A Complete Axiomatisation for Divergence Preserving Branching Congruence of Finite-State Behaviours

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Abstract—We present an equational inference system for finite-state expressions, and prove that the system is sound and complete with respect to divergence preserving branching congruence, closing a problem that has been open since 1993. The inference system refines Rob van Glabbeek’s simple and elegant complete axiomatisation for branching bisimulation congruence of finite-state behaviours by joining four simple axioms after dropping one axiom which is unsound under the more refined divergence sensitive semantics.

I. INTRODUCTION

Over the years the notion of bisimulation which was proposed by Park and popularized by the work of Milner emerges as a very important foundation for concurrency theory. Based on this notion, many interesting equivalence and congruence relations are introduced and studied. Rob van Glabbeek gave a fairly complete list of these equivalences in [5] and [6].

For a bisimulation based congruence relation on a set of expressions, an interesting question is whether there is an equational inference system, or axiomatisation, which infers exactly the pairs of equal expressions. Even in the cases where the equivalence relation is decidable, an inference system of this nature can convey valuable information about the rationale behind the equalities. For finite-state expressions (expressions which can only generate finitely many states but may generate infinite behaviours), Milner pioneered this line of research, and proposed complete axiomatisations for strong bisimulation congruence [1] and observational bisimulation congruence [2]. Following Milner’s work, Walker [4] and Lohrey et al. [13] proposed complete axiomatisations for variations of observational bisimulation congruence which take divergence behaviour into account, and van Glabbeek proposed a complete axiomatisation for branching bisimulation congruence [11].

The notion of branching bisimulation was introduced by van Glabbeek and Weijland in [10], in which a refined notion of divergence preservation is introduced and used to define divergence-preserving version of the corresponding bisimulation equivalence and congruence. So, in [11] van Glabbeek posed the following natural question: to find a complete axiomatisation for the divergence-preserving version of branching bisimulation congruence. The problem remains open until today, although Chen and Lu [14] and Fu [16] proposed complete axiomatisations for divergence-preserving semantics for sub-languages of finite-state behaviours. In this paper, we propose an axiomatisation for finite-state behaviours, and prove its soundness and completeness with respect to divergence-preserving branching bisimulation congruence.

The divergence-preserving branching bisimulation equivalence with the corresponding congruence is unique in that it is the finest possible bisimulation equivalence which abstracts from internal moves. Thus, a complete axiomatisation for the congruence could serve as a core theory which can be readily extended to axiomatisations for other bisimulation based congruences by adding new axioms.

For the completeness proof, we use Milner’s [1] framework of set of guarded equations, with the following difference: instead of using the product construction of equation sets proposed by Milner, we use the quotient construction which was introduced by Grabmayer and Fokkink in [19] and independently by Liu and Yu in [20].

The paper is organized as follows. In the next section we settle the preliminaries, including the definitions and properties of the equivalence and congruence relations. In section III we present the inference system and prove its soundness for divergence-preserving branching congruence. In section IV we introduce the notion of standard sum, and prove a standardization result: every expression can be proven equal to a standard sum. In section V we study standard equation systems (SES), and prove the quotient theorem, i.e. equivalent formal variables of an SES have common provable solution in a related guarded equation system. Using the result of section V, the completeness of the inference system is proved in section VI. Then, we conclude in section VII.
II. Expressions, Divergence-Preserving Semantics

Let $V$ be an infinite set of variables, $A$ be an infinite set of visible actions, $\tau$ be the invisible action or silent move ($\tau \not\in A$). We write $A_\tau$ for $A \cup \{\tau\}$. Consider the set $E$ of process expressions, given by the following BNF rules:

$$E \ ::= \ 0 \mid X \mid a.E \mid E + E \mid \mu X.E$$

where $a \in A_\tau$, $X \in V$. The precise meaning of the expressions will be given by operational semantics later. Here we provide the following explanations for the syntax, which may help to understand the intuitive meaning of the expressions:

* $0$ is the expression which is not capable of any action;
* $a.E$ is a prefix expression which first performs the action $a$ and then proceeds as $E$;
* $E + F$ is a non-deterministic expression which is capable of actions from $E$ and $F$;
* $\mu X.E$ is a recursion which behaves as $E$ except that whenever $X$ is encountered in an execution then the rest behaviour is as $\mu X.E$.

We assume the usual notion of free and bound occurrence of variables with respect to the variable binder $\mu$, write $FV(E)$ for the set of free variables of $E$, and write $E[F/X]$ for the resulting expression obtained by (capture free) substitution of $F$ for (free occurrences of) $X$ in $E$. For a set of variables $\{X_1, \ldots, X_n\}$, we write $E[F_1/X_1, \ldots, F_n/X_n]$ for the simultaneous (capture free) substitution of $F_i$ for $X_i$ in $E$. Sometimes we will also use set notation to write simultaneous substitution of $F_i$ for $X_i$ in $E$ for each $i \in I$ as $E[F_i/X_i | i \in I]$, where $I$ is an index set. The notation $E[F_i/X_i | i \in I]$ can also be used standing alone to represent the intended substitution. Note that $E[E_1/X_1 \{E_2/X_2\}]$ stands for the expression obtained by successive substitution of $E_1$ for the free occurrences of $X_1$ in $E$, and then $E_2$ for the free occurrences of $X_2$ in $E[E_1/X_1]$. We write $E \equiv F$ when $E, F$ are syntactically identical expressions.

The operational semantics of expressions is given by a transition relation $\rightarrow$ and a binary relation $\triangleright$ between expressions and variables defined as follows.

**Definition 2.1:** The transition relation $\rightarrow \subseteq E \times A_\tau \times E$ is the smallest relation such that (we write $E \overset{a}{\rightarrow} E'$ for $(E,a,E') \in \rightarrow)$:

1. $a.E \overset{a}{\rightarrow} E$;
2. If $E_1 \overset{a}{\rightarrow} E'$ then $E_1 + E_2 \overset{a}{\rightarrow} E'$;
3. If $E_2 \overset{a}{\rightarrow} E'$ then $E_1 + E_2 \overset{a}{\rightarrow} E'$;
4. If $E(\mu X.E/X) \overset{a}{\rightarrow} E'$ then $\mu X.E \overset{a}{\rightarrow} E'$.

The relation $\triangleright \subseteq E \times V$ is the smallest relation such that (we write $E \triangleright X$ for $(E,X) \in \triangleright)$:

1. $X \triangleright X$;
2. If $E_1 \triangleright X$ then $E_1 + E_2 \triangleright X$;
3. If $E_2 \triangleright X$ then $E_1 + E_2 \triangleright X$;
4. If $E(\mu Y.E/X) \triangleright X$ then $\mu Y.E \triangleright X$.

Also we write $\triangleright \rightarrow$ for $(\overset{a}{\rightarrow})^*$ and $\triangleright \Rightarrow$ for $\triangleright \rightarrow$ (in this paper we write $R^*$ for the reflexive and transitive closure of a binary relation $R \subseteq E \times E$, and write $R_1 R_2$ for the composition of binary relations $R_1$ and $R_2$ when $R_1$’s codomain and $R_2$’s domain are both $E$).

In an expression $E$, an occurrence of a variable $X$ is said guarded if the occurrence is within a subexpression $a.E'$ of $E$ where $a \neq \tau$. $X$ is said guarded in $E$ if every free occurrence of $X$ in $E$ is a guarded occurrence.

To see some examples, $\tau.X + a.0 \triangleright X$, and if $X \not\in FV(E)$ and $a \neq \tau$, then $X$ is guarded in $\tau.(E + a.(F + \tau.X))$.

For $E \in E$, an infinite $\tau$-run from $E$ is an infinite sequence of expressions $E_0 E_1 \ldots E_i \ldots$ such that $E_0 \equiv E$ and $E_{i+1} \overset{\tau}{\rightarrow} E_i$ for $i > 0$.

Next we state some lemmas which are needed later.

**Lemma 2.2:** Let $E, F, H \in E, a \in A_\tau, X \in V$. Then:

1) If $H \{E/X\} \overset{a}{\rightarrow} F$, then either there is $H'$ such that $H \overset{a}{\rightarrow} H'$ and $F \equiv H'\{E/X\}$, or $H \triangleright Y$, $Y \not\in E$ and $E \triangleright Y$;
2) If $H \{E/X\} \triangleright Y$, then either $H \triangleright Y$, or $H \triangleright X$ and $E \triangleright Y$;
3) If $H \overset{a}{\rightarrow} H'$, then $H \{E/X\} \overset{a}{\rightarrow} H'\{E/X\}$;
4) If $H \triangleright X$ and $E \overset{a}{\rightarrow} F$ then $H \{E/X\} \overset{a}{\rightarrow} F$.

The proof of this lemma can be found in [11] (Lemma 4).

**Lemma 2.3:** Let $X$ be a variable, $H \in E$. Then $X$ occurs unguarded in $H$ if and only if $H \triangleright X$.

**Proof** It is easy to prove by induction on the rules defining $\rightarrow$ and $\triangleright$ the following:

1) If $H \triangleright X$ then $X$ occurs unguarded in $H$;
2) If $H \overset{\tau}{\rightarrow} H'$ and $X$ occurs unguarded in $H'$ then $X$ occurs unguarded in $H$.

Note that $H \triangleright X$ is $H \Rightarrow \tau X$. Then the “if” direction can be proved by an easy induction on the length of transition sequence $\Rightarrow$, and the “only if” direction can be proved by using Lemma 2.2 and analyzing the structure of $H$.

**Lemma 2.4:** Let $E, F, X \in E, a \in A_\tau, X, W \in V$.

1) $\mu X.E \overset{a}{\rightarrow} F$ if and only if there is $E' \in E$ such that $E \overset{a}{\rightarrow} E'$ and $F \equiv E'\{X.E/X\}$;
2) $\mu X.E \triangleright W$ if and only if $E \triangleright W$ and $W, X$ are different variables.

**Proof.** The proof of 1) can be found in [13] (Lemma 6). The proof of 2) is straightforward.

To present the definition of divergence-preserving branching bisimulation, we first define a number of functions on binary relations, and study the relationships of these functions.

**Definition 2.5:** For a binary relation $R \subseteq E \times E$, define binary relations $S(R), B(R), B^*(R)$, and $B^0(R)$ as follows:

1) $S(R)$ is a binary relation such that $(E, F) \in S(R)$ if:
   a) whenever $E \overset{a}{\rightarrow} E'$, then there exists $F'$ such that $F \overset{a}{\rightarrow} F'$ and $(E', F') \in R$;
   b) whenever $E \triangleright X$, then $F \triangleright X$.
2) $B(R)$ is a binary relation such that $(E, F) \in B(R)$ if:
   a) whenever $E \overset{a}{\rightarrow} E'$, then either $a = \tau$ and there exists $F'$ s.t. $F \overset{a}{\rightarrow} F'$ and $(E', F') \in R$,
   or there exist $F', F''$ s.t. $F \Rightarrow F', F'' \Rightarrow F'' \overset{a}{\rightarrow} F'$ and $(E', F') \in R$;
   b) whenever $E \triangleright X$, then there exists $F'$ such that $F \Rightarrow F', (E, F') \in R$, and $F' \triangleright X$. 

3) \( B'(R) \) is a binary relation such that \((E, F) \in B'(R) \) if:
   a) whenever \( E \xrightarrow{\tau} E' \), then either \( \alpha = \tau \) and there exists \( F' \) s.t. \( F \xrightarrow{\alpha} F' \) and \((E', F') \in R \), or there exist \( F', F'' \) s.t. \( F \xrightarrow{\alpha} F' \) and \((E', F'), (E', F'') \in R \);
   b) whenever \( E \xrightarrow{\tau} X \), then there exists \( F' \) such that \( F \xrightarrow{\tau} F' \) and \((E, F') \in R \), and \( F' \xrightarrow{\tau} X \).

4) \( B^\Delta(R) \) is a binary relation such that \((E, F) \in B^\Delta(R) \) if:
   a) \((E, F) \in B(R) \);
   b) whenever \( EE_1 \ldots E_i \ldots \) is an infinite \( \tau \)-run from \( E \), then there exist \( E_j \) on the \( \tau \)-run and \( F' \) such that \( F \xrightarrow{\tau} F' \) and \((E_j, F') \in R \).

\( S, B, B^\Delta \) will be used to define corresponding bisimulation equivalences. \( B' \) will be introduced in a divergence preserving bisimulation verification technique. Note that \( B' \) differs from \( B \) only in one place where \( B' \) requires \( \tau \rightarrow \tau \) (in which at least one \( \tau \rightarrow \) step must be made) instead of the transition sequence \( \rightarrow \). The result of this subtle change made a big difference as \( B'(R) \) becomes a subset of not only \( B(R) \) but also \( B^\Delta(R) \) for any binary relation \( R \), as shown by the following lemma.

Lemma 2.6: Let \( R \subseteq E \times E \). Then
\[ S(R) \subseteq B'(R) \subseteq B^\Delta(R) \subseteq B(R). \]

Proof. \( S(R) \subseteq B'(R) \) and \( B^\Delta(R) \subseteq B(R) \) can be checked immediately from the definitions. To see \( B'(R) \subseteq B^\Delta(R) \), note that from the definitions one easily observes \( B'(R) \subseteq B(R) \).

Then for all \((E, F) \in B'(R) \), whenever \( EE_1 \ldots E_i \ldots \) is an infinite \( \tau \)-run from \( E \), the transition \( E \xrightarrow{\tau} E_1 \) would find some \( F' \) such that \( F \xrightarrow{\tau} F' \) and \((E_1, F') \in R \), meeting the requirement of being a member of \( B^\Delta(R) \).

Definition 2.7: A binary relation \( R \subseteq E \times E \) is a strong\( \) bisimulation if \( R \) is symmetric and \( R \subseteq S(R) \).

A binary relation \( R \subseteq E \times E \) is a branching\( \) bisimulation if \( R \) is symmetric and \( R \subseteq B(R) \).

A binary relation \( R \subseteq E \times E \) is a divergence-preserving branching\( \) bisimulation if \( R \) is symmetric and \( R \subseteq B^\Delta(R) \).

Define three binary relations, called strong\( \), branching\( \), divergence-preserving branching\( \) bisimilarity, and written \( \approx_b, \approx_b^\Delta \) respectively, as follows

\[ \sim = \bigcup \{ R \mid R \text{ is a strong \textit{\sim}} \}, \]
\[ \approx_b = \bigcup \{ R \mid R \text{ is a branching \textit{\approx_b}} \}, \]
\[ \approx_b^\Delta = \bigcup \{ R \mid R \text{ is a div.-pre. branching \textit{\approx_b^\Delta}} \}. \]

For \( \sim, \approx_b, \approx_b^\Delta \), we have the following justification.

Theorem 2.8: \( \sim, \approx_b, \approx_b^\Delta \) are equivalence relations.

Moreover

1) \( \sim \) is the coarsest strong bisimulation;
2) \( \approx_b \) is the coarsest branching bisimulation;
3) \( \approx_b^\Delta \) is the coarsest divergence-preserving branching bisimulation. Moreover, it is the coarsest equivalence relation which is a branching bisimulation and which preserves divergence, i.e. for two equivalent expressions if one has an infinite \( \tau \)-run within its equivalence class then so is the other.

Proof. 1), 2) are well known ([8], [10]). See [17] for 3). □

Note that \( \approx_b \) does not respect divergence. A simple example to show this is \( \mu \chi. (\tau.X + a.0) \approx_b \tau.a.0 \), where the left expression has an infinite \( \tau \)-run, while the right expression has not. Thus \( \approx_b^\Delta \) provides an alternative when divergence needs to be taken into account. There are different presentations of divergence-preserving branching bisimilarity. It was called branching bisimilarity with explicit divergence in [10] and [15], and called complete branching bisimilarity in [17]. When first introduced in [10], it was defined as the coarsest equivalence relation which is a branching bisimulation and which preserves divergence. 3) of Theorem 2.8 shows that the present definition gives the same relation. In [15], a condition similar to condition b) in 4) of Definition 2.7 for matching a divergent run requires that \( F' \) is found after exactly one step of \( \tau \) action. The condition b) in 4) of Definition 2.7 is from [17], which allows \( F'' \) to be found after one or more steps of \( \tau \) action. The discrepancy does not affect the resulting bisimilarity. One advantage of the weaker condition is a weaker divergence-preserving obligation in checking divergence-preserving branching bisimulation.

It is also worth noting that the divergence preserving condition for \( \approx_b^\Delta \) is more strict than that required in the divergence preserving relations introduced earlier in [4] and [13], in that the earlier works does not concern about the different equivalence classes that passed through by a divergent run.

Lemma 2.9: A strong bisimulation is a progressing branching\( \) bisimulation, which in turn is a divergence-preserving branching\( \) bisimulation, which in turn is a branching\( \) bisimulation.

Proof. Immediately follows from Lemma 2.6. □

The idea of progressing branching bisimulation comes from the notion of progressing weak bisimulation in [8]. where by applying the standard fixed-point definition as in Definition 2.7, the notion of progressing weak bisimulation results in an equivalence relation which is a congruence. While this is not the case for progressing branching bisimulation, i.e. progressing branching bisimulation equivalence defined in similar fashion is not a congruence. Here we refrain from introducing a new equivalence, but instead, with Lemma 2.9 we can use progressing branching bisimulation as a tool to establish divergence-preserving branching bisimulation, and it turned out to suit this role very well.

Proposition 2.10: \( \sim \subseteq \approx_b^\Delta \subseteq \approx_b \).

Proof. Immediately follows from Lemma 2.9. □

Proposition 2.11: Let \( B(E) = \{ (E, E) \mid E \in E \} \). Then \( B(E) \) is a divergence-preserving branching\( \) bisimulation.

Proof. Easy to check. □

Proposition 2.11 can be strengthened to state that \( B(\approx_b^\Delta) \) is a divergence-preserving branching\( \) bisimulation, which can be a very useful technique in establishing divergence-preserving branching\( \) bisimulation, since the troublesome condition about
infinite $\tau$-run in $B^h$ is avoided. However Proposition 2.11 is sufficient for the following development.

**Lemma 2.12:** Let $E, F \in \mathcal{E}$. Then $\tau.(E+F) + F \cong_{b}^h E+F$. 

**Proof.** It is easy to verify that $(\tau.(E+F)+F, E+F) \in B(Id_{\mathcal{E}})$. By Proposition 2.11 $B(Id_{\mathcal{E}})$ is a divergence-preserving branching bisimulation, thus $\tau.(E+F) + F \cong_{b}^h E+F$ follows from Definition 2.7. □ 

**Lemma 2.13:** Let $E, F \in \mathcal{E}, X \in \mathcal{V}$. If $E \cong_{b}^h F$ then $E \triangleright X$ if and only if $F \triangleright X$.

**Proof.** Straightforward from Definition 2.5 and 2.7. □

Divergence-preserving branching bisimilarity $\cong_{b}^h$ is not a congruence on $\mathcal{E}$. For a simplest counter example, note that and $a.0 \cong_{b}^h \tau.a.0$ while $a.0 + b.0 \not\cong_{b}^h \tau.a.0 + b.0$ when $a, b$ are different non-$\tau$ actions. This is solved in [18] by adding a rootedness condition.

**Definition 2.14:** Two expressions $E$ and $F$ are rooted divergence-preserving branching bisimilar, notation $E \cong_{b}^R F$, if the following hold:

1) whenever $E \xrightarrow{\tau} E'$ then $F \xrightarrow{\tau} F'$ with $E' \cong_{b}^h F'$; 
2) whenever $F \xrightarrow{\tau} F'$ then $E \xrightarrow{\tau} E'$ with $E' \cong_{b}^h F'$; 
3) $E \triangleright X$ if and only if $F \triangleright X$.

**Proposition 2.15:** $\cong_{b}^h \subseteq \cong_{b}^R$.

**Proof.** Immediately follows from the definitions. □

The following theorem shows that $\cong_{b}^R$ is a congruence relation, thus from now on we call it divergence-preserving branching congruence.

**Theorem 2.16:** $\cong_{b}^R$ is a congruence on $\mathcal{E}$, i.e. if $E \cong_{b}^R F$ then $a.E \cong_{b}^R a.F$, $E + D \cong_{b}^R F + D$, and $\mu X E \cong_{b}^R \mu X F$ for arbitrary $a \in A, \tau \in \mathcal{E}, X \in \mathcal{V}$.

**Proof.** Only $\mu X E \cong_{b}^R \mu X F$ needs a proof ($E \cong_{b}^R F$ assumed), all the rest are easy. The divergence-preserving nature of the relation made it much harder to prove than expected. A detailed proof is presented in [18]. □

Rob van Glabbeek et al. also proved in [13] that $\cong_{b}^R$ is the weakest congruence that implies divergence-preserving branching bisimilarity.

We close this section by introducing two versions of the very useful up-to technique. The notion of strong bisimulation up to $\sim$ is well known ([13]), while that of strong bisimulation up to $\sim_{b}^h$ is new.

**Definition 2.17:** A binary relation $R \subseteq \mathcal{E} \times \mathcal{E}$ is a strong bisimulation up to $\sim$ if it is symmetric and $R \subseteq S(\sim R \sim)$. A binary relation $R \subseteq \mathcal{E} \times \mathcal{E}$ is a strong bisimulation up to $\sim_{b}^h$ if it is symmetric and moreover the following hold for all $(E, F) \in R$: 

1) whenever $E \xrightarrow{\tau} E'$ then there exists $F'$ such that $F' \xrightarrow{\tau} F$ and $(E', F') \in R$; 
2) whenever $E \xrightarrow{\alpha} E'$ for $\alpha \neq \tau$ then there exists $F'$ such that $F' \xrightarrow{\tau} F$ and $(E', F') \in \approx_{b}^h R \approx_{b}^h$; 
3) whenever $E \triangleright X$ then $F \triangleright X$.

**Lemma 2.18:** Let $R \subseteq \mathcal{E} \times \mathcal{E}$.

1) If $R$ is a strong bisimulation up to $\sim$, then $R \subseteq \sim$.
2) If $R$ is a strong bisimulation up to $\sim_{b}^h$, then $R \subseteq \approx_{b}^h$.

**Proof.** 1) is proved in [3]. For 2), we first show that $\approx_{b}^h R \approx_{b}^h$ is a divergence-preserving branching bisimulation. Once this is done, then $R$ is a strong bisimulation up to $\approx_{b}^h$ implies that $\approx_{b}^h R \approx_{b}^h \subseteq \approx_{b}^h$.

First it is easy to see that because $R$ is symmetric then so is $\approx_{b}^h R \approx_{b}^h$. We need the following simple property of $\approx_{b}^h$ which is easy to establish: whenever $E \approx_{b}^h F$ and $E \rightarrow F'$, then there exists $G'$ such that $G \rightarrow G'$ and $F' \approx_{b}^h G'$.

Suppose $E \approx_{b}^h R \approx_{b}^h F$, and $E \rightarrow F'$, we shall show that:

(A) either $a = \tau$ and there exists $F'$ such that $F \rightarrow F'$, $E \approx_{b}^h R \approx_{b}^h F'$ and $E' \approx_{b}^h R \approx_{b}^h F'$; 
(B) or there exist $F', F''$ such that $F \rightarrow F'$, $F' \approx_{b}^h F''$, such that $E \approx_{b}^h R \approx_{b}^h F'$, $E' \approx_{b}^h R \approx_{b}^h F''$.

According to the meaning of relation composition, there exist $G, H \in \mathcal{E}$ such that $E \approx_{b}^h G$, $(G, H) \in R$, and $H \approx_{b}^h F$. By the branching bisimulation property of $\approx_{b}^h$, for the transition $E \rightarrow F'$, either of the following must hold:

(a) $a = \tau$ and there exists $G'$ such that $G \rightarrow G'$, $E \approx_{b}^h G'$, and $E' \approx_{b}^h G'$; 
(b) there exist $G', G''$ such that $G \rightarrow G'$, $G' \approx_{b}^h G''$, such that $E \approx_{b}^h R \approx_{b}^h F'$, $E' \approx_{b}^h R \approx_{b}^h F''$.

We will show that (a) implies (A), and (b) implies either (A) or (B) to fulfill the above proof obligation.

If (a) is the case, from $G \rightarrow G'$, according to case 1) in the definition of strong bisimulation up to $\approx_{b}^h$, there exists $H'$ such that $H \rightarrow H'$ and $(G', H') \in R$, and hence because $H \approx_{b}^h F$, there exists $F'$ such that $F \rightarrow F'$ and $H' \approx_{b}^h F'$. To summarize, in this case we find $F'$, such that $F \rightarrow F'$, and $E \approx_{b}^h R \approx_{b}^h F'$, and $E' \approx_{b}^h R \approx_{b}^h F'$, that is to say (A) holds.

If (b) is the case, from $G \rightarrow G'$ there exists $H'$ such that $H \rightarrow H'$ and $(G', H') \in R$, and since $R$ is a strong bisimulation up to $\approx_{b}^h$, the transition $G' \rightarrow G''$ implies that there exists $H''$ such that $H' \rightarrow H''$ and $G'' \approx_{b}^h R \approx_{b}^h H''$ (of cause it holds when $a = \tau$). Now because $H \approx_{b}^h F$, from the move $H \rightarrow H'$ there must exist $F_0$ such that $F \rightarrow F_0$ and $H' \approx_{b}^h F_0$. Now from $H' \rightarrow H''$, then since $H' \approx_{b}^h F_0$, either $a = \tau$ and there exists $F'$ such that $F \rightarrow F'$, and $H' \approx_{b}^h F'$ and $H'' \approx_{b}^h F''$, in this case we obtain $F \rightarrow F'$ such that $E \approx_{b}^h R \approx_{b}^h F'$ and $E' \approx_{b}^h R \approx_{b}^h F''$, that is to say (A) holds, or there exist $F', F''$ such that $F \rightarrow F'$, $F' \rightarrow F''$ and $H' \approx_{b}^h F'$, $H'' \approx_{b}^h F''$, in this case we find $F \rightarrow F'$ and $F' \rightarrow F''$ such that $E \approx_{b}^h R \approx_{b}^h F'$ and $E' \approx_{b}^h R \approx_{b}^h F''$, that is to say (B) holds. To summarize, (b) implies either (A) or (B).
there exists $E_i$ on the $\tau$-run from $E$ with $E_i \equiv_0^\circ G_j$. Then by the
definition of strong bisimulation up to $\equiv_0^\circ$ there exists an
infinite $\tau$-run $HH_1 \cdots H_j \ldots$ such that for each $H_j$ on the run,
$G_j$ on the corresponding position of the $\tau$-run from $G$ satisfies
$(G_j, H_j) \in R$. Then because $H \not\equiv_0^\circ F$, for this infinite $\tau$-run
from $H$ there exists a $H_k$ on the $\tau$-run from $H$ and also exists
$F'$ such that $F \xrightarrow{\tau} \rightarrow F'$ and $H_k \approx_0^\circ F'$, and with $H_k$
we can find $G_k$ on the $\tau$-run from $G$ such that $(G_k, H_k) \in R$,
and with $G_k$ we can find some $E_i$ on the $\tau$-run from $E$ such
that $E_i \approx_0^\circ G_k$. To summarize, we find $E_i$ on the $\tau$-run from $E$
and $F'$ such that $F \xrightarrow{\tau} \rightarrow F'$ and $E_i \approx_0^\circ R \approx_0^\circ F'$. □

III. THE INFERENCE SYSTEM AND ITS SOUNDNESS

In this section we present our inference system for $\equiv_0^\circ$ and
prove its soundness. The following is the set of axioms and
rules of the inference system, besides the rules for equational
reasoning (reflexivity, symmetry, transitivity, and substituting
equal for equal):

$S1$ $E + F = F + E$
$S2$ $E + (F + G) = (E + F) + G$
$S3$ $E + E = E$
$S4$ $E + 0 = E$
$B$ $a.(\sigma(E + F) + F) = a.(E + F)$
$R0$ $\mu X.E = \mu Y.(E[/Y/X])$ ($Y \notin FV(\mu X.E)$)
$R1$ $\mu X.E = E[\mu X.E/X]$
$R2$ If $F = \mu X.E$ then $E = \mu X.E$

We write $\vdash E = F$ if $E = F$ can be inferred from the above
axioms and rules through equational reasoning. The aim of this
section is to establish the soundness of the inference system
with respect to $\equiv_0^\circ$, i.e. Theorem 3.5.

S1-S4 are familiar axioms which appear in axiomatisations
for all bisimulation based congruences. It is easy to show that
S1-S4 are sound with respect to $\sim$ (3), hence are also sound
with respect to $\equiv_0^\circ$ here by proposition 2.15.

$B$ is the branching axiom which was first introduced in 10
and proved sound with respect to (divergence blind) branching
congruence. The following lemma proves that B is also sound
with respect to $\equiv_0^\circ$.

Lemma 3.1: Let $E, F \in \mathcal{E}, a \in Act_\tau$. Then

$$a.(\sigma(E + F) + F) \equiv_0^\circ a.(E + F).$$

Proof. Directly follows from Lemma 2.12 □

R0 is the axiom of $\alpha$-conversion, which is known to be sound
with respect to $\sim$, hence also sound with respect to $\equiv_0^\circ$ because of proposition 2.15.

R1-R3 are three rules for recursion introduced in 11, and
can be proved sound with respect to $\sim$, hence R1 and R3 are
also sound with respect to $\equiv_0^\circ$. As an equational rule with an

equality as premise, the soundness of R2 with respect to $\equiv_0^\circ$
does not immediately follow from its soundness with respect to $\sim$,
therefore we need the following lemma for the soundness of R2 here.

Lemma 3.2: If $X$ is guarded in $E$, $F \equiv_0^\circ E[\mu X.E]$, then $F \equiv_0^\circ \mu X.E$.

Proof. Construct the following relation:

$S = \{(H[\mu X.E], X, Y) \mid H \in \mathcal{E}, X \text{ is guarded in } H\}$. We show that $S \cup S^{-1}$ is a strong
bisimulation up to $\equiv_0^\circ$. Once this is done, then $(E[\mu X.E], E[\mu X.E]) \in S \cup S^{-1}$,
it follows from Lemma 2.18 that $E[\mu X.E] \equiv_0^\circ E[\mu X.E]$, and then $F \equiv_0^\circ E[\mu X.E] \equiv_0^\circ E[\mu X.E] \equiv_0^\circ \mu X.E$.

To show that $S \cup S^{-1}$ is a strong bisimulation up to $\equiv_0^\circ$, suppose $(H[\mu X.E], H[\mu X.E]) \in S$ and $H[\mu X.E] \rightarrow_a L$. Then since $X$ is guarded in $H$, according to 1 of Lemma 2.2 it must be that $H \rightarrow_a H'$ and $L \equiv H'[\mu X.E]$, thus
$H[\mu X.E] \rightarrow_a H'[\mu X.E]$ if $a = \tau$. In $X$ must still be guarded in $H'$, then $(H'[\mu X.E], H'[\mu X.E]) \in S$. If $a \neq \tau$, then $X$ could be unguarded in $H'$. However, since $X$ is guarded in $E$, in this case $X$ is still guarded in $H'[\mu X.E]$, and $H'[\mu X.E] \equiv_0^\circ H'[\mu X.E]$; $H'[\mu X.E] \in S \cup S^{-1}$, $H'[\mu X.E] = H'[\mu X.E]$ for this infinite branch,
thus $(H'[\mu X.E], H'[\mu X.E]) \in \equiv_0^\circ (S \cup S^{-1})$. Also since $X$ is guarded in $H$, it easily follows from 2) of Lemma 2.2 that if $H[F/X] \rightarrow Y$ then $H[\mu X.E/X] \rightarrow Y$.

If $(H[\mu X.E], H[\mu X.E]) \in S^{-1}$, in the same way we can show that the conditions 1),2),3) in Definition 2.17 are satisfied. So $S \cup S^{-1}$ is a strong bisimulation up to $\equiv_0^\circ$. □

R4 as an axiom was first introduced in 11 for eliminating
$\tau$’s in front of unguarded occurrences of bound variables. For
$\equiv_0^\circ$ however, R4 alone is not enough to eliminate all such $\tau$’s.
So here we introduce R5 to work together with R4. Intuitively, R5 means that, the presence of unguarded occurrences of $X$
in $E$ implies a $\tau$-circle going back to the recursion represented
by $X$, thus the inner $\tau$-loop for the recursion represented by $Y$
can be eliminated without changing the divergent behaviour
of the expression. The following lemma states the soundness of
R4 and R5. The proof uses the technique of progressing
branching bisimulation.

Lemma 3.3: Let $E, F, G \in \mathcal{E}, X, Y \in \gamma, E \equiv X$, then

1) $\mu X.(\sigma(E + F) + G) \equiv_0^\circ \mu X.(\sigma(E + F) + G)$;
2) $\mu X.(\sigma.Y + E + F) \equiv_0^\circ \mu X.(\sigma.Y + E + F)$.

Proof. To prove 1), let $L = \mu X.(\sigma(E + F) + G)$ and
$R = \mu X.(\sigma.E + G)$, and let $S$ be the following relation:

$$\{(H[L/X], H[R/X]) \mid H \in \mathcal{E}\} \cup$$

$$\{(E[L/X], E[R/X] + F[R/X])$$

$$\rightarrow \tau.E[L/X] + F[L/X], E[R/X] + F[R/X])\}.$$ We show that $S \cup S^{-1}$ is a progressing branching bisimulation.

By Lemma 2.4 $L \rightarrow_a L'$ if and only if

1) either $G \rightarrow G'$ and $L' \equiv G'[L/X]$,
2) or $a = \tau$ and $L' \equiv \tau.E[L/X] + F[L/X]$.
For the same reason $R \xrightarrow{a} R'$ if and only if
1) either $G \xrightarrow{a} G'$ and $R' \equiv G'(R/X)$,
2) or $a = \tau$ and $R' \equiv E(R/X) + F(R/X)$.

Because $E \vDash X$, together with $L \xrightarrow{\sigma} \tau.E(L/X) + F(L/X)$ and $R \xrightarrow{\tau} E(R/X) + F(R/X)$, the following follows from Lemma 2.3

1) $E(L/X) \xrightarrow{\tau} \tau.E(L/X) + F(L/X)$, and
2) $E(R/X) \xrightarrow{\tau} \tau.E(R/X) + F(R/X)$.

From these observation, it is easy to check that $S \cup S^{-1}$ is a progressing branching bisimulation, hence a div.-pres. branching bisimulation by Lemma 2.9 and $S \cup S^{-1} \subseteq \equiv_b^\circ$.

Moreover, whenever $L \xrightarrow{\alpha} L'$ then there exists $R'$ such that $R \xrightarrow{\alpha} R'$ with $(L', R') \in S$, and whenever $R \xrightarrow{\alpha} R'$ then there exists $L'$ such that $L \xrightarrow{\alpha} L'$ with $(R', L') \in S^{-1}$, and $L \triangleright Y$ if and only if $R \triangleright Y$ for any $Y \in \mathcal{E}$, thus $L = \equiv_b^\circ R$.

To prove 2), let $L_X = \mu X. (\tau.Y.E(\tau.X/X) + F)$, $L_Y = \mu Y. (\tau.Y.E(\tau.X/X) + F)$, $R_X = \mu X. (\tau.Y.E(\tau.X/X) + F)$, and $R_Y = \mu Y. (\tau.Y.E(\tau.X/X) + F)$, and let $S$ be the following relation:

$\{(H(L_X/X), H(R_X/X)) \mid | H \in \mathcal{E} \} \cup \{(H(L_X/X, L_Y/Y), H(R_X/X, R_Y/Y)) \mid | H \in \mathcal{E} \}$

We show that $S \cup S^{-1}$ is a progressing branching bisimulation.

By Lemma 2.4 $L_X \xrightarrow{\alpha} L'$ if and only if
1) either $F \xrightarrow{\alpha} F'$ and $L' \equiv E'(L_X/X)$,
2) or $a = \tau$ and $L'$ is just $L_Y$.

And $L_Y \xrightarrow{\alpha} L'$ if and only if
1) either $a = \tau$ and $L'$ is just $L_Y$.
2) or $E \xrightarrow{\alpha} E'$ and $L' \equiv E'(L_X/X, L_Y/Y)$.
3) or $L_X \xrightarrow{\alpha} L'$, in which case either $F \xrightarrow{\alpha} F'$ and $L' \equiv F'(L_X/X)$ or $a = \tau$ and $L'$ is just $L_Y$.

For the same reason $R \xrightarrow{\alpha} R'$ if and only if
1) either $F \xrightarrow{\alpha} F'$ and $R' \equiv F'(R_X/X)$,
2) or $a = \tau$ and $R'$ is just $R_Y$.

And $R_Y \xrightarrow{\alpha} R'$ if and only if
1) either $E \xrightarrow{\alpha} E'$ and $R' \equiv E'(L_X/X, L_Y/Y)$,
2) or $R_X \xrightarrow{\alpha} R'$, in which case either $F \xrightarrow{\alpha} F'$ and $R' \equiv F'(R_X/X) \text{ or } a = \tau \text{ and } R' \equiv R_Y$.

Because $E \vDash X$, together with $R_X \xrightarrow{\tau} R_Y$, it follows from Lemma 2.3 that $E(R_X/X, R_Y/Y) \xrightarrow{\tau} \rightarrow R_Y$, thus $R_X \xrightarrow{\tau} \rightarrow R_Y$. From these observation, it is easy to check that $S \cup S^{-1}$ is a progressing branching bisimulation, hence a div.-pres. branching bisimulation, and $S \cup S^{-1} \subseteq \equiv_b^\circ$.

Moreover, whenever $L_X \xrightarrow{\alpha} L'$ then there exists $R'$ such that $R_X \xrightarrow{\alpha} R'$ with $(L', R') \in S$, and whenever $R \xrightarrow{\alpha} R'$ then there exists $L'$ such that $L_X \xrightarrow{\alpha} L'$ with $(R', L') \in S^{-1}$, and $L \triangleright Z$ if and only if $R_X \triangleright Z$ for any $Z \in \mathcal{V}$, thus $L_X = \equiv_b^\circ R_X$.

A key idea proposed by Milner in [1] for equations is to transform arbitrary expressions into guarded ones, i.e. expressions in which every recursive subexpression is a guarded recursion, so that the fixed point induction rule R2 can be applied to derive equality of semantically equivalent expressions. Determined by its divergence preserving nature, a major difficulty in an axiomatisation for $\equiv_b^\circ$ is that unguarded recursions cannot be eliminated completely like in the axiomatisation for $\equiv_b$. Thus, in order to use the full power of the fixed point induction rule R2, careful manipulation of unguarded recursions is called for, and axioms R6, R7, and R8 are exactly for that purpose. The intuition for R6 is that, the left hand side expression is a recursion starting with a $\tau$, while the right hand side expression always perform a $\tau$ before recursion, so they should be doing the same thing. R7 roughly says that double loop is the same as a single loop, which intuitively makes sense. The intuition for R8 is that here the successive recursion for X and Y effectively defines X and Y as the same behaviour, thus interchanging the two variables should not affect the overall behaviour. The following lemma states the soundness of B6, R7, and R8 with respect to $=\equiv_b^\circ$. Surprisingly it turned out that these axioms are even sound with respect to the stronger congruence $\sim$.

**Lemma 3.4**: Let $E, F \in \mathcal{E}$, $X, Y \in \mathcal{V}$. Then

1) $\mu X.\tau.E.\tau.X/X \sim \mu X.\tau.E(\tau.X/X)/X$,
2) $\mu X.\tau.X/\mu Y.(\tau.Y+E) \sim \mu X.\mu Y.(\tau.Y+E)$,
3) $\mu X.\mu Y.(\tau.X+E+F) \sim \mu X.\mu Y.(\tau.Y+E+F)$.

**Proof**: To prove 1), let

$S = \{(H(L_X/X, L_Y/Y), H(R_X/X, R_Y/Y)) \mid | H \in \mathcal{E} \}$

Then $S \cup S^{-1}$ is a strong bisimulation up to $\sim$, thus $S \subseteq \sim$. Take $X$ as $H$, then $(\mu X.\tau.E, \tau.X.E(\tau.X/X)) \in S$, thus $\mu X.\tau.E \sim \tau.X.E(\tau.X/X)$.

To prove 2), let

$S = \{(H(L_X/X, L_Y/Y), H(R_X/X, R_Y/Y)) \mid | H \in \mathcal{E} \}$

where

$L_X = \mu X.\tau.X+\mu Y.(\tau.Y+E)$, $L_Y = \mu Y.(\tau.Y+E)$,

$R_X = \mu X.\mu Y.(\tau.Y+E)$, $R_Y = \mu Y.(\tau.Y+E)$.

Note that $R_X \sim R_Y$, then $S \cup S^{-1}$ is a strong bisimulation up-to $\sim$, thus $S \subseteq \sim$. Take $X$ as $H$ to have $(L_X, R_X) \in S$, hence $\mu X.\tau.X+\mu Y.(\tau.Y+E) \sim \mu X.\mu Y.(\tau.Y+E)$.

To prove 3), let

$S = \{(H(L_X/X, L_Y/Y), H(R_X/X, R_Y/Y)) \mid | H \in \mathcal{E} \}$

where

$L_X = \mu X.\mu Y.(\tau.X+E+F)$, $L_Y = \mu Y.(\tau.X+E)$,

$R_X = \mu X.\mu Y.(\tau.Y+E)$, $R_Y = \mu Y.(\tau.Y+E)$.

Note that $R_X \sim R_Y$, then $S \cup S^{-1}$ is a strong bisimulation up-to $\sim$, thus $S \subseteq \sim$. Take $X$ as $H$ to obtain $(L_X, R_X) \in S$, hence $\mu X.\mu Y.(\tau.X+E+F) \sim \mu X.\mu Y.(\tau.Y+E+F)$.

With these lemmas, finally we have the following soundness theorem for the inference system with respect to $=\equiv_b^\circ$.

**Theorem 3.5**: For $E, F \in \mathcal{E}$, if $\vdash E = F$ then $E = F$.

**Proof**: Since $=\equiv_b^\circ$ is an equivalence relation, equational reasoning preserves soundness. Also by Theorem 2.16 $=\equiv_b^\circ$ is a congruence, thus the inference rule of substituting equal for
equal preserves soundness. We also know the soundness of $S1$-$S4$, $B$, $R0$-$R7$. Thus if $\vdash E = F$ then $E = F$. □

Before closing this section we prove two useful derived rules. Milner’s inference system in [2] included the following axiom T1: $a.\tau.E = a.E$. The following theorem shows that with B, T1 can be derived from the present axiomatisation.

**Theorem 3.6:** Let $E \in \mathcal{E}$. Then:

$$T_1 \vdash a.\tau.E = a.E.$$ 

**Proof.** Thus T1 can be used as a derived rule in the inference system.

The next derived rule is D0 as stated in Theorem 3.8 To prove it we need to prove the following lemma.

**Lemma 3.7:** Let $E, E' \in \mathcal{E}, X \in \mathcal{V}$.

1. If $E \xrightarrow{a} E'$ then $\vdash E = E + a.E'$;
2. If $E \triangleright X$ then $\vdash E = E + X$.

**Proof.** By straightforward induction on the set of rules defining the relations $\rightarrow$ and $\triangleright$ (Definition 2.1).

**Theorem 3.8:** For $E, F \in \mathcal{E}, X \in \mathcal{V}$, the following holds: D0 if $E \triangleright X$ then $\vdash \mu X.(\tau.E + F) = \mu X.(\tau.(X + E) + F)$. The following is a proof of D0:

$$\vdash \mu X.(\tau.E + F) = \mu X.(\tau.(X + E) + F) \quad \text{(Theorem 3.8)}$$

**Definition 4.1:** Let $E$ be an expression, define the loop expression $\tau^*E$ as $\mu X.((\tau X + E)\) where $X \notin FV(E)$. In [13], the loop expression $\tau^*E$ was included into the basic syntax of expressions (where the notation used is $\Delta(E)$), and played a key role in defining (guarded) standard forms for all.

The loop operator $\tau^*$ applied on an expression $E$ obtains the loop expression $\tau^*E$ which can choose to perform $\tau$ without changing its state or choose to perform actions of $E$. In [13], the loop expression $\tau^*E$ was included into the basic syntax of expressions (where the notation used is $\Delta(E)$), and played a key role in defining (guarded) standard forms for all.

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**Definition 4.1:** Let $E$ be an expression, define the loop expression $\tau^*E$ as $\mu X.((\tau X + E)\) where $X \notin FV(E)$. In [13], the loop expression $\tau^*E$ was included into the basic syntax of expressions (where the notation used is $\Delta(E)$), and played a key role in defining (guarded) standard forms for all.

**Theorem 4.2:** The following equalities can be derived from the inference system.

$$D1 \vdash \tau^*E = \tau.\tau^*E + E;$$
$$D2 \vdash \tau^*E = \tau^*E + E;$$
$$D3 \vdash \mu X.(\tau.(X + E) + F) = \mu X.\tau^*(E + F) + E;$$
$$D4 \vdash \mu X.(\tau.(X+\tau^*(E + F)) + G) = \mu X.\tau^*(E + F) + G;$$
$$D5 \vdash \tau^*(\tau^*(E + F) + G) = \tau^*(E + F) + G;$$
$$D6 \vdash \tau^*(\tau^*E) = \tau^*E.$$ 

**Proof.** D1 is a special instance of R1.

The following is a proof of D2:

$$\vdash \tau^*E = \tau.\tau^*E + E \quad \text{D1}$$
$$= \tau.\tau^*E + E \quad \text{S3}$$
$$= \tau^*E + E \quad \text{D1}$$

The following is a proof of D3, where $Y$ is a variable which occurs neither free in $E$ nor in $F$:

$$\vdash \mu X.(\tau.(X + E) + F) = \mu X.\mu Y.(\tau.(Y + E) + F) \quad \text{R1}$$
$$= \mu X.\mu Y.(\tau.Y + E + F) \quad \text{R8}$$
$$= \mu X.\mu Y.(\tau.Y + F + E) \quad \text{R4}$$
$$= \mu X.(\tau.Y + F + E + F) \quad \text{R6}$$

The following is a proof of D4, where $Y$ is a variable which does not occur free in $E, F, G$:

$$\vdash \mu X.\tau^*(E + \tau.(X + F) + G) + \tau.(X + F) + G \quad \text{D3}$$
$$= \mu X.\tau^*(E + \tau.\tau^*(E + F) + G) + \tau.(X + F) + G \quad \text{R5}$$
$$= \mu X.\tau^*(E + \tau.(X + F) + G) + \tau.(X + F) + G \quad \text{R1}$$
$$= \mu X.\tau^*(E + \tau.(X + F) + G) + \tau.(X + F) + G \quad \text{R4, S1}$$
$$= \mu X.\tau^*(E + \tau.(X + F) + G) + \tau.(X + F) + G \quad \text{S1, S2}$$

The following is a proof of D5, where $X, Y$ are different variables which occur neither free in $E$ nor in $F$:

$$\vdash \tau^*(\tau^*(E + F) + G)$$
$$= \mu X.\tau^*(\tau.\tau^*(E + F) + F) \quad \text{Definition 4.1}$$
$$= \mu X.\tau^*(\tau^*(E + F) + G)$$
$$= \mu X.\tau^*(\tau^*(E + F) + G)$$

The following is a proof of D6, where $X, Y$ are different variables which occur neither free in $E$ nor in $F$: 

$$\vdash \tau^*(\tau^*(E + F) + G)$$
$$= \mu X.\tau^*(\tau.\tau^*(E + F) + F) \quad \text{Definition 4.1}$$

The following is a proof of D7, where $X, Y$ are different variables which occur neither free in $E$ nor in $F$: 

$$\vdash \tau^*(\tau^*(E + F) + G)$$
$$= \mu X.\tau^*(\tau.\tau^*(E + F) + F) \quad \text{Definition 4.1}$$

The following is a proof of D8, where $X, Y$ are different variables which occur neither free in $E$ nor in $F$: 

$$\vdash \tau^*(\tau^*(E + F) + G)$$
$$= \mu X.\tau^*(\tau.\tau^*(E + F) + F) \quad \text{Definition 4.1}$$
Lemma 4.5: Let $E, F \in \mathcal{E}, X \in \mathcal{V}$. If $E$ is guarded, $E \triangleright X$, and $X$ is fully exposed in $E$, then there exists $E_1$ such that $X$ is guarded in $E_1$, and $\vdash \mu X.(\tau.E + F) = \mu X.(\tau.(X + E_1) + F)$.

Proof. The proof is by induction on the structure of $E$.

If $E$ is 0, then $E \nless X$, and in this case the claim holds vacuously.

If $E$ is a variable, then since $E \triangleright X$ the variable must be $X$, we can take 0 as $E_1$, then $X$ is guarded in $E_1$ and the equality holds and $E_1$ is a guarded expression.

If $E$ is a prefix form $a.E'$, since $E \triangleright X$, it must be that $a = \tau$ and $E' \nless X$. Since $X$ is fully exposed in $E$, $X$ must be fully exposed in $E'$. Then

$$\vdash \mu X.(\tau.E + F) = \mu X.(\tau.(E' + F))$$

Let $E \in \mathcal{E}, X \in \mathcal{V}$. If $E$ is a guarded expression, then there exists a guarded expression $E'$ such that $X$ is fully exposed in $E'$ and $\vdash E = E'$.

Proof. The proof is by induction on the structure of $E$. Here we look at the only non trivial case where $E$ is a recursion $\mu Y.F$. Since $\mu Y.F$ is a guarded expression, either it is a loop expression so $E = \tau.Y + F_1$ where $F_1$ is a guarded expression, or $Y$ is guarded in $F$ which itself is a guarded expression.

In the first case, by the induction hypothesis there exists a guarded expression $F_1'$ such that $X$ is fully exposed in $F_1'$ and $\vdash F_1 = F_1'$. Then $\mu Y.(\tau.Y + F_1')$ (which is a loop expression) is guarded and $X$ is fully exposed in $\mu Y.(\tau.Y + F_1')$ and $\vdash \mu Y.F = \mu Y.(\tau.Y + F_1')$. In the second case, by the induction hypothesis there exists a guarded expression $F'$ such that $X$ is fully exposed in $F'$ and $\vdash F = F'$. Since $Y$ is guarded in $F$ and $F = \mu Y.F'$ (follows from $\vdash F = F'$), $Y$ must be guarded also in $F'$. Then $X$ is fully exposed in $F'$ implies that $X$ is fully exposed in $\mu Y.F'/Y'$ since those unexposed unguarded occurrences of $X$ in $\mu Y.F'$ becomes guarded occurrences in $F'(\mu Y.F'/Y')$. Clearly $F'(\mu Y.F'/Y')$ is guarded since $F'$ is guarded. Now $\vdash \mu Y.F' = F'(\mu Y.F'/Y')$, thus $\vdash \mu Y.F = F'(\mu Y.F'/Y')$ and $F'(\mu Y.F'/Y')$ is the $E'$ we need in this case.
Here we only look at the case where \( E \) is a recursion \( \mu X.F \); all other cases are simple. By the induction hypothesis there exist guarded expressions \( E_1, \ldots, E_n \) and variables \( W_1, \ldots, W_n \) such that \( \vdash F = \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j \). Let \( S \) be the set of summands in \( \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j \), then the elements of \( S \) can be divided into four groups:

1) those of the form \( a.H \) where \( X \) occurs unguarded in \( a.H \);
2) those of the form \( a.H \) where \( X \) is guarded in \( a.H \);
3) those variable \( W \) which is not \( X \);
4) \( X \) (if it is one of the summands).

Let \( G \) be the sum of all expressions in the second and third groups above, then \( G \) is a standard guarded sum and \( X \) is guarded in \( G \), and let \( \tau.F_1, \ldots, \tau.F_k \) be the expressions in the first group (since \( X \) occurs unguarded in these summands, the prefix must be \( \tau \)), then \( \vdash \mu X.F = \mu X.(\sum_{i=1}^k \tau.F_i + G) \) (using R3 to eliminate \( X \) in the fourth group if needed). Since each \( F_i \) is a guarded expression, by Lemma 4.3 there is a guarded expression \( F'_i \) such that \( \vdash F_i = F'_i \), thus \( \vdash \mu X.F = \mu X.(\sum_{i=1}^k \tau.F'_i + G) \). Also note that since the \( \tau.F_i \)’s are from the first group, \( X \) occurs unguarded in \( F_i \), so \( F'_i \triangleright X \) (Lemma 2.3). Then \( F'_i \triangleright X \) follows from Lemma 2.13. Now apply Lemma 4.5 \( k \) times we find guarded expressions \( F'_1, \ldots, F'_k \) such that \( X \) is guarded in all \( F'_i \)’s and

\[
\vdash \mu X.(\sum_{i=1}^k X.F'_i + G) = \mu X.(\sum_{i=1}^k \tau.X.(X + F'_i) + G).
\]

\[
\vdash \mu X.F = \mu X.(\sum_{i=1}^k \tau.X.(X + F'_i) + G) = \mu X.(\tau.(X + \sum_{i=1}^k F'_i)) + G = \mu X.(\tau.\tau.(\sum_{i=1}^k F'_i + G) + G) = \tau.\tau.(\sum_{i=1}^k F'_i + \sum_{j=1}^m W_j) + G(L/X) + G(L/X) + G(L/X)
\]

where \( L \equiv \mu X.(\tau.\tau.(\sum_{i=1}^k F'_i + \sum_{j=1}^m W_j) + G) \). Now \( L \) is a guarded expression since \( G \) and \( F'_1, \ldots, F'_k \) are guarded expressions and \( X \) is guarded in \( G \) and \( F'_1, \ldots, F'_k \). Moreover since \( G \) is a standard guarded sum, so are \( G(L/X) \) and \( \tau.\tau.(\sum_{i=1}^k F'_i + \sum_{j=1}^m W_j) + G(L/X) \) and \( G(L/X) \).

V. QUOTIENT OF STANDARD EQUATION SYSTEM

Definition 5.1: A recursive equation system \( S \) is a finite set of equations

\[
\{ X_i = F_i \mid i = 1, \ldots, n \}
\]

where \( X_1, \ldots, X_n \in \mathcal{V} \) are \( n \) different variables, called the formal variables of \( S \), and \( F_i \in \mathcal{E} \) for \( i = 1, \ldots, n \).

For \( E \in \mathcal{E}, E \) is said to provably solve (or satisfy) the recursive equation system \( S \) above for variable \( X_k \) with \( 1 \leq k \leq n \) if there are expressions \( E_i \) for \( i = 1, \ldots, n \) with \( E \) being \( E_k \), such that \( \vdash E_i = F_i \) for \( i = 1, \ldots, n \) with \( E \) being \( E_k \), such that \( \vdash E \).

Let \( X, Y \) be two formal variables of a recursive equation system \( S \), \( Y \) is said \( S \)-unguarded for \( X \), written \( X \triangleright S Y \), if \( Y \) occurs unguarded in the defining expression of \( X \) in \( S \), i.e. \( X = F_X + S + F_X X \). A recursive equation system is said guarded if \( \triangleright S \) is a well-founded relation between the formal variables of \( S \).

Theorem 5.2: Let \( S \) be a recursive equation system, then for every formal variable of \( S \) there is a provable solution. Moreover, if \( S \) is guarded, and both \( D, E \) are provable solutions of \( S \) for the same formal variable \( X \), then \( \vdash E = D \), i.e. every guarded recursive equation system has unique solution up to provability.

Proof. See [2], where only axioms valid in our inference system were used. □

A particular kind of recursive equation system is standard equation system.

Definition 5.3: A standard equation system (or standard equation set), noted SES, is a guarded recursive equation system in which the right hand side of each equation has either of the following forms

1) \( \sum_{i=1}^n a_i X_i + \sum_{j=1}^m W_j \)

2) \( \tau.(\sum_{i=1}^n a_i X_i + \sum_{j=1}^m W_j) \)

where the \( X_i \)’s are formal variables of \( S \), and \( W_j \)’s are not formal variables of \( S \).

Definition 5.4: Let \( X, X' \in \mathcal{V} \) be two formal variables of a guarded recursive equation system \( S \).

\( X \) and \( X' \) are said to have equivalent solutions of \( S \) if whenever \( E, E' \in \mathcal{E} \) are provable solutions of \( S \) for \( X \) and \( X' \) respectively then \( E \equiv_S E' \), and in this case we write \( X \equiv_S X' \).

\( S \) is said to have common provable solution for \( X \) and \( X' \) if there exists \( E \in \mathcal{E} \) which provable solves \( S \) for \( X \) as well as for \( X' \).

A formal variable \( X \) of \( S \) is called a bottom variable if whenever \( X \triangleright_S Y \) for some formal variable \( Y \) of \( S \) then \( X \neq_S Y \).

The following lemma shows that when \( S \) is guarded, then \( \equiv_S \) is (as the symbol suggests) an equivalence relation between the formal variables of \( S \).

Lemma 5.5: Let \( S \) be a guarded recursive equation system.

1) \( \equiv_S \) is an equivalence relation between the formal variables of \( S \);
2) for any formal variable \( X \) of \( S \), \( [X] \) contains a bottom variable, where \( [X] \) is the \( \equiv_S \)-equivalence class containing \( X \).

Proof. We first prove 1). Since \( S \) is guarded, Theorem S2 guarantees unique solution for every formal variable \( X \), which implies reflexivity of \( \equiv_S \). Symmetry is obvious by the symmetric phrase in the definition. Transitivity easily follows from the transitivity of \( \equiv \).

We now turn to 2). By the guardedness of \( S \) we know that \( \triangleright_S \) is a well-founded relation between formal variables of \( S \). We prove the lemma by well-founded induction on \( \triangleright_S \). If \( X \) is a bottom element, then of course \( [X] \) contains \( X \) which is a bottom element. If \( X \) is not a bottom element, then there is a formal variable \( X' \) such that \( x' \in [X] \) and \( X \triangleright_S X' \). By the induction hypothesis \( [X'] \) contains a bottom element, thus \( [X] \) contains a bottom element since \( [X] = [X'] \).

The notion of bottom variable resembles the notion of bottom elements of branching bisimulation equivalence class introduced in [7].

The main purpose of this section is to use the quotient construction to prove Theorem 5.16 which states that the \( \equiv_S \)-equality between formal variables of a standard equation system \( S \) implies common solution for the formal variables of

\[
\begin{align*}
(1) & \quad \sum_{i=1}^n a_i X_i + S + \sum_{j=1}^m W_j \\
(2) & \quad \tau.(\sum_{i=1}^n a_i X_i + \sum_{j=1}^m W_j)
\end{align*}
\]

where the \( X_i \)’s are formal variables of \( S \), and \( W_j \)’s are not formal variables of \( S \).
a related equation system. We need some preparation to prove the theorem. The next three lemmas state some important properties of bottom variables of SES.

**Lemma 5.6:** Let $S$ be an SES, $X = F_X \in S$. If $X$ is a bottom variable and $F_X \overset{a}{\rightarrow} X'$, then $X' \not\equiv_S X$.

**Proof.** In this case $X \not\equiv_S X'$, and since $X$ is a bottom variable, $X' \not\equiv_S X$. □

**Lemma 5.7:** Let $S$ be an SES, $X, Y$ be two formal variables of $S$ with $X = F_X, Y = F_Y \in S$. If $X$ is a bottom variable and $X \equiv_S Y$, then

1. whenever $F_Y \not\overset{a}{\rightarrow} W$ then $F_X \not\overset{a}{\rightarrow} W$;
2. whenever $F_Y \overset{a}{\rightarrow} Y'$ where either $Y' \not\equiv_S X$ or $a \neq \tau$, then there exists $X'$ such that $F_X \overset{a}{\rightarrow} X'$ and $X' \not\equiv_S Y'$.

**Proof.** Let $X_1, \ldots, X_n$ be the formal variables of $S$, $S = \{X_1 = F_{X_1}, \ldots, X_n = F_{X_n}\}$, $D_{X_1}, \ldots, D_{X_n}$ be a set of provable solutions of $S$ for $X_1, \ldots, X_n$ respectively, i.e. $\vdash D_{X_i} = F_{X_i}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}$ holds for $i = 1, \ldots, n$ (according to Theorem [5.2] there exist such solutions). Without loss of generality let $X, Y$ be $X_1$ and $X_2$ respectively, thus $X_1$ is a bottom variable. In particular, $D_{X_1} = F_{X_1}\{D_{X_1}/X_1 \mid i = 1, \ldots, n\}$ and $D_{X_2} = F_{X_2}\{D_{X_2}/X_2 \mid i = 1, \ldots, n\}$ follow from the soundness of the inference system, and the condition $X_1 \equiv_S X_2$ forces $D_{X_1} \approx_D D_{X_2}$ to hold, thus

$$F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\} \approx_D F_{X_2}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}.$$

First we prove the following basic fact: whenever

$$X_1 \overset{a}{\rightarrow} X_2 \text{ with } D_{X_i} \equiv_D D_{X_i}, \text{ then } G \equiv F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\} \Rightarrow G \equiv F_{X_2}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}.$$

We can prove this by induction on the length of the $\overset{a}{\rightarrow}$ transition sequence from $F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}$ to $G$. If the length is zero, then $G$ does not make a move from $F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}$, clearly in this case $G \equiv F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}$ holds. If the length is not zero, then there is $G'$ such that $X_1 \overset{a}{\rightarrow} G'$, $G' \Rightarrow G$ and the number of transitions from $G'$ to $G$ is one less than that from $F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}$, and since $D_{X_1}$ provably solves $S$, $D_{X_1} = F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\}$ follows from the soundness of the inference system, thus

$$F_{X_1}\{D_{X_i}/X_i \mid i = 1, \ldots, n\} \approx_D G \equiv D_{X_1} \approx_D G' \Rightarrow G \equiv D_{X_1} \approx_D G \equiv D_{X_1}.$$
Thus by 5.15, when \( X \approx_S X' \) and \( X \) is a bottom variable, with the \( \approx_S \)-respecting property of the substitution, any summand of \( F_X^1(\{E \} | Y \in V) \) is a summand of \( F_X^1(\{E \} | Y \in V) \).

Thus by S3
\[
\vdash F_X^1(\{E \} | Y \in V) = F_X^1(\{E \} | Y \in V) + F_X^1(\{E \} | Y \in V).
\]

**Definition 5.11:** For a recursive equation set \( S \), define a new recursive equation set \( \tau(S) = \{ X = \tau.X | X = F_X \in S \} \).

Then \( \tau(S) \) has the same set of formal variables as \( S \).

**Lemma 5.12:** Let \( S \) be a recursive equation set, \( X, Y \) be formal variables of \( S \). Then \( X \tau \approx_S Y \) if and only if \( X \tau \approx_{\tau(S)} Y \).

**Proof.** Obvious. □

**Lemma 5.13:** Let \( S \) be an SES. Then \( \tau(S) \) is guarded.

**Proof.** Note that an SES is a guarded recursive equation set, then the claim immediately follows from Lemma 5.12. □

**Proposition 5.14:** Let \( S \) be an SES, \( X \) be one of the formal variables. If \( E \in E \) is a provable solution of \( S \) for \( X \) then \( \tau.E \) is a provable solution of \( \tau(S) \) for \( X \).

**Proof.** It easily follows from T1 (Theorem 3.6). □

We need the following lemma about successive substitution applied on a simple sum \( F \) (Definition 4.6). The lemma also holds for general expression \( F \), which is the well-known substitution lemma (for example Lemma 2.1 in [9]). The simplified version is sufficient for our purpose, and is easy to establish.

**Lemma 5.15:** Let \( F \) be a simple sum, \( X_1, \ldots, X_n \) be \( n \) variables, \( E_1, \ldots, E_n \) be \( n \) expressions, \( Z_1, \ldots, Z_m \) be \( m \) variables which do not occur in \( F \), \( N_1, \ldots, N_m \) be \( m \) expressions. Then
\[
\begin{align*}
F[E_1/X_1, \ldots, E_n/X_n] & \equiv F[E_1(N_1/Z_1, \ldots, N_m/Z_m)/X_1, \ldots, E_n(N_1/Z_1, \ldots, N_m/Z_m)/X_n] \\
\end{align*}
\]

**Proof.** Can be proved by an easy induction on the number of summands in \( F \). □

We now arrive at the main result of this paper, we name it quotienting theorem because the proof uses the quotient construction of an SES.

**Theorem 5.16:** (Quotienting) Let \( S \) be an SES, \( X, X' \) be two formal variables of \( S \). If \( X \approx_S X' \), then \( \tau(S) \) has common provable solution for \( X \) and \( X' \).

**Proof.** Let \( V \) be the set of formal variables of \( S \). Since \( S \) is an SES which is a guarded recursive equation system, \( \approx_S \) is an equivalence relation on \( V \), we can assume that \( \approx_S \) partitions \( V \) into \( n \) equivalence classes \( C_1, \ldots, C_n \). According to Lemma 5.7 there exists a bottom variable in each equivalence class, thus we can assume that \( X_1, \ldots, X_n \) are \( n \) designated bottom variables such that \( X_i \in C_i \) with \( X_i = F_X \in S \) for \( i = 1, \ldots, n \). For \( X \in V \), we define the index of \( X \), written \( \iota(X) \), such that \( \iota(X) = i \) if \( X \in C_i \). Let \( Z_1, \ldots, Z_n \) be \( n \) variables which are not variables occurring in any equation in \( S \). We construct the following \( \approx_S \)-quotient equation system of \( S \):
\[
S/\approx_S = \{ Z_i = G_i | i = 1, \ldots, n \}
\]

where \( G_i = F_X \{Z_i(Y)/Y \in V \} \). Then \( S/\approx_S \) is a recursive equation system, and according to Theorem 5.2, there exist \( n \) expressions \( B_1, \ldots, B_n \) such that for \( i = 1, \ldots, n \) we have the following equality:
\[
\vdash B_i = G_i \{B_1/Z_1, \ldots, B_n/Z_n\}. \quad (1)
\]

With this equality, we have that for each \( i = 1, \ldots, n \):
\[
\vdash B_i = G_i \{B_1/Z_1, \ldots, B_n/Z_n\} \quad (1)
\]

where \( \iota(X) \), i.e. \( X_i \) is the designated bottom variable in \( [X] \) and \( X_i = F_X \in S \). To prove this we discuss four cases according to whether \( X \) is a bottom variable and whether \( F_X \) is a loop expression by using Proposition 5.10 (note that \( \{B_1(Y)/Y | Y \in V\} \) is clearly a \( \approx_S \)-respecting substitution).

If \( X \) is a bottom variable and \( F_X \) is not a loop expression, then by Lemma 5.8 \( F_X \) is not a loop expression. So
\[
\vdash \tau.F_X \{B_i(Y)/Y | Y \in V\}
\]

is a recursive equation system, and according to Theorem 5.2, there exist \( n \) expressions \( B_1, \ldots, B_n \) such that for \( i = 1, \ldots, n \) we have the following equality:
\[
\vdash B_i = F_X \{B_i(Y)/Y | Y \in V\}. \quad (2)
\]

Our next step is to prove that for each \( X \in V \) with \( X = F_X \in S \) it holds that
\[
\vdash \tau.F_X \{B_i(Y)/Y | Y \in V\} = \tau.F_X \{B_i(Y)/Y | Y \in V\}, \quad (3)
\]

where \( \iota(X) \), i.e. \( X_i \) is the designated bottom variable in \( [X] \) and \( X_i = F_X \in S \). To prove this we discuss four cases according to whether \( X \) is a bottom variable and whether \( F_X \) is a loop expression by using Proposition 5.10 (note that \( \{B_1(Y)/Y | Y \in V\} \) is clearly a \( \approx_S \)-respecting substitution).
\[ \tau.\tau^* (F^0_X \{ B_i(y)/y \mid y \in V \} + F^1_X \{ B_i(y)/y \mid y \in V \}) \]
\[ = \tau. F_X \{ B_i(y)/y \mid y \in V \} \]
\[ = F^0_X \{ B_i(y)/y \mid y \in V \} + F^1_X \{ B_i(y)/y \mid y \in V \} \]
\[ \text{4) of prop. 5.10 and S4} \]

If \( X \) is not a bottom variable and \( F_X \) is a loop expression, then \( F_X \) is also a loop expression (otherwise, since \( X \) is a bottom variable, \( F_X \) cannot be a loop expression). Now
\[ \vdash \tau. F_X \{ B_i(y)/y \mid y \in V \} \]
\[ = \tau. \tau^* (F^0_X \{ B_i(y)/y \mid y \in V \} + F^1_X \{ B_i(y)/y \mid y \in V \}) \]
\[ \text{2) of proposition 5.10} \]
\[ \vdash \tau. \tau^* (\tau. B_i(x) + F^1_X \{ B_i(y)/y \mid y \in V \}) \]
\[ = \tau. \tau^* (\tau. B_i(x)) \]
\[ \text{identity} \]

Now we are prepared to prove the completeness of the axiomatisation.

**Theorem 6.1:** Let \( E \in E \). If \( E \) is guarded, then there is a standard equation system \( S \) with a formal variable \( X \), such that \( E \) is a provable solution of \( S \) for \( X \).

**Proof.** It is proved by induction on the structure of \( E \).

(i) \( E \equiv 0 \). Take \( S \) to be the single equation \( X = 0 \).

(ii) \( E \equiv W \). Take \( S \) to be the single equation \( X = W \).

(iii) \( E \equiv a \cdot E' \). By the induction hypothesis \( E' \) provably solves a standard equation set \( S' \) for variable \( X' \). Then add the equation \( X = a \cdot X' \) to \( S' \) to form \( S \), \( E \) provably solves \( S \) for \( X \), and \( S \) is a standard equation system.

(iv) \( E \equiv E' + E'' \). By the induction hypothesis \( E' \) provably solves a standard equation set \( S' \) for \( X' \) with \( X' = F' \in S' \), and \( E'' \) provably solves a standard equation set \( S'' \) for \( X'' = F'' \in S'' \) (assume that the formal variables of \( S' \) are distinct from those of \( S'' \)). There are four cases to discuss according to the forms of \( F' \) and \( F'' \). If both \( F' \) and \( F'' \) are standard sums then \( S' \cup S'' \) and add \( X = F' + F'' \) to form \( S \) (with \( X \) distinct from the formal and free variables of \( S' \) and \( S'' \)). If \( F' \) is a standard sum while \( F'' \) \( = \tau \cdot G'' \) for some standard sum \( G'' \), then take \( S' \cup S'' \) and add the equation \( X = F' + \tau. X'' + G'' \) to form \( S \) (with \( X \) distinct from the formal and free variables of \( S' \) and \( S'' \)). Likewise for the case where \( F'' \) is a standard sum while \( F'' \) is in loop form. If \( F' \equiv \tau \cdot G' \) and \( F'' \equiv \tau \cdot G'' \) for standard sums \( G' \) and \( G'' \), then take \( S' \cup S'' \) and add \( X = \tau \cdot X' + G' + \tau. X'' + G'' \) to form \( S \) (with \( X \) distinct from the formal and free variables of \( S' \) and \( S'' \)). With \( D1 \) it is easy to see that in all the cases \( S \) is a standard equation set and that \( E \) provably solves \( S \) for \( X \).

(v) \( E \equiv \mu W'. E' \), with \( W' \) guarded in \( E' \). By the induction hypothesis \( E' \) provably solves a standard equation set \( S' \) for variable \( X' \) with \( X' = H \in S' \). We discuss two cases according to whether \( H \) is a standard sum or not. If \( H \) is a standard sum, take \( \{ Y = G[H'/W'] \mid Y \in G \in S' \} \) to form \( S \), and if \( H \equiv \tau \cdot H' \) where \( H' \) is a standard sum, take \( \{ Y = G[\tau \cdot X' + H'/W'] \mid Y \in G \in S' \} \) to form \( S \). Since \( W' \) is not a formal variable of \( S' \), in both cases \( S \) is a standard equation set. Also \( E \) provably solves \( S \) for \( X \) (in the second case \( D1 \) is used to show this).

(vi) \( E \equiv \tau \cdot E' \). By the induction hypothesis \( E' \) provably solves a standard equation set \( S' \) for \( X' \) with \( X' = F' \in S' \). If \( F' \) is a standard sum, then add the equation \( X = \tau \cdot F' \) to \( S' \) to form \( S \), \( E \) provably solves \( S \) for \( X \), and \( S \) is a standard equation system. If \( F' \equiv \tau \cdot F'' \) where \( F'' \) is a standard sum, then add the equation \( X = F'' \) to form \( S \). According to \( D6 \), \( E \) provably solves \( S \), and \( S \) is a standard equation set. □

**Lemma 6.2:** (Promotion) Let \( E, F \in E \) be guarded expressions. If \( E \approx F \), then \( \tau. E \equiv \tau. F \).

**Proof.** According to Theorem 6.1 there exist standard equation systems \( S_1 \) with a formal variable \( X \) and \( S_2 \) with a formal variable \( Y \) (assume they have disjoint sets of formal variables) such that \( E \) and \( F \) provably solve \( S_1 \) for variable \( X \) and \( S_2 \) for variable \( Y \) respectively. Then it is clear that \( S_1 \cup S_2 \) is an SES
with formal variables \( X \) and \( Y \), and that \( E \) and \( F \) provably solve \( S_1 \cup S_2 \) for \( X \) and \( Y \), respectively. Then according to Proposition 5.14 \( \tau.E \) and \( \tau.F \) provably solve \( \tau(S_1 \cup S_2) \) for \( X \) and \( Y \), respectively. If \( E \equiv \_0^F \), then \( X \approx_{S_1 \cup S_2} Y \), and according to Theorem 5.2 \( X \) and \( Y \) have common solution in \( \tau(S_1 \cup S_2) \), i.e. there is an expression \( B \) which provably solves \( \tau(S_1 \cup S_2) \) for \( X \) as well as for \( Y \). Since \( S_1 \cup S_2 \) is an SES which is guarded, by Lemma 5.13 \( \tau(S_1 \cup S_2) \) is also guarded, according to Theorem 5.2 it has unique solution. Now both \( \tau.E \) and \( B \) provably solve \( \tau(S_1 \cup S_2) \) for \( X \) so \( \tau.E = \tau.F \), and both \( \tau.F \) and \( B \) provably solve \( \tau(S_1 \cup S_2) \) for \( Y \) so \( \vdash \tau.F = B \), hence \( \vdash \tau.E = \tau.F \). □

Theorem 6.3: Let \( E, F \in \mathbf{E} \). If \( E =_b^Y F \) then \( \vdash E = F \).

**Proof.** First we show that in this case \( \vdash E + F = F \). By Theorem 4.7 there exist guarded expressions \( E_1, \ldots, E_n \), and variables \( W_1, \ldots, W_m \), such that

\[
\vdash E = \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j. \tag{A}
\]

Thus in order to prove \( \vdash E + F = F \) we only need to show that \( \vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j + F = F \), and we will do this by induction on \( m + n \). If \( m + n = 0 \), by \( S_4 \) obviously this holds. If \( m > 0 \), then \( \vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j \triangleright \vdash F \). Since \( \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j \triangleright \vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j = b \vdash E =_b^Y F \) by (A) and the soundness of the proof system and \( \vdash F \), it follows that \( F \triangleright \vdash W_m \). Thus \( \vdash F = F + W_m \) follows from Lemma 5.7. Now

\[
\vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j + F = F + W_m + F = F + W_m.
\]

Lemma 5.7.

If \( n > 0 \), then \( \vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j \triangleright \vdash a_n E_n \). By (A) and the soundness of the proof system, \( \vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j = b \vdash F \), it follows that \( F \triangleright \vdash a_n F' \) with \( F' \triangleright \vdash E_n \). Thus \( \vdash F + a_n F' \) follows from Lemma 5.7. Now

\[
\vdash \sum_{i=1}^n a_i E_i + \sum_{j=1}^m W_j + F = F + a_n F'.
\]

Lemma 5.7.

In the same way we can show \( \vdash E + F = F \), hence

\[
\vdash E = E + F = F.
\]

**VII. CONCLUSION AND FUTURE WORK**

In this paper we presented a complete axiomatisation for divergence-preserving branching congruence of finite-state behaviours. Also, along the way of proving soundness we identified three techniques for establishing divergence-preserving bisimulation equivalence and congruence: the \( B(\approx_b^Y) \) technique (proof of Lemma 5.12), the progressing branching bisimulation technique (proof of Lemma 5.13), and the strong bisimulation up to \( \approx_b^Y \) technique (proof of Lemma 5.2). Since they help to relieve one off the burden of showing divergence preservation, these techniques enrich the theory of divergence-preserving branching bisimulation, and could be useful in other works. In [12] Aceto et al. studied complete axiomatisations for (divergence-blind) weak congruence, delay congruence, and \( \eta \)-congruence besides branching congruence. These other congruences also have corresponding divergence-preserving version similar to divergence-preserving branching congruence. We hope that the result of this paper may help to establish sound and complete axiomatisations for these divergence-preserving congruences.

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