Running Hubble constant from the SNe Ia Pantheon sample?

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Abstract. The mismatch between different independent measurements of the expansion rate of the Universe is known as the Hubble constant ($H_0$) tension, and it is a serious and pressing problem in cosmology. We investigate this tension considering the dataset from the Pantheon sample, a collection of 1048 Type Ia Supernovae (SNe Ia) with a redshift range $0 < z < 2.26$. We perform a binned analysis in redshift to study if the $H_0$ tension also occurs in SNe Ia data. Hence, we build equally populated subsamples in three and four bins, and we estimate $H_0$ in each bin considering the $\Lambda$CDM and $w_0w_a$CDM cosmological models. We perform a statistical analysis via a Markov Chain Monte Carlo (MCMC) method for each bin. We observe that $H_0$ evolves with the redshift, using a fit function $H_0(z) = \tilde{H}_0(1 + z)^{-\alpha}$ with two fitting parameters $\alpha$ and $\tilde{H}_0$. Our results show a decreasing behavior of $H_0$ with $\alpha \sim 10^{-2}$ and a consistency with no evolution between 1.2 $\sigma$ and 2.0 $\sigma$. Considering the $H_0$ tension, we extrapolate $H_0(z)$ until the redshift of the last scattering surface, $z = 1100$, obtaining values of $H_0$ consistent in 1 $\sigma$ with the cosmic microwave background (CMB) measurements by Planck. Finally, we discuss possible $f(R)$ modified gravity models to explain a running Hubble constant with the redshift, and we infer the form of the scalar field potential in the dynamically equivalent Jordan frame.

INTRODUCTION

The Hubble constant tension is one of the biggest open problems of modern cosmology. Several independent measurements of the actual expansion rate of the Universe, the Hubble constant $H_0$, provide incompatible values. The status of all these discrepancies and theoretical attempts is summarized in a review [1]. It should be noted that this inconsistency occurs between measurements referred to the early Universe, and those based on local probes (for instance, Cepheids or Type Ia supernovae - SNe Ia) in the late Universe. This tension has become more and more serious, little by little more precise measurements have reduced gradually the error bars. More precisely, there is a discrepancy in 4.4 $\sigma$ between the Planck data [2], $H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and the value obtained from Cepheids [3], $H_0 = 74.03 \pm 1.42 \text{ km s}^{-1} \text{ Mpc}^{-1}$. This tension could point out inconsistencies in Planck data or local probes or, alternatively, it could be a signal for a new cosmology.
Regarding the latter point, modified gravity theories are studied to try to solve open problems in cosmology, such as the Hubble constant tension or the nature of dark energy, which cannot be fully explained in General Relativity (GR) and the standard ΛCDM cosmological model. In particular, the $f(R)$ modified gravity theories [1] [2] provide a generalization of GR, including an extra geometrical degree of freedom given by a function $f$ of the Ricci scalar $R$, which implies a cosmological dynamics that differs from GR. The scalar-tensor theories in the so-called Jordan frame [1] [5] [6] are dynamically equivalent to the $f(R)$ proposals; within this framework, the additional scalar degree of freedom is provided by a scalar field, which is non-minimally coupled to the metric.

Since the $H_0$ tension concerns measurements referred to the early and late Universe, in this work we investigate the possibility of a hidden evolutionary effect that could imply a running Hubble constant with the redshift [7] [8] [9] [10], focusing on the data analysis of the Pantheon sample [11]. In this regard, we follow a binned approach of the Pantheon sample in three and four redshift bins to extract the values of $H_0$ in each bin. Then, we observe an unexpected decreasing trend of $H_0$ with the redshift, and we interpret this extra degree of freedom via a modified gravity scenario.

This work is organized as follows: in Sec. 1 we recall briefly some basic notions about cosmology using SNe Ia datasets; in Sec. 2 we introduce the $f(R)$ extended models in the Jordan frame; in Sec. 3 we follow a binning approach to address the Hubble constant tension within the redshift range of the Pantheon sample; in Sec. 4 we discuss our results and the theoretical interpretations in the $f(R)$ modified cosmology.

We adopt the metric signature $(-, +, +, +)$, and we set the speed of light $c = 1$. The Einstein constant is denoted with $\chi \equiv 8\pi G$, and $G$ is the Newton constant.

**TYPE I A SUPERNOVA COSMOLOGY**

SNe Ia are usually regarded as standard candles since they are characterized by their uniformity in the absolute magnitude profiles. Hence, they are very useful to estimate cosmic distances. In this regard, we recall the definition of the theoretical distance modulus of a SN:

$$\mu_{th} = m_{th} - M = 5 \log_{10} d_L(z) + 25,$$

where $m_{th}$ and $M$ are the apparent and absolute magnitude of a SN, respectively, and $d_L$ is the luminosity distance, which depends on a specific cosmological model. In a flat geometry, $d_L$ is given by [12]

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')}.$$

Considering a dark energy component in which the equation of state parameter $w$ evolves with the redshift $z$, i.e. $w = w(z)$, and neglecting relativistic components in the late Universe, the Hubble function $H(z)$ contained in Eq. (2) is written as

$$H(z) = H_0 \sqrt{\Omega_{m0} (1 + z)^3 + \Omega_{DE0} \exp \left[ 3 \int_0^z \frac{dz'}{1 + w(z')} \right] \frac{dz'}{1 + z'}},$$

where $\Omega_{m0}$, $\Omega_{DE0} \equiv 1 - \Omega_{m0}$ are the cosmological density parameters of the matter and dark energy components, respectively, at the present redshift $z = 0$. Note that if $w = -1$, a cosmological constant $\Lambda$ is reproduced, and the standard ΛCDM scenario is recovered. Differently, considering in Eq. (3) the $w_0w_a$CDM model, given by the Chevalier-Polarski-Linder (CPL) parameterization [13] [14] $w(z) = w_0 + w_a \times z / (1 + z)$ with parameters $w_0$ and $w_a$, we end up in a slowly evolving dark energy scenario slightly different from a cosmological constant.

Moreover, it should be noted that the Hubble constant $H_0$ is degenerate with $M$, as you can see easily combining Eqs. (1) [2] and (3). Specifically, $M$ is calibrated to $-19.35$ in the Pantheon sample such that $H_0 = 70.0 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [11].

To discriminate between several cosmological models, it is useful to compare $\mu_{th}$ in Eq. (1) with the observed distance modulus $\mu_{obs}$, which is obtained from SNe Ia data,

$$\mu_{obs} = m_B - M + \alpha x_1 - \beta c + \Delta M + \Delta B,$$

where

1. $\alpha = 5 \times 10^{-4}$
2. $\beta = 3 \times 10^{-3}$
3. $\Delta M = 0.15$ units
4. $\Delta B = 0.15$ units

The expression for $H(z)$, Eq. (3), is then used to find the Hubble constant $H_0$. This involves calculating the integral $\int_0^z \frac{dz'}{1 + w(z')}$, which represents the slowing down of the expansion of the Universe as $z$ increases.
where $m_B$ is the $B$-band apparent magnitude, and $M$ is the absolute magnitude in the $B$-band $z_1$ of a SN with $c = 0$ and $x_1 = 0$. SNe parameters are the color $c$ and the stretch $x_1$, while $\alpha$ and $\beta$ are coefficients. Finally, $\Delta M$ is a distance correction related to the host-galaxy mass, and $\Delta B$ is a bias correction. The Pantheon sample data is available in the repository by Scolnic et al. (2018) [11] (https://github.com/dscolnic/Pantheon).

To perform a statistical analysis, we define $\Delta \mu = \mu_{\text{obs}} - \mu_{\text{th}}$ as the difference between the distance moduli in Eqs. (1) and (4). Furthermore, the $\chi^2$ is built as

$$\chi^2 = \Delta \mu^T C^{-1} \Delta \mu,$$

where $C = C_{\text{sys}} + D_{\text{stat}}$ is the full covariance matrix. The $C_{\text{sys}}$ matrix involves systematic errors, while $D_{\text{stat}}$ is a diagonal matrix that contains statistical errors for each SN, due to peculiar velocities, photometry, bias, lensing, and intrinsic scatter [11]. Referring to the Pantheon sample, $\Delta \mu$ represents a vector with 1048 components, while $C$, $C_{\text{sys}}$ and $D_{\text{stat}}$ are $1048 \times 1048$ square matrices.

**F(R) MODIFIED COSMOLOGY IN THE JORDAN FRAME**

Within the framework of the $f(R)$ modified gravity [4, 5], the gravitational Lagrangian density becomes a function $f(R)$ of the scalar curvature $R$, involving an additional scalar degree of freedom with respect to GR. Adopting the metric formalism, it can be noted that the generalized field equations are of fourth-order in the metric [5], and if $f(R) = R$ one might obtain the Einstein-Hilbert equations in GR. The scalar-tensor theories in the so-called Jordan frame [4, 5, 6] are useful to rewrite $f(R)$ models in a dynamically equivalent form, which provides field equations of lower order. The total action in the Jordan frame is written as

$$S_f = \frac{1}{2X} \int d^4x \sqrt{-g} \left[ f(R) - V(\phi) \right] + S_M(g_{\mu\nu}, \psi),$$

where $g$ is the determinant of $g_{\mu\nu}$ the metric tensor, $S_M$ is the matter contribution to the total action, and $\psi$ is referred to the matter fields. Moreover, in this paradigm, the extra degree of freedom given by $f(R)$ is expressed as a scalar field $\phi$, defined as $\phi = f'(R) = df/dR$, which is non-minimally coupled to the metric, and it is governed by the scalar field potential $V(\phi) = \phi R(\phi) - f(R(\phi))$.

The field equations in the Jordan frame for a flat FLRW geometry in the late Universe, considering a pressureless matter component and neglecting relativistic species, are given by the following equation system [6]:

$$H^2 = \frac{\chi \rho}{3 \phi} - H \frac{\dot{\phi}}{\phi} + \frac{V(\phi)}{6 \phi},$$

$$\ddot{a} = -\frac{\chi \rho}{6 \phi} + \frac{V(\phi)}{6 \phi} - H \frac{\dot{\phi}}{\phi} - \frac{1}{2} \frac{\ddot{\phi}}{\phi},$$

$$3\dot{\phi}^2 - 2V(\phi) + \phi \frac{dV}{d\phi} + 9H \dot{\phi} = \chi \rho,$$

where $\rho(t)$ is the matter energy density, and a dot denotes a time derivative. The first equation above is the modified Friedmann equation, the second one is the modified acceleration equation, while the third equation describes the scalar field dynamics. Note that all these equations are now of second order in the equivalent formalism in the Jordan frame, but the disadvantage is the introduction of a non-minimally coupling between the scalar field and the metric.

It should be emphasized that the gravitational coupling constant is restated in the Jordan frame. More specifically, looking at Eq. (7a), $G_{\text{eff}} = G/\phi$ may be regarded as an effective gravitational coupling. Modified gravity models are very interesting to explore alternative scenarios for the cosmic acceleration in the late Universe without a true cosmological constant (see the models by [5, 15, 16, 17]).

Finally, following the Hu-Sawicki formalism [15], we introduce auxiliary variables $y_H$ and $y_R$ that are useful to write the luminosity distance in the $f(R)$ gravity. We write the Hubble parameter and Ricci scalar as

$$H^2 = m^2 \left( (1 + z)^3 + y_H \right), \quad R = m^2 \left( 3(1 + z)^3 + y_R \right),$$

where $m$ describes the scalar field dynamics. Note that all these equations are now of second order in the equivalent formalism in the Jordan frame, but the disadvantage is the introduction of a non-minimally coupling between the scalar field and the metric.
where the two dimensionless variables \( y_H \) and \( y_R \) encompass extra contributions with respect to the matter component in the \( \Lambda \)CDM model. We have defined \( m^2 = \chi \rho_0 / 3 = H_0^2 / \Omega_{m0} \) with \( \rho_0 \) present matter density. Note that if \( y_H \) is simply a constant, the first relation is nothing more the Friedmann equation in the \( \Lambda \)CDM model. In the context of \( f(R) \) gravity, instead, \( y_H \) and \( y_R \) evolve and their dynamics strongly depend on the form of the \( f(R) \) function or, equivalently, the scalar field potential \( V(\phi) \). It is possible to rewrite the generalized Friedmann equation (7a) and the Ricci scalar in terms of \( y_H \) and \( y_R \) and their derivatives to obtain a first-order differential equation system (see [15]), which can be solved numerically. Then, we can write the luminosity distance in a \( f(R) \) model

\[
d_L(z) = \frac{(1 + z)}{H_0} \int_0^z \frac{d\tilde{z}}{\sqrt{\Omega_{m0}(y_H(\tilde{z}) + (1 + \tilde{z})^3)}}
\]

in terms of the deviation \( y_H(z) \), combining Eqs. (2) and (8).

**Binned Analysis**

We decide to follow a binned approach using the Pantheon sample dataset to evaluate if the Hubble constant evolves with the redshift [9,10], motivated by the fact the independent measurements of SNe Ia in each bin have shown a tension.

Therefore, we have split the redshift range of the Pantheon sample in bins each having the same number \( N \) of SNe Ia. More precisely, considering three redshift bins, we have \( N \approx 349 \) SNe for each bin, and the redshift ranges for three bins are \( 0.01 < z < 0.18, 0.18 < z < 0.34, \) and \( 0.34 < z < 2.26 \). In the case of four bins, instead, \( N = 262 \) in each bin and the redshift ranges are: \( 0.01 < z < 0.13, 0.13 < z < 0.25, 0.25 < z < 0.42, \) and \( 0.42 < z < 2.26 \). Then, we have built the subvectors \( \Delta \mu \) with \( N \) SNe, according to the binning division and redshift order. We have also divided the full covariance matrix \( C \) into \( N \times N \) submatrices for each redshift bin. In this regard, concerning simply the statistical contribution \( D_{	ext{stat}} \) in \( C \), each diagonal element of the \( D_{	ext{stat}} \) matrix is related to a single SN, hence we can quickly build the submatrices starting from \( D_{	ext{stat}} \). However, if we include the systematic errors, we need to consider also \( C_{	ext{sys}} \), which is not a diagonal matrix. Thus, we used a customized code to select \( C_{	ext{sys}} \) elements involved only with the SNe within the redshift bin taken into account. Hence, we built properly the submatrices, considering both statistical and systematic errors related to the SNe for each bin.

To focus only on \( H_0 \) in a one-dimensional analysis and constrain it in each bin, we decide to fix the value of the other cosmological parameters. Thus, we assume the fiducial value [11] \( \Omega_{m0} = 0.298 \) for a flat \( \Lambda \)CDM model, and \( \Omega_{m0} = 0.308 \) with \( w_0 = -1.009 \) and \( w_a = -0.129 \) for a flat \( w_0w_a \)CDM model for all the redshift ranges. Moreover, we need to fix the absolute magnitude \( M \) of SNe Ia in each bin. Bearing in mind the degeneracy between \( M \) and \( H_0 \), we decide to calibrate \( M \) to obtain a value of \( H_0 = 73.5 \text{ km s}^{-1} \text{ Mpc}^{-1} \) in the first redshift bins, according to measurements of the local probes at low redshifts.

Thus, we perform a preliminary analysis for both \( \Lambda \)CDM and \( w_0w_a \)CDM models to obtain the respective value of \( M \) for each binning division in the first redshift bin, which is characterized by a mean redshift \( z \ll 1 \). To do this, we minimize the \( \chi^2 \) in Eq. (3). Then, we use the MCMC methods to sample a posterior distribution. Finally, \( M \) is obtained for each binning division (see Table I). After this preliminary analysis, we fix \( M \) in the other bins. One might expect that also \( H_0 \) assumes the same values in all the redshift bins, due to its degeneracy with \( M \).

Now, we can approach the main part to check the values of \( H_0 \) after a binned analysis of the Pantheon Sample in three and four bins for both the \( \Lambda \)CDM and \( w_0w_a \)CDM models. We use again the minimization of \( \chi^2 \), as illustrated in the preliminary analysis, and in the MCMC we set the priors: \( 60 \text{ km s}^{-1} \text{ Mpc}^{-1} < H_0 < 80 \text{ km s}^{-1} \text{ Mpc}^{-1} \). Finally, we have extracted the values of \( H_0 \) for the all bins. To investigate an evolution of \( H_0 \) with the redshift \( z \), we apply a non-linear fit [9] of \( H_0(z) \) written as

\[
H_0(z) = \frac{\hat{H}_0}{(1 + z)^\alpha},
\]

with two fitting parameters, \( \hat{H}_0 \) and \( \alpha \). Note that \( \hat{H}_0 = H_0(z = 0) \). Moreover, if \( \alpha \neq 0 \), we have an evolutionary trend, otherwise \( H_0 \) is a constant.
TABLE I. Fitting parameters ($H_0$ and $\alpha$) of $H_0(z)$ (Eq. (10)) and extrapolated values at high redshifts, after a binned analysis of the Pantheon sample, focusing on a flat $\Lambda$CDM model (upper part) and a flat $w_0w_a$CDM model (lower part). The compatibility of $\alpha$ with zero (no evolution) is expressed in terms of 1 $\sigma$ in the fourth column. The new absolute magnitude $M$, which provides $H_0 = 73.5$ km s$^{-1}$ Mpc$^{-1}$ in the first bins, is listed in the fifth column. Finally, the extrapolated values of $H_0(z)$ at the redshift of the most distant galaxies, $z = 11.09$, and the last scattering surface, $z = 1100$, are listed in the last two columns. All the errors are presented in 1 $\sigma$.

| Bins (km s$^{-1}$ Mpc$^{-1}$) | $H_0$ (km s$^{-1}$ Mpc$^{-1}$) | $\alpha$ | $\frac{\sigma}{\alpha}$ | $M$ (lower part) | $H_0 (z = 11.09)$ (km s$^{-1}$ Mpc$^{-1}$) | $H_0 (z = 1100)$ (km s$^{-1}$ Mpc$^{-1}$) |
|-------------------------------|-------------------------------|----------|--------------------------|-----------------|------------------------------------------|------------------------------------------|
| 3 73.577 ± 0.106               | 0.009 ± 0.004                 | 2.0      | −19.245 ± 0.006          | 72.000 ± 0.805  | 69.219 ± 2.159                         |
| 4 73.513 ± 0.142               | 0.008 ± 0.006                 | 1.2      | −19.246 ± 0.004          | 71.975 ± 1.020  | 69.272 ± 2.737                         |

FIGURE 1. Evolution of $H_0(z)$ with redshift according to Eq. (10) as a result of a binned analysis of the Pantheon sample for a flat $\Lambda$CDM model (two upper plots) and a flat $w_0w_a$CDM model (two lower plots).

The fitting parameters and their respective errors in 1 $\sigma$ are listed in Table I. Note that, for instance, the $\alpha$ parameter is consistent with zero, namely no evolution, in 2.0 $\sigma$ in the $\Lambda$CDM model using three bins. In Fig. 1 we can observe a slowly and unexpected decreasing trend for $H_0(z)$ in the $\Lambda$CDM model and also in the $w_0w_a$CDM model. It should be noted that all the $\alpha$ coefficients are mutually compatible in 1 $\sigma$, therefore pointing out a reliable decreasing trend of $H_0$ with the redshift in the Pantheon sample.

If the observed trend of $H_0(z)$ is intrinsic and does not depend on a specific sample, one may ask what happens with other local probes or in the early Universe at very high redshifts. Hence, we extrapolate the
fit function of $H_0(z)$ at the redshift of the most distant galaxies, $z = 11.09$ [18], and at the redshift of the last scattering surface, $z = 1100$, to compare the latter values of $H_0(z)$ with the one inferred from Planck measurements for the CMB. We find that the extrapolated values (see Table I) are consistent within 1 $\sigma$ with the value of $H_0$ obtained with the Planck measurements for both the $\Lambda$CDM and $w_0w_a$CDM models regardless the number of bins.

**INTERPRETATION OF THE RESULTS IN THE F(R) MODIFIED COSMOLOGY**

The evolution of the Hubble constant with the redshift needs a physical interpretation. In this section, we do not focus on possible astrophysical reasons (see [9]), but we investigate our results in modified gravity theories.

The Hubble constant must be a constant by definition, and its observed evolution could be a signal of a wrong framework, for instance, a possible hidden function of the redshift, which has not been taken into account so far. As already discussed in [9, 19], a varying Einstein constant $\chi = 8\pi G$ (or a varying Newton constant $G$) may in principle lead to an evolution of $H_0(z)$. If we look at the function of $H_0(z)$ in Eq. (10), we require an effective Einstein constant $\chi \sim (1 + z)^{-2} \alpha$ to preserve constant the present critical density $\rho_{c0} = 3H_0^2/\chi$.

We recall that an effective Einstein constant is naturally obtained in the Jordan frame (Sec. 2) through the rescaling: $\chi \to \chi/\phi$. The extra degree of freedom in the Jordan frame, i.e. the presence of a non-minimally coupled scalar field $\phi$, which depends on the redshift, provides indeed an evolving Einstein constant. Moreover, another reason to focus on the Jordan frame to account for the observed trend of $H_0(z)$ lies in the modified Friedmann equation (7a). Indeed, the Hubble parameter $H(z)$ is related to matter density $\rho$ via an effective Einstein constant, hence the dynamics of the scalar field $\phi$ may imply an evolution of the Hubble constant with redshift. More specifically, we require the following ansatz

$$\phi(z) = (1 + z)^{2\alpha}$$

(11)

to account for $H_0(z)$ in Eq. (10). These concepts suggest that an extra degree of freedom with respect to GR might imply the unexpected trend of $H_0(z)$.

We do not mention so far which scalar field potential we should have in the Jordan frame, mimicking a cosmological constant in a slow-roll and at the same time providing an effective Hubble constant. Considering the cosmological dynamics in the Jordan frame, we can infer $V(\phi)$ [10], if we assume the behavior of $H_0(z)$ in Eq. (10) and the ansatz above for $\phi(z)$ in Eq. (11). Therefore, the modified Friedmann equation (7a) allows us to write $V(\phi)$ as

$$V(\phi) = 6(1 - 2\alpha) \left(\frac{dz}{dt}\right)^2 \phi^{1-1/\alpha} - 6m^2 \phi^{3/2\alpha},$$

(12)

where we have used the standard definition of redshift, and $\rho \sim (1 + z)^3$ for a matter component. Now, we need to specify the term $dz/dt$, which is given by

$$\frac{dz}{dt} = -(1 + z)H(z).$$

(13)

Here, we can not simply use the extended Friedmann equation (7a) to replace $H(z)$, because we should set a specific scalar field potential to solve $H(z)$. Then, we decide to impose the Hubble parameter

$$H(z) = \frac{\dot{H}_0}{(1 + z)^{\alpha}} \sqrt{\Omega_{m0} (1 + z)^3 + 1 - \Omega_{m0}},$$

(14)

as suggested from our binned analysis in Sec. 3, although its analytical form is different from the Hubble parameter obtained using the modified Friedmann equation. In other words, to obtain $V(\phi)$, we impose in the Jordan frame dynamics the same physical effect observed from the redshift binned analysis of the Pantheon sample.
Hence, combining Eqs. (12), (13) and (14), we obtain the final expression of the potential (10):

$$V(\phi) \approx m^2 \left[ (3 - 2\alpha) \left( \frac{-R}{18m^2} \right)^{3/2} - (1 + \ln 18) R + 18m^2 \frac{1 - \Omega_{0m}}{\Omega_{0m}} \right] + O(\alpha^2).$$

Expanding this expression for $\alpha \sim 0$, according to the values of $\alpha$ from the binned analysis (see Table I), we obtain (10):

$$f(R) \approx \left( R - 6m^2 \frac{1 - \Omega_{0m}}{\Omega_{0m}} \right) + \frac{2}{3} \alpha \left[ R \ln \left( \frac{-R}{m^2} \right) - (1 + \ln 18) R + 18m^2 \frac{1 - \Omega_{0m}}{\Omega_{0m}} \right] + O(\alpha^2).$$

It should be emphasized that the first term above is precisely the gravitational Lagrangian density in GR with a cosmological constant $\Lambda = 3m^2 (1 - \Omega_{0m}) / \Omega_{0m}$, where we used $m^2 = \tilde{H}_0^2 \Omega_{0m}$. Moreover, it is clear that the first-order term in $\alpha$ in Eq. (17) gives the deviation from GR. Hence, the $\alpha$ coefficient stresses corrections to GR.

The expressions inferred for the scalar field potential in Eq. (15), and the corresponding $f(R)$ function in Eq. (17), are viable in the late Universe since in all the calculations we do not include relativistic components. However, it could be interesting to test this form of $V(\phi)$ using other probes in the late Universe.

The proposed discussions above show a possible interpretation of our results coming from the binned analysis, and we suggest a candidate for $V(\phi)$, assuming the evolution of $H_0(z)$.

**FIGURE 2.** Scalar field potential $V(\phi)$ in the Jordan frame given by Eq. (15) and inferred from the evolution of $H_0(z)$ with redshift (10). The quantity $V(\phi)/m^2$ is dimensionless. Note the presence of a flat region of the potential for $0 < z \lesssim 0.3$ or $\phi \lesssim 1.005$. 

$\phi$ denotes the scalar field.
CONCLUSIONS

In this work, performing a redshift binned analysis of the Pantheon sample of SNe Ia, we have seen that there is a slow evolution of the Hubble constant $H_0$ with the redshift $z$, described by Eq. (10) with a parameter $\alpha$ that is consistent with zero (no evolution) between 1.2 $\sigma$ and 2.0 $\sigma$. The evolutionary behavior of $H_0(z)$ occurs regardless of the binning division both in the $\Lambda$CDM and $w_0w_a$CDM models. It seems that the Hubble constant tension also emerges locally in the narrow redshift range $0 < z < 2.26$ of the Pantheon sample of SNe Ia.

Then, extrapolating the fit function of $H_0(z)$ (10) at higher redshifts, we obtain values of $H_0$ that are remarkably consistent within 1 $\sigma$ with the CMB measurements by Planck. Our results could point out an intrinsic trend of $H_0(z)$, which might imply the mismatch between independent measurements of the Hubble constant referred to the early and late Universe.

We remark that we perform a one-dimensional analysis to obtain constraints only on $H_0$, following [9]. Actually, in [10] we extend the analysis, and we obtain similar results, considering more than one variable in the MCMC methods, for instance $H_0$ and $\Omega_m$ to avoid fixing all the cosmological parameters except $H_0$.

Our analysis could suggest the presence of a hidden astrophysical effect, which is the reason for the evolution of $H_0(z)$ and has not been taken into account so far. Alternatively, the observed effect may point out that we need a cosmology beyond the $\Lambda$CDM paradigm. We have discussed the $f(R)$ modified gravity theories, and we suggest a possible form for the scalar field potential (15) in the Jordan frame, and the respective $f(R)$ function (17). Furthermore, we mention that a new binned analysis of the Pantheon sample could be worthwhile, using a specific $f(R)$ model and the modified version of the luminosity distance [9] to test new physics.

Concerning the future perspectives, to explore further the possible evolution of cosmological parameters, it could be very useful to include in our analysis also probes at high redshifts, such as the Gamma-Ray Bursts (GRBs), which in principle could allow to enlarge the redshift range of the actual Hubble diagram [20, 21, 22, 23, 24, 25]. This work could be regarded as a preliminary study for the application of GRBs in cosmology, using, for instance, the prompt-afterglow relations with the plateau emission [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42], similarly to the magnetar model [43, 44, 45].

In conclusion, we have presented a new perspective to address the Hubble constant tension through a binning approach, thus we foster investigations on the astrophysical parameters of SNe Ia and further theoretical discussions.

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