Phases of a matrix model with non-pairwise index contractions

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Recently a matrix model with non-pairwise index contractions has been studied in the context of the canonical tensor model, a tensor model for quantum gravity in the canonical formalism. This matrix model also appears in the same form with different ranges of parameters and variables, when the replica trick is applied to the spherical $p$-spin model ($p=3$) in spin glass theory. Previous studies of this matrix model suggested the presence of a continuous phase transition around $R \sim N^2/2$, where $N$ and $R$ designate its matrix size $N \times R$. This relation between $N$ and $R$ intriguingly agrees with a consistency condition of the tensor model in the leading order of $N$, suggesting that the tensor model is located near or on the continuous phase transition point and therefore its continuum limit is automatically taken in the $N \to \infty$ limit. In the previous work, however, the evidence for the phase transition was not satisfactory due to the slowdown of the Monte Carlo simulations. In this work, we provide a new setup for Monte Carlo simulations by integrating out the radial direction of the matrix. This new strategy considerably improves the efficiency, and allows us to clearly show the existence of the phase transition. We also present various characteristics of the phases, such as dynamically generated dimensions of configurations, cascade symmetry breaking and a parameter zero limit, and discuss their implications for the canonical tensor model.

1. Introduction

Quantization of gravity is one of the most challenging fundamental problems in physics, and various approaches to this problem have been proposed so far. These include sophisticated applications of the renormalization group procedure to general relativity [1] as well as approaches that use discretization of spacetime in the definition of theory, for instance the approaches in Refs. [2–5], matrix models [6–10], and tensor models [11–14]. One of the goals of these discretized approaches is to show the emergence of a macroscopic spacetime as a continuous manifold, with general relativity emerging as the effective description of dynamics. This is still a challenging goal for any of these approaches.

In this paper, we study the dynamics of a matrix model which contains non-pairwise index contractions [15,16]. This matrix model has only the $O(N) \times S_R$ symmetry for the index spaces of the matrix variable $\phi^i_a$ ($a = 1, 2, \ldots, N, \ i = 1, 2, \ldots, R$), where $O(N)$ denotes the orthogonal group and $S_R$ the symmetric group. The symmetry is not enough to diagonalize an arbitrary matrix, and therefore this matrix model is not solvable by the methods usually employed to solve matrix models [6–10] or rectangular matrix models [17–19]. Our matrix model can also be regarded as a vector model of $R$ vector variables, but our setup is different from the exactly solved ones in Refs. [20,21].
The background motivation for our matrix model comes from the fact that this model has an intimate connection [15,16] to an exact wave function [22] of a tensor model in the Hamilton formalism, which we call the canonical tensor model [23,24]. It has previously been found that this wave function peaks around Lie-group symmetric configurations of the tensor-variable of the model [25,26]. This is encouraging towards potential emergence of a spacetime as mentioned above, because Lie-group symmetries, such as Lorentz, deSitter, gauge and so on, are ubiquitous in the universe. However, it is still difficult to show whether the peaks contain configurations which can be interpreted as some sort of spacetimes, for instance, in the manner described for a classical treatment in Ref. [27]. Understanding the properties of the dynamics of our matrix model will potentially provide useful insights about the relation between the wave function of the tensor model and spacetime emergence.

It is an intriguing coincidence that a matrix model with the same form has previously appeared in the context of spin glasses. It is obtained when the replica trick is applied to the spherical $p$-spin model ($p = 3$) [28,29] for spin glasses, where $R$ designates the replica number. However, the physics of the spin glass and that of our model will be largely different, because the parameter and variable regions of interests are different from each other. In the spin glass case, the replica number $R$ is taken to the limit $R \to 0$ as part of the replica trick, while our interest is rather in the limit $R \sim N^2/2 \to \infty$, the reason for which comes from the consistency of the tensor model, as explained in more depth in Sect. 7. In addition, the coupling parameters (called $\lambda$ in later sections) of the models have opposite signs, and the spin glass case has a spherical constraint, $\phi^i_a \phi^i_a = 1$, for each $i$. Considering these differences, it seems necessary to analyze our model independently from the spin glass case.

In the previous paper [16], Monte Carlo simulations of the model were performed with the usual Metropolis update method. This has revealed various interesting characteristics of the model. However, there was an issue which affects the reliability of the Monte Carlo simulations: For some values of the parameters important to the study of the properties of the model, the iterative updates in the radial direction of the matrix variable were too slow to reach thermodynamic equilibrium in a reasonable amount of time. For instance, it could not be determined with confidence whether the transition is a phase transition or just a crossover, since the parameters could not be tuned to make the transition more evident. The major improvement of the present paper is that we integrate out the troublesome radial direction before performing the numerical simulations, obtaining a model essentially defined on a compact manifold (the hypersphere). In addition, the more efficient Hamiltonian Monte Carlo method is employed instead of the more straightforward Metropolis algorithm. This replacement of the model drastically improves the efficiency of the simulations, and we have successfully obtained much more evident results than previously.

We summarize below the properties of the transition and the phases derived from the numerical results:

- The transition becomes sharper as $N$ is taken larger. This implies the transition is a phase transition in the thermodynamic limit. We have not observed any discrete behavior of observables around the transition point, implying that the transition is continuous.
- The value of $R$ at the transition point, which we call the critical value $R_c$, is a little smaller than $(N + 1)(N + 2)/2$, which was previously obtained by the perturbative analytic computations in Refs. [15,16]. The critical value $R_c$ is better approximated by $R_c \sim (N + 1)(N + 2)/2 - N + 2$ in our numerical results, where the parameters of the model are taken in the range $k/\lambda \gtrsim O(10^{-10})$ and $N \lesssim 12$. 
The data were inconclusive with respect to the decision of which of the two options is correct, as will be discussed later on.
to some \( k/\lambda \)-dependent values as \( N \) increases. In Sect. 6.3, the \( k/\lambda \to +0 \) limit is discussed. Its behaviour severely differs in the two phases. In Sect. 6.4, the geometry of dominant configurations is discussed. In particular, the dimensions take minimum values at the transition point. In Sect. 6.5, symmetry breaking in a cascade manner for \( R \geq R_c \) is shown. In Sect. 6.6, the slowdown of iterative updates in our simulations is discussed. This appears to occur quickly as \( k/\lambda \) becomes smaller at \( k/\lambda \lesssim O(10^{-8}) \) in our simulations, but a quantitative investigation shows that this is a smooth change, implying that there is no phase transition to another phase with characteristic of slow dynamics. In Sect. 7, the implications of the numerical results to the tensor model are discussed. In Sect. 7.1, the coincidence of the transition point with a consistency condition of the tensor model is discussed. In Sect. 7.2, the behavior of the dimensions is explained from the symmetry–peak relation argued in Refs. [25,26]. In Sect. 7.3, the normalizability of the wave function of the tensor model is discussed.

The last section is devoted to a summary and future prospects.

2. The matrix model and the setup for simulations

The matrix model we consider in this paper is defined by the partition function,

\[
Z_{N,R}(\lambda, k) := \int_{\mathbb{R}^{NR}} \prod_{a=1}^{N} \prod_{i=1}^{R} d\phi_a^i \exp \left[ -\lambda \sum_{i,j=1}^{R} U_{ij}(\phi) - k \sum_{i=1}^{R} U_{ii}(\phi) \right],
\]

where \( \phi_a^i (a = 1, 2, \ldots, N, i = 1, 2, \ldots, R) \) denotes the matrix variable, the integration is over the whole \( NR \)-dimensional real space, and the coupling parameters, \( k \) and \( \lambda \), are assumed to be positive real for the convergence of the integral as will be explained in more detail below. Here

\[
U_{ij}(\phi) := (\phi_a^i \phi_a^j)^3,
\]

where the repeated lower indices are assumed to be summed over. Throughout this paper, repeated lower indices always appear pairwise, and we assume the common convention that they are summed over, unless otherwise stated. On the other hand, the upper indices are triply or sixfold contracted in Eq. (1), and summation over them will always be written explicitly.

The matrix model (1) has the \( O(N) \times S_R \) symmetry, where \( O(N) \) denotes the orthogonal group transformation in the \( N \)-dimensional vector space of the lower index and \( S_R \) denotes the permutation symmetry for the upper index values \( \{1, 2, \ldots, R\} \). The \( O(N) \times S_R \) symmetry is generally not enough to diagonalize the matrix \( \phi_a^i \), and therefore the model cannot exactly be solved by the well-known methods often applied to the usual matrix models [6–10] or rectangular matrix models [17–19]. Because of the \( O(N) \) symmetry, the model can also be regarded as a vector model [20,21] with the multiplicity of vectors labeled by the upper index. In fact, our model can be solved in the \( N \to \infty \) limit with finite \( R \) [15], as in the vector models [20,21] and in the spherical \( p \)-spin model [28,29]. However, this solution is not so useful, because our major interest is the vicinity of the phase transition point with \( R \sim N^2/2 \), as will be explained later.

The first term of the exponent of the matrix model (1) is positive semi-definite, since

\[
\sum_{i,j=1}^{R} U_{ij}(\phi) = \sum_{i,j=1}^{R} (\phi_a^i \phi_a^j)^3 = \left( \sum_{i=1}^{R} \phi_b^i \phi_c^i \right) \left( \sum_{j=1}^{R} \phi_b^j \phi_c^j \right) \geq 0.
\]

In fact, the equality on the furthest right is satisfied by various configurations, including straightforward ones like \( \phi_a^1 = -\phi_a^2, \ldots \). Moreover, when \( R \) is larger than a certain value, there will be a
Thus $k > 0$ assures the convergence of the integral (1) for general cases, while it will be shown later that the $k/\lambda \to +0$ limit can be taken if $R < R_c$.

As explained above, the second term in the exponent of Eq. (1) acts as a regularization of the integral. A term with the same role existed in the previous studies of the model [15, 16], but had a different, namely quadratic, form, $\sum_{i=1}^{R} \phi_i^1 \phi_i^1$. The main reason for the choice of this quadratic form was that in this case the action (the exponent) has the standard form used in perturbative computations. It is however not necessary to take a quadratic term as a regularization term for the perturbative computations, as we will review in Sect. 4.3. The present choice of \( \phi_i^1 \) can then be integrated out in a straightforward way, as we will perform below. Since the radial direction was the main source of the difficulties in the previous simulations [16], the present choice will ease the deadlock of the simulations.

Now let us divide \( \phi_i^1 \) into the radial and the angular coordinates, \( \phi_i^1 = r \tilde{\phi}_i^1 \), where \( r \) denotes the radial coordinate and \( \tilde{\phi}_i^1 \) denotes the angular coordinates. Putting this reparametrization into Eq. (1) and integrating over \( r \), one obtains

$$Z_{N,R}(\lambda, k) = \int_{S^{NR-1}} d\tilde{\phi} \int_0^\infty dr r^{NR-1} \exp \left\{ -\left[ \lambda \sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k \sum_{i=1}^{R} U_{ii}(\tilde{\phi}) \right] r^6 \right\}$$

$$= \frac{1}{6} \Gamma \left( \frac{NR}{6} \right) \int_{S^{NR-1}} d\tilde{\phi} \left[ \lambda \sum_{i,j=1}^{R} U_{ij}(\tilde{\phi}) + k \sum_{i=1}^{R} U_{ii}(\tilde{\phi}) \right]^{-(NR)/6},$$

(4)

where \( \Gamma(\cdot) \) is the gamma function, \( S^{NR-1} \) denotes the unit \( NR - 1 \) dimensional sphere and \( d\tilde{\phi} \) denotes the volume element on the \( S^{NR-1} \). We will use (4) as the weight of our simulations, where the variables are only the angular ones. Our implementation of the Hamiltonian Monte Carlo method for this system will briefly be explained in Sect. 5.

Finally, let us comment about the relation between the matrix model (1) and the spherical \( p \)-spin model for spin glasses, while we leave the relation to the canonical tensor model for Sect. 7. The partition function of the spherical \( p \)-spin model for \( p = 3 \) is given by

$$Z_{p=3}(P) := \int_{\phi_1,\phi_2} d\phi \exp (-P_{abc} \phi_1 \phi_2 \phi_1),$$

(5)

with a random real coupling \( P_{abc} \) to simulate a spin glass system. Considering \( R \) replicas of the same system in the replica trick and simulating the random coupling by a Gaussian distribution \( e^{-\alpha P_{abc} P_{abc}} \)

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2 A simple counting of degrees of freedom implies that the dimension of the solution space of \( \sum_{i=1}^{R} \phi_i^1 \phi_i^1 \phi_i^1 = 0 \) will be given by \( NR - N(N + 1)(N + 2)/6 \), where the former counts the degrees of freedom of \( \phi_i^1 \) and the latter the number of independent conditions. Therefore, in general for \( R > (N + 1)(N + 2)/6 \), the solutions to the equality will exist continuously.

3 This was implicitly carried out in Refs. [15, 16] as well.
with positive $\alpha$, one obtains

$$
\int_{\mathbb{R}^p} \prod_{a,b,c=1}^N dP_{abc} e^{-\alpha P_{abc}} P_{abc} [Z_{p\text{-spin}}(P)]^R = \mathcal{N} \int_{\phi_a^i \phi_a^i = 1} \prod_{i=1}^R d\phi_i^j \exp \left[ \frac{1}{4\alpha} \sum_{i,j=1}^R U_{ij}(\phi) \right], \quad (6)
$$

where $\mathcal{N}$ is an overall coefficient. The right-hand side has a similar form as (1), but there are two major differences. There are the restrictions, $\phi_i^i \phi_i^i = 1$ for each $i$, which assure the finiteness of the integration, taking the role of the second term in the exponent of Eq. (1). The other difference is that the coefficient of the exponent has the inverse sign compared to (1). This physically means that the dominant configurations will be largely different between the matrix model (1) and that of the spherical $p$-spin model. In addition, the $R \to 0$ limit is finally taken as part of the replica trick, while this is not necessary in the matrix model (1) itself. We are rather interested in the dependence on $R$ of the system, especially in the regime $R \sim N^2/2$, as we will see later. In particular, the last relation requires $R \to \infty$ in the thermodynamic limit $N \to \infty$, which is opposite to the spin glass case.

### 3. Expectation values of observables

For convenience, let us first slightly generalize the definition of the partition function (1) of the matrix model to

$$
Z_{N,R}(\Lambda) := \int_{\mathbb{R}^{NR}} \prod_{a=1}^N \prod_{i=1}^R d\phi_i^a \exp \left[ - \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\phi) \right]. \quad (7)
$$

We assume the symmetric matrix coupling $\Lambda$ is taken such that the integral is convergent. This includes the original case (1) with $\Lambda = \Lambda^{\lambda,k}$, where

$$
\Lambda_{ij} := \lambda + k \delta_{ij} \quad \text{(8)}
$$

with positive $\lambda, k$.

Let us introduce

$$
z_{N,R}(\Lambda, \beta) := \int_{S^{NR-1}} d\tilde{\phi} \left[ \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]^{-\beta},
$$

$$
z_{N,R}(\Lambda, \beta, \mathcal{O}) := \int_{S^{NR-1}} d\tilde{\phi} \mathcal{O}(\tilde{\phi}) \left[ \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]^{-\beta}, \quad (9)
$$

where $U_{ij}(\tilde{\phi}) := (\tilde{\phi}_i^i \tilde{\phi}_j^j)^3$ and $\mathcal{O}(\tilde{\phi})$ is an arbitrary observable expressed as a function of $\tilde{\phi}_a^i$. As derived in Sect. 3, by integrating over $r$, the partition function (7) can be expressed by

$$
Z_{N,R}(\Lambda) = \frac{1}{6} \Gamma(NR) z_{N,R}(\Lambda, \Delta_{NR}), \quad (10)
$$

where $\Delta_{NR} := NR/6$. From Eq. (10), the expectation value of an observable $\mathcal{O}(\tilde{\phi})$ is given by

$$
\langle \mathcal{O}(\tilde{\phi}) \rangle = \frac{z_{N,R}(\Lambda, \Delta_{NR}, \mathcal{O})}{z_{N,R}(\Lambda, \Delta_{NR})}. \quad (11)
$$

These are the observables which are directly obtained in our Monte Carlo simulations for the angular variables.
There are other kinds of observables that are expressed as functions of $\phi_a^i$. The difference is just the normalization of $\phi_a^i$. Let us introduce a weight $[\cdot]$ which counts the multiplicity of $\phi_a^i$ contained in an observable that is assumed to be a homogeneous function of $\phi_a^i$. For example, the weight of $\phi_a^i\phi_a^j$ is given by $[\phi_a^i\phi_a^j] = 2$.

Let us consider an observable $\mathcal{O}(\phi)$ with weight $w$. This can be rewritten as $\mathcal{O}(\phi) = \mathcal{O}(\tilde{\phi}) r^w$ by the reparametrization $\phi_a^i = r \tilde{\phi}_a^i$ with the radial and angular variables. Then the expectation value is given by

$$
\langle \mathcal{O}(\phi) \rangle = \frac{1}{Z_{N,R}(\Lambda)} \int_{\mathbb{R}^{NR}} \prod_{a=1}^N \prod_{i=1}^R d\phi_a^i \mathcal{O}(\phi) \exp \left[ - \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\phi) \right] = \frac{1}{Z_{N,R}(\Lambda)} \int_{S^{NR-1}} d\tilde{\phi} \int_0^\infty dr \ r^{NR-1+w} \mathcal{O}(\tilde{\phi}) \exp \left[ -r^6 \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]
$$

$$
= \frac{\Gamma(\Delta_{NR} + w/6) z_{N,R}(\Lambda, \Delta_{NR} + w/6, \mathcal{O})}{\Gamma(\Delta_{NR}) z_{N,R}(\Lambda, \Delta_{NR})} \left\langle \mathcal{O}(\tilde{\phi}) \left[ \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]^{-w/6} \right\rangle,
$$

where we have used $z_{N,R}(\Lambda, \Delta + w/6, \mathcal{O}) = z_{N,R}(\Lambda, \Delta, \mathcal{O} \left[ \sum_{ij} \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]^{-w/6})$, which follows from the definition (9). This formula relates the expectation values of the observables of the matrix model (1) expressed by $\phi_a^i$ with those in our Monte Carlo simulations for the angular variables $\tilde{\phi}_a^i$. This formula is used for deriving some results in Sect. 6.

### 4. Analytic computations by a perturbative method

In the previous papers Refs. [15,16], the authors introduced a function defined by

$$
f_{N,R}(\Lambda) := \frac{1}{V_{S^{NR-1}}} \int_{S^{NR-1}} d\tilde{\phi} \ \exp \left[ - \sum_{i,j=1}^R \Lambda_{ij} (\tilde{\phi}_a^i \tilde{\phi}_a^j)^3 \right],
$$

where $V_{S^{NR-1}}$ designates the volume of the unit sphere, $\int_{S^{NR-1}} d\tilde{\phi}$, for the normalization $f_{N,R}(\Lambda = 0) = 1$. This function is related to the partition function (7) of the matrix model by

$$
Z_{N,R}(\Lambda) = V_{S^{NR-1}} \int_0^\infty dr \ r^{NR-1} f_{N,R}(\Lambda, r^6).
$$

A merit of introducing the function (13) is that it is well-defined for arbitrary complex values of $\Lambda_{ij}$ since the integral region in Eq. (13) is compact, meaning that it is an entire function of $\Lambda_{ij}$ [15]. Therefore, if the whole perturbative series expansion in $\Lambda_{ij}$ of this function is obtained, this is convergent for any $\Lambda_{ij} \neq \infty$, and hence determines the function completely in the whole complex region of $\Lambda_{ij}$. This means that, in principle, the dynamics of the matrix model (1) can be determined as precisely as one desires by taking into account more terms of the perturbative series expansion of $f_{N,R}(\Lambda)$. This is in contrast with the partition function $Z_{N,R}(\lambda, k)$, which is singular at $\lambda = 0$ or $k = 0$ and merely an asymptotic perturbative series expansion of it in $\lambda$ or $k$ can be obtained.

The perturbative computations of $f_{N,R}(\Lambda)$ using Feynman diagrams have been performed in Refs. [15,16]. This can be done by rewriting the integrals $\int_{S^{NR-1}} d\tilde{\phi} \tilde{\phi}_a^i \tilde{\phi}_a^j \tilde{\phi}_a^k$, which appear when
the integrand in Eq. (13) is expanded in $\tilde{\phi}$, with familiar computations using Wick contractions. The final result in the leading order is given by

$$f_{N,R}^{\text{leading}}(\Lambda) = \prod_{e_{\Lambda}} h_{N,R}(e_{\Lambda}),$$  
(15)

where the product is over the eigenvalues of the matrix $\Lambda_{ij}$ with degeneracies taken into account, and

$$h_{N,R}(t) := (1 + 12 \gamma^3 t)^{-\frac{1}{2} (N+4)(N-1)} \left[ 1 + 6 (N+4) \gamma^3 t \right]^{-N/2}$$  
(16)

with

$$\gamma^3 := \frac{\Gamma(NR/2)}{8 \Gamma[(NR/2)+3]}.$$  
(17)

In the case of $\Lambda^{\lambda,k}_{ij} = \lambda + k \delta_{ij}$, the eigenvalues are $k + \lambda R$ for the eigenvector $(1,1,\ldots,1)$ and $k$ for all the other vectors transverse to that. Therefore

$$f_{N,R}^{\text{leading}}(\Lambda^{\lambda,k}t) = h_{N,R}[(k + \lambda R) t] \left[ h_{N,R}(k t) \right]^{R-1}.$$  
(18)

To obtain formulas for expectation values of observables, let us introduce

$$g_{N,R}(\Lambda, \beta) := \int_0^{\infty} dt \ t^{\beta-1} f_{N,R}(\Lambda t).$$  
(19)

From Eq. (13), one finds

$$g_{N,R}(\Lambda, \beta) = \frac{\Gamma(\beta)}{V_{SNR-1}} \int_{SNR-1} d\tilde{\phi} \left[ \sum_{i,j=1}^R \Lambda_{ij} (\tilde{\phi}_i \tilde{\phi}_j)^3 \right]^{-\beta}.$$  
(20)

Therefore, it has a relation with (9) as

$$g_{N,R}(\Lambda, \beta) = \frac{\Gamma(\beta)}{V_{SNR-1}} z_{N,R}(\Lambda, \beta).$$  
(21)

Then, by comparing with the results in Sect. 3, the correlation functions of $U_{ij}(\tilde{\phi}) := (\tilde{\phi}_i \tilde{\phi}_j)^3$ for the angular variables can be expressed as

$$\langle U_{i_1j_1}(\tilde{\phi}) \cdots U_{i_Mj_M}(\tilde{\phi}) \rangle := \frac{\int_{SNR-1} d\tilde{\phi} \ U_{i_1j_1}(\tilde{\phi}) \cdots U_{i_Mj_M}(\tilde{\phi}) \left[ \sum_{i,j=1}^R \Lambda_{ij} (\tilde{\phi}_i \tilde{\phi}_j)^3 \right]^{-\Delta_{NR}}}{\int_{SNR-1} d\tilde{\phi} \left[ \sum_{i,j=1}^R \Lambda_{ij} (\tilde{\phi}_i \tilde{\phi}_j)^3 \right]^{-\Delta_{NR}}} = (-1)^M \frac{\partial}{\partial \Lambda_{i_1j_1}} \cdots \frac{\partial}{\partial \Lambda_{i_Mj_M}} g_{N,R}(\Lambda, \Delta_{NR} - M).$$  
(22)

Therefore, by combining with Eqs. (15) [or (18)] and (19) and numerically integrating over $t$, one can compute the correlation functions of $U_{ij}(\tilde{\phi})$ in the leading order of the analytic perturbative computation.
Let us next consider the correlation functions of $U_{ij}(\phi)$ for the variable $\phi^i_a$. We can use the formula (12), where each $U_{ij}(\phi)$ has weight $w = 6$. We obtain

$$
\langle U_{ij}(\phi) \cdots U_{imjm}(\phi) \rangle = \frac{\Gamma(\Delta_{NR} + M)}{\Gamma(\Delta_{NR})} \left[ \sum_{i,j=1}^{R} \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]^{-M}
$$

$$
= \frac{(-1)^M}{g_{N,R}(\Lambda, \Delta_{NR})} \frac{\partial}{\partial \Lambda_{ij}} \cdots \frac{\partial}{\partial \Lambda_{imjm}} g_{N,R}(\Lambda, \Delta_{NR}).
$$

This gives the correlation functions of $U_{ij}(\phi)$ in the leading order from the analytic perturbative method.

Later we consider an observable given by $U_d(\phi) = \sum_{i=1}^{R} U_{ii}(\phi) = \sum_{i=1}^{R} (\phi_i^a \phi^a_i)^3$. In our actual case with $\Lambda = \Lambda^{a,k}$ given in Eq. (8), the derivatives in Eqs. (22) and (23) for the observable can be performed by $\partial/\partial k$. Therefore the correlation functions are given by

$$
\langle (U_d(\phi))^M \rangle = \frac{(-1)^M}{g_{N,R}(\Lambda^{a,k}, \Delta_{NR})} \frac{\partial^M}{\partial k^M} g_{N,R}(\Lambda^{a,k}, \Delta_{NR} - M),
$$

$$
\langle (U_d(\phi))^M \rangle = \frac{(-1)^M}{g_{N,R}(\Lambda^{a,k}, \Delta_{NR})} \frac{\partial^M}{\partial k^M} g_{N,R}(\Lambda^{a,k}, \Delta_{NR}).
$$

These formulas are used in Sect. 6.2, where the numerical results are compared with the analytical ones.

5. Hamiltonian Monte Carlo method for angular variables

In this paper, we use the Hamiltonian Monte Carlo method [30] for the numerical simulations. This method upgrades the configuration space of some integral to a phase space by introducing conjugate variables, creates a Hamilton system and (locally) solves the equations of motion in order to find new, more remote, candidates for the Metropolis update. The last process is called leapfrog, which consists of a sequence of discrete jumps from one phase space location to another. While it is enough to approximately solve the classical equation of motion for presenting update candidates, the time reversal symmetry and the conservation of phase space volume must be exactly satisfied under the discrete jumps for a correct sampling of configurations. For a flat configuration space, these conditions are easily satisfied by alternately sequencing the following two processes:

(i) $\delta q_i = \epsilon p_i$, $\delta p_i = 0$,

(ii) $\delta q_i = 0$, $\delta p_i = -\epsilon \frac{\partial V(q)}{\partial q_i}$,

where $(q_i, p_i)$ designates the phase space variables indexed by $i$, $\epsilon$ is the size of one jump and $V(q)$ is the Gibbs potential for a weight $\exp[-V(q)]$. Observe that (i) is a free motion in a flat space, and only (ii) takes effects from $V(q)$. One can easily check that each of the two jumps satisfies the conservation of the phase space volume, $\det [\partial (q_i + \delta q_i), \partial (p_i + \delta p_i) / \partial q_j, \partial p_j] = 1$, thanks to the fact that $q_i$ and $p_i$ do not jump simultaneously. The time reversal symmetry is also satisfied, since $(q_i + \delta q_i, p_i + \delta p_i) \rightarrow (q_i, p_i)$ when $\epsilon$ is replaced with $-\epsilon$.

When the configuration space $q_i$ is constrained to a non-flat sub-manifold embedded in a flat space, a free motion corresponding to (i) is generally a simultaneous jump of $p_i$ and $q_i$, since the tangent space of the sub-manifold containing $p_i$ changes along $q_i$. In such a case, finding an appropriate jump
corresponding to (i) satisfying the two necessary conditions above is generally a difficult problem. An obvious solution to an appropriate jump is to exactly solve the classical equation of the free (geodesic) motion on the sub-manifold \[31\]. This is possible when a sub-manifold is simple enough to allow us to obtain such exact solutions. In our case, the embedded manifold is a unit hypersphere, which gives the constraints \(\sum_i q_i^2 = 1\) and \(\sum_i q_i p_i = 0\), and the jump describing the exact free (geodesic) motion on the sphere is given by

\[
(i') \begin{pmatrix} q_i' \\ p_i' \end{pmatrix} = \begin{pmatrix} \cos \theta \\ -|p| \sin \theta \end{pmatrix} \begin{pmatrix} q_i \\ p_i \end{pmatrix},
\]

where \(|p| = \sqrt{\sum_i p_i^2}\) and \(\theta = \epsilon |p|\). The second jump (ii) does not contain a jump in \(q_i\), therefore there are no difficult issues, and it can just be replaced by

\[
(ii') \delta q_i = 0, \quad \delta p_i = -\epsilon \frac{\partial V(q)}{\partial q_i} + \epsilon q_i \sum_j q_j \frac{\partial V(q)}{\partial q_j},
\]

where the additional term takes into account the constraint \(\sum_i q_i p_i = 0\).

In our present case (4), the coordinates \(\tilde{\phi}_i \) are constrained on a unit sphere \(\sum R_i = 1\), \(\tilde{\phi}_i \tilde{\phi}_i = 1\), and we employ these jumps (i’) and (ii’). The potential energy can be read from Eq. (4) as

\[
V(\tilde{\phi}_i) = \Delta_{NR} \log \left[ \sum_{i,j=1}^R \Lambda_{ij} U_{ij}(\tilde{\phi}) \right]
\]

with \(\Lambda = \Lambda^{\lambda,k}\).

6. Results of Monte Carlo simulations

In this section, we summarize the results of our Hamiltonian Monte Carlo simulations from several viewpoints. Since the overall factor of the exponent of Eq. (1) can be absorbed in the rescaling of \(\phi_i\), we set \(\lambda = 1\) in all the simulations, leaving \(N, R,\) and \(k\) as variable parameters. Errors were estimated by the jackknife method, described for example in Ref. [32]. We took the leapfrog numbers to be about 1000–10000, depending on the amount necessary to reach sufficient accuracy explained in Sect. 6.6, and the step sizes were tuned so that the acceptance rates were about 80–99 percent, which were a little higher than the commonly taken ones because of the reason explained in Sect. 6.6. Parallel tempering [33] was also used in some of the computations for some data which systematically study \(k\)-dependencies. However, as will be explained further in Sect. 6.6, parallel tempering did not seem to essentially affect the expectation values computed.

In the following subsections, we show the results of the simulations of the expectation values of various observables depending on the purposes. The observables are taken to be invariant under the \(O(N) \times S_R\) symmetry.

6.1. Phase transition point

There are various observables which can be used to study the location of the phase transition. We will present one example for \(\tilde{\phi}_i\) and another for \(\phi_i\).

The observable we first consider is

\[
\mathcal{O}_1 := N \sum_{i,j=1}^R (\tilde{\phi}_i \tilde{\phi}_j)^2.
\]
An important reason for considering this observable is that this has the natural normalization factor \(N\). This factor is determined by the uncorrelated case, in which each \(\tilde{\phi}_i^a\) is regarded as an equally independent variable. More precisely, the uncorrelated case corresponds to
\[
\langle \tilde{\phi}_i^a \tilde{\phi}_j^b \rangle_{\text{uncorrelated}} \sim \delta_{ab} \delta_{ij} / (RN)
\]
up to sub-leading corrections in \(N\) and \(R\) by taking into account the constraint \(\sum_i R \tilde{\phi}_i^a \tilde{\phi}_i^a = 1\).
Under this assumption,
\[
\langle O_1 \rangle_{\text{uncorrelated}} = N \sum_{i,j=1 \atop i \neq j}^R \langle \tilde{\phi}_i^a \tilde{\phi}_j^b \rangle_{\text{uncorrelated}} \sim N \sum_{i,j=1 \atop i \neq j}^R \langle \tilde{\phi}_j^b \rangle_{\text{uncorrelated}} \langle \tilde{\phi}_i^b \rangle_{\text{uncorrelated}} \sim 1,
\]
where we have ignored sub-leading corrections in \(N\) and \(R\).

Figure 1 shows the results of the Monte Carlo simulations for \(\langle O_1 \rangle\). The normalization factor \(R_c\) for the horizontal axes is chosen as \(R_c = (N + 1)(N + 2)/2 - N + 2\). The perturbative computations in the leading order predict the transition point to be at \(R_c = (N + 1)(N + 2)/2\) [15,16]. However, for the data shown in the left-hand panel for \(k = 10^{-8}\), it is better to take \(R_c = (N + 1)(N + 2)/2 - N + 2\) to locate all the peaks near \(R/R_c = 1\). The values of \(\langle O_1 \rangle\) approach 1 as \(R\) takes more distant values from \(R_c\), implying that the correlations become more independent there. On the other hand, the values of \(\langle O_1 \rangle\) at the peaks become larger for larger \(N\). This can be checked more clearly in Fig. 2. This means that the correlation becomes larger at the transition point for larger \(N\), which is a typical signature of a continuous phase transition.

The right-hand panel of Fig. 1 shows the dependence of \(\langle O_1 \rangle\) on \(k\) for \(N = 10\). The dependence on \(k\) seems little for \(R \lesssim R_c\), as we will discuss more of this aspect in Sect. 6.3. On the other hand, at \(R \gtrsim R_c\), \(\langle O_1 \rangle\) seems to become larger as \(k\) becomes smaller, slightly shifting the locations of the peaks to the right. This implies that the correlations become larger for smaller \(k\) and the critical value \(R_c\) depends not only on \(N\) but also on \(k\) as well. The last statement implies that what we have taken as \(R_c\) above cannot be considered to be a correct expression that is valid for the general values of the parameters, but can at most be considered to be an approximate expression valid for our parameter range \(N \lesssim 12\) and \(10^{-10} \lesssim k \lesssim 10^{-8}\).
Fig. 2. $\langle O_1 \rangle$ from the simulations are plotted against $N$ with $R = (N+1)(N+2)/2 - N + 2$.

Fig. 3. The results of the Monte Carlo simulations for $\langle O_2 \rangle$. The horizontal axes are $R/R_c$, where $R_c = (N+1)(N+2)/2 - N + 2$.

Let us next turn to the observable

$$O_2 := \sum_{i=1}^{R} \phi_i^a \phi_i^a = r^2. \quad (31)$$

From the formula (12), by setting the weight $w = 2$ and noting $O_2(\tilde{\phi}) = 1$ identically, we obtain

$$\langle r^2 \rangle = \frac{\Gamma (\Delta_{NR} + 1/3)}{\Gamma (\Delta_{NR})} \left( \sum_{i,j=1}^{R} \Lambda_{ij} U_{ij}(\tilde{\phi}) \right)^{-1/3} \quad (32)$$

with $\Lambda = \Lambda^{\lambda,k}$. The results of the simulations for $\langle r^2 \rangle$ are plotted in Fig. 3. The figures clearly show that the two phases are characterized by $\langle r^2 \rangle \sim 0$ for $R < R_c$ and $\langle r^2 \rangle > 0$ for $R > R_c$, respectively. The transition becomes sharper as $N$ becomes larger or $k$ becomes smaller. $\langle r^2 \rangle$ changes continuously at $R \sim R_c$, supporting the claim that the transition is continuous.

6.2. Comparison with the perturbative computation

In this section, we compare the results of the simulations with the analytic perturbative computation in the leading order, which was reviewed in Sect. 4. In particular, we see that the analytic computation does not explain the peaks of the correlations of $\tilde{\phi}_i^a$, which was shown in Sect. 6.1. We find clear
deviations between them around the phase transition point $R \sim R_c$, while they converge as $R$ takes distant values from $R_c$.

To see this we consider the observables $U_d(\tilde{\phi}) := \sum_{i=1}^{R} U_{ii}(\tilde{\phi})$ and $U_d(\phi) := \sum_{i=1}^{R} U_{ii}(\phi)$, whose formulas of analytic computation are given in Eq. (24). The explicit values are obtained by performing the numerical integration of (19) with (18) contained in Eq. (24) for $M = 1$. On the other hand, we compare these with $\langle U_d(\tilde{\phi}) \rangle$ and

$$
\langle U_d(\phi) \rangle = \Delta_{NR} \left( \frac{U_d(\tilde{\phi})}{\sum_{i,j=1}^{R} \Lambda_{ij}^{\lambda,\kappa} U_{ij}(\tilde{\phi})} \right),
$$

(33)

from the simulations, where we have used (12) for $w = 6$.

Figure 4 shows the comparison between the Monte Carlo results and the analytic computations for $\langle U_d(\phi) \rangle$. There exist systematic deviations in the vicinity of $R = R_c$, as was previously reported in Ref. [16]. For $R > R_c$, the deviations quickly converge as $R$ leaves $R_c$. For $R < R_c$, they slowly converge as $R$ becomes smaller.

One can see similar deviations for the $\langle U_d(\tilde{\phi}) \rangle$. Figure 5 plots the ratio $\langle U_d(\tilde{\phi}) \rangle / \langle U_d(\tilde{\phi}) \rangle_{\text{pert}}$ between the Monte Carlo results and the perturbative analytic computations. Indeed the ratio deviates from 1 in the vicinity of the transition point. The deviations at the peaks become larger as $k$ becomes smaller. On the other hand, as shown in Fig. 6, it seems that the deviations increase with $N$ for $k < 10^{-8}$, but this is not clear for $k \geq 10^{-8}$. We cannot rule out the possibility that they actually converge in the large $N$ limit to some values which increase with the decrease of $k$.  

![Figure 4](https://example.com/fig4.png)

**Fig. 4.** The comparison between the results of the Monte Carlo simulation and the analytic perturbative computation of $\langle U_d(\phi) \rangle$ for $N = 10$ and $k = 10^{-8}$. The blue dots with error bars are the Monte Carlo results, and the red lines are the analytic results. The horizontal axes are $R/R_c$, where $R_c = (N + 1)(N + 2)/2 - N + 2$. The right-hand panel magnifies the region $R/R_c < 0.5$ in the left-hand panel.

![Figure 5](https://example.com/fig5.png)

**Fig. 5.** The ratio $\langle U_d(\tilde{\phi}) \rangle / \langle U_d(\tilde{\phi}) \rangle_{\text{pert}}$ between the Monte Carlo and the perturbative analytic results. The horizontal axes are $R/R_c$, where $R_c = (N + 1)(N + 2)/2 - N + 2$. 

Fig. 6. The ratio $\langle U_d(\tilde{\phi}) \rangle / \langle U_d(\tilde{\phi}) \rangle_{\text{pert}}$ from the simulations are plotted against $N$ for $R = (N + 1)(N + 2)/2 - N + 2$.

Fig. 7. The values of $\langle U_d(\phi) \rangle$ from the simulations are plotted against $-\log_{10} k$ for $N = 10$, $R = 45$, which belongs to the region $R < R_c$. The reason for a slightly larger error for $k = 10^{-8}$ data point may come from the trapping in the narrow region explained in Sect. 6.6.

From the comparisons above, we conclude that the perturbative analytic computation in the leading order does not correctly reproduce the behavior of the matrix model in the vicinity of the transition point. As was previously performed in Ref. [16], the situation does not essentially improve, even if we take into account the next leading order corrections to the analytic computation.

6.3. $k/\lambda \to +0$ limit

In this subsection, we focus on the $k/\lambda \to +0$ limit of the matrix model (1). There are a few reasons to study this. One is the characterization of the phases separated at $R = R_c$; we find different limits for each phase at $R > R_c$ and $R < R_c$. Another is its relevance to the tensor model. The behavior determines whether the wave function is normalizable or not. This will be discussed in Sect. 7.

First, let us show that the limit $k/\lambda \to +0$ converges in the phase $R < R_c$. This can be seen by looking at the behavior of the expectation values of observables. Figure 7 shows the result of the simulation about the behavior of $\langle U_d(\phi) \rangle$ in Eq. (33) against $k$ for a case with $R < R_c$. As can be seen in the figure, the expectation value approaches a constant value in the $k \to +0$ limit. In fact, similar convergence can be observed also for other observables in other cases with $R < R_c$. 

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Let us discuss the consequence of this behavior on the free energy defined by $F_{N,R}(\lambda, k) := -\log Z_{N,R}(\lambda, k)$. By taking the derivative of Eq. (1) with respect to $k$, we obtain

$$\frac{\partial}{\partial k} F_{N,R}(\lambda, k) = \langle U_d(\phi) \rangle. \quad (34)$$

Therefore, as a function of $k$, $F_{N,R}(\lambda, k)$ can be determined by studying the $k$-dependence of $\langle U_d(\phi) \rangle$ and performing integration:

$$F_{N,R}(\lambda, k) = \int_{k_0}^{k_1} dk \langle U_d(\phi) \rangle + F_{N,R}(\lambda, k_0). \quad (35)$$

In particular, $\lim_{k \to +0} \langle U_d(\phi) \rangle$ will determine $\lim_{k \to +0} F_{N,R}(\lambda, k)$.

Let us discuss the behavior of the free energy in $k/\lambda \to +0$. By performing the rescaling of the variable $\phi_i \to \lambda^{-1/6} \phi_i$ in Eq. (1), one obtains

$$F_{N,R}(\lambda, k) = U_0^d k + p_{N,R}(k/\lambda) + \frac{N R}{6} \log \lambda. \quad (36)$$

In the region $R < R_c$, there is a finite limit of $\lim_{k \to +0} \langle U_d(\phi) \rangle$ as shown above. Considering (35) and (36), the behavior of the free energy is obtained as

$$F_{N,R}(\lambda, k) = U_0^d k + p_{N,R}(k/\lambda) + \frac{N R}{6} \log \lambda \quad \text{for } k/\lambda \sim +0 \text{ and } R < R_c, \quad (37)$$

where $U_0^d := \lim_{k \to +0} \langle U_d(\phi) \rangle|_{\lambda = 1}$, and $p_{N,R}(k/\lambda)$ is smaller than $k/\lambda$ in order and has a finite limit $p_{N,R}(+0)$. We comment that this finiteness was proven analytically for $R = 2$ and any $N$ previously in Ref. [16]. This finiteness of $F_{N,R}(\lambda, k)$ in the $k \to +0$ limit is non-trivial, as discussed in Sect. 2.

On the other hand, for $R > R_c$, the simulations show that $\langle U_d(\phi) \rangle$ diverges in the $k \to +0$ limit. An interesting matter is that, instead, $k \langle U_d(\phi) \rangle$ converges in the $k \to +0$ limit, as can be seen from Fig. 8. This implies that, from Eq. (35), $F_{N,R}(\lambda, k)$ logarithmically diverges in the limit $k/\lambda \to +0$.

Let us discuss this divergence of the free energy in more detail. As we have seen in Fig. 8, if we take $k$ small enough, $k \langle U_d(\phi) \rangle$ can be regarded as its limiting value, $\lim_{k \to +0} k \langle U_d(\phi) \rangle$. By assuming this for the $N = 10$ data in the large $R$ region in the left-hand panel of Fig. 4, and fitting a linear function of $R$ for the data in the region $R > 1.4 \cdot R_c$, one obtains

$$k \langle U_d(\phi) \rangle|_{k=10^{-8}} \simeq 1.66 \cdot (R - 65.9). \quad (38)$$

---

In an appendix therein.
This curiously agrees very well with what can be obtained by putting \( N = 10 \) to a hypothetical expression for the right-hand side,

\[
\frac{N}{6} \left[ R - \frac{(N + 1)(N + 2)}{2} \right] = \frac{NR}{6} - \frac{#P}{2} \tag{39}
\]

where \( #P := N(N + 1)(N + 2)/6 \) is the number of independent components of a symmetric three-index tensor \( P_{abc} \). We have performed similar analyses for \( N = 5, 7 \) cases and have found good matches with the hypothesis (39). Assuming the hypothesis and recalling the form (36), we obtain

\[
F_{N,R}(\lambda, k) = \tilde{U}_d^0 \log(k/\lambda) + \frac{NR}{6} \log \lambda + \tilde{p}_{N,R}(k/\lambda) \quad \text{for } k/\lambda \sim +0 \text{ and } R > R_c, \tag{40}
\]

where \( \tilde{U}_d^0 := \lim_{k \to +0} k \langle U_d(\phi) \rangle, \tilde{p}_{N,R}(k/\lambda) \) is smaller than \( \log(k/\lambda) \) in order and

\[
\tilde{U}_d^0 = \frac{NR}{6} - \frac{#P}{2} + \delta \tilde{U}_d^0 \tag{41}
\]

with \( \delta \tilde{U}_d^0 \) being sub-leading in large \( R \). Note that \( \tilde{U}_d^0 \geq 0 \) due to \( \langle U_d(\phi) \rangle > 0 \), and therefore \( \delta \tilde{U}_d^0 \) must take positive values in the range \( R_c < R < (N + 1)(N + 2)/2 \).

Here it is a non-trivial question whether \( \tilde{p}_{N,R}(k/\lambda) \) has a finite limit \( \tilde{p}_{N,R}(+0) \). For example, a slow correction of the order of \( \sim 1/\log(k) \) to \( k \langle U_d(\phi) \rangle \) for \( k \sim +0 \) leads to a double logarithmic divergence of \( \tilde{p}_{N,R}(+0) \). However, as shown in Fig. 8, the data points of \( k \langle U_d(\phi) \rangle \) can be fitted very well with a correction of order \( \sqrt{k} \), and there is no good motivation for introducing such slow corrections. Therefore it would be reasonable to assume \( \tilde{p}_{N,R}(+0) \) exists as a finite value. We also comment that the hypothesis (39) is nothing but what can be obtained from the perturbative computation in the leading order [15], as the coincidence in the left-hand panel of Fig. 4 shows. Therefore \( \delta \tilde{U}_d^0 \) is a correction beyond the leading order perturbative computation.

The difference between the behavior of the free energy (37) and (40) characterizes the two phases separated by \( R = R_c \). These formulas will be used in Sect. 7.3, where we will discuss the normalizability of the wave function of the tensor model.

6.4. Geometric properties

In Sect. 6.2, we have found the deviation between the results of the simulations and the perturbative analytic results in the vicinity of the phase transition point. This suggests that some non-perturbative configurations are important in the vicinity of the phase transition point. In this subsection, to discuss the characteristics of the configurations around the phase transition point, we study the distributions of the vectors \( \phi_d^i \) \( (i = 1, 2, \ldots, R) \) in the \( N \)-dimensional vector space associated to the lower index. These vectors define a point cloud with \( R \) points, where \( \phi_d^i \) for each \( i \) determines the location of each point in the \( N \)-dimensional vector space. We study the dimensions of such point clouds. It turns out that the dimensions depend on the parameters of the matrix model. In particular, the dimensions take the smallest values at the transition point as functions of \( R \).

To study the dimension of a point cloud we use angle distributions among the vectors. The angle between two vectors, say \( \phi_a^i \) and \( \phi_b^j \) \( (i \neq j) \), in the \( N \)-dimensional vector space is given by

\[
\text{ang}(\phi^i, \phi^j) := \arccos \left( \frac{\phi_a^i \phi_a^j}{\sqrt{\phi_a^i \phi_a^j \phi_b^i \phi_b^j}} \right). \tag{42}
\]
Fig. 9. Examples of fitting (43) to the histograms of the mutual angles among $\phi^i$ values from the actual data. The horizontal axes represent the angle $\theta$. The parameters are $N = 10$, $k = 10^{-8}$ with $R = 60$ (left) and $R = 70$ (right), respectively. Fitting is performed only for the 3/5 portion around the center ($\theta \sim \pi/2$), ignoring 1/5 portions on each side. The fitted values of dimensions are $d = 2.4$ and $d = 4.7$, respectively, in these cases.

Fig. 10. The $R$-dependence of the dimension for $N = 10$, $k = 10^{-8}$. It takes the lowest value at the transition point. $R_c = (N + 1)(N + 2)/2 - N + 2$ with $N = 10$. Errors are not estimated in this plot.

When the vectors approximately form a rotationally symmetric $d$-dimensional point cloud, the distribution of the angles will approximately be given by

$$\rho(\theta) \, d\theta = N \sin^{d-2}(\theta) \, d\theta,$$

(43)

where $\theta$ designates the angle, and $N$ is a normalization factor. This formula can easily be obtained by radially projecting points to the unit sphere $S^{d-1}$, and computing infinitesimal volumes associated with given mutual angles. The dimensions of point clouds can be computed by fitting the formula (43) to the angle distributions obtained from the data.

In Fig. 9, we show two examples of the fitting. As shown in the figure, the fitting is generally quite good for high dimensions but not so much for low dimensions. The reason behind this is that the point clouds cannot be characterized as a single dimensional object but are a mixture of objects with different dimensions, as we discuss in Sect. 7.2. Yet, to characterize the configurations in terms of dimensions, we perform the fitting restricted to a portion around the center, namely $\theta \sim \pi/2$, because there exist dominant numbers of data in this region. In this sense, the dimension is merely a qualitative characterization, but it still gives a fairly interesting observable: The dimension takes the lowest value at the phase transition point as a function of $R$. For instance, this can be observed for $N = 10$, $k = 10^{-8}$ in Fig. 10.
Fig. 11. Left: The collection of the point clouds obtained from the simulation with $N = 10$, $R = 57$, and $k = 10^{-8}$. The point cloud from each datum of $\phi_i$ is projected into the three-dimensional space and the collection over all the data are plotted. For the projection, PCA is used to take three major directions out of $N$ dimensions. Right: The corresponding density plot. The shape is like a squashed rugby ball, which may be regarded as an object with a dimension between 2 and 3.

Fig. 12. The $k$-dependence of the dimensions of the configurations from the data of $N = 10$ and $R$ shown in the figure. The data for $R = 80$ do not converge well for small $k$ due to the difficulty of the simulations explained in Sect. 6.6. Errors are not estimated in this plot.

It is instructive to directly see a point cloud itself. A point cloud exists in an $N$-dimensional space, but if its dimension is lower than three, one can project it into a three-dimensional space by extracting the main three extending directions through principal component analysis (PCA). Figure 11 shows a collection of a number of such projected point clouds, which have been sampled from the simulation with $N = 10$, $R = 57$, $k = 10^{-8}$. According to Fig. 10, the point cloud has a dimension of nearly two in this case, and we indeed find an approximately two-dimensional object which has the shape of a squashed rugby ball, as shown in the right-hand panel of Fig. 11.

Finally, let us discuss the $k$-dependence of the dimension. The general behavior is that the dimensions decrease with the decrease of $k$ and converge to limiting values, as is shown in Fig. 12.
As shown in Sect. 6.1, the phase at \( R > R_c \) is characterized by large values of \( \langle r^2 \rangle \). Since a non-vanishing value of \( \phi_\alpha^i \) breaks the \( O(N) \) symmetry associated to the lower index vector space, the phase at \( R > R_c \) will be characterized by symmetry breaking. In this subsection, we will study this aspect.

Let us consider one of the Lie algebra generators \( T_{ab} \) of \( SO(N) \). The size of the breaking of \( T_{ab} \) by a vector \( \phi_\alpha^i \) will be characterized by the size of the vector \( T_{ab} \phi_\alpha^i \). By considering its square and summing over all the vectors, the breaking by a configuration can be characterized by \( \sum_{i=1}^{R} T_{ab} T_{ab} \phi_\alpha^i \phi_\alpha^i \). Thus the natural quantity to study is

\[
M_{mn'} := T_{ab}^{(m)} T_{ab}^{(m')} \sum_{i=1}^{R} \phi_\alpha^i \phi_\alpha^i,
\]

where \( T_{ab}^{(m)} \) is a basis of the Lie algebra generators of \( SO(N) \) with the normalization \( T_{ab}^{(m)} T_{ab}^{(m')} = 2 \delta_{mm'} \) for later convenience. Note that the definition of \( M_{mn'} \) conserves the \( S_R \) symmetry for the upper index.

An \( O(N) \)-invariant observable which can be obtained from \( M_{mn'} \) is the set of the eigenvalues of the matrix \( M \). For an arbitrary \( \phi_\alpha^i \), one can diagonalize \( L_{ab} := \sum_i T_{ab} \phi_\alpha^i \phi_\alpha^i \) by an \( SO(N) \) transformation. Then it is straightforward to prove that the eigenvalues of \( M \) are given by

\[
eg(M) = \{ e_a^b + e_b^a | a, b = 1, 2, \ldots, N, \ a < b \},
\]

where \( e_a^b \) are the eigenvalues of the matrix \( L \).

Figure 13 gives two examples of the eigenvalues \( \neg(M) \). \(^6\) In the panels, the eigenvalues are plotted in ascending order along the horizontal direction. In the case of the left-hand panel, one can find an interesting stair-like pattern of the eigenvalues. This pattern means that the original \( SO(N) \) symmetry

\(^5\) This can be proven by explicitly taking the basis, \( T^{(w)}_{ab} := \delta_{ab} \delta_{\mu \nu} - \delta_{ab} \delta_{\mu \nu} \).

\(^6\) We computed the matrix \( L \) from the actual Monte Carlo datas of \( \tilde{\phi} \) by regarding \( \phi_\alpha^i \phi_\alpha^i = [\Gamma(\Delta_{SR} + 1/3) / \Gamma(\Delta_{SR})] \tilde{\phi}^a \tilde{\phi}^a [\sum_{ij=1}^{R} \Lambda_{ij} \tilde{U}_i(\tilde{\phi})]^{-1/3} \), following the relation (12) with \( w = 2 \).
Fig. 14. The mean eigenvalues $\langle e_g(M) \rangle$ are plotted for the data of $R = 60$ (left) and of $R = 80$ (right), respectively, with $N = 10, k = 10^{-8}$.

Fig. 15. The mean eigenvalues $\langle e_g(M) \rangle$ against $R/R_c$ for $N = 10, k = 10^{-8}$, where $R_c = (N+1)(N+2)/2 - N + 2$. The eigenvalues are plotted vertically at each $R$. For clear distinction, the points are colored according to the numbers of the generators of the symmetries. Each symmetry is broken when the corresponding eigenvalues leave the horizontal axis.

is hierarchically broken to $SO(N - 1), SO(N - 2), \ldots$. In fact, the horizontal locations of the steps agree with the numbers of the generators of these symmetries. On the other hand, in the case of the right-hand panel, all the symmetries are broken with no obvious hierarchical structure.

Since the pattern above generally fluctuates over the samples of $\phi_i^a$, in a simulation, we consider an average, $\langle e_g(M) \rangle$. The precise definition of this quantity is as follows: We run a simulation with a certain choice of parameters; for each sample of $\phi_i^a$ in a simulation, we compute eigenvalues $e_g(M)$ and order them in ascending order; then we take mean values of each entry over all the data of the simulation. Figure 14 shows $\langle e_g(M) \rangle$ computed from the simulations respectively for $R = 60$ (left) and for $R = 80$ (right) with $N = 10, k = 10^{-8}$.

Figure 15 shows the dependence of $\langle e_g(M) \rangle$ over the change of $R$ for $N = 10, k = 10^{-8}$. The eigenvalues start to increase from $R \sim 0.9R_c$ with the increase of $R$. The symmetry breaking occurs one by one: first $SO(10)$, then $SO(9)$, and so on, until finally all the symmetries are broken at $R \sim 1.3R_c$. In the figure, one can find some gaps between the eigenvalues in the vicinity of $R \sim R_c$. They correspond to the differences of the step heights, which, for example, exist in left-hand panel of Fig. 14. As $R$ becomes larger, the gaps gradually disappear, approaching the situation in the right-hand panel of Fig. 14.

The above symmetry breaking in a cascade manner is consistent with the results in the previous subsections. When $R < R_c$, since $\langle \phi^2 \rangle$ is small, there is no symmetry breaking. As $R$ increases from
$R \sim R_c$, the vectors $\phi^i_\alpha$ ($i = 1, 2, \ldots, R$) start to take larger values and fill a subspace, the dimension of which increases with the increase of $R$. Since the subspace breaks part of the $SO(N)$ symmetry, depending on its dimensions, more symmetries are broken with the increase of $R$.

Let us comment about the fate of the discrete symmetry $\phi^i_\alpha \to -\phi^i_\alpha \forall i$. For $N = \text{odd}$, this corresponds to the $Z_2$ subgroup of the $O(N)$ symmetry. The quantity $\sum_{i=1}^{R} \phi^i_\alpha / R$ is not invariant under the discrete symmetry, and therefore the expectation value of its square, $\langle \sum_{i,j=1}^{R} \phi^i_\alpha \phi^j_\alpha / R^2 \rangle$, will provide a good quantity to measure its breaking.\footnote{To balance the normalization with that of $T^{(m)\alpha\beta}$, we put a factor of $1/R$.} It has turned out that the expectation values computed from the simulation data stay small in the order $\lesssim O(1)$ over the range, and we have not observed any signatures of its breaking.

### 6.6. Slowdown of Monte Carlo updates

In the previous work [16], we encountered a rather serious difficulty of the Monte Carlo simulation: For $R \gtrsim R_c$ and small $k$, the step sizes of the simulations had to be tuned very small for reasonable acceptance rates of Metropolis updates, but then the updates of configurations were too slow for the system to reach thermodynamic equilibria within our runtimes. Therefore, in this paper, we have improved the strategy: We integrate out the radial direction of the model and use the so-called Hamiltonian Monte Carlo method for simulations. Indeed, the new strategy drastically improves the efficiency of the simulations, but we still encounter the slowdown for smaller $k$, which is however several orders of magnitude smaller than that in the previous work. This implies that this slowdown is an intrinsic property of the model, which is independent from methods of simulations, and would even suggest a possibility of the presence of a transition to a new phase characterized by slow dynamics. However, in this subsection we will show that the latter possibility is unlikely, and the system in the phase at $R > R_c$ is rather like a fluid with a viscosity which continuously grows for smaller $k$.

The speed of updates can be quantified by the mean value of distances between neighboring configurations in a sequence of updates, $\tilde{\phi}^i_\alpha (1)$, $\tilde{\phi}^i_\alpha (2)$, $\ldots$, $\tilde{\phi}^i_\alpha (M + 1)$:

$$
\langle (\delta \tilde{\phi})^2 \rangle := \frac{1}{M} \sum_{m=1}^{M} \left| \tilde{\phi}(m + 1) - \tilde{\phi}(m) \right|^2,
$$

where $|X|^2 := \sum_{i=1}^{R} X_{i\alpha} X_{i\alpha}$. In the ideal maximum situation that each entry of the sequence is independent from the others, $\langle (\delta \tilde{\phi})^2 \rangle = 2$ because of the normalization $|\tilde{\phi}(m)|^2 = 1$.

Figure 16 shows the dependence of $\langle (\delta \tilde{\phi})^2 \rangle$ against the value of $k$ for $R = 45$ (left) and $R = 80$ (right), respectively, with $N = 10$. In the simulations, the step sizes, namely the value $\epsilon$ in Sect. 5, are properly chosen for reasonable acceptance rates for each $k$, while the other parameters of simulations, such as leapfrog numbers, are fixed.\footnote{The acceptance rates are typically around from 80 to 99 percent in our simulations.} The $R = 45$ case keeps the ideal values around $\log_{10} 2 \sim 0.3$ throughout the shown range of $k$. On the other hand, the $R = 80$ case has a rapid decrease of the speed with the decrease of $k$ at $k \lesssim 10^{-6}$.

The speed of updates defined above is dependent on the parameters of the simulation such as step size, leapfrog number, and even the frequency at which the data is saved, and is therefore not a quantity intrinsic to the model. For instance, the starting point of decreasing, $k \sim 10^{-6}$, has no physical meaning, since this can easily be changed by taking different simulation parameters.
Fig. 16. The average speed of updates, \( \langle (\delta \phi)^2 \rangle \), is plotted for \( R = 45 \) (left) and \( R = 80 \) (right), respectively, with \( N = 10 \). In the simulations, the ideal maximum situation (\( \log_{10} 2 \sim 0.3 \)) is realized for \( R = 45 \), while there is a rapid decrease for \( R = 80 \) with the decrease of \( k \) for \( k \lesssim 10^{-6} \).

Fig. 17. Left: The values of \( \epsilon \) for the simulation of \( N = 10, R = 80 \). Right: The rescaled speed of updates.

However, in the data above, the only parameter which is varied is the step size \( \epsilon \) among different values of \( k \), and it is therefore meaningful to compare the data for different values of \( k \) by rescaling \( \langle (\delta \phi)^2 \rangle / \epsilon^2 \) to cancel the obvious dependence on \( \epsilon \). The left-hand panel of Fig. 17 plots the values of \( \epsilon \) taken for the simulation of \( N = 10, R = 80 \), and the right-hand panel is for the corrected values, \( \langle (\delta \phi)^2 \rangle / \epsilon^2 \). In the right-hand panel, leaving aside the irrelevant ideal region \( k \gtrsim 10^{-6} \), one can see that the values are almost flat in the region \( k \lesssim 10^{-6} \) with no essential change. This implies that the system is similar up to the rescaling among different values of \( k \).

We also used parallel tempering [33] in addition to the Hamiltonian Monte Carlo method for taking some data which systematically study \( k \)-dependencies. The exchanges of configurations were performed among neighboring values of \( k \), typically taking \( k = 10^{-n} \) \((n = 2, 3, \ldots, 11)\), with common values of the other parameters. In the region \( R > R_c \), as \( R \) increases from \( R_c \), the exchange rate quickly reduces for the above choices of \( k \). Therefore, parallel tempering does not seem effective in solving the slow update problem, which exists at \( R \gtrsim R_c \) for small \( k \). On the other hand, the exchange rate is high for small \( k \) at \( R \lesssim R_c \), which can easily be understood by the presence of the well-defined \( k/\lambda \to +0 \) limit at \( R < R_c \), as discussed in Sect. 6.3; the sets of configurations are similar among different values of \( k \), when \( k \) is small enough. However, we did not observe any major differences between the data with or without parallel tempering. This would imply that there are no major isolated dominant configurations which can only be reached by employing parallel tempering. All in all, we have not observed any essential improvement by employing parallel tempering in addition to the Hamiltonian Monte Carlo method.

Another interesting aspect of our actual Hamiltonian Monte Carlo simulation is that a relatively smaller choice of the step size \( \epsilon \) seems to give better sampling, and we even took such small values
that acceptance rates were nearly 1. This seems to be in contradiction with the more common situation that larger $\epsilon$ with a reasonable acceptance rate, such as several 10 percent, would give better sampling. However, this apparent contradictory aspect could be explained in the following manner in our case. The positive semi-definite (2) of the first term in the exponent of the matrix model (1) implies that the dominant configurations for small $k$ are around $\sum_{i=1}^{R} \phi_a^i \phi_b^i \phi_c^i \sim 0$, and this condition becomes tighter as $\phi_a^i$ can take larger values when $k$ is smaller. Therefore, the space of dominant configurations can be illustrated as in Fig. 18; the dominant configuration space is broad in the small $\phi_a^i$ region, but it becomes narrower as $\phi_a^i$ becomes larger. Here, we also assume that dominant configurations are connected, as suggested in the previous paragraph. Assuming the dominant configuration space as shown in the figure, the updates with relatively smaller $\epsilon$ will smoothly visit the narrow region as well as the broad region. On the other hand, sampling with relatively larger $\epsilon$ mainly stays within the broad region, occasionally jumps to the narrow region, and is trapped for a while to compensate the low possibility of visiting the narrow region. We have actually observed such trapping to occur more frequently for relatively larger values of $\epsilon$. This occasional trapping damages the quality of sampling and it generally takes longer to obtain a dataset with smaller errors.

Let us summarize this subsection. We encountered the slowdown of the Monte Carlo updates in the region $R \gtrsim R_c$ with small $k$. The speed of updates becomes slower as $k$ becomes smaller, but the dependence is continuous and is subject to the explanation with obvious rescaling. Therefore we have not observed any qualitative changes of the system under the change of the value of $k$, and it is unlikely that there is a phase transition to a new phase with characteristics of slow dynamics. Rather it seems that the system continues to behave like a fluid with a viscosity which continuously grows for smaller $k$ in the region $R \gtrsim R_c$.

7. Implications for the tensor model

In this section, we discuss the implications of the results of the simulations for the aforementioned tensor model in the canonical formalism, the canonical tensor model [23,24].

7.1. Phase transition point and the consistency of the tensor model

The wave function of the canonical tensor model, which is obtained by solving a number of first-class constraints which the wave function must satisfy, has the following form [22],

$$\Psi(P) = \left[ \int_{\mathbb{R}^N} \prod_{a=1}^{N} d\phi_a \exp \left( I^P_{abc} \phi_a^i \phi_b^i \phi_c^i \right) \Ai \left( \kappa \phi_a^i \phi_a^i \right) \right]^{\frac{\lambda_H}{2}},$$

(47)

where $P_{abc}$, a real symmetric tensor, is the configuration variable of the tensor model, $\lambda_H = (N + 2)(N + 3)/2$, $I$ is the imaginary unit (so $I^2 = -1$), $\Ai(\cdot)$ designates the Airy Ai function and $\kappa$ is a
real constant in the tensor model.\textsuperscript{9} It is particularly important that \( \lambda_H \) is determined by the hermiticity condition for the Hamiltonian constraint of the tensor model, and therefore must have this particular dependence on \( N \). Physically, the sign of the parameter \( \kappa \) is supposed to be opposite to that of the cosmological constant, based on the argument relating a mini-superspace approximation of general relativity (GR) to the tensor model with \( N = 1 \) \cite{[34]} (see an appendix of Ref. [16] for more details).

The simplest observable for the physical state represented by the wave function (47) would be given by

\[
\langle \Psi | e^{-\alpha \hat{P}_{abc} \hat{P}_{abc}} | \Psi \rangle = \int_{\mathbb{R}^{#P}} \prod_{a,b,c=1 \atop a \leq b \leq c}^{N} dP_{abc} e^{-\alpha P_{abc} P_{abc}} | \Psi(P) |^2
\]

\[
= N \alpha^{-#P/2} \int_{\mathbb{R}^{NR}} \prod_{a=1 \atop \lambda \geq 0}^{N} \prod_{i=1 \atop \lambda \geq 0}^{R} d\phi^i_a \exp \left[ -\frac{1}{4\alpha} \sum_{i,j=1}^{R} U_{ij}(\phi) \right] \prod_{i=1}^{R} Ai (\kappa \phi^i_1 \phi^i_2), \quad (48)
\]

where \( R = \lambda_H \), and we have introduced replicas \( \phi^i_a \) (\( i = 1, 2, \ldots, \lambda_H \)) to replace the power coming from that of Eq. (47) in the first line and performed the Gaussian integration over \( P_{abc} \). Here \( \alpha \) is an arbitrary positive number and \( N \) is an unimportant factor independent from \( \alpha \) and \( #P = N(N+1)(N+2)/6 \), i.e. the number of independent components of \( P_{abc} \).

The system (48) is complicated due to the presence of the Airy functions. However, when \( \kappa \) is taken to be positive, which physically corresponds to a negative cosmological constant, the Airy function \( Ai (\kappa \phi^i_1 \phi^i_2) \) is a function that rapidly decays with the increase of \( \phi^i_1 \phi^i_2 \). Therefore, as an interesting simplification, we could replace the Airy function by a rapidly decreasing function with a simpler form. In particular, to make the correspondence to the matrix model (1), we consider a simplified wave function,

\[
\Psi_{\text{simple}}(P) = \left[ \int_{\mathbb{R}^{N}} \prod_{a=1}^{N} d\phi^i_a \exp (I P_{abc} \phi^i_a \phi^i_b \phi^i_c - k (\phi^i_a \phi^i_a)^3) \right]^{R/2} \quad (49)
\]

with \( R = \lambda_H \) and a positive \( k \) by performing the replacement \( Ai (\kappa \phi^i_1 \phi^i_2) \rightarrow \exp [-k (\phi^i_1 \phi^i_2)^3] \) in Eq. (47). With the observable mentioned above, this leads to

\[
\langle \Psi | e^{-\alpha \hat{P}_{abc} \hat{P}_{abc}} | \Psi \rangle \sim \langle \Psi_{\text{simple}} | e^{-\alpha \hat{P}_{abc} \hat{P}_{abc}} | \Psi_{\text{simple}} \rangle = \int_{\mathbb{R}^{#P}} \prod_{a,b,c=1 \atop a \leq b \leq c}^{N} dP_{abc} e^{-\alpha P_{abc} P_{abc}} | \Psi_{\text{simple}}(P) |^2
\]

\[
= N \alpha^{-#P/2} \int_{\mathbb{R}^{NR}} \prod_{a=1 \atop \lambda \geq 0}^{N} \prod_{i=1 \atop \lambda \geq 0}^{R} d\phi^i_a \exp \left[ -\frac{1}{4\alpha} \sum_{i,j=1}^{R} U_{ij}(\phi) - k \sum_{i=1}^{R} (\phi^i_1 \phi^i_2)^3 \right]
\]

\[
= N \alpha^{-#P/2} Z_{N,R} \left( \frac{1}{4\alpha}, k \right), \quad (50)
\]

where \( R = \lambda_H \).

\textsuperscript{9} The equations are given by the physical state conditions, \( \hat{\mathcal{H}}_x | \Psi \rangle = 0 \) and \( \hat{\mathcal{J}}_{ab} | \Psi \rangle = 0 \), where \( \hat{\mathcal{H}}_x \) and \( \hat{\mathcal{J}}_{ab} \) are the quantized first-class constraints of the tensor model. See an appendix of Ref. [16] for more thorough, compact explanations.
One important matter in the relation (50) between the tensor and matrix models is that the parameter $R$ of the matrix model (1) is related to $N$ by $R = \lambda_H = (N + 2)(N + 3)/2$. What is striking is that this value agrees with the critical value $R_c \sim (N + 1)(N + 2)/2 - N + 2$ in the leading order of $N$. Because of the ambiguity of the approximate relation (50) it is difficult to make exact statements about the exact location of the tensor model. However, we could say that the tensor model is at least in the vicinity of (probably/possibly a little above) or even possibly exactly at the continuous phase transition point of the matrix model. This is quite intriguing, because our common knowledge tells us that continuum theories can often be obtained by taking continuum limits around continuous phase transition points in discretized theories. We could say that the consistency condition of the tensor model automatically puts the tensor model at the location where a continuum limit may be feasible, though it is currently difficult to conclude this because of the ambiguity contained in the simplification above.

7.2. Dimensions and symmetries of the configurations

In this subsection we will discuss the results of the simulations concerning dimensions and symmetries obtained in Sect. 6. For this purpose we refer to a property of the wave function (47) that the peaks (ridges) of the wave function are located on the values of $P_{abc}$ which are invariant under Lie-group transformations. This symmetry highlighting phenomenon has been found in Refs. [25,26], where the qualitative argument was given as follows. The integration (47) is of an integrand which oscillates rather widely due to the pure imaginary cubic function in the exponent. Therefore, for a “generic” value of $P_{abc}$, the contributions from different integration spots generally have different phases and mutually cancel among themselves so that the total amount of integration does not take a large value. However, at the location where $P_{abc}$ is invariant under a representation $H$ of a Lie group, $P_{abc} = h_a^d h_b^e h_c^f P_{d'b'c'} (\forall h \in H)$, the integration along the gauge orbit $h_a^d \phi_{d'} (\forall h \in H)$ contributes coherently in Eq. (47), and the wave function potentially takes a large value compared to the value at a “generic” location. This is indeed realized and has concretely been shown for some tractable cases in Refs. [25,26].

The above qualitative argument will hold at least partially after the simplification (49), since we can expect a similar coherence phenomenon in this case, too. Then, from the relation (50), the symmetry highlighting phenomenon of the wave function explained above will also be present in the matrix model (1) [16]. Note that this will be valid for general values of $R$, since the constraint $R = \lambda_H = (N + 2)(N + 3)/2$ coming from the consistency of the tensor model has nothing to do with the equalities in Eq. (50). In the relation (50), the contribution of a peak with a Lie group representation $H$ in the second line will correspond on the matrix model side to the contributions of $N$-dimensional vectors $\phi_i^d (i = 1, 2, \ldots, R)$ being distributed along a gauge orbit $h_a^d \phi_{d'} (\forall h \in H)$. In the simulation data, such distributed vectors will appear as a point cloud, which has been discussed in Sect. 6.4. This point cloud will have the dimension of the Lie group representation, and will break the part of the $SO(N)$ symmetry which is not commutative with $H$. Generally, the wave function contains a number of peaks with various Lie group representations, and therefore the point cloud will be that of a mixture of various gauge orbits. This mixed structure will induce a non-obvious pattern of symmetry breaking, which would be consistent with the hierarchical symmetry breaking in Sect. 6.5.

To get more information from the behavior obtained in Sect. 6, let us rewrite the second line in Eq. (50) as
Fig. 19. A possible profile of $\tilde{\Psi}(\tilde{P}\varphi^3)$, depending on the Lie-group symmetry of $\tilde{P}_{abc}$.

\[
\int_{\mathbb{R}^p} \prod_{a,b,c} P_{abc} e^{-\alpha P_{abc}P_{abc}} \left\{ \int_{\mathbb{R}^N} \prod_{a=1}^N d\phi_a \exp \left[ I P_{abc} \phi_a \phi_b \phi_c - k(\phi_a \phi_a)^3 \right] \right\}^R
= N' \int_0^\infty dPP^{3-P-NR/3-1} e^{-\alpha P^2} \int_{S^{N-1}} d\tilde{P} \left\{ \int_{\mathbb{R}^N} \prod_{a=1}^N d\varphi_a \exp \left[ I \tilde{P}_{abc} \varphi_a \varphi_b \varphi_c - k(\varphi_a \varphi_a)^3 / P^2 \right] \right\}^R
= N' \int_0^\infty dPP^{3-NR/3-1} e^{-\alpha kP^2} \int_{S^{N-1}} d\varphi \varphi^{N-1} \tilde{\Psi} \left( \tilde{P}\varphi^3 \right) e^{-\varphi^3 / P^2} \right\}^R, \tag{51}
\]

where $N$, $N'$ are some unimportant coefficients. Here, in the second line, we have separated $P_{abc}$ into the radial and angular variables, $P_{abc} = P \tilde{P}_{abc}$, where $P = \sqrt{P_{abc}P_{abc}}$, and have introduced a rescaled variable, $\varphi_a = P^{1/3} \phi_a$. Then, in the last line, we have rescaled $P^2 \rightarrow kP^2$, have divided $\varphi_a$ into the radial and angular variables, $\varphi_a = \varphi \tilde{\varphi}_a$ with $\varphi = \sqrt{\varphi_a \varphi_a}$, and have introduced

\[
\tilde{\Psi} \left( \tilde{P}\varphi^3 \right) := \int_{S^{N-1}} d\tilde{\varphi} e^{i\tilde{P}_{abc} \tilde{\varphi}_a \tilde{\varphi}_b \tilde{\varphi}_c}. \tag{52}
\]

The function $\tilde{\Psi}(\tilde{P}\varphi^3)$ will have a number of peaks at Lie-group symmetric $\tilde{P}_{abc}$. On such a peak, the value of $\tilde{\Psi}(\tilde{P}\varphi^3)$ will generally become smaller as $\varphi$ increases, because the oscillation of the integrand in Eq. (52) will become wilder. In the following paragraphs we will further argue that the $\varphi$-dependence of $\tilde{\Psi}(\tilde{P}\varphi^3)$ qualitatively depends on the symmetry of $\tilde{P}_{abc}$ as in Fig. 19 to explain the dimensional behavior in Fig. 10; namely, for $\tilde{P}_{abc}$ symmetric under higher-dimensional Lie-groups, $\tilde{\Psi}(\tilde{P}\varphi^3)$ takes larger values at small $\varphi$ but quickly decays with $\varphi$, while it takes smaller values at small $\varphi$ but slowly decays with $\varphi$ for $\tilde{P}_{abc}$ symmetric under lower-dimensional Lie-groups.

To see how the dimensional behavior in Fig. 10 can be explained by the profile in Fig. 19, let us first consider $R < (N + 1)(N + 2)/2$. In this case, the power of $P$ in the integrand of the last line of Eq. (51) is positive, and therefore the integral over $P$ will be over the range $0 \leq P \leq 1 / \sqrt{\alpha k}$ with some preference to larger $P$. As $k$ is taken smaller, the region of larger $\varphi$ becomes more dominant in the integral of Eq. (51), making the peaks associated with lower dimensional Lie-groups more dominant than higher dimensional ones. Then the increase of the power $R$ in Eq. (51) will enhance the peaks of lower dimensional Lie-groups. This explains the decrease of the dimensions with the increase of $R$ in the region $R < R_c$ in Fig. 10.

Let us next consider $R > (N + 1)(N + 2)/2$. In this case the power of $P$ in the integrand of Eq. (51) is negative, and $P$ will have a preference to smaller values as $R$ increases. Then, the last term
in Eq. (51) will bound the range of \( \varphi \) in the integration, as \( R \) increases. Because of the profile in Fig. 19, an increase of \( R \) will enhance the peaks with higher-dimensional Lie groups, explaining the increase of dimensions in the region \( R > R_c \) in Fig. 10.

7.3. Normalizability of the wave function of the tensor model

From the physical point of view, it would be interesting to discuss the norm of the wave function of the tensor model. If the wave function of the tensor model successfully contains a ridge which represents a spacetime with an infinite extension in time, the norm of the wave function will linearly diverge along the time direction. Thus, the norm of the wave function has a connection to the question concerning time in the tensor model.

As an approximation or as an example case study similar to the actual case, we discuss the norm of the simplified wave function (49). More precisely, we study the \( \alpha \to +0 \) limit of the relation (50):

\[
\lim_{\alpha \to +0} \int_{\mathbb{R}^p} dP_{abc} e^{-\alpha P_{abc}^aP_{abc}^b} |\Psi_{\text{simple}}(P)|^2 = \lim_{\alpha \to +0} N \alpha^{-\#P/2} Z_{N,R} \left( \frac{1}{4\alpha}, k \right),
\]

where \( N \) is an unimportant factor independent of \( \alpha \).

When \( R < R_c \), by putting (37) with \( \lambda = 1/(4\alpha) \) on the right-hand side of Eq. (53), the dominant \( \alpha \) dependence in the \( \alpha \to +0 \) limit is obtained as

\[
\alpha^{-\#P/2} Z_{N,R} \left( \frac{1}{4\alpha}, k \right) \sim \alpha^{-\#P/2}. 
\]

This concludes that the norm diverges for this case, because the critical value \( R_c \) satisfies \( R_c < (N + 1)(N + 2)/2 \), as has been shown numerically.

On the other hand, when \( R > R_c \), by putting Eq. (40) with \( \lambda = 1/(4\alpha) \) on the right-hand side, we obtain

\[
\alpha^{-\#P/2} Z_{N,R} \left( \frac{1}{4\alpha}, k \right) \sim \alpha^{-\delta \tilde{U}_d}, 
\]

under the assumption that \( \tilde{p}_{N,R}(+0) \) is finite, which has been supported from the data in Sect. 6.3. This is divergent in the limit \( \alpha \to +0 \) in the range \( R_c < R < (N + 1)(N + 2)/2 \), since \( \delta \tilde{U}_d > 0 \), as discussed in Sect. 6.3. On the other hand, since \( R = \lambda_H = (N + 2)(N + 3)/2 \) of the tensor model is in the region \( R > (N + 1)(N + 2)/2 \) and the values of \( \delta \tilde{U}_d \) are not known there, we cannot currently determine whether the simplified wave function of the tensor model is normalizable or not, or how rapidly this diverges if it does. As explained in Sect. 6.3, this is beyond the leading order perturbative computation.

The simplification (49) of the real wave function (47) is to approximate the case with a positive \( \kappa \), which corresponds to a negative cosmological constant. Therefore, it is an interesting future study to determine \( \delta \tilde{U}_d \) to answer the physical question concerning the emergence of time in the tensor model for a negative cosmological constant. Note also that the above discussion deals with finite \( N \), and therefore taking \( N \to \infty \) would also require more study on this matter.

8. Summary and future prospects

In this paper, we have numerically studied a matrix model with non-pairwise index contractions by Monte Carlo simulations. The matrix model has an intimate connection to the canonical tensor
model, a tensor model for quantum gravity in the Hamilton formalism [23, 24], and also has a similar structure as a matrix model that appears in the replica trick of the spherical $p$-spin model ($p = 3$) for spin glasses [28, 29]. The matrix model has previously been analyzed by a few analytic methods and Monte Carlo simulations in Refs. [15, 16], which suggested the presence of a continuous phase transition around $R \sim N^2/2$. This relation between $N$ and $R$ is particularly interesting, because this agrees with a consistency condition of the tensor model in the leading order of $N$, implying that the tensor model is automatically located exactly at or near a continuous phase transition point. However, in the previous works the evidence for the phase transition was not very clear. In this paper we have presented a new set up for Monte Carlo simulations by first integrating out the radial direction, and have studied the model by employing the more efficient Hamiltonian Monte Carlo method. We have obtained considerable improvement of the efficiency of the simulations, and have found a rather sharp continuous phase transition around $R = R_c \sim (N + 1)(N + 2)/2 - N + 2$. We have also studied various properties of the phase transition and the two phases: the dimensions of the configurations take the smallest values at the transition point; the phase at $R > R_c$ is characterized by cascade symmetry breaking; and the $k/\lambda \to +0$ limit is convergent in one phase and diverges in the other.

We have also discussed some implications for the canonical tensor model. In particular, the most striking is the coincidence mentioned above between the location of the continuous phase transition point and the consistency condition of the tensor model in the leading order of $N$. A well-known fact is that continuum theories can often be obtained by taking continuum limits near continuous phase transition points. This means that the tensor model seems to automatically put itself at the location where it is possible to find a sensible continuum limit. We have also discussed the wave function of the tensor model by using the connection between the matrix model and an approximation of a known exact wave function of the tensor model. In particular, we have provided a qualitative argument for the dependence on $R$ of both the dimension of the preferred class of configurations and the observed symmetry breaking patterns, using the intimate connection between Lie-group symmetries and peaks of the wave function that has been investigated before [25, 26].

While we have numerically obtained a rather clear picture of the phase structure of the matrix model, we are still seriously lacking analytic understanding. As shown in Sect. 6.2, there seem to exist essential differences between the numerical results and the analytic perturbative results performed in Refs. [15, 16]. Moreover, other than the qualitative argument given in Sect. 7.2, a more rigorous understanding of the behavior of the dimensions and the symmetry breaking would be desirable. Analytic understanding is also necessary to discuss the continuum limit discussed in the previous paragraph, since taking a large $N$ limit while simultaneously tuning $R$ and $k$ is difficult to do exclusively through numerical methods. Developing an analytical non-perturbative understanding is an important future direction to understand the dynamics of the matrix model.

Though this paper has given several clear pieces of evidence for the phase transition in the matrix model and has explained its interesting connection to the canonical tensor model, various things still need to be explored in order to understand more about the canonical tensor model through matrix models of this sort. Most importantly, the simplification of the wave function discussed in Sect. 7.1 by approximating the Airy function for $\kappa > 0$ by a conveniently chosen damping function does not explain whether the obtained results are universal under different choices of damping functions. Moreover, from a physical point of view we would like to explore the $\kappa < 0$ case corresponding to the positive cosmological constant, rather than the $\kappa > 0$ case corresponding to the negative cosmological constant. In the case of $\kappa < 0$, the Airy function becomes oscillatory, and the dynamics will most
likely be different from the $\kappa > 0$ case. Since this case suffers from the notorious sign problem, it is technically very challenging. It would also be an interesting future direction to apply new Monte Carlo methods developed to analyze various other systems suffering from sign problems to our case, as well as to develop analytical treatment.

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