Growing Directed Networks: Estimation and Hypothesis Testing

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Based only on the information gathered in a snapshot of a directed network, we present a formal way of checking if the proposed model is correct for the empirical growing network under study. In particular, we show how to estimate the attractiveness, and present an application of the model presented in [1] to the scientific publications network from the ISI dataset.

PACS numbers: 05.65.+b, 89.75.Kd, 87.23.Ge, 02.50.r, 02.50.Cw

The study of networks has attracted many scientists during the last decade. The most studied directed growing networks are the WWW network [2], where each node represents a web page and the hyper-links (references to other web pages) represents the directed edges or links, and the scientific papers network [3], where each paper is a node, and its references the directed links. Many of the theoretical approaches try to determine whether a particular model provides a “good description” of the dynamics of the empirical growing network under study, paying special attention to the stationary state. Empirical growing networks shows that this state is characterized by the fact that it has a degree distribution with a power law tail, where the degree of a vertex is defined as the total number of its connections, and a large clustering coefficient, which is a measure of how much connected are the neighborhoods nodes of random selected node. These two descriptive statistical measures characterized in good way the network topology.

Typically, the way of checking if a model mimics the real growing network (once the clustering coefficient is near the empirical) is comparing the limit degree distribution with the empirical degree one, paying special attention in the tail of the distribution. In [1] it was shown that this measure is not a good one, since many different models can give the same tail (typically scale invariant). This way, two other informal checks were suggested [1] in order to have a first idea whether the model works well. The fist one is based on the relation between the variance of the out-degree random variable, $D_{\text{out}}$ and the in-out degree covariance, $Cov(D_{\text{in}},D_{\text{out}}) = E(D_{\text{in}}D_{\text{out}}) - E(D_{\text{in}})E(D_{\text{out}})$. Where $D_{\text{in}}$ is the in-degree of a randomly selected node. The second informal check is based on the relation between the tail of the in degree distribution and the tail of the out degree one. In [1] these two relation were studied in detail for the growing model shown in Fig. 1.

In this letter, we try to go further this informal checks proposing a way to test whether a particular model describes well the empirical directed network. Besides, we test the model presented in [1] to see if it describe the scientific citation network.

Let us now describe the growing network process (Fig. 1) presented in [1]: as time evolves new nodes with $D_{\text{out}}$ number of out-links appear, which connect to the existing nodes according to some probability law (uniform, preferential linking, etc.). $D_{\text{out}}$ is a random variable (or the number of out-links of a randomly selected node) with an arbitrary distribution, $P(D_{\text{out}} = j) = p_j$ with $j \in N$. For this model, the limit (in/out) degree distribution ($\nu_{\text{in}}^{k}$, $\nu_{\text{in}}^{k}$, $\nu_{\text{in}}^{k}$, $\nu_{\text{in}}^{k}$) was computed. Under preferential linking with attractiveness they get:

$$\nu_{\text{out}}^{k} = p_k$$

$$\nu_{\text{in}}^{k} = \sum_{j=1}^{\infty} p_j \Psi(j + k + A, 3 + \delta)$$

$$\nu_{\text{deg}}^{k} = \Psi(k + A, 3 + \delta) \sum_{j=1}^{k} p_j \Psi(j + A, 2 + \delta)$$

$$\nu_{\text{in},out}^{n-k,k} = \nu_{\text{deg},out}^{n-k,k} = \frac{\Psi(n + A, 3 + \delta)}{\Psi(k + A, 2 + \delta)} p_k$$

where $\Psi(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_{0}^{1} t^{a-1}(1-t)^{b-1} dt$, $E_o = \sum_{k=1}^{\infty} kp_k$, and $\delta = A/E_o$. One advantage of explicitly knowing the dependence of the $\nu_{\text{in}}^{k}$, $\nu_{\text{deg}}^{k}$ and $\nu_{\text{in},deg}^{k}$ distributions as functions of the out-degree distribution is that it is now possible to estimate the attractiveness using just the empirical marginal distributions. Let us suppose that we are given a growing network with a large number of nodes, whose distribution we assume close to the limit measure. We take a snapshot and from the information.
gathered in the picture we estimate: 1) the out-degree distribution \( p_k \) by its empirical law, 
\[
\hat{p}_k = \frac{X^k_{out}}{\sum_{k \in N} X^k_{out}},
\]
where \( X^k_{out} \) is the number of nodes with \( k \) out-links at the time the picture was taken, 2) similarly, the in-degree distribution \( \nu^j_{in} \) by \( \hat{\nu}^j_{in} = \frac{X^j_{in}}{\sum_{j \in N} X^j_{in}} \), and 3) \( E_o \), by \( \hat{E}_o = \sum k \hat{p}_k \). On the other hand, for each model, \( \nu^k_{in} \) be can also estimated from the computed limit in-degree distribution (eq. 1(b) for the model under study). In this case, we define \( \hat{\nu}^k_{in,A} \) as \( \nu^k_{in} \) of eq. 1(b) after replacing \( p_j \) by \( \hat{p}_j \) and \( \delta \) by \( A/E_o \). Evidently, \( \hat{\nu}^k_{in,A} \) depends on \( A \), and the proposed consistent attractiveness estimator \( \hat{A} \) becomes:
\[
\hat{A} = \arg\min_{A \in \mathbb{R}_{>0}} \{ \max_{k \in N} | F_{\nu_{in,A}}(k) - F_{\hat{\nu}_{in}}(k) | \} \quad (2)
\]
where \( F_{\nu_{in}}(k) \) is the cumulative distribution, \( F_{\nu_{in,A}}(k) = \sum_{j=0}^k \nu^j_{in} \). \( \hat{A} \) is a minimum-distance (for the distribution) estimator, but there exist many other methods for estimating \( A \) such as maximum likelihood or moment method. The advantage of the first method is discussed below. Alternatively, the \( A \) estimation can be carried out using the degree and the out-degree information \( (\hat{A} = \arg\min_{A \in \mathbb{R}_{>0}} \{ \max_{k \in N} | F_{\nu_{out,A}}(k) - F_{\hat{\nu}_{out}}(k) | \}) \), or the joint information. Clearly, an estimator based in the joint distribution is better since there exist different joint distributions with the same marginal ones. Now, we test the model to see if it mimics the scientific citation network. The model (Fig. 1) seems to have all the real ingredients of the real dynamics of the growing network. Each node represent a scientific publication and the directed links the citations. The main point that can be criticized is the attachment probability law: in the model it depends on the degree (number of papers that cite a random selected paper plus the number of citations of this article) of each node. We discuss another attachment law below.

Fig. 2 shows the citation distribution of all scientific publications in 1981, from the ISI dataset, which were cited between 1981 and 1997 (see [4]). Clearly, this represent the in-degree distribution for a growing network process. Unfortunately the out-degree distribution \( p_k \), the number of cites in a randomly selected paper, has not been reported, making a plug-in (see eq. 1) approach to test the growing model impossible. Nevertheless, we adopt the following strategy: we assume a geometric out-degree distribution \( p_k = p(1-p)^k \) with \( k \in \mathbb{N} \), a preferential linking with attractiveness attachment law, and estimate \( A \) and \( p \) by eq. 2. Clearly, the empirical out-degree distribution may not fall in any parametric family, however a good estimated in-degree distribution will be a very useful result, since the in-degree distribution is a theoretical calculation based on the out one. The \( T \) statistic achieves its minimum when \( A = 0 \) and \( p = 0.14 \) (see dashed line). If we are less ambitious, and disregard the first values of the distribution, but require a good match from the 10\( ^{th} \) citation onwards, we obtain that \( p = 0.1 \) and \( A = 0 \) work remarkably well. Note that the theoretical curve (solid line) is extremely similar to the empirical one in almost the whole range of the probability. Moreover, in this case the mean out-degree \( (E_o) \) is 9, which seems to be a good guess for the average number of cites for all the scientific publications.

As we discussed before, the attachment probability law can be criticized. Perhaps in a better model this law must depend on the in-degree (not on the degree) of each node. For this case, using the property 1 introduced in [1], it is very easy to compute the limit distributions:
\[
\nu^k_{out} = p_k \quad (a)
\]
\[
\nu^k_{in} = \frac{\Psi(k + A, 2 + \delta)}{\Psi(A, 1 + \delta)} \quad (b)
\]
\[
\nu^k_{deg} = \frac{1}{\Psi(A, 1 + \delta)} \sum_{j=0}^k p_j \Psi(k - j + A, 2 + \delta) \quad (c)
\]
\[
\nu^j_{in, out} = \nu^j_{deg, out} = \nu^j_{in} \nu^k_{out} \quad (d)
\]
where \( k,j \in \mathbb{N} \). This case is specially easy to solve because, for a randomly selected node, the number of out-links \( (D_{out}) \) and the number of in-links \( (D_{in}) \) are independent random variables \( (\nu^j_{in, out} = \nu^j_{in} \nu^k_{out}) \). Note that for this attachment law the attractiveness must be
strictly positive, since if \( A = 0 \) we get that the limit in-degree probability is \( \nu_k = \delta_{k=1} \). This result is easy to understand: new papers appear but they can not cite \((A = 0)\) papers without previous citations, and in this way the scientific network will be formed by almost all papers with zero citations and only a few very cited. Clearly, in the limit this goes to a delta function. Moreover, for \( k \gg 1 \), \( \nu_k \) behaves as \( k^{-(2+\delta)} \). Now, in the same way as before, we will find \( A \) and \( \delta \) such that \( T \) is minimum. This happens for \( A = 2.71 \), and \( \delta = 0.55 \) where \( T = 0.108 \) (see Fig. 3). If we are less are less ambitious, and disregard the first values of the distribution, but now require a good match from the 30th citation onwards, we obtain that \( \delta = 1 \) and \( A = 11 \) match well the tail.

Although we know that for both attachment probability laws presented the power exponent of the tail of the in-degree distribution grows linear with \( A \) (\( k^{-3+A/E_o} \) or \( k^{-2+A/E_o} \)), we do not have much intuition of what is the role of attractiveness over the in-degree distribution. In order to try to understand this, in Fig. 3 we show \( \nu_{in}^k \) for different values of the attractiveness. The role of the attractiveness is to make flatten the first values of the in-degree distribution. The same type of behavior is observed for the case where the law of attachment depends on the degree of the node (data not shown). Once we have the figure, the interpretation is immediate since for greater \( A \) the difference between selecting a node with \( \delta = 2 \) citations will be very similar to the one with 1 or 3 citations (\( (A-k) \nu_{in}^k/(E_o + A) \)).

In general a growing network model can be separated in two parts: A) the model in its own \((e.g. \text{in each temporal step a new node with } D_{out} \text{ number of links is aggregated})\), and B) the attachment probability law for the new links. We have previously shown two models that differ only on part B). It is important to check whether a proposed model \( A+B \) describes well the empirical data. If we are convinced that the part A) is correct for describing an empirical growing network, we can test if the attachment probability law is for example preferential linking \( \text{(or uniform)} \). In the case where you are convinced that the attachment law has some known law \( \text{(part B correct)} \), we can test if the part A) of the model is adequate. In the other case, it is also possible to test both hypothesis \( \text{(the model)} \), but if \( H_o \) is rejected we can not determine which \( A \) or \( B \) or both) is the incorrect.

One advantage of the proposed estimator in eq. [2] is that it is now very easy to test any of the hypothesis previously mentioned. For example \( H_o \); the real growing network has an underlying link attachment law that is preferential with attractiveness. This hypothesis can be tested \( [10] \) with the usual statistic, \( T = \max_{k \in E_o} |F_{\nu_{in},A}(k) - F_{\nu_{in}}(k)| \). This test is one of the main result presented here.

Another important issue is to be able to rank the models. For example, in \( [11] \) a nice growing model was proposed for the WWW dynamics. In this model, with probability \( p \) a new node with only one out-link is aggregated, or a new directed link from an existing node is created \( \text{(with probability } (1-p) \text{)} \). This is the part A) of the model, and the following constitute the part B) of the model. The new link from the new node is attached by preferential linking with attractiveness for the in-degree, \( \pi_{in}^k = \frac{k+\lambda}{k+\lambda + A} \nu_k \) (in \( [11] \) they use \( \lambda \) instead of \( A \) and works with rates and not with probability). And the selection of the new created link, has two independent events: 1) the selection, by preferential linking with attractiveness for the out-degree, of the origin, 2) and the selection of the target by preferential linking with attractiveness \( \text{(with a different parameter from the previous one)} \) for the in-degree. An alternative model can be the one proposed for the scientific network \( \text{(see Fig. 1)} \), where now the nodes represent the web pages and the links the hyper-links. Clearly, both models have some weak points. In the first one \( [1] \), people can not put more than one hyper-link when they are constructing their own web page \( \text{(later some new links can appear)} \). The second model \( \text{(Fig. 1)} \) do not have this inconvenient, but is “static”, once the hyper-links are fixed they can not be changed. Probably, a mixed model between both be more realistic. How to compare or rank these model is a relevant question in order to approach to the “real model”. There exit many statistical measures that do this job, in particular there is an extent bibliography for Hidden Markov chain problems, and also for linear regression models. We propose a very simple ranking variable that is the value of \( T \) using
the (cumulative) joint distribution. The best model is the one that has the minimum value of $T$. Much work must be done to understand how must be the penalization (if it is necessary) for models with many parameters.

In summary, we discussed: 1) how to estimate model parameters, showing an application to the scientific publications network, and 2) a way of checking whether a proposed model is correct, based on the limit (joint) in and out degree distribution. This way, the results presented here shed some light on the problem of estimating the underlying attachment law, ranking models, and test models in a general way.

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