The stability index for dynamically defined
Weierstrass functions

C.P. Walkden and T. Withers*

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Abstract

Let \( \hat{T} : X \times \mathbb{R} \to X \times \mathbb{R} \) given by \( \hat{T}(x,t) = (Tx, g_x(t)) \) be a skew-product dynamical system where \( T : X \to X \) is a mixing conformal expanding map and, for each \( x \in X \), \( g_x : \mathbb{R} \to \mathbb{R} \) is an affine map of the form \( g_x(t) = -f(x) + \lambda(x)^{-1}t \). Under a suitable contraction hypotheses on \( \lambda \) there exists a measurable function \( u : X \to \mathbb{R} \) such that graph(\( u \)) = \{ \( (x,u(x)) \) \( | \) \( x \in X \) \} is \( \hat{T} \)-invariant and divides \( X \times \mathbb{R} \) into two regions, \( \mathbb{B}^+ \) and \( \mathbb{B}^- \), consisting of points that are repelled under iteration by \( \hat{T} \) to \( \pm \infty \). These two regions act as basins of attraction to \( \pm \infty \) in the sense of Milnor. The two basins have a complicated local structure: a neighbourhood of a point \( (x,t) \in \mathbb{B}^+ \) will typically intersect \( \mathbb{B}^- \) in a set of positive measure. The stability index (as introduced by Podvigina and Ashwin [PA] for general Milnor attractors) is the rate of polynomial decay of the measure of this intersection.

We calculate the stability index at typical points in \( X \times \mathbb{R} \). We also perform a multifractal analysis of the level sets of the stability index.

§1 Introduction and statement of results

Given a dynamical system \( T : X \to X \), a Milnor attractor [Mi] is a closed invariant set \( A \) for which the basin of attraction \( \mathbb{B}(A) \) (defined to be the set of points \( x \in X \) for which the omega-limit set \( \omega(x) \subseteq \mathbb{B}(A) \)) has positive measure and there is no strictly smaller closed set \( A' \subset A \) for which \( \mathbb{B}(A') = \mathbb{B}(A) \). We note that \( \mathbb{B}(A) \) is not required to be an open set. If \( T \) has two, or more, attractors \( (A_1, A_2, \text{say}) \) then the basins may have a complicated local structure: given a point \( x \in \mathbb{B}(A_1) \), any neighbourhood of \( x \) may intersect \( \mathbb{B}(A_2) \) in a set of positive measure. The notions of riddled basins and intermingled basins were introduced, following numerical observations, in [AYYK]. A basin is riddled if its complement intersects every ball in a set of positive measure; two basins are intermingled if each ball that intersects one basin in a set of positive measure also intersects the other basin in a set of positive measure. The study of riddled basins has attracted considerable interest at the interface of dynamical systems and physics [G, AYYK, OSAKY, SO, SH, K2 for example].

In [PA], the notion of stability index for a basin was introduced (see also [G]). The stability index measures the extent to which a basin \( \mathbb{B} \) is riddled at a given point. Specifically, let \( B_r(x) := \{ y \in X \mid d(x,y) < r \} \). For a given measure \( \mu \), define the stability index \( \sigma(x) \) to be \( \sigma(x) := \sigma^+(x) - \sigma^-(x) \) where

\[
\sigma^+(x) = \lim_{r \to 0} \frac{1}{\log r} \log \left( \frac{m(B_r(x)) \cap \mathbb{B}}{m(B_r(x))} \right), \quad \sigma^-(x) = \lim_{r \to 0} \frac{1}{\log r} \log \left( 1 - \left( \frac{m(B_r(x)) \cap \mathbb{B}}{m(B_r(x))} \right) \right) \tag{1}
\]

Skew-products provide a class of particularly rich, yet tractable, examples of dynamical system. In [MG], skew-products of the form \( \hat{T} : [0,1]^2 \to [0,1]^2 \), \( \hat{T}(x,t) = (Tx, h(x,t)) \), where

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$T$ is a skewed doubling map and $h$ belongs to a family of piecewise linear transformations are considered. There are two attractors: $\mathbb{B}^- = X \times \{0\}$, $\mathbb{B}^+ = X \times \{1\}$ and the stability index at a.e. point of the form $(x, 0)$ is calculated.

In [K1], skew-products of the form $\hat{T} : X \times [0, 1] \to X \times [0, 1]$ of the form $\hat{T}(x, t) = (Tx, h(x, t))$ where $T$ is an invertible hyperbolic map, and $h(x, t) = h_1(x)h_2(t)$ with $h_2(t)$ a strictly increasing $C^{1+\alpha}$ concave function. Again $X \times \{0\}$ and $X \times \{1\}$ are attractors and the stability index at Lebesgue a.e. point of the form $(x, 0)$ is calculated.

Results similar to those above were put into the context of thermodynamic formalism, in [K2]. Here, skew-products of the form $\hat{T} : [0, 1]^2 \to [0, 1]^2$, $\hat{T}(x, t) = (Tx, g_x(t))$ are considered where the map $T$ is assumed to be a Markov expanding map and, for each $x \in [0, 1]$, $g_x : [0, 1] \to [0, 1]$ is a diffeomorphism with negative Schwartzian derivative. There are three invariant graphs $u^- \leq u^c \leq u^+$ and the graphs of $u^-$ and $u^+$ are attractors with riddled basins. The stability index at almost every (with respect to an appropriate measure) point $(x, t)$ is calculated.

In [M0  K1  K2], the stability index is typically found to be the ratio of two Lyapunov exponents (corresponding to the exponential rate of contraction in the fibre direction and the Lyapunov exponent of the base transformation) multiplied by a constant related to the asymptotic distribution of the invariant graph.

In this paper, we consider skew products over $C^{1+\alpha}$ conformal expanding maps $T : X \to X$ and with fibre $\mathbb{R}$ where the dynamics in the fibre direction is affine. Specifically, for $\alpha$-Holder continuous functions $f : X \to \mathbb{R}^+$ and $\lambda : X \to \mathbb{R}^+$ we define

$$\hat{T} : X \times \mathbb{R} \to X \times \mathbb{R}, \hat{T}(x, t) = (Tx, -f(x) + \lambda(x)^{-1}t) =: (Tx, g_x(t)).$$

Note that, by replacing $T$ by $T^2$, we can assume without loss of generality that $\hat{T}$ is orientation-preserving in each fibre. Note that, for a fixed $t \in \mathbb{R}$, the map $x \mapsto g_x(t)$ is $\alpha$-Hölder. We define $g^n_x(t)$ by $T^n(x, t) =: (T^n x, g^n_x(t))$.

Under appropriate contraction hypotheses on $\lambda$ that ensure that $\hat{T}$ expands in the fibre direction, skew products of the form (2) possess an invariant graph, namely a function $u : X \to \mathbb{R}$ such that graph$(u) = \{(x, u(x)) \mid x \in X\}$ is $\hat{T}$-invariant. Under our hypotheses, $u$ will be measurable but not continuous. The graph of $u$ divides $X \times \mathbb{R}$ into two regions (up to a set of measure zero), one consisting of points that are repelled to $+\infty$ in the fibre direction under iteration by $\hat{T}$ and the other consisting of points repelled to $-\infty$. We regard $X \times \{-\infty\}$ and $X \times \{\infty\}$ as attractors for the skew-product $\hat{T}$. We define

$$\mathbb{B}^+ = \{(x, t) \in X \times \mathbb{R} \mid \lim_{n \to \infty} g^n_x(t) = \infty\}, \quad \mathbb{B}^- = \{(x, t) \in X \times \mathbb{R} \mid \lim_{n \to \infty} g^n_x(t) = -\infty\}$$

and refer to these as the basins of attraction to $\pm \infty$, respectively.

We define the stability index of these basins as follows. We let $B_r(x, t) := B_r(x) \times [t-r, t+r] \subset X \times \mathbb{R}$ be a neighbourhood of $(x, t) \in X \times \mathbb{R}$. Let $\mu$ denote an appropriate $T$-invariant probability measure on $X$ and let $m$ denote Lebesgue measure on $\mathbb{R}$. Define

$$\Sigma^+_\mu x(t) := \frac{\mu \times m(B_r(x, t) \cap \mathbb{B}^+)}{\mu \times m(B_r(x, t))}, \quad \Sigma^-_\mu x(t) := \frac{\mu \times m(B_r(x, t) \cap \mathbb{B}^-)}{\mu \times m(B_r(x, t))}$$

and

$$\sigma^+_\mu (x, t) = \lim_{r \to 0} \frac{\log \Sigma^+_\mu x(t)}{\log r}, \quad \sigma^-_\mu (x, t) = \lim_{r \to 0} \frac{\log \Sigma^-_\mu x(t)}{\log r}.$$
Typically, the invariant graph $u$ can be written in the form

$$u(x) = \sum_{n=0}^{\infty} \lambda^{n+1} f(T^n x)$$

where $\lambda^{n+1}(x) := \lambda(x)\lambda(Tx)\cdots\lambda(T^n x)$ and $\lambda^0(x) := 1$. As a particular example, take $T(x) = bx \mod 1$ (where $b \in \mathbb{N}, b \geq 2$), $\lambda(x) = \lambda \in (0,1)$, $\lambda b > 1$, $f(x) = \cos 2\pi x$ then $u(x) = \sum_{n=0}^{\infty} \lambda^{n+1} \cos 2\pi b^n x$, the classical Weierstrass function. (Note that, as $\lambda < 1$, this function is continuous and so graph($u$) divides $X \times \mathbb{R}$ into two open sets and neither basin is riddled with the other.) For this reason, we call functions of the form (5) dynamically defined Weierstrass functions.

Assuming that $\lambda$ has a negative Lyapunov exponent for an appropriate measure $\mu$, Stark [SU] (cf. also [HNW]) proved that an invariant graph $u$ exists and is given by (5) $\mu$-a.e. Generically, $u$ is not continuous and is only measurable. However graph($u$) still divides (mod 0) $X \times \mathbb{R}$ into two basins corresponding to attractors at $+\infty$ and $-\infty$.

The hypotheses we impose on the base dynamics and on the skew product are as follows.

(H1) The base dynamical system $T : X \to X$ is a $C^{1+\alpha}$ conformal expanding map or a uniformly expanding $C^{1+\alpha}$ Markov map of the interval. We assume that $T$ is topologically mixing.

(H2) The function $f : X \to \mathbb{R}$ is $\alpha$-Hölder continuous and $f > 0$.

(H3) The function $\lambda : X \to \mathbb{R}$ is $\alpha$-Hölder continuous and $\lambda > 0$. Moreover, there exists an equilibrium state $\mu$ corresponding to a Hölder continuous potential $\phi$ such that $\int \log \lambda \, d\mu < 0$ and a $T$-invariant probability measure $\zeta$ such that $\int \log \lambda \, d\zeta > 0$. We assume without loss of generality that $\phi$ is normalised so that the pressure $P(\phi) = 0$. (Equilibrium states and pressure are defined in §2.2)

(H4) The skew product is partially hyperbolic: $m(\lambda m(|T'|)^{\alpha} \geq \kappa^{-1} > 1$. (Here $m(h) = \inf_{x \in X} h(x)$.)

The invariant measure $\zeta$ need not be an equilibrium state; indeed, $\zeta$ could be a Dirac point mass at a fixed point for $T$.

We shall show that, under (H1)–(H4), an invariant graph $u$ exists $\mu$-a.e., and we calculate the stability index at $\mu$-a.e. point. We also calculate the multifractal spectrum of the stability index.

As a specific example that satisfies (H1)–(H4), take $T(x) = 2x \mod 1$ to be the doubling map. Let

$$f(x) = \frac{2 + \sin 2\pi x}{5}, \quad \lambda(x) = \frac{4}{5} + \frac{\cos 2\pi x}{4}.$$

Take $\mu$ to be Lebesgue measure and $\zeta$ to be the Dirac point mass at 0. Then $\int \log \lambda \, d\mu < -0.24$ and $\int \log \lambda \, d\zeta = \log 21/20 > 0$. Hypothesis (H4) is satisfied as $m(\lambda m(|T'|) = 1.1$. Figure 1 illustrates the structure of the invariant graph $u$ and the basins.

In the example illustrated in Figure 1 it appears that the proportion of $X \times \mathbb{R}$ occupied by $\mathbb{B}^-$ decreases as we move towards $+\infty$. More specifically, for a fixed $t > 0$, consider the horizontal section $X(t) = \{ x \in X \mid (x, t) \in \mathbb{B}^- \}$. We prove that under (H1)–(H4), $\mu(X(t))$ decays polynomially fast as $t \to \infty$.

**Theorem 1.1**

Assume that (H1)–(H3) hold and recall that $\phi$, the potential for the equilibrium state $\phi$, is normalised so that the pressure $P(\phi) = 0$. Then $\lim_{t \to \infty} -\log \mu(X(t))/\log t = s^*$ where $s^* > 0$ is the unique positive solution to the pressure equation $P(\phi + s \log \lambda) = 0$. 

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We will also see that $s^*$ is related to the $L^p(\mu)$ class of the invariant graph $u$ (Lemma 4.3). We call the constant $s^*$ the Loynes exponent (cf. [L]).

Our main result is the following.

**Theorem 1.2**

Assume that (H1)–(H4) hold.

(i) The basin $B^+$ is riddled with $B^-$ and for $\mu$-a.e. $x \in X$ and all $t > u(x)$ we have

$$\sigma_\mu(x,t) = -\sigma^-_\mu(x,t) = \frac{s^* \int \log \lambda \, d\mu}{\int \log |T'| \, d\mu} < 0$$

where $s^*$ is as in Theorem 1.1.

(ii) The basin $B^-$ is not riddled with with $B^+$. Indeed, for $\mu$-a.e. $x \in X$ and all $t < u(x)$, there exists $r_0 > 0$ such that for all $0 < r < r_0$

$$\mu \times m(B_r(x,t) \cap B^-) = \mu \times m(B_r(x,t))$$

so that $\sigma_\mu(x,t) = \infty$.

(iii) For $\mu$-a.e. $x$, we have $\sigma_\mu(x,u(x)) = 0$.

In particular (and in contrast to the class of skew-products considered in [K2]), under (H1)–(H4), $B^+$ is not intermingled with $B^-$. 

Note that, in the case that $m$ denotes Lebesgue measure, (1) is (in the limit as $r \to 0$) the density of $B^-$ at $x$, by the Lebesgue Density Theorem. Thus the stability index is a form of local dimension, and this motivates many of the arguments.

We give a multifractal analysis of the Hausdorff dimension of the level sets of the stability index. Define

$$K_\mu(\sigma) = \{ x \in X \mid \sigma_\mu(x,t) = -\sigma \text{ for all } t > u(x) \}$$

to be the level sets of the stability index $\sigma_\mu(x,t)$. We are interested in the Hausdorff dimension of $K_\mu(\sigma)$. We first state a special case when $X = [0,1]$ and $\mu$ is the SRB measure.

**Proposition 1.3**

Assume (H1)–(H4) and assume that $T$ is a mixing uniformly expanding Markov map of the interval. Let $\mu$ be the SRB measure for $T$. Define $S(q)$ by $P(-S(q) \log |T'| + q s^* \log \lambda) = 0$
(here $s^*$ is as in Theorem 1.1). Let $\mu_q$ denote the equilibrium state with potential $-S(q)\log |T'| + qs^*\log \lambda$ and let $\sigma(q) = -s^* \int \log \lambda \, d\mu_q / \int \log |T'| \, d\mu_q$. Then $\mu_0 = \mu$, $S(0) = 1$, $S(1) = 1$ and $S(q)$ is a real analytic, strictly convex function.

(i) We have that
\[
\dim_H \left\{ x \in X \mid \sigma_\mu(x,t) = \frac{s^* \int \log \lambda \, d\mu}{\int \log |T'| \, d\mu} \text{ for all } t > u(x) \right\} = 1.
\]

(ii) There exists a unique $q^* \in (0, 1)$ such that $\int \log \lambda \, d\mu_{q^*} = 0$.

(iii) The functions $\sigma \mapsto \dim_H K_\mu(\sigma)$ and $q \mapsto S(q)$ form a Legendre transform pair. In particular, $\dim_H K_\mu(\sigma(q)) = S(q) - q\sigma(q)$ for $q \in (-\infty, q^*)$.

More generally we have

**Theorem 1.4**

Assume (H1)–(H4). Define $S(q)$ by $P(-S(q)\log |T'| + qs^*\log \lambda) = 0$ (here $s^*$ is as in Theorem 1.1), let $\mu_q$ denote the equilibrium state with potential $-S(q)\log |T'| + qs^*\log \lambda$ and let $\sigma(q) = -s^* \int \log \lambda \, d\mu_q / \int \log |T'| \, d\mu_q$. Then $S(0) = \dim_H X$ and $S(q)$ is a real analytic, strictly convex function.

(i) There exists a unique $q^* \in \mathbb{R}$ such that $\int \log \lambda \, d\mu_{q^*} = 0$.

(ii) The functions $\sigma \mapsto \dim_H K_\mu(\sigma)$ and $q \mapsto S(q)$ form a Legendre transform pair. In particular, $\dim_H K_\mu(\sigma(q)) = S(q) - q\sigma(q)$ for $q \in (-\infty, q^*)$.

A similar result to Proposition 1.3 was obtained in [SH]. Here, a skew-product acting on $[0,1] \times [0,2]$ similar to that in [MO] (with $T$ a skewed tent map equipped with Lebesgue measure and a piecewise linear skewing function) was considered. The set $[0,1] \times \{0\}$ is a Milnor attractor, and the multifractal structure of the stability index for points of the form $(x,0)$ is calculated and is of the form in Figure 2.

**§2 Preliminary definitions and results**

**§2.1 Conformal expanding maps**

Let $M$ be a smooth Riemannian manifold. Let $T : M \to M$ be $C^{1+\alpha}$ and conformal. Suppose that $X \subset M$ is a compact $T$-invariant set such that

(i) there exists $C > 0$ and $\theta \in (0,1)$ such that $\|d_xT^n(v)\| \geq C\theta^{-n}\|v\|$ for all $x \in X$, all $v \in T_xM$ and all $n \geq 0$;

(ii) there exists an open neighbourhood $U$ of $X$ such that $X = \{x \in U \mid T^n(x) \in U \text{ for all } n \geq 0\}$.

We consider $T : X \to X$ and we assume (without loss of generality) that $T$ is mixing. As $T$ is conformal, we write $d_xT(x) = T'(x)O(x)$ where $O(x)$ is orthogonal and $T' : X \to \mathbb{R}$. We also use an adapted Riemannian metric and take, without loss of generality, $C = 1$ in (i). Note that there exists $r_0 > 0$ such that, for any $x \in X$, $T$ restricted to $B_{4r_0}(x)$ has well-defined inverses.

We can also consider the case when $X = [0,1]$ and $T : X \to X$ is a topologically mixing uniformly expanding Markov map. In this case, there is a partition $\{t_i \mid 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1\}$ such that, with $I_j = (t_j,t_{j+1})$, we have that if $T(I_i) \cap I_j \neq \emptyset$ then $T(I_i) \supset I_j$; (ii) for each $j$, $T|_{I_j}$ is a $C^{1+\alpha}$ diffeomorphism onto its image and $|T'(x)| \geq \theta^{-1} > 1$ for some $\theta \in (0,1)$. Note that if $x \neq t_j$ then there exists $r_0 = r_0(x) > 0$ such that $T$ restricted to $B_{4r_0}(x)$ has well-defined inverses.
A cover $\mathcal{R} = \{R_1, \ldots, R_k\}$ of $X$ by closed subsets of is said to be a Markov partition if (i) each $R_j$ is the closure of its interior, (ii) $\text{int} R_i \cap \text{int} R_j = \emptyset$ for $i \neq j$, and (iii) for each $j$, $T(R_j)$ is the union of sets in $\mathcal{R}$. It is well-known that $T$ possesses Markov partitions with arbitrarily small diameters. When we calculate the stability index at a point $(x, t)$, we choose $r_0$ as above and then choose a Markov partition $\mathcal{R}$ to have diameter smaller than $r_0$.

We write $S_nh(x) := \sum_{j=0}^{n-1} h(T^j x)$. The following estimate is well-known.

**Lemma 2.1**

Let $r_0$ be such that $T$ restricted to $B_{r_0}(z)$ has well-defined inverse branches. Let $n > 0$ and let $\tau$ be any branch of $T^{-n}$. Then for all $x, y, z \in \tau(B_{r_0}(z))$ we have $d(T^j x, T^j y) \leq \theta^{n-j}$ for $0 \leq j \leq n-1$. Moreover, let $h : X \to \mathbb{R}$ be Hölder continuous. Then there exists $C_h > 0$ such that, whenever $x, y, z \in \tau(B_{r_0}(z))$ then $|S_nh(x) - S_nh(y)| \leq C_h$.

In particular, applying Lemma 2.1 to log $\lambda$ and log $|T'|$ (and writing $C_\lambda$, $C_T$ in place of $C_{\log \lambda}$, $C_{\log |T'|}$, respectively), we have that for all $n$ and all $x, y, z \in \tau(B_{r_0}(z))$

$$C_\lambda^{-1} \leq \frac{\lambda^n(x)}{\lambda^n(y)} \leq C_\lambda, \quad C_T^{-1} \leq \prod_{j=0}^{n-1} \frac{|T'(T^j x)|^{-1}}{|T'(T^j y)|^{-1}} \leq C_T. \quad (6)$$

We let $[i_0, \ldots, i_n] := \{x \in X \mid T^j x \in I_j \text{ for } j = 0, 1, \ldots, n\}$ and define this to be a cylinder of rank $n$. If $x \notin \bigcup_{n=0}^{\infty} T^{-n} \partial \mathcal{R}$ then we write $A_n(x)$ to be the unique cylinder of rank $n$ that contains $x$.

We recap the definition of a Moran cover [PW]. Given $r > 0$ define $n_r(x)$ to be the unique integer such that

$$\prod_{j=0}^{n_r(x)} |T'(T^j x)|^{-1} - r \leq \prod_{j=0}^{n_r(x)-1} |T'(T^j x)|^{-1}. \quad (7)$$

Fix $x$ and consider the cylinder set $A_{n_r(x)}(x)$. Then $x \in A_{n_r(x)}(x)$. If $y \in A_{n_r(x)}(x)$ and $n_r(y) \leq n_r(x)$ then $A_{n_r(x)}(x) \subset A_{n_r(y)}(x)$. Let $A_{(r)}(x)$ denote the largest (in diameter) cylinder such that $x \in A_{(r)}(x)$ and $A_{(r)}(x) = A_{n_r(y)}(x)$ for some $y \in A_{n_r(x)}(x)$ and $A_{n_r(x)}(x) \subset A_{(r)}(x)$ for all $z \in A_{(r)}(x)$. The sets $A_{(r)}(x)$ (as $x$ varies) either coincide or are disjoint except at their endpoints. They form a partition $\mathcal{U}_r$ of $X$ which we call a Moran cover. We enumerate the sets in $\mathcal{U}_r$ as $\{A_{n_r(x_i)}(x_i) \mid 1 \leq i \leq \ell_r\}$. We note that it follows from (7) that

$$r \|T'\|^{-1}_\infty \leq \prod_{j=0}^{n_r(x)} |T'(T^j x)|^{-1} \quad (8)$$

and that $r \|T'\|^{-1}_\infty \leq \text{diam} A_{n_r(x_i)}(x_i) < r$ for $1 \leq i \leq \ell_r$.

A Moran cover forms the most efficient cover of $X$ by cylinders of diameter no greater than $r$. An important property of Moran covers is the following. There exists $M > 0$ such that, for any $x \in X$ and $r > 0$ sufficiently small, the number of sets in $\mathcal{U}_r$ that have non-empty intersection with $B_r(x)$ is bounded above by $M$. We call $M$ the Moran multiplicity factor.

In general, $\sup \{n_r(x_i) - n_r(x_j) \mid 1 \leq i, j \leq \ell_r\}$ tends to infinity as $r \to 0$. Below, we will only consider elements of $\mathcal{U}_r$ that cover a ball $B_r(x)$. With this restriction, we have the following lemma.

**Lemma 2.2**

Fix $x \in X$ and choose a Markov partition as above. Choose the elements of $\mathcal{U}_r$ that have non-empty intersection with $B_r(x)$, with labelling chosen so that

$$B_r(x) \subset \bigcup_{i=1}^{M} A_{n_r(x_i)}(x_i),$$

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where $M$ is the Moran multiplicity factor. Moreover, there exists $L$, independent of $r$, such that $|n_r(x_i) - n_r(x_k)| \leq L$ for all $1 \leq i, k \leq M$.

**Proof.** The existence of the Moran multiplicity factor is derived in [P, §20].

Choose $k$ such that $n_r(x_k) \leq n_r(x_i)$ for $1 \leq i \leq M$. By (6), (7) and (8) we have

$$(C_T \|T'\|_\infty)^{-1} \leq \frac{\prod_{j=0}^{n_r(x_j)} |T'(T^j x_i)|^{-1}}{\prod_{j=0}^{n_r(x_k)} |T'(T^j x_k)|^{-1}} \leq \frac{\prod_{j=0}^{n_r(x_i)} |T'(T^j x_i)|^{-1}}{\prod_{j=0}^{n_r(x_k)} |T'(T^j x_k)|^{-1}} = \frac{\prod_{j=n_r(x_k)+1}^{n_r(x_i)} |T'(T^j x_i)|^{-1}}{\prod_{j=n_r(x_k)+1}^{n_r(x_i)} |T'(T^j x_k)|^{-1}} \leq C_T \|T'\|_\infty.$$  

Hence $(C_T \|T'\|_\infty)^{-1} \leq \|T'\|_\infty^{n_r(x_k) - n_r(x_i)}$. It follows that $n_r(x_i) - n_r(x_k) \leq \log(C_T \|T'\|_\infty)^{-1} / \log \|T'\|_\infty$.

**§2.2 Thermodynamic formalism**

Let $\phi : X \to \mathbb{R}$ be continuous. The *pressure* of $\phi$, $P(\phi)$, is defined to be

$$P(\phi) := \sup \left\{ h_\nu(T) + \int \phi \, d\nu \mid \nu \text{ is a } T\text{-invariant probability measure} \right\} \quad (9)$$

where $h_\nu(T)$ is the entropy of $T$ with respect to $\nu$.

If $\phi$ is H"older continuous then there exists a unique measure $\mu = \mu_\phi$, the *equilibrium state with potential* $\phi$, that achieves this supremum. The measure $\mu$ satisfies the *Gibbs property*, namely that there exists $C_\mu > 0$ such that for all $x \in X$ and all $n \in \mathbb{N}$

$$\frac{1}{C_\mu} \leq \frac{\mu(A_n(x))}{\exp(\langle S_n \phi(x) - nP(\phi) \rangle)} \leq C_\mu. \quad (10)$$

By replacing $\phi$ by $\phi - P(\phi)$ there is no loss in assuming that $P(\phi) = 0$ (it is clear that $\mu_\phi - P(\phi) = \mu_\phi$); we say that $\phi$ is *normalised* if $P(\phi) = 0$.

We need the following distortion bound for measures that satisfy the Gibbs property. Note that for any cylinder $A_n(x)$ of rank $n$, the restriction $T|_{A_n(x)} : A_n(x) \to X$ is a bijection.

**Lemma 2.3**

There exists $D > 1$ such that the following property holds. Suppose $B \subset B_r(x)$ is a Borel subset and $r < r_0$. Let $A_{n_r(x)}(x_j)$, $1 \leq j \leq M$, be a Moran cover for $B_r(x)$ and suppose that the indexing is chosen so that $A_{n_r(x)}(x_1) \subset B_r(x)$. Then

$$D^{-1} \mu(T^{n_r(x_1)} B) \leq \frac{\mu(B)}{\mu\left(\bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j)\right)} \leq D \mu(T^{n_r(x_1)} B). \quad (11)$$

**Proof.** As cylinders in a Moran cover overlap only on their boundaries (which have zero $\mu$-measure), we have that $\mu(\bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j)) = \sum_{j=1}^{M} \mu(A_{n_r(x_j)}(x_j))$. We also note that as $r < r_0$ then $\bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \subset B_{4r_0}(x)$ and we can apply Lemma 2.1.

Let $1 \leq i, j \leq M$ and suppose that $n_r(x_i) < n_r(x_j)$. By Lemma 2.2 we can write $n_r(x_j) = n_r(x_i) + k$ with $k \leq L$, where $L$ is independent of $r$. As $\phi$ is normalised we have that $\phi(x) < 0$; indeed, $-\|\phi\|_\infty \leq \phi(y) \leq -m(\phi)$ for all $y \in X$. By (10) we have

$$C_\mu^{-1} e^{-L\|\phi\|_\infty} e^{S_{n_r(x_j)} \phi(x_j)} \leq C_\mu^{-1} e^{S_{n_r(x_j)} \phi(x_j)} \leq \mu(A_{n_r(x_j)}(x_j)) \leq C_\mu e^{S_{n_r(x_j)} \phi(x_j)} \leq C_\mu e^{S_{n_r(x_j)} \phi(x_j)}.$$  

We also note from Lemma 2.1 that there exists a constant $C_\phi > 0$ such that

$$C_\phi^{-1} \leq \frac{e^{S_{n_r(x_j)} \phi(x_j)}}{e^{S_{n_r(x_j)} \phi(x)}} \leq C_\phi. \quad (13)$$
From (12), (13) it follows that there exists a constant $C_{\phi,\mu}$ independent of $r$ such that for all $1 \leq j \leq M$ we have

$$C_{\phi,\mu}^{-1}e^{S_{n_r(x_j)}\phi(x)} \leq \mu(A_{n_r(x_j)}(x_j)) \leq C_{\phi,\mu}e^{S_{n_r(x_j)}\phi(x)}.$$ 

Hence

$$MC_{\phi,\mu}^{-1}e^{S_{n_r(x_j)}\phi(x)} \leq \mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right) \leq MC_{\phi,\mu}e^{S_{n_r(x_j)}\phi(x)}.$$ 

We first observe that (11) holds when $I \subset B_r(x)$ is a cylinder of sufficiently large rank. To see this, let $I = A_{n_r(x_1)}(y) \subset B_r(x)$. By (10) we have that $\mu(I) \leq C_{\mu} \exp S_{n_r(x_1)}\phi(y) = C_{\mu} \exp S_{n_r(x_1)}\phi(y) \exp S_{\phi}(T^{n_r(x_1)}y)$. Now $T^{n_r(x_1)}I$ is a cylinder of rank $p$ containing $T^{n_r(x_1)}y$, hence $C_{\mu}^{-1}\exp S_{\phi}(T^{n_r(x_1)}y) \leq \mu(T^{n_r(x_1)}I) \leq C_{\mu}\exp S_{\phi}(T^{n_r(x_1)}y)$. By Lemma 2.1 and (10), for an appropriate constant $C_{\phi} > 0$, we have that

$$\mu(I) \leq C_{\mu}^2 e^{S_{n_r(x_1)}\phi(y)} \exp S_{\phi}(T^{n_r(x_1)}y) \leq C_{\phi}^2 C_{\phi} e^{S_{n_r(x_1)}\phi(x)} \mu(T^{n_r(x_1)}I) \leq M^{-1} C_{\phi,\mu} C_{\mu}^2 C_{\phi} \mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right) \mu(T^{n_r(x_1)}I).$$

The lower bound follows similarly.

Now let $I, J \subset B_r(x)$ be disjoint cylinders of rank at least $n_r(x_1)$. Then $T^{n_r(x_1)}I$ and $T^{n_r(x_1)}J$ are disjoint cylinders. It is straightforward to check that (11) holds when $B = I \cup J$.

Now let $B \subset B_r(x)$ be a Borel subset. Let $\varepsilon > 0$. Choose a finite union of cylinders $C$ of rank at least $n_r(x_1)$ such that $\mu(C \Delta T^{n_r(x_1)}B) < \varepsilon$. Let $C' \subset B_r(x)$ be such that $T^{n_r(x_1)}C' = C$; note that $C'$ is a finite union of cylinders of rank at least $n_r(x_1)$ and that $\mu(C' \Delta B) < \varepsilon$. Then

$$\frac{\mu(B)}{\mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right)} \leq \frac{\mu(C)}{\mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right)} + \varepsilon \leq D\mu(T^{n_r(x_1)}C) + \varepsilon \frac{\mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right)}{\mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right)} \leq D\mu(T^{n_r(x_1)}B) + \left( D + \frac{1}{\mu \left( \bigcup_{j=1}^{M} A_{n_r(x_j)}(x_j) \right)} \right) \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, the right-hand side of (11) holds for $B$. The left-hand side follows similarly. \hfill \square

Given $s > 0$ we define the transfer operator $L_s$ by

$$L_s w(x) = \sum_{y: T^y x = x} e^{\phi(y)} e^{s \log \lambda(y)} w(y).$$

(14)

Define $p(s) := P(\phi + s \log \lambda)$. It is well-known that $L_s$ has spectral radius $e^{p(s)}$. As $\phi$ is normalised we have $p(0) = 0$. After possibly adding a coboundary to $\phi$, we can assume that $L_s 1 = 1$, where $1$ denotes the constant function. Note that, as $\phi$ is normalised, we must have that $\phi(x) < 0$ for all $x \in X$.

It is well-known that there is a Banach space $B$ of functions, which contain the constants, such that $L_s : B \rightarrow B$ has $e^{p(s)}$ as a simple maximal eigenvalue and the remainder of the spectrum is contained inside a disc of radius $\gamma_s < e^{p(s)}$. In particular, we can write $L_s^n = e^{np(s)} \pi_s + O(\gamma_s^n)$ where $\pi_s$ is a projection operator.
§2.3 Stability index

It is clear from (3) that \( \sigma^\pm_\mu(x,t) \geq 0 \), including the possibility that it is infinite. We make the following remark.

**Lemma 2.4**

Suppose that \( \sigma^+_\mu(x,t) \) exists and \( \sigma^-_\mu(x,t) > 0 \) or is infinite. Then \( \sigma^+_\mu(x,t) \) exists and \( \sigma^-_\mu(x,t) = 0 \).

**Proof.** Note that \( \log r < 0 \) if \( 0 < r < 1 \). Suppose first that \( \sigma^+_\mu(x,t) = \delta > 0 \). Then, provided \( r > 0 \) is sufficiently small, we have that \( \Sigma^+_{\mu,r}(x,t) > \varepsilon/2 \). As \( \mu \times m(\text{graph}(u)) = 0 \), we have that \( \Sigma^-_{\mu,r}(x,t) = 1 - \Sigma^+_{\mu,r}(x,t) > 1 - r\delta/2 \). Hence

\[
\frac{1}{\log r} \log \Sigma^-_{\mu,r}(x,t) < \frac{\log(1 - r\delta/2)}{\log r}.
\]

Letting \( r \to 0 \), the claim follows. A similar argument holds when \( \sigma^+_\mu(x,t) = \infty \). \( \square \)

§3 Structure of the invariant graph

We define (formally) \( u(x) \) by

\[
u = \sum_{n=0}^{\infty} \lambda^{n+1}(x) f(T^n x).
\]

Let \( X_u \) denote the set of points \( x \in X \) for which there exists \( C(x) > 0 \eta > 0 \) and \( N(x) \in \mathbb{N} \) such that \( \lambda^n(x) < C(x)e^{-\eta n} \) for all \( n \geq N(x) \). We will only consider \( u \) to be defined on \( X_u \) (but see Proposition 3.1).

We define \( g^n_x(t) \) by iterating (2), specifically \( T^n(x,t) = \left(T^n x, g^n_x(t)\right) \). It is straightforward to see that

\[
g^n_x(t) = -\sum_{j=0}^{n-1} \lambda^{n-1-j}(x)^{-1} f(T^n x) + \lambda^n(x)^{-1} t = \lambda^n(x)^{-1} \left( -\sum_{j=0}^{n-1} \lambda^j(x) f(T^n x) + t \right).
\]

We introduce the notation

\[
S_{n,\lambda}(x) := \sum_{j=0}^{n-1} \lambda^{j+1}(x) f(T^n x)
\]

so that \( T^n(x,t) = \left(T^n x, \lambda^n(x)^{-1} (-S_{n,\lambda}(x) + t)\right) \).

**Proposition 3.1**

Let \( \nu \) be an ergodic invariant measure such that \( \int \log \lambda \, d\nu < 0 \). Then \( \nu(X_u) = 1 \). Moreover, \( u \) is \( \nu \)-measurable, \( \text{graph}(u) = \{(x,u(x)) \mid x \in X_u\} \) is \( \hat{T} \)-invariant, and \( u \) is unique in the sense that if \( v \) is a \( \nu \)-measurable function with a \( \hat{T} \)-invariant graph then \( v = u \) \( \nu \)-a.e.

**Proof.** That \( \nu(X_u) = 1 \) follows immediately from Birkhoff’s Ergodic Theorem.

Let \( u_n(x) = \sum_{j=0}^{n-1} \lambda^{j+1}(x) f(T^n x) \). If \( x \in X_u \) then there exists \( \eta > 0 \) such that for all sufficiently large \( n \) we have \( \lambda^n(x) < C(x)e^{-\eta n} \). Then \( |S_{n+1,\lambda}(x) - S_{n,\lambda}(x)| = \lambda^{n+1}(x) f(T^n x) \leq \|f\|_{\infty} C(x)e^{-\eta(n+1)} \) and it follows that \( u_n \) is Cauchy, and so converges.

That \( X_u \) is \( T \)-invariant and \( \text{graph}(u) \) is \( \hat{T} \)-invariant are straightforward calculations.

To prove uniqueness, suppose that \( v \) is \( \nu \)-measurable and has a \( \hat{T} \)-invariant graph. Then \( v(x) - u(x) = \lambda^n(x)(v(T^n x) - u(T^n x)) \). As \( v - u \) is measurable, there exists a constant \( C_1 > 0 \) and a set \( V \) of positive \( \nu \)-measure such that \( (v - u)(x) < C_1 \) for all \( x \in V \). By ergodicity, for \( \nu \)-a.e. \( x \in V \) there is a subsequence such that \( T^n x \in V \) as \( \lambda^n(x) \to 0 \) \( \nu \)-a.e. it follows that \( u(x) = v(x) \) \( \nu \)-a.e. \( \square \)
Under the hypotheses of Proposition \[3.1\] it follows from [HNW] that \( u \) is continuous if and only if there exists a continuous function \( r \) such that \( f(x) = r(Tx) - \lambda(x)^{-1}r(x) \) and that generically this does not happen. We shall see below in Corollary \[4.3\] that, under hypotheses (H1)-(H3), the function \( u \) is never continuous.

We now prove that the graph of \( u \) determines the boundary between the two basins.

**Proposition 3.2**

Suppose \( x \in X_u \) so that \( u(x) \) exists. Then \((x,t) \in \mathbb{B}^+\) if and only if \( t > u(x) \) and \((x,t) \in \mathbb{B}^-\) if and only if \( t < u(x) \).

**Proof.** Recall that \( f, \lambda > 0 \) so that \( u(x) > 0 \). Suppose \( t < u(x) \) and write \( t = u(x) - \delta_x(t) \). Provided that \( n \) is sufficiently large we have \( 0 < \sum_{j=n+1}^{\infty} \lambda^{j+1}(x)f(T^j x) < \delta_x(t)/2 \). Hence

\[
g_x^n(t) = \frac{\lambda^n(x)^{-1}(-S_n,\lambda f(x) + u(x) - \delta_x(t))}{\sum_{j=n+1}^{\infty} \lambda^{j+1}(x)f(T^j x) - \delta_x(t)} \leq -\frac{\delta_x(t)}{2} \lambda^n(x)^{-1}
\]

so that \((x,t) \in \mathbb{B}^-\), noting that \( \lambda^n(x)^{-1} \to \infty \) as \( n \to \infty \) by the definition of \( X_u \). The argument for \( t > u(x) \) is analogous. \( \Box \)

§4 A thermodynamic Loynes exponent

For \( s \geq 0 \) recall that \( p(s) = P(\phi + s \log \lambda) \) where \( P \) denotes the topological pressure. It is well-known that \( p(s) \) is a convex analytic function of \( s \).

**Lemma 4.1**

Assume (H1)–(H3). Then there exists a unique \( s^* > 0 \) such that \( p(s^*) = 0 \). Moreover, \( p'(s^*) > 0 \) and \( p'(s) \) is strictly increasing on an open interval \((\underline{s}, \bar{s})\) that contains \( s^* \).

**Proof.** Recall from [11] that if \( \phi \) is Hölder continuous, \( P(\phi) = 0 \) and has equilibrium state \( \mu \) and \( \psi \) is Hölder continuous then \( \partial P(\phi + t\psi)/\partial t|_{t=0} = \int \psi d\mu \). Moreover \( \partial^2 P(\phi + t\psi)/\partial t^2|_{t=0} \geq 0 \) with equality if and only if \( \psi \) is cohomologous to a constant.

First note that \( p(0) = 0 \) as \( \phi \) is normalised. By the above we have that \( p'(0) = \int \log \lambda d\mu < 0 \). As \( \int \log \lambda d\zeta > 0 \), we see that \( \log \lambda \) cannot be cohomologous to a constant. Hence \( p(s) \) is strictly convex.

By the variational principle [9], \( p(s) = \sup \{h_\nu(T) + \int \phi d\nu + s \int \log \lambda d\nu \} \) where the supremum is taken over all \( T \)-invariant probability measures \( \nu \). Hence \( p(s) \geq h_\xi(T) + \int \phi d\zeta + s \int \log \lambda d\zeta \). It follows that \( p(s) \to \infty \) as \( s \to \infty \) as \( \int \log \lambda d\zeta \).

As \( p(s) \) is analytic and convex, it follows that there is a unique \( s^* > 0 \) such that \( p(s^*) = 0 \). Moreover, \( p'(s^*) > 0 \). Hence there is an interval \((\underline{s}, \bar{s})\) containing \( s^* \) on which \( p'(s) > 0 \). As \( p \) is convex, \( p' \) is non-decreasing. To see that \( p' \) is strictly increasing on \((\underline{s}, \bar{s})\), suppose for a contradiction that \( p'(s) = 0 \) on a subinterval of \((\underline{s}, \bar{s})\). Then \( p'(s) = 0 \) for all \( s \), by analytic continuation, implying that \( p'(s) \) is constant for all \( s \); this contradicts \( p'(0) < 0, p'(s^*) > 0 \).

The goal of this section is to prove the following result.

**Proposition 4.2**

Assume that (H1)–(H3) hold. Let \( u \) be the \( \mu \)-a.e. defined invariant graph for \( T \). Let \( s^* > 0 \) be the unique positive solution to \( p(s) = 0 \). Then

\[
\lim_{M \to \infty} \frac{-\log \mu \{x \in X \mid u(x) > M\}}{\log M} = s^*.
\]
Before proving Proposition 4.2, we relate the constant $s^*$ to the regularity of the invariant graph and also prove Theorem 1.1.

**Corollary 4.3**
Assume that (H1)–(H3) hold. Let $u$ be the $\mu$-a.e. defined invariant graph for $\hat{T}$. Then $u \in L^p(\mu)$ if $p < s^*$ and $u \not\in L^p(\mu)$ if $p > s^*$.

**Proof.** Recall that $u > 0$ $\mu$-a.e. Let $U_n := \{x \in X \mid u(x)^p > n\} = \{x \in X \mid u(x) > n^{1/p}\}$. Note that

$$\bigcup_{n=0}^{\infty} U_{n+1} \times [n, n+1] \subset \{(x, t) \in X \times \mathbb{R} \mid 0 \leq t < u(x)\} \subset \bigcup_{n=0}^{\infty} U_n \times [n, n+1].$$

Let $\varepsilon > 0$. Provided $n$ is sufficiently large implies that $n^{-(s^*+\varepsilon)/p} \leq \mu(U_n) \leq n^{-(s^*-\varepsilon)/p}$. Hence $\int u \, d\mu = \mu \times m\{(x, t) \in X \times \mathbb{R} \mid 0 \leq t < u(x)\} < \infty$ if $\sum_{n=0}^{\infty} n^{-(s^*-\varepsilon)/p} < \infty$. Hence $u \in L^p(\mu)$ if $p < s^*$. Similarly, $u \not\in L^p(\mu)$ if $p > s^*$.

Hence if $s^* < 1$ then $u$ will not be integrable; however, $u$ is always log-integrable.

**Corollary 4.4**
Assume that (H1)–(H3) hold. Then $\log^+ u := \max\{0, \log u\} \in L^1(\mu)$.

**Proof.** Let $V_n := \{x \in X \mid \log u(x) > n\} = \{x \in X \mid u(x) > e^n\}$. By Proposition 4.2, $\mu(V_n) < e^{-ns^*/2}$ provided $n$ is sufficiently large. Note that

$$\text{graph}(\log^+ u) \subset \bigcup_{n=0}^{\infty} V_n \times [n, n+1]$$

and that $\log^+ u$ is positive. Noting that $\sum_{n=0}^{\infty} \mu(V_n) < \infty$ as $\sum_{n=0}^{\infty} e^{-ns^*/2} < \infty$, the claim follows.

**Proof of Theorem 1.1.** By Proposition 4.2, we can write $X(t) = \{x \in X \mid u(x) > t\}$ and the result follows immediately from Proposition 4.2.

We now prove Proposition 4.2; the arguments follow those in [K1, K2]. We establish the limsup and liminf in [18] separately.

**Lemma 4.5**
Let $\delta \in (0, s^*)$. Then there exists $\delta_0 = \delta_0(s) > 0$ such that if $\delta < (0, \delta_0)$ then there exists a constant $C = C(\delta, s) > 0$ with the following property: for all $M > 0$ and all $n \in \mathbb{N}$ we have

$$\mu\left(\left\{x \in X \mid M^{-1}\lambda^n(x) \geq e^{-2n\delta}\right\}\right) \leq CM^{-s}e^{-ns\delta}.$$

**Proof.** As $s < s^*$, choose $\delta > 0$ such that $p(s) + 4s\delta < 0$. Note that, by the spectral radius theorem, $\int L^s_u \, d\mu \leq Ce^n(p(s)+4s\delta)$ for some constant $C > 0$. Then

$$\mu(\{x \in X \mid M^{-1}\lambda^n(x) \geq e^{-2n\delta}\}) = \mu(\{x \in X \mid M^{-s}e^{\lambda^n \log \lambda(x)} \geq e^{2s\delta n}\}) \leq M^{-s}e^{2s\delta n} \int e^{\lambda^n \log \lambda(x)} \, d\mu = M^{-s}e^{2s\delta n} \int L^s_u \, d\mu = CM^{-s}e^{-s\delta n}.$$

\[\square\]
Proof. Let \( M > 1 \). As \( u(x) = \sum_{n=0}^{\infty} \lambda^{n+1}(x)f(T^n x) \) \( \mu \)-a.e. we have that
\[
\mu (\{ x \in X \mid u(x) > M \}) \leq \mu \left( \left\{ x \in X \mid \sum_{n=0}^{\infty} \lambda^{n+1}(x) > M \| f \|_{\infty}^{-1} \right\} \right) =: \mu (\Delta).
\]
Note that \( M \| f \|_{\infty}^{-1} = M \| f \|_{\infty}^{-1}(1 - e^{-\delta}) \sum_{n=0}^{\infty} e^{-\delta n} \). Let \( \tilde{M} = M \| f \|_{\infty}^{-1}(1 - e^{-\delta}) \). Hence
\[
\Delta = \left\{ x \in X \mid \tilde{M}^{-1} \sum_{n=0}^{\infty} \lambda^{n+1}(x) > \sum_{n=0}^{\infty} e^{-\delta n} \right\}.
\]
If \( x \in \Delta \) then there must exist \( n \geq 0 \) such that \( \tilde{M}^{-1} \lambda^{n+1}(x) > e^{-\delta n} \). From this observation and Lemma 4.5 we have
\[
\mu (\Delta) \leq \sum_{n=0}^{\infty} \mu \left( \left\{ x \in X \mid \tilde{M}^{-1} \lambda^{n+1}(x) > e^{-\delta n} \right\} \right) \leq \sum_{n=0}^{\infty} \tilde{M}^{-s} C e^{-\delta/2} n^{s} \leq C' M^{-s}
\]
on summing the geometric series, for some constant \( C' = C'(s, \delta) > 0 \). Hence \( \mu (\{ x \in X \mid u(x) > M \}) \leq C' M^{-s} \). Taking logs, dividing by \( -\log M \) and taking the liminf as \( M \to \infty \) gives that the left-hand side of (13) is at least \( s \). As this is true for any \( s < s^* \), the result follows.

We now prove the limsup in (18). This makes use of the fact that \( f > 0 \). The following large deviations theorem due to Plachky and Steinebach [PS] is true far more generally and we state it in the setting that we shall use it.

**Theorem 4.7 ([PS])**

Let \((s, \bar{s})\) be an open interval containing \( s^* \) and suppose that, for \( s \in (s, \bar{s}) \), \( p(s) \) is a differentiable function with \( p'(s) \) strictly monotone. Suppose that
\begin{enumerate}[(i)]
  \item \( \int e^{s \log \lambda^n(x)} \, d\mu < \infty \) for all \( s \in [0, \bar{s}] \),
  \item we have
  \[
  \lim_{n \to \infty} \frac{1}{n} \int e^{s \lambda^n(x)} \, d\mu = p(s)
  \]
  for all \( s \in (s, \bar{s}) \).
\end{enumerate}

Then
\[
\lim_{n \to \infty} \frac{1}{n} \log \mu (\{ x \in X \mid \log \lambda^n(x) > np'(s^*) \}) = p(s^*) - s^* p'(s^*).
\]

We check that our setting does indeed satisfy the hypotheses of Theorem 4.7. As \( p(s) \) is convex and not linear, \( p'(s) \) is strictly increasing. Hypothesis (i) of Theorem 4.7 holds trivially as \( \lambda \) is continuous, hence bounded. We need only check the convergence in (20). To see this, simply note that
\[
\lim_{n \to \infty} \frac{1}{n} \log \int e^{s \lambda^n(x)} \, d\mu = \lim_{n \to \infty} \frac{1}{n} \log \int L^n_{\pi_s} 1 \, d\mu = \lim_{n \to \infty} \frac{1}{n} \log \int e^{np(s)} \pi_s 1 + O(\gamma_n) \, d\mu = p(s).
\]

We can now apply Theorem 4.7 to complete the proof of Proposition 4.2.
Lemma 4.8
Assume that (H1)–(H3) hold. Then
\[
\limsup_{M \to \infty} \frac{-\log \mu(\{x \in X \mid u(x) > M\})}{\log M} \leq s^*.
\]

Proof. Recall that both \( f, \lambda \) are positive. First note that, for any \( n \) and any \( 0 \leq m \leq n - 1 \), we have \( u(x) \geq S_{n,\lambda} f(x) \geq \lambda^m(x) m(f) \).

Let \( \alpha = p'(s^*) > 0 \). For each \( M \), choose \( m \) such that
\[
\alpha(m - 1) + \log \alpha + m f \leq \log M < \alpha m + m f.
\]

Then \( \mu(\{x \in X \mid u(x) > M\}) \geq \mu(\{x \in X \mid \lambda^m(x) m(f) > M\}) \geq \mu(\{x \in X \mid \log \lambda^m(x) > \alpha m\}) \).

Hence
\[
\frac{- \log \mu(\{x \in X \mid u(x) > M\})}{\log M} \leq \frac{- \log \mu(\{x \in X \mid \log \lambda^m(x) > \alpha m\})}{\log \alpha(m - 1) + \log m f} = \frac{-1 \log \mu(\{x \in X \mid \log \lambda^m(x) > \alpha m\})}{\alpha m} \leq \frac{m f}{\alpha(m - 1) + \log m f}
\]
and the lemma follows from Theorem 4.7 by letting \( M \), equivalently \( m \), tend to \( \infty \).

§5 Stability index

We are now in a position to prove Theorem 1.2.

§5.1 The upper basin

To calculate the stability index for points in \( B^+ \) we first prove that the exponential fibre-wise growth rate for a.e. point above the graph is given by the Lyapunov exponent of \( \lambda \).

Lemma 5.1
Assume that (H1)–(H3) hold. For \( \mu \)-a.e. \( x \in X \) and for all \( t > u(x) \) we have
\[
\lim_{n \to \infty} \frac{1}{n} \log g_x^n(t) = - \int \log \lambda \, d\mu.
\]

Proof. For \( \mu \)-a.e. \( x \in X \), \( u(x) \) is given by (15). For such an \( x \), let \( \delta_x(t) = t - u(x) > 0 \). Choose \( N \) such that for all \( n \geq N \) we have
\[
0 \leq \sum_{j=n}^{\infty} \lambda^{j+1}(x)f(T^j x) < \delta_x(t).
\]

Hence
\[
g_x^n(t) = \lambda^n(x)^{-1}(S_{n,\lambda} f(x) + t)
\]
\[
= \lambda^n(x)^{-1} \left( \sum_{j=n}^{\infty} \lambda^{j+1}(x)f(T^j x) + \delta_x(t) \right)
\]
so that \( \lambda^n(x)^{-1} \delta_x(t) \leq g_x^n(t) \leq 2 \lambda^n(x)^{-1} \delta_x(t) \). Taking logarithms, dividing by \( n \) and letting \( n \to \infty \) then gives (21).

We require the following lemma.
Lemma 5.2
Let \( x \in X_u \). For \( t > u(x) \) for which \( \sigma^+_{\nu}(x,t) \) exists we have

\[
\liminf_{r \to 0} \frac{1}{\log r} \log \left( \frac{\mu(\{y \in B_r(x) \mid u(y) > t - r\})}{\mu(B_r(x))} \right) \leq \sigma^+_{\nu}(x,t).
\]

\[
\leq \limsup_{r \to 0} \frac{1}{\log r} \log \left( \frac{\mu(\{y \in B_r(x) \mid u(y) > t + r\})}{\mu(B_r(x))} \right).
\]

Proof. First note that

\[
\{y \in B_r(x) \mid u(y) > t + r\} \times [t - r, t + r]
\]

\[
\subset B_r(x,t) \cap \mathbb{B}^-
\]

\[
\subset \{y \in B_r(x) \mid u(y) > t - r\} \times [t - r, t + r].
\]

Noting that \( \mu \times m(B_r(x,t)) = \mu(B_r(x)) \times 2r \) and taking logs we obtain

\[
\log \left( \frac{\mu(\{y \in A_n(x) \mid u(y) > t + r\})}{\mu(B_r(x))} \right) \leq \log \Sigma^+_{\nu}(x,t) \leq \log \left( \frac{\mu(\{y \in B_r(x) \mid u(y) > t - r\})}{\mu(B_r(x))} \right).
\]

Dividing by \( \log r \) (noting that \( \log r < 0 \)) then gives the result.

The following bounded distortion estimate allows us to move between different points in \( A_n(x) \). Note that it is here that we require the partial hyperbolicity assumption (H4).

Lemma 5.3
Assume that (H1)–(H4) hold. Let \( x, y \in A_n(x) \). Then there exists \( C_{f,\lambda} > 0 \), independent of \( x, y, n, \) such that

\[
|S_{n,\lambda} f(x) - S_{n,\lambda} f(y)| \leq C \lambda^n(x).
\]

Proof. We write \( |h|_\alpha := \sup_{x,y} |h(x) - h(y)|/d(x,y)^\alpha \) for the Hölder semi-norm of \( h \).

Note that by Lemma 2.1 we have

\[
|\lambda^{j+1}(x)f(T^j x) - \lambda^{j+1}(y)f(T^j y)| \leq \lambda^{j+1}(x) |f(T^j x) - f(T^j y)| + |f(T^j y)| |\lambda^{j+1}(x) - \lambda^{j+1}(y)|
\]

\[
\leq |f|_\alpha |\theta^{(n-j)}\lambda^{j+1}(x)| + \|f\|_\infty |\lambda^{j+1}(x) - \lambda^{j+1}(y)|.
\]

We can bound

\[
|\lambda^{j+1}(x) - \lambda^{j+1}(y)| \leq \sum_{i=0}^j \lambda^{-i}(T^{i+1} y)|\lambda(T^i x) - \lambda(T^i y)|\lambda^i(x).
\]

(23)

As \( \log \lambda \) is \( \alpha \)-Hölder, we have

\[
|\log \lambda^{-i}(T^{i+1} x) - \log \lambda^{-i}(T^{i+1} y)| \leq \sum_{k=0}^{j-i-1} |\log \lambda|_\alpha d(T^{k+i+1} x, T^{k+i+1} y)^\alpha \leq \sum_{k=0}^\infty |\log \lambda|_\alpha \theta^{ka} = D
\]

where \( D > 0 \) is independent of \( x, y, i, j, n \). Hence

\[
e^{-D} \leq \frac{\lambda^{j-i}(T^{i+1} x)}{\lambda^{j-i}(T^{i+1} y)} \leq e^D.
\]

As \( \lambda \) is \( \alpha \)-Hölder continuous, we have \( |\lambda(T^i x) - \lambda(T^i y)| \leq |\lambda|_\alpha \theta^{a(n-i)} \). We can then bound

\[
|\lambda^{j+1}(x) - \lambda^{j+1}(y)| \leq e^D |\lambda|_\alpha m(\lambda) \lambda^{j+1}(x) \sum_{i=0}^j \theta^{a(n-i)}.
\]
Noting that \( \sum_{i=0}^{j} \theta^{\alpha(n-i)} \leq (\theta^{-\alpha} - 1)^{-1} \theta^{\alpha(n-j)} \) we have
\[
|\lambda_{i+1}^{j}(x) - \lambda_{i+1}^{j}(y)| \leq C \lambda_{i+1}^{j}(x) \theta^{\alpha(n-j)}
\]
for some constant \( C > 0 \) independent of \( x, y, j, n \).
Hence
\[
|\lambda_{i+1}^{j}(x) f(T^j x) - \lambda_{i+1}^{j}(y) f(T^j y)| \leq \sum_{j=0}^{n-1} C' \theta^{\alpha(n-j)} \lambda_{i+1}^{j}(x)
\]
for some constant \( C' \) independent of \( x, y, n \).
Recall that by (H4) we have \( \sum_{j=0}^{n} \theta^{\alpha(n-j)} \lambda_{i+1}^{j}(x) \leq \sum_{j=0}^{n} \theta^{\alpha(n-j)} \lambda_{i+1}^{j}(x) \leq C \lambda_{i+1}^{j}(x) \theta^{\alpha(n-j)} \)
\[
\sum_{j=0}^{n-1} C' \theta^{\alpha(n-j)} \lambda_{i+1}^{j}(x) \leq C' \lambda_{i+1}^{j}(x) \sum_{j=0}^{n-1} \theta^{\alpha(n-j)} m(\lambda)^{-(n-j-1)} \leq C' \lambda_{i+1}^{j}(x) \sum_{j=0}^{n-j} \lambda_{i+1}^{j}(x) \leq C \lambda_{i+1}^{j}(x)
\]
for some constant \( C_{f,\lambda} \), summing the geometric progression. \( \square \)

We can now obtain the following bounded distortion estimate.

**Lemma 5.4**
Assume that (H1)–(H4) hold. Suppose that \( x \in X_u \) and \( t > u(x) \). Then there exists \( K \geq 1 \), depending on \( x, t \), such that for all sufficiently large \( n \) and all \( y \in A_n(x) \) we have
\[
K^{-1} \leq \frac{g_x^n(t)}{g_x^n(u(x))} \leq K.
\]

**Proof.** First note that as \( u(x) > 0 \) and \( g_x^n(\cdot) \) is orientation preserving, we have \( g_x^n(t) > g_x^n(u(x)) = u(T^n x) > 0 \).
Recall from (16) that
\[
\frac{g_x^n(t)}{g_x^n(u(x))} = \frac{\lambda^n(x)^{-1} (-S_{n,\lambda} f(x) + t)}{\lambda^n(y)^{-1} (-S_{n,\lambda} f(y) + t)}.
\]
By (6), \( C_{\lambda}^{-1} \leq \lambda^n(x)^{-1} / \lambda^n(y)^{-1} \leq C_{\lambda} \).
Let \( t - u(x) = \delta_x(t) > 0 \). Then
\[
-S_{n,\lambda} f(x) + t = -S_{n,\lambda} f(x) + u(x) + \delta_x(t) = \sum_{j=n}^{\infty} \lambda^{j+1} f(T^j x) + \delta_x(t) > \delta_x(t).
\]
Provided \( n \) is sufficiently large we have that \( \sum_{j=n}^{\infty} \lambda^{j+1} f(T^j x) < \delta_x(t) \). Hence \( -S_{n,\lambda} f(x) + t \leq 2\delta_x(t) \).

By Lemma 5.3, we have
\[
-S_{n,\lambda} f(y) + t = -S_{n,\lambda} f(y) + S_{n,\lambda} f(x) - S_{n,\lambda} f(x) + u(x) + \delta_x(t) \geq -C_{f,\lambda} \lambda^n(x) + \sum_{j=n}^{\infty} \lambda^{j+1} f(T^j x) + \delta_x(t) \geq -C_{f,\lambda} \lambda^n(x) + \delta_x(t).
\]

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As $x \in X_u$, we have $\lambda^n(x) \to 0$ as $n \to \infty$. Hence provided $n$ is sufficiently large then $-S_{n,\lambda} f(y) + t \geq \delta_x(t)/2$. Similarly,

$$-S_{n,\lambda} f(y) + t \leq C_{f,\lambda} \lambda^n(x) + \sum_{j=n}^{\infty} \lambda^{j+1}(x) f(T^j x) + \delta_x(t).$$

By choosing $n$ sufficiently large we can assume that $C_{f,\lambda} \lambda^n(x) < \delta_x(t)$ and $\sum_{j=n}^{\infty} \lambda^{j+1}(x) f(T^j y) \leq \delta_x(t)$. Hence $-S_{n,\lambda} f(y) + t < 3\delta_x(t)$.

Hence

$$\frac{1}{3} \leq -S_{n,\lambda} f(x) + t \leq 4.$$

This suffices to prove the lemma.

We can now calculate the stability index for points above the graph.

**Lemma 5.5**

Assume that (H1)–(H4) hold. For $\mu$-a.e. $x \in X$ and all $t > u(x)$ we have

$$\sigma^-_{\mu}(x, t) = \frac{-s^* \int \log \lambda d\mu}{\int \log |T'| d\mu}.$$

**Proof.** We first prove that $\sigma^-_{\mu}(x, t) \leq -s^* \int \log \lambda d\mu/ \int \log |T'| d\mu$ for $\mu$-a.e. $x \in X$.

Given $x \in X_u$, choose $r_0$ as in §2.1 and choose a Markov partition with diameter no more than $r_0$. We assume that $r < r_0$.

For each $r$, let $A_{n_r(x_j)}(x_j)$, $1 \leq j \leq M$ be a Moran cover of $B_r(x)$. Let $N = \min\{n_r(x_j) \mid 1 \leq j \leq M\}$. As $\text{diam} A_{n_r(x_j)}(x_j) < r$ for each $j$ and $\text{diam} B_r(x) = 2r$, without loss of generality we can choose the indexing so that $A_{n_r(x_j)}(x_1) \subset B_r(x) \subset \bigcup_{j=1}^k A_{n_r(x_j)}(x_j) \subset B_{2r}(x)$. Note that $x \in A_{n_r(x_1)}(x_1)$.

Let $t^+ > t$. Then $t^+ > t + r \in \mathbb{B}^+$ provided that $r$ is sufficiently small. Note that

$$\mu(\{y \in X \mid u(y) > t^+\}) \leq \mu(\{y \in X \mid u(y) > t + r\}).$$

We have that

$$\frac{\mu(\{y \in B_r(x) \mid u(y) > t^+\})}{\mu(B_r(x))} \geq \frac{\mu(\{y \in A_{n_r(x_1)}(x_1) \mid u(y) > t^+\})}{\mu(\bigcup_{j=1}^M A_{n_r(x_j)}(x_j))}.$$

By Lemma 2.3 we see that

$$\frac{\mu(\{y \in A_{n_r(x_1)}(x_1) \mid u(y) > t^+\})}{\mu(\bigcup_{j=1}^M A_{n_r(x_j)}(x_j))} \geq D^{-1} \mu(T^n_r(x_1) \{y \in A_{n_r(x_1)}(x_1) \mid u(y) > t^+\}).$$

We claim that

$$T^n_r(x_1) \{y \in A_{n_r(x_1)}(x_1) \mid u(y) > t^+\} \supset \{z \in A_{n_r(x_1)} \setminus N(T^n_r(x_1)) \mid u(z) > Kg_{x_1}^{n_r(x_1)}(t^+)\}.$$

To see this, let $z \in X$ and suppose $u(z) > Kg_{x_1}^{n_r(x_1)}(t^+)$. There exists a unique $y \in A_{n_r(x_1)}(x_1)$ such that $T^n_r(x_1)(y) = z$. Note that $x_1, y$ are in the same cylinder of rank $n_r(x_1)$; hence by Lemma 5.4, we have that $u(T^n_r(x_1))(y) = u(z) \geq Kg_{x_1}^{n_r(x_1)}(t^+) > Kg_{x_1}^{n_r(x_1)}(t^+)$. As $g_{x_1}^{n_r(x_1)}(\cdot)$ is orientation preserving, we have that $u(y) > t^+$.

Hence

$$\frac{\mu(\{y \in A_{n_r(x_1)}(x_1) \mid u(y) > t^+\})}{\mu(\bigcup_{j=1}^M A_{n_r(x_j)}(x_j))} \geq D^{-1} \mu(\{z \in X \mid u(z) > Kg_{x_1}^{n_r(x_1)}(t^+)\}).$$
The above, together with Lemma 5.4, gives that
\[
\frac{\mu(\{y \in B_r(x) \mid u(y) > t^+\})}{\mu(B_r(x))} \geq D^{-1} \mu(\{z \in X \mid u(z) > K^2 g_x^{n_r(x)}(t^+)\}).
\]

For convenience, write \( n_r := n_r(x_1) \) and note that \( n_r \to \infty \) as \( r \to 0 \). Dividing the above by \( \log r \) (again, noting that \( \log r < 0 \)) it follows from Lemma 5.2 that
\[
\sigma_{t^+}(x, t) \leq \limsup_{r \to 0} \frac{1}{\log r} \log \left( \frac{\mu(\{y \in B_r(x) \mid u(y) > t + r\})}{\mu(B_r(x))} \right)
\leq \limsup_{r \to 0} \frac{1}{\log r} \log D^{-1} \mu(\{z \in X \mid u(z) > K^2 g_x^{n_r}(t^+)\}).
\]

We split the right-hand side of (25) as
\[
\frac{n_r}{-\log r} \times \frac{\log K g_x^{n_r}(t^+)}{n_r} \times \frac{-\log D^{-1} \mu(\{z \in X \mid u(z) > K^2 g_x^{n_r}(t^+)\})}{\log K^2 g_x^{n_r}(t^+)}
\]

It follows from (7) and Birkhoff’s Ergodic Theorem that for \( \mu \)-a.e. \( x \)
\[
\lim_{r \to 0} \frac{-\log r}{n_r} = \lim_{n \to \infty} \frac{-1}{n} S_n \log |T'(T^j x)| = -\int \log |T'| \, d\mu
\]

That \( n_r^{-1} \log K^2 g_x^{n_r}(t^+) \to -\int \log \lambda \, d\mu \) as \( n_r \to \infty \) for \( \mu \)-a.e. \( x \in X \) follows from Lemma 5.1. As \( t^+ > u(x) \), Proposition 3.2 implies that \( g_x^{n_r}(t^+) \to \infty \) as \( r \to 0 \). By Proposition 4.2 we have that
\[
-\log D^{-1} \mu(\{z \in X \mid u(z) > K^2 g_x^{n_r}(t^+)\}) \to s^*
\]
for \( \mu \)-a.e. \( x \in X \).

Hence \( \sigma_{t^+}(x, t) \leq -s^* \int \log \lambda \, d\mu / \int \log |T'| \, d\mu \) for \( \mu \)-a.e. \( x \in X \).

The argument for the lower bound on \( \sigma_{t^+}(x, t) \) is similar. Let \( t^- < t \). Then \( t^- < t - r \) provided that \( r \) is sufficiently small. We have
\[
\mu(\{y \in X \mid u(y) > t^-\}) \geq \mu(\{y \in X \mid u(y) > t - r\})
\]

We have that
\[
\frac{\mu(\{y \in B_r(x) \mid u(y) > t^-\})}{\mu(B_r(x))} \leq \frac{\mu\left( \left\{ y \in \bigcup_{j=1}^M A_{n_r(x)}(x_j) \mid u(y) > t^- \right\} \right)}{\mu(A_{n_r(x)}(x_1))} = \sum_{j=1}^M \frac{\mu\left( \left\{ y \in A_{n_r(x)}(x_j) \mid u(y) > t^- \right\} \right)}{\mu(A_{n_r(x)}(x_1))}.
\]

An argument similar to that in the proof of Lemma 2.3 shows that there exists \( D > 1 \), independent of \( r \), such that
\[
\frac{\mu(\{y \in A_{n_r(x)}(x_j) \mid u(y) > t^-\})}{\mu(A_{n_r(x)}(x_1))} \leq D \mu(T^{n_r(x_j)} \{y \in A_{n_r(x)}(x_j) \mid u(y) > t^-\}).
\]

We claim that
\[
T^{n_r(x_j)} \{y \in A_{n_r(x)}(x_j) \mid u(y) > t^-\} \subset \{z \in X \mid u(z) > K^{-1} g_x^{n_r(x_j)}(t^-)\}.
\]

To see this, let \( y \in A_{n_r(x)}(x_j) \) be such that \( u(y) > t^- \). Let \( z = T^{n_r(x_j)}(y) \). Then, as \( g_y^{n_r(x_j)}(\cdot) \) is orientation preserving and using Lemma 5.4, we have \( u(z) = u(T^{n_r(x_j)} y) = g_y^{n_r(x_j)}(u(y)) > g_y^{n_r(x_j)}(t^-) \geq K^{-1} g_x^{n_r(x_j)}(t^-) \).
Hence we have
\[
\frac{\mu\{y \in B_r(x) \mid u(y) > t_+\}}{\mu(B_r(x))} \leq D \sum_{j=1}^{M} \mu \left( \left\{ z \in X \mid u(z) > K^{-1} g_{x}^{n_{x}(x)}(t_+) \right\} \right)
\]
so that, by Lemma 5.2
\[
\sigma_{\mu}^{-}(x, t) \geq \liminf_{r \to 0} \frac{1}{\log r} \log \left( \frac{\mu\{y \in B_r(x) \mid u(y) > t - r\}}{\mu(B_r(x))} \right) \leq \limsup_{r \to 0} \frac{1}{\log r} \log D \sum_{j=1}^{M} \mu \left( \left\{ z \in X \mid u(z) > K^{-1} g_{x}^{n_{x}(x)}(t_+) \right\} \right).
\]

Arguing as in the estimates following (25) we see that for each \( j, 1 \leq j \leq M \) and for \( \mu \)-a.e. \( x \in X \),
\[
\lim_{r \to 0} \frac{1}{\log r} \log \mu \left( \left\{ z \in X \mid u(z) > K^{-1} g_{x}^{n_{x}(x)}(t_+) \right\} \right) = -s \int \frac{\log \lambda d\mu}{\log |T'| d\mu}.
\]
Hence \( \sigma_{\mu}^{-}(x, t) \geq -s \int \frac{\log \lambda d\mu}{\int \log |T'| d\mu} \) for \( \mu \)-a.e. \( x \in X \).

By Lemma 2.4 we have \( \sigma_{\mu}^{+}(x, t) = 0 \). This proves Theorem 1.2(i).

§5.2 The lower basin

We now prove Theorem 1.2(ii). We remark that the partial hyperbolicity condition (H4) is not needed for this result.

Lemma 5.6
Assume that (H1)–(H3) hold. For \( \mu \)-a.e. \( x \in X \) and all \( t < u(x) \), there exists \( r_0 > 0 \) such that for all \( 0 < r < r_0 \) we have \( \mu \times m(B_r(x, t) \cap \mathbb{B}^{-}) = \mu \times m(B_r(x, t)) \).

Proof. Suppose \( x \) is such that \( u(x) \) is defined. Let \( t < u(x) \) and define \( \delta_{x}(t) = u(x) - t > 0 \). Choose \( n \) such that
\[
\sum_{j=n}^{\infty} \lambda^{j+1}(x) f(T^{j} x) \leq \frac{\delta_{x}(t)}{3}.
\]
As \( f, \lambda \) are continuous, we can choose \( r' > 0 \) such that if \( y \in B_{r'}(x) \), then
\[
|S_{n, \lambda} f(y) - S_{n, \lambda} f(x)| < \frac{\delta_{x}(t)}{3}.
\]
Hence for \( \mu \)-a.e. \( y \in B_{r'}(x) \) we have
\[
u(y) \geq S_{n, \lambda} f(y) \geq S_{n, \lambda} f(x) - \frac{\delta_{x}(t)}{3} = u(x) - \sum_{j=n}^{\infty} \lambda^{j+1}(x) f(T^{j} x) - \frac{\delta_{x}(t)}{3} \geq u(x) - 2 \frac{\delta_{x}(t)}{3} = t + \frac{\delta_{x}(t)}{3} > t.
\]
By Proposition 3.2 \((y, t) \in \mathbb{B}^{-}\). Hence, for \( r < r_0 := \max\{r', \delta_{x}(t)/3\} \) we have that \( \mu \times m(B_r(x, t) \cap \mathbb{B}^{-}) = \mu \times m(B_r(x, t)) \).

Hence \( \Sigma_{\mu, r}^{-}(x, t) = 1 \) and \( \Sigma_{\mu, r}^{+}(x, t) = 0 \) provided \( r < r_0 \). Hence, by convention, \( \sigma_{\mu}^{-}(x, t) = 0 \) and \( \sigma_{\mu}^{+}(x, t) = \infty \).

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§5.3 On the graph

We prove Theorem 1.2(iii). We need the following version of Lemma 5.2.

**Lemma 5.7**
Let \( x \in X_u \).

(i) If \( \sigma^-_{\mu}(x, u(x)) \) exists then
\[
\sigma^-_{\mu}(x, u(x)) \leq \limsup_{r \to 0} \frac{1}{\log r} \log \left( \frac{\mu(\{ y \in B_r(x) \mid u(y) > u(x) \})}{\mu(B_r(x))} \right).
\]

(ii) If \( \sigma^+_{\mu}(x, u(x)) \) exists then
\[
\sigma^+_{\mu}(x, u(x)) \leq \limsup_{r \to 0} \frac{1}{\log r} \log \left( \frac{\mu(\{ y \in B_r(x) \mid u(y) < u(x) \})}{\mu(B_r(x))} \right).
\]

**Proof.** We prove (i). First note that
\[
\{ y \in B_r(x) \mid u(y) > u(x) \} \times [u(x) - r, u(x)] \subset B_r(x, u(x)) \cap B^-.
\]
Noting that \( m([u(x) - r, u(x)]) = r \) and that \( \mu \times m(B_r(x, t)) = \mu(B_r(x)) \times 2r \) we have
\[
\frac{\mu(\{ y \in B_r(x) \mid u(y) > u(x) \})}{2 \mu(B_r(x))} \leq \Sigma^-_{\mu,r}(x, u(x)).
\]
Hence
\[
1 \frac{1}{\log r} \log \left( \frac{\mu(\{ y \in B_r(x) \mid u(y) > u(x) \})}{2 \mu(B_r(x))} \right) \geq \frac{\log \Sigma^-_{\mu,r}(x, u(x))}{\log r}
\]
and the result follows by taking the limsup.

The proof of (ii) is analogous, noting that \( \{ y \in B_r(x) \mid u(y) < u(x) \} \times [u(x), u(x) + r] \subset B_r(x, t) \cap B^+ \).

The following lemma is a straightforward consequence of Birkhoff’s Ergodic Theorem and Proposition 4.2.

**Lemma 5.8**
We have that \( u(T^n x) \) is unbounded for \( \mu \)-a.e. \( x \in X \).

**Proof.** Let \( A_N = \{ x \in X \mid u(x) > N \} \). By Proposition 4.2 for all sufficiently large \( N \) we have \( N^{-3s'/2} < \mu(A_N) < N^{-s'/2} \); in particular, \( \mu(A_N) > 0 \) for all \( N > N_0 \), say. By Birkhoff’s Ergodic Theorem, the set \( X_N := \{ x \in X \mid T^n x \in A_N \text{ for infinitely many } n \} \) has full \( \mu \)-measure. Then \( \mu(\bigcap_{N=N_0}^{\infty} X_N) = 1 \) and consists of points \( x \) for which \( u(T^n x) \) is unbounded.

We can now calculate \( \sigma^-_{\mu}(x, u(x)) \).

**Proposition 5.9**
Assume (H1)–(H4) hold. Then \( \sigma^-_{\mu}(x, u(x)) = 0 \) for \( \mu \)-a.e. \( x \in X \).

**Proof.** As \( \sigma^-_{\mu}(x, u(x)) \) is non-negative, it suffices to show that \( \sigma^-_{\mu}(x, u(x)) \leq 0 \) for \( \mu \)-a.e. \( x \in X \).
By Lemma 2.3,
\[
\mu(\{y \in B_r(x) \mid u(y) > u(x)\}) \geq \frac{\mu(\{y \in A_{n_r}(x_1) \mid u(y) > u(x)\})}{\mu(B_r(x))} \geq D^{-1} \mu(\mathcal{A}) \mu(\mathcal{B})
\]

We now calculate \( D^{-1} \mu(\{y \in A_{n_r}(x_1) \mid u(y) > u(x)\}) \).

Let \( n_r = n_r(x_1) \) and note that \( n_r \to \infty \) as \( r \to 0 \). We claim that

\[
T^{n_r}\{y \in A_{n_r}(x) \mid u(y) > u(x)\} \supset \{z \in X \mid u(z) > C_{f,\lambda} + C_{\lambda} u(T^{n_r} x)\}
\]

where \( C_{f,\lambda}, C_{\lambda} \) are as in (22), (6), respectively. To see this, let \( z \) be such that \( u(z) > C_{f,\lambda} + C_{\lambda} u(T^{n_r} x) \). As \( T |_{A_{n_r}(x)} : A_{n_r}(x) \to X \) is a bijection, for each \( z \in X \) there is a unique \( y \in A_{n_r}(x) \) for which \( T^{n_r} y = z \). Recall that \( u(x) = S_{n_r,\lambda} f(x) + \lambda^{n_r} u(T^{n_r} x) \). Hence

\[
u(y) - u(x) = S_{n_r,\lambda} f(y) - S_{n_r,\lambda} f(x) + \lambda^{n_r} (y)u(T^{n_r} y) - \lambda^{n_r} (x)u(T^{n_r} x)
\]

\[
\geq -C_{f,\lambda} \lambda^{n_r} (y) + \lambda^{n_r} (y)u(T^{n_r} y) - \lambda^{n_r} (x)u(T^{n_r} x) \hspace{1cm} \text{by (22)}
\]

\[
\geq -C_{f,\lambda} \lambda^{n_r} (y) + \lambda^{n_r} (y)u(T^{n_r} y) - C_{\lambda} \lambda^{n_r} (y)u(T^{n_r} x) \hspace{1cm} \text{by (6)}
\]

\[
\lambda^{n_r} (y) (-C_{f,\lambda} + u(z) - C_{\lambda} u(T^{n_r} x))
\]

As \( \lambda > 0 \), it follows that \( u(y) > u(x) \). This proves (26).

Hence

\[
\mu(\{y \in B_r(x) \mid u(y) > u(x)\}) \geq D^{-1} \mu(\{z \in X \mid u(z) > C_{f,\lambda} + C_{\lambda} u(T^{n_r} x)\})
\]

Hence

\[
\sigma_{\mu}(x, u(x)) \leq \limsup_{r \to 0} \frac{1}{\log r} \log D^{-1} \mu(\{z \in X \mid u(z) > C_{f,\lambda} + C_{\lambda} u(T^{n_r} x)\})
\]

\[
= \limsup_{r \to 0} \frac{1}{\log r} \log \mu(\{z \in X \mid u(z) > C_{f,\lambda} + C_{\lambda} u(T^{n_r} x)\})
\]

We bound and split the right-hand side of (27) as

\[
\frac{n_r}{-\log r} \times \frac{\log^+(C_{f,\lambda} + C_{\lambda} u(T^{n_r} x))}{n_r} \times \frac{\log \mu(\{z \in X \mid u(z) > C_{f,\lambda} + C_{\lambda} u(T^{n_r} x)\})}{\log (C_{f,\lambda} + C_{\lambda} u(T^{n_r} x))}
\]

By Lemma 5.8 for \( \mu \)-a.e. \( x \in X \) we can choose a sequence \( n'_k \to \infty \) such that \( C_{f,\lambda} + C_{\lambda} u(T^{n'_k} x) \to \infty \). Hence, by Proposition 4.2 the third term in (27) converges to \( s^* \) as \( n'_k \to \infty \).

By Corollary 4.4 \( \int u \), and so \( \log^+(C_{f,\lambda} + C_{\lambda} u) \), is integrable. It is then a well-known corollary of Birkhoff’s Ergodic Theorem that

\[
\lim_{n \to \infty} \frac{1}{n} \log^+(C_{f,\lambda} + C_{\lambda} u(T^n x)) = 0
\]

for \( \mu \)-a.e. \( x \in X \). Finally, the first term in (27) converges \( \mu \)-a.e. to \( 1/\int \log |T'| d\mu \) by (7).

Hence \( \sigma_{\mu}^-(x, u(x)) = 0 \) for \( \mu \)-a.e. \( x \in X \).

We now calculate \( \sigma_{\mu}^+(x, u(x)) \).

**Proposition 5.10**
Assume (H1)–(H4) hold. Then \( \sigma_{\mu}^+(x, u(x)) = 0 \) for \( \mu \)-a.e. \( x \in X \).
Proof. Again, it suffices to show that \( \sigma^\pm(x,u(x)) \leq 0 \text{ \( \mu \)} - \text{a.e.} \).

Analogously to the proof of Proposition 5.9 we can bound

\[
\frac{\mu\{y \in B_r(x) \mid u(y) < u(x)\}}{\mu(B_r(x))} \geq D^{-1} \mu\{z \in X \mid u(z) < C_f,\lambda + C_\lambda u(T^n x)\}
\]

so that

\[
\sigma^+_r(x,u(x)) \leq \limsup_{r \to 0} \frac{1}{\log r} \log \mu\{z \in X \mid u(z) < C_f,\lambda + C_\lambda u(T^n x)\}.
\]

We split the limit inside the limsup as

\[
\frac{n_r}{\log r} \times \frac{\log^+ (C_f,\lambda + C_\lambda u(T^n x))}{n_r} \times \left( \frac{\log \mu\{z \in X \mid u(z) < C_f,\lambda + C_\lambda u(T^n x)\}}{\log (C_f,\lambda + C_\lambda u(T^n x))} \right).
\]

As in the proof of Proposition 5.9, the first two terms of (28) converge to \(-1/\int \log |T'| d\mu\) and 0, respectively, for \( \mu \)-a.e. \( x \in X \).

By Lemma 5.8 for \( \mu \)-a.e. \( x \in X \) we can choose a sequence \( n'_k \to \infty \) such that \( M_{n'_k} := C_f,\lambda + C_\lambda u(T^{n'_k} x) \to \infty \). By Proposition 1.2 provided \( M \) is sufficiently large, we have \( \mu\{z \in X \mid u(z) > M\} \geq M^{-\gamma s/2} \). Hence

\[
\frac{\log \mu\{z \in X \mid u(z) < M_{n'_k}\}}{\log M_{n'_k}} = \frac{\log \left(1 - \mu\{z \in X \mid u(z) > M_{n'_k}\}\right)}{\log M_{n'_k}} \leq \frac{\log \left(1 - M_{n'_k}^{-\gamma s/2}\right)}{\log M_{n'_k}}
\]

which converges to 0 as \( n'_k \to \infty \).

Hence \( \sigma^+_r(x,u(x)) = 0 \).

\( \square \)

§ 6 The multifractal spectrum of the stability index

Recall that we define \( K_\mu(\sigma) = \{x \in X \mid \sigma_\mu(x,t) = -\sigma \text{ for all } t > u(x)\} \). From the proof of Theorem 1.2 we see that

\[
K_\mu(\sigma) = \left\{ x \in X \mid \lim_{n \to \infty} \frac{s^S \log \lambda(x)}{-S_n \log |T^n(x)|} = \sigma \right\}.
\]

Thus the Hausdorff dimension of the level sets of the stability index can be analysed by invoking multifractal analysis, as described in [PW, P] for example.

We first recall multifractal analysis as it is formulated in [PW]. Let \( \psi \) be Hölder continuous and suppose that \( \log \psi \) is normalised (so that \( P(\log \psi) = 0 \)). Define \( S(q) \) by \( P(-S(q) \log |T'| + q \log \psi) = 0 \) and let \( \mu_q \) denote the equilibrium state with potential \( -S(q) \log |T'| + q \log \psi \). Let \( \sigma(q) = -S'(q) = \int \log \psi \, d\mu_q/\int \log |T'|^{-1} \, d\mu_q \). Suppose that \( \log |T'| \) is not cohomologous to \( \log \psi \) plus a constant. Then \( S(q) \) is a strictly convex analytic function and is the Legendre transform of the function \( f(\sigma) = \text{dim}_H\{x \in X \mid \lim_{n \to \infty} S_n \log \psi(x)/S_n \log |T^n(x)|^{-1} = \sigma\} \), so that \( f(\sigma(q)) = S(q) + q\sigma(q) \). Moreover, \( f(\sigma(q)) \) is defined on the interval \( [\sigma(\infty), \sigma(-\infty)] \).

Finally, \( S(q) \) is the Hentschel–Proccacia dimension spectrum. We remark that an analysis of the proofs shows that only the last statement requires \( \log \psi \) to be normalised. We briefly sketch why \( f(\sigma(q)) \) is the Legendre transform of \( S(q) \). Note that \( \mu_q(K_\mu(\sigma(q))) = 1 \). Let \( x \in K_\mu(\sigma(q)) \) then \( \prod_{j=0}^{n-1} \phi(T^j x) \sim \prod_{j=0}^{n-1} |T'(T^j x)|^\sigma(q) \). Let \( A_n(x) \) is a cylinder of diameter approximately \( r \). Then, by (10), \( \mu_q(A_n(x)) \sim \prod_{j=0}^{n-1} |T'(T^j x)|^{-S(q)} \phi(T^j x)^\sigma(q) \sim \prod_{j=0}^{n-1} |T'(T^j x)|^{-S(q)} \phi(T^j x)^\sigma(q) \). Hence one would expect typical points in \( K_\mu(\sigma(q)) \) to have local dimension \( S(q) + q\sigma(q) \).

When considering the multifractal structure of \( K_\mu(\sigma) \), we recall that we require \( \mu_q \) to be such that the invariant graph \( u \) is defined \( \mu_q \)-a.e. Thus we require \( \int \log \lambda \, d\mu_q < 0 \). This places an additional restriction on the set of \( q \) for which the multifractal spectrum is defined.

We prove Proposition 1.3 and Theorem 1.4 below.
§6.1 The SRB measure

We first consider the case when $T$ is a uniformly expanding Markov map on $[0, 1]$ and $\mu$ is the SRB measure. In this case, $\mu$ has potential $\phi = -\log |T'|$. Note that $P(-\log |T'|) = 0$. The Lyapunov exponent $s^*$ is defined by $P(-\log |T'| + s^* \log \lambda) = 0$.

Define $S(q)$ by $P(-S(q) \log |T'| + q s^* \log \lambda) = 0$. That $S(q)$ is well-defined follows by defining $\phi(q, r) = -r \log |T'| + q s^* \log \lambda$, noting that $\partial P(\phi(q, r))/\partial r = -\int \log |T'| d\nu \neq 0$ for an appropriate measure $\nu$, and using the implicit function theorem. Standard arguments involving the analyticity of pressure show that $S(q)$ is analytic. Note that $S(0) = 1$ and $S(1) = 1$ (as $s^*$ is the Lyapunov exponent). Let $\mu_q$ be the equilibrium state with potential $-S(q) \log |T'| + q s^* \log \lambda$.

Differentiating $P(-S(q) \log |T'| + q s^* \log \lambda) = 0$ with respect to $q$ shows that $S'(q) = -s^* \int \log \lambda d\mu_q/\int \log |T'| d\mu_q$. Differentiating $P(-S(q) \log |T'| + q s^* \log \lambda) = 0$ twice with respect to $q$ and using a standard result from [R1] shows that $S''(q) \geq 0$ with equality if and only if $s^* \log \lambda$ and $S'(q) \log |T'|$ are cohomologous up to a constant. Recall if two functions $f, g$ are cohomologous up to a constant $c$ then $\int f d\nu = \int g d\nu + c$ for any $T$-invariant measure $\nu$. Note that, by (H3), $\int s^* \log \lambda d\zeta > 0$ and $\int s^* \log \lambda d\mu < 0$. However, as $\log |T'| \geq 0$, we have $\int S'(q) \log |T'| d\zeta$ and $\int S'(q) \log |T'| d\mu$ have the same sign (or are zero, if $S'(q) = 0$). Hence $S''(q) > 0$ and so $S(q)$ is a strictly convex function.

As $S(q)$ is strictly convex, $S(0) = 1$ and $S(1) = 1$, there exists a unique $q^* \in (0, 1)$ such that $S'(q^*) = 0$. If $q < q^*$ then $\int \log \lambda d\mu_q < 0$. Hence $\mu_q(X_u) = 1$ so that $u$ is defined $\mu_q$-a.e.

Let $\sigma(q) = -S'(q) = -s^* \int \log \lambda d\mu_q/\int \log |T'| d\mu_q$. Then standard arguments from [PW] (sketched above) show that $\dim_H K_\mu(\sigma(q)) = S(q) + q \sigma(q)$, the Legendre transform of $S(q)$, and that this is defined for $q \in (-\infty, q^*)$. This is illustrated in Figure 2 below.

When $q = 0$ we have that $\mu_q = \mu$, the SRB measure. Hence

$$\dim_H \left\{ x \in X \mid \sigma_{\mu}(x, t) = \frac{s^* \int \log \lambda d\mu}{\int \log |T'| d\mu} \text{ for all } t > u(x) \right\} = 1.$$ 

![Figure 2: The multifractal spectrum of the stability index in the case of the SRB measure.](image)

§6.2 The general case

Assume that (H1)–(H4) hold and that $\mu$ is an arbitrarily equilibrium state corresponding to a Hölder potential. As above, define $S(q)$ by $P(-S(q) \log |T'| + q s^* \log \lambda) = 0$ and let $\mu_q$ denote the equilibrium state with potential $-S(q) \log |T'| + q s^* \log \lambda$. As in §6.1, $S(q)$ is well-defined, strictly convex and $S'(q) = s^* \int \log \lambda d\mu_q/\int \log |T'| d\mu_q$. Noting that $S(0)$ solves the pressure equation $P(-S(0) \log |T'|)$ we see that $S(0) = \dim_H X$. 

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We first show that there exists $q \in \mathbb{R}$ such that $\int \log \lambda \, d\mu_q < 0$. We remark that we do not necessarily have that $q \geq 0$.

The following is proved in [Si] (we note that [Si] assumes $\log \lambda$ to be such that $P(\log \lambda) = 0$, however the proof can be easily modified to hold without this assumption).

**Lemma 6.1 ([Si])**

Assume that (H1)–(H3) hold. Let $S(q)$ be defined as above. Then

$$S(q) = \inf \left\{ \frac{h(\nu) + qs^* \int \log \lambda \, d\nu}{\int \log |T'| \, d\nu} \mid \nu \text{ is a } T\text{-invariant probability measure} \right\}$$

where $h(\nu)$ denotes the entropy of $T$ with respect to $\mu$.

Lemma 6.1 implies the following result (cf. [Sc]).

**Lemma 6.2**

Assume that (H1)–(H3) hold. Then there exists $q \in \mathbb{R}$ such that $\int \log \lambda \, d\mu_q < 0$.

**Proof.** Suppose for a contradiction that $\int \log \lambda \, d\mu_q \geq 0$ for all $q \in \mathbb{R}$. Then $S'(q) \geq 0$ for all $q \in \mathbb{R}$. As $S$ is strictly convex, it follows that $\alpha_0 := \inf_{q \in \mathbb{R}} S'(q) = \lim_{q \to -\infty} S'(q) \geq 0$. We show that this cannot happen. Recall that if $m$ is any $T$-invariant probability measure then $h_m(T) \leq h\text{top}(T)$, the topological entropy of $T$. By ([9], Lemma 6.1), $S(q) = (h(\mu_q) + qs^* \int \log \lambda \, d\mu_q) / \int \log |T'| \, d\mu_q$.

Let $\varepsilon > 0$. Choose $q < 0$ such that $\alpha_0 < S'(q) < \alpha_0 + \varepsilon$. Then

$$q \alpha_0 > \frac{qs^* \int \log \lambda \, d\mu_q}{\int \log |T'| \, d\mu_q} - q \varepsilon$$

$$> S(q) - \frac{h(\mu_q)}{\int \log |T'| \, d\mu_q} - q \varepsilon$$

$$\geq \inf \left\{ \frac{h(\nu) + qs^* \int \log \lambda \, d\nu}{\int \log |T'| \, d\nu} \right\} - \frac{h\text{top}(T)}{\int \log |T'| \, d\mu} - q \varepsilon$$

where both infima are taken over all $T$-invariant probability measures $\nu$. Dividing by $q$, letting $q \to -\infty$ and noting that $\varepsilon > 0$ is arbitrary, we have that $\alpha_0 \leq \inf \{s^* \int \log \lambda \, d\nu / \int \log |T'| \, d\nu\}$ where the infimum is taken over all $T$-invariant probability measures.

Taking $\nu = \mu$, by (H3) we see that $\alpha_0 < 0$. Hence there exists $\mu_q$ such that $S'(q) < 0$, a contradiction. \qed

Repeating the above argument with $q > 0$ and letting $q \to \infty$ shows that $\sup_{q \in \mathbb{R}} S'(q) = \lim_{q \to \infty} S'(q) \geq \sup \{s^* \int \log \lambda \, d\nu / \int \log |T'| \, d\nu\}$ where the last supremum is taken over all $T$-invariant probability measures. Taking $\nu = \zeta$, we see that $S'(q) > 0$ for all sufficiently large $q$.

As $S(q)$ is strictly convex, we have $S'(q)$ is increasing. Hence there exists a unique $q^* \in \mathbb{R}$ such that $S'(q^*) = 0$. Note that if $q < q^*$ then $\int \log \lambda \, d\mu_q < 0$; hence, for $q < q^*$, we have that $\mu_q(X_u) = 1$ so that the invariant graph $u$ is defined $\mu_q$-a.e. Let $\sigma(q) = -s^* \int \log \lambda \, d\mu_q / \int \log |T'| \, d\mu_q$. Then standard arguments from [PW] (and sketched above) show that $\dim_H K_\mu(\sigma(q)) = S(q) + q \sigma(q)$, the Legendre transform of $S(q)$, and that this is defined for $q \in (-\infty, q^*)$. The two cases are illustrated in Figure 3.

The Hausdorff dimension of $\{x \in X \mid \sigma_\mu(x, t) = s^* \int \log \lambda \, d\mu / \int \log |T'| \, d\mu \text{ for all } t > u(x)\}$ is given by the unique $q \in (-\infty, q^*)$ for which $S'(q) = s^* \int \log \lambda \, d\mu / \int \log |T'| \, d\mu$. (In general, unless $\mu = \mu_q$ for some $q$, i.e. $\phi$ is cohomologous to $-S(q) \log |T'| + q s^* \log \lambda$, then we cannot expect to find a closed form for $q$.)
Figure 3: The multifractal spectrum of the stability index in the general case when (i) $q^* > 0$, (ii) $q^* < 0$.

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Charles Walkden, School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, U.K., email: charles.walkden@manchester.ac.uk

Tom Withers, School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, U.K., email: tomwithers1991@hotmail.co.uk