UNIFORM DISCRETENESS OF THE HOLOMONY VECTORS OF TRANSLATION SURFACES

CHENXI WU

Abstract. We answered a question by Barak Weiss on the uniform discreteness of the holonomy vectors of translation surfaces.

For a translation surface $M$, let $S_M \subset \mathbb{R}^2$ be the set of holonomy vectors of all saddle connections of $M$. The growth rate of $S_M$ has been studied by many, including [Vee89], [Mas90], [EM01], and the distribution of angles has been studied by [ACL12], [ACL13] and others. The following is a plot of $S_M$ where $M$ is a lattice surface in $H(2)$ with discriminant 13.

A way of describing uniformity of a set is the concept of Deloné sets, which is different from the above properties and defined as follows:

Definition 1. [Sen06] A subset $A$ of a metric space $(X, d)$ is a Deloné set if it satisfies:

1. $A$ is relatively dense, i.e. there is $R > 0$ such that any ball of radius $R$ in $X$ contains at least one point in $A$.

2. $A$ is uniformly discrete, i.e. there is $r > 0$ such that for any two distinct points $x, y \in A$, $d(x, y) > r$. 

arXiv:1408.4828v2 [math.DS] 22 Aug 2014
Barak Weiss asks for which translation surface $M$, $S_M$ is a Delone set. Here, we will show that:

**Theorem 1.** If $M$ is a lattice surface then $S_M$ is never a Delone set. There exists translation non-lattice surface $M$ for which $S_M$ is a Delone set.

We will show that if $M$ is a non-arithmetic lattice surface, $S_M$ can not be uniformly discrete. Combining it with the result that $S_M$ for a square-tiled surface $M$ can not be relatively dense, we can conclude that when $M$ is a lattice surface, $S_M$ can not be a Delone set. Furthermore, we will show that there are surfaces whose $S_M$ are Delone.

**Theorem 2.** If $M$ is a non-arithmetic lattice surface, $S_M$ is not uniformly discrete.

*Proof.* Let $r > 0$ be given, let $M$ is a non arithmetic lattice surface, by Vee89 we can choose a periodic direction $\gamma$ of $M$ such that in this direction $M$ is decomposed into cylinders $C_1, \ldots, C_n$, and the width of all these cylinders are no larger than $r/4$. Because $M$ is not square-tiled, there exist two numbers $i$ and $j$ such that the quotient of the circumferences of $C_i$ and $C_j$ is not in $\mathbb{Q}$. Denote the holonomy vectors of periodic geodesics correspond to $C_i$ and $C_j$ by $l$ and $l'$. Let $h$ and $h'$ be the holonomy of two saddle connections $\alpha_0$ and $\beta_0$ crossing $C_i$ and $C_j$ respectively. Because the image of $\alpha_0$ and $\beta_0$ after $n$ Dehn twists in cylinder $C_i$ and $C_j$ are still saddle connections of $M$, for any integer $n$, $h + nl \in S_M$ and $h' + nl' \in S_M$. 


Let \( h = h_1 + h_2 \) and \( h' = h'_1 + h'_2 \), where \( h_1, h'_1 \) are in direction \( \gamma \), and \( h_2, h'_2 \) are in direction \( \gamma^\perp \). Because the width of \( C_i \) and \( C_j \) are no larger than \( r/4 \) by assumption, \( ||h_2 - h'_2|| < r/2 \). Because \( h_1, h'_1, l \) and \( l' \) are all in the same direction, we can write \( h_1 = al \), \( h'_1 = bl \), \( l' = \lambda l \). Because the quotient \( \lambda \) of \( l' \) and \( l \) is irrational, \( \{m + m'\lambda : m, m' \in \mathbb{R}\} \) is dense in \( \mathbb{R} \), so there exists a pair of integers \( n_0 \) and \( n'_0 \) such that \( |a-b+n_0-n'_0\lambda| < \frac{r}{2||l||} \), therefore \( ||(h_1+n_0l)-(h'_1+n'_0l'|| < r/2 \), therefore \( ||(h+n_0l)-(h'+n'_0l'|| < r/2+r/2 = r \), so \( S_M \) contains two points whose distance is less than any \( r > 0 \), thus \( S_M \) is not uniformly discrete. \( \square \)

The above argument also works on those completely periodic surfaces such that in any given periodic direction, there are at least two closed geodesics whose holonomy are not related by a rational multiple. Barak Weiss also showed that the above argument implies that any orbit in \( \mathbb{R}^2 \) under the action of two parabolic elements in \( SL(2,\mathbb{R}) \) is either contained in a lattice in \( \mathbb{R}^2 \) or not uniformly discrete.

For completeness, we also give a treatment of the square-tiled case below.

**Lemma 3.** For any positive integer \( N \), \( \{(p,q) \in \mathbb{Z}^2 : \gcd(p,q) \leq N\} \) is not relatively dense in \( \mathbb{R}^2 \).
Proof. (HS71) Given $R > 0$, choose integer $n > 2R$, and $n^2$ prime numbers $p_{i,j}$, $1 < i, j < n$ larger than $N$. Let $q_i = \prod_j p_{i,j}$, $q_j' = \prod_i p_{i,j}$. By Chinese remainder theorem there is an integer $x$ such that $x \equiv -i \mod q_i$, and an integer $y$ such that $y \equiv -j \mod q_j'$; hence for any two positive integers $i, j \leq n$, $(x + i, y + j) \not\in \{(p, q) \in \mathbb{Z}^2 : \gcd(p, q) > N\}$, i.e. there is a ball in $\mathbb{R}^2 - \{(p, q) \in \mathbb{Z}^2 : \gcd(p, q) \leq N\}$ of radius $R$.

Now we use the above lemma to show that if the surface $M$ is square-tiled, $S_M$ cannot be relatively dense.

**Theorem 4.** If $M$ is a square-tiled lattice surface then $S_M$ is not relatively dense in $\mathbb{R}^2$.

**Proof.** If $M$ is square-tiled, it can assume that it is tiled by $N \times 1 \times 1$ squares, i.e. an $n$-fold branched cover of $T = \mathbb{R}^2 / \mathbb{Z}^2$ branched at $(0, 0)$, hence the holonomy of any saddle connection is in $\mathbb{Z} \times \mathbb{Z}$. For any pair of coprime integers $(p, q)$, let $\gamma$ be the closed geodesic in $T$ starting at $(0, 0)$ in $(p, q)$-direction. The length of $l$ is $\sqrt{p^2 + q^2}$. The preimage of $\gamma$ in $M$ is a graph $\Gamma$ of volume $N\sqrt{p^2 + q^2}$. Any saddle connection of $M$ in $(p, q)$-direction is a path on $\Gamma$ without self intersection, hence the length of such a saddle connection can not be greater than $N\sqrt{p^2 + q^2}$. Hence, the holonomy $(sp, sq)$, $s \in \mathbb{Z}$ of such a saddle connection must satisfy $|s| \leq N$. In other words, $S_M \subset \{(p, q) \in \mathbb{Z}^2 : \gcd(p, q) \leq N\}$, so $S_M$ is not relatively dense in $\mathbb{R}^2$.

Finally, when $M$ is not completely periodic, $S_M$ can be Deloné set. This finishes the proof of Theorem 1.

**Example 1.** Let $M_1$ be the branched double cover of $\mathbb{R}^2 / \mathbb{Z}^2$ branched at points $(0, 0), (\sqrt{2} - 1, \sqrt{3} - 1)$. Consider a $\mathbb{Z}^2$-cover $\tilde{M}$ which is a branched double cover of $\mathbb{R}^2$ branched at $U = \mathbb{Z}^2$ and at $V = \mathbb{Z}^2 + (\sqrt{2} - 1, \sqrt{3} - 1)$, where the deck group action is by translation. Then saddle connections on $M_1$ are lifted to $\mathbb{Z}^2$-orbits of saddle connections on $\tilde{M}$, hence $S_{M_1}$ is the same as $S_M$, which is the set of holonomies of line segments linking two points in $W = U \cup V$ which do not pass through any other point in $W$. If a line segment links two points in $U$, its slope must be rational or $\infty$, hence it would not pass through any point in $V$. Furthermore, it does not pass through any other point in $U$ if and only if its holonomy is a pair of coprime integers. The same is true for line segments linking two points in $V$. On the other hand, given any point $p \in U$ and any point $q \in V$, a line segment from $p$ to $q$ has irrational slope hence can not pass through any other point in $U$ or $V$, hence the holonomy of such line segment can be any vector in $\mathbb{Z}^2 + (\sqrt{2} - 1, \sqrt{3} - 1)$. Similarly the holonomies of saddle connections from $V$ to $U$ are $\mathbb{Z}^2 - (\sqrt{2} - 1, \sqrt{3} - 1)$. Hence $S_{M_1} = \{(a, b) \in \mathbb{Z}^2 : \gcd(a, b) = 1\} \cup \mathbb{Z}^2 + (\sqrt{2} - 1, \sqrt{3} - 1) \cup \mathbb{Z}^2 - (\sqrt{2} - 1, \sqrt{3} - 1)$, which is a Deloné set.

**Questions:**

(1) Can the set of holonomy vectors of all saddle connections of a non-arithmetic lattice surface be relatively dense?
(2) Is there any characterization of a flat surface $M$ such that $S_M$ is Deloné, relatively dense or uniformly discrete in general?

(3) Is there a surface $M$ which is not a branched cover of the torus so that $S_M$ is Deloné?

References

[AC12] Jayadev S Athreya and Jon Chaika. The distribution of gaps for saddle connection directions. *Geometric and Functional Analysis*, 22(6):1491–1516, 2012.

[ACL13] Jayadev S Athreya, Jon Chaika, and Samuel Lelievre. The gap distribution of slopes on the golden l. arXiv preprint arXiv:1308.4203, 2013.

[EM01] Alex Eskin and Howard Masur. Asymptotic formulas on flat surfaces. *Ergodic Theory and Dynamical Systems*, 21(02):443–478, 2001.

[HS71] Fritz Herzog and BM Stewart. Patterns of visible and nonvisible lattice points. *American Mathematical Monthly*, pages 487–496, 1971.

[Mas90] Howard Masur. The growth rate of trajectories of a quadratic differential. *Ergodic Theory and Dynamical Systems*, 10(01):151–176, 1990.

[Sen06] Marjorie Senechal. What is a quasicrystal. *Notice of the AMS*, 53:886–887, 2006.

[Vee89] William A Veech. Teichmüller curves in moduli space, eisenstein series and an application to triangular billiards. *Inventiones mathematicae*, 97(3):553–583, 1989.