Existence of Rosseland equation

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Abstract The global boundness, existence and uniqueness are presented for the kind of Rosseland equation with a small parameter. This problem comes from conduction-radiation coupled heat transfer in the composites; it’s with coefficients of high order growth and mixed boundary conditions. A linearized map is constructed by fixing the function variables in the coefficients and the right-hand side. The solution to the linearized problem is uniformly bounded based on De Giorgi iteration; it is bounded in the Hölder space from a Sobolev-Campanato estimate. This linearized map is compact and continuous so that there exists a fixed point. All of these estimates are independent of the small parameter. At the end, the uniqueness of the solution holds if there is a big zero-order term and the solution’s gradient is bounded. This existence theorem can be extended to the nonlinear parabolic problem.

Keywords: nonlinear elliptic equation, well-posedness, fixed point, mixed boundary conditions, without growth conditions, Rosseland equation

MSC(2000): 35J60, 47H10

1 Introduction

Our original motivation is the Rosseland equation in the conduction-radiation coupled heat transfer [1,2]. Find \((u_\varepsilon - u_b) \in W^{1,2}_0(G)\) (Definition 2.3), such that

\[
\int_G a_{ij}(u_\varepsilon(x), x, x_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \int_{\Gamma} \alpha(u_\varepsilon - u_{gas}) \varphi = \int_G f(u_\varepsilon(x), x, x_\varepsilon) \varphi, \quad \forall \varphi \in W^{1,2}_0(G),
\]

where \(a_{ij} = k_{ij}(x, \frac{x}{\varepsilon}) + 4u_\varepsilon^3 b_{ij}(x, \frac{x}{\varepsilon});\) \((k_{ij}), (b_{ij})\) are symmetric positive definite; \(k_{ij}(x, y), b_{ij}(x, y)\) are 1-periodic in \(y\). The small parameter \(\varepsilon\) is the period of the composite structure. \(\Gamma\) is the natural boundary part of \(\partial G\). There may be no ellipticity for \(A = (a_{ij})\) without considering physical conditions; uniform estimates independent of \(\varepsilon\) are also needed. This open problem (the existence theory for the equation with coefficients like \(k + 4u^3b\), without \(\varepsilon\)) was proposed by Laitinen in 2002 (Remark 3.4 [3]).

There are several steps: firstly describe the physical conditions and find a suitable temperature interval by the global boundness in \(L^\infty\) (Lemma 3.1); then construct a linearized map with a fixed point in this interval (Theorem 3.4); the fixed point is unique if there is a big zero-order term and the solution’s gradient is bounded (Theorem 4.3).

The novelty is we don’t need any growth conditions in [4]: this method can be used for coefficients like \(k + 4u^m b, \forall m > 0\). More specifically, \(\forall C_1, C_2, 0 < C_1 \leq C_2,\)

\[
A(u_\varepsilon(x), x, x_\varepsilon) \in [C_3, C_4], \quad \text{if } u_\varepsilon \in [C_1, C_2]; \quad 0 < C_3 = C_3(C_1), C_4 = C_4(C_1, C_2). \quad (1.1)
\]

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Our main tool is the regularity established by Griepentrog and Recke in the Sobolev-Campanato space [5]. Their work asserted that linear elliptic equation of second order with non-smooth data ($L^\infty$-coefficients, Lipschitz domain, regular sets, non-homogeneous mixed boundary conditions) has a unique solution in $C^3(\overline{\Omega})$; this Hölder norm smoothly depends on the data.

Note that the well-posedness is still valid if the ellipticity is a priori known or only Dirichlet boundary condition is considered (the famous De Giorgi-Nash estimate holds; see Theorem 8.29 [4]). We present a local gradient estimate for a simplified problem (only the righthand side is nonlinear) in Lemma 4.1; it can be used in the error estimate of the homogenization [6]. All of these results can be extended to the nonlinear parabolic equation if we use the regularity in the parabolic Sobolev-Morrey space [7].

Throughout this paper, $C, C_i$ denote positive constants independent of the solution and the small parameter $\varepsilon$. The unit cell $Y = (0, 1)^n$. $B(x, r)$ is the open ball of radius $r$ centered at $x$. $\varphi \in [C_0, C_1]$ means that $\varphi \in L^\infty$ in the relevant domain if without confusion and $C_0 \leq \varphi \leq C_1$. For a real symmetric matrix $A = (a_{ij}(u(x), x, y))$, $A \in [C_0, C_1]$ implies

$$a_{ij}(u(x), x, y)\xi_i\xi_j \geq C_0|\xi|^2, \quad \sum |a_{ij}(u(x), x, y)|^2 \leq C_1^2, \quad \forall (x, y) \in \Omega \times Y, \xi \in \mathbb{R}^n.$$  

$\|\varphi\|_q$ is an abbreviation of the norm in the relevant $L^q$ space. $T_{min}, T_{max}$ are positive physical constants (the range of the environmental temperature); $0 < T_{min} \leq T_{max}$.

2 Regular sets, Campanato space and model problem

5 Conclusions

The well-posedness is given for the Rosseland equation with a small parameter $\varepsilon$. The physical conditions are included in (A1)-(A5). Based on the boundness in $L^\infty$, we construct a closed convex set $[T_{min}, T_*]$. Then, we prove the linearized map is compact and continuous from the Sobolev-Campanato estimate established by Griepentrog and Recke. So there exists a fixed point; the solution to the original nonlinear problems has almost the same estimates as the linear one. These estimates are independent of the small parameter. So there is a subsequence which converges in $C^0(\overline{\Omega})$ (or $H^1(\Omega)$), if $\varepsilon \to 0$. A local gradient estimate of the solution is given for a simplified problem; it can be used to the error estimate of the same type of equation’s homogenization. The uniqueness is also based on a linearized map; see (??). Similar results on the nonlinear parabolic problem based on the same method and Sobolev-Morrey estimate [7] will appear elsewhere.

Acknowledgements This work is supported by National Natural Science Foundation of China (Grant No. 90916027). The authors thank the referees for their careful reading and helpful comments.

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