ON THE GALOIS GROUPS OF THE DUALIZING COVERINGS FOR PLANE CURVES

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Abstract. Let $C_1$ be an irreducible component of a reduced projective curve $C \subset \mathbb{P}^2$ defined over the field $\mathbb{C}$, $\deg C_1 \geq 2$, and let $T$ be the set of lines $l \subset \mathbb{P}^2$ meeting $C$ transversally. In the article, we prove that for a line $l_0 \in T$ and any two points $P_1, P_2 \in C_1 \cap l_0$ there is a loop $l_t \subset T$, $t \in [0, 1]$, such that the movement of the line $l_0$ along the loop $l_t$ induces the transposition of the points $P_1, P_2$ and the identity permutation of the other points of $C \cap l_0$.

Introduction

Let $C_i \subset \mathbb{P}^2$, $1 \leq i \leq k$, be irreducible reduced curves defined over the field $\mathbb{C}$, $\deg C_i = d_i \geq 2$, and $C = C_1 \cup \cdots \cup C_k$, $d = \deg C = d_1 + \cdots + d_k$. Denote by $\nu : \overline{C} \to C$ the normalization of the curve $C$ and consider a point $p \in C$ and its image $P = \nu(p) \in \mathbb{P}^2$. Choose homogeneous coordinates $(x_1, x_2, x_3)$ in $\mathbb{P}^2$ such that $P = (0, 0, 1)$. We can choose a local parameter $t$ in a complex analytic neighborhood $U \subset \overline{C}$ of the point $p$ such that the regular map $\nu$ is given by

$$x_1 = \sum_{i=s_p}^{\infty} a_it^i, \quad x_2 = t^{s_p}, \quad x_3 = 1,$$  \hspace{1cm} (1)

where $a_{s_p} \neq 0$ and $s_p > r_p \geq 1$. The integer $r_p$ is called the multiplicity of the germ $\nu(U)$ of the curve $C$ at $P = \nu(p)$, the line $l_p = \{x_1 = 0\}$ is called a tangent line to $C$ at $P$, and the integer $s_p$ is called the tangent multiplicity of the germ $\nu(U)$ at $P$.

Let $\hat{C} \subset \hat{\mathbb{P}}^2$ be the dual curve to the curve $C$ (the curve $\hat{C}$ consists of the tangents $l_p$, $p \in \overline{C}$, to $C$). The graph of the correspondence between $C$ and $\hat{C}$ is a curve $\hat{C}$ (the so called Nash blow-up of $C$) in $\mathbb{P}^2 \times \hat{\mathbb{P}}^2$ which lies in the incidence variety $I = \{(P, l) \in \mathbb{P}^2 \times \hat{\mathbb{P}}^2 \mid P \in l\}$,

$$\hat{C} = \{(\nu(p), l_p) \in I \mid p \in C \text{ and } l_p \text{ is the tangent line to } C \text{ at } \nu(p) \in C\}.$$

In the sequel, by $L_p \subset \hat{\mathbb{P}}^2$, we denote the line dual to the point $\nu(p) \in C \subset \mathbb{P}^2$.

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Let \( \text{pr}_1 : \mathbb{P}^2 \times \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) and \( \text{pr}_2 : \mathbb{P}^2 \times \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2 \) be the projections to the factors, \( X = \text{pr}_1^{-1}(C) \cap I \), and \( f' : X \to \tilde{\mathbb{P}}^2 \) the restriction of \( \text{pr}_2 \) to \( X \). Obviously, \( f'^{-1}(l) \) consists of the points \((P, l) \in \mathbb{P}^2 \times \tilde{\mathbb{P}}^2 \) such that \( P \in C \cap l \) and hence \( \deg f' = \deg C = d \).

Denote by \( \nu' : Z \to X \) the normalization of \( X \) and by \( f = f' \circ \nu' : Z \to \tilde{\mathbb{P}}^2 \). We have \( \deg f = d \). We call \( f \) the dualizing covering for \( C \subset \mathbb{P}^2 \). Obviously, the variety \( Z \) is isomorphic to the fibre product \( \tilde{C} \times_C X \) of the normalization \( \nu : \tilde{C} \to C \) and the projection \( \text{pr}_1 : X \to C \). The projection \( \text{pr}_1 : X \to C \) gives on \( X \) a structure of a ruled surface and it induces a ruled structure on \( Z \) over the curve \( \tilde{C} \), \( \rho : Z \to \tilde{C}, \rho^{-1}(p) := F_p \simeq \mathbb{P}^1 \) for \( p \in \tilde{C} \). Note that the curve \( \tilde{C} \) is a section of this ruled structure, where \( \tilde{C} = \nu'^{-1}(\tilde{C}) \subset Z \), and the image \( f(F_p) \) of a fibre \( F_p \) is the line \( L_p \subset \tilde{\mathbb{P}}^2 \) dual to the point \( \nu(p) \in C \subset \mathbb{P}^2 \).

Denote by \( \overline{\mathcal{B}} \subset \tilde{\mathbb{P}}^2 \) the branch locus of \( f \), choose a point \( l_0 \in \tilde{\mathbb{P}}^2 \setminus \overline{\mathcal{B}} \), and number the points of \( f^{-1}(l_0) \). In this case the covering \( f \) induces a homomorphism \( f_* : \pi_1(\tilde{\mathbb{P}}^2 \setminus \overline{\mathcal{B}}, l_0) \to \Sigma_d \) from the fundamental group \( \pi_1(\mathbb{P}^2 \setminus \overline{\mathcal{B}}, l_0) \) to the symmetric group \( \Sigma_d \) acting on the fibre \( f^{-1}(l_0) \). The image, \( Im f_* := G \subset \Sigma_d \), is called the Galois group of the covering \( f \).

**Theorem 1.** Let \( f : Z \to \tilde{\mathbb{P}}^2 \) be the dualizing covering for a reduced curve \( C \subset \mathbb{P}^2 \), \( \deg C = d \), and \( C_1, \ldots, C_k \) the irreducible components of \( C \), \( \deg C_i = d_i \geq 2 \). Then the Galois group of \( f \) is \( G \simeq \Sigma_{d_1} \times \cdots \times \Sigma_{d_k} \).

The following theorem describes properties of dualizing coverings.

**Theorem 2.** Let \( C \) be as in Theorem 1 and \( f : Z \to \tilde{\mathbb{P}}^2 \) the dualizing covering for \( C \). Then \( Z \) is a non-singular surface consisting of \( k \) irreducible components and \( f \) is a degree \( d \) finite covering.

The branch locus of \( f \) is \( \overline{\mathcal{B}} = \tilde{C} \cup \tilde{\mathcal{L}} \), where \( \tilde{\mathcal{L}} = \bigcup_{r_p \geq 2} L_p \), \( L_p \) are the lines dual to the points \( \nu(p) \in C \) and the union is taken over all \( p \in \tilde{C} \) for which the multiplicity \( r_p \geq 2 \).

The ramification locus of \( f \) is \( \overline{\mathcal{R}} = \tilde{C} \cup \tilde{\mathcal{F}} \), where \( \tilde{\mathcal{F}} = \bigcup_{r_p \geq 2} F_p \) and the union is taken over all \( p \in \tilde{C} \) for which \( r_p \geq 2 \).

The local degree \( \deg_q f \) of \( f \) at a point \( q = F_p \cap \tilde{C} \) is equal to the tangent multiplicity \( s_p \), and \( \deg_q f = r_p \) at all points \( q \in F_p \setminus \tilde{C} \). For all points \( q \in \tilde{C} \setminus \tilde{\mathcal{F}} \), the local degree \( \deg_q f = 2 \).

For given reduced projective curve \( C \subset \mathbb{P}^2 \), \( \deg C = n \), let \( T_C \) be the set of lines \( l \subset \mathbb{P}^2 \) meeting \( C \) transversally. Let \( l_t \subset T_C, t \in [0, 1] \), be a loop and let \( l_0 \cap C = \{P_1, \ldots, P_n \} \). Then the movement of the line \( l_0 \) along the loop \( l_t \) defines \( n \) paths \( \psi_i(t) = l_t \cap C \subset \mathbb{P}^2, i = 1, \ldots, n \), starting and ending at the points.
$P_1, \ldots, P_n$ and, consequently, induces a permutation of the points $P_1, \ldots, P_n$ called the monodromy of the points $P_1, \ldots, P_n$ along the loop $l_t$ (the start point $P_i = \psi_i(0)$ of the path $\psi_i(t)$ maps to the end point $\psi_i(1) \in l_0 \cap C$).

**Corollary 1.** Let $C_1$, $\deg{C_1} \geq 2$, be an irreducible component of a reduced curve $C \subset \mathbb{P}^2$. For a line $l_0 \subset T_C$ and any two points $P_1, P_2 \in C_1 \cap l_0$ there is a loop $l_t \subset T_C$, $t \in [0, 1]$, such that the monodromy along the loop $l_t$ is the transposition of the points $P_1, P_2$ and the identity permutation of the other points in $C \cap l_0$.

The proof of Theorems 1, 2 and Corollary 1 will be given in Section 3. In sequel, we will assume that each finite set $I$ is a subset of an integer segment $[1, d] = \{1, 2, \ldots, d\}$, so that $\Sigma_I \subset \Sigma_d$.

Let $J = \{I_1, \ldots, I_k\}$ be a partition of the segment $[1, d]$. The partition $J$ defines an embedding of the group $\Sigma_J := \Sigma_{I_1} \times \cdots \times \Sigma_{I_k}$ into $\Sigma_d$.

We say that a partition $J$ of $[1, d]$ is invariant under the action of a subgroup $G \subset \Sigma_d$ if $g(I_j) = I_j$ for all $g \in G$ and all $I_j \in J$.

Let $J_1 = \{I_{i_1, 1}, \ldots, I_{i_1, k_1}\}$, $i = 1, 2$, be two partitions of $[1, d]$. We say that the partition $J_1$ is thinner than $J_2$ (resp., $J_2$ is thicker than $J_1$) and write $J_1 \preceq J_2$ if for each $j$, $1 \leq j \leq k_1$, there is $t(j)$ such that $I_{j, t(j)} \subset I_{i_2, t(j)}$. For any two partitions $J_i = \{I_{i, 1}, \ldots, I_{i, k_i}\}$, $i = 1, 2$, denote by $J_1 \oplus J_2$ the thinnest partition of $[1, d]$ such that $J_1 \preceq J_1 \oplus J_2$ and $J_2 \preceq J_1 \oplus J_2$.

**Claim 1.** Let $G$ be a subgroup of $\Sigma_d$ and let $J_i = \{I_{i, 1}, \ldots, I_{i, k_i}\}$, $i = 1, 2$, be two partitions of $[1, d]$ such that $\Sigma_{J_i} \subset G$ for $i = 1, 2$. Then $\Sigma_{J_1 \oplus J_2} \subset G$.

**Proof.** Obvious. \hfill $\Box$

It follows from Claim 1 that for each subgroup $G$ of $\Sigma_d$ there is the thickest partition of $[1, d]$ (denote it by $J_G$) such that $\Sigma_{J_G} \subset G$.

Let $\sigma = c_1 \cdot \ldots \cdot c_n \in \Sigma_d$ be the factorization of $\sigma$ into the product of cycles with disjoint orbits. The number $n_\sigma = n$ will be called the length of cycle factorization.
Lemma 1. Let $H$ be a subgroup of $\Sigma_d$ generated by a set of transpositions and a permutation $\sigma$, and let $\sigma = c_1 \cdot \ldots \cdot c_n$ be the factorization of $\sigma$ into the product of cycles with disjoint orbits. Assume that for each $i$, $1 \leq i \leq n$, there is a partition $J_i = \{I_{i,1}, \ldots, I_{i,k_i}\}$ of $[1,d]$ invariant under the action of $\sigma$ and such that

(i) for each $I_{i,j} \in J_i$ there is at most one cycle $c_{m(i,j)}$ entering into the factorization of $\sigma$ such that the cycle $c_{m(i,j)}$ acts non-trivially on $I_{i,j}$,

(ii) the cycle $c_i$ acts non-trivially on $I_{i,1}$ and the length of the cycle $c_i$ is strictly less than the cardinality of $I_{i,1}$,

(iii) the group $H$ acts transitively on $I_{i,1}$.

Then $H = \Sigma_{J_H}$ and, in particular, $H$ is generated by transpositions.

Proof. Consider a set $I_{i,1}$. Let $l_{i,1}$ be its cardinality and let $l_i$ be the length of the cycle $c_i$. We have $l_i < l_{i,1}$ and it follows from (i) and (iii) that there exists a transposition $\tau \in H \cap \Sigma_{I_{i,1}}$ such that it commutes with $c_j$ if $j \neq i$ and it transposes an element entering in the cycle $c_i$ and an element of $I_{i,1}$ which does not enter in $c_i$. Without loss of generality, we can assume that $I_{i,1} = \{1,2,\ldots,l_{i,1}\}$, $c_i = (1,2,\ldots,l_i)$, and $\tau = (l_i, l_i + 1)$. Therefore

$$\sigma^{-j} \tau \sigma^j = (l_i - j, l_i + 1) \in H$$

for $j = 1, \ldots, l_i - 1$ and hence $H \cap \Sigma_{I_{i,1}} = \Sigma_{l_{i,1} + 1}$, since the subgroup of $H$, generated by the transpositions $\sigma^{-j} \tau \sigma^j$, $j = 0,1,\ldots,l_i - 1$, acts transitively on the set $\{1,2,\ldots,l_{i,1} + 1\}$. If we apply conditions (i) - (iii) $l$ times, where $l = l_{i,1} - l_i$, we obtain that $\Sigma_{I_{i,1}} \subset H$ for each $i$ and hence $\sigma$ is a product of some transpositions belonging to $H$. \qed

The following proposition is an easy consequence of Claim $\square$ and Lemma $\square$.

Proposition 1. Let $G$ be a subgroup of $\Sigma_d$ generated by some set of transpositions and by permutations $\sigma_1, \ldots, \sigma_m$. Assume that for each $i$, $1 \leq i \leq m$, there are partitions $J_{i,j}$ of $[1,d]$, $1 \leq j \leq n_{\sigma_i}$, such that the subgroup $H_i$ of $G$, generated by transpositions and by $\sigma_i$, and the partitions $J_{i,j}$ satisfy the conditions of Lemma $\square$. Then $G = \Sigma_{J_G}$.

Corollary 2. Let $G \subset \Sigma_d$ satisfy the conditions of Proposition $\square$. Assume that there is a partition $J = \{I_1, \ldots, I_k\}$ of $[1,d]$ such that $G$ leaves invariant the partition $J$ and acts transitively on each $I_j \in J$, $1 \leq j \leq k$. Then $J = J_G$ and $G = \Sigma_J$. 


2. Coverings

By a covering we understand a branched covering, that is a finite morphism \( f : Z \to Y \) from a normal projective surface \( Z \) onto a non-singular irreducible projective surface \( Y \). To each covering \( f \) we associate the branch locus \( B \subset Y \), the ramification locus \( R \subset f^{-1}(B) \subset Z \), and the unramified part \( Z \setminus f^{-1}(B) \to Y \setminus B \) (which is the maximal unramified subcovering). As is usual for unramified coverings of degree \( d \), there is a homomorphism \( f_* \) which acts from the fundamental group \( \pi_1(Y \setminus B, p_0) \) to the symmetric group \( \Sigma_d \) acting on the points of \( f^{-1}(p_0) \). The homomorphism \( f_* \) (called monodromy of \( f \)) is defined by \( f \) uniquely if we number the points of \( f^{-1}(p_0) \); reciprocally, according to Grauert-Remmert-Riemann-Stein Extension Theorem (see, for example, [2]) the conjugacy class of \( f_* \) defines \( f \) up to an isomorphism. The image \( G \subset \Sigma_d \) of \( f_* \) is a transitive subgroup of \( \Sigma_d \) if \( Z \) is irreducible and in general case the number of connected components of \( Z \) is equal to the number of orbits of the action of \( G \) on \( f^{-1}(p_0) \).

An element \( \gamma_q, q \in B \setminus \text{Sing}B \), of the fundamental group \( \pi_1(Y \setminus B, p_0) \) is called a geometric generator if it is represented by a loop \( \Gamma_q \) of the following form. To define \( \Gamma_q \), let \( L \subset Y \) be a curve meeting \( B \) transversely at \( q \) and let \( S^1 \subset L \) be a circle of small radius with center at \( q \). The choice of an orientation on \( Y \) defines an orientation on \( S^1 \). Then \( \Gamma_q \) is a loop consisting of a path \( l \) in \( Y \setminus B \) joining \( p_0 \) with a point \( q_1 \in S^1 \), the loop \( S^1 \) (with positive direction) starting and ending at \( q_1 \), and a return path to \( p_0 \) along \( l \) in the opposite direction (of course, we must note that a geometric generator \( \gamma_q \) depends not only on \( q \), but it depends also on the choice of the path \( l \)). Note that if \( Y \) is simply connected then \( \pi_1(Y \setminus B, p_0) \) is generated by geometric generators.

In the sequel, we will assume that the covering \( f \) satisfies some additional conditions. The first of them is:

\((R_0)\) If for an irreducible component \( B_i \) of the branch curve \( B \) the image \( f_*(\gamma_{q_i}) \) of a geometric generator \( \gamma_{q_i} \in \pi_1(Y \setminus B_i, p_0) \), \( q_i \in B_i \), is not a transposition, then \( B_i \) is a smooth curve and \( f|_{\overline{R}_{i,j}}: \overline{R}_{i,j} \to B_i \) is an isomorphism for all \( j, 1 \leq j \leq n \), where \( \overline{R} \cap f^{-1}(B_i) = \overline{R}_{i,1} \cup \ldots \cup \overline{R}_{i,n} \) is the decomposition of \( \overline{R} \cap f^{-1}(B_i) \) into the union of irreducible components.

Let \( r_{i,j} \) be the ramification multiplicity of \( f \) along \( \overline{R}_{i,j} \) (that is, the local degree of \( f \) at a generic point of \( \overline{R}_{i,j} \)), then the cycle type of the permutation \( f_*(\gamma_{q_i}) \in \Sigma_d \) is \((r_{i,1}, \ldots, r_{i,n})\).

Let for \( B_1 \) the image \( f_*(\gamma_{q_1}) \) be not a transposition and let \( \overline{R}_1, \ldots, \overline{R}_m \) be the irreducible components of \( \overline{R} \cap f^{-1}(B_1) \). For each point \( o \in \overline{B}_1 \cap \text{Sing}B \) let us choose a very small (in complex analytic topology) neighbourhood \( W_o \subset Y \) of
the point $o$. Denote by $B_1 := \overline{B_1} \setminus \bigcup_{o \in \text{Sing}\overline{B}} W_o$ and $R_j := \overline{R_j} \cap f^{-1}(B_1)$, where $\overline{W_o}$ is the closure of $W_o$ in $Y$. The following Lemma is well known.

**Lemma 2.** There are neighbourhoods $U_1 \subset Y$ and $V_j \subset Z$, $j = 1, \ldots, n$, such that

(i) $U_1 \cap \overline{B} = B_1$ and $V_j \cap \overline{R_j} = R_j$,

(ii) $U_1$ is biholomorphic to $B_1 \times D_1$ and $V_j$ is biholomorphic to $R_j \times D_2$, where $D_1 = \{u_1 \in \mathbb{C} \mid |u_1| < 1\}$ and $D_2 = \{v_1 \in \mathbb{C} \mid |v_1| < 1\}$ are discs in $\mathbb{C}$,

(iii) the restriction of $f$ to each $V_j$ is proper and $f(V_j) = U_1$,

(iv) if $u_2$ is a local parameter on $B_1$ at a point $p \in B_1$ and $v_{2,j} = f_{|R_j}^{-1}(u_2)$ is the local parameter at the point $q_j = f_{|R_j}^{-1}(p)$ on $R_j$, then $f : V_j \to U_1$ is given by $u_1 = v_1^{2i}$ and $u_2 = v_{2,j}$ in a neighbourhood of the point $q_j$.

Consider a neighbourhood $U_1$ the existence of which is claimed in Lemma 2. Let $\text{pr} : U_1 \to B_1$ be the projection defined by bi-holomorphic isomorphism $U_1 \simeq B_1 \times D_1$. Let us choose a point $q_1 \in B_1$, a point $p_1 \in \text{pr}^{-1}(q_1) \simeq D_1$ lying in the circle $\delta = \{q_1\} \times \{u_1 \in \mathbb{C} \mid |u_1| = \frac{1}{2}\} \subset \{q_1\} \times D_1 \subset U_1 \setminus B_1$ (let, for definiteness, $u_1 = \frac{1}{2}$ at the point $p_1$), and a path $l_1 \subset Y \setminus \overline{B}$ connecting the points $p_0$ and $p_1$. The choice of $l_1$ defines homomorphisms $\text{im}_{l_1} : \pi_1(U_1 \setminus B_1, p_1) \to \pi_1(Y \setminus \overline{B}, p_0)$ and $\varphi_1 = f_* \circ \text{im}_{l_1} : \pi_1(U_1 \setminus B_1, p_1) \to \Sigma_d$. Denote by $H_{B_1}$ the image $\varphi_1(\pi_1(U_1 \setminus B_1, p_1))$ in $G$. Let $\gamma_{q_1} \in \pi_1(U_1 \setminus B_1, p_1)$ be a geometric generator represented by the circle $\delta$ (the circuit in positive direction). The cycle type of the permutation $\sigma_1 = \varphi_1(\gamma_{q_1})$ is $(r_{1,1}, \ldots, r_{1,n})$.

Let a set $S = \{o_1, \ldots, o_n\}$ be the intersection of $\overline{B_1}$ and $\text{Sing}\overline{B}$. For a point $o_i \in S$ we choose a small (in complex analytic topology) simply connected neighbourhood $U_{o_i} \subset Y$ of the point $o_i$ such that the number $k_i$ of the connected components $V_{o_i,1}, \ldots, V_{o_i,k_i}$ of $f^{-1}(U_{o_i})$ is equal to the number of points belonging to $f^{-1}(o_i)$. In addition, $U_{o_i}$ can be chosen so that $\overline{W_{o_i}} \subset U_{o_i}$, where $W_{o_i}$ is the neighbourhood of $o_i$ used in the definition of the neighbourhood $U_1$.

Let us choose points $q_{o_i} \in B_1 \cap U_{o_i}$ and paths $l'_{o_i} \subset B_1$ connecting, respectively, the point $q_1$ with the points $q_{o_i}$. Let $l_{o_i} = \{p \in U_1 \setminus B_1 \mid \text{pr}(p) \in l'_{o_i}, u_1(p) = \frac{1}{2}\}$ be paths connecting the point $p_1$, respectively, with points $p_{o_i} = \text{pr}^{-1}(q_{o_i}) \cap U_1 \cap U_{o_i}$ (without loss of generality, we can assume that $\text{pr}^{-1}(q) \subset U_{o_i}$ if $q \in B_1 \cap U_{o_i}$). Denote by $\tilde{l}_{o_i}$ the composition of paths $l_1$ and $l_{o_i}$ connecting the point $p_0$ with the point $p_{o_i}$.

The path $\tilde{l}_{o_i}$ defines homomorphisms $\text{im}_{\tilde{l}_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \to \pi_1(Y \setminus \overline{B}, p_0)$ and $\varphi_{l_{o_i}} = f_* \circ \text{im}_{\tilde{l}_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \to \Sigma_d$. Denote by $H_{o_i}$ the image $\varphi_{l_{o_i}}(\pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}))$ in $G$. 
Similarly, if $U = U_1 \cup (\bigcup_{o_i \in S} U_{o_i})$, then the path $l_1$ defines homomorphisms $\text{im}_{l_1} : \pi_1(U \setminus (U \cap \overline{B}), p_1) \to \pi_1(Y \setminus \overline{B}, p_0)$ and $\varphi = f_* \circ \text{im}_{l_1} : \pi_1(U \setminus (U \cap \overline{B}), p_1) \to \Sigma_d$, and the paths $l_{o_i}$ define homomorphisms $\text{im}_{l_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \to \pi_1(U \setminus (U \cap \overline{B}), p_1)$ and $\psi_{o_i} = f_* \circ \text{im}_{l_{o_i}} : \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i}) \to \Sigma_d$. Denote by $H_{\overline{B}_1}$ the image $\varphi(\pi_1(U \setminus (U \cap \overline{B}), p_1))$ in $G$. It is easy to see that $\varphi_{l_{o_i}} = \psi_{l_{o_i}}$. Therefore, $H_{B_1} \subset H_{\overline{B}_1}$ and $H_{o_i} \subset H_{\overline{B}_1}$.

Let $\gamma_{q_{o_i}} \in \pi_1(U_{o_i} \setminus (U_{o_i} \cap \overline{B}), p_{o_i})$ be a geometric generator represented by the circle $\delta_{o_i} = \{q_{o_i}\} \times \{u_1 \in \mathbb{C} \mid |u_1| = \frac{1}{2}\} \subset \{q_{o_i}\} \times D_1 \subset U_{o_i} \setminus \overline{B}$ (the circuit in positive direction). It is easy to see that $\text{im}_{l_{o_i}}(\gamma_{q_{o_i}}) = \gamma_{q_i}$. Therefore $\varphi_{l_{o_i}}(\gamma_{q_{o_i}}) = \sigma_1$.

Consider the restriction of $f$ to each $V_{o_i, m}$, $f_{i,m} = f|_{V_{o_i,m}} : V_{o_i,m} \to U_{o_i}$. Denote by $d_{i,m}$ the degree of $f_{i,m}$, $d = d_{i,1} + \cdots + d_{i,k_i}$. By construction, for the point $\overline{v}_{i,m} = V_{o_i,m} \cap f^{-1}(o_i)$ we have $\deg_{\overline{v}_{i,m}} f = d_{i,m}$.

The numbering of the points of $f^{-1}(p_0)$ and the path $\overline{l}_0$ define a numbering of the points of $f^{-1}(p_{o_i})$. Then the decomposition $f^{-1}(U_{o_i}) = V_{o_i,1} \cup \cdots \cup V_{o_i,k_i}$ defines a partition $J_i = \{I_{i,1}, \ldots, I_{i,k_i}\}$ of $[1, d]$, $j \in I_{i,m}$ if and only if $\overline{l}_j \in f^{-1}(p_{o_i}) \cap V_{o_i,m}$. By construction, the group $H_{o_i}$ leaves invariant the partition $J_i$ and acts transitively on each $I_{i,m} \in J_i$. In particular, the action of $\sigma_1$ leaves invariant the partition $J_i$.

Assume that if $\overline{B}_j$ is an irreducible component of the branch locus $\overline{B}$ of a covering $f$ such that $f_*(\gamma_j)$ is not a transposition, then $f$ satisfies the following conditions:

1. (R1) For each $o_i \in \overline{B}_j \cap \text{Sing} \overline{B}$ and each $V_{o_i,m}$ there is at most one irreducible component $\overline{R}_k \subset f^{-1}(\overline{B}_j)$ of the ramification locus of $f$ which intersects with $V_{o_i,m}$.

2. (R2) For each $\overline{R}_k \subset f^{-1}(\overline{B}_j)$ there is $o_i \in \overline{B}_j \cap \text{Sing} \overline{B}$ and $m$ such that $\overline{R}_k \cap V_{o_i,m} \neq \emptyset$ and $r_k < d_{i,m}$.

3. (R3) If $\overline{R}_k \cap V_{o_i,m} \neq \emptyset$ and $\overline{R}$ is another ramification curve of $f$ such that $\overline{R} \cap V_{o_i,m} \neq \emptyset$, then for a point $q \in f(\overline{R})$ the image $f_*(\gamma_q)$ of a geometric generator $\gamma_q$ is a transposition in $\Sigma_d$.

**Lemma 3.** Let $f$ and its branch curve $B_1$ satisfy conditions (R0) – (R3), and let $H$ be a subgroup of $H_{\overline{B}_1}$ generated by $\sigma_1$ and the transpositions belonging to $H_{\overline{B}_1}$. Then $H = \Sigma_{J_H}$.

**Proof.** Let $\sigma = \sigma_1 = \varphi_1(\gamma_{p_1})$ where $\gamma_{p_1}$ is the geometric generator defined above. Then it is easy to see that condition (R1) implies that $H$ and $\sigma$ satisfy condition (i) from Lemma 1. Similarly, it follows from conditions (R2) and (R3) that $H$ and $\sigma$ satisfy conditions (ii) and (iii) from Lemma 1. Therefore, $H = \Sigma_{J_H}$. □
Proposition 2. Let $Z_1, \ldots, Z_k$ be the irreducible components $f : Z \to Y$ be a ramified covering of a simply connected surface $Y$. Assume that the branch locus $B$ of $f$ satisfies conditions $(R_0) - (R_3)$. Then the Galois group $G$ of $f$ is isomorphic to $\Sigma_{d_1} \times \cdots \times \Sigma_{d_k}$, where $d_i = \deg f|Z_i$.

Proof. The decomposition $Z = Z_1 \sqcup \cdots \sqcup Z_k$ defines a partition $J = \{I_1, \ldots, I_k\}$ of the set $f^{-1}(p_0)$. The group $G$ leaves invariant the partition $J$ and acts transitively on each $I_j \subset J$. Therefore Proposition 2 follows from Lemma 3 and Corollary 2. □

3. Proof of Theorem 1, 2 and Corollary 1

We use notations defined in Introduction and Section 2.

3.1. Proof of Theorem 2. Denote by $\overline{C}_i = \nu^{-1}(C_i)$ the irreducible components of $\overline{C}$, $1 \leq i \leq k$.

Obviously,

$$Z \simeq \{(p, l) \in \overline{C} \times \hat{P}^2 \mid p \in \overline{C}, \nu(p) \in l\}$$

and it is easy to see that

$$Z_i \simeq \{(p, l) \in \overline{C}_i \times \hat{P}^2 \mid p \in \overline{C}_i, \nu(p) \in l\}$$

are the irreducible components of the surface $Z$.

Let $t$ be a local parameter in a small neighbourhood $U \subset \overline{C}$ of a point $p \in \overline{C}$ and let the normalization $\nu$ be given in $U$ by

$$x_1 = \phi_1(t), \ x_2 = \phi_2(t), \ x_3 = \phi_3(t).$$

(2)

If $(y_1, y_2, y_3)$ are homogeneous coordinates in $\hat{P}^2$ dual to the coordinates $(x_1, x_2, x_3)$ in $\mathbb{P}^2$, then the surface $Z$, in the neighbourhood $U \times \hat{P}^2 \subset \overline{C} \times \hat{P}^2$, is given by equation

$$y_1\phi_1(t) + y_2\phi_2(t) + y_3\phi_3(t) = 0.$$  

In particular, if $\nu$ is given by equations (1), that is,

$$\phi_1 = \sum_{i=s_p}^{\infty} a_it^i, \ \phi_2 = t^{r_p}, \ \phi_3 = 1,$$

(3)

then $Z \cap (U \times \hat{P}^2)$ lies in $U \times \mathbb{C}^2$, where $\mathbb{C}^2 = \{y_1 \neq 0\}$ is the affine plane in $\hat{P}^2$, and $Z \cap (U \times \mathbb{C}^2)$ is given by equation

$$\sum_{i=s_p}^{\infty} a_it^i + z_2t^{r_p} + z_3 = 0,$$

(4)

where $z_i = y_i/y_1, \ i = 2, 3$. Therefore $Z$ is a smooth surface and $(t, z_2)$ are coordinates in $Z \cap (U \times \mathbb{C}^2)$.  

The restriction of the covering $f$ to $Z \cap (U \times \mathbb{C}^2)$,
$$f_U : Z \cap (U \times \mathbb{C}^2) \rightarrow \mathbb{C}^2,$$
is the restriction of the projection $(t, z_2, z_3) \mapsto (z_2, z_3)$, therefore it is given by
$$z_2 = z_2, \quad z_3 = -(\sum_{i=s_p} a_i t^i + z_2 t^{r_p}). \quad (5)$$
Its Jacobian is equal
$$J(f_U) = -t^{r_p-1}(\sum_{i=s_p} ia_i t^{i-r_p} + r_p z_2).$$
Therefore $f_U$ is ramified along a curve $R$ given by equation
$$1/r_p \sum_{i=s_p} a_i t^{i-r_p} + z_2 = 0 \quad (6)$$
with multiplicity two and along the fibre $F_p = \{ t = 0 \}$ with multiplicity $r_p$ if $r_p \geq 2$ and hence $f(F_p) = L_p \subset \mathbb{P}^2$ is a component of the branch locus of $f$ if $r_p \geq 2$. Note also that $R$ is a section of the ruled surface $Z \cap (U \times \mathbb{C}^2) \rightarrow \mathbb{C}^2$ and, in addition, it is the unique section contained in the ramification locus.
Therefore to show that $R$ is a germ of the curve $\tilde{C}$, we can assume that the image $\nu(p)$ is a smooth point of $C$, that is, we can assume that $r_p = 1$ and $\phi_2(t) = t$. Then $R$ is given by $\phi'_1(t) + z_2 = 0$ and the restriction of $f_U$ to $R$ is given by
$$y_1 = 1, \quad y_2 = -\phi'_1(t), \quad y_3 = -\phi_1(t) + t\phi'_1(t). \quad (7)$$
Everyone easily check that that equations (7) together with the equations
$$x_1 = \phi_1(t), \quad x_2 = t, \quad x_3 = 1$$
(defining the germ $\nu(U)$ of $C$) is a parametrization of $\tilde{C} \subset \mathcal{C}$ over $\nu(U)$.

To count the local degree $\deg_q f_U$ of the covering $f_U : Z \cap (U \times \mathbb{C}^2) \rightarrow \mathbb{C}^2$ at the point $q = (0, 0, 0)$, first of all, note that the curve $\{ z_2 = 0 \}$ does not belong to the ramification locus of $f_U$, since $a_{sp} \neq 0$ in equation (6). Next, let us choose a new parameter $t_1$ such that $t_1^{sp} = \sum_{i=s_p} a_i t^i$, then $f_U$ is given by equations of the form
$$z_2 = z_2, \quad z_3 = -(t_1^{sp} + z_2 \sum_{i=r_p} c_i t_1^i) \quad (8)$$
and to count $\deg_q f_U$, it suffices to count the number of points belonging to $f_U^{-1}((z_{2,0}, z_{3,0}))$, where a point $(z_{2,0}, z_{3,0}) \in \text{Im} f_U$ is such that $z_{2,0} = 0$, $z_{3,0} \neq 0$, and $z_{3,0}$ is sufficiently close to zero. It follows from equations (8) that this number is equal to $s_p$. \qed
3.2. **Proof of Theorem** 1. By Theorem 2, the branch locus $\overline{B}$ of the dualizing covering $f : Z \to \mathbb{P}^2$ consists of the curve $\tilde{C}$ and the lines $L_p$ for which $r_p \geq 2$, the ramification locus $\overline{R}$ consists of the curve $\tilde{C}$ and the fibres $F_p$ with $r_p \geq 2$. Each line $L_p$ and each fibre $F_p$ are smooth and $F_{p_1} \cap F_{p_2} = \emptyset$ for $p_1 \neq p_2$. Next, the restriction $f_{|F_p} : F_p \to L_p$ of $f$ to $F_p$ is an isomorphism and the restriction $f_0 : \tilde{C} \to \tilde{C}$ of $f$ to $\tilde{C}$ is a bi-rational map, and $f$ is ramified along $\tilde{C}$ with multiplicity two. Therefore $f_*(\gamma_1)$ is a transposition for any geometric generator $\gamma_1 \in \pi_1(\mathbb{P}^2 \setminus \overline{B}, l_0)$, $l \in \tilde{C}$, and hence the dualizing covering $f$ and its branch locus $\overline{B}$ satisfy conditions ($R_0$) and ($R_1$) (see Section 2). Next, if $F_p \subset \overline{R}$ then the point $q = F_p \cap \tilde{C}$ belongs to $f^{-1}(\text{Sing} \overline{B})$ and $s_p = \deg_q f > r_p$, that is, $f$ and its branch locus $\overline{B}$ satisfy conditions ($R_2$) and ($R_3$). Now Theorem 1 follows from Proposition 2. 

3.3. **Proof of Corollary** 1. First of all, note that if a line $l \subset \mathbb{P}^2$ is an irreducible component of the curve $C$ then for each loop $l_t \subset T_C$ starting at $l_0$ the monodromy defined by $l_t$ leaves fixed the point $l \cap l_0$.

Let $C_1, \ldots, C_n$ be the irreducible components of $C$ and let $\deg C_i = d_i \geq 2$ for $i = 1, \ldots, k$ and $\deg C_i = 1$ for $i > k$. Denote by $\text{Sing} C$ the set of singular points of $C$, by $L_{\text{Sing}} = \bigcup_{P \in \text{Sing} C} L_P \subset \mathbb{P}^2$, by $C' = C_1 \cup \cdots \cup C_k$. Let $f' : Z \to \mathbb{P}^2$ be the dualizing covering for $C'$ and $\overline{B}$ its branch locus (see Theorem 2). Then it is easy to see that $T_C = \mathbb{P}^2 \setminus (\overline{B} \cup L_{\text{Sing}})$.

We have $f^{-1}(l_0) = \{(p_1, l_0), (p_2, l_0), \ldots, (p_d, l_0)\}$, where $p_i = \nu^{-1}(P_i)$, $l_0 \cap C' = \{P_1, P_2, \ldots, P_t\}$, and $P_1, P_2 \in C_1$.

The embedding $i : T_C \hookrightarrow \mathbb{P}^2 \setminus \overline{B}$ defines an epimorphism $i_* : \pi_1(T_C, l_0) \to \pi_1(\mathbb{P}^2 \setminus \overline{B}, l_0)$. By Theorem 1 there is an element $\gamma' \in \pi_1(\mathbb{P}^2 \setminus \overline{B}, l_0)$ such that $f'_* (\gamma') = (1, 2) \in \Sigma_d$, where $(1, 2)$ is the transposition transposing the points $(p_1, l_0)$ and $(p_2, l_0)$. Let $l_t$ be a loop representing an element $\gamma \in \pi_1(T_C, l_0)$ such that $i_* (\gamma) = \gamma'$. It is easy to see that the loop $l_t$ is a desired one.

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