Presentations of groups with even length relations

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ABSTRACT
We study the properties of groups that have presentations in which the generating set is a fixed set of involutions and all additional relations are of even length. We consider the parabolic subgroups of such a group and show that every element has a factorization with respect to a given parabolic subgroup. Furthermore, we give a counterexample, using a cluster group presentation, which demonstrates that this factorization is not necessarily unique.

1. Introduction

A presentation of a group is a concise method of defining a group in terms of generators and relations. In special cases, much information about the corresponding group can be extracted from a given presentation [12]. Coxeter presentations are a classical example of this [2].

Recall that a pair \((W, S)\), where \(S = \{s_1, \ldots, s_n\}\) is a non-empty finite set and \(W\) is a group, is called a Coxeter system if \(W\) has a group presentation with generating set \(S\) subject to relations of the form \((s_is_j)^{m(i,j)}\), for all \(s_i, s_j \in S\), with \(m(i,j) = 1\) if \(i = j\) and \(m(i,j) \geq 2\) otherwise, where no relation occurs on \(s_i\) and \(s_j\) if \(m(i,j) = \infty\) [11, Section 5.1]. Such a group, \(W\), is called a Coxeter group.

An outline of the rich history of research into these groups is given in [3, Historical Note]. In particular, it is well known that the finite Coxeter groups can be classified via their Coxeter graphs and the class of finite Coxeter groups is precisely the class of finite reflection groups [3, Chapter VI, Section 4, Theorem 1;5,6]. The applications of Coxeter groups are widespread throughout algebra [3], analysis [9], applied mathematics [4], and geometry [7]. However, the many combinatorial properties of Coxeter groups make them an interesting topic of research in their own right (see [2]).

For example, for any subset \(I \subseteq S\), \(W_I\) denotes the subgroup of \(W\) generated by \(I\). Any subgroup of \(W\) which can be obtained in this way is called a parabolic subgroup of \(W\) [11, Section 5.4]. It is known that the coset \(wW_I\) contains a unique element of minimal length for each \(w \in W\), meaning that we can choose a distinguished coset representative of \(wW_I\). It follows that there exists a unique factorization of each element with respect to a given parabolic subgroup [2, Proposition 2.4.4]. In certain cases, the reduced expressions for these distinguished coset
representatives for a maximal parabolic subgroup can be described explicitly; see for example [13, Corollary 3.3; 15].

In this paper, we will consider group presentations which generalize the Coxeter case by allowing generating sets of infinite size and any relations that have even length. We show that a variation of this property of Coxeter groups holds for any group, $G$, which has a group presentation of this type.

In particular, we can define a parabolic subgroup of $G$ in an analogous way to the Coxeter case and prove that there exists a (not necessarily unique) factorization of each element of $G$ with respect to a given parabolic subgroup. We also give a counterexample using cluster group presentations (in the sense of [17, Definition 1.2]) showing that, in contrast to the Coxeter case, this factorization is not unique in general.

Some of these results form part of the author’s Ph.D. thesis [18], carried out at the University of Leeds. The paper will proceed in the following way. In Section 2, we establish basic properties of the length function on $G$ and provide a more detailed summary of the main result of this paper. In Section 3, we prove our first main result: that there exists a factorization of each element of $G$ with respect to a given parabolic subgroup. In Section 4, we construct an example in which the factorization with respect to a given parabolic subgroup is not unique.

2. Group presentations with even length relations

Let $G$ be a group arising from a group presentation $\langle X | R \rangle$, where $X$ may be finite or infinite. By the definition of the presentation of a group, any element $w$ of $G$ can be written as

$$w = x_1^{a_1} x_2^{a_2} \ldots x_r^{a_r},$$

where $r \in \mathbb{N}, x_j \in X$ and $a_j = \pm 1$, for all $1 \leq j \leq r$. The length of $w \in G$, $l(w)$, is the smallest $r$ such that $w$ has an expression of this form and a reduced expression of $w$ is any expression of $w$ as a product of $l(w)$ elements of $X \cup X^{-1}$ (where $X^{-1} = \{x^{-1} : x \in X\}$ is a copy of $X$). We refer to $l$ as the length function on $G$. Taking an index set $I$, if $R$ is a set of relations of the form $u_i = v_i$ for $i \in I$, with $u_i, v_i \in F(X)$, where $F(X)$ denotes the free group on $X$, then the length of the relation $u_i = v_i$ is given by the length of the word $u_i v_i^{-1}$ in $F(X)$.

Analogously to the Coxeter case, we define a parabolic subgroup of $G$.

**Definition 2.1.** For $I \subseteq X$, we denote by $G_I$ the subgroup of $G$ generated by $I$. A (standard) parabolic subgroup of $G$ is a subgroup of the form $G_I$ for some $I \subseteq X$.

Moreover, for each $I \subseteq X$ we define the following sets.

$$G^I = \{ w \in G : l(wx) > l(w) \forall x \in I \};$$

$$I^G = \{ w \in G : l(xw) > l(w) \forall x \in I \}.$$

In Section 3, we will prove our main result:

**Proposition 2.2.** Let $G$ be a group generated by a fixed set, $X$, of involutions subject only to relations of even length. For $I \subseteq X$, let $G_I$ denote the subgroup of $G$ generated by $I$ and $G^I = \{ w \in G : l(wx) > l(w) \forall x \in I \}$. Then every element $w \in G$ has a factorization

$$w = ab,$$

where $a \in G^I, b \in G_I$ and $l(w) = l(a) + l(b)$.

This result is a strengthening of [18, Proposition 7.2.4]. By comparison, we can see that this result is similar to [2, Proposition 2.4.4] for Coxeter groups. However, unlike the Coxeter case, we will show in Section 4 that the factorization for elements of $G$ with respect to a given parabolic subgroup is not necessarily unique or determined by minimal length elements of the coset $wG_I$. 
3. Proof of main result

To begin, we prove some basic results for the length function on a group with an arbitrary group presentation \( \langle X|R \rangle \).

**Lemma 3.1.** If \( x \in X \) and \( w \in G \) then \( l(xw) < l(w) \) if and only if there exists a reduced expression of \( w \) beginning in \( x^{-1} \).

**Proof.** Let \( l(xw) < l(w) \). Suppose \( xw = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \) is a reduced expression, where \( a_j \in \mathbb{Z} \) for all \( 1 \leq j \leq r \). Then \( w = x^{-1} x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \) is an expression of \( w \) of length \( r+1 \). Moreover, this expression must be reduced otherwise \( l(w) \leq r \), contradicting that \( l(xw) < l(w) \).

Conversely, if there exists a reduced expression of \( w \) beginning in \( x^{-1} \), say \( w = x^{-1} x_2^{a_2} \cdots x_r^{a_r} \), where \( a_j \in \mathbb{Z} \) for all \( 2 \leq j \leq r \), then \( xw = x_2^{a_2} \cdots x_r^{a_r} \) and so \( l(xw) \leq r-1 < l(w) \). \( \square \)

For the remaining results, we let \( G \) be a group with group presentation \( \langle X|R \rangle \) which satisfies the following conditions.

(I) \( X \) is a fixed set of involutions. That is, for each \( x \in X, x^2 = e \).

(II) Every relation in \( R \) has finite order. In this case, the surjective homomorphism properties. Below, we consider the length function on \( G \). The first result is a consequence of Lemma 3.1.

**Lemma 3.2.** Let \( I = X \setminus \{ x \} \) for some \( x \in X \) and take \( w \in G \) such that \( w \neq e \). Then \( w \in \langle I \rangle G \) if and only if all reduced expressions of \( w \) begin in \( x \).

**Proof.** If \( w \in \langle I \rangle G \) has a reduced expression beginning in \( y \) for some \( y \in X \) such that \( y \neq x \) then \( l(yw) < l(w) \), contradicting that \( w \in \langle I \rangle G \). Conversely, suppose \( w \in G \) is such that all reduced expressions of \( w \) begin in \( x \). For any \( y \in X \) such that \( l(yw) < l(w) \) there exists a reduced expression of \( w \) beginning in \( y \), by Lemma 3.1. Hence \( y = x \) and so \( w \in \langle I \rangle G \). \( \square \)

**Remark 3.3.** It follows from Lemma 3.2 that \( w \in G^I \) if and only if all reduced expressions of \( w \) end in \( x \).

The following result is analogous to [11, Proposition 5.1] for Coxeter groups.

**Proposition 3.4.** There exists a surjective homomorphism \( \varepsilon : G \to \{ \pm 1 \} \) defined by \( \varepsilon : x \mapsto -1 \) for each \( x \in X \). It follows that the order of each generator is 2.

**Remark 3.5.** Suppose the group presentation \( \langle X|R \rangle \) satisfies only condition (II) and each generator has finite order. In this case, the surjective homomorphism \( \varepsilon \) exists and the order of each generator \( x \in X \) of \( G \) is even.

In [11, Section 5.2], the length function on a Coxeter group is defined along with five basic properties. Below, we consider the length function on \( G \).

As for the Coxeter case, each element of the generating set \( X \) of \( G \) is an involution and so any element \( w \) of \( G \) can be written in the form \( w = x_1 x_2 \cdots x_r \) for \( x_j \in X \). So the length, \( l(w) \), of \( w \in G \), will be the smallest \( r \) such that \( w = x_1 x_2 \cdots x_r \).

**Lemma 3.6.** For all \( w_1, w_2 \in G \) and \( x \in X \) the following properties hold.

1. \( l(w_1) = l(w_1^{-1}) \).
2. \( l(w_1) = 1 \) if and only if \( w_1 \in X \).
3. \( l(w_1 w_2) \leq l(w_1) + l(w_2) \).
4. \( l(w_1 w_2) \geq l(w_1) - l(w_2) \).
(5) \( l(w_1) - 1 \leq l(w_1x) \leq l(w_1) + 1 \).
(6) \( l(w_1x) = l(w_1) + 1 \) and \( l(xw_1) \neq l(w_1) \)

**Proof.** The proof is analogous to the Coxeter case, see [11, Section 5.2].

**Remark 3.7.** The properties (1), (3) and (4) in Lemma 3.6 hold for a group with arbitrary presentation.

It is easy to see that (2) and (6) fail for the trivial group given by the presentation \( \langle x \mid x = e \rangle \).

However, the property

\[ (2^*) \ w_1 \in X \cup X^{-1} \implies l(w_1) \leq 1 \]

holds for any group with group presentation \( \langle X \mid R \rangle \). We can then apply \((2^*)\), (3) and (4) to prove that (5) also holds for a group with arbitrary presentation.

We are now ready to prove Proposition 2.2.

**Proof of Proposition 2.2.** We proceed by induction on \( l(w) \). If \( l(w) = 1 \) then \( w = x \), for some \( x \in X \). If \( x \in I \), then we choose \( a = e, b = x \) and, by Lemma 3.6(2), this is the desired factorization. If \( x \notin I \), then we claim that \( l(xy) > l(x) \) for all \( y \in I \) and so we choose \( a = x, b = e \). Taking any \( y \in I \), if \( l(xy) < l(x) \) then \( l(xy) = 0 \) as \( l(x) = 1 \), by Lemma 3.6(2). Thus, \( xy = e \). It follows that \( x = y \in I \), contradicting the fact that \( x \notin I \). As \( l(xy) \neq l(x) \) by Lemma 3.6(6), it must be that \( l(xy) > l(x) \).

Suppose \( l(w) = r \geq 1 \) and that the statement holds for every element of \( G \) of shorter length. If \( w \in G^I \) then we choose \( b = e \) and \( a = w \). Similarly, if \( w \in G_I \) then we choose \( a = e \) and \( b = w \). So we need only consider the case when \( w \notin G^I \) and \( w \notin G_I \).

As \( w \notin G_I \), there exists \( x \in I \) such that \( l(wx) < l(w) \). By Lemma 3.6(5), \( l(wx) = l(w) - 1 < r \).

By induction, there exists \( a' \in G^I \) and \( b' \in G_I \) such that \( wx = a'b' \) and

\[ l(wx) = l(w) - 1 = l(a') + l(b'). \]

Let \( a = a' \) and \( b = b'x \). Then \( ab = a'b'x = (wx)x = wx^2 = w \). It remains to show that \( l(b'x) = l(b') + 1 \), giving \( l(a) + l(b) = l(a') + l(b') = l(a') + l(b') + 1 = l(wx) + 1 = l(w) \).

We assume, for a contradiction, that \( l(b'x) < l(b') \). That is, by Lemma 3.6(5), \( l(b'x) = l(b') - 1 \). By the above, \( wx = a'b' \), so \( w = a'b'x \), and \( l(wx) = l(a') + l(b') \). Thus

\[ l(w) = l(a'b'x) \leq l(a') + l(b'x) \]
\[ = l(a') + (l(b') - 1) \]
\[ = ((l(a') + l(b')) - 1 \]
\[ = l(wx) - 1 \]
\[ < l(wx), \]

contradicting the fact that \( l(wx) < l(w) \). Since \( l(b'x) \neq l(b') \) by Lemma 3.6(6), we have \( l(b'x) > l(b') \) and so \( l(b'x) = l(b') + 1 \) by Lemma 3.6(5). Therefore, \( l(w) = l(a) + l(b) \). Finally, we note that \( a = a' \in G^I \) and, as \( x, b' \in G_I \), we have that \( b \in G_I \). Thus, we have obtained the required factorization of \( w \).

\[ \square \]

**Remark 3.8.** By applying Proposition 2.2 to \( w^{-1} \), it can be shown that, for any \( I \subseteq X \), every element \( w \in G \) has a factorization \( w = ab \) for some \( a \in G_I, b \in I^G \) such that \( l(w) = l(a) + l(b) \).

**Remark 3.9.** Proposition 2.2 does not hold in general for groups with an arbitrary group presentation. A counterexample is given by the Klein four-group, \( V = \{ e, i, j, k \} \) [19, Section 44.5], which has group presentation:
\[ \mathcal{V} = \langle i, j, k | i^2 = j^2 = k^2 = ijk = e \rangle. \]

Taking \( w = j \), this is a unique reduced expression of \( w \). We have \( \mathcal{V}_{\{i\}} = \{e, i\} \) and so \( j \notin \mathcal{V}_{\{i\}} \). However, \( ji = k \) and so \( l(ji) = l(j) \), meaning \( w \notin \mathcal{V}_{\{i\}} \). Thus, \( j \) has no reduced factorization with respect to \( \mathcal{V}_{\{i\}} \).

As stated in [2, Proposition 2.4.4], in the Coxeter case this factorization exists and is furthermore unique. The element in the factorization lying in the set \( W^I \) can be shown to be the unique element of \( wW_I \) of minimal length [11, Proposition 1.10]. The uniqueness of these minimal length coset elements distinguish them as coset representatives and they are referred to as the minimal coset representatives [11, Section 1.10]. Thus, the set \( wW_I \cap W^I \) contains only one element, namely the minimal coset representative of \( wW_I \). The uniqueness of the factorization for elements of Coxeter groups is a consequence of the Deletion Condition and is not a property that is transferable to the factorizations of elements in \( G \) with respect to given parabolic subgroup, \( G_I \). A counterexample proving this will be given in the next section.

However, in the cases when \( I = X \) and \( |I| = 1 \) the factorization of all elements of the group with respect to the corresponding parabolic subgroup will be unique.

**Lemma 3.10.** If \( I = X \) or \( I = \{x\} \) for some \( x \in X \), then for all \( w \in G \), \( wG_I \cap G^I \) contains a unique element. Hence the factorization of each element of \( G \) with respect to \( G_I \) is unique.

**Proof.** If \( I = X \) then \( G_I = G \) and \( G^I = \{e\} \), thus \( wG_I \cap G^I = \{e\} \).

If \( I = \{x\} \) for some \( x \in X \), then \( G_I = \{e, x\} \), as \( X \) is a fixed set of involutions generating \( G \), and \( wG_I = \{w, wx\} \) for any \( w \in G \). By Lemma 3.6(6), \( l(wx) = l(w)\pm 1 \) meaning that exactly one of \( w \) or \( wx \) is an element of \( G^I \) and thus the unique element in \( wG_I \cap G^I \).

We conclude that, in both cases, the factorization of any \( w \in G \) with respect to \( G_I \) is unique as if \( w = ab = a'b' \) where \( a, a' \in G^I \) and \( b, b' \in G_I \), then \( a, a' \in wG_I \cap G^I \). Thus, \( a = a' \) and consequently \( b = b' \). \( \square \)

### 4. Non-uniqueness of factorizations

In this section, we present a counterexample demonstrating that the factorizations, shown to exist by Proposition 2.2, for elements of a group with a presentation whose generators are involutions and whose relations are of even length are not necessarily unique.

Specifically, we consider the finite group, \( G \), arising from the group presentation \( \langle t_1, t_2, t_3 | R \rangle \), where \( R \) is the following set of relations:

(a) \( t_1^2 = t_2^2 = t_3^2 = e \) (each generator is an involution).
(b) \( t_1t_2t_1 = t_2t_1t_2, t_1t_3t_1 = t_3t_1t_3, t_2t_3t_2 = t_3t_2t_3 \) (the braid relations).
(c) \( t_1t_2t_3t_1 = t_2t_3t_1t_2 = t_3t_1t_2t_3 \) (the cycle relation).

We note that the above group presentation of \( G \) satisfies conditions (I) and (II), thus Proposition 2.2 holds for elements of \( G \) with respect to a given parabolic subgroup.

**Remark 4.1.** This group presentation is a cluster group presentation. Cluster groups are groups defined by presentations arising from cluster algebras. It was first shown in [1] and then in [10] that a group presentation could be associated to a quiver appearing in a seed of a cluster algebra of finite type and that the corresponding group is invariant under mutation of the quiver [1, Theorem 5.4; 10, Lemma 2.5]. For detailed definitions of cluster algebras and quiver mutation, see [14, Chapter 2]. It is the group presentation based on the work done in [10], that was considered more generally in [17], where due to the context, the corresponding groups were labeled
cluster groups. Each quiver appearing in a cluster algebra of finite type is mutation-equivalent to an oriented Dynkin diagram [8, Theorem 1.4] and the corresponding cluster group presentation is precisely a Coxeter presentation. Consequently, a cluster group associated to a mutation-Dynkin quiver is isomorphic to the finite reflection group of the same Dynkin type. The presentation defined above is the cluster group presentation associated to the quiver of mutation-Dynkin type \( A_3 \) in Figure 1.

It was shown in [17, Lemma 3.10] that an isomorphism between a cluster group associated to a mutation-Dynkin quiver of type \( A_n \) and the symmetric group on \( n+1 \) elements, denoted by \( \Sigma_{n+1} \), can be constructed explicitly from the quiver. It follows from [17, Lemma 3.10] that the following map defines an isomorphism between \( G \) and \( \Sigma_4 \).

**Lemma 4.2.** [16, Proposition 3.4; 17, Lemma 3.10]. There exists an isomorphism \( \pi : G \to \Sigma_4 \) given by

\[
\begin{align*}
\pi : t_1 &\mapsto (1,2), \\
\pi : t_2 &\mapsto (2,3), \\
\pi : t_3 &\mapsto (2,4).
\end{align*}
\]

From Lemma 4.2, we conclude that \( G \) contains 24 distinct elements. Moreover, we can use this isomorphism to determine that \( r,t,t,t_r,t_k,t,t_t,t_1, t_2, t_3, t_1 t_2 t_3, t_2 t_1 t_3 \) for all combinations of pairwise distinct \( i, j, k \). Using this together with the group relations, we construct the Cayley graph of \( G \) with respect to this presentation, which is displayed in Figure 2.

Our goal is to choose a subset, \( I \), of the generating set of \( G \) such that we can find an element of \( G \) with two distinct factorizations with respect to the corresponding parabolic subgroup, thus proving the factorizations shown to exist in Proposition 2.2 are not necessarily unique. Due to Lemma 3.10, \( I \) must contain two distinct elements.

Take \( I = \{t_1, t_2\} \). From the Cayley graph of \( G \) (Figure 2) we have

\[ G^I = \{c, t_3, t_1 t_3, t_2 t_3, t_1 t_2 t_3, t_2 t_1 t_3\}. \]

For \( w = t_2 t_3 t_1 t_2 \), we have \( a = t_2 t_3 \in G^I \) and \( b = t_1 t_2 \in G^I \). Using the Cayley graph of \( G \) in Figure 2, we observe that each of these are all reduced expressions and so \( a \) and \( b \) yield a factorization of \( w \) as given in Proposition 2.2. Moreover, we can apply the cycle relation to \( w \) to obtain another reduced expression of \( w, w = t_1 t_2 t_3 t_1 \). Let \( a' = t_1 t_2 t_3 \in G^I \) and \( b' = t_1 \in G^I \). As this expression for \( a' \) is a subexpression of a reduced expression of \( w \), it must be reduced. From Lemma 4.2 we conclude that \( t_i \neq e \) for \( i = 1, 2, 3 \), thus \( t_1 \) is also reduced. Clearly, as \( a, a' \) and \( b, b' \) are of different lengths, these are distinct pairs of elements in \( G^I \) and \( G_n \) respectively. Thus, we obtain distinct factorizations \( w = ab = a'b' \) where \( l(w) = l(a) + l(b) = l(a') + l(b') \) for \( a, a' \in G^I \) and \( b, b' \in G^I \).

This counterexample demonstrates that for a group \( G \) with presentation \( \langle X | R \rangle \) satisfying conditions (1) and (2), unlike in the Coxeter case, it is possible for the set \( wG_I \cap G^I \) to have more than one element for some \( I \subseteq X \) and \( w \in G \). In the counterexample above, two distinct elements in this set are given by \( a = t_2 t_3 \) and \( a' = t_1 t_2 t_3 \) for \( w = t_2 t_3 t_1 t_2 \).

Furthermore, for a Coxeter group, \( W \), the unique factorization of an element \( w \in W \) with respect to a parabolic subgroup, \( W_p \), is determined by the unique minimal length element of the cost \( wW_f \) [2, Corollary 2.4.5]. In the more general case, the same example used to demonstrate that the factorization is not necessarily unique can also be used to show that minimal length elements of \( wG_I \) do not necessarily yield factorizations of \( w \).
Indeed, from the Cayley graph in Figure 2, we have

\[ G_I = \{ e, t_1, t_2, t_1 t_2, t_1 t_2 t_1, t_1 t_2 t_1 t_2, t_2 t_1 t_2 t_1, t_2 t_1 t_2 t_1 t_2, t_2 t_1 t_2 t_1 t_2 t_1 \}. \]

Choosing \( w = t_1 t_2 t_3 \in G_I \), we observe from the Cayley graph that this is a unique reduced expression of \( w \) and so the only factorization with respect to \( G_I \) is obtained by taking \( a = w \) and \( b = e \). However,

\[ wG_I = \{ t_1 t_2 t_3, t_1 t_2 t_3 t_1, t_1 t_2 t_3 t_1 t_2, t_1 t_2 t_3 t_1 t_2 t_1, t_1 t_2 t_3 t_1 t_2 t_1 t_2, t_1 t_2 t_3 t_1 t_2 t_1 t_2 t_1 t_2, t_1 t_2 t_3 t_1 t_2 t_1 t_2 t_1 t_2 t_1 t_2 t_1 \}. \]

Thus, no factorization of \( w \) can be obtained from the minimal length element, \( t_2 t_3 \), of \( wG_I \).

The property that this factorization is unique for elements of \( W \), and determined by the unique minimal length element of \( wW_I \), is a result of the Deletion Condition [11, Theorem 1.7]. The Deletion Condition is a characterizing result for Coxeter groups. That is, if \( W \) is a group and \( S \) a set of involutions generating \( W \), then \((W, S)\) has the Deletion Condition if and only if \((W, S)\) is a Coxeter system [2, Theorem 1.5.1]. Considering the proof of [11, Proposition 1.10(c)], we see that the factorizations of \( w \) with respect to \( G_I \) as taken in the above counterexample, are no longer unique or determined by the minimal length elements of \( wG_I \) because, without the Deletion Condition, we can no longer omit certain factors from a non-reduced expression of \( w \) and leave \( w \) unchanged.
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