Rethinking Connes’ approach to the standard model of particle physics via non-commutative geometry

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Abstract
Connes’ non-commutative geometry (NCG) is a generalization of Riemannian geometry that is particularly apt for expressing the standard model of particle physics coupled to Einstein gravity. In a previous paper, we suggested a reformulation of this framework that is: (i) simpler and more unified in its axioms, and (ii) allows the Lagrangian for the standard model of particle physics (coupled to Einstein gravity) to be specified in a way that is tighter and more explanatory than the traditional algorithm based on effective field theory. Here we explain how this same reformulation yields a new perspective on the symmetries of a given NCG. Applying this perspective to the NCG traditionally used to describe the standard model we find, instead, an extension of the standard model by an extra $U(1)_{B-L}$ gauge symmetry, and a single extra complex scalar field $\sigma$, which is a singlet under $SU(3)_C \times SU(2)_L \times U(1)_Y$, but has $B - L = 2$. This field has cosmological implications, and offers a new solution to the discrepancy between the observed Higgs mass and the NCG prediction.

Introduction
Connes’ non-commutative geometry (NCG) [1, 2] is a generalization of Riemannian geometry which also provides a particularly apt framework for expressing and geometrically reinterpreting the action for the standard model of particle physics, coupled to Einstein gravity [3–12] (for an introduction, see [13, 14]). In a recent paper [15], we suggested a simple reformulation of the NCG framework, and pointed out three key advantages of this reformulation: (i) it unifies many of the traditional NCG axioms into a single, simpler axiom; (ii) it immediately yields a further generalization, from non-commutative to non-associative geometry [16]; and (iii) it resolves a key problem with the traditional NCG construction of the standard model, thereby making the NCG construction tighter and more explanatory than the traditional one based on effective field theory [17].

Here we report the discovery of three crucial and unexpected consequences of the reformulation in [15]. (i) First, it yields a new notion of the natural symmetry associated to any non-commutative space, and the action functional that lives on that space. (ii) Second, when we work out the realization of this symmetry for the non-commutative geometry used to describe the standard model of particle physics we find that the usual $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry is augmented by an extra $U(1)_{B-L}$ factor. (iii) Third, as a consequence of this additional gauge symmetry, we find the standard model field content must be augmented by the following two fields: a $U(1)_{B-L}$ gauge boson $G_{\mu}$, and a single complex scalar field $\sigma$ which is an $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet and has charge $B - L = 2$.

The scalar field $\sigma$ has important phenomenological implications. (i) First, although the traditional NCG construction of the standard model predicted an incorrect Higgs mass ($m_h \approx 170$ GeV), several recent works [18–21] have explained that an additional real singlet scalar field $\sigma$ can resolve this problem, and also restore the stability of the Higgs vacuum. Our $\sigma$ field, although somewhat different (since it is complex, and charged under $B - L$), solves these same two problems for exactly the same reasons (as may be seen in the $U(1)_{B-L}$ gauge where $\sigma$ is real). (ii) Furthermore, precisely this field content (the standard model, extended by a right-handed neutrino in each generation of fermions, plus a $U(1)_{B-L}$ gauge boson $G_{\mu}$, and a complex scalar field $\sigma$ that is a singlet
under $SU(3)_C \times SU(2)_L \times U(1)_Y$ but carries $B - L = 2$) has been previously considered \cite{22, 23} because it provides a minimal extension of the standard model that can account for several cosmological phenomena that may not be accounted for by the standard model alone: namely, the existence of dark matter, the cosmological matter–antimatter asymmetry, and the scale invariant spectrum of primordial curvature perturbations.

### Symmetries in NCG: a new perspective

Before turning to NCG, let us recap three key points about ordinary Einstein gravity. (i) A geometry is specified by two pieces of information: a manifold $\mathcal{M}$ (which specifies the differential topology), and a metric $g_{\mu
u}$ (which specifies the geometrical information—distances, angles, curvature). (ii) To this geometry, we then assign the Einstein–Hilbert action $S = M^2_n \int d^4x \; g^{1/2}R$. (iii) The symmetries of this theory are the diffeomorphisms of $\mathcal{M}$: automorphisms of the differential topology that leave $S$ invariant. Now let us explain how to recast and extend these three statements in the NCG context.

(i) In NCG, a geometry is traditionally specified by a so-called ‘real spectral triple’ $[A, H, D, J, \gamma]$, consisting of a $*$-algebra $A$, a hermitian operator $D$, a hermitian unitary operator $\gamma$, and an anti-unitary operator $J$, all of which act as operators on a Hilbert space $H$ and are constrained to satisfy a list of axioms that relate them to one another (see sections 1 and 2 in \cite{13} for an introduction). Reference \cite{15} shows that these various elements naturally fuse to form a new algebra $B$. This reformulation in terms of $B$ is more unified (in the sense that many traditional NCG axioms then follow from the single requirement that $B$ is an associative $*$-algebra) and more general (in the sense that the new formalism continues to cohere even when the underlying algebras, $A$ and $B$, are taken to be non-associative). In this generalization of ordinary differential geometry, we can think of $B$ as carrying the information about the differential topology, while $D$ carries the information about the metric.

(ii) To this geometry, we assign the so-called spectral action $S = \text{Tr} \{ f(D/A) \} + \langle h | D | h \rangle$, where $A$ is a constant (with units of energy), $f(x)$ is a function from $\mathbb{R} \rightarrow \mathbb{R}$ that vanishes sufficiently rapidly for $x \gg 1$, and $h$ is an arbitrary element of $H$ (see section 3 in \cite{13} for details)\footnote{Here we have written the fermionic action as $\langle h | D | h \rangle$ for simplicity, but the precise form depends on the KO dimension—see \cite{10} and section 16.2 in \cite{24}.}.

(iii) Now we want to characterize the symmetries of this theory, and how those symmetries act on the fields. Here we come to this paper’s new contribution: the reformulation in terms of $B$ sheds new light on this issue.

To begin, we briefly recap how $B$ is constructed (for details, see \cite{15}). First, from $A$, we generate $\Omega A = \Omega^2 A \oplus \Omega^3 A \oplus \Omega^4 A \oplus \ldots$, the differential graded $*$-algebra of forms over $A$. Then, from $\Omega A$ and $H$, we construct the $*$-algebra $B = \Omega A \otimes H$ by equipping its elements $b = \omega + h$ and $b' = \omega' + h'$ with the product

$$bb' = (\omega + h)(\omega' + h') = \omega\omega' + oh' + ho' + hh'$$

and the anti-automorphism

$$b^* = \omega^* + Jh,$$  

where $\omega\omega' \in \Omega A$ is the product inherited from $\Omega A$, while $oh' \in H$ and $ho' \in H$ are bilinear products that define the left-action and right-action of $\Omega A$ on $H$. Notice that $B$, like $\Omega A$, is also a graded algebra, where we may think of $H$ as $\Omega^0 A$. (The interesting consequences of this simple observation will be explained in \cite{25}.)

Now we can easily characterize the relevant symmetries: as in the Riemannian case, they are the automorphisms of the differential topology that leave the action invariant. In the NCG context, the automorphisms of the differential topology are simply the automorphisms of the graded $*$-algebra $B$—i.e. the invertible linear transformations $\alpha : B \rightarrow B$ that also preserve the grading, product and $*$-operation on $B$:

$$\alpha = \alpha_0 \oplus \alpha_1 \oplus \alpha_2 \oplus \ldots \quad (\alpha_0 : \Omega^n A \rightarrow \Omega^n A),$$  

$$\alpha (bb') = \alpha (b) \alpha (b'),$$  

$$\alpha (b^*) = \alpha (b)^*.$$  

The fact that diffeomorphisms are either orientation-preserving or orientation-reversing translates to the NCG condition that $\alpha_0 : \Omega^n A \rightarrow \Omega^n A$ (i.e. $\alpha_0 : H \rightarrow H$) either commutes or anti-commutes with the orientation $\gamma$.
\[ \alpha_\infty \gamma \alpha_\infty^{-1} = \pm \gamma, \quad (2d) \]
while the condition that the spectral action is invariant translates to the requirement that \( \alpha_\infty \) is unitary
\[ \alpha_\infty^* = \alpha_\infty^{-1}. \quad (2e) \]

Next we translate our conditions (2a)-(2c) on the automorphism \( \alpha = e^\theta \) into conditions on its infinitesimal generator, the derivation \( \delta \):
\[ \delta = \delta_0 \oplus \delta_\downarrow \oplus \delta_\uparrow \oplus \ldots \quad (\delta_0: \Omega^0 A \to \Omega^0 A), \quad (3a) \]
\[ \delta(bb') = \delta(b)b' + b\delta(b'), \quad (3b) \]
\[ \delta(b^*) = \delta(b)^*, \quad (3c) \]
\[ [\delta_\infty, \gamma] = 0, \quad (3d) \]
\[ \delta_\infty = -\delta_\infty. \quad (3e) \]

So far, we have characterized the \textit{classical} symmetries associated to a given NCG; these will generate the symmetries of the corresponding classical gauge theory obtained from the spectral action. In order for these gauge symmetries to remain consistent at the quantum level, they must also be anomaly free. If \( \delta_\infty \) denotes a basis for the space of all operators \( \delta_\infty \) obtained by satisfying the restrictions (3), then anomaly freedom corresponds to the additional constraint
\[ \text{Tr} \left[ \gamma \delta_\infty \left\{ \delta_\downarrow, \delta_\uparrow \right\} \right] = 0 \quad (4) \]
for any basis elements \( \delta_\downarrow, \delta_\uparrow \) and \( \delta_\infty \)—see equation (20.81) in [28]. In contrast to the classical constraints (3), we do not know if the quantum constraint (4) has a more fundamental geometric interpretation in our formalism.

**Application to the standard model: I. Symmetries and fermion charges**

Let us apply this formalism to the NCG traditionally used to describe the standard model of particle physics (coupled to Einstein gravity). The detailed spectral triple \( \{ A, H, D, J, \gamma \} \) (which may be fused into an algebra \( B \)) is reviewed pedagogically in [13]. It is the product of two triples: the canonical Riemannian triple \( \{ A_\gamma, H_\gamma, D_\gamma, J_\gamma, \gamma \} \) (which may be fused into an algebra \( B_\gamma \)), and the finite triple \( \{ A_\gamma, H_\gamma, D_\gamma, J_\gamma, \gamma \} \) (which may be fused into an algebra \( B_\gamma \)). The derivations \( \delta \) of \( B \) will involve two types of contributions: those coming from the derivations \( \delta_\downarrow \) of \( B_\gamma \) and those coming from the derivations \( \delta_\uparrow \) of \( B_\eta \). Here we focus on the derivations \( \delta_\downarrow \) and their implications. In a subsequent paper [26], we treat the derivations \( \delta_\uparrow \) and their implications.

See [15] for a succinct introduction to the finite spectral triple \( \{ A_\gamma, H_\gamma, D_\gamma, J_\gamma, \gamma \} \), and the corresponding associator graded \(*\)-algebra \( B_\gamma = \Omega^0 A_\gamma \otimes \Omega^\infty A_\gamma \otimes \ldots \). We will stick to the notation used there.

We would like to find all symmetries of this geometry. As explained in the previous section, this is done by finding all derivations \( \delta: B_\gamma \to B_\gamma \) satisfying equations (3) and (4). First focus on the subalgebra \( A_\gamma = \Omega^0 A_\gamma \) and its derivations \( \delta_\downarrow: \Omega^0 A_\gamma \to \Omega^0 A_\gamma \) since \( \Omega^0 A_\gamma \) is a finite-dimensional semi-simple associative \(*\)-algebra, its general derivation is given by \( \delta_0 = L_\alpha - R_\alpha [27] \), where \( a = -a^* \) is any anti-hermitian element of \( A_\gamma \) and \( L_\alpha \) and \( R_\alpha \) denote, respectively, the left-action and right-action of \( a: L_\alpha \omega = a\omega, R_\alpha \omega = \omega a \). We can extend this to a derivation on \( B_\gamma \) by taking \( \delta_\downarrow = L_\alpha - R_\alpha \) (for all \( \alpha = 0, 1, 2, \ldots \)). To display these derivations more explicitly, let us denote an element of the algebra \( A_\gamma = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \) by \( a = (\lambda, \mu, m) \), where \( \lambda \in \mathbb{C} \) is a complex number, \( \mu \in \mathbb{H} \) is a quaternion, and \( m \in M_3(\mathbb{C}) \) is a \( 3 \times 3 \) complex matrix. We can split the anti-hermitian elements of \( A_\gamma \) into 3 pieces: namely (i) \( a_1 = (\lambda, 0, \mu_3) \), where \( \lambda \in \mathbb{C} \) and \( \mu \in \mathbb{H} \) are pure imaginary and \( I_3 \) is the \( 3 \times 3 \) identity matrix, (ii) \( a_2 = (0, q, 0) \), where \( q \) is a general anti-hermitian \( 2 \times 2 \) matrix, and (iii) \( a_3 = (0, 0, m) \), where \( m \) is a general traceless anti-hermitian \( 3 \times 3 \) matrix. Demanding that the corresponding symmetry generators \( \delta_\infty \in \{ L_\alpha - R_\alpha \} \) are anomaly free (4) yields the additional restriction \( \mu = -\lambda/3 \). The \( \delta_\infty \) are block diagonal; if, following [15], we label the subspaces of \( H_\gamma \) as \( \{ L_R, Q_R, L_I, Q_I, L_L, Q_L \} \), the blocks are
\[ \delta_\infty^{(1)} = \left\{ \gamma^{(0)}_R \otimes \gamma^{(0)}_L \otimes I_3, \gamma^{(1)}_R \otimes \gamma^{(0)}_L \otimes I_3, \gamma^{(2)}_R \otimes I_3, \gamma^{(0)}_I \otimes I_3, \gamma^{(0)}_I \otimes I_3 \right\}, \quad (5a) \]
\[ \delta_\infty^{(2)} = \left\{ 0, 0, q, q \otimes I_3, 0, 0, q \otimes I_3 \right\}, \quad (5b) \]
\[ \delta_\infty^{(3)} = \left\{ 0, I_2 \otimes m, 0, I_2 \otimes m, 0, I_2 \otimes m \right\}, \quad (5c) \]
where

\[
\begin{align*}
\gamma_L^{(1)} &= 2\lambda \left( -\frac{1}{2} 0 \right), \\
\gamma_R^{(1)} &= 2\lambda \left( 0 0 \right),
\end{align*}
\]

\[
\begin{align*}
\gamma_L^{(q)} &= 2\lambda \left( \frac{1}{6} 0 \right), \\
\gamma_R^{(q)} &= 2\lambda \left( 0 -\frac{1}{3} \right).
\end{align*}
\]  

(6)

In other words, from the derivations \( \delta_h = L_a - R_a \) we precisely obtain the generators \( \delta_{\infty}^{(1)}, \delta_{\infty}^{(2)} \) and \( \delta_{\infty}^{(3)} \) of the familiar standard model gauge group \( U(1)_Y \times SU(2)_L \times SU(3)_C \), with the right- and left-handed leptons and quarks transforming in their familiar representations.

But \( \delta_h = L_a - R_a \) is not the most general possible extension of \( \delta_h = L_a - R_a \); more generally, we can take \( \delta_h = L_a - R_a + T_n \), where the linear operators \( T_n \) may be non-zero for \( n \geq 1 \), as long as they satisfy

\[
T_m : \Omega^n A_\gamma \rightarrow \Omega^n A_\gamma,
\]

\[
T_{m+n}(\alpha_m \omega_n) = (T_m \alpha_m) \omega_n + \alpha_m (T_n \omega_n),
\]

(7a)

\[
T_m \left( \alpha_m^* \right) = (T_m \alpha_m)^*,
\]

(7b)

\[
[T_\alpha, T_\beta] = 0,
\]

(7c)

\[
T_{\infty} = -T_\infty,
\]

(7d)

for any \( \alpha_m \in \Omega^n A_\gamma \) and \( \omega_n \in \Omega^n A_\gamma \). If we specialize to the case \( m = 0, n = \infty \) or \( m = \infty, n = 0 \), and use the fact that \( T_0 = 0 \), equations (7b) and (7c) become

\[
[T_\alpha, L_a] = [T_\alpha, R_a] = 0,
\]

(7b’)

\[
[T_\alpha, B_{\beta}] = 0,
\]

(7c’)

where \( L_a \) and \( R_a \) denote the left or right action of any element \( a \in A_{\gamma} \) on \( H \). It is straightforward to check that the most general matrix \( \delta_{\infty} = T_\infty \) which satisfies the constraints (7) along with the anomaly constraint (4) is diagonal, and given by a general linear combination of the hypercharge generator \( \delta_{\infty}^{(1)} \) (5a) and another generator

\[
\delta_{\infty}^{(1)} = \left\{ x_l, x_q \otimes I_3, \tilde{x}_l, \tilde{x}_q \otimes I_3 \right\},
\]

(8)

where

\[
x_l = i \left( -\frac{1}{2} 0 \right), \quad x_q = i \left( \frac{1}{2} 0 \right).
\]

(9)

This \( \delta_{\infty}^{(1)} \) generates an extra \( U(1) \) symmetry: it is nothing but \( U(1)_{B-L} \) (baryon minus lepton number). The full gauge symmetry associated to the algebra \( B_{\beta} \) is thus \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \).

**Application to the standard model: II. Bosonic fields and charges**

Under an automorphism \( \alpha : B \rightarrow B \), the Dirac operator \( D : H \rightarrow H \) must transform covariantly:

\[
D \rightarrow D' = \alpha_\delta D \alpha_\delta^{-1} \approx D + [D, \delta_\alpha],
\]

as in ordinary gauge theory, by inspecting the fluctuation term \([D, \delta_\alpha]\), we can read off the 'connection' terms which must be added to \( D \) in order to make it covariant. In this paper, \( \delta_\infty \) means \( \delta_{\infty, \delta}(x) \) (as explained above, this paper focuses on the derivations \( \delta_\delta \) of \( B_\delta \) obtained in the previous section, while the derivations \( \delta_\delta \) of \( B_\delta \), which relate to local Lorentz invariance, are treated in a subsequent paper [26]). The Dirac operator \( D \) on the product space is the sum of two terms, \( D = D_\gamma \otimes I_F + \gamma_\nu \otimes D_\gamma \), where

\[
D_\gamma = \gamma^\nu V_\nu
\]

is the ordinary curved space Dirac operator, while \( D_\gamma \) is a finite dimensional Hermitian matrix (see [13]); thus its fluctuation has two terms as well:

\[\text{This}\; \delta_{\infty}^{(1)} \text{generates an extra } U(1) \text{ symmetry: it is nothing but } U(1)_{B-L}. \text{The full gauge symmetry associated to the algebra } B_{\beta} \text{ is thus } SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}. \]

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\[\text{This}\; \delta_{\infty}^{(1)} \text{generates an extra } U(1) \text{ symmetry: it is nothing but } U(1)_{B-L}. \text{The full gauge symmetry associated to the algebra } B_{\beta} \text{ is thus } SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}. \]
\[ D, \delta_\omega = \left[ D_\mu \otimes \mathbb{1}_F, \delta_\omega \right] + \left[ \mathbb{X} \otimes D_F, \delta_\omega \right], \] (10)

Although it may be expressed in unfamiliar notation, the first \((D_\mu)\) term on the right-hand side of (10) is nothing but the familiar term that, in ordinary gauge theory, forces one to introduce a gauge field \(A^\mu(x)\) corresponding to each generator \(\mu\) of the gauge group (in this case, \(SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}\)) in order to make the derivatives transform covariantly (see [13] for more details). In an analogous way, the second \((D_F)\) term on the right-hand side of (10) forces us to add extra fields; but, whereas the first term involves the regular curved space Dirac operator \(D_\mu = \gamma^\mu \partial_\mu\), and thus induces fields \(\gamma^\mu A_\mu\) with a spacetime index \(\mu\), the second term involves the finite matrix \(D_F\), with no spacetime index, and thus induces fields with no spacetime index—i.e. scalar fields (again, see [13]). This is one of the most important advantages of Connes’ approach: the gauge fields and scalar fields and their properties emerge hand in hand, from a single formula, as an inevitable consequence of covariance (in constrast to the standard approach, where the gauge fields and their properties emerge this way, but the scalar fields and their properties do not, and must instead just be added to the theory by hand). Let us now compute the \((D_F)\) term in (10) and inspect the result.

As explained in [15], there are only four matrices \(D_F\) compatible with the associative algebra \(B_F\). The one which is relevant to describing the standard model is given (in the basis \([L_R, Q_R, L_L, Q_L, \bar{L}_R, \bar{Q}_R, \bar{L}_L, \bar{Q}_L]\)) by

\[
D_F = \begin{pmatrix}
0 & 0 & Y_l^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Y_q^T & 0 & 0 & 0 & 0 \\
0 & Y_l & 0 & 0 & 0 & 0 & 0 & 0 \\
m & 0 & 0 & 0 & 0 & Y_l^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Y_q^T & 0 \\
0 & 0 & 0 & \gamma_l & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\] (11)

where \(Y_l\) and \(Y_q\) are arbitrary \(2 \times 2\) matrices that act on the doublet indices in the lepton and quark sectors, respectively, \(m = \text{diag}[M, 0]\) is a \(2 \times 2\) diagonal, and for brevity we have written \(\gamma_l, \gamma_q\) in place of \(\otimes \mathbb{1}_F\). Thus, if we calculate the fluctuation \(D \rightarrow D' \approx D - \left[ D_F \otimes \mathbb{1}_F, \delta_\omega \right]\), where \(\delta_\omega = \delta_\omega^{(1)}(x) + \delta_\omega^{(2)}(x) + \delta_\omega^{(3)}(x) + \alpha(x)\delta_\omega^{(4)}\), we find \(Y_\mu, Y_q\) and \(m\) transform as

\[
Y_l' = Y_l - Y_l q_l(x) + q(x) Y_l, \quad (12a)
\]

\[
Y_q' = Y_q - Y_q q_q(x) + q(x) Y_q, \quad (12b)
\]

\[
m' = m + 2i\alpha(x) m, \quad (12c)
\]

where \(q_l(x) = \text{diag}[\lambda(x), \bar{\lambda}(x)]\). From this, we read off that, to make \(D\) covariant, \(Y_l\) and \(Y_q\) and \(m\) must be promoted to fields

\[
Y_l \rightarrow \begin{pmatrix} Y_l \phi_1 \ Y_l \psi_1 \end{pmatrix}, \quad Y_q \rightarrow \begin{pmatrix} Y_q \phi_2 \ Y_q \psi_2 \end{pmatrix},
\]

and \(m \rightarrow \text{diag}[\sigma, 0]\), where \([\phi_1(x), \phi_2(x)]\) and \([\psi_1(x), \psi_2(x)]\) are scalar fields that transform as \(SU(2)_L\) doublets, with hypercharge \(y = +1/2\) and \(y = -1/2\), respectively, and \(\sigma(x)\) transforms with charge +2 under \(U(1)_{B-L}\), but is a singlet under \(SU(3)_C \times SU(2)_L \times U(1)_Y\). Finally, as explained in [15], one can choose the embedding of \(\mathbb{C}\) in \(\mathbb{H}\) so that \([\psi_1, \psi_2] = [-\phi_2, \phi_1]\); in this way, instead of obtaining a 2-Higgs doublet model, one obtains a single Higgs doublet \([\phi_1, \phi_2]\).

**Discussion**

In our previous paper [15], we found a reformulation of NCG that simplified and unified the mathematical axioms while, at the same time, resolving a problem with the NCG construction of the standard model Lagrangian, by precisely eliminating 7 terms which had previously been problematic. In this paper, we show that this same reformulation leads to a new perspective on the gauge symmetries associated to a given NCG, uncovering some that were previously missed. In particular, when we apply our formalism to the NCG traditionally used to describe the standard model of particle physics, we find a new \(U(1)_{B-L}\) gauge symmetry (and, correspondingly, a new \(B - L\) gauge boson). This, in turn, implies the existence of a new complex Higgs field \(\sigma\) that is a singlet under \(SU(3)_C \times SU(2)_L \times U(1)_Y\) but transforms with charge +2 under \(U(1)_{B-L}\), allowing...
it to form a Majorana-like Yukawa coupling $\sigma \nu R \nu R$ with two right-handed neutrinos (so that, if it obtains a large VEV, it induces see-saw masses for the neutrinos). It is striking, on the one hand, that this precise extension of the standard model has been previously considered in the literature [22, 23] on the basis of its cosmological advantages; and, on the other hand, that the new field $\sigma$ can resolve a previous discrepancy between the observed Higgs mass and the NCG prediction [18–21]. Note that in the previous works [18–21] which introduced the $\sigma$ field for this purpose, it was a real field, and a gauge singlet. By contrast, from the perspective presented here, the fact that $\sigma$ is complex, and transforms under $-U(1)_{B-L}$, is the key to its existence: had it been real, it would not have been induced by the covariance argument of the previous section.

It is important to carefully reconsider the phenomenological and cosmological implications of the standard model extension which we have landed on here, especially in light of the extra constraints imposed by the spectral action. This is an exciting topic for future work.

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