Determination of the Riemann modulus and sheet resistivity by a six-point generalization of the van der Pauw method

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Abstract

A six-point generalization of the van der Pauw method is presented. The method is applicable for 2D homogeneous systems with an isolated hole. A single measurement performed on the contacts located arbitrarily on the sample edge allows us to determine the specific resistivity and a dimensionless parameter related to the hole, known as the Riemann modulus. The parameter is invariant under conformal mappings of the sample shape. The hole can be regarded as a high resistivity defect. Therefore, the method can be applied for the experimental determination of the sample inhomogeneity.

Keywords: resistivity, van der Pauw method, thin samples, sheet resistance

(Some figures may appear in colour only in the online journal)

1. Introduction

The four-point probe electric measurement is a fundamental tool in material science, semiconductor industries, physics, and geology [1]. We focus on the famous van der Pauw method of measuring the resistivity of a flat, isotropic, homogeneous sample with contacts located at its edge [2, 3]. The power of this method, following from the conformal invariance, lies in the ability of measuring sheet resistance (the ratio of specific resistivity and thickness) without any information about the positions of the contacts along the edge. A similar method, based on the Thompson and Lampard theorem [4], is applicable to a particular class of capacitors in which the cross-capacitance per unit length is independent of the size [5–7]. In fact, the van der Pauw method and the Thompson and Lampard theorem are closely related and follow from the solution of the Laplace equation in 2D systems with well-defined boundary conditions [8]. There are many papers studying the influence of inhomogeneities or defects on the van der Pauw measurements [9–12]. To the best of our knowledge the proposed corrections are not conformally invariant and depend on the specific shape of the sample. An interesting generalization of the van der Pauw method for contacts positioned away from the boundary was presented and tested by Lim et al [13].

In the original concept of van der Pauw, two four-probe resistances are measured on a flat, homogeneous sample without an isolated hole. The van der Pauw-type measurements can be extended on doubly-connected samples (samples with a hole) [14]. Such samples are characterized by a dimensionless parameter known as the Riemann modulus [15]. Any 2D sample with an isolated hole can be transformed by conformal mapping into an annulus (see [16], p 255) with inner radius \( r_{\text{in}} \) and outer radius \( r_{\text{out}} \), and the Riemann modulus \( \mu = r_{\text{out}}/r_{\text{in}} \). The Riemann modulus is an invariant of conformal mappings (i.e. it has the same value for conformally equivalent samples). Since a sample with a hole is a natural extension of a homogeneous sample without an isolated hole (considered originally by van der Pauw) the Riemann modulus of the sample without holes is equal to infinity. Recently it was shown [17] how to determine the sheet resistivity and Riemann modulus for a doubly-connected...
sample of arbitrary shape. A pair of four-probe resistances measured for various locations of probes fall inside a certain region which is confined by some boundaries. The essence of the method [17] was the observation that some extreme values of the resistances show a well-defined correlation. The shape of the correlation curve depends on both the Riemann modulus and the specific resistivity in a single measurement.

2. Six-point method

Let us consider a conductive medium with four different point contacts located at points $\alpha \beta \gamma \delta$. A current $i_{\alpha \beta}$ enters the sample at contact $\alpha$ and leaves at contact $\beta$, while the potential $V_{\gamma \delta}$ is measured between contacts $\gamma$ and $\delta$. The resistance for contacts $\alpha \beta \gamma \delta$ is defined as $R_{\alpha \beta \gamma \delta} = V_{\alpha \beta \gamma \delta} / i_{\alpha \beta}$. We further consider only ohmic media, e.g. currents are linear functions of the potentials. It was shown recently [14] that in a special case of the four contacts located at the edge of an annulus with outer radius $r_{\text{out}}$ and inner radius $r_{\text{inn}}$ shown in figure 1(a)

$$R_{\alpha \beta \gamma \delta} = \lambda \ln \frac{G(\gamma - \alpha, h)G(\delta - \beta, h)}{G(\gamma - \beta, h)G(\delta - \alpha, h)},$$

where $h$, is a geometric parameter defined in [17] and $\lambda = \rho(\pi d)$ is proportional to the specific resistivity $\rho$ of a sample with thickness $d$. The function $G(\phi, h)$ is defined as

$$G(\phi, h) := \sin \frac{\phi}{2} \prod_{j=1}^{\infty} \left(1 - \frac{\cos \phi}{\cosh jh}\right).$$

The geometric parameter $h$ is directly related to the so-called Riemann modulus of a doubly-connected region. Any doubly-connected region can be conformally mapped into an annulus with outer radius $r_{\text{out}}$ and inner radius $r_{\text{inn}}$. The ratio $\mu = r_{\text{out}} / r_{\text{inn}}$ ($\mu > 1$), known as the Riemann modulus, is invariant with respect to conformal transformations. The parameter $h = 2\mu a$ for an annulus. The function $G$ is even and periodic:

$$G(-\phi, h) = G(\phi, h), \quad G(\phi + 2\pi, h) = G(\phi, h).$$

Taking into account the evenness of $G$ we easily see that $R_{\alpha \beta \gamma \delta} = R_{\beta \gamma \delta \alpha} = R_{\gamma \delta \alpha \beta} + R_{\delta \alpha \beta \gamma} + R_{\alpha \beta \gamma \delta} = 0$, which is a particular case of the general reciprocity theorem [5].

Let us consider the same annulus with six contacts located at arbitrary angles of the same edge, shown schematically in figure 1(b). In our convention $\alpha \beta \gamma \delta \epsilon \zeta$ the positions of the contacts at the vertices of the inscribed hexagon in figure 1(b) result in nine four-probe resistances, as shown schematically in figure 1(c). Six of them are obtained by cyclic permutations and are abbreviated by $r_{ij}$, $i = 1, \ldots, 6$ while three others are abbreviated by $r_{ij}$, where the pair of indices $(i, j) = (1, 4), (2, 5), (3, 6), (6, 3)$ corresponds to the location of the contacts at the opposite edges of the hexagon. It is clear that one consequence of our definition is $r_{ij} = r_{ji}$. The nine resistances shown in figure 1(c) depend on the geometrical parameter $h$, resistivity $\lambda$, and five independent differences $\phi_i$. Thus, we obtain the following overdetermined system of nine equations for seven unknowns:

$$r_{ij} = \lambda \ln \frac{G(\phi_i + \phi_{j-1} + 1, h)G(\phi_i + \phi_{j+1}, h)}{G(\phi_i, h)G(\phi_{j-1} + \phi_{j+1}, h)},$$

$$r_{i,i+3} = \lambda \ln \frac{G(\phi_i + \phi_{i+1} + \phi_{i+2} + 1, h)G(\phi_i + \phi_{i+1}, h)}{G(\phi_i - 1 + \phi_{i+2}, h)G(\phi_i + \phi_{i+2}, h)}.$$

where we use $\phi_{i+6} = \phi_i$ in order to define $\phi_i$ for indices outside the range $1, \ldots, 6$. The same rule can be applied to $r_i$ and $r_{i,j}$ when needed. Taking into account $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 + \phi_6 = 2\pi$ and (3) we have

$$G(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 = 2\pi)$$

$$G(\phi_{i+3} + \phi_{i+1} + 1, h) = G(\phi_i + \phi_{i-1} + \phi_{i-2} - 2, h)$$

$$G(\phi_{i+3} + \phi_{i+4} + \phi_{i+5} - h) = G(\phi_i + \phi_{i+1} + \phi_{i+2} + h).$$

Hence, $r_{i,i+3} = r_{i+3,i}$, which perfectly agrees with the above physical argument.

The currents and potentials in (1) are invariant under any conformal transformation. Therefore, equations (4) are valid
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for a sample of any shape with an isolated hole. Having measured resistances \( r_i \) for the arbitrary positions of the contacts, the parameters \( h \), \( \lambda \), and \( \phi_i \), can be obtained by solving a set of nonlinear algebraic equations (4). Since the set (4) is overestimated, the convenient method of solution is the minimization of

\[
\chi^2 = \sum_{i=1}^{6} (r_i - r_i^{\text{exp}})^2 + \sum_{j=1}^{3} (r_{i,j+3} - r_{i,j+3}^{\text{exp}})^2
\]  

(6)

with respect to seven independent parameters: \( h \), \( \lambda \), and six angles \( \phi_i \) constrained by \( \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 = 2\pi \).
The symbols $r_i^{\text{exp}}$ and $r_{ij}^{\text{exp}}$ were used for the abbreviation of the experimentally measured four-probe resistances.

It is worth mentioning that in the analogous five-point method, one gets five independent resistances and six unknowns. Therefore, the system is underdetermined and one cannot determine all the required quantities. Thus, the presented six-point method, although resulting in an overdetermined system of equations (4), is an optimal one.

3. Experimental results

Our samples were prepared from commercially available, technical quality bronze foil with a thickness $(0.271 \pm 0.003)$ mm, the form of the disc with a radius $(146.1 \pm 0.2)$ mm, and an annulus with outer radius $r_{\text{out}} = 124.5 \pm 0.5$ mm and inner radius $r_{\text{inn}} = 74.3 \pm 0.3$ mm. The geometrical parameter for the annulus depends on the ratio of the radii [14], and in our case $h = 2\ln(r_{\text{out}}/r_{\text{inn}}) = 1.03 \pm 0.01$. The resistivity to thickness ratio was determined by the standard van der Pauw method (for the disc) and also by the envelope method (proposed in [17] for the annulus) giving the same value $\lambda = (0.3489 \pm 0.0003)$ mΩ.

The six-point method was verified on the annulus by comparing the values of $\varphi_i$ obtained by the minimization of (6) and the experimental positions of the contacts located at the outer boundary, as shown schematically in figure 1(b). In order to simplify the comparison, the positions of the contacts were parametrized by one parameter $\varphi$, see figure 1(d). The complete set of experimental results is presented in figure 2. We see a very good agreement of the experimental data with the theoretical dependencies given by equation (4). Note that for the specific configuration of the contacts shown in figure 1(d) our theory predicts: $r_1 = r_5$ and $r_2 = r_3 = r_4$. Moreover, $r_1$ and $r_5$ should vanish for $\varphi = 2\pi/5$ (i.e. $\varphi_6 = 0$). All these properties can also be observed in figure 2. There are clear differences between the results shown in figures 2(a) and 4(b). Therefore, the presented method is sensitive enough for the detection of a hole. In the case of the configuration from figure 1(d), where the positions of the contacts are parametrized by a single value $\varphi$, we expect that all five values $\varphi_i$ ($i = 1,2,\ldots,5$) minimizing (6) are equal to $\varphi$. The sets of solutions for different values of $\varphi$ are presented in figure 3. We see that for $\varphi < 0.3$ the agreement with the theory is far from being perfect (it corresponds to configurations where all six contacts are located near each other, in a quarter of the circle). The highest precision is obtained when at least five contacts are approximately equally spaced on the edge of the annulus ($\varphi = 2\pi/6$). The obtained value of $h$ agrees (for $\varphi > 0.3$) with geometrical value $h$ determined from the ratio of the radii.

The six-point method was also applied to the sample with an irregular shape. Nine resistances were measured for arbitrary positions of the contacts (shown on the ordinate in figure 3(c) and geometrical parameter $h$ and five unknown angles were obtained from (5). Next, the predicted resistances obtained were drawn on the abscissa in figure 3(c) showing perfect agreement.

Figure 3. (a) Geometrical parameter $h$ obtained as a solution of the nonlinear system (4) by the minimization of (6) for six contacts located at the edge of the annulus under experimental conditions $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 2\pi/5\varphi$. (b) Angles $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$, $\varphi_5$ between contacts estimated by the minimization of (6). For each value of $\varphi$ (ordinate) there are five estimated values (abscissa) shown by dots (some of them overlap). (c) Comparison of resistances $r_i$, $r_{ij}$ measured on an irregular sample with theoretical predictions, where $h$, $\lambda$, and the five angles are obtained from the minimization of (6). The solid lines in (b) and (c) are guides to the eye corresponding to the ideal agreement between the experimental values and the estimation based on the minimization of (6).

4. Discussion

One of the most obvious applications of the presented method is the determination of the parameter of geometrical type known as the Riemann modulus by the measurement of the resistances between contacts located at the sample edge. One may expect not too many applications for the determination of the Riemann modulus of the sample with a hole. Nevertheless, there is at least one important case. A small hole can be regarded as a...
high resistivity defect in an otherwise homogeneous sample. Therefore, the hole is mimicking a class of defects. A small hole can be considered as a small deviation from the homogeneity, and the procedure of measuring the Riemann modulus can be used for the quantitative characteristic of the sample quality. A perfectly homogeneous sample without a hole should yield a Riemann modulus equal to infinity.

An important question is about the sensitivity of the method in the presence of a small hole. We have performed some initial experiments and we observed a clear dependence of the Riemann modulus on the hole size when the hole was relatively large. However, in the case of a small hole located at the sample center, the experimental observations were disturbed, most probably by sample inhomogeneity, by the thickness distribution, and, as a consequence, the distribution of the lambda parameter, e.g. the ratio of specific resistivity and the sample thickness. The experiments are in progress.

The presented formalism is applicable to 2D systems with an isolated hole. However, this does not mean that only thin, flat samples can be measured. A sample made of a thin layer forming a cylindrical [14] or conical surface can be measured by positioning six contacts on one edge, as shown schematically in figures 4(a) and (b), yielding the Riemann modulus and a sheet resistivity.

Also, a class of samples of arbitrary thickness (figures 4(c)–(f)) can also be considered as 2D systems. The six-point method can be applied by using the knife-edge contacts shown schematically in figure 4(c). Therefore, a large class of elements—thick cylinders and thick cylinders with a cylindrical hole (symmetric or eccentric, see figures 4(e) and (f). Also, the thick cuboid (with a hole or without a hole) shown in figure 4(d) can be measured by knife-shaped contacts. It is worth mentioning that one of the possible applications is the testing of the large and thick steel sheets produced industrially by rolling. These large and heavy objects can be tested when stored in stacks, where only the sheet edges are easily accessible. Another obvious application area is an option of six-point measurements in scientific laboratory equipment using the standard van der Pauw method. In this way the sample quality could be tested simultaneously.

There are non-destructive methods of investigation of local resistance by four probes located far from the sample edge, see [18] and the references cited therein. In contrast to these measurements, the proposed six-point method is global, e.g. the degree of inhomogeneity depends on the whole sample. In other words, in a single measurement the hole can be detected irrespective of their location. Therefore, the presented method is complementary and not competitive to the abovementioned technique. The advantage of the six-point method is a single, well-defined quantity, the parameter \( h \) in (4) considered as the inhomogeneity degree.

5. Summary

We have presented and verified experimentally an extension of the van der Pauw method. The extension is valid for 2D, flat, homogeneous samples with an isolated, single hole. In the method six contacts are located at arbitrary positions at the sample edge and by measurement of the resistances between the contacts one is able to determine the geometrical parameter known as the Riemann modulus and the resistivity to thickness ratio. The geometrical parameter is invariant under conformal transformation.

The method was verified experimentally on a sample with a shape of annulus where the positions of the contacts were used for comparison between the measured and predicted values of the geometrical parameter. Full agreement was achieved. The results presented in figures 2(a) and (b) clearly show that the method is sensitive to the presence of an isolated hole.

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