NEW EXAMPLES OF DETERMINANT DIVISIBILITY SEQUENCES

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Abstract. In this paper we consider divisibility sequences obtained from square matrices. We work with of matrix divisibility sequences associated to a semigroup and arising from endomorphisms of an affine space. We prove that determinant divisibility sequences originated from powers of square matrices are generalized Lucas sequences.

1. Introduction

By the divisibility sequence we mean in this paper a sequence \( \{d_n\}_{n \in \mathbb{N}} \) of integers such that if \( n|m \) then \( d_n|d_m \). One of the most famous divisibility sequence is the Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34,... which arise from linear recurrence: \( F_n = F_{n-1} + F_{n-2} \). This is an example of the Lucas sequences: \( L_n = \alpha^n - \beta^n \), where \( \alpha, \beta \) are the roots of some quadratic polynomial over \( \mathbb{Z} \). See [2] for a complete classification of linear recurrence divisibility sequences and [5], [6] for introduction to other divisibility sequences. In this paper we discuss properties of certain matrix divisibility sequences. We follow the approach initiated in [1].

2. Matrix divisibility sequence

Let \( S \) be a commutative ring with 1. Let \( M_r(S) \) be a ring of \( r \times r \) matrices with entries in \( S \). By a divisor class of a matrix \( M \in M_r(S) \) we mean a coset \( GL_r(S) \cdot M \) of \( M \) with respect to the natural left action of \( GL_r(S) \). We say that matrix \( M \in M_r(S) \) divides a matrix \( N \in M_r(S) \) if there exists a \( Q \in M_r(S) \) such that \( N = QM \). If \( M \) divides \( N \), then any element of the divisor class of \( M \) also divides \( N \). Let \( (\Gamma, \cdot) \) denote a semigroup. A divisibility sequence of matrices over a commutative ring \( S \), indexed by \( \Gamma \), is a collection of matrices \( \{M_\alpha\}_{\alpha \in \Gamma} \) in \( M_r(S) \), such that if \( \alpha \) divides \( \beta \) in \( \Gamma \), then \( M_\alpha \) divides \( M_\beta \) in \( M_r(S) \). If \( \{M_\alpha\}_{\alpha \in \Gamma} \) is a divisibility sequence of matrices, then by the multiplicativity of the determinant \( \{det(M_\alpha)\}_{\alpha \in \Gamma} \) is a divisibility sequence of elements of the ring \( S \).

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We fix a faithful representation:
\[\cdot : \Gamma \leftrightarrow \text{End}(A^m_S) : \alpha \to [\alpha]\]
of \(\Gamma\) into the group of endomorphisms of affine \(m\)-dimensional space \(A^m_S\) over \(S\).

**Definition 2.1.** Let \(x \in A^r_S\). The matrix divisibility sequence associated to \((\Gamma, \cdot)\) is the sequence of Jacobians \(\{J_\alpha(x)\}_{\alpha \in \Gamma}\) which are \(r \times r\) matrices with \((i, j)\)-entry given by partial differentials:
\[J_\alpha(x)_{i,j} := \frac{\partial((\alpha)(x))_i}{\partial x^j},\]
where \((\alpha)(x)_i\) is an \(i\)th entry of the value of the endomorphism \([\alpha]\) on \(x\). The associated determinant divisibility sequence is defined by \(\{\det(J_\alpha(x))\}_{\alpha \in \Gamma}\).

### 3. Main Result

**Theorem 3.1.** Let \(X \in GL_r(\mathbb{Z})\) and \(\lambda_1, \ldots, \lambda_r\) be eigenvalues of \(X\). Then for every \(n \geq 1\):
\[(3.1)\quad D_n = n^2 [\det X]^{n-1} \prod_{1 \leq i < j \leq r} \left( \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} \right)^2\]
is an integer and the sequence \(\{D_n\}_{n \in \mathbb{N}}\) is a determinant divisibility sequence.

**Proof:** Let \(X, Y, Z\) be square \(s \times s\) matrices. Assume that entries of matrices \(Y\) and \(Z\) are functions of entries of the matrix \(X\). Then the following matrix derivative formula holds ([4]):
\[(3.2)\quad \frac{d(YZ)}{dX} = (I \otimes Y) \frac{dZ}{dX} + (Z^t \otimes I) \frac{dY}{dX},\]
where \(\otimes\) means the Kronecker product, \(I\) is the identity matrix of rank \(s\) and \(A^t\) means the transpose matrix of \(A\). In addition we will use property of the Kronecker product:
\[(3.3)\quad (A \otimes C)(B \otimes D) = AB \otimes CD\]
for any square matrices \(A, B, C, D\) of size \(s \times s\). From now on we fix \(\Gamma = \mathbb{N}\). Consider the group \(G\) of all invertible \(s \times s\) matrices with the embedding:
\[G \to \mathbb{A}^{s^2} : \begin{bmatrix} X_{11} & \cdots & X_{1s} \\ \vdots & \ddots & \vdots \\ X_{s1} & \cdots & X_{ss} \end{bmatrix} \mapsto (X_{11}, \ldots, X_{1s}, \ldots, X_{s1}, \ldots, X_{ss})\]
We define the endomorphism \([n]\) for \(n \in \mathbb{N}\). Let \(X := [X_{ij}] \in G\) and respectively \(X^n := [X_{kl}] \in G\), where we treat \(X_{kl}\) as functions of \(X_{ij}\), for \(1 \leq i, j, k, l \leq s\). We define \([n] : \mathbb{A}^s \to \mathbb{A}^s\) as
\[
[n](X_{11}, \ldots, X_{1s}, \ldots, X_{s1}, \ldots, X_{ss}) = (\bar{X}_{11}, \ldots, \bar{X}_{1s}, \ldots, \bar{X}_{s1}, \ldots, \bar{X}_{ss}).
\]
Using (3.2) we compute Jacobians of the \(n\)-th power of the matrix \(X\)
\[
J_n = \frac{d(X^n)}{dX} = \frac{d(X^{n-1}X)}{dX} = (I \otimes X^{n-1}) \frac{dX}{dX} + (X^t \otimes I) \frac{dX^{n-1}}{dX}.
\]
By induction and the property (3.3) of the Kronecker product we get:
\[
J_n = \sum_{k=0}^{n-1} (X^t)^k \otimes X^{n-1-k}.
\]
Assume that \(X \in GL_n(\mathbb{Z})\) is a diagonalizable matrix. Then \(X = PDP^{-1}\), for some \(P \in GL_n(\mathbb{C})\) and a diagonal matrix \(D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_s\}\). The set of matrices with distinct eigenvalues is dense in the set of all square matrices \(M_n(\mathbb{C})\) with respect to the topology of \(\mathbb{C}^s\). Hence we can assume that \(X\) has different eigenvalues. Therefore
\[
J_n = \sum_{k=0}^{n-1} ((PD(P^{-1})^t)^k \otimes (PD(P^{-1})^{n-1-k}) = \sum_{k=0}^{n-1} (P^{-1})^t(D^k)^t(P)^t \otimes PD^{n-1-k}P^{-1} =
\]
\[
= ((P^{-1})^t \otimes P) \left[ \sum_{k=0}^{n-1} (D^k \otimes D^{n-1-k}) \right] (P^t \otimes P^{-1})
\]
and the determinant divisibility sequence is of the form:
\[
D_n = \det J_n = \det \left[ \sum_{k=0}^{n-1} (D^k \otimes D^{n-1-k}) \right].
\]
Since \(D^k \otimes D^{n-1-k}\) is a diagonal matrix whose diagonal consists of terms \(\lambda_i^k\lambda_j^{n-1-k}\), we conclude that:
\[
D_n = n^2 \prod_{l=0}^{s} \lambda_l^{n-1} \prod_{1 \leq i \neq j \leq s} \sum_{k=0}^{n-1} \lambda_i^k\lambda_j^{n-1-k} = n^2 [\det X]^{n-1} \prod_{1 \leq i < j \leq s} \left( \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} \right)^2.
\]
The last term on the right hand side is a product of values of symmetric polynomials computed at eigenvalues of the matrix \(X\). The Galois group of the splitting field of the characteristic polynomial of \(X\) acts trivially on these algebraic integers, hence \(D_n \in \mathbb{Z}\). For any \(n, m\) such that \(n\) divides \(m\) we have \([\det X]^n[\det X]^m\) and \((\lambda_i^n - \lambda_j^n)((\lambda_i^m - \lambda_j^m))\). Therefore, the sequence \(D_n\) is a divisibility sequence. For equal eigenvalues we compute \(D_n\) using the exact form of symmetric polynomials instead of their fractional expression.
If \(X\) is not a diagonalizable matrix, then instead of the matrix \(D\) we consider an upper-triangular matrix obtained from the Jordan form of \(X\) with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_r\) on the diagonal and eventually we get the same sequence \(D_n\).
4. Examples

1) Let $X \in \text{GL}_2(\mathbb{Z})$ and $a = \text{tr}X$, $b = \text{tr}^2X - 4\det X$. Then using Theorem 1 we obtain the sequence presented in [1], example 4.3:

$$D_n = \frac{n^2}{b}[\det X]^{n-1} \left( \frac{a + \sqrt{b}}{2} \right)^n - \left( \frac{a - \sqrt{b}}{2} \right)^n$$

2) Let $X \in \text{GL}_3(\mathbb{Z})$ and $b = -\text{tr}X$, $c = X_{11} + X_{22} + X_{33}$, $d = -\det X$. The discriminant of the characteristic polynomial of $X$ is $\Delta = (4\Delta_0^3 - \Delta_1^2)/27$, where $\Delta_0 = b^2 - 3c$ and $\Delta_1 = 2b^3 - 9bc + 27d$. We obtain the divisibility sequence defined by:

$$D_n = \frac{n^2d^{n-1}}{\Delta} \prod_{i=1}^{3} \left[ \frac{1}{3} \left( b + \epsilon^i A + \epsilon^{2i} \Delta_0 \bar{A} \right)^n - \left( b + \epsilon^{i+1} A + \epsilon^{2i+2} \Delta_0 \bar{A} \right)^n \right]^2,$$

where $A = \sqrt[3]{(\Delta_1 + \sqrt{-27\Delta})/2}$, $\bar{A} = \sqrt[3]{(\Delta_1 - \sqrt{-27\Delta})/2}$ and $\epsilon$ is a fixed primitive cube root of unity.

3) It is easy to compute values of $D_n$ for any square matrix $X$. The matrix $X = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix} \in \text{GL}_3(\mathbb{Z})$ gives the following divisibility sequence:

| $n$ | $d_n$ | factorization of $d_n$ |
|-----|-------|-------------------------|
| 1   | 1     | 1                       |
| 2   | 100   | $2^5 5^2$               |
| 3   | 6561  | $3^8$                   |
| 4   | 193600| $2^6 5^2 11^2$          |
| 5   | 808201| $2^9 31^2$              |
| 6   | 189612900 | $2^2 3^8 5^2 17^2$     |
| 7   | 50131657801 | $41^2 43^2 127^2$     |
| 8   | 4096576000000 | $2^{12} 5^6 11^2 23^2$ |
| 9   | 159625511221401 | $3^{14} 53^2 109^2$ |
| 10  | 1865976489302500 | $2^2 5^4 29^2 31^6$ |
| 11  | 31583922467632921 | $131^2 857^2 1583^2$ |
| 12  | 21985833099924302400 | $2^8 3^5 11^2 17^2 71^2 109^2$ |
| 13  | 2370466451421685365841 | $1637^2 4057^2 733^2$ |
| 14  | 118070682478980566428900 | $2^5 41^2 43^3 83^2 127^2$ |
| 15  | 23622553669723766871090801 | $3^8 29^2 31^2 2969^2 7109^2$ |
| 16  | 84956038709284864000000 | $2^{18} 5^6 11^2 23^2 47^2 383^2$ |
4) The matrix $X = \begin{bmatrix} -1 & 2 & 4 & -1 \\ 0 & 1 & -2 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in Gl_4(\mathbb{Z})$ gives the following divisibility sequence:

| $n$ | $d_n$ | factorization of $d_n$ |
|-----|-------|-------------------------|
| 1   | 1     | 1                       |
| 2   | 65536 | $2^{16}$                |
| 3   | 1     | 1                       |
| 4   | 281474976710656 | $2^{48}$ |
| 5   | 18448995933652254721 | $4295229439^2$ |
| 6   | 18013780039499776 | $2^{16}7^274897^2$ |
| 7   | 79223326847881056061239459841 | $281466386710529^2$ |
| 8   | 5194832314440011219064571543158784 | $2^{60}7^423^259561^2$ |
| 9   | 5775028020578736368542570774529 | $37^2701^2292993041329^2$ |
| 10  | 2229666183092939970266587262959037621272576 | $2^{16}19^43449^44295229439^2$ |
| 11  | 1463330673647120201450844900178197550156472647681 | $32363^27282397^25132726390881^2$ |
| 12  | 91328172579326327868701556304335790376407269376 | $2^{48}7^213^210177^274897^2259691^2$ |
| 13  | 627422831076804852924579197717363301022335434456089620481 | $3^{18}3769^215053^227205307^22607270173^2$ |
| 14  | 41240607345768667405443309242704726434308782374158659336863744 | $2^{16}13^2794009^227304061^2281466386710529^2$ |
| 15  | 980617555463239147044270484791252942607177394577588251525121 | $17489^24295229439^2131825214490835791^2$ |
| 16  | 1765121615339370515604475310366412659400104668637242611924881667927310336 | $2^{72}7^823^259561^220394769^2288208447^2$ |
References:

[1] Gunther Cornelissen, Jonathan Reynolds, Matrix divisibility sequences, Acta Arith. 156 (2012), 177-188

[2] J.P. Bezivin, A. Ptheo, A.J. van der Poorten, A full characterization of divisibility sequences, Amer. J. of Math. 112 (1990), 985-1001;

[3] Rachel Shipsey, Elliptic divisibility sequences, Ph.D. thesis, Goldsmiths College, University of London, see home-pages.gold.ac.uk/rachel, 2000

[4] Mike Brooks: The Matrix Reference Manual, [http://www.ee.ic.ac.uk/~staff/dmb/matrix/intro.html](http://www.ee.ic.ac.uk/~staff/dmb/matrix/intro.html)

[5] Joseph H. Silverman, Generalized greatest common divisors, divisibility sequences, and Vojtas conjecture for blowups, Monatsh. Math. 145 (2005), no. 4, 333-350.

[6] Patrick Ingram, Elliptic divisibility sequences over certain curves, J. Number Theory 123 (2007), no. 2, 473-486.

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