In this work we suggest that higher-dimensional modifications to the matter content in FRW spacetimes can be obtained not only, as first considered by Ponce de Leon, referring to “moving” 4D hypersurfaces non-orthogonal to the time-dependent extra dimension of an embedding 5D manifold, but also referring to “fixed” 4D hypersurfaces orthogonal to a suitable scalar function which defines a static foliation of the 5D manifold and takes the role of the extra dimension in a suitable coordinate system. Results obtained in each approach crucially depend on the method used to identify the 4D metric of our brane universe from the 5D metric of the bulk manifold.

PACS numbers: 04.50.-h, 04.20.Cv

Keywords: Brane theory; FRW models.

1. Introduction

Recently Ponce de Leon \(^1,^2,^3\) showed that our observable universe can be devised as a dynamic four-dimensional hypersurface which depends explicitly on the evolution of the time-dependent extra dimension of the embedding five-dimensional manifold and is non-orthogonal to it. As a consequence it is possible to construct a four-dimensional model which predicts higher-dimensional modifications to the energy-momentum tensor as obtained by the usual form of the Friedmann-Robertson-Walker (FRW) metric. However, since there is more than one way for embedding a four-dimensional spacetime in a given five-dimensional manifold, the results obtained crucially depend on the method used to identify the 4D metric from the 5D one. In this paper we show that higher-dimensional
modifications to “conventional” FRW spacetimes can be obtained not only referring to “moving” hypersurfaces non-orthogonal to the extra dimension but also considering “fixed” hypersurfaces orthogonal to a suitable scalar function which defines a static covariant foliation of the bulk and takes the role of the extra dimension in a suitable coordinate system. We consider a five-dimensional manifold embedding a homogeneous and isotropic universe and described by a 5D metric with a time-dependent extra dimension and utilize the geometric construction performed by Sehara and Wesson \(^4\)–\(^6\) to obtain the foliation of the manifold by static 4D hypersurfaces. Then we transform the previous 5D metric into a metric where the new extra coordinate does not depend on the new time and verify that also on each fixed leaf of the foliation the induced 4D metric predicts higher-dimensional modifications with respect to the usual FRW line element. Finally we apply our approach to a well-known five-dimensional metric found by Ponce de Leon \(^7\) and discuss the results obtained in the different models.

Conventions. Throughout the paper the 5D metric signature is taken to be \((+,+,+,-,\varepsilon)\) where \(\varepsilon\) can be +1 or −1 depending on whether the extra dimension is spacelike or timelike, while the choice of the 4D metric signature is \((+,+,+,-)\). Bulk indices will be denoted by capital Latin letters and brane indices by lower Greek letters. Finally we use units where \(c = G = 1\).

2. From a time-dependent extra dimension to static 4D hypersurfaces

Our homogeneous and isotropic universe is envisaged as embedded in a five-dimensional manifold \(M\) covered by an arbitrary system of coordinates \(x^A = (r, \vartheta, \varphi, t, y)\). The 5D line element will be written in the usual form as

\[
\begin{align*}
  ds^2_5 &= a^2(t,y) \, d\sigma^2_k - n^2(t,y) \, dt^2 + \varepsilon b^2(t,y) \, dy^2 \\
  d\sigma^2_k &= \frac{dr^2}{1 - k \, r^2} + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right)
\end{align*}
\]  

(1)

where

\[
  d\sigma^2_k = \frac{dr^2}{1 - k \, r^2} + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right)
\]  

(2)

is the metric for a spherically symmetric space with curvature index \(k = +1, 0, -1\). In a common approach our universe is identified with a hypersurface \(\Sigma_{y_0} : y = y_0 = \text{constant}\) which is orthogonal to the extra dimension. On the hypersurface \(\Sigma_{y_0}\), introducing the proper
time $T$ by means of $T = \int n(t, y_0) \, dt$, the line element (1) can be reduced to the usual form of the FRW metric

$$ds_4^2 = a^2(T, y_0) \, d\sigma_k^2 - dT^2$$

However this embedding is not unique. Ponce de Leon assumed that our universe is generated on a moving hypersurface $\Sigma_f : y = f(t)$ which is not orthogonal to the extra dimension because its normal vector

$$\nu_A = \sqrt{\epsilon n b} \left(0, 0, 0, -\frac{df}{dt}, 1\right)$$

is not tangent to the $y$-lines. The metric induced on $\Sigma_f$ is

$$ds_4^2 = a^2(t, f(t)) \, d\sigma_k^2 - \left[n^2(t, f(t)) - \epsilon b^2(t, f(t)) \left(\frac{df(t)}{dt}\right)^2\right] \, dt^2$$

The function $f(t)$ which determines the hypersurface $\Sigma_f$ is the solution to

$$n^2(t, f(t)) - \epsilon b^2(t, f(t)) \left(\frac{df(t)}{dt}\right)^2 = 1$$

Now the scale factor depends on $f(t)$ so the effective matter content on $\Sigma_f$ is not the same as in “conventional” FRW models: the evolution of the extra dimension carries higher dimensional modifications to four-dimensional general relativity. In this paper, utilizing the geometric construction performed by Sehara and Wesson $^{4-6}$ to obtain a covariant foliation of a five-dimensiona manifold $M$, we show that such modifications are obtained also in four-dimensional hypersurfaces $\Sigma_\ell$ which are constant and orthogonal to a scalar function $\ell(t, y)$ which takes the role of the extra dimension in a suitable coordinate system. Let us consider the five-dimensional manifold $M$ given by Eq. (1) when the extra dimension $y = f(t)$ is time-dependent. The geometric construction of Refs. 4-6, which here we briefly recall, introduces a scalar function $\ell = \ell(t, y)$ which defines a foliation of the higher-dimensional manifold $M$ with hypersurfaces $\Sigma_\ell$ given by $\ell = \text{constant}$. Each hypersurface $\Sigma_\ell$ corresponds to a four-dimensional spacetime and is assumed to have a normal vector given by

$$n_A = \epsilon \Phi \, \frac{\partial \ell}{\partial x^A}, \quad n_A n^A = \epsilon$$

The scalar $\Phi$ which normalizes $n^A$ is known as the lapse function. The projector tensor $h_{AB}$ from the bulk to the hypersurfaces is

$$h_{AB} = g_{AB} - \epsilon n_A n_B$$
This tensor is symmetric and orthogonal to \( n^A \). Each hypersurface \( \Sigma_\ell \) is mapped by a 4D coordinate system \( \{ \tilde{x}^\alpha \} \). The four basis vectors

\[
e^A_\alpha = \frac{\partial x^A}{\partial \tilde{x}^\alpha} \quad \text{with} \quad n_A e^A_\alpha = 0
\]

(9)

are tangent to the \( \Sigma_\ell \) hypersurfaces and orthogonal to \( n_A \). These basis vectors can be used to project 5D objects onto \( \Sigma_\ell \) hypersurfaces. The induced metric on the \( \Sigma_\ell \) hypersurfaces is given by

\[
h_{\alpha\beta} = e^A_\alpha e^B_\beta g_{AB} = e^A_\alpha e^B_\beta h_{AB}
\]

(10)

Clearly \( \{ \tilde{x}^\alpha, \ell \} \) defines an alternative coordinate system to \( \{ x^\alpha, y \} \) on \( M \). Moreover 5D vectors are decomposed into the sum of a part tangent to \( \Sigma_\ell \) and a part normal to \( \Sigma_\ell \). For \( dx^A \) it results

\[
dx^A = e^A_\alpha d\tilde{x}^\alpha + (N^\alpha e^A_\alpha + \Phi n^A) d\ell
\]

(11)

The 4D vector \( N^\alpha \) is called the shift vector and it describes how the \( \{ \tilde{x}^\alpha \} \) coordinate system changes as one moves from a given \( \Sigma_\ell \) hypersurface to another. The 5D line element (1) can then be rewritten as

\[
ds^2_5 = h_{\alpha\beta} (d\tilde{x}^\alpha + N^\alpha d\ell) (d\tilde{x}^\beta + N^\beta d\ell) + \varepsilon \Phi^2 d\ell^2
\]

(12)

We choose on \( M \) the coordinate system \( \tilde{x}^A = \{ \tilde{x}^\alpha, \ell \} \) alternative to \( x^A = \{ x^\alpha, y \} \) maintaining unchanged the spatial coordinates \( r, \vartheta, \varphi \) but changing the time coordinate from \( t \) to \( \tau \), namely \( \tilde{x}^A = (r, \vartheta, \varphi, \tau, \ell) \), so we have to consider the following transformation in the \( (t, y) \) hyperplane:

\[
t = t(\tau, \ell), \quad y = y(\tau, \ell)
\]

(13)

Then after obtaining from the diffeomorphism (11) the foliation parameters \( \Phi \) and \( N^\alpha \) the line element (12) becomes

\[
ds^2_5 = a^2 d\sigma^2_k - \left[ n^2 \left( \frac{\partial t}{\partial \tau} \right)^2 - \varepsilon b^2 \left( \frac{\partial y}{\partial \tau} \right)^2 \right] \left[ d\tau + \frac{n^2 \left( \frac{\partial t}{\partial \ell} \right) \left( \frac{\partial \ell}{\partial y} \right) + \varepsilon b^2 \left( \frac{\partial y}{\partial \ell} \right) \left( \frac{\partial \ell}{\partial t} \right)}{n^2 \left( \frac{\partial t}{\partial \tau} \right) \left( \frac{\partial \tau}{\partial y} \right) + \varepsilon b^2 \left( \frac{\partial y}{\partial \tau} \right) \left( \frac{\partial \tau}{\partial \ell} \right)} \right] d\ell^2
\]

\[
+ \varepsilon n^2 b^2 \frac{\partial \ell}{\partial \ell} \left( \frac{\partial \ell}{\partial \ell} \right) + \left( \frac{\partial y}{\partial \ell} \right) \left( \frac{\partial \ell}{\partial y} \right) \right] \frac{1}{n^2 \left( \frac{\partial \tau}{\partial y} \right)^2 - \varepsilon b^2 \left( \frac{\partial \tau}{\partial t} \right)^2} d\ell^2
\]

(14)
Here the functions $a$, $n$ and $b$ depend on $\tau$ and $\ell$. Let us notice that the line element (14) can be reduced on a hypersurface $\Sigma_{\ell_0}: \ell = \ell_0 = \text{constant}$ to the usual form of the FRW metric requiring the condition

$$\left[ n^2(t(\tau), y(\tau)) \left( \frac{dt(\tau)}{d\tau} \right)^2 - \epsilon b^2(t(\tau), y(\tau)) \left( \frac{dy(\tau)}{d\tau} \right)^2 \right] = 1$$

which in our approach takes the place of Eq. (6) and shows that the functions $t(\tau)$ and $y(\tau)$ are not independent but, as already found in a similar context in Ref. 3, they can be parametrized by one function $F(\tau)$ of the proper time $\tau$. The condition $n_An^A = \epsilon$ gives

$$\Phi^2 = \frac{n^2 b^2}{n^2 \left( \frac{\partial \ell}{\partial t} \right)^2 - \epsilon b^2 \left( \frac{\partial \ell}{\partial y} \right)^2}$$

so looking at the expression of $\Phi^2$ in Eq. (14) it must be

$$\left( \frac{\partial t}{\partial \tau} \right) \left( \frac{\partial \ell}{\partial t} \right) + \left( \frac{\partial y}{\partial \tau} \right) \left( \frac{\partial \ell}{\partial y} \right) = 1$$

Moreover the condition $e^A_{\alpha}n_A = 0$ gives

$$\left( \frac{\partial t}{\partial \tau} \right) \left( \frac{\partial \ell}{\partial t} \right) + \left( \frac{\partial y}{\partial \tau} \right) \left( \frac{\partial \ell}{\partial y} \right) = 0$$

To satisfy the constraints (17) and (18) we begin choosing, between all the possible transformations, the following one

$$\begin{cases} t = F(\tau) \cosh (\sqrt{\epsilon} \psi) + (\ell - \ell_0) \sqrt{\epsilon} \sinh (\sqrt{\epsilon} \psi) \\ y - y_0 = F(\tau) \frac{\sinh (\sqrt{\epsilon} \psi)}{\sqrt{\epsilon}} + (\ell - \ell_0) \cosh (\sqrt{\epsilon} \psi) \end{cases}$$

where $\psi$ is a constant. One can verify that the constraint (18) is satisfied solving the partial differential equation for the function $\ell = \ell(t, y)$ which, in a simple form, is

$$\ell = \ell_0 - \frac{\sinh (\sqrt{\epsilon} \psi)}{\sqrt{\epsilon}} t + \cosh (\sqrt{\epsilon} \psi) (y - y_0)$$

while the constraint (17) becomes an identity after substituting in it the derivatives of $\ell(t, y)$. Eq. (20) shows that $\ell = \text{constant}$ implies that here $y(t)$ is a linear function of $t$. The value of $F(\tau)$ can be determined once are known the metric coefficients $n$ and $b$. Finally we can write the higher dimensional line element as
\[
ds^2_5 = a^2 \, d\sigma_k^2 - (n^2 \cosh^2(\sqrt{\epsilon} \psi) - b^2 \sinh^2(\sqrt{\epsilon} \psi)) \left[ \left( \frac{dF}{d\tau} \right)^2 \right] \, d\tau
\]

\[
- \frac{(n^2 - b^2)}{(n^2 \cosh^2(\sqrt{\epsilon} \psi) - b^2 \sinh^2(\sqrt{\epsilon} \psi))} \, d\ell \right]^2 + \epsilon \frac{n^2 b^2}{(n^2 \cosh^2(\sqrt{\epsilon} \psi) - b^2 \sinh^2(\sqrt{\epsilon} \psi))} \, d\ell^2
\]

We notice that while on \( \Sigma_f \) it was \( \frac{dy}{dt} \neq 0 \) now on \( \Sigma_\ell \) it results \( d\ell/d\tau = 0 \), so the normal vector to \( \Sigma_\ell \) is tangent to the \( \ell \)-lines. The metric induced on the hypersurface \( \Sigma_{\ell_0} \) is

\[
ds^2_4 = a^2 \, d\sigma_k^2 - (n^2 \cosh^2(\sqrt{\epsilon} \psi) - b^2 \sinh^2(\sqrt{\epsilon} \psi)) \left( \frac{dF}{d\tau} \right)^2 \, d\tau^2 \quad (22)
\]

Clearly on \( \Sigma_{\ell_0} \) we have that \( a, n \) and \( b \) depend only on \( \tau \) through the function \( F(\tau) \). The line element (22) can be reduced to the usual form of the FRW metric requiring that

\[
[n^2(F) \cosh^2(\sqrt{\epsilon} \psi) - b^2(F) \sinh^2(\sqrt{\epsilon} \psi)] \left( \frac{dF}{d\tau} \right)^2 = 1 \quad (23)
\]

which, with the initial condition \( F(\tau)|_{\tau=0} = 0 \), provides the unknown function \( F(\tau) \). We notice that the left-hand side of (23) is clearly greater than zero when \( \epsilon = -1 \) but when \( \epsilon = 1 \) one has to discuss the sign of the term enclosed within square brackets. The FRW metric (3) is recovered from (22) in the particular case \( \psi = 0 \) because, as can be checked using Eqs. (19) and (23), in this case it results \( y = y_0 \) and \( T = \int n(F)dF = \tau \).

3. A comparison between induced metrics on \( \Sigma_{y_0}, \Sigma_{\ell_0} \) and \( \Sigma_f \)

To see more in detail how our model works and to make the comparison between the induced metrics on the various hypersurfaces above defined, we shall consider the well-known five-dimensional metric found by Ponce de Leon

\[
ds^2_5 = A^2 \left( \frac{t}{L} \right)^{2/\alpha} \left( \frac{y}{L} \right)^{2/(1-\alpha)} \, d\sigma_0^2 - \left( \frac{y}{L} \right)^2 \, dt^2 + \left( \frac{\alpha}{1 - \alpha} \right)^2 \left( \frac{t}{L} \right)^2 \, dy^2 \quad (24)
\]

where \( A \) and \( L \) are constant lengths and \( \alpha \) is a constant dimensionless parameter different from 0 and 1. This metric is a solution to the five-dimensional Einstein equations in vacuum, it is flat \( (k = 0) \) in ordinary three-space and has a space-like \( (\epsilon = +1) \) extra dimension. Equation (24) is one of the classes of solutions obtained in Ref. 7 for cosmological models in a Kaluza-Klein theory; it was worked out by Wesson 8 to discuss the details of a FRW
model and it was generalized by Rippl, Romero and Tavakol to study lower-dimensional gravity. Since then, many other cosmological solutions and their associated matter properties have been derived and the whole analysis about the embedding of four-dimensional general relativity in five dimension goes back to the Campbell theorem which was rediscovered by Romero, Tavakol and Zalaletdinov. First we recall some results obtained projecting the metric (24) on the hypersurface \( \Sigma_{y_0} \) where the line element is

\[
\begin{align*}
d_{s_1}^2 &= A^2 \left( \frac{y_0}{L} \right)^{2/(1-\alpha)} \left( \frac{T}{y_0} \right)^{2/\alpha} d\sigma_0^2 - dT^2
\end{align*}
\]

here \( T = (y_0/L) t \) is the FRW proper time. Consequently the pressure \( p \), the density \( \rho \), the equation of state \( p/\rho = w \) of the induced matter, the gravitational density \( \rho_g = 3p + \rho \) and the deceleration parameter \( q \) are given by

\[
\begin{align*}
8\pi p &= \frac{2\alpha - 3}{\alpha^2 T^2}, & 8\pi \rho &= \frac{3}{\alpha^2 T^2}, & \frac{p}{\rho} &= \left( \frac{2\alpha - 3}{3} \right), & \rho_g &= \frac{3(\alpha - 1)}{4\pi \alpha^2 T^2}, & q &= \alpha - 1
\end{align*}
\]

So models with \( \alpha \in (0, 1) \) describe an accelerating universe with exotic matter, while models with \( \alpha > 1 \) have ordinary matter satisfying the strong energy condition. Models with \( \alpha < 0 \) are excluded because they imply a contracting universe. The present-day age \( T_0 \) of an universe emerging from a big bang is given, as explicitly first pointed out in Ref. [11], by

\[
T_0 = \frac{H_0^{-1}}{1 + \tilde{q}(T_0)}
\]

where \( H_0 \) and \( \tilde{q}(T_0) \) are respectively the present-day values of the Hubble parameter and of the average deceleration parameter which is given by

\[
\tilde{q}(T_0) = \frac{1}{T_0} \int_0^{T_0} q(T) dT
\]

If the conjecture \( H_0 T_0 = 1 \) is valid then the average deceleration parameter must be zero when averaged after a long interval of time, which means that the universe evolves through a cascade of accelerating/decelerating regimes. Our aim in this paper is to show that higher-dimensional modifications to FRW spacetimes can be obtained also on 4D static hypersurface, so we shall not treat here the cross-over from decelerate to accelerate cosmic expansion. Now we give our results obtained projecting the metric (24) on the hypersurface \( \Sigma_{y_0} \). Equation (23) for the function \( F(\tau) \) becomes

\[
\frac{\cosh^2(\psi)}{L^2} \left[ \frac{(1 - 2\alpha)}{(1 - \alpha)^2} \sinh^2(\psi) F^2(\tau) + 2 y_0 \sinh(\psi) F(\tau) + y_0^2 \right] \left( \frac{dF(\tau)}{d\tau} \right)^2 = 1
\]
Before solving Eq. (29) we have to discuss the sign of the binomial in \( F(\tau) \) enclosed within square brackets. Hereafter we shall assume that \( \psi, \, y_0 \) and \( L \) are all positive and finite constants and we shall require that \( F(\tau) \) increases with \( \tau \). The binomial is greater than zero: i) if \( 0 < \alpha \leq 1/2 \) for all \( F(\tau) > 0 \); ii) if \( 1/2 < \alpha < 1 \) for \( F(\tau) < (y_0/\sinh(\psi))(1-\alpha)/(2\alpha-1) \); iii) if \( \alpha > 1 \) for \( F(\tau) < (y_0/\sinh(\psi))(\alpha - 1) \). Once \( F(\tau) \) has been found the scale factor for the metric (24) becomes

\[
a(\tau) = A \left( \frac{F(\tau)}{L} \tanh(\psi) \right)^{1/\alpha} \left( \frac{y_0}{L} + \frac{F(\tau)}{L} \sinh(\psi) \right)^{1/(1-\alpha)}
\]  

(30)

Now the scale factor depends on \( F(\tau) \) so the effective matter content on \( \Sigma_{\ell_0} \) is not the same as in “conventional” FRW models. It is apparent that when \( \alpha > 1/2 \) the scale factor can not become greater than a particular amount in time, so models with \( \alpha > 1/2 \) can describe a particular stage of the evolution of the universe, for example its early evolution. Two consecutive stages can be joined by using appropriate junction conditions.\(^{1,2}\) Equation (29) can be easily integrated, however in the case \( \alpha \neq 1/2 \) it is not possible to explicitly obtain \( F(\tau) \) as a function of \( \tau \) so one must use an approximate expression for it. Let us begin considering the value \( \alpha = 1/2 \) for which the function \( F(\tau) \) can be exactly obtained. It is worth noticing that Seahra and Wesson\(^{12}\) discussing the structure of the big bang from higher-dimensional embeddings mentioned that the \( \alpha = 1/2 \) cosmology is the only case for which the Ponce de Leon metric (24) is well defined. From Eq. (29) we have

\[
F(\tau) = \frac{y_0}{2 \sinh(\psi)} \left[ (1 + \kappa \tau)^{2/3} - 1 \right]
\]  

(31)

where \( \kappa = (3L/y_0^2) \tanh(\psi) \). The scale factor is

\[
a(\tau) = A \left[ \frac{1}{4 \tanh(\psi)} \left( \frac{y_0}{L} \right)^2 \right]^2 \left[ (1 + \kappa \tau)^{4/3} - 1 \right]^2
\]  

(32)

Pressure and density are

\[
8\pi p = -\frac{16}{9} \kappa^2 \left[ \frac{9 (1 + \kappa \tau)^{4/3} - 1}{(1 + \kappa \tau)^{2/3} \left[ (1 + \kappa \tau)^{4/3} - 1 \right]^2} \right] ^2, \quad 8\pi \rho = \frac{64}{3} \kappa^2 \left[ \frac{(1 + \kappa \tau)^{2/3}}{(1 + \kappa \tau)^{4/3} - 1} \right]^2
\]  

(33)

and the equation of state of the effective matter is

\[
\frac{p}{\rho} = -\frac{3}{4} \left( 1 - \frac{1}{9 (1 + \kappa \tau)^{4/3}} \right)
\]  

(34)

Gravitational density and deceleration parameter are

\[
\rho_g = -\frac{2}{3\pi} \left[ \frac{5 (1 + \kappa \tau)^{4/3} - 1}{(1 + \kappa \tau)^{2/3} \left[ (1 + \kappa \tau)^{4/3} - 1 \right]^2} \right], \quad q = -\frac{5}{8} \left( 1 - \frac{1}{5 (1 + \kappa \tau)^{4/3}} \right)
\]  

(35)
In the case \( \alpha = 1/2 \) we have therefore an accelerating universe with exotic matter. Before making comparison with observers in the hypersurfaces \( \Sigma_{y_0} \) and \( \Sigma_f \) we recall that they use different clocks and that the relations between the proper times \( T, t \) and \( \tau \) are given by

\[
T = \frac{y_0}{L} t = \frac{y_0}{L} F(\tau) \cosh(\psi)
\]  
(36)

From the results found on \( \Sigma_{y_0} \) it is apparent that there are higher-dimensional modifications to the FRW spacetime (25). The usual FRW description is however approximately recovered in a period close enough to the initial time \( \tau = 0 \). Starting again from the metric (24) modifications to the FRW spacetime (25) were first obtained by Ponce de Leon.\(^1\)\(^2\) In particular on the hypersurface \( \Sigma_f \) when for \( \alpha = 1/2 \) it results

\[
f(t) = \frac{L}{2Kt} (1 + K^2 t^2)
\]  
(37)

\[
a(t) = \frac{A}{(2C)^2} (1 + K^2 t^2)^2
\]  
(38)

\[
8\pi p = -\frac{8K^2(1 + 5K^2 t^2)}{(1 + K^2 t^2)^2}, \quad 8\pi \rho = \frac{48K^4 t^2}{(1 + K^2 t^2)^2}, \quad \frac{p}{\rho} = -\frac{5}{6} \left(1 + \frac{1}{5K^2 t^2} \right)
\]  
(39)

\[
\rho_g = -\frac{3}{\pi} \frac{(1 + 3K^2 t^2)}{(1 + K^2 t^2)}, \quad q = -\frac{3}{4} \left(1 + \frac{1}{3K^2 t^2} \right)
\]  
(40)

where \( C \) is a dimensionless constant coming from the integration of Eq. (6) and \( K = C/L \). We have again an accelerating universe with exotic matter but the higher-dimensional modifications are different from those found on \( \Sigma_{\ell_0} \). Equation (38) gives \( a(\tau)|_{\tau=0} \neq 0 \) in disagreement with FRW models, however this feature can be put away\(^1\)\(^2\) making use of the dominant energy condition which requires \( Kt \geq 1 \). When \( \alpha \neq 1/2 \) we integrate Eq. (29) but now \( F(\tau) \) cannot be explicitly given as a function of \( \tau \) to obtain \( a(\tau) \) on \( \Sigma_{\ell_0} \). We shall therefore consider approximate expressions for \( F(\tau) \) which can be obtained from (29) using power series expansions in the particular cases when \( F(\tau)/L \ll 1 \) (early evolution of the universe) and when \( F(\tau)/L \gg 1 \) (late evolution of the universe). After the values of \( F(\tau) \) have been found we shall obtain the corresponding approximate values of the scale factor \( a(\tau) \) and of the other quantities of interest by a series expansion in the proper time both in the early universe \( \kappa \tau \ll 1 \) and in the late universe \( \kappa \tau \gg 1 \). Finally, the expressions of \( F(\tau), a, p, \rho, p/\rho, \rho_g, q \) are given in Appendix A when \( \kappa \tau \ll 1 \) and in Appendix B when \( \kappa \tau \gg 1 \). Since the quantities of physical interest can be derived from the knowledge of the
scale factor, to compare between our results and those obtained on Σ_{y_0} and Σ_f we recall that the expression of the scale factor on Σ_{y_0} is

\[ a(T) = A \left( \frac{y_0}{L} \right)^{1/(1-\alpha)} \left( \frac{T}{y_0} \right)^{1/\alpha} \] (41)

and on Σ_f is

\[ a(t) = \frac{A}{(2C)^{1/\alpha}} \left( \frac{t}{L} \right)^{(1-|1-\alpha|/(1-\alpha))/\alpha} \left[ 1 + C^2 \left( \frac{t}{L} \right)^{2(1-\alpha)/\alpha} \right]^{1/(1-\alpha)} \] (42)

It is apparent from Appendix A that the usual FRW description is approximately recovered in a period close enough to the initial time \( \tau = 0 \). Later on, both Appendixes show that when \( \alpha \neq 1/2 \) there are on Σ_{\ell_0} higher-dimensional modifications to the “conventional” FRW spacetime (25) and that these modifications are different from the ones found on Σ_f. As emphasized in Refs. 1-2 the different results coming from different values of \( \alpha \) represent the same spacetime in another parametrization so if the value of \( \alpha \) is allowed to change in order to have a not constrained equation of state, then one can study a more realistic model of the universe by joining metrics with different values of \( \alpha \) across a time-constant hypersurface. Such a calculation is, however, beyond the scope of this paper.

4. Conclusion

We have shown that, starting from a 5D given metric which has a large time-dependent extra dimension and describes a homogeneous universe, it is possible to consider an embedded static hypersurface Σ_{\ell_0} where exist higher-dimensional modifications to the energy-momentum tensor as obtained on a static FRW hypersurface Σ_{y_0} and also different from those found on a dynamic hypersurface Σ_f. As a working example, we applied our model to the line element (24) which can be considered the generalization of the flat FRW cosmological metric to five dimensions so we have to check whether the obtained results are in agreement with the present-day observational data, in particular with the fact that our universe is now in accelerated expansion. Irrespective of the value assumed by the quantity \( \kappa \tau \) at the present age \( \tau_0 \) of the universe, we see that when \( \alpha < 1 \) it results \( q < 0 \) but if we limit the range of \( \alpha \) to \( \alpha \leq 1/2 \), then we have that \( q \in (-1, -1/2) \) which is consistent with recent constraints on accelerating universe combined with various cosmological probes. 13 It remains to fix the value of \( \kappa \tau \) today, but having choosen to associate \( \kappa \tau \ll 1 \) and \( \kappa \tau \gg 1 \)
respectively to the early and to the late universe, we choose $\kappa \tau_0 = 1$ as an acceptable value. Therefore we must use the exact solution corresponding to $\alpha = 1/2$ and find that today the deceleration parameter is $q_0 = -0.575$ while the equation of state parameter is $w_0 = -0.717$, results which are in accordance with the estimated mean values of these parameters in the actual universe. Finally, in the hypotheses that the free parameter $\alpha$ in the considered line element (24) can be a function of the time $t$ and that the values $\kappa \tau \ll 1$ and $\kappa \tau \gg 1$ correspond respectively to the early (post-inflationary) and to the late universe, one might associate different values of $\alpha$ to the various eras of the evolving universe. So, for example, when $1 < \alpha < 3$ the primordial matter behaves similar to ordinary matter and the expansion is slowing down, but when $0 < \alpha < 1$ the matter has exotic properties, and the expansion is speeding up. It will be interesting to apply the approach we suggest in this paper to other solutions of modern cosmologies with extra dimensions.

Appendix A: Results on $\Sigma_{\ell_0}$ when $\alpha > 1/2$

When $\alpha > 1/2$ we obtain in the early universe ($\kappa \tau \ll 1$

\[ F(\tau) \approx \frac{y_0}{\sinh(\psi)} \left[ \sqrt{1 + \frac{2}{3} \kappa \tau - 1} \right] \]

\[ a(\tau) \approx A \frac{1}{(3 \tanh(\Psi))^{1/\alpha}} \left( \frac{y_0}{L} \right)^{1/\alpha} \left( \frac{y_0}{L} \right)^{1/\alpha} \left[ 1 - \frac{3\alpha - 1}{6\alpha(\alpha - 1)} \kappa \tau \right] \]

\[ 8\pi p \approx \frac{2\alpha - 3}{\alpha^2 \tau^2} \left[ 1 + \frac{3\alpha - 1}{2(\alpha - 3)(\alpha - 1)} \kappa \tau \right] \]

\[ 8\pi \rho \approx \frac{3}{\alpha^2 \tau^2} \left[ 1 - \frac{3\alpha - 1}{3(\alpha - 1)} \kappa \tau \right] \]

\[ \frac{p}{\rho} \approx \frac{2\alpha - 3}{3} \left[ 1 + \frac{2\alpha(3\alpha - 1)}{3(2\alpha - 3)(\alpha - 1)} \kappa \tau \right] \]

\[ \rho_g \approx \frac{3(\alpha - 1)}{4\pi \alpha^2 \tau^2} \left[ 1 + \frac{3\alpha - 1}{3(\alpha - 1)^2} \kappa \tau \right] \]

\[ q \approx (\alpha - 1) \left[ 1 + \frac{\alpha(3\alpha - 1)}{3(\alpha - 1)^2} \kappa \tau \right] \]

Appendix B: Results on $\Sigma_{\ell_0}$ when $\alpha < 1/2$

When $\alpha < 1/2$ we obtain in the late universe ($\kappa \tau \gg 1$

\[ F(\tau) \approx \frac{y_0}{\sinh(\psi)} \left( \frac{1 - \alpha}{1 - 2\alpha} \right) \left[ \sqrt{1 + \frac{2(1 - 2\alpha)^{3/2}}{3(1 - \alpha)^3} \kappa \tau - 1} \right] \]
\( a(\tau) \approx A \left[ \frac{(1 - \alpha)}{\tanh^{2(1 - \alpha)}(\Psi) \sqrt{1 - 2\alpha}} \left( \frac{y_0}{L} \right)^2 \right]^{1/(2\alpha(1 - \alpha))} \left( \kappa \tau \right)^{1/(2\alpha(1 - \alpha))} \left[ 1 - \frac{\sqrt{3(1 - 3\alpha(1 - \alpha))}}{\sqrt{2\alpha ((1 - \alpha) \sqrt{1 - 2\alpha})^{3/2}} \sqrt{\kappa \tau}} \right] \) \( (B2) \)

\( 8\pi p \approx -\frac{(3 - 4\alpha(1 - \alpha))}{4\alpha^2(1 - \alpha)^2 \tau^2} \left[ 1 - \frac{3\sqrt{6(1 - \alpha(1 - \alpha))}(1 - 3\alpha(1 - \alpha))}{(3 - 4\alpha(1 - \alpha)) \sqrt{(1 - \alpha) (1 - 2\alpha)^{3/2}}} \frac{1}{\sqrt{\kappa \tau}} \right] \) \( (B3) \)

\( 8\pi \rho \approx \frac{3}{4\alpha^2(1 - \alpha)^2 \tau^2} \left[ 1 + \frac{\sqrt{6(1 - 3\alpha(1 - \alpha))}}{\sqrt{(1 - \alpha) (1 - 2\alpha)^{3/2}}} \frac{1}{\sqrt{\kappa \tau}} \right] \) \( (B4) \)

\( \frac{p}{\rho} \approx -\left( \frac{3 - 4\alpha(1 - \alpha)}{3} \right) \left[ 1 + \frac{\sqrt{6\alpha(1 - 3\alpha(1 - \alpha))}}{3 - 4\alpha(1 - \alpha)} \sqrt{(1 - \alpha) (1 - 2\alpha)^{3/2}} \frac{1}{\sqrt{\kappa \tau}} \right] \) \( (B5) \)

\( \rho_g \approx -\frac{3(\alpha^2 + (1 - \alpha)^2)}{16\pi \alpha^2(1 - \alpha)^2 \tau^2} \left[ 1 - \frac{\sqrt{3(1 - 3\alpha(1 - \alpha))(2 - 3\alpha(1 - \alpha))}}{\sqrt{2(1 - 2\alpha(1 - \alpha)) \sqrt{(1 - \alpha) (1 - 2\alpha)^{3/2}}} \frac{1}{\sqrt{\kappa \tau}} \right] \) \( (B6) \)

\( q \approx -(1 - 2\alpha(1 - \alpha)) \left[ 1 + \frac{\sqrt{3\alpha(1 - 3\alpha(1 - \alpha))}}{\sqrt{2(1 - 2\alpha(1 - \alpha)) \sqrt{(1 - \alpha) (1 - 2\alpha)^{3/2}}} \frac{1}{\sqrt{\kappa \tau}} \right] \) \( (B7) \)

[1] J. Ponce de Leon, *Mod. Phys. Lett. A* **21**, 947 (2006) [gr-qc/0511067].

[2] J. Ponce de Leon, *Int. J. Mod. Phys. D* **15**, 1237 (2006) [gr-qc/0511150].

[3] J. Ponce de Leon, *JHEP* 0903:052 (2009) [gr-qc/0902.2270].

[4] S. S. Seahra, *Phys. Rev. D* **65**, 124004 (2002) [gr-qc/0204032].

[5] S. S. Seahra, *Phys. Rev. D* **68**, 104027 (2003) [gr-qc/0309081].

[6] S. S. Seahra and P. S. Wesson, *Class. Quantum Grav.* **20**, 1321 (2003) [gr-qc/0302015].

[7] J. Ponce de Leon, *Gen. Rel. Grav.* **20**, 539 (1988).

[8] P. S. Wesson, *Astrophys. J.* **394**, 19 (1992).

[9] S. Rippl, C. Romero and R. Tavakol, *Class. Quant. Grav.* **12**, 2411 (1995) [gr-qc/9511016].

[10] C. Romero, R. Tavakol and R. Zalaletdinov, *Gen. Rel. Grav.* **28**, 365 (1996).

[11] J. A. S. Lima, [astro-ph/0708.3414].

[12] S. S. Seahra and P. S. Wesson, *Class. Quantum Grav.* **19**, 1139 (2002) [gr-qc/0204048].

[13] J. Lu, L. Xu, M. Liu and Y. Gui, *Eur. Phys. J. C* **58**, 311 (2008) [astro-ph/0812.3209].