1. Introduction

Let $V$ be a complex vector space, $G$ be a reductive group acting on $V$ and $\omega$ be a $G$-character. Geometric invariant theory produces a semiprojective Deligne-Mumford (DM) stack $X := [V//G]$. Quasimap invariants are Gromov-Witten type enumerative invariants associated to GIT quotients. These invariants exploit the fact that geometry of $X$ can be captured by the much simpler space $V$ together with the group action on it: quasimap is a particular type of map to the Artin stack $[V/G]$ which generically maps to the $\omega$-stable locus. It is known that under a certain choice of stability condition ($\epsilon = 0^+$ stability) genus zero quasimap graph spaces are particularly simple - the domain curve must be irreducible. Corresponding invariants can often be computed explicitly.

There exists a generalization of quasimaps allowing to capture enumerate geometry of a critical locus of a function on $X$. This generalization is called Gauge Linear Sigma Model (GLSM) following the paper of Witten 32. Mathematical theory have been developed recently by several groups of people: 13, 7, 14. Data of GLSM consists of the GIT quotient data together with a holomorphic $G$-invariant function $W : V \rightarrow \mathbb{C}$ called superpotential that is weighted homogeneous with respect to additional $\mathbb{C}_R^*$-action on $V$ that commutes with the $G$-action. GLSM depends on the stability condition $\omega$ in a locally-constant way. Namely, the space of characters decomposes into chambers where the GIT quotient $X$ and the associated GLSM are different. Such different models are sometimes called phases of the GLSM associated to different choices of $\omega$. The simplest example is so-called Landau-Ginzburg/Calabi-Yau correspondence 9,10. It is a manifestation of the fact that a Calabi-Yau hypersurface in a (weighted) projective
space and certain FJRW theory are different phases of the same GLSM. Relation between different phases of GLSM is called GIT stability wall crossing. This wall crossing is directly related to Crepant Transformation Conjecture \cite{25,27} for the underlying GIT quotients, e.g. \cite{11}.

GIT stability wall crossing for abelian GLSMs is studied in \cite{2}. In \cite{2} we compute genus-zero invariants of abelian GLSMs directly and show that they have Mellin-Barnes type integral representation of the form proposed in \cite{19}. (In \cite{30} genus-zero invariants of a large class of GLSMs are computed via relation to quasimap invariants of the underlying GIT quotients.) Usually, when doing this type of wall crossing, people work with cohomology-valued Givental $I$-functions or associated D-modules. Instead, we work with central charges because they are very convenient in the setting of wall crossing and mirror symmetry.

In this paper we study genus-zero $K$-theoretic quasimap GLSM invariants. GIT stability wall crossing for $K$-theoretic invariants is much less studied and to our knowledge remains a mystery beyond some particular cases. Notably, there are results in holomorphic symplectic smooth case \cite{1,24,33} and in the smooth balanced case \cite{16}. In particular, the classical Landau-Ginzburg/Calabi-Yau correspondence has not been worked out. In fact, computations in \cite{31} showed that it should break even for the quintic threefold. Here we consider the general GLSM with one-dimensional gauge group $\mathcal{X} = [V/\mathbb{C}^*]$. This includes the classical Landau-Ginzburg/Calabi-Yau correspondence and many features of the general case, but is technically simpler. We note that our result also implies the wall crossing statement for the underlying GIT if we put the superpotential and $R$-charges to zero.

We take the approach of \cite{2}. First, we define elliptic central charges (Definition \ref{def:central_charge}) that are generalizations of the D-brane central charges in quantum cohomology, see \cite{20}. A central charge can be thought of as a component of the $I$-function associated to a B-type brane, that is a coherent sheaf for a GIT or a matrix factorization in the case of GLSMs. In both cases the B-branes factor through the corresponding $K$-theory. Our elliptic central charges depend on an elliptic cohomology class which provide a natural generalization of $K$-theory in the triality of generalized cohomology theories: cohomology, $K$-theory and elliptic cohomology associated to the formal group laws on $\mathbb{C}, \mathbb{C}^*$ and elliptic curve $E = \mathbb{C}^*/q^2$. In holomorphic symplectic setting the importance of elliptic cohomology was understood in \cite{1} in constructions of elliptic stable envelopes. We also discuss appropriate modifications of the standard elliptic cohomology constructions in the orbifold case and construct the orbifold elliptic Chern character, see Definition \ref{def:orbifold_elliptic_chern_character}. The orbifold elliptic Chern character takes twisted sectors into account similarly to how the usual orbifold Chern character \cite{22} does.

One important advantage of central charges is that they have natural integral representations of Mellin-Barnes and Euler type. We show that our elliptic central charges also have integral representation called solid torus partition functions, $K$-theoretic $I$-functions and elliptic central charges have additional parameter called the level structure \cite{25,29}. It turns out that appropriate choice of the level structure is crucial for the existence of integral representations.

The solid torus should be thought of as a product of a disk and a circle. The circle appears due to the relation of $K$-theory of $\mathcal{X}$ to cohomology of the loop space of $\mathcal{X}$. Solid torus partition functions have very simple representation and are morally elliptic central charges for counts of formal disks into the critical locus of $W$ in the Artin stack $[V/G]$ (see also the discussion of vertex functions in \cite{24}, especially Section 3.2). This relation is much more transparent in $K$ theory than in cohomology.

Representation of central charges in terms of solid torus partition functions is closely related to the interpolation problem in elliptic cohomology (see Theorem \ref{thm:interpolation}). We follow the approach to elliptic cohomology \cite{15} and conventions of \cite{1,23,24}. Let $T$ be a torus acting on $V$, which induces a $T$-action on the GIT quotient $\mathcal{X}$. Then $T$-equivariant elliptic cohomology of a point is a product of elliptic curves $Ell_T(pt) = T/q^{\text{cochar}(T)}$. $T$-equivariant elliptic cohomology of $\mathcal{X}$ and $[V/G]$ are projective schemes over $Ell_T(pt)$. Stable locus inclusion $\mathcal{X} \subset [V/G]$ induces an inclusion map on the level of elliptic cohomology

$$Ell_T(\mathcal{X}) \subset Ell_T([V/G]) = Ell_{T \times G}(pt).$$

An elliptic cohomology class on $\mathcal{X}$ is a section of a line bundle over $Ell_T(\mathcal{X})$. Such a section can be written in terms of theta functions. Solid torus partition function construction involves interpolation of this section to a section of a line bundle $\mathcal{L}$ over $Ell_{T \times G}(pt)$. If we enlarge the base $Ell_T(pt)$ by another elliptic curve $E_0^T$, this interpolation exists and is unique under a degree restriction on $\mathcal{L}$ due to a basic fact about line bundles over abelian varieties. This additional degree of freedom corresponds to the Kähler variable - variable that counts the degree in the $K$-theoretic $I$-function. The degree condition on the line bundle $\mathcal{L}$ is called an elliptic grade restriction rule since it is a natural generalization of the classical grade restriction rule. See, for example \cite{2,3,12,18,19} for more information of the classical grade restriction rule.
In Theorem 4.3 we show that elliptic central charge of an elliptic cohomology class has a solid torus partition function representation that uses interpolation of that elliptic cohomology class. As a simple consequence we obtain the wall crossing statement (Proposition 4.6). Wall crossing accounts to interpolating (elliptic cohomology class or corresponding central charge) from one chamber and restricting it to another. Thus, analytic continuation of elliptic central charges is compatible with the elliptic version of the grade restriction rule. We also show that elliptic central charges satisfy a certain q-difference equation which we call quantum q-difference equation which is a q-analog of the Picard-Fuchs differential equation for the usual central charges.

We also discuss generalization of Landau-Ginzburg/Calabi-Yau correspondence. We study elliptic wall crossing between degree $r$ hypersurface in an $n - 1$-dimensional projective space (geometric phase) and a pair $([\mathbb{C}^n/\mu_r], W)$ for a superpotential $W$ with a unique fixed point at the origin (Landau-Ginzburg phase). When $\mathcal{O}_{\mathbb{P}^{n-1}}(-r)$ and $[\mathbb{C}^n/\mu_r]$ are related by a crepant transformation when $n = r$. This implies crepant wall crossing for the usual central charges. In particular, any central charge on the one side can be represented as analytic continuation of a central charge on the other and vice versa.

For the elliptic central charges the situation changes: analogous property holds if $n = r^2$ for an appropriate choice of the level structure whereas for $n = r$ the Landau-Ginzburg phase contains more elliptic central charges than the geometric phase. This has the following geometric explanation. Crepant transformation guarantees that K-theories of the respective phases have the same dimension and one can construct an isomorphism using Fourier-Mukai transforms that reduce to the usual grade restriction rule. K-theoretic I-functions and elliptic central charges are instead related to K-theory of inertia stack or elliptic cohomology correspondingly.

1.1. Notations.

1. $\mu_r \simeq \mathbb{Z}/r\mathbb{Z}$ is a group of $r$-th roots of unity.
2. $E[r] \simeq (\mathbb{Z}/r\mathbb{Z})^2$ is a group of $r$-torsion points on the elliptic curve $E = \mathbb{C}^*/q^2$.
3. Let $x \in X$. Then $[x] \in X/\sim$ denotes an equivalence class of $x$.
4. For a given real number $x$ we denote its floor $[x]$ which is the largest integer not greater than $x$, and fractional part $\{x\} = x - [x]$.
5. $pt \simeq \text{Spec}(\mathbb{C})$ denotes a point.
6. Let $T \simeq (\mathbb{C}^*)^n$ be an algebraic torus with coordinates $t_1, \ldots, t_n$. Its character lattice is isomorphic to $\mathbb{Z}^n$. Let $\alpha \in \mathbb{Z}^n$. Then we use the notation $t^\alpha$ in the corresponding character, one-dimensional representation of $T$ defined by this character and corresponding element of $K_T(pt)$. In particular, $\bigoplus c_i t^{\alpha_i}$, $c_i \in \mathbb{Z}$ denotes a $T$-representation and $\sum c_i t^{\alpha_i}$ - the corresponding K-theory class. Analytic functions in $t$ can be thought of as elements of completed K-theory by power series expansion.
7. Let $G$ act on $X$ and $g \in G$. Then $X^g$ is a subset (scheme/stack) of $X$ fixed by $g$.
8. If $H$ is a $\mathbb{Z}$-module and $k$ is a field, then $H_k := H \otimes_{\mathbb{Z}} k$.

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2. Geometry of the model.

In this paper we consider Gauged Linear Sigma Models (GLSM) with the 1-dimensional gauge group $G$. Recall that the GLSM data is given by the following 5-tuple: $(V, G, \mathbb{C}^*_R, W, \omega)$. Let $G \simeq \mathbb{C}^*$. Then we can write $V = \bigoplus_{i=1}^N s^{D_i}$, where $s^{D_i}$ denotes a character of $G$ and $D_i \in \mathbb{Z}$. Without loss of generality we assume that

1. All $D_i \neq 0$.
2. $D_1, \ldots, D_{N_+} > 0$, and $D_{N_+ + 1}, \ldots, D_N < 0$. We also denote $N_- = N - N_+$.

In order to make the discussion cleaner we also assume that $0 < N_+ < N$, so that the model has two nontrivial phases $\pm \omega > 0$. We also introduce the notation $V = V_+ \oplus V_-$, where the summands consist of positive (respectively negative) characters: $V_+ = \bigoplus_{i=1}^{N_+} s^{D_i}$, $V_- = \bigoplus_{i=N_++1}^N s^{D_i}$.

The ambient spaces of both phases are total spaces of bundles over weighted projective spaces.

1. $X_+ := [V /_1 G] = \bigoplus_{i > N_+} \mathcal{O}(D_i) \to \mathbb{P}[D_1 : \ldots : D_{N_+}]$,

where $\mathbb{P}[D_1 : \ldots : D_{N_+}]$ is a weighted projective space.

2. $X_- := [V /_{-1} G] = \bigoplus_{i \leq N_+} \mathcal{O}(-D_i) \to \mathbb{P}[-D_{N_+ + 1}, \ldots, -D_N]$,
The classical example is when $D_1 = D_2 = \cdots = D_n = 1$ and $D_n = -r$. In this case we have $X_+ = \mathcal{O}_{\mathbb{P}^{n-1}}(-r)$ and $X_- = [\mathbb{C}^n/\mu_r]$. In the plus phase the GLSM describes quasimaps to a degree $r$ hypersurface in $\mathbb{P}^{n-1}$ and in the minus phase FJRW theory of $([\mathbb{C}^n/\mu_r], W_-)$ where $W_-$ is the restriction of superpotential. In particular, if $n = r$, then the resulting GLSMs are Calabi-Yau. Below we consider the case $\omega > 0$ and denote $X = X_+$ for concreteness since the other case $\omega < 0$ is exactly the same.

Let also $T$ be a maximal torus in $GL(V)$ with coordinates $a_1, \ldots, a_N$ such that $V = \bigoplus a_i s^{D_i}$. Below we consider a $\prod_i D_i$-cover $T'$ of $T$ so that $a_i^{1/D_i}$ are coordinates on $T'$. Roots of equivariant variables $a_i$ should be understood in this sense.

2.2. Inertia stack. $X_+$ has $N_+$ torus fixed points $pt_i$, $i \leq N_+$. Each of the points is a stacky point if $D_i > 1$:

\[ pt_i = \left[ (0 \times \cdots \times 0 \times \mathbb{C}^* \times 0 \times \cdots 0) / \mathbb{C}^* \right] \simeq [pt/\mu_{D_i}]. \]

Let $\Sigma$ be the toric fan of $X_+$. Set of cones is in one-to-one correspondence with the set of anticones $A_+$. The correspondence is $\sigma = \sum_{i \in I} R_{\geq 0} u_i \to I = [1..N] \setminus I'$. Each cone corresponds to a closed substack $X_+(\sigma) \subset X_+$ defined by

\[ X_+(\sigma) = \left[ \{0\}^{I'} \times \mathbb{C}^I \right]^{ss}/G. \]

Generic stabilizer $G_\sigma$ of $X_+(\sigma)$ is

\[ G_\sigma = \{ g \in G \mid \{0\}^{I'} \times \mathbb{C}^I \text{ is fixed under } g \}. \]

If $\tau \subset \sigma$, then $X_+(\sigma) \subset X_+(\tau)$ and therefore $G_\sigma \subset G_\tau$. The set $G_\sigma$ is naturally bijective to the set $Box_\sigma$ [13] Section 4], so we will use $Box_\sigma$ and $G_\sigma$ interchangeably. Define also Box $= \bigcup_{\sigma \in \Sigma} Box_\sigma$. This is a set of group elements in $G$ that have nontrivial stabilizers on $V$ and thus produce stacky subvarieties in $X_+$. For a $v \in Box$ denote by $g(v) \in G$ the corresponding group element. In our case $Box = \bigcup_{i \leq L_+} \mu_{D_i+1} \subset \mathbb{C}^* \simeq G$.

The inertia stack of a toric orbifold can be described using the box set:

\[ \mathcal{I}X = \bigcup_{v \in Box} X_{+,v}, \quad X_{+,v} \simeq [(V^{ss})_v^{g(v)}/G]. \]

We denote the natural embedding $X_v \to X$ by $\iota_v$.

Given a line bundle $L \to X_+$ and $v \in Box$ we have $g(v) \cdot L = e^{2\pi \sqrt{-1}c} L$ for a unique $0 < c < 1$. We define $age_v(L) := c$ and $age_v(L_1 \oplus L_2) = age_v(L_1) + age_v(L_2)$.

2.2. Cohomology, K-theory and Elliptic cohomology. All 3 cohomology theories satisfy Kirwan surjectivity. Let $u_i := c_1(a_i) \in H^2_F(pt)$ and $p = c_1(s) \in H^2_G(pt)$. Then the cohomology and K-theory rings of $X_+$ with complex coefficients are given by

\[ H^*_T(X_+)_\mathbb{C} = \frac{\mathbb{C}[u,p]}{\left( \prod_{i \leq N_+} (D_i p - u_i) \right)}, \]

\[ K_T(X_+)_\mathbb{C} = \frac{\mathbb{C}[s^\pm]^{a_i^{1/s^{-D_i}}}}{\left( \prod_{i \leq N_+} (1 - a_i^{1/s^{-D_i}}) \right)}. \]

We look at both cohomology and K-theory as structure sheaves over their (affine) spectra. Corresponding spectra are cut out by equations $\prod_{i \leq N_+} (D_i p - u_i) = 0$ and $\prod_{i \leq N_+} (1 - a_i^{1/s^{-D_i}})$ in $\text{Spec}(H^*_T \times G(pt)_\mathbb{C}) \simeq \text{Lie}(T) \oplus \text{Lie}(G) \simeq \mathbb{C}^{N+1}$ and $\text{Spec}K_T(X_+)(pt)_\mathbb{C} \simeq T \times G \simeq (\mathbb{C}^*)^{N+1}$ respectively. All these schemes are defined over the spectrum of $T$-equivariant cohomology/K-theory of a point.

Elliptic cohomology $\text{Ell}_{T \times G}(pt)$ is a projective scheme, and elliptic cohomology classes are sections of sheaves over this scheme. In particular, we define $E \simeq \mathbb{C}^*/q^2$ and for a torus $T$ we define $E_T \simeq T/q^{\text{cochar}(T)} \simeq E_{\text{rank}(T)}$. Then

\[ \text{Ell}_{T \times G}(pt) \simeq E_T \times E_G. \]

Equivariant elliptic cohomology of $X_+$ is a scheme that is cut out in $\text{Ell}_{T \times G}(pt)$ by the equation

\[ \prod_{i \leq N_+} \theta(a_i^{-1}, s^{-D_i}) = 0, \]
where the theta function is defined as
\[
\theta(x) = \phi(x)\phi(q/x) = \prod_{i \geq 0} (1 - q^i x)(1 - q^{i+1}/x).
\]

The equation (10) geometrically is a zero locus of the section \( \prod_{i \leq N_+} \theta(a_i^{-1}s^{-D_i}) \) of the line bundle \( \Theta(V_+) \simeq \Theta(V_-) \). This section is an (equivariant) elliptic Euler class of \( V_+ \).

Geometrically all 3 theories are transverse intersections of irreducible hypersurfaces, e.g. \( 1 - a_i^{-1}s^{-D_i}, i \leq N_+ \) in K-theory. This corresponds to the fixed point decomposition. Taking a finite cover of \( T \) to allow roots of equivariant variables we get:
\[
(1 - a_i^{-1}s^{D_i}) = \prod_{\zeta \in \mu_{D_i}} (1 - \zeta a_i^{-1/D_i}s),
\]
where \( K([pt/\mu_{D_i}]) \simeq \mu_{D_i} \).

Explicitly, we have
\[
\chi(\mathcal{F}) = \sum_{k \leq N_+} \sum_{\zeta \in \mu_{D_k}} \text{Res}_{s^{-\zeta a_k^{-1/D_k}}} \frac{\mathcal{F}}{\lambda^*V_+^\vee} = -(\text{Res}_0 + \text{Res}_\infty) \frac{\mathcal{F}}{\lambda^*V_+^\vee},
\]
where we used that sum of residues of a rational differential form on a Riemann sphere is 0.

K-theoretic vertex function (I-function) takes values in the inertial K-theory \( K_T(\mathcal{X}_+) \) which is the correct object to consider once we have elliptic cohomology.

Let \( \mathcal{F} \in K_T(\mathcal{X}_+) \) such that \( \mathcal{F}|_{\mathcal{X}_0} = \mathcal{F}_0 \). Then the Euler characteristic of \( \mathcal{F} \) is the sum of Euler characteristics over all the components of inertia stack:
\[
\chi(\mathcal{F}) := \sum_v \chi(\mathcal{X}_v, \mathcal{F}_v).
\]

Explicitly, we have
\[
\chi(\mathcal{F}) = \sum_{v \in \text{Box}} \sum_{i \leq N_+} \sum_{\zeta \in \mu_{D_i}} \text{Res}_{s^{-\zeta a_i^{-1/D_i}}} \frac{\mathcal{F}_v}{(1 - a_j^{-1}s^{-D_j})} |_{s = \zeta a_j^{-1/D_j}}.
\]

3. Higgs branch: quasimap counting

3.1. LG quasimaps. GLSM quasimaps are called Landau-Ginzburg quasimaps. Let \( \Gamma \subset \text{GL}(V) \) be a subgroup generated by \( G \) and \( \mathbb{C}_R^* \). We view projection \( \Gamma \rightarrow \mathbb{C}_w^* \) as a character \( \chi \) of \( \Gamma \). Such a map is captured by the commutative diagram:
\[
\begin{array}{c}
\mathbb{C} \\
\downarrow f \\
[V/\Gamma] \\
\downarrow \\
B\Gamma \\
\downarrow \\
B\mathbb{C}_w^* \\
\end{array}
\]
\[
\begin{array}{c}
P \\
\downarrow \omega_{\log} \\
B\mathbb{C}_w^*
\end{array}
\]
In the diagram above \( \mathbb{C}_w^* = \Gamma/G \). Let us unwrap the data for \( \mathcal{C} \simeq \mathbb{P}(a : 1) \). The orbifold domain is required because the target space can have orbifold singularities, so we need at least one marked (possibly stacky) point.

### 3.2. Geometry of the teardrop.

We represent
\[
\mathbb{P}(a : 1) = \{(x, y) \in \mathbb{C}^2 \setminus \{0\} \mid (x, y) \sim (\lambda^a x, \lambda y)\}.
\]
The orbifold point is \( [\infty] := [1 : 0] \) and the other pole \( [0] := [0 : 1] \) is a usual point. This geometry is sometimes called “teardrop” geometry.

We also consider a \( \mathbb{C}_q^* \) action on the teardrop:
\[
(x, y) \rightarrow (qx, y) \sim (x, yq^{-1/a}).
\]
Then \([0]\) and \([\infty]\) are the fixed points of this action. In what follows we will study representation structure of bundles over the teardrop. We assign the weights 0 and \(-1/m\) to variables \(x, y\) in \( \mathbb{C}[x, y] \) (which is equivalent to a choice of linearization with respect to \( \mathbb{C}_q^* \) action). The variable \( q \) is called the loop variable.

Let \( \mathcal{O}_{\mathbb{P}(a, 1)}(1) \) denote the orbibundle with the total space
\[
(x, y, z) \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C} \mid (x, y, z) \sim (\lambda^a x, \lambda y, \lambda z)\}.
\]
We upgrade it to a \( \mathbb{C}_q^* \) equivariant line bundle as
\[
\mathcal{O}_{\mathbb{P}(a, 1)}(n) \to \mathcal{O}_{\mathbb{P}(a, 1)}((n \text{ mod } a)[\infty] + [n/a][0]).
\]
We also use the following notations for the floor function and fractional part: \( \lfloor x \rfloor \) is the smallest integer smaller than \( x \) and \( \{x\} = x - [x] \).

We will be interested in cohomology and Euler characteristic of these sheaves. Consider the maps
\[
\mathbb{P}[a : 1] \xrightarrow{\pi} \mathbb{P}^1 \to pt,
\]
where \( \pi \) is the projection to the coarse moduli space. We introduce the notation:
\[
\mathcal{O}(n/a) = \mathcal{O}_{\mathbb{P}^1(n/a)} := \mathcal{O}_{\mathbb{P}[a, 1]}((n \text{ mod } a)[\infty] + [n/a][0])
\]
Sections of \( \mathcal{O}(n/a) \) are weighted polynomials of degree \( n \) of the form
\[
\sum_{i=0}^{\lfloor n/a \rfloor} c_i x^i y^{n-a_i},
\]
so
\[
H^0(\mathbb{P}^1, \mathcal{O}(n/a)) = \sum_{i=0}^{\lfloor n/a \rfloor} q^{-(n/a)-i},
\]
Using the Cech complex we can compute the \( \mathbb{C}_q^* \)-equivariant Euler characteristic:
\[
\chi_{\mathbb{C}_q^*}(\mathcal{O}(n/a)) = \frac{q^{\lfloor -n/a \rfloor}}{1 - q^{-1}} + \frac{q^{-n/a}}{1 - q} = \sum_{k \geq 0} q^{\lfloor -n/a \rfloor} - k - \sum_{k \geq 0} q^{-n/a - 1 - k} = \begin{cases} \sum_{k=0}^{\lfloor n/a \rfloor} q^{-\lfloor n/a \rfloor - k}, & n \geq 0, \\ 0, & n \in [-a, 0) \\ \sum_{k=0}^{-\lfloor n/a \rfloor - 2} q^{k+1-\{n/a\}}, & n < -a. \end{cases}
\]
Note that if \( n \in [-a, -1] \), then \( H^0 = H^1 = 0 \).

Another way to compute the Euler characteristic is by the equivariant localization on the coarse moduli \( \mathbb{P}^1 \):
\[
\chi_{\mathbb{C}_q^*}(\mathbb{P}^1, \mathcal{O}(n/a)) = \frac{\mathcal{O}(n/a)|_{[0]}}{\Lambda^* N_{[0]}} + \frac{\mathcal{O}(n/a)|_{[\infty]}}{\Lambda^* N_{[\infty]}} = \frac{q^{-\{n/a\}}}{1 - q^{-1}} + \frac{q^{n/a}}{1 - q}.
\]
3.3. Maps of formal disks. We can formally consider domain as a $D := \text{Spf} \mathbb{C}[y]$ with $\mathbb{C}^*_q$ action $y \to q^{-1/a} y$. This can be thought of as a (formal) open chart around $[1 : 0] = [\infty] \in \mathbb{P}^1[a : 1]$. We have
\begin{equation}
\Gamma(O_D(n/a)) = \sum_{k \geq 0} c_k y^{n+ka},
\end{equation}
so
\begin{equation}
\chi_{\mathbb{C}^*_q}(D, O(n/a)) = \frac{q^{-n/a}}{1 - q}.
\end{equation}

3.4. Stacky loop space. Part of the data of a LG quasimap is a $G$-bundle $P$. Its degree is $P_i[C] \in H_2(B\Gamma^i)$, $H_2(B\Gamma^i)_Q \simeq H_2(BG)_Q \oplus H_2(B\mathbb{C}P^1)_Q \simeq \text{cochar}_Q(G) \oplus \mathbb{Q}$. The quasimap condition fixes the second component to be just one, so that we have $\beta_\Gamma = (\beta, 1)$. Let $P_\beta \to \mathbb{P}[a : 1]$ be the corresponding principal $\Gamma$-bundle.

Define quantum version of the space $V$:
\begin{equation}
V_\beta := \mathbb{R}\Gamma(P_\beta \times_\Gamma V), \quad V_\beta := H^0(V_\beta), \quad W_\beta := H^1(V_\beta).
\end{equation}
Morally, $V_\beta, W_\beta$ are deformation and obstruction spaces for LG quasimaps. In particular, the quantum loop space with obstruction bundle is defined as
\begin{equation}
X_\beta := [V_\beta \sslash_\omega G], \quad \text{Ob}_\beta := [V_\beta^{\omega-ss} \oplus W_\beta/G].
\end{equation}
We also define the subspace of the quasimaps non-singular at infinity $X_\beta^\circ$. Let
\begin{equation}
V_\beta^\circ := \{s \in V_\beta \mid s(1 : 0) \in V^{\omega-ss}\}.
\end{equation}
We note that the last condition is independent of the trivialization of $P_\beta \times_\Gamma V$ because the unstable locus is a cone in $V$. Then $X_\beta^\circ := [V_\beta^\circ \sslash_\omega G]$. Advantage of using this space is the fact that there is an evaluation map at infinity:
\begin{equation}
ev_\infty : X_\beta^\circ \to X_{\nu(\beta)}.
\end{equation}
We note that due to the fact that we consider $C$ of genus 0 with one marked point and $\epsilon = 0^+$ condition our quasimap moduli space is very simple, in particular, it is an explicit smooth DM-stack and the obstruction bundle is a vector (orbi-)bundle.

Let us compute $V_\beta, W_\beta$ as $\mathbb{C}^*_q$-representations. As a $T_{(a)} \times G_s \times \mathbb{C}^*_q$-representation we have:
\begin{equation}
V = \sum_{i=1}^N a_i s^{D_i} u_i q_i/2.
\end{equation}
For the quantum version we have compute
\begin{equation}
V_\beta = \sum_{i=1}^N H^0\left(\mathbb{P}^1, \mathcal{O}(D_i \beta - q_i/2)\right),
\end{equation}
\begin{equation}
W_\beta = \sum_{i=1}^N H^1\left(\mathbb{P}^1, \mathcal{O}(D_i \beta - q_i/2)\right).
\end{equation}
Let
\begin{equation}
d_i(\beta) := D_i \beta - q_i/2.
\end{equation}
Using the formulas for the Euler characteristic from above from we compute:
\begin{equation}
V_\beta = \sum_{i=1}^N \left(a_i s^{D_i} \sum_{k=0}^{\lfloor d_i(\beta) \rfloor} q^{-i-\{d_i(\beta)\}}\right),
\end{equation}
\begin{equation}
W_\beta = \sum_{i=1}^N \left(a_i s^{D_i} \sum_{k=0}^{-\lfloor d_i(\beta) \rfloor - 2} q^{i+1-\{d_i(\beta)\}}\right).
\end{equation}
Note, that $\{d_i(\beta)\} = \text{age}_{\nu(\beta)}(U_i^{-1})$.

Define a vector subspace $Z_\beta \subset V$ by the equation
\begin{equation}
Z_\beta = \bigcap_{i, d_i(\beta) \in \mathbb{Z}_{\geq 0}} \{x_i = 0\}.
\end{equation}
Value of a LG quasimap of degree $\beta$ must be in this subspace. We also define
\begin{equation}
F_\beta := [Z_\beta \parallel G] \subset \mathcal{X}_\beta.
\end{equation}

We use $\mathbb{C}_q^*$-localization to compute the vertex function. $(\mathcal{X}_\beta^q)^{\mathbb{C}_q^*}$ consists of constant maps. Moreover, we have
\begin{equation}
ev_{\infty} : (\mathcal{X}_\beta^q)^{\mathbb{C}_q^*} \simeq F_\beta \subset \mathcal{X}_{v(\beta)},
\end{equation}
where the map $\beta \to v(\beta)$ is a natural map constructed as follows. $\beta \in H^2(BG)_{\mathbb{Q}} \simeq \text{cochar}(G)_{\mathbb{Q}}$ defines a group element $g(\beta) \in G$ by the formula $e^{2\pi \sqrt{-1}\beta}$. $\mathcal{X}_{v(\beta)}$ is the inertia stack component fixed by $g(\beta)$.

### 3.5. Virtual localization and $I$-function
$\mathbb{C}_q^*$ acts fiberwise on bundles over $F_\beta$.
\begin{equation}
\begin{array}{l}
\text{Def}_\beta := T_{\mathcal{X}_\beta^q}|_{F_\beta} = \text{Def}_f + \text{Def}_m,
\end{array}
\end{equation}
where the fixed part $f$ and the moving part $m$ denote the subspaces of weight zero and nonzero under the $\mathbb{C}_q^*$ action.

We define the the vertex function or K-theoretic $I$-function as a pushforward of the localized virtual structure sheaf:
\begin{equation}
I^K := \sum_{\beta \in \text{Eff}} \frac{\theta(z^{-1})}{\theta(q^{-\beta}s^{-1}z^{-1})} z^{\lfloor \beta \rfloor} (\ev_{\infty})^\bullet \left( \frac{O_{F_\beta}}{\mathbb{C}_q^* N_{vfr}} \right) \cdot O_{\mathcal{X}_{v(\beta)}},
\end{equation}
where $s \in G$ is a variable dual to $z$ and
\begin{enumerate}
\item The summation range Eff is a set of effective curve classes. In our case it is
\[\text{Eff} = \bigcup_{i \leq N_+} \frac{1}{D_i} \mathbb{Z}_{\geq 0},\]
where the summation is over the fixed points in $\mathcal{X}_+$.
\item The virtual structure sheaf of $F_\beta$ defined as
\begin{equation}
O_{F_\beta} \otimes \mathbb{C}_q^* \text{Ob}_f.
\end{equation}
\item Formula \[37\] to see that $W_\beta$ does not have a fixed part, so it is equal to the usual structure sheaf $O_{F_\beta}$.
\item $N_{vfr} = \text{Def}_m - \text{Ob}_m$ is a virtual normal bundle.
\item $O_{\mathcal{X}_{v(\beta)}}$ is the structure sheaf of the corresponding component of the inertia stack of $\mathcal{X}$.
\end{enumerate}

### Remark 3.1.
The significance of the prefactor in theta functions and floor function in the power will become clear later. For now we note the following relevant facts:
\begin{enumerate}
\item The function
\begin{equation}
\frac{\theta(z^{-1})}{\theta(q^{-\beta}s^{-1}z^{-1})}
\end{equation}
should be thought as an elliptic version of $z^{\lfloor \beta \rfloor} s^{-\ln(z)/\ln(q)}$ which is the usual prefactor in the $I$-functions.
\item Let $T_z$ be an operator $f(z) \to f(qz)$. We have the following relation:
\begin{equation}
T_z \left( \frac{\theta(z^{-1})}{\theta(q^{-\beta}s^{-1}z^{-1})} z^{\lfloor \beta \rfloor} \right) = (q^\beta s) \cdot \frac{\theta(z^{-1})}{\theta(q^{-\beta}s^{-1}z^{-1})} z^{\lfloor \beta \rfloor}.
\end{equation}
This is the same transformation property as
\begin{equation}
T_z (z^{\beta} s^{-\ln(s)/\ln(q)}) = (q^\beta s) \cdot z^{\beta} s^{-\ln(s)/\ln(q)}.
\end{equation}
\end{enumerate}

### Remark 3.2.
In the definition of the $I$-function the exact form of the prefactor with theta functions is not important. Indeed, every meromorphic section $s$ of the same line bundle over $\mathcal{E}_G \times \mathcal{E}_G$ would be as good as $\theta(z^{-1})/\theta(q^{-\beta}s^{-1}z^{-1})$ for the results below (this section must be compatible with orbifold elliptic Chern character, see definition \[3.3\]). This line bundle is degree zero over each component and is nontrivial. More precisely, it is defined by the transformation properties in the remark above. However, fractional powers and logarithms will ruin the integral representation below. In fact, one can even choose a different nontrivial degree 0 over each component line bundle, but then the $q$-difference equations satisfied by the $I$-function will change as well.
In order to compute the I-function we compute the virtual normal bundle first. Below we will use the following formulas:

\begin{equation}
\Lambda^* \left( \frac{q^x}{1 - q} \right) = \Lambda^* \left( \sum_{k \geq 0} q^{x+k} \right) = \prod_{k \geq 0} (1 - q^{x+k}) = \phi(q^x),
\end{equation}

and

\begin{equation}
\Lambda^* \left( \frac{q^x}{1 - q^{-1}} \right) = \Lambda^* \left( - \sum_{k \geq 0} q^{x+1+k} \right) = \prod_{k \geq 0} \frac{1}{(1 - q^{x+1+k})} = \frac{1}{\phi(q^{x+1})}.
\end{equation}

Moreover,

\begin{equation}
\chi_{\mathbb{C}^*} (\mathbb{P}^1, \mathcal{O}(x)) = \frac{\phi(q^{-x})}{\phi(q^{1-x})},
\end{equation}

\begin{equation}
\det (a \chi_{\mathbb{C}^*} (\mathbb{P}^1, \mathcal{O}(x))) = \begin{cases} 
(q^{-x} - x^{(x+1)}_x q^{-\frac{(x+1)(x+2)}{2}}), & x > 0, \\
(q^{-x} - x^{(x+1)}_x q^{-\frac{(x+1)(x+2)}{2}}), & x < -1,
\end{cases}
\end{equation}

We can rewrite the last formula using theta functions:

\begin{equation}
\det (a \chi_{\mathbb{C}^*} (\mathbb{P}^1, \mathcal{O}(x))) = \frac{\theta(-aq^{-x})}{\theta(-aq^{-1}(x))} = \Theta(1, \frac{\theta(1-q^{-x})}{\theta(aq^{-1})}).
\end{equation}

Recall that

\begin{equation}
\iota_v : X_v \to \mathcal{X}
\end{equation}

is the embedding of the inertia stack component $X_v$ into $\mathcal{X}$.

We have:

\begin{equation}
\text{Def} - \text{Ob} = \sum_i \iota_v^* a_i s_i D_i \left( q^{-d_i(\beta)} - q^{-1} \right) = \sum_i \iota^*_v a_i s_i D_i \left( q^{-d_i(\beta)} - q^{-1} \right) = \sum_{i, d_i(\beta) \in \mathbb{Z}_{\geq 0}} a_i s_i D_i.
\end{equation}

Then using that \{-x\} = 1 - \{x\} if $x \notin \mathbb{Z}$ and \{-x\} = \{x\} if $x \in \mathbb{Z}$ we can write the moving part as:

\begin{equation}
(\text{Def} - \text{Ob})_m = \sum_i \iota_v^* a_i s_i D_i \left( q^{-d_i(\beta)} - q^{-1} \right) = \sum_{i, d_i(\beta) \in \mathbb{Z}_{\geq 0}} a_i s_i D_i.
\end{equation}

The (virtual) bundles $N_{\mathcal{X}/X_v(\beta)}$ and $(\text{Def} - \text{Ob})_m$ are defined on $\mathcal{F}_\beta$, but actually they are pullbacks of bundles from $\mathcal{X}$ via the embedding $\mathcal{F}_\beta \subset \mathcal{X}_v(\beta)$. Thus, we can use the formula $\iota_v^* = \alpha/\Lambda^* N_i$ to compute the pushforwards. Abusing notations by denoting the bundles on $X_v(\beta)$ by the same letter we compute:

\begin{equation}
\left( c_{\mathbb{C}^*} \right)^* \left( \frac{\mathcal{O}_{\mathcal{F}_\beta}}{\lambda^* (\text{Def}_m - \text{Ob}_m)} \right) = \frac{\Lambda^* N_{\mathcal{F}_\beta/X_v(\beta)}}{\Lambda^* (\text{Def}_m - \text{Ob}_m)} \otimes \Lambda^* N_{\mathcal{F}_\beta/X_v(\beta)}^\vee = \prod_i \frac{\phi(t_v^* a_i^{-1} s_i - D_i q_{-d_i(\beta)})}{\phi(t_v^* a_i^{-1} s_i - D_i q_{-d_i(\beta)})}.
\end{equation}

Therefore, the vertex function is

\begin{equation}
I^K = \sum_{\beta \in \text{Eff}} \frac{\theta(z^{-1})}{(q^{-\beta} s_{-1} z^{-1})} z^\beta \prod_i \frac{\phi(t_v^* a_i^{-1} s_i - D_i q_{-d_i(\beta)})}{\phi(t_v^* a_i^{-1} s_i - D_i q_{-d_i(\beta)})} \cdot \mathcal{O}_{X_v(\beta)}.
\end{equation}

It is more convenient to work with the $\mathcal{H}^K$-function:

\begin{equation}
\mathcal{H}^K := I^K \cdot \Gamma_K = \sum_{\beta \in \text{Eff}} \frac{\theta(z^{-1})}{(q^{-\beta} s_{-1} z^{-1})} z^\beta \prod_i \phi(t_v^* a_i^{-1} s_i - D_i q_{-d_i(\beta)}) \cdot \mathcal{O}_{X_v(\beta)},
\end{equation}

where we introduced the K-theoretic Gamma-class

\begin{equation}
\hat{\Gamma}_K := \sum_{i \in \text{box}} \prod_{i \leq N} \phi(t_v^* a_i^{-1} s_i D_i q_{-d_i(\beta)}) \cdot \mathcal{O}_{X_i}.
\end{equation}
It is tempting to use the following interpretation of the \( H^K \)-function. One gets the formula \([56]\) directly from the virtual localization formula \([42]\) if we replace the Euler characteristic of maps of spheres to the one of formal maps of disks:

\[
\chi_{\mathbb{P}^1}(\mathbb{O}_B(x)) = \chi_{\mathbb{P}^1}(\mathbb{D}, \mathcal{O}(x)), \quad \frac{q^{-x}}{1-q} + \frac{q^{-\{x\}}}{1-q} \to \frac{q^{-x}}{1-q}.
\]

### 3.6 Level structure.

In quantum K-theory there is another way to modify the I-function which is called the level structure. Let \( \mathcal{C}_{X_\beta} \) be the universal curve and \( f_\mathcal{C} : \mathcal{C}_{X_\beta} \to [V/\Gamma] \) be the universal section. We can pullback elements of \( K_T([V/\Gamma]) \) to the universal curve and push them to the moduli space via projection. Let \( R = \sum_{l,n,m} R_{l,n,m} a^l s^m w^n \in K_T([V/\Gamma]) \approx K_T \mathcal{X}_\beta \) (pt), where \( a^l, s^m \) and \( w^n \) denote the characters of \( T \times G \times \mathbb{C}_w \) and \( R_{l,n,m} \in \mathbb{Z} \). We define the level \( R \) K-theoretic I-function to be

\[
I^R_R := \sum_{\beta \in \text{Eff}} (-1)^{\sigma_R(\beta)} \begin{pmatrix} \theta(z^{-1}) \\ \theta(q^{-s} z^{-1}) \end{pmatrix}_{\mathcal{C}_{X_\beta}} \left( \frac{\mathcal{O}_{F_{\beta}} \otimes \mathcal{O}_{\mathbb{P}^1}(\mathcal{O}(m\beta-n))}{\Lambda^* N^*_\text{vir}(\mathcal{C})} \right) \cdot \mathcal{O}_{\mathcal{C}_{X_\beta}},
\]

where \( \sigma_R(\beta) := \sum_{l,n,m} R_{l,n,m} |m\beta-n| \). In order to compute the I-function we first compute \( \text{det} f_\mathcal{C}^* R \). We compute

\[
f_\mathcal{C}^* R = \sum_{l,n,m} R_{l,n,m} a^l \chi_{\mathbb{P}^1}(\mathbb{O}(m\beta-n)),
\]

and

\[
\text{det} f_\mathcal{C}^* R = \prod_{l,m,n} \frac{\theta(-a^l s^m q^{-m\beta+n})}{\theta(-a^l s^m q^{1-(m\beta-n)})} = (-1)^{\sigma_R(\beta)} \prod_{l,m,n} \frac{\theta(a^l s^m q^{-m\beta+n})}{\theta(a^l s^m q^{1-(m\beta-n)})} \prod_{l,m,n} (-a^l s^m)_{R_{l,m,n}} \times \frac{\theta(a^l s^m q^{-m\beta+n})}{\theta(a^l s^m q^{1-(m\beta-n)})},
\]

where we used the relation \( 1 - \{x\} = (-x) \) if \( x \notin \mathbb{Z} \). Notice that in the cohomological limit \( q \to 1 \), \( a \to 1 \) this becomes trivial. In K-theory this expression is elliptic in nature.

We also define the level \( R \) K-theoretic Gamma class:

\[
\hat{\Gamma}_K.R := \sum_{v \in \text{Box}} \prod_{g(v) = e^{2\pi i/m_{v}/2}} (-t_v a^l s^m)_{-R_{l,m,n}} \prod_{i \leq N} \phi(t_v a_i^{-1} s^{-D_i} q^{-\text{age}(U_i)}) \cdot \mathcal{O}_v.
\]

In the most important case \( R = V^+ = \sum_{i \leq N} a_i^{-1} s^{-D_i} \), the product over \( l, n, m \) reduces to \( \prod_{i \leq N} \phi(t_v a_i^{-1} s^{-D_i} q^{-\text{age}(U_i)}) \) and

\[
\hat{\Gamma}_{K,V^+} = \sum_{v \in \text{Box}} \prod_{g(v) = e^{2\pi i/m_{v}/2}} (-t_v a_i s^{D_i})_{-R_{l,m,n}} \prod_{i \leq N} \phi(t_v a_i^{-1} s^{-D_i} q^{-\text{age}(U_i)}) \cdot \mathcal{O}_v.
\]

Correspondingly, the level \( R \) H-function is

\[
H^K_R := I^K_R \cdot \hat{\Gamma}_{K,R}.
\]

### 3.7 Central charges.

Similarly to the cohomological case, the natural object from the mirror symmetry point of view is not the I-function itself, but rather its pairing with some B-model branes. In the cohomological case such an object was the central charge. Recall, that if \( \mathcal{H} \) is a cohomological \( H \)-function \([21]\), then one can construct a central charge of \([B] \in K(\mathcal{X})\) or more generally of \( B \in D^K(\mathcal{X})\):

\[
Z(B, R) := \langle \mathcal{H}, \text{ch}([B]^\vee) \rangle,
\]

where in the orbifold case we use orbifold intertia stack valued Chern character and orbifold cohomology pairing \([5]\). In the K-theory we need to upgrade the space of branes \( K(\mathcal{X}) \) and the pairing. It turns out, that it is most natural to upgrade the K-theory \( K(\mathcal{X}) \) to some version of (equivariant) elliptic cohomology \([17]\).
Elliptic branes. Let $\mathcal{L}$ be a line bundle over $E_T \times E_G$. Then its total space can be represented as

$$\mathbb{C}_\chi(\mathcal{L}) \times T \times G/q^{\text{cochar}(T \times G)},$$

where $\chi(\mathcal{L})$ is a character of $G \times T$ that specifies the action on $\mathbb{C}$. (Meromorphic) sections of $\mathcal{L}$ can be represented by quasiperiodic functions on $T \times G$ such that for a given cocharacter $\sigma \in \text{cochar}(T \times G)$:

$$f(q^nt) = \sigma(t)^{-1}q^{-\frac{|\alpha|^2}{2}}(-q^{-\frac{1}{2}})^{\langle \sigma, \chi \rangle}f(t).$$

Note that $\pi_1(E_T \times E_G)$ acts on the sections by rescaling according to the formula above. We want to fix an identification of sections of $\mathcal{L}$ with quasiperiodic functions. We call an elliptic brane $\mathcal{E}$ a section of a line bundle $\mathcal{L} \to \text{Ell}_T(\mathcal{X})$ with such an identification. Abusing notations below we assume that every elliptic cohomology class is promoted to an elliptic brane, i.e., it corresponds to a fixed quasiperiodic functions.

As an example consider an elliptic curve $E$ with coordinate $x$ and the line bundle $\mathcal{L} = O([1]) \to E_x$. Then theta functions $\theta(q^n x)$ for different $n$ define the same section of this bundle. The transformation property of the theta function:

$$\theta(q^n x) = (-x)^{-n}q^{-\frac{n(n-1)}{2}}\theta(x) = x^{-n}q^{-\frac{n^2}{2}}(-q^{-\frac{1}{2}})^n\theta(x)$$

is a particular case of the general transformation $\theta(q^n)$. A choice of an elliptic brane structure is a choice of $n \in \mathbb{Z}$ in this case.

Elliptic Chern character. Recall, that $\text{Ell}_T(\mathcal{X})$ is a scheme over $\text{Ell}_T(pt) \simeq E_T \simeq T/q^{\text{cochar}(T)}$. Kirwan surjectivity implies

$$\text{Ell}_T(\mathcal{X}) = \{ \prod_{i \leq N^+} \theta(a_i^{-1}s^{-D_i}) = 0 \} \subset \text{Ell}_T \times G(pt).$$

Elliptic cohomology elements on $\mathcal{X}$ are sections of line bundles $\mathcal{L} \to \text{Ell}_T(\mathcal{X})$. Sections of such a bundle can be represented by theta functions.

Consider the following diagram:

$$\begin{array}{ccc}
\text{Spec}(K_T(\mathcal{X})) & \xrightarrow{\text{ch}_{K \rightarrow E}} & \text{Ell}_T(\mathcal{X}) \\
\downarrow & & \downarrow \\
T \mod q^{\text{cochar}(T)} & \rightarrow & E_T
\end{array}$$

where

$$\text{ch}_{K \rightarrow E} : \text{Spec}(K_T(pt)) \rightarrow \text{Ell}_T(pt)$$

is the natural map making the diagram commute. This map is induced by the map

$$\text{Spec}(K_T([V/G])) \simeq T \times G \mod q \rightarrow E_T \times E_G \simeq \text{Ell}_T([V/G]).$$

Let $\mathcal{L} \to \text{Ell}_T(\mathcal{X})$ be a line bundle. Sections of such line bundles are elliptic cohomology elements. Then we can use pull-back via $\text{ch}_{K \rightarrow E}$ to construct elliptic version of Chern character map:

$$(\text{ch}_{K \rightarrow E})^* : \mathcal{L} \rightarrow (\text{ch}_{K \rightarrow E})^* \mathcal{L}.$$ 

Moreover, $\Gamma(\mathcal{O}_{\text{Spec}(K_T(\mathcal{X}))})$ is isomorphic to $\hat{K}_T(\mathcal{X})$, but the isomorphism is not canonical. The notation $\hat{K}_T(\mathcal{X})$ denotes completion in the variable $q$. It might be necessary because the map $\text{ch}_{K \rightarrow E}$ is of infinite index. In other words, theta functions are transcendental functions.

A choice of elliptic brane structure on sections of $\mathcal{L}$ provides an embedding $\Gamma(\text{ch}_{K \rightarrow E})^* \mathcal{L} \subset \hat{K}_T(\mathcal{X})$ by sending a section of $\mathcal{L}$ to the quasiperiodic function on $\text{Spec}(K_T(\mathcal{X})) \subset T \times G$ representing it. Thus, we get

$$(\text{ch}_{K \rightarrow E})^* : \Gamma(\mathcal{L}) \rightarrow \Gamma((\text{ch}_{K \rightarrow E})^* \mathcal{L}) \subset \hat{K}_T(\mathcal{X}).$$

Orbifold elliptic Chern character. In the case of DM stacks it is more natural to define the image of the Chern character map to be the inertial K-theory $\hat{K}(\mathcal{I}\mathcal{X})$ instead of the usual one. This is parallel to how the usual Chern character for DM stacks maps to the orbifold cohomology instead of the usual cohomology. This is consistent with the fact that

$$\text{Spec}(H^*(\mathcal{I}pt/\mu_r)) = pt, \quad \text{Spec}(K(\mathcal{I}pt/\mu_r)) \simeq \mu_r, \quad \text{Ell}(\mathcal{I}pt/\mu_r) \simeq E[r].$$

So if we want Chern character maps to be isomorphisms, we need to add twisted sectors to their image.
Recall, that we have $\mathcal{I}\mathcal{X} \simeq \bigsqcup_{v \in \Box} \mathcal{X}_v$, and each $v$ corresponds to $g = g(v) \in G$ such that $\mathcal{X}_v = [V^g / \omega G]$. Let $\langle g \rangle \simeq \mu_r$ be a subgroup in $G$ generated by $g$, where $r = \text{ord}(g)$. Let us choose a logarithm of $g$ such that $\log(g)/2\pi \sqrt{-1} \in \frac{1}{r} \mathbb{Z} \cap [0, 1)$. We define the action of $\tilde{\mu}_r \simeq \frac{1}{r} \mathbb{Z}$ on $G$ by $k \to q^{k \log(g)/2\pi \sqrt{-1}}$. This reduces to the action of $\mu_r$ on $E_G$.

More abstractly we have

$$\text{Ell}_{\langle g \rangle}(pt) \subset \text{Ell}_{T \times G}(pt).$$

In particular, since $\text{Ell}_{T \times G}(pt)$ is an abelian variety and $\text{Ell}_{\langle g \rangle}(pt)$ is its abelian subgroup, so $\text{Ell}_{\langle g \rangle}(pt)$ acts on $\text{Ell}_{T \times G}(pt)$. This action preserves $\text{Ell}_T(\mathcal{X}_v)$, where the latter is cut out in $\text{Ell}_{T \times G}(pt)$ by the equation:

$$\prod_{i \leq \mathcal{N}_+ \text{ age}_c(U_i) = 0} \theta(U_i) = 0.$$ 

The equation is preserved by the action due to the condition $\text{age}_c(U_i) = 0$ and quasiperiodicity of theta functions. This provides the action of $\text{Ell}_{\langle g \rangle}(pt) \simeq E[\text{ord}(g)]$ on $\text{Ell}_T(\mathcal{X}_v)$.

We have $E[\text{ord}(g)] \simeq \mu_{\text{ord}(g)}^2$ with generators $[g]$ and $[q^{h(g)/2\pi \sqrt{-1}}]$ where the brackets denote an equivalence class modulo $q^\mathbb{Z}$. As discussed above, we can upgrade it to the action of $E[\text{ord}(g)] = \mu_r \times \frac{1}{r} \mathbb{Z}$ on quasiperiodic functions on $\text{Spec}(K_T(\mathcal{X})) \subset T \times G$. We denote this action by $(l, k) \to g^{l q^{k \log(g)/2\pi \sqrt{-1}}}$.

Now we are ready to define the orbifold elliptic Chern character:

**Definition 3.3.** Let $\mathcal{E}$ be a $T$-equivariant elliptic brane on $\mathcal{X}$. Then we define its elliptic Chern character to be

$$\text{ch}_{\text{orb}}^{E \to K}(\mathcal{E}) := \sum_{v \in \Box} (g^{-\log(g)/2\pi \sqrt{-1}})^*(\text{ch}_{K \to E}^*|\mathcal{E}) \cdot \mathcal{O}_{\mathcal{X}_v}.$$ 

**Remark 3.4.** The construction is parallel to the usual orbifold Chern character [22], but is formulated in a different language.

Let $\mathcal{X}$ be a smooth DM stack as above and $\mathcal{I}\mathcal{X} \simeq \bigsqcup_{v \in \Box} \mathcal{X}_v$ be its inertia stack. Let $\mathcal{L} \to \mathcal{X}$ be an orbifold line bundle. Then

$$\text{ch}_{\text{orb}}(\mathcal{L}) = \sum_{v \in \Box} e^{2\pi \sqrt{-1} \text{age}_c(\mathcal{L})} \cdot \text{ch}(\mathcal{L}|_{\mathcal{X}_v}) \cdot \mathbb{1}_v^H \in H^*_{\text{orb}}(\mathcal{X}),$$

where $0 \leq \text{age}_c(\mathcal{L}) < 1$ is defined such that eigenvalue of $g(v)$ on $\mathcal{L}|_{\mathcal{X}_v}$ is $\exp(2\pi \sqrt{-1} \text{age}_c(\mathcal{L}))$ and $\mathbb{1}_v^H$ is one in the corresponding twisted sector.

Let $\exp : \text{Lie}(T) \to T$ be the exponential map. Consider the diagram

$$\begin{array}{ccc}
\text{Spec}(H^*_T(\mathcal{X})) & \xrightarrow{\text{ch}_{H \to K}^*} & \text{Spec}(K_T(\mathcal{X})) \\
\downarrow & & \downarrow \\
\text{Lie}(T) & \xrightarrow{\exp} & T
\end{array}$$

Then usual Chern character is

$$\text{ch}(V) = (\text{ch}_{H \to K}^*)^* V,$$

where we interpret $K(\mathcal{X})$ as $\Gamma(\mathcal{O}_{\text{Spec}(K_T(\mathcal{X}))})$. Now, let $\mathcal{X}_v$ be an inertia stack component. There is a natural $\langle g(v) \rangle$-action on $K(\mathcal{X}_v)$ since all sheaves on $\mathcal{X}_v$ are representations of the generic stabilizer of a point on $\mathcal{X}_v$.

Then we can write the orbifold Chern character for $V$ as

$$\text{ch}_{\text{orb}}(V) = \sum_{v \in \Box} (\text{ch}_{H \to K}^* g^{-1}(v)^* (V|_{\mathcal{X}_v}) \cdot \mathbb{1}_v^H.$$ 

**Remark 3.5.** We can also define the Chern character $\text{ch}_{H \to E} = \text{ch}_{K \to E} \circ \text{ch}_{H \to K}$ and use it to pull back elliptic cohomology classes to $H^*(\mathcal{X})$. The orbifold version is constructed analogously and takes values in the double inertia stack $H^*(\mathcal{I}\mathcal{X})$.

**Example 3.6.** Let $\mathcal{X} \simeq [\mathbb{C}^3/\mu_3]$, where the action is diagonal. Then we have

$$K(\mathcal{X}) \simeq \mathbb{C}[x^\pm]/\langle (1 - x^3) \rangle, \quad \text{Ell}(\mathcal{X}) \simeq \{ \theta(x^3) = 0 \} \subset E.$$
Let $\mathcal{E} = \theta(x)$ be an elliptic brane on $X$. Then
\begin{equation}
\text{ch}^*_K(\theta(x)) = \theta(x) = \theta(e^{2\pi \sqrt{-1}/3}) \cdot \mathcal{O}_{x_2^2 + \frac{x_1}{3}} + \theta(e^{4\pi \sqrt{-1}/3}) \cdot \mathcal{O}_{e^{1/3}x_3},
\end{equation}
where in the right we view $\theta(x)$ as an element of $\mathbb{C}[x^4]/(1 - x^3)$ and in the last equation we used the isomorphism $\mathbb{C}[x^4]/(1 - x^3)$ with the set of functions on the solution of equation $1 - x^3 = 0$. $\mathcal{O}_{x_2^2 + \frac{x_1}{3}}, \mathcal{O}_{e^{1/3}x_3}$ denote the skyscraper sheaves at corresponding points.

Elliptic Chern character is
\begin{equation}
\text{ch}^{E \rightarrow K}(\theta(x)) = \theta(x) \cdot 1_0 + \theta(q^{1/3}x) \cdot 1_1 + \theta(q^{2/3}x) \cdot 1_{2/3},
\end{equation}
where $1_v = \mathcal{O}_{[C^3/\mu^3]}_v$.

**Elliptic central charges.**

**Definition 3.7.** We define the K-theoretic central charge of an elliptic brane $\mathcal{E} \in Ell_T(X_+)$ of level $R \in K_T([V/\Gamma])$:
\begin{equation}
Z^K(\mathcal{E}, R) := \chi(H^0_R \otimes \text{ch}^{E \rightarrow K}(\mathcal{E})),
\end{equation}

### 4. Solid torus partition function

Let $E_G$ with coordinate $z$ be an elliptic curve dual to $E_G$. Recall that $z$ is the Kähler variable, that is $z$ counts the degree of quasi maps. It naturally appears as an interpolation parameter for elliptic cohomology classes from $Ell_T(X_+)$ to $Ell_T([V/G]) = Ell_{T \times G}(pt)$. Addition of this parameter also guarantees that the interpolation exists under certain degree restrictions.

**Theorem 4.1 (Elliptic grade restriction rule).** Let $\mathbb{L} \rightarrow Ell_{T \times G}(pt) \times E_G$ be a line bundle nontrivial over the second component $E_G$. Then the following interpolation problem has a unique solution for generic $z$
\begin{equation}
0 \rightarrow \mathbb{L} \otimes \Theta(-V_+) \rightarrow \mathbb{L} \otimes \mathcal{O}_{Ell_{T \times G}(pt)} \rightarrow i^*\mathbb{L} \otimes \mathcal{O}_{Ell_T(X_+)} \rightarrow 0
\end{equation}
if $\deg_G(\mathbb{L}) = \deg_G(V_+)$.\footnote{If $\mathbb{L} \otimes \Theta(-V_+)$ over elliptic curve $Ell_G(pt)$ has trivial $H^0$ and $H_1$. Line bundle over elliptic curve of degree $0$ has nontrivial cohomology if and only if it is trivial. Since $\mathbb{L}$ is nontrivial over $E_G$, the line bundle $\mathbb{L} \otimes \Theta(-V_+)$ is nontrivial for generic $z$.}

**Proof.** Consider the long exact sequence corresponding to (86):
\begin{equation}
0 \rightarrow H^0(\mathbb{L} \otimes \Theta(-V_+)) \rightarrow H^0(\mathbb{L} \otimes \mathcal{O}_{Ell_{T \times G}(pt)}) \rightarrow H^0(i^*\mathbb{L} \otimes \mathcal{O}_{Ell_T(X_+)})) \rightarrow H^1(\mathbb{L} \otimes \Theta(-V_+)) \rightarrow \cdots
\end{equation}
The line bundle $\mathbb{L} \otimes \Theta(-V_+)$ over elliptic curve $Ell_G(pt)$ has trivial $H^0$ and $H^1$. Line bundle over elliptic curve of degree $0$ has nontrivial cohomology if and only if it is trivial. Since $\mathbb{L}$ is nontrivial over $E_G$, the line bundle $\mathbb{L} \otimes \Theta(-V_+)$ is nontrivial for generic $z$.\hfill $\Box$

**Remark 4.2.** Recall the usual grade restriction rule in our case (e.g. [2]). Consider projection
\begin{equation}
K_T([V/G]) \rightarrow K_T(X_+).
\end{equation}
This map is surjective (due to Kirwan surjectivity) but not injective. The other direction map can be thought as an interpolation of $\mathbb{B}_+ \in K_T(X_+) \simeq \Gamma(\mathcal{O}_{\operatorname{Spec}(K_T)})$ to $K_{T \times G}(pt) \simeq \Gamma(\mathcal{O}_{T \times G})$. The interpolation problem has a solution but it is not unique. A regular function on $T \times G$ is a linear combination of $T \times G$-characters. Let $K_{T \times G}(pt)|_{[L,L+D_+]}$ be a vector subspace spanned by $G$-characters in the range $[L,L+D_+ - 1] \subset \mathbb{Z} \simeq \operatorname{char}(G)$.

**Grade restriction rule states that solution to the interpolation problem becomes unique on $K_{T \times G}(pt)|_{[L,L+D_+]}$, where**
\begin{equation}
D_+ = \sum_{1 \leq N_+} D_i.
\end{equation}

Let us pick $a \in T$. In figure [1] the infinite cylinder denotes $G \simeq \mathbb{C}^* \simeq \operatorname{Spec}(K_{T \times G}(pt))|_a$. Then $\operatorname{Spec}(K_{T \times G}(X_+)|_a) = \bigcup_i D_i \subset \mathbb{C}^*$. Interpolation problem is equivalent to finding a regular function $f$ on $\mathbb{C}^*$ that takes given values at $p_i$. We can compactify $\mathbb{C}^*$ to $\mathbb{P}^1$. The result is called compactified K-theory [23]. The problem of finding a function $f$ transforms to finding a line bundle $\mathbb{L} \rightarrow \mathbb{P}^1$ together with a local trivialization and its section $f$ such that the section takes required values at $p_i$. Then given a local trivialization of $\mathbb{L}$ the interpolation exists and is unique for $f \in \Gamma(\mathbb{P}^1, \mathbb{L})$ for $\mathbb{L} = \mathcal{O}(\sum_i p_i - \lambda[0] + \lambda[\infty]),$ where $\lambda \in \cochar_G(\mathbb{C}^*) \simeq \mathbb{Q}$ is a generic cocharacter. The twist by $\lambda$ does not change the degree of the line bundle but shifts its polytope (that is an interval in $\mathbb{R}$) by $\lambda$ such that the number of integral points inside coincides with its length.

The middle section of the cylinder is a fundamental domain of the elliptic curve $E = E_G \simeq Ell_G(pt)|_a$. Interpolation problem is then to find a line bundle $\mathcal{L} \rightarrow \mathcal{E}$ with a section $s$ such that in a given trivialization $s(p_i)$ are fixed numbers. It is equivalent to finding a quasiperiodic function on $G$ with prescribed values at $p_i$.\footnote{If $\mathbb{L} \otimes \Theta(-V_+)$ over elliptic curve $Ell_G(pt)$ has trivial $H^0$ and $H_1$. Line bundle over elliptic curve of degree $0$ has nontrivial cohomology if and only if it is trivial. Since $\mathbb{L}$ is nontrivial over $E_G$, the line bundle $\mathbb{L} \otimes \Theta(-V_+)$ is nontrivial for generic $z$.}
We can compute solution to the interpolation problem \[ (86) \] explicitly. Let \( k \leq N_+ \) and \( \zeta \in E[D_k] \). Then \( \mathcal{O}_{\text{Ell}(\mathcal{X}_+)} \) is spanned over \( \mathcal{O}_{\text{Ell}(pt)} \) by the restriction of \[ (90) \]

\[
\tilde{E}_z^{(k,\zeta)} := \frac{\theta(z^{1/D_k} s z^{-1})}{\theta(z^{-1}) \theta(\zeta a_k^{1/D_k} s)} \prod_{i \leq N_+} \theta(a_i s D_i)
\]

to \( \mathcal{X}_+ \). \( \mathcal{E}_{\text{T}}(\mathcal{X}_+) = \{ \prod_{i \leq N_+} \theta(a_i s D_i) = 0 \} \subset E_T \times E_G \) is a union of \( \sum_{i \leq N_+} D_i^2 \) copies of \( E_T \). Then \( \mathcal{E}_+^{(k,\zeta)} |_{\mathcal{X}_+} \) is nonzero only on one of these copies cut out by the equation \[ (91) \]

\[
\theta(\zeta a_k^{1/D_k} s) = 0.
\]

We have \[ (92) \]

\[
\tilde{E}_z^{(k,\zeta)} \in \Gamma(E_T \times E_G, \mathcal{U}(k,\zeta) \otimes \Theta(V_+)),
\]

where Poincare line bundle \( \mathcal{U}(k,\zeta) \) is a bundle with the meromorphic section \[ (93) \]

\[
\frac{\theta(\zeta a_k^{1/D_k} s z^{-1})}{\theta(\zeta a_k^{1/D_k} z^{-1}) \theta(\zeta a_k^{1/D_k} s)}.
\]

These sections have a problem that they have different transformation properties with respect to \( z \) for different \( (k, \zeta) \). We upgrade them to have the same transformation properties, that is to be sections of the same bundle over \( E_T \times E_G \times E_G^\vee \). Let \[ (94) \]

\[
E_z^{(k,\zeta)} := \frac{\theta(z^{-1})}{\theta(\zeta a_k^{1/D_k} z^{-1})} \tilde{E}_z^{(k,\zeta)} = \frac{\theta(\zeta a_k^{1/D_k} s z^{-1})}{\theta(\zeta a_k^{1/D_k} z^{-1}) \theta(\zeta a_k^{1/D_k} s)} \prod_{i \leq N_+} \theta(a_i s D_i).
\]

Then \[ (95) \]

\[
E_z^{(k,\zeta)} \in \Gamma(E_T \times E_G, \mathcal{U} \otimes \Theta(V_+)),
\]

where Poincare line bundle \( \mathcal{U} \to E_G \times E_G^\vee \) has a meromorphic section \[ (96) \]

\[
\frac{\theta(sz^{-1})}{\theta(s) \theta(z^{-1})}.
\]

The price to pay for this simplification is that \[ (97) \]

\[
E_z^{(k,\zeta)} |_{\mathcal{X}_+}
\]

is not a trivial bundle over the elliptic curve of Kahler variables \( E_G^\vee \). More explicitly, the dependence on the Kahler variables is the same as \[ (98) \]

\[
\frac{\theta(z^{-1})}{\theta(s^{-1} z^{-1})}.
\]

**Theorem 4.3** (Central charge is equal to the solid torus partition function). Let \( \mathcal{X}_+ \) be as above and \( \mathcal{E}_+ \in \Gamma(\mathcal{O}_{\text{Ell}(\mathcal{X}_+)} \) and \( \mathcal{L} \simeq \mathcal{U} \otimes \Theta(V_+) \) satisfy the condition of Theorem 4.4. Then if \( Z^K(\mathcal{E}_+, V^\vee_+) \) converges then \[ (99) \]

\[
Z^K(\mathcal{E}_+, V^\vee_+) = \frac{1}{2\pi i} \oint_{c_\varepsilon} \frac{ds}{s} \Gamma_q \cdot E_z = \text{tr}(\Gamma_q \otimes \text{ch}_{K \to E}(E_z)),
\]

where \( \text{tr} \) denotes the integration over the maximal compact subgroup of \( G \), the \( K \)-theoretic Gamma factor is \[ (100) \]

\[
\Gamma_q = \prod_{i=1}^N \phi(a_i q^{1/2}),
\]

and \( E_z \) is a unique solution to the interpolation problem \[ (101) \]

\[
0 \to \mathcal{L} \otimes \Theta(-V_+) \to \mathcal{L} \otimes \mathcal{O}_{\text{Ell}^{\times} G(pt)} \to i^* \mathcal{L} \otimes \mathcal{O}_{\text{Ell}(\mathcal{X}_+)} \to 0.
\]
for

\[ (102) \quad \mathcal{E}_+ \cdot \frac{\theta(z^{-1})}{\theta(s^{-1}z^{-1})} \in \Gamma(i^* L \otimes \mathcal{O}_{\text{El}_{T}(x_+)}). \]

The contour \( \mathcal{C}_s \) is a circle which is a shifted compact part of \( G \). The shift is chosen to separate the poles from \( \phi(V_+^\vee) \) and \( \phi(V_+^\vee) \).

Moreover, if \( Z^K(\mathcal{E}_+, V_+^\vee) \) does not converge, then the equality \( (99) \) should be understood as an asymptotic expansion as \( z \to 0 \).

**Proof.** We prove the theorem by computing the integral by residues and matching them with the contributions from the central charge.

Let us deform the integration contour in the direction \( s \to 0 \). Then the contour hit the poles of

\[ (103) \quad \frac{1}{\phi(V_+^\vee)} = \prod_{i \leq N_+} \phi(a_i^{-1}s^{-D_i}q^{0/2}). \]

Notice that \( \mathcal{E}_z \) does not have any poles in \( s \). The poles appear at

\[ (104) \quad s = s_{k, \beta, m} = a_k^{1/D_k} q^\beta e^{2\pi i m \zeta_{1/k}}, \quad k \leq N_+, \quad \beta \in \frac{1}{D_k} \mathbb{Z}_{\geq 0}. \]

Assume \( \Gamma_q \cdot \mathcal{E}_z \to 0, \quad s \to 0 \) on the complement to the neighbourhood of the poles. Then we have

\[ (105) \quad \oint_{\mathcal{C}_s} \frac{ds}{s} \Gamma_q \cdot \mathcal{E}_z = \sum_{k \leq N_+} \sum_{\beta \in \mathbb{Z}_{\geq 0}/D_k} \sum_{0 \leq m \leq D_k-1} \text{Res}_{s=s_{k, \beta, m}} \frac{1}{s} \oint_{\mathcal{C}_s} \frac{ds}{s} \Gamma_q \cdot \mathcal{E}_z. \]

In order to rewrite it as localization for the orbifold Euler characteristic we first shift \( s \to sq^3 \):

\[ (106) \quad \oint_{\mathcal{C}_s} \frac{ds}{s} \Gamma_q \cdot \mathcal{E}_z = \sum_{k \leq N_+} \sum_{\beta \in \mathbb{Z}_{\geq 0}/D_k} \sum_{0 \leq m \leq D_k-1} \text{Res}_{s=s_{k, \beta, m}} \frac{1}{s} \oint_{\mathcal{C}_s} \frac{ds}{s} \Gamma_q \cdot \mathcal{E}_z(q^3s). \]

In order to compare with the Euler characteristic formula \( (16) \) we multiply and divide by the ideal of \( K_T(\mathcal{X}_{\text{orb}}) : \prod_{i, d_i(\beta) \in \mathbb{Z}}(1 - U_i) \)

\[ (107) \quad \frac{1}{\Pi_{i \leq N} \phi(a_i^{-1}s^{-D_i}q^{-d_i(\beta)})} = \prod_{i \leq N} \phi(U_i^{-1}) \prod_{i, d_i(\beta) \in \mathbb{Z}}(1 - U_i)^{1 \over i} \]

Then we transform the elliptic brane factor. Since

\[ (108) \quad \mathcal{E}_z \in \Gamma(\mathcal{U} \otimes \Theta(V_+)), \]

the \( q^n \)-shift of the elliptic brane is

\[ (109) \quad \mathcal{E}_z(s\zeta^n) = z^n \prod_{i \leq N_+} \theta(a_i^{-1}s^{-D_i}q^{-d_i(\beta)}) \cdot \mathcal{E}_z(s\zeta) = z^n \prod_{i \leq N_+} \theta(a_i^{-1}s^{-D_i}q^{-d_i(\beta)}) \cdot \theta(z^{-1}) \cdot \mathcal{E}_+. \]

We further use the identity \( \{-x\} = 1 - \{x\}, \quad x \notin \mathbb{Z} \) to rewrite the last factor:

\[ (110) \quad \prod_{i \leq N_+} \frac{\theta(a_i^{-1}s^{-D_i}q^{-d_i(\beta)})}{\theta(a_i^{-1}s^{-D_i}q^{1-d_i(\beta)})} = \prod_{i, d_i(\beta) \in \mathbb{Z}} (-U_i)^{-1} \prod_{i \leq N_+} \frac{\theta(a_i^{-1}s^{-D_i}q^{-d_i(\beta)})}{\theta(a_i^{-1}s^{-D_i}q^{1-d_i(\beta)})}. \]

Collecting the results we obtain:

\[ (111) \quad \frac{1}{2\pi i} \oint_{\mathcal{C}_s} \frac{ds}{s} \Gamma_q \mathcal{E}_z = \sum_{k \leq N_+} \sum_{\beta} \sum_{m} z^{[\beta]} \text{Res}_{s=a_k^{1/D_k}e^{2\pi i m \zeta_{1/k}}} \prod_{i, d_i(\beta) \in \mathbb{Z}}(1 - U_i^{-1}) \frac{\theta(z^{-1})}{\theta(q^{-2}(s^{-1}z^{-1}))} \prod_{i, d_i(\beta) \in \mathbb{Z}}(1 - U_i^{-1}) \]

Using the formulas \( (16), (64) \) we can represent the integral \( (105) \) as

\[ (112) \quad \sum_{\beta \in \text{Eff}} z^{[\beta]} \frac{\theta(z^{-1})}{\theta(q^{-2}(s^{-1}z^{-1}))} \chi \left( (\mathcal{H}_+^K)_{\beta} \otimes \text{ch}_{\text{orb}}^{E \to K}(\mathcal{E}_+) \right) = \chi(\mathcal{H}_+^K) \otimes \text{ch}_{\text{orb}}^{E \to K}(\mathcal{E}_+). \]
Remark 4.4. We do not have to require $L$ to be unique solution of the interpolation problem. More generally, let $L \in \mathcal{U} \otimes \Theta(R)$ for $R \in K_T([V/G])$. Define
\begin{equation}
L_+ = L \cdot \left. \frac{\theta(s^{-1}z^{-1})}{\theta(z^{-1})} \right|_{E_{\text{Ell}(\mathcal{X})}},
\end{equation}
where mero means that the section might have poles in $z$. By definition $L_+$ is some solution to the interpolation problem for $L_+ \theta(z^{-1})/\theta(s^{-1}z^{-1})$. Then assuming convergence we have
\begin{equation}
Z^K(L_+, R) = \frac{1}{2\pi i} \oint_{\mathcal{C}_s} \frac{ds}{s} \Gamma_q \cdot L_+.
\end{equation}
The only place where the proof above changes is the transformation factor computation for $L_+$ which is consistent with the definition of level $R$-structure.

Last theorem provides a way to relate objects in the phases $\mathcal{X}_\pm$ with objects on $[V/G]$. By invoking this relation twice we can obtain the wall crossing statement:

**Definition 4.5.** Let $L = \mathcal{U} \otimes \Theta(V_+)$, $L_+ \in \Gamma(\mathcal{O}_{E\text{Ell}(\mathcal{X})})$ and $L_- \in \Gamma(L)$ is the unique interpolation of
\begin{equation}
L_+ \cdot \left. \frac{\theta(z^{-1})}{\theta(s^{-1}z^{-1})} \right|_{\mathcal{X}_+}.
\end{equation}
Then
\begin{equation}
L_- := L_\mid_{\mathcal{X}_-} \cdot \left. \frac{\theta(z^{-1})}{\theta(s^{-1}z^{-1})} \right|_{\mathcal{X}_-} \in \Gamma(\mathcal{O}_{E\text{Ell}(\mathcal{X})})
\end{equation}
is called the wall crossing brane of $L_+$. Note that it is a meromorphic section in $z$ as opposed to $L_+$ that does not depend on $z$ at all.

Note that multiplication by $\theta(z^{-1})/\theta(s^{-1}z^{-1})$ and interpolation above do not commute.

The following proposition is a consequence of definition above and theorem 4.3 applied to both $\mathcal{X}_+$ and $\mathcal{X}_-$.  

**Proposition 4.6 (Elliptic Wall Crossing).** Let $L_+ \in \Gamma(\mathcal{O}_{E\text{Ell}(\mathcal{X})})$ and $L_- \in \Gamma(\mathcal{O}_{E\text{Ell}(\mathcal{X})})$ be its wall crossing brane. Then $Z^K(L_+, V_+)$ is the analytic continuation of $Z^K(L_+, V_+)$ to the region $z \to \infty$ if it converges. Otherwise it is an asymptotic expansion in $z \to \infty$.

4.1. **Quantum difference equation.** K-theoretic I-function satisfies a certain difference equation that can be thought of as a generalization of Picard-Fuchs differential equation or Dubrovin (quantum) differential equation. This is evident in the Coulomb branch representation. Without loss of generality let $\mathcal{X} = \mathcal{X}_+$. Let $T_s: s \to qs$ and $T_z: z \to qz$ be q-difference operators acting on $s$ and $z$ respectively. By the previous section we know that
\begin{equation}
Z^K(L_+, V_+) = \sum_{k \leq N_s} \sum_{\beta \in \mathbb{Z}_d} \sum_{n=1}^{d_E \cdot E_s} \text{res}_{s \to \zeta^{-1}_k \cdot q^{-1}} ds \frac{1}{s} T_s(\Gamma_q \cdot L_-),
\end{equation}
where we expanded summation in $n$ to all integers (integrand does not have poles at $n \in \mathbb{Z}_{\leq 0}$). By changing integration variable $s \to qs$ we can write
\begin{equation}
Z^K(L_+, V_+) = \sum_{k \leq N_s} \sum_{\beta \in \mathbb{Z}_d} \sum_{n=1}^{d_E \cdot E_s} \text{res}_{s \to \zeta^{-1}_k \cdot q^{-1}} \text{res}_{s \to \zeta^{-1}_k \cdot q^{-1}} ds \frac{1}{s} T_s(\Gamma_q \cdot L_-).
\end{equation}

Let us compute the action of $T_s$ on the integrand. We use
\begin{align}
T_s \phi(x) &= \frac{\phi(x)}{1-x}, & T_s \theta(x) &= (-x)^{-1} \theta(x), \\
\phi(q^n U) &= \frac{\phi(U)}{(U; q)_n}, \\
\theta(q^n U) &= U^{-n} q^{-\frac{n^2}{2}} (-q^{-\frac{1}{2}})^{-n} \cdot \theta(U), & n &> 0, \\
\phi(q^{-n} U) &= (q^{-n} U) q_n \cdot \phi(U), \\
\theta(q^{-n} U) &= U^n q^{-\frac{n^2}{2}} (-q^{-\frac{1}{2}})^n \cdot \theta(U), & n &> 0.
\end{align}
Let \( V_i := U_i q^{n_i/2} \). Then

\[
(118) \quad T_s \gamma_q = T_s \frac{1}{\phi(V')} = \frac{1}{\prod_i \phi(q^{-D_i} V_i)} = \frac{\prod_{i > N_+} (V_i; q)^{-D_i}}{\prod_{i \leq N_+} (q^{-D_i} V_i; q)^{D_i}} \cdot \gamma_q.
\]

Moreover,

\[
(119) \quad \mathcal{E}_z \in \Gamma(\mathcal{U} \otimes \Theta(V_+)).
\]

Operator \( T_s \) is scalar on sections of both \( \mathcal{U} \), \( \Theta(V_+) \). We have

\[
(120) \quad T_s \frac{\theta(sz^{-1})}{\theta(s) \theta(z^{-1})} = z, \quad T_z \frac{\theta(sz^{-1})}{\theta(s) \theta(z^{-1})} = s
\]

\[
(121) \quad T_s \prod_{i \leq N_+} \theta(V_i) = \prod_{i \leq N_+} \theta(q^{-D_i} V_i) = \prod_{i \leq N_+} V_i^{D_i} q^{-\frac{D_i^2}{2}} (-q^{-\frac{1}{2}}) \cdot \prod_{i \leq N_+} \theta(V_i).
\]

Using this we find

\[
(122) \quad T_s (\Gamma_q \cdot \mathcal{E}_z) = z \prod_{i \leq N_+} \frac{q^{-\frac{D_i^2}{2}} (-V_i)^{D_i}}{\prod_{k=0}^{D_i} (1 - q^{k} V_i)} \prod_{i > N_+} (V_i; q)^{-D_i} \cdot (\Gamma_q \cdot \mathcal{E}_z),
\]

where

\[
(123) \quad q^{-\frac{D_i^2}{2}} (-V_i)^{D_i} \prod_{k=1}^{D_i} (1 - q^{-k} V_i) = q^{-\frac{D_i^2}{2}} (-V_i)^{D_i} \prod_{k=1}^{D_i} q^{-k} (-V_i) (1 - q^k V_i^{-1}) = \prod_{k=0}^{D_i-1} (1 - q^{1+k} V_i^{-1}).
\]

Thus, we can simplify (122):

\[
(124) \quad T_s (\Gamma_q \cdot \mathcal{E}_z) = z \prod_{i > N_+} \frac{(V_i; q)^{-D_i}}{\prod_{i \leq N_+} (q V_i^{-1}; q)^{D_i}} \cdot (\Gamma_q \cdot \mathcal{E}_z).
\]

Using the the second part of (120) we can produce \( U_i \) and \( V_i \)

\[
(125) \quad a_i^{-1} (T_z)^{-D_i} \cdot \mathcal{E}_z = U_i \mathcal{E}_z, \quad a_i^{-1} q^{n_i/2} (T_z)^{-D_i} \cdot \mathcal{E}_z = V_i \mathcal{E}_z.
\]

Therefore, we can write \( T_s (\Gamma_q \cdot \mathcal{E}_z) \) in terms of \( T_z \) as well:

\[
(126) \quad \prod_{i \leq N_+} \prod_{k=1}^{D_i} (1 - q^{-k} a_i T_z^{D_i}) z^{-1} T_s (\Gamma_q \cdot \mathcal{E}_z) = \prod_{i > N_+} \prod_{k=0}^{D_i-1} (1 - q^{1+k} a_i^{-1} T_z^{-D_i}) \cdot (\Gamma_q \cdot \mathcal{E}_z),
\]

or commuting the difference operator in the left hand side with \( z^{-1} \):

\[
(127) \quad z^{-1} \prod_{i \leq N_+} \prod_{k=1}^{D_i} (1 - q^{-k} a_i T_z^{D_i}) T_s (\Gamma_q \cdot \mathcal{E}_z) = \prod_{i > N_+} \prod_{k=0}^{D_i-1} (1 - q^{1+k} a_i^{-1} T_z^{-D_i}) \cdot (\Gamma_q \cdot \mathcal{E}_z),
\]

We can use this relation and formula (116) to get a q-difference equation on \( Z^K (\mathcal{E}_+, V_+) \). so

\[
(128) \quad \begin{bmatrix} \prod_{i \leq N_+} \prod_{k=1}^{D_i} (1 - q^{-k} a_i T_z^{D_i}) - z \prod_{i > N_+} \prod_{k=0}^{D_i-1} (1 - q^{1+k} a_i^{-1} T_z^{-D_i}) \end{bmatrix} Z^K (\mathcal{E}_+, V_+) = 0.
\]

This equation is called the quantum difference equation and is analogous to Picard-Fuchs differential equation in the cohomological case.

**Order of QqDE and elliptic cohomology.** The order of this equation is \( \max(\sum_{i \leq N_+} D_i^2; \sum_{i > N_+} D_i^2) \). This is in contrast with the cohomological case (quantum differential equation) where the order is \( \max(\sum_{i \leq N_+} D_i, \sum_{i > N_+} (-D_i)) \). Geometrically the reason is clear. In cohomology each fixed point \([pt/\mu_r]\) contributes \( \dim(H^*([pt/\mu_r])) = \dim(K([pt/\mu_r])) = r \) solutions to the equation meanwhile in K-theory the same fixed point contributes \( \dim(K([pt/\mu_r])) = \dim(O_{Ell([pt/\mu_r])}) = r^2 \) solutions.
5. Example: degree $r$ hypersurface in $\mathbb{P}^{n-1}$

Consider the corresponding GLSM data: $D_1, \ldots, D_{N_e} = 1$, $D_N = -r$. We have $X_+ \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-r)$ and $X_- \simeq [\mathbb{C}^{N_e} / \mu_r]$. The R-charge weights can be set as $(0, \ldots, 0, 1)$ and an example of the superpotential is $p(x_1^r + \cdots + x_N^r)$, where $(x_1, \ldots, x_{N_e}, p)$ are coordinates on $V \simeq \mathbb{C}^{N_e}$.

In particular,

$$
Ell_T(X_+) = \{ \prod_{i=1}^{N_e} \theta(a_i s) = 0 \} \subset E_T \times E_G.
$$

This is a degree $N_+$ cover over $E_T$. Whereas

$$
Ell_T(X_-) = \{ \theta(a_N s^r) = 0 \} \subset E_T \times E_G.
$$

This is a degree $r^2$ cover over $E_T$. We see that dimensions of elliptic cohomology of $X_+$ and $X_-$ (over $Ell_T(pt)$) coincide only if $N_+ = r^2$. This is in contrast with the K-theory where the crepant condition is $N_+ = r$.

Geometric phase. Consider the following elliptic branes on $[V/G]$:

$$
\mathcal{E}_z^{(+,k)} := \frac{\theta(qa_ksz^{-1})}{\theta(qakz^{-1})} \prod_{i \leq N_+, i \neq k} \theta(qa_is) \in \Gamma(\mathcal{U} \otimes \Theta(V_+)).
$$

Restricting to the fixed points $pt_j = \{ s = a_j^{-1} \}$ for generic $z$ we get

$$
\mathcal{E}_z^{(+,k)}|_{pt_j} = 0, \quad j \neq k
$$

Thus, these branes are supported on the fixed points and provide a basis (over $\mathcal{O}_{E_T}$) of elliptic branes. The solid torus partition function is

$$
Z^K(\mathcal{E}_z^{(+,k)}, V_+^\vee) = \frac{1}{2\pi i} \oint_{\mathcal{E}_z} ds \prod_{i \neq k} \frac{\phi(qa_is)}{\phi(a^{-1}_i s^* q)} \cdot \frac{\theta(qa_ksz^{-1})}{\theta(qakz^{-1})}\phi(a^{-1}_k s^{-1}) =
$$

$$
= \sum_{n \geq 0} \frac{\text{Res}}{s \rightarrow a^{-1}_n q^n} \left[ \frac{ds}{s \phi(a^{-1}_k s^{-1})} \right] \prod_{i \neq k} \phi(q^{n+1}a_i a^{-1}_k) \theta(q^{n+1}z).
$$

Now we compute

$$
\frac{\theta(q^{n+1}z)}{\theta(qakz)} = (-z)^{-n} q^{-\frac{n(n+1)}{2}} \frac{\theta(z^{-1})}{\theta(a^{-1}_k z^{-1})}
$$

and

$$
\text{Res}_{s \rightarrow a^{-1}_n q^n} \left[ \frac{ds}{s \phi(q^{-n} s)} \right] = - \text{Res}_{s \rightarrow 1} \left[ \frac{ds}{s \phi(q^{-n} s)} \right] = - \frac{1}{\phi(q)(q^{-n}; q)_n} \text{Res}_{s \rightarrow 1} \left[ \frac{ds}{s(1-s)} \right] = (-1)^n \phi(q)^{-1} q^{-\frac{n(n+1)}{2}} (q; q)_n.
$$

Thus, the central charge is

$$
Z^K(\mathcal{E}_z^{(+,k)}, V_+^\vee) = \phi(q)^{-1} \sum_{n \geq 0} z^n \prod_{i \neq k} \phi(q^{n+1}a_i a^{-1}_k) \theta(z^{-1}) (q; q)_n.
$$

It satisfies the $Qq$DE in the geometric phase:

$$
\left[ \prod_{i \leq n} (1 - a_i T_z) - z \prod_{k=0}^{n-1} (1 - q^{k+1}a^{-1}_N T^k_z) \right] Z^K(\mathcal{E}_+, V_+) = 0,
$$

which can be also checked by a direct computation.
Landau-Ginzburg phase. In the other phase we can choose the following set of branes:
\[
E_z^{(-\zeta)} := \frac{\theta(\zeta^{-1} a_N^{1/r} s^{-1} z)}{\theta(\zeta^{-1} a_N^{1/r} z)} \in \Gamma(\mathcal{U} \otimes \Theta(V_-)),
\]
where \([\zeta] \in E[r]\). Restricting to the twisted sectors \(s = \xi a_N^{1/r}\) of the unique fixed point we obtain
\[
E_z^{(-\zeta)}|_{pt} = 0, \quad \zeta \neq \xi
\]
(139)
We can also simplify the expression above using
\[
\theta'(q^{-n}) = -(q^{-1}; q^{-1})_n \phi(\theta) = (-1)^{n+1} q^{-\frac{n(n+1)}{2}} (q; q)_n \phi(\theta).
\]
(140)
The solid torus partition function is
\[
Z^K(E_z^{(-\zeta)}), V^\vee) = \frac{1}{2\pi i} \oint_{C,s} ds \left[ \frac{\phi(a_N^{1/r} s^{-1})}{\theta(\zeta^{-1} a_N^{1/r} s^{-1} z)} \right] = \sum_{n>0} \frac{\phi(\zeta^{-1} a_N^{1/r} s^{-1} z)}{\theta(\zeta^{-1} a_N^{1/r} s^{-1})} = \prod_{i \leq N_+} \frac{\phi(\zeta^{-1} a_N^{1/r} q^n)}{\theta(\zeta^{-1} a_N^{1/r} z)}.
\]
(141)
We compute:
\[
\text{Res}_{s \to \zeta^{-1} a_N^{1/r} q^{-n}} \left[ \frac{ds}{s \theta(\zeta^{-1} a_N^{1/r} s^{-1})} \right] = \text{Res}_{s \to 1} \left[ \frac{ds}{s \theta(q^n)} \right] = q^{\frac{n}{2}} (-q^{-\frac{1}{2}})^n \text{Res}_{s \to 1} \frac{ds}{s \theta(s)} = -q^{\frac{n}{2}} (-q^{-\frac{1}{2}})^n \phi(q)^{-2}.
\]
(142)
Then
\[
Z^K(E_z^{(-\zeta)}), V^\vee) = -\phi(q)^{-2} \sum_{n>0} z^{-n} \frac{\phi(\zeta^{-1} a_N^{1/r} q^n)}{\prod_{i \leq N_+} \phi(\zeta^{-1} a_N^{1/r} q^n)} \frac{\theta(z)}{\theta(\zeta^{-1} a_N^{1/r} z)}.
\]
(143)
It satisfies the QDE:
\[
\left[ \prod_{k=1}^r (1 - q^{k-r} a_N T_z^{-r}) - z \prod_{i \leq N_+} (1 - a_i^{-1} T_z^{-1}) \right] Z^K(E_z^{(-\zeta)}), V^\vee) = 0.
\]
(144)
As we see, these equations are of order max\(\{a^2, N_+\}\) as opposed to order max\(\{r, n\}\) of the usual Picard-Fuchs equation. For the quintic threefold this phenomenon was noticed, for example in [15]. The reason being that dim\(K[\mathcal{C}^+_{/\mu_+}]\) = 31. The reason being that dim\(K[\mathcal{C}^+_{/\mu_+}]\) = r as opposed to dim\(K[\mathcal{C}^+_{/\mu_+}]\) = r.

Wall Crossing We can also deform the contour in (133) to \(s \to \infty\). Wall-crossing of the branes \(E_z^{(+, k)}\) has the form
\[
\frac{\theta(s^{-1} z^{-1})}{\theta(z^{-1})} E_z^{(+, k)} = \frac{\theta(s^{-1} z^{-1})}{\theta(z^{-1})} \prod_{i \leq N_+, i \neq k} \theta(q a_i s) \in \Gamma(O_{Ell_{x+} E_z^k}),
\]
(15)
and wall-crossing of \(E_z^{(-, k)}\) are
\[
\frac{\theta(s^{-1} z^{-1})}{\theta(z^{-1})} E_z^{(-, k)} = \frac{\theta(q s z)}{\theta(z)} \frac{\theta(\zeta^{-1} a_N^{1/r} s^{-1} z)}{\theta(\zeta^{-1} a_N^{1/r} z)} \frac{\theta(a_N s^{-r})}{\theta(\zeta^{-1} a_N^{1/r} s^{-1} z)} = \prod_{i \leq N_+} \frac{\theta(q a_i s)}{\theta(a_i s^{-r})} \in \Gamma(O_{Ell_{x+} E_z^k}).
\]
(16)

APPENDIX A. ELLIPTIC COHOMOLOGY

There are various constructions of elliptic cohomology that serve different purposes. We follow expositions of [15] and [123, 24] and refer the reader there for more detailed exposition of the subject. Our notations are closer to [123, 24] rather than [15]. Here we present a very brief introduction to the subject.

Given a Lie group \(G\) (which is abelian in this paper), equivariant cohomology and K-theory of a smooth DM stack are rings of \(H_G(pt)\) (respectively \(K_G(pt)\)) modules. Equivariant cohomology is not a ring of modules but rather a (sheaf on a) projective scheme, so it is more convenient to use the dual language of spectra. For example, spectrum of equivariant cohomology (K-theory) is an affine scheme over \(\text{Lie}(G) \simeq \text{Spec}(H_G(pt))\) (respectively \(G \simeq \text{Spec}(K_G(pt))\)).
Let $E \simeq \mathbb{C}^*/q^{\mathbb{Z}}$ be an elliptic curve, where we can think of $q$ as either a complex number or a formal parameter. In the study of elliptic cohomology the modularity properties (that correspond to the $q$-dependence) play an important role, but we do not study it here. We define $G$-equivariant elliptic cohomology of a point to be a projective scheme:

$$\text{Ell}_G(pt) = E_G := \text{Bun}_G(E'^{\vee}).$$

Two main examples for this paper are $G = T \simeq (\mathbb{C}^*)^n$ and $G = \mathbb{Z}_n$:

$$E_T \simeq T/q^{\text{cochar}(T)}, \quad E_{\mu_n} = E[n],$$

where $E[n]$ is a group of $n$-torsion points on $n$, or simply $n$-th roots of unity on $E$. For a general smooth DM stack $X$ elliptic cohomology is a projective scheme

$$\text{Ell}_G(X) \xrightarrow{\pi_X} \text{Ell}_G(pt).$$

One should think of the scheme above as generalization of $\text{Spec}(K_G(X)) \to \text{Spec}(K_G(pt))$.

Elements of elliptic cohomology are sections of line bundles over $\text{Ell}_G(X)$. In particular, the pushforward of structure sheaf $(\pi_X)_*_X \mathcal{O}_{\text{Ell}_G(X)}$ is a sheaf of $\mathcal{O}_{\text{Ell}_G}$-modules that satisfies a set of axioms for equivariant generalized cohomology theories [15].

One can define $\text{Ell}_G(X)$ either axiomatically or provide explicit construction [17]. Another useful fact is that elliptic cohomology fits into the natural diagram:

$$\begin{array}{ccc}
\text{Spec}(H(T))^\text{chH} \to \text{Spec}(K_T(X)) & \text{cochar}(T) & \text{Ell}_T(X) \\
\text{Lie}(T) \quad \exp & \quad \mod q^{\text{cochar}(T)} & \quad E_T
\end{array}$$

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