1. INTRODUCTION

Interval preference relations (IPRs) have been widely used in uncertain group decision making (GDM) to represent decision makers' (DMs') preference over alternatives. When encountering complex or emergent situations, DMs cannot express their complete preference information on alternatives and often present an incomplete (sparse) form of judgment because of knowledge reserve, information mastery, environment impact, and so on. Incomplete IPRs can be managed by two approaches. One is filling in incomplete values based on consistency [1–6], and the other is ranking alternatives directly with known elements [7–9]. Although the latter preserves the true preferences of DMs, it cannot guarantee the consistency between individuals, thus resulting in the distortion of decision making. Based on consistency, the iterative algorithm [10–13] and the optimisation model [4,14–17] are two typical methods to complete missing values. Based on the iterative algorithm, although it is easy to change the original preference relations and the convergence speed is slow, the process is robust and the consistency is good. Furthermore, the optimisation model with missing parameter constraints can obtain the optimal solutions of incomplete values.

The consistency index is typically used for measuring the consistency level of an individual IPR; subsequently, an iterative algorithm is established to achieve an acceptable level of consistency, such as in an interval fuzzy preference relation (IFPR) [18,19], interval intuitionistic preference relation [20–22], linguistic preference relation [23,24], and hesitant fuzzy preference relation (FPR) [25,26]. The compatibility [27–32] is used similar to the consistency index. To ensure the efficiency and consensus [11,33–35], GDM typically aggregates individual preference relations into a collective preference relation, which is often obtained by the weighted averaging operator [36,37], ordered weighted averaging operator [6,10,38], and weighted geometric averaging operator [39,40], followed by a consensus to measure the difference among all individuals.

In traditional GDM, an incomplete value is often determined by minimising the deviation between the incomplete value and a supplementary value obtained by the consistency property. However, the new supplementary value may not necessarily match the original preference information, and the DM cannot measure the authenticity between the supplementary value and original missing information. Belief degrees in the uncertainty theory proposed by Liu [41] can solve this problem. Moreover, when handling the interval information, only the two end points of the interval are used in the operation, and the internal information of the interval is
completely ignored, which easily results in decision distortion caused by the discretisation operation of the intervals.

In fact, the pairwise comparison of alternatives is located in an uncertain interval range in the IPRs, which is an uncertain estimation based on subjective experience. It can be understood as the uncertainty distribution (UD) of the subjective preference of the DM [41–47]: for the linear uncertainty distribution (LUD), every value of the DM’s preference in the interval is equally possible; for the normal uncertainty distribution (NUD), the preference value obeys the NUD under a certain belief degree, and so on. As a mathematical system dedicated to researching the belief degrees of experts, uncertainty theory [41] provides a new solution to GDM problems.

Let $\Gamma$ be a nonempty set, and let $\mathcal{L}$ be a $\sigma$-algebra over $\Gamma$, and let $M$ be an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, M)$ is called an uncertainty space.

**Axiom 1.** $M(\Gamma) = 1$ for the universal set $\Gamma$.

**Axiom 2.** $M(\lambda) + M(\lambda^c) = 1$ for any event $\lambda$.

**Axiom 3.** For each countable sequence of events $\lambda_1, \lambda_2, \ldots$, we have $M\left(\bigcup_{i=1}^{n} \lambda_i\right) \leq \sum_{i=1}^{n} M(\lambda_i)$.

**Axiom 4.** Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $M$ is an uncertain measure satisfying $M\left(\prod_{k=1}^{\infty} \lambda_k\right) = \bigwedge_{k=1}^{\infty} M_k(\lambda_k)$, where $\lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

**Definition 3.** An uncertain variable is a function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers. Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables, and let $f$ be a real-valued measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.

**Definition 4.** The UD $\Phi$ of an uncertain variable $\xi$ is defined by $\Phi(x) = M[\xi \leq x]$ for any real number $x$. $\Phi(x)$ is a monotone increasing function and $0 \leq \Phi(x) \leq 1$ (Figure 1).

From Axiom 2, we have $M[\xi \leq x] + M[\xi > x] = 1$, then $M[\xi > x] = 1 - \Phi(x)$. When the UD $\Phi$ is continuous, we also have $M[\xi < x] = M[\xi < x] = \Phi(x), M[\xi > x] = 1 - \Phi(x)$.

**Definition 5.** An UD $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to $x$ at which $0 < \Phi(x) < 1$, and $\lim_{x \rightarrow -\infty} \Phi(x) = 0$, $\lim_{x \rightarrow +\infty} \Phi(x) = 1$.

**Definition 6.** Let $\xi$ be an uncertain variable with regular UD $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertain distribution (IUD) of $\xi$. A function $\Phi^{-1} : (0, 1) \rightarrow R$ is the IUD of an uncertain variable $\xi$ if and only if $M[\xi \leq \Phi^{-1}(\alpha)] = \Phi(\Phi^{-1}(\alpha)) = \alpha$, for all $\alpha \in (0, 1)$.

**Definition 7.** An uncertain variable $\xi$ is called linear if it has a LUD $\Phi(x)$:

$$
\Phi(x) = \begin{cases} 
0 & x \leq a \\
\frac{x-a}{b-a} & a \leq x \leq b \\
1 & x \geq b
\end{cases}
$$

(1)

denoted by $\xi \sim \mathcal{L}(a, b)$ where $a$ and $b$ are real numbers with $a < b$ (Figure 2).
The IUD of linear uncertain variable (LUV) \( \xi \) is \( \Phi^{-1}(\alpha) = (1 - \alpha) a + \alpha b \) (Figure 3). LUD is regular.

**Theorem 1.** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with regular UDs \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If \( f(\xi_1, \xi_2, \ldots, \xi_n) \) is strictly increasing with respect to \( \xi_1, \xi_2, \ldots, \xi_m \), and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \ldots, \xi_n \), then \( f(\xi_1, \xi_2, \ldots, \xi_n) \) has an IUD \( \Psi^{-1}(\alpha) \). \( \Psi^{-1}(\alpha) \) is defined as follows:

\[
f(\Phi^{-1}_1(\alpha), \ldots, \Phi^{-1}_m(\alpha), \Phi^{-1}_{m+1}(1 - \alpha), \ldots, \Phi^{-1}_n(1 - \alpha))
\]

**2.2. Several Preference Relations**

Let \( X = \{x_i \} (i \in N = \{1, 2, \ldots, n \}, n \geq 2 \) be a nonempty set of alternatives, \( D = \{d_i \} (t \in M = \{1, 2, \ldots, m \}) \) be the set of DMs.

**2.2.1. Fuzzy preference relations**

**Definition 8.** [48] A FPR \( R = (r_{ij})_{n \times n} \) is characterised by a function \( \mu_R : X \times X \rightarrow [0, 1] \), where \( \mu_R(x_i, x_j) = r_{ij} \) indicates the preference intensity with which alternative \( x_i \) is preferred over \( x_j \). \( R \) is additive reciprocal, if

\[
r_{ij} + r_{ji} = 1, \quad r_{ii} = 0.5, \quad i, j \in N
\]

\( r_{ij} = 0.5 \) indicates that there is no difference between \( x_i \) and \( x_j \); \( r_{ij} > 0.5 \) represents that \( x_i \) is preferred to \( x_j \); \( r_{ij} < 0.5 \) depicts that \( x_j \) is better than \( x_i \).

**Definition 9.** [49] The FPR \( R = (r_{ij})_{n \times n} \) is additively consistent, if it satisfies

\[
r_{ij} = r_{ik} + r_{kj} - 0.5, \quad i, j, k \in N
\]

**2.2.2. Interval fuzzy preference relations**

**Definition 10.** [50] Let \( \widetilde{a}_1 = [a^+_1, a^-_1] \), \( \widetilde{a}_2 = [a^+_2, a^-_2] \) be any two positive intervals.

(a) \( \widetilde{a}_1 \ominus \widetilde{a}_2 = [a^+_1, a^-_1] \ominus [a^+_2, a^-_2] = [a^+_1 - a^-_2, a^-_1 - a^+_2] \)

(b) \( \widetilde{a}_1 \ominus \widetilde{a}_2 = [a^+_1, a^-_1] \ominus [a^+_2, a^-_2] = [a^-_1 - a^+_2, a^-_1 - a^+_2] \)

(c) \( \lambda\widetilde{a}_1 = \lambda [a^+_1, a^-_1] = \begin{cases} [\lambda a^+_1, \lambda a^-_1] & \lambda \geq 0 \\ [\lambda a^-_1, \lambda a^+_1] & \lambda < 0 \end{cases} \)

**Definition 11.** [51] Let \( \widetilde{R} = (\tilde{r}_{ij})_{n \times n} \) be a FPR. \( \widetilde{R} \) is called an IFPR if \( \tilde{r}_{ij} = [r^+_{ij}, r^-_{ij}] \) and \( \tilde{r}_{ji} = [r^+_{ji}, r^-_{ji}] \) satisfy

\[
r^+_{ij} + r^-_{ij} = r^+_{ji} + r^-_{ji} = 1
\]

\[
r^+_{ii} = r^-_{ii} = 0.5
\]

\( r^+_{ij} \geq r^-_{ij} \geq 0 \), Eq. (5) is the additive reciprocity of \( \widetilde{R} \).

**Definition 12.** [1] Let \( \widetilde{R} = (\tilde{r}_{ij})_{n \times n} \) be an IFPR. \( \widetilde{R} \) is additively consistent if

\[
\begin{cases}
  r^+_{ik} + r^-_{kj} = r^+_{ij} + 0.5 \\
  r^+_{ki} + r^-_{jk} = r^+_{ji} + 0.5
\end{cases}, \quad i < k < j, \quad i, j, k \in N
\]

is true. Additionally, it can be represented as \( \tilde{r}_{ik} \oplus \tilde{r}_{kj} = \tilde{r}_{ij} \oplus [0.5, 0.5] \).

**2.2.3. Incomplete fuzzy preference relations**

**Definition 13.** [10] Let \( \widetilde{R} = (\tilde{r}_{ij})_{n \times n} \) be a FPR. If at least an unknown preference value \( r_{ij} \) exists in \( \widetilde{R} \), then \( \widetilde{R} \) is called an incomplete FPR. The incomplete FPR \( R = (r_{ij})_{n \times n} \) can be completed based on the additive consistency if \( n - 1 \) nonleading diagonal preference values are known.

**3. FPRs BASED ON UNCERTAINTY THEORY**

**3.1. Uncertain Preference Relations**

**Definition 14.** Let \( \widetilde{R} = (\mathcal{L}(r^+_{ij}, r^-_{ij}))_{n \times n} \) be a non-negative matrix, where \( \tilde{r}_{ij} \) is an uncertain variable. The UD of \( \tilde{r}_{ij} \) is \( \Phi_{ij} \), and the IUD is \( \Phi_{ij}^{-1} \). \( \widetilde{R} \) is an UPR if it satisfies

\[
\Phi_{ij}^{-1}(\beta) + \Phi_{ji}^{-1}(1 - \beta) = 1
\]

\[
\Phi_{ii}^{-1}(\beta) = 0.5
\]

for any \( \beta \) in \([0, 1]\), \( i, j \in N \).
Eq. (8) is the additive reciprocity of $\bar{R}$. The judgment element $\bar{r}_{ij}$ in an UPR indicates the degree to which the alternative $x_i$ is superior to $x_j$, $\Phi_{ij}^1(\beta) = \beta$ denotes that there is no difference between $x_i$ and $x_j$; $\Phi_{ij}^1(\beta) > 0.5$ indicates that $x_j$ is superior to $x_i$; $\Phi_{ij}^1(\beta) < 0.5$ depicts that $x_j$ is better than $x_i$.

For example, if $\bar{r}_{ij}$ obeys the LUD, $\bar{r}_{ij} \sim \mathcal{L}(r_{ij}', r_{ij})$, $\bar{r}_{ii} \sim \mathcal{L}(0, 1)$, then $\bar{R} = (\bar{r}_{ij})_{n \times n}$ is called a linear uncertainty preference relation.

According to the additive reciprocity, $(1 - \beta)r_{ij}' + \beta r_{ij}' + \beta(1 - r_{ij}') + (1 - \beta)(1 - r_{ij}') = 1$ (Figure 4). When $\beta = 0$, $r_{ij}' + (1 - r_{ij}') = 1$. When $\beta = 1$, $r_{ij}' + (1 - r_{ij}') = 1$. These are the definitions of additive reciprocity of IFPRs. That is, when $\beta$ moves in $[0, 1]$, $\Phi_{ij}^1(\beta)$ corresponds to each value in the interval.

**Definition 15.** Let $\bar{R} = (\bar{r}_{ij})_{n \times n}$ be an UPR. $\bar{R}$ is additively consistent if it satisfies

$$\Phi_{ik}^1(\beta) + \Phi_{ji}^1(\beta) = \Phi_{ij}^1(\beta) + 0.5$$

(10)

for any $\beta$ in $[0, 1]$, $i < k < j, i, j, k \in N$.

### 3.2. Uncertain Chance Constrained Programming Model

Let $\bar{R}' = (\bar{r}_{ij}')_{n \times n}$ be the incomplete preference relation given by the DM $d_t$ ($t \in M$). The DMs are independent. $\bar{r}_{ij}$ is an uncertain variable and its UD is $\Phi_{ij}$. We have $\Phi_{ij}^1(\beta) + \Phi_{ji}^1(1 - \beta) = 1$, $\Phi_{ij}^1(\beta) = 0.5$ for any $\beta$ in $[0, 1]$. $\bar{R}$, $\bar{r}_{ij}$, and $\bar{r}_{ij}'$ are obtained from Eq. (10).

We have two sets of equations, one for diagonal elements and the other for incomplete values. $\bar{r}_{ij}'$ is determined using the mean value, and the incomplete values in the lower triangular matrix can be obtained from the additive reciprocity.

For higher accuracy, the deviation between the ideal value $\bar{r}_{ij}'$ and the estimated value $\bar{r}_{ij}$ should be as small as possible. An UCCPM is established as follows:

$$\min \sum_{i < j} \varepsilon_{ijt} \quad (M - 1)$$

Figure 4 | Additive reciprocity of inverse LUD.
Therefore, the equivalent model of \((M - 1)\) represented by an inverse distribution is as follows:

\[
\min \sum_{i,j,t=1}^{n} \varepsilon_{ijt} \quad (M - 2)
\]

\[
\begin{align*}
\varepsilon_{ijt} \geq \Phi_{ijt}^{1}(\alpha) - \\
\frac{1}{n-2} \sum_{k=1,k \neq ij,t}^{n} \left[ \Phi_{ikt}^{1}(1-\alpha) + \Phi_{kjt}^{1}(1-\alpha) - 0.5 \right] \\
\varepsilon_{ijt} \leq \Phi_{ijt}^{1}(1-\alpha) - \\
\frac{1}{n-2} \sum_{k=1,k \neq ij,t}^{n} \left[ \Phi_{ikt}^{1}(\alpha) + \Phi_{kjt}^{1}(\alpha) - 0.5 \right] \\
\Phi_{ijt}^{1}(\vartheta) + \Phi_{ijt}^{1}(1-\vartheta) = 1, \vartheta \in [0, 1] \\
\varepsilon_{ijt} \geq 0, i < j, j \in N, t \in M
\end{align*}
\]

\(s.t.
\]

\[
\begin{align*}
\Phi_{ijt}^{1}(\vartheta) + \Phi_{ijt}^{1}(1-\vartheta) & = 1, \vartheta \in [0, 1]\\
\varepsilon_{ijt} & \geq 0, i < j, j \in N, t \in M
\end{align*}
\]

\[
\Rightarrow \left\{ \begin{array}{l}
\Phi_{ijt}^{1}(\vartheta) + \Phi_{ijt}^{1}(1-\vartheta) = 1, \vartheta \in [0, 1]\\
\varepsilon_{ijt} \geq 0, i < j, j \in N, t \in M
\end{array} \right.
\]

Let \(\bar{r}_{ijt}\) be a LUV, \(\bar{r}_{ijt} \sim \mathcal{L}(\tilde{r}_{ijt}, \tilde{r}_{ijt}')\). \((M - 2)\) equals to

\[
\Rightarrow (1-\alpha)\bar{r}_{ijt} + \alpha \bar{r}_{ijt}' - \frac{1}{n-2} \sum_{k=1,k \neq ij,t}^{n} \left[ (1-\alpha)\bar{r}_{ikt} + \alpha \bar{r}_{ikt}' + (1-\alpha)\bar{r}_{kjt} + \alpha \bar{r}_{kjt}' - 0.5 \right] \]

\((M - 2)\) equals to

\[
\Rightarrow \frac{1}{n-2} \sum_{k=1,k \neq ij,t}^{n} \left[ (1-\alpha)\bar{r}_{ikt} + \alpha \bar{r}_{ikt}' + (1-\alpha)\bar{r}_{kjt} + \alpha \bar{r}_{kjt}' - 0.5 \right] 
\]

Therefore, the equivalent model of \((M - 2)\) is

\[
\min \sum_{i,j,t=1}^{n} \varepsilon_{ijt} \quad (M - 3)
\]

\[
\begin{align*}
\varepsilon_{ijt} & \geq (1-\alpha)\bar{r}_{ijt} + \alpha \bar{r}_{ijt}' - \\
\frac{1}{n-2} \sum_{k=1,k \neq ij,t}^{n} \left[ (1-\alpha)\bar{r}_{ikt} + \alpha \bar{r}_{ikt}' + (1-\alpha)\bar{r}_{kjt} + \alpha \bar{r}_{kjt}' - 0.5 \right] \\
\varepsilon_{ijt} & \leq (1-\alpha)\bar{r}_{ijt} + \alpha \bar{r}_{ijt}' - \\
\frac{1}{n-2} \sum_{k=1,k \neq ij,t}^{n} \left[ (1-\alpha)\bar{r}_{ikt} + \alpha \bar{r}_{ikt}' + (1-\alpha)\bar{r}_{kjt} + \alpha \bar{r}_{kjt}' - 0.5 \right] \\
\left(1-\beta\right)\bar{r}_{ijt} + \beta \bar{r}_{ijt}' + \beta \bar{r}_{ijt}' + (1-\beta)\bar{r}_{ijt} & = 1 \\
\beta \in [0, 1], \varepsilon_{ijt} & \geq 0, i < j, i, j \in N, t \in M
\end{align*}
\]

\(s.t.
\]

Theorem 2. The linear equivalent model of UCCPM \((M - 2)\) is \((M - 3)\).

3.3. Consistency Analysis of UPRs

3.3.1. Individual additive consistency analysis

Let \(\bar{R}^{'}\) be the associated additively consistent UPR of \(\bar{R}\). \(\bar{R}^{'}\) is an uncertain variable and its UD is \(\Phi^{'}\), IUD is \(\Phi^{'}^{'}\). When the completed UPR \(\bar{R}^{'}\) is equal to \(\bar{R}^{'}\), then \(\bar{R}^{'}\) is additively consistent. When \(i < j\), the deviations of the upper triangular matrix and lower triangular matrix between \(\bar{R}^{'}\) and \(\bar{R}^{'}\) are respectively defined as follows:

\[
d \left( \bar{r}_{ijt}, \bar{r}_{ijt}' \right) = |\Phi_{ijt}^{'}(\beta) - \Phi_{ijt}^{'}(1-\beta)| \quad (11)
\]

\[
d \left( \bar{r}_{ijt}, \bar{r}_{ijt}' \right) = |\Phi_{ijt}^{'}(\beta) - \Phi_{ijt}^{'}(1-\beta)| \quad (12)
\]

The smaller the value of \(d\), the closer is \(\bar{R}^{'}\) to \(\bar{R}^{'}\). Additionally, the consistency level of \(\bar{R}^{'}\) is higher. Considering that

\[
d \left( \bar{r}_{ijt}, \bar{r}_{ijt}' \right) = |\Phi_{ijt}^{'}(\beta) - \Phi_{ijt}^{'}(1-\beta) - \left| -1 - \Phi_{ijt}^{'}(\beta) \right| + \Phi_{ijt}^{'}(\beta) - \Phi_{ijt}^{'}(1-\beta)|
\]

the deviations of the upper triangular matrix and lower triangular matrix can be expressed in the same formula; thus, we will no longer calculate the deviation of the lower triangular matrix separately.

Definition 16. [16] Let \(\bar{R}^{'} = (\bar{r}_{ijt}')_{n \times n}\) be the associated additively consistent UPR of \(\bar{R}\). The additive consistency index (ACI) of \(\bar{R}^{'}\) is defined as follows:

\[
ACI \left( \bar{R}^{'} \right) = 1 - \frac{1}{n(n-1)} \sum_{ij=1}^{n} d \left( \bar{r}_{ijt}, \bar{r}_{ijt}' \right) \quad (13)
\]

The larger the \(ACI \left( \bar{R}^{'} \right)\), the higher the consistency level of \(\bar{R}^{'}\). If and only if \(d \left( \bar{r}_{ijt}, \bar{r}_{ijt}' \right) = 0\), \(ACI(\bar{R}^{'}) = 1\) and \(\bar{R}^{'}\) is fully additively consistent.

3.3.2. Adjustment of inconsistent UPRs

Definition 17. Let \(\tilde{R}^{'} = (\tilde{r}_{ijt}')_{n \times n}\) be the associated additively consistent UPR of \(\bar{R}\). Suppose the UD of \(\tilde{R}^{'}\) is \(\tilde{r}_{ijt}^{'}, \tilde{r}_{ijt}^{'}\) is an improved UPR, if it satisfies

\[
\Phi_{ijt}^{'}(\beta) = \theta \Phi_{ijt}^{'}(\beta) + (1-\theta) \Phi_{ijt}^{'}(1-\beta) \quad (14)
\]

Theorem 3. Let \(\tilde{R}^{'}\) be an improved UPR and \(\tilde{R}^{'}\) be a complete UPR. Therefore, we can derive that \(ACI(\tilde{R}^{'}) > ACI(\tilde{R}^{'})\).
Proof.  
\[
d (\tilde{r}_{ijt}, \tilde{r}_{ijt}^{'}) = |\Phi_{ijt}^{1} (\beta) - \Phi_{ijt}^{1!} (1 - \beta)| \\
= |\delta \Phi_{ijt}^{1} (\beta) + (1 - \delta) \Phi_{ijt}^{1!} (1 - \beta) - \Phi_{ijt}^{1} (1 - \beta)| \\
= \delta |[\Phi_{ijt}^{1} (\beta) - \Phi_{ijt}^{1!} (1 - \beta)]| \\
= \delta |[\Phi_{ijt}^{1} (\beta) - \Phi_{ijt}^{1!} (1 - \beta)]| \\
= \delta |d (\tilde{r}_{ijt}, \tilde{r}_{ijt}^{'})| \\
< d (\tilde{r}_{ijt}, \tilde{r}_{ijt}^{'}) \\
\]
Thus, \(ACI (\tilde{R}) > ACI (\tilde{R})\).

Corollary 1. Let \(\tilde{R}^{(h)} = (\tilde{r}_{ijt}^{(h)})_{h\in D}\) be the \(h\)th improved UPR. After an adjustment, the consistency index of the UPR is better than that of the previous one.

Proof. Let \(\tilde{R}^{(h)}_{n(h-1)} = (\tilde{r}_{ijt}^{(h-1)})\) be the additively consistent UPR with UD \(\Phi_{ijt}^{(h-1)}\) and IUD \(\Phi_{ijt}^{(h-1)}\).

\[
d (\tilde{r}_{ijt}^{(h)}, \tilde{r}_{ijt}^{(h-1)}) = |\Phi_{ijt}^{(h-1)} (\beta) - \Phi_{ijt}^{(h-1)} (1 - \beta)| \\
= \delta |[\Phi_{ijt}^{(h-1)} (\beta) - \Phi_{ijt}^{(h-1)} (1 - \beta)]| \\
= \delta |d (\tilde{r}_{ijt}^{(h)}, \tilde{r}_{ijt}^{(h-1)})| \\
\]
Since \(\delta \in (0, 1), d (\tilde{r}_{ijt}^{(h-1)}, \tilde{r}_{ijt}^{(h-1)})\) are known constants, the deviation after each adjustment is smaller than that of the previous one.

In Corollary 1, after \(h_0\) number of iterations, a certain threshold \(Z_0\) is obtained. When \(d (\tilde{r}_{ijt}^{(h_0)}, \tilde{r}_{ijt}^{(h_0)}) < Z_0\), we consider the additive consistency of \(\tilde{R}^{(h_0)}\) to be acceptable.

3.3.3. Consensus analysis

To improve the level of consensus in GDM, all individual FPRs are typically aggregated to obtain a collective FPR. The latter is used to obtain the individual preference relation which deviates significantly from the consensus [4, 52, 53]. Combining the uncertainty theory and the aggregation operator, a consensus index which measures the deviation between an individual UPR and a collective UPR is introduced.

Let \(\tilde{R}^{\ast} = (\tilde{r}_{ijc})_{i\in D}\) be the collective UPR of all individual DMs. \(\tilde{r}_{ijc}\) is an uncertain variable with an UD \(\Phi_{ijc}\). For any \(\beta \in [0, 1]\), we have \(\Phi_{ijc}^{1} (\beta) + \Phi_{ijc}^{1!} (1 - \beta) = 1, \Phi_{ijc}^{1} (\beta) = 0.5\).

Definition 18. [16] Let \(\tilde{R}^{\ast} = (\tilde{r}_{ijc})_{i\in D}\) be the UPR of \(d_i\), and its IUD is \(\Phi_{ijc}^{1}\). The consensus index of \(\tilde{R}^{\ast}\) is defined as follows:

\[
COI (\tilde{R}, \tilde{R}^{\ast}) = 1 - \frac{1}{n(n - 1)} \sum_{i,j=1}^{n} d (\tilde{r}_{ijt}, \tilde{r}_{ijc}) \\
\]
(15)

The smaller the \(d (\tilde{r}_{ijt}, \tilde{r}_{ijc})\), the larger the consensus index of \(\tilde{R}\). If and only if \(d (\tilde{r}_{ijt}, \tilde{r}_{ijc}) = 0\), \(COI (\tilde{R}, \tilde{R}^{\ast}) = 1\), and the individuals reach full consensus.

Since \(d (\tilde{r}_{ijt}, \tilde{r}_{ijc}) = |\Phi_{ijc}^{1} (\beta) - \Phi_{ijc}^{1!} (1 - \beta)|\), we have

\[
d (\tilde{r}_{ijt}, \tilde{r}_{ijc}) = |\Phi_{ijc}^{1} (\beta) - \Phi_{ijc}^{1!} (1 - \beta)| = |1 - \Phi_{ijc}^{1} (1 - \beta) - [1 - \Phi_{ijc}^{1!} (\beta)]| \\
= |\Phi_{ijc}^{1} (1 - \beta) + \Phi_{ijc}^{1!} (\beta)| \\
= |\Phi_{ijc}^{1} (\beta) - \Phi_{ijc}^{1!} (1 - \beta)| \\
= d (\tilde{r}_{ijt}, \tilde{r}_{ijc}) \\
\]
with \(i < j\). The deviations of the upper triangular matrix and lower triangular matrix between \(\tilde{R}\) and \(\tilde{R}^{\ast}\) can be expressed in the same formula.

- IHWA Operator

Meng and Chen [16] propose an IHWA operator to calculate the elements of a collective FPR based on the importance of DMs (or criteria) and ordered positions. Extending the operator to interval UPRs, we propose the following definition of the collective UPR.

Definition 19. [16] An IHWA operator with dimension \(n\) is a mapping IHWA: Let \(\tilde{R}^{n} \rightarrow \tilde{R}\) be defined on the set of the second arguments of two tuples \((u_1, a_1), (u_2, a_2), \ldots, (u_n, a_n)\) with a set of order-inducing variables \(\{u_i\}_{i \in N}\), denoted by

\[
IHWA_{\lambda, \nu} ((u_1, a_1), (u_2, a_2), \ldots, (u_n, a_n)) \\
= \sum_{l=1}^{n} v_{l} A_{l}^{(i)}(u_1, a_1, \ldots, a_n) \\
\]
(16)
where \((\cdot)\) is a permutation on \(u_i\) such that \(u_i(\lambda)\) is the \(j\)th smallest value of \(u_i\), \(v = (v_1, v_2, \ldots, v_n)^T\) is the weight vector on the ordered set \(O = \{1, 2, \ldots, n\}\), and \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T\) is the weight vector on object set \(A = \{a_1, a_2, \ldots, a_n\}\).

Definition 20. Let \(\tilde{R} = (\tilde{r}_{ijc})_{i\in D}\) be the UPR of DM \(d_i\), \(v = (v_1, v_2, \ldots, v_n)^T\) and \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T\) be the respective weight vectors on the ordered set \(O\) and on the DM set \(D = \{d_1, d_2, \ldots, d_m\}\). Then, the collective UPR \(\tilde{R} = (\tilde{r}_{ijc})_{i\in D}\) is defined by

\[
IHWA (ACI (\tilde{R}^{\ast}), \Phi_{ijc}^{1}) = \left\langle ACI (\tilde{R}^{\ast}), \Phi_{ijc}^{1} \right\rangle \ldots \\
\]
(17)
\[
= \sum_{k=1}^{m} \sum_{l=1}^{n} v_{k} A_{k}^{(i)} \Phi_{ijc}^{1} = \Phi_{ijc}^{1} \\
\]
\( \Phi_{ij}^1 \) represents the IUD of \( \vec{R}_i \), and \( \Phi_{ij}^{k} \) represents the IUD of \( \vec{R}_i \). \( u_i \) is represented as the ACI. (.) is a permutation on \( ACI(\vec{R}_i) \) (\( k \in M \)) such that \( ACI(\vec{R}_i)^{(k)} \) is the \( k \)th smallest value of \( ACI(\vec{R}_i) \).

Theorem 4. Let \( \vec{R}_i \) be the UPR of DM \( d_i \), and \( \vec{R} = (\vec{r}_{ijc})_{n \times n} \) be the collective UPR; then, the consistency index of the collective UPR is no less than the minimum value of any individual consistency index. That is, \( ACI(\vec{R}) \geq \min_{1 \leq i \leq n} ACI(\vec{R}_i) \).

Proof. From Eq. (13), we have

\[
ACI(\vec{R})
= 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} d(\vec{r}_{ijc}, \vec{r}_{ijc}')
= 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} [\Phi_{ijc}((\beta) - \Phi_{ijc}^{-1}(1 - \beta)]
= 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \left[ \sum_{p=1}^{n-1} \sum_{p \neq i,j} \left[ \Phi_{ipk}(1 - \beta) + \Phi_{pjk}(1 - \beta) \right] + 0.5 \right]
= \sum_{k=1}^{m} \sum_{j=1}^{n} \sum_{i=1}^{n} \left[ 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \left[ \Phi_{ijc}(\beta) - \Phi_{ijc}^{-1}(1 - \beta)] \right] \right]
\geq \min_{1 \leq i, j \leq n} \left\{ 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \left[ \sum_{p=1}^{n-1} \sum_{p \neq i,j} \left[ \Phi_{ipk}(1 - \beta) + \Phi_{pjk}(1 - \beta) \right] + 0.5 \right] \right\}
= \min_{1 \leq i, j \leq n} \left\{ 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \left[ \sum_{p=1}^{n-1} \sum_{p \neq i,j} \left[ \Phi_{ipk}(1 - \beta) + \Phi_{pjk}(1 - \beta) \right] + 0.5 \right] \right\}
= \min_{1 \leq i, j \leq n} \left\{ \sum_{i,j=1}^{n} d(\vec{r}_{ijc}, \vec{r}_{ijc}') \right\}
= \min_{1 \leq i, j \leq n} ACI(\vec{R}_i)
\]

\[
\text{Definition 21.} \ [16] \text{Let } \vec{R} = (\vec{r}_{ijc})_{n \times n} \text{ be the UPR of DM } d_i; \lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T \text{ be the weight vector of DM set } D = \{d_1, d_2, ..., d_m\}; \bar{R} = (\vec{r}_{ijc})_{m \times n} \text{ be the collective UPR. Then, the weighted averaging consensus index is defined by}
\]

\[
\overline{\text{COI}}(\vec{R}) = \sum_{i=1}^{m} \lambda_i \text{COI}(\vec{R}, \vec{R}_i)
\]

\[
\text{Theorem 5. Let } \omega \text{ be the consensus threshold value, } \vec{R} \vec{(h,a+1)} = (\vec{r}_{ijc})_{n \times n} (g \in Z) \text{ be the } (g+1) \text{th improved individual UPR for } \text{COI}(\vec{R}, \vec{R}) \leq \omega, \Phi_i^{(h,a+1)} \text{ be the IUD of } \vec{R} \vec{(h,a+1)}, \text{ and } \Phi_i^{(h,a)} \text{ be the IUD of } \vec{R} \vec{(h,a)}. \text{ When } \Phi_i^{(h,a+1)}(\beta) = \partial \Phi_i^{(h,a)}(\beta) + (1 - \partial) \Phi_i^{(h,a)}(1 - \beta), \partial \in (0,1), \text{ we have}
\]

\[
\text{COI}(\vec{R}, \vec{R}) > \text{COI}(\vec{R} \vec{(h,a+1)}, \vec{R}(h,a))
\]
Step 1. Determine the preference relations and weight vectors.
Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an independent UPR of the DM $d_i$, $v = (v_1, v_2, ..., v_n)^T$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T$ be the weight vectors on the ordered set $O$ and on the DM set $D = \{d_1, d_2, ..., d_m\}$, respectively.

Step 2. Complete the incomplete UPRs.
Use UCCPM $(M - 3)$ to compute the incomplete values in individual preference relation $\tilde{R}$. Additionally, use $\tilde{R}$ to represent the completed individual preference relation.

Step 3. Calculate the individual consistency index.
Let $\tilde{R} = (\tilde{r}_{ij})_{10n \times n}$ be the $h$th improved individual preference relation, $h = 0, \tilde{R} = \tilde{R}$. The individual ACI threshold value is $\lambda$ ($\lambda \in [0, 1]$). If $ACI (\tilde{r}_{ij}) \geq \lambda$, then proceed to Step 5; otherwise, proceed to the next step.

Step 4. Adjust inconsistent individual UPRs.
Let $\tilde{R}^{(1)} = \tilde{R}$. If $ACI (\tilde{r}_{ij}) < \lambda$, let $\tilde{R}^{(k+1)} = \tilde{R}^{(k)}$. Adjust inconsistent individual UPRs.

Step 5. Calculate the individual consensus index.
Let $\omega$ be the weighted averaging consensus index threshold value, $g$ be the number of iterations, and $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be the $g$th improved individual UPR for $\tilde{R}^{(g-1)} = \tilde{R}$. When $g = 1$, we have $\tilde{R} = \tilde{R}$. Use the IHWA operator to calculate the collective UPR $\tilde{R}^{(g)}$. Use Eq. (15) to calculate the individual consensus index $COI (\tilde{r}_{ij}) (\tilde{r}_{ij})$

Step 6. Calculate the weighted averaging consensus index.
Use Eq. (18) to calculate the weighted averaging consensus index. If $COI (\tilde{r}_{ij}) \geq \omega$, proceed to Step 7; otherwise, proceed to Step 8.

Step 7. Adjust the individual UPRs.
Let $\tilde{R} = \tilde{R}^{(g)}$. If $ACI (\tilde{r}_{ij}) < \lambda$, let $\tilde{R}^{(g+1)} = \tilde{R}^{(g)}$ be the collective UPR improved for $ACI (\tilde{r}_{ij}) < \lambda$. When $s = 0$, $\tilde{R}^{(s)} = \tilde{R}$. If $ACI (\tilde{r}_{ij}) \geq \lambda$, then $\tilde{R}^{(s)}$ is additively consistent; otherwise, proceed to Step 9.

Step 9. Adjust inconsistent collective preference relation.
Let $\tilde{R}^{(s+1)}$ be the IUD of $\tilde{R}^{(s)}$. If $ACI (\tilde{r}_{ij}) < \lambda$, let $\tilde{R}^{(s+1)} = \tilde{R}^{(s)}$. Adjust inconsistent collective preference relation.

3.4. Illustrative Example
Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of alternatives, $D = \{d_1, d_2, d_3\}$ be the set of DMs, $\tilde{R} = (\tilde{r}_{ij})_{4 \times 4}$ be the independent UPR of $d_i$ ($i \in N = \{1, 2, 3, 4\}$) be the independent UPR of $d_i$ ($i \in M = \{1, 2, 3\}$). $\tilde{r}_{ij}$ is a LUV, i.e., $\tilde{r}_{ij} = L(\tilde{r}_{ij})$, and an UD $\tilde{r}_{ij}$. Let $\tilde{R} = (\tilde{r}_{ij})_{4 \times 4}$ be the collective UPR, $\tilde{r}_{ij} = L(\tilde{r}_{ij})$, and $\tilde{r}_{ij}$ be its UD function. For any $\beta$ ($\beta \in [0, 1]$), we have $\Phi^{i_1}(\beta) + \Phi^{i_2}(1 - \beta) = 1$, and $\Phi^{i_1}(\beta) = 0.5$. The incomplete UPRs $\tilde{R}^1, \tilde{R}^2, \tilde{R}^3$ provided separately by the three DMs are as follows:

$$
\begin{bmatrix}
L(0.5, 0.5) & L(0.6, 0.8) & L(x_{13}^{1}, 0.75) & L(0.4, 0.7) \\
L(0.2, 0.4) & L(0.5, 0.5) & L(x_{23}^{1}, x_{23}^{2}) & L(0.35, 0.55) \\
L(0.25, x_{31}^{1}) & L(x_{32}^{2}, x_{32}^{3}) & L(0.5, 0.5) & L(0.3, x_{34}^{1}) \\
L(0.3, 0.6) & L(0.45, 0.65) & L(x_{33}^{4}, 0.7) & L(0.5, 0.5)
\end{bmatrix}
$$

$$
\begin{bmatrix}
L(0.5, 0.5) & L(x_{13}^{2}, 0.75) & L(0.5, 0.75) & L(x_{14}^{1}, x_{14}^{2}) \\
L(0.3, x_{21}^{1}) & L(x_{32}^{1}, x_{32}^{3}) & L(0.5, 0.5) & L(0.3, x_{34}^{1}) \\
L(0.25, 0.5) & L(x_{33}^{4}, 0.7) & L(0.5, 0.5) & L(0.7, 0.8) \\
L(x_{41}^{1}, x_{41}^{2}) & L(0.2, 0.45) & L(0.2, 0.3) & L(0.5, 0.5)
\end{bmatrix}
$$
\[
\left( L(0.5,0.5) \ L(0.1,0.3) \ L(x_{13}',x_{13}') \ L(0.3',x_{14}') \ L(0.7,0.9) \ L(0.5,0.5) \ L(0.2,0.4) \ L(x_{24}',x_{24}') \ L(x_{11}',x_{11}') \ L(0.6,0.8) \ L(0.5,0.5) \ L(0.7,0.8) \ L(x_{41}',0.7) \ L(x_{42}',x_{42}') \ L(0.2,0.3) \ L(0.5,0.5) \right)
\]

Step 1. Let the belief degree \( \alpha = 0.8 \). Based on (M – 3), the complete values are estimated. In \( R^1 \), \( x_{13}' = 0.75, x_{23}' = 0.58, x_{23}' = 0.37 \). In \( R^2 \), \( x_{14}' = 0.7, x_{14}' = 0.875, x_{14}' = 0.785, x_{14}' = 0.34 \). In \( R^3 \), \( x_{13}' = 0.025, x_{13}' = 0.025, x_{14}' = 0.3, x_{14}' = 0.575, x_{24}' = 0.575 \).

Step 2. Let the individual ACI threshold value \( \lambda \) be 0.95. Since the actual deviation is an interval with the same end points, it is treated as a real number. Based on Eq. (13), the individual ACI is calculated. \( ACI\left(1\right) = 0.952, ACI\left(2\right) = 0.932, \) and \( ACI\left(3\right) = 0.953 \).

Let \( \delta = 0.8 \) and the adjusted ACI \( ACI\left(F(1)\right) = 0.955, R^1 \) is defined as follows:

\[
\left( L(0.5,0.5) \ L(0.7,0.7) \ L(0.52,0.71) \ L(0.88,0.88) \ L(0.3,0.3) \ L(0.5,0.5) \ L(0.34,0.34) \ L(0.57,0.76) \ L(0.29,0.48) \ L(0.65,0.66) \ L(0.5,0.5) \ L(0.24,0.43) \ L(0.23,0.25) \ L(0.5,0.5) \right)
\]

Step 3. Let the consensus threshold value \( \omega \) be 0.85. The weight vector on the set of DMs \( D = \{d_1, d_2, d_3\} \) is \( \lambda = (\lambda_1, \lambda_2, \lambda_3)^T = \left(\frac{1}{3} \ 1 \ \frac{1}{3}\right)^T \), and the weight vector on the ordered set \( O = \{1,2,3\} \) is \( v = (v_1 \ v_2 \ v_3)^T = \left(\frac{1}{9} \ \frac{3}{9} \ \frac{5}{9}\right)^T \). As \( ACI\left(R^1\right) \) < \( ACI\left(R^2\right) \), the weights of \( R^1, R^2 \), and \( R^3 \) are \( \frac{1}{9}, \frac{3}{9}, \frac{5}{9} \), respectively. The collective UPR \( R' = (\bar{\tau}_c)_{4 \times 4} \) is defined as follows:

\[
\left( L(0.5,0.5) \ L(0.49,0.58) \ L(0.38,0.49) \ L(0.63,0.67) \ L(0.42,0.51) \ L(0.5,0.5) \ L(0.32,0.39) \ L(0.55,0.68) \ L(0.51,0.62) \ L(0.61,0.68) \ L(0.5,0.5) \ L(0.68,0.74) \ L(0.33,0.37) \ L(0.33,0.45) \ L(0.26,0.32) \ L(0.5,0.5) \right)
\]

Step 4. From Eq. (15), \( COI\left(1, R'\right) = 0.767 \), \( COI\left(2, R'\right) = 0.904 \), \( COI\left(3, R'\right) = 0.778 \). From Eq. (18), \( COI\left(R\right) = \frac{1}{5} \times 0.767 + \frac{1}{5} \times 0.904 + \frac{1}{5} \times 0.778 = 0.816 < \omega \).

Step 5. As \( COI\left(R^1, R'\right) < \omega \), \( COI\left(R^2, R'\right) < \omega \), \( \lambda \), and \( R^3 \) must be adjusted. The ultimate individual UPRs are as follows:

\[
\left(\begin{array}{cccc}
L(0.5,0.5) & L(0.58,0.67) & L(0.61,0.61) & L(0.47,0.64) \\
L(0.33,0.42) & L(0.5,0.5) & L(0.65,0.65) & L(0.49,0.64) \\
L(0.39,0.39) & L(0.5,0.5) & L(0.5,0.5) & L(0.5,0.5) \\
L(0.36,0.53) & L(0.45,0.55) & L(0.53,0.56) & L(0.5,0.5)
\end{array}\right)
\]

Let the ACI threshold value of the collective UPR be \( \delta = \lambda = 0.95; \) then, \( ACI\left(R'\right) = 0.982 > 0.95 \), the priority problem is written separately, and this indicates that the individuals have reached the optimal consensus.

To verify the feasibility and efficiency of the proposed method, comparative analysis is conducted using existing methods to estimate the missing values and calculate ACI.

\[
\left(\begin{array}{cccc}
L(0.5,0.5) & L(0.46,0.56) & L(0.43,0.44) & L(0.47,0.57) \\
L(0.44,0.54) & L(0.5,0.5) & L(0.44,0.44) & L(0.51,0.58) \\
L(0.56,0.57) & L(0.6,0.6) & L(0.6,0.6) & L(0.56,0.6) \\
L(0.43,0.53) & L(0.42,0.49) & L(0.4,0.44) & L(0.5,0.5)
\end{array}\right)
\]

Meng et al. [54] propose the concepts of quasi intervals with its additive consistency presentation. It is independent of the permutation of object labels and considers the additive consistency of lower and upper endpoints of IPRs simultaneously. Further, Meng et al. [55] discover that the additive consistency of quasi intervals is included in Krečiš [56] which is more flexible.

Using model (26) in [54] whose solutions of missing values has the highest additive consistency level with respect to known values and model (M-3) in [55] separately, the results are shown in Table 1. Although the proposed method is dependent on the labels of objects, the estimated values of proposed method are all included in the results of [54,55] when the belief degree \( \alpha = 0.8 \). It lessens uncertainty and has higher consistency level. Meanwhile, [54,55] are based on the interval with end points transformation
which ignores internal values. The proposed method treats subjective preference as certain UD which handles the interval preference collectively. Furthermore, when the belief degree is higher, the stricter the requirement for deviation between estimated values and ideal values is, the greater the influence on ACI is. When end points of missing values are equal, the estimated values will not change whatever the belief degree changes.

4. CONCLUSION

Based on the LUD and its consistency condition, the algorithm to fill in the incomplete values and the optimisation of group consistency of completed UPRs are investigated in this study.

The main contributions of this study are as follows:

• An UCCPM is introduced to calculate the missing values in incomplete UPRs, which allows DMs to measure the confidence level of deviation between the supplementary values and the original incomplete information and guarantees the effectiveness of estimated values via a belief degree. It also proves that the operation of incomplete UPRs is an extension of that of traditional IPRs under a certain belief degree.

• A novel distance measure and the ACI of incomplete UPRs are proposed to calculate the consistency and consensus degree of preference relations based on LUD. They are also used to improve the consistency and consensus index of UPRs iteratively.

• The interval preference is treated collectively by obeying the LUD, which avoids the decision distortion and discretisation operation of intervals in the traditional interval operation.

Our future research will focus on two aspects. We discuss the situation of independent DMs in current work. If social relationships of individuals are considered, then GDM can be more scientific. Besides the interval preference information of DMs obeys a LUD, it may obey a NUD, zigzag uncertainty distribution, or lognormal uncertainty distribution, etc. Nonlinear distributions of other types of preference relations with their multiplicative consistency indices will be further investigated.

CONFLICT OF INTEREST

Authors have no conflict of interest to declare.

AUTHORS’ CONTRIBUTIONS

The study is guided by Zaiwu Gong and written by all authors.

### Table 1

| Methods                  | Consistency                  | Missing Values | ACI     |
|--------------------------|------------------------------|----------------|---------|
| Method of Meng et al.    | Quasi intervals additive consistency | 0.6 0.3 0.63 0.1 0.43 0.893 |         |
| Method of Meng et al.    | Krejčí’s additive consistency | 0.65 0.6 0.6 0.25 0.4 0.918 |         |
| The proposed method with | UPRs additive consistency    | 0.75 0.62 0.62 0.55 0.43 0.919 |         |
| The proposed method with | UPRs additive consistency    | 0.75 0.65 0.65 0.55 0.3 0.909 |         |
| The proposed method with | UPRs additive consistency    | 0.75 0.65 0.65 0.55 0.3 0.909 |         |

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