Exact Nonperturbative Unitary Amplitudes for $1 \Rightarrow 8$ Transitions in a Field Theoretic Model

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ABSTRACT

We present a quantum mechanical model with an infinite number of (discrete) degrees of freedom, which can serve as a laboratory for multiparticle production in a collision. There is a cubic coupling between modes without, however, any problems associated with unstable ground states. The model is amenable to precise numerical calculations of nonperturbative $1 \rightarrow N$ transition amplitudes. On an ordinary workstation, time and memory limitations effectively restrict $N$ to be $\leq 8$, and we present results for this case. We find (1) that there is reasonable period of time for which there is a constant rate for the $1 \Rightarrow 8$ transition; (2) at the end of the linear period, the eight particle amplitude attains a maximum value $|A_8|_{\text{max}}$ which is about $3 - 4$ orders of magnitude larger than the comparable amplitude for excitation of the $N = 8$ state in the anharmonic oscillator; (3) for values of the coupling in the region where the Born approximation fails, the amplitude is much larger than the naive estimates $A_8 \simeq \exp(-1/g^2)$ or $\exp(-8)$; it is more like $A_8 \sim \exp(-0.20/g^2)$. 
The calculation of tree-level amplitudes for the production of $N$ bosons at fixed momenta indicates a failure of perturbation theory when $N$ is large. For a quartic coupling $\frac{1}{4}g^2\phi^4$, the $1 \to N$ amplitude $A_{1\to N} \sim g^{N-1}N!$ for large $N$. [1,2] Thus, perturbation theory may become untrustworthy when $N \sim 1/g$. The origin of the problem is simply the overcoming of the $g^N$ suppresion by the enhancement factor $N!$ which originates in the coherent addition of the $O(N!)$ graphs obtained by permutation of the momenta. The coherence has been established in the non-relativistic region both in the scalar field theory [2] and in the standard model, including the production of $W$’s and $Z$’s [3]. For theories with a continuous momentum spectrum in an infinite volume, the transition rate for large $E,N$, fixed $E/N$ is $\Gamma \sim |A_{1\to N}|^2/N!$, which violates unitarity for $N \gtrsim 1/g^2$. Of course, it is clear that for $N$ somewhere in this range, loops will contribute equally to the tree graph: each tree may generate $O(N^2)$ final state scatterings, at a cost of $g^2$ for each. Depending on the coherence of the loop corrections, these may become unsuppressed when $N \sim 1/g$ or $N \sim 1/g^2$.

Subsequent to these studies, there have been several papers written which argue for an exponential suppression of the amplitude. That is, if we define the ‘holy grail’ function $F$ [4] via

$$A_{1\to N} \sim e^{F/g^2},$$

then exponential suppression means that for $N \gtrsim 1/g^2$,

$$F(N,g^2) \sim O(-1) \quad \text{or} \quad O(-Ng^2).$$

One approach has been based on finding a bound for the $|\text{ground state}\rangle \to |N\rangle$ transition for the anharmonic well in Schrödinger mechanics; either a calculation based on the $WKB$ approximation [5] or a variational study [6] shows that the overlap matrix elements $\langle N|x^2|0\rangle$ (or $\langle N|x|0\rangle$) are exponentially suppressed, with an $F$ which is negative and monotonically decreasing as a function of the variable $\lambda = Ng^2$. ($g^2$ is the coefficient of the quartic coupling.) Thus, for fixed $N$ (which
we will be constrained to work with), $F$ in the quantum mechanical model is a decreasing function of $g^2$. In a second approach [7,8], a bound on $A_{1 \rightarrow N}$ is claimed to exist on the basis of a Lipatov-type analysis. Here again, the bound is stated to be exponential: $F \sim O(-1)$ for $N \gtrsim 1/g^2$.

In our view, there are uncertainties in the application of both these approaches to the problem at hand. In the first, we have no argument with the bound in terms of quantum mechanics per se. However, since the physics of the q.m. problem involves the excitation of the single $N$’th level, while the multiparticle process addresses the production of $N$ different low-energy quanta (for fixed momenta), some hesitancy may be justified before adopting for the multiparticle problem the exponential suppression from the q.m. bound. With respect to the Lipatov-type analysis, at least in the approach of Ref. [8], we have found ourselves uncertain in extrapolating the results from fixed numbers of legs to a large number $N \sim 1/g^2$; without being critical, we wish to observe the suppression (should it exist) in a much more direct manner; that is, we would really like to see how unitarity for $A_{1 \rightarrow N}$ gets restored through rescattering. If there is exponential suppression, we would like to find the actual value of $F$.

In this work, we propose a simple field-theoretic model in which exact unitary amplitudes for transitions from 1 to $N$ particles can be calculated. In principle, these can be done exactly analytically; in practice, we will resort to a numerical calculation. Even numerically, we cannot go much beyond a $1 \Rightarrow 8$ transition; and even this process will require the solution of 1336 coupled first order differential equations. Nevertheless, the resulting amplitude will be an exact, unitary, nonperturbative solution of coupled time-dependent Schrödinger equations; and the Born approximation will exhibit (for $N = 8$) the $g^{N-1} N!$ blowup of the scalar tree.

The model may be conveniently defined through its Hamiltonian

$$H = \sum_{k=1}^{\infty} a_k^\dagger a_k + \frac{1}{2} g \mathcal{P} \sum_{j,k=1}^{\infty} a_j^\dagger a_j a_k a_k \mathcal{P} + \text{h.c.}$$  \hspace{1cm} (3)

The modes labeled by $i,j,k$ will be called “momenta”, and the action of the her-
mitean projection operator $\mathcal{P}$ on a state vector $|\psi\rangle$ is as follows:

$$\mathcal{P} |\psi\rangle = 0$$

if there is more than one quantum in the state with any given momentum

$$\mathcal{P} |\psi\rangle = |\psi\rangle$$

otherwise.

$\mathcal{P}$ has been introduced in order to mimic the infinite volume effect of field theory (in box normalization): namely, we do not generally concern ourselves with amplitudes for transitions to states with more than one particle in a given (discrete) momentum state. Thus, we exclude “laser” effects, with their attendant $\sqrt{n_i}$ factors in matrix elements. In this sense, these are “hard core” bosons: there is a fermionic exclusion principle without imposing anticommutation relations and antisymmetrization. Other than that, $H$ resembles a $\phi^3$ field theory in a cavity, with no momentum dependence to the unperturbed energies. It is also a kind of matrix model.

It is an important consequence of (3) that the momentum operator

$$P = \sum_{k=1}^{\infty} k \, a_k^\dagger a_k$$

(4)

is a constant of the motion. Thus, the Hilbert space factorizes into subspaces with definite $P$. Because of the positivity of all of the momenta, these will be finite dimensional subspaces. In considering the subspaces with definite $P$, we may conveniently think in terms of a maximal state, namely an $N$-particle state with momenta $k = 1, 2, \ldots N$. This state will have a total momentum $P = N(N + 1)/2$. We will our study to such sectors, labeled by $N$, which have this maximal state in their spectrum.
In general, the total number of states for a given \( P \) can be obtained as the exponent of \( x^P \) in the expansion of the generating function

\[
\prod_{j=1}^{\infty} (1 + x^j) = \sum_{P=0}^{\infty} N_P x^P .
\] (5)

If we consider the symmetric trees corresponding to \( N = 2, 4, 8, 16 \ldots \) \( (P = 3, 10, 36, 136 \ldots) \), we find that the number of states to be considered are 2, 10, 668, 7 215 644,... respectively. Even going from \( N = 8 \) to \( N = 9 \) means increasing the subspace from 668 to 2048 states. Thus, in a practical decision which is based on CPU time considerations, we restrict the present study to the case \( N = 8 \).

For a given \( N \), there will be states with \( n = 1, 2, \ldots N \) particles. The cubic interaction \( gV \) will couple a state \( |n, r\rangle \) to states \( |n + 1, s\rangle \) and \( |n - 1, s\rangle \), where \( r, s \) label the different states with a given number of particles. For example, with \( N = 8 \) \( (P = 36) \), \( V \) will couple the 4-particle state \( |1, 3, 8, 24\rangle \) to various 5-particle states (such as \( |1, 3, 8, 11, 13\rangle \) but not \( |1, 3, 8, 12, 12\rangle \)) and 3-particle states (such as \( |1, 11, 24\rangle \)).

The energy difference between (unperturbed) states coupled by \( V \) is always ±1. Thus, if we work in the interaction representation, so that the the amplitude to be in the (unperturbed) state \( |n, r\rangle \) at time \( t \) is \( A^r_n(t) \), then the Schrödinger dynamics is governed by the coupled equations, for \( n = 1, 2 \ldots N \) and \( A_0 = A_{N+1} = 0 \):

\[
i \dot{A}^r_n(t) = g \left( \sum_s \langle n, r|V|n-1, s\rangle \ e^{it} A^s_{n-1}(t) \right. \\
\left. + \sum_s \langle n, r|V|n+1, s\rangle \ e^{-it} A^s_{n+1}(t) \right) .
\] (6)

For the case \( N = 8 \), we have calculated explicitly the number of states for each \( n \), their momentum content, and the 668 \( \times \) 668 transition matrix (whose elements
are $g$ or 0). For example, the number of states for $n = 1, \ldots, 8$ are (1, 17, 91, 206, 221, 110, 21, 1) respectively. We have solved these equations subject to the boundary condition

$$A_1(0) = 1, \quad A_n(0) = 0, \quad n \neq 1.$$  \hspace{1cm} (7)

The Born (tree) approximation is obtained from (6) by keeping only the coupling of $n$ to $n - 1$ (no rescattering). This set of equations can be solved analytically. For the amplitude $A_8$ (there is only one), we obtain

$$A_8 = 427, 206 \frac{g^7}{7!} \left(1 - e^{it}\right)^7,$$  \hspace{1cm} (8)

where the first numerical factor is the result of successive matrix multiplications leading from the single one-particle state $|36\rangle$ to the single 8-particle state $|1 \ldots 8\rangle$. It is an “entropy” factor, since this multiplication counts the number of paths between these states. From (8) we find

$$|A_8^{\text{Born}}|_{\text{max}} \simeq 0.27 \ g^7 \ 8!,$$ \hspace{1cm} (9)

a relation in accordance with the tree level result in the $\phi^3$ theory [2]. We note that the Born amplitude will exceed the unitary bound when $g^2 > 0.07$; this is then an upper limit on $g^2$ for weak coupling in the case of $N = 8$.

**Numerical Results.** In integrating the coupled 1336 (real) Eqs. (6), we have used an adaptive Runge-Kutta routine, and have found that unitarity is satisfied to better than 1 part in $10^7$ for all times relevant to the discussion. In Fig. 1 we show the probability $|A_8(t)|^2$ vs. $t$ for the “critical” coupling $g^2 = 0.07$. Note that $|A_8|^2$ has a maximum value of about $3 \times 10^{-4}$, and enjoys a linear rise (constant transition rate) for a reasonable amount of time. We find for all $g^2$’s of interest,

$$\left(\frac{d}{dt}|A_8(t)|^2\right)_{\text{max}} \sim |A_8(t)|^2_{\text{max}},$$  \hspace{1cm} (10)

with the unit of time set by the basic frequency, $\omega = 1$. Thus, for ease of comparison with the quantum mechanical results, we will phrase the discussion in terms of
$|A_8(t)|_{\text{max}}$, and compare this with the variational bound to the transition matrix element $\langle 8|x|0 \rangle$ of ref. [6]. For the sake of general interest, we show in Fig. 2 some of the long time behavior of the probability of staying in the original state. It can be seen that even in the nonperturbative regime, there is a residual quasi-periodicity which we expect will dissipate over very long times.

In Fig. 3, we present a plot of $|A_8|_{\text{max}}$ vs. $g^2$. For comparison, the dotted line shows the variational upper bound on the amplitude $\langle 8|x|0 \rangle$ in ref. [6] for the anharmonic oscillator with potential $\frac{1}{2}x^2 + \frac{1}{4}g^2x^4$. Without particular justification, we have identified this $g^2$ with ours. In the weak coupling regime, there are $3 - 4$ orders of magnitude difference between these amplitudes. It should be noted that both our 8-particle state $|1, 2, 3, 4, 5, 6, 7, 8 \rangle$ and the oscillator state $|8 \rangle$ are normalized to unity. The difference between the dynamics becomes more apparent when the $g^2$ behavior of the amplitudes (for fixed $N = 8$) is compared. In Fig. 4, we have plotted the holy grail function $F$ vs. $g^2$, with the variational upper limit from the anharmonic oscillator [6] (for fixed $N = 8$) shown for comparison. Again, one may note the great difference in behavior of the two amplitudes. We may also note that the Lipatov-type analysis might predict $F \sim O(-1)$ for $g^2 \simeq 0.05 - 0.10$, the region where the tree approximation breaks down. We find $F \simeq -0.20$ in that region. While it is difficult to claim that this indicates a lack of exponential suppression, it is nevertheless a possible hint that any suppression, should it exist, is weaker than believed. In a subsequent work [9] in which an extension to arbitrary $N$ is presented, it will be seen that exponential suppression is extremely weak ($|F| \ll 1$) in the weak coupling limit.

Conclusions. We have proposed a model, reminiscent of $\phi^3$ field theory, in which $1 \to N$ transition amplitudes can be calculated in principle. CPU time and disc storage limitations have limited the present considerations to a maximum of $N = 8$. In contrast to quantum mechanical calculations in the anharmonic oscillator model, the “$N$” in the present case refers to $N$ distinct low energy quanta rather than the excitation of a single high energy level. We have found that the Born approximation to the amplitude attains a maximum value of $\sim g^7 8!$, the expected
behavior from the analysis of tree graphs in a $g\phi^3$ theory. As a consequence, unitarity breaks down for $g^2 \simeq 0.07$. For a range of $g^2$ near this value, the holy grail function $F \simeq -0.20$ and shows only slow $g^2$ dependence. This is in total contrast to the case of the anharmonic oscillator. The amplitude is also about 200 times greater than $e^{-N}$, $N = 8$, another suggested form of exponential suppression. There is a hint in this work that any exponential suppression of $A_{1 \rightarrow N}$ which exists is considerably weaker than that expected ($F \simeq -1$) from Lipatov type arguments. In another work, one of us [H.G.] will present an extension of this model to arbitrary $N$.

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**FIGURE CAPTIONS**

*Fig. 1.* The time dependence of the probability for transition from the 1-particle to the 8-particle state, for $g^2 = 0.07$. This is the value of the coupling at which the Born approximation for $A_{1 \rightarrow 8}$ violates unitarity.

*Fig. 2.* Time dependence of the probability to stay in the 1-particle state, $g^2 = 0.07$.

*Fig. 3.* Solid line: Maximum amplitude attained for reaching the 8-particle state, as a function of $g^2$. Dashed line: Upper bound on matrix element $\langle 8 | x | 0 \rangle$ as a function of $g^2$, from ref. [6].

*Fig. 4.* Same as Fig. 3, but for the holy grail function $F$ defined in Eq. (1).
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