BF Theories and 2-knots

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Abstract
We discuss the relations between (topological) quantum field theories in 4 dimensions and the theory of 2-knots (embedded 2-spheres in a 4-manifold). The so-called BF theories allow the construction of quantum operators whose trace can be considered as the higher-dimensional generalization of Wilson lines for knots in 3-dimensions. First-order perturbative calculations lead to higher dimensional linking numbers, and it is possible to establish a heuristic relation between BF theories and Alexander invariants. Functional integration-by-parts techniques allow the recovery of an infinitesimal version of the Zamolodchikov tetrahedron equation, in the form considered by Carter and Saito.

1 Introduction

In a seminal work, Witten [1] has shown that there is a very deep connection between invariants of knots and links in a 3-manifold and the quantum Chern–Simons field theory.

One of the questions that is possible to ask is whether there exists an analog of Witten’s ideas in higher dimensions. After all, knots and links are defined in any dimension (as embeddings of \(k\)-spheres into a \((k+2)\)-dimensional space), and topological quantum field theories exist in 4 dimensions [2, 3].

The question, unfortunately, is not easy to answer. On one hand, the theory of higher dimensional knots and links is much less developed; the relevant invariants are, for the time being, more scarce than in the theory of ordinary knots and links. As far as 4-dimensional topological field theories are concerned, only the general BRST structure has been thoroughly discussed; perturbative calculations at least in the non-abelian case, do not appear to have been carried out. Even the quantum observables have not been clearly defined (in the non-abelian case). Moreover, Chern–Simons theory in 3 dimensions benefitted
greatly from the results of (2-dimensional) conformal field theory. No analogous help is available for 4-dimensional topological field theories.

In this paper we make some preliminary considerations and proposals concerning 2-knots and topological field theories of the BF type. Specifically we propose a set of quantum observables associated to surfaces embedded in 4-space. These observables can be seen as a generalization of ordinary Wilson lines in 3 dimensions. Moreover, the framework in which these observables are defined fits with the picture of 2-knots as “movies” of ordinary links (or link-diagrams) considered by Carter and Saito (see references below). It is also possible to find heuristic arguments that may relate the expectation values of our observables in the BF theory with the Alexander invariants of 2-knots.

As far as perturbative calculations are concerned, they are much more complicated then in the lower dimensional case. Nevertheless it is possible to recover a relation between first-order calculations and (higher-dimensional) linking numbers, which parallels a similar relation for ordinary links in a 3-dimensional space.

Finally, in 4 dimensions it is also possible to apply functional integration-by-parts techniques. In this case, these techniques produce a solution of an infinitesimal Zamolodchikov tetrahedron equation, consistent with the proposal made by Carter, Saito and Lawrence.

2 BF theories and their geometrical significance

We start by considering a 4-dimensional Riemannian manifold $M$, a compact Lie group $G$ and a principal bundle $P(M, G)$ over $M$. If $A$ denotes a connection on such a bundle, then its curvature will be denoted by $F_A$ or simply by $F$. The space of all connections will be denoted by the symbol $\mathcal{A}$ and the group of gauge transformations by the symbol $\mathcal{G}$. The group $G$, with Lie Algebra $\text{Lie}(G)$, will be, in most cases, the group $SU(n)$ or $SO(n)$.

Looking for possible topological actions for a 4-dimensional manifold, we may consider ([2],[3]???) the (Gibbs measures of the):

1. “Chern” action

$$\exp\left(-\kappa \int_M \text{Tr}_\rho(F \wedge F)\right), \quad (1)$$

and of the

2. “BF” action

$$\exp\left(-\lambda \int_M \text{Tr}_\rho(B \wedge F)\right), \quad (2)$$
In the formulas above \( \kappa \) and \( \lambda \) are coupling constants, \( \rho \) is a given representation of the group \( G \) and of its Lie algebra. The curvature \( F \) is a 2-form on \( M \) with values in the associated bundle \( P \times_{Ad} \text{Lie}(G) \) or, equivalently, a tensorial 2-form on \( P \) with values in \( \text{Lie}(G) \).

The field \( B \) in eqn (2) is assumed (classically) to be a 2-form of the same nature as \( F \). We shall see though, that the quantization procedure may force \( B \) to take values in the universal enveloping algebra of \( \text{Lie}(G) \). It is then convenient to discuss the geometrical aspects of a more general \( BF \) theory, where \( B \) is assumed to be a 2-form on \( M \) with values in the associated bundle \( P \times_{Ad} \mathcal{U}G \) where \( \mathcal{U}G \) is the universal enveloping algebra of \( \text{Lie}G \). In other words \( B \) is a section of the bundle:

\[
\Lambda^2(T^*M) \otimes (P \times_{Ad} \mathcal{U}G).
\]

The space of forms with values in \( P \times_{Ad} \mathcal{U}G \) will be denoted by the symbol \( \Omega^*(M, \text{ad}\mathcal{P}) \): this space naturally contains \( \Omega^*(M, \text{ad}\mathcal{P}) \), the space of forms with values in \( P \times_{Ad} \text{Lie}(G) \).

In order to obtain an ordinary form on \( M \) from an element of \( \Omega^*(M, \text{ad}\mathcal{P}) \), one needs to take the trace with respect to a given representation \( \rho \) of \( \text{Lie}(G) \). Notice that the wedge product for forms in \( \Omega^*(M, \text{ad}\mathcal{P}) \) is obtained by combining the product in \( \mathcal{U}G \) with the exterior product for ordinary forms.

We now mention some examples of \( B \)-fields that have a nice geometrical meaning:

1. Let \( LM \to M \) be the frame bundle and let \( \theta \) the soldering form. It is a \( \mathbb{R}^n \)-valued 1-form, given by the identity map on the tangent space of \( M \). For any given metric \( g \) we consider the reduced \( SO(n) \)-bundle of orthonormal frames \( O_g M \). A vielbein in physics is defined as the \( \mathbb{R}^n \)-valued 1-form on \( M \) given by \( \sigma^*\theta \), for a given section \( \sigma : M \to O_g M \). We now define the 2-form \( B \in \Omega^2(M, \text{ad}O_gM) \) as

\[
B \equiv \theta \wedge \theta^T|_{O_gM},
\]

where \( T \) denotes transposition. In the corresponding \( BF \) action, the curvature \( F \) is the curvature of a connection in the orthonormal frame bundle \( O_gM \). This action is known to be (classically) equivalent to the Einstein action, written in the vielbein formalism.

2. The manifold \( \mathcal{A} \) is an affine space with underlying vector space \( \Omega^1(M, \text{ad}P) \). We denote by \( \eta \) the 1-form on \( \mathcal{A} \) given by the identity map on the tangent space of \( \mathcal{A} \), i.e. on \( \Omega^1(M, \text{ad}P) \). From this we can obtain the 2-form on \( \mathcal{A} \)

\[
B \equiv \eta \wedge \eta,
\]

with values in \( \Omega^2(M, \text{ad}\mathcal{P}) \). More explicitly, \( B \) can be defined as

\[
B(a, b)(X, Y)|_{\mathcal{P}, \mathcal{A}} = a(X)b(Y) - a(Y)b(X) - b(X)a(Y) + b(Y)a(X),
\]

with values in \( \Omega^2(M, \text{ad}\mathcal{P}) \). More explicitly, \( B \) can be defined as

\[
B(a, b)(X, Y)|_{\mathcal{P}, \mathcal{A}} = a(X)b(Y) - a(Y)b(X) - b(X)a(Y) + b(Y)a(X),
\]
where $a, b \in T_A A; \ X, Y \in T_p P$.

We now want to show that the $BF$ action considered in the last example is connected with the ABJ anomaly in 4 dimensions. On the principal $G$–bundle

$$P \times A \longrightarrow M \times A$$

we can consider the tautological connection defined by the following horizontal distribution:

$$\text{Hor}_{p,A} = \text{Hor}_p^A \oplus T_A A$$

where $\text{Hor}_p^A$ denotes the horizontal space at $p \in P$ with respect to the connection $A$. The connection form of the tautological connection is given simply by $A$, seen as a $(1,0)$-form on $P \times A$. Its curvature form is given by

$$F_{p,A} = (F_A)_p + \eta_A.$$  \hfill (6)

Let us now use $Q_l$ to denote an irreducible ad-invariant polynomial on $\text{Lie}(G)$ with $l$ entries. Integrating the Chern-Weil form corresponding to such polynomials over $M$ we obtain (for $l = 2, 3, 4$):

$$\int_M Q_2(\mathcal{F}, \mathcal{F}) = k_2 \int_M Tr_p(F_A \wedge F_A)$$

where $B$ is defined as in eqns (3,4), and $k_l$ are normalization factors. In eqn (10) the wedge product also includes the exterior product of forms on $A$.

As Atiyah and Singer [4] pointed out, one can consider together with (5) the following principal $G$-bundle

$$P \times A \longrightarrow M \times A$$

Any connection on the bundle $A \longrightarrow A/G$ determines a connection on the bundle (11) and hence on the bundle (5). When we consider such connection on

1Strictly speaking, in this case, one should consider only either the space of irreducible connections and the gauge group, divided by its center, or the space of all connections and the group of gauge transformations which give the identity when restricted to a fixed point of $M$. We always assume that one of the above choices is made.
instead of the tautological one, then the left-hand side of eqn (8) becomes the term which generates, via the so called “descent equation”, the ABJ anomaly in 4 dimensions [4, 5]. Hence the $BF$ action (9) is a sort of gauge-fixed version of the generating term for the ABJ anomaly.

In any $BF$-action one has to integrate forms defined on $M \times \mathcal{A}$. Hence we can consider different kinds of symmetries [4, 6]:

1. “connection-invariance”: this refer to forms on $M \times \mathcal{A}$ which are closed: when they are integrated over cycles of $M$, they give closed forms on $\mathcal{A}$;

2. “gauge invariance”: this refer to forms which are defined over $M \times \mathcal{A}$, but are pullbacks of forms defined over $M \times \mathcal{A}/G$;

3. “diffeomorphism-invariance”: this can be considered whenever we have an action of $Diff(M)$ over $P$ and over $\mathcal{A}$. In particular we can consider diffeomorphism-invariance when $P$ is a trivial bundle, or when $P$ is the bundle of linear frames over $M$. In the latter case

$$
\frac{P \times \mathcal{A}}{Diff(M)} \rightarrow \frac{M \times \mathcal{A}}{Diff(M)}
$$

(12)

is a principal bundle.

4. diffeomorphism/gauge invariance: this refers to the action of $Aut \mathcal{P}$, i.e. the full group of automorphisms of the principal bundle $P$ (including the automorphisms which do not induce the identity map on the base manifold). This kind of invariance imply gauge invariance and diffeomorphism-invariance when the latter one is defined.

For instance the Chern action (1) is both connection- and diffeomorphism/gauge-invariant, while the $BF$ action (2) is gauge-invariant and its connection-invariance depends on further specifications. In particular the $BF$ action of example 2) is connection-invariant, while the action of example 1) is not. More precisely, in example 1), the integrand of the $BF$ action defines a closed 2-form on $O_g M \times \mathcal{A}$, i.e., we have

$$
\delta_A \left( \int_M \text{Tr} \{ F \wedge [\theta \wedge \theta^T] \} \right) = 0
$$

only when the covariant derivative $d_A \theta$ is zero; in other words, the critical connections are the Levi-Civita connections.

\footnote{In fact one should really consider only the group of diffeomorphisms of $M$ which strongly fix one point of $M$. Also, instead of $LM \times \mathcal{A}$ it is possible to consider $\mathcal{O}_M \times \mathcal{A}^{\text{metric}}$, namely the space of all orthonormal frames paired with the corresponding metric connections (see e.g. [3]).}
When \( B \) is given by 3, then for any 2-cycle \( \Sigma \) in \( M \)
\[
\int_{\Sigma} \text{Tr}_B = \int_{\Sigma} Q_2(\mathcal{F} \wedge \mathcal{F})
\]
is a closed 2-form on \( \mathcal{A} \) and so we have a map
\[
\mu : Z_2(M) \longrightarrow Z^2(\mathcal{A}),\tag{13}
\]
where \( Z_2(\mathcal{Z}^2) \) denotes 2-cycles (2-cocycles). When we replace the curvature of the tautological connection with the curvature of a connection in the bundle (11), then we again obtain a map (13), but now this map descends to a map
\[
\mu : H_2(M) \longrightarrow H^2(\mathcal{A}/G).\tag{14}
\]
The map (14) is the basis for the construction of the Donaldson polynomials [7].

As a final remark we would like to mention \( BF \) theories in arbitrary dimensions. When \( M \) is a manifold of dimension \( d \), then we can take the field \( B \) to be a \((d-2)\)-form, again with values in the bundle \( P \times_{\text{Ad}} U \). The form (3) makes then perfectly sense in any dimension.

A special situation occurs when the group \( G \) is \( SO(d) \). In this case there is a linear isomorphism
\[
\Lambda^2(\mathbb{R}^d) \longrightarrow \text{Lie}(SO(d)),\tag{15}
\]
defined by
\[
e_i \wedge e_j \longrightarrow (E^i_j - E^j_i),
\]
where \( \{e_i\} \) is the canonical basis of \( \mathbb{R}^d \) and \( E^i_j \) is the matrix with entries \( (E^i_j)^m_n = \delta_{m,i} \delta_{n,j} \). The isomorphism (15) has the following properties:

1. It is an isomorphism of inner product spaces, when the inner product in \( \text{Lie}(SO(d)) \) is given by \( (A, B) \equiv (-1/2) \text{Tr} AB \).

2. The left action of \( SO(d) \) on \( \mathbb{R}^n \) corresponds to the adjoint action on \( \text{Lie}(SO(d)) \).

One can then consider a principal \( SO(d) \)-bundle and an associated bundle \( E \) with fiber \( \mathbb{R}^d \). In this special case we can consider a different kind of \( BF \)-theory, where the field \( B \) is assumed to be a \((d-2)\)-form with values in the \((d-2)\)th exterior power of \( E \). In this case the corresponding \( BF \) action is given by:
\[
\exp(-\lambda \int_M B \wedge F),\tag{16}
\]
where the wedge product combines the wedge product of forms on \( M \) with the wedge product in \( \Lambda^*(\mathbb{R}^d) \). The action (16) is invariant under gauge transformations. When the \( SO(d) \)-bundle is the orthonormal bundle and the field \( B \) is the \((d-2)\)nd exterior power of the soldering form, then the corresponding \( BF \) action (16) gives the (classical) action for gravity in \( d \) dimensions, in the so-called Palatini (first-order) formalism [8, 9].
3 2-knots and their quantum observables

In Witten–Chern–Simons \[1\] theory we have a topological action on a 3-dimensional manifold \(M^3\), and the observables correspond to knots (or links) in \(M^3\). More precisely to each knot we associate the trace of the holonomy along the knot in a fixed representation of the group \(G\), or “Wilson line.” In 4 dimensions we have at our disposal the Lagrangians considered at the beginning of the previous section. It is natural to consider as observables, quantities (higher-dimensional Wilson lines) related to 2-knots.

Let us recall here that while an ordinary knot is a 1-sphere embedded in \(S^3\) (or \(\mathbb{R}^3\)), a 2-knot is a 2-sphere embedded in \(S^4\) or in the 4-space \(\mathbb{R}^4\). A generalized 2-knot in a four-dimensional closed manifold \(M\), can be defined as a closed surface \(\Sigma\) embedded in \(M\). Two 2-knots (generalized or not) will be called equivalent if they can be mapped into each other by a diffeomorphism connected to the identity of the ambient manifold \(M\).

The theory of 2-knots (and 2-links) is less developed than the theory of ordinary knots and links. For instance it is not known whether one can have an analogue of the Jones polynomial for 2-knots. On the other hand, one can define Alexander invariants for 2-knots (see e.g. \[10\]).

The problem we would like to address here is whether there exists a connection between 4-dimensional field theories and (invariants of) 2-knots. Namely, we would like to ask whether there exists a generalization to 4 dimensions of the connection established by Witten between topological field theories and knot invariants in 3 dimensions.

Even though we are not able to show rigorously that a consistent set of non-trivial invariants for 2-knots can be constructed out of 4-dimensional field theories, we can show that there exists a connection between \(BF\) theories in 4 dimensions and 2-knots. This connection involves, in different places, the Zamolodchikov tetrahedron equation as well as self-linking and (higher-dimensional) linking numbers.

In Witten’s theory one has to functionally integrate an observable depending on the given (ordinary) knot (in the given 3-manifold) with respect to a topological Lagrangian. Here we want to do something similar, namely we want to functionally integrate different observables depending on a given embedded surface \(\Sigma\) in \(M\) with respect to a path-integral measure given by the \(BF\)-action.

The first (classical) observable we want to consider is

\[
\text{Tr}_p \exp \left( \int_{\Sigma} \text{Hol}(A, \gamma_{y,x})B(y)\text{Hol}(A, \gamma_{z,y}) \right), \quad y \in \Sigma.
\]

(17)

Here by \(\text{Hol}(A, \gamma_{y,x})\) we mean the holonomy with respect to the connection \(A\) along the path \(\gamma\) joining two given points \(y\) and \(x\) belonging to \(\Sigma\). Expression \[3\]We assume to have chosen once for all a reference section for the (trivial) bundle \(P|_{\Sigma}\). Hence the holonomy along a path is defined by the comparison of the horizontally lifted path and the path lifted through the reference section.

\[7\]
1. depends on the choice of a map assigning to each $y \in \Sigma$ a path $\gamma$ joining $x$ and $z$ and passing through $y$; more precisely, it depends on the holonomies along such paths. One expects that the functional integral over the space of all connections will integrate out this dependency. More precisely, we recall that the (functional) measure over the space of connections is given, up to a Jacobian determinant, by the measure over the paths over which the holonomy is computed. See in this regard the analysis of the Wess-Zumino-Witten model made by Polyakov and Wiegmann \[11\].

2. is gauge invariant if we consider only gauge transformations that are the identity at the points $x$ and $z$. In the special case when $z$ and $x$ coincide, then (17) is gauge invariant without any restriction.

The functional integral of (17) can be seen as the vacuum expectation value of the trace (in the representation $\rho$) of the quantum surface operator denoted by $O(\Sigma; x, z)$. In other words we set:

$$O(\Sigma; x, z) \equiv \exp\left( \int_{\Sigma} \text{Hol}(A, \gamma_{y,x}) B(y) \text{Hol}(A, \gamma_{z,y}) \right), \quad y \in \Sigma;$$

where, in order to avoid a cumbersome notation, we did not write the explicit dependency on the paths $\gamma$, and we did not use different symbols for the holonomy and its quantum counterpart (quantum holonomy). Moreover in (18) a time-ordering symbol has to be understood.

We point out that we will also allow the operator (18) to be defined for any embedded surface $\Sigma$ closed or with boundary. In the latter case the points $x$ and $z$ are always supposed to belong to the boundary.

As far as the role of the paths $\gamma$ considered above is concerned, we recall that, in a 3-dimensional theory with knots, a framing is a (smooth) assignment of a tangent vector to each point of the knot. In four dimensions, instead, we assign a curve to each point of the 2-knot. We will refer to this assignment as a (higher-dimensional) framing. We will also speak of a framed 2-knot.

### 4 Gauss constraints

In order to study the quantum theory corresponding to (2), we first consider the canonical quantization approach. We take the group $G$ to be $SU(n)$ and consider its fundamental representation, hence we write the field $B$ as $\hat{B} + \sum_a B^a R^a$. Here $\hat{B}$ is a multiple of 1 and $\{R^a\}$ is an orthonormal basis of $\text{Lie}(G)$. In the $BF$ action we can therefore disregard the $\hat{B}$ component of the field $B$.

We choose a time-direction $t$ and write in local coordinates $\{t, x\} \equiv \{t, x^1, x^2, x^3\}$:

$$d = dt + dx, \quad A = A_0 dt + A_x = A_0 dt + \sum_i A_idx^i \quad (i = 1, 2, 3),$$
\[ F_A = \sum_i (d_t A_i) \wedge dx^i + (d_x A_0) \wedge dt + \sum_i [A_i, A_0] dx^i \wedge dt + \sum_{i < j} F_{i,j} dx^i \wedge dx^j \]

\[ B = \sum_{i < j} B_{i,j} dx^i \wedge dx^j + B_{i,0} dt \wedge dx^i. \]

In local coordinates the action is given by:

\[ \mathcal{L} = \int_M Tr(B \wedge F A) \]

\[ \approx \int_{M^3} \sum_{i,j,k} Tr(B_{i,j} d_t A_k + B_{i} F_{j,k} dt + A_0(DB)_{i,j,k}) dx^i \wedge dx^j \wedge dx^k (19) \]

where \( D \) denotes the covariant 3-dimensional derivative with respect to the connection \( A_x \).

We first consider the pure BF-theory, namely, a BF theory where no embedded surface is considered. We perform a Legendre transformation for the Lagrangian \( \mathcal{L} \); the conjugate momenta to the fields \( B_i \) and \( A_0 \), namely \( \frac{\partial \mathcal{L}}{\partial (d_t B_i)} \) and \( \frac{\partial \mathcal{L}}{\partial (\partial_t A_0)} \), are zero, so we have the primary constraints:

\[ \frac{\partial \mathcal{L}}{\partial B_i} = \sum_{j,k} \epsilon_{i,j,k}(F A_x)_{j,k} = 0, \quad \frac{\partial \mathcal{L}}{\partial A_0} = \sum_{i,j,k} \epsilon_{i,j,k}(DB)_{i,j,k} = 0. \quad (20) \]

We do not have secondary constraints since the Hamiltonian is zero, hence the time evolution is trivial. At the formal quantum level the constraints (20) will be written as:

\[ (F A_x)_{op} |\phi_{\text{physical}}\rangle = 0, \quad (DB)_{op} |\phi_{\text{physical}}\rangle = 0, \quad (21) \]

where \( \text{op} \) stands for “operator” and the vectors \( |\phi_{\text{physical}}\rangle \) span the physical Hilbert space of the theory.

We now consider the expectation value of \( Tr_o O(\Sigma; x, x) \) (see definition (18)). We still want to work in the canonical formalism, the result will be that instead of the constraints (20), we will have a source represented by the assigned surface \( \Sigma \).

In order to represent the surface operator (18) as a source-action term to be added to the BF-action, we assume that we have a current (singular 2-form) \( J_{\text{sing}} \) concentrated on the surface \( \Sigma \), of the form:

\[ J_{\text{sing}} = \sum_a J_a^{\text{sing}} R^a, \]

where again \( R^a \) is an orthonormal basis of \( \text{Lie}(G) \). We assume in general that \( J_{\text{sing}} \) can take values in \( \mathcal{U} G \).

\[ ^4\text{Hence by definition } J_{\text{sing}} \text{ has no component in } \mathfrak{g} \subset \mathcal{U}(G). \]
We are now in position to write the operator (18) as:

\[
O(\Sigma; z, z) = \exp\left(\int_M \hat{\text{tr}}[\text{Hol}(A, \gamma_{y,z})B(y)\text{Hol}(A, \gamma_{z,y}) \wedge J_{\text{sing}}]\right),
\]

(22)

where for any \(A, A' \in U\mathcal{G}\), \(A = \sum_a A^a R^a\), \(A' = \sum_a A'^a R^a\) we have set \[\hat{\text{tr}}(AA') = \sum_a A^a A'^a.\]

From (22) it follows that the component of \(J_{\text{sing}}\) in \(\text{Lie}(G)\) does not give any significant contribution. We now assume \(J_{\text{sing}} \in \text{Lie}(G) \otimes \text{Lie}(G)\). In this way we neglect possible higher order terms in \(\text{Lie}(G) \otimes^k \subset U\mathcal{G}\), but this is consistent with our semi-classical treatment.

With the above notation and taking into account the BF action as well, the observable to be functionally integrated is:

\[
\text{Tr}_\rho \exp\left(\int_M \hat{\text{tr}}\left\{[\text{Hol}(A, \gamma_{y,z})B(y)\text{Hol}(A, \gamma_{z,y})] \wedge [J_{\text{sing}} - \text{Hol}(A, \gamma_{z,y})^{-1} F(y)\text{Hol}(A, \gamma_{y,z})^{-1}]\right\}\right).
\]

(23)

At this point the introduction of the singular current allows us to formally represent a theory with sources concentrated on surfaces, as a pure BF theory with a new “curvature” given by the difference of the previous curvature and the currents associated to such sources. From eqn (22) we conclude moreover that \(J_{\text{sing}}\) as a 2-form should be such that \(J_{\text{sing}}(y) \wedge d^2\sigma(y) \neq 0, y \in \Sigma\), where \(d^2\sigma(y)\) is the surface 2-form of \(\Sigma\).

Now we choose a time direction and proceed to an analysis of the Gauss constraints as in the pure BF theory. We assume that locally the four manifold \(M\) is given by \(M^3 \times I\) (\(I\) being a time-interval) and the intersection of the 3-dimensional manifold \(M^3 \times \{t\}\) with the surface \(\Sigma\) is an ordinary link \(L_t\).

In a neighborhood of \(y \in \Sigma\) the current \(J_{\text{sing}}\) will look like:

\[
J_{\text{sing}}^a(x; y) = \delta^{(2)}(x - y) R^a dx_1^2 \wedge dx_1^2, \quad y \in \Sigma,
\]

(24)

where \(\Pi\) is a plane orthogonal to \(\Sigma\) at \(y\) (with coordinates \(x_1^1\) and \(x_1^2\)) and \(\delta^{(n)}\) denotes the \(n\)-dimensional delta function.

The Gauss constraints imply that in the above neighborhood of \(y \in \Sigma\) we have:

\[
F^a(x) = \text{Hol}(A, \gamma_{z,y}) J_{\text{sing}}^a(x; y)\text{Hol}(A, \gamma_{y,z}); \quad y \in \Sigma, x \in \Pi
\]

(25)

\[\text{In particular } \hat{\text{tr}} \text{ coincides with } \text{tr} \text{ when } A, A' \in \text{Lie}(G).\]
where $\gamma$ is a loop in $\Sigma$ based at $z$ and passing through $y$. The right-hand side of (25) is also equal to
\[
\frac{\partial \text{Hol}(A, \gamma_{z,y,z})}{\partial A^a(x)}; \quad y \in \Sigma, x \in \Pi.
\]

Notice that the previous analysis implies that the components of the curvature $F^a$ (and consequently the components of the connection $A^a$) must take values in a noncommutative algebra, at least when sources are present. More precisely the components $F^a$ (as well as $A^a$) should be proportional to $R^a \in \text{Lie}(G)$. Given the fact that $B^a$ and $A^a$ are conjugate fields, we can conclude that the (commutation relations of the) quantum theory will require $B^a$ to be proportional to $R^a$ as well.

As in the lower-dimensional case (ordinary knots (or links) as sources in a 3-dimensional theory), the restriction of the connection $A$ to the plane $\Pi$ looks (in the approximation for which $\text{Hol}(A, \gamma_{z,x,z}) = 1$) like
\[
A^a(x)|_{\Pi} = R^a \; d \log(x - y)
\] (26)
for a given $y \in \Sigma$ (here $d$ is the exterior derivative).

In order to have some geometrical insight into the above connection, we consider a linear approximation, where the surface $\Sigma$ can be approximated by a collection of 2-dimensional planes in $\mathbb{R}^4$, i.e., by a collection of hyperplanes in $\mathbb{C}^2$. These hyperplanes, and the hyperplane $\Pi$ considered above, are assumed to be in general position. This means that the intersection of the hyperplane $\Pi$ with any other hyperplane is given by a point.

The quantum connection (26) on $\Pi$ gives a representation of the first homotopy group of the manifold of the arrangements of hyperplanes (points) in $\Pi$, i.e., more simply, of the (pure) braid group. It is then natural to ask whether the same is true in 4 dimensions, namely whether the 4-dimensional connection, corresponding to the (critical) curvature (25) (in the linear approximation defined above), is related to the higher braid group. We will discuss this point further elsewhere; let us mention only that the 4-dimensional critical connection is related to the existence of a higher dimensional version of the Knizhnik-Zamolodchikov equation associated with the representation of the higher braid groups, as suggested by Kohno.

5 Path integrals and the Alexander invariant of a 2-knot

In the previous section we worked specifically in the canonical approach. When instead we consider a covariant approach, and compute the expectation values,
with respect to the $BF$ functional measure only, we can write:

$$\langle \text{Tr}_\rho \mathcal{O}(\Sigma; x, x) \rangle =$$

$$\int \mathcal{D}A \mathcal{D}B \text{Tr}_\rho \exp \left( \text{tr} \int_M \left[ B(x) \wedge (F(x) - \text{Hol}(A, \gamma_{x,z}) J_{\text{sing}}(x)) \right] \right) \quad (27)$$

Again, in the approximation in which $\text{Hol}(A, \gamma_{x,z}) = 1$,

$$\langle \text{Tr}_\rho \mathcal{O}(\Sigma; x, x) \rangle \approx \int \mathcal{D}A \mathcal{D}B \text{Tr}_\rho \exp \left( \text{tr} \int_M \left[ B(x) \wedge (F(x) - J_{\text{sing}}(x)) \right] \right) \quad (28)$$

A rough (and formal) estimate of the previous expectation value (28) can be given as follows.

By integration over the $B$ field we obtain something like

$$\langle \text{Tr}_\rho \mathcal{O}(\Sigma; x, x) \rangle \approx \dim(\rho) \int \mathcal{D}(A) \delta(F - J_{\text{sing}}), \quad (29)$$

using a functional Dirac delta. By applying the standard formula for Dirac deltas evaluated on composite functions, we heuristically obtain:

$$\langle \text{Tr}_\rho \mathcal{O}(\Sigma; x, x) \rangle \approx \dim(\rho) \left| \text{det}(D_{A_0}|_{M\setminus\Sigma}) \right|^{-1}, \quad (30)$$

where $\text{det}(D_{A_0}|_{M\setminus\Sigma})$ denotes the (regularized) determinant of the covariant derivative operator with respect to a background flat connection $A_0$ on the space $M \setminus \Sigma$.

It is then natural to interpret the expectation value (30) (with the Faddeev-Popov ghosts included) as the analytic (Ray-Singer) torsion for the complement of the 2-knot $\Sigma$ (see [16, 17]). This torsion is related to the Alexander invariant of the 2-knot [18].

6 First-order perturbative calculations and higher-dimensional linking numbers

Let us now consider the expectation value of the quantum surface observable $\text{Tr}_\rho \mathcal{O}(\Sigma; x, x)$, with respect to the total $BF + \text{Chern}$ action. The two fields involved in the quantum surface operator are $A^a_\mu$ (the connection) and $B^a_{\mu, \nu}$ (the $B$-field) with $a = 1, \cdots, \dim(G)$ and $\mu, \nu = 1, \cdots, 4$. The connection $A$ is present in the quantum surface operator via the holonomy of the paths $\gamma_{y,x}$ and $\gamma_{x,y}$ ($y \in \Sigma$) as in definitions (17) and (18).

\[\text{We omit the ghosts and gauge fixing terms \cite{3}, since they are not relevant for a rough calculation of \cite{27}.}\]
At the first-order approximation in perturbation theory (with a background field and covariant gauge), we can write the holonomy as

\[ \text{Hol}(A, \gamma_{y,x}) \approx 1 + \kappa \int_{\gamma_{y,x}} \sum_{\mu} A_\mu(z) dz^\mu + \cdots. \]

At the same order, the only relevant part of the $BF$ action is the kinetic part $(B \wedge dA)$, so we get the following Feynman propagator:

\[
\langle A^a_\mu(x)B^b_\nu(y) \rangle = \frac{2i}{\lambda} \frac{1}{4\pi^2} \sum_{\mu, \nu, \rho, \tau} \epsilon^{\mu \nu \rho \tau} \frac{(x^\tau - y^\tau)}{|x - y|^4}.
\]

In order to avoid singularities, we perform the usual point-splitting regularization. This is tantamount as lifting the loops $\gamma_{x,y,x}$ in a neighborhood of $y$. The lifting is done in a direction which is normal to the surface $\Sigma$.

The complication here is that the loop $\gamma$ itself (based at $x$ and passing through $y$) depends on the point $y$. While the general case seems difficult to handle, we can make simplifying choices which appear to be legitimate from the point of view of the general framework of quantum field theory. For instance we can easily assume that when we assign to a point $y$ the loop $\gamma_{x,y,x}$ based at $x$ and passing through $y$, then the same loop will be assigned to any other point $y'$ belonging to $\gamma$.

Moreover we can consider an arbitrarily fine triangulation $T$ of $\Sigma$ and take into consideration only one loop $\gamma_T$ which has non empty intersection with each triangle of $T$. This approximation will break gauge-invariance, which will be, in principle, recovered only in the limit when the size of the triangles go to zero. (For a related approach see [19].) The advantage of this particular approximation is that we have only to deal with one loop $\gamma_T$, which by point-splitting regularization is then completely lifted from the surface. We call this lifted path $\gamma^r_T$, where $r$ stands for regularized.

Finally we get (to first-order approximation in perturbation theory and up to the normalization factor $\langle 1 \rangle^7$):

\[
\langle \text{Tr}_\rho O(\Sigma; x, x) \rangle \approx \dim(\rho) \left[ 1 + C_2(\rho) \frac{2i \kappa}{\lambda} \frac{1}{4\pi^2} \sum_{\mu, \nu, \rho, \tau} \epsilon^{\mu \nu \rho \tau} \int_\Sigma dx^\mu dx^\nu \int_{\gamma^r_T} dy^\rho \frac{(x^\tau - y^\tau)}{|x - y|^4} + \cdots \right].
\]

where $C_2(\rho)$ is the trace of the quadratic Casimir operator in the representation $\rho$. When $\Sigma$ is a 2-sphere, then \([32]\) is given by:

\[
\dim(\rho) \left[ 1 + C_2(\rho) \frac{2i \kappa}{\lambda} \mathcal{L}(\gamma^r_T, \Sigma) + \cdots \right]
\]

where $\mathcal{L}(\gamma^r_T, \Sigma)$ is the (higher-dimensional) linking number of $\Sigma$ with $\gamma^r_T$. \([20]\)

\[ \text{While the normalization factor for a pure (4-dimensional) BF theory appears to be trivial \([17]\), in a theory with a combined BF + Chern action we have a contribution coming from the Chern action. This contribution has been related to the first Donaldson invariant of M \([4]\).} \]
7 Wilson “channels”

Let Σ again represent our 2-knot (surface embedded in the 4-manifold \(M\)). Carter and Saito \[21, 22\] inspired by a previous work by Roseman \[23\] describe a 2-knot Σ (in fact an embedded 2-sphere) in 4-space by the 3-dimensional analogue of a knot-diagram: they project down Σ to a 3-dimensional space, keeping track of the over- and under-crossings. One of the coordinates of this 3-space is interpreted as “time.” The intersection of this 3-dimensional diagram of a 2-knot with a plane (at a fixed time) gives an ordinary link-diagram, while the collection of all these link diagrams at different times (“stills”) gives a “movie” representing the 2-knot.

In the Feynman formulation of field theory, we start by considering the surface Σ embedded in the 4-manifold \(M\) directly, not its projection. At the local level \(M\) will be given by \(M^3 \times I\) (\(I\) being a time-interval), so that the intersection of the 3-dimensional manifold \(M^3 \times \{t\}\) with Σ gives an ordinary link \(L_t\) (for all times \(t\)) in \(M^3\). In order to connect quantum field theory with the standard theory of 2-knots, we assume more simply that \(M\) is the 4-space \(\mathbb{R}^4\) and that Σ is a 2-sphere. One coordinate axis in \(\mathbb{R}^4\) is interpreted as time, and the intersection of Σ with 3-space (at fixed times) is given by ordinary links, with the possible exception of some critical values of the time parameter.

In other words, we have a space-projection \(\pi_t : \mathbb{R}^4 \rightarrow \mathbb{R}^3\) for each time \(t\), so that \(L_t(\Sigma) \equiv \pi_t^{-1}(\mathbb{R}^3) \cap \Sigma\) is, for non-critical times, an ordinary link. For any time interval \((t_1, t_2)\) we will use also the notation \(\Sigma_{(t_1, t_2)} \equiv \bigcup_{t \in (t_1, t_2)} \pi_t^{-1}(\mathbb{R}^3) \cap \Sigma\).

We consider a set \(\mathcal{T}\) of (noncritical) times \(\{t_i\}_{i=1, \ldots, n}\). We denote the components of the corresponding links \(L_{t_i}\) by the symbols \(K_{j(i)}\), \(j = 1, \ldots, s(i)\).

Here and below we will write \(j(i)\) simply to denote the fact that the index \(j\) ranges over a set depending on \(i\). We choose one base-point \(x_{j(i)}(i)\) in each knot \(K_{j(i)}\) and we denote by \(\Sigma_{j(i), j(i+1)}\) the surface contained in Σ whose boundary includes \(K_{j(i)}\) and \(K_{j(i+1)}\). We then assign a framing to \(\Sigma_{j(i), j(i+1)}\) as follows: for each point \(p\) in the interior of \(\Sigma_{j(i), j(i+1)}\) we associate a path with endpoints \(x_{i(i)}(i)\) and \(x_{i+1(i+1)}\) passing through \(p\). To the surface \(\Sigma_{j(i), j(i+1)}\) with boundary components \(K_{j(i)}\) and \(K_{j(i+1)}\), we associate the operator:

\[
\text{Hol}(A, K_{j(i)})_{x_{i(i)}} O(\Sigma_{j(i), j(i+1)}; x_{i(i)}, x_{i+1(i+1)}) \text{Hol}(A, K_{j(i+1)})_{x_{i+1(i+1)}},
\]

where the subscript in the symbol \(\text{Hol}(A, \ldots)\) denotes the base-point.

We refer to the 2-knot Σ equipped with the set \(\mathcal{T}\) of times as the (temporally) sliced 2-knot. For the sliced 2-knot, we define the Wilson channels to be the operators of the form
\[
O(\Sigma_{(-\infty,t_1)}) \, \text{Hol}(A, K^1_i)_{x_1^i} \cdots \text{Hol}(A, K^{j(i)}_{i})_{x_{j(i)}^i}
\]

\[
O(\Sigma^{j(i),j(i+1)}_i; x_{j(i)}^i, x_{j(i+1)}^{i+1}) \, \text{Hol}(A, K^{j(i+1)}_{i+1})_{x_{j(i+1)}^{i+1}} \cdots O(\Sigma(t_n, +\infty); x_n^{j(n)}, +\infty).
\]

(34)

We assume the following requirements for the Wilson channels:

1. the operators which appear in a Wilson channel are chosen in such a way that a surface operator is sandwiched between knot-operators only if the relevant knots belong to the boundary of the surface;

2. the paths that are included in the surface operators \(O(\Sigma^{j(i),j(i+1)}_i)\) are not allowed to touch the boundaries of the surface, except at the initial and final point.

Finally, to the “sliced” 2-knot we associate the product of the traces of all possible Wilson channels. This product can be seen as a possible higher-dimensional generalization of the Wilson operator for ordinary links in 3-space that was considered by Witten.

We would like now to compare the Wilson channel operators associated to a sliced 2-knot \(\Sigma\) with the operator

\[
O(\Sigma; -\infty, +\infty)
\]

considered in Section 5 (no temporal slicing involved). The main difference concerns the prescriptions that are given involving the paths \(\gamma\)’s that enter the definition of (35) (or, equivalently, the framing of the 2-knot). The Wilson channels are obtained from (35) by:

1. choosing a finite set of times \(T \equiv \{t_i\}\), and

2. requiring the paths \(\gamma\)’s to follow one of the components of the link \(L_t\), for each \(t \in T\).

Finally, recall that in the Wilson channel operators, each of the paths \(\gamma\)’s is forbidden to follow two separate components of the same link \(L_t\), for any time \(t \in T\). This last condition can be understood by noticing that in order to follow both the \(j\)th and the \(j'\)th components of the link \(L_{t_i}\), for some \(i, j(i), j'(i)\), one path should follow the \(j\)th component, then enter the surface \(\Sigma_{i}^{j(i),j(i+1)}\) (which is assumed to be equal to \(\Sigma_{i}^{j(i),j'(i+1)}\)), and finally go back to the \(j'\)th component, thus contradicting a (local) causality condition.
The results of the previous section imply that, given a temporally sliced 2-knot Σ, we can construct quantum operators by considering all the Wilson channels relevant to surfaces Σ_{j(i)j(i+1)}^{(i)}, i \leq l (i \geq l) and links L_{t_i} for i < l (i > l). Let us denote these quantum operators by the symbols

\[ W^{\text{in}}(\Sigma, t_l), \quad (W^{\text{out}}(\Sigma, t_l)). \] (36)

They can be seen as a sort of convolution, depending on Σ, of all the quantum holonomies associated to the links L_{t_i} for i < l (i > l).

Ordinary links are closure of braids, and one may describe the surface which generates the different links at times \{t_i\}, as a sequence (movie) of braids [24]. But a link is also obtained by closing a tangle, and the time-evolution of a link (represented by a surface) can also be described in terms of 2-tangles, i.e. movies of (ordinary) tangles [24]. 2-tangles form a (rigid, braided) 2-category [25]. The 2-tangle approach (with its 2-categorical content) may reveal itself as a useful one for quantum field theory.

The initial tangle is called the source and the final tangle is called the target. In our case, say, the source represents L_{t_i} while the target represents L'_{t_i}, for l' > l. The quantum counterpart of the the source and target is represented by the quantum holonomies associated to the corresponding links, while the quantum counterpart of the 2-tangle is represented by the quantum surface operator relevant to the surface Σ_{(t_l,t_{l'})}.

One can think to the quantum surface operators as acting on quantum holonomies (by convolution), but it is difficult to do finite calculations and write this action explicitly. What we can do instead is to make some infinitesimal calculations using integration-by-parts techniques in Feynman path integrals.

Generally speaking, in a finite time evolution \(t_l \rightarrow t_{l'}\), the quantum surface operator (defined by the surface with boundaries L_{t_l} and L'_{t_{l'}}) will map the operator \(W^{\text{in}}(\Sigma, t_l)\) into the operator \(W^{\text{in}}(\Sigma, t_{l'})\). Let us now consider what can be seen as the “derivative” of this map. Namely, let us see what happen when we consider the quantum operator corresponding to a surface bordering two links L_{t_l} and L'_{t_{l'}} that differ by an elementary change. What the functional integration-by-parts rules will show, is that one can interchange a variation of the surface operator with a variation of the link (and consequently of its quantum holonomy). So in order to compute the “derivative” of the surface operator, we may consider variations of the link L_{t_l}.

In order to describe elementary changes of links, recall that braids or tangles (representing links at a fixed time) can be seen as collection of oriented strings, while 2-tangles describe the interaction of those strings [26]. We consider now two links that differ from each other only in a (small) region involving three strings. Let us number these strings with numbers 1, 2, 3. The string 1 crosses under the string 2, then the string 2 crosses under the string 3 and the string 1
crosses under the string 3. We denote by the symbols $a$ and $b$ the part of each string that precedes and, respectively, follows the first crossing, as shown in Fig. 1.

The integration-by-parts rules for the $BF$ theory are as follows. We consider a knot $K_{j}^{i(l)}$, which we denote here by $C$ in order to simplify the notation, and a point $x \in C$. By the symbol $\delta x$ we mean the variation (of $C$) obtained by inserting an infinitesimally small loop $c_{x}$ in $C$ at the point $x$. We obtain:

$$
\delta x \left( \text{Hol}(A, C)_x \right) \exp \left( -\lambda \int_M \text{Tr}(B \wedge F) \right) = \sum_{\mu\nu} d^2 c_{x}^{\mu\nu} \text{Hol}(A, C)_x F^a_{\mu\nu} R^a \exp \left( -\lambda \int_M \text{Tr}(B \wedge F) \right)
$$

$$
= -\frac{1}{2\lambda} \sum_{\mu\nu\rho\tau} \epsilon_{\mu\nu\rho\tau} d^2 c_{x}^{\mu\nu} \text{Hol}(A, C)_x R^a \frac{\delta}{\delta B^a_{\rho\tau}(x)}, \quad (37)
$$

where $d^2 c_{x}$ denotes the surface 2-form of the surface bounded by the infinitesimal loop $c_{x}$, $R^a$ denotes as usual an orthonormal basis of Lie$(SU(n))$ and the trace is meant to be taken in the fundamental representation. The second equality has been obtained by functional integration by parts; in this way we produced the functional derivative with respect to the $B$-field. The integration-by-parts rules of the $BF$ theory in 4 dimensions completely parallel the integration-by-parts rules of the Chern–Simons theory in 3 dimensions [27]. When we apply the functional derivative $\frac{\delta}{\delta B^a_{\rho\tau}(x)}$ to a surface operator $O(\Sigma', x', x'')$ for a surface
\(\Sigma'\) whose boundary includes \(C\) we obtain:

\[
\frac{\delta}{\delta B_{\rho\tau}^a(x)} O(\Sigma' ; x', x'') = R^a \delta^{(4)}(x - x') d^2 \sigma^{\rho\tau} O(\Sigma' ; x', x''),
\]

where \(d^2 \sigma'\) is the surface 2-form of \(\Sigma'\).

From eqn (37) we conclude that only the variations \(c_x\) for which \(d^2 \sigma' \wedge d^2 c_x \neq 0\) matter in the functional integral. But these variations are exactly the ones for which the small loop \(c_x\) is such that the surface element \(d^2 c_x\) is transverse to the knot \(C\) (as in [27]).

Now we come back to the link \(L_t\) with the three crossings as in Fig. 1. In order to consider the more general situation we assume that the three strings in Fig. 1 belong to different components \(K_{1l}^1, K_{2l}^2, K_{3l}^3\) of \(L_t\) that will evolve into three different components \(K_{1l+1}^1, K_{2l+1}^2, K_{3l+1}^3\) of \(L_{t+1}\). By performing three such small variations at each one of the crossings of the three strings considered in Fig. 1, we can switch each under-crossing into an over-crossing. Now as in [27] we compare the expectation value of the original configuration with the configuration with the three crossings switched. Such comparison, via the aid of eqns (37, 38) and the Fierz identity [27], leads to the conclusion

\[
(\delta O)| \cdots W_l \cdots |0\rangle = \mathcal{R} | \cdots W_l |0\rangle.
\]

Here \(W_l\) is a short notation for the operator

\[
W_l \equiv \text{Hol}(A, K_{1l}^1)_{x_1l} O(\Sigma_1^{1,1}; x_1^1, x_{1+1}^1) \text{Hol}(A, K_{1l+1}^1)_{x_{1+1}^1} \\
\otimes \text{Hol}(A, K_{2l}^2)_{x_2l} O(\Sigma_2^{2,2}; x_2^2, x_{2+1}^2) \text{Hol}(A, K_{2l+1}^2)_{x_{2+1}^2} \\
\otimes \text{Hol}(A, K_{3l}^3)_{x_3l} O(\Sigma_3^{3,3}; x_3^3, x_{3+1}^3) \text{Hol}(A, K_{3l+1}^3)_{x_{3+1}^3};
\]

the dots in (39) involve the operators \(W_{\text{in}}(\Sigma, t_l)\) and \(W_{\text{out}}(\Sigma, t_l)\) (see [28]), the vector space associated to the string \(j\) is given by a tensor product \(V^j_a \otimes V^j_b\), and finally \(\mathcal{R}\) is a suitable representation of the following operator:

\[
(1 + (1/\lambda) \sum_a R^a \otimes R^a + \cdots)_{1a, 2a} (1 + (1/\lambda) \sum_a R^a \otimes R^a + \cdots)_{1b, 3a} \\
(1 + (1/\lambda) \sum_a R^a \otimes R^a + \cdots)_{2b, 3b}.
\]

\(^8\) We are considering here only the small variations of \(\Sigma'\), so we can disregard the holonomies of the paths \(\gamma\)'s in the surface operators; the only relevant contribution to be considered is the one given by by the field \(B = \sum B^a R^a\).

\(^9\) Details will be discussed elsewhere.
Now we recall that $1 + (1/\lambda) \sum a R^a \otimes R^a$ is the infinitesimal approximation of a quantum Yang-Baxter matrix $R$. Hence the calculations of this section suggest that, at the finite level, the solution of the Zamolodchikov tetrahedron equation which may be more relevant to topological quantum field theories, is the one considered in [28, 29], namely:

$$S_{123} = R_{1a,2a} R_{1b,3a} R_{2b,3b},$$

where $R$ is a solution of the quantum Yang-Baxter equation.

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