Dualities and inhomogeneous phases in dense quark matter with chiral and isospin imbalances in the framework of effective model

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It has been shown in \cite{2}--\cite{3} in the framework of Nambu–Jona-Lasinio model with the assumption of spatially homogeneous condensates that in the large-\(N_c\) limit (\(N_c\) is the number of quark colours) there exist three dual symmetries of the thermodynamic potential, which describes dense quark matter with chiral and isospin imbalances. The main duality is between the chiral symmetry breaking and the charged pion condensation phenomena. There have been a lot of studies and hints that the ground state could be characterized by spatially inhomogeneous condensates, so the question arises if duality is a rather deep property of the phase structure or just accidental property in the homogeneous case. In this paper we have shown that even if the phase diagram contains phases with spatially inhomogeneous condensates, it still possesses the property of this main duality. Two other dual symmetries are not realized in the theory if it is investigated within an inhomogeneous approach to a ground state. From various earlier studied aspects of QCD phase diagram of isospin asymmetric matter with possible inhomogeneous condensates, in the present paper the unified picture and full phase diagram of isospin imbalanced dense quark matter have been assembled. Acting on this diagram by a dual transformation, we obtained, in the framework of an approach with spatially inhomogeneous condensates and without any calculations, a full phase diagram of chirally asymmetric dense medium. This example shows that the duality is not just entertaining mathematical property but an instrument with very high predictivity power. The obtained phase diagram is quite rich and contains various spatially inhomogeneous phases.

I. INTRODUCTION

The phase diagram of strongly interacting matter (QCD phase diagram) is now one of the open questions in the Standard Model (SM) of elementary particle physics. The knowledge of QCD phase structure is essential for the understanding of such physical phenomena as compact stars and laboratory experiments on relativistic heavy-ion collisions. While the properties of matter at finite temperatures and vanishing densities are now understood quite well thanks to first principle lattice calculations and heavy-ion collisions, there is no consensus yet on the QCD phase structure at finite chemical potentials. In this case lattice simulations are not possible up to now and most of our understanding comes from calculations performed within effective models. One of the most used models is Nambu–Jona-Lasinio (NJL) model \cite{1} (see for review, e.g., in Refs. \cite{2}).

Besides the temperature and baryon chemical potential it is interesting to take into account nonzero isotopic and chiral chemical potentials. Isotopic (isospin) chemical potential allows us to consider systems with isospin imbalance (different numbers of \(u\) and \(d\) quarks). Chiral chemical potential accounts for a chiral imbalance (different average numbers of right-handed and left-handed quarks). In fact, there are physical systems with these imbalances, for example, matter inside neutron stars or in the fireball after heavy-ion collisions.

Recently, it was found a number of strong arguments, corroborated by model calculations, implied that at large and intermediate densities the phases with spatially inhomogeneous order parameters (condensates) can exist in different branches of modern physics. If ground state of a theory is inhomogeneous, then some of its spatial symmetries are spontaneously broken. This phenomenon is not a quite new concept. For example, the idea of density waves in nuclear matter was already discussed in 1960 by Overhauser \cite{3}. Moreover, in the 1970s there was much activity related to inhomogeneous pion condensation \cite{4}, first considered by Migdal \cite{5}. Spatially inhomogeneous phases in nucleon matter were also studied in \cite{6} in terms of sigma model for nucleon and meson fields. These phases in the form of charge and spin density waves are not exotic in condensed matter physics as well \cite{7}. In particular, the crystalline phases have been considered long time ago for superconductors by Larkin and Ovchinnikov \cite{8} as well as by Fulde and Ferrell \cite{9} (usually, the phase is called LOFF phase), etc. And more recently crystalline inhomogeneous phases in color superconducting quark matter was discussed in Ref. \cite{10}. Deryagin, Grigoriev, and Rubakov have shown that at high densities in the limit of infinite number of colors \(N_c\) the QCD ground state might be inhomogeneous and anisotropic so that the QCD ground state has the structure of a spatial standing wave \cite{11}. And of course spatially nonuniform condensates are intensively studied in the framework of effective theories such as NJL\textsubscript{4} model \cite{12} or different \((1+1)\)-dimensional theories with four-fermionic interactions \cite{13}. The modern state of investigations of dense baryonic matter in the framework of the spatially inhomogeneous condensate approach is presented in the reviews \cite{14,15}.

Unfortunately, general analytical methods which can be used in order to establish the inhomogeneity of the ground state of an arbitrary system are absent. So different ansatzes for spatially inhomogeneous condensates are used. For
example, for charged pion and/or superconducting condensates usually the single-plane-wave LOFF ansatz is employed \cite{10,17}. The simplest possible spatially inhomogeneous ansatz for chiral condensate in quantum field theories is an analog of the spin-density waves in condensed matter systems, and it is called “chiral density wave” (CDW) \cite{18}. Being an analytically treatable case, CDW ansatz has been the object of intense investigations during the course of the last 25 years and provides us with an excellent prototype for many generic features of inhomogeneous condensation in dense matter (see, e.g., Refs. \cite{19,27}). Of course, there can be a more favorable form of the inhomogeneous chiral condensate that minimizes thermodynamic potential even more effectively, but using CDW ansatz we can at least conclude that system favours inhomogeneity in some regions. More interesting fact is that recently it was argued in Refs \cite{27} that at small temperatures and rather high baryon densities the so-called Quarkyonic phase can be realized in QCD. This phase is described locally by the chiral condensate also in the form of chiral density waves. Furthermore, some realistic parity doublet hadron models, see, e.g., in Ref. \cite{28}, predict the CDW phase of dense nuclear matter.

Duality is another interesting possible property of dense matter, which permits to relate different regions of its phase diagram. For example, previously in Refs. \cite{29} it has been found the duality correspondence between chiral symmetry breaking (CSB) and superconductivity phenomena in low-dimensional NJL-type models extended by baryon and chiral chemical potentials. It means that if inside some definite region of a phase diagram the CSB phase is realized, then in a dually conjugated region the superconductivity must be observed, and vice versa. Note that this kind of duality is based on the so-called Pauli-Gursey symmetry between CSB and superconducting channels of the initial NJL-type Lagrangians both in two- and three-dimensional spacetimes \cite{29}. Moreover, just due to the Pauli-Gursey symmetry, it is also possible to assert that if in these systems a particular spatially inhomogeneous phase is realized, then dually conjugated phase also exists and spatially inhomogeneous.

In our previous papers \cite{30,32,33} the phase diagram of dense baryon matter with chiral and isospin imbalances has been considered in the framework of the massless NJL model with two quark flavors (see Lagrangian (1) below) both in (3+1)- and (1+1)-dimensional spacetime. The model is designed to describe CSB and charged pion condensation phenomena of dense matter. Using a homogeneous ansatz for chiral and pion condensates, we have proved in Refs \cite{30,32,33} that in the large-$N_c$ limit, where $N_c$ is the number of colors, there is a duality correspondence (symmetry) between CSB and charged pion condensation (CPC) phenomena within this effective model. However, the Pauli-Gursey-type symmetry between CSB and charged pion channels is absent in the model Lagrangian. So this kind of duality can be considered only as a dynamical effect, manifested in the symmetry of the thermodynamic potential (TDP) in the leading $1/N_c$-order of the model with respect to some dual (see below) transformations both of order parameters and chemical potentials. Since Pauli-Gursey-type symmetry of the Lagrangian is absent, the fact that there is a dual symmetry of the TDP, calculated in the framework of a homogeneous approach to condensates in the leading $1/N_c$-order, does not at all mean that there should be a duality between spatially inhomogeneous CSB and CPC phases. Hence, a special study is needed to check this possibility. \cite{1}

The proposed work is devoted to the solution of this problem. In our paper isotopically and chirally asymmetric dense baryon matter is investigated, as in Refs \cite{32,33}, in the framework of (3+1)-dimensional NJL model. However, in contrast to Refs \cite{32,33}, here we take into account the possibility of spatially inhomogeneous both chiral and charged pion condensates. For the chiral condensate we use the CDW ansatz, while the single-plane-wave LOFF ansatz is used for the charged pion one. Our main result is the conclusion that in the leading order of the large-$N_c$ expansion the NJL$_4$ model predicts the duality between these inhomogeneous phases. The phase structure itself with inhomogeneous condensates is not studied numerically here and it could be the subject of the future work. But the other method of obtaining information of the phase diagram has been used, namely, the studied duality was employed. We note that earlier some aspects of the phase diagram of isotopically asymmetric dense baryon matter have been studied in the framework of inhomogeneous condensate approach, and it turns out that it is possible to combine the findings of these studies together and draw full phase diagram. Then, applying just to this phase diagram the dual mapping, we obtain the full phase diagram of chirally asymmetric dense baryon matter. The obtained phase diagram is quite rich and contains various inhomogeneous phases, both inhomogeneous chiral symmetry breaking phase and inhomogeneous charged pion condensation phase. This example shows that the duality is not just interesting and entertaining mathematical property but a potent instrument with very high predictive capabilities.

The paper is organized as follows. First, in Sec. II a (3+1)-dimensional massless NJL model with two quark flavors ($u$ and $d$ quarks) that includes four kinds of chemical potentials, $\mu_B, \mu_1, \mu_5, \mu_5$, is presented. Here spatially inhomogeneous CDW and single-plane-wave LOFF ansatzes respectively for chiral and charged pion condensates are introduced, and the expression for the TDP of the system in the leading $1/N_c$-order is obtained. In Sec. III we show that in the homogeneous approach to condensates the TDP is invariant with respect to three dual symmetries. One of them, $D_H$ \cite{21}, corresponds to the duality of the phase structure of the model between CSB and CPC phenomena. Just this symmetry of the TDP is realized within an inhomogeneous approach. However, two other dual symmetries of the TDP have no analogs in the inhomogeneous approach to the investigation of the ground state of the system. In Sec. IV we show that using only the duality property of dense quark matter, it is possible, basing on the well-known

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1 Earlier, it was found in the paper \cite{10} that in the framework of the (3+1)-dimensional NJL model (1) the spatially inhomogeneous, e.g., CPC phase can be realized in dense isotopically asymmetric matter at some chemical potential region. Then, if we had confidence in the existence of a duality between nonuniform CSB and CPC phases, we could predict the existence of an inhomogeneous CSB phase in the dually conjugated region of chemical potentials.
(μ_1, μ)-QCD phase diagram, to construct the (μ_15, μ)-phase portrait without any numerical studies.

II. THE MODEL AND THERMODYNAMIC POTENTIAL

We study a phase structure of the two flavored massless (3+1)-dimensional NJL model with several chemical potentials. Its Lagrangian, which is symmetrical under global color SU(N_c) group, has the form

\[ L = \bar{q} \left[ \gamma^\nu i \partial_\nu + \frac{\mu_B}{3} \gamma^0 + \frac{\mu_I}{2} \tau_3 \gamma^0 + \mu_5 \gamma^0 \gamma^5 + \frac{\mu_{15}}{2} \tau_3 \gamma^0 \gamma^5 \right] q + \frac{G}{N_c} \left[ (\bar{q}q)^2 + (\bar{q}i\gamma^5 \tau_3 q)^2 \right] \tag{1} \]

and describes dense quark matter with two massless u and d quarks, i.e. q in (1) is the flavor doublet, q = (q_u, q_d)^T, where q_u and q_d are four-component Dirac spinors as well as color N_c-plets (the summation in (1) over flavor, color, and spinor indices is implied); \( \tau_k \ (k = 1, 2, 3) \) are Pauli matrices. The Lagrangian (1) contains baryon \( \mu_B \), isospin \( \mu_I \), chiral isospin \( \mu_{15} \) and chiral \( \mu_5 \) chemical potentials. In other words, this model is able to describe the properties of quark matter with nonzero baryon \( n_B \), isospin \( n_I \), chiral isospin \( n_{15} \) and chiral \( n_5 \) densities which are the quantities, thermodynamically conjugated to chemical potentials \( \mu_B, \mu_I, \mu_{15} \) and \( \mu_5 \), respectively.

The quantities \( n_B, n_I \) and \( n_{15} \) are densities of conserved charges, which correspond to the invariance of Lagrangian (1) with respect to the abelian \( U_B(1), U_I(1) \) and \( U_{15}(1) \) groups, where

\[ U_B(1) : q \rightarrow \exp(i\alpha/3)q; \ U_I(1) : q \rightarrow \exp(i\alpha \tau_3/2)q; \ U_{15}(1) : q \rightarrow \exp(i\alpha \tau_5/2)q. \tag{2} \]

So we have from (2) that \( n_B = \bar{q}q \delta q/3, n_I = \bar{q}\gamma^0 \tau_3 q/2 \) and \( n_{15} = \bar{q}\gamma^0 \tau_3 q/2 \). We would like also to remark that, in addition to (2), Lagrangian (1) is invariant with respect to the electromagnetic \( U_Q(1) \) group,

\[ U_Q(1) : q \rightarrow \exp(iQ\alpha)q, \tag{3} \]

where \( Q = \text{diag}(2/3, -1/3) \). However, the chiral chemical potential \( \mu_5 \) does not correspond to a conserved quantity of the model (1). It is usually introduced in order to describe a system on the time scales, when all chirality changing processes are finished in the system, and it is in the state of thermodynamical equilibrium with some fixed value of the chiral density \( n_5 \) [33]. The ground state expectation values of \( n_B, n_I, n_{15} \) and \( n_5 \) can be found by differentiating the thermodynamic potential of the system (1) with respect to the corresponding chemical potentials.

In our previous paper [33] it was shown that if an approach with spatially homogeneous condensates is applied to the model (1), then thermodynamic potential (TDP) of the system obeys some symmetry, which is manifested in the duality between CSB and CPC phenomena of dense quark matter (see below). The goal of the present paper is to show that this duality must also manifest itself in the approach with inhomogeneous condensates. At the same time, for chiral and charged pion condensates we use the CDW and single-plane-wave LOFF ansatzes, respectively.

To find the TDP of the system, we use a semibosonized version of the Lagrangian (1), which contains composite bosonic fields \( \sigma(x) \) and \( \pi_a(x) \) (a = 1, 2, 3) (in what follows, we use the notations \( \nu = \mu_B/3, \nu = \mu_I/2 \) and \( \nu_5 = \mu_{15}/2 \)):

\[ \bar{L} = \bar{q} \left[ \gamma^\nu i \partial_\nu + \mu_0 \gamma^0 + \nu_3 \gamma^0 + \mu_5 \gamma^0 \gamma^5 + \nu_3 \tau_3 \gamma^0 \gamma^5 - \sigma - i\gamma^5 \pi_a \right] q - \frac{N_c}{4G} \left[ \sigma^a + \pi_a \pi_a \right]. \tag{4} \]

In (4) and below the summation over repeated indices is implied. From the auxiliary Lagrangian (1) one gets the equations for the bosonic fields

\[ \sigma(x) = -2 \frac{G}{N_c} (\bar{q}q); \quad \pi_a(x) = -2 \frac{G}{N_c} (\bar{q}i\gamma^5 \tau_{3a} q). \tag{5} \]

Note that the composite bosonic field \( \pi_3(x) \) can be identified with the physical \( n_0(x) \)-meson field, whereas the physical \( \pi^\pm(x) \)-meson fields are the following combinations of the composite fields, \( \pi^\pm(x) = (\pi_1(x) \mp i\pi_2(x))/\sqrt{2} \). Obviously, the semibosonized Lagrangian \( \bar{L} \) is equivalent to the initial Lagrangian (1) when using the equations (4). Furthermore, it is clear from (2) and footnote [2] that the composite bosonic fields \( \pi_a(x) \) are transformed under the isospin \( U_I(1) \) and axial isospin \( U_{15}(1) \) groups in the following manner:

\[ U_I(1) : \quad \sigma \rightarrow \sigma; \quad \pi_3 \rightarrow \pi_3; \quad \pi_1 \rightarrow \cos(\alpha) \pi_1 + \sin(\alpha) \pi_2; \quad \pi_2 \rightarrow \cos(\alpha) \pi_2 - \sin(\alpha) \pi_1, \]
\[ U_{15}(1) : \quad \pi_1 \rightarrow \pi_1; \quad \pi_2 \rightarrow \pi_2; \quad \sigma \rightarrow \cos(\alpha) \sigma + \sin(\alpha) \pi_3; \quad \pi_3 \rightarrow \cos(\alpha) \pi_3 - \sin(\alpha) \sigma. \tag{6} \]

Starting from the theory (1), one obtains in the leading order of the large \( N_c \)-expansion (i.e. in the one-fermion loop approximation) the following path integral expression for the effective action \( S_{\text{eff}}(\sigma, \pi_a) \) of the bosonic \( \sigma(x) \) and \( \pi_a(x) \) fields:

\[ \exp(iS_{\text{eff}}(\sigma, \pi_a)) = N' \int [dq] [dq] \exp \left( i \int \bar{L} d^4x \right), \]
where
\[ S_{\text{eff}}(\sigma, \pi_a) = -N_c \int d^4x \left[ \frac{\sigma^2 + \pi_a^2}{4G} \right] + \tilde{S}_{\text{eff}} \]  \tag{7}
and \( N' \) is a normalization constant. The quark contribution to the effective action, i.e. the term \( \tilde{S}_{\text{eff}} \) in Eq. (7), is given by:
\[ \exp(i\tilde{S}_{\text{eff}}) = N' \int [dq][dq] \exp \left( i \left\{ \bar{q} \left[ \gamma^\mu \partial_\mu + \mu \gamma^0 + \nu \tau_3 \gamma^0 + \mu_5 \gamma^0 \gamma^5 + \nu_5 \tau_3 \gamma^0 \gamma^5 - \sigma - i\gamma^5 \pi_a \tau_a \right] q \right\} d^4x \right). \tag{8} \]

The ground state expectation values \( \langle \sigma(x) \rangle \) and \( \langle \pi_a(x) \rangle \) of the composite bosonic fields are determined by the saddle point equations,
\[ \frac{\delta S_{\text{eff}}}{\delta \sigma(x)} = 0, \quad \frac{\delta S_{\text{eff}}}{\delta \pi_a(x)} = 0, \tag{9} \]
where \( a = 1, 2, 3 \). It is clear from Eq. (8) that if \( \langle \sigma(x) \rangle \neq 0 \) and/or \( \langle \pi_3(x) \rangle \neq 0 \), then the axial isospin \( U_A(1) \) symmetry of the model is spontaneously broken down, whereas at \( \langle \pi_1(x) \rangle \neq 0 \) and/or \( \langle \pi_2(x) \rangle \neq 0 \) we have a spontaneous breaking of the isospin \( U_I(1) \) symmetry. Since in the last case the ground state expectation values, or condensates, both of the field \( \pi^+(x) \) and of the field \( \pi^-(x) \) are nonzero, this phase is usually called the charged pion condensation (CPC) phase. It is easy to see from Eq. (9) that the nonzero condensates \( \langle \pi_{1,2}(x) \rangle \) or \( \langle \pi^\pm(x) \rangle \) are not invariant with respect to the electromagnetic \( U_Q(1) \) transformations (3) of the flavor quark doublet. Hence in the CPC phase the electromagnetic \( U_Q(1) \) invariance of the model (1) is also broken spontaneously, so superconductivity is an unavoidable property of the CPC phase.

In vacuum, i.e., in the state corresponding to an empty space with zero particle density and zero values of the chemical potentials \( \mu, \nu, \mu_5 \) and \( \nu_5 \), the quantities \( \langle \sigma(x) \rangle \) and \( \langle \pi_a(x) \rangle \) do not depend on space coordinate \( x \). However, in a dense medium, when some of the chemical potentials are nonzero quantities, the ground state expectation values of bosonic fields might have a nontrivial dependence on spatial coordinates. In particular, in this paper we use the following spatially inhomogeneous CDW ansatz for chiral condensate and the single-plane-wave LOFF ansatz for charged pion condensates (for simplicity we suppose that wavevectors of the inhomogeneous condensates are directed along the \( x^1 \) coordinate axis):
\[ \langle \sigma(x) \rangle = M \cos(2kx^1), \quad \langle \pi_3(x) \rangle = M \sin(2kx^1), \]
\[ \langle \pi_1(x) \rangle = \Delta \cos(2k'x^1), \quad \langle \pi_2(x) \rangle = \Delta \sin(2k'x^1), \tag{10} \]
where gaps \( M, \Delta \) and wavevectors \( k, k' \) are constant dynamical quantities. In fact, they are coordinates of the global minimum point of the TDP \( \Omega(M, k, k', \Delta) \). In the leading order of the large-\( N_c \) expansion it is defined by the following expression:
\[ \int d^4x \Omega(M, k, k', \Delta) = -\frac{1}{N_c} S_{\text{eff}}\{\sigma(x), \pi_a(x)\}|_{\sigma(x) = \langle \sigma(x) \rangle, \pi_a(x) = \langle \pi_a(x) \rangle}, \tag{11} \]
which gives
\[ \int d^4x \Omega(M, k, k', \Delta) = \int d^4x \frac{M^2 + \Delta^2}{4G} + \frac{i}{N_c} \ln \left( \int [dq][dq] \exp \left( i \int d^4x \bar{q} \tilde{D}q \right) \right), \tag{12} \]
where
\[ \bar{q} \tilde{D}q = \bar{q} \left[ \gamma^\mu \partial_\mu + \mu \gamma^0 + \nu \tau_3 \gamma^0 + \mu_5 \gamma^0 \gamma^5 + \nu_5 \tau_3 \gamma^0 \gamma^5 - M \exp(2i\gamma^5 \tau_3 kx^1) \right] q \]
\[ - \Delta (\bar{q} \gamma^5 q) e^{-2ik'x^1} - \Delta (\bar{q} \gamma^5 q) e^{2ik'x^1}. \]
(Remember that in this formula \( q \) is indeed a flavor doublet, i.e. \( q = (q_u, q_d)^T \).) To proceed, let us introduce in Eqs. (12)–(13) the new quark doublets, \( \psi \) and \( \bar{\psi} \), by the so-called Weinberg (or chiral) transformation of these fields (31, 32), \( \psi = \exp(i\gamma^5kx^1 + i\gamma^5k'x^1)q \) and \( \bar{\psi} = \bar{q} \exp(i\gamma^5kx^1 - i\gamma^5k'x^1) \). Since this transformation of quark fields does not change the path integral measure in Eq. (12), the expression (12) for the TDP is easily transformed to the following one:
\[ \int d^4x \Omega(M, k, k', \Delta) = \int d^4x \frac{M^2 + \Delta^2}{4G} + \frac{i}{N_c} \ln \left( \int [d\bar{\psi}][d\psi] \exp \left( i \int d^4x \bar{\psi} \tilde{D}\psi \right) \right), \tag{14} \]

3 Strictly speaking, performing Weinberg transformation of quark fields in Eq. (12), one can obtain in the path integral measure a factor, which however does not depend on the dynamical variables \( M, \Delta, k, \) and \( k' \). Hence, we ignore this unessential factor in the following calculations. Note that only in the case when there is an interaction between spinor and gauge fields there might appear a nontrivial, i.e. dependent on dynamical variables, path integral measure, generated by Weinberg transformation of spinors. This unobvious fact follows from the investigations by Fujikawa [31].
where instead of the \(x\)-dependent Dirac operator \(\tilde{D}\) a new \(x\)-independent operator \(D = i\gamma^\mu \partial_\mu + \mu\gamma^0 + \nu\gamma^5 + \mu_5\gamma^0\gamma^5 + \nu_5\gamma^0\gamma^5 + \tau_3\gamma^0\gamma^5 + \tau_3\gamma^1\gamma^k + \tau_3\gamma^1k' - M - i\gamma^5\Delta\tau_1\) appears. In this case path integral can be evaluated and one get for the TDP an expression that reads

\[
\Omega(M, \Delta, k, k') = \frac{M^2 + \Delta^2}{4G} + i \frac{\text{Tr}_{sf} \ln D}{\int d^4x} = \frac{M^2 + \Delta^2}{4G} + i \int \frac{d^4p}{(2\pi)^4} \ln \det \tilde{D}(p),
\]

where

\[
\tilde{D}(p) = \gamma_p + \mu\gamma^0 + \nu\gamma^5 + \mu_5\gamma^0\gamma^5 + \nu_5\gamma^0\gamma^5 - M + \gamma^5k + \gamma^1k';
\]

\[
A = \gamma_p + \mu\gamma^0 + \nu_5\gamma^0\gamma^5 - M - \gamma^5k - \gamma^1k';
\]

\[
B = \gamma_p - \mu\gamma^0 + \mu_5\gamma^0\gamma^5 - \nu_5\gamma^0\gamma^5 - M - \gamma^5k + \gamma^1k';
\]

\[
U = V = -i\gamma^5\Delta,
\]

so the quantity \(\tilde{D}(p)\) from (16) is indeed a 8×8 matrix whose determinant in Eq. (15) can be calculated on the basis of the following general relations

\[
\text{Det} \tilde{D}(p) \equiv \text{det} \left( \begin{array}{cc} A & U \\ V & B \end{array} \right) = \text{det}[-VU + VAV^{-1}B] = \text{det}[\Delta^2 + \gamma^5A\gamma^5B] = \text{det}[BA - BUB^{-1}V].
\]

III. DUALITY BETWEEN CSB AND CPC PHENOMENA

A. Spatially homogeneous approach to condensates

In this case the wave vectors \(k\) and \(k'\) in the inhomogeneous ansatzes (10) are zero by assumption and, as a result, for the quantity (18) one can get the following expression (for details see Ref. 33)

\[
\text{Det} \tilde{D}(p) = (\eta^4 - 2a\eta^2 + b\eta + c) (\eta^4 - 2a\eta^2 + b\eta + c) \equiv P_+ (p_0) P_-(p_0),
\]

where \(\eta = p_0 + \mu, |\eta| = \sqrt{p_1^2 + p_2^2 + p_3^2}\) and

\[
a_\pm = M^2 + \Delta^2 + (|\eta| \pm \mu_5)^2 + \nu_5^2, \quad b_\pm = \pm 8(|\eta| \pm \mu_5)\nu_5;
\]

\[
c_\pm = a_\pm - 4\nu_5^2 (M^2 + (|\eta| \pm \mu_5)^2) - 4\nu_5^2 (\Delta^2 + (|\eta| \pm \mu_5)^2) - 4\nu_5^2.
\]

One can notice that in this case the expression (19) and hence the TDP (15) are invariant with respect to the so-called duality transformation \(\mathcal{D}_H\) of the order parameters and chemical potentials,

\[
\mathcal{D}_H : \quad M \leftrightarrow \Delta, \quad \nu \leftrightarrow \nu_5.
\]

Other parameters such as \(\mu, \mu_5\) and the coupling constant \(G\) do not change under this transformation. It means that if at \(\mu, \mu_5, \nu = A, \nu_5 = B\) the global minimum of the TDP lies at the point \((M = M_0, \Delta = \Delta_0)\), then at \(\mu, \mu_5, \nu = B, \nu_5 = A\) it is at the point \((M = M_0, \Delta = \Delta_0)\). The results of the paper [33] support this conclusion. In particular, it was shown there that if at some fixed values of chemical potentials, e.g., the homogeneous CSB phase is realized, then in the dually conjugated region of the chemical potentials, i.e. at \(\nu \leftrightarrow \nu_5\) and unchanged values of \(\mu\) and \(\mu_5\), the homogeneous CPC phase must be observed in the system and vice versa.

The duality similar to Eq. (21), i.e. the duality between CSB and CPC phenomena, is also observed between phase structures of gauge theories with different gauge groups and matter content in the framework of the so-called orbifold equivalence formalism in the large-\(N_c\) limit [57].

In addition to invariance of the TDP with respect to the duality transformation (21), in the case of homogeneous approach to the condensates there are two other transformations of chemical potentials and order parameters, \(\mathcal{D}_{HM}\) and \(\mathcal{D}_{H\Delta}\), which we call constrained dual transformations, that leave the TDP unchanged. Indeed, one can check (see in Ref. [32]) that under the constraint \(\Delta = 0\) and at fixed values of \(\mu, \nu\) the TDP (15) at \(k, k' = 0\) is invariant with respect to the permutation \(\mu_5 \leftrightarrow \nu_5\). It is the so-called constrained dual transformation \(\mathcal{D}_{HM}\). Whereas at \(M = 0\) and at fixed values of \(\mu, \nu_5\) the TDP (14) at \(k, k' = 0\) is invariant under the permutation \(\mu_5 \leftrightarrow \nu\) and \(\mu_5 \leftrightarrow \nu_5\). It is the so-called constrained dual transformation \(\mathcal{D}_{H\Delta}\). The symmetry of the TDP with respect to \(\mathcal{D}_{HM}\) (with respect to \(\mathcal{D}_{H\Delta}\)) means that if at some values of the chemical potentials the CSB phase (the CPC phase) is realized, then at \(\mu_5 \leftrightarrow \nu_5\) (at \(\mu_5 \leftrightarrow \nu\)) the same phase will be observed, if dynamically or due to other reasons the charged pion condensate \(\Delta\) is equal to zero (the chiral condensate \(M\) is equal to zero) in the system. Hence, in the case of a homogeneous approach to condensates the symmetry of the TDP under the constrained \(\mathcal{D}_{HM}\) and \(\mathcal{D}_{H\Delta}\) dual transformations can also be useful in relating phase structure of the model between dually conjugated regions of the chemical potentials.
B. Duality in inhomogeneous case

Let us now discuss the possibility of the duality between CSB and CPC phenomena when spatially inhomogeneous approach in the form (10) to condensates is used, i.e. we suppose that \( k \neq 0, k' \neq 0 \). In this case, using any program of analytic calculations, it is also possible to obtain an exact analytical expression for the quantity (15) in the form of the 8-th order polynomial,

\[
\text{Det} \overline{D}(p) = a_8 p^8 + a_7 p^7 + a_6 p^6 + a_5 p^5 + a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0,
\]

where \( p = p_0 + \mu, a_8 = 1, a_7 = 0 \) and

\[
a_6 = -4 \left( k^2 + k^2 + \Delta^2 + \nu^2 + M^2 + \mu_5^2 + \nu_5^2 + |\vec{p}|^2 \right), \quad a_5 = 16 \left( k' k \mu_5 + \mu_5 \nu_5 - (k' \nu + k \nu_5) p_1 \right),
\]

\[
a_4 = 2 \left( 3 k'^2 + 3 k^2 + 2 k^2 \Delta^2 + k^2 \Delta^2 + \nu^2 + 3 M^2 + \mu_5^2 + \nu_5^2 + p_1^2 + 3 p_2^2 + 3 p_3^2 \right) +
\]

\[
2k^2 \left( 3 \Delta^2 + \nu^2 + M^2 + \mu_5^2 + \nu_5^2 + p_1^2 + 3 p_2^2 + 3 p_3^2 \right) + 3 \left( \Delta^2 + M^2 \right)^2 + 3 \nu^4 + 3 \nu^6 +
\]

\[
+6 \left( M^2 + \Delta^2 \right) |\vec{p}|^2 + 3 \mu_5^4 + 3 |\vec{p}|^4 + 2 |\vec{p}|^2 |\vec{p}|^2 \right) + 6 \Delta^2 \nu^2 + 6 M^2 \nu_5^2 +
\]

\[
24 k \nu (\mu_5 p_1 - k' \nu_5) + 24 k' \nu_5 (\mu_5 p_1 - k \nu) + 24 k_5 p_1 (k' \nu_5 + k \nu) +
\]

\[
2 \Delta |\vec{p}|^2 + 2 M^2 \nu^2 + 2 (\nu^2 + \nu_5^2) \mu_5 + 12 \mu_5^2 (\Delta^2 + M^2) + 2 (\nu^2 + \nu_5^2) |\vec{p}|^2 +
\]

\[
2 \nu^2 \left( 3 \Delta^2 + M^2 + \mu_5^2 + \nu_5^2 + |\vec{p}|^2 \right) + 2 \nu^2 \left( \Delta^2 + M^2 + \mu_5^2 + \nu_5^2 + |\vec{p}|^2 \right),
\]

\[
a_3 = 32 \left( k'^3 (\nu p_1 - k \mu_5) + k^3 (\nu_5 p_1 - k' \nu_5) + k^2 (\nu \mu_5 \nu_5 - k \nu_5 p_1) + k^2 (\nu \mu_5 \nu_5 - k' \nu p_1) -
\]

\[
k' k \mu_5 \Delta^2 - \nu^2 + M^2 + \mu_5^2 - \nu_5^2 - p_1^2 + p_2^2 + p_3^2 - \nu_5 \mu_5 (\Delta^2 + \nu^2 + M^2 + \mu_5^2 + \nu_5^2 - |\vec{p}|^2) -
\]

\[
k' \nu_5 (\Delta^2 + \nu^2 + M^2 - \mu_5^2 - \nu_5^2 + |\vec{p}|^2) + k \mu_5 (\Delta^2 - \nu^2 + M^2 - \mu_5^2 - \nu_5^2 + |\vec{p}|^2) \right), ...
\]

We do not give here exact analytical expressions for the coefficients \( a_{0,1,2} \) of the polynomial (22), since they are too extensive and take up too much space. Nevertheless, it is possible to check that all the coefficients \( a_i \) of the polynomial (22) are invariant with respect to the following duality transformation \( D_I \) of the chemical potentials, absolute values \( \Delta, M \) and wavevectors \( k, k' \) of the condensates (10)

\[
D_I : \quad M \leftrightarrow \Delta, \quad \nu \leftrightarrow \nu_5, \quad k \leftrightarrow k'.
\]

(The invariance of \( a_{3,4,5,6} \) with respect to the dual transformation (26) is directly seen from Eqs (23)-(25).) As a result one can find that the whole TDP \( \Omega(M, \Delta, k, k') \) (15) of the system is also invariant under the dual transformation \( D_I \). It means that in the leading order of the large-\( N_c \) approximation the duality correspondence between CSB and CPC phenomena, found in (33) for homogenous case, is valid even in the case if in the system the phases with spatially inhomogeneous condensates are realized. The dual invariance (26) of the TDP allows one to perform also a dual mapping of some well-known QCD phase diagrams in order to predict a phase structure of the system under influence of more exotic external conditions, such as chiral asymmetry, etc (see below in the section IV).

In contrast, in the framework of an inhomogeneous approach to condensates in the form (10), we were unable to find analogues of two other dual symmetries, the constrained \( D_{H\Delta} \) and \( D_{HM} \) symmetries, which are inherent in the model under consideration within a spatially homogeneous approach to condensates (see in Refs. [33] and/or the end of the previous subsection III A). Indeed, assuming that in Eqs. (23)-(25) \( \Delta = 0 \) and \( k' = 0 \) (or \( M = 0 \) and \( k = 0 \)). We see that these coefficients of the polynomial (22) are not invariant with respect to transposition \( \nu_5 \leftrightarrow \mu_5 \) (or \( \nu \leftrightarrow \mu_5 \)). Hence, the analog of the dual symmetry \( D_{HM} \) (or dual symmetry \( D_{H\Delta} \)) of the TDP in the homogenous case is not realized in the case, when an inhomogeneous approach to condensates is used. (In fact, in the case of inhomogeneous condensates it is necessary to use some other transformations. Indeed, in this approach there arise usually some spurious (unphysical) terms in the TDP (15).) So it cannot be considered as a physical TDP of the system. To overcome this difficulty, one should use a more physical regularization procedure or apply to TDP the subtraction procedure using the rule presented, e.g., in Ref. [31] (see there Eqs (47)-(48)). According to it, we can construct the physical TDP \( \Omega^{phys}(M, \Delta, k, k') \). It turns out that \( \Omega^{phys} \) is invariant under the dual transformation \( D_I \) (26). However, both \( \Omega^{phys}(M, \Delta = 0, k, k') \) and \( \Omega^{phys}(M = 0, \Delta, k = 0, k') \) are not invariant with respect to the transpositions \( \nu_5 \leftrightarrow \mu_5 \) and \( \nu \leftrightarrow \mu_5 \), correspondingly, i.e. the constrained dual symmetries are not the properties of the TDP in the case of spatially inhomogeneous approach to condensates.)

C. Duality and the physical point

Though the chiral limit is an excellent approximation to QCD, one knows that in reality the current quark masses are nonzero. Let us now consider briefly the situation with nonzero current quark mass \( m_0 \) (physical point). The
way that one can deal with CDW and/or single plane wave ansatz for charged pion condensate at the physical point is the same as in [20]. In this case the Lagrangian looks like

\[ L = \bar{q} \left[ \gamma^\mu i \partial_\mu + \frac{\mu_B}{3} \gamma^0 + \frac{\mu_1}{2} \tau_3 \gamma^0 + \mu_5 \gamma^0 \gamma^5 + \frac{\mu_5}{2} r_3 \tau_3 \gamma^0 \gamma^5 - m_0 \right] q + \frac{G}{N_c} \left[ (\bar{q}q)^2 + (\bar{q}u^5 \tau q)^2 \right]. \]  

(27)

If current quark mass \( m_0 \neq 0 \), then the ansatz (10) should be transformed to the following one

\[ (\sigma(x)) = M \cos(2kx^1) - m_0, \quad (\pi_3(x)) = M \sin(2kx^1), \]
\[ (\pi_1(x)) = \Delta \cos(2k'x^1), \quad (\pi_2(x)) = \Delta \sin(2k'x^1), \]  

(28)

where gaps \( M, \Delta \) and wavevectors \( k, k' \) are the same quantities. Using this ansatz in the definition (11), one can obtain, instead of Eq. (14), the following expression for the TDP \( \Omega(M,k,k') \) in the leading large-\( N_c \) order

\[ \int d^4x \Omega(M,\Delta,k,k') = \int d^4x \frac{M^2 - 2m_0 M \cos(2kx^1) + m_0^2 + \Delta^2}{4G} + iTr_{sfx} \ln D, \]  

(29)

where \( D \) is the same \( x \)-independent Dirac operator as in Eq. (13). The averaging over spacetime coordinates in Eq. (29) supposes that \( \int d^4x \cos(2kx^1) = 0 \) if \( k \neq 0 \) and \( \int d^4x \cos(2kx^1) = \int d^4x \) if \( k = 0 \). So, similar to the Ref. [20], we get

\[ \Omega(M,\Delta,k,k') = \frac{M^2 - 2m_0 M \delta_{k,0} + m_0^2 + \Delta^2}{4G} + \int \frac{d^4p}{(2\pi)^4} \ln \text{Det} \overline{D}(p), \]  

(30)

where \( \overline{D}(p) \) is the momentum space representation of the Dirac operator \( D \) (see in Eq. (10)). One can see that the only difference compared to the case of zero current quark mass is one term that is proportional to delta symbol. Hence, if the chiral condensate is homogeneous, then it is easy to see that the duality \( D_H \) is no longer the exact symmetry of the TDP, but still one can show that it is a rather good approximation. \footnote{In more detail, the influence of \( m_0 \neq 0 \) on the duality between CSB and CPC phenomena is investigated in [22] and it is shown there that approximate duality \( D_H \) of the phase diagram is observed in the framework of the NJL4 model, and it is a quite good approximation.}

But if duality between inhomogeneous phases is concerned, the duality is exact even in the case of nonzero current quark mass \( m_0 \), which is a rather interesting in itself.

### IV. STRENGTH OF DUALITY AND CHIRALLY ASYMMETRIC QCD PHASE DIAGRAM

Recall that chiral asymmetry is a relatively recent phenomenon, so the QCD phase diagram under this condition has not been studied in detail so far. But investigations of the QCD-phase structure at \( \mu_5, \mu_{I5} = 0 \) were performed in the presence of its isospin asymmetry, i.e. at \( \mu_1 = 2\mu \neq 0 \). Let us discuss how it is possible to obtain information about the phase structure of chirally asymmetric dense quark matter (at \( \mu_5 \neq 0 \) and/or \( \mu_{I5} = 2\mu_5 \neq 0 \)), applying the so-called duality mapping procedure to the well-known QCD-phase diagram with zero chiral asymmetry (at \( \mu_5 = 0 \) and \( \nu_5 = 0 \)).

First, let us discuss what one knows about the QCD (\( \nu, \mu \))-phase diagram (dense quark matter with isospin asymmetry). In the framework of a homogeneous ansatz for condensates the QCD phase diagram has been studied in many approaches and models and can be described schematically in the following way (see, e.g., the phase diagram of Fig. 3 in the paper [10]). If the values of isospin chemical potential \( \mu_1 \) is larger than pion mass \( m_\pi \), i.e. at \( \nu > m_\pi/2 \), there is homogeneous charged pion condensation (CPC) phase at rather small values of \( \mu \). But if \( \mu_1 \) is less than pion mass, \( \nu < m_\pi/2 \), then homogeneous CSB phase with zero baryon density (it is the so-called vacuum state) is realized for rather small \( \mu \). But for rather large \( \mu \) the normal quark matter (NQM) phase is arranged (for arbitrary values of \( \nu \)), in which baryon density is nonzero and chiral symmetry is broken.

This is the sketch of the phase diagram in the physical situation of nonzero current quark mass \( m_0 \). In the chiral limit (zero current quark mass) it is even simpler for in this case the homogeneous PC phase is realized at any nonzero values of isospin chemical potential provided that the value of the quark number chemical potential \( \mu \) is not too large.

If one considers only homogeneous CSB phase, then the phase diagram was obtained, e.g., in [22, 41, 41] (similar finding has been obtained in [21], where this situation was considered in the chiral limit in quark meson model with quarks). It was found that at rather small values of \( \nu \) (\( \nu < 60 \text{ MeV} \)) and for \( \mu > 300 \text{ MeV} \) there can exist a region of inhomogeneous CSB phase, namely CDW ansatz has been considered in these papers, and it was shown that through solitonic modulations are energetically favored against CDW ansatz, it turned out that these changes of ansatz have only mild influence on the phase structure in our model. These calculations were done for simplicity in the chiral limit but probably it is a good approximation, the case of non-zero current quark mass in the case of zero isospin asymmetry was considered in [22] and qualitative picture stays the same. The only changes are that the critical point
FIG. 1. Combined schematic \((\nu, \mu)\)-phase diagram at \(\mu_5 = 0\) and \(\nu_5 = 0\). Here CPC denotes the homogeneous charged pion condensation phase, ICPC denotes the inhomogeneous charged pion condensation phase, CSB and ICSB denotes homogeneous and inhomogeneous phase with broken chiral symmetry, NQM is the normal quark matter phase, where charged pion condensate is zero and quarks have small masses.

FIG. 2. \((\nu_5, \mu)\)-phase diagram at \(\nu = 0\) and \(\mu_5 = 0\). This plot is duality conjugated one to a phase diagram of Fig. 1. All the notations are the same as in Fig. 1.

shifts towards smaller temperatures and larger quark chemical potentials with increase of current quark mass and the region of inhomogeneous phase shrinks because its lower border shifts to the larger values of baryon chemical potential. On the other hand, the possibility of the existence of a spatially inhomogeneous charged pion condensation (ICPC) phase of quark matter has been investigated in the framework of NJL model, e.g., in Ref. [16], where the situation of inhomogeneous CPC and homogeneous CSB phases has been considered. It was shown there that ICPC phase, in which pion condensate exists in a single-plane-wave form, can be realized at a rather high value of \(\nu, \nu \gtrsim 400\) MeV.

Let us now try to connect these three situations to get full \((\nu, \mu)\) phase diagram. In [41] where ICSB and CPC phases have been considered the CDW was not found in the regions where CPC phase was considered to be in homogeneous case, and CPC and ICPC phases have been found in [? ] (At larger values of \(\nu\) the region occupied by ICSB phase decreases but the phase continues to be present at the phase diagram up to the values of \(\nu = 60\) MeV and probably even higher values, there are no plots at larger values in [40, 41]. It is not clear how far ICSB phase goes to larger values of \(\nu\) but it seems it goes over values of \(\mu = 0.3\) GeV almost to the point \(\nu \approx 200\). And vise versa in [? ] there has not been found ICPC phase at rather small values of \(\nu\) where ICSB phase is realized in [41]. And if one assumes that there is no mixed phase with ICSB and ICPC condensates then one can attach this figures and sketch the whole \((\nu, \mu)\) phase diagram in inhomogeneous case. If there is a mixed phase with ICSB and ICPC condensates then phase diagram can become even more complicated and some regions of normal quark matter phase, homogeneous or inhomogeneous phases can be exchanged to the mixed phase (inhomogeneous) and the inhomogeneous phases can become only larger. Putting together the results of the study of the QCD phase diagram, performed in the above mentioned papers [16, 21, 22, 40, 41], etc. both in spatially homogeneous and inhomogeneous approaches to order parameters (condensates), as well as in different areas of chemical potential values, and our above arguments one can imagine schematically the following \((\nu, \mu)\)-phase portrait of dense quark matter with isotopic asymmetry that is depicted in Fig 1. Note that it corresponds to quark matter, in which chiral asymmetry is absent \((\mu_5 = 0, \nu_5 = 0)\) and quarks are massive, \(m_0 \neq 0\).

Now, let us try to get some information about the phase diagram of dense quark matter when it has chiral asymmetry (in the most general case it means that nonzero values of \(\mu_5\) and \(\nu_5\) should be taken into account in the model). Moreover, we suppose that its phase structure is investigated in the framework of a spatially inhomogeneous approach to condensates. If \(\mu_5 \neq 0\), then there is only one way, namely the numerical analysis of the TDP (30) or (15). This task is rather complicated and has not been solved yet. However, if \(\mu_5 = 0\) but \(\nu_5 \neq 0\), then in order to understand how the phase portrait of the model looks, it is not necessary to carry out a numerical study of the TDP (15). In this case it is sufficient to perform a dual mapping of the phase diagram of Fig. 1 in order to obtain an approximate \((\nu_5, \mu)\)-phase portrait at \(\nu = 0\) and \(\mu_5 = 0\).

Recall that in the chiral limit and in the leading order of the large-\(N_c\) expansion there is an exact dual symmetry between CSB and CPC phenomena predicted by the NJL model (1) (see in Refs [22, 32]). As a result, there is an

\[ As it was shown in the paper [39] (see there Appendix B), finite spatial areas with nonzero chiral isospin densities, i.e. with \(\nu_5 \neq 0\), might exist inside compact (neutron) stars due to the chiral separation effect.\]
(exact) dual correspondence between some phase diagrams, the consequence of which is the possibility to obtain some phase diagrams of the model without cumbersome numerical calculations, simply acting by the dual transformations \( D_H \) or \( D_I \) (see below) on the previously obtained other phase portraits. However, if \( m_0 \neq 0 \), then the duality is only approximate symmetry of the TDP, it is observed in the NJL model when some of the chemical potentials are intermediate, i.e., greater than \( m_\pi \) (see in Ref. [39]). And the dual correspondence between phase portraits is absent when chemical potentials are small, i.e., \( \lesssim m_\pi \).

Hence, to get the approximate \((\nu_5, \mu)\)-phase diagram of the model \( \nu_5 = 0 \) at \( \nu = 0 \) and \( \mu_5 = 0 \), one just need to take the \((\nu, \mu)\)-phase diagram at \( \nu_5 = 0 \) and \( \mu_5 = 0 \) and make the following transformations of it: (i) exchange \( \nu \) to the axis \( \nu_5 \), (ii) outside the region \( \omega = \{(\nu, \mu) : \nu \leq m_\pi, \mu \leq 300 \text{ MeV}\} \) perform the following renaming of the phases ICSB \( \leftrightarrow \) ICPC, CSB \( \leftrightarrow \) CPC, and NQM phase stays intact here, and (iii) the phase that lies in the region \( \bar{\omega} = \{(\nu_5, \mu) : \nu_5 \leq m_\pi, \mu \leq 300 \text{ MeV}\} \) of the \((\nu_5, \mu)\)-phase diagram. The obtained phase diagram is shown in Fig. 2 and it is called dually \( D_I \) conjugated to a phase diagram of Fig. 1. Since we suppose that current quark mass \( m_0 \) is nonzero (in this case the dual symmetry between CSB and CPC phenomena is only approximate one [39]), the dual \( D_I \) mapping of the diagram of Fig. 1, i.e. the \((\nu_5, \mu)\)-phase diagram of Fig. 2, presents only an approximate schematic phase portrait of the model (as far as inhomogeneous phases are concerned it is exact). But, nevertheless, it is enough to make several conclusions and predictions about the properties of dense medium with chiral asymmetry.

The obtained phase diagram of Fig. 2 is quite rich and contains an inhomogeneous CSB phase as well as inhomogeneous CPC phase. It is clear that (inhomogeneous) charged pion condensation phenomenon can be created in the system at a rather small value of \( \nu_5 \) even at zero value of the chemical potential \( \nu \) (for rather high values of quark number chemical potential \( \mu \)). Finally, we see that ICPC phase in Fig. 2 is located in the region corresponding to rather high values of \( \mu \) and, most likely, with nonzero baryon densities. Hence, the chiral isotopic asymmetry in the form of \( \nu_5 \neq 0 \) promotes the creation of the CPC phenomenon in dense quark matter. Earlier, this effect was established in the framework of a spatially homogeneous approach to condensates [32, 33, 39] both in the chiral limit and at \( m_0 \neq 0 \), and in the present paper we generalize this conclusion to inhomogeneous case, in addition. Let us add a fly in the ointment. One cannot say about the presence of the ICPC phase for sure because the duality for homogeneous condensates is an approximate one and it does not work very well in the region of small values of \( \nu_5 \) and \( \nu \), and the consideration of [41] have not included the PC phase that is pushed to the values of larger \( \nu \) and does not stand in the way and they have not compared the inhomogeneous CSB phase with the homogeneous PC phase in the chiral limit, but in the duality conjugated case the homogeneous CSB phase (analog of homogeneous PC phase in the previous case) is not pushed to the larger values of \( \nu_5 \) and can stand in the way. But we know for sure that there is a local minimum point corresponding to ICPC phase (ICPC phase is boosted by chiral imbalance) and there is another one corresponding to CSB phase, to determine which one is the GMP one needs to employ calculation and it cannot be studied in terms of duality only. If one assumes that ICPC phase is the GMP then one can see that the chiral isospin chemical potential \( \nu_5 \) generates the charged pion condensation phenomenon even better in the inhomogeneous case and if not one can say that inhomogeneous pion condensation phase is favoured over the homogeneous one but still chiral symmetry breaking phase is real vacuum of the system. There could be homogeneous PC phase below the ICPC phase due to duality but from calculations [39] it is known that there is no homogeneous PC there. One can also see that chiral isospin chemical potential \( \nu_5 \) can lead to the ICSB phase at values of baryon chemical potential around 0.2 GeV (see Fig. 2). At first glance one can think that the baryon chemical potential is rather small here, but due to rather large values of chiral isospin chemical potential, at least part of this phase can possess non-zero baryon density. One can see that at non-zero values of \( \nu_5 \) there seems to be very rich phase diagram featuring as ICSB at rather large values of \( \nu_5 \) as probably ICPC phase at small and moderate values of \( \nu_5 \).

This example shows that duality between CSB and CPC phenomena is not just an interesting mathematical artifact, but a powerful tool in scrutinizing the QCD phase diagram. One knows nothing about the phase structure of inhomogeneous condensate in QCD with chiral isospin chemical potential, does not know even whether it is favoured anywhere in the phase diagram at all, and can get the full phase diagram only due to the use of duality.

V. SUMMARY AND CONCLUSIONS

In this paper dense quark matter with isospin and chiral imbalance and dualities (symmetries) of its phase diagram are considered in the framework of (3+1)-dimensional NJL model (1) in the case of spatially inhomogeneous approach to chiral and charged pion condensates.

Earlier, the phase structure of this model has been studied in details in Refs. [32, 33] in the context of spatially homogeneous approach to condensates and in the chiral limit, \( m_0 = 0 \). In particular, it was shown there that in the leading large-\( N_c \) order the TDP of the system is invariant under three different dual transformations, \( D_H \), \( D_H \Delta \) and \( D_{HM} \) (see in the present section [11A]). One of them, \( D_H \), is realized on the phase portrait of the model as a duality correspondence between CSB and CPC phases. It means that if at some fixed values of \( \mu, \mu_5, \mu_1 = A \) and \( \mu_5 = B \), e.g., the CSB (or the charged PC) phase is realized in the model, then at the dually conjugated values of the chemical potentials, i.e. at the same values of \( \mu \) and \( \mu_5 \), but at the permuted values of other chemical potentials, \( \mu_1 = B \) and
\( \mu_{15} = A \), the CPC (or the CSB) phase must be arranged. So, it is enough to know the phase structure of the model at \( \mu_1 < \mu_{15} \), in order to establish the phase structure at \( \mu_1 > \mu_{15} \). Knowing condensates and other dynamical and thermodynamical quantities of the system, e.g., in the CSB phase, one can then obtain the corresponding quantities in the dually conjugated charged PC phase of the model, by simply performing there the duality transformation, \( \mu_1 \leftrightarrow \mu_{15} \). Two other symmetries of the TDP, \( D_{HM} \) and \( D_{H\Delta} \), can also impose some restrictions on the shape of the CSB and CPC phases, respectively (see in [32]).

Note that similar dualities have also been considered in the framework of universality principle (large-\( N_c \) orbifold equivalence) of phase diagrams in QCD and QCD-like theories in the limit of large \( N_c \) [35, 38]. In particular, it was noted there that in the chiral limit QCD at \( \mu_{15} \neq 0 \) might be equivalent to QCD at \( \mu_1 \neq 0 \), etc (see remarks in Sec. 4 of [37]). Since the Lagrangian (1) itself does not have a symmetry that would automatically lead to the dual symmetries of its phase portrait, an interesting question arise whether the duality of the phase portrait obtained in the large-\( N_c \) limit is a deep property of the theory described by Lagrangian (1) or just an accidental feature. In order to get some hints in this direction, the discussed duality between CSB and CPC phenomena has been established both within the homogeneous and inhomogeneous approaches to condensates, but only in the framework of the NJL\(_2\) model [30, 31] (but in this case the dualities in inhomogeneous and homogeneous case is quite similar in terms of proving them, and duality in inhomogeneous case is much easier to show). To confirm these results and to ensure that the duality and related phenomena are intrinsic also to (3+1)-dimensional variant of the model (1), in the present paper we study the possibility of dual symmetries of its thermodynamic potential using, in contrast to Ref. [33], a more extended approach based on the spatially inhomogeneous condensates.

In this paper we have obtained in the leading 1/\( N_c \) order an exact expression [15] of the thermodynamic potential of the model (1), when for chiral and charged pion condensates the CDW and single-plane-wave LOFF ansatzes are used, respectively (see Eq. (10)). A detailed analysis of the phase structure of the model was not carried out in this case. However, we were able to establish that the TDP [15] of the system possesses the dual symmetry \( D_{I_1} \) [20], which necessarily leads to a duality between CSB and CPC phenomena. Hence, in the chiral limit both in the homogeneous and more extended spatially inhomogeneous approaches to the ground state of the NJL\(_4\) system (1), we observe the duality between CSB and CPC phenomena. So, in our opinion, this type of duality is not an artifact of the method of investigation, but the true property of a chirally and isotopically asymmetric dense medium described by the NJL\(_4\) Lagrangian (1).

Moreover, it is known that when non-zero current quark masses is included to the consideration (at the physical point) the duality is not exact, though it is a good approximation [39]. When one considers the duality in the inhomogeneous case (between inhomogeneous phases) then the duality is exact even at the physical point.

It is necessary to bear in mind that in the model (1) an arbitrary dual invariance of its TDP calculated in the approach with homogeneous condensates is not automatically transferred to the case of inhomogeneous condensates. Indeed, the duality between CSB and CPC are realized in the model (1) in both approaches, however other dual symmetries, \( D_{HM} \) and \( D_{H\Delta} \), of the TDP [15] at \( k = 0 \) and \( k' = 0 \) are not observed in the case with inhomogeneous condensates, i.e. at \( \mu_{15} = 0 \), etc (see remarks in [37]).

In this paper we have not studied the phase portrait in the framework of (3+1)-dimensional massless NJL model itself, but we have shown that even if the phase diagram contains phases with nonzero inhomogeneous condensates, it possesses the property of duality (dual symmetry). We showed that in terms of TDP in the leading order of the large-\( N_c \) approximation.

In the literature, the \((\nu, \mu)\)-QCD phase diagram has been studied very intensively and it is understood well in homogeneous case. The various aspects of the \((\nu, \mu)\)-phase diagram with possible inhomogeneous condensates were investigated in [16, 40, 41], etc. It has been shown that it is possible to use these shreds and combine them into one unified picture and draw full \((\nu, \mu)\)-phase diagram in inhomogeneous case (see in Fig. 1). When this interesting thing has been completed, from this assembled phase diagram the phase diagram in a completely different situation has been obtained, namely \((\nu\delta, \mu)\)-diagram of chirally asymmetric QCD matter (see in Fig. 2). It has been shown that at nonzero values of \(\nu\delta\) there is a very rich phase diagram featuring both the ICSB phase at rather high values of \(\nu\delta\) and the ICPC phase at small and moderate values of \(\nu\delta\). The phase diagram of Fig. 2 was obtained only by using the duality between CSB and CPC phenomena which is an inherent property of dense quark matter. This instance shows that the duality is not just entertaining mathematical gaud and interesting, but useless mathematical property. In our opinion, it is a potent instrument with very high predictivity power.

It has also been hinted that in inhomogeneous case CPC phase is generated in dense quark matter even by infinitesimally small values of chiral isospin chemical potential \(\nu\delta\). Qualitatively, the same behaviour has been predicted in the framework of (1+1)-dimensional NJL model, this concurrence once more consolidates the confidence that NJL\(_2\)

\[ \text{As a counterexample, we can bring the duality between CSB and superconductivity in some (1+1)- and (2+1)-dimensional theories [29]. But there the original Lagrangians are invariant with respect to the so-called Pauli-Gursey transformation, which transforms the chiral interaction channel into a superconducting one, and vice versa. As a result, a duality between these phenomena appears on the phase portrait.} \]
model can be used as a legit laboratory for the qualitative simulation of specific properties of QCD.

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