Abstract

In this article we completely determine the spectrum for uniformly resolvable decompositions of the complete graph $K_v$ into $r$ 1-factors and $s$ classes containing only copies of $h$-suns.

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1 Introduction

Given a collection $\mathcal{H}$ of graphs, an $\mathcal{H}$-decomposition of a graph $G$ is a decomposition of the edge set of $G$ into subgraphs (called blocks) isomorphic to some element of $\mathcal{H}$. Such a decomposition is said to be resolvable if it is possible to partition the blocks into classes $\mathcal{P}_i$ (often referred to as parallel classes) such that every vertex of $G$ appears in exactly one block of each $\mathcal{P}_i$; a class is called uniform if every block of the class is isomorphic to the same graph from $\mathcal{H}$. A resolvable $\mathcal{H}$-decomposition of $G$ is sometimes also referred to as an $\mathcal{H}$-factorization of $G$, and a class can be called an $\mathcal{H}$-factor of $G$. The case where $\mathcal{H} = \{K_2\}$ (a single edge) is known as a 1-factorization; for $G = K_v$ it is well known to exist if and only if $v$ is even. A single class of a 1-factorization, that is a pairing of all vertices, is also known as a 1-factor or perfect matching.

Uniformly resolvable decompositions of $K_v$ have been studied in [3, 7, 10, 12, 13, 17, 19] and [20]. Moreover when $\mathcal{H} = \{G_1, G_2\}$ the question of the existence of a uniformly resolvable decomposition of $K_v$ into $r > 0$ classes of $G_1$ and $s > 0$ classes of $G_2$ have been studied in the case in which the number $s$ of $G_2$-factors is maximum. Rees and Stinson [18] have solved the case $\mathcal{H} = \{K_2, K_3\}$; Hoffman and Schellenberg [9] the case $\mathcal{H} = \{K_2, C_k\}$; Dinitz, Ling and Danziger [4] the case $\mathcal{H} = \{K_2, K_4\}$; Küçükkılıç, Milici and Tuza [10] the case $\mathcal{H} = \{K_3, K_{1,3}\}$; Küçükkılıç, Lo Faro, S. Milici and Tripodi [11] the case $\mathcal{H} = \{K_2, K_{1,3}\}$.

1.1 Definitions and notation

An $h$-sun ($h \geq 3$) is a graph with $2h$ vertices $\{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_h\}$, consisting of an $h$-cycle $C_h = (a_1, a_2, \ldots, a_h)$ and a 1-factor $\{(a_1, b_1), (a_2, b_2), \ldots, (a_h, b_h)\}$; in what follows we will denote the $h$-sun by $S(C_h) = (a_1, a_2, \ldots, a_h; b_1, b_2, \ldots, b_h)$ or $S(C_h)$. An $h$-sun is also called a crown graph [5]. The spectrum problem for a $h$-sun system of order $v$ have been solved for $h = 3, 4, 5, 6, 8, 16, 18, 19$. Moreover cyclic $h$-sun systems of order $v$ have been studied in [5, 6, 23].

Let $C_{mn}$ denote the graph with vertex set $\bigcup_{i=1}^m X^i$, with $|X^i| = n$ for $i = 1, 2, \ldots, m$ and $X^i \cap X^j = \emptyset$ for $i \neq j$, and edge set $\{\{u, v\} : u \in X^i, v \in X^j, |i-j| \equiv 1 \pmod{m}\}$.

In this paper we study the existence of a uniformly resolvable decomposition of $K_v$ having $r$ 1-factors and $s$ classes containing only $h$-suns; we will use the notation $(K_2, S(C_h))$-URD($v; r, s$) for such a uniformly resolvable decomposition of $K_v$. Further, we will use the notation $(K_2, S(C_h))$-URGDD($r, s$) of
\(C_{m(n)}\) to denote a uniformly resolvable decomposition of \(C_{m(n)}\) into \(r\) 1-factors and \(s\) classes containing only \(h\)-suns.

## 2 Necessary conditions

In this section we will give necessary conditions for the existence of a uniformly resolvable decomposition of \(K_v\) into \(r\) 1-factors and \(s\) classes of \(h\)-suns.

**Lemma 2.1.** If there exists a \((K_2, S(C_h))\)-URD\((v; r, s)\), \(s > 0\), then \(v \equiv 0 \pmod{2^h}\) and \(s \equiv 0 \pmod{2}\).

**Proof.** Assume that there exists a \((K_2, S(C_h))\)-URD\((v; r, s)\), \(s > 0\). By resolvability it follows that \(v \equiv 0 \pmod{2^h}\). Counting the edges of \(K_v\) we obtain

\[
\frac{rv}{2} + \frac{(2h)s v}{2h} = \frac{v(v-1)}{2}
\]

and hence

\[
r + 2s = (v-1). \tag{1}
\]

Denote by \(R\) the set of \(r\) 1-factors and by \(S\) the set of \(s\) parallel classes of \(h\)-suns. Since the classes of \(R\) are regular of degree 1, we have that every vertex \(x\) of \(K_v\) is incident with \(r\) edges in \(R\) and \((v-1) - r\) edges in \(S\). Assume that the vertex \(x\) appears in \(a\) classes with degree 3 and in \(b\) classes with degree 1 in \(S\). Since

\[
a + b = s \quad \text{and} \quad 3a + b = v - 1 - r,
\]

the equality (1) implies that

\[3a + b = 2(a + b) \Rightarrow a = b\]

and hence \(s = 2a\). This completes the proof. \(\square\)

Given \(v \equiv 0 \pmod{2^h}\), \(h \geq 3\), define \(J(v)\) according to the following table:

| \(v\)                  | \(J(v)\)                                                                 |
|------------------------|---------------------------------------------------------------------------|
| 0 \((\text{mod } 4h)\) | \((3 + 4x, \frac{v}{2} - 2x), x = 0, 1, \ldots, \frac{v-4}{4}\)           |
| 2\(h\) \((\text{mod } 4h), h \text{ even},\) | \((3 + 4x, \frac{v}{4} - 2x), x = 0, 1, \ldots, \frac{v-4}{4}\)           |
| 2\(h\) \((\text{mod } 4h), h \text{ odd},\) | \((1 + 4x, \frac{v}{2} - 2x), x = 0, 1, \ldots, \frac{v-2}{2}\)           |

Table 2: The set \(J(v)\).

Since a \((K_2, S(C_h))\)-URD\((v; v-1, 0)\) exists for every \(v \equiv 0 \pmod{2}\), we focus on \(v \equiv 0 \pmod{2^h}, h \geq 3\).
Lemma 2.2. If there exists a \((K_2, S(C_h))-\text{URD}(v; r, s)\) then \((r, s) \in J(v)\).

Proof. Assume there exists a \((K_2, S(C_h))-\text{URD}(v; r, s)\). Lemma 2.1 and Equation (1) give \(s \equiv 0 \pmod{2}\) and \(r \equiv (v - 1) \pmod{4}\) and so

- if \(v \equiv 0 \pmod{4h}\), then \(r \equiv 3 \pmod{4}\),
- if \(v \equiv 2h \pmod{4h}\), \(h\) even, then \(r \equiv 3 \pmod{4}\),
- if \(v \equiv 2h \pmod{4h}\), \(h\) odd, then \(r \equiv 1 \pmod{4}\).

Letting \(r = a + 4x\), \(a = 1\) or \(3\), in Equation (1), we obtain \(2s = (v - 1) - a - 4x\); since \(s\) cannot be negative, and \(x\) is an integer, the value of \(x\) has to be in the range as given in the definition of \(J(v)\).

Let now \(\text{URD}(v; K_2, S(C_h)) := \{(r, s) : \exists \ (K_2, S(C_h))-\text{URD}(v; r, s)\}\). In this paper we completely solve the spectrum problem for such systems, i.e., characterize the existence of uniformly resolvable decompositions of \(K_v\) into \(r\) 1-factors and \(s\) classes of \(h\)-suns by proving the following result:

Main Theorem. For every \(v \equiv 0 \pmod{2h}\), \(\text{URD}(v; K_2, S(C_h)) = J(v)\).

3 Small cases and basic lemmas

Lemma 3.1. \(\text{URD}(6; K_2, S(C_3)) \supseteq J(v)\).

Proof. The case \((5, 0)\) is trivial. For the case \((1, 2)\), let \(V(K_6)=\mathbb{Z}_6\) and the classes listed below:

\[
\{(0,1,2; 5,4,3)\}, \{(3,5,4;0,1,2)\}, \{(0,4), \{1,3\}, \{2,5\}\}.
\]

\(\square\)

Lemma 3.2. \(\text{URD}(12; K_2, S(C_3)) \supseteq J(v)\).

Proof. The case \((11, 0)\) is trivial. For the remaining cases, let \(V(K_{12})=\mathbb{Z}_{12}\) and the classes listed below:

- \((3, 4)\):
  \[
  \{(0,4,8; 10,2,7),(1,5,9; 11,3,6)\}, \{(2,6,10; 8,0,5),(3,7,11;9,1,4)\},
  \{(5,11; 9,2,6),(1,4,10; 8,3,7)\}, \{(2,7,9; 11,0,4),(3,6,8; 10,1,5)\},
  \{(0,1), \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}, \{10,11\},
  \{(0,2), \{1,3\}, \{4,7\}, \{5,6\}, \{8,11\}\}, \{9,10\},
  \{(0,3), \{1,2\}, \{4,6\}, \{5,7\}, \{8,10\}, \{9,11\}\};
  \]

- \((7, 2)\):
  \[
  \{(0,4,8; 10,2,7),(1,5,9; 11,3,6)\}, \{(2,6,10; 8,0,5),(3,7,11;9,1,4)\},
  \{(0,1), \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}, \{10,11\},
  \{(0,2), \{1,3\}, \{4,7\}, \{5,6\}, \{8,11\}\}, \{9,10\},
  \]

\(\square\)
\[
\{\{0, 3\}, \{1, 2\}, \{4, 6\}, \{5, 7\}, \{8, 10\}, \{9, 11\}\}, \\
\{\{0, 5\}, \{1, 10\}, \{2, 11\}, \{3, 4\}, \{6, 8\}, \{7, 9\}\}, \\
\{\{0, 7\}, \{1, 8\}, \{2, 5\}, \{3, 10\}, \{4, 9\}, \{6, 11\}\}, \\
\{\{0, 9\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 10\}, \{5, 11\}\}, \\
\{\{0, 11\}, \{1, 4\}, \{2, 9\}, \{3, 6\}, \{5, 8\}, \{7, 10\}\}.
\]

\[\square\]

**Lemma 3.3.** There exists a \((K_2, S(C_h))-\text{URGDD}(r, s)\) of \(C_{h(2)}\), for every \((r, s) \in \{\{0, 2\}, \{4, 0\}\}\).

**Proof.** Consider the sets \(X^i = \{a_i, b_i\}\), for \(i = 1, 2, \ldots, h\), and take the classes listed below (where we assume \(h + 1 = 1\)):

- **(0, 2):**
  \[\{(a_1, a_2, \ldots, a_h; b_2, b_3, \ldots, b_h, b_1), (b_1, b_2, \ldots, b_h; a_2, a_3, \ldots, a_h, a_1)\};\]

- **(4, 0), \(h\) even:**
  \[\{(a_{1+2i}, a_{2+2i}); b_{1+2i}, b_{2+2i} : i = 0, 1, \ldots, \frac{h}{2} - 1\}, \]
  \[\{(a_{2+2i}, a_{3+2i}); b_{2+2i}, b_{3+2i} : i = 0, 1, \ldots, \frac{h}{2} - 1\}, \]
  \[\{(a_{1+i}, b_{2+i}) : i = 0, 1, \ldots, h - 1\}, \]
  \[\{(a_{2+i}, b_{1+i}) : i = 0, 1, \ldots, h - 1\};\]

- **(4, 0), \(h\) odd:**
  \[\{(a_{1+2i}, a_{2+2i}); b_{2+2i}, b_{3+2i} : i = 0, 1, \ldots, \frac{h-5}{2} \} \cup \{(a_{h-2}, a_{h-1}); a_h, b_{h-1}\}, \]
  \[\{(a_{2+2i}, a_{3+2i}); b_{1+2i}, b_{2+2i} : i = 0, 1, \ldots, \frac{h-3}{2} \} \cup \{(a_1, b_h)\}, \]
  \[\{(a_{2+i}, b_{1+i}) : i = 0, 1, \ldots, h - 3\} \cup \{(a_1, a_h); (b_{h-1}, b_h)\}, \]
  \[\{(a_{1+i}, b_{2+i}) : i = 0, 1, \ldots, h - 1\}.\]

\[\square\]

### 4 Main Results

**Lemma 4.1.** For every \(v \equiv 0 \pmod{4h}\), \(h \geq 3\), \(\text{URD}(v; K_2, S(C_h)) \supseteq J(v)\).

**Proof.** Let \(v = 4ht\). The case \(h = 3\) and \(t = 1\) corresponds to a \((K_2, S(C_3))-\text{URD}(12; r, s)\) which follows by Lemma 3.2. For \(t \geq 2\), start with a \(C_h\)-factorization \(P_1, P_2, \ldots, P_l\), \(l = ht - 1\), of \(K_{2ht} - F\) which comes from [9] and give weight 2 to each point of \(X\). Fixed any integer \(0 \leq x \leq l\), for each \(h\)-cycle \(C\) of \(x\) parallel classes place on \(C \times \{1, 2\}\) a copy of a \((K_2, S(C_h))-\text{URGDD}(4, 0)\) of \(C_{h(2)}\), while for each \(h\)-cycle \(C\) of the remaining classes place a copy of a \((K_2, S(C_h))-\text{URGDD}(0, 2)\) of \(C_{h(2)}\) (the input designs are from Lemma 3.3); for each edge \(e \in F\) consider a 1-factorization of \(K_4\) on \(e \times \{1, 2\}\). The result is a resolvable decomposition of \(K_v\) into \(3 + 4x\) 1-factors and \(\frac{h-1}{2} - 2x\) classes of \(h\)-suns.

\[\square\]
Lemma 4.2. For every \( v \equiv 2h \pmod{4h} \), \( h \geq 3 \) even, \( \text{URD}(v; K_2, S(C_h)) \supseteq J(v) \).

Proof. Let \( v = 2h + 4ht \). Starting with a \( C_h \)-factorization of \( K_{h+2ht} - F \), which comes from [9], the assertion follows by a similar argument as in Lemma 4.1. \( \square \)

Lemma 4.3. For every \( v \equiv 2h \pmod{4h} \), \( h \geq 3 \) odd, \( \text{URD}(v; K_2, S(C_h)) \supseteq J(v) \).

Proof. Let \( v = 2h + 4ht \). The case \( h = 3 \) and \( t = 0 \) corresponds to a \((K_2, S(C_3))\)-URD \((6; 1, 2)\) which follows by Lemma 3.1. For \( t \geq 1 \), start with a \( C_h \)-factorization \( P_1, P_2, \ldots, P_l \), \( l = ht + \frac{h-1}{2} \), of \( K_{h+2ht} \) which comes from [1] and give weight 2 to each point of \( X \). Fixed an integer \( 0 \leq x \leq l \), for each \( h \)-cycle \( C \) of \( x \) parallel classes place on \( C \times \{1, 2\} \) a copy of a \((K_2, S(C_h))\)-URGDD\((4, 0)\) of \( C(h)_2 \), while for each \( h \)-cycle \( C \) of the remaining classes place a copy of a \((K_2, S(C_h))\)-URGDD\((0, 2)\) of \( C(h)_2 \) (the input designs are from Lemma 3.3). If we consider also the 1-factor consisting of the edges \( \{x_1, x_2\} \) for \( x \in X \), the result is a resolvable decomposition of \( K_v \) into \( 1 + 4x \) 1-factors and \( \frac{v^2}{2} - 2x \) classes of \( h \)-suns. \( \square \)

Combining Lemmas 4.1, 4.2, 4.3 we obtain our main theorem.

Theorem 4.4. For each \( v \equiv 0 \pmod{2h} \), \( \text{URD}(v; K_2, S(C_h)) = J(v) \).

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