FOLIATION DIVISORIAL CONTRACTION BY THE Sasaki-Ricci FLOW ON SASAKIAN FIVE-MANIFOLDS

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Abstract. Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian 5-manifold with finite cyclic quotient foliation singularities of type \(\frac{1}{r}(1, a)\). First, we derive the foliation minimal model program by applying the resolution of cyclic quotient foliation singularities. Secondly, based on the study of local model of resolution of foliation singularities, we prove the foliation canonical surgical contraction or the foliation extremal ray contraction under the Sasaki-Ricci flow. As a consequence, we prove a Sasaki analogue of analytic minimal model program with the Kaehler-Ricci flow due to Song-Tian and Song-Weinkove.

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2020 Mathematics Subject Classification. Primary 53E50, 53C25; Secondary 53C12, 14E30.
Key words and phrases. Folation singularities, Folation canonical surgical contraction, Folation extremal ray contraction, Sasaki-Ricci flow, Transverse Mori program, Folation minimal model program.

*Shu-Cheng Chang and ***Chin-Tung Wu are partially supported in part by the MOST of Taiwan.
A.2. Minimal Model Program on Compact Quasi-Regular Sasakian 5-Manifolds

1. Introduction

Sasakian geometry is very rich as the odd-dimensional analogous of Kähler geometry. A Sasaki-Einstein \((2n + 1)\)-manifold is to say that its Kaehler cone is a Calabi-Yau \((n + 1)\)-fold. Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n + 1)\)-manifold. If the orbits of the Reeb vector field \(\xi\) are all closed, and hence circles, then integrates to an isometric \(U(1)\) action on \((M, g)\). Since it is nowhere zero this action is locally free; that is, the isotropy group of every point in \(M\) is finite. If the \(U(1)\) action is in fact free then the Sasakian structure is said to be regular. Otherwise, it is said to be quasi-regular. Thus the space of leaves of the canonical \(U(1)\)-fibration will have orbifold singularities (cf section 3). In general, it is said to be irregular if the orbits of are not all closed.

In particular, Sasaki-Einstein 5-manifolds provide interesting examples of the AdS/CFT correspondence. On the other hand, The class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism due to Smale-Barden [S], [B]. Then, in this paper, it is our goal to focus on a classification of compact quasi-regular Sasakian 5-manifolds according to the global properties of the Reeb \(U(1)\)-fibration.

Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian 5-manifold. Then by the first structure theorem, \(M\) is a principal \(S^1\)-orbibundle (V-bundle) over an orbifold \(Z = (Z, \Delta)\) which is also a \(Q\)-factorial, polarized, normal projective orbifold surface such that there is an orbifold Riemannian submersion

\[
\pi : (M, g, \omega) \to (Z, h, \omega_h)
\]

and

\[
(1.1) \quad K_M^T = \pi^*(K_Z^{orb}) = \pi^*(\varphi^*(K_Z + [\Delta])).
\]

If the orbifold structure of the leave space \(Z\) is well-formed (c.f. section 3), then the orbifold canonical divisor \(K_Z^{orb}\) and canonical divisor \(K_Z\) are the same and thus

\[
K_M^T = \pi^*(\varphi^*(K_Z)).
\]

In a such case, there is the Sasaki analogue of Mori’s minimal model program with respect to \(K_Z\) in a such compact quasi-regular Sasakian 5-manifold. More precisely, one can ask the following so-called foliation minimal model program:

Is there a foliation \((-1)\)-curve? One of Mori program ([KMM], [M], [KM]) is to replace this criterion with the one dictated by the transverse canonical divisor \(K_M^T\) of \(M\): Does the transverse canonical divisor have nonnegative intersection

\[
K_M^T \cdot V \geq 0
\]

with any invariant submanifold \(V\) on \(M\) which is a Sasakian 3-submanifold. In other words,

is \(K_M^T\) nef?
If $K^T_M$ is not nef, there is an extremal transverse contraction map which turns out not only to generalize the Sasaki analogue of Castelnuovo's contractibility criterion but also to provide decisive information on the global structures of the end results of the foliation minimal model program. Then an end result of foliation MMP starting from $M$ is a transverse Mori fiber space if and only if there exists a nonempty open set $U \subset M$ such that for any $S^1$-fibre in $U$, there is an irreducible invariant 3-submanifold $V$ passing through such a fibre $S^1$ with $K^T_M \cdot V < 0$. We refer to Proposition 11. Moreover, the Mori's minimal model program in birational geometry can be viewed as the complex analogue of Thurston's geometrization conjecture which was proved via Hamilton's Ricci flow with surgeries on 3-dimensional Riemannian manifolds by Perelman ([P1], [P2], [P3]). Likewise, there is a conjecture picture by Song-Tian ([ST]) that the Kaehler-Ricci flow should carry out an analytic minimal model program with scaling on projective varieties. Recently, Song and Weinkove ([SW1]) established the above conjecture on a projective algebraic surface.

The Sasaki–Ricci flow is introduced by Smoczyk–Wang–Zhang ([SWZ]) to study the existence of Sasaki $\eta$-Einstein metrics on Sasakian manifolds. They showed that the flow has the longtime solution and asymptotic converges to a Sasaki $\eta$-Einstein metric when the basic first Chern class is negative or null. It can be viewed as a Sasaki analogue of Cao’s result ([Cao]) for the Kaehler–Ricci flow.

In view of the previous discussions, it is natural to conjecture that the Sasaki-Ricci flow will carry out an analytic foliation minimal model program with scaling on quasi-regular Sasakian 5-manifolds as well. In this paper, we deal with the case of its orbifold structure $(Z, \Delta)$ of the leave space $Z$ is well-formed which has the codimension two fixed point set of every non-trivial isotropy subgroup with no branch divisors ($\Delta = \emptyset$).

In section 3, we first derive the following result concerning its foliation cyclic quotient singularities on a compact quasi-regular Sasakian 5-manifold:

**Theorem 1.** Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and $Z$ be its leave space of the characteristic foliation. Then $Z$ is a $Q$-factorial normal projective algebraic orbifold surface satisfying

1. if its leave space $(Z, \emptyset)$ has at least codimension two fixed point set of every non-trivial isotropy subgroup. That is to say $Z$ is well-formed, then $Z$ has isolated singularities of a finite cyclic quotient of $C^2$ and the action is

$$\mu_z : (z_1, z_2) \rightarrow (\zeta^a z_1, \zeta^b z_2),$$

where $\zeta$ is a primitive $r$-th root of unity. We denote the cyclic quotient singularity by $\frac{1}{r}(a, b)$ with $(a, r) = 1 = (b, r)$. In particular, the action can be rescaled so that every cyclic quotient singularity corresponds to a $\frac{1}{r}(1, a)$-type singularity with $(r, a) = 1, \zeta = e^{2\pi i}$. In particular, it is klt (Kawamata log terminal) singularities. Moreover, the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called foliation cyclic quotient singularities of type

$$\frac{1}{r}(1, a).$$
at a singular fibre $S^1_1$ in $M$.

(2) if its leave space $(Z, \Delta)$ has the codimension one fixed point set of some non-trivial isotropy subgroup. Then the action is

$$\mu_{Z_r} : (z_1, z_2) \rightarrow (e^{2\pi i a_1 r_1} z_1, e^{2\pi i a_2 r_2} z_2),$$

for some positive integers $r_1, r_2$ whose least common multiplier is $r$, and $a_i, i = 1, 2$ are integers coprime to $r_i, i = 1, 2$. Then the foliation singular set contains some 3-dimensional submanifolds of $M$. More precisely, the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called the Hopf $S^1$-orbibundle over a Riemann surface $\Sigma_r$.

In particular, it is a foliation $(-1)$-curve (Theorem 5) at the regular point if $r = a = 1$ as in (1) of Theorem 1. Then we have the following definition ([Cu]):

**Definition 1.** Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold with foliation singularities of type $\frac{1}{r}(1, a)$. A foliation $(-1)$-curve $V$ in $M$ is said to be floating if $V$ is entirely contained in the smooth locus of $M$ with respective to the foliation $F_\xi$. Then $M$ is said to be minimal if it has no floating foliation $(-1)$-curves.

Secondly, a detail work on the local model of resolution of foliation cyclic quotient singularities and a Castelnuovo’s contraction theorem, we have the following Sasaki analogue of Castelnuovo’s contraction Theorem on Sasakian 5-Manifolds as in section 3:

**Theorem 2.** (Sasaki Castelnuovo’s Contraction Theorem) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold with foliation singularities of type $\frac{1}{r}(1, a)$ and $V$ a floating foliation $(-1)$-curve. Then there exists a transverse birational morphism $f : M \rightarrow N$ that contracts $V \in S^3 \times D^2 \subset M$ to a regular fibre $S^3_\xi \in S^3 \times D^4 \subset N$ and it is an isomorphism outside of $V$. $N$ is again a compact quasi-regular Sasakian 5-manifold. In particular, every proper transverse birational morphism between compact regular Sasakian 5-manifolds can be factored in a sequence of contractions of foliation $(-1)$-curves.

In section 4, we work on some basic facts of the Sasaki analogue of the Kaehler-Ricci flow through singularities due to Song-Tian ([ST]). In particular, we have the following definition for the foliation canonical surgical contraction on a compact quasi-regular Sasakian 5-manifold.

**Definition 2.** We say that the solution $g(t)$ of the Sasaki-Ricci flow on a compact Sasakian 5-manifold $M$ performs a floating foliation canonical surgical contraction if the following holds: There exist distinct floating foliation $(-1)$-curves $V_1, ..., V_k$ of $M$, a compact Sasakian 5-manifold $N$ and a divisorial contraction $\psi : M \rightarrow N$ with $\psi(V_i) = S^1_1 \subset N$ and $\psi : M \setminus \cup_{i=1}^k V_i \rightarrow N \{S^1_1, ..., S^1_k\}$ a basic transverse biholomorphism onto $N \{S^1_1, ..., S^1_k\}$ such that:

1. The metrics $g(t)$ converge to a smooth Sasakain metric $g_T$ on $M \setminus \cup_{i=1}^k V_i$, as $t \rightarrow T^-$, smoothly on compact subsets of $M \setminus \cup_{i=1}^k V_i$.
2. $(M, g(t))$ converges to a unique compact metric space $(N, d_T)$ in the Gromov-Hausdorff sense as $t \rightarrow T^-$. In particular, $(N, d_T)$ is homeomorphic to the
Sasakian 5-manifold $N$. Here $d_T$ is defined to be the metric on $N$ by extending $(\psi^{-1})^*g_T$ to be zero on $\{S^1_1, ..., S^1_k\}$.

3. There exists a unique maximal smooth solution $g(t)$ of the Sasaki-Ricci flow on $N$ for $t \in (T, T_N)$ with $T < T_N \leq \infty$, such that $g(t)$ converges to $(\psi^{-1})^*g_T$ as $t \to T^+$ smoothly on compact subsets of $N \setminus \{S^1_1, ..., S^1_k\}$.

4. $(N, g(t))$ converges to $(N, d_T)$ in the Gromov-Hausdorff sense as $t \to T^+$.

In the final section, we will apply the results of local model of resolution of foliation singularities as in section 3 to prove the foliation canonical surgical contraction on a compact quasi-regular Sasakian 5-manifold $M$.

Along the lines of the arguments in [SW2], we first deal with the case of floating foliation canonical surgical contraction:

**Theorem 3.** Let $g(t)$ be a smooth solution of the Sasaki-Ricci flow on a compact quasi-regular Sasakian 5-manifold $M$ with the foliation singularities of type $1/(1, a)$ for $t \in [0, T)$ and assume $T < \infty$. Suppose there exists a blow-down map $\psi : M \to N$ contracting disjoint floating foliation $(-1)$-curves $V_1, ..., V_k$ on $M$ with $\psi(V_i) = S^1_i \subset N$, for a smooth compact Sasakian 5-manifold $(N, \omega_N)$ such that the limiting transverse Kaehler class satisfies

$$[\omega_0]_B - TC^B(M) = [\psi^*\omega_N]_B. \tag{1.2}$$

Then the Sasaki-Ricci flow $g(t)$ performs a foliation canonical surgical contraction with respect to the data $V_1, ..., V_k$, $N$ and $\psi$.

By applying Theorem 3 to the minimal resolution of $1/(1, a)$-type foliation singularities of $M$ as in Theorem 6, we have

**Corollary 1.** Let $g(t)$ be a smooth solution of the Sasaki-Ricci flow on a compact regular Sasakian 5-manifold $M$ for $t \in [0, T)$ and assume $T < \infty$. Suppose there exists a divisorial contraction $\psi : M \to N$ contracting disjoint foliation $(-1)$-curves $V_1, ..., V_k$ on $M$ with $\psi(V_i) = S^1_i \subset N$, for a smooth compact Sasakian 5-manifold $(N, \omega_N)$ such that the limiting transverse Kaehler class satisfies

$$[\omega_0]_B - TC^B(M) = [\psi^*\omega_N]_B. \tag{4}$$

Then the Sasaki-Ricci flow $g(t)$ performs a foliation canonical surgical contraction with respect to the data $V_1, ..., V_k$, $N$ and $\psi$.

As a consequence of Theorem 3 and Proposition 11, we have our main results in the paper on the analytic foliation minimal model program with scaling in a compact quasi-regular Sasakian 5-manifold.

**Theorem 4.** Let $(M, \xi, \omega_0)$ be a compact quasi-regular Sasakian 5-manifold with the foliation singularities of type $1/(1, a)$. Then there exists a unique maximal Sasaki-Ricci flow $\omega(t)$ on $M_0, M_1, ..., M_k$ starting at $(M, \omega_0)$ with floating foliation canonical surgical contractions of a finite number of disjoint floating foliation $(-1)$-curves or foliation extremal ray contractions of foliation $K^*_M$-negative curves $\psi_i : M_{i-1} \to M_i$. In addition, we have

1. Either $T_k < \infty$ and the flow $\omega(t)$ collapses in the sense that $Vol_\xi(M_k, \omega(t)) \to 0$
as \( t \to T_k^- \).

(a) there exists a contraction
\[
\varphi : M_k \to \text{pt},
\]
then \( K^T_M < 0 \) and thus \( M_k \) is transverse minimal Fano and the foliation space \( M_k/\mathcal{F}_k \) is minimal log del Pezzo surface of at worst \( \frac{1}{r}(1,a) \)-type singularities and Picard number one, or

(b) there exists a fibration
\[
\varphi : M_k \to \Sigma_h,
\]
then \( M_k \) is an \( S^1 \)-orbibundle of a rule surface over Riemann surfaces \( \Sigma_h \) of genus \( h \).

(2) \( T_k = \infty \) and \( M_k \) is nef : \( \psi_i = \psi_k \), and \( M_k \) has at worst foliation cyclic quotient singularities (orbifold singularities) and has no foliation \( K^T_M \)-negative curves.

**Corollary 2.** Let \((M, \xi, \omega_0)\) be a compact regular Sasakian 5-manifold with a smooth transverse Kähler metric \( \omega_0 \). Then there exists a unique maximal Sasaki-Ricci flow \( \omega(t) \) on \( M_0, M_1, \ldots, M_k \) with foliation canonical surgical contractions starting at \((M, \omega_0)\). Moreover, each foliation canonical surgical contraction corresponds to a divisorial contraction \( \psi_i : M_{i-1} \to M_i \) of a finite number of disjoint foliation \((-1)\)-curves on \( M_i \). In addition, we have

(1) Either \( T_k < \infty \) and the flow \( \omega(t) \) collapses in the sense that
\[
\text{Vol}_\xi(M_k, \omega(t)) \to 0,
\]
as \( t \to T_k^- \). Then \( M_k \) is transverse birational to a regular Sasakian sphere \( S^5 \) or \( (S^2 \times S^3) \) or \( S^2 \hat{\times} S^3 \) or \( \Sigma_h \times S^3 \) or \( \Sigma_h \hat{\times} S^3 \).

(2) Or \( T_k = \infty \) and \( M_k \) has no foliation \((-1)\)-curves.

In this paper, we assume that \( M \) is a compact quasi-regular transverse Sasakian manifold and the space \( Z \) of leaves is well-formed which means its orbifold singular locus and algebro-geometric singular locus coincide, equivalently \( Z \) has no branch divisors. However when its leave space \( Z \) is not well-formed, the orbifold structure \((Z, \Delta)\) of the leave space \( Z \) has the codimension one fixed point set of some non-trivial isotropy subgroup. Then the corresponding singularities in \((M, \eta, \xi, \Phi, g)\) is the Hopf \( S^1 \)-orbibundle over a Riemann surface \( \Sigma_h \). The orbifold canonical divisor \( K^\text{orb}_Z \) and canonical divisor \( K_Z \) are related by
\[
K^\text{orb}_Z = \varphi^*(K_Z + [\Delta])
\]
and then
\[
K^T_M = \pi^*(K^\text{orb}_Z).
\]
There is a Sasaki analogue of Log minimal model program ([KM]) with respect to \( K_Z + [\Delta] \). Thus we conjecture that there is a Sasaki analogue of analytic Log minimal model program with respect to \( K_Z + [\Delta] \) via the conical Sasaki-Ricci flow which is the odd dimensional counterpart of the conical Kähler-Ricci flow ([LZ], [Shen]). We hope to address this issue in the near future.
Acknowledgement. The authors would like to thank Yi-Sheng Wang for sharing his ideas on the construction of foliation singularities as in the section three.

2. Sasakian Geometry

In this section, we will give some preliminaries on structures theorems for Sasakian structures and the orbifold structure of its leave space, the foliation normal local coordinates, the basic cohomology and its Type II deformation. We refer to [BG], [FOW], and references therein for some details.

2.1. Sasakian Structure and Orbifold Structure of Its Leave Space. Let 
\((M, g, \nabla)\) be a Riemannian \((2n+1)\)-manifold. \((M, g)\) is called Sasaki if the cone 
\((C(M), \overline{g}) := (\mathbb{R}^+ \times M, \ dr^2 + r^2 g)\)
such that \((C(M), \overline{g}, J, \omega)\) is Kaehler with 
\[\omega = \frac{1}{2} i \partial \overline{\partial} r^2.\]
The function \(\frac{1}{2} r^2\) is hence a global Kaehler potential for the cone metric. As \(\{r = 1\} = \{1\} \times M \subset C(M)\), we may define 
\[\xi = J(r \frac{\partial}{\partial r})\]
and the Reeb vector field \(\xi\) on \(M\)
\[\xi = J\left(\frac{\partial}{\partial r}\right).\]
Also 
\[\overline{\eta}(\cdot) = \frac{1}{2} \overline{g}(\xi, \cdot)\]
and the contact 1-form \(\eta\) on \(TM\)
\[\eta(\cdot) = g(\xi, \cdot)\]
Then \(\xi\) is the killing vector field with unit length such that 
\[\eta(\xi) = 1 \text{ and } d\eta(\xi, X) = 0.\]
The tensor field of type\((1, 1)\), defined by \(\Phi(Y) = \nabla_Y \xi\), satisfies the condition 
\[\nabla_X \Phi(Y) = g(\xi, Y)X - g(X, Y)\xi\]
for any pair of vector fields \(X\) and \(Y\) on \(M\). Then such a triple \((\eta, \xi, \Phi)\) is called a Sasakian structure on a Sasakian manifold \((M, g)\). Note that the Riemannian curvature satisfying the following 
\[R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi\]
for any pair of vector fields \(X\) and \(Y\) on \(M\). In particular, the sectional curvature of every section containing \(\xi\) equals one.

The first structure theorem on Sasakian manifolds states that
Proposition 1. (Ru, Sp) Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian manifold of dimension \(2n+1\) and \(Z\) denote the space of leaves of the characteristic foliation \(\mathcal{F}_\xi\) (just as topological space). Then

(i) \(Z\) carries the structure of a Hodge orbifold \(Z = (Z, \Delta)\) with an orbifold Kaehler metric \(h\) and Kaehler form \(\omega\) which defines an integral class \([p^*\omega]\) in \(H^2_{\text{orb}}(Z, Z)\) in such a way that \(\pi: (M, g) \to (Z, h)\) is an orbifold Riemannian submersion, and a principal \(S^1\)-orbibundle \((V\text{-bundle})\) over \(Z\). Furthermore, it satisfies \(\frac{1}{2}d\eta = \pi^*(\omega)\). The fibers of \(\pi\) are geodesics.

(ii) \(Z\) is also a \(Q\)-factorial, polarized, normal projective algebraic variety.

(iii) \((M, \xi, g)\) is Sasaki-Einstein if and only if \((Z, h)\) is Kaehler-Einstein with scalar curvature \(4n(n+1)\).

(iv) If \((M, \eta, \xi, \Phi, g)\) is regular then the orbifold structure is trivial and \(\pi\) is a principal circle bundle over a smooth projective algebraic variety.

(v) As real cohomology classes, there is a relation between the basic Chern class and orbifold Chern class

\[
c_k^B(M) := c_k(F_\xi) = \pi^*c_k^{\text{orb}}(Z).
\]

Conversely, let \(\pi: M \to Z\) be a \(U(1)\)-orbibundle over a compact Hodge orbifold \((Z, h)\) whose first Chern class is an integral class defined by \([\omega_Z]\), and \(\eta\) be a 1-form with \(\frac{1}{2}d\eta = \pi^*\omega_Z\). Then \((M, \pi^*h + \eta \otimes \eta)\) is a Sasakian manifold if all the local uniformizing groups inject into the structure group \(U(1)\).

On the other hand, the second structure theorem on Sasakian manifolds states that

Proposition 2. (Ru) Let \((M, g)\) be a compact Sasakian manifold of dimension \(2n+1\). Any Sasakian structure \((\xi, \eta, \Phi, g)\) on \(M\) is either quasi-regular or there is a sequence of quasi-regular Sasakian structures \((M, \xi_i, \eta_i, \Phi_i, g_i)\) converging in the compact-open \(C^\infty\)-topology to \((\xi, \eta, \Phi, g)\). In particular, if \(M\) admits an irregular Sasakian structure, it admits many locally free circle actions.

2.2. The Foliated Normal Coordinate. Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n+1)\)-manifold with \(g(\xi, \xi) = 1\) and the integral curves of \(\xi\) are geodesics. For any point \(p \in M\), we can construct local coordinates in a neighbourhood of \(p\) which are simultaneously foliated and Riemann normal coordinates (GKN). That is, we can find Riemann normal coordinates \(\{x, z^1, z^2, \cdots, z^n\}\) on a neighbourhood \(U\) of \(p\) such that \(\frac{\partial}{\partial x} = \xi\) on \(U\). Let \(\{U_\alpha\}_{\alpha \in A}\) be an open covering of the Sasakian manifold and

\[
\pi_\alpha: U_\alpha \to V_\alpha \subset \mathbb{C}^n
\]

submersion such that \(\pi_\alpha \circ \pi_\beta^{-1}: \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)\) is biholomorphic. On each \(V_\alpha\), there is a canonical isomorphism

\[
d\pi_\alpha : D_p \to T_{\pi_\alpha(p)}V_\alpha
\]

for any \(p \in U_\alpha\), where \(D = \ker \xi \subset TM\). Since \(\xi\) generates isometries, the restriction of the Sasakian metric \(g\) to \(D\) gives a well-defined Hermitian metric \(g^T_\alpha\) on \(V_\alpha\). This Hermitian metric in fact is Kähler. More precisely, let \(z^1, z^2, \cdots, z^n\) be the local holomorphic coordinates on \(V_\alpha\). We pull back these to \(U_\alpha\) and still write the same. Let \(x\) be the coordinate along the leaves with \(\xi = \frac{\partial}{\partial x}\). Then we have the
foliation local coordinate \( \{x, z^1, z^2, \ldots, z^n\} \) on \( U_\alpha \) and \( (D \otimes \mathbb{C}) \) is spanned by the form
\[
Z_\alpha = \left( \frac{\partial}{\partial z^\alpha} - \theta \left( \frac{\partial}{\partial z^\alpha} \right) \frac{\partial}{\partial x} \right), \quad \alpha = 1, 2, \ldots, n.
\]
Moreover
\[
\Phi = i \left( \frac{\partial}{\partial z^j} + i h_j \frac{\partial}{\partial x} \right) \otimes dz^j + \text{conj}
\]
and
\[
\eta = dx - i h_j dz^j + i h_j d\bar{z}^j.
\]
Here \( h \) is basic: \( \frac{\partial h}{\partial x} = 0 \) and \( h_j = \frac{\partial h}{\partial z^j}, h_{jl} = \frac{\partial^2 h}{\partial z^j \partial \bar{z}^l} \) with
\begin{equation}
(2.1) \quad \text{The normal coordinate: } h_j(p) = 0, h_j(p) = \delta^j_i, dh_j(p) = 0.
\end{equation}

A frame
\[
\left\{ \frac{\partial}{\partial x}, Z_j = \left( \frac{\partial}{\partial z^j} + i h_j \frac{\partial}{\partial x} \right), \quad j = 1, 2, \ldots, n \right\}
\]
and the dual
\[
\{\eta, dz^j, j = 1, 2, \ldots, n\}
\]
with
\[
[Z_i, Z_j] = [\xi, Z_j] = 0.
\]
Since \( i(\xi) d\eta = 0 \),
\[
d\eta(Z_\alpha, \overline{Z_\beta}) = d\eta(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}).
\]
Then the Kähler 2-form \( \omega^T_\alpha \) of the Hermitian metric \( g^T_\alpha \) on \( V_\alpha \), which is the same as the restriction of the Levi form \( d\eta \) to \( \widetilde{D}_n^\alpha \), the slice \( \{x = \text{constant}\} \) in \( U_\alpha \), is closed. The collection of Kähler metrics \( \{g^T_\alpha\} \) on \( \{V_\alpha\} \) is so-called a transverse Kähler metric. We often refer to \( d\eta \) as the Kähler form of the transverse Kähler metric \( g^T \) in the leaf space \( \widetilde{D}^n \).

The Kähler form \( d\eta \) on \( D \) and the Kähler metric \( g^T \) is define such that
\[
g = g^T + \eta \otimes \eta
\]
and
\[
g^T_{ij} = g^T \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right) = d\eta \left( \frac{\partial}{\partial z^i}, \Phi \frac{\partial}{\partial \bar{z}^j} \right) = 2h_{ij}.
\]
In terms of the normal coordinate, we have:
\[
g^T = g^T dz^i d\bar{z}^j, \omega = d\eta = 2i h_{ij} dz^i \wedge d\bar{z}^j.
\]

The transverse Ricci curvature \( \text{Ric}^T \) of the Levi-Civita connection \( \nabla^T \) associated to \( g^T \) is
\[
\text{Ric}^T = \text{Ric} + 2g^T
\]
and
\[
R^T = R + 2n.
\]
The transverse Ricci form \( \rho^T \)
\[
\rho^T = \text{Ric}^T (J \cdot, \cdot) = -i R_{ij} dz^i \wedge d\bar{z}^j
\]
with
\[ R_{ij}^T = -\frac{\partial_2}{\partial z^i \partial \bar{z}^j} \log \det(g_{\alpha \beta}^T) \]
and it is a closed basic \((1,1)\)-form
\[ \rho_T = \rho + 2d\eta. \]

2.3. Basic Cohomology and Type II Deformation in a Sasakian Manifold.

**Definition 3.** Let \((M, \eta, \xi, \Phi, g)\) be a Sasakian \((2n + 1)\)-manifold. Define
A \( p \)-form \( \gamma \) is called basic if
\[ i(\xi)\gamma = 0 \]
and
\[ \mathcal{L}_\xi \gamma = 0. \]

Let \( \Lambda^p_B \) be the sheaf of germs of basic \( p \)-forms and \( \Omega^p_B \) be the set of all global sections of \( \Lambda^p_B \). It is easy to check that \( d\gamma \) is basic if \( \gamma \) is basic. Set \( d_B = d|_{\Omega^p_B} \).

Then \( d_B : \Omega^p_B \to \Omega^{p+1}_B \).

We then have the well-defined operators
\[ d_B := \partial_B + \overline{\partial}_B \]
with
\[ \partial_B : \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B \]
and
\[ \overline{\partial}_B : \Lambda^{p,q}_B \to \Lambda^{p,q+1}_B. \]

Then for \( d_B^c := \frac{i}{2}(\overline{\partial}_B - \partial_B) \), we have
\[ d_Bd_B^c = i\overline{\partial}_B \overline{\partial}_B, \quad d_B^2 = (d_B^c)^2 = 0. \]

The basic Laplacian is defined by
\[ \Delta_B := d_Bd_B^* + d_B^*d_B. \]

Then we have the basic de Rham complex \( (\Omega^*_B, d_B) \) and the basic Dolbeault complex \( (\Omega^{p,*}_B, \overline{\partial}_B) \) and its cohomology group \( H^*_B(M, \mathbb{R}) \) ([EKA]).

**Definition 4.** (i) We define the basic Cohomology of the foliation \( F_\xi \) by
\[ H^*_B(F_\xi) := H^*_B(M, \mathbb{R}). \]

Then by transverse Hodge decomposition and transverse Serré duality
\[ H^{p,q}_B(F_\xi) \simeq H^{q,p}_B(F_\xi) \]
and the cohomology of the leaf space \( Z = M/U(1) \) to this basic cohomology of the foliation:
\[ H^{*}_\text{orb}(Z, \mathbb{R}) = H^*_B(F_\xi) := H^*_B(M, \mathbb{R}). \]

(ii) ([34], Theorem 7.5.17) : Define the basic first Chern class \( c_1^B(M) \) by
\[ c_1^B = [\rho_T]_B \]
and the transverse Einstein (Sasaki $\eta$-Einstein) equation up to a $D$-homothetic deformation
$$[\rho^T]_B = \kappa [d\eta]_B, \kappa = -1, 0, 1.$$  

Basic $k$-th Chern class $c^B_k(M)$ is represented by a closed basic $(k,k)$-form $\gamma_k$ which is determined by the formula:
$$\det(I_n - \frac{1}{2\pi i} \Omega^T) = 1 + \gamma_1 + \ldots + \gamma_k.$$  

Here $\Omega^T$ is the curvature 2-form of type basic $(1,1)$ with respect to the transverse connection $\nabla^T$.

**Example 1.** Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n+1)$-manifold. If $g^T$ is a transverse Kaehler metric on $M$, then $h_\alpha = \det((g^\alpha_{ij})^T)$ on $U_\alpha$ defines a basic Hermitian metric on the canonical bundle $K^T_M$. The inverse $(K^T_M)^{-1}$ of $K^T_M$ is sometimes called the anti-canonical bundle. Its basic first Chern class $c^B_1((K^T_M)^{-1})$ is called the basic first Chern class of $M$ and is often denoted by $c^B_1(M)$. Then it follows from the previous result that $c^B_1(M) = [\text{Ric}^T(\omega)]_B$ for any transversal Kaehler metric $\omega$ on a Sasakian manifold $M$.

**Definition 5.** We define Type II deformations of Sasakian structures $(M, \eta, \xi, \Phi, g)$ by fixing the $\xi$ and varying $\eta$. That is, for $\varphi \in \Omega^0_B$, define
$$\tilde{\eta} = \eta + d^c_B \varphi,$$
then
$$d\tilde{\eta} = d\eta + d_B d^c_B \varphi = d\eta + i\partial_B \partial_{\overline{B}} \varphi$$
and
$$\tilde{\omega} = \omega + i\partial_B \partial_{\overline{B}} \varphi.$$  

Note that we have the same transversal holomorphic foliation ($\xi$ is fixed) but with the new Kaehler structure on the Kaehler cone $C(M)$ and new contact bundle $\tilde{D} : \tilde{\omega} = dd^c\tilde{r}, \tilde{r} = re^\varphi$. The same holomorphic structure : $r \frac{\partial}{\partial r} = \tilde{r} \frac{\partial}{\partial \tilde{r}}; \xi = J(\frac{\partial}{\partial r})$ and $\xi + ir \frac{\partial}{\partial \varphi} = \tilde{\xi} = i\tilde{\Phi}(\xi)$ is the holomorphic vector field on $C(M)$. Moreover, we have
$$\tilde{\Phi} = \Phi - \xi \otimes (d^c_B \varphi) \circ \Phi,$$
and
$$L_{\xi} \tilde{\Phi} = L_{\xi} \Phi = 0.$$  

3. Blow-up and Resolution of Foliation Singularities

3.1. Foliation Singularities. In this section, parts of notions are due to the papers of Boyer-Galicki ([BG]) and Wang ([W]). Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian manifold of dimension five and $Z$ denote the space of leaves of the characteristic foliation $\mathcal{F}_\xi$. It follows from the first structure theorem as in Proposition 1 that $Z$ is a compact normal, orbifold surface locally given by charts written as quotients of smooth coordinate charts. That is, $Z$ can be covered by open charts $Z = \cup U_i$. The orbifold charts on $(Z, U_i, \varphi_i)$ is defined by the local
uniformizing systems $((\tilde{U}_i, G_i, \varphi_i))$ centered at the point $z_i$, where $G_i$ is the local uniformizing finite group acting on a smooth complex space $\tilde{U}_i$ such that

$$\varphi_i : \tilde{U}_i \to U_i = \tilde{U}_i / G_i$$

is the biholomorphic map. A point $x$ of complex orbifold $Z$ whose isotropy subgroup $\Gamma_x \neq \text{Id}$ is called a singular point. Those points with $\Gamma_x = \text{Id}$ are called regular points. The set of singular points is called the orbifold singular locus or orbifold singular set, and is denoted by $\Sigma_{\text{orb}}(Z)$. Let $\Gamma \subset \text{GL}(2, \mathbb{C})$ be a finite subgroup. Then the quotient space $\mathbb{C}^2 / \Gamma$ is smooth if and only if $\Gamma$ is a reflection group which fixes a hyperplane in $\mathbb{C}^2$. By the first structure theorem, the underlying complex orbifold space $Z = (Z, U_i)$ is a normal, orbifold variety with the algebro-geometric singular set $\Sigma(Z)$. Then $\Sigma(Z) \subset \Sigma_{\text{orb}}(Z)$ and it follows that $\Sigma(Z) = \Sigma_{\text{orb}}(Z)$ if and only if none of the local uniformizing groups $\Gamma_i$ of the orbifold $Z = (Z, U_i)$ contain a reflection. Moreover if some $\Gamma_i$ contains a reflection, then the reflection fixes a hyperplane in $\tilde{U}_i$ giving rise to a ramification divisor on $\tilde{U}_i$ and a branch divisor on $Z$. Then we have the following definition regarding orbifold singular locus and algebro-geometric singular locus.

**Definition 6.** (i) The branch divisor $\Delta$ of an orbifold $Z = (Z, \Delta)$ is a $\mathbb{Q}$-divisor on $Z$ of the form

$$\Delta = \sum_\alpha (1 - \frac{1}{m_\alpha})D_\alpha,$$

where the sum is taken over all Weil divisors $D_\alpha$ that lie in the orbifold singular locus $\Sigma_{\text{orb}}(Z)$, and $m_\alpha$ is the gcd of the orders of the local uniformizing groups taken over all points of $D_\alpha$ and is called the ramification index of $D_\alpha$.

(ii) The orbifold structure $Z = (Z, \Delta)$ is called well-formed if the fixed point set of every non-trivial isotropy subgroup has codimension at least two. Then $Z$ is well-formed if and only if its orbifold singular locus and algebro-geometric singular locus coincide, equivalently $Z$ has no branch divisors.

**Example 2.** For instance, the weighted projective $\mathbb{C}P(1; 4; 6)$ has a branch divisor $\frac{1}{2}D_0 = \{z_0 = 0\}$. But $\mathbb{C}P(1; 2; 3)$ is a unramified well-formed orbifold with two singular points, $\{0; 1; 0\}$ with local uniformizing group the cyclic group $\mathbb{Z}_2$, and $\{0; 0; 1\}$ with local uniformizing group $\mathbb{Z}_3$.

Let $(M, \xi, \alpha_\xi)$ be a compact quasi-regular Sasakian 5-manifold and the leave space $Z := M/S^1$ be an orbifold Kaehler surface. Denote by the quotient map

$$\pi : M \to Z$$

as before, and call such a 5-manifold $(M, \xi)$ an $S^1$-orbibundle. More precisely, $M$ admits a locally free, effective $S^1$-action

$$\alpha_\xi : S^1 \times M \to M$$

such that $\alpha_\xi(t)$ is orientation-preserving, for every $t \in S^1$. Since $M$ is compact, $\alpha_\xi$ is proper and the isotropy group $\Gamma_p$ of every point $p \in M$ is finite.
Definition 7. The principal orbit type $M_{\text{reg}}$ corresponds to points in $(M, \xi, \alpha_\xi)$ with the trivial isotropy group and $M_{\text{reg}} \to M_{\text{reg}}/S^1$ is a principle $S^1$-bundle. Furthermore, the orbit $S^1_p$ of a point $p \in M$ is called a regular fiber if $p \in M_{\text{reg}}$, and a singular fiber otherwise. In this case, $M_{\text{sing}}/S^1 \simeq \Sigma_{\text{orb}}(Z)$.

The Proof of Theorem 1:

Proof. Since each orbit $S^1_p$ in an $S^1$-orbibundle $(M, \xi, \alpha_\xi)$ admits an $S^1$-invariant neighborhood $U_p$ such that it is $S^1$-equivalent to a standard $S^1$-structure $V_p := S^1 \times \mathbb{C}^2/\mu_p$, where $\mu_p$ is a linear representation of $\Gamma_p$ of $p$ on the vector space $\mathbb{C}^2$, and the $S^1$-action on $\mathbb{C}^2$ is induced by the Lie group structure of $S^1$. Since $\alpha$ is effective and $\mu_p$ is faithful. If in addition $\mathbb{C}^2$ is endowed with a complex structure, one can enhance the notion of a standard $S^1$-structure, and define the standard complex $S^1$-structure which is the standard $S^1$-structure $S^1 \times \mathbb{C}^2/\mu_p$ with $\mu_p$ complex linear. Note that, up to conjugation, a representation $\mu_p$ of $\mathbb{Z}_r$ on $\mathbb{C}^2$ is conjugate to

$$\mathbb{Z}_r \to \mathbb{C}^2$$

$$\gamma \mapsto \begin{bmatrix} e^{2\pi i \frac{a_1}{r}} & 0 \\ 0 & e^{2\pi i \frac{a_2}{r}} \end{bmatrix}$$

for some positive integers $r_1, r_2$ whose least common multiplier is $r$, and $a_i, i = 1, 2$ are integers coprime to $r_i$, $i = 1, 2$. If $r_1, r_2 < r$, then $(M, \xi, \alpha_\xi)$ has a non-discrete set of fibers; more precisely, $M_{\text{sing}}$ contains some 3-dimensional submanifolds of $M$. In particular, in the case where $M_{\text{sing}}$ is discrete, we have

$$r_1 = r_2 = r.$$ 

Up to change of generator, the representation $\mu_p$ can be normalized as follows:

(3.1) $\mathbb{Z}_r \to \mathbb{C}^2$

$$\gamma \mapsto \begin{bmatrix} e^{2\pi i \frac{a_1}{r}} & 0 \\ 0 & e^{2\pi i \frac{a_2}{r}} \end{bmatrix}$$

for some $a$ coprime to $r$. Then Theorem 1 follows easily. \hfill \Box

Now we come out with the following definition.

Definition 8. Given $p \in M_{\text{sing}}$, the singular fiber $S^1_p$ is a foliation singularity of type $\frac{1}{r}(1,a)$ at $p$ with $r > 0, a$ two integers and $a$ coprime with $r$, if there exists an isomorphism $\Gamma_p \simeq \mathbb{Z}_r = \langle \gamma \rangle$ such that $\mu_p$ can be identified with the representation (3.1), up to conjugation. This corresponds to the orbifold structure $Z$ is well-formed where the fixed point set of every non-trivial isotropy subgroup has codimension at least two.

Since $M - M_{\text{reg}}$ is a union of smooth submanifolds of $M$. The present paper concerns mainly the case where $M_{\text{sing}} = M - M_{\text{reg}}$ is a discrete, and hence finite set.
3.2. Local Model of $\frac{1}{k}(1, 1)$-type Foliation Singularities. Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and $S^1_p$ be a discrete singular fiber of type $\frac{1}{k}(1, a)$ at $p \in M$. Then there is an $S^1$-equivariant neighborhood $U_p$ of $S^1_p$ and $S^1$-equivalent to a standard complex $S^1$-structure $S^1 \times \mathbb{C}^2/\mathbb{Z}_r$. In this subsection, we examine the $S^1$-structure of a neighborhood of a singular fiber of $\frac{1}{k}(1, a)$ type. Then, by the slice theorem as discussed in the previous subsection, there is an $S^1$-equivariant neighborhood $U_p$ of $S^1_p$, $S^1$-equivalent to a standard complex $S^1$-structure $S^1 \times \mathbb{C}^2/\mathbb{Z}_r$. The $S^1$-action on $S^1 \times \mathbb{C}^2$ is induced by the $S^1$-action on $S^1 \times \mathbb{C}^2$ given by

$$S^1 \times S^1 \times \mathbb{C}^2 \to S^1 \times \mathbb{C}^2$$

$$(t, w, z_1, z_2) \mapsto (tw, z_1, z_2).$$

Now we consider the local diffeomorphism

$$\pi : S^1 \times \mathbb{C}^2 \to S^1 \times \mathbb{C}^2$$

$$(w, z_1, z_2) \mapsto (w^k, w^{-1}z_1, w^{-a}z_2)$$

which descends to a diffeomorphism from $S^1 \times \mathbb{C}^2/\mathbb{Z}_r$ to $S^1 \times \mathbb{C}^2$. Via the diffeomorphism, one can identify the neighborhood $U_p$ with $S^1 \times \mathbb{C}^2$ with the $S^1$-action on $S^1 \times \mathbb{C}^2$ given by

$$(3.2) \quad S^1 \times S^1 \times \mathbb{C}^2 \to S^1 \times \mathbb{C}^2$$

$$(t, u, v_1, v_2) \mapsto (tu^k, t^{-1}v_1, t^{-a}v_2).$$

The above identification also yields an $S^1$-equivalence between a closed tubular neighborhood of $S^1_p$ and $S^1 \times D^4$, where $D^4$ is the 4-ball in $\mathbb{C}^2$, and the $S^1$-action on $S^1 \times D^4$ is given by

$$(3.3) \quad S^1 \times S^1 \times D^4 \to S^1 \times D^4$$

$$(t, u, v_1, v_2) \mapsto (tu^k, t^{-1}v_1, t^{-a}v_2).$$

Now we are ready to consider the resolution of a singular fiber $S^1_p$ of type $\frac{1}{k}(1, 1)$ at $p \in M$. Consider the space $S(-k)$ given by

$$\{(z_1, z_2) \in \mathbb{C} \times S^3 \mid z_1u^k_2 = z_2u^k_1\},$$

where $S^3$ is the standard sphere in $\mathbb{C}^2$ and oriented using inward normal. $S(-k)$ admits a natural free $S^1$-action given by the restriction of

$$(3.4) \quad \alpha_k : S^1 \times \mathbb{C}^2 \times S^3 \to \mathbb{C} \times S^3$$

$$(t, z_1, z_2, u_1, u_2) \mapsto (z_1, z_2, t^{-1}u_1, t^{-1}u_2).$$

$S(-k)$ can be identified with $\mathbb{C} \times S^3$ via the diffeomorphism

$$(3.5) \quad \mathbb{C} \times S^3 \to S(-k)$$

$$(z, (w_1, w_2)) \mapsto (zw^k_1, zw^k_2, w_1, w_2)$$
which induces a free $S^1$-action $\beta_k$ on $\mathbb{C} \times S^3$:

(3.6) \[ \beta_k : S^1 \times \mathbb{C} \times S^3 \to \mathbb{C} \times S^3 \]
\[ (t, z, w_1, w_2) \mapsto (t^k z, t^{-1} w_1, t^{-1} w_2). \]

We remark that the quotient of the embedding $S(-k) \subset \mathbb{C}^2 \times S^3$ by $\alpha_0$ is the embedding $O(-k) \subset \mathbb{C}^2 \times \mathbb{C}P^1$, and furthermore, there is a pullback diagram

\[
\begin{array}{ccc}
\overline{M} \supset S(-k) & \cong & \mathbb{C} \times S^3 \\
\downarrow & & \downarrow \pi_3 \\
Z \supset O(-k) & \cong & \mathbb{C}P^1 \\
\end{array}
\]

Here $\pi_2$ is the projection onto the second factor, and $H$ is the Hopf fibration. Note that $D^2 \times S^3 \simeq \mathbb{C} \times S^3$ can be identified with the subspace of $\mathbb{C}^2 \times \mathbb{C}P^1$ given by

\[ \{ (z_1, z_2), (u_1, u_2) | z_1 u_2^k - z_2 u_1^k = 0 \} \]

which is the coordinate for the canonical line bundle $S(-k)$ over $S^3$ and $\overline{M}$ can be realized by projecting down the first three coordinates to with the subspace of $\mathbb{C}^2 \times \mathbb{C}P^1$ given by

\[ \{ (z_1, z_2, [l_1, l_2]) | z_1 l_2^k - z_2 l_1^k = 0 \} \]

which is the coordinate for the canonical line bundle $O(-k)$ over $\mathbb{C}P^1$.

Now, to resolve the singular fiber $S^1_p$ amounts to replacing $U$ with $S(-k)$. To see how this is done, we observe that there is an $S^1$-equivariant embedding

(3.8) \[ \Psi : S^1 \times (\mathbb{C}^2 - \{0,0\}) \to S(-k) \subset \mathbb{C}^2 \times S^3 \]
\[ (s, z_1, z_2) \mapsto (sz_1^k, sz_2^k, \frac{z_1}{\sqrt{|z_1|^2 + |z_2|^2}}, \frac{z_2}{\sqrt{|z_1|^2 + |z_2|^2}}). \]

The complement of the image in $\tilde{U}$ is precisely the 3-sphere $\{0,0\} \times S^3$ where $\alpha_0$ restricts to give the negative Hopf fibration $-H$.

We define $(M', \alpha')$ to be the new $S^1$-orbibundle given by gluing $M - S^1_p$ and $S(-k)$ via the identification between $U - S^1_p$ and $S(-k) - \{0,0\} \times S^3$ given in above with $\alpha'$ induced by $\alpha_0$ and $\alpha_k$. In other words, we remove the singular fiber by replacing it with a negative Hopf fibration. It follows that

**Lemma 1.** $(M', \alpha')$ is again a compact quasi-regular Sasakian 5-manifold.

### 3.2.1. The Coordinates on $S(-k)$

Let $U_i, i = 1, 2$ be two copies of $\mathbb{C} \times \mathbb{C} \times S^1$ denoted by $(x_i, v_i, s_i)$ the coordinate of a point in $U_i$. Consider the embedding

\[
\begin{array}{ccc}
U_1 & \to & S(-k) \\
(x_1, v_1, s_1) & \mapsto & (v_1, v_1 x_1^k, (\frac{s_1}{\sqrt{1+|x_1|^2}}, \frac{sx_1}{\sqrt{1+|x_1|^2}}))
\end{array}
\]

and

\[
\begin{array}{ccc}
U_2 & \to & S(-k) \\
(x_2, v_2, s_2) & \mapsto & (v_2 x_2^k, v_2, (\frac{s_2}{\sqrt{1+|x_2|^2}}, \frac{sx_2}{\sqrt{1+|x_2|^2}}))
\end{array}
\]

which induce coordinate charts covering $S(-k)$ with the transition function given by

\[ x_2 = x_1^{-1}, v_2 = v_1 x_1^k, s_2 = s_1 \frac{x_1}{x_2}. \]
Note that the quotients $\overline{U}_i$ of $U_i, i = 1, 2,$ give us the standard coordinates of $\mathcal{O}(-k)$, namely,

$$\overline{U}_1 \to \mathcal{O}(-k), \quad (x_1, v_1) \mapsto (v_1, v_1 x_1^k, [1, x_1])$$

and

$$\overline{U}_2 \to \mathcal{O}(-k), \quad (x_2, v_2) \mapsto (v_2 x_2^k, v_2, [x_2, 1]),$$

where $[l_1, l_2]$ is the homogeneous coordinate of a point in $\mathbb{CP}^1$. In particular, we have $x_1 = \frac{l_1}{l_2}; x_2 = \frac{l_1}{l_2}$, and $v_i, i = 1, 2$ correspond to vectors in a fiber of $\mathcal{O}(-k)$.

### 3.2.2. Distance to the Zero Section

Let $S^3_0$ be the zero section of $\pi_2 : S(-k) \subset \mathbb{C}^2 \times S^3 \to S^3$ and endow $S(-k)$ with the metric induced by the standard metric on $\mathbb{C}^2 \times S^3$. Then the distance square $\text{dist}^2(y, S^3)$ between a point $y \in S(-k)$ and $S^3$ expressed in terms of the coordinates charts is

$$\text{dist}^2(y, S^3_0) = (1 + |x_i|^{2k})|v_i|^2,$$

where $y = (x_i, v_i, s_i) \in U_i, i = 1, 2$.

Now we define the section $s$ of $[V]$ over $S(-k)$ by

$$s_i : U_i \to \mathbb{C}, \quad s_i = v_i.$$

Note that $s$ is vanishing along the exceptional basic divisor $V$ and the hermitian metric $h$ on $[V]$ such that

$$h_i = \frac{|l_1|^{2k} + |l_2|^{2k}}{|l_i|^{2k}} = (1 + |x_i|^{2k})$$

on $U_i$. If furthermore the composition

$$S(-k) \approx \mathbb{C} \times S^3 \xrightarrow{\pi_2} S^3 \xrightarrow{\downarrow -H} \mathbb{CP}^1$$

send $y$ to $[l_1, l_2]$, then

$$(3.9) \quad \text{dist}^2(y, S^3) = \left(\frac{|l_1|^{2k} + |l_2|^{2k}}{|l_i|^{2k}}\right)|v_i|^2 := |s_i|^2_h$$

where $y \in U_i$.

### 3.2.3. Distance to the Singular Fiber

The standard metric on $\mathbb{C}^2 \times S^3$ also induces a metric on $U - S^3_p = S^1 \times (\mathbb{C}^2 - \{(0, 0)\})$ via

$$S^1 \times (\mathbb{C}^2 - \{(0, 0)\}) \xrightarrow{t \mapsto} S(-k) = \mathbb{C} \times S^3 \subset \mathbb{C}^2 \times S^3.$$

In particular, given a point $x = (s, z_1, z_2)$, the image $\Psi \circ \tau(t)$ of the ray $\tau(t) = (s, tz_1, tz_2), 0 \leq t \leq 1$ is a geodesic

$$(sz_1^k, sz_2^k, \frac{z_1}{\sqrt{|z_1|^2 + |z_2|^2}}, \frac{z_2}{\sqrt{|z_1|^2 + |z_2|^2}}) \subset \mathbb{C}^2 \times S^3$$
away from zero, so the distance square to the singular fiber is
\[(3.10) \quad \text{dist}^2(x, S^1_p) = |z_1|^{2k} + |z_2|^{2k} = r_{S^1_p}^{2k}.
\]

Furthermore, \(\Psi \circ \tau(t)\) is a geodesic from \(\Psi(x) \in S(-k)\) to the zero section \(S^3_0\). Then
\[(3.11) \quad \Psi^* (\text{dist}^2(\Psi(x), S^3)) = \text{dist}^2(x, S^1_p).
\]

### 3.2.4. Foliation Blow-up and Sasaki Castelnuovo’s Contraction Theorem.

As in (3.7) with \(k = 1\), we pull back a negative Hopf fibration \(-H\) along the canonical disk bundle \(\pi_d : O(-1) \to CP^1\), we get a \(S^1\)-bundle \(S(-1)\) over \(O(-1)\), which is necessarily diffeomorphic to \(S^3 \times D^2\), since the only \(D^2\)-bundle over \(S^3\) is the trivial one:
\[M \supset S(-1) = S^3 \times D^2 \quad \rightarrow \quad S^3 \quad \text{via} \quad S^1_p \subset S^1 \times D^4 \subset M \quad \Downarrow \quad \pi \]
\[(3.12) \quad Z \supset O(-1) \quad \rightarrow \quad CP^1 \quad \text{via} \quad z_p = \pi(p) \in Z.
\]

Here \(\phi : Z \rightarrow Z\) is a blow-up of \(Z\) at a nonsingular point \(z_p = \pi(p) \in Z\).

As the first consequence of previous subsections (cf Lemma 1, et c), it follows that we have the following foliation blow-up along a regular fibre \(S^1_p\) of type \(\frac{1}{k}(1,1)\) with \(k = 1\):

**Theorem 5.** Let \((M, \eta, \xi, \Phi, g)\) be a compact regular Sasakian 5-manifold and \(Z\) denote the space of leaves of the characteristic foliation which is a smooth projective surface with the regular \(S^1\)-principal bundle over \(M\). Then

1. There is a compact regular Sasakian 5-manifold \(\overline{M}\) which is obtained by gluing
   \[M - \mathcal{N}(S^1_p)\]
   and \(S^3 \times D^2\) along their boundary via an orientation-preserving diffeomorphism
   \[f : S^3 \times S^1 \simeq \partial(S^3 \times D^2) \rightarrow -\partial(M - \mathcal{N}(S^1_p)) \simeq S^3 \times S^1.
   \]
   In other words, we have the blow-up map \(\psi\)
   \[\psi : \overline{M} \rightarrow M\]
   and \(\overline{M}\) is called the foliation blowing up along a regular fibre \(S^1_p\) at a point \(p \in M\)
   \[\overline{M} = (M - \mathcal{N}(S^1_p)) \cup_f S^3 \times D^2\]

2. For the irreducible transverse exceptional divisor \(V \in \text{Exc}(\psi)\) and the irreducible exceptional divisor \(E \in \text{Exc}(\phi)\), the foliation \((-1)\)-curve \(V\) of such a regular fibre \(S^1_p\) at a point \(p \in M\) is a regular \(S^1\)-principal bundle over a compact Riemann sphere \(S^2\). Moreover, \(E\) is biholomorphic to \(CP^1\) and \(V\) is transverse biholomorphic to \(S^3\). Furthermore it follows from (A.2) and (A.4) that
   \[E \cdot E = -1\]
and then
\[ V : V = -1. \]

(3) For the foliation \((-1)\)-curve blow-up map \(\psi : M \to M\), it follows from (3.9), (3.10) and (3.11) that
\[ (3.13) \quad \psi^* r_{S^1_p}^2 = |s|^2_h. \]

In general, for the foliation \((-k)\)-curve contraction map, we have
\[ (3.14) \quad \psi^* r_{S^1_p}^{2k} = |s|^2_h. \]

In particular, let \(X\) and \(Y\) be smooth projective surfaces with the Castelnuovo’s Contraction map \(\phi : X \to Y\), we have the following lifting transverse contraction morphism \(\psi : M \to N\) via the regular \(S^1\)-principal bundles \(\pi_M\) and \(\pi_N\) as in
\[ \begin{array}{ccc}
M \supset V & \xrightarrow{\psi} & N \supset S^1_p \\
\downarrow \pi_M & \circlearrowleft & \downarrow \pi_N \\
X \supset E & \xrightarrow{\phi} & Y \ni \pi_N(p). 
\end{array} \]

Then, as the consequence of Theorem 5, we have the Sasaki analogue of Castelnuovo’s Contraction Theorem 2 on Sasakian Five-Manifolds.

**Remark 1.** In general for a Sasakian 3-manifold, it is an \(S^1\)-Seifert 3-manifold and the Euler characteristic \(e\) of the Seifert bundle is nonzero if it is closed. For a Seifert bundle, the geometry is determined by the Euler characteristic \(\chi\) of the base 2-orbifold (\(\text{[Gel]}\)):

\[
\begin{array}{cccc}
\chi > 0 & S^2 \times \mathbb{E} & \chi = 0 & H^2 \times \mathbb{E} \\
\chi < 0 & \mathbb{E}^3 = \mathbb{E}^2 \times \mathbb{E} & e > 0 & S^3 \\
e = 0 & \text{Nil} & e \neq 0 & SL(2, \mathbb{R})
\end{array}
\]

**Example 3.** As an example, \(S^5\) is equipped with a free \(S^1\)-action given by the Hopf bration:
\[ S^5 \xrightarrow{H} \mathbb{CP}^2. \]

Then the blowing up along a regular fibre \(S^1_p\) in \(S^5\) via the the gluing map gives us the \(S^1\)-bundle
\[ S^5\#(S^3 \times S^2) \xrightarrow{\pi} \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \]
or
\[ S^5\#(S^3 \tilde{\times} S^2) \xrightarrow{\pi} \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}. \]

In the case \(N\) is simply-connected, the blowing up along a regular fibre \(S^1_p\) is either \(N\#(S^3 \times S^2)\) or \(N\#(S^3 \tilde{\times} S^2)\), twisted \(S^3\)-bundle over \(S^2\), a non-spin manifold. More precisely, since
\[ \pi_1(\text{Diff}(S^3)) \cong \pi_1(SO(4)) \cong \mathbb{Z}_2, \]
the blowing up along \(S^1_p\) can produce at most two \(S^1\)-manifolds, depending on the gluing map. As an example, \(S^5\) is equipped with a free \(S^1\)-action given by the
negative Hopf fibration. Then blowing up along $S^1_p$ via $f = \text{id}$ gives us the $S^1$-bundle

$$M \#(S^3 \times S^2) \rightarrow S^5$$
$$\downarrow \quad \downarrow -H$$
$$\text{CP}^2 \# \text{CP}^2 \rightarrow \text{CP}^2.$$

On the other hand, if we take the gluing map $f$ to be

$$[\cos t \quad -\sin t \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]$$

then

$$M \#(S^3 \times S^2) \rightarrow S^5$$
$$\downarrow \quad \downarrow -H$$
$$\text{CP}^2 \# \text{CP}^2 \rightarrow \text{CP}^2.$$

3.3. Reid’s Model and Resolution of $\frac{1}{r}(1, a)$-Type Foliation Singularities.
In general, for a foliation singularity of type $\frac{1}{r}(1, a)$, we can reduce to a foliation singularity of type $\frac{1}{r}(1, 1)$ in this subsection. More precisely, given $p \in M_{\text{sing}}$, the singular fiber $S^1_p$ is a foliation singularity of type $\frac{1}{r}(1, a)$ at $p$, one can measure its complexity by the Hirzebruch-Jung continued fraction.

**Definition 9.** Given $p \in M_{\text{sing}}$, the singular fiber $S^1_p$ is foliation singularity of type $\frac{1}{r}(1, a)$. The Hirzebruch Jung continued fraction $[b_1, \ldots, b_l]$ of $\frac{r}{a}$ is defined by

$$\frac{r}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}.$$ 

Then we define the length of the singular fiber to be $l$. Note that the fraction can be calculated by the recursive formula:

$$r = ab_1 - a_1,$$
$$a = a_1b_2 - a_2,$$
$$a_i = a_{i+1}b_{i+2} - a_{i+2},$$

(3.15)

In the paper of [W, section 2], Wang presents a construction that reduce a foliation singularity of type $\frac{1}{r}(1, a)$ to a foliation singularity of type $\frac{1}{a}(1, a_1)$ with $r = ab_1 - a_1$. More precisely, we consider the five manifold $C \times S^3$ equipped with the $S^1$-action

$$\delta_a : S^1 \times (C \times S^3) \rightarrow C \times S^3$$

$$(x, y_1, y_2) \mapsto (t^r x, t^{-a} y_1, t^{-a} y_2),$$

where $(x, y_1, y_2) \in C \times S^3 \subset C^3$. $(C \times S^3, \delta_a)$ has a singular fiber of type $\frac{1}{a}(1, a_1)$ with $a_1 = [b_2, \ldots, b_l]$ and $\frac{1}{a} = [b_1, b_2, \ldots, b_l]$. 

Now, given a Sasakian manifold \((M, \alpha), p \in M_{\text{sing}}\), the singular fiber \(S^1_p\) is of type \(\frac{1}{r}(1, a)\) of length \(l\). Then there exists an \(S^1\)-equivariant neighborhood \(\hat{N}(S^1_p)\) of \(S^1_p\) such that \(\hat{N}\) is \(S^1\)-equivariant to \((N, \nu)\) with \(N \simeq S^1 \times C^2\) and \(\nu\) given by

\[
\begin{align*}
S^1 \times N & \to N \\
(t, u, v_1, v_2) & \mapsto (t^{r}u, t^{-1}v_1, t^{-a}v_2).
\end{align*}
\]

For the \(S^1\)-equivariant embedding

\[
\Psi : (\hat{N}(S^1_p) - S^1_p) \simeq (N - S^1 \times (0, 0)) \to C \times S^3
\]

\[
(u, v_1, v_2) \mapsto (u|v|, \frac{v_1}{|v|}, \frac{v_2}{|v|}),
\]

where \(v = (v_1, v_2)\), we glue the manifold

\[
(M', \alpha') = (M - \hat{N}(S^1_p), \alpha) \cup (C \times S^3, \delta_a).
\]

Then resulting new Sasakian manifold \((M', \alpha')\) has all singular fibers the same as \((M, \alpha)\) except the singular fiber \(S^1_p\) is now replaced with a (singular) fiber of smaller length. Moreover, we have

\[M' \simeq M \# S^3 \times S^2\]

topologically.

Furthermore, it follows from Reid’s model [R] on Kaehler surfaces that the gluing map \(\Psi\) descends to the gluing maps (2.9) and (2.10) of the paper [W, section 2.2]. Then we have the following the minimal resolution of the general \(\frac{1}{r}(1, a)\)-type foliation singularities of \(M\) which is compatible with the minimal resolution in case of Kaehler surfaces.

**Theorem 6.** Let \((M, \xi, g)\) be a compact quasi-regular Sasakian 5-manifold and \(Z\) denote the space of leaves of the characteristic foliation which is a normal projective orbifold surface of \(\frac{1}{r}(1, a)\)-type singularities. Let \(\phi : \overline{Z} \to Z\) be a resolution of singularities of \(Z\) for a nonsigular projective variety \(\overline{Z}\). Then there exist a lifting \(\psi : \overline{M} \to M\) and an \(\pi : \overline{M}^5 \to \overline{Z}\) the regular \(S^1\)-principal bundle over \(\overline{Z}\) such that the following diagram is commutative:

\[
\begin{align*}
\overline{M}^5 & \ni V_i \xrightarrow{\psi} M^5 \ni S^1_p \\
\downarrow \pi & \circ \downarrow \pi \\
\overline{Z} & \ni E_i \xrightarrow{\phi} Z \ni \pi(p),
\end{align*}
\]

where the irreducible transverse exceptional divisors \(V_i \in \text{Exc}(\psi)\) and the irreducible exceptional divisors \(E_i \in \text{Exc}(\phi)\). \(\psi\) is defined to be the resolution of \(\frac{1}{r}(1, a)\)-type foliation singularities from \(\overline{M}\) to \(M\). Moreover, the Hirzebruch Jung continued fraction

\[
\frac{r}{a} = [b_1, \ldots, b_l]
\]

gives the information on the foliation resolution \(\psi : \overline{M} \to M\) of \(M\). The exceptional foliation \((-b_1)\)-curves form a chain of \(\{V_1, \ldots, V_l\}\) such that each \(V_i\) has self
intersection

\[ V_i^2 = -b_i \]

for every \( i = 1, \ldots, l \) and \( V_i \) intersects another foliation curve \( V_j \) transversely only if \( j = i - 1 \) or \( j = i + 1 \). In particular, for a foliation cyclic quotient singularity of type \( \frac{1}{k} (1, 1) \), we have \( \frac{1}{k} = [k] : \) the foliation \((-k)\)-curve.

**Proof.** Note that, for the quasi-regular Sasakian structure on \( M \) with foliation singularities of type \( \frac{1}{k} (1, a) \) in which the well-formed leave (foliation) space \( Z \) has orbifold singularities as same as the algebraic singularities

\[ \pi^*(K_Z) = K^T_M; \quad \overline{\pi}^*(K_Z) = K^T_{M} \]

and

\[ \psi^* \circ \pi^* = \overline{\pi}^* \circ \phi^*. \]

It follows from (3.21) that

\[ \overline{\pi}^*(K_Z) = \overline{\pi}^*(\phi^*(K_Z)) + \sum_i a_i \overline{\pi}^*( [E_i] ), \]

where the sum is over the irreducible exceptional divisors \( E_i \in \text{Exc}(\phi) \subset \overline{Z} \), \( \pi(p) \in Z \) such that

\[ \overline{\pi}(V_i) = E_i \]

with \( S^1_p \subset M, \quad p \in M. \)

**i) Foliation singularity of type \( \frac{1}{k} (1, 1) \):** It is the most simple case. For a resolution of singularities of type \( \frac{1}{k} (1, 1) \) in \( Z \)

\[ \phi : \overline{Z} \to Z \]

with a nonsingular projective surface \( \overline{Z} \) and \( E \) is the exceptional curve of such resolution. Then, over the singular point \( \pi(S^1_p) \), the exceptional \((-k)\)-curve \( E \) has self intersection

\[ E^2 = -k. \]

Then, from the previous section construction for \( S(-k) \) in \( \overline{M}^5 \) as in subsection 3.3, it follows that there exists a lifting \( \psi : \overline{M} \to M \) as in Theorem 5. Thus there is an \( \overline{\pi} : \overline{M} \to \overline{Z} \) the regular \( S^1 \)-principal bundle over \( \overline{Z} \) such that the following diagram is commutative

\[ \begin{array}{ccc}
\overline{M}^5 & \supset & S(-k) \approx C \times S^3 \quad \pi_3 \quad S^3 \approx V \\
\downarrow & & \quad \psi|_V \\
\overline{Z} & \supset & O(-k) \quad \pi_4 \quad \mathbb{C}P^1 \approx \quad \phi|_E \quad \pi_M(p) \in Z.
\end{array} \]

Hence the exceptional foliation \((-k)\)-curve \( V \) has self intersection

\[ V^2 = -k. \]

**ii) Foliation singularity of type \( \frac{1}{k} (1, a) \):** The similar situation as in (i). Let \( \phi : \overline{Z} \to Z \)
be a resolution of singularities of type $\frac{1}{r}(1, a)$ in $Z$ with a nonsingular projective surface $\overline{Z}$ and $E_i$ be the exceptional curves of such resolution. Then it follows from M. Reid ([R]) that the Hirzebruch Jung continued fraction

$$\frac{r}{a} = [b_1, \cdots, b_l]$$

gives the information on the resolution $\phi : \overline{Z} \rightarrow Z$. More precisely, over the singular point $\pi(S^1_p)$, the exceptional curves form a chain of $\{E_1, \cdots, E_l\}$ such that each $E_i$ has self intersection $E_i^2 = -b_i$ for every $i = 1, \ldots, l$ and the $(-b_i)$-curve $E_i$ intersects another $(-b_j)$-curve $E_j$ transversely only if $j = i - 1$ or $j = i + 1$.

Again we consider the resolution of foliation singularities of type $\frac{1}{r}(1, a)$

$$\psi : \overline{M} \rightarrow M$$

in $M$ with a regular Sasakian 5-manifold $\overline{M}$. Let $V_i$ be the exceptional foliation curves of $\psi$ lifting from $E_i$ via the diagram (3.20) such that $\pi(V_i) = E_i$

Then, by the previous construction (3.18) of foliation singularity of type $\frac{1}{r}(1, a)$ over the singular fiber $S^1_p$ as in subsection 3.4., the exceptional foliation curves form a chain of $\{V_1, \cdots, V_l\}$ such that each $V_i$ has self intersection $V_i^2 = -b_i$

for every $i = 1, \ldots, l$ and the foliation $(-b_i)$-curve $V_i$ intersects another foliation $(-b_j)$-curve $V_j$ transversely only if $j = i - 1$ or $j = i + 1$. We refer to (A.2), (A.1) and [CHLW, (6.3)] for some details.

Here we come out some definition on the $\frac{1}{r}(1, a)$-type foliation singularities of $M$:

**Definition 10.** Let $(M, \xi, g)$ be a compact quasi-regular Sasakian 5-manifold and $Z$ denote the space of leaves of the characteristic foliation which is a normal projective orbifold surface of $\frac{1}{r}(1, a)$-type singularities. Let $\psi : \overline{M} \rightarrow M$ be a minimal resolution of $\frac{1}{r}(1, a)$-type foliation singularities from $\overline{M}$ to $M$. If

$$K^T_{\overline{M}} = \psi^*(K^T_M) + \sum a_i[V_i]_B,$$

where the sum is over the irreducible transverse exceptional divisors $V_i \in \text{Exc}(\psi)$ over $S^1_p \subset M$ and the $a_i$ are rational numbers, called the discrepancies. Then the $\frac{1}{r}(1, a)$-type foliation singularities of $M$ are called:

1. terminal if $a_i > 0$ for all $i$.
2. canonical if $a_i \geq 0$ for all $i$.
3. log terminal if $a_i > -1$ for all $i$.
4. log canonical if $a_i \geq -1$ for all $i$. 
4. The Sasaki-Ricci Flow Through Singularities

In this section, we warm up some basic facts of the Sasaki analogue of the Kaehler-Ricci flow through singularities due to Song-Tian ([ST]).

4.1. The Sasaki-Ricci Flow. By a $\partial_B \bar{\nabla}_B$-Lemma ([EKA]) in the basic Hodge decomposition, there is a basic function $F : M \to \mathbb{R}$ such that $\rho^T(x, t) = \kappa d\eta(x, t) = d_B \bar{\partial}_B F$. 

We focus on finding a new $\eta$-Einstein Sasakian structure $(M, \xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ with

$$\tilde{\eta} = \eta + d_B^c \varphi, \varphi \in \Omega_B^0$$

and

$$\tilde{g}^T = (g^T + \varphi \tilde{\gamma})dz^i d\bar{z}^j = 2i(h_{ij} + \frac{1}{2} \varphi_{ij})dz^i \wedge d\bar{z}^j$$

such that

$$\tilde{\rho}^T = \kappa d\tilde{\eta}.$$ 

Hence

$$\tilde{\rho}^T - \rho^T = \kappa d_B^c \varphi - d_B^c \bar{\partial}_B F$$

and it follows that

$$\frac{\det(g^T + \varphi \tilde{\gamma})}{\det(g^T)} = e^{-\kappa \varphi + F}.$$ 

This is a Sasakian analogue of the Monge-Ampere equation for the orbifold version of Calabi-Yau Theorem (([EKA])).

Now we consider the Sasaki-Ricci flow on $M \times [0, T)$

$$(4.2) \frac{d}{dt} g^T(x, t) = -(\text{Ric}^T(x, t) - \kappa g^T(x, t))$$

or

$$\frac{d}{dt} d\eta(x, t) = -(\rho^T(x, t) - \kappa d\eta(x, t)).$$

It is equivalent to consider

$$(4.3) \frac{d}{dt} \varphi = \log \det(g^T_{\alpha \overline{\alpha}} + \varphi_{\alpha \overline{\alpha}}) - \log \det(g^T_{\alpha \overline{\alpha}}) + \kappa \varphi - F.$$ 

Note that, for any two Sasakian structures with the fixed Reeb vector field $\xi$, we have

$$Vol(M, g) = Vol(M, g')$$

and

$$\tilde{\omega}^n \wedge \eta = i^n \det(g^T_{\alpha \overline{\alpha}} + \varphi_{\alpha \overline{\alpha}})dz^1 \wedge d\bar{z}^1 \wedge ... \wedge dz^n \wedge d\bar{z}^n \wedge dx.$$ 

With all the above discussions, let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian manifold of dimension $2n + 1$ and $Z$ denote the space of leaves of the characteristic foliation $\mathcal{F}_\xi$. There is an orbifold Riemannian submersion, and a principal $S^1$-orbibundle (V-bundle) $\pi : (M, g, \omega) \to (Z, h, \omega_h)$ with $\omega = \pi^*(\omega_h)$. Now for the natural projection

$$(4.4) \Pi : (C(M), \bar{g}, J, \bar{\omega}) \to (Z, h, \omega_h)$$
with $\Pi|_{(M,g,\omega)} = \pi$, then we have the volume form of the Kähler cone metric on the cone $C(M)$:

$$\overline{\omega}^{n+1} = r^{2n+1}(\Pi^*\omega_h)^n \wedge dr \wedge \eta$$

and the volume form of the Sasaki metric on $M$:

$$(4.5) \quad i_{\overline{\omega}} \overline{\omega}^{n+1} = (\Pi^*\omega_h)^n \wedge \eta.$$  

4.2. Basic Cohomology Characterization of the Maximal-Time Solution. Let $(M,\xi_0,\eta_0,\Phi_0,g_0,\omega_0)$ be a compact quasi-regular Sasakian $(2n+1)$-manifold and its leave space $Z$ of the characteristic foliation be well-formed. We consider a solution $\omega = \omega(t)$ of the Sasaki-Ricci flow

$$(4.6) \quad \frac{\partial}{\partial t} \omega(t) = -\text{Ric}^T(\omega(t)), \quad \omega(0) = \omega_0.$$  

As long as the solution exists, the cohomology class $[\omega(t)]_B$ evolves by

$$\frac{\partial}{\partial t} [\omega(t)]_B = -c^B_1(M), \quad [\omega(0)]_B = [\omega_0]_B,$$

and solving this ordinary differential equation gives

$$[\omega(t)]_B = [\omega_0]_B - tc^B_1(M).$$

We see that a necessary condition for the Sasaki-Ricci flow to exist for $t > 0$ such that

$$[\omega_0]_B - tc^B_1(M) > 0.$$  

This necessary condition is in fact sufficient. In fact we define

$$T_0 := \sup\{t > 0 | [\omega_0]_B - tc^B_1(M) > 0\}.$$  

That is to say that

$$(4.7) \quad [\omega_0]_B - T_0c^B_1(M) \in \overline{C^B_M}$$

which is a nef class.

For a representative $\chi \in -C^B_1(M)$, we can fix the adapted measure $\Omega$ on the leave space $Z$ and then a volume form $\Omega \wedge \eta_0$ on $(M,\xi_0,\eta_0,\Phi_0,g_0,\omega_0)$ such that

$$(4.8) \quad \Omega \wedge \eta_0 = (\sqrt{-1})^n F(z_1, \ldots, z_n) dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n \wedge dx$$

with

$$\sqrt{-1} \partial_B \overline{\partial}_B \log F = -\text{Ric}^T(\Omega) = \chi$$

and

$$\int_M \Omega \wedge \eta_0 = \int_M \omega_0^n \wedge \eta_0.$$  

We choose a reference (transverse) Kähler metric

$$\tilde{\omega}_t := \omega_0 + t\chi.$$
Then the corresponding transverse parabolic Monge-Ampere equation for the basic function \( \varphi(x, t) \) to (4.6) on \( M \times [0, T_0) \) is

\[
\begin{align*}
\partial_t \varphi(x, t) &= \log \left( \frac{(\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \right), \\
\hat{\omega}_t &= \omega_0 + t \chi, \\
\sqrt{-1} \partial \bar{\partial} \log \Omega &= \chi, \\
\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi &> 0, \\
\varphi(0) &= 0.
\end{align*}
\]

(4.9)

Based on [SW1], [T], [CHLW] and references therein as in the Kaehler case, we have the following cohomological characterization for the maximal solution of the Sasaki-Ricci flow:

**Theorem 7.** There exists a unique maximal solution \( \omega(t) \) of the Sasaki-Ricci flow (4.6) on \( M \times [0, T_0) \) for \( t \in [0, T_0) \).

### 4.3. The Weak Sasaki–Ricci Flow

Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian 5-manifold with finite cyclic quotient foliation singularities of type \( \frac{1}{r}(1, a) \). Such singularities are rather mild and they do not become worse after divisorial contractions are performed in the foliation minimal model program we considered. More precisely, \( \overline{M} \rightarrow M \) be a resolution of foliation singularity, then the pullback of any volume measure \( \Omega \wedge \eta \) on \( M \) is \( L^p \)-integrable on the nonsingular model \( \overline{M} \) for some \( p > 1 \).

**Definition 11.** Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian 5-manifold with finite cyclic quotient foliation singularities of type \( \frac{1}{r}(1, a) \) and \( H^T \) be a big and semi-ample \( Q \)-divisor with a basic transverse birational morphism \( \Psi_{mH^T} : M \rightarrow \mathbb{C}P^{N_m} \) induced by the linear system \( |mH^T| \) for some \( m >> 0 \). Let \( \overline{\varphi} = \frac{1}{m} \Psi^*(\omega_{FS}) \in [H^T] \) and an volume measure \( \Omega \wedge \eta \) on \( M \), where \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{C}P^{N_m} \) and define for \( p \in (1, \infty] \), \( \varphi \) is the basic function

\[
\mathcal{PSH}_p(M, \overline{\varphi}, \Omega \wedge \eta) = \{ \varphi \in \mathcal{PSH}(M, \overline{\varphi}) \cap L^\infty | (\overline{\varphi} + i \partial \bar{\partial} \varphi)^n \wedge \Omega \wedge \eta \}
\]

and

\[
K_{H^T_{0, p}}(M) = \{ \overline{\varphi} + i \partial \bar{\partial} \varphi | \varphi \in \mathcal{PSH}_p(M, \overline{\varphi}, \Omega \wedge \eta) \}.
\]

Now we can define the weak Sasaki-Ricci flow on a compact quasi-regular Sasakian 5-manifold for which its leave space \( Z \) is normal projective surface with mild singularities.

**Definition 12.** Let \((M, \xi, \eta_0)\) be a compact quasi-regular Sasakian 5-manifold and its leave space \( Z \) be normal projective variety with finite cyclic quotient singularities of type \( \frac{1}{r}(1, a) \) which are klt singularities and \( H^T \) be a big and semi-ample \( Q \)-divisor so that \( H^T + tK^T_M \) is ample for a small \( t \in Q^+ \) such that

\[
T_0 = \sup \{ t > 0 \mid H^T + tK^T_M \text{ nef} \} > 0.
\]

A family of closed basic semi-positive \((1, 1)\)-currents \( \omega(t, \cdot) \) on \( M \) for \( t \in [0, T_0) \) are said to be a solution of the weak Sasaki–Ricci flow starting \( \omega_0 \in K_{H^T, p}(M) \) for some \( p > 1 \) if the following conditions hold

\[
\frac{\partial}{\partial t} \omega(x, t) = (\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n \wedge \Omega_0 \wedge \eta_0,
\]

(4.10)
(1) $\omega \in C^\infty((0, T_0) \times M_{reg})$ and $\varphi \in L^\infty([0, T] \times M)$,
(2) $\omega(t, \cdot)$ satisfies

\[
\begin{cases}
\frac{\partial}{\partial t}\omega = -Ric^T(\omega) & \text{on } (0, T_0) \times M_{reg}, \\
\omega(0) = \omega_0 & \text{on } M.
\end{cases}
\]

Let $(M, \xi, \eta_0)$ be a compact quasi-regular Sasakian 5-manifold and the space of leaves $Z$ be normal projective surface with klt singularities and $H^T$ be a big and semi-ample $Q$-divisor. There exists a basic transverse holomorphic map $\Psi : M \to (\mathbb{CP}^N, \omega_{FS})$ defined by the basic transverse holomorphic section $\{s_0, s_1, \ldots, s_N\}$ of $H^0(M, (K_M^T)^m)$ which is $S^1$-equivariant with respect to the weighted $C^*$ action in $\mathbb{C}^{N+1}$ with $N = \dim H^0(M, (K_M^T)^m) - 1$ for a large positive integer $m$ and $\hat{\omega}_\infty = \frac{1}{m}\Psi^*(\omega_{FS}) \in [H^T]$ and an adapted measure $\Omega_Z$ on $Z$ as in [ST, Theorem 4.3] with $\Omega = \pi^*(\Omega_Z)$, where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{CP}^{N_m}$. We also denote

$\chi = \sqrt{-1}\partial B\bar{\partial}B \log \Omega \in -c_1^B(M)$

and

$\hat{\omega}_t = \hat{\omega}_\infty + t\chi.$

The Sasaki-Ricci flow is equivalent to the following Monge-Ampere flow for the basic function $\varphi(x, t)$:

\[
\begin{cases}
\frac{\partial}{\partial t}\varphi(x, t) = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial B\bar{\partial}B \varphi)^n}{\Omega^\wedge \eta_0}, \\
\hat{\omega}_t + \sqrt{-1}\partial B\bar{\partial}B \varphi > 0, \\
\varphi(0) = \varphi_0.
\end{cases}
\]

(4.11)

In order to define the Monge-Ampere flow (4.11) on $M$, one might want to lift the equation to the regular Sasakian manifold $\overline{M}$ and its foliation space $\overline{Z}$ is a smooth projective surfaces. However, $\hat{\omega}_Z$ is not Kaehler on $\overline{Z}$ and $\Omega_Z$ in general vanishes or blow up along $E_i \in Exc(f)$, so $\hat{\omega}$ is not transverse Kaehler on $\overline{M}$ and $\Omega$ in general vanishes or blow up along $V_i \in Exc(f)$. Then the lifted equation is degenerate near the exceptional locus $Exc(f)$. We have to perturb the Monge–Ampere equation (4.11) to (4.12) and obtain uniform estimates so that the equation descend to $M$.

**Theorem 8.** Let $(M, \xi, \eta_0)$ be a compact quasi-regular Sasakian 5-manifold and its leave space $Z$ be normal projective variety with finite cyclic quotient singularities of type $\frac{1}{p}(1, a)$ which are klt singularities and $\varphi_0 \in \mathcal{PSH}_p(M, \hat{\omega}, \Omega)$ for some $p > 1$. Define the minimal resolution of foliation singularities of $M$ to be $\overline{f} : \overline{M} \to M$. Then the Monge-Ampere flow on $\overline{M}$ defined by

\[
\begin{cases}
\frac{\partial}{\partial t}\overline{\varphi}(x, t) = \log \frac{\overline{f}^* \overline{\omega}_t + \sqrt{-1}\partial B\bar{\partial}B \overline{\varphi}}{\overline{f}^* \Omega^\wedge \eta_0}, \\
\overline{f}^* \overline{\omega}_t + \sqrt{-1}\partial B\bar{\partial}B \overline{\varphi} > 0, \\
\overline{\varphi}(0) = \overline{f}^* \varphi_0.
\end{cases}
\]

(4.12)

has a unique solution

\[
\overline{\varphi} \in C^\infty((0, T_0) \times \overline{M}\setminus V) \cap C^0([0, T_0) \times \overline{M}\setminus V)
\]

such that

\[
\overline{\varphi} \in L^\infty(\overline{M}) \cap \mathcal{PSH}(\overline{M}, \overline{f}^* \omega_t),
\]
for all $t \in [0, T_0)$. Moreover, $\varphi$ is constant along each fibre of $\tilde{f}$, and so $\varphi$ descends to a unique solution 

$$\varphi \in C^{\infty}((0, T_0) \times M_{\text{reg}}) \cap C^0([0, T_0) \times M_{\text{reg}})$$

of the Monge–Ampere flow (4.11) such that 

$$\varphi \in C^0(M) \cap \mathcal{P}S\mathcal{H}(M, \omega_t)$$

for each $t \in [0, T_0)$.

**Proof.** As notions in Proposition 1, for a transverse Kaehler metric $\omega_Z$ on the foliation space $Z = M/\mathcal{F}_\xi$ such that

(i) $\omega_Z \in C^{\infty}((0, T_0) \times Z_{\text{reg}})$ and $\varphi \in L^\infty([0, T] \times Z)$,

(ii) $\omega_Z(t, \cdot)$ satisfies

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \omega_Z = -\text{Ric}(\omega_Z) \quad \text{on } (0, T_0) \times Z_{\text{reg}}, \\
\pi^*(\omega_Z(0)) = \omega_0 \quad \text{on } Z.
\end{array} \right.$$ 

(4.13)

It follows from Proposition 1 that

$$\pi : (M, \xi, \eta, g) \to (Z, h, \omega_h)$$

is an orbifold Riemannian submersion and a principal $S^1$-orbibundle ($V$-bundle) over $Z$ with

$$\frac{1}{2}d\eta = \pi^*(\omega).$$

Then the basic transverse Kaehler form

$$\omega = \pi^*(\omega_Z)$$

satisfies the flow (4.10).

Let $\tilde{f} : \overline{Z} \to Z$ be a minimal resolution of singularities of $Z$ for a nonsingular projective variety $\overline{Z}$ and $\tilde{\pi} : \overline{M} \to \overline{Z}$ is the regular $S^1$-principal bundle over $\overline{Z}$ and regular Sasakian manifold $\overline{M}$. The minimal resolution of foliation singularities of $M$ to be $\tilde{f} : \overline{M} \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\overline{M} & \supset & V_i \\
\downarrow \tilde{\pi} & \circlearrowleft & \downarrow \pi \\
\overline{Z} & \supset & E_i \\
\end{array} \xrightarrow{\tilde{f}} \begin{array}{ccc}
M & \supset & S^1_p \\
\downarrow \pi & \circlearrowleft & \downarrow \pi \\
Z & \supset & \pi(p).
\end{array}$$

(4.14)

Then we have

$$K^T_M = \tilde{f}^* (K^T_M) + \sum_i a_i V_i,$$

where $V_i \in \text{Exc}(\tilde{f}) \subset \overline{M}$ over $S^1_p \subset M$ and $a_i > -1$ for all $i$.

As in [ST, Theorem 4.2], they lifted (4.13) to the minimal resolution $\overline{Z}$ of singularities of $Z$ to have the desired estimates. Here we have to lift the Monge–Ampere equation (4.11) to (4.12). Thus we obtain the estimates via the diagram (4.14). □

As a consequence, we have the following existence theorem for the solution of the weak Sasaki-Ricci flow (4.10).
Corollary 3. ([ST Theorem 4.3]) Let \((M, \xi, \eta_0)\) be a compact quasi-regular Sasakian 5-manifold and its leave space \(Z\) be normal projective surface with finite cyclic quotient orbifold singularities of type \(\frac{1}{r}(1, a)\) which are klt singularities. Let \(H_T\) be an ample \(\mathbb{Q}\)-divisor such that

\[
T_0 = \sup \{ t > 0 \mid H_T + tK^T_M \text{ nef} \} > 0.
\]

If \(\omega_0 \in K^p_{H_T}(M)\) for some \(p > 1\), then there exists a unique solution \(\omega(t, \cdot)\) of the weak Sasaki–Ricci flow (4.10) for \(t \in [0, T_0)\). Moreover, \(\omega(t, \cdot)\) is a smooth transverse orbifold Kaehler-metric on \(M\) for \(t \in (0, T_0)\) and so the weak Sasaki-Ricci flow becomes the smooth orbifold Sasaki-Ricci flow on \(M\) immediately when \(t > 0\).

4.4. The Sasaki-Ricci Flow Through Divisorial Contractions. Now based on the proofs of Theorem 3, Corollary 3 and [ST Theorem 5.3], the weak Sasaki-Ricci flow (4.10) can be continued through divisorial contractions on a compact quasi-regular Sasakian 5-manifold \(M\).

Proposition 3. Let \((M, \xi, \eta_0)\) be a compact quasi-regular Sasakian 5-manifold and its leave space \(Z\) be normal projective surface with finite cyclic quotient orbifold singularities of type \(\frac{1}{r}(1, a)\) which are klt singularities. Let \(H_T\) be an ample \(\mathbb{Q}\)-divisor such that

\[
T_0 = \sup \{ t > 0 \mid H_T + tK^T_M \text{ nef} \} > 0.
\]

is the first singular time and \(L^T_{T_0} := H_T + tK^T_M\) is the semi-ample basic divisor which induces a divisorial contraction

\[
\psi_{(L^T_{T_0})^m} : M \to N \subset \mathcal{P}(H^0_B(M, (L^T_{T_0})^m)),
\]

for some \(m >> 1\). If \(\omega_0 \in K^p_{H_T}(M)\) for some \(p > 1\), Let \(\omega(t, \cdot)\) be the unique solution \(\omega(t, \cdot)\) of the weak Sasaki–Ricci flow (4.10) for \(t \in [0, T_0)\) if \(\omega_0 \in K^p_{H_T}(M)\) for some \(p > 1\), starting with \(\omega_0 \in K^p_{H_T}(M)\) for some \(p > 1\). Then there exists \(\omega_N(t, \cdot)\) such that

1. \(\omega_N \in K^p_{L^T_{T_0}^p}(N) \cap C^\infty(N_{\text{reg}}\setminus \psi(\text{Exc}(\psi)))\) for some \(p' > 1\).
2. \(\omega(t, \cdot) \to \psi^*\omega_N\) in \(C^\infty(M_{\text{reg}}\setminus \text{Exc}(\psi))\)-topology as \(t \to T_0^-\).
3. There exists a unique solution \(\omega(t, \cdot)\) of the weak Sasaki-Ricci flow on \(N\) starting with \(\omega_N\) at \(T_0\) for \(t \in (T_0, T_N)\) with \(T_0 < T_N \leq \infty\), such that

\[
\omega(t, \cdot) \to \omega_N\text{ in }C^\infty(M_{\text{reg}}\setminus \text{Exc}(\psi))\text{-topology as }t \to T_0^+.
\]

Therefore the weak Sasaki-Ricci flow can be uniquely continued on \(N\) starting with \(\omega_N\) at \(T_0\).

In particular, we have Definition 2 for the floating foliation canonical surgical contraction on a compact quasi-regular Sasakian 5-manifold.
5. Foliation MMP with Scaling via the Sasaki-Ricci Flow

Since the Reeb vector field and the transverse holomorphic structure are both invariant, all the quantities in this section are only involved with the transverse Kähler structure $\omega(t)$ and basic tensors. Hence, under the Sasaki-Ricci flow, when one applies for instance, the maximal principle and foliation resolutions etc, the expressions involved behave essentially the same as the Kähler-Ricci flow.

5.1. Canonical Surgical Contraction for Floating Foliation $(-1)$-Curves.

In this subsection, primarily along the lines of the arguments in [SW2], we are ready to give the proof of Theorem 3. We write the Sasaki-Ricci flow (4.6) as a transverse parabolic complex Monge-Ampère equation. First, using the assumption (4.2), define a family of reference transverse Kähler metrics $\hat{\omega}_t$ for $t \in [0,T_0)$ by

$$\hat{\omega}_t = \frac{1}{T_0}((T_0 - t)\omega_0 - t\psi^*\omega_N) \in [\omega(t)]_B = [\omega_0]_B - tc^B_1(M).$$

We can fix the adapted measure $\Omega$ on the leave space $Z$ and then a volume form $\Omega \wedge \eta_0$ on $M$ such that

$$\sqrt{-1}\partial_B\bar{\partial}_B \log \Omega = \frac{\partial}{\partial t} \hat{\omega}_t = -\frac{1}{T_0}(\omega_0 + \psi^*\omega_N) \in c^B_1(M), \int_M \Omega \wedge \eta_0 = 1.$$  

Then the corresponding transverse parabolic Monge-Ampère equation for the basic function $\varphi = \varphi(t)$ to (4.9) on $M \times [0,T_0)$ is

$$\frac{\partial}{\partial t} \varphi = \log \frac{\hat{\omega}_t + \sqrt{-1}\partial_B\bar{\partial}_B \varphi^2 \wedge \eta_0}{\Omega \wedge \eta_0}, \quad \varphi(0) = 0.$$  

5.1.1. Key Estimates for the Sasaki-Ricci Flow. In the following, by using the Sasaki analogue of Koaira Lemma (CHLW, Proposition 3.3) under assumption (1.2) and (A.3), we prove the main estimates for the Sasaki-Ricci flow under the assumptions of Theorem 3. In particular, the estimates of the first two lemmas are essentially contained in [CHLW].

**Lemma 2.** There is a uniform constant $C$ depending only on $(M,\omega_0)$ such that the solution $\varphi = \varphi(t)$ of (5.3) satisfies, for $t \in [0,T_0)$,

$$\|\varphi\|_{L^\infty} \leq C, \quad \varphi' = \partial \varphi / \partial t \leq C \text{ and } \omega^2 \wedge \eta_0 \leq C \Omega \wedge \eta_0.$$  

As $t \to T_0^-$, $\varphi(t)$ converges pointwise on $M$ to a bounded basic function $\varphi_T$ satisfying

$$\omega_{T_0} := \hat{\omega}_0 + \sqrt{-1}\partial_B\bar{\partial}_B \varphi_{T_0} \geq 0,$$

and $\omega(t)$ converges weakly in the sense of currents to the closed positive $(1,1)$ current $\omega_{T_0}$.

We have the $C^\infty$ estimates for the solution $\omega(t)$ of the Sasaki-Ricci flow on compact subsets of $M \setminus V$ as following.

**Lemma 3.** With the assumptions of Theorem 3, the solution $\omega = \omega(t)$ of the Sasaki-Ricci flow (4.6) satisfies

(i) There exists a uniform constant $c > 0$ such that

$$\omega \geq c \psi^*\omega_N.$$  

For every compact subset $K \subset M \setminus V$, there exist constants $C_{K,k}$ for $k = 0, 1, 2, ...$, such that
\begin{equation}
\|\omega(t)\|_{C^k(K,\omega_0)} \leq C_{K,k}. \tag{5.5}
\end{equation}
(iii) The closed $(1,1)$ current $\omega_{T_0}$, given in Lemma 2, is a smooth transverse Kähler metric on $M \setminus V$.
(iv) As $t \to T_0^-$, the metrics $\omega(t)$ converge to $\omega_{T_0}$ in $C^\infty$ on compact subsets of $M \setminus V$.

Then it follows from Lemma 3 that part (1) in the Definition 2 of foliation canonical surgical contraction holds.

We now compare $\psi^*\omega_N$ with a fixed transverse Kähler metric $\omega_0$ on $M$ by using the description of the neighborhood of $V$ in the blow-up map
$$\psi: M \to N.$$ 
As the notion in section 2, for $p = (x_i, v_i, s_i) \in U_i \subset \mathbb{C} \times \mathbb{C} \times S^1$, $i = 1, 2$, the composition
$$S(-1) \cong \mathbb{C} \times S^3 \xrightarrow{\pi_3} S^3 \xrightarrow{H} \mathbb{C}P^1$$
send $p$ to $[l_1, l_2]$, we have $x_1 = \frac{l_1}{l_2}$; $x_2 = \frac{l_2}{l_1}$, and $v_i$, $i = 1, 2$ correspond to vectors in a fiber of $\mathcal{O}(-1)$. Now $s$ is the section of $[V]$ over $S(-1)$ given by
$$s_i : U_i \to \mathbb{C}, s_i = v_i$$
and $h$ is the Hermitian metric on $[V]$ such that
$$h_i = (\frac{|l_1|^2 + |l_2|^2}{|l_i|^2}) = (1 + |x_i|^2)$$
on $U_i$. Then we have
\begin{equation}
\text{dist}^2(p, S^3_0) = \left(\frac{|l_1|^2 + |l_2|^2}{|l_i|^2}\right)|v_i|^2 := |s|^2_h \tag{5.6}
\end{equation}
On the other hand, the distance square to the singular fiber $S^3_0$ is
\begin{equation}
\text{dist}^2(p, S^3_0) = |z_1|^2 + |z_2|^2 := r^2_{S^1}. \tag{5.7}
\end{equation}
If $\psi: \mathbb{C} \times S^3 \to S^1 \times \mathbb{C}^2$ is the composition of
$$S(-k) \cong \mathbb{C} \times S^3 \xrightarrow{\pi_3} S^3 \to S^1 \times \mathbb{C}^2 \supset S^1 \times D^4,$$
it follows from (5.6) and (5.7) that the function $|s|^2_h$ on $\psi^{-1}(S^1 \times D^4_{1/2})$ is given by
\begin{equation}
\psi^*r^2_{S^1} = |s|^2_h, \tag{5.8}
\end{equation}
and then write
$$|s|^2_h(x) = r^2_{S^1} = |z_1|^2 + |z_2|^2$$
for $\psi(x) = (s, z_1, z_2)$, $s \in S^1$. Hence the transverse Ricci curvature of $h$ is given by
$$R^{	au}_{h} = -\frac{1}{2\pi} \partial B \overline{\partial B} \log(|z_1|^2 + |z_2|^2)$$
on $\psi^{-1}(S^1 \times D^4_{1/2} \setminus \{0\})$.

We have the following lemmas.
Lemma 4. For sufficiently small $\varepsilon_0 > 0$,

$$\omega_M := \psi^*\omega_N - \varepsilon_0 R^T_h$$

is a transverse Kähler form on $M$. Furthermore, in term of Sasaki normal coordinate (2.7)

$$\omega_M := \psi^*\omega_N + \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^2 (\delta_{ij} - \frac{\bar{\omega}_j}{s_i}) dz_i \wedge d\bar{z}_j$$

on $\psi^{-1}(S^1 \times D^4_{1/2} \setminus \{0\})$.

Lemma 5. There exist positive constants $C_1, C_2$ such that

$$\psi^*\omega_N \leq \omega_M \leq \frac{C_1}{|s|_h^2} \psi^*\omega_N$$

and

$$\frac{C_2}{|s|_h^2} \psi^*\omega_N \leq \omega_0 \leq \frac{C_1}{|s|_h^2} \psi^*\omega_N.$$

Proof. We observe that $\omega_M$ and $\omega_0$ are uniformly equivalent transverse Kähler metrics on $M$. Hence (5.11) follows easily from (5.10). Thus it suffices for us to prove (5.10) in $\psi^{-1}(S^1 \times D^4_{1/2} \setminus \{0\})$. By using (5.2), the first inequality of (5.10) follows from the fact that if $X^i$ is any $T^{1,0}$ vector then by the Cauchy-Schwarz inequality,

$$\sum_{i,j=1}^2 \frac{\bar{z}_j z_j}{s_i} X^i \overline{X^j} = \left( \sum_{i=1}^2 \frac{\bar{z}_i}{s_i} X^i \right) \left( \sum_{j=1}^2 \frac{z_j}{s_j} \overline{X^j} \right) \leq |X|^2 = \sum_{i,j=1}^2 \delta_{ij} X^i \overline{X^j}.$$

The second inequality of (5.10) follows from the fact that $\bar{z}_i z_j$ is semi-positive definite.

Lemma 6. There exists $\delta > 0$ and a uniform constant $C$ such that for $\omega = \omega(t)$ a solution of the Sasaki-Ricci flow (4.6) satisfies

$$\omega \leq \frac{C}{|s|_h^2} \psi^*\omega_N \text{ and } \omega \leq \frac{C}{|s|_h^2(1-\delta)} \omega_0.$$

Proof. From Lemma 5 we have

$$|s|_h^2 tr_{\psi^*\omega_N} \omega \leq C tr_{\omega_0} \omega.$$ 

We fix $0 < \varepsilon \leq 1$ and will apply the maximum principle to the quantity

$$Q_\varepsilon = \log tr_{\omega_0} \omega + A \log(|s|_h^{2+2\varepsilon} tr_{\psi^*\omega_N} \omega) - A^2 \varphi,$$

on $M \setminus V$ where $A$ is a constant to be determined later. It follows that at any fixed time $t$, $Q_\varepsilon(x,t)$ tends to negative infinity as $x \in M$ tends to $V$. Suppose there exists $(x_0,t_0) \in M \setminus V \times (0,T_0)$ with $\sup_{M \setminus V \times (0,T_0)} Q_\varepsilon = Q_\varepsilon(x_0,t_0)$. For any fixed transverse Kähler metric $\bar{\omega}$ on $M$, if $\omega = \omega(t)$ solves the Sasaki-Ricci flow, we have the following estimate like the Kähler-Ricci flow

$$\left( \frac{\partial}{\partial t} - \Delta_{\bar{B}} \right) \log tr_{\bar{\omega}} \omega$$

$$\leq \frac{1}{tr_{\bar{\omega}} \omega} \left( g_{TT}^T \bar{R}^T_i \bar{T}^T_{kT} + g_{T\bar{T}} \bar{\bar{\varphi}} T_{kT}^T \bar{T}^T_{j\bar{T}} g_{kT}^T \nabla_T g_{jT}^T - \frac{|\nabla_T tr_{\bar{\omega}} \omega|^2}{tr_{\bar{\omega}} \omega} \right)$$

$$
\leq \tilde{C} tr_{\bar{\omega}} \omega,
$$
for $\tilde{C}$ depending only on the lower bound of the transverse bisectional curvature of $g^T$. At $(x_0, t_0)$ we have
\begin{equation}
0 \leq \left( \frac{\partial}{\partial t} - \Delta_B \right) Q_\varepsilon \leq C_1 \text{tr}_\omega \omega_0 - \text{Attr}_\omega (A\tilde{\omega}_{t_0} - (1 + \varepsilon)R_h^T - C_2 \psi^* \omega_N) - A^2 \log \frac{\omega^2 \wedge \eta_0}{\omega_0^2 \wedge \eta_0} + 16A,
\end{equation}
for some uniform constants $C_1$, $C_2$. Here we apply (5.13) with $\tilde{\omega} = \omega_0$ and then $\tilde{\omega} = \psi^* \omega_N$ at the point $(x_0, t_0)$. From Lemma 1 and the definition of $\tilde{\omega}_t$ we can choose $A$ sufficiently large and independent of $\varepsilon$, $t_0$ so that
\[ A(A\tilde{\omega}_{t_0} - (1 + \varepsilon)R_h^T - C_2 \psi^* \omega_N) \geq (C_1 + 1)\omega_0. \]
Then we have
\[ (A^2 \log \frac{\omega^2 \wedge \eta_0}{\omega_0^2 \wedge \eta_0} + \text{tr}_\omega \omega_0)(x_0, t_0) \leq 16A, \]
which implies that
\[ (\text{tr}_\omega \omega_0)(x_0, t_0) \leq C. \]
Since the volume form $\omega \wedge \eta_0$ is uniformly bounded from above by Lemma 2
\[ (\text{tr}_\omega \omega_0)(x_0, t_0) \leq (\text{tr}_\omega \omega_0)(x_0, t_0) \left( \frac{\omega^2 \wedge \eta_0}{\omega_0^2 \wedge \eta_0} \right)(x_0, t_0) \leq C. \]
From (5.12), we obtain
\[ (|s|^2 \text{tr}_{\psi^* \omega_N} \omega)(x_0, t_0) \leq C. \]
Since $\varphi$ is uniformly bounded, we can get
\[ Q_\varepsilon \leq C, \]
for $C$ independent of $\varepsilon$. Letting $\varepsilon \to 0$ and applying (5.12) again we get
\[ \omega \leq \frac{C}{|s|^4_h} \psi^* \omega_N. \]
Using the inequality $\text{tr}_\omega \omega \leq C \text{tr}_{\psi^* \omega_N} \omega$ in Lemma 3, we also have
\[ \log (|s|^2 \text{tr}_\omega \omega)^{A+1} \leq C, \]
this yields that $\text{tr}_\omega \omega \leq \frac{C}{|s|^4_h}$ for $\delta = \frac{1}{A+1} > 0$, and also $\omega \leq \frac{C}{|s|^4_h} \psi^* \omega_N$.

For $(s, z_1, z_2) \in S^1 \times D^4$, we consider the transverse holomorphic vector field
\[ \sum_{i=1}^2 z_i \frac{\partial}{\partial z_i}, \]
on $S^1 \times D^4$, which defines a transverse holomorphic vector field $X$ on $\psi^{-1}(S^1 \times D^4) \subset M$ via $\psi$ and we extend $X$ to be a smooth $T^{1,0}$ vector field on $M$. We then have the following lemma.

**Lemma 7.** For $\omega = \omega(t)$ be a solution of the Sasaki-Ricci flow, we have the estimate
\begin{equation}
|X|^2_\omega \leq C|s|_h.
\end{equation}
for a uniform constant $C$. Locally, in $S^1 \times D^4 \setminus \{0\}$ we have
\begin{equation}
|W|^2_{g^T} \leq \frac{C}{|s|^4_h},
\end{equation}
for $W = \sum_{i=1}^2 \frac{1}{r_{s_1}} (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i})$ the unit transverse radial vector field with respect to $g_{Eucl}^T$, where $z_i = x_i + \sqrt{-1} y_i$. 
Proof. By the expression (5.9) of $\omega_M$, we have

$$|X|_{\omega_M}^2 = |X|_{\omega_N}^2$$

in $S^1 \times D_1^4$. It follows that $|X|_{\omega_N}^2$ is uniformly equivalent to $|s|^2 = r_{S^1}^2$ in $S^1 \times D_1^4$. Hence there exists a positive constant $C(t)$ depending on $t$ such that

$$\frac{1}{C(t)}|s|^2 \leq |X|_{\omega_N}^2 \leq C(t)|s|^2. \tag{5.17}$$

Now define a transverse Kähler metric $\tilde{\omega}_N$ on $N$ by $\tilde{\omega}_N = \omega_{\text{Eucl}}$ on $S^1 \times D_1^4$, and extending in an arbitrary way to be a smooth transverse Kähler metric on $N$. For small $\varepsilon > 0$, we apply the maximum principle to the quantity

$$Q_\varepsilon = \log(|X|_{\tilde{\omega}_N}^{2+2\varepsilon} \tr_{\tilde{\omega}_N} \omega) - t.$$ 

For any fixed $t$, using (5.12), (5.17) and $\tilde{\omega}_N$ is uniformly equivalent to $\omega_N$, we know that $(|X|_{\tilde{\omega}_N}^{2+2\varepsilon} \tr_{\tilde{\omega}_N} \omega)(x, t)$ tends to zero as $x$ tends to $V$ and thus $Q_\varepsilon(x, t)$ tends to negative infinity. Since $Q_\varepsilon$ is uniformly bounded from above on the complement of $S^1 \times D_1^4 \{0\}$, thus $Q_\varepsilon$ may attain its maximum at a point in $S^1 \times D_1^4 \{0\}$. So we assume that at some point $(x_0, t_0) \in S^1 \times D_1^4 \{0\} \times (0, T)$ with $\sup_{(M \setminus V) \times (0, t_0)} Q_\varepsilon = Q_\varepsilon(x_0, t_0)$. By (5.13), we have

$$\left(\frac{\partial}{\partial t} - \Delta_B\right) \tr_{\tilde{\omega}_N} \omega \leq 0, \tag{5.18}$$

since the transverse bisectional curvature of $\tilde{\omega}_N$ is zero. Using the Cauchy-Schwarz inequality to get

$$\left(\frac{\partial}{\partial t} - \Delta_B\right) \log |X|_{\tilde{\omega}_N}^2 = \frac{1}{|X|_{\tilde{\omega}_N}^2} \left(-g^{ij} \frac{\partial \omega}{\partial t} (\partial_i X)(\partial_j X) + \frac{|\nabla^T |X|_{\tilde{\omega}_N}^2|^2}{|X|_{\tilde{\omega}_N}^2} \right) \leq 0. \tag{5.19}$$

All these (5.18) and (5.19) yield that

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_B\right) Q_\varepsilon(x_0, t_0) < 0.$$ 

This implies $Q_\varepsilon$ is uniformly bounded from above. Letting $\varepsilon \to 0$ we obtain

$$|X|_{\omega_N}^2 \tr_{\tilde{\omega}_N} \omega \leq C,$$

for some uniform constant $C$. By Lemma 5 we have $|X|_{\omega_N}^2 \tr_{\omega_N} \omega \leq C$, and since $|X|_{\omega_N}^2 \leq |X|_{\omega_0}^2 \tr_{\omega_N} \omega$, this gives $|X|_{\omega_0}^2 \leq C |X|_{\omega_0}^2$ and (5.13) follows from the fact that $|X|_{\omega_0}^2$ is uniformly equivalent to $|s|^2$ in $S^1 \times D_1^4$.

Since $2 \Re(\frac{1}{r_{S^1}^2} X) = W$ in $S^1 \times D_1^4 \{0\}$ and hence (5.16) follows from (5.15). \qed

From Lemma 7 we have the bound of the lengths of spherical and transverse radial paths in $S^1 \times D_1^4 \{0\}$. For $0 < r_{S^1} < 1/2$, the diameter of the product sphere $S^1 \times S^3_{r_{S^1}}$ in $S^1 \times D_1^4$ with the metric induced from $\omega$ is uniformly bounded from above, independent of $r_{S^1}$. For any $(s_0, z_1, z_2) \in S^1 \times D_1^4 \{0\}$, the length of a transverse radial path $\gamma(u) = (s_0, u z_1, u z_2)$ for $u \in (0, 1]$ with respect to $\omega$ is uniformly bounded from above by $C r_{S^1}$ with $r_{S^1} = \sqrt{\sum z_i^2}$. Hence the diameter of $S^1 \times D_1^4 \{0\}$ with respect to $\omega$ is uniformly bounded from above and

$$\text{diam}_{g^T(t)} M \leq C. \tag{5.20}$$

We then obtain a diameter bound of $M$ with respect to $g^T(t)$. 

5.1.2. Gromov-Hausdorff Convergence as $t \to T_0^-$. In this subsection, we will derive the part (2) in the Definition 2 of floating foliation canonical surgical contraction under the assumption (1.2). For the simplicity, we assume that there is only one exceptional foliation $(-1)$-curve $V$.

More precisely, we will show that $(M, g^T(t))$ converges to $(N, d_{T_0})$ in the sense of Gromov-Hausdorff as $t \to T_0^-$ and $\omega(t)$ converges in $C^\infty$ on any compact subset of $M \setminus V$ to a smooth transverse Kähler metric $\omega_T$ on $M \setminus V$. We will first define a compact metric space $(N, d_{T_0})$ and show that $(M, g(t))$ converges in the Gromov-Hausdorff sense to this metric space.

Recall that $(M, \eta, \xi, \Phi, g)$ is a compact quasi-regular Sasakian manifold, then by the first structure theorem on Sasakian manifolds, $M$ is a principal $S^1$-orbibundle (V-bundle) over $Z$ which is also a $Q$-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion $\pi_M : (M, g, \omega) \to (Z, h, \omega_h)$ with

$$g = g^T + \eta \otimes \eta$$

such that

$$g^T = \pi_M^*(h); \quad \frac{1}{2}d\eta = \pi_M^*(\omega_h).$$

The orbit $\xi_x$ is compact for any $x \in M$, we then define the transverse distance function as

$$d^T(x_1, x_2) \triangleq d_g(\xi_{x_1}, \xi_{x_2}),$$

where $d_g$ is the distance function defined by the Sasaki metric $g$. Then

$$d^T_{M}(x_1, x_2) = d_h(\pi_M(x_1), \pi_M(x_2)).$$

We define a transverse ball $B_{\xi, g}(x, r)$ as follows:

$$B_{\xi, g}(x, r) = \{\bar{x} : d^T_{M}(x, \bar{x}) < r\} = \{y : d_h(\pi_M(x), \pi_M(\bar{x})) < r\}.$$

Since $M \setminus V$ can be identified with $N \setminus S^1_{y_0}$ via the map $\psi$. Abusing notation, we denote $g_{T_0}$ for the smooth Kähler metric $(\psi^{-1})^*g_{T_0}$ on $N \setminus S^1_{y_0}$ and extend $g_{T_0}$ to a nonnegative $(1, 1)$-tensor $\bar{g}_{T_0}$ on the whole space $N$ by setting $\bar{g}_{T_0}|_{S^1_{y_0}}(\cdot, \cdot) = 0$. Since $N$ is a $S^1$-bundle over $Y$, there is a Riemannian submersion

(5.21)

$$\pi_N : (N, \bar{g}_{T_0}, \omega_{T_0}) \to (Y, k_{T_0}, \omega_k)$$

such that

$$\pi_N^*(\omega_k) = \omega_{T_0}.$$

We define a distance function $d^T_{T_0}$ on $N$ as follows: For $\bar{g}^T_{T_0} := \bar{g}_{T_0}(\pi_N(\cdot), \pi_N(\cdot))$,

$$d^T_{T_0}(y_1, y_2) = \inf \int_0^1 \sqrt{\bar{g}^T_{T_0}(\gamma'(s), \gamma'(s))} ds$$

where the infimum is taken over all piecewise smooth paths $\gamma : [0, 1] \to N$ with $\gamma(0) = y_1$ and $\gamma(1) = y_2$ such that

$$d^T_{T_0}(y_1, y_2) = d_{\bar{g}_{T_0}}(\pi_N(y_1), \pi_N(y_2)).$$

Then

$$d^T_{T_0} = \pi_N^*(d_{k_{T_0}}),$$

where $d_{k_{T_0}}$ is the distance with respect to $k_{T_0}$ on $Y$ as in (5.21).
Adapting from section 3, it follows from the previous set-up that there is a commutative diagram
\[(5.22)\]
\[
\begin{array}{ccc}
S(-1) \approx \mathbb{C} \times S^3 & \xrightarrow{\pi_3} & S^3 \subset (M, g^T, \omega, d^T) \\
\downarrow \pi & \downarrow \pi_M & \downarrow \pi_N \\
O(-1) & \xrightarrow{\pi_4} & CP^1 \subset (Z, h, \omega_h, d_h) \rightarrow \pi_N(y_0) \in (Y, k_{T_0}, d_{k_{T_0}}).
\end{array}
\]
Here \(d_h\) is the distance with respect to \(h\) on \(Z\). Note that
\[(5.23)\]
\[d^T = \pi_M^*(d_h) \quad \text{and} \quad \omega = \pi_M^*(\omega_h).\]
Hence, in view of the commutative diagram (5.22), all the estimates as in [SW2] can be adapted on \(Z, Y\) and then on \(M, N\) via (5.23). More precisely, it follows from [SW2, Lemma 3.2] that

**Lemma 8.** There exists a uniform constant \(C\) such that for any points \(x_1, x_2 \in V \subset (M, g^T, \omega, d^T)\)
\[d_T(x_1, x_2) \leq C(T - t)^{\frac{1}{2}}\]
for any \(t \in [0, T_0]\).

Note that when \(r\) small enough, \(B_{\xi, \pi_{T_0}}(y, r)\) is a trivial \(S^1\)-bundle over the geodesic ball \(B_h(\pi_N(y), r)\). Then it follows from [SW2, Lemma 3.3], Lemma 8 and (5.20) that

**Lemma 9.** For any \(\varepsilon > 0\), there exist \(\delta\) and \(T \in [0, T_0]\) such that
\[\text{diam}_{d_{T_0}}^T B_{\xi, \pi_{T_0}}(y, \delta) < \varepsilon\]
and
\[\text{diam}_{\omega(T)}^{-1}(B_{\xi, \pi_{T_0}}(y, \delta)) < \varepsilon\]
for all \(t \in [0, T_0]\).

Combining Lemma 9 and [CHLW, Theorem 6.2], it follows that

**Proposition 4.** \((M, g^T(t))\) converges to \((N, d_{T_0}^T)\) and \((M, g(t))\) converges to \((N, d_{T_0})\) in the sense of Gromov-Hausdorff as \(t \to T_0^-\). In particular, \((N, d_{T_0})\) is compact metric space homeomorphic to the Sasakian 5-manifold \(N\).

This establishes the part (2) in the Definition 2 of floating foliation canonical surgical contraction under the assumption (1.2).

5.1.3. **Higher Order Estimates for \(\omega(t)\) as \(t \to T_0^+\).** Under the assumption of Theorem 3, we know that \((M, g(t))\) converges to \((N, d_{T_0})\) in the sense of Gromov-Hausdorff as \(t \to T_0^-\) and \(\omega(t)\) converges in \(C^\infty\) on any compact subset of \(M \setminus \bigcup_{i=1}^k V_i\) to a smooth transverse Kähler metric \(\omega_T\) on \(M \setminus \bigcup_{i=1}^k V_i\). In particular, we already have \(C^\infty\) estimates for \(\omega(t)\) away from the floating foliation \((-1)\)-curves \(V_i\) as \(t\) approaches \(T_0^+\). However, to understand how the Sasaki-Ricci flow can be continued past the singular time we need more precise estimates.

We assume as before that there is only one exceptional foliation \((-1)\)-curve \(V\). From Lemmas 3, 5 and 6 there exist positive constants \(\delta\) and \(C\) such that
\[(5.24)\]
\[
\frac{|s|^2}{C} \omega_0 \leq \omega(t) \leq \frac{C}{|s|^{2(1-s)}} \omega_0.
\]
For simple we assume that $|s|_h \leq 1$ on $M$. Following the same method in [PSS], we define an endomorphism $H = H(t)$ of the transverse tangent bundle by $H^i_i = g^T_o T^j (g^T_o)^{j\ell} g^T_\ell g^T_j$ and consider the quantity $S = S(t)$ given by

$$S = |\nabla^T \log H|^2,$$

here the transverse covariant derivative $\nabla^T$ and the norm $| \cdot |$ are taken with respect to the evolving metric $g^T$.

**Proposition 5.** There exist positive constants $\alpha$ and $C$ such that for $t \in [0, T)$,

$$S \leq \frac{C}{|s|_h^{1/2}}. \quad (5.25)$$

**Proof.** Following the same computation as in [PSS], we have

$$\left( \frac{\partial}{\partial t} - \Delta_B \right) S \leq -|\nabla^T \nabla^T \log H|^2 - |\nabla^T \nabla^T \log H|^2 + CS + C|\nabla^T \operatorname{Rm}^T(g^T_0)|^2, \quad (5.26)$$

where $\operatorname{Rm}^T(g^T_0)$ is the transverse Riemannian curvature of $g^T$. Using (5.24)

$$|\nabla^T \operatorname{Rm}^T(g^T_0)|^2 \leq C|s|_h^{-K}(S + 1), \quad (5.27)$$

for a positive constant $K$. By the Cauchy-Schwarz inequality we have

$$|\nabla^T S| \leq S^{1/2}(|\nabla^T \nabla^T \log H| + |\nabla^T \nabla^T \log H|), \quad (5.28)$$

and the inequalities

$$|\nabla^T |s|_h^{4K}| \leq C|s|_h^{3K}, \quad |\Delta_B |s|_h^{4K}| \leq C|s|_h^{3K}, \quad (5.29)$$

where we are choosing $K$ sufficiently large. Combining (5.26), (5.27), (5.28) and (5.29), we obtain

$$\left( \frac{\partial}{\partial t} - \Delta_B \right)(|s|_h^{4K} S) \leq -|s|_h^{4K}(|\nabla^T \nabla^T \log H|^2 + |\nabla^T \nabla^T \log H|^2) + C|s|_h^{2K} S$$

$$+ C|s|_h^{3K}S^{1/2}(|\nabla^T \nabla^T \log H| + |\nabla^T \nabla^T \log H|) + C \leq C(1 + |s|_h^{2K} S). \quad (5.30)$$

Moreover, we know that $\operatorname{tr}_\omega \omega$ satisfies

$$\left( \frac{\partial}{\partial t} - \Delta_B \right) \operatorname{tr}_\omega \omega \leq -g^T g^T_\ell g^T_j (g^T_0)^{j\ell} \frac{0}{g^T_0} g^T_0 T^T = C|s|_h^{-K} - \frac{1}{C}|s|_h^{K} S - \frac{1}{2} g^T_0 T^T g^T_\ell g^T_j \frac{0}{g^T_0} g^T_0 T^T g^T_\ell g^T_j \frac{0}{g^T_0} g^T_0 T^T,$$

where we used (5.24) again. Here $\nabla^T$ denotes the transverse covariant derivative with respect to $g^T_0$. Since

$$|\nabla^T \operatorname{tr}_\omega \omega|^2 \leq (\operatorname{tr}_\omega \omega) g^T_0 T^T g^T_\ell g^T_j \frac{0}{g^T_0} g^T_0 T^T g^T_\ell g^T_j \frac{0}{g^T_0} g^T_0 T^T,$$
we have
\[
\left(\frac{\partial}{\partial t} - \Delta_B\right)(|s|_h^K \text{tr}_\omega \omega) \\
\leq -\frac{1}{e} |s|_h^{2K} S + C - 2 \text{Re}(\nabla^T |s|_h^K \cdot \nabla^T \text{tr}_\omega \omega) \\
- \frac{1}{2} |s|_h^K g_0^{Tk} g^0 \overline{\tau} g^0 \tau \sigma_i g_k q i g_l q j g^0 g^n m
\leq -\frac{1}{e} |s|_h^{2K} S + C,
\]
where
\[
2 \text{Re}(\nabla^T |s|_h^K \cdot \nabla^T \text{tr}_\omega \omega) \leq C + \frac{1}{e} |\nabla^T |s|_h^K|^2 |\nabla^T \text{tr}_\omega \omega|^2
\leq C + \frac{1}{2} |s|_h^K g_0^{Tk} g^0 \overline{\tau} g^0 \tau \sigma_i g_k q i g_l q j g^0 g^n m,
\]
which follows from (5.24) and (5.31).

By applying the maximum principle to \( Q = |s|_h^{4K} S + A |s|_h^K \text{tr}_\omega \omega - B t \), for constants \( A \) and \( B \) being chosen sufficiently large, from (5.30) and (5.32) we obtain
\[
\left(\frac{\partial}{\partial t} - \Delta_B\right) Q < 0.
\]
This gives a uniform upper bound for \( Q \) and (5.24) follows. \( \square \)

We have the estimates of curvature of \( g^T \) and all its higher order estimates as in [SW2].

**Proposition 6.** For each integer \( m \geq 0 \) there exist \( C_m, \alpha_m > 0 \) such that for \( t \in [0, T_0) \),
\[
|(|\nabla^T_{\nabla_R})^m Rm^T(g^T(t))| \leq \frac{C_m}{|s|_h^\alpha_m} \quad \text{and} \quad |(|\nabla^T_{\nabla_R})^m g^T(t)|_{g^0} \leq \frac{C_m}{|s|_h^\alpha_m},
\]
where \( \nabla^T_{\nabla_R} = \frac{1}{2}(\nabla^T + \nabla^R) \) and \( \nabla^R \) denote the real covariant derivative of \( g^T \) and \( g^0_T \), respectively.

5.1.4. **Continuing the Sasaki-Ricci Flow at \( T_0 \).** Now we will show how to continue the Sasaki-Ricci flow past time \( T_0 \) on the manifold \( N \) by following the methods in [SW2]. First we know that \( (M, g^T(t)) \) converges to \( (N, d^T_{T_0}) \) in the sense of Gromov-Hausdorff as \( t \to T_0^- \). We explain how one can continue the Sasaki-Ricci flow through the singularity. We replace \( M \) with the manifold \( N \) at the singular time \( T_0 \). Again, we assume for simplicity that we have only one exceptional floating foliation \((1)-\text{curve} \ V \).

Write \( \omega_{T_0} = \psi^* \omega_N \), where \( \omega_N \) is the smooth transverse Kähler metric on \( N \). From Lemma 2 there is a closed positive \((1,1)\) current \( \omega_{T_0} \) and a bounded basic function \( \varphi_{T_0} \) with
\[
\omega_{T_0} = \omega_{T_0} + \sqrt{-1} \partial_B \overline{\partial}_B \varphi_{T_0} \geq 0.
\]
Moreover from Lemma 3, \( \varphi(t) \) converges to \( \varphi_{T_0} \) pointwise on \( M \) and smoothly on any compact subset of \( M \setminus V \). Since \( \varphi_{T_0} \) is constant on \( V \), there exists a bounded function \( \phi_{T_0} \) on \( N \) and is smooth on \( N \setminus S^1 \) such that \( \psi^* \phi_{T_0} = \varphi_{T_0} \). We then define a closed positive \((1,1)\) current \( \omega' \) on \( N \) by
\[
\omega' = \omega_N + \sqrt{-1} \partial_B \overline{\partial}_B \phi_{T_0} \geq 0.
\]
This implies $\omega'$ is smooth on $N \setminus S^1$ which satisfies $\omega_{T_0} = \psi^* \omega'$. It follows from [Kol2, Kol3] and [SW2] Lemma 5.1 and Lemma 5.2] that there exists $p > 1$ such that
\begin{equation}
\frac{\omega^m \wedge \eta_N}{\omega^m_N \wedge \eta_N} \in L^p(N)
\end{equation}
and then $\phi_{T_0}$ is a bounded basic continuous function on $N$, which is smooth on $N \setminus S^1$, such that $\varphi_{T_0} = \psi^* \phi_{T_0}$.

Then, by using the method as in section 4 and [ST], we construct a solution of the Sasaki-Ricci flow on $N$ starting at $\omega'$. First one fix a smooth closed $(1, 1)$ form $\chi \in c_1^B(N)$. There exists $T' > T_0$, this $T'$ is strictly less than the maximal time $T_N$, such that the closed $(1, 1)$ form
\begin{equation}
\tilde{\omega}_{t,N} := \omega_N + (t - T) \chi
\end{equation}
is a transverse Kähler metric for $t \in [T_0, T')$. We use the metrics $\tilde{\omega}_{t,N}$ as our reference metrics for the Sasaki-Ricci flow as it continues on $N$. Fix a smooth adapted measure $\Omega_N$ and then a volume form $\Omega_N \wedge \eta_N$ on $N$ satisfying
\begin{equation*}
\sqrt{-1} \partial_B \bar{\partial} B \log \Omega_N = \frac{\partial}{\partial t} \tilde{\omega}_{t,N} = \chi \in c_1^B(N),
\end{equation*}
for $t \in [T_0, T')$.

Next by using the method as in section 4 and [ST], we will now construct a family of basic functions $\phi_{T_0, \varepsilon}$ on $N$ which converge to $\phi_{T_0}$. For $\varepsilon > 0$ sufficiently small and $K$ fixed be sufficiently large, define a family of volume forms $\Omega_\varepsilon \wedge \eta_N$ on $N$ by
\begin{equation*}
\Omega_\varepsilon \wedge \eta_N = (\psi|^{-1}_{M,N})^*(\frac{|\omega_1^*|^2}{|\omega_h^*| + \varepsilon}) \wedge \eta_N + \varepsilon \Omega_N \wedge \eta_N \quad \text{on } N \setminus S^1,
\end{equation*}
and $(\Omega_\varepsilon \wedge \eta_N)|_{S^1} = \varepsilon (\Omega_N \wedge \eta_N)|_{S^1}$. We then define the basic functions $\phi_{T_0, \varepsilon}$ to be solutions of the transverse complex Monge-Ampère equations
\begin{equation}
(\omega_N + \sqrt{-1} \partial_B \bar{\partial} B \phi_{T_0, \varepsilon})^2 \wedge \eta_N = C_\varepsilon \Omega_\varepsilon \wedge \eta_N; \sup(\phi_{T_0, \varepsilon} - \phi_{T_0}) = \sup(\phi_{T_0} - \phi_{T_0, \varepsilon})
\end{equation}
with $C_\varepsilon \int_N \Omega_\varepsilon \wedge \eta_N = \int_N \omega_N^2 \wedge \eta_N$. Such $\phi_{T_0, \varepsilon}$ exist and are unique due to El Kacimi-Alaoui (EKA). These solution functions $\phi_{T_0, \varepsilon}$ are continuous on $N$ and smooth on $N \setminus S^1$. Furthermore, it follows from (5.36) and [Kol2] that
\begin{equation*}
||C_\varepsilon \Omega_\varepsilon \wedge \eta_N \wedge \eta_N \Omega_N \wedge \eta_N ||_{L^1(N)} \to 0 \quad \text{as } \varepsilon \to 0
\end{equation*}
and thus
\begin{equation*}
||\phi_{T_0, \varepsilon} - \phi_{T_0}||_{L^\infty(N)} \to 0 \quad \text{as } \varepsilon \to 0.
\end{equation*}

Now let $\varphi_\varepsilon = \varphi_\varepsilon(t)$ to be the basic functions which are solutions of the transverse parabolic complex Monge-Ampère equations
\begin{equation*}
\frac{\partial}{\partial t} \varphi_\varepsilon = \log \frac{\tilde{\omega}_{1,N} + \sqrt{-1} \partial_B \bar{\partial} B \varphi_\varepsilon}{\omega_N \wedge \eta_N}, \quad \varphi_\varepsilon(T_0) = \phi_{T_0, \varepsilon},
\end{equation*}
for $t \in [T_0, T']$. It follows from [SW2] Proposition 5.1 and section 4 that there exists a basic function $\varphi$ in $C^0([T_0, T'] \times N) \cap C^\infty((T_0, T') \times N)$ such that $\varphi_\varepsilon \to \varphi$ in $L^\infty([T_0, T'] \times N)$ as $\varepsilon \to 0$, such a $\varphi$ is continuous on $[T_0, T'] \times N$ and smooth on
any compact subset of $(T_0, T') \times N$. This basic function $\varphi$ satisfies the transverse complex Monge-Ampère equation

$$
\frac{\partial}{\partial t} \varphi = \log \left( \frac{\bar{\omega}_{t,N} + \sqrt{-1} \partial_B \bar{\partial}_B \varphi}{\Omega_N} \right)^{2 \wedge \eta_N}, \quad \varphi(T_0) = \phi_{T_0}, \quad t \in (T_0, T')
$$

in $C^0([T_0, T'] \times N) \cap C^\infty((T_0, T') \times N)$. Then we define for $t \in [T_0, T']$,

$$
\omega_{t,\varepsilon}(t) := T_0 \omega_N + \sqrt{-1} \partial_B \bar{\partial}_B \varphi_{t,\varepsilon},
$$

and

$$
\omega_{\varepsilon}(t) = \bar{\omega}_{t,N} + \sqrt{-1} \partial_B \bar{\partial}_B \varphi_{\varepsilon}.
$$

We thus obtain estimates on $\omega_{T_0,\varepsilon}$ and $\omega_{\varepsilon}$ and its volume form $\omega_{\varepsilon}^2 \wedge \eta_N$ for $t \in [T_0, T']$ as Lemmas 3, 5 and 6.

**Lemma 10.** There exist positive constants $\alpha$ and $C$, independent of $\varepsilon$, such that

$$
|s|^{2a} \omega_N \leq \omega_{T,\varepsilon} \leq C |s|^{2a} \omega_N
$$

and also

$$
|s|^{2a} \omega_N \leq \omega_{\varepsilon} \leq C |s|^{2a} \omega_N \quad \text{and} \quad \omega_{\varepsilon}^2 \wedge \eta_N \leq C |s|^{2a} \Omega_N \wedge \eta_N
$$

on $[T_0, T'] \times (N \setminus S^1)$. Here we write $(\psi_{M,V}^{-1})^* |s|^2_h$ on $N \setminus S^1$ as $|s|^2_h$. For a fixe l positive integer $L$, for each integer $0 \leq m \leq L$ there exist $C_m$, $\alpha_m > 0$ such that

$$
|((\nabla_R^T)^m g^T(t))_{g_R}| \leq \frac{C_m}{|s|^2_h} \quad \text{and} \quad |((\nabla_R^T)^m g^T(t))_{g_R}| \leq \frac{C_m}{|s|^2_h},
$$

where $\nabla_R^T$ denote the real covariant derivative with respect to the fixed metric $g_R^T$.

We can now prove that the Sasaki-Ricci flow can be smoothly connected at time $T_0$ between $[0, T_0) \times M$ and $(T_0, T'] \times N$, outside $T_0 \times S^1 \cong T_0 \times V$ via the map $\psi$.

Recall that for $t \in [T_0, T']$, $\varphi(t)$ is the limit of $\varphi_{\varepsilon}$ as $\varepsilon \to 0$ and as in (5.38), the metric

$$
\omega = \bar{\omega}_{t,N} + \sqrt{-1} \partial_B \bar{\partial}_B \varphi
$$

is the solution of the Sasaki-Ricci flow on $N$

$$
\frac{\partial}{\partial t} \omega = -\text{Ric}^T(\omega) \quad \text{with} \quad \omega(T_0) = \omega',
$$

for $t \in (T_0, T')$. We define

$$
Z = ([0, T_0) \times M) \cup (T_0 \times N \setminus S^1) \cup ((T_0, T'] \times N).
$$

Consider a family of metrics $\omega(t, x)$, for $(t, x) \in Z$. We define what is $\omega$ is smooth on $Z$. If $(t, x) \in [0, T_0) \times M$ or $(t, x) \in (T_0, T'] \times N$, then we have $\omega$ to be smooth in the usual sense. Recall that we have the estimates for $\omega(t, x)$ on $[0, T_0) \times M \setminus V$ by (5.33) and on $[T_0, T'] \times N \setminus S^1$ from (5.40). When $(t, x) = (T_0, x) \in T_0 \times N \setminus S^1 \cong T_0 \times M \setminus V$, we take a small neighborhood $U$ of $x$ in $M \setminus V$ and consider $\omega$ as a metric on $(T_0 - \delta, T_0 + \delta) \times U$, for some $\delta > 0$. We say $\omega$ is smooth at $(T_0, x)$ if $\omega$ is smooth at $(T_0, x)$ in the $(T_0 - \delta, T_0 + \delta) \times U$. It follows from Lemma 10 that $\omega = \omega(t)$ satisfies the Sasaki-Ricci flow (5.38) and is smooth a time $T_0$ in the sense above. Hence the metric $\omega = \omega(t)$ is a smooth solution of the Sasaki-Ricci flow in the space-time region $Z$.

This establishes the part (3) in the Definition 2 of floating foliation canonical surgical contraction under the assumption (1.2).
5.1.5. Gromov-Hausdorff Convergence as \( t \to T_0^+ \). To complete the proof of Theorem \( 3 \) it remains to show that \((N, \omega(t))\) converges in the Gromov-Hausdorff sense to \((N, d_T)\) as \( t \to T_0^+ \).

**Lemma 11.** There exists \( \delta > 0 \) and a uniform constant \( C \) such that for \( \omega = \omega(t) \) a solution of the Sasaki-Ricci flow \((5.38)\) satisfies
\[
\omega \leq C \left( \frac{\omega_N}{(\psi|_{M \setminus V})^*|\psi|_h^2} \right) \quad \text{and} \quad \omega \leq C(\psi|_{M \setminus V})^{-1} \left( \frac{\omega_0}{|\psi|_h^{2(1-\delta)}} \right),
\]
for \( t \in [T_0, T'] \). Here \( \omega_0 \) is the initial metric on \( M \).

**Proof.** The arguments of the proof involved is similar to the estimate as in Lemma \( 6 \) we omit it. We refer to \([SW2, Proposition 6.1]\) for some details as in the Kähler-Ricci flow. \( \square \)

Finally, it follows by the arguments of earlier sections that \((N, \omega(t))\) converges in the Gromov-Hausdorff sense to \((N, d_T)\) as \( t \to T_0^+ \). This establishes the part (4) in the Definition \( 2 \) of floating foliation canonical surgical contraction under the assumption \((1.2)\).

This completes the proof of Theorem \( 3 \).

5.2. Analytic Foliation Minimal Model Program with Scaling. As a consequence of Theorem \( 3 \) and Proposition \( 11 \) we will prove our main result in the paper on the analytic foliation minimal model program with scaling in a compact quasi-regular Sasakian 5-Mmanifold. We will follow the lines of the arguments of the proof of Proposition \( 11 \) via the Sasaki-Ricci flow.

**The proof of Theorem \( 4 \):**

**Proof.** Let \((M, \xi, \omega_0)\) be a compact quasi-regular Sasakian 5-manifold with a smooth transverse Kähler metric \( \omega_0 \). In view of the cohomological characterization of the maximal solution of the Sasaki-Ricci flow \((4.6)\), we start with a pair \((M, H^T)\), where \( M \) is a Sasakian manifold with an ample basic divisor \( H^T \). Let
\[
T_0 = \sup\{ t > 0 \mid H^T + tK_M^T \text{ is nef} \}.
\]
Denote
\[
L_0^T := H^T + T_0K_M^T
\]
which is a basic \( Q \)-line bundle and semi-ample. In fact, it follows from Kleiman criterion that
\[
mL_0^T - T_0K_M^T
\]
is ample and then nef and big for some sufficiently large \( m \). Then by Kawamata criterion for base-point free, we have the semi-ample for \( L_0^T \).

Next we define a subcone
\[
R := NE(M)_{K_M^T < 0} \cap (L_0^T)^\perp
\]
which is a foliation extremal ray \( R \) with the generic choice of \( H^T \). For \( V \) with \( 0 = L_0^T \cdot V \), we have
\[
K_M^T \cdot V = -\frac{1}{T_0}(H^T \cdot V) < 0.
\]
That is the map $\Psi : M \rightarrow N$ induced from $(L_0^T)^m$ contract all foliation curves whose class lies in the foliation extremal ray $R$ with

$$L_0^T \cdot V = 0.$$ 

and

$$K_M^T \cdot V < 0.$$

We observe that $K_M^T \cdot V = 0$ as $T_0 \rightarrow \infty$.

Now we consider a sequence of contarctions $g(t)$ on the manifolds $M_0, M_1, \ldots, M_k$ on the time intervals $[0, T_0), (T_1, T_2), \ldots, (T_{k-1}, T_k)$. Denote $L_{-1}^T = H^T$ as above and

$$L_i^T := L_{i-1}^T + t_i K_{M_i}^T.$$

Then the nef class $L_i^T$ is semi-ample, there exists a map $\psi : M_l \rightarrow N$ where $N$ is a quasi-regular Sasakian 5-manifold with foliation cyclic quotient singularities. The exceptional locus of $\psi$ is a sum of irreducible foliation curves $W = \sum W_i$ with

$$W_i \cdot L_i^T = 0$$

and

$$K_M^T \cdot W_i < 0.$$

(I) If $L_i^T$ is big : 

(i) If all foliation curves $W = \sum W_i$ do not pass the singularity on $M_l$ : We show that the Sasaki-Ricci flow will perform a canonical surgical contraction in $M_l$ as in Theorem 3. Since $L_i^T$ is big and all foliation curves $W_i$ do not pass the singularity on $M_i$, thus by index theorem we have $W_i^2 < 0$, and by Adjunction Formula, all foliation curves $W_i$ are foliation $(-1)$-curves and floating on $M_i$. On the other hand,

$$W_i \cdot L_i^T = 0 \quad \text{and} \quad W_j \cdot L_i^T = 0, \; i \neq j$$

and

$$W_i^2 = -1 = W_j^2.$$

Then

$$(W_i + W_j) \cdot L_i^T = 0$$

and index theorem again

$$(W_i + W_j)^2 < 0.$$

Hence

$$W_i \cdot W_j = 0$$

and $\{W_i\}$ is a finite number of disjoint floating foliation $(-1)$-curves in $M_l$. Thus by Sasaki analogue of Castelnuovo’s contraction theorem (Theorem 2), Thus $\psi$ is a map blowing down the basic exceptional curves $W_i$. It follows from Proposition 10 that $L_i^T$ is the pull-back of an ample line bundle over $M_l$, we obtain (1.2) for some transverse $\omega_{M_i}$ on $M_l$. It follows from Theorem 3 that the Sasaki-Ricci flow $g(t)$ performs a foliation canonical surgical contraction with respect to the data $W_1, \ldots, W_k$, $N$ and $\psi$.

(ii) If some of $W_i := \Gamma$ pass the foliation singularity of type $\frac{1}{r}(1, a)$ : We show that the Sasaki-Ricci flow will perform foliation extremal contractions of foliation
Following above notions, it follows from Theorem 6 that we can have the minimal resolutions of foliation singularities of $M_l$ and $N$
\[ \varphi : \tilde{M}_l \to M_l \quad \text{and} \quad \tilde{\varphi} : \tilde{N} \to N. \]

Our goal is to find $\tilde{\psi} : \tilde{M}_l \to \tilde{N}$ and $\psi : M_l \to N$ such that the following diagram is commutative:
\[ \begin{array}{c}
\tilde{M}_l \supset V_i, \quad \tilde{\Gamma} \\
\uparrow \varphi \\
M_l \supset \Gamma, \quad S^1_p
\end{array} \quad \begin{array}{c}
\tilde{N} \\
\downarrow \tilde{\varphi} \\
N \ni S^1_q
\end{array} \]

Suppose $\tilde{M}_l$ and $\tilde{N}$ are not isomorphic; then as they are both regular Sasakian manifolds, by Theorem 2, there must exist a foliation $(-1)$-curve $\tilde{\Gamma}$ such that $\tilde{\varphi} \circ \tilde{\psi}(\tilde{\Gamma}) = S^1_q$. Let $\Theta$ be a set of the foliation curves in $\tilde{M}_l$ such that $\Theta = \tilde{\psi}^{-1} \circ \tilde{\varphi}^{-1}(S^1_q)$ and $D^T = \{V_i\}$ be the exceptional locus of $\varphi$ at the singular fibre $S^1_p$. Then the Hirzebruch Jung continued fraction
\[ r = \frac{a}{b} = [b_1, \ldots, b_m] \]
say that
\[ V_i^2 = -b_i. \]

Now there is at most one such a foliation exceptional curve $V_j$ for some $j$ so that $V_j \cdot \tilde{\Gamma} \neq \emptyset$ with $V_j^2 = -b_j$. Then
\[ (5.43) \]
\[ K^T_{M_l} \cdot \Gamma < 0 \]
and $\Gamma$ passed the foliation singularity of type $\frac{1}{r}(1, a)$. As in section 4, we lift the Sasaki-Ricci flow to the foliation minimal resolution $\tilde{M}_l$ of $M_l$ by lifting the Monge–Ampere equation (4.11) to (4.12), we thus obtain uniform estimates as in subsection 5.1. (cf Theorem 8 and Proposition 3) so that the Sasaki-Ricci flow (4.12) performs a canonical surgical contraction of floating foliation $(-1)$-curves
\[ \tilde{\psi}_l : \tilde{M}_l \to \tilde{N}_l \]
by Corollary 1. Furthermore, the canonical surgical contraction maps $V_i$ onto $V'_i$ so that
\[ V'_i \cdot V'_i = -b_i \]
for all $i \neq j$ and
\[ V'_j \cdot V'_j = -(b_j - 1). \]

Finally, by (5.43) and (A.4), it descends to perform foliation extremal contractions of foliation $K^T_{M_l}$-negative curves
\[ \psi_l : M_l \to M_{l+1} \]
with at worst singularity type $\frac{1}{r_{\tilde{N}_l}}(1, a_{\tilde{N}_l})$ so that the Hirzebruch Jung continued fraction
\[ \frac{r_{\tilde{N}_l}}{a_{\tilde{N}_l}} = [b_1, \cdots, (b_j - 1), \cdots, b_m] \]
and it will be \( \frac{r_{\tilde{N}_l}}{a_{\tilde{N}_l}} = [b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_m] \) if \((b_j - 1) = 1\).

Now take \( \tilde{\psi}_l = \tilde{\psi} \) and \( \psi_l = \psi \) into (5.42) with \( N = M_{l+1} \) and \( \tilde{N} = \tilde{N}_l \); it follows that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{M}_l \supset \tilde{\Gamma}, V_j & \xrightarrow{\tilde{\psi}_l} & \tilde{N}_l \supset V'_j \\
\downarrow \varphi & \sim & \downarrow \tilde{\varphi}_l \\
M_l \supset \Gamma, S^1_p & \xrightarrow{\psi_l} & M_{l+1}.
\end{array}
\]

Then we are done.

Finally, it follows that the divisorial contractions end with either

\[ T_k = \infty \]

and \( M_k \) is nef which has no foliation \( K_{M_k}^T \)-negative curves or

\[ L_k^T \text{ is not big} \]

on \( M_k \).

Note that similar method can be applied to a finite number of cyclic quotient foliation singularities.

**II** If \( L_k^T \text{ is not big} : \) In this situation, we have

\[ \text{Vol} M = \int_M \omega^2(t) \wedge \eta_0 \to (\epsilon_1^B(L_k^T))^2 = 0 \]

as \( t \to T_k^\leftarrow \) and we can not have a canonical surgical contraction. However since it is semi-ample, as in Proposition \( \text{[1]} \) we have

(i) there exists a transverse morphism

\[ \phi : M_k \to pt, \]

then \( K_{M_k}^T < 0 \) and thus \( M_k \) is transverse minimal Fano and the foliation space \( M_k/F_\xi \) is minimal log del Pezzo surface of at worst \( \frac{1}{r}(1, a) \)-type singularities. Or

(ii)

\[ \phi : M_k \to \Sigma_h, \]

then \( M_k \) is an \( S^1 \)-orbibundle of a rule surface over Riemann surfaces \( \Sigma_h \) of genus \( h \).

\[ \square \]

**Appendix A. Foliation Minimal Model Program**

**A.1. Basic Holomorphic Line Bundles, Basic Divisors on Sasakian Manifolds.** For a completeness, we will address basic holomorphic line bundles, basic divisors over Sasakian manifolds. We refer to [BG], [M], and references therein for some details.

Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n + 1)\)-manifold \( M \). We define \( D = \ker \eta \) to be the holomorphic contact vector bundle of \( TM \) such that

\[ TM = D \oplus < \xi > = T^{1,0}(M) \oplus T^{0,1}(M) \oplus < \xi >. \]

Then its associated strictly pseudoconvex CR \((2n + 1)\)-manifold to be denoted by \((M, T^{1,0}(M), \xi, \Phi)\).
Definition 13. (Ta) Let \((M, T^{1,0}(M))\) be a strictly pseudoconvex CR \((2n+1)\)-manifold and \(E \to M\) a \(C^\infty\) complex vector bundle over \(M\). A pair \((E, \overline{\partial}_h)\) is a CR-holomorphic vector bundle if the differential operator
\[
\overline{\partial}_h : \Gamma^\infty(E) \to \Gamma^\infty(T^{0,1}(M)^* \otimes E)
\]
is defined by
(i) \[
\overline{\partial}_Z(fs) = (\overline{\partial}_h f)(Z) \otimes s + f \overline{\partial}_Z s,
\]
(ii) \[
\overline{\partial}_Z \overline{\partial}_W s - \overline{\partial}_W \overline{\partial}_Z s - \overline{\partial}_{[Z, W]} s = 0,
\]
for any \(f \in C^\infty(M) \otimes \mathbb{C}\), \(s \in \Gamma^\infty(E)\) and \(Z, W \in \Gamma^\infty(T^{1,0}(M))\).

The condition (ii) of the definition means that \((0, 2)\)-component of the curvature operator \(R(E)\) is vanishing when \(E\) admits a connection \(D\) whose \((0, 1)\)-part is the operator \(\overline{\partial}_h\) as in the following Lemma.

Lemma 12. Let \((M, T^{1,0}(M), \theta)\) be a strictly pseudoconvex CR \((2n+1)\)-manifold and \((E, \overline{\partial}_h)\) a CR-holomorphic vector bundle over \(M\). Let \(h = \langle , \rangle_h\) be a Hermitian structure in \(E\). Then there exists a unique (Tanaka) connection \(D\) in \(E\) such that
(i) \[
D_Z s = (\overline{\partial}_h s) Z,
\]
(ii) \[
Z < s_1, s_2 >_h = < D_Z s_1, s_2 >_h + < s_1, D_Z s_2 >_h,
\]
(iii) The \((0, 2)\)-component of the curvature operator \(\Theta(E)\) is vanishing. Here \(\Theta(E) := D^2 s\).

Definition 14. Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n+1)\)-manifold. A CR-holomorphic vector bundle \((E, \overline{\partial}_h)\) over \(M\) is a basic transverse holomorphic vector bundle over \((M, T^{1,0}(M))\) if there exists an open cover \(\{U_\alpha\}\) of \(M\) and the trivializing frames on \(U_\alpha\), such that its transition functions are matrix-valued basic CR functions. The trivializing frames is called the basic transverse holomorphic frame.

Example 4. Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n+1)\)-manifold. Then, with respect to the trivializing frames
\[
\left\{ \frac{\partial}{\partial x_j}, Z_j = \left( \frac{\partial}{\partial z_j} + i h_j \frac{\partial}{\partial x_j} \right), \quad j = 1, 2, \cdots, n \right\}
\]
the transition functions of such frames are basic transverse holomorphic functions, that is \(h\) is basic. Thus \(T^{1,0}(M)\) is a basic transverse holomorphic vector bundle. Moreover, the canonical (determinant) bundle \(K^T_M\) of \(T^{1,0}(M)\) is a basic transverse holomorphic line bundle whose transition functions are given by \(t_{\alpha \beta} = \det(z_{\beta j}/z_{\alpha j})\) on \(U_\alpha \cap U_\beta\), where \((x, z_1^\alpha, \ldots, z_n^\alpha)\) is the normal coordinate on \(U_\alpha\).
Definition 15. (i) Let \((M, \eta, \xi, \Phi, g)\) be a Sasakian \((2n + 1)\)-manifold and \(L\) be a basic transverse holomorphic bundle over \(M\). A basic transverse holomorphic section \(s\) of \(L\) is a collection \(\{s_\alpha\}\) of CR-holomorphic maps \(s_\alpha : U_\alpha \to \mathbb{C}\) satisfying the transformation rule \(s_\alpha = t_{\alpha\beta}s_\beta\) on \(U_\alpha \cap U_\beta\). The transition function \(t_{\alpha\beta}\) is basic. A basic Hermitian metric \(h\) on \(L\) is a collection \(\{h_\alpha\}\) of smooth positive functions \(h_\alpha : U_\alpha \to \mathbb{R}\) satisfying the transformation rule

\[
h_\alpha = |t_{\alpha\beta}|^2h_\beta
\]
on \(U_\alpha \cap U_\beta\). Given a basic transverse holomorphic section \(s\) and a Hermitian metric \(h\), we can define the pointwise norm squared of \(s\) with respect to \(h\) by

\[
|s_\alpha|^2 = h_\alpha s_\alpha \overline{s_\alpha}
on U_\alpha. The reader can check that \(|s_\alpha|^2\) is a well-defined function on \(M\).

(ii) A Hermitian metric is called a basic hermitian metric if \(h_\alpha\) is basic. It always exists if \(L\) is a basic transverse holomorphic line bundle.

(iii) We define the curvature \(R^T_h\) of a basic Hermitian metric \(h\) on \(L\) to be the basic closed \((1, 1)\)-form on \(M\) given by

\[
R^T_h = -\frac{\sqrt{-1}}{2\pi} \partial_B \overline{\partial_B} \log h_\alpha
\]
on \(U_\alpha\). This is well-defined. The basic first Chern class \(c^B_1(L)\) of \(L\) to be the cohomology class \(\left[R^T_h\right]_B \in H^{1,1}_B(M, \mathbb{R})\). Since any two basic Hermitian metrics \(h, h'\) on \(L\) are related by \(h' = e^{-\phi}h\) for some smooth basic function \(\phi\), we see that \(R^T_{h'} = R^T_h + \frac{\sqrt{-1}}{2\pi} \partial_B \overline{\partial_B} \phi\) and hence \(c^B_1(L)\) is well-defined, independent of choice of basic Hermitian metric \(h\). We say that \((L, h)\) is positive if the curvature \(R^T_h\) is positive definite at every \(p \in M\).

Example 5. Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n + 1)\)-manifold. If \(g^T\) is a transverse Kaehler metric on \(M\), then \(h_0 = \det((g^T_{ij}))\) on \(U_\alpha\) defines a basic Hermitian metric on the canonical bundle \(K^T_M\). The inverse \((K^T_M)^{-1}\) of \(K^T_M\) is sometimes called the anti-canonical bundle. Its basic first Chern class \(c^B_1((K^T_M)^{-1})\) is called the basic first Chern class of \(M\) and is often denoted by \(c^B_1(M)\). Then it follows from the previous result that \(c^B_1(M) = [\text{Ric}^T(\omega)]_B\) for any transversal Kaehler metric \(\omega\) on a Sasakian manifold \(M\).

Definition 16. (i) Let \((L, h)\) be a basic transverse holomorphic line bundle over a Sasakian manifold \((M, \eta, \xi, \Phi, g)\) with the basic hermitian metric \(h\). We say that \(L\) is very ample if for any ordered basis \(z = (s_0, ..., s_N)\) of \(H^0_B(M, L)\), the map \(i_z : M \to \mathbb{CP}^N\) given by

\[
i_z(x) = [s_0(x), ..., s_N(x)]
\]
is well-defined and an embedding which is \(S^1\)-equivariant with respect to the weighted \(\mathbf{C}^*\) action in \(\mathbb{C}^{N+1}\) as long as not all the \(s_i(x)\) vanish. We say that \(L\) is ample if there exists a positive integer \(m_0\) such that \(L^m\) is very ample for all \(m \geq m_0\).

(ii) \(L\) is a semi-ample basic transverse holomorphic line bundle if there exists a basic Hermitian metric \(h\) on \(L\) such that \(R^T_h\) is a nonnegative \((1, 1)\)-form. In fact, there exists a foliation basepoint-free holomorphic map

\[
\Psi : M \to (\mathbb{CP}^N, \omega_{FS})
\]
defined by the basic transverse holomorphic section \( \{ s_0, s_1, \ldots, s_N \} \) of \( H_B^0(M, L^m) \) which is \( S^1 \)-equivariant with respect to the weighted \( C^* \) action in \( \mathbb{C}^{N+1} \) with \( N = \dim H_B^0(M, L^m) - 1 \) for a large positive integer \( m \) and

\[
0 \leq \frac{1}{m} \Psi^*(\omega_{FS}) = \tilde{\omega}_\infty \in c_1^B(L).
\]

There is a Sasakian analogue of Kodaira embedding theorem on a compact quasi-regular Sasakian \((2n + 1)\)-manifold due to [RT], and [HLM] :

**Proposition 7.** Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian \((2n + 1)\)-manifold and \((L, h)\) be a basic transverse holomorphic line bundle over \(M\) with the basic hermitian metric \(h\). Then \(L\) is ample if and only if \(L\) is positive.

**Definition 17.** (i) First we say that a subset \(V\) of a (quasi-regular) Sasakian manifold \((M^{2n+1}, \eta, \xi, \Phi, g)\) an **invariant (Sasakian) submanifold (with or without singularities)** of dimension \(2n - 1\). if \(\xi\) is tangent to \(V\) and \(\Phi TV \subset TV\) at all points of \(V\) and is locally given as the zero set \(\{ f = 0 \}\) of a locally defined basic \(CR\) holomorphic function \(f\). In general, \(V\) may not be a submanifold. Denote by \(V^{reg}\) the set of points \(p \in V\) for which \(V\) is a submanifold of \(M\) near \(p\).

We say that \(V\) is irreducible if \(V^{reg}\) is connected. A transverse divisor \(D^T\) on \(M\) is a formal finite sum \(\sum \alpha_i V_i\) where \(\alpha_i \in \mathbb{Z}\) and each \(V_i\) is an irreducible invariant submanifold of dimension \(2n - 1\). We say that \(D^T\) is effective if the \(\alpha_i\) are all nonnegative. The support of \(D^T\) is the union of the \(V_i\) for each \(i\) with \(\alpha_i \neq 0\).

(ii) Given a transverse divisor \(D^T\), we define an associated line bundle as follows. Suppose that \(D^T\) is given by local defining basic functions \(f_\alpha\) (vanishing on \(D^T\) to order 1) over an open cover \(\mathcal{U}_\alpha\). Define transition functions \(f_\alpha = t_{\alpha \beta} f_\beta\) on \(\mathcal{U}_\alpha \cap \mathcal{U}_\beta\). These are basic \(CR\) holomorphic and nonvanishing in \(\mathcal{U}_\alpha \cap \mathcal{U}_\beta\), and satisfy

\[
t_{\alpha \beta} t_{\gamma \alpha} = 1; \quad t_{\alpha \beta} t_{\gamma \beta} = t_{\alpha \gamma}.
\]

Write \([D^T]\) for the associated basic line bundle, which is well-defined independent of choice of local defining functions.

(iii) One can define

\[
(A.1) \quad L_M \cdot V = \int_V R^T_h \wedge \eta
\]

for all invariant Sasakian 3-manifold \(V\) in \(M\). \(h\) is a basic Hermitian metric on the basic line bundle \(L_M\). From (ii), for a compact Sasakian 5-manifold \(M\), a transverse divisor \(D^T\) defines an element of \(H^{1,1}_{\partial B}(M, \mathbb{R})\) by \(D^T \to [R^T_h] \in H^{1,1}_{\partial B}(M, \mathbb{R})\) for a basic Hermitian metric on the associate basic line bundle \([D^T]\), and we define

\[
\alpha \cdot \beta = \int_M \alpha \wedge \beta \wedge \eta
\]

for \(\alpha, \beta \in H^{1,1}_{\partial B}(M, \mathbb{R})\). Then for an invariant 3-manifold \(V_i\) which is both a foliation curve and a transverse divisor, the \(V \cdot V\) is well-defined and we may write \(V^2\) instead of \(V \cdot V\).

**Remark 2.** (Gei) Note that the Sasakain 3-manifold \(V\) is either canonical, anticanonical or null. \(V\) is up to finite quotient a regular Sasakian 3-manifold, i.e., a
circle bundle over a Riemann surface of positive genus. In the positive case, $V$ is covered by $S^3$ and its Sasakian structure is a deformation of a standard Sasakian structure.

**Definition 18.** (i) We say that a basic line bundle $L$ is nef if $L \cdot V \geq 0$ for any invariant Sasakian 3-manifold $V$ in $M$. In particular if $M$ is quasi-regular, then $V$ is the $S^1$-oribundle over the curve $C$ in $Z$ so that

$$L_Z \cdot C = \int_C R_{h_Z} \geq 0.$$  

Here $c^B(L_M) = \pi^*c_1^{orb}(L_Z)$ and $h_Z$ is the hermitian metric in the corresponding line bundle $L_Z$. Define

$$C^B_M = \{[\alpha]_B \in H^1_B(M, \mathbb{R}) \mid \exists \omega > 0 \text{ such that } [\omega]_B = [\alpha]_B\}.$$  

Then we can also define a class $[\alpha]_B$ called nef class if $[\alpha]_B \in C^B_M$ and a class $[\alpha]_B$ called big if

$$\int_M \alpha^n \wedge \eta > 0.$$  

(ii) If the Sasakian manifold $(M^{2n+1}, \eta, \xi, \Phi, g)$ has the canonical basic line bundle $K^T_M$ nef, then we say that $M$ is a smooth transverse minimal model. If $M$ has $K^T_M$ big, then we say that $M$ is of general type.

**A.2. Minimal Model Program on Compact Quasi-Regular Sasakian 5-Manifolds.** We will focus on the proof of foliation minimal model program on a compact quasi-regular Sasakian 5-manifold with the foliation singularitie of type $\frac{1}{r}(1, a)$.

Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and the space $Z$ of leaves be normal projective orbifold with finite cyclic quotient singularities of type $\frac{1}{r}(1, a)$ which are klt singularities. It follows from the first structure theorem for Sasakian structures that

$$K^T_M = \pi^*(K^{orb}_Z).$$

On the other hand, the leave space $Z$ is well-formed and the orbifold canonical divisor $K^{orb}_Z$ and canonical divisor $K_Z$ are the same, then via the $S^1$-orbibundle

$$\pi : (M, g) \to (Z, \omega),$$

we have the following Sasaki analogue of basepoint-free theorem, rationality theorem, cone and contraction theorem ([KM], [KMM], [M]):

**Proposition 8.** (Foliation Base-point free Theorem) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and the space $Z$ of leaves be normal projective orbifold with finite cyclic quotient singularities of type $\frac{1}{r}(1, a)$ which are klt singularities. Suppose that $L^T$ is a nef Cartier basic line bundle and $aL^T - K^T_M$ is nef and big Cartier basic line bundle for some $a \in \mathbb{N}$. Then $[mL^T]$ is transverse basepoint-free for $m >> 0$. More precisely, there exists a $S^1$-equivariant foliation basepoint-free holomorphic map

$$\Psi_{[mL^T]} : M \to \mathcal{P}(H^0_B(M, (L^T)^m))$$
which is $S^1$-equivariant with respect to the weighted $\mathbb{C}^*$ action in $\mathbb{C}^{N+1}$ with $N = \dim H^0(M, L_m) - 1$.

**Proof.** Note that the leave space $Z$ is a normal projective orbifold surface with finite cyclic quotient singularities of type $\frac{1}{r}(1, a)$ which are $klt$, Suppose that $L^T$ is a nef Cartier basic line bundle and $aL^T - K^T_M$ is nef and big Cartier basic line bundle for some $a \in \mathbb{N}$. On the other hand by applying Proposition 1, there exists a Riemannian submersion, $S^1$-orbibundle $\pi : M \to Z$, such that

$$K^T_M = \pi^*(K^\text{orb}_Z) = \pi^*(K_Z)$$

and

$$\pi^*(L) = L^T.$$

Then $L$ is a nef Cartier line bundle and $aL - K_Z$ is nef and big Cartier line bundle. Therefore by Kawamata base-point free theorem, there is a basepoint-free holomorphic map

$$\psi_{|mL|} : Z \to \mathcal{P}(H^0(Z, L^m)).$$

Define

$$\Psi_{|mL^T|} = \psi_{|mL|} \circ \pi$$

such that

$$\Psi_{|mL^T|} : M \to \mathcal{P}(H^0_B(M, (L^T)^m)).$$

It follows that $\Psi_{|mL^T|}$ is a $S^1$-equivariant foliation basepoint-free holomorphic map with respect to the weighted $\mathbb{C}^*$ action in $\mathbb{C}^{N+1}$ with $N = \dim H^0(M, L_m) - 1$ for a large positive integer $m$. □

**Proposition 9.** (Rationality Theorem) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and the space $Z$ of leaves be normal projective orbifold with finite cyclic quotient singularities of type $\frac{1}{r}(1, a)$ which are $klt$ singularities. Suppose that $K^T_M$ is not nef. Let $a(Z)$ be an integer such that $a(Z)K^T_M$ is Cartier. Let $H^T$ be a nef and big Cartier basic divisor and define

$$r = \sup\{t \in \mathbb{R} : H^T + tK^T_M \text{ is nef} \}.$$

Then $r$ is a rational number of the form $\frac{p}{q}$ with

$$0 < q \leq a(Z)(\dim Z + 1).$$

A basic 1-cycle $V$ on $M$ is a formal finite sum $V = \sum a_i V_i$, for $a_i \in \mathbb{Z}$ and $V_i$ is the irreducible invariant Sasakian 3-manifold. We denote by $N_1(M)_\mathbb{Z}$ the space of 1-cycles modulo numerical equivalence. Write

$$N_1(M)_\mathbb{Q} = N_1(M)_\mathbb{Z} \otimes \mathbb{Q} \quad \text{and} \quad N_1(M)_\mathbb{R} = N_1(M)_\mathbb{Z} \otimes \mathbb{R}.$$

Then write $NE(M)$ for the cone of effective elements of $N_1(M)_\mathbb{R}$ and $\overline{NE(M)}$ for its closure. Furthermore, a basic divisor $D^T$ is ample if and only if

$$D^T \cdot V > 0$$

for all nonzero $V \in \overline{NE(M)}$. It is the Kleiman criterion for the ample line bundle. Also we define the Picard number

$$\rho(M) := \dim N_1^1(M) \leq \dim H^2_B(M, \mathbb{R}) < \infty.$$
Proposition 10. (Foliation Contraction Theorem) Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian 5-manifold and the space \(Z\) of leaves be normal projective orbifold surface with finite cyclic quotient singularities of type \(\frac{1}{r}(1, a)\) which are klt singularities. Then there exists a countable collection of Sasakain 3-spheres \(\{V_i\}\) such that

\[
0 < -K^T_M \cdot V_i \leq 2 \dim Z
\]

and

\[
\text{NE}(M) = \text{NE}(M)_{K^T_M \geq 0} + \sum a_i |V_i|, \quad a_i \geq 0.
\]

The rays \(a_i |V_i|\) are locally discrete in the half space \(\{K^T_M < 0\}\). If \(R \in \text{NE}(M)\) is a \(K^T_M\)-negative foliation extremal ray such that

\[
R = \text{NE}(M)_{K^T_M < 0} \cap (L^T)^\perp
\]

for some nef basic line bundle \(L^T\) which can be chose by

\[
\pi^* L = L^T
\]

for a nef line bundle \(L\) over \(Z\). Then there is a unique foliation extremal ray contraction

\[
\psi_{(L^T)m} = \text{cont}_V : M \to N
\]

for some \(m \gg 1\) such that an irreducible foliation curve \(V \subset M\) is mapped to a leave \(S^1\) by \(\psi\) if and only if \([V]_B \in R\). Furthermore, \(L^T = \psi^* A\) for some basic ample line bundle on \(N\).

Proof. Since \(Z\) is a normal projective orbifold surface with finite cyclic quotient singularities of type \(\frac{1}{r}(1, a)\) which are klt singularities, it follows ([KM]) that there exists a nef and semi-ample line bundle \(L\) over \(Z\) such that if \(R_Z \in \text{NE}(Z)\) is a \(K^Z\)-negative extremal ray with

\[
R_Z = \text{NE}(Z)_{K^Z < 0} \cap (L)^\perp.
\]

Then there is a unique extremal ray contraction

\[
\psi_{(L)m} = \text{cont}_C : Z \to Y \subset \mathcal{P}(H^0(Z, mL))
\]

for some \(m \gg 1\) such that an irreducible curve \(C \subset Z\) is mapped to a point by \(\psi_{(L)m}\) if and only if \([C] \in R_Z\).

Now for a Riemannian submersion, \(S^1\)-orbibundle \(\pi : M \to Z\), we choose \(L^T\) such that

\[
\pi^* L = L^T
\]

ann thus \(L^T\) is a basic nef and semi-ample line bundle over \(M\) such that there is a unique \(S^1\)-equivariant foliation extremal ray contraction

\[
\psi_{(L^T)m} = \text{cont}_V : M \to N \subset \mathcal{P}(H^0_B(M_k, (L^T)^m))
\]

with

\[
\psi_{(L^T)m} = \psi_{(L)m} \circ \pi
\]

for which an irreducible foliation curve \(V \subset M\) is mapped to a leave \(S^1\) by \(\psi_{(L^T)m}\) if and only if

\[
[V]_B \in R = \text{NE}(M)_{K^M < 0} \cap (L^T)^\perp.
\]

\(\square\)
Now we are ready to prove the following foliation minimal model program:

**Proposition 11.** Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian 5-manifold with finite cyclic quotient foliation singularities of type \(\frac{1}{r}(1, a)\). Then there exists a finite sequence of foliation extremal ray contractions

\[
\psi_i : M_{i-1} \to M_i, \quad i = 1, \ldots, k
\]
such that every \(M_i\) is a Sasakian manifold having at worst foliation cyclic quotient singularities and for every \(\psi_i\) one of the following holds:

1. **Foliation divisorial contraction:** (The locus of the foliation extremal ray is an irreducible basic divisor): \(\psi_i\) is a foliation divisorial contraction of a foliation curve \(V\) with \(V^2 < 0\), and the Picard number satisfies
   \[
   \rho(M_i) = \rho(M_{i-1}) - 1.
   \]

2. **Foliation fibre contraction** (transverse Mori fibre space): (The locus of the foliation extremal ray is \(M_{i-1}\)):
   \(\psi_i\) is a singular fibration such that either
   - (a) there is a map \(f : M_k \to \text{pt}\), then \(K_T^{M_k} < 0\) and thus \(M_k\) is transverse minimal Fano and the leave space \(Z_k\) is minimal log del Pezzo surface of \(\frac{1}{r}(1, a)\)-type singularities and Picard number one.
   - (b) \(f : M_k \to \Sigma_h\), then \(Z_k\) is a rule surface over Riemann surfaces \(\Sigma_h\) of genus \(h\) and thus \(M_k\) is an \(S^1\)-orbibundle over \(Z_k\). Or
   (3) \(M_k\) is nef: \(\psi_i = \psi_k\), and \(M_k\) has at worst foliation cyclic quotient singularities and has no foliation \(K_T^{M_i}\)-negative curves.

**Proof.** Let \(S^1_p\) be the singular fibre of type \(\frac{1}{r}(1, a), p \in M\). Let \(\psi : M \to N\) be a proper transverse birational morphism. By applying Theorem 6 and [Cu], we consider the minimal resolution of foliation singularities of \(M\) to be \(\phi : M^\prime \to M\) and \(\phi^\prime : N^\prime \to N\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
(M, \Gamma) & \xrightarrow{\psi} & (N, S^1_q) \\
\downarrow \phi & \circ & \downarrow \phi^\prime \\
(M, \Gamma, S^1_p) & \xrightarrow{\psi^\prime} & (N, S^1_q).
\end{array}
\]

Since \(\overline{M}\) and \(\overline{N}\) are regular Sasakian manifolds, by Theorem 2 the transverse birational morphism \(\overline{\psi}\) can be factored into a sequence of ordinary blow ups. Suppose \(\overline{M}\) and \(\overline{N}\) are not isomorphic, there must exist a foliation \((-1)\)-curve \(\Gamma\) such that \(\overline{\phi^\prime} \circ \overline{\psi}\(\Gamma) = S^1_q\), for \(q \in N\). Let \(\Theta\) be a set of foliation curves such that \(\Theta = \overline{\psi} \circ \overline{\phi^\prime}^{-1}(S^1_q)\) and \(D_T = \{V_i\}\) be the exceptional locus of \(\phi\) at the singular fibre \(S^1_p\). Then the Hirzebruch Jung continued fraction

\[
\frac{r}{a} = [b_1, \ldots, b_l]
\]
say that

\[
V_i^2 = -b_i.
\]
(i) If \( V_i \cdot \bar{\Gamma} = \emptyset \), for all \( i \), then \( \bar{\Gamma} \) is not exceptional for \( \varphi \) and its birational transform \( \Gamma \) is a floating foliation \((-1)\)-curve on \( M \). Then by Theorem 2 we obtain a transverse birational morphism \( \psi' : M \to M_1 \) corresponding to the blow up of a smooth point with exceptional locus given by \( \Gamma \). Thus \( \psi \) factors through \( \psi' \) and we obtain a transverse birational morphism \( M_1 \to N \).

(ii) If \( V_j \cdot \bar{\Gamma} \neq \emptyset \) for some \( j \) (at most one such a foliation exceptional curve), then \( V_j^2 = -b_j \) and \( \Gamma \) is passing the singular fibre \( S^1_p \) of the foliation singularities of type \(-1\) and \( K^M_1 \cdot \Gamma < 0 \). Moreover, \( \bar{\Gamma} \) is also a foliation \((-1)\)-curve on \( M \). Again by Theorem 2, we obtain a transverse birational morphism \( \psi'' : M \to M_1 \) corresponding to the transverse extremal ray contraction of \( \bar{\Gamma} \) into a singular fibre of foliation singularities of at worst type \( -\frac{1}{r_{M_1}}(1, a_{M_1}) \) such that

\[
\frac{r_{M_1}}{a_{M_1}} = [b_1, \ldots, (b_i - 1), \ldots, b_l].
\]

Again \( \psi \) factors through \( \psi' \) and we obtain a transverse birational morphism \( M_1 \to N \). Similar method can be applied to a finite number of cyclic quotient foliation singularities.

On the other hand, since for divisorial extremal contraction, we have \( \rho(M_i) = \rho(M_{i-1}) - 1 \), then after a finite number of foliation extremal ray contractions, we end up

\[
M = M_0 \xrightarrow{\psi_1} M_1 \xrightarrow{\psi_2} M_2 \xrightarrow{\psi_3} \cdots \xrightarrow{\psi_h} M_k
\]

with \( M_k \) nef or

by [11, Lemma 2.1.27] and \( L^T \) is semi-ample over \( M_k \) (see Proposition [10] to see that there exist a morphism

\[
\lambda : Z_k \to \mathcal{P}(H^0(Z_k, mL))
\]

such that \( f = \lambda \circ \pi \) is a \( S^1 \)-equivariant transverse morphism

\[
f : M_k \to \mathcal{P}(H^0_B(M_k, mL^T)),
\]

with \( L \) is semi-ample over \( Z_k \) for \( \pi^*L = L^T \).

Then \( Y := h(Z_k) \) is either a point or a nonsingular curve.

(a). If \( Y \) is a point, then \( K_{\text{orb}}^r(Z_k) < 0 \) and \( K_{\text{Mk}}^T < 0 \). Hence \( M_k \) is transverse Fano.

(b). If \( Y = \Sigma_h \) is a non-singular curve, then a generic fiber \( C \) of \( \lambda \) is a smooth curve and then foliation curve \( V \) over \( C \) satisfying \( L_k \cdot C = 0 \) and \( K_{\text{orb}}^r Z_k \cdot C < 0 \) as well as \( L_k^T \cdot V = 0 \) and \( K_{\text{Mk}}^T \cdot V < 0 \). It follows from the adjunction formula that \( C \) is a rational curve and thus \( V \) is covered by the three sphere \( S^3 \). Hence \( Z_k \) is a rule surface over Riemann surfaces \( \Sigma_h \) of genus \( h \) and thus \( M_k \) is an \( S^1 \)-orbibundle over \( Z_k \).

Remark 3. Let \((Z, 0)\) be a log del Pezzo orbifold surface with no branch divisors (well-formed). It follows from [BG, Theorem 4.7.14] that the total space of an \( S^1 \)-orbibundle over \( Z \) is diffeomorphic to some \( k(S^2 \times S^3) \) for some \( k = 0, 1, \ldots \) \( (k = 0 \) means \( S^5 \)).

Corollary 4. Let \((M, \eta, \xi, \Phi, g)\) be a compact regular Sasaki 5-manifold. Then there exists a transverse morphism \( \psi : M \to M' \), which is a composition of blowing
down foliation \((-1)\)-curves and a morphism \(\varphi : M' \to N\) such that one of the following holds:

1. **Nef**: \(M' \cong N\) is a compact regular Sasakian 5-manifold with \(K_T^N\) nef;
2. **Mori fibre space**:
   - (a) \(N\) is a Riemann surface \(\Sigma_h\) and \(M' = \Sigma_h \times \mathbb{S}^3\) or \(X_{\infty, h} = \Sigma_h \tilde{\times} \mathbb{S}^3\) or
   - (b) \(N\) is a point and \(M'\) is isomorphic to \(\mathbb{S}^5\).

As a consequence, we have

**Corollary 5.** Let \((M, \eta, \xi, \Phi, g)\) be a compact simply connected regular Sasakian 5-manifold.

1. If the Sasakian structure is positive, then \(M = \mathbb{S}^5\) (\(k = 0\)); or \(k(\mathbb{S}^2 \times \mathbb{S}^3)\) or \(N = X_{\infty, 0} \# (k - 1)(\mathbb{S}^2 \times \mathbb{S}^3), 1 \leq k \leq 8\).
2. If the Sasakian structure is indefinite (\(K_T^M\) is nef), then \(M = k(\mathbb{S}^2 \times \mathbb{S}^3)\); or \(M = X_{\infty, 0} \# (k - 1)(\mathbb{S}^2 \times \mathbb{S}^3), 1 \leq k\).

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