Controlled Algebra for Simplicial Rings and Algebraic K-theory

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Abstract

We develop a version of controlled algebra for simplicial rings. This generalizes the methods which lead to successful proofs of the algebraic K-theory isomorphism conjecture (Farrell-Jones Conjecture) for a large class of groups. This is the first step to prove the algebraic K-theory isomorphism conjecture for simplicial rings. We show that the category in question has the structure of a Waldhausen category and discuss its algebraic K-theory.

We lay emphasis on detailed proofs. Highlights include the discussion of a simplicial cylinder functor, the glueing lemma, a simplicial mapping telescope to split coherent homotopy idempotents, and a direct proof that a weak equivalence of simplicial rings induces an equivalence on their algebraic K-theory. Because we need a certain cofinality theorem for algebraic K-theory, we provide a proof and show that a certain assumption, sometimes omitted in the literature, is necessary. Last, we remark how our setup relates to ring spectra.

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1 Introduction

Controlled algebra is a powerful tool to prove statements about the algebraic $K$-theory of a ring $R$. While early on it was used in [PW85] to construct a non-connective delooping of $K(R)$—a space such that $\pi_i(K(R)) = K_{i-1}(R)$—it is a crucial ingredient in recent progress of the so-called Farrell-Jones Conjecture. Our aim here is to construct for a simplicial ring $R$, and a so-called “control space” $X$, a category of “controlled simplicial $R$-modules over a $X$”. It should be regarded as a generalization of controlled algebra from rings to simplicial rings.

The category of “controlled simplicial modules” supports a homotopy theory which is formally very similar to the homotopy theory of CW-complexes. In particular we have a “cylinder object” which yields a notion of homotopy and therefore the category has homotopy equivalences. Waldhausen nicely summarized a minimal set of axioms to do homotopy theory in his notion of a Waldhausen category, called “category with cofibrations and weak equivalences in [Wal85]. He did this to define algebraic $K$-theory of such a category. Our category satisfies Waldhausen’s axioms, which is our main result:

**Theorem.** Let $X$ be a control space and $R$ a simplicial ring. The category of controlled simplicial modules over $X$, $C(X; R)$, together with the homotopy equivalences and a suitable class of cofibrations is a “category with cofibrations and weak equivalences” in the sense of Waldhausen ([Wal85]. Therefore Waldhausen’s algebraic $K$-theory of $C(X; R)$ is defined.

The category has a cylinder functor and it satisfies Waldhausen’s cylinder axiom, his saturation axiom and his extension axiom.

In fact, for $G$ a (discrete) group, there is a $G$-equivariant version, $C^G(X; R)$, of this theorem, which is crucial for applications to the Farrell-Jones Conjecture. Maybe surprisingly the $G$-equivariant version of the theorem is not more difficult to prove than its non-equivariant counterpart. It is stated in Section 3.1 as Theorem 3.1.4 and Section 6 is devoted completely to its proof.

It is well-known that if a category has infinite coproducts of one objects, its algebraic $K$-theory vanishes. As $C^G(X; R)$ suffers from this, we define a full subcategory of finite objects $C^G_f(X; R)$. It behaves from the homotopy theoretic
point of view like finite CW-complexes. From the algebraic point of view it corresponds to finitely generated free modules. Corresponding to projective modules we define the full subcategory of homotopy finitely dominated objects $C^G_{hfd}(X; R)$ of $C^G(X; R)$. (An object $X$ is homotopy finitely dominated if there is a finite object $A$ and maps $r: A \to X$, $i: X \to A$ such that $r \circ i \simeq \text{id}_X$.)

**Theorem.** Both $C^G_f(X; R)$ and $C^G_{hfd}(X; R)$ are Waldhausen categories, with the inherited structure from $C^G(X; R)$. They still have a cylinder functor and satisfy the saturation, extension and cylinder axiom.

Further the inclusion $C^G_f(X; R) \to C^G_{hfd}(X; R)$ induces an isomorphism

$$K_i(C^G_f(X; R)) \to K_i(C^G_{hfd}(X; R))$$

for $i \geq 1$ and an injection for $i = 0$.

The category $C^G_{hfd}(X; R)$ the closest analogy to the idempotent completion of an additive category. We show in the appendix that indeed idempotents and “coherent” homotopy idempotents split up to homotopy in $C^G_{hfd}(X; R)$.

We think that both $C^G_f(X; R)$ and $C^G_{hfd}(X; R)$ are basic ingredients to attack the Farrell-Jones Conjecture for simplicial rings.

### 1.1 Results of independent interest

In this article we need to discuss several topics which might be of interest for readers who are not interested in our main theorems. Here is a guide for these topics.

#### 1.1.1 Controlled algebra for discrete rings

We explain in Subsection 9.1 that the constructions here specialize to a construction of a category of controlled modules over a (discrete) ring. Readers who are interested in controlled algebra for rings can read Sections 2.2, 2.3 and the relevant part of 2.5 as well as 9.1. This gives in very few pages a construction of a category of controlled modules. We think our category is technically nicer than the model described in [BFJR04], because it is e.g. functorial in the control space and has an obvious forgetful functor to free modules. Otherwise the categories are interchangeable.

#### 1.1.2 Establishing a Waldhausen structure and the glueing lemma

A basic result in the homotopy theory of topological spaces is the Glueing Lemma: Assume that $D_i$ is pushout of $C_i \leftarrow A_i \rightarrow B_i$ for $i = 0, 1$, where $\rightarrow$ denotes a cofibration. Assume we have maps $\varphi_A: A_0 \to A_1$ etc., which form a map of pushout diagrams. If $\varphi_A, \varphi_B, \varphi_C$ are homotopy equivalences, then $\varphi_D$ is one. This is not obvious, as the homotopy inverse of $\varphi_D$ is not induced by the homotopy inverses of the other maps.
Waldhausen made the Glueing Lemma into one of the axioms of a Waldhausen category (called a “category with cofibrations and weak equivalences” in [Wal85]). Proofs that a given category satisfies Waldhausen’s axioms are usually omitted in the literature. Section 6 contains a detailed proof that our category $C^G(X; R)$ satisfies Waldhausen’s axioms. Because in $C^G(X; R)$ the weak equivalences are homotopy equivalences, which one can define once one has a cylinder functor, the proofs might be helpful for readers who seek for proofs in related situation.

1.1.3 A Cofinality Theorem for algebraic K-theory Let $\mathcal{B}$ the category of finitely generated projective modules over a (discrete) ring $R$ and $\mathcal{A}$ the subcategory of free modules. It is well-known that the algebraic K-theory of $\mathcal{B}$ differs from the one of $\mathcal{A}$ only in degree 0. A way to describe this is to say that

\[ K(\mathcal{A}) \to K(\mathcal{B}) \to "K_0(\mathcal{B})/K_0(\mathcal{A})" \]

is a homotopy fiber sequence of connective spectra, where the last term is the Eilenberg-MacLane spectrum of the group $K_0(\mathcal{B})/K_0(\mathcal{A})$ in degree 0. There are statements in the literature providing such a homotopy fiber sequence when $\mathcal{A}$ and $\mathcal{B}$ satisfy a list of conditions, e.g. in [Wei13, TT90]. We show these miss an essential assumption and provide counterexamples, as well as a proof of such a cofinality theorem, in Subsection 8.2. (Note that the above example of free and projective modules is just an illustration. To apply the theorem to finite and projective modules we would need to replace them by suitable categories of chain complexes first, as they do not have mapping cylinders.)

1.1.4 A simplicial mapping telescope In topological spaces on can form a mapping telescope of a sequence $A_0 \to A_1 \to A_2 \ldots$ by glueing together the mapping cylinder of the individual maps. It can be used to show that a space which is dominated by a CW-complex is homotopy equivalent to a CW-complex, see e.g. [Hat02, Proposition A.11]. We need an analogue in our category $C^G(X; R)$. Because it is a simplicial category, and the homotopies are simplicial, a lot more care is required. We construct a simplicial mapping telescope in Appendix A. For this we define an analogue of Moore homotopies and provide the necessary tools to deal with them. Our results are summarized as Theorem [A.2.1]. We also define what we call a coherent homotopy idempotent and use the mapping telescope to show these split up to homotopy in $C^G(X; R)$. We only use a few formal properties of $C^G(X; R)$ to derive that result. We expect this construction to work in other settings to split idempotents there. But because we have no further examples of such categories we refrained from providing an axiomatic framework in which the Theorem would hold.

1.1.5 Weak equivalences of simplicial rings and algebraic K-theory A map $f: R \to S$ of simplicial rings is a weak equivalence if it one on the
geometric realization of the underlying simplicial sets. Such a map induces an equivalence $K(R) \to K(S)$ on algebraic K-theory. Usually this is proved by using a plus-construction description of $K(R)$ (e.g. in [Wal78, Proposition 1.1]). Here we provide a proof which only uses Waldhausen’s Approximation Theorem. The proof shows that $f$ induces a weak equivalence on the algebraic K-theory of the categories of controlled modules, for which we do not have a plus-construction description. Note however, that [Wal78, Proposition 1.1] provides the stronger statement that an $n$-connected map induces an $n + 1$-connected map on K-theory. We currently have no analogue of this for controlled modules over simplicial rings.

1.2 The idea of control

Let us now sketch the construction of $C^G(X; R)$. For simplification we assume that $G$ is the trivial group and $X$ arises from a metric space $(X, d)$, for example $\mathbb{R}^n$ with the euclidean metric. The complete and precise definitions can be found in Section 2.  

As simplicial $R$-module $M$ is generated by a set $\odot_R M = \{e_i\}_{i \in I} \subseteq \coprod_n M_n$ if every $R$-submodule $M' \subseteq M$ which contains $\{e_i\}_{i \in I}$ is equal to $M$. The idea is now to label each of the chosen generators $e_i$ of $M$ by an element $\kappa(e_i)$ of $X$ and require that maps respect the labeling “up to an $\alpha > 0$”. More precisely, a controlled simplicial $R$-module over $X$ is a simplicial $R$-module $M$, a set of generators $\odot_R M$ of $M$ and a map $\kappa_M : \odot_R M \to X$. A morphism $f : (M, \odot_R M, \kappa_M) \to (N, \odot_R N, \kappa_N)$ of controlled simplicial $R$-modules is a map $f : M \to N$ of simplicial $R$-modules such that there is an $\alpha \in \mathbb{R}_{>0}$ such that for each $e \in \odot_R M$ we have that $f(e) \subseteq N$ is contained in an $R$-submodule generated by elements $e' \in \odot_R N$ with $d(\kappa_N(e'), \kappa_M(e)) \leq \alpha$.

There are two problems with the objects here: First, we want to have the generators as few relations as possible. This is the case for cellular $R$-modules, when $\odot_R M$ is a set cells of $M$. We define this notion in Section 2.1. Second, the boundary maps in $M$ should behave well with respect to the labels in the control space. A quick way of requiring that is that $\text{id}_M : (M, \odot_R M, \kappa_M) \to (M, \odot_R M, \kappa_M)$ should be controlled. This is a condition on $(M, \odot_R M, \kappa_M)$. We restrict to such modules which are controlled. This defines $C(X; R)$ for $X$ a metric space. The general notion is carefully introduced in Section 2.

The category $C(X; R)$ relates to “ordinary” controlled modules of e.g. [PW85] or [BFJR04] like chain complexes of free modules relate to projective modules, or like CW-complexes relate to projective $\mathbb{Z}$-modules. For $M$ a simplicial $R$-module and $\mathbb{Z}[\Delta^1]$ the free simplicial abelian group on the 1-simplex define $M[\Delta^1] = M \otimes_{\mathbb{Z}} \mathbb{Z}[\Delta^1]$. If $(M, \odot_R M, \kappa_M)$ is a controlled simplicial $R$-module, $M[\Delta^1]$ is also one, canonically. This is the cylinder which yields the homotopy theory in $C(X; R)$.
1.3 Structure of this article

The proof of our main theorem is quite involved as we need to develop the homotopy theory in $C^G(X; R)$ from scratch. We therefore split this article into two main parts. The first, Sections 2 to 4 provides only definitions without any proofs, such that we can state our main theorems as soon as possible. We hope that this makes it easier for the reader to grasp the main definitions of this article, compared to when the definitions would be scattered over the rather long proofs.

The second part, Sections 5 to 8 and the appendix provide the proofs and all intermediate definitions and theorems we need. There does not seem to exist an established way to verify the axioms of a Waldhausen category in the literature, apart from trivial cases, although they are surely well-known. Therefore we provide a reasonable level of detail. Most of the proofs are rather formal once we established the Relative Horn-Filling Lemma 6.2.1. We hope the level of detail is helpful in case one wants to transfer the proofs here to other settings.

Section 9 gives some applications. We briefly elaborate on the relation of this work to the Farrell-Jones Conjecture and to controlled algebra for (discrete) rings. We give a construction of non-connective delooping of the algebraic K-theory of a simplicial ring without any proofs. We add a remark on ring spectra. Appendix A constructs a simplicial mapping telescope and proves the main Theorem A.2.1 about them, which is used to analyse idempotents and coherent homotopy idempotents in $C^G(X; R)$.

1.4 Contents of the Sections 2 to 8

In Section 2 we concisely review simplicial rings and simplicial modules, as well as the idea of control. We define the category $C^G(X; R)$. Section 3 defines the Waldhausen structure on $C^G(X; R)$. Then we introduce the finiteness conditions of finite, homotopy finite, and homotopy finitely dominated modules. Each of these gives us a full subcategory of $C^G(X; R)$. We show that the full subcategories of these are naturally Waldhausen categories. In Section 4 we state that the category of finite and homotopy finite modules have the same algebraic K-theory, and the algebraic K-theory of the homotopy finitely dominated ones differ one at $K_0$. We also compare the K-theory of our category for weakly equivalent rings.

The second part, Sections 5 to 8 contains the proofs for the previous sections and some elaborations. First we state some initial results on simplicial modules and controlled maps between them in Section 5. We are rather brief there. The result provided should be enough to make it possible for the experienced reader to verify all statements we made in Section 3. (Most of the statements in Section 3 are definitions anyway.)

Section 6 verifies the axioms of a Waldhausen category for $C^G(X; R)$ for
the cofibrations and weak equivalences we defined in Section 3.1. We give
careful and complete proofs. The key ingredient is the Relative Horn-Filling
Lemma 6.2.1. Further important results are the establishing of a Cylinder
Functor 6.1.3, the Glueing Lemma 6.4 and the Extension Axiom for the
homotopy equivalences 6.5.

Section 7 discusses the different finiteness conditions. This proves the
results of Section 3.2, i.e., it establishes that each of the full subcategories of
finite, homotopy finite and homotopy finitely dominated modules are again
Waldhausen categories which satisfy all the extra axioms we listed.

In Section 8 we switch to algebraic K-theory and prove comparison theorems
of the algebraic K-theory of the aforementioned categories. As an important
part we prove a cofinality theorem 8.2.1 for algebraic K-theory. It is stated
as an exercise in [TT90] and as Corollary V.2.3.1 in [Wei13], but we show
that a crucial assumption is missing there and prove the correct statement.
Last, we give a direct proof that a weak equivalence of simplicial rings gives
an equivalence on algebraic K-theory of controlled modules.

1.5 Previous results

In [PW85] Pedersen and Weibel first used controlled modules to construct a
non-connective delooping of the algebraic K-theory space of a (discrete) ring.
In [Vog90], Vogell used the idea of control to construct a category related
to Waldhausen’s algebraic K-theory of spaces $A(X)$, which is homotopically
flavored. Unfortunately Vogell does not provide any details on why his category
is a Waldhausen category. Later Weiss [Wei02] gave a quick construction of
a category similar to Vogell’s one, but he also does not give a proof of the
Waldhausen structure. Weiss’ definitions inspired the definitions we use here.

With regard to discrete rings Controlled algebra was developed with the
applications to the Farrell-Jones Conjecture in mind. A fundamental result
is in [CP97], which constructs a highly useful fiber sequence on algebraic K-
theory spaces, arising solely from control spaces. The most recent incarnation
of controlled algebra is described in [BFJR04] which describes the category
which is used in the most recent approaches to the Farrell-Jones Conjecture.
[Ped00] contains a nice survey of the area at the time of its writing.

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1.7 Conventions

We sometimes use the property that for a diagram in a category

\[ \begin{array}{ccc}
I & \to & II \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\end{array} \]

the whole diagram $I + II$ is a pushout if $I$ and $II$ are pushouts and $II$ is a pushout if $I$ and $I + II$ are pushouts. The dual version is proved in [Bor94, I.2.5.9], the third possible implication does not hold in general.

The set of natural numbers $\mathbb{N}$ contains zero. All rings have a unit.

2 Simplicial modules and control

2.1 Basic definitions

We assume familiarity with the theory of simplicial sets. A good reference is [GJ99].

2.1.1 Simplicial modules

We recall the definition of simplicial modules. $\Delta$ is always the simplicial category $\{[n] \mid n \in \mathbb{N}\}$. A simplicial abelian group is a functor $\Delta^{\text{op}} \to \text{Ab}$, similar a simplicial ring is a functor $\Delta^{\text{op}} \to \text{Rings}$. There are obvious generalizations of the notions of left and right modules and tensor products.

We introduce a notation. For a simplicial set $A$ let $Z[A]$ be the free simplicial abelian group on $A$. For $M$ a simplicial left $R$-module define $M[A]$ as the simplicial left $R$-module $M \otimes_{\mathbb{Z}} Z[A]$. For $M, N$ simplicial left $R$-modules define $\text{HOM}_R(M, N)$ as the simplicial abelian group $\{[n] \mapsto \text{Hom}_R(M[\Delta^n], N)\}$.

2.1.2 Cellular modules

We call the simplicial left $R$-module $R[\partial \Delta^n]$ an $n$-cell and $R[\partial \Delta^n]$ the boundary of an $n$-cell. We say $M$ arises from $M'$ by attaching an $n$-cell if $M$ is isomorphic to the pushout $M' \cup_{R[\partial \Delta^n]} R[\Delta^n]$. Like a CW-complex in topological spaces, a cellular $R$-module relative to a submodule $A$ is a module $M$ together with a filtration of $R$-submodules $M^i, i \geq -1$ with $M^{-1} = A$ and $\bigcup M^i = M$ such that $M^i$ arises from $M^{i-1}$ by attaching $i$-cells. We call the map $A \to M$ a cellular inclusion. The composition of two cellular inclusions is again a cellular inclusion. A simplicial
left $R$-module is called cellular if $* \to M$ is a cellular inclusion. $\to\to$ denotes cellular inclusions.

For our setting we will always remember the attaching maps of the cells to $M$ and call this a cellular structure on $M$. This gives and can be reconstructed from an element $e_n \in M_n$ for each $n$-cell of $M$, where $M_n$ denotes the set of $n$-simplices of $M$. This gives a set $\circ R M \subseteq \bigcup_n M_n$ to which we refer as the cells of $M$. As $R$-module, $M$ can have many different cellular structures and we do not require maps to respect them.

**Lemma 2.1.3.** Let $A \to B$ be an inclusion of simplicial sets. Let $M$ be a cellular module. Then $M[A] \to M[B]$ is a cellular inclusion.

If $M \to N$ is a cellular inclusion, then $M[A] \to N[A]$ is a cellular inclusion.

**2.1.4 Finiteness conditions** Similar to the case of CW-complexes or simplicial sets we call a cellular module finite if it has only finitely many cells and finite-dimensional if it has only cells of finitely many dimensions.

**2.1.5 Dictionary** We compare the notions introduced in this section to the corresponding notions of “ordinary”, or “discrete” rings.

| simplicial $R$-modules, $R$ simplicial ring | discrete $R$-modules, $R$ discrete ring |
|--------------------------------------------|---------------------------------------|
| cellular module                            | free module                           |
| cellular structure                         | choice of a basis                     |
| cellular inclusion                         | direct summand with free complement   |
| $M[A]$ ($A$ a simplicial set)              | $\bigoplus_{a \in A} M$ ($A$ a set)   |
| $\bigsqcup_i R[\Delta^n]$ of finite dimension | $\bigoplus_i R$                      |
| finite                                     | finite dimensional                    |

Table 1: Dictionary simplicial rings and modules.

**2.2 Control spaces**

**Definition 2.2.1.** Let $X$ be a topological Hausdorff space. A morphism control structure on $X$ consists of a set $\mathcal{E}$ of subsets $E$ of $X \times X$ (i.e., relations on $X$), called the morphism control conditions. We require:

1. For $E, E' \in \mathcal{E}$ there is an $E'' \in \mathcal{E}$ such that $E \circ E' \subseteq E''$ where “$\circ$” is the composition of relations.
2. For $E, E' \in \mathcal{E}$ there is an $E'' \in \mathcal{E}$ such that $E \cup E' \subseteq E''$.
3. Each $E \in \mathcal{E}$ is symmetric, i.e., $(x, y) \in E$ $\iff$ $(y, x) \in E$. 

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4. The diagonal \( \Delta \subseteq X \times X \) is a subset of each \( E \in \mathcal{E} \).

The topology is only relevant for finiteness conditions later.

2.2.2 Thickennings

For \( U \subseteq X \) and \( E \in \mathcal{E} \) we call \( U^E = \{ x \in X \mid \exists y \in U : (x, y) \in E \} \) the \( E \)-thickening of \( X \).

**Definition 2.2.3.** Given \( X \) and a morphism control structure \( \mathcal{E} \) on \( X \). An object support structure on \((X, \mathcal{E})\) is a set \( F \) of subsets \( F \) of \( X \), called the object support conditions. We require:

1. For \( F, F' \in F \) there is an \( F'' \in F \) such that \( F \cup F' \subseteq F'' \).
2. For \( F \in F \) and \( E \in \mathcal{E} \) there is an \( F''' \in F \) such that \( F^E \subseteq F''' \).

2.2.4 In all applications we can close up both conditions under taking subsets, i.e., require if \( E \in \mathcal{E} \) and \( \Delta \subseteq E' \subseteq E \) then \( E' \in \mathcal{E} \), and if \( F' \subseteq F \) and \( F \in F \) then \( F'' \in F \). We call the triple \((X, \mathcal{E}, F)\) a control space. If \( F = \{X\} \) we often leave it out of the notation.

2.2.5 Maps

A map of control space \((X_1, \mathcal{E}_1, F_1) \rightarrow (X_2, \mathcal{E}_2, F_2)\) is a (not necessarily continuous) map \( f : X_1 \rightarrow X_2 \) such that for each \( E_1 \in \mathcal{E}_1 \) and \( F_1 \in F_1 \) there are \( E_2 \in \mathcal{E}_2 \) and \( F_2 \in F_2 \) with \((f \times f)(E_1) \subseteq E_2 \) and \( f(F_1) \subseteq F_2 \).

We give the most important examples, see [BFJR04, Section 2.3] for more.

**Example 2.2.6 (metric control).** Let \( X \) have a metric \( d \). Then \( \mathcal{E}_d = \{ E \mid \text{there is an } \alpha \text{ such that } E = \{(x, y) \mid d(x, y) \leq \alpha\} \} \) is a morphism control structure on \( X \).

**Example 2.2.7 (continuous control).** Let \( Z \) be a topological space and \([1, \infty)\) the half-open interval with closure \([1, \infty]\). Define a morphism control structure \( \mathcal{E}_{cc} \) on \( X := Z \times [1, \infty) \) as follows. \( E \) is in \( \mathcal{E}_{cc} \) if it is symmetric and

1. For every \( x \in Z \) and each neighborhood \( U \) of \( x \times \infty \) in \( Z \times [1, \infty] \) there is a neighborhood \( V \subseteq U \) of \( x \times \infty \) in \( Z \times [1, \infty] \) such that \( E \cap ((X \setminus U) \times V) = \emptyset \).
2. \( p_{[1, \infty)} \times p_{[1, \infty)}(E) \in \mathcal{E}_d([1, \infty]), \) where \( d \) is the standard euclidean metric on \([1, \infty]\) and \( p_{[1, \infty)} \) is the projection to \([1, \infty)\).

**Example 2.2.8 (compact support).** Let set \( F \subseteq X \) be in \( F \) if it is compact. These are the compact object support conditions. They are object support conditions for \((X, \mathcal{E}_d)\) where \( X \) is a proper metric space (closed balls are compact) or for the continuous control conditions \( \mathcal{E}_{cc} \) on \( Z \times [1, \infty) \).
2.3 Controlled simplicial modules

2.3.1 Cellular submodules We required cellular $R$-modules to come with a chosen cellular structure $\odot_R M$. A cellular submodule is an $R$-submodule $M'$ of $M$ which is generated by a subset of $\odot_R M$. In particular we have an inclusion $\odot_R M' \subseteq \odot_R M$ induced by $M' \hookrightarrow M$.

**Definition 2.3.2.** For a set of simplices $Q \subseteq \bigcup M_n$ define $\langle C \rangle_M$ as the smallest cellular submodule of $M$ containing $Q$.

We abbreviate $\langle \{e\} \rangle_M$ by $\langle e \rangle_M$ or $\langle e \rangle$.

2.3.3 Modules over a space For a control space $(X, \mathcal{E}, \mathcal{F})$ define a general module over $X$ to be a cellular module $(M, \odot_R M, \kappa_R)$ together with a map $\kappa_R : \odot_R M \to X$. (We followed [Wei02] in the notation.)

**Definition 2.3.4 (Controlled module).** A controlled $R$-module over $X$ is a general $R$-module $(M, \odot_R M, \kappa_R)$ over $X$ such that there are $E \in \mathcal{E}$, $F \in \mathcal{F}$ with:

1. For all $e \in \odot_R M$ and $e' \in \langle e \rangle_M$ we have $(\kappa_R(e), \kappa_R(e')) \in E$.
2. $\kappa_R(\odot_R M) \subseteq F$.

We say $(M, \kappa_R)$ is $E$-controlled and has support in $F$, and often leave $\kappa_R$ understood.

**Definition 2.3.5 (Controlled maps).** A map $(M, \kappa^M_R) \to (N, \kappa^N_R)$ of controlled modules is a map $f : M \to N$ of simplicial $R$-modules such that there is an $E \in \mathcal{E}$ and for all $e \in \odot_R M$, $e' \in \odot_R \langle f(e) \rangle_N$ we have $(\kappa^M_R(e), \kappa^N_R(e')) \in E$.

We say $f$ is $E$-controlled. We just say $f$ is controlled if we do not want to specify the $E$.

2.3.6 Composition If $f, f_1, f_2 : M \to M'$ and $g : M' \to M''$ are controlled maps (of controlled modules), then $g \circ f$ and $f_1 + f_2$ are controlled.

2.3.7 The category of controlled modules For $(X, \mathcal{E}, \mathcal{F})$ a control space the controlled $R$-modules over $X$ together with the controlled maps between them form a category which we denote by $\mathcal{C}(X, \mathcal{E}, \mathcal{F}; R)$. We will usually abbreviate it by $\mathcal{C}(X; R)$, $\mathcal{C}(X)$, $\mathcal{C}(X, \mathcal{E}, \mathcal{F})$ or $\mathcal{C}$.

If $M$ is a controlled module over $X$ and $A$ a simplicial set then $M[A]$ is canonically a controlled module over $X$. This is functorial in $A$ and $M$. 
2.3.8 Cellular inclusion of controlled modules  Define a cellular inclusion of controlled modules to be a map \((M, \kappa^M_R) \to (N, \kappa^N_R)\) such that \((M, \partial_R M) \to (N, \partial_R N)\) is a cellular inclusion of simplicial \(R\)-modules and the inclusion \(i: \partial_R M \hookrightarrow \partial_R N\) satisfies \(\kappa^M_R = \kappa^N_R \circ i\). This is the right notion of a subobject in \(\mathcal{C}\).

If \(A \hookrightarrow B\) is an inclusion of simplicial sets, then \(M[A] \to M[B]\) is a cellular inclusion of controlled modules. If \(M \to N\) is a cellular inclusion of controlled modules, then \(M[A] \to N[A]\) is one.

2.4 A kind of an adjunction

2.4.1 Controlled filtration on the HOM-space  Let \(M, N \in \mathcal{C}(X, E, F), E \in E\). Define \(\text{Hom}^E_R(M, N)\) as the subset of maps \(f: M \to N\) in \(\mathcal{C}(X)\) which are \(E\)-controlled. Similar define \(\text{HOM}^E_R(M, N)\) as the sub-simplicial set of \(E\)-controlled maps \(M[\Delta^n] \to N\). Boundaries and degeneracies respect \(E\), so this is a well-defined simplicial subset of \(\text{HOM}_R(M, N)\). Define \(\text{HOM}^E_R(M, N)\) as \(\bigcup_{E \in E} \text{HOM}^E_R(M, N)\).

2.4.2 Uncontrolled adjunction and its controlled counterparts  If \(A\) is a simplicial set, we have an adjunction

\[
\text{Hom}_R(M[A], N) \cong \text{Hom}_{sSet}(A, \text{HOM}_R(M, N))
\]

in simplicial \(R\)-modules. This restricts to a bijection

\[
\text{Hom}^E_R(M[A], N) \cong \text{Hom}_{sSet}(A, \text{HOM}^E_R(M, N)).
\]

If \(A\) is a finite simplicial set, we have a bijection \(\text{Hom}^E_R(M[A], N) \cong \text{Hom}_{sSet}(A, \text{HOM}^E_R(M, N))\) which is natural in \(A, M\) and \(N\).

2.5 G-equivariance

Let \(G\) be a (discrete) group. All notions above generalize in a straightforward way to \(G\)-equivariant versions:

2.5.1 G-equivariant cellular modules  An action of \(G\) on a simplicial \(R\)-module \(M\) is a group homomorphism \(\rho: G \to \text{Aut}_R(M)\). The action is a called cell-permuting if it induces an action on \(\partial_R M\). An action is free if it is cell-permuting and the action on \(\partial_R M\) is free. If \(M, N\) are simplicial \(R\)-modules with \(G\)-actions \(\rho_M, \rho_N\) a map \(f: M \to N\) of simplicial \(R\)-modules is \(G\)-equivariant if for each \(g \in G\) the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\rho_M(g) \downarrow & & \downarrow \rho_N(g) \\
M & \xrightarrow{f} & N
\end{array}
\]

is commutative.
commutes. A $G$-equivariant map $L \to M$ is a cellular inclusion, if it is one after forgetting the $G$-action.

If $M$ is a simplicial $R$-module with $G$-action and $A$ a simplicial set then $M[A]$ has a $G$-action by the functoriality in $M$.

2.5.2 $G$-equivariant control spaces

A control space $(X, \mathcal{E}, \mathcal{F})$ is $G$-equivariant if $X$ has a continuous $G$-action such that $gE = E$ (diagonal action) and $gF = F$ for all $g \in G$, $E \in \mathcal{E}$, $F \in \mathcal{F}$. A free control space is one where the action of $G$ on $X$ is free. The examples of control spaces in Section 2.2 have $G$-equivariant analogues, see [BFJR04, 2.7, 2.9, 3.1, 3.2].

2.5.3 The category of $G$-equivariant controlled modules

Let $(X, \mathcal{E}, \mathcal{F})$ be a free $G$-equivariant control space. A controlled simplicial $R$-module with $G$-action over $X$ is a controlled module $(M, \diamondsuit_R M, \kappa_R)$ with cell-permuting $G$-action such that $\kappa_R$ is $G$-equivariant. A morphism $(M, \kappa_R) \to (N, \kappa_R)$ of such modules is a $G$-equivariant morphism $M \to N$ which is controlled over $X$. Denote the category of these as $\mathcal{C}^G(X, \mathcal{E}, \mathcal{F}; R)$. We use abbreviations like $\mathcal{C}^G$, etc. All further definitions of Section 2.3 transfer to $\mathcal{C}^G$.

2.5.4 The $G$-equivariant kind of adjunction

The adjunction between $M[-]$ and $\text{HOM}_R(M, -)$ and its controlled counterparts generalizes to the $G$-equivariant setting. Denote by $\text{Hom}_R(M, N)^G$ and $\text{HOM}_R(M, N)^G$ the subset of $\text{Hom}_R(M, N)$, resp. subspace of $\text{HOM}_R(M, N)$, of $G$-equivariant maps. The adjunctions of 2.4.2 restrict to adjunctions

$$\text{Hom}_R(M[A], N)^G \cong \text{Hom}_{sSet}(A, \text{HOM}_R(M, N)^G)$$

and

$$\text{Hom}_R^E(M[A], N)^G \cong \text{Hom}_{sSet}(A, \text{HOM}_R^E(M, N)^G).$$

Similarly, if $A$ is a finite simplicial set there is a bijection $\text{Hom}_R^E(M[A], N)^G \cong \text{Hom}_{sSet}(A, \text{HOM}_R^E(M, N)^G)$.

3 Waldhausen categories of controlled modules

In the following $(X, \mathcal{E}, \mathcal{F})$ is always a free $G$-equivariant control space which we abbreviate as $X$. We choose a simplicial ring $R$ for this section. We put some additional structure on $\mathcal{C}^G(X; R)$. We will always work in this category in this section.

3.1 $\mathcal{C}^G(X, \mathcal{E}, \mathcal{F}; R)$ as a Waldhausen category

First we make $\mathcal{C}^G(X; R)$ into a Waldhausen category, called “category with cofibrations and weak equivalences” in [Wal85]. We will use the definitions of
category with cofibrations, category with weak equivalences, cylinder functor and the saturation, cylinder and extension axiom from there. We will give detailed proofs of the statements below in Section 6.

3.1.1 Cofibrations

Define a map $f: M \to N$ in $C^G(X)$ to be a cofibration if there are isomorphisms $\alpha: M' \to M$ and $\beta: N \to N'$ in $C^G(X)$ such that $\beta \circ f \circ \alpha$ is a cellular inclusion. Note that $\alpha, \beta$ do not need to preserve the cellular structures, so the notion is independent of chosen cellular structures. The compositions of cofibrations is a cofibration. We also denote cofibrations by $\hookrightarrow$. If $A \hookrightarrow B$ is a cofibration and $A \to C$ a map then the pushout $B \cup_A C$ exists and $C \to B \cup_A C$ is a cofibration. If $A \to C$ has been a cellular inclusion, then this pushout can be chosen canonically, in particular functorially. This makes $C^G(X; R)$ into a category with cofibrations.

3.1.2 Cylinders

Consider the simplicial set $\Delta^1$, call it the interval. It comes with inclusions $i_0, i_1: \text{pt} \to \Delta^1$ and a projection $p: \Delta^1 \to \text{pt}$. For $M$ in $C^G(X; R)$ this induces the corresponding cellular inclusions $M \to M[\Delta^1]$ and a projection $M[\Delta^1] \to M$ which makes $M[\Delta^1]$ into a cylinder object. For a map $f: A \to B$ define $T(f)$ as $A[\Delta^1] \cup_{i_1} B$. This is functorial in the arrow category and therefore gives cylinder functor on $C^G(X; R)$ in the sense of Waldhausen [Wal85, 1.6].

3.1.3 Weak equivalences

Two maps $f, g: A \to B$ are homotopic if there is a homotopy $H: A[I] \to B$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. This gives rise to the obvious notion of homotopy equivalence.

Theorem 3.1.4. The subcategory of homotopy equivalences in $C^G(X; R)$ forms a category of weak equivalences, in particular it satisfies the gluing lemma. It also satisfies the saturation axiom and the extension axiom. The cylinder functor satisfies the cylinder axiom with respect to these weak equivalences.

Note that this category is too big, it has an Eilenberg-Swindle. But it contains interesting full subcategories, which we discuss next.

3.1.5 A remark on the proofs

Let us interrupt for a remark on the proofs. The main tool for the proofs are the adjunctions of 2.5.4 and a careful analysis of the control conditions in the settings, often accompanied by an induction over the cells. Here is a prototype of such a proof. We need to show that we have horn-filling in our category. In particular the following (simplified) lemma should hold.

Lemma. Given a map $M[\Lambda^n] \to P$. Then there is an extension to a map $M[\Delta^n] \to P$. 

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Proof. By the adjunction the situation is equivalent to finding a (dotted) lift in the diagram

\[
\Lambda^n_i \longrightarrow \text{HOM}^E_R(M, P) \\
\downarrow \text{HOM}^E_R(M, P) \\
\Delta^n \rightarrow \text{HOM}^E_R(M, P)
\]

But the simplicial set \(\text{HOM}^E_R(M, P)\) is in fact a simplicial abelian group, hence the lift exists by the Kan-property, i.e., it is fibrant.

The general proofs are considerably more complicated and quite long if carried out in detail. We will devote Section 6 to them.

3.2 Finiteness conditions

We define full subcategories of \(\mathcal{C}G(X; R)\) by specifying conditions on the objects.

3.2.1 (Locally) finite controlled modules Here we use the topology on \(X\). Let \((M, \circ_R M, \kappa_R)\) be a controlled module over \(X\). We say that \(M\) is locally finite if for each \(x \in X\) there is a neighborhood \(U\) of \(x\) in \(X\) such that \(\kappa_R^{-1}(U)\) is a finite subset of \(\circ_R M\). Then \(M\) is called finite if it is locally finite and finite-dimensional. Denote the full subcategory of finite module by \(\mathcal{C}G_f^f(X; R)\). It inherits the structure of a Waldhausen category from \(\mathcal{C}G(X; R)\). It satisfies the saturation and extension axiom and has a cylinder functor satisfying the cylinder axiom. Also, it turns out that cofibrations are isomorphic in \(\mathcal{C}G_f(X; R)\) to cellular inclusions. The proof needs the Hausdorff-property of \(X\).

3.2.2 Homotopy finite controlled modules An object \(M \in \mathcal{C}G(X; R)\) is homotopy finite if there is a homotopy equivalence \(M \sim M'\) such that \(M'\) is a finite module. We denote the full subcategory of homotopy finite modules by \(\mathcal{C}G_{hf}^f(X; R)\). Similar to \(\mathcal{C}G_f(X; R)\) it inherits the structure of a Waldhausen category. It has a cylinder functor satisfying the cylinder axiom. The saturation and extension axiom hold.

3.2.3 Homotopy finitely dominated controlled modules An object \(M \in \mathcal{C}G(X; R)\) is homotopy finitely dominated if it is a strict retract of a homotopy finite object. We denote the full subcategory of homotopy finitely dominated modules by \(\mathcal{C}G_{hfd}(X; R)\). Similar to \(\mathcal{C}G_f(X; R)\) it inherits the structure of a Waldhausen category. It has a cylinder functor satisfying the cylinder axiom. The saturation and extension axiom hold. Homotopy finitely dominated modules can equivalently be characterized by being a retract up to homotopy of a finite object.
4 Algebraic $K$-theory of controlled modules

4.1 Connective $K$-theory

Let us make explicit that we can get algebraic $K$-theory out of the defined Waldhausen categories.

**Definition 4.1.1** (Algebraic $K$-theory of categories of controlled modules). Let $G$ be a group, $(X, E, F)$ be a free $G$-equivariant control space. Let $R$ be a simplicial ring. Define the algebraic $K$-theory spectrum of the category with cofibrations and weak equivalences $C^G_f(X, R, E, F)$ as the connective spectrum

$$K(wC^G_f(X, R, E, F))$$

where $K$ is Waldhausen’s algebraic $K$-theory of spaces [Wal85]. We define similar the algebraic $K$-theory of $C^G_{hf}$ and $C^G_{hfd}$.

**Remark 4.1.2.** In [Wal85] Waldhausen defines the $K$-theory as a space and then constructs a delooping, i.e., an $\Omega$-spectrum. This is what we use here, because for the Cofinality Theorem it is more convenient to work with spectra. See also [TT90, 1.5.3].

**Theorem 4.1.3** (Different finiteness conditions). Let $(X, E, F)$ be a control space and $R$ a simplicial ring.

1. The inclusion $C^G_f(X, R, E, F) \rightarrow C^G_{hf}(X, R, E, F)$ is exact and induces a homotopy equivalence on $K$-Theory.

2. The inclusion $C^G_{hf}(X, R, E, F) \rightarrow C^G_{hfd}(X, R, E, F)$ is exact and induces an isomorphism on $K_n$ for $n \geq 1$ and an injection $K_0(C^G_{hf}) \rightarrow K_0(C^G_{hfd})$.

4.1.4 Separations of variables The categories $C^G(G/1, \{G \times G\}, \{G\}; R)$ and $C(pt, \{pt\}, \{pt\}; R[G])$ are equivalent. Both are equivalent to the category of cellular $R[G]$-modules. The equivalences respect the finiteness conditions $f$, $hf$ and $hfd$.

**Corollary 4.1.5.** The algebraic $K$-theory of $C^G_{hfd}(G/1, \{G \times G\}, \{G\}; R)$ is homotopy equivalent to the algebraic $K$-theory of the simplicial ring $R[G]$.

4.1.6 Change of rings The constructions are functorial in change of ring maps $f: R \rightarrow S$. If $f$ is a weak equivalence, then the induced map $C^G_f(X; R) \rightarrow C^G_f(X; S)$ is an equivalence on algebraic $K$-theory. Here $?$ can be $f$, $hf$, or $hfd$. 

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5 Proofs I: About control

Section 2 introduced our basic categories of controlled simplicial modules. Most results are straightforward to check. Therefore we discuss only a few important lemmas and leave the rest to the reader. For the structure of a Waldhausen category which we introduced in Section 3 we will provide much more detailed proofs in Section 6.

5.1 Cellular structure

5.1.1 Proof of Lemma 2.1.3

**Lemma 5.1.2 (Lemma 2.1.3).** Let $A \hookrightarrow B$ be an inclusion of simplicial sets. Let $M$ be a cellular module. Then $M[A] \to M[B]$ is a cellular inclusion.

If $M \hookrightarrow N$ is a cellular inclusion, then $M[A] \to N[A]$ is a cellular inclusion.

**Proof.** We need to show that $M[A]$ respects cellular inclusion in both variables. First, the pushout of a cellular inclusion is a cellular inclusion. Also sequential colimits and coproducts of cellular inclusions are cellular inclusions. Further, $M[-]$ commutes with colimits as it is the composition of two left adjoint functors. Therefore it suffices to consider the case $M[\partial \Delta^n] \to M[\Delta^n]$. This map factors over $M[\partial \Delta^n] \cup_{L_i[\partial \Delta^n]} L_i[\Delta^n]$ for each $i$, where $L_i$ is the $i$-skeleton of $M$. Using that $-[\Delta^n]$ commutes with pushouts one can show that the map $M[\partial \Delta^n] \cup_{L_i[\partial \Delta^n]} L_i[\Delta^n] \to M[\partial \Delta^n] \cup_{L_{i+1}[\partial \Delta^n]} L_{i+1}[\Delta^n]$ is a cellular inclusion. Then $M[\partial \Delta^n] \to M[\Delta^n]$ is the sequential colimit of these maps.

For the second part it suffices to consider the case of attaching one $R$-cell. Then it follows from $R(\partial \Delta^n)[A] \to R(\Delta^n)[A]$ being a cellular inclusion. Namely applying $-[A]$ to the pushout “attaching an $n$-cell” gives again a pushout with desired map being the pushout of the map above.

5.2 Controlled maps

We denote the support of a controlled module by $\text{supp}(M)$. We have the following more precise statement about the control of maps.

**Lemma 5.2.1.**

1. If $f : M \to M'$ is $E$-controlled and $g : M' \to M''$ is $E'$-controlled, then $g \circ f$ is $E' \circ E$-controlled.

2. If $f_1, f_2 : M \to M'$ are $E_1$-, resp. $E_2$-controlled and $E_1 \cup E_2 \subseteq E_3$, then $f_1 + f_2$ is $E_3$-controlled.

3. If $M$ is an $E$-controlled module, then $\text{id}_M$ is $E$-controlled.
Proof. Let \( e \in \diamond R M \). Then \( \text{supp}(\langle f(e) \rangle) \subseteq \{ \kappa^M(e) \}^E \). Further for \( e' \in \diamond R \langle f(e) \rangle \) we have \( \text{supp}(\langle g(e') \rangle) \subseteq \{ \kappa^{M'}(e') \}^{E'} \). By minimality \( \langle (g \circ f)(e) \rangle \rangle_{M''} \subseteq \langle g(\langle f(e) \rangle) \rangle_{M''} \), so its support is contained in \( \{ \kappa^M(e) \}^{E''} \).

Further \( \langle (f_1 + f_2)(e) \rangle \subseteq \langle (f_1)(e) \rangle \cup \langle (f_2)(e) \rangle \). So its support is contained in \( \{ \kappa^M(e) \}^{E_1} \cup \{ \kappa^M(e) \}^{E_2} \subseteq \{ \kappa^M(e) \}^{E_3} \). The third part is clear.

We say that a map \( f: M \rightarrow N \) of cellular \( R \)-modules is 0-controlled if it induces a map \( \diamond RM \rightarrow \diamond RN \) and \( \kappa^M_R = \kappa^N_R \circ f \). Cellular inclusions are 0-controlled. The name 0-controlled is a misuse of notation, such a map has the control of its image. However, \( g \circ f \) has the control of \( g \) if \( f \) is 0-controlled.

Lemma 5.2.2. Let \( (M, \kappa_R) \) be an \( E \)-controlled \( R \)-module. Let \( A \) be a simplicial set. Then \( M[A] \) can be made canonically into an \( E \)-controlled \( R \)-module.

Further, each map \( A \rightarrow B \) of simplicial sets induces a 0-controlled map \( M[A] \rightarrow M[B] \).

Proof. From Lemma 2.1.3 it follows that each cell \( e \) of \( M[A] \) arises from exactly on cell \( p(e) \) of \( M \). Define \( \kappa^{M[A]}(e) := \kappa^M(e) \). It makes \( M[A] \) into an \( E \)-controlled module: For \( e \in M[A] \) we have \( p(e) \in \diamond R M \). Then \( e \in \langle p(e) \rangle[A] \subseteq M[A] \), and \( \langle p(e) \rangle[A] \) is supported on \( \{ \kappa(e) \}^E \). This shows the first part.

Another way to describe the control map is to note that the map \( M[A] \rightarrow M[pt] \cong M \) is 0-controlled. A cell of \( M \) is given by a map \( R[\Delta^n] \rightarrow M \). Hence for each cell of \( M \) we get a commutative diagram

\[
\begin{array}{ccc}
M[A] & \xrightarrow{f} & M[B] \\
\uparrow & & \uparrow \\
R[\Delta^n][A] & \longrightarrow & R[\Delta^n][B]
\end{array}
\]

which shows that \( f \) maps cells to cells. As \( f \) commutes with the map to \( M[pt] \) this shows that \( f \) is 0-controlled.

6 Proofs II: Controlled simplicial modules as a Waldhausen category

In this section we establish that \( C^G \) is indeed a Waldhausen category, thus proving Subsection 3.1. We will use the definitions from 3.1 without further notice.

It would have been great if we could have followed an established pattern to show that \( C^G(X; R) \) is a Waldhausen category, but the author is not aware of worked-out proofs of the structure of a Waldhausen category in the literature in elementary terms. If the category in question is a subcategory of cofibrant
objects of a Quillen model category, one basically gets the structure of a Waldhausen category for free, and lots of examples in the literature are of that form. But unfortunately there does not seem to be a suitable Quillen model category which contains our category of controlled modules. In fact, neither general pushouts, nor infinite unions exists in general in any of our categories of controlled modules.

Therefore we prove all results directly. The hardest part is to prove the glueing lemma. We follow a strategy the author learned from a proof of Waldhausen of the proof of the glueing lemma for topological spaces.

Sometimes we are brief or do not comment on statements which are easy to prove. We expect the experienced reader to be able to fill the gaps easily, but otherwise refer to the author’s thesis [Ull11] which has even more details. Note that the thesis works with a slightly different definition of controlled module.

6.1 The Waldhausen structure

We assume familiarity with Section 1.1 to 1.6 of [Wal85] and use the language from there freely.

Lemma 6.1.1 (Pushouts along cellular inclusions). Let \((A, \kappa^A_R) \rightarrow (B, \kappa^B_R)\) be a cellular inclusion in \(C^G\), let \(f: (A, \kappa^A_R) \rightarrow (C, \kappa^C_R)\) be any controlled map in \(C^G\). The pushout \(D := C \cup_A B\),

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^{f} & & \downarrow \\
C & \leftarrow & D
\end{array}
\]

of simplicial \(R\)-modules can be chosen canonically and further it has a canonical structure of an object \((D, \kappa^D_R)\) in \(C^G\). Further \((C, \kappa^C_R) \rightarrow (D, \kappa^D_R)\) is a cellular inclusion.

Hence \(C^G\) has canonical pushouts along cellular inclusions.

Proof. The category of simplicial \(R\)-modules can be equipped with canonical pushouts. For example, in the above situation one could take the coproduct of \(B\) and \(C\) and divide out the relations from \(A\). We discuss the cellular structure.

Let \(e\) be a cell in \(B\) not in \(A\) with attaching map \(\alpha\). This gives a cell in \(D\) with attaching map \(f \circ \alpha\). This way one shows that \(C \rightarrow D\) is cellular, and therefore so is \(D\). Hence there is a canonical isomorphism \(\circ_R D \cong \circ_R C \cup (\circ_R B \setminus \circ_R A)\). Define \(\kappa^D_R : \circ_R D \rightarrow X\) via that isomorphism. We get the control conditions of Table 2. As \(\varphi\) is \(G\)-equivariant, \(D\) is a controlled \(G\)-equivariant cellular \(R\)-module. 

\(\blacksquare\)
control conditions we have | control conditions we get
---|---
$A, B$ $E_B$-controlled | $D$ $E_C \cup E_f \circ E_B$-controlled
$C$ $E_C$-controlled | $	ilde{f}$ $E_f \circ E_B$-controlled
$f$ $E_f$-controlled | $g$ $E$-controlled
$g_C, g_B$ $E$-controlled

Table 2: Control conditions on pushouts along cellular inclusions in $C^G$.

Having a canonical pushout is important for the functoriality of the cylinder functor, which we will introduce later. There are no canonical pushouts along cofibrations in $C^G$, but as cofibrations are isomorphic to cellular inclusions, pushouts along cofibrations also exist in $C^G$.

6.1.2 The subcategory of cofibrations One can use the lemma above to show that the composition of cofibrations is again a cofibration. Isomorphisms are cofibrations, the map $* \to M$ from the trivial module to any controlled module $M$ is a cofibration. Lemma 6.1.1 immediately implies that pushouts along cofibrations exist. These are the axioms of a category with cofibrations in the sense of [Wal85, 1.1], which are therefore satisfied by $C^G$.

Note that in our setting a retract of a cofibration can not be a cofibration in general, as pushouts along such maps do not need to be cellular. This is already true for discrete rings and free modules.

6.1.3 The Cylinder Functor We defined for a map $f: M \to N$ in $C^G$ the cylinder functor as $T(f) := A[\Delta^1] \cup_{i_0} B$. We will outline how to verify that this indeed gives a cylinder functor in the sense of [Wal85, 1.6]. If we need to refer to the object $T(f)$, we call it the mapping cylinder of $f$.

$T$ gives a functor from $\text{Arc}^G$ into diagrams in $C^G$, taking $f: A \to B$ to a commutative diagram

$$
A \xrightarrow{i_0} T(f) \xleftarrow{i_1} B.
$$

(1)

Here $i_0$ is called the front inclusion, $i_1$ is called the back inclusion and $p$ is called the projection. Waldhausen requires the following two axioms to be satisfied.
1. (Cyl 1) Front and back inclusion assemble to an exact functor

\[
\text{Ar} C \rightarrow \mathcal{F}_1 C
\]

\[f \mapsto (\iota_0 \lor \iota_1 : A \lor B \rightarrow T(f)).\]

2. (Cyl 2) \(T(* \rightarrow A) = A\) for every \(A \in C\) and the projection and the back inclusion are the identity map on \(A\).

Here \(\text{Ar} C\) is the arrow category of \(C\) and \(\mathcal{F}_1 C\) is the full subcategory of \(\text{Ar} C\) with objects the cofibrations. Both can be made into categories with cofibrations, with the cofibrations of \(\mathcal{F}_1 C\) being slightly non-obvious. See [Wal85, 1.1] for the precise definitions and the notion of an exact functor. We use the notion “\(A \lor B\)” from [Wal85] for the coproduct of \(A\) and \(B\). We will first exclude the weak equivalences from the discussion.

(Cyl 2) is directly verified using \(*[\Delta^1] = *\) and choosing the right canonical pushouts along \(* \rightarrow *\).

**Lemma 6.1.4.** Front and back inclusion give a functor \(\text{Ar}^G C \rightarrow \mathcal{F}_1^G C\),

\[f \mapsto (A \lor B \rightarrow T(f)).\]

**Proof.** The only thing to show is that \(A \lor B \rightarrow T(f)\) is a cellular inclusion. Consider the diagram

\[
\begin{array}{ccc}
* & \longrightarrow & A \\
\downarrow & \downarrow I & \\
A & \longrightarrow & A \lor A \hookrightarrow A[\Delta^1] \\
\downarrow f & \Downarrow \Pi & \Downarrow \Pi \Pi \\
B & \longrightarrow & A \lor B \longrightarrow T(f)
\end{array}
\]

Here \(A \lor A \rightarrow A[\Delta^1]\) is the cellular inclusion \(i_0 \lor i_1 : A[0] \lor A[1] = A[0 \Pi 1] \rightarrow A[\Delta^1]\). We claim that every possible square is a pushout along a cellular inclusion. \(I\) is a pushout square by definition, as well as \(I + \Pi\). It follows that \(\Pi\) is one. Further \(\Pi + \Pi\Pi\) is a pushout square by definition of \(T(f)\), so \(\Pi\Pi\) is one. Hence the lower map \(A \lor B \rightarrow T(f)\) is a cellular inclusion by Lemma 6.1.1.

**Lemma 6.1.5.** The functor of Lemma 6.1.4 is exact.

**Proof.** We need to show that the functor respects the structure of a category with cofibrations. Pushouts and the zero object are defined pointwise in \(\text{Ar}^G C\) and \(\mathcal{F}_1^G C\). Therefore \(A \lor B\) and \(T(f)\) commute with pushouts. So we only have to show that the functor maps cofibrations to cofibrations.
Let us briefly recall the cofibrations in $\text{Ar}^G$ and $\mathcal{F}_1^G$, cf. [Wal85, Lemma 1.1.1]. For notation let

$$
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow f & & \downarrow f' \\
B & \rightarrow & B'
\end{array}
$$

be a map in $\text{Ar}^G$ from $A \rightarrow B$ to $A' \rightarrow B'$. It is a cofibration in $\text{Ar}^G$ if both horizontal maps are cofibrations. The category $\mathcal{F}_1^G$ is the full subcategory of $\text{Ar}^G$ with objects being the cofibrations in $C^G$. Hence if $f$ and $f'$ are cofibrations, the diagram also shows a map in $\mathcal{F}_1^G$. It is a cofibration in $\mathcal{F}_1^G$ if $A \rightarrow A'$ and $A' \cup_A B \rightarrow B'$ are cofibrations in $C^G$. (It follows that $B \rightarrow B'$ is a cofibration.) See [Wal85, Lemma 1.1] for details and a proof that the composition of cofibrations in $\mathcal{F}_1^G$ is again a cofibration.

We have to show that for a map (2) which is a cofibration in $\text{Ar}^G$ the maps $A \cup B \rightarrow A' \cup B'$ and $(A' \cup B') \cup_{A \cup B} T(f) \rightarrow T(f')$ are cofibrations in $C^G$. As functors respect isomorphisms we can assume that all cofibrations are cellular inclusions.

So assume we have a diagram (2) where the vertical maps are cellular inclusions. We can factor (2) into

$$
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow f & & \downarrow f' \\
B & \rightarrow & B'
\end{array}
$$

It suffices to check each map individually. The map $A \cup B \rightarrow A \cup B'$ is a pushout along the cofibration $B \rightarrow B'$, similar for $A \cup B' \rightarrow A' \cup B'$. Hence both are cofibrations.

Recalling that by Definition $T(f)$ is the pushout $B \cup_f A[\Delta^1]$

$$
\begin{array}{ccc}
A & \rightarrow & A[\Delta^1] \\
\downarrow f & & \downarrow T(f) \\
B & \rightarrow & T(f)
\end{array}
$$

we see that $T(f^*)$ is the pushout

$$
\begin{array}{ccc}
B & \rightarrow & T(f) \\
\downarrow & & \downarrow \\
B' & \rightarrow & T(f^*)
\end{array}
$$
and hence (by “canceling A” by a similar pushout argument as in the proof of Lemma 6.1.4) it is the pushout

\[
\begin{array}{ccc}
A \cup B & \longrightarrow & T(f) \\
\downarrow & & \downarrow \\
A \cup B' & \longrightarrow & T(f^*)
\end{array}
\]

so the map \((A \cup B') \cup_{A \cup B} T(f) \to T(f^*)\) is an isomorphism and therefore a cellular inclusion. Using the canceling argument for \(B'\) we can write the other map

\[
(A' \cup B') \cup_{A \cup B} T(f^*) \to T(f')
\]
as

\[
A' \cup_{A[1]} A[\Delta^1] \cup f^* B' \to A'[\Delta^1] \cup f_1 B'.
\]

(3)

Here the first object is a cylinder where we glued in spaces at both sides. But because \(A \to A'\) is a cellular inclusion so is \(A'[0] \cup_{A[0]} A[\Delta^1] \cup_{A[1]} A'[1] \to A'[\Delta^1]\). We have the commutative diagram

\[
\begin{array}{ccc}
A[1] & \longrightarrow & A'[0] \cup_{A[0]} A[\Delta^1] \\
\downarrow & & \downarrow \\
A'[1] & \longrightarrow & A'[0] \cup_{A[0]} A[\Delta^1] \cup_{A[1]} A'[1] \times A'[\Delta^1] \\
\downarrow & & \downarrow \\
B' & \longrightarrow & A'[0] \cup_{A[0]} A[\Delta^1] \cup f^* B' \longrightarrow A'[\Delta^1] \cup f_1 B'
\end{array}
\]

where every square and in particular the lower right one is a pushout (by the same reasoning as in the proof of Lemma 6.1.4). The lower right horizontal map is the map (3). Hence using Lemma 6.1.1 one last time it follows that the map (3) is a cellular inclusion.

Weak equivalences in \(\text{Ar}^G\) and \(\mathcal{F}_1 \text{C}^G\) are defined pointwise. As \(A \to A[\Delta^1]\) respects homotopy equivalences, the cylinder functor respects homotopy equivalences, too.

### 6.2 The homotopy extension property and the glueing lemma

We want to prove the glueing lemma for the homotopy equivalences in \(\text{C}^G(X; R)\). The main ingredient is the relative homotopy extension property, which we will prove first. The glueing lemma is then a relatively formal consequence, but the proof is a bit lengthy and in the end relies on the fact that the glueing
lemma holds in a category of cofibrant objects. So we will show that $C^G(X; R)$ is a category of cofibrant object in the sense of [GJ99] I.8.

Recall that the $i$th horn $\Lambda^n_i \subseteq \Delta^n$ is $\partial \Delta^n$ minus the $i$th face, see e.g. [GJ99] I.1, p. 6.

**Lemma 6.2.1** (Relative Horn-Filling). Let $M, P \in C^G$. Let $A$ be a cellular submodule of $M$, let $A^n_i \subseteq \Delta^n$ be a horn. Any controlled maps $A[\Delta^n] \to P$ and $M[A^n_i] \to P$ which agree on $A[A^n_i]$ can be extended to a controlled map $M[\Delta^n] \to P$.

If $M$ is $E_M$-controlled and both maps to $P$ are $E_f$-controlled, then the extended map can be chosen to be $E_f \circ E_M$-controlled.

**Proof.** First we prove the claim of the lemma if $G = \{1\}$. It suffices to produce a controlled retraction $r: M[\Delta^n] \to M[A^n_i] \cup A[A^n_i] A[\Delta^n]$ with section the inclusion.

Let $B_k := A \cup M_k$, where $M_k$ is the submodule of $M$ generated by all cells of dimension $\leq k$. We do induction over $k$. We assume the following induction hypothesis:

1. There is a retraction $g_k: M[A^n_i] \cup B_k[A^n_i] B_k[\Delta^n] \to M[A^n_i] \cup A[A^n_i] A[\Delta^n]$.  
2. For each $e_0 \in \circ R M$ the map $g_k$ restricts to  

$$\langle e_0 \rangle_M [\Delta^n] \cap (M[A^n_i] \cup B_k[A^n_i] B_k[\Delta^n]) \xrightarrow{g_k} \langle e_0 \rangle_M [\Delta^n] \cap (M[A^n_i] \cup A[A^n_i] A[\Delta^n]) \quad (4)$$

The second condition is needed to gain enough control. It is important that the condition holds for all cells of $M$ and not only the ones from $B_k$. We abbreviate $N_k := (M[A^n_i] \cup B_k[A^n_i] B_k[\Delta^n])$. Hence $g_k$ is a map $N_k \to N_{-1}$.

For $k = -1$ the induction hypothesis is satisfied because $g_{-1} = \text{id}$.

So assume the induction hypothesis holds for $k - 1$. As $B_k$ arises from $B_{k-1}$ by attaching cells of dimension $k$ it suffices to treat the case of attaching one cell $e_i$ as cells of the same dimension can be attached independently.

We get a diagram

$$\begin{array}{ccc} R[\Delta^k \times \Lambda^n_i \cup \partial \Delta^k \times \Delta^n] & \xrightarrow{\partial e_i} & B_k[A^n_i] \cup B_{k-1}[A^n_i] B_{k-1}[\Delta^n] \xrightarrow{g_{k-1}} N_{-1} \\ \downarrow & & \downarrow \\ R[\Delta^k \times \Delta^n] & \xrightarrow{e_i} & B_k[\Delta^n] \quad (5) \end{array}$$

where we want to find the dashed lift to get $g_k$. Here $R$ is the module $R[\Delta^0]$ over $\kappa^M(e)$. But as the square is a pushout, as one checks, it suffices to find a lift $R[\Delta^k \times \Delta^n] \to N_{-1}$. The lower horizontal map $e_i$ of course factors over $\langle e \rangle_M [\Delta^n]$, so does the upper horizontal one. By the induction hypothesis
then $g_{k-1} \circ \partial e_\ast \subseteq \langle e \rangle_M [\Delta^n] \cap N_{-1}$. We want to find a lift $R[\Delta^k \times \Delta^n] \rightarrow \langle e \rangle_M [\Delta^n] \cap N_{-1}$.

By the adjunction from section 2.5.4 it suffices to find a lift in the diagram of simplicial sets

$$
\Delta^k \times \Lambda_i^n \cup \partial \Delta^k \times \Delta^n \rightarrow \text{HOM}_R^E(R, \langle e \rangle_M [\Delta^n] \cap N_{-1}).
$$

But

$$
\text{HOM}_R^E(R, \langle e \rangle_M [\Delta^n] \cap N_{-1}) = \langle e \rangle_M [\Delta^n] \cap N_{-1}
$$

as $\langle e \rangle_M$ has bounded support. Such a lift exists as the vertical inclusion arises by repeated horn-filling (cf. [GJ99, p. 18/19]) and $\langle e \rangle_M [\Delta^n] \cap N_{-1}$ is an abelian group and hence Kan. This gives a lift $R[\Delta^k \times \Delta^n] \rightarrow \langle e \rangle_M [\Delta^n] \cap N_{-1}$, which extends $g_{k-1}$ to $g_k : N_k \rightarrow N_{-1}$.

Note that $g_k$ restricts to a map $\langle e \rangle [\Delta^n] \rightarrow \langle e \rangle_M [\Delta^n] \cap N_{-1}$ which is exactly the second condition of the induction hypothesis for $k$ if $e_0 = e \in \partial_R M$. For general $e_0 \in \partial_R M$ consider first the case $e \notin \partial_R \langle e_0 \rangle_M$. Then $\langle e_0 \rangle_M [\Delta^n] \cap N_k \subseteq \langle e_0 \rangle_M [\Delta^n] \cap N_{k-1}$ and the induction hypothesis follows from the induction hypothesis for $k - 1$. Otherwise $\langle e \rangle_M \subseteq \langle e_0 \rangle_M$ and then $g_k$ restricts to

$$
\langle e_0 \rangle_M [\Delta^n] \cap N_k = (\langle e_0 \rangle_M [\Delta^n] \cap N_{k-1}) \cup \langle e \rangle_M [\Delta^n] \\
\rightarrow (\langle e_0 \rangle_M [\Delta^n] \cap N_{-1}) \cup (\langle e \rangle_M [\Delta^n] \cap N_{-1}) \subseteq \langle e_0 \rangle_M [\Delta^n] \cap N_{-1}
$$

Setting $r := \text{colim}_{k \rightarrow \infty} g_k$ yields the retraction, which has the property that $r(\langle e \rangle_M [\Delta^n]) \subseteq \langle e \rangle_M [\Delta^n] \cap N_{-1}$, so it is $E_M$-controlled when $E_M$ is the control of $M$.

If $G \neq \{1\}$ we can choose the above lifts equivariantly, e.g. by constructing first a lift for one cell in a $G$-orbit and then extending equivariantly. This shows the general case.

As a special case it follows that cofibrations have the homotopy extension property, i.e., homotopies on $A$ can be extended to $M$. The usual arguments show that being homotopic is an equivalence relation, and that the homotopy equivalences satisfy the 2-out-of-3 property, i.e., the saturation axiom.

We need a little bit more of homotopy theory.

**Definition 6.2.2 (Deformation retraction).** Let $i : A \rightarrow M$ be a cellular inclusion in $C^G$, i.e., we can consider $A$ as a submodule of $M$. $A$ is a
deformation retract of $M$ if there is a map $r : M \to A$ such that $r \circ i$ is $\text{id}_A$ and $i \circ r$ is homotopic to $\text{id}_M$ relative $A$.

The map $i$ is called the inclusion and $r$ is called the retraction or deformation retraction.

For $f : A \to B$ the target $B$ is a retract of the mapping cylinder $T(f)$.

**Lemma 6.2.3.** $p : T(f) \to B$ is even a deformation retraction.

**Proof.** We only have to prove that $i_1 \circ p : T(f) \to T(f)$ is homotopic relative $B$ to $\text{id}_{T(f)}$. Recall that $T(f)$ is defined as the pushout of $B \leftarrow A[1] \rightarrow A[\Delta^1]$. We see that $i_1 \circ p$ is induced by $p_1 : A[\Delta^1] \rightarrow A[1] \rightarrow A[\Delta^1]$. It suffices to give a homotopy from the identity to $A[\Delta^1] \rightarrow A[1] \rightarrow A[\Delta^1]$ which is relative to $A[1]$.

But there is a well-known map $\tilde{H} : \Delta^1 \times \Delta^1 \to \Delta^1$ of simplicial sets inducing such a map. Thus the homotopy $H$ which is induced by $\tilde{H}$ is a homotopy relative to $A[1]$ which induces the desired homotopy.

The lemma obviously implies that $p$ is a homotopy equivalence, hence the cylinder functor satisfies the cylinder axiom of [Wal85, 1.6]. For the next part we need a diagram language.

### 6.2.4 Describing maps by diagrams

In the following we will often have to describe maps of the form $A[\Delta^1 \times \Delta^1] \to B$ or similar. We give concise ways to describe them. The simplicial set $\Delta^1 \times \Delta^1$ comes from a simplicial complex, so it suffices to give compatible maps on the 0-, 1- and 2-simplices.

We use the pictures

```
  . . ,  . . ,  . . ,  . . ,  . . , etc.
```

to denote simplicial subsets of $\Delta^1 \times \Delta^1$ which are generated by the shown 1- and 2-simplices. A 2-simplex is in the subset if its boundary is. The dots are only drawn to specify the corresponding subset and are only in the subset if they are a boundary. As an example, we write a map

$$A[\Delta^1 \times \{0, 1\}] \cup A[0 \times \Delta^1] \to B \quad \text{as} \quad A[\ . . . , \ . . . ] \to B.$$  

We can use the same kind of diagrams for other simplicial sets like $\Delta^1 \cup \Delta^0 \leftrightarrow \Delta^1$ and products of them.

If we want to concisely describe such maps we often draw diagrams like the ones above and write the maps into it. Here are some examples. The left picture below shows a homotopy $H$ from $\alpha$ to $\beta$, the middle one shows a horn $A[\Delta_2^0] \to B$, and the right one the map $A[\Delta_2^1] \to B$ which arises by filling the horn in the middle. $\text{Tr}$ denotes a constant ("Trivial") homotopy.

```
\alpha \rightarrow H \rightarrow \beta
\alpha \rightarrow \text{Tr} \rightarrow \beta
\alpha \rightarrow H \rightarrow \beta
```

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The diagram shows how one can prove the symmetry of the relation “homotopic” by horn-filling. We sometimes call \( H \) the “inverse homotopy” to \( H \). It is usually not unique.

Sometimes we leave out the decorations for vertices, as they are uniquely determined by the decorations on the arrows, and draw dots instead. We usually leave out the decoration for the 2-simplices as well, as the actual maps are usually less important for us. All this works for more complicated simplicial sets as long as we can draw diagrams for them.

6.2.5 Rectifying homotopy commutative diagrams If we have the homotopy commutative diagram on the left below, we can turn it into a strictly commutative diagram on the right below.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
\ast & \xrightarrow{g'} & P
\end{array}
\quad \sim \quad
\begin{array}{ccc}
A & \xrightarrow{\iota_0} & T(f) \\
\downarrow{h} & & \downarrow{g'} \\
\ast & \xrightarrow{} & P
\end{array}
\]

Define \( g': A[\Delta^1] \cup B \to P \) as induced by the homtopy and by \( g \). The back inclusion \( \iota_1: B \to T(f) \) is a homotopy equivalence and \( g = g' \circ \iota_1 \).

If \( f: A \to B \) is a homotopy equivalence, then the two-out-of-three property implies that then \( i: A \to T(f) \) is a homotopy equivalence.

Proposition 6.2.6. If \( f: A \to B \) is a homotopy equivalence, then \( A \) is even a deformation retract of \( T(f) \) via \( \iota_0 \).

We prove the proposition in the rest of this section. It is an adaption of the corresponding proof for topological spaces which the author learned from F. Waldhausen [Wal, pp. 140ff.]. Let \( g \) be the homotopy inverse of \( f \), then we have a homotopy commutative diagram \( \text{id}_A = g \circ f \). By the argument above we get a map \( T(f) \to A \). This is the retraction \( r \).

Lemma 6.2.7. The map \( \text{id}_A \xrightarrow{\iota_0} A \xrightarrow{\iota_0} T(f) \) is homotopic to the identity.

Proof. By Lemma 6.2.3 the composition \( T(f) \xrightarrow{\iota_0} B \xrightarrow{\iota_1} T(f) \) is a homotopic to the identity. So we pre- and postcompose \( T(f) \to A \to T(f) \) with \( T(f) \to B \to T(f) \) and get a map which is homotopic to it. This can be written as

\[
\begin{array}{ccc}
T(f) & \xrightarrow{r} & A \\
\downarrow{\iota_1} & \downarrow{g} & \downarrow{f} \\
B & \xrightarrow{p} & B \\
\downarrow{p} & \downarrow{\iota_1} & \downarrow{p} \\
T(f) & \xrightarrow{} & T(f)
\end{array}
\]
with compositions identified as \( f \) and \( g \). But \( f \circ g \) is homotopic to \( \text{id}_B \) by assumption. So we are left with

\[
\begin{array}{ccc}
B & \xrightarrow{\text{id}} & B \\
\uparrow_p & & \downarrow_{\iota_1} \\
T(f) & & T(f)
\end{array}
\]

which is homotopic to \( \text{id}_{T(f)} \) again by Lemma 6.2.3. Being homotopic is an equivalence relation so \( \iota_0 \circ r \) is homotopic to \( \text{id}_{T(f)} \). □

The homotopy does not need to be relative to \( A \), but we can improve it as follows. Let \( s := \iota_0 \circ r : T(f) \to A \to T(f) \) and let \( H \) be the homotopy from \( \text{id}_{T(f)} \) to \( s \) we get by Lemma 6.2.7. We have \( s \circ \iota_0 = \iota_0 \) as well as \( \text{id}_{T(f)} \circ \iota_0 = \iota_0 \) so on the endpoints \( H \) is relative to the cellular inclusion \( \iota_0 : A \to T(f) \). We want to make the whole homotopy relative to \( A \), i.e., \( A[\Delta^1] \xrightarrow{\iota_0[\Delta^1]} T(f)[\Delta^1] \xrightarrow{H} T(f) \) should be equal to \( A[\Delta^1] \xrightarrow{\iota_0} A \xrightarrow{\iota_0} T(f) \).

**Lemma 6.2.8.** Let \( s \) be the map \( T(f) \xrightarrow{r} A \xrightarrow{\iota_0} T(f) \). There is a homotopy relative \( A \) from the identity on \( T(f) \) to \( s \).

**Proof.** (We use the diagram notation of 6.2.4) \( A \) is a retract of \( T(f) \) and \( \iota_0 : A \to T(f) \) has the homotopy extension property. We will use this homotopy extension property to construct a certain map \( T(f)[\Delta^1 \times \Delta^1] \to T(f) \) which restricted to \( 1 \times \Delta^1 \) will be the desired homotopy from \( \text{id}_{T(f)} \) to \( s \) relative to \( A \).

Note that \( s \) is an idempotent, i.e., \( s^2 = s \). We use the notation from above. The proof will proceed as follows. We will prescrybe the map \( T(f)[\Delta^1 \times \Delta^1] \to T(f) \) on the subspace \( A[\Delta^1 \times \Delta^1] \to T(f)[\Delta^1 \times \Delta^1] \) and on the top, bottom and left part of \( \Delta^1 \times \Delta^1 = \cdots \), i.e., on \( T(f)[\cdots \cdots] \). Then we check that the two maps are compatible. This will give a map

\[
T(f)[\cdots \cdots] \cup A[\cdots \cdots] \to T(f)
\]

which can be extended by the homotopy extension property to the desired map \( T(f)[\cdots \cdots] \to T(f) \).

Both maps will be constructed from the same map, which we describe first. Horn-filling gives for any map \( T(f)[\cdots \cdots] \to T(f) \) a map \( T(f)[\cdots \cdots] \to T(f) \), in particular we get for the first diagram below the second one, where \( \overline{H} \) is the “inverse homotopy”. Extending this as in the third diagram below gives a map \( G : T(f)[\Delta^1 \times \Delta^1] \to T(f) \).

\[
\begin{array}{ccc}
\vdots & T_f & \vdots \\
\uparrow & \uparrow & \uparrow \\
\vdots & \vdots & \vdots \\
\end{array}
\quad
\begin{array}{ccc}
\vdots & T_f & \vdots \\
\uparrow & \uparrow & \uparrow \\
\vdots & \vdots & \vdots \\
\end{array}
\quad
\begin{array}{ccc}
\vdots & T_f & \vdots \\
\uparrow & \uparrow & \uparrow \\
\vdots & \vdots & \vdots \\
\end{array}
\]

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Define the map $A[Δ^1 × Δ^1] → T(f)$ as the restriction of $G$ to $A[Δ^1 × Δ^1]$. Define the map $T(f)[0 × Δ^1] → T(f)$ as

\[
\begin{array}{ccc}
\bullet & \xrightarrow{H_{0,s}} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{-M_{1}} & \bullet
\end{array}
\]

so on the $\bullet \bullet \bullet$-part it is the restriction of $G$, but on the upper part $\bullet \bullet \bullet$ we replace the homotopy $H$ by $H \circ s$. This replacement is crucial for the proof.

We check that these maps are compatible. First $H$ is a homotopy from $s$ to $id$, hence $H \circ s$ is a homotopy from $s^2$ to $s$; but $s^2 = s$ so it agrees with $H$ on the upper left vertex. Second, restricted to $A$ the map $s$ is the inclusion $ι_0: A → T(f)$, hence $H \circ s \circ ι_0 = H \circ ι_0$. So this glues to a map

\[
T(f)[0 × Δ^1] ∪ A[0 × Δ^1] → T(f).
\]

This can be interpreted as a map $T(f)[0 × Δ^1] → T(f)$ together with a homotopy on the submodule $T(f)[0 × \{0,1\}] ∪ A[0 × Δ^1]$. So using the homotopy extension property we get map $T(f)[Δ^1 × Δ^1] → T(f)$. This map in turn defines a homotopy when restricting along $T(f)[1 × Δ^1] → T(f)[Δ^1 × Δ^1]$ (which is $T(f)[1 × Δ^1] → T(f)[Δ^1 × Δ^1]$). This homotopy starts at the identity, ends at the map’s and is the constant homotopy on $A$. Hence it is the desired homotopy.

6.2.9 Alternative proof Another way to prove this statement is the following idea, which I owe to a discussion with Wolfgang Steimle. Assume we have cofibrations $A ↪ X, A ↪ Y$ and a map $f: X → Y$ respecting the inclusions.

**Lemma 6.2.10.** Assume that $f$ has a homotopy inverse. Then $f$ has a homotopy inverse under $A$. This is, the inverse and all homotopies respect the inclusion of $A$.

Here is a sketch of a proof. We can assume that $f$ is homotopic to $id_X$. We have to find a left-inverse to $f$, such that the composition is homotopic to $id_X$ relative to $A$. We use our diagram language. Let $H$ be the homotopy $f ↪ id_X$. Restricted to $A$ it is a homotopy $id_A ↪ id_A$. Using homotopy extension we get the dotted map in the diagram below and define $g$ as the shown composition.
Let $S$ be the simplicial set $\cdots \times \cdots$. We want to get a map $X[S] \to X$ via horn-filling. We prescribe it on $X[S']$ and $A[S]$ as shown below, where $S'$ is the subset of $S$ shown on the left.

Filling it to $X[S] \to X$ and restricting to $X[\cdots \times 1]$ we get the desired homotopy from $id$ to $g \circ f$ which is constant on $A$. The same argument gives a left-inverse to $g$, hence $f$ has a two-sided inverse.

We keep the original proof because it is more explicit. Also it generalizes directly in subsequent work, when we treat germs.

### 6.3 Pushouts of weak equivalences

**Lemma 6.3.1.** Let

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

be a pushout diagram in $C^G$ where $A \to C$ is a cofibration and a homotopy equivalence. Then $B \to D$ is a homotopy equivalence.

This is a key result on the way to prove the Glueing Lemma for homotopy equivalences. We remark that almost exactly the same proof works if we assume that $A \to B$ is a cofibration instead of $A \to C$.

We can factor $f : A \to C$ into $A \to T(f) \to C$. Taking the pushouts along the cellular inclusion $A \to T(f)$ and along the cofibration $A \to C$ gives a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
T(f) & \longrightarrow & Q \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

and the induced map $Q \to D$ completes the lower square to a pushout square.

The following Lemma shows that both maps $B \to Q$ and $Q \to D$ are homotopy equivalences, so their composition $B \to D$ is one.
Lemma 6.3.2. In the situation above the following holds.

1. The map $B \to Q$ is a homotopy equivalence.
   (This uses that $A \to C$ is a homotopy equivalence.)
2. The map $Q \to D$ is a homotopy equivalence.
   (This uses that $A \to C$ is a cofibration.)

Proof of part 1. By Proposition 6.2.6 $A$ is a deformation retract of $T(f)$ via the cellular inclusion $A \hookrightarrow T(f)$. One checks directly that then $B$ is a deformation retract of $Q$. Hence in particular $B \to Q$ is a homotopy equivalence.

Proof of part 2. Written out $Q \to D$ is the map

$$B \cup_{A[0]} A[\Delta^1] \cup_{A[1]} C \to B \cup_A C$$

induced by $A[\Delta^1] \to A$. We have to construct a homotopy inverse for this map. We will construct a homotopy equivalence $A[\Delta^1] \cup_{A[1]} C \to C$ and its inverse which is relative to $i_0^A: A[0] \hookrightarrow A[\Delta^1] \cup_{A[1]} C$, resp. to $j_A: A \hookrightarrow C$, hence glues along $A[0] \to B$ to the desired homotopy equivalence

$$B \cup_{A[0]} A[\Delta^1] \cup_{A[1]} C \xrightarrow{\simeq} B \cup_A C,$$

as $-[\Delta^1]$ commutes with pushouts. We therefore have to construct for $e: A[\Delta^1] \cup_{A[1]} C \to C$ (induced by $A[\Delta^1] \to A$) maps

$$g: C \to A[\Delta^1] \cup_{A[1]} C$$

and homotopies

$$H: C[\Delta^1] \to C$$

$$G: (A[\Delta^1] \cup_{A[1]} C)[\Delta^1] \to A[\Delta^1] \cup_{A[1]} C$$

with the properties

$$H_0 = e \circ g \quad \quad H_1 = \text{id}$$
$$G_0 = \text{id} \quad \quad G_1 = g \circ e$$
$$G \circ i_0^A[\Delta^1] = i_0^A \quad \quad H \circ j_A[\Delta^1] = j_A$$
$$g \circ j_A = i_0^A \quad \quad e \circ i_0^A = j_A.$$

Using the homotopy extension property of the cofibration $j_A: A \hookrightarrow C$ there is a retraction $R: C[\Delta^1] \to A[\Delta^1] \cup_{A[1]} C$. Define $g$ as the composition

$$C \xrightarrow{i_0^C} C[\Delta^1] \xrightarrow{R} A[\Delta^1] \cup_{A[1]} C.$$
We get \( g \circ j_A = i_0^A \).
Define \( H \) as the composition \( e \circ R : C[\Delta^1] \to A[\Delta^1] \cup A[1] \to C \). One checks that \( H \) is a homotopy from \( e \circ g \) to \( \text{id}_C \) relative to \( A \).

For the other composition consider the commutative diagram

\[
\begin{array}{ccc}
A[\Delta^1] \cup A[1] & \xrightarrow{e} & C \\
\downarrow j & & \downarrow g \\
C[\Delta^1] & \xrightarrow{R} & A[\Delta^1] \cup A[1]
\end{array}
\]

where dashed map is the projection to \( C[0] \). It is homotopic relative \( C[0] \) to the identity. This gives a homotopy \( G \) from the identity to the composition \( g \circ e \), using that \( R \) is a retraction for \( j \). One checks that \( G \) is relative to \( A[0] \).

This shows that \( e \) is a homotopy equivalence and therefore makes \( Q \to D \) into one. \( \square \)

6.4 The Glueing Lemma

The Glueing Lemma is the following statement.

**Lemma 6.4.1.** If we have the diagram in \( C^G \)

\[
\begin{array}{ccc}
B & \xleftarrow{\alpha} & A \xrightarrow{\beta} C \\
\downarrow & & \downarrow \\
B' & \xleftarrow{\alpha'} & A' \xrightarrow{\beta'} C'
\end{array}
\]

with \( A \hookrightarrow B \) and \( A' \hookrightarrow B' \) cofibrations and all three vertical arrows are homotopy equivalences, then the induced map

\[
B \cup_A C \to B' \cup_{A'} C'
\]

on the pushouts is also a homotopy equivalence.

**Proof.** It it shown in Lemma II.8.8 in [GJ99, p. 127] that a category of cofibrant objects satisfies the Glueing Lemma. We recall that notion from [GJ99, p. 122]. It was first introduced by Kenneth Brown in [Bro73], where he treats the dual version.

A category of cofibrant objects is a category \( \mathcal{D} \) which satisfies the following axioms.

0. The category contains all finite coproducts.

1. The 2-out-of-3 property holds for weak equivalences.

2. The composition of cofibrations is a cofibration, isomorphisms are cofibrations.
3. Pushout diagrams of the form

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow^i & & \downarrow^i_* \\
C & \longrightarrow & D
\end{array}
\]

exist when \(i\) is a cofibration. In this case \(i_*\) is a cofibration which is additionally a weak equivalence if \(i\) is one.

4. For each object there is a cylinder object.

5. For each \(X\) the unique map \(* \rightarrow X\) from the initial object is a cofibration.

The notion of a cylinder object in [GJ99, p. 123] is slightly different from our notion, but if the Cylinder Axiom [6.2.3] holds our Cylinder Functor applied to the identity yields a cylinder object in the sense of [GJ99, p. 123].

We have shown that these axioms hold for \(\mathcal{C}G\). Hence the glueing lemma holds.

Summarized, we have established that \(\mathcal{C}^G(X, \mathcal{E}, \mathcal{F}; R)\) has the structure of a category with cofibrations and weak equivalences, where the cofibrations are isomorphic to cellular inclusions and the weak equivalences are the homotopy equivalences. That is, we finished the proof of Theorem [3.1.4].

6.5 The Extension Axiom

Next we want to prove the extension axiom for our category \(\mathcal{C}^G(X; R)\). We will need to use explicitly that we can add maps. Unlike the results in the previous sections, the extension axiom does not hold in Waldhausen’s category of spaces over a point, see [Wal85, 1.2].

Let \(\mathcal{C}\) be a category with cofibrations. A cofiber sequence in \(\mathcal{C}\) is a sequence \(A \rightarrow B \rightarrow C\) in \(\mathcal{C}\) where \(A \rightarrow B\) is a cofibration and \(B \rightarrow C\) is isomorphic to the map \(B \rightarrow B/A := B \cup_A \ast\).

A subcategory \(\mathcal{C}\) of weak equivalences of \(\mathcal{C}\) satisfies the Extension Axiom if for each map of cofiber sequences

\[
\begin{array}{ccc}
A & \rightarrow & B \rightarrow C \\
\downarrow^{f_A} & & \downarrow^{f_B} \\
A' & \rightarrow & B' \rightarrow C'
\end{array}
\]

where \(f_A\) and \(f_C\) are weak equivalences the map \(f_B\) is a weak equivalence. Sometimes \(B\) (resp. \(f_B\)) is called an extension of \(A\) by \(C\) (resp. of \(f_A\) by \(f_C\)).

We first need a relative homotopy lifting property. We directly prove a more general horn-filling property.
Lemma 6.5.1 (Horn-filling relative to a map). Let $A \to M$ be a cellular inclusion in $\mathcal{C}^G$. Let $U \to P$ also be a cellular inclusion in $\mathcal{C}^G$ and let $P \to Q := P/U$ the quotient map. Then $A \to M$ has the relative horn-filling property with respect to $P \to Q$. This means, given a horn $\Lambda^n_i \subseteq \Delta^n$ and a solid commutative diagram of controlled maps

$$
\begin{array}{ccc}
M[\Lambda^n_i] \cup A[\Delta^n] & \to & P \\
\downarrow & & \downarrow \\
M[\Delta^n] & \to & Q
\end{array}
$$

then the dashed lift exists.

Specializing to $n = 1, i = 1$ gives the homotopy lifting property with respect to $P \to Q$. The proof proceeds similarly to the proof of Lemma 6.2.1. It is not stated there in the full generality, as we will need the generalized version only in this section. So we choose to keep the already quite complicated proof of Lemma 6.2.1 a little bit simpler.

We will need the following extra ingredient: Any surjective map $B \to C$ of simplicial abelian groups is a Kan fibration. This follows e.g. from [GJ99, Corollary V.2.7, p. 263]. Consequently, for a cellular inclusion of simplicial $R$-modules $A \to B$, the map $B \to B/A$ is a Kan fibration of simplicial sets.

Proof of Lemma 6.5.1. The proof is very similar to the proof of Lemma 6.2.1, but more involved. The main point is that we need to find a lift relative to the map $P \to Q$. To still keep the control, we have to strengthen the induction hypothesis.

We first treat the case $G = \{1\}$. Let $B_k := A \cup M_k$, where $M_k$ is the submodule of $M$ generated by all cells of dimension $\leq k$. We do induction over $k$. We abbreviate $N_k := (M[\Lambda^1] \cup B_k[\Lambda^1])$, $N_\infty := M[\Delta^n] = \bigcup_k N_k$. We have to find a lift in the diagram (which also fixes our notation for the maps)

$$
\begin{array}{ccc}
N_{\infty} & \stackrel{f}{\longrightarrow} & P \\
\downarrow & & \downarrow \\
N_{-1} & \to & M[\Lambda^1] \cup A[\Delta^n] \\
\downarrow & & \downarrow \\
N_{\infty} & \stackrel{h}{\longrightarrow} & Q
\end{array}
$$

We need to be able to restrict $p$ “locally”, such that it is still a fibration. It suffices that we construct “locally” maps which are surjections of abelian simplicial groups after forgetting control and $R$-module structure. We make the following choices. For each $e_Q \in \varphi_R Q$ choose an $\vartheta(e_Q) \in \varphi_R P$ with $p(\vartheta(e_Q)) = e_Q$. Such a map $\vartheta : \varphi_R Q \to \varphi_R P$ exists as $\varphi_R P \cong \varphi_R Q \cup \varphi_R U$.

We assume the following induction hypothesis:

---

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1. There is a map $g_k : N_k \to P$ which extends $f$ over $h$, i.e., is a partial lift in the diagram (8).

2. For each $e_0 \in \odot_R M$ the map $g_k$ restricts to

$$\langle e_0 \rangle_M [\Delta^n] \cap N_k \xrightarrow{g_k} \langle f((\langle e_0 \rangle_M [\Delta^n] \cap N_{-1})_P \cup \bigcup \left\{ \langle \vartheta(e_Q) \rangle_P \mid e_Q \in \odot_R \langle h((\langle e_0 \rangle_M [\Delta^n])_Q \right\} \right.$$  

(9)

The second condition implies that $g_k$ is $E_f \circ E_M \cup E_h \circ E_P$-controlled. Roughly speaking it ensures that the lift does not hit a module which is uncontrollably large. Here is a reason for why it has to be at least that size. First we must allow a cell $e_0$ to at least hit the image of $f$ of the part of the cell intersecting $N_{-1}$. Second, the cell hits certain elements in $Q$, so we must have possible lifts for all of them.

We do induction over $k$. We can attach cells of the same dimension independently, so we only treat the case of attaching one cell $e$ of dimension $k$. As before the left square of the following diagram is a pushout.

We can replace the middle column by $N_{k-1} \to N_k$ and the diagram remains commutative and the left square a pushout. So we only have to find a lift in the outer diagram of (10). We abbreviate

$$P_f(e) := \langle f((\langle e \rangle_M [\Delta^n] \cap N_{-1})_P$$

$$P_h(e) := \bigcup \left\{ \langle \vartheta(e_Q) \rangle_P \mid e_Q \in \odot_R \langle h((\langle e \rangle_M [\Delta^n])_Q \right\.$$  

(10)

Both are cellular submodules of $P$. We get a factorization of the outer diagram of (10)

by the induction hypothesis and it suffices to find a lift in the left diagram. By the fundamental lemma, and because the middle column in the diagram

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has bounded support on some \( \{x\}^E \), it suffices to find a (dashed) lift in the diagram of simplicial sets

\[
\begin{array}{c}
\Delta^k \times \Lambda^n_i \cup \partial \Delta^k \times \Delta^n \\
\downarrow \\
\Delta^k \times \Delta^n \\
\downarrow \\
\langle h(\langle e \rangle_M [\Delta^n]) \rangle_Q
\end{array} \xrightarrow{Pf(e) \cup Ph(e)} \\
\downarrow \\
\langle h(\langle e \rangle_M [\Delta^n]) \rangle_Q
\]

Such a lift exists if the right map is a Kan Fibration. But as it is a homomorphism of simplicial abelian groups it suffices to show that it is surjective. But \( Ph(e) \rightarrow \langle h(\langle e \rangle_M [\Delta^n]) \rangle_Q \) is already surjective by construction, as \( Ph(e) \) has exactly one cell \( e_P \) for each cell \( e_Q \) of \( \langle h(\langle e \rangle_M [\Delta^n]) \rangle_Q \) and by definition of \( \vartheta \) the cell \( e_P \) is mapped to \( e_Q \). This gives the lift \( g_k \), and by construction it satisfies the first condition of the induction hypothesis for \( k \).

The second condition is satisfied for \( e \) by construction and for \( e_0 \) with \( e \notin \langle e_0 \rangle_M \) by the induction hypothesis for \( k - 1 \). Otherwise \( \langle e \rangle_M \subseteq \langle e_0 \rangle_M \) which implies \( Pf(e) \subseteq Pf(e_0) \) and \( Ph(e) \subseteq Ph(e_0) \). Then \( g_k \) restricted to \( \langle e_0 \rangle_M [\Delta^n] \) factors as

\[
\langle e_0 \rangle_M [\Delta^n] \cap N_k = \langle e_0 \rangle_M [\Delta^n] \cap (N_{k-1} \cup \langle e \rangle_M [\Delta^n])
\]

\[
\xrightarrow{g_k} Pf(e_0) \cup Ph(e_0) \cup Pf(e) \cup Ph(e) = Pf(e_0) \cup Ph(e_0)
\]

Therefore the second condition is also satisfied.

If \( G \neq \{1\} \) we can choose the above lifts equivariantly, e.g. by constructing first a lift for one cell in a \( G \)-orbit and then extending equivariantly. This shows the general case.

\[\square\]

**Lemma 6.5.2 (Extension axiom).** Let

\[
\begin{array}{c}
A \\
\downarrow \sim \downarrow \sim
\end{array} \xrightarrow{} \\
\begin{array}{c}
B \\
\downarrow \sim \downarrow \sim
\end{array} \xrightarrow{} \\
\begin{array}{c}
C \\
\end{array}
\]

be a map of cofiber sequences in \( C^G \). Assume that \( A \rightarrow A' \) and \( C \rightarrow C' \) are homotopy equivalences. Then \( B \rightarrow B' \) is a homotopy equivalence.

**Proof.** We can factor the vertical maps functorially by using the Cylinder Functor. As a Cylinder Functor is exact it respects the cofiber sequences. We
get a diagram

\[
\begin{array}{c}
A \rightarrow B \rightarrow C \\
\sim \quad \sim \\
T_A \rightarrow T_B \rightarrow T_C \\
\sim \quad \sim \quad \sim \\
A' \rightarrow B' \rightarrow C'
\end{array}
\]

By Proposition 6.2.6 \( A \) and \( C \) are deformation retracts of \( T_A \) and \( T_C \), respectively, with the inclusions being the left and the right vertical upper maps. What remains to be shown is that the vertical upper middle map is a homotopy equivalence. This is proved in Lemma 6.5.3 below, where it is shown that \( B \) is a deformation retract of \( T_B \).

\textbf{Lemma 6.5.3.} Assume we have a cofiber sequence \( A \rightarrow B \rightarrow \mathbb{B} \) in \( C^G \) where \( \mathbb{B} = B/A \) for brevity. Suppose we have a diagram

\[
\begin{array}{c}
A \rightarrow B \rightarrow \mathbb{B} \\
\sim \quad \sim \\
T_A \rightarrow T_B \rightarrow T_{\mathbb{B}}
\end{array}
\]

in \( C^G \) where the horizontal lines are cofiber sequences and the vertical arrows are cellular inclusions. Suppose that \( A \) and \( \mathbb{B} \) are deformation retracts of \( T_A \) and \( T_{\mathbb{B}} \) with inclusions the left and right vertical maps. Then \( B \) is a deformation retract of \( T_B \) with inclusion the middle vertical map.

We prove a slightly stronger statement than Lemma 6.5.3:

\textbf{Lemma 6.5.4.} Assume that we are in the situation of Lemma 6.5.3. Let \( D_0 \) be a cellular submodule of \( D \). Then each controlled map \( (D,D_0) \rightarrow (T_B,B) \) of pairs in \( C^G \) is controlled homotopic relative \( D_0 \) to a map into \( B \).

\textbf{Proof of Lemma 6.5.3 using 6.5.4.} By Lemma 6.5.4 the map id: \( (T_B,B) \rightarrow (T_B,B) \) is controlled homotopic relative \( B \) to a map \( T_B \rightarrow B \). This is the desired deformation retraction.

\textbf{Remark 6.5.5 (Toy situation).} Assume we have a commutative diagram of abelian groups

\[
\begin{array}{c}
A \rightarrow B \rightarrow \mathbb{B} \\
\downarrow \quad \downarrow \\
A' \rightarrow B' \rightarrow \mathbb{B}
\end{array}
\]
where the horizontal lines are short exact sequences. Assume that the outer maps are surjective. We want to show that the middle map is surjective. The proof proceeds exactly like the proof of Lemma 6.5.4, but is easier. We give it to help with the general proof.

Let $\alpha$ be an element in $B'$. We will denote the constructed elements by consecutive Greek letters and denote projections to the quotient by a bar. So $\bar{\alpha}$ is an element in $B'$. As $B \to B'$ is surjective there is an element $\beta$ in $B$ which maps to $\bar{\alpha}$. As $B \to B$ is surjective there is an element $\gamma$ in $B$ which maps to $\bar{\beta} \in B'$. The elements $f_B(\gamma)$ and $\alpha$ do not need to be equal in $B'$, but they become equal when projected to $\bar{B}'$, so $\alpha - f_B(\gamma)$ factors through $A' \to B'$. As $A \to A'$ is surjective there is an element $\delta$ in $A$ which maps to $\alpha - f_B(\gamma)$ in $B'$. Hence, considered in $B$, $f_B(\delta + \gamma)$ equals $\alpha$.

Lemma 6.5.1 applies to the maps $B \to B$ and $T_B \to T_B$, so we have the relative homotopy lifting property with respect to these maps.

**Proof of Lemma 6.5.4.** Let $\alpha: (D, D_0) \to (T_B, B)$ be a controlled map. This gives a map $\bar{\alpha}$ into $(\bar{B}, T_{\bar{B}})$. As $\bar{B}$ is a deformation retract of $T_{\bar{B}}$ we get a homotopy $\bar{H}: D[\Delta^1] \to T_{\bar{B}}$ from $\bar{\alpha}$ to a map into $\bar{B}$ which is constant on $D_0$. It comes from the deformation of $(\bar{B}, T_{\bar{B}})$ precomposed with $\alpha$. Lemma 6.5.1 applies to the map $T_B \to T_{\bar{B}}$. So we get a lift $H$ of $\bar{H}$, relative to $\alpha$ and $D_0$.

\[
\begin{array}{ccc}
D_0[\Delta^1] \cup D[0] & \xrightarrow{\alpha} & T_B \\
\downarrow & & \downarrow \\
D[\Delta^1] & \xrightarrow{H} & T_{\bar{B}}
\end{array}
\]

This is a homotopy from $\alpha$ to a better map, call it $\beta: D \to T_B$. However, $\beta$ might not yet factor through $B$ in which case the lemma would follow. But composition with $T_B \to T_{\bar{B}}$ gives a map $\bar{\beta}$ to $T_{\bar{B}}$ which factors through $\bar{B}$.

\[
\begin{array}{ccc}
B & \xrightarrow{\bar{\beta}} & \bar{B} \\
\downarrow & & \downarrow \\
D & \xrightarrow{\beta} & T_B \\
\end{array}
\]

Using Lemma 6.5.1 again this time for $B \to \bar{B}$ and the constant homotopy of $\bar{\beta}$ in $\bar{B}$ we get some lift of $\bar{\beta}$ to $B$, call it $\gamma$.

\[
\begin{array}{ccc}
* & \xrightarrow{\gamma} & \bar{B} \\
\downarrow & & \downarrow \\
D & \xrightarrow{\bar{\beta}} & \bar{B}
\end{array}
\]
It follows that the difference $\beta - \gamma: D \to T_B$ is zero when composed with $T_B \to T_B$. Hence it factors through $T_A$. As the restrictions of $\beta$ and $\gamma$ to $D_0$ both lie in $B$ the restriction of $\beta - \gamma$ to $D_0$ factors through $A$. So $\beta - \gamma$ gives a map $(D, D_0) \to (T_A, A)$. We can show the situation by the following commuting diagrams.

$$
\begin{array}{cccc}
T_A & \longrightarrow & T_B & \\
\uparrow & & \uparrow & \\
\beta - \gamma & \longrightarrow & \beta - \gamma & \\
D & \longrightarrow & D_0 & \longrightarrow & D
\end{array}
$$

Hence, as $A \to T_A$ is a deformation retraction, there is a homotopy $G$ relative to $D_0$ of $\beta - \gamma$ to a map into $A$. It comes from the deformation of $(A, T_A)$ precomposed with $\beta - \gamma$. Call the resulting map $\delta: D \to A$. Via the inclusion $(T_A, A) \to (T_B, B)$ the map $G$ can be viewed as a homotopy to $T_B$ with:

$$
G: \quad D[\Delta^1] \to T_B
$$

$$
G|_0 = \beta - \gamma
$$

$$
G|_1 = \delta
$$

$$
G|_{D_0[\Delta^1]} = \beta - \gamma|_{D_0}
$$

Therefore $G + \gamma: D[\Delta^1] \to T_B$ is a homotopy from $\beta$ to $\delta + \gamma$, where $\delta$ and $\gamma$ factor through $B$ so the sum also factors through $B$. Furthermore the homotopy is constant on $D_0$. Concatenating the two homotopies $H$ and $G$ thus gives a homotopy relative $D_0$ from $\alpha$ to a map into $B$. This is what we wanted to show.

Note that all maps above are in fact in $C^G$, because maps in $C^G$ form an abelian group and being homotopic relative a subspace is an equivalence relation in $C^G$.

\section{Proofs III: Finiteness conditions}

\subsection{Finiteness conditions}

Let $(X, \mathcal{E}, \mathcal{F})$ be a $G$-equivariant control space. We have shown that the category $C^G(X, \mathcal{E}, \mathcal{F}; R)$ has the structure of a category with cofibrations and weak equivalences. Also, the weak equivalences satisfy the saturation and extension axiom. We have a cylinder functor which satisfies the cylinder axiom.

A full subcategory $C^G_\ell$ of $C^G(X; R)$ inherits all that structure if the following two conditions as satisfied:

\begin{enumerate}
\item[(C1)] For $C \leftarrow A \to B$ in $C^G_\ell$ the pushout is in $C^G_\ell$.
\item[(C2)] For $A$ in $C^G_\ell$, $A[\Delta^1]$ is in $C^G_\ell$.
\end{enumerate}

We show that the finite, homotopy finite and homotopy finitely dominated objects satisfy these conditions.
7.2 Finite modules

There is the obvious notion of a set over $X$ and controlled maps of sets over $X$. Let $(M, \diamond_R M, \kappa)$ be a controlled module over $X$. Our prime example of a set over $X$ is $(\diamond_R M, \kappa)$. If $(M, \kappa^R_1)$ and $(M, \kappa^R_2)$ are controlled modules over $X$ such that $(\diamond_R M, \kappa^R_1)$ and $(\diamond_R M, \kappa^R_2)$ are controlled isomorphic as sets over $X$ then $(M, \kappa^R_1)$ and $(M, \kappa^R_2)$ are controlled isomorphic.

A controlled module $(M, \diamond_R M, \kappa)$ over $X$ is locally finite if the set $(\diamond_R M, \kappa)$ is locally finite over $X$, i.e., each $x \in X$ has a neighborhood $U$ with $\kappa^{-1}(U) \subseteq \diamond_R M$ being finite.

**Remark 7.2.1.** Note that modules isomorphic to finite modules do not need to be finite again, if the control space is not “good”. Take as example $\mathbb{R} \setminus \{0\}$ with metric control.

A control space $(X, E, F)$ is called proper, if for each compact subset $K$ and $E \in E$, $F \in F$ we have that $(F \cap K)^E \cap F$ is contained in a compact set. If the control space is proper then modules isomorphic to finite modules are again finite, see also Remark 7.2.4.

We denote the full subcategory of finite objects by $C^G_f$. We want to show that it is indeed a category with cofibrations and weak equivalences. Is is clear that $A[\Delta^1]$ is again finite if $A$ is finite. The main part is to show that the pushout of $C \leftarrow A \rightarrow B$ exists when $A, B, C$ are finite modules and $A \rightarrow B$ is a cofibration in $C^G$. If we know that it is isomorphic to a cellular inclusion of finite modules we are done.

But that is not obvious, we mentioned above that not every module which is isomorphic to a finite module needs to be finite again. Further, the problem is not only that we could change the map $\kappa$ to $X$, but we could have a different cellular structure.

**Lemma 7.2.2.** Let $f': A \rightarrow B$ be a cofibration in $C^G_f$. Then it is isomorphic to a cellular inclusion in $C^G_f$.

The proof is fairly complicated when done in detail. The first step is the following:

By definition $f'$ is only isomorphic to a cellular inclusion in $C^G$, which does not need to be in $C^G_f$ by the Remark above.

By Lemma 6.1.1 the pushout of $f'$ along $\text{id}_A$ can be chosen as

$$
\begin{array}{ccc}
A & \xrightarrow{f'} & B \\
\downarrow \text{id}_A & & \downarrow \\
A & \xrightarrow{f} & D
\end{array}
$$

such that $f$ is a cellular inclusion. Then $D$ is isomorphic to the finite module $B$, but need not be finite itself. We show in the next lemma that $D$ can indeed be made into a finite module. That lemma finishes the proof.
Lemma 7.2.3. Let $f : A \to D$ be a cellular inclusion in $C^G$ such that $A$ is finite and $(D, \kappa^D)$ is isomorphic to a finite module $(B, \kappa^B)$. Then there is a control map $\pi^D : \circ_R D \to X$ such that $(D, \pi)$ is a finite module which is isomorphic to $(D, \kappa^D)$.

It follows that $A \to (D, \pi)$ is isomorphic to a cellular inclusion in $C^G$.

Proof. We only sketch the proof. The difficult part is, of course, that $D$ and $B$ might have different cellular structures.

We have to find for $(\varphi_R D, \kappa^D)$ a set over $X$ which is controlled isomorphic to it and locally finite. We will that in two steps, first improve $\kappa_0 := \kappa^D$ to $\kappa_1$, and then to $\kappa_2$. All maps $\kappa_1, \kappa_2 : \circ_R D \to X$ are controlled isomorphic to $\kappa_0$ and “improve” $\kappa_0$, in particular $\kappa_2$ is a locally finite set over $X$. Then we can take $\pi := \kappa_2$. We prove the non-equivariant case first, i.e., assume $G = \{ e \}$.

First we define $\kappa_1$ such that its image is contained in the image of $\kappa^B$. As $(B, \kappa^B)$ and $(D, \kappa_0)$ are controlled isomorphic there is an $E \in \mathcal{E}$ such that for each $x \in \circ_R D$ there is an $x(e) \in \operatorname{Im} \kappa^B \subseteq X$ such that $(\kappa_0(e), x(e)) \in E$. Set $\kappa_1(e) := x(e)$.

Set

$$T := \{ x \in X \mid \kappa_1^{-1}(x) \text{ is infinite} \}.$$ 

As $X$ is Hausdorff we have that $(D, \kappa_1)$ is finite if and only if $T$ is empty. So $T$ are the “trouble points”. We change $\kappa_1$ on $\kappa_1^{-1}(T)$. We can proceed degreewise. The rough idea is, that if $\kappa_1^{-1}(T)$ is infinite in that degree, it needs to come from an infinite submodule of $B$.

Let $\theta : B \to D$ be the controlled isomorphism. So we change $\kappa_1$ on, say, $d \in \circ_R D$ to map to $\kappa_B(b)$ for some $b \in \circ_R B$ with $d \in \langle \theta(b) \rangle_D$. Careful checking shows that the changed map is locally finite, at least in that degree. Also, the new map is controlled isomorphic to the old one.

For $G$ being non-trivial, we can make all choices $G$-equivariant and are done. \( \square \)

Remark 7.2.4. The proof of the Lemma implies the following for the control space $X$ if $T$ was not empty: There are points $x \in X$ and $E \in \mathcal{E}$ such that $\{ x \}^E$ is not contained in a compact subset. Namely $i^2_n$ must hit infinitely many cells of $B$ over points in $\{ x \}^E$, but $B$ is locally finite. In particular $X$ is not a proper control space in the sense of Section 2.2.

If $X$ and $Y$ are $G$-equivariant control spaces, then any $G$-equivariant map $f : X \to Y$ induces a functor $C^G(X) \to C^G(Y)$. For the finite objects we have the following obvious criterion.

Lemma 7.2.5. Let $\varphi : (X, \mathcal{E}_X, \mathcal{F}_X) \to (Y, \mathcal{E}_Y, \mathcal{F}_Y)$ be a map of control spaces which maps locally finite sets over $X$ to locally finite sets over $Y$. Then $\varphi$ induces a functor $C^G(Y, R, \mathcal{E}_Y, \mathcal{F}_Y) \to C^G(X, R, \mathcal{E}_X, \mathcal{F}_X)$. \( \square \)

Remark 7.2.6. Note that inclusions of subspaces do not map locally finite sets to locally finite sets in general. A counterexample is the inclusion $\mathbb{R} \setminus \{0\} \to \mathbb{R}$. 41
However closed inclusions do map locally finite sets to locally finite set and hence do induce a functor of categories of controlled modules.

### 7.3 Homotopy finite objects

Showing that homotopy finite objects form a Waldhausen category follows formally from the facts that the finite ones are an exact subcategory. It is follows directly from the cylinder axiom that the cylinder functor of homotopy finite objects is again homotopy finite. Assume that $C \leftarrow A \rightarrow B$ is a diagram of homotopy finite objects and $A \rightarrow B$ a cofibration. To show that the pushout is again homotopy finite one uses factorizations given by the cylinder functor and the glueing lemma several times to obtain a homotopy equivalences to the pushout of a diagram of finite objects. Here is a detailed proof:

**Proof.** So assume that there are finite objects $A', B', C'$ weakly equivalent to $A, B, C$. Note that we have inverses for weak equivalences, which we will use freely. Below we denote mapping cylinders by $M_A, M_B$, etc. and cofibrations by $\rightarrow$.

We get a chain of maps of diagrams. In the following the arrows marked with $\bullet \rightarrow$ are defined by composition. The first step is

\[
\begin{array}{ccc}
C & \leftarrow & A \\
\downarrow & & \downarrow \sim \\
C & \leftarrow & A' \\
\downarrow & & \downarrow \\
C & \leftarrow & A' \\
\downarrow & & \downarrow \\
M_B
\end{array}
\]

where $A'$ is finite and $M_B$ is the mapping cylinder of $A' \rightarrow A \rightarrow B$, which still is homotopy finite. Next we get a map

\[
\begin{array}{ccc}
C & \leftarrow & A' \\
\downarrow & & \downarrow \sim \\
C' & \leftarrow & A' \\
\downarrow & & \downarrow \\
C' & \leftarrow & A' \\
\downarrow & & \downarrow \\
M_B
\end{array}
\]

by $C'$ being homotopy finite. Then take

\[
\begin{array}{ccc}
C' & \leftarrow & A' \\
\downarrow & & \downarrow \sim \\
M_{C'} & \leftarrow & A' \\
\downarrow & & \downarrow \\
M_{C'} & \leftarrow & A' \\
\downarrow & & \downarrow \\
M_B
\end{array}
\]

with $M_{C'}$ being the cylinder of $A' \rightarrow C'$ which is finite as $A'$ and $C'$ are finite.
Finally we get a map

\[
\begin{array}{c}
M_C' \xleftarrow{A'} \rightarrow M_B \\
\downarrow \downarrow \downarrow \\
M_C' \xleftarrow{A'} \rightarrow B'
\end{array}
\]

as \( M_B \) is weakly equivalent to \( B' \). Using the Glueing Lemma four times gives that \( C \cup_A B \) is weakly equivalent to the finite object \( M_C' \cup_A' B' \).

\[
\square
\]

### 7.4 Homotopy finitely dominated objects

We give a three different characterizations of homotopy finitely dominated objects.

**Definition 7.4.1.** Let \( M, M' \) be objects in \( C^G \).

1. \( M \) is called a retract of \( M' \) if there are maps \( i: M \rightarrow M', r: M' \rightarrow M \) such that \( r \circ i = \text{id}_M \).

2. \( M \) is called a homotopy retract of \( M' \), or dominated by \( M' \), if there are maps \( i: M \rightarrow M', r: M' \rightarrow M \) and a homotopy \( H: M[\Delta^1] \rightarrow M \) from \( r \circ i \) to \( \text{id}_M \).

**Lemma 7.4.2.** Let \( A \in C^G \). Then the following are equivalent.

1. \( A \) is a homotopy retract of a finite module \( A' \).

2. \( A \) is a retract of a homotopy finite module \( A'' \).

3. \( A \) is a homotopy retract of a homotopy finite module \( A''' \).

**Proof.** Clearly (1) \( \Rightarrow \) (3) and (2) \( \Rightarrow \) (3) hold. We show (3) \( \Rightarrow \) (1) first.

As \( A''' \) is homotopy finite, there is a finite module \( B \) and maps \( f: A''' \rightarrow B \), \( g: B \rightarrow A''' \) such that \( g \circ f \simeq \text{id}_{A'''} \), so \( A \) is a homotopy retract of \( B \) via \( A \xrightarrow{f} A''' \xrightarrow{g} B \) and \( B \xrightarrow{t'} A''' \xrightarrow{f} B \).

Now we show (1) \( \Rightarrow \) (2). We have maps \( i: A \rightarrow A', r: A' \rightarrow A \) with \( r \circ i \simeq \text{id}_A \). We can make the homotopy commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A' \\
\downarrow \text{id} & & \downarrow r \\
A & & A
\end{array}
\]

into a strict commutative one, namely

\[
\begin{array}{ccc}
A & \xrightarrow{i} & T(i) \\
\downarrow \text{id} & & \downarrow \text{id} \\
A & & A
\end{array}
\]

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Hence $A$ is a retract of $T(i)$ and as $T(i) \sim A'$ is a homotopy equivalence, $T(i)$ is homotopy finite.

Lemma 7.4.3. $c^G_{hId}$ is a category with cofibrations and weak equivalences. It has a Cylinder Functor satisfying the Cylinder Axiom and the class of weak equivalences satisfies the Extension and the Saturation Axiom.

Proof. Again we only show (C1) and (C2) from before. Assume that $A, B, C$ are retracts of homotopy finite objects $A', B', C'$. Note that we can make the co-retraction into a cofibration by replacing $A'$ with the mapping cylinder of $A \to A'$, so we will assume that the co-retractions $i_A, i_B, i_C$ are actually cofibrations.

We want to show that $C \cup_A B$ is a retract of a homotopy finite object. We reduce this to the case where $A$ is homotopy finite. Consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & B \\
\downarrow & & \downarrow \\
A' & \circlearrowleft & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{i_A} & B
\end{array}
\]

(As before $\circlearrowleft$ denotes a map defined by composition.) We can factor the horizontal maps into cofibrations simultaneously using the Cylinder Functor. We obtain

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & M \sim B \\
\downarrow & & \downarrow \\
A' & \circlearrowleft & M \sim B \\
\downarrow & & \downarrow \\
A & \xrightarrow{i_A} & M \sim B
\end{array}
\]

There and in all following diagrams the composition of the vertical arrows is always the identity, which holds in the diagram above by the functoriality of the Cylinder Functor. By the Glueing Lemma $C \cup_A M$ is weakly equivalent to $C \cup_A B$. Then the diagram, extended by $C$,

\[
\begin{array}{ccc}
C & \xleftarrow{i_C} & A \xrightarrow{i_A} M \\
\downarrow & & \downarrow \\
C & \circlearrowleft & A' \xrightarrow{i_A} M \\
\downarrow & & \downarrow \\
C & \xleftarrow{i_C} & A \xrightarrow{i_A} M
\end{array}
\]
shows that $C \cup_A M$ is a retract of $C \cup_{A'} \overline{M}$. We are done if we show that $C \cup_{A'} \overline{M}$ is finitely dominated. As $M$ is homotopy equivalent to the homotopy finitely dominated object $B$, Lemma 7.4.2 shows that $\overline{M}$ is again a retract of a homotopy finite module $M'$.

Now we can use that we have co-retractions $C \hookrightarrow C', \overline{M} \twoheadrightarrow M'$ with $C', M'$ homotopy finite objects, which are also cofibrations. This gives a commuting retraction diagram

$$
\begin{array}{cccc}
C & \leftarrow & A' & \hookrightarrow \overline{M} \\
\downarrow & & \downarrow & \\
C' & \leftarrow & A' & \hookrightarrow \overline{M}'
\end{array}
$$

where we want to emphasize, that the map $A' \hookrightarrow M'$, defined by composition, is a cofibration. Thus $C \cup_{A'} \overline{M}$ is a retract of $C' \cup_{A'} \overline{M}'$, which is homotopy finite, as being a pushout of homotopy finite objects along a cofibration.

For (C2), $A[\Delta^1]$ is dominated by $A'[\Delta^1]$. \hfill \square

8 Proofs IV: Connective algebraic $K$-theory of categories of controlled simplicial modules

8.1 Algebraic K-theory

In the last section we showed that the categories $C^G_f, C^G_{hf}$ and $C^G_{hfd}$ are all categories with cofibrations and weak equivalences, so we can use Waldhausen’s $S_*$-construction from [Wal85] to produce an algebraic $K$-Theory spectrum $K(C^G_f)$ and therefore also the corresponding infinite loop space. Define $K_n(C^G_f)$ for $n \geq 0$ as the $n$th homotopy group $\pi_n K(C^G_f)$. This algebraic $K$-Theory spectrum is always connective so we do not assign any name to its negative homotopy groups.

Remark 8.1.1. There is a slight set-theoretical problem, as $C^G_{a'}$ is not a small category according to our definition but it needs be one to apply the $K$-theory construction. However, we take the usual approach (see e.g. [Wal85], Remark before 2.1.1) and fix a suitable large set-theoretical small category of simplicial $R$-modules to begin with. Then all the categories we consider are again small. (We could get such a category by fixing a large cardinal and require all elements to lie in it.) We will assume such a choice from now on.

We want to show that $K_n(C^G_f)$, $K_n(C^G_{hf})$ and $K_n(C^G_{hfd})$ agree for $n \geq 1$. For this we need a cofinality theorem first.
8.2 A cofinality theorem

For comparison of homotopy finite and homotopy finitely dominated objects we need a cofinality result.

**Theorem 8.2.1** (Waldhausen-Thomason cofinality). Let $\mathcal{A}$ and $\mathcal{B}$ be Waldhausen categories. Suppose that $\mathcal{A}$ is a full subcategory of $\mathcal{B}$ which satisfies the following conditions

1. $\mathcal{A}$ is a Waldhausen subcategory: $f : X \to B$ in $\mathcal{A}$ is a cofibration if and only if it is a cofibration in $\mathcal{B}$ with cokernel in $\mathcal{A}$.

2. A map in $\mathcal{A}$ is a weak equivalence if and only if it is one in $\mathcal{B}$.

3. $\mathcal{A}$ is saturated: Every object in $\mathcal{B}$ which is weakly equivalent to an object in $\mathcal{A}$ is itself in $\mathcal{A}$.

4. $\mathcal{A}$ is closed under extensions: If $X \to Y \to Z$ is a cofiber sequence in $\mathcal{B}$ and $X, Z$ are in $\mathcal{A}$, then $Y$ is in $\mathcal{A}$.

5. $\mathcal{B}$ has mapping cylinders satisfying the cylinder axiom and $\mathcal{A}$ is closed under them.

6. $\mathcal{A}$ is cofinal in $\mathcal{B}$: For every object $X$ in $\mathcal{B}$ there is an object $\overline{X}$ in $\mathcal{B}$ such that $X \lor \overline{X}$ is in $\mathcal{A}$.

Then

$$K(\mathcal{A}) \to K(\mathcal{B}) \to "K_0(\mathcal{B})/K_0(\mathcal{A})"$$

is a homotopy fiber sequence of connective spectra. Here "$K_0(\mathcal{B})/K_0(\mathcal{A})$" denotes the Eilenberg-MacLane spectrum with the group $K_0(\mathcal{B})/K_0(\mathcal{A})$ in degree 0.

**Remark 8.2.2** (Similar results). The result is inspired from Thomason-Trobaugh [TT90, Exercise 1.10.2] and Vogell [Vog90, Theorem 1.6]. However, we have slightly different assumptions. In particular in [TT90] the saturatedness assumption is missing. We will provide a counterexample in Section 8.2.5. Parts of the proof were also inspired by Weibel’s K-Book [Wei13, Corollary V.2.3.1 and prerequisites], which unfortunately suffers from the same missing assumption, see again Section 8.2.5. Staffeldt, [Sta89, Thm. 2.1], has a similar result as ours in the context of exact categories. It is strictly less general because it only treats isomorphisms as weak equivalences.

**Remark 8.2.3**. Instead of proving the theorem directly, we prove a lemma about $K_0$ and then rely on the cofinality theorems of Thomason-Trobaugh [TT90, Thm. 10.1] and Waldhausen’s strict cofinality theorem [Wal85, 1.5.9]. (The latter has an implicit assumption that the subcategory is full, see 8.2.6.)

The following lemma will provide a crucial step.
Lemma 8.2.4. Assume we are in the same situation as in Theorem 8.2.1. Assume further that \( A \to B \) induces a surjection \( K_0(A) \to K_0(B) \). Then \( A \) is strictly cofinal in \( B \) in the sense of \([\text{Wal85}, 1.5.9]\), i.e., for each \( B \in B \) there is an \( A \in A \) such that \( B \vee A \in A \). (In this situation we do not need the cylinder functor assumption, but all the other ones are used.)

In the situation of the lemma Waldhausen’s strict cofinality theorem applies and shows that \( K(A) \simeq K(B) \).

Proof. Recall (e.g. from \([\text{TT90}, 1.5.6]\)) that \( K_0(A) \) is the abelian group generated by isomorphism classes of objects \([A] \), \( A \in A \) with relations

1. \([A] = [B]\) if there is a weak equivalence \( A \xrightarrow{\sim} B \)
2. \([A] + [C] = [B]\) if there is a cofiber sequence \( A \rightarrowtail B \twoheadrightarrow C \).

Let \( K'_0(A) \) be the group where we ignore the weak equivalences and consider only split cofiber sequences. That is, it is the group generated by isomorphism classes of objects \([A] \), \( A \in A \) with relation

1. \([A] + [C] = [A \vee C]\) for \( A,C \in A \).

We do the same for \( B \). From the inclusion \( i: A \to B \) we get a (solid) commutative diagram

\[
\begin{array}{ccc}
K'_0(A) & \longrightarrow & K'_0(B) \\
\downarrow & & \downarrow \\
K_0(A) & \longrightarrow & K_0(B)
\end{array}
\]

which we can extend to the cokernels \( G \) and \( G' \) as shown. We claim \( G' \to G \) is an isomorphism.

First, \( G \) is the abelian group with generators \([B]\) for \( B \in B \) and relations

1. \([A] + [C] = [B]\) if there is a cofiber sequence \( A \rightarrowtail B \twoheadrightarrow C \).
2. \([A] = 0\) if \( A \in A \).
3. \([A] = [B]\) if there is a weak equivalence \( A \xrightarrow{\sim} B \)

We claim that \( 3 \) is redundant: By cofinality there is an \( \overline{A} \in B \) such that \( A \vee \overline{A} \in A \). By the Glueing Lemma \( A \vee \overline{A} \xrightarrow{\sim} B \vee \overline{A} \) is still a weak equivalence. By saturation therefore \( B \vee \overline{A} \) is also in \( A \). Therefore, by \( 2 \), \([A \vee \overline{A}] = 0\) and \([B \vee \overline{A}] = 0\) in \( G \). By \( 1 \) then \([A] = -[\overline{A}] = [B]\) in \( G \), which implies \( 3 \).

Further, \( G' \) is the abelian group with generators \([B]\) for \( B \in B \) and relations

1. \([A] + [B] = [A \vee B]\).
2. \([A] = 0\) if \( A \in A \).
We show that the stronger relation (1) for \( G \) comes from \( G' \) and therefore the groups are isomorphic. Let \( A \to B \to C \) be a cofiber sequence in \( B \). Then, by cofinality, there are \( \overline{A}, \overline{C} \) in \( B \) such that \( \overline{A} \lor A \) and \( \overline{C} \lor C \) are in \( A \). Then

\[
A \lor \overline{A} \to \overline{B} \lor A \lor C \to C \lor \overline{C}
\]

(11)
is a cofiber sequence in \( B \) by the pushout axiom, and the first and last object are in \( A \). By being closed under extensions, therefore the middle term \( B \lor A \lor C \) is also in \( A \). It follows that in \( G' \) we have

\[
[B] + [\overline{A}] + [\overline{C}] = 0
\]

\[
[A] + [\overline{A}] = 0
\]

\[
[C] + [\overline{C}] = 0
\]

and therefore \([B] = [A] + [C]\) in \( G' \). If follows \( G \cong G' \).

We now prove that \( A \subseteq B \) is strictly cofinal. If \( K_0(A) \to K_0(B) \) is a surjection, \( G \) and therefore \( G' \) is trivial. Hence \( K_0(A) \to K_0(B) \) is surjective. Let \( B \in B \), then there is an \( A \in A \) such that \([A] = [B] \in K_0(B)\). Now \( K_0(B) \) is just a group completion with respect to \( \lor \) after taking isomorphism classes. Therefore \([A] = [B]\) if and only if there is a \( C \in B \) with \( A \lor C \cong B \lor C \). (This is the usual algebraic argument.) As \( A \) is cofinal, there is a \( \overline{C} \in B \) such that \( C \lor \overline{C} \in A \). Therefore

\[
A \lor (C \lor \overline{C}) \cong B \lor C \lor \overline{C}.
\]

with, of course, \( B \lor C \lor \overline{C} \in A \). This shows strict cofinality. \( \square \)

**Proof of Thm. 8.2.1.** We want to apply [TT90, 1.10.1]. For the convenience of the reader we quote the result:

1.10.1. **Cofinality Theorem.** Let \( vB \) be a Waldhausen category with a cylinder functor satisfying the cylinder axiom. Let \( G \) be an abelian group, and \( \pi: K_0(vB) \to G \) an epimorphism. Let \( B^w \) be the full subcategory of those \( B \) in \( B \) for which the class \([B]\) in \( K_0(vB) \) has \( \pi(B) = 0 \) in \( G \). Make \( B^w \) a Waldhausen category with \( v(B^w) = B^w \cap v(B) \), \( co(B^w) = B^w \cap co(B) \). Let “\( G \)” denote \( G \) considered as Eilenberg-MacLane spectrum whose only non-zero homotopy group is \( G \) in dimension 0.

Then there is a homotopy fibre sequence

\[
K(vB^w) \to K(vB) \to \text{“}G\text{”}
\]

Define \( G \) as \( K_0(B)/K_0(A) \). So we get the above fiber sequence and in particular \( K_0(vB^w) = \ker K_0(vB) \to G \). Clearly the inclusion \( A \to B \) factors as \( A \to B^w \),

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as \( \pi[A] = 0 \) for \( A \in \mathcal{A} \) by definition. It suffices to show that \( K(A) \rightarrow K(\mathcal{B}^w) \) is a weak equivalence. Like in \( \mathcal{B} \), \( \mathcal{A} \) is cofinal in \( \mathcal{B}^w \), as one can choose the same complement. Then \( \mathcal{A} \subseteq \mathcal{B}^w \) satisfies the assumptions of Theorem 8.2.1. But \( K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B}^w) \) is surjective, so by Lemma 8.2.4, \( \mathcal{A} \) is strongly cofinal in \( \mathcal{B}^w \). By Waldhausen’s strict cofinality Theorem [Wal85, 1.5.9] there is a homotopy equivalence \( K(\mathcal{A}) \rightarrow K(\mathcal{B}^w) \). This shows that we get the desired homotopy fiber sequence
\[
K(A) \rightarrow K(B) \rightarrow \text{"}K_0(B)/K_0(A)\text{"}.
\]

8.2.5 Saturated is necessary: A counterexample The assumption (3) in Theorem 8.2.1 “saturatedness” in Theorem 5.1 is necessary. Here is a counterexample, which I owe to a discussion with Chuck Weibel. Let \( \mathcal{C} \) be the following Waldhausen category:

1. Objects are finite pointed sets \( X \) with decomposition \( X = A \vee B \vee C \). (Think of the elements as being colored, while the basepoint is black.)

2. Morphisms are maps \( A \vee B \vee C \rightarrow A' \vee B' \vee C' \) such that they restrict to maps \( A \rightarrow A' \), \( B \rightarrow A' \vee B' \), \( C \rightarrow A' \vee C' \). That is, you can change any color to \( A \) or map to the basepoint, or do not change the color.

3. Let Cofibrations be split injections. That is maps \( i: X \rightarrow X' \) with \( p: X' \rightarrow X \) such that \( p \circ i = id \). It follows that \( X' \cong X \vee Y \) for some \( Y \) in \( \mathcal{C} \). This is, they come from the direct sum “\( \vee \)” in pointed sets.

4. Let a weak equivalence be a bijection of pointed sets.

This category has \( K_0(\mathcal{C}) \cong \mathbb{Z} \), as each object is weakly equivalent to an object \( A \vee * \vee * \).

Consider the full subcategory \( \mathcal{B} \) of objects \( A \vee B \vee C \) with \( |A| = |C| \). This is a cofinal subcategory in \( \mathcal{C} \). It is not saturated: While \( A \vee * \vee C \) is equivalent to \( (A \vee C) \vee * \vee * \) in \( \mathcal{C} \), the latter is not in \( \mathcal{B} \). It satisfies all the other assumptions of Theorem 5.1 except for the cylinder functor (v), so Lemma 8.2.4 would apply and show that \( K_0(\mathcal{B}) \subseteq K_0(\mathcal{C}) \). However, one sees that \( K_0(\mathcal{B}) = \mathbb{Z} \oplus \mathbb{Z} \), with generators represented by \( A \vee * \vee C \) and \( * \vee B \vee * \), as all weak equivalences in \( \mathcal{B} \) are isomorphisms. Hence \( K_0(\mathcal{B}) \rightarrow K_0(\mathcal{C}) \) cannot be an injection.

This counterexample shows that the saturatedness assumption is necessary and missing in Exercise 1.10.2 in [TT90], as well as in Corollary V.2.3.1 in [We13]. The latter is deduced in [We13] from Theorem II.9.4 through a chain along exercise II.9.14 (“Grayson’s trick”), Theorem IV.8.9 and Remark IV.8.9.1, (which states \( K_0(\mathcal{B}) = K_0(\mathcal{C}) \) is equivalent to \( \mathcal{B} \) being strictly cofinal in \( \mathcal{C} \)). The gap is in the proof of Theorem II.9.4, which claims that the proof of Lemma II.7.2 applies verbatim. The above counterexample in particular applies to Theorem II.9.4.

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8.2.6 Fullness is necessary  For completeness, let us remark that the fullness assumption in [8.2.1] is also necessary: In Waldhausen’s cofinality theorem [Wal85, 1.5.9] he does not mention that one needs to assume that the subcategory $\mathcal{A} \subseteq \mathcal{B}$ is full. There is a counterexample due to Inna Zakharevich [Zak10] which shows that one has to assume it.

8.3 Change of finiteness conditions

We turn to the proofs for Subsection 4.1. We need Waldhausen’s Approximation Theorem, which we recall for convenience.

**Definition 8.3.1** (Approximation Property [Wal85, 1.6],[TT90, 1.9.1]). Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor of categories with cofibrations and weak equivalences. $F$ has the Approximation Property if the following two axioms hold.

(App 1) A map $f$ in $\mathcal{A}$ is a weak equivalence if (and only if) its image $F(f)$ in $\mathcal{B}$ is a weak equivalence.

(App 2) Given any object $A$ in $\mathcal{A}$ and a map $x: F(A) \to B$ in $\mathcal{B}$ there exists a map $a: A \to A'$ in $\mathcal{A}$ and a weak equivalence $x': F(A') \to B$ in $\mathcal{B}$ such that the triangle

\[
\begin{array}{ccc}
F(A) & \xrightarrow{x} & B \\
F(a) \downarrow & & \downarrow \\
F(A') & \xrightarrow{x'} & B \\
\end{array}
\]

commutes.

**Theorem 8.3.2** (Approximation Theorem [Wal85, 1.6.7],[TT90, 1.9.1]). Let $\mathcal{A}, \mathcal{B}$ be categories with cofibrations and weak equivalences which satisfy the Saturation Axiom. Assume $\mathcal{A}$ has a Cylinder Functor satisfying the Cylinder Axiom. Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor with the Approximation Property. Then $F$ induces an equivalence

\[
K(F): K(w\mathcal{A}) \to K(w\mathcal{B})
\]

on connective algebraic $K$-theory spectra.

We used Thomason-Trobaugh’s remark in [TT90, 1.9.1] that we can use a weaker version of the approximation property. In [Wal85] there is the further requirement in (App 2) that $a$ is a cofibration, which we can always arrange due to the existence of a Cylinder Functor.

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1. [http://mathoverflow.net/questions/23515/cofinal-inclusions-of-waldhausen-categories](http://mathoverflow.net/questions/23515/cofinal-inclusions-of-waldhausen-categories)

“Consider the following example. Let $\mathcal{C}$ be the category of pairs of pointed finite sets, whose morphisms $(A,B) \to (A',B')$ are pointed maps $A \vee B \to A' \vee B'$, and let $\mathcal{B}$ be the category of pairs of pointed finite sets whose morphisms $(A,B) \to (A',B')$ are pairs of pointed maps $A \to B$ and $A' \to B$. We make $\mathcal{C}$ a Waldhausen category by defining the weak equivalences to be the isomorphisms, and the cofibrations to be the injective maps. $\mathcal{B}$ is clearly cofinal in $\mathcal{C}$, but $K_0(\mathcal{B}) = \mathbb{Z} \times \mathbb{Z}$, while $K_0(\mathcal{C}) = \mathbb{Z} \times \mathbb{Z}$.”
Proof of Proposition 4.1.3. To prove (1) we use Waldhausen’s Approximation Theorem 8.3.2 and apply it to the inclusion functor. We check the conditions. First, all our categories satisfy the Saturation Axiom and have a Cylinder functor satisfying the Cylinder Axiom.

A map is a homotopy equivalence in $C^G_f$ if and only if it is one in $C^G_{hf}$, so (App 1) is satisfied.

So given $A \in C^G_f$, $B \in C^G_{hf}$ and a map $f: A \to B$. For $B$ there is by definition a finite object $B_f \in C^G_f$ which is homotopy equivalent to $B$, i.e., there are maps $g: B_f \to B$ and $\eta: B \to B_f$ with both compositions being homotopic to the identity. Define $j: A \to B_f$ as $j := \eta \circ f$. Then $g \circ j$ is homotopic to $f$. Using the cylinder functor (and Remark 6.2.5) we can rectify the homotopy commutative diagram on the left below to the strict commutative diagram on the right:

As $B_f \to T(j)$ is a homotopy equivalence, by the Saturation Axiom $H$ is a homotopy equivalence. This shows (App 2) and therefore (1).

For (2) we use the Waldhausen-Thomason cofinality Theorem 8.2.1. We have to check the conditions (1) to (6). Most of them are clear or shown in the previous sections.

In particular a map $A \to A'$ in $C^G_{hf}$ which is a cofibration in $C^G_{hf,d}$ is a cofibration in $C^G_{hf}$, and therefore its quotient is again in $C^G_{hf}$. This is (1). By definition $C^G_{hf}$ is full in $C^G_{hf,d}$. A map in the former is a weak equivalence if and only if it is in the latter. Similar the cylinder functor is just inherited. This shows (2) and (5). An object homotopy equivalent to an object in $C^G_{hf}$ is homotopy finite, hence itself in $C^G_{hf}$, this shows (3).

We are left with first showing the cofinality (6) and then that $C^G_{hf}$ is closed under extensions in $C^G_{hf,d}$ (4).

For $B \in C^G_{hf,d}$ there is an $A \in C^G_{hf}$ such that $B$ is a retract of $A$, i.e., there are maps $r: A \to B$, $i: B \to A$ such that $r \circ i = \text{id}_B$. By replacing $A$ with $T(i)$ we can assume that $i$ is a cofibration, hence there is a cofiber sequence

$$B \xrightarrow{i} A \xrightarrow{p} C := A/B.$$  

The retraction $r: A \to B$ and the map $* \to C$ give a map $A \to B \vee C$, and $* \to B$ and $A \to C$ give another one. The sum of these maps makes the
diagram

\[
\begin{array}{c}
\text{commutative and both rows are cofiber sequences. By the Extension Axiom 6.5.2 the map } A \to B \vee C \text{ is a homotopy equivalence, hence } B \vee C \in C^G_{hf}.
\end{array}
\]

This show cofinality.

Next we need to show that \(C^G_{hf}\) is closed under extensions in \(C^G_{hfd}\). So let

\[
A \to B \to C
\]

be a cofiber sequence in \(C^G_{hfd}\) with \(A, C \in C^G_{hf}\) and \(B \) (“the extension of \(A\) by \(C\)” in \(C^G_{hfd}\). As \(C^G_{hf}\) is cofinal there is a \(B' \in C^G_{hfd}\) such that \(B \vee B' \in C^G_{hf}\).

Then

\[
A \to B \vee B' \to C \vee B'
\]

is a cofiber sequence with \(A, B \vee B' \in C^G_{hf}\), hence the quotient \(C \vee B'\) is in \(C^G_{hf}\) by the Glueing Lemma. Similar but easier we get cofiber sequences

\[
C \to C \vee B' \to B'
\]

showing \(B' \in C^G_{hf}\) and

\[
B' \to B \vee B' \to B
\]

showing \(B \in C^G_{hf}\), what we wanted to show.

The Cofinality Theorem 8.2.1 therefore gives us a homotopy fiber sequence of connective spectra

\[
K(C^G_{hf}) \to K(C^G_{hfd}) \to K_0(C^G_{hfd})/K_0(C^G_{hf})
\]

As \(\pi_n(K_0(C^G_{hfd})/K_0(C^G_{hf})) = 0\) for \(n \neq 0\) part (2) of the proposition follows. \(\square\)

Remark 8.3.3. In view of the proposition one can consider \(C^G_{hf}\) as kind of “idempotent completion” of \(C^G_{hf}\). (Recall that for algebraic \(K\)-theory of rings the idempotent completion [Pre03, 2.B, p.61] of the category of finitely generated free \(R\)-modules gives the category of finitely generated projective \(R\)-modules, which has the correct \(K_0\), cf. also e.g. [CP97].)

Corollary A.2.2 from the appendix shows that idempotents and certain homotopy idempotents split in \(C^G_{hfd}\). The author does not know if every homotopy idempotent splits in \(C^G_{hfd}\). Hence it is not clear that \(K_0(C^G_{hf})\) is the “correct” group from this point of view. However, the difference and interplay between \(K_0(C^G_{hf})\) and \(K_0(C^G_{hfd})\) is crucial in later work to construct a non-connective delooping of \(K(C^G_{?})\) for all \(? = f, hf, hfd\). The deloopings will then be equivalent non-connective \(K\)-theory spectra for all three finiteness conditions.
8.4 Easy examples

If $X$ is a point, the category $\mathcal{C}(X, R)$ is just the category of simplicial $R$-modules. We use the definition of algebraic $K$-Theory of simplicial rings we from [Wal85 2.3]. Then $\mathcal{C}_{\text{hfd}}^G(G/1, R)$ is equivalent (as category with cofibrations and weak equivalences) to the category of homotopy finitely dominated $R[G]$-modules. Corollary 4.1.5 follows.

8.5 Change of rings

Let $f : R \to S$ be a map of simplicial rings.

If $M$ is a cellular $R$-module then $S \otimes_R M$ is a cellular $S$-module and we get a natural bijection $\otimes_R M \cong \otimes_S (S \otimes_R M)$ which makes $S \otimes_R M$ into a controlled $S$-module. This construction respects all finiteness conditions and cofibrations, so we get an exact functor $S \otimes_R -$.

Theorem 8.5.1 (Change of rings). Let $f : R \to S$ be map of simplicial rings which is a weak equivalence. Then $f$ induces a map $\mathcal{C}_f^G(X, R) \to \mathcal{C}_f^G(X, S)$ which is an equivalence on algebraic $K$-Theory.

For technical reasons we assume all modules to be finite-dimensional in this section. Therefore we only have this theorem for the finiteness condition $f$. Note that the theorem for $hf$ and $hfd$, except the $K_0$-part of $hfd$, is already implied by the Theorem 8.5.1 using Proposition 4.1.3.

The proof takes the rest of this section, we need some preparations first. A map of simplicial rings which is a weak equivalence of the underlying simplicial sets is called a weak equivalence of simplicial rings for short.

Lemma 8.5.2. Let $R \to S$ be a weak equivalence of simplicial rings and $P$ a cellular (uncontrolled) $R$-module. Let $\eta : P \to \text{res}_R S \otimes_R P$ be the unit of the adjunction between the induction $S \otimes_R -$ and the restriction $\text{res}_R$. Then $\eta$ is a weak equivalence of simplicial $R$-modules and in particular a homotopy equivalence of simplicial sets.

Proof. This follows from the Glueing Lemma and induction over the dimension of $P$. We get a pushout-diagram

\[
\begin{array}{ccc}
\coprod R[\Delta^n] & \rightarrow & \coprod R[\partial \Delta^n] \\
\downarrow \cong & & \downarrow \cong \\
\coprod S[\Delta^n] & \rightarrow & \coprod S[\partial \Delta^n] \\
\end{array}
\]

where the vertical maps are weak equivalences of simplicial $R$-modules, hence by the Glueing Lemma for simplicial $R$-modules (cf. [GJ99 II.8.12;II.2.14]) the pushout $P_n \to S \otimes_R P_n$ is a weak equivalence. We have $P = \bigcup_n P_n$ and the $n$-skeleton of $P$ and $P_n$ agree. Also the $n$-skeleton of $S \otimes_R P$ and $S \otimes_R P_n$.
agree and $S \otimes_R P = \bigcup_n S \otimes_R P_n$. Therefore $P \to S \otimes R P$ is a weak equivalence. As simplicial abelian groups are fibrant as simplicial sets the weak equivalence is a homotopy equivalence of simplicial sets. \hfill \qed

**Lemma 8.5.3.** Let $M, P \in C_a^G(X, R)$. Let $i: A \subseteq M$ be a cellular submodule. Assume $M$ is finite-dimensional. Let $g: A \to P$ and $\hat{f}: S \otimes_R M \to S \otimes_R P$ be maps such that the diagram

\[
\begin{array}{ccc}
S \otimes_R A & \to & S \otimes_R P \\
S \otimes_R M & \downarrow & \downarrow \\
S \otimes_R P
\end{array}
\]

commutes. Then there is a map $f: M \to P$ such that $\hat{f}$ is homotopic to $S \otimes_R f$ relative to the cellular submodule $S \otimes_R A$ of $S \otimes_R M$.

**Proof.** Assume $M, \hat{f}, g, P$ are $E$-controlled. We do induction over the dimension of cells of $M$ which are not in $A$. As usual it suffices to consider only one cell. Let $e: R[\Delta^n] \to M$ be attached to $A$ via $\partial: R[\partial \Delta^n] \to A$.

Looking at the smallest submodules containing $e, \partial e$ and $\hat{f}(S \otimes_R e)$ we get the following commutative diagram. (We denote by $S \otimes_R e$ the cell in $S \otimes_R M$ corresponding to $e$ via the isomorphism $\circ_R M \cong \circ_S (S \otimes_R M)$.)

\[
\begin{array}{ccc}
S \otimes_R \langle \partial e \rangle_A & \to & \langle \hat{f}(S \otimes_R e) \rangle_{S \otimes_R P} \\
\downarrow & & \downarrow \\
S \otimes_R \langle e \rangle_M
\end{array}
\]

Because everything is $E$-controlled the support of every module is contained in $\{\kappa(e)\}^E$. Note that $\langle \hat{f}(S \otimes_R e) \rangle_{S \otimes_R P}$ is isomorphic to $S \otimes_R \eta$ for $P' \subseteq P$ a cellular $R$-submodule. Using the adjunction $S \otimes_R -$ and restriction $\text{res}_R$ we obtain the solid commutative diagram below.

\[
\begin{array}{ccc}
R[\partial \Delta^n] & \to & \langle \partial e \rangle_A \\
\downarrow & & \downarrow \\
R[\Delta^n] & \to & \langle e \rangle_M \\
\downarrow & & \downarrow \eta \\
R[\Delta^n] & \to & \text{res}_R S \otimes_R P'
\end{array}
\]

Here $\eta$ is the unit of the adjunction. We want to find a lift up to homotopy relative to $\langle \partial e \rangle_A$ in the solid diagram. We can extend the diagram to the left by the dashed square, which is a pushout square. Hence, using the adjunction
$R[-]$ and forgetful functor, it suffices to construct a lift up to homotopy relative to $\partial \Delta^n$ in the diagram of simplicial sets

![Diagram](image)

There is a lift because by Lemma 8.5.2 $\eta$ is a homotopy equivalence of simplicial sets. With more effort we can arrange that we lift to a map $f : \Delta^n \to P'$ with $f \circ i = g$ and $\eta \circ f$ is homotopic to $\tilde{f}$ with the homotopy begin constant when restricted via $i$.

So we get a map $\langle \varepsilon \rangle_M \xrightarrow{f} P'$ such that $S \otimes_R f$ is homotopic to $\tilde{f} : S \otimes_R \langle \varepsilon \rangle_M \to S \otimes_R P'$ relative to $S \otimes_R (\partial \varepsilon)_{A'}$. As $S \otimes_R P'$ has support on $\{ \kappa_R(\varepsilon)^E \}$ the map and the homotopy are $E$-controlled.

So, assuming the first cells of $M$ which are not in $A$ are of dimension $n$, we can use this procedure and the homotopy extension property to produce a map $S \otimes_R M \to S \otimes_R P$ which satisfies the assumption of the lemma for an $A' = A \cup sk_n M$. Induction and the finite-dimensionality of $M$ finishes the proof.

**Proof of Theorem 8.5.1** We want to apply Waldhausen’s Approximation Theorem 8.3.2 to the functor $F := S \otimes_R -$ : $C_f^G(X,R) \to C_f^G(X,S)$. We prove (App 1) first. Let $\alpha : M \to M'$ be a map in $C_f^G(X,R)$ such that $S \otimes_R \alpha$ is a homotopy equivalence in $C_f^G(X,S)$. By Lemma 8.5.3 there is a map $\beta' : M' \to M$ such that the homotopy inverse $\beta : S \otimes_R M' \to S \otimes_R M$ of $S \otimes_R \alpha$ in $C_f^G(X,S)$ is homotopic to $S \otimes_R \beta'$. Hence there is a homotopy $H : S \otimes_R M[\Delta^1] \to S \otimes_R M$ from $S \otimes_R \text{id}_R$ to $S \otimes_R (\beta' \circ \alpha)$ in $C_f^G(X,S)$ which is homotopic relative to $M[\partial \Delta^1]$ to a homotopy $S \otimes_R H'$ where $H'$ is a homotopy from $\text{id}_R$ to $\beta' \circ \alpha$, using Lemma 8.5.3 again. Vice versa for $\alpha \circ \beta'$, so $\alpha$ is also a homotopy equivalence in $C_f^G(X,R)$.

For (App 2) take $M \in C_f^G(X,R)$ and $N \in C_f^G(X,S)$ and a map $f : S \otimes_R M \to N$. Assume that it is a cellular inclusion by taking the mapping cylinder. We show that $N$ is homotopy equivalent relative $S \otimes_R M$ to a module $S \otimes_R M'$, with $M' \in C_f^G(X,R)$.

Assume that the $n$-skeleton of $S \otimes_R M$ and $N$ agree. Let $N^{n+1}$ be the $(n+1)$-skeleton of $N$ relative to $S \otimes_R M$, i.e., $N^{n+1} = sk_{n+1} N \cup M$. Then $N^{n+1}$ is the pushout

$$S[\coprod \Delta^n] \xleftarrow{S[\coprod \partial \Delta^n]} S[\coprod \Delta^{n+1}] \xrightarrow{\varepsilon^{n+1}} S \otimes_R M.$$
where $\varphi^{n+1}$ is the attaching map for the cells. By Lemma 8.5.3 there is a map $\psi^{n+1} : R\prod \partial \Delta^{n+1} \to M$ such that $S \otimes_R \psi^{n+1}$ is homotopic to a $\varphi^{n+1}$. Call the homotopy $H^{n+1}$. Applying the Glueing Lemma to the diagram (where all vertical maps are homotopy equivalences)

\[
\begin{array}{ccc}
S[\prod \Delta^{n+1}] & \leftarrow & S[\prod \partial \Delta^{n+1}] \\
\downarrow & & \downarrow \\
S[\prod \Delta^{n+1}][\Delta^1] & \leftarrow & S[\prod \partial \Delta^{n+1}][\Delta^1] \\
\downarrow & & \downarrow \\
S \otimes_R R[\prod \Delta^{n+1}] & \leftarrow & S \otimes_R R[\prod \partial \Delta^{n+1}] \\
\end{array}
\]

shows that the pushout of the first row is homotopy equivalent to the pushout of the last row. (This is a simplicial version of the topological fact that homotopic attaching maps yield homotopy equivalent CW-complexes.) Choose such a homotopy equivalence $\xi$. In the last row $S \otimes_R -$ commutes with the pushout, define $\mathcal{M}$ as the pushout of

\[
R[\prod \Delta^{n+1}] \leftarrow R[\prod \partial \Delta^{n+1}] \xrightarrow{\psi^{n+1}} M.
\]

Then take the pushout along $\xi : N^{n+1} \to S \otimes_R \mathcal{M}$ and the inclusion $N^{n+1} \hookrightarrow N$ to obtain $\mathcal{N}$:

\[
\begin{array}{c}
N^{n+1} \xrightarrow{\xi} N \\
\downarrow \cong \downarrow \cong \\
S \otimes_R \mathcal{M} \xrightarrow{\mathcal{f}} \mathcal{N}
\end{array}
\]

Now the $(n + 1)$-skeleton of $S \otimes_R \mathcal{N}$ isomorphic to $\mathcal{N}$ via $\mathcal{f}$. By induction and because $N$ is finite-dimensional we get a diagram

\[
\begin{array}{c}
S \otimes_R M \xrightarrow{\mathcal{f}} N \\
\downarrow \cong \\
S \otimes_R \mathcal{M} \xrightarrow{\cong} \mathcal{N}^1
\end{array}
\]

which we can make into the desired diagramm

\[
\begin{array}{c}
S \otimes_R M \xrightarrow{\mathcal{f}} N \\
S \otimes_R \mathcal{M}^2
\end{array}
\]
using a homotopy inverse for the right map and defining $M^2$ as the mapping cylinder of $M \to M^1$ to make the diagram strictly commutative. This proves (App 2). The theorem follows by the Approximation Theorem 8.3.2. □

9 Applications

We outline some application and the relevance of our category $C^G(X; R)$. We will not provide proofs because they require considerably more technology.

9.1 Controlled algebra for discrete rings

Each (discrete) ring $R_d$ can be made into a simplicial ring taking the constant functor $[n] \mapsto R_d$ where all structure maps are the identity. Then a simplicial module over $R_d$ is essentially, by the Dold-Kan-Theorem (see [GJ99]), a non-negatively graded chain complex. (More precisely there is an adjunction between simplicial modules and non-negative chain complexes and this adjunction is an equivalence on homotopy categories.) Even more, a cellular simplicial $R_d$-module with cells only in dimension 0 is just a free $R_d$-module.

Therefore we can look at the subcategory of 0-dimensional controlled $R_d$-modules, we suppress the control space in the following. This has no longer the nice homotopical properties, but it is an additive category. In fact, this is essentially the category of controlled $R_d$-modules of e.g. [BFJR04] or [PW85]. (There is a small technical difference to [BFJR04] in the definition of morphisms, but that does not affect the algebraic K-theory of that category.)

Therefore the category $C^G(X; R)$ we present here can be viewed as a homotopical generalization of the category of controlled $R_d$-modules. Unfortunately it comes with a price: The arguments to treat $C^G(X; R)$ get more involved. There are a lot cases where we need to invoke Waldhausen’s approximation theorem where in the case of discrete modules and rings it would sufficient to prove two categories at hand are equivalent. However, we believe that most arguments have an analogue for $C^G(X; R)$.

9.2 The Farrell-Jones Conjecture

9.2.1 Statement and Significance Let $R$ be a ring or a simplicial ring and $G$ a (discrete) group. The Farrell-Jones Conjecture provides a “calculation” of $K_n(R[G])$, $n \in \mathbb{Z}$, the algebraic K-theory of the group ring $R[G]$, in terms of the algebraic K-theory of $R$ and the “geometry” of the group $G$. More precisely, it claims that the so-called assembly map

$$H^G_n(E_{VC}G; K_R) \to K_n(R[G])$$

is an isomorphism for every $n \in \mathbb{Z}$. Here the right-hand side is the (non-connective) algebraic K-theory of the group ring $R[G]$, while the left-hand side
is the $G$-equivariant homology theory with coefficients in the $G$-equivariant non-connective K-theory spectrum, evaluated at the classifying space of $G$ for virtual cyclic subgroups. We refrain from discussing more details, as the Farrell-Jones Conjecture is not our main focus in this article and refer to [Bar13] or the slightly outdated survey [LR05].

The Farrell-Jones Conjecture implies a plethora of other usually long-standing conjectures. This includes the vanishing of the Whitehead Group for torsionfree groups and the Borel conjecture about the rigidity of aspherical manifolds. We refer to [LR05, BLR08b] for details. Therefore it is interesting to know the Farrell-Jones Conjecture for as many rings $R$, called the “coefficients”, and groups $G$ as possible.

9.2.2 Status and Proofs There is recent and ongoing progress on class of groups for which the Farrell-Jones Conjecture is known. Recent approaches in fact prove a more general version, the “Farrell-Jones Conjecture with wreath products”, see Section 6 of [BLRR13]. Also, that version even allows any additive category $A$ as coefficients. If $A$ is the category of finitely generated free $R$-modules, we obtain back the version we stated above.

Recent work by Bartels, Farrell, Lück, Reich, Rüping, Wegner, Wu and others shows the “Farrell-Jones Conjecture with wreath products and coefficient in an additive category” for large classes of groups, most recently for $GL_n(\mathbb{Z})$ an some related groups in [BLRR13], solvable Baumslag-Solitar groups in [FW13] and, more generally, solvable groups in [Weg13].

All recent proofs have in common, that they start by translating the Farrell-Jones Conjecture to a problem in the algebraic K-theory of controlled algebra, a strategy first formulated in this way in [BLR08a].

9.2.3 A reformulation in terms of controlled algebra Let $Z$ be a $G$-CW-complex. We want to make $X \times G \times [1, \infty)$ into a $G$-control space. Recall the continuous control conditions $\mathcal{E}_{cc}$ on $X \times [1, \infty)$ from Example 2.2.7. We can “pull back” this morphism control conditions along the projection $p: X \times G \times [1, \infty) \to X \times [1, \infty)$ by setting

$$p^{-1}\mathcal{E}_{cc} := \{(p \times p)^{-1}(E) \mid E \in \mathcal{E}_{cc}\}.$$  

Then $p^{-1}\mathcal{E}_{cc}$ is a morphism control structure on $X \times G \times [1, \infty)$. We get object support conditions by setting

$$\mathcal{F}_{Gc}(X \times G) := \{G.K \times [1, \infty) \mid K \subseteq X \times G \text{ is compact}\}$$

where $G.K$ is the $G$-orbit of $K$. If $X$ is $E_{\mathcal{VC}}G$, the classifying space for $G$ (cf. [DS77] I.6) and the family $\mathcal{VC}$ of virtually cyclic subgroups, we obtain a category

$$\mathcal{S}^G = \mathcal{C}^G_f(E_{\mathcal{VC}}G \times G \times [1, \infty), p^{-1}\mathcal{E}_{cc}, \mathcal{F}_{Gc}; R)$$

58
of controlled simplicial $R$-modules. As explained above, for a discrete ring $R_d$ there is a similar category of discrete controlled modules which we call $O_G^d$ for brevity. As it is an additive category, its algebraic K-theory is defined.

**Theorem** ([BLR08a, 3.8]). $K_i(O_G^d) = 0$ for all $i \in \mathbb{N}$ if and only if the Farrell-Jones Conjecture holds for $G$.

Hence one can use controlled algebra and manipulation of the control space to prove the Farrell-Jones Conjecture. This is in fact the strategy carried out by recent proofs.

For simplicial rings an analogue of the theorem holds by unpublished work of the author in [Ull11]. Thus this article should be viewed as first step to carry out the successful program of proving the Farrell-Jones Conjecture for discrete rings in the settings of simplicial rings.

### 9.3 Non-connective algebraic K-theory

There is a second direct application we can give, but again not prove here. Take the control space $(\mathbb{R}^n, E_d)$ arising from the euclidean metric. Then there is a map

$$K(R) \rightarrow \Omega^n K(C_f(\mathbb{R}^n; R))$$

of connective K-theory spectra which is an isomorphism on $\pi_i$ for $i \geq 1$ and an injection on $\pi_0$. This deloops $K(R)$, that means the $K(C_f(\mathbb{R}^n; R))$ for varying $n$ can be made into a spectrum which may have interesting negative homotopy groups and where the positive homotopy group are the ones of $K(R)$. This is the first construction of a non-connective $K$-theory spectrum for simplicial rings. It generalizes the delooping construction of $K(R_d)$ of [PW85]. However, is it known that $\pi_i K(R) = K_i(\pi_0 R)$ for $i = 0, 1$. Because a Bass-Heller-Swan theorem is expected to hold for algebraic K-theory of simplicial rings this means that the negative algebraic K-groups of a simplicial ring are just the ones of the discrete ring $\pi_0 R$. But of course having spectrum is more information than just knowing its homotopy groups.

The proofs of both applications need considerably more technology in $C^G(X; R)$, namely a notion of germs, which were developed in [Ull11]. We will come back to these in later work.

### 9.4 Ring spectra

There are generalizations of rings for which the Farrell-Jones Conjecture should give interesting results with implications to manifold theory. The details are hard to explain in brief, but homotopy theorist know since a long time that the so-called “ring spectra” provide a natural generalization of rings, and simplicial rings are an intermediate step between rings and ring spectra. Algebraic K-theory can be defined for ring spectra, this done in [EKMM97]. The statement
of the Farrell-Jones Conjecture makes sense for connective ring spectra as coefficients. In fact, when Farrell and Jones in [FJ93, FJ87] originally stated and proved a version of their conjecture (for a certain class of groups), they also treated the case of pseudoisotopies, which is more or less the case where the sphere spectrum, the “initial ring spectrum”, are the coefficients.

We hope that the theory presented here can be adapted for ring spectra. We do not want to go into details, but let us remark that in the 1990’s a bunch of different models for ring spectra and categories of modules over ring spectra where discovered. The main models are symmetric spectra [HSS00], orthogonal spectra [MMSS01] and S-modules [EKMM97]. The category of S-modules is special among these as it has the nice property that it has a model structure such that every object is fibrant and [EKMM97, III.2] provides a nice theory of cellular objects. Unfortunately the category is rather hard to define. It looks like a suitable candidate to carry our the program presented here, but certainly a lot of work still needs to be done for that.

A A simplicial mapping telescope

A map η: K → K in C^G_a is called a homotopy idempotent if η^2 is homotopic to η. Here we provide the necessary tools we need about homotopy idempotents in this and later work. This gives some insight into the category C^G_{hfd}(X; R), for any control space X and simplicial ring R.

We defined the category with cofibrations and weak equivalences C^G_a = C^G_a(X, R, E, F) for a control space (X, E, F) and a simplicial ring R in Section 2.5.3.

A.1 Coherent homotopy idempotents

Some parts of Theorem A.2.1 below need an extra assumption on the idempotent, which we will define now.

**Definition A.1.1.** A homotopy idempotent η: K → K with homotopy H from η^2 to η is called coherent if there is a map G: K[Δ^1 x Δ^1] → K whose restrictions to the boundary look as in the following diagram

Note that we used the diagram language of 6.2.4 to describe the map. We will use it in following without further comment. If η^2 = η then η is coherent.

**Lemma A.1.2.** If η arises from a homotopy domination, it is coherent.
Proof. The assumption means that there are maps \( i : K \rightarrow L, p : L \rightarrow K \) such that \( \eta = p \circ i \) and \( i \circ p \simeq \text{id} \) via a homotopy \( H' \). Then \( \eta \) is an idempotent with homotopy \( p \circ H' \circ i \) from \( \eta^2 \) to \( \eta \).

The coherence homotopy \( G \) can be given by the composition

\[
K[\Delta^1 \times \Delta^1] \cong K[\Delta^1][\Delta^1] \xrightarrow{[i][\Delta^1]} L[\Delta^1][\Delta^1] \xrightarrow{H'[\Delta^1]} L[\Delta^1] \xrightarrow{H'} K \xrightarrow{p} K.
\]

We will prove that every coherent homotopy idempotent in \( C^G_\alpha \) splits up to homotopy.

Remark A.1.3. The author does not know if every homotopy idempotent in \( C^G_\alpha \) is coherent. For the topological case it is known that there are unpointed homotopy idempotents of infinite-dimensional CW-complexes which do not split, however every pointed homotopy idempotent as well as every homotopy idempotent of finite-dimensional CW-complexes splits, see [HH82].

A.2 Existence and properties of a mapping telescope

The results we show in this appendix are summarized in the following Theorem. Its proof will take the rest of this appendix.

**Theorem A.2.1.** Let \( \eta : K \rightarrow K \) be a homotopy idempotent in \( C^G_\alpha(X) \). There is a construction \( \text{Tel}(\cdot) \) which assigns to any homotopy idempotent \( \eta \) an object \( \text{Tel}(\eta) \) in \( C^G_\alpha(X) \). It has the following properties.

1. There is a cellular inclusion \( \iota : K \rightarrow \text{Tel}(\eta) \)

2. Let

\[
\begin{align*}
A & \xrightarrow{\mu} A \\
\downarrow f & \quad \downarrow f \\
K & \xrightarrow{\eta} K
\end{align*}
\]

be a strict commutative diagram of homotopy idempotents. Then \( f \) induces a map \( f_* : \text{Tel}(\mu) \rightarrow \text{Tel}(\eta) \). This is functorial in \( f \). In particular if \( f \) is an isomorphism then \( \text{Tel}(f) \) is an isomorphism.

3. If \( \eta, \mu : K \rightarrow K \) are homotopic homotopy idempotents then there is a homotopy equivalence \( \text{Tel}(\eta) \simeq \Rightarrow \text{Tel}(\mu) \).

4. Consider the telescope \( \text{Tel}(\text{id}_K) \) of the homotopy idempotent \( \text{id}_K : K \rightarrow K \). There is a map \( \text{Tel}(\text{id}_K) \rightarrow K \)

which is a homotopy equivalence.
5. All maps in (2) to (4) are relative to $\iota: K \to \text{Tel}(\eta)$, i.e., they commute with this cellular inclusion.

6. From (2) we get for $\mu = \eta = f$ an induced map $\eta_*: \text{Tel}(\eta) \to \text{Tel}(\eta)$. This map is a homotopy equivalence. If $\eta$ is coherent, $\eta_*$ is homotopic to id.

7. If $\eta$ is coherent then there is a map $c: \text{Tel}(\eta) \to K$ such that $\iota \circ c$ is homotopic to $\eta_*: \text{Tel}(\eta) \to \text{Tel}(\eta)$ and hence, by (6), to the identity on $\text{Tel}(\eta)$. Therefore $\text{Tel}(\eta)$ is a homotopy retract of $K$. Further $c \circ \iota$ is homotopic to $\eta$ itself.

**Corollary A.2.2** (Coherent homotopy idempotents split). Let $\eta: K \to K$ be a coherent homotopy idempotent in $C_G^a$. Then there is a $B \in C_G^a$ such that $K$ is homotopy equivalent to $\text{Tel}(\eta) \vee B$. Moreover under this equivalence $\eta$ corresponds to the projection $\text{pr}: \text{Tel}(\eta) \vee B \to \text{Tel}(\eta) \to \text{Tel}(\eta) \vee B$, i.e., there is a homotopy commutative diagram

$$
\begin{array}{ccc}
K & \xrightarrow{f} & \text{Tel}(\eta) \vee B \\
\downarrow & & \downarrow \text{pr} \\
K & \xrightarrow{f} & \text{Tel}(\eta) \vee B
\end{array}
$$

where $f$ is the homotopy equivalence $K \xrightarrow{\simeq} \text{Tel}(\eta) \vee B$.

**Proof.** We know by A.2.1 (7) that $\text{Tel}(\eta)$ is a homotopy retract of $K$. We can make the homotopy commutative diagram

$$
\begin{array}{ccc}
\text{Tel}(\eta) & \xrightarrow{c} & K \\
\downarrow \text{id} & \searrow \iota & \downarrow \text{id} \\
\text{Tel}(\eta) & \xleftarrow{\text{inc}} & \text{Tel}(\eta)
\end{array}
$$

\begin{array}{ccc}
\text{Tel}(\eta) & \xrightarrow{s} & T(c) \\
\downarrow & \downarrow \text{id} & \downarrow \text{id} \\
\text{Tel}(\eta) \vee B & \xrightarrow{\text{pr}} & \text{Tel}(\eta) \vee B
\end{array}
$$

which is a strict commutative diagram, where $T(c)$ is the mapping cylinder of $c$ and inc is a cellular inclusion. Take the cofiber of inc and call it $B$. The sum of the retraction $T(c) \to \text{Tel}(\eta)$ and the quotient map $T(c) \to B$ gives a map $s: T(c) \to \text{Tel}(\eta) \vee B$ (using that $C^G$ is an additive category). The map makes the diagram of cofiber sequences

$$
\begin{array}{ccc}
\text{Tel}(\eta) & \xrightarrow{T(c)} & B \\
\downarrow & \downarrow s & \downarrow \\
\text{Tel}(\eta) \vee B & \xrightarrow{\text{pr}} & B
\end{array}
$$
commutative and the Extension Axiom 6.3.2 shows that $s$ is a homotopy equivalence. This gives the homotopy equivalence $f: K \to T(c) \to \text{Tel}(\eta) \vee B$.

By [A.2.1 (7)] the map $\eta: K \to K$ factorizes up to homotopy as $c \circ i: K \to \text{Tel}(\eta) \to K$. Hence the upper triangle in

\[
\begin{array}{ccc}
K & \xrightarrow{\eta} & K \\
\downarrow{i} & \searrow{c} & \downarrow{}
\end{array}
\quad \begin{array}{ccc}
\text{Tel}(\eta) & \xrightarrow{T(c)} & B \\
\downarrow{} & \downarrow{} & \downarrow{}
\end{array}
\]

is homotopy commutative, whereas the lower one commutes strictly. It follows that $K \xrightarrow{\eta} K \to T(c) \to B$ is homotopic to the zero map.

Further $K \to T(c) \to \text{Tel}(\eta)$ equals $i$, hence by adding homotopies the map $f \circ \eta: K \to K \xrightarrow{\simeq} T(c) \xrightarrow{\simeq} \text{Tel}(\eta) \vee B$

is homotopic to $K \xrightarrow{i} \text{Tel}(\eta) \to \text{Tel}(\eta) \vee B$. As

\[
\begin{array}{ccc}
K & \xrightarrow{\eta} & \text{Tel}(\eta) \vee B \\
\downarrow{} & \downarrow{} & \downarrow{}
\end{array}
\begin{array}{ccc}
\text{Tel}(\eta) \vee B & \xrightarrow{\eta \vee 0_B} & \text{Tel}(\eta) \vee B \\
\downarrow{} & \downarrow{} & \downarrow{}
\end{array}
\]

is strictly commutative by [A.2.1 (2) and (5)], where $0_B$ denotes the zero map on $B$, and as $\eta \simeq \text{id}$ by [A.2.1 (6)] it follows that

\[
\begin{array}{ccc}
K & \xrightarrow{\simeq} & \text{Tel}(\eta) \vee B \\
\downarrow{} & \downarrow{} & \downarrow{}
\end{array}
\begin{array}{ccc}
\text{Tel}(\eta) \vee B & \xrightarrow{\text{id} \vee 0_B} & \text{Tel}(\eta) \vee B \\
\downarrow{} & \downarrow{} & \downarrow{}
\end{array}
\]

is homotopy commutative and the claim follows.

The proof of Theorem A.2.1 is a bit involved. Because we need to handle concatenation of homotopies well, we need an analogue of “Moore homotopies”, i.e., homotopies where we allow the “intervals” to have different lengths. We will introduce it in the next section.

### A.3 Simplicial Intervals

We start by defining what we will mean by an interval in the category of simplicial sets. This is no common notion there. The name is chosen to stress the analogies to the topological setting. Basically we want a nice formal description of simplicial set of the form $\to \leftarrow$ etc.
**Definition A.3.1** (Simplicial intervals).

1. Let $i \in \mathbb{N}$. A one-point simplicial set $I(i)$, $I(i)_k = \{i\}$, together with a bijection $l: I(i)_0 \to \{i\}$ from its zero simplices is called a point at $i$ or interval of length 0 from $i$ to $i$.

2. An interval of length 1 from $i$ to $(i+1)$, denoted $I(i,i+1)$, is a simplicial set isomorphic to $\Delta^1$ together with a bijection of its zero simplices to the set $\{i, i+1\}$, $l: I(i,i+1)_0 \to \{i, i+1\}$. The map $l$ is called the labeling.

3. Let $i, j \in \mathbb{N}$, $i + 2 \leq j$. An interval of length $(j-i)$ from $i$ to $j$ is a simplicial set $I(i,j)$ together with a bijection $l: I(i,j)_0 \to \{i, i+1, \ldots, j\}$ such that there is a pushout diagram

$$
\begin{array}{ccc}
I(j-1) & \rightarrow & I(j-1,j) \\
\downarrow & & \downarrow \\
I(i,j-1) & \rightarrow & I(i,j)
\end{array}
$$

(13)

where the maps are compatible with the labelings and $I(j-1) \rightarrow I(j-1,i)$, $I(j-1) \rightarrow I(i,j-1)$ are the obvious inclusions.

4. The standard interval from $i$ to $(i+1)$ is the simplicial set $\Delta^1$ together with the labeling $l(0) = i$, $l(1) = i + 1$. The standard interval from $i$ to $j$ for $i + 2 \leq j$ is the simplicial set arising from the standard interval from $i$ to $j-1$ by the pushout (13) with $I(j-1,j)$ being the standard interval of length 1.

5. An interval $I(i,j)$ from $i$ to $j$ is called ordered if it is isomorphic to the standard interval from $i$ to $j$ and the isomorphism respects the labeling.

**Remark A.3.2.** We sometimes draw pictures for intervals. The standard interval is $0 \rightarrow 1$. The four intervals for $I(0,2)$ are the following ones:

\[
\begin{align*}
0 \rightarrow 1 \rightarrow 2 & \quad 0 \rightarrow 1 \leftarrow 2 \\
0 \leftarrow 1 \rightarrow 2 & \quad 0 \leftarrow 1 \leftarrow 2.
\end{align*}
\]

That the notion $I(i,j)$ is ambiguous is intentional, as we want to allow all those cases.

**A.3.3 Concise notation** We often just write $I(i,j)$ for an interval from $i$ to $j$ leaving all the other data understood. For $A \in C^G_0$ we also often abbreviate $A[I(i,j)]$ as $A[i,j]$ and $A[I(i)]$ as $A[i]$, slightly misusing notation.
A.3.4 The infinite interval Define an simplicial set $I(i, \infty)$ to be an interval from $i$ to $\infty$ if it is the filtered colimit (or union) of intervals $I(i, j)$ for $j \to \infty$. It is called ordered if each of the $I(i, j)$ is.

A.4 Long Homotopies

Our notion of interval gives rise to a notion of homotopy.

Definition A.4.1 (Long Homotopy). Let $I(0, j)$ be an interval from $0$ to $j$. Let $f_0, f_j: A \to B$ be two maps in $C^G_a$. A (long) homotopy from $f_0$ to $f_j$ is a map $H: A[I(0, j)] \to B$ such that the restriction to $A[0]$ is $f_0$ and the restriction to $A[j]$ is $f_j$. We say that $H$ has length $j$.

Example A.4.2. If $f: A \to B$ is a map in $C^G_a$ and $I(0, i)$ any interval we always have the constant or trivial homotopy $\text{Tr}: A[0, i] \to B$ induced by the map $A[0, i] \to A \to B$. We also define it for $i = 0$ and therefore call the map $\text{Tr}: A[0, 0] = A[0] = A \to B$ the trivial homotopy of length 0.

A.4.3 Ordinary and long homotopies Every homotopy in the usual sense in a long homotopy of length 1. Every long homotopy gives a homotopy in the usual sense by the Kan property. This is not functorial, which is the reason why we need to consider long homotopies. We will omit the “long” in the following.

A.4.4 Concatenation of intervals If $I(0, i)$ and $I(0, j)$ are intervals we define the concatenation $I(0, i) \sqcup I(0, j)$ to be the pushout

\[
\begin{array}{ccc}
I(i) & \longrightarrow & I(i, i + j) \\
\downarrow & & \downarrow \\
I(0, i) & \longrightarrow & I(0, i) \sqcup I(0, j)
\end{array}
\]

where $I(i, i + j)$ is defined as a “relabeling” of $I(0, j)$, replace the labeling $l$ of $I(0, j)$ by $l(k) = i + k$.

A.4.5 Concatenation of homotopies Homotopies which agree on the start resp. endpoint can be concatenated. For $H_1: A[0, i] \to B$, $H_2: A[0, j] \to B$ with $H_1|_{A[i]} = H_2|_{A[0]}$ define the concatenation

$H_1 \sqcup H_2: A[0, i + j] \to B$

as the map induced by the identification on the pushout $I(0, i) \sqcup I(0, j)$. The concatenation of homotopies is strictly associative.
A.4.6 Inverse homotopies If $I(0,j)$ is an interval, define the reversed interval $\overline{I}(0,j)$ as the same simplicial set with the labeling $l$ replaced by $\overline{l}(k) := j - l(k)$. If $H : A[I(0,j)] \to B$ is a homotopy the inverse homotopy $\overline{H}$ is the obvious map $\overline{H} : A[\overline{I}(0,j)] \to B$.

If $j = 1$ and $I(0,j)$ is an ordered interval we draw the homotopy as $\overset{H}{\longrightarrow}$ and the inverse homotopy as $\overset{\overline{H}}{\longleftarrow}$.

Lemma A.4.7 (Concating a homotopy and its inverse). Let $H : A[0,1] \to B$ be a homotopy. The concatenation $H \boxtimes \overline{H}$ is homotopic, relative boundary, to the constant (or trivial) homotopy $\text{Tr} : A[0,2] \to A \to B$.

Proof. Assume that $I(0,1)$ is the standard interval, the other case proceeds similar. The homotopy $A[0,2][\Delta^1] \to B$ is given by the left diagram below. It is constructed by glueing the 2-simplices together which are shown on the right. These arise from the 2nd degeneracy map $\Delta^2 \to \Delta^1$.

A.4.8 Variations of the previous lemma In the proof of the lemma we gave a homotopy from the constant homotopy to the given one. We can give one in the other direction by a similar proof where we use a map $A[\Delta^2] \to B$ arising from the Kan extension property.

Of course also $\overline{H} \boxtimes H$ is homotopic to the trivial homotopy. The lemma also holds if we allow an interval of length $n$ instead of length 1. To prove this one does induction over $n$ and starts building the homotopy from the middle, patching in in each induction step some of four trivial homotopies of the remaining homotopies $H'$ of length 1, like

These techniques will work in the more complicated situations later, hence we will tacitly only draw the diagrams for length 1 homotopies in the following.

A.4.9 Intervals of different length are homotopy equivalent For $A \in C_\omega^G$ the modules $A[0,i]$, ($i \in \mathbb{N}$), are homotopy equivalent to $A$ and the homotopies are relative to the endpoints. This means, that the restriction of both homotopies to the endpoints gives the constant homotopy there. This allows us to glue the homotopy equivalences together later. The result will follow from the following lemma.
Lemma A.4.10. Let $Λ^2_i$ be the $i$th horn and $d_i$ the $i$th face of $Δ^2$. Then $A[d_i]$ and $A[Λ^2_i]$ are homotopy equivalent relative the 0-simplices of $d_i$. The homotopy equivalence can be chosen to be one of the maps $A[Λ^2_i] \to A[d_i]$ which induced by collapsing one 1-simplex.

Proof. Consider the composition

$$A[Λ^2_i] \to A[Δ^2] \to A[d_i]$$

where the first map is the inclusion and the last map (and hence the composition) can be induced by any map collapsing a 1-simplex not equal to $d_1$. It is not hard to see that the first map has a deformation retraction by horn-filling. The second map is induced by a deformation retraction of simplicial sets. Therefore the composition is a homotopy equivalence relative to the 0-simplices of $d_i$.

Corollary A.4.11. Let $I(0, 1)$ be the standard interval and $I(0, i)$ any interval. Then we have a homotopy equivalence relative endpoints

$$A[0, 1] \simeq A[0, i].$$

Proof. Lemma A.4.10 implies $A[0, 2] \simeq A[0, 1]$ relative endpoints if there is a projection $I(0, 2) \to I(0, 1)$. It also implies $A[\to] \simeq A[\leftarrow]$ relative endpoints by the chain $A[\to] \simeq A[\to\leftarrow] \simeq A[\leftarrow]$ of homotopy equivalences relative endpoints. The corollary follows by induction.

A.4.12 Homotopies of infinite length In the following we assume the infinite interval $I[0, \infty)$ to be ordered for simplicity (cf. A.3.4). We abbreviate $A[I[0, \infty)]$, $A \in \mathcal{C}_a^g$, as $A[0, \infty)$. We suggestively call a map $A[0, \infty) \to B$ a homotopy of infinite length. From such a homotopy we want to get a homotopy of length 1. In general, this is of course impossible. But if the homotopy is “convergent” in the sense below this can be done.

Lemma A.4.13 (Convergent Homotopy Lemma). Let $H : A[0, \infty) \to B$ be a convergent homotopy, this means we assume:

1. There is a filtration $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq A$ by cellular submodules such that $\bigcup A_i = A$.

2. For each $A_i$ there is an $n_i$ such that $H|_{A_i[n_i, \infty)}$ is the constant homotopy $\text{Tr}$ (cf. A.4.2).

Then there exists a homotopy $G : A[Δ^1] \to B$ with $G|_{A[0]} = H|_{A[0]}$ and $G|_{A_i[1]} = H|_{A_i[n_i]}$.

Remark A.4.14. Recall that $A_i[n_i]$ and $A_i[n_i, \infty)$ denote obvious cellular submodules of $A[0, \infty)$. We can and will assume in the proof that $n_{i+1} \geq n_i$.

This lemma is well-known in the topological case. $G$ may be called the “limit” of the homotopy $H$. 67
Proof of Lemma A.4.13. Recall that we assumed $I(0, \infty)$ to be an ordered interval. We first enlarge $I(0, \infty)$ to a new simplicial set $\hat{I}(0, \infty)$ by filling some horns.

The subsimplicial set $I(0, 2)$ is isomorphic to the horn $\Lambda^2_1$. We take the pushout of $\Delta^2 \leftarrow \Lambda^2_1 \rightarrow I(0, \infty)$ and call it $\hat{I}(0, 2)$. It has an extra 1-simplex with boundaries $I(0)$ and $I(2)$ in $I(0, \infty)$, which we call $(0 \rightarrow 2)$.

In $\hat{I}(0, 2)$ the 1-simplices $(0 \rightarrow 2)$ and $I(2, 3)$ constitute a horn $\Lambda^2_1$. Like before we define $\hat{I}(0, 3)$ to be the following pushout

$$
\Lambda^2_1 \rightarrow \Delta^2 \rightarrow \hat{I}(0, 2) \rightarrow \hat{I}(0, 3)
$$

Again it has an extra 1-simplex with boundaries $I(0)$ and $I(3)$ in $I(0, \infty)$, which we call $(0 \rightarrow 3)$. Now we proceed by induction and define $\hat{I}(0, \infty)$ as the filtered colimit

$$
\hat{I}(0, \infty) := \bigcup_n \hat{I}(0, n).
$$

Figure 1 sketches a picture of $\hat{I}(0, \infty)$ with $I(0, \infty)$ being the bottom line.

We have $A[0, \infty] = \bigcup_n A[0, n]$, with $A[0, n]$ and $A[0, \infty]$ being abbreviations for $A[I(0, n)]$ and $A[I(0, \infty)]$, respectively. Note that $A[0, n]$ arises from $A[0, n - 1]$ by horn-filling. We want to construct a certain map $\hat{H} : A[0, \infty) \rightarrow B$ which extends $H$.

We do induction over $i$. Assume that we have constructed a homotopy $G_i : A_i[0, \infty) \rightarrow B$ which extends $H : A_i[0, \infty) \rightarrow B$ and has the property that $G_i|_{A_i[(0 \rightarrow n) \rightarrow (0 \rightarrow n+1)]} = G_i|_{A_i[(0 \rightarrow n)]}$ for all $n \geq n_i$. By iterating the relative horn-filling property 6.2.1 we can extend $G_i$ to a map $A_{i+1}[0, n_{i+1}] \cup A_i[0, \infty) \rightarrow B$.

For $n \geq n_{i+1}$ we do not want to apply the relative horn-filling property as we need special fillings. By assumption $\hat{H}|_{A_{i+1}[n,n+1]}$ is the constant homotopy $\text{Tr}$. Hence the (relative) horn spanned by $A_{i+1}[\{0 \rightarrow n\}]$ and $A_{i+1}[n, n + 1]$, given by $A_{i+1}[\Lambda^n_1] \rightarrow A_{i+1}[0, n] \cup A_i[0, \infty)$, can be filled in the following way

$$
\begin{array}{c}
\Lambda^n_1 \\
\downarrow \\
I(0, 2)
\end{array} \rightarrow \begin{array}{c}
\Delta^n \\
\downarrow \\
I(0, 3)
\end{array}
$$

Figure 1: A sketch of $\hat{I}(0, \infty)$.
where $X$ is the homotopy coming from the previous horn-fillings. This defines a map $G_{i+1}: A_{i+1}[0, \infty)$ with $G_{i+1}[A_{i+1}[0 \rightarrow n)] = G_{i+1}[A_{i+1}[0 \rightarrow n_{i+1}])$ for all $n \geq n_{i+1}$. This shows the induction step.

Taking the colimit over $G_i$ we get a map $\hat{H}: A^\infty \rightarrow B$. We now define $G_{j}: A_j[\Delta^1] \rightarrow B$ as the restriction of $\hat{H}$ (or equivalently $G_j$) to $A_j[[0 \rightarrow n_j]]$, i.e., to the 1-simplex from 0 to $n_j$. This is compatible with the inclusion $A_j \rightarrow A_{j+1}$ and thus the colimit over $j$ gives the desired homotopy $G: A[\Delta^1] \rightarrow B$.

**Corollary A.4.15.** The map $i: A \rightarrow A[0, \infty)$ has a deformation retraction, so in particular $i$ is a homotopy equivalence.

**Proof.** The map $[0, \infty) \rightarrow 0$ induces a retraction $r: A[0, \infty) \rightarrow A$ for $i$. We have to prove that the composition $i \circ r: A[0, \infty) \rightarrow A \rightarrow A[0, \infty)$ is homotopic to the identity. We use the Convergent Homotopy Lemma A.4.13. Define the convergent homotopy $H: A[0, \infty)[0, \infty) \rightarrow A[0, \infty)$ as the map induced by the map $\min(i,j)$

where we use that map $[0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is determined on the 0-simplices. We regard $j$ as the homotopy direction. This map has the following properties:

1. For $j = 0$ it is the projection to 0, hence the map $i \circ r$.
2. For any $j \geq i$ the map $A[0,i][j] \rightarrow A[0,i]$ is the identity.
3. $\bigcup_i A[0,i]$ is a filtration of $A[0,\infty)$ by cellular modules.

Now the Convergent Homotopy Lemma A.4.13 applies and hence we get a homotopy $G: A[0,\infty)[\Delta^1] \rightarrow A[0,\infty)$ from $i \circ r$ to the identity.

**A.5 On the mapping telescope**

In the following $I = I(0, i)$ is always an interval and $f: A \rightarrow A$ a map in $C^G$.

**A.5.1 Long mapping cylinder** Define the mapping cylinder $M^I(f)$ for $I$ of $f$ like for $I = \Delta^1$ in 6.1.3. We have the analogous notions of front and back inclusion as well as projection. We modified the notation because the mapping cylinder will play a slightly different role than in 6.1.3 and because we need to keep track of the interval.
**Definition A.5.2** (Long Mapping Telescope). If $I = I(0, 1)$ define $\text{Tel}^I(f)$, the mapping telescope of $f: A \to A$ for the interval $I$ as the pushout

$$
\begin{array}{ccc}
\coprod_{i=1}^\infty A[i] \coprod \coprod_{i=0}^\infty A[i+1] & \xrightarrow{\iota_0 \coprod \iota_1} & \coprod_{i=0}^\infty M^{I(i,i+1)}(f) \\
\downarrow c & & \downarrow \text{Tel}^I(f) \\
\coprod_{i=1}^\infty A[i] & \xrightarrow{\iota_1} & \text{Tel}^I(f)
\end{array}
$$

In the diagram $\iota_1$ is the back inclusion of $A[i+1]$ into $M^{I(i,i+1)}(f)$, $\iota_0$ the front inclusion, both from the $i$th summand to the $i$th summand. $c$ is the map which maps the summand $A[i]$ to $A[i]$.

For a general interval $I$ the definition works analogously, if we delete “[i]” etc. in the diagram above. (Which makes it less sugestive.)

**A.5.3 Front inclusion**  The front inclusion into the first mapping cylinder $\iota_0 A: M^I(f)$ (which is not used in the diagram above) gives a map

$$
\iota: A \to \text{Tel}^I(f)
$$

which is a called the front inclusion of the mapping cylinder.

**Remark A.5.4.** The telescope consists of infinitely many mapping cylinders plugged together on the right. Each mapping cylinder has the same interval structure. We used that the countable coproducts exists in $\mathbb{C}_G^a$.

Using Corollaries [A.4.11] and [A.4.15] we obtain the following lemma.

**Lemma A.5.5.** Recall $\Delta^1$ is the standard interval.

1. The mapping cylinders for $I$ and $\Delta^1$ are homotopy equivalent relative to the front and the back inclusion: $M^I(f) \simeq M^{\Delta^1}(f) = T(f)$.

2. The mapping telescopes for $I$ and $\Delta^1$ are homotopy equivalent: $\text{Tel}^I(f) \simeq \text{Tel}^{\Delta^1}(f)$.

3. The telescope $\text{Tel}^I(\text{id}_A)$ of the identity is homotopy equivalent to $A$, the homotopy equivalence is given by the projection to $A$.

Each map $I \to \Delta^1$ respecting the endpoints can be chosen to induce the first two homotopy equivalences.

**Remark A.5.6.** While we can give the homotopy equivalences quite explicit, the inverse is not canonical and not easy to write down as we used the Kan Extension property to construct it.
A.5.7 First part of Theorem A.2.1 We can apply this section to an idempotent \( \eta : K \to K \). Define

\[ \text{Tel}(\eta) := \text{Tel}^I(\eta) \]

Then Theorem A.2.1 [1] and [2] are satisfied, by the definition of \( \iota \) in A.5.3 and the functoriality of mapping cylinders and pushouts along cellular inclusions.

A.5.8 Homotopy commutative squares For the rest of Theorem A.2.1 we need to know more about homotopy commutative diagrams. They will induce a map of (long) mapping telescopes, but it is only “functorial”, in some sense we make precise, if we allow to change the intervals.

Definition A.5.9 (Homotopy commutative square). A square in \( C^G_a \)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{a} & & \downarrow{a} \\
B & \xrightarrow{g} & B
\end{array}
\] (15)

is homotopy commutative if there is an interval \( I = I(0,i) \) and a specified homotopy \( H^a : A[0,i] \to B \) which goes from \( g \circ a \) to \( a \circ f \). This should mean \( H^a|_{A[0]} = g \circ a \) and \( H^a|_{A[i]} = a \circ f \).

Remark A.5.10. The homotopy of a homotopy commutative square always goes from the lower left corner to the upper right, it is helpful to visualize this as

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{a} & \nearrow{H^a} & \downarrow{a} \\
B & \xrightarrow{g} & B
\end{array}
\]

when thinking about the homotopies. We chose the direction of the homotopy such that it will fit together with our definition of mapping cylinder.

The next observation is central for the rest of the proof.

Lemma/Definition A.5.11 (Stacking squares). Homotopy commutative squares can be composed (stacked). Given two homotopy commutative squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{a} & & \downarrow{a} \\
B & \xrightarrow{g} & B \\
\downarrow{b} & & \downarrow{b} \\
C & \xrightarrow{h} & C
\end{array}
\] (16)
with homotopies $H^a, H^b$ using intervals $I^a, I^b$, then composed (stacked) square

$$
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
C & \xrightarrow{h} & C
\end{array}
$$

is homotopy commutative with homotopy

$$(H^b \circ a[I^b]) \Box (b \circ H^a) : A[I^b \Box I^b] \to C. \quad \square$$

A.5.12 **Stacking is associative** Composition (stacking) of homotopy commutative squares is strictly associative, because concatenation of homotopies is. The length of the homotopies add. If the square is strictly commutative we can and hence will assume that the "homotopy" has length 0.

Note that we only “compose in one direction” of the two possible directions in which the square could be “stacked”. The reason is simply, that this is the only case we are interested in.

**Lemma/Definition A.5.13** (Homotopy commutative squares and mapping cylinders). Let $I$ be an interval. A homotopy commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & B
\end{array}
$$

with homotopy $H : A[I] \to B$ induces a map called $(H, a)_* : M^I(f) \to B$ such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\iota_0} & M^I(f) & \xleftarrow{\iota_1} & A \\
\downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{g} & B & \xleftarrow{id_B} & B
\end{array}
$$

commutes (strictly). Here $\iota_0$ is the front and $\iota_1$ the back inclusion. Each such diagram determines uniquely the homotopy of (17).

**Proof.** The pushout of the strictly commutative diagram

$$
\begin{array}{ccc}
A[I] & \xleftarrow{\iota_1} & A[i] & \xrightarrow{f} & A \\
\downarrow & & \downarrow & & \downarrow \\
B & \xleftarrow{id_B} & B & \xrightarrow{id_B} & B
\end{array}
$$
gives the map \((H, a)_*: M^I(f) \to B\) and then diagram (18) commutes. Conversely taking the map \(A[I] \to M^I(f) \to B\) gets back the homotopy \(H\) and the commutativity of (18) shows that \(H\) makes the square (17) homotopy commutative.

**A.5.14 Induced map on longer mapping cylinders** Let \(J\) be another interval. Taking the mapping cylinder with \(J\) of the rows of the left-hand square of (18) gives a map \(\alpha[J] \Box (H, a)_*: M^J \Box I(f) \to M^J(g)\) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\epsilon_0} & M^J \Box I(f) & \xleftarrow{\epsilon_1} & A \\
\downarrow a & & \downarrow a & & \downarrow a \\
B & \xrightarrow{\epsilon_0} & M^J(g) & \xleftarrow{\epsilon_1} & B
\end{array}
\]  

(19)

commutes. We call the induced maps the cylinder maps of the homotopy commutative diagram, resp. the cylinder maps with respect to \(J\). Basically, we make the cylinders longer by glueing in \(A[J]\).

**A.5.15 Composition of induced map** We want to compose the map \((H, a)_*\), which should correspond to stacking homotopy commutative squares. Stacking squares makes the homotopies longer, so we will not have a composition on the nose, but only after changing the first map. The composition is visualized in Figure 2 below.

**Definition A.5.16 (Composition).** Given maps \(f: A \to A\), \(g: B \to B\) and \(h: C \to C\) as well as \(a: A \to B\) and \(b: B \to C\). Assume we have cylinder maps \((H^a, a)_*: M^I(f) \to B\) and \((H^b, b)_*: M^I(g) \to C\) like in Definition A.5.13 satisfying diagrams like (18). Define the “composition” \((H^a, a)_* \Box (H^b, b)_*: M^I(a[I] \Box (H^a, a)_*) \to M^I(b[I] \Box (H^b, b)_*)\) as

\[
M^I(b[I] \Box (H^a, a)_*) \xrightarrow{\alpha[I] \Box (H^a, a)_*} M^I(a[I] \Box (H^b, b)_*) \xrightarrow{(H^a, a)_* \Box (H^b, b)_*} C
\]

More generally let \(J\) be another interval. Assume we have cylinder maps with respect to \(J\)

\[
a[J] \Box (H^a, a)_*: M^J \Box I^n(f) \to M^J(g) \quad \text{and} \quad b[J] \Box (H^b, b)_*: M^J \Box I^n(g) \to M^J(h)
\]

Define the “composition” as

\[
\left(a[J] \Box (H^a, a)_*\right) \Box \left(b[J] \Box (H^b, b)_*\right):
\]

\[
M^J \Box I^n(f) \xrightarrow{a[J] \Box a[I^n] \Box (H^a, a)_*} M^J \Box I^n(g) \xrightarrow{b[J] \Box (H^b, b)_*} M^J(h)
\]
Lemma A.5.17. Given two homotopy commutative squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow a & & \downarrow a \\
B & \xrightarrow{g} & B \\
\end{array}
\begin{array}{ccc}
B & \xrightarrow{g} & B \\
\downarrow b & & \downarrow b \\
C & \xrightarrow{h} & C \\
\end{array}
\]

with homotopies \(H^a: A[I] \to B\), \(H^b: B[I] \to C\). Then the cylinder map of the stacked homotopy commutative square (cf. A.5.11)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & b \circ a \\
\downarrow b \circ a & & \downarrow b \circ a \\
C & \xrightarrow{h} & C \\
\end{array}
\]

is equal to the “composition” of the cylinder maps of the individual squares, i.e.,

\[
(\left(H^b \circ a[I^b]\right) \Box (b \circ H^a), b \circ a) = (H^b, b) \circ (a[I^b] \Box (H^a, a))
\]

The same is true for cylinder maps with respect to \(J\).

Proof. We have to check the equality of two maps \(M^J \Box I^a \Box I^b(f) \to M^J(h)\). Figure 2 shows the situation. With its help for the bookkeeping the equality can be checked directly. \(\square\)

A.5.18 Induced maps on telescopes Everything from A.5.13 on transfers immediately to mapping telescopes, by gluing the parts together. In particular if we have homotopy commutative squares like in A.5.17 we obtain a maps

\[
(H^a, a)_*: \text{Tel}^J \Box I^a(f) \to \text{Tel}^J(g),
\]

\[
(H^b, b)_*: \text{Tel}^J \Box I^b(g) \to \text{Tel}^J(h).
\]

and their “composition”

\[
(H^b, b)_* \Box (H^a, a)_*: \text{Tel}^J \Box I^b \Box I^a(f) \to \text{Tel}^J(h)
\]

which is the same as the induced map of the stacking of the homotopy commutative squares. The map \((H, a)_*\) commutes with the front inclusion \(\iota\) from A.5.3

We can specialize to \(H\) being the trivial homotopy of length 0 and \(J := \Delta^1\). Then we get the strict functoriality of \(\text{Tel}^J(-)\) from Theorem A.2.1 (2).

One the other hand we can consider the square given by \(f = g\) and \(a = \text{id}_A\) being homotopy commutative with trivial homotopy \(\text{Tr}: A[I] \to A\), for \(I\) any interval. The resulting map \((\text{Tr}, \text{id})_*: \text{Tel}^J \Box I(f) \to \text{Tel}^J(f)\) is induced by the projection \(\Delta^1 \Box I \to \Delta^1\) mapping \(I\) to \(I(1) \subseteq \Delta^1\).
A.5.19 A homotopy criterion

We need a criterion when two homotopy commutative squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{a} & & \downarrow{a} \\
B & \xrightarrow{g} & B
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{\tilde{a}} & & \downarrow{\tilde{a}} \\
B & \xrightarrow{g} & B
\end{array}
\]

with homotopies \(H^a\) and \(H^{\tilde{a}}\) induce homotopic maps on mapping telescopes.

We make the following assumptions.

1. \(H^a\) and \(H^{\tilde{a}}\) have the same length and are indexed over the same interval \(I = I(0, i)\). We can arrange this by extending with trivial homotopies.

2. There is a homotopy \(H: A[J] \to B\) from \(a\) to \(\tilde{a}\).
3. There is “2-homotopy” \( G: A[I][J] \to B \) from \( H^a \) to \( H^\tilde{a} \) which restricts on \( I(0) \times J \) to \( g \circ H \) and on \( I(i) \times J \) to \( H \circ f[J] \).

The last condition can be visualized for \( I = J = \Delta^1 \) by writing \( G: A[\Delta^1 \times \Delta^1] \to B \) in our diagram language as

\[
\begin{array}{c}
g \circ a \xrightarrow{H^a} a \circ f \\
g \circ \tilde{a} \xrightarrow{H^\tilde{a}} \tilde{a} \circ f
\end{array}
\]

So \( G \) can also be viewed as a homotopy from \( g \circ H \) to \( H \circ f[J] \).

**Lemma A.5.20 (Homotopy criterion).** If the above conditions are satisfied the two induced maps \((H^a, a)_*, (H^\tilde{a}, \tilde{a})_*: \text{Tel}^{\Delta^1 \square I}(f) \to \text{Tel}^{\Delta^1}(g)\) are homotopic. The homotopy is \((G, H)_*: \text{Tel}^{\Delta^1 \square I}(f)[J] \to \text{Tel}^{\Delta^1}(g)\).

**Proof.** Interpreting \( G \) as homotopy from \( g \circ H \) to \( H \circ f[J] \) gives a homotopy commutative square

\[
\begin{array}{ccc}
A[J] & \xrightarrow{f[J]} & A[J] \\
\downarrow H & & \downarrow H \\
B & \xrightarrow{g} & B
\end{array}
\]

with homotopy \( (A[J])[I] \to B \). We get an induced map \((G, H)_*: \text{Tel}^{\Delta^1 \square I}(f[J]) \to \text{Tel}^{\Delta^1}(g)\). As the telescope is a colimit it commutes with adjoining an interval, hence we can write the domain of the induced map as \( \text{Tel}^{\Delta^1 \square I}(f)[J] \). Therefore \((G, H)_*\) is a homotopy. We leave it to the reader to check that it is the desired one.

**Remark A.5.21.** Lemma [A.5.20] is a main tool in the following to analyze maps between telescopes. Thanks to the lemma we only need give diagrams like (20) to prove that certain maps on telescopes are homotopic. To simplify the diagram we will usually assume that all intervals have length 1 and are ordered. In [A.4.8] we explained how to get to longer intervals from this.

**Lemma A.5.22 (Homotopic maps between telescopes).** Let \( f, g: A \to A \) be homotopic maps. Then \( \text{Tel}^{\Delta^1}(f) \) and \( \text{Tel}^{\Delta^1}(g) \) are homotopy equivalent. The homotopy equivalences are relative to the front inclusions.

**Proof.** Let \( H: A[I] \to A \) be the homotopy from \( g \) to \( f \). We get two homotopy
commutative squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{id} & & \downarrow{id} \\
A & \xrightarrow{g} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A & \xrightarrow{g} & A \\
\downarrow{id} & & \downarrow{id} \\
A & \xrightarrow{f} & A
\end{array}
\]

with homotopies \(H: A[I] \to A\) and \(\overline{H}: A[I] \to A\) (where the latter is the “inverse” homotopy, cf. Section \ref{sec:2.3}). This gives maps \((H, \text{id})_*: \text{Tel}^\Delta \circ I (f) \to \text{Tel}^{\Delta^1}(g)\) and \((\overline{H}, \text{id})_*: \text{Tel}^\Delta \circ I (g) \to \text{Tel}^{\Delta^1}(f)\). The “composition”

\[
(H, \text{id})_* \circ (H, \text{id})_*: \text{Tel}^\Delta \circ I (f) \to \text{Tel}^{\Delta^1}(f)
\]

(cf. \ref{sec:5.18}) is induced by the homotopy commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{id} & & \downarrow{id} \\
A & \xrightarrow{f} & A
\end{array}
\]

with homotopy \(\overline{H} \circ H: A[I] \to A\). By Lemma \ref{sec:4.7} that homotopy is homotopic relative endpoints to the trivial homotopy, hence Lemma \ref{sec:5.20} shows that \((\overline{H}, \text{id})_* \circ (H, \text{id})_*\) is homotopic to \((\text{Tr}, \text{id})_*\). The same holds for the other composition.

As \((\text{Tr}, \text{id})_*: \text{Tel}^\Delta \circ I (f) \to \text{Tel}^{\Delta^1}(f)\) is a homotopy equivalence induced by the projection \(\Delta^1 \circ I \to \Delta^1\) we get two homotopy commutative triangles

\[
\begin{array}{ccc}
\text{Tel}^{\Delta^1}(f) & \xrightarrow{\sim} & \text{Tel}^{\Delta^1}(f) \\
(H, \text{id})_* \downarrow & & \downarrow \varphi \\
\text{Tel}^{\Delta^1}(g) & & \text{Tel}^{\Delta^1}(g)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Tel}^{\Delta^1}(g) & \xrightarrow{\sim} & \text{Tel}^{\Delta^1}(g) \\
(\overline{H}, \text{id})_* \downarrow & & \downarrow \psi \\
\text{Tel}^{\Delta^1}(f) & & \text{Tel}^{\Delta^1}(f)
\end{array}
\]

where \(\varphi\) is defined using a chosen homotopy inverse of the horizontal map and \(\psi\) similarly. We claim both compositions of these maps are homotopic to the identity. Together with these triangles we get a large diagram

\[
\begin{array}{ccc}
\text{Tel}^{\Delta^1}(f) & \xrightarrow{\sim} & \text{Tel}^{\Delta^1}(f) \\
(H, \text{id})_* \downarrow & & \downarrow \varphi \\
\text{Tel}^{\Delta^1}(g) & \xrightarrow{\sim} & \text{Tel}^{\Delta^1}(g)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Tel}^{\Delta^1}(g) & \xrightarrow{\sim} & \text{Tel}^{\Delta^1}(g) \\
(\overline{H}, \text{id})_* \downarrow & & \downarrow \psi \\
\text{Tel}^{\Delta^1}(f) & & \text{Tel}^{\Delta^1}(f)
\end{array}
\]

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where the square is strictly commutative. The left vertical composition is the “composition” \((\vec{H}, \text{id})_\ast \otimes (H, \text{id})_\ast\) which is homotopic to \((\text{Tr}, \text{id})_\ast\), which is exactly the composition of the upper horizontal maps. It follows that \(\psi \circ \varphi \simeq \text{id}\) and similar for the other composition.

As the homotopy inverses in the definition of \(\psi\) and \(\varphi\) can be chosen to respect the front inclusion by Lemma \[\text{A.5.5}\] and all other maps and homotopies are relative to it \(\varphi\) is a homotopy equivalence relative to the front inclusion.

**A.5.23 The shift map** Recall that \(\text{Tel}^I(f)\) is a quotient of \(\bigsqcup_{n \in \mathbb{N}} M^I(f)\).

The map taking the \(n\)th component to the \((n + 1)\)st component is compatible with the quotient, hence induces a map \(\text{Tel}^I(f) \to \text{Tel}^I(f)\), which we will call the *shift map* and denote it by \(\text{sh}\).

**Lemma A.5.24.** Let \(I\) be an interval, \(f: A \to A\) a self-map. The maps \(\text{sh}\) and \((\text{Tr}, f)_\ast\) from \(\text{Tel}^I(f)\) to \(\text{Tel}^I(f)\) are homotopy inverse:

\[(\text{Tr}, f)_\ast \circ \text{sh} = \text{sh} \circ (\text{Tr}, f)_\ast \simeq \text{id}: \text{Tel}^I(f) \to \text{Tel}^I(f)\]

*Sketch of proof.* The first equality is clear. For the homotopy one restricts the map of telescopes to a map \(M^I(f) \to M^I(f) \cup_A M^I(f)\), which maps into the second summand. Then one can construct a simplicial homotopy from this map to the map “inclusion of the first summand”. Namely, the two inclusions \(I \to I \sqcup I\) give two maps \(A[I] \to A[I \sqcup I]\) which are homotopic by “sliding”. The desired homotopy arises from this homotopy. The details are left to the reader.

**A.5.25 Telescopes of coherent homotopy idempotents** Parts (1) to (5) of Theorem \[\text{A.2.1}\] follow from the previous discussion. We will give a summary later on, but for the moment note that we have not even used that \(f: A \to A\) is a homotopy idempotent. For the following we need to consider coherent homotopy idempotents.

**Lemma A.5.26.** Let \(\eta: K \to K\) be a coherent homotopy idempotent in \(\mathcal{C}_a^G\).

Then the induced map \(\eta_\ast = (\text{Tr}, \eta)_\ast: \text{Tel}^\Delta^1(\eta) \to \text{Tel}^\Delta^1(\eta)\) is not only a homotopy equivalence but even homotopic to the identity \(\text{id}\).

*Proof.* We show \(\eta_\ast \circ \eta_\ast \simeq \eta_\ast\), then by Lemma \[\text{A.5.24}\] we have \(\eta_\ast \circ \text{sh} \simeq \text{id}\) and therefore \(\eta_\ast \simeq \eta_\ast \circ \eta_\ast \circ \text{sh} \simeq \eta_\ast \circ \text{sh} \simeq \text{id}\) and we are done.

So assume that \(H: A[I] \to A\) is the homotopy from \(\eta^2\) to \(\eta\) and for simplicity assume \(I = \Delta^1\). As \(\eta\) is coherent we have a diagram

\[
\begin{array}{c}
\eta \circ H \\
H \circ \eta \\
\end{array}
\]

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where $X$ is just a name to denote the restriction to the diagonal. Using that diagram we can build the diagram

\[
\begin{array}{ccccccc}
\eta^3 & \xrightarrow{\text{Tr}} & \eta^3 & \leftarrow & \eta^3 \\
\downarrow_{\eta \circ H} & & \downarrow_{X} & & \downarrow_{X} & & \downarrow_{H \circ \eta[I]} \\
\eta^2 & \xrightarrow{H} & \eta & \xleftarrow{H} & \eta^2 \\
\downarrow_{\eta \circ \text{Tr}} & & \downarrow_{\eta} & & \downarrow_{\eta} & & \downarrow_{\text{Tr} \circ \eta[I]} \\
\eta^2 & \xrightarrow{\text{Tr}} & \eta^2 & \leftarrow & \eta^2 \\
\end{array}
\]

which gives a 2-homotopy $G$ such that Lemma [A.5.20] shows that $(G, H)_*$ is a homotopy from $(\text{Tr}, \eta^2)_*$ to $(\text{Tr}, \eta)_*$ as maps $\text{Tel}^\Delta \sqcup I \sqcup T(\eta) \to \text{Tel}^\Delta(\eta)$. But $\Delta^1 \sqcup I \sqcup T \to \Delta^1$ induces a homotopy equivalence on telescopes such that the triangle for $\eta_*$

\[
\begin{array}{ccc}
\text{Tel}^\Delta \sqcup I \sqcup T(\eta) & \xrightarrow{\sim} & \text{Tel}^\Delta(\eta) \\
\downarrow_{\eta_*} & & \downarrow_{\eta_*} \\
\text{Tel}^\Delta(\eta) & & \text{Tel}^\Delta(\eta)
\end{array}
\]

as well as the one for $\eta^2_*$, commutes strictly. Therefore also the maps on the cylinder of the same lengths are homotopic. \qed

**Lemma A.5.27.** Let $\eta: K \to K$ be a coherent homotopy idempotent in $\mathcal{C}_a^G$. Then there is a map $c: \text{Tel}^\Delta(\eta) \to K$ such that the composition $\iota \circ c$ with the inclusion $\iota: K \to \text{Tel}^\Delta(\eta)$ is homotopic to the identity on $\text{Tel}^\Delta(\eta)$, whereas the other composition $c \circ \iota$ is homotopic to $\eta: K \to K$.

**Proof.** Let $H: A[I] \to A$ be the homotopy from $\eta^2$ to $\eta$. We get two homotopy commutative squares

\[
\begin{array}{ccc}
K & \xrightarrow{\text{id}} & K \\
\downarrow_{\eta} & & \downarrow_{\eta} \\
K & \xrightarrow{\eta} & K
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
K & \xrightarrow{\eta} & K \\
\downarrow_{\eta} & & \downarrow_{\eta} \\
K & \xrightarrow{\text{id}} & K
\end{array}
\]

with homotopies $H: A[I] \to A$ and $\overline{H}: A[\overline{I}] \to A$. Hence we get two induced maps

\[
(H, \eta)_*: \text{Tel}^\Delta \sqcup I(\text{id}_K) \to \text{Tel}^\Delta(\eta) \\
(\overline{H}, \eta)_*: \text{Tel}^\Delta \sqcup \overline{T}(\eta) \to \text{Tel}^\Delta(\text{id}_K).
\]

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Consider the “composition” (A.5.16)

\[(H, \eta)_* \otimes (H, \eta)_* : \text{Tel}^{\Delta^1 \square I \square T}(\eta) \to \text{Tel}^{\Delta^1}(\eta)\]

which by Lemma A.5.17 is equal to \((H \circ \eta[I] \square \eta \circ H, \eta)^*\). As the homotopy idempotent is coherent we have a map \(A[I \times I] \to A\) which is on the boundary of \(I^2\):

\[
\begin{array}{ccc}
\eta \circ H & \to & H \\
\downarrow & & \downarrow \\
H \circ \eta[I] & \to & H
\end{array}
\]

Thus by pasting two copies of the above square together as shown below we get the 2-homotopy \(G\)

\[
\begin{array}{c}
\eta^3 \to \eta^2 \to \eta^2 \to \eta^3 \\
\downarrow & \downarrow & \downarrow \\
\eta^2 \to \eta^2 \to \eta^2 \\
\eta^2 \to \eta^2 \to \eta^2
\end{array}
\]

from \(H \circ \eta[I] \square \eta \circ H\) to \(\text{Tr}\) and by Lemma A.5.20 \((G, H)^*\) gives a homotopy from the composition \((H, \eta)_* \otimes (H, \eta)_*\) to the map \((\text{Tr}, \eta)_*\). Similar the other “composition” \((H, \eta)_* \otimes (H, \eta)_*\) is homotopic to \((\text{Tr}, \eta)_* : \text{Tel}^{\Delta^1 \square T \square I}(\text{id}_K) \to \text{Tel}^{\Delta^1}(\text{id}_K)\) using the 2-homotopy

\[
\begin{array}{c}
\eta^2 \to \eta^2 \to \eta^2 \\
\downarrow & \downarrow & \downarrow \\
\eta^2 \to \eta^2 \to \eta^2 \\
\eta^2 \to \eta^2 \to \eta^2
\end{array}
\]

where \(X\) is the diagonal in diagram (21) and the upper left and right triangles are also from (21).

Now we make \((H, \eta)_*\) and \((H, \eta)_*\) into maps of telescopes of the same length as in the proof of Lemma A.5.22. Define \(\bar{c}\) and \(\bar{t}\) by choosing a homotopy inverse in the top row of the following diagrams

\[
\begin{array}{c}
\text{Tel}^{\Delta^1 \square T}(\eta) \xrightarrow{\bar{c}} \text{Tel}^{\Delta^1}(\eta) \\
\downarrow \quad \text{(H, } \eta)_* \\
\text{Tel}^{\Delta^1}(\text{id}_K)
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\text{Tel}^{\Delta^1 \square I}(\text{id}_K) \xrightarrow{\bar{t}} \text{Tel}^{\Delta^1}(\text{id}_K) \\
\downarrow \quad \text{(H, } \eta)_* \\
\text{Tel}^{\Delta^1}(\eta)
\end{array}
\]

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A similar argument as in the proof of Lemma A.5.22 using a big triangle shows that $\tilde{c} \circ \tilde{\iota}$ is homotopic to $\eta_* : \text{Tel}^{\Delta^1} (\text{id}_K) \to \text{Tel}^{\Delta^1} (\text{id}_K)$ and $\tilde{\iota} \circ \tilde{c}$ is homotopic to $\eta_* : \text{Tel}^{\Delta^1} (\eta) \to \text{Tel}^{\Delta^1} (\eta)$. Lemma A.5.26 shows that on $\text{Tel}^{\Delta^1} (\eta)$ the map $\eta_*$ is homotopic to the identity.

Now $\iota_{\text{id}_K} : K \to \text{Tel}^{\Delta^1} (\text{id}_K)$ is a homotopy equivalence and even an inclusion for a deformation retraction $pr$ by Lemma A.5.5. Set $c := pr \circ \tilde{c} : \text{Tel}^{\Delta^1} (\eta) \to K$, note $\iota \circ c = \iota \circ \iota_{\text{id}_K} \circ pr \circ \tilde{c}$ is homotopic to $\iota \circ \tilde{c}$ and hence to $\text{id}_{\text{Tel}^{\Delta^1} (\eta)}$ and $c \circ \iota = pr \circ \tilde{c} \circ \iota_{\text{id}_K}$ is homotopic to $\eta : K \to K$. This shows the lemma.

A.5.28 Proof of Theorem A.2.1 We proved all parts of Theorem A.2.1. We defined $\text{Tel}(\eta)$ in A.5.7 and settled (A.2.1(1)) and (A.2.1(2)). The inclusion $\iota : K \to \text{Tel}(\text{id}_K)$ is a homotopy equivalence (A.2.1(1)) by Lemma A.5.5 and homotopic maps gives homotopy equivalent telescopes (A.2.1(3)) by Lemma A.5.22. The compatibility with the inclusion (A.2.1(5)) was noted along the lemmas.

A homotopy idempotent induces a homotopy equivalence on its own telescope by Lemma A.5.24 and a coherent homotopy idempotent even induces a map homotopic to the identity by Lemma A.5.26 (A.2.1(6)). Finally the retraction up to homotopy to the inclusion (A.2.1(7)) is provided in Lemma A.5.27.

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