SHARP COMMUTATOR ESTIMATES VIA HARMONIC EXTENSIONS

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ABSTRACT. We give an alternative proof of several sharp commutator estimates involving Riesz transforms, Riesz potentials, and fractional Laplacians. Our methods only involve harmonic extensions to the upper half-space, integration by parts, and trace space characterizations.

The commutators we investigate are Jacobians, more generally Coifman-Rochberg-Weiss commutators, Chanillo’s commutator with the Riesz potential, Coifman-Meyer or Kato-Ponce-Vega type commutators, and the Da Lio-Riviè re three-term commutators. We also give a new limiting $L^1$-estimate for a double commutator of Coifman-Rochberg-Weiss-type, and several intermediate estimates.

The beauty of our method is that all those commutator estimates, which are originally proven by various specific methods or by general paraproduct arguments, can be obtained purely from integration by parts and trace theorems. Another interesting feature is that in all these cases the “cancellation effect” responsible for the commutator estimate simply follows from the product rule for classical derivatives and can be traced in a precise way.

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1. Introduction: Jacobian estimates

In this work we propose an alternative method for proving a large class of sharp and intermediate commutator estimates. The method is based on harmonic extensions to the upper half space \( \mathbb{R}^{n+1}_+ \), integration by parts, and trace space characterizations.

To illustrate the main ideas, let us first consider Jacobians \( \det(\nabla u) \) for a map \( u : \mathbb{R}^n \to \mathbb{R}^n \). Jacobians appear naturally in geometric analysis, they are infinitesimal deformations of the space under the map \( u \), as one sees, e.g., in the change of variables formula. From the point of view of harmonic analysis, they are very special forms of commutators, as was discovered in the 1990s by Coifman, Lions, Meyer, and Semmes [11].

Fine estimates on Jacobians have proven to be crucial in particular to the theory of geometric partial differential equations. Just to name a few examples: the harmonic map equation [24, 33], the prescribed mean curvature equation [47, 23, 1, 33], or the conformally parametrized surface equation [31]. The reason is that the determinant structure acting on gradients leads to “compensated compactness” and “integrability by compensation”-effects. These fine Jacobian estimates had been observed in relation with Wente’s inequality [32, 47, 4, 45, 30]. Finally, the above mentioned seminal work of Coifman, Lions, Meyer, and Semmes [11] drew the connection to commutator estimates by Coifman, Rochberg, and Weiss [14] from the 1970s. In particular, the following estimate holds.

**Theorem 1.1** (Jacobian estimate, [11, 42]). Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( u = (u^1, \ldots, u^n) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n) \). Then

\[
\int_{\mathbb{R}^n} \varphi \det(\nabla u) \lesssim [\varphi]_{BMO} \|\nabla u\|_{L^n(\mathbb{R}^n)}^n.
\]

Also, the following intermediate estimate holds: let \( 0 < s_i < 1, 1 < p_i < \infty \) for \( i = 0, \ldots, n \) be such that

\[
\sum_{i=0}^n s_i = n, \quad \sum_{i=0}^n \frac{1}{p_i} = 1
\]

then

\[
\int_{\mathbb{R}^n} \varphi \det(\nabla u) \lesssim [\varphi]_{W^{s_0,p_0}} [u^1]_{W^{s_1,p_1}} \ldots [u^n]_{W^{s_n,p_n}}.
\]

For a definition of the norms we refer to Section 9. The first estimate (1.1) is due to [11], where it is proven that \( \det(\nabla u) \) belongs to the Hardy space if \( u \in W^{1,n}(\mathbb{R}^n) \). An alternative proof in [11] rewrites the Jacobian in the form of the Coifman-Rochberg-Weiss commutator [14], see Theorem 4.1. The second estimate (1.3) is due to Sickel and Youssfi [42], who use Littlewood-Paley decomposition and paraproducts.

The main discovery of the present work is that one can obtain Theorem 1.1 – and several other commutator estimates (see below) – by an integration by parts formula and trace theorems: one needs to interpret the involved functions as \( \mathbb{R}^n \times \{0\} \)-boundary values of
harmonic functions on \( \mathbb{R}^{n+1}_+ \). We illustrate this method with a new proof first of (1.3) and then of (1.1).

**Proof of intermediate estimate (1.3).** Let \( \Phi : \mathbb{R}_+^{n+1} \to \mathbb{R} \), \( U : \mathbb{R}_+^{n+1} \to \mathbb{R}^n \) be a harmonic extension to \( \mathbb{R}_+^{n+1} \) of \( \varphi \) and \( u \), respectively:

\[
\begin{align*}
\Delta_{\mathbb{R}_+^{n+1}} \Phi &\equiv (\Delta_x + \partial_t) \Phi = 0 \quad \text{in} \ \mathbb{R}_+^{n+1}, \\
\Phi(x, 0) &= \varphi(x) \quad \text{in} \ \mathbb{R}^n. \\
\Delta_{\mathbb{R}_+^{n+1}} U &= 0 \quad \text{in} \ \mathbb{R}_+^{n+1}, \\
U(x, 0) &= u(x) \quad \text{in} \ \mathbb{R}^n.
\end{align*}
\]

(1.4)

For the following to work we will choose zero-boundary data at infinity, so \( \Phi \) and \( U \) are given explicitly by the Poisson formula (2.1), \( \Phi := P_1^t \varphi \) and \( U := P_1^t u \).

The integration-by-parts formula\(^1\) gives us

\[
C := \left| \int_{\mathbb{R}^n} \varphi \det_{n \times n}(\nabla_{\mathbb{R}^n} u) \right| = \left| \int_{\mathbb{R}_+^{n+1} \times (n+1)} \det (\nabla_{\mathbb{R}^{n+1}} \Phi, \nabla_{\mathbb{R}^{n+1}} U) \right|.
\]

(1.5)

This beautiful observation\(^2\) was used by Brezis and Nguyen [5] who gave an elegant argument for estimates in terms of \( [\Phi]_{\text{Lip}} \). However, if we aim for \( W^{s,p} \)-norms (and later the BMO-norm), we need to argue more carefully and distribute weights in the \((n+1)\)-th direction, i.e. \( t \)-weights.

Namely, in view of (1.2) and Hölder’s inequality,

\[
C \lesssim \left( \int_{\mathbb{R}^n} \int_0^\infty \left| t^{1 - \frac{1}{p_0} - s_0} \nabla \Phi(x,t) \right|^{p_0} \, dt \, dx \right)^{\frac{1}{p_0}}
\]

\[
\cdot \left( \int_{\mathbb{R}^n} \int_0^\infty \left| t^{1 - \frac{1}{p_1} - s_1} \nabla U^1(x,t) \right|^{p_1} \, dt \, dx \right)^{\frac{1}{p_1}}
\]

\[
\vdots
\]

\[
\cdot \left( \int_{\mathbb{R}^n} \int_0^\infty \left| t^{1 - \frac{1}{p_n} - s_n} \nabla U^n(x,t) \right|^{p_n} \, dt \, dx \right)^{\frac{1}{p_n}}.
\]

\(^1\)The easiest way to see this might be via Stokes’ theorem for differential forms on \( \mathbb{R}^{n+1}_+ \). Observe that the boundary \( \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\} \). Thus

\[
\left| \int_{\mathbb{R}^{n+1}_+} d\Phi \wedge dU^1 \wedge dU^2 \ldots \wedge dU^n \right| = \left| \int_{\mathbb{R}^n} \varphi \, du^1 \wedge du^2 \ldots \wedge du^n \right|.
\]

\(^2\)Here is where the “compensation effect” enters: in (1.5) the derivatives are uniformly distributed to all functions. Exactly this distribution of derivatives for Jacobian structures was observed by L. Tartar via the Fourier transform. He used this to give a proof for Wente’s inequality [45]. See also the introduction of [37] where this strategy is applied to the Da Lio-Rivi`ere three-term commutator.
Now the trace theorems for harmonic functions, see Proposition 9.2 below, yields
\[
\left( \int_{\mathbb{R}^n_+} t^{1-\frac{1}{p_0}} |\nabla \Phi(x, t)|^{p_0} \, dt \, dx \right)^{\frac{1}{p_0}} \approx [\varphi]_{W^{s_0, p_0}(\mathbb{R}^n)},
\]
and for \( i = 1, \ldots, n, \)
\[
\left( \int_{\mathbb{R}^n_+} t^{1-\frac{1}{p_i}} |\nabla U^i(x, t)|^{p_i} \, dt \, dx \right)^{\frac{1}{p_i}} \approx [u^i]_{W^{s_i, p_i}(\mathbb{R}^n)}.
\]
Thus we have shown that estimate (1.3) holds. \( \square \)

The BMO-estimate (1.1) is a little more delicate: an additional integration by parts is needed and the trace-estimates are more involved, see Proposition 9.4.

**Proof of BMO-estimate (1.1).** As in (1.4), let \( \Phi \) and \( U \) be harmonic extensions of \( \varphi \) and \( u \), respectively. Again we have
\[
C := \left| \int_{\mathbb{R}^n} \det(\nabla u, \ldots, \nabla u^n) \, \varphi \right| = \left| \int_{\mathbb{R}^n_+^{n+1}} \det(\nabla_{\mathbb{R}^{n+1}} U^1, \ldots, \nabla_{\mathbb{R}^{n+1}} U^n, \nabla_{\mathbb{R}^{n+1}} \Phi) \right|.
\]
Integrating by parts in \( t \)-direction, we can add an additional derivative \( \partial_t \), and obtain
\[
(1.6) \quad C = \left| \int_{\mathbb{R}^n_+^{n+1}} t \partial_t \det(\nabla_{\mathbb{R}^{n+1}} U^1, \ldots, \nabla_{\mathbb{R}^{n+1}} U^n, \nabla_{\mathbb{R}^{n+1}} \Phi) \right|.
\]
Here we used that the harmonic extensions \( U \) and \( \Phi \) satisfy
\[
\lim_{t \to \infty} t |\nabla U(x, t)|^n |\nabla \Phi(x, t)| = \lim_{t \to 0} t |\nabla U(x, t)|^n |\nabla \Phi(x, t)| = 0,
\]
see Lemma 9.1. Next we claim that
\[
(1.7) \quad C \lesssim \int_{\mathbb{R}^n_+} t |\nabla_{\mathbb{R}^{n+1}} U|^{n-1} |\nabla_{\mathbb{R}^{n+1}} \nabla_x U| |\nabla_{\mathbb{R}^{n+1}} \Phi|.
\]
That is, we can ensure that a second derivative in \( x \) hits a term different from \( \Phi \) – this can be seen as a second compensation effect. Once we have (1.7), the BMO-estimate follows from trace theorems (see Proposition 10.2 for \( s = 1 \)) and we conclude that
\[
C \lesssim [\varphi]_{BMO(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^s(\mathbb{R}^n)} \approx [\varphi]_{BMO(\mathbb{R}^n)} \|\nabla u\|_{L^s(\mathbb{R}^n)}.
\]
Let us prove (1.7). When the derivative \( \partial_t \) in (1.6) hits one of the \( \nabla_{\mathbb{R}^{n+1}} U^i \), \( i = 1, \ldots, n, \) we simply observe that the harmonicity \( \partial_t U = -\Delta_x U \) implies
\[
|\nabla_{\mathbb{R}^{n+1}} U| \lesssim |\nabla_{\mathbb{R}^{n+1}} \nabla_x U|,
\]
which leads to an estimate as in (1.7).
It remains to consider the case when the derivative \( \partial_t \) hits \( \nabla_{R^{n+1}} \Phi \). Renaming the variables \((z_1, \ldots, z_{n+1}) = (x_1, \ldots, x_n, t)\) we have

\[
C_2 := \left| \int_{R^{n+1}} t \det(\nabla_{R^{n+1}} U^1, \ldots, \nabla_{R^{n+1}} U^n, \partial_t \nabla_{R^{n+1}} \Phi) \right|
\]

\[
\leq \sum_{i_1, \ldots, i_n = 1}^{n+1} \sum_{k = 1}^{n} \int_{R^{n+1}} z_{n+1} \partial_{z_{i_1}} U^1 \cdots \partial_{z_{i_n}} U^n \partial_{z_{n+1}} \partial_{z_k} \Phi
\]

\[
+ \sum_{i_1, \ldots, i_n = 1}^{n+1} \int_{R^{n+1}} z_{n+1} \partial_{z_{i_1}} U^1 \cdots \partial_{z_{i_n}} U^n \partial_{z_{n+1}} \partial_{z_{\ell}} \Phi
\]

By harmonicity \( \partial_{z_{n+1}} \Phi = -\sum_{\ell = 1}^{n} \partial_{z_{\ell}} \partial_{z_{\ell}} \Phi \).

\[
C_2 \leq \sum_{i_1, \ldots, i_n = 1}^{n+1} \sum_{k = 1}^{n} \int_{R^{n+1}} z_{n+1} \partial_{z_{i_1}} U^1 \cdots \partial_{z_{i_n}} U^n \partial_{z_{n+1}} \partial_{z_k} \Phi
\]

\[
+ \sum_{i_1, \ldots, i_n = 1}^{n+1} \sum_{\ell = 1}^{n} \int_{R^{n+1}} z_{n+1} \partial_{z_{i_1}} U^1 \cdots \partial_{z_{i_n}} U^n \partial_{z_{n+1}} \partial_{z_{\ell}} \Phi
\]

Integrating by parts, in \( z_k \) in the first term, and in \( z_{\ell} \) in the second term, we find

\[
C_2 \leq \sum_{i_1, \ldots, i_n = 1}^{n+1} \sum_{k = 1}^{n} \left| \int_{R^{n+1}} z_{n+1} \partial_{z_{i_1}} \left( \partial_{z_{i_1}} U^1 \cdots \partial_{z_{i_n}} U^n \right) \partial_{z_{n+1}} \Phi \right|
\]

\[
+ \sum_{i_1, \ldots, i_n = 1}^{n+1} \sum_{\ell = 1}^{n} \left| \int_{R^{n+1}} z_{n+1} \partial_{z_{i_1}} \left( \partial_{z_{i_1}} U^1 \cdots \partial_{z_{i_n}} U^n \right) \partial_{z_{n+1}} \Phi \right|
\]

No boundary terms appear in the above integration-by-parts in \( z_k \) and \( z_{\ell} \) direction, since \( k, \ell \leq n \) and the harmonic extension decays to zero sufficiently fast as \( |x| \to \infty \), see Proposition 9.1. We conclude that

\[
C_2 \lesssim \int_{R^{n+1}} t \left| \nabla_{R^{n+1}} U \right|^{n-1} \left| \nabla_{R^{n+1}} \nabla_x U \right| \left| \nabla_{R^{n+1}} \Phi \right|.
\]

This establishes (1.7) and consequently (1.1) is proven. \( \square \)

To summarize: by an harmonic extension, integration by parts, and then trace-space characterizations we obtain the full strength of the Jacobian estimate, Theorem 1.1, by Coifman, Lions, Meyer, and Semmes [11].

**Outline of the paper.** In Section 2 we introduce the harmonic extension via the generalized Poisson-operator.

In the remaining sections we use the ideas presented above to show several commutator estimates. Most of them have been proven before, some of the intermediate estimates seem
to be new or known only to some experts. Let us remark that some of those estimates here (that we could not find in the literature) have been announced in [40], and were proven in the arxiv-version of that paper via paraproducts.

We will treat the following:

- Section 3: The div-curl estimate by Coifman, Lions, Meyer, and Semmes.
- Section 4: The Coifman-Rochberg-Weiss commutator estimate for Riesz transforms.
- Section 5: The Chanillo commutator estimate for Riesz potentials (of order < 1).
- Section 6: Coifman-Meyer and Kato-Ponce-Vega type commutator estimates.
- Section 7: The Da Lio-Riviè re three-term commutator estimate.
- Section 8: \(L^1\)-estimate for a double-commutator of Coifman-Rochberg-Weiss type.

The last-mentioned \(L^1\)-estimate of Section 8 seems to be new. In some sense it is a limit-version of the Coifman-Rochberg-Weiss-Theorem, Theorem 4.1.

This work is partially thought as an invitation to the reader: we do not expect that the above list of examples is exhaustive. It should be possible to obtain several more estimates with our method. In particular it would be interesting to see if one obtains sharp limit space estimates as in [3, 28]. Also we treat only differential orders < 2, sometimes even < 1. Since there are higher order extensions to local operators, see [48, 34], it should in principle be also possible to obtain higher order commutator estimates.

In the last two sections we collect the used trace-space characterization. We propose to use them as black boxes. In Section 9 we gather estimates on the Poisson operator and identification of trace spaces; in Section 10 we state resulting trace-inequalities. The proofs can mostly be found in the literature, in particular Stein’s books. We indicate the relevant arguments in the appendix.

**Comparison to Littlewood-Paley decomposition.** There is, of course, a general technique for proving almost any commutator estimate: Littlewood-Paley decompositions and paraproducts. The advantage of the method discovered here is that the deep harmonic analysis facts are concentrated in the trace characterization results and those can be used as a black box – see Section 9 and 10. Moreover, the *cancellation effects* responsible for the commutator estimates follow from very simple product-rules and can be traced exactly. This is different from the paraproduct approach which – while being a stronger and more general technique – is also much more involved, seemingly messy, and less accessible to the non-expert.

Let us also remark that the methods presented here generalize to estimates in Besov- and Triebel-Lizorkin spaces – some trace theorems (which are estimates on Poisson-type potentials, see Section 2) actually can be seen from the identification of Besov- and Triebel spaces, see in particular the recent work by Bui and Candy [8].
Limits of our method. Generally, the extension method seems to be useful, if the resulting extended operator has a product rule. As we shall see, this is the case for Riesz transforms \( R_i \) and \( s/2 \)-Laplacians \((-\Delta)^{\frac{s}{2}}\). For example \( \Delta(\phi v) \) is easy to compute, while \((-\Delta)^{\frac{s}{2}}(uv)\) is more complicated. When the operator – in our case the fractional Laplacian \((-\Delta)^{\frac{s}{2}}\) and Riesz transforms \( R_i \) – are replaced with more general operators (e.g. general multipliers, Calderón-Zygmund kernels), then the extension does not simplify the situation – the extended expression may not enjoy an easily computable product rule. Then indeed the general argument of Littlewood-Paley theory and para products seems more appropriate.

Similar to the Littlewood-Paley decomposition and paraproducts, the extension technique presented here seems to be essentially intrinsically linear. It does not seem to work that well for nonlinear commutators, see e.g. [38, 39]. Also, Fourier-transform based arguments like Littlewood-Paley theory but also our method do not seem to be well-suited to obtain pointwise commutator estimates, as e.g. the ones introduced by the second-named author in [36], see also [18, 2, 40].

Finally, let us also remark that interpreting commutators via the extension is not new itself, see, e.g., [41] and references within – but we are not aware that it is observed anywhere that one can prove sharp commutator estimates for the operators we consider here in this way.

2. Extension via the Poisson operator

For smooth, compactly supported functions \( f \in C_c^\infty (\mathbb{R}^n, \mathbb{R}^m) \) the Poisson extension operator \( P_t^s \) for \( s > 0 \) is given by

\[
P_t^s f(x) := C_{n,s} \int_{\mathbb{R}^n} \frac{t^s}{(|x-z|^2 + t^2)^{n+2}} f(z) \, dz.
\]

Often we will denote \( F^s(x,t) := P_t^s f(x) \). For \( s = 1 \) the operator \( P_t^1 \) is the usual Poisson operator and \( F^1 \) is the harmonic extension of \( f \) to \( \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty) \). More precisely,

\[
\begin{align*}
\Delta_{\mathbb{R}^{n+1}} F^1(x,t) &\equiv (\partial_t + \Delta_x) F^1(x,t) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\
\lim_{t \to 0} -\partial_t F^1(x,t) &= c (-\Delta)^{\frac{s}{2}} f(x) \quad \text{on } \mathbb{R}^n, \\
\lim_{t \to 0} F^1(x,t) &= f(x) \quad \text{on } \mathbb{R}^n, \\
\lim_{||(x,t)|| \to \infty} F^1(x,t) &= 0.
\end{align*}
\]
For $s \in (0, 2)$ the generalized Poisson operator $P_t^s$ the function $F(x, t) := P_t^s f(x)$ satisfies

$$
\begin{align*}
\text{div}_{\mathbb{R}^{n+1}}(t^{1-s}\nabla F(x, t)) &= 0 \quad \text{in } \mathbb{R}^{n+1}, \\
\lim_{t \to 0} -t^{1-s} \partial_t F(x, t) &= c (-\Delta)^{\frac{s}{2}} f(x) \quad \text{on } \mathbb{R}^n, \\
\lim_{t \to 0} F(x, t) &= f(x) \quad \text{on } \mathbb{R}^n, \\
\lim_{|x(t)| \to \infty} F^s(x, t) &= 0.
\end{align*}
$$

(2.3)

The operator $P_t^s$ is sometimes called Poisson-Bessel kernel, see Marias [29]. The boundary identification $\lim_{t \to 0} -t^{1-s} \partial_t F = c (-\Delta)^{\frac{s}{2}} f$ is due to Caffarelli and Silvestre [9]. Here $(-\Delta)^{\frac{s}{2}}$ denotes the fractional Laplacian on $\mathbb{R}^n$, defined as the operator with Fourier symbol $c |\xi|^s$. More precisely, denote with $\mathcal{F}$, $\mathcal{F}^{-1}$ the Fourier transforms and its inverse, respectively. Then $(-\Delta)^{\frac{s}{2}}$ is defined as

$$
(-\Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}(c |\xi|^s \mathcal{F} f).
$$

Many function spaces involving functions $f : \mathbb{R}^n \to \mathbb{R}$ can be characterized by function spaces on the $\mathbb{R}^{n+1}$-function $F^s(x, t)$. Several of those characterizations can be found in Section 9. Resulting estimates of $\mathbb{R}^{n+1}$-integrals involving Poisson extended functions can be found in Section 10.

### 3. The Coifman-Lions-Meyer-Semmes estimate

The estimate (3.1) below is the general div-curl estimate\(^3\) that was proven by Coifman-Lions-Meyer-Semmes in [11]. We could not find the intermediate estimate in the literature, although it is known to some experts.

**Theorem 3.1** (Coifman-Lions-Meyer-Semmes). Assume that $f, \varphi \in C_\infty^\infty(\mathbb{R}^n)$, $g \in C_\infty^\infty(\mathbb{R}^{n}, \mathbb{R}^n)$ and

$$
\text{div}(g) = \sum_{i=1}^{n} \partial_i g_i = 0.
$$

If $1 < p_1, p_2 < \infty$, $1 \leq q_1, q_2 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $\frac{1}{q_1} + \frac{1}{q_2} = 1$, then

$$
\int_{\mathbb{R}^n} \sum_{i=1}^{n} \partial_i f : g_i \varphi \lesssim [\varphi]_{\text{BMO}} \|\nabla f\|_{L^{(p_1,q_1)}(\mathbb{R}^n)} \|g\|_{L^{(p_2,q_2)}(\mathbb{R}^n)}
$$

Moreover we have the following intermediate estimate. If $s_1 + s_2 + s_3 = 2$, $0 < s_1, s_2, s_3 < 1$, and $1 < p_1, p_2, p_3 < \infty$, $1 \leq q_1, q_2, q_3 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$, then

$$
\int_{\mathbb{R}^n} \sum_{i=1}^{n} \partial_i f : g_i \varphi \lesssim \|(-\Delta)^{\frac{s_2}{2}} \varphi\|_{L^{(p_1,q_1)}} \|(-\Delta)^{\frac{s_2}{2}} f\|_{L^{(p_2,q_2)}} \|I^{1-s_3} g\|_{L^{(p_3,q_3)}}
$$

\(^3\)In [11] it is shown that the div-curl-term belongs to the Hardy-space which by the Hardy-BMO-duality is equivalent to this estimate.
To prove this, it is convenient to use the language of differential forms. Let \( f \in C_c^\infty(\mathbb{R}^n) \) and \( g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \) so that
\[
\text{div}(g) = \sum_{i=1}^n \partial_i g^i = 0.
\]
If we interpret \( g \) as an \((n-1)\)-form, \( g \in C_c^\infty(\bigwedge^{n-1} \mathbb{R}^n) \), by the Poincaré Lemma on differential forms, we find an \((n-2)\)-form \( h \in C_c^\infty(\bigwedge^{n-2} \mathbb{R}^n) \) so that
\[
\sum_{i=1}^n \partial_i f \ g^i = df \wedge dh.
\]
The div-curl estimate by Coifman-Lions-Meyer-Semmes, Theorem 3.1 is then equivalent to the following estimate.

**Theorem 3.2.** Let \( \ell \in \{0, \ldots, n-2\} \). Assume that \( f \in C_c^\infty(\bigwedge^{\ell} \mathbb{R}^n) \), \( h \in C_c^\infty(\bigwedge^{n-\ell-2} \mathbb{R}^n) \) and \( \varphi \in C_c^\infty(\mathbb{R}^n) \). If \( 1 < p_1, p_2 < \infty \), \( 1 \leq q_1, q_2 \leq \infty \) and \( \frac{1}{p_1} + \frac{1}{p_2} = 1 \), \( \frac{1}{q_1} + \frac{1}{q_2} = 1 \), then
\[
(3.1) \quad \int_{\mathbb{R}^n} df \wedge dh \varphi \lesssim [\varphi]_{\text{BMO}} \| \nabla f \|_{L^{(p_1, q_1)}(\mathbb{R}^n)} \| \nabla h \|_{L^{(p_2, q_2)}(\mathbb{R}^n)}
\]
Moreover we have the following intermediate estimate. If \( s_1 + s_2 + s_3 = 2 \), \( 0 < s_1, s_2, s_3 < 1 \), and \( 1 < p_1, p_2, p_3 < \infty \), \( 1 \leq q_1, q_2, q_3 \leq \infty \) such that \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \), \( \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1 \), then
\[
(3.2) \quad \int_{\mathbb{R}^n} df \wedge dh \varphi \lesssim \| (-\Delta)^{\frac{s_1}{2}} \varphi \|_{L^{(p_1, q_1)}(\mathbb{R}^n)} \| (-\Delta)^{\frac{s_2}{2}} f \|_{L^{(p_2, q_2)}(\mathbb{R}^n)} \| (-\Delta)^{\frac{s_3}{2}} h \|_{L^{(p_3, q_3)}(\mathbb{R}^n)}
\]
Let us explain the norms appearing in (3.1) and (3.2). Any \( \ell \)-form \( f \) is of the form
\[
f = \sum_{1 \leq i_1 < \ldots < i_\ell \leq n} f_{i_1, \ldots, i_\ell} \, dx^{i_1} \wedge \ldots \wedge dx^{i_\ell}.
\]
We say that \( f \in C_c^\infty(\bigwedge^{\ell} \mathbb{R}^n) \), if \( f_{i_1, \ldots, i_\ell} \in C_c^\infty(\bigwedge^{\ell} \mathbb{R}^n) \) for all \( 1 \leq i_1 < \ldots < i_\ell \leq n \). In this canonical way all function spaces extend to function spaces on \( \ell \)-forms. In particular,
\[
\| \nabla f \|_{L^{(p, q)}(\mathbb{R}^n)} := \sum_{1 \leq i_1 < \ldots < i_\ell \leq n} \| \nabla f_{i_1, \ldots, i_\ell} \|_{L^{(p, q)}(\mathbb{R}^n)},
\]
and
\[
\| (-\Delta)^{\frac{s}{2}} f \|_{L^{(p, q)}(\mathbb{R}^n)} := \sum_{1 \leq i_1 < \ldots < i_\ell \leq n} \| (-\Delta)^{\frac{s}{2}} f_{i_1, \ldots, i_\ell} \|_{L^{(p, q)}(\mathbb{R}^n)}.
\]

**Proof of Theorem 3.2.** We extend \( f, h, \varphi \) harmonically to \( \mathbb{R}^{n+1}_+ \), that is we let
\[
\Phi(x, t) := P_t^1 \varphi(x), \quad F(x, t) := P_t^1 f(x), \quad H(x, t) := P_t^1 h(x).
\]
By Stokes’ theorem on differential forms,
\[
C := \left| \int_{\mathbb{R}^n} df \wedge dh \varphi \right| = \left| \int_{\mathbb{R}^{n+1}_+} dF \wedge dH \wedge d\Phi \right|.
\]
The second claim, the intermediate estimate (3.2), follows from Proposition 10.1 and

\[ C \lesssim \int_{\mathbb{R}^{n+1}_+} |\nabla_{\mathbb{R}^{n+1}} F| \ |\nabla_{\mathbb{R}^{n+1}} H| \ |\nabla_{\mathbb{R}^{n+1}} \Phi|. \]

In order to show (3.1), we integrate-by-parts in \( t \) (observe the decay as \( t \to \infty \), see (9.1)),

\[ C := \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t (dF \wedge dH \wedge d\Phi) \right|. \]

We claim that

\[ C \lesssim \int_{\mathbb{R}^{n+1}_+} \left( |\nabla_x \nabla_{\mathbb{R}^{n+1}} F| \ |\nabla_{\mathbb{R}^{n+1}} H| + |\nabla_{\mathbb{R}^{n+1}} F| \ |\nabla_x \nabla_{\mathbb{R}^{n+1}} H| \right) \ |\nabla_{\mathbb{R}^{n+1}} \Phi|. \tag{3.3} \]

For this, renaming the coordinates on \( \mathbb{R}^{n+1}_+ \) from \((x_1, \ldots, x_n, t)\) to \((z_1, \ldots, z_{n+1})\)

\[ df \wedge dh \wedge d\varphi = \sum_{i,j,k=1}^{n} \sum_{I,J} \partial_{z_i} F_I \partial_{z_j} H_J \partial_{z_k} \Phi \ dz^i \wedge dz^j \wedge dz^k, \]

where the sum is over all multiindices \( I \) and \( J \) which are of the form \( I = (i_1, i_2, \ldots, i_\ell) \) for some \( i_1 < i_2 < \ldots < i_\ell \), and \( J = (j_1, j_2, \ldots, j_{n-2-\ell}) \) for some \( j_1 < j_2 < \ldots < j_{n-2-\ell} \).

Consequently,

\[
\partial_t (dF \wedge dH \wedge d\Phi) = \sum_{i,j,k=1}^{n} \sum_{I,J} \partial_{z_{i+1}} (\partial_{z_i} F_I \partial_{z_j} H_J) \partial_{z_k} \Phi \ dz^i \wedge dz^j \wedge dz^k
+ \sum_{i,j}^{n} \sum_{k=1}^{n} \partial_{z_i} F_I \partial_{z_j} H_J \partial_{z_k} \partial_{z_{n+1}} \Phi \ dz^i \wedge dz^j \wedge dz^k
+ \sum_{i,j}^{n} \partial_{z_i} F_I \partial_{z_j} H_J \partial_{z_{n+1}} \partial_{z_{n+1}} \Phi \ dz^i \wedge dz^j \wedge dz^{n+1}.
\]

Observe that \( \partial_t F_I = -\sum_{\ell=1}^{n} \partial_{z_{\ell+1}} \partial_{z_{\ell}} F_I \) and likewise for \( H \), for the first term \( I \) we have

\[
\left| \int_{\mathbb{R}^{n+1}_+} I \right| \lesssim \int_{\mathbb{R}^{n+1}_+} \left( |\nabla_x \nabla_{\mathbb{R}^{n+1}} F| \ |\nabla_{\mathbb{R}^{n+1}} H| + |\nabla_{\mathbb{R}^{n+1}} F| \ |\nabla_x \nabla_{\mathbb{R}^{n+1}} H| \right) \ |\nabla_{\mathbb{R}^{n+1}} \Phi|. \]
As for the second term $II$, since $k < n + 1$ the variable $z_k = x_k$, and we can integrate by parts in $z_k$,

$$
\left| \int_{\mathbb{R}^{n+1}_+} II \right| = \left| \int_{\mathbb{R}^{n+1}_+} \sum_{i,j} \sum_{k=1}^{n+1} \partial_{z_k} \left( \partial_{z_i} F_I \partial_{z_j} H_J \right) \partial_{z_{n+1}} \Phi \ dz_i \wedge dz_j \wedge dz_k \wedge dz_l \right|
$$

$$
\lesssim \int_{\mathbb{R}^{n+1}_+} \left( |\nabla_x \nabla_{\mathbb{R}^{n+1}_+} F| |\nabla_{\mathbb{R}^{n+1}_+} H| + |\nabla_x \nabla_{\mathbb{R}^{n+1}_+} F| |\nabla_x \nabla_{\mathbb{R}^{n+1}_+} H| \right) |\partial_I \Phi|.
$$

Finally, for $III$, again by harmonicity of $\Phi$ we have $\partial_{z_{n+1}} \partial_{z_{n+1}} \Phi \equiv \partial_I \Phi = -\sum_{\ell=1}^{n} \partial_{z_{\ell}} \partial_{z_{n+1}} \Phi$, and thus

$$
\left| \int_{\mathbb{R}^{n+1}_+} III \right| \leq \sum_{\ell=1}^{n} \int_{\mathbb{R}^{n+1}_+} \sum_{i,j} \sum_{k=1}^{n+1} \partial_{z_k} \left( \partial_{z_i} F_I \partial_{z_j} H_J \right) \partial_{z_{n+1}} \Phi \ dz_i \wedge dz_j \wedge dz_k \wedge dz_l \right|
$$

$$
\lesssim \int_{\mathbb{R}^{n+1}_+} \left( |\nabla_x \nabla_{\mathbb{R}^{n+1}_+} F| |\nabla_{\mathbb{R}^{n+1}_+} H| + |\nabla_x \nabla_{\mathbb{R}^{n+1}_+} F| |\nabla_x \nabla_{\mathbb{R}^{n+1}_+} H| \right) |\nabla_x \Phi|.
$$

Consequently we have shown (3.3) and Theorem 3.2 is proven.

\[ \square \]

4. The Coifman-Rochberg-Weiss Commutator

We turn to the Coifman-Rochberg-Weiss theorem [14]. We only prove it for Riesz transforms $(R_i)_{i=1}^n$ acting on functions in $\mathbb{R}^n$, while the theorem is actually true for all Calderon-Zygmund operators. Recall that the Riesz transforms are defined as $R_i = \partial_i I^1$.

**Theorem 4.1 (Coifman-Rochberg-Weiss [14]).** For any smooth and compactly supported $f, g \in C^\infty_c(\mathbb{R}^n)$ and any $i = 1, \ldots, n$ we define the commutator

$$
[R_i, \varphi](g) := R_i(\varphi g) - \varphi R_i(g).
$$

Then, with constants depending only on $p$ and the dimension,

$$
\| [R_i, \varphi](g) \|_{L^p(\mathbb{R}^n)} \lesssim [\varphi]_{BMO} \| g \|_{L^p(\mathbb{R}^n)}
$$

From the proof below, one can also obtain intermediate estimates, see Theorem 6.1.

Let us remark on the relation to Theorem 3.1. Jacobian estimates and div-curl estimates are special cases of the Coifman-Rochberg-Weiss commutator theorem. Indeed, let us illustrate this for the two-dimensional situation: take $u : \mathbb{R}^2 \to \mathbb{R}^2$ and consider the Jacobian $\det(\nabla u^1, \nabla u^2)$. The following facts are important:

$$
R_2 \partial_1 f = R_1 \partial_2 f
$$

and

$$
\int_{\mathbb{R}^n} g R_i f = -\int_{\mathbb{R}^n} R_i g f.
$$
Then, since \( \partial_t = \mathcal{R}_i \circ (-\Delta)^{\frac{1}{2}} \),
\[
\int_{\mathbb{R}^2} \varphi \text{ det}(\nabla u^1, \nabla u^2)
\]
\[
= \int_{\mathbb{R}^2} \varphi \left( \mathcal{R}_1(-\Delta)^{\frac{1}{2}} u^1 \partial_2 u^2 - \mathcal{R}_2(-\Delta)^{\frac{1}{2}} u^1 \partial_1 u^2 \right)
\]
\[
= - \int_{\mathbb{R}^2} (-\Delta)^{\frac{1}{2}} u^1 \left( [\mathcal{R}_1, \varphi](\partial_2 u^2) - [\mathcal{R}_2, \varphi](\partial_1 u^2) \right).
\]

Similar arguments hold for determinants and \text{div-curl} products of any dimension. Thus indeed, as was discovered in [11], Theorem 3.1 is a special case of Theorem 4.1.

\textbf{Proof of Theorem 4.1.} As before, let \( \Phi, F, \) and \( G \) denote the harmonic extension to \( \mathbb{R}^{n+1}_+ \) of \( \varphi, f, g, \) respectively.

With an abuse of notation we write \( \tilde{\mathcal{R}}_i[F](x, t) := P_t^1 \mathcal{R}_i f \). A word of warning: this object \( \tilde{\mathcal{R}}_i \) acting on \( \mathbb{R}^{n+1} \) is not the actual Riesz transform on \( \mathbb{R}^{n+1} \). Indeed its symbol \( \sigma(\tilde{\mathcal{R}}_i)(\xi, t) \) is \( \xi_i/|\xi| \) as opposed to the symbol of the \( \mathbb{R}^{n+1} \)-Riesz transform \( \xi_i/(\sqrt{\xi^2 + t^2}) \). Thus \( \tilde{\mathcal{R}}_i \) is not even a Hörmander-type multiplier operator on \( \mathbb{R}^{n+1} \) (those multipliers are continuous away from the origin), but a rough Marcinkiewicz-multiplier (multipliers which are possibly singular at the coordinate axes).

We use the integration-by-parts formula in \( t \), using the decay of the harmonic extensions from Lemma 9.1. Then we have
\[
C := \left| \int_{\mathbb{R}^n} \varphi f \mathcal{R}_i[g] + \varphi \mathcal{R}_i[f] g \right| = \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t \left( \Phi F \tilde{\mathcal{R}}_i[G] + \Phi \tilde{\mathcal{R}}_i[F] G \right) \right|.
\]
We claim that
\[
(4.2) \quad C \lesssim \max_{F \in \{\Phi, F, F\}} \max_{G \in \{G, G, G\}} \int_{\mathbb{R}^{n+1}_+} t |\nabla_{\mathbb{R}^{n+1}} \Phi| \left( |\tilde{F}| |\nabla_{\mathbb{R}^{n+1}} \tilde{G}| + |\tilde{G}| |\nabla_{\mathbb{R}^{n+1}} \tilde{F}| \right).
\]
In words: one derivative hits \( \Phi \), the other one hits \( F \) or \( G \).

Once we have (4.2), Proposition 10.2 implies
\[
C \lesssim [\varphi]_{\text{BMO}} \max_{f \in \{\Phi, F, F\}} \max_{g \in \{G, G, G\}} \| \tilde{g} \|_{L^p(\mathbb{R}^n)} \| \tilde{f} \|_{L^{p'}(\mathbb{R}^n)} \lesssim [\varphi]_{\text{BMO}} \| g \|_{L^p(\mathbb{R}^n)} \| f \|_{L^{p'}(\mathbb{R}^n)},
\]
the last inequality is the boundedness of Riesz transforms on \( L^p(\mathbb{R}^n) \) for any \( 1 < p < \infty \). This estimate implies (4.1) by duality.

Now we establish (4.2). Computing the derivatives \( \partial_{tt} \) we have three terms to consider: Firstly, the term
\[
C_1 := \left| \int_{\mathbb{R}^{n+1}_+} t \left( \partial_t \Phi \partial_t (F \tilde{\mathcal{R}}_i[G]) + \partial_t \Phi \partial_t (\tilde{\mathcal{R}}_i[F] G) \right) \right|
\]
can directly be estimated as in (4.2). Secondly, since $\partial_{tt} \Phi = -\Delta_x \Phi = -\nabla_x \cdot \nabla_x \Phi$, an integration by parts in $x$-direction

$$\mathcal{C}_2 := \left| \int_{\mathbb{R}^{n+1}_+} t \left( \partial_{tt} \Phi (F \mathcal{R}_i[G] + \mathcal{R}_i[F] G) \right) \right| = \left| \int_{\mathbb{R}^{n+1}_+} t \left( \nabla_x \Phi \cdot \nabla_x (F \mathcal{R}_i[G] + \mathcal{R}_i[F] G) \right) \right|,$$

which again can be estimated as required for (4.2).

Finally, it remains to find an estimate of the form (4.2) for

$$\mathcal{C}_3 := \left| \int_{\mathbb{R}^{n+1}_+} t \Phi \partial_{tt} \left( F \mathcal{R}_i[G] + \mathcal{R}_i[F] G \right) \right|.$$

For this we need some rules on the interplay of Riesz transform and derivatives. Those can be computed, e.g., from the exponential representation of the Poisson potential,

$$F(x, t) = \tilde{c} e^{-t\sqrt{-\Delta} f}.$$

For some constant $c \in \mathbb{R}$,

(4.4) \hspace{1cm} \partial_t \mathcal{R}_i F = -c \partial_{x_i} F,$$

(4.5) \hspace{1cm} \Delta \mathcal{R}_i F = c \partial_t \partial_{x_i} F,$$

and

(4.6) \hspace{1cm} \partial_{tt} \mathcal{R}_i F = -c \partial_t \partial_{x_i} F.$$

For sake of overview we may assume (by renormalizing $\mathcal{R}_i$) that $c = 1$.

One cancellation effect for the estimate (4.3) appears here:

$$\partial_t \left( F \partial_t \mathcal{R}_i[G] + \partial_t \mathcal{R}_i[F] G \right) = -\partial_{x_i} \partial_t (FG).$$

Moreover, using harmonicity, $\partial_{tt} F = -\Delta_x F$, (observe that everything commutes with $\mathcal{R}_i$)

$$\partial_t \left( \partial_t F \mathcal{R}_i[G] + \mathcal{R}_i[F] \partial_t G \right) = -\Delta_x F \mathcal{R}_i[G] - \mathcal{R}_i[F] \Delta_x G$$

$$+ \Delta_x \mathcal{R}_i[F] G + F \Delta_x \mathcal{R}_i[G]$$

$$- \partial_{x_i} \left( \partial_t F G + F \partial_t G \right),$$

and with a second cancellation effect

$$= -\nabla_x \cdot \left( \nabla_x F \mathcal{R}_i[G] - \mathcal{R}_i[F] \nabla_x G \right)$$

$$+ \nabla_x \cdot \left( \nabla_x \mathcal{R}_i[F] G + F \nabla_x \mathcal{R}_i[G] \right)$$

$$- \partial_{x_i} \left( \partial_t F G + F \partial_t G \right).$$
Thus, we have shown that
\[
\partial_t \left( F \tilde{R}_i [G] + \tilde{R}_i [F] G \right) = - \partial_{x_i} \partial_t (FG) \\
- \nabla_x \cdot \left( \nabla_x F \tilde{R}_i [G] - \tilde{R}_i [F] \nabla_x G \right) \\
+ \nabla_x \cdot \left( \nabla_x \tilde{R}_i [F] G + F \nabla_x \tilde{R}_i [G] \right) \\
- \partial_{x_i} (\partial_t FG + F \partial_t G).
\]

Plugging this into (4.3) and performing an integration by parts in x-direction (no boundary terms appear in x-direction), we see the estimate of the form (4.2). □

5. Chanillo-type commutator of Riesz Potentials

For \( s \in (0,1) \) we also obtain an extension of the results of Coifman-Rochberg-Weiss to Riesz potentials. In [10], Chanillo showed the following theorem on commutators of Riesz potential and pointwise multiplication,
\[
[I^s, \varphi](f) := I^s(\varphi f) - \varphi I^s f.
\]

**Theorem 5.1** (Chanillo). Let \( s \in (0, n) \) then
\[
\|[I^s, \varphi](f)\|_{L^q(\mathbb{R}^n)} \lesssim \|\varphi\|_{BMO} \|f\|_{L^p(\mathbb{R}^n)},
\]
where \( 1 < p < \frac{n}{s} \) and
\[
(5.1) \quad \frac{1}{q} = \frac{1}{p} - \frac{s}{n}.
\]

By duality, setting \( u := I^s f \), Theorem 5.1 is a consequence of the following proposition. It is stated for \( s \in [0,1) \). With little extra work (iterating the integration-by-parts procedure) one can extend this to \( s \in [0,2) \). For higher order \( s \) one first needs a suitable higher-order extension replacing the one by Caffarelli and Silvestre [9]. This is done in [48, 34]. So we think it is likely to obtain the full Theorem 5.1 with this method, but we will make no attempt to prove this here.

**Proposition 5.2.** Let \( s \in [0,1) \), and \( p, q \) as in Theorem 5.1, \( q' = \frac{q}{q-1} \) then
\[
\int_{\mathbb{R}^n} \left( (-\Delta)^{s} u v - u (-\Delta)^{s} v \right) \varphi \lesssim \|\varphi\|_{BMO} \|(-\Delta)^{s} u\|_{L^p} \|(-\Delta)^{s} v\|_{L^{q'}}.
\]

**Proof.** Let \( U(x,t) := P_t^s u(x) \), \( V(x,t) := P_t^s v(x) \), \( \Phi(x,t) := P_t^s \varphi(x) \) the Caffarelli-Silvestre extension, as in (2.3). Then with the integration-by-parts formula in \( t \),
\[
C := \left| \int_{\mathbb{R}^n} \left( (-\Delta)^{s} u v - u (-\Delta)^{s} v \right) \varphi \right| = c \left| \int_{\mathbb{R}^{n+1}} \partial_t \left( t^{1-s} (\partial_t U V - U \partial_t V) \Phi \right) \right|.
\]
By two cancellation effects and since \( \partial_t (t^{1-s} \partial_t U) = -t^{1-s} \Delta_x U \),
\[ \partial_t (t^{1-s} (\partial_t U V - U \partial_t V)) = -t^{1-s} (\Delta_x U V - U \Delta_x V) = -t^{1-s} \nabla_x \cdot (\nabla_x U V - U \nabla_x V). \]
Thus,
\[ C \lesssim \int_{\mathbb{R}^{n+1}} t^{1-s} (\nabla_x U V - U \nabla_x V) \nabla_x \Phi \bigg| + \int_{\mathbb{R}^{n+1}} t^{1-s} (\partial_t U V - U \partial_t V) \Phi_t \bigg| \]
By our assumption \( s < 1 \), and as we shall see below the first term is already in a good shape and can be estimated by Proposition 10.2. The second term needs one more step, because with \( \partial_t U \) and Proposition 10.2 we only get an estimate in terms of \((-\Delta)^\frac{\nu}{2} u\) for \( \nu < s \). So we use again the integration-by-parts in \( t \),
\[ \int_{\mathbb{R}^{n+1}} t^{1-s} (\partial_t U V - U \partial_t V) \Phi_t = \frac{1}{s} \int_{\mathbb{R}^{n+1}} t^s \partial_t (t^{1-s} (\partial_t U V - U \partial_t V) t^{1-s} \Phi_t) \bigg| \]
Again we observe a cancellation,
\[ t^s \partial_t (t^{1-s} (\partial_t U V - U \partial_t V) t^{1-s} \Phi_t) = -t^{2-s} (\Delta_x U V - U \Delta_x V) \Phi_t - t^{2-s} (\partial_t U V - U \partial_t V) \Delta_x \Phi. \]
With yet another integration by parts in \( x \)-direction, since \( \Delta_x = \nabla_x \cdot \nabla_x \) we arrive at
\[ C \lesssim \int_{\mathbb{R}^{n+1}} t^{1-s} \big( |\nabla_x U| |V| + |U| |\nabla_x V| \big) |\nabla_{\mathbb{R}^{n+1}} \Phi| \]
\[ + \int_{\mathbb{R}^{n+1}} t^{2-s} \big( |\nabla_{\mathbb{R}^{n+1}} \nabla_x U| |V| + |U| |\nabla_{\mathbb{R}^{n+1}} \nabla_x V| \big) |\nabla_{\mathbb{R}^{n+1}} \Phi| \]
\[ + \int_{\mathbb{R}^{n+1}} t^{2-s} |\nabla_{\mathbb{R}^{n+1}} U| |\nabla_{\mathbb{R}^{n+1}} V| |\nabla_{\mathbb{R}^{n+1}} \Phi|. \]
By Proposition 10.2,
\[ C \lesssim [\varphi]_{BMO} \left( \|(-\Delta)^{\frac{\nu}{2}} u\|_{L^p} \|v\|_{L^{p'}} + \|u\|_{L^q} \|(-\Delta)^{\frac{\nu}{2}} v\|_{L^{q'}} \right). \]
We conclude by the relation between \( p \) and \( q \) (5.1) and Sobolev-inequality:
\[ \|v\|_{L^{p'}} \lesssim \|(-\Delta)^{\frac{\nu}{2}} v\|_{L^{q'}}, \quad \|u\|_{L^q} \lesssim \|(-\Delta)^{\frac{\nu}{2}} u\|_{L^p}. \]
\[ \square \]

6. Coifman-Meyer and Kato-Ponce-Vega type commutator estimate

Now we treat commutators in terms of Hölder norms, namely we consider
\[ [(-\Delta)^{\frac{\nu}{2}}, g](f) = (-\Delta)^{\frac{\nu}{2}}(gf) - g(-\Delta)^{\frac{\nu}{2}} f, \]
and its (notrivial) zero-order version
\[ [\mathcal{R}_i, g](f) = \mathcal{R}_i(gf) - g\mathcal{R}_i f. \]
The estimates below are probably most close to the Coifman-Meyer commutator estimates and Kato-Ponce-Vega type estimates, see [13, Theorem 1], [12, 26, 27]. In some sense, the
limit cases of the commutator estimates below is the Coifman-Rochberg-Weiss theorem, Theorem 4.1.

**Theorem 6.1.** Let $s \in (0, 1]$ and $p \in (1, \infty)$. Then, for $\sigma \in [s, 1]$,
\begin{equation}
\|[(−\Delta)^{\frac{s}{2}}, g](f)\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{C^s} \|I^{σ−s} f\|_{L^p(\mathbb{R}^n)}.
\end{equation}

Also, for $q_1, q_2, p \in (1, \infty)$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$, $\sigma \in [s, 1]$,
\begin{equation}
\|[(−\Delta)^{\frac{s}{2}}, g](f)\|_{L^p(\mathbb{R}^n)} \lesssim \|−\Delta\|_{L^q(\mathbb{R}^n)} \|I^{σ−s} f\|_{L^q(\mathbb{R}^n)}.
\end{equation}

For $\sigma < 1$, any $i = 1, \ldots, n$,
\begin{equation}
\|[\mathcal{R}_i, g] f\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{C^s} \|I^{σ} f\|_{L^p(\mathbb{R}^n)}.
\end{equation}

Also, for $q_1, q_2, p \in (1, \infty)$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$, $\sigma \in [0, 1]$,
\begin{equation}
\|[\mathcal{R}_i, g] f\|_{L^p(\mathbb{R}^n)} \lesssim \|−\Delta\|_{L^q(\mathbb{R}^n)} \|I^{σ} f\|_{L^q(\mathbb{R}^n)}.
\end{equation}

**Proof of (6.1) and (6.2).** Let again $F(x, t) := P_t^s f(x)$, and $G, H$ likewise be the $P_t^s$-extension of $g, h$. Integration by parts gives
\begin{equation}
\mathcal{C} := \int_{\mathbb{R}^n} g f (−\Delta)^{\frac{s}{2}} h − g (−\Delta)^{\frac{s}{2}} f h,
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^n} \partial_t \left( G \left( F t^{1−s} \partial_t H - t^{1−s} \partial_t F H \right) \right).
\end{equation}

We claim that
\begin{equation}
\mathcal{C} \lesssim \int_{\mathbb{R}^{n+1}_+} t^{2−s} |\nabla_{\mathbb{R}^{n+1}} G| \left( |\nabla_{\mathbb{R}^{n+1}} F| |H| + |\nabla_{\mathbb{R}^{n+1}} H| |F| \right)
\end{equation}
\begin{equation}
+ \int_{\mathbb{R}^{n+1}_+} t^{2−s} |\nabla_x G| |\nabla_{\mathbb{R}^{n+1}} F| |\nabla_{\mathbb{R}^{n+1}} H|.
\end{equation}

Once we confirm this, we argue with Proposition 10.3 for (6.1) and with Proposition 10.1 for (6.2). Taking in the resulting estimate the supremum over all $h$ with $\|h\|_{L^{p'}} \leq 1$ we obtain (6.1) and (6.2), respectively.

It remains to show (6.5). Observe a first cancellation
\begin{align}
\partial_t \left( G \left( F t^{1−s} \partial_t H - t^{1−s} \partial_t F H \right) \right)
&= t^{1−s} G_t (F H_t - F_t H) + G \left( F \partial_t (t^{1−s} \partial_t H) − \partial_t (t^{1−s} \partial_t F) H \right),
\end{align}
and since $\partial_t (t^{1−s} F_t) = −t^{1−s} ∆_x F$,
\begin{align}
&= t^{1−s} G_t (F H_t - F_t H) + G t^{1−s} \left( F ∆_x H - ∆_x F H \right).
\end{align}

With another cancellation in the second term,
\begin{align}
&= t^{1−s} G_t (F H_t - F_t H) + G t^{1−s} \nabla_x \cdot (F \nabla_x H - \nabla_x F H)
\end{align}
Using integration-by-parts in $x$ we decompose $C = C_1 + C_2$ with

$$C_1 := \int_{R^{n+1}_+} t^{1-s} G_t (F H_t - F_t H), \quad C_2 := \int_{R^{n+1}_+} t^{1-s} \nabla_x G \cdot (\nabla_x H - \nabla_x F H).$$

As for the second term, integration by parts in $t$-direction, gives

$$C_2 = -\frac{1}{2 - s} \int_{R^{n+1}_+} t^{2-s} \partial_t (\nabla_x G \cdot (F \nabla_x H - \nabla_x F H)).$$

Now the only term that is not already of a form needed for (6.5) is the case where the $\partial_t$ hits $\nabla_x H$ or $\nabla_x F$. But then we perform another integration-by-parts in $x$-direction,

$$\int_{R^{n+1}_+} t^{2-s} \nabla_x G \cdot (F \nabla_x H_t - \nabla_x F_t H)$$

$$= -\int_{R^{n+1}_+} t^{2-s} \Delta_x G (F H_t - F_t H) - \int_{R^{n+1}_+} t^{2-s} \nabla_x G \cdot (\nabla_x F H_t - F_t \nabla_x H).$$

This is clearly of the form needed for (6.5). Thus $C_2$ is estimated.

As for $C_1$, integration by parts tells us

$$C_1 = \int_{R^{n+1}_+} t^{1-s} G_t (F H_t - F_t H), = -\frac{1}{s} \int_{R^{n+1}_+} t^s \partial_t (t^{1-s} G_t (F t^{1-s} H_t - t^{1-s} F_t H)).$$

Now, a second cancellation happens, since $F_t t^{1-s} H_t - t^{1-s} F_t H = 0$,

$$= \frac{1}{s} \int_{R^{n+1}_+} t^{2-s} \Delta_x G (F H_t - F_t H) + \frac{1}{s} \int_{R^{n+1}_+} t^{2-s} G_t (F \Delta_x H - \Delta_x F H).$$

and for the second term a further cancellation $0 = \nabla F \cdot \nabla H - \nabla F \cdot \nabla H$,

$$= \frac{1}{s} \int_{R^{n+1}_+} t^{2-s} \Delta_x G (F H_t - F_t H) + \frac{1}{s} \int_{R^{n+1}_+} t^{2-s} G_t \nabla \cdot (F \nabla H - \nabla F H).$$

Integrating by parts in $x$ we obtain an estimate of the form (6.5), and (6.1) is proven.

**Proof of (6.3) and (6.4).** Let $F, G, \Phi$ be the harmonic extension of $f, g, \varphi$. By duality it suffices to show

$$C := \int_{R^n} g f \mathcal{R}_i [\varphi] + g \mathcal{R}_i [f] \varphi \lesssim \left\{ [g]_{C^{\alpha}} \| I^{s} f \|_{L^p(R^n)} \| \varphi \|_{L^{p'}(R^n)} \right\} \| (-\Delta)^{\frac{s}{2}} g \|_{L^q(R^n)} \| I^{s} f \|_{L^{q_2}(R^n)} \| \varphi \|_{L^{p'}(R^n)}.$$

We estimated $C$ in the proof of Theorem 4.1 (note that the role of $\Phi$ and $G$ are exchanged there). Setting

$$C := \int_{R^{n+1}_+} t \partial_t \left( G F \mathcal{R}_i \Phi + G \mathcal{R}_i F \Phi \right),$$

and we have

$$C \lesssim \max_{F \in \{ F, \mathcal{R}_i F \}} \max_{\Phi \in \{ \Phi, \mathcal{R}_i \Phi \}} \int_{R^{n+1}_+} t |\nabla_{R^{n+1}} G| \left( |\nabla_{R^{n+1}} F| \| \Phi \| + |F| \| \nabla_{R^{n+1}} \Phi \| \right).$$
The claim follows now from Proposition 10.3 and Proposition 10.1. \qed

7. Da Lio-Rivièrem three-term commutator

For $s > 0$ let

$$H_s(f, g) := (-\Delta)^{\frac{s}{2}} (fg) - f (-\Delta)^{\frac{s}{2}} g - f (-\Delta)^{\frac{s}{2}} g.$$  

This commutator has been estimated by Kenig-Ponce-Vega \cite{27} in $L^p$-spaces, see also \cite{21, 3}. In \cite{17, 16} Da Lio and Rivièrem used this operator and showed that it appears as natural replacement for the Jacobian structure for $1/2$-harmonic maps. See also \cite{37, 15, 40} for higher order analogues and extensions. In \cite[Theorem 1.2]{17} Da Lio and Rivièrem proved the following three-term commutator estimate

**Theorem 7.1** (Da Lio-Rivièrem \cite{17}).

$$\|(-\Delta)^{\frac{s}{2}} H_1(f, g)\|_{H^1(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} g\|_{L^2(\mathbb{R}^n)}.$$  

Here, $H^1(\mathbb{R}^n)$ is the Hardy-space.

They used the theory of Triebel-Lizorkin spaces and paraproducts. Extending their techniques, the following was shown in \cite{40} (for a proof see the arxiv-version). Again one needs a lengthy computation with Triebel spaces and paraproducts. In particular the $s = 1$-case was somewhat unexpected and required special care. Now it just follows from integration by parts.

**Lemma 7.2.** For any $s \in (0, 1]$, $p \in (1, \infty)$, $p' = \frac{p}{p-1}$, $q \in [1, \infty]$, $q' = \frac{q}{q-1} \in [1, \infty]$,

\begin{equation}
\int_{\mathbb{R}^n} H_s(f, g)(-\Delta)^{\frac{s}{2}} \varphi \lesssim [\varphi]_{BMO} \|(-\Delta)^{\frac{s}{2}} a\|_{L^{p, q}(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} b\|_{L^{p', q'}(\mathbb{R}^n)}.
\end{equation}

In particular, by the duality of Hardy-space $H^1$ and BMO,

$$\|(-\Delta)^{\frac{s}{2}} ((-\Delta)^{\frac{s}{2}} (ab) - a(-\Delta)^{\frac{s}{2}} (b) - b(-\Delta)^{\frac{s}{2}} a)\|_{H^1} \lesssim \|(-\Delta)^{\frac{s}{2}} a\|_{L^{p, q}} \|(-\Delta)^{\frac{s}{2}} b\|_{L^{p', q'}}.$$  

**Proof for $s = 1$.** Assume that $s = 1$, and let $A$, $B$, $\Phi$ be the harmonic extensions as in (2.2) of $a$, $b$, $\varphi$, respectively.

We set

$$C := \left| \int_{\mathbb{R}^n} H_1(f, g)(-\Delta)^{\frac{1}{2}} \varphi \right| = \left| \int_{\mathbb{R}^n} f g (-\Delta) \varphi - (-\Delta)^{\frac{1}{2}} f g (-\Delta)^{\frac{1}{2}} \varphi - f (-\Delta)^{\frac{1}{2}} g (-\Delta)^{\frac{1}{2}} \varphi \right|.$$  

We show the following estimate from which the claim follows via Proposition 10.2.

\begin{equation}
|C| \lesssim \int_{\mathbb{R}^n} t |\nabla_{\mathbb{R}^{n+1}} \Phi| (|\nabla_{\mathbb{R}^{n+1}} A| |\nabla_{\mathbb{R}^{n+1}}^2 B| + |\nabla_{\mathbb{R}^{n+1}}^2 A| |\nabla_{\mathbb{R}^{n+1}} B|) .
\end{equation}
To obtain (7.2) we use the integration-by-parts in $t$,

$$
C = \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t (A B \partial_t \Phi - A \partial_t B \partial_t \Phi - \partial_t A B \partial_t \Phi) \right|
= \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t ((A B) \partial_t \Phi - \partial_t (A B) \partial_t \Phi) \right|.
$$

In the next step a cancellation occurs. By the product-rule for $\partial_t$,

$$
= \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t ((A B) \partial_t \Phi - \partial_t (A B) \partial_t \Phi) \right|.
$$

Due to the harmonicity of the extensions (2.2) we may replace $\partial_t \Phi$ by $-\Delta_x \Phi$, and then use integration by parts on $\Delta_x$ which does not give boundary values since it is in tangential direction,

$$
= \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t ((A B) (-\Delta_x) \partial_t \Phi - \partial_t (A B) \partial_t \Phi) \right|
= \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t (\Delta_{x,t} (A B) \partial_t \Phi) \right|.
$$

Since $A$ and $B$ are harmonic, $\Delta_{x,t} (A B) = 2 \nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B$, and thus

$$
= 2 \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t (\nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B \partial_t \Phi) \right|
\leq 2 \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t (\nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B) \partial_t \Phi \right| + 2 \left| \int_{\mathbb{R}^{n+1}_+} t \nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B \partial_t t \Phi \right|.
$$

Again replacing $\partial_t \Phi$ by $-\Delta_x \Phi$ and using integration by parts in $x$ for the second term, we arrive at

$$
= 2 \left| \int_{\mathbb{R}^{n+1}_+} t \partial_t (\nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B) \partial_t \Phi \right| + 2 \left| \int_{\mathbb{R}^{n+1}_+} t \nabla_x (\nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B) \cdot \nabla_x \Phi \right|.
$$

This proves (7.2). \qed

Proof for $s < 1$. Assume that $s < 1$. Set $\tilde{\varphi} := (-\Delta)^{\frac{s}{2}} \varphi$, and let $A, B, \tilde{\Phi}$ be the $s$-harmonic extensions of $a, b, \tilde{\varphi}$, respectively. That is

$$A(x,t) = P^s_t a(x), \quad B(x,t) = P^s_t b(x), \quad \tilde{\Phi}(x,t) = P^s_t \tilde{\varphi}(x),$$

where $P^s_t$ is the Caffarelli-Silvestre Poisson operator as in (2.3).
This time we aim for the following estimate:

$$C \lesssim \int_{\mathbb{R}^{n+1}^+} t^{1-s} |\Phi| |\nabla_x A| |\nabla_x B|$$

(7.3)

$$+ \int_{\mathbb{R}^{n+1}^+} t^{2-s} |\Phi| \left( |\nabla_x \nabla_{\mathbb{R}^{n+1}} A| |\nabla_{\mathbb{R}^{n+1}} B| + |\nabla_{\mathbb{R}^{n+1}} A| |\nabla_x \nabla_{\mathbb{R}^{n+1}} B| \right)$$

$$+ \int_{\mathbb{R}^{n+1}^+} t^{3-s} |\nabla_{\mathbb{R}^{n+1}} \tilde{\Phi}| \left( |\nabla_{\mathbb{R}^{n+1}} \nabla_x A| |\nabla_{\mathbb{R}^{n+1}} B| + |\nabla_{\mathbb{R}^{n+1}} A| |\nabla_{\mathbb{R}^{n+1}} \nabla_x B| \right)$$

Observe that $\tilde{\Phi} = P_t(-\Delta)^{\frac{3}{2}} \varphi$, thus Proposition 10.2 applied to (7.3) implies (7.1) for $s < 1$.

It remains to establish (7.3). We use integration-by-parts in $t$ and the representation of $(-\Delta)^{\frac{3}{2}} a = c \lim_{t \to 0} t^{1-s} \partial_t A$ from (2.3),

$$C := \left| \int_{\mathbb{R}^n} ((-\Delta)^{\frac{3}{2}} (ab) - a(-\Delta)^{\frac{3}{2}} b - (-\Delta)^{\frac{3}{2}} a \ b) \varphi \right|$$

$$= \left| \int_{\mathbb{R}^{n+1}^+} \partial_t \left( t^{1-s} AB \partial_t \tilde{\Phi} - t^{1-s} A \partial_t B \tilde{\Phi} - t^{1-s} \partial_t A B \tilde{\Phi} \right) \right|$$

$$= \left| \int_{\mathbb{R}^{n+1}^+} \partial_t \left( t^{1-s}(AB) \partial_t \tilde{\Phi} - t^{1-s} \partial_t (AB) \tilde{\Phi} \right) \right| .$$

Again we use the product rule for $\partial_t$ and have a cancellation

$$= \left| \int_{\mathbb{R}^{n+1}^+} (AB) \partial_t \left( t^{1-s} \partial_t \tilde{\Phi} \right) - \partial_t (t^{1-s} \partial_t (AB)) \tilde{\Phi} \right| .$$

Since $\partial_t \left( t^{1-s} \partial_t \tilde{\Phi} \right) = -t^{1-s} \Delta_x \tilde{\Phi}$ and with an integration by parts in $x$,

$$= \left| \int_{\mathbb{R}^{n+1}^+} L_s(AB) \tilde{\Phi} \right| ,$$

where we set

$$L_s(AB) := t^{1-s} \Delta_x (AB) + \partial_t (t^{1-s} \partial_t (AB)) .$$

By (2.3), $L_s(A) = L_s(B) = 0$. On the other hand, we have the product rule

$$L_s(AB) - L_s(A) B - A L_s(B) = 2t^{1-s} \nabla_{\mathbb{R}^{n+1}} A \cdot \nabla_{\mathbb{R}^{n+1}} B .$$

Consequently,

$$C \leq 2 \left| \int_{\mathbb{R}^{n+1}^+} t^{1-s} \nabla_x A \cdot \nabla_x B \tilde{\Phi} \right| + 2 \left| \int_{\mathbb{R}^{n+1}^+} t^{1-s} \partial_t A \partial_t B \tilde{\Phi} \right| .$$
The first term is already of the form in (7.3). As for the second term, we use the following integration by parts formula in $t$-direction
\[ \int_0^\infty f(t) \, dt = -\frac{1}{s} \int_0^\infty t^s \partial_t \left( t^{1-s} f(t) \right) \, dt. \]
Thus,
\[ \int_{\mathbb{R}^{n+1}} t^{1-s} \partial_t A \partial_t B \tilde{\Phi} \]
\[ = \frac{1}{s} \int_{\mathbb{R}^{n+1}} t^s \partial_t \left( t^{1-s} \partial_t A t^{1-s} \partial_t B \tilde{\Phi} \right) \]
\[ = -\frac{1}{s} \int_{\mathbb{R}^{n+1}} t^{2-s} \left( \Delta_x A \partial_t B \tilde{\Phi} \right) - \int_{\mathbb{R}^{n+1}} t^{2-s} \left( \partial_t A \Delta_x B \tilde{\Phi} \right) \]
\[ + \frac{1}{s} \int_{\mathbb{R}^{n+1}} t^{2-s} \partial_t A \partial_t B \partial_t \tilde{\Phi} \]
\[ = -\frac{1}{s} \int_{\mathbb{R}^{n+1}} t^{2-s} \left( \Delta_x A \partial_t B \tilde{\Phi} \right) - \int_{\mathbb{R}^{n+1}} t^{2-s} \left( \partial_t A \Delta_x B \tilde{\Phi} \right) \]
\[ + \frac{1}{2s^2} \int_{\mathbb{R}^{n+1}} t^{2s} \partial_t \left( t^{1-s} \partial_t A t^{1-s} \partial_t B t^{1-s} \partial_t \tilde{\Phi} \right) \]
Now we finish by integrating by parts if $\partial_t$ hits $t^{1-s} \partial_t \tilde{\Phi}$. \qed

We also show another estimate which was obtained in [40, (5.29)] with para-product arguments.

**Theorem 7.3.** For any $s \in (0, 1]$, $p \in (1, \infty)$,
\[ \| H_s(f, \varphi) \|_{L^p(\mathbb{R}^n)} \lesssim \| (-\Delta)^{s/2} f \|_{L^p(\mathbb{R}^n)} \| \varphi \|_{BMO} \]
Also we have an intermediate estimate: for any $t \in (0, s)$, $p, p_1, p_2 \in (1, \infty)$, $q, q_1, q_2 \in [1, \infty]$ such that
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \]
it holds that
\[ \| H_s(f, \varphi) \|_{L^{(p,q)}(\mathbb{R}^n)} \lesssim \| (-\Delta)^{s/2} f \|_{L^{(p_1,q_1)}(\mathbb{R}^n)} \| (-\Delta)^{s/2} \varphi \|_{L^{(p_2,q_2)}(\mathbb{R}^n)} \]

**Proof.** We only show the BMO-estimate, the intermediate estimate follows with the same argument using Proposition 10.1 instead of Proposition 10.2.

By duality we need to show
\[ C := \left\| \int_{\mathbb{R}^n} f \varphi (-\Delta)^{s/2} g - (-\Delta)^{s/2} f \varphi \, g - f \varphi (-\Delta)^{s/2} \varphi \, g \right\|_{L^p(\mathbb{R}^n)} \lesssim [\varphi]_{BMO} \| (-\Delta)^{s/2} f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}. \]
Letting $F(x,t) := P_t^s f(x)$, $G(x,t) := P_t^s g(x)$ and $\Phi(x,t) := P_t^s \varphi(x)$, an integration by parts in $t$ gives
\[
C \lesssim \left| \int_{\mathbb{R}_+^{n+1}} \partial_t \left( t^{1-s} F \right) \partial_t G - t^{1-s} \partial_t F \left( t^{1-s} \partial_t G \right) \right|.
\]

We compute,
\[
\partial_t \left( t^{1-s} F \right) \partial_t G = t^{1-s} \left( \Delta_x (F \Phi) G - F \Phi \Delta_x G \right)
- 2 \nabla_x F \cdot \nabla_x \Phi G - 2 t^{1-s} \partial_t F \partial_t \Phi G.
\]
The first term integrates to zero when integrating in $x$,
\[
\int_{\mathbb{R}_+^{n+1}} t^{1-s} \left( \Delta_x (F \Phi) G - F \Phi \Delta_x G \right) = 0.
\]
So we have
\[
C \lesssim \left| \int_{\mathbb{R}_+^{n+1}} t^{1-s} \nabla_x F \cdot \nabla_x \Phi G \right| + \left| \int_{\mathbb{R}_+^{n+1}} t^{1-s} \partial_t F \partial_t \Phi G \right|.
\]

For $s < 1$, the first term already can be estimated by Proposition 9.4
\[
C_1 := \left| \int_{\mathbb{R}_+^{n+1}} t^{1-s} \nabla_x F \cdot \nabla_x \Phi G \right| \lesssim [\varphi]_{BMO} \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} \|g\|_{L^{p'}}.
\]

For $s = 1$, by another integration-by-parts in $t$-direction,
\[
C_1 = \left| \int_{\mathbb{R}_+^{n+1}} t \partial_t (\nabla_x F \cdot \nabla_x \Phi G) \right| \lesssim \int_{\mathbb{R}_+^{n+1}} t \left| \nabla_{\mathbb{R}_+^{n+1}} \Phi \right| \left( |\nabla_{\mathbb{R}_+^{n+1}} F| |G| + |\nabla_{\mathbb{R}_+^{n+1}} F| |\nabla_{\mathbb{R}_+^{n+1}} G| \right).
\]
Indeed, the only term not immediately in this constellation can be transformed into the right form by an integration-by-parts in $x$-direction
\[
\int_{\mathbb{R}_+^{n+1}} t \nabla_x F \cdot \nabla_x \partial_t \Phi G = - \int_{\mathbb{R}_+^{n+1}} t \nabla_x \cdot (\nabla_x F G) \partial_t \Phi.
\]
Thus also for $s = 1$, again with the help of Proposition 9.4,
\[
C_1 \lesssim [\varphi]_{BMO} \|(-\Delta)^{\frac{1}{2}} f\|_{L^p} \|g\|_{L^{p'}}.
\]

For the remaining term
\[
C_2 := \left| \int_{\mathbb{R}_+^{n+1}} t^{1-s} \partial_t F \partial_t \Phi G \right|
\]
with an integration-by-parts in $t$-direction,
\[
C_2 = \frac{1}{s} \left| \int_{\mathbb{R}_+^{n+1}} t^s \partial_t \left( t^{1-s} \partial_t F t^{1-s} \partial_t \Phi G \right) \right|.
\]
In view of $\partial_t(t^{1-s}\partial_t F) = -ct^{1-s}\Delta_x F$, this can be estimated by
\[
C_2 \lesssim \int_{\mathbb{R}^{n+1}_+} t^{2-s} |\nabla_{\mathbb{R}^{n+1}} \Phi| \left( |\nabla_{\mathbb{R}^{n+1}} \nabla_x F| \cdot |G| + |\nabla_{\mathbb{R}^{n+1}} F| \cdot |\nabla_{\mathbb{R}^{n+1}} G| \right).
\]
Indeed, the only term not in this form can be treated as above,
\[
\int_{\mathbb{R}^{n+1}_+} t^{2-s} \partial_t F \cdot t^{1-s} \Delta_x \Phi G = -\int_{\mathbb{R}^{n+1}_+} t^{2-s} \nabla_x \Phi \cdot \nabla_x (\partial_t F G).
\]
We conclude with Proposition 9.4.

\[\square\]

8. $L^1$-estimate for a double-commutator

The Coifman-Rochberg-Weiss theorem, Theorem 4.1, fails on $L^1$. More generally, it seems
that there is no reason that an $L^1$-analogue of the Riesz-transform estimates in (6.3) holds.
As an application of our techniques, we show here a replacement estimate that estimates
the commutators of those commutators in $L^1$. Denote with $\mathcal{H}$ the Hilbert transform (i.e.
the one-dimensional Riesz transform $\mathcal{R}_1$).

Theorem 8.1. For $s_1, s_2 \in (0, 1)$ and $s_1 + s_2 = 1$ and any $p \in (1, \infty)$, $q \in [1, \infty]$ we have
\[
\left\| [f, \mathcal{H}]((-\Delta)^{\frac{s_1}{2}} g) - [g, \mathcal{H}]((-\Delta)^{\frac{s_2}{2}} f) \right\|_{L^1(\mathbb{R})} \lesssim \|(-\Delta)^{\frac{s_1}{2}} f\|_{L^{(p, q)}(\mathbb{R})} \|(-\Delta)^{\frac{s_2}{2}} g\|_{L^{(p', q')}(\mathbb{R})}.
\]

and
\[
\left\| \mathcal{H} \left( [f, \mathcal{H}]((-\Delta)^{\frac{s_1}{2}} g) + [g, \mathcal{H}]((-\Delta)^{\frac{s_2}{2}} f) \right) \right\|_{L^1(\mathbb{R})} \lesssim \|(-\Delta)^{\frac{s_1}{2}} f\|_{L^{(p, q)}(\mathbb{R})} \|(-\Delta)^{\frac{s_2}{2}} g\|_{L^{(p', q')}(\mathbb{R})}.
\]

Proof of (8.1). Let
\[
C := \left| \int_{\mathbb{R}} \left( [f, \mathcal{H}](\Delta)^{\frac{s_1}{2}} g - [g, \mathcal{H}](\Delta)^{\frac{s_2}{2}} f \right) \varphi \right|
= \left| \int_{\mathbb{R}} f \mathcal{H}(\Delta)^{\frac{s_1}{2}} g \varphi + f (\Delta)^{\frac{s_1}{2}} g \mathcal{H}\varphi - \mathcal{H}(\Delta)^{\frac{s_1}{2}} f g \varphi - (\Delta)^{\frac{s_2}{2}} f g \mathcal{H}\varphi \right|.
\]

For the theorem to be proven, by duality, it suffices to show
\[
C \lesssim \|(-\Delta)^{\frac{s_1}{2}} f\|_{L^{(p, q)}(\mathbb{R})} \|(-\Delta)^{\frac{s_2}{2}} g\|_{L^{(p', q')}(\mathbb{R})} \|\varphi\|_{L^{\infty}(\mathbb{R})}.
\]
Let $F := P^1_t f$, $G := P^1_t g$, $\Phi := P^1_t \varphi$ be the respective harmonic extensions. Then, as
above, via integration by parts in $t$,
\[
C \lesssim \left| \int_{\mathbb{R}^2_+} \partial_t \left( F \mathcal{H}G_t \Phi + F G_t \mathcal{H}\Phi - \mathcal{H}F_t G \Phi - F_t G \mathcal{H}\Phi \right) \right|.
\]
Recall the rules for derivatives of the harmonic extensions of Hilbert transforms:
\[
\mathcal{H}F_t = -F_x, \quad F_t = \mathcal{H}F_x.
\]
Then
\[ C = \left| \int_{\mathbb{R}^2} \partial_t \left( -F G_x \Phi + F G_t \tilde{H} \Phi + F_x G \Phi - F_t G \tilde{H} \Phi \right) \right|. \]

We compute
\[ \mathcal{I} := \partial_t \left( -F G_x \Phi + F G_t \tilde{H} \Phi + F_x G \Phi - F_t G \tilde{H} \Phi \right) \]
\[ = -\partial_t \left( -F G_x \Phi + F_x G \Phi \right) + \partial_t \left( F G_t \tilde{H} \Phi - F_t G \tilde{H} \Phi \right) \]
and with a cancellation in the second term,
\[ = (-F_t G_x \Phi + F_x G_t \Phi) + (-F G_{xt} \Phi + F_{xt} G \Phi) + (-F G_x \Phi_t + F_x G \Phi_t) \]
\[ + \left( F G_{tt} \tilde{H} \Phi - F_{ttt} G \tilde{H} \Phi \right) + \left( F G_t \tilde{H} \Phi_t - F_t G \tilde{H} \Phi_t \right) \]

We use (8.4), the fact that \( \partial_{tt} F = -\partial_{xx} F \),
\[ = (-F_t G_x \Phi + F_x G_t \Phi) + (-F G_{xt} \Phi + F_{xt} G \Phi) + (-F G_x \Phi_t + F_x G \Phi_t) \]
\[ + \left( -F G_{xx} \tilde{H} \Phi + F_{xx} G \tilde{H} \Phi \right) + (F G_t \tilde{H} \Phi_t + F_t G \Phi_t) \]

Next, again with the help of (8.4),
\[ -F G_{xx} \tilde{H} \Phi = -(F G_x \tilde{H} \Phi)_x + F_x G_x \tilde{H} \Phi + F G_x \Phi_t \]
\[ F_{xx} G \tilde{H} \Phi = (F_x G \tilde{H} \Phi)_x - F_x G_x \tilde{H} \Phi - F_x G \Phi_t \]

Plugging this into (8.5), more terms cancel,
\[ \mathcal{I} = (-F_t G_x \Phi + F_x G_t \Phi) + (-F G_{xt} \Phi + F_{xt} G \Phi) \]
\[ + \left( (F_x G \tilde{H} \Phi)_x - (F G_x \tilde{H} \Phi)_x \right) + (-F G_t \Phi_x + F_t G \Phi_x) . \]

We repeat this strategy with
\[ -F G_{xt} \Phi = -(F G_t \Phi)_x + F_x G_t \Phi + F G_t \Phi_x, \]
\[ F_{xt} G \Phi = (F_t G \Phi)_x - F_t G_x \Phi - F_t G \Phi_x. \]

This we plug into (8.6), and arrive at
\[ \mathcal{I} = 2 (F_x G_t \Phi - F_t G_x \Phi) + \left( F_t G \Phi - F G_t \Phi + F_x G \tilde{H} \Phi - F_G \tilde{H} \Phi \right)_x . \]

The second term vanishes when integrating in \( x \), and thus
\[ (8.7) \quad C = \left| \int_{\mathbb{R}^2} \mathcal{I} \right| = 2 \left| \int_{\mathbb{R}^2} \det(\nabla \mathbb{R}^2 F, \nabla \mathbb{R}^2 G) \Phi \right| . \]

With Proposition 10.1 we obtain (8.3). \( \square \)
Remark 8.2. In (8.7), having the determinant structure, one might hope to use the Hardy-BMO duality, in form of Theorem 1.1, to obtain (in view of Proposition 9.5) an estimate only in terms of \([\varphi]_{BMO}\) instead of \(\|\varphi\|_{L^\infty}\). If that was the case, then in Theorem 8.1 we had a Hardy-space bound instead of merely the \(L^1\) bound. However, we were not able to do this, the reason being that the integral is on the half-space and for a reflection argument we would need to estimate \(\frac{1}{|t|}\Phi(|t|, x)\) in \(BMO\). However, even though \(\Phi(|t|, x)\) is in \(BMO\), see Proposition 9.5, there is no reason \(\frac{1}{|t|}\Phi(|t|, x)\) belongs to \(BMO\) as well.

Proof of (8.2). As in the proof of (8.1) let \(F := P^1_t f, G := P^1_t g, \Phi := P^1_t \varphi\) be the respective harmonic extensions. Then,

\[
C := \left| \int \left( [f, \mathcal{H}][(-\Delta)^{\frac{1}{2}} g] + [g, \mathcal{H}][(-\Delta)^{\frac{1}{2}} f] \right) \mathcal{H}\varphi \right|
\]

This time we compute,

\[
\mathcal{I} := \partial_t \left( F \tilde{\mathcal{H}} G_t \tilde{\mathcal{H}} \Phi - F G_t \Phi + \tilde{\mathcal{H}} F_t G \tilde{\mathcal{H}} \Phi - F_t G \Phi \right)
\]

\[
= \partial_t \left( -(F G)_x \tilde{\mathcal{H}} \Phi \right) + \partial_t \left( -(F G)_t \Phi \right)
\]

\[
= - \left( F G_\Phi \right)_{xt} + \partial_t \left( (F G) \Phi_t - (F G)_t \Phi \right)
\]

\[
= - \left( F G_\Phi \right)_{xt} + (F G) \Phi_{tt} - (F G)_{tt} \Phi
\]

Thus, since \((FG)_{xx} + (FG)_{tt} = 2\nabla_{\mathbb{R}^2} F \cdot \nabla_{\mathbb{R}^2} G\), we have

\[
(8.8) \quad C = 2 \int_{\mathbb{R}^2_+} \nabla_{\mathbb{R}^2} F \cdot \nabla_{\mathbb{R}^2} G \cdot \Phi.
\]

We conclude as for (8.1). \(\square\)

Remark 8.3. Actually, our computations show that the left-hand sides in (8.1) and (8.2) are essentially the same estimates, more precisely,

\[
\left\| [f, \mathcal{H}][(-\Delta)^{\frac{1}{2}} g] - [g, \mathcal{H}][(-\Delta)^{\frac{1}{2}} f] \right\|_{L^1(\mathbb{R})} = \left\| \mathcal{H} \left( [f, \mathcal{H}][\mathcal{H}(-\Delta)^{\frac{1}{2}} g] + [\mathcal{H} g, \mathcal{H}][(-\Delta)^{\frac{1}{2}} f] \right) \right\|_{L^1(\mathbb{R})}.
\]

To see this, replace in (8.8) \(g\) with \(\tilde{G} := \tilde{\mathcal{H}} G\). Then in view of (8.4), the equation (8.8) becomes (8.7),

\[
2 \left| \int_{\mathbb{R}^2_+} \nabla_{\mathbb{R}^2} F \cdot \nabla_{\mathbb{R}^2} G \cdot \Phi \right| = 2 \left| \int_{\mathbb{R}^2_+} \det(\nabla_{\mathbb{R}^2} F, \nabla_{\mathbb{R}^2} \tilde{G}) \cdot \Phi \right|.
\]
9. Trace theorems

Characterizations of function spaces via the Poisson potential have a long tradition, see in particular Stein’s books [43, 44]. In this section we gather such characterizations for the Poisson operator. We postpone references to literature and proofs to Appendix A.

Recall the definition of the Poisson extension operator from Section 2. Let $F^s(x, t) = P_t^s f(x)$ for some $s \in (0, 2)$ and $f \in C_c^\infty(\mathbb{R}^n)$. With $\mathcal{M}f$ we denote the Hardy-Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{B \ni x} |B|^{-1} \int_B |f| \, dz$$

where the supremum is over balls $B$ containing $x$.

**Proposition 9.1** (Pointwise estimates). We have for any $k \in \mathbb{N}_0$,

\begin{align*}
(9.1) \quad & \sup_{(x, t) \in \mathbb{R}^n_{+1}} t^{n+k} |\nabla_{\mathbb{R}^n_{+1}}^k F^s(x, t)| \leq C_s \|f\|_{L^1(\mathbb{R}^n)}, \\
(9.2) \quad & \sup_{(x, t) \in \mathbb{R}^n_{+1}} t^k |\nabla_{\mathbb{R}^n_{+1}}^k F^s(x, t)| \leq C_s \|f\|_{L^\infty(\mathbb{R}^n)}.
\end{align*}

Also, we have the following estimates in terms of the maximal function,

\begin{align*}
(9.3) \quad & \sup_{(y, t) : |y-x|<t} |F^s(y, t)| \leq C_s \mathcal{M}f(x).
\end{align*}

For any $s \leq 1$,

\begin{align*}
(9.4) \quad & \sup_{(y, t) : |y-x|<t} |t^{1-s} \partial_y F^s(y, t)| \leq C_s \mathcal{M}((-\Delta)^{\frac{s}{2}} f)(x).
\end{align*}

Finally, denoting $\nabla^s = \nabla I^{1-s} = \mathcal{R} (-\Delta)^{\frac{s}{2}}$ what is sometimes called a “fractional gradient” (i.e. the vectorial Riesz transform $\mathcal{R}$ applied the $(-\Delta)^{\frac{s}{2}}$), we have

\begin{align*}
(9.5) \quad & \sup_{(y, t) : |y-x|<t} |t^{1-s} \nabla_x F^s(y, t)| \leq C_s \mathcal{M}(\nabla^s f)(x).
\end{align*}

Finally, for any $\sigma > 0$,

\begin{align*}
(9.6) \quad & \sup_{t>0} t^\sigma |P_t^s f|(x) \lesssim \mathcal{M}(I^\sigma f)(x),
\end{align*}

where $I^\sigma$ is the Riesz potential.

As usual, the norm of the Lebesgue-spaces $L^p$ is defined as

$$\|f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f|^p \right)^{\frac{1}{p}} \quad p \in [1, \infty),$$

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|.$$

A finer scale than Lebesgue-spaces are the Lorentz-spaces $L^{(p,q)}$, $q \in [1, \infty]$ – see, e.g., [25, 46, 20]. For $q = p$ they are the same as Lebesgue spaces, $L^{(p,p)} = L^p$. They are defined...
as follows. For measurable functions \( f : \mathbb{R}^n \to \mathbb{R} \) the *decreasing rearrangement* \( f^*(t) \), \( t > 0 \), is defined as
\[
f^*(t) := \inf \{ s > 0 : \mathcal{L}^n(\{|f| > s\}) \leq t \}.
\]
Here, \( \mathcal{L}^n \) denotes the Lebesgue measure. The *Lorentz-space norm* \( \| \cdot \|_{L(p,q)} \) is given by
\[
\|f\|_{L(p,q)} := \begin{cases} 
\left( \frac{\int_0^\infty (t^{1/p} f^*(t))^{q \frac{dt}{t}}}{t} \right)^{1/q} & \text{if } p, q \in [1, \infty), \\
\sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty.
\end{cases}
\]

The fractional Sobolev spaces \( W^{\nu,p} \), \( \nu \in (0,1) \), have the seminorm
\[
[f]_{\dot{W}^{\nu,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\nu p}} \, dx \, dy \right)^{\frac{1}{p}}.
\]
If \( p \neq 2 \), another fractional Sobolev space, sometimes denoted with \( \dot{H}^{\nu,p} \) is defined via the seminorm \( \|(-\Delta)^{\frac{\nu}{2}} f\|_{L^p(\mathbb{R}^n)} \).

We turn to characterizations for Sobolev spaces. In the following, it is crucial to observe the different orders up to which the characterization holds. The general rule is: the order of the derivative on the extension \( F^*(x,t) \) has to be strictly larger than the order of the Sobolev space we want to characterize. However, and this is very important to observe when \( s \neq 1 \), the \( t \)-direction derivatives \( t^{1-s} \partial_t \) count only as being “of order \( s \)”.

This is by construction of the Poisson potentials \( P_t^s \): they are supposed to satisfy \( \lim_{t \to 0} t^{1-s} \partial_t F^s(x,t) = c(-\Delta)^{\frac{s}{2}} f \).

**Proposition 9.2** (Fractional Sobolev spaces). *The following holds whenever \( p \in (1, \infty) \), \( q \in [1, \infty] \).

For \( s \in (0,2) \), \( \nu \in (0,1) \),
\[
(9.7) \quad \left( \int_{\mathbb{R}^n} \int_0^\infty |t^{1-\frac{s}{p}-\nu} \nabla_x F^s(x,t)|^p \, dt \, dx \right)^{\frac{1}{p}} \approx [f]_{\dot{W}^{\nu,p}(\mathbb{R}^n)}.
\]
For \( s \in (0,2) \), \( \nu \in (0,2) \),
\[
(9.8) \quad \left( \int_{\mathbb{R}^n} \int_0^\infty |t^{\frac{2}{p}-\nu} \nabla^2_x F^s(x,t)|^p \, dt \, dx \right)^{\frac{1}{p}} \approx [f]_{\dot{W}^{\nu,p}(\mathbb{R}^n)}.
\]
For \( s \in (0,2) \), \( \nu \in (0,1) \), \( \nu < s \),
\[
(9.9) \quad \left( \int_{\mathbb{R}^n} \int_0^\infty |t^{1-\frac{s}{p}-\nu} \partial_t F^s(x,t)|^p \, dt \, dx \right)^{\frac{1}{p}} \approx [f]_{\dot{W}^{\nu,p}(\mathbb{R}^n)}.
\]
For \( s \in (0,2) \), \( \nu \in [0,s) \), \( \nu < 1 \),
\[
(9.10) \quad \left\| x \mapsto \left( \int_0^\infty |t^{\frac{s}{p}-\nu} \nabla_x F^s(x,t)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \|(-\Delta)^{\frac{s}{2}} f\|_{L^{(p,q)}(\mathbb{R}^n)}.
\]
For $s \in (0, 2)$, $\nu \in [0, 1)$, $\nu < s$

$$\left\| x \mapsto \left( \int_0^\infty |t^{\frac{3}{2} - \nu} \partial_t F_s^\nu(x, t)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \|(-\Delta)^{\frac{\nu}{2}} f\|_{L^{(p,q)}(\mathbb{R}^n)}.$$ (9.11)

For $s \in (0, 2)$, $\nu \in [0, 2)$,

$$\left\| x \mapsto \int_{\mathbb{R}^n} \left( \int_0^\infty |t^{\frac{3}{2} - \nu} \nabla_x F_s^\nu(x, t)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \|(-\Delta)^{\frac{\nu}{2}} f\|_{L^{(p,q)}(\mathbb{R}^n)}.$$ (9.12)

The estimates (9.10), (9.11), (9.12) also hold for $\nu < 0$ with $(-\Delta)^{\frac{\nu}{2}} f$ replaced by the Riesz potential $I^{[\nu]} f$.

We also record the following characterizations in terms of so-called nontangential square functions.

**Proposition 9.3** (Square function estimates). For $1 < p < \infty$, $q \in [1, \infty]$ and $s \in (0, 1]$, and any $\nu \in [0, s)$,

$$\left\| x \mapsto \left( \int_{(y, t) : |y-x| < t} t^{1-2\nu-n} |\partial_t F_s^\nu(y, t)|^2 \, dy \, dt \right)^{\frac{1}{2}} \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \|(-\Delta)^{\frac{\nu}{2}} f\|_{L^{(p,q)}(\mathbb{R}^n)}.$$ (9.13)

If we replace $|\partial_t F_s^\nu(y, t)|$ by $|\nabla_x F_s^\nu(y, t)|$ this estimate holds for any $\nu \in [0, 1)$.

Moreover, for $s \in (0, 1]$, $\nu \in (0, 1+s)$,

$$\left\| x \mapsto \left( \int_{(y, t) : |y-x| < t} t^{3-2\nu-n} |\nabla_x \nabla_{\mathbb{R}^{n+1}} F_s^\nu(y, t)|^2 \, dy \, dt \right)^{\frac{1}{2}} \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \|(-\Delta)^{\frac{\nu}{2}} f\|_{L^{(p,q)}(\mathbb{R}^n)}.$$ (9.14)

Next, we consider BMO. Denoting $(f)_B \equiv f_B := |B|^{-1} \int_B f$ the mean value integral on $B$, the BMO-seminorm is given by

$$[f]_{BMO} = \sup_{B \subset \mathbb{R}^n} \int_B |f - (f)_B|,$$ (9.15)

where the supremum is over balls $B \subset \mathbb{R}^n$. There is a well-known relation between BMO and certain Carleson-measures on $\mathbb{R}^{n+1}$. This takes the following form.

**Proposition 9.4** (BMO-Characterization). For $s \in (0, 2)$,

$$[f]_{BMO(\mathbb{R}^n)} \approx \sup_{B \subset \mathbb{R}^n} \left( |B|^{-1} \int_{T(B)} t |\nabla_{\mathbb{R}^{n+1}} F_s^\nu(y, t)|^2 \, dy \, dt \right)^{\frac{1}{2}}.$$ (9.16)

Here the supremum is over balls $B \subset \mathbb{R}^n$ and $T(B)$ is the "tent" over $B$ in $\mathbb{R}^n$, i.e. $T(B_r(x_0)) = \{(x, t) \in \mathbb{R}^{n+1} : |x - x_0| < r - t\}.$
As an interesting observation, we also state the following result which treats even reflection of the harmonic extension.

**Proposition 9.5.** Let $f \in C_c^\infty(\mathbb{R}^n)$. We set $F^e$ to be the harmonic extension to $\mathbb{R}^{n+1}_+$ evenly reflected to a function on $\mathbb{R}^{n+1}_+$. That is,

$$F^e(x, t) := P_{|t|}^1 f(x).$$

Then we have

$$[F^e]_{BMO(\mathbb{R}^{n+1})} \lesssim \|f\|_{BMO(\mathbb{R}^n)}.$$

We turn to Hölder- and Lipschitz spaces. We denote the Hölder semi-norm by, for $\nu > 0$

$$[f]_{C^\nu(\mathbb{R}^n)} := \begin{cases} 
\sup_{x \neq y \in \mathbb{R}^n} \frac{(|\nabla^{|\nu|} f(x) - \nabla^{|\nu|} f(y)|)}{|x - y|^{\nu - |\nu|}} & \text{if } \nu \notin \mathbb{N} \\
\|\nabla^\nu f\|_{L^\infty} & \text{if } \nu \in \mathbb{N}
\end{cases}$$

As usual we will denote $[f]_{\text{Lip}} = [f]_{C^1}$.

**Proposition 9.6 (Hölder spaces).** For $\nu \in (0, s)$,

$$\sup_{(x, t) \in \mathbb{R}^{n+1}_+} t^{1-\nu}|\partial_t F^s(x, t)| \approx [f]_{C^\nu(\mathbb{R}^n)}.$$ (9.17)

and for any $\nu \in (0, 1]$,

$$\sup_{(x, t) \in \mathbb{R}^{n+1}_+} t^{1-\nu}|\nabla_x F^s(x, t)| \lesssim [f]_{C^\nu(\mathbb{R}^n)}.$$ (9.18)

For sake of completeness, let us also mention that one can characterize the full range of Besov- and Triebel-Lizorkin spaces in terms of Poisson-type potentials $P^s_t(-\Delta)^{\frac{\alpha}{2}} f$ and $P^s_t\nabla f$. This follows from a general characterization of those spaces via convolution operators, recently given by Bui and Candy in [8]. Again for $s \neq 1$, the maximal differential order of the spaces which can be characterized is different depending on whether we use $\partial_t F^s(x, t)$ or $\nabla_x F^s(x, t)$.

**Theorem 9.7.** For any $\beta > \alpha$, $\beta > 0$, $0 < p, q < \infty$, $s > 0$,

$$\|f\|_{B^{\alpha}_{p,q}} \approx \left( \int_0^\infty \left( \int_{\mathbb{R}^n} |t^{-\frac{1}{q} - \alpha + \beta} P^s_t(-\Delta)^{\frac{\alpha}{2}} f(x)|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{1}{q}}$$

$$\|f\|_{F^{\alpha}_{p,q}} \approx \left( \int_0^\infty \left( \int_{\mathbb{R}^n} |t^{-\frac{1}{q} - \alpha + \beta} P^s_t(-\Delta)^{\frac{\alpha}{2}} f(x)|^q \, dx \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}}.$$

If $\alpha < 1$,

$$\|f\|_{B^{\alpha}_{p,q}} \approx \left( \int_0^\infty \left( \int_{\mathbb{R}^n} |t^{-\frac{1}{q} - \alpha + 1} \nabla_x P^s_t f(x)|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{1}{q}}.$$
we obtain the following estimates. The proofs can be found in Appendix B. Recall that \( I^s \) denotes the Riesz potential, the inverse of \((-\Delta)^{\frac{s}{2}}\).

First we consider estimates with \( L^p \)-spaces.

**Proposition 10.1** \((L^p\)-estimates). Let \( s \in (0,1) \). Take any \( p_i \in (1,\infty) \) and \( q_i \in [1,\infty] \), for \( i \in \{1,2,3\} \), and such that

\[
\sum_{i=1}^{3} \frac{1}{p_i} = \sum_{i=1}^{3} \frac{1}{q_i} = 1.
\]

Denote with \( F^s(x,t) = P^s_t f(x) \), \( G^s(x,t) = P^s_t g(x) \), \( H^s(x,t) := P^s_t h(x) \). For \( s_1, s_2 \in [0,s) \) and \( s_3 \geq 0 \),

\[
\int_{\mathbb{R}^{n+1}_+} t^{1-s_1-s_2+s_3} |\nabla_{\mathbb{R}^{n+1}} F| |\nabla_{\mathbb{R}^{n+1}} G| |H| \lesssim \|(-\Delta)^{\frac{s_1}{2}} f\|_{L(p_1,q_1)} \|(-\Delta)^{\frac{s_2}{2}} g\|_{L(p_2,q_2)} \|I^{s_3} h\|_{L(p_3,q_3)},
\]

and for \( s_1 \in [0,s) \) and \( s_2, s_3 \geq 0 \),

\[
\int_{\mathbb{R}^{n+1}_+} t^{1-s_1+s_2+s_3} |\nabla_{\mathbb{R}^{n+1}} F| |\nabla_{\mathbb{R}^{n+1}} G| |H| \lesssim \|(-\Delta)^{\frac{s_1}{2}} f\|_{L(p_1,q_1)} \|I^{s_2} g\|_{L(p_2,q_2)} \|I^{s_3} h\|_{L(p_3,q_3)}.
\]

If \( \nabla_{\mathbb{R}^{n+1}} F \) is replaced with \( \nabla_x F \) then \( s_1 \in [0,1) \) is allowed in the above two estimates.
For $s_1 \in (0, 1 + s)$, and $s_2, s_3 \geq 0$.

$$\int_{\mathbb{R}^{n+1}} t^{2-s_1-s_2-s_3} |\nabla_x \nabla_{\mathbb{R}^{n+1}} F| |\nabla_{\mathbb{R}^{n+1}} G| |H| \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L(p_1,q_1)} \|(-\Delta)^{\frac{s}{2}} g\|_{L(p_2,q_2)} \|I^{s_3} h\|_{L(p_3,q_3)},$$

If $s_2 < 0$ the last estimate still holds with $(-\Delta)^{\frac{s}{2}} g$ replaced by $I^{s_2} g$.

In all terms above we may replace $|H|$ by $t|\nabla_{\mathbb{R}^{n+1}} H|$.

All estimates also hold if $(p_3, q_3) = (\infty, \infty)$.

Next, we list estimates involving the BMO-norm.

**Proposition 10.2** (BMO-estimates). Let $\ell \geq 1$, $s \in (0, 1]$. We have the following estimates for $F^s_t(x, t) = P_t^s f_t(x)$, $G^s(x, t) = P_t^s g(x)$, and $\Phi^s(x, t) = P_t^s \varphi(x)$.

Assume that $p_i \in (1, \infty)$, $q_i \in [1, \infty]$, for $i \in \{0, \ldots, \ell\}$ such that

$$\sum_{i=0}^{\ell} \frac{1}{p_i} = \sum_{i=0}^{\ell} \frac{1}{q_i} = 1.$$

Then

$$\int_{\mathbb{R}^{n+1}} t^{1+(1-s)(\ell+1)} |\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)| |\nabla_x \nabla_{\mathbb{R}^{n+1}} G^s(x, t)| |\nabla_{\mathbb{R}^{n+1}} F^s_t(x, t)| \ldots |\nabla_{\mathbb{R}^{n+1}} F^s_\ell(x, t)| d(x, t)
\lesssim [\varphi]_{BMO} \|(-\Delta)^{\frac{s}{2}} g\|_{L(p_0,q_0)} \|(-\Delta)^{\frac{s}{2}} f_1\|_{L(p_1,q_1)} \ldots \|(-\Delta)^{\frac{s}{2}} f_\ell\|_{L(p_\ell,q_\ell)}.$$

Also, for $\nu \in [0, s)$

$$\int_{\mathbb{R}^{n+1}} t^{1-\nu} |\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)| |\partial_t G^s(x, t)| |F^s_t(x, t)| \ldots |F^s_\ell(x, t)| d(x, t)
\lesssim [\varphi]_{BMO} \|(-\Delta)^{\frac{s}{2}} g\|_{L(p_0,q_0)} \|f_1\|_{L(p_1,q_1)} \ldots \|f_\ell\|_{L(p_\ell,q_\ell)}.$$

The last estimate also holds

- if we replace $|\partial_t G^s(x, t)|$ with $|\nabla_x G^s(x, t)|$ for any $\nu \in [0, 1]$.
- if we replace $|\partial_t G^s(x, t)|$ with $t|\nabla_x \nabla_{\mathbb{R}^{n+1}} G^s(x, t)|$ for any $\nu \in [0, 1+s]$.
- if we replace $|F^s_t(x, t)|$ with $t|\nabla_{\mathbb{R}^{n+1}} F^s_t(x, t)|$ for any $\nu \in [0, s]$.

All the above estimate also hold if we replace $|\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)|$ with $|t^{s-1}P^s_t((-\Delta)^{\frac{s}{2}} \varphi)|$ or $|t^s \nabla_{\mathbb{R}^{n+1}} F^s_t((-\Delta)^{\frac{s}{2}} \varphi)|$.

Lastly, we state estimates involving the Hölder-norm.

**Proposition 10.3** (Hölder-space-estimates). Let $\ell \geq 1$, $s \in (0, 1]$. We have the following estimates for $F^s(x, t) = P^s_t f(x)$, $G^s(x, t) = P^s_t g(x)$, and $\Phi^s(x, t) = P^s_t \varphi(x)$. Assume that $p_1, p_2 \in (1, \infty)$, $q_1, q_2 \in [1, \infty]$ such that

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1.$$
For $\nu \in (0, s)$, $s_1 \in (0, s)$, $s_2 \geq 0$,

$$\int_{\mathbb{R}^{n+1}} t^{1-\nu-s_1+s_2} |\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)| |\nabla_{\mathbb{R}^{n+1}} G^s(x, t)| |F^s(x, t)| d(x, t)$$

$$\lesssim \|\varphi\|_{L^0} \|(-\Delta)^{\frac{\nu}{2}} g\|_{L^{(p_1, q_1)}} \|I^{s_2} f\|_{L^{(p_2, q_2)}}$$

- For $\nu = 0$ we replace $[\varphi]_{C^\nu}$ with $\|\varphi\|_{L^\infty}$.
- For $s_1 = 0$ we replace $\|(-\Delta)^{\frac{s_1}{2}} g\|_{L^{(p_1, q_1)}}$ with $\|g\|_{L^{(p_1, q_1)}}$.
- If we replace $|\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)|$ with $|\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)|$ we can take $\nu \in (0, 1]$.
- If we replace $|\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)|$ with $|\nabla_{\mathbb{R}^{n+1}} \nabla_x \Phi^s(x, t)|$, we may take $\nu \in (0, 1+s)$.
- If we take $s_1 < 0$, then $\|(-\Delta)^{\frac{s_1}{2}} g\|_{L^{(p_1, q_1)}}$ needs to be replaced with $\|I^{s_1} g\|_{L^{(p_1, q_1)}}$.
- We may replace $|F^s(x, t)|$ with $t|\nabla_{\mathbb{R}^{n+1}} F^s(x, t)|$.

Appendix A. Proofs and Literature for Section 9

A.1. The Fourier transform of the Poisson potential. We recall that the (generalized) Poisson potential $P^s_t f$ is given as a convolution operator $P^s_t f = p^s_t * f$, where the kernel $p^s_t$ is a Bessel-potential kernel,

$$p^s_t(z) := \frac{t^s}{(|z|^2 + t^2)^{\frac{n+2}{2}}} = t^{-n} p^s_1(z/t).$$

A direct computation gives

$$\|p^s_t\|_{L^1(\mathbb{R}^n)} = C \quad \forall t > 0, \quad \|p^s_t\|_{L^\infty(\mathbb{R}^n)} = C t^{-n}.$$  

To apply the characterization for Triebel spaces of Bui and Candy [8] one needs to find the growth of the Fourier transform $\mathcal{F}$ of $p^s_t$.

The case $s = 1$ is well known, $\mathcal{F}(p^1_t)(\xi) = e^{-ct|\xi|}$. Indeed, the conditions $(\partial_t + \Delta_x)(p^1_t * f) = 0$ and $p^1_t * f \big|_{t=0} = f$ are transformed into an ordinary differential equation under the Fourier transform in $x$-variables. Namely, $\sigma(t) := \mathcal{F}(p^1_t)(\xi)$ has to satisfy the equation

$$\begin{cases} 
\partial_t \sigma(t) - c|\xi|^2 \sigma(t) = 0 & t \in \mathbb{R}_+ \\
\sigma(0) = 1.
\end{cases}$$

In this sense, some authors write $P^1_t = e^{-ct\sqrt{-\Delta}}$.

For $s \neq 1$ this is more involved. Observe that $P^s_t \neq e^{-ct(-\Delta)^{\frac{s}{2}}}$. That extension $\bar{F}(x, t) := e^{-t(-\Delta)^{\frac{s}{2}}} f$ is in principle possible as well, is simpler and has the right boundary behavior. But its major, and for our purpose crucial, disadvantage is that the extended objects $\bar{F}$ do not satisfy a local equation, but rather the nonlocal equation $(\partial_t + (-\Delta)^{\frac{s}{2}}) F = 0$. 


In our case, as introduced by Caffarelli and Silvestre \cite{CaffarelliSilvestre}, \( P_t^s \) is a Bessel potential. The following calculations for \( s \neq 1 \) can be found, e.g., in \cite[Proposition 7.6]{Boreu}. We have

\[
F(p_t^s)(\xi) = c_s \int_0^\infty \frac{\lambda^{\frac{s}{2}} e^{-\lambda^{\frac{s}{2}}|\xi|}}{\lambda} d\lambda.
\]

Here \( c > 0 \) is a uniform constant, and \( c_s \) depends only on dimension and \( s \).

In \cite[Proposition 7.6]{Boreu}, one can also find the following estimates: for any multiindex \( \kappa \),

\[
|\partial_\xi^\kappa F(p_t^s)(\xi)| \lesssim \max\{1, |\xi|^{s-|\kappa|}\} \quad \text{for } |\xi| \leq 2,
\]

\[
|\partial_\xi^\kappa F(p_t^s)(\xi)| \lesssim e^{-c|\xi|} \quad \text{for } |\xi| > 2,
\]

Moreover, setting \( q_t^s := \left( \partial_t p_t \right)_{|t|=1} \),

\[
|\partial_\xi^\kappa F(q_t^s)(\xi)| \lesssim \max\{|\xi|^{s-|\kappa|}, 1\} \quad \text{for } |\xi| \leq 2.
\]

A.2. The pointwise estimates: Proposition 9.1.

Proof of (9.1), (9.2). Estimates (9.1) and (9.2) follow from a direct computation using convolution estimates. \( \square \)

Proof of (9.3). Estimate (9.3) follows from \cite[II, §2.1, Proposition, p. 57]{Takeda}, since \( P_t^s f = p_t^s \ast f \), with

\[
p_t^s(z) = c \frac{1}{(1 + |z|^2)^{\frac{n+s}{2}}},
\]

a kernel which is bounded, radial, and in \( L^1(\mathbb{R}^n) \). \( \square \)

Proof of (9.4). For \( s = 1 \) observe that \( \partial_t P_t = c(-\Delta)^{\frac{1}{2}} P_t \) (which follows from \( F(p_t^1)(\xi) = e^{-t|\xi|} \)). Thus (9.4) follows from (9.3).

The case \( s \neq 1 \) requires more work. We use the representation

\[
t^{1-s} \partial_t P_t f(x) = c \int_{\mathbb{R}^n} (|x-z|^2 + t^2)^{\frac{2-s-n}{2}} \Delta f(z) \, dz.
\]

To see (A.3), one can use the Fourier representation in (A.2). Alternatively, we solve an initial value problem for an ordinary differential equation: By (2.3),

\[
\partial_t (t^{1-s} \partial_t P_t f) = -t^{1-s} P_t \Delta_x f,
\]

so both sides of (A.3) solve the same equation. Moreover at \( t = 0 \), both sides of (A.3) coincide: since \( |x-z|^{2-s-n} \) is the kernel of the Riesz potential \( I^{2-s} \),

\[
\lim_{t \to 0} t^{1-s} \partial_t P_t f(x) = c(-\Delta)^{\frac{1}{2}} f(x) = \lim_{t \to 0} \int_{\mathbb{R}^n} (|x-z|^2 + t^2)^{\frac{2-s-n}{2}} \Delta f(z) \, dz.
\]

The relation (A.3) is now established, since both sides of (A.3) solve the same equation in \( t \) and have the same initial datum at \( t = 0 \).
Now we set \( g(z) := (-\Delta)^{\frac{s}{2}} f(tz) \). Note that \( \mathcal{M}g(x) = c\mathcal{M}(-\Delta)^{\frac{s}{2}} f(x) \). (A.3) then follows once we can show
\[
(A.4) \sup_{|x-y|<1} c \int_{\mathbb{R}^n} (|y - z|^2 + 1)^{\frac{2-s-n}{2}} (-\Delta)^{\frac{s}{2}} g(z) \, dz \lesssim \mathcal{M}g(x).
\]
To obtain (A.4) we use
\[
(A.5) (-\Delta)^{\frac{s}{2}} (|x|^2 + 1)^{\frac{2-s-n}{2}} = c(|x|^2 + 1)^{\frac{-2-s}{2}}
\]
This equation might look surprising at first – in particular one might think its the wrong 'homogeneity' if one thinks of \((-\Delta)^{\frac{s}{2}}\) as the s-derivative. But \((-\Delta)^{\frac{s}{2}}\) behaves more like the Laplacian \(\Delta\), indeed we suggest to check this formula for \(\gamma = 2\). The computation (A.5) can be found in Dyda, Kuznetsov, Kwasnicki’s [19, Corollary 1] who compute several explicit fractional Laplacians in terms of the Meijer G-function. As the authors informed us, special cases of (A.5) appear in the work of Samko, see for example [35]. It is also possible to obtain (A.5) from the Fourier representation (A.2).

To obtain (A.4) from (A.5) simply integrate by parts
\[
\sup_{|x-y|<1} c \int_{\mathbb{R}^n} (|y - z|^2 + 1)^{\frac{2-s-n}{2}} \nabla f(z) \, dz
\]
Now the kernel \((|y - z|^2 + 1)^{-\frac{(2-s)-n}{2}}\) is bounded and belongs to \(L^1\), so it falls into the realm of Stein’s [44, II, §2.1, Proposition, p. 57]. This proves (9.4). \(\square\)

Proof of (9.5). For (9.5), a rougher estimate than (A.5) suffices,
\[
(A.6) (-\Delta)^{\frac{s}{2}} (|x|^2 + 1)^{-\frac{n+s}{2}} \leq (|x|^2 + 1)^{-\frac{n+1}{2}}.
\]
Then
\[
\sup_{|x-y|<1} c \int_{\mathbb{R}^n} (|y - z|^2 + 1)^{-\frac{n+s}{2}} \nabla f(z) \, dz
\]
\[\lesssim \sup_{|x-y|<1} c \int_{\mathbb{R}^n} (|y - z|^2 + 1)^{-\frac{n+1}{2}} |\nabla f(z)| \, dz.
\]
Again we can conclude with Stein’s [44, II, §2.1, Proposition, p. 57]. \(\square\)

Proof of (9.6). We have
\[
t^\sigma P_t^s f = \int_{\mathbb{R}^n} t^{-n} \kappa \left( \frac{x - y}{t} \right) I^\sigma f(y) \, dy,
\]
where
\[
\kappa(z) = (-\Delta)^{\frac{s}{2}} \frac{1}{(|z|^2 + 1)^{\frac{n+s}{2}}}.
\]
For $\sigma \geq 0$, $\kappa \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is radial, and we conclude again with [44, II, §2.1, Proposition, p. 57].

A.3. Proof of Propositions 9.2, 9.3, 9.4.

Proof of Proposition 9.2. Proposition 9.2 follows from the Besov- and Triebel space characterization by Bui-Candy [8, Theorem 1.1, Theorem 1.3]. To ensure the “Cancellation condition (C1)” in their article, one needs to use the growth estimates from Section A.1.

Proof of Proposition 9.3. The claim follows by estimates on so-called (non-tangential) square functions. More precisely, we use [44, Chapter I, §8.23, p.46]. There it is shown that

$$
\left\| x \mapsto \int_{(y,t): |y-x|<t} t^{-1-n} |q_t * f|^2 dy dt \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \| f \|_{L^{(p,q)}(\mathbb{R}^n)},
$$

where $q_t = t^{-n}q(z/t)$ and $q$ is suitably growing radial kernel, belongs to $L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} q = 0$. In particular $t\partial_t p_t^s$ satisfies these conditions, and thus

$$
\left\| x \mapsto \int_{(y,t): |y-x|<t} t^{-1-n} |\partial_t F^s(x,t)|^2 dy dt \right\|_{L^{(p,q)}(\mathbb{R}^n)} \approx \| f \|_{L^{(p,q)}(\mathbb{R}^n)},
$$

More generally, with help of the representation (A.3), we may find a suitable $q$ when $0 \leq \nu < s$ such that

$$
t^{1-\sigma} \partial_t p_t^s * f = q_t * (-\Delta)^{\frac{\nu}{2}} f.
$$

This leads to (9.13). (9.14) follows by the same argument.

Proof of Propososition 9.4. We refer to [44, Theorem 3. Chapter IV, §4.3., p.159] together with the remark on the kernel in [44, Chapter IV, §4.4.3., p.165].

This result can also be recovered via the Poisson characterizations of Triebel-Lizorkin spaces by [8], using the Triebel-Lizorkin space characterization of BMO.

A.4. Reflected harmonic extensions: Proof of Proposition 9.5. In [7, Appendix 3] Brezis and Nirenberg, together with Mironescu, show that the harmonic extension of a $VMO$-function defined on the boundary of a bounded domain $\partial \Omega$ extends to a $VMO$-function in $\Omega$. Their definition of $BMO(\Omega)$, [7, §II.1, Definition 1, p.313], however excludes balls that intersect the boundary. In the following we adapt their proof to our situation.

From now on we denote with $F(x,t) \equiv F^e(x,t) := P_{[t]}^1 f$ the harmonic extension to $\mathbb{R}^{n+1}_+$ of $f : \mathbb{R}^n \to \mathbb{R}$ reflected evenly across $\mathbb{R}^n$. 

\[ \begin{array}{c}
\]
The first step is to replace $F$ by another function which easier to compute. For $x \in \mathbb{R}^n$, $t \in (0, \infty)$ we pick the ball $B_t(x) \subset \mathbb{R}^n$ and define

$$G(x, t) := \int_{B_t(x)} f \equiv (f)_{B_t(x)}.$$ 

This is possible due to the embedding $L^\infty \subset BMO$ and the following Lemma, cf. [7, Lemma A3.1.].

**Lemma A.1.** There is a uniform constant $c \in \mathbb{R}$ such that

$$\sup_{(x, t) \in \mathbb{R}^{n+1}} \left| c P_t[f(x) - \int_{B_t(x)} f] \right| \lesssim \|f\|_{BMO(\mathbb{R}^n)}.$$ 

In other words,

$$\|F - G\|_{L^\infty(\mathbb{R}^{n+1})} \leq \|f\|_{BMO(\mathbb{R}^n)}.$$

**Proof.** Pick $c := (P_t^1[1])^{-1} \in \mathbb{R}$ such that

$$|c P_t f(x) - (f)_{B_t(x)}| \lesssim \int_{\mathbb{R}^n} \frac{t}{(|x - z|^2 + t^2)^{n+1/2}} |f(z) - (f)_{B_t(x)}| \, dz.$$ 

Now we split the integration domain into $B_t(x)$ and annuli

$$\lesssim \int_{B_t(x)} |f(z) - (f)_{B_t(x)}| \, dz + \sum_{k=1}^{\infty} \int_{B_{2^k t}(x) \setminus B_{2^{k-1} t}(x)} \frac{t}{(|x - z|^2 + t^2)^{n+1/2}} |f(z) - (f)_{B_t(x)}| \, dz.$$ 

Estimating the kernel in these domains we have

$$\lesssim \int_{B_t(x)} |f(z) - (f)_{B_t(x)}| \, dz + \sum_{k=1}^{\infty} 2^{-k} \int_{B_{2^k t}(x)} |f(z) - (f)_{B_t(x)}| \, dz.$$ 

On the first term we use the definition of BMO, in the second term we want to do the same and thus introduce $(f)_{B_{2^k t}(x)}$

$$\lesssim [f]_{BMO} + \sum_{k=1}^{\infty} 2^{-k} \int_{B_{2^k t}(x)} |f(z) - (f)_{B_{2^k t}(x)}| \, dz + \sum_{k=1}^{\infty} 2^{-k} |(f)_{B_{2^k t}(x)} - (f)_{B_t(x)}| \, dz.$$ 

Now we can estimate the second term again with the BMO-term and the sum converges. For the third term we write the difference of mean values as a telescoping sum,

$$\lesssim [f]_{BMO} + \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{-k} |(f)_{B_{2^j t}(x)} - (f)_{B_{2^{j-1}} t}(x)|$$

Again we estimate by the BMO-norm, and are left with

$$\lesssim [f]_{BMO} + \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{-k} [f]_{BMO}.$$
To see that this sum converges, we use the Fubini theorem for series. Namely,
\[ \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{-k} = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} 2^{-k} = \sum_{j=1}^{\infty} 2^{-j} < \infty. \]

\[ \blacksquare \]

To measure the BMO-norm, in Definition 9.15 one can replace the balls with other objects such as squares, cylinders. To this end, in \( \mathbb{R}^{n+1} \) we consider the following cylinders
\[ \tilde{D}_{\rho}^{n+1}(x_0, t_0), := B_{\rho}^{n}(x_0) \times (t_0 - \rho, t_0 + \rho) \quad x_0 \in \mathbb{R}^{n}, t_0 \in \mathbb{R}. \]

The following estimate was proven in [7, §II.3, Lemma 7, p.327]. It treats the case when the cylinder is away from the boundary.

**Lemma A.2** (Estimates away from the boundary). The following holds:
\[ \sup_{t_0 \in \mathbb{R}} \sup_{x_0 \in \mathbb{R}^n} \rho > 0 : 2\rho < |t_0| \int_{\tilde{D}_{\rho}^{n+1}(x_0, t_0)} |G - (G)_{\tilde{D}_{\rho}^{n+1}(x_0, t_0)}| \leq C_n \|f\|_{BMO(\mathbb{R}^n)}. \]

**Proof.** Fix \( x_0 \in \mathbb{R}^{n}, t_0 \in \mathbb{R} \) and \( \rho > 0 \). Set
\[ I := \int_{\tilde{D}_{\rho}^{n+1}(x_0, t_0)} |G - (G)_{\tilde{D}_{\rho}^{n+1}(x_0, t_0)}| = \int_{D_{\rho}^{n+1}(x_0, t_0)} |G(x_1) - G(x_2)| d(s_1) d(s_2). \]

For \( s_1, s_2 \in (t_0 - \rho, t_0 + \rho), x_1, x_2 \in B_{\rho}^{n}(x_0) \) we have
\[ B_{|s_1|}^{n}(x_1), B_{|s_2|}^{n}(x_2) \subset B_{|t_0|+2\rho}(x_0). \]

Consequently, in view of [6, Lemma A.4, p. 36] which states that for \( A \subset B \),
\[ |\int_{A} g - \int_{B} g| \lesssim \frac{|B|}{|A|} \int_{B} |g - \int_{B} g|, \]
we have
\[ |G(|s_1|, x_1) - (f)_{B_{|t_0|+2\rho}(x_0)}| \lesssim \frac{|B_{|t_0|+2\rho}(x_0)|}{|B_{|s_1|}^{n}(x_1)|} \|f\|_{BMO(\mathbb{R}^n)}. \]

With the assumption \( |t_0| > 2\rho \),
\[ \frac{|B_{|t_0|+2\rho}(x_0)|}{|B_{|s_1|}^{n}(x_1)|} \leq C_n \left( 2 \frac{|t_0| + 2\rho}{\max\{\rho, |t_0|\}} \right)^n \leq 6^n C_n. \]

Consequently,
\[ |G(|s_1|, x_1) - (f)_{B_{|t_0|+2\rho}(x_0)}|, |G(|s_2|, x_2) - (f)_{B_{|t_0|+2\rho}(x_0)}| \lesssim \|f\|_{BMO(\mathbb{R}^n)}. \]

Plugging this into \( I \), we obtain \( I \lesssim \|f\|_{BMO(\mathbb{R}^n)}. \)

Since we want to find an BMO-estimate up to (and over the) boundary \( \mathbb{R}^n \times \{0\} \), we need to accompany Lemma A.2 with an estimate close to the boundary \( \mathbb{R}^n \times \{0\} \). Namely we have
Lemma A.3 (Close to the boundary). For any $\Lambda > 0$ the following holds.

$$
\sup_{t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n} \sup_{\rho > 0, |t_0| \leq \Lambda \rho} \int_{\mathcal{D}_\rho^{n+1}(x_0, t_0)} |G - (G)_{\mathcal{D}_\rho^{n+1}(x_0, t_0)}| \leq C_n (\Lambda + 2)^n [f]_{BMO(\mathbb{R}^n)}.
$$

Proof. Fix $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $\rho > 0$. Set

$$
\mathcal{I} := \int_{\mathcal{D}_\rho^{n+1}(x_0, t_0)} |G - (G)_{\mathcal{D}_\rho^{n+1}(x_0, t_0)}|
$$

$$
\lesssim \rho^{-2(n+1)} \int_{t_0 - \rho}^{t_0 + \rho} \int_{t_0 - \rho}^{t_0 + \rho} \int_{B_\rho(x_0)} \int_{B_\rho(x_0)} |G(y_1, s_1) - G(y_2, s_2)| \, dy_1 \, dy_2 \, ds_1 \, ds_2.
$$

Now

$$
|G(y_1, s_1) - G(y_2, s_2)| \leq \int_{B_1(0)} \int_{B_1(0)} |f(y_1 + |s_1|z_1) - f(y_2 + |s_2|z_2)| \, dz_1 \, dz_2.
$$

Consequently, with Fubini

$$
\int_{B_\rho(x_0)} \int_{B_\rho(x_0)} |G(y_1, s_1) - G(y_2, s_2)| \, dy_1 \, dy_2
$$

$$
\lesssim \int_{B_1(0)} \int_{B_1(0)} \int_{B_\rho(x_0)} \int_{B_\rho(x_0)} |f(y_1 + |s_1|z_1) - f(y_2 + |s_2|z_2)| \, dy_1 \, dy_2 \, dz_1 \, dz_2.
$$

Next, by substitution,

$$
\int_{B_\rho(x_0)} \int_{B_\rho(x_0)} |f(y_1 + |s_1|z_1) - f(y_2 + |s_2|z_2)| \, dy_1 \, dy_2
$$

$$
= \int_{B_\rho(x_0 + |s_1|z_1)} \int_{B_\rho(x_0 + |s_2|z_2)} |f(y_1) - f(y_2)| \, dy_1 \, dy_2.
$$

Now if $z_1, z_2 \in B_1(0)$, $s_1, s_2 \in (t_0 - \rho, t_0 + \rho)$

$$
B_\rho(x_0 + |s_1|z_1), B_\rho(x_0 + |s_2|z_2) \subset B_{t_0 + 2\rho}(x_0).
$$

We thus have

$$
\mathcal{I} \lesssim \rho^{-2n} \int_{B_{t_0 + 2\rho}(x_0)} \int_{B_{t_0 + 2\rho}(x_0)} |f(y_1) - f(y_2)| \, dy_1 \, dy_2,
$$

and with $|t_0| \leq \Lambda \rho$,

$$
\lesssim \rho^{-2n} \int_{B_{(\Lambda + 2)^2}(x_0)} \int_{B_{(\Lambda + 2)^2}(x_0)} |f(y_1) - f(y_2)| \, dy_1 \, dy_2.
$$

Finally we use the definition of BMO, (9.15), and have

$$
\lesssim (\Lambda + 2)^n [f]_{BMO}.
$$

Now we have all the ingredients for the proof of Proposition 9.5.
Proof. From Lemma A.2 and Lemma A.3 we obtain
\[ [G]_{\text{BMO}(\mathbb{R}^{n+1})} \lesssim [f]_{\text{BMO}(\mathbb{R}^n)}. \]

With the help of the embedding \( L^\infty \subset \text{BMO} \) and Lemma A.1 we obtain
\[ [F^c]_{\text{BMO}(\mathbb{R}^{n+1})} \leq 2\|F^c - G\|_{L^\infty} + [G]_{\text{BMO}(\mathbb{R}^{n+1})} \lesssim [f]_{\text{BMO}(\mathbb{R}^n)}. \]

Proposition 9.5 is proven.

A.5. Proof of Proposition 9.6.

Proof of (9.17). In Stein’s [43, V, §4.2, Proposition 7, p.142] the following is proven for any \( \nu < s \).
\[ \|f\|_\infty + \sup_{t>0} \sup_{x \in \mathbb{R}^n} ||t^{1-\nu} \partial_t P^s_t f|| \approx \|f\|_\infty + [f]_{C^{0,\nu}(\mathbb{R}^n)} \]

Indeed, it is proven for \( P^1_t f \), but this easily extends to \( P^s_t f \). Apply this equation to
\[ f_k(x) := k^{-\nu} f(kx), \]
and one has
\[ k^{-\nu} \|f\|_\infty + \sup_{t>0} \sup_{x \in \mathbb{R}^n} ||t^{1-\nu} \partial_t P^s_t f|| \approx k^{-\nu} \|f\|_\infty + [f]_{C^{0,\nu}(\mathbb{R}^n)}. \]

Letting \( k \to \infty \), we obtain (9.17).

Proof of (9.18). Since \((|\cdot|^2+1)^{-\frac{n+1}{2}}\) is integrable, with Hölder inequality
\[ \|\nabla_x P^s_t f(x)\| \lesssim \int_{\mathbb{R}^n} \frac{t^s}{(|x-z|^2 + t^2)^{\frac{n+1}{2}}} |\nabla f(z)| \, dz \lesssim \|\nabla f\|_{L^\infty}. \]

This shows (9.18) for \( \nu = 1 \).

Let us now more generally consider any \( 0 < \nu \leq 1 \),
\[ \|\nabla_x P^s_t f(x)\| = \left| \int_{\mathbb{R}^n} t^s \nabla_z \left( |x-z|^2 + t^2 \right)^{-\frac{n+1}{2}} f(z) \, dz \right|. \]

Now observe that \( \int_{\mathbb{R}^n} \nabla_z \left( |x-z|^2 + t^2 \right)^{-\frac{n+1}{2}} = 0 \), and thus
\[ = \left| \int_{\mathbb{R}^n} t^s |x-z|^\nu \nabla_z \left( |x-z|^2 + t^2 \right)^{-\frac{n+1}{2}} \frac{f(z) - f(x)}{|x-z|^\nu} \, dz \right|. \]

Since \( \nu \leq 1 \),
\[ \lesssim [f]_{C^\nu} \left| \int_{\mathbb{R}^n} t^s |z|^{\nu} \nabla_z \left( |z|^2 + t^2 \right)^{-\frac{n+1}{2}} \, dz \right|. \]

If we set \( \kappa(z) := |z|^\nu \nabla_z (|z|^2 + 1)^{-\frac{n+1}{2}}, \)
\[ = [f]_{C^\nu} t^{\nu-1} \|\kappa\|_{L^1(\mathbb{R}^n)}. \]

Since \( \kappa \) is integrable whenever \( \nu < 1 + s \), we have shown
\[ \|\nabla_x P^s_t f(x)\| \lesssim t^{\nu-1} [f]_{C^\nu}. \]

Proposition 9.6 is proven.
A.6. **On Theorem 9.7.** The theorem follows from the work by Bui and Candy [8, Theorem 1.1, Theorem 1.3]. As mentioned above, in particular the “Cancellation condition (C1)” has to be ensured, but this can be checked with the explicit representation and estimates for the Fourier transform of the Poisson kernel in Section A.1. □

**APPENDIX B. PROOFS AND LITERATURE FOR SECTION 10**

We need an extension of the $L^\infty$-$L^1$-Hölder-inequality on $\mathbb{R}^{n+1}_+$. We have the following estimate between Carleson-measures and square functions, which can be found in [44, IV, §4.4, Proposition, p. 162].

**Lemma B.1.**

$$\int_{\mathbb{R}^{n+1}_+} F(x, t) G(x, t) \, d(x, t) \lesssim \sup_{B \subset \mathbb{R}^n \text{ balls}} \left( |B|^{-1} \int_{T(B)} t |F(y, t)|^2 \, dy \, dt \right)^{\frac{1}{2}} \int_{\mathbb{R}^n} \left( \int_{|x-y| < t} |G(y, t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{\frac{1}{2}} \, dx$$

where the supremum is over balls $B \subset \mathbb{R}^n$ and $T(B)$ is the “tent” over $B$ in $\mathbb{R}^n$, i.e. $T(B, (x_0)) = \{(x, t) \in \mathbb{R}^{n+1}_+ : |x - x_0| < r - t\}$.

If $F$ is chosen to be $\nabla_{\mathbb{R}^{n+1}} P^s \Phi$, then we can use the BMO-characterization of $f$ in Proposition 9.4, and have the following corollary.

**Corollary B.2.** Let $F^s(x, t) := P^s \Phi(x)$, $s \in (0, 2)$. Then

$$\int_{\mathbb{R}^{n+1}_+} |\nabla_{\mathbb{R}^{n+1}} F^s(x, t)||G(x, t)| \, d(x, t) \lesssim [f]_{BMO} \int_{\mathbb{R}^n} \left( \int_{|x-y| < t} |G(y, t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{\frac{1}{2}} \, dx.$$

**Proof of Proposition 10.1.** For the first two estimates, observe that with (9.6)

$$\sup_{t > 0} t^{s_3} |H(x, t)| \lesssim \mathcal{M}|I^{s_3} h|(x).$$

The estimates then follow from Hölder’s inequality and (9.10), (9.12), (9.11), respectively. □

**Proof of Proposition 10.2.** Denote the square function

$$\mathcal{S}(H)(x) := \left( \int_{|x-y| < t} |H(y, t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Let $W(x, t) := t^{2-s} \nabla_x \nabla_{\mathbb{R}^{n+1}} G^s$. Then, in view of Corollary B.2 and (9.4),

$$\int_{\mathbb{R}^{n+1}_+} t^{1+(1-s)(t+1)} |\nabla_{\mathbb{R}^{n+1}} \Phi(x, t)| |\nabla_x \nabla_{\mathbb{R}^{n+1}} G^s(x, t)| |\partial_t F_1^s(x, t)| \ldots |\partial_t F_\ell^s(x, t)| \, d(x, t)$$

$$\lesssim [\varphi]_{BMO} \int_{\mathbb{R}^n} \mathcal{M}((-\Delta)^{\frac{s}{2}} f_1)(x) \ldots \mathcal{M}((-\Delta)^{\frac{s}{2}} f_\ell)(x) \mathcal{S}(W)(x)$$
If we had to work with $\nabla_{\mathbb{R}^{n+1}} F^s(x, t)$ then we would also use (9.5) and obtain the same only with $\mathcal{M}((-(\Delta)^{\frac{\gamma}{2}} f_i)(x)$ replaced by $\mathcal{M}((-(\Delta)^{\frac{\gamma}{2}} f_i)(x) + \mathcal{M}(\nabla f_i)(x)$.

With Hölder’s inequality and the maximal theorem (here we use that $p_i \in (1, \infty)$),

$$\lesssim [\varphi]_{BMO} \| (-(\Delta)^{\frac{\gamma}{2}} f_1 \|_{L(p_1, q_1)} \cdots \| (-(\Delta)^{\frac{\gamma}{2}} f_\ell \|_{L(p_\ell, q_\ell)} \| S(W) \|_{L(p_0, q_0)}.$$  

We conclude the first estimate of Proposition 10.2 with (9.14). The other estimates follow the same way.

**Proof of Proposition 10.3.** In view of (9.17) and (9.18),

$$\mathcal{I} := \int_{\mathbb{R}^{n+1}} t^{1-n-s_1 + s_2} |\nabla_{\mathbb{R}^{n+1}} \Phi^s(x, t)| |\nabla_{\mathbb{R}^{n+1}} G^s(x, t)| |F^s(x, t)| d(x, t)$$

$$\lesssim [\varphi]_{C^0} \int_{\mathbb{R}^{n+1}} t^{s_2 - s_1} |\nabla_{\mathbb{R}^{n+1}} G^s(x, t)| |F^s(x, t)| d(x, t).$$

With Hölder inequality we find

$$\lesssim [\varphi]_{C^0} \| (-(\Delta)^{\frac{s_2}{2}} g \|_{L(p_1, q_1)} \| x \mapsto \left( \int_{t=0}^\infty \left| t^{s_2 - \frac{s_1}{2}} F^s(x, t) \right|^2 dt \right) \frac{1}{2} \|_{L(p_2, q_2)}$$

and with another Hölder inequality and in view of (9.11) and (9.10),

$$\lesssim [\varphi]_{C^0} \| (-(\Delta)^{\frac{s_2}{2}} g \|_{L(p_1, q_1)} \| x \mapsto \left( \int_{t=0}^\infty \left| t^{s_2 - \frac{s_1}{2}} F^s(x, t) \right|^2 dt \right) \frac{1}{2} \|_{L(p_2, q_2)}.$$

Now

$$t^{s_2 - \frac{s_1}{2}} F^s(x, t) = t^{s_2 - \frac{s_1}{2}} p_i^s \ast f(x) = t^{s_2 - \frac{s_1}{2}} (-(\Delta)^{\frac{s_1}{2} p_i^s} \ast I^{s_1} f(x) = t^{-\frac{s_1}{2}} \kappa_i \ast I^{s_2} f(x),$$

for

$$\kappa(z) = (-(\Delta)^{\frac{s_2}{2}} p_i^s) (z),$$

and $\kappa_i(z) = t^{-n} \kappa(z/t)$. Observe that $\kappa$ is integrable and $\int_{\mathbb{R}^n} \kappa = 0$, since $\kappa$ is a (fractional) derivative. Thus,

$$\| x \mapsto \left( \int_{t=0}^\infty \left| t^{s_2 - \frac{s_1}{2}} F^s(x, t) \right|^2 dt \right) \frac{1}{2} \|_{L(p_2, q_2)}$$

This is a (tangential) square function as in [44, Chapter I, §6.3, (20), p. 27], and with [44, Chapter I, §8.23, p. 46] we conclude

$$\| x \mapsto \left( \int_{t=0}^\infty \left| t^{s_2 - \frac{s_1}{2}} F^s(x, t) \right|^2 dt \right) \frac{1}{2} \|_{L(p_2, q_2)} \lesssim \| I^{s_2} f \|_{L(p_0, q_0)}.$$ 

This shows the main estimate in Proposition 10.3, the other estimates follow by variations of the above argument. \qed
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