F-theory Yukawa Couplings and Supersymmetric Quantum Mechanics

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Abstract
The localized fermions on the intersection curve Σ of D7-branes, are connected to a $N = 2$ supersymmetric quantum mechanics algebra. Due to this algebra the fields obey a global $U(1)$ symmetry. This symmetry restricts the proton decay operators and the neutrino mass terms. Particularly, we find that several proton decay operators are forbidden and the Majorana mass term is the only one allowed in the theory. A special SUSY QM algebra is studied at the end of the paper. In addition we study the impact of a non-trivial holomorphic metric perturbation on the localized solutions along each matter curve. Moreover, we study the connection of the localized solutions to an $N = 2$ supersymmetric quantum mechanics algebra when background fluxes are turned on.

Introduction

F-theory [1–84], has received a prominent role lately, due to the fact that GUTs can be consistently constructed and well founded, within F-theory’s wide theoretical framework. It is an 12-dimensional theory that consists of toroidal elliptic fibrations over Calabi-Yau manifolds. D7 branes are essential to the theory, since the D7 branes are located on the $T^2$ fiber. The modulus of the torus is a varying parameter and is related to the axio-dilaton. Thereupon, we can say that F-theory is an UV completion of type IIB superstring theory with 7-branes. For comprehensive reviews on the formulation of F-theory GUT’s see [6–10,20,23].

One of it’s most interesting outcomes, is that within F-theory we can produce many phenomenological features of GUTs with gravity being excluded from the theoretical apparatus (for recent work on realistic F-theory GUTs models see [4-9,11-67,70-73]). Moreover,

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certain phenomenological features that was not possible to be realized within the perturbative framework of superstring theories, for example the couplings $5_H \times 10_M \times 10_M$ for $SU(5)$, or the spinor $16$ representation of $SO(10)$, now can be consistently incorporated to the phenomenological outcomes of the theory.

Complex manifolds singularities play a critical role in F-theory phenomenology [1, 4-12], with gauge groups realized by the geometry of singularities. Additionally, $N = 1$, $D = 4$ supersymmetric gauge theories arise when F-theory is compactified on Calabi-Yau fourfolds [3, 19, 12].

One of the most important features that a UV completion of the Standard Model must somehow explain is the hierarchical structure of fermion masses and mixings. In F-theory GUTs much work has been done towards this direction [4, 12, 16, 24, 27, 38, 49, 79, 80] and also in order to explain the neutrino sector, see [81] and also [49, 51, 80, 82, 83]. Yukawa couplings in F-theory are obtained by calculating overlapping integrals of three matter curves wave functions over a complex surface $S$. Hence Yukawas depend drastically on the local structure of the theory, near the intersection point of the three matter curves (nonetheless the global structure of the theory affects the normalization of the wave functions).

In this work we shall consider the localized fields that are generated on the intersection curve $\Sigma$ of D7-branes without external gauge fluxes, and also the Yukawa couplings generated by the intersection of three matter curves. The fields that have localized solutions along the matter curve $\Sigma$, are connected [89] to an $N = 2$ supersymmetric quantum mechanics algebra [85, 86] and the number of the zero modes is connected to the Witten index of the susy algebra. We find that every localized field on the matter curve obeys a hidden global $U(1)$ symmetry. We shall require that this symmetry holds even at the intersection point of three matter curves. The conditions that must hold in order this to happen, pose some restrictions on various proton decay operators and on the operators that give masses to the neutrinos. Furthermore, we study the impact of a certain type of susy quantum algebra on the Yukawa coupling that gives mass to the top quark. The results are interesting, since, the imposed conditions result to a form of wave functions with de-localized Higgs. Moreover, we shall include the effects of a gravitational backreaction on the complex surface $S$, in terms of linear perturbations of the Euclidean metric. We conclude that the spectral problems of the perturbed and unperturbed system are identical, due to the topological invariance of the index of the corresponding operators. We shall examine all the matter curves. Finally, we shall check whether we can relate a SUSY QM algebra to the fermions localized along the three matter curves, in the case we introduce constant background gauge fluxes.

This paper is organized as follows: In section 1 we describe in brief the F-theory setup we shall use, that is, D7-branes intersections, matter curves and the eight dimensional Super-Yang-Mills (SYM) theory. In section 2 we give in short, a self-contained review of the supersymmetric quantum mechanics algebra. In section 3 we connect the localized solutions of the BPS equations of motion, to an $N = 2$ supersymmetric quantum mechanics algebra and also we study the impact of a non-trivial linear perturbation of the metric on the localized solutions. In section 4 we examine the localized solutions of the fermionic system under the influence of background gauge fluxes. In section 5 we study the impact of a certain type of susy quantum algebra on the top quark Yukawa coupling. In section 6 we
study the $U(1)$ symmetries and the restrictions these imply to the proton decay operators and to the neutrino mass operators. Finally in section 7 we present the conclusions.

1 Localized Fermions on D7-Branes Intersections

We shall consider F-theory compactifications on a Calabi-Yau fourfold. This manifold is an elliptic $K3$ fibration over a complex dimension two surface $S$. Locally the theory can be described by the worldvolume of an ADE type D7 brane wrapping $R^{1,3} \times S$ over the Calabi-Yau fourfold. The resulting $d = 4$ theory is an $N = 1$ supersymmetric theory \[4, 12\]. Our analysis is based mostly on references \[4, 12\].

The physics of the D7-branes wrapping $S$ can be described in terms of an $D = 8$ twisted Super Yang-Mills on $R^{3,1} \times S$. The supersymmetric multiplets contain the gauge field plus a complex scalar $\varphi$ and the set of adjoint fermions $\eta, \psi, \chi$. We parameterize the complex surface $S$ using the local coordinates $(z_1, z_2)$. Then the supermultiplets are:

\[
A = A_\mu dx^\mu + A_m dz^m + A_\bar{m} d\bar{z}^m, \quad \varphi = \varphi_{12} dz^1 \wedge dz^2
\]

and additionally,

\[
\psi_a = \psi_{a1} dz^1 + \psi_{a2} dz^2, \quad \chi_a = \chi_{a12} dz^1 \wedge dz^2
\]

with $a = 1, 2$ and $m = 1, 2$.

The gauge multiplet $(A_\mu, \eta)$ together with the chiral multiplets $(A_\bar{m}, \psi_\bar{m})$ and $(\varphi_{12}, \chi_{12})$, plus their complex conjugates constitute the $N = 1, D = 4$ supersymmetric theory.

Omitting the kinetic terms (we shall use the kinetic terms later on in this article), the bilinear in fermions part of the action is,

\[
I_F = \int_{R^{1,3} \times S} dx^4 \text{Tr} \left( \chi \wedge \partial A \psi + 2 i \sqrt{2} \omega \wedge \partial A \eta \wedge \psi + \frac{1}{2} \psi \wedge [\varphi, \psi] + \sqrt{2} \eta [\bar{\varphi}, \chi] + \text{h.c.} \right)
\]

with $\omega$ is the fundamental Kähler form of the complex surface $S$. The variation of $\eta, \psi$ and $\chi$, yields the equations of motion \[4,12\]:

\[
\omega \wedge \partial A \psi + \frac{i}{2} [\bar{\varphi}, \chi] = 0
\]

\[
\bar{\partial} A \chi - 2 i \sqrt{2} \omega \wedge \partial \eta - [\varphi, \psi] = 0
\]

\[
\bar{\partial} A \psi - \sqrt{2} [\bar{\varphi}, \eta] = 0
\]

Before we proceed in details on how to find zero modes, we review in brief some issues, regarding the supersymmetric quantum mechanics algebra which we shall frequently use in the subsequent sections.

2 $N = 2$ Supersymmetric Quantum Mechanics Algebra

Consider a quantum system, described by a Hamiltonian $H$ and characterized by the set \{\(H, Q_1, ..., Q_N\)}, with $Q_i$ self-adjoint operators. The quantum system is called supersymmetric, if,

\[
\{Q_i, Q_j\} = H \delta_{i,j}
\]
with $i = 1, 2, \ldots N$. The $Q_i$ are the supercharges and the Hamiltonian “$H$” is called SUSY Hamiltonian. The algebra (5) describes the $N$-extended supersymmetry with zero central charge. Owing to the anti-commutativity, the Hamiltonian can be written as,

$$H = 2Q_1^2 = Q_2^2 = \ldots = 2Q_N^2 = \frac{2}{N} \sum_{i=1}^{N} Q_i^2. \quad (6)$$

A supersymmetric quantum system $\{H, Q_1, \ldots, Q_N\}$ is said to have unbroken supersymmetry, if its ground state vanishes, that is $E_0 = 0$. In the case $E_0 > 0$, that is, for a positive ground state energy, susy is said to be broken.

In order supersymmetry is unbroken, the Hilbert space eigenstates must be annihilated by the supercharges,

$$Q_i |\psi_j^0\rangle = 0 \quad (7)$$

for all $i, j$.

The $N = 2$ algebra ("$N = 2$ SUSY QM", or "SUSY QM" thereafter) consists of two supercharges $Q_1$ and $Q_2$ and a Hamiltonian $H$, which obey the following,

$$\{Q_1, Q_2\} = 0, \quad H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2 \quad (8)$$

We use the complex supercharge $Q$ and it’s adjoint $Q^\dagger$ defined as,

$$Q = \frac{1}{\sqrt{2}} (Q_1 + iQ_2), \quad Q^\dagger = \frac{1}{\sqrt{2}} (Q_1 - iQ_2) \quad (9)$$

which satisfy the following equations,

$$Q^2 = Q^\dagger 2 = 0 \quad (10)$$

and also are related to the Hamiltonian as,

$$\{Q, Q^\dagger\} = H \quad (11)$$

A very important operator that is inherent to the definition of a SUSY QM system is the Witten parity, $W$, which, for a $N = 2$ algebra, is defined as,

$$\{W, Q\} = \{W, Q^\dagger\} = 0, \quad [W, H] = 0 \quad (12)$$

and satisfies,

$$W^2 = I \quad (13)$$

The main and important use of the operator $W$ is that, by using it, we can span the Hilbert space $\mathcal{H}$ of the quantum system to positive and negative Witten parity spaces, defined as, $\mathcal{H}^\pm = P^\pm \mathcal{H} = \{|\psi\rangle : W|\psi\rangle = \pm|\psi\rangle\}$. Therefore, the quantum system Hilbert space $\mathcal{H}$ is decomposed into the eigenspaces of $W$, hence $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Since each operator acting on the vectors of $\mathcal{H}$ can be represented by $2N \times 2N$ matrices, we use the representation:

$$W = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (14)$$
with $I$ the $N \times N$ identity matrix. Recalling that $Q^2 = 0$ and $\{Q, W\} = 0$, the supercharges take the form,

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$

which imply,

$$Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad Q_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix}$$

The $N \times N$ matrices $A$ and $A^{\dagger}$, are generalized annihilation and creation operators with $A$ acting as $A : \mathcal{H}^- \to \mathcal{H}^+$ and $A^{\dagger}$ as, $A^{\dagger} : \mathcal{H}^+ \to \mathcal{H}^-$. In the representation (14), (15), (16) the quantum mechanical Hamiltonian $H$, can be cast in a diagonal form,

$$H = \begin{pmatrix} AA^{\dagger} & 0 \\ 0 & A^{\dagger}A \end{pmatrix}$$

We denote $n_\pm$ the number of zero modes of $H_\pm$. The Witten index for Fredholm operators is defined as,

$$\Delta = n_- - n_+$$

When the Witten index is non-zero integer, supersymmetry is unbroken and in the case the Witten index is zero, if $n_+ = n_- = 0$ supersymmetry is broken, while if $n_+ = n_- \neq 0$ supersymmetry is unbroken.

The Fredholm index of the operator $A$ and the Witten index are related as,

$$\Delta = \text{ind} A = \dim \ker A - \dim \ker A^{\dagger} = \dim \ker A^{\dagger}A - \dim \ker AA^{\dagger} = \dim \ker H_- - \dim \ker H_+$$

We shall consider only Fredholm operators.

3 Intersecting Matter Curves, Localized Fermions and Supersymmetric Quantum Mechanics

The localized fields on each matter curve on $S$ are related to a SUSY QM algebra, as shown in [89]. In order to make this article self contained we review the basic facts (for details see [89]). The localized fermion fields exist on a matter curve $\Sigma$ which is the intersection of the complex surfaces $S$ and $S'$. In order to preserve $N = 1$ supersymmetry in $D = 4$, the theory defined on $R^{1,3} \times \Sigma$ must be $D = 6$ twisted super Yang-Mills [4,12].

Solving the $D = 8$ equations of motion for the twisted fermions we find how localized fermion matter on $\Sigma$ results from zero modes of the $D = 8$ bulk theory. Consider three matter curves denoted as $\Sigma_i$, with $i = 1, 2, 3$. Each matter curve has a group $G_i$, that on the intersection point further enhances to a higher group $G_p$. A non-trivial background for the adjoint scalar is required in order to extract the localized fermionic solutions of the eight dimensional theory on $S$ [4,12], which is equal to [4,12]:

$$\langle \varphi \rangle = m^2 z_1 Q_1 + m^2 z_2 Q_2$$

(20)
In the above, $Q_1$ and $Q_2$ are the $U(1)$ generators that are included in the enhancement group $G_p$ at the intersection point, and “$m_1$” and “$m_2$” are mass scales related to the F-theory scale $M_*$. Taking $m_1 = m_2 = m$ will simplify things but will not change the results.

The three matter curves can intersect at a point which is $(z_1, z_2) = (0, 0)$. The adjoint vacuum expectation value $\langle 20 \rangle$ resolves the $G_p$ singularity at the intersection point. The three different curves $\Sigma_1, \Sigma_2, \Sigma_3$ are defined by the loci $z_1 = 0$, $z_2 = 0$ and $z_1 + z_2 = 0$ respectively. Note that each curve represents a fermion under the $U(1)$ charges, the curves can be classified according to the table,

| matter curve $\Sigma_i$ | $(q_1, q_2)$ | surface locus $z_i = 0$ |
|-------------------------|--------------|--------------------------|
| $\Sigma_1$              | $(q_1, 0)$   | $z_1 = 0$                |
| $\Sigma_2$              | $(0, q_2)$   | $z_2 = 0$                |
| $\Sigma_3$              | $(-q_1, -q_2)$ | $z_1 + z_2 = 0$          |

Table 1: Charge Classification of the three matter curves

We assume that the Kähler form of $S$ is the canonical form,

$$\omega = i \frac{1}{2} (dz^1 \wedge \bar{dz}^1 + dz^2 \wedge \bar{dz}^2)$$

(21)

The coordinates $z_1$ and $z_2$ that parameterize $S$, describe the intersection $\Sigma$ in transverse and tangent directions respectively. With $\omega$ as in (21) and neglecting the $z_2$ derivatives, the equations of motion can be written as [4, 12]:

$$\sqrt{2} \partial_1 \eta - m^2 z_1 q_1 \psi_2 = 0$$
$$\partial_1 \psi_1 - m^2 \bar{z}_1 q_1 \chi = 0$$

(22)

where $(q_1, q_2)$ are the $U(1)$ charges of the fermions belonging to an irreducible representation $(R, q_1, q_2)$ of $G_S \times U(1)_1 \times U(1)_2$ (note that $Q_1$ is the $U(1)_1$ generator and $Q_2$ is the $U(1)_2$ generator). Taking the adjoint vacuum expectation value $\langle 20 \rangle$ the equations of motion can be cast as:

$$\partial_2 \psi_2 + \partial_1 \psi_1 - m^2 (\bar{z}_1 q_1 + z_2 q_2) \chi = 0$$
$$\bar{\partial}_1 \chi - m^2 (z_1 q_1 + \bar{z}_2 q_2) \bar{\psi}_1 = 0$$
$$\bar{\partial}_2 \chi - m^2 (z_1 q_1 + \bar{z}_2 q_2) \bar{\psi}_2 = 0$$

(23)

### 3.1 Localized fermion around $z_1 = 0$

The curve $\Sigma_1$, corresponds to $q_2 = 0$. The fermions localized at $z_1 = 0$ are obtained by (23) and are equal to [12]:

$$\psi_2 = 0, \quad \chi_1 = f(z_2) e^{-q_1 m^2 |z_1|^2}, \quad \bar{\psi}_1 = -\chi.$$
with \( f(z_2) \) a \( z_2 \)-dependent holomorphic function. We can connect a \( N = 2 \) SUSY QM algebra to this matter curve. Indeed, we can define the matrix \( D_1 \) and also \( D_1^\dagger \) as follows,

\[
D_1 = \begin{pmatrix}
\partial_1 & -m^2 z_1 q_1 \\
-m^2 \bar{z}_1 q_1 & \bar{\partial}_1
\end{pmatrix}
\]

and,

\[
D_1^\dagger = \begin{pmatrix}
\bar{\partial}_1 & -m^2 \bar{z}_1 q_1 \\
-m^2 z_1 q_1 & \partial_1
\end{pmatrix}
\]

acting on,

\[
\begin{pmatrix}
\psi_1 \\
\chi_1
\end{pmatrix}
\]

(27)

The solutions of the equations of motion (23) with \( \psi_\bar{1} \) and \( \chi_1 \) the zero modes of \( D_1 \). The Fredholm index \( I_{D_1} \), of the operator \( D_1 \), is equal to,

\[
\text{ind} I_{D_1} = \dim \ker (D_1^\dagger) - \dim \ker (D_1)
\]

(28)

which is equal to the number of zero modes of \( D_1 \) minus the number of zero modes of \( D_1^\dagger \).

Using \( D_1 \) we can define the \( N = 2 \) supersymmetric quantum mechanical system by defining the supercharges \( Q \) and \( Q^\dagger \),

\[
Q = \begin{pmatrix}
0 & D_1 \\
0 & 0
\end{pmatrix}
\quad Q^\dagger = \begin{pmatrix}
0 & D_1^\dagger \\
0 & 0
\end{pmatrix}
\]

(29)

Also the Hamiltonian of the system can be written,

\[
H = \begin{pmatrix}
D_1 D_1^\dagger & 0 \\
0 & D_1^\dagger D_1
\end{pmatrix}
\]

(30)

The above matrices obey, \( \{Q, Q^\dagger\} = H \), \( Q^2 = 0 \), \( Q^\dagger^2 = 0 \). Like so, the Witten index of the \( N = 2 \) supersymmetric quantum mechanics system, is related to the index \( I_{D_1} \) of the operator \( D_1 \). Indeed we have \( I_{D_1} = -\Delta \), because,

\[
I_{D_1} = \dim \ker D_1^\dagger - \dim \ker D_1 = \dim \ker D_1 D_1^\dagger - \dim \ker D_1^\dagger D_1 = -\text{ind} D_1 = -\Delta = n_- - n_+
\]

(31)

with \( n_- \) and \( n_+ \) defined in the previous section. Accordingly, the zero modes of the operators \( D_1 \) and \( D_1^\dagger \) are related to the zero modes of the operators \( D_1 D_1^\dagger \) and \( D_1^\dagger D_1 \). Additionally, the zero modes of the operators \( D_1 D_1^\dagger \) and \( D_1^\dagger D_1 \) can be classified to parity positive and parity negative solutions according to their Witten parity.

Note that the SUSY QM structure exists if \( \psi_2 = 0 \) on this matter curve. Moreover, SUSY is unbroken, since \( I_{D_1} \neq 0 \) (the operator \( D_1^\dagger \) has no localized zero modes).
3.2 Localized fermion around $z_2 = 0$

Along the curve $\Sigma_2$, we have $q_1 = 0$ and the fermions are peaked around $z_2 = 0$. The localized solutions to the equations of motion (23) read:

$$\psi_2 = -\chi, \quad \chi_2 = g(z_2) e^{-q_2 m^2 |z_1|^2}, \quad \psi_1 = 0.$$  \hspace{1cm} (32)

with $g(z_1)$ an arbitrary holomorphic function of $z_1$. The $N = 2$ SUSY QM algebra can be defined in terms of the $D_2$ matrix, which is equal to:

$$D_2 = \begin{pmatrix} \frac{\partial_2}{2} & -m^2 \bar{z}_2 q_2 \\ -m^2 z_2 q_2 & \frac{\partial_2}{2} \end{pmatrix}$$ \hspace{1cm} (33)

acting on

$$\begin{pmatrix} \psi_2 \\ \chi_2 \end{pmatrix}$$ \hspace{1cm} (34)

3.3 Localized fermion around $z_1 + z_2 = 0$

The matter curve $\Sigma_3$, corresponds to generic charges $q_1$ and $q_2$. Performing the transformations:

$$w = z_1 + z_2, \quad \psi_w = \frac{1}{2} (\psi_1 + \psi_2)$$
$$u = z_1 - z_2, \quad \psi_u = \frac{1}{2} (\psi_1 - \psi_2)$$ \hspace{1cm} (35)

the equations of motion (23) can be written:

$$2 \partial_w \psi_w + 2 \partial_u \psi_u - \frac{m^2}{2} \left( \bar{w} (q_1 + q_2) + \bar{u} (q_1 - q_2) \right) \chi = 0$$ \hspace{1cm} (36)
$$2 \bar{\partial}_w \chi - m^2 \left( w (q_1 + q_2) + u (q_1 - q_2) \right) \psi_w = 0$$
$$2 \bar{\partial}_u \chi - m^2 \left( w (q_1 + q_2) + u (q_1 - q_2) \right) \psi_u = 0$$

When $\psi_u = 0$, an $N = 2$ SUSY QM algebra underlies the fermion system, defined in terms of the matrices $D_3$ and $D_3^\dagger$ as:

$$D_3 = \begin{pmatrix} \partial_w & -\frac{m^2}{2} \left( \bar{w} (q_1 + q_2) + \bar{u} (q_1 - q_2) \right) \\ -m^2 \left( w (q_1 + q_2) + u (q_1 - q_2) \right) & 2 \bar{\partial}_w \end{pmatrix}$$ \hspace{1cm} (37)

and,

$$D_3^\dagger = \begin{pmatrix} 2 \bar{\partial}_w & -m^2 \left( \bar{w} (q_1 + q_2) + \bar{u} (q_1 - q_2) \right) \\ -\frac{m^2}{2} \left( w (q_1 + q_2) + u (q_1 - q_2) \right) & 2 \partial_w \end{pmatrix}$$ \hspace{1cm} (38)
acting on,
\[
\begin{pmatrix}
\psi_{\bar{w}} \\
\chi_w
\end{pmatrix}
\] (39)

Then, the fermionic localized solutions to the new equations of motion around \( z_1 + z_2 = 0 \) are:
\[
\psi_{\bar{w}} = \frac{1}{\sqrt{2}} \chi, \quad \chi_w = g(u) e^{-\frac{m^2}{\sqrt{2}} |w|^2}, \quad \psi_{\bar{w}} = 0.
\] (40)

We therefore conclude that each matter curve corresponds to an underlying \( N = 2 \) SUSY QM algebra. In turn, each SUSY algebra can be constructed using the operators \( D_1, D_2 \) and \( D_3 \) respectively, the zero modes of which correspond to the solutions of (23).

### 3.4 Gravitational Backreaction on the Base Manifold-Metric Perturbations

In the previous section we chose the canonical form for the metric that describes \( S \). However, the surface \( S \) is more like a base space of the Calabi-Yau threefold and not a divisor \([24]\). Therefore there is no way to know what metric describes precisely the base space \( S \), hence there is some freedom in the choice of the metric on \( S \). The metric adopted in the previous section is the simplest case and describes perfectly the case for which the system is fully described by a Super Yang-Mills theory, and gravity is decoupled, as we previously noted. However we are free to choose another metric that incorporates the gravitational backreaction of the surface \( S \) on the system. Note that the volume of \( S \) gives the gauge coupling of the effective four-dimensional GUT \([92]\). In this section we shall put the previous section’s index problem, into a different context, by perturbing the metric of the complex surface \( S \) in the following way:
\[
ds^2 = (1 + \epsilon f_1(z_1)) dz_1 \otimes d\bar{z}_1 + (1 + \epsilon f_2(z_2)) dz_2 \otimes d\bar{z}_2
\] (41)

Using the above metric, the Kähler form is written as follows,
\[
\omega = \frac{i}{2} (1 + \epsilon f_1(z_1)) dz_1 \wedge d\bar{z}_1 + \frac{i}{2} (1 + \epsilon f_2(z_2)) dz_2 \wedge d\bar{z}_2
\] (42)

The corresponding equations of motion for the fermionic fields are:
\[
\begin{align*}
(1 + \epsilon f_1(z_1)) \partial_2 \psi_2 + (1 + \epsilon f_2(z_2)) \partial_1 \psi_1 - m^2 (\bar{z}_1 q_1 + \bar{z}_2 q_2) \chi &= 0 \\
\partial_1 \chi - m^2 (z_1 q_1 + z_2 q_2) \psi_1 &= 0 \\
\partial_2 \chi - m^2 (z_1 q_1 + z_2 q_2) \psi_2 &= 0
\end{align*}
\] (43)

By looking at the equations of motion (43), we can generally say that the form of the localized solutions along each matter curve will have a more evolved dependence on all the local coordinates that parameterize the complex surface \( S \). By looking equation (41) we can see that the functions \( f_1, f_2 \) have a holomorphic dependence on their coordinates. There is a particular reason for using holomorphic functions, which is the fact that the solutions of the equations of motions (wave functions) are the sections of holomorphic
line bundles along the loci \( z_1 = 0, \ z_2 = 0 \) and \( z_1 + z_2 = 0 \). In this section we shall study if the holomorphic linear perturbation of the metric \( \text{(41)} \) modifies the spectral problem of the operator corresponding to each matter curve. However we shall not be interested in the particular form of the localized wave functions that solve the equations of motion. Additionally, due to the lack of knowledge of the global geometry that describes the compact complex threefold (also since the local geometry around the singularity affects the Standard Model physics), and in order to avoid theoretical inconsistencies, we assume that the functions \( f_1 \) and \( f_2 \) are decreasing functions of their arguments.

### 3.4.1 The matter curve \( z_1 = 0 \)

Let us start with the matter curve \( z_1 = 0 \), which means that \( q_2 = 0 \). By using the holomorphic perturbation of the metric \( \text{(41)} \) we can see that, the whole problem is a perturbation of the one that corresponds to the canonical metric. Indeed, as can be easily checked, localized solutions can exist if \( \bar{\psi}^2 = 0 \) (the situation is similar to the un-perturbed case). Then, by setting \( q_2 = 0 \), the equations of motion corresponding to the matter curve \( z_1 = 0 \), are:

\[
\begin{align*}
(1 + \epsilon f_2(z_2, \bar{z}_2)) \partial_1 \psi_1 - m^2 \bar{z}_1 q_1 \chi &= 0 \\
\bar{\partial}_1 \chi - m^2 \bar{z}_1 q_1 \bar{\psi}_1 &= 0 \quad (44)
\end{align*}
\]

which can be recast as,

\[
\begin{align*}
\partial_1 \psi_1 - \frac{m^2 \bar{z}_1 q_1}{(1 + \epsilon f_2(z_2, \bar{z}_2))} \chi &= 0 \\
\bar{\partial}_1 \chi - m^2 \bar{z}_1 q_1 \bar{\psi}_1 &= 0 \quad (45)
\end{align*}
\]

Performing a perturbation expansion and keeping terms linear to the expansion parameter \( \epsilon \) we obtain:

\[
\begin{align*}
\partial_1 \psi_1 - m^2 \bar{z}_1 q_1 (1 - \epsilon f_2(z_2, \bar{z}_2)) \chi &= 0 \\
\bar{\partial}_1 \chi - m^2 \bar{z}_1 q_1 \bar{\psi}_1 &= 0 \quad (46)
\end{align*}
\]

Clearly, the zero modes of the above equation \( \text{(53)} \) correspond to the zero modes of the matrix:

\[
D_{1\epsilon} = \begin{pmatrix}
\partial_1 & -m^2 \bar{z}_1 q_1 (1 - \epsilon f_2(z_2, \bar{z}_2)) \\
-m^2 \bar{z}_1 q_1 & \bar{\partial}_1
\end{pmatrix} \quad (47)
\]

We can write \( D_{1\epsilon} = D_1 + C \), with \( D_1 \) as in equation \( \text{(25)} \) and \( C \) being the matrix:

\[
C = \begin{pmatrix}
0 & m^2 \bar{z}_1 q_1 \epsilon f_2(z_2, \bar{z}_2) \\
0 & 0
\end{pmatrix} \quad (48)
\]

There exists a theorem in the mathematical literature that guarantees invariance of the index of Fredholm operators under odd perturbations of Fredholm type \([87,88,91]\). Particularly the theorem states:
* Let $Q$ be a Fredholm operator and $C$ be an odd operator. Then, $Q+C$ is a Fredholm operator then the indices of the two operators are equal, i.e.:

$$\text{ind}(D_1 + C) = \text{ind}D_1$$

(49)

We must note that an odd operator is defined as a matrix that anti-commutes with the Witten operator, $W$, that is $\{W, C\} = 0$. Using the notation we introduced in section 3, the matrix $W$ is equal to:

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(50)

It can be easily seen that the matrix $C$, defined in equation (62) is odd (using the terminology of the theorem), since it anti-commutes with $W$. Therefore the indices of the two matrices $D_1 + C$ and $D_1$ are equal, that is,

$$\text{ind}(D_1 + C) = \text{ind}D_1$$

(51)

As a consequence of the aforementioned results, the Witten index of the composite operator $D_1 + C$ is equal to the Witten index of the operator $D_1$. A direct implication of the equality of the two indices is that the spectral problem of the two operators is the same. This does not necessarily imply that the zero modes of $D_1$ is equal to the zero modes of the operator $D_1 + C$, but it certainly implies that the net number of the zero modes corresponding to the operators and their adjoint are equal. This is of particular importance since it gives us the opportunity to study more evolved cases and investigate more difficult aspects of these problems, such as the spectral asymmetry of the operators.

The above result does not change if we include higher orders of $\epsilon$ in the matrix $C$. Indeed, the matrix $C$ would then be:

$$C = \begin{pmatrix} 0 & m^2 \bar{z}_1 q_1 \epsilon f_2(z_2, \bar{z}_2) - m^2 \bar{z}_1 q_1 \epsilon^2 f_2^2(z_2, \bar{z}_2) + \ldots \\ 0 & 0 \end{pmatrix}$$

(52)

which still satisfies the theorem above.

Note that the situation we studied in this section can be much more difficult in the case a background flux is turned on. In that case, the restrictions on Kähler form are more stringent, since the Kähler form must satisfy the D-term equation:

$$i[\phi, \bar{\phi}] + 2\omega \wedge F^{1,1} + *sD = 0$$

(53)

where in the above $F^{1,1}$ stands for the flux.

### 3.4.2 The matter curve $z_2 = 0$

In the case of the $z_2 = 0$ matter curve, we have $q_2 = 0$. As a result of the holomorphicity of the function $f_1(z_1)$, in order to solve the equations of motion, we must set $\psi_1 = 0$ just in the non-perturbed case. Then, the equations of motion are written,

$$\partial_2 \psi_2 - m^2 \bar{z}_2q_2(1 - \epsilon f_2(z_2, \bar{z}_2)) \chi = 0$$

(54)

$$\bar{\partial}_2 \chi - m^2 \bar{z}_2q_2\psi_2 = 0$$

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As in the \( z_1 = 0 \) case, we can write \( D_{2e} = D_2 + C \), with \( D_{2e} \) being,
\[
D_{2e} = \begin{pmatrix}
\partial_{\bar{z}_2} & -m^2 \bar{z}_2 q_2 (1 - \epsilon f_1(z_1, \bar{z}_1)) \\
-m^2 \bar{z}_2 q_2 & \partial_{\bar{z}_2}
\end{pmatrix}
\]
and \( D_2 \) as in equation (33). In this case the matrix \( C \) is equal to:
\[
C = \begin{pmatrix}
0 & m^2 \bar{z}_2 q_2 \epsilon f_1(z_1, \bar{z}_1) \\
0 & 0
\end{pmatrix}
\]
Both the matrices \( C \) and \( D_{2e} \) satisfy the requirements of the theorem we used previously, therefore we also have in this case:
\[
\text{ind}(D_2 + C) = \text{ind}D_{2e}
\]

### 3.4.3 The Higgs curve \( z_1 + z_2 = 0 \)

The case \( z_1 + z_2 = 0 \) is much more evolved than the previous two cases. Using the transformations (35), equation (43) can be cast as:
\[
(2 + \epsilon f_1 + \epsilon f_2) \partial_{\bar{w}} \psi_{\bar{w}} + (2 + \epsilon f_1 + \epsilon f_2) \partial_{\bar{u}} \psi_{\bar{u}}
\]
\[
+ (\epsilon f_2 - \epsilon f_1)(\partial_{\bar{w}} \psi_{\bar{w}} + \partial_{\bar{u}} \psi_{\bar{u}}) - \frac{m^2}{2}(\bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2))\chi = 0
\]
\[
2 \partial_{\bar{w}} \chi - m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) \psi_{\bar{w}} = 0
\]
\[
2 \partial_{\bar{u}} \chi - m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) \psi_{\bar{u}} = 0
\]

In this case the theorem we presented previously does not find application, since the complex derivatives are interrelated. The only case that the theorem can find application is when \( f_1 = f_2 \). Nevertheless, the last case corresponds to a trivial (coordinate independent) deformation of the metric, thus it is a perturbative constant shift. In the same way as in the un-perturbed \( z_1 + z_2 = 0 \) case, when \( \psi_{\bar{u}} = 0 \) and \( f_1 = f_2 = f \), the above equation can be cast as:
\[
(2 + 2\epsilon f) \partial_{\bar{w}} \psi_{\bar{w}} - \frac{m^2}{2}(\bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2))\chi = 0
\]
\[
2 \partial_{\bar{w}} \chi - m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) \psi_{\bar{w}} = 0
\]

Following the same steps as previously, we obtain:
\[
2 \partial_{\bar{w}} \psi_{\bar{w}} - (1 - \epsilon f) \frac{m^2}{2}(\bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2))\chi = 0
\]
\[
2 \partial_{\bar{w}} \chi - m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) \psi_{\bar{w}} = 0
\]

The zero modes of the above equation are the zero modes of the matrix:
\[
D_{we} = \begin{pmatrix}
2 \partial_{\bar{w}} & \frac{m^2}{2}(\bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2))(1 - \epsilon f) \\
-m^2 \bar{z}_1 q_1 & \partial_{\bar{w}}
\end{pmatrix}
\]
Likewise, we can write $D_{we} = D_w + C_w$, with $D_w$ as in equation (57) and $C_w$:

$$C_w = \left( \begin{array}{ccc} 0 & \frac{m^2}{f}(\bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2))\epsilon f \\ 0 & 0 \end{array} \right)$$  \hspace{1cm} (62)

Therefore applying the theorem for the two matrices, we have:

$$\text{ind}(D_w + C) = \text{ind}D_{we}\hspace{1cm} (63)$$

Hence, the indices of the two operators are equal.

The results of this section are very important since, in virtue of the theorem, the net number of the zero modes of the metric-perturbed fermionic system is equal to the net number of the zero modes that the Euclidean metric-fermionic system has. Nevertheless we know that the solutions exist, but this theorem tells us nothing on how these perturbed solutions behave. Before we close this section, we must note that in the case we perform a non-holomorphic perturbation of the Euclidean metric, the solutions of the equation of motion are not the ones that appeared in this section. Indeed, let us take for example the matter curve $z_1 = 0$, for which a non-holomorphic perturbation of the metric would result to three wave functions—solutions to the equation of motion, namely $\chi, \psi_1$ and $\psi_2$. The solutions $\psi_1$ and $\psi_2$ are given as functions of $\chi$, which in turn is a perturbation of the gaussian profile solution. For a specific example of this type, see for example reference [92].

4 Yukawa Couplings in the Presence of Constant Background Gauge Fluxes and SUSY QM

The situation of the fermionic system without background gauge fluxes is very useful but we can get only one non-trivial Yukawa coupling [12]. In order to obtain the hierarchies of the quark masses and the appropriate mixing of the quark and lepton matter fields, the wave functions we found in section 3 must be appropriately distorted [12]. This distortion can be caused by the appearance of background gauge fields. It is proven that when the gauge fluxes are field dependent, then reasonable agreement with the observed mass hierarchies and mixings can be achieved [12]. In this section we shall add non-trivial background gauge fluxes and study whether the resulting localized fields on each matter curve on $S$ are related to an $N = 2$ SUSY QM algebra. We shall follow reference [12]. Trying to find localized solutions along the matter curve, when the gauge fields have a local coordinate dependence can be quite difficult. We shall confine ourselves to the case where the gauge fields are constant and independent from the coordinates $z_1, z_2$.

In general, the total flux can be written as follows [12]:

$$\mathcal{F} = FQ + F^{(1)}Q_1 + F^{(2)}Q_2$$  \hspace{1cm} (64)

In the above equation, $\mathcal{F}$ is the total flux, $F$ is the $U(1)$ bulk gauge flux, with generator $Q$, and $F^{(1)}, F^{(2)}$ are the fluxes along the matter curves $z_1$ and $z_2$ respectively (with
generators $Q_1$ and $Q_2$ as we saw in section 3). The corresponding gauge potentials are $A$, $A^{(1)}$, and $A^{(2)}$, respectively, with,

$$A = qA + q_1A^{(1)} + q_2A^{(2)}$$  \hspace{1cm} (65)$$

In the above, $q$ stands for the total $U(1)$ bulk charge, $q_1$ is the $U(1)$ charge along the matter curve $z_1$ and the and $q_2$ is the $U(1)$ charge along the matter curve $z_2$. The bulk flux breaks the initial $G_s$ gauge symmetry to $\Gamma_s \times U(1)$, and the fermions transform to a representation $R$ which a direct sum of irreducible representations labelled as $(q,q_1,q_2)$. In the general case, and if we consider only diagonal components of the gauge flux, the bulk flux can be written [12]:

$$F = F_{1\bar{1}}dz_1 \wedge d\bar{z}_1 + F_{2\bar{2}}dz_2 \wedge d\bar{z}_2$$  \hspace{1cm} (66)$$

and the $U(1)$'s along the matter curves are taken to be:

$$F^{(1)} = F_{2\bar{2}}^{(1)}dz_2 \wedge d\bar{z}_2, \quad F^{(2)} = F_{1\bar{1}}^{(2)}dz_1 \wedge d\bar{z}_1$$  \hspace{1cm} (67)$$

Hence, if the adjoint vacuum expectation value $\langle \phi \rangle$ is the same as in equation (20), the equations of motion for the charged fermionic fields are [12]:

$$(\partial_2 - iA_2)\psi_2 + (\partial_1 - iA_1)\psi_1 - m^2(\bar{z}_1 q_1 + \bar{z}_2 q_2)\chi = 0$$  \hspace{1cm} (68)$$

$$(\partial_2 - iA_2)\chi - m^2(z_1 q_1 + z_2 q_2)\psi_2 = 0$$

$$(\partial_2 - iA_2)\chi - m^2(z_1 q_1 + z_2 q_2)\psi_1 = 0$$

In the constant gauge flux case, we take:

$$F = 2iMdz_1 \wedge d\bar{z}_1 + 2iNdz_2 \wedge d\bar{z}_2$$  \hspace{1cm} (69)$$

with $M$, $N$, real constants. The fluxes along the matter curves are then equal to:

$$F^{(1)} = 2iN^{(1)}dz_2 \wedge d\bar{z}_2, \quad F^{(2)} = 2iM^{(2)}dz_1 \wedge d\bar{z}_1$$  \hspace{1cm} (70)$$

where $N^{(1)}$ and $M^{(2)}$ real constants. Therefore, the gauge potentials are equal to:

$$A = iM(z_1 dz_1 - \bar{z}_1 d\bar{z}_1) + iN(z_2 dz_2 - \bar{z}_2 d\bar{z}_2)$$  \hspace{1cm} (71)$$

$$A^{(1)} = iN^{(1)}(z_2 d\bar{z}_2 - \bar{z}_2 dz_2)$$

$$A^{(2)} = iM^{(2)}(z_1 d\bar{z}_1 - \bar{z}_1 dz_1)$$

Consequently, the total gauge potential is equal to:

$$A = i(qM + q_2M^{(2)})(z_1 dz_1 - \bar{z}_1 d\bar{z}_1) + i(qN + q_1N^{(1)})(z_2 dz_2 - \bar{z}_2 d\bar{z}_2)$$  \hspace{1cm} (72)$$

Performing a suitable gauge transformation of the form,

$$A = \hat{A} + d\Omega$$  \hspace{1cm} (73)$$
we can set $A_1=0$ and $A_2=0$ in equation (68) and work with the hatted fields. Indeed, equation (68) can simplified to:

$$\left( \partial_2 - i \hat{A}_2 \right) \hat{\psi}_2 + (\partial_1 - i \hat{A}_1) \hat{\psi}_1 - m^2 (\bar{z}_1 q_1 + \bar{z}_2 q_2) \hat{\chi} = 0$$

$$\tilde{\partial}_1 \hat{\chi} - m^2 (z_1 q_1 + z_2 q_2) \hat{\psi}_1 = 0$$

$$\tilde{\partial}_2 \hat{\chi} - m^2 (z_1 q_1 + z_2 q_2) \hat{\psi}_2 = 0$$

with $\chi = e^{i\Omega} \hat{\chi}$, $\psi_1 = e^{i\Omega} \hat{\psi}_1$ and $\psi_2 = e^{i\Omega} \hat{\psi}_2$. Supposing that the gauge field is coordinate independent, the total gauge potential reads:

$$\hat{A} = -2i M \bar{z}_1 dz_1 - 2i N \bar{z}_2 dz_2$$

and $\hat{A}_1 = -2i M \bar{z}_1$ and $\hat{A}_2 = -2i N \bar{z}_2$. The gauge parameter $\Omega$ is in this case:

$$\Omega = i (M|z_1|^2 + N|z_2|^2)$$

Working in the gauge we chose above makes the calculation of the wave functions (and hence of the corresponding gauge invariant properties such as Yukawa couplings) simpler [12]. This gauge is referred to as holomorphic gauge [12].

### 4.1 The Matter Curve $z_1 = 0$

Let us study here the first matter curve $z_1 = 0$. By taking $q_2 = 0$, the localized solutions in this case are [12]:

$$\hat{\psi}_1 = -\frac{\lambda_1}{q_1 m_2} \hat{\chi}, \quad \hat{\chi} = g(z_2) e^{-\lambda_1 m^2 |z_1|^2}, \quad \hat{\psi}_2 = 0.$$  

(77)

with $\lambda_1$ equal to:

$$\lambda_1 = -M + q_1 m^2 \sqrt{1 + \frac{M^2}{q_1^2 m^4}}$$

(78)

The above solutions correspond to the following equations of motion:

$$(\partial_1 - i \hat{A}_1) \hat{\psi}_1 - m^2 z_1 q_1 \hat{\chi} = 0$$

$$\tilde{\partial}_1 \hat{\chi} - m^2 z_1 q_1 \hat{\psi}_1 = 0$$

(79)

It is very easy to prove that we can associate an $N = 2$ SUSY QM algebra corresponding to the equations of motion (79). Indeed, following the steps of section 3 we define the matrix $D_{A_1}$ and also $D_{A_1}^\dagger$ as follows,

$$D_{A_1} = \begin{pmatrix} \partial_1 - i \hat{A}_1 & -m^2 \bar{z}_1 q_1 \\ -m^2 z_1 q_1 & \tilde{\partial}_1 \end{pmatrix}$$

(80)

and,

$$D_{A_1}^\dagger = \begin{pmatrix} \tilde{\partial}_1 + i \hat{A}_1 & -m^2 z_1 q_1 \\ -m^2 \bar{z}_1 q_1 & \partial_1 \end{pmatrix}$$

(81)
acting on,
\[
\begin{pmatrix}
\hat{\psi}_1 \\
\hat{\chi}
\end{pmatrix}
\]  
(82)

In the above \( \hat{A}_1 = -2iM\hat{z}_1 \). The matrix \( D_{\hat{A}_1} \) has no zero modes, while the matrix \( D_{A_1} \) has solutions the functions of equation (77). Therefore the Fredholm index \( I_{D_A} \), of the operator \( D_{A_1} \), is equal to,
\[
\text{ind}I_{D_A} = \dim \ker(D_{\hat{A}_1}^\dagger) - \dim \ker(D_{A_1})
\]  
(83)

for which clearly \( \text{ind}I_{D_A} \neq 0 \). From the last we conclude that SUSY is unbroken. The \( N = 2 \) supersymmetric quantum mechanical system can be defined using the super-charges \( Q_A \) and \( Q_A^\dagger \):
\[
Q = \begin{pmatrix} 0 & D_{A_1} \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ D_{A_1}^\dagger & 0 \end{pmatrix}
\]  
(84)

Furthermore, the Hamiltonian can be written as:
\[
H = \begin{pmatrix} D_{A_1}D_{\hat{A}_1} & 0 \\ 0 & D_{\hat{A}_1}^\dagger D_{A_1} \end{pmatrix}
\]  
(85)

Finally, the Witten index of the SUSY QM algebra, is \( I_{D_A} = -\Delta \).

We can see that the constant background gauge fluxes do not spoil the SUSY QM algebra that underlies the fermionic system of the flux-less case. The algebra itself is of-course different but still SUSY is unbroken.

4.2 The Matter Curve \( \hat{z}_2 = 0 \)

In the case of the \( \hat{z}_2 = 0 \) matter curve, the equations of motion are (for \( q_1 = 0 \)):
\[
(\partial_2 - iA_2)\hat{\psi}_2 - m^2 \hat{z}_2q_2\hat{\chi} = 0
\]
\[
\bar{\partial}_2\hat{\chi} - m^2 \hat{z}_2q_2\hat{\psi}_2 = 0
\]

with \( A_2 = -2iN\hat{z}_2 \). The localized solutions are:
\[
\hat{\psi}_2 = -\frac{\lambda_2}{q_2m^2}\hat{\chi}, \quad \hat{\chi} = g(z_1)e^{-\lambda_2|z_2|^2}, \quad \hat{\psi}_1 = 0
\]  
(87)

with \( \lambda_2 \) equal to:
\[
\lambda_2 = -N + q_2m^2\sqrt{1 + \frac{N^2}{q_2^2m^4}}
\]  
(88)

The \( N = 2 \) SUSY QM algebra is built on the matrices:
\[
D_{A_2} = \begin{pmatrix} \partial_2 - iA_2 & -m^2 \hat{z}_2q_2 \\ -m^2 \hat{z}_2q_2 & \bar{\partial}_2 \end{pmatrix}
\]  
(89)
and,

\[ D_{A_2}^\dagger = \begin{pmatrix} \bar{\partial}_2 + iA_2 & -m^2 z_2 q_2 \\ -m^2 z_2 q_2 & \partial_2 \end{pmatrix} \] (90)

acting on,

\[ \begin{pmatrix} \hat{\psi}_2 \\ \hat{\chi} \end{pmatrix} \] (91)

We shall not pursue this case further, since it is identical with the previous \( z_1 = 0 \). The result is that a \( N = 2 \) unbroken SUSY QM algebra underlies the system.

As for the \( z_1 + z_2 = 0 \) case, it is much more difficult to handle, compared to the other two cases. Performing the transformation (35), the equations of motion (79) can be cast in the form:

\[ 2 \partial \bar{\psi} + 2 \partial \bar{\psi} + (N(\bar{w} - \bar{u}) - M(\bar{w} + \bar{u})) \hat{\psi} = 0 \] (92)

\[ 2 \bar{\partial} \bar{\psi} - m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) \hat{\bar{\psi}} = 0 \]

\[ 2 \bar{\partial} \hat{\bar{\psi}} - m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) \hat{\psi} = 0 \]

It is not easy to relate the above fermionic system to an \( N = 2 \) SUSY QM algebra. Perhaps central charges must be included to this \( N = 2 \) algebra. Such a behavior kind of surprised us, because we expected all the localized fermion solutions to have the same, central charge free, \( N = 2 \) SUSY QM algebra. It seems that this is not the case. We shall not pursue this issues further.

### 5 Yukawa couplings in the absence of gauge fluxes and \( N = 2 \) SUSY QM algebra.

In the previous we found that when a matter curve has localized zero modes, we can built a \( N = 2 \) SUSY QM algebra from the system. In most cases localization occurs when, one of the fields that exist on the D7 brane intersection vanishes. By looking the equations of motion (36), it is natural to make the equations of motion look like the following,

\[ (2 \partial \bar{\psi} + 2 \partial \bar{\psi} + m^2 (w(q_1 + q_2) + u(q_1 - q_2)) \chi = 0 \] (93)

\[ (2 \bar{\partial} \bar{\psi} + 2 \bar{\partial} \bar{\psi} - m^2 (w(q_1 + q_2) + u(q_1 - q_2)) \chi = 0 \]

This is clarified by looking the \( \chi \) derivative, which is \( 2\bar{\partial} \bar{\psi} + 2 \partial \bar{\psi} \). In the equations of motion (93), the derivative that acts on \( \chi \) is the conjugate derivative of the one that acts on \( \psi + \psi \). Let us see when this is possible and what would be the implications of this construction.

---

1. Recall the SUSY QM algebras for the matter curves \( z_1 = 0 \) and \( z_2 = 0 \)
An $N = 2$ SUSY QM algebra can be built based on (93), by using the matrix:
\[
D = \begin{pmatrix}
2 \partial_w + 2 \partial_u & -m^2 \left( \bar{w}(q_1 + q_2) + \bar{u}(q_1 - q_2) \right) \\
-m^2 \left( w(q_1 + q_2) + u(q_1 - q_2) \right) & 2 \bar{\partial}_w + 2 \bar{\partial}_u
\end{pmatrix}
\]
acting on:
\[
\begin{pmatrix}
\psi \bar{w} + \bar{\psi} u \\
\chi \bar{w}
\end{pmatrix}
\]
It is obvious that, by using (94), we can construct the matrix $D^\dagger$ and the rest of the SUSY algebra, such as the Hamiltonian and so on.

In order the equations of motion (36) to be identical to (93) the following condition must be imposed on the fields $\psi \bar{w}$ and $\bar{\psi} u$,
\[
\partial_w \psi \bar{w} = \partial_u \bar{\psi} u
\]
The implications of the above condition to the case of the three matter curves are quite interesting. For the $\Sigma_1$ ($z_1 = 0$) matter curve, since $\psi_2 = 0$, relation (96) would imply $\partial_1 \psi_1 = 0$. The curve $\Sigma_2$ ($z_2 = 0$) has localized solutions when $\psi_1 = 0$, and in conjunction with (96) we get the condition $\partial_1 \psi_2 = 0$. In the same vain, one has for the $\Sigma_3$ ($z_1 + z_2 = 0$) curve $\partial_w \psi_\bar{w} = 0$. The conditions $\partial_1 \psi_2 = 0$ and $\partial_2 \psi_1 = 0$ imply that $\psi_1$ is a function only of $z_1$ (thus has no $z_2$ dependence) and $\psi_2$ is a function only of $z_2$. This in turn would imply that the functions $f(z_2)$ and $g(z_1)$ defined in relations (24), (32) and (40) are constant functions, that is, $f(z_2) = c_1$ and $g(z_1) = c_2$, with $c_1$ and $c_2$ arbitrary constants. Furthermore, the condition $\partial_w \psi_\bar{w} = 0$, implies that $\psi_\bar{w}$ is a constant function, say $\psi_\bar{w} = c_3$. We summarize:

\[
\begin{align*}
\partial_w \psi_\bar{w} = \partial_u \bar{\psi} u & \quad \Rightarrow \quad \text{Matter Curve } z_1 = 0 \rightarrow f(z_2) = c_1 \\
\text{Matter Curve } z_2 = 0 \rightarrow g(z_1) = c_2 \\
\text{Matter Curve } z_1 + z_2 = 0 \rightarrow \psi_\bar{w} = c_3
\end{align*}
\]
The three conditions we just presented, are very much related to the calculation of Yukawa couplings, when we have constant Higgs wave function and absence of non-constant fluxes [12].

As we saw earlier, the Yukawa coupling, in terms of the three matter wave functions reads:
\[
Y = M^4 \int_S d^2 z_1 d^2 z_2 \psi_1 \psi_2 \phi
\]
In the presence of background fluxes, the Yukawa coupling is given by the overlapping integral [31],
\[
Y^{ij} \sim \int_S z_1^{3-i} z_2^{3-j} e^{M_i i z_1 z_2} f(z_1, z_2)
\]
with $f(z_1, z_2)$ containing the gaussian profiles of the localized fermions along the matter curves. When the $M_{i,j}$ is constant or zero, there is a $U(1) \times U(1)$ symmetry, under which the coordinates are invariant,
\[
z_1 \rightarrow e^{ia_1} z_1 \quad z_2 \rightarrow e^{ia_2} z_2
\]
In that case, all Yukawas other than the $Y_{33}$, vanish. These gauge symmetries are broken when $M_{i,j}$ has a non trivial gauge dependence, which happens when background fluxes are turned on. This case is particularly interesting, since in this way a hierarchical fermion Yukawa matrix is obtained but we shall not pursue these issues further.

In the absence of fluxes, the fermionic matter functions are equal to:

$$
\psi_1 = f(z_2)e^{-q_1m^2|z_1|^2}
$$

$$
\psi_2 = g(z_1)e^{-q_2m^2|z_2|^2}
$$

In the special case that $\phi = \text{const}$, the wave function of the Higgs field is unlocalized, which means that the Higgs field lives in the bulk rather than localized on a matter curve. In the absence of non-constant fluxes, we have $f(z_2) = g(z_1) = 1$ and it can be proved that in this case, the only non-vanishing Yukawa is the $Y_{33}$. This coupling gives mass to the heaviest lepton and quark generations, and is equal to,

$$
Y_{33} \sim M_4^4 \int_S d^2 z_1 d^2 z_2 e^{-q_1m^2|z_1|^2} e^{-q_2m^2|z_2|^2}.
$$

Consequently, we see that the conditions (97), imposed by the $N = 2$ SUSY QM of the system described by the equations of motion (93), are the same with the conditions that the matter curves wave functions satisfy, in order to built Yukawa couplings in the absence of non-constant fluxes with non-localized Higgs. Indeed the two cases are identical when $c_3 = c_2 = 1$. This type of Yukawa couplings is usually found in type IIB and F-theory compactified on non del-Pezzo surfaces [12].

Before closing this section we discuss an important issue. Spacetime supersymmetry and supersymmetric quantum mechanics are not the same, nevertheless the connection is profound, since extended (with $N = 4, 6, ...$) supersymmetric quantum mechanics models describe the dimensional reduction to one (temporal) dimension of $N = 2$ and $N = 1$ Super-Yang Mills models [90]. A serious question rises at this point. By looking the $N = 2$ SUSY QM algebra supercharges [13], one could thing that it is intended to embed any intersection curve in the same sort of $N = 2$ SUSY QM algebra. In general, a sort of $N = 2$ supersymmetry is rather unexpected, since the intersection of three D7-branes, breaks four-dimensional supersymmetry down to $N = 1$, and so the $N = 2$ structure mentioned above is lost. However, this is not true, since supersymmetry and supersymmetric quantum mechanics is not the same. Indeed, the $N = 2$ SUSY QM supercharges do not generate spacetime supersymmetry. By the same token, the supersymmetry in supersymmetric quantum mechanics, does not relate fermions and bosons. The SUSY QM supercharges do not generate transformations between fermions and bosons. These supercharges generate transformations between two orthogonal eigenstates of a Hamiltonian, eigenstates that are classified according to their Witten parity. Hence, this flow of the $N = 2$ breaking argument is not true, due to the non-spacetime structure of SUSY QM.
6 Global $U(1)$ Symmetries Along Matter Curves, Yukawa Couplings, Proton Decay Operators and Neutrino Mass Operators

The $N = 2$ supersymmetric quantum mechanics algebra is invariant under an R-symmetry. Likewise, the Hamiltonian is also invariant under this symmetry \[85\]. Actually, the superalgebra (5) and (6) is invariant under the transformation,

\[
\left( \begin{array}{c} Q'_1 \\ Q'_2 \end{array} \right) = \left( \begin{array}{cc} \cos a & \sin a \\ -\sin a & \cos a \end{array} \right) \cdot \left( \begin{array}{c} Q_1 \\ Q_2 \end{array} \right)
\] (103)

with $a$ an arbitrary constant. Furthermore, the complex supercharges $Q$ and $Q^\dagger$ are transformed under a global $U(1)$ transformation:

\[
Q' = e^{ia} Q, \quad Q'^\dagger = e^{-ia} Q^\dagger
\] (104)

This R-symmetry is also a symmetry of the Hilbert states corresponding to the subspaces $H^+$ and $H^-$. Thus, the eigenfunctions of $H_+ = AA^\dagger$ and $H_- = A^\dagger A$, are invariant under this $U(1)$-symmetry, namely,

\[
|\psi'\rangle = e^{i\beta_+} |\psi\rangle, \quad |\psi''\rangle = e^{i\beta_-} |\psi\rangle
\] (105)

It is clear that the parameters $\beta_+$ and $\beta_-$ are global parameters with $\beta_+ \neq \beta_-$. Consistency with relation (104) requires that $a = \beta_+ - \beta_-$. For our purposes we shall use only the symmetry $|\psi'\rangle = e^{i\beta_+} |\psi\rangle$. The implications of this symmetry are quite interesting, since this implies that the localized fields on each matter curve are invariant under this symmetry. Let us see this for the $z_1 = 0$ matter curve. Due to this $U(1)$ symmetry, the $Q$ and $Q^\dagger$ supercharges of equation (29) are invariant under the transformation (104). Consequently the eigenfunctions of $D_1$,

\[
\left( \begin{array}{c} \psi_1 \\ \chi_1 \end{array} \right)
\] (106)

are invariant under the following transformation,

\[
\left( \begin{array}{c} \psi_1 \\ \chi_1 \end{array} \right)' = e^{ib_+} \left( \begin{array}{c} \psi_1 \\ \chi_1 \end{array} \right)
\] (107)

In the same way, the localized fields on the second fermion matter curve $z_2$, are invariant under,

\[
\left( \begin{array}{c} \psi_2 \\ \chi_2 \end{array} \right) = e^{ib_2+} \left( \begin{array}{c} \psi_2 \\ \chi_2 \end{array} \right)
\] (108)

Finally, the localized fields on the Higgs matter curve $z_1 + z_2 = 0$, are invariant under,

\[
\left( \begin{array}{c} \psi_w \\ \chi_w \end{array} \right)' = e^{ib_3+} \left( \begin{array}{c} \psi_w \\ \chi_w \end{array} \right)
\] (109)

Note that the fields on each matter curve have the same transformation properties. For convenience, we gather the results in the following table,
The localized fields on each matter curve are invariant under this $U(1)$ symmetry presented above. However, certain conditions must hold. Indeed, if this symmetry is an actual symmetry of the localized fields, then the action (plus kinetic terms) for the localized fields must be invariant under this $U(1)$ symmetry. Let us examine for example the matter curve $z_1 = 0$. The localized fields action reads (recall $\eta = \psi_2 = 0$),

$$
I_L = \int_{R^{1,3} \times S} d^4x \text{Tr} \left( \chi_1 \wedge \partial_A \psi_1 + \frac{1}{2} \psi_1 \wedge [\phi, \psi_1] + \text{h.c.} \right)
$$

(110)

In addition the kinetic terms are of the form,

$$
\int_{R^{1,3} \times S} \psi_1^\dagger \psi_1 d^4x, \quad \int_{R^{1,3} \times S} \chi_1^\dagger \chi_1 d^4x
$$

(111)

It is obvious that under the $U(1)$ transformation,

$$
\chi_1'^\dagger = e^{ib_+} \chi_1, \quad \psi_1'^\dagger = e^{ib_+} \psi_1
$$

(112)

the kinetic terms (111) are invariant. Still, the action (110) cannot be invariant unless,

$$
e^{2ib_+} = 1
$$

(113)

Also, we suppose that the field $\phi$ is not affected by the $U(1)$-symmetry. The condition (113) implies that $b_+ = \pi n$, with $n = 0, 1, 2, ...$

But the fields $\chi_1$ and $\phi$ belong to the same susy multiplet, thus we would expect that this $U(1)$-symmetry should be a symmetry of the whole action. It turns out that in order the localized fields have this $U(1)$-symmetry, the $\phi$ field must not transform under this symmetry. The implications of this symmetry are quite interesting, at least phenomenologically, as we shall see.

### 6.1 Proton Decay Operators, Dirac and Majorana Neutrino Masses

Let us recall how Yukawas are constructed within F-theory [49][12][14][16][21][51]. In F-theory, Yukawa couplings are considered to be overlapping integrals of the three matter curves wave functions over $S$. The matter curves are the two fermionic and the one corresponding to the Higgs. Let the wave functions that describe each fermionic matter curve be, $\psi_1, \psi_2$ describing $\Sigma_1$ and $\Sigma_2$ respectively. Owing to the $N = 1$ supersymmetry of the four dimensional theory, the wave function of the Higgs curve, $\phi$, is the same with
the function $\psi_w$, corresponding to the $z_1 + z_2 = 0$ curve, as we saw earlier. Then, in terms of the three wave functions, the Yukawa coupling reads:

$$Y = M^4 \int_S d^2z_1 d^2z_2 \psi_1 \psi_2 \phi$$  \hspace{1cm} (114)$$

The Yukawa couplings give masses to fermions, therefore these couplings are most welcome in the theoretical setup of the local model. As we saw, each localized fermion field corresponding to the two matter curves $z_1$ and $z_2$, obeys a SUSY QM $U(1)$-symmetry, different for each matter curve (see table 2). Yukawa couplings describe couplings between a quark and a righthanded quark or between a lepton and a righthanded lepton. We denote the lepton fields with the field operators $L = (N,E)$ and also with $E^c$ the right handed one. Additionally, the quark fields are represented by $Q = (U,D)$ and their righthanded counterparts, $U^c, D^c$. The Yukawa's stem from the superpotential and are of the form $^{12,16}$,

$$W_{Yuk} = Y^{U} QU^c H^u + Y^{D} QD^c H^d + Y^{L} LE^c H^d$$  \hspace{1cm} (115)$$

We require the wave functions $\psi_1$ and $\psi_2$ in equation (114) to describe a lepton field and it’s righthanded field, or a quark field and it’s righthanded field, respectively. This means that leptons and quarks must be assigned to different matter curves. This situation cannot be true in all local geometrical GUT setups, like in the case of $SU(5)$, but can be true in some cases, like in the flipped $SU(5)$ $^{49}$ construction. Let the transformations of $\psi_1$ and $\psi_2$ be that of table 2, that is,

$$\psi_1' = e^{ib_+} \psi_1, \hspace{0.5cm} \psi_2' = e^{ib_2+} \psi_2$$  \hspace{1cm} (116)$$

Due to equation (113), the parameter $b_+$ is equal to $b_+ = n \pi$, with $n = 0, 1, 2, ...$ and similarly, $b_2+ = m \pi$, with $m = 0, 1, 2, ...$.

In order the Yukawa coupling to be invariant under this combined action of the $U(1)$’s, we easily find that the parameters $b_+$ and $b_2+$ must be related as follows,

$$b_+ = -b_2+$$  \hspace{1cm} (117)$$

On that account, we conclude that fermions belonging to a quark or lepton family and their righthanded fermions (corresponding to different matter curves), must have opposite transformation properties under the SUSY QM-$U(1)$ symmetry, in order the Yukawa couplings are invariant under this symmetry. Note that this $U(1)$-symmetry is not a result coming from the local geometric features of the surface $S$. It comes from the SUSY QM algebra that the localized fields obey. The outcomes of the two conditions (113) and (117), are quite interesting phenomenologically. We consider first proton decay operators. The proton decay operators are unwanted terms coming from the action. The 4-dimensional proton decay operators are,

$$W_{4a} \sim LLE^c \hspace{0.5cm} W_{4b} \sim QD^c L \hspace{0.5cm} W_{4c} \sim U^c D^c D^c$$  \hspace{1cm} (118)$$
Moreover, the 5-dimensional proton decay operators are,

\[ W_{5a} \sim \frac{1}{M}QQQL \quad W_{5b} \sim \frac{1}{M}LLH_uH_u, \quad W_{5c} \sim \frac{1}{M}U^c U^c D^c E^c \]  

(119)

The SUSY QM \( U(1) \)-symmetry restricts the proton decay operators, as is obvious by looking the constraints (113) and (117). Let us see which operators are allowed subject to the SUSY QM \( U(1) \)-symmetry in detail. We study first the dimension-4 operators. The operator \( W_{4a} \) (see relation (118)) is not allowed since although the \( LL \) part is invariant (same fermions, see (113)), the \( E^c \) gives a total \( e^{ia} \) factor to the term. Likewise, \( W_{4b} \) is not invariant, since, although the \( QD^c \) part is invariant (fermion and corresponding righthanded fermion, see 117), the leptons \( L \) have different transformation properties from the quarks. The term \( W_{4c} \) is not allowed because, although the \( D^c D^c \) is invariant, the \( U^c \) gives an overall exponential factor to the term. Hence, the dimension-4 proton decay operators of relation (118) are not allowed in the theory, if the SUSY QM \( U(1) \)-symmetry is obeyed by the fermion fields localized on the matter curves.

Let us now check the dimension-5 operators. The operator \( W_{5a} \) is not invariant under the \( U(1) \). Indeed, although the \( QQ \) part is invariant (same fermions, see (113)), the \( E^c \) gives a total \( e^{ia} \) factor to the term. Likewise, \( W_{5b} \) is not invariant, since, although the \( QD^c \) part is invariant (fermion and corresponding righthanded fermion, see 117), the leptons \( L \) have different transformation properties from the quarks. The term \( W_{5c} \) is not allowed because, although the \( D^c D^c \) is invariant, the \( U^c \) gives an overall exponential factor to the term. Hence, the dimension-4 proton decay operators of relation (118) are not allowed in the theory, if the SUSY QM \( U(1) \)-symmetry is obeyed by the fermion fields localized on the matter curves.

6.2 Neutrino Masses and SUSY QM \( U(1) \)-Invariance

The minimal \( SU(5) \) F-theory GUT predicts Dirac and Majorana neutrino masses [81]. Indeed, by integrating out massive Kaluza-Klein modes, generates higher dimensional operators that give phenomenologically acceptable masses for neutrinos. Particularly, the Majorana mass F-term is of the form [81],

\[ \int d\theta^2 H_u L H_u L \frac{1}{A_{UV}} \]  

(120)

When the Higgs field develops a vacuum expectation value, the above term yields a Majorana mass for the neutrinos. The Majorana mass term (120) is clearly invariant under the SUSY QM \( U(1) \) symmetry because the term \( LL \) is invariant (same fermions) and the Higgs fields are not affected at all.

In the Dirac scenario, the D-term, generated by integrating out massive Kaluza-Klein modes on the Higgs curves,

\[ \int d\theta^4 H_d^1 L N_R \frac{1}{A_{UV}} \]  

(121)

gives a Dirac mass to the neutrinos. The field \( N_R \) describes the right-handed neutrino. The peculiarity of \( N_R \) is due to that, the right handed neutrino localizes on curves normal to the GUT-seven brane [81], a fact that put’s in question the local description concept of F-theory GUTs. Still, normal curves can form part of a consistent local model [81]. The
Dirac mass term (121) is not invariant under the SUSY QM $U(1)$, since from the term $H_d^T L N_R$, only the field $L$ is transformed under the $U(1)$. Thus we can see that only the Majorana mass terms is favored in the scenario we presented.

7 Conclusions

In this article we found that the fields localized at $D7$ branes intersections are closely connected to an $N = 2$ SUSY QM algebra. Particularly, each matter curve corresponds to a different algebra and due to this algebra, a global $U(1)$-symmetry underlies the system. In view of this symmetry, the localized fields on each matter curve satisfy certain conditions which we classified in Table 2. Furthermore, since the Yukawa couplings are important to GUT phenomenology, they must be invariant under this $U(1)$. This condition, in conjunction with the table 2 transformations, classifies the fermion transformations as in the following table,

| Term   | $e^{ib_2+}$ | $e^{ia_2+}$ | $e^{ib_2+}$ | $U(1)$-Invariant |
|--------|--------------|--------------|--------------|-----------------|
| $L$    | $e^{ia_2+}$  |              |              | $U(1)$-Invariant |
| $LL$   |              | $e^{ia_2+}$  |              | $U(1)$-Invariant |
| $E^c$  | $e^{ib_2+}$  |              |              | $U(1)$-Invariant |
| $Q$    |              |              | $e^{ia_2+}$  | $U(1)$-Invariant |
| $QQ$   | $e^{ib_2+}$  |              |              | $U(1)$-Invariant |
| $D^c$  | $e^{ib_2+}$  |              |              | $U(1)$-Invariant |
| $U^c$  | $e^{ib_2+}$  |              |              | $U(1)$-Invariant |
| $L E^c$|              | $U(1)$-Invariant |
| $Q D^c$| $U(1)$-Invariant |
| $Q U^c$| $U(1)$-Invariant |
| $U^c U^c$| $U(1)$-Invariant |

Table 3: $U(1)$-Classification of various terms

Owing to the above transformation properties, we found restrictions on the proton decay operators, many of which are not allowed. Moreover, this $U(1)$ SUSY QM symmetry restricts the neutrino mass operators. Particularly we found that only the Majorana mass terms are allowed in our scenario.

We must mention that there are much more elaborated and geometry inspired techniques to restrict proton decay operators (see for example [6–9, 20, 23]), such as monodromies, but we do not discuss these here.

Moreover, the requirement of an $N = 2$ SUSY QM on a special system leads to specific conditions on the fermion fields and (due to supersymmetry) on the Higgs boson. As we found, these conditions are met in the construction of the Yukawa coupling that gives mass to the top quark, with a delocalized bulk Higgs.

In order to obtain the correct hierarchies and mixing of the matter fields, external non-trivial background fluxes must be turned on. We examined if the $N = 2$ SUSY QM
algebra still underlies the localized fermionic solutions. We studied the constant flux case and we found that the SUSY algebra still holds for the two matter curves, namely $z_1 = 0$ and $z_2 = 0$. However, for the Higgs curve, namely $z_1 + z_2 = 0$, things are different. It seems that the algebra is not a $N = 2$ without central charge SUSY QM algebra. We hope to address this problem in the future, but it kind of surprised us. The surprise is due to the fact that the adjoint vacuum expectation value $<\phi>$ remains the same as in the flux-less case, so we did not expect things to change so drastically.

Finally we performed a holomorphic perturbation of the metric that describes the complex surface $S$ and we studied how the perturbation modifies the net number of the zero modes that the un-perturbed system has. We found that, due to a theorem characteristic for Fredholm operators, the operators that describe the perturbed and un-perturbed systems have equal indices. We checked the validity of the theorem, for every matter curve and Higgs curve. Unfortunately, this theorem does not gives us information on the specific form of the wave functions.

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**References**

[1] C. Vafa, Evidence for F-Theory, Nucl. Phys. B 469 (1996) 403

[2] David R. Morrison, Cumrun Vafa, Compactifications of F theory on Calabi-Yau threefolds 1, Nucl. Phys. B473 (1996) 74

[3] David R. Morrison, Cumrun Vafa, Compactifications of F theory on Calabi-Yau threefolds 2, Nucl.Phys. B476 (1996) 437

[4] C. Beasley, J. J. Heckman and C. Vafa, GUTs and Exceptional Branes in F-theory - I, JHEP 0901 (2009) 058

[5] C. Beasley, J. J. Heckman and C. Vafa, GUTs and Exceptional Branes in F-theory - II: Experimental Predictions, JHEP 0901, 059 (2009)

[6] T. Weigand, Lectures on F-theory compactifications and model building, Class. Quant. Grav. 27 (2010) 214004

[7] R. Donagi and M. Wijnholt, Model Building with F-Theory, [arXiv:0802.2969] [hep-th]

[8] Frederik Denef, Les Houches Lectures on Constructing String Vacua, [arXiv:0803.1194]

[9] Jonathan J. Heckman, Particle Physics Implications of F-theory, [arXiv:1001.0577] [hep-th]
[10] D. Klevers, Holomorphic Couplings In Non-Perturbative String Compactifications, \texttt{arXiv:1106.6259} [hep-th]

[11] Ron Donagi, Martijn Wijnholt, Higgs Bundles and UV Completion in F-Theory, \texttt{arXiv:0904.1218} [hep-th]

[12] A. Font, L.E. Ibanez, Matter wave functions and Yukawa couplings in F-theory Grand Unification, JHEP 0909 (2009) 036

[13] Ralph Blumenhagen, Thomas W. Grimm, Benjamin Jurke, Timo Weigand, F-theory uplifts and GUTs, JHEP 0909 (2009) 053

[14] Tianjun Li, SU(5) and SO(10) Models from F-Theory with Natural Yukawa Couplings, Phys. Rev. D81 (2010) 065018

[15] A Note on Local GUT Models in F-Theory Ching-Ming Chen, Yu-Chieh Chung, Nucl. Phys. B824 (2010) 273

[16] Yukawa Structure from U(1) Fluxes in F-theory Grand Unification, A. Font, L.E. Ibanez, JHEP 0902 (2009) 016

[17] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, Global F-theory GUTs, Nucl. Phys. B 829 (2010) 325

[18] Tianjun Li, Dimitri V. Nanopoulos, Joel W. Walker, F-ast Proton Decay, \texttt{arXiv:0910.0860}

[19] J. J. Heckman, A. Tavanfar and C. Vafa, The Point of E8 in F-theory GUTs, JHEP 1008 (2010) 040 27

[20] Ron Donagi, Martijn Wijnholt, Breaking GUT Groups in F-Theory, \texttt{arXiv:0808.2223} [hep-th]

[21] Jeffrey A. Harvey, Andrew B. Royston, Gauge/Gravity duality with a chiral N=(0,8) string defect, JHEP 0808 (2008) 006

[22] A. Sebbar, M.B. Sedra, On ADE quiver models and F-theory compactification, A. Belhaj, Jorgen Rasmussen, J. Phys. A39 (2006) 9339

[23] M. Bershadsky, Kenneth A. Intriligator, S. Kachru, David R. Morrison, V. Sadov, Cumrun Vafa, Geometric singularities and enhanced gauge symmetries, Nucl. Phys. B481 (1996) 215

[24] H. Hayashi, T. Kawano, R. Tatar and T. Watari, Codimension-3 Singularities and Yukawa Couplings in F-theory, Nucl. Phys. B, 823 (2009) 47

[25] J. Marsano, N. Saulina and S. Schafer-Nameki, Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs, JHEP 0908 (2009) 046
[43] Kang-Sin Choi, Jihn E. Kim, Codes of all elementary particles, arXiv:1012.0847 [hep-ph]

[44] Andres Collinucci, Raffaele Savelli, On Flux Quantization in F-Theory, arXiv:1011.6388 [hep-th]

[45] Joseph Marsano, Hypercharge Flux, Exotics, and Anomaly Cancellation in F-theory GUTs, arXiv:1011.2212 [hep-th]

[46] Ching-Ming Chen, Yu-Chieh Chung, On F-theory $E_6$ GUTs, arXiv:1010.5536 [hep-th]

[47] Sebastian Franco, Gonzalo Torroba, Gauge theories from D7-branes over vanishing 4-cycles, JHEP 1101 (2011) 017

[48] Volker Braun, Discrete Wilson Lines in F-Theory, arXiv:1010.2520 [hep-th]

[49] G.K. Leontaris, G.G. Ross, Yukawa couplings and fermion mass structure in F-theory GUTs, arXiv:1009.6000 [hep-th]

[50] Martijn Wijnholt, F-theory and unification, Fortsch. Phys. 58 (2010) 846

[51] Radu Tatar, Yukawa couplings and neutrinos in F-theory compactifications, Fortsch. Phys. 58 (2010) 900

[52] Eric Kuflik, Joseph Marsano, Comments on Flipped SU(5) (and F-theory), arXiv:1009.2510 [hep-ph]

[53] Yu-Chieh Chung, On Global Flipped SU(5) GUTs in F-theory, arXiv:1008.2506 [hep-th]

[54] Jacek Pawelczyk, F-theory inspired GUTs with extra charged matter, Phys. Lett. B697 (2011) 75

[55] Jonathan J. Heckman, Cumrun Vafa, An Exceptional Sector for F-theory GUTs, Phys. Rev. D83 (2011) 026006

[56] Ching-Ming Chen, Yu-Chieh Chung, Flipped SU(5) GUTs from $E_6$ Singularity in F-theory, arXiv:1005.5728 [hep-th]

[57] Ching-Ming Chen, Johanna Knapp, Maximilian Kreuzer, Christoph Mayrhofer, Global SO(10) F-theory GUTs, JHEP 1010 (2010) 057

[58] Hirotaka Hayashi, Teruhiko Kawano, Yoichi Tsuchiya, Taizan Watari, More on Dimension-4 Proton Decay Problem in F-theory - Spectral Surface, Discriminant Locus and Monodromy, Nucl. Phys. B840 (2010) 304

[59] Kang-Sin Choi, Tatsuo Kobayashi, Towards the MSSM from F-theory, Phys. Lett. B693 (2010) 330
[60] Tianjun Li, James A. Maxin, Dimitri V. Nanopoulos, F-Theory Grand Unification at the Colliders, arXiv:1002.1031 [hep-ph]

[61] G.K. Leontaris, N.D. Tracas, Gauge coupling flux thresholds, exotic matter and the unification scale in F-SU(5) GUT, Eur. Phys. J. C67 (2010) 489

[62] Bjorn Andreas, Gottfried Curio, From Local to Global in F-Theory Model Building, J. Geom. Phys. 60 (2010) 1089

[63] Loriano Bonora, Raffaele Savelli, Non-simply-laced Lie algebras via F theory strings, JHEP 1011 (2010) 025

[64] A. P. Braun, S. Gerigk, A. Hebecker, H. Triendl, D7-Brane Moduli vs. F-Theory Cycles in Elliptically Fibred Threefolds, Nucl. Phys. B836 (2010) 1

[65] Marco Billo, Laurent Gallot, Alberto Lerda, Igor Pesando, F-theoretic versus microscopic description of a conformal N=2 SYM theory, JHEP 1011 (2010) 041

[66] Murad Alim, Michael Hecht, Hans Joekers, Peter Mayr, Adrian Mertens, Masoud Soroush, Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry, arXiv:1010.0977 [hep-th]

[67] Mirjam Cvetiè, Inaki Garcia-Etxebarria, James Halverson, On the computation of non-perturbative effective potentials in the string theory landscape: IIB/F-theory perspective, arXiv:1009.5386

[68] Thomas W. Grimm, Tae-Won Ha, Albrecht Klemm, Denis Klevers, Five-Brane Superpotentials and Heterotic F-theory Duality. Nucl.Phys.B 838, (2010) 458; Thomas W. Grimm, Tae-Won Ha, Albrecht Klemm, Denis Klevers, Computing Brane and Flux Superpotentials in F-theory Compactifications JHEP 1004015 (2010); Thomas W. Grimm, Tae-Won Ha, Albrecht Klemm, Denis Klevers, The D5-brane effective action and superpotential in N=1 compactifications, Nucl. Phys. B816 (2009) 139

[69] Andres Collinucci, New F-theory lifts, JHEP 0908076 (2009)

[70] Thomas W. Grimm, The N=1 effective action of F-theory compactifications, Nucl. Phys. B845 (2011) 48

[71] Luis J. Boya, Arguments for F-theory, Mod. Phys. Lett. A21 (2006) 287

[72] Adrian Clingher, John W. Morgan, Mathematics underlying the F theory / Heterotic string duality in eight-dimensions, Commun. Math. Phys. 254 (2005) 513

[73] G. Aldazabal, Luis E. Ibanez, F. Quevedo, A.M. Uranga, D-branes at singularities: A Bottom up approach to the string embedding of the standard model, JHEP 0008 (2000) 002

[74] Paul S. Aspinwall, Sheldon H. Katz, David R. Morrison, Lie groups, Calabi-Yau threefolds, and F theory, Adv. Theor. Math. Phys. 4 (2000) 95
[75] Duiliu-Emanuel Diaconescu, Govindan Rajesh, Geometrical aspects of five-branes in heterotic / F theory duality in four-dimensions, JHEP 9906 (1999) 002

[76] Michael Bershadsky, Andrei Johansen, Tony Pantev, Vladimir Sadov, On four-dimensional compactifications of F theory. Nucl. Phys. B505 (1997) 165

[77] Adil Belhaj, Luis J. Boya, Antonio Segui, Holonomy Groups Coming From F-Theory Compactification, Int. J. Theor. Phys. 49 (2010) 681

[78] Yu-Chieh Chung, Abelian Gauge Fluxes and Local Models in F-Theory, JHEP 1003 (2010) 006

[79] Fernando Marchesano, Luca Martucci, Non-perturbative effects on seven-brane Yukawa couplings, Phys. Rev. Lett. 104 (2010) 231601

[80] Hirotaka Hayashi, Teruhiko Kawano, Yoichi Tsuchiya, Taizan Watari, Flavor Structure in F-theory Compactifications, JHEP 1008 (2010) 036

[81] Vincent Bouchard, Jonathan J. Heckman, Jihye Seo, Cumrun Vafa, F-theory and Neutrinos: Kaluza-Klein Dilution of Flavor Hierarchy, JHEP 1001 (2010) 061

[82] Radu Tatar, Yoichi Tsuchiya, Taizan Watari, Right-handed Neutrinos in F-theory Compactifications, Nucl. Phys. B823, 1 (2009)

[83] Lisa Randall, David Simmons-Duffin, Quark and Lepton Flavor Physics from F-Theory, [arXiv:0904.1584]

[84] J. C. Callaghan, S. F. King, G. K. Leontaris, Graham G. Ross, Towards a Realistic F-theory GUT, [arXiv:1109.1399] Magnetized E3-brane instantons in F-theory, Massimo Bianchi, A. Collinucci, L. Martucci, [arXiv:1107.3732] A. P. Braun, A. Collinucci, R. Valandro, G-flux in F-theory and algebraic cycles [arXiv:1107.5337] M. Cvetic, I. Garcia-Etxebarria, J. Halverson, Global F-theory Models: Instantons and Gauge Dynamics, JHEP1101, 073 (2011)

[85] G. Junker, “Supersymmetric Methods in Quantum and Statistical Physics”, Springer, 1996

[86] M. Combescure, F. Gieres, M. Kibler, J. Phys. A: Math. Gen. 37 (2004) 10385

[87] V. K. Oikonomou, Witten index and superconducting strings, Mod. Phys. Lett. A, 25 (2010) 2611

[88] “The Dirac Equation”, Bernd Thaller, Springer 1992

[89] V.K. Oikonomou, F-theory and the Witten Index, Nucl. Phys. B850, 273 (2011)

[90] A. Pashnev, F. Toppan, J. Math. Phys. 42, 5257 (2001)

[91] F. Gesztesy, B. Simon, J. Funct. Anal. 79, 91 (1988)
[92] J.P. Conlon, E. Palti, "Aspects of Flavour and Supersymmetry in F-theory GUTs", JHEP, 1001, 029 (2010)