THE ROLE OF CONNECTEDNESS IN THE STRUCTURE AND THE ACTION OF GROUP OF ISOMETRIES OF LOCALLY COMPACT METRIC SPACES

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ABSTRACT. By proving that, if the quotient space Σ(X) of the connected components of the locally compact metric space (X, d) is compact, then the full group I(X, d) of isometries of X is closed in C(X, X) with respect to the pointwise convergence topology, i.e., that I(X, d) coincides in this case with its Ellis’ semigroup, we complete the proof of the following:

Theorem
(a): If Σ(X) is not compact, I(X, d) need not be locally compact, nor act properly on X.
(b): If Σ(X) is compact, I(X, d) is locally compact but need not act properly on X.
(c): If, especially, X is connected, the action (I(X, d), X) is proper.

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1. Introduction

Transformation groups have been proven an effective tool to investigate the structure of locally compact spaces. Since we know more about locally compact groups than about locally compact spaces, it is reasonable to use special properties of groups and their actions in order to understand the structure of spaces on which they act. In this direction, we treat in this note the following twofold question:

Let (X, d) be a locally compact metric space and I(X, d) its group of isometries. When (a) is I(X, d) locally compact (always with respect to the pointwise convergence topology), and (b) does it act properly on X?

In the case where X is connected the first question is answered affirmatively in [1] (cf. also [2, Ch. I, Th. 4.7]). The non–connected case is investigated in [3]. It turned out in this case, that the compactness of Σ(X), the quotient–space of the connected components of X, is a topological property, independent of the each time considered admissible metric, which ensures the local compactness of $\overline{I(X,d)}$, the closure of $I(X,d)$ in $C(X,X)$, the space of the continuous maps $X \to X$ endowed with the pointwise convergence topology, for all admissible metrics d. The question whether I(X, d) is closed in C(X, X) remained open. In this note we fill this gap (cf. also [4]), i.e., we show that in the case where Σ(X) is compact I(X, d) coincides with its Ellis’ semigroup, completing the proof of the following:

Theorem Let (X, d) be a locally compact metric space. Denote by Σ(X) the space of the connected components of X, and by I(X, d) its group of isometries. Then

1. If Σ(X) is not compact, I(X, d) need not be locally compact, nor act properly on X.
2. If Σ(X) is compact, then
   (a): I(X, d) is locally compact,
(b): the action \((I(X,d),X)\) is not always proper, and
(c): in the special case where \(X\) is connected, the action \((I(X,d),X)\) is proper.

For the sake of completeness, we give short and slightly improved proofs of already published by the authors partial results, crucial for a unified proof of the above theorem. Our treatment is based on the sets \((x,V_x) = \{g \in I(X,d) : gx \in V_x\}\), where \(V_x\) is a neighborhood of \(x \in X\). These sets form a neighborhood subbases for the identity, for the pointwise convergence topology, the natural topology of \(I(X,d)\).

2. Generalities

2.1 The following simple examples answer 1 and 2(b) of the above theorem.

Example Let \(X = \mathbb{Z}\) with the discrete metric. Obviously \(\Sigma(X)\) is not compact. It can be easily seen that \(I(X,d)\) is the group of the bijections of \(\mathbb{Z}\), which is not locally compact with respect to the pointwise convergence topology, therefore it cannot act properly on a locally compact space.

Example Let \(X = Y \cup \{(1,0)\} \subset \mathbb{R}^2\) where \(Y = \{(0,y) : y \in \mathbb{R}\}\), and \(d = \min\{1,\delta\}\), where \(\delta\) denotes the Euclidean metric. As we shall see in §3, by Theorem 3.7, \(I(X,d)\) is locally compact; however the action \((I(X,d),X)\) is not proper, because the isotropy group of \((1,0)\) is not compact, since it contains the translations of \(Y\). So, the action of \(I(X,d)\) on \(X\) is not proper, even if \(X\) has two components.

Since the sets \((x,V_x)\) as above form a neighborhood subbases for the identity for the pointwise convergence topology in \(I(X,d)\), the following condition is necessary for the local compactness of \(I(X,d)\):

\[ \left(\text{a)}\right): \text{There exist } x_i \in X, \ i = 1, \ldots, m \text{ such that } \bigcap_{i=1}^{m} (x_i, V_{x_i}) \text{ is relatively compact in } C(X,X). \]

This condition becomes also sufficient, if additionally the following condition is satisfied.

\[ \left(\text{b)}\right): I(X,d) \text{ is closed in } C(X,X). \]

So, to prove the local compactness of \(I(X,d)\), we have to ensure that both of the above conditions are satisfied.

3. The Local Compactness of \(I(X,d)\)

The following is crucial for the investigation of the conditions 2.1(a) and (b):

3.1 Lemma Let \((X,d)\) be a locally compact metric space, \(F \subseteq I(X,d)\), and
\[ K(F) = \{ x \in X : F(x) = \{ fx : f \in F \} \text{ is relatively compact} \}. \]

Then \(K(F)\) is an open and closed subset of \(X\).

Proof. Let \(x \in K(F)\) and \(A\) be a relatively compact neighborhood of \(\overline{F(x)}\). Let \((B,\varepsilon)\) denote the \(\varepsilon\)-neighborhood of \(B \subseteq X\), and \(\varepsilon\) be such that \(S(F(x),\varepsilon) \subseteq A\). It is easily seen that \(S(x,\varepsilon) \subseteq K(F)\), hence \(K(F)\) is open.
On the other hand, if $K(F) \ni x_n \to x$ and $x_n \in S(x, \eta)$, there are $f_i \in F$ such that

$$\overline{F(x_n)} \subseteq \bigcup_{i=1}^{k} f_i(S(x, 2\eta)).$$

Since

$$d(fx, f_ix_n) \leq d(fx, fx_n) + d(fx_n, f_ix) + d(f_ix, f_ix_n) = d(x, x_n) + d(fx_n, f_ix) + d(x, x_n),$$

if $i \in \{1, \ldots, k\}$ is such that $fx_n \in f_i(S(x, 2\eta))$, then $d(fx, f_ix_n) < 4\eta$. Hence

$$fx \in S(f_i, 4\eta) = f_i(S(x, 4\eta)) \subseteq f_i(S(x, 5\eta)).$$

Therefore $F(x) \subseteq \bigcup_{i=1}^{k} f_i(S(x, 5\eta))$, which means that $F(x)$ is relatively compact, if $S(x, 5\eta)$ is.

3.2 Remark In the sequel we assume that $\Sigma(X)$ is compact in the quotient topology via the natural map $q : X \to \Sigma(X)$. Note that $\Sigma(X)$ is a $T_1$-space, and need not be Hausdorff. Nevertheless

$X$ is separable, hence second countable; so sequences are adequate in $C(X, X)$.

The proof is similar to the lengthy one in [3] (see also [4, Appendix 2]).

3.3 Lemma Let $(X, d)$ be a locally compact metric space with compact space of connected components $\Sigma(X)$. Then condition 2.1(a) is satisfied.

Proof. Let $V_x$ be a relatively compact neighborhood of $x \in X$. Then

$$(x, V_x) = \{g \in I(X, d) : gx \in V_x\}$$

is a neighborhood of the identity in $I(X, d)$. By Lemma 3.1, $K((x, V_x))$ is not empty, open and contains whole components of $X$, therefore $q(K((x, V_x)))$ is an open subset of $\Sigma(X)$. Since $\Sigma(X)$ is compact, there are $x_i, i = 1, \ldots, m$, such that the corresponding $q(K((x_i, V_{x_i})))$’s cover $\Sigma(X)$. This means that $X = \bigcup_{i=1}^{m} K((x_i, V_{x_i}))$, i.e., the neighborhood $F = \bigcap_{i=1}^{m} (x_i, V_{x_i})$ of the identity has the property: $F(x)$ is relatively compact in $X$, for every $x \in X$, therefore, by Ascoli’ s theorem, $F$ is relatively compact in $C(X, X)$.

3.4 Now we proceed to prove that under assumption that $\Sigma(X)$ is compact, $I(X, d)$ is a closed subspace of $C(X, X)$. Because of Remark 3.2, the elements $f$ of the boundary of $I(X, d)$ in $C(X, X)$ are limits of sequences $\{f_n \in I(X, d), n \in \mathbb{N}\}$. Obviously, such an $f$ preserves $d$; so the question is when $f$ is surjective. If $\Sigma(X)$ is not compact this is not always true:

Example Let $X = \mathbb{Z}$ with the discrete metric. If $f_n(z) = z$ for $-n < z < 0$, $f(-n) = 0$, and $f_n(z) = z + 1$ otherwise then $f_n \to f$, where $f(z) = z$ for $z < 0$, and $f(z) = z + 1$ for $z \geq 0$. Hence each $f_n$ is an isometry, but $f$ is not surjective since $0 \notin f(\mathbb{Z})$.

3.5 Lemma If $\Sigma(X)$ is compact and $I(X, d) \ni f_n \to f$, then $f(X)$ is open and closed in $X$. 
**Proof.** By Lemma 3.1, it suffices to show that $f(X) = K(F)$, where $F = \{f_n^{-1}, n \in \mathbb{N}\}$. Indeed, since $d(f_n(x), f(x)) = d(x, f_n^{-1}(f(x)))$, we have $f_n^{-1}(f(x)) \to x$, so (since $X$ is locally compact) $f(x) \in K(F)$, for every $x \in X$. Now, if $y \in K(F)$, we may assume $f_n^{-1}(y) \to x$ for some $x \in X$, because $F(y)$ is relatively compact in $X$, hence $f(x) = y$.

**3.6 Proposition** If $(X, d)$ is a locally compact metric space, and $\Sigma(X)$ is compact, then $I(X, d)$ is closed in $C(X, X)$.

**Proof.** Let $I(X, d) \ni f_n \to f \in C(X, X)$. We prove that $f$ is surjective. Let $y \in X$. We denote by $S_x$ the connected component containing $x \in X$, and by $S_n$ the component of $f_n^{-1}y$. If $\{S_n, n \in \mathbb{N}\}$ has any constant subnet $\{S_{n_i}, i \in I\}$, then $S_{n_i} = S_0$, for some $S_0 \in \Sigma(X)$. Hence $S_{f_{n_i}}y = S_0$, so $f_{n_i}(S_0) = S_y$, for every $i \in I$. Therefore $y \in f(X)$.

Suppose that $\{S_n, n \in \mathbb{N}\}$ has not any constant subnet. By the compactness of $\Sigma(X)$, there exists a subnet $\{S_{n_i}, i \in I\}$ of $\{S_n, n \in \mathbb{N}\}$ such that $S_{n_i} \to S$, for some $S \in \Sigma(X)$. With the above notation, the following is true:

**Claim.** There exists a subsequence $\{S_{k_i}, k_i \in \mathbb{N}\}$ of $\{S_n, n \in \mathbb{N}\}$ such that there are $x_{k_i} \in S_{k_i}$ with $x_{k_i} \to x_0$, for some $x_0 \in X$.

**Proof.** If not, $R = (\bigcup_{n=1}^{\infty} S_n) \setminus S$ is closed in $X$. Indeed, let $R \ni y_m \to y \in X$. If $y_m \in (\bigcup_{n=1}^{m_0} S_n) \setminus S$ for $m > m_0$, then a subsequence of $\{y_m, m \in \mathbb{N}\}$ is contained in some $S_i$ for some $i \in \{1, \ldots, m_0\}$, therefore $y \in S_i \subset R$, as required. If this is not the case, we construct a subsequence $\{y_{m_p}, p \in \mathbb{N}\}$ of $\{y_m, m \in \mathbb{N}\}$ in the following way: We correspond to $S_1$ the point $y_{m_1} \in S_{n_1}$, with $n_1 > 1$ and $d(y_{m_1}, y) < 1$, to $(\bigcup_{n=1}^{m_1} S_n) \setminus S$ the point $y_{m_2} \in S_{n_2}$ with $n_2 > n_1$ and $d(y_{m_2}, y) < \frac{1}{2}$, and so on. Obviously, $S_{n_p} \ni y_{m_p} \to y$, a contradiction.

Hence $R$ is closed in $X$, from which follows $S \subset X \setminus R$, and $X \setminus R$ is open and contains entire components, so $S_{n_i} \subset X \setminus R$, eventually. Therefore $S_{n_i} = S$, a contradiction, since we have assumed that $\{S_n, n \in \mathbb{N}\}$ has not any constant subnet.

According to the Claim, we have $S_k \ni x_{k_i} \to x_0 \in X$, where $S_k = S_{f_{k_i}}y = f_{k_i}^{-1}S_y$, from which follows $x_{k_i} = f_{k_i}^{-1}y_{k_i}$ for some $y_{k_i} \in S_y$. Then

$$d(y_{k_i}, f(x_0)) \leq d(y_{k_i}, f_k x_0) + d(f_k x_0, f(x_0)) = d(f_{k_i}^{-1}y_{k_i}, x_0) + d(f_k x_0, f(x_0)) \to 0,$$

therefore $f(x_0) \in S_y$, which means that $S_y \cap f(X) \neq \emptyset$ and, by Lemma 3.5, $S_y \subset f(X)$, hence $y \in f(X)$, and $f$ is surjective.

**3.7 Theorem** If $\Sigma(X)$ is compact then $I(X, d)$ is locally compact.

**Proof.** It follows from Lemma 3.3 and Proposition 3.6, since both conditions 2.1(a) and (b) are satisfied.

4. THE PROPERNESS OF THE ACTION $(I(X, d), X)$

In this short section, applying the methods used previously, we give a complete proof of the following:
Proposition If \((X, d)\) is locally compact and connected, then \(I(X, d)\) is locally compact and the action \((I(X, d), X)\) is proper.

Proof. Since \(X\) is connected \(G = I(X, d)\) is locally compact by Theorem 3.7. So, we have to show that, for every \(x, y \in X\), there are neighborhoods \(U_x, U_y\) of \(x\) and \(y\) respectively such that

\[
(U_x, U_y) := \{g \in G : (gU_x) \cap U_y \neq \emptyset\}
\]

is relatively compact in \(G\). Let \(U_x = S(x, \varepsilon)\) and \(U_y = S(y, \varepsilon)\) be such that \(S(y, 2\varepsilon)\) is relatively compact. Then, for \(g \in (U_x, U_y)\) and \(z \in U_x\) with \(gz \in U_y\), we have

\[
d(gx, y) \leq d(gx, gz) + d(gz, y) = d(x, z) + d(gz, y) < 2\varepsilon,
\]

therefore \(g \in F = \{g \in G : gx \in S(y, 2\varepsilon)\}\). Then \(x \in K(F)\), and, according to Lemma 3.1, \(K(F)\) coincides with the connected space \(X\). From this and Ascoli’s theorem it follows that \(F\) is relatively compact in \(C(X, X)\). So \((U_x, U_y) \subseteq F\) is relatively compact in \(C(X, X)\), hence in \(G\), because \(G\) is closed (cf. Proposition 3.6).

This proves the Proposition and completes the proof of the Theorem in the Introduction.

5. Final Remark

Using the same arguments we can prove that if \(X\) is a locally compact metrizable space, then \(I(X, d)\) is locally compact for all admissible metrics \(d\), provided that the space \(Q(X)\) of the quasicomponents of \(X\) is compact (cf. [3]). Recall that the quasicomponent of a point is the intersection of all open and closed sets which contain it. Our exposition is given via \(\Sigma(X)\) because we regard that the condition “\(\Sigma(X)\) is compact” is a topologically more natural condition than “\(Q(X)\) is compact”, although it is more restrictive: There are locally compact metric spaces with compact \(Q(X)\) and non compact \(\Sigma(X)\) as the following example shows:

Example The space of the connected components of the locally compact space

\[
X = \left( \bigcup_{n=1}^{\infty} \left\{ \left( \frac{1}{n}, y \right) \mid y \in [-1, 1] \right\} \right) \cup \{(0, y) \mid y \in [-1, 0]\} \cup \left( \bigcup_{k=1}^{\infty} I_k \right) \subseteq \mathbb{R}^2,
\]

where

\[
I_k = \{(0, y) \mid y \in \left( \frac{1}{k+1}, \frac{1}{k} \right)\}, \quad k \in \mathbb{N}^*,
\]

is not compact, because the sequence \(\{I_k\} \subseteq \Sigma(X)\) does not have a convergent subsequence in \(\Sigma(X)\). On the contrary, \(Q(X)\) is compact, because the quasicomponent of the point \((0, -1)\) consists of the set \(\{(0, y) \mid y \in [-1, 0]\}\) and the intervals \(I_k, k \in \mathbb{N}^*\).

So the compactness of \(\Sigma(X)\) is not necessary for the local compactness of \(I(X, d)\).
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