Scaling of quantum Zeno dynamics in many-body systems

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Abstract
We study the quantum Zeno effect (QZE) in a many-body system, namely the Lipkin–Meshkov–Glick model, coupled to a central qubit. Our result shows that in order to observe QZE in the symmetry-broken phase of the model, the frequency of the projective measurement should be of comparable order to that of the system sizes. However, in the polarized phase of the model, the QZE can be easily observed by frequent measurements.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Quantum Zeno effect (QZE) refers to the suppression of the unitary time evolution of a quantum system provided by external perturbations such as measurement or interaction with the environment [1–7]. It is a phenomenon which is intrinsically related to the projection postulate in quantum mechanics. For an isolated quantum system, its state vector undergoes an unitary evolution according to the Schrödinger equation. At the very beginning, the survival probability for the system to remain in the initial state changes quadratically with the elapsed time, while it decreases exponentially in the long time limit. If a measurement is performed to check whether the system is still in the initial state or not after a certain time, the projection postulate states that the system’s state will collapse to the initial state immediately after the measurement. Therefore, when successive measurements are performed at a time interval within the quadratic decaying region, the unitary evolution of the system is found to be suppressed. In the limit of continuous measurement, the survival probability of the system remains unity and the dynamic evolution of the system is frozen.

The possibility of such an effect to be found in an unstable quantum system was first pointed out by Fonda et al [8] and Degasperis et al [9]. They suggested that the decay rate of an unstable system may be dependent on the frequency of intermediate measurements.
Later, Misra et al., who gave the effect its name, presented a general formalism concerning the semigroups to analyze the QZE [1]. Experimentally, the QZE was first manifest in a radio-frequency transition between two laser-cooled beryllium ion ground-state levels [10].

In recent decades, QZE has been studied intensively within the content of quantum optics. Among those analyses, the systems under consideration are only of a few levels in which the QZE can be easily observed by frequent measurements. However, little attention has been paid so far to the possibility of observing QZE in quantum many-body systems. Primary motivations of our work are to investigate the QZE from the viewpoint of condensed matter physics and to obtain criteria for analyzing how frequent the measurements should be compared to the system size in order to observe the QZE in many-body systems.

Mathematically speaking, we are interested in the scaling behavior of the survival probability of the system’s initial state in the short-time limit under the influence of an external perturbation. We reviewed the formulation in obtaining the survival probability from the perturbation form of the Loschmidt echo. The leading term of the survival probability is nothing but the fluctuation in the interaction Hamiltonian. Specifically, we take the Lipkin–Meshkov–Glick (LMG) model as an example and calculate the leading term of the survival probability analytically and obtain its scaling behavior. In our analysis, we allowed the model to couple to a central qubit. The significance of the central qubit is just to provide an external perturbation in the Hamiltonian and to allow us to study how the system’s evolution is affected by the perturbation in the short-time regime. A point to note here is that the central qubit only serves as a source of external perturbation and how it evolves is not of interest to us. Moreover, we would like to point out that the leading term of the survival probability in our study is just the same as the linear response in the study of quench dynamics [11–13].

The paper is organized as follows. In section 2, we present a general mathematical formulation for obtaining the leading term of the survival probability of the system under external perturbations in the short-time limit. Then in section 3, we take the Lipkin–Meshkov–Glick (LMG) model as an example to illustrate the criteria for observing the QZE, in terms of the size dependence of the leading term on the survival probability in the short-time limit. Our analysis shows that the criteria for observing the QZE in the frequency of measurement are different in the two phases of the model. Finally, a summary is given in section 4.

2. Mathematical formulation

Consider a quantum system interacting with the environment, the Hamiltonian of the whole setup can be generally written as

$$H = H_0 + \delta H_I,$$

where $H_0$ describes the initial Hamiltonian of the system and the environment such that $H_0|m\rangle = E_m|m\rangle$. Here $\{|m\rangle\}$ is a set of orthogonal bases constructed from the tensor product of the eigenstate of the system and that of the environment and $E_m$ is the corresponding eigenenergy. $H_I$ is the coupling between the system and the environment with a small parameter $\delta$ denoting the coupling strength.

Suppose the setup is initially in a state $|\psi(0)\rangle = |m\rangle$. It is allowed to evolve freely for a time $\tau$. According to the Schrödinger equation, the state at time $\tau$ is given by $|\psi(\tau)\rangle = e^{-iH\tau}|m\rangle$ and the probability of finding the setup in the initial state, i.e. the survival probability, is given by

$$P(\tau) \equiv |p(\tau)|^2 = |\langle \psi(0)|\psi(\tau)\rangle|^2 = |\langle m|e^{-i(H_0+\delta H_I)\tau}|m\rangle|^2.$$ (2)
For small $\delta$, one can expand $p(\tau)$ around $\delta = 0$ and keep terms up to the second order. We have

$$p(\tau) \approx p(\tau)|_{\delta=0} + \delta \frac{\partial p(\tau)}{\partial \delta}|_{\delta=0} + \frac{\delta^2}{2} \frac{\partial^2 p(\tau)}{\partial \delta^2}|_{\delta=0},$$

(3)

with

$$\frac{\partial p(\tau)}{\partial \delta}|_{\delta=0} = (-i \tau) e^{-i \tau E_n} H_{nm}^{nn},$$

(4)

and

$$\frac{\partial^2 p(\tau)}{\partial \delta^2}|_{\delta=0} = -2 e^{-i \tau E_n} \sum_{n \neq m} \frac{1 - \cos[(E_n - E_m)\tau]}{(E_n - E_m)^2} |H_{nm}^{nn}|^2 e^{-i \tau (E_n - E_m)} - e^{-i \tau E_n} |H_{nm}^{nn}|^2 \tau^2,$$

(5)

where $H_{nm}^{nn} = \langle n|H|m\rangle$. Keeping terms to the lowest order of $\delta$, equation (2) becomes

$$P(\tau) \approx 1 - 2\delta^2 \sum_{n \neq m} |H_{nm}^{nn}|^2,$$

(6)

which is just the perturbative form of the Loschmidt echo [14]. The decay of the Loschmidt echo and its relation to quantum phase transitions (QPTs) [15] has been studied in a number of many-body systems [16–19]. Different from previous studies, we focus on the short-time response in the dynamics of the system to external perturbations. In the short-time limit, equation (6) gives

$$P(\tau) \approx 1 - \delta^2 \tau^2 \sum_{n \neq m} |H_{nm}^{nn}|^2.$$  

(7)

Now, suppose successive measurements are performed at every time interval $\tau$, and the system is allowed to evolve freely in-between consecutive measurements. The probability of finding the system in the initial state after a finite duration $\Delta t = Q\tau$, where $Q$ is the number of measurements, is

$$P_{\Delta t} = |P(\tau)|^Q \approx 1 - \Delta t^2 \delta^2 \frac{\chi}{Q},$$

(8)

where

$$\chi \equiv \sum_{n \neq m} |H_{nm}^{nn}|^2 = \langle m|H|m\rangle - \langle m|H|m\rangle^2,$$

(9)

which is the fluctuation in the interaction Hamiltonian. Note that the projection postulate has been implanted in the first equality in equation (8). From equation (8), one can realize that there exists a competition between $\chi$ and $Q$. For a few-level system, since $\chi$ is finite, we have $P_{\Delta t} \to 1$ under the case of continuous measurements ($Q \to \infty$). In other words, the system would remain in the initial state if the measurements are performed continuously. However, for a quantum many-body system, $\chi$ may not be finite but exhibits a power law dependence on the system size. In the thermodynamic limit, $\chi$ is also infinite, and the second term in equation (8) could not be simply ignored even if $Q \to \infty$. The key motivation of our work is to obtain the scaling behavior of $\chi$ in quantum many-body systems. From there, one can then predict the behavior of $\chi$ in the thermodynamic limit and draw criteria for studying how large $Q$ should be in comparison to the system size in order to observe the QZE in the system.

In the following, we take the LMG model coupled to a central qubit as an example. Using equation (9), we calculated $\chi$ explicitly and extracted its size dependence. One can then determine the criteria to analyze how large $Q$ should be in order to observe the QZE in the model.
3. Analysis on the Lipkin–Meshkov–Glick model

3.1. The model

The LMG model consists of a cluster of spins mutually interacting with each other in the $xy$ plane. All the spins are embedded in an external field in the $z$-direction. The Hamiltonian of the model reads

$$ H_{\text{LMG}} = -\frac{1}{N} \sum_{i<j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - \lambda \sum_i \sigma_i^z, \quad (10) $$

where $\gamma$ is the parameter indicating the anisotropy in the interaction and $\lambda$ is the strength of the external field. The pre-factor $1/N$ is to ensure a finite energy per spin in the thermodynamic limit. In the following, we focused on the ferromagnetic case where $\lambda > 0$ and the case for $0 < \gamma < 1$. In the thermodynamic limit, the model exhibits a quantum phase transition at $\lambda = 1 [20]$. Later study also showed that four distinct regions in the parameter space are distinguished if one considers the whole range of real values of $\lambda$ and $\gamma [21]$.

Introducing the total spin operator $S = \sum_i \sigma_i^z/2$, where $\kappa = x, y, z$, and $S_+ = (S_x + S_y)/2$ and $S_- = (S_x - S_y)/2i$, equation (10) can be rewritten as

$$ H_{\text{LMG}} = -2\lambda S_z - \frac{1}{2N} (1 - \gamma) (S_x^2 + S_y^2) - \frac{1}{N} (1 + \gamma) \left( S_z^2 - S_z^2 - \frac{N}{2} \right). \quad (11) $$

Obviously, the above Hamiltonian commutes with $S^z$. The Hilbert space is then divided into subspaces of a particular value of $S$. The ground state lies in the subspace of $S = N/2$. Moreover, the model also has an advantage in computation. As the Hamiltonian is tri-diagonalized in the $S_z$ basis, it can be exactly diagonalized numerically up to very large system size through standard packages.

In the large $N$ limit, the model’s Hamiltonian can be diagonalized analytically through a semiclassical approach following [22]. We outline the procedure in the following.

(i) 1/$N$ expansion of the Holstein–Primakoff (HP) transformation. The transformation maps the spin operators into boson creation and annihilation operators. Note that for $\lambda < 1$, the spins lie somewhere between the $xy$ plane and the $z$-axis. Before applying the HP transformation, a rotation is needed to bring the $z$-axis along the direction of the classical magnetization. This is done by

$$ \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} \tilde{S}_x \\ \tilde{S}_y \\ \tilde{S}_z \end{pmatrix}, \quad (12) $$

where $\alpha = 0 \text{ for } \lambda > 1$ so that $S = \tilde{S}$, and $\alpha = \cos^{-1} \lambda$ for $\lambda < 1$.

The HP transformation is defined as

$$ \tilde{S}_z = S - a^\dagger a = N/2 - a^\dagger a, \quad (13) $$

$$ \tilde{S}_+ = (2S - a^\dagger a)^{1/2}a = \sqrt{N}(1 - a^\dagger a/N)^{1/2}a, \quad (14) $$

$$ \tilde{S}_- = a^\dagger (2S - a^\dagger a)^{1/2} = \sqrt{N}a^\dagger (1 - a^\dagger a/N)^{1/2}, \quad (15) $$

where $a$ and $a^\dagger$ are boson annihilation and creation operators, respectively, satisfying the commutation relation $[a, a^\dagger] = 1$ and $[a, a] = [a^\dagger, a^\dagger] = 0$. In the second equality, we take $S$ to be $N/2$ by just considering the low excitation spectrum. Also, for large $N$ and keeping terms of order $(1/N)^{-1}$, $(1/N)^{-1/2}$ and $(1/N)^0$, it is sufficient to approximate $(1 - a^\dagger a/N)^{1/2}$ by 1 in the transformation. The transformed Hamiltonian then has a quadratic form and can be diagonalized through standard methods.
(ii) Bogoliubov transformation. The transformation is given by
\[ a = \sinh (\theta) \ b + \cosh (\theta) \ b^\dagger, \]
\[ a^\dagger = \cosh (\theta) \ b^\dagger + \sinh (\theta) \ b, \]
where \( b \) and \( b^\dagger \) are also boson annihilation and creation operators, respectively. They satisfy the same commutation relation as that of \( a \) and \( a^\dagger \). The diagonalization condition is given by
\[ \tanh (2 \theta) = - \frac{2 \Gamma}{\Delta}. \]
The diagonalized Hamiltonian takes the form of
\[ H_{\text{LMG}} = \epsilon_0 N + \epsilon_1 - \Delta/2 + \sqrt{\Delta^2 - 4 \Gamma^2} (b^\dagger b + 1/2), \]
where for \( \lambda < 1 \),
\[ \begin{cases} 
\epsilon_0 = - \frac{1 + \lambda^2}{2} \\
\epsilon_1 = \frac{1 - \lambda^2}{2} \\
\Delta = 2 - \lambda^2 - \gamma \\
\Gamma = \frac{\gamma - \lambda^2}{2}.
\end{cases} \] (20)
For \( \lambda > 1 \),
\[ \begin{cases} 
\epsilon_0 = - \lambda \\
\epsilon_1 = 0 \\
\Delta = 2\lambda - 1 - \gamma \\
\Gamma = - \frac{1 - \gamma}{2}.
\end{cases} \] (21)
Let \(| n \rangle \) be a set of eigenkets of the diagonalized Hamiltonian such that \( b^\dagger b | n \rangle = n | n \rangle \) with \( n = 0, 1, 2, \ldots \). The ground state is given by the vacuum state \(| G \rangle = | 0 \rangle \). (22)

3.2. Quantum Zeno dynamics in the model
Let us consider a central qubit interacting transversely with the LMG model. The corresponding Hamiltonian reads
\[ H = H_{\text{LMG}} - \delta \sum_i | e \rangle \langle e | \sigma^z_i, \]
where \( H_{\text{LMG}} \) is given by equation (10) and
\[ H_I = - \sum_j \sigma^z_j. \] (24)
\(| g \rangle \) and \(| e \rangle \) denote the ground state and the excited state of the central qubit, respectively, and \( \delta \) is a small parameter indicating the strength of the coupling between the system and the central qubit. This model is similar to the one set up by Quan and his collaborators to study the decay of the Loschmidt echo in the one-dimensional transverse-field Ising model and the relation to QPTs. They found that the decay of the Loschmidt echo is greatly enhanced at the critical
point of the Ising model. In the following, we followed a similar approach as used by Quan et al, but focused on the short-time regime of the Loschmidt echo.

Without loss of generality, we assume that the LMG model is initially in some state $|\phi(0)\rangle$ and the central qubit is in a superposition state $c_g|g\rangle + c_e|e\rangle$, where the coefficients satisfy $|c_g|^2 + |c_e|^2 = 1$. The initial state of the whole setup is thus $|\psi(0)\rangle = |\phi(0)\rangle \otimes (c_g|g\rangle + c_e|e\rangle)$. The evolution of the LMG model splits into two branches and the state vector of the setup at a later time $\tau$ is given by $|\psi(\tau)\rangle = c_g e^{-iH_{\text{LMG}}\tau}|\phi(0)\rangle \otimes |g\rangle + c_e e^{-iH_{\text{LMG}}+\Delta H_I\tau}|\phi(0)\rangle \otimes |e\rangle$.

For simplicity, we suppose that the LMG model is initially in the ground state $|0\rangle$ and the central qubit is in the excited state, i.e. $c_g = 0$ and $c_e = 1$. The survival probability of the system as given in equation (2) becomes

$$P(\tau) = |\langle G| e^{-i(H_{\text{LMG}}+\Delta H_I)\tau} |G\rangle|^2.$$  

In the eigenbasis of the diagonalized Hamiltonian, we have for $\lambda > 1$,

$$H_{(\lambda>1)} = -N + 1 + \cosh(2\theta) + 2 \cosh(2\theta)b^\dagger b + \sinh(2\theta)(b^{\dagger 2} + b^2),$$

and

$$\chi_{(\lambda>1)} = \frac{(1-\gamma)^2}{2(\lambda-1)(\lambda-\gamma)}.$$  

For $\lambda < 1$,

$$H_{(\lambda<1)} = \lambda [-N - 1 + \cosh(2\theta) + 2 \cosh(2\theta)b^\dagger b + \sinh(2\theta)(b^{\dagger 2} + b^2)] + \sqrt{1-\lambda^2}\sqrt{N(\cosh \theta + \sinh \theta)(b^\dagger + b)}$$

and

$$\chi_{(\lambda<1)} = \frac{\lambda^2(\gamma-\lambda^2)^2}{2(1-\gamma)(1-\lambda^2)} + N\sqrt{(1-\gamma)(1-\lambda^2)}.$$  

We then have

$$\chi \sim \begin{cases} N & \text{for } \lambda < 1, \\ N^0 & \text{for } \lambda > 1. \end{cases}$$  

$\chi$ exhibits different scaling behavior in the two phases of the LMG model. Interestingly, $\chi$ is intensive in the polarized phase of the model. This may be understood by the fact that in this phase, the ground state of the model is given by the configuration in which spins are polarized along the external field and does not posses any long-range correlations. The physics here is equivalent to that of a localized single particle, and we thus have $\chi$ being intensive.

Figure 1 shows a plot of $\chi$ and the normalized $\chi$ as a function of $\lambda$ for the case of $\gamma = 0.5$ and $\gamma = 0$. These are the numerical results obtained from the exact diagonalization. As one can observe from the figure, the curves of $\chi$ for various system sizes overlap with each other for $\lambda > 1$, while the curves of $\chi/N$ overlap for the case of $\lambda < 1$. This is consistent with our analytical analysis presented above.

To interpret the results obtained, we argue that in order to observe the QZE in the symmetry-broken phase, the number of measurements $Q$ has to be of comparable order of $N$. In a thermodynamic system, this condition is hardly realized. However, in the polarized phase where $\chi$ is independent of the system size, the QZE is much easier to observe in comparison to that in the symmetry-broken phase.

A remark of this section is that in the analytical analysis we presented above, we have only kept terms to the lowest order of $N$ in the $1/N$ expansion of the HP transformation. The higher order terms become more important as one gets closer to the critical point of the model. The discrepancy decreases with increasing system size. To obtain more accurate results near
the critical point, one has to keep the higher order terms or to carry out the analysis by studying the Majorana polynomial roots \[21\]. In this section, our concern is the scaling behavior of \(\chi\) which is not close to the critical point. The analytical analysis presented so far remains valid as long as one stays within the same phase of the model.

### 3.3. \(\chi\) around the critical point of the model

From figure 1, one can observe that \(\chi\) also shows some abnormal behavior around the quantum critical point of the LMG model. Figure 2 shows a plot of the first derivative of the normalized \(\chi\) with respect to \(\lambda\) near the vicinity of the critical point for \(\gamma = 0.5\). \(d(\chi/N)/d\lambda\) exhibits a minimum near \(\lambda = 1\), and the value of \(\lambda\) where this minimum occurs comes closer and closer to the critical point as the system size increases. The minimum value amplitude also increases with the system size. From the slope of the straight line in the inset of the figure, we find that

\[
\frac{d(\chi/N)}{d\lambda} \sim N^{0.324\pm0.001}.
\]

A similar result is obtained for the case of \(\gamma = 0\). From the slope of the straight line shown in figure 3, we have

\[
\frac{d(\chi/N)}{d\lambda} \sim N^{0.328\pm0.001},
\]

which is consistent with the case for \(\gamma = 0.5\) up to two digits. In the thermodynamic limit, we also expected it to be divergent at the critical point and it may be used as a seeker for quantum phase transitions.
Figure 2. First derivative of the normalized $\chi$ with respect to $\lambda$, as a function of $\lambda$, in the vicinity of the critical point for the LMG model coupled to a central qubit ($\gamma = 0.5$). The inset shows the scaling behavior of the minimum of $d(\chi/N)/d\lambda$. The slope of the straight line gives $0.324 \pm 0.001$.

Figure 3. First derivative of the normalized $\chi$ with respect to $\lambda$, as a function of $\lambda$, in the vicinity of the critical point for the LMG model coupled to a central qubit ($\gamma = 0$). The inset shows the scaling behavior of the minimum of $d(\chi/N)/d\lambda$. The slope of the straight line gives $0.328 \pm 0.001$.

4. Summary

In this paper, we have investigated the QZE from the viewpoint of condensed matter physics. We have obtained the scaling behavior of $\chi$ in the LMG model. We found that in the symmetry-broken phase of the LMG model, the frequency of the projective measurement should be of comparable order to that of the system size in order to observe the QZE. However, in the polarized phase of the model, the QZE can be easily observed by frequent measurement.

Besides, we also found that the derivative of $\chi/N$ shows some non-trivial scaling with the system size around the critical point of the model. One then expected it to be divergent.
in the thermodynamic limit and able to be used to identify the quantum phase transition (QPT) in the model. We would also like to remind the reader that $\chi$ is just the quantum fluctuation in the interaction Hamiltonian. Meanwhile, quantum phase transitions are also driven by quantum fluctuations. Detailed study on $\chi$ may provide further insight into the role of quantum fluctuation played in quantum phase transitions.

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References

[1] Misra B and Sudarshan E C G 1977 J. Math. Phys. 18 765
[2] Chiu C B, Sudarshan E C G and Misra B 1977 Phys. Rev. D 16 520
[3] Ghirardi G C, Omero C, Weber T and Rimini A 1979 Nuovo Cimento A 52 421
[4] Peres A 1980 Am. J. Phys. 48 931
[5] Kraus K 1981 Found. Phys. 11 547
[6] Joos E 1984 Phys. Rev. D 29 1626
[7] Home D and Whitaker M A B 1986 J. Phys. A: Math. Gen. 19 1847
[8] Fonda L, Ghirardi G C, Rimini A and Weber T 1973 Nuovo Cimento A 15 689
[9] Degasperis A, Fonda L and Ghirardi G C 1974 Nuovo Cimento A 21 471
[10] Itano W M, Heinzen D J, Bollinger J J and Wineland D J 1990 Phys. Rev. A 41 2295
[11] Dziarmaga J 2010 Adv. Phys. 59 1063
[12] Zurek W H, Dorner U and Zoller P 2005 Phys. Rev. Lett. 95 105701
[13] Dziarmaga J 2005 Phys. Rev. Lett. 95 245701
[14] Zhang J F et al 2009 Phys. Rev. A 79 012305
[15] Sachdev S 1999 Quantum Phase Transition (Cambridge: Cambridge University Press)
[16] Quan H T, Song Z, Liu X F, Zanardi P and Sun C P 2006 Phys. Rev. Lett. 96 140604
[17] Rossini D et al 2007 Phys. Rev. A 75 032333
[18] Rossini D et al 2008 Phys. Rev. A 77 052112
[19] Rossini D et al 2007 J. Phys. A: Math. Theor. 40 8033
[20] Botet R, Jullien R and Pfeuty P 1982 Phys. Rev. Lett. 49 478
Botet R and Jullien R 1983 Phys. Rev. B 28 3955
[21] Ribeiro P, Vidal J and Mosseri R 2007 Phys. Rev. Lett. 99 050402
Ribeiro P, Vidal J and Mosseri R 2008 Phys. Rev. E 78 021106
[22] Dusuel S and Vidal J 2005 Phys. Rev. B 71 224420
Dusuel S and Vidal J 2007 Phys. Rev. B 76 174519