TURÁN INEQUALITIES AND ZEROS OF ORTHOGONAL POLYNOMIAL

ILIA KRASIKOV

Abstract. We use Turán type inequalities to give new non-asymptotic bounds on the extreme zeros of orthogonal polynomials in terms of the coefficients of their three term recurrence. Most of our results deal with symmetric polynomials satisfying the three term recurrence

$$p_{k+1}(x) = (x-a_k)p_k(x) - c_k p_{k-1}(x), \quad b_k, c_k > 0,$$

with the initial conditions

$$p_{-1} = 0, \quad p_0 = 1,$$

and let $x_{1k} < \ldots < x_{kk}$ be the zeros of $p_k(x)$. We are interested in finding uniform bounds on the extreme zeros, that is an interval $I = [A(k), B(k)]$, such that $A(k) < x_{1k} < x_{kk} < B(k)$, in terms of the coefficients of the recurrence. Such a setting arises naturally if one deals with the family depending on parameters, as in the case of classical Jacobi and Laguerre polynomials, and is seeking for bounds uniform in all the parameters involved. The main aim of this paper is to show that the classical Turán inequality

$$T_k(P, x) = p_k^2(x) - p_{k-1}(x)p_{k+1}(x) \geq 0,$$

and its analogues (abbreviated as $TI$ in the sequel), provide a convenient tool for tackling the problem. It is known that (2) holds for some families of orthogonal polynomials including Laguerre, Jacobi and some other polynomials (see [18] and the references therein).

At present there are two general approaches to the problem, one is based on the chain sequences [6] and another exploiting the Rayleigh quotient to find the extreme eigenvalues of the corresponding Jacobi matrix. The last one was discovered independently by G. Freud [5] and V.I. Levenshtein (see [13], where the references on the earlier papers of the author are given), and yields the following elegant representation for the extreme zeros.

$$x_{1k} = \min \left( \sum_{i=0}^{k-1} a_i x_i^2 - 2 \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_i c_{i+1}} \right),$$

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\begin{align}
\sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_{i+1}},
\end{align}

where the extrema are taken over all (or only over positive) \(x_0, x_1, \ldots, x_{k-1}\), subjected to \(\sum_{i=0}^{k-1} x_i^2 = 1\). For symmetric polynomials (i.e. when \(p_k(-x) = (-1)^k p_k(x)\)) and the monic normalization, the case we mainly deal with in this paper, the recurrence (1) can be rewritten as

\begin{align}
p_{k+1}(x) = x p_k(x) - c_k p_{k-1}(x),
\end{align}

and (3), (4) become

\begin{align}
x_{kk} = \max \left( \sum_{i=0}^{k-1} a_i x_i^2 + 2 \sum_{i=0}^{k-2} x_i x_{i+1} \sqrt{b_{i+1}} \right),
\end{align}

Thus \(x_{kk}\) as a function of the vector \((c_1, c_2, \ldots)\), possesses many nice properties, e.g. it is clearly subadditive, continuous, increasing with \(k\). This makes transparent many otherwise puzzling questions concerning the behaviour of the extreme zeros. The most striking result obtained via (3), (4) is due to A. Máté, P. Nevai and V. Totik \[15\] and states that if \(c, \delta > 0\) are fixed and \(c_k = c^2 k^{2 \delta} (1 + o(k^{-2/3}))\), then

\begin{align}
x_{kk} k^{-\delta} = 2c - c \cdot 3^{-1/3} (2\delta)^{2/3} i_1 k^{-2/3} + o(k^{-2/3}),
\end{align}

where \(i_1 = 3.3721\ldots\) is the smallest zero of the Airy function.

If \(x_{kk}\) is represented by a series in decreasing powers of \(k\), it is naturally to distinguish between the first and the second order bounds, e.g. \(2c\) and \(2c(1 - K \delta^{2/3} k^{-2/3})\) in (7), where to maintain the uniformity in \(k\) we will allow a weaker constant \(K\) than the exact asymptotic one. Whenever the first order bounds can be obtained rather easily, say, by replacing \(2x_i x_{i+1}\) in (3) by \(x_i^2 + x_{i+1}^2\), or by similar elementary arguments \[11, 13\], the second order estimates uniform in all the parameters, which are the main subject of this paper, were found only recently for the case of classical orthogonal polynomials \[8\] - \[12\].

It is worth also noticing that it is rather easy to extract bounds on the zero \(x_{kk}^*\) corresponding to the perturbed recurrence \(p_{k+1}(x) = x p_k(x) - c_k (1 + \epsilon k) p_{k-1}(x)\), provided one knows the result for \(\epsilon = 0\). Indeed, \(\sum_{i=0}^{k-2} x_i x_{i+1} \leq 1\), and thus, \(x_{kk}^*\) is in the interval \((1 \pm \epsilon) x_{kk}\), for \(|\epsilon_k| < \epsilon < 1\). For example it would be enough to establish (7) for \(c_k = c^{2} k^{2 \delta}\), the general case with the extra factor \((1 + o(k^{-2/3}))\), follows from the above arguments. In this paper we will not state explicitly such obvious generalizations, but the reader should keep them in mind.

Our approach to the problem is quite different from the above two and based on the following observation \[7\]. Let \(f = f(x)\) and \(g = g(x), \ |\deg(f) - \deg(g)| \leq 1\), be two real polynomials with only real interlacing zeros. Suppose also that we know a (preferably quadratic) form \(\sum_{i=0}^{m} A_i(x) f^{m-i} g^i \geq 0\), which is indefinite in \(f\) and \(g\) viewed as formal variables. Then one can routinely obtain bounds on the extreme zeros of \(f\) and \(g\) similar to (7) but, of course, with a weaker constant instead of \(i_1\). The existence of such a form is far from being obvious and in fact the main difficulty is to find an appropriated one. For classical orthogonal polynomials, when a second order differential equation is known, one may choose \(f = p_k, \ g = p_k', \) and the quadratic form obtained from the Laguerre inequality \(f'^2 - f'' > 0\), or its higher
order generalizations \cite{7, 8, 9, 12}. In the discrete case $f = p_k(x + 1), g = p_k(x)$, play a similar role \cite{7, 10, 11}.

In this paper we will show that the required forms, in particular these giving second order bounds, may be obtained directly from TI via the three term recurrence. To this end we will establish two new sets of TI yielding second order bounds. The following theorem is one of our main results.

**Theorem 1.** Let $p_k$ be a symmetric polynomial satisfying \cite{6} and suppose that $c_k$ are nondecreasing, then
\[
x_{kk} < 2 \sqrt{c_{k-1}}, \ k \geq 2.
\]
Moreover if $d_i = \frac{c_i - c_{i-1}}{c_i} \geq 0, \ c_0 = 0$, satisfy
\[
\frac{d_i}{2(1 + d_i)} < d_{i+1} < \frac{d_i(1 + 2 \sqrt{d_i} + 2d_i)}{1 + d_i}, \ i = 1, 2, \ldots;
\]
then for $k \geq 2$,
\[
x_{kk} < 2 \sqrt{c_k \left(1 - \frac{d_{2/3}^2}{d_{k+1}^{2/3} + d_{k+1}^{1/3}}\right)}, \ k \geq 2.
\]

As a corollary we obtain the following uniform version of (7).

**Theorem 2.** Let $c_k = c^2 k^{2\delta}$, where $c > 0, \delta \geq 0$, are fixed, then
\[
x_{kk} k^{-\delta} < 2c \sqrt{1 - \frac{\delta^{2/3}}{(k + \frac{1}{2})^{1/3} + \delta^{1/3}}}, \ k \geq 2.
\]

It would be important to have lower bounds corresponding to (9). A trivial one $x_{kk} > \sqrt{c_{k-1}}$ readily follows from (6) (see also \cite{6, 11} in this connection). One hardly could expect that (9) is sharp for a faster than polynomial rate of growth of $c_k$, although formally Theorem 1 allows $c_k$ to be of order about $e^{k^2/2}$. It would be also interesting to obtain similar results for polynomials orthogonal on $[-1, 1]$ and $[0, \infty)$.

The paper is organized as follows. In the next section we survey some known TI which will be used in section 3 for obtaining bounds on the extreme zeros. In the last section we establish two new sets of TI and give second order bounds for a vast class of orthogonal polynomials satisfying \cite{6} with a nondecreasing sequence $c_k$.

## 2. Turán Inequalities

The reason for study TI in the theory of orthogonal polynomials is that they have a few quite important applications. For example, under appropriate restrictions $T_k(P, x)$ converges uniformly on compact subsets of $(-1, 1)$ to $\frac{2x\alpha'(x)}{\pi \alpha(x)}$, where $\alpha'(x)$ is the absolutely continuous part of the corresponding orthogonality measure on $[-1, 1]$. The existing technique enables one to obtain similar limiting expressions for higher order analogues of TI as well, e.g. the right hand side of inequality \cite{17} below tends to $\frac{\delta(1 - x^2)}{\pi \alpha'(x)}$, see \cite{3} and the references therein.

Today there are two almost independent theories related to TI, the first dealing with the validity of \cite{14} for different families of orthogonal polynomials and goes back to the pioneer work of Turán \cite{19}. Another one is motivated by the theory of entire
functions, in particular by the Riemann hypothesis. In the last case much more precise higher order generalization of (2) are known, but the orthogonal polynomials must have a generating function of a very special type. A survey of this theory and the relevant references can be found in [2], [3], [4].

Progress in the first direction was recently summarized by R. Szwarc [18], who established the following result.

**Theorem 3.** (i) Let $\mathcal{P}$ be a family of symmetric polynomials orthogonal on $[-1, 1]$, where three term recurrence (10) is normalized by

$$b_k + c_k = 1, \quad c_0 = 0, \quad c_{k+1} > 0, \quad b_k > 0,$$

that is by $p_k(1) = 1, \quad p_k(-1) = (-1)^k$. Then

$$T_k(\mathcal{P}, x) \geq 0, \quad |x| \leq 1,$$

provided one of the following conditions holds

(i$_a$) $c_k$ is nondecreasing and $c_k \leq \frac{1}{2}$;

(i$_b$) $c_k$ is nonincreasing and $c_k \geq \frac{1}{2}$.

(ii) Let $\mathcal{P}$ be a family of polynomials orthogonal on $[0, \infty)$, which are normalized by $p_k(0) = 1$, with the three term recurrence

$$xp_k(x) = -b_k p_{k+1}(x) + (b_k + c_k) p_k(x) - c_k p_{k-1}(x),$$

where $b_0 = 1, c_0 = 0, b_k > 0, c_{k+1} > 0$. Suppose $b_k$ and $c_k$ are nondecreasing, then

$$T_k(\mathcal{P}, x) \geq 0, \quad x \geq 0,$$

provided one of the following conditions holds

(ii$_a$) $c_k \leq b_k, \quad c_k - c_{k-1} \geq b_k - b_{k-1};$

(ii$_b$) $c_k \geq b_k, \quad c_k - c_{k-1} \leq b_k - b_{k-1}.$

R. Szwarc [18] also obtained similar yet rather technical conditions which guarantee the validity of (2) for a general nonsymmetric polynomial with the orthogonality measure supported on $[-1, 1]$. For the sake of simplicity we did not state them here. It seems nothing is known in the case of nonsymmetric polynomials orthogonal on the whole real axis. On the other hand the case of symmetric polynomials orthogonal on $(\infty, \infty)$ is almost trivial.

**Theorem 4.** Let $\mathcal{P}$ be a family of orthogonal polynomials satisfying (2) with a nondecreasing sequence $\{c_k\}_k$. Then $T_k(\mathcal{P}, x) \geq 0$.

**Proof.** The result follows by $T_0 = 1$, and an easy to check identity

$$T_{k+1}(\mathcal{P}, x) = c_k T_k(\mathcal{P}, x) + (c_{k+1} - c_k) p_k^2(x).$$

□

A few higher order generalization of (2) are known. To state them we shall consider the Laguerre- Pólya class of functions which consists of real polynomials with only real zeros and real entire functions

$$F(z) = ce^{-\alpha z^2 + \beta z} z^r \prod_i (1 - z/z_i) e^{z/z_i},$$

where $\alpha \geq 0, \quad c, \beta, r$ are real, $r$ is a nonnegative integer and $\sum_i z_i^{-2}$ is convergent.

Suppose now that a family $\mathcal{U}$ of real functions $u_k = u_k(x), \quad k = 0, 1, \ldots$, has for
some values of $x$ a generating function of the Laguerre-Pólya class,

$$\sum_{k=0}^{\infty} \frac{u_k z^k}{k!} = F(z). \tag{15}$$

An instructive example is provided by the binary Krawtchouk polynomials $u_k = K^n_k(x)$, $n$ is a positive integer, having the generating function

$$\sum_{i=0}^{\infty} K^n_i(x) z^i = (1-z)^x (1+z)^{n-x},$$

which satisfy (15) for $x = 0, 1, \ldots, n$. Another examples are given by ultraspherical $C_k^{(\lambda)}$, Hermite $H_k$, and Laguerre $L_k^{(\alpha)}$ polynomials [4], with

$$u_k = \frac{C_k^{(\lambda)}(x)}{C_k^{(\lambda)}(1)}, \lambda > -\frac{1}{2}, \quad -1 \leq x \leq 1;$$

$$u_k(x) = H_k(x), \quad -\infty < x < \infty;$$

$$u_k = \frac{L_k^{(\alpha)}(x)}{L_k^{(\alpha)}(0)}, \quad \alpha > -1, \quad x \geq 0.$$

In the following theorem the first part belongs to M. Patrick [17] and the second one to J. Maćić [14] (the extension of it to the whole Laguerre-Pólya class is due to D.K. Dimitrov [4]).

**Theorem 5.** For those values of $x$ for which (15) holds

$$T_k^{(m)}(U, x) = \frac{1}{2} \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} u_{k-m+j} u_{k+m-j} \geq 0, \quad m = 0, 1, \ldots, \tag{16}$$

and

$$S_k(U, x) = 4(u_k^2 - u_{k-1} u_{k+1})(u_{k+1}^2 - u_k u_{k+2}) - (u_k u_{k+1} - u_{k-1} u_{k+2})^2 \geq 0. \tag{17}$$

Notice that $T_k^{(1)}(U, x)$ is just $T_k(U, x)$. The inequality (17) can be viewed as a refinement of (2) and is intimately connected with so-called Newton inequalities [16]. To apply (16,17) to orthogonal polynomials it would be important to to restate the condition for their validity in terms of the coefficients of (11). For $T_k^{(2)}$ and recurrence [5] this will be accomplished in the last section. The corresponding question for $S_k$ remains open.

### 3. Extreme zeros

In this section we give some first order bounds on the extreme zeros which can be deduced from Theorems 3 and 4. We also show that $S_k(U, x)$ (which gives a fourth degree form), yields second order bounds for Hermite polynomials. The inequality $T_k^{(2)} \geq 0$, will be considered in more details in the next section.

First, we describe simple geometric arguments which enable one to deduce bounds on the extreme zeros from a given form. Let $p = p(x)$, $q = q(x)$ be two real polynomials, $\deg(p) = k \geq 2$, $\deg(q) = k - 1$, with only real interlacing zeros $x_1, \ldots, x_k$, and $y_1, \ldots, y_{k-1}$, respectively, $x_i < y_i < x_{i+1}$. Suppose that for $x \in$
(M, N), M < x_1 < x_k < N, where M and N can be finite or infinite, there exists a nonnegative form

$$\sum_{i=0}^{m} A_i(x) q^i p^{m-i} \geq 0, \ m \geq 2, \text{ even},$$

where A_i(x) are certain functions defined in all the points of (M, N). Introducing the function $t = t(x) = q/p$, we rewrite it as

$$Q(t, x) = \sum_{i=0}^{m} A_i(x) t^i \geq 0.$$

Since $\lim_{x \to \pm \infty} t(x) = 0$, and the zeros of p and q are interlacing then $t(x)$ consists of two hyperbolic $B_0, B_k$ and $k-1$ cotangent-shaped decreasing branches $B_1, ..., B_{k-1}$, where $B_i$ is defined for $x_i < x < x_{i+1}$, $x_0 = -\infty$, $x_{k+1} = \infty$. A function $t_0 = t_0(x)$ will be called an $(i, j)$-transversal if

(i) $t_0$ is continuous on $[M, N]$,

(ii) $t_0$ intersects each of the branches $B_i$, of $t$ for $i \leq l \leq j$.

(iii) there is an open interval $I \subset [M, N]$ such that $Q(t_0, x) \leq 0$ iff $x \in [M, N] \setminus I$.

Obviously, if an $(i, j)$-transversal exists then $[x_{i+1}, x_j] \subset I$, and to get bounds on $x_{i+1}, x_j$ one needs just to find the extreme roots of the equation $Q(t_0, x) = 0$, on $[M, N]$. Note that any continuous function intersects all the cotangent-shaped branches $B_1, ..., B_{k-1}$, and is, if (iii) holds, a $(1, k-1)$-transversal, thus giving bounds on $x_2$ and $x_{k-1}$. For example, $t_0 = cx$, is a $(0, k)$-transversal for $c > 0$, and a $(1, k-1)$-transversal for $c \leq 0$, provided it satisfies (iii). As we will see, in many cases the condition (iii) is automatically fulfilled and moreover, a naive choice of $t_0$ as a solution of $\frac{\partial Q(t, x)}{\partial t} = 0$, that is as the function providing the minimum to $Q(t, x)$, does work. The situation is especially simple for quadratic forms, the case we mainly exploit here. Then $Q(t, x) = A_0 + A_1 t + A_2 t^2$, and one may try

$$t_0 = -\frac{A_1}{2A_2}, \text{ with } Q(t_0, x) = \frac{4A_0A_2 - A_1^2}{4A_2^2}.$$

In the rest of the paper we will use $t = p_{k-1}/p_k$, and with one explicitly stated exception, $t_0$ will be chosen as a solution of $\frac{\partial Q(t, x)}{\partial t} = 0$.

The simplest way to obtain the required quadratic form $Q(t, x)$ for orthogonal polynomials is to express $p_{i-1}, p_i,$ and $p_{i+1}$ in $T_i$, $|k-\ell| \leq 1$, via $p_{k-1}, p_k$ by the three term recurrence. In this case one gets three (slightly different) bounds on the zeros, we present just one of them in the theorem below. But already for $T_{k+2}$, the above expression for $t_0$ may have singularities and our arguments are not applicable without certain restrictions on the coefficients (see Lemma 7 below).

Using Theorems 3 and 4 to guarantee the corresponding $TI$ we get the following first order bounds.

**Theorem 6.** (i) Let $p_k$ be a symmetric polynomial satisfying (5) and suppose that $c_k$ are nondecreasing, then

$$x_{kk} < 2\sqrt{c_{k-1}}, \ k \geq 2.$$

(ii) Let $p_k$ be a symmetric polynomial orthogonal on $[-1, 1]$ satisfying (6) and (10), then

$$|x_{ik}| < 2\sqrt{b_k c_k},$$

where $i = 1, ..., k$, if $c_k$ is nondecreasing and $c_k \leq \frac{1}{2}$; and $i = 2, ..., k - 1$, if $c_k$ is nonincreasing and $c_k \geq \frac{1}{2}$. 

(iii) Let $p_k$ be a polynomial orthogonal on $[0, \infty)$ satisfying \((1)\). If $b_k$ and $c_k$ are nondecreasing, then
\[
(\sqrt{b_k} - \sqrt{c_k})^2 < x_{2,k} < x_{k,k} < (\sqrt{b_k} + \sqrt{c_k})^2,
\]
provided $c_k \leq b_k$, $c_k - c_{k-1} \geq b_k - b_{k-1}$; and
\[
(\sqrt{b_k} - \sqrt{c_k})^2 < x_{1,k} < x_{k,k} < (\sqrt{b_k} + \sqrt{c_k})^2,
\]
provided $c_k \geq b_k$, $c_k - c_{k-1} \leq b_k - b_{k-1}$.

Proof. (i) By \((1)\) we get
\[
Q(t,x) = c_k^{-1}p_k^{-2}T_{k-1}(\mathcal{P}, x) = 1 - xt + c_{k-1}t^2 \geq 0.
\]
In our case $M = N = \infty$, and $t_0 = \frac{x}{2c_{k-1}}$, is clearly a $(0,k)$-transversal. Finally
\[
4c_{k-1}^2Q(t_0,x) = 4c_{k-1} - x^2,
\]
the result follows.

(ii) Substituting $p_{k+1}$ from \((1)\) (in our case $a_k = 0$), we have for $[M,N] = [-1,1]$
\[
Q(t,x) = b_k T_k(\mathcal{P}, x) = b_k - xt + c_k t^2 \geq 0,
\]
with $t_0 = \frac{x}{2c_k}$, and $4b_c c_k Q(t_0,x) = 4b_k c_k - x^2$. Obviously, $t_0$ is a $(1,k-1)$-transversal. Finally, using the normalization $p_0(1) = 1$, $p_1(-1) = (-1)^i$, and $b_k + c_k = 1$, one can check that $t_0(-1) \leq t(-1)$, and $t_0(1) \geq t(1)$, only if $c_k \leq \frac{1}{2}$
Thus, for $c_k \leq \frac{1}{2}$, $t_0$ intersects all the branches of $t$ and hence is a $(0,k)$-transversal.

(iii) Substituting $p_{k+1}$ from \((1)\) we obtain with $[M,N] = [0,\infty]$
\[
b_k p_k^{-2} T_k(\mathcal{P}, x) = b_k - (b_k + c_k - x) t + c_k t^2 \geq 0.
\]
This yields $t_0 = \frac{b_k + c_k - x}{2c_k}$, which is at least a $(1,k-1)$-transversal, and
\[
4b_k c_k \Delta(x) = \left( x - (\sqrt{b_k} - \sqrt{c_k})^2 \right) ( (\sqrt{b_k} + \sqrt{c_k}^2 - x ) \geq 0.
\]
Notice that the polynomials here are normalized so that the sign of the leading coefficient of $p_k$ is $(-1)^k$, and so the branches of $t(x)$ are up-side-down in comparison with the previous cases. Thus, to guarantee the intersection of $t_0$ with $B_k$ one should check
\[
\lim_{x \to \infty} t(x) = -1 > \lim_{x \to \infty} t_0(x) = -\infty,
\]
hence $t_0$ is a $(1,k)$-transversal. On the other hand to be a $(0,k)$-transversal it should satisfy $t_0(0) = \frac{b_k + c_k}{2c_k} \leq p_k(0) = 1$. This is the case only if $b_k \leq c_k$. This complete the proof.

Note that similar bounds can be obtained for nonsymmetric polynomials orthogonal on a finite interval. The corresponding $TI$ are given in \((13)\). The restrictions we have to impose in cases (ii) and (iii) to obtain bounds on all the zeros reflect the real situation. An easy example is provided by ultraspherical and Laguerre polynomials with small parameters. Roughly speaking, the reason why the obtained bounds exclude one or both extreme zeros is that these zeros are too close to the ends of the interval of orthogonality and the used inequalities do not have enough precision to distinguish between them.

The simplest, seemingly open question, arising naturally in connection with the above theorem is

Suppose that $p_k$ satisfies \((4)\), what is the maximal rate of growth of $c_k$ such that
\[
\lim_{k \to \infty} \frac{x_{kk}}{2\sqrt{c_{k-1}}} = 1?
\]
The asymptotic (7) shows that this is true for the polynomial growth. On the other hand it is almost certainly wrong for exponential $c_k \sim c^k$, $c > 1$. The following lemma supports this claim.

**Lemma 7.** Let $p_k$ be a symmetric polynomial satisfying (6) and suppose that $\frac{3}{4}c_k < c_{k-1} \leq c_k$. Then

$$x_{kk} < 2\sqrt{c_{k-2}}, \quad k \geq 3.$$ 

**Proof.** We consider

$$Q(t, x) = c_{k-2}c_{k-1}T_{k-2}(P, x) = (c_{k-1} - (c_{k-1} - c_{k-2})x^2)t^2 - x(2c_{k-2} - c_{k-1})t + c_{k-2} \geq 0.$$ 

In view of Theorem (i) we can choose $[M, N] = [-2\sqrt{c_{k-1}}, 2\sqrt{c_{k-1}}]$. Then

$$t_0 = \frac{x(2c_{k-2} - c_{k-1})}{2(c_{k-1} - (c_{k-1} - c_{k-2})x^2)},$$

by the assumption $\frac{3}{4}c_k < c_{k-1} \leq c_k$, therefore $t_0$ is a continuous function on $[M, N]$ and thus a $(0, k)$–transversal. Finally

$$Q(t_0, x) = \frac{(4c_{k-2} - x^2)c_{k-1}^2}{4(c_{k-1} - (c_{k-1} - c_{k-2})x^2)},$$

and the result follows. \(\square\)

Similar but more involved calculations with $T_{k-3}$ instead of $T_{k-2}$ yield $x_{kk} < 2\sqrt{c_{k-3}}$, provided $\frac{5+\sqrt{5}}{2}c_k < c_{k-1} \leq c_k$, we omit the details.

Now we will show that using (17) which yields a fourth degree form, one can obtain much sharper second order bounds. We will consider the simplest case of monic Hermite polynomials $H_k$ defined by (5) with $c_k = k/2$. The corresponding asymptotic for $x_{kk}$ given by (7) is

$$x_{kk} = \sqrt{2k - 2^{-1/3} - 1/3}1_{i}k^{-1/6} \approx \sqrt{2k} - 1.65 \cdot k^{-1/6}.$$ 

Putting $u_k = H_k$, $t = H_{k-1}/H_k$, in (17), we get

$$Q(t, x) = 4S_k(P, x)u_k^{-4} = k^2(2k - x^2)t^4 - 2kx(1 + 4k - 2x^2)t^3 +$$

$$(4k + x^2)(2k + 1 - x^2 - 1)t^2 - 4x(4k + 3 - 2x^2)t + 4(2k + 2 - x^2) \geq 0.$$ 

Choosing the same $t_0 = \frac{1}{2}$, as for the case of the quadratic form given by $T_k$ and calculating $Q(t_0, x)$ we get that all the zeros of $p_k$ satisfy

$$8k^2(k + 1) - (6k + 1)(2k + 1)x^2 + (6k + 2)x^4 - x^6 \geq 0.$$ 

This equation has the only positive root $x$ being the sought bound,

$$x = \frac{(m^2 - 1)^2\sqrt{m^4 + 4m^2 + 1}}{3\sqrt{3m^3}} = \sqrt{2k - 2^{-7/6}k^{-1/6} + O(k^{-5/6})},$$

where $m = 2^{-1/6}(\sqrt{27k + 2} + \sqrt{27k})^{1/3}$. Thus we get $2^{-7/6} \approx 4/9$, instead of 1.65. Notice that the result can be slightly improved by solving the system $Q(t, x) = 0$, $\frac{\partial Q(t, x)}{\partial t} = 0$, exactly. This yields $\sqrt{\frac{4k - 3k^{1/3} + 1}{2}} \approx \sqrt{2k} - 0.53k^{-1/6}$, we omit the details.
4. Bounds from higher order Turán inequalities

In this section we will consider only the symmetric case where we put for convenience \( c_0 = 0 \). First, we will establish sufficient conditions for the validity of the inequality

\[
T_k^{(2)} = T_k^{(2)}(p, x) = 3p_k^2 - 4p_{k-1}p_{k+1} + p_{k-2}p_{k+2} \geq 0,
\]
in terms of the recurrence and present the corresponding bounds on the extreme zeros. Next, we will show how to modify \( T_k^{(1)} \) to obtain second order bounds for a vast class of nondecreasing sequences \( c_k \). In particular, we will prove Theorems 1 and 2.

**Theorem 8.** Let \( \{c_k\}_{k=1}^{\infty} \) be a nondecreasing positive sequence such that

\[
(20) \quad c_{k-1} - 3c_k + 3c_{k+1} - c_{k+2} \geq 0.
\]

Then for \( k \geq 2 \),

\[
(21) \quad c_{k-1}p_k^{-2} T_k^{(2)} = c_k(4c_{k-1} - x^2)t^2 - x(4c_{k-1} - c_k + c_{k+1} - x^2)t + 3c_{k-1} + c_{k+1} - x^2 \geq 0.
\]

**Proof.** We have the following directly checked identity

\[
(22) \quad T_{k+1}^{(2)} = c_{k-1}T_k^{(2)} + (c_{k+2} + 3c_k - 4c_{k-1})T_k + (c_{k-1} - 3c_k + 3c_{k+1} - c_{k+2})p_k^2.
\]

Now the result follows by the induction on \( k \) and

\[
T_2^{(2)} = (c_0 - 3c_1 + 3c_2 - c_3)x^2 + 3c_1^2 + c_1c_3 > 0.
\]

\( \square \)

**Remark 1.** If we set \( c_k = \sum_{i=1}^{k} \delta_i \), then the conditions (20) can be rewritten as \( \delta_i \geq 0, \delta_i - 2\delta_i + \delta_{i+1} < 0 \), i.e. \( \delta_i \) should be a nonnegative concave function of \( i \).

**Theorem 9.** Let \( c_k \) satisfy the conditions (20) of Theorem 3, then all the zeros of \( p_k \) are confined between the smallest and the largest real zeros of the equation

\[
(23) \quad F(x) = x^6 - 2(4c_{k-1} + c_k + c_{k+1})x^4 + (16c_k^2 - 1 + (c_k + c_{k+1})^2 + 4c_{k-1}(5c_k + 2c_{k+1}))x^2 - 16c_kc_{k-1}(3c_{k-1} + c_{k+1}) = 0.
\]

**Proof.** We have by (21),

\[
Q(t, x) = c_{k-1}p_k^{-2} T_k^{(2)} = 3c_{k-1} + c_{k+1} - x^2 - x(4c_{k-1} - c_k + c_{k+1} - x^2)t + c_k(4c_{k-1} - x^2)t^2 \geq 0.
\]

Clearly, \( x^2 < 4c_{k-1} \), and we can choose \([M, N] = [-2\sqrt{c_{k-1}}, 2\sqrt{c_{k-1}}] \). Now

\[
t_0 = \frac{x(4c_{k-1} - c_k + c_{k+1} - x^2)}{2c_k(4c_{k-1} - x^2)},
\]

is a \((0, k)-\) transversal and any zero \( x \) satisfies

\[
Q(t_0, x) = -\frac{F(x)}{4c_k(4c_{k-1} - x^2)} > 0,
\]
yielding the required result. \( \square \)
To show that (24) indeed yields second order bounds we again consider the monic Hermite polynomials $H_k(x)$. The conditions of Lemma 8 are fulfilled as $c_k = k/2$. Solving (24) we get

$$x_{kk} < \sqrt{2k - \frac{(1 + (\sqrt{k} + \sqrt{k-1})^{2/3})^2 - 2(\sqrt{k} + \sqrt{k-1})^{2/3}}{2(\sqrt{k} + \sqrt{k-1})^{2/3}}} = \sqrt{2k-1} - 2^{-5/3}(2k-1)^{-1/6} + O(k^{-5/6}).$$

Now we will establish a new TI which is valid for a vast class of sequences $c_k$. Given $\mathcal{P} = \{p_k\}$, define $\Delta \mathcal{P} = \{q_k\}$, by $q_k(x) = p_{k+1}(x) - c_k p_{k-1}(x)$. The following identity can be checked directly.

**Lemma 10.**

$$T_{k+1}(\Delta \mathcal{P}, x) = c_k T_k(\Delta \mathcal{P}, x) + 2c_k \mu_k T_k(\mathcal{P}, x) + G,$$

where

$$p_k^{-2} G = 2c^2_k(2c_{k+2} - 2c_k - \mu_k)t^2 - 2xc_k(3c_{k+2} - 2c_{k+1} - c_k - \mu_k)t +$$

$$x^2(2c_{k+2} - 3c_{k+1} + c_k) + 4c_{k+1}(c_{k+1} - c_k) - 2c_k \mu_k,$$

$$\mu_k = 2(c_{k+1} - c_k) + \frac{1}{2} \left( \sqrt{c_{k+1} - c_k} - \sqrt{2c_{k+2} - 3c_{k+1} + c_k} \right)^2.$$

**Theorem 11.** Let $\{c_k\}_{k=1}^{\infty}$ be a nondecreasing sequence satisfying for $k = 1, 2, ...$, the following conditions

(24) $$2c_{k+2} - 3c_{k+1} + c_k \geq 0,$$

(25) $$(c_{k+1} - c_k)(\sqrt{c_{k+1} - c_k} + \sqrt{2c_{k+2} - 3c_{k+1} + c_k}) \geq$$

$$\sqrt{c_k} |c_{k+2} - 2c_{k+1} + c_k|.$$

Then

(26) $$p_k^{-2} T_k(\Delta \mathcal{P}, x) = c_k(4c_k - x^2)t^2 - x(2c_{k+1} + 2c_k - x^2)t + 4c_{k+1} - x^2 \geq 0.$$

**Proof.** As $T_k(\mathcal{P}, x) \geq 0$, by $c_{i+1} \geq c_i$, and $T_1(\Delta \mathcal{P}, x) = (2c_2 - 3c_1)x^2 + 4c_1^2 > 0$, by (24) it is left to show that $G \geq 0$. For, consider

$$H = p_k^{-2} G = 2c^2_k(2c_{k+2} - 2c_k - \mu_k)t^2 - 2xc_k(3c_{k+2} - 2c_{k+1} - c_k - \mu_k)t +$$

$$x^2(2c_{k+2} - 3c_{k+1} + c_k) + 4c_{k+1}(c_{k+1} - c_k) - 2c_k \mu_k,$$

The coefficient at $t^2$ is positive, hence it is left to check that the discriminant of this quadratic in $t$ is nonpositive, what yields

$$c_k(c_{k+2} - 2c_{k+1} + c_k)^2 - (c_{k+1} - c_k)^2(\sqrt{c_{k+1} - c_k} + \sqrt{2c_{k+2} - 3c_{k+1} + c_k})^2 \leq 0,$$

and (21), (26) follow.

Practically the conditions of the above theorem are much less restrictive than these of Theorem 8. Yet formally (20) does not follow from (24), (26), as the example $c_k = k^2 + c$, shows.

The conditions (24) and (26) are rather complicated but can be simplified by the substitution $c_k/c_{k-1} = 1 + d_k$, $d_k > 0$, giving respectively

(27) $$d_{k+1} \geq \frac{d_k}{2(1 + d_k)},$$

(28) $$d_k(\sqrt{d_k} + \sqrt{2d_k d_{k+1} + 2d_{k+1} - d_k}) \geq |d_k d_{k+1} + d_{k+1} - d_k|.$$
Lemma 12. The conditions of Theorem 11 hold if \( d_k > 0 \), and

\[
\frac{d_k}{2(1 + d_k)} < \frac{d_k(1 + 2\sqrt{d_k} + 2d_k)}{1 + d_k},
\]

Proof. Putting in (27), (28) \( d_{k+1} = \frac{d_k(1+2y^2+y^2)}{1+d_k}, \ y \geq -1/2, \) we get

\[
1 + 2y + 2y^2 \geq \frac{1}{2}, \quad 4d_k^2(1 + y)^2(d_k - y^2) > 0,
\]

and the result follows. \( \square \)

Now we are in the position to state the main result of the paper.

Theorem 13. Suppose that \( d_i = \frac{c_i - c_{i-1}}{c_i} \geq 0 \), satisfy (27), (28). Then

\[
x_k^2 < 4c_k \left(1 - 6^{-4/3}d_{k+1}^{2/3} \left((v + 9)^{1/3} - (v - 9)^{1/3}\right)^2\right)
\]

where \( v = \sqrt{6d_{k+1} + 81} \).

Proof. Let \( [M, N] = [-2\sqrt{c_k}, 2\sqrt{c_k}] \), and consider \( Q(t, x) = P_k^{-2}T_k(\Delta P, x) \) given by (28). Then we find

\[
t_0 = \frac{x(2c_{k+1} + 2c_k - x^2)}{2c_k(4c_k - x^2)},
\]

and \( t_0 \) is a \((0, k)\)-transversal. Calculating \( Q(t_0, x) \) we conclude that all the zeros of \( p_k \) satisfy

\[
x^6 - 4(c_{k+1} + 2c_k)x^4 + 4(c_{k+1} + c_k)(c_{k+1} + 5c_k)x^2 - 64c_k^2c_{k+1} < 0
\]

The corresponding equation has only two real roots giving the required bounds, namely

\[
x_k^2 = 4c_k \left(1 - 6^{-4/3}d_{k+1}^{2/3} \left((v + 9)^{1/3} - (v - 9)^{1/3}\right)^2\right).
\]

To show that there are no other roots we calculate its discriminant which is

\[-2^{28}c_{k+1}c_k^4(c_{k+1} - c_k)^8(2c_{k+1} + 25c_k)^2.
\]

As it does not change the sign, provided \( c_{k+1} > c_k > 0 \), the number of real zeros is the same for any such a choice of \( c_k, c_{k+1} \). Choosing \( c_k = 1, c_{k+1} = 2 \), we obtain the test equation \( x^6 - 16x^4 + 84x^2 - 128 = 0 \), having only two real roots. \( \square \)

Now Theorem 13 is a direct consequence of the following claim.

Lemma 14. In the conditions of Theorem 13

\[
x_k^2 < 4c_k \left(1 - \frac{d_{k+1}^{2/3}}{(2^{1/3} + d_{k+1}^{1/3})^2}\right).
\]

Proof. It is enough to show that

\[
(v + 9)^{1/3} - (v - 9)^{1/3} > \frac{6^{2/3}}{2^{1/3} + d_{k+1}^{1/3}},
\]

which is transformed into \( y > \frac{y}{1+y-y^2} \), by the substitution \( d_{k+1} = \frac{2(1-y^3)^3}{y^3}, 0 < y \leq 1 \). \( \square \)
Finally, Theorem 2 follows from (30) with $d_{k+1} = (1 + \frac{1}{k})^{2\delta} - 1$. For we observe that $\frac{d_{k+1}^{k^{1/3}}}{\sqrt{d_{k+1}^{2k^{1/3}}}}$ is an increasing function in $d_{k+1}$ and and the result follows by applying the elementary inequality

$$(1 + \frac{1}{k})^{2\delta} - 1 \geq \frac{2\delta}{k + \frac{1}{2}}, \quad \delta \geq 0.$$ 

Moreover, $k + \frac{1}{2}$ may be replaced by $k$ for $\delta \geq \frac{1}{2}$.

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