Is the composite fermion state of Graphene a Haldane’s Chern insulator?

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The properties of interacting electrons in a half-filled Landau level are well addressed by the Halperin-Lee-Read (HLR) picture of non-interacting composite fermions. We discuss how such a prescription can be applied to a lattice system. Introducing a Chern-Simons’s gauge field with even charge, we construct a many-body wavefunction whose effective single particle Hamiltonian has the action of the HLR theory above half filling of Graphene’s honeycomb lattice with next-nearest neighbor hoppings. Within our construction, we demonstrate that half-filled Landau level in Graphene leads to a state that is described by composite fermions populating Haldane’s Chern insulator model. We propose experiments to detect the Haldane’s gap and the implications to fractional quantum Hall states.

Introduction: Since the discovery of the “1/3-plateau” in resistivity in the 2DEG Al-Ga-As[1], the fractional quantum Hall effect (fQHE) has since puzzled the researchers. Roughly three decades of experimentation has revealed a multitude of the fractional states between filling fractions ν = 0 and ν = 1 of the lowest Landau level. This has been seen in 2DEGs[2,3] with patterns consistent with n/(pn±1) (the Jain series[4]; where p is an even integer and n ∈ {1, 2, ...}), as well as in Dirac systems[5–8] with new features owing to valley degree of freedom in Graphene[9,10]. While the fQHE are only seen with odd denominators, even denominator states are even more subtle and exotic displaying anyonic behavior[11].

Despite the experimental progress, theoretical understanding still has some catching up to do. Major advancements were provided by the formulation of the variational wavefunction by Laughlin[12], the composite fermion theory by Jain[1,13], the Chern-Simons’s (CS) gauge theory of the fQHE[14–16], topological order in fQHE states[17], effect of spin degree of freedom[18], to name a few. While most of these studies laid emphasis on odd-denominators, the physics of even denominators were explored in the famous Halperin-Lee-Read (HLR) theory of the half-filled Landau level[19], the Dirac composite fermion[20], the vortex metal state at even denominators[21]. The half-filled Landau level has itself been an intense subject of investigation[22–27].

A question that has never been answered regarding composite fermion state is what is the flux distribution in a lattice for the corresponding fQH states? In this work, we address precisely this and prescribe a scheme for studying the effects of a CS gauge theory in lattice systems. As a demonstration, we will apply this scheme to Graphene accounting for the next nearest-neighbor (nnn) hopping. Our theory is in fact a two-component lattice generalization of the HLR theory for the single-component half-filled Landau level. We demonstrate, at a lattice level, that the flux distribution within a unit cell that arises from the external field, and that of the CS field, are different. The former is uniformly distributed across the unit cell and we refer to this type of distribution as the Maxwell type. The latter results in a modulation of the flux within the unit cell in way which can be seen as a superposition of a Maxwell type component φMW (which contributes to the total flux through the unit cell) and a intra-unit cell component, φH, such that the net contribution of φH towards the flux through the unit cell is zero and hence is only responsible for modulation. The flux φMW is an orbital component that can be accounted for by minimal coupling of a gauge field (that is given by the 2+1D CS action), where as φH is an intra-unit cell property that should be accounted for independently in the Hamiltonian (the closest analogy is the spin-field coupling of Dirac electrons). We provide the means to arrive at the flux distribution and thus derive the effective low energy Hamiltonians at various filling fractions. We note that most theories addressing Graphene ignore the fact the nnn hopping is about r ~ 0.1 times the nearest-neighbor hopping[26,27]. Accounting for r, we explicitly demonstrate that near half-filling of the lowest Landau level, i.e., the ν = 1/2 state, the low energy theory of carriers hopping on Graphene’s honeycomb lattice can be described by composite fermions populating a Haldane’s Chern insulator[28]. Thus we predict that the ν = 1/2 composite fermion sea in Graphene is topologically nontrivial, possessing a Haldane’s gap in the bulk excitation at the Dirac points of Graphene, and exhibits chiral edge states. We suggest experiments to detect these features. As an outlook, we provide the flux distributions corresponding to various fQHE states of the lattice, and suggest revisiting these states from the current perspective.

Peierls’ substitution with nnn Graphene: The tight binding Hamiltonian for Graphene with nnn hopings can be written in the momentum space as

$$H = -t \begin{pmatrix} r T^A_k & T_k \\ T^B_k & -r T^B_k \end{pmatrix}.$$  \hspace{1cm} \hspace{1cm} (1)

Here, t and rt denote the nearest neighbor (nn) and nnn
hopping amplitudes respectively, and
\[ T_k^A = T_k^B = e^{i k \cdot a_1} + e^{i k \cdot a_2} + e^{i k \cdot a_3} + \text{c.c.,} \]
\[ T_k = e^{i k \cdot e_1} + e^{i k \cdot e_2} + e^{i k \cdot e_3}, \quad (2) \]
where \( e_1 = \frac{a}{\sqrt{3}}(\frac{\sqrt{3}}{2}, -\frac{1}{2}) \), \( e_2 = \frac{a}{\sqrt{3}}(0, 1) \), \( e_3 = -\frac{a}{\sqrt{3}}(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}) \) (the translations from A to B atoms of Graphene), and \( a_1 = a(1, 0) \), \( a_2 = a(\frac{1}{2}, \frac{\sqrt{3}}{2}) \), \( a_3 = a(\frac{1}{2}, -\frac{\sqrt{3}}{2}) \) (the lattice translation vectors), where \( a \) is the constant. Observe that \( T_k^A = T_k^B - 3 \) and \( T_k^L = T_k^T - 3 \).

When an external field \( B \) is applied, one employs the Peierls’ substitution and either resorts to a Hofstadter like scheme\[^2\] to obtain the spectrum; or one performs an expansion of \( H \) around a high symmetry point \( (k_0) \) in the Brillouin zone(BZ) where the chemical potential is expected to lie, in powers of \( k \) (expected to lie, in powers of \( \delta \)). The total Hamiltonian is then of the form
\[ H = E_0 + \sum_{n \geq 1} \Phi_n \left( \frac{k_0 n}{\sqrt{m}} \right) e^{i k_0 n} \psi_n / \sqrt{m}. \]
\[ \phi \]
where \( \psi_n \) is the wavefunction for the \( n \)th sublattice A. The presence of \( \phi \) affects the relative weights of the respective sublattices.

When \( r \neq 0 \), the first thing we note is that upon momentum elongation \( k \rightarrow k_0 + p \), \( T_k^{A} + p \neq T_k^{B} + p - 3 \). This is because,
\[ e^{i p \cdot e_1} e^{-i p \cdot e_2} = e^{i p \cdot a_3} e^{-i 2 \pi \phi_B / \phi_0}, \quad (3) \]
where \( \phi_B = B a^2 / 4 \sqrt{3} \), the flux through the small triangle in Fig. 1 and \( \phi_0 = 2 \pi / e \) the flux quantum \( (h = 1) \). One is first left with a choice of using either \( T_k^{A} + p \) or \( T_k^{B} + p - 3 \) in the Hamiltonian.

This ambiguity is removed by noting that the two choices reflect the two translations from \( A \rightarrow A \): directly \( (T_k^{A}) \) or via \( B \left( T_k^B \right) \), which must not commute as it encloses a flux \( \phi_B \). The choice of \( T_k^{A} + p T_k^{B} + p \) is consistent with \( \phi_B = 0 \) and hence we can conclude that the correct choice is using \( T_k^{A} + p \). In other words, to obtain a Maxwell type flux distribution in a lattice, the momentum elongation must be carried out on the translations corresponding to every allowed hop on the lattice. Carrying this procedure out and expanding the Hamiltonian around the \( K \)-point of Graphene \( [k_0 = \frac{1}{2}(\frac{\sqrt{3}}{2}, 0)] \) to \( O(p^2) \), we find
\[ H_B(r) = -t \left[ r k^2 \sigma_0 - \tilde{p}_x \sigma_x - \tilde{p}_y \sigma_y \right], \quad (4) \]
where \( \tilde{p}_i \equiv \sqrt{3} a p_i / 2 \). This effective low energy Hamiltonian can be exactly solved. The energy spectrum is given by
\[ \frac{E^\pm}{-t} = 2 r \phi_B n \pm \sqrt{2 \phi_B n + r^2 \phi_B^2}, \quad (5) \]
where \( \phi_B \equiv 3 \sqrt{3} e \phi_B \). The wavefunction for the \( A/B \) components for the quantum number \( n \) with the gauge choice of \( A = -B y \hat{x} \) are:
\[ \psi_B^n = \Phi_n(r), \quad \psi_A^n = \Phi_{n-1}(r)/(c \pm \sqrt{1 + c^2}) \]
where \( c = r \sqrt{\phi_B / 2n} \), and
\[ \Phi_n(r) = \frac{1}{\sqrt{2^{n+1} n! \sqrt{n}}} e^{i k x r - \phi_B^2 / 2 H_n(y)}. \quad (6) \]
Here \( y \equiv (y - k_B^B) / B \), \( \phi_B^B = 1 / Bc \), \( k \) is a quantum number corresponding to translations along \( x \), and \( n \in \{1, 2, \ldots\} \). For \( n = 0 \), we have \(-E/t = \phi_B \) and the wavefunction components are \( \psi_B^n = \Phi_B(r) \) and \( \psi_A^n = 0 \).

At the \( K' \) point where \( k_0 = \frac{1}{2} (\frac{4 \pi}{3}, 0) \), our low energy Hamiltonian is the same as \( H_B \) with \( p_x \rightarrow -p_x \). This leads to the same spectrum however, \( \psi_B^{K'} \rightarrow \psi_A^{K'} \) and \( \psi_A^{K'} \rightarrow -\psi_B^{K'} \). Thus the \( n = 0 \) Landau level at \( K' \) point only has sublattice \( A \) occupied. The choice of \( \phi_B / \phi_0 \ll 1 \).

At finite \( r \), there is an interesting situation that arises specifically when \( \phi_B = 0 / 6 \) (see Fig. 1). The total flux though the unit cell is \( \phi_0 \) meaning the phase accumulation around the unit cell is \( 2 \pi \) which restores the translation invariance to the lattice, although the internal hoppings still acquire a Peierls’ phase. The distribution of this phase is demonstrated in the Supplementary material(SM). The effective Hamiltonian (at \( K \) point) is
\[ H_B(r) = -t \left[ r k^2 \sigma_0 - \tilde{p}_x \sigma_x - \tilde{p}_y \sigma_y + r \phi_0 \sigma_z \right], \quad (7) \]
The spectrum is \(-E^\pm / t = r k^2 \pm \sqrt{k^2 + r^2 \phi_B^2} \), with a spectral gap of \( 2 r \phi_0 [t] \).

At such a value of the external field the system becomes equivalent to the Haldane’s model for Graphene\[^2\]. Given the lattice constant of Graphene \( a = 2.46 \AA \), this phenomenon is expected to happen when \( 6 \phi_B = \frac{2 \pi}{3} a^2 = \phi_0 \), resulting in an unrealistic magnetic field of \( B \approx 8 \times 10^3 \)T. Another idealization in arriving at Haldane’s model this way is that this analysis applies to non-interacting electrons in a Graphene, which makes the situation less probable to be realized as we expect strong correlation effects to take over at large fields.

However, below we will show that the Haldane’s model can be realized for composite fermions at much weaker fields when the lattice analog of the HLR theory due to interactions is considered. But before we make this connection, we describe how the CS field affects the lattice and the flux distribution within the unit cell.

The CS field in \( \mu \mu \mu \) Graphene: Consider a Hamiltonian \( H \) and its wavefunction \( \Psi \) satisfying \( H(r) \Psi(r) = E \Psi(r) \), where \( H = H_{kin} + H_{int} \). The many body generalization of this Hamiltonian would be \( \hat{H}(\{r\}) \hat{\Psi}(\{r\}) = E \hat{\Psi}(\{r\}) \). The set of coordinates \( \{r_1, \ldots, r_N\} \) for \( N \) particles. If \( H_{int} = 0 \), then \( \hat{\Psi}(\{r\}) \) is nothing but the Slater determinant constructed out of \( \Psi(r) \), the single particle states of \( H(r) \). However, when \( H_{int} \neq 0 \) such that the problem cannot be treated perturbatively, one can propose a variational solution to the
FIG. 1: Flux distribution in a unit cell of Graphene in uniform external field, CS field, external+CS field. The CS field consists of a Maxwell part $\phi_{MW}$ and a modulated part $\phi_H (= \phi_{MW})$ that does not contribute to the total flux through the unit cell. The total flux through the unit cell is $6(\phi_H + \phi_{MW})$ when external field and CS is present. The extreme right is a schematic spectrum of composite fermions in Graphene at half-filling, consisting of a bulk gap and an edge state.

The problem by introducing composite fermions in terms of original electrons coupled via a CS phase

$$\Psi(\{r\}) = e^{iA(r)} \psi_c(\{r\}), \quad A_{\{r\}} = \kappa \sum_{r' \neq r} \theta_{r'r}, \quad (8)$$

where $\theta_{r'r} = \text{arg}(r' - r)$, and $\kappa$ is an even integer to retain fermionic statistics of the resulting composite particles. It then follows that:

$$H(\{r\})\Psi(\{r\}) = E\Psi(\{r\}),$$

$$\Rightarrow \quad H(\{r\})e^{iA(\{r\})} \psi_c(\{r\}) = e^{iA(\{r\})} E\psi_c(\{r\}),$$

$$\Rightarrow \quad e^{-iA(\{r\})} H(\{r\})e^{iA(\{r\})} \psi_c(\{r\}) = E\psi_c(\{r\}),$$

$$\Rightarrow \quad H(\{r\}, A(\{r\}))\psi_c(\{r\}) = E\psi_c(\{r\}),$$

MFA $\Rightarrow \sum_{\{r\}} H_{\text{kin}}(\{r\}, A(\{r\}))\psi_c^d(\{r\}) = E\psi_c^d(\{r\}), \quad (9)$

where $A_{\{r\}}(\{r\}) = \frac{1}{2} \sum_{r'} A_{r'}$ and sl stands for slater determinant. The standard flux-smeared mean-field approximation $[19]$ (MFA) leads to $H(\{r\}, A(\{r\}) \rightarrow \sum_{\{r\}} H_{\text{kin}}(\{r\}, A(\{r\}).$ This amounts to changing the local $A_{\{r\}}$ to a local single-particle $A^{MW}(\{r\}$. Equivalently, the CS magnetic field defined by $B(\{r\}) \equiv \nabla \times A(\{r\}) = \phi_0 \kappa \sum_{r' \neq r} \delta(r' - r)$ is approximated by a uniform $B = \phi_0 \kappa \rho_{2D} = \nabla \times A^{MW}(\{r\). This is the many-body version of the field theories considered in Refs. $[12, 19]$. This formalism has been used in Refs. $[30, 54]$ to treat hard-core bosons as fermions with an odd $\kappa$ leading to chiral spin-liquid behaviour in honeycomb and Kagome lattices. The non interacting single particle Hamiltonian resulting after MFA is $H[\{r\}, A^{MW}(\{r\]$ whose eigenfunctions are the composite fermions.

We iterate the point made in Refs. $[31, 34]$ that since the CS field produces a flux that is bound to a particle, the flux enclosed within the space of the particles in a unit cell must be zero. The net flux that arises from the MFA must then be re-distributed to the part of the unit cell that does not include any internal hops. Thus, the regions bounded by internal hops that are entirely within the unit cell (defined as loops having at the most one shared edge with external cells) must enclose zero flux. This means that in addition to a non-zero flux through the unit cell (which is accounted for my $\phi_{MW}$ in Fig. 1), we need a component $\phi_H = \phi_{MW}$ that ensure the flux through any internal loop is zero (see Fig. 1 in the SM for an extended construction). In the lattice, this translates to the fact that the direct $A \rightarrow A$ translation and the one mediated through a $B$ atom commute (which is different from the case of uniform field). Also note that while $\phi_{MW}$ can be accounted for by momentum elongation, $\phi_H$ cannot, and this results in an internal degree of freedom which is to be seen as arising from the many-body aspect of the problem. This results in the effective Hamiltonian (at the $K$ point) to be

$$H_{\text{CS}}(\{r\}) = \sum_{p} \left[ \vec{p} \cdot \sigma_0 - \vec{p} \cdot \sigma_x - \vec{p} \cdot \sigma_y + r\phi_H \sigma_z \right], \quad (10)$$

where $\vec{p} = \sqrt{3}d[-i\vec{\sigma}_x + eA^{MW}(\{r\})/2,$ and $\phi_H = 3\sqrt{3}e\phi_H$. Further, $[\vec{p}_x, \vec{p}_y] = 3\sqrt{3}eA^{MW}$. Since the case of CS requires $\phi_{MW} = \phi_H$, it is possible to compact the Eq. (10) to

$$H_{\text{CS}}(\{r\}) = -t \sum_{\{\sigma\}} \left[ r \{\vec{p} \cdot \sigma\}^2 - \vec{p} \cdot \sigma \right], \quad (11)$$

where the dot-product is 2-dimensional. This means that the Hamiltonian for nnn Graphene subject to a CS field can be written as $H_{\text{CS}} = -t [H_{K0} + p]^2 - tH_{K0} + p$. The important constraint arising form the CS nature of the problem is that the flux through the unit cell is $\phi_{MW} = 2\pi \kappa \rho_{2D}$ where $\rho_{2D}$ is the number of atoms per unit cell and $\nu_l$ is deviation from half-filling per site. The energy spectrum is:

$$E_{\pm}^{\pm} = 2r[\phi_{MW} |n \pm \sqrt{2}| \phi_{MW} |n], \quad (12)$$

where $\phi_{MW} = 3\sqrt{3}e\phi_{MW}$, $n \in \{1, 2, ...\}$ and the wavefunctions are $\psi_{B} = \Phi_n(\{r\}$ and $\psi_{A} = \pm \Phi_{n-1}(\{r\}. For n = 0, E = 0 and $\psi_B = \Phi_0(\{r\}$ and $\psi_A = 0$. At $K'$ point $p_x \rightarrow -p_x$ and $\psi_B \rightarrow \psi_K^A$, $\psi_A \rightarrow -\psi_K^B$. In Ref. $[33]$, $\kappa$ formed spontaneously due to interactions in a flat band. Here we have another flat band scenario (the partially filled LLL) and we impose that $\kappa$ is formed due to interactions. However to arrive at the flat band, we must first account for the large external field. In what follows we discuss precisely this idea and show that this is the analog of the HLR theory for a two component system (Graphene with nnn hopping in this case).
Composite Fermions in Haldane’s model: Having argued that $\kappa$ can be non-zero due to many-body effects in a flat band, we have to now construct a theory of the composite fermions (as introduced above) in the presence of an external field. The ansatz of $\kappa$ being even for our lattice formalism is the same one used in the composite fermion and HLR theories. The difference with standard HLR is that we now have to account for three types of fluxes within the unit cell: $\phi_B$, $\phi_{MW}$ and $\phi_H$. The first two produce a field that grows with area and can be accounted for by momentum elongation, while $\phi_H$ needs to be introduced at the level of matrix elements for the Hamiltonian (see SM for detailed construction). The resulting theory is a fermion model populating Haldane’s model coupled to the CS action that was generated from flux attachment. This “statistical” CS term is not different from the one obtained in previous theories in the literature\cite{24,25}. The distinguishing feature however is the flux distribution within the unit cell which is shown in Fig. 1. The resulting low-energy Hamiltonian around the $K$-point is:

$$H_{HLR}(\mathbf{r}) = -t \left[ \hat{p}^2 \sigma_0 - \hat{p}_x \sigma_x - \hat{p}_y \sigma_y + r \phi_h \sigma_z \right],$$

where $\mathbf{p} = -i \partial_x + e\mathbf{A}_B + e\mathbf{A}_{MW}$, $\nabla_x \times \mathbf{A}_B = \mathbf{B}$, which leads to the flux $\phi_B$, and $\nabla_x \times \mathbf{A}_{MW} = \mathbf{B}_{MW}$ which leads to the flux $\phi_{MW} = \phi_H = \phi_0 \kappa n_{uc} \nu_l$ from the induced CS field. Here $\kappa = -2 \text{sgn}(B)$, where $\text{sgn}(B)$ simply indicates that the sign of $\kappa$ is such that the induced field opposes the external field. Just like in the HLR theory, the net ‘orbital’ field (resulting from the vector potential) experienced by the composite fermions is $\mathbf{B}_{eff} = \mathbf{B} + \mathbf{B}_{MW} = \phi_0 \rho_{2D} (1/\nu + \kappa) \mathbf{B} = \mathbf{B} (1 - 2\nu)$. But unlike the HLR theory, the two component nature of the problem, causes the composite fermions to experience an additional field that acts oppositely on the two atoms in the unit cell. This field is denoted by the flux $\phi_H$ and is analogous to the ‘spin’ coupling of the Maxwell field to the true spin-up and spin-down fermions. The resulting spectrum is:

$$E_{\pm} = 2r|\phi_b + \phi_{mw}| n \pm \sqrt{2|\phi_b + \phi_{mw}| n + r^2 \phi_b^2},$$

where $n \in \{1, 2, \ldots\}$, and for $n = 0$ only $E^+$ is present. The wavefunctions are similar to the solutions for uniform field with $c \to \hat{c} = r \sqrt{\phi_b/2|\phi_b + \phi_{mw}|/\phi_b}$. Note that Eq. (13) is different from Eq. (10) in the definition of $\mathbf{p}$. Due to the presence of the external field, we cannot compact the Hamiltonian with a $(\sigma \cdot \mathbf{p})^2$ term which yields $\hat{p}^2 \sigma_0 + 3\sqrt{3e}(\phi_{MW} + \phi_B) \sigma_z$ (the two translations do not commute).

There are a number of points that need emphasis:

- The resulting flux distribution in Fig. 1 at $\nu = 1/2$ (with $\phi_{mw} = -\phi_b$) suggests that this is indeed the distribution considered by Haldane.$^{23}$

- Just like the Landau-problem, Eq. (14) is to be seen as a solution to the composite fermions in a weak $\mathbf{B}_{eff}$, which is realized by small departures from half-filling.

Noting that $\phi_B = \phi_0 (n_{uc} \nu_l)/6\nu$ and $\phi_{MW} = \phi_0 (n_{uc} \nu_l) \kappa/6$, treating $\phi_B$ and $\nu_l$ as external control variables, the $\nu = 1/2$ condition is realized whenever $\nu_l$ and $\mathbf{B}$ satisfy $\phi_0 \kappa (n_{uc} \nu_l) = \sqrt{3} Ba^2/2$. Departures from $\nu = 1/2$ can thus be achieved by slightly changing the field or $\nu_l$ (which can be achieved by changing the chemical potential). This mismatch creates the effective field that the composite fermions are subjected to. In fact, for small deviations such that $\phi_b = -\phi_{mw} + \delta$ ($\delta \ll \phi_b$), we get $-E_{\pm}/t \approx \pm r\phi_b (1 \pm 2n|\delta|/\phi_b)$.

![Fig. 2: Flux distribution in a unit cell of Graphene for various FQHE states given by the Jain series.](image)

We wish to point out that our composite fermion theory is an HLR analog for a two component system (in this case it turns out to be a Dirac system). An important characteristic of the composite fermion state in Graphene is the following: it implies formation of the Haldane’s Chern insulator populated by composite fermions, which, in turn, are coupled to the fluctuating “statistical” CS gauge field with the coefficient $\propto 1/2\pi\kappa$. The low-energy field theory description of this state can be obtained upon integration over fermionic degrees of freedom. This will yield an additional CS term, which, combined with the statistical term will have a coefficient $\propto (1/2\pi\kappa + 2\text{sgn}(B)/8\pi) = 0$. The low-energy theory is thus of Maxwell type, preserving time reversal symmetry. This is not quite the Dirac composite fermion construction by Son.$^{21}$ for a non-Dirac system in a magnetic field. Such a construction for nnn Graphene is set a future goal and is beyond the scope of this work.
Implication for fractional quantum Hall states: This prescription also works for any other fQHE state. Consider the Jain series: \(\nu/(2n+1)\) and \((n+1)/(2n+1)\). The flux combination \(\phi_0 + \phi_{nw} = \pm \phi_0/(2n+1)\). The resulting flux distributions for some of the fQHE states are shown in Fig. 2. The low energy Hamiltonian for small deviations around each \(n\) can be derived by appropriately enlarging the unit-cell (as in the original Hofstadter construction\[29\]): with the special care that \(\phi_H\) has to be accounted for every matrix element of the Hamiltonian for the enlarged unit cell.

Experimental consequences: A naive application of single-component HLR theory would have predicted that at the \(\nu = 1/2\) state in Graphene must have a Fermi-surface(FS) corresponding to non-zero \(v_t\). Our theory suggests that in addition to the FS, we must see a spectral gap in the bulk consistent with the Haldane’s model. For a density of \(\rho_{2D} = \bar{\rho} \times 10^4\text{cm}^{-2}\), this gap is \(\approx r\bar{\rho}l/\text{meV}\) (where \(r\) must be in eV); and this happens at \(B \approx 8\bar{\rho}\). In Graphene, \(t \approx 3\text{eV}\) and \(r \approx 0.1\)\[28\]. If \(\bar{\rho}\) is tuned from \(1 - 10\), then for \(B \approx 8-50\), we should see a Haldane gap of 0.3-3 meV. Further, \(v \approx 1 \times 10^4\text{ms}^{-1}\) implies \(E_F \approx 8\sqrt{\bar{\rho}}\text{meV}\) which will range from 8-24meV. The energy scale suggests that the Haldane gap can be probed by LDOS measurements via STM. Due to correspondence with Haldane’s model, the composite fermion bands are topological with Chern number \(\pm 1\) and thus have chiral edge states which can be measured via tunneling into the edge\[34–38\] and via measuring the quantized thermal transport due to a Luttinger liquid description of edge states.\[39\].

Following our theory, we also predict that when \(r > 1/6\), the minimum of the spectrum at the \(\Gamma\)-point is non-dispersive (see SM for derivation). This result can be verified for fermionic/bosonic cold atoms on honeycomb lattice at very low densities.

Conclusion: We have demonstrated that when the variational scheme that was applied to fermionization of bosons in flat bands in Ref. 34 is applied to a flat band of fermions, we get the composite fermion theory of fQH states. We showed that this theory, applied to Graphene with mn hopping, leads to a model of composite fermions living on Haldane’s model of Graphene. This perspective not only provides an internal flux distributions within the unit cell but also the effective wave-functions for fQH states. This unified picture of treating bosonic and fermionic systems can lead a substantial progress in materials modelling and searching for new quantum phenomena.

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For definiteness, we shall work in the symmetric gauge where $A(\mathbf{r}) = -(y \mathbf{x} + x \mathbf{y}) B/2$. The phase accumulated on a bond upon traversing in the direction $g_n$ is given by $A_{g_n} = \int_{g_n} A \cdot d\mathbf{l}$. In the lattice problem, we may treat $A$ to be constant for a unit translation of $g_n$, where $g_n \in \{e_{1,2,3}, a_{1,2,3}\}$. The directions of these vectors are illustrated in Fig. 3. This leads to $A_{g_n} = A_{g_n} + (-y g_x + x g_y) B a / 2$, where $A_{g_n}$ refers to phase accumulation that can be gauged out and is the same contribution from every lattice point. Thus, $A_{e_1} + A_{e_2} + A_{e_3} = 0$ and $A_{a_1} = A_{a_2} + A_{a_3}$. Once an origin is picked, we can start assigning $A^{a/b}$ to denote phase accumulation originating from atom $a/b$ of the lattice. Thus for a given bond along $g_n$, the phase can be written as $A_{g_n} = A_{g_n}^{a/b} + \Delta A_{g_n}$, where $A_{g_n}^{a/b}$ denotes the value of the bond at the origin, and $\Delta A_{g_n}$ denotes the value at a bond separated from the origin by $(\Delta x, \Delta y)$. These quantities are expressed as (in the symmetric gauge):

\[
\Delta A_{a_1} = -\phi_{MW} \left( \frac{\sqrt{3} \Delta y}{a} \right),
\]

\[
\Delta A_{a_2} = \phi_{MW} \left( \frac{3 \Delta x - \sqrt{3} \Delta y}{a} \right),
\]

\[
\Delta A_{a_3} = -\phi_{MW} \left( \frac{3 \Delta x + \sqrt{3} \Delta y}{a} \right),
\]

\[
\Delta A_{e_1} = -\phi_{MW} \left( \frac{\Delta x}{a} + \frac{\sqrt{3} \Delta y}{a} \right),
\]

\[
\Delta A_{e_2} = \phi_{MW} \left( \frac{2 \Delta x}{a} \right),
\]

\[
\Delta A_{e_3} = -\phi_B \left( \frac{\Delta x}{a} - \frac{\sqrt{3} \Delta y}{a} \right).
\]

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**PEIERLS PHASES FOR UNIFORM AND CHERN SIMONS’ FIELDS**

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To initialize the bonds starting from the \(a/b\) atoms belonging to the lattice point at the origin:

\[
\begin{align*}
\tilde{A}_{a_1}^a &= A_{a_1} + \phi_H, \\
\tilde{A}_{a_2}^a &= A_{a_2} - \phi_H, \\
\tilde{A}_{a_3}^a &= A_{a_3} - \phi_H, \\
\tilde{A}_{a_1}^b &= A_{a_1} + \phi_{MW} - \phi_H, \\
\tilde{A}_{a_2}^b &= A_{a_2} + 2\phi_{MW} + \phi_H, \\
\tilde{A}_{a_3}^b &= A_{a_3} - \phi_{MW} + \phi_H, \\
\tilde{A}_{e_1}^b &= A_{e_1}, \\
\tilde{A}_{e_2}^b &= A_{e_2} + \phi_{MW}, \\
\tilde{A}_{e_3}^b &= A_{e_3} - \phi_{MW}.
\end{align*}
\]

And finally \(A_{g_n} \equiv \mathbf{A}' \cdot \mathbf{g}_n\), where \(\mathbf{A}'\) is some vector which will be gauged out and can be set to zero for our purposes. The extended scheme is summarized in Fig. 3. Note that any field that grows with area will be treated as \(\phi_{MW}\) and the internal modulation will be treated as \(\phi_H\).

**EFFECTIVE HAMILTONIANS**

Following the prescription in the above section, are able to assign the phases to every matrix element involved in the Hamiltonian. The Hamiltonian of nearest neighbor Graphene in an external field can be derived from the lattice Hamiltonian as \(H \to H_B\) where

\[
\begin{align*}
H_{B1}^{11} &= r \{3 + 2 [\cos(k_{0,1} - k_{0,2} + p_1 - p_2) \\
&\quad + \cos(k_{0,2} - k_{0,3} + p_2 - p_3) \\
&\quad + \cos(k_{0,3} - k_{0,1} + p_3 - p_1)]\}, \\
H_{B2}^{12} &= r \{3 + 2 [\cos(k_{0,1} - k_{0,2} + p_1 - p_2) \\
&\quad + \cos(k_{0,2} - k_{0,3} + p_2 - p_3) \\
&\quad + \cos(k_{0,3} - k_{0,1} + p_3 - p_1)]\}, \\
H_{B1}^{12} &= T_{k_0 + p}, \\
H_{B2}^{21} &= T^{\ast}_{k_0 + p},
\end{align*}
\]

where \(x_i = \mathbf{x} \cdot \mathbf{e}_i\), \(\mathbf{k}_0\) is a point about which the semi-classical momentum elongation is carried out. It is understood that \(H_B\) is to be expanded to \(O(p^2)\). Here the elongated momentum \(p\) involves \(\mathbf{A}^B\) such that \(\nabla \times \mathbf{A}^B = \mathbf{B}\).

When we consider the case of a Chern-Simons’ (CS) field, we have to account for \(\phi_H\). The Hamiltonian matrix elements look like

\[
\begin{align*}
H_{CS}^{11} &= r \{3 + 2 [\cos(k_{0,1} - k_{0,2} + p_1 - p_2 - e\phi_H) \\
&\quad + \cos(k_{0,2} - k_{0,3} + p_2 - p_3 - e\phi_H) \\
&\quad + \cos(k_{0,3} - k_{0,1} + p_3 - p_1 - e\phi_H)]\}, \\
H_{CS}^{22} &= r \{3 + 2 [\cos(k_{0,1} - k_{0,2} + p_1 - p_2 + e\phi_H) \\
&\quad + \cos(k_{0,2} - k_{0,3} + p_2 - p_3 + e\phi_H) \\
&\quad + \cos(k_{0,3} - k_{0,1} + p_3 - p_1 + e\phi_H)]\}, \\
H_{CS}^{12} &= T_{k_0 + p}, \\
H_{CS}^{21} &= T^{\ast}_{k_0 + p}.
\end{align*}
\]

Here the elongated momentum \(p\) involves \(\mathbf{A}^{MW}\) such that \(\nabla \times \mathbf{A}^{MW} = \mathbf{B}^{MW}\). Note also that the \(a/b\) atoms experience different signs of the flux \(\phi_H\) although they originate from the same field \(\mathbf{A}^{MW}\).

To address the HLR case, we have to account for the external field and the CS field. This results in a very similar looking Hamiltonian as in Eq. 31 but with \(p\) containing \(\mathbf{A}^B\) and \(\mathbf{A}^{MW}\), and \(\phi_H\) is still only generated from \(\mathbf{A}^{MW}\). The flux \(\phi_B\) is generated from \(\mathbf{A}^B\) and the flux \(\phi_{MW}\) is generated from \(\mathbf{A}^{MW}\).

When \(\phi_B = -\phi_{MW} = \phi_h\), we realize the Haldane’s model of flux distribution in the unit cell. The extended scheme of the flux distribution shown in Fig. 1 of the main text (MT), is shown in Fig. 4 in this text.
FIG. 3: Extended scheme outlining the phase accumulation in addition to $\int \mathbf{A} \cdot d\mathbf{l}$ along each bond. The field $\mathbf{A}$ is written in symmetric gauge. The figure is split into (a) and (b) for clarity. The flux $\phi_{MW}$ adds up with area, whereas the flux $\phi_H$ denotes flux modulation within the unit cell such that net contribution from $\phi_H$ to the unit cell is zero. The presence of $\phi_H$ does not break translational symmetry.

SPECTRUM AROUND $\Gamma$-POINT

At the $\Gamma$-point the effective low energy Hamiltonian for all cases discussed in the text is given by

$$H^\Gamma = -t \begin{pmatrix} r(9 - 2\bar{p}^2) & 3 - \frac{1}{3}\bar{p}^2 \\ 3 - \frac{1}{3}\bar{p}^2 & r(9 - 2\bar{p}^2) \end{pmatrix}. \tag{32}$$
The spectrum is given by

\[ E_{\pm}^{\pm} = 9r \pm 3 - 2 \left( r \pm \frac{1}{6} \right) (2n + 1)\phi_b, \]  

(33)

and the wavefunctions are \( \psi_A^{\pm} = \pm \psi_B^{\pm} = \Phi_n(r) \), with \( n \in \{0, 1, 2, \ldots\} \). Thus for \( r < 1/6 \), the energy of the lowest Landau level, which can occur at any \( n \), linearly increases with field. However, for \( r > 1/6 \), the spectrum in Eq. (33) suggests that the energy goes down. This is indeed what happens for the uniform field case. However, for the CS case with \( r > 1/6 \), the minimum of the energy for any \( \{n, \phi_{mw}\} \) is actually locked at \( E = -|t|/4r \). This means that the lowest energy level of the spectrum is dispersionless with the field \( \phi_b \). This is not evident from Eq. (33) due to the fact that we expanded to \( O(p^2) \).

This minimum is guaranteed because the Hamiltonian is of the form \( rG_p^2 + G_p \). This means the eigenvectors of \( G_p \) are also the eigenvectors of \( G_p^2 \), with squared eigenvalues. The quadratic form suggests that there is a minimum of the energy for any eigenvalue. For \( r < 1/6 \), this minimum energy scales with the field as \( E = (1/6 - r)\phi_b \), but for \( r > 1/6 \) the minimum is at \( E = t/4r \) for infinite pairs of \( \{n, \phi_{mw}\} \). Note that this effect of CS field also applies to the \( K/K' \) point analysis. This we should expect a dispersionless band around the center of the spectrum. These cases are demonstrated in Fig. 5.
FIG. 5: The Hofstadter spectrum for a CS field acting on Graphene with \( r = 0.11 \) (left) and \( r = 0.22 \) (right). Observe the linear scaling and the flatness of the lowest energy level in the two plots.