DYNAMICAL BEHAVIOR OF A SYSTEM MODELING WAVE BIFURCATIONS WITH HIGHER ORDER VISCOSITY

TONG LI*, XIAOYAN WANG†, AND JINGHUA YAO‡

Abstract. We obtain new bifurcation diagrams from a system of partial differential equations modeling wave bifurcations through deriving its full dynamics on the center manifold(s). Specific parametrization, cancellation given by \( \int_{-\pi}^{\pi} e^{ikx} \, dx = 0 \) for nonzero integer \( k \) and comparisons with the authors’ former works are devised, used, and made respectively. During the process, several subtle and deeper points in the study of dynamics driven by partial differential equations are demonstrated through complete computations and careful comparisons.

Keywords: Spectrum; Resolvent; Half pitchfork bifurcation; Limit cycle bifurcation; Center manifold; Symmetry; Implicit function theorem.

Mathematics Subject Classification(2010): 37G; 35P; 34G; 34B

1. Introduction

The current work is a continuation of our former works [34, 22] on dynamical behaviors of solutions to partial differential equations with symmetries. Here our primary goals include the study of new bifurcation pictures based on a different spectral scenario, the demonstration of a second viewpoint of computing the dynamics on center manifold, the illustration of a mechanism to get limit cycle, and a comparison between the current work and our former works on equivariant Hopf bifurcations. The current study together with our former studies in [34, 22] covers the generic bifurcation dynamics for the system (1.1) below and demonstrate two different ways to do center manifold computations. One common feature in our work here and [34, 22] is that we use Fourier analysis to decompose the problem to get admissible critical configuration points for our bifurcation analysis. This feature has similarity with E. Hopf’s study [15] on fluid turbulence and with L. Rayleigh’s study [29] on fluid stability. Meanwhile, we hope that our work could also shed some light on the vanishing viscosity method. With these goals in mind,
we will adopt a similar organizational structure as in our former works for the ease of comparisons. We also made an effort to make the exposition be in an elementary way and in a begin-to-end thoughts order for the ease of reading.

1.1. The system to be studied. Let us first recall the system we are going to study and some physical background (see also [34, 22]). In this work, we continue to study the following class of systems in continuum mechanics

\[
\begin{aligned}
\partial_t \tau - \partial_x u &= -a \partial^4_x \tau, \\
\partial_t u - \partial_x \sigma(\tau) &= -\delta \partial^2_x u - \varepsilon \partial^4_x u
\end{aligned}
\] (1.1)

on the spatial periodic domain \( \mathbb{R}^1 / [-M, M] \) where \( M \) is any positive constant. In system (1.1), \( \tau = \tau(x, t) \) and \( u = u(x, t) \) are the scalar unknown functions. The scalar function \( \sigma(\tau) \) is usually called flux function in mathematics and pressure law in physics. The three parameters \( a, \delta, \varepsilon \) are called diffusion coefficients. We emphasize that the domain is \( \mathbb{R}^1 / [-M, M] \), which means that we have periodic boundary conditions.

Without loss of generality, we can always consider the case \( M = \pi \) and \( \varepsilon = 1 \) or else we can do the following two successive scalings and renamings given by

\[
\begin{aligned}
t &\mapsto \bar{t} = \frac{\pi}{M} t, \quad x \mapsto \bar{x} = \frac{\pi}{M} x, \quad a \mapsto \bar{a} = \frac{\pi^3}{M^3} a, \\
\delta &\mapsto \bar{\delta} = \frac{\pi}{M} \delta, \quad \varepsilon \mapsto \bar{\varepsilon} = \frac{\pi^3}{M^3} \varepsilon.
\end{aligned}
\]

and

\[
\begin{aligned}
t &\mapsto \tilde{t} = \varepsilon t, \quad x \mapsto \tilde{x} = x, \quad u \mapsto \tilde{u} = u, \quad \tau \mapsto \tilde{\tau} = \varepsilon \tau, \\
a &\mapsto \tilde{a} = \varepsilon^{-1} a, \quad \sigma(\tau) \mapsto \tilde{\sigma}(\tilde{\tau}) = \varepsilon^{-1} \sigma(\varepsilon^{-1} \tilde{\tau}) = \varepsilon^{-1} \sigma(\tau), \quad \delta \mapsto \tilde{\delta} = \varepsilon^{-1} \delta.
\end{aligned}
\]

Meanwhile, we observe that any constant state \((\tau_0, u_0)\) is a solution to system (1.1). We consider the constant state \((0, 0)\) without loss of generality. This is obvious by first choosing \((\tau_0, u_0)\) in the physical range and then using the invariance of system (1.1) in the translation group actions \( u \mapsto u + h \) for any \( h \in \mathbb{R}^1 \) and redefining \( \sigma(\tau) \) by \( \sigma(\tau_0 + \tau) \).

Systems of form (1.1) are generic in classical continuum mechanics ([11, 7, 10] and the references therein) as they describe Newton’s Second Law of motion, in gas dynamics, for example the \( p \)-system (see in particular Chapter 2 of Dafermos [7] and Nishida [27]). When \( M = \infty \), systems of form (1.1) are also connected with the Kuramoto-Sivashinsky and related systems when we seek traveling waves solutions (see [10, 12, 19, 20, 30, 31]).
If we regard the term \((-a\partial_4^4 \tau, -\varepsilon\partial_4^4 x u)\) as vanishing viscosity term for the system (1.1) and study the process \(a \to 0^+, \varepsilon \to 0^+\), this is the vanishing viscosity method in partial differential equations. Of course, this requires \(a > 0\) and \(\varepsilon > 0\). However, in our work here, we are not restricted to the case with \(a > 0\) and \(\varepsilon > 0\) which gives dissipation. We will see later that as long as \((a_c, \delta_c)\) is an admissible critical configuration point, we can get bifurcation dynamics under necessary conditions to be specified later. If there are admissible critical configuration point \((a_c, \delta_c)\) such that \(a_c > 0\) after scalings and renaming, we can scale back and consider the vanishing viscosity limit. Our study in particular indicates that in such cases we should avoid the parameter ranges which lead to bifurcations during the limiting process when we use the vanishing viscosity method.

1.2. Main Results. In order to state our main results clearly, we first introduce the admissible critical configuration set \(A(k_0)\) for an arbitrary nonzero integer \(k_0\). Actually, we will see that \(A(k_0) = A(-k_0)\). Hence we may regard \(k_0\) as a positive integer in the remaining of the paper.

**Definition 1.1.** For a given positive integer \(k_0\), we say that the pair \((a_c, \delta_c) \in \mathbb{R}^2\) is an admissible critical configuration point if the following three conditions are fulfilled:
(a) \(a_c k_0^4 (k_0^2 - \delta_c) + \sigma'(0) = 0\).
(b) for any nonzero integer \(k\) such that \(|k| \neq |k_0|\) and nonzero real number \(\omega\),
\[
(a_c k^4 (k^2 - \delta_c k^2) + \sigma'(0) - \omega^2) + i[(a_c + 1)k^4 - \delta_c k^2] \neq 0.
\]
(c) \((a_c + 1)k_0^2 - \delta_c \neq 0\).
Consequently, \(A(k_0)\) is the collection of all admissible critical configuration points \((a_c, \delta_c)\) associate with \(k_0\).

Now we are in a position to state our main results.

**Theorem 1.2.** Let \((a_c, \delta_c)\) be any admissible critical configuration point, \(\mu = \mu(\nu) := (\delta_c - k_0^2)\nu_1 + a_c \nu_2\) be the bifurcation parameter with \(\mu(a_c, \delta_c, k_0) = 0\). Then system (1.1) or equivalently (2.1) admits a center manifold reduction and either undergoes a half pitchfork bifurcation or bifurcates a limit cycle from the zero solution \((\tau, u) = (0, 0)\) near \(\mu = 0\) in the Hilbert space \(Y\) consisting of functions
in $L^2_{\mathrm{per}}(-\pi, \pi)$ with zero mean over one period. The full dynamics on the center manifold(s) has the following form

$$\frac{d}{dt} A(t) = \mathfrak{a} \mu A(t) + \mathfrak{b} A^2(t) A^*(t) + O(|A|^5)$$

where $A = A(t) \in \mathbb{C}^1$ and the complex conjugate pair $(A(t), A(t)^*)$ gives the coordinate of the center space and represents the dynamics for $(\tau, u)$ in the center manifold and $\nu = (\nu_1, \nu_2)$, the constants $\mathfrak{a}, \mathfrak{b}$ are given by

$$\nu_1 = a - a_c, \ \nu_2 = \delta - \delta_c,$$

$$\mathfrak{a} := a(k_0, a_c, \delta_c) = \frac{k_0^4}{(a_c + 1)k_0^2 - \delta_c},$$

$$\mathfrak{b} := \mathfrak{b}(k_0, a_c, \delta_c, \sigma''(0), \sigma'''(0)) = \frac{1}{(a_c + 1)k_0^2 - \delta_c} \left[ \frac{\sigma''(0)^2}{6a_c k_0(21k_0^2 - 5\delta_c)} - \frac{\sigma'''(0)}{2} \right].$$

The $O(|A|^5)$ term is given by $\sum_{j=2}^m c_j |A|^j + O(|A|^{2m+3})$ if $\sigma(\cdot) \in C^{2m+1}$ around 0.

The corresponding bifurcation diagram is given by

**Theorem 1.3.** Let $(a_c, \delta_c)$ be any admissible critical configuration point and $\mu$ be as above and small. Parameterizing the solution by $\mu$, the system (1.1) or equivalently system (4.2) or (4.6) either undergoes a supercritical (resp. subcritical) half pitchfork bifurcation around $\mu = 0$ when $\mathfrak{b} < 0$ (resp., $\mathfrak{b} > 0$), more precisely, the following properties hold in a neighborhood of 0 of $\mathbb{R}^1$ for sufficiently small $\mu$:

(i) If $\mathfrak{a}\mathfrak{b} < 0$ (resp., $\mathfrak{a}\mathfrak{b} > 0$), the differential equation has precisely one trivial equilibrium $U = 0$ for $\mu < 0$ (resp., $\mu > 0$). This equilibrium is stable when $\mathfrak{b} < 0$ and unstable when $\mathfrak{b} > 0$.

(ii) If $\mathfrak{a}\mathfrak{b} < 0$ (resp., $\mathfrak{a}\mathfrak{b} > 0$), the differential equation possesses, for $\mu > 0$ (resp., $\mu < 0$), the trivial solution $U = 0$ and one nontrivial equilibrium $U = U_\epsilon$ parametrized by $\epsilon = \mu^{1/2}$ and the dependence of $U$ on $\epsilon$ is smooth in a neighborhood of $(0,0)$, and $U_\epsilon = O(\epsilon)$, i.e., of magnitude $O(|\mu|^{1/2})$. The nontrivial equilibrium is stable if $\mathfrak{b} < 0$ and unstable if $\mathfrak{b} > 0$,

or bifurcates a limit cycle around $\mu = 0$ when $\mathfrak{b} \neq 0$ as follows:

(iii) If $\mathfrak{a}\mathfrak{b} < 0$ (resp., $\mathfrak{a}\mathfrak{b} > 0$), the differential equation has precisely one trivial equilibrium $U = 0$ for $\mu < 0$ (resp., $\mu > 0$). This equilibrium is stable when $\mathfrak{b} < 0$ and unstable when $\mathfrak{b} > 0$. 

(iv) If $\alpha \beta < 0$ (resp., $\alpha \beta > 0$), the differential equation possesses, for $\mu > 0$ (resp., $\mu < 0$), the trivial solution $U = 0$ and one nontrivial limit cycle, denoted by $\Gamma_\epsilon$, with diameter $R_\epsilon$ parametrized by $\epsilon = \mu^{1/2}$. The dependence of $R_\epsilon$ on $\epsilon$ is smooth in a neighborhood of 0, and $R_\epsilon = O(\epsilon)$ measured in the $Y$-norm, i.e., of magnitude $O(|\mu|^{1/2})$. The nontrivial limit cycle is stable if $\beta < 0$ and unstable if $\beta > 0$.

See Section 6.6 and Section 7 for other results and more explanations. Here we do not seek to list all the results and subtle points in this introductory section as some of them should be demonstrated in the process of computations. However, we remark that both bifurcation pictures are interesting and new in the current partial differential equations setting: the first bifurcation scenario is not that of a pitchfork bifurcation but a half pitchfork bifurcation we call here, while the limit cycle bifurcation we call here can be thought of as a result of rotation of the half pitchfork bifurcation, in other words, due to the $SO(2)$-symmetry (see Section 7.2). The above simple yet interesting mechanism of getting limit cycles through symmetry group actions from equilibria is among what we want to illustrate. This mechanism is of course different from the well-known Hopf bifurcation mechanism as the spectrum passes the imaginary axis of the complex plane through the real axis in our current work. Meanwhile, we emphasize that we get the dynamics on the center manifold to arbitrary orders allowed by the smoothness of the flux functions. Hence we get the full dynamics on the center manifold in view of the unique approximation property of the center manifold reduction function(s). Therefore we completely classify the current dynamical behaviors of the system in which the third order term in the flux also plays an important role. This is in sharp contrast to our former study on the equivariant Hopf bifurcation case in which the third order term only enters the angular equations. This is also different from the usual studies in which approximations up to order 2 are often made. Besides, the computations are different from our former methods in which normal form theory are involved.

We refer the reader to the works of Bressan [3, 4], Carr [6], Chicone [9], Golubitsky-Stewart-Schaeffer [13], Haragus and Iooss [8], Henry [14], Iooss and Adelmeyer [16], Jia and Sverak [17, 18], Ma and Wang [24, 25], Nakanishi and
Schlag \cite{Schlag2015} and the references therein for similar and related studies in a variety of settings.

**Convention.** We will use “∗” to denote “complex conjugate”, i.e., for \( z \in \mathbb{C} \), \( z^* \) means the complex conjugate of \( z \); “\( \int f \, dx \)” means “\( \int_{-\pi}^{\pi} f \, dx \)” for the integrand \( f \); for two nonnegative quantities, “\( A \lesssim B \)” means “\( A \leq C B \)” for some constant \( C > 0 \). For \( U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \), \( V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) \( \in \mathbb{C}^2 \), “\( \langle U, V \rangle \)” means “\( \int_{-\pi}^{\pi} u_1 v_1^* + u_2 v_2^* \, dx \)”.

We also adopt the standard big \( O \) “\( O \)” and small o “\( o \)” notations for limiting processes. For a vector \( V \), we use \( v_j \) or \( v^{(j)} \) to represent its components. We use “[·, ·]” to denote commutator: \([F,G] = FG - GF\) for \( F, G \) being functions, symbols or operators.

For a linear operator \( L : X \mapsto X \) on some Banach space \( X \), we use \( \sigma(L) \), \( \rho(L) \) to denote its spectrum set and resolvent set. Further, \( \sigma(L) = \sigma_c(L) \cup \sigma_s(L) \cup \sigma_u(L) \), i.e., \( \sigma(L) \) is the union of the center spectrum set \( \sigma_c(L) \), the stable spectrum set \( \sigma_s(L) \) and the unstable spectrum set \( \sigma_u(L) \). The associated space decomposition is \( X = X_c \cup X_s \cup X_u \) where \( X_c := \left( \frac{1}{2\pi i} \oint_{\gamma_c} (\lambda - L)^{-1} \, d\lambda \right) \) \( X \) is the center space, \( X_s := \left( \frac{1}{2\pi i} \oint_{\gamma_s} (\lambda - L)^{-1} \, d\lambda \right) \) \( X \) is the stable space, \( X_u := \left( \frac{1}{2\pi i} \oint_{\gamma_u} (\lambda - L)^{-1} \, d\lambda \right) \) \( X \) is the unstable space; The hyperbolic space is given \( X_h := X_s \cup X_u \). The \( \gamma_j \) above is any closed simple curve in the complex plane containing only spectrum of type \( j = c, s, u \).

### 2. Functional Analytic Setting

Now we start our analysis from the functional analytic setting.

First, we write the system \((1.1)\) in the form of nonlinear perturbation system. For this purpose, we need to regard \((\tau, u)\) as the perturbation variable around the state \((0, 0)\) and write \((1.1)\) as the following system

\[
\begin{align*}
\partial_t \tau - \partial_x u &= -a \partial_x^4 \tau \\
\partial_t u - \sigma'(0) \partial_x \tau &= -\delta \partial_x^2 u + \partial_x \left( \frac{\sigma''(0)}{2} \tau^2 + \frac{\sigma'''(0)}{6} \tau^3 + \Gamma(\tau) \right)
\end{align*}
\]

(2.1)

where \( \Gamma(\tau) := \sigma(\tau) - \sigma(0) - \frac{\sigma''(0)}{2} \tau^2 - \frac{\sigma'''(0)}{6} \tau^3 \) and \( \Gamma(\tau) = \mathcal{O}(|\tau|^4) \) when \(|\tau|\) is small. We shall keep in mind that the system \((2.1)\) is equivalent to the system \((1.1)\).
Second, we write the nonlinear perturbation system \((2.1)\) into operator equation form. For this purpose, we denote 
\[ U = \begin{pmatrix} \tau \\ u \end{pmatrix} \] and \( \mathcal{L}, \mathcal{N} \) as follows:

\[ \mathcal{L} := \begin{pmatrix} -a \partial_x^4 \\ \sigma'(0) \partial_x \\ -\delta \partial_x^2 - \partial_x^4 \end{pmatrix} \] \hspace{1cm} (2.2)

and

\[ \mathcal{N} \left( \begin{pmatrix} \tau \\ u \end{pmatrix} \right) := \begin{pmatrix} \partial_x \left( \frac{\sigma''(0)}{2} \tau^2 + \frac{\sigma'''(0)}{6} \tau^3 + \Gamma(\tau) \right) \end{pmatrix} \] \hspace{1cm} (2.3)

With the above notations, the nonlinear perturbation system can be written symbolically as

\[ \partial_t U = \mathcal{L} U + \mathcal{N}(U). \] \hspace{1cm} (2.4)

In order to emphasize the linear operator and nonlinear term, we may also write the system \((1.1)\) as

\[ \partial_t \left( \begin{pmatrix} \tau \\ u \end{pmatrix} \right) = \begin{pmatrix} -a \partial_x^4 \\ \sigma'(0) \partial_x \\ -\delta \partial_x^2 - \partial_x^4 \end{pmatrix} \begin{pmatrix} \tau \\ u \end{pmatrix} + \begin{pmatrix} \partial_x \left( \frac{\sigma''(0)}{2} \tau^2 + \frac{\sigma'''(0)}{6} \tau^3 + \Gamma(\tau) \right) \end{pmatrix}. \] \hspace{1cm} (2.5)

Third, we need to decide the space triplet under which we work and achieve our specific goals. The choice of working space triplet is nontrivial (See also Section 7). Here \( \mathcal{L} \) is a fourth order linear differential operator on the periodic domain \( \mathbb{R}/[-\pi, \pi] \). We may consider it as a linear operator on the space \( L^2_{\text{per}}(-\pi, \pi) \) with domain \( H^4_{\text{per}}(-\pi, \pi) \). Meanwhile, if we seek solutions \((\tau, u)\) on the periodic Sobolev spaces, the quantities \( \int \tau \, dx \) and \( \int u \, dx \) are conserved due to the conservative form of the original system \((1.1)\) as follows:

\[ \partial_t \left( \begin{pmatrix} \tau \\ u \end{pmatrix} \right) = \partial_x \left( \sigma(\tau) - \delta \partial_x u - \partial_x^2 u \right). \] \hspace{1cm} (2.6)

Taking into account of the above considerations and the requirement of center manifold theory, we will first (but also see Section 7) work on the space triplet \( Z \subset Y \subset X \) given below:

\[ X := \{ U \in L^2_{\text{per}}(-\pi, \pi); \int_{-\pi}^{\pi} U \, dx = 0 \}, \]

\[ Y = X = \{ U \in L^2_{\text{per}}(-\pi, \pi); \int_{-\pi}^{\pi} U \, dx = 0 \}, \]

\[ Z := \{ U \in H^4_{\text{per}}(-\pi, \pi); \int_{-\pi}^{\pi} U \, dx = 0 \}. \]
It is important to notice that the mean zero restriction which comes naturally from the conservative structure of system (1.1) has influence in the spectra of the linear operator $L(a, \delta)$. See also Remark 5.3.

3. Spectral Analysis I

With the functional analytic preparations above, we are now ready to study the spectra of the linear operator $L$ and give explanations of the admissible configuration set associated with an arbitrarily fixed positive integer $k_0$. Based on our choice of space triplet, we shall regard

$$L = L(a, \delta) := \begin{pmatrix} -a\partial_x^4 & \partial_x \\ \sigma'(0)\partial_x & -\delta\partial_x^2 - \partial_x^4 \end{pmatrix}$$

(3.1)

as a linear operator on the space $X$ with domain $Z$ to study its spectra. To this end, we can proceed by Fourier analysis as we are working on periodic domains. After Fourier transformation, the differential operator is represented by

$$M_k = \begin{pmatrix} -ak^4 & ik \\ \sigma'(0)ki & \delta k^2 - k^4 \end{pmatrix}, \quad k \in \mathbb{Z}, \ k \neq 0.$$

Therefore, we have

$$\sigma(L) = \bigcup_{k \in \mathbb{Z}, k \neq 0} \sigma(M_k).$$

The mode $k = 0$ is not included in the above union because we have mean 0 restriction in the definition of $X$. The eigenvalues $\lambda$ of $M_k$ for $k \neq 0$ are given by

$$\det(\lambda - M_k) = \det \begin{pmatrix} \lambda + ak^4 & -ik \\ -\sigma'(0)ki & \lambda - \delta k^2 + k^4 \end{pmatrix}$$

$$= (\lambda + ak^4)(\lambda - \delta k^2 + k^4) + \sigma'(0)k^2$$

$$= \lambda^2 + ((a + 1)k^4 - \delta k^2)\lambda + ak^4(k^4 - \delta k^2) + \sigma'(0)k^2$$

$$= 0.$$

Before further analysis, we first notice that in the formula for the eigenvalues of $M_k$, i.e.,

$$\lambda^2 + ((a + 1)k^4 - \delta k^2)\lambda + ak^4(k^4 - \delta k^2) + \sigma'(0)k^2 = 0, \quad (3.2)$$

$k$ enters the equation through $k^2$, which in particular implies $\sigma(M_k) = \sigma(M_{-k})$. There is no surprise here as this is a direct consequence of the $O(2)$-symmetry exhibited by the system (1.1) (see Section 7).
Now we explain the admissible critical configuration sets $A(k_0)$ associated with a nonzero (positive) integer $k_0$ in this paper. As we have studied the equivariant Hopf bifurcation spectral scenario in [34, 22], we study here the spectral scenario that there is one and only one spectrum curve crosses the imaginary axis in $\mathbb{C}^1$ through the origin with the purpose of covering the generic cases.

Now let us fix the nonzero (positive) integer $k_0$. To achieve the above spectrum crossing scenario based on the wave number $k_0$, necessarily we need that $M_{k_0}$ contributes zero spectrum for the linear operator $L$ while all the other $M_k$ for $|k| \neq k_0$ do not. In other words, we require the following necessary conditions (a) and (b) in (3.2):

(a) for the nonzero integer $k_0$,

$$a_c k_0^4 (k_0^2 - \delta_c) + \sigma'(0) = 0.$$

(b) for any nonzero integer $k$ such that $|k| \neq |k_0|$, and $\omega \in \mathbb{R}^1 - \{0\}$,

$$[a_c k^4 (k^2 - \delta_k^2) + \sigma'(0) - \omega^2] + i[(a_c + 1)k^4 - \delta_c k^2] \neq 0.$$

The above conditions (a) and (b) give the necessary conditions for the spectral crossing scenario. We write condition (b) in the above form for the ease of verifying consistency. As we may expect or guess, it is not sufficient for our bifurcation analysis as we need the crossing to be transverse in order to complete the bifurcation analysis. It turns out that conditions (a), (b) together with the following nondegeneracy condition (c) are sufficient for our purpose:

(c) for the above $k_0$, $(a_c + 1)k_0^2 - \delta_c \neq 0$.

The reasons that we call condition (c) a nondegeneracy condition lie in the following two observations: (1) the condition $(a_c + 1)k_0^2 - \delta_c \neq 0$ means that $M_{k_0}$ does not contribute repeated zero spectrum for $L$ as $(a + 1)k^4 - \delta k^2 = ((a + 1)k^2 - \delta)k^2$ is the coefficient of the first order term in the spectral variable $\lambda$ in (3.2); (2) when we compute the dual kernel of the linear operator $L(a_c, \delta_c)$ during the reduction procedure, $(a_c + 1)k_0^2 - \delta_c$ appears in the denominator, see Remark 6.1.

We may notice that the expression $a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0)$ also appears as a denominator during the reduction procedure, see in particular (6.7). However, we do not need to worry about if $a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0)$ is nonzero or not.
Actually, under conditions $(a)$, $(b)$ and $(c)$, the expression \( a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0) \) is automatically nonzero, which is a direct consequence of the paragraph above (6.7) (see also Remark 6.3). Therefore, in the definition of \( A(k_0) \), it is equivalent no matter we include or do not include the following condition $(d)$:

\[(d) \ a_c(2k_0)^4((2k_0)^2 - \delta_c) + \sigma'(0) \neq 0.\]

Before we do further spectrum analysis, we need first verify that there exists flux function \( \sigma(\tau) \) such that \( A(k_0) \) is nonempty for some nonzero (positive) integer \( k_0 \), i.e., $(a)$ $(b)$ and $(c)$ are consistent for some \( k_0 \) as above, which is enough for us to carry out the remaining parts of the program. Actually, we can easily show that there exist flux functions \( \sigma(\tau) \) such that \( A(k_0) \) is nonempty for any nonzero (positive) integer \( k_0 \). For this purpose, we claim:

*(Consistency) conditions $(a)-(d)$ are consistent.*

This is easily seen by considering the special case \( \sigma'(0) = 0 \). Then $(a)$ and $(d)$ require that \( a_c \neq 0 \) and \( \delta_c = k_0^2 \). As long as \( a_c \neq 0 \) and \( \delta_c = k_0^2 \), $(a)$, $(c)$ and $(d)$ are satisfied. We can then adjust \( a_c > 0 \) to satisfy $(b)$, which is easy. In particular if \( k_0 = \pm 1 \), $(b)$ is satisfied since \( (a_c + 1)k_0^4 - \delta_c k_0^2 = a_c \neq 0 \). Notice that we have no requirement on the sign of \( a_c \) here. Alternatively, we can let \( a_c < -1 \) to satisfy $(b)$. In particular, the operator \( L(a_c, \delta_c) \) is not a sectoral operator if \( a_c < -1 \).

Another easy way to see the consistency is to choose \( \delta_c = 0 \), \( a_c \neq 0 \), $-1$ and \( a_c k_0^8 + \sigma'(0) = 0 \).

Next we give several remarks based on the above analysis and the whole paper.

**Remark 3.1.** *From the analysis in the above two paragraphs, we see that \( A(k_0) \) is generically nonempty for any nonzero integer \( k_0 \) and the conditions $(a)$, $(b)$ and $(c)$ are actually rather mild though we may regard on first seeing that they impose strong constraints on the flux functions or the parameters.*

**Remark 3.2.** *Without condition $(c)$, we can still show the existence of center manifold reduction but we have some problems in computation as this appears as a denominator.*

**Remark 3.3.** *As long as \( a_c \neq 0 \), the resolvent estimate in Lemma 5.2 is valid, which will be sufficient to guarantee the existence of center manifold. \( a_c \neq 0 \) as long as \( (a_c, \delta_c) \in A(k_0) \), which is easily seen from conditions $(a)$ and $(d)$.***
4. Spectrum Analysis II

From now on, we let \((a_c, \delta_c)\) be an admissible critical configuration point. Due to the spectral preparations in the last section, we are now on a sound foundation to do the bifurcation analysis. In this section, we will introduce the bifurcation system and make some spectral preparations.

First, we introduce several notations:

\[
\nu = (\nu_1, \nu_2) := (a - a_c, \delta - \delta_c), \quad \mu = \mu(\nu) := (\delta_c - k_0^2)\nu_1 + a_c\nu_2. \tag{4.1}
\]

Technically, we can do the bifurcation analysis for the following three cases: (1) \(\delta\) varies around \(\delta_c\) with \(a = a_c\) being fixed; (2) \(a\) varies around \(a_c\) with \(\delta = \delta_c\) being fixed; (3) \((a, \delta)\) varies around \((a_c, \delta_c)\). Formally, case (3) covers cases (1) and (2) and we will first do the analysis based on case (3). However, there are points to be remarked about cases (1) and (2) from both the mathematical and applicational points of view, see Section 7.

Second, we isolate the bifurcation parameter \(\mu\) to get the bifurcation system. For this purpose, we write system (2.1) in the following form

\[
\partial_t U = L(a_c, \delta_c)U + \mathcal{L}(a, \delta) - L(a_c, \delta_c))U + \mathcal{N}(U), \tag{4.2}
\]

where we have the obvious identifications:

\[
L(a_c, \delta_c) = \left(\begin{array}{cc}
-a_c\partial_x^4 & \partial_x \\
\sigma'(0)\partial_x & -a_c\partial_x^2 - \partial_x^4
\end{array}\right), \tag{4.3}
\]

\[
\mathcal{L}(a, \delta) - L(a_c, \delta_c) = \left(\begin{array}{cc}
-\nu_1\partial_x^4 & 0 \\
0 & -\nu_2\partial_x^2
\end{array}\right) \tag{4.4}
\]

and

\[
\mathcal{N}(U) = \left(\partial_x \left(\frac{\sigma''(0)}{2}\tau^2 + \frac{\sigma'''(0)}{6}\tau^3 + \Gamma(\tau)\right)\right) \tag{4.5}
\]

Now we introduce \(R(U, \nu) := R_{11}(U, \nu) + R_{20}(U, U) + R_{30}(U, U, U) + \tilde{R}(U)\) where

\[
R_{11}(U, \nu) := \left(\begin{array}{c}
-\nu_1\partial_x^4(U^{(1)}) \\
-\nu_2\partial_x^2(U^{(2)})
\end{array}\right), \quad R_{20}(U, V) := \left(\begin{array}{c}
\frac{\sigma''(0)}{2}\partial_x(U^{(1)}V^{(1)})
\end{array}\right), \tag{4.6}
\]

\[
R_{30}(U, V, W) := \left(\begin{array}{c}
\frac{\sigma'''(0)}{6}\partial_x(U^{(1)}V^{(1)}W^{(1)})
\end{array}\right), \quad \tilde{R}(U) := \left(\begin{array}{c}
0
\end{array}\right) \tag{4.7}
\]

and write the nonlinear perturbation system (4.2) around \(\mu = 0\) as

\[
\partial_t U = L(a_c, \delta_c)U + R(U, \nu). \tag{4.6}
\]
The above equation (4.6) is the bifurcation equation or system.

Finally, we make some spectral preparations for the use of center manifold theory.

**Lemma 4.1.** \( \sigma_c(\mathcal{L}(a_c, \delta_c)) = \{0\} \).

*Proof.* This is a direct consequence of the definition of the admissible critical configuration set \( \mathcal{A}(k_0) \).

Concerning the spectrum of \( \sigma(\mathcal{L}(a_c, \delta_c)) \), we also have the following lemma:

**Lemma 4.2.** There exists a positive constant \( \gamma > 0 \), such that \( \sup \{ \Re \lambda; \lambda \in \sigma_s(\mathcal{L}(a_c, \delta_c)) \} < -\gamma \) and \( \inf \{ \Re \lambda; \lambda \in \sigma_u(\mathcal{L}(a_c, \delta_c)) \} > \gamma \).

*Proof.* We need to consider the distribution of the roots of equation (3.2) for \( k \neq k_0 \). By symmetry, we just need consider the case \( |k_0| \neq k \in \mathbb{N} \). By Lemma 4.1, we know that roots of equation (3.2) for \( k \neq k_0 \) do not lie in the imaginary axis. Hence we just need to show that there is no accumulation of spectra to the imaginary axis when \( k \to +\infty \). This is obvious by writing down the solutions explicitly through quadratic formula: the real parts of the roots of (3.2) can only tend to \( \pm \infty \).

**Remark 4.3.** \( \bigcup_{0 \neq k_0 \in \mathbb{Z}} \mathcal{A}(k_0) \) contains all the admissible critical configuration points. Our bifurcation analysis will be done in particular around \( \nu = (\nu_1, \nu_2) = (0, 0) \). The role of the parameter \( \mu \) will be self-evident after we get the reduced dynamics (see (6.12)).

5. **Existence of Parameter-Dependent Center Manifold**

In this section, we show the existence of parameter-dependent center manifold for the bifurcation system (4.6) or equivalently the system (4.2) or (1.1). The main ingredient remaining to show is a resolvent estimate. A similar estimate was first done in [34]. However, we will reproduce it here to see the roles of the defining conditions in \( \mathcal{A}(k_0) \) and correct some misprints in [34] and also for completeness. Meanwhile, we will add symmetries into consideration in order to make comparisons with our former works (see Section 7).

Let us first state a version of the parameter-dependent center manifold theorem with group actions.
Theorem 5.1. (Parameter dependent center manifold theorem with symmetries, see [8, 13, 34]) Let the inclusions in the Banach space triplet $Z \subset Y \subset X$ be continuous. Consider a differential equation in a Banach space $X$ of the form

$$\frac{du}{dt} = Lu + R(u, \nu)$$

and assume that

1. (Assumption on linear operator and nonlinearity) $L: Z \rightarrow X$ is a bounded linear map and for some $k \geq 2$, there exist neighborhoods $V \subset Z$ and $V_\nu \subset \mathbb{R}^m$ of $(0, 0)$ such that $R \in C^k(V_u \times V_\nu, Y)$ and $R(0, 0) = 0$, $D_u R(0, 0) = 0$.

2. (Spectral decomposition) there exists some constant $\gamma > 0$ such that

   $$\inf \{ \Re \lambda; \lambda \in \sigma_u(L) \} > \gamma, \quad \sup \{ \Re \lambda; \lambda \in \sigma_s(L) \} < -\gamma,$$

   and the set $\sigma_c(L)$ consists of a finite number of eigenvalues with finite algebraic multiplicities.

3. (Resolvent estimates) For Hilbert space triplet $Z \subset Y \subset X$, assume there exists a positive constant $\omega_0 > 0$ such that $i\omega \in \rho(L)$ for all $|\omega| > \omega_0$ and $\|(i\omega - L)^{-1}\|_{X \rightarrow X} \lesssim \frac{1}{|\omega|^\alpha}$ for some $\alpha \in [0, 1)$.

   For Banach space triplet, we need further $\|(i\omega - L)^{-1}\|_{Y \rightarrow X} \lesssim \frac{1}{|\omega|^\alpha}$.

Then there exists a map $\Psi \in C^k(Z_c, Z_h)$ and a neighborhood $O_u \times O_\nu$ of $(0, 0)$ in $Z \times \mathbb{R}^m$ such that

1. (Tangency) $\Psi(0, 0) = 0$ and $D_u \Psi(0, 0) = 0$.

2. (Local flow invariance) the manifold $M_0(\nu) = \{ u_0 + \Psi(u_0, \nu); u_0 \in Z_c \}$ has the properties. (i) $M_0(\nu)$ is locally invariant, i.e., if $u$ is a solution satisfying $u(0) \in M_0(\nu) \cap O_\nu$ and $u(t) \in O_u$ for all $t \in [0, T]$, then $u(t) \in M_0(\nu)$ for all $t \in [0, T]$; (ii) $M_0(\nu)$ contains the set of bounded solutions staying in $O_u$ for all $t \in \mathbb{R}^1$, i.e., if $u$ is a solution satisfying $u(t) \in O_u$ for all $t \in \mathbb{R}^1$, then $u(0) \in M_0(\nu)$.

3. (Symmetry) Moreover, if the vector field is equivariant in the sense that there exists an isometry $\mathcal{T} \in \mathcal{L}(X) \cap \mathcal{L}(Z)$ which commutes with the vector field in the original system,

   $$[\mathcal{T}, L] = 0, \quad [\mathcal{T}, L] = 0,$$

then $\Psi$ commutes with $\mathcal{T}$ on $Z_c$: $[\Psi, \mathcal{T}] = 0$. 

It is obvious to see that in the above Theorem 5.1, the linear operator is not required to be sectorial. Another subtle point is that the center manifolds are generally not unique but the center manifold reduction function can be approximated uniquely up to any finite order (hence we use “manifold” and “manifolds” interchangeably) as long as the nonlinearity has enough smoothness (see for example, Theorem 2.5 in page 35 and Theorem 10 in page 120 of [6]). More insight can be drawn here. In particular, this uniqueness of approximation to any finite order enables us to get full dynamics on center manifold through center manifold reduction. Of course, the distinct properties of the systems under consideration should be taken into account. We have the following lemma regarding system \( (4.6) \).

**Lemma 5.2.** (Existence of parameter-dependent center manifolds) For system \( (4.6) \), there exists a map \( \Psi \in C^k(Z_c \times \mathbb{R}^1, Z_h) \), with
\[
\Psi(0, 0) = 0, \quad D_U \Psi(0, 0) = 0,
\]
and a neighborhood of \( \mathcal{O}_U \times \mathcal{O}_\nu \) of \( (0, 0) \) such that for \( \nu \in \mathcal{O}_\mu \), the manifold
\[
\mathcal{M}_0(\nu) := \{ U_0 + \Psi(U_0, \nu); \ U_0 \in Z_c \}
\]
is locally invariant and contains the set of bounded solutions of the nonlinear perturbation system in \( \mathcal{O}_U \) for all \( t \in \mathbb{R} \).

**Proof.** From the definition of the operator \( \mathcal{L}(a_c, \delta_c) \) and \( R(U, \nu) \) in \( (4.6) \), we know that the assumptions (1) and (2) in Theorem 5.1 on the linear operator and nonlinearity hold. A subtle point here is that the highest order of derivatives in \( R(U, \nu) \) is four, which is allowed by our space triplet choice \( Z \subset Y \subset X \) in Section 2 and Theorem 5.1. The spectrum decomposition assumption is a direction consequence of the analysis in Section 4, see Lemma 4.1 and Lemma 4.2. It is obvious that \( i\omega \in \rho(\mathcal{L}(a_c, \delta_c)) \) for \( |\omega| > 0 \). To show the resolvent estimate in the current Hilbert space triplet setting, we write \( (i\omega - \mathcal{L}(a_c, \delta_c))U = \hat{U} \)
for \( \hat{U} = \begin{pmatrix} \tau \\ u \end{pmatrix} \in X \) and \( U = \begin{pmatrix} \tau \\ u \end{pmatrix} \in Z \) and show that \( \|U\|_X \lesssim \frac{1}{|\omega|}\|\hat{U}\|_X \) for \( |\omega| > \omega_0 > 0 \) where \( \omega_0 \gg 1 \) is a large constant. Without loss of generality, we just need to prove for \( \omega \geq \omega_0 \). Multiplying the equation \( (i\omega - \mathcal{L}(a_c, \delta_c))U = \hat{U} \) by \( U^* \), integrating over \([ -\pi, +\pi ]\) and integrating by parts, we arrive at
\[
i\omega|\tau|^2_{L^2} + a_c|\partial_x^2 \tau|^2_{L^2} - \int \partial_x u \tau^* = \int \hat{\tau} \tau^*;
\]
\[
\int -\sigma'(0)\partial_x\tau u^* + i\omega|u|^2_L - \delta_c \int |\partial_x u|^2 + \int |\partial_x^2 u|^2 = \int \tilde{u}u^*.
\]

Taking the imaginary and real parts respectively, we see
\[
\omega|\tau|^2_L = Im \int \partial_x u\tau^* + Im \int \tilde{\tau}\tau^*,
\]
\[
\omega|u|^2_L = \sigma'(0) Im \int \partial_x\tau u^* + Im \int \tilde{u}u^*
\]
and
\[
a_c|\partial_x^2 \tau|^2_L = Re \int \partial_x u\tau^* + Re \int \tilde{\tau}\tau^*,
\]
\[
-\delta_c|\partial_x u|^2_L + \int |\partial_x^2 u|^2 = Re \int \tilde{u}u^* + Re \int \sigma'(0)\partial_x\tau u^*.
\]

From the imaginary part equations, we get by using elementary inequalities, Fourier analysis and mean zero property for elements in the space \(X\), that
\[
\omega|\tau|^2_L \lesssim \frac{1}{\omega}(|\partial_x u|^2_L + |\tilde{\tau}|^2_L) \leq \frac{1}{\omega}(|\partial_x^2 u|^2_L + |\tilde{\tau}|^2_L),
\]
\[
\omega|u|^2_L \lesssim \frac{1}{\omega}(|\partial_x\tau|^2_L + |\tilde{\tau}|^2_L) \leq \frac{1}{\omega}(|\partial_x^2 \tau|^2_L + |\tilde{\tau}|^2_L).
\]

In view of \(a_c \neq 0\) (see Remark (3.3)), we obtain from the real part equations that
\[
|\partial_x^2 \tau|^2_L \lesssim |\tau|^2_L(|\tilde{\tau}|^2 + |\partial_x u|^2_L),
\]
\[
|\partial_x^2 u|^2_L \lesssim |u|^2_L(|\tilde{\tau}|^2 + |\partial_x\tau|^2_L) + |\partial_x u|^2_L.
\]

Notice that (5.4) is valid for any real numbers \(\delta_c\). In particular, \(\delta_c\) can be 0. By interpolation, we know that for any \(\epsilon > 0\), the following holds
\[
|\partial_x u|^2_L \leq \epsilon|\partial_x^2 u|^2_L + C(\epsilon)|u|^2_L.
\]

We may pick \(\epsilon\) so small that we can conclude from (5.4) that
\[
|\partial_x^2 u|^2_L \lesssim |u|^2_L(|\tilde{\tau}|^2 + |\partial_x\tau|^2_L) + |u|^2_L.
\]

By interpolation in \(|\partial_x\tau|^2_L\),
\[
|\partial_x\tau|^2_L \lesssim \epsilon|\partial_x^2 \tau|^2_L + C(\epsilon)|\tau|^2_L,
\]
we can get from (5.3) and (5.4) that
\[
\begin{cases}
|\partial_x^2 \tau|^2_L \lesssim |\tau|^2_L + |\tilde{\tau}|^2_L + \epsilon|\partial_x^2 u|^2_L + |u|^2_L, \\
|\partial_x^2 u|^2_L \lesssim |u|^2_L + |\tilde{\tau}|^2_L + \epsilon|\partial_x^2 \tau|^2_L + \epsilon|\partial_x^2 u|^2_L.
\end{cases}
\]
Adding the two equations in (5.5) together and choosing \( \epsilon \) smaller if necessary, we get

\[
|\partial_x^2 \tau|_{L^2}^2 + |\partial_x^2 u|_{L^2}^2 \lesssim |\tau|_{L^2}^2 + |u|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + |\tilde{u}|_{L^2}^2. \tag{5.6}
\]

Now we have in view of (5.1), (5.2) and (5.6) that

\[
\omega |\tau|_{L^2}^2 \lesssim \frac{1}{\omega} |\tilde{\tau}|_{L^2}^2 + \frac{1}{\omega} |\partial_x u|_{L^2}^2
\lesssim \frac{1}{\omega} |\tilde{\tau}|_{L^2}^2 + \frac{1}{\omega} \left( |\tau|_{L^2}^2 + |u|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + |\tilde{u}|_{L^2}^2 \right).
\]

\[
\omega |u|_{L^2}^2 \lesssim \frac{1}{\omega} |\tilde{u}|_{L^2}^2 + \frac{1}{\omega} |\partial_x \tau|_{L^2}^2
\lesssim \frac{1}{\omega} |\tilde{u}|_{L^2}^2 + \frac{1}{\omega} \left( |\tau|_{L^2}^2 + |u|_{L^2}^2 + |\tilde{\tau}|_{L^2}^2 + |\tilde{u}|_{L^2}^2 \right).
\]

Adding the above two inequalities together, we get

\[
(-\frac{1}{\omega} + \omega)|U|_{L^2}^2 \lesssim \frac{1}{\omega} |\tilde{U}|_{L^2}^2,
\]

which implies

\[
|U|_{L^2}^2 \lesssim \frac{1}{\omega(\omega - \frac{1}{\omega})} |\tilde{U}|_{L^2}^2 = \frac{1}{\omega^2 - 1} |\tilde{U}|_{L^2}^2 \lesssim \frac{1}{\omega^2} |\tilde{U}|_{L^2}^2.
\]

for \( \omega \geq \omega_0 \) large. Hence we arrive at the estimate of desired form \( |U|_{L^2} \lesssim \frac{1}{\omega} |\tilde{U}|_{L^2} \). \( \square \)

**Remark 5.3.** In general, the inequalities in (5.1) and (5.2) would not be true if \( \int \tau \, dx \neq 0 \) and \( \int u \, dx \neq 0 \). And now we see that the mean zero assumption in the choice of space triplet is required not only by the conservative form of system (1.1) but also by the above resolvent estimate which is necessary for the existence of center manifold.

### 6. Dynamics on Center Manifold

In this section, we compute and analyze the dynamics on the center manifold for our bifurcation system (4.6). As we want to demonstrate some subtle points during the process and also to make the exposition clear, we divide this section into subsections.
6.1. Center space and parametrization. First, we compute the center space of the operator $\mathcal{L}(a_c, \delta_c)$. Recall that
\[
\mathcal{L}(a_c, \delta_c) := \begin{pmatrix} -a_c \partial_x^4 & \partial_x \\ \sigma'(0) \partial_x & -\delta_c \partial_x^2 - \partial_x^4 \end{pmatrix}.
\]
For the wave number $k_0$, letting $\xi = e^{ik_0 x} V = e^{ik_0 x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, we get in Fourier side the system satisfied by $V$:
\[
\begin{pmatrix} -a_c k_0^4 & ik_0 \\ \sigma'(0)i k_0 & \delta_c k_0^2 - k_0^4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6.1}
\]
which is equivalent to
\[
\begin{cases}
-a_c k_0^4 v_1 + ik_0 v_2 = 0 \\
\sigma'(0)i k_0 v_1 + (\delta_c k_0^2 - k_0^4) v_2 = 0.
\end{cases} \tag{6.2}
\]
From condition $(a)$ in the definition of $\mathcal{A}(k_0)$, we know that
\[
\det \begin{pmatrix} -a_c k_0^4 & ik_0 \\ \sigma'(0)i k_0 & \delta_c k_0^2 - k_0^4 \end{pmatrix} = k_0^2 [a_c k_0^4 (k_0^2 - \delta_c) + \sigma'(0)] = 0
\]
and get
\[
a_c k_0^4 v_1 = ik_0 v_2, \quad i.e., \quad v_2 = \frac{a_c k_0^4}{ik_0} v_1 = -ia_c k_0^3 v_1.
\]
Consequently, we can choose
\[
\xi = e^{ik_0 x} \begin{pmatrix} 1 \\ -ia_c k_0^3 \end{pmatrix}.
\]
By conjugacy, for the wave number $-k_0$, we seek solutions of the form $e^{-ik_0 x} V = e^{-ik_0 x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and get that
\[
\xi^* = e^{-ik_0 x} \begin{pmatrix} 1 \\ i a_c k_0^3 \end{pmatrix}.
\]
Then the center space $Z_h$ which is nothing but ker $\mathcal{L}(a_c, \delta_c)$ can be parametrized by
\[
Z_h = \{ A\xi + A^* \xi^*; \ A \in \mathbb{C}^1 \}.
\]
This parametrization will be important for the later computations.
6.2. The dual Kernel $\mathcal{L}^*(a_c, \delta_c)$. Next, we compute the kernel of the conjugate operator $\mathcal{L}^*(a_c, \delta_c)$. By simple checking based on the definition of conjugate operator, we get

$$\mathcal{L}^*(a_c, \delta_c) := \begin{pmatrix} -a_c \partial_x^4 & -\sigma'(0) \partial_x \\ \partial_x & -\delta_c \partial_x^2 - \partial_x^4 \end{pmatrix}. $$

Seeking elements of the form $\eta = \kappa e^{i k_0 x} V = \kappa e^{i k_0 x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in the dual kernel where $\kappa$ is a renomalization constant to be picken, we get

$$\begin{pmatrix} -a_c k_0^4 & \sigma'(0) i k_0 \\ -i k_0 & \delta_c k_0^2 - k_0^4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6.3}$$

which is equivalent to

$$\begin{cases} -a_c k_0^4 v_1 - \sigma'(0) i k_0 v_2 = 0 \\ -i k_0 v_1 + (\delta_c k_0^2 - k_0^4) v_2 = 0. \end{cases} \tag{6.4}$$

Since $\det \begin{pmatrix} -a_c k_0^4 & \sigma'(0) i k_0 \\ \sigma'(0) i k_0 & \delta_c k_0^2 - k_0^4 \end{pmatrix} = 0$, we get the relation between $v_1$ and $v_2$ as

$$-a_c k_0^4 v_1 - \sigma'(0) i k_0 v_2 = 0.$$

Since $a_c \neq 0$, we know $v_1 = \frac{\sigma'(0) i k_0}{a_c k_0^4} v_2 = -i \frac{\sigma'(0)}{a_c k_0^4} v_2 = -i k_0 (\delta_c - k_0^2) v_2$. Consequently, we get

$$\eta = \kappa e^{i k_0 x} \begin{pmatrix} -i k_0 (\delta_c - k_0^2) \\ 1 \end{pmatrix}. $$

For computational convenience, we choose $\kappa$ such that $\langle \eta, \xi \rangle = 1$. By direct computation, we get

$$\langle \eta, \xi \rangle = \langle \kappa e^{i k_0 x} \begin{pmatrix} -i k_0 (\delta_c - k_0^2) \\ 1 \end{pmatrix}, e^{i k_0 x} \begin{pmatrix} 1 \\ -i a_c k_0^3 \end{pmatrix} \rangle$$

$$= \int \kappa e^{i k_0 x} \begin{pmatrix} -i k_0 (\delta_c - k_0^2) \\ 1 \end{pmatrix} e^{-i k_0 x} \begin{pmatrix} 1 \\ i a_c k_0^3 \end{pmatrix}$$

$$= 2\pi \left[-i k_0 (\delta_c - k_0^2) + i a_c k_0^3 \right] \kappa$$

$$= 2\pi i k_0 [(a_c + 1) k_0^2 - \delta_c] \kappa$$

$$= 1,$$

which suggests that

$$\kappa = \left(2\pi i k_0 [(a_c + 1) k_0^2 - \delta_c] \right)^{-1}.$$
and
\[ \eta = \left(2\pi ik_0[(a_c + 1)k_0^2 - \delta_c]\right)^{-1}e^{ik_0x}\left(-ik_0\delta_c - k_0^2\right). \]

By our choice of \( \kappa \) and conjugacy of inner product, we have the following duality numerical relations:
\[ \langle \eta, \xi \rangle = 1, \langle \eta^*, \xi^* \rangle = 1, \langle \eta^*, \xi \rangle = 0, \langle \eta, \xi^* \rangle = 0. \]

Also, we have obtained by now
\[ \ker L^*(a_c, \delta_c) = \{ B\eta + B^*\eta^*; B \in \mathbb{C}^1 \}. \]

We end this subsection by the following remark:

**Remark 6.1.** The denominator in \( \kappa \) which is the coefficient of the first order term in the characteristic equation (3.2) up to a nonzero constant involving \( k_0^2 \) is not zero for \( (a_c, \delta_c) \in A(k_0) \).

### 6.3. The projection operator \( \mathbb{P} \)

In this subsection, we will introduce the projection operator \( \mathbb{P} \) to make the remaining computations elegant and efficient. To do so, we decompose the space triplet \( Z \subset Y \subset X \) as follows:
\[ Z = Z_c \oplus Z_h, Y = Y_c \oplus Y_h, X = X_c \oplus X_h, \]
where the subindices “\( c \)” and “\( h \)” stand for center and hyperbolic spaces respectively and
\[ Z_c = Y_c = X_c = \{ A\xi + A^*\xi^*; A \in \mathbb{C}^1 \}. \]

For an element \( U \in Z \) or in \( Y, X \), we decompose it according to the above space decompositions as
\[ U = A\xi + A^*\xi^* + (U - A\xi - A^*\xi^*) \]
\[ = \langle \eta^*, U \rangle \xi + \langle \eta, U \rangle \xi^* + (U - \langle \eta^*, U \rangle \xi - \langle \eta, U \rangle \xi^*) \]
\[ = \mathbb{P}U + (1 - \mathbb{P})U \]
where the projection operator \( \mathbb{P} \) is defined as
\[ \mathbb{P} \cdot = \langle \eta^*, \cdot \rangle \xi + \langle \eta, \cdot \rangle \xi^*. \]
6.4. The center manifold reduction function. In this subsection, we will compute the second order approximation of the center manifold reduction function $\Psi$. By now, we have shown the existence of center manifold and parametrized the center space of $\mathcal{L}(a_c, \delta_c)$ in terms of the complex conjugate pair $(A, A^*)$. Hence we can correspondingly decompose an element $U = U(x, t)$ by $U = A(t)\xi + A(t)\xi^* + \Psi(A, A^*)$ where $A = A(t) \in \mathbb{C}^1$ is a complex function in time variable $t$ and $\Psi(A, A^*) \in Y_k$ is the center manifold function.

For brevity, we define

$$R_{20}(A, A^*) := R_{20}(A(t)\xi + A(t)\xi^*, A(t)\xi + A(t)\xi^*)$$

and

$$R_{30}(A, A^*) := R_{20}(A(t)\xi + A(t)\xi^*, A(t)\xi + A(t)\xi^*, A(t)\xi + A(t)\xi^*).$$

The second order approximation of the center manifold function $\Psi(A, A^*)$ is given by $\Psi(A, A^*) = (-\mathcal{L}(a_c, \delta_c))^{-1}(1 - \mathbb{P})R_{20}(A, A^*) + O(|A|^3)$. Simple computation yields

$$R_{20}(A, A^*) = \left(\frac{\sigma''(0)}{2}\partial_x[(Ae^{ik_0x} + A^*e^{-ik_0x})^2]\right)$$

$$= ik_0\sigma''(0) \left(\begin{array}{cc} 0 & 0 \\ e^{2ik_0x} - e^{-2ik_0x}A^2 & 0 \end{array}\right)$$

$$= ik_0\sigma''(0) \left(\begin{array}{cc} 0 & 0 \\ e^{2ik_0x}A^2 - e^{-2ik_0x}A^*^2 \end{array}\right)$$

We can easily check

$$\langle \eta^*, R_{20}(A, A^*) \rangle = \int \kappa e^{-ik_0x} \left(\begin{array}{c} ik_0(\delta_c - k^2_0) \\ 1 \end{array}\right) \cdot [(-ik_0\sigma''(0))[e^{2ik_0x}(0 - e^{-2ik_0x})^2 \left(\begin{array}{c} 0 \\ A^2 \end{array}\right) - e^{-2ik_0x}(0 - e^{-2ik_0x})^2 \left(\begin{array}{c} 0 \\ A^*^2 \end{array}\right)]$$

$$= \int \kappa \left(\begin{array}{c} ik_0(\delta_c - k^2_0) \\ 1 \end{array}\right) \cdot [(-ik_0\sigma''(0))[e^{ik_0x}(0 - e^{-3ik_0x})^2 \left(\begin{array}{c} 0 \\ A^2 \end{array}\right) - e^{-3ik_0x}(0 - e^{-3ik_0x})^2 \left(\begin{array}{c} 0 \\ A^*^2 \end{array}\right)]$$

$$= 0.$$ 

Similarly, we can also check that $\langle \eta, R_{20}(A, A^*) \rangle = 0$. Hence we conclude

$$\mathbb{P}R_{20}(A, A^*) = \langle \eta^*, R_{20}(A, A^*) \rangle \xi + \langle \eta, R_{20}(A, A^*) \rangle \xi^* = 0.$$ 

Consequently,

$$\Psi(A, A^*) = (-\mathcal{L}(a_c, \delta_c))^{-1}(1 - \mathbb{P})R_{20}(A, A^*) = (-\mathcal{L}(a_c, \delta_c))^{-1}R_{20}(A, A^*) + O(|A|^3).$$
To proceed further, we compute \((-\mathcal{L}(a_c, \delta_c))^{-1} R_{20}(A, A^*)\) which we define by \(\Phi(A, A^*)\). As \(R_{20}(A, A^*)\) is in the hyperbolic space, the equality
\[\quad (-\mathcal{L}(a_c, \delta_c))^{-1} R_{20}(A, A^*) = \Phi(A, A^*)\]
is equivalent to that
\[\quad -\mathcal{L}(a_c, \delta_c) \Phi(A, A^*) = R_{20}(A, A^*),\]

hence can be written specifically as
\[
-\mathcal{L}(a_c, \delta_c) \Phi(A, A^*) = i k_0 \sigma''(0) [e^{2ik_0x} \begin{pmatrix} 0 \\ A^2 \end{pmatrix} - e^{-2ik_0x} \begin{pmatrix} 0 \\ A^*2 \end{pmatrix}] \quad (6.5)
\]

Due to the form in the right hand side of the above equation and the fact that \(\Phi(A, A^*)\) is real-valued, we should assume that
\[\quad \Phi(A, A^*) = i(e^{2ik_0x} V - e^{-2ik_0x} V^*),\]

where \(V \in \mathbb{C}^2\) is to be determined. The above observation enables us to carry out the current program and it is a crucial point.

Now we observe that (6.5) is equivalent to that
\[\quad -\mathcal{L}(a_c, \delta_c) e^{2ik_0x} V = k_0 \sigma''(0) e^{2ik_0x} \begin{pmatrix} 0 \\ A^2 \end{pmatrix}.\]

To solve for the complex vector \(V \in \mathbb{C}^2\), we write the above equation specifically as
\[
\begin{pmatrix}
16a_c k_0^4 & -2ik_0 \\
-2ik_0 \sigma'(0) & -4\delta_c k_0^2 + 16k_0^4
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
k_0 \sigma''(0) A^2
\end{pmatrix}. \quad (6.6)
\]

Before we solve this elementary system of algebraic equation, let us first claim that
\[\det \begin{pmatrix}
16a_c k_0^4 & -2ik_0 \\
-2ik_0 \sigma'(0) & -4\delta_c k_0^2 + 16k_0^4
\end{pmatrix} \neq 0.\]

The above claim is a simple consequence of the facts that \(k_0 \sigma''(0) e^{2ik_0x} \begin{pmatrix} 0 \\ A^2 \end{pmatrix} \in X_h\), \(\Phi(A, A^*) \in X_h\), and \(-\mathcal{L}(a_c, \delta_c)|_{X_h}\) is invertible. Actually, we can also directly compute it out as
\[\det \begin{pmatrix}
16a_c k_0^4 & -2ik_0 \\
-2ik_0 \sigma'(0) & -4\delta_c k_0^2 + 16k_0^4
\end{pmatrix} = (2k_0)^2[\sigma'(0) + a_c(2k_0)^4((2k_0)^2 - \delta_c)] \neq 0\]
at any admissible critical configuration pairs. Now simple computations yield

\[
\begin{align*}
v_1 &= \frac{i\sigma''(0)A^2}{2[\sigma''(0) + a_c(2k_0)^4((2k_0)^2 - \delta_c)]} - \frac{8a_c k_0^4(21k_0^2 - 5\delta_c)}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \nonumber \\
v_2 &= \frac{i\sigma''(0)A^2}{2[\sigma''(0) + a_c(2k_0)^4((2k_0)^2 - \delta_c)]} + \frac{8a_c k_0^4(21k_0^2 - 5\delta_c)}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \nonumber 
\end{align*}
\tag{6.7}
\]

Noticing the relation \(a_c k_0^4 (21k_0^2 - \delta_c) + \sigma''(0) = 0\), we get

\[
\sigma''(0) + a_c (2k_0)^4((2k_0)^2 - \delta_c) = 3a_c k_0^4 (21k_0^2 - 5\delta_c) \neq 0.
\]

Then we can simplify the expressions of \(v_1\) and \(v_2\) further as

\[
\begin{align*}
v_1 &= \frac{i\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \\
v_2 &= \frac{8a_c k_0^4\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c). 
\end{align*}
\tag{6.8}
\]

To avoid ambiguity and for later reference, we will denote from now on that

\[
\phi_1 = \frac{i\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)}, \quad \phi_2 = \frac{8a_c k_0^4\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)}.
\]

To sum up, we have computed that

\[
\Phi(A, A^*) = i(e^{2ik_0x} \left( \frac{i\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \right) - e^{-2ik_0x} \left( \frac{-i\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \right)),
\]

and

\[
\Psi(A, A^*) = i(e^{2ik_0x} \left( \frac{-i\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \right) - e^{-2ik_0x} \left( \frac{-i\sigma''(0)A^2}{6a_c k_0^4(21k_0^2 - 5\delta_c)} k_0^4 (21k_0^2 - 5\delta_c) \right)) + O(|A|^3).
\]

We will again end this subsection with remarks:

**Remark 6.2.** The form of \(\Phi(A, A^*)\) is inherited from the symmetry of the system \(\mathcal{A}\).

**Remark 6.3.** The claim right before the equation \((6.7)\) in particular shows that condition \((d)\) is a consequence of conditions \((a), (b)\) and \((c)\) in the definition of \(\mathcal{A}(k_0)\).
6.5. **Dynamics on center manifold.** In this subsection, we compute the reduced dynamics for the system (1.1) or equivalently the system (4.2) or (4.6).

By our specific parametrization of the center space through complex-conjugate coordinates, the full dynamics of the system (1.1) or equivalently the system (4.2) or (4.6) on center manifold is given by

\[
\frac{d}{dt}(A \xi + A^* \xi + \Psi(A, A^*)) = \mathcal{L}(a, \delta)(A \xi + A^* \xi + \Psi(A, A^*)) + \mathcal{N}(A \xi + A^* \xi + \Psi(A, A^*))
\]

\[
= \mathcal{L}(a, \delta)(A \xi + A^* \xi) + R_{20}(A \xi + A^* \xi + \Psi(A, A^*), A \xi + A^* \xi + \Psi(A, A^*))
\]

\[
+ R_{30}(A \xi + A^* \xi + \Psi(A, A^*), A \xi + A^* \xi + \Psi(A, A^*), A \xi + A^* \xi + \Psi(A, A^*))
\]

\[
+ \Psi(A, A^*) + \tilde{R}(A \xi + A^* \xi + \Psi(A, A^*))
\]

In view of flow invariance and by direct computations through expanding the left-hand side or directly referring to page 245 of [25] (or pages 56-62 of [24]), we know that the second order approximation of the dynamics on the center manifold is given by

\[
\frac{d}{dt}(A(t) \xi + A(t) \xi^*) = \mathbb{P}\mathcal{L}(a, \delta)(A \xi + A^* \xi + \Phi(A, A^*)) + \mathbb{P}R_{20}(A \xi + A^* \xi + \Phi(A, A^*)) + \sum_{j=2}^{3} \mathbb{P}R_{j0}(A, A^*)
\]

(6.9)

Next, we proceed to compute the dynamics on the center manifold. During the procedure, we make use of the trivial fact that \( \int_{-\pi}^{\pi} e^{ikx} \, dx = 0 \) for any nonzero integer \( k \). Later, we will also see that the second approximation together with the latter fact will give enough insight for the full dynamics on the center manifold(s).

First, we compute \( \mathbb{P}R_{20}(A, A^*) \). From former analysis, we know that \( R_{20} \in \mathbb{Z}_h \), which yields \( \mathbb{P}R_{20}(A, A^*) = 0 \).

Second, we compute \( \mathbb{P}R_{20}(A \xi + A^* \xi^* + \Phi(A, A^*), A \xi + A^* \xi^* + \Phi(A, A^*)) \). To proceed, we first observe that

\[
(A \xi + A^* \xi^* + \Phi(A, A^*))^{(1)} = Ae^{ik_0x} + A^*e^{-ik_0x} + i(e^{2ik_0x}\phi_1 - e^{-2ik_0x}\phi_1^*)
\]

\[
\partial_x(A \xi + A^* \xi^* + \Phi(A, A^*))^{(1)} = ik_0(Ae^{ik_0x} - A^*e^{-ik_0x}) - 2k_0(e^{2ik_0x}\phi_1 + e^{-2ik_0x}\phi_1^*)
\]

Consequently, we have
\[ (A\xi + A^*\xi^*) + \Phi(A, A^*) \right)^{(1)} \frac{\partial}{\partial x} (A\xi + A^*\xi^*) + \Phi(A, A^*) = Q_1 + Q_2 + Q_3 + Q_4 \]

where

\[ Q_1 = ik_0 (Ae^{i k_0 x} + A^*e^{-i k_0 x})(Ae^{i k_0 x} - A^*e^{-i k_0 x}) = ik_0 (A^2 e^{2i k_0 x} - A^2 e^{-2i k_0 x}); \]

\[ Q_2 = -2k_0 (Ae^{i k_0 x} + A^*e^{-i k_0 x})(e^{2i k_0 x}\phi_1 + e^{-2i k_0 x}\phi_1^*) = -2k_0 (A\phi_1 e^{3i k_0 x} + A\phi_1^* e^{i k_0 x} + A^*\phi_1 e^{-i k_0 x} + A^*\phi_1^* e^{-3i k_0 x}); \]

\[ Q_3 = -k_0 (e^{2i k_0 x}\phi_1 - e^{-2i k_0 x}\phi_1^*)(Ae^{i k_0 x} - A^*e^{-i k_0 x}) = -k_0 (A\phi_1 e^{3i k_0 x} - A^*\phi_1 e^{i k_0 x} - A^*\phi_1^* e^{-i k_0 x} + A^*\phi_1^* e^{-3i k_0 x}); \]

\[ Q_4 = -2ik_0 (e^{2i k_0 x}\phi_1 - e^{-2i k_0 x}\phi_1^*)(e^{2i k_0 x}\phi_1 + e^{-2i k_0 x}\phi_1^*) = -2ik_0 (e^{4i k_0 x}\phi_1^2 - e^{-4i k_0 x}\phi_1^*); \]

Now we compute a specific form of \( R_{20}(A\xi + A^*\xi^*) + \Phi(A, A^*), A\xi + A^*\xi^* + \Phi(A, A^*) \) as follows

\[ R_{20}(A\xi + A^*\xi^*) + \Phi(A, A^*) = (0) \]

\[ = \sigma''(0) \left( (A\xi + A^*\xi^*) + \Phi(A, A^*) \right)^2 = \sigma''(0) \left( (A\xi + A^*\xi^*) + \Phi(A, A^*) \right)^{(1)} \]

\[ = \sigma''(0) \left( Q_1 + Q_2 + Q_3 + Q_4 \right). \]

Now, it is easy to observe that

\[ \langle \eta^*, \begin{pmatrix} 0 \\ Q_1 \end{pmatrix} \rangle = 0, \quad \langle \eta^*, \begin{pmatrix} 0 \\ Q_4 \end{pmatrix} \rangle = 0; \]

\[ \langle \eta^*, \begin{pmatrix} 0 \\ Q_2 \end{pmatrix} \rangle = \int_{-\pi}^{\pi} (-ke^{-ik_0 x})(-2k_0 A^*\phi_1 e^{ik_0 x}) \, dx = 4\pi\kappa A^*\phi_1; \]
\[
\langle \eta^*, \left( \frac{0}{Q_j} \right) \rangle = \int_{-\pi}^{\pi} (-\kappa e^{-ik_0x})(k_0A^*\phi_1 e^{ik_0x}) \, dx = -2\pi \kappa A^*\phi_1.
\]

From the expression of \( Q_j \) for \( 1 \leq j \leq 4 \) and the above expression on \( R_{20}(A \xi + A^*\xi^* + \Phi(A, A^*), A \xi + A^*\xi^* + \Phi(A, A^*)) \), we obtain

\[
\mathbb{P} R_{20}(A \xi + A^*\xi^* + \Phi(A, A^*), A \xi + A^*\xi^* + \Phi(A, A^*)) \ni \\
= \langle \eta^*, R_{20}(A \xi + A^*\xi^* + \Phi(A, A^*)) \xi + \langle \eta, R_{20}(A \xi + A^*\xi^* + \Phi(A, A^*)) \xi^* \\
= \sigma''(0) \sum_{j=1}^{4} \langle \eta^*, \left( \frac{0}{Q_j} \right) \xi + \sigma''(0) \sum_{j=1}^{4} \langle \eta, \left( \frac{0}{Q_j} \right) \xi^* \\
= 2\pi \sigma''(0)(\kappa A^*\phi_1 \xi + \kappa^* A\phi^*_1 \xi^*)
\]

Third, we compute \( \mathbb{P} R_{30}(A, A^*) \). First, we note that

\[
R_{30}(A, A^*) = \left( \frac{\sigma''(0)}{6} \partial_x((A \xi + A^*\xi^*)^{(1)^3}) \right) \\
= \left( \frac{\sigma''(0)}{2} (A \xi + A^*\xi^*)^{(1)} \partial_x((A \xi + A^*\xi^*)^{(1)}) \right) \\
= \left( \frac{i k_0 \sigma''(0)}{2} (Ae^{ik_0x} + A^*e^{-ik_0x})(A^2e^{2ik_0x} - A^2e^{-2ik_0x})) \right) \\
= \left( \frac{i k_0 \sigma''(0)}{2} (A^3e^{3ik_0x} - AA^2e^{-3ik_0x} + A^2A^*e^{ik_0x} - A^3e^{-ik_0x}) \right)
\]

Now, we have

\[
\langle \eta^*, R_{30}(A, A^*) \rangle = \langle \eta^*, \left( \frac{i k_0 \sigma''(0)}{2} (-AA^2e^{-ik_0x}) \right) \rangle \\
= \int_{-\pi}^{\pi} (-\kappa e^{ik_0x})(\frac{i k_0 \sigma''(0)}{2} A^2 A^*e^{ik_0x}) \, dx \\
= -i\pi \kappa \sigma''(0)k_0A^2A^*.
\]

Consequently, we obtain

\[
\mathbb{P} R_{30}(A, A^*) = \langle \eta^*, R_{30}(A, A^*) \rangle \xi + \langle \eta, R_{30}(A, A^*) \rangle \xi^* \\
= -i\pi \kappa \sigma''(0)k_0A^2A^*\xi - i\pi \kappa \sigma''(0)k_0AA^2\xi^*
\]

where we have used the fact that \( i\kappa \) is real.
Fourth, we compute the remaining term
\[ \mathbb{P}(\mathcal{L}(a, \delta) - \mathcal{L}(a_c, \delta_c))(A\xi + A\xi^* + \Phi(A, A^*)). \]

For this purpose, we recall that
\[ \mathcal{L}(a, \delta) - \mathcal{L}(a_c, \delta_c) = \begin{pmatrix} -\nu_1 \partial_x^4 & 0 \\ 0 & -\nu_2 \partial_x^2 \end{pmatrix}, \]

We note that
\[ \partial_x^4(A\xi + A^*\xi^* + \Phi(A, A^*))^{(1)} = \partial_x^4(Ae^{ik_0 x} + A^*e^{-ik_0 x} + i(e^{2ik_0 x}\phi_1 - e^{-2ik_0 x}\phi_1^*)) \]
\[ = k_0^4Ae^{ik_0 x} + k_0^4A^*e^{-ik_0 x} + i((2k_0)^4e^{2ik_0 x}\phi_1 - (2k_0)^4e^{-2ik_0 x}\phi_1^*) \]
and
\[ \partial_x^2(A\xi + A^*\xi^* + \Phi(A, A^*))^{(2)} = \partial_x^2[Ae^{ik_0 x}(-ia_ck_0^3) + A^*e^{-ik_0 x}(ia_ck_0^3) + i(e^{2ik_0 x}\phi_2 - e^{-2ik_0 x}\phi_2^*)] \]
\[ = ia_ck_0^5Ae^{ik_0 x} - ia_ck_0^5A^*e^{-ik_0 x} + i[-(2k_0)^2e^{2ik_0 x}\phi_2 + (2k_0)^2e^{-2ik_0 x}\phi_2^*]. \]

Hence, we can obtain
\[ \langle \eta^*, (\mathcal{L}(a, \delta) - \mathcal{L}(a_c, \delta_c))(A\xi + A\xi^* + \Phi(A, A^*)) \rangle \]
\[ = \langle \eta^*, \begin{pmatrix} -\nu_1 \partial_x^4(A\xi + A^*\xi^* + \Phi(A, A^*))^{(1)} \\ -\nu_2 \partial_x^2(A\xi + A^*\xi^* + \Phi(A, A^*))^{(2)} \end{pmatrix} \rangle \]
\[ = \langle \eta^*, e^{-ik_0 x} \begin{pmatrix} (-\nu_1)k_0^4A^* \\ (-\nu_2)(-ia_ck_0^5A^*) \end{pmatrix} \rangle \]
\[ = 2\pi(i\kappa)k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]A, \]

which yields
\[ \mathbb{P}(\mathcal{L}(a, \delta) - \mathcal{L}(a_c, \delta_c))(A\xi + A\xi^* + \Phi(A, A^*)) = 2\pi(i\kappa)k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2](A\xi + A^*\xi^*). \]

(6.10)

To sum up, we get the reduced dynamics on the center manifold given by the following equation on \( A = A(t) \in \mathbb{C}^1 \)
\[ \frac{d}{dt}A(t) = 2\pi(i\kappa)k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]A(t) + 2\pi\sigma''(0)\kappa A^*(t)\phi_1 - i\pi\kappa\sigma''(0)k_0A^2(t)A^*(t) + O(\lvert A \rvert^4) \]
\[ = \frac{1}{(a_c + 1)k_0^2 - \delta_c} \left( k_0^5[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]A(t) + \left[ \frac{\sigma''(0)}{6a_ck_0^2(21k_0^2 - 5\delta_c)} - \frac{\sigma''(0)}{2} \right]A(t)^2A(t)^* \right) + O(\lvert A \rvert^4), \]
and its complex conjugate equation on \( A^* \). In next subsection, we will strengthen the dynamics to be (6.15). However, we shall remark here that the above system is already enough for us to do the bifurcation analysis.

### 6.6. Analysis of the dynamics on center manifold.

To analyze the dynamics of the system (4.6), it is sufficient to analyze the \( A(t) \) equation or equivalently \( A^*(t) \) equation. If we introduce polar coordinates \( A(t) = r(t)e^{i\theta(t)} \), we get after standard computations that

\[
\begin{align*}
\frac{dr}{dt} &= \frac{1}{(a_c+1)k_0^2 - \delta_c} \left( k_0^4[(\delta_c - k_0^2)\nu_1 + a_c\nu_2]r(t) + \left[ \frac{\sigma''(0)^2}{6a_c k_0^2(21k_0^2 - 5\delta_c)} - \frac{\sigma'''(0)}{2} \right] r(t)^3 \right) + O(|r|^4), \\
\frac{d\theta}{dt} &= 0.
\end{align*}
\]

When we approximate the center manifold to an arbitrary order for sufficient smooth flux function \( \sigma(\tau) \), using the cancellation given by \( \int_{-\pi}^{\pi} e^{ikx} dx = 0 \) for any nonzero integer \( k \), we obtain that the higher order terms (\( \geq 4 \)) in the center manifold dynamics appear as \( AP(|A|^2) \) where \( P(\cdot) \) is a one variable polynomial with in general complex coefficients and without constant term and first order term. As a demonstration to make this point clear, we can consider typical terms to appear in the \( A(t) \) equation, say \( \langle \eta, PR_{n0}(A, A^*) \rangle \) where \( R_{n0}(A, A^*) = R_{n0}(A\xi + A^*\xi^*, ..., A\xi + A^*\xi^*) \). In view of the definitions of \( \eta, \xi \) and the inner product, we easily see that \( \langle \eta, PR_{n0}(A, A^*) \rangle = 0 \) for \( n \) even and \( \langle \eta, PR_{n0}(A, A^*) \rangle = \Omega A^kA^{k+1} = \Omega |A|^{2k}A \) for \( n = 2k + 1 \) for \( k \geq 2 \) for some number \( \Omega \).

Denote \( a := a(k_0, a_c, \delta_c) = \frac{k_0^4}{(a_c+1)k_0^2 - \delta_c} \) which is nonzero and real, and

\[
\mathbb{b} := \mathbb{b}(k_0, a_c, \delta_c, \sigma''(0), \sigma'''(0)) = \frac{1}{(a_c+1)k_0^2 - \delta_c} \left[ \frac{\sigma''(0)^2}{6a_c k_0^2(21k_0^2 - 5\delta_c)} - \frac{\sigma'''(0)}{2} \right].
\]

Now it is natural to choose the bifurcation parameter \( \mu \) as

\[
\mu = \mu(\nu) = \mu(\nu_1, \nu_2) = (\delta_c - k_0^2)\nu_1 + a_c\nu_2.
\]

Due to the above analysis, we conclude that the full \( A(t) \) dynamics on center manifold are actually given by

\[
\frac{d}{dt} A(t) = a\mu A + \mathbb{b}|A|^2A + O(|A|^5)
\]
in which the $O(|A|^5)$ term is given by $AP(|A|^2)$. If further $\sigma(\tau)$ is assumed to be smooth (which is a technical assumption), then $P(z) = \sum_{j \geq 2} c_j z^j$ with $c_j$ being complex in general.

Using subindices $r$ and $i$ to represent real and imaginary parts respectively, the dynamics of equation (6.13) can always be written in polar coordinates as

$$\begin{align*}
\frac{d}{dt} r(t) &= a_r \mu r + b_r r^3 + O(r^5) \\
\frac{d}{dt} \theta(t) &= a_i \mu r + b_i r^3 + O(r^5)
\end{align*}$$

(6.14)

where the $O(r^5)$ term in the $\frac{d}{dt} r(t)$-equation is given by $\sum_{j \geq 2} c_j r^{2j+1}$ while the $O(r^5)$ term in the $\frac{d}{dt} \theta(t)$-equation is given by $\sum_{j \geq 2} c_j i r^{2j+1}$ if $\sigma(\tau)$ is smooth.

Noticing that $a$ and $b$ are real, i.e., $a_r = a, a_i = 0$ and $b_r = b, b_i = 0$, we know that the above system takes the following form

$$\begin{align*}
\frac{d}{dt} r(t) &= f(r, \mu) \\
\frac{d}{dt} \theta(t) &= g(r)
\end{align*}$$

(6.15)

where $f(r, \mu) := a_r \mu r + b_r r^3 + O(|r|^5)$ and $g(r) = \sum_{j \geq 2} c_j i r^{2j+1} = O(r^5)$. Notice that $f(r, \mu)$ and $g(r)$ are both real polynomials.

Let us first analyze the radial equation, which is independent of the angular equation. Consider the radial equation $f(r, \mu) = a_r \mu r + b_r r^3 + O(|r|^5) = 0$. From our former analysis, we can write $f(r, \mu)$ as $f(r, \mu) = r h(r^2, \mu)$ where $h(r, \mu) = a_r \mu + b_r r^2 + o(|\mu| + r^2)$ with $h$ being at least a $C^1$-function. First, we observe that 0 is always an equilibrium. Noticing that $h(0, 0) = 0$ and $\frac{\partial h}{\partial r^2} |_{(0, 0)} = a \neq 0$, we conclude by the implicit function theorem that $\mu$ is a function of $r^2$ in a neighborhood of 0, i.e., $\mu = g(r^2)$ with $g(0) = 0$ for some function $g$ in a neighborhood of $(0, 0)$. In view that $h(0, 0) = 0$, we know that the Taylor expansion of $g$ is $\mu = g(r^2) = -\frac{b}{a} r^2 + o(r^4)$. Then we obtain that there is a curve of nontrivial equilibria in the $(\mu, r)$-plane that has a second order tangency at $(0,0)$ to the graph of $\mu = -\frac{b}{a} r^2 + o(r^4)$.

Moreover, for the truncated equation $f_0(r, \nu) := a_r \mu r + b_r r^3 = 0$, we observe that $r = 0$ is always a solution. Meanwhile, $r = \sqrt{-\frac{a \mu}{b}} > 0$, i.e., $\mu = -\frac{b}{a} r^2$ is another solution in the parameter range such that $-\frac{a \mu}{b} > 0$. In the parameter range $-\frac{a \mu}{b} < 0$, 0 is the only solution. The truncated differential equation $\frac{dr}{dt} = a_r r + b_r r^3$ can be solved explicitly: $r^2(t) = \frac{a \mu t^2}{\mu - \frac{b}{a} + \mu \frac{b}{a}(e^{-2a \mu t} - 1)}$. 


The above two paragraphs show that the truncated equation and the full equation have the same number of equilibria in a neighborhood of origin \( \mu = 0 \), which are \( o(|\mu|^{1/2}) \)-close to each other. Another fact is that the dynamics of the \( r \)-equation is exactly that of a standard pitchfork bifurcation if we allow \( r < 0 \). As in our case here \( r(t) = |A(t)| \), there is only one bifurcated nonzero equilibrium, which is different from the standard pitchfork bifurcation with two bifurcated equilibria. Hence, we may regard the dynamic here as \textit{half pitchfork bifurcation}.

Now, we analyze the angular equation, which is easy from the form of the equation itself. We need to distinguish two different dynamics: (i) \( g(r) = 0 \), in other words, all the \( c_j \) are real and the angular equation is trivial, (ii) there is some \( c_j \neq 0 \). Next we let the nontrivial equilibrium of radial equation be \( r_\mu \).

If the angular equation is nontrivial, then \( \theta(t) - \theta_0 = g(r_\mu)t \mod 2\pi = O(|\mu|^{5/2}) \) and any nontrivial equilibrium \( r_\mu \) corresponds to a \textit{limit cycle} with radius \( r_\mu \) and angular speed \( O(|\mu|^{5/2}) \), which is slow in order.

From the above analysis, we content ourselves to end this section with the following theorem from which we could make conclusions for our original system (1.1):

\textbf{Theorem 6.4.} The differential equation (6.15) undergoes a supercritical (resp. subcritical) half pitchfork bifurcation around \( \mu = 0 \) when \( b < 0 \) (resp., \( b > 0 \)) when the angular equation is trivial. More precisely, the following properties hold in a neighborhood of 0 of \( \mathbb{R}^1 \) for sufficiently small \( \mu \):

(i) If \( ab < 0 \) (resp., \( ab > 0 \)), the differential equation has precisely one trivial equilibrium \( r = 0 \) for \( \mu < 0 \) (resp., \( \mu > 0 \)). This equilibrium is stable when \( b < 0 \) and unstable when \( b > 0 \).

(ii) If \( ab < 0 \) (resp., \( ab > 0 \)), the differential equation possesses, for \( \mu > 0 \) (resp., \( \mu < 0 \)), the trivial solution \( r = 0 \) and one nontrivial equilibrium \( r_\epsilon \) parametrized by \( \epsilon = \mu^{1/2} \) and the dependence of \( r \) on \( \epsilon \) is smooth in a neighborhood of 0, and \( r_\epsilon = O(\epsilon) \), i.e., of magnitude \( O(|\mu|^{1/2}) \). The nontrivial equilibrium is stable if \( b < 0 \) and unstable if \( b > 0 \).

Otherwise, the differential equation (6.15) bifurcates a limit circle around \( \mu = 0 \) when \( b \neq 0 \) as follows:
(iii) If $a\beta < 0$ (resp., $a\beta > 0$), the differential equation has precisely one trivial equilibrium $r = 0$ for $\mu < 0$ (resp., $\mu > 0$). This equilibrium is stable when $\beta < 0$ and unstable when $\beta > 0$.

(iv) If $a\beta < 0$ (resp., $a\beta > 0$), the differential equation possesses, for $\mu > 0$ (resp., $\mu < 0$), the trivial solution $r = 0$ and one nontrivial limit cycle $R$ with radius $r_\epsilon$ and parametrized by $\epsilon = \mu^{1/2}$ and the dependence of $r$ on $\epsilon$ is smooth in a neighborhood of $0$, and $r_\epsilon = O(\epsilon)$, i.e., of magnitude $O(|\mu|^{1/2})$. The nontrivial limit cycle is stable if $\beta < 0$ and unstable if $\beta > 0$.

7. Conclusions

In this section, we discuss some other subtle points given by the following three subsections.

7.1. Isolation, Bifurcation Parameters and Measurement. In our bifurcation analysis, we always isolate the bifurcation parameters in our way of computing dynamics on the center manifold, which is natural. Here we introduced the bifurcation parameter $\mu$ through the bifurcation vector $\nu \in \mathbb{R}^2$. Mathematically, the results involving this parameter vector take care of variations of the control parameters $a$ and $\delta$ in one go. Though wide enough, these results are not easy to measure due to the change of two parameters, and also not sharp mathematically in certain cases. We consider the following two cases:

(i) If we fixed $a_c$ and consider the dynamics of $\partial_\nu U = L(a_c, \delta_c)U + (L(a, \delta) - L(a_c, \delta_c))U + N(U)$ when $\delta$ varies around $\delta_c$. Since $L(a, \delta_c) - L(a_c, \delta_c) = \begin{pmatrix} 0 & 0 \\ 0 & -(\delta - \delta_c)\partial_\nu^2 \end{pmatrix}$, then it is better for the ease of measurement to introduce the bifurcation parameter $\Gamma_2 := a_c k_0^4 (k_0^2 - \delta) + \sigma'(0)$. The relation of $\Gamma_2$ and $\nu_2$ is given by $\Gamma_2 = -a_c k_0^4 \nu_2$ through the difference $a_c k_0^4 (k_0^2 - \delta_c) + \sigma'(0) = 0$ and $a_c k_0^4 (k_0^2 - \delta) + \sigma'(0) = \Gamma_2$.

In this case, we have $R(U, \Gamma_2) = \begin{pmatrix} 0 & 0 \\ 0 & -(\delta - \delta_c) \end{pmatrix} U + N(U) \in H_{\text{per}}^2 (\pi, \pi)$. As a consequence, we can choose space triplet $Z \subset Y \subset X$ as $Z = H_{\text{per}}^4 (-\pi, \pi)$, $Y = H_{\text{per}}^2 (-\pi, \pi)$ and $X = L_{\text{per}}^2 (-\pi, \pi)$. Then the corresponding bifurcation occurs in $Y = H_{\text{per}}^2 (-\pi, \pi)$. This is sharp.

(ii) Similarly, if we fixed $\delta_c$ and consider the dynamics of $\partial_\nu U = L(a_c, \delta_c)U + (L(a, \delta) - L(a_c, \delta_c))U + N(U)$ when $a$ varies around $a_c$. Since $L(a, \delta_c) - L(a_c, \delta_c) = \begin{pmatrix} -(a - a_c) \partial_\nu^4 & 0 \\ 0 & 0 \end{pmatrix}$, then it is better for the ease of measurement to introduce the
bifurcation parameter $\Gamma_1 := ak_0^4(k_0^2 - \delta_c) + \sigma'(0)$. The relation of $\Gamma_1$ and $\nu_1$ is given by $\Gamma_1 = k_0^4(k_0^2 - \delta_c)\nu_1$ through the difference $a_c k_0^4(k_0^2 - \delta_c) + \sigma'(0) = 0$ and $ak_0^4(k_0^2 - \delta_c) + \sigma'(0) = \Gamma_1$. In this case, we have $R(U, \Gamma_2) = \begin{pmatrix} 0 & 0 \\ 0 & -\delta - \delta_c \end{pmatrix} U + N(U) \in L^2_{per}(-\pi, \pi)$. As a consequence, we can not upgrade the space triplet choice $Z \subset Y \subset X$ as in (i) above but shall choose $Z = H^4_{per}(-\pi, \pi)$, $Y = X = L^2_{per}(-\pi, \pi)$. Of course, the bifurcation occurs in $Y = L^2_{per}(-\pi, \pi)$ in this case. A drawback is that the observational bifurcation parameter $\Gamma_1$ is degenerate if $k_0^2 = \delta_c$ which is allowed.

7.2. Symmetry and Computations. For our system (4.6), we can verify that it is equivariant under the $O(2)$-group action (hence under $SO(2)$-group action) entirely similarly as in [22] (see Proposition 5.1 and Section 9 of [22]). In the current paper, we see that symmetry does play its role and the effect is embodied by the fact $A(k_0) = A(-k_0)$, by the form of the eigenvalue equation (3.2) for $M_k$ where $k$ enters the equation through $k^2$, by the parametrization of center space, during the computation of the dynamics on center manifold (for example, the form of $\Phi(A, A^*)$), by the paragraph below (6.11). However, the computations of the dynamics on center manifold in the current work did not refer to normal form theory. The reason is that the representation of the linear operator $\mathcal{L}(a_c, \delta_c)$ on the center space coordinated through the conjugate pair $(A, A^*)$ is the two by two zero matrix $0_{2 \times 2}$. We refer the reader to Theorem 6.2 in [22] or Theorem 7.11 in [34] for this point. Though we can not use the normal form theory to help us to do computations, it is interesting to compare the $O(2)$-equivariant Hopf bifurcations in [34, 22] and the bifurcations here. In particular, both $\sigma'(0)$ and $\sigma''(0)$ are allowed to be zero in the current paper, which means the important fact that hyperbolicity and genuine nonlinearity (see page 90 of [4] for the two definitions; see also [5, 21]) are not necessary for the current study.

7.3. Choice of spaces. This can be regarded as a remark for Section 2 and a continuation of Subsection 7.1 and we content ourselves in touching only several points among those we can make. In the following discussion, we talk about infinitely dimensional dynamics. There are two main ways of choosing working spaces in the literature: one is a space pair, say $D \subset X$, and the other is a space triplet $Z \subset Y \subset X$ as we adopted. In the former, one usually choose $D$ as the domain
of the linear operator on a Banach space $X$ which is properly large. While in the latter, one may choose $Z$ to be a subspace of the domain of the linear operator on the large ambient Banach space $X$ while $Y$ to be the interpolation space which takes care of the nonlinearities in a specific problem. Consequently, bifurcations occurs in $X$ in the former and in $Y$ in the latter. Obviously, the former way of choosing working spaces can be regarded as a special case of the latter one. It is due to this elementary fact that the space triplet choice way usually locates more precisely the spaces in which bifurcation dynamics occur. Carefully checking the proofs in the center manifold theory, we also see that the theory based on space pairs is more effective if the linear operators involved are sectorial (which is the case for dissipative partial differential equations) while the theory established with space triplets can deal with linear operators that are not sectorial. Hence the latter has more applications besides dissipative partial different equations.

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*Department of Mathematics, The University of Iowa, Iowa City, IA, 52242, USA
E-mail address: tong-li@uiowa.edu

†Department of Mathematics, Indiana University, Bloomington, IN, 47405, USA
E-mail address: wang264@indiana.edu

‡Corresponding author, Department of Mathematics, The University of Iowa, Iowa City, IA, 52242, USA
E-mail address: jinghua-yao@uiowa.edu