Mesh Currents and Josephson Junction Arrays

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Abstract

A simple but accurate mesh current analysis is performed on a XY model and on a SIMF model to derive the equations for a Josephson junction array. The equations obtained here turn out to be different from other equations already existing in the literature. Moreover, it is shown that the two models come from a unique hidden structure.

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1 Introduction

This is an analysis of a planar electric network made of Josephson junctions connected by superconducting wires, biased by external currents in the presence of a perpendicular external static magnetic field. Such a circuit is usually named Josephson array and it is interesting for technological and theoretical reasons. It is modeled by a system of coupled nonlinear dynamical equations, and its physics is based on the behavior of supercurrents interacting directly each other and also by induced magnetic flux. When there is no external bias, starting from any initial condition the supercurrents reach an equilibrium state and then circulate forever without dissipation, sometimes trapping flux in the network meshes and accommodating themselves in characteristic patterns which also depend on the external magnetic field. In the presence of bias these currents can rearrange and adjust to the new external conditions, up to a certain bias limit. If this limit is exceeded by the bias, an out of equilibrium dissipative state is reached and flux patterns can move through. The circuit then emits a radiation originated in its elements: if synchronization sets up, this radiation can be constructively coherent and much more energetic than the radiation from a single junction. The exploitation of this synchronization mechanism is the technical reason for the interest for such a system, which allows making usable the feeble but very high frequency and easily tunable radiation emitted from a single Josephson junction.

The theoretical interest is in the understanding of the complex dynamics underlying the dance of the flux patterns and the action of the magnetic field on both static and dynamic state. To achieve this result, the attention was focused on the choice of the model to be used to describe this system. As pointed out in Ref.[1], two seemingly different classes of models have been selected: the first one is referred to as XY model and the second one can be referred to as self-induced magnetic field (SIMF) model. The XY model derives from field theory [2] and indirectly explains many array phenomena, but is limited to systems in which array inductance is negligible. The SIMF model takes into account the induced flux interaction and then should be valid in every situation, including the case when the inductance goes smoothly to zero. Exactly at zero inductance it should merge with the XY model, but this cannot be proved because of the presence of a parameter singularity. The SIMF model is more difficult to be studied analytically than the XY model, but in the form studied in Ref.[3] it has a rare direct experimental...
evidence [4,5]. It is anyway thought to be fundamental [6] to explain some peculiar features in giant Shapiro steps and other related phenomena. In Ref.[7] a compact formalization of the SIMF model was given as an explicit dynamical system. In Ref.[8] an attempt was made to study analytically the parameter region where the two models should merge, but the singularity in the inductance parameter makes the analysis difficult. A numerical investigation of this parameter region was made in Ref.[6], but a numerical simulation drawback is the difficulty to get a global understanding of how the transition works, being limited only to finite parameter ranges. Ref.[6] is also a source for more references.

In this work two main conclusions about these models are reached. A simple but careful derivation of the equations leads to systems which are different from those existing in the previous literature. Moreover, XY and SIMF models derive from the same set of equations and the study of the transition region is easier than it was thought before. The plan of the paper is to work out a simple reference example in Section 2 and to derive in a general way the equations for a XY and a SIMF rectangular network in Section 3. Then in Section 4 the source of the difference will be briefly discussed. In Section 5 the system from which both models derive will be shown.

2 Four circuits as an example

In principle, a Josephson array can be built laying a rectangular grid of superconducting stripes on an insulating substrate so that cells surrounded by superconducting wires are formed. A rectangular array has $N_r \times N_c$ cells, where $N_r$ indicates the number of cells in one row and $N_c$ the number of cells in one column. In the examples of Fig.1 the cells are square and each circuit consists in two cells. In circuit a) conventions are displayed: nodes are labeled by the two row and column indices $(r, c)$ and reference directions for the currents are chosen. Orientated currents will be called $i_{v,r,c}$ and orientated gauge invariant phases will be called $\varphi_{v,r,c}$, where $r$ and $c$ refer to the node where they originate from, index $v = 0$ is for horizontal branches and $v = 1$ for vertical branches. Meshes are labeled only with the two indices $(r, c)$ of their bottom left hand corner and orientated clockwise. External bias currents are called $I_{r,c}$ where the indices $r$ and $c$ refer to the node where the
Figure 1: Example circuits and notation

current is injected. The Josephson junctions considered here are proximity effect junctions, which can be obtained cutting the branch wire in the center, and they are indicated by crosses. Small boxes will be explained later. The current $i$ flowing along a branch containing a junction is described by the resistively shunted (RSJ) model \[9\] as

$$i = i_n + i_s = \frac{1}{R} \frac{d}{dt} \varphi + i_c \sin \varphi$$ \hspace{1cm} (1)

in terms of the superconductor order parameter phase difference $\varphi$ between the edges of the junction: $R$ is a phenomenological resistance and $i_c$ is the maximum supercurrent that can flow in the junction. The first $i_n$ term arises from the voltage $V = \frac{\hbar}{2e} \varphi$ which sets in due to a.c. Josephson effect when $\varphi$ varies in time, the second $i_s$ term describes the $V = 0$ supercurrent that flows by Cooper pairs tunneling. To treat $\varphi$ as a real variable and not as a distribution, an assumption of very low temperature conditions is made here.

Three sets of equations will now be written \[9\] for each of the circuits b), c), d) and e) in Fig.1. Since circuits b), d) and e) have junctions in every branch they will be named pure Josephson networks while circuit c) will be named hybrid, lacking junctions in some branches.

When currents circulate in an array, each wire induces a magnetic field proportional to the current $i$ flowing in the wire itself. If this magnetic field
is not too strong, each cell is threaded only by the self-induced magnetic flux \( \phi_{r,c} \) generated by the closed current formed by the currents \( i_{v,r,c} \) circulating in the wires along the borders of the mesh \((r,c)\), plus a possible external magnetic flux \( \phi_{r,c}^{ext} \). If \( \phi_{r,c}^{tot} \) is the total flux,
\[
\phi_{r,c}^{tot} = \phi_{r,c}^{ext} - \phi_{r,c}
\] (2)

For mesh \((1,1)\) and \((1,2)\) in circuit \(b\) induced fluxes are
\[
\begin{align*}
\phi_{1,1} &= L(i_{1,1,1} + i_{0,2,1} - i_{1,1,2} - i_{0,1,1}) \\
\phi_{1,2} &= L(i_{1,1,2} + i_{0,2,2} - i_{1,1,3} - i_{0,1,2})
\end{align*}
\] (3)

where \(L\) is the cell self-inductance. In the circuits in Fig.1 some branches have small boxes attached to indicate generation of flux, while branches without boxes generate negligible flux. If a junction is present in a wire, it limits to \(i_c\) the current that can flow in it; if the junction critical current \(i_c\) is very low, the wire contributes to induced flux only with the \(i_n\) normal term, which at equilibrium is null. In fact, junctions not enclosed in boxes have critical currents \(i'_c\) negligible in comparison to others.

In this aspect hybrid array \(c\) and pure array \(d\) are analogous: in \(c\) vertical branches contributions to the flux are negligible in comparison to contributions from unbroken horizontal wires. In \(d\) junctions with very low \(i'_c < i_c\) are present in vertical branches and their induced flux is negligible in comparison to that from horizontal branches. Then for circuits \(c\) and \(d\)
\[
\begin{align*}
\phi_{1,1} &= L(i_{0,2,1} - i_{0,1,1}) \\
\phi_{1,2} &= L(i_{0,2,2} - i_{0,1,2})
\end{align*}
\] (4)

while in circuit \(e\)
\[
\phi_{1,1} = \phi_{1,2} = 0
\] (5)

which could be naively seen as a \(L \to 0\) limit of Eq.(3). Eq.(3), (4) and (5) are different instances of the first set of equations. Circuits \(b\), \(c\) and \(d\) are SIMF models while circuit \(e\) is a prototypical XY model.

The second set of equations comes from the principle that the directed sum of gauge invariant phase differences around a mesh is related to the induced flux threading the mesh \[3\]:
\[
\begin{align*}
\phi_{1,1} &= -(\varphi_{1,1,1} + \varphi_{0,2,1} - \varphi_{1,1,2} - \varphi_{0,1,1}) + f_{1,1} \\
\phi_{1,2} &= -(\varphi_{1,1,2} + \varphi_{0,2,2} - \varphi_{1,1,3} - \varphi_{0,1,2}) + f_{1,2}
\end{align*}
\] (6)
where \( f_{r,c} = 2\pi n_{r,c} + \phi_{r,c}^{ext} \) are called for historical reasons *frustrations* and \( n_{r,c} \) are integers. In \( \phi_{1,1}, \phi_{1,2}, \phi_{r,c}^{ext} \) a coefficient \( \frac{4\pi}{h} \) has been absorbed, where \( e \) is the electron charge and \( h \) the Plank constant. The sums \( (\varphi_{1,1,1} + \varphi_{0,2,1} - \varphi_{1,1,2} - \varphi_{0,1,1} + \phi_{1,1} - \phi_{r,c}^{ext}) \) and \( (\varphi_{1,1,2} + \varphi_{0,2,2} - \varphi_{1,1,3} - \varphi_{0,1,2} + \phi_{1,2} - \phi_{r,c}^{ext}) \) are named fluxoids and Eq.(8) states that fluxoids are equal to \( 2\pi \) times an arbitrary integer \( n \).

The third set of equations implements Kirchhoff current conservation law for nodes in an electric network. The oriented currents sum to zero for each node as

\[
\begin{align*}
-i_{1,1,1} - i_{0,1,1} + I_{1,1} &= 0 \\
i_{1,1,1} - i_{0,2,1} + I_{2,1} &= 0 \\
i_{0,2,1} + i_{1,1,2} - i_{0,2,2} + I_{2,2} &= 0
\end{align*}
\]

which is valid for every circuit.

From these three sets of equations a nonlinear differential system for each circuit is going to be derived, where only one phase derivative will appear explicitly for each equation. Such a system is called explicit dynamical system. For simplicity, from now until differently stated, it will be assumed that external fluxes \( \phi_{r,c}^{ext} \) and integers \( n_{r,c} \) are zero, and \( R = 1 \). A bias current \( \gamma \) enters from the upper nodes and leaves the circuit from the lower nodes as \(-\gamma\): then \( I_{2,1} = I_{2,2} = I_{2,3} = \gamma \) and \( I_{1,1} = I_{1,2} = I_{1,3} = -\gamma \). System (7) consists of 6 linear equations in 7 unknowns, but only 5 equations are linearly independent. Two free parameters \( I_{1,1}^{a} \) and \( I_{1,2}^{a} \) are introduced, named mesh currents, to solve it:

\[
\begin{align*}
i_{1,1,1} &= I_{1,1}^{a} - \gamma \\
i_{1,1,2} &= -I_{1,1}^{a} + I_{1,2}^{a} - \gamma \\
i_{1,1,3} &= -I_{1,2}^{a} - \gamma \\
i_{0,1,1} &= -I_{1,1}^{a} \\
i_{0,1,2} &= -I_{1,2}^{a} \\
i_{0,2,1} &= I_{1,1}^{a} \\
i_{0,2,2} &= I_{1,2}^{a}
\end{align*}
\]

Direct substitution in system (7) works as a proof. Induced fluxes in systems (8) and (4) can be expressed in terms of Eq.(8) too:

\[
\begin{align*}
(3) \rightarrow \begin{cases} 
\phi_{1,1} = L(4I_{1,1}^{a} - I_{1,2}^{a}) \\
\phi_{1,2} = L(-I_{1,1}^{a} + 4I_{1,2}^{a})
\end{cases}, \quad (4) \rightarrow \begin{cases} 
\phi_{1,1} = 2LI_{1,1}^{a} \\
\phi_{1,2} = 2LI_{1,2}^{a}
\end{cases}
\end{align*}
\]

An immediate remark has to be made here: the system from (3) relates the self-induced flux \( \phi_{r,c} \) with all the possible mesh currents whereas the system
from (8) relates the flux $\phi_{r,c}$ only to the mesh current $I^a_{r,c}$ bearing the same indices $(r, c)$. These systems can be inverted to give

$$\begin{align*}
(3) \rightarrow \begin{cases} 
I^a_{2,1} = \frac{1}{4L} \left( 4\phi_{1,1} + \phi_{1,2} \right) \\
I^a_{2,2} = \frac{1}{4L} \left( \phi_{1,1} + 4\phi_{1,2} \right)
\end{cases},
(10) \rightarrow \begin{cases} 
I^a_{1,1} = \frac{1}{4L} \phi_{1,1} \\
I^a_{1,2} = \frac{1}{2L} \phi_{1,2}
\end{cases}
$$

and such mesh currents can be inserted in Eq.(8), which now relates branch currents $i_{v,r,c}$ to self-induced fluxes $\phi_{r,c}$.

The differential equations for the circuit $b)$ can now be written substituting in system (8) Eq.(11) for the l.h.s. and the first block of Eq.(11) for the r.h.s., where $\phi_{r,c}$ are obtained in turn from Eq.(8)

$$b) \begin{cases} 
\dot{\phi}_{0,1,1} + i_c \sin(\phi_{0,1,1}) = \\
\frac{1}{8\gamma L} (4\phi_{0,1,1} + \phi_{0,1,2} - 4\phi_{1,1,1} + 3\phi_{1,1,2} + \phi_{1,1,3} - 4\phi_{0,2,1} - \phi_{0,2,2}) \\
\dot{\phi}_{0,1,2} + i_c \sin(\phi_{0,1,2}) = \\
\frac{1}{8\gamma L} (\phi_{0,1,1} + 4\phi_{0,1,2} - \phi_{1,1,1} - \phi_{1,1,2} + 4\phi_{1,1,3} - 4\phi_{0,2,1} - \phi_{0,2,2}) \\
\dot{\phi}_{1,1,1} + i_c \sin(\phi_{1,1,1}) = \\
\frac{1}{8\gamma L} (-4\phi_{0,1,1} - \phi_{0,1,2} + 4\phi_{1,1,1} - 3\phi_{1,1,2} - \phi_{1,1,3} + 4\phi_{0,2,1} + \phi_{0,2,2}) - \gamma \\
\dot{\phi}_{1,1,2} + i_c \sin(\phi_{1,1,2}) = \\
\frac{1}{8\gamma L} (-3\phi_{0,1,1} - 3\phi_{0,1,2} - 3\phi_{1,1,1} + 6\phi_{1,1,2} - 3\phi_{1,1,3} - 3\phi_{0,2,1} + 3\phi_{0,2,2}) - \gamma \\
\dot{\phi}_{1,1,3} + i_c \sin(\phi_{1,1,3}) = \\
\frac{1}{8\gamma L} (\phi_{0,1,1} + 4\phi_{0,1,2} - \phi_{1,1,1} + 3\phi_{1,1,2} + 4\phi_{1,1,3} - \phi_{0,2,1} - 4\phi_{0,2,2}) - \gamma \\
\dot{\phi}_{0,2,1} + i_c \sin(\phi_{0,2,1}) = \\
\frac{1}{8\gamma L} (-4\phi_{0,1,1} - \phi_{0,1,2} + 4\phi_{1,1,1} - 3\phi_{1,1,2} - \phi_{1,1,3} + 4\phi_{0,2,1} + \phi_{0,2,2}) \\
\dot{\phi}_{0,2,2} + i_c \sin(\phi_{0,2,2}) = \\
\frac{1}{8\gamma L} (-\phi_{0,1,1} - 4\phi_{0,1,2} + \phi_{1,1,1} + 3\phi_{1,1,2} - 4\phi_{1,1,3} + \phi_{0,2,1} + 4\phi_{0,2,2})
\end{cases}
$$

This result is remarkably different in the form of the interaction among phases from those in Ref.[7] and [13]. Here in each equation are present all the phases of the system, while there in each equation is present only a subset of the whole set of phases.

The differential equations for the hybrid circuit $c)$ are obtained in a slightly different way, because orizontal branch currents $i_{0,r,c}$ are not described by Eq.(4). In this case branches are considered as containing inductances and not as simple shorts. Substituting $\phi_{r,c}$ from Eq.(11) in the second block of Eq.(11) gives $I^a_{r,c}$ in terms of $i_{v,r,c}$. In turn $I^a_{r,c}$ are inserted in system (8). A set of 4 relations $i_{0,1,1} = -\frac{1}{2}(i_{0,2,1} - i_{0,1,1}), i_{0,2,1} = \frac{1}{2}(i_{0,2,1} - i_{0,1,1}),$
\( i_{0,1,2} = -\frac{1}{2}(i_{0,2,2} - i_{0,1,2}) \) and \( i_{0,2,2} = \frac{1}{2}(i_{0,2,2} - i_{0,1,2}) \) is obtained and it is satisfied iff \( i_{0,1,1} = -i_{0,2,1} \) and \( i_{0,1,2} = -i_{0,2,2} \). This implies that the 6 equations in system (7) are reduced to the 3 equations

\[
\begin{cases}
\dot{i}_{1,1,1} = i_{0,2,1} - \gamma \\
\dot{i}_{1,1,2} = i_{0,1,1} - \gamma \\
\dot{i}_{1,1,3} = i_{0,2,2} - \gamma
\end{cases}
\]

(11)

where Eq. (11) is to be applied only to the l.h.s. terms. Eq. (9) for circuit c) has to be modified in \( \varphi_{1,1} = -(\varphi_{1,1,1} - \varphi_{1,1,2}) \) and \( \varphi_{1,2} = -(\varphi_{1,1,2} - \varphi_{1,1,3}) \), because only two junctions are present in each cell. Using this relation gives

\[
c) \begin{cases}
\dot{\varphi}_{1,1,1} + i_c \sin(\varphi_{1,1,1}) = -\frac{1}{2L} (\varphi_{1,1,1} - \varphi_{1,1,2}) - \gamma \\
\dot{\varphi}_{1,1,2} + i_c \sin(\varphi_{1,1,2}) = \frac{1}{2L} (\varphi_{1,1,1} - 2\varphi_{1,1,2} + \varphi_{1,1,3}) - \gamma \\
\dot{\varphi}_{1,1,3} + i_c \sin(\varphi_{1,1,3}) = \frac{1}{2L} (\varphi_{1,1,2} - \varphi_{1,1,3}) - \gamma
\end{cases}
\]

This result is the same as that in Ref.[3], where it was derived for an infinite stripe of cells. For this circuit, in contrast to the circuit b), in each equation is present only a subset of all possible phases. This is clearly due to the fact that in every mesh of this circuit the self-induced flux \( \varphi_{r,c} \) is proportional only to the \( I_{r,c} \) with the same indices \( (r, c) \).

The differential equations for the pure array d) are derived in the same way as for circuit b), substituting in system (8) Eq. (11) for the l.h.s. and the second block of Eq. (11) for the r.h.s., where in turn \( \varphi_{r,c} \) are obtained from Eq. (11):
This system has the same polarization scheme as systems b) and c) and a limited interaction among phases: only in the differential equation for phase $\varphi_{1,1,2}$ all the other phases appear. This is again due to the proportionality of mesh currents and induced fluxes. It seems having never been discussed in the literature, even though it has an interaction equal to that attributed to circuit b) in Ref.[8].

The differential equations for the last circuit e) are obtained in another different way. In this case induced fluxes are zero, so it is not possible to use them to eliminate the mesh currents in system (8). Anyway, Eq.(8) is still valid and can be differentiated. Inserting Eq.(11) in system (8) together with differentiated Eq.(12) gives a system that can be rewritten as

$$
\begin{align*}
\varphi_{0,1,1} + i_e \sin(\varphi_{0,1,1}) &= -I_{1,1}^a \\
\varphi_{0,1,2} + i_e \sin(\varphi_{0,1,2}) &= -I_{1,2}^a \\
\varphi_{1,1,1} + i_e \sin(\varphi_{1,1,1}) &= I_{1,1}^a - \gamma \\
\varphi_{1,1,2} + i_e \sin(\varphi_{1,1,2}) &= -I_{1,1}^a + I_{1,2}^a - \gamma \\
\varphi_{1,1,3} + i_e \sin(\varphi_{1,1,3}) &= -I_{1,2}^a - \gamma
\end{align*}
$$

Eliminating the last two equations allows to obtain explicitly the mesh currents that are now expressed in terms of sine functions of phases:

$$
\begin{align*}
I_{1,1}^a &= \frac{i_e}{15} (-4 \sin(\varphi_{0,1,1}) - \sin(\varphi_{0,1,2}) + 4 \sin(\varphi_{1,1,1}) + \\
&\quad -3 \sin(\varphi_{1,1,2}) - \sin(\varphi_{1,1,3}) + 4 \sin(\varphi_{0,2,1}) - \sin(\varphi_{0,2,2})) \\
I_{1,2}^a &= \frac{i_e}{15} (- \sin(\varphi_{0,1,1}) - 4 \sin(\varphi_{0,1,2}) + \sin(\varphi_{1,1,1}) + \\
&\quad + \sin(\varphi_{1,1,2}) - 4 \sin(\varphi_{1,1,3}) + \sin(\varphi_{0,2,1}) + 4 \sin(\varphi_{0,2,2}))
\end{align*}
$$

After inserting them in Eq(8), an explicit system of 7 differential equations (of which only 5 are independent) results. To give a synthetic idea of its structure, only two equations are presented here:

$$
\begin{align*}
\varphi_{0,1,1} + i_e \sin(\varphi_{0,1,1}) &= \frac{i_e}{15} (4 \sin(\varphi_{0,1,1}) + \sin(\varphi_{0,1,2}) - 4 \sin(\varphi_{1,1,1}) + \\
&\quad +3 \sin(\varphi_{1,1,2}) + \sin(\varphi_{1,1,3}) - 4 \sin(\varphi_{0,2,1}) - \sin(\varphi_{0,2,2})) \\
\vdots \\
\varphi_{0,2,2} + i_e \sin(\varphi_{0,2,2}) &= \frac{i_e}{15} (- \sin(\varphi_{0,1,1}) - 4 \sin(\varphi_{0,1,2}) + \sin(\varphi_{1,1,1}) + \\
&\quad +3 \sin(\varphi_{1,1,2}) - 4 \sin(\varphi_{1,1,3}) + \sin(\varphi_{0,2,1}) + 4 \sin(\varphi_{0,2,2}))
\end{align*}
$$

Remarkably, this system is very different from the previous ones, having no term linear in $\varphi_{v,r,c}$. It has the same polarization scheme as the other circuits.
and the same coefficients as b) in front of the interaction terms, which are now sine functions of phases instead of being simply phases. It is definitely not obtained as a direct $L \to 0$ limit from system b).

3 General derivation of the equations

One of the two main points of this paper is that a careful derivation of the equations for the circuits b), d) and e) leads to systems of equations different from those already existing in the literature. In the previous example it was shown that a series of manipulations leads to an interaction that has a range longer than what precedently thought. This point will be discussed more explicitly in the next section. In Section 5 it will be shown that, even though the three circuits seem to be rather different systems, their equations can be derived from a unique hidden structure and that it is misleading to consider the limit $L \to 0$ directly from equations like those just obtained. To discuss more easily the physics involved, two steps will be followed. First, the generalization of the preceding derivation will be illustrated, then a new set of variables will be discussed. In these variables it will be possible to smoothly handle the limit $L \to 0$.

For any $N_r \times N_c$ array the relation

$$n_m = n_b - n_n + 1$$

holds, where $n_m = N_r N_c$ is the number of meshes, $n_b = 2N_r N_c + N_r + N_c$ is the number of branches and $n_n = (N_r + 1)(N_c + 1)$ is the number of nodes. Using a matrix notation, the current vector $i = \{i_{v,r,c}\} = \{i_k, k = 1, \ldots, n_b\}$, the phase vector $\varphi = \{\varphi_{v,r,c}\} = \{\varphi_k, k = 1, \ldots, n_b\}$, the mesh current vector $I^a = \{I^a_{r,c}\} = \{I_k, k = 1, \ldots, n_m\}$, the external bias current vector $T_a = \{T_{r,c}\} = \{I_k, k = 1, \ldots, n_m\}$ and the frustration vector $f = \{f_{r,c}\} = \{f_k, k = 1, \ldots, n_m\}$ are introduced. In current and phase vectors the elements are progressively ordered by an index $k$ obtained from the indices of their elements as $k = (2N_c + 1)(r - 1) + v N_c + c$, $r = 1, \ldots, N_r$, $c = 1, \ldots, N_r + v$: for example, current $i_{11,2}$ of circuit a) becomes $i_4$. In words, starting from the bottom left hand corner and following the row of nodes, first come the horizontal branches then come the vertical branches, and then next node row is scanned. In frustration and external current vectors the elements are progressively ordered by an index $k = N_c (r - 1) + c$,
where $r = 1 \ldots , N_r$ and $c = 1 \ldots , N_c$, and the same order is assigned to the meshes, considered as rings clockwise oriented. Two matrices $A_a$ and $M_a$ are now built [10]. The matrix $A_a$, with $n_n$ rows and $n_b$ columns, describes the Kirchhoff law for current conservation at the nodes. Its element $(A_a)_{m,n}$ is 1 if in the $m$-th node the $n$-th current enters the node, it is $-1$ if the current leaves it, it is 0 otherwise. The matrix $M_a$, with $n_m + 1$ rows and $n_b$ columns, describes the sums around meshes. Its element $(M_a)_{m,n}$ is 1 if in the $m$-th mesh the $n$-th current belongs to the mesh and is oriented in the same direction of the mesh, it is $-1$ if the current is oriented in the opposite direction and 0 otherwise. The matrix $M_a$ has one row more than $n_m$ because it includes the so called external mesh, which follows the rule that $(M_a)_{n_b+1,n}$ is $-1$ if the $n$-th current is on the boundary of the array and is oriented clockwise following the boundary ring, 1 if it is oriented in the opposite direction and 0 otherwise. It can be proved using network theory [10] that rows of $A_a$ and $M_a$ are mutually orthogonal, and this is also seen by direct inspection. Since rank$(A_a) = n_n - 1$ and rank$(M_a) = n_m$, from Eq.(12) rank$(A_a) + \text{rank}(M_a) = n_b$ follows, which means that $A_a$ and $M_a$ divide the $n_b$ dimensional space in two parts. Deleting a row in each of the two matrices does not change their rank: as a convention the last row will be deleted in $A_m$ to form a new matrix $A$, and the last row will be deleted in $M_m$ to form a new matrix $M$. These two matrices have maximum rank and the property

$$
\begin{align*}
&MAT = 0, \quad AMT = 0 \\
&M^T(MM^T)^{-1}M + A^T(AA^T)^{-1}A = 1
\end{align*}
$$

having $AA^T$ and $MM^T$ non-zero determinant. They appear to be dual in the sense of linear algebra and the quantities

$$
\begin{align*}
&M^T(MM^T)^{-1}M = K \\
&A^T(AA^T)^{-1}A = \overline{K}
\end{align*}
$$

can be seen as projectors on the $n_b$ dimensional space. The induced flux vector can be introduced as

$$
\phi_{r,c} = M^Li
$$

where the self-induced flux matrix $M^L$ expresses a relation analogous to Eq.(3) or Eq.(4). This matrix is built from $M$ substituting in $M$ for every non-zero entry the branch inductance $\lambda_{m,n}$ if the entry is 1 and $-\lambda_{m,n}$
if the entry is \(-1\). Eq.(3) is recovered when all \(\lambda_{m,n}\) are equal to \(L\), that is \(M^L = LM\). In the general case \(M^L A^T \neq 0\) and \(A(M^L)^T \neq 0\). The last element is deleted from vector \(T_a\) to form vector \(T\). Since \(T_a\) elements obey current conservation law \(\sum_k I_k = 0\), the deleted element is always obtained as \(I_{n_m} = -\sum_{k=1}^{n_m} I_k\). In this notation equations Eq.(7), Eq.(6) and Eq.(1) can be rewritten as

\[
\begin{align*}
\begin{cases}
Ai = T \\
M^L i + M \varphi = f \\
\varphi + i_c \sin_k(\varphi)_k = i
\end{cases}
\end{align*}
\]

where in \(\sin_k(\varphi)_k\) the label \(k\) is to remind the vectorial form. It is stressed that notation \(F \sin_k(G)_k\) indicates that the generic row \(k\) of matrix \(G\) appears as the argument of the \(k\)-th element of a vector defined as \(\sin_k(G)_k = \{\sin(G_t), t = 1, \ldots, n_b\}\), on which matrix \(F\) operates: e.g., if \(F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \)

and \(G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\) then \(F \sin_k(G)_k = \begin{pmatrix} F_{1,1} \sin G_1 + F_{1,2} \sin G_2 \\ F_{2,1} \sin G_1 + F_{2,2} \sin G_2 \end{pmatrix}\). Clearly, \(k\) is not an index. The equations for circuit \(b\) are now quickly rederived. The general solution of \(Ai = T\) is the sum of the solution of the homogeneous equation \(Ax = 0\) and a particular solution of \(Ax = T\). Using Eq.(13), \(i = M^T I^a + A^T D\) where \(D = (AA^T)^{-1} T\). Operating with \(M^L\) on both sides of this equation results in \(M^L i = M^L M^T I^a + M^L A^T D\), Eq.(3), that after inversion gives \(I^a = (M^L M^T)^{-1} M^L i - (M^L M^T)^{-1} M^L A^T D\), Eq.(10). From the second equation of system (16) it follows that \(M^L i = f - M \varphi\), Eq.(3), which inserted in the equation for \(I^a\) gives \(I^a = - (M^L M^T)^{-1} M \varphi + (M^L M^T)^{-1} f - (M^L M^T)^{-1} M^L A^T D\). Finally, using the Josephson equation and defining

\[
\begin{align*}
\begin{cases}
K = M^T(MM^T)^{-1} M \\
K^L = M^T(M^L M^T)^{-1} M \\
K^{LL} = M^T(M^L M^T)^{-1} M^L
\end{cases}
\end{align*}
\]

the equation

\[
\varphi + i_c \sin_k(\varphi)_k = -K^L \varphi + M^T(M^L M^T)^{-1} f - (K^{LL} - 1) A^T D
\]

for systems \(b\) and \(d\) is recovered. A comment about the conditions under which \(M^L M^T\) can be inverted is needed. As a general rule \(\det(M^L M^T) \neq 0\) if at least \(n_m\) branch inductances are non-zero and every mesh has at least
one branch with non-zero inductance. This is the case of circuits b) and d) but not of circuit e). Moreover, it can be shown that for any array dimension no element of $K$ is zero, so that $K$ generates an infinite range interaction among phases.

In the case of circuit e) starting equations are

$$
\begin{align*}
  Ai &= T \\
  M\varphi &= f \\
  \dot{\varphi} + i_c \sin_k(\varphi)_k &= i 
\end{align*}
$$

If $f = 0$ it follows that $M \dot{\varphi} = 0$. Operating with $M$ on the Josephson equation gives $M \dot{\varphi} = M i_c \sin_k(\varphi)_k + Mi$, where the l.h.s. member is zero. Inserting in this equation the solution $i = MT I^a + AT D$ gives $Mi \sin_k(\varphi)_k = MM^T I^a$, where Eq. (13) has been used. After inversion, this results in $I^a = (MM^T)^{-1} Mi_c \sin_k(\varphi)_k$. Using again the Josephson equation $\dot{\varphi} + i_c \sin_k(\varphi)_k = MT I^a + AT D$ and Eq. (13) the equation for e) quickly follows:

$$
\dot{\varphi} + i_c \sin_k(\varphi)_k = i_c K \sin_k(\varphi)_k + AT D
$$

If $f \neq 0$ this method doesn’t work, because it gives again the same equation without any trace of $f$. The use of $M\varphi = f$ as a constraint is critical. When differentiated this equation is always $M \dot{\varphi} = 0$ independently of $f$. This problem can be avoided introducing another variable $p$ as already done with $I^a$. This variable will be called cut phase, in analogy with network theory terminology. More precisely, the equation $M\varphi = f$ can be solved by the substitution $\varphi = AT p + MT s$ where $s = (MM^T)^{-1} f$. The quantity $M \dot{\varphi}$ is still zero but at the end of the calculation the result is

$$
AT \dot{p} = (K - 1) i_c \sin_k(A^T p + M^T (MM^T)^{-1} f)_k + AT D
$$

which can be multiplied by $A$ and inverted to give

$$
\dot{p} = -(AA^T)^{-1} A i_c \sin_k(A^T p + M^T (MM^T)^{-1} f)_k + (AA^T)^{-1} T
$$

where Eq. (13) has been used. After solving this differential system, phases $\varphi$ are recovered as $\varphi(t) = AT p(t) + MT s$. This new variable $p$ seems to better suit to the $L = 0$ problem, transforming it in a system of independent and explicit differential equations. Unlike the case of Eq. (16) it is not possible to eliminate $p$ at the end of the calculation, as it was did with the intermediate
variable $I^a$. The reason stands on the fact that system (19) is constrained by the $n_m$ static equations $M\varphi = f$ and only $n_b - n_m$ variables are left independent, so that the whole dynamics depends only on these new variables. Applying this new point of view to the circuit $e$), the system of 5 independent explicit differential equations in the 5 $p_l$ ($l = 1, \ldots, 5$) variables

$$
\frac{\dot{p}_1}{i_c} = \begin{pmatrix}
 9 & 6 & 6 & 3 & 6 & 6 & 9 \\
-2 & 7 & 2 & 6 & 7 & 2 & 8 \\
-1 & -4 & 1 & 3 & 11 & 1 & 4 \\
5 & 5 & -5 & 0 & 5 & 10 & 10 \\
1 & 4 & -1 & -3 & 4 & -1 & 11 \\
\end{pmatrix}
\begin{pmatrix}
\sin(-p_1 + p_2) \\
\sin(-p_2 + p_3) \\
\sin(-p_1 + p_4) \\
\sin(-p_2 + p_5) \\
\sin(-p_3) \\
\sin(-p_4 + p_5) \\
\sin(-p_5) \\
\end{pmatrix}
\gamma
$$

(23)

results.

This idea suggests a way to write the general system in a form suitable to take the $L \to 0$ limit in a smooth way, but before undertaking this task the difference between Eq.(18) and other existing equations will be discussed.

4 Mesh currents and induced fluxes

It is stated in Eq.(6) from Ref.[11] that for an array where only the self-inductance of each lattice mesh is retained, and the mutual inductance among cells is ignored, the relation between the total flux $\phi_{ij}$ in one mesh and the mesh current of the same mesh $I_{ij}$ is

$$
\phi_{ij}^{\text{tot}} = \phi_{\text{ext}} - L'I_{ij}^a
$$

(24)

This means that the self-induced flux is proportional to the mesh current. This equation is used also in Ref.[3], [7] and [12] and implicitly in Ref.[13]. In the present work it has been shown for hybrid circuit $c$), which is a particular $1 \times 2$ case of the $1 \times \infty$ array of [3], that Eq.(24) is correct because in this particular case the induced fluxes are indeed proportional to the mesh currents. From the second block of Eq.(10) it is seen that Eq.(24) is also valid for the pure circuit $d$), but it is not in general valid, e.g. when the magnetic fields induced by the four branch currents surrounding a cell are all of the same strenght. This happens for example in circuit $b$), as it is seen from
the first block of Eq.(10). This is the source of the difference between the
equation for circuit \( b \) derived here and the equations derived using Eq.(24).
Yet, there is a case in which Eq.(24) can be used also for circuits similar to
\( b \) where all the four branches contribute in the same way to the flux. Now
it will be shown that in the formalism developed in this paper it is easy to
work out an interaction in which Eq.(24) is anyway valid. To achieve this
goal, the mutual inductance among meshes has to be taken into account.

The task is to link mesh currents and mutually induced fluxes in such a
way that

\[
\phi_{r,c} = M^{MI}i = M^{MI}M^T I^a = p I^a_{r,c}
\]

where in analogy with Eq.(15) \( M^{MI} \) is a mutually induced flux matrix
and \( p \) is an adjustable constant. For circuit \( b \) in the example of Fig.4,

\[
M^{MI} = \left( \begin{array}{cccc}
-\lambda^{SI} & -\lambda^{FN} & \lambda^{SI} & (-\lambda^{SI} + \lambda^{FN}) \\
-\lambda^{FN} & -\lambda^{SI} & \lambda^{FN} & (-\lambda^{FN} + \lambda^{SI}) \\
\lambda^{SI} & \lambda^{FN} & \lambda^{SI} & \lambda^{FN} \\
\end{array} \right)
\]

where \( \lambda^{SI} \) is a self-inductance and \( \lambda^{FN} \) is a first neighbour mutual inductance,
and

\[
M = \left( \begin{array}{cccc}
-1 & 0 & 1 & -1 \\
0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{array} \right)
\]

Requiring that \( M^{MI}M^T = p 1 \), where 1 is the unitary matrix, gives the system

\[
\begin{align*}
4\lambda^{SI} - \lambda^{FN} &= 1 \\
4\lambda^{FN} - \lambda^{SI} &= 0 \\
4\lambda^{FN} - \lambda^{SI} &= 0 \\
4\lambda^{SI} - \lambda^{FN} &= 1
\end{align*}
\]

which is solved by \( \lambda^{FN} = \frac{\lambda^{SI}}{4} \) and \( p = \frac{15}{4} \). When such an interaction is
taken into account, Eq.(24) is recovered. This model can be called MIMF
(Mutually Induced Magnetic Flux) model. From this result an immediate
consideration follows. In a circuit like \( b \) with only self-inductance taken into
account, the interaction among phases is global: every phase of the system
appears in each differential equation. It is interesting to realize that a local
phase interaction typical of a circuit like \( d \), where in each equation only
a small subset of the phases appears, can be obtained introducing in \( b \) a
global mutual inductance coupling. In this sense interaction range in mutual
inductance among cells and interaction range in coupling among phases are dual.

5 The $L \to 0$ limit

To focus attention on the structure of the limit, it will be assumed that in Eq. (13) $M^L = LM$, which means that circuit $b$ will now be studied. Equation $M(Li + \varphi) = f$ can be solved by setting $q = Li + \varphi$ where $q = ATp + MTs$. Equation $Ai = T$ is solved by $i = MTIa + ATD$, and merging these two relations gives

$$\varphi = ATp - LM^T Ia + MTs - LA^TD$$

Inserting this definition in the Josephson relation gives

$$AT \dot{p} - LM^T Ia + i_c \sin k(ATp - LM^T Ia + MTs - LA^TD)k = MTIa + ATD$$

In a block matrix form, defining the matrices

$$B^T = (AT, -LM^T), \quad B^Ia = (0, MT)$$

and the vector $c = (p, Ia)$, Eq. (27) is rewritten as

$$B^T \dot{c} = -i_c \sin k(B^Tc + MTs - LA^TD)k + B^Ia c + AT D$$

which after inversion is the differential system

$$\begin{cases}
\dot{p} = -(AA^T)^{-1}A i_c \sin k(ATp - LM^T Ia + MTs - LA^TD)k + D \\
L \dot{I}a = (MM^T)^{-1}M i_c \sin k(ATp - LM^T Ia + MTs - LA^TD)k - Ia \\
D = (AA^T)^{-1}T, \quad s = (MM^T)^{-1}f
\end{cases}$$

In these variables the limit $L \to 0$ is easier to handle than in a form like

$$\varphi + i_c \sin(\varphi) = \frac{1}{L} \varphi$$

In the first row of Eq. (30) the interaction between $p$ and $LIa$ smoothly vanishes and at $L = 0$ Eq. (22) is recovered. In the second row the dynamics gets smoothly frozen and at $L = 0$ a static constraint $Ia = (MM^T)^{-1}i_c M$
\( \sin_k(A^T p + M^T s)_k \) is imposed to the space of the degrees of freedom (d.o.f.). The variables \( p \) dynamically decouple from \( I^a \) and \( I^a \) depend statically on \( p \). Multiplying this equations by \( MM^T \) gives \( Mi = Mi_c \sin_k(A^T p + M^T s)_k \) which going back to \( \phi \) is the condition \( M(i - i_c \sin(\phi)) = 0 \) that follows from \( M \phi = 0 \). The \( n_b \) dimensional d.o.f. space is separated in two sectors, the \( n_m \) dimensional mesh currents sector and the \( n_b - n_m \) dimensional cut phases sector, and dynamics survives only in the cut phases sector. This is a nice example of what could be called a parametric dynamical bifurcation, in analogy with the usual parametric Hopf-like bifurcation \([14]\) of dynamical system theory. When this differential system is to be studied by perturbation theory a singular problem is obtained, but in this form the stiffness is less harmful. When \( L \) is very small but not zero, the time variable \( t \) in the derivative \( \frac{d}{dt} I^a \) can be replaced by \( t/L \) to use a dilated time scale. In this scale the \( p \) appear almost frozen. The almost slaved \( I^a \) tend to follow the dynamics of the \( p \), yet display a second faster but strongly damped dynamics. Moreover, the behaviour of Eq.(30) under \( L \rightarrow 0 \) limit is physically satisfactory, because it is hardly believable that changing smoothly the inductance of these circuits, e.g. making them bigger, should give some sharp effect only and exactly at \( L = 0 \). When \( M^L \) instead of \( LM \) is used, notation becomes heavier, but it can be shown that only \( I^a_k \) belonging to meshes with zero inductances around the cell become constraints while the remaining \( I^a_k \) keep on taking part in the dynamics.

Eq.(30) hides a nice asymmetry or duality between external currents and external magnetic fluxes. To make it explicit, for \( L \neq 0 \) the change of variables is made

\[
\begin{align*}
\{ & u = p - LD \\
& w = -LI^a + s
\end{align*}
\tag{32}
\]

Multiplying the first row of Eq.(30) by \( A^T \) and the second row by \( M^T \), and substituting in it Eq.(32), gives

\[
\begin{align*}
A^T \dot{u} &= -K i_c \sin_k(A^T u + LMT w)_k + A^T D \\
M^T \dot{w} &= -Ki_c \sin_k(A^T u + LMT w)_k - \frac{1}{L} M^T w + \frac{1}{L} M^T s
\end{align*}
\tag{33}
\]

and taking away again \( A^T \) from the first row and \( M^T \) from the second row gives

\[
\begin{align*}
\dot{u} &= -(AA^T)^{-1} A i_c \sin_k(A^T u + LMT w)_k + D \\
\dot{w} &= -(MM^T)^{-1} Mi_c \sin_k(A^T u + LMT w)_k - \frac{1}{L} w + \frac{1}{L} s
\end{align*}
\tag{34}
\]
In this form it is clear that external currents $D$ act as a bias only for the phase sector while external magnetic fluxes $s$ act as a bias only for the mesh current sector. This is consistent with the physics of the system. Phases can grow indefinitely for suitable current biases and cannot be stopped by magnetic counterfields. Mesh currents instead are subject to a confining potential and an external bias cannot let them grow indefinitely, it can just modify their confined dynamics. In this case, it is seen from Eq. (34) that only an external magnetic field and not a bias current can change the position of mesh current minima, which are the flux dance steps.

6 Conclusions

The first conclusion is that in a mesh analysis a Josephson junction array SIMF model is described by an equation like Eq. (18). For a circuit like $b$) the interaction range of phases is infinite so that it may be not necessary to take into account mutual inductance effects to explain finer system features like those studied in Ref. [3] or in Ref. [7]. The second conclusion is that the $L \to 0$ limit of the SIMF model is actually the XY model and with Eq. (30) this limit can be studied analytically in a way easier than thought before.

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