Geometry and physics of today

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Abstract

The "geometry", in the sense of the classical differential geometry of smooth manifolds (CDG), is put under scrutiny from the point of view of Abstract Differential Geometry (ADG), along with resulting, thereby, potential physical consequences, in what, in particular, concerns physical "gauge theories", when the latter are viewed as being, anyway, of a "geometrical character". Yet, "physical geometry", in connection with physical laws and the associated with them, within the context of ADG, "differential" equations (whence, no background spacetime manifold is needed thereat), are also under discussion.

"αεί ο θεός γεωμετρεῖ" 
(ː"eternally the God geometrizes"

1. By looking at the previous famous utterance (attributed to Plato, according to Plutarch, see e.g. D.E. Smith [31: p. 88, ft. 4]) as in the above frontispiece, while taking also into account our nowadays conception of Physics, we can say that;

\[(1.1) \quad \text{"physical geometry" is the outcome of the physical laws.}\]

In this regard, one might also refer here, for instance, still to M. Faraday, as he is quoted by H. Weyl [35: p. 169], in that [emphasis below is ours]:

\[(1.2) \quad \text{\ldots not the field should derive its meaning through its association with matter, but, conversely, \ldots particles of matter are \ldots singularities of the field.}\]
Now, by looking at the technical correspondence/association,

\[(1.3) \quad \text{physical law} \leftrightarrow \mathcal{A}\text{-connection},\]

one realizes that (1.1) might also be construed, as an equivalent analogue of the implication;

\[\mathcal{A}\text{-connection} (: \text{physical law}) \Rightarrow \text{curvature} \quad (: \text{“geometry”, alias, “shaping”}).\]

Consequently, still to repeat (1.1), thus, said it otherwise, one concludes that;

\[(1.5) \quad \text{it is actually the physical laws, that make, what we might call (physical) “geometry”}.\]

Of course, we take for granted, concerning the above terminology, the meaning of the technical term, “\(\mathcal{A}\text{-connection}”\), for which we refer thus, for instance, to A. Mallios [12], or even to [13], [17].

Now, it is worthwhile to comment here, a bit more, on the inverted commas, put above on the word,

\[(1.6) \quad \text{geometry}.\]

Indeed, the same are meant, as well as, hint, therein, at the technical and also fundamental, in point of fact, issue, which the aforesaid (Greek) word contains in itself; namely, the entanglement of ourselves, in that point of view, or even, the manner we look at that notion, as this is implemented/understood, exactly, by the second component of the same word (the latter being, in effect, a \textit{composed one}, that is, the Greek verb, “\textit{metrō}” (: measure). Accordingly, any time we refer to/use that notion, by definition, viz. by the real essence of the same word,

the term “\textit{geometry}” does not actually correspond to/means something physical (: \textit{real}), but, simply, \textit{a model of ours, pertaining to} the description of reality (in whatever sense of the latter concept).

Furthermore, it is still appropriate to remind us, at this point, of A. Einstein’s maxim, in that;
“Time and space are modes by which we think, not conditions in which we live”.

See thus, for instance, Yu.I. Manin [24: p. 71], as well as, within actually the same vein of ideas, (1.29) in the sequel. [Emphasis in (1.8) above is ours, as it will also be the case, occasionally, in quotations, throughout the sequel]. Yet, we mention here the relevant remarks of P.G. Bergmann [3: p. 33], in that,

“Einstein ... did not consider geometrization of physics a foremost or even a meaningful objective...”

(I am indebted here to I. Raptis for bringing to my attention the previous citation of Bergmann). Yet, the same author, as above (loc. cit.), insists in that, what is of importance here is,

“... not a geometric formulation or picturization but a ... fusing of the mathematical structures intended to represent physical fields.”

We remark here that the above are still in accord with (1.1) or (1.5) in the preceding. Thus, we are led again, herewith, to a

“relational aspect” of what we might call, “physical geometry”.

In other words, we thus arrive at something, which is more close to what, as we still mentioned above, we have already said by (1.5). Furthermore, this same aspect is also akin to what we may understand, as we shall see later on, when speaking of “geometry”, determined by “differential” equations, yet, the “solution space” of the latter. The same might still be conceived, even, as the source(!) of the “cartesian point of view”; however, see also (1.14) in the sequel, concerning that perspective, within the present abstract (thus, space-independent (!)) setting.

So, still, within the aforesaid context (see also e.g. (1.11)), we can further say that;

“geometrization” of physics means, in point of fact, “arithmetization” of the same, for our “geometry” is, in effect, “arithmetical”, that is, “cartesian”(!), in character, hence, not a physical (: natural) one!

Consequently, one comes to realize that,
the previous association becomes thus more natural, to the extent that it is more “relational”(!), in nature.

However, what is also here of a particular significance, concerning the whole subject matter of the present work, the preceding point of view, as in (1.12), is actually meant in an

entirely “space independent” way, that is, not in a “cartesian-wise” manner,

as this also will become clear, along with the terminology applied herewith, through the subsequent discussion. That is, in other words, based on the abstract formalism of the same technique of Abstract Differential Geometry (ADG), one is able to

formulate “differential” equations without having the need to resort to any background (“cartesian”–“newtonian”, so to say) “space”, to work with.

This latter situation might be, in point of fact, as we shall see in the sequel, of paramount significance for problems of quantum gravity, when the same problems are viewed from the standard perspective, viz. from that one of the classical differential geometry of smooth (: $C^\infty$-)manifolds (CDG).

So, in accordance with (1.11), one gets, indeed, at a “leibnizian”, so to say, point of view, that is, by following Leibniz himself,

we should find a “geometrical calculus” that operates directly on the “geometrical objects” without the intervention of coordinates.

In this regard, we may even remark here, anyway, that the latter function, as above (: coordinates) is, for that matter,

“... an act of violence”.

See thus H. Weyl [36: 90]. On the other hand, concerning (1.17), cf., for instance, N. Bourbaki [5: Chapt. I; p. 161, ft. 1]. Furthermore, within the same context, one has here the relevant remarks of B. Riemann, in that;

“Specifications of mass [: measurements] require an independence of quantity from position, which can happen in more than one way”.
Cf., for example, A. Mallios [14: (1.3)]. Thus, in toto, the preceding sustain, indeed, the aspect that:

\begin{equation}
\text{the description of the physical laws, something that could also include the quantum régime, as well, should be made in such a manner, that no supporting space, or even space scaffolding (: framework), essentially contributing to that description, is to be included in our “calculations” (: rationale); hence, the latter have thus to be entirely independent of any notion of “space” of the aforesaid type.}
\end{equation}

Now, the previous aspect of “description of physical laws”, as in (1.20), can, in point of fact, be conceived, as just referring to the very “geometrical calculus” à la Leibniz (cf. (1.17)), hence, to this same “geometry”(!), in that respect, in the sense of Leibniz, or even, to its “relational point of view”, according to (1.11). Furthermore, the same perspective of

\begin{equation}
\text{“geometry”, as “description(study) of physical laws”,}
\end{equation}

leads, of course, simply, to the aspect of,

\begin{equation}
\text{doing “geometry”, via “differential” equations (a fact that actually goes back to René Descartes himself: “Analytic Geometry”),}
\end{equation}

as exactly hinted at, already, by (1.12) in the foregoing. Now, we are just going to comment further on the latter aspect, as appeared, within the present abstract setup of ADG, straightforwardly, by the next Section, making thus also still, more clear, our previous remarks in (1.16), as above.

2. **“Differential” equations in the setting of ADG. Functoriality.**— As already mentioned above, our aim, by the following discussion, is virtually to clarify (1.16), and to look also at further consequences thereat:

Thus, to start with, we can certainly remark that, one of the most effective methods, thus far, of describing physical laws has been, of course, that one, provided by “differential equations”; hence, the foremost applications thereof of the (classical) differential geometry (CDG, indeed, Calculus(!), yet, of “the glittering trappings of Analysis”, to recall here G.D. Birkhoff; see, for instance, A. Weinstein [34: p.1, ft.2]).
However, the latter (viz. the classical) way of describing physical laws contains in itself, already, the seeds of the defaults, that exactly should be avoided, just by virtue of our previous remarks, as in (1.20). Indeed, by the very characters of the classical theory (CDG), its whole machinery (mechanism) is entirely rooted on the supporting space (viz. on the “locally euclidean” smooth manifold). Accordingly, simply, as a result of (1.20), one concludes that:

\[(2.1)\]

the notion of a (locally euclidean–smooth–)manifold proves thus not to be the appropriate one(!), in order to describe physical laws (: the “reality”) to the extent, at least, that the latter refer to the quantum deep, as well.

In this context, we may still recall herewith, the relevant comments of A. Einstein himself, pertaining, in point of fact, to the

\[(2.2)\]

inappropriateness of the manifold concept for physical reality(!).

See, for instance, A. Mallios [16: (1.6)]. On the other hand, one can further say that,

the aforementioned drawback of the notion of smooth manifold in problems connected with the quantum deep is mainly due, not only (!) to the way, we consider arising the “differential-geometric” mechanism, within the context of CDG (see thus, however, (3.3) in the sequel), but,

\[(2.3)\]

\[(2.3.1)\]

much more, because we still keep, as a “working framework”, the whole “space”, viz. the entire smooth manifold itself,

by further looking at it, even locally, as domain of definition of what we define, as “differentiable functions”.

Now, the latter point of view, as mentioned in (2.3) above, proves to be, by concrete working examples we present below, a quite unnatural way of trying to apply the “differential geometric mechanism” of CDG, its character being, in point of fact, entirely algebraic(!), as we are still going to clarify in the sequel. Furthermore, it is this same aspect, as above, where we are usually confronted with an extremely pestilential anomaly of the classical theory, pertaining, in particular, to the quantum
deep, this being thus, indeed, the main source of “infinities” (: “singularities”)! Notwithstanding, all these anomalies, without actually being real ones (!) (cf. thus the aforementioned examples, as presented by the ensuing discussion).

On the other hand, we further illuminate the situation that appears, within the quantum framework, when looking at it from the point of view of the abstract theory, summarizing thus briefly the relevant conclusions into the following.

**Scholium 2.1.**— When looking at the fundamental of quantum theory, in conjunction with potential applications in that context of (differential) geometry, one actually realizes that;

\[
\text{we usually associate numbers (à la Descartes) to a space that, in effect, does not exist(!), in the sense, at least, we ascribe to it, that “spatial perspective” of ours being, in point of fact, always cartesian(!), something, of course, which is not in accord with our (experimental) knowledge, as it concerns the quantum régime.}
\]

Thus, we are, indeed, trapped here, by our own perspective, due actually to our pre-existent assumption, pertaining, as a matter of fact, to the manner we consider our “calculus” (hence, of course, that same instrumental issue of (classical) differential geometry, as well) is virtually arrived, this being thus, according to the classical theory, “locally euclidean”, viz. “newtonian”, in nature (: manifold ↔ spacetime).

Therefore, as a consequence, we are thus unable to apply the classical (: newtonian) aspect of differential geometry in the “quantum deep”, due mainly to the emergence of the so-called “singularities”, and other relevant anomalies. [As already said, several times in the preceding, the latter phenomenon being actually due to the particular type of our (“smooth”) functions involved, that “smoothness” being, in turn, a direct outcome of the sort of “space” (: locally euclidean) we use!].

Consequently, once more,
it is not the functions we use (viz., when considering them, as carriers of the “differential geometric-mechanism”, from the point of view of ADG, that is, so to say, in the “leibnizian” perspective of the latter term), which are inappropriate, concerning the quantum deep, but, simply, the “space”, on which the said functions are supposed to be defined, such a space, as that one we try to apply (viz. the “locally euclidean” one), being virtually non-existent(!), in that context, and not only this, given that,

an “arithmetical space”, as it actually is the standard “euclidean/cartesiant space”, which we usually employ in the classical theory (: CDG), is not, of course, “physical” (: real)!!, as this is, in effect, realized when, in particular, referring to the “quantum deep”. [We thus get, in that context, even an “experimental” (: concrete) ascertainment of the ineffectiveness of our (spatial) model!!]. That is, the “space” model, we usually ascribe in our physical theories, to what we actually understand, as “physical space”, is entirely a numerical one”. In that context, we are influenced, of course, from our own successes, so far, in the macroscopic world. This model, however, collapses when confronted with the quantum deep.

So, in other words, we are thus entrapped, in that respect, by the particular success, thus far, of the aforesaid point of view (: the classical one), in what, namely, especially concerns our experience/applications “in the large”.

Now, within the same vein of ideas, and still, in connection with the (categorical), in effect, correspondence amongst “space” and functions, one actually has, in that respect, the following “identification”,

\[(2.8) \quad \text{functions } \rightleftharpoons \text{space, “Gel’fand duality”}\]

which we may also call (already depicted above), Gel’fand duality”, a fact strikingly pointed out, in its full generality, by the language of the theory of (especially, non-normed) Topological Algebras (see, for instance, A. Mallios [TA; p. 223, Theorem 1.2, as well as, p. 227, Theorem 2.1]).
Now, as already hinted at in the foregoing, and which will also be considered, by the ensuing discussion, the previous situation, has nothing to do, in effect, with the mechanism itself of the aforesaid classical theory (: differential geometry), the same machinery being essentially “leibnizian”(!), in nature, as this, indeed, has been pointed out, by what we may call “Abstract Differential Geometry” (: ADG); see thus A. Mallios [13], as well as, [17].

In toto, the preceding represent the way one may look at what we usually understand, nowadays, as “space” (speaking, of course, in terms, of what we call “mathematical physics”). True, the previous thoughts are actually the outcome of our experience derived from ADG, while the same still supplies potential applications in problems of quantum relativity, as the latter has been explained already in other places (see, for instance, A. Mallios [17], as well as, A. Mallios–I. Raptis [20], [21]). So it is this entirely new (axiomatic) perspective of ADG, pertaining to the inherent mechanism of the classical differential geometry (: CDG), which provides several potential applications, while the same mechanism proves, very likely, to be also in accord with the “spatial” situation, one is confronted with in the quantum deep, as already hinted at, by the foregoing discussion; in this regard, see also e.g. A. Mallios–E.E. Rosinger [23], along with A. Mallios–I. Raptis [21]. On this latter aspect we are still going to present, however, some further illuminating comments, through the subsequent discussion, as well.

On the other hand, by looking at the whole classical set-up from the point of view of ADG, we can still point out here that, by complete contrast with the situation, which usually dominates the classical case,

the framework of ADG does not, in principle, depend on any background “space” (: carrier, think e.g. of “space-time” for the classical domain),

that would contribute to its “differential” equipment, the latter being thus entirely rooted on \(\mathcal{A}(\lambda)\), our “generalized arithmetics”, alias, “sheaf of coefficients”.

Yet, the latter issue in (2.6), as above, constitutes, in point of fact, still,

the quintessence of the quantum field-theoretic character of ADG.

Indeed, the whole set-up of ADG becomes, by its very definition, susceptible of
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formulating our equations in a quantum field-theoretic manner,

viz. quantum-relativistically! In this connection, see also our previous relevant remarks in A. Mallios [14: (9.8), (9.23), along with Section 11 therein]. Yet, to put the above subject matter still in an equivalent way, we can further remark here that;

it is actually we, who describe the (physical) laws, read, “differential” equations, by means of our “arithmetics”, thus, for the case at issue, through the (C-algebra) sheaf \( \mathcal{A} \), while the same machinery (calculus”, à la Leibniz, or even “differential geometry”) is still based on \( \mathcal{A} \), and not on any background “space”, at all(!!), as it was classically the case. Of course, this latter fact may be of paramount importance, when one is confronted with problems of quantum gravity.

In this regard, one may still refer to the relevant comments of J. Baez [2: beginning of Preface], in that (emphasis below is ours):

“A fundamental problem with quantum ... gravity ... is that in ... general relativity there is no background geometry to work with: the geometry of spacetime itself becomes a dynamical variable”.

On the other hand, the aforementioned (see (2.9), (2.12))

independence of the “differential” mechanism of ADG from any background space, gives to that mechanism the possibility to be considered, as, a “variable” entity too, the same being, by its very construction, entirely based on (reduced to) \( \mathcal{A} \); therefore, what we also understand, as “differential” geometry (: “geometrical calculus”, à la Leibniz), entailed thereof, becomes still a “variable”, as well.

Furthermore, that also appears fundamental, herewith, the same “geometrical calculus”, hence, the concomitant “geometry” too, becomes simply “relational”, referring thus directly to the “geometrical objects” (in our case, vector sheaves) themselves, without the interference, of any “space”, in the classical sense of the latter term.

In this connection, we also recall, for convenience, technically speaking, that: We suppose herewith that we are thinking, in terms of an (abstract) “differential
setting”, based on a given “differential triad”,

\[ (\mathcal{A}, \partial, \Omega) \]  

over an (arbitrary, in general) topological space \( X \), base space of all the sheaves involved, throughout. Now, within that context, a “geometrical object”, thus, for instance, an elementary particle, can be associated with what we call a Yang-Mills field, viz. a pair

\[ (\mathcal{E}, D), \]  

consisting of a vector sheaf \( \mathcal{E} \) on \( X \) and an \( \mathcal{A} \)-connection \( D \) on \( \mathcal{E} \); see e.g. A. Mallios [16: (3.2), (3.3)], or even [17: Chapt. VI]. It is actually in terms of such pairs, as above, that “differential” equations, in the framework of ADG are referred (loc. cit.).

Thus, within the above set-up, we can further refer here to a fundamental principle, in effect, of the whole machinery, thus far, of Abstract Differential Geometry (: ADG), in that;

\[ (\mathcal{E}, D), \]

everything, that we want to ascribe to a pair \((\mathcal{E}, D)\), as above, is virtually reduced to a similar condition/assumption for the pair \((\mathcal{A}, \partial)\), see (2.15), yet, occasionally, under appropriate (in principle, only(!)) topological hypotheses for \( X \) (see also the subsequent comments).

As already noted before, the context of (2.17) exhibits, in point of fact, the “Leitmotiv” that actually dominates the very technique of ADG; see thus A. Mallios [VS], or even [17]. On the other hand, the same ensures also the

\[ “covariance” \] of the whole setting of ADG, with respect to \( \mathcal{A} \).

Thus, the “variance” here is always relative to our own “arithmetic”, or even (generalized) domain of coefficients”, yet, “structure sheaf” \( \mathcal{A} \) (by assumption, a unital commutative \( \mathbb{C} \)-algebra sheaf on \( X \), cf. (2.15)), which, for that matter, is, of course, the case, as well; so, strictly speaking, it is actually we always, who measure(!)/calculate, while, and this is also of a particular importance, as already pointed out in the preceding, this whole framework/calculations of ADG, without
actually leaning upon any background “space” (: carrier), as, for instance, “space-time”(!) of the classical case.

On the other hand, we can further say that;

\[
\text{(2.19)} \quad \text{physical laws are always “functorial”}. \\
\]

Of course, in point of fact, we “abuse language” here, when referring to the above statement, as explained by the following.

**Note 2.1.**— Looking at the sense, we actually use the term, “functorial” as in (2.19) above, and also in conjunction with (1.2) in the preceding, we should still remark here that;

\[
\text{(2.20)} \quad \text{the aforesaid term is always meant, with respect, in effect, to us(!), viz. relative to (our “generalized arithmetics”) } A. \\
\]

Therefore, what we actually consider in (2.19) is, in point of fact, the manifestation of the physical laws!

Now, this goes, of course, hand in hand, with (or even, it is something that is, in point of fact, an equivalent expression of) the “principle of general covariance”. Accordingly, by considering now, as we did it in the preceding, *differential equations, as expressing physical laws* (see (1.5), (1.12)), we can realize that, indeed,

\[
\text{(2.21)} \quad \text{differential equations should be, by their very definitions, “functorial”, in nature! Consequently, their formulation should be made, in terms of “functorial objects”, as well.} \\
\]

Now, by the last term, when speaking in technical language, we mean, of course, something that, by definition, is \(\mathcal{A}\)-invariant, alias a “tensor”, in the sense that it respects our “arithmetics” \(\mathcal{A}\).

Furthermore, (2.19) can still be construed, as an outcome of (1.2), in conjunction with (1.5) in the preceding. Thus, by further considering (see also (2.20), as above) the
physical laws, as the manifestation of the (deepest physical) dynamics ("causality"), one comes to the conclusion that;

(2.22.1) "dynamics" should be "functorial", as well,

whenever we actually effectuate it (viz. the physical law, cf. also (1.4)). Therefore, this very realization of it (by us(!), of course) becomes "functorial", or even "tensorial" too, hence, the same physis of the curvature ("geometry"), see also (1.4), as before.

Now, by further commenting on our last conclusion, as above, we still recall that, according to our axiomatics,

the curvature (: field strength) is the manifestation (effectuation) of the "identification" (correspondence, cf. also (1.4)),

(2.23) (2.23.1) \( \text{dynamics} (: \text{"causality"}) \leftrightarrow (A-)\text{connection} \),

therefore (see also (2.22.1)), the tensorial (functorial, cf. (2.20)) aspect of the curvature.

In this connection, we can still note that the aforementioned functorial/tensorial character of the curvature, in the sense, of course, of (2.20), being always the outcome (field strength) of a given "field" (: A-connection, see, for instance, A. Mallios [16: (3.15)] or even [14: (3.21.1)] is further expressed, by the familiar relation,

(2.24) \( \nabla \rho = 0 \),

yet, equivalently (precisely speaking, in terms of the formalism of ADG), by the relation;

(2.24) \( \mathcal{D}_{\mathcal{H}om(\mathcal{E},\mathcal{E}^*)}(\tilde{\rho}) = 0 \),

where we still have;

(2.26) \( \mathcal{H}om(\mathcal{E},\mathcal{E}^*) = \mathcal{E}^* \otimes_A \mathcal{E}^* = (\mathcal{E} \otimes_A \mathcal{E})^* \).

See A. Mallios [VS: Chapt. VII; p. 165, (8.70), along with Chapt. IV; p. 302, Theorem 6.1 and p. 305: (6.16)]; thus, we have herewith the so-called, classically, "Levi-Civita identity". By further referring to the above notation, we consider therein a
given Yang-Mills field

\[(\mathcal{E}, D),\]

see loc. cit., Chapt. IX; p. 244, along with (2.15), as above, while \(\rho\) stands there for a Riemannian \(A\)-metric on \(\mathcal{E}\), “compatible with \(D\)” (ibid., Chapt. VII; Section 8). It is worth noticing here that the previous condition on the pair

\[(D, \rho),\]

as above, is actually the upshot of a similar assumption for the standard pair,

\[(A, \partial),\]

cf. (2.14), under appropriate supplementary conditions on the items involved herewith, these being in the case of \(X\), only topological ones (cf. thus (2.16) in the preceding); yet, in that context, see also A. Mallios [VS: Chapt. VII; p. 168, Theorem 9.1: Fundamental lemma of Riemannian vector sheaves]. Accordingly, we further understands here that (:

“physical significance” of (2.25)),

\[(2.30)\]

\[\text{to “realize” the curvature, one has to “compare” it with something else!}\]

We terminate the present Section with the subsequent remarks of N. Bohr, as quoted e.g., by S.Y. Auyang [1: p. 229]), referring to the way one actually has to look at the Nature; indeed, with the same remarks the foregoing rationale and related remarks thereon are really in accord, as it actually concerns our relevance, with respect to the observed physical laws, which, technically speaking, as it was pointed out in the preceding, is expressed, in effect, through the “structure sheaf” \(A\), independently of any surrounding/supporting “space”. Thus, according to the aforementioned remarks (emphasis below is ours),

\[(2.31)\]

\[\text{“It is wrong to think that the task of physics is to point out how nature is. Physics concerns what we can say about nature.”}\]

Consequently, to follow in that context the favorite parlance of A. Einstein himself, we thus always “describe”, hence, not explain (!), the physical applications
of every day life being, therefore, simply, consequences of the former (descriptions), as above(!). Yet, within the same vein ideas, we may still quote, herewith, L. Wittgenstein [38: p. 17], in that;

\[
\text{"Physics does not explain anything; it simply describes concomitant cases".}
\]

(Emphasis above is ours). Therefore, as already emphasized in the preceding, we do not actually explain “anything”, through Physics, as it concerns the physical laws (: physis), but, just, describe/study their consequence(!); notwithstanding, as a consequence, however, of the latter function, it undoubtedly seems (cf. applications) that,

\[
\text{we do understand, nevertheless, several times and, of course, always, to a certain extent(!), the way that these laws work!}
\]

3. ADG, as applicable in the quantum deep.– Our purpose, by the ensuing discussion, as the title of this Section indicates, is to further clarify the way one can look at a potential application of ADG in the quantum régime, thus, in point of fact, of the very mechanism of the classical (: "newtonian") differential geometry, very effective(!), for that matter, so far, however, now, within the aforesaid domain, but, already from the point of view of ADG (viz. axiomatically), thus, freed from its “beautiful shackles” (C.J. Isham); indeed, it is proved that the latter obstacles are due, simply, to the entanglement, according to the classical theory, of the same mechanism with the “locally euclidean” nature of that theory (in effect, much more, because of the maintenance of the whole “smooth setting”, as a working framework, in this context, cf. also (2.3) in the preceding), the latter being also considered, in view of the same standard theory (CDG), the only source(!), within that context, of the all powerful (infinitesimal/integral) Calculus, hence, of the classical differential-geometric machinery, as well. So, it is here exactly that a supreme didagma of ADG comes just to the foreground, in fact;

\[
\text{the differential-geometric mechanism of the classical differential geometry (CDG)–being, in effect, of a leibnizian character–can, equally well, be supplied, by other sources, apart from a “locally euclidean” space/(smooth) manifold, its existence being thus independent of any such “space”}.\]
Furthermore, as already pointed out in the preceding (see, for instance, the quoted citations, in that context, of Einstein, Feynman, Isham), a “space”, as in the latter part of (3.1), together with its “differential set-up”, is entirely out of the question for the quantum deep(!).

On the other hand, by further commenting, within the preceding vein of ideas, on the basis of our experience from ADG, as exposed above, we realize that one can virtually interrelate well-known phenomena in the past with still existing tendencies in quantum physics of today:

Thus, the heuristic opposition of Einstein against Quantum Field Theory (as “the other Einstein”, see e.g. J. Stachel [33: p. 283, 285]) might also be viewed, apart from other physical reasons, still, as an outcome of the failure of classical differential geometry –hence, in particular, of general relativity too– as it concerns the way the inherent in that theory (differential) “Calculus” is supplied, to cope with problems of the quantum theory. Indeed, we can further say that, looking at the same classical (: “newtonian) manner of definition of the “derivative”,

Einstein was demanding, within that framework, to abandon, even the notion of continuity(!) in physics, having thus, instead, to invent a “purely algebraic physics” (loc. cit., p. 285); therefore, in particular, as, of course, we can say, a (purely) algebraic analysis(!), as well.

In this connection, we can certainly refer here, as already done in the preceding, to the relevant remarks thereof of R.P. Feynman [7: p. 166]. as well as, to those of C.J. Isham [9: p. 393] (in this regard, see also e.g. A. Mallios [16]), concerning, namely, the ineffectiveness of the classical differential geometry, accordingly, of that one of a smooth (: $C^\infty$)-manifold too, within the quantum régime (yet, see (2.3) in the foregoing, along with our discussion in the subsequent Section 4).

On the other hand, the pertinence, in that context, of ADG to confronting with problems of quantum gravity still lies in its algebraic (viz. “leibnizian”, so to say) character: Indeed, the whole edifice of ADG is, by its very construction, sheaf-theoretic, sheaf theory being, of course, of an algebraic nature (see, for instance, H. Grauert–R. Remmert [9: p. VII]). Thus, ADG might also be construed, as an
algebraic ("leibnizian") manner of presenting the fundamentals of the classical differential geometry, while, at the same time, still getting, as an outstanding outcome (see, for example, (2.22), as well as, (2.1) in the preceding), the possibility of working, without any resort to a background "space", in the classical sense of the latter term, as for instance, to a "space-time continuum"(!), as it happens, instead, in the standard theory.

Certainly, the significance of the aforementioned two issues of ADG cannot be underestimated, while the same might be, in effect, quite well, what A. Einstein himself, by 1935 already, was looking for (see, for instance, still, J. Stachel [33: p. 285], as above).

4. Particular potential applications of ADG in the quantum régime.

We start, by presenting, within the framework of ADG, the relevant theory of Elemér E. Rosinger, pertaining to "generalized functions", whose algebra (sheaf), in particular, the "foamy" one, can be used, as a "sheaf of coefficients", defining thus, appropriately, a corresponding herewith "differential triad", basic ingredient to having a set-up in developing the mechanism of ADG (see (2.19) in the foregoing). For similar previous accounts, see also A. Mallios–E.E. Rosinger [22], [23], as well as, A. Mallios [17: Chapt. IX; Section 5].

However, before we come to the relevant exposition, it is still to be noticed, herewith, a fact of a particular significance, referring to the very structure of ADG (cf. (4.1) below, along with Subsection 4.(b) in the sequel), as it concerns two important special cases of the general theory of ADG, we are going to consider, by the subsequent discussion; the same are also characteristic of the way, one may have a "differential-geometric mechanism", in the sense of ADG, different, in character, from the classical manner of obtaining it (viz., via smooth manifolds, but, see also (4.1), along with (4.11) below). So it is, indeed quite useful (yet, rather, necessary(!)) to make the following remarks. That is,
even, if we take, as the base space of the sheaves involved, within the abstract context of ADG, a (smooth) manifold $X$, in the standard sense of this term (cf. thus the ensuing two Subsections below), its rôle (as the source of Calculus) is actually transferred now to the “sheaf of coefficients”, $A$. Yet, this is very organic, since it is essentially we(!), who make the calculations/experiments, based on our own “arithmetics”, viz. for the case in hand, again, via the algebra sheaf $A$.

Therefore,

the manifold $X$, as in (3.1) above, is just viewed, simply, as a particular topological space, being, of course, by its very definition, paracompact (Hausdorff); the latter condition is certainly, otherwise, very useful, indeed, when referring to cohomological issues: sheaf cohomology is, for that matter, apart from sheaf theory itself, the other fundamental ingredient of ADG. Yet, it may still happen that the topology of $X$ be chosen quite different from the initial, viz. the standard topology of the manifold $X$, i.e., the “locally euclidean” one); see, for instance, “Sorkin’s topology” in Subsection 4.(b) below.

However, as we shall see, by the ensuing discussion, the particular cases we look at in the sequel, do have, so to say, a

$$\text{(4.3)} \quad \text{newtonian spark (!),}$$

that is, something of a “starting point”, that will become better clear, by the subsequent rationale. Notwithstanding, as we shall also realize, in that context,

$$\text{(4.4)} \quad \text{this does not affect, at all(!), the “leibnizian” character of the mechanism of ADG,}$$

as the latter is inherently afforded, by the same two particular subsequent examples of the general theory.

The preceding certainly constitutes a fundamental special issue of paramount importance, indeed, for potential applications; the same could still be worthwhile to be viewed axiomatically(!), as well, contributing thus to our knowledge, as it
concerns the whole character of the general theory. In this regard, see also our previous account thereof, already in A. Mallios [17: Chapter IX; Section 5].

**Note 4.1.**— By still referring to our previous issue in (4.3), as we shall see, by the ensuing examples the so-called therein “newtonian spark” not only supplies the “structure sheaf” \( \mathcal{A} \), by the “spark” (fuse) of its “differential” mechanism, but what is, in effect, herewith of a particular importance, is that one assures, in that context, the validity of Poincaré Lemma, indeed, of an extraordinary importance of the whole mechanism of ADG. Thus, one can complete (4.3), by actually setting the equivalence:

\[
\text{“newtonian spark”} \iff \text{Poincaré Lemma}. \tag{4.5}
\]

So here again one realizes the fitness of

\[
\text{replacing of the “geometric character”, locally(!), of classical analysis, by cohomological issues.} \tag{4.6}
\]

However, more on this we shall see in the pertinent places below.

Thus, we come now to examine our first Example, pertaining to the situation described by (4.3), (4.5) above, straightforwardly, by the ensuing Subsection:

**4.(a). Rosinger’s algebra sheaf.**— Here the aforementioned already “newtonian spark”, as in (4.3) above, is nothing more, as we shall presently see, right below, than the classical

\[
\text{“} dx \text{”} \tag{4.7}
\]

of the standard theory of \( \mathcal{C}^\infty \)-manifolds. Thus, the above classical “\( dx \)” is, for the case at issue, prolonged, true, it is, in point of fact, “promoted”(!), so to speak, to an abstract,

\[
\text{“} \partial \text{”} \tag{4.8}
\]

in the sense of ADG (see, for instance, (2.14) in the preceding), defined now on an algebra sheaf (Rosinger’s), containing the standard one \( \mathcal{C}^\infty_X \), viz. the \( \mathbb{C} \)-algebra sheaf (of germs of \( \mathbb{C} \)-valued smooth functions \( \mathbb{R} \)-valued functions could also be considered,
of course]) on a given manifold $X$. Indeed, we can still say, in anticipation, that Rosinger’s algebra sheaf $\mathcal{A}_{nd}$, and, in extenso $\mathcal{A}_{foam}$ (see (4.20), (4.21) in the sequel) contain much more than $\mathcal{C}^\infty_X$ of the classical theory (cf. thus (4.14) below).

We depict the above, by the following diagram, whose notation will become more clear, through the ensuing discussion. Thus, we have;

$$
\begin{array}{ccc}
\mathcal{C}^\infty_X & \xrightarrow{d} & \Omega^1_X \\
\cap & & \cap \\
\mathcal{A}_{nd} & \xrightarrow{\partial} & \Omega^1_{nd} \equiv \Omega^1
\end{array}
$$

(4.9)

We proceed, by explaining the notation applied in (4.9); thus,

(4.10) $\mathcal{A}_{nd} \equiv \mathcal{A}$,

stands therein for Rosinger’s algebra sheaf a $\mathcal{C}$-algebra sheaf on $X$, the latter space being, by assumption, an open subset of $\mathbb{R}^k$. However, since the whole theory is, in point of fact, of a local nature, one may consider, instead, $\mathbb{R}^k$ just locally, that is, we can assume that $X$ is a smooth (: $\mathcal{C}^\infty$-)manifold. Notwithstanding, for simplicity’s sake, we adopt, throughout, that

(4.11) $X$ is open in $\mathbb{R}^k$.

Accordingly, $X$ being, by its very definition, a metrizable space, one concludes, in particular, that

(4.12) $X$ is a paracompact Hausdorff (topological) space.

See, for instance, J. Dugundji [6: p. 186, Theorem 5.3].

Now, Rosinger’s algebra sheaf $\mathcal{A} \equiv \mathcal{A}_{nd}$, as in (4.10) above, is actually an appropriate (cf. (4.13) in the sequel) quotient of a functional (algebra) sheaf: thus, technically speaking, it is defined, as a quotient of a functional (algebra) presheaf, the latter being proved, in particular, to be a “complete” one, therefore (J. Leray), a
sheaf. Yet, the corresponding, in that context, quotient algebras are defined, modulo a suitable (2-sided) ideal ("Rosinger's ideal"), which is essentially characterized, by what we may consider, as "Rosinger's asymptotic vanishing condition"; in particular, the latter is defined, via a

\[ \text{closed nowhere dense (hence, the subindex "nd", appeared in (4.10)) subset } \Gamma \text{ of } X, \text{ the same ideal consisting thus of those functions/elemenets of the (local section) algebras concerned, that vanish "eventually" (w.r.t. a parameter involved, a natural number, index) on any relatively compact subset of the complement of } \Gamma. \]

Concerning the precise definition of the preceding, we refer to A. Mallios [17: Chapt. IX; Section 5], or even (: to A. Mallios-E.E. Rosinger [22: p. 236; (2.2)]. Yet, by further looking at the same sheaf (4.10), as above, and also complementing the information we have through (4.9), we still note that we actually get, by the very definition of (4.10) (cf., for instance, E.E. Rosinger [29: p. 8; (1.2.15), (1.2.16), along with p. 367, (2)]),

\[ C_\infty^X \subsetneq \mathcal{D}'_X \subseteq \mathcal{A} \equiv \mathcal{A}_{nd}. \]  

Here the middle term in (4.14) denotes the sheaf (of germs) of Schwartz distributions on \( X \), viewed, as a \( \mathbb{C} \)-vector space sheaf on \( X \) (loc. cit., (5.19)).

On the other hand, the "basic differential operator"

\[ \partial : \mathcal{A} \rightarrow \Omega^1, \]

that one has to define, according to the general theory of ADG, see, for instance, (2.14) in the preceding, or even in A. Mallios [13: Chapt. VI; Section 1], is here provided by the presence of the first member in (4.14), that is, locally, by that one of a \((\mathbb{C})\)-algebra of the form

\[ C_\infty^\infty(U), \text{ with } U \text{ open in } X \subseteq \mathbb{R}^k, \]

(see also (4.11)), that virtually constitutes, within the present context, the "newtonian spark", hinted at in (4.3). Thus, the basic differential \( \partial \), as in (4.15), in now defined coordinate-wise, along the classical patterns, since the basic constituents of
the Rosinger’s algebra (pre)sheaf are (local sections of) cartesian product algebras of the form,

\[(C^\infty(U))^N,\]

with \(U\), as in (4.16), which then are “quotiented”, according to (4.11). Of course, the previously coordinate-wise (classically!) defined differential passes to the quotient. For technical details see A. Mallios [17: Chapt. IX; Subsection 5.(b)], or even to A. Mallios-E.E. Rosinger [22]. Thus, the overall moral, that is here, concluded according to the general principles of ADG, is the following;

Starting from any basic “differential triad”, in the sense of ADG (even a classical one, as e.g. a “locally euclidean one, this is the case, here-with, we can then perform any (functorial) operation, provided within the category of differential triads, to get thus at a new one [occasionally, more useful/flexible than the initially given one!].

Thus, by referring, in particular, to Rosinger’s algebra sheaf, as above, and the associated with it differential triad, we remark that, in view of (4.18), what we actually consider, in that context, is:

i) to take a denumerable cartesian product of the standard (newtonian-cartesian) differential triad

\[(C^\infty_X, d, \Omega^1),\]

as well as,

ii) to take, in particular, a pertinent quotient of the above, modulo Rosinger’s ideal, as indicated by (4.13).

In this connection, the aforesaid categorical treatment of ADG, has been occasionally considered already in A. Mallios [13: Chapt. VI; Sections 5, 6], as well as, in [17: Chapt. I; Section 5.(e), 5.(f): “pull-back” functor]; yet, an analogous fuller and systematic categorical study of differential triads has been recently supplied by the relevant work of M. Papatriantafillou [25], [26], [27].

On the other hand, one gets at an immense generalization of the above, by considering, in place of \(A_{ad}\), what we may call a Rosinger’s multi-foam algebra
sheaf, along with the associated differential triad,

(4.20) \((B_{\Lambda,J}, \partial, \Omega^1)\);

here the space \(X\), base of the sheaves concerned, is still given by (4.11), while the sheaf on \(X\) appeared in the first member of (4.20) is again a pertinent quotient of the \((\mathbb{C}-)\)algebra

(4.21) \(\mathcal{C}^\infty(X)^\Lambda\),

with \(\Lambda\) an upwards directed set, modulo an analogously defined (2-sided) ideal of the same algebra, with respect to a given (upwards) directed family \(J\) of “residual” subsets of \(X\), the “singularity-sets” of \(X\) (viz. those \(A \subseteq X\), with \(\overline{A} = X\)), the applied terminology, herewith, being hinted at potential physical applications: See A. Mallios-E.E. Rosinger [23], as well as, A. Mallios [17: Chapt. IX; Section 6]. Of course, the singularity-sets, as above, generalize the notion of nowhere dense sets, considered by (4.13) in the preceding. Hence, the increase of the types of “singularities”, one can cope with, in the framework of ADG, as explained in the foregoing.

Now, the same moral, that dominates our previous comments in (4.18), is, in point of fact, as we shall presently see in the sequel, the prevalent point of view also in the ensuing example, referring to another potential application of the very technique of ADG in problems of quantum gravity.

4.(b). Finitary incidence algebra sheaves.-- Similarly to the preceding Example 4.(a), here too, as already said, for that matter, one starts again from a smooth \((\mathcal{C}^\infty-)\)manifold \(X\), that still, for simplicity’s sake, we assume that it is just an open subset of the euclidean space \(\mathbb{R}^k\) (see (4.11)). However, as we shall see, this important (: very restrictive(!), otherwise) hypothesis will finally be used, only(!) in connection with (4.5)(!), as that was the case in the foregoing, as well: Thus,

(4.22) no “global use/presence” of the euclidean or even locally euclidean space is made, at all(!).

This important fact, indeed, ensures actually, the associated method, as it concerns, at least, its differential-geometric nature, its potential versatility.
Now, following R. Sorkin [32], one chooses the *locally finite open coverings* of $X$ (recall that the latter space is here also *paracompact Hausdorff*, see e.g. (4.12) in the preceding), while one further considers on the set $X$ the *topology generated by* such *locally finite open coverings of* $X$, as above. In this connection we also recall, for occasional use, in relation with ADG, that;

\begin{equation}
\text{the local frames of a given vector sheaf on a paracompact (Hausdorff)}
\end{equation}

space $X$ constitute a *cofinal subset of the locally finite open coverings* of $X$.

See [VS: Chapt. IV; p. 325, (8.42), along with Chapt. II: p. 127; (4.9)].

Now, the previous topological spaces, that are associated with locally finite open coverings of $X$, are further endowed, à la Sorkin (loc. cit.), with appropriate partial orders, becoming thus “*posets*”, alias, “*fintoposets*”, in the terminology of I. Raptis [28] (see also A. Mallios-I. Raptis [29]). On the other hand, these toposets are further suitably associated with certain finite-dimensional associative (non-abelian) linear $\mathbb{C}$-algebras, the so-called “*incidence Rota algebras*” (loc. cit.). The same algebras are further sheafified, the resulting sheaves leading finally to appropriate “*differential triads*”, in the sense, of course, that this notion is used by ADG (see [VS: Vol. II]). Here again, as it also was the case in our previous example in Subsection 4.(a) above, it is of a crucial significance the

\begin{equation}
\text{possibility of using the item connected with what we have called in the preceding, “newtonian spark” (cf. thus (4.5)).}
\end{equation}

As it was pointed out therein, the latter issue is the “*source*” of the “*differential mechanism*”, that one is supplied with, yet within the present context too,

\begin{equation}
\text{without employing, in effect, the euclidean, or even locally euclidean nature of the origin of that particular “spark”, in the way, at least, we are used to do it in the classical theory, thus far!}
\end{equation}

However, for the technical details thereof, we refer to the relevant work of A. Mallios-I. Raptis [20], along with that one of the same authors in [21]. We have thus herewith still another realization of the fact, being, in point of fact, a *fundamental moral of ADG* (see also A. Mallios [14]), that;
when we try to apply (differential) geometrical methods, more so in the quantum deep, it seems more natural to apply an analytic (:= algebraic) way (with symbols –recall here, for instance, “Feynman diagrams”– viz. a “Leibnizian” manner of looking at the things, in focus), not that one of the standard theory (:= “spatial-newtonian”).

Yet, what actually leads to the same thing,

it is quite natural to try to concoct, at each particular case, under consideration, the appropriate “differential geometric”-machinery (viz. “differential triad”), to cope with the problem at issue.

In toto, we could also mention herewith, a basic moral of ADG, in what actually concerns Quantum Field Theory. That is,

we should not relate any (quantum) field theory with the existence of an ad hoc given “continuum” (:= “space-time manifold”, whatsoever); this, of course, to the extent, at least, that we wish to apply therein (classical) differential geometry (CDG), since, in that context, the preponderant and really instrumental issue is, in effect, the relevant (differential-geometric) technique and not(!) the underlying space.

So, in other words, it is important to afford, in that context, a “differential-geometric” machinery, irrespective of the way the latter might have been displayed (cf., for instance, the preceding two examples), while, in any case, this particular way, “spatial”, or not (loc. cit.), should not intervene in the whole process, this being especially significant, when referred to the quantum régime (see also the relevant comments already in (1.19) in the preceding).

Indeed, in this regard, we can still remark that,

as it concerns the “infinitely small” (Feynman), the (differential) “geometry”, in the way, at least, that we use to look at it (viz. in the “newtonian-cartesian” one), is no more valid(!), since the same –namely, the “geometry” becomes –in point of fact, appears to us –in that deep, more “physical”(!), as it always is, for that matter, viz. “relational” (:= algebraic-analytic)!
Exactly at this point, we might also recall the quite relevant remarks here of D.R. Finkelstein [8: p. 155], in that (emphasis below is ours);

\begin{equation}
(4.30)
\text{“Physics was dominated by the Cartesian epistemology until the quantum theory.”}
\end{equation}

Relate the above with our previous considerations in Scholiun 2.1 in the preceding.

Yet, as a further illumination of the point of view of the whole formalism of ADG, we have to point out/clarify, herewith, once more, two fundamental issues of the aforesaid perspective, that also provide a potential outstanding application of the above formalism to ever present problems, thus far, of quantum gravity. That is, we have to note, in that context, that:

i) One can employ ADG, as a (differential) “geometry”, in the classical sense of the latter term, \textit{even in the quantum deep(!)}, provided, of course we accept the following correspondence/ “identification” (: axiomatic),

\begin{equation}
(4.31) \quad \text{fields} \longleftrightarrow \text{vector sheaves},
\end{equation}

that is, in other words what we have already called elsewhere “Selesnick’s correspondence” (see, for instance, A. Mallios [17: Chapt. II], for a detailed account of this subject matter).

ii) The same “geometry”, as above (viz. always, within the framework of ADG), \textit{can still be construed, as a “dynamical variable”, as well (see (2.6), in conjunction with (2.13), as well as, with (2.12)).

On the other hand, another technical issue, that should also be pointed out in this regard, is that, \textit{it, very likely, seems} that;

\begin{equation}
(4.32.1) \quad \text{“quantize analysis”),}
\end{equation}

as it concerns, in particular, its topological-linear character (this being the source of the Calculus), since the inherent/deeper nature of the same (: of the “analysis”), namely, the “algebraic”, or even the, so to say, “leibnizian” one, is already, viz., by its very definition, “quantized”!

Yet, by further commenting on our last claim, as above, we still note that;
(4.33) there is no, in effect, according to the same definitions, any “infinite” in (pure) algebra!

So it is, therefore, in “geometry”/topology (viz. in the so-called, “infinite”(!), a consequence, in fact, of the latter perspective), that we are, actually, entangled, when confronting with the quantum deep (: “small distances”). Consequently, our systematic endeavor, up to this day, in one way or another, to succeed in getting an appropriate “algebraization” of the whole scenario!

5. Scholium (: more on the “newtonian spark”).— We usually curve a linear structure, by “localizing” it (manifolds); in point of fact, this is a quite general device, referring, irrespective of the dimension (finite or infinite), to the (topological) vector space-model of our (cartesian) “geometry”. In the case of Analysis, an extraordinary issue, in that context, is that the classical Calculus that traditionally was hospitalized in (even, emanated from) topological vector space-structures (: euclidean spaces) still survived after this transport, a sine qua non, of course, of the justification, for that matter, of the previous movement, suggested, indeed, by particular important applications. Notwithstanding, a fundamental moral of the whole issue of ADG is that;

(5.1) the real corner-stone of the previous total enterprize is, in effect, what we have already called in the foregoing, the “newtonian spark”, in that context, a fact that might also be paralleled with the famous archimedean demand, for a pedestal (: “Δῶσο μοι πᾶ στῶ καὶ τῶν γὰρ κυνήγων—“give me somewhere to stand and I shall move the earth”).

That is, in other words, following now Leibniz, in what actually concerns (classical) differential geometry (CDG), what one virtually needs is to provide (according to ADG) the appropriate, concerning the particular problem, at issue, “differential-geometric mechanism”(!).

Furthermore, what is here of a particular significance, having also important potential applications (even, very likely(!), in quantum gravity too), is that:
the aforesaid “differential-geometric mechanism”, in the sense of ADG, does not actually depend, at all(!), on any space, as it was the case, so far, for the classical theory (CDG), the same mechanism referred now directly to the (“geometric”) objects, that live on the “space”.

Indeed, the latter issue in the above remarks, as in (5.2), is, most likely, what already Leibniz, at his time, was looking for! (See, for instance, N. Bourbaki [4: Chapt. I; Note historique, p. 161, ft. 1], or even A. Mallios [14: (2.1), along with comments following it]).

Thus, by further commenting on (5.2), we can still say, based also on our previous considerations in A. Mallios [15], that;

What one actually perceives appears to be the “sheafification” of a “local aspect/information” pertaining to the particular subject matter in focus. Besides,

the way we get a “local information”, may, in principle, be entirely different, in character, from the mechanism ( : inherent law–“physical”/relational procedure), which governs (hence, the manner too, we should essentially employ the aforesaid “sheafification”, viz. the global aspect of) that local information.

The above explains too what one essentially encounters, in connection with what we have called in the preceding “newtonian spark”.

Now, the replacement of a “field”

(cf. (2.15)), by its corresponding “Heisenberg ( : “matrix”) picture”, viz. by the “field”

(see A. Mallios [14: (9.20)], along with A. Mallios [17: Chapt. VII; (5.8), (5.11)]), hence, via its “principal sheaf” version,

(5.6) \( \langle \text{Aut}\mathcal{E}, D_{\text{End}\mathcal{E}}|_{\text{Aut}\mathcal{E}} \rangle \),
as well, may still be viewed, as being in accord with the “impossibility of having a “relativistic quantum field”, defined at a point” (!); see, for instance, N.N. Bogolubov et al. [4: p. 282, §10.4, p. 283, Theorem (Wightman) 10.6].

On the other hand, (5.3.1), as above, might also be construed, as another effectuation of the classical “local commutativity”, or “microscopic causality” “micro-causality” yet, “principle of relativistic microcausality”, or even “Einstein’s locality”.

On the other hand, by further meditating, a bit more, on our previous scholium in (5.3.1), we can actually reformulate it, as well, by remarking, in particular, that:

\[ \text{the deeper (algebraic) mechanism that might be inherent in (}\text{esoteric of}) \text{ a given local information (alias, of given local data), may, in general, be, quite well, independent of the way, one has drawn this information (}\text{the local data, concerned).} \]

Yet, in connection with the above remarks in (5.7) and our issue in (4.3), one may recall, in this regard, Wittgenstein’s motto [37: p. 74; 6.54], in that;

\[ \text{“... [one must]... throw away the ladder after he has climbed up it.”} \]

Now, as a fundamental spinoff of the above, one can still conceive, for instance, in that context, the classical (Machian) perspective of an

\[ \text{“action at a distance”(!).} \]

6. Concluding remarks (the “continuum”)— The purpose of this final section is to make clear, once more, that:

\[ \text{the notion of the “continuum”, as a “foundational element” is not actually the case, when physically speaking, at least (!) (and not only (!), see e.g. (6.5) in the sequel).} \]

Now, the inverted commas put on the word continuum, as above, refer, of course, to the way we usually understand that notion in the familiar terminology of the classical theory, where, in point of fact, we wish to ascribe to it a physical substance, that is, equivalently, to endow it with a physical meaning. And just hear one has the crux of the problem: That is,
we are actually influenced by our mathematical terminology–conception, in what virtually concerns the word “continuum”, i.e., the “cartesian”, in point of fact, perspective of the so-called “space-time”.

Thus, in other words, we make the following identifications:

(6.3) \[ \text{“physical space”} \leftrightarrow \text{mathematical “space”/“continuum”, viz. some } \mathbb{R}^n, \]
as a (finite dimensional) topological vector space.

However, it is exactly the above identifications, that is really the source of the problems: Indeed, as we have already remarked in other places (cf., for instance, A. Mallios [14: (1.4), or even (3.1)]),

(6.4) \[ \text{“physical space” is what virtually constitutes it, that is, in other words, what we may call, à la Leibniz, the “geometrical objects” themselves, that make up, what in effect, we perceive, as “space”, in the large, as well as, in the small.} \]

Therefore, in that respect, the substance of the “physical space”, as above, is thus discrete/granular, hence not at all corresponding to something “continuous”, viz. not-discrete, when physically/conceptually speaking. On the other hand, when mathematically speaking, a set is already, by its very definition, being thus “point-wise determined”, absolutely “discrete”, in character!

Thus, by referring to the mathematical notion of the “continuum”, as an \( \mathbb{R}^n \), \( n \in \mathbb{N} \),

(6.5) \[ \text{we note that the so-called “continuum” is, technically speaking, viz. as a mathematical term, our own definition of an } \mathbb{R}^n \ (n \in \mathbb{N}), \text{ as already said, viewed herewith not just, as a discrete set, as it actually is, for that matter, but now, as a topological (vector) space, this particular (mathematical) “structure” on (the set) } \mathbb{R}^n \text{ being also the source of the (newtonian) Calculus!} \]

In this connection, we are thus influenced, by our own mathematical experience of the concept of the “continuum”, in the way we defined it, as above, that is, as a particular finite dimensional (Hausdorff) topological vector space, a point of view that we also attribute, in turn, to what we actually perceive, as a “physical space”, this being further construed, as another “continuum”, this time, however, as a physical one (!),
based rather on a “dynamically/kinematically” ascribed description of the (physical) world; alas, something here in complete conflict with our actual (: experimental) experience, as it virtually concerns, at least, the quantum régime (see also, for instance, (4.30) in the preceding).

Now, in this context, the previous items:

\[ (6.6) \]

“dynamical-kinematical description” of the (physical) world, differential equations-theoretic point of view, Calculus, and “space-time continuum” are, in effect, intimately related and, in point of fact, tautosemous, in substance.

Strictly speaking, as a matter of fact,

*Calculus* is the source/cause of the first two items, as above, while, in turn, the same (Calculus) is the spin-off, as already said, of the newtonian-cartesian, so far, definition of the “space”, that is, of the so-called “geometrical” perception of it, yet, the outcome of the same “space-time continuum” $\equiv \mathbb{R}^n$ (cf. (6.5)).

On the other hand, the above differential part of the Calculus, viz. “differentiation of functions”, in principle, presupposes “good (– (: smooth) differentiable)–functions”, something that essentially depends on the “local behavior” of the functions concerned; hence, a fact that directly refers to the local nature of the domain of definition of the same functions, that is, to the local structure (: “geometry”) of the “euclidean space”, $\mathbb{R}^n$, itself. Consequently,

\[ (6.8) \]

we are actually entangled with the way the differential calculus (: “differentiation”, as a mechanism) is supplied (cf. (6.5)), therefore, the type of the “differentiable” functions that are thereby involved, or, in other words, that are “locally” defined on that particular “space” (extremely important, as well as, effective (!), anyhow, concerning the classical theory).

However, the applications of the same differential calculus, as above, in the domain of (classical) differential geometry, as a means of study (: working instrument) in that particular discipline, namely, that what we have already considered in the preceding, a “differential-geometric machinery”, yet, in other words, a “geometrical calculus”,...
à la Leibniz (see 1.17)), refers, in point of fact, to the very “geometrical objects” (Leibniz, loc. cit.), the same being actually (Leibniz, ibid., Riemann, see [14: (1.3)]) independent of any “space”, in the sense, at least, of (6.5), as above!

On the other hand, it is reasonable to think that,

the very character of what we may call

(6.9.1) “physical space” (see also (1.1)) is, in principle, the same, both in the large, as well as, in the small.

We are thus led to a dissonance, by applying our usual classical representation of the physical space (in the large), as an $\mathbb{R}^n$, irrespective, of course, of the tremendous success, thus far, of the latter perspective, when realizing, on the other hand (see also, for instance, (2.2), as well as, (4.30) in the preceding), that the same (physical) “space” is virtually quite different from what we are confronted with, when looking at the quantum régime, as it concerns the aforesaid classical perspective; see also A. Mallios [14: (8.10), along with (8.11)].

Consequently, the appeared inconveniences (: “singularities”), regarding, of course, applications of classical differential geometry, as a means of study, in that context, of the physical space/geometry” in the small, that is, to say, physical laws/“fields” (see also loc. cit., (3.21.1)) at the “quantum resolution”.

Thus, the “physical space”, as a whole, yet, according to recent advances in theoretical physics, concerning, in particular, the quantum deep, does not seem to be the usual “space-time” manifold, in the sense of the classical differential geometry–theory of smooth ($C^\infty$-)manifolds. Indeed, it appears that we have therein,

something foamy, very singular, or even something like what we may call, a “singularity manifold”, to refer, in that respect, to a rather recent utterance of R. Penrose, pertaining to a “true theory of quantum gravity”, by replacing the “present concept of spacetime at a singularity”. (See also, for instance, A. Mallios [14: (10.8), along with the subsequent discussion therein]).

Thus, concerning the “infinitely small”, or else “quantum resolution”,

the “geometry”, in the way we usually look at it (viz. in the “newtonian-cartesian” manner), is no more valid (!), in that context, since, at that deep, the same becomes thus, even to our senses (!), more “physical”, as, in point of fact, it always is, for that matter (see also (1.1) in the preceding, along with (6.9) above), viz. “relational” (: algebraic-analytic); the latter aspect is still a fundamental issue, in effect, of our experience, thus far, from ADG, as well!

Indeed, as already advocated in several places in the preceding, working within the context of ADG, we are able to look at fundamental concepts of physics, as, for instance, particles/fields (see also A. Mallios [16: (3.2)]) etc, without being compelled to stick to such “technical” notions, as e.g. “space-time” (!). In this regard, see also, for instance, (1.8) in the preceding, yet, loc. cit. (1.6).

Yet, in this connection, we are thus very likely, led to conclude that the entanglement of the “manifold” perspective in nowadays physics, especially, in the quantum domain, is to be attributed, in effect, to the relation of the former with the notion of “differential”; indeed, the latter is the main function, that is actually applied in our relevant rationale, in that context, while finally, as an upshot of the classical theory (: differential geometry), we are usually of the opinion that the same manifold concept is thus the unique(!) source of the notion of a “derivative”, covariant or not! Therefore, the significance, hereupon of the new proposal, as this is, provided, by ADG: That is, once more, we realize that

(6.12) to have a connection, we do not actually need a “manifold”, even if we momentarily borrow from such a concept the “infinitesimal (: “newtonian”) spark! The latter constitutes, in point of fact, the central moral of the entire study of ADG.

Thus, by looking at the standard question (cf., for instance, R.W. Sharpe [30: p. 2, ft. 2]),

(6.13) “what geometry on a manifold supports physics?”,

we can combine it now with the fundamental moral of ADG (see e.g. (6.12), as above, along with A. Mallios [14: (1.2), (3.21.2)]), that, in point of fact,
(6.14) differential geometry means, in effect, connection.

Consequently, blending the previous two aspects, as in (6.13) and (6.14), we are thus led to a response to (6.13), in the sense of affording a pertinent choice of “\( A \)”, more precisely speaking, the associated with it “differential triad” and the concomitant “differential-geometric mechanism”, à la ADG, suitable to the particular problem at issue; see thus, for instance, Subsections 4.(a), 4.(b) in the preceding, along with A. Mallios [12: p. 174; concluding remarks]. In this regard, see also the latest relevant account in A. Mallios–I. Raptis [21].

6.(a). ADG vis-à-vis a Unified Field Theory. — The quite ambitious(!) title of the present Subsection is rooted, in point of fact, on our previous comments in (6.9.1) and on the very essence of the point of view of the same Abstract Differential Geometry (ADG), as the latter can be applied, in that context, in conjunction, for instance, with Rosinger’s theory of “generalized functions”: The technical part of the aforesaid scheme, hinted at herewith, has been already expounded in A. Mallios-E.E. Rosinger [22], [23]; in this connection, see also our previous discussion in Subsection 4.(a) in the foregoing, along with A. Mallios [16: (5.20), (5.21)], as well as, [14: Sections 6, 8; see, in particular, (8.8) therein, or even (8.11), yet, Section 10]. Moreover, cf. also A. Mallios [17: Chapt. IX; Sections 5, 6, along with Section 10 therein, see e.g. (10.29)].

Now, as already said, the preceding just hint at a potential confrontation with the second issue in the title of this Subsection, through the machinery of ADG, that is, to say, in terms of the techniques of the classical differential geometry, being, however, freed now from the ever disturbing/pestilential “singularities”, and the like, of the classical approach to the problem at issue. Of course, this is due here, as already explained, throughout the preceding, to the absence, according to ADG, of any supporting “space”, that would also exclusively supply (: generate) the “differential-geometric” machinery employed, in that context (see the relevant citations, as before), a situation inherent, in effect, in the classical theory (CDG; cf. the previous Section 5, along with the concluding remarks above, preceding the present Subsection).
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