BIQUOTIENT ACTIONS ON UNIPOTENT LIE GROUPS

ANNETT PÜTTMANN

Abstract. We consider pairs $(V, H)$ of subgroups of a connected unipotent complex Lie group $G$ for which the induced $V \times H$-action on $G$ by multiplication from the left and from the right is free. We prove that this action is proper if the Lie algebra $\mathfrak{g}$ of $G$ is 3-step nilpotent. If $\mathfrak{g}$ is 2-step nilpotent, then there is a global slice of the action that is isomorphic to $\mathbb{C}^n$. Furthermore, a global slice isomorphic to $\mathbb{C}^n$ exists if $\dim V = 1 = \dim H$ or $\dim V = 1$ and $\mathfrak{g}$ is 3-step nilpotent. We give an explicit example of a 3-step nilpotent Lie group and a pair of 2-dimensional subgroups such that the induced action is proper but the corresponding geometric quotient is not affine.

1. Introduction and generic situation

By $G$ we always denote a connected, simply connected complex unipotent Lie group, i.e. an algebraic subgroup of a group of upper triangular matrices. Any pair $(V, H)$ of complex algebraic subgroups $V, H \subset G$ defines a natural $V \times H$ action on $G$ by $(v, h)g = vgh^{-1}$. Equivalently, we can discuss the $V$-action given by $v.gH = vgh$ on the homogeneous space $G/H$, which is isomorphic to $\mathbb{C}^{\dim G - \dim H}$ as a complex algebraic variety. We consider only free $V \times H$-actions on $G$.

Denote by $\mathfrak{g}$, $\mathfrak{v}$, and $\mathfrak{h}$ the Lie algebras of $G$, $V$, and $H$, respectively. We will frequently use the descending central series $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \ldots \supset \mathfrak{g}^{(j+1)} := [\mathfrak{g}, \mathfrak{g}^{(j)}] \supset \ldots \supset \{0\}$. A nilpotent Lie algebra $\mathfrak{g}$ is said to be $l$-step nilpotent if $l$ is the smallest integer such that $\mathfrak{g}^{(l)} = \{0\}$.

Since the only compact algebraic subgroup of $V$ is the trivial group and the exponential map $\exp : \mathfrak{g} \to G$ is an isomorphism, the isotropy groups $V_x$ are compact for all $x \in G/H$ iff the isotropy groups $V_x$ are trivial for all $x \in G/H$ iff $\text{Ad}(g)(v) \cap \mathfrak{h} = \{0\}$ for all $g \in G$.

Lipsman conjectured [2] that the $V \times H$-action on $G$ is proper if it is free. In fact, Nasrin proved [3] Lipsman’s conjecture to be true for 2-step nilpotent Lie groups $G$. We improve this result showing that Lipsman’s conjecture is true if $\mathfrak{g}^{(3)} = \{0\}$, and that there is a global slice isomorphic to $\mathbb{C}^{\dim G - \dim H - \dim V}$, if $\mathfrak{g}^{(2)} = \{0\}$.

There is a counterexample to Lipsman’s conjecture presented by Yoshino [7] with $\mathfrak{g}^{(4)} = \{0\}$. Essentially the same example serves to show that there is a free non-proper affine $\mathbb{C}^2$-action on $\mathbb{C}^5$, which is just the smallest member of a series, $n \geq 5$, of free non-proper affine $\mathbb{C}^2$-actions on $\mathbb{C}^n$ [5]. We construct a 3-step nilpotent Lie algebra $\mathfrak{g}$ and 2-dimensional subalgebras $\mathfrak{h}$ and $\mathfrak{v}$, such that the induced $V$-action on $G/H$ is Winkelmann’s [6] the free affine proper $\mathbb{C}^2$-action on $\mathbb{C}^6$ without global slice.

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We can not answer the question of whether Lipsman’s conjecture is true if one of the subgroups is one-dimensional. But a global slice isomorphic to \( C^n \) exists if \( \dim V = 1 = \dim H \) or \( \dim V = 1 \) and \( g \) is 3-step nilpotent.

Let us start by looking at the generic case which turns out to be easy: We call a basis \( B = \{(X_1, \ldots, X_n)\} \) of \( g \) a Levi-Malcev basis, if it is compatible with the descending central series, i.e., \( B \cap g^{(j)} \) is a basis of \( g^{(j)} \) for all \( j \). Given subalgebras \( v, h \subset g \) a vector space decomposition \( g = v \oplus h \) is called Levi-Malcev decomposition if there are bases of the subspaces \( v, h, \) and \( s \) such that their union is a Levi-Malcev basis of \( g \). Note that there is a Levi-Malcev decomposition if and only if \( \pi_j(v \cap g^{(j)}) \cap \pi_j(h \cap g^{(j)}) = \{0\} \) for all \( j \), where \( \pi_j : g^{(j)} \to g^{(j)}/g^{(j+1)} \) denotes the canonical homomorphism from \( g^{(j)} \) to the commutative Lie algebra \( g^{(j)}/g^{(j+1)} \). If two subspaces \( h, v \subset g \) are generically chosen, then there exists a Levi-Malcev decomposition \( g = v \oplus s \oplus h \).

**Lemma 1.** If there is a Levi-Malcev decomposition \( g = v \oplus s \oplus h \), then \( \text{Ad}(g)(v) \cap h = \{0\} \) for all \( g \in G \).

**Proof.** If \( X \in g^{(j)} \), then \( \text{Ad}(g)(X) - X \in g^{(j+1)} \) for all \( g \in G \). \( \square \)

Since the exponential map \( \exp : g \to G \) is an isomorphism, the action can be pulled-back onto the Lie algebra. We use the notation \( \exp(X \ast Y) = (\exp X)(\exp Y) \) for \( X, Y \in g \).

**Proposition 1.** If there is a Levi-Malcev decomposition \( g = v \oplus s \oplus h \), then \( S := \exp s \subset G \) is a global slice of the \( V \times H \)-action on \( G \).

**Proof.** If \( X \notin g^{(j+1)} \), then \( \pi_j(X \ast Y) = \pi_j(X + Y) \) for all \( Y \in g \). \( \square \)

**Corollary 1.** There is a global slice of the \( H \)-action on \( G \). In particular, \( G/H \cong C^{\dim G - \dim H} \).

**Proof.** Arbitary vector space decompositions \( \pi_j(g^{(j)}) = s_j \oplus \pi_j(h \cap g^{(j)}) \) imply a Levi-Malcev decomposition \( g = s \oplus h \). Then \( S := \exp s \subset G \) is a global slice of the \( H \)-action on \( G \). \( \square \)

## 2. Reductions using normal subgroups

If \( N \triangleleft G \) is a normal subgroup, then \( G/N \) is again a connected simply connected unipotent Lie group. Furthermore, there is the induced action of the subgroups \( V/(V \cap N) \) and \( H/(H \cap N) \) on \( G/N \). In particular, if \( Z(G) \) is the center of \( G \), then \( V \cap Z(G) \) and \( H \cap Z(G) \) are central subgroups of \( G \). Therefore, the \( V \times H \)-action on \( G \) is equivalent to the \( V/(V \cap Z(G)) \times H/(H \cap Z(G)) \)-action on \( G/((Z(G) \cap V) \times (Z(G) \cap H)) \). Consequently, we can assume \( V \cap Z(G) = H \cap Z(G) = \{e\} \) and \( v \cap g^{(l-1)} = h \cap g^{(l-1)} = \{0\} \), which might be a weaker condition.

**Lemma 2.** Let \( N \triangleleft G \) be a normal subgroup of \( G \). If the induced \( V \times H \)-action on \( G/N \) is free, then any local slice \( S \subset G/N \) gives a local slice \( SN \subset G \).

**Proof.** If there are \( v \in V, h \in H, s, s' \in S, \) and \( n, n' \in N \) such that \( vsnh^{-1} = s'n' \), then there exists \( n'' \in N \) such that \( vsnh^{-1} = s' \), since \( N \) is normal. \( \square \)

Let \( G_1 \triangleleft G \) be a normal subgroup that contains \( V \) and \( H \). Any global slice \( S \) of the \( G_1 \)-action on \( G \) by left-multiplication defines an isomorphism \( \Psi : S \times G_1 \to G \),
(s, g_1) \mapsto sg_1. Now, the V \times H\text{-}action on G can be regarded as a family of V \times H\text{-}actions on G_1 parameterized by s \in S:

\((v, h)(s, g_1) = \Psi^{-1}(vsg_1h^{-1}) = \Psi^{-1}(s^{-1}vs)g_1h^{-1} = (s, s^{-1}vsg_1h^{-1})\).

**Lemma 3.** Let \(G_1 \triangleleft G\) be a normal subgroup that contains H and V. If there is an element \(Y_0 \in g\) such that \(\text{ad}(Y_0)(v) \subset v\) and \(g = \langle Y_0 \rangle_C \oplus g_1\) is a Levi-Malcev decomposition, then the \(V \times H\text{-}action on G\) is proper iff the \(V \times H\text{-}action on G_1\) is proper. In that case, \(V\backslash G/H \cong (V\backslash G_1/H) \times \C\).

**Proof.** The condition \(\text{ad}(Y_0)(v) \subset v\) implies \(\text{Ad}(\exp(y_0Y_0))(v) = e^{y_0\text{ad}(Y_0)}(v) = v\) for all \(y_0 \in C\), i.e., we have a trivial family of \(V \times H\)-actions on \(G_1\).

**Corollary 2.** Let \(n := \pi_0^{-1}(\pi_0(h) \cap \pi_0(v))\), \(v_0 := n \cap v\), \(h_0 := n \cap h\), \(V_0 := \exp v_0\), and \(H_0 := \exp h_0\). Let \(g_1 \subset g\) be a maximal subspace that satisfies \(n \subset g_1\), \(g_1 \cap h = h_0\), and \(g_1 \cap v = v_0\). The \(V \times H\text{-}action on G\) is proper iff the \(V_0 \times H_0\text{-}action on G\) is proper. In that case, \(V\backslash G/H \cong V_0\backslash (\exp g_1)H_0\).

**Proof.** The subspace \(g_1\) is an ideal, since it contains \(g^{(1)}\). Note that V and H can be interchanged in Lemma 3. We therefore assume \(\pi_0(h) \subset \pi_0(v)\). Choose Lie algebra elements \(X_1, \ldots, X_m \in v\) such that \(\dim s = m\) and \(g = s \oplus g_1\) is a Levi-Malcev decomposition for subspace \(s := \langle X_1, \ldots, X_m \rangle_C\). Now, \(\{\exp(x_1X_1) \cdots \exp(x_mX_m) : x_j \in C\} =: S\) is a global slice for the \(G_1\text{-}action on G\) be left multiplication.

**Lemma 4.** Let \(N \triangleleft G\) be a normal subgroup. If \(S_N \subset G/N\) is a global slice of the \(V/(V \cap N) \times H/(H \cap N)\text{-}action on G/N\) and \(S \subset G\) is a global slice of the \((V \cap N) \times (H \cap N)\text{-}action on G\), then \(S \cap S_N N\) is global slice of the \(V \times H\text{-}action on G\).

**Proof.** The map \(V \times (S \cap S_N N) \times H \to G, (v, s, g) \mapsto vsg\), is an isomorphism, because \((V \cap N)S_N(N \cap H) = S_NN\) and \(S/N = G/N\).

3. **Special cases**

If G is commutative, then any vector space decomposition \(g = v \oplus s \oplus h\) is a Levi-Malcev decomposition and defines a global slice \(S := \exp s\) of the \(V \times H\text{-}action.

**3.1. 2-step nilpotent Lie algebras.** Applying the results of section 2 we can assume that there are \(X_1, \ldots, X_m \in g \setminus g^{(1)}\) and \(Z_1, \ldots, Z_m \in g^{(1)}\) such that \(\{X_1, \ldots, X_m\}\) is a basis of \(h\) and \(\{X_1 + Z_1, \ldots, X_m + Z_m\}\) is a basis of \(v\) if \(g\) is 2-step nilpotent. In particular, \(v, h,\) and \(h \oplus g^{(1)}\) are commutative.

**Proposition 2.** If \(g\) is 2-step nilpotent, then there exists a global slice \(S \subset G\) of the \(V \times H\text{-}action on G\) that is algebraically isomorphic to \(\C^{\dim G - \dim H - \dim V}\).
**Proof.** Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ be any Levi-Malcev decomposition. Then $\exp \mathfrak{s}$ is a global slice of the $H$-action on $G$. The induced $V$-action on $\mathfrak{s} \cong G/H$ is given by

$$
\exp(\sum_{j=1}^{m} t_j (X_j + Z_j)).Y = \exp(\sum_{j=1}^{m} t_j (X_j + Z_j)) \exp(Y) \exp(\sum_{j=1}^{m} -t_j X_j)
$$

$$= \text{Ad}(\exp(\sum_{j=1}^{m} t_j X_j))(Y) + \sum_{j=1}^{m} t_j Z_j
$$

$$= \exp(\text{ad}(\sum_{j=1}^{m} t_j X_j))(Y) + \sum_{j=1}^{m} t_j Z_j
$$

which is an affine action of the commutative group $\mathbb{C}^m$ on $\mathfrak{s} \cong \mathbb{C}^{\dim G-m}$ of degree one, i.e., the expression $\exp(\sum_{j=1}^{m} t_j (X_j + Z_j)).Y$ is linear in the variables $t_j$.

By [5] this action has a global slice that is algebraically isomorphic to $\mathbb{C}^{\dim \mathfrak{s}-m}$. □

### 3.2. 3-step nilpotent Lie algebras

If $\mathfrak{g}$ is 3-step nilpotent, we can assume that $\mathfrak{v} \cap \mathfrak{g}^{(2)} = \mathfrak{h} \cap \mathfrak{g}^{(2)} = \{0\}$ and $\pi_0(\mathfrak{v}) = \pi_0(\mathfrak{h})$. Choose $X_1, \ldots, X_m \in \mathfrak{h} \setminus \mathfrak{g}^{(1)}$ such that $((\pi_0(X_1), \ldots, \pi_0(X_m)))$ is a basis of $\pi_0(\mathfrak{h}) = \pi_0(\mathfrak{v})$. There are $Z_1, \ldots, Z_m \in \mathfrak{g}^{(1)}$ such that $X_j + Z_j \in \mathfrak{v}$ for $1 \leq j \leq m$. Let us consider the subalgebras $\mathfrak{h}_0 \subset \mathfrak{h}$ and $\mathfrak{v}_0 \subset \mathfrak{v}$ generated by the elements $X_j \in \mathfrak{h}, j = 1, \ldots, m$, and $X_j + Z_j \in \mathfrak{v}, j = 1, \ldots, m$, respectively. Note that $[\mathfrak{h}_0, \mathfrak{h} \cap \mathfrak{g}^{(1)}] = [\mathfrak{v}_0, \mathfrak{v} \cap \mathfrak{g}^{(1)}] = \{0\}$, since these commutators are in $\mathfrak{g}^{(2)}$ and $V$ and $H$ are assumed to have trivial intersection with the center of $G$. Hence, there is a Lie algebra decomposition $\mathfrak{v} = \mathfrak{v}_0 \oplus \mathfrak{v}_1$ with $\mathfrak{v}_1 \subset \mathfrak{g}^{(1)}$.

By construction there is an isomorphism $\phi : \mathfrak{v}_0 \to \mathfrak{h}_0, X_j + Z_j \mapsto X_j$. The set $\mathfrak{n} := \{Y_0 + Y_1, \phi(Y_0) + Y_0 : Y_j \in \mathfrak{v}_j, Y \in \mathfrak{h} \cap \mathfrak{g}^{(1)} \} \subset \mathfrak{v} \oplus \mathfrak{h}$ is an ideal isomorphic to $\mathfrak{v} \oplus \mathfrak{v} \cap \mathfrak{g}^{(1)}$. The resulting action of the normal subgroup $\exp \mathfrak{n} \subset V \times H$ is affine of degree two, since $X \ast Y = X + Y + \frac{1}{2}[X, Y]$ if $X \in \mathfrak{g}^{(1)} = \mathfrak{g}^{(2)}$, i.e., $\delta^3(x_j) = 0$ for all $j$ and for any locally nilpotent derivation $\delta \in \mathfrak{n}$ for any choice of affine coordinates $x_j$ on $\mathfrak{g}$.

**Lemma 5.** A free affine action of a connected simply connected complex unipotent Lie group $N$ on $\mathbb{C}^m$ of degree two is proper.

**Proof.** Since the action is affine, there are coordinates $x_1, \ldots, x_m$ of $\mathbb{C}^m$ such that the action map $N \times \mathbb{C}^m \to \mathbb{C}^m, (n, x) \mapsto n.x$, defines a representation of the Lie algebra $\mathfrak{n}$ on $\mathbb{C}[x_1, \ldots, x_m]$ by locally nilpotent triangular derivations such that $\delta(x_k) \in \{(1, x_1, \ldots, x_{k-1}) \} \subset \mathfrak{n}$ for all $\delta \in \mathfrak{n}$ and all $k = 1, \ldots, m$. Given a pair $(x_0, \delta_0) \in \mathbb{C}^m \times \mathfrak{n}$ we can assume $x_j(x_0) = 0$ for all $j = 1, \ldots, m$ and $\delta_0(x_k) = c_0$ and $|c_0| \neq 0$ for some $k \in \{1, \ldots, m\}$, since the action is free. Consequently, there is a neighborhood $U_0$ of $(x_0, \delta_0)$ in $\mathbb{C}^m \times \mathfrak{n}$ such that $|\delta(x_k)(x)| > \frac{1}{2} c_0$ for all $(x, \delta) \in U_0$.

We have to show that the map $\Phi : N \times \mathbb{C}^m \to \mathbb{C}^m \times \mathbb{C}^m, (n, x) \mapsto (n, x)$, is proper, i.e., for any compact set $K \subset \mathbb{C}^m \times \mathbb{C}^m$ the preimage $\Phi^{-1}(K)$ is compact. Since the action is of degree two,

$$x_j(\exp(\delta), x) = (\exp(\delta), x_j)(x) = x_j(x) - \delta(x_j)(x) + \frac{1}{2} \delta^2(x_j)(x)
$$

for all $\delta \in \mathfrak{n}, x \in \mathbb{C}^m, j = 1, \ldots, m$. Note that if $|x, n.x| \leq R$ and $n = \exp \delta$, then $h_j(\delta, x) := |1 - \delta(x_j)(x) + \frac{1}{2} \delta^2(x_j)(x)| \leq 2R$ for all $j = 1, \ldots, m$. We will prove that the set $\{(x, \delta) \in \mathbb{C}^m \times \mathfrak{n} : |x| < R, h_j(x, \delta) \leq 2R \forall j\}$ is bounded. For any point $(x_0, \delta_0) \in \mathbb{C}^m \times \mathfrak{n}$ satisfying $|x_0| \leq R$ and $|\delta_0| = 1$ we will find a neighborhood $U_0$
and a number $S_{(x_0,h_0)}$ such that $h_j(x, \alpha \delta) \leq 2R$ for $(x, \delta) \in U_0$ and for all $j$ implies $|\alpha| \leq S_{(x_0,\delta_0)}$.

For $\alpha \in \mathbb{R}^\geq 0$ and $(x, \delta) \in U_0$,

$$| - \alpha \delta(x_k)(x) + \frac{1}{4} \alpha^2 \delta^2(x_k)(x) | = | \alpha | - \delta(x_k)(x) + \frac{1}{2} \alpha \delta^2(x_k)(x) | \geq \alpha (| \delta(x_k)(x) | - \frac{1}{4} \alpha | \delta^2(x_k)(x) | ) \geq \alpha (\frac{4}{9} c_0 - \frac{1}{4} \alpha | \delta^2(x_k)(x) | ).$$

Since $0 = \delta^3(x_k) = \sum_{j<k} a_j(\delta) \delta^2 x_j$ for all $\delta \in n$ and $h_j(x, \alpha \delta) \leq 2R$,

$$\alpha \delta^2(x_k) = \alpha \sum_{j<k} \delta(c_0(\delta) + a_j(\delta)x_j) = \sum_{j<k} a_j(\delta) \alpha \delta x_j \leq \sum_{j<k} a_j(\delta) (\alpha \delta x_j - \frac{1}{2} \alpha^2 \delta^2 x_j) \leq 2R \sum_{j<k} a_j(\delta).$$

The neighborhood $U_0$ can be shrunk further to obtain $\sum_{j<k} a_j(\delta) \leq \frac{c_0}{4R}$ for all $(x, \delta) \in U_0$, because $\sum_{j<k} a_j(\delta)(\delta) = 0$. Hence, $\frac{1}{2} \alpha \delta^2(x_k)(x) \leq \frac{1}{4} c_0$. This gives the upper bound $S_{(x_0,\delta_0)} = \frac{4K}{c_0} \geq |\alpha|$.

**Corollary 3.** If $g$ is 3-step nilpotent, then the $V \times H$-action on $G$ is proper.

**Proof.** By the previous Lemma the $N$-action on $G$ is proper. This means that there are local slices $S_\alpha \subset G$ of the $N$-action such that the sets $U_\alpha := N.S_\alpha$ cover $G$. Let $G^{(1)} := \exp(g^{(1)})$. The $N$-action on $G/G^{(1)}$ is trivial and $(V \times H)/N \cong H/(H \cap G^{(1)})$. Let $\pi_0(g) = s_0 \oplus \pi_0(h)$ be any vector space decomposition. Then $U_\alpha := \exp(\pi_0^{-1}(s_0)) \cap U_\alpha$ are $N$-invariant subsets and $\tilde{S}_\alpha := U_\alpha/N$ are local slices of the $V \times H$-action on $G$.

To conclude this section we give an example of a 3-step nilpotent Lie algebra $g$ and two subalgebras $n, h \subset g$ such that the corresponding free action of $V \times H$ on $G$ does not have a global (holomorphic or algebraic) slice. It was shown in [6] that the two commuting derivations $\delta' = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + (1 + y_1) \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_1}$ and $\delta = y_2 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial z_1}$ define a proper free affine $\mathbb{C}^2$-action on $\mathbb{C}^6$ that has a geometric quotient that is neither affine nor Stein.

**Lemma 6.** There is a Lie bracket on the complex vector space generated by $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ such that the only non-vanishing commutators of basis elements are $[X_1, Y_2] = Y_3, [X_2, Y_1] = Y_3 + Z_2, [X_2, Y_2] = Y_4$, and $[X_2, Z_2] = Z_1$.

**Proof.** We have to check the Jacobi-identity: The elements $Y_3, Y_4$, and $Z_1$ are in the center,

$$[x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2 + z Z_2, [x'_1 X_1 + x'_2 X_2 + y'_1 Y_1 + y'_2 Y_2 + z' Z_2, x''_1 X_1 + x''_2 X_2 + y''_1 Y_1 + y''_2 Y_2 + z'' Z_2]$$

$$= [x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2, (x'_2 y''_1 - x''_2 y'_1) Z_2] = x_2 (x'_2 y''_1 - x''_2 y'_1) Z_1,$$

and

$$x'(xy'' - x'' y) - x''(xy' - x' y) = xx'y'' - x'x''y - xx''y + x'x''y = x(x'y'' - x'' y).$$

$\square$
Lemma 8. The Lie algebra constructed in Lemma 6 is 3-step nilpotent. Its complex subspaces \( \mathfrak{h} = \langle X_1, X_2 \rangle_{\mathbb{C}} \) and \( \mathfrak{v} = \langle X_1 + Z_1, X_2 + Z_2 \rangle_{\mathbb{C}} \) are subalgebras. The induced \( V \)-action on \( G/H = \exp((Y_1, Y_2, Y_3, Z_1, Z_2)_{\mathbb{C}}) \cong \mathbb{C}^6 \) is the \( \mathbb{C}^2 \)-action on \( \mathbb{C}^6 \) given by \( \delta \) and \( \delta' \).

Note the similarity to the construction of a pair of counterexamples to Lipsman's conjecture in [7]. The smallest example of a free, affine, non-proper action of a unipotent group on some \( \mathbb{C}^n \) is given by the \( \mathbb{C}^2 \)-action on \( \mathbb{C}^5 \) generated by the two derivations \( \delta_1 = \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2} \) and \( \delta_2 = y_1 \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_4} + \frac{\partial}{\partial y_5} \), see [8], [5]. The complex subspaces \( \mathfrak{h} = \langle X_1, X_2 \rangle_{\mathbb{C}} \) and \( \mathfrak{v} = \langle X_1 + Z_1, X_2 + Z_2 \rangle_{\mathbb{C}} \) are subalgebras of the 4-step nilpotent Lie algebra \( \mathfrak{g} \) constructed in Lemma 7. The induced \( V \)-action on \( G/H \cong \exp((Y_1, Y_2, Y_3, Z_1, Z_2)_{\mathbb{C}}) \cong \mathbb{C}^5 \) is the \( \mathbb{C}^2 \)-action generated by \( \delta_1 \) and \( \delta_2 \), i.e., the statement of Corollary 3 is not true for 4-step nilpotent Lie algebras.

Lemma 7. There is a Lie bracket on the vector space generated by \( X_1, X_2, Y_1, Y_2, Y_3, Z_1, Z_2 \) such that the only non-vanishing commutators of basis elements are \([X_1, Y_1] = Y_2, [X_1, Y_2] = Y_3, [X_1, Y_3] = Z_2, \) and \([X_2, Y_1] = Z_1 \).

Proof. We verify the Jacobi-identity using the same argument as in Lemma 6. \( \square \)

3.3. Induced \( \mathbb{C} \)-actions on \( G/H \). Note that dim \( H = 2 = \text{dim} \mathbb{C} \) in the explicit examples of the previous section. In fact, dim \( H \), dim \( \mathbb{C} \) \( \geq 2 \) in all known examples for which \( V \backslash G/H \cong \mathbb{C}^N \). Let us look more closely at the case where one of the subgroups, say \( V \), is 1-dimensional.

Using the results of section 2 we can assume that there are \( X_0, \ldots, X_m \in \mathfrak{g} \setminus \mathfrak{g}^{(l-1)} \) and \( Z_0 \in \mathfrak{g}^{(l-1)} \) such that \( X_0 + Z_0 \) generates \( \mathfrak{v} \) and \( ([X_0, \ldots, X_m]) \) is a Levi-Malcev basis of \( \mathfrak{h} \) if \( \text{dim} \mathfrak{v} = 1 \). Let \( \mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h} \) be a Levi-Malcev decomposition satisfying \( Z_0 \in \mathfrak{s} \). Recall that a \( \exp \mathfrak{s} \) is a global slice of the \( H \)-action on \( G \) and the induced \( V \)-action on \( G/H \cong \mathfrak{s} \) is given by

\[
e^{t(X_0+Z_0)}Y = \pi_H(\text{Ad}(e^{tX_0})(Y) + tZ_0) = \pi_H(e^{t\text{ad}(X_0)}(Y) + tZ_0),
\]

where \( \pi_H : \mathfrak{g} \to \mathfrak{g}/H = \mathfrak{s} \) denotes the quotient map of the \( H \)-action on \( \mathfrak{g} \). Consider the linear map \( \text{ad}(X_0) : \mathfrak{g} \to \mathfrak{g} \). Let \( r \) be the largest \( j \) such that \( \text{ad}(X_0)(\mathfrak{g}) \subset \mathfrak{g}^{(j)} \).

Lemma 8. If \( r \geq l - 1 \) or \( \text{dim} \ H = 1 \), then the \( V \times H \)-action on \( G \) is proper and there is a global slice that is algebraically isomorphic to \( \mathbb{C}^{\text{dim} \ G - \text{dim} \ H - 1} \).

Proof. Since \( \mathfrak{h} \) has trivial intersection with \( \mathfrak{g}^{(l-1)} \), in both cases, the global slice \( \mathfrak{s} \) of the \( H \)-action on \( \mathfrak{g} \) is \( \text{ad}(X_0) \) invariant and contains the subspace \( \langle Z_0 \rangle_{\mathbb{C}} \). Consequently, the induced \( V \)-action on \( \mathfrak{g}/H \cong \mathfrak{s} \) is given by

\[
e^{t(X_0+Z_0)}Y = e^{t\text{ad}(X_0)}(Y) + tZ_0,
\]

which is a free affine \( \mathbb{C} \)-action on \( \mathfrak{s} \).

Since \( \text{ad}(X_0)(\mathfrak{s}) \) is a vector space, that does not contain \( \mathbb{C}Z_0 \) and \( \mathbb{C}X_0 \), there is a vector space decomposition \( \mathfrak{s} = \mathfrak{s}_0 \oplus \mathbb{C}Z_0 \) such that \( \text{ad}(X_0)(\mathfrak{s}_0) \subset \mathfrak{s}_0 \). Hence, \( \mathfrak{s}_0 \) is a global slice of the \( V \)-action on \( \mathfrak{s} \).

Lemma 9. If \( V \) is contained in a normal commutative subgroup \( N \), then the \( V \times H \)-action on \( G \) is proper and there is a global slice isomorphic to \( \mathbb{C}^{\text{dim} \ G - \text{dim} \ H - 1} \).

Proof. By Lemma 4 we can assume that \( H \subset N \), because \( V/(V \cap N) \) is trivial and the \( H/(H \cap N) \)-action on \( G/N \) admits a global slice isomorphic to some \( \mathbb{C}^n \). Let \( \mathfrak{n} = \mathfrak{s}_1 \oplus \mathfrak{h} \) be a Levi-Malcev decomposition. Let \( S_0 \subset G \) be a global slice of the
The normal commutative subgroup $V$-action on $H$ action. Note that $\log S$ is a slice of the action of $G$. Assume that $\log S$ is a slice using Lemma 3 with position 3. Let $G$ slice using Lemma 3 with assertion 3.

Let $G$ satisfy $\dim G = \dim H - 1$.

**Proposition 3.** If $\dim V = 1$ and $g^{(3)} = \{0\}$, then the $V \times H$-action on $G$ is proper and there is a global slice isomorphic to $\mathbb{C}^{\dim G - \dim H - 1}$.

**Proof.** The Lie algebra $g^{(1)}$ is commutative, since $[g^{(j)}, g^{(k)}] \subset g^{(j+k+1)}$. We can assume that $\mathfrak{h} \cap g^{(2)} = \{0\}$, $\pi_0(\mathfrak{g}) = \pi_0(\mathfrak{v})$, $\mathfrak{h} = \langle X_0, \ldots, X_m \rangle \mathbb{C}$, and $\mathfrak{v} = \langle X_0 + Z_0 \rangle \mathbb{C}$ where $Z_0$ is central.

The element $Z_0$ is not contained in the vector space $\mathfrak{s}_2 := \text{ad}(X_0)(g^{(1)})$, since $[X_0, Y_1] = Z_0$ would imply $\text{Ad}(e^{-Y_1})(X_0 + Z_0) = X_0$ contradicting $\text{Ad}(G)(\mathfrak{v}) \cap \mathfrak{h} = \{0\}$. Let $S_2 := \exp \mathfrak{s}_2$. Now, the $V \times H$-action on $G/S_2$ is free if and only if the $V \times H$-action on $G$ is free, because $\text{ad}(X_0)(\mathfrak{g}) \cap \mathfrak{h}$ is contained in $g^{(1)}$ and $[\text{ad}(X_0)(\mathfrak{g}) \cap \mathfrak{h}, g^{(1)}] = \{0\}$. □

The following examples illustrates the construction of the global slice in Proposition 3. Let $G$ be the group of upper triangular $(4 \times 4)$-matrices and

$$H = \left\{ \left( \begin{array}{cccc} 1 & 0 & 0 & y_3 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & x_0 \\ 0 & 0 & 0 & 1 \end{array} \right) : x_0, x_1 \in \mathbb{C} \right\}, \quad V = \left\{ \left( \begin{array}{cccc} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) : t \in \mathbb{C} \right\}$$

The normal commutative subgroup

$$N = \left\{ \left( \begin{array}{cccc} 1 & 0 & 0 & y_3 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & x_0 \\ 0 & 0 & 0 & 1 \end{array} \right) : y_3, z, x_0, x_1 \in \mathbb{C} \right\}$$

contains $H$ and $V$ as subgroups. The set

$$S_0 = \left\{ \left( \begin{array}{cccc} 1 & 0 & 0 & y_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) : y_1, y_2 \in \mathbb{C} \right\}$$

is a slice of the action of $N$ on $G$ by left-multiplication, i.e., $S_0 \times N \rightarrow G$, $(s_0, n) \mapsto s_0 n$ is an isomorphism. The decomposition $n = \{x_0 = x_1 = 0\} \oplus \mathfrak{h}$ defines a slice $S_1 := \{x_0 = x_1 = 0\}$ of the $H$-action on $N$. Using the coordinates $y_1, y_2, y_3, z$, the $V$-action on $G/H \cong S_0 \times S_1$ is given by the derivation $\delta = -y_1 \frac{\partial}{\partial y_3} + (1 - y_1 y_2) \frac{\partial}{\partial z}$, because

$$\left( \begin{array}{cccc} 1 - y_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -y_2 \\ 0 & 0 & 0 & 1 \end{array} \right) \left( \begin{array}{cccc} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{cccc} 1 & 0 & 0 & t - y_1 y_2 t \\ 0 & 1 & 0 & t y_2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{array} \right).$$

Hence, $\delta(z - y_2 y_3) = 1$ and $S := \{z = y_2 y_3\} \subset S_0 \times S_1$ is a global slice of the $V$-action. Note that $\log S \subset \mathfrak{g}$ is non-linear. Alternatively, we can construct the global slice using Lemma 3 with $G' = \{y_2 = 0\}$ and Lemma 8, because $X_1 \in g \setminus g^{(1)}$.

Let us summarize the properties of the smallest example of a triple $H, V \subset G$ satisfying $\text{Ad}(G)(V) \cap H = \{e\}$ and $\dim V = 1$ that can not be handled by the methods presented so far:

1. $g^{(3)} \neq \{0\}$, $\text{Ad}(X_0)(g) \subset g^{(l-1)}$,
2. There is a basis $\langle (X_0, \ldots, X_m) \rangle$ of $\mathfrak{h}$ and $Z_0 \subset g^{(l-1)}$ such that $X_i \in g \setminus g^{(l-1)}$ and $\mathfrak{v} = (X_0 + Z_0) \mathbb{C}$.
3. For all $Y \in g \setminus (\pi_0^{-1}(\pi_0(\mathfrak{h})))$, $\text{Ad}(Y)(\mathfrak{h}) \not\subset \mathfrak{h}$ and $[Y, X_0] \neq 0$. 

We finally come back to the explicit examples of the previous section that are related to free affine \( \mathbb{C}^2 \)-actions on some \( \mathbb{C}^n \). The quotient \( \mathbb{C}^n/V_0 \) of a subgroup \( \mathbb{C} \cong V_0 \subset \mathbb{C}^2 \) and the induced \( \mathbb{C} \cong \mathbb{C}^2/V_0 \)-action on \( \mathbb{C}^n/V \) can be easily calculated. It would be helpful to know if these \( \mathbb{C} \)-actions, that are well understood, can arise in our context, i.e., if they are equivalent to the \( V \)-action on \( G/H \), where \( V \) and \( H \) are subgroups of a unipotent Lie group \( G \). Unfortunately there is no method available to decide this question or to construct \( G \), \( H \) and \( V \) from a given \( \mathbb{C} \)-action.

If we choose an \( \text{ad}(X_0) \)-invariant subspace \( s_0 \subset s \) such that \( g = (Z_0)c \oplus s_0 \oplus h \) is a Levi-Malcev decomposition as in Lemma 8 and compatible coordinates \( y_j \) of \( s_0 \) and \( z_0 \), then the \( \mathbb{C} \)-action on \( g/H \) is given by a triangular derivation \( \delta \) and \( \delta z_0 = 1 + P \), where \( P \) is a polynomial in the coordinates \( y_j \) without constant and linear terms. Let the depth \( d(y_j) = d \) be defined by \( Y_j \in \mathfrak{g}^{(d-1)} \setminus \mathfrak{g}^{(d)} \). We define the degree \( d(P) := \max_i \sum_j \kappa_j d(y_j) \) of a polynomial \( P = \sum_i a_\kappa \prod_j y_j^{\kappa_j} \in \mathbb{C}[s_0] \) as the usual notion of degree weighted by the depth of the variables. Since \( [\mathfrak{g}^{(d)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(d+k+1)} \), we have the inequality \( d(y_j) \geq d(\delta(y_j)) + d(x_0) \). We can estimate the size of \( G \) assuming that a given \( \mathbb{C} \)-action arises as the induced \( V \)-action on \( G/H \) using the (linear Levi-Malcev) coordinates. For example, the quotient of the action generated by \( \delta_2 \) on \( \mathbb{C}^5 \) is \( \{z_2 = 0\} \cong \mathbb{C}^4 \). The induced action generated by \( \delta_1 \) on this quotient is given by the derivation \(-y_1^2 \frac{\partial}{\partial y_2} - y_1 y_2 \frac{\partial}{\partial y_3} + (1 - y_1 y_3) \frac{\partial}{\partial z_1} \). We recursively obtain \( d(y_1) = 1, d(y_2) \geq 3, d(y_3) \geq 5 \), and \( d(z_1) \geq 7 \). This means that \( \mathfrak{g}^{(6)} \neq \{0\} \) for the corresponding nilpotent Lie algebra \( \mathfrak{g} \).

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Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum
E-mail address: annett.puettmann@rub.de