An Alternative Derivation of the Gaussian Noise model

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Abstract

By extending the results in Bononi 2012, we provide here a complete alternative derivation of Turin’s Gaussian Noise (GN) model for dual-polarization dispersion uncompensated coherent optical links. This paper contains the lecture notes used by the authors first on July 19, 2012, and then again on June 6, 2016 at the University of Parma.

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I. INTRODUCTION

Goal of this paper is to provide an alternative derivation of the results appearing in P. Johannisson’s et al. [2] on (a generalization of) the well-known frequency-domain GN model introduced by Turin’s group in [3], i.e., a closed-form formula for the power spectral density (PSD) of the nonlinear interference (NLI) at the output of a dispersion uncompensated (DU) coherent system. The main proof was already reported in [1], and we here make it more clear by providing more detail. The novelty here is that we also show that the removal of the extra “phase term” appearing in the output PSD formula in [2] is due to a change of variable that tracks the average nonlinear phase and yields exactly Turin’s GN formula [3].

Ref. [2] first proved that the GN model is indeed a first-order regular perturbation (RP1) model fed by independent Gaussian spectral lines. Given the equivalence of RP1 and the Volterra series model proved in [4], these results also explain the coincidence of the Volterra series and the GN model approaches to the study of DU coherent system.

II. FREQUENCY DOMAIN NLI RP1 SOLUTION

We start from the dual-polarization (DP) single-channel first-order Regular Perturbation (RP1) solution of the Manakov-based dispersion-managed nonlinear Schroedinger equation (DMNLSE) [5, Appendix 2]:

\[ \tilde{U}(L, f) = \tilde{U}(0, f) + \tilde{U}_p(L, f) \tag{1} \]

where \( L \) is the total link length, and the NLI perturbation field is

\[ \tilde{U}_p(L, f) = -jP_0 \int_{-\infty}^{\infty} K(f_1, f_2) \tilde{U}(0, f + f_1) \tilde{U}^\dagger(0, f + f_1 + f_2) \tilde{U}(0, f + f_2) df_1 df_2 \tag{2} \]

where:
i) boldface fields are 2x1 vectors containing the Fourier transforms (denoted by a tilde) of the X and Y polarizations in the transmitter polarization frame of reference; a dagger stands for transposition and conjugation; and the DP field power is normalized to an arbitrary reference power $P_0$ (which in [2] is chosen as the per-polarization average power);

ii) the un-normalized scalar frequency kernel is defined as (Cfr. [2] eq. (45) and [3] eq. (25)):

$$K(F) = \int_0^L \gamma'(s)G(s)e^{-jC(s)(2\pi)^2F} ds$$

(3)

where $F = f_1 f_2$ is the product of two frequencies, $\gamma' = \frac{2}{\eta} \gamma$ with $\gamma$ the fiber nonlinear coefficient, $G(s)$ the line power gain from $z = 0$ to $z = s$, and $C(s) = -\int_0^s \beta_2(z)dz$ is the cumulated dispersion in the transmission fibers (with dispersion coefficient $\beta_2$) up to coordinate $s$. We do not normalize here the frequency axis, as done instead in [2]. In [3] we use the normalized kernel

$$\tilde{\eta}(F) = \frac{K(F)}{K(0)}$$

which then at $F = 0$ equals 1. The nonlinear phase referred to power $P_0$ is

$$\Phi_{NL} = P_0 K(0).$$

Hence by multiplying and dividing by $K(0)$ we can recast (2) as

$$\tilde{U}_p(L, f) = -j \Phi_{NL} \int_{-\infty}^{\infty} \tilde{\eta}(f_1 f_2) \tilde{U}(0, f + f_1) \tilde{U}^*(0, f + f_1 + f_2) \tilde{U}(0, f + f_2) df_1 df_2$$

(4)

From [4], the X component of the RP1 solution writes explicitly as

$$\tilde{U}_{x,p}(L, f) = -j \Phi_{NL} \int_{-\infty}^{\infty} \tilde{\eta}(f_1 f_2) \tilde{U}_x(0, f + f_1) \tilde{U}_x^*(0, f + f_1 + f_2) \tilde{U}_x(0, f + f_2) df_1 df_2 +$$

$$\int_{-\infty}^{\infty} \tilde{\eta}(f_1 f_2) \tilde{U}_y(0, f + f_1) \tilde{U}_y^*(0, f + f_1 + f_2) \tilde{U}_y(0, f + f_2) df_1 df_2$$

(5)

where the first line gives the self-phase modulation (SPM) of X on X, while the second line gives the intra-channel cross-polarization modulation (I-XPolM) of Y on X. A perfectly dual expression for component Y is obtained by exchanging the indices $x$ and $y$.

### III. Gaussian Assumption and Johannisson’s Result

In [2], [3] the key assumption is that the input fields are the sum of independent spectral lines:

$$\tilde{U}_x(0, f) = \sqrt{P_0} \sum_{k=-\infty}^{\infty} \xi_k \sqrt{G_x(k f_0)} \delta(f - k f_0)$$

$$\tilde{U}_y(0, f) = \sqrt{P_0} \sum_{k=-\infty}^{\infty} \zeta_k \sqrt{G_y(k f_0)} \delta(f - k f_0)$$

with $\xi_k$ and $\zeta_k$ independent identically distributed standard (i.e. zero-mean unit variance) circular complex Gaussian random variables (RV). Such signals do have a per-polarization power spectral density $\tilde{G}_{x,y}(f)$ (normalized to $P_0$) in the limit $f_0 \to 0$ [3]. After long statistical averaging calculations, the authors get the power spectral density of the $\tilde{U}_{x,y}(L, f)$ RV as per eq. (89) of reference [2]. Note that our PSD $\tilde{G}(f)$ is normalized such that $G(f) \equiv P_0 G(f)$, where $G$ is the un-normalized PSD per polarization. Also, $P_x = P_0 \int_{-\infty}^{\infty} \tilde{G}_x(f) df$ and $P_y = P_0 \int_{-\infty}^{\infty} \tilde{G}_y(f) df$. We report in our notation the result in [2]:

$$\tilde{G}_{x,y}(f) = P_x \{ 2 \int_{-\infty}^{\infty} |K((f_1 - f)(f_2 - f))|^2 \tilde{G}_x(f_1)\tilde{G}_x(f_2)\tilde{G}_x(f_1 + f_2 - f) df_1 df_2$$

$$+ \int_{-\infty}^{\infty} |K((f_1 - f)(f_2 - f))|^2 \tilde{G}_x(f_1)\tilde{G}_y(f_2)\tilde{G}_y(f_1 + f_2 - f) df_1 df_2$$

$$+ K(0)^2 \tilde{G}_x(f) \left( 4 \int_{-\infty}^{\infty} \tilde{G}_x(f) df \right)^2 + 4 \int_{-\infty}^{\infty} \tilde{G}_x(f) df \int_{-\infty}^{\infty} \tilde{G}_y(f) df + \left( \int_{-\infty}^{\infty} \tilde{G}_y(f) df \right)^2 \}$$

(6)
and a dual expression for $Y$ is obtained by swapping $x \leftrightarrow y$. Recall that $\hat{G}_{x,p}(f)$ is the NLI PSD, normalized by $P_0$.

An equivalent form of (6) using the normalized kernel is the following

$$\hat{G}_{x,p}(f) = \Phi_{NL}^2 \left\{ 2 \int_{-\infty}^{\infty} \left| \tilde{\eta}(f_1 f_2) \right|^2 \hat{G}_x(f + f_1)\hat{G}_x(f + f_2)\hat{G}_x(f + f_1 + f_2) df_1 df_2 
+ \int_{-\infty}^{\infty} \left| \tilde{\eta}(f_1 f_2) \right|^2 \hat{G}_y(f + f_1)\hat{G}_y(f + f_2)\hat{G}_y(f + f_1 + f_2) df_1 df_2 
+ \hat{G}_x(f) \left\{ 4 \left( \int_{-\infty}^{\infty} \hat{G}_x(f) df \right)^2 
+ 4 \int_{-\infty}^{\infty} \hat{G}_x(f) df \int_{-\infty}^{\infty} \hat{G}_y(f) df 
+ \left( \int_{-\infty}^{\infty} \hat{G}_y(f) df \right)^2 \right\} \right\}$$

(7)

which better shows the formal parallel with the field equation (5): the field double integral in $x$ and a dual expression for $Y$ is obtained by swapping kernel-field-field $\leftrightarrow$ kernel-field-field $\leftrightarrow$ PSD-PSD.

Regarding the input $X,Y$ fields $U_x(0,t), U_y(0,t)$:

1) they are wide-sense stationary (WSS);
2) they are jointly Gaussian processes.

Regarding assumption 1), we plan to exploit the following extension of result (6, p. 418, eq. (12-76)):

**Theorem 1**

Consider the jointly WSS stochastic processes $x(t)$ and $y(t)$, and let

$$\tilde{X}(f) \equiv \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt$$

the Fourier transform of $x$ (in the mean-square (MS) sense), and $\tilde{Y}(f)$ is similarly defined. Let their cross power spectral density (PSD) be $G_{xy}(f) = \mathcal{F}[R_{xy}(\tau)] = \mathcal{F}[E[x(t + \tau)y^*(t)]]$. Then

$$E[\tilde{X}(f)\tilde{Y}^*(u)] = G_{xy}(f)\delta(u - f) \quad \square$$

(8)

As a byproduct, we also have

$$E[\tilde{X}(f)\tilde{X}^*(u)] = G_x(f)\delta(u - f).$$

This theorem thus shows that the Fourier transform of any MS-integrable WSS process is nonstationary white noise, and thus the *spectral lines of its Fourier transform are uncorrelated*. This was one of the key assumptions about the input field in (2), (3), which therefore was tantamount to assuming a WSS input field in the time domain.

Regarding assumption 2), we plan to exploit the following result, known as the complex Gaussian moment theorem (CGMT), a generalization to complex variables of Isserlis theorem (7), (8):

**Theorem 2**

Let $U_1, U_2, \ldots, U_{2k}$ be zero-mean jointly circular complex Gaussian random variables. Then

$$E[U_1^* U_2^* \ldots U_k^* U_{k+1} U_{k+2} \ldots U_{2k}] = \sum_p E[U_1^* U_p]E[U_2^* U_q] \ldots E[U_k^* U_r]$$

(9)

where $\sum_p$ denotes a summation over the $k!$ possible permutations $(p, q, \ldots, r)$ of indices $(k+1, k+2, \ldots, 2k) \quad \square$
For instance,

$$E[U_1^* U_2 U_3^* U_4 U_5 U_6] = E[U_1^* U_4]E[U_2^* U_6]E[U_3^* U_6] + E[U_1^* U_4]E[U_2^* U_6]E[U_3^* U_6] + E[U_1^* U_6]E[U_2^* U_4]E[U_3^* U_5] + E[U_1^* U_6]E[U_2^* U_5]E[U_3^* U_4] + E[U_1^* U_6]E[U_3^* U_4]E[U_2^* U_5] + E[U_1^* U_6]E[U_3^* U_5]E[U_2^* U_4].$$

(scheme: 1, 2, 3 containing the conjugate terms stay at their place. Terms 4, 5, 6 get all possible permuted positions).

Let’s now start the new proof. We are interested in the PSD \( \hat{G}_{x,p}(f) \) of the NLI field \( U_{x,p}(L, t) = F^{-1} [\tilde{U}_{x,p}(L, f)] \). By theorem 1 we have:

$$E[\tilde{U}_{x,p}(L, f) \tilde{U}_{x,p}^*(L, u)] = \hat{G}_{x,p}(f) \delta(u - f).$$

and we wish to get an expression for \( \hat{G}_{x,p}(f) \) from the left hand side, which can be explicitly calculated using (5):

$$E[\tilde{U}_{x,p}(L, f) \tilde{U}_{x,p}^*(L, u)] =$$

$$\Phi_{NL}^*$$

$$E[\int_{-\infty}^{\infty} \tilde{\eta}(f_1 f_2) \tilde{U}_x(0, f + f_1) \tilde{U}_x^*(0, f + f_1 + f_2) \tilde{U}_x(0, f + f_2) +$$

$$\tilde{U}_x(0, f + f_1) \tilde{U}_y^*(0, f + f_1 + f_2) \tilde{U}_y(0, f + f_2)] df_1 df_2$$

$$\int_{-\infty}^{\infty} \tilde{\eta}(f_3 f_4) \tilde{U}_x^*(0, u + f_3) \tilde{U}_x(0, u + f_4) +$$

$$\tilde{U}_y^*(0, u + f_3) \tilde{U}_y(0, u + f_4)] df_3 df_4$$

$$\{ E[\tilde{U}_x(0, f + f_1) \tilde{U}_x^*(0, f + f_1 + f_2) \tilde{U}_x(0, f + f_2) \tilde{U}_x^*(0, u + f_3) \tilde{U}_x(0, u + f_3 + f_4)] +$$

$$E[\tilde{U}_x(0, f + f_1) \tilde{U}_y^*(0, f + f_1 + f_2) \tilde{U}_y(0, f + f_2) \tilde{U}_y^*(0, u + f_3) \tilde{U}_y(0, u + f_3 + f_4)] +$$

$$E[\tilde{U}_x(0, f + f_1) \tilde{U}_x^*(0, f + f_1 + f_2) \tilde{U}_x(0, f + f_2) \tilde{U}_x^*(0, u + f_3) \tilde{U}_x(0, u + f_3 + f_4)] +$$

$$E[\tilde{U}_x(0, f + f_1) \tilde{U}_y^*(0, f + f_1 + f_2) \tilde{U}_y(0, f + f_2) \tilde{U}_y^*(0, u + f_3) \tilde{U}_y(0, u + f_3 + f_4)] \}.$$ (12)

The above expectations of circular Gaussian RVs can now be obtained by the following Theorem 3, which exploits Theorems 1 and 2 and is proved in Appendix 1:

**Theorem 3**

For jointly stationary circular complex Gaussian zero-mean processes \( A(t), B(t), C(t), D(t), E(t), F(t) \) we have the general formula

$$E \left[ \hat{A}(f + f_1) \hat{B}^*(f + f_1 + f_2) \hat{C}(f + f_2) \hat{D}^*(u + f_3) \hat{E}(u + f_3 + f_4) \hat{F}^*(u + f_4) \right] =$$

$$G_{ab}(f + f_1)G_{cd}(f + f_2)G_{ef}(f + f_4) \delta(f_2) \delta(f_3) +$$

$$G_{ab}(f + f_1)G_{cd}(f + f_3)G_{ef}(f + f_4) \delta(f_2) \delta(f_4) +$$

$$G_{ab}(f + f_2)G_{cd}(f + f_4)G_{ef}(f + f_3) \delta(f_1) \delta(f_3) +$$

$$G_{ab}(f + f_2)G_{cd}(f + f_3)G_{ef}(f + f_4) \delta(f_1) \delta(f_4) +$$

$$G_{ab}(f + f_3)G_{cd}(f + f_4)G_{ef}(f + f_2) \delta(f_1 - f_2) \delta(f_2 - f_3) +$$

$$G_{ab}(f + f_3)G_{cd}(f + f_1)G_{ef}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) \delta(f_3 - f_2)$$

$$\delta(u - f) \quad \Box \quad (14)$$

(scheme: b, d, f containing the conjugate terms stay at their place. Terms a, c, e get all possible permuted positions.

Arguments of terms in products must be the same, hence the deltas)

We next apply the general formula (14) to the expectations in (13) to get:
First expectation:

\[
E \left[ \vec{U}_x(0, f + f_1) \vec{U}_x(0, f + f_1 + f_2) \vec{U}_x(0, f + f_2) \vec{U}_y(0, u + f_3) \vec{U}_y(0, u + f_3 + f_4) \vec{U}_y^*(0, u + f_4) \right] = \\
\delta(u - f) \left[ G_{xx}(f + f_1) G_{yx}(f) G_{yy}(f + f_4) \delta(f_2) \delta(f_3) + \\
G_{xy}(f + f_1) G_{yx}(f) G_{yy}(f) \delta(f_2) \delta(f_3) + \\
G_{xy}(f + f_2) G_{yx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{xy}(f + f_2) G_{yx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{xy}(f + f_1 + f_2) G_{yx}(f + f_1) G_{yx}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) + \\
G_{xy}(f + f_1 + f_2) G_{yx}(f + f_1) G_{yx}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) \right]
\]

(15)

where \( G_{xx} \equiv \hat{G}_x \).

Second expectation:

\[
E \left[ \vec{U}_x(0, f + f_1) \vec{U}_x(0, f + f_1 + f_2) \vec{U}_y(0, f + f_2) \vec{U}_y^*(0, u + f_3) \vec{U}_y(0, u + f_3 + f_4) \vec{U}_y^*(0, u + f_4) \right] = \\
\delta(u - f) \left[ G_{xy}(f + f_1) G_{yx}(f) G_{yy}(f + f_4) \delta(f_2) \delta(f_3) + \\
G_{xy}(f + f_1) G_{yx}(f) G_{yy}(f + f_4) \delta(f_2) \delta(f_3) + \\
G_{xy}(f + f_2) G_{yx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{xy}(f + f_2) G_{yx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{xy}(f + f_1 + f_2) G_{yx}(f + f_1) G_{yx}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) + \\
G_{xy}(f + f_1 + f_2) G_{yx}(f + f_1) G_{yx}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) \right]
\]

where \( G_{yy} \equiv \hat{G}_y \), and assuming uncorrelated X and Y (i.e., \( G_{yx} \equiv 0 \)) we get

\[
E \left[ \vec{U}_x(0, f + f_1) \vec{U}_y^*(0, f + f_1 + f_2) \vec{U}_y(0, f + f_2) \vec{U}^*_y(0, u + f_3) \vec{U}_y(0, u + f_3 + f_4) \vec{U}^*_y(0, u + f_4) \right] = \\
\delta(u - f) \left[ G_{yy}(f + f_2) G_{xx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{yy}(f + f_1 + f_2) G_{xx}(f + f_1) G_{yy}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) \right].
\]

(16)
Third expectation:

\[
E \left[ \begin{array}{c}
\bar{U}_x(0, f + f_1) \bar{U}^*_x(0, f + f_1 + f_2) \bar{U}_x(0, f + f_2) \bar{U}^*_x(0, u + f_3) \bar{U}_y(0, u + f_3 + f_4) \bar{U}^*_y(0, u + f_4)
\end{array} \right] = \\
\delta(u - f) \left[ G_{xx}(f + f_1) G_{xx}(f) G_{yy}(f + f_4) \delta(f_2) \delta(f_3) + \\
G_{ab}(f + f_1) G_{ab}(f) G_{ef}(f + f_4) \\
G_{yx}(f + f_1) G_{yx}(f + f_3) G_{xy}(f) \delta(f_2) \delta(f_4) + \\
G_{ab}(f + f_1) G_{ad}(f + f_3) G_{ef}(f) \\
G_{xx}(f + f_2) G_{xx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{cb}(f + f_2) G_{cb}(f + f_3) G_{ef}(f + f_4) \\
G_{yx}(f + f_1 + f_2) G_{xx}(f + f_1) G_{xy}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) + \\
G_{ab}(f + f_1 + f_2) G_{cb}(f + f_1) G_{ef}(f + f_2) \\
G_{yx}(f + f_1 + f_2) G_{xx}(f + f_2) G_{xy}(f + f_1) \delta(f_4 - f_1) \delta(f_3 - f_2) \right]
\]

and assuming uncorrelated X and Y (i.e., \(G_{yx} \equiv 0\)) we get

\[
E \left[ \begin{array}{c}
\bar{U}_x(0, f + f_1) \bar{U}^*_x(0, f + f_1 + f_2) \bar{U}_x(0, f + f_2) \bar{U}^*_x(0, u + f_3) \bar{U}_y(0, u + f_3 + f_4) \bar{U}^*_y(0, u + f_4)
\end{array} \right] = \\
\delta(u - f) \left[ G_{xx}(f + f_1) G_{xx}(f) G_{yy}(f + f_4) \delta(f_2) \delta(f_3) + \\
G_{cb}(f + f_1) G_{cb}(f) G_{ef}(f + f_4) \\
G_{yx}(f + f_1) G_{yx}(f + f_3) G_{xy}(f) \delta(f_2) \delta(f_4) + \\
G_{cb}(f + f_1) G_{ad}(f + f_3) G_{ef}(f) \\
G_{xx}(f + f_2) G_{xx}(f) G_{yy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{cb}(f + f_2) G_{cb}(f + f_3) G_{ef}(f + f_4) \\
G_{yx}(f + f_1 + f_2) G_{xx}(f + f_1) G_{xy}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) + \\
G_{cb}(f + f_1 + f_2) G_{cb}(f + f_1) G_{ef}(f + f_2) \\
G_{yx}(f + f_1 + f_2) G_{xx}(f + f_2) G_{xy}(f + f_1) \delta(f_4 - f_1) \delta(f_3 - f_2) \right].
\]

(17)

Fourth expectation:

\[
E \left[ \begin{array}{c}
\bar{U}_x(0, f + f_1) \bar{U}^*_x(0, f + f_1 + f_2) \bar{U}_y(0, f + f_2) \bar{U}^*_x(0, u + f_3) \bar{U}_x(0, u + f_3 + f_4) \bar{U}^*_x(0, u + f_4)
\end{array} \right] = \\
\delta(u - f) \left[ G_{xy}(f + f_1) G_{yx}(f) G_{xx}(f + f_4) \delta(f_2) \delta(f_3) + \\
G_{cb}(f + f_1) G_{cb}(f) G_{ef}(f + f_4) \\
G_{yx}(f + f_1) G_{yx}(f + f_3) G_{xy}(f) \delta(f_2) \delta(f_4) + \\
G_{cb}(f + f_1) G_{ad}(f + f_3) G_{ef}(f) \\
G_{yx}(f + f_2) G_{xx}(f) G_{xy}(f + f_4) \delta(f_1) \delta(f_3) + \\
G_{cb}(f + f_2) G_{cb}(f + f_3) G_{ef}(f + f_4) \\
G_{yx}(f + f_1 + f_2) G_{xx}(f + f_1) G_{xy}(f + f_2) \delta(f_3 - f_1) \delta(f_4 - f_2) + \\
G_{cb}(f + f_1 + f_2) G_{cb}(f + f_1) G_{ef}(f + f_2) \\
G_{yx}(f + f_1 + f_2) G_{xx}(f + f_2) G_{xy}(f + f_1) \delta(f_4 - f_1) \delta(f_3 - f_2) \right]
\]
and assuming uncorrelated \( X \) and \( Y \) (i.e., \( G_{yx} \equiv 0 \)) we get

\[
E \left[ \hat{U}_x(0, f + f_1) \hat{U}_y^*(0, f + f_1 + f_2) \hat{U}_y(0, f + f_2) \hat{U}_x(0, u + f_3 + f_4) \hat{U}_x^*(0, u + f_4) \right] = \\
\delta(u - f) \left[ G_{yy}(f + f_2) \hat{G}_{xx}(f + f_4) \delta(f_1) \delta(f_3) + G_{yy}(f + f_2) \hat{G}_{xx}(f + f_3) \hat{G}_{xx}(f + f_4) \delta(f_1) \delta(f_4) \right].
\]

(18)

Substitution of (15), (16), (17), (18) into (13) finally gives

\[
\frac{E[\hat{U}_{x,p}(L, f) \hat{U}_{x,p}^*(L, u)]}{\Phi_{NL}^2} = \delta(u - f) \left\{ \int \cdots \int \int \delta(f_1) \delta(f_4) \delta(f_3) \delta(f_2) \delta(f_1) \delta(f_3) \delta(f_2) \delta(f_4) \hat{G}_x(f) \hat{G}_y(f + f_4) \delta(f_2) \delta(f_1) \delta(f_3) \delta(f_2) \delta(f_4) \delta(f_3) \delta(f_2) \delta(f_4) \right\}.
\]

From (11), the term multiplying \( \delta(u - f) \) must be the desired PSD.

Each of the 6 lines in the above quadruple integral provides 2 integrands whose integral we must calculate:

1.1)

\[
\left[ \hat{G}_x(f) \left\{ \int \hat{G}_x(f + f_1) \left[ \int \hat{G}_x(f + f_1) \delta(f_2) \delta(f_3) \right] \cdots \right\} \right] \cdots
\]

1.2)

\[
\left[ \hat{G}_x(f) \left\{ \int \hat{G}_x(f + f_1) \left[ \int \hat{G}_x(f + f_1) \delta(f_2) \delta(f_3) \right] \cdots \right\} \right] \cdots
\]
2.1) 
\[
\begin{align*}
\mathcal{H}_{x}(f) & = \int_{-\infty}^{\infty} \hat{G}_{x}(f + f_{2}) (\int_{-\infty}^{\infty} \tilde{\eta}(f_{1} f_{2}) \delta(f_{1}) df_{1}) df_{2} = \\
& = \int_{-\infty}^{\infty} \hat{G}_{x}(f) \left[ \int \tilde{\eta}(f_{1} f_{2}) \delta(f_{1}) df_{1} \right] df_{2} = \\
& = \int_{-\infty}^{\infty} \hat{G}_{x}(f) \left[ \int \tilde{\eta}(f_{3} f_{4}) \delta(f_{3}) df_{3} \right] df_{4} = \\
& = \left[ \int \hat{G}_{x}(f + f_{2}) df_{2} \right] \left[ \int \hat{G}_{x}(f + f_{4}) df_{4} \right] = \\
& = \hat{G}_{x}(f) |\tilde{\eta}(0)|^{2} \left[ \int \hat{G}_{x}(f + f_{2}) df_{2} \right] \left[ \int \hat{G}_{x}(f + f_{4}) df_{4} \right] = \\
& = \hat{G}_{x}(f) |\tilde{\eta}(0)|^{2} \frac{P_{x}}{F_{0}}
\end{align*}
\]

2.2) 
\[
\begin{align*}
\mathcal{H}_{x}(f) & = \int_{-\infty}^{\infty} \hat{G}_{x}(f + f_{2}) \delta(f_{1}) df_{1} df_{2} = \\
& = \int_{-\infty}^{\infty} \hat{G}_{x}(f) \left[ \int \tilde{\eta}(f_{1} f_{2}) \delta(f_{1}) df_{1} \right] df_{2} = \\
& = \int_{-\infty}^{\infty} \hat{G}_{x}(f) \left[ \int \tilde{\eta}(f_{3} f_{4}) \delta(f_{4}) df_{4} \right] df_{3} = \\
& = \left[ \int \hat{G}_{x}(f + f_{2}) df_{2} \right] \left[ \int \hat{G}_{x}(f + f_{4}) df_{4} \right] = \\
& = \hat{G}_{x}(f) |\tilde{\eta}(0)|^{2} \left[ \int \hat{G}_{x}(f + f_{2}) df_{2} \right] \left[ \int \hat{G}_{x}(f + f_{4}) df_{4} \right] = \\
& = \hat{G}_{x}(f) |\tilde{\eta}(0)|^{2} \frac{P_{x}}{F_{0}}
\end{align*}
\]

3.1) 
\[
\begin{align*}
\mathcal{H}_{x}(f) & = \int_{-\infty}^{\infty} \hat{G}_{x}(f + f_{1} + f_{2}) \delta(f_{1}) df_{1} df_{2} = \\
& = \int \tilde{\eta}(f_{1} f_{2}) \hat{G}_{x}(f + f_{1} + f_{2}) \left[ \int \tilde{\eta}(f_{3} f_{4}) \delta(f_{3} - f_{1}) df_{3} \right] df_{4} = \\
& = \int \tilde{\eta}(f_{1} f_{2}) \hat{G}_{x}(f + f_{1} + f_{2}) \left[ \int \delta(f_{3} - f_{1}) df_{3} \right] \left[ \int \tilde{\eta}(f_{3} f_{4}) \delta(f_{4} - f_{2}) df_{4} \right] df_{1} df_{2} = \\
& = \hat{G}_{x}(f) |\tilde{\eta}(f_{1} f_{2})|^{2} \frac{P_{x}}{F_{0}}
\end{align*}
\]

Each pair of delta removes two integrals, so that the PSD turns out to be (first two lines above produce first 4 lines, third line above produces 5th line, 4th line above produces 6th and 7th lines, and final line above produces...
the last two lines):

\[
\frac{\hat{G}_{xy,p}(f)}{\Phi_{NL}^2} = |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_1)\hat{G}_x(f)\hat{G}_x(f + f_4)df_1df_4
\]

\[
+ |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_1)\hat{G}_x(f + f_2)\hat{G}_x(f)df_1df_2
\]

\[
+ |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_1)\hat{G}_x(f + f_3)\hat{G}_x(f)df_1df_3
\]

\[
+ |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_1)\hat{G}_x(f + f_4)\hat{G}_x(f)df_1df_4
\]

\[
+ |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_2)\hat{G}_x(f)\hat{G}_x(f)df_2df_3
\]

\[
+ 2 \int_{-\infty}^{\infty} |\hat{\eta}(f_1f_2)|^2 \hat{G}_x(f + f_1 + f_2)\hat{G}_x(f + f_1)\hat{G}_x(f + f_2)df_1df_2
\]

\[
+ |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_y(f + f_2)\hat{G}_x(f)\hat{G}_y(f + f_4)df_2df_4
\]

\[
+ \int_{-\infty}^{\infty} |\hat{\eta}(f_1f_2)|^2 \hat{G}_y(f + f_1 + f_2)\hat{G}_x(f + f_1)\hat{G}_y(f + f_2)df_1df_2
\]

\[
+ 2 \left( |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_1)\hat{G}_x(f)\hat{G}_y(f + f_4)df_1df_4 + |\hat{\eta}(0)|^2 \int_{-\infty}^{\infty} \hat{G}_x(f + f_2)\hat{G}_x(f)\hat{G}_y(f + f_4)df_2df_4 \right)
\]

In summary, considering that by construction \( \hat{\eta}(0) = 1 \), we have:

\[
\frac{\hat{G}_{xy,p}(f)}{\Phi_{NL}^2} = 2 \int_{-\infty}^{\infty} |\hat{\eta}(f_1f_2)|^2 \hat{G}_x(f + f_1 + f_2)\hat{G}_x(f + f_1)\hat{G}_x(f + f_2)df_1df_2
\]

\[
+ \int_{-\infty}^{\infty} |\hat{\eta}(f_1f_2)|^2 \hat{G}_y(f + f_1 + f_2)\hat{G}_x(f + f_1)\hat{G}_y(f + f_2)df_1df_2
\]

\[
+ \hat{G}_x(f) \left( 4\left(\frac{P_x}{P_0}\right)^2 + 4\frac{P_x}{P_0}\frac{P_y}{P_0} + 1\left(\frac{P_y}{P_0}\right)^2 \right)
\]

which confirms Johannisson’s equation (7) and completes the desired alternative proof.

V. INTERMEDIATE SUMMARY

We have presented an alternative derivation of Johannisson’s et al. [2] result based on the new method in [1]. We first remark that both the result in [2] and our new approach are able to deal with correlated X and Y, although this feature was not exploited here. Next we note that we did not have to assume independent input spectral lines: this comes naturally from the stationarity of the input process. Finally, the truly critical assumption in the model in [2], [3] is therefore the assumption of Gaussianity at any \( z \) during propagation, which is implicit in the assumption of a Gaussian input process, and the fact that the “forcing terms” in the RP equation are the linearly distorted signals at any \( z \), which thus remain Gaussian.
Therefore the true limit of the GN model in [2], [3] is that indeed starting from a non-Gaussian spectrum such as the one of a digitally modulated signal it takes some finite propagation in a non-infinite dispersion line to approximately get both a Gaussian spectrum and a Gaussian-like time-domain signal.

VI. ROLE OF THE LAST LINE IN [19]

We now focus the attention of the last “phase term” in the main result of [2], i.e. our equation (19). We note that
\[
\int_{-\infty}^{\infty} \tilde{G}_x(f) df = P_x/P_0, \int_{-\infty}^{\infty} \tilde{G}_y(f) df = P_y/P_0,
\]

hence the last term is
\[
\left(4 \int_{-\infty}^{\infty} \tilde{G}_x(f) df \right)^2 + 4 \int_{-\infty}^{\infty} \tilde{G}_x(f) df \int_{-\infty}^{\infty} \tilde{G}_y(f) df + 1 \left( \int_{-\infty}^{\infty} \tilde{G}_y(f) df \right)^2 = \left( \frac{P_x}{P_0} \right)^2 + 2 \left( \frac{P_x}{P_0} \frac{P_y}{P_0} + \left( \frac{P_y}{P_0} \right)^2 \right) = \left( \frac{2P_x + P_y}{P_0} \right)^2.
\]

Going back to un-normalized kernels and PSDs, we can finally write the sought nonlinear interference PSD as
\[
G_{x,p}(f) = K(0)^2 \left\{ 2 \int_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 G_x(f + f_1 + f_2) G_x(f + f_1) G_x(f + f_2) df_1 df_2 \right. \\
+ \int_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 G_y(f + f_1 + f_2) G_y(f + f_1) G_y(f + f_2) df_1 df_2 \\
\left. + K(0)^2 \left( 2P_x + P_y \right)^2 G_x(f) \right\}.
\]

The last line can be interpreted as the PSD of the first-order regular perturbation (RP1) solution of a system whose “exact” output is:
\[
U_x(L,t) = U_x(0,t) e^{-j\gamma \int_0^L \tilde{G}(s) ds} \left( 2P_x + P_y \right) = \text{U}_x(0,t) \left( 1 - j \gamma \sqrt{P_0} \int_0^L \tilde{G}(s) ds \right) \left( 2P_x + P_y \right) = U_x(0,t) - jK(0) \left( 2P_x + P_y \right) U_x(0,t) \\
\]

so that the PSD of \(U_{x,p}(L,t)\) is \(K(0)^2 \left( 2P_x + P_y \right)^2 G_x(f)\). Note the surprising average nonlinear phase rotation by an effective power \(2P_x + P_y\) which is larger than the actual average power in the fiber \(P_x + P_y\). This is one of the new unexpected features of dispersion uncompensated systems.

In such DU systems we are thus naturally lead to postulate an ansatz of the DP dispersion managed nonlinear Schroedinger equation (DMNLSE) of the kind
\[
\tilde{\mathbf{A}}(z, f) = \sqrt{P_0 \tilde{G}(z)} e^{j \int \tilde{G}(s) ds} \left[ \begin{array}{c} \tilde{U}_x(z, f) e^{-j\gamma \int_0^L \tilde{G}(s) ds} \left( 2P_x + P_y \right) \\
\tilde{U}_y(z, f) e^{-j\gamma \int_0^L \tilde{G}(s) ds} \left( 2P_x + P_y \right) \end{array} \right]
\]

which we call the dual-polarization enhanced regular perturbation (DP-ERP) ansatz (from a similar ERP definition in [4]). We prove below as an exercise that this change of variables removes the unwanted spectral components in the last line of equation (20). That’s what tacitly also Turin’s group does in their detailed GN model derivation [9].

\[\text{1} Although the authors in [3] present in their Appendix B an appealing heuristic justification of their Gaussian signal assumption, still their invoking the central limit theorem at their equation (37) is not rigorous. They would conclude that any digitally modulated signal with any number of levels has a Gaussian Fourier transform (which in turn implies the time-domain signal itself is Gaussian), which is clearly not the case.
VII. ANALYSIS

The procedure reported below re-derived from scratch the DM-NLSE and the RP1 solution in dual polarization and in the correct phase-reference.

The Manakov Nonlinear Schroedinger equation (M-NLSE) in engineering notation writes in the frequency domain as \[ \text{(21)} \]

\[
\frac{\partial \tilde{A}(z, f)}{\partial z} = \frac{g(z) - j\omega^2 \beta_2(z)}{2} \tilde{A}(z, f) + \int_{-\infty}^{\infty} \left[ \begin{array}{c} \tilde{A}(z, f + f_1) \tilde{A}^\dagger(z, f + f_1 + f_2) \\ \tilde{A}(z, f + f_1 + f_2) \tilde{A}^\dagger(z, f + f_2) \end{array} \right] df_1 df_2 \\
- j\gamma(z) \left[ \begin{array}{c} \left( \tilde{A}_x^*(z, f + f_1 + f_2) \tilde{A}_x(z, f + f_1) + \tilde{A}_y^* \tilde{A}_y(z, f + f_1) \right) \\ \left( \tilde{A}_x^*(z, f + f_1 + f_2) \tilde{A}_x(z, f + f_2) + \tilde{A}_y^* \tilde{A}_y(z, f + f_1) \right) \end{array} \right]
\]

where \( \omega = 2\pi f \), and \( f \) is the frequency normalized to the baud rate. The input modulated field \( \tilde{A}_0(f) \) may be pre-chirped to give \( \tilde{A}(0, f) \equiv \tilde{A}_0(f)e^{j\frac{\pi}{2} \xi_{\text{pre}}} \), where \( \xi_{\text{pre}} \) is the pre-compensation. Now make the change of variable

\[
\begin{bmatrix} \tilde{A}_x \\ \tilde{A}_y \end{bmatrix}(z, f) = \sqrt{P_0} \begin{bmatrix} \tilde{U}_x(z, f) e^{j\int_0^z \gamma(s)G(s)ds (2P_x + P_y)} \\ \tilde{U}_y(z, f) e^{j\int_0^z \gamma(s)G(s)ds (2P_y + P_x)} \end{bmatrix}
\]

where \( P_0 \) is a reference normalizing power, \( G(z) = e^{j\int_0^z \gamma(s)ds} \) is the power gain at \( z \), and \( C(z) \equiv \int_0^z \beta_2(s)ds \) is the cumulated dispersion up to \( z \). The change can also be written more explicitly as

\[
\begin{align*}
\tilde{A}_x(z, f) &= \sqrt{P_0 G(z)} \tilde{U}_x(z, f) e^{j\psi_x(f^2)} \\
\tilde{A}_y(z, f) &= \sqrt{P_0 G(z)} \tilde{U}_y(z, f) e^{j\psi_y(f^2)}
\end{align*}
\]

where we defined

\[
\begin{align*}
\psi_x(f^2) &= j \frac{C(z)(2\pi f)^2}{2} - j\phi_x \\
\psi_y(f^2) &= j \frac{C(z)(2\pi f)^2}{2} - j\phi_y
\end{align*}
\]

\[
\begin{align*}
\phi_x(z) &= \int_0^z \gamma(s)G(s)ds \cdot (2P_x + P_y) \\
\phi_y(z) &= \int_0^z \gamma(s)G(s)ds \cdot (2P_y + P_x)
\end{align*}
\]

Differentiating Eq. (22) gives

\[
\frac{\partial \tilde{A}_x}{\partial z} = \sqrt{P_0 G(z)} \left( \frac{\partial \tilde{U}_x}{\partial z} + \left[ \frac{g(z) - j\omega^2 \beta_2(z)}{2} - j\gamma(z)G(z)(2P_x + P_y) \right] \tilde{U}_x \right) e^{j\psi_x(f^2)}
\]

and a dual equation holds for \( \tilde{A}_y \). Substituting into Eq. (21) we get for the X component:

\[
\sqrt{P_0 G(z)} \left( \frac{\partial \tilde{U}_x}{\partial z} + \left[ \frac{g(z) - j\omega^2 \beta_2(z)}{2} + j\gamma(z)G(z) \right] \tilde{U}_x \right) e^{j\psi_x(f^2)}
\]

\[
\begin{align*}
&= \frac{g(z) - j\omega^2 \beta_2(z)}{2} \tilde{U}_x(z, f) \sqrt{P_0 G(z)} e^{j\psi_x(f^2)} \\
&- j\gamma(z) \int_{-\infty}^{\infty} df_1 df_2 P_0 G(z) \\
&\left( e^{j\psi_x((f+f_1+f_2)^2)} \tilde{U}_x^*(z, f + f_1 + f_2)e^{j\psi_y((f+f_1+f_2)^2)} \tilde{U}_y(z, f + f_2) + e^{j\psi_y((f+f_1+f_2)^2)} \tilde{U}_y^*(z, f + f_2)e^{j\psi_x((f+f_1+f_2)^2)} \tilde{U}_x(z, f + f_1) \right)
\end{align*}
\]
Dividing both sides by $\sqrt{P_0 G(z)} e^{\psi_0(f^2)}$ and simplifying, one gets:

$$\frac{\partial \tilde{U}_x}{\partial z} + j \gamma(z) G(z)(2P_x + P_y) \tilde{U}_x = -j \gamma(z) \int_{-\infty}^{\infty} df_1 df_2 P_0 G(z) e^{-\psi_0(f^2)}$$

$$(e^{\psi_0((f + f_1 + f_2)^2)} \tilde{U}_x^*(z, f + f_1 + f_2)) \tilde{U}_x(z, f + f_2) + e^{\psi_0((f + f_1 + f_2)^2)} \tilde{U}_y^*(z, f + f_1 + f_2) \tilde{U}_y(z, f + f_2) + e^{\psi_0((f + f_1)^2)} \tilde{U}_x(z, f + f_1).$$

Now let’s collect all the exponents of $e^{(\cdot)}$ together in the r.h.s.. For the first term we have the exponent

$$-\psi_x(f^2) + \psi_y((f + f_1 + f_2)^2) \ast \psi_y((f + f_2)^2) + \psi_y((f + f_1)^2) = -\left\{ \frac{C(z) \omega^2}{2} \right\} - j \phi_x - \left\{ \frac{C(z)(\omega + \omega_1 + \omega_2)^2}{2} \right\} - j \phi_y + \left\{ \frac{C(z)(\omega + \omega_1)^2}{2} \right\} - j \phi_x =$$

$$\frac{C(z)}{2} \left( -\omega^2 - (\omega + \omega_1 + \omega_2)^2 + (\omega + \omega_2)^2 + (\omega + \omega_1)^2 \right) = -C(z) \omega_1 \omega_2$$

where we used the relation

$$[\omega^2 - (\omega + \omega_1)^2 - (\omega + \omega_1)^2 + (\omega + \omega_1 + \omega_2)^2] = 2 \omega_1 \omega_2. \quad (25)$$

For the second term we have the exponent

$$-\psi_y(f^2) + \psi_y((f + f_1 + f_2)^2) \ast \psi_y((f + f_2)^2) + \psi_y((f + f_1)^2) = -\left\{ \frac{C(z) \omega^2}{2} \right\} - j \phi_y - \left\{ \frac{C(z)(\omega + \omega_1 + \omega_2)^2}{2} \right\} - j \phi_y + \left\{ \frac{C(z)(\omega + \omega_1)^2}{2} \right\} - j \phi_y =$$

$$\frac{C(z)}{2} \left( -\omega^2 - (\omega + \omega_1 + \omega_2)^2 + (\omega + \omega_2)^2 + (\omega + \omega_1)^2 \right) = -C(z) \omega_1 \omega_2$$

Hence finally we get the Manakov DMNLSE (M-DNMLSE) for $X$:

$$\frac{\partial \tilde{U}_x(z, f)}{\partial z} = +j \gamma(z) G(z)(2P_x + P_y) \tilde{U}_x(z, f) - j \gamma(z) G(z) P_0 \int_{-\infty}^{\infty} e^{-jC(z) \omega_1 \omega_2}$$

$$(\tilde{U}_x^*(z, f + f_1 + f_2) \tilde{U}_x(z, f + f_2) + \tilde{U}_y^*(z, f + f_1 + f_2) \tilde{U}_y(z, f + f_2)) \tilde{U}_x(z, f + f_1) df_1 df_2 \quad (26)$$

and swapping indices $x$ and $y$ we get the dual equation for $Y$. In red I have highlighted the main difference from the traditional result.

A. Dual Polarization first-order Enhanced Regular Perturbation (DP-ERP1)

The output field $\tilde{U}(L, f)$ is obtained as the following formal explicit solution of (26):

$$\tilde{U}_x(L, f) - \tilde{U}_x(0, f) = j \int_0^L \phi_x(z) \tilde{U}_x(s, f) ds$$

$$- j P_0 \int_{-\infty}^{\infty} \int_0^L \gamma(s) G(s) e^{-jC(s) \omega_1 \omega_2} ds.$$

$$(\tilde{U}_x^*(s, f + f_1 + f_2) \tilde{U}_x(s, f + f_2) + \tilde{U}_y^*(s, f + f_1 + f_2) \tilde{U}_y(s, f + f_2)) \tilde{U}_x(s, f + f_1) ds df_1 df_2. \quad (27)$$

For systems where the spectrum of the propagating normalized signal $\tilde{U}(z, \omega)$ does not significantly vary along $z$ we see that the exact solution (27) may be approximated by the first-order regular perturbation (DP-ERP1) solution, which we write for the $X$ component:

$$\tilde{U}_x(L, f) = \tilde{U}_x(0, f) + j \phi_x(L) \tilde{U}_x(0, f)$$

$$(2P_x + P_y) K(0)$$

$$- j P_0 \int_{-\infty}^{\infty} \mathcal{K}(f_1 f_2) (\tilde{U}_x^*(0, f + f_1 + f_2) \tilde{U}_x(0, f + f_2) +$$

$$\tilde{U}_y^*(0, f + f_1 + f_2) \tilde{U}_y(0, f + f_2) \tilde{U}_x(0, f + f_1) df_1 df_2 \quad (28)$$
and a dual expression holds for $Y$ by swapping indices $x \leftrightarrow y$, where we defined

$$K(F) \triangleq \int_0^L \gamma(s)G(s)e^{-jC(s)(2\pi)^2 F ds}$$

(29)

as the un-normalized frequency kernel, a function of only the product $F = f_1 f_2$. We may also define the (cumulated) nonlinear phase as

$$\Phi_{NL} = P_0 K(0) \quad [\text{rad}]$$

(30)

and the normalized (scalar) frequency kernel, or briefly the frequency kernel as

$$\tilde{n}(F) = \frac{K(F)}{K(0)} = \frac{\int_0^L \gamma(s)G(s)e^{-jC(s)(2\pi)^2 F ds}}{\int_0^L \gamma(s)G(s)ds}$$

(31)

so that one finally gets a convenient form of the RP1 perturbation as:

$$\frac{\hat{U}_{x,p}(L, f)}{-j\Phi_{NL}} = -A(f) + B(f)$$

with

$$A(f) \equiv \hat{P}_{T_x}\hat{U}_x(0, f)$$

and

$$B(f) \equiv \iint_{-\infty}^{\infty} \hat{n}(f_1 f_2)(\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_2) +$$

$$\hat{U}_y^*(0, f + f_1 + f_2)\hat{U}_y(0, f + f_2))\hat{U}_x(0, f + f_1)d f_1 d f_2$$

where $\hat{P}_x = P_x/P_0, \hat{P}_y = P_y/P_0$, and $\hat{P}_{T_x} = (2\hat{P}_x + \hat{P}_y)$.

Now we must repeat the long calculations used to verify Joanniss’ result in [1], and check if indeed the NLI power spectral density obtained from the DP-ERP1 solution does not have the unwanted “phase” components, third line in [20]. We need

$$\frac{E[\hat{U}_{x,p}(L, f)\hat{U}_{x,p}^*(L, u)]}{\Phi_{NL}^2} = E[A(f)A^*(u)] - 2\operatorname{Re}(E[B(f)A^*(u)]) + E[B(f)B^*(u)]$$

and in [1] we already calculated $E[B(f)B^*(u)]$, while

$$E[A(f)A^*(u)] = \hat{P}_{T_x}^2 E[\hat{U}_x(0, f)\hat{U}_x^*(0, u)]$$

$$= \hat{P}_{T_x}^2 \hat{G}_x(f)\delta(f - u)$$

where we used Theorem 1 in [1]. We finally have to evaluate

$$E[B(f)A^*(u)] = \iint_{-\infty}^{\infty} df_1 df_2 \hat{n}(f_1 f_2) \cdot$$

$$\hat{P}_{T_x}\{E[\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_2)\hat{U}_x(0, f + f_1)\hat{U}_x^*(0, u)] +$$

$$E[\hat{U}_y^*(0, f + f_1 + f_2)\hat{U}_y(0, f + f_2)\hat{U}_x(0, f + f_1)\hat{U}_x^*(0, u)]\}.$$

For 4 generic complex Gaussian RVs, theorem 2 in [1] specializes to

$$E[U_1^* U_2^* U_3 U_4] = E[U_1^* U_3]E[U_2^* U_4] + E[U_1^* U_4]E[U_2^* U_3].$$

Hence

$$E[\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_2)\hat{U}_x(0, f + f_1)\hat{U}_x^*(0, u)] =$$

$$\frac{E[\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_2)] E[\hat{U}_x^*(0, u)\hat{U}_x(0, f + f_1)]}{\hat{G}_x(f + f_2)\delta(f_1)}$$

$$+ \frac{E[\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_1)] E[\hat{U}_x^*(0, u)\hat{U}_x(0, f + f_2)]}{\hat{G}_x(f + f_1)\delta(u - f_1)}$$

$$+ \frac{E[\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_2)] E[\hat{U}_x^*(0, u)\hat{U}_x(0, f + f_1)]}{\hat{G}_x(f + f_2)\delta(f_2)}$$

$$+ \frac{E[\hat{U}_x^*(0, f + f_1 + f_2)\hat{U}_x(0, f + f_1)] E[\hat{U}_x^*(0, u)\hat{U}_x(0, f + f_2)]}{\hat{G}_x(f + f_1)\delta(u - f_2)}$$

$$\hat{G}_x(f + f_2)\hat{G}_x(f)\delta(f_1)\delta(u - f) + \hat{G}_x(f + f_1)\delta(f_2)\hat{G}_x(f)\delta(u - f).$$
and
\[
E[\tilde{U}_y^*(0, f + f_1 + f_2)\tilde{U}_y(0, f + f_2)\tilde{U}_y(0, f + f_1)\tilde{U}_y^*(0, u)] =
\frac{G_x(f + f_2)\delta(f_1)}{0} + \frac{G_x(f + f_1)\delta(u - f_1)}{0}
\]
\[
E[\tilde{U}_y^*(0, f + f_1 + f_2)\tilde{U}_y(0, f + f_2)] E[\tilde{U}_x^*(0, u)\tilde{U}_x(0, f + f_1)] =
\tilde{G}_x(f + f_2)\tilde{G}_x(f)\delta(f_1)\delta(u - f).
\]

Thus
\[
E[B(f)A^*(u)] = \delta(u - f) \int_{-\infty}^{\infty} \tilde{\eta}(f_1 f_2) \tilde{P}_{T_x} |\tilde{G}_x(f + f_2)\tilde{G}_x(f)\delta(f_1) +
\tilde{G}_x(f + f_1)\delta(f_2)\tilde{G}_x(f) + \tilde{G}_y(f + f_2)\tilde{G}_x(f)\delta(f_1)|df_1 df_2
\]
\[
= \delta(u - f) \tilde{G}_x(f) \tilde{P}_{T_x} \tilde{\eta}(0)(2 \int_{-\infty}^{\infty} \tilde{G}_x(f_2) df_2 + \int_{-\infty}^{\infty} \tilde{G}_y(f_2) df_2)
\]
\[
= \delta(u - f) \tilde{G}_x(f) \tilde{P}_{T_x}(2 \tilde{P}_x + \tilde{P}_y).
\]

Finally from \[1\] we already know that
\[
E[B(f)B^*(u)] = \delta(u - f)\{2 \int_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \tilde{G}_x(f + f_1 + f_2)\tilde{G}_x(f + f_1)\tilde{G}_x(f + f_2) df_1 df_2
\]
\[
+ \int_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \tilde{G}_y(f + f_1 + f_2)\tilde{G}_x(f + f_1)\tilde{G}_y(f + f_2) df_1 df_2
\]
\[
+ \tilde{G}_x(f)(2\tilde{P}_x + \tilde{P}_y)^2\}.
\]

Therefore, getting rid of \(\delta(u - f)\) we have
\[
\frac{\tilde{G}_{x,y}(f)}{\Phi_{NL}^2} = \{ \tilde{P}_{T_x}^2 \tilde{G}_x(f)
\]
\[
-2\tilde{G}_x(f)\tilde{P}_{T_x}(2\tilde{P}_x + \tilde{P}_y)
\]
\[
+ \tilde{G}_x(f)(2\tilde{P}_x + \tilde{P}_y)^2\}
\]
\[
+ 2 \int_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \tilde{G}_x(f + f_1 + f_2)\tilde{G}_x(f + f_1)\tilde{G}_x(f + f_2) df_1 df_2
\]
\[
+ \int_{-\infty}^{\infty} |\tilde{\eta}(f_1 f_2)|^2 \tilde{G}_y(f + f_1 + f_2)\tilde{G}_x(f + f_1)\tilde{G}_y(f + f_2) df_1 df_2
\]

and we recognize that the red curly bracket is zero since by definition \(\tilde{P}_{T_x} \equiv (2\tilde{P}_x + \tilde{P}_y)\). Hence indeed the DP-ERP ansatz removes the unwanted “phase” terms in the NLI PSD.

VIII. CONCLUSIONS

This report is conceptually very important, as it explains the following apparent contradiction about the RP1 solution of the DMNLSE:

It is known from \[4\] that the RP1 solution largely overestimates the true DMNLSE solution, and holds only at extremely small nonlinear phases. How can it then accurately reproduce the nonlinear interference (NLI) power spectral density, as claimed by Turin’s group \[3\]?

The explanation is the following: if instead of the RP1 solution we use the ERP1 solution \[4\], and as extended here to dual polarization and called the DP-ERP1 solution, then the numerical predictions are very close to the true field, and thus also the NLI variance predictions are close to real. Moreover, the DP-ERP1 predicted NLI PSD is exactly that reported by Turin’s group \[3\], without the puzzling “phase term” in the RP1 result in \[2\].
APPENDIX 1

In this Appendix we prove Theorem 3 in the text. Assuming jointly stationary circular complex Gaussian zero-mean processes $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $F(t)$, we have by using (8) in (10):

$$T \triangleq E \left[ \tilde{A}(f + f_1)\tilde{B}^*(f + f_1 + f_2)\tilde{C}(f + f_2)\tilde{D}^*(u + f_3)\tilde{E}(u + f_3 + f_4)\tilde{F}^*(u + f_4) \right] =$$

$$E[\tilde{B}^*(f + f_1 + f_2)\tilde{A}(f + f_1)] E[\tilde{D}^*(u + f_3)\tilde{C}(f + f_2)] E[\tilde{F}^*(u + f_4)\tilde{E}(u + f_3 + f_4)] +$$

$$G_{eb}(f + f_1)\delta(f_2) G_{ed}(f + f_1 + f_2)\delta(u + f_3 - f_2) G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_1 + f_2)\delta(f_1) G_{ed}(u + f_3 + f_4)\delta(u + f_4 - f - f_2) G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_1 + f_2)\delta(f_1) G_{ed}(u + f_3 + f_4)\delta(u + f_4 - f - f_2) G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_1 + f_2)\delta(f_1) G_{ed}(u + f_3 + f_4)\delta(u + f_4 - f - f_2) G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

thus

$$T = G_{eb}(f + f_1)G_{cd}(f + f_2)G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_1 + f_2)G_{cd}(u + f_3 + f_4)G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_2)G_{cd}(f + f_1 + f_2)G_{ef}(u + f_3 + f_4)\delta(f_1)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_2)G_{cd}(u + f_3 + f_4)G_{ef}(f + f_1 + f_2)\delta(f_1)\delta(f_2)\delta(f_4)\delta(u + f_4 - f - f_2) +$$

$$G_{eb}(f + f_2)G_{cd}(u + f_3 + f_4)G_{ef}(f + f_1 + f_2)\delta(f_1)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_2)G_{cd}(f + f_1 + f_2)G_{ef}(f + f_1 + f_2)\delta(f_1)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

$$G_{eb}(f + f_1 + f_3 + f_4)G_{cd}(f + f_2 + f_3 + f_4)G_{ef}(f + f_1 + f_2)\delta(f_1)\delta(f_2)\delta(f_3)\delta(u + f_3 - f - f_2) +$$

Now we use the sampling property of the delta to write, e.g. for the first line where $f_2 = 0$ and $f_3 = 0$,

$$G_{eb}(f + f_1)G_{cd}(f + f_2)G_{ef}(u + f_3 + f_4)\delta(f_2)\delta(f_3)\delta(u + f_3 - f) =$$

and e.g. for the last line where $u + f_4 = f + f_1$ and $u + f_3 = f + f_2$ which we add up to get

$$u + f_3 + f_4 = (f - u) + f + f_1 + f_2$$

whence

$$f + f_1 + f_2 - u - f_3 - f_4 = u - f$$

so that the last line writes as

$$G_{eb}(u + f_3 + f_4)G_{cd}(f + f_2 + f_3 + f_4)G_{ef}(f + f_1 + f_2 + f_3 + f_4)\delta(u + f_4 - f - f_1)\delta(u + f_3 - f - f_2)\delta(u + f_3 - f)\delta(u + f_3 - f - f_2)\delta(u - f) =$$

$$G_{eb}(f - u) + f + f_1 + f_2)G_{cd}(f + f_2 + f_3 + f_4)G_{ef}(f + f_1 + f_2 + f_3 + f_4)\delta(u + f_4 - f - f_1)\delta(u + f_3 - f - f_2)\delta(u + f_3 - f)\delta(u - f)$$

$$G_{eb}(f + f_1 + f_2)G_{cd}(f + f_2 + f_3 + f_4)G_{ef}(f + f_1 + f_2)\delta(f_4 - f_1)\delta(f_3 - f_2)\delta(u - f) =$$

We therefore get

$$T = G_{eb}(f + f_1)G_{cd}(f + f_2)G_{ef}(f + f_4)\delta(f_2)\delta(f_3)\delta(u - f) +$$

$$G_{eb}(f + f_1)G_{cd}(f + f_3 + f_4)G_{ef}(f + f_4)\delta(f_2)\delta(f_4)\delta(u - f) +$$

$$G_{eb}(f + f_2)G_{cd}(f + f_1 + f_2)G_{ef}(f + f_4)\delta(f_1)\delta(f_3)\delta(u - f) +$$

$$G_{eb}(f + f_2)G_{cd}(f + f_3 + f_4)G_{ef}(f + f_4)\delta(f_1)\delta(f_4)\delta(u - f) +$$

$$G_{eb}(f + f_1 + f_2)G_{cd}(f + f_2 + f_3 + f_4)G_{ef}(f + f_4)\delta(f_1)\delta(f_3 - f_2)\delta(u - f).$$
whence the final form \[ (14) \] given in Theorem 3.