Measures of self-blocking system with infinite space

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Abstract. In this paper, we study an infinite series queueing system with self-blocking phenomenon. Poisson arrivals and exponential service times are assumed. We develop the structured generator matrix to compute steady-state probabilities of the self-blocking system with infinite space by matrix-geometric method. The stability condition of the system is obtained in closed-form. We also present some performance measures including mean number in the system, and blocking probability, etc. The characteristics of the system with different service order are discussed as well.

1. Introduction
Traditionally, most of the published studies of series queues deal with the type of problems in which customers must be served by a series of service stations. This kind of systems is known as “tandem queues”. In this paper, we investigate a special self-blocking queueing system which consists of only two service stations, as shown in Figure 1. In this system, a customer is forced to enter station-2 directly if both service stations are available. A completed service is defined as when a customer is fully served at either station-1 or station-2. It is apparent that we can use this self-blocking queueing system to model the operations in a gas station or other similar queueing systems. This kind of configuration in the gas station is popular in the metropolitan area due to the land use restriction.

![Figure 1. Scheme of self-block queueing system.](image)

In the field of tandem queues, Hunt [1] first studied tandem queues with blocking phenomenon in sequential arrays of waiting lines with Poisson arrivals and exponential service times. Avi-Itzhak [2] provided an analysis about a two-stage tandem queue with blocking, arbitrary input and regular service times. The waiting time distributions and other characteristics of such queueing system can be obtained by some properties. Avi-Itzhak and Yadin [3] investigated a queueing system consisting of a sequence of two service stations with no intermediate queue. They derived the moment generating functions of the steady-state queueing times as well as the generating functions of the steady-state number of customers in the various parts of the system under assumptions of Poisson process of arrivals and arbitrarily distributed service times. Neuts [4] presented a series system with a finite intermediate waiting room. He demonstrated that this system can be studied in terms of an imbedded semi-Markov process.

Performance and approximation analysis of tandem queues can be found in the following works. Langaris and Conolly [5] conducted an analysis on the waiting time of a two-stage queueing system with blocking. The survey of closed queueing networks with blocking was provided by Onvural [6]. Gomez-Corral [7] focused on the embedded process at departures of a tandem queueing system with...
blocking and Markovian arrival process. Gomez-Corral and Martos [8] studied the influence of several flows of signals on the performance evaluation of a two-stage tandem system with blocking.

In this paper, we investigate the performance measures of a system with two service units operating independently in series simultaneously. The phenomenon of self-blocking occurs at the scenario when the station-1 has completed the service, but the station-2 is still serving for the other customer. We study the impacts of the performance measures of the system by switching different service orders. Therefore, we can suggest a better disposition strategy for the self-blocking queueing system according to our research. In addition, we derive the stability condition of the system with infinite capacity.

2. Problem formulation and notations
The assumptions of the model are described as follows. Suppose that a queueing system which consists of two independent service units in series operating simultaneously, as shown in Figure 1. Suppose that each customer enters to the system in accordance with Poisson process with mean arrival rate $\lambda$. Each customer is forced to move to the station-2 if both service stations are available. The service is complete when a customer is fully served at either station-1 or station-2. This means that the customer can depart the system when the service is performed completely at either station. The main characteristic of the service process in this queueing system is self-blocking, which occurs at the scenario when the station-1 has completed the service, but the station-2 is still serving for the other customer. A queue with infinite capacity is allowed in front of station-1. We also assume that no waiting space exists at station-2. It is assumed that each service station can serve only one customer at a time and the service rate is independent of the number of customers. The service is performed on the first come first serve (FCFS) discipline. In addition, the time to serve a customer of station-1 and station-2 is exponentially distributed with mean service time $1/\mu_1$ and $1/\mu_2$, respectively.

$\lambda$ Mean arrival rate of Poisson arrivals
$\mu_1$ Mean service rate of the station-1
$\mu_2$ Mean service rate of the station-2
$P_{n_1, n_2, n_3}$ The steady-state probability $P_{n_1, n_2, n_3}$ of $n_1$ customers in the station-2 and $n_2$ customers in the station-1 and $n_3$ customers in the queue.

3. Steady-state equations and structured generator matrix
The quasi-birth-death process of the self-blocking queueing system with infinite capacity is shown in Figure 2. The infinite infinitesimal steady-state equations of the self-blocking queueing system are given by:

$$\lambda P_{0,0,0} = \mu_1 P_{0,1,0} + \mu_2 P_{1,0,0} + \mu_2 P_{1,0,0},$$

(1)

$$(\lambda + \mu_2) P_{1,0,0} = \lambda P_{0,0,0} + \mu_1 P_{0,1,0} + \mu_2 P_{1,1,0},$$

(2)

$$(\lambda + \mu_1 + \mu_2) P_{1,1,0} = \lambda P_{0,0,0} + \mu_1 P_{0,2,0} + \mu_2 P_{1,1,2},$$

(3)

$$(\lambda + \mu_2) P_{1,0,0} = \mu_1 P_{1,1,0},$$

(4)

$$(\lambda + \mu_1) P_{0,1,0} = \mu_2 P_{1,1,0},$$

(5)

$$(\lambda + \mu_1 + \mu_2) P_{1,1,i+1} = \lambda P_{1,1,i} + \mu_1 P_{0,1,i+2} + \mu_2 P_{1,1,i+2}, \quad i \geq 0.$$
\[(\lambda + \mu_2)P_{1, b, i+1} = \lambda P_{1, b, i} + \mu_2 P_{1, i, i+1}, \quad i \geq 0,\]

\[(\lambda + \mu_1)P_{0, l, i+1} = \lambda P_{0, l, i} + \mu_1 P_{1, i, i+1}, \quad i \geq 0.\]

\[B_{0,0} \begin{bmatrix} -\lambda & \lambda \\ \mu_2 & -(\lambda + \mu_2) \end{bmatrix}, \quad B_{1,0} = \begin{bmatrix} 0 & 0 \\ \mu_2 & 0 \end{bmatrix}, \quad B_{2,0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{0,1} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ \mu_2 & 0 \end{bmatrix}, \quad \mu \begin{bmatrix} 0 \\ \mu_2 \end{bmatrix}.\]

Figure 2. The quasi-birth-death process of the self-block system with infinite capacity.

We construct the structured generator matrix according to the quasi-birth-death process of the system. The structured generator matrix can be written in the following block form:

\[
Q = \begin{bmatrix}
B_{0,0} & B_{0,1} & 0 & 0 & 0 & 0 & L \\
B_{1,0} & A_1 & A_0 & 0 & 0 & 0 & L \\
B_{2,0} & A_2 & A_1 & A_0 & 0 & 0 & L \\
0 & A_1 & A_2 & A_1 & A_0 & 0 & L \\
0 & 0 & A_3 & A_2 & A_1 & L \\
M & M & M & L & M & M & M & M & M & O
\end{bmatrix}_{\ell}\]

3.1. Matrix-geometric method and stability conditions

Let us define \(P = [P_0, P_1, P_2, \ldots] \) as the steady-state probability vector corresponding to the structured generator matrix \(Q\). Therefore, we can yield \(P\) by solving \(PQ = 0\), while obeying the normalization condition, \(P1 = 1\). The global balance equations of the quasi-birth-death process are considered:

\[
P_0 B_{0,0} + P_1 B_{1,0} + P_2 B_{2,0} = 0,
\]

\[
P_0 B_{0,1} + P_1 A_1 + P_3 A_3 = 0,
\]

\[
P_1 A_0 + P_{i+3} A_3 = 0, \quad i \geq 1.
\]

A constant rate matrix \(R\) can be introduced that leads to the matrix-geometric equation:
\( \mathbf{P}_i = \mathbf{P}_{i-1} \mathbf{R} = \mathbf{P}_{i-1} \mathbf{R}^{i-1}, \quad i > 1. \)  

Eq. (12) can be simplified to 
\[ \mathbf{A}_0 + \mathbf{R} \mathbf{A}_1 + \mathbf{R}^2 \mathbf{A}_2 = 0. \]

The normalization condition can be written as: 
\[ \sum_{i=0}^{\infty} \mathbf{P}_i \mathbf{1} = 1. \]

Using some derived results referred in Neuts [4], we can obtain an important normalization condition equation constrained to Eqs. (14) and (15):
\[ \mathbf{P}_0 \mathbf{1} + \mathbf{P}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{1} = 1. \]

We evaluate the rate matrix, \( \mathbf{R} \), iteratively by successive substitution. To obtain \( \mathbf{P}_i \), Eqs. (10), (11) and (16) are taken into account
\[ (\mathbf{P}_0 \mathbf{1}) \begin{pmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1}^* \\ \mathbf{B}_{1,0} + \mathbf{R} \mathbf{B}_{2,0} & (\mathbf{A}_1 + \mathbf{R}^2 \mathbf{A}_2)^* \end{pmatrix} \begin{pmatrix} 1 \\ (\mathbf{I} - \mathbf{R})^{-1} \mathbf{1} \end{pmatrix} = (0,1). \]

where \( (,)^* \) indicates that the last column of the included matrix is removed to avoid linear dependency. The removed column is replaced by the normalization condition. Equation (17) is solved for evaluating \( \mathbf{P}_0 \) and \( \mathbf{P}_1 \).

In order to confirm whether the quasi-birth-death process is tractable by the matrix-geometric method, the stability of the queueing system must be verified by checking the stability condition of Neuts [4]:
\[ \mathbf{P}_\lambda \mathbf{A}_0 \mathbf{1} < \mathbf{P}_\lambda \mathbf{A}_1 \mathbf{1} + 2(\mathbf{P}_\lambda \mathbf{A}_2 \mathbf{1}), \]

where \( \mathbf{P}_\lambda \) is the steady-state probability vector corresponding to the generator matrix \( \mathbf{A} \).

Thus, the stability conditions can be concluded as the following inequalities:
\[ \rho < \frac{2}{\frac{\mu_1}{\lambda} + \frac{\mu_2}{\mu_1} + \frac{\mu_2}{\mu_1}} = \frac{2}{3}, \]

where \( \rho = \frac{\lambda}{\mu_1 + \mu_2} \).

3.2. Performance measures

Performance measures of the self-blocking queueing system with infinite capacity are defined as:

(1) Mean number of customers in the system
\[ L = (\mathbf{P}_{1,0,0} + \mathbf{P}_{0,1,0} + \mathbf{P}_{1,0,0}) + \sum_{n=1}^{\infty} (\mathbf{P}_{1,0,n-1} + \mathbf{P}_{1,1,n-2} + \mathbf{P}_{0,1,n-1}) \mathbf{n}_g \]

(2) Mean waiting time in the system (Little’s Law)
\[ W = \frac{L}{\lambda}. \]

(3) Blocking probability of the customer in the station-1
4. Numerical results

We perform numerical experiments in this section. We first present how stability conditions change as the ratio of the service rate of the station-1 to the service rate of the station-2 varies from 0.1 to 10, as shown in Figure 3. We observe that the order of service stations doesn’t affect the stability condition which can be verified by the symmetric figure to the ratio $\frac{\mu_1}{\mu_2} = 1$. It is noted that when the service rate of the station-1 is equal to the service rate of the station-2, the stability condition for the self-blocking queueing system must be less than $\frac{2}{3}$. Next, we inspect the trends of blocking probability and mean number of customers in the system as the mean arrival rate increases. Therefore, we fix the system parameters at $\mu_1 = 1$, $\mu_2 = 1$ and vary the values of $\rho$ from 0.01 to 0.66, as shown in Figure 4. We observe that the mean number of customers in the system increases slowly for low values of $\rho$, then it increases very rapidly as $\rho$ approaches to 0.66. The increasing trends of blocking probability as a function of $\rho$ are also shown in Figure 4.

Next, we study the impact of different service orders that cause the distinct performances of the self-blocking queueing system. Comparisons with different service orders on mean waiting time in the system and mean service time of the system and blocking probability are given in the following special case: setting different service orders for $\mu_1 = 2$, $\mu_2 = 1$, $\mu_1 = 1$, $\mu_2 = 2$ and vary the values of $\rho$ from 0.01 to 0.56.

Intuitively, if we set the higher service rate for the station-2, the performance on blocking probability of the self-blocking queueing system is going to be less than that of higher service rate of
the station-1. We define service order called obverse order, if we dispose higher service rate for the station-2 (i.e. $\mu_2 > \mu_1$). Otherwise, if the service rate of the station-2 is lower than that of the station-1, it is called reverse order (i.e. $\mu_2 < \mu_1$).

We observed that the performances of the system placed in the obverse order on mean waiting time in the system, mean service time of the system and blocking probability are less than that of the system placed in the reverse order, as shown in Figure 5 ~ Figure 6, respectively. Percentage change relative to the obverse order on mean waiting time of the system is presented in the red line of Figure 5. From Figure 6 of the case we have studied above, it is noted that if we want to optimize the performances of the self-blocking queueing system, it is better to dispose the obverse order for the self-blocking system. We can increase the efficiency for the self-blocking system which means maintain lower performances on the mean number of customers in the system, mean waiting time in the system, mean service time of the system and blocking probability by disposing the service order with obverse way.

5. Conclusions
In this paper, we study a series type queueing system with self-blocking phenomenon. We have applied matrix-geometric method to evaluate steady-state probabilities of the system with infinite capacity. We have also derived stability conditions of the system for both service stations with same service rates or with different service rates.

Numerical results of performance measures of the self-blocking queueing system indicate that it is better to adjust the service station-2 with higher service rate than that of the service station-1 in order to effectively reduce the mean number of customers in the system, mean waiting time in the system, and blocking probability of the self-blocking queueing system.

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