On the performance of optimal double circulant even codes

T. Aaron Gulliver∗ and Masaaki Harada†

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Abstract

In this note, we investigate the performance of optimal double circulant even codes which are not self-dual, as measured by the decoding error probability in bounded distance decoding. To do this, we classify the optimal double circulant even codes that are not self-dual which have the smallest weight distribution for lengths up to 72. We also give some restrictions on the weight enumerators of (extremal) self-dual [54, 27, 10] codes with shadows of minimum weight 3. Finally, we consider the performance of extremal self-dual codes of lengths 88 and 112.

1 Introduction

A (binary) [n, k] code C is a k-dimensional vector subspace of Fn, where F2 denotes the finite field of order 2. All codes in this note are binary. The parameter n is called the length of C. The weight wt(x) of a vector x ∈ Fn is the number of non-zero components of x. A vector of C is called a codeword. The minimum non-zero weight of all codewords in C is called the minimum weight of C and an [n, k] code with minimum weight d is called an [n, k, d]

∗Department of Electrical and Computer Engineering, University of Victoria, P.O. Box 1700, STN CSC, Victoria, BC, Canada V8W 2Y2. email: agullive@ece.uvic.ca
†Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980–8579, Japan. email: mharada@m.tohoku.ac.jp
code. A code with only even weights is called even. Two codes are equivalent if one can be obtained from the other by a permutation of coordinates.

Let \( C \) be an \([n, k, d]\) code. Throughout this note, let \( A_i \) denote the number of codewords of weight \( i \) in \( C \). The sequence \((A_0, A_1, \ldots, A_n)\) is called the weight distribution of \( C \). A code \( C \) of length \( n \) is said to be formally self-dual if \( C \) and \( C^\perp \) have identical weight enumerators, where \( C^\perp \) is the dual code of \( C \). A code \( C \) is isodual if \( C \) and \( C^\perp \) are equivalent, and \( C \) is self-dual if \( C = C^\perp \). A self-dual code is an even isodual code, and an isodual code is a formally self-dual code. There are formally self-dual even codes which are not self-dual. One reason for our interest in formally self-dual even codes is that for some lengths there are formally self-dual even codes with larger minimum weights than any self-dual code of that length. Double circulant codes are a remarkable class of isodual codes.

The question of decoding error probabilities was studied by Faldum, La-
fuente, Ochoa and Willems [9] for bounded distance decoding. Let \( C \) and \( C' \) be \([n, k, d]\) codes with weight distributions \((A_0, A_1, \ldots, A_n)\) and \((A'_0, A'_1, \ldots, A'_n)\), respectively. Suppose that symbol errors are independent and the symbol error probability is small. Then \( C \) has a smaller decoding error probability than \( C' \) if and only if

\[
(1) \quad (A_0, A_1, \ldots, A_n) \prec (A'_0, A'_1, \ldots, A'_n),
\]

where \( \prec \) means the lexicographic order, that is, there is an integer \( s \in \{0, 1, \ldots, n\} \) such that \( A_i = A'_i \) for all \( i < s \) but \( A_s < A'_s \) [9, Theorem 3.4]. We say that \( C \) performs better than \( C' \) if (1) holds. By making use of [9, Theorem 3.4], Bouyuklieva, Malevich and Willems [2] investigated and compared the performance of extremal doubly even and singly even self-dual codes.

In this note, we consider the performance of optimal double circulant even codes which are not self-dual using [9, Theorem 3.4]. To do this, we classify the optimal double circulant even codes that are not self-dual which have the smallest weight distribution for lengths up to 72. For \((2n, d) = (32, 8), (36, 8), (38, 8), (40, 8), (46, 10), (52, 10), (56, 12), (60, 12), (62, 12), (64, 12), (66, 12)\) and \((68, 12)\), we demonstrate that there is an optimal double circulant even \([2n, n, d]\) code \( C \) which is not self-dual such that \( C \) performs better than any self-dual \([2n, n, d]\) code. We also give some restrictions on weight enumerators of (extremal) self-dual \([54, 27, 10]\) codes having shadows of minimum weight 3. Finally, we consider the performance of extremal self-dual codes of lengths 88 and 112.
2 Double circulant codes

An $n \times n$ circulant matrix has the form:

$$
\begin{pmatrix}
  r_1 & r_2 & r_3 & \cdots & r_n \\
  r_n & r_1 & r_2 & \cdots & r_{n-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  r_2 & r_3 & r_4 & \cdots & r_1
\end{pmatrix}
$$

so that each successive row is a cyclic shift of the previous one. A pure double circulant code and a bordered double circulant code have generator matrices of the form:

(2) \hspace{1cm} \begin{pmatrix} I_n & R \end{pmatrix}

and

(3) \hspace{1cm} \begin{pmatrix} 
  \alpha & 1 & \cdots & 1 \\
  1 \\
  I_n & \vdots & R' \\
  1
\end{pmatrix}

respectively, where $I_n$ is the identity matrix of order $n$, $R$ (resp. $R'$) is an $n \times n$ (resp. $n - 1 \times n - 1$) circulant matrix, and $\alpha \in \mathbb{F}_2$. These two families are called double circulant codes. Since we consider only even codes in this note, $\alpha = 0$ if $n$ is even and $\alpha = 1$ if $n$ is odd.

It is a fundamental problem to classify double circulant codes over binary and nonbinary fields as well as finite rings, for modest lengths, up to equivalence. There has been significant research on finding double circulant self-dual codes with the large minimum weights (see e.g., [12], [13], [14], [16]). Beyond self-dual codes, few results on the classification of double circulant codes are known (see e.g., [10]). One reason for this is that the classification of double circulant codes which are not self-dual is much more difficult than double circulant self-dual codes.

In this note, we consider codes $C$ satisfying the following conditions:

(C1) $C$ is a pure (resp. bordered) double circulant even code of length $2n$ which is not self-dual.

(C2) $C$ has the largest minimum weight $d_P$ (resp. $d_B$) among pure (resp. bordered) double circulant codes of length $2n$ which are not self-dual.
(C3) $C$ has the smallest weight distribution $(A_0, A_1, \ldots, A_n)$ under the lexicographic order $\prec$ among pure (resp. bordered) double circulant even codes of length $2n$ and minimum weight $d_P$ (resp. $d_B$) which are not self-dual.

We say that a double circulant even code which is not self-dual is optimal if it has the largest minimum weight among all double circulant even codes of that length which are not self-dual.

The following lemma is trivial.

**Lemma 1.** If $C$ is a pure double circulant even code of length $2n$ with generator matrix (2), then every row of $R$ has odd weight. If $C$ is a bordered double circulant even code of length $2n$ with generator matrix (3), then every row of $R'$ has even weight.

## 3 Weight enumerators

The weight enumerator $W$ of a formally self-dual even code of length $2n$ can be represented as an integral combination of Gleason polynomials (see [21]), so that

$$W = \sum_{j=0}^{\lfloor n/4 \rfloor} a_j (1 + y^2)^{n-4j} \{y^2(1 - y^2)^2\}^j,$$

for some integers $a_j$ with $a_0 = 1$. Note that the weight enumerators of formally self-dual even codes of length $2n$ as well as self-dual codes of length $2n$ can be expressed using (4). This is one of the reasons why we compare the performance of optimal double circulant even codes which are not self-dual with self-dual codes having the largest minimum weight.

For the following parameters

$$\quad (2n, d) = (32, 8), (34, 8), (36, 8), (38, 8), (40, 8), (42, 10), (44, 10), (46, 10), (48, 10), (50, 10), (52, 10), (54, 10), (56, 12), (58, 12), (60, 12), (62, 12), (64, 12), (66, 12), (68, 12), (70, 12), (72, 14),$$

the possible weight enumerators $W_{2n,d} = \sum_{i=0}^{2n} A_i y^i$ of a formally self-dual even $[2n, n, d]$ code can be determined as follows. The coefficients $A_i$ ($i = 0, d, d + 2, d + 4, d + 6$) are listed in Table 1, where $a, b, c$ are integers. For
(2n, d) = (32, 8), (34, 8), (36, 8), (38, 8), (42, 10), (44, 10) and (46, 10), the weight enumerator \( W_{2n,d} \) is completely determined by only \( A_d \). For \((2n, d) = (40, 8), (48, 10), (50, 10), (52, 10), (54, 10), (56, 12), (56, 12), (58, 12), (60, 12)\) and \((62, 12)\), the weight enumerator \( W_{2n,d} \) is completely determined by \( A_d \) and \( A_{d+2} \). For the remaining values of \((2n, d)\), the weight enumerator \( W_{2n,d} \) is completely determined by \( A_d, A_{d+2} \) and \( A_{d+4} \).

Table 1: Possible weight enumerators \( W_{2n,d} \)

| \((2n, d)\) | \(A_0\) | \(A_d\) | \(A_{d+2}\) | \(A_{d+4}\) | \(A_{d+6}\) |
|-------------|--------|--------|--------|--------|--------|
| (32, 8)     | 1      |         | 4960 - 8a | -3472 + 28a | 34720 - 56a |
| (34, 8)     | 1      |         | 4114 - 7a | 2516 + 20a | 29172 - 28a |
| (36, 8)     | 1      |         | 3366 - 6a | 6630 + 13a | 30600 - 8a |
| (38, 8)     | 1      |         | 2717 - 5a | 9177 + 7a | 35910 + 5a |
| (40, 8)     | 1      |         | -4a + b   | 32110 + 2a - 10b | -54720 + 12a + 45b |
| (42, 10)    | 1      |         | 26117 - 9a | -10455 + 35a | 286713 - 75a |
| (44, 10)    | 1      |         | 21021 - 8a | 19712 + 26a | 250778 - 40a |
| (46, 10)    | 1      |         | 16744 - 7a | 38709 + 18a | 249458 - 14a |
| (48, 10)    | 1      |         | -6a + b   | 207552 + 11a - 12b | -606441 + 4a + 66b |
| (50, 10)    | 1      |         | -5a + b   | 166600 + 5a - 11b | -271950 + 15a + 54b |
| (52, 10)    | 1      |         | -4a + b   | 132600 - 10b | -41990 + 20a + 43b |
| (54, 10)    | 1      |         | -3a + b   | 104652 - 4a - 9b | 107406 + 20a + 33b |
| (56, 12)    | 1      |         | -8a + b   | 1343034 + 24a - 14b | -5765760 - 24a + 91b |
| (58, 12)    | 1      |         | -7a + b   | 1067838 + 16a - 13b | -3224452 + 77b |
| (60, 12)    | 1      |         | -6a + b   | 843030 + 9a - 12b | -1454640 + 16a + 64b |
| (62, 12)    | 1      |         | -5a + b   | 660858 + 3a - 11b | -270940 + 25a + 52b |
| (64, 12)    | 1      |         | -4a + b   | -2a - 10b + c | 8707776 + 28a + 41b - 16c |
| (66, 12)    | 1      |         | -3a + b   | -6a - 9b + c | 6874010 + 26a + 31b - 15c |
| (68, 12)    | 1      |         | -2a + b   | -9a - 8b + c | 5393454 + 20a + 22b - 14c |
| (70, 12)    | 1      |         | -a + b    | -11a - 7b + c | 4206125 + 11a + 14b - 13c |
| (72, 14)    | 1      |         | -6a + b   | 7a - 12b + c | 56583450 + 28a + 62b - 18c |

4 Performance of double circulant even codes

A classification of optimal double circulant even codes was given in [10] for lengths up to 30. For lengths \(2n\) with \(32 \leq 2n \leq 72\), by determining the largest minimum weights \(d_P\) (resp. \(d_B\)), our exhaustive search found all distinct pure (resp. bordered) double circulant even codes satisfying the conditions (C1)–(C3). This was done by considering all \(n \times n\) circulant matrices
$R$ in (2) (resp. $n - 1 \times n - 1$ circulant matrices $R'$ in (3)) satisfying the condition given in Lemma 1. Since a cyclic shift of the first row for a code defines an equivalent code, the elimination of cyclic shifts substantially reduced the number of codes which had to be checked further for equivalence to complete the classification. Then Magma [1] was employed to determine code equivalence which completed the classification.

In Table 2, we list the numbers $N_P$ and $N_B$ of inequivalent pure and bordered double circulant even codes satisfying the conditions (C1)–(C3), respectively. In Table 2 we also list $d_P$, $A_{d_P}$, $d_B$ and $A_{d_B}$. For the pure and bordered double circulant even codes satisfying the conditions (C1)–(C3), the first rows of $R$ in (2) and $R'$ in (3) are listed in Tables 3 and 4, respectively. The minimum weights $d$ and $(A_d, A_{d+2}, A_{d+4})$ are also listed in the tables.

Table 2: Double circulant even codes satisfying (C1)--(C3)

| $2n$ | $d_P$ | $A_{d_P}$ | $N_P$ | $d_B$ | $A_{d_B}$ | $N_B$ | $d_{SD}$ | $A_{d_{SD}}$ |
|------|-------|-----------|-------|-------|-----------|-------|----------|-------------|
| 32   | 8     | 348       | 2     | 8     | 300       | 1     | 8        | 364         |
| 34   | 8     | 272       | 15    | 8     | 272       | 10    | 6        | -           |
| 36   | 8     | 153       | 4     | 8     | 153       | 3     | 8        | 225         |
| 38   | 8     | 76        | 4     | 8     | 72        | 1     | 8        | 171         |
| 40   | 8     | 25        | 1     | 8     | 38        | 2     | 8        | 125         |
| 42   | 10    | 1680      | 2     | 10    | 1682      | 1     | 8        | -           |
| 44   | 10    | 1144      | 2     | 10    | 1267      | 3     | 8        | -           |
| 46   | 10    | 851       | 1     | 10    | 858       | 2     | 10       | 1012        |
| 48   | 10    | 480       | 1     | 10    | 575       | 1     | 12       | 17296       |
| 50   | 10    | 325       | 1     | 10    | 356       | 1     | 10       | 196         |
| 52   | 10    | 156       | 1     | 10    | 150       | 1     | 10       | 250         |
| 54   | 10    | 27        | 1     | 10    | 52        | 1     | 10       | 7-135       |
| 56   | 12    | 4060      | 1     | 10    | 3         | 1     | 12       | 4606-8190   |
| 58   | 12    | 3161      | 1     | 12    | 3227      | 1     | 10       | -           |
| 60   | 12    | 2095      | 1     | 12    | 2146      | 1     | 12       | 2555        |
| 62   | 12    | 1333      | 1     | 12    | 1290      | 1     | 12       | 1860        |
| 64   | 12    | 544       | 1     | 12    | 806       | 1     | 12       | 1312        |
| 66   | 12    | 374       | 1     | 12    | 480       | 1     | 12       | 858         |
| 68   | 12    | 136       | 1     | 12    | 165       | 1     | 12       | 442-486     |
| 70   | 12    | 35        | 1     | 14    | 12172     | 1     | 12-14    | -           |
| 72   | 14    | 8064      | 1     | 14    | 8190      | 1     | 12-16    | -           |

To compare the performance of the optimal double circulant even codes which are not self-dual as measured by the decoding error probability with
bounded distance decoding, we list the largest minimum weight $d_{SD}$ among self-dual codes of length $2n$. We also list the smallest number $A_{SD}$ of code-words of minimum weight $d_{SD}$ among self-dual codes of length $2n$ and minimum weight $d_{SD}$, when $d_P = d_{SD}$ or $d_B = d_{SD}$, along with the references. From Table 2 we have the following results concerning the performance of optimal double circulant even codes which are not self-dual.

**Theorem 2.** Suppose that

$$(2n, d) = (32, 8), (36, 8), (38, 8), (40, 8), (46, 10), (52, 10),$$

$$(56, 12), (60, 12), (62, 12), (64, 12), (66, 12), (68, 12).$$

Then there is an optimal double circulant even $[2n, n, d]$ code $C$ which is not self-dual such that $C$ performs better than any self-dual $[2n, n, d]$ code.

**Remark 3.**

1. For $2n = 34, 42, 44, 58$, there is a double circulant even code $C$ of length $2n$ which is not self-dual such that $C$ has a larger minimum weight than any self-dual code of length $2n$.

2. Up to equivalence, there is a unique extremal doubly even self-dual $[48, 24, 12]$ code [18]. There is no double circulant even $[48, 24, d]$ code with $d \geq 12$.

3. There is a self-dual $[50, 25, 10]$ code $C$ such that $C$ performs better than any double circulant even $[50, 25, 10]$ code which is not self-dual [20].

4. There is a double circulant even $[70, 35, 14]$ code. The largest minimum weight among currently known self-dual codes of length 70 is 12. The weight enumerator of an extremal self-dual $[70, 35, 14]$ code $C$ is uniquely determined and the number of codewords of weight 14 is 11730 [8]. If there is such a code $C$, then $C$ performs better than any double circulant even $[70, 35, 14]$ code which is not self-dual.

5. There is a double circulant even $[72, 36, 14]$ code. The largest minimum weight among currently known self-dual codes of length 72 is 12. The existence of an extremal doubly even self-dual $[72, 36, 16]$ code is a long-standing open question [24] (see also [22, Section 12]).
At the end of this section, we examine the equivalence of some codes in Tables 3 and 4. We verified by Magma [1] that there is no pair of equivalent codes among \( P_{34,i} \) \((i = 1, 2, \ldots, 15)\) and \( B_{34,i} \) \((i = 1, 2, \ldots, 10)\), and that there is no pair of equivalent codes among \( P_{36,i} \) \((i = 1, 2, 3, 4)\) and \( B_{36,i} \) \((i = 1, 2, 3)\).

A formally self-dual even [70, 35, 14] code \( C_{70} \) can be found in [11]. The code \( C_{70} \) is equivalent to some bordered double circulant codes, and \( C_{70} \) and \( B_{70} \) have identical weight enumerators [11]. Hence, \( B_{70} \) must be equivalent to \( C_{70} \).

5 Self-dual codes of length 54

In this section, we consider the remaining case. Table 2 gives rise to a natural question, namely, is there a self-dual [54, 27, 10] code such that \( A_{10} < 27? \) A self-dual [54, 27, 10] code \( C \) and its shadow \( S \) (see [5] for the definition of shadows), have the following possible weight enumerators \( W_i(C) \) and \( W_i(S) \), respectively \((i = 1, 2)\):

\[
\begin{align*}
W_1(C) &= 1 + (351 - 8\beta)y^{10} + (5031 + 24\beta)y^{12} + \cdots, \\
W_1(S) &= \beta y^7 + (2808 - 10\beta)y^{11} + \cdots, \\
W_2(C) &= 1 + (351 - 8\beta)y^{10} + (5543 + 24\beta)y^{12} + \cdots, \\
W_2(S) &= y^3 + (-12 + \beta)y^7 + (2874 - 10\beta)y^{11} + \cdots,
\end{align*}
\]

where \( \beta \) is an integer with \( 0 \leq \beta \leq 43 \) for \( i = 1 \) and \( 12 \leq \beta \leq 43 \) for \( i = 2 \) [5]. Self-dual codes exist with \( W_1(C) \) for \( \beta = 0, 1, \ldots, 20, 22, 26 \) and self-dual codes exist with \( W_2(C) \) for \( \beta = 12, 13, \ldots, 22, 24, 26, 27 \) (see [3], [5], [19], [25], [27]). Note that the smallest number \( A_{10} \) among currently known self-dual codes of length 54 and minimum weight 10 is 135.

**Proposition 4.** If there is a self-dual [54, 27, 10] code \( C \) such that \( A_{10} < 27 \), then \( C \) has weight enumerator \( W_1(C) \) with \( 41 \leq \beta \leq 43 \).

**Proof.** Let \( C \) be a self-dual [54, 27, 10] code such that \( A_{10} < 27 \). Suppose that \( C \) has weight enumerator \( W_2(C) \). From the assumption, \( \beta > 12 \). Thus, there is a vector of weight 7 in the shadow \( S \). Let \( x_1 \) and \( x_2 \) be vectors of weights 3 and 7 in \( S \), respectively. Since the sum of two distinct vectors of \( S \) is a codeword of \( C \), \( x_1 + x_2 \) must be a codeword of weight 10. From the coefficient of \( y^7 \) in \( W_2(S) \), we have

\[
351 - 8\beta \geq \beta - 12 \\
40 \geq \beta.
\]
Table 3: Pure double circulant even codes satisfying (C1)–(C3)

| Code  | First row                      | $d$ | $(A_d, A_{d+2}, A_{d+4})$ |
|-------|-------------------------------|-----|--------------------------|
| $P_{32,1}$ | (11001011100110101) | 8   | (348, 2176, 6272)        |
| $P_{32,2}$ | (11101101000100111) | 8   | (348, 2176, 6272)        |
| $P_{34,1}$ | (11111110001000100) | 8   | (272, 2210, 7956)        |
| $P_{34,2}$ | (11000001111011010) | 8   | (272, 2210, 7956)        |
| $P_{34,3}$ | (11110101101101100) | 8   | (272, 2210, 7956)        |
| $P_{34,4}$ | (11110011110110101) | 8   | (272, 2210, 7956)        |
| $P_{34,5}$ | (10001011101100000) | 8   | (272, 2210, 7956)        |
| $P_{34,6}$ | (10001100110010100) | 8   | (272, 2210, 7956)        |
| $P_{34,7}$ | (11101101101100000) | 8   | (272, 2210, 7956)        |
| $P_{34,8}$ | (10100101100011110) | 8   | (272, 2210, 7956)        |
| $P_{34,9}$ | (10100100110010001) | 8   | (272, 2210, 7956)        |
| $P_{34,10}$ | (10101010011111000) | 8   | (272, 2210, 7956)        |
| $P_{34,11}$ | (10001100001011100) | 8   | (272, 2210, 7956)        |
| $P_{34,12}$ | (11010010010011110) | 8   | (272, 2210, 7956)        |
| $P_{34,13}$ | (10001100001111010) | 8   | (272, 2210, 7956)        |
| $P_{34,14}$ | (11011011000111101) | 8   | (272, 2210, 7956)        |
| $P_{34,15}$ | (11100001101010011) | 8   | (272, 2210, 7956)        |
| $P_{36,1}$ | (10101110111000000) | 8   | (153, 2448, 8619)        |
| $P_{36,2}$ | (11110000100001011) | 8   | (153, 2448, 8619)        |
| $P_{36,3}$ | (10010111010100010) | 8   | (153, 2448, 8619)        |
| $P_{36,4}$ | (10001011101011100) | 8   | (153, 2448, 8619)        |
| $P_{38}$ | (11110000100010110) | 8   | (76, 2337, 9709)         |
| $P_{40}$ | (10101111111110010) | 8   | (25, 2080, 10360)        |
| $P_{42,1}$ | (10000110111110010) | 10  | (1680, 10997, 48345)     |
| $P_{42,2}$ | (10101001010111011) | 10  | (1680, 10997, 48345)     |
| $P_{44}$ | (10011111111100001) | 10  | (1144, 11869, 49546)     |
| $P_{46}$ | (1100101101011100000) | 10  | (851, 10787, 54027)      |
| $P_{48}$ | (11111000101111101110100) | 10  | (480, 10384, 53664)      |
| $P_{50}$ | (1001000010100011101) | 10  | (325, 8650, 55200)       |
| $P_{52}$ | (10001010001010110100001) | 10  | (156, 7267, 53690)       |
| $P_{54}$ | (11100000010110101100000) | 10  | (27, 6030, 49545)        |
| $P_{56}$ | (1001100011110110111111010) | 12  | (4060, 49420, 293874)    |
| $P_{58}$ | (1101100001010000000011011010) | 12  | (3161, 41142, 292407)    |
| $P_{60}$ | (100001011011100010011010001) | 12  | (2095, 37320, 263205)    |
| $P_{62}$ | (0010100111011100111111010000000) | 12  | (1333, 30597, 254975)    |
| $P_{64}$ | (101010001110011111111101010000000) | 12  | (544, 34304, 115756)     |
| $P_{66}$ | (101000010010000101110110101000000) | 12  | (374, 20163, 208308)     |
| $P_{68}$ | (1010001011011010110101011001000000) | 12  | (136, 15606, 176936)     |
| $P_{70}$ | (010111011101001101110000111000000) | 12  | (35, 11550, 151130)      |
| $P_{72}$ | (101101101101001101011111100010000) | 14  | (8064, 127809, 1202464)  |
Table 4: Bordered double circulant even codes satisfying (C1)–(C3)

| Code | First row | $d$ | $(A_d, A_{d+2}, A_{d+4})$ |
|------|-----------|-----|--------------------------|
| $B_{32}$ | $(100101010001111)$ | 8 | $(300, 2560, 4928)$ |
| $B_{34,1}$ | $(101101010001101)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,2}$ | $(111011100010110)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,3}$ | $(101010011101110)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,4}$ | $(100110011011101)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,5}$ | $(111001001101101)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,6}$ | $(110110110101000)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,7}$ | $(100010010011101)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,8}$ | $(111000011111011)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,9}$ | $(111000011110111)$ | 8 | $(272, 2210, 7956)$ |
| $B_{34,10}$ | $(100101001101011)$ | 8 | $(272, 2210, 7956)$ |
| $B_{36,1}$ | $(110011101101111)$ | 8 | $(153, 2448, 8619)$ |
| $B_{36,2}$ | $(110111000011101)$ | 8 | $(153, 2448, 8619)$ |
| $B_{36,3}$ | $(110011001101101)$ | 8 | $(153, 2448, 8619)$ |
| $B_{38}$ | $(110000111101010)$ | 8 | $(72, 2357, 9681)$ |
| $B_{40,1}$ | $(110000011110100)$ | 8 | $(38, 2014, 10526)$ |
| $B_{40,2}$ | $(100101110011101)$ | 8 | $(38, 2014, 10526)$ |
| $B_{42}$ | $(101111101110101)$ | 10 | $(1682, 10979, 48415)$ |
| $B_{44,1}$ | $(1010100000111010)$ | 10 | $(1267, 10885, 52654)$ |
| $B_{44,2}$ | $(111000011110101)$ | 10 | $(1267, 10885, 52654)$ |
| $B_{44,3}$ | $(110001011111101)$ | 10 | $(1267, 10885, 52654)$ |
| $B_{46,1}$ | $(110100100111001)$ | 10 | $(858, 10738, 54153)$ |
| $B_{46,2}$ | $(111100111110111)$ | 10 | $(858, 10738, 54153)$ |
| $B_{48}$ | $(110100101000100)$ | 10 | $(575, 9752, 55453)$ |
| $B_{50}$ | $(111100111001101)$ | 10 | $(356, 8524, 55036)$ |
| $B_{52}$ | $(101000100111001)$ | 10 | $(356, 8524, 55036)$ |
| $B_{54}$ | $(11101011000000011)$ | 10 | $(52, 5876, 50156)$ |
| $B_{56}$ | $(1011100001010000100)$ | 10 | $(3, 4545, 45477)$ |
| $B_{58}$ | $(11011001001101110111011)$ | 12 | $(3227, 40950, 293463)$ |
| $B_{60}$ | $(101001111101101111101)$ | 12 | $(1290, 30850, 254428)$ |
| $B_{62}$ | $(111010000111101000000001)$ | 12 | $(1290, 30850, 254428)$ |
| $B_{64}$ | $(100001010101011011011100000)$ | 12 | $(806, 25358, 226982)$ |
| $B_{66}$ | $(101011011111001111110110101)$ | 12 | $(480, 19848, 203112)$ |
| $B_{68}$ | $(100111101011101101010110010)$ | 12 | $(165, 15620, 176099)$ |
| $B_{70}$ | $(1101000111011001101111100000000)$ | 14 | $(12172, 147390, 1352811)$ |
| $B_{72}$ | $(1001111010111100111001101011000)$ | 14 | $(8190, 126952, 1204560)$ |
Hence, we have that $A_{10} \geq 31$, which is a contradiction. It follows from $A_{10} < 27$ that $41 \leq \beta \leq 43$ in $W_1(C)$. \hfill \Box

Hence, the above question can be refined as follows.

Question 5. Is there a self-dual $[54, 27, 10]$ code which has weight enumerator $W_1(C)$ with $41 \leq \beta \leq 43$?

As a consequence of the proof of the above proposition, we have the following.

Corollary 6. If there is a self-dual $[54, 27, 10]$ code with weight enumerator $W_2(C)$, then $\beta \in \{12, 13, \ldots, 40\}$.

6 Performance of extremal self-dual codes of lengths 88 and 112

The performance of extremal doubly even and singly even self-dual codes was compared in [2, Section 4] for lengths $24k + 8$ ($k = 1, 2, 3, 4$) and $24k + 16$ ($k = 1, 2$). In this section, we consider the case for lengths $24k + 16$ ($k = 3, 4$).

An extremal doubly even self-dual code of length 88 has the following weight enumerator (see [21]):

$$1 + 32164y^{16} + 6992832y^{20} + 535731625y^{24} + \cdots.$$  

There are at least 470 inequivalent extremal doubly even self-dual codes of length 88 (see [13]). An extremal singly even self-dual code of length 88 was given in [17] which has the following weight enumerator:

$$1 + 18436y^{16} + 268928y^{18} + 3493248y^{20} + 267717065y^{24} + \cdots.$$  

Hence, there is an extremal singly even self-dual code $C$ of length 88 such that $C$ performs better than any extremal doubly even self-dual code of that length.

An extremal doubly even self-dual code of length 112 has the following weight enumerator (see [21]):

$$1 + 355740y^{20} + 95307030y^{24} + 10847290300y^{28} + \cdots.$$  

11
An extremal doubly even self-dual code of length 112 was found in [15]. By [5, Theorem 5], the possible weight enumerators $W_1(C)$ and $W_1(S)$ of an extremal singly even self-dual code $C$ of length 112 and its shadow $S$ are:

$$
\begin{align*}
W_1(C) &= 1 + (157388 + 16a)y^{20} + (3125056 - 64a)y^{22} \\
&\quad + (52740406 - 160a)y^{24} + \cdots, \\
W_1(S) &= y^4 + (-2002 + a)y^{16} + (428099 - 20a)y^{20} + \cdots, \\
W_2(C) &= 1 + (157388 + 16a)y^{20} + (3431232 + 1024b - 64a)y^{22} \\
&\quad + (48040246 - 10240b - 160a)y^{24} + \cdots, \\
W_2(S) &= y^8 + (-24 - b)y^{12} + (276 + 22b + a)y^{16} \\
&\quad + (394680 - 231b - 20a)y^{20} + \cdots, \\
W_3(C) &= 1 + (157388 + 16a)y^{20} + (3431232 + 1024b - 64a)y^{22} \\
&\quad + (47974710 - 10240b - 160a)y^{24} + \cdots, \\
W_3(S) &= -by^{12} + (22b + a)y^{16} + (396704 - 231b - 20a)y^{20} + \cdots,
\end{align*}
$$

where $a, b$ are integers. Currently, it is not known whether there is an extremal singly even self-dual code of length 112.

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