This paper analyzes, for a multi-particle system of spin-1/2 particles, the consequences of replacing the Poincaré group as fundamental symmetry group by the de Sitter group SO(3,2). The flat-space approximation of the de Sitter group by the Poincaré group defines a superselection rule, which correlates spin and momentum of particles. This correlation can be formulated as an interaction between two particles, which exhibits properties of the electromagnetic interaction.

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I. INTRODUCTION

Assume that the basic symmetry group of particle physics is not the Poincaré group P(3,1) but the de Sitter group SO(3,2). Let \( l_{ab} \), \( a, b = 0, \ldots, 4 \), be the generators of SO(3,2) pseudo-rotations within a representation of SO(3,2). Let \( s^{ab} \) be 4x4-matrices, built from Dirac matrices,

\[
\begin{align*}
  s_{\mu\nu} &:= \frac{1}{2} \sigma_{\mu\nu}, \quad s_{4\mu} := \frac{1}{2} \gamma_{\mu}, \quad \mu, \nu = 0, \ldots, 3 \\
\end{align*}
\]

with the commutation and anticommutation relations

\[
\begin{align*}
  \sigma_{\mu\nu} &= \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] \quad \text{and} \quad \{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu} \\
\end{align*}
\]

\( s_{ab} \) and \( l_{ab} \) satisfy the commutation relations of SO(3,2)

\[
\begin{align*}
  [l_{\mu\nu}, l_{\rho\sigma}] &= -i[g_{\mu\rho} l_{\nu\sigma} - g_{\mu\sigma} l_{\nu\rho} + g_{\nu\sigma} l_{\mu\rho} - g_{\nu\rho} l_{\mu\sigma}] \\
  [l_{4\mu}, l_{4\nu}] &= -ig_{44} l_{\mu\nu} \\
  [l_{\mu\nu}, l_{4\rho}] &= ig_{\nu\rho} l_{4\mu} - g_{\mu\rho} l_{4\nu}
\end{align*}
\]

with \( g_{ab} = \text{diag} (+1, -1, -1, -1, +1) \). Then the equation

\[
(2s^{ab}l_{ab} - m) \psi = 0
\]

(factor 2 is added for convenience) is a SO(3,2) generalization of the Dirac equation. The operators

\[
j_{ab} = l_{ab} + s_{ab}
\]

correspond to the total pseudo-angular momentum of solutions of (6).

The contraction limit \[1\] approximates SO(3,2) by P(3,1) in the neighborhood of a given point in space-time. The operators \( l_{4\mu} \) are then approximated by momentum operators \( p_{\mu} \) of P(3,1). The operators \( l_{\mu\nu} \) generate transformations of the common Lorentz subgroup SO(3,1) of SO(3,2) and P(3,1).

Consider now a multi-particle state space, formed by direct products of solutions of (6). The total pseudo-angular momentum of the system is given by

\[
J_{ab} = \sum j_{ab}
\]
If the total state is a pure state, then the constant operator (Casimir operator)

\[ J^{ab}J_{ab} = c\text{-number} \]  

(9)
defines a superselection rule. In the contraction limit it is replaced by an analogous relation for the square of the total momentum \( P_\mu \)

\[ P^\mu P_\mu = M^2. \]  

(10)

It will be shown that the contraction limit delivers a second superselection rule that correlates spin and momentum. It has its origin in invariant terms with the structure of \( s^{ab}l_{ab} \), which is, not least, responsible for the existence of the generalized Dirac equation (6).

II. GROUP CONTRACTION

Group contraction has been defined by E. Inönü and E. P. Wigner [1], as a flat-space approximation for the neighborhood of a given point in space-time. Consider the representation of the generators \( l_{ab} \) by differential operators acting on wave functions on the pseudo-sphere \( g^{ab}x_a x_b = 1 \)

\[ l_{ab} = i (x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a}) \]  

(11)

If, in a neighborhood of the point \( P = (0,0,0,0,1) \), we rescale the coordinates \( x_a \) by replacing

\[ x_\mu = \frac{1}{R} \xi_\mu \text{ and } x_4 = \xi_4, \]  

(12)

(11) can be written in the form

\[ l_4_\mu = i (R \xi_4 \frac{\partial}{\partial \xi^\mu} - \frac{1}{R} \xi_\mu \frac{\partial}{\partial \xi^4}) \]  

(13)

and

\[ l_\mu_\nu = i (\xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}). \]  

(14)

For \( R \to \infty \), the first term in (13) is of order \( R^3 \), the second of order \( R^{-1} \) and \( l_{\mu\nu} \) of order \( R^0 \). When \( R \to \infty \), any neighborhood of \( P \), expressed in coordinates \( \xi_\mu \), is ‘contracted’ towards the point \( P \). At the same time the operator \( l_{4\mu} \) is approximated by commuting operators

\[ p_\mu = i R \frac{\partial}{\partial \xi^\mu}. \]  

(15)

To get rid of the factor \( R \), \( p_\mu \) is rescaled while performing the limit. This is done by replacing \( \xi_\mu \) by \( R \xi'_\mu \). According to (12) the new coordinates \( \xi'_\mu \) will be called \( x_\mu \) again. But now \( x_\mu \) are coordinates in tangential space-time. Then \( p_\mu \) adopt the structure of generators of translations in tangential space-time

\[ p_\mu = i \frac{\partial}{\partial x^\mu}. \]  

(16)

III. SUPERSELECTION RULES

With (15) as an approximation to \( l_{4\mu} \), we can arrange the terms of the Casimir operator (9) with respect to their order of \( R \). When we ignore terms that are of orders smaller than \( R^3 \), we obtain

\[ p^\mu p_\mu + 2p^\mu p'_\mu + p'^\mu p'^\mu + \cdots + \gamma^\mu p_\mu + \gamma^\nu p_\mu + \gamma'^\mu p'_\mu + \gamma'^\nu p'_\mu + \cdots = c\text{-number}. \]  

(17)
In the first line, we find the Casimir operator $P^\mu P_\mu$ of the Poincaré group. In the second line, we find the contraction of the invariant form $S^{ab} L_{ab}$, where $S^{ab} = \sum s^{ab}$ and $L_{ab} = \sum l_{ab}$. It is evident that the first and the second line are separately invariant with respect to transformations of the Poincaré group.

Now consider a multi-particle system of spin-1/2 particles, whose individual momenta add up to a total momentum $P$. For this system $P^\mu P_\mu$ is a c-number. We can combine this number with the c-number on the right-hand side of (17). Then we obtain another constant expression

$$\gamma^\mu (p_\mu + p'_\mu + \cdots) + \gamma^\mu (p'_\mu + p_\mu + \cdots) + \cdots = \text{c-number},$$

which is valid for all $R$, especially for $R \to \infty$, up to contributions of orders less than $R^3$.

In this way, the contraction limit produces two superselection rules. Rule I concerns, as expected, the constancy of the Casimir operator $P^\mu P_\mu$ of the Poincaré group. Rule II correlates spin and momentum within a multi-particle system.

For a single particle system, rule II reduces to the Dirac equation

$$(\gamma^\mu p_\mu - m) \psi = 0.$$  

Under the aspect of the basic SO(3,2) symmetry, this is the correct extension of the single-particle Dirac equation to a multi-particle system.

**IV. INTERACTION TERM IN FOCK SPACE**

To further analyze the restrictions imposed by the correlation terms in (18), we make use of standard Fock space methods, and treat the correlation terms as perturbation to the ‘free’ parts of (18). The ‘free’ parts are defined by the terms $\gamma^\mu p_\mu$. They are easily converted into a Fock space formulation, following the usual ‘second quantization’ of the Dirac field. We refer to standard textbooks (see e.g. [2]). The field operator of the Dirac field (taken from this reference) has the form

$$\psi(x) = (2\pi)^{-\frac{3}{2}} \int d^3p \left( b_s(p) u_s(p) e^{-ipx} + d^\dagger_s(p) v_s(p) e^{ipx} \right).$$

A similar expression defines the Dirac adjoint operator $\bar{\psi}(x)$. $b^\dagger_s(p), b_s(p)$ are electron emission and absorption operators, $d^\dagger_s(p), d_s(p)$ are the corresponding operators for positrons. They satisfy the anticommutation relations of the Dirac field.

Unlike the usual approach to a multi-particle system of Dirac particles, we are confronted with mixed terms that correlate the states of two particles at a time. These correlation terms $\gamma^\mu p_\mu$ of (18) can be written (on the time-cut $t = t'$) as a Fock space operator

$$\int d^3x d^3x' \bar{\psi}(x') \gamma^\mu \psi(x) \bar{\psi}(x') p_\mu' \psi(x').$$

Using the decomposition of the field operators (20) into emission and absorption operators, the contributions to (21) take on the form

$$\ldots b^\dagger (p + k) \gamma^\mu b(p) b^\dagger (p' - k') p'_\mu b(p') \ldots.$$  

(For simplicity we have omitted the factors $u_s, v_s$ and the time dependencies.) When we evaluate (22) for any two-particle state with a given total momentum $P$, then only terms with $k = k'$ will deliver a contribution.

If we use the correlation operator in a perturbation calculation, then the requirement of momentum conservation is satisfied by only using terms with $k = k'$. We can easily convince ourselves that any other term would violate momentum conservation. Hence, the condition $k = k'$ is required and sufficient to ensure momentum conservation in a perturbation calculation.

Therefore, we are not only allowed to, but, in fact, are forced to drop all other terms in the decomposition (22). By collecting all contributions that belong to the same $p$ and $k$, we obtain

$$\ldots b^\dagger (p + k) \gamma^\mu b(p) \kappa_\mu (k) \ldots$$

with

$$\kappa_\mu (k) := \int dV (p') b^\dagger (p' - k) p'_\mu b(p').$$
(Again time dependencies and spin functions are omitted.) $dV(p')$ indicates a summation over all terms that contribute to a given $k$. The same consideration is valid for the positron operators $d^j(p')$, $d(p)$ and mixed terms.

23 and 24 are not yet suited for a standard perturbation calculation. The reason is that this perturbation term is ‘nonlinear’, in the sense that it is defined as a product of two Fock space operators, built from the same field operators. In classical and quantum mechanics it is common practice to ‘linearize’ a two-body problem by the introduction of a potential, which stands for the ‘forces’ between the bodies. The two-body problem is then reduced to the generally simpler one of a single body moving within a potential.

23 and 24 already have a form that strongly suggests, how a similar linearization can be achieved within our two-particle problem. Consider the action of the operator $\kappa_\mu(k)$ in 24. It ‘absorbs’ a particle with momentum $p'$ and ‘recreates’ this particle with momentum $p' - k$. In this way a momentum $k$ is ‘emitted’. $\kappa_\mu(k)$ then acts as a placeholder for $k$ and ‘transfers’ this momentum to 23, where it is ‘absorbed’.

In the following we will reproduce these features by a quantum mechanical potential. The essential step will be, to split up the aforementioned emission/absorption process, and assign specific emission and absorption operators $a_\mu(k)$ and $a_\mu(k')$ to each part of the process. These operators, applied to the same vacuum state as $b(p)$, $b^\dagger(p)$, shall emit and absorb quanta of momentum $k$. As such they have to satisfy the commutation relations

$$[a_\mu(k), a_\mu(k')] = \delta_{\mu\nu} \delta(k - k') .$$

(25)

Just as $\kappa_\mu(k)$ they shall transform like a 4-momentum.

These operators will take over the placeholder role in 23, if we make sure that, together, they act in the same way as $\kappa_\mu(k)$. This is achieved by the requirement that, under inclusion of the placeholder momentum $k$, energy-momentum is conserved at each ‘vertex’ 23. The commutation relations 23 finally ensure that any combination of $a_\mu(k)$ and $a_\mu(k')$ that, on the time-cut, does not satisfy the requirement $k = k'$, is eliminated by multiplication to the vacuum state, either on the right or left side of a matrix element.

By this step we have linearized the perturbation term by adding a new ‘vector field’ generated by $a_\mu(k)$, $a_\mu(k)$.

Below we will also use the following operators, known from the conventional formulation of quantum electrodynamics (see e.g. 2),

$$A_j(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2k^0}} \left( a_j(k) e^{-ikx} + a_j^\dagger(k) e^{ikx} \right) ,$$

(26)

$$j = 1, 2, 3,$$ and

$$A_0(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2k^0}} i \left( a_0(k) e^{-ikx} + a_0^\dagger(k) e^{ikx} \right) .$$

(27)

$k^0$ shall be determined by the requirement of energy-momentum conservation, when these operators are evaluated within two-particle states. This leads to an ‘off-shell’ behavior, which is well-known from standard perturbation theory 6.

In 26 we now add the proper space-time dependencies to the emission and absorption operators. By replacing $\kappa_\mu$ in 23 by $e a_\mu$ we write

$$\ldots \ e b^\dagger(p + k) e^{-i(p + k)x} \gamma^\mu b(p) e^{-ipx} a_\mu(k) e^{-ikx} \ldots .$$

(28)

$e$ is a normalization factor that must be added, since 24 fixes a normalization of $a_\mu(k)$, $a_\mu(k)$, which cannot be expected to be the same as of $\kappa_\mu(k)$. Notice that the correct space-time dependency of $a_\mu(k)$ is determined by 24, when the proper time dependencies are added to $b(p')$ and $b(p')$.

After inserting the spin functions $u_s(p)$ and $v_s(p)$, these terms and the corresponding positron and mixed terms, in combination with $a_\mu(k)$ and $a_\mu(k)$, add up to a Fock space operator in the form

$$e \int d^3x : \bar{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x) .$$

(29)

Here :: stand for normal ordering of emission and absorption operators (all emission operators left of all absorption operators). This operator has the form of the interaction term of quantum electrodynamics (QED) with a coupling constant $e$. 
V. DISCUSSION

We have found that, within the framework of perturbation theory, the interaction term of QED is identical to a Fock space representation of the correlation terms of \([\text{15}]\). Therefore, quantum electrodynamics can be considered as a formulation in Fock space of superselection rule \(\Pi\). Since QED is central to particle physics, this strongly suggests that the Hilbert space of particle physics has a basic symmetry defined by the de Sitter group \(\text{SO}(3,2)\), rather than by the Poincaré group. This symmetry can be approximated with high precision, but not replaced, by a Poincaré symmetry. As a consequence of the \(\text{SO}(3,2)\) symmetry, spin-1/2 particles exhibit an inherent property of electromagnetic interaction.

In deriving the interaction term, we have linearized a two-body problem by introducing a quantum mechanical potential. The potential has not been obtained by a formal ‘quantization rule’, applied to a classical potential, but rather by explicit construction on the quantum mechanical level. This provides valuable insights into the mechanism of ‘interacting quantized fields’.

The perturbation algorithm of QED calculates transition amplitudes between ‘incoming’ states \((t \to -\infty)\) and ‘outgoing’ states \((t \to \infty)\). Our derivation of the interaction term leads to the following interpretation of its function within this algorithm: Applied to incoming states with momenta \(p\) and \(p'\), it develops an entangled two-particle state. This state comprises all combinations of individual momenta that add up to the same value of \(p + p'\). The entanglement is built up in such a way that correlations between spin and momentum of different particles are established, in compliance with superselection rule \(\Pi\). By a measurement, which means a projection onto ‘outgoing’ states, transition amplitudes are obtained. These connect the incoming states to other combinations within the entangled two-particle state.

Notice that entanglement basically is a non-local phenomenon. Nevertheless, after the introduction of the potential, the interaction term is strictly local. This is a direct consequence of energy-momentum conservation at the ‘vertex’.

In our approach ‘intermediate photons’ appear as ‘pseudo-particles’, manifesting correlations within a multi-particle system. They are auxiliary elements within an algorithm, introduced ad hoc to simplify the algorithm. These elements describe correlation, but definitely not the physical ‘creation’ and ‘annihilation’ of photons as independent physical entities. Consequently, there is also no ‘virtual pair production’ from ‘intermediate photons’. Actually, there are the same commutation functions as in the conventional formulation of QED, so-called ‘contractions’, which in Feynman graphs are symbolized by photon and fermion lines. They are responsible for the correct transfer of energy-momentum through the iterated interaction term, with the important consequence of relativistic causality in the space-time domain. Within our approach they are obtained in a transparent way as part of the perturbation algorithm, which does not leave room for any other interpretation. This means, there is no ‘metamorphosis’ of plain Fock space operators to ‘physical fields’, when the interaction term ‘is switched on’. But the most significant consequence is this: After the interaction ‘is switched on’, a two-fermion system is still a two-fermion system and can be completely described and understood within the framework of elementary Fock space methods.

The consequences for the physical interpretation of some typical Feynman diagrams were discussed in \([\text{15}]\). In the same reference, an estimate of the numerical value of the coupling constant \(\varepsilon\) was obtained by evaluating the volume element \(dV(p')\) of \([\text{24}]\). It reproduces Wyler’s formula \([\text{4}]\) of the fine-structure constant.

Ignoring contributions to \([\text{24}]\) of orders smaller than \(R^{1}\) means ignoring terms with the structure of \(\gamma^\mu \gamma^\nu\) and \(x^\mu \partial^\nu \partial^\mu\). An tentative interpretation of these highly interesting terms was given in \([\text{15}]\).

[1] E. Inönü and E. P. Wigner, Proc. Nat. Acad. Sci. USA 39, 510 (1953).
[2] G. Scharf, Finite Quantum Electrodynamics, (Springer, Berlin Heidelberg New York, 1989 and 1995).
[3] W. Smilga, “Spin foams, causal links and geometry-induced interaction”, available as \[\text{hep-th/0403137}\]
[4] A. Wyler, C. R. Acad. Sc. Paris 269, 743 (1969).
[5] Of course, no physical creation or annihilation of particles is associated with these processes.
[6] In the ‘free radiation field’ \(k_0\) is ‘on-shell’: \(k_0 = |k|\).
[7] The symmetry of space-time is a different story.
[8] This is not in contradiction to the experimental ‘observation’ of photons: we do observe the interaction of a photon with a detector, but never a photon ‘itself’.
[9] Many textbooks tell us the contrary, namely that interacting relativistic field theories always deal with an unlimited number of particles. The reason for this opinion is a misinterpretation of Fock space operators, taking the words ‘emission’ and ‘absorption’ too literal. In the past we have too readily adopted this interpretation, obviously because it mirrors what we believe to experience in ‘daily live’.