A Stronger Multiple Exchange Property for M♮-concave Functions

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Abstract

The multiple exchange property for matroid bases has recently been generalized for valuated matroids and M♮-concave set functions. This paper establishes a stronger form of this multiple exchange property that imposes a cardinality condition on the exchangeable subset. The stronger form immediately implies the defining exchange property of M♮-concave set functions, which was not the case with the recently established multiple exchange property without the cardinality condition.

Keywords: Discrete convex analysis, Matroid, Exchange property, Combinatorial optimization

1 Introduction

The concept of M♮-concave functions in discrete convex analysis \[3, 8, 9, 12\] has found applications in mathematical economics and game theory; see \[8, Chapter 11\], \[15, 16\], and recent survey papers \[10, 14\]. M♮-concavity of a set function \(f\) is defined in terms of the exchange property that, for any subsets \(X, Y\) and any element \(i \in X \setminus Y\), at least one of (i) and (ii) holds, where (i) \(f(X) + f(Y) \leq f(X \setminus \{i\}) + f(Y \cup \{i\})\) or (ii) there exists some \(j \in Y \setminus X\) such that \(f(X) + f(Y) \leq f((X \setminus \{i\}) \cup \{j\}) + f((Y \cup \{i\}) \setminus \{j\})\).

It has been shown recently in \[11\] that an M♮-concave set function \(f\) has the multiple exchange property that, for any subsets \(X, Y\) and a subset \(I \subseteq X \setminus Y\), there exists a subset \(J \subseteq Y \setminus X\) such that \(f(X) + f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)\). This result has an economic significance that the gross substitutes (GS) condition of Kelso and Crawford \[5\] is in fact equivalent to the strong no complementarities (SNC) condition of Gul and Stacchetti \[4\]. In the special case of M-concave functions, this multiple exchange property gives a quantitative generalization of a classical result in matroid theory (\[6, 13, Section 39.9a\]) that the basis family of a matroid enjoys the multiple exchange property, which says that, for two bases \(X\) and \(Y\) in a matroid and a subset \(I \subseteq X \setminus Y\), there exists a subset \(J \subseteq Y \setminus X\) such that \((X \setminus I) \cup J\) and \((Y \setminus J) \cup I\) are both bases.

The objective of this paper is to establish a stronger form of the multiple exchange property that imposes a cardinality condition \(|J| \leq |I|\) on the exchangeable subset \(J\). The stronger form immediately implies the defining exchange property of M♮-concave set functions, which is not the case with the multiple exchange property of \[11\] without the cardinality condition. The results are described in Section 2 and two alternative proofs are given in Sections 3 and 4.

2 Results

Let \(N\) be a finite set, say, \(N = \{1, 2, \ldots, n\}\). For a function \(f : 2^N \to \mathbb{R} \cup \{-\infty\}\), \(\text{dom } f\) denotes the effective domain of \(f\), i.e., \(\text{dom } f = \{X \mid f(X) > -\infty\}\).
A function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ with $\text{dom} f \neq \emptyset$ is called $M^t$-concave if, for any $X, Y \in \text{dom} f$ and $i \in X \setminus Y$, we have (i) $X - i \in \text{dom} f$, $Y + i \in \text{dom} f$, and
\[
f(X) + f(Y) \leq f(X - i) + f(Y + i),
\]
or (ii) there exists some $j \in Y \setminus X$ such that $X - i + j \in \text{dom} f$, $Y + i - j \in \text{dom} f,$ and
\[
f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j).
\]

Here we use short-hand notations $X = X \setminus \{i\}$, $Y = Y \cup \{i\}$, $X - i + j = (X \setminus \{i\}) \cup \{j\}$, and $Y + i - j = (Y \cup \{i\}) \setminus \{j\}$. This property is referred to as the exchange property. The exchange property can be expressed more compactly as:

\[
(M^t-\text{EXC}) \quad \text{For any } X, Y \subseteq N \text{ and } i \in X \setminus Y, \text{ we have}
\]
\[
f(X) + f(Y) \leq \max_{j \in Y \setminus X} \left[ f(X - i) + f(Y + i) \right],
\]
where $(-\infty) + a = a + (-\infty) = (-\infty) + (-\infty) = -\infty$ for $a \in \mathbb{R}$, $-\infty \leq -\infty$, and a maximum taken over an empty set is defined to be $-\infty$.

The multiple exchange property means the following more general form of $(M^t-\text{EXC})$:

\[
(M^t-\text{EXC}_m) \quad \text{For any } X, Y \subseteq N \text{ and } I \subseteq X \setminus Y, \text{ we have}
\]
\[
f(X) + f(Y) \leq \max_{J \subseteq Y \setminus X} \left[ f((X \setminus I) \cup J) + f((Y \setminus J) \cup I) \right].
\]

Here we may specify any subset $I$, rather than a single element $i$, in $X \setminus Y$, and we can always find an exchangeable subset $J \subseteq Y \setminus X$. It has recently been shown [11] that $(M^t-\text{EXC})$ and $(M^t-\text{EXC}_m)$ are equivalent.

**Theorem 1** ([11]). A function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ is $M^t$-concave if and only if it has the multiple exchange property $(M^t-\text{EXC}_m)$.

The content of this theorem lies in the implication “$(M^t-\text{EXC}) \Rightarrow (M^t-\text{EXC}_m)$.” It is emphasized, however, that “$(M^t-\text{EXC}_m) \Rightarrow (M^t-\text{EXC})$” is not obvious and a separate proof is needed also for this direction, though the proof [11] Section 5.2] is straightforward.

In this paper we are interested in a stronger form of the multiple exchange property, in which an additional condition $|I| \leq |J|$ is imposed on the exchangeable subset $J$:

\[
(M^t-\text{EXC}_m^*) \quad \text{For any } X, Y \subseteq N \text{ and } I \subseteq X \setminus Y, \text{ we have}
\]
\[
f(X) + f(Y) \leq \max_{J \subseteq Y \setminus X, |J| \leq |I|} \left[ f((X \setminus I) \cup J) + f((Y \setminus J) \cup I) \right].
\]

The following theorem, the main result of this paper, states that $(M^t-\text{EXC})$ implies the stronger multiple exchange property $(M^t-\text{EXC}_m^*)$ with cardinality requirement.

**Theorem 2.** Every $M^t$-concave function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$ has the stronger multiple exchange property $(M^t-\text{EXC}_m^*)$ with cardinality requirement.

**Proof.** Two alternative proofs are given in Sections 3 and 4. The first proof is a self-contained direct proof, being a refinement of the argument in [11] for (the only-if part of) Theorem [1], whereas the second makes use of (the only-if part of) Theorem [1] through a transformation of an $M^t$-concave function to a valuated matroid. □ □

The stronger form $(M^t-\text{EXC}_m^*)$ immediately implies $(M^t-\text{EXC})$ as a special case with $|I| = 1$, whereas $(M^t-\text{EXC}_m^*)$ obviously implies $(M^t-\text{EXC}_m)$. Therefore, we obtain the equivalence of the three exchange properties as a corollary of Theorems [1] and [2].

**Corollary 1.** For a function $f : 2^N \to \mathbb{R} \cup \{-\infty\}$, the three conditions $(M^t-\text{EXC})$, $(M^t-\text{EXC}_m)$, and $(M^t-\text{EXC}_m^*)$ are equivalent.
3 The first proof of Theorem \[2\]

In this section we give a self-contained direct proof of Theorem \[2\]. This is a refinement of the argument in \[11\] for (the only-if part of) Theorem \[1\].

The proof is based on the Fenchel-type duality theorem in discrete convex analysis (\[17\] Theorem 3.1, \[8\], Theorem 8.21 (1)), which is stated below in a form convenient for our use.

**Theorem 3** (Fenchel-type duality). Let \(f_1, f_2 : 2^N \to \mathbb{R} \cup \{-\infty\} \) be \(M^2\)-concave functions, and \(g_1, g_2 : \mathbb{R}^N \to \mathbb{R}\) be their (convex) conjugate functions defined by \(g_i(q) = \max_{J \subseteq N} \{f_i(J) - \sum_{j \in J} q_j\} \) \((i = 1, 2)\) for \(q \in \mathbb{R}^N\). Then \[\]

\[
\max_{J \subseteq N} \{f_1(J) + f_2(J)\} = \inf_{q \in \mathbb{R}^N} \{g_1(q) + g_2(-q)\},
\]

where the maximum on the left-hand side is defined to be \(-\infty\) if \(f_1 \cap \text{dom} f_2 = \emptyset\). If \(f_1\) and \(f_2\) are integer-valued, the vector \(q\) can be restricted to integers.

We also need the following consequence of the exchange property (\(M^2\)-EXC).

**Lemma 1.** If \(f\) satisfies (\(M^2\)-EXC), then, for any \(X, Y\) with \(|X| \leq |Y|\) and \(i \in X \setminus Y\), there exists \(j \in Y \setminus X\) such that \(f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)\).

**Proof.** This is a direct translation of the exchange property (ii) of (\(M^2\)-EXC) given in \[12\] Theorem 4.2 for \(M^2\)-convex function on \(\mathbb{Z}^N\). \(\square\)

We prove Theorem 2 in Sections 3.1 to 3.3. In Section 3.1 the stronger multiple exchange property (\(M^2\)-EXC\(_m\)) is reformulated in terms of the conjugate functions by the Fenchel-type duality. The submodularity of the conjugate functions is revealed in Section 3.2 and the dual objective function is evaluated in Section 3.3.

### 3.1 Translation to the conjugate functions

Let \(f : 2^N \to \mathbb{R} \cup \{-\infty\}\) be an \(M^2\)-concave function, \(X, Y \in \text{dom} f\) and \(I \subseteq X \setminus Y\). To express the size constraint with bound \(k\), we define \(\beta(J; k) = 0\) if \(|J| \leq k\) and \(\beta(J; k) = -\infty\) otherwise. We shall prove

\[
f(X) + f(Y) \leq \max_{I \subseteq Y \setminus X} \{f((X \setminus I) \cup J) + f((Y \setminus J) \cup I) + \beta(J; |I|)\}, \tag{3.2}
\]

which is equivalent to (2.5). With the notations

\[
C = X \cap Y, \quad X_0 = X \setminus Y = X \setminus C, \quad Y_0 = Y \setminus X = Y \setminus C, \tag{3.3}
\]

\[
f_1(J) = f((X \setminus I) \cup J) \quad (J \subseteq Y_0), \tag{3.4}
\]

\[
f_1(J) = f_1(J) + \beta(J; |I|) = f((X_0 \setminus I) \cup C \cup J) \quad (J \subseteq Y_0), \tag{3.5}
\]

\[
f_2(J) = f((Y \setminus J) \cup I) = f(I \cup C \cup (Y_0 \setminus J)) \quad (J \subseteq Y_0), \tag{3.6}
\]

the inequality (3.2) is rewritten as

\[
f(X) + f(Y) \leq \max_{J \subseteq Y_0} \{f_1(J) + f_2(J)\}. \tag{3.7}
\]

**Lemma 2.** (1) \(\text{dom} f_1, \text{dom} f_1, \text{and} \text{dom} f_2\) are nonempty.
(2) \(f_1, f_1, \text{and} f_2\) are \(M^2\)-concave functions.

\(^1\)The assumption \(\text{dom} g_1 \cap \text{dom} g_2 \neq \emptyset\) in \[8\] Theorem 8.21 (1) is satisfied, since \(\text{dom} g_1 = \text{dom} g_2 = \mathbb{R}^N\).
Proof. (1) We prove \( \text{dom} \tilde{f}_1 \neq \emptyset \) and \( \text{dom} f_1 \neq \emptyset \) by showing

there exists \( J \subseteq Y \setminus X \) such that \((X \setminus J) \cup J \in \text{dom} f \) and \( |J| \leq |I| \) \hspace{1cm} (3.8)

by induction on \( |I| \). If \( |I| = 0 \), (3.8) holds trivially with \( J = \emptyset \). Suppose \( |I| \geq 1 \) and \( I = I' + i \) with \( i \not\in I' \). By the induction hypothesis there exists \( J' \subseteq Y \setminus X \) such that \((X \setminus I') \cup J' \in \text{dom} f \) and \(|J'| \leq |I'| \). By \((M^3\text{-EXC})\) for \((X', Y, i)\), (i) \( X' - i \in \text{dom} f \) or (ii) there exists \( j \in Y \setminus X' \subseteq Y \setminus X \) such that \( X' - i + j \in \text{dom} f \). In case (i), we set \( J = J' + j \) to obtain \(|J| = |J'| + 1 \leq |I'| + 1 = |I| \) and \((X \setminus I) \cup J = (X \setminus (I' + i)) \cup J' = X' - i \in \text{dom} f \). In case (ii), we set \( J = J' + j \) to obtain \(|J| = |J'| + 1 \leq |I'| + 1 = |I| \) and \((X \setminus I) \cup J = (X \setminus (I' + i)) \cup (J' + j) = X' - i + j \in \text{dom} f \). Thus (3.8) is proved.

To prove \( \text{dom} f_2 \neq \emptyset \), we show

there exists \( J \subseteq Y \setminus X \) such that \((Y \setminus I) \cup I \in \text{dom} f \) \hspace{1cm} (3.9)

by induction on \(|I| \) (by almost the same argument as above). If \(|I| = 0 \), (3.9) holds trivially with \( J = \emptyset \). Suppose \(|I| \geq 1 \) and \( I = I' + i \) with \( i \notin I' \). By the induction hypothesis there exists \( J' \subseteq Y \setminus X \) such that \( Y' := (Y \setminus J') \cup I' \in \text{dom} f \). By \((M^3\text{-EXC})\) for \((X', Y', i)\), (i) \( Y' - i \in \text{dom} f \) or (ii) there exists \( j \in Y' \setminus X' \subseteq Y \setminus X \) such that \( Y' - i + j \in \text{dom} f \). In case (i), we set \( J = J' \) to obtain \((Y \setminus J) \cup J = (Y \setminus J') \cup (I' + i) = Y' - i \in \text{dom} f \). In case (ii), we set \( J = J' + j \) to obtain \((Y \setminus J) \cup I = (Y \setminus J') \cup (I' + i) = Y' + j \in \text{dom} f \). Thus (3.9) is proved.

(2) For \( f_1 \) and \( f_2 \), the \( M^3 \)-concavity is easy to see from \((M^3\text{-EXC})\) of \( f \). Then the function \( \tilde{f}_1 \), being a restriction of \( f_1 \), is also \( M^3 \)-concave. \( \Box \) \hspace{1cm} \( \Box \)

Consider the (convex) conjugate functions of \( \tilde{f}_1 \) and \( f_2 \) given by

\[
\tilde{g}_1(q) = \max_{J \subseteq Y} \{ \tilde{f}_1(J) - q(J) \}, \hspace{1cm} (q \in \mathbb{R}^N),
\]

\[
g_2(q) = \max_{J \subseteq Y} \{ f_2(J) - q(J) \}, \hspace{1cm} (q \in \mathbb{R}^N),
\]

where \( q(J) = \sum_{j \in J} q_j \). By Theorem 3, the desired inequality \( \tilde{g}_1 + g_2 \) can be rewritten as

\[
f(X) + f(Y) \leq \inf_{q \in \mathbb{R}^N} \{ \tilde{g}_1(q) + g_2(-q) \}. \hspace{1cm} (3.12)
\]

### 3.2 Submodularity

To compute \( \tilde{g}_1(q) + g_2(-q) \) in (3.12), we relate \( \tilde{g}_1 \) and \( g_2 \), respectively, to

\[
\tilde{g}(p) = \max_{Z \subseteq N} \{ f(Z) + \beta(Z; |X|) - p(Z) \}, \hspace{1cm} (p \in \mathbb{R}^N),
\]

\[
g(p) = \max_{Z \subseteq N} \{ f(Z) - p(Z) \}, \hspace{1cm} (p \in \mathbb{R}^N).
\]

We use notation \( f[-p](Z) = f(Z) - p(Z) \) for \( Z \subseteq N \).

Since \( f(Z) + \beta(Z; |X|) \) and \( f(Z) \) are \( M^3 \)-concave, the conjugacy theorem in discrete convex analysis (8 Theorems 8.4, (8.10)], [9 Theorem 3.4]) shows that both \( \tilde{g} \) and \( g \) are \( L^3 \)-convex functions on \( \mathbb{R}^N \). In particular, they are submodular:

\[
\tilde{g}(p) + \tilde{g}(p') \geq \tilde{g}(p \lor p') + \tilde{g}(p \land p') \hspace{1cm} (p, p' \in \mathbb{R}^N), \hspace{1cm} (3.15)
\]

\[
g(p) + g(p') \geq g(p \lor p') + g(p \land p') \hspace{1cm} (p, p' \in \mathbb{R}^N), \hspace{1cm} (3.16)
\]

where \( p \lor p' \) and \( p \land p' \) denote, respectively, the vectors of component-wise maximum and minimum of \( p \) and \( q \).

For our proof we need the following form of submodularity across \( \tilde{g} \) and \( g \).

**Lemma 3.** \( \tilde{g}(p) + g(q) \geq \tilde{g}(p \land q) + g(p \lor q) \hspace{1cm} (p, q \in \mathbb{R}^N). \)
Proof. It follows from Lemma 4 below and (3.16) that
\[ \hat{g}(p) - \hat{g}(p \land q) \geq g(p) - g(p \land q) \geq g(p \lor q) - g(q). \]
which is equivalent to the claim. \qed 

Lemma 4. For any \( p, q \in \mathbb{R}^N \) with \( p \geq q \), it holds\footnote{\ref{3.17} means a kind of strong quotient relation.}
\[ \hat{g}(p) - \hat{g}(q) \geq g(p) - g(q). \] \hspace{1cm} (3.17)

Proof. The assertion (3.17) is equivalent to the monotonicity of \( \hat{g}(p) - g(p) \) in \( p \). To prove this it suffices to show that for each \( q \in \mathbb{R}^N \) there exists a positive number \( \varepsilon(q) > 0 \) such that (3.17) holds for all \( p \in \mathbb{R}^N \) of the form
\[ p = q + \alpha \chi_k \] \hspace{1cm} (3.18)
with \( 0 \leq \alpha < \varepsilon(q) \), where \( \chi_k \) denotes the \( k \)th unit vector for \( k \in N \). We will show that the minimum of the nonzero absolute values of \( f[-q](Z_1) + f[-q](Z_2) - f[-q](Z_3) - f[-q](Z_4) \) over all \( Z_1, Z_2, Z_3, Z_4 \subseteq N \) serves as such \( \varepsilon(q) \). We define
\[ \varepsilon(q) = \min \{|f[-q](Z_1) + f[-q](Z_2) - f[-q](Z_3) - f[-q](Z_4)| \neq 0 | Z_1, Z_2, Z_3, Z_4 \subseteq N \}. \]
Recalling (3.13) and (3.14), denote \( m = |X| \) and take \( U \) and \( W \) such that
\[ g(p) = f(U) - p(U), \quad \hat{g}(q) = f(W) - q(W), \quad |W| \leq m. \]
We choose such \( U, W \) with minimum \( |W \setminus U| \). Then (3.17) is rewritten as
\[ [f(U) - p(U)] + [f(W) - q(W)] \leq \hat{g}(p) + g(q). \] \hspace{1cm} (3.19)
This inequality can be shown as follows.

- If \( |U| \leq m \), we have \( f(U) - p(U) \leq \hat{g}(p) \) by (3.13) as well as \( f(W) - q(W) \leq g(q) \) by (3.14). Hence (3.19) holds.

- If \( W \subseteq U \), then \( p(U) + q(W) \geq p(W) + q(U) \) by \( p \geq q \), and hence\[ [f(U) - p(U)] + [f(W) - q(W)] \leq [f(W) - p(W)] + [f(U) - q(U)] \leq \hat{g}(p) + g(q), \]which shows (3.19).

- The remaining case, where \( |U| > m \) and \( W \setminus U \neq 0 \), is excluded by the minimality of \( |W \setminus U| \), as shown below.

Suppose that \( |U| > m \) and \( W \setminus U \neq 0 \). Then \( |U| > m \geq |W| \). Take any \( i \in W \setminus U \), which is possible since \( W \setminus U \neq 0 \). By Lemma 3 there exists \( j \in U \setminus W \) such that
\[ f(W) + f(U) \leq f(W - i + j) + f(U + i - j) = f(W') + f(U'), \] \hspace{1cm} (3.20)
where \( W' = W - i + j \) and \( U' = U + i - j \). Note that \( |W'| = |W| \leq m. \)

- Case of \( k \notin W \setminus U \): Since \( i \neq k \), we have \( p_i = q_i \) and \( p_j \geq q_j \). Then, by (3.20), we have\[ [f(U) - p(U)] + [f(W) - q(W)] \leq [f(U + i - j) - p(U + i - j)] + [f(W - i + j) - q(W - i + j)] = [f(U') - p(U')] + [f(W') - q(W')]. \]
Since \( |W'| = |W| \leq m \), this means \( f(W') - q(W') = f(W) - q(W) \) as well as \( f(U') - p(U') = f(U) - p(U) \), whereas \( W' \setminus U = (W \setminus U) - i \). This is a contradiction to the minimality of \( |W \setminus U| \).
• Case of \( k \in W \setminus U \): We choose \( i = k \) in (3.20) and rewrite (3.20) as

\[
f[-q](W) + f[-q](U) \leq f[-q](W') + f[-q](U').
\] (3.21)

Here we have

\[
f[-q](W') \leq f[-q](W),
\] (3.22)

\[
f[-q](U') - \alpha = f[-p](U') \leq f[-p](U) = f[-q](U)
\] (3.23)

by the definitions of \( W \) and \( U \), (3.18), \( k \in U' \), and \( k \notin U \). Hence the difference of both sides of (3.21) is at most \( \alpha \), whereas \( \alpha < \epsilon(q) \). Hence we have equality in (3.21), and therefore \( f[-q](W') = f[-q](W) \) in (3.22). This is a contradiction to the minimality of \( |W \setminus U| \), since \( |W' \setminus U| < |W \setminus U| \).

\[\square\] \[\square\]

### 3.3 Evaluation of the Fenchel dual

The desired inequality (3.12) follows from the following lemma, whose proof uses Lemma 3.

**Lemma 5.** For any \( q \in \mathbb{R}^I \), we have \( \tilde{g}_1(q) + g_2(-q) \geq f(X) + f(Y) \).

**Proof.** For a vector \( q \in \mathbb{R}^I \) we define \( p^{(1)}, p^{(2)} \in \mathbb{R}^N \) by

\[
p^{(1)}_i = p^{(2)}_i = \begin{cases} q_i & (i \in Y_0), \\ -M & (i \in C), \\ +M & (i \in N \setminus (X \cup Y)), \end{cases}
\]

where \( M \) is a sufficiently large positive number.

The maximizer \( Z \) of \( \tilde{g}(p) \) in (3.13) for \( p = p^{(1)} \) must avoid \( I \) and include \( (X_0 \setminus I) \cup C \). Hence \( Z = (X_0 \setminus I) \cup C \cup J \) for some \( J \subseteq Y_0 \), and then

\[
|Z| \leq |X| \iff |J| \leq |I|,
\]

\[
p^{(1)}(Z) = -M(|X_0 \setminus I| + |C|) + q(J).
\]

Therefore, we have

\[
\tilde{g}(p^{(1)}) = \max_{Z \subseteq N} \{ f(Z) + \beta(Z; |X|) - p^{(1)}(Z) \}
= \max_{J \subseteq Y_0} \{ f((X_0 \setminus I) \cup C \cup J) + \beta(J; |I|) - q(J) \} + M(|X_0 \setminus I| + |C|)
= \tilde{g}_1(q) + M(|X_0 \setminus I| + |C|).
\] (3.24)

The maximizer \( Z \) of \( g(p) \) in (3.14) for \( p = p^{(2)} \) must include \( I \cup C \) and avoid \( X \setminus (I \cup C) \). Hence \( Z = I \cup C \cup (Y_0 \setminus J) \) for some \( J \subseteq Y_0 \), and then

\[
p^{(2)}(Z) = -M(|I| + |C|) + q(Y_0 \setminus J).
\]

Therefore, we have

\[
g(p^{(2)}) = \max_{Z \subseteq N} \{ f(Z) - p^{(2)}(Z) \}
= \max_{J \subseteq Y_0} \{ f(I \cup C \cup (Y_0 \setminus J)) + q(J) \} - q(Y_0) + M(|I| + |C|)
= g_2(q) - q(Y_0) + M(|I| + |C|).
\] (3.25)
By adding (3.24) and (3.25) we obtain
\[ g_1(q) + g_2(-q) = \tilde{g}(p^{(1)}) + g(p^{(2)}) - M(|X| + |C|) + q(Y_0). \] (3.26)

By Lemma 3 we have
\[ \tilde{g}(p^{(1)}) + g(p^{(2)}) \geq \tilde{g}(p^{(1)} \land p^{(2)}) + g(p^{(1)} \lor p^{(2)}). \] (3.27)

Since
\[
\begin{align*}
(p^{(1)} \lor p^{(2)})_i &= (p^{(1)} \land p^{(2)})_i = \begin{cases} 
q_i & (i \in Y_0), \\
-M & (i \in C), \\
+M & (i \in N \setminus (X \cup Y)), 
\end{cases} \\
(p^{(1)} \lor p^{(2)})_i &= -(p^{(1)} \land p^{(2)})_i = +M & (i \in X_0),
\end{align*}
\]

we have
\[ \tilde{g}(p^{(1)} \land p^{(2)}) \geq f(X) + M|X|, \] (3.28)
\[ g(p^{(1)} \lor p^{(2)}) \geq f(Y) - q(Y_0) + M|C|, \] (3.29)

where (3.28) follows from (3.13) with \( Z = X \) and (3.29) follows from (5.14) with \( Z = Y. \) The combination of (3.26), (3.27), (3.28), and (3.29) yields the desired inequality \( g_1(q) + g_2(-q) \geq f(X) + f(Y). \)

We have thus completed the proof of Theorem 2.

Remark 3.1. For an integer-valued function \( f : 2^N \to \mathbb{Z} \cup \{-\infty\}, \) the above proof can be made purely discrete. In particular, the integrality in the Fenchel-type duality in Theorem 3 allows us to assume \( p \) and \( q \) to be integer vectors. In the proof of Lemma 4 we assume \( p = q + \chi_k, \) with \( \alpha = 1 \) in (3.18). At the end of the proof of Lemma 4 in the case where \( |U| > m \) and \( k \in W \setminus U, \) the inequalities (3.21), (3.22), and (3.23) together with integrality yield at least one of the following: (i) \( f[-q](W') = f[-q](W) \) and (ii) \( f[-p](U') = f[-p](U). \) This is a contradiction to the minimality of \( |W \setminus U|, \) since in case (i) we can replace \( W \) to \( W' \) to obtain \( |W' \setminus U| < |W \setminus U|, \) and in case (ii) we can replace \( U \) to \( U' \) to obtain \( |W \setminus U'| < |W \setminus U|. \)

4 The second proof of Theorem 2

The second proof transforms a given \( M^3\)-concave function \( f \) to an \( M \)-concave function (valuated matroid) \( \hat{f} \), and then applies the only-if part of Theorem 1 to \( \hat{f} \) in its special case for \( M \)-concave functions.

A function \( f : 2^N \to \mathbb{R} \cup \{-\infty\} \) with \( \text{dom } f \neq \emptyset \) is called an \( M \)-concave function (valuated matroid [1][2]) if, for any \( X, Y \subseteq N \) and \( i \in X \setminus Y, \) it holds that
\[ f(X) + f(Y) \leq \max_{j \in Y \setminus X} (f(X - i + j) + f(Y + i - j)). \] (4.1)

We can also say that an \( M \)-concave function is nothing but an \( M^3\)-concave function \( f \) such that dom \( f \) consists of equi-cardinal subsets, i.e., \( |X| = |Y| \) for any \( X, Y \in \text{dom } f. \) Therefore, Theorem 1 in this special case shows that every \( M \)-concave function has the multiple exchange property (\( M^3\)-EXC\(_m\)) with the additional condition \( |J| = |I|. \)

Let \( f : 2^N \to \mathbb{R} \cup \{-\infty\} \) be an \( M^3\)-concave function. Denote by \( r \) and \( s \) the maximum and minimum, respectively, of \( |X| \) for \( X \in \text{dom } f, \) and define \( S = \{n + 1, n + 2, \ldots, n + (r - s)\} \) and \( \hat{N} = N \cup S = \{1, 2, \ldots, \hat{n}\}, \) where \( \hat{n} = n + (r - s). \) Define \( \hat{f} : 2^\hat{N} \to \mathbb{R} \cup \{-\infty\} \) by
\[ \hat{f}(Z) = \begin{cases} 
f(Z \cap N) & (|Z| = r), \\
-\infty & \text{(otherwise)}. 
\end{cases} \] (4.2)
That is, for \( X \subseteq N \) and \( U \subseteq S \), we have \( \hat{f}(X \cup U) = f(X) \) if \( |U| = r - |X| \). By Lemma 6 below, \( \hat{f} \) is an M-concave function.

Suppose that we are given \( X, Y \in \text{dom} \ f \) and a subset \( I \subseteq X \setminus Y \). Take any \( U, W \subseteq S \) with \( |U| = r - |X| \) and \( |W| = r - |Y| \). Then \( X \cup U, Y \cup W \in \text{dom} \hat{f} \) and \( I \subseteq (X \cup U) \setminus (Y \cup W) \). By Theorem 1 for \( \hat{f} \), there exists \( J \subseteq Y \setminus X \) and \( V \subseteq W \setminus U \) such that

\[
\hat{f}(X \cup U) + \hat{f}(Y \cup W) \\
\leq \hat{f}((X \setminus I) \cup (U \cup V)) + \hat{f}((Y \setminus J) \cup (W \setminus V)),
\]

which implies \( f(X) + f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I) \). Since \( \text{dom} \hat{f} \) consists of equi-cardinal sets, we must have \( |I| = |J| + |V| \), which shows \( |I| \geq |J| \).

**Lemma 6.** For an \( M^2 \)-concave function \( f \), the function \( \hat{f} \) in (4.2) is \( M \)-concave.

**Proof.** Let \( X, Y \in \text{dom} f \) and \( U, W \subseteq S \) with \( |U| = r - |X| \) and \( |W| = r - |Y| \). The exchange property for \( \hat{f} \) amounts to the following:

- For any \( i \in X \setminus Y \) there exists \( j \in Y \setminus X \) with (4.3) or \( j \in W \setminus U \) with (4.4), where
  \[
  \hat{f}(X \cup U) + \hat{f}(Y \cup W) \leq \hat{f}((X - i) \cup J) + \hat{f}((Y - j) \cup W), \tag{4.3}
  \]
  \[
  \hat{f}(X \cup U) + \hat{f}(Y \cup W) \leq \hat{f}((X - i) \cup (U + j)) + \hat{f}((Y + i) \cup (W - j)). \tag{4.4}
  \]

- For any \( i \in U \setminus W \) there exists \( j \in Y \setminus X \) with (4.5) or \( j \in W \setminus U \) with (4.6), where
  \[
  \hat{f}(X \cup U) + \hat{f}(Y \cup W) \leq \hat{f}((X + j) \cup (U - i)) + \hat{f}((Y - j) \cup (W + i)), \tag{4.5}
  \]
  \[
  \hat{f}(X \cup U) + \hat{f}(Y \cup W) \leq \hat{f}((X - i) \cup (U + j)) + \hat{f}(Y \cup (W + i - j)). \tag{4.6}
  \]

The exchange properties above can be shown as follows. For any \( i \in X \setminus Y \), we have (2.1) or (2.2). In case of (2.2) we obtain (4.3). In case of (2.1) we obtain (4.4) for any \( j \in W \setminus U \), if \( W \setminus U \) is nonempty. If \( W \setminus U \) is empty, then \( |X| \leq |Y| \) and we have (4.4) by Lemma 1. Next, take any \( i \in U \setminus W \). If \( W \setminus U \) is nonempty, (4.6) holds for any \( j \in W \setminus U \). If \( W \setminus U \) is empty, we have \( |U| > |W| \) and hence \( |X| < |Y| \).

Then Lemma 7 below shows (4.5). □ □

**Lemma 7.** If \( f \) satisfies (\( M^2 \)-EXC), then, for any \( X, Y \) with \( |X| < |Y| \), there exists \( j \in Y \setminus X \) such that \( f(X) + f(Y) \leq f(X + j) + f(Y - j) \).

**Proof.** This is a direct translation of the exchange property (i) of (\( M^2 \)-EXC\(_p\)) given in [12, Theorem 4.2] for \( M^2 \)-convex function on \( \mathbb{Z}^N \). □ □

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