DIFFERENCE OF MODULAR FUNCTIONS AND THEIR CM VALUE
FACTORIZATION
TO APPEAR IN TRANS. AMS

TONGHAI YANG AND HONGBO YIN

Abstract. In this paper, we use Borcherds lifting and the big CM value formula of
Bruinier, Kudla, and Yang to give an explicit factorization formula for the norm of
Ψ(d_1 + \sqrt{d_1}) - Ψ(d_2 + \sqrt{d_2})$, where Ψ is the $j$-invariant or the Weber invariant $\omega_2$. The
j-invariant case gives another proof of the well-known Gross-Zagier factorization formula
of singular moduli, while the Weber invariant case gives a proof of the Yui-Zagier conjecture
for $\omega_2$. The method used here could be extended to deal with other modular functions
on a genus zero modular curve.

Contents

1. Introduction 1
2. Borcherds lifting and the Big CM value formula 4
3. Product of modular curves and its diagonal divisor 16
4. Gross and Zagier’s singular moduli factorization formula 21
5. The Yui-Zagier conjecture for $\omega_i$ 23
References 32

1. Introduction

In 1980s, Gross and Zagier discovered a beautiful factorization formula for the singular
moduli [GZ85] in preparation to their well-known Gross-Zagier formula. It was extended
slightly by Dorman [Dor88] which can be stated as follows (see Remark 4.1).

Theorem 1.1. (Gross-Zagier, Dorman) Let $E_i = \mathbb{Q}(\sqrt{d_i})$ be two imaginary quadratic
fields of fundamental discriminants $d_i$ with $(d_1, d_2) = 1$, let $F = \mathbb{Q}(\sqrt{D})$ with $D = d_1d_2$
and $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Let $j(\tau)$ be the well-known $j$-invariant. Then

$$\sum_{[\alpha] \in \text{Cl}(E_i)} \log |j(\tau_{a_1}) - j(\tau_{a_2})|^w = \sum_{t = m + \sqrt{D} \in \mathcal{O}_F \, \text{p inert in } E/F} \sum_{|m| < \sqrt{D}} \frac{1 + \text{ord}_p(t\mathcal{O}_F)}{2} \rho(tp^{-1}) \log(N(p)).$$

2000 Mathematics Subject Classification. 14G35, 14G40, 11G18, 11F27.
The first author is partially supported by a NSF grant DMS-1500743.
Here \( w_i \) is the number of roots of unity in \( E_i \), and for an integral ideal \( \mathfrak{a} \) of \( F \)
\[
\rho(\mathfrak{a}) = |\{ \mathfrak{A} \subset \mathcal{O}_E : N_{E/F}(\mathfrak{A}) = \mathfrak{a}\}|.
\]

Finally, for an integral ideal \( \mathfrak{a}_i \) of \( E_i \) with
\[
\mathfrak{a}_i = \mathbb{Z}a_i + \mathbb{Z} \frac{b_i + \sqrt{d_i}}{2}, \quad a_i = N(\mathfrak{a}_i),
\]
its associated CM point is \( \tau_{\mathfrak{a}_i} = \frac{b_i + \sqrt{d_i}}{2a_i} \).

This gives a beautiful factorization formula for \( N(j(\frac{d_1 + \sqrt{d}}{2}) - j(\frac{d_2 + \sqrt{d}}{2})) \) (up to sign). In particular, the biggest prime factor of this norm is less than or equal to \( D/4 \), extremely small comparing to the norm. The first few examples of this phenomenon were discovered by Berwick in 1920s \([\text{Ber}27]\). For example, one has
\[
\begin{align*}
j(1 + \sqrt{-163}) - j(1 + \sqrt{-2}) &= -2^{18}3^35^323^329^3 = -262537412640768000, \\
j(1 + \sqrt{-163}) - j(i) &= -2^{6}3^67^211^219^2127^2163 = -262537412640769728.
\end{align*}
\]

In 1997, Yui and Zagier \([\text{YZ}97]\) defined a mysterious CM value \( f(\frac{d + \sqrt{d}}{2}) \) via the three Weber functions of level 48 (when \( d \equiv 1 \) (mod 8) and \( 3 \nmid d \)) and proved that it is defined over the Hilbert class field of \( \mathbb{Q}(\sqrt{d}) \), and claimed that its Galois conjugates are the CM values at other CM points of the same discriminant \( d \) with some modifications, which was later proved by Alice Gee using Shimura’s reciprocity law. In addition, Yui and Zagier gave a conjectural factorization formula for the norm of \( f(\frac{d_1 + \sqrt{d}}{2}) \) \(- f(\frac{d_2 + \sqrt{d}}{2}) \) similar to the Gross-Zagier factorization formula for any positive integer \( a \mid 24 \). For example, when \( a = 24 \), the conjecture can be restated as follows.

**Conjecture 1.2.** Let the notation be as in Theorem \([L1]\) and assume further \( d_1 \equiv d_2 \equiv 1 \) (mod 8). Let
\[
\omega_2(\tau) = 2^{12}q \prod_{n>0} (1 + q^n)^{24} = 2^{12} \cdot \frac{\Delta(2\tau)}{\Delta(\tau)}
\]
be the Weber modular function for \( \Gamma_0(2) \). Then
\[
\sum_{[\mathfrak{a}_i] \in \text{Cl}(E_i)} \log |\omega_2(\tau_{\mathfrak{a}_i}) - \omega_2(\tau_{\mathfrak{a}_j})|^2 = \sum_{t = \frac{d+\sqrt{d}}{2}} \sum_{\text{inert in } F/E} \frac{1 + \text{ord}_p(t\mathcal{O}_F)}{2} \rho(tp^{-1}p^{-2}) \log(N(p)).
\]

Here \( p_i \) is the unique prime ideal of \( F \) above 2 such that \( \text{ord}_{p_i}(t\mathcal{O}_F) \geq 1 \), and for each ideal class \([\mathfrak{a}_i] \in \text{Cl}(E_i)\), we choose a representative \( \mathfrak{a}_i \) integral with norm prime to 2, i.e.,
\[
\mathfrak{a}_i = \mathbb{Z}a_i + \mathbb{Z} \frac{b_i + \sqrt{d_i}}{2}, \quad \text{with } 2 \nmid a_i, \quad a_i > 0. \quad \tau_{\mathfrak{a}_i} = \frac{b_i + \sqrt{d_i}}{2a_i}.
\]

They provided some numerical evidence in their paper. Notice that the biggest prime factor of this norm is less than or equal to \( D/16 \). In this paper, we will prove this conjecture.
Theorem 1.3. Conjecture \(L_2\) is true.

In his 2006 thesis [Sch09], Schofer used regularized theta lifting to generalize the Gross-Zagier factorization formula to small CM values of the so-called Borcherds products on the orthogonal Shimura varieties of type \((n,2)\). Bruinier and Yang generalized it to big CM values of Hilbert modular forms (which are Borcherds products) over a real quadratic field [BY06]. More recently, Bruinier, Kudla, and Yang ([BKY12]) generalized it to big CM values of Borcherds products on Shimura varieties of orthogonal type \((n,2)\), following Schofer and [BY09]. On a different track, Lauter, Goren, and Viray have used geometric methods to generalize the Gross-Zagier formula to Igusa’s \(j\)-invariants for genus two curves, which have important applications to genus two curve cryptosystem ([GL07], [GL12], [LV15]). Yang also proved Lauter’s conjecture on Igusa’s \(j\)-invariants by combining the result in [BY06] with his work on arithmetic intersection [Yan13]. The big CM value formula in [BKY12] has also been used to prove certain cases of the Colmez conjecture ([Yan10a], [Yan10b], [Yan13], [BHK+]), and the average Colmez conjecture ([AGHMP]). Dongxi Ye is extending the result to other modular curves of genus zero [Ye17].

This paper is the first part of our effort to prove Yui and Zagier’s conjectural formula using the big CM value formula.

The general idea is as follows. Let \(\Gamma\) be a congruence subgroup such that the compactification of \(X_\Gamma = \Gamma \setminus \mathbb{H}\) has genus zero, and let \(\Psi\) be a generator of the function field of \(X_\Gamma\), which is a modular function for \(\Gamma\). Then the difference function \(\Psi(z_1) - \Psi(z_2)\) is a two variable modular function on \(X_\Gamma \times X_\Gamma\) with divisor being the diagonal divisor. We view \(X_\Gamma \times X_\Gamma\) as an orthogonal Shimura variety of type \((2,2)\) associated to \((V = M_2(\mathbb{Q}), \mathbb{Q} = N \det)\) for some positive integer \(N\). One can show that the diagonal divisor is a special divisor on the product \(X_\Gamma \times X_\Gamma\) so that \(\Psi(z_1) - \Psi(z_2)\) has a chance to be a Borcherds lifting (product) of some weakly holomorphic modular forms ([Bor98], [Bru02]). The first task is to find a weakly holomorphic modular form, if any, whose Borcherds lifting is the difference \(\Psi(z_1) - \Psi(z_2)\) ([Bor98], [Bru02], see Section 2). There are two complications even with Bruinier’s converse results ([Bru02], [Bru14]). First, when \(N > 1\), Bruinier’s converse theorem does not apply. Second, there are two variable modular functions whose divisors are only supported on the boundary, so it is not enough to compare the divisors of the Borcherds product with our function only in the open Shimura variety. We also need to understand their boundary behavior. The Borcherds product expansion is important in this aspect. In this paper, we are only successful in this step for the Weber functions \(\omega_i\) (Section 5) but not for the more interesting Weber functions \(f_i\) of level 48.

The second task is to identify a pair of Heegner points \((\tau_{a_1}, \tau_{a_2})\) with a big CM point on \(X_\Gamma \times X_\Gamma\) associated to the CM number field \(E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})\) in the sense of Bruinier, Kudla, and Yang in [BKY12]. This is done in Section 3. The third task is to apply the big CM value formula in [BKY12] (assuming that \(\Psi\) is a Borcherds lifting) to provide the expected formula. One serious problem (for the Yui-Zagier conjecture) is that the big CM cycle in [BKY12] is likely bigger in size than the ideal class groups used in Yui-Zagier’s conjectural formula in general. One might need to use Shimura’s reciprocity law to analyze the Galois action on the values as in [Gee99] to solve the problem. In the case of \(\omega_2\), the condition \(d_i \equiv 1 \pmod{8}\) allows us to choose an embedding from \(E_i\) to \(GL_2(\mathbb{Q})\) so that the
ideal class group maps into $X_0(2)$ nicely. Another minor complication (interesting feature) is the explicit computation of the Fourier coefficient of the derivatives of some incoherent Eisenstein series since Schwartz functions are not factorizable in the $\omega_2$ case (Section 5).

Here is the organization of this paper. In Section 2, we review Borcherds lifting, Borcherds product expansion ([Bor98], [Bru02]), and the big CM value formula ([BKY12]). In Section 3, we identify the product $X_\Gamma \times X_\Gamma$ of two copies of a modular curve with a Shimura variety of orthogonal type $(2,2)$ and identify its big CM points with pairs of the CM points on the modular curve $X_\Gamma$. In Section 4, we reprove Theorem 1.1 using the big CM value formula. In Section 5, we first identify $\omega_2(z_1) - \omega_2(z_2)$ with a Borcherds lifting of some explicit weakly holomorphic modular forms, and then use the big CM value formula to prove Theorem 1.3.

Acknowledgement: The authors thank Dongxi Ye for carefully reading the earlier drafts and giving us very valuable feedback and correction. They thank the anonymous referee for his/her suggestion and comments. Part of the work was done when Yin visited Department of Mathematics at UW-Madison during 2014-15 and when Yang visited the Morninside Center of Mathematics at Beijing during summer of 2015. The authors thank both institutes for providing excellent working conditions for them.

2. Borcherds lifting and the Big CM value formula

2.1. Borcherds lifting and Borcherds product expansion. In this subsection, we review the beautiful work of Borcherds in details using slightly different convention and notation for our purpose. Let $(V, Q)$ be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and let $L$ be an even integral lattice, i.e., $Q(x) = \frac{1}{2}(x, x) \in \mathbb{Z}$ for $x \in L$. Let $L' = \{y \in V : (x, y) \in \mathbb{Z}, \text{ for } x \in L\} \supset L$ be its dual. We assume in this paper that $n$ is even for simplicity. Let $H = \text{GSpin}(V)$, and let $\mathbb{D}$ be the oriented negative 2-planes in $V_\mathbb{R}$. Then for a compact open subgroup $K$ of $H(\mathbb{A}_f)$, there is a Shimura variety $X_K$ defined over $\mathbb{Q}$ such that

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathbb{D} \times H(\mathbb{A}_f)/K).$$

We will identify $X_K$ with $X_K(\mathbb{C})$ in this section. We assume that $K$ fixes $L$ and acts on $L'/L$ trivially. The Hermitian symmetric domain $\mathbb{D}$ has two other useful forms. Let

$$\mathcal{L} = \{w \in V_\mathbb{C} : (w, w) = 0, \quad (w, \bar{w}) < 0\}.$$  

Then one has an isomorphism

$$\mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}, \quad w = u + iv \mapsto \mathbb{R}u + \mathbb{R}(-v).$$

This isomorphism gives a complex structure on $\mathbb{D}$, and we can view $\mathcal{L}$ as a line bundle over $\mathbb{D}$—the tautological line bundle. It descends to a line bundle $\mathcal{L}_K$ over $X_K$—the line bundle of modular forms of weight 1 on $X_K$. Finally, given an isotropic element $\ell \in V$, choose another element $\ell' \in V$ such that $(\ell, \ell') = 1$, and let $V_0 = (\mathbb{Q}\ell + \mathbb{Q}\ell')^\perp$. Then we have a tube domain (associated to $(\ell, \ell')$):

$$\mathcal{H} = H_{\ell, \ell'} = \{z = x + iy \in V_0, \mathbb{C} : Q(y) < 0\}.$$
The map
\[ w = w_{\ell,w} : \mathcal{H} \to \mathcal{L}, \quad w(z) = z + \ell - (Q(z) + Q(\ell))\ell \]
gives an isomorphism \( \mathcal{H}_{\ell,w} \cong \mathcal{L}/\mathbb{C}^\times \), and actually a nowhere vanishing section of the line bundle \( \mathcal{L} \). We emphasize that \( w \) depends on the choice of the primitive isotropic vector \( \ell \) and the subspace \( \mathbb{Q}\ell + \mathbb{Q}\ell' \), but not \( \ell' \). Furthermore, this map \( w \) induces an action of \( \Gamma = K \cap H(\mathbb{Q})^+ \) on \( \mathcal{H} \) and an automorphy factor \( j(\gamma, z) \) characterized by the following identity:
\[ (2.2) \quad \gamma w(z) = \nu(\gamma)j(\gamma, z)w(\gamma z). \]

Here \( H(\mathbb{R})^+ \) is the identity component of \( H(\mathbb{R}) \), and \( H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+ \), \( \nu(\gamma) \) is the spinor norm of \( \gamma \). This action preserves the two connected components of \( \mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \). A (meromorphic) function \( \Psi \) on \( \mathcal{H}^+ \) is called a (meromorphic) modular form for \( \Gamma \) of weight \( k \) if
\[ (2.3) \quad \Psi(\gamma z) = j(\gamma, z)^k \Psi(z). \]

Alternatively, it is a section of the line bundle \( \mathcal{L}_K^k \) over \( X_K \).

For a vector \( x \in V \) with \( Q(x) > 0 \), and \( h \in H(\mathbb{A}_f) \). Let
\[ H_x = \{ h \in H : h(x) = x, \quad \mathcal{D}_x = \{ z \in \mathcal{D} : (x, z) = 0 \}, \quad \text{and} \quad K_{x,h} = H_x(\mathbb{A}_f) \cap hKh^{-1}. \]

Then the map
\[ H_x(\mathbb{Q}) / (\mathcal{D}_x \times H_x(\mathbb{A}_f)/K_{x,h}) \to X_K(\mathbb{C}), \quad [z, h_1] \mapsto [z, h_1h] \]
gives a divisor \( Z(x, h) \) in \( X_K \). It is actually defined over \( \mathbb{Q} \). For a rational number \( m > 0 \) and \( \phi \in S(V_f) \), if there is an \( x \in V \) with \( Q(x) = m \), we define, following Kudla [Kud97a], the weighted special divisor
\[ Z(m, \phi) = \sum_{h \in H_x(\mathbb{A}_f) \cap H(\mathbb{A}_f)/K} \phi(h^{-1}x)Z(x, h). \]

When there is no \( x \in V \) with \( Q(x) = m \), we simply set \( Z(m, \phi) = 0 \).

Associated to the quadratic space \( V \) is a reductive dual pair \( (\text{SL}_2, O(V)) \) and a Weil representation \( \omega = \omega_{V,\psi} \) of \( \text{SL}_2(\mathbb{A}) \) on \( S(V_h) = S(V_f) \otimes S(V_\infty) \), where \( V_f = V \otimes_{\mathbb{Q}} \mathbb{A}_f \) and \( V_\infty = V \otimes_{\mathbb{Q}} \mathbb{Q}_\infty = V \otimes_{\mathbb{Q}} \mathbb{R} \). Embed \( \text{SL}_2(\mathbb{Z}) \) into \( \text{SL}_2(\hat{\mathbb{Z}}) \) diagonally, and let \( S_L \subset S(\mathbb{A}_f) \) be the subspace of Schwartz functions \( \phi \) which is supported on \( \hat{L}' = L' \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \) and is \( \hat{L}\)-translation invariant, i.e., \( \phi(x) \) depends only on \( x \mod \hat{L} \). Then
\[ S_L = \oplus_{\mu \in L'/L} \mathbb{C}\phi_{\mu}, \quad \phi_{\mu} = \text{Char}(\mu + \hat{L}). \]

It is easy to check that \( S_L \) is \( \text{SL}_2(\mathbb{Z}) \)-invariant under the Weil representation \( \omega \), we denote this representation \( \omega_L \). One has by definition
\[ (2.4) \quad \omega_L(n(b))\phi_{\mu} = e(-bQ(\mu))\phi_{\mu}, \quad b \in \mathbb{Z}, \]
\[ \omega_L(w)\phi_{\mu} = e\left(\frac{n-2}{8}([L' : L])^{-\frac{1}{2}}\right) \sum_{\nu \in L'/L} e((\mu, \nu))\phi_{\nu}. \]

Here
\[ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
and we have used the fact
\[ \psi_f(x) = \psi_\infty(-x) = e(-x) \]
when \( x \in \mathbb{Q} \). We also write
\[ m(a) = \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right). \]
If we identify \( S_L \cong \mathbb{C}[L'/L] = \bigoplus_{\mu \in L'/L} \mathbb{C} e_\mu \) via \( \phi_\mu \mapsto e_\mu \) and let \( \rho_{L_-} \) be the Weil representation in [Bor98] (also [Bru02]) associated to the quadratic lattice \( L_- \), where \( L_- = L \) but with quadratic form \( Q_-(x) = -Q(x) \), then one sees immediately
\[ (2.5) \quad \omega_L = \rho_{L_-}. \]

Recall that a meromorphic function \( f : \mathbb{H} \to S_L \) is called a weakly holomorphic modular form of weight \( k \) with respect to \( SL_2(\mathbb{Z}) \) and \( \omega_L \) if it satisfies the following conditions.

(i) One has \( f |_{k,\omega_L} \gamma = f \) for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), where
\[ f |_{k,\omega_L} \gamma(\tau) = (c\tau + d)^{-k} \omega_L(\gamma)^{-1} f(\tau). \]
(ii) There is a \( S_L \)-valued Fourier polynomial
\[ P_f(\tau) = \sum_{\mu \in L'/L} \sum_{n \leq 0} c(n, \mu) q^n \phi_\mu \]
such that \( f(\tau) - P_f(\tau) = O(e^{-\varepsilon v}) \) as \( v \to \infty \) for some \( \varepsilon > 0 \).

The Fourier polynomial \( P_f \) is called the principal part of \( f \). We denote the vector space of these weakly holomorphic modular forms by \( M^!_{k,\omega_L} \). The Fourier expansion of any \( f \in M^!_{k,\omega_L} \) is of the form
\[ (2.6) \quad f(\tau) = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Q}} c(n, \mu) q^n \phi_\mu \]
With this notation, we define
\[ (2.7) \quad Z(f) = \sum_{n>0,\mu \in L'/L} c(-n, \mu) Z(n, \mu). \]

Here \( Z(m, \mu) = Z(m, \phi_\mu) \). Let \( S_L^\vee \) be the dual space of \( S_L \)—the space of linear functionals on \( S_L \), and let \( \{ \phi_\mu^\vee \} \) be the dual basis in \( S_L^\vee \) of the basis \( \{ \phi_\mu \} \) of \( S_L \). Recall that the Siegel theta function (for \( (z, h) \in X_K \))
\[ \theta_L(\tau, z, h) = \sum_\mu \theta(\tau, z, h, \phi_\mu) \phi_\mu^\vee \]
is an \( S_L^\vee \)-valued holomorphic modular form of weight 0 for \( SL_2(\mathbb{Z}) \) and \( \omega_L^\vee \) defined as follows (see [BY09 Section 2] or [Kud03] for details). For \( z \in \mathbb{D} \), consider the orthogonal decomposition:
\[ V_\mathbb{R} = z \oplus z^\perp, \quad x = x_z + x_z^\perp. \]
Then for \( \phi \in S(V_f) \) and \((z,h) \in X_K\), one defines
\[
\theta(\tau, z, h, \phi) = v \sum_{x \in V} \phi(h^{-1}x) e(\tau Q(x_z) + \bar{\tau} Q(x_\bar{z})).
\]

Here \( v = \text{Im}(\tau) \) is the imaginary part of \( \tau \). Notice that \( \theta(\tau, z, 1, \phi_\mu) = \overline{\theta(\tau, z, \mu)} \) in comparison with Borcherds' Siegel theta functions.

We consider the regularized theta integral
\[
\Phi(z, h, f) = \int_{F} \langle f(\tau), \theta_L(\tau, z, h) \rangle d\mu(\tau) = \int_{F} \sum_{\mu \in L' / L} f_\mu(\tau) \theta(\tau, z, h, \phi_\mu) d\mu(\tau)
\]
for \( z \in \mathbb{D} \) and \( h \in H(A_f) \). Here \( F \) is the standard domain for \( \text{SL}_2(\mathbb{Z}) \) \( \backslash \mathbb{H} \), and we write
\[
f(\tau) = \sum_{\mu \in L' / L} f_\mu(\tau) \phi_\mu.
\]

The integral is regularized as in [Bor98], that is, \( \Phi(z, h, f) \) is defined as the constant term in the Laurent expansion at \( s = 0 \) of the function
\[
\lim_{T \to \infty} \int_{F_T} \langle f(\tau), \theta_L(\tau, z, h) \rangle v^{-s} d\mu(\tau).
\]
Here \( F_T = \{ \tau \in \mathbb{H}; |u| \leq 1/2, |\tau| \geq 1, \text{ and } v \leq T \} \) denotes the truncated fundamental domain and the integrand
\[
\langle f(\tau), \theta_L(\tau, z, h) \rangle = \sum_{\mu \in L' / L} f_\mu(\tau) \theta(\tau, z, h, \phi_\mu)
\]
is the pairing of \( f \) with the Siegel theta function, viewed as a linear functional on the space \( S_L \). We remark that our regularized theta integral \( \Phi(z, h, f) \) is exactly the same with the one in [Bor98] and [Bru02] when \( h = 1 \).

The following is the first part of [Bor98, Theorem 13.3] (see also [Bru02, Theorem 3.22]) in our setting.

**Theorem 2.1.** Let \( f(\tau) = \sum c(m, \mu) q^m \phi_\mu \in M_{1-\frac{\omega_L}{2}} \) be a weakly holomorphic modular form of weight \( 1-\frac{\omega_L}{2} \) for \( \text{SL}_2(\mathbb{Z}) \) and \( \omega_L \), and assume that \( c(m, \mu) \in \mathbb{Z} \) for \( m < 0 \). Then there is a meromorphic modular form \( \Psi(z, h, f) \) of weight \( k = c(0, 0)/2 \) on \( X_K \) (with some characters) such that

1. one has
\[
\text{div}(\Psi(z, h, f)^2) = Z(f) = \sum_{m > 0, \mu \in L' / L} c(-m, \mu) Z(m, \mu).
\]
Here we count \( Z(m, \mu) \) with multiplicity 2 or 1 depending on whether \( 2\mu \in L \) or not.

2. One has
\[
-\log \| \Psi(z, h, f) \|_{\text{Pet}}^4 = \Phi(z, h, f).
\]
Here \( \| \|_{\text{Pet}} \) is a suitably normalized Petterson norm.
To describe the Borcherds product expansion formula for $\Psi(z, h, f)$, we need some preparation. First, it works in each connected component. By the strong approximation theorem, one has

$$H(A_f) = \prod H(\mathbb{Q}^+ h_j K),$$

so

$$X_K = \coprod X_{\Gamma_j} = \coprod \Gamma_j \backslash \mathbb{D}^+,$$

where $\Gamma_j = H(\mathbb{Q}^+ \cap h_j K h_j^{-1}$ and $\mathbb{D}^+$ is one of the two connected components of $\mathbb{D}$. In this decomposition, one has

$$Z(m, \phi_\mu) = \sum_j Z_{L_j}(m, \mu_j),$$

where $L_j = h_j L = h_j \hat{L} \cap V$, and $\mu_j \in L'_j/L_j$ with $\mu_j - h_j \mu \in \hat{L}_j$, and

$$Z_{L_j}(m, \mu_j) = \{ z \in \mathbb{D}^+ : (z, x) = 0 \text{ for some } x \in \mu_j + L_j, Q(x) = m \}.$$

In the following, we will stick with the irreducible component $X_{\Gamma} = \Gamma \backslash \mathbb{D}^+$ and the lattice $L$. The other components are the same.

Assume that $V$ has an isotropic line $\mathbb{Q} \ell$ (a cusp). We assume that $\ell \in L$ is primitive, i.e., $L \cap \mathbb{Q} \ell = \mathbb{Z} \ell$. Choose $\ell' \in L'$ with $(\ell, \ell') = 1$. Assume further $(\ell, L) = N_\ell \mathbb{Z}$ and choose $\xi \in L$ with $(\ell, \xi) = N_\ell$. Let $M = (\mathbb{Q} \ell + \mathbb{Q} \ell')^\perp \cap L$, and let

$$L'_0 = \{ x \in L' : (\ell, x) \equiv 0 \mod (N_\ell) \} \supset L.$$

Then there is a projection

$$p: L'_0 \to M', \quad p(x) = x_M + \frac{(x, \ell)}{N_\ell} \xi_M,$$

where $x_M$ and $\xi_M$ are the orthogonal projections of $x, \xi \in V$ to $M_\mathbb{Q} = M \otimes \mathbb{Q}$. The projection $p$ has the nice property $p(L) \subset M$ although it is not an orthogonal projection anymore (see [Bru02, Pages 40-41]). So it induces a projection from $L'_0/L$ to $M'/M$.

Next, we define the Weyl chamber for

$$f = \sum f_\mu \phi_\mu = \sum c(m, \mu) q^m \phi_\mu \in M'_1 - \frac{1}{4} \omega_L.$$

Define

$$f_M(\tau) = \sum_{\lambda \in M'/M} f_\lambda(\tau) \phi_{\lambda, M} = \sum c_M(m, \lambda) q^m \phi_{\lambda, M},$$

where $\phi_{\lambda, M} = \text{Char}(\lambda + \hat{M})$,

$$f_\lambda(\tau) = \sum_{\substack{\mu \in L'_0/L \\text{ such that } p(\mu) = \lambda}} f_\mu(\tau).$$

Then $f_M$ is a $S_M$-valued modular form by Borcherds [Bor98, Theorem 5.3] with Weil representation $\omega_M$. 
Let $\text{Gr}(M)$ be the set of negative lines in $M_R$ (the Grassmannian), which is a real manifold of dimension $n - 1$ (as $M$ has signature $(n - 1, 1)$). For $\lambda \in M'/M$ and $m \in \mathbb{Q}$ with $m \equiv Q(\lambda) \pmod{1}$, let

$$Z_M(m, \lambda) = \{z \in \text{Gr}(M) : (z, x) = 0 \text{ for some } x \in \lambda + M, Q(x) = m\},$$

which is either empty or a real divisor of $\text{Gr}(M)$. The Weyl chamber associated to a weakly holomorphic form $f \in M_{1 - \frac{1}{2}\omega_L}$ is the connected components of (see [Bru02 Page 88])

$$\text{Gr}(M) = \bigcup_{\mu \in L_0/L} \bigcup_{m \in Q(\mu) + \mathbb{Z}} Z_M(m, p(\mu)).$$

Given a Weyl chamber $W$ associated to $f$, we define its Weyl vector $\rho(W, f) = \rho(W, f_M) \in M'$ following Borcherds as follows ([Bor98 Section 10.4], see also [Bru02 Page 88]). Let $\bar{W}$ be the closure of $W$. If $M \cap \bar{W}$ is anisotropic, it was defined in [Bor98 Section 9] with correction and extension given recently in [BS17 Section 5]. We don’t need it in this paper and refer to [BS17] for details. When $M \cap \bar{W}$ is isotropic, choose an isotropic $\ell_M \in M \cap \bar{W}$ and $\ell'_M \in M'$ with $(\ell_M, \ell'_M) = 1$. Let $P = M \cap (\mathbb{Q}\ell_M + \mathbb{Q}\ell'_M)^\perp$, which is positive definite of rank $n - 2$. Similar to the projection $p$ from $L_0/L$ to $M'/M$, one has also a projection $\tilde{p}$ from $M_0'/M$ to $P'/P$ defined in the same way. For the same reason, we have the weakly holomorphic modular form $\tilde{f}_P$ (coming from $f_M$). Define

\begin{align*}
(2.15) \quad &\rho_{\ell'_M} = \text{constant term of } \theta_P(\tau)f_P(\tau)E_2(\tau)/24, \\
(2.16) \quad &\rho_{\ell_M} = -\rho_{\ell'_M}Q(\ell'_M) - \frac{1}{4} \sum_{\lambda \in M_0'/M \atop p(\lambda) = 0 + P} c_M(0, \lambda)\text{B}_2((\lambda, \ell'_M)) \\
&\quad - \frac{1}{2} \sum_{\gamma \in P'} \sum_{\lambda \in M_0'/M \atop p(\lambda) = \gamma + P} c_M(-Q(\gamma), \lambda)\text{B}_2((\lambda, \ell_M)), \\
(2.17) \quad &\rho_P = -\frac{1}{2} \sum_{\gamma \in P' \cap M' \atop (\gamma, W) > 0} c_M(-Q(\gamma), \gamma), \\
(2.18) \quad &\rho(W, f) = \rho_P + \rho_{\ell'_M}\ell_M + \rho_{\ell'_M}\ell'_M.
\end{align*}

Here

$$E_2 = 1 - 24 \sum_{n > 0} \sigma_1(n)q^n$$

is the weight 2 Eisenstein series, $\text{B}_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$ is the second Bernoulli polynomial of $\{x\}$, where $0 \leq \{x\} = x - [x] < 1$ is the fractional part of $x$.

Now we can state the beautiful product expansion formula of Borcherds as follows in the signature $(n, 2)$ case ([Bor98 Theorem 13.3], see also [Bru02 Theorem 3.22]):

**Theorem 2.2.** (Borcherds) Let the notation be as above. Let $W$ be a Weyl chamber of $f$ whose closure contains $\ell_M$. Then the memomorphic automorphic form $\Psi(z, f) = \Psi(z, 1, f)$
has an infinite product expansion near the cusp \( \mathbb{Q} \ell \) (more precisely, when \( \text{Im}(z) \in W \) with \( -Q(\text{Im}(z)) \) sufficiently large).

\[
\Psi(z, f) = Ce((z, \rho(W, f))) \prod_{\lambda \in M'} \prod_{\mu \in \mathbb{L}_0/L, (\lambda, \lambda) > 0, \mu \in \lambda + M} [1 - e((\lambda, z) + (\mu, \ell'))]^{c(-Q(\lambda), \mu)}.
\]

Here \( C \) is a constant with absolute value

\[
(2.19) \quad \left| \prod_{\delta \in \mathbb{Z}/\mathbb{N}_{\ell}, \delta \neq 0} (1 - e(\frac{\delta}{N_\ell})) \right|^{c(0, \frac{4}{N_\ell})}.
\]

**Proof.** (sketch) We derive the formula from [Bor98, Theorem 13.3]. Let \( L_- = \mathbb{L} \) with quadratic form \( Q_-(x) = -Q(x) \) so that \( L_- \) has signature \( (2, n) \), for which we can apply Borcherds’ theorem. We use subscript \( - \) to indicate the corresponding notation in Borcherds. First notice that the symmetric domain \( \mathbb{D}_- = \mathbb{D} \) and the tautological bundle \( L_- = \mathcal{L} \). Since \( (\ell, \ell') = 1 \), one has \((-\ell, \ell')_- = 1\). So the tube domains \( \mathcal{H}_{\ell, \ell'} \) and \( \mathcal{H}_{-\ell, \ell', -} \) are the same too. Furthermore, for \( z \in \mathcal{H}_{\ell, \ell'} = \mathcal{H}_{-\ell, \ell', -} \), one has

\[
w_-(z) = z + \ell - (Q_-(z) + Q_-(\ell'))(-\ell) = w(z).
\]

Notice that \( f| \left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) = \( f \) implies

\[
(2.20) \quad c(m, \mu) = c(m, -\mu), \quad \text{and} \quad c_M(m, \lambda) = c_M(m, -\lambda).
\]

Since \( (\ell_M, \ell'_M) = 1 \), one has \( (\ell_M, -\ell'_M)_- = 1 \). Using [Bor98, Theorem 10.4] (a minor mistake there missing the \( \frac{1}{4} \) summation part), one checks that \( \rho_{\ell,-} = \rho_{\ell'} \) (as \( \theta_{P,-} = \theta_P \)), and

\[
\rho_{\ell,-} = -\rho_{\ell,-}Q_-(\ell'_M) + \frac{1}{4} \sum_{\lambda \in M'_0/M, \chi(\lambda) = 0} c_M(0, \lambda)B_2((\lambda, -\ell'_M)_-)
\]

\[
\quad + \frac{1}{2} \sum_{\gamma \in P'} \sum_{\lambda \in M'_0/M, \chi(\lambda) > 0, \mu(\lambda) = \gamma + P} c_M(Q_-(\gamma), \lambda)B_2((\lambda, -\ell'_M)_-)
\]

\[
= \rho_{\ell,M}Q(\ell'_M) + \frac{1}{4} \sum_{\lambda \in M'_0/M, \chi(\lambda) = \gamma + P} c_M(0, \lambda)B_2((\lambda, \ell'_M))
\]

\[
\quad + \frac{1}{2} \sum_{\gamma \in P'} \sum_{\lambda \in M'_0/M, \chi(\lambda) > 0, \mu(\lambda) = \gamma + P} c_M(-Q(\gamma), \lambda)B_2((\lambda, \ell'_M))
\]

\[
= -\rho_{\ell,M}.
\]

In the last identity, we substitute \( \gamma \) by \(-\gamma\) and \( \lambda \) by \(-\lambda\), and apply \( (2.20) \). Similarly, one checks \( \rho_{P,-} = -\rho_P \). So Borcherds’ Weyl vector

\[
\rho(W, f)_- = \rho_{P,-} + \rho_{\ell,-} - \rho_{\ell,M}(-\ell'_M) = -\rho(W, f).
\]
So \[ \text{Bor98, Theorem 13.3} \] gives for \( z \in \mathcal{H}_{\ell, \ell'} \)

\[ \Psi(z, f) = C e((z, \rho(W, f)) \prod_{\lambda \in M'} \prod_{\lambda, W > 0, p(\mu) = \lambda + M} [1 - e((\lambda, z) + (\mu, \ell'))] e(Q_{-}(\lambda, \mu)) \]

\[ = C e((z, \rho(W, f)) \prod_{\lambda \in M'} \prod_{\lambda, W > 0, p(\mu) = \lambda + M} [1 - e((\lambda, z) + (\mu, \ell'))] e(-Q(\lambda, \mu)) \]

as claimed. Here we again substitute \( \lambda \) and \( \mu \) by \(-\lambda \) and \(-\mu \), and apply \((2.20)\).

\[ \square \]

Remark 2.3. It is worthwhile to make a few remarks to clear up some (potentially confusing) differences in different versions.

1. The sign difference in the formula above and the formula in \[ \text{Bor98, Theorem 13.3} \] (and \[ \text{Bru02, Theorem 3.22} \]) is due to the fact that they use \( \mathcal{L}_{-} \) (signature \((2, n)\)) while we use \( \mathcal{L} \).
2. The condition \( p(\mu) \in \lambda + M \) here is a more explicit reinterpretation of Borcherds’ condition \( \mu| M = \lambda \) given by Bruinier \( \text{[Bru02, Theorem 3.22]} \).
3. The constant \( C \) can be taken as the product in \((2.19)\) at a given cusp. However, once it is fixed, the constants at other cusps are determined by this constant (they are in the same connected component).
4. The conditions that \( n \geq 3 \) and that \( M \) is isotropic in \[ \text{Bru02} \] was for convenience and not necessary.
5. The neighborhood near the cusp \( \mathcal{Q}_{\ell} \) where the product formula is valid can be made precise. We refer to \[ \text{Bru02, Theorem 3.22} \] for details.
6. At different cusps, the product formulas look different. This is similar to the different Fourier expansions of a modular form at different cusps.

2.2. Big CM cycles, incoherent Eisenstein series, and the big CM value formula. Let \( E \) be a CM number field of degree \( n + 2 \) with the maximal totally real subfield \( F \). Let \( \sigma_i, 1 \leq i \leq \frac{n}{2} + 1 \) be distinct real embeddings of \( F \). Choose an element \( \alpha \in F \) with \( \sigma_{\frac{n}{2} + 1}(\alpha) < 0 \) and \( \sigma_i(\alpha) > 0 \) for all \( 1 \leq i \leq \frac{n}{2} \), and let \( W = E \) with the \( F \)-quadratic form \( Q_F(z) = \alpha z\bar{z} \). Let \( W_\mathcal{Q} = E \) with the \( \mathcal{Q} \)-quadratic form

\[ Q_\mathcal{Q}(z) = \text{tr}_{F/\mathcal{Q}} Q_F(z) = \text{tr}_{F/\mathcal{Q}}(\alpha z\bar{z}) \]

Notice that \( (W_\mathcal{Q}, Q_\mathcal{Q}) \) is a \( \mathcal{Q} \)-quadratic space of signature \((n, 2)\). Now we assume that \( (W_\mathcal{Q}, Q_\mathcal{Q}) \cong (V, Q) \), where \((V, Q)\) is a given \( \mathcal{Q} \)-quadratic space of signature \((n, 2)\). Write \( n_0 = \frac{n}{2} + 1 \). Then we have

\[ V_\mathbb{R} \cong \bigoplus_{1 \leq i \leq n_0} W_{\sigma_i}, \]

where \( W_{\sigma_i} = W \otimes_{F, \sigma_i} \mathbb{R} \) has signature \((2, 0)\) or \((0, 2)\) according to \( 1 \leq i < n_0 \) or \( i = n_0 \). The negative two plane \( W_{\sigma_{n_0}} \) gives rise to two ‘big’ CM points \( z_{\sigma_{n_0}}^\pm \), which turns out to be defined over a finite extension of \( \sigma_{n_0}(F) \). Define an algebraic torus \( T \) over \( \mathbb{Q} \) by the
Then $T$ is a maximal torus in $H = \text{GSpin}(V)$ (thus the name big CM points and big CM cycles). It is known ([BKY12, Section 2]) that

$$Z(W, z_{\sigma_{n_0}}^\pm) = \{z_{\sigma_{n_0}}^\pm\} \times (T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T), \quad K_T = K \cap T(\mathbb{A}_f)$$

is a zero cycle in $X_K$ defined over $F$, called a big CM cycle of $X_K$. Let $Z(W)$ be the formal sum of all its Galois conjugates (counting multiplicity), which is a big CM cycle of $X_K$ over $\mathbb{Q}$. We refer to [BKY12, Section 2] for a more precise definition and basic properties of this cycle.

Associated to this quadratic space and the additive adelic character $\psi_F = \psi \circ \text{tr}_{F/Q}$ is a Weil representation $\omega = \omega_{\psi_F}$ of $\text{SL}_2(\mathbb{A}_F)$ (and thus $T(\mathbb{A}_f)$) on $S(W(\mathbb{A}_F)) = S(V(\mathbb{A}_Q))$. Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of $F$ associated to $E/F$, then $\chi = \chi_W$ is also the quadratic Hecke character of $F$ associated to $W$, and there is a $\text{SL}_2(\mathbb{A}_F)$-equivariant map

$$\lambda = \prod \lambda_v : S(W(\mathbb{A}_F)) \rightarrow I(0, \chi), \quad \lambda(\phi)(g) = \omega(g)\phi(0).$$

Here $I(s, \chi) = \text{Ind}_{B_{p_F}}^{\text{SL}_2(\mathbb{A}_F)} \chi | \cdot |^{s}$ is the principal series, whose sections (elements) are smooth functions $\Phi$ on $\text{SL}_2(\mathbb{A}_F)$ satisfying the condition

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g, s), \quad b \in \mathbb{A}_F, \quad a \in \mathbb{A}_F^\times.$$

Here $B = NM$ is the standard Borel subgroup of $\text{SL}_2$. Such a section is called factorizable if $\Phi = \otimes \Phi_v$ with $\Phi_v \in I(s, \chi_v)$. It is called standard if $\Phi|_{\text{SL}_2(\mathbb{A}_F) \text{SO}_2(\mathbb{R})^{n_0}}$ is independent of $s$. For a standard section $\Phi \in I(s, \chi)$, its associated Eisenstein series is defined as

$$E(g, s, \Phi) = \sum_{\gamma \in B_F \backslash \text{SL}_2(F)} \Phi(\gamma g, s)$$

for $\Re(s) \gg 0$.

For $\phi \in S(V_f) = S(W_f)$, let $\Phi_f$ be the standard section associated to $\lambda_f(\phi) \in I(0, \chi_f)$. For each real embedding $\sigma_i : F \hookrightarrow \mathbb{R}$, let $\Phi_{\sigma_i} \in I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}}) = I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}})$ be the unique ‘weight one’ eigenvector of $\text{SL}_2(\mathbb{R})$ given by

$$\Phi_{\sigma_i}(n(b)m(a)k_{\theta}) = \chi_{C/R}(a)|a|^{s+1}e^{i\theta},$$

for $b \in \mathbb{R}$, $a \in \mathbb{R}^\times$, and $k_{\theta} = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \in \text{SO}_2(\mathbb{R})$. We define for $\vec{\tau} = (\tau_1, \ldots, \tau_{n_0}) \in \mathbb{H}^{n_0}$

$$E(\vec{\tau}, s, \phi) = N(\vec{v})^{-\frac{1}{2}}E(g_{\vec{\tau}}, s, \Phi_f \otimes (\otimes_{1 \leq i \leq n_0} \Phi_{\sigma_i})),$$
where $\bar{v} = \text{Im}(\tau)$, $N(\bar{v}) = \prod v_i$, and $g_\tau = (n(u_i)m(\sqrt{v_i}))_{1 \leq i \leq n_0}$. It is a (non-holomorphic) Hilbert modular form of scalar weight 1 for some congruence subgroup of $\text{SL}_2(\mathcal{O}_F)$. Following [BKY12], we further normalize

$$E^*(\bar{v}, s, \phi) = \Lambda(s+1, \chi) E(\bar{v}, s, \phi),$$

where $\partial_F$ is the different of $F$, $d_{E/F}$ is the relative discriminant of $E/F$, and

$$\Lambda(s, \chi) = A^\flat (\pi^{-\frac{\sigma}{2}} \Gamma(s+\frac{1}{2}))^{n_0} L(s, \chi), \quad A = N_{F/Q}(\partial_F d_{E/F}).$$

The Eisenstein series is incoherent in the sense that $\Phi = \otimes \Phi_v$ is in the image of $\lambda$ on $S(C)$, where $C$ is an incoherent system of quadratic spaces over $F_v$, given by $C_v = W_v$ for all places $v$ except the one $v = \sigma_{n_0}$. This incoherence forces $E^*(\bar{v}, 0, \phi) = 0$ automatically.

**Proposition 2.4.** ([BKY12 Proposition 4.6]) Let $\phi \in S(V_F) = S(W_F)$. For a totally positive element $t \in F^+_\times$, let $a(t, \phi)$ be the $t$-th Fourier coefficient of $E^*(\bar{v}, 0, \phi)$ and write the constant term of $E^*(\bar{v}, 0, \phi)$ as

$$\phi(0) \Lambda(0, \chi) \log N(\bar{v}) + a_0(\phi).$$

Let

$$E(\tau, \phi) = a_0(\phi) + \sum_{n \in \mathbb{Q}_{>0}} a_n(\phi) q^n$$

where (for $n > 0$)

$$a_n(\phi) = \sum_{t \in F^+_\times, t_{\text{tr}} F/Q t = n} a(t, \phi).$$

Here $F^+_\times$ consists of all totally positive elements in $F$. Then, writing $\tau^\Delta = (\tau, \ldots, \tau)$ for the diagonal image of $\tau \in \mathbb{H}$ in $\mathbb{H}^{n_0}$,

$$E^*(\tau^\Delta, 0, \phi) - E(\tau, \phi) - \phi(0) (\frac{n}{2} + 1) \Lambda(0, \chi) \log v$$

is of exponential decay as $v$ goes to infinity. Moreover, for $n > 0$

$$a_n(\phi) = \sum_p a_{n,p}(\phi) \log p$$

with $a_{n,p}(\phi) \in \mathbb{Q}(\phi)$, the subfield of $\mathbb{C}$ generated by the values $\phi(x)$, $x \in V(\mathcal{A}_f)$.

**Remark 2.5.** There is a minor mistake in [BKY12 Proposition 4.6]) about the constant. The corrected form is

$$E^*_0(\bar{v}, 0, \phi) = \phi(0) \Lambda(0, \chi) \log N(\bar{v}) + a_0(\phi)$$

(i.e., $a_0(\phi)$ might not be a multiple of $\phi(0)$). Direct calculation gives

$$E^*_0(\bar{v}, s, \phi) = \phi(0) \Lambda(s+1, \chi)(N(\bar{v}))^\sharp - (N(\bar{v}))^{-\sharp} \Lambda(s, \chi) W_{0,f}(s, \phi)$$

where (when $\phi$ is factorizable)

$$W_{0,f}(s, \phi) = \prod_{p \mid \infty} W_{0,p}(s, \phi_p) = \prod_{p \mid \infty} A_p^{-\frac{\gamma}{2}} \frac{s + 1, \chi}{\gamma(Q_p_s)} W_{0,p}(s, \phi_p)$$
is the product of re-normalized local Whittaker functions (see (2.25)). With this notation, one has

\begin{equation}
\tag{2.24}
a_0(\phi) = -\tilde{W}_0^f(0, \phi) - 2\phi(0)\Lambda'(0, \chi).
\end{equation}

Notice that \(a(t, \phi_\mu) = 0\) automatically unless \(\mu + \hat{L}\) represents \(t\), i.e., \(t - Q_F(\mu) \in \partial^{-1}F_F\).

The following is a special case of the main CM value formula of Bruinier, Kudla, and Yang (\cite[Theorem 5.2]{BKY12}).

**Theorem 2.6.** Let

\[ f(\tau) = \sum_{\mu \in L'/L} f_\mu(\tau)\phi_\mu = \sum c(m, \mu)q^m\phi_\mu \in M_{1-\frac{2}{T}}^!\omega_L \]

with \(c(0, 0) = 0\), and let \(\Psi(z, f)\) be its Borcherds lifting. Then

\[ -\log |\Psi(Z(W), f)|_4^1 = C(W, K) \left( \sum_{\mu \in L'/L, m \geq 0 \atop m \equiv Q(\mu) \pmod{1}} c(-m, \mu)a_m(\phi_\mu) \right). \]

Here

\[ C(W, K) = \frac{\deg(Z(W, z_{\alpha_2}^\pm))}{\Lambda(0, \chi)}. \]

To compute the \(t\)-th Fourier coefficient \(a(t, \phi)\) of \(E_{\tau'}^*(\tau, 0, \phi)\), one may assume that \(\phi = \otimes\phi_p\) is factorizable by linearity. Write for \(t \neq 0\)

\[ E_{\tau'}^*(\tau, s, \phi) = \prod_{\text{p}} W_{t, \text{p}}^*(s, \phi) \prod_{j=1}^{n_0} W_{t, \sigma_j}^*(\tau_j, s, \Phi_{\sigma_j}), \]

where

\[ W_{t, \text{p}}^*(s, \phi) = |A|^\frac{s+1}{2} L_p(s+1, \chi_p) W_{t, \text{p}}(s, \phi) \]

for finite prime \(\text{p}\) with

\begin{equation}
\tag{2.25}
W_{t, \text{p}}(s, \phi) = \int_{F_\text{p}} \omega(wn(b))(\phi_\text{p})(0)|a(wn(b))|^\text{p} \psi_\text{p}(-tb) \, db,
\end{equation}

and for infinite prime \(\sigma_j\)

\[ W_{t, \sigma_j}^*(\tau_j, s, \Phi_{\sigma_j}) = v_j^{-1/2} \pi^{-\frac{s+2}{2}} \Gamma\left(\frac{s+2}{2}\right) \int_{\mathbb{R}} \Phi_{\sigma_j}(wn(b)g_{\tau_j}, s) \psi(-bt) \, db. \]

Here \(A\) is defined in (2.23) and \(|a(q)|_p = |a|_p\) if \(q = n(b)m(a)k\) with \(k \in \text{SL}_2(O_p)\).

The following proposition is well-known and is recorded here for reference. Recall \(W = E\) with \(Q_F(z) = \alpha z\bar{z}\), \(\alpha \in F^\times\).

**Proposition 2.7.** For a totally positive number \(t \in F^+,\) let

\[ \text{Diff}(W, t) = \{ \text{p} : W_{\text{p}} \text{ does not represent t} \} \]

be the so-called ‘Diff’ set of Kudla. Then \(|\text{Diff}(W, t)|\) is finite and odd. Moreover,
(1) If $|\text{Diff}(W,t)| > 1$, then $a(t,\phi) = 0$.
(2) If $\text{Diff}(W,t) = \{p\}$, then $W_{t,p}^*(0,\phi) = 0$, and
$$a(t,\phi) = (-2i)^{n_0} W_{t,p}^*(0,\phi) \prod_{q \in \mathcal{P}} W_{t,q}^*(0,\phi).$$

(3) When $p \nmid \alpha A$ is unramified in $E/F$, and $\phi_p = \text{Char}(\mathcal{O}_E_p)$, $W_{t,p}^*(s,\phi) = 0$ unless $t \in \partial_F^{-1}$. In this case, one has
$$W_{t,p}^*(0,\phi) = 0 \quad \text{if} \quad p \text{ is split in } E,$$
$$W_{t,p}^*(0,\phi) = 1 + \text{ord}_p(t\sqrt{D}) \quad \text{if} \quad p \text{ is inert in } E.$$ 
Here $\gamma(W_p)$ is the local Weil index (a $8$-th root of unity) associated to the Weil representation. Moreover, in this case, $W_{t,p}^*(s,\phi) = 0$ if and only if $\text{ord}_p(t\sqrt{D})$ is odd and $p$ is inert in $E$. In such a case, one has
$$W_{t,p}^*(0,\phi) = 1 + \text{ord}_p(t\sqrt{D}) \log N(p).$$

(4) One has for $1 \leq j \leq n_0$
$$W_{t,\sigma_j}^*(\tau,0,\Phi_{\sigma_j}) = -2ie(t\tau), \quad t > 0$$
and
$$W_{0,\sigma_j}^*(\tau,s,\Phi_{\sigma_j}) = -i\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) v^{-\frac{s}{2}}.$$

Proof. (sketch) The Diff set is first defined by Kudla in [Kud97b]. In our case, the incoherent collection of $F_0$-quadratic spaces is $\{C_v\}$ where $C_v = W_v$ for $v \neq \sigma_{n_0}$ and $C_v$ positive definite. The archimedian places are not in the Diff set as $t$ is totally positive. Let $\psi_F'(x) = \psi_F(x\sqrt{D})$ and $W' = W$ with $F$-quadratic form $Q_F'(x) = \sqrt{D}Q_F(x) = x\bar{x}$. Then one has as Weil representations on $S(W_f) = S(W'_f)$:
$$\omega_{W,\psi_F} = \omega_{W',\psi_F'},$$
and thus the Whittaker functions have the following relation
$$W_{t,p}^{\psi_F}(s,\phi) = |\sqrt{D}|^{\frac{s}{2}} W_{t,v\sqrt{D},p}^{\psi'}(s,\phi)$$
for each prime $p$ of $F$. Recall that $W_{t,p}^*(0,\phi) = 0$ if $p \in \text{Diff}(W,t)$. So (1) is obvious. Claim (3) follows from [Yam05] Proposition 2.1. Claim (4) is a special case of [KRY99] Proposition 2.6. Claim (2) follows from
$$E_t^*(\bar{\tau},s,\phi) = \prod_{p|\infty} W_{t,p}^*(s,\phi) \prod_{j=1}^{n_0} W_{t,\sigma_j}^*(\tau_j,s,\Phi_{\sigma_j})$$
and (4).
3. Product of modular curves and its diagonal divisor

3.1. Product of modular curves as a Shimura variety of orthogonal type (2, 2).

Let \( N \) be a positive integer, and let \( V = M_2(\mathbb{Q}) \) with the quadratic form \( Q(X) = N \det X \).

Let \( H \) be the algebraic group over \( \mathbb{Q} \)

\[
H = \{(g_1, g_2) \in \text{GL}_2 \times \text{GL}_2 : \det g_1 = \det g_2 \}.
\]

Then \( H \cong \text{GSpin}(V) \) and acts on \( V \) via

\[
(g_1, g_2)X = g_1Xg_2^{-1}.
\]

One has the exact sequence

\[
1 \to \mathbb{G}_m \to H \to SO(V) \to 1.
\]

Recall the Hermitian symmetric domain \( \mathbb{D} \) and the tautological line bundle \( \mathcal{L} \) in Section 2.

For a tube domain, take an isotropic matrix \( \ell = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \in L \) and \( \ell' = \left( \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) \in V \) with \( (\ell, \ell') = 1 \). Then the associated tube domain is

\[
\mathcal{H}_{\ell, \ell'} = \{ \left( \begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right) : y_1y_2 > 0 \}, \quad y_i = \text{Im}(z_i),
\]

together with

\[
w : \mathcal{H}_{\ell, \ell'} \to \mathcal{L}, \quad w\left( \begin{smallmatrix} z_1 & 0 \\ 0 & -z_2 \end{smallmatrix} \right) = \left( \begin{smallmatrix} z_1 & -Nz_1z_2 \\ z_2 & \bar{z}_2 \end{smallmatrix} \right).
\]

Now the following proposition is clear.

**Proposition 3.1.** Define

\[
w_N : \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \to \mathcal{L}, \quad w_N(z_1, z_2) = \left( \begin{smallmatrix} z_1 & -z_1z_2 \\ 1 & z_2 \end{smallmatrix} \right) = Nw\left( \begin{smallmatrix} z_1 & 0 \\ 0 & -z_2 \end{smallmatrix} \right),
\]

and

\[
pr : \mathcal{L} \to \mathbb{D}, \quad x + iy \mapsto z = \mathbb{R}x + \mathbb{R}(-y).
\]

Then \( pr \) gives an isomorphism between \( \mathcal{L}/\mathbb{C}^\times \) and \( \mathbb{D} \), and the composition \( pr \circ w \) gives an isomorphism between \( \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \) and \( \mathbb{D} \). Moreover, \( w_N \) is \( H \)-equivariant, where \( H \subset \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}) \) acts on \( \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \) via the usual linear fraction transformations:

\[
(g_1, g_2)(z_1, z_2) = (g_1(z_1), g_2(z_2)),
\]

and acts on \( \mathcal{L} \) and \( \mathbb{D} \) naturally via its action on \( V \). Moreover, one has

\[
(3.1) \quad (g_1, g_2)w_N(z_1, z_2) = \frac{(c_1z_1 + d_1)(c_2z_2 + d_2)}{\nu(g_1, g_2)}w_N(g_1(z_1), g_2(z_2)),
\]

where \( \nu(g_1, g_2) = \det g_1 = \det g_2 \) is the spin character of \( H = \text{GSpin}(V) \). So

\[
j(g_1, g_2, z_1, z_2) = (c_1z_1 + d_1)(c_2z_2 + d_2).
\]

For a congruence subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \), let \( X_\Gamma \) be the associated open modular curve over \( \mathbb{Q} \) such that \( X_\Gamma(\mathbb{C}) = \Gamma\backslash \mathbb{H} \). Assume \( \Gamma \supset \Gamma(M) \) for some integer \( M \geq 1 \). Let

\[
\nu : \mathbb{A}^\times \to \text{GL}_2(\mathbb{A}), \quad \nu(d) = \text{diag}(1, d).
\]
Let \( K(\Gamma) \) be the product of \( \nu(\hat{\mathbb{Z}}^\times) \) and the preimage of \( \Gamma/\Gamma(M) \) in \( \text{GL}_2(\hat{\mathbb{Z}}) \) (under the map \( \text{GL}_2(\hat{\mathbb{Z}}) \to \text{GL}_2(\mathbb{Z}/M) \)). Let \( K = (K(\Gamma) \times K(\Gamma)) \cap H(\mathbb{A}_f) \). Then one has by the strong approximation theorem

\[ X_K \cong X_\Gamma \times X_\Gamma. \]

In this way, we have identified the product of two copies of a modular curve \( X_\Gamma \) with a Shimura variety \( X_K \). We will fix this \( K \) for a given congruence subgroup \( \Gamma \) in this paper. The tautological line bundle \( L \) descends to a line bundle \( L_K = K \backslash \mathcal{L} \) of modular forms of 2 variables of weight \( (1, 1) \) by \( \text{(3.1)} \).

Let \( L \) be an even integral lattice of \( V \), and let \( L' \) be its dual with respect to the quadratic form \( Q \). We assume that \( \Gamma \times \Gamma \) acts on \( L'/L \) trivially. Then for \( \mu \in L'/L \) and a rational number \( m > 0 \) (and \( m \equiv Q(\mu) \pmod{1} \)), the associated special divisor \( Z(m, \mu) = Z_K(m, \mu) \) is given in this special case by

\[ Z(m, \mu) = (\Gamma \times \Gamma) \backslash \{(z_1, z_2) \in \mathbb{H}^2 : w_N(z_1, z_2) \perp x \text{ for some } x \in \mu + L, Q(x) = m\}. \]

Alternatively, \( Z(m, \mu) \) is the sum of \( Z(x) \), where \( x \in \mu + L \) with \( Q(x) = m \) modulo the action of \( \Gamma \times \Gamma \). Here \( Z(x) \) is the subvariety of \( X_K \) given by \( x^\perp \) (of signature \( (1, 2) \)):

\[ Z(x) = (\Gamma \cap x^{-1} \Gamma x) \backslash \mathbb{H} \cong (\Gamma \times \Gamma) \backslash \{(xz, z) : z \in \mathbb{H}\}, \quad [z] \mapsto [xz, z]. \]

The linear combinations of these divisors \( Z(m, \mu) \) are called the special divisors of \( X_K \).

**Lemma 3.2.** Let \( \Gamma = \Gamma(N) \) and \( L = M_2(\mathbb{Z}) \) with \( Q(X) = N \det X \). For each \( \gamma \in \text{SL}_2(\mathbb{Z}) \), let

\[ Z_N(\gamma) = \{(\gamma z, z) \in X_{\Gamma(N)} \times X_{\Gamma(N)} : z \in X_{\Gamma(N)} \} \subset X_{\Gamma(N)} \times X_{\Gamma(N)}. \]

Then \( Z_N(\gamma) = Z(\frac{1}{N}, \frac{1}{N} \gamma + L) \) is a special divisor of \( X_K \).

We denote by \( X(N) \) the compactification of \( X_{\Gamma(N)} \) (to be compatible with usual definition of \( X(N) \))

**Proof.** If \( x \in \frac{1}{N} \gamma + L \) with \( Q(x) = N \det x = 1/N \), then \( Nx \in \gamma + NL \) and \( \det(Nx) = 1 \). So \( Nx \in \text{SL}_2(\mathbb{Z}) \), and \( Nx\gamma^{-1} = \gamma_1 \in \Gamma(N) \), and \( x = \gamma_1(\frac{1}{N} \gamma) \). This implies

\[ Z(\frac{1}{N}, \frac{1}{N} \gamma + L) = Z_N(\gamma). \]

\[ \square \]

**Corollary 3.3.** Let \( X_\Gamma^\Delta \) be the diagonal embedding of \( X_\Gamma \) into \( X_\Gamma \times X_\Gamma \). The \( X_\Gamma^\Delta \) is a special divisor of \( X_\Gamma \times X_\Gamma \) in the following sense. Assume \( \Gamma \supset \Gamma(N) \), we take \( L = M_2(\mathbb{Z}) \) with \( Q(X) = N \det X \). Then the preimage of \( X_\Gamma^\Delta \) in \( X_{\Gamma(N)} \times X_{\Gamma(N)} \) is equal to

\[ \sum_{\gamma \in \Gamma/\Gamma(N)} Z_N(\gamma) \]

in the notation of Lemma 3.2.
3.2. Products of CM cycles as big CM cycles. For \( j = 1, 2 \), let \( E_j = \mathbb{Q} ( \sqrt{d_j} ) \) with ring of integers \( \mathcal{O}_j = \mathbb{Z}[\frac{d_j + \sqrt{d_j}}{2}] \) of discriminant \( d_j < 0 \) with \( (d_1, d_2) = 1 \). In this subsection, we describe how to view a pair of CM points \((\tau_1, \tau_2) \in X_F \times X_F\) associated to \( E_1 \) and \( E_2 \) as a big CM point in \( X_K \) in the sense of [BKY12]. For this purpose, let \( E = E_1 \otimes_{\mathbb{Q}} E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \) with ring of integers \( \mathcal{O}_E = \mathcal{O}_1 \otimes_{\mathbb{Z}} \mathcal{O}_2 \). Then \( E \) is a biquadratic CM number field with real quadratic subfield \( F = \mathbb{Q}(\sqrt{D}) \) and \( D = d_1 d_2 \).

We define \( W = E \) with the \( F \)-quadratic form \( Q_F(x) = \frac{N_{x \bar{x}}}{\sqrt{D}} \). Let \( W_Q = W \) with the \( \mathbb{Q} \)-quadratic form \( Q_Q(x) = \text{tr}_{F/\mathbb{Q}} Q_F(x) \). Let \( \sigma_1 = 1 \) and \( \sigma_2 = \sigma \) be two real embeddings of \( F \) with \( \sigma_j(\sqrt{D}) = (-1)^{j-1} \sqrt{D} \). Then \( W \) has signature \((0, 2)\) at \( \sigma_2 \) and \((2, 0)\) at \( \sigma_1 \) respectively, and so \( W_Q \) has signature \((2, 2)\). Choose a \( \mathbb{Z} \)-basis of \( \mathcal{O}_E \) as follows

\[
\begin{align*}
\epsilon_1 &= 1 \otimes 1, \quad \epsilon_2 = -\frac{d_1 + \sqrt{d_1}}{2} = -\frac{d_2 + \sqrt{d_2}}{2}, \\
\epsilon_3 &= \frac{d_2 + \sqrt{d_2}}{2}, \quad \epsilon_4 = \epsilon_2 \epsilon_3.
\end{align*}
\]

We will drop \( \otimes \) when there is no confusion. Then it is easy to check that

\[
(W_Q, Q_Q) \cong (V, Q) = (M_2(\mathbb{Q}), N \text{ det}), \quad \sum x_i \epsilon_i \mapsto (x_3, x_1, x_2, x_3) .
\]

We will identify \((W_Q, Q_Q)\) with the quadratic space \((V, Q) = (M_2(\mathbb{Q}), N \text{ det})\). Under this identification, the lattice \( L = M_2(\mathbb{Z}) \) becomes \( \mathcal{O}_E \). Then one can check that the maximal torus \( T \) in (2.21) can be identified with ([HY12], [BKY12] Section 6)

\[
T(R) = \{(t_1, t_2) \in (E_1 \otimes_{\mathbb{Q}} R)^\times \times (E_2 \otimes_{\mathbb{Q}} R)^\times : t_1 \bar{t}_1 = t_2 \bar{t}_2\},
\]

for any \( \mathbb{Q} \)-algebra \( R \), and the map from \( T \) to \( \text{SO}(W) \) is given by \((t_1, t_2) \mapsto t_1/t_2 \). The map from \( T \) to \( H \) is explicitly given as follows. Define the embedding

\[
\iota_j : E_j \to M_2(\mathbb{Q}), \quad \iota_j(r)(\epsilon_{j+1}, \epsilon_1)^t = (re_{j+1}, re_1)^t.
\]

Then \( \iota = (\iota_1, \iota_2) \) gives the embedding from \( T \) to \( H \).

Extend the two real embeddings of \( F \) into a CM type \( \Sigma = \{\sigma_1, \sigma_2\} \) of \( E \) via

\[
\sigma_1(\sqrt{d_i}) = \sqrt{d_i} \in \mathbb{H}, \quad \sigma_2(\sqrt{d_1}) = \sqrt{d_1}, \quad \sigma_2(\sqrt{d_2}) = -\sqrt{d_2}.
\]

Since \( W_{\sigma_2} = W \otimes_{F, \sigma_2} \mathbb{R} \subset V_{\mathbb{R}} \) has signature \((0, 2)\), it gives two points \( z_{\sigma_2}^\pm \) in \( \mathbb{D} \). In this case, the big CM cycles in Section 2.2 become

\[
Z(W, z_{\sigma_2}^\pm) = \{ z_{\sigma_2}^\pm \} \times T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K_T \subset Z^2(X_K),
\]

and

\[
Z(W) = Z(W, z_{\sigma_2}^+) + \sigma(Z(W, z_{\sigma_2}^-)).
\]

For simplicity, we will denote \( z_{\sigma_2} \) for \( z_{\sigma_2}^+ \).

Lemma 3.4. On \( \mathbb{H}_{\pm, 2} \), one has \( z_{\sigma_2} = (\tau_1, \tau_2) \in \mathbb{H}^2 \) and \( z_{\sigma_2}^- = (\bar{\tau}_1, \bar{\tau}_2) \in (\mathbb{H}^-)^2 \), where

\[
\tau_j = \frac{d_j + \sqrt{d_j}}{2}.
\]
Proof. In the decomposition
\[ V_\mathbb{R} = V \otimes_{\mathbb{Q}} \mathbb{R} = W_{\sigma_1} \oplus W_{\sigma_2}, \quad W_{\sigma_1} = E \otimes_{F,\sigma_1} \mathbb{R} \cong \mathbb{C}, \quad r \mapsto \sigma_i(r), \]
the \( \mathbb{R} \)-basis \( \{e_i, i = 1, 2, 3, 4\} \) becomes
\[ e_1 = (1, 1), \quad e_2 = (-\bar{\tau}_1, -\bar{\tau}_1), \quad e_3 = (\bar{\tau}_2, \bar{\tau}_2), \quad \text{and} \quad e_4 = (-\bar{\tau}_1 \bar{\tau}_2, -\bar{\tau}_1 \bar{\tau}_2). \]
The negative two plane \( W_{\sigma_2} \) representing \( z_{\sigma_2}^\pm \) has an \( \mathbb{R} \)-orthogonal basis
\[ u = (0, \sqrt{|d_2|}), \quad \text{and} \quad v = (0, \sqrt{d_2}) \in W_{\sigma_2} \subset V_\mathbb{R}. \]
One checks
\[
u = -\frac{D - \sqrt{D}}{2|d_1|} e_1 - \frac{d_2}{\sqrt{|d_1|}} e_2 + \frac{d_1}{\sqrt{|d_1|}} e_3 + \frac{2}{\sqrt{|d_1|}} e_4 = \left( \frac{d_1}{\sqrt{|d_1|}} \frac{d_2}{\sqrt{|d_1|}} \right).
\]
and
\[ v = \frac{\sqrt{d_2}(\sqrt{|d_1|} + \sqrt{d_2})}{2} e_1 + \frac{\sqrt{d_2}}{\sqrt{|d_1|}} e_2 - e_3 = \left( \frac{-1}{\sqrt{|d_1|}} \frac{\sqrt{d_2}}{\sqrt{|d_1|}} \right). \]
So
\[ u - iv = \frac{2}{\sqrt{|d_1|}} \left( \frac{\tau_1 - \bar{\tau}_1 \bar{\tau}_2}{1} \right) = \frac{2}{N(\tau_1, \tau_2)} w_N(\tau_1, \tau_2), \]
and
\[ u + iv = \frac{2}{\sqrt{|d_1|}} w_N(\tau_1, \tau_2) \]
as claimed. \( \square \)

Lemma 3.5. Let \( K_j = \mathfrak{i}_j^{-1}(K(\Gamma)) \) and let \( \text{Cl}(K_j) = E_j^\times \backslash E_{j,f}^\times /K_j \) be the associated class group of \( E_j \). Then there is an injection
\[ p' : T(\mathbb{Q}) \setminus T(\mathbb{A}_f) /K_T \rightarrow \text{Cl}(K_1) \times \text{Cl}(K_2) \]
with image
\[ \text{IM}(p') = \{(C_1, C_2) \in \text{Cl}(K_1) \times \text{Cl}(K_2) : \exists \ t_j \in E_{j,f}^\times \text{ with } C_j = [t_j], \ t_1 \bar{t}_1 = t_2 \bar{t}_2 \}
= \{(C_1, C_2) \in \text{Cl}(K_1) \times \text{Cl}(K_2) : \exists \text{ fractional ideals } a, \text{ with } C_j = [a_j], N(a_1) = N(a_2) \}. \]

Proof. Clearly \( p' \) is a group homomorphism. We first check that \( p' \) is injective. Assume \( [t_1, t_2] \in \ker p' \), write \( t_j = g_j k_j \) with \( g_j \in E_{j,f}^\times \) and \( k_j \in K_j \). Then \( t_1 \bar{t}_1 = t_2 \bar{t}_2 \) implies
\[ \frac{g_1 \bar{g}_1}{g_2 \bar{g}_2} = \frac{k_2 \bar{k}_2}{k_1 \bar{k}_1} \in \mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}, \]
so \((g_1, g_2) \in T(\mathbb{Q})\), and \(k_2 \bar{k}_2 = k_1 \bar{k}_1\). This implies

\[(k_1, k_2) \in K_T = \{(t_1, t_2) \in T(\mathbb{A}_f) : (t_1(t_1), t_2(t_2)) \in K_\Gamma = K(\Gamma) \times K(\Gamma)\}.\]

So \((t_1, t_2) \in T(\mathbb{Q})K_T\). The first formula for \(\text{IM}(p)\) is the definition. To show the second formula, assume \(N(\mathfrak{a}_1) = N(\mathfrak{a}_2)\). Let \(t_j \in E_{j,f}\) such that its associated ideal is \(\mathfrak{a}_j\). Then \(t_1 \bar{t}_1 = t_2 \bar{t}_2 u\) for some \(u \in \hat{\mathbb{Z}}^\times\). When \(p \nmid d_j, u_p = w_p \bar{w}_p\) for some \(w_p \in \mathcal{O}_{E_{j,p}}^\times\). So we can decompose \(u = u_1^{-1} u_2\) such that \(u_j = w_j \bar{w}_j \in N_{E_{j}/\mathbb{Q}} \bar{\mathcal{O}}_{E_{j}}^\times\). Replacing \(t_j\) by \(t_j w_j\), we find \(t_j \in E_{j,f}^\times\) such that \(t_1 \bar{t}_1 = t_2 \bar{t}_2\) and \([t_j] = [\mathfrak{a}_j]\).

Let \(H_j\) be the class field of \(E_j\) associated to \(K_j\) and \(H = H_1 H_2\), the composition of \(H_1\) and \(H_2\). By the complex multiplication theory, the point \([z_{\sigma_2}] \in X_K\) is defined over \(H\). Moreover, one has a natural map induced by \(t_j\) in (3.4)

\[(6.6) \quad t_j : \text{Cl}(K_j) \to X_\Gamma = \text{GL}_2(\mathbb{Q}) \backslash \mathbb{H}_\Gamma \times \text{GL}_2(\mathbb{A}_f)/K(\Gamma), \quad t_j([t^{-1}]) = [\tau_j, t_j(t^{-1})] = \tau_{j}^{\sigma_j}.\]

Here \(\sigma_j \in \text{Gal}(H_j/E_j)\) is associated to \([t]\) by class field theory. The last identity is Shimura’s reciprocity law (see for example [Yan16]). We will also write \(\tau_{j}^{\sigma_1} = \tau_{j}^{\sigma_2}\) in ideal language where \([\mathfrak{a}_j] \in \text{Cl}(K_j)\) corresponds to the idele class of \(t\). Now the following two propositions are clear.

**Proposition 3.6.** Let \((t_1, t_2) \in T(\mathbb{A}_f)\), and let \(\sigma_j \in \text{Gal}(H_j/E_j)\) be the associated Galois element (to \(t_j\)) via the Artin map. Then

\[\left[z_{\sigma_2}, (t_1^{-1}, t_2^{-1})\right] = [\tau_{1}^{\sigma_1}, \tau_{2}^{\sigma_2}].\]

**Proposition 3.7.** Assume \((d_1, d_2) = 1\). Then

\[
Z(W, z_{\sigma_2}) = \sum_{([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \text{IM}(p')} [\tau_{1}^{\sigma_1}, \tau_{2}^{\sigma_2}],
\]

\[
Z(W, z_{\bar{\sigma}_2}) = \sum_{([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \text{IM}(p')} [(-\tau_{1})^{\sigma_1}, (-\tau_{2})^{\sigma_2}],
\]

\[
Z(W) = \sum_{([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \text{IM}(p')} \left([\tau_{1}^{\sigma_1}, \tau_{2}^{\sigma_2}] + [(-\tau_{1})^{\sigma_1}, (-\tau_{2})^{\sigma_2}] + [\tau_{1}^{\sigma_1}, (-\tau_{2})^{\sigma_2}] + [(-\tau_{1})^{\sigma_1}, (-\tau_{2})^{\sigma_2}]\right).
\]

The following lemma will be used later.

**Lemma 3.8.** Assume again \((d_1, d_2) = 1\). Let \(\text{Cl}(E_j)\) be the ideal class group of \(E_j\). Let \(C_j \in \text{Cl}(E_j)\) be an ideal class for each \(j = 1, 2\). Then there is an ideal \(\mathfrak{a}_j \in C_j\) such that \(N(\mathfrak{a}_1) = N(\mathfrak{a}_2)\). In particular, when \(K_j = \bar{\mathcal{O}}_{E_j}\) in Lemma 3.7 then the map \(p'\) is an isomorphism.

**Proof.** We first show \(H_1 \cap H_2 = \mathbb{Q}\). Let \(p\) be a rational prime, then \(p \nmid d_1\) or \(p \nmid d_2\). When \(p \nmid d_j, p\) is unramified in \(H_j\) and thus in \(H_1 \cap H_2\). So every prime \(p\) is unramified in \(H_1 \cap H_2\), and thus \(H_1 \cap H_2 = \mathbb{Q}\). This implies

\[\text{Gal}(H/\mathbb{Q}) \cong \text{Gal}(H_1/\mathbb{Q}) \times \text{Gal}(H_2/\mathbb{Q}).\]
So there is $\sigma \in \text{Gal}(H/\mathbb{Q})$ such that $\sigma|H_j = \sigma_{C_j}$. In particular, $\sigma \in \text{Gal}(H/E)$, which is abelian. By the class field theory, there is an ideal $a$ of $E$ such that $\sigma_a = \sigma$. Let $a_j = N_{E/E_j}a$. Then $\sigma|H_j = \sigma_{a_j}$, and $N(a_1) = N(a_2) = N(a)$. Moreover, one has $C_j = [a_j]$. □

4. Gross and Zagier’s singular moduli factorization formula

We will give a different proof of Gross and Zagier’s factorization formula (Theorem 1.1) in this section. For this, we take $L = M_2(\mathbb{Z})$ with $Q(X) = \det X$, and $W = E$ with $Q_F(x) = \frac{x^2}{\sqrt{D}}$, where $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and $F = \mathbb{Q}(\sqrt{D})$ are as in Section 3. In this case, the lattice $L \cong \mathcal{O}_E$ is unimodular.

**Proof of Theorem 1.1** Recall the identification at the beginning of Section 3 of the product $X_0(1) \times X_0(1)$ of modular curves with the orthogonal Shimura surface of signature $(2, 2)$ and the isotropic vectors $\ell = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$ and $\ell' = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ used for the identification. We also use them as in Theorem 2.2 for Borcherds product expansion. Write

$$j(\tau) - 744 = \sum_{m \geq -1} c(m)q^m,$$

then Borcherds proved in [Bor95]

$$j(z_1) - j(z_2) = \Psi(j(\tau) - 744)$$

which can be checked easily by Theorem 2.2. Notice that $\text{Cl}(K_i) = \text{Cl}(E_i)$ is the ideal class group of $E_i$ and $j(-\tau_i) = j(\tau_i)$. So the map $p'$ in Lemma 3.5 is an isomorphism, and

$$\sum_{(z_1, z_2) \in Z(W)} \log |j(z_1) - j(z_2)| = 4 \sum_{[a_i] \in \text{Cl}(E_i)} \log |j(\tau_1^{a_{1_i}}) - j(\tau_2^{a_{2_i}})|$$

$$= 4 \sum_{[a_i] \in \text{Cl}(E_i)} \log |j(\tau_{a_1}) - j(\tau_{a_2})|.$$ 

Here

$$\tau_{a_i} = \frac{b_i + \sqrt{d_i}}{2a_i}, \quad \text{if } a_i = [a_i, \frac{b_i + \sqrt{d_i}}{2}].$$

So one has by Theorem 2.6

$$-4 \sum_{[a_i] \in \text{Cl}(E_i)} \log |j(\tau_{a_1}) - j(\tau_{a_2})|^4 = C(W, K)a_1(\phi)$$

with $\phi = \text{Char}(\hat{\mathcal{O}}_E)$, and

$$C(W, K) = \frac{|Z(W, \sigma_2^\pm)|}{\Lambda(0, \chi_{E/F})} = \frac{2h(E_1)h(E_2)}{\Lambda(0, \chi_{E_1/Q})\Lambda(0, \chi_{E_2/Q})} = \frac{w_1w_2}{2},$$

where $h(E_i)$ is the class number of $E_i$. By Proposition 2.7 one has

$$a_1(\phi) = \sum_{t \in \mathcal{O}_F^{-1}, \text{totally positive}} a(t, \phi).$$
When \(|\text{Diff}(W,t)| > 1\), \(a(t,\phi) = 0\). When \(\text{Diff}(W,t) = \{p\}\), \(p\) is inert in \(E/F\) and \(\text{ord}_p(t\sqrt{D})\) is even, Proposition \(2.7\) implies

\[
a(t,\phi) = -4 \frac{1 + \text{ord}_p(t\sqrt{D})}{2} \rho(t\sqrt{D}p^{-1}) \prod_{q<\infty} \gamma(W_q) \log(N(p)),
\]

since

\[
\prod_{q \neq p} \frac{W^*_q(0,\phi)}{\gamma(W_q)} = \prod_{q \neq p} \rho_q(t\sqrt{D}) = \prod_{q} \rho_q(t\sqrt{D}p^{-1}) = \rho(t\sqrt{D}p^{-1}).
\]

Here we used the fact that \(\rho_p(t\sqrt{D}p^{-1}) = 1\) when \(p \in \text{Diff}(W,t)\). Next, \(\gamma(W_{\sigma_1}) = -i = -\gamma(W_{\sigma_2})\) implies

\[
\prod_{q<\infty} \gamma(W_q) = \prod_{\text{all primes } v} \gamma(W_v) = 1.
\]

So

\[
a(t,\phi) = -2(1 + \text{ord}_p(t\sqrt{D}))\rho(t\sqrt{D}p^{-1}) \log(N(p)).
\]

Notice that the right hand side in the above identity is automatically zero if we replace \(p\) by other inert primes in \(E/F\) since \(\rho(t\sqrt{D}q^{-1}) = 0\). So we have always

\[
a(t,\phi) = -2 \sum_{p \text{ inert in } E/F} (1 + \text{ord}_p(t\sqrt{D}))\rho(tp^{-1}\partial_F) \log(N(p)).
\]

Putting everything together, and replacing \(t\sqrt{D}\) by \(t\), we obtain the theorem.

**Remark 4.1.** It is easy to check that our formula coincides with \([GZ85, (7.1)]\) and thus their main formula. Indeed,

\[
\sum_{a|O_F} \chi_{E/F}(a) \log N(a) = - \sum_{p \text{ inert in } E/F} \frac{(1 + \text{ord}_p(tO_F))}{2} \rho(tp^{-1}) \log(N(p))
\]

for \(t = \frac{m+\sqrt{D}}{2} \in O_F\) with \(|m| < \sqrt{D}\). To see it, for any fixed integral ideal \(b\) of \(F\), define

\[
f(b) = \sum_{a|b} \chi_{E/F}(a) \log N(a).
\]
Write \( b = \prod_{i=1}^{n} p_i^{e_i} \) with \( e_i > 0 \). Assume \( p_1 \) is inert in \( E/F \) and \( e_1 \) is odd, write \( b_1 = b p_1^{-e_1} \). Then (recall \( \chi_{E/F}(p_1) = -1 \)).

\[
\begin{align*}
\sum_{a_1 | b_1} (-1)^j \chi_{E/F}(a_1) & \left( \sum_{a_1 | b_1} \chi_{E/F}(a_1) \right) \\
& = -1 + e_1 \sum_{a_1 | b_1} \chi_{E/F}(a_1) \\
& = -1 + e_1 \sum_{i=2}^{n} \chi_{E/F}(p_i)^{e_i} \\
& = -1 + e_1 \rho(b_1) \log N(p_1) \\
& = -1 + e_1 \rho(b p_1^{-1}) \log N(p_1).
\end{align*}
\]

In particular, if there is another \( p_i \) (\( i > 1 \)) inert in \( E/F \) with \( e_i \) odd, then \( \rho(b p_1^{-1}) = 0 \) and \( f(b) = 0 \). In our case,

\[
tO_F = \prod_{i=1}^{n} p_i^{e_i}.
\]

Then \( p_i \in \text{Diff}(W, t/\sqrt{D}) \) if and only if \( p_i \) is inert in \( E/F \), and \( e_i \) is odd. When

\[ |\text{Diff}(W, t/\sqrt{D})| > 1, \]

the above argument shows \( f(tO_F) = 0 \), and (4.1) holds as the right hand side of (4.1) is also zero. When \( \text{Diff}(W, t/\sqrt{D}) = \{p\} \), say, \( p = p_1 \), one has

\[
f(tO_F) = -\frac{1 + e_1}{2} \rho(b_1) N(p_1).
\]

The right hand side of (4.1) equals this value too. So (4.1) holds.

5. The Yui-Zagier conjecture for \( \omega_i \)

5.1. Borcherds product for \( \omega_2(z_1) - \omega_2(z_2) \). In this section, let

\[
L = \left( \begin{array}{cc}
\frac{z_1}{2} & \frac{z_2}{2} \\
\frac{z_2}{2} & \frac{z_1}{2}
\end{array} \right)
\]

and \( \Gamma = \Gamma_0(2) \) in this section. It acts on \( L'/L \) trivially, where

\[
L'/L = \left\{ \mu_0 = 0, \mu_1 = e_{21}, \mu_2 = \frac{1}{2} e_{12}, \mu_3 = \mu_1 + \mu_2 \right\}.
\]

Here \( e_{ij} \) is the \( 2 \times 2 \) matrix with the \( (i, j) \) entry 1 and all other entries 0. It is easy to check \( Z(1, \mu_0) = X_{\Gamma_0(2)}^{\Delta} \) in the open variety \( X_K = X_{\Gamma_0(2)} \times X_{\Gamma_0(2)} \).
Take the primitive isotropic vector \(\ell = -e_{12} \in L\) and the vector \(\ell' = e_{21} \in L'\) with \((\ell, \ell') = 1\). Since \((\ell, L) = 2\mathbb{Z}\), we choose \(\xi = 2\ell' \in L\) with \((\ell, \xi) = 2\). In this case,
\[
L'_0 = \{x \in L' : (x, \ell) \equiv 0 \pmod{2}\} = \left\{\left(\frac{a}{2c}, \frac{b}{d}\right) : a, b, c, d \in \mathbb{Z}\right\}, \quad L'_0/L = \{0, \mu_2\}.
\]
One has also
\[
M = L \cap (\mathbb{Q}\ell + \mathbb{Q}\ell') = \{m(a, b) = (\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}) : a, b \in \mathbb{Z}\},
\]
which is self-dual. So the projection \(p\) from \(L'_0/L\) to \(M'/M\) is zero. We further choose \(\ell_M = e_{11}\) and \(\ell'_M = e_{22}\) with \((\ell_M, \ell'_M) = 1\), so \(P = 0\). Finally for a weakly holomorphic modular form \(f \in M'_{0, \omega L}\) with \(\eta(2z) = 2\), \(M = f_{\mu_0} + f_{\mu_2}\).

Now Theorem 2.2 gives the following proposition in this special case.

**Proposition 5.1.** Let
\[
f(\tau) = \sum_{m, \mu} c(m, \mu)q^m \phi_\mu \in M'_{0, \omega L}.
\]
Then there is a meromorphic modular form of two variables \(\Psi(z, z_2, f)\) for \(\Gamma_0(2) \times \Gamma_0(2)\) of parallel weight \(\frac{c(0,0)}{2}\) with the following product expansion near the cusp \(\mathbb{Q}\ell\), with respect to a Weyl chamber \(W\) whose closure contains \(\ell_M = (z_1, z_2)\):
\[
\Psi(z, f) = Ce((\rho(W), f), z)) \prod_{(m, n) \in \mathbb{Z}^2} (1 - q_1^n q_2^m)^{c(mn, 0)} (1 + q_1^n q_2^m)^{c(mn, \mu_2)}.
\]
Here \(q_j = e(z_j)\), and \(|C| = 2^{\frac{c(0,0)}{2}}\).

**Proposition 5.2.** (1) Let \(M'_{0, \omega L}\) be the subspace of \(M'_{0, \omega L}\) consisting of constant vector \(f = \sum a_i \phi_\mu\). Then it is of dimension 2 with a basis \(\{\phi_{\mu_0} + \phi_{\mu_1}, \phi_{\mu_0} + \phi_{\mu_2}\}\).

(2) One has
\[
\Psi(z, \phi_{\mu_0} + \phi_{\mu_1}) = \eta(z_1)\eta(z_2),
\]
\[
\Psi(z, \phi_{\mu_0} + \phi_{\mu_2}) = \sqrt{2} \eta(2z_1)\eta(2z_2),
\]
\[
\Psi(z, \phi_{\mu_2} - \phi_{\mu_1}) = \frac{1}{\sqrt{2}} f_2(z_1) f_2(z_2).
\]
Here \(f_2(z) = \omega_2(z) = \sqrt{2^{\frac{n(z_2)}{n(z)}}}\) is also a famous Weber function.

**Proof.** Recall that \(\text{SL}_2(\mathbb{Z})\) is generated by \(n(1) = (1 \ 1)\) and \(w = (0 \ 1 \ 1 \ 0)\).
\[
n(1)(f) = a_0 e_0 + a_1 e_1 + a_2 e_2 - a_3 e_3 = f
\]
if and only if \( a_3 = 0 \). Next, assuming \( a_3 = 0 \), then
\[
w(f) = \sum a_i \omega_L(w)(e_i) = \frac{1}{2} \left[ \left( \sum a_i \right) e_0 + \sum_{i=1}^{3} (a_0 + a_i - \sum_{j \neq i} a_j) e_i \right] = f
\]
if and only if \( a_0 = a_1 + a_2 \). This proves (1). In such a case, \( f = a_1(e_0 + e_1) + a_2(e_0 + e_2) \).

To prove (2), notice that
\[
\text{Gr}(M) = \{ \mathbb{R} (a_0 \ 0 \ -1) : a > 0 \} \cong \mathbb{R}_{>0}.
\]
Since \( f = a_1(e_0 + e_1) + a_2(e_0 + e_2) \) has no negative term, one sees that \( \text{Gr}(M) \) has only one Weyl chamber, i.e., itself with respect to \( f \). A vector \( \lambda = (m \ 0 \ n) \) satisfies \( (\lambda, W) > 0 \) if and only \( m, n \geq 0 \) but not both equal to 0. One also has
\[
\rho(W, f) = \frac{2a_2 + a_1}{24} (\ell_M + \ell_M).
\]

Now the proposition is clear from Theorem 2.2 if we just take \( C = 2^{e(0,\mu_2)/2} \).

**Proposition 5.3.** Let
\[
f = 12(\phi_{\mu_2} - \phi_{\mu_1}) + \sum_{\gamma \in \Gamma_0(2) / \text{SL}_2(\mathbb{Z})} (212 \omega_2^{-1} + 12) |\gamma \omega_L(\gamma)^{-1}\phi_0| \in M_{0,\omega_L}^!.
\]
Then (5.1)
\[
c(0, \mu_0) = c(0, \mu_1) = c(0, \mu_3) = 0, \quad c(0, \mu_2) = 24,
\]
and
\[
\Psi(z, f) = \omega_2(z_1) - \omega_2(z_2).
\]

**Proof.** Direct calculation gives
\[
f = (q^{-1} - 98028q - 10749952q^2 - 432133182q^3 + \cdots) \phi_{\mu_0} + (-98296q - 10747904q^2 - 432144384q^3 + \cdots) \phi_{\mu_1} + (24 - 98296q - 10747904q^2 - 432144384q^3 + \cdots) \phi_{\mu_2} + (4096q^2 + 1228800q^3 + 74244096q^4 + \cdots) \phi_{\mu_3}.
\]
In particular, (5.1) holds, and \( Z(f) = Z(1, \mu_0) = X_{\Gamma_0(2)}^A \) in \( X_K \). This implies that
\[
g(z_1, z_2) = \frac{\Psi(z_1, z_2, f)}{\omega_2(z_1) - \omega_2(z_2)}
\]
has no zeros or poles in the open Shimura variety \( X_K \), i.e., its divisor is supported on the boundaries \( \{ P \} \times X_0(2) \) and \( X_0(2) \times \{ P \} \), where \( P \) runs through the cusps 0 and \( \infty \) of \( X_0(2) \). We now use Borcherds product expansion to show that \( g(z_1, z_2) \) has no zeros or poles on the boundaries, and thus has to be a constant.

The weakly holomorphic form \( f \) gives rise to two Weyl chambers
\[
\text{Gr}(M) - Z_M(1, \mu_0) = W^\pm,
\]
where
\[
W^\pm = \{ \mathbb{R} (a_0 \ 0 \ -1) : a^\pm 1 > 1 \}.
\]
Lemma 5.4. Assume 

\[ m + n \geq 0, \quad n \geq 0, \quad \text{and} \quad m^2 + n^2 > 0. \]

Direct calculation using (2.15)–(2.18) gives the Weyl vector:

\[ \rho(W^+, f) = -\frac{1}{24} \left( c(0, \mu_0) + c(0, \mu_2) \right) \ell_M + \left( -c(-1, \mu_0) + \frac{c(0, \mu_0) + c(0, \mu_2)}{24} \right) \ell_M = -\varepsilon_{11}. \]

We can take the constant

\[ C = -2^{c(0, \mu_2)/2} = -2^{12}. \]

One has by Proposition 5.1

\[ \Psi(z, f) = -2^{12} q_2 (1 - q_1 q_2^{-1}) \prod_{m,n \geq 0, m+n>0} (1 - q_1^{m} q_2^{n})^{c(mn,0)} (1 + q_1^{m} q_2^{n})^{c(mn,\mu_2)} \]

\[ = 2^{12} (q_1 - q_2) \prod_{m,n \geq 0} (1 - q_1^{m} q_2^{n})^{c(mn,0)} (1 + q_1^{m} q_2^{n})^{c(mn,\mu_2)}. \]

This product formula shows that \( \Psi(z, f) \) has no zeros or poles along the boundary \( \{ \infty \} \times X_{\Gamma_0(2)} \) and \( X_{\Gamma_0(2)} \times \{ \infty \} \).

Since \( \omega_2(z_1) - \omega_2(z_2) \) has the same property, \( g(z_1, z_2) \) has no zeros or poles in these boundaries. Fixing a \( z_2 \in \mathbb{H} \), the function \( g(z_1, z_2) \) of \( z_1 \) has then only zeros or poles at the cusp \( \{ 0 \} \) in \( X_0(2) \), and is thus independent of \( z_1 \): \( g(z_1, z_2) = g(z_2) \). This implies that \( g(z_2) \) has only zeros or poles at the cups 0, and is thus a constant \( g(z_2) = A \). Therefore,

\[ \Psi(z_1, z_2, f) = A (\omega_2(z_1) - \omega_2(z_2)). \]

Comparing the leading coefficients on both sides, one sees that \( A = 1 \). \( \square \)

5.2. Proof of Theorem 1.3. Now we start to prove Theorem 1.3. Under the isomorphism

\( (M_2(\mathbb{Q}), \det) \cong (E, \text{tr}_{E/\mathbb{Q}} \frac{xE}{\sqrt{D}}), \quad (x_4, x_1 x_2) \mapsto \sum x_i e_i, \)

one has

\[ L \cong \mathbb{Z} + \mathbb{Z} \frac{D + \sqrt{D}}{2} + \mathbb{Z} \frac{-d_1 + \sqrt{d_1}}{2} + \mathbb{Z} \frac{d_2 + \sqrt{d_2}}{2}, \]

which is of index 2 in \( \mathcal{O}_E \), but is not an \( \mathcal{O}_F \)-lattice unfortunately. By Proposition 5.3, we have

\[ \omega_2(z_1) - \omega_2(z_2) = \Psi(z, f). \]

Lemma 5.4. Assume \( d_j \equiv 1 \) (mod 8). Then

\[ \iota_j^{-1}(K(\Gamma_0(2))) = \hat{\mathcal{O}}_{E_j}^\times. \]

Proof. We work the case \( j = 2 \). Then case \( j = 1 \) is the same. For \( r = x + y \frac{d_2 + \sqrt{d_2}}{2} \in E_2^\times \),

one has

\[ \iota_2(r) = \left( x + dy \frac{d_2 + \sqrt{d_2}}{2} \right). \]

So \( \iota_2(r) \in K(\Gamma_0(2)) \) if and only if \( y \in 2\hat{\mathbb{Z}} \). This implies

\[ \iota_2^{-1}(K(\Gamma_0(2))) = (\hat{\mathbb{Z}} + 2\hat{\mathcal{O}}_{E_2})^\times. \]
Since $d_2 \equiv 1 \pmod{8}$, 2 is split in $E_2$ and

$$\mathcal{O}_{E_2}^\times = \mathbb{Z}_2^\times \times \mathbb{Z}_2^\times = (1 + 2\mathcal{O}_{E_2}).$$

So

$$(\hat{\mathbb{Z}} + 2\hat{\mathcal{O}}_E) = \hat{\mathcal{O}}_E^\times.$$  

□

This lemma and Lemma 3.8 implies that the class projection $p'$ in Lemma 3.5 is an isomorphism. By Proposition 3.6 one has

$$\tau_{a_j} = b_j + \sqrt{d_j} 2a_j \quad \text{if } a_j = [a_j, b_j + \sqrt{d_j}]_2, 2 \nmid a_j.$$

On the other hand,

$$\omega_2(-\tau_j) = \omega_2(\tau_j - d_j) = \omega_2(\tau_j).$$

So one has again by Proposition 3.7

$$\sum_{(z_1, z_2) \in Z(W)} \log |\omega_2(z_1) - \omega_2(z_2)| = 4 \sum_{[a_j] \in \text{Cl}(E_j)} \log |\omega_2(\tau_{a_1}) - \omega_2(\tau_{a_2})|.$$  

So we have by Theorem 2.6

$$-4 \sum_{[a_j] \in \text{Cl}(E_j)} \log |\omega_2(\tau_{a_1}) - \omega_2(\tau_{a_2})|^4 = C(W, K)[a_1(\phi) + 24a_0(\tilde{\phi})] = 2[a_1(\phi) + 24a_0(\tilde{\phi})],$$  

with $\phi = \text{Char}(\hat{L})$, and $\tilde{\phi} = \text{Char}(\mu_2 + \hat{L})$. Here

$$C(W, K) = \frac{\deg Z(W, z_{a_2}^\pm)}{\Lambda(0, \chi)} = \frac{w_1w_2}{2} = 2.$$  

Now Theorem 1.3 follows from the following lemma which we will prove in next subsection.

**Lemma 5.5.** Let the notation be as above. Then

1. $a_0(\tilde{\phi}) = 0$,
2. $a_1(\phi) = -4 \sum_{m^2 \equiv D \pmod{16}} \sum_{|m| < \sqrt{D}, \text{odd}} \frac{1 + \text{ord}_p(t\mathcal{O}_F)}{2} \rho(tp^{-1}p^{-2}) \log(N(p)).$
5.3. Whittaker functions and Proof of Lemma 5.5.

Lemma 5.6. Let $W = \mathbb{Q}_2^2$ with the quadratic form $Q(x) = \alpha^{-1}x_1x_2$ with $\alpha \in \mathbb{Z}_2^\times$. For $a = 0, 1$, let

$$M_a = \{(x_1, x_2) \in \mathbb{Z}_2^2 : x_1 + x_2 \equiv a \pmod{2}\}$$

and

$$\varphi_a = \text{Char}(M_a), \quad \tilde{\varphi}_a = \text{Char}(\langle \frac{1}{2}, \frac{1}{2} \rangle + M_a).$$

Let $\psi$ be an unramified additive character of $\mathbb{Q}_2$.

1. When $a = 0$, the local Whittaker function $W_{ta}(s, \varphi_a) = 0$ unless $t \in \mathbb{Z}_2$, and

$$\frac{W_{ta}(s, \varphi_0)}{\gamma(W)} = \begin{cases} \frac{1}{2} & \text{if } t \in \mathbb{Z}_2^\times, \\ \frac{1}{2} - 2^{-s} + (1 - 2^{1-s}) \sum_{n=1}^{\infty} 2^{-ns} & \text{if } t \in 2\mathbb{Z}_2, \end{cases}$$

where $o(t) = \text{ord}_2 t$. In particular,

$$\frac{W_{ta}(0, \varphi_0)}{\gamma(W)} = \begin{cases} \frac{1}{2} & \text{if } o(t) = 0, \\ \frac{o(t)-1}{2} & \text{if } o(t) \geq 1. \end{cases}$$

2. When $a = 1$, the local Whittaker function $W_{ta}(s, \varphi_a) = 0$ unless $t \in \mathbb{Z}_2$, and

$$\frac{W_{ta}(s, \varphi_1)}{\gamma(W)} = \begin{cases} \frac{1}{2}(1 - 2^{-s}) & \text{if } t \in \mathbb{Z}_2^\times, \\ \frac{1}{2}(1 + 2^{-s}) & \text{if } t \in 2\mathbb{Z}_2. \end{cases}$$

In particular,

$$\frac{W_{ta}(0, \varphi_1)}{\gamma(W)} = \begin{cases} 0 & \text{if } o(t) = 0, \\ 1 & \text{if } o(t) \geq 1. \end{cases}$$

3. One has

$$W_{ta}(s, \tilde{\varphi}_a) = 0$$

unless $t - \frac{1+2a}{4} \in \mathbb{Z}_2$, in which case it is the constant $\frac{1}{2} \gamma(W)$. In particular,

$$W_0(s, \tilde{\varphi}_a) = 0.$$

Proof. (sketch) By the definition and unfolding, one has

$$\frac{W_{ta}(s, \varphi_a)}{\gamma(W)} = \int_{\mathbb{Q}_2} J_a(b) \psi(-tb)|a(wn(b))|^s \, db$$

$$= \int_{\mathbb{Z}_2} J_a(b) \psi(-tb) \, db + \sum_{n \geq 1} 2^n \int_{\mathbb{Z}_2^\times} J_a(2^{-n}b) \psi(-2^{-n}tb)|a(wn(2^{-n}b))|^s \, db,$$

where

$$J_a(b) = \int_{M_a} \psi(bx_1x_2) \, dx_1 dx_2.$$
Then one checks
\[ J_1(b) = \frac{1}{2} \text{Char}(\frac{1}{2} \mathbb{Z}_2)(b), \]
\[ J_0(b) = \begin{cases} \frac{1}{2} & \text{if } b \in \mathbb{Z}_2, \\ 0 & \text{if } b \in \frac{1}{2} \mathbb{Z}_2^\times, \\ |b|^{-1} & \text{if } b \notin \frac{1}{2} \mathbb{Z}_2, \end{cases} \]
and
\[ |a(wn(b))| = \min(1, |b|^{-1}). \]

Now a direct calculation proves (1) and (2). For (3), one has similarly
\[ W_{ta}(s, \tilde{\varphi}_a) = \int_{\mathbb{Q}_2} \tilde{J}_a(b) \psi(-tb)|a(wn(b))|^s \, db, \]
where
\[ \tilde{J}_a(b) = \int_{\mathbb{Q}_2} \psi(bx_1x_2) \, dx_1 \, dx_2 = \tilde{J}_a^{(0)}(b) + \tilde{J}_a^{(1)}(b). \]

Here (after a simple substitution)
\[ 4 \tilde{J}_a^{(j)}(b) = \int_{\mathbb{Q}_2} \psi(b(\frac{1}{2} + j + 2y_1)(\frac{1}{2} - j + a + 2y_2)) \, dy_1 \, dy_2 \]
\[ = \psi((\frac{1}{2} + j)(\frac{1}{2} - j + a)b) \text{Char}(\mathbb{Z}_2)(b). \]

So
\[ \frac{4W_{ta}(s, \tilde{\varphi}_a)}{\gamma(W)} = \sum_{j=0}^1 \int_{\mathbb{Q}_2} \psi((\frac{1}{2} + j)(\frac{1}{2} - j + a)b) \psi(-tb) \, db \]
\[ = 2 \int_{\mathbb{Q}_2} \psi((\frac{1}{4} + 2a + t) - t)b) \, db \]
\[ = 2 \text{Char}(\frac{1}{4} + 2a + \mathbb{Z}_2)(t). \]

□

To compute \( a_1(\phi) \) and \( a_0(\tilde{\phi}) \), we keep the notation in the proof of Theorem 1.3. Recall that
\[ L = \mathbb{Z} + \mathbb{Z} \frac{D + \sqrt{D}}{2} + \mathbb{Z} \frac{-d_1 + \sqrt{d_1}}{2} + \mathbb{Z} \frac{d_2 + \sqrt{d_2}}{2} \]
is not an \( \mathcal{O}_F \)-lattice as \( \frac{D + \sqrt{D}}{2} \notin L \). So \( \phi \) and \( \tilde{\phi} \) are not factorizable over primes of \( F \), instead one has only
\[ \phi = \phi_2 \prod_{p|2} \phi_p \quad \text{and} \quad \tilde{\phi} = \tilde{\phi}_2 \prod_{p|2} \phi_p, \]
where \( \phi_p = \text{Char}(\mathcal{O}_{E,p}) \) for a prime (ideal) \( p \) of \( F \) not dividing 2, \( \phi_2 = \text{Char}(L_2) \), and \( \phi_2 = \text{Char}(\frac{1}{2} + L_2) \). So we need to take special care at the local calculation at \( p = 2 \). We focus on \( \phi \) and \( a_1(\phi) \) first.

Our assumption implies also that 2 splits in \( E \) completely. Write

\[
2\mathcal{O}_F = p_1p_2, \quad p_i\mathcal{O}_E = \mathfrak{P}_i\mathfrak{Q}_i.
\]

Let \( \sqrt{d} \in \mathbb{Z}_2 \) and \( \sqrt{d_i} \in \mathbb{Z}_2 \) be some prefixed square roots of \( D \) and \( d_i \) respectively with \( \sqrt{d_1}\sqrt{d_2} = -\sqrt{D} \). We identify \( F_{p_i}, E_{\mathfrak{P}_i}, \) and \( E_{\mathfrak{Q}_i} \) with \( \mathbb{Q}_2 \) as follows.

\[
F_{p_i} \cong \mathbb{Q}_2, \quad \sqrt{D} \mapsto (-1)^{i-1}\sqrt{D},
\]

\[
E_{\mathfrak{P}_i} \cong \mathbb{Q}_2, \quad \sqrt{D} \mapsto (-1)^{i-1}\sqrt{D}, \sqrt{d_i} \mapsto \sqrt{d_i},
\]

\[
E_{\mathfrak{Q}_i} \cong \mathbb{Q}_2, \quad \sqrt{D} \mapsto (-1)^{i-1}\sqrt{D}, \sqrt{d_i} \mapsto -\sqrt{d_i}.
\]

With this identification, we can check that \( L_2 = L \otimes_{\mathbb{Z}} \mathbb{Z}_2 \) is given by

\[
L_2 = \{ x = (x_1, x_2, x_3, x_4) \in E_{\mathfrak{P}_1} \times E_{\mathfrak{Q}_1} \times E_{\mathfrak{P}_2} \times E_{\mathfrak{Q}_2} \cong \mathbb{Q}_2^4 : x_i \in \mathbb{Z}_2, \sum x_i = 2\mathbb{Z}_2 \},
\]

with quadratic form

\[
Q(x) = \frac{x_1x_2}{\sqrt{D}} - \frac{x_3x_4}{\sqrt{D}} = Q_{p_1}(x_1, x_2) + Q_{p_2}(x_3, x_4).
\]

The embedding from \( L \) to \( L_2 \) is given by

\[
x \mapsto (\sigma_1(x), \sigma_1(\bar{x}), \sigma_2(x), \sigma_2(\bar{x}))
\]

where \( \sigma_1(\sqrt{d_i}) = \sqrt{d_i} \) and \( \sigma_2(\sqrt{d_i}) = (-1)^i\sqrt{d_i} \). So

\[
L_2 = (M_0 \times M_0) \cup (M_1 \times M_1)
\]

where \( M_a \) is given as in Lemma 5.6. This implies

\[
\phi_2 = \text{Char}(L_2) = \phi_{p_1,0}\phi_{p_2,0} + \phi_{p_1,1}\phi_{p_2,1},
\]

where \( \phi_{p_i,a} \) is \( \varphi_a \) in Lemma 5.6. Correspondingly, we have

\[
\phi = \phi_0 + \phi_1, \quad a(t, \phi) = a(t, \phi_0) + a(t, \phi_1),
\]

where \( \phi_i = \phi_{p_i, i} \phi_{p_2, i} \prod_{p \nmid 2} \phi_p \). Now Proposition 2.7 and the proof of Theorem 1.1 give

\[
(5.4) \quad a(t, \phi_i)
\]

\[
= -4 \sum_{p \text{ inert in } E/F} \frac{1 + \text{ord}_p(t\sqrt{D})}{2} \rho^{(2)}(t p^{-1} \partial_F) \prod_{j=1}^2 W_{i\bar{\mathbb{Z}}_{\mathfrak{P}_j}}(0, \phi_{p_j, i}) \gamma(W_{p_j}) \log(N(p)).
\]

Here \( \psi'(x) = \psi_F(x/\sqrt{D}) \) and

\[
\rho^{(2)}(a) = \prod_{p \nmid 2} \rho_p(a)
\]

as in the proof of Theorem 1.1.
Lemma 5.7. Assume again $d_1 \equiv d_2 \equiv 1 \pmod{8}$. Let $t \in \mathcal{O}_F$ with $\text{tr}_{F/Q}(t) = 1$. Then there is a unique prime ideal $p_t$ with $t\sqrt{D} \in p_t$. Moreover,

$$W_{t\sqrt{D},p_1}^*(0, \phi_{p_1,1}) W_{t\sqrt{D},p_2}^*(0, \phi_{p_2,1}) = 0,$$

and

$$W_{t\sqrt{D},p_1}^*(0, \phi_{p_1,0}) W_{t\sqrt{D},p_2}^*(0, \phi_{p_2,0}) = \frac{1 + \text{ord}_p(t\sqrt{D})}{2} \rho_p(t\sqrt{D}p_t^{-2}) \log(N(p)).$$

Proof. Write $t = \frac{m + \sqrt{D}}{2\sqrt{D}} \in \mathcal{O}_F$. Recall the two natural embeddings $\sigma_i : F \hookrightarrow F_{p_i}, i = 1, 2$. Since $\sigma_1(t\sqrt{D})\sigma_2(t\sqrt{D}) = \frac{m^2 - D}{4} \equiv 0 \pmod{2}, \sigma_1(t\sqrt{D}) - \sigma_2(t\sqrt{D}) \equiv 1 \pmod{2},$ one sees that exactly one of $\text{ord}_{p_i}(\sigma_i(t\sqrt{D}))$ is positive while the other one is zero. For simplicity, let $p_t = p_1$ with $\text{ord}_{p_1}(t\sqrt{D}) \geq 1,$ and $\text{ord}_{p_2}(t\sqrt{D}) = 0$. Then Lemma 5.6 implies $W_{t\sqrt{D},p_2}^*(0, \phi_{p_2,1}) = 0.$

The same lemma also implies (recall $L(1, \chi_{p_i}) = 2$)

$$W_{t\sqrt{D},p_2}^*(0, \phi_{p_2,0}) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \text{ord}_p(t\sqrt{D}) - 1 = \rho_p(t\sqrt{D}p_t^{-2}).$$

Now, one has by Lemma 5.4 and (5.1)

$$a(t, \phi_1) = 0,$$

$$a(t, \phi_0) = -4 \sum_{p \text{ inert in } E/F} \frac{1 + \text{ord}_p(t\sqrt{D})}{2} \rho(tp\partial_Fp_t^{-2}) \log(N(p)).$$

Here $p_t$ is the only prime ideal of $F$ above 2 with $t\sqrt{D} \in p_t$. Replacing $t$ by $t/\sqrt{D}$, one obtains for $t = \frac{m + \sqrt{D}}{2\sqrt{D}} \in \mathcal{O}_F$ with $|m| < \sqrt{D}$

$$a(t/\sqrt{D}, \phi) = -4 \sum_{p \text{ inert in } E/F} \frac{1 + \text{ord}_p(t\mathcal{O}_F)}{2} \rho(tp\mathcal{O}_Fp_t^{-2}) \log(N(p)).$$

The condition $\rho(tp\mathcal{O}_Fp_t^{-2}) \neq 0$ implies $t \in p_t^2$ and so

$$N(t) = \frac{m^2 - D}{4} \equiv 0 \pmod{4}, \text{ i.e. } m^2 \equiv D \pmod{16}.$$
This proves the second identity in Lemma 5.5

\[ a_1(\phi) = \sum_{t = \frac{m + \sqrt{D}}{2}} \sum_{p \text{ inert in } E/F} \frac{1 + \ord_p(tO_F)}{2} \rho(tp^{-1}p^{-2}) \log(N(p)). \]

Now we prove \( a_0(\tilde{\phi}) = 0 \). The same argument as above gives

\[ \tilde{\phi} = \tilde{\phi}_2 \prod_{p \nmid 2} \phi_p, \]

and

\[ \tilde{\phi}_2 = \tilde{\phi}_{p,0} \phi_{p,0} + \tilde{\phi}_{p,1} \phi_{p,1} \]

with \( \tilde{\phi}_{p,i} \) being \( \tilde{\phi}_a \) in Lemma 5.6. So

\[ W_{0,2}(s, \tilde{\phi}_2) = \frac{1}{2} \prod_{a=0}^1 \prod_{i=0}^1 W_{0,p}(s, \tilde{\phi}_{p,i}) = 0 \]

by Lemma 5.6. This implies

\[ W_{0,f}(s, \tilde{\phi}) = 0 \]

and thus \( a_0(\tilde{\phi}) = 0 \) by Remark 2.5 (and \( \phi(0) = 0 \)). This proves Lemma 5.5 and thus Theorem 1.3.

**Remark 5.8.** When \( d_i \equiv 1 \pmod{8} \) are not satisfied, the big CM value formula will still give a factorization formula for the CM values of \( \omega_1(z_1) - \omega_2(z_2) \) although the summation will be over the ring class group of \( E_i \) with conductor 2 when \( d_i \equiv 5 \pmod{8} \) (see Lemma 5.4). We leave the details to the reader.

**Remark 5.9.** The Weber function \( \omega_2 \) has two companions \( \omega_1(\tau) = w(\omega_2) \) and \( \omega_0(\tau) = \omega_1(\tau + 1) \). So the results on \( \omega_2 \) can easily be transferred to its companions \( \omega_0 \) and \( \omega_1 \).

**References**

[AGHMP] F. Andreatta, E. Goren, B. Howard, and K. Madapusi-Pera. Faltings heights of abelian varieties with complex multiplication. *preprint.*

[Ber27] W. E. H. Berwick. Modular invariants. *Proc. London Math. Soc.*, 28, 1927.

[BHK+] J. H. Bruinier, B. Howard, S. S. Kudla, M. Rapoport, and T. H. Yang. Modularity of an arithmetic generating series. *preprint.*

[BKY12] J. H. Bruinier, S. S. Kudla, and T. H. Yang. Special values of Green functions at big CM points. *Int. Math. Res. Not. IMRN*, (9):1917–1967, 2012.

[Bor95] R. E. Borcherds. Automorphic forms on \( \text{O}_{s+2,2}(\mathbb{R}) \) and infinite products. *Invent. Math.*, 120(1):161–213, 1995.

[Bor98] R. E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998.

[Bru02] J. H. Bruinier. *Borcherds products on \( \text{O}(2, 1) \) and Chern classes of Heegner divisors*, volume 1780 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[Bru14] J. H. Bruinier. On the converse theorem for Borcherds products. *J. Algebra*, 397:315–342, 2014.
[BS17] J. H. Bruinier and M. Schwagenscheidt. Algebraic formulas for the coefficients of mock theta functions and Weyl vectors of Borcherds products. *J. Algebra*, 478:38–57, 2017.

[BY06] J. H. Bruinier and T. H. Yang. CM-values of Hilbert modular functions. *Invent. Math.*, 163(2):229–288, 2006.

[BY09] J. H. Bruinier and T. H. Yang. Faltings heights of CM cycles and derivatives of $L$-functions. *Invent. Math.*, 177(3):631–681, 2009.

[Dor88] D. R. Dorman. Special values of the elliptic modular function and factorization formulae. *J. Reine Angew. Math.*, 383:207–220, 1988.

[Gee99] A. Gee. Class invariants by Shimura’s reciprocity law. *J. Théor. Nombres Bordeaux*, 11(1):45–72, 1999. Les XXèmes Journées Arithmétiques (Limoges, 1997).

/GL07] E. Z. Goren and K. E. Lauter. Class invariants for quartic CM fields. *Ann. Inst. Fourier (Grenoble)*, 57(2):457–480, 2007.

[GL12] E. Z. Goren and K. E. Lauter. Genus 2 curves with complex multiplication. *Int. Math. Res. Not. IMRN*, (5):1068–1142, 2012.

[GZ85] B. H. Gross and D. B. Zagier. On singular moduli. *J. Reine Angew. Math.*, 355:191–220, 1985.

[HY12] B. Howard and T. H. Yang. *Intersections of Hirzebruch-Zagier divisors and CM cycles*, volume 2041 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[Kud97a] S. S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997.

[Kud97b] S. S. Kudla. Central derivatives of Eisenstein series and height pairings. *Ann. of Math. (2)*, 146(3):545–646, 1997.

[LV15] K. Lauter and B. Viray. An arithmetic intersection formula for denominators of Igusa class polynomials. *Amer. J. Math.*, 137(2):497–533, 2015.

[Sch09] J. Schofer. Borcherds forms and generalizations of singular moduli. *J. Reine Angew. Math.*, 629:1–36, 2009.

[Yan05] T. H. Yang. CM number fields and modular forms. *Pure Appl. Math. Q.*, 1(2, part 1):305–340, 2005.

[Yan10a] T. H. Yang. An arithmetic intersection formula on Hilbert modular surfaces. *Amer. J. Math.*, 132(5):1275–1309, 2010.

[Yan10b] T. H. Yang. The Chowla-Selberg formula and the Colmez conjecture. *Canad. J. Math.*, 62(2):456–472, 2010.

[Yan13] T. H. Yang. Arithmetic intersection on a Hilbert modular surface and the Faltings height. *Asian J. Math.*, 17(2):335–381, 2013.

[Yan16] Tonghai Yang. Rational structure of $X(N)$ over $\mathbb{Q}$ and explicit Galois action on CM points. *Chin. Ann. Math. Ser. B*, 37(6):821–832, 2016.

[Ye17] D. X. Ye. Difference of a hauptmodul for $\gamma_0(n)$. *Preprint*, 478:38–57, 2017.

[YZ97] N. Yui and D. Zagier. On the singular values of Weber modular functions. *Math. Comp.*, 66(220):1645–1662, 1997.

---

**Department of Mathematics, University of Wisconsin Madison, Van Vleck Hall, Madison, WI 53706, USA**

*E-mail address*: thyang@math.wisc.edu

**Academy of Mathematics and Systems Science, Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing 100190**

*E-mail address*: yinhb@math.ac.cn