The Nonlinear Schroedinger equation: solitons dynamics

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Abstract

In this paper we investigate the dynamics of solitons occurring in the nonlinear Schroedinger equation when a parameter $h \to 0$. We prove that under suitable assumptions, the soliton approximately follows the dynamics of a point particle, namely, the motion of its barycenter $q_h(t)$ satisfies the equation

$$\ddot{q}_h(t) + \nabla V(q_h(t)) = H_h(t)$$

where

$$\sup_{t \in \mathbb{R}} |H_h(t)| \to 0 \text{ as } h \to 0.$$ 

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1 Introduction

Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time.

By soliton we mean an orbitally stable solitary wave so that it has a particle-like behavior (for the definition of orbital stability we refer e.g. to [1], [2], [5] etc.).

In this paper we will be concerned with the dynamics of solitons relative to a class of nonlinear Schroedinger equations (NSE).

Let us consider the following Cauchy problem relative to the NSE:

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \Delta \psi + \frac{1}{2\hbar^\alpha} W'(h^\gamma |\psi|) \frac{\psi}{|\psi|} \]  \tag{1}

\[ \psi(0, x) = \frac{1}{\hbar^\gamma} U \left( \frac{x - q_0}{\hbar^\beta} \right) e^{i\hbar^\gamma \cdot x} \]  \tag{2}

where

\[ \beta = 1 + \frac{\alpha - \gamma}{2} \]

and \( U : \mathbb{R}^N \to \mathbb{R} \), \( N \geq 2 \), is a positive, radially symmetric solution of the static nonlinear Schroedinger equation

\[ -\Delta U + W'(U) = 2\omega U \]  \tag{3}

with

\[ \|U\|_{L^2} = \sigma \]  \tag{4}
Direct computations show that a solution of (1), (2) is given by

\[ \psi(t, x) = \frac{1}{h^{\gamma}} U \left( \frac{x - q_0 - vt}{h^{\beta}} \right) e^{\frac{i}{h} (v \cdot x - Et)} \]  

(5)

with

\[ E = \frac{1}{2} \nu^2 + \frac{\omega}{h^{\alpha - \gamma}} \]

Moreover if the problem (1), (2) is well posed this is the unique solution.

We can interpret this result saying that the barycenter \( q(t) \) of the solution of (1, 2) defined by

\[ q(t) = \frac{\int_{\mathbb{R}^N} x |\psi(t, x)|^2 dx}{\int_{\mathbb{R}^N} |\psi(t, x)|^2 dx} \]  

(6)

satisfies the Cauchy problem

\[
\begin{cases}
\ddot{q} = 0 \\
q(0) = q_0 \\
\dot{q}(0) = v
\end{cases}
\]

The aim of this paper is to investigate what happens if the problem is perturbed namely to investigate the problem

\[
\begin{align*}
&\frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2} \Delta \psi + \frac{1}{2h^{\alpha}} W'(h^\gamma |\psi|) \frac{\psi}{|\psi|} + V(x) \psi = 0 \\
&\psi(0, x) = \varphi_h(x)
\end{align*}
\]

(P\( h \))

where

\[ \varphi_h(x) = \left[ \frac{1}{h^{\gamma}} (U + w_0) \left( \frac{x - q_0}{h^{\beta}} \right) \right] e^{\frac{i}{h} (v \cdot x)} \]  

(7)

and \( w_0 \) is small, namely there is a constant \( C \) such that

\[ \| w_0 \|_{H^1} \leq Ch^{\alpha - \gamma} \]

\[ \int_{\mathbb{R}^N} V(x) |w_0(x)|^2 dx < Ch^{\alpha - \gamma} \]

Also we assume that

\[ \| U + w_0 \|_{L^2} = \| U \|_{L^2} = \sigma \]

We make the following assumptions:
(i) the problem \( [P_h] \) has a unique solution

\[
\psi \in C^0(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N))
\]  

(sufficient conditions can be found in Kato [14], Cazenave [6], Ginibre-Velo [11]; see Remark 2).

(ii) \( W : \mathbb{R}^+ \to \mathbb{R} \) is a \( C^3 \) function which satisfies the following assumptions:

\[
W(0) = W'(0) = W''(0) = 0
\]  

\[
|W''(s)| \leq c_1|s|^{q-2} + c_2|s|^{p-2} \quad \text{for some} \ 2 < q \leq p < 2^*. \]  

\[
W(s) \geq -c|s|^\nu, \ c \geq 0, \ 2 < \nu < 2 + \frac{4}{N} \quad \text{and} \ s \ \text{large}
\]  

\[
\exists s_0 \in \mathbb{R}^+ \ \text{such that} \ W(s_0) < 0 \quad \text{(W3)}
\]

(iii) \( V : \mathbb{R}^N \to \mathbb{R} \) is a \( C^2 \) function which satisfies the following assumptions:

\[
V(x) \geq 0; \quad \text{(V0)}
\]

\[
|\nabla V(x)| \leq V(x)^b \quad \text{for} \ |x| > R_1 > 1, b \in (0,1); \quad \text{(V1)}
\]

\[
V(x) \geq |x|^a \quad \text{for} \ |x| > R_1 > 1, a > 1. \quad \text{(V2)}
\]

The main result of this paper is the following theorem:

**Theorem 1.** Assume that (i), (ii) and (iii) hold and that

\[
\alpha > \gamma \quad \text{(CRUCIAL ASSUMPTION)}
\]

Then the barycenter \( q_h(t) \) of the solution of the problem \( [P_h] \) satisfies the following Cauchy problem:

\[
\begin{cases}
\ddot{q}_h(t) + \nabla V(q_h(t)) = H_h(t) \\
q_h(0) = q_0 \\
\dot{q}_h(0) = v
\end{cases}
\]

where

\[
\sup_{t \in \mathbb{R}} |H_h(t)| \to 0 \ as \ h \to 0
\]

Let us discuss the set of our assumptions.
Remark 2. About the assumption (i), we recall a result on the global existence of solutions of the Cauchy problem \( P_h \) (see [6, 11, 14]). Assume \((W_1), (W_2)\) and \((W_3)\) for \(W\). Let \(D(A)\) (resp. \(D(A^{1/2})\)) denote the domain of the selfadjoint operator \(A\) (resp. \(A^{1/2}\)) where

\[
A = -\Delta + V : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N).
\]

If \(V \geq 0, V \in C^2\) and \(|\partial^2 V| \in L^\infty\) and the initial data \(\psi(0, x) \in D(A^{1/2})\) then there exists the global solution \(\psi\) of \(P_h\) and

\[
\psi(t, x) \in C^0(\mathbb{R}, D(A^{1/2})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N)).
\]

Furthermore, if \(\psi(0, x) \in D(A)\) then

\[
\psi(t, x) \in C^0(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N)).
\]

In this case, since \(D(A) \subset H^2(\mathbb{R}^N)\), (i) is satisfied.

Remark 3. The conditions \((W_0)\) and \((V_0)\) are assumed for simplicity; in fact they can be weakened as follows

\[
W''(0) = E_0
\]

and

\[
V(x) \geq E_1.
\]

In fact, in the general case, the solution of the Schroedinger equation is modified only by a phase factor.

Remark 4. In [2] the authors prove that if (ii) holds equation (1) admits orbitally stable solitary waves having the form (2). In particular the authors show that, under assumptions \((W_1), (W_2)\) and \((W_3)\), for any \(\sigma\) there exists a minimizer \(U(x) = U_\sigma(x)\) of the functional

\[
J(u) = \int \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dx
\]

on the manifold \(S_\sigma := \{u \in H^1, ||u||_{L^2} = \sigma\}\). Such a minimizer satisfies eq.(3) where \(2\omega\) is a Lagrange multiplier. We will call ground state solution a minimizer radially symmetric around the origin. We recall that by a well known result of Gidas, Ni, and Nirenberg [10], any positive solution of eq.(3) is radially symmetric around some point.
Remark 5. We set

\[ u_h(x) = h^{-\gamma} U \left( \frac{x}{h^\beta} \right) \]

where \( U \) is a ground state solution. Now we establish a relation between \( \alpha, \beta \) and \( \gamma \) in order to have stationary solution of (1) of the form \( \psi(t, x) = u_h(x) e^{i\frac{\omega}{h} t} \), namely, \( u_h(x) \) is a solution of the equation

\[ -h^2 \Delta u_h + \frac{1}{h^\alpha} W'(h^\gamma u_h) = 2\omega_h u_h. \]

In fact, replacing \( u_h \) by its explicit expression, we get

\[ -h^{2-\gamma} \Delta \left[ U \left( \frac{x}{h^\beta} \right) \right] + \frac{1}{h^\alpha} W' \left( U \left( \frac{x}{h^\beta} \right) \right) = 2\omega_h h^{-\gamma} U \left( \frac{x}{h^\beta} \right) \]

and hence, by rescaling the variable \( x \),

\[ -h^{2-\gamma-2\beta+\alpha} \Delta U(x) + W'(U(x)) = 2\omega_h h^{\alpha-\gamma} U(x). \]

Thus, it is sufficient to take

\[ \beta = 1 + \frac{\alpha - \gamma}{2} \tag{9} \]

and

\[ \omega_h = \frac{\omega}{h^{\alpha-\gamma}} \tag{10} \]

to obtain the claim. In the following we always assume (9).

Remark 6. The assumption (V2) is necessary if we want to identify the position of the soliton with the barycenter (6). Let us see why. Consider a soliton \( \psi(x) \) and a perturbation

\[ \psi_d(x) = \psi(x) + \varphi(x - d), \quad d \in \mathbb{R}^N \]

Even if \( \varphi(x) \ll \psi(x) \), when \( d \) is very large, the “position” of \( \psi(x) \) and the barycenter of \( \psi_d(x) \) are far from each other. In Lemma 23, we shall prove that this situation cannot occur provided that (V2) hold. In a paper in preparation, we give a more involved notion of barycenter of the soliton and we will be able to consider other situations.

Remark 7. We will give a rough explanation of the assumption \( \alpha > \gamma \) which, in this approach to the problem, is crucial. In section 2.1 we will show that the energy \( E_h \) of a soliton \( \psi_h \) is composed by two parts: the internal energy \( J_h \) and the dynamical energy \( G \). The internal energy is a kind of binding energy that prevents the soliton from splitting, while the dynamical energy
is related to the motion and it is composed of potential and kinetic energy.

As $h \to 0$, we have that (see section 2.1)

$$J_h(\psi_h) \cong h^{N\beta - \alpha - \gamma}$$

and

$$G(\psi_h) \cong ||\psi_h||^2 \cong h^{N\beta - 2\gamma}$$

Then, we have that

$$\frac{G(\psi_h)}{J_h(\psi_h)} \cong h^{\alpha - \gamma}$$

So the assumption $\alpha - \gamma > 0$ implies that, for $h \ll 1$, $G(\psi_h) \ll J_h(\psi_h)$, namely the internal energy is bigger than the dynamical energy. This is the fact that guarantees the existence and the stability of the travelling soliton for any time.

As far we know, this is the first paper in which there is a result of type Th. 1 for all times $t \in \mathbb{R}$. However there are other results which compare the motion of the soliton with the solution of the equation $\ddot{X}(t) + \nabla V(X(t)) = 0$ for $t \in [0, T]$ for some constant $T < \infty$.

Earlier results for pure power nonlinearity and bounded external potential are in [4]. The authors have shown that if the initial data is close to $U(\frac{x - q_0}{h}) e^{i v_0 \cdot c} h$ in a suitable sense then the solution $\psi_h(t, x)$ of $(P_h)$ satisfies for $t \in [0, T]$

$$\left\| \frac{1}{h^N} |\psi_h(t, x)|^2 - \frac{1}{h^N} \int_{\mathbb{R}^N} |\psi_h(t, x)|^2 dx \delta_X(t) \right\|_{C^{1*}} \to 0 \text{ as } h \to 0. \quad (11)$$

Here $C^{1*}$ is the dual of $C^1$ and $X(t)$ satisfies $\frac{1}{2} \ddot{X}(t) = \nabla V(X(t))$ with $X(0) = q_0$, $\dot{X}(0) = v_0$.

In related papers [15] and [16] there are slight generalizations of the above result. Using a similar approach, Marco Squassina [19] described the soliton dynamics in an external magnetic potential.

In [7] and [8] the authors study the case of bounded external potential $V$ respectively in $L^\infty$ or confining.

A result comparable with Theorem 24 is contained in [8]. The authors assume the existence of a stable ground state solution with a null space non degeneracy condition of the equation

$$- \Delta \eta_\mu + \mu \eta_\mu + W'(\eta_\mu) = 0. \quad (12)$$

The authors define a parameter $\varepsilon$ which depends on $\mu$ and on other parameters of the problem. Under suitable assumptions they prove that there exists
$T > 0$ such that, if the initial data $\psi^0(x)$ is very close to $e^{ip_0(x-a_0)+i\gamma_0 \eta_{\mu_0}(x-a_0)}$ the solution $\psi(t, x)$ of problem $(P_1)$ with initial data $\psi^0$ is given by

$$\psi(t, x) = e^{ip(t)(x-a(t))} + w(t) \tag{13}$$

with $||w||_{H^1} \leq \varepsilon$, $\dot{p} = -\nabla V(a) + o(\varepsilon^2)$, $\dot{a} = 2p + o(\varepsilon^2)$ with $0 < t < T$ for $\varepsilon$ small.

The main differences with our result are the following. First of all we do not have any limitation on the time $t$. Also, we have an explicit estimate on $\ddot{q}$ (which roughly speaking correspond to $\ddot{a}$). Our assumption on the nonlinearity $W$ are explicit (namely ($W_1$), ($W_2$), ($W_3$)) and we do not require the null space condition which are, in general, not easy to verify.

### 1.1 Notations

In the next we will use the following notations:

- $\text{Re}(z), \text{Im}(z)$ are the real and the imaginary part of $z$
- $B(x_0, \rho) = \{x \in \mathbb{R}^N : |x - x_0| \leq \rho\}$
- $B(x_0, \rho)^C = \mathbb{R}^N \setminus B(x_0, \rho)$
- $S_\sigma = \{u \in H^1 : ||u||_{L^2} = \sigma\}$
- $J^c_h = \{u \in H^1 : J_h(u) < c\}$
- $U_q(x) = U(x - q)$
- $\partial_t \psi = \frac{\partial}{\partial t} \psi$
- $|\partial^\alpha V(x)| = \sup_{i_1, \ldots, i_\alpha} \left| \frac{\partial^\alpha V(x)}{\partial x_{i_1} \cdots \partial x_{i_\alpha}} \right|$ where $\alpha \in \mathbb{N}$, $i_1, \ldots, i_\alpha \in \{1, \ldots, N\}$
- $I_{\sigma^2} = \inf_{u \in H^1, \int u^2 = \sigma^2} J(u) = m$

### 2 General features of NSE

Equation $(P_h)$ is the Euler-Lagrange equation relative to the Lagrangian density

$$\mathcal{L} = ih \partial_t \psi \bar{\psi} - \frac{h^2}{2} |\nabla \psi|^2 - W_h(\psi) - V(x) |\psi|^2 \tag{14}$$

where, in order to simplify the notation we have set

$$W_h(\psi) = \frac{1}{h^{\alpha + \gamma}} W(h^{\gamma} |\psi|)$$
Sometimes it is useful to write $\psi$ in polar form

$$\psi(t, x) = u(t, x)e^{iS(t, x)/\hbar}. \quad (15)$$

Thus the state of the system $\psi$ is uniquely defined by the couple of variables $(u, S)$. Using these variables, the action $S = \int \mathcal{L} dx dt$ takes the form

$$S(u, S) = -\int \left[ \frac{\hbar^2}{2} |\nabla u|^2 + W_h(u) + \left( \partial_t S + \frac{1}{2} |\nabla S|^2 + V(x) \right) u^2 \right] dx dt \quad (16)$$

and equation $\mathcal{P}_h$ becomes:

$$-\frac{\hbar^2}{2} \Delta u + W'_h(u) + \left( \partial_t S + \frac{1}{2} |\nabla S|^2 + V(x) \right) u = 0 \quad (17)$$

$$\partial_t (u^2) + \nabla \cdot (u^2 \nabla S) = 0 \quad (18)$$

### 2.1 The first integrals of NSE

Noether’s theorem states that any invariance for a one-parameter group of the Lagrangian implies the existence of an integral of motion (see e.g. [9]).

Now we describe the first integrals which will be relevant for this paper, namely the energy and the ”hylenic charge”.

**Energy** The energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; it has the following form

$$E_h(\psi) = \int \left[ \frac{\hbar^2}{2} |\nabla \psi|^2 + W_h(\psi) + V(x) |\psi|^2 \right] dx \quad (19)$$

Using (15) we get:

$$E_h(\psi) = \int \left( \frac{\hbar^2}{2} |\nabla u|^2 + W_h(u) \right) dx + \int \left( \frac{1}{2} |\nabla S|^2 + V(x) \right) u^2 dx \quad (20)$$

Thus the energy has two components: the *internal energy* (which, sometimes, is also called *binding energy*)

$$J_h(u) = \int \left( \frac{\hbar^2}{2} |\nabla u|^2 + W_h(u) \right) dx \quad (21)$$

and the *dynamical energy*

$$G(u, S) = \int \left( \frac{1}{2} |\nabla S|^2 + V(x) \right) u^2 dx \quad (22)$$
which is composed by the kinetic energy \( \frac{1}{2} \int |\nabla S|^2 u^2 dx \) and the potential energy \( \int V(x)u^2 dx \).

By our assumptions, the internal energy is bounded from below and the dynamical energy is positive.

**Hylenic charge** Following [1] the *hylenic charge*, is defined as the quantity which is preserved by by the invariance of the Lagrangian with respect to the action

\[
\psi \mapsto e^{i\theta} \psi.
\]

For equation \((P_h)\) the charge is nothing else but the \(L^2\) norm, namely:

\[
\mathcal{H}(\psi) = \int |\psi|^2 dx = \int u^2 dx
\]

Now we study the rescaling properties of the internal energy and the \(L^2\) norm of a function \(u(x)\) having the form

\[
u(x) := h^{-\gamma} v \left( \frac{x}{h^\beta} \right)
\]

We have

\[
||u||^2_{L^2} = h^{-2\gamma} \int v \left( \frac{x}{h^\beta} \right)^2 dx = h^{N\beta-2\gamma} \int v(\xi)^2 d\xi = h^{N\beta-2\gamma} ||v||^2_{L^2}.
\]

and

\[
J_h(u) = \int \frac{h^2}{2} |\nabla u|^2 + \frac{1}{h^{\alpha+\gamma}} W(h^\gamma u) dx = \int \frac{h^{2-2\gamma}}{2} |\nabla_x v \left( \frac{x}{h^\beta} \right)|^2 + \frac{1}{h^{\alpha+\gamma}} W \left( v \left( \frac{x}{h^\beta} \right) \right) dx = \int \frac{h^{N\beta+2-2\gamma-2\beta}}{2} |\nabla \xi v(\xi)|^2 + h^{N\beta-\alpha-\gamma} W(v(\xi)) d\xi = h^{N\beta-\alpha-\gamma} \int \frac{1}{2} |\nabla \xi v(\xi)|^2 + W(v(\xi)) d\xi = h^{N\beta-\alpha-\gamma} J_1(v),
\]

using the fundamental relation (9).

**Remark 8.** If we choose \(N\beta - 2\gamma = 0\), the \(L^2\) norm does not change by rescaling. This implies that the dynamical energy \(G\), for \(h\) small, changes very little.
2.2 The swarm interpretation

In this section we will suppose that the soliton is composed by a swarm of particles which follow the laws of classical dynamics given by the Hamilton-Jacobi equation. This interpretation will permit us to give an heuristic proof of the main result.

First of all let us write NSE with the usual physical constants $m$ and $\hbar$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{1}{2} W'(\psi) + V(x)\psi.$$ 

Here $m$ has the dimension of mass and $\hbar$, the Plank constant, has the dimension of action.

In this case equations (17) and (18) become:

$$-\frac{\hbar^2}{2m} \Delta u + \frac{1}{2} W'(u) + \left( \partial_t S + \frac{1}{2m} |\nabla S|^2 + V(x) \right) u = 0$$  \hspace{1cm} (23)

$$\partial_t (u^2) + \nabla \cdot \left( u^2 \frac{\nabla S}{m} \right) = 0$$  \hspace{1cm} (24)

The second equation allows us to interprete the matter field to be a fluid composed by particles whose density is given by

$$\rho_\hbar = u^2$$

and which move in the velocity field

$$\mathbf{v} = \frac{\nabla S}{m}.$$  \hspace{1cm} (25)

So equation (24) becomes the continuity equation:

$$\partial_t \rho_\hbar + \nabla \cdot (\rho_\hbar \mathbf{v}) = 0.$$  

If

$$-\frac{\hbar^2}{2m} \Delta u + \frac{1}{2} W'(u) \ll u,$$  \hspace{1cm} (26)

equation (23) can be approximated by the eikonal equation

$$\partial_t S + \frac{1}{2m} |\nabla S|^2 + V(x) = 0.$$  \hspace{1cm} (27)

This is the Hamilton-Jacobi equation of a particle of mass $m$ in a potential field $V$. 

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If we do not assume (26), equation (27) needs to be replaced by

$$ \partial_t S + \frac{1}{2m} |\nabla S|^2 + V + Q(u) = 0 $$

(28)

with

$$ Q(u) = - \left( \frac{\hbar^2}{m} \right) \Delta u + \frac{W'(u)}{2u}.$$ 

The term $Q(u)$ can be regarded as a field describing a sort of interaction between particles.

Given a solution $S(t, x)$ of the Hamilton-Jacobi equation, the motion of the particles is determined by eq. (25).

### 2.3 An heuristic proof

In this section we present an heuristic proof of the main result. This proof is not at all rigorous, but it helps to understand the underlying Physics.

If we interpret $\rho_H = u^2$ as the density of particles then

$$ H = \int \rho_H dx $$

is the total number of particles. By (28), each of these particle moves as a classical particle of mass $m$ and hence, we can apply to the laws of classical dynamics. In particular the center of mass defined in (6) takes the following form:

$$ q(t) = \frac{\int x \rho_H dx}{\int m \rho_H dx} = \frac{\int x \rho_H dx}{\int \rho_H dx}. $$

(29)

The motion of the barycenter is not affected by the interaction between particles (namely by the term (28)), but only by the external forces, namely by $\nabla V$. Thus the global external force acting on the swarm of particles is given by

$$ \overrightarrow{F} = - \int \nabla V(x) \rho_H dx. $$

(30)

Thus the motion of the center of mass $q$ follows the Newton law

$$ \overrightarrow{F} = M \ddot{q}, $$

(31)

where $M = \int m \rho_H dx$ is the total mass of the swarm; thus by (29), (30) and (31), we get

$$ \ddot{q}(t) = - \frac{\int \nabla V \rho_H dx}{m \int \rho_H dx} = - \frac{\int \nabla Vu^2 dx}{m \int u^2 dx}. $$
If we assume that the \( u(t,x) \) and hence \( \rho_{\mathcal{H}}(t,x) \) is concentrated in the point \( q(t) \), we have that
\[
\int \nabla V u^2 dx \cong \nabla V(q(t)) \int u^2 dx
\]
and so, we get
\[
m \ddot{q}(t) \cong - \nabla V(q(t)).
\]
Notice that the equation \( m \ddot{q}(t) = - \nabla V(q(t)) \) is the Newtonian form of the Hamilton-Jacobi equation (27).

3 Preliminary results

In this section we prove two results which are the base of the main theorem. They have some interest in themselves and require less assumptions than the final theorem.

3.1 Existence and dynamics of barycenter

We recall the definition of barycenter of \( \psi \)
\[
q_h(t) = \frac{\int_{\mathbb{R}^N} x|\psi(t,x)|^2 dx}{\int_{\mathbb{R}^N} |\psi(t,x)|^2 dx}
\]
(32)

The barycenter is not well defined for all the functions \( \psi \in H^1(\mathbb{R}^N) \). Thus we need the following result:

**Theorem 9.** Let \( \psi(t,x) \) be a global solution of \( \{P_h\} \) such that \( \psi(t,x) \in C(\mathbb{R}, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N)) \) with initial data \( \psi(0,x) \) such that
\[
\int_{\mathbb{R}^N} |x||\psi(0,x)|^2 dx < +\infty.
\]

Then the map \( q_h(t) : \mathbb{R} \to \mathbb{R}^N \), given by (32) is well defined.

**Proof.** We show that \( |\cdot|^{1/2}|\psi(t,\cdot)| \in L^2(\mathbb{R}^N) \) for any \( t \), using a regularization argument.

We set
\[
k_\varepsilon(t) = \int_{\mathbb{R}^N} e^{-2\varepsilon|\cdot|} |x||\psi(t,x)|^2 dx.
\]
Since $\psi$ is a solution of $(P_h)$, we have
\[
k'_\varepsilon(t) = \int_{\mathbb{R}^N} e^{-2\varepsilon|x|} \left| x \right| \left[ \partial_t \psi \bar{\psi} - \psi \partial_t \bar{\psi} \right] = 2 \operatorname{Im} \left( \int i \left| x \right| \partial_t \psi \bar{\psi} e^{-2\varepsilon|x|} \right) =
\]
\[= \hbar \operatorname{Im} \left( \int \nabla \psi \nabla (|x| \bar{\psi} e^{-2\varepsilon|x|}) \right) + \frac{1}{\hbar} \operatorname{Im} \left( \int 2 |x| V |\psi|^2 e^{-2\varepsilon|x|} \right) +
\]
\[+ \frac{1}{\hbar} \operatorname{Im} \left( \int \frac{|x|}{\hbar^\alpha} W_{-2\varepsilon|x|} \right) =
\]
\[= \hbar \operatorname{Im} \left( \int \nabla \psi \nabla (|x| e^{-2\varepsilon|x|}) \right) + h \operatorname{Im} \left( \int \bar{\psi} \nabla \psi (|x| e^{-2\varepsilon|x|}) \right) =
\]
\[= h \operatorname{Im} \left( \int \bar{\psi} \nabla \psi \cdot \frac{x}{|x|} e^{-2\varepsilon|x|} (1 - 2\varepsilon|x|) \right),
\]
so we have
\[
|k'_\varepsilon(t)| \leq \int_{\mathbb{R}^N} |\bar{\psi}| ||\nabla \psi|| \leq ||\psi(t, \cdot)||_{L^2} ||\nabla \psi(t, \cdot)||_{L^2}, \tag{33}
\]
then by (33) we get
\[
k_\varepsilon(t) = k_\varepsilon(0) + \hbar \operatorname{Im} \left( \int_0^t \int \bar{\psi} \nabla \psi \cdot \frac{x}{|x|} e^{-2\varepsilon|x|} (1 - 2\varepsilon|x|) dx dt \right) \leq
\]
\[\leq ||\sqrt{|x|} \psi(0, x)||_{L^2}^2 + \int_0^t ||\psi(t, \cdot)||_{L^2} \|
\]
\[||\nabla \psi(t, \cdot)||_{L^2} dt.
\]
By Fatou Lemma, when $\varepsilon \to 0$ we get $| \cdot |^{1/2} |\psi(t, \cdot)| \in L^2(\mathbb{R}^N)$ for any $t \geq 0$.
So the map $q(t) : [0, \infty) \to \mathbb{R}^N$ is well defined.

\[\square\]

**Theorem 10.** The map $q_h(t) : \mathbb{R} \to \mathbb{R}^N$, given by (32) is $C^1$ and
\[
\dot{q}_h(t) = \operatorname{Im} \left( h \int_{\mathbb{R}^N} \bar{\psi}(t, x) \nabla \psi(t, x) dx \right) \tag{34}
\]
Moreover if $\psi(t, x) \in C(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N))$ then $q_h(t)$ is $C^2$ and
\[
\ddot{q}_h(t) = \frac{\int_{\mathbb{R}^N} V(x) \nabla \psi(t, x)^2 dx}{||\psi(t, x)||_{L^2}^2}. \tag{35}
\]
**Proof.** We have

\[ \dot{q}_h(t) = \frac{h \text{Im} \left( \int_{\mathbb{R}^N} \bar{\psi} \nabla \psi \right)}{||\psi(t, x)||_{L^2}^2} \]

We use the same regularization argument of Th. 9. We set

\[ K^i_\varepsilon(t) = \int_{\mathbb{R}^N} e^{-2\varepsilon |x|} |x_i| \psi^2 dx \]

and again we find in the same way that

\[
\frac{d}{dt} K^i_\varepsilon(t) = h \text{Im} \left( \int_{\mathbb{R}^N} \bar{\psi} \nabla \psi \cdot e_i e^{-2\varepsilon |x|} \right) - h \text{Im} \left( \int_{\mathbb{R}^N} \bar{\psi} \nabla \psi \cdot \frac{x}{|x|} 2\varepsilon x_i e^{-2\varepsilon |x|} \right)
\]

where \( e_i \) is the \( i \)-th vector of the canonical base of \( \mathbb{R}^N \). So, there exists a constant \( c > 0 \) such that

\[
\left| \frac{d}{dt} K^i_\varepsilon(t) \right| \leq c ||\psi(t, \cdot)||_{L^2} ||\nabla \psi(t, \cdot)||_{L^2},
\]

Then we have that

\[ K^i_\varepsilon(t) = K^i_\varepsilon(0) + h \text{Im} \left( \int_0^t \int_{\mathbb{R}^N} \bar{\psi} \nabla \psi \cdot e_i e^{-2\varepsilon |x|} - \int_{\mathbb{R}^N} \bar{\psi} \nabla \psi \cdot \frac{x}{|x|} 2\varepsilon x_i e^{-2\varepsilon |x|} \right). \]

Using the dominated convergence theorem, when \( \varepsilon \to 0 \) we have

\[ \int x_i |\psi(t, x)|^2 dx = \int x_i |\psi(0, x)|^2 dx + \int_0^t h \text{Im} \left( \int \bar{\psi} \nabla \psi \cdot e_i \right) dt, \]

so, for all \( i \) we have

\[ \frac{d}{dt} \int_{\mathbb{R}^N} x_i |\psi|^2 dx = h \text{Im} \left( \int \bar{\psi} \nabla \psi \cdot e_i \right). \]  

(36)

This proves the first part of the theorem.

Next we prove that \( \ddot{q}(t) = \frac{\int_{\mathbb{R}^N} V(x) \nabla |\psi(t, x)|^2 dx}{||\psi(t, x)||_{L^2}^2} \) under the supplementary assumption \( \psi \in C^1(\mathbb{R}, H^1) \).
By this assumption we have that \( \tilde{\psi}(t, x) \nabla \psi(t, x) \in C^1(\mathbb{R}, L^1(\mathbb{R}^N)) \). Thus

\[
\ddot{q}_h(t) = \frac{h \int_{\mathbb{R}^N} \text{Im}(\partial_t \overline{\psi}(t, x) \nabla \psi(t, x)) dx}{||\psi(t, x)||_{L^2}^2} = \frac{h \int_{\mathbb{R}^N} \text{Im}(\partial_t \overline{\psi}(t, x) \nabla \psi(t, x) + \overline{\psi}(t, x) \partial_t \nabla \psi(t, x)) dx}{||\psi(t, x)||_{L^2}^2} = \frac{h \int_{\mathbb{R}^N} \text{Im}(\partial_t \overline{\psi}(t, x) \nabla \psi(t, x) + \overline{\psi}(t, x) \nabla \partial_t \psi(t, x)) dx}{||\psi(t, x)||_{L^2}^2}
\]

\[
= \frac{2\text{Re} \left( \int_{\mathbb{R}^N} i h \partial_t \psi(t, x) \nabla \overline{\psi}(t, x) dx \right)}{||\psi(t, x)||_{L^2}^2} = \frac{\text{Re} \left( \int_{\mathbb{R}^N} \left[ -h^2 \Delta \psi + 2V \psi + \frac{1}{h^\alpha} W'(|\psi|) \frac{\psi}{|\psi|} \right] \nabla \overline{\psi} dx \right)}{||\psi(t, x)||_{L^2}^2}.
\]

We have, for all \( i = 1, \ldots, N \),

\[
\text{Re} \left( \int_{\mathbb{R}^N} W'(|\psi|) \frac{\psi}{|\psi|} \partial_{x_i} \overline{\psi} \right) = \int_{\mathbb{R}^N} \partial_{x_i} W(|\psi|) = 0,
\]

because \( W(|\psi|) \in L^1(\mathbb{R}^N) \) and \( W'(|\psi|) \partial_{x_i} \overline{\psi} \in L^1 \) because \( \psi(t, \cdot) \in H^2 \).

In the same way we have

\[
\text{Re} \left( \int_{\mathbb{R}^N} -\Delta \psi \partial_{x_i} \overline{\psi} \right) = \int_{\mathbb{R}^N} \partial_{x_i} |\nabla \psi|^2 = 0.
\]

Thus

\[
\ddot{q}_h(t) = \frac{2\text{Re} \left( \int_{\mathbb{R}^N} V \psi \nabla \overline{\psi} dx \right)}{||\psi(t, x)||_{L^2}^2} = \frac{\text{Re} \left( \int_{\mathbb{R}^N} V(x) |\psi(t, x)|^2 dx \right)}{||\psi(t, x)||_{L^2}^2}. \tag{37}
\]

We point out that \( V|\psi| \in L^1 \) because \( \psi \) is a global solution with \( \psi \in H^2 \), \( \partial_t \psi \in L^2 \).

\textit{Step 3.} Conclusion.
Let $\psi(t, x) \in C^0(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$. We define a function $\gamma(t, x) \in C^0(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1)$ as

$$
\gamma(t, x) = \int_{\mathbb{R}^N} \varphi(x - \xi) \psi(t, \xi) d\xi
$$

(38)

where $\varphi(\xi) = \varphi(|\xi|)$ is a positive smooth function with compact support in $|\xi| < \lambda$, with $\int_{\mathbb{R}^N} \varphi(\xi) d\xi = 1$.

Fixed $t$ we have that

$$
\gamma(t, x) \to \psi(t, x) \text{ in } H^2(\mathbb{R}^N) \text{ for } \lambda \to 0;
$$

$$
\partial_t \gamma(t, x) \to \partial_t \psi(t, x) \text{ in } L^2(\mathbb{R}^N) \text{ for } \lambda \to 0,
$$

and the convergence is uniform for every compact set in $\mathbb{R}$.

Furthermore, using that $\psi$ is a global solution in $C^0(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$ we have that $V(x)\psi(t, x) \in C^0(\mathbb{R}, L^2)$ and that, fixed $t$

$$
V(x)\gamma(t, x) \to V(x)\psi(t, x) \text{ in } L^2(\mathbb{R}^N) \text{ for } \lambda \to 0;
$$

again, the convergence is uniform for every compact set in $\mathbb{R}$.

We have that $\gamma(t, x)$ solve the following differential equation

$$
i\hbar \frac{\partial \gamma(t, x)}{\partial t} = \frac{-\hbar^2}{2}\Delta \gamma(t, x) + \frac{1}{2}W'(\gamma(t, x)) \frac{\gamma(t, x)}{\gamma(t, x)} + V(x)\gamma(t, x) + r(\gamma(t, x))$$

where, $r(\gamma(t, x)) \to 0$ in $L^2$, for all $t$, as $\lambda \to 0$, uniformly on every compact set in $\mathbb{R}$. Thus we have, proceeding as in Step 3,

$$
\frac{d}{dt} \int_{\mathbb{R}^N} \text{Im}(\gamma(t, x)\overline{\gamma}(t, x)) = 
$$

$$
= \int_{\mathbb{R}^N} V(x)\nabla|\gamma(t, x)|^2 + 2\text{Re}(r(\gamma(t, x))\nabla\overline{\gamma}(t, x)) dx. \quad (39)
$$

We have

$$
\int_{\mathbb{R}^N} \text{Im}(\gamma(t, x)\overline{\gamma}(t, x)) = \int_{\mathbb{R}^N} \text{Im}(\gamma(0, x)\overline{\gamma}(0, x)) +
$$

$$
+ \int_{0}^{t} \int_{\mathbb{R}^N} V(x)\nabla|\gamma(s, x)|^2 + 2\text{Re}(r(\gamma(s, x))\nabla\overline{\gamma}(s, x)) dx ds \quad (40)
$$
and, for all $s$,

$$\int_{\mathbb{R}^N} V(x) \nabla |\gamma_\lambda(s, x)|^2 + 2\text{Re}(r(\gamma_\lambda(s, x)) \nabla \bar{\gamma}(s, x)) \, dx \to \int_{\mathbb{R}^N} V(x) \nabla |\psi(s, x)|^2$$

as $\lambda \to 0$. Finally,

$$\int_{\mathbb{R}^N} V(x) \nabla |\gamma_\lambda(s, x)|^2 + 2\text{Re}(r(\gamma_\lambda(s, x)) \nabla \bar{\gamma}(s, x)) \, dx \leq \left\|V(x)\gamma_\lambda(s, \cdot)\right\|_{L^2} \left\|\gamma_\lambda(s, \cdot)\right\|_{H^1} + \left\|(r(\gamma_\lambda(s, \cdot))\right\|_{L^2} \left\|\gamma_\lambda(s, \cdot)\right\|_{H^1}. \quad (41)$$

And because $V(x)\gamma_\lambda(s, \cdot) \to V(x)\psi(s, \cdot)$ in $L^2$, $r(\gamma_\lambda(s, \cdot)) \to 0$ in $L^2$ and $\gamma_\lambda(s, \cdot) \to \psi(s, \cdot)$ in $H^1$ uniformly in $s$ on every compact set, we have that for some constant $C$

$$\sup_{s \in [0,t]} \int_{\mathbb{R}^N} V(x) \nabla |\gamma_\lambda(s, x)|^2 + 2\text{Re}(r(\gamma_\lambda(s, x)) \nabla \bar{\gamma}(s, x)) \, dx \leq C. \quad (42)$$

By (41) we get

$$\int_0^t \int_{\mathbb{R}^N} V(x) \nabla |\gamma_\lambda(s, x)|^2 \, dx \, ds + \int_0^t \int_{\mathbb{R}^N} 2\text{Re}(r(\gamma_\lambda(s, x)) \nabla \bar{\gamma}(s, x)) \, dx \, ds \to \int_0^t \int_{\mathbb{R}^N} V(x) \nabla |\psi(s, x)|^2 \, dx \, ds. \quad (43)$$

Furthermore we know

$$\int_{\mathbb{R}^N} \text{Im}(\bar{\gamma}_\lambda(t, x) \gamma_\lambda(t, x)) \to \int_{\mathbb{R}^N} \text{Im}(\bar{\psi}(t, x) \psi(t, x)); \quad (44)$$

$$\int_{\mathbb{R}^N} \text{Im}(\bar{\gamma}_\lambda(0, x) \gamma_\lambda(0, x)) \to \int_{\mathbb{R}^N} \text{Im}(\bar{\psi}(0, x) \psi(0, x)). \quad (45)$$

At this point by (40), (43), (44) and (45) we get

$$\int_{\mathbb{R}^N} \text{Im}(\bar{\psi}(t, x) \psi(t, x)) =$$

$$= \int_{\mathbb{R}^N} \text{Im}(\bar{\psi}(0, x) \psi(0, x)) + \int_0^t \int_{\mathbb{R}^N} V(x) \nabla |\psi(s, x)|^2 \, dx \, ds \quad (46)$$

that concludes the proof.

We have the following corollary
Corollary 11. Assume \( (V_1) \) and the assumptions of the previous theorem; then

\[
\ddot{q}_h(t) = -\int_{\mathbb{R}^N} \nabla V(x) |\psi(t, x)|^2 dx \quad \frac{\|\psi(t, x)\|_{L^2}^2}{\|\psi(t, x)\|_{L^2}^2}.
\]

Proof. By \((V_1)\), we have that

\[
\left| \int_{\mathbb{R}^N} \nabla V(x) |\psi(t, x)|^2 dx \right| \leq \int_{\mathbb{R}^N} V^b(x) |\psi(t, x)|^2 dx \leq C_1 \int_{\mathbb{R}^N} V(x) |\psi(t, x)|^2 dx \leq C_2 G(\psi) < +\infty
\]

where \( G \) is the dynamical energy \((22)\). Thus, we can integrate by parts and we have that

\[
\int_{\mathbb{R}^N} V(x) \nabla |\psi(t, x)|^2 dx = -\int_{\mathbb{R}^N} \nabla V(x) |\psi(t, x)|^2 dx
\]

\[\square\]

Remark 12. If we use the polar form \((15)\), \((34)\) and \((47)\) take the more meaningful form respectively:

\[
\dot{q}_h(t) = \frac{\int_{\mathbb{R}^N} \nabla S u^2 dx}{\int_{\mathbb{R}^N} u^2 dx}
\]

\[
\ddot{q}_h(t) = -\frac{\int_{\mathbb{R}^N} \nabla V(x) u^2 dx}{\int_{\mathbb{R}^N} u^2 dx}
\]

They can be interpreted as follows: \(\dot{q}_h(t)\) is the average momentum (remember \((25)\) and that \(m = 1\)); \(\ddot{q}_h(t)\) equals the average force, since \(\bar{F} \approx -\nabla V\) (see \((30)\)).

3.2 Concentration results

In this section we prove a concentration property of the solution of \((P_h)\) with initial data \((7)\); more exactly, we prove that for any time \(t \in \mathbb{R}\), this solution is a "bump" of radius less than some constant \("R\). In order to prove this
result, it is sufficient to assume that problem \( P_h \) admits global solutions
\( \psi(t, x) \in C(\mathbb{R}, H^1(\mathbb{R}^N)) \) which satisfy the conservation of the energy and of the \( L^2 \) norm, namely it is not necessary to assume the regularity \( (8) \).

For some nonlinearities \( W \), it is possible that the ground state solution is not unique. In any case we have the following result:

**Proposition 13.** Let \( U \) be a ground state solution of \( (3) \). Then, for \( |x| > 1 \) and \( N \geq 2 \)

\[
U(x) < \frac{C}{|x|^\frac{N-2}{2}}
\]

where \( C \) is a constant which does not depend on \( U \).

**Proof.** By a well known inequality due to Strauss [20] we have

\[
0 < U(x) \leq C_N \frac{||U||_{H^1}}{|x|^\frac{N}{2}} \text{ for } |x| \geq \alpha_N \text{ a.e.} \tag{48}
\]

where \( C_N \) and \( \alpha_N \) depend only on the dimension \( N \). Moreover there exists a constant \( C_m \), such that \( ||U||_{H^1} \leq C_m \) for any \( U \) minimizer of \( \inf_{u \in S_\sigma} J(u) = m \). In fact we have the following inequality

\[
||u||_{L^\nu} \leq b_\nu \|u\|_{L^2}^{1 - \frac{N}{2} + \frac{\nu}{N}} ||\nabla u||_{L^2}^{\frac{N}{2} - \frac{\nu}{N}} \tag{49}
\]

for some constant \( b_\nu \). Then, by \( (4) \)

\[
||U||_{L^\nu} \leq b_\nu \sigma^\nu \left| 1 - \frac{N}{2} + \frac{\nu}{N} \right| ||\nabla U||_{L^2}^{\frac{N}{2} - \frac{\nu}{N}} \tag{50}
\]

By assumption \( (W_2) \) and by \( (50) \) we have

\[
m = J(U) \geq \frac{1}{2} ||\nabla U||^2 - c_1 U^2 - c_2 U^\nu dx \geq \\
\geq \frac{1}{2} \int ||\nabla U||^2 - c_3 \left( \int ||\nabla U||^2 \right)^\nu N - \frac{N}{2} - c_1 \sigma^2
\]

for some constant \( c_3 \). If \( 0 < \nu \frac{N}{2} - N < 2 \), namely \( 2 < \nu < 2 + \frac{4}{N} \), we have the claim.

□

**Lemma 14.** For any \( \varepsilon > 0 \), there exists an \( \hat{R} = \hat{R}(\varepsilon) \) and a \( \delta = \delta(\varepsilon) \) such that, for any \( u \in J^{m+\delta} \cap S_\sigma \), we can find a point \( \hat{q} = \hat{q}(u) \in \mathbb{R}^N \) such that

\[
\frac{1}{\sigma^2} \int_{\mathbb{R}^N \setminus B(\hat{q}, \hat{R})} u^2(x) dx < \varepsilon. \tag{51}
\]
Proof. Firstly we prove that for any \( \varepsilon > 0 \), there exists a \( \delta \) such that, for any \( u \in J^{m+\delta} \cap S_\sigma \), we can find a point \( \hat{q} = \hat{q}(u) \in \mathbb{R}^N \) and a radial ground state solution \( U \) such that
\[
||u(x) - U(x - \hat{q})||_{H^1} \leq \varepsilon.
\] (52)

We argue by contradiction. Suppose that there exists an \( \varepsilon > 0 \) and a sequence \( \{u_n\}_n \) such that \( ||u_n||_{L^2} = \sigma \), \( J(u_n) \to m \) and, for any \( q \in \mathbb{R}^N \) and for each \( U \) ground state solution it holds
\[
\varepsilon < ||u(x) - U(x - q)||_{H^1}.
\] (53)

By the Ekeland principle we can assume that \( \{u_n\} \) is a Palais Smale sequence for \( J \) on \( S_\sigma \), that is, there exists \( \{\lambda_n\} \) such that
\[
-\Delta u_n + W'(u_n) - \lambda_n u_n \to 0 \quad \text{as} \quad n \to \infty.
\] (54)

By [2, Proposition 11] up to a subsequence we have that \( \lambda_n \to \bar{\lambda} < 0 \). So we get
\[
-\Delta u_n + W'(u_n) - \bar{\lambda} u_n \to 0; \quad J(u_n) - \bar{\lambda} \int u_n^2 \to m - \bar{\lambda} \sigma^2.
\] (55, 56)

As a consequence of the Concentration Compactness principle [17, 18], we can describe the behavior of this P.S. sequence. We use the Splitting Lemma (see [21], [2]) and we get
\[
u_n = \sum_{j=1}^k U_j^j(x - q_j^n) + w_n \quad \text{with} \quad w_n \to 0 \quad \text{in} \quad H^1
\] (57)
\[
\sigma^2 = \sum_{j=1}^k ||U_j^j(x - q_j^n)||_{L^2}^2
\] (58)
\[
\sum_{j=1}^k J(U_j(x - q_j^n)) = m = I_\sigma^2
\] (59)

where \( U_j^j \) are solutions of \( -\Delta u + W'(u) = \bar{\lambda} u \) and \( q_j^n \in \mathbb{R}^N \).

Here \( I_\mu^2 = \min_{||u||_{L^2} = \mu^2} J(u) \). We recall (see [17]) that for any \( \mu \in (0, \rho) \) we have
\[
I_{\mu^2} < I_{\mu^2 + I_{\rho^2 - \mu^2}}.
\] (60)
We verify that in (57)-(59) it is $k = 1$. We assume $k = 2$. Suppose that $||U^1||_{L^2} = \mu^2 < \sigma^2$. Then, by (60), we have a contradiction because

$$I_\sigma^2 < I_{\mu^2} + I_{\sigma^2 - \mu^2} \leq J(U^1) + J(U^2) = I_\sigma^2. \quad (61)$$

For the case $k > 2$ we argue analogously.

Thus we have, up to subsequence,

$$u_n(x) = U(x - q_n) + w_n \quad w_n \to 0 \text{ in } H^1 \quad (62)$$

for some $U$ radial ground state solution, and (62) contradicts (53).

At this point, given $\varepsilon$, there exist a point $\hat{q} = \hat{q}(u) \in \mathbb{R}^N$ and a radial ground state solution $U$ such that

$$u(x) = U(x - \hat{q}) + w \text{ and } ||w||_{H^1} \leq C\varepsilon. \quad (63)$$

Now, we choose $\hat{R}$ such that

$$\frac{1}{\sigma^2} \int_{\mathbb{R}^N \setminus B(0, \hat{R})} U^2(x) dx < C\varepsilon \quad (64)$$

for all $U$ radial ground state solutions. This is possible because, if $U$ is a radial minimizer of $J(u)$ on $S_\sigma$, then, as showed in following Remark 13,

$$U(x) \leq \frac{C(m, N)}{x^{N-1}} \text{ for } |x| >> 1, \ N \geq 2, \quad (65)$$

the constant $C(m, N)$ depending only on the dimension $N$ and on $m = \inf_{u \in S_\sigma} J(u)$.

We get

$$\frac{1}{\sigma^2} \int_{B(\hat{q}, \hat{R})^C} u^2(x) dx < \frac{1}{\sigma^2} \int_{B(\hat{q}, \hat{R})^C} U^2(x - \hat{q}) dx + \frac{1}{\sigma^2} \int_{B(\hat{q}, \hat{R})^C} w^2 + 2wU dx$$

$$= \frac{1}{\sigma^2} \int_{B(0, \hat{R})^C} U^2(x) dx + \frac{1}{\sigma^2} \int_{B(\hat{q}, \hat{R})^C} w^2 + 2wU dx. \quad (66)$$

By (63), (64), (66) we get the claim. We notice also the $\hat{R}$ does not depend on $u, U$.

We can describe now the concentration properties of the solution of $(P_\lambda)$. 

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Lemma 15. For any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ and a $\hat{R} = \hat{R}(\varepsilon)$ such that for any $\psi(t, x)$ solution of (\(P_h\)) with $|h^{-\gamma}\psi(t, h^\beta x)| \in J^{m+\delta} \cap S_\sigma$ for all $t$ there exists a $\hat{q}_h(t) \in \mathbb{R}^N$ for which

$$\frac{1}{\sigma^2 h^{N\beta - 2\gamma}} \int_{\mathbb{R}^N \setminus B(\hat{q}_h(t), \hat{R} h^\beta)} |\psi(t, x)|^2 dx < \varepsilon. \quad (67)$$

Here $\hat{q}_h(t)$ depends on $\varepsilon$ and $\psi(t, x)$.

**Proof.** Fixed $h$ and $t$, we set $v(\xi) = |h^{-\gamma}\psi(t, h^\beta \xi)|$. So we have

$$m < J(v) \leq m + \delta \quad \text{and} \quad ||v||_{L^2} = \sigma.$$  

So, by Lemma 14 we have that there exist an $\hat{R} > 0$ and a $\bar{q} = \bar{q}(v)$ such that

$$\varepsilon > \frac{1}{\sigma^2} \int_{\mathbb{R}^N \setminus B(\bar{q}, \hat{R})} |v(\xi)|^2 d\xi \quad (68)$$

By a change of variable we obtain

$$\varepsilon > \frac{1}{\sigma^2} \int_{\mathbb{R}^N \setminus B(\bar{q}, \hat{R})} |v(\xi)|^2 d\xi = \frac{1}{\sigma^2 h^{N\beta - 2\gamma}} \int_{\mathbb{R}^N \setminus B(\hat{q}_h(t), \hat{R} h^\beta)} |\psi(t, x)|^2 dx, \quad (69)$$

where $\hat{q}_h(t)$ depends on $\varepsilon, h, t$ and $\psi$, while $\hat{R}$ depends only by $\varepsilon$.

We give now some results about the concentration property of the solutions $\psi(t, x)$ of the problem (\(P_h\)). Given $K > 0$ $q \in \mathbb{R}^N$, $h > 0$ we call

$$B_h^{K, q} = \left\{ \begin{array}{l}
\psi(0, x) = u_h(0, x) e^{i S_h(0, x)} \\
\text{with } u_h(0, x) = h^{-\gamma} [ (U + w) \left( \frac{x - q}{h^\beta} \right) ] \\
U \text{ is a radial ground state solution} \\
||U + w||_{L^2} = ||U||_{L^2} = \sigma \text{ and } ||w||_{H^1} < Kh^{\alpha - \gamma} \\
||\nabla S_h(0, x)||_{L^\infty} \leq K \text{ for all } h \\
\int_{\mathbb{R}^N} V(x) u_h^2(0, x) dx \leq Kh^{N\beta - 2\alpha}.
\end{array} \right\} \quad (70)$$

the set of admissible initial data.
Remark 16. The condition $\|w\|_{H^1} \leq Kh^{\alpha-\gamma}$ can be weakened. Indeed in the proof of the theorem we need $J(U+w) \leq m+Kh^{\alpha-\gamma}$, which is implied by $\|w\|_{H^1} \leq Kh^{\alpha-\gamma}$. We prefer to refer to the strongest but simpler hypotheses to simplify the statement of the main theorem.

Remark 17. In Theorem 1 we assume $S_h(0,x) = v \cdot x$ which is more stronger than $\|\nabla S_h(0,x)\|_{L^\infty} \leq K$ to simplify the statement and for a better physical interpretation.

Finally, we can prove the main result of this section.

Theorem 18. Assume $V \in L^\infty_{loc}$ and (\ref{V}). Fix $K > 0$, $q \in \mathbb{R}^N$. Let $\alpha > \gamma$.

For all $\varepsilon > 0$, there exists $\hat{R} > 0$ and $h_0 > 0$ such that, for any $\psi(t,x)$ solution of \((P_h)\) with initial data $\psi(0,x) \in B_{h}^{K,q}$ with $h < h_0$, and for any $t$, there exists $\hat{q}(t) \in \mathbb{R}^N$ for which

$$\frac{1}{||\psi(t,x)||_{L^2}^2} \int_{\mathbb{R}^N \setminus B(\hat{q}(t),\hat{R}h^{\beta})} |\psi(t,x)|^2 dx < \varepsilon.$$  \hspace{1em} (71)

Here $\hat{q}(t)$ depends on $\psi(t,x)$.

Proof. By the conservation law, the energy $E_h(\psi(t,x))$ is constant with respect to $t$. Then we have

$$E_h(\psi(t,x)) = E_h(\psi(0,x))$$

$$= J_h(u_h(0,x)) + \int_{\mathbb{R}^N} u_h^2(0,x) \left[ \frac{\|\nabla S_h(0,x)\|^2}{2} + V(x) \right] dx$$

$$\leq J_h(u_h(0,x)) + \frac{K}{2} \sigma^2 \alpha^{2N-2}\gamma + Kh^{N\beta-2\gamma}$$

$$= h^{N\beta-\alpha-\gamma} J(U+w) + h^{N\beta-2\gamma}C$$

where $C$ is a suitable constant. Now, by rescaling, and using that $\psi(0,x) \in B_{h}^{K,q}$, we obtain

$$E_h(\psi(t,x)) = h^{N\beta-\gamma-\alpha} J(U+w) + Ch^{N\beta-2\gamma}$$

$$\leq h^{N\beta-\gamma-\alpha}(m+Kh^{\alpha-\gamma}) + Ch^{N\beta-2\gamma}$$

$$= h^{N\beta-\gamma-\alpha}(m+Kh^{\alpha-\gamma} + Ch^{\alpha-\gamma}) = h^{N\beta-\gamma-\alpha}(m+h^{\alpha-\gamma}C_1)$$  \hspace{1em} (72)

where $C_1$ is a suitable constant. Thus

$$J_h(u_h(t,x)) = E_h(\psi(t,x)) - G_h(\psi(t,x))$$

$$= E_h(\psi(t,x)) - \int_{\mathbb{R}^N} \left[ \frac{\|\nabla S_h(t,x)\|^2}{2} + V(x) \right] u_h(t,x)^2 dx$$

$$\leq h^{N\beta-\gamma-\alpha}(m+h^{\alpha-\gamma}C_1)$$  \hspace{1em} (73)
because $V \geq 0$. By rescaling the inequality (73) we get

$$J(h^{-\gamma}u_h(t, h^\beta x)) \leq m + h^{\alpha-\gamma}C_1 \quad (74)$$

So, if $\alpha > \gamma$, for $h$ small we can apply Lemma 15 and we get the claim.

\[\square\]

4 The final result

4.1 Barycenter and concentration point

In this paragraph, we estimate the distance between the concentration point and the barycenter of a solution $\psi(t, x)$ for a potential satisfying hypothesis $[V_0]$ and $[V_2]$. Hereafter, fixed $K > 0$, we assume that $\psi(t, x)$ is a global solution of the Schrödinger equation $(P_h)$, $\psi(t, x) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$, with initial data $\psi(0, x) \in B^K_{h,q}$ with $B^K_{h,q}$ given by (70).

Lemma 19. There exists a constant $L > 0$ such that

$$0 \leq \frac{1}{h^{N\beta-2\alpha}} \int_{\mathbb{R}^N} V(x)u_h^2(t, x)dx \leq L \quad \forall t \in \mathbb{R}.$$

Proof. At first we notice that $||h^{-\gamma}u_h(t, h^\beta x)||^2_{L^2} = ||h^{-\gamma}u_h(0, h^\beta x)||^2_{L^2} = ||U+w||^2_{L^2} = \sigma^2$. Thus

$$J_h(u_h(t, x)) = h^{N\beta-\gamma-\beta}J(h^{-\gamma}u_h(t, h^\beta x)) \geq h^{N\beta-\gamma-\beta}m. \quad (75)$$

By (72), there exist a constant $L$ such that

$$E_h(\psi(t, x)) \leq h^{N\beta-\gamma-\alpha}m + Lh^{N\beta-2\gamma}. \quad (76)$$

Finally,

$$\int_{\mathbb{R}^N} V(x)u_h^2(t, x)dx = E_h(\psi(t, x)) - J_h(u_h(t, x)) - \int_{\mathbb{R}^N} \frac{\vert\nabla S\vert^2}{2}u_h^2(t, x)dx$$

$$\leq E_h(\psi(t, x)) - J_h(u_h(t, x)) \leq h^{N\beta-\gamma-\alpha}m + Lh^{N\beta-2\gamma} - h^{N\beta-\gamma-\beta}m = Lh^{N\beta-2\gamma}$$

that concludes the proof.

\[\square\]
Remark 20. By Lemma 19 we get, for any $R_2 \geq R_1$ ($R_1$ given in (V2)) and for any $t \in \mathbb{R}$ the following inequality

$$L \geq \frac{1}{h^{N\beta-2\gamma}} \int_{|x| \geq R_2} V(x) u_h^2(t, x) dx$$

and

$$\geq \frac{1}{h^{N\beta-2\gamma}} \int_{|x| \geq R_2} |x|^a u_h^2(t, x) dx \geq \frac{R_2^{a-1}}{h^{N\beta-2\gamma}} \int_{|x| \geq R_2} |x| u_h^2(t, x) dx$$

(77)

Lemma 21. There exists a constant $K_1$ such that

$$|q_h(t)| \leq K_1$$

for $t \in \mathbb{R}$.

By Lemma 19 and Remark 20 we have that

$$\left| \int_{\mathbb{R}^N} x u_h^2(t, x) dx \right| \leq \int_{|x| \geq R_1} |x| u_h^2(t, x) + \int_{|x| < R_1} |x| u_h^2(t, x)$$

$$\leq R_1 \int_{\mathbb{R}^N} u_h^2(t, x) dx + \frac{L}{R_1} \sigma^2 R_2^a$$

So, using the definition of $q_h(t)$ we have

$$|q_h(t)| \leq R_1 + \frac{L}{R_1^{a-1} \sigma^2} = K_1,$$

(78)

for some $K_1 > 0$.

Remark 22. By the inequality (77) in Remark 20 we have also that, for any $R_2 \geq R_1$,

$$\frac{\int_{|x| \geq R_2} u_h^2(t, x) dx}{\int_{\mathbb{R}^N} u_h^2(t, x) dx} \leq \frac{L}{\sigma^2 R_2}$$

for all $t \in \mathbb{R}$.

Hereafter, we always choose $R_2$ large enough to have

$$\frac{L}{\sigma^2 R_2^a} < \frac{1}{2}$$

(79)

Now we show the boundedness of the concentration point $\hat{q}_h(t)$ defined in Lemma (15).

Lemma 23. Given $0 < \varepsilon < 1/2$, and $R_2$ as in the previous remark.

We get

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1. \( \sup_{t \in \mathbb{R}} |\hat{q}_h(t)| < R_2 + \hat{R}(\varepsilon)h^\beta < R_2 + 1 \), for all \( h < \bar{h} \) and \( \delta < \bar{\delta} \) small enough.

2. \( \sup_{t \in \mathbb{R}} \left| q_h(t) - \hat{q}_h(t) \right| < \frac{3L}{\sigma^2 R_3} + 3R_3\varepsilon + \hat{R}(\varepsilon)h^\beta \), for any \( R_3 \geq R_2 \), and for all \( h \) small enough.

**Proof.**

**Step 1.** We prove the boundedness of the concentration point \( \hat{q}_h(t) \).

By the Theorem 18, with \( \varepsilon < 1/2 \), and by Remark 22, it is obvious that the ball \( B(\hat{q}_h(t), \hat{R}(\varepsilon)h^\beta) \) is not contained in the set \( \mathbb{R}^N \setminus B(0, R_2) \), and we have

\[ B(\hat{q}_h(t), \hat{R}(\varepsilon)h^\beta) \subset B(0, R_2 + 2\hat{R}(\varepsilon)h^\beta). \]  

(80)

Because \( \hat{R}(\varepsilon) \) does not depend on \( h \), we can assume \( h \) so small that \( 2\hat{R}(\varepsilon)h^\beta < 1 \). Then

\[ |\hat{q}_h(t)| < R_2 + 2\hat{R}(\varepsilon)h^\beta < R_2 + 1; \]

(81)

\[ B(\hat{q}_h(t), \hat{R}(\varepsilon)h^\beta) \subset B(0, R_2 + 1); \]  

(82)

This concludes the proof of the first claim.

**Step 2.** We estimate the difference between the barycenter and the concentration point.

We have

\[ \left| q_h(t) - \hat{q}_h(t) \right| = \left| \frac{\int_{\mathbb{R}^N} (x - \hat{q}_h(t))u_h^2(t, x)dx}{\int_{\mathbb{R}^N} u_h^2(t, x)dx} \right| \]

(83)

and we split the integral in three parts, with \( R_3 \geq R_2 \):

\[ I_1 = \left| \frac{\int_{\mathbb{R}^N \setminus B(0, R_3)} (x - \hat{q}_h(t))u_h^2(t, x)dx}{\int_{\mathbb{R}^N} u_h^2(t, x)dx} \right|; \]

\[ I_2 = \left| \frac{\int_{A_2} (x - \hat{q}_h(t))u_h^2(t, x)dx}{\int_{\mathbb{R}^N} u_h^2(t, x)dx} \right| \quad \text{where } A_2 = B(0, R_3) \setminus B(\hat{q}_h(t), \hat{R}(\varepsilon)h^\beta); \]

\[ I_3 = \left| \frac{\int_{A_3} (x - \hat{q}_h(t))u_h^2(t, x)dx}{\int_{\mathbb{R}^N} u_h^2(t, x)dx} \right| \quad \text{where } A_3 = B(0, R_3) \cap B(\hat{q}_h(t), \hat{R}(\varepsilon)h^\beta). \]
It’s trivial that $I_3 \leq \hat{R}(\varepsilon) h^\beta$. By Lemma 23 and by Theorem 18 we have

$$I_2 \leq [2R_3 + 1] \varepsilon. \quad (84)$$

By Step 1 and Remark 22 we have

$$\left| \hat{q}_h(t) \right| \frac{\int_{\mathbb{R}^N \setminus B(0,R_?)} u_h^2(t,x) dx}{\int_{\mathbb{R}^N} u_h^2(t,x) dx} < (R_3 + 1) \frac{L}{\sigma_2 R_3^\alpha} < \frac{2L}{\sigma_2 R_3^\alpha-1}. \quad (85)$$

Also, by Remark 20

$$\frac{\int_{\mathbb{R}^N \setminus B(0,R_3)} |x| u_h^2(t,x) dx}{\int_{\mathbb{R}^N} u_h^2(t,x) dx} \leq \frac{L}{\sigma_2 R_3^\alpha-1}, \quad (86)$$

hence

$$I_1 \leq \frac{3L}{\sigma_2 R_3^\alpha-1}. \quad (87)$$

Concluding, we have that

$$\left| q_h(t) - \hat{q}_h(t) \right| < \frac{3L}{\sigma_2 R_3^\alpha-1} + 3R_3 \varepsilon + \hat{R}(\varepsilon) h^\beta, \quad (88)$$

for all $t \in \mathbb{R}$.

□

We notice that $R_1, R_2$ and $R_3$ defined in this section do not depend on $\varepsilon$.

4.2 Equation of the travelling soliton

We prove that the barycenter dynamics is approximatively that of a point particle moving under the effect of an external potential $V(x)$.

**Theorem 24.** Assume that $V$ satisfies $(V_0), (V_1), (V_2)$. Given $K > 0$, $q \in \mathbb{R}^N$, let $\psi(t,x) \in C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1)$ be a global solution of equation $(P_h)$, with initial data in $B_h^{K,q}$, $h < h_0$. Then we have

$$\ddot{q}_h(t) + \nabla V(q_h(t)) = H_h(t) \quad (89)$$

with $\|H_h(t)\|_{L^\infty}$ goes to zero when $h$ goes to zero.

**Proof.** We know by Theorem 9 that

$$\ddot{q}_h(t) + \frac{\int_{\mathbb{R}^N} \nabla V(x) u_h^2(t,x) dx}{\int_{\mathbb{R}^N} u_h^2(t,x) dx} = 0 \quad (90)$$

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Hence we have to estimate the function

\[ H_h(t) = [\nabla V(\hat{q}_h(t)) - \nabla V(q_h(t))] + \int_{\mathbb{R}^N} \frac{[\nabla V(x) - \nabla V(\hat{q}_h(t))] u_h^2(t, x) dx}{\int_{\mathbb{R}^N} u_h^2(t, x) dx}. \]  

(91)

We set

\[ M = \max_{|\alpha| \leq K_1 + R_2 + 1} |\partial^\alpha V(\tau)| \]  

(92)

where \( K_1 \) is defined in Lemma 21 and \( R_2 \) is defined in Remark 22.

By Lemma 21 and Lemma 23 we get

\[ |\nabla V(\hat{q}_h(t)) - \nabla V(q_h(t))| \leq \max_{i, j = 1, \ldots, N} \left| \frac{\partial^2 V(\tau)}{\partial x_i \partial x_j} \right| |\hat{q}_h(t) - q_h(t)| \leq M \left[ \frac{3L}{\sigma^2 R_3^2} + 3R_3 \varepsilon + \hat{R}(\varepsilon) h^\beta \right], \]  

(93)

for any \( R_3 \geq R_2 \).

To estimate

\[ \frac{\int_{\mathbb{R}^N} [\nabla V(x) - \nabla V(\hat{q}_h(t))] u_h^2(t, x) dx}{\int_{\mathbb{R}^N} u_h^2(t, x) dx} \]

we split the integral three parts.

\[ L_1 = \frac{\int_{B(\hat{q}_h(t), R(\varepsilon) h^\beta)} |\nabla V(x) - \nabla V(\hat{q}_h(t))| u_h^2(t, x) dx}{\int_{\mathbb{R}^N} u_h^2(t, x) dx}; \]

\[ L_2 = \frac{\int_{\mathbb{R}^N \setminus B(\hat{q}_h(t), R(\varepsilon) h^\beta)} |\nabla V(x)| u_h^2(t, x) dx}{\int_{\mathbb{R}^N} u_h^2(t, x) dx}; \]

\[ L_3 = \frac{\int_{\mathbb{R}^N \setminus B(\hat{q}_h(t), R(\varepsilon) h^\beta)} |\nabla V(\hat{q}_h(t))| u_h^2(t, x) dx}{\int_{\mathbb{R}^N} u_h^2(t, x) dx}. \]

By the Theorem 18 and by Lemma 23 we have \( L_3 < M \varepsilon. \)
By (82) and Lemma 23 we have
\[
L_1 \leq \beta h^\beta \int_{B(\hat{q}_h(t), \hat{R}(\varepsilon) h^d)} \frac{\partial^2 V(\tau)}{\partial x_i \partial x_j} \hat{R}(\varepsilon) h^\beta u_h^2(t, x) dx
\]
\[
\int_{\mathbb{R}^N} u_h^2(t, x) dx
\]
\[
\leq M \hat{R}(\varepsilon) h^\beta.
\]

Now, using hypothesis (V), equation (82), Theorem 18 and Remark 20, we have
\[
\int_{B(0, R_2 + 1) \setminus B(\hat{q}_h(t), \hat{R}(\varepsilon) h^d)} \left| \nabla V(x) \right| \frac{u_h^2(t, x)}{||u_h(t, \cdot)||^2_{L^2}} \leq M \varepsilon.
\]

where \( b \in (0, 1) \) is defined in (V). Furthermore, again by Theorem 18 we have
\[
\int_{B(0, R_2 + 1) \setminus B(\hat{q}_h(t), \hat{R}(\varepsilon) h^d)} \left| \nabla V(x) \right| \frac{u_h^2(t, x)}{||u_h(t, \cdot)||^2_{L^2}} dx \leq M \varepsilon + \left[ \frac{L}{\sigma^2} \right]^b \varepsilon^1 - b.
\]

Concluding
\[
\text{L}_2 \leq M \varepsilon + \left[ \frac{L}{\sigma^2} \right]^b \varepsilon^1 - b + M \hat{R}(\varepsilon) h^\beta.
\]
Finally, by (93) and (97) we have

$$|H_h(t)| \leq \frac{3LM}{\sigma^2 R_3^{-1}} + \left[ \frac{L}{\sigma^2} \right]^b \varepsilon^{1-b} + M(2 + 3R_3)\varepsilon + 2M\hat{R}(\varepsilon)h^\beta. \quad (98)$$

At this point we can have $\sup_{t} |H_h(t)|$ arbitrarily small choosing firstly $R_3$ sufficiently large, secondly $\varepsilon$ sufficiently small, and finally $h$ small enough.

$\square$

**Proof of Theorem 1.** By Theorem 24 we get immediately the proof of Theorem 1.

$\square$

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