Geometric approach to the local
Jacquet-Langlands correspondence
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Abstract. In this paper, we give a purely geometric approach to the local Jacquet-Langlands correspondence for GL(n) over a p-adic field, under the assumption that the invariant of the division algebra is 1/n. We use the ℓ-adic étale cohomology of the Drinfeld tower to construct the correspondence at the level of the Grothendieck groups with rational coefficients. Moreover, assuming that n is prime, we prove that this correspondence preserves irreducible representations. This gives a purely local proof of the local Jacquet-Langlands correspondence in this case. We need neither a global automorphic technique nor detailed classification of supercuspidal representations of GL(n).

1 Introduction
Let $F$ be a $p$-adic field, i.e., a finite extension of $\mathbb{Q}_p$. Let $n \geq 1$ be an integer and $D$ a central division algebra over $F$ such that $\dim_F D = n^2$. The famous local Jacquet-Langlands correspondence gives a natural bijective correspondence between irreducible discrete series representations of $\text{GL}_n(F)$ and irreducible smooth representations of $D^\times$. Let us recall its precise statement. Write $\text{Irr}(D^\times)$ for the set of isomorphism classes of irreducible smooth representations of $D^\times$. We denote by $\text{Disc}(\text{GL}_n(F))$ the set of isomorphism classes of irreducible discrete series representations of $\text{GL}_n(F)$. For $\rho \in \text{Irr}(D^\times)$ (resp. $\pi \in \text{Disc}(\text{GL}_n(F))$), we denote the character of $\rho$ (resp. $\pi$) by $\theta_\rho$ (resp. $\theta_\pi$). Here $\theta_\rho$ is a locally constant function on $D^\times$, and $\theta_\pi$ is a locally integrable function on $\text{GL}_n(F)$ which is locally constant on $\text{GL}_n(F)^\text{reg}$, the set of regular elements of $\text{GL}_n(F)$. The precise statement of the local Jacquet-Langlands correspondence is the following:

Theorem 1.1 (the local Jacquet-Langlands correspondence) There exists a unique bijection

$$\text{JL}: \text{Irr}(D^\times) \overset{\cong}{\longrightarrow} \text{Disc}(\text{GL}_n(F))$$
satisfying the following character relation: for every regular element \( h \) of \( D^\times \),
\[
\theta_{\rho}(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h),
\]
where \( g_h \) is an arbitrary element of \( \text{GL}_n(F) \) whose minimal polynomial is the same as that of \( h \).

The original proof of this theorem, due to Deligne-Kazhdan-Vigneras [DKV84] and Rogawski [Rog83], was accomplished by using a global automorphic method. In some cases, more explicit local studies can be found in [Hen93], [BH00], [BH05], which are based on the theory of types. However, apart from the case of \( \text{GL}(2) \), a purely local proof of Theorem 1.1 seems not to be known yet (cf. [Hen06, a comment after Theorem 2]).

In this article, under the assumption that the invariant of \( D \) is \( 1/n \), we will give a geometric approach to construct the bijection \( JL \) above. Let \( \text{R}(D^\times) \) be the Grothendieck group of finite-dimensional smooth representations of \( D^\times \), and \( \overline{\text{R}}(\text{GL}_n(F)) \) the Grothendieck group of finite length smooth representations of \( \text{GL}_n(F) \) “modulo induced representations” (for a precise definition, see [Kaz86]). It is known that the classes of elements of \( \text{Irr}(D^\times) \) (resp. \( \text{Disc} (\text{GL}_n(F)) \)) form a basis of \( R(D^\times) \) (resp. \( \overline{\text{R}}(\text{GL}_n(F)) \)). Put \( R(D^\times)_Q = R(D^\times) \otimes \mathbb{Z}_Q \) and \( \overline{\text{R}}(\text{GL}_n(F))_Q = \overline{\text{R}}(\text{GL}_n(F)) \otimes \mathbb{Z}_Q \). The main theorems of this article are the following:

**Theorem 1.2 (Theorem 6.6)** We can construct the following two homomorphisms geometrically:

\[
JL: R(D^\times)_Q \rightarrow \overline{\text{R}}(\text{GL}_n(F))_Q, \quad LJ: \overline{\text{R}}(\text{GL}_n(F))_Q \rightarrow R(D^\times)_Q.
\]

These two maps are inverse to each other, and satisfy the character relations
\[
\theta_{\rho}(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h), \quad \theta_{\pi}(g_h) = (-1)^{n-1}\theta_{LJ(\pi)}(h)
\]
for every regular \( h \in D^\times \).

**Theorem 1.3 (Theorem 6.10)** If \( n \) is prime, then \( JL \) induces a bijection \( JL: \text{Irr}(D^\times) \rightarrow \text{Disc} (\text{GL}_n(F)) \).

Theorem 1.3 provides a purely local proof of Theorem 1.1 in the case above. In particular, the local Jacquet-Langlands correspondence for \( \text{GL}_2(F) \) and \( \text{GL}_3(F) \) are fully recovered.

The geometric object we use is the Drinfeld tower for \( \text{GL}_n(F) \). It is a tower of rigid spaces over (a disjoint union of) the \((n-1)\)-dimensional Drinfeld upper half space \( \mathbb{P}^{n-1}_F \setminus \bigcup_H H \), where \( H \) runs through hyperplanes of \( \mathbb{P}^{n-1}_F \) defined over \( F \) (for more detailed explanation, see Section 2). Thanks to extensive studies by many people (cf. [Har97], [HT01], [Boy09], [Dat07]), it is now well-known that the local Jacquet-Langlands correspondence is realized in the \( \ell \)-adic cohomology \( H_{\text{Dr}} \) of the Drinfeld tower. The methods in the works cited above are again global and automorphic. However, there is also a purely local study of the cohomology due
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to Faltings [Fal94]. He began with an irreducible smooth representation $\rho$ of $D^\times$, and investigated the $\rho$-isotypic part $H_{\text{Dr}}[\rho]$ of $H_{\text{Dr}}$ by means of the Lefschetz trace formula. By this method, he succeeded to observe that the character relation in Theorem 1.1 appears naturally in $H_{\text{Dr}}[\rho]$. By using this result, we can give the map $JL$ in Theorem 1.2.

To construct the inverse map $LJ$, we need to consider the opposite direction; we begin with an irreducible discrete series representation $\pi$ of $GL_n(F)$ and investigate the “$\pi$-isotypic part” of $H_{\text{Dr}}$. More precisely, we should consider the alternating sum of the extension groups $\sum_j (-1)^j \text{Ext}^j(H_{\text{Dr}}, \pi)$, since $\pi$ is neither projective nor injective in general. To study it, we apply the method introduced in [Mie11]; namely, we use local harmonic analysis, such as transfer of orbital integrals. Furthermore, to prove Theorem 1.3 the non-cuspidality result obtained in [Mie10b] plays a crucial role.

Since our approach is entirely geometric, it is natural to expect that a similar argument may give interesting consequences for the mod-$\ell$ Jacquet-Langlands correspondence (cf. [Dat11]). The author also expects that our strategy can be extended to other Rapoport-Zink spaces, especially the Rapoport-Zink space for $\text{GSp}(4)$. He hopes to deal with these problems in future works.

We sketch the outline of this paper. In Section 2 we recall the definition of the Drinfeld tower and results in [Fal94]. We use the framework of [Dat00] to deal with finitely generated representations systematically. In Section 3, we study the alternating sum of the extension groups $\sum_j (-1)^j \text{Ext}^j(H_{\text{Dr}}, \pi)$ by means of local harmonic analysis. In Section 4, we apply another deep result of Faltings on the comparison of the Lubin-Tate tower and the Drinfeld tower. It provides a very important finiteness result on $H_{\text{Dr}}$. Under this finiteness, results in Section 2 and Section 3 can be written in a very simple form. After short preliminaries on representation theory in Section 5, finally in Section 6, we construct the maps $JL$ and $LJ$ in Theorem 1.2 by using the $\ell$-adic cohomology of the Drinfeld tower, and investigate their properties.

Acknowledgment The author would like to thank Matthias Strauch for very helpful discussions. He is also grateful to Tetsushi Ito and Sug Woo Shin for their valuable comments.

Notation For a totally disconnected locally compact group $H$, let $\text{Irr}(H)$ be the set of isomorphism classes of irreducible smooth representations of $H$. We denote the Grothendieck group of finitely generated (resp. finite length) smooth $H$-representations by $K(H)$ (resp. $R(H)$). Put $R(H)_\mathbb{Q} = R(H) \otimes \mathbb{Q}$. For a finite-dimensional smooth representation $\sigma$ of $H$, write $\theta_\sigma$ for the character of $\sigma$. It is a locally constant function on $H$. If moreover a Haar measure on $H$ is fixed, we denote by $\mathcal{H}(H)$ the Hecke algebra of $H$, namely, the abelian group of locally constant compactly supported functions on $H$ with convolution product. Put $\overline{\mathcal{H}}(H) = \mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)] = \mathcal{H}(H)_H$ (the $H$-coinvariant quotient).
Let $F$ be a $p$-adic field and $O$ its ring of integers. We denote the normalized valuation of $F$ by $v_F$ and the cardinality of the residue field of $O$ by $q$. Fix a uniformizer $\varpi$ of $O$. Denote the completion of the maximal unramified extension of $O$ by $\hat{O}$ and the fraction field of $\hat{O}$ by $F$.

Throughout this paper, we fix an integer $n \geq 1$. Let $D$ be the central division algebra over $F$ with invariant $1/n$, and $O_D$ its maximal order. Fix a uniformizer $\Pi \in O_D$ such that $\Pi^n = \varpi$.

For simplicity, put $G = \text{GL}_n(F)$. We denote by $G_{\text{reg}}$ (resp. $G_{\text{ell}}$) the set of regular (resp. regular elliptic) elements of $G$. Write $Z_G$ for the center of $G$. We apply these notations to other groups. For example, we write $(D^x)_{\text{reg}}$ for the set of regular elements of $D^x$. As in Theorem 1.1 for $h \in (D^x)_{\text{reg}}$, let $y_h$ be an element of $G_{\text{ell}}$ whose minimal polynomial is the same as that of $h$. Such an element always exists, and is unique up to conjugacy. Moreover, $h \mapsto y_h$ induces a bijection between conjugacy classes in $(D^x)_{\text{reg}}$ and those in $G_{\text{ell}}$. Therefore, to $g \in G_{\text{ell}}$ we can attach an element $h_g \in (D^x)_{\text{reg}}$ whose minimal polynomial is the same as that of $g$.

For a smooth $G$-representation $\pi$ of finite length, we denote by $\theta_\pi$ the distribution character of $\pi$. It is a locally integrable function on $G$ which is locally constant on $G_{\text{reg}}$.

We identify $F^x$ with $Z_G$ and $Z_{D^x}$. Then, we can consider the quotient groups $G/\varpi^Z$ and $D^x/\varpi^Z$ under a discrete subgroup $\varpi^Z$ of $F^x$. We regard $\text{Irr}(D^x/\varpi^Z)$ (resp. $\text{Irr}(G/\varpi^Z)$) as a subset of $\text{Irr}(D^x)$ (resp. $\text{Irr}(G)$). Similarly, $R(D^x/\varpi^Z)$ (resp. $R(G/\varpi^Z)$) is regarded as a submodule of $R(D^x)$ (resp. $R(G)$). Write $\text{Disc}(G)$ for the set of isomorphism classes of irreducible discrete series representations of $G$. Put $\text{Disc}(G/\varpi^Z) = \text{Disc}(G) \cap \text{Irr}(G/\varpi^Z)$.

Fix Haar measures on $G$ and $D^x$. We endow $\varpi^Z$ with the counting measure and consider the quotient measures on $G/\varpi^Z$ and $D^x/\varpi^Z$. For $\varphi \in \mathcal{H}(G/\varpi^Z)$ and $g \in G_{\text{ell}}$, put $O_g^{G/\varpi^Z}(\varphi) = \int_{G/\varpi^Z} \varphi(x^{-1} gx) dx$ (the orbital integral). It is well-known that this integral converges. Similarly, for $\varphi' \in \mathcal{H}(D^x/\varpi^Z)$ and $h \in D^x$, put $O_h^{D^x/\varpi^Z}(\varphi') = \int_{D^x/\varpi^Z} \varphi'(y^{-1} hy) dy$.

For a field $k$, we denote its algebraic closure by $\overline{k}$. Let $\ell$ be a prime which is invertible in $O$. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Every representation is considered over $\mathbb{C}$.

2 Drinfeld tower

Let us briefly recall the definition of the Drinfeld tower. For more detailed description, see [Dri76], [BC91], [RZ96, Chapter 3].

First of all, fix a special formal $O_D$-module $X$ of $O_F$-height $n^2$ over $\mathbb{F}_q$. It is well-known that such $X$ is unique up to $O_D$-isogeny.

We denote by $\text{Nilp}$ the category of $\hat{O}$-schemes on which $\varpi$ is locally nilpotent. Consider the functor $\mathcal{M}_D$, from $\text{Nilp}$ to the category of sets that maps $S$ to the set of isomorphism classes of pairs $(X, \rho)$ consisting of
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- a special formal $\mathcal{O}_D$-module $X$ over $S$,
- and an $\mathcal{O}_D$-quasi-isogeny $\rho: X \otimes_{\mathbb{Q}_q} \overline{S} \to X \otimes_{S} \overline{S}$,
where we put $\overline{S} = S \otimes_{\mathcal{O}} \overline{\mathbb{F}}_q$. Then $\mathcal{M}_{\text{Dr}}$ is represented by a formal scheme locally of finite type over $\mathcal{O}$. We denote the formal scheme by $\mathcal{M}_{\text{Dr}}$ again, and the rigid generic fiber of $\mathcal{M}_{\text{Dr}}$ by $M_{\text{Dr}}$. It is known that $M_{\text{Dr}}$ is the disjoint union of countable copies of the $(n-1)$-dimensional Drinfeld upper half space.

For an integer $m \geq 0$, let $M_{\text{Dr},m}$ be the rigid space classifying $\Pi^m$-level structures on the universal formal $\mathcal{O}_D$-module over $M_{\text{Dr}}$. It is a finite étale Galois covering of $M_{\text{Dr}}$ with Galois group $(\mathcal{O}_D/(\Pi^m))^\times$. The projective system $\{M_{\text{Dr},m}\}_{m \geq 0}$ is called the Drinfeld tower. We can define a natural right action of $\mathcal{O}$-finite-dimensional smooth representations $\sigma$ of $\Pi_\infty$. Moreover, there exist compact open subgroups $\mathcal{O}_\Lambda$ of $M_{\text{Dr}}$ indexed by the set $\Lambda$ of vertices of the Bruhat-Tits building of $\text{PGL}_n(F)$. This covering satisfies the following properties:

- The covering $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ consisting of quasi-compact open subsets, indexed by the set $\Lambda$ of vertices of the Bruhat-Tits building of $\text{PGL}_n(F)$.
- $\mathcal{U}$ this covering satisfies the following properties:

(a) For $g \in G/\mathcal{O}$, $U_\lambda \cdot g = U_{g^{-1}\lambda}$.
(b) For each $\lambda \in \Lambda$, there exist only finitely many $\lambda' \in \Lambda$ satisfying $U_\lambda \cap U_{\lambda'} \neq \emptyset$.
(c) For each $\lambda \in \Lambda$, $K_\lambda = \{g \in G/\mathcal{O} \mid U_\lambda \cdot g = U_\lambda\}$ is a compact open subgroup of $G/\mathcal{O}$.

For a finite subset $I \subset \Lambda$, put

$$ U_I = \bigcap_{\lambda \in I} U_\lambda, \quad K_I = \{g \in G/\mathcal{O} \mid U_I \cdot g = U_I\}. $$

For an integer $r \geq 0$, set $\Lambda_r = \{I \subset \Lambda \mid \# I = r + 1, U_I \neq \emptyset\}$. Then, by the three properties above, we have the following:
For each \( r \geq 0 \), the set of \( G/\wp^{\mathbb{Z}} \)-orbits in \( \Lambda_r \) is finite.

- We have \( \Lambda_r = \emptyset \) for sufficiently large \( r \).
- For every finite subset \( I \subseteq \Lambda \), \( K_I \) is a compact open subgroup of \( G/\wp^{\mathbb{Z}} \).

Take a system of representatives \( I_{r,1}, \ldots, I_{r,N_r} \) of \((G/\wp^{\mathbb{Z}})\setminus \Lambda_r\) and put \( K_{r,i} = K_{I_{r,i}} \).

Let \( U_m = \{ U_{m,\lambda} \}_{\lambda \in \Lambda} \) be the covering obtained as the inverse image of \( \Omega \). For a finite subset \( I \subseteq \Lambda \), put \( U_{m,I} = \bigcap_{\lambda \in I} U_{m,\lambda} \). Then we have the \( \check{\text{C}} \text{ech} \) spectral sequence

\[
E_1^{r,s} = \bigoplus_{I \subseteq \Lambda_r} H^s_c(U_{m,I} \otimes_{\mathbb{F}} \overline{\mathbb{F}}_\ell) \implies H^{r+s}_{D_r, m, I}.
\]

Put

\[
V_{m,r,i}^s = H^s_c(U_{m,I_{r,i}} \otimes_{\mathbb{F}} \overline{\mathbb{F}}_\ell).
\]

It is a finite-dimensional smooth representation of \( K_{r,i} \times D^x/\wp^{\mathbb{Z}} \) and vanishes for \( s > 2n \) (cf. [Hub96, Proposition 5.5.1, Proposition 6.3.2]). We can easily observe that \( E_1^{r,s} \) is isomorphic to \( \bigoplus_{s=1}^{N_r} \text{c-Ind}_{K_{r,i}}^{G/\wp^{\mathbb{Z}}} V_{m,r,i}^s \) as a \( G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}} \)-representation. Therefore \( E_1^{r,s} \) is a finitely generated \( G/\wp^{\mathbb{Z}} \)-representation, and vanishes for all but finitely many \((r, s)\). Hence \( H_{D_r, m}^i \) is finitely generated as a \( G \)-module (cf. [Ber84, Remarque 3.12]). Moreover, in \( K(G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}}) \) we have

\[
\sum_i (-1)^i [H_{D_r, m}^i] = \sum_{r, s} (-1)^{-r+s} E_1^{r,s} = \sum_{r, s} \sum_{i=1}^{N_r} (-1)^{-r+s} \text{c-Ind}_{K_{r,i}}^{G/\wp^{\mathbb{Z}}} V_{m,r,i}^s.
\]

This concludes the proof.

**Definition 2.2** We denote by \( \eta_m \) the image of \( \sum_i (-1)^i [H_{D_r, m}^i] \) under the rank map

\[
\text{Rk}: K(G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}}) \to \overline{\mathcal{H}}(G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}})
\]

(cf. [Dat00, 1.2]). For \( h \in D^x \), we define \( \eta_{m,h} \in \overline{\mathcal{H}}(G/\wp^{\mathbb{Z}}) \) by

\[
\eta_{m,h}(g) = \int_{D^x/\wp^{\mathbb{Z}}} \eta_m(g, h^{-1}hh') dh'.
\]

Using the expression of \( \sum_i (-1)^i [H_{D_r, m}^i] \) in Proposition 2.1, we can give more explicit description of \( \eta_m \) and \( \eta_{m,h} \).

**Proposition 2.3** For \( m \geq 0 \) and \( h \in D^x \), define \( \tilde{\eta}_m \in \mathcal{H}(G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}}) \) and \( \tilde{\eta}_{m,h} \in \mathcal{H}(G/\wp^{\mathbb{Z}}) \) by

\[
\tilde{\eta}_m = \sum_{\nu=1}^N \varepsilon_{\nu} \theta_{\sigma_{m,\nu}^\vee}(-, h), \quad \tilde{\eta}_{m,h} = \sum_{\nu=1}^N \varepsilon_{\nu} \theta_{\sigma_{m,\nu}^\vee}(-, h),
\]

where \(-\)\(^\vee\) denotes the contragredient, and \( \theta_{\sigma_{m,\nu}^\vee}(-, h) \) is regarded as a function on \( G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}} \) (resp. \( G/\wp^{\mathbb{Z}} \)) by setting \( \theta_{\sigma_{m,\nu}^\vee}(g, h) = 0 \) for \( g \notin K_{\nu} \).

Then, the image of \( \tilde{\eta}_m \) (resp. \( \tilde{\eta}_{m,h} \)) in \( \overline{\mathcal{H}}(G/\wp^{\mathbb{Z}} \times D^x/\wp^{\mathbb{Z}}) \) (resp. \( \overline{\mathcal{H}}(G/\wp^{\mathbb{Z}}) \)) coincides with \( \eta_m \) (resp. \( \eta_{m,h} \)).
Proof. The assertion for \( \tilde{\eta}_{m,h} \) immediately follows from that for \( \tilde{\eta}_m \). Thus it suffices to prove the following:

Let \( K \) be a compact open subgroup of \( G/\omega^Z \). For every finite-dimensional smooth representation \( \sigma \) of \( K \times D^x/\omega^Z \), the image of \( \text{vol}(K)^{-1}\theta_{\sigma^V} \) in \( \overline{H}(G/\omega^Z \times D^x/\omega^Z) \) coincides with \( \text{Rk}(\text{c-Ind}_K^G\omega^Z \sigma) \).

Since the image of \( [\sigma] \) under the rank map \( \text{Rk} : K(K \times D^x/\omega^Z) \to \overline{H}(K \times D^x/\omega^Z) \) is \( \text{vol}(K \times D^x/\omega^Z)^{-1}\theta_{\sigma^V} \), this claim follows from the commutative diagram below (cf. [Dat00] proof of Theorem 1.6):

\[
\begin{array}{ccc}
K(K \times D^x/\omega^Z) & \xrightarrow{\text{Rk}} & \overline{H}(K \times D^x/\omega^Z) \\
\text{c-Ind}_K^G\omega^Z & \downarrow & \downarrow \\
K(G/\omega^Z \times D^x/\omega^Z) & \xrightarrow{\text{Rk}} & \overline{H}(G/\omega^Z \times D^x/\omega^Z).
\end{array}
\]

In [Fal94], Faltings investigated the function \( \tilde{\eta}_m \) above by means of the Lefschetz trace formula. His results can be summarized in the following theorem.

**Theorem 2.4** Let \( g \in G^{\text{reg}} \) and \( h \in D^x \).

i) If \( g \) is elliptic, then we have

\[
O_g^{G/\omega^Z}(\eta_{m,h}) = \# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\omega^Z) = n \cdot \# \{ a \in D^x/\omega^Z(1 + \Pi^m\mathcal{O}_D) \mid hah_g^{-1} = a \}.
\]

ii) If \( g \) is not elliptic, then we have \( \int_{Z(g) \setminus G} \eta_{m,h}(x^{-1}gx)dx = 0 \), where \( Z(g) \) denotes the centralizer of \( g \).

**Proof.** Let us briefly recall the proof in [Fal94]. We use the notation in the proof of Proposition [2.1].

First consider the case where \( g \) is elliptic. Then, we can find a finite subset \( \Lambda_g \subseteq \Lambda \) such that \( g\Lambda_g = \Lambda_g \) and \( gU_\lambda \cap U_\lambda = \emptyset \) for \( \lambda \in \Lambda \setminus \Lambda_g \). Put \( U_{m,g} = \bigcup_{\lambda \in \Lambda_g} U_{m,\lambda} \). Then \( U_{m,g} \) is quasi-compact smooth and \( (g^{-1}, h^{-1}) : U_{m,g} \to U_{m,g} \) has no fixed point on the boundary of \( U_{m,g} \). Therefore we can apply the Lefschetz trace formula for this endomorphism (for a general theory of the Lefschetz trace formula for rigid spaces, see [Mie10a]). Noting that every fixed point of \( (g^{-1}, h^{-1}) : M_{\text{Dr},m}/\omega^Z \to M_{\text{Dr},m}/\omega^Z \) lies in \( U_{m,g} \), we obtain the following equality:

\[
\sum_i (-1)^i \text{Tr}((g^{-1}, h^{-1}) ; H_c^i(U_{m,g} \otimes \mathcal{F}, \mathcal{O}_D)) = \# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\omega^Z).
\]

By the \( \check{\text{C}} \)ech spectral sequence, we can easily show that the left hand side is equal to \( O_g^{G/\omega^Z}(\eta_{m,h}) \). The right hand side can be computed by using the period map, as in [Str08] §2.6. The result is

\[
\# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\omega^Z) = n \cdot \# \{ a \in D^x/\omega^Z(1 + \Pi^m\mathcal{O}_D) \mid hah_g^{-1} = a \}.
\]
It is slightly different from \[\text{[Fal94, Theorem 1]}, because our \(M_{D_r/\mathcal{O}}\) is the disjoint union of \(n\) copies of \(\Omega\) considered in \[\text{[Fal94]. Rather, it is compatible with \[\text{[Str08, Theorem 2.6.8]. This concludes the proof of \(i)\].}

To prove \(ii)\), apply the same argument to \(M_{D_{r,m}/\Gamma}\), where \(\Gamma\) is a sufficiently small discrete torsion-free cocompact subgroup of \(Z(g)\).

**Corollary 2.5** For \(\rho \in \text{Irr}(D^{\times}/\mathcal{O})\), \(\text{Hom}_{D^{\times}}(\rho, H_{D_r}^{i})\) is a finitely generated \(G/\mathcal{O}\)-representation by \[\text{[Mie11, Lemma 5.2]}. The image of \(\sum_i (-1)^i [\text{Hom}_{D^{\times}}(\rho, H_{D_r}^{i})]\) under the map

\[
\text{Rk}^\vee: K(G/\mathcal{O}) \xrightarrow{\text{Rk}} \mathcal{H}(G/\mathcal{O}) \xrightarrow{\vee} \mathcal{C}^\infty(G^{\text{crys}})
\]

coincides with \(g \mapsto n\theta_\rho(h_g^{-1}).\) Recall that for \(f \in \mathcal{H}(G/\mathcal{O})\) the locally constant function \(f^\vee\) on \(G^{\text{crys}}\) is given by \(f^\vee(g) = \int_{G/\mathcal{O}} f(xg^{-1}x^{-1})dx\) (cf. \[\text{[Dat00, p. 190]}).

**Proof.** Take a sufficiently large integer \(m \geq 0\) so that \(\rho|_{1+\Pi^m\mathcal{O}_D}\) is trivial. Then we have \(\text{Hom}_{D^{\times}}(\rho, H_{D_r}^{i}) = \text{Hom}_{D^{\times}}(\rho, (H_{D_r}^{i})^{1+\Pi^m\mathcal{O}_D}) = \text{Hom}_{D^{\times}}(\rho, H_{D_{r,m}}^{i}).\)

It is easy to see that the following diagram is commutative:

\[
\begin{array}{ccc}
K(G/\mathcal{O}) \times D^{\times}/\mathcal{O} & \xrightarrow{\text{Rk}} & \mathcal{H}(G/\mathcal{O}) \times D^{\times}/\mathcal{O} \\
\text{Hom}_{D^{\times}}(\rho, -) \downarrow & & \downarrow (*) \\
K(G/\mathcal{O}) & \xrightarrow{\text{Rk}} & \mathcal{H}(G/\mathcal{O}),
\end{array}
\]

where (*) is given by

\[
f \mapsto \left(g \mapsto \int_{D^{\times}/\mathcal{O}} f(g, h) \theta_\rho(h)dh\right).
\]

Therefore, the image of \(\sum_i (-1)^i [\text{Hom}_{D^{\times}}(\rho, H_{D_{r,m}}^{i})]\) under \(\text{Rk}^\vee\) can be calculated as follows:

\[
g \mapsto \int_{G/\mathcal{O}} \int_{D^{\times}/\mathcal{O}} \eta_m(xg^{-1}x^{-1}, h) \theta_\rho(h)dhdx
\]

\[
= \frac{1}{\text{vol}(D^{\times}/\mathcal{O})} \int_{G/\mathcal{O}} \int_{D^{\times}/\mathcal{O}} \int_{D^{\times}/\mathcal{O}} \eta_m(xg^{-1}x^{-1}, h) \theta_\rho(h'h'h'^{-1})dh'dh'dx
\]

\[
= \frac{1}{\text{vol}(D^{\times}/\mathcal{O})} \int_{D^{\times}/\mathcal{O}} O_{g^{-1}}(\eta_m, h) \theta_\rho(h)dh
\]

\[
= \frac{n}{\text{vol}(D^{\times}/\mathcal{O})} \int_{D^{\times}/\mathcal{O}} \# \{ a \in D^{\times}/\mathcal{O} (1+\Pi^m\mathcal{O}_D) \mid hah_g = a \} \theta_\rho(h)dh
\]

\[
= \frac{n}{\text{#}(D^{\times}/\mathcal{O}(1+\Pi^m\mathcal{O}_D))} \times \sum_{h \in D^{\times}/\mathcal{O}(1+\Pi^m\mathcal{O}_D)} \# \{ a \in D^{\times}/\mathcal{O}(1+\Pi^m\mathcal{O}_D) \mid hah_g = a \} \theta_\rho(h)
\]
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\[ n \quad \text{as desired.} \]

This completes the proof. \[ \square \]

3 Some harmonic analysis

In this section, we use Theorem 2.4 to investigate the virtual \( D^x \)-representation \( \sum_{i,j \geq 0} (-1)^{i+j} \text{Ext}^j_{G/\Sigma}(H_{\text{Dr}}, \pi) \). In the Lubin-Tate case, a similar study is carried out in [Mie11]. For \( m \geq 0, \) denote by \( K'_m \) the image of \( 1 + \Pi^m \mathcal{O}_D \) in \( D^\times/\Sigma^\times \).

**Lemma 3.1** For a smooth representation \( V \) of \( G/\Sigma^\times \times D^\times/\Sigma^\times \) and a smooth representation \( \pi \) of \( G/\Sigma^\times \), we have \( \text{Ext}^j_{G/\Sigma^\times}(V, \pi)^{K'_m} \cong \text{Ext}^j_{G/\Sigma^\times}(V^{K'_m}, \pi) \). Here \( \text{Ext}^j_{G/\Sigma^\times} \) is taken in the category of smooth \( G/\Sigma^\times \)-representations.

**Proof.** First we prove that there exist a smooth \( G/\Sigma^\times \times D^\times/\Sigma^\times \)-representation \( P \) which is projective as a \( G/\Sigma^\times \)-representation and a \( G/\Sigma^\times \times D^\times/\Sigma^\times \)-equivariant surjection \( P \rightarrow V \). Take a \( G/\Sigma^\times \)-equivariant surjection \( P' \rightarrow V \rightarrow 0 \) from a projective \( G/\Sigma^\times \)-representation \( P' \). Put \( P = P' \otimes \mathbb{C}_\infty(D^\times/\Sigma^\times) \). Then \( P \) is a smooth \( G/\Sigma^\times \times D^\times/\Sigma^\times \)-representation which is projective as a \( G/\Sigma^\times \)-representation, and a surjection \( P \rightarrow V \) is naturally induced.

Therefore, we can take a \( G/\Sigma^\times \times D^\times/\Sigma^\times \)-equivariant resolution \( P_\bullet \rightarrow V \rightarrow 0 \) of \( V \) such that \( P_i \) is projective as a smooth \( G/\Sigma^\times \)-representation. Since \( P_i^{K'_m} \) is a direct summand of \( P_i \) as a \( G/\Sigma^\times \)-representation, \( P_i^{K'_m} \) is also projective. Thus \( P_\bullet^{K'_m} \rightarrow V^{K'_m} \rightarrow 0 \) gives a projective resolution of \( V^{K'_m} \). Hence we have

\[
\text{Ext}^j_{G/\Sigma^\times}(V, \pi)^{K'_m} = \text{Ext}^j_{G/\Sigma^\times}(V^{K'_m}, \pi),
\]

as desired. \[ \square \]

**Corollary 3.2** For every \( m \geq 0 \) and \( \pi \in \text{Irr}(G/\Sigma^\times) \), we have \( \text{Ext}^j_{G/\Sigma^\times}(H_{\text{Dr}}, \pi)^{K'_m} \cong \text{Ext}^j_{G/\Sigma^\times}(H_{\text{Dr},m}, \pi) \).

It is finite-dimensional and vanishes if \( j \geq n \). In particular, \( \text{Ext}^j_{G/\Sigma^\times}(H_{\text{Dr},m}^{(j')}^{sm}) \) is an admissible representation of \( D^\times/\Sigma^\times \) vanishes if \( j \geq n \), where \((-)^{sm}\) denotes the set of \( D^\times/\Sigma^\times \)-smooth vectors.

**Proof.** Lemma 3.1 tells us that

\[
\text{Ext}^j_{G/\Sigma^\times}(H_{\text{Dr}}, \pi)^{K'_m} = \text{Ext}^j_{G/\Sigma^\times}((H_{\text{Dr}})^{K'_m}, \pi) = \text{Ext}^j_{G/\Sigma^\times}(H_{\text{Dr},m}, \pi).
\]

By Proposition 2.1 and [SS97, Corollary II.3.3], it is finite-dimensional and vanishes if \( j \geq n \). \[ \square \]
Remark 3.3 Later (Corollary 4.3) we will prove that $\operatorname{Ext}^j_{G/\mathcal{O}}(H^i_{\text{Dr}}, \pi)$ is in fact a finite-dimensional smooth $D^\times$-representation.

The character of $\sum_{i,j \geq 0} (-1)^{i+j} \operatorname{Ext}^j_{G/\mathcal{O}}(H^i_{\text{Dr}}, \pi)$ can be computed by $\eta_{m,h}$ introduced in the previous section:

**Proposition 3.4** For every $\pi \in \text{Irr}(G/\mathcal{O})$ and $h \in D^\times/\mathcal{O}$, we have

$$
\sum_{i,j \geq 0} (-1)^{i+j} \operatorname{Tr}(h; \operatorname{Ext}^j_{G/\mathcal{O}}(H^i_{\text{Dr}}, \pi)) = \operatorname{Tr}(\eta_{m,h}; \pi)
$$

**Proof.** First note that $V \mapsto \sum_{j \geq 0} (-1)^j \operatorname{Tr}(h, \operatorname{Ext}^j_{G/\mathcal{O}}(V, \pi))$ induces a homomorphism $K(G/\mathcal{O}) \to C$ of abelian groups. Therefore, by Proposition 2.1 and Proposition 2.3, we have only to show the following:

Let $K$ be a compact open subgroup of $G/\mathcal{O}$. For every finite-dimensional smooth representation $\sigma$ of $K \times D^\times/\mathcal{O}$ and $h \in D^\times$, we have

$$
\sum_{j \geq 0} (-1)^j \operatorname{Tr}(h; \operatorname{Ext}^j_{G/\mathcal{O}}(\text{c-Ind}_K^G \sigma, \pi)) = \operatorname{Tr}(\theta_{\sigma^\vee}(-, h); \pi)
$$

Since $\text{c-Ind}_K^G \sigma$ is a projective $G/\mathcal{O}$-representation, $\operatorname{Ext}^j_{G/\mathcal{O}}(\text{c-Ind}_K^G \sigma, \pi) = 0$ for $j \geq 1$. Take an open normal subgroup $K_1 \subset K$ such that $\sigma|_{K_1}$ is trivial. Then the left hand side can be computed as follows:

$$
\sum_{j \geq 0} (-1)^j \operatorname{Tr}(h; \operatorname{Ext}^j_{G/\mathcal{O}}(\text{c-Ind}_K^G \sigma, \pi)) = \operatorname{Tr}(h; \operatorname{Hom}_G(\text{c-Ind}_K^G \sigma, \pi))
$$

$$
= \frac{1}{\#(K/K_1)} \sum_{g \in K/K_1} \operatorname{Tr}((g^{-1}, h^{-1}); \sigma) \operatorname{Tr}(g; \pi_{K_1})
$$

$$
= \frac{1}{\operatorname{vol}(K)} \operatorname{Tr}(\theta_{\sigma^\vee}(-, h); \pi).
$$

This completes the proof.

**Lemma 3.5** For every $g \in G^{\text{ad}}$, $h \in D^\times$ and an integer $m \geq 0$, we have

$$
O^G_{g/\mathcal{O}}(\eta_{m,h}) = nO^{D^\times/\mathcal{O}}_{h_g}(\frac{1_{h_K'^m}}{\operatorname{vol}(K_m')}),
$$

where $1_{h_K'^m}$ denotes the characteristic function of $hK_m'$.

**Proof.** By Theorem 2.4 ii), we obtain

$$
O^G_{g/\mathcal{O}}(\eta_{m,h}) = n \cdot \# \{ a \in (D^\times/\mathcal{O})/K_m' \mid hah_g^{-1} = a \}
$$

$$
= \frac{n}{\operatorname{vol}(K_m')} \int_{D^\times/\mathcal{O}} 1_{hK_m'}(ah_ga^{-1}) da = nO^{D^\times/\mathcal{O}}_{h_g}(\frac{1_{h_K'^m}}{\operatorname{vol}(K_m')}).
$$
Next recall the definition of a transfer of a test function.

**Definition 3.6** For \( \varphi \in \mathcal{H}(G) \) and \( \varphi^D \in \mathcal{H}(D^\times) \), we say that \( \varphi^D \) is a transfer of \( \varphi \) if

\[
\int_{D^\times/\omega^\times} \varphi^D(y^{-1}h_gy)dy = (-1)^{n-1} \int_{G/\omega^\times} \varphi(x^{-1}gx)dx
\]

for every \( g \in G^{\text{ell}} \).

We know that if \( \varphi \in \mathcal{H}(G) \) is supported on \( G^{\text{ell}} \), then it has a transfer \( \varphi^D \in \mathcal{H}(D^\times) \) (cf. [Mie11, Lemma 3.2]). The following lemma is obvious:

**Lemma 3.7** Assume that \( \varphi \in \mathcal{H}(G) \) is supported on \( G^{\text{ell}} \) and let \( \varphi^D \in \mathcal{H}(D^\times) \) be its transfer. Put

\[
\varphi_\omega(g) = \sum_{i \in \mathbb{Z}} \varphi(\omega^i g), \quad \varphi^D_\omega(h) = \sum_{i \in \mathbb{Z}} \varphi^D(\omega^i h).
\]

Then \( \varphi_\omega \in \mathcal{H}(G/\omega^\times) \), \( \varphi^D_\omega \in \mathcal{H}(D^\times/\omega^\times) \) and \( O_{h_g}^{D^\times/\omega^\times}(\varphi^D_\omega) = (-1)^{n-1}O_g^{G/\omega^\times}(\varphi_\omega) \) for every \( g \in G^{\text{ell}} \).

**Theorem 3.8** Assume that \( \varphi \in \mathcal{H}(G) \) is supported on \( G^{\text{ell}} \) and let \( \varphi^D \in \mathcal{H}(D^\times) \) be its transfer. Then, for every \( \pi \in \text{Irr}(G/\omega^\times) \) we have

\[
\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D; \text{Ext}^j_{G/\omega^\times}(H^i_{Dr}, \pi)^{sm}) = (-1)^{n-1}n \text{Tr}(\varphi; \pi).
\]

**Proof.** Let \( \varphi_\omega \) and \( \varphi^D_\omega \) be as in the previous lemma. Then clearly we have

\[
\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D; \text{Ext}^j_{G/\omega^\times}(H^i_{Dr}, \pi)^{sm}) = \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D_\omega; \text{Ext}^j_{G/\omega^\times}(H^i_{Dr}, \pi)^{sm}),
\]

\[
(-1)^{n-1}n \text{Tr}(\varphi; \pi) = (-1)^{n-1}n \text{Tr}(\varphi_\omega; \pi).
\]

Thus we may replace \( \varphi \) and \( \varphi^D \) by \( \varphi_\omega \) and \( \varphi^D_\omega \), respectively.

Take \( m \geq 0 \) such that \( \varphi^D_\omega \) is \( K'_m \)-invariant, and write

\[
\varphi^D_\omega = \sum_{h \in J} a_h \frac{1}{\text{vol}(K'_m)}
\]

where \( J \) is a finite subset of \( D^\times/\omega^\times \) and \( a_h \in \mathbb{C} \). By Corollary 3.2 and Proposition 3.4 we have

\[
\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D_\omega; \text{Ext}^j_{G/\omega^\times}(H^i_{Dr}, \pi)^{sm}) = \sum_{i,j \geq 0, h \in J} (-1)^{i+j}a_h \text{Tr}(h; \text{Ext}^j_{G/\omega^\times}(H^i_{Dr,m}, \pi))
\]

\[
= \sum_{h \in J} a_h \int_{G/\omega^\times} \eta_{m,h}(g)\theta_\pi(g)dg.
\]
By Theorem 2.4 ii) and Weyl’s integral formula (cf. [Kaz86, Theorem F]), we have
\[
\int_{G/\mathcal{O}} \eta_{m,h}(g) \theta_{\pi}(g) dg = \sum_{T} \frac{1}{\# W_T} \int_{T^{\text{reg}}/\mathcal{O}} D(t) O^G_{t}(\eta_{m,h}) \theta_{\pi}(t) dt,
\]
where \( T \) runs through conjugacy classes of elliptic maximal tori of \( G \), \( W_T \) denotes the rational Weyl group of \( T \) and \( D(t) \) denotes the Weyl denominator (cf. [Rog83, p. 185]). The measure \( dt \) on \( T/\mathcal{O} \) is normalized so that the volume of \( T/\mathcal{O} \) is one.

Lemma 3.5 and Lemma 3.7 tell us that
\[
\sum_{h \in J} a_h O^G_{t}(\eta_{m,h}) = \sum_{h \in J} n a_h O^{D^x}_{h\text{t}} \left( \frac{1}{\text{vol}(K^i_m)} \right) = n O^{D^x}_{h\text{t}} (\varphi_D) = (-1)^{n-1} n O^G_{t}(\varphi) \]
for every \( t \in T^{\text{reg}} \). By Weyl’s integral formula again, we have
\[
\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_D; \text{Ext}^j_{G/\mathcal{O}}(H^i_{\text{LT}}; \pi)^{\text{sm}}) = (-1)^{n-1} n \sum_{T} \frac{1}{\# W_T} \int_{T^{\text{reg}}/\mathcal{O}} D(t) O^G_{t}(\varphi) \theta_{\pi}(t) dt = (-1)^{n-1} n \int_{G/\mathcal{O}} \varphi(g) \theta_{\pi}(g) dg = (-1)^{n-1} n \text{Tr}(\varphi; \pi),
\]
as desired.

4 Faltings isomorphism

Here we freely use the notation in [Mie11, Section 2]. We need the following deep theorem due to Faltings ([Fal02], see also [FGL08] for more detailed exposition):

**Theorem 4.1** We have a \( G \times D^x \)-equivariant isomorphism \( H^i_{\text{Dr}} \cong H^i_{\text{LT}} \) for every \( i \).

Note that the proof of Faltings’ theorem does not require automorphic method. It gives the following very important finiteness result on \( H^i_{\text{Dr}} \).

**Corollary 4.2** The \( G \)-representation \( H^i_{\text{Dr}} \) is admissible.

**Proof.** Put \( K_m = \text{Ker}(\text{GL}_n(\mathcal{O}) \to \text{GL}_n(\mathcal{O}/(\mathcal{O}^m))) \). Since
\[
(H^i_{\text{LT}})^{K_m} = H^i((M_m/\mathcal{O}) \otimes F, \mathcal{O}_F, \mathcal{O}_F)
\]
is finite-dimensional, \( H^i_{\text{LT}} \) is an admissible representation of \( G \). Thus, by Theorem 4.1, \( H^i_{\text{Dr}} \) is also an admissible representation of \( G \).
**Corollary 4.3** For every $\pi \in \text{Irr}(G/\varpi^2)$ and integers $i, j \geq 0$, $\text{Ext}^j_{G/\varpi^2}(H_{Dr}^i, \pi)$ is a finite-dimensional smooth representation of $D^\times$. Moreover, $\text{Ext}^j_{G/\varpi^2}(H_{Dr}^i, \pi) = 0$ if $j \geq n$.

**Proof.** Let $s$ be the cuspidal support of $\pi$, and $H^i_{Dr,s}$ be the $s$-component of $H^i_{Dr}$. Clearly we have $\text{Ext}^j_{G/\varpi^2}(H^i_{Dr}, \pi) = \text{Ext}^j_{G/\varpi^2}(H^i_{Dr,s}, \pi)$.

Let us observe that $H^i_{Dr,s}$ is a finitely generated $G$-representation. Since $H^i_{Dr,s}$ is an admissible $G$-representation by Corollary 4.2, it is $\mathcal{Z}(G)$-admissible, where $\mathcal{Z}(G)$ denotes the Bernstein center of $G$ (cf. [Ber84, §3.1]). Therefore, [Ber84, Corollaire 3.10] tells us that $H^i_{Dr,s}$ is finitely generated. In particular, there exists a compact open subgroup $K' \subset D\times$ which acts on $H^i_{Dr,s}$ trivially.

Therefore, by [SS97, Corollary II.3.3], $\text{Ext}^j_{G/\varpi^2}(H^i_{Dr,s}, \pi)$ is finite-dimensional and vanishes if $j \geq n$. The natural action of $D^\times$ on $\text{Ext}^j_{G/\varpi^2}(H^i_{Dr,s}, \pi)$ is smooth, since the action of $K'$ is trivial. This completes the proof. □

**Definition 4.4** For $\pi \in \text{Irr}(G/\varpi^2)$, put $H^i_{Dr}[\pi] = \sum_{i,j \geq 0} (-1)^{i+j} \text{Ext}^j_{G/\varpi^2}(H^i_{Dr}, \pi)$ in $R(G/\varpi^2)$.

We can consider the character $\theta_{H^i_{Dr}[\pi]}$ of $H^i_{Dr}[\pi]$. Theorem 3.8 can be written in the following way:

**Theorem 4.5** For every $\pi \in \text{Irr}(G/\varpi^2)$ and $h \in (D^\times)^{\text{reg}}$, we have

$$\theta_{H^i_{Dr}[\pi]}(h) = n \theta_{\pi}(g_h).$$

**Proof.** Theorem 3.8 says that $\text{Tr}(\varphi^D; H^i_{Dr}[\pi]) = (-1)^{n-1} n \text{Tr}(\varphi; \pi)$. We can use exactly the same method as in the proof of [Mie11, Theorem 4.3]. □

The following is another consequence of Theorem 4.1.

**Corollary 4.6** For every $\rho \in \text{Irr}(D^\times/\varpi^2)$, $H^i_{Dr}[\rho] = \text{Hom}_{D^\times}(H^i_{Dr}, \rho)^{sm}$ is a smooth $G$-representation of finite length. Moreover $H^i_{Dr}[\rho]$ is isomorphic to $\text{Hom}_{D^\times}(\rho, H^i_{Dr})^\vee$.

**Proof.** By Proposition 2.1, Corollary 4.2 and [Mie11, Lemma 5.2], $H^i_{Dr}[\rho]$ is a smooth $G$-representation of finite length. In the proof of [Mie11, Lemma 5.2], a $G$-equivariant injection $H^i_{Dr}[\rho] \hookrightarrow \text{Hom}_{D^\times}(\rho, H^i_{Dr})^\vee$ is constructed. It is easy to see that it is actually an isomorphism. □

**Definition 4.7** For $\rho \in \text{Irr}(D^\times/\varpi^2)$, put $H^i_{Dr}[\rho] = \sum_i (-1)^i H^i_{Dr}[\rho]$ in $R(G/\varpi^2)$. 

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We can consider the character $\theta_{H^{\text{Dr}}[\rho]}$ of $H^{\text{Dr}}[\rho]$. Corollary 2.5 can be written in the following way:

**Theorem 4.8** For every $\rho \in \text{Irr}(D^\times/\mathbb{Z})$ and $h \in (D^\times)^{\text{reg}}$, we have

$$\theta_{H^{\text{Dr}}[\rho]}(gh) = n\theta_{\rho}(h).$$

**Proof.** We denote the natural homomorphism $R(G/\mathbb{Z}) \rightarrow K(G/\mathbb{Z})$ by $\text{EP}$. By [Dat07, Lemma 3.7], the composite of

$$R(G/\mathbb{Z}) \xrightarrow{\text{EP}} K(G/\mathbb{Z}) \xrightarrow{\text{Rk}} H(G/\mathbb{Z}) \xrightarrow{\vee} C^\infty(G^{\text{ell}})$$

coincides with $\pi \mapsto \theta_{\pi}|_{G^{\text{ell}}}$ (it was originally proved in [SS97, Theorem III.4.23]). Therefore, by Corollary 2.5 and Corollary 4.6 we have

$$\theta_{H^{\text{Dr}}[\rho]}(g) = ((\vee \circ \text{Rk} \circ \text{EP})(H^{\text{Dr}}[\rho]))(g) = n\theta_{\rho}(h_g)$$

for every $g \in G^{\text{ell}}$. Hence $\theta_{H^{\text{Dr}}[\rho]}(gh) = n\theta_{\rho}(h)$ for every $h \in (D^\times)^{\text{reg}}$, as desired. 

**Remark 4.9** The proof of [SS97, Theorem III.4.23] seems to use [Kaz86, Theorem 0], whose proof relies on global technique. However, Theorem 4.1 and [Mie11, Theorem 4.3] give an alternative proof of Theorem 4.8 which does not involve any global argument.

## 5 Complements on representation theory

For locally constant class functions $\varphi_1, \varphi_2$ on $G^{\text{ell}}/\mathbb{Z}$, put

$$\langle \varphi_1, \varphi_2 \rangle_{\text{ell}} = \sum_T \frac{1}{\#W_T} \int_{T/\mathbb{Z}} D(t)\varphi_1(t)\overline{\varphi_2(t)} \, dt,$$

where $T$ runs through conjugacy classes of elliptic maximal tori of $G$. Other notations are also the same as in the proof of Theorem 3.8.

For locally constant functions $\phi_1, \phi_2$ on $D^\times/\mathbb{Z}$, put

$$\langle \phi_1, \phi_2 \rangle = \int_{D^\times/\mathbb{Z}} \phi_1(h)\overline{\phi_2(h)} \, dh,$$

where the measure $dh$ is normalized so that the volume of the compact group $D^\times/\mathbb{Z}$ is one.

These two pairings are compatible, in the sense of the following lemma:

**Lemma 5.1** Let $\varphi_1, \varphi_2$ be locally constant class functions on $G^{\text{ell}}/\mathbb{Z}$, and $\phi_1, \phi_2$ locally constant class functions on $D^\times/\mathbb{Z}$. Assume that $\varphi_i(gh) = \phi_i(h)$ for every $h \in (D^\times)^{\text{reg}}$. Then, we have

$$\langle \varphi_1, \varphi_2 \rangle_{\text{ell}} = \langle \phi_1, \phi_2 \rangle.$$
Geometric approach to the local Jacquet-Langlands correspondence

Proof. Clear from Weyl's integral formula for $D^\times$.

The following orthogonality relation of characters is very important for our work.

**Proposition 5.2** For $\pi_1, \pi_2 \in \text{Disc}(G/\mathcal{O})$, we have

$$\langle \theta_{\pi_1}, \theta_{\pi_2} \rangle_\text{ell} = \begin{cases} 1 & \pi_1 \cong \pi_2, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** Let $\omega_1, \omega_2$ be the central characters of $\pi_1, \pi_2$, respectively. Then they are unitary, since $F^\times/\mathcal{O}^\times$ is compact. If $\omega_1 = \omega_2$, then the lemma follows immediately from [Rog83, Lemma 5.3]. Otherwise,

$$\int_{T/\mathcal{O}} D(t)\theta_{\pi_1}(t)\overline{\theta_{\pi_2}(t)} \, dt = \int_{T/\mathcal{O}} \left( \int_{ZG/\mathcal{O}} D(tz)\theta_{\pi_1}(tz)\overline{\theta_{\pi_2}(tz)} \, dz \right) dt$$

$$= \int_{T/\mathcal{O}} \left( \int_{ZG/\mathcal{O}} \omega_1(z)\overline{\omega_2(z)} \, dz \right) D(t)\theta_{\pi_1}(t)\overline{\theta_{\pi_2}(t)} \, dt = 0,$$

as desired. \hfill \square

**Remark 5.3** The proof of [Rog83, Theorem 5.3] (for example, [DKV84, §A.3, §A.4]) seems to need a global argument, such as Howe's conjecture due to Clozel. However, at least if $n$ is prime, we can give a purely local proof of Proposition 5.2 as follows. Here we use freely the notation which will be introduced in the next section.

Note that, if $n$ is prime, then any irreducible discrete series representation of $G$ is either a twisted Steinberg representation or supercuspidal (cf. [Zel80, Theorem 9.3]). First assume that both $\pi_1$ and $\pi_2$ are twisted Steinberg representations, and write $\pi_1 = \text{St}_{\chi_1}$ and $\pi_2 = \text{St}_{\chi_2}$, where $\chi_1$ and $\chi_2$ are characters of $F^\times/\mathcal{O}^\times$. Then, by Lemma §5.1 and Lemma §6.7, we have

$$\langle \theta_{\text{St}_{\chi_1}}, \theta_{\text{St}_{\chi_2}} \rangle_\text{ell} = \langle \theta_{\chi_1 \circ \text{Nrd}}, \theta_{\chi_2 \circ \text{Nrd}} \rangle = \int_{F^\times/\mathcal{O}^\times} \chi_1(z)\overline{\chi_2(z)} \, dz = \begin{cases} 1 & \chi_1 = \chi_2, \\ 0 & \chi_1 \neq \chi_2. \end{cases}$$

Since $\chi_1 \neq \chi_2$ implies $\text{St}_{\chi_1} \ncong \text{St}_{\chi_2}$ (their characters are different), we have the orthogonality relation in this case.

Next assume that $\pi_2$ is supercuspidal. Then, by [DKV84, §A.3.e, §A.3.g], we can find a matrix coefficient $\phi \in \mathcal{H}(G/\mathcal{O})$ of $\pi_2$ satisfying the following:

(a) $O_{G/\mathcal{O}}^\times(\phi) = \overline{\theta_{\pi_2}(g)}$ for $g \in G^{\text{ell}}$ and $\int_{Z(G)} \phi(x^{-1}gx) \, dx = 0$ for $g \in G^{\text{res}} \setminus G^{\text{ell}}$.

(b) $\text{Tr}(\phi; \pi_1) = 1$ and $\text{Tr}(\phi; \pi_1) = 0$ for $\pi_1 \in \text{Disc}(G/\mathcal{O})$ with $\pi_1 \ncong \pi_2$.

By (a) and Weyl’s integral formula, we have

$$\text{Tr}(\phi; \pi_1) = \int_{G/\mathcal{O}} \phi(g)\theta_{\pi_1}(g) \, dg = \sum_T \frac{1}{\#W_T} \int_{T/\mathcal{O}} D(t)O_{G/\mathcal{O}}^\times(\phi)\theta_{\pi_1}(t) \, dt$$

$$= \sum_T \frac{1}{\#W_T} \int_{T/\mathcal{O}} D(t)\overline{\theta_{\pi_2}(t)} \theta_{\pi_1}(t) \, dt = \langle \theta_{\pi_1}, \theta_{\pi_2} \rangle_\text{ell}.$$

Therefore (b) gives the desired orthogonality relation.
6 Local Jacquet-Langlands correspondence

Let $R_I(G)$ be the submodule of $R(G)$ generated by the image of parabolically induced representations (cf. [Kaz86]) and put $\overline{R}(G) = R(G)/R_I(G)$. It is known that $\overline{R}(G)$ is a free $\mathbb{Z}$-module with a basis $\{[\pi] \mid \pi \in \text{Disc}(G)\}$ (cf. [Dat10], Lemma 2.1.4). We regard Disc$(G)$ as a subset of $\overline{R}(G)$. Recall that the character of a parabolically induced representation vanishes on $G^{\text{ell}}$. Therefore, $\pi \mapsto \theta_\pi |_{G^{\text{ell}}} \text{ induces a map } \overline{R}(G) \to C^\infty(G^{\text{ell}})$.

Moreover, we denote by $\overline{R}(G/\omega^Z)$ the image of $R(G/\omega^Z)$ in $\overline{R}(G)$. It is easy to see that $\{[\pi] \mid \pi \in \text{Disc}(G/\omega^Z)\}$ gives a basis of $\overline{R}(G/\omega^Z)$. Set $\overline{R}(G/\omega^Z)_Q = \overline{R}(G/\omega^Z) \otimes \mathbb{Q}$ and $\overline{R}(G)_Q = \overline{R}(G) \otimes \mathbb{Q}$.

Lemma 6.1 The homomorphism $\overline{LJ} : R(G/\omega^Z) \to R(D^\times/\omega^Z)_Q$ given by

$$\pi \mapsto \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\pi] \quad \text{for } \pi \in \text{Irr}(G/\omega^Z)$$

factors through $\overline{R}(G/\omega^Z)$.

Proof. Let $P$ be a proper parabolic subgroup of $G$ with Levi factor $M$ and $\sigma$ an irreducible smooth representation of $M$ on which $\omega \in Z_M$ acts trivially. By Theorem 4.5 the character of $\overline{LJ}(\text{Ind}_P^G \sigma)$ on $(D^\times)^\text{reg}$ is given by $h \mapsto (-1)^{n-1} \theta_{\text{Ind}_P^G \sigma}(g_h) = 0$. Since $(D^\times)^\text{reg}$ is dense in $D^\times$, it vanishes for every $h \in D^\times$. Thus $\overline{LJ}(\text{Ind}_P^G \sigma) = 0$ by linear independence of characters. Since the kernel of $R(G/\omega^Z) \to \overline{R}(G/\omega^Z)$ is generated by such representations as $\text{Ind}_P^G \sigma$, we conclude the proof.

The following is the main construction in this paper.

Definition 6.2 We define two homomorphisms

$$JL : R(D^\times/\omega^Z)_Q \to \overline{R}(G/\omega^Z)_Q, \quad LJ : \overline{R}(G/\omega^Z)_Q \to R(D^\times/\omega^Z)_Q$$

by

$$JL(\rho) = \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\rho], \quad LJ(\pi) = \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\pi]$$

for $\rho \in \text{Irr}(D^\times/\omega^Z)$ and $\pi \in \text{Irr}(G/\omega^Z)$. The latter map is well-defined by the previous lemma.

Proposition 6.3 i) We have the character relations

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every $\rho \in \text{Irr}(D^\times/\omega^Z)$, $\pi \in \text{Irr}(G/\omega^Z)$ and $h \in (D^\times)^\text{reg}$.

ii) For every $\rho, \rho' \in \text{Irr}(D^\times/\omega^Z)$ and $\pi, \pi' \in \text{Irr}(G/\omega^Z)$, we have

$$\langle \theta_{JL(\rho)}, \theta_{JL(\rho')} \rangle_{\text{ell}} = \langle \theta_\rho, \theta_{\rho'} \rangle_{\text{ell}}, \quad \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle = \langle \theta_\pi, \theta_{\pi'} \rangle_{\text{ell}}, \quad \langle \theta_{JL(\rho)}, \theta_{\pi} \rangle_{\text{ell}} = \langle \theta_\rho, \theta_{LJ(\pi)} \rangle_{\text{ell}}$$.
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iii) Two maps \( \text{JL} \) and \( \text{LJ} \) are inverse to each other.

iv) The map \( \text{JL} \) is compatible with character twists. Namely, for a character \( \chi \) of \( F^\times \) which is trivial on \( \mathbb{C}^\times \), we have \( \text{JL}(\rho \otimes (\chi \circ \text{Nrd})) = \text{JL}(\rho) \otimes (\chi \circ \det) \). The same holds for \( \text{LJ} \).

v) The map \( \text{JL} \) preserves central characters. Namely, for \( \rho \in \text{Irr}(D^\times / \mathbb{C}^\times) \), write \( \text{JL}(\rho) = \sum_{\pi \in \text{Disc}(G/\mathbb{C})} a_\pi [\pi] \). Then, every \( \pi \) with \( a_\pi \neq 0 \) has the same central character as \( \rho \). The same holds for \( \text{LJ} \).

Proof. i) is clear from Theorem 1.5 and Theorem 1.8. ii) follows from i) and Lemma 5.1.

Prove iii). For \( \pi \in \text{Disc}(G/\mathbb{C}) \), write \( \text{JL}(\text{LJ}(\pi)) = \sum_{\pi' \in \text{Disc}(G/\mathbb{C})} a_{\pi'}[\pi'] \).

Then, by ii) and Proposition 5.2 we have
\[
a_{\pi'} = \langle \theta_{\text{JL}(\text{LJ}(\pi))}, \theta_{\pi'} \rangle_{\text{ell}} = \langle \theta_{\text{LJ}(\pi)}, \theta_{\text{LJ}(\pi')} \rangle = \langle \theta_{\pi}, \theta_{\pi'} \rangle_{\text{ell}}.
\]

Therefore, \( a_{\pi'} = 1 \) if \( \pi' = \pi \), and \( a_{\pi'} = 0 \) otherwise. In other words, \( \text{JL}(\text{LJ}(\pi)) = \pi \). Thus we have \( \text{JL} \circ \text{LJ} = \text{id} \). Similarly we can prove that \( \text{LJ} \circ \text{JL} = \text{id} \).

For iv), it suffices to show that \( \text{LJ} \) is compatible with character twists. Let \( \pi \in \text{Irr}(G/\mathbb{C}) \) and \( \chi \) be a character of \( F^\times \) which is trivial on \( \mathbb{C}^\times \). Then, for every \( h \in (D^\times)^{\text{reg}}, \) we have
\[
\theta_{\text{LJ}(\pi)}(h) = (-1)^{n-1}\theta_{\pi}(h) = (-1)^{n-1}\chi(\det h)\theta_{\pi}(h) = \chi(\text{Nrd} h)\theta_{\pi}(h) = \theta_{\text{LJ}(\pi)}(h) = \theta_{\text{LJ}(\pi)}(h).
\]

Since \( (D^\times)^{\text{reg}} \) is dense in \( D^\times \), we have \( \theta_{\text{LJ}(\pi)}(\chi(\text{det})) = \theta_{\text{LJ}(\pi)}(\chi(\text{Nrd})) \). By linear independence of characters, we conclude that \( \text{LJ}(\pi) \otimes (\chi \circ \text{det}) = \text{LJ}(\pi) \otimes (\chi \circ \text{Nrd}) \).

Finally we prove v). Write \( \text{JL}(\rho) = \sum_{\pi \in \text{Disc}(G/\mathbb{C})} a_{\pi} [\pi] \). By Proposition 5.2, \( a_{\pi} = \langle \theta_{\text{JL}(\rho)}, \theta_{\pi} \rangle_{\text{ell}} \). Assume that the central character of \( \pi \in \text{Disc}(G/\mathbb{C}) \) is different from that of \( \rho \). Then, we have
\[
a_{\pi} = \langle \theta_{\text{JL}(\rho)}, \theta_{\pi} \rangle_{\text{ell}} = \sum_{T} \frac{1}{\# W_T} \int_{T/\mathbb{C}} D(t)\theta_{\text{JL}(\rho)}(t)\overline{\theta_{\pi}(t)}dt = 0
\]
in the same way as in the proof of Proposition 5.2. Therefore \( \text{JL} \) preserves central characters. By this result and ii), we have \( \langle \theta_{\rho}, \theta_{\text{LJ}(\pi)} \rangle = 0 \) unless \( \rho \) and \( \pi \) have the same central character. This means that the coefficient of \( \rho \in \text{Irr}(D^\times / \mathbb{C}) \) in \( \text{LJ}(\pi) \) is zero unless the central character of \( \rho \) is the same as that of \( \pi \). Namely, \( \text{LJ} \) preserves central characters.

By twisting, we can extend \( \text{JL} \) and \( \text{LJ} \) to maps between \( R(D^\times) \) and \( \overline{\text{R}}(G) \).

**Proposition 6.4** There exists a unique extension of \( \text{JL} \) to a homomorphism from \( R(D^\times)_Q \) to \( \overline{\text{R}}(G)_Q \) which is compatible with character twists. Similarly, we have a unique extension of \( \text{LJ} \) to a homomorphism \( \overline{\text{R}}(G)_Q \rightarrow R(D^\times)_Q \) which is compatible with character twists. We denote them by \( \text{JL} \) and \( \text{LJ} \) again. These are inverse to each other, satisfy the same character relations as in Proposition 6.3 ii), and preserve central characters.
Proof. For $\rho \in \text{Irr}(D^\times)$, let $\omega_\rho$ be its central character. Take $c \in \mathbb{C}^\times$ such that $c^n = \omega_\rho(z)$, and consider the character $\chi_c: z \mapsto e^{\rho(z)}$ of $F^\times$. Then $\rho \otimes (\chi_c^{-1} \circ Nrd) \in \text{Irr}(D^\times/\mathbb{Z})$. Extend $JL$ to $R(D^\times)_Q \rightarrow \overline{R}(G)_Q$ by

$$JL(\rho) = JL(\rho \otimes (\chi_c^{-1} \circ Nrd)) \otimes (\chi_c \circ \det).$$

By Proposition 6.3 iv), it is independent of the choice of $c$. Moreover, we can easily observe that it is the unique extension of the original $JL$ which is compatible with character twists. Similarly, we can uniquely extend $LJ$ to a map $\overline{R}(G)_Q \rightarrow R(D^\times)_Q$ compatible with character twists.

By using Proposition 6.3 v), we can easily check that the extended $JL$ and $LJ$ are inverse to each other. The remaining parts are also immediate consequences of Proposition 6.3 i), v).

Next we will observe the uniqueness of the maps $JL$, $LJ$ satisfying the character relations.

**Proposition 6.5**

i) Let $JL': R(D^\times)_Q \rightarrow \overline{R}(G)_Q$ be a homomorphism satisfying the character relation $\theta_\rho(h) = (-1)^{n-1} \theta_{JL'(\rho)}(g_h)$ for every $h \in (D^\times)^{\text{reg}}$. Then we have $JL' = JL$.

ii) Let $LJ': \overline{R}(G)_Q \rightarrow R(D^\times)_Q$ be a homomorphism satisfying the character relation $\theta_\pi(h) = (-1)^{n-1} \theta_{LJ'(\pi)}(h)$ for every $h \in (D^\times)^{\text{reg}}$. Then we have $LJ' = LJ$.

**Proof.** To prove i), it suffices to show that $LJ \circ JL' = id$. By the character relation, we have $\theta_{LJ(JL'(\rho))}(h) = \theta_\rho(h)$ for every $\rho \in \text{Irr}(D^\times)$ and $h \in (D^\times)^{\text{reg}}$. Thus we can conclude that $LJ(JL'(\rho)) = \rho$ by linear independence of characters for $D^\times$. For ii), prove $LJ' \circ JL = id$ by a similar argument.

So far, we have obtained the following theorem:

**Theorem 6.6** We can construct the following two homomorphisms geometrically:

$$JL: R(D^\times)_Q \rightarrow \overline{R}(G)_Q, \quad LJ: \overline{R}(G)_Q \rightarrow R(D^\times)_Q.$$  

These two maps are inverse to each other, and satisfy the character relations

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every $h \in (D^\times)^{\text{reg}}$. They are characterized by these character relations. Moreover, $JL$ and $LJ$ are compatible with character twists, and preserve central characters.

Let $B \subset G$ be the Borel subgroup consisting of upper triangular matrices. Recall that the Steinberg representation $\text{St}$ is the unique irreducible quotient of the unnormalized induction $\text{Ind}_B^G 1$ from the trivial character $1$ on $B$. For a character $\chi$ of $F^\times$, put $\text{St}_\chi = \text{St} \otimes (\chi \circ \det)$. A representation of the form $\text{St}_\chi$ is called a twisted Steinberg representation. It is an irreducible discrete series representation of $G$. The following lemma is very well-known:
Lemma 6.7 We have \( \theta_{\chi \circ \text{Nrd}}(h) = (-1)^{n-1}\theta_{\text{St}_\chi}(gh) \) for a character \( \chi \) of \( F^\times \) and \( h \in (D^\times)\text{reg} \).

Proof. In \( \overline{\text{R}}(G) \), we have \([\text{St}_\chi] = (-1)^{n-1}[\chi \circ \det]\) (cf. [Dat07, Remarque 2.1.14]). As the character of a parabolically induced representation vanishes on \( G^{\text{ell}} \), we have

\[
\theta_{\text{St}_\chi}(gh) = (-1)^{n-1}\chi(\det gh) = (-1)^{n-1}\chi(\text{Nrd} h) = (-1)^{n-1}\theta_{\chi \circ \text{Nrd}}(h),
\]

as desired. \( \square \)

Corollary 6.8 For a character \( \chi \) of \( F^\times \), we have \( J\ell(\chi \circ \text{Nrd}) = \text{St}_\chi \) and \( L\ell(\text{St}_\chi) = \chi \circ \text{Nrd} \).

Proof. By Lemma 6.7, we have \( \theta_{L\ell(\text{St}_\chi)} = \theta_{\chi \circ \text{Nrd}} \). Linear independence of characters tells us that \( L\ell(\text{St}_\chi) = \chi \circ \text{Nrd} \). \( \square \)

The following is a consequence of the non-cuspidality result in [Mic10b]:

Proposition 6.9 For an irreducible supercuspidal representation \( \pi \) of \( G \), write

\[
L\ell(\pi) = \sum_{\rho \in \text{Irr}(D^\times)} a_{\rho}[\rho].
\]

Then we have \( a_\rho \geq 0 \) for every \( \rho \).

Proof. We may assume that \( \pi \in \text{Irr}(G/\varpi\mathbb{Z}) \). Since \( \pi \) is injective in the category of smooth \( G/\varpi\mathbb{Z} \)-representations, \( \text{Ext}^j_{G/\varpi\mathbb{Z}}(H^i_{\text{Dr}t}, \pi) = 0 \) unless \( j = 0 \). By Theorem 4.1 and [Mic10b, Theorem 3.7], we have \( \text{Hom}_{G/\varpi\mathbb{Z}}(H^i_{\text{Dr}t}, \pi) = 0 \) unless \( i = n - 1 \). Therefore we have \( L\ell(\pi) = n^{-1}[\text{Hom}_{G/\varpi\mathbb{Z}}(H^{n-1}_{\text{Dr}t}, \pi)] \). This concludes the proof. \( \square \)

Now we can prove the local Jacquet-Langlands correspondence for prime \( n \).

Theorem 6.10 Assume that \( n \) is a prime number. Then \( J\ell \) induces a bijection

\[ J\ell: \text{Irr}(D^\times) \xrightarrow{\cong} \text{Disc}(G) \]

satisfying the character relation \( \theta_\rho(h) = (-1)^{n-1}\theta_{J\ell(\rho)}(gh) \) for every \( h \in (D^\times)\text{reg} \).

Proof. For simplicity, we denote by \( \text{Cusp}(G/\varpi\mathbb{Z}) \) the subset of \( \text{Disc}(G/\varpi\mathbb{Z}) \) consisting of supercuspidal representations. By Theorem 6.6 it suffices to show the following:

(a) For \( \rho \in \text{Irr}(D^\times/\varpi\mathbb{Z}) \), \( J\ell(\rho) \in \text{Disc}(G/\varpi\mathbb{Z}) \).
(b) For \( \pi \in \text{Disc}(G/\varpi\mathbb{Z}) \), \( J\ell(\pi) \in \text{Irr}(D^\times/\varpi\mathbb{Z}) \).
By (a) and Proposition 5.2, there exists some \( \rho \) such that

\[
a_\pi = a_{\text{st}_\chi} = \langle \theta_{\text{LL}(\rho)}, \theta_{\text{st}_\chi} \rangle_{\text{ell}} = \langle \theta_\rho, \theta_{\text{LL}(\text{st}_\chi)} \rangle = \langle \theta_\rho, \theta_{\chi \circ \text{Nrd}} \rangle = 0.
\]

Since \( J_L(\rho) \neq 0 \), there exists at least one \( \pi \in \text{Cusp}(G/\wp^2) \) satisfying \( a_\pi \neq 0 \). Let us observe that such \( \pi \) is unique. Assume that there exist \( \pi, \pi' \in \text{Cusp}(G/\wp^2) \) such that \( a_\pi \) and \( a_{\pi'} \) are non-zero. Then, we have \( \langle \theta_{\text{LL}(\pi)}, \theta_\rho \rangle = \langle \theta_\pi, \theta_{\text{LL}(\rho)} \rangle_{\text{ell}} = a_\pi \neq 0 \), and similarly \( \langle \theta_{\text{LL}(\pi')}, \theta_\rho \rangle \neq 0 \). In other words, if we write

\[
L_J(\pi) = \sum_{\varrho \in \text{Irr}(D^*/\wp^2)} b_\varrho[q], \quad L_J(\pi') = \sum_{\varrho \in \text{Irr}(D^*/\wp^2)} b'_{\varrho}[q],
\]

then \( b_\varrho = a_\pi \) and \( b'_{\varrho} = a_{\pi'} \) are non-zero. On the other hand, Proposition 6.9 tells us that \( b_\varrho \geq 0 \) and \( b'_{\varrho} \geq 0 \) for every \( \varrho \). Thus, by Proposition 6.3 ii), we conclude that

\[
\langle \theta_\pi, \theta_{\pi'} \rangle_{\text{ell}} = \langle \theta_{\text{LL}(\pi)}, \theta_{\text{LL}(\pi')} \rangle = \sum_{\varrho} b_\varrho b'_{\varrho} > 0,
\]

which is equivalent to \( \pi \cong \pi' \) by Proposition 5.2. Now we have \( J_L(\rho) = a_\pi[\pi] \) for some \( \pi \in \text{Cusp}(G/\wp^2) \). Moreover, the argument above tells us that \( a_\pi \geq 0 \). By Proposition 5.2 and Proposition 6.3 ii), we have

\[
1 = \langle \theta_\rho, \theta_\rho \rangle = \langle \theta_{\text{LL}(\rho)}, \theta_{\text{LL}(\rho)} \rangle_{\text{ell}} = a_\pi^2 \langle \theta_\pi, \theta_\pi \rangle_{\text{ell}} = a_\pi^2.
\]

Hence we conclude that \( a_\pi = 1 \) and \( J_L(\rho) = \pi \in \text{Disc}(G/\wp^2) \).

Next prove (b). Write \( L_J(\pi) = \sum_{\rho \in \text{Irr}(D^*/\wp^2)} b_\rho[q] \). Then, since \( J_L \) and \( L_J \) are inverse to each other, we have \( \pi = \sum_{\rho \in \text{Irr}(D^*/\wp^2)} b_\rho[J_L(\rho)] \), and thus

\[
1 = \sum_{\rho \in \text{Irr}(D^*/\wp^2)} b_\rho[J_L(\rho), \theta_\pi]_{\text{ell}}.
\]

By (a) and Proposition 5.2, there exists \( \rho \in \text{Irr}(D^*/\wp^2) \) such that \( \pi = J_L(\rho) \). Then \( L_J(\pi) = \rho \in \text{Irr}(D^*/\wp^2) \), as desired.

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