THE NORMAL HULL AND COMMUTATOR GROUP FOR NONCONNECTED GROUP SCHEMES

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Abstract. We prove that there is a well behaved notion of normal hull for smooth algebraic group schemes over a field and that the commutator group \((G, H)\) is well defined for \(H \subset G\) smooth, even when both of them are not connected.

If \(G\) is an abstract group and \(H \subset G\) is a subgroup, then the normal hull \(H^G\) of \(H\) in \(G\) can be equivalently defined as the smallest normal subgroup of \(G\) containing \(H\) and as the subgroup of \(G\) generated by all conjugates of \(H\), that is

\[
H^G = \langle \bigcup_{g \in G} gHg^{-1} \rangle.
\]

If now \(G\) is a group scheme over a field \(k\) and \(H \subset G\) is a subgroup scheme, then the two notions can be generalized as follows: one on one hand we denote again by \(H^G\) the smallest normal subgroup scheme of \(G\) containing \(H\), on the other we can define the fppf subgroup-sheaf of \(G\) given by

\[
\tilde{H}^G(R) = \{ l \in G(R) \mid \exists R \to R' \text{ fppf}, l \in H(R') \} = \bigcup_{g \in G(R')} gH(R')g^{-1}
\]

for every \(k\)-algebra \(R\). We want to prove that the two notions coincide, that is \(H^G(R) = \tilde{H}^G(R)\) for every \(k\)-algebra \(R\), under the assumptions that \(H\) and \(G\) are smooth algebraic group schemes.

Notations. For \(H, K\) two abstract subgroups of an abstract group \(G\), we denote by \((H, K)\) their commutator, that the subgroup of \(G\) spanned by \([h, k] = hkh^{-1}k^{-1}\) for \(h \in H\) and \(k \in K\). With the abbreviation fppf we mean faithfully flat and locally of finite presentation.

All group schemes are over \(\text{Spec } k\) for some (non necessarily algebraically closed) field \(k\) and by algebraic we mean of finite type over \(\text{Spec } k\).

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1. A THEOREM OF SCHUR

In this section all groups are abstract groups.

Lemma 1.1. Let \(H\) be a subgroup of a group \(G\). Then

i) The group \((G, H)\) is normal in \(G\);

ii) For every inner automorphism \(\sigma\) of \(G\), we have that \((G, H) = (G, \sigma(H))\)

iii) The normal hull of \(H\) in \(G\) is equal to \(H \cdot (G, H)\).

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Proof. For every \( g, c \in G \) and \( h \in H \) we have that \( c[g, h]c^{-1} = [cg, h][h, c] \), which proves (i). If \( \sigma \) is the inner automorphism induced by \( c \), we have that \([g, c \sigma^{-1}] = [c(c^{-1}gc)c^{-1}, c \sigma^{-1}] = c\sigma g^{-1}gc, \sigma^{-1} \) which is in \((G, H)\) as the latter is normal hence \((G, \sigma(H)) \subset (G, H)\), hence the equality in (ii). As \((G, H)\) is normal, the group \( H \cdot (G, H) \) is well defined. As every \([g, h]\) is contained in the normal hull of \( H \), we have the inclusion \( H \cdot (G, H) \subset H^G \), on the other hand for every \( h \in H \) and \( g \in G \) we have that \( ghg^{-1} = [g, h]h^{-1} \) is in \((G, H) \cdot H = H \cdot (G, H)\), hence the equality.

A classical theorem of Schur states that if the center of a group \( G \) has finite index in \( G \), then the group of commutators is finite. We need a slight generalization of this theorem:

**Proposition 1.2.** If \( H \) is a subgroup of \( G \) such that its centralizer \( C_G(H) \) has finite index, then \((G, H)\) is finite.

**Proof.** We follow the proof of Schur’s theorem as presented in [Dix67, Prop. 5.21-5.24]. First notice that by Poincaré’s theorem the normal core of \( W \) is finite in \( G \) as well, that we denote \( n \). Hence for every \( g \in G \) one has that \( g^n \in C_G(H) \). In particular, if \( g \in G \) and \( h \in H \) we have that

\[
[g, h]^{n+1} = [g, h][g, h]^n = ghg^{-1}h^{-1}[g, h]^n = ghg^{-1}g \cdot [g, h]^{n-1} = [g, h^2][gh^{-1}, h].
\]

Moreover, it is easy to see that the normal core of \( C_G(H) \) is included into \( C_G(\sigma(H)) \), for every inner automorphism \( \sigma \) of \( G \). Therefore more in general the following holds: for every \( h \in H \), \( g \in G \) and \( \sigma \) inner automorphism of \( G \):

\[
[g, \sigma(h)]^{n+1} = [g, \sigma(h)^2][\sigma(h)g\sigma(h)^{-1}, \sigma(h)].
\]

As \( C_G(H) \) has finite index, there are finitely many commutators \([g, h]\), actually there are at most \( n^2 \) commutators. Furthermore, there are at most \( n^3 \) commutators of the form \([g, \sigma(h)]\) with \( h \in K \) and \( \sigma \) an inner automorphism of \( G \). By the previous lemma, \([g, \sigma(h)] \in (G, H)\), hence \((G, H)\) is spanned by the \([g, \sigma(h)]\).

Then we claim that every element in \((G, H)\) can be written as a product of at most \( n^4 \) commutators of the form \([g, \sigma(h)]\); indeed assume that \( c = c_1 \ldots c_r \), with \( r > n^4 \), then there is a commutator \( \dot{c} \) occurring at least \( n+1 \) times, assume that \( \dot{c} = c_i \), then

\[
c = \dot{c}^{-1}c_1 \dot{c}^{-1}c_2 \dot{c}^{-1} \ldots \dot{c}^{-1}c_{i-1} \dot{c} \dot{c}_i \ldots \dot{c} \dot{c}_r.
\]

As the conjugate of a commutator of the form \([g, \sigma(h)]\) for some inner automorphism \( \sigma \) is again a commutator of the same form, we can assume that \( c = c_{n+1} \ldots c_r \) but by (2) this means that we can write it as a product of \( r-1 \) commutators of the form \([g, \sigma(h)]\) and by induction we have that we can always assume that \( r \leq n^4 \). 

\[\square\]

2. Commutators and normal hulls

The following is a classical result:

**Proposition 2.1** ([SGA3, VI, §7 Prop. 7.1] or [DG70, II, §5, Prop. 4.9]). Let \( G \) be a group scheme locally of finite type. Let \( H, K \) two smooth subgroup schemes, with \( H \) irreducible and \( K \) of finite type. Then there exists a unique smooth irreducible subscheme of \( G \) such that for any \( k \)-algebra \( R \),

\[
(H, K)(R) = \{g \in G(R) \mid \exists R \to R' \text{ fppf, } g \in (H(R'), K(R'))\},
\]
in particular for any \( k' \) algebraically closed
\[
(H, K)(k') = (H(k'), K(k')).
\]
Moreover, there exist \( n \) such that every element in \((H, K)\) is the product of at most \( n \) commutators.

Building on this result, we can prove the main result of this paper:

**Theorem 2.2.** Let \( G \) be a smooth algebraic group scheme and \( H \subset G \) be a smooth subgroup scheme, then the group functor \( \tilde{H}^G(R) \) is representable by a smooth subgroup scheme of \( G \), namely \( H^G \).

**Proof.** By Proposition \([\mathbb{L}] \), \((H, G_0)\) is a closed smooth subscheme of \( G \) and represents the functor
\[
(H, G_0)(R) = \{ g \in G(R) \mid \exists R \to R' \text{ fppf}, g \in (H(R'), G_0(R')) \}.
\]
It is easy to see that \( H \) normalizes \((H, G_0)\), hence
\[
H \cdot (H, G_0)(R) = \{ g \in G(R) \mid \exists R \to R' \text{ fppf}, g \in H(R') \cdot (H, G_0)(R') \}
\]
is represented by a closed subgroup scheme of \( G \) which is smooth by \([\mathbb{L}] \), VI.B.87 Cor. 7.1.1. Moreover \( H(R) \subset (H \cdot (H, G_0))(R) \subset H(R)^{G(R)} \) therefore without loss of generality we can substitute \( H \) with \( H \cdot (H, G_0) \). The latter, though, is normalized by \( G_0 \): if \( g_0, g_1 \in G_0(R) \) and \( l, h \in H(R) \)
\[
g_0 h [g_1, l] g_0^{-1} = \underbrace{g_0 h g_0^{-1}}_{\in H \cdot (H, G_0)} \underbrace{[g_0 g_1, l] [l, g_0]}_{\in (H, G_0)}.\]

In particular we can assume without loss of generality that \( H \) is normalized by \( G_0 \). We can apply the same arguments as before to \((G_0, H)\): it is smooth and irreducible, hence \((G, (G_0, H))\) is smooth and irreducible, and by Lemma \([\mathbb{L}1]\) it is normal in \( G \). In particular the group \((G_0, H) \cdot (G, (G_0, H))\) is well defined, but by Lemma \([\mathbb{L}1] iii\) the latter is simply \((G_0, H)^G\). Note that in particular there exists \( n_1, n_2 \) such that every element of \((G_0, H)^G\) is the product of \( n_1 \) commutators in \((G_0, H)\) and \( n_2 \) commutators in \((G, (G_0, H))\).

Fix now a separable closure \( k^s \) of \( k \) and consider the quotient
\[
\tilde{G}(k^s) = G(k^s)/(G_0(k^s), H(k^s))^G(k^s)
\]
and let \( \tilde{H}(k^s) \) be the image of \( H(k^s) \) in \( \tilde{G}(k^s) \). Then the image of \( G_0(k^s) \) is in the centralizer \( C_{\tilde{G}(k^s)}(\tilde{H}(k^s)) \), in particular the latter has finite index. Hence
\[
(\tilde{G}(k^s), \tilde{H}(k^s)) = (G(k^s), H(k^s))/(G_0(k^s), H(k^s))^G(k^s) \]
is finite, that is
\[
(G_0(k^s), H(k^s))^G(k^s) \subset (G(k^s), H(k^s))
\]
has finite index \( d \). As \((G_0(k^s), H(k^s))^G(k^s)\) is closed, so is \((G(k^s), H(k^s))\). In particular it corresponds to the closed points of a normal subgroup of \( G \), defined over \( k^s \), that we denote \((G, H)\). Note that \((G, H)(k^s) = (G(k^s), H(k^s))\), hence it is stable under \( \text{Gal}(k^s/k) \), that is \((G, H)\) is defined over \( k \) (see also \([\mathbb{S}GA3], VI.B.87\) Lemma 7.7], noting that one only uses that \( k \) is algebraically closed to ensure that \( k \)-points are dense).

To show that \((G, H)\) represents the functor
\[
(G, H)(R) = \{ g \in G(R) \mid \exists R \to R' \text{ fppf}, g \in (G(R'), H(R')) \},
\]
note that every element of \((G, H)\) is the product of at most \( N = n_1 + n_2 + d \) commutators of the form \([g, h]\), but then we can use the methods of \([\mathbb{D}G70], II.5', Prop. 4.8\):
\((G \times H)^N \to (G, H)\) is dominant hence flat over some open \(U \subset (G \times H)^N\). Hence 
\(U \times U \to (G, H) \times (G, H) \overset{\eta}{\to} (G, H)\) is flat and surjective. Let \(R\) be any \(k\)-
algebra and \(g : \text{Spec } R \to (G, H)\) a \(R\)-point. Let \(X = \text{Spec } R \times_{(G, H)} (U \times U)\), then 
the projection \(X \to \text{Spec } R\) is fppf, hence so is \(\varphi : \text{Spec}(\prod R_i) \to \text{Spec } R\), where 
\(\cup \text{Spec } R_i\) is an open affine covering of \(X\). Hence \(g \circ \varphi \in (G, H)(\prod R_i)\) factors 
through \(U \times U\), that is it is a product of commutators in \((G(\prod R_i), H(\prod R_i))\).
Conversely, \(G \times H \to G\) given by \((g, h)\) to \(hgh^{-1}g^{-1}\) factors through \((G, H)\), giving 
the reverse inclusion.

By Lemma 1.1 \((G, H)\) is normal in \(G\) and \((G, H) \cdot H\) is a smooth subscheme of 
\(G\) representing \(\hat{H}_G\), hence by minimality \(H^G = (G, H) \cdot H\).

**Corollary 2.3.** Let \(G\) be an algebraic group scheme and let \(H \subset G\) be a smooth 
subgroup scheme, then the functor 
\[(H, G)(R) = \{g \in G(R) \mid \exists R \to R' \text{ fppf} , g \in (H(R'), G(R'))\}\]
is representable by a closed smooth subscheme of \(G\).

**Proof.** It suffice to note that for every \(R\) the inclusion \((H(R), G(R)) \subset (H(R) \cdot 
(G_0(R), H(R)), G(R))\) is actually an equality: \(H \cdot (G_0, H)\) is generated by all 
conjugates of \(H\) under \(G_0\), in particular it suﬃces to show that \([g_0h_{g_0^{-1}}, g] \in 
(H(R), G(R))\) for every \(g_0 \in G_0(R), h \in H(R)\) and \(g \in G(R)\). But by Lemma 1.1, \(H(R), G(R))\) is normal in \(G(R)\), in particular \([g_0h_{g_0^{-1}}, g] \in (H(R), G(R))\) if and 
only if \(g_0^{-1}[g_0h_{g_0^{-1}}, g]g_0 = [h, g_0^{-1}gg_0] \in (H(R), G(R))\). We showed in the proof of 
the main theorem that 
\(R \mapsto \{g \in G(R) \mid \exists R \to R' \text{ fppf} , g \in (H(R'), (G_0(R'), H(R'))\}\)
is representable, hence so must be \((H, G)(R)\). \(\square\)

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