded in irreducible representations $Q$ of the size $|Q| = m|R|:

$$ R^{\otimes m} = \sum_{Q^m | R|} \mathcal{M}_{RQ} \otimes Q $$

where $\mathcal{M}_{RQ}$ is the space of intertwining operators $R^{\otimes m} \rightarrow Q$ of dimension $\dim(\mathcal{M}_{RQ}) = N_{RQ}$.

- Finally, $\hat{R}_j$ are quantum $R$-matrices. Originally

$$ \hat{R}_j = I \otimes \ldots \otimes I \otimes R \otimes I \otimes \ldots \otimes I $$

acts on $R^{\otimes m}$, and is associated with the crossings of strands $j$ and $j+1$. In the expansion (4), i.e. on the irreducible representations its action is proportional to unity on each $Q$, and, hence, reduces to an $N_{RQ} \times N_{RQ}$ matrix acting on $\mathcal{M}_{RQ}$, which we denote through $\hat{R}_j$ (omitting indices $RQ$ which it actually depends on). These matrices with different $j$ are related by orthogonal ”mixing matrices” $U_j$,

$$ \hat{R}_j = \hat{U}_j \hat{R}_1 \hat{U}_j^{-1} $$

\footnote{Our calculus is based on the approach by [13], though that of [14] is, by essence, also very close.}
made from the Racah coefficients of $SU_q(\infty)$. Reduction to the Racah coefficients provides a natural decomposition of $\hat{U}_t$ into a product involving $m-2$ matrices (called $\hat{U}, \hat{V}, \hat{W}, \hat{Y}, \hat{Z}$ in the present text, where $m = 5, 6, 7$). In fact, for given $m$ and $R$, in the product $R^{\otimes m}$ there is a diagram, at most, with $ml(R)$ lines (where $l(R)$ denotes the number of lines in the Young diagram $R$), thus, the representation theory of $SU_q(ml(R))$ is sufficient. However, one can use the "mirror" symmetry under simultaneous changing $q \to -\frac{1}{q}$ and transposing the Young diagram $Q \to \tilde{Q}$ in order to reduce even this group. For instance, for $R = [1] = \square$ one suffices to consider $SU_q([\frac{m}{2}])$, where $[\ldots]$ denotes the integer part. In particular, $SU_q(2)$ is enough for $R = \square$ and $m = 2, 3, 4$. For $m = 5, 6$ one needs $SU_q(3)$, where the necessary Racah coefficients were recently evaluated in [11]. For $m = 7$ one needs to know the Racah coefficients for $SU_q(4)$, however, for a simple hook diagram $[4,1,1,1]$, when they are immediate, (16).

Eq. (1) defines the extended HOMFLY polynomials [8], which depend on infinitely many time variables $\{p_k\}$. This allows one to consider them as a kind of a knot theory $\tau$-functions, though they belong to the ordinary (free-fermion) KP/Toda family only for the torus knots. Moreover, the extended polynomials depend on the braid representation of a knot, and the knot invariants (conventional HOMFLY polynomials) arise when the time variables are restricted to the 1-dimensional topological locus

$$p_k = p_k^s \equiv \frac{A^k - A^{-k}}{q^k - q^{-k}}$$

(7)

where the Schur functions reduce to the quantum dimensions:

$$S_Q\{p^s\} = \prod_{(k,l) \in Q} \{Aq^{k-l}\}_{\{q^{h_{k,l}}\}}$$

(8)

$h_{k,l}$ being a hook length and $\{x\} \equiv x - x^{-1}$. Then, the HOMFLY polynomial

$$H_R^E(A|q) = \left(q^{4\kappa_{R}}A^{R} \right)^{-\sum_{ij}a_{ij}} \hat{M}_R^E\{p^s\}$$

(9)

with the cut-and-join-operator eigenvalue $\kappa_R = \sum_k (k-1)r_k$, does not depend on the braid representation $B$ of the knot/link $\mathcal{K}$. In order to obtain the Wilson loop average for the gauge group $SU(N)$ one should further put $A = q^N$.

2 Racah coefficients for $[1] \times S \times [1] \to Q$

Associativity of the tensor product:

$$(R_1 \times R_2) \times R_3 = R_1 \times (R_2 \times R_3) = \sum_Q \mathcal{M}_Q^{R_1 R_2 R_3} \times Q$$

(10)

implies that there are two linear dependent bases in $\mathcal{M}_Q^{R_1 R_2 R_3}$:

$$(R_1 \times R_2) \times R_3 = \left(\sum_T \mathcal{M}_T^{R_1 R_2 R_3} \times T\right) \times R_3 = \left(\sum_T \mathcal{M}_T^{R_1 R_2 R_3} \times \mathcal{M}^Q_T \times Q\right)$$

(11)

$$R_1 \times (R_2 \times R_3) = R_1 \times \left(\sum_T \mathcal{M}_T^{R_2 R_3} \times T'\right) = \left(\sum_T \mathcal{M}_T^{Q R_1 T} \times \mathcal{M}_T^{R' R_2 R_3} \times Q\right)$$

(12)

The two bases are related by the Racah matrix

$$U_{R_1 R_2 R_3 Q}^{T T'}$$

(13)

When $R_1 = R_3 = [1]$, this matrix is at most $2 \times 2$, since the two boxes can be added to $R_2$ to form a given $Q$ in at most two different ways (differing by permutation).
As suggested in [9] and confirmed in [11], these orthogonal matrices have the form (we change the notation \( R_2 \rightarrow S \) to simplify the formulas)

\[
U_{[1]S[1]Q} = \begin{pmatrix}
-u_{SQ} & \varepsilon_S \sqrt{1 - u_{SQ}^2} \\
\sqrt{1 - u_{SQ}^2} & \varepsilon_S u_{SQ}
\end{pmatrix}
\]

(14)

where \( \varepsilon_S = -1 \) for \( S = [1] \) and \( \varepsilon_S = +1 \) for all other \( S \), while \( u_{SQ}^{-1} \) is a \( q \)-integer:

\[
u_{SQ}^{-1} = \frac{1}{[k_{SQ}]} , \quad \sqrt{1 - u_{SQ}^2} = \sqrt{[k_{SQ} - 1][k_{SQ} + 1]} /[k_{SQ}]
\]

(15)

where \([x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \left\{ \frac{q^x}{(q)} \right\} \) (the square brackets are also used to define the Young diagrams, but this should not cause confusion because they appear in different contexts). Remarkably, \([k]^2 - 1 = [k + 1][k - 1]\).

According to [9, 11], the actual values of \( k_{SQ} \) are:

**Q-doublets descending from \( S \):**

| \( S \) | \( Q \) | \( k_{SQ} \) |
|---|---|---|
| 1 | 21 | 2 |
| 2 | 31 | 3 |
| 11 | 211 | 3 |
| 3 | 41 | 4 |
| 21 | 32 | 2 |
| 221 | 2 |
| 311 | 4 |
| 4 | 51 | 5 |
| 31 | 42 | 3 |
| 321 | 2 |
| 411 | 5 |
| 22 | 321 | 4 |

| \( S \) | \( Q \) | \( k_{SQ} \) |
|---|---|---|
| 5 | 61 | 6 |
| 41 | 52 | 4 |
| 421 | 2 |
| 511 | 6 |
| 32 | 43 | 2 |
| 421 | 5 |
| 331 | 3 |
| 311 | 421 | 3 |
| 4111 | 6 |

The general \( SU_q(3) \)-formula looks like

\[
k_{SQ} = s_i - s_j + j - i
\]

(17)

where \( i < j \) are the numbers of lines in the Young diagram where the boxes are added and \( s_i > s_j \) are the lengths of these lines. In particular, if one starts from a 1-hook diagram with \( s_1 = |S| - l + 1 \) and \( s_2 = s_3 = \ldots = s_l = 1 \) and puts the new boxes to the ends obtaining a new 1-hook diagram, then the simple rule \( k_{SQ} = |S| - l + 1 + l + 1 - 1 = |S| + 1 = |Q| - 1 \) works. Here \( |S| \) is the depth (or the level) of the Racah coefficient, see the next section. This rule is immediately generalized to higher rank groups, for instance, the hook Young diagram (the boxed number in the table) would require \( SU_q(4) \).

Formula (17) can be derived by "the brute force" following the way described in detail in [9]. It is done in [11]. However, one can guess it from its values at \( q = 1 \) and "check" by comparing the HOMFLY polynomials for 6- and 7-strands (i.e. for groups \( SU_q(3) \) and \( SU_q(4) \)) with known results.

The mixing matrices \( \hat{U}_j \) in (6) with \( j = 1, \ldots , m - 1 \), are products of involving \( j - 1 \) elementary Racah matrices, which in their turn are made from such \( 2 \times 2 \) blocks complemented by unit matrices with the sign factors \( v_{SQ} = \pm 1 \), depending on \( S \) and \( Q \):
Q-singlets descending from $S$:

|   |   |   |
|---|---|---|
| 1 | 3 | 1 |
| 111 |   |   |
| 2 | 4 | 1 |
| 22 | 1 |   |
| 211 |   | -1 |
| 11 | 31 | 1 |
| 22 |   | -1 |
| 3 | 5 | 1 |
| 32 | 1 |   |
| 311 |   | -1 |
| 21 | 41 | 1 |
| 111 | 311 | 1 |
| 221 |   | -1 |

$S$ | $Q$ | $v_{SQ}$
|---|---|---|
| 4 | 6 | 1 |
| 42 |   | 1 |
| 411 |   | -1 |
| 31 | 51 | 1 |
| 33 |   | 1 |
| 32 | 42 | 1 |
| 33 |   | -1 |
| 311 | 511 | 1 |
| 331 |   | 1 |
| 322 |   | 1 |
| 32 | 52 | 1 |
| 322 |   | 1 |
| 322 |   | -1 |

In general, the unit matrices correspond to the case when the order of adding the boxes can not be changed, with $v_{SQ} = 1$ when the boxes are added to the same line and $v_{SQ} = -1$ when the boxes are added to the neighbor lines of equal lengths. This fact does not depend on the rank of the gauge group.

3 Explicit construction of mixing matrices

In this section we describe a very simple and obvious procedure to build up an arbitrary mixing matrix. It is provided in the form of a product of elementary Racah matrices, each made out of $1 \times 1$ and $2 \times 2$ blocks with entries listed in the tables (16) and (18). $SU_q(L)$ group theory is needed only to find the entries of these tables. Once this is done, the problem of finding the fundamental HOMFLY polynomials will be solved completely. So far, our reliable knowledge is sufficient to describe all $m \leq 7$ braids. Explicit formulas for the mixing matrices are given below in s.4 and the Appendices, here we briefly describe the way to construct them.

3.1 Decomposition of mixing matrices

The mixing matrix $\hat{U}_j$ converts the $\hat{R}$ matrix, acting on the first pair of [1] in the product $[1] \times^m$ into that acting on the $j$-th pair:

\[
\begin{array}{c}
[1] \times [1] \times [1] \times \ldots \\
\hat{R}_1
\end{array}
\rightarrow
\begin{array}{c}
[1] \times [1] \times \ldots \times [1] \times [1] \times \ldots \\
\hat{R}_j
\end{array}
\]  

(19)

$\hat{R}_1$ intertwines the first two [1]'s in the product and does not affect any other, i.e. in order to define $\hat{R}_1$ it is important to multiply at the beginning the first two [1]'s and after that all other can be multiplied in an arbitrary order. For $\hat{R}_j$ the first to be multiplied is the $j$-th pair, and all other do not matter.

In other words, in order to convert $\hat{R}_1$ into $\hat{R}_2$ it is enough to consider the Racah transform

\[
(1 \times 1) \times 1 \xrightarrow{U} 1 \times (1 \times 1)
\]

(from now on, we omit the square brackets to simplify the formulas, at least, a little). Similarly, to convert $\hat{R}_1$ into $\hat{R}_3$ one needs the three steps:

\[
(1 \times 1) \times 1 \times 1 \cong (1 \times 1) \times 1 \xrightarrow{U} (1 \times (1 \times 1)) \times 1
\]

\[
\downarrow V
\]

\[
1 \times 1 \times (1 \times 1) \cong 1 \times (1 \times (1 \times 1)) \xleftarrow{U} 1 \times (1 \times (1 \times 1))
\]

The sign $\cong$ means that the two expressions are equivalent from the point of view of the action of $\hat{R}_1$ or $\hat{R}_3$.  
When we move the bracket $(1 \times S) \times 1 \rightarrow 1 \times (S \times 1)$, we say that we do this at the depth $|S|$ ($|S|$ denotes the number of boxes in the Young diagram $S$; since here we obtain $S$ from products of representations $[1]$, it comes from $[1]^{[6][S]}$). Following [9], the transition of depth 1 is denoted by $U$, that of the depth 2 by $V$, and we use $W$, $Y$ and $Z$ for the depths 3, 4 and 5. Thus, the mixing matrix $\hat{U}_2$ is of the type $U$, $\hat{U}_3$ is the combination of three Racah matrices $UVU$, $\hat{U}_4$ will be $UVWVU$ and so on. In fact, there are combinations with different order of the Racah matrices for a $\hat{U}_j$ with $j \geq 3$, but the Racah matrices with depth difference exceeding one commute and, thus, all the seemingly different representations are in fact the same.

### 3.2 Representation tree

The main object that we need in order to construct the Racah matrices $U, V, W, Y, Z, \ldots$ is the representation tree:

```
      1
   1 x 1
   /
 /  \
/    \  
1 x 1 x 1
   /
/  \
/    \  
1 x 1 x 1 x 1
   /
/  \
/    \  
Y W Y Y V Y Y W Y ...
```

We actually need this tree at least up to the level $1 \times 7$, but the space is not enough to present it here, in what follows we draw also a fragment of the tree at the next levels.

### 3.3 The structure of Racah matrices

#### 3.3.1 Depth one ($U$)

The first Racah matrix, $U$, appears at the level of $1 \times 1 \times 1$, and it mixes the two underlined representations $[21]$. This matrix is $2 \times 2$ and explicitly given by (16) with $k_{[1],[21]} = 2$, this $2 = \text{depth} + 1$. The same $2 \times 2$ matrix will mix all the descendants of these two $[21]$ at lower levels: the two underlined $[31]$, the two underlined $[22]$ and the two underlined $[211]$. There is still one other $[31]$ and one other $[211]$ at level 4, which are not underlined, and are not affected by the matrix $U$, which has $v_{[1],[3]} = v_{[1],[11]} = 1$ at the corresponding positions: see eq.(24) below. The same $2 \times 2$ blocks with the same $k_{[1],[21]} = 2$ will appear in the $U$-matrices for all other descendants of the two $[21]$ at lower levels. If the same representations appear, which are the descendants of $[3]$ and $[11]$, the corresponding entries of $U$-matrices are $1 \times 1$ and equal to $v_{[3],[3]} = 1$ and $v_{[1],[11]} = 1$.

To finalize the notational agreements, $U_{[31]}$ has three rows and three columns, corresponding to the three appearances of $[31]$ in the representation tree, and they are ordered just as in the tree: the first remains unaffected by $U$, the second and the third are mixed. For $[211]$ the $2 \times 2$ block would involve the first two rows and columns, but we do not actually need this mixing matrix, because the contribution of $[211]$ to the HOMFLY polynomial is the mirror image of the $[31]$ contribution.

#### 3.3.2 Depth two ($V$)

The second (depth-two) Racah matrix, $V$, describes the transition

\[
\left( [1] \otimes \left( [2] + [11] \right) \right) \otimes [1] \overset{V}{\rightarrow} [1] \otimes \left( \left( [2] + [11] \right) \otimes [1] \right)
\]

This means that $[2]$ and $[11]$ are not affected by $V$, and so are all their descendants: there is a "wall" separating representations, which can be mixed by $V$: it is the vertical double line in the picture, labeled by $V$. This means that at the level $1 \times 4$ only the two overlined $[31]$ and the two overlined $[211]$ can be mixed, this time in the $[31]$ sector the $2 \times 2$ block involves the first two rows and columns. And also the corresponding $k_{[2],[31]} = 3$ is now different. The same $V$ with the same $k_{[2],[31]} = 3$ will mix all the descendants of these two $[31]$ at all the
lower levels. All other representations in the left half of the tree will remain intact under V (at most, change
sign, if the corresponding $v_{[2], Q} = -1$ (according to (18), this happens only for $Q = [211])$.

3.3.3 Depth three (W)

The depth-three Racah matrix W is associated with the transition

$$
\left(1 \times \left[\begin{array}{c}
[3] + 2 \cdot [21] + \{111\}
\end{array}\right]\right) \times [1] \xrightarrow{W} [1] \times \left(\left[\begin{array}{c}
[3] + 2 \cdot [21] + \{111\}\end{array}\right] \times [1]\right)
$$

and there are now three W-"walls" separating representations, which can not be mixed by W (one of the W-
walls coincides with the V-wall). To see what can be mixed by W, we draw now the next levels, but only for the
left half of the representation tree:

```

2
  3
   4
    5
     Z
      Z
        Y
          Y
            Z
              Z
                Z
                  Y
                    V
```

Now it is clear that W-mixed at the fifth level are the two double-underlined [41] to the left of the W-wall,
and three underlined pairs to the right of it: [32], [311] and [221], and this will generate exactly the same
W-mixing of all their descendants at lower levels.

The new thing is that the numbers $k_{SQ}$ at the same depth can now be different. Deviation from the simple
rule $k = \text{depth} + 1$ occurs when mixed are descendants of a non-hook diagram, namely [22]. The general rule
of this deviation is the only remaining uncertainty in description of the mixing matrices, but in this particular
case it is explicitly done in the table (16). Note that the two hook diagrams [311] are not the descendants of [22],
therefore, their mixing obeys the rule $k = |S| + 1$. The two non-hook diagrams [32] and [221] are
[22]-descendants, and for them the $k$-numbers are smaller.

Another new thing is that at level five the mixing occurs for the first time between the descendants of [21]
only, namely [32], [311] and [221]. Since there are two [21] in the representation tree, this means that there are
two copies of these mixing pairs, and the Racah matrices contain two identical $2 \times 2$ blocks, not a single such
block, as it happened at levels three and four. However, at level five these two blocks are still identical.

The last new phenomenon is the occurrence of $2 \times 2$ blocks with two or more different $k$-numbers within
one and the same Racah matrix. This happens when the same representation can appear both in the product
of two hook and of non-hook diagrams. For the first time this happens at level 6, where one and the same
representation, namely [42] or [321], appears in two copies at different sides of the wall, and this wall can be
either W or Y. Therefore, the four Racah matrices $W_{[42]}$, $Y_{[42]}$, $W_{[411]}$, $Y_{[411]}$ can contain $2 \times 2$ blocks with
different $k$-numbers, and this is indeed the case, see Appendix 1.

3.3.4 Depth four and five (Y and Z)

These can be analyzed in exactly the same way, but nothing essentially new happens, just matrices become
bigger and bigger. In general, say, the hook representation $[k, 1^{m-k}]$ appears $\frac{(m-1)!}{(k-1)! (m-k)!}$ times in the expansion
of $[1]^{\times m}$, i.e. for $m$ strands. The mixing matrices for symmetric hooks are the biggest, but they are easily predictable: all the $k$-numbers for all hook diagrams depend only on the depth: $k_{\text{hook}} = |S| + 1 = |Q| - 1$. This
fact allows us to define the mixing matrices for the representation [4111]: the only one for 7 strands, which
is not controlled by representation theory of $SU_q(3)$, and for the "honest" calculation $SU_q(4)$ is needed. It is
needed anyway for the non-hook diagrams with four columns, which contribute to calculations for $m \geq 8$. To
have the complete description of the mixing matrices (of their Racah constituents, to be exact) one needs a generic formula for all the elements of the tables (16) and (18). It can easily happen that formula (17) remains valid for \( m \geq 8 \), but this remains to be checked. Knot theory itself does not help to make a check, because non-hook diagrams do not contribute in the case of the torus knots (in the fundamental representation), and not much is known independently about non-torus knots for \( m \geq 8 \) as we already mentioned there is none of this kind in [3].

4 Explicit formula for extended HOMFLY polynomials

We are now ready to provide explicit expressions for the HOMFLY polynomials (1): it remains to substitute into (2) concrete expressions for \( \hat{R}_i \) through \( \hat{R} = \hat{R}_1 \) and Racah matrices. We also use an operation which changes \( q \rightarrow -\frac{1}{q} \) and transposes the Young diagram \( Q \rightarrow \hat{Q} \). This symmetry effectively used in [15, 16, 17] was named mirror in [18]. It not only simplifies the formulas, it allows one to restrict consideration to the Young diagrams with no more than \( m/2 \) (rather than \( m \)) columns, for which the Racah coefficients are provided by the theory of \( SU_q(m/2) \) rather than \( SU_q(m) \) group (of course \( m/2 \rightarrow (m+1)/2 \) for \( m \) odd).

\[ m = 2 \]

\[
\mathcal{H}_2^{(a)} = q^a S_{[2]} + \left(-\frac{1}{q}\right)^a S_{[\bar{2}]} = q^a S_{[2]} + \text{(mirror)}
\]

\[ m = 3 \]

\[
\mathcal{H}_3^{(a_1 b_1 | a_2 b_2 | \ldots)} = q \sum_i (a_i + b_i) S_{[3]} + \text{(mirror)} + \left( \text{Tr}_{2 \times 2} \prod_i \left( \hat{R}^{a_i}_{[2]} \hat{U}_{[2]}^{a_i} \hat{R}^{b_i}_{[2]} \hat{U}^{b_i}_{[2]} \right) \right) \cdot S_{[\bar{2}]} \tag{21}
\]

In this particular case \( \text{(mirror)} = \left(-\frac{1}{q}\right) \sum_i (a_i + b_i) S_{[\bar{1}1]} \) and

\[
\hat{R}_{[2]} = \begin{pmatrix} q & -1 \end{pmatrix}, \quad \hat{U}_{[2]} = \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & -\frac{1}{2} \end{pmatrix} \tag{22}
\]

\[ m = 4 \]

\[
\mathcal{H}_4^{(a_1 b_1 c_1 | a_2 b_2 c_2 | \ldots)} = q \sum_i (a_i + b_i + c_i) S_{[4]} + S_{[3]} \cdot \text{Tr}_{3 \times 3} \prod_i \left( \hat{R}^{a_i}_{[3]} \hat{U}_{[3]}^{a_i} \hat{R}^{b_i}_{[3]} \hat{V}^{b_i}_{[3]} \hat{R}^{c_i}_{[3]} \hat{V}^{c_i}_{[3]} \hat{U}^{\dagger}_{[3]} \hat{V}^{\dagger}_{[3]} \right) \tag{23}
\]

Now \( \text{(mirror)} \) contains contributions from two diagrams \( Q = [211] \) and \( Q = [1111] \), while

\[
\hat{R}_{[3]} = \begin{pmatrix} q & -1 \end{pmatrix}, \quad \hat{U}_{[3]} = \begin{pmatrix} 1 & -\frac{1}{2} & -\sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad \hat{V}_{[3]} = \begin{pmatrix} -\frac{1}{3} & \sqrt{\frac{2}{3}} & \frac{2}{3} \\ \sqrt{\frac{2}{3}} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{3}{1} \end{pmatrix} \tag{24}
\]

and

\[
\hat{R}_{[2]} = \begin{pmatrix} q & -1 \end{pmatrix}, \quad \hat{U}_{[2]} = \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & -\frac{1}{2} \end{pmatrix}, \quad \hat{V}_{[2]} = \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{25}
\]
\[ m=5 \]
\[ \mathcal{H}_{l_1l_2r_1r_2}[a_1a_2,a_1a_2,a_2a_2,...] = q^{\sum_i(a_i+b_i+c_i+d_i)}S_{[5]} + \]
\[ + S_{[41]} \cdot \text{Tr}_{4 \times 4} \prod_i \left( \hat{R}_{[41]}^{a_i} \hat{U}_{[41]} \hat{R}_{[41]}^{b_i} \hat{V}_{[41]} \hat{W}_{[41]} \hat{U}_{[41]}^\dagger \hat{V}_{[41]}^\dagger \right) + \]
\[ + S_{[32]} \cdot \text{Tr}_{3 \times 3} \prod_i \left( \hat{R}_{[32]}^{a_i} \hat{R}_{[32]}^{b_i} \hat{V}_{[32]} \hat{W}_{[32]} \hat{U}_{[32]}^\dagger \hat{V}_{[32]}^\dagger \hat{W}_{[32]}^\dagger \hat{U}_{[32]}^\dagger \right) + \text{ (mirror)} + \]
\[ + S_{[311]} \cdot \text{Tr}_{6 \times 6} \prod_i \left( \hat{R}_{[311]}^{a_i} \hat{R}_{[311]}^{b_i} \hat{V}_{[311]} \hat{W}_{[311]} \hat{V}_{[311]}^\dagger \hat{W}_{[311]}^\dagger \hat{V}_{[311]}^\dagger \hat{W}_{[311]}^\dagger \hat{V}_{[311]}^\dagger \hat{W}_{[311]}^\dagger \hat{V}_{[311]}^\dagger \hat{W}_{[311]}^\dagger \hat{V}_{[311]}^\dagger \hat{W}_{[311]}^\dagger \hat{V}_{[311]}^\dagger \hat{W}_{[311]}^\dagger \right) (26) \]

This (mirror) contains contributions from three diagrams \( Q = [2111], [221] \) and \([11111] \), while

\[ \hat{R}_{[41]} = \begin{pmatrix} q & q & -\frac{1}{q} \\ q & q & -\frac{1}{q} \\ 1 & 1 & 1 \end{pmatrix}, \quad \hat{U}_{[41]} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 1 \end{pmatrix} \]

\[ \hat{V}_{[41]} = \begin{pmatrix} 1 \\ -\frac{1}{3} \\ \frac{\sqrt{2}[4]}{3} \\ \frac{\sqrt{2}[4]}{3} \end{pmatrix}, \quad \hat{W}_{[41]} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{\sqrt{3}[5]}{2} \end{pmatrix} \]

\[ \hat{R}_{[32]} = \begin{pmatrix} q & q & -\frac{1}{q} \\ q & q & -\frac{1}{q} \\ -\frac{1}{3} & \frac{\sqrt{2}[4]}{3} \end{pmatrix}, \quad \hat{U}_{[32]} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 1 \end{pmatrix} \]

\[ \hat{V}_{[32]} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}[4]}{3} \\ \frac{\sqrt{2}[4]}{3} & \frac{1}{3} \\ 1 & 1 \end{pmatrix}, \quad \hat{W}_{[32]} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 1 \end{pmatrix} \]
and

\[
\hat{R}_{[311]} = \begin{pmatrix}
q & q & \frac{1}{q} & q \\
q & \frac{1}{q} & q & q \\
\frac{1}{q} & q & q & \frac{1}{q} \\
\frac{1}{q} & q & \frac{1}{q} & q \\
\end{pmatrix}, \quad \hat{U}_{[311]} = \begin{pmatrix}
1 & -\frac{1}{\sqrt{2}} & -\sqrt{3} & 0 \\
-\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\
-\sqrt{3} & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

\[
\hat{V}_{[311]} = \begin{pmatrix}
-\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 1 \\
\frac{\sqrt{2}}{3} & -\frac{1}{3} & 1 & -1 \\
\frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{1}{3} & 1 \\
\frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{1}{3} & 1 \\
\end{pmatrix}, \quad \hat{W}_{[311]} = \begin{pmatrix}
-1 & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\
-\frac{1}{4} & 1 & 0 & 0 \\
\frac{\sqrt{3}}{4} & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

m=6

Here the sum goes over four representations [51], [42], [411], [33] plus the contributions of their mirrors [211111], [2211], [3111], [222] and of the symmetric diagram $Q = [321]$. The contribution of [111111] is explicitly mentioned in the first line as a mirror of [6]: these both representations enter with multiplicities one and no mixing matrices are involved. All other relevant matrices are explicitly listed in Appendix 1.

m=7

In this case, the formula looks exactly like (30), only one more piece with $\hat{R}^I_Q$ and $Z_Q$ is added:

\[
\mathcal{H}_{\Omega}^{(a_1b_1c_1d_11 \ldots a_2b_2c_2d_22 \ldots)} = q \sum_i (a_i+b_i+c_i+d_i+f_i) S_{[6]} + (\text{mirror}) + \\
+ \sum Q S_Q \cdot \text{Tr} \prod \left( \hat{R}^I_Q \hat{U}_Q \hat{R}^b_Q \hat{V}_Q \hat{R}^c_Q \hat{W}_Q \hat{U}_Q \hat{R}^d_Q \hat{Y}_Q \hat{R}^e_Q \hat{W}_Q \hat{U}_Q \hat{R}^f_Q \hat{Y}_Q \hat{R}^I_Q \hat{W}_Q \hat{U}_Q \hat{R}^I_Q \hat{W}_Q \hat{U}_Q \hat{R}^I_Q \hat{W}_Q \hat{U}_Q \hat{R}^I_Q \hat{W}_Q \hat{U}_Q \hat{R}^I_Q \hat{W}_Q \right)
\]

The mixing matrices are up to $20 \times 20$, they are are explicitly listed in Appendix 2.
5 Skein relations

All matrices $\hat{R}_Q$ have the same eigenvalues $\lambda = q$ and $\lambda = -\frac{1}{q}$, which both satisfy

$$\lambda - \lambda^{-1} = q - q^{-1}$$

(32)

Therefore, each matrix satisfies

$$\hat{R}_Q - \hat{R}_Q^{-1} = \left(q - \frac{1}{q}\right) \cdot I_Q$$

(33)

This implies simple difference equations, which for $R = 1$ looks the same for all coefficients (2) and, hence, the extended HOMFLY polynomial (1), considered as a function of any of the braid parameters $a_{ij}$ in (3) satisfies

$$\mathcal{H}_0^{(\ldots a_{ij}+1 \ldots)}(p) - \mathcal{H}_0^{(\ldots a_{ij}^{-1} \ldots)}(p) = \left(q - \frac{1}{q}\right) \mathcal{H}_0^{(\ldots a_{ij} \ldots)}(p)$$

(34)

Note that this skein relation holds for the extended HOMFLY polynomials, with arbitrary values of time variables $\{p_k\}$ beyond the topological locus (7), for arbitrary parameters $a_{ij}$, but only for the fundamental representation $R = \Box = [1]$. If complemented by the symmetry condition

$$\mathcal{H}_0^R(q^1|p) = \mathcal{H}_0^R(q^{-1}|p)$$

(35)

this skein relation can be used to recursively find all the HOMFLY polynomials for all braids. The ordinary skein relation [20] for the HOMFLY polynomials $H^K(A|q)$ on the topological locus follows immediately:

$$A \cdot H_0^{(\ldots a_{ij}+1 \ldots)}(q|A) - \frac{1}{A} \cdot H_0^{(\ldots a_{ij}^{-1} \ldots)}(q|A) = \left(q - \frac{1}{q}\right) H_0^{(\ldots a_{ij} \ldots)}(q|A)$$

(36)

where the extra powers of $A$ at the l.h.s. arise from the correcting factor in (9) (note that $|\Box| = 1$ and $\tau_0 = 0$).

6 Applications of the 5-, 6- and 7-strand formulas

Eq.(29) is the only new one in the 5-strand case as compared to [9], but it allows one to study arbitrary 5-strand knots. We consider two applications: to the torus knots and to all 5-strand knots with 9 crossings listed in [3] (see the Tables below).

Similarly, with eqs.(30) and (31) we list all 6-strand knots with 10 crossings listed in [3] (also see the Tables below). For an illustrative purpose we also list in the Tables an example of answer for a 7-strand knot from [3].

6.1 Torus knots $[5, n]$

For the torus knots and links $a_{ij} = 1$ for $i \leq n$ and all other $a_{ij} = 0$. These knots/links are denoted $T[m,n]$, where $m$ is the number of strand (it is a knot if $m$ and $n$ are mutually prime and is an $l$-component link if $l$ is the greatest common divisor of $m$ and $n$). In particular, for $m = 5$ this means that (26) reduces to

$$\mathcal{H}_0^{[5,n]} = \sum_{Q=5} \mathcal{S}_Q \cdot \text{Tr}_{N_1 \times N_1} \left( \hat{R}_Q \hat{U}_Q \hat{R}_Q \hat{V}_Q \hat{R}_Q \hat{W}_Q \hat{R}_Q \hat{U}_Q \hat{R}_Q \hat{V}_Q \hat{R}_Q \hat{W}_Q \hat{R}_Q \hat{U}_Q \right)^n$$

(37)

From the explicit form of the constituent matrices it is easy to find the eigenvalues of the composite matrices $\mathcal{R}_Q$:

$$\det_{N_1 \times N_1} \left( \mathcal{R}_Q - \lambda \cdot I \right) = \begin{cases} \\
\lambda^5 - q^{10} & \text{for } Q = [41] \\
\lambda^5 - q^4 & \text{for } Q = [32] \\
(\lambda - 1)(\lambda^5 - 1) & \text{for } Q = [311] \\
\lambda^5 - q^{-4} & \text{for } Q = [221] \\
\lambda^5 - q^{-10} & \text{for } Q = [2111] \\
\end{cases}$$

(38)
This implies that for \( n \neq 5 \), i.e. for the torus knots,
\[
H_{5}^{[5,n]} = q^{4n}S_{[5]} + q^{-2n}S_{[41]} + S_{[31]} - q^{-2n}S_{[211]} + q^{-4n}S_{[1111]}
\]  
(39)
and for \( n \geq 5 \), i.e. for the torus 5-component links
\[
H_{5}^{[5,n]} = q^{4n}S_{[5]} + 4q^{2n}S_{[41]} + 5q^{4n}S_{[32]} + 6S_{[31]} + 5q^{-4n}S_{[221]} + 4q^{-2n}S_{[211]} + q^{-4n}S_{[1111]}
\]  
(40)
in full accordance with the Adams rule of [19]\(^3\).

In order to obtain \( H_{5}^{[5,n]} \), one now has to multiply (39) and (40) by \( A^{-4n} \) in accordance with (9) and use values (8) of the Schur functions at the topological locus (7). These are

\[
S_{[5]}(p^*) = \frac{\{A\} \{A\} \{A\} \{A\} \{A\} \{A\} \{A\}}{\{q\} \{q^2\} \{q^3\} \{q^4\} \{q^5\}} \\
S_{[41]}(p^*) = \frac{\{A/q\} \{A\} \{A\} \{A\} \{A\}}{\{q^2\} \{q^3\} \{q^4\} \{q^5\}} \\
S_{[31]}(p^*) = \frac{\{A/q\} \{A\} \{A\} \{A\} \{A\} \{A\}}{\{q^2\} \{q^3\} \{q^4\} \{q^5\}} \\
S_{[211]}(p^*) = \frac{\{A/g\} \{A\} \{A\} \{A\} \{A\} \{A\}}{\{q^2\} \{q^3\} \{q^4\} \{q^5\}} \\
S_{[221]}(p^*) = \frac{\{A\} \{A\} \{A\} \{A\} \{A\}}{\{q^2\} \{q^3\} \{q^4\} \{q^5\}} \\
S_{[1111]}(p^*) = \frac{\{A\} \{A\} \{A\} \{A\} \{A\}}{\{q^2\} \{q^3\} \{q^4\} \{q^5\}}
\]  
(43)

6.2 Knots and links from the katlas tables

The 5-strand and 6-strand formulas (26) and (30) enable to calculate the HOMFLY polynomials for arbitrary 5- and 6-strand knots straightforwardly. Coefficients of the character expansion for the HOMFLY polynomials for all the 5-strand knots with 9 crossings and for all 6-strand knots with 10 crossings are given in the Tables below. We also give there coefficients of this expansion for 7-strand knot 12\_0_{125}.

6.3 Cabling of 2 and 3-strand links

Like it was done in [17], one can now reproduce various formulas for the colored HOMFLY polynomials by the cabling procedure applied to the fundamental representations. Namely, the 5-strand links can describe the 2-strand links, where one component is in [2] or [11], and the other one is in [3], [21] or [111]. Similarly, the 6-strand knots can be used for cabling of the 2-strand knots in representation [3], [21], [111] and of the 3-strand knots in representations [2] or [11]. The last option is most interesting, because it tests the 3-strand formulas of [17], the simplest case of non-torus knots, where the Rosso-Jones formula [19] is unapplicable.

In this case, the projectors from the cable of the knot to irrep are (see also [17]; the projectors are definitely not uniquely defined, these were just the simplest ones made out of zero and one insertions of \( \mathcal{R} \))
\[
\mathcal{H}^{(a_1 b_1 | a_2 b_2 | \ldots)}_{[2]}(p) = \frac{1}{(1 + q^2)^3} \sum_{i,j,k=0,1} q^{i+j+k} \mathcal{H}^{(i)j(k)0)(01100 | 11000)^{n_1} | (00111 | 00110)^{n_1} | (01100 | 11000)^{n_1} | (01100 | 11000)^{n_1} | \ldots} \{p\}
\]  
(47)
\[
\mathcal{H}^{(a_1 b_1 | a_2 b_2 | \ldots)}_{[3]}(p) = \frac{q^6}{(1 + q^2)^3} \sum_{i,j,k=0,1} \left( -\frac{1}{q} \right)^{i+j+k} \mathcal{H}^{(i)j(k)0)(01100 | 11000)^{n_1} | (00111 | 00110)^{n_1} | (01100 | 11000)^{n_1} | (01100 | 11000)^{n_1} | \ldots} \{p\}
\]

\(^3\)The Adams rule consists of constructing a set of coefficients \( C_{R_k}^Q \) via the decomposition of the product
\[
\prod_{i=1}^{l} S_{R_i} (p_k^{(m)}) = \sum_{Q^{m} \sum R_i} C_{R_k}^Q S_Q (p_k)
\]  
(41)
where \( p_k^{(m)} \equiv p_{mk} \), \( m \) and \( l \) are integers and the sign \( \ast \) implies that the sum runs over all representations of the size \( m \sum_i |R_i| \). With these coefficients, the colored HOMFLY polynomial for the torus link \( T[m, n] \) with \( l \) components (i.e. \( l \) is the greatest common divisor of \( m \) and \( n \)) is
\[
\mathcal{H}^{[5,n]}_{(R_k)} = \sum_{Q^{m} \sum R_i} C_{R_k}^Q S_Q (p_k)
\]  
(42)
The braiding at the r.h.s. can be directly read from the cabling picture (these are elementary braiding patterns in terms of the integers $a_{ij}$ for the 6-strand braid, i.e., for instance, $(00011|00110)$ means that first the 4th and 5th strands cross, then the 5th and 6th, the 3d and 4th and, finally, the 4th and 5th ones):

$$
\begin{align*}
& a : (01100|11000) \\
& b : (00011|00110)
\end{align*}
$$

Making use of the 6-strand formula (30) for the r.h.s. and equally explicit expressions for the colored 3-strand braids from [17] for the l.h.s., one can easily check that these equalities are indeed true for various choices of the braiding numbers $a_1, b_1, a_2, b_2, a_3, \ldots$

As to the 5-strand example, the HOMFLY polynomials for the bi-colored 2-strand links $H_{[2][3]}^{(a_1|a_2|a_3|\ldots)}(p)$ with even number of parameters $a_1, a_2, \ldots = \pm 1$ can be evaluated with the help of the Rosso-Jones formula [19], it is valid also for the extended HOMFLY polynomials. On the other hand, it can be extracted by cabling from the 5-strand formula (26). Thus,

$$
(01100|11010|00110) \\
(00110|11100|11000)
$$

The projectors $\pi$ are derived as follows.

From [17], the projector on $[2]$ is $\pi_{[2]} = \frac{q^i}{1+q^2}$ with $i = 0, 1$. That on $[11]$ is just its mirror $\pi_{[11]} = \frac{(q^i)^2-1}{1+q^2}$.

This follows from the relation

$$
\pi_{[2]} = \alpha + \beta + S_{[2]}(p) = \alpha(S_{[2]} + S_{[11]}) + \beta(qS_{[2]} - q^{-1}S_{[11]})
$$

which implies that $\alpha = \frac{1}{q_{[2]}} = \frac{1}{1+q^2}$ and $\beta = \frac{1}{q_{[2]}} = \frac{q^i}{1+q^2}$.

Similarly, for $\pi_{[3]}$ one can make a choice

$$
S_{[3]}(p) = \alpha(S_{[3]} + 2S_{[21]} + S_{[111]}) + \beta(qS_{[3]} + (q - q^{-1})S_{[21]} - q^{-1}S_{[111]}) + \gamma(q^2S_{[3]} - S_{[21]} + q^{-2}S_{[111]})
$$

so that $\alpha = \frac{1}{q_{[3][2]}}$, $\beta = \frac{q^i+2}{q_{[2][3]}}$, $\gamma = \frac{1}{q_{[3]}}$ and

$$
\pi_{[3]}^{jk} = \frac{1}{q^2[2][3]}(\delta^j_0 \delta^k_0 + (2 + q^2)\delta^j_1 \delta^k_0 + q^2[2]\delta^j_1 \delta^k_1)
$$

Likewise for the projection on the mirror representation $[111]$, one gets $\alpha = \frac{q^i}{[2][3]}$, $\beta = \frac{q^i-2}{[2][3]}$, $\gamma = \frac{1}{[3]}$ and for the projection on $[21]$: $\alpha = \frac{1}{[2][3]}$, $\beta = \frac{q^i-1}{[2][3]}$, $\gamma = \frac{1}{[3]}$. Once again, there is a big freedom in choosing the projectors, and this is just one possibility, not distinguished in any way.

In the same way, one can use the 6- and 7-strand formulas from this paper to represent the 2-strand links, bi-colored as [2][4], [2][5], [3][3] and [3][4].
7 Conclusion

To conclude, we represented the extended HOMFLY polynomials (Wilson loop averages in 3d Chern-Simons theory) in the fundamental representation for arbitrary 5-, 6- and 7-strand braids as linear combinations of respectively seven, eleven and fifteen Schur functions with $q$-dependent coefficients, which are traces of at most $6 \times 6$, $16 \times 16$ and $20 \times 20$ explicitly listed matrices. Parameters (3) of the braid enter as powers of the diagonal constituents of these matrices. These formulas immediately reproduce all the 5-, 6- and 7-strand formulas in [3]. Application to torus knots and links $[5, n]$ in s.6.1 illustrates the possibility to describe arbitrary series of knots/links by evaluating the eigenvalues of the corresponding composite $\mathcal{R}$-matrices.

Actually, the result of this paper can be far more ambitious: if the simple rule eq.(17) (which does not depend on $q$ and, hence, can be read off not only from the quantum groups, but from the Lie groups as well) remains true beyond $SU_q(3)$, then we provided a complete description of the fundamental HOMFLY polynomials for all braids with any number $m$ of strands. Moreover, this description provides an explicit function of the braiding numbers, and not only for ordinary, but for the extended HOMFLY polynomials as well.

Given the importance of this result, we summarize briefly the steps of the construction.

- The Turaev-Reshetikhin description of the HOMFLY polynomials is projected on the space of intertwining operators which directly provides their character expansion naturally continued to the infinite-dimensional space of the time variables $\{p_k\}$.

- The expansion coefficients are traces of products of the diagonal $\hat{\mathcal{R}}$-matrices with generically known entries dictated by the eigenvalues of the cut-and-join operator $W_2$ (certain symmetric group characters) and the orthogonal mixing matrices.

- The mixing matrices are products of the Racah matrices.

- For the fundamental representation the Racah matrices are made from the $2 \times 2$ blocks; this decomposition is immediately read off from the representation tree and can be easily programmed.

- The cosines of angles of the $2 \times 2$ rotation matrices are inverse quantum integers. These integers are the only non-trivial parameters to calculate.

- An explicit calculation of the Racah coefficients for $SU_q(2)$ and $SU_q(3)$ quantum groups provides a formula (17) for these integers, which is sufficient to describe all the 6-strand braids.

- For the hook diagrams the answer for the integers is clearly universal, this allows us to conjecture the answer for representation $[4111]$, the only one appearing in the 7-strand formulas, which needs $SU_q(4)$ for being honestly calculated.

- In fact, eq.(17) looks so nice that it can easily remain true for all bigger quantum groups and thus provides an answer for braids with arbitrary number of strands. If this is the case, one gets a complete description of all extended HOMFLY polynomials for all braids.

- The only restriction is to the fundamental representation. However, since the formulas are true for any braids, the cabling procedure is actually straightforward as demonstrated in sec.6.3. Still, the formulas obtained by cabling seem more complicated than the direct counterparts of our results in the fundamental case, and development of a similar formalism for colored knots is highly desirable (see [17, 16] for the first results in this direction).

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| knot | \((a_1, b_1, c_1, d_1|a_2, b_2, c_2, d_2|a_3, b_3, c_3, d_3, \ldots)\) | \(S_7^1\) | \(S_7^1^{11}\) | \(S_7^1^{111}\) |
|------|---------------------------------|-----------------|-------------------|
| 92   | \((-3, -1, 0, 0), -1, -1, -1, 0, 0, -1, 0, 1, 1\) | \(q^{-6}\) | \(-q^2 + 4 - q^{-2} + 6 q^{-4} - 3 q^{-6}\) | \(-q^8 + 3 q^{-6} - 2 q^2 + 6 - 9 q^{-2} + 8 q^{-4} - 5 q^{-6} + 2 q^{-8}\) |
| 95   | \((-2, -1, 0, 0), -1, -1, -1, 0, 0, -1, 1, 0, 1\) | \(q^{-6}\) | \(-2 q^2 + 7 - 10 q^{-2} + 8 q^{-4} - 4 q^{-6}\) | \(-q^8 + 3 q^{-6} + 4 q^2 + 9 - 12 q^{-2} + 4 q^{-4} - 6 q^{-6} + 2 q^{-8}\) |
| 98   | \((-2, -1, 0, 0), -1, -1, -1, 0, 0, -1, 1, 0, 1\) | \(q^{-2}\) | \(-q^4 + 3 q^2 + 5 + 4 q^{-2} - q^{-4} - 2 q^{-6} + q^{-8}\) | \(-q^8 + 3 q^{-6} + 6 q^2 + 10 q^{-2} - 12 + 10 q^{-2} - 6 q^{-6} + 2 q^{-8}\) |
| 912  | \((-2, 1, 0, 0), -1, -1, 0, 0, -1, 0, -1, 1\) | \(q^{-4}\) | \(-q^2 - 2 q^{-2} + 3 + 3 q^{-4} - 2 q^2 - 2 q^{-4}\) | \(-q^8 + 2 q^6 + 3 q^4 - 4 q^2 + 3 + 2 q^2 - 2 q^{-2} + q^4\) |
| 914  | \((2, -1, 0, 0), 1, 1, 0, 1, 0, -1, 0, 1\) | \(q^4\) | \(q^8 - 4 q^6 + 4 q^4 - 2 q^2 + 1 + 2 q^{-2} - q^{-4}\) | \(q^8 - q^6 + q^4 - 7 + 8 q^{-2} - 6 q^4 - 3 q^6 - 6 q^{-6} + q^{-8}\) |
| 915  | \((3, 1, 0, 0), -1, -1, 0, 1, 0, 0, 1, 0, -1\) | \(q^4\) | \(-q^8 + 4 q^6 + 4 q^4 - 2 q^2 + 1 + 2 q^{-2} - q^{-4}\) | \(3 q^4 + 8 q^2 - 12 + 12 q^{-2} + 8 q^4 - 4 q^6 + q^{-8}\) |
| 919  | \((-1, -1, 0, 0), -1, -1, 0, 0, 1, 1, 1, 0, -1\) | \(q^4\) | \(-q^8 + 5 q^6 + 6 q^4 - 4 q^2 + 2 q^{-2} - q^{-4}\) | \(-q^8 - 2 q^6 + 2 q^4 + 2 q^2 - 6 + 7 q^{-2} - 6 q^4 - 3 q^6 - q^{-8}\) |
| 921  | \((2, 1, 0, 0), -1, -1, 0, 1, 0, -1, 0, -1, 0\) | \(q^{-4}\) | \(-q^8 + 3 q^6 + 5 + 3 q^{-2} - 2 q^6 - q^8\) | \(-q^8 + 4 q^6 - 4 q^4 + 13 q^2 - 15 + 13 q^{-2} - 9 q^{-4} + 4 q^{-6} - q^{-8}\) |
| 935  | \((-2, -1, 0, 0), -1, -1, 0, 0, -1, 0, 0, 1, 0\) | \(q^{-6}\) | \(-q^4 + q^2 + 3 - 8 q^{-2} + 10 q^{-4} - 7 q^{-6} + q^{-8}\) | \(-q^8 + 2 q^6 - 4 q^4 + 2 q^2 + 4 - 9 q^{-2} + 11 q^{-4} - 9 q^{-6} + 5 q^{-8} - q^{-10}\) |
| 937  | \((-2, 1, 0, 0), -1, -1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1\) | \(q^4\) | \(-q^8 + 7 q^6 - 15 q^4 + 15 - 10 q^{-2} + 3 q^4\) | \(-q^{10} + 4 q^8 - 11 q^6 + 22 q^4 - 30 q^2 + 31 - 25 q^{-2} + 14 q^{-4} - 5 q^{-6} + q^{-8}\) |
| 939  | \((2, 1, 0, 0), -1, -1, 0, -1, 0, 0, -1, 0, 0, -1, 0, 1\) | \(q^4\) | \(-q^8 + 7 q^6 - 12 q^4 + 11 q^2 - 6 q^{-4}\) | \(-q^{10} + 3 q^8 + 4 q^6 + 4 q^4 + 4 q^2 - 6 + 6 q^{-2} - 6 q^{-4} + 3 q^{-6} - q^{-8}\) |
| 941  | \((2, 1, 0, 0), -1, -1, 0, 0, -2, 0, 1, 0, 1, 0, 1, 0, 1, 0, -1, 1\) | \(q^2\) | \(-2 q^8 + 7 q^6 - 12 q^4 + 11 q^2 - 6 q^{-4}\) | \(-q^{10} + 4 q^8 + 8 q^6 - 10 q^4 + 9 q^2 - 3 - 3 q^{-2} + 5 q^{-4} - 4 q^{-6} - 4 q^{-8}\) |
The Table of HOMFLY polynomials for all 6-strand knots with 10 crossings

The contribution of the remaining diagrams is obtained with help of mirror symmetry

\[ q \mapsto -\frac{1}{q} \quad S_6^* \mapsto S_6^{111111} \quad S_5^* \mapsto S_5^{21111} \quad S_4^* \mapsto S_4^{2211} \quad S_4^{111} \mapsto S_4^{3111} \quad S_3^* \mapsto S_3^{222} \]

| knot | \((a_1, b_1, c_1, d_1, e_1, f_1)\) | \(S_6^*\) | \(S_5^*\) | \(S_4^*\) |
|------|-------------------------------|-----------------|-------------|-------------|
| 10_1 | \((-2, -1, 0, 0, 0, 0, 0, -1, -1)\) | \(q^{-3}\) | \(3q^3 - 7q + 5q^{-1} - q^{-3} - 2q^{-5} + q^{-7}\) | \(q^9 - q^7 - 4q^5 + 14q^3 - 24q + 25q^{-1} - 16q^{-3} + 6q^{-5} - q^{-7}\) |
| 10_2 | \((-2, 0, 0, 0, 0, 0, 0, 0, -1)\) | \(q^{-3}\) | \(2q^5 - 4q^3 + 2q - q^{-3} - q^{-5} + q^{-7}\) | \(q^9 - 3q^7 + 5q^5 - 4q^3 - 3q + 10q^{-1} - 10q^{-3} + 5q^{-5} - q^{-7}\) |
| 10_3 | \((-1, 1, 0, 0, 0, 0, 0, -1, -1)\) | \(q^{-3}\) | \(q^5 - 5q + 7q^{-1} - 6q^{-3} + 2q^{-5}\) | \(q^9 - 4q^7 + 7q^5 - 5q^3 - 3q + 11q^{-1} - 13q^{-3} + 9q^{-5} - 4q^{-7} + q^{-9}\) |
| 10_4 | \((-1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, -1)\) | \(q^{-3}\) | \(q^5 - 5q + 7q^{-1} - 6q^{-3} + 2q^{-5}\) | \(q^9 - 5q^7 + 13q^5 - 21q^3 + 21q - 11q^{-1} + 4q^{-5} - 3q^{-7} + q^{-9}\) |
| 10_5 | \((-1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, -1, -1)\) | \(q^{-3}\) | \(2q^5 - 6q^3 + 6q - 3q^{-1} - q^{-3} + q^{-5}\) | \(q^9 - 4q^7 + 6q^5 - 4q^3 - 5q + 16q^{-1} - 20q^{-3} + 15q^{-5} - 6q^{-7} + q^{-9}\) |

| knot | \(S_4^{111}\) | \(S_4^{211}\) | \(S_4^{321}\) |
|------|-------------|-------------|-------------|
| 10_1 | \(-6q^5 + 16q^3 - 20q + 19q^{-1} - 14q^{-3} + 9q^{-5} - 3q^{-7}\) | \(q^7 - 6q^5 + 12q^3 - 16q + 16q^{-1} - 12q^{-3} + 7q^{-5} - 3q^{-7} + q^{-9}\) | \(-2q^9 + 10q^7 - 24q^5 + 41q^3 - 52q + 52q^{-1} - 41q^{-3} + 24q^{-5} - 10q^{-7} + 2q^{-9}\) |
| 10_2 | \(-2q^7 + 2q^5 + q^3 - 2q + 4q^{-1} - 5q^{-3} + 6q^{-5} - 3q^{-7}\) | \(q^7 - 4q^5 + 6q^3 - 6q^1 - 6q^{-3} + 5q^{-5} - 3q^{-7} + q^{-9}\) | \(-q^9 + 4q^7 - 8q^5 + 12q^3 - 14q + 14q^{-1} - 12q^{-3} + 8q^{-5} - 4q^{-7} + q^{-9}\) |
| 10_3 | \(q^9 - 5q^7 + 5q^5 + q^3 - 4q + 6q^{-1} - 8q^{-3} + 10q^{-5} - 6q^{-7} + q^{-9}\) | \(q^9 - 3q^7 + 3q^5 - q^3 + 2q + 5q^{-1} - 5q^{-3} + 2q^{-5}\) | \(q^7 - 4q^5 + 9q^3 - 13q + 13q^{-1} - 9q^{-3} + 4q^{-5} - q^{-7}\) |
| 10_4 | \(q^9 - 6q^7 + 12q^5 - 15q^3 + 17q - 14q^{-1} + 7q^{-3} + 2q^{-5} - 4q^{-7} + q^{-9}\) | \(q^9 - 4q^7 + 8q^5 - 11q^3 + 12q - 9q^{-1} + 4q^{-3} - q^{-5}\) | \(q^9 - 5q^7 + 13q^5 - 24q^3 + 32q - 32q^{-1} + 24q^{-3} - 13q^{-5} + 5q^{-7} - q^{-9}\) |
| 10_5 | \(q^9 - 4q^7 + 4q^5 + 2q^3 - 7q + 10q^{-1} - 12q^{-3} + 14q^{-5} - 9q^{-7} + 2q^{-9}\) | \(q^7 - 2q^5 + 3q^3 - 6q + 10q^{-1} - 11q^{-3} + 9q^{-5} - 5q^{-7} + q^{-9}\) | \(-2q^9 + 8q^7 - 16q^5 + 22q^3 - 24q + 24q^{-1} - 22q^{-3} + 16q^{-5} - 8q^{-7} + 2q^{-9}\) |
The HOMFLY polynomial for an example of 7-strand knot

The contribution of the remain diagrams is obtained with help of mirror symmetry

\[ q \mapsto \frac{1}{q}, \quad S^*_7 \mapsto S^*_{1111111}, \quad S^*_61 \mapsto S^*_{2111111}, \quad S^*_52 \mapsto S^*_{2211111}, \quad S^*_511 \mapsto S^*_3111111, \]

| \( S^*_7 \)                      | 1                              |
|---------------------------------|---------------------------------|
| \( S^*_61 \)                    | \( q^6 - 5q^2 + 7 - 5q^{-2} + q^{-6} \) |
| \( S^*_52 \)                    | \( q^{10} - 3q^8 + 16q^6 - 41q^4 + 54 - 41q^{-2} + 16q^{-4} - 3q^{-8} + q^{-10} \) |
| \( S^*_511 \)                   | \( q^{10} - 2q^8 - 6q^6 + 23q^4 - 35q^2 + 39 - 35q^{-2} + 23q^{-4} - 6q^{-6} - 2q^{-8} + q^{-10} \) |
| \( S^*_43 \)                    | \( -q^{10} + 7q^8 - 25q^6 + 56q^4 - 89q^2 + 104 - 89q^{-2} + 56q^{-4} - 25q^{-6} + 7q^{-8} - q^{-10} \) |
| \( S^*_421 \)                   | \( q^{12} - 9q^{10} + 35q^8 - 82q^6 + 135q^4 - 174q^2 + 188 - 174q^{-2} + 135q^{-4} - 82q^{-6} + 35q^{-8} - 9q^{-10} + q^{-12} \) |
| \( S^*_4111 \)                  | \( q^{12} - 9q^{10} + 29q^8 - 48q^6 + 56q^4 - 62q^2 + 65 - 62q^{-2} + 56q^{-4} - 48q^{-6} + 29q^{-8} - 9q^{-10} + q^{-12} \) |
| \( S^*_331 \)                   | \( q^{12} - 8q^{10} + 29q^8 - 66q^6 + 112q^4 - 151q^2 + 166 - 151q^{-2} + 112q^{-4} - 66q^{-6} + 29q^{-8} - 8q^{-10} + q^{-12} \) |

knot 12a0125

\[(a_1, b_1, c_1, d_1, e_1, f_1|a_2, b_2, c_2, d_2, e_2, f_2|a_3, b_3, c_3, d_3, e_3, f_3)\]

(1, −1, −1, 0, 0, 0|1, 0, −1, −1, 1, −1|0, −1, 1, 2, 1, −1)
Appendix 1. Mixing and $\mathcal{R}$-matrices for the 6-strand formula (30)

$5 \times 5$ mixing matrices for $[5, 1]$

$$\hat{\mathcal{R}}_{[5,1]} = \begin{pmatrix} q & q & q \\ q & q & -\frac{1}{q} \end{pmatrix}, \quad \hat{U}_{[5,1]} = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & \sqrt{\frac{3}{2}} & -\frac{1}{2} \\ -\sqrt{\frac{3}{2}} & -\frac{1}{2} \\ \sqrt{\frac{3}{2}} & \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \hat{V}_{[5,1]} = \begin{pmatrix} 1 & 1 & \sqrt{\frac{2}{4}} \\ -\frac{1}{3} & \frac{\sqrt{2}}{4} & \frac{1}{3} \\ \frac{3}{3} \end{pmatrix}$$

$$\hat{W}_{[5,1]} = \begin{pmatrix} 1 & \frac{1}{4} & \frac{\sqrt{3} \sqrt{5}}{4} \\ -\frac{\sqrt{3} \sqrt{5}}{4} & 1 \\ \frac{3}{4} \end{pmatrix}, \quad \hat{Y}_{[5,1]} = \begin{pmatrix} 1 & 1 & \frac{\sqrt{4} \sqrt{6}}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

$5 \times 5$ mixing matrices for $[3, 3]$

$$\hat{\mathcal{R}}_{[3,3]} = \begin{pmatrix} q & q & -\frac{1}{q} \\ q & -\frac{1}{q} & q \\ -\frac{1}{q} & q & -\frac{1}{q} \end{pmatrix}, \quad \hat{U}_{[3,3]} = \begin{pmatrix} 1 & \frac{1}{2} & \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \hat{V}_{[3,3]} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{\sqrt{2} \sqrt{4}}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\hat{W}_{[3,3]} = \begin{pmatrix} 1 & \frac{1}{2} & \sqrt{\frac{3}{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad \hat{Y}_{[3,3]} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$
$9 \times 9$ mixing matrices for $[4, 2]$:

$$\hat{R}_{[4, 2]} = \begin{pmatrix} q & q & \frac{-1}{q} \\
q & q & \frac{-1}{q} \\
\frac{-1}{q} & q & \frac{-1}{q} \\
\end{pmatrix}, \quad \hat{U}_{[4, 2]} = \begin{pmatrix} 1 & 1 & \sqrt{\frac{3}{2}} \\
\frac{-1}{2} & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\
\frac{-\sqrt{3}}{2} & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\
\end{pmatrix},$$

$$\hat{V}_{[4, 2]} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{\sqrt{2[4]}}{3} \\
\frac{-\sqrt{2[4]}}{3} & \frac{1}{3} & \frac{\sqrt{2[4]}}{3} \\
\frac{-\sqrt{2[4]}}{3} & \frac{1}{3} & \frac{-\sqrt{2[4]}}{3} \\
\end{pmatrix},$$

$$\hat{W}_{[4, 2]} = \begin{pmatrix} 1 & 1 & 1 & 1 \\
\frac{-1}{4} & \frac{-\sqrt{3[5]}}{4} & \frac{1}{4} & \frac{-\sqrt{3[5]}}{4} \\
\frac{1}{4} & \frac{\sqrt{3[5]}}{4} & \frac{-1}{4} & \frac{-\sqrt{3[5]}}{4} \\
\frac{1}{4} & \frac{\sqrt{3[5]}}{4} & \frac{-1}{4} & \frac{-\sqrt{3[5]}}{4} \\
\end{pmatrix}.$$


$$\hat{Y}_{[4,2]} = \begin{pmatrix} 1 & \frac{\sqrt{2}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{8}}{3} \\ \frac{1}{3} & 1 & \frac{\sqrt{2}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{4}}{3} & 1 & \frac{\sqrt{2}}{3} & \frac{\sqrt{4}}{3} \\ \frac{\sqrt{4}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{2}}{3} & 1 & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & 1 \end{pmatrix},$$

**10 × 10 mixing matrices for [4,1,1]**

$$\hat{R}_{[4,1,1]} = \begin{pmatrix} q & q & q & \frac{1}{q} & \frac{1}{q} \\ q & q & q & \frac{1}{q} & \frac{1}{q} \\ q & q & q & \frac{1}{q} & \frac{1}{q} \\ \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} \\ \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} & \frac{1}{q} \end{pmatrix}, \quad \hat{U}_{[4,1,1]} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$\hat{V}_{[4,1,1]} = \begin{pmatrix} 1 & \frac{\sqrt{2}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{8}}{3} \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{4}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{4}}{3} & \frac{\sqrt{4}}{3} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{8}}{3} & \frac{\sqrt{8}}{3} & \frac{\sqrt{8}}{3} & \frac{\sqrt{8}}{3} & \frac{\sqrt{8}}{3} \end{pmatrix}.$$
$\hat{W}_{[4,1,1]} = \begin{pmatrix}
\frac{1}{4} & \frac{\sqrt{3}[0]}{4} \\
\frac{\sqrt{3}[0]}{4} & 1 \\
\frac{1}{4} & \frac{\sqrt{3}[0]}{4} \\
\frac{\sqrt{3}[0]}{4} & 1 \\
\frac{1}{4} & \frac{\sqrt{3}[0]}{4} \\
\frac{\sqrt{3}[0]}{4} & 1 \\
\frac{1}{4} & \frac{\sqrt{3}[0]}{4} \\
\frac{\sqrt{3}[0]}{4} & 1
\end{pmatrix}$,

$\hat{Y}_{[4,1,1]} = \begin{pmatrix}
-1 & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5} & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5} & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5} & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5} & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5} & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5} & \frac{\sqrt{4}[0]}{5} \\
-\frac{1}{5}
\end{pmatrix}$,

$16 \times 16$ mixing matrices for $[3,2,1]$

$\hat{R}_{[3,2,1]} = \begin{pmatrix}
q & q & -\frac{1}{q} & q & -\frac{1}{q} & q & -\frac{1}{q} & q & -\frac{1}{q} & q & -\frac{1}{q} & q & -\frac{1}{q} & q & -\frac{1}{q}
\end{pmatrix}$.
\[
U_{[3,2,1]} = \begin{pmatrix}
1 & \frac{-1}{2} & \sqrt{3} \\
\sqrt{3} & \frac{-1}{2} & 1 \\
\frac{-1}{2} & \sqrt{3} & 1 \\
\end{pmatrix}
\]

\[
V_{[3,2,1]} = \begin{pmatrix}
-\frac{1}{3} & \sqrt{\frac{4}{3}} \\
\sqrt{\frac{4}{3}} & \frac{1}{3} \\
\frac{1}{3} & \sqrt{\frac{4}{3}} \\
\end{pmatrix}
\]
\[ W_{[3,2,1]} = \begin{pmatrix}
1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2}
\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2}
\frac{\sqrt{3}}{2} & \frac{1}{2} & 1
\end{pmatrix}^{-1} \]
\( \hat{Y}_{[3,2,1]} = \begin{pmatrix}
-\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{5|3|}}{2} & \frac{\sqrt{5|3|}}{4} & \frac{\sqrt{5|3|}}{4} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\
\end{pmatrix} \)
\[ V_{s2} = \begin{pmatrix} 1 & 1 & -c_3 & c_3 \\ -c_3 & c_3 & 1 & 1 \end{pmatrix} \]

\[ W_{s2} = \begin{pmatrix} 1 & -c_4 & c_4 \\ c_4 & -c_4 & 1 \end{pmatrix} \]

\[ \hat{V}_{s2} = \begin{pmatrix} 1 & -c_5 & c_5 \\ -c_5 & c_5 & 1 \end{pmatrix} \]

\[ \hat{W}_{s2} = \begin{pmatrix} 1 & -c_4 & c_4 & c_4 \end{pmatrix} \]

\[ V_{s11} = \begin{pmatrix} 1 & 1 & -c_2 & -c_2 \\ c_2 & -c_2 & 1 \end{pmatrix} \]

\[ W_{s11} = \begin{pmatrix} 1 & -c_4 & c_4 & c_4 \end{pmatrix} \]
\[
\begin{pmatrix}
1 & 1 \\
-\frac{1}{x^2} & -\frac{1}{x^2} & 1 \\
\frac{1}{x^2} & \frac{1}{x^2} & -\frac{1}{x^2} & -\frac{1}{x^2} & 1 \\
\end{pmatrix}
\]
\[ v_{421} = \begin{pmatrix} 1 & -c_3 & s_3 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ -c_3 & 1 & -c_3 & s_3 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ s_3 & -c_3 & 1 & -c_3 & s_3 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & s_3 & 1 & -c_3 & s_3 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ -c_3 & s_3 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ s_3 & -c_3 & s_3 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & s_3 & 1 & -c_3 & s_3 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & s_3 & 1 & -c_3 & s_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ -c_3 & s_3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ W_{421} = \begin{pmatrix}
-c_4 & s_4 & 1 & 1 \\
-s_4 & c_4 & -c_2 & s_2 \\
-s_2 & c_2 & -c_4 & s_4 \\
s_2 & c_2 & -s_4 & c_4 \\
\end{pmatrix} \]
\[ Z_{421} = \begin{pmatrix}
-2 & 2 & 2 & 2 & 2
-2 & -5 & -5 & -5 & -5
-2 & -5 & -5 & -5 & -5
-2 & -5 & -5 & -5 & -5
-2 & -5 & -5 & -5 & -5
\end{pmatrix}\]
\[
\mathbf{u}_{4111} = \begin{pmatrix}
1 & -e_2 & -e_2 & -e_2 \\
-2 & -e_2 & -e_2 & -e_2 \\
-1 & -e_2 & -e_2 & -e_2 \\
-1 & -e_2 & -e_2 & -e_2 \\
\end{pmatrix}
\]

\[
\hat{\mathbf{u}}_{4111} = \begin{pmatrix}
1 & -e_3 & -e_3 & -e_3 \\
-3 & -e_3 & -e_3 & -e_3 \\
-1 & -e_3 & -e_3 & -e_3 \\
-1 & -e_3 & -e_3 & -e_3 \\
\end{pmatrix}
\]

\[
\hat{\hat{\mathbf{u}}}_{4111} = \begin{pmatrix}
1 & -e_4 & -e_4 & -e_4 \\
-4 & -e_4 & -e_4 & -e_4 \\
-1 & -e_4 & -e_4 & -e_4 \\
-1 & -e_4 & -e_4 & -e_4 \\
\end{pmatrix}
\]

\[
\hat{\hat{\hat{\mathbf{u}}}}_{4111} = \begin{pmatrix}
1 & -e_5 & -e_5 & -e_5 \\
-5 & -e_5 & -e_5 & -e_5 \\
-1 & -e_5 & -e_5 & -e_5 \\
-1 & -e_5 & -e_5 & -e_5 \\
\end{pmatrix}
\]

\[
\hat{\hat{\hat{\hat{\mathbf{u}}}}}_{4111} = \begin{pmatrix}
1 & -e_6 & -e_6 & -e_6 \\
-6 & -e_6 & -e_6 & -e_6 \\
-1 & -e_6 & -e_6 & -e_6 \\
-1 & -e_6 & -e_6 & -e_6 \\
\end{pmatrix}
\]
