Transferring Davey's Theorem on Annihilators in Bounded Distributive Lattices to Modular Congruence Lattices and Rings

Claudia MUREŞAN*
University of Bucharest
Faculty of Mathematics and Computer Science
Academiei 14, RO 010014, Bucharest, Romania
E-mails: cmuresan@fmi.unibuc.ro, c.muresan@yahoo.com

September 5, 2017

Abstract

Congruence lattices of semiprime algebras from semi–degenerate congruence–modular varieties fulfill the equivalences from B. A. Davey's well–known characterization theorem for $m$–Stone bounded distributive lattices; moreover, changing the cardinalities in those equivalent conditions does not change their validity. I prove this by transferring Davey's Theorem from bounded distributive lattices to such congruence lattices through a certain lattice morphism and using the fact that the codomain of that morphism is a frame. Furthermore, these equivalent conditions are preserved by finite direct products of such algebras, and similar equivalences are fulfilled by the elements of semiprime commutative unitary rings and, dualized, by the elements of complete residuated lattices.

2010 Mathematics Subject Classification: primary: 08A30; secondary: 08B10, 03G10.
Keywords: congruence, (semi–degenerate, congruence–modular, congruence–distributive) variety, commutator, annihilator, (Stone, semiprime) algebra.

1 Introduction

In [15, 16], I have transferred [5, Theorem 1] from bounded distributive lattices to residuated lattices, by using the reticulation of a residuated lattice. In [9], G. Georgescu and I have constructed the reticulation for algebras from congruence–modular varieties. I have noticed that the kind of transfer from [15, 16] can be made in this general context, but referring to the lattices of congruences of the algebras from such varieties instead of their elements and enforcing some restrictions: the varieties must be semi–degenerate and the algebras in question must be semiprime. It turns out that the transfer of these properties doesn’t even necessitate the reticulation, but only part of its construction from [9], and it produces further equivalences, because changing the cardinality in those conditions forms other properties which are equivalent to those conditions. The present work contains two main results, the first of which is Theorem 2.29 below, stating the equivalences fulfilled by the congruence lattices of such algebras, which I am proving along with some related results, such as the fact that, if the congruence lattice of such an algebra satisfies the equivalent conditions from Theorem 2.29 then the reticulation of that algebra satisfies those conditions, as well, and the fact that those conditions are preserved by finite direct products of such algebras. While the structure of the proof of Theorem 2.29 resembles the one I have made for residuated lattices in [15, 16], the auxiliary results for that proof do not hold in the present context, which, in turn, produces new auxiliary results, so all the following results are new and original, excepting only the ones cited from other works or mentioned as being either well known or immediate from well–known properties.

The natural question that arises is whether other algebras fulfill analogues of Theorem 2.29 or [5, Theorem 1] for elements instead of congruences, as is the case for residuated lattices. I prove that semiprime commutative unitary rings do, in the second main result of the present paper: Theorem 3.32. It also turns out that complete residuated lattices fulfill the dual of Theorem 2.29 expressed for elements, because, for such residuated lattices,
changing the cardinalities in the equivalent conditions from the analogue of Davey’s Theorem for residuated lattices from [13, 16] produces other properties equivalent to those conditions. Of course, both residuated lattices, which are congruence–distributive, thus semiprime, and form a semi–degenerate variety, and semiprime commutative unitary rings, which are semiprime algebras from the semi–degenerate congruence–modular variety of commutative unitary rings, fulfill those equivalences for congruences, too.

Section 2 contains Theorem 2.29 and the related results on congruence lattices. Section 3 contains a brief presentation of the situation in residuated lattices, the analogue of Theorem 2.29 for the elements of semiprime commutative unitary rings instead of the congruences from congruence lattices of semiprime algebras from semi–degenerate congruence–modular varieties, and some related results. Section 4 contains a brief layout of some directions for future research.

2 Transferring Davey’s Theorem to Congruence Lattices

For any set $S$, $|S|$ shall denote the cardinality of $S$ and by $|S| < \infty$ we shall specify the fact that $S$ is finite. And, throughout this paper, $m$ shall be an infinite cardinal, arbitrary but fixed. For brevity, instead of treating separate cases, we shall often use the fact that, in any bounded poset, $\bigvee 0$ is the minimum and $\bigwedge \emptyset$ is the maximum of that poset. Throughout this paper, we shall designate any algebra by its underlying set, unless there is danger of confusion.

Let $(L, \lor, \land, 0, 1)$ be an arbitrary bounded lattice. $(\text{Id}(L), \lor, \land, \emptyset, L)$ shall be the complete lattice of the ideals of $L$, which is well known to be distributive exactly when $L$ is distributive. Let $a \in L$ and $U, V \subseteq L$. We shall denote by $(U)$ the ideal of $L$ generated by $U$, by $[a] = \{a\}$ the principal ideal of $L$ generated by $a$, by Ann($U$) the annihilator of $U$ and by Ann($a$) the annihilator of $a$ in $L$: Ann($U$) = $\{x \in L \mid (\forall u \in U) (x \land u = 0)\}$ and Ann($a$) = Ann($\{a\}$) = $\{x \in L \mid x \land a = 0\}$. Whenever specifying $L$ is necessary, we shall denote $(U)_L$, Ann$_L(U)$ and Ann$_L(a)$ instead of $(U)$, $[a]$, Ann($U$) and Ann($a$), respectively. Obviously, Ann($a$) = Ann($[a]$), $U \subseteq \text{Ann}(\text{Ann}(U))$ and Ann($U$) = $\bigcap_{u \in U} \text{Ann}(u)$, hence Ann($U$) $\cap$ Ann($V$) = $\bigcap_{u \in U} \text{Ann}(u) \cap \bigcap_{v \in V} \text{Ann}(v)$.

Obviously, for any $x \in L$, Ann($\{x\}$) = Ann($x$). It is straightforward that, if $L$ is distributive, then Ann($U$) is an ideal of $L$. Recall that $L$ is called a compact lattice iff all its elements are compact. Notice that, in a compact lattice $L$, the join of any non–empty family $U \subseteq L$ equals the join of a finite non–empty subfamily of $U$.

We shall denote by $B(L)$ the set of the complemented elements of the bounded lattice $L$. If $L$ is distributive, then $B(L)$ is the Boolean center of $L$, which is a Boolean sublattice of $L$. We shall call $B(L)$ the Boolean center of $L$ regardless of whether $L$ is distributive. We shall call $L$ a Stone lattice iff, for all $a \in L$, there exists an $e \in B(L)$ such that Ann($a$) = $\{e\}$. We shall call $L$ a strongly Stone lattice iff, for all $U \subseteq L$, there exists an $e \in B(L)$ such that Ann($U$) = $\{e\}$. Trivially, if $L$ is strongly Stone, then $L$ is Stone. Since Ann($U$) = $\bigcap_{u \in U} \text{Ann}(u)$ for any $U \subseteq L$ and $\bigcap_{i \in I} [a_i] = (\bigvee_{i \in I} a_i)$ for any non–empty family $(a_i)_{i \in I} \subseteq L$ having a join, it follows that the converse holds if $B(L)$ is closed w.r.t. arbitrary joins.

Now let $M$ be a bounded lattice and $f : L \to M$ be a surjective lattice morphism. Then it is straightforward that the map $I \to f(I)$ is a surjective lattice morphism from Id($L$) to Id($M$), which fulfills $f([a]) = (f(a))$ for all $a \in L$. Moreover, this lattice morphism preserves arbitrary joins. Indeed, if $(J_i)_{i \in I}$ is a family of ideals of $L$, then $f(\bigvee_{i \in I} J_i) \supseteq f(\bigcup_{i \in I} J_i) = \bigcup_{i \in I} f(J_i)$, thus $f(\bigvee_{i \in I} J_i) \supseteq \bigvee_{i \in I} f(J_i)$.

For the arbitrary bounded lattice $L$, let us denote by: Ann($L$) = $\{\text{Ann}_L(U) \mid U \subseteq L\}$, PAnn($L$) = $\{\text{Ann}_L(a) \mid a \in L\}$, P2Ann($L$) = $\{\text{Ann}_L(\text{Ann}_L(a)) \mid a \in L\}$ and 2Ann($L$) = $\{\text{Ann}_L(\text{Ann}_L(U)) \mid U \subseteq L\}$. Regarding the conditions below on subsets of $L$, note that they are trivially fulfilled by $\emptyset$, because Ann$_L(\emptyset)$ = $L = \{1\}_L$ and Ann$_L(\text{Ann}_L(\emptyset))$ = Ann$_L(L)$ = $\{0\} = \{0\}_L$ and, clearly, $0, 1 \in B(L)$. Here, I am specifying
that is: for any finite nonzero cardinality $\kappa$, having a meet, 

(1) $\kappa, \leq \kappa$ holds if

(2) $\kappa, L$ is a strongly Stone lattice;

(3) $\kappa, \L$ is a Stone lattice and $\mathcal{B}(L)$ is a $\kappa$–complete Boolean sublattice of $L$;

(4) $\kappa, L$ is a Stone lattice and $\mathcal{B}(L)$ is a Boolean sublattice of $L$;

(5) $\kappa, L$ is a complete Boolean sublattice of $L$;

Remark 2.2. The converse holds if $\mathcal{B}(L)$ is a complete Boolean sublattice of $L$, case in which, therefore, $L$ is Stone iff $L$ is strongly Stone, that is: for any finite nonzero cardinality $\kappa$, (1) $\kappa, L$ is equivalent to (1)$_{\kappa, L}$, and thus also to (1)$_{\kappa, L}$.

Also, for any family $(U_i)_{i \in I}$ of subsets of $L$: $\mathcal{A}_L(U) = \bigcup_{i \in I} \mathcal{A}_L(U_i) = \bigcap_{i \in I} \mathcal{A}_L(U_i) = \bigcup_{i \in I} \mathcal{A}_L(U_i) = \bigcap_{i \in I} \mathcal{A}_L(U_i)$.

Now assume that $L$ is distributive. Then, for all $n \in \mathbb{N}^*$ and all $u_1, \ldots, u_n \in L$, $\bigcap_{i=1}^n \mathcal{A}_L(u_i) = \mathcal{A}_L(\bigvee_{i=1}^n u_i)$. Indeed, for all $k \in \overline{1, n}$, $u_k \leq \bigvee_{i=1}^n u_i$, hence $\mathcal{A}_L(\bigvee_{i=1}^n u_i) \subseteq \mathcal{A}_L(u_k)$.
Corollary 2.4. \( \text{Ann}_L(\bigvee_{i=1}^n u_i) \subseteq \bigcap_{i=1}^n \text{Ann}_L(u_i) \). If \( x \in \bigcap_{i=1}^n \text{Ann}_L(u_i) \), then \( x \wedge (\bigvee_{i=1}^n u_i) = \bigvee_{i=1}^n (x \wedge u_i) = 0 \), thus \( x \in \text{Ann}_L(\bigvee_{i=1}^n u_i) \), so the converse inclusion holds, as well. Therefore \( \text{FA}n(L) = P\text{Ann}(L) \), from which it is easy to see that, for any \( i \in \{1,5\} \) and any finite nonzero cardinality \( \kappa \), \( (i)_L \) is equivalent to \((i)_{\infty,L}\). It is immediate that, for any finite nonzero cardinality \( \kappa \), \((1)_L, (2)_L, (3)_L, (4)_L \) and \((5)_L\) are equivalent.

If, moreover, \( L \) is a frame, then, for any \( 0 \neq u \subseteq L \), \( \bigcap_{u \in U} \text{Ann}_L(u) = \text{Ann}_L(U) = \text{Ann}_L(\bigvee_{u \in U} u) \). Indeed, for all \( a \in U \), \( a \leq \bigvee_{u \in U} u \), hence \( \text{Ann}_L(\bigvee_{u \in U} u) \subseteq \text{Ann}_L(a) \), therefore \( \text{Ann}_L(\bigvee_{u \in U} u) \subseteq \bigcap_{u \in U} \text{Ann}_L(u) \). If \( x \in \bigcap_{u \in U} \text{Ann}_L(u) \), then \( x \wedge (\bigvee_{u \in U} u) = \bigvee_{u \in U} (x \wedge u) = 0 \), thus \( x \in \text{Ann}_L(\bigvee_{u \in U} u) \), so the converse inclusion holds, as well. Therefore, in a frame \( L \), \( \text{Ann}_L(L) = P\text{Ann}(L) \), from which it is easy to see that, for any \( i \in \{1,5\} \) and any finite nonzero cardinality \( \kappa \), \((i)_L \) is equivalent to \((i)_{\infty,L}\), and it also follows that \( 2\text{Ann}(L) = P2\text{Ann}(L) \subseteq \text{Ann}(L) = P\text{Ann}(L) \), so that the second part of condition \((4)_L \) is fulfilled for any nonzero cardinality \( \kappa \), and thus \((4)_L, (4)_{\infty,L}, (4)_L \) and \((iv)_L \) are equivalent.

**Corollary 2.3.** If \( L \) is a bounded distributive lattice, then:

- **conditions** \((1)_L, (2)_L, (3)_L, (4)_L \) and \((5)_L \) are equivalent;
- for any nonzero cardinality \( \kappa \), \((1)_L, (2)_L, (3)_L, (4)_L \) and \((5)_L \) are equivalent;
- for any finite nonzero cardinality \( \kappa \), \((1)_L, (2)_L, (3)_L, (4)_L \) and \((5)_L \) are equivalent;
- if, moreover, \( L \) is a frame, then, for any nonzero cardinality \( \kappa \) and any \( h, i, j \in \{1,5\} \), \((h)_L, (i)_L, (j)_L \) are equivalent; in particular, \((1)_L \) is equivalent to \((2)_{\infty,L} \), so that: \( L \) is a Stone lattice if \( L \) is a strongly Stone lattice.

Throughout this paper, all algebras shall be non–empty, \( C \) shall be a semi–degenerate congruence–modular equational class of algebras of the same type and \( A \) shall be an arbitrary algebra from \( C \). \( (\text{Con}(A), \lor, \cap, \Delta_A, \nabla_A) \) shall be the complete modular lattice of the congruences of \( A \), with \( \Delta_A = \{(a,a) \mid a \in A \} \) and \( \nabla_A = A^2 \), so that the set of the proper congruences of \( A \) is \( \text{Con}(A) \setminus \{\nabla_A\} \). \([\cdot,\cdot]_A : (\text{Con}(A))^2 \rightarrow \text{Con}(A) \) shall be the commutator of \( A \). Recall that \([\cdot,\cdot]_A \) is commutative, smaller than the intersection, increasing in both arguments and distributive in both arguments with respect to arbitrary joins [6]. Following [6], if \( \phi \) is a proper congruence of \( A \), then we call \( \phi \) a prime congruence iff, for all \( \theta, \zeta \in \text{Con}(A) \), \([\theta,\zeta]_A \subseteq \phi \) implies \( \theta \subseteq \phi \) or \( \zeta \subseteq \phi \). We shall denote by \( \text{Spec}(A) \) the set of the prime congruences of \( A \).

Since \( C \) is semi–degenerate, it follows that \( C \) has no skew congruences [8, Theorem 8.5, p. 85] and, for any member \( M \) of \( C \), \( \nabla_M \) is a compact congruence of \( M \) [12] and each proper congruence of \( M \) is included in a prime congruence [1, Theorem 5.3], thus, if \( M \) is non–trivial, that is \( |M| > 1 \), then \( \text{Spec}(M) \) is non–empty. Recall that the compact congruences of an algebra are exactly its finitely generated congruences.

Following [9], for all \( \theta \in \text{Con}(A) \), we shall denote by \( \rho_A(\theta) \) the radical of \( \theta \): \( \rho_A(\theta) = \bigcap \{ \phi \in \text{Spec}(A) \mid \theta \subseteq \phi \} \), and by \( \equiv_A \) the binary relation on \( \text{Con}(A) \) defined by: for any \( \theta, \zeta \in \text{Con}(A) \), \( \theta \equiv_A \zeta \) iff \( \rho_A(\theta) = \rho_A(\zeta) \). We have proven in [9], that \( \equiv_A \) is an equivalence on \( \text{Con}(A) \), and we have denoted by \( \lambda_A : \text{Con}(A) \rightarrow \text{Con}(A)/\equiv_A \) the canonical surjection. Moreover, we have proven that \( \equiv_A \) is a congruence of the lattice \( \text{Con}(A) \), thus \( \text{Con}(A)/\equiv_A \lor \cap \lambda_A(\Delta_A), \lambda_A(\nabla_A) \) is a bounded lattice and \( \lambda_A \) is a surjective lattice morphism, where \( \lambda_A(\theta) \lor \lambda_A(\zeta) = \lambda_A(\theta \lor \zeta) \) and \( \lambda_A(\theta) \land \lambda_A(\zeta) = \lambda_A(\theta \land \zeta) \) for all \( \theta, \zeta \in \text{Con}(A) \). We have also proven that \( \lambda_A(\theta \cap \zeta) = \lambda_A([\theta,\zeta]_A) \) for all \( \theta, \zeta \in \text{Con}(A) \), from which, by using the distributivity of the commutator with respect to the join, it immediately follows that the bounded lattice \( \text{Con}(A)/\equiv_A \) is distributive. Moreover, since \( \text{Con}(A) \) is a complete lattice and \([\cdot,\cdot]_A \) is distributive w.r.t. arbitrary joins, it follows that \( \text{Con}(A)/\equiv_A \) is a complete lattice in which the meet is distributive w.r.t. arbitrary joins, that is \( \text{Con}(A)/\equiv_A \) is a frame (see also [9]). Hence, from Corollary 2.3 we obtain:

**Corollary 2.4.** For any nonzero cardinality \( \kappa \) and any \( h, i, j \in \{1,5\} \), \((h)_{\kappa,\text{Con}(A)/\equiv_A}, (i)_{\infty,\text{Con}(A)/\equiv_A} \) and \((j)_{\text{Con}(A)/\equiv_A} \) are equivalent.
We have proven in [9] that:

- for any $\theta \in \text{Con}(A)$, $\lambda_A(\theta) = \lambda_A(\nabla A)$ iff $\theta = \nabla A$; indeed, $\lambda_A(\theta) = \lambda_A(\nabla A)$ iff $\rho_A(\theta) = \rho_A(\nabla A) = \bigcap \theta = \nabla A$ iff no prime congruence of $A$ includes $\theta$ iff $\theta = \nabla A$;

- $\mathcal{B}(\text{Con}(A))$ is a Boolean sublattice of $\text{Con}(A)$, in which the commutator coincides to the intersection.

We call $A$ a semiprime algebra iff $\rho_A(\Delta A) = \Delta A$. If $C$ is congruence–distributive, then $\rho_A(\theta) = \theta$ for all $\theta \in \text{Con}(A)$, thus $A$ is semiprime (see, also, [9]).

Throughout the rest of this paper, the algebra $A$ shall be semiprime. Then, as we have proven in [9]:

- $(1^\circ)$ $\mathcal{B}(\text{Con}(A)/\equiv_A) = \mathcal{B}(\text{Con}(A))/\equiv_A$;

- $(2^\circ)$ for all $\theta \in \text{Con}(A)$: $\lambda_A(\theta) \in \mathcal{B}(\text{Con}(A)/\equiv_A)$ iff $\theta \in \mathcal{B}(\text{Con}(A))$;

- $(3^\circ)$ for all $\theta \in \text{Con}(A)$, $\lambda_A(\theta) = \lambda_A(\delta A)$ iff $\theta = \delta A$; indeed, $\lambda_A(\theta) = \lambda_A(\delta A)$ implies $\theta \subseteq \rho_A(\theta) = \rho_A(\delta A) = \delta A$, thus $\theta = \delta A$; the converse implication is trivial;

- $(4^\circ)$ for all $\theta, \zeta \in \text{Con}(A)$, $[\theta, \zeta] = \Delta A$ iff $\theta \cap \zeta = \delta A$; indeed, by $(3^\circ)$: $[\theta, \zeta] = \Delta A$ iff $\lambda_A([\theta, \zeta] = \lambda_A(\Delta A)$ iff $\lambda_A(\theta \cap \zeta) = \lambda_A(\Delta A)$ iff $\theta \cap \zeta = \delta A$.

I shall make repeated use of the surjectivity of $\lambda_A : \text{Con}(A) \to \text{Con}(A)/\equiv_A$, without mentioning it; the same goes for the remarks from this paper and the results I am recalling from [9], excepting $(1^\circ)$, $(2^\circ)$, $(3^\circ)$ and $(4^\circ)$.

By Corollary 2.3 in the particular case when $\text{Con}(A)$ is distributive, conditions $(1)_{\text{Con}(A)}$, $(5)_{\text{Con}(A)}$ are equivalent, if $k$ is a nonzero cardinality, then conditions $(1)_{k,\text{Con}(A)}$, $(5)_{k,\text{Con}(A)}$ are equivalent, and, if $k$ is finite, then conditions $(1)_{k,\text{Con}(A)}$, $(5)_{k,\text{Con}(A)}$, $(1)_{\infty,\text{Con}(A)}$, $(5)_{\infty,\text{Con}(A)}$ are equivalent. An example of a semi–degenerate congruence–modular variety which is not congruence–distributive is the variety of commutative unitary rings.

Throughout the rest of this section, unless there is danger of confusion, all annihilators of elements or subsets of $\text{Con}(A)$ respectively $\text{Con}(A)/\equiv_A$, shall be in the lattice $\text{Con}(A)$, respectively $\text{Con}(A)/\equiv_A$, and the same shall go for generated ideals.

**Remark 2.5.** By $(4^\circ)$, for all $\theta \in \text{Con}(A)$ and all $\Omega \subseteq \text{Con}(A)$, $\text{Ann}(\theta) = \{\zeta \in \text{Con}(A) \mid \theta \cap \zeta = \delta A\} = \{\zeta \in \text{Con}(A) \mid [\theta, \zeta] = \Delta A\}$ and $\text{Ann}(\Omega) = \{\zeta \in \text{Con}(A) \mid (\forall \omega \in \Omega) (\omega \cap \zeta = \delta A\} = \{\zeta \in \text{Con}(A) \mid (\forall \omega \in \Omega) ([\omega, \zeta] = \Delta A)\}$.

**Remark 2.6.** Any annihilator of $\text{Con}(A)$ is an ideal of $\text{Con}(A)$. Indeed, let $\omega \in \text{Con}(A)$. Then $\delta A \cap \omega = \delta A$, thus $\delta A \subseteq \text{Ann}(\omega)$, so $\text{Ann}(\omega) \neq \emptyset$. Now let $\theta, \zeta \in \text{Con}(A)$. If $\zeta \in \text{Ann}(\omega)$ and $\theta \subseteq \zeta$, then $\theta \cap \omega \subseteq \zeta \cap \omega = \delta A$, thus $\theta \in \text{Ann}(\omega)$. If $\theta, \zeta \in \text{Ann}(\omega)$, then $[\omega, \theta \cap \zeta] = [\omega, \theta] \cup [\omega, \zeta] = \delta A \cup \delta A = \delta A$, hence $\theta \cap \zeta \in \text{Ann}(\omega)$. Therefore $\text{Ann}(\omega) \in \text{Id}(\text{Con}(A))$. Hence, for any $\Omega \subseteq \text{Con}(A)$, $\text{Ann}(\Omega) = \{\omega \in \text{Con}(A) \mid (\forall \omega \in \Omega) (\text{Ann}(\omega))\}$.

**Remark 2.7.** Of course, the direct image of $\lambda_A : \text{Con}(A) \to \text{Con}(A)/\equiv_A$ preserves arbitrary unions of subsets of $\text{Con}(A)$. It also preserves arbitrary intersections, because, if $(\Omega_i)_{i \in I}$ is a family of subsets of $\text{Con}(A)$, then $\lambda_A(\bigcap_{i \in I} \Omega_i) = (\bigcap_{i \in I} \Omega_i)/\equiv_A = (\bigcap_{i \in I} \Omega_i)/\equiv_A = (\bigcap_{i \in I} \lambda_A(\Omega_i))$. Since $\lambda_A : \text{Con}(A) \to \text{Con}(A)/\equiv_A$ is a surjective lattice morphism, it follows that the map $I \to \lambda_A(I)$ is a surjective lattice morphism from $\text{Id}(\text{Con}(A))$ to $\text{Id}(\text{Con}(A)/\equiv_A)$ which fulfills $\lambda_A([\theta]) = (\lambda_A(\theta))$ for all $\theta \in \text{Con}(A)$ and preserves arbitrary joins.

**Lemma 2.8.** (i) If $L$ is a bounded distributive lattice, then $\text{Ann}_L(U) = \text{Ann}_L([U]_L)$ for all $U \subseteq L$.

(ii) For all $\Omega \subseteq \text{Con}(A)$, $\text{Ann}(\Omega) = \text{Ann}(\Omega)$.

**Proof.** (i) $U \subseteq ([U]_L)$, thus $\text{Ann}_L([U]_L) \subseteq \text{Ann}_L(U)$. If $x \in \text{Ann}_L(U)$ and $u \in ([U]_L$, then $u \leq u_1 \lor \ldots \lor u_n$ for some $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U$, thus $x \lor u \leq x \lor (u_1 \lor \ldots \lor u_n) = (x \lor u_1) \lor \ldots \lor (x \lor u_n) = 0 \lor \ldots \lor 0 = 0$, thus $x \in \text{Ann}_L([U]_L)$ since $u$ is arbitrary in $([U]_L$, so the converse inclusion holds, as well.

(ii) $\Omega \subseteq ([\Omega]_L)$, thus $\text{Ann}_L(\Omega) \subseteq \text{Ann}_L(\Omega)$. Now let $\theta \in \text{Ann}(\Omega)$ and $\omega \in (\Omega)$, so that $\omega \subseteq \omega_1 \lor \ldots \lor \omega_n$ for some $n \in \mathbb{N}$ and $\omega_1, \ldots, \omega_n \in \Omega$, thus $[\theta, \omega] \subseteq [\theta, \omega_1 \lor \ldots \lor \omega_n] = [\theta, \omega_1] \lor \ldots \lor [\theta, \omega_n] = \Delta A \lor \ldots \lor \Delta A = \Delta A$, thus $\theta \in \text{Ann}(\Omega)$ since $\omega$ is arbitrary in $\Omega$, so the converse inclusion holds, as well. \qed
Lemma 2.9. If $L$ is a bounded distributive lattice, then:

- for any family $(I_k)_{k \in K} \subseteq \text{Id}(L)$, $\bigcap_{k \in K} \Ann_L(I_k) = \Ann_L(\bigvee_{k \in K} I_k)$;
- for any family $(a_k)_{k \in K} \subseteq L$ having a join, $\bigcap_{k \in K} \Ann_L(a_k) = \Ann_L(\bigvee_{k \in K} a_k)$.

Proof. By Lemma 2.8, $\bigcap_{k \in K} \Ann_L(I_k) = \Ann_L(\bigcup_{k \in K} I_k) = \Ann_L(\bigvee_{k \in K} I_k)$, hence, if $\bigvee_{k \in K} a_k$ exists, then $\bigcap_{k \in K} \Ann_L(a_k) = \bigcap_{k \in K} \Ann_L((a_k)_L) = \Ann_L(\bigvee_{k \in K} (a_k)_L) = \Ann_L(\bigvee_{k \in K} a_k)$. \qed

Lemma 2.10. 

- For any family $(I_k)_{k \in K} \subseteq \text{Id}(\Con(A))$, $\bigcap_{k \in K} \Ann(I_k) = \Ann(\bigvee_{k \in K} I_k)$.
- For any family $(\theta_k)_{k \in K} \subseteq \Con(A)$, $\bigcap_{k \in K} \Ann(\theta_k) = \Ann(\bigvee_{k \in K} \theta_k)$.

Proof. Same as the proof of Lemma 2.9, but using (ii) from Lemma 2.8 instead of (i). \qed

Lemma 2.11. 

- For all $\theta \in \Con(A)$, $\Ann(\lambda_A(\theta)) = \lambda_A(\Ann(\theta))$.
- For all $\Omega \in \Con(A)$, $\Ann(\lambda_A(\Omega)) = \lambda_A(\Ann(\Omega))$.

Proof. $\Ann(\lambda_A(\theta)) = \{\lambda_A(\zeta) \mid \zeta \in \Con(A), \lambda_A(\theta) \wedge \lambda_A(\zeta) = \lambda_A(\Delta_A)\} = \{\lambda_A(\zeta) \mid \zeta \subseteq \Con(A), \lambda_A(\theta \cap \zeta) = \lambda_A(\Delta_A)\} = \{\lambda_A(\zeta) \mid \zeta \subseteq \Con(A), \theta \cap \zeta = \Delta_A\} = \lambda_A(\Ann(\theta))$, by (3'). Therefore $\lambda_A(\Ann(\Omega)) = \lambda_A(\bigcap_{\omega \in \Omega} \Ann(\omega)) = \bigcap_{\omega \in \Omega} \lambda_A(\Ann(\omega)) = \bigcap_{\omega \in \Omega} \Ann(\lambda_A(\omega)) = \bigcap_{x \in \lambda_A(\Omega)} \Ann(x) = \Ann(\lambda_A(\Omega))$. \qed

Lemma 2.12. For all $\theta \in \Con(A)$ and $\alpha \in \mathcal{B}(\Con(A))$, $\lambda_A(\theta) \leq \lambda_A(\alpha)$ iff $\theta \subseteq \alpha$.

Proof. Assume that $\lambda_A(\theta) \leq \lambda_A(\alpha)$, so that $\lambda_A(\theta \cup \alpha) = \lambda_A(\theta) \vee \lambda_A(\alpha) = \lambda_A(\alpha) \in \lambda_A(\mathcal{B}(\Con(A))) = \mathcal{B}(\Con(A)/\equiv_A)$ by (1°), hence $\theta \cup \alpha \in \mathcal{B}(\Con(A))$ by (2°), and thus $\theta \cup \alpha = \alpha$, again by (1°), so $\theta \subseteq \alpha$. The fact that $\lambda_A$ is order-preserving proves the converse implication. \qed

Lemma 2.13. For all $\Gamma, \Omega \subseteq \Con(A)$, $\gamma \in \Con(A)$ and $\alpha \in \mathcal{B}(\Con(A))$:

(i) $\lambda_A(\gamma) \in \lambda_A(\Ann(\Omega))$ iff $\gamma \in \Ann(\Omega)$;
(ii) $\lambda_A(\Gamma) \subseteq \lambda_A(\Ann(\Omega))$ iff $\Gamma \subseteq \Ann(\Omega)$;
(iii) $\lambda_A(\Ann(\Gamma)) = \lambda_A(\Ann(\Omega))$ iff $\Ann(\Gamma) = \Ann(\Omega)$;
(iv) $\lambda_A(\langle \alpha \rangle) = \lambda_A(\Ann(\Omega))$ iff $\langle \alpha \rangle = \Ann(\Omega)$.

Proof. (i) Assume that $\lambda_A(\gamma) \in \lambda_A(\Ann(\Omega)) = \Ann(\lambda_A(\Omega))$ according to Lemma 2.11, so that, for all $\omega \in \Omega$, $\lambda_A(\gamma \cap \omega) = \lambda_A(\gamma) \wedge \lambda_A(\omega) = \lambda_A(\Delta_A)$, thus, by (3°), $\gamma \cap \omega = \Delta_A$, hence $\gamma \in \Ann(\Omega)$. The converse implication is trivial.

(ii) By (i).

(iii) By (ii), $\lambda_A(\Ann(\Gamma)) = \lambda_A(\Ann(\Omega))$ iff $\lambda_A(\Ann(\Gamma)) \subseteq \lambda_A(\Ann(\Omega))$ and $\lambda_A(\Ann(\Omega)) \subseteq \lambda_A(\Ann(\Gamma))$ iff $\Ann(\Gamma) \subseteq \Ann(\Omega)$ and $\Ann(\Omega) \subseteq \Ann(\Gamma)$ iff $\Ann(\Gamma) = \Ann(\Omega)$.

(iv) Assume that $\lambda_A(\langle \alpha \rangle) = \lambda_A(\Ann(\Omega))$, which implies $\langle \alpha \rangle \subseteq \Ann(\Omega)$ by (ii). Now let $\theta \in \Ann(\Omega)$, so that $\lambda_A(\theta) \in \lambda_A(\Ann(\Omega)) = \lambda_A(\langle \alpha \rangle) = \lambda_A(\langle \alpha \rangle)$, hence $\theta \in \langle \alpha \rangle$ by Lemma 2.12, thus $\Ann(\Omega) \subseteq \langle \alpha \rangle$, therefore $\langle \alpha \rangle = \Ann(\Omega)$. The converse implication is trivial. \qed

Proposition 2.14. For any nonzero cardinality $\kappa$, the properties $(1)_{\kappa, \Con(A)}$ and $(1)_{\kappa, \Con(A)/\equiv_A}$ are equivalent.
Proof. For the converse implication, assume that \((1)_{\kappa,\text{Con}(A)/\equiv_A}\) is satisfied and let \(\emptyset \neq \Omega \subseteq \text{Con}(A)\) having \(|\Omega| \leq \kappa\). Then \(\lambda_A(\Omega) \leq |\Omega| \leq \kappa\), hence, by \((1^\circ)\), Lemma 2.11 and Lemma 2.13 \((iv)\), there exists an \(\alpha \in \mathcal{B}(\text{Con}(A))\) such that \(\lambda_A(\text{Ann}(\Omega)) = \text{Ann}(\lambda_A(\Omega)) = (\lambda_A(\alpha)) = \lambda_A((\alpha))\), therefore \(\text{Ann}(\Omega) = (\alpha)\).

For the direct implication, assume that \((1)_{\kappa,\text{Con}(A)/\equiv_A}\) is satisfied and let \(U \subseteq \text{Con}(A)/\equiv_A\) with \(|U| \leq \kappa\). For each \(u \in U\), there exists an \(\omega_u \in \text{Con}(A)\) such that \(\lambda_A(\omega_u) = u\). If we denote by \(\Omega = \{\omega_u \mid u \in U\} \subseteq \text{Con}(A)\), then \(\lambda_A(\Omega) = U\) and \(|\Omega| = |U| \leq \kappa\), hence, by Lemma 2.11 and \((1^\circ)\), \(\text{Ann}(U) = \text{Ann}(\lambda_A(\Omega)) = \lambda_A(\text{Ann}(\Omega)) = \lambda_A((\alpha)) = (\lambda_A(\alpha))\) for some \(\alpha \in \mathcal{B}(\text{Con}(A))\), so that \(\lambda_A(\alpha) \in \mathcal{B}(\text{Con}(A)/\equiv_A)\).

**Proposition 2.15.** \(\text{Con}(A)\) is a Stone lattice iff \(\text{Con}(A)/\equiv_A\) is a Stone lattice.

Proof. If \(\text{Con}(A)\) is a Stone lattice, then, for all \(\theta \in \text{Con}(A)\), there exists an \(\alpha \in \mathcal{B}(\text{Con}(A))\) such that \(\text{Ann}(\theta) = (\alpha),\) so that \(\lambda_A(\alpha) \in \mathcal{B}(\text{Con}(A)/\equiv_A)\) and \(\text{Ann}(\lambda_A(\theta)) = \lambda_A(\text{Ann}(\theta)) = \lambda_A((\alpha)) = (\lambda_A(\alpha))\) by \((1^\circ)\) and Lemma 2.11 hence \(\text{Con}(A)/\equiv_A\) is a Stone lattice.

If \(\text{Con}(A)/\equiv_A\) is a Stone lattice, then, by \((1^\circ)\), Lemma 2.11 and Lemma 2.13 \((iv)\), for all \(\theta \in \text{Con}(A)\), there exists an \(\alpha \in \mathcal{B}(\text{Con}(A))\) such that \(\lambda_A(\text{Ann}(\theta)) = \lambda_A((\alpha)) = (\lambda_A(\alpha))\), hence \(\text{Ann}(\theta) = (\alpha)\), therefore \(\text{Con}(A)\) is a Stone lattice.

**Proposition 2.16.** For any nonzero cardinality \(\kappa\), the properties \((2)_{\kappa,\text{Con}(A)}\) and \((2)_{\kappa,\text{Con}(A)/\equiv_A}\) are equivalent.

Proof. By Proposition 2.15 and \((1^\circ)\), which ensures us that the Boolean algebras \(\mathcal{B}(\text{Con}(A))\) and \(\mathcal{B}(\text{Con}(A)/\equiv_A)\) are isomorphic.

**Proposition 2.17.** \(\text{Con}(A)\) is a strongly Stone lattice iff \(\text{Con}(A)/\equiv_A\) is a strongly Stone lattice.

Proof. Same as the proof of Proposition 2.15.

**Remark 2.18.** Since \(\text{Con}(A)/\equiv_A\) is a frame, from Corollary 2.4 we get that: the lattice \(\text{Con}(A)/\equiv_A\) is Stone iff it is strongly Stone.

**Corollary 2.19.** The lattice \(\text{Con}(A)\) is Stone iff \(\text{Con}(A)/\equiv_A\) is strongly Stone iff \(\text{Con}(A)/\equiv_A\) is Stone iff \(\text{Con}(A)/\equiv_A\) is strongly Stone.

Let us consider the following conditions:

\(\begin{align*}
(p\text{ann})_L \quad & \text{PAnn}(L) \text{ is a sublattice of } \text{Id}(L) \text{ such that the map } a \mapsto \text{Ann}_L(a) \text{ is a lattice anti–morphism from } L \text{ to } \text{PAnn}(L); \\
(p2\text{ann})_L \quad & \text{P2Ann}(L) \text{ is a sublattice of } \text{Id}(L) \text{ such that the map } a \mapsto \text{Ann}_L(\text{Ann}_L(a)) \text{ is a lattice morphism from } L \text{ to } \text{P2Ann}(L).
\end{align*}\)

**Lemma 2.21.**

(i) The map \(P \mapsto \lambda_A(P)\) from \(\text{PAnn}(\text{Con}(A))\) to \(\text{PAnn}(\text{Con}(A)/\equiv_A)\) is an order isomorphism.

(ii) For all \(\theta, \zeta \in \text{Con}(A)\): \(\text{Ann}(\theta \land \zeta) = \text{Ann}(\theta) \lor \text{Ann}(\zeta)\) iff \(\text{Ann}(\lambda_A(\theta) \land \lambda_A(\zeta)) = \text{Ann}(\lambda_A(\theta)) \lor \text{Ann}(\lambda_A(\zeta))\).

(iii) \((p\text{ann})_{\text{Con}(A)}\) is equivalent to \((p\text{ann})_{\text{Con}(A)/\equiv_A}\).

(iv) If the equivalent conditions from \((iii)\) are fulfilled, then the map from \((iii)\) is a lattice isomorphism.

Proof. By Lemma 2.11, this restriction of the direct image of \(\lambda_A\) takes \(\text{Ann}(\theta)\) to \(\text{Ann}(\lambda_A(\theta))\) for all \(\theta \in \text{Con}(A)\), thus it is well defined and surjective. By Lemma 2.13 \((ii)\), it is also injective. By Lemma 2.13 \((iv)\), for all \(\theta, \zeta \in \text{Con}(A)\), \(\text{Ann}(\theta) \subseteq \text{Ann}(\zeta)\) iff \(\lambda_A(\text{Ann}(\theta)) \subseteq \lambda_A(\text{Ann}(\zeta))\), therefore this bijection and its inverse preserve order. So this map is an order isomorphism, which is a restriction of the lattice morphism \(I \mapsto \lambda_A(I)\) from \(\text{Id}(\text{Con}(A))\) to \(\text{Id}(\text{Con}(A)/\equiv_A)\).

By \((i)\), the map \(P \mapsto \lambda_A(P)\) from \(\text{PAnn}(\text{Con}(A))\) to \(\text{PAnn}(\text{Con}(A)/\equiv_A)\) preserves all joins and does its inverse. Lemmas 2.10 and 2.11 show that \(\text{PAnn}(\text{Con}(A))\) and \(\text{PAnn}(\text{Con}(A)/\equiv_A)\) always are inferior subsemilattices of \(\text{Id}(\text{Con}(A))\), respectively \(\text{Id}(\text{Con}(A)/\equiv_A)\). From \((i)\) it follows that \(\text{PAnn}(\text{Con}(A))\) is closed with respect to the join from the lattice \(\text{Id}(\text{Con}(A))\) iff \(\text{PAnn}(\text{Con}(A)/\equiv_A)\) is closed with respect to the join from the lattice \(\text{Id}(\text{Con}(A)/\equiv_A)\), and, if they are closed with respect to the join, then, for all \(\theta, \zeta \in \text{Con}(A)\), \(\text{Ann}(\theta) \lor \text{Ann}(\zeta) = \text{Ann}(\theta \land \zeta)\) iff \(\lambda_A(\text{Ann}(\theta) \lor \text{Ann}(\zeta)) = \lambda_A(\text{Ann}(\theta \land \zeta))\), which in turn is equivalent to
Ann(λ_A(θ)) ∨ Ann(λ_A(ζ)) = Ann(λ_A(θ ∧ ζ)) by Lemma 2.11 and the fact that the direct image of λ_A preserves the joins of ideals. Therefore PAnn(Con(Α)) is a sublattice of Id(Con(Α)) if PAnn(Con(Α)/≡_Α) is a sublattice of Id(Con(Α)/≡_Α), with the expressions for the joins above.

Lemma 2.22. (i) The map Q ↦ λ_A(Q) from P2Ann(Con(Α)) to P2Ann(Con(Α)/≡_Α) is an order isomorphism.

(ii) (p2ann)_{Con(Α)} is equivalent to (p2ann)_{Con(Α)/≡_Α}.

(iii) If the equivalent conditions from (i) are fulfilled, then the map from (i) is a lattice isomorphism.

Proof. (i) By Lemma 2.11, this restriction of the direct image of λ_A takes Ann(Ann(θ)) to Ann(Ann(λ_A(θ))) for all θ ∈ Con(Α), thus it is well defined and surjective. By Lemma 2.13 (i), for any θ, ζ ∈ Con(Α), λ_A(Ann(Ann(θ))) = λ_A(Ann(Ann(ζ))) if Ann(Ann(θ)) = Ann(Ann(ζ)), so this map is also injective. By Lemma 2.21 (ii), and the surjectivity of the map (3), hence this bijection and its inverse preserve order, so this map is an order isomorphism, which is a restriction of the lattice morphism I ↦ λ_A(I) from Id(Con(Α)) to Id(Con(Α)/≡_Α).

(ii) and (iii) follow from (i) in the same way in which properties (i), (iii) and (iv) from Lemma 2.21 follow from (i). Another way to prove these facts is to notice that the equivalences in Lemma 2.21 (iii), hold if we replace the congruences by sets of congruences, and also apply Lemmas 2.29 and 2.10.

Proposition 2.23. For any nonzero cardinality κ, the properties (3)_{κ,Con(Α)} and (3)_{κ,Con(Α)/≡_Α} are equivalent.

Proof. By Lemma 2.22 and the fact that the map from Lemma 2.22 (i), composed with the map from (3)_{κ,Con(Α)} equals the map from (3)_{κ,Con(Α)/≡_Α} composed with the canonical surjective lattice morphism from Con(Α) to Con(Α)/≡_Α.

Proposition 2.24. Properties (iv)_{Con(Α)} and (iv)_{Con(Α)/≡_Α} are equivalent.

Proof. By Lemma 2.21 (iii), and the surjectivity of the map λ_A : Con(Α) → Con(Α)/≡_Α.

 Remark 2.25. Since Con(Α)/≡_Α is a frame, 2Ann(Con(Α)/≡_Α) ⊆ Ann(Con(Α)/≡_Α) = PAnn(Con(Α)/≡_Α), which means that the second part of condition (4)_{κ,Con(Α)/≡_Α} is fulfilled for any nonzero cardinality κ.

Lemma 2.26. Ann(Con(Α)) = PAnn(Con(Α)).

Proof. By Remark 2.25 and Lemma 2.11 for any Ω ⊆ Con(Α), λ_A(Ann(Ω)) = Ann(λ_A(Ω)) = Ann(λ_A(θ)) = λ_A(Ann(θ)) for some θ ∈ Con(Α), so that Ann(Ω) = Ann(θ) by Lemma 2.13 (iii).

Proposition 2.27. For any nonzero cardinality κ, the properties (iv)_{Con(Α)}, (4)_{κ,Con(Α)}, (4)_{κ,Con(Α)/≡_Α} and (4)_{Con(Α)} are equivalent.

Proof. By Lemma 2.20, 2Ann(Con(Α)) ⊆ Ann(Con(Α)) = PAnn(Con(Α)), which means that the second condition in (4)_{κ,Con(Α)} is fulfilled for any nonzero cardinality κ, so that conditions (iv)_{Con(Α)}, (4)_{κ,Con(Α)}, (4)_{κ,Con(Α)/≡_Α} and (4)_{Con(Α)} are equivalent.

Proposition 2.28. For any nonzero cardinality κ, the properties (5)_{κ,Con(Α)} and (5)_{κ,Con(Α)/≡_Α} are equivalent.

Proof. Assume that (5)_{κ,Con(Α)} is fulfilled and let U ⊆ Con(Α)/≡_Α with |U| ≤ κ. For each u ∈ U, there exists an ω_u ∈ Con(Α) such that λ_A(ω_u) = u. Let Ω = {ω_u | u ∈ U} ⊆ Con(Α). Then |Ω| = |U| ≤ κ and λ_A(Ω) = U, thus Ann(Ω) ∨ Ann(Ann(Ω)) = Con(Α) and hence, by Lemma 2.14, Ann(U) ∨ Ann(Ann(U)) = Ann(λ_A(Ω)) ∨ Ann(Ann(λ_A(Ω))) = λ_A(Ann(Ω)) ∨ Ann(Ann(Ω)) = λ_A(Ann(Ω) ∨ Ann(Ann(Ω))) = λ_A(Con(Α)) = Con(Α)/≡_Α, so the direct implication holds.

Now assume that (5)_{κ,Con(Α)/≡_Α} is fulfilled and let Ω ⊆ Con(Α) with |Ω| ≤ κ. Then λ_A(Ω) ⊆ Con(Α)/≡_Α and |λ_A(Ω)| ≤ |Ω| ≤ κ, hence Ann(λ_A(Ω)) ∨ Ann(Ann(λ_A(Ω))) = Con(Α)/≡_Α = λ_A(Con(Α)), so that λ_A(\nabla_A) ∈ Ann(Ann(λ_A(Ω))) and hence \nabla_A = \nabla_A(\nabla_A(\nabla_A)) = λ_A(\nabla_A(\nabla_A(\nabla_A))) = λ_A(\nabla_A(\nabla_A)) = Con(Α)/≡_Α, the converse implication holds, as well.
Theorem 2.29. For any nonzero cardinality $\kappa$ and any $h,j \in \mathbb{I} \cup \mathbb{J}$, conditions (iv)$_{\text{Con}(A)}$, $(h)_{\text{Con}(A)}$, $(i)_{<\infty,\text{Con}(A)}$ and $(j)_{\text{Con}(A)}$ are equivalent.

Proof. By Corollary 2.3 and Propositions 2.14, 2.16, 2.23, 2.27 and 2.28.

Open problem 2.30. Determine what kinds of bounded modular lattices are congruence lattices of semiprime algebras from semi-degenerate congruence-modular varieties. It will follow that the equivalences in the last statement in Corollary 2.3 hold for all those kinds of bounded modular lattices.

Corollary 2.31. Let $\kappa$ be a nonzero cardinality. Then: the equivalent conditions (iv)$_{\text{Con}(A)}, (1)_{\kappa,\text{Con}(A)}, \ldots, (\nu)_{\text{Con}(A)}$ are isomorphic lattices, therefore the reticulation of $A$ is homeomorphic to the prime spectrum of congruences of $A$, with respect to the Stone topologies. It is well known that, if two bounded distributive lattices have homeomorphic prime spectra of ideals, then they are isomorphic lattices, therefore the reticulation of $A$ is unique up to a lattice isomorphism (or dual isomorphism, if we also consider the variant of the reticulation with the property above for filters). This is our construction of the reticulation of $A$ from [9]: $\text{L}(A) = \mathcal{K}(A)/\equiv_A$, where $\mathcal{K}(A)$ is the set of the compact elements of the lattice $\text{Con}(A)$, thus $\text{L}(A) = \text{Con}(A)/\equiv_A$ if the lattice $\text{Con}(A)$ is compact, in particular if $\text{Con}(A)$ is a finite lattice, in particular if $A$ is finite. Following [9], I am denoting the restriction $\lambda_A|_{\mathcal{K}(A)}: \mathcal{K}(A) \to \text{L}(A)$ of the canonical surjective lattice morphism $\lambda_A: \text{Con}(A) \to \text{Con}(A)/\equiv_A$ by $\lambda_A$, as well. $\text{L}(A)$ is a bounded sublattice of $\text{Con}(A)/\equiv_A$, thus a bounded distributive lattice, hence, from Corollaries 2.23 and 2.24 and the obvious fact that a complete sublattice of a frame is a frame, we obtain:

Corollary 2.32. • Conditions (1)$_{\text{L}(A)}, \ldots, (\nu)_{\text{L}(A)}$ are equivalent. For any nonzero cardinality $\kappa$, $(1)_{\kappa,\text{L}(A)}, \ldots, (\nu)_{\kappa,\text{L}(A)}$ are equivalent. For any finite nonzero cardinality $\kappa$, $(1)_{\kappa,\text{L}(A)}, \ldots, (\nu)_{\kappa,\text{L}(A)}$ are equivalent.

• If $\text{L}(A)$ is a frame, in particular if $\text{L}(A)$ is a complete sublattice of $\text{Con}(A)/\equiv_A$, in particular if $\text{L}(A) = \text{Con}(A)/\equiv_A$, in particular if $\text{Con}(A)$ is a compact lattice, then, for any nonzero cardinality $\kappa$ and any $h,i,j \in \mathbb{I} \cup \mathbb{J}$, $(i)_{\text{L}(A)}$, $(h)_{\kappa,\text{L}(A)}$, $(i)_{<\infty,\text{L}(A)}$ and $(j)_{\text{L}(A)}$ are equivalent, so, in particular, (1)$_{\text{L}(A)}$ is equivalent to (2)$_{<\infty,\text{L}(A)}$, which means that: $\text{L}(A)$ is a Stone lattice iff $\text{L}(A)$ is a strongly Stone lattice.

Corollary 2.33. If $\text{L}(A) = \text{Con}(A)/\equiv_A$, in particular if $\text{Con}(A)$ is a compact lattice, in particular if $\text{Con}(A)$ is finite, in particular if $A$ is finite, then: $\text{Con}(A)$ is Stone iff $\text{Con}(A)$ is strongly Stone iff $\text{L}(A)$ is Stone iff $\text{L}(A)$ is strongly Stone.

Remark 2.34. Let $M$ be a bounded sublattice of $L$ and $U \subseteq L$. Then it is straightforward that $\text{Ann}_L(U) \cap M \subseteq \text{Ann}_M(U \cap M)$, $(U \cap M)_L \cap M = (U \cap M)_M$ and, if $L$ is distributive, then $(U|_L \cap (V|_L = (U \cap V|_L).

Lemma 2.35. If $L$ is a bounded distributive lattice, $M$ is a bounded sublattice of $L$ and $U \subseteq M$, such that $\text{Ann}_L(U) \cap \text{Ann}_L(M(U)) = L$, then $\text{Ann}_M(U) \cap \text{Ann}_M(M(U)) = M$.

Proof. If $U$ is as in the hypothesis, then $\text{Ann}_M(U) \cup \text{Ann}_M(M(U)) \supseteq (\text{Ann}_L(U) \cap M) \cup (\text{Ann}_L(M(U)) \cap M) \supseteq (\text{Ann}_L(U) \cap M) \cap (\text{Ann}_L(M(U)) \cap M) = (\text{Ann}_L(U) \cap \text{Ann}_L(M(U))) \cap M = (\text{Ann}_L(U) \cup \text{Ann}_L(M(U))) \cap M = (U \cap M) \cap M = L$.

Lemma 2.36. If $L$ is a bounded distributive lattice and $M$ is a bounded sublattice of $L$, then, for any nonzero cardinality $\kappa$, $(5)_{\kappa,L}$ implies $(5)_{\kappa,M}$.

Proof. Assume that $(5)_{\kappa,L}$ is fulfilled, and let $U \subseteq M \subseteq L$ with $|U| \leq \kappa$, so that $\text{Ann}_L(U) \cap \text{Ann}_L(M(U)) = L$, hence $\text{Ann}_M(U) \cap \text{Ann}_M(M(U)) = M$ by Lemma 2.35.
Proposition 2.37. If \( L \) is a bounded distributive lattice, \( M \) is a bounded sublattice of \( L \), \( \kappa \) is a nonzero cardinality and the equivalent conditions \((1)_{\kappa,L}, \ldots, (5)_{\kappa,L}\) are fulfilled, then the equivalent conditions \((1)_{\kappa,M}, \ldots, (5)_{\kappa,M}\) are fulfilled.

Proof. By Corollary 2.3 and Lemma 2.36.

Corollary 2.38. For any nonzero cardinality \( \kappa \), if the equivalent conditions \((iv)_{\text{Con}(A)}, (1)_{\kappa, \text{Con}(A)}, \ldots, (5)_{\kappa, \text{Con}(A)}\), \((1)_{\kappa, \mathcal{L}(A)}, (5)_{\kappa, \mathcal{L}(A)}\) are fulfilled, then conditions \((iv)_{\mathcal{L}(A)}, (1)_{\kappa, \mathcal{L}(A)}, \ldots, (5)_{\kappa, \mathcal{L}(A)}\) (not necessarily equivalent) are fulfilled.

Remark 2.39. Obviously, if \( L \) and \( M \) are isomorphic bounded lattices, then \((iv)_L\) is equivalent to \((iv)_M\), for any \( i \in \{1, 5\}, (i)_L \) is equivalent to \((i)_M\), \((i)_{<\kappa,L}\) is equivalent to \((i)_{<\kappa,M}\) and, for any nonzero cardinality \( \kappa \), \((i)_{\kappa,L}\) is equivalent to \((i)_{\kappa,M}\).

Remark 2.40. If \( L \) and \( M \) are bounded distributive lattices, then it is immediate that: \( B(L \times M) = B(L) \times B(M) \) and, for all \( a \in L \) and all \( b \in M \), \( \text{Ann}_{L \times M}((a, b)) = \{(x, y) \mid x \in \text{Ann}_L(a), y \in \text{Ann}_M(b)\} = \text{Ann}_L(a) \times \text{Ann}_M(b) \); since, for any \( x \in L \) and \( y \in M \), \((x, y) \leq (a, b)\) means that \( x \leq a \) and \( y \leq b \), it follows that \( ((a, b))_{L \times M} = \{(x, y) \mid x \in (a)_L, y \in (b)_M\} = (a)_L \times (b)_M \). And, for any set \( I \) and any families \((a_i)_{i \in I} \subseteq L \) and \((b_i)_{i \in I} \subseteq M \), there exists \( \bigvee_{i \in I} (a_i, b_i) \) in \( L \times M \) iff there exist \( a_i \) in \( L \) and \( b_i \) in \( M \), and, if these joins exist, then \( \bigvee_{i \in I} (a_i, b_i) = \left( \bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i \right) \); the same goes for arbitrary meets. From this, it is easy to obtain that, for any nonzero cardinality \( \kappa \), \((2)_{\kappa, L \times M}\) is fulfilled iff both \((2)_{\kappa,L}\) and \((2)_{\kappa,M}\) are fulfilled, and hence, by Corollary 2.3.

Proposition 2.41. For any nonzero cardinality \( \kappa \): condition \((iv)_{L \times M}\), respectively the equivalent conditions \((1)_{\kappa,L \times M}, \ldots, (5)_{\kappa,L \times M}\), respectively \((1)_{<\kappa,L \times M}, \ldots, (5)_{<\kappa,L \times M}\), respectively \((1)_{L \times M}, \ldots, (5)_{L \times M}\), are fulfilled iff condition \((iv)_L\), respectively the equivalent conditions \((1)_{\kappa,L}, \ldots, (5)_{\kappa,L}\), respectively \((1)_{<\kappa,L}, \ldots, (5)_{<\kappa,L}\), respectively \((1)_{L}, \ldots, (5)_{L}\), as well as condition \((iv)_M\), respectively the equivalent conditions \((1)_{\kappa,M}, \ldots, (5)_{\kappa,M}\), respectively \((1)_{<\kappa,M}, \ldots, (5)_{<\kappa,M}\), respectively \((1)_{M}, \ldots, (5)_{M}\), are fulfilled.

Throughout the rest of this section, \( B \) shall be a semiprime algebra from \( C \). Then:

Remark 2.42. \( \text{Con}(A \times B) \) is isomorphic to \( \text{Con}(A) \times \text{Con}(B) \) and, as we have proven in [9], \( A \times B \) is semiprime, \( \text{Con}(A \times B)/\equiv_{A \times B} \) is isomorphic to \( \text{Con}(A)/\equiv_A \times \text{Con}(B)/\equiv_B \) and \( \mathcal{L}(A \times B) \) is isomorphic to \( \mathcal{L}(A) \times \mathcal{L}(B) \).

Corollary 2.43.\( \bullet \) For any nonzero cardinality \( \kappa \): the equivalent conditions \((iv)_{\text{Con}(A \times B), \equiv_{A \times B}^*}, (1)_{\kappa, \text{Con}(A \times B), \equiv_{A \times B}^*}, \ldots, (5)_{\kappa, \text{Con}(A \times B), \equiv_{A \times B}^*}\), \((1)_{\kappa, \mathcal{L}(A \times B), \equiv_{A \times B}^*}, \ldots, (5)_{\kappa, \mathcal{L}(A \times B), \equiv_{A \times B}^*}\) are fulfilled iff the equivalent conditions \((iv)_{\text{Con}(A), \equiv_A^*}, (1)_{\kappa, \text{Con}(A), \equiv_A^*}, \ldots, (5)_{\kappa, \text{Con}(A), \equiv_A^*}\), \((1)_{\kappa, \mathcal{L}(A), \equiv_A^*}, \ldots, (5)_{\kappa, \mathcal{L}(A), \equiv_A^*}\) are equivalent conditions \((1)_{\kappa, \mathcal{L}(A) \times \mathcal{L}(B), \equiv_{A \times B}^*}, \ldots, (5)_{\kappa, \mathcal{L}(A) \times \mathcal{L}(B), \equiv_{A \times B}^*}\)

\( \bullet \) For any nonzero cardinality \( \kappa \): condition \((iv)_{\mathcal{L}(A \times B)}\), respectively the equivalent conditions \((1)_{\kappa, \mathcal{L}(A \times B)}, \ldots, (5)_{\kappa, \mathcal{L}(A \times B)}\), respectively \((1)_{<\kappa, \mathcal{L}(A \times B)}, \ldots, (5)_{<\kappa, \mathcal{L}(A \times B)}\), respectively \((1)_{\mathcal{L}(A \times B)}, \ldots, (5)_{\mathcal{L}(A \times B)}\), as well as condition \((iv)_{\mathcal{L}(B)}\), respectively the equivalent conditions \((1)_{\kappa, \mathcal{L}(B) \times \mathcal{L}(B)}, \ldots, (5)_{\kappa, \mathcal{L}(B) \times \mathcal{L}(B)}\), respectively \((1)_{<\kappa, \mathcal{L}(B) \times \mathcal{L}(B)}, \ldots, (5)_{<\kappa, \mathcal{L}(B) \times \mathcal{L}(B)}\), respectively \((1)_{\mathcal{L}(B) \times \mathcal{L}(B)}, \ldots, (5)_{\mathcal{L}(B) \times \mathcal{L}(B)}\), are fulfilled.

Proof. By Remark 2.42, Proposition 2.41 and Corollary 2.3.

Corollary 2.44. For any nonzero cardinality \( \kappa \): the equivalent conditions \((iv)_{\text{Con}(A \times B)}, (1)_{\kappa, \text{Con}(A \times B)}, \ldots, (5)_{\kappa, \text{Con}(A \times B)}, (1)_{\kappa, \mathcal{L}(A \times B)}, \ldots, (5)_{\kappa, \mathcal{L}(A \times B)}\) are fulfilled iff the equivalent condition \((iv)_{\text{Con}(A)}, (1)_{\kappa, \text{Con}(A)}, \ldots, (5)_{\kappa, \text{Con}(A)}, (1)_{\kappa, \mathcal{L}(A)}, \ldots, (5)_{\kappa, \mathcal{L}(A)}\) as well as the equivalent conditions \((iv)_{\text{Con}(B)}, (1)_{\kappa, \text{Con}(B)}, \ldots, (5)_{\kappa, \text{Con}(B)}, (1)_{\kappa, \mathcal{L}(B)}, \ldots, (5)_{\kappa, \mathcal{L}(B)}\), are fulfilled.

Proof. By Corollaries 2.43 and 2.33.
3 Transferring Davey’s Theorem to Commutative Unitary Rings

Let \((T, \vee, \wedge, \cdot, \to, 0, 1)\) be a residuated lattice, which means that \((T, \vee, \wedge, 0, 1)\) is a bounded lattice, that we shall denote by \(S\), \((T, \cdot, 1)\) is a commutative monoid and \(\to\) is a binary operation on \(T\) which fulfills the law of \textit{residuation}: for all \(a, b, c \in T\), \(a \leq b \to c\) if \(a \circ b \leq c\). Let us denote by \(S’\) the dual of the bounded lattice \(S\). See more about residuated lattices in \([7, 13, 17]\).

Residuated lattices form a semi–degenerate congruence–distributive variety, hence they are semiprime and thus their congruence lattices fulfill Theorem 2.29. Let us see that, like residuated lattices, commutative unitary rings fulfill Davey’s Theorem for elements, too.

Theorem 3.1. \([15, \text{Theorem } 5.2.6], [16, \text{Theorem } 3.13]\) \textit{Conditions \((1)_{m,S’}, (2)_{m,S’}, (3)_{m,S’}, (4)_{m,S’}\) and \((5)_{m,S’}\) are equivalent.}

In \([15, 16]\), I have proven Theorem 3.1 by transferring the dual of Theorem 2.1 from bounded distributive lattices to residuated lattices through the reticulation functor for residuated lattices. With the notation for the reticulation from Section 2, the construction from \([9]\) identifies \(L(T)\), up to a lattice isomorphism, as the bounded lattice \(PFilt(T)\) of the principal filters of \(T\), whose dual is a frame if \(T\) is complete. Since, for any nonzero cardinality \(\kappa\) and any \(i \in \{\infty, 5\}\), \((\iota\gamma)^{i}_{\kappa,S'}\) is equivalent to \((\iota\gamma)^{i}_{\kappa,D}\), where \(D\) is the dual of \(PFilt(T)\), according to \([15, 16]\) and the above, it follows that:

Theorem 3.2. If \(T\) is a complete residuated lattice and \(S’\) is the dual of the underlying bounded lattice of \(T\), then, for any nonzero cardinality \(\kappa\) and any \(h, i, j \in \{\infty, 5\}\), conditions \((iv)^{i}_{S'}, (h)^{i}_{\kappa,S'}, (i)^{\infty}_{\kappa,S'}\) and \((j)^{S'}\) are equivalent; in particular, \(T\) is co–Stone if \(T\) is strongly co–Stone.

Remark 3.3. If the commutator of \(A\) is associative, as it is, for example, in any commutative unitary ring, then \(\text{Con}(A)\) is a residuated lattice, in which \([\cdot, \cdot]_{A}\) is the multiplication, according to \([9]\) (see also \([4]\)), thus, from the above and the fact that \(\text{Con}(A)\) is a complete lattice, it follows that both the lattice \(\text{Con}(A)\) and its dual fulfill the equivalences in Theorem 2.29. Note that the commutator is not always associative \([8]\).

Commutative unitary rings form a semi–degenerate congruence–modular equational class, thus their congruence lattices fulfill Theorem 2.29. Let us see that, similarly to what happens in (complete) residuated lattices, they also fulfill an analogue of Theorem 2.29 for elements instead of congruences.

Commutative unitary rings form a semi–degenerate congruence–modular variety, thus their congruence lattices fulfill Theorem 2.29. Let us see that, like residuated lattices, commutative unitary rings fulfill Davey’s Theorem for elements, too.

Let \(\langle R, +, \cdot, 0, 1 \rangle\) be a commutative unitary ring, \((\text{Id}(R), \vee = +, \cap, \{0\}, R)\) be the bounded modular lattice of the ideals of \(R\), \(\text{Spec}_{R}(R)\) the set of the prime ideals of \(R\) and \(\nu_{R} : \text{Id}(R) \to \text{Con}(R)\) the canonical lattice isomorphism: for all \(I \in \text{Id}(R)\), \(\nu_{R}(I) = \{(x, y) \in I^{2} \mid x - y \in I\}\). Note that, since \(\nu_{R}\) is an order isomorphism, it preserves arbitrary intersections. Recall that \(\text{Spec}_{R}(R) = \{P \in \text{Id}(R) \setminus \{R\} \mid (\forall I, J \in \text{Id}(R)) (I \cdot J \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P)\} \text{ and } \nu_{R}(I), \nu_{R}(J)) = \nu_{R}(I \cdot J)\) for all \(I, J \in \text{Id}(R)\), from which it is easy to deduce that \(\nu_{R}(\text{Spec}_{R}(R)) = \text{Spec}(R)\). For every \(U \subseteq R\), \(\langle U \rangle_{R}\) shall be the ideal of \(R\) generated by \(U\), so, for each \(x \in R\), \(\langle \{x\} \rangle_{R} = xR\). Let \(\text{PId}(R)\) and \(\text{FGId}(R)\) be the set of the principal ideals of \(R\) and that of the finitely generated ideals of \(R\), respectively. It is straightforward that, for all \(x, a, b \in R\), \(\nu_{R}(xR) = C_{G_{R}}(a, b, \text{Id}(R)R)\) and \(\nu_{R}(C_{G_{R}}(a, b)) = C_{G_{R}}(a, b, \text{Id}(R)\{0\})\), hence \(\nu_{R}(\text{PId}(R)) = \text{PCon}(R)\) and thus \(\nu_{R}(\text{FGId}(R)) = K(R)\). Recall that, if we denote by \(E(R)\) the set of the idempotents of \(R\), then \(E(R), \vee, \wedge = \cdot, \neg, 0, 1\) is a Boolean algebra, where, for every \(e, f \in E(R)\), \(\neg e = 1 - e\) and \(e \wedge f = (\neg e \wedge \neg f) = 1 - (1 - e) \cdot (1 - f)\).

Let \(\mathcal{L}(R) = (K(R)\{0\}, \vee, \wedge, 0 = \lambda_{R}(\Delta_{R}) = \Delta_{R}R, 1 = \lambda_{R}(\overline{R}) = \overline{R}R)\) be the reticulation of \(R\), as constructed in \([9]\) (see the notations in Section 2). Let \(R^{*} : R \to R^{*}\) be the reticulation of \(R\) and the reticulation function, respectively, as constructed in \([2, 3]\) (see also \([14, 18]\)): if we denote, for each \(I \in \text{Id}(R)\), by \(\sqrt{I} = \{P \in \text{Spec}_{R}(R) \mid I \subseteq P\}\) the radical of \(I\), and by \(\sim_{\text{Id}(R)} = \{I, J \in \text{Id}(R)^{2} \mid \sqrt{I} = \sqrt{J}\}\), then \(\sim_{R}\) is a congruence of the lattice \(\text{Id}(R)\), so \(\text{Id}(R)\sim_{R}\) is a bounded lattice and the canonical surjection \(\nu_{R} : \text{Id}(R) \to \text{Id}(R)\sim_{R}\) is a lattice morphism that fulfills: \(\nu_{R}(I \cdot J) = \nu_{R}(I \cap J)\) for all \(I, J \in \text{Id}(R)\), therefore the bounded lattice \(\text{PId}(R) = \{\text{Id}(R)/\sim_{R} \cap \{0\}\} = \{0\}/\sim_{R}\) is distributive, since multiplication is distributive w.r.t. the join in \(\text{Id}(R)\). As shown in \([2, 3]\), \(R^{*}\) has the prime spectrum of ideals.
homeomorphic to the prime spectrum of congruences of \( R \), w.r.t. the Stone topologies, hence \( R^* \) is a reticulation of \( R \). The reticulation function \( \mu_R : R \to R^* \) is defined, in \([2]\), by \( \mu_R(x) = \nu_R(xR) = xR/\sim_R \) for all \( x \in R \).

For any \( U \subseteq R \), we denote the annihilator of \( U \) by \( \text{Ann}_R(U) \); so \( \text{Ann}_R(U) = \{ x \in R \mid (\forall u \in U) (u \cdot x = 0) \} \in \text{Id}(R) \). For any \( a \in R \), we also denote \( \text{Ann}_R(a) = \text{Ann}_R(\{a\}) = \{ x \in R \mid u \cdot a = 0 \} \). Let us denote by \( \text{Ann}(R) = \{ \text{Ann}_R(U) \mid U \subseteq R \} \) and by \( 2\text{Ann}(R) = \{ \text{Ann}_R(\text{Ann}_R(U)) \mid U \subseteq R \} \).

\[ \text{Lemma 3.5.} \]

- For any \( U \subseteq R \), \( \text{Ann}_R(U) = \text{Ann}_R(\langle U \rangle_R) \).
- For any \( V \subseteq \text{Id}(R) \), \( \bigwedge_{I \in V} \text{Ann}_R(I) = \text{Ann}_R(\bigvee_{I \in V} I) \).

\[ \text{Proof.} \] Clearly, for any \( W \subseteq V \subseteq R \), we have \( \text{Ann}_R(W) \subseteq \text{Ann}_R(V) \), hence \( \text{Ann}_R(\langle U \rangle_R) \subseteq \text{Ann}_R(U) \). But the converse inclusion holds, as well, since, given any \( a \in \langle U \rangle_R \) and any \( x \in \text{Ann}_R(U) \), we have \( a = a_1 u_1 + \ldots + a_n u_n \) for some \( n \in \mathbb{N}^* \), \( a_1, \ldots, a_n \in R \) and \( u_1, \ldots, u_n \in U \), so that \( x \cdot u_1 = \ldots = x \cdot u_n = 0 \), therefore \( x \cdot u = 0 \), so \( x \in \text{Ann}_R(\langle U \rangle_R) \). Thus \( \text{Ann}_R(U) = \text{Ann}_R(\langle U \rangle_R) \). Hence \( \bigwedge_{I \in V} \text{Ann}_R(I) = \bigwedge_{a \in I} \text{Ann}_R(a) = \bigcap_{a \in \bigcup_{I \in V} I} \text{Ann}_R(a) \).

Regarding the results from \([2]\) I am using, note that, since \( R \) is commutative, it follows that \( R \) is quasicommutative, thus, by \([2] \text{ Theorem 3}\), \( R \) fulfills condition (*). It also follows that:

\[ \text{Lemma 3.6.} \] \([2] \text{ Lemma, p. 1861}\] For all \( I \in \text{Id}(R) \), there exists a \( K \in \text{FGId}(R) \) such that \( K \subseteq I \) and \( \sqrt{K} = \sqrt{I} \).

\[ \text{Proposition 3.7.} \]

(i) \( R^* = \text{Id}(R)/\sim_R = \text{FGId}(R)/\sim_R \);

(ii) \( \varphi_R : R^* \to \text{Con}(R)/\equiv_R \), for all \( I \in \text{Id}(R) \), \( \varphi_R(I/\sim_R) = \nu_{\gamma}(I)/\equiv_R \), is a lattice isomorphism;

(iii) \( \text{Con}(R)/\equiv_R = K(R)/\equiv_R = \mathcal{L}(R) \).

\[ \text{Proof.} \] \([3] \text{ Lemma 3.5}\) says that, for all \( I \in \text{Id}(R) \), there exists a \( K \in \text{FGId}(R) \) such that \( K \subseteq I \) and \( K/\sim_R = I/\sim_R \), therefore \( R^* = \text{Id}(R)/\sim_R = \text{FGId}(R)/\sim_R \).

\([4]\) The map \( \varphi_R \) is defined by: \( \varphi_R(\nu_{\gamma}(I)) = \lambda_R(\nu_{\gamma}(I)) \) for all \( I \in \text{Id}(R) \), hence it makes the following diagram commutative, where the second equality in the bottom row will follow shortly:

\[
\begin{array}{ccc}
\text{FGId}(R) & \subseteq & \text{Id}(R) \\
\nu_R | \text{FGId}(R) & \downarrow & \downarrow \nu_R \\
\text{FGId}(R)/\sim_R = \text{Id}(R)/\sim_R = R^* & \xrightarrow{\varphi_R} & \text{Con}(R)/\equiv_R = K(R)/\equiv_R = \mathcal{L}(R) \\
\varphi_R & \downarrow & \downarrow \lambda_R \\
\nu_{\gamma} & \downarrow & \downarrow \lambda_R \\
\varphi_R & \downarrow & \downarrow \lambda_R \\
\end{array}
\]

For all \( I, J \in \text{Id}(R) \), we have: \( I/\sim_R = J/\sim_R \) if \( \sqrt{I} = \sqrt{J} \) if \( \nu_{\gamma}(\sqrt{I}) = \nu_{\gamma}(\sqrt{J}) \) if \( \nu_{\gamma}(\{ P \in \text{Spec}_\text{Id}(R) \mid I \subseteq P \}) = \nu_{\gamma}(\{ P \in \text{Spec}_\text{Id}(R) \mid J \subseteq P \}) \) if \( \nu_{\gamma}(\{ P \in \text{Spec}_\text{Id}(R) \mid I \subseteq P \}) = \nu_{\gamma}(\{ P \in \text{Spec}_\text{Id}(R) \mid J \subseteq P \}) \)

\[ \text{Remark 3.8.} \] Since \( \mathcal{B}(\text{Con}(R)) \) is a Boolean sublattice of \( \text{Con}(R) \) and \( \text{Con}(R) \) is isomorphic to \( \text{Id}(R) \), it follows that \( \mathcal{B}(\text{Id}(R)) \) is a Boolean sublattice of \( \text{Id}(R) \).
Corollary 3.9. \( R^* \) and \( \mathcal{L}(R) \) are frames and \( \varphi_R : \text{FGId}(R)/\sim_R = \text{Id}(R)/\sim_R = R^* \to \text{Con}(R)/\equiv_R = \mathcal{K}(R)/\equiv_R = \mathcal{L}(R) \) is a lattice (and thus a frame) isomorphism, and, if we take, in the diagram above in the class of bounded lattices, the restrictions of the bounded lattice morphisms to the Boolean centers, then we get the following commutative diagram in the class of Boolean algebras, in which \( \psi_R : \text{Con}(R)/\equiv_R \to \mathcal{B}(\mathcal{L}(R)) \)
\( \varphi_R : \mathcal{B}(R^*) \to \mathcal{B}(\mathcal{L}(R)) \) are Boolean isomorphisms:

\[
\begin{array}{ccc}
\text{B(Con}(R)) & \xrightarrow{\lambda_R} & \text{B(L}(R)) \\
\text{B(Id}(R)) & \xrightarrow{\iota_R} & \text{B(Con}(R)) \\
\text{B(R^*)} & \xrightarrow{\varphi_R} & \text{B(L}(R))
\end{array}
\]

Proof. By Remak 3.8 Proposition 3.7 and the fact that \( \text{Con}(R)/\equiv_R \) is a frame, along with (1°).

So the bounded distributive lattices \( \mathcal{L}(R) \) and \( R^* \) are isomorphic, which was to be expected, since they have homeomorphic prime spectra of ideals w.r.t. the Stone topologies.

\( R \) is called a \textit{Baer ring} iff, for any \( a \in R \), there exists an \( e \in E(R) \) such that \( \text{Ann}_R(a) = eR \). By analogy to the case of bounded lattices, we shall call \( R \) a \textit{strongly Baer ring} iff, for any \( U \subseteq R \), there exists an \( e \in E(R) \) such that \( \text{Ann}_R(U) = eR \). Following [2, 3], we call \( R \) a \textit{semiprime ring} iff \( \sqrt{\{0\}} = \{0\} \), which is equivalent to \( \Delta_R = \iota_R(\{0\}) = \iota_R(\bigcap_{P \in \text{Spec}_R(R)} P) = \bigcap_{P \in \text{Spec}_R(R)} \iota_R(P) = \bigcap_{\phi \in \text{Spec}(R)} \phi = \rho_R(\Delta_R) \), which means that \( R \) is a semiprime algebra.

Remark 3.10. [2, 3] Let \( I \in \text{Id}(R) \). Then: \( I/\sim_R = \top \) iff \( \nu_R(I) = \nu_R(R) \) iff \( \sqrt{\{0\}} = \sqrt{\{0\}} = R \) iff no prime ideal of \( R \) includes \( I \) iff \( I = R \). If \( R \) is semiprime, then we also have the following equivalences: \( I/\sim_R = \bot \) iff \( \nu_R(I) = \nu_R(\{0\}) \) iff \( I \subseteq \sqrt{\{0\}} = \{0\} \) iff \( I = \{0\} \).

Remark 3.11. By Corollary 3.9 and Remarks 2.8 and 2.7 \( \text{Ann}(\text{Id}(R)) \subseteq \text{Id}(\text{Id}(R)) \), and the canonical surjective lattice morphism \( \nu_R : \text{Id}(R) \to R^* = \text{Id}(R)/\sim_R \) preserves arbitrary intersections and joins and fulfills: \( (I)_{\text{Id}(R)}/\sim_R = (I/\sim_R)_{R^*} \) for any \( I \in \text{Id}(R) \).

Corollary 3.12. For any nonzero cardinality \( \kappa \) and any \( h, i, j \in \{0, 1\} \), \( (iv)_{R^*} \), \( (h)_{R^*} \), \( (i)_{\kappa, R^*} \) and \( (j)_{R^*} \) are equivalent.

Proof. By Corollaries 2.8 and 3.9.

Let us consider the following conditions on \( R \), where \( \kappa \) is an arbitrary nonzero cardinality:

1. \( \kappa, R \) for each \( U \subseteq R \) with \( |U| \leq \kappa \), there exists an \( e \in E(R) \) such that \( \text{Ann}_R(U) = eR \);
2. \( \kappa, U \) for each finite \( U \subseteq R \), there exists an \( e \in E(R) \) such that \( \text{Ann}_R(U) = eR \);
3. \( R \) a strongly Baer ring;
4. \( \kappa, R \) a Baer ring and \( E(R) \) is a \( \kappa \)-complete Boolean algebra;
5. \( \kappa, R \) a Baer ring and \( E(R) \) is a Boolean algebra;
6. \( R \) a Baer ring and \( E(R) \) is a complete Boolean algebra;
7. \( \kappa, R \) \( 2\text{Ann}(R) \) is a \( \kappa \)-complete Boolean sublattice of \( \text{Id}(R) \) such that \( I \mapsto \text{Ann}_R(\text{Ann}_R(I)) \) is a lattice morphism from \( \text{Id}(R) \) to \( 2\text{Ann}(R) \);
8. \( \kappa, R \) \( 2\text{Ann}(R) \) is a Boolean sublattice of \( \text{Id}(R) \) such that \( I \mapsto \text{Ann}_R(\text{Ann}_R(I)) \) is a lattice morphism from \( \text{Id}(R) \) to \( 2\text{Ann}(R) \);
9. \( R \) a complete Boolean sublattice of \( \text{Id}(R) \) such that \( I \mapsto \text{Ann}_R(\text{Ann}_R(I)) \) is a lattice morphism from \( \text{Id}(R) \) to \( 2\text{Ann}(R) \);
10. \( \kappa, R \) for all \( I, J \in \text{Id}(R) \), \( \text{Ann}_R(I \cap J) = \text{Ann}_R(I) \lor \text{Ann}_R(J) \), and, for each \( U \subseteq R \) with \( |U| \leq \kappa \), there exists a finite subset \( S \subseteq R \) such that \( \text{Ann}_R(\text{Ann}_R(U)) = \text{Ann}_R(S) \);
11. \( \kappa, R \) for all \( I, J \in \text{Id}(R) \), \( \text{Ann}_R(I \cap J) = \text{Ann}_R(I) \lor \text{Ann}_R(J) \), and, for each finite \( U \subseteq R \), there exists a finite subset \( S \subseteq R \) such that \( \text{Ann}_R(\text{Ann}_R(U)) = \text{Ann}_R(S) \);
12. \( R \) for all \( I, J \in \text{Id}(R) \), \( \text{Ann}_R(I \cap J) = \text{Ann}_R(I) \lor \text{Ann}_R(J) \), and, for each \( U \subseteq R \), there exists a finite subset \( S \subseteq R \) such that \( \text{Ann}_R(\text{Ann}_R(U)) = \text{Ann}_R(S) \);
13. \( (iv)_{R} \) for all \( I, J \in \text{Id}(R) \), \( \text{Ann}_R(I \cap J) = \text{Ann}_R(I) \lor \text{Ann}_R(J) \).
(5) $\kappa_R$ for each $U \subseteq R$ with $|U| \leq \kappa$, $\text{Ann}_R(U) \vee \text{Ann}_R(\text{Ann}_R(U)) = R$;
(3) $\kappa_{\leq \kappa} R$ for each finite $U \subseteq R$, $\text{Ann}_R(U) \vee \text{Ann}_R(\text{Ann}_R(U)) = R$;
(5) $R$ for each $U \subseteq R$, $\text{Ann}_R(U) \vee \text{Ann}_R(\text{Ann}_R(U)) = R$.

Remark 3.13. Clearly, the properties from the second paragraph of Remark 2.2 hold for $R$ instead of $L$, too. For the motivation behind the change of the second part of conditions (4) $\kappa_L$, (4) $\kappa_{\leq \kappa} L$ and (4) $L$ into what we see above in conditions (4) $\kappa_R$, (4) $\kappa_{\leq \kappa} R$ and (4) $R$, compare Lemma 2.20 above to Lemma 3.20 below.

Throughout the rest of this paper, $R$ shall be semiprime.

Lemma 3.14. (i) [2] Lemma, p. 1863] $\mu_R \mid_{E(R)}: E(R) \to \mathcal{B}(R^*)$ is a Boolean isomorphism.

(ii) [1N Theorem 2.6] $R$ is a Baer ring iff $R^*$ is a Stone lattice.

Proposition 3.15. For any nonzero cardinality $\kappa$, conditions (2) $\kappa_R$ and (2) $\kappa_{\kappa R}$ are equivalent.

Proof. By Lemma 3.14 (i) and (ii).

Lemma 3.16. $\nu_R \mid_{B(\text{Id}(R))}: B(\text{Id}(R)) \to B(\text{Id}(R))/\sim_R = B(\text{Id}(R))/\sim_R = B(R^*)$ is a Boolean isomorphism.

Proof. By (1) and Corollary 3.9.

Lemma 3.17. For any $V, W \subseteq \text{Id}(R), I \in \text{Id}(R)$ and $D \in B(\text{Id}(R))$:

(i) $\text{Ann}_R(I/\sim_R) = \text{Ann}_R(I)/\sim_R$ and $\text{Ann}_R^*(W/\sim_R) = \text{Ann}_R^*(W)/\sim_R$;

(ii) $\text{Ann}_R^*(V/\sim_R) = \text{Ann}_R^*(V)/\sim_R$ iff $\text{Ann}_R^*(D/\sim_R) = \text{Ann}_R^*(D)/\sim_R$ iff $\text{Ann}_R^*(V) = \text{Ann}_R^*(D)$;

(iii) $(D/\sim_R) = \text{Ann}_R^*(W/\sim_R)$ iff $(D/\sim_R) = \text{Ann}_R^*(W)/\sim_R$ iff $(D/\sim_R) = \text{Ann}_R^*(W)/\sim_R$.

Proof. By Corollary 3.9 and Lemma 2.11.

Lemma 3.18. For any $U \subseteq R$ and any $I \in \text{Id}(R)$:

- $\text{Ann}_R(I) = \text{Ann}_R(I)$ and $\text{Ann}_R^*(U/\sim_R) = \text{Ann}_R^*(U)$;
- $\text{Ann}_R^*(\mu_R(I)) = \text{Ann}_R^*(\mu_R(I)) = \text{Ann}_R^*(\mu_R(U)/\sim_R) = \text{Ann}_R^*(\mu_R(U))$;
- $\text{Ann}_R^*(\mu_R(U))$ is $\text{Ann}_R^*(\mu_R(U))$ and $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$;
- $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$.

Proof. Let $J \in \text{Id}(R)$. Then: $J \in (\text{Ann}_R(I))$ iff $J \subseteq \text{Ann}_R(I)$ iff $x \in \text{Ann}_R(I)$ for all $x \in J$ and all $y \in I$ iff $J \cdot I = \{0\}$ iff $J \in (\text{Ann}_R(I))$, hence $\text{Ann}_R(I) = (\text{Ann}_R(I))$, thus $\text{Ann}_R(I) = \text{Ann}_R(I)$ and $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$.

By Lemma 3.17 (i) and Remark 3.11, $\text{Ann}_R^*(I/\sim_R) = \text{Ann}_R^*(I)/\sim_R = \text{Ann}_R^*(I)/\sim_R$. By Lemma 3.5 and Remark 3.11, $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$, hence, by the above, Lemma 3.17 (i), Remark 3.11 and Lemma 3.5, $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$. Therefore $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$ and $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$.

Hence $\text{Ann}_R(I) = \text{Ann}_R^*(I)$ and $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$. Hence, by the above, $\text{Ann}_R(I) = \text{Ann}_R^*(I)$ and $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$. Therefore $\text{Ann}_R^*(\mu_R(U)) = \text{Ann}_R^*(\mu_R(U))$.
Lemma 3.19. For any $U, V \subseteq R$ and any $W \subseteq R^*$:

(i) $\Ann_R(\mu_R(U)) = \Ann_R(\mu_R(V))$ iff $\Ann_R(\langle U \rangle_R/\sim_R) = \Ann_R(\langle V \rangle_R/\sim_R)$ iff $\Ann_R(U)/\sim_R = \Ann_R(V)/\sim_R$; 

(ii) there exists a finite subset $S \subseteq \langle U \rangle_R$ such that $\Ann_R(U) = \Ann_R(S)$; 

(iii) there exists a finite subset $S \subseteq R$ such that $\Ann_R(W) = \Ann_R(\mu_R(S))$.

Proof. By Lemma 3.18 and Lemma 3.17 (iii), $\Ann_R(\mu_R(U)) = \Ann_R(\mu_R(V))$ iff $\Ann_R(\langle U \rangle_R/\sim_R) = \Ann_R(\langle V \rangle_R/\sim_R)$ iff $\Ann_R(U)/\sim_R = \Ann_R(V)/\sim_R$ which is equivalent both to $(\Ann_R(U)/\sim_R) = (\Ann_R(V)/\sim_R)$, and to $(\Ann_R(U))/\sim_R = (\Ann_R(V))/\sim_R$, which in turn are equivalent to $\Ann_R(U)/\sim_R = \Ann_R(V)/\sim_R$ and to $(\Ann_R(U))/\sim_R = (\Ann_R(V))/\sim_R$, respectively, the latter of which is equivalent to $\Ann_R(U) = \Ann_R(V)$.

By Proposition 3.7 (i), for an appropriate finite subset $S \subseteq \langle U \rangle_R$, $\langle U \rangle_R/\sim_R = \langle S \rangle_R/\sim_R$, thus $\Ann_R(\langle U \rangle_R/\sim_R) = \Ann_R(\langle S \rangle_R/\sim_R) = \Ann_R(U) = \Ann_R(S)$ by (iii) for an appropriate finite subset $S \subseteq \langle U \rangle_R$, $\langle U \rangle_R/\sim_R = \langle S \rangle_R/\sim_R$, hence $\Ann_R(U) = \Ann_R(S)$ by Lemma 3.17 (iii).

Let $W \subseteq R^*$, so that $W = \{I_k/\sim_R \mid k \in K\}$ for some $(I_k)_{k \in K} \subseteq \Id(R)$. As pointed out in Corollary 3.9 $R^*$ is a frame, thus $\Ann_R(W) = \Ann_R(\bigvee_{k \in K} I_k)$ implies both to (Ann $\langle S \rangle_R/\sim_R = \Ann_R(U)/\sim_R$, hence $\Ann_R(U) = \Ann_R(S)$ by (iii) for an appropriate finite subset $S \subseteq \langle U \rangle_R$, $\langle U \rangle_R/\sim_R = \langle S \rangle_R/\sim_R$, hence $\Ann_R(U) = \Ann_R(S)$ by Lemma 3.17 (iii).

Let $W \subseteq R^*$, so that $W = \{I_k/\sim_R \mid k \in K\}$ for some $(I_k)_{k \in K} \subseteq \Id(R)$. As pointed out in Corollary 3.9 $R^*$ is a frame, thus $\Ann_R(W) = \Ann_R(\bigvee_{k \in K} I_k)$ implies both to (Ann $\langle S \rangle_R/\sim_R = \Ann_R(U)/\sim_R$, hence $\Ann_R(U) = \Ann_R(S)$ by (iii) for an appropriate finite subset $S \subseteq \langle U \rangle_R$, $\langle U \rangle_R/\sim_R = \langle S \rangle_R/\sim_R$, hence $\Ann_R(U) = \Ann_R(S)$ by Lemma 3.17 (iii).

Proof. If $n \in \mathbb{N}^*$, $u_1, \ldots, u_n \in R$ and, for each $i \in \{1, \ldots, n\}$, $\Ann_R(u_i) = e_i R$ for some $e_i \in E(R)$, then $\Ann_R(\langle u_1, \ldots, u_n \rangle) = \bigcap_{i=1}^n \Ann_R(u_i) = \bigcap_{i=1}^n e_i R = e_1 R \cap \cdots \cap e_n R = (e_1 \cap \cdots \cap e_n) R$, and $e_1 \cap \cdots \cap e_n = e_1 \land \cdots \land e_n \in E(R)$.

(iii) Let $W \subseteq R^*$. By Lemma 3.19 (iii), there exists a finite subset $S \subseteq R$ such that $\Ann_R(W) = \Ann_R(\mu_R(S))$. If (1) is fulfilled, then $\Ann_R(S) = e R$ for some $e \in E(R)$. By Lemma 3.18 and Lemma 3.14 (i), it follows that $\Ann_R(W) = \Ann_R(\mu_R(S)) = (\Ann_R(S))/\sim_R = (e R)/\sim_R = (\mu_R(e))/\sim_R = (\mu_R(e)/e R)$, and $\mu_R(e) \in B(R^*)$.

Let $U \subseteq R$, so that $\mu_R(U) \subseteq R^*$. If (1) is fulfilled, then $\Ann_R(\mu_R(U)) = \{f/\sim_R \mid f \in B(R^*)\}$.

By Lemma 3.14 (i), $f = \mu_R(e) = e R/\sim_R = \nu_R(e R)$ for some $e \in E(R)$, so that $e R \in B(\Id(R))$ by Lemma 3.18. By Lemma 3.18 and Lemma 3.17 (iii), $\Ann_R(U)/\sim_R = \Ann_R(\mu_R(U)) = (e R)/\sim_R = (\nu_R(e R))/\sim_R$, hence $\Ann_R(U)/\sim_R = (\Ann_R(U))/\sim_R = (\Ann_R(U))/\sim_R$, that is $\Ann_R(U)/\sim_R = (\Ann_R(U))/\sim_R$, so that $\Ann_R(U) = e R$.

By (i), (ii) and (iii).

Remark 3.22. By Lemma 3.5 $\Ann_R(\langle I \rangle_R) = \Ann_R(I) = \Ann_R(\mu_R(\langle I \rangle_R)$ and $2\Ann_R = \Ann_R(\Ann_R(I))/I \in \Id(R))$.

Let us consider the following conditions on $R$:

(ann)$_R$ $\Ann_R(\langle I \rangle_R)$ is a sublattice of $\Id(R)$ such that the map $I \mapsto \Ann_R(I)$ is a lattice anti–morphism from $\Id(R)$ to $\Ann_R$;

(2ann)$_R$ $2\Ann_R(\langle I \rangle_R)$ is a sublattice of $\Id(R)$ such that the map $I \mapsto \Ann_R(\Ann_R(I))$ is a lattice morphism from $\Id(R)$ to $2\Ann_R$.

Lemma 3.23. (i) The map $\Ann_R(\langle I \rangle_R) \mapsto \Ann_R(\Ann_R(\langle I \rangle_R))$ from $\Ann_R(I)$ to $P\Ann_R(\Id(R))$ is an order isomorphism.
(ii) The map \( \text{Ann}_R(\text{Ann}_R(I)) \mapsto \text{Ann}_{\text{Id}(R)}(\text{Ann}_{\text{Id}(R)}(I)) \) \((I \in \text{Id}(R))\) from \(2\text{Ann}(R)\) to \(\text{P}2\text{Ann}(\text{Id}(R))\) is an order isomorphism.

**Proof.** Clearly, these maps are surjective and order–preserving. By Lemma 3.5 they are completely defined. By Lemma 3.18 the map from \((\bullet)\) is well defined and injective. By Lemma 3.18, Lemma 3.17 \((\bullet)\), and the injectivity of the map from \((\bullet)\), it follows that the map from \((\bullet)\) is well defined and injective. Hence these maps are bijective. Clearly, their inverses are order–preserving, as well.

**Lemma 3.24.** (i) The map \( \text{Ann}_{\text{Id}(R)}(I) \mapsto \text{Ann}_{\text{Id}(R)}(I)/\sim_R = \text{Ann}_{R^*}(I/\sim_R) \) \((I \in \text{Id}(R))\) from \(\text{PAnn}(\text{Id}(R))\) to \(\text{PAnn}(R^*)\) is an order isomorphism.

(ii) For all \(I, J \in \text{Id}(R)\): \(\text{Ann}_{\text{Id}(R)}(I \cap J) = \text{Ann}_{\text{Id}(R)}(I) \vee \text{Ann}_{\text{Id}(R)}(J)\) iff \(\text{Ann}_{R^*}(I/\sim_R \wedge J/\sim_R) = \text{Ann}_{R^*}(I/\sim_R) \vee \text{Ann}_{R^*}(J/\sim_R)\).

(iii) \((\text{pann})_{\text{Id}(R)}\) is equivalent to \((\text{pann})_{R^*}\), and, if they are fulfilled, then the map from \((\bullet)\) is a lattice isomorphism.

**Proof.** By Lemma 2.24 and Corollary 3.9 with the equality in \((\bullet)\) holding by Lemma 3.18.

**Lemma 3.25.** (i) The map \( \text{Ann}_R(I) \mapsto \text{Ann}_{R^*}(I/\sim_R) \) \((I \in \text{Id}(R))\) from \(\text{Ann}(R)\) to \(\text{PAnn}(R^*)\) is an order isomorphism.

(ii) For all \(I, J \in \text{Id}(R)\): \(\text{Ann}_R(I \cap J) = \text{Ann}_R(I) \vee \text{Ann}_R(J)\) iff \(\text{Ann}_{R^*}(I/\sim_R \wedge J/\sim_R) = \text{Ann}_{R^*}(I/\sim_R) \vee \text{Ann}_{R^*}(J/\sim_R)\).

(iii) \((\text{ann})_R\) is equivalent to \((\text{pann})_{R^*}\), and, if they are fulfilled, then the map from \((\bullet)\) is a lattice isomorphism.

**Proof.** By Remark 3.22, Lemma 3.23 \((\bullet)\), Lemma 3.24 and the fact that the map from \((\bullet)\) is the composition of the map from Lemma 3.24 \((\bullet)\), with the map from Lemma 3.26 \((\bullet)\).

**Lemma 3.26.** (i) The map \( \text{Ann}_{\text{Id}(R)}(\text{Ann}_{\text{Id}(R)}(I)) \mapsto \text{Ann}_{\text{Id}(R)}(\text{Ann}_{\text{Id}(R)}(I))/\sim_R = \text{Ann}_{R^*}(\text{Ann}_{R^*}(I/\sim_R)) \) \((I \in \text{Id}(R))\) from \(\text{P2Ann}(\text{Id}(R))\) to \(\text{P2Ann}(R^*)\) is an order isomorphism.

(ii) \((\text{p2ann})_{\text{Id}(R)}\) is equivalent to \((\text{p2ann})_{R^*}\), and, if these conditions are fulfilled, then the map from \((\bullet)\) is a lattice isomorphism.

**Proof.** By Lemma 2.22 and Corollary 3.9 with the equality in \((\bullet)\) holding by Lemma 3.18.

**Lemma 3.27.** (i) The map \( \text{Ann}_R(\text{Ann}_R(I)) \mapsto \text{Ann}_{R^*}(\text{Ann}_{R^*}(I/\sim_R)) \) \((I \in \text{Id}(R))\) from \(2\text{Ann}(R)\) to \(\text{P2Ann}(R^*)\) is an order isomorphism.

(ii) \((\text{2ann})_R\) is equivalent to \((\text{p2ann})_{R^*}\), and, if these conditions are fulfilled, then the map from \((\bullet)\) is a lattice isomorphism.

**Proof.** By Remark 3.22, Lemma 3.23 \((\bullet)\), Lemma 3.26 and the fact that the map from \((\bullet)\) is the composition of the map from Lemma 3.26 \((\bullet)\), with the map from Lemma 3.28 \((\bullet)\).

**Proposition 3.28.** For any nonzero cardinality \(\kappa\), the properties \((3)_{\kappa,R}\) and \((3)_{\kappa,R^*}\) are equivalent.

**Proof.** By Lemma 3.27 and the fact that the map from Lemma 3.27 \((\bullet)\), composed with the map from condition \((3)_{\kappa,R}\) equals the map from condition \((3)_{\kappa,R^*}\) composed with the canonical surjective lattice morphism from \(\text{Id}(R)\) to \(\text{Id}(R)/\sim_R = R^*\).

**Proposition 3.29.** For any nonzero cardinality \(\kappa\), conditions \((4)_{\kappa,R}\), \((4)_{<\infty,R}\), \((4)_R\), \((iv)_R\) and \((iv)_{R^*}\) are equivalent.

**Proof.** By Lemma 3.25 \((\bullet)\), \((iv)_R\) is equivalent to \((iv)_{R^*}\). By Lemma 3.20 \((\text{Ann}_R(S) \mid S \subseteq R, |S| < \infty) = \text{Ann}(R) \geq 2\text{Ann}(R)\), which means that, for any nonzero cardinality \(\kappa\), the second property in \((4)_{\kappa,R}\) is fulfilled, so that conditions \((4)_{\kappa,R}\), \((4)_{<\infty,R}\), \((4)_R\) and \((iv)_R\) are equivalent.

**Lemma 3.30.** (i) \(B(\text{Id}(R)) = \{eR \mid e \in E(R)\}\).
(ii) If \( U \subseteq R \), then \( U \cap \text{Ann}_R(U) \subseteq \{0\} \). If \( I \in \text{Id}(R) \), then \( I \cap \text{Ann}_R(I) = \{0\} \).

(iii) If \( U \subseteq R \) such that \( \text{Ann}_R(U) \vee \text{Ann}_R(\text{Ann}_R(U)) = R \), then \( \text{Ann}_R(U) = eR \) for some \( e \in E(R) \).

**Proof.** This statement may be known, but, for the sake of completeness, I am deriving it from the statements in this paper. Let \( \pi_R = (\nu_R |_{B(\text{Id}(R))}) \circ \mu_R |_{\kappa(R)} : E(R) \to B(\text{Id}(R)) \) be the composition of the inverse of the Boolean isomorphism from Corollary 3.10 to the Boolean isomorphism from Lemma 3.14. For any \( e \in E(R) \), since \( \mu_R(e) = eR/\sim_R = \nu_R(eR) \), it follows that \( \pi_R(e) = eR \). Hence \( B(\text{Id}(R)) = \pi_R(E(R)) = \{eR | e \in E(R)\} \).

(ii) If \( U \subseteq R \) and \( x \in U \cap \text{Ann}_R(U) \), then \( x \cdot x = 0 \), so that \( x = 0 \) since a semiprime commutative unitary ring has no nonzero nilpotents [11, p.125, 126]. Now, if \( I \in \text{Id}(R) \), then \( 0 \in I \cap \text{Ann}_R(I) \).

(iii) By (ii), it follows that \( \text{Ann}_R(U) \in B(\text{Id}(R)) \), having \( \text{Ann}_R(\text{Ann}_R(U)) \) as a complement, so that \( \text{Ann}_R(U) = eR \) for some \( e \in E(R) \) by (i).

**Proposition 3.31.** For any nonzero cardinality \( \kappa \):

(i) \((5)_{\kappa,R}\) implies \((1)_{\kappa,R}\);

(ii) \((5)_{<\infty,R}\) implies \((5)_R^*\);

(iii) \((5)_R^*\) implies \((5)_R\);

(iv) \((5)_{\kappa,R}, (5)_{<\infty,R}, (5)_R\), and \((5)_R^*\) are equivalent.

**Proof.** By Lemma 3.30 (iii).

(ii) Let \( W \subseteq R^* \), so that, by Lemma 3.19 (iii), \( \text{Ann}_R(W) = \text{Ann}_R^*(\mu_R(S)) \) for some finite subset \( S \subseteq R \). If \((5)_{<\infty,R}\) is fulfilled, then \( \text{Ann}_R(S) \lor \text{Ann}_R(\text{Ann}_R(S)) = R \). By Lemma 3.18 it follows that \( \text{Ann}_R^*(W) \lor \text{Ann}_R^*(\text{Ann}_R^*(W)) = \text{Ann}_R^*(\mu_R(S)) \lor \text{Ann}_R^*(\text{Ann}_R^*(\mu_R(S))) = (\text{Ann}_R(S)/\sim_R^*) \lor (\text{Ann}_R(S)/\sim_R^*) = ((\text{Ann}_R(S) \lor \text{Ann}_R(\text{Ann}_R(S))) / \sim_R^*) = (R/\sim_R^*)^* = (R^*)^* = R^* \).

(iii) If \( U \subseteq R \), so that \( \mu_R(S) \subseteq R^* \), thus, if \((5)_R^*\) is fulfilled, then, by Lemma 3.18 \( R^* = \text{Ann}_R^*(\mu_R(U)) \lor \text{Ann}_R^*(\text{Ann}_R^*(\mu_R(U))) = (\text{Ann}_R(U)/\sim_R^*) \lor (\text{Ann}_R(U)/\sim_R^*) = ((\text{Ann}_R(U) \lor \text{Ann}_R(\text{Ann}_R(U))) / \sim_R^*) = R \) by Remark 3.10 (v) By (i), (ii), (iii), Proposition 3.21, and Corollary 3.12.

**Theorem 3.32.** For any nonzero cardinality \( \kappa \) and any \( h, i, j \in \mathbb{T}_5 \):

(i) \((iv)_R^*, (h)_{\kappa,R}, (i)_{<\infty,R}\) and \((j)\) are equivalent;

(ii) \((iv)_{\text{Id}(R)}^*, (h)_{\kappa,\text{Id}(R)}, (i)_{<\infty,\text{Id}(R)}\) and \((j)_{\text{Id}(R)}\) are equivalent;

(iii) the conditions from (i) are equivalent to those from (ii) and to the equivalent conditions \((iv)_{R^*}, (h)_{\kappa,R^*}, (i)_{<\infty,R^*}\) and \((j)_{R^*}\).

In particular, \( R \) is a Baer ring iff \( R \) is a strongly Baer ring iff \( \text{Id}(R) \) is a Stone lattice iff \( \text{Id}(R) \) is a strongly Stone lattice iff \( R^* \) is a Stone lattice iff \( R^* \) is a strongly Stone lattice.

**Proof.** By Corollaries 3.9 and 3.12 and Propositions 3.21, 3.15, 3.28, 3.29, and 3.31.

Throughout the rest of this paper, \( S \) shall be a semiprime commutative unitary ring.

**Remark 3.33.** By Corollary 3.30 and Remark 3.24, \( R \times S \) is semiprime, \( \text{Id}(R \times S) \cong \text{Id}(R) \times \text{Id}(S) \) and \((R \times S)^* \cong R^* \times S^* \).

**Corollary 3.34.** For any nonzero cardinality \( \kappa \) and any \( h, i, j \in \mathbb{T}_5 \): the equivalent conditions \((iv)_{R \times S}, (h)_{\kappa,R \times S}, (i)_{<\infty,R \times S}\) and \((j)_{R \times S}\) are fulfilled iff the equivalent conditions \((iv)_R, (h)_{\kappa,R}, (i)_{<\infty,R}\) and \((j)_R\), as well as the equivalent conditions \((iv)_S, (h)_{\kappa,S}, (i)_{<\infty,S}\) and \((j)_S\), are fulfilled.

Concerning the semiprimality condition, recall that it does not need to be enforced for residuated lattices or bounded distributive lattices because they are congruence–distributive and thus semiprime.
4 Conclusions

It may be possible to extend Theorem 2.29 to other kinds of congruence lattices, possibly of algebras from non-modular commutator varieties. If the congruence lattices of the algebras from such varieties cover entire varieties of bounded lattices, then all lattices from those varieties fulfill [5, Theorem 1]. The study of such extensions of Davey’s Theorem remains a theme for future research.

Another important research theme is finding more classes of algebras in which, given an appropriate setting (regarding definitions for annihilators and a Boolean center), Davey’s Theorem holds not only for congruences, but also for elements, as in the case of bounded distributive lattices, residuated lattices and commutative unitary rings.

References

[1] P. Agliano, Prime Spectra in Modular Varieties, Algebra Universalis 30 (1993), 581–597.
[2] L. P. Belluce, Spectral Spaces and Non–commutative Rings, Communications in Algebra 19, Issue 7 (1991), 1855–1865.
[3] L. P. Belluce, Spectral Closure for Non–commutative Rings, Communications in Algebra 25, Issue 5 (1997), 1513–1536.
[4] J. Czelakowski, Additivity of the Commutator and Residuation, Reports on Mathematical Logic 43 (2008), 109–132.
[5] B. A. Davey, m–Stone Lattices, Canadian Journal of Mathematics 24, No. 6 (1972), 1027–1032.
[6] R. Freese, R. McKenzie, Commutator Theory for Congruence–modular Varieties, London Mathematical Society Lecture Note Series 125, Cambridge University Press, 1987.
[7] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Studies in Logic and The Foundations of Mathematics 151, Elsevier, Amsterdam/ Boston /Heidelberg /London /New York /Oxford /Paris /San Diego/ San Francisco /Singapore /Sydney /Tokyo, 2007.
[8] G. Georgescu, I. Voiculescu, Some Abstract Maximal Ideal–like Spaces, Algebra Universalis 26 (1989), 90–102.
[9] G. Georgescu, C. Mureșan, The Reticulation of a Congruence–modular Algebra, arXiv: 1706.04270 [math.RA].
[10] J. Kaplansky, Commutative Rings, First Edition: University of Chicago Press, 1974; Second Edition: Polygonal Publishing House, 2006.
[11] J. E. Kist, Two Characterizations of Commutative Baer Rings, Pacific Journal of Mathematics 50, No. 1, 1974.
[12] J. Kollár, Congruences and One–element Subalgebras, Algebra Universalis 9, Issue 1 (December 1979), 266–267.
[13] A. Iorgulescu, Algebras of Logic as BCK Algebras, Editura ASE, Bucharest, 2008.
[14] A. Joyal, Le Théorème de Chevalley – Tarski et Remarques sur l’algèbre Constructive, Cahiers Topol. Géom. Différ., 16 (1975), 256–258.
[15] C. Mureșan, Algebras of Many–valued Logic. Contributions to the Theory of Residuated Lattices, Ph. D. Thesis, 2009.
[16] C. Mureșan, Co–Stone Residuated Lattices, Annals of the University of Craiova, Mathematics and Computer Science Series 40 (2013), 52–75.
[17] D. Piciu, Algebras of Fuzzy Logic, Editura Universitaria Craiova, Craiova, 2007.
[18] H. Simmons, Reticulated Rings, Journal of Algebra 66, Issue 1 (September 1980), 169–192.