How Ampère could have derived the Lorentz Transformations

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(Dated: August 30, 2018)

Lorentz Transformations of Special Relativity are derived from two postulates: the first is the Principle of Relativity, while the postulate of invariance of the velocity of light, used in usual derivations, is replaced by a law of electro-magneto-statics and invariance of electrical charge. Our derivation does not require the assumption of regularity conditions of the transformations, such as linearity and continuity required by other derivations. The level of the needed mathematics and physical concepts makes the proposed derivation suitable for Secondary School.

PACS numbers: 03.30.+p; 01.40.-d

I. INTRODUCTION

In Special Relativity, Lorentz Transformations are usually derived from two postulates: the Principle of Relativity stating that “The laws of physics are the same in all inertial frames”, and the law of the invariance of velocity of light (Einstein’s second postulate) \(^\text{[2]}\). The literature shows that Lorentz Transformations can be derived without this second postulate; a partial review can be found in \(^\text{[2]}\).

In his approach to “relativity without light”, Mermin \(^\text{[2]}\) derives the relativistic addition law for parallel velocities directly from the Principle of Relativity and “a few simple assumptions of smoothness and symmetry”. Starting from the same assumptions of Mermin, Singh \(^\text{[5]}\) derives the Lorentz Transformations. The pedagogical advantage of these approaches is that the invariance of light velocity, with its paradoxical character, is not imposed \textit{ab initio}, but it is obtained as a consequence of the theory. However, Mermin himself acknowledges that his approach entails an “higher level of analysis”, hence it “is unavailable for a general educational physics course, but as an introduction to special relativity for physics majors”.

Another approach is due to Sen \(^\text{[5]}\). He was able to obtain essentially the same results of Mermin by using only simple algebra and some assumptions of regularity of the functions involved in his derivations. For this reason this approach can be used for introducing the Special Relativity to the students at an introductory level.

We have to remark that all these derivations do not single out Lorentz Transformations as the only transformations consistent with their postulates. Actually, what can be stated is that the function \(w(u, v)\) expressing the additional law of velocities \(u\) and \(v\) is \(w(u, v) = (u + v)/(1 + Kuv)\), where \(K\) is a non-negative constant. Therefore we have a large range of possibilities: if \(K = 0\) then we get the Galilean Transformations; if \(K > 0\) then we get a theory in which there is an invariant velocity \(c_K = 1/\sqrt{K}\). Thus, what is the right transformation law cannot be decided on the basis of their assumptions only.

In this work we propose a derivation of Lorentz Transformations, and hence of Special Relativity, without making use of invariance of light velocity. Similarly to the approaches above outlined, we assume the Principle of Relativity and a few simple assumptions of symmetry and reciprocity; nevertheless, our derivation is different. We impose that a given law of electro-magneto-statics, whose empirical validity was known since Ampère \(^\text{[2]}\), holds in all inertial frames according to the Principle of Relativity. In so doing we can directly derive Lorentz Contraction and thereby Lorentz Transformations in their complete form without indeterminate parameters, as the only transformations consistent with the validity of that physical law in all inertial frames. In our derivation, furthermore, it is not necessary making use of regularity assumptions such as linearity or continuity. For this reason, our derivation lies on (empirical) physics rather than on mathematics.

A central point of the proposed derivation is to single out this particular law. It is obtained by reformulating the conditions for equilibrium of the classical electro-magneto-statics without resorting to the notion of force, in a particular ideal experimental situation. The main contribution to discovery of this rule was given by Ampère, who established its empirical validity independently of any underlying theory.

Section II is devoted to establish the above mentioned law, denoted by \((L, 3)\). Furthermore, it is proved that Principle of Relativity implies that the lengths orthogonal to be relative motion between inertial frames are in-

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II. A LAW OF ELECTRO-MAGNETO-STATICS

In our derivation of Lorentz Transformations we need to impose that, according to Principle of Relativity, a same law holds in two different reference frames. In the present section, first we show that Principle of Relativity implies that lengths orthogonal to the direction of the relative motion between two inertial frames are invariant; second we assume that electrical charge is invariant, and finally establish a law of electro-magneto-statics submitted to Principle of Relativity and which can be expressed without the concept of force.

A. Invariance of transversal lengths

Let us consider two inertial frames $\Sigma_1$ and $\Sigma_2$, with the $x$ axes oriented as in Fig. 1 and with relative constant velocity $v = (v, 0, 0)$. Let us suppose that two material wires are parallel to the direction of the relative motion, and that one wire is at rest with respect to $\Sigma_1$, while the other is at rest with respect to $\Sigma_2$. By $r_1$ and $r_2$ we denote the distances between the two wires with respect to $\Sigma_1$ and $\Sigma_2$, respectively. Between these distances, which are the measures of lengths orthogonal to the direction of the relative motion,

$$r_1 \leq r_2,$$

parallel to the direction of the relative motion, one at rest with respect to $\Sigma_1$, and the other at rest with respect to $\Sigma_2$, satisfy the following relation.

The Principle of Relativity implies that (L.1) holds in every inertial frame $\Sigma_1$. Now, let us denote frames $\Sigma_1$ by $\Sigma''$, frame $\Sigma_2$ by $\Sigma'$, distance $r_2$ by $r'$ and distance $r_1$ by $r''$. After this rewording we see that $\Sigma''$, $r'$ and $r''$, with respect to $\Sigma'$, satisfy the conditions that (L.1) requires to $\Sigma_2$, $r_1$ and $r_2$ with respect to $\Sigma_1$. In particular, one of the wires is at rest with respect to $\Sigma'$ (namely, the wire at rest with respect to $\Sigma_2$) and the other is at rest with respect to $\Sigma''$ (Fig. 2). Therefore law (L.1) applies, and we have to conclude that $r' \leq r''$; hence, $r_2 \leq r_1$ holds together with $r_1 \leq r_2$, so that $r_1 = r_2$.

If case (ii) were realized, we would reach the same conclusion, by using the same argument.

B. Invariance of electrical charge

Electrical charge is assumed to be invariant. The quite familiar experience that if a solid body, with no (net) charge is heated, then its (net) charge remains zero provides an argument showing invariance of charge. Indeed, the increase of the temperature provokes an increase of the average kinematic energy of the particles constituting the body, i.e. electrons and nuclei. But due to the lower mass of electrons, their velocity increases much more than the velocity of nuclei. If the charge were dependent on the velocity, the change of charge due to electrons should overcome the change due to nuclei, and the body would acquire a (net) electrical charge. But this phenomenon has never been observed.

C. The empirical law

Let us consider two parallel wires which carry a stationary electrical current $i$ and an uniform charge density $\lambda$. One of the laws of Electro-magneto-statics establishes that
The function
\[ \phi(\lambda, i) = \frac{\lambda^2}{2\pi\epsilon_0 r} + \frac{\mu_0 i^2}{2\pi r} \]

is the density of the force acting on each wire.

Unfortunately, this physical law can be interpreted only by making resort to the notion of Newtonian force. Since the effects of a Newtonian force follow from the second Newton law
\[ \text{force} = \text{mass} \times \text{acceleration} \]
the validity of (1.2) could be verified only by implicitly assuming (NL). To avoid such a further undesired assumption, we shall extract from (1.2) another physical law, with a more poor physical content than (1.2), but which can be expressed without using undesired concepts, such as that of force, and whose empirical validity can be tested independently of any theory.

Let us consider a device \( D \) consisting of an uniform distribution of identical springs, each of them acting on both wires. If the springs are suitably manufactured, the action of the device establishes equilibrium between the wires. An implication of (1.2) is that such an equilibrium is broken if \( \lambda \) and \( i \) change their values in such a way that also the value of \( \phi \) turns out to be modified. This statement can be re-formulated as follows.

(L.2) If the action of device \( D \) yields equilibrium for both the two pair of values \( (\lambda, i) \) and \( (\hat{\lambda}, \hat{i}) \), then
\[ \phi(\hat{\lambda}, \hat{i}) = \phi(\lambda, i) \]

Statement (L.2) is an empirical implication of the laws of electromagnetostatics, which is expressed without the need to interpret function \( \phi \) in terms of force. Principle of Relativity implies that (L.2) must be considered as a physical law which holds in all inertial frames.

Remark 1. The equilibrium could be obtained by means of a device different from \( D \), for instance by replacing the springs with a greater number of weaker ones, without affecting the validity of (L.2). Now we show that the device can be designed in such a way to be invariant.

Given the two wires considered above in frame \( \Sigma \), let us consider another frame \( \Sigma' \) which moves with respect to frame \( \Sigma \) with a constant velocity \( v \) in the direction parallel to the wires (Fig. 3). Our device \( \mathcal{I} \) consists of two distributions of springs, \( D_1 \) and \( D_2 \). The first one, \( D_1 \), is made up of identical springs, each of them acting on both wires, uniformly distributed with density \( \rho_1 \), at rest with respect to frame \( \Sigma \). Second distribution, \( D_2 \), is made up of springs which move with velocity \( v \) (hence they are at rest with respect to \( \Sigma' \)), identical to each other, but different from the spring of \( D_1 \), uniformly distributed with density \( \rho_2 \) with respect to \( \Sigma \). Hence, with respect to \( \Sigma \) device \( \mathcal{I} \) consists of a distribution at rest with density \( \rho_1 \) and another distribution with density \( \rho_2 \) which moves with velocity \( v \).

Now, the value \( \rho_2 \) of the density of \( D_2 \) with respect to \( \Sigma \) is chosen in such a way that its value \( \rho_2' \) with respect to \( \Sigma' \) turns out to be equal to \( \rho_1 \). This means that if a distribution at rest has density \( \rho_1 \), then it has density \( \rho_2 \) with respect to a frame where it moves with velocity \( v \). As a consequence, with respect to \( \Sigma' \), device \( \mathcal{I} \) consists of a distribution at rest with density \( \rho_1 \) and another distribution with density \( \rho_2 \) and velocity \( v \) (Fig. 4).

Therefore, as regards to the densities, device \( \mathcal{I} \) composed by \( D_1 \) and \( D_2 \) appears to \( \Sigma \) identical to that seen by \( \Sigma' \), apart from an exchange of the roles of \( D_1 \) and \( D_2 \). Like the density \( \rho_2 \), the value of any other physical magnitude of \( D_2 \), which determines its action on the wires, can be chosen in such a way that device \( \mathcal{I} = D_1 + D_2 \) appears to \( \Sigma' \) identical to that seen by \( \Sigma \), after an exchange in the roles of \( D_1 \) and \( D_2 \). For instance, suppose that the value of one of these magnitudes, say \( C \), is \( c_1 \) for the springs of \( D_1 \) with respect to \( \Sigma \). Then we choose the springs of \( D_2 \) with a value \( c_2 \) of \( C \) with respect to \( \Sigma \) such that its value \( c_2' \) with respect to \( \Sigma' \) satisfies \( c_2' = c_1 \), which implies \( c_2' = c_2 \) (in fig.4 value \( c_1 \) (resp., \( c_2 \)) is revealed by the white (resp. gray) colour of the spring). The invariance of \( \mathcal{I} \) is completed by the fact that the distance \( r \) between the wires is invariant, as proved in section II.A.

Now, according to remark 1, we state the following law (L.3), obtained from (L.2) by replacing device \( D \) by the invariant device \( \mathcal{I} \).
(L.3) If the action of device $\mathcal{I}$ yields equilibrium for both the two pair of values $(\lambda, i)$ and $(\hat{\lambda}, \hat{i})$, then
\[ \phi(\lambda, i) = \phi(\lambda, i). \]

III. LORENTZ TRANSFORMATIONS

In this section we shall establish which are the transformation laws consistent with the Principle of Relativity and law (L.3). To do this, we derive a relation between the lengths observed by two different frames of reference. While the lengths orthogonal to the direction of relative motion are invariant, in III.A we show that the lengths parallel to the direction of relative motion are not invariant.

A. Lorentz contraction

We consider two wires with uniform electrical charge, at rest in $\Sigma$. Let us introduce another reference frame $\Sigma'$ that moves with respect $\Sigma$ with velocity $v$ in the same direction of the wires. For realizing the equilibrium we consider a device $\mathcal{I}$, like that designed in sect. II.C, manufactured in such a way that it is invariant. Since the equilibrium is realized in both $\Sigma$ and $\Sigma'$ by means the same device, by (L.3) we have to conclude that
\[ \phi(\lambda, i) = \phi(\lambda', i') \] (2)
where $\lambda$ (i) and $\lambda'$ (i') are the values of density charge (current) with respect $\Sigma$ and $\Sigma'$ respectively. Since the wires in $\Sigma$ are at rest, we have $i = 0$. In $\Sigma'$ the current is produced by the motion of the wires, therefore $i' = \lambda' v$. Then (2) becomes
\[ \frac{\lambda^2}{2\pi\epsilon_0 r} = \frac{\lambda'^2}{2\pi\epsilon_0 r} - \frac{\mu_0 \lambda'(\lambda' v)^2}{2\pi r} \]
which implies $\lambda^2 = \lambda'^2(1 - \epsilon_0 \mu_0 v^2)$; therefore, if we set $\epsilon_0 \mu_0 = \frac{1}{c^2}$, we have
\[ \lambda = \lambda' \sqrt{1 - \frac{v^2}{c^2}} \] (3)
This result says that the charge density is not invariant.

Now we consider a piece of wire of length $L$ carrying a charge $\delta Q$ respect to $\Sigma$. With respect to $\Sigma'$, this same piece of wire has a length $L'$ and carries a charge $\delta Q' = \delta Q$. Then
\[ \lambda = \frac{\delta Q}{L} \quad \text{and} \quad \lambda' = \frac{\delta Q'}{L'} = \frac{\delta Q}{L'}. \]

Therefore, (3) becomes
\[ \frac{\delta Q}{L} = \frac{\delta Q'}{L'} \sqrt{1 - \frac{v^2}{c^2}} \]
which leads to
\[ L' = L \sqrt{1 - \frac{v^2}{c^2}} \] (4)
This relation is known as Contraction of Lorentz.

B. From Lorentz contraction to Lorentz transformations

Now we derive Lorentz Transformations from Lorentz Contraction. Our aim is to obtain the law of motion of a particle $P$ with respect to $\Sigma'$ when this law is known in $\Sigma$. We proceed as follows:

(a) first, we consider the case in which particle $P$ is at rest in $\Sigma$ on the $x$ axis and we use Lorentz Contraction to derive its law of motion in $\Sigma'$;

(b) the result obtained in (a) is generalized to a particle at rest in any spatial point of $\Sigma$.

(c) by means of the results of (b), we derive the law of motion in $\Sigma'$ when the particle moves in $\Sigma$ according to any known law.

Step (a). If particle $P$ is at rest in the point of coordinate $\xi$ of the $x$ axis with respect to $\Sigma$, its motion with respect to $\Sigma'$ is described by the “world line” $(\tau', \xi'(\tau'))$, where $\xi'(\tau')$ is the $x$ coordinate of the particle at time $\tau'$ with respect to $\Sigma'$. The particle moves with respect to $\Sigma'$ with a velocity $-v$.

The value $\xi$ represents the length $l$ of the segment $[0, \xi]$ on the spatial $x$ axis of $\Sigma$. This length $l$ is related to the length $l'$ of this same segment with respect to $\Sigma'$ just by Lorentz Contraction
\[ l' = l \sqrt{1 - \frac{v^2}{c^2}}. \] (5)
But $l'$ is also the difference between the coordinates of $P$ and of the origin of $\Sigma$ with respect to $\Sigma'$, which are $\xi'(\tau')$ and $-v\tau'$:
\[ l' = \xi'(\tau') - (-v\tau') = \xi'(\tau') + v\tau'. \] (6)
Therefore, by equating (5) and (6) we get
\[ \xi'(\tau') = \xi \sqrt{1 - \frac{v^2}{c^2}} - v\tau'. \] (7)
This relation is the law of motion with respect to $\Sigma'$ in the case (a). The same argument can be used to show that if a particle is at rest in point $\xi'$ with respect to $\Sigma'$, then its world line $(\tau, \xi(\tau))$ with respect to $\Sigma$ is given by
\[ \xi(\tau) = \xi' \sqrt{1 - \frac{v^2}{c^2}} + v\tau. \] (8)
Step (b). Now we consider a particle $P$ at rest in the point of coordinates $(\xi, \eta, \zeta)$ with respect to $\Sigma$. Our aim is to find the world line $(\tau', \xi'(\tau'), \eta'(\tau'), \zeta'(\tau'))$ with respect to $\Sigma'$. Let us imagine a parallelepiped at rest in $\Sigma$ with a vertex in the origin of $\Sigma$ three edges lying along the axes $(x, y, z)$, and particle $P$ in the vertex with the greatest distance from the origin. With respect to $\Sigma'$ the coordinates $(\xi'(\tau'),\eta'(\tau'),\zeta'(\tau'))$ of $P$ at time $\tau'$ coincide with those of this last vertex. The coordinates $\eta'(\tau')$ and $\zeta'(\tau')$ are the lengths of the edges of the parallelepiped orthogonal to the relative motion, and therefore are invariant. For the $x$ coordinate, we can repeat the argument of step (a); thus, with respect to $\Sigma'$, particle $P$ moves according to

\[
\begin{align*}
\xi'(\tau') & = \xi \sqrt{1 - \frac{v^2}{c^2}} - vt', \\
\eta'(\tau') & = \eta, \\
\zeta'(\tau') & = \zeta.
\end{align*}
\]

(9)

Reciprocally if a particle is at rest in $\Sigma'$ its laws motion with respect to $\Sigma$ are

\[
\begin{align*}
\xi(\tau) & = \xi' \sqrt{1 - \frac{v^2}{c^2}} + vt, \\
\eta(\tau) & = \eta(\tau'), \\
\zeta(\tau) & = \zeta(\tau').
\end{align*}
\]

(10)

Step (c). Now we let particle $P$ move in an arbitrary way with respect to $\Sigma$; suppose that at time $t$ it is in the point $(x, y, z)$. Let us imagine a particles $A$ at rest with respect to $\Sigma$ and another particles $B$ at rest in $\Sigma'$, such that both $A$ and $B$ collide with our particle just in the space-time point $(t, x, y, z)$. Our assumption is simply that this threefold collision occurs also in $\Sigma'$ in a space-time point denoted by $(t', x', y', z')$. Since $B$ is at rest in $\Sigma'$ its position is just the location of the impact, i.e. $(x', y', z')$. The space-time point of the collision must belong to the world line of particle $A$ in $\Sigma'$. Therefore by (9) we have

\[
\begin{align*}
x' & = x \sqrt{1 - \frac{v^2}{c^2}} - vt', \\
y' & = y, \\
z' & = z.
\end{align*}
\]

(11)

and by (10)

\[
\begin{align*}
x & = x' \sqrt{1 - \frac{v^2}{c^2}} + vt, \\
y' & = y, \\
z' & = z.
\end{align*}
\]

(12)

By rewriting (11) and (12) in explicit form, we get the usual form of Lorentz Transformations:

\[
\begin{align*}
x' & = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \\
y' & = y, \\
z' & = z, \\
t' & = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.
\end{align*}
\]

(13)

IV. CONCLUSIONS

The heart of the present derivation is law \((L.3)\). Since this rule is obtained from \((L.2)\), which has not an interpretation independent of the concept of force, the legitimacy of its use could be questioned. However, these kinds of doubts are ruled out by the following arguments. Once function $\phi$ in \((L.3)\) is interpreted as empirical mean to establish conditions for the equilibrium, its empirical validity can be experimentally verified without making reference to any underlying theory. This is the method followed by Ampère in establishing the “Théorie des phénomènes électro-dynamiques, uniquement déduite de l’expérience” \([7]\).

Ampère him-self wrote: “The main advantage of the formulas so established [...] is that of remaining independent of the hypotheses, both of them used by their authors in the research of the formulas, and also of the hypotheses that in the future replace the former. [...] Whatever the physical cause one wants attribute to the phenomena made by such an [electro-dynamical] action, the formula obtained by my-self will be always the expression of real facts. [...] The adopted [method] which led me to the desired results [...] consists in verifying, by means of experience, that an electrical conductor rests in equilibrium under equal forces [...]”

Clariﬁed the legitimacy of the arguments presented in our derivation of Lorentz Transformations, we would now to emphasize the advantage that our approach carry by a didactics and pedagogical point of view. The derivations of Lorentz Transformations without light produce that the second postulate of Einstein is as a direct consequence of the theory and, for this reason, it can be easily accepted by the student that frequently reject concepts that cannot be observed and verified experimentally. In addition, the proposed approach allows to obtain the Lorentz Transformations as the only transformations consistent with the starting assumptions. Other derivations without light, on the contrary, exhibit that there are only two possible equations of transformations, one corresponding to the old Galilean-Newtonian transformation laws and the other corresponding to the standard Lorentz ones without specifying the value of the velocity of light. The starting assumptions are very few, do not concerning linearity and continuity of the functions of transformations; the linearity and continuity are properties that the student can directly verify. The use of physical concepts and laws already known by students enables teachers to use this approach in an introductory physics class as well, and furthermore in a Secondary School in which the Lorentz Transformation are presented, but not derived, as new laws of transformations to be substituted to the Galilean Transformation in order to take into account the time dilatation and length contractions and other effects which follow from the Relativity Theory.
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