Mapping class groupoids and Thompson’s groups

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Abstract

We concoct a uniform treatment of mapping class groups with Thompson’s groups and to generalize them. We give a description of the outer automorphism group of free groups as the isotropy group of a groupoid, akin to the mapping class groupoid of Penner and Mosher. We also illustrate some arithmetic aspects of these groupoids.

Keywords Mapping class group, mapping class groupoid, outer automorphism of a free group, Thompson’s groups, Universal Ptolemy group, circle homeomorphisms, tree boundary, flip, modular graph, modular group.

1 Introduction

Let \( S = S_n^g \) (\( n \geq 1 \)) be an oriented surface of genus \( g \) and with \( n \geq 1 \) punctures. We assume that \( g \) and \( n \) are finite but this hypothesis will be relaxed later.

Our aim in this paper is twofold. Firstly, we give a unified picture for

- \( \text{Mod}(S) \) (mapping class group of \( S \))
- \( T \) (Thompson’s group under its guise as Penner’s universal Ptolemy group)
- \( F \) (Thompson’s group)

and simultaneously generalize them. These groups are the isotropy groups of one disconnected groupoid \( \Pi MG \), which we call the \textit{fundamental modular groupoid}. One has

\[
\text{Mod}(S) < \text{MCG}(S) < \Pi MG < \Pi MG
\]  

\((1)\)

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where $\text{MCG}(S)$ is Penner’s mapping class groupoid of $S$ under its form defined by Mosher, see [9], [10]. The groupoid $\Pi\text{MG}$ is a certain completion of $\Pi\text{MG}$ that we introduce later in the text.

Secondly, we introduce a bigger disconnected groupoid $\text{OMG}$ called the outer modular groupoid, which contains $\Pi\text{MG}$ as a subgroupoid with the same set of objects. Isotropy subgroups of $\text{OMG}$ gives a unified picture for

- $\text{Out}(\mathcal{F}_d)$ (outer automorphism groups of free groups)
- $\mathcal{V}$ (Thompson’s group)

and simultaneously generalizes them. One has

$$\text{Out}(\pi_1(S)) < \text{OMG}(S) < \text{OMG} < \overline{\text{OMG}}$$  \hspace{1cm} (2)

where $\text{OMG}(S)$ is the connected subgroupoid of $\text{OMG}$ which contains $\text{MCG}(S)$ and $\overline{\text{OMG}}$ is a certain completion of $\text{OMG}$ that we introduce later in the text.

Hence, these groupoids fits in the following diagram

$$\begin{array}{ccc}
\Pi\text{MG} & < & \text{OMG} \\
\lor & & \lor \\
\Pi\text{MG} & < & \text{OMG} \\
\lor & & \lor \\
\text{MCG}(S) & < & \text{OMG}(S) \\
\lor & & \lor \\
\text{Mod}(S) & < & \text{Out}(\pi_1(S))
\end{array}$$  \hspace{1cm} (3)

As we shall see, the column on the left is related to the group $\text{Homeo}^0(S^1)$ of homeomorphisms of the circle $S^1 := \mathbb{R} \cup \{\infty\}$ preserving the set of rationals and the one on the right is related to the group $\text{Homeo}(\partial F)$ of homeomorphisms of $\partial F$; the boundary of the planar trivalent tree (it is a Cantor set).

**Notation.** We use sans serif fonts $\mathbb{Z}, \mathbb{T}, \ldots$ to denote groups, bold letters $\mathbf{X}, \mathbf{S}, \ldots$ to denote groupoids and categories and finally calligraphic letters $\mathcal{F}, \mathcal{G}, \ldots$ to denote graphs.

## 2 Mapping Class Groupoids

Here we recall the definitions of the mapping class groupoids introduced by Penner [10] as exposed by Mosher [9]. Let $S = S^g_n$ be an orientable surface of genus $g$ and with $n \geq 1$ punctures. Denote by $\text{Mod}(S)$ the mapping class group of $S$ which is defined to be the group of orientation-preserving homeomorphisms of $S$ preserving
setwise the set of free homotopy classes of simple loops around the punctures of $S$ modulo isotopies.

In this section, we assume that $S$ is of finite type, i.e. $g, n < \infty$.

**Definition 2.1.** An *ideal arc* of $S$ is an embedded arc connecting punctures in $S$, which is not homotopic to a point relative to punctures. An *ideal cell decomposition of $S$* is a collection of ideal arcs so that each region complementary to arcs is a polygon with vertices among the punctures. A maximal ideal cell composition is called an *ideal triangulation*.

Denote the set of triangulations of $S$ modulo isotopy as

$$\text{TRN}(S) := \{\text{triangulations of } S\}/\text{isotopies}$$

and those with a distinguished oriented edge (doe) modulo isotopy as

$$\overrightarrow{\text{TRN}}(S) := \{\text{triangulations of } S \text{ with a doe}\}/\text{isotopies}$$

$\text{TRN}(S)$ and $\overrightarrow{\text{TRN}}(S)$ are countably infinite discrete sets.

Due to the existence of triangulations with automorphisms, the obvious $\text{Mod}(S)$-action on $\text{TRN}(S)$ is not free. On the other hand, the $\text{Mod}(S)$-action on $\overrightarrow{\text{TRN}}(S)$ is free [9]. Suppose that $J$ is a set with a freely acted upon by a group $H$ from the left. If this action is free, then we can associate to this action a groupoid $[H \backslash J]$, whose objects are $H$-orbits of $J$ and morphisms are $H$-orbits of $J \times J$, i.e.

$$\text{Obj}([H \backslash J]) := H \backslash J = \{Hx : x \in J\},$$

$$\text{Mor}_{[H \backslash J]}(Hx, Hy) := \{H(x', y') | Hx' = Hx, Hy' = Hy\}.$$  \hspace{1cm} (7)

The composition $H(x, y) \circ H(y, z)$ is defined as $H(x, z)$. Isotropy groups of this groupoid are isomorphic to $H$.

**Definition 2.2.** The mapping class groupoid $\text{MCG}(S)$ is the groupoid associated to the free $\text{Mod}(S)$-action on $\overrightarrow{\text{TRN}}(S)$, i.e.

$$\text{MCG}(S) := \overrightarrow{\text{TRN}}(S)/\text{Mod}(S).$$

Thus, the isotropy group of any element in $\text{MCG}(S)$ is isomorphic to $\text{Mod}(S)$. The object set of $\text{MCG}(S)$ is the set

$$|\overrightarrow{\text{TRN}}(S)| := \overrightarrow{\text{TRN}}(S)/\text{Mod}(S)$$

which consists of combinatorial types of triangulations of $S$ with a doe. This set is finite if $S$ is of finite type. Morphisms of $\text{MCG}(S)$ are the $\text{Mod}(S)$-orbits

$$\text{Mor}_{\text{MCG}}(|T, \bar{e}|, |T', \bar{e}'|) = \text{Mod}(S)([T, \bar{e}], [T', \bar{e}']),$$

where $|T, \bar{e}| \in \overrightarrow{\text{TRN}}(S)$ denotes the isotopy type of the triangulation-with-a-doe $(T, \bar{e})$ and $|T, \bar{e}| \in \text{Obj}(\text{MCG})$ denotes its combinatorial type.
2.1 The dual picture: spines

A (topological) graph $\Gamma$ is a one-dimensional CW-complex comprised of vertices $V(\Gamma)$ and open edges $E(\Gamma)$; a ribbon graph or fatgraph is a topological graph together with a cyclic ordering of edges emanating from each vertex.

Let $\Gamma \hookrightarrow S$ be an embedding of a topological graph $\Gamma$. We say that $\Gamma \hookrightarrow S$ is a spine of $S$ if it is dual to an ideal cell decomposition $T$ of $S$. In other words, the image of $\Gamma \hookrightarrow S$ is obtained by putting the vertices of $\Gamma$ inside the cells of $T$, picking some paths connecting these vertices by passing through a unique ideal arc of the cell decomposition, and finally by identifying the edges of $\Gamma$ by these paths.

Every spine $\Gamma \hookrightarrow S$ is a strong deformation retract of $S$. Note that $\Gamma$ acquires via the embedding $\Gamma \hookrightarrow S$ a natural ribbon graph structure from the orientation of $S$. Note also that a spine dual to an ideal triangulation is a trivalent ribbon graph.

Denote the set of isotopy classes of trivalent ribbon graph spines of $S$ as $\text{SPN}(S) := \{ \phi : \Gamma \hookrightarrow S : \text{is a spine} \}/\text{isotopy}.$

and denote the set of isotopy classes of trivalent ribbon graph spines with a doe as $\overline{\text{SPN}}(S) := \{ \phi : (\Gamma, \vec{e}) \hookrightarrow S : \text{is a spine with a doe} \}/\text{isotopy}.$

$\text{SPN}(S)$ and $\overline{\text{SPN}}(S)$ are countably infinite discrete sets. Now, $\text{Mod}(S)$ acts by post-composition on $\text{SPN}(S)$, and it acts freely on $\overline{\text{SPN}}(S)$. Therefore we have the associated groupoid $[\overline{\text{SPN}}(S)/\text{Mod}(S)]$, whose object set is the set of combinatorial types of trivalent ribbon graph spines

$|\overline{\text{SPN}}(S)| := \overline{\text{SPN}}(S)/\text{Mod}(S)$.

This is a finite set if $S$ is of finite type. Morphisms of $[\overline{\text{SPN}}(S)/\text{Mod}(S)]$ are the orbits

$\text{Mor}_{\text{SPN}}(|\Gamma, \vec{e}|, |\Gamma, \vec{e}'|) = \text{Mod}(S)(|\Gamma, \vec{e}| \hookrightarrow S, |\Gamma', \vec{e}'| \hookrightarrow S),$ 

where $|\Gamma, \vec{e}| \hookrightarrow S \in \overline{\text{SPN}}(S)$ denotes the isotopy type of the spine-with-a-doe $(\Gamma, \vec{e}) \hookrightarrow S$ and $|\Gamma, \vec{e}|$ denotes its combinatorial type.

By duality, we have bijections

$\text{SPN}(S) \leftrightarrow \text{TRN}(S), \quad \overline{\text{SPN}}(S) \leftrightarrow \overline{\text{TRN}}(S)$

which are compatible with the $\text{Mod}(S)$-action. Hence we have the duality isomorphism

$\text{MCG}(S) = [\overline{\text{TRN}}(S)/\text{Mod}(S)] \simeq [\overline{\text{SPN}}(S)/\text{Mod}(S)]$.

From now on, we shall use the second description of $\text{MCG}(S)$, i.e. we will consider it to be the groupoid $[\overline{\text{SPN}}(S)/\text{Mod}(S)]$. 

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2.2 Flips

Given an ideal triangulation $\mathcal{T}$ of $S$ with an arc $f$ of this triangulation, one obtains a new ideal triangulation by replacing the given arc by the other diagonal of the quadrilateral formed by the neighbouring arcs as depicted in Figure 2. This operation is well-defined on the set of isotopy classes and is called a flip. This defines an involution of the set $\text{TRN}^*(S)$ of pairs of isotopy classes $[\mathcal{T}, f]$, where $\mathcal{T} \in \text{TRN}(S)$ and $f$ is an arc of $\mathcal{T}$. By duality there is an operation on trivalent ribbon graph spines, which replaces an H-shaped part of a spine $\mathcal{G}$ of $S$ by an I-shaped graph, thereby producing a new spine $\mathcal{G}'$ of $S$ as depicted in Figure 2. This operation is also called a flip, which defines the dual involution

$$\phi : \text{SPN}^*(S) \to \text{SPN}^*(S),$$

$\text{SPN}^*(S)$ being the set of isotopy classes of pairs $[(\mathcal{G}, f) \hookrightarrow S]$, where $\mathcal{G} \hookrightarrow S$ is a spine and $f$ is an edge of $\mathcal{G}$.

If $f$ is an edge of $[\mathcal{G} \hookrightarrow S] \in \text{SPN}(S)$ and $\phi([\mathcal{G} \hookrightarrow S], f) = ([\mathcal{G}' \hookrightarrow S], f')$ then define

$$\phi_f([\mathcal{G} \hookrightarrow S]) := [\mathcal{G}' \hookrightarrow S]$$

(13)

Since $\phi$ commutes with the $\text{Mod}(S)$-action it induces an involution

$$\phi : |\text{SPN}^*(S)| \to |\text{SPN}^*(S)|,$$

(14)

where $|\text{SPN}^*(S)|$ is the set of combinatorial types of spines with an arc. Moreover, $\phi$ induces an obvious pairing between the (oriented) edges of $\mathcal{G}$ and $\mathcal{G}'$:

$$\hat{\phi}_f : E(\mathcal{G}) \longrightarrow E(\mathcal{G}').$$

(15)
We have the following combinatorial result:

**Lemma 2.3.** (Whitehead [12] see also [7].) Any two elements of \( TRN(S) \) are connected via a finite sequence of flips. Equivalently, elements of \( SPN(S) \) are connected via a finite sequence of flips.

**Flips of spines with a doe.** The flip of a trivalent ribbon graph spine \([ (G, \vec{e}) \hookrightarrow S ] \) with a doe is a flip of the underlying spine which maps the does in the obvious manner: if doe is not the flipped edge, then it is simply preserved; if it is, then rotated clockwise. This defines a map

\[
\Phi : SPN^*(S) \to SPN^*(S),
\]

which is of order four. Observe that the \( \text{Mod}(S) \)-orbit

\[
\text{Mod}(S)([(G, \vec{e}) \hookrightarrow S], [(G', \phi(f)(\vec{e})) \hookrightarrow S])
\]

is an element of \( \text{MCG}(S) \). By Whitehead’s lemma \( \text{MCG}(S) \) is generated by these elements (again called flips) and “doe moves”, i.e. orbits

\[
\text{Mod}(S)([(G, \vec{e}) \hookrightarrow S], [(G, \vec{e}') \hookrightarrow S])
\]

(see also [9]). In fact, with the exception \( S_0^3 \) and \( S_1^1 \), the groupoid \( \text{MCG}(S) \) is already generated by flips, see [12]. This generation is not free, i.e. there are some relations obeyed by flips. An example is the famous pentagon relation, see [12].

### 3 The fundamental modular groupoid

#### 3.1 Combinatorial graphs as non-Hausdorff spaces

We shall model the combinatorial type of a topological graph \( G \) by a non-Hausdorff topological space \( |G| \) whose set of points is \( E(G) \cup V(G) \). Open sets of \( |G| \) consists of the stars, i.e. sets of the form \( v \cup s(v) \), where \( v \in V(G) \) is a vertex and \( s(v) \) is the set of edges incident to \( v \). The points of \( G \) from \( E(G) \) will be called the ‘edges’ and those from \( V(G) \) will be called the ‘vertices’ of \( |G| \). Note that a ribbon graph structure on \( G \) induces a similar structure on \( |G| \), i.e. a cyclic ordering of \( s(v) \) for every vertex \( v \). Moreover, if \( G \) is trivalent then so is \( |G| \), i.e. \( |s(v)| = 3 \) for every vertex \( v \). A modular graph is a topological space homeomorphic to a space \( |G| \), where \( G \) is a trivalent ribbon graph, together with the ribbon graph structure inherited from \( G \). Denote the set of modular graphs by

\[
|MGR| := \{ |G| : G \text{ is a trivalent ribbon graph} \} \text{ (modulo homeomorphisms)}
\]
and those with a doe by
\[ |\text{MGR}| : \{(|\mathcal{G}|, \vec{e}) : |\mathcal{G}| \text{ is a modular graph and } \vec{e} \text{ is a doe of } |\mathcal{G}| \}. \] (20)

The set of modular graphs that has an embedding in \( S \) will be denoted by \(|\text{MGR}(S)|\). The set \(|\text{MGR}^\to(S)|\) is defined likewise. There are bijections
\[ |\text{MGR}(S)| \leftrightarrow |\text{SPN}(S)|, \quad |\text{MGR}^\to(S)| \leftrightarrow |\text{SPN}^\to(S)| \] (21)

There is the canonical ‘retraction’ map
\[ f_G : \mathcal{G} \to |\mathcal{G}|, \] (22)
which sends vertices to vertices and each \( x \in e \in E(\mathcal{G}) \) to the point \( e \in |\mathcal{G}| \). The fundamental groupoid \( \Pi_1^{\mathcal{G}} \) is defined in the usual way, and \( f_G \) induces a morphism
\[ f_G^* : \Pi_1^{\mathcal{G}} \to \Pi_1^{|\mathcal{G}|}. \] (23)
This morphism has a homotopy inverse
\[ f_G^\#: \Pi_1^{|\mathcal{G}|} \to \Pi_1^{\mathcal{G}}, \] (24)
defined as follows: If a path starts and ends at a vertex, then lift it to \( \mathcal{G} \) in the obvious way. If one or both endpoints lies on an edge, then lift it to the obvious path on \( \mathcal{G} \), whose corresponding endpoint(s) is the midpoint of the corresponding edge. Note that \( f_G^\# \) is not induced by some continuous map \(|\mathcal{G}| \to \mathcal{G} \).

Obviously, \( f_G^\# \circ f_G^* \) is the identity morphism. On the other hand, the morphism \( f_G^* \circ f_G^\# \) is homotopic to the identity, via
\[ \gamma \in \Pi_1^{\mathcal{G}}(x, y) \to \alpha_t \cdot \Pi_1^{\mathcal{G}}(x_t, y_t) \cdot \beta_t, \] (25)
where \( x_t = x - (x - 1/2)t \) moves towards the midpoint of the edge containing \( x \), \( y_t = y - (y - 1/2)t \) moves towards the midpoint of the edge containing \( y \), \( \alpha_t(s) = x_t(1-s) + sx \) is a path from \( x_t \) to \( x \) inside the edge containing \( x \), and \( \beta_t(s) = y(1-s) + sy \) is a path from \( y \) to \( y_t \) inside the edge containing \( y \).

(Note that this homotopy restricts to identity on the subgroupoid of \( \Pi_1^{\mathcal{G}} \) which consists of paths whose endpoints lies on vertices or midpoints of \( \mathcal{G} \), i.e. it may be considered as a deformation retract on the fundamental groupoid level.)

Hence, the groupoids \( \Pi_1^{\mathcal{G}} \) and \( \Pi_1^{|\mathcal{G}|} \) are equivalent under a uniquely determined homotopy. Denote by
\[ \tilde{\Pi}_1^{|\mathcal{G}|} < \Pi_1^{|\mathcal{G}|} \] (26)
the subgroupoid of paths starting and terminating at an edge of \(|\mathcal{G}|\). In fact, the subgroupoid \( \tilde{\Pi}_1^{|\mathcal{G}|} \) is equivalent to its ambient groupoid \( \Pi_1^{|\mathcal{G}|} \). Note that \( \tilde{\Pi}_1 \) is not
the fundamental groupoid of some topological space. Note that its elements can be represented as sequences of edges of $|\mathcal{G}|$,

$$\gamma = (a_1, a_2, \ldots, a_k)$$

such that $a_i, a_{i+1}$ are incident for each $i = 1, \ldots, k - 1$. With these generators, $\tilde{\Pi}_1^{[\mathcal{G}]}$ is generated by paths of length 2, subject to relations

$$(a, a) = (a), \quad (a, b) \cdot (b, a) = (a) \quad \text{and} \quad (a, b) \cdot (b, c) \cdot (c, a) = (a).$$

Note that $(a)$ is the identity of $\tilde{\Pi}_1^{[\mathcal{G}]}(a, a)$; $(b, a)$ is the inverse of $(a, b)$ and that the last relator can be read as:

$$(a, b) \cdot (b, c) = (a, c).$$

We call $\tilde{\Pi}_1^{[\mathcal{G}]}$ the graph fundamental groupoid of $|\mathcal{G}|$ as opposed to $\Pi_1^{[\mathcal{G}]}$ which is the usual fundamental groupoid of $|\mathcal{G}|$ from topology. They have the same inertia, $\pi_1(\mathcal{G})$.

### 3.2 A groupoid of groupoid isomorphisms

Now let $\mathcal{G} \hookrightarrow S$ be a spine of $S$. Then since $\mathcal{G}$ is a deformation retract of $S$, there is an equivalence of groupoids $\Pi_1^\mathcal{G}$ and $\Pi_1^S$. If $\mathcal{G}' \hookrightarrow S$ then we have another equivalence $\Pi_1^{\mathcal{G}'} \equiv \Pi_1^S$, and so there is an equivalence

$$\Pi_1^\mathcal{G} \equiv \Pi_1^{\mathcal{G}'} \iff \Pi_1^{[\mathcal{G}]} \equiv \Pi_1^{[\mathcal{G}']} \iff \tilde{\Pi}_1^{[\mathcal{G}]} \equiv \tilde{\Pi}_1^{[\mathcal{G}]},$$

and the latter equivalence is in fact an isomorphism. If $\mathcal{G}' \hookrightarrow S$ and $\mathcal{G} \hookrightarrow S$ are isotopic, then $\Pi_1^{[\mathcal{G}]} = \Pi_1^{[\mathcal{G}]}$, i.e. the induced isomorphism $\tilde{\Pi}_1^{[\mathcal{G}]} \to \tilde{\Pi}_1^{[\mathcal{G}]}$ is the identity. Define

$$\Pi_{\text{SPN}}(S) := \{\text{embedding-induced isomorphisms } \tilde{\Pi}_1^{[\mathcal{G}]} \to \tilde{\Pi}_1^{[\mathcal{G}]}\}$$

and define $\overrightarrow{\Pi_{\text{SPN}}}(S)$ to be its version with does. We have bijections

$$\text{SPN}(S) \leftrightarrow \Pi_{\text{SPN}}(S), \quad \text{SPN}^\wedge(S) \leftrightarrow \overrightarrow{\Pi_{\text{SPN}}}(S),$$

and the groupoid $\text{MCG}(S)$ associated to the $\text{Mod}(S)$-action on $\text{SPN}^\wedge(S)$ is isomorphic to the groupoid associated to the $\text{Mod}(S)$-action on $\overrightarrow{\Pi_{\text{SPN}}}(S)$. Consider now a flip inside the groupoid $\text{MCG}(S) = [\text{SPN}(S)/\text{Mod}(S)]$:

$$\text{Mod}(S)([[\mathcal{G}, \vec{e}] \hookrightarrow S], [[\mathcal{G}', \phi(f)(\vec{e})] \hookrightarrow S]).$$

What is the associated element $\phi^*_f$ of $[\overrightarrow{\Pi_{\text{SPN}}}(S)/\text{Mod}(S)]$?
Lemma 3.1. The isomorphism $\phi^\ast_\ast : \tilde{\Pi}^{[\mathcal{G}]}_1 \to \tilde{\Pi}^{[\mathcal{G}']}_1$ sends the generator $(x,y)$ to the generator of $\tilde{\Pi}^{[\mathcal{G}]}_1$ to the generator $(\hat{\phi}_f(x), \hat{\phi}_f(y))$ of $\tilde{\Pi}^{[\mathcal{G}']}_1$.

Proof. If none of $x, y$ is the flipped edge, then this is clear. Otherwise consider Figure 3, where for each edge $x$ of $\mathcal{G}$, we denote the edge $\hat{\phi}_f(x)$ of $\mathcal{G}'$ by $x'$. In particular, $\phi^\ast_\ast$ has the following effect on generators:

\[
\begin{array}{c|c|c}
(a,f) & (a',f') & (f,a) \\
(b,f) & (a',f') & (f,b) \\
(c,f) & (a',f') & (f,c) \\
(d,f) & (a',f') & (f,d) \\
\end{array}
\]

(34)

To compute the effect of $\phi^\ast_\ast$ on other elements of $\tilde{\Pi}^{[\mathcal{G}]}_1$, one has for example

\[
\phi^\ast_\ast((a,f), (f,c)) = \phi^\ast_\ast((a,f) \cdot (f,c)) = \phi^\ast_\ast((a,f)) \cdot \phi^\ast_\ast((f,c)) = (a',f') \cdot (f',c') = (a',c').
\]

(35)

The last equality follows from the presentation of $\tilde{\Pi}^{[\mathcal{G}']}_1$. Note that $(a,f) \cdot (f,c) \neq (a,c)$ inside $\tilde{\Pi}^{[\mathcal{G}]}_1$; indeed $(a,c)$ is not even an element of this groupoid.

Note that Lemma 3.1 is word for word true for isomorphisms induced by doe moves. Consider therefore the groupoid

Definition 3.2. The fundamental modular groupoid of $S$ is the groupoid $\PiMG(S)$ whose object set is $|\text{MGR}(S)|$. Morphisms of $\PiMG(S)$ are defined as

\[
\text{Mor}_{\PiMG}([\mathcal{G}], [\mathcal{G}'], (e, e')) := \left\{ \text{isomorphisms } \tilde{\Pi}^{[\mathcal{G}]}_1 \to \tilde{\Pi}^{[\mathcal{G}']}_1 \text{ induced by finite sequences of flips & doe moves and mapping } e \text{ to } e' \right\}.
\]

(36)

The groupoid $\PiMG(S)$ has the same object set as $\text{MCG}(S)$. By construction we have the following result:
Theorem 3.3. Association of isomorphisms to flips and doe moves is a strict
groupoid isomorphism $\text{MCG}(S) \to \Pi\text{IMG}(S)$, inducing isomorphisms between
the isotropy groups:

$$\text{Mod}(S) \simeq \text{Aut}_{\text{MCG}}(|\mathcal{G}|, \vec{e}) \to \text{Aut}_{\Pi\text{IMG}}(\tilde{\Pi}_{1}^{[\mathcal{G}]}) \tag{37}$$

Since groupoid automorphisms induces outer automorphisms of isotropy groups,
we have morphisms

$$\text{Aut}_{\Pi\text{IMG}}(|\mathcal{G}|, \vec{e}) \to \text{Out}(\tilde{\Pi}_{1}^{[\mathcal{G}]}) \to \text{Out}(\pi_{1}(|\mathcal{G}|, f)) \tag{38}$$

for each edge $f$ of $|\mathcal{G}|$. Composing this with (37), for each $|\mathcal{G}|$ we get a group homomorphism

$$\text{Mod}(S) \to \text{Out}(\pi_{1}(|\mathcal{G}|, f)), \tag{39}$$

which is injective though far from being surjective as will be explained now.

3.3 Punctures

Recall that spines $\mathcal{G} \hookrightarrow S$ are endowed with a ribbon graph structure induced from
the orientation of $S$. This induces a similar structure on $|\mathcal{G}|$. Define a puncture to be a maximal loop in $\tilde{\Pi}_{1}^{[\mathcal{G}]}$ which turns always in the left direction defined by this
ribbon graph structure. If this loop is finite, we say that the puncture is finite. Denote by $|\mathcal{G}|^0$ the set of punctures of $|\mathcal{G}|$. A quick experiment shows that flips preserves punctures, i.e. the set $|\mathcal{G}|^0$ is sent to the set $|\mathcal{G}′|^0$ under a flip. Since the
same is true for doe moves, morphisms of $\Pi\text{IMG}$ preserve this extra structure, i.e.
we may revise the sequence (38) as

$$\text{Mod}(S) \simeq \text{Aut}_{\Pi\text{IMG}}(|\mathcal{G}|, \vec{e}) \to \text{Out}^o(\tilde{\Pi}_{1}^{[\mathcal{G}]}) \to \text{Out}^o(\pi_{1}(|\mathcal{G}|, f)) \tag{40}$$

where $\text{Aut}^o(\tilde{\Pi}_{1}^{[\mathcal{G}]})$ is the group of groupoid isomorphisms preserving the set of punctures and $\text{Out}^o(\pi_{1}(|\mathcal{G}|, f))$ is the group of puncture-preserving automorphisms of the fundamental group. Since $\text{Mod}(S)$ is known to be isomorphic to the group of outer automorphisms of $\pi_{1}(S)$ preserving the conjugacy classes of loops around the punctures, the composition of these morphisms defines an isomorphism

$$\text{Mod}(S) \to \text{Out}^o(\pi_{1}(|\mathcal{G}|, f)), \tag{41}$$

for each pair $|\mathcal{G}|, \vec{e}$ with a doe and for each edge $f$ of $|\mathcal{G}|$. 
4 The fundamental modular groupoid

In the previous section, under the hypothesis that $S$ is a surface of finite type (of finite genus and with finitely many punctures), we have introduced the fundamental modular groupoid $\Pi MG(S)$ of $S$. Since we have also shown that this groupoid is isomorphic to the groupoid associated to the $\text{Mod}(S)$-action on $\overrightarrow{\text{TRN}}(S)$ or equivalently on $\overrightarrow{\text{SPN}}(S)$, this is just another version of the mapping class groupoid $\text{MCG}(S)$. On the other hand, Def. 3.2 can be restated solely in terms of the non-Hausdorff space $|\mathcal{G}|$, without referring to the surface $S$ or its type, as follows:

Definition 4.1. The fundamental modular groupoid is the disconnected groupoid $\Pi MG$ whose object set is $|\text{MGR}|$. Morphisms of $\Pi MG$ are same as in Def. 3.2:

$$\text{Mor}_{\Pi MG}((|\mathcal{G}|, \vec{e}), (|\mathcal{G'}|, \vec{e'})) := \begin{cases} \text{isomorphisms } \overrightarrow{\Pi_1^{\mathcal{G}}} \to \overrightarrow{\Pi_1^{\mathcal{G'}}} \text{ induced by} \\ \text{finite sequences of flips & doe moves} \\ \text{and mapping } \vec{e} \text{ to } \vec{e'} \end{cases}.$$  \hspace{1cm} (42)

In the definition, $\mathcal{G}$ can be a connected graph with at most countably many edges. Since there are uncountably many such graphs, the $\Pi MG$ has uncountably many objects. Since finite sequences of flips & doe moves are at most countable in number, $\Pi MG$ has uncountably many connected components. If $\mathcal{G}$ is a trivalent graph, and $\vec{e}$ is a doe, then by $\Pi MG(|\mathcal{G}|, \vec{e})$ we denote the connected component of $\Pi MG$ containing $(|\mathcal{G}|, \vec{e})$. Thanks to doe moves, this connected component is same as $\Pi MG(|\mathcal{G'}|, \vec{e'})$, where $\vec{e'}$ is any other doe, and we will unambiguously denote both as $\Pi MG(|\mathcal{G}|)$ or simply as $\Pi MG(\mathcal{G})$. Finally, if $S$ is a surface of finite type, then by $\Pi MG(S)$ we denote $\Pi MG(\mathcal{G})$ where $G \hookrightarrow S$ is any spine of $S$. This notation is compatible with our previous definition of $\Pi MG(S)$.

Figure 4: Graphs in different flip orbits (connected components of $\Pi MG$) with $g = 0$, $n = \infty$

Since the isotropy group of $\Pi MG(S)$ is isomorphic to $\text{Mod}(S)$ for a surface $S$ of finite type, the isotropy groups of connected components of $\Pi MG$ gives an uncountable number of analogues of mapping class groups. Naturally, these will not be finitely generated nor presented in general. Figure 4 exhibits graphs from three different connected components (whose isotropy groups may well be finitely presented).
**Question.** Given some graph \( G \) as in Figure 4, find a presentation of the isotropy group of the groupoid \( \PiMG(G) \).

**Definition 4.2.** The isotropy group of the groupoid \( \PiMG(G) \) will be called the *mapping class group* of the graph \( G \) and will be denoted \( \text{Mod}(G) \).

For what follows, note that our definition of a puncture (maximal left-turning path) makes sense for boundary punctures emerging in infinite graphs as well.

**Definition 4.3.** The *completed fundamental modular groupoid* is the groupoid \( \PiMG \) whose object set is \( \overleftarrow{\text{MGR}} \). Morphisms of \( \PiMG \) are defined as

\[
\text{Mor}_{\PiMG}((|G|, \vec{e}), (|G'|, \vec{e}')) := \left\{ \begin{array}{l}
\text{all isomorphisms } \tilde{\Pi}_1^{|G|} \to \tilde{\Pi}_1^{|G'|} \\
\text{preserving the set of punctures}
\end{array} \right\} \quad \text{and mapping } \vec{e} \text{ to } \vec{e}' \quad (43)
\]

We follow the same notations for \( \PiMG(G) \) as the notations we introduced for \( \PiMG(G) \). In particular, \( \text{Mod}(G) \) means the isotropy group of \( \PiMG(G) \). Since flips and doe moves preserve the puncture structure, one has

\[
\forall G \quad \PiMG(G) < \PiMG(G) \implies \PiMG < \PiMG \quad (44)
\]

4.1 Farey tree and the universal Ptolemy group

Fundamental groups of the graphs in Figure 4 are not finitely generated. It is more interesting to start with infinite graphs of finite topology (i.e. with finitely generated fundamental groups). The simplest such graph is the Farey tree \( F \), which is the plane trivalent tree with no terminal vertices. Note that its automorphism group \( \text{Aut}(F) \) is the free product, \( \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z} \simeq \text{PSL}_2(\mathbb{Z}) \), generated by an order-2 rotation around an edge and an order-3 rotation around a vertex adjacent to this edge. The set of punctures of \( F \) is in a canonical bijection with \( \mathbb{Q} \), see e.g. [14].

Given any pair of does, there exists an \( F \)-automorphism sending one doe to the other. Hence, the groupoid \( \PiMG(F) \) has just one element, i.e. it is already a group. \( F \) being simply connected, \( \PiMG(F) \) has trivial isotropy, and the last element of the sequence (40) vanish, yielding the sequence

\[
\text{Mod}(F) = \text{Aut}_{\PiMG}((|F|, \vec{e})) \to \text{Aut}_0(\tilde{\Pi}_1^{|F|}) \to 1 \quad (45)
\]

Since \( F \) is simply connected, there exists a unique path connecting any pair of edges, and the groupoid \( \tilde{\Pi}_1^{|F|} \) is identified with the pair groupoid \( E(F) \times E(F) \).

**Theorem 4.4.** ([10], [8], [13]) The group \( \text{Mod}(F) < \text{Aut}_0(E(F) \times E(F)) \) is the group \( \text{PPSL}_2(\mathbb{Z}) \) of piecewise-\( \text{PSL}_2(\mathbb{Z}) \) homeomorphisms of the circle.
As a group of circle homeomorphisms, $\text{PPSL}_2(\mathbb{Z})$ is conjugate Thompson’s group $\mathcal{T}$ via the Minkowski question mark function, see [8]. See [3] for more details about Thompson’s group. As we exhibit in the next section, hereby we introduce a family of common generalizations of these groups and the mapping class groups.

The puncture preserving piece of $\text{Aut}(\tilde{\Pi}_1^{[\mathcal{F}]})$ is much smaller:

**Theorem 4.5.** $\text{Mod}(\mathcal{F})$ is the group $\text{Homeo}^Q(S^1)$ of homeomorphisms of the circle sending the set $Q \cup \{\infty\}$ to itself.

Since any order-preserving map $Q \cup \{\infty\} \to Q \cup \{\infty\}$ is extrapolated by a unique circle homeomorphism, $\text{Homeo}^Q(S^1)$ can be equivalently described as the group of automorphisms of the cyclically ordered set $Q \cup \{\infty\}$, i.e. the set of order-preserving bijections $Q \cup \{\infty\} \to Q \cup \{\infty\}$.

**Proof.** This is because the closure of $\text{PPSL}_2(\mathbb{Z})$ under the compact-open topology inside $\text{Homeo}(S^1)$ is $\text{Homeo}^Q(S^1)$. \(\square\)

Note that the group $\overline{\text{Mod}}(\mathcal{F}) \simeq \text{Homeo}^Q(S^1)$ is isomorphic to Penner’s completion of the universal Ptolemy group, which is an avatar of Thompson’s group $\mathcal{T}$ from [10], see also [12].
Problem. Describe the elements of $\text{PGL}_2(\mathbb{Q}) < \text{Homeo}^0(\mathbb{S}^1)$ as elements of $\text{Mod}(\mathcal{F})$, i.e. as sequences of flips, starting from the simplest cases $x \to 2x$ and $x \to x + 1/2$. This is an algorithmic question; see e.g. [6]. As a special case of this problem, one may consider the elements of $\text{PSL}_2(\mathbb{Q})$ or by some more special subgroups of it.

4.2 Charks and other infinite graphs with finite topology

This subsection is a report about an on-going work with Ayberk Zeytin [15].

The Farey tree is the simply connected trivalent planar graph with a trivial fundamental group. What is the next infinite graph with the smallest fundamental group? In fact, there are two rooted trees $\mathcal{F}_2$ and $\mathcal{F}_3$ with fundamental groups isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Graphs with fundamental groups isomorphic to $\mathbb{Z}$ provides the next simplest case. They can be described as follows. Consider a planar trivalent graph without terminal vertices and having precisely one cycle. Therefore it consists of a bunch of infinite branches attached to this main cycle. If all branches are inside (or outside) this main cycle, then we denote this graph by $\mathcal{F}_\infty$. If not, we call it a chark. It turns out that charks correspond to Gauss class groups in a very natural manner [14]; moreover, every chark with a doe represents a unique binary quadratic form in the class represented by the chark. Every chark lives on a conformal annulus which is determined uniquely by the chark.

The main observation concerning charks is that the flip action is transitive on them. Hence the groupoid $\Pi \text{MG}(\mathcal{C})$ is the same for any chark $\mathcal{C}$. Our preliminary investigations suggest that $\text{Mod}(\mathcal{C}) \simeq T \times T$ and that $\text{Mod}(\mathcal{F}_2), \text{Mod}(\mathcal{F}_3), \text{Mod}(\mathcal{F}_\infty)$ are both isomorphic to the Thompson group $F$.

For other infinite ribbon graphs $\mathcal{G}$ with a finite topology, $\text{Mod}(\mathcal{G})$ will be a hybrid of mapping class groups of finite type punctured surfaces and Thompson’s groups. It seems to be an interesting task to study them, because their world is quite rich [15]. They admit a canonical representation on arithmetic surfaces of finite genus and with a finite (or zero) number of punctures and finite (non-zero) number of boundary components. There is one boundary component for each set.
of branches attached to a same left-turning cycle in the same direction.

5 Outer Modular Groupoids

Shuffles and Outer Groupoids. Recall that a ribbon graph is a graph with a cyclic ordering of edges emanating from each one of its vertices. A shuffle of a trivalent ribbon graph $G$ is the operation of reversing this orientation at a given vertex, as in the figure below:

![Figure 7: A shuffle.](image)

A shuffle of a trivalent ribbon graph with a doe is defined likewise. Since shuffles do not modify the combinatorial graph underlying the ribbon graph, they act trivially on fundamental groupoids. However, they change the genus and the number of punctures of $G$. By applying flips and shuffles to $G$, we can obtain every trivalent ribbon graph whose fundamental group is isomorphic to $\pi_1(G)$, provided that $\pi_1(G)$ is finitely generated. Hence we may define the following (disconnected) groupoid:

**Definition 5.1.** The outer modular groupoid of $G$ is the groupoid $\text{OMG}$ whose object set is $|\text{MGR}|$. Morphisms of $\text{OMG}$ are defined as

$$\text{Mor}_{\text{OMG}}((|G|, \vec{e}), (|G'|, \vec{e}')) := \left\{ \begin{array}{l} \text{isomorphisms } \tilde{\Pi}_1^{\{|G|\}} \to \tilde{\Pi}_1^{\{|G'|\}} \text{ induced by finite sequences of flips, shuffles & doe moves} \\ \text{and mapping } \vec{e} \text{ to } \vec{e}' \end{array} \right\}$$

(46)

It follows immediately from the definition that $\text{PIMG} < \text{OMG}$ and that these two groupoids share the same set of objects. By $\text{OMG}(G)$, we denote the connected component of $\text{OMG}$ containing the graph $G$ and by $\text{Out}(G)$ we denote its isotropy group, which we call the outer automorphism group of the graph $G$. Similarly, $\text{OMG}(S)$ denotes the connected component of $\text{OMG}$ containing a spine of $S$. This connected component is same for every surface $S_g^n$ with $d = 6q + 3n - 6$ which we denote as $\text{OMG}(d)$. Hence, $\text{OMG}(d)$ contains every $\text{PIMG}(S_{g_r}^n)$ with...
\(d = 6g + 3n - 6\). Recall that \(\pi_1(S^n_g)\) is isomorphic to \(F_d\), the free group of rank \(d\), and

\[d = 6g + 3n - 6 \implies \text{Out}(\pi_1(S^n_g)) \simeq \text{Mod}(S^n_g) < \text{Aut}_{\text{OMG}}(S^n_g)\]  

\textbf{Theorem 5.2.} The isotropy groups of \(\text{OMG}(d)\) are isomorphic to \(\text{Out}(\pi_1(S^n_g))\).

A proof strategy consists in relating \(\text{OMG}\) to the outer space \([4]\).

The following result is also from \([15]\).

\textbf{Theorem 5.3.} The isotropy group of \(\text{OMG}(\mathcal{F})\) is isomorphic to Thompson’s group \(V\).

In parallel to the study of the groups \(\text{Mod}(\mathcal{G})\), it seems to be of interest to study the isotropy groups \(\text{Out}(\mathcal{G})\) of the groupoids \(\text{OMG}(\mathcal{G})\), especially when \(\mathcal{G}\) has a finite topology.

It is appropriate to give the following definition:

\textbf{Definition 5.4.} The completed outer modular groupoid is the groupoid \(\overline{\text{OMG}}\) whose object set is \(|\overrightarrow{\text{MGR}}|\). Morphisms of \(\overline{\text{OMG}}\) are defined as

\[
\text{Mor}_{\overline{\text{OMG}}}((|\mathcal{G}|, \vec{e}), (|\mathcal{G}'|, \vec{e}')) := \left\{\text{all isomorphisms } \overrightarrow{\Pi}_1^{(|\mathcal{G}|)} \to \overrightarrow{\Pi}_1^{(|\mathcal{G}'|)} \text{ mapping } \vec{e} \text{ to } \vec{e}' \right\}.
\]  

(48)

Note that the groupoid \(\overline{\text{OMG}}(\mathcal{F})\) is in fact a group since it has just one object.

\textbf{Theorem 5.5.} One has

\[
\text{Homeo}(|\mathcal{F}|) < \overline{\text{OMG}}(\mathcal{F}) \simeq \text{Sym}(\mathcal{E}(\mathcal{F})) \simeq \text{Sym}(\text{PSL}_2(\mathbb{Z})) < \text{Homeo}(\partial\mathcal{F})\]  

where \(\partial\mathcal{F}\) is the boundary of the Farey tree.

Note that \(\mathcal{E}(\mathcal{F})\) as a set is naturally identified with the modular group \(\text{PSL}_2(\mathbb{Z})\) and \(\overrightarrow{\Pi}_1^{(|\mathcal{F}|)}\) is naturally identified with \(\text{PSL}_2(\mathbb{Z}) \times \text{PSL}_2(\mathbb{Z})\).

\textit{Proof.} The automorphism group of the pair groupoid \(X \times X\) is the symmetric group on \(X\). Hence, the full automorphism group of \(\overrightarrow{\Pi}_1^{(|\mathcal{F}|)}\) is the symmetric group on \(\mathcal{E}(\mathcal{F})\), which includes the group \(\text{Homeo}(|\mathcal{F}|)\) (i.e. the group of graph automorphisms respecting the planar structure or not) as a proper subgroup. \(\square\)

Note that \(\text{Homeo}(|\mathcal{F}|)\) is a locally compact uncountable group, and the subgroup of automorphisms that fix a given edge is a profinite group.
6 Miscellaneous Structures

6.1 The vertical structure of IIMG

For more details about this section explaining the connection with dessins, see [13].

Recall that the object set of the groupoids IIMG and OMG is the set of modular graphs with a doe, \(|MGR|\). This set carries a second structure which we view as its ‘vertical’ structure: the category whose objects set is \(|MGR|\) and whose morphisms are covering maps (in the usual topological sense) of modular graphs with a doe. It is equivalent to the category of properly discontinuous free \(PSL_2(\mathbb{Z})\)-actions on the space \(F\); or alternatively to the poset of torsion-free subgroups of \(PSL_2(\mathbb{Z})\) under inclusion. We denote this latter category as SUB\(^{tf}\) (the superscript \(tf\) means: ‘torsion-free’). The set of cosets of torsion-free subgroups of \(PSL_2(\mathbb{Z})\) is in bijection with the set of modular graphs with two does.

To sum up, there is a natural correspondence

\[
|MGR| \leftrightarrow \text{SUB}^{tf}
\]  
(50)

(and in fact the most practical way to explicitly construct the left-hand side of the above correspondence is to construct it as the right hand-side). Any pair of (finite index) subgroups in \(PSL_2(\mathbb{Z})\) is included in a subgroup of \(PSL_2(\mathbb{Z})\) (of finite index) which makes \(|MGR|\) into a projective system. Note this categorical property endows the disconnected groupoids IIMG and OMG with a sort of connectedness property.

One may take the limit of this projective system with respect to finite covers:

\[
\mathcal{X} := \limleft\downarrow |MGR|.
\]  
(51)

If we restrict ourselves only to finite graphs inside \(|MGR|\), then the limit space \(\mathcal{X}_f\) we obtain is a compact space which is a retraction of the punctured solenoid of Penner [10]. In its entirety, \(\mathcal{X}\) is a directed system of solenoids, which are non-compact except the final object \(\mathcal{X}_f\). Systems of finite coverings of infinite graphs of finite topology are also interesting in this context. Moreover, the \(PSL_2(\mathbb{Z})\)-action on \(|MGR|\) defined below induces a \(PSL_2(\mathbb{Z})\)-action on \(\mathcal{X}\) and on \(\mathcal{X}_f\).

Via path lifting each covering \(\psi : (\mathcal{H}, \vec{h}) \to (\mathcal{G}, \vec{g})\) gives rise to an embedding \(\tilde{\Pi}^{|G|}_1 \to \tilde{\Pi}^{|H|}_1\). These two structures (groupoid and poset) on \(|MGR|\) are intertwined in other ways. Given a covering \(\psi : (\mathcal{H}, \vec{h}) \to (\mathcal{G}, \vec{g})\), one may try to lift a flip \(\phi_f : (\mathcal{G}, \vec{g}) \to (\mathcal{G}', \vec{g}')\) by flipping all of the edges \(\psi^{-1}(f)\) simultaneously, though this makes sense only if the lifted flips commute and the sequence of flips so obtained converge in some sense. This is the case for some but not all of the flips. Since \(F\) is the universal cover of every modular graph \(\mathcal{G}\), one may in particular
want to embed $\text{Mod}(S) < \text{IMG} \to \text{IMG}(F)$ this way. It may be that both difficulties can be overcome in this case, though we haven’t checked the details.

6.2 The $\text{PSL}_2(\mathbb{Z})$-action

The group $\text{PSL}_2(\mathbb{Z})$ acts on $\text{SUB}^{tf}$ by conjugation. Normal subgroups are among the fixed elements of this action. The quotient set is the set of subgroups of $\text{PSL}_2(\mathbb{Z})$ modulo conjugation, which we denote as $\text{SUB}^{tf^*}$.

We may transfer this action to $|\text{MGR}|$ via the correspondence (50). By using an isomorphism $\text{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, the order-2 generator acts by flipping the doe and the order-3 generator rotates the doe around a vertex of degree 3 (in some fixed sense). Modular graphs with symmetry are fixed points of this action. The quotient set, which we denote $|\text{MGR}|$ is the set of modular graphs (without doe). The map

$$|\text{MGR}| \to |\text{MGR}|$$

is the doe-forgetting map.

In a similar way, $\text{PSL}_2(\mathbb{Z})$ acts on the sets $|\text{SPN}|$ and $|\text{TRN}|$, the quotients being the sets $|\text{SPN}|$ and $|\text{TRN}|$ respectively.

This action on objects defines an action on the groupoids $\text{IMG}$, $\text{IMG}^*$, $\text{OMG}$, and $\text{OMG}^*$, whose orbit groupoids we denote respectively by $\text{IMG}^*$, $\text{IMG}^*$, $\text{OMG}^*$ and $\text{OMG}^*$. We give the definition of $\text{IMG}^*$, others are defined likewise:

**Definition 6.1.** $\text{IMG}^*$ has the object set $|\text{MGR}|$. Its morphisms are defined as

$$\text{Mor}_{\text{IMG}}(|G|, |G'|) := \left\{ \text{isomorphisms } \tilde{\Pi}_{1}^{[G]} \to \tilde{\Pi}_{1}^{[G']} \text{ induced by finite sequences of flips & doe moves} \right\}. \quad (53)$$

See [1] and [2] for some details on group actions on groupoids.

6.3 $\text{IMG}$-representation on punctures

Recall that flips and moves maps punctures to punctures. Consider the map $\psi$ sending each element of $|\text{MGR}|$ to its set of punctures and each morphism of $\text{IMG}$ to the bijection it induces on punctures. We have an exact sequence

$$1 \to \text{PUR} \to \text{IMG} \to \text{PUN} \quad (54)$$

where PUN is the category of sets of punctures with bijections as morphisms and PUR is the kernel of $\psi$: it is the category whose object set is $|\text{MGR}|$ and whose morphisms are fundamental groupoid isomorphisms that keep each puncture fixed.
In fact, since $\Pi MG$ consists of puncture-preserving isomorphisms, we have a representation

$$1 \to \text{PUR} \to \Pi MG \to \text{PUN}$$

(55)

If $\mathcal{G} \in |MGR|$ is finite, then the isotropy of $\text{PUR}(\mathcal{F})$ is the pure mapping class group. The case of infinite $\mathcal{G}$ is most interesting when $\mathcal{G}$ has no finite punctures. For instance, for the sequence

$$1 \to \text{PUR}(\mathcal{F}) \to \Pi MG(\mathcal{F}) \to \text{PUN}(\mathcal{F}),$$

(56)

the puncture part $\text{PUN}(\mathcal{F})$ has a unique object which is identified with $\mathbb{Q}$, the pure part $\text{PUR}(\mathcal{F})$ is trivial, and the middle term is isomorphic to Homeo$^\infty(\mathcal{S}^1)$. Note that we can always lift $\text{PUN}(\mathcal{G})$ to $\text{PUN}(\mathcal{F})$. As for the sequence

$$1 \to \text{PUR}(\mathcal{F}) \to \Pi MG(\mathcal{F}) \to \text{PUN}(\mathcal{F}),$$

(57)

the middle term is $\text{PPSL}_2(\mathbb{Z})$ and the remaining terms are the same. For other infinite elements of $|MGR|$ without finite punctures (e.g. charks), the isotropy group of the groupoid $\text{PUR}(\mathcal{G})$ is the mapping class group of the compact surface obtained from the ambient surface of $\mathcal{G}$ by filling in the boundary components by discs. To see this, we consider the realization of $\mathcal{G}$ on an arithmetic hyperbolic surface (see [13]) $\mathcal{R}$. The boundary components of $\mathcal{G}$ are then identified by boundary circles of $\mathcal{R}$, such that the boundary punctures are dense inside these circles. Since each element of $\text{PUR}(\mathcal{G})$ fixes these punctures pointwise, the corresponding homeomorphism of $\mathcal{R}$ extends continuously to the boundary as the identity map and from there it extends to the compact surface obtained $\mathcal{R}$ by filling the boundary circles with discs.

**Question.** Therefore for infinite graphs of finite topology $\mathcal{G}$, we may view the ambient surface as having parametrized boundary circles, and $\text{PUN}(\mathcal{G})$ controls these boundary parametrizations. Is it possible to cook up a topological quantum field theory from this data? If we glue two such surfaces along their boundaries to get some compact surface without boundary, is this surface an arithmetic surface? If yes, what is its defining modular graph? If no, what is its field of definition?

### 6.4 Forgetting the ribbon graph structure of OMG

Denote by $|\text{CMGR}|$ the set of combinatorial modular graphs with a doe, i.e. modular graphs with a doe and without the ribbon graph structure and denote by $|\text{MGR}|$ the set of combinatorial modular graphs, i.e. modular graphs without a doe and without the ribbon graph structure. There are forgetful maps

$$|\text{MGR}| \longrightarrow |\text{CMGR}| \longrightarrow |\text{CMGR}|$$

(58)
and

\[ |\overrightarrow{MGR}| \rightarrow |MGR| \rightarrow |CMGR|. \quad (59) \]

Considering the composed map (which forgets the doe and the ribbon structure), we obtain a morphism

\[ OMG \rightarrow OMG^{**}, \quad (60) \]

where \( OMG^{**} \) is the groupoid with object set \( |CMGR| \) and whose morphisms are fundamental groupoid isomorphisms generated by flips (doe moves and shuffles act trivially).

### 6.5 Torsion subgroups

Recall that the object set of our categories \( |\overrightarrow{MGR}| \) is equivalent to the directed set \( \text{SUB}^{tf} \) of torsion-free subgroups of \( PSL_2(\mathbb{Z}) \). One can extend much of this paper to the object set equivalent to the category \( \text{SUB} \) of all subgroups of the modular group. The best way to do this is to consider bipartite graphs (as topological spaces), with vertices of degree 3 and 2. In this setting the notion of a doe also is more natural as it becomes simply the choice of a distinguished edge of the graph. These graphs may have terminal vertices of two types, corresponding to the conjugacy classes of 2-torsion and 3-torsion elements in the subgroup. Their category have been denoted \( \text{Cov} \) in [13]. Their fundamental groupoids must be taken in the orbifold (or stack) sense. In particular these groupoids contains paths of order 2 and three. This requires the extension of the notion of a flip to graphs with terminal vertices. This way, e.g. if \( \mathcal{T} \) is a planar tree with some terminal vertices then these constructions permits one to define its mapping class group \( \text{Mod}(\mathcal{T}) \).

### 6.6 Topologies on groupoids and their boundaries

In order to express everything in the right setting, we should have proved that \( \Pi IMG \) (respectively \( OMG \)) is the completion of \( \Pi IMG \) (respectively \( OMG \)) under an appropriate topology; giving sense to infinite sequences of flips and twists. We defer these tasks to [15].

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