Two-sample tests for high-dimensional covariance matrices using both difference and ratio

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Abstract: By borrowing strengths from the difference and ratio between two sample covariance matrices, we propose three tests for testing the equality of two high-dimensional population covariance matrices. One test is shown to be powerful against dense alternatives, and the other two tests are suitable for general cases, including dense and sparse alternatives, or the mixture of the two. Based on random matrix theory, we investigate the asymptotical properties of these three tests under the null hypothesis as the sample size and the dimension tend to infinity proportionally. Limiting behaviors of the new tests are also studied under various local alternatives. Extensive simulation studies demonstrate that the proposed methods outperform or perform equally well compared with the existing tests.

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1. Introduction

Testing whether two populations share the same covariance matrix is of fundamental interest in statistical inference, as many statistical procedures, such as the linear discriminant analysis, require the task of the equality test of covariance matrices. Let $x_{k1}, x_{k2}, \ldots, x_{kN_k}$ be an independent and identically distributed sample from the $k$th $p$-dimensional population with mean vector $\mu_k$ and covariance matrix $\Sigma_k$, where $N_k$ is the sample size, $k = 1, 2$. Let $n_k = N_k - 1$ and $y_{nk} = p/n_k$. The sample covariance matrix of $\Sigma_k$ is given by $S_k = n_k^{-1} \sum_{i=1}^{N_k} (x_{ki} - \bar{x}_k)(x_{ki} - \bar{x}_k)^T$, where $\bar{x}_k = N_k^{-1} \sum_{i=1}^{N_k} x_{ki}$ is the sample mean of the $k$th population, $k = 1, 2$. We are interested in testing the following hypothesis

$$H_0 : \Sigma_1 = \Sigma_2 \quad \text{versus} \quad H_1 : \Sigma_1 \neq \Sigma_2. \quad (1.1)$$

In the conventional low-dimensional setting where $p$ is relatively small compared with the sample sizes, such a testing problem has been well studied. See, for example, [16], [8], [13], [7], [12] and [1]. In the high-dimensional setting where $p$ is large relative to $N_1$ and $N_2$, several different approaches have been proposed to address the failure of classical methods, see [4, 14, 15, 10, 5, 19, 21], and among others. In particular, many of the existing testing procedures are developed based on $\text{tr}(\Sigma_1 - \Sigma_2)^2$, the square of the Frobenius norm of $\Sigma_1$ and $\Sigma_2$. For example, under the Gaussian population assumption, [14] developed an unbiased estimator by correcting the bias of the sample statistic $T_d = \text{tr}(S_1 - S_2)^2$ in estimating $\text{tr}(\Sigma_1 - \Sigma_2)^2$. [11] considered a $U$-statistic for $\text{tr}(\Sigma_1 - \Sigma_2)^2$ and showed that their proposed test can yield desirable performance in various situ-
Two-sample tests for high dimensional covariance matrices

137

ations, especially when the two population matrices have dense differences. [21] also studied the limiting behavior of $T_d$, under general populations including both Gaussian and non-Gaussian distributions. On the other hand, known as a relative measure, the ratio measure, as given by $\text{tr}(\Sigma_1 \Sigma_2^{-1} - I_p)^2$ with $I_p$ denoting the identity matrix, can also characterize the relative discrepancy between $\Sigma_1$ and $\Sigma_2$. When $S_2$ is invertible, a natural estimator of $\text{tr}(\Sigma_1 \Sigma_2^{-1} - I_p)^2$ is $T_r = \text{tr}(S_1 S_2^{-1} - I_p)^2$. Unfortunately, such a test statistic has been seldom studied. This may be due to the reason that the statistic $T_r$ involves the inverse of $S_2$, which requires the dimension $p$ does not exceed the sample size $N_2$; and when $p$ is close to $N_2$, the behavior of $T_r$ is complicated. In fact, the relative performance between the tests $T_d$ and $T_r$ varies case by case: as will be shown in the simulation studies as well as through the theoretical power analyses, any one of $T_d$ and $T_r$ cannot dominate the other in terms of statistical power under all scenarios.

This paper makes several contributions to the literature. (1) Firstly, we investigate the relative performances of $T_d$ and $T_r$ through theoretical power analyses in the high-dimensional setting. In particular, the statistic $T_r$ is not well studied in literature. To adaptively borrow strengths from $T_d$ and $T_r$, we develop a new testing procedure through establishing the joint limiting null distribution of $T_d$ and $T_r$ using random matrix theory. The proposed method is able to perform nearly the same as the better one of $T_d$ and $T_r$ under all scenarios. (2) Since $T_d$ and $T_r$ only target on dense alternatives where there are many small differences between $\Sigma_1$ and $\Sigma_2$, they may be not powerful for sparse alternatives where there are only few but large differences between $\Sigma_1$ and $\Sigma_2$. Our second contribution is to propose another two testing procedures, by combining $T_d$, $T_r$, and a maximum norm statistic, to maintain high power for testing the equality of two high-dimensional covariance matrices under both dense and sparse alternatives. To the best of our knowledge, there is no existing approach to two-sample testing problems with high-dimensional covariance matrices using the combination of three statistics. (3) The theoretical properties of the proposed tests based on different combination procedures are extensively investigated in the paper, which sheds lights on the pros and cons of different procedures for combining multiple statistics.

The rest of this paper is organized as follows. Section 2 establishes the joint limiting null distribution of $T_d$ and $T_r$, and introduces three novel procedures for testing the equality of two covariance matrices. Section 3 investigates the power functions of the proposed tests under representative alternatives. Section 4 presents various simulation results to demonstrate the performance of the proposed methods. All technical details are presented in the Appendix.

2. Main results

Before presenting the main results, we first introduce some basic notations. The empirical spectral distribution (ESD) is defined as

$$F^A = p^{-1} \sum_{j=1}^{p} \delta_{\lambda_j},$$
where \( A \) is a \( p \times p \) non-negative definite matrix, \( \{\lambda_j, 1 \leq j \leq p\} \) are the eigenvalues of \( A \) and \( \delta_a \) denotes the Dirac mass at point \( a \). Define the operator as

\[
A \circ B = (a_{ij}b_{ij})_{i=1,\ldots,m,j=1,\ldots,n}
\]

with \( m \times n \) matrices \( A = (a_{ij})_{i=1,\ldots,m,j=1,\ldots,n} \) and \( B = (b_{ij})_{i=1,\ldots,m,j=1,\ldots,n} \).

The following theorem establishes the joint limiting null distribution of the two statistics

Theorem 2.1. Under Assumptions A-B and the null hypothesis \( H_0 \) in (1.1), let \( \Sigma_1 = \Sigma_2 = \Sigma \), then

\[
\begin{pmatrix}
T_d - \mu_0 - \mu_{10} \\
T_r - \mu_{20}
\end{pmatrix}
\xrightarrow{d} N\left(0_2, \begin{pmatrix}
\sigma_{110} & \sigma_{120} \\
\sigma_{210} & \sigma_{220}
\end{pmatrix}
\right),
\]

where \( \bar{S}_1 = n_1^{-1} \sum_{i=1}^{N_1} (x_{1i} - \bar{x}_1)(x_{1i} - \bar{x}_1)^T \), \( \bar{S}_2 = n_2^{-1} \sum_{i=1}^{N_2} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)^T \),

where \( \bar{x}_1 = N_1^{-1} \sum_{i=1}^{N_1} x_{1i} \) and \( \bar{x}_2 = N_2^{-1} \sum_{i=1}^{N_2} x_{2i} \) are the sample means. Let the aforementioned two statistics be

\[
T_d = \text{tr}(S_1 - S_2)^2, \quad T_r = \text{tr}(S_1 S_2^{-1} - I_p)^2,
\]

where \( I_p \) is the \( p \times p \) identity matrix.

Next, we impose the following two assumptions, which are commonly used in random matrix theory, for studying the limiting behaviors of the considered statistics throughout the paper.

- **Assumption A.** The random vector \( x_{ki} \) satisfies the independent component structure \( x_{ki} = \mu_k + \Sigma_k^{1/2} w_{ki} \), where \( w_{ki} = (w_{k1i}, \ldots, w_{kpi})^T \), the elements \( \{w_{kli}, k = 1, 2; l = 1, \ldots, p; i = 1, \ldots, N_k\} \) are independent and identically distributed with \( \text{E} w_{kli} = 0, \text{E} w_{kli}^2 = 1 \) and \( \beta_k = \text{E} w_{kli}^4 - 3 \). Moreover, \( \Sigma_k > 0 \) has a bounded spectral norm. The ESD of \( \Sigma_k \) converges weakly to a limit spectral distribution (LSD) \( L_k \) for \( k = 1, 2 \).

- **Assumption B.** The dimension \( p \) and \( n_1, n_2 \) grow to infinity under the convergence regime

\[
y_{n_1} = p/n_1 \to y_1 \in (0, +\infty), \quad y_{n_2} = p/n_2 \to y_2 \in (0, 1),
\]

where \( n_1 = N_1 - 1 \) and \( n_2 = N_2 - 1 \).

In Assumption B, we require the data dimension \( p \) to be less than \( n_2 \) so that the sample covariance matrix of the second population is invertible.

### 2.1. Joint limiting null distribution of \( T_d \) and \( T_r \)

The following theorem establishes the joint limiting null distribution of the two statistics \( T_d \) and \( T_r \). Let \( L(x) \) be the LSD of \( \Sigma_1 = \Sigma_2 = \Sigma \) under the null hypothesis in (1.1).

**Theorem 2.1.** Under Assumptions A-B and the null hypothesis \( H_0 \) in (1.1), let \( \Sigma_1 = \Sigma_2 = \Sigma \), then

\[
\begin{pmatrix}
T_d - \mu_0 - \mu_{10} \\
T_r - \mu_{20}
\end{pmatrix}
\xrightarrow{d} N\left(0_2, \begin{pmatrix}
\sigma_{110} & \sigma_{120} \\
\sigma_{210} & \sigma_{220}
\end{pmatrix}
\right),
\]

where \( \bar{x}_1 = N_1^{-1} \sum_{i=1}^{N_1} x_{1i} \) and \( \bar{x}_2 = N_2^{-1} \sum_{i=1}^{N_2} x_{2i} \) are the sample means.
where \( \mathbf{0}_2 = (0, 0)^T \),

\[
\begin{align*}
\mu_0 &= n_1^{-1} \text{tr}\mathbf{S}_1 + n_2^{-1} \text{tr}\mathbf{S}_2, \\
m_{10} &= \int x dL(x), \\
m_{20} &= \int x^2 dL(x), \\
\mu_{10} &= (y_{n_1} + y_{n_2})p^{-1} \text{tr}\mathbf{\Sigma}^2 + (\beta_1 y_{n_1} + \beta_2 y_{n_2})p^{-1} \text{tr}(\mathbf{\Sigma} \circ \mathbf{\Sigma}), \\
\sigma_{110} &= 4(y_1 + y_2)^2m_{20}^2, \\
\sigma_{120} &= \sigma_{210} = \frac{8y_2(y_1 + y_2)^2 + 4y_1^2 + 4y_2^2}{(1 - y_2)^2} + \frac{8y_1y_2 + 8y_3^2}{(1 - y_2)^3}m_{10}^2, \\
\sigma_{220} &= \frac{8y_1^2 + 16y_2^2}{(1 - y_2)^2} + \frac{4y_2^2 + 4y_1^2y_2 + 64y_1y_2^2}{(1 - y_2)^6} \\
&+ \frac{8y_1y_2 + 56y_1^2y_2^2 + 48y_2^2 + 8y_1y_2}{(1 - y_2)^7} + \frac{8y_2^2 + 24y_2^2 + 4y_2^2}{(1 - y_2)^8} \\
&+ 4(\beta_1 y_1 + \beta_2 y_2) \left[ \frac{(y_1 + y_2)^2}{(1 - y_2)^2} + \frac{2y_2(y_1 + y_2)^2}{(1 - y_2)^3} + \frac{y_2^2}{(1 - y_2)^6} \right].
\end{align*}
\]

**Remark 2.1.** From Hölder’s inequality, \( m_{20} \geq m_{10}^2 \) and \( \beta_k \geq -2 \) for \( k = 1, 2 \). Thus,

\[
\sigma_{110}\sigma_{220} - \sigma_{120}\sigma_{210} \geq 16 \left[ \frac{(4y_2^2 + 2y_1y_2)(y_1 + y_2)^4 + 4y_1^2y_2^2(y_1 + y_2)^2 + y_1y_2^2}{(1 - y_2)^6} \right.
\]

\[
+ \left( \frac{2y_2^3 + 4y_2^2}{y_1 + y_2} \right) + 4y_1y_2^2(y_1 + y_2)^3 + 2y_1^2y_2^2 \right] 
\]

\[
+ \left( \frac{4y_1y_2^3 + 4y_1y_2^2}{y_1 + y_2} \right) + y_1^2y_2^2(y_1 + y_2) + 3y_1^2y_2^2 \right] 
\]

\[
+ \left( \frac{14y_2^4}{y_1 + y_2} \right) + 8y_2^4 + 4y_1y_2^2(y_1 + y_2)^2 \right] 
\]

\[
+ \left( \frac{y_2^3}{y_1 + y_2} \right) + 2y_1^3y_2 + 3y_1^2y_2^2 + 2y_1y_2^3 \right] 
\]

\[
+ \left( \frac{4y_2^3 + 4y_2^2}{y_1 + y_2} \right) + y_2^2(y_1 + y_2) + 2y_1y_2 \right] \right] \geq 0.
\]

Therefore, the covariance matrix \( \begin{pmatrix} \sigma_{110} & \sigma_{120} \\ \sigma_{210} & \sigma_{220} \end{pmatrix} \) is positive definite.

Theorem 2.1 shows that the joint limiting distribution of \( T_d \) and \( T_r \) is a bivariate normal distribution under the null hypothesis. Its proof is provided.
Theorem 2.2. Under the conditions of Theorem 2.1, the weakly consistent positively-correlated.

\[ \sigma_{\hat{h}_{10}} = \frac{p^{-1}\text{tr}(S \circ S)}{n_1 + n_2}, \]
\[ \hat{m}_{10} = p^{-1}\text{tr}S, \quad \hat{m}_{20} = p^{-1}\text{tr}(S^2 - (n_1 + n_2)^{-1}\text{tr}^2S). \]

**Theorem 2.2.** Under the conditions of Theorem 2.1, the weakly consistent estimators of \( \mu_{10}, \sigma_{110}, \sigma_{120}, \sigma_{210} \) and \( \sigma_{220} \) are as follows:

\[
\hat{\mu}_{10} = (y_{n_1} + y_{n_2})\hat{m}_{20} + (\beta_1 y_{n_1} + \beta_2 y_{n_2})\hat{h}_{10},
\]
\[
\hat{\sigma}_{110} = 4(y_{n_1} + y_{n_2})^2\hat{m}_{20}^2,
\]
\[
\hat{\sigma}_{120} = \frac{8y_{n_1}^3 + 16y_{n_1}^2y_{n_2}}{(1 - y_{n_2})^3} + \frac{8y_{n_1}y_{n_2}^2 + 4y_{n_1}^2 + 4y_{n_2}^2 + 8y_{n_1}y_{n_2} + 64y_{n_1}y_{n_2}^2}{(1 - y_{n_2})^6}
+ \frac{8y_{n_1}y_{n_2}^2 + 56y_{n_1}y_{n_2}^2 + 48y_{n_2}^3 + 8y_{n_1}y_{n_2}}{(1 - y_{n_2})^6} + \frac{8y_{n_2}^5 + 24y_{n_2}^3 + 4y_{n_2}^2}{(1 - y_{n_2})^6}
+ 4(\beta_1 y_{n_1} + \beta_2 y_{n_2}) \left[ \frac{2y_{n_1} (y_{n_1} + y_{n_2} - y_{n_1} y_{n_2} - y_{n_2}^2)}{(1 - y_{n_2})^4} 
+ \frac{y_{n_1} (y_{n_1} + y_{n_2} - 1) - y_{n_1} y_{n_2} - y_{n_2}^2}{(1 - y_{n_2})^4} \right] v_{10p}
+ \frac{y_{n_1} (y_{n_1} + y_{n_2} - 1) - y_{n_1} y_{n_2} - y_{n_2}^2}{(1 - y_{n_2})^4} v_{20p}
+ \frac{16y_{n_1} (y_{n_1} + y_{n_2} - 1)}{(1 - y_{n_2})} v_{30p} + 8y_{n_1} v_{40p}
+ 4y_{n_2}^2 \left[ \frac{2y_{n_1} (y_{n_1} + y_{n_2} - y_{n_1} y_{n_2}) + \beta_1 y_{n_1} + \beta_2 y_{n_2}}{(1 - y_{n_2})^4} \right] v_{10p}
+ \frac{16y_{n_2}^2 (y_{n_1} + y_{n_2} - 1)}{(1 - y_{n_2})} v_{10p} v_{20p} + 16y_{n_1} v_{10p} v_{30p}
+ 4y_{n_1} v_{20p} + 8y_{n_1}^2 v_{10p} v_{20p},
\]

where

\[
v_{10p} = \frac{1}{p} \left[ \frac{y_{n_2}}{(1 - y_{n_2})^2} + \frac{\beta_2 y_{n_2}}{(1 - y_{n_2})}, \right],
\]
\[
v_{20p} = \frac{1}{p} \left[ \frac{y_{n_2}^2 + 3y_{n_2}}{(1 - y_{n_2})^4} - \frac{\beta_2 (y_{n_2}^2 - 3y_{n_2})}{(1 - y_{n_2})^3}, \right].
\]
Two-sample tests for high dimensional covariance matrices

\[ v_{30p} = \frac{1}{p} \left[ \frac{y_n^3 + 9y_n^2 + 6y_n}{(1 - y_n)^6} + \frac{6\beta_2y_n}{(1 - y_n)^5} \right], \]

\[ v_{40p} = \frac{1}{p} \left[ \frac{y_n^4 + 18y_n^3 + 35y_n^2 + 10y_n}{(1 - y_n)^8} + \frac{10\beta_2(y_n^2 + y_n)}{(1 - y_n)^7} \right]. \]

Based on the Slutsky’s theorem, we further have that the random vector

\[ \left( \frac{T_d - \mu_0 - \hat{\mu}_{10}}{\sqrt{\hat{\sigma}_{110}}}, \frac{T_r - \mu_{20}}{\sqrt{\hat{\sigma}_{220}}} \right)^T \]

converges in distribution to \( N \left( \begin{pmatrix} 0_2, \begin{pmatrix} 1 & \hat{\rho} \\ \hat{\rho} & 1 \end{pmatrix} \end{pmatrix} \right) \) with \( \rho = \sigma_{120}/\sqrt{\sigma_{110}\sigma_{220}}. \)

**Remark 2.2.** Instead of simply replacing \( y_1 \) and \( y_2 \) in \( \sigma_{220} \) respectively with \( y_{n1} \) and \( y_{n2} \), the expression of \( \sigma_{220} \) has additional terms involving \( v_{0p}, i = 1,2,3,4. \) This is because even if the order of \( v_{0p} \) is \( o(1) \), the value of \( v_{0p} \) is much greater than \( \rho \) when \( p \) is small. For more details, one can refer to the proof of Theorem 2.2 in Appendix.

### 2.2. Three test procedures

For testing the hypothesis (1.1), three test procedures are proposed as follows:

**Test 1:** Let the first statistic be

\[ T_{dr} = \max \{|T_d - \mu_0 - \hat{\mu}_{10}|/\sqrt{\hat{\sigma}_{110}}, |T_r - \mu_{20}|/\sqrt{\hat{\sigma}_{220}|}. \] (2.1)

That is, \( T_{dr} \) is constructed by the maximum absolute value of the standardized statistics \( T_d \) and \( T_r \). For a given test level \( \alpha \), the rejection region is

\[ \{x_{11}, \ldots, x_{1N_1}, x_{21}, \ldots, x_{2N_2} : T_{dr} > t_\alpha \}, \] (2.2)

where the critical value \( t_\alpha \) is obtained by

\[ \alpha = 1 - \int_{-t_\alpha}^{t_\alpha} \int_{-t_\alpha}^{t_\alpha} f(x_d, x_r)dx_ddx_r, \] (2.3)

with \( f(x_d, x_r) \) being the density of \( N \left( \begin{pmatrix} 0_2, \begin{pmatrix} 1 & \hat{\rho} \\ \hat{\rho} & 1 \end{pmatrix} \end{pmatrix} \right) \) and \( \hat{\rho} = \sigma_{120}/\sqrt{\sigma_{110}\sigma_{220}}. \)

**Test 2:** \( T_d \) and \( T_r \) are powerful to measure the dense differences between \( \Sigma_1 \) and \( \Sigma_2 \), so is \( T_{dr} \). To enhance the power of the test procedure (2.2) for sparse alternatives, we use a theoretical result of [5]. Under Conditions (C1), (C2) (or (C2*)) and (C3) of [5] (see Appendix) and the null hypothesis \( H_0 \), for any \( t \in \mathbb{R}, \)

\[ P(T_x - 4\log p + \log \log p \leq t) \to \exp \left( -\frac{1}{\sqrt{8\pi}} \exp \left( -\frac{t}{2} \right) \right), \] (2.4)

as \( N_1, N_2, p \to \infty \), where

\[ T_x = \max_{1 \leq i, j \leq p} \frac{(s_{1i}t_j - s_{2i}t_j)^2}{\theta_{1i}t_j/n_1 + \theta_{2i}t_j/n_2}, \] (2.5)

\[ y_{n1}^i, y_{n2}^i, \ldots, y_{n1}^p, y_{n2}^p \] are the sample mean vectors of \( x_{1i} \) and \( x_{2i} \), respectively.
with

\[ \hat{\theta}_{kl1l2} = n_k^{-1} \sum_{i=1}^{N_k} \{(x_{kl1i} - \bar{x}_{kl1})(x_{kl2i} - \bar{x}_{kl2}) - s_{kl1l2}\}^2, \]

\[ S_1 = (s_{s1l1l2})_{l1,l2=1,...,p} \quad \text{and} \quad S_2 = (s_{s2l1l2})_{l1,l2=1,...,p}. \]

Borrowing the idea of [6] and [21], let the second test statistic be

\[ T_{drx_1} = T_{dr} + p^2 I(T_{x} > s(N_1, N_2, p)), \]

where \( I(\cdot) \) is the indicator function, and \( s(N_1, N_2, p) \) is a pre-specified threshold depending on the sample sizes \( N_1, N_2 \) and the dimension \( p \). Specifically, with a carefully selected threshold \( s(N_1, N_2, p) \), the second term in \( T_{drx_1} \) converges to zero under the null hypothesis, and it takes effect as long as \( T_{x} \) detects a strong signal. As a result, the first term \( T_{dr} \) plays a dominant role, and the second term serves as a power enhancer for screening sparse disturbances between the two covariance matrices. Therefore, such a weighted statistic is able to adaptively combine the information from \( T_d, T_r \) and \( T_x \). The corresponding rejection region is

\[ \{ x_{11}, ..., x_{1N_1}, x_{21}, ..., x_{2N_2} : T_{drx_1} > t_\alpha \}, \quad (2.6) \]

with \( t_\alpha \) obtained from (2.3).

**Test 3:** To incorporate the information of \( T_{dr} \) and \( T_x \), we propose another statistic \( T_{drx_2} \) as follow,

\[ T_{drx_2} = \max\{ T_{dr}, c_\alpha T_x \}, \]

where \( c_\alpha = t_{\alpha/2}/q_{\alpha/2} \), \( t_{\alpha/2} \) satisfies the equation (2.3) with \( \alpha \) replaced by \( \alpha/2 \), and

\[ q_{\alpha/2} = -\log(8\pi) - 2 \log \log(1 - \alpha/2)^{-1} + 4 \log p - \log \log p. \]

The rejection region is

\[ \{ x_{11}, ..., x_{1N_1}, x_{21}, ..., x_{2N_2} : T_{drx_2} > t_{\alpha/2} \}. \quad (2.7) \]

The statistic \( T_{drx_2} \) has a similar form as \( T_{dr} \), but incorporates the additional contribution from the statistic \( T_x \).

**Theorem 2.3.** We have the following results:

- Under the conditions of Theorem 2.1, the test based on \( T_{dr} \) has an asymptotic significance level \( \alpha \);
- Under the conditions of Theorem 2.1 and Conditions (C1), (C2) (or (C2*)) and (C3) of [5], if the threshold \( s(N_1, N_2, p) \) satisfies \( s(N_1, N_2, p) - 4 \log p \geq 0 \), the test based on \( T_{drx_1} \) has an asymptotic significance level \( \alpha \);
- Under the conditions of Theorem 2.1 and Conditions (C1), (C2) (or (C2*)) and (C3) of [5], the size of the test based on \( T_{drx_2} \) is asymptotically equal to or less than the significance level \( \alpha \).
Theorem 2.3 states that all of the three proposed methods can maintain the nominal test level asymptotically. According to Theorem 2.3, in this paper, we take

\[ s(N_1, N_2, p) = \{(\log \log (N_1/2 + N_2/2) - 1)^2 + 4\} (\log p - \log \log p/4) + q \] (2.8)

with

\[ \exp\{-8(\pi)^{-1/2} \exp(-q/2)\} = 0.99. \]

It is easy to show that such a specification satisfies \( s(N_1, N_2, p) - 4\log p \geq 0 \), thus the result of \( T_{drx_2} \) in Theorem 2.3 holds.

**Remark 2.3.** Last, several remarks are in place for comparing \( T_{drx_1} \) and \( T_{drx_2} \). Both of the two tests borrow information from the maximum norm statistic \( T_x \) for power enhancement under the sparse alternatives. But the role of \( T_x \) functions in different ways: \( T_{drx_1} \) directly adds the contribution from \( T_x \) to the main term \( T_{dr} \) without changing the critical value; \( T_{drx_2} \) integrates the information from \( T_{drx_1} \) and \( T_x \) using a similar idea as the Tippett’s combination test \[18, 9\], and it is essentially a multiple-testing procedure that requires multiplicity adjustments. Since \( t_{\alpha/2} > t_\alpha \), when the main signals come from \( T_{dr} \), it can be shown that the power of \( T_{drx_1} \) is greater than that of \( T_{drx_2} \). On the other hand, to guarantee the asymptotic unbiasedness of \( T_{drx_1} \), the contribution of \( T_x \) to \( T_{drx_1} \) is heavily penalized. As a result, \( T_{drx_2} \) sometimes is more powerful than \( T_{drx_1} \) under sparse alternatives. These results are all confirmed in the simulation studies.

### 3. Power analysis

From Theorems 2.1-2.2, the rejection regions of the tests based on the statistics \( T_d \) and \( T_r \) are defined as

\[ \{x_{11}, \ldots, x_{1N_1}, x_{21}, \ldots, x_{2N_2} : |T_d - \mu_0 - \mu_{10}|/\sqrt{\hat{\sigma}_{110}} > z_{1-\alpha/2}\} \] (3.1)

and

\[ \{x_{11}, \ldots, x_{1N_1}, x_{21}, \ldots, x_{2N_2} : |T_r - \mu_{20}|/\sqrt{\hat{\sigma}_{220}} > z_{1-\alpha/2}\}, \] (3.2)

where \( z_{1-\alpha/2} \) is the \((1-\alpha/2)\) quantile of \( N(0, 1) \).

In this section, we study the asymptotic powers of \( T_d, T_r, T_{dr}, T_{drx_1} \) and \( T_{drx_2} \) under the following alternative sets:

\[
\begin{align*}
\Pi_1 &= \left\{ (\Sigma_1, \Sigma_2) : \Sigma_2 = \tau_p \Sigma_1, \tau_p = p/(p + a_1), a_1 > 0 \text{ is a constant} \right\}, \\
\Pi_2 &= \left\{ (\Sigma_1, \Sigma_2) : \Sigma_2 = \Sigma_1 + \frac{a_2}{p} J_p, \Sigma_1 = I_p, a_2 > 0 \text{ is a constant} \right\}, \\
\Pi_3 &= \left\{ (\Sigma_1, \Sigma_2) : \max_{1 \leq i_1 \leq i_2 \leq p} \frac{(\sigma_{1i_1} - \sigma_{2i_1}/n_1 + \theta_{i_2}/n_2 - 16 \log p)^2}{\theta_{i_1} + \theta_{i_2}/n_2} \geq 16 \log p \right\},
\end{align*}
\]

where \( J_p \) is a \( p \times p \) matrix with all elements being 1, and \( \theta_{i_1} = \text{Var}(x_{k1i_1} - \mu_{k1}) \) for \( k = 1, 2 \). The first two sets correspond to local dense alternatives, and the last set includes sparse alternatives as a special case.
3.1. Power analysis in the dense alternative set \( \Pi_1 \)

The following theorem gives the joint limiting distribution of \( T_d \) and \( T_r \) when \( (\Sigma_1, \Sigma_2) \in \Pi_1 \).

**Theorem 3.1.** Under Assumptions A-B and \( (\Sigma_1, \Sigma_2) \in \Pi_1 \), we have

\[
\left( \frac{T_d - \mu_0 - \mu_{11}}{T_r - \mu_{21}} \right) \xrightarrow{d} N \left( \mathbf{0}_2, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right),
\]

where

\[
m_{11} = \int x dL_1, \quad m_{21} = \int x^2 dL_1,
\]

\[
\mu_{11} = \frac{(y_{n_1} + y_{n_2} r_p^2)}{2} \text{tr} \Sigma_1^2 + (\beta_1 y_{n_1} + \beta_2 y_{n_2} r_p^2) \text{tr} (\Sigma_1 \circ \Sigma_1),
\]

\[
\mu_{21} = \frac{2}{p} \left( \frac{y_{n_1}}{(1 - y_{n_2})^2} + \frac{y_{n_2}^2 + 3y_{n_2}}{(1 - y_{n_2})^3} - \frac{2 \beta_2 y_{n_2}}{(1 - y_{n_2})^3} \right),
\]

\[
\sigma_{111} = \frac{4(y_1 + y_2)^2 m_{21}^2}{n_{11}},
\]

\[
\sigma_{121} = \sigma_{211} = \frac{8y_2(y_1 + y_2)^2 + 4y_1 + 4y_2^2}{(1 - y_2)^2} + \frac{8y_1 y_2 + 8y_2^2}{(1 - y_2)^3} m_{11}^2.
\]

**Proposition 3.1.** Under the conditions of Theorem 3.1, we have

(I) For the test based on \( T_d \),

\[
P(|T_d - \mu_0 - \hat{\mu}_{10}|/\sqrt{\sigma_{110}} > z_{1-\alpha/2}) \to \alpha;
\]

(II) For the test based on \( T_r \),

\[
P(|T_r - \mu_{20}|/\sqrt{\sigma_{220}} > z_{1-\alpha/2})
\]

\[
\to 1 - \Phi(z_{1-\alpha/2} - \Delta_1) - \Phi(-z_{1-\alpha/2} - \Delta_1) > \alpha,
\]

where \( \Phi(\cdot) \) is the distribution function of \( N(0, 1) \) and

\[
\Delta_1 = \frac{2\alpha_1}{\sqrt{\sigma_{220}}} \left[ \frac{y_1 + y_2}{(1 - y_2)^2} + \frac{y_2}{(1 - y_2)^3} \right];
\]

(III) For the test based on \( T_{dr} \),

\[
P(T_{dr} > t_\alpha) \to 1 - \int_{-t'_\alpha - \Delta_1}^{t'_\alpha - \Delta_1} \int_{-t'_\alpha}^{t'_\alpha} f'(x_d, x_r) dx_d dx_r > \alpha,
\]
where \( f'(x_d, x_r) \) is the density of \( N \left( \begin{array}{c} 0_2 \\ \rho_1 \rho_1 \end{array} \right) \) with \nabla \begin{align*}
\rho_1 &= \frac{\sigma_{121}}{\sqrt{\sigma_{111}\sigma_{220}}}, \\
\alpha &= 1 - \int_{-t'_\alpha}^{t'_\alpha} \int_{-t'_\alpha}^{t'_\alpha} f'(x_d, x_r) dx_d dx_r; \quad (3.3)
\end{align*}

(IV) For the test based on \( T_{drx_1} \),
\[
\lim_{p \to \infty} P(T_{drx_1} > t_\alpha) \geq \lim_{p \to \infty} P(T_{dr} > t_\alpha) > \alpha;
\]

(V) For the test based on \( T_{drx_2} \),
\[
\lim_{p \to \infty} P(T_{drx_2} > t_{\alpha/2}) \geq 1 - \int_{-t'_{\alpha/2} - \Delta_1}^{t'_{\alpha/2} - \Delta_1} \int_{-t'_{\alpha/2} - \Delta_1}^{t'_{\alpha/2} - \Delta_1} f'(x_d, x_r) dx_d dx_r > \alpha/2,
\]

where \( t'_{\alpha/2} \) satisfies the equation (3.3) with replacing \( \alpha \) by \( \alpha/2 \).

Proposition 3.1 shows that the test based on \( T_d \) suffers from low power if \((\Sigma_1, \Sigma_2) \in \Pi_1\). Since the asymptotical power functions of the tests based on \( T_r, T_{dr}, T_{drx_1} \) and \( T_{drx_2} \) are increasing functions of \( \Delta_1 \), these tests will enjoy high powers if \( \Delta_1 \) is large enough.

### 3.2. Power analysis in the dense alternative set \( \Pi_2 \)

The following theorem gives the joint limiting distribution of \( T_d \) and \( T_r \) when \((\Sigma_1, \Sigma_2) \in \Pi_2\).

**Theorem 3.2.** Under Assumptions A-B and \((\Sigma_1, \Sigma_2) \in \Pi_2\), if \( \beta_2 = 0 \), then

\[
\left( \begin{array}{c} T_d - \mu_0 - \mu_{12} \\ T_r - \mu_{22} \end{array} \right) \xrightarrow{d} N \left( \begin{array}{c} 0_2 \\ \sigma_{112} \sigma_{122} \sigma_{212} \sigma_{222} \end{array} \right),
\]

where

\[
\begin{align*}
\mu_{12} &= a_2^2 + y_{n_1} + y_{n_2} + \beta_1 y_{n_1}, \\
\mu_{22} &= p \left[ -\frac{2}{(1-y_{n_2})^2} + \frac{y_{n_1}}{(1-y_{n_2})^3} + \frac{1}{(1-y_{n_2})^4} + p
\right] + \frac{a_2^2 - 2a_2(1 + a_2)(y_{n_1} + y_{n_2})}{(1-y_{n_2})^2(1 + a_2)^2} - \frac{2a_2 y_{n_2}}{(1-y_{n_2})^3(1 + a_2)} - \frac{2y_{n_2}}{(1-y_{n_2})^2} + \frac{y_{n_1} + 2y_{n_1} y_{n_2}}{(1-y_{n_2})^3} + \frac{y_{n_2}^2 + 3y_{n_2}^2 + \beta_1 y_{n_1}}{(1-y_{n_2})^5}, \\
\sigma_{112} &= 4(y_1 + y_2)^2, \\
\sigma_{122} &= \sigma_{212} = \frac{8y_2(y_1 + y_2)^2 + 4y_1^2 + 4y_2^2}{(1-y_2)^2} + \frac{8y_1 y_2 + 8y_2^3}{(1-y_2)^4}, \\
\sigma_{222} &= \frac{8y_1^3 + 16y_1^2 y_2^2}{(1-y_2)^5} + \frac{4y_1^2 + 40y_1^2 y_2^2 + 64y_1 y_2^2}{(1-y_2)^6}.
\end{align*}
\]
Proposition 3.2. Under the conditions of Theorem 3.2, we have

(I) For the test based on $T_d$, 

$$P(|T_d - \mu_0 - \hat{\mu}_{10}|/\sqrt{\hat{\sigma}_{110}} > z_{1-\alpha/2})$$

$$\rightarrow 1 - \left[ \Phi(z_{1-\alpha/2} - \Delta_2) - \Phi(-z_{1-\alpha/2} - \Delta_2) \right] > \alpha,$$

where $\Delta_2 = a_2^2/[2(y_1 + y_2)];$

(II) For the test based on $T_r$, 

$$P(|T_r - \mu_{20}|/\sqrt{\hat{\sigma}_{220}} > z_{1-\alpha/2})$$

$$\rightarrow 1 - \left[ \Phi(z_{1-\alpha/2} - \Delta_3) - \Phi(-z_{1-\alpha/2} - \Delta_3) \right] \geq \alpha,$$

where

$$\Delta_3 = \frac{1}{\sqrt{\hat{\sigma}_{222}}} \left[ a_2^2 - 2a_2(1 + a_2)(y_1 + y_2) \right] - \frac{2a_2y_2}{(1 - y_2)^2(1 + a_2)^2}; \]$$

(III) For the test based on $T_{dr}$, 

$$P(T_{dr} > t_\alpha) \rightarrow 1 - \int_{-t_\alpha - \Delta_2}^{t_\alpha - \Delta_2} \int_{-t_\alpha - \Delta_3}^{t_\alpha - \Delta_3} f^*(x_d, x_r)dx_d dx_r > \alpha,$$

where $f^*(x_d, x_r)$ is the density of $N\left(0_2, \begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix} \right)$ with

$$\rho_2 = \sigma_{122}/\sqrt{\sigma_{112} \sigma_{222}}, \quad \alpha = 1 - \int_{-t_\alpha}^{t_\alpha} \int_{-t_\alpha}^{t_\alpha} f^*(x_d, x_r)dx_d dx_r; \quad (3.4)$$

(IV) For the test based on $T_{drx_1}$,

$$\lim_{p \to \infty} P(T_{drx_1} > t_\alpha) \geq \lim_{p \to \infty} P(T_{dr} > t_\alpha) > \alpha;$$

(V) For the test based on $T_{drx_2}$,

$$\lim_{p \to \infty} P(T_{drx_2} > t_{\alpha/2}) \geq 1 - \int_{-t_{\alpha/2} - \Delta_3}^{t_{\alpha/2} - \Delta_3} \int_{-t_{\alpha/2} - \Delta_2}^{t_{\alpha/2} - \Delta_2} f^*(x_d, x_r)dx_d dx_r > \alpha/2,$$

where $t_{\alpha/2}$ satisfies the equation (3.4) with parameter $\alpha/2$ instead of $\alpha$. 
Proposition 3.2 shows that the tests based on $T_d, T_r, T_{dr}$ and $T_{drx_i}$ are all asymptotically unbiased if $(\Sigma_1, \Sigma_2) \in \Pi_2$ and $\beta_2 = 0$. Apparently, $\Delta_2$ is an increasing function of $a_2$, the tests based on $T_d, T_r, T_{dr}$, and $T_{drx_i}$ will enjoy high powers when $a_2$ is large enough. However, with the increase of $a_2$, $\Delta_3$ will converge to $\left(1 - \frac{2(y_1 + y_2)}{(1 - y_2)^2} - \frac{2y_2}{(1 - y_2)^3}\right)$. Thus, the test based on $T_r$ will suffer from low power in the case of small $\Delta_3$. For example, when $y_1 = 0.5 - \frac{(2y_2 - y_2^2)}{(1 - y_2)}$ and $a_2 \to \infty$, it follows that the power of the test based on $T_r$ converges to $\alpha$.

3.3. Power analysis in the alternative set $\Pi_3$

The following proposition gives some results on the power functions of the tests based on $T_{drx_1}$ and $T_{drx_2}$ under the alternative set $\Pi_3$. Let

$$
\Sigma_d = \Sigma_1 - \Sigma_2, \quad \Sigma_w = \frac{n_1}{n_1 + n_2} \Sigma_1 + \frac{n_2}{n_1 + n_2} \Sigma_2.
$$

**Proposition 3.3.** When $(\Sigma_1, \Sigma_2) \in \Pi_3$, we have

(I) Under Assumptions A-B and Conditions (C2) or (C2*) in [5], if $p^{-1} \text{tr} \Sigma_d$, $p^{-1} \text{tr} \Sigma_w$ and $p^{-1} \text{tr} (\Sigma_w^2)$ converge and

$$
\max_{1 \leq i_1 \leq i_2 \leq p} \frac{(\sigma_{i_1 i_2} - \sigma_{i_1 i_2})^2}{\theta_{i_1 i_2} n_1 + \theta_{i_1 i_2} n_2} \geq 2s(N_1, N_2, p) + 8 \log p,
$$

$$
P(T_{drx_1} > t_\alpha) \to 1;
$$

(II) Under Conditions (C2) or (C2*) in [5],

$$
P(T_{drx_2} > t_{\alpha/2}) \to 1.
$$

It can be seen from Proposition 3.3 that the power functions of the tests based on $T_{drx_1}$ and $T_{drx_2}$ will tend to 1 if some entries of $\Sigma_d$ are large enough.

4. Simulation studies

We perform extensive simulation studies to examine the finite-sample performance of the proposed new tests. The observations are drawn from $x_{ki} = \Sigma_k^{1/2} w_{ki}$, where $\{w_{ki}, k = 1, 2; l = 1, \ldots, p; i = 1, \ldots, N_k\}$ are independent and identically distributed (i.i.d.) from the Gaussian population $N(0, 1)$ or the Gamma population $\text{Gamma}(4, 2) - 2$. We evaluate our proposed tests under four different scenarios for $\Sigma_1$ and $\Sigma_2$.

- **Scenario 1.**
  The population covariance matrix is $\Sigma_k = \Gamma_k \Gamma_k^T$, where $\Gamma_k = I_p + \theta_k (u_{i_1 i_2})_{i_1, i_2=1}^p$ for $k = 1, 2$, and $\{u_{i_1 i_2}, i_1, i_2 = 1, \ldots, p\}$ are i.i.d. from Unif
The empirical size with \( \theta_1 = \theta_2 = 0 \) and the empirical power with \( (\theta_1, \theta_2) = (0, 1) \), which leads to a dense alternative. This data-generating model was used in [21].

- **Scenario 2.**
  We take \( \Sigma_k = (\rho_k |l_1 - l_2|)^p \)\(_{l_1,l_2=1} \), where \( |\rho_k| < 1 \) for \( k = 1, 2 \). When \( \rho_1 = \rho_2, \Sigma_1 = \Sigma_2 \), \( \Sigma_1 - \Sigma_2 \) has relatively dense signals. We evaluate the empirical size with \( \rho_1 = \rho_2 = 0.5 \) and the empirical power with \( (\rho_1, \rho_2) = (0.5, 0.55) \).

- **Scenario 3.**
  Denote \( C = ((0.1|l_1 - l_2| + 0.2|l_1 - l_2|)/2)^p \)\(_{l_1,l_2=1} \). Let \( U \) be a \( p \times p \) symmetric matrix having only four nonzero entries from \( \text{Unif}(0.4, 0.6) \) in its upper triangle. The locations of these four nonzero entries are selected randomly from the upper triangle of \( U \). For \( k = 1, 2 \), we take the population covariance matrix as

\[
\Sigma_k = C + \delta_0 I + \theta_k U,
\]

where \( \delta_0 = \min \{ \lambda_{\text{min}}(C + U), \lambda_{\text{min}}(C) \} \} + 0.05 \). As \( \theta_1 = \theta_2 = 0, \Sigma_1 = \Sigma_2 \). As \( (\theta_1, \theta_2) = (0, 1), \Sigma_1 - \Sigma_2 \) is extremely sparse. We evaluate the empirical size with \( \theta_1 = \theta_2 = 0 \) and the empirical power with \( (\theta_1, \theta_2) = (0, 1) \). This scenario was considered in [5].

- **Scenario 4.**
  Denote \( C = ((0.5 - |l_1 - l_2|)/10)I(|l_1 - l_2| \le 4)^p \)\(_{l_1,l_2=1} + 0.5I_p \). Let \( U \) be a \( p \times p \) symmetric matrix having \( [p^2/4] \) entries from \( \text{Unif}(0, 0.04) \) in its upper triangle and \( E \) be a \( p \times p \) symmetric matrix having only one entry \( \log p/10 \) in its upper triangle, where \( [x] \) denotes the smallest integer not less than \( x \). The locations of the nonzero entries in both \( U \) and \( E \) are selected randomly from their upper triangle. For \( k = 1, 2 \), the population covariance matrix is taken as

\[
\Sigma_k = C + \delta_0 I_p + \theta_k (U + E),
\]

where \( \delta_0 = \min \{ \lambda_{\text{min}}(C + U + E), \lambda_{\text{min}}(C) \} \} + 0.05 \). When \( \theta_1 = \theta_2 = 0, \Sigma_1 = \Sigma_2 \). When \( (\theta_1, \theta_2) = (0, 1), \Sigma_1 - \Sigma_2 \) has a mixture of dense and sparse signals. We evaluate the empirical size with \( \theta_1 = \theta_2 = 0 \) and the empirical power with \( (\theta_1, \theta_2) = (0, 1) \).

The above four scenarios all satisfy the conditions listed in Theorem 2.3 with the justifications provided in Appendix. We compare our new tests with two existing tests LC [11] and CLX [5]. The nominal significant level for all the tests is set at \( \alpha = 0.05 \). Based on 10,000 replications under each scenario, the empirical sizes for the Gaussian and Gamma populations are reported in Tables 1-4, and the empirical powers for each of the considered methods are exhibited Figures 1-2. For reference, we also provide detailed numerical values for power comparison in Tables A.1-A.4 of Appendix.

It is observed that all the tests can maintain the nominal level for both Gaussian and Gamma populations. For power comparisons, relative performance of the seven tests depend on the different scenarios. In Scenarios 1-2, the test CLX
fails to detect the dense but small disturbances and yields low empirical testing powers. The test based on $T_d$ performs better than the test based on $T_r$ in Scenario 1; while the test based on $T_r$ is more powerful than the test based on $T_d$ in Scenario 2. Since the proposed three tests based on $T_{dr}$, $T_{drx_1}$ and $T_{drx_2}$ all have the advantages of the tests based on $T_d$ and $T_r$, these three tests can maintain competitive testing powers in Scenarios 1-2. Specifically, consistent with Remark 2.3, $T_{drx_1}$ delivers higher powers than $T_{drx_2}$ in these two scenarios. Scenario 3 examines the performance of the considered tests under a sparse alternative, as expected, the tests based on $T_d$, $T_r$ and $T_{dr}$ fail to detect the sparse signals and have low powers. By contrast, the tests based on $T_{drx_1}$ and $T_{drx_2}$ are adaptive and thus yield high powers. Scenario 4 is a hybrid alternative that has a mixture of dense and sparse signals, in this case, the proposed tests based on $T_{drx_1}$ and $T_{drx_2}$ have better performance than the other methods in
Comparison of empirical powers (in percentages) of the proposed tests with existing tests under the considered four scenarios when the observations are sampled from a Gamma population. 

$(p, N_1, N_2)$ is varied from $(40,80,120)$ to $(320,480,480)$, and x-axis takes values corresponding to the first column in Tables 1-4.

In terms of empirical powers, sparse differences exist in last two scenarios, the test based on $T_{\text{drx}_2}$ is more powerful than that using $T_{\text{drx}_1}$. In summary, the simulation studies show that the proposed three tests can effectively borrow strengths from each of the individual statistics in signal detection, and the nominal testing level is well maintained under all methods.

Appendix

The appendix includes three sections: Section A.1 gives Conditions (C1), (C2) (or (C2*)) and (C3) of [5], Section A.2 justifies the conditions of the four scenarios considered in the simulation studies, and Section A.3 provides the proofs of some theorems and propositions.
Table 1
Comparison of empirical sizes (in percentages) of the proposed tests with the tests LC and CLX under Scenario 1 when the observations are from the Gaussian and Gamma population.

| No. | p   | (N1, N2) | T_{11} | T_{12} | T_{1r} | T_{dr} | LC   | T_{drx} x | CLX | T_{dx} x |
|-----|-----|----------|--------|--------|--------|--------|------|------------|-----|----------|
| 1   | 40  | (80, 120)| 5.42   | 6.06   | 6.11   | 5.84   | 5.03 | 6.54       | 5.45 |          |
| 2   | 40  | (120, 120)| 5.09  | 6.09   | 6.23   | 5.53   | 4.85 | 6.69       | 6.01 |          |
| 3   | 40  | (120, 160)| 5.33  | 5.52   | 5.51   | 5.49   | 4.88 | 5.99       | 5.24 |          |
| 4   | 80  | (120, 160)| 5.13  | 6.35   | 6.05   | 5.32   | 4.81 | 6.55       | 5.74 |          |
| 5   | 80  | (160, 240)| 4.94  | 5.42   | 5.29   | 5.42   | 5.21 | 5.63       | 5.2  |          |
| 6   | 80  | (240, 240)| 5.01  | 5.52   | 5.59   | 5.31   | 4.69 | 5.77       | 5.21 |          |
| 7   | 160 | (240, 240)| 5.26  | 6.07   | 5.98   | 5.15   | 4.08 | 6.26       | 5.29 |          |
| 8   | 160 | (240, 240)| 5.28  | 6.48   | 6.27   | 5     | 4.65 | 6.62       | 5.51 |          |
| 9   | 160 | (240,320)| 5.11  | 5.38   | 5.51   | 5.49   | 4.88 | 5.99       | 5.24 |          |
| 10  | 320 | (240,480)| 5.12  | 5.43   | 5.46   | 5.28   | 4.38 | 5.6        | 4.9  |          |
| 11  | 320 | (320,480)| 4.61  | 5.37   | 5.33   | 4.76   | 4.71 | 5.48       | 4.97 |          |
| 12  | 320 | (480,480)| 4.79  | 5.59   | 5.5    | 4.72   | 4.62 | 5.59       | 4.92 |          |

A.1. Review of Conditions (C1), (C2) (or (C2*)) and (C3) of [5]

Before giving the Conditions (C1), (C2) (or (C2*)) and (C3) of [5], we introduce some basic notations. For two sequences of real numbers \( \{a_n\} \) and \( \{b_n\} \), write \( a_n \asymp b_n \) if there exist constants \( C > c > 0 \) such that \( c|b_n| \leq |a_n| \leq C|b_n| \) for all sufficiently large \( n \). Let \( N = \max\{N_1, N_2\} \).

(C1). Assume that there is a positive constant \( \alpha_0 \) and a subset \( \Upsilon \subset \{1, 2, \ldots, p\} \) with \( \text{Card}(\Upsilon) = o(p) \) such that \( \max_{1 \leq l_1 \leq l_2 \leq p, l_2 \not\in \Upsilon} s_{l_2}(\alpha_0) = o(p^\gamma) \) for all \( \gamma > 0 \), where \( s_{l_2} \) is the cardinality of the set of indices \( l_1 \) \((l_1 \neq l_2)\) that are highly correlated with variable \( l_2 \) in population 1 or 2, as given by

\[
s_{l_2} = s_{l_2}(\alpha_0) := \text{card}\{l_1 : |\rho_{1 l_1 l_2}| \geq (\log p)^{-1-\alpha_0} \text{ or } |\rho_{2 l_1 l_2}| \geq (\log p)^{-1-\alpha_0}\}.
\]
Table 2
Comparison of empirical sizes (in percentages) of the proposed tests with the tests LC and CLX under Scenario 2 when the observations are from the Gaussian and Gamma population.

| No. | p   | \((N_1,N_2)\) | \(T_d\) | \(T_r\) | \(T_{dr}\) | LC  | CLX | \(T_{drx}\) | \(T_{drx2}\) |
|-----|-----|---------------|--------|--------|---------|-----|-----|------------|-------------|
| 1   | 40  | (80,120)      | 5.75   | 6.11   | 6.53    | 6.59| 5   | 7.08       | 6.23        |
| 2   | 40  | (120,120)     | 6.12   | 6.23   | 6.74    | 6.48| 4.45| 7.2        | 5.92        |
| 3   | 40  | (120,160)     | 5.85   | 6.13   | 6.39    | 6.5 | 4.63| 6.8        | 5.87        |
| 4   | 80  | (120,160)     | 5.31   | 6.01   | 5.97    | 5.65| 4.7 | 5.93       | 5.1         |
| 5   | 80  | (240,240)     | 5.37   | 5.64   | 5.82    | 5.79| 4.83| 7.2        | 5.92        |
| 6   | 80  | (240,320)     | 4.84   | 5.35   | 5.57    | 5.65| 4.7 | 6.14       | 5.26        |
| 7   | 160 | (160,240)     | 5.17   | 5.56   | 5.44    | 5.13| 4.6 | 6.4        | 5.29        |
| 8   | 160 | (240,240)     | 4.85   | 5.46   | 5.98    | 5.98| 4.45| 8.19       | 6.49        |
| 9   | 160 | (240,320)     | 4.78   | 5.35   | 5.57    | 5.65| 4.5 | 8.12       | 6.12        |
| 10  | 320 | (240,480)     | 5.19   | 5.72   | 5.41    | 5.32| 4.75| 8.19       | 6.49        |
| 11  | 320 | (320,480)     | 5.09   | 5.72   | 5.41    | 5.32| 4.75| 8.19       | 6.49        |
| 12  | 320 | (480,480)     | 5.09   | 5.72   | 5.41    | 5.32| 4.75| 8.19       | 6.49        |

In addition, there is a sequence of numbers \(\Lambda_{p,r}\) such that \(\text{card}(\Lambda(\Lambda(r))) \leq \Lambda_{p,r} = o(p)\) for some constant \(0 < r < 1\), where

\[
\Lambda(r) = \{1 \leq l_1 \leq p : |\rho_{l_1,l_2}| \geq r \text{ or } |\rho_{l_2,l_1}| \geq r \text{ for some } l_2 \neq l_1\},
\]

that is, \(\Lambda(r)\) is the set of highly correlated variable incidences.

(C2). Assume that \(\log p = o(N^{1/5})\) and \(N_1 \approx N_2\). There exist constants \(\eta > 0\) and \(K > 0\) satisfying the following moment conditions

\[
E \exp(\eta(x_{1l1} - \mu_{1l})^2/\sigma_{1ll}) \leq K, \\
E \exp(\eta(x_{2l1} - \mu_{2l})^2/\sigma_{2ll}) \leq K, \text{ for } l = 1, \cdots, p.
\]

Additionally, for some constants \(\tau_1 > 0\) and \(\tau_2 > 0\),

\[
\min_{1 \leq l_1 \leq l_2 \leq p} \frac{\theta_{1l_1l_2}}{\sigma_{1l_1l_1}\sigma_{1l_2l_2}} \geq \tau_1 \quad \text{and} \quad \min_{1 \leq l_1 \leq l_2 \leq p} \frac{\theta_{2l_1l_2}}{\sigma_{2l_1l_1}\sigma_{2l_2l_2}} \geq \tau_2. \quad (A.1)
\]
Two-sample tests for high dimensional covariance matrices

Comparison of empirical sizes (in percentages) of the proposed tests with the tests LC and CLX under Scenario 3 when the observations are from the Gaussian population.

| No. | p     | \((N_1, N_2)\) | \(T_{11}\) | \(T_{12}\) | \(T_{21}\) | \(T_{22}\) | LC  | CLX | \(T_{d1x1}\) | \(T_{d2x2}\) |
|-----|-------|----------------|----------|----------|----------|----------|-----|-----|--------------|--------------|
| 1   | 40    | \((80,120)\)  | 4.96     | 5.86     | 5.95     | 5.32     | 5.39| 6.4 | 5.89         |              |
| 2   | 40    | \((120,120)\) | 5.44     | 6.18     | 6.18     | 5.46     | 4.83| 6.57| 5.6          |              |
| 3   | 40    | \((120,160)\) | 5.52     | 6.03     | 6.17     | 5.8      | 4.42| 6.56| 5.58         |              |
| 4   | 80    | \((120,160)\) | 5.29     | 6.23     | 6.09     | 5.3      | 4.86| 6.44| 5.77         |              |
| 5   | 80    | \((160,240)\) | 5.3      | 5.76     | 5.55     | 4.96     | 4.79| 5.84| 4.95         |              |
| 6   | 80    | \((240,240)\) | 5.31     | 5.51     | 5.5      | 5.29     | 5.24| 5.89| 5.27         |              |
| 7   | 160   | \((160,240)\) | 4.71     | 6.17     | 5.94     | 5.12     | 4.9 | 6.18| 5.74         |              |
| 8   | 160   | \((240,240)\) | 4.87     | 6.02     | 6.17     | 4.9      | 4.41| 6.5 | 5.43         |              |
| 9   | 160   | \((240,320)\) | 5.11     | 5.67     | 5.65     | 4.84     | 4.53| 5.89| 4.89         |              |
| 10  | 320   | \((240,480)\) | 5.03     | 5.73     | 5.57     | 5.26     | 4.48| 5.74| 4.9          |              |
| 11  | 320   | \((320,480)\) | 4.99     | 5.45     | 5.62     | 5.49     | 4.36| 5.73| 5.01         |              |
| 12  | 320   | \((480,480)\) | 5.05     | 5.58     | 5.48     | 4.93     | 4.84| 5.59| 5.31         |              |

(C2*). Suppose that condition (A.1) holds, and that \(N_1 \asymp N_2\) and \(p \leq c_1 N^{\gamma_0}\) for some \(\gamma_0, c_1 > 0\). Furthermore, the following moment conditions hold

\[
\begin{align*}
E(|x_{1l1} - \mu_{1l1}|/\sigma_{1l1}^{1/2} |^{4\gamma_0+4+\epsilon} & \leq K, \\
E(|x_{2l1} - \mu_{2l1}|/\sigma_{2l1}^{1/2} |^{4\gamma_0+4+\epsilon} & \leq K, \text{ for } l = 1, \ldots, p,
\end{align*}
\]

for some constants \(\epsilon > 0\) and \(K > 0\).

(C3). For any \(l_1, l_2, l_3, l_4 \in \{1, 2, \ldots, p\}\), and for some constants \(\kappa_1, \kappa_2 \geq \frac{1}{3}\),

\[
\begin{align*}
E(x_{1l11} - \mu_{1l1})(x_{1l21} - \mu_{1l2})(x_{1l31} - \mu_{1l3})(x_{1l41} - \mu_{1l4}) &= \kappa_1 (\sigma_{1l1 l_2}^2 \sigma_{1l3 l_4}^2 + \sigma_{1l1 l_3} \sigma_{1l2 l_4} + \sigma_{1l1 l_4} \sigma_{1l2 l_3}), \\
E(x_{2l11} - \mu_{2l1})(x_{2l21} - \mu_{2l2})(x_{2l31} - \mu_{2l3})(x_{2l41} - \mu_{2l4}) &= \kappa_2 (\sigma_{2l1 l_2}^2 \sigma_{2l3 l_4}^2 + \sigma_{2l1 l_3} \sigma_{2l2 l_4} + \sigma_{2l1 l_4} \sigma_{2l2 l_3}).
\end{align*}
\]
Comparison of empirical sizes (in percentages) of the proposed tests with the tests LC and CLX under Scenario 4 when the observations are from the Gaussian and Gamma population.

| No. | p   | (N₁, N₂) | Tₐ   | Tᵣ   | Tᵣx₁ | Tᵣx₂ |
|-----|-----|----------|------|------|------|------|
|     |     |          |      |      |      |      |
|     |     | Gaussian population |      |      |      |      |
| 1   | 40  | (80,120) | 6.17 | 5.73 | 6.7  | 6.53 |
| 2   | 40  | (120,120)| 5.65 | 6.26 | 6.46 | 6.19 |
| 3   | 40  | (120,160)| 5.48 | 5.77 | 5.95 | 6.2  |
| 4   | 80  | (120,160)| 5.76 | 6.18 | 6.23 | 5.54 |
| 5   | 80  | (160,240)| 5.97 | 5.77 | 6.13 | 6   |
| 6   | 80  | (240,240)| 5.36 | 5.88 | 5.93 | 5.56 |
| 7   | 160 | (160,240)| 5.14 | 6.09 | 5.83 | 5.35 |
| 8   | 160 | (240,240)| 5.16 | 6.39 | 6.33 | 5.7 |
| 9   | 160 | (240,320)| 5.12 | 5.47 | 5.48 | 5.22 |
| 10  | 320 | (240,480)| 4.88 | 5.05 | 4.94 | 4.3  |
| 11  | 320 | (320,480)| 5.24 | 5.46 | 5.53 | 5.48 |
| 12  | 320 | (480,480)| 5.44 | 5.82 | 5.86 | 5.28 |

| No. | p   | (N₁, N₂) | Tₐ   | Tᵣ   | Tᵣx₁ | Tᵣx₂ |
|-----|-----|----------|------|------|------|------|
|     |     | Gamma population |      |      |      |      |
| 1   | 40  | (80,120) | 7.27 | 6.82 | 7.91 | 7.03 |
| 2   | 40  | (120,120)| 6.83 | 7.22 | 7.84 | 6.74 |
| 3   | 40  | (120,160)| 7.36 | 6.82 | 7.95 | 7.46 |
| 4   | 80  | (120,160)| 6.02 | 6.46 | 6.57 | 6.33 |
| 5   | 80  | (160,240)| 5.94 | 5.94 | 6.53 | 6.02 |
| 6   | 80  | (240,240)| 5.96 | 5.96 | 6.43 | 5.99 |
| 7   | 160 | (160,240)| 5.29 | 6.36 | 6.2 | 5.74 |
| 8   | 160 | (240,240)| 5.7 | 6.94 | 6.67 | 5.62 |
| 9   | 160 | (240,320)| 5.49 | 5.76 | 5.85 | 5.41 |
| 10  | 320 | (240,480)| 5.65 | 5.85 | 5.97 | 5.49 |
| 11  | 320 | (320,480)| 5.08 | 5.62 | 5.66 | 5.01 |
| 12  | 320 | (480,480)| 5.13 | 6.03 | 5.69 | 5.45 |

A.2. Verification of the considered simulation scenarios

In what follows, we verify our simulation settings in the simulation part satisfy the three conditions as shown in Appendix A.1. For easiness of presentation, some notations are given:

- Let $\Sigma_{1k} = (\sigma_{1kl}l_1, l_2=1,\ldots,p$ and $\Sigma_{2k} = (\sigma_{2kl}l_1, l_2=1,\ldots,p$ be the population covariance matrices in Scenarios $k = 1, 2, 3, 4$;
- Let $R_{1k} = (\rho_{1kl}l_1, l_2=1,\ldots,p$ and $R_{2k} = (\rho_{2kl}l_1, l_2=1,\ldots,p$ be the population correlation matrices in Scenarios $k = 1, 2, 3, 4$;
- Let $r_{1kl}$ and $r_{2kl}$ denote the $l$th column of the matrices $\Sigma_{1k}^{1/2}$ and $\Sigma_{2k}^{1/2}$, respectively;
- Let $w = (w_1, \ldots, w_p)$ denote the vector with elements being i.i.d. from $N(0, 1)$ or $\text{Gamma}(4, 2) - 2$. 
In our simulation studies, under the null hypothesis \( \Sigma_{1k} = \Sigma_{2k} \), the data are i.i.d. from \( \Sigma_{1k}^{1/2} w \).

**1.** All \( \Sigma_{1k} \)'s in Scenarios 1-4 satisfy Condition (C1) under the null hypothesis \( H_0 \).

For a fixed constant \( \alpha_0 > 0 \), define
\[
s_{klz} = s_{klz}(\alpha_0) := \text{card}\{l_1 : |\rho_{1kl_1l_2}| \geq (\log p)^{-1-\alpha_0} \text{ or } |\rho_{2kl_1l_2}| \geq (\log p)^{-1-\alpha_0}\}
\]
and
\[
s'_{klz} = s'_{klz}(\alpha_0) := \text{card}\{l_1 : |\sigma_{1kl_1l_2}| \geq (\log p)^{-1-\alpha_0} \text{ or } |\sigma_{2kl_1l_2}| \geq (\log p)^{-1-\alpha_0}\}.
\]
For \( 0 < r < 1 \), define the sets
\[
\Lambda_k(r) = \{1 \leq l_1 \leq p : |\rho_{1kl_1l_2}| \geq r \text{ or } |\rho_{2kl_1l_2}| \geq r \text{ for some } l_2 \neq l_1\}
\]
and
\[
\Lambda'_k(r) = \{1 \leq l_1 \leq p : |\sigma_{1kl_1l_2}| \geq r \text{ or } |\sigma_{2kl_1l_2}| \geq r \text{ for some } l_2 \neq l_1\}.
\]

Since all diagonal elements of \( \Sigma_{1k} \) are greater than or equal to 1, we have \( \rho_{1kl_1l_2} \leq \sigma_{1kl_1l_2} \), which indicates that \( s_{klz} \leq s'_{klz} \) and \( \text{card}(\Lambda_k(r)) \leq \text{card}(\Lambda'_k(r)) \). Therefore, we only need to show that \( s'_{klz} \) and \( \text{card}(\Lambda'_k(r)) \) satisfy Condition (C1) for \( k = 1, 2, 3, 4 \).

- In Scenario 1, when \( l_1 \neq l_2 \), \( \sigma_{11l_1l_2} = 0 \). For any fixed constant \( \alpha_0 > 0 \) and \( 0 < r < 1 \), we have \( s'_{1l2} = 1 \) and \( \text{card}(\Lambda'_1(r)) = 0 \). Thus, Condition (C1) is satisfied.
- In Scenario 2, when \( l_1 \neq l_2 \), \( \sigma_{12l_1l_2} = 0.5 |l_1 - l_2| \). When we take \( r = 0.5 \), we have \( \text{card}(\Lambda'_2(r)) = 0 \), indicating Condition (C1) is satisfied. For any \( \gamma > 0 \), any fixed constant \( \alpha_0 > 0 \) and sufficiently large \( p \), we have \( 0.5 p^{\gamma/2} \leq (\log p)^{-1-\alpha_0} \), which indicates that \( \max_{1 \leq l_2 \leq p} s'_{2l2} \leq 2p^{\gamma/2} + 1 = o(p^{\gamma}) \), that is, \( s'_{2l2} \) satisfies Condition (C1).
- In Scenario 3, when \( l_1 \neq l_2 \), \( \sigma_{13l_1l_2} = (0.1 |l_1 - l_2| + 0.2 |l_1 - l_2|)/2 \). Mimicking the discussion in Scenario 2, we can show that \( s'_{3l2} \) and \( \text{card}(\Lambda'_3(r)) \) satisfy Condition (C1).
- In Scenario 4, when \( l_1 \neq l_2 \), \( \sigma_{14l_1l_2} = (0.5 - |l_1 - l_2|/10) I(||l_1 - l_2|| \leq 4) \), where \( I(\cdot) \) is the indicator function. When \( r = 0.4 \), we have \( \text{card}(\Lambda'_4(r)) = 0 \), which obviously satisfies Condition (C1). For any fixed constant \( \alpha_0 > 0 \), we have \( \max_{1 \leq l_2 \leq p} s'_{4l2} \leq 9 \), which also satisfies Condition (C1).

**2.** The data of group 1 are i.i.d. from \( \Sigma_{1k}^{1/2} w \), and those of group 2 are i.i.d. from \( \Sigma_{2k}^{1/2} w \), \( k = 1, 2, 3, 4 \). We show that \( \Sigma_{1k}^{1/2} w \) and \( \Sigma_{2k}^{1/2} w \) satisfy Condition (C2*), regardless \( \{w_1, \ldots, w_p\} \) are i.i.d. from \( N(0, 1) \) or \( \text{Gamma}(4, 2) \). When \( \gamma_1 = 1 \), \( c_1 = y_1 \) and \( \epsilon = 2 \) in Condition (C2*) for \( k = 1, 2, 3, 4 \), we have
\[
E|\mathbf{r}_{ikl}^T w / \sigma_{1kl}^{1/2}|^{4 + \epsilon} = E|\mathbf{r}_{ikl}^T w / \sigma_{1kl}^{1/2}|^{10} = E(\mathbf{w}^T \mathbf{r}_{ikl} \mathbf{r}_{ikl}^T \mathbf{w}^T) / \sigma_{1kl}^{3}. \]
From Lemma 2.7 in [2] and \( r_{1kl}^T r_{1kl} = \sigma_{1kl}^2 \), we have
\[
E(w^T r_{1kl} r_{1kl}^T w)^5 \\
\leq 16E|w^T r_{1kl} r_{1kl}^T w - \text{tr}(r_{1kl} r_{1kl}^T)|^5 + 16|\text{tr}(r_{1kl} r_{1kl}^T)|^5 \\
\leq 16K_5((E|w_1|^4 \text{tr}(r_{1kl} r_{1kl}^T r_{1kl} r_{1kl}^T))^5/2 + E|w_1|^{10} |\text{tr}(r_{1kl} r_{1kl}^T)|^5) \\
= 16K_5((E|w_1|^4)^{5/2} + E|w_1|^{10}) (r_{1kl} r_{1kl}^T)^5 + 16(r_{1kl} r_{1kl}^T)^5 \\
= 16K_5((E|w_1|^4)^{5/2} + E|w_1|^{10}) \sigma_{1kl}^2 + 16\sigma_{1kl}^5,
\]
where \( K_5 \) is a constant. No matter \( w_1 \) is from \( N(0, 1) \) or \( \text{Gamma}(4, 2) - 2 \), \( E|w_1|^4 \) and \( E|w_1|^{10} \) are bounded. Therefore, it can be shown that
\[
E|w_{1kl}^T w|^{2^5/4} \leq K \text{ for all } l.
\]
Similarly, for \( k = 1, 2, 3, 4 \), we can also prove
\[
E|w_{2kl}^T w|^{2^5/4} \leq K \text{ for all } l.
\]
Let \( \theta_{1kl, l2} = \text{Var}(r_{1kl}^T w r_{1kl}^T w) \), from the equation (A.76), we have
\[
\text{Var}(r_{1kl}^T w r_{1kl}^T w) = E(w^T r_{1kl} r_{1kl}^T w - r_{1kl}^T r_{1kl})^2 \\
= \text{tr}(r_{1kl} r_{1kl}^T r_{1kl}^T r_{1kl}^T) + \text{tr}(r_{1kl} r_{1kl}^T r_{1kl}^T r_{1kl}^T) + \beta_w \sum_{i=1}^p (r_{1kl}^T r_{1kl}^T)_{ii}^2 \\
= \sigma_{1kl, l2}^2 + \sigma_{1kl, l2}^2 + \beta_w \sum_{i=1}^p (r_{1kl}^T r_{1kl}^T)_{ii}^2,
\]
where \((r_{1kl}^T r_{1kl})_{ii}\) denotes the \(i\)th diagonal element of matrix \( r_{1kl}^T r_{1kl}^T\) and \( \beta_w = Ew_1^4 - 3 \geq 0 \). Therefore, we have
\[
\min_{1 \leq i_1 \leq i_2 \leq p} \frac{\theta_{1kl, l2}}{\sigma_{1kl, l2} \sigma_{1kl, l2}} \geq 1.
\]
Similarly, denoting \( \theta_{2kl, l2} = \text{Var}(r_{2kl}^T w r_{2kl}^T w) \), we also have
\[
\min_{1 \leq i_1 \leq i_2 \leq p} \frac{\theta_{2kl, l2}}{\sigma_{2kl, l2} \sigma_{2kl, l2}} \geq 1.
\]
\( \textbf{(3)} \) When \( \{w_1, \ldots, w_p\} \) are i.i.d. from \( N(0, 1) \), we will prove that \( \Sigma_{1k}^{1/2} w \) satisfies Condition (C3). When \( \{w_1, \ldots, w_p\} \) are i.i.d. from \( \text{Gamma}(4, 2) - 2 \), we show that \( \Sigma_{1k}^{1/2} w \) does not satisfy Condition (C3).
For any \(l_1, l_2, l_3, l_4 \in \{1, 2, \cdots, p\}\), from the equation (A.76), we have

\[
E(r_{kl1}^T w r_{kl2}^T w r_{kl3}^T w r_{kl4}^T w) = E(w^T r_{kl1} r_{kl2}^T w w^T r_{kl3}^T r_{kl4}^T w) = \text{tr}(r_{kl1}^T r_{kl2}^T) \text{tr}(r_{kl3}^T r_{kl4}^T) + \beta w \sum_{i=1}^{p} (r_{kl1} r_{kl2}^T)_{ii} (r_{kl3} r_{kl4}^T)_{ii} + \sum_{i=1}^{p} (r_{kl1} r_{kl2}^T)_{ii} (r_{kl3} r_{kl4}^T)_{ii}.
\]

- When \(\{w_1, \cdots, w_p\}\) are i.i.d. from \(N(0, 1)\), we have \(\beta w = 0\). That is, Condition (C3) is satisfied;
- When \(\{w_1, \cdots, w_p\}\) are i.i.d. from \(\text{Gamma}(4, 2) - 2\), we have \(\beta w = 1.5\). That is, Condition (C3) is not satisfied.

However, by Proposition 1 in [5], under \(H_0\) and Condition (C2) or (C2*), we have

\[
P(T_x - 4 \log p + \log \log p \geq c_\alpha) \leq -\log(1 - \alpha) + o(1),
\]

where \(c_\alpha\) is the \(1 - \alpha\) quantile of the Type I extreme value distribution. It indicates that even without Conditions (C1) and (C3), the test based on \(T_x\) can effectively control the Type I error. This is the reason that when \(\{w_1, \cdots, w_p\}\) are i.i.d. from \(\text{Gamma}(4, 2) - 2\) with Condition (C3) being violated, the empirical test sizes are still satisfactory.

A.3. Proofs of some theorems, propositions and lemmas

This section is divided into three subsections. Subsection A.3.1 gives some preparations. Subsection A.3.2 contains the proofs of theorems and propositions. Lemmas used in this paper are placed in Subsection A.3.3.

A.3.1. Preparatory works for proving Theorems 2.1, 3.1 and 3.2

The preparatory works include two steps. The skeletons of the two steps are as follows:

**Skeleton of Step 1.** Define the linear combination \(T\) of \(T_d - \mu_0\) and \(T_r\) as

\[
T = \omega_1 (T_d - \mu_0) + \omega_2 T_r,
\]

where \(\omega_1\) and \(\omega_2\) are two arbitrary constants. In this step, we will obtain

\[
T = T_A + T_B + \omega_1 \mu_1 + \omega_2 \mu_2 + o_p(1),
\]

where \(T_A\) and \(T_B\) are given in (A.14) and (A.15) and \(\mu_1\) and \(\mu_2\) are given in (A.12) and (A.13).
Skeleton of Step 2. Let $E_0(\cdot)$ denote expectation, $E_j(\cdot)$ denote the conditional expectation with respect to the $\sigma$-field generated by $w_{21}, \ldots, w_{2j}$ and $E_{2,i}(\cdot)$ denote the conditional expectation with respect to the $\sigma$-field generated by $w_{21}, \ldots, w_{2N_2}, w_{11}, \ldots, w_{11}$. Then we have

$$T_A + T_B = \sum_{j=1}^{N_2} (E_j - E_{j-1})T_A + \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1})T_B,$$

that is, $T_A + T_B$ is a sum of the martingale difference sequence

$$\{(E_j - E_{j-1})T_A, (E_{2,i} - E_{2,i-1})T_B, j = 1, \ldots, N_2, i = 1, \ldots, N_1\}.$$

Therefore, based on the central limit theorem of martingale difference sequences, it suffices to consider

$$\sum_{j=1}^{N_2} E_j - 1[(E_j - E_{j-1})T_A]^2] + \sum_{i=1}^{N_1} E_{2,i-1}[(E_{2,i} - E_{2,i-1})T_B]^2].$$

Routine algebra shows that

$$\sum_{j=1}^{N_2} E_j - 1[(E_j - E_{j-1})T_A]^2] + \sum_{i=1}^{N_1} E_{2,i-1}[(E_{2,i} - E_{2,i-1})T_B]^2]$$

$$= \omega_1^2 \sigma_{11p} + 2\omega_1\omega_2(\sigma_{12p} + \sigma_{22p}) + \omega_2^2(\sigma_{12p}^2 + \sigma_{22p}^2) + o_p(1),$$

where $\sigma_{11p}$, $\sigma_{12p}^2$, $\sigma_{12p}^2$, and $\sigma_{22p}^2$ are given in (A.24), (A.25), (A.26), (A.27) and (A.28), respectively. Under the conditions of Theorems 2.1, 3.1 and 3.2, based on the expressions of $\sigma_{11p}$, $\sigma_{12p}^2$, $\sigma_{12p}^2$, and $\sigma_{22p}^2$, we will prove the Theorems 2.1, 3.1 and 3.2.

Step 1: Letting $\omega_1$ and $\omega_2$ be two arbitrary constants, define

$$T = \omega_1(T_d - \mu_0) + \omega_2T_r$$

$$= \omega_1[\text{tr}S_1^2 - 2\text{tr}(S_1S_2) + \text{tr}S_2^2 - n_1^{-1}\text{tr}^2S_1 - n_2^{-1}\text{tr}^2S_2]$$

$$+ \omega_2[\text{tr}(S_1S_2^{-1})^2 - 2\text{tr}(S_1S_2^{-1}) + p]$$

$$= \omega_1[\text{tr}S_1^2 - 2\text{tr}(S_1S_2) + \text{tr}S_2^2 - n_1^{-1}(\text{tr}^2S_1 - \text{tr}^2\Sigma_1) - n_1^{-1}\text{tr}^2\Sigma_1]$$

$$+ \omega_2[\text{tr}^2S_2 - \text{tr}^2\Sigma_2] - n_2^{-1}\text{tr}^2\Sigma_2]$$

$$+ \omega_2[\text{tr}(S_1S_2^{-1})^2 - 2\text{tr}(S_1S_2^{-1}) + p].$$

Denote the noncentralized sample covariance matrices as

$$B_k = N_k^{-1} \sum_{i=1}^{N_k} \Sigma_k^{1/2}w_{ki}w_{ki}^T\Sigma_k^{1/2}, \quad k = 1, 2.$$

Letting $\gamma_{ki} = (1/\sqrt{N_k})w_{ki}$, we have

$$S_k = B_k - n_k^{-1} \sum_{i,j \neq j} \Sigma_k^{1/2}\gamma_{ki}\gamma_{kj}^T\Sigma_k^{1/2}, \quad k = 1, 2. \quad (A.3)$$
Based on (A.3), we obtain \( \text{tr}S_1 = \text{tr}B_1 - n_1^{-1} \sum_{i \neq j} \gamma_{ij}^T \Sigma_1 \gamma_{ij} \). Under Assumptions A-B, from (A.76), we have \( n_1^{-1} \sum_{i \neq j} \gamma_{ij}^T \Sigma_1 \gamma_{ij} = o_p(1) \) and

\[
E(\text{tr}B_1 - \text{tr} \Sigma_1)^2 = N_1^{-1}[2\text{tr} \Sigma_1^2 + \beta_1 \text{tr}(\Sigma_1 \circ \Sigma_1)],
\]

(A.4)

which yields that \( n_1^{-1}(\text{tr}S_1 - \text{tr} \Sigma_1)^2 = o_p(1) \). Then we have

\[
n_1^{-1}(\text{tr}^2S_1 - \text{tr}^2 \Sigma_1) = 2n_1^{-1} \text{tr} \Sigma_1(\text{tr}S_1 - \text{tr} \Sigma_1) + o_p(1).
\]

Similarly, we also have

\[
n_2^{-1}(\text{tr}^2S_2 - \text{tr}^2 \Sigma_2) = 2n_2^{-1} \text{tr} \Sigma_2(\text{tr}S_2 - \text{tr} \Sigma_2) + o_p(1).
\]

Let

\[
T^* = \omega_1[\text{tr}S_1^2 - 2\text{tr}(S_1S_2) + \text{tr}S_2^2 - 2n_1^{-1} \text{tr} \Sigma_1(\text{tr}S_1 - \text{tr} \Sigma_1)
\]

\[
- n_1^{-1} \text{tr}^2 \Sigma_1 - 2n_2^{-1} \text{tr} \Sigma_2(\text{tr}S_2 - \text{tr} \Sigma_2) - n_2^{-1} \text{tr}^2 \Sigma_2]
\]

\[
+ \omega_2[\text{tr}(S_1S_2^{-1})^2 - 2\text{tr}(S_1S_2^{-1}) + p].
\]

Rewrite

\[
T^* = T^* - E(T^*|S_2) + E(T^*|S_2),
\]

where

\[
E(T^*|S_2) = \omega_1[\text{Etr}S_1^2 - 2\text{tr}(S_2 \Sigma_1) + \text{tr}S_2^2 - n_1^{-1} \text{tr}^2 \Sigma_1
\]

\[
- 2n_2^{-1} \text{tr} \Sigma_2(\text{tr}S_2 - \text{tr} \Sigma_2) - n_2^{-1} \text{tr}^2 \Sigma_2]
\]

\[
+ \omega_2[E(\text{tr}(S_1S_2^{-1}))^2|S_2] - 2E(\text{tr}(S_1S_2^{-1})|S_2) + p]
\]

and

\[
T^* - E(T^*|S_2) = \omega_1[\text{tr}S_1^2 - \text{Etr}S_1^2 - 2(\text{tr}(S_1S_2) - \text{tr}(S_2 \Sigma_1))
\]

\[
- 2n_1^{-1} \text{tr} \Sigma_1(\text{tr}S_1 - \text{tr} \Sigma_1)]
\]

\[
+ \omega_2[\text{tr}(S_1S_2^{-1})^2 - E(\text{tr}(S_1S_2^{-1}))^2|S_2]
\]

\[
- 2(\text{tr}(S_1S_2^{-1}) - E(\text{tr}(S_1S_2^{-1})|S_2))].
\]

From (A.3) and (A.78), we have

\[
\text{Etr}S_1^2 = n_1^{-1} \text{tr}^2 \Sigma_1 + \text{tr} \Sigma_1^2 + n_1^{-1} \text{tr} \Sigma_1^2 + \beta_1 n_1^{-1} \text{tr}(\Sigma_1 \circ \Sigma_1) + o(1)
\]

(A.5)

and

\[
\text{Etr}S_2^2 = n_2^{-1} \text{tr}^2 \Sigma_2 + \text{tr} \Sigma_2^2 + n_2^{-1} \text{tr} \Sigma_2^2 + \beta_2 n_2^{-1} \text{tr}(\Sigma_2 \circ \Sigma_2) + o(1).
\]

(A.6)

When \( \Sigma_1 \) is invertible, we denote \( \Gamma = \Sigma_1^{-1/2} \Sigma_1^{1/2} \) and \( \tilde{S}_2 = \Sigma_1^{-1/2} S_2 \Sigma_1^{-1/2} \).

Under Assumptions A-B, if \( T_p = \Gamma T_p^T \) and the ESD \( H_p \) of \( T_p \) satisfies the Assumptions c-d-f in Subsection A3.3, we obtain that \( \|S_2^1\| \) is almost everywhere
bounded. Therefore, from (A.5) and (A.6), we have

\[
E(T^*|S_2) = \omega_1 [\text{tr}S_2^2 - \text{Etr}S_2^2 - 2(\text{tr}(S_2\Sigma_1) - \text{tr}(\Sigma_2\Sigma_1))] \\
-2n_1^{-1}\text{tr}(\Sigma_2|\text{tr}(\Sigma_2) + \text{Etr}S_2^2) \\
-2\text{tr}(\Sigma_2\Sigma_1) - n_1^{-1}\text{tr}^2\Sigma_1 - n_2^{-1}\text{tr}^2\Sigma_2 \\
+\omega_2[n_1^{-1}\text{tr}^2(S_2^{-1}\Sigma_1) + \text{tr}(S_2^{-1}\Sigma_1)] \\
-2\text{tr}(S_2^{-1}\Sigma_1) + n_1^{-1}\text{tr}(S_2^{-1}\Sigma_1)^2 \\
+\beta_1n_1^{-1}\text{tr}(\Sigma_1^{1/2}S_2^{-1}\Sigma_1^{1/2}\circ\Sigma_1^{1/2}S_2^{-1}\Sigma_1^{1/2}) + p + o_p(1)
\]

\[
= \omega_1 [\text{tr}(\bar{S}_2\Sigma_1)^2 - \text{Etr}(\bar{S}_2\Sigma_1)^2 - 2(\text{tr}(\bar{S}_2\Sigma_1^2) - \text{tr}(\Sigma_2\Sigma_1))] \\
-2n_1^{-1}\text{tr}(\Sigma_2(\text{tr}(\bar{S}_2\Sigma_1) - \text{tr}(\Sigma_2) + \text{tr}(\Sigma_1 - \Sigma_2)^2 + n_1^{-1}\text{tr}(\Sigma_1^2) \\
+n_2^{-1}\text{tr}(\Sigma_1^2 + \beta_1n_1^{-1}\text{tr}(\Sigma_1\circ\Sigma_1) + \beta_2n_2^{-1}\text{tr}(\Sigma_2\circ\Sigma_2))] \\
+\omega_2[n_1^{-1}\text{tr}^2(\bar{S}_2^{-1} - n_1^{-1}(\text{Etr}\bar{S}_2^{-1})^2 + \text{tr}(\bar{S}_2^{-1})^2 - 2(\text{tr}(\bar{S}_2^{-1} - \text{Etr}\bar{S}_2^{-1}) + n_1^{-1}(\text{Etr}\bar{S}_2^{-1})^2 + \text{Etr}(\bar{S}_2^{-1})^2 \\
-2\text{Etr}\bar{S}_2^{-1} + n_1^{-1}\text{tr}(\bar{S}_2^{-1})^2 + \beta_1n_1^{-1}\text{tr}(\bar{S}_2^{-1}\circ\bar{S}_2^{-1}) + p + o_p(1).
\]

Based on the proof of Theorem 2.1 in [20], we know that the limiting distribution of \(\text{tr}\bar{S}_2^{-1} - \text{Etr}\bar{S}_2^{-1}\) is normal and

\[
\text{Etr}(\bar{S}_2^{-1})^i = pc_i + \xi_i + o(1), \quad i = 1, 2, \quad (A.7)
\]

where

\[
c_i = \int f_i(x) dF^{y_{n_2},H_p}(x) \quad (A.8)
\]

and

\[
\xi_i = -\frac{1}{2\pi i} \int_C f_i(z) \frac{y_2 \int m_{y_2}(z)t^2(1 + tm_{y_2}(z))^{-3}dH(t)}{1 - y_2 \int m_{y_2}(z)^2(1 + tm_{y_2}(z))^{-3}dH(t)} dz \\
-\frac{\beta_2}{2\pi i} \int_C f_i(z) \frac{y_2 \int m_{y_2}(z)t^2(1 + tm_{y_2}(z))^{-3}dH(t)}{1 - y_2 \int m_{y_2}(z)^2(1 + tm_{y_2}(z))^{-2}dH(t)} dz, \quad (A.9)
\]

with \(f_1(x) = 1/x, f_2(x) = 1/x^2, F^{y_{n_2},H_p}\) and \(m_{y_2}(z)\) being the corresponding Marchenko-Pastur distribution of index \((y_{n_2}, H_p)\) and Stieltjes transform of the companion \(LSD F^{y_{n_2},H}\), \(H\) being the \(LSD\) of \(H_p\), and \(C\) being closed contours in the complex plane enclosing the support of the \(LSD F^{y_{n_2},H}\). Thus, from (A.7), we obtain

\[
n_1^{-1}\text{tr}^2\bar{S}_2^{-1} - n_1^{-1}(\text{Etr}\bar{S}_2^{-1})^2 = 2y_{n_1}c_1(\text{tr}\bar{S}_2^{-1} - \text{Etr}\bar{S}_2^{-1}) + o_p(1).
\]

Therefore,

\[
E(T^*|S_2) = \omega_1 [\text{tr}(\bar{S}_2\Sigma_1)^2 - \text{Etr}(\bar{S}_2\Sigma_1)^2 - 2(\text{tr}(\bar{S}_2\Sigma_1^2) - \text{tr}(\Sigma_2\Sigma_1))] \\
-2n_1^{-1}\text{tr}(\Sigma_2(\text{tr}(\bar{S}_2\Sigma_1) - \text{tr}(\Sigma_2) + \text{tr}(\Sigma_1 - \Sigma_2)^2 + n_1^{-1}\text{tr}(\Sigma_1^2) \\
+n_2^{-1}\text{tr}(\Sigma_1^2 + \beta_1n_1^{-1}\text{tr}(\Sigma_1\circ\Sigma_1) + \beta_2n_2^{-1}\text{tr}(\Sigma_2\circ\Sigma_2))
\]

\[
+\beta_1n_1^{-1}\text{tr}(\Sigma_1^{1/2}\Sigma_1^{1/2}\circ\Sigma_1^{1/2}\Sigma_1^{1/2}) + p + o_p(1)
\]

\[
= \omega_1 [\text{tr}(\bar{S}_2\Sigma_1)^2 - \text{Etr}(\bar{S}_2\Sigma_1)^2 - 2(\text{tr}(\bar{S}_2\Sigma_1^2) - \text{tr}(\Sigma_2\Sigma_1))] \\
-2n_1^{-1}\text{tr}(\Sigma_2(\text{tr}(\bar{S}_2\Sigma_1) - \text{tr}(\Sigma_2) + \text{tr}(\Sigma_1 - \Sigma_2)^2 + n_1^{-1}\text{tr}(\Sigma_1^2) \\
+n_2^{-1}\text{tr}(\Sigma_1^2 + \beta_1n_1^{-1}\text{tr}(\Sigma_1\circ\Sigma_1) + \beta_2n_2^{-1}\text{tr}(\Sigma_2\circ\Sigma_2))] \\
+\omega_2[n_1^{-1}\text{tr}^2(\bar{S}_2^{-1} - n_1^{-1}(\text{Etr}\bar{S}_2^{-1})^2 + \text{tr}(\bar{S}_2^{-1})^2 - 2(\text{tr}(\bar{S}_2^{-1} - \text{Etr}\bar{S}_2^{-1}) + n_1^{-1}(\text{Etr}\bar{S}_2^{-1})^2 + \text{Etr}(\bar{S}_2^{-1})^2 \\
-2\text{Etr}\bar{S}_2^{-1} + n_1^{-1}\text{tr}(\bar{S}_2^{-1})^2 + \beta_1n_1^{-1}\text{tr}(\bar{S}_2^{-1}\circ\bar{S}_2^{-1}) + p + o_p(1).
\]
Denote \(\tilde{S}_1 = N_1^{-1} \sum_{i=1}^{N_1} (\mathbf{w}_1 - \bar{\mathbf{w}}_1)(\mathbf{w}_{1i} - \bar{\mathbf{w}}_1)^T\), we have

\[
T^* - \mathbb{E}(T^* | \mathbf{S}_2) = \omega_1 [\text{tr} (\tilde{S}_1 \Sigma_1) - \text{Etr}(\tilde{S}_1 \Sigma_1)]^2 - 2(\text{tr}(\tilde{S}_1 \Sigma_2 \Sigma_1) - \text{tr}(\tilde{S}_1 \Sigma_1))
-2n_1^{-1} \text{tr}(\tilde{S}_1 (\text{tr}(\tilde{S}_1 \Sigma_1) - \text{tr}(\Sigma_1)))
+\omega_2 [\text{tr}(\tilde{S}_1 \Sigma_1) - \text{Etr}(\tilde{S}_1 \Sigma_1)]^2 - 2(\text{tr}(\tilde{S}_1 \Sigma_2) - \text{Etr}(\tilde{S}_1 \Sigma_1)) + o_p(1)
\]

Denote \(\tilde{B}_1 = N_1^{-1} \sum_{i=1}^{N_1} \mathbf{w}_{1i} \mathbf{w}_{1i}^T\) and \(\tilde{B}_2 = N_2^{-1} \sum_{i=1}^{N_2} \mathbf{w}_{2i} \mathbf{w}_{2i}^T\). Using the proof of Theorem 2.1 in [20] again, we have

\[
T^* - \mathbb{E}(T^* | \mathbf{S}_2) = \omega_1 [\text{tr} (\tilde{B}_1 \Sigma_1) - \text{Etr}(\tilde{B}_1 \Sigma_1)]^2 - 2(\text{tr}(\tilde{B}_1 \Sigma_2 \Sigma_1) - \text{tr}(\tilde{B}_1 \Sigma_1))
-2n_1^{-1} \text{tr}(\tilde{B}_1 (\text{tr}(\tilde{B}_1 \Sigma_1) - \text{tr}(\Sigma_1)))
+\omega_2 [\text{tr}(\tilde{B}_1 \Sigma_1) - \text{Etr}(\tilde{B}_1 \Sigma_1)]^2 - 2(\text{tr}(\tilde{B}_1 \Sigma_2) - \text{Etr}(\tilde{B}_1 \Sigma_1)) + o_p(1)
\]

According to the discussion following Assumptions a-b-c-d-f in Subsection A.3.3, we know that \(|\tilde{B}_2|\) and \(|\tilde{B}_2^{-1}|\) are almost everywhere bounded. Since

\[
\tilde{S}_2 = \frac{N_2}{n_2} (\tilde{B}_2 - N_2 \Gamma \gamma_2 \Gamma^T)
\]

and

\[
\tilde{S}_2^{-1} = \frac{n_2}{N_2} (\tilde{B}_2^{-1} + \frac{1}{1 - N_2 \gamma_2^T \Gamma^T \tilde{B}_2^{-1} \Gamma \gamma_2} - N_2 \tilde{B}_2^{-1} \Gamma \gamma_2 \Gamma^T \tilde{B}_2^{-1})
\]

based on (A.76), (A.78), Lemma 5.2 and Lemma 5.3 of [20], we have

\[
\text{tr}(\tilde{B}_1 \Sigma_1 \tilde{S}_2 \Sigma_1) - \text{tr}(\tilde{S}_2 \Sigma_1^2) = \text{tr}(\tilde{B}_1 \Sigma_1 \tilde{B}_2 \Sigma_1) - \text{tr}(\tilde{S}_2 \Sigma_1^2) + o_p(1),
\]

\[
\text{tr}(\tilde{B}_1 \tilde{S}_2^{-1}) - \text{E}(\text{tr}(\tilde{B}_1 \tilde{S}_2^{-1}) | \tilde{S}_2) = \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1}) - \text{E}(\text{tr}(\tilde{B}_1 \tilde{B}_2^{-1}) | \tilde{B}_2) + o_p(1)
\]
Therefore, we have
\begin{align*}
\text{tr}(\tilde{B}_1S_2^{-1})^2 - E(\text{tr}(\tilde{B}_1S_2^{-1})^2|S_2) &= \text{tr}(\tilde{B}_1B_2^{-1})^2 - E(\text{tr}(\tilde{B}_1B_2^{-1})^2|B_2) + o_p(1).
\end{align*}

Therefore, we get
\begin{align*}
T^* - E(T^*|S_2) &= \omega_1[\text{tr}(\tilde{B}_1\Sigma_1)^2 - E\text{tr}(\tilde{B}_1\Sigma_1) ]^2 \\
&-2\text{tr}(\tilde{B}_1\Sigma_1\tilde{B}_2\Sigma_1) - \text{tr}(\tilde{B}_2\Sigma_1) ) \\
&-2n_1^{-1}\text{tr}\Sigma_1(\text{tr}(\tilde{B}_1\Sigma_1) - \text{tr}\Sigma_1) ] \\
&+\omega_2[\text{tr}(\tilde{B}_1B_2^{-1}) - E(\text{tr}(\tilde{B}_1B_2^{-1})|B_2) \\
&-2(\text{tr}(\tilde{B}_1B_2^{-1})^2 - E(\text{tr}(\tilde{B}_1B_2^{-1})^2|B_2) ) + o_p(1).
\end{align*}

Denote
\begin{align*}
\mu_1 &= \text{tr}(\Sigma_1 - \Sigma_2) + n_2^{-1}\text{tr}\Sigma_1^2 + n_2^{-1}\text{tr}\Sigma_2^2 \\
&+\beta_1n_1^{-1}\text{tr}(\Sigma_1 \circ \Sigma_1) + \beta_2n_2^{-1}\text{tr}(\Sigma_2 \circ \Sigma_2), \\
\mu_2 &= p(y_{n1}c_1^2 - 2c_1 + c_2) + p + 2y_{n1}c_1\xi_1 - 2\xi_1 + \xi_2 \\
&+n_1^{-1}\text{tr}(\tilde{S}_2^{-1})^2 + \beta_1n_1^{-1}\text{tr}(\tilde{S}_2^{-1} \circ \tilde{S}_2^{-1}), \\
T_A &= \omega_1[\text{tr}(\tilde{B}_2\Sigma_1)^2 - E\text{tr}(\tilde{B}_2\Sigma_1)^2 - 2(\text{tr}(\tilde{B}_2\Sigma_1^2) - \text{tr}(\Sigma_2\Sigma_1)) \\
&-2n_2^{-1}\text{tr}\Sigma_2(\text{tr}(\tilde{B}_2\Sigma_1) - \text{tr}\Sigma_2) ] \\
&+\omega_2[2(y_{n1}c_1 - 1)(\text{tr}\tilde{B}_2^{-1} - E\text{tr}\tilde{B}_2^{-1}) + \text{tr}\tilde{B}_2^{-1} - E\text{tr}\tilde{B}_2^{-1})^2]
\end{align*}

and
\begin{align*}
T_B &= \omega_1[\text{tr}(\tilde{B}_1\Sigma_1)^2 - E\text{tr}(\tilde{B}_1\Sigma_1)^2 - 2(\text{tr}(\tilde{B}_1\Sigma_1\tilde{B}_2\Sigma_1) - \text{tr}(\tilde{B}_2\Sigma_1^2)) \\
&-2n_1^{-1}\text{tr}\Sigma_1(\text{tr}(\tilde{B}_1\Sigma_1) - \text{tr}\Sigma_1) ] \\
&+\omega_2[\text{tr}(\tilde{B}_1B_2^{-1}) - E(\text{tr}(\tilde{B}_1B_2^{-1})^2|B_2) \\
&-2(\text{tr}(\tilde{B}_1B_2^{-1})^2 - E(\text{tr}(\tilde{B}_1B_2^{-1})^2|B_2) )].
\end{align*}

Therefore, we have
\begin{align*}
T &= T_A + T_B + \omega_1\mu_1 + \omega_2\mu_2 + o_p(1).
\end{align*}

Based on Slutsky’s Theorem, we only need to derive the limiting distribution of $T_A + T_B$.

**Step 2:** We have
\begin{align*}
T_A + T_B &= \sum_{j=1}^{N_2} (E_j - E_{j-1})T_A + \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1})T_B,
\end{align*}
where
\[
\sum_{j=1}^{N_2} (E_j - E_{j-1}) T_A
\]
\[
= \omega_1 \left[ \sum_{j=1}^{N_2} (E_j - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1)^2 - 2 \sum_{j=1}^{N_2} (E_j - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1^2) \right]
+ \omega_2 \left[ 2(y_n, c_1 - 1) \sum_{j=1}^{N_2} (E_j - E_{j-1}) \text{tr}(\tilde{B}_2^{-1}) + \sum_{j=1}^{N_2} (E_j - E_{j-1}) \text{tr}(\tilde{B}_2^{-1})^2 \right]
\]

and
\[
\sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1}) T_B
\]
\[
= \omega_1 \left[ \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1)^2 - 2 \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1 \tilde{B}_2 \Sigma_1) \right]
- 2n_1^{-1} \text{tr} \Sigma_1 \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1)
+ \omega_2 \left[ \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1})^2 - 2 \sum_{i=1}^{N_1} (E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1}) \right].
\]

Based on the central limit theorem of martingale difference sequences, it suffices to consider
\[
\sum_{j=1}^{N_2} E_{j-1} \left[ ((E_j - E_{j-1}) T_A)^2 \right] + \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) T_B)^2 \right].
\]

From (A.71)-(A.74) in Lemma A.1, (A.88)-(A.91) in Lemma A.4 and (2.5) in the Theorem 2.1 of [20], we obtain
\[
\sigma_A = \sum_{j=1}^{N_2} E_{j-1} \left[ ((E_j - E_{j-1}) T_A)^2 \right]
= \omega_2^2 \left[ 4(n_2^{-1} \text{tr} \Sigma_2)^2 \eta_{33} + 4 \eta_{44} + \eta_{55} \right]
+ 8(n_2^{-1} \text{tr} \Sigma_2) \eta_{34} - 4(n_2^{-1} \text{tr} \Sigma_2) \eta_{35} - 4 \eta_{45} \\
+ 2 \omega_1 \omega_2 \left[ -4(y_n, c_1 - 1)(n_2^{-1} \text{tr} \Sigma_2) \eta_{13} + 4(y_n, c_1 - 1) \eta_{14} \right]
+ 2(y_n, c_1 - 1) \eta_{15} - 2(n_2^{-1} \text{tr} \Sigma_2) \eta_{23} - 2 \eta_{24} + \eta_{25} \\
+ \omega_2^2 \left[ 4(y_n, c_1 - 1)^2 \eta_{11} + 2 \eta_{22} + 4(y_n, c_1 - 1) \eta_{12} \right].
\]
when \(i, j \in \{1, 2\}\) and \(p \to \infty\),
\[
\eta_{ij} \rightarrow -\frac{1}{2\pi^2} \int_{C_1} \int_{C_2} \frac{f_i(z_1)f_j(z_2)}{(m_{g_1}(z_1) - m_{g_2}(z_2))^2} \, dm_{g_1}(z_1) \, dm_{g_2}(z_2) \tag{A.16}
- \frac{y_2 \beta_2}{4\pi^2} \int_{C_1} \int_{C_2} f_i(z_1)f_j(z_2) \left[ \int \frac{t^2 dH(t)}{(m_{g_2}(z_1)t + 1)^2(m_{g_2}(z_2)t + 1)^2} \right] \, dm_{g_1}(z_1) \, dm_{g_2}(z_2),
\]
where \(f_1(x) = 1/x\) and \(f_2(x) = 1/x^2\), \(C_1\) and \(C_2\) are closed contours in the complex plan enclosing the support of the LSD \(F_{g_1, H}\), and \(C_1\) and \(C_2\) are nonoverlapping,
\[
\eta_{33} = \sum_{j=1}^{N_2} E_{j-1}[(E_j - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1)]^2 = \frac{1}{N_2} [2\text{tr} \Sigma_2^2 + \beta_2 \text{tr}(\Sigma_2 \circ \Sigma_2)],
\]
\[
\eta_{44} = \sum_{j=1}^{N_2} E_{j-1}[(E_j - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1^2)]^2
= \frac{1}{N_2} [2\text{tr}(\Sigma_1 \Sigma_2)^2 + \beta_2 \text{tr}(\Sigma_1^2 \Sigma_2^2 \circ \Sigma_2^1 \Sigma_2^1 \circ \Sigma_2^{1/2} \Sigma_1^1 \Sigma_2^{1/2})],
\]
\[
\eta_{55} = \sum_{j=1}^{N_2} E_{j-1}[(E_j - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1)]^2
= \frac{4}{N_2} \text{tr}^2 \Sigma_2[2\text{tr} \Sigma_2^2 + \beta_2 \text{tr}(\Sigma_2 \circ \Sigma_2)] + \frac{8}{N_2} \text{tr} \Sigma_2[2\text{tr} \Sigma_2^3 + \beta_2 \text{tr}(\Sigma_2 \circ \Sigma_2^2)]
+ \frac{4}{N_2} [2\text{tr} \Sigma_2^2 + \beta_2 \text{tr}(\Sigma_2^2 \circ \Sigma_2^2)] + \frac{4}{N_2} \text{tr}^2 \Sigma_2^2 + o_p(1),
\]
\[
\eta_{13} = \sum_{j=1}^{N_2} E_{j-1}[(E_j - E_{j-1}) \text{tr} \tilde{B}_2^{-1}][(E_j - E_{j-1}) \text{tr} \tilde{B}_2 \Sigma_1)]
= -\frac{2}{(1 - y_{N_2})} N_2^{-1} \text{tr} \Sigma_1 - \frac{\beta_2}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2) + o_p(1),
\]
\[
\eta_{14} = \sum_{j=1}^{N_2} E_{j-1}[(E_j - E_{j-1}) \text{tr} \tilde{B}_2^{-1}][(E_j - E_{j-1}) \text{tr} \tilde{B}_2 \Sigma_2^2)]
= -\frac{2}{(1 - y_{N_2})} N_2^{-1} \text{tr} \Sigma_1^2 - \frac{\beta_2}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2^2 \Sigma_1 \Sigma_2^{1/2})
+ o_p(1),
\]
\[
\eta_{15} = \sum_{j=1}^{N_2} E_{j-1}[(E_j - E_{j-1}) \text{tr} \tilde{B}_2^{-1}][(E_j - E_{j-1}) \text{tr} \tilde{B}_2 \Sigma_1^2)]
= -\frac{4}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2) - \frac{2\beta_2}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2^2)
- \frac{2\beta_2}{(1 - y_{N_2})} N_2^{-1} \text{tr} \Sigma_2 N_2^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2) + o_p(1),
\]
\[ \eta_{23} = \sum_{j=1}^{N_2} E_{j-1}[[((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2^{-1}))((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2 \Sigma_1))]] \\
= -\frac{4}{(1 - y N_2)^2} N_2^{-1} \operatorname{tr}(\Sigma_2^2 \Sigma_2^{-1}) - \frac{4}{(1 - y N_2)^3} N_2^{-1} \operatorname{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1} \operatorname{tr} \Sigma_1 \\
- \frac{2\beta_2}{(1 - y N_2)^2} N_2^{-1} \operatorname{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2) \\
- \frac{2\beta_2}{(1 - y N_2)^3} N_2^{-1} \operatorname{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1} \operatorname{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2) + o_p(1), \]

\[ \eta_{24} = \sum_{j=1}^{N_2} E_{j-1}[[((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2^{-1}))((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2 \Sigma_1^2))]] \\
= -\frac{4}{(1 - y N_2)^2} N_2^{-1} \operatorname{tr}(\Sigma_2^3 \Sigma_2^{-1}) - \frac{4}{(1 - y N_2)^3} N_2^{-1} \operatorname{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1} \operatorname{tr} \Sigma_1^2 \\
- \frac{2\beta_2}{(1 - y N_2)^2} N_2^{-1} \operatorname{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2^2) \\
- \frac{2\beta_2}{(1 - y N_2)^3} N_2^{-1} \operatorname{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1} \operatorname{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2^2) \\
+ o_p(1), \]

\[ \eta_{25} = \sum_{j=1}^{N_2} E_{j-1}[[((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2^{-1}))((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2 \Sigma_1 \Sigma_2))]] \\
= -\frac{8}{(1 - y N_2)^2} N_2^{-1} \operatorname{tr}(\Sigma_1 \Sigma_2^{-1} \Sigma_1 \Sigma_2) \\
- \frac{8}{(1 - y N_2)^3} N_2^{-2} \operatorname{tr}(\Sigma_2 \Sigma_1^{-1}) \operatorname{tr}(\Sigma_1 \Sigma_2) \\
+ \frac{4}{(1 - y N_2)^2} (N_2^{-1} \operatorname{tr} \Sigma_1)^2 \\
- \frac{4\beta_2}{(1 - y N_2)^2} N_2^{-1} \operatorname{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2^2) \\
- \frac{4\beta_2}{(1 - y N_2)^3} N_2^{-1} \operatorname{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1} \operatorname{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_2^2) \\
+ o_p(1), \]

\[ \eta_{34} = \sum_{j=1}^{N_2} E_{j-1}[((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2 \Sigma_1))((E_j - E_{j-1})\operatorname{tr}(\tilde{B}_2 \Sigma_1^2))] \\
= \frac{1}{N_2} [2 \operatorname{tr}(\Sigma_1 \Sigma_2^2) + \beta_2 \operatorname{tr}(\Sigma_2 \circ \Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2})] + o_p(1), \]
From (A.81), (A.82), (A.83) and (A.84) in Lemma A.2, we have
\[
\eta_{15} = \sum_{j=1}^{N_2} E_{j-1} \{(E_{j} - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1)\}((E_{j} - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1)^2) \]
\[
= \frac{2}{N_2} \text{tr}(\Sigma_2[2\text{tr}(\Sigma_2^2 + \beta_2 \text{tr}(\Sigma_2 \circ \Sigma_2)] + \frac{2}{N_2} [2\text{tr}(\Sigma_2^2 + \beta_2 \text{tr}(\Sigma_2 \circ \Sigma_2)] + o_p(1),
\]
\[
\eta_{45} = \sum_{j=1}^{N_2} E_{j-1} \{(E_{j} - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1^2)\}((E_{j} - E_{j-1}) \text{tr}(\tilde{B}_2 \Sigma_1^2)^2) \]
\[
= \frac{2}{N_2} \text{tr}(\Sigma_2[2\text{tr}(\Sigma_1 \Sigma_2^2) + \beta_2 \text{tr}(\Sigma_2 \circ \Sigma_2^{1/2} \Sigma_1 \Sigma_2^{1/2})]
\]
\[
+ \frac{2}{N_2} [2\text{tr}(\Sigma_1 \Sigma_2^3) + \beta_2 \text{tr}(\Sigma_2^2 \circ \Sigma_2^{1/2} \Sigma_1 \Sigma_2^{1/2})] + o_p(1).
\]

From (A.81), (A.82), (A.83) and (A.84) in Lemma A.2, we have
\[
p^{-1} \text{tr}(\tilde{B}_2^{-1}) = d_{1p} + o_p(1), \quad p^{-1} \text{tr}(\tilde{B}_2^{-1})^2 = d_{2p} + o_p(1),
\]
\[
p^{-1} \text{tr}(\tilde{B}_2^{-1})^3 = d_{3p} + o_p(1), \quad p^{-1} \text{tr}(\tilde{B}_2^{-1})^4 = d_{4p} + o_p(1)
\]
and
\[
p^{-1} \text{tr}(\tilde{B}_2^{-1} \circ \tilde{B}_2^{-1}) = l_{1p} + o_p(1),
\]
\[
p^{-1} \text{tr}(\tilde{B}_2^{-1} \circ (\tilde{B}_2^{-1})^2) = l_{2p} + o_p(1),
\]
\[
p^{-1} \text{tr}((\tilde{B}_2^{-1})^2 \circ (\tilde{B}_2^{-1})^2) = l_{3p} + o_p(1),
\]

where
\[
d_{1p} = \frac{1}{1 - y_N} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}), \quad (A.17)
\]
\[
d_{2p} = \frac{1}{1 - y_N} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})^2
\]
\[
+ \frac{y_N}{(1 - y_N)^2} (p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}))^2, \quad (A.18)
\]
\[
d_{3p} = \frac{1}{1 - y_N} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})^3
\]
\[
+ \frac{3y_N}{(1 - y_N)^2} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}) p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})^2
\]
\[
+ \frac{2y_N^2}{1 - y_N} (p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}))^3, \quad (A.19)
\]
\[
d_{4p} = \frac{1}{1 - y_N} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})^4
\]
\[
+ \frac{4y_N}{(1 - y_N)^2} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}) p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})^3
\]
\[
+ \frac{2y_N^2}{1 - y_N} (p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}))^2. \quad (A.20)
\]
\[ \text{Then from (A.71)-(A.75) in Lemma A.1, we have} \]

\[ \sigma_B = \sum_{i=1}^{N_1} E_{2, i-1} \left[ \left( (E_{2, i} - E_{2, i-1}) T_B \right)^2 \right] \]

\[ = \omega_1^2 [4(n_1^{-1} \tr \Sigma_1)^2 \theta_{11} + 12 \theta_{11} + 8(n_1^{-1} \tr \Sigma_1)\theta_{13} - 4(n_1^{-1} \tr \Sigma_1)\theta_{15} - \theta_{44}] \]

\[ + 2 \omega_1 \omega_2 [4(n_1^{-1} \tr \Sigma_1)\theta_{12} - 2(1^{-1} \tr \Sigma_1)\theta_{14} + 4 \theta_{23} - 2 \theta_{25} - 2 \theta_{34} + \theta_{45}] \]

\[ + \omega_2^2 [4 \theta_{22} + \theta_{44} - \theta_{24}], \]

where

\[ \theta_{11} = \sum_{i=1}^{N_1} E_{2, i-1} \left[ \left( (E_{2, i} - E_{2, i-1}) \tr (\tilde{B}_1 \Sigma_1) \right)^2 \right] = \frac{1}{N_1} [2 \tr \Sigma_1^2 + \beta_1 \tr (\Sigma_1 \circ \Sigma_1)], \]

\[ \theta_{22} = \sum_{i=1}^{N_1} E_{2, i-1} \left[ \left( (E_{2, i} - E_{2, i-1}) \tr (\tilde{B}_2 \Sigma_2^{-1}) \right)^2 \right] \]

\[ = \frac{1}{N_1} [2 \tr (\tilde{B}_2^{-1})^2 + \beta_1 \tr (\tilde{B}_2^{-1} \circ \tilde{B}_2^{-1})] = y_{N_1} (2d_{2p} + \beta_1 l_{1p}) + o_p(1), \]

\[ \theta_{33} = \sum_{i=1}^{N_1} E_{2, i-1} \left[ \left( (E_{2, i} - E_{2, i-1}) \tr (\tilde{B}_1 \Sigma_1 \tilde{B}_2 \Sigma_1) \right)^2 \right] \]

\[ = \frac{1}{N_1} [2 \tr (\Sigma_2 \tilde{B}_2 \Sigma_1 \tilde{B}_2 \Sigma_1) + \beta_1 \tr (\Sigma_1 \tilde{B}_2 \Sigma_1 \circ \Sigma_1 \tilde{B}_2 \Sigma_1)] \]

\[ = \frac{2}{N_1} \tr \left( \Sigma_2 \Sigma_1 \Sigma_2 \right) + \frac{2}{N_1} \tr \left( \Sigma_1 \Sigma_2 \right)^2 \]
\[
\theta_{44} = \sum_{i=1}^{N_1} E_{2,i-1} \left[ \left( (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\bar{B}_2^{-1}) \right)^2 \right] \\
= \frac{4}{N_1^3} \text{tr}^2(\bar{B}_2^{-1}) \left[ 2\text{tr}(\bar{B}_2^{-1})^2 + \beta_1 \text{tr}(\bar{B}_2^{-1} \circ \bar{B}_2^{-1}) \right] + \frac{4}{N_1^2} \text{tr}^2(\bar{B}_2^{-1})^2 \\
+ \frac{8}{N_1} \text{tr}(\bar{B}_2^{-1}) \left[ 2\text{tr}(\bar{B}_2^{-1})^3 + \beta_1 \text{tr}(\bar{B}_2^{-1} \circ (\bar{B}_2^{-1})^2) \right] \\
+ \frac{4}{N_1} \left[ 2\text{tr}(\bar{B}_2^{-1})^4 + \beta_1 \text{tr}((\bar{B}_2^{-1})^2 \circ (\bar{B}_2^{-1})^2) \right] \\
= 4g_{N_1} d_{ip}^2 (2d_{2p} + \beta_1 l_{1p}) + 4g_{N_1} d_{ip}^2 + 8g_{N_1} d_{ip} (2d_{3p} + \beta_1 l_{2p}) + 4g_{N_1} (2d_{4p} + \beta_1 l_{3p}) + o_p(1),
\]

\[
\theta_{55} = \sum_{i=1}^{N_1} E_{2,i-1} \left[ \left( (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\Sigma_1) \right)^2 \right] \\
= \frac{4}{N_1^3} \text{tr}^2(\Sigma_1) \left[ 2\text{tr}(\Sigma_1^2 + \beta_1 \text{tr}(\Sigma_1 \circ \Sigma_1)) \right] + \frac{8}{N_1^2} \text{tr}(\Sigma_1 \circ (\Sigma_1 \circ \Sigma_1)) \\
+ \frac{4}{N_1} \left[ 2\text{tr}(\Sigma_1^3 + \beta_1 \text{tr}(\Sigma_1 \circ \Sigma_1^2)) \right] + \frac{4}{N_1} \text{tr}^2(\Sigma_1^2) + o_p(1),
\]

\[
\theta_{12} = \sum_{i=1}^{N_1} E_{2,i-1} \left[ \left( (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\Sigma_1) \right) (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\bar{B}_2^{-1}) \right] \\
= \frac{1}{N_1} \left[ 2\text{tr}(\bar{B}_2^{-1} \circ (\Sigma_1)) \right] + \beta_1 \text{tr}(\bar{B}_2^{-1} \circ \Sigma_1),
\]

\[
\theta_{13} = \sum_{i=1}^{N_1} E_{2,i-1} \left[ \left( (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\Sigma_1) \right) (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\bar{B}_2\Sigma_1) \right] \\
= \frac{1}{N_1} \left[ 2\text{tr}(\bar{B}_2 \circ (\Sigma_1) \circ (\Sigma_1)) \right] + \beta_1 \text{tr}(\bar{B}_1 \circ (\Sigma_1 \circ \Sigma_1)) \\
= \frac{1}{N_1} \left[ 2\text{tr}(\Sigma_1 \circ (\Sigma_1) \circ (\Sigma_1)) \right] + \beta_1 \text{tr}(\Sigma_1 \circ (\Sigma_1 \circ \Sigma_1)) + o_p(1),
\]

\[
\theta_{14} = \sum_{i=1}^{N_1} E_{2,i-1} \left[ \left( (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\Sigma_1) \right) (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\bar{B}_2^{-1})^2 \right] \\
= \frac{2}{N_1^2} \text{tr}(\bar{B}_2^{-1}) \left[ 2\text{tr}(\bar{B}_2^{-1} \circ (\Sigma_1)) \right] + \beta_1 \text{tr}(\bar{B}_2^{-1} \circ \Sigma_1) \\
+ \frac{2}{N_1} \text{tr}(\bar{B}_2^{-1} \circ (\Sigma_1)) + \beta_1 \text{tr}(\bar{B}_2^{-1} \circ (\Sigma_1)^2) + o_p(1),
\]

\[
\theta_{15} = \sum_{i=1}^{N_1} E_{2,i-1} \left[ \left( (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\Sigma_1) \right) (E_{2,i} - E_{2,i-1}) \text{tr}(\bar{B}_1\Sigma_1^2) \right] \\
= \frac{2}{N_1^2} \text{tr}(\Sigma_1 \circ (\Sigma_1 \circ (\Sigma_1))) + \beta_1 \text{tr}(\Sigma_1 \circ (\Sigma_1 \circ (\Sigma_1))) + o_p(1),
\]

\[
\frac{\beta_1}{N_1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{1/2} \circ (\Sigma_1^{1/2} \Sigma_2^{1/2}) + o_p(1),
\]
\[ \begin{align*}
\theta_{23} &= \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1}))((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1 \tilde{B}_2 \Sigma_1)) \right] \\
&= \left[ 2 \text{tr}(\tilde{B}_2^{-1} \Sigma_1 \tilde{B}_2 \Sigma_1) + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1 \tilde{B}_2 \Sigma_1) \right], \\
\theta_{24} &= \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1}))((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1})^2) \right] \\
&= \frac{2}{N_1} \text{tr}(\tilde{B}_2^{-1}) \left[ 2 \text{tr}(\tilde{B}_2^{-1})^2 + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ \tilde{B}_2^{-1}) \right] \\
&\quad + \frac{2}{N_1} \text{tr}(\tilde{B}_2^{-1})^3 + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ (\tilde{B}_2^{-1})^2) \\
&= 2 \beta_1 d_1 p(2d_2p + \beta_1 l_1p) + 2 \gamma N_1(2d_3p + \beta_1 l_2p) + o_p(1), \\
\theta_{25} &= \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1}))((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1))^2 \right] \\
&= \frac{2}{N_1} \text{tr}(\Sigma_1)[2 \text{tr}(\tilde{B}_2^{-1} \Sigma_1) + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1)] \\
&\quad + \frac{2}{N_1} \text{tr}(\tilde{B}_2^{-1} \Sigma_1^2) + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1^2)] + o_p(1), \\
\theta_{34} &= \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1 \tilde{B}_2 \Sigma_1))((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1})^2) \right] \\
&= \frac{2}{N_1} \text{tr}(\tilde{B}_2^{-1}) \left[ 2 \text{tr}(\tilde{B}_2^{-1} \Sigma_1 \tilde{B}_2 \Sigma_1) + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1 \tilde{B}_2 \Sigma_1) \right] \\
&\quad + \frac{2}{N_1} \text{tr}(\tilde{B}_2^{-1})^3 + \beta_1 \text{tr}((\tilde{B}_2^{-1})^2 \circ \Sigma_1 \tilde{B}_2 \Sigma_1) \right] + o_p(1), \\
\theta_{35} &= \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1 \tilde{B}_2 \Sigma_1))((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1)^2) \right] \\
&= \frac{2}{N_1} \text{tr}(\Sigma_1)[2 \text{tr}(\tilde{B}_2 \Sigma_1^2) + \beta_1 \text{tr}((\Sigma_1 \circ \tilde{B}_2 \Sigma_1)] \\
&\quad + \frac{2}{N_1} \text{tr}(\Sigma_1^2) + \beta_1 \text{tr}(\Sigma_1 \circ \Sigma_1^2)] + o_p(1), \\
\theta_{45} &= \sum_{i=1}^{N_1} E_{2,i-1} \left[ ((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \tilde{B}_2^{-1})^2)((E_{2,i} - E_{2,i-1}) \text{tr}(\tilde{B}_1 \Sigma_1)^2) \right] \\
&= \frac{4}{N_1} \text{tr}(\tilde{B}_2^{-1}) \text{tr}(\tilde{B}_1 \Sigma_1)[2 \text{tr}(\tilde{B}_2^{-1} \Sigma_1) + \beta_1 \text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1)] \\
\end{align*} \]
After sorting and calculation, we obtain

\[
\sigma_{11p} = 4(n_2^{-1}\text{tr}\Sigma_2)^2\eta_{33} + 4\eta_{44} + \eta_{55} + 8(n_2^{-1}\text{tr}\Sigma_2)\eta_{34} \\
- 4(n_2^{-1}\text{tr}\Sigma_2)\eta_{35} - 4\eta_{45} + 4(n_1^{-1}\text{tr}\Sigma_1)^2\theta_{11} + 4\theta_{33} \\
+ \theta_{55} + 8(n_1^{-1}\text{tr}\Sigma_1)\theta_{13} - 4(n_1^{-1}\text{tr}\Sigma_1)\theta_{15} - 4\theta_{35} \\
= \frac{4}{N_1^2}\text{tr}^2\Sigma_1^2 + \frac{4}{N_2^2}\text{tr}^2\Sigma_2^2 + \frac{8}{N_1N_2}\text{tr}^2(\Sigma_1\Sigma_2) \\
+ \frac{4}{N_1}[2\text{tr}(\Sigma_1(\Sigma_2 - \Sigma_1)\Sigma_1\Sigma_2) \\
+ \beta_1\text{tr}(\Sigma_1^{1/2}(\Sigma_2 - \Sigma_1)\Sigma_1^{1/2} \circ \Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})] \\
+ \frac{4}{N_1}[2\text{tr}(\Sigma_1^2(\Sigma_1 - \Sigma_2)) + \beta_1\text{tr}(\Sigma_1^2 \circ \Sigma_1^{1/2}(\Sigma_1 - \Sigma_2)\Sigma_1^{1/2}] \\
+ \frac{4}{N_2}[2\text{tr}(\Sigma_1\Sigma_2(\Sigma_1 - \Sigma_2)\Sigma_2) \\
+ \beta_2\text{tr}(\Sigma_2^{1/2}(\Sigma_1 - \Sigma_2)\Sigma_2^{1/2} \circ \Sigma_2^{1/2}\Sigma_1\Sigma_2^{1/2})] \\
+ \frac{4}{N_2}[2\text{tr}(\Sigma_2(\Sigma_2 - \Sigma_1)\Sigma_2^2) + \beta_2\text{tr}(\Sigma_2^2 \circ \Sigma_2^{1/2}(\Sigma_2 - \Sigma_1)\Sigma_2^{1/2})] + o_p(1),
\]

\[
\sigma_{12p} = -4(y_{n_1-1}1)(n_2^{-1}\text{tr}\Sigma_2)\eta_{13} - 4(y_{n_1-1}1)\eta_{14} \\
+ 2(y_{n_1-1}1)\eta_{15} - 2(n_2^{-1}\text{tr}\Sigma_2)\eta_{23} - 2\eta_{24} + \eta_{25} \\
= 2(y_{n_1-1}1)[\frac{4y_{N_2}^2}{(1 - y_{N_2})}p^{-1}\text{tr}\Sigma_1p^{-1}\text{tr}\Sigma_2 \\
+ \frac{4y_{N_2}}{(1 - y_{N_2})}p^{-1}\text{tr}(\Sigma_1(\Sigma_1 - \Sigma_2)) \\
+ \frac{2\beta_2y_{N_2}}{(1 - y_{N_2})}p^{-1}\text{tr}(\Sigma_1^{1/2}\Sigma_2^{-1}\Sigma_1^{1/2} \circ \Sigma_2^{1/2}(\Sigma_1 - \Sigma_2)\Sigma_2^{1/2})] \\
+ \frac{8y_{N_2}^2}{(1 - y_{N_2})^2}p^{-1}\text{tr}\Sigma_2p^{-1}\text{tr}(\Sigma_1^2\Sigma_2^{-1}) \\
+ \frac{8y_{N_2}^3}{(1 - y_{N_2})^3}p^{-1}\text{tr}\Sigma_1p^{-1}\text{tr}\Sigma_2p^{-1}\text{tr}(\Sigma_1\Sigma_2^{-1}) \\
+ \frac{4y_{N_2}^2}{(1 - y_{N_2})^2}(p^{-1}\text{tr}\Sigma_1)^2 + \frac{8y_{N_2}}{(1 - y_{N_2})}p^{-1}\text{tr}(\Sigma_1\Sigma_2^{-1}\Sigma_1(\Sigma_1 - \Sigma_2))]
\]
and

\[
\sigma_{12}^2 = 4(n_1^{-1} \text{tr} \Sigma_1) \theta_{12} - 2(n_1^{-1} \text{tr} \Sigma_1) \theta_{14} + 4 \theta_{23} - 2 \theta_{25} - 2 \theta_{34} + \theta_{45}
\]

\[
= 4(y_N d_{1p} - 1) N_1^{-1} \left[ 2 \text{tr} (\tilde{\Sigma}_2^{-1} \tilde{\Sigma}_1^2) - 2 \text{tr}(\tilde{\Sigma}_2^{-1} \tilde{\Sigma}_1 \tilde{\Sigma}_2 \Sigma_1) \right. \\
\left. + \beta_1 \text{tr} (\tilde{\Sigma}_2^{-1} \circ \Sigma_2) - \beta_1 \text{tr}(\tilde{\Sigma}_2^{-1} \circ \Sigma_1 \tilde{\Sigma}_2 \Sigma_1) \right] \\
+ 4 N_1^{-2} \left[ 2 \text{tr}((\tilde{\Sigma}_2^{-1})^2 \Sigma_1^2) - 2 \text{tr}((\tilde{\Sigma}_2^{-1})^2 \Sigma_1 \tilde{\Sigma}_2 \Sigma_1) + \beta_1 \text{tr}((\tilde{\Sigma}_2^{-1})^2 \circ \Sigma_1^2) \\
- \beta_1 \text{tr}((\tilde{\Sigma}_2^{-1})^2 \circ \Sigma_1 \tilde{\Sigma}_2 \Sigma_1) \right] + 4 N_1^{-2} \text{tr}^2(\tilde{\Sigma}_2^{-1} \Sigma_1) + o_p(1).
\]

Based on (A.81) and (A.82) in Lemma A.2 and (A.86) and (A.87) in Lemma A.3, we have

\[
N_2^{-1} \text{tr} (\tilde{\Sigma}_2^{-1} \Sigma_1) = \frac{1}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_1^2 \Sigma_2^{-1}) + o_p(1),
\]

\[
N_2^{-1} \text{tr} (\tilde{\Sigma}_2^{-1} \Sigma_2^2) = \frac{1}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_2^2 \Sigma_2^{-1}) + o_p(1),
\]

\[
N_2^{-1} \text{tr} (\tilde{\Sigma}_2^{-1} \Sigma_2 \Sigma_1) = \frac{1}{(1 - y_{N_2})} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2 \Sigma_1 \Sigma_2^{-1}) \\
- \frac{1}{(1 - y_{N_2})} (N_2^{-1} \text{tr}(\Sigma_1))^2 + o_p(1),
\]

\[
N_2^{-1} \text{tr} ((\tilde{\Sigma}_2^{-1})^2 \Sigma_1^2) = \frac{1}{(1 - y_{N_2})^2} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2 \Sigma_1^2 \Sigma_2^{-1}) \\
+ \frac{1}{(1 - y_{N_2})^3} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2 \Sigma_1 \Sigma_2^2) N_2^{-1} \text{tr}(\Sigma_1^3 \Sigma_2^{-1}) \\
+ o_p(1),
\]

\[
N_2^{-1} \text{tr} ((\tilde{\Sigma}_2^{-1})^2 \Sigma_2 \Sigma_1) = \frac{1}{(1 - y_{N_2})^2} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2 \Sigma_1 \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1}) \\
+ \frac{1}{(1 - y_{N_2})^3} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2 \Sigma_1 \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1}) \\
- \frac{2}{(1 - y_{N_2})^2} N_2^{-1} \text{tr}(\Sigma_1^2 \Sigma_2^{-1}) N_2^{-1} \text{tr}(\Sigma_1) \\
- \frac{1}{(1 - y_{N_2})^3} N_2^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})(N_2^{-1} \text{tr}(\Sigma_1))^2 + o_p(1)
\]
and

\[
N_2^{-1}\text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1^2) = \frac{1}{1 - N_2} N_2^{-1}\text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^2) + o_p(1),
\]

\[
N_2^{-1}\text{tr}(\tilde{B}_2^{-1} \circ \Sigma_1 \tilde{B}_2 \Sigma_1)
= \frac{1}{1 - N_2} N_2^{-1}\text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2}) + o_p(1),
\]

\[
N_2^{-1}\text{tr}((\tilde{B}_2^{-1})^2 \circ \Sigma_1^2)
= \frac{1}{(1 - N_2)^2} N_2^{-1}\text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^2)
+ \frac{1}{1 - N_2} N_2^{-1}\text{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1}\text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^2) + o_p(1),
\]

\[
N_2^{-1}\text{tr}((\tilde{B}_2^{-1})^2 \circ \Sigma_1 \tilde{B}_2 \Sigma_1)
= \frac{1}{(1 - N_2)^2} N_2^{-1}\text{tr}(\Sigma_1 \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2})
+ \frac{1}{1 - N_2} N_2^{-1}\text{tr}(\Sigma_1 \Sigma_2^{-1}) N_2^{-1}\text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^{1/2} \Sigma_2 \Sigma_1^{1/2}) + o_p(1).
\]

Therefore, we get

\[
\sigma_{12p}^2 = 4(y_{Ni} d_{ip} - 1) \left( \frac{2y_{Ni} y_{N_2}}{(1 - N_2)} (p^{-1} \text{tr} \Sigma_1)^2 \right) + \left( \frac{2y_{Ni}}{(1 - N_2)} p^{-1} \text{tr}(\Sigma_1 (\Sigma_1 - \Sigma_2) \Sigma_1^{-1}) \right)
+ \left( \frac{\beta_1 y_{Ni}}{(1 - N_2)} p^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^{1/2} (\Sigma_1 - \Sigma_2) \Sigma_1^{1/2}) \right)
+ 4 \left( \frac{2y_{Ni} y_{N_2}}{(1 - N_2)} p^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1}) p^{-1} \text{tr} \Sigma_1 \right)
+ \left( \frac{2y_{Ni} y_{N_2}^2}{(1 - N_2)^2} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})(p^{-1} \text{tr} \Sigma_1)^2 \right)
+ \left( \frac{2y_{Ni}^2}{(1 - N_2)^2} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1} \Sigma_1^{-1} (\Sigma_1 - \Sigma_2)) \right)
+ \left( \frac{2y_{Ni} y_{N_2}}{(1 - N_2)^2} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})(p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1} \Sigma_1 (\Sigma_1 - \Sigma_2))) \right)
+ \left( \frac{\beta_1 y_{Ni}}{(1 - N_2)^2} p^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^{1/2} (\Sigma_1 - \Sigma_2) \Sigma_1^{1/2}) \right)
+ \left( \frac{\beta_1 y_{Ni} y_{N_2}}{(1 - N_2)^2} p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1})(p^{-1} \text{tr}(\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} \circ \Sigma_1^{1/2} (\Sigma_1 - \Sigma_2) \Sigma_1^{1/2})) \right)
+ \left( \frac{4y_{N_1}^2}{(1 - N_2)^2} (p^{-1} \text{tr}(\Sigma_1 \Sigma_2^{-1}))^2 \right) + o_p(1).
\]
Finally, we have
\[
\sigma^2_{22p} = 4(y_n, c_1 - 1)^2 \eta_{11} + \eta_{22} + 4(y_n, c_1 - 1) \eta_{12} + o_p(1) \tag{A.27}
\]
and
\[
\sigma^2_{22p} = 4\theta_{22} + \theta_{44} - 4\theta_{24} \tag{A.28}
\]
\[
= 4y_N, (y_N, d_{1p} - 1)^2 (2d_{2p} + \beta_1 t_{1p}) + 8y_N, (y_N, d_{1p} - 1)(2d_{3p} + \beta_1 t_{3p}) + 4y_N, (2d_{4p} + \beta_1 t_{3p}) + 4y_N, d_{2p}^2 + o_p(1). \tag{A.29}
\]

Next, we consider \(\mu_2\), according to (A.11) and (A.13), we have
\[
\mu_2 = p(y_n, c_1^2 - 2c_1 + c_2) + p + 2y_n, c_1 \xi_1 - 2\xi_1 + \xi_2 + y_n, d_{2p} + \beta_1 y_n, t_{1p} + o_p(1). \tag{A.30}
\]

A.3.2. Proofs of the theorems and propositions

The Proof of Theorem 2.1. Now we prove the Theorem 2.1, because all quantities are computed under \(H_0\), we add 0 to the subscripts of these quantities, for example, we use \(\mu_{10}\) instead of \(\mu_1\) in this subsection. Under \(H_0\), assuming that \(\Sigma_1 = \Sigma_2 = \Sigma\), from (A.12), we have
\[
\mu_{10} = (y_n + y_n)^p^{-1} \Sigma^2 + (\beta_1 y_n + \beta_2 y_n)^p^{-1} \text{tr}(\Sigma \circ \Sigma).
\]

When \(\Sigma_1 = \Sigma_2 = \Sigma\), we have \(\Gamma = \Sigma_1^{-1/2} \Sigma_2^{1/2} = \Pi_p\) and \(\Gamma_p = \Pi^\top = \Pi_p\), which satisfy the Assumptions c-d-f when we treat \(\hat{\Sigma}_2\) as \(\hat{\Sigma}_n\) in Subsection A.3.3, \(H_p\) and \(H\), the ESD and LSD of \(\Gamma_p\), are \(\delta_1\), the \(F^{y_n, H_p}\) has an explicit density function,
\[
\frac{1}{2\pi x y_n^2} \sqrt{(b_{y_n} - x)(x - a_{y_n})}, \quad a_{y_n} \leq x \leq b_{y_n}, \tag{A.31}
\]
which is the seminal Marčenko-Pastur law with index \(y_{n2}\) and support \([a_{y_{n2}}, b_{y_{n2}}]\), where \(a_{y_{n2}} = (1 - \sqrt{y_{n2}})^2\), \(b_{y_{n2}} = (1 + \sqrt{y_{n2}})^2\), from (A.8), we have
\[
c_{10} = \frac{1}{(1 - y_{n2})}, \quad c_{20} = \frac{1}{(1 - y_{n2})^3}. \tag{A.32}
\]

Based on (A.9), (A.16) and the Proposition A.1. in the Appendix of [20], we get
\[
\xi_{10} = \frac{y_{n2}}{(1 - y_{n2})^2} + \frac{\beta_2 y_{n2}}{1 - y_{n2}}, \tag{A.33}
\]
\[
\xi_{20} = \frac{y_{n2}^2 + 3y_{n2}}{(1 - y_{n2})^4} + \frac{\beta_2(-y_{n2}^2 + 3y_{n2})}{(1 - y_{n2})^3}. \tag{A.34}
\]
and

\[ \eta_{110} = \frac{2y_2}{(1 - y_2)^4} + \frac{2\beta_2 y_2}{(1 - y_2)^2}, \quad (A.34) \]
\[ \eta_{120} = \frac{4y_2(1 + y_2)}{(1 - y_2)^6} + \frac{2\beta_2 y_2}{(1 - y_2)^4}, \quad (A.35) \]
\[ \eta_{220} = \frac{4y_2(2y_2^2 + 5y_2 + 2)}{(1 - y_2)^8} + \frac{4\beta_2 y_2}{(1 - y_2)^6}. \quad (A.36) \]

From (A.17)-(A.23), we have

\[ d_{10p} = \frac{1}{(1 - y_{N_2})} \rightarrow \frac{1}{(1 - y_2)} = d_{10}, \quad (A.37) \]
\[ d_{20p} = \frac{1}{(1 - y_{N_2})^3} \rightarrow \frac{1}{(1 - y_2)^3} = d_{20}, \quad (A.38) \]
\[ d_{30p} = \frac{1 + y_{N_2}}{(1 - y_{N_2})^5} \rightarrow \frac{1 + y_2}{(1 - y_2)^5} = d_{30}, \quad (A.39) \]
\[ d_{40p} = \frac{y_{N_2}^2 + 3y_{N_2} + 1}{(1 - y_{N_2})^7} \rightarrow \frac{y_2^2 + 3y_2 + 1}{(1 - y_2)^7} = d_{40} \quad (A.40) \]

and

\[ l_{10p} = \frac{1}{(1 - y_{N_2})^2} \rightarrow \frac{1}{(1 - y_2)^2} = l_{10}, \quad (A.41) \]
\[ l_{20p} = \frac{1}{(1 - y_{N_2})^4} \rightarrow \frac{1}{(1 - y_2)^4} = l_{20}, \quad (A.42) \]
\[ l_{30p} = \frac{1}{(1 - y_{N_2})^6} \rightarrow \frac{1}{(1 - y_2)^6} = l_{30}. \quad (A.43) \]

Therefore, from (A.29), (A.31), (A.32), (A.33), (A.38) and (A.41), we have

\[ \mu_{20} = p(y_{n_1}c_{10}^2 - 2c_{10} + c_{20}) + p + 2y_{n_1}c_{10}\xi_{10} - 2\xi_{10} + \xi_{20} + y_{n_1}d_{20p} + \beta_1 y_{n_1} l_{10p} + o_p(1) \]
\[ = p - \frac{2}{(1 - y_{n_2})^2} + \frac{y_{n_1}}{(1 - y_{n_2})^2} + \frac{1}{(1 - y_{n_2})^3} + p \\
- \frac{2y_{n_2}}{(1 - y_{n_2})^2} + \frac{y_{n_1} + 2y_{n_1}y_{n_2}}{(1 - y_{n_2})^3} + \frac{y_{n_2}^2 + 3y_{n_2}}{(1 - y_{n_2})^4} \\
+ \frac{2\beta_2 y_{n_2} + \beta_1 y_{n_2} + \beta_2 y_{n_2}}{(1 - y_{n_2})^2} + \frac{2\beta_2 y_{n_2}}{(1 - y_{n_2})^3} + o_p(1). \]

Letting \( m_{10} = \int x dL(x) \) and \( m_{20} = \int x^2 dL(x) \), based on (A.24), (A.25), (A.26), (A.31) and (A.37), we get

\[ \sigma_{110p} = 4[(y_{N_1} + y_{N_2})p^{-1}\Sigma^2]^2 + o_p(1) \rightarrow 4(y_1 + y_2)^2 m_{20}^2 = \sigma_{110} \]
and

\[
\begin{align*}
\sigma_{120p}^1 &= \left\{ \frac{8(y_{N_1} + y_{N_2})^2 y_{N_2}^2 + 4y_{N_2}^4}{(1 - y_{N_2})^2} + \frac{8y_{N_2}^3}{(1 - y_{N_2})^3} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1), \\
\sigma_{120p}^2 &= \left\{ \frac{8(y_{N_1} + y_{N_2})^2 y_{N_1} y_{N_2} + 4y_{N_1}^3}{(1 - y_{N_2})^2} + \frac{8y_{N_1} y_{N_2}}{(1 - y_{N_2})^3} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1),
\end{align*}
\]

which yields that

\[
\sigma_{120p} = \sigma_{120p}^1 + \sigma_{120p}^2
\]

\[
= \left\{ \frac{8y_{N_1}^2 (y_{N_1} + y_{N_2})^2 + 4y_{N_1}^2 y_{N_2}}{(1 - y_{N_2})^2} + \frac{8y_{N_1} y_{N_2}}{(1 - y_{N_2})^3} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1)
\]

\[
\rightarrow \left\{ \frac{8y_1^2 (y_1 + y_2)^2 + 4y_1^2 y_2 + 8y_1 y_2^3}{(1 - y_2)^2} + \frac{8y_1 y_2}{(1 - y_2)^3} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1)
\]

\[
= \left\{ \frac{8y_1^3 + 16y_1^2 y_2 + 4y_1^2 y_2 + 8y_1 y_2^3}{(1 - y_2)^5} + \frac{8y_1 y_2}{(1 - y_2)^6} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1)
\]

\[
\sigma_{220p} = \sigma_{220p}^1 + \sigma_{220p}^2
\]

\[
= \left\{ \frac{8y_1^3 + 16y_1^2 y_2 + 4y_1^2 y_2 + 8y_1 y_2^3}{(1 - y_2)^5} + \frac{8y_1 y_2}{(1 - y_2)^6} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1)
\]

\[
\rightarrow 4(y_1 d_{d_0} - 1)^2(y_1 + y_2 d_{d_0} + \beta_1 y_1 l_{l_0})
\]

\[
+ 4(y_1 d_{d_0} - 1)(y_1 + y_2 d_{d_0} + 2\beta_1 y_1 l_{l_0})
\]

\[
+ \eta_{1220} + 4y_1(2d_{d_0} + \beta_1 l_{l_0}) + 4y_1^2 d_{d_0}
\]

\[
= \left\{ \frac{8y_1^3 + 16y_1^2 y_2 + 4y_1^2 y_2 + 8y_1 y_2^3}{(1 - y_2)^5} + \frac{8y_1 y_2}{(1 - y_2)^6} \right\}(p^{-1}\text{tr}\Sigma)^2 + o_p(1)
\]

\[
\rightarrow 4(\beta_1 y_1 + \beta_2 y_2)
\]

\[
\left\{ \frac{(y_1 + y_2)^2}{(1 - y_2)^4} + \frac{2y_1(y_1 + y_2)}{(1 - y_2)^5} + \frac{y_2^2}{(1 - y_2)^6} \right\} = \sigma_{220p}
\]

Therefore, under the conditions of Theorem 2.1, based on the central limit theorem of martingale difference sequences, we have

\[
T_A + T_B \overset{d}{\rightarrow} N(0, \omega_1^2 \sigma_{110} + 2\omega_1 \omega_2 \sigma_{120} + \omega_2^2 \sigma_{220}).
\]

Thus, we complete the proof of Theorem 2.1.

The Proof of Theorem 2.2. In this subsection, we prove the Theorem 2.2. Under the Assumptions A-B and $H_0$, denoting $\Sigma_1 = \Sigma_2 = \Sigma$ and $S = (n_1 + n_2)^{-1}(n_1 S_1 + n_2 S_2)$, from (A.3), (A.4) and (A.76), we have

\[
p^{-1}\text{tr}\Sigma = p^{-1}\text{tr}\Sigma + o_p(1),
\]

\[
p^{-1}\text{tr}\Sigma^2 = p^{-1}\text{tr}\Sigma^2 + (n_1 + n_2)^{-1}p^{-1}\text{tr}^2\Sigma + o_p(1),
\]

\[
p^{-1}\text{tr}(S \circ S) = p^{-1}\text{tr}(\Sigma \circ \Sigma) + o_p(1),
\]
which implies that

\[ p^{-1} \text{tr} S^2 - (n_1 + n_2)^{-1} \text{tr}^2 S = p^{-1} \text{tr} \Sigma^2 + o_p(1). \]

According to the discussion before (A.17)-(A.20), we find that \( d_{10}, d_{20}, d_{30}, d_{40} \) are the limits of the estimators of \( p^{-1} \text{tr} \overline{B}_2^{-1}, p^{-1} \text{tr} (\overline{B}_2^{-1})^2, p^{-1} \text{tr} (\overline{B}_2^{-1})^3, p^{-1} \text{tr} (\overline{B}_2^{-1})^4 \), respectively. When \( \Sigma_1 = \Sigma_2 = \Sigma \), that is, \( \Gamma = \Sigma_1^{-1/2} \Sigma_2^{-1/2} = \mathbf{I}_p \), we know \( \text{tr} \overline{B}_2^{-1}, \text{tr} (\overline{B}_2^{-1})^2, \text{tr} (\overline{B}_2^{-1})^3, \text{tr} (\overline{B}_2^{-1})^4 \) are the linear spectral statistics of the noncentralized sample covariance matrix \( \overline{B}_2 \) with the population covariance matrix \( \mathbf{I}_p \). Therefore, based on the proof of Theorem 2.1 and the Proposition A.1. of [20], for \( i = 1, 2, 3, 4 \), we have

\[ p^{-1} (\text{tr}(\overline{B}_2^{-1})^i - \text{Etr}(\overline{B}_2^{-1})^i) = o_p(1) \tag{A.45} \]

and

\[ p^{-1} \text{Etr}(\overline{B}_2^{-1})^i = d_{i0p} + v_{i0p} + o(1), \tag{A.46} \]

where

\[
\begin{align*}
v_{10p} &= \frac{1}{p} \left[ \frac{y_{n_2}^2 + \beta_2 y_{n_2}}{(1 - y_{n_2})^2} \right], \\
v_{20p} &= \frac{1}{p} \left[ \frac{y_{n_2}^4 + 3y_{n_2}^2 + \beta_2 (y_{n_2}^2 - 3y_{n_2})}{(1 - y_{n_2})^4} \right], \\
v_{30p} &= \frac{1}{p} \left[ \frac{y_{n_2}^6 + 9y_{n_2}^4 + 6y_{n_2}^2 + 6\beta_2 y_{n_2}}{(1 - y_{n_2})^6} \right], \\
v_{40p} &= \frac{1}{p} \left[ \frac{y_{n_2}^8 + 18y_{n_2}^6 + 35y_{n_2}^4 + 10y_{n_2}^2 + 10\beta_2 (y_{n_2}^2 + y_{n_2})}{(1 - y_{n_2})^8} \right].
\end{align*}
\]

Therefore, when we substitute \( d_{i0p} + v_{i0p} \) for \( d_{i0} \) in (A.44) for \( i = 1, 2, 3, 4 \), we can get the the expression of \( \hat{\sigma}_{20} \). Thus, we complete the proof of Theorem 2.2.

**The Proof of Theorem 2.3.** In this subsection, we prove the Theorem 2.3. First, under the assumptions of Theorem 2.1, based on Theorem 2.1 and Theorem 2.2, we have

\[
P(T_{dt} > t_\alpha) = P(\max\{ |T_0 - \hat{\mu}_0|/\sqrt{\hat{\sigma}_{110}}, |T_t - \hat{\mu}_0|/\sqrt{\hat{\sigma}_{220}} \} > t_\alpha) = 1 - P( |T_0 - \hat{\mu}_0|/\sqrt{\hat{\sigma}_{110}} \leq t_\alpha, |T_t - \hat{\mu}_0|/\sqrt{\hat{\sigma}_{220}} \leq t_\alpha) \rightarrow \alpha,
\]

which implies that the test based on \( T_{dt} \) is asymptotically with the test level \( \alpha \). Secondly, under the conditions of Theorem 2.1 and the Conditions (C1), (C2) (or (C2*)) and (C3) of [5], we have

\[
P(T_{dtx2} > t_{\alpha/2}) = P(\max\{ T_{dt}, c_x T_t \} > t_{\alpha/2}) \leq P(T_{dt} > t_{\alpha/2}) + P(T_{x} > q_{\alpha/2}) \rightarrow \alpha/2 + \alpha/2 = \alpha,
\]
which implies that the test based on $T_{\text{drx}_2}$ is asymptotically equal to or less than $\alpha$. Finally, under the conditions of Theorem 2.1 and the Conditions (C1), (C2) (or (C2*) and (C3) of [5], when the threshold $s(N_1, N_2, p) - 4\log p \geq 0$, from (2.4), we have

$$P(T_x - 4\log p + \log \log p \leq s(N_1, N_2, p) - 4\log p + \log \log p) \to 1.$$ 

Therefore, we have

$$P(T_{\text{dr}} > t_\alpha) \leq P(T_{\text{drx}_1} > t_\alpha) = P(T_{\text{dr}} + p^2 I(T_x > s(N_1, N_2, p)) > t_\alpha) \leq P(T_{\text{dr}} > t_\alpha) + P(p^2 I(T_x > s(N_1, N_2, p)) > 0) = P(T_{\text{dr}} > t_\alpha) + 1 - P(p^2 I(T_x > s(N_1, N_2, p)) = 0) = P(T_{\text{dr}} > t_\alpha) + 1 - P(T_x \leq s(N_1, N_2, p)) \to \alpha + 1 - 1 = \alpha.$$ 

It follows that the test based on $T_{\text{drx}_1}$ is asymptotically with the test level $\alpha$. Thus, we complete the proof of Theorem 2.3.

The Proof of Theorem 3.1. In this subsection, we prove the Theorem 3.1. Similar with the proof of Theorem 2.1, because all quantities are computed under $(\Sigma_1, \Sigma_2) \in \Pi_1$, we add 1 to the subscripts of these quantities. When $(\Sigma_1, \Sigma_2) \in \Pi_1$, we have

$$\Sigma_1 - \Sigma_2 = a_1/(p + a_1)\Sigma_1, \quad \Sigma_2^{-1} = \tau_p^{-1}\Sigma_1^{-1}.$$ 

(A.47)

Then we have

$$\text{tr}(\Sigma_1 - \Sigma_2)^2 = a_1^2/(p + a_1)^2\text{tr} \Sigma_1^2 = o(1),$$ 

from (A.12), we obtain

$$\mu_{11} = n_1^{-1}\text{tr} \Sigma_1^2 + n_2^{-1}\text{tr} \Sigma_2^2 + \beta_1n_1^{-1}\text{tr}(\Sigma_1 \circ \Sigma_1) + \beta_2n_2^{-1}\text{tr}(\Sigma_2 \circ \Sigma_2) + o(1) = (y_{n_1} + y_{n_2}n_p^2)p^{-1}\text{tr} \Sigma_1^2 + (\beta_1y_{n_1} + \beta_2y_{n_2}n_p^2)p^{-1}\text{tr}(\Sigma_1 \circ \Sigma_1) + o(1).$$

When $\Sigma_2 = \tau_p \Sigma_1$, we have $\Gamma_1 = \Sigma_1^{-1/2} \Sigma_2^{-1/2} = \tau_p^{1/2} \Gamma_p$ and $\Gamma_{1p} = \Gamma_1 \Gamma_p^T = \tau_p I_p$, which also satisfy the Assumptions c-d-f when we treat $B_{2n}$ as $B_n$ in section A.3.3, $H_{1p}$ and $H_1$, the $ESD$ and $LSD$ of $T_{1p}$, are $\tau_p \delta_1$ and $\delta_1$, respectively. Therefore, for $i, j \in \{1, 2\}$, from (A.8), (A.9) and (A.16), we get

$$c_{11} = \tau_p^{-i} c_{00}, \quad \xi_{11} = \xi_{00}, \quad \eta_{i j 1} = \eta_{i j 0}.$$ 

(A.48)
From (A.47) and (A.17)-(A.23), we have

\[
\begin{align*}
  d_{11p} &= \frac{1}{(1 - y_{N_2})}\frac{1}{\tau_p} \to \frac{1}{1 - y_2} = d_{10}, \quad (A.49) \\
  d_{21p} &= \frac{1}{(1 - y_{N_2})}\frac{1}{\tau_p^2} \to \frac{1}{1 - y_2^2} = d_{20}, \quad (A.50) \\
  d_{31p} &= \frac{1 + y_{N_2}}{\tau_p^3} \to \frac{1 + y_2}{1 - y_2^3} = d_{30}, \quad (A.51) \\
  d_{41p} &= \frac{y_{N_2} + 3y_{N_2} + 1}{(1 - y_{N_2})\tau_p^4} \to \frac{y_2^2 + 3y_2 + 1}{(1 - y_2)^4} = d_{40} \quad (A.52)
\end{align*}
\]

and

\[
\begin{align*}
  l_{11p} &= \frac{1}{(1 - y_{N_2})}\frac{1}{\tau_p^2} \to \frac{1}{1 - y_2^2} = l_{10}, \quad (A.53) \\
  l_{21p} &= \frac{1}{(1 - y_{N_2})}\frac{1}{\tau_p^3} \to \frac{1}{1 - y_2^3} = l_{20}, \quad (A.54) \\
  l_{31p} &= \frac{1}{(1 - y_{N_2})}\frac{1}{\tau_p^4} \to \frac{1}{1 - y_2^4} = l_{30}. \quad (A.55)
\end{align*}
\]

Therefore, from (A.29), (A.31), (A.32), (A.33), (A.48), (A.50) and (A.53), we have

\[
\begin{align*}
  \mu_{21} &= p(y_{n_1}c_{11}^2 + 2c_{11} + c_{21}) + p + 2y_{n_1}c_{11}\xi_{11} - 2\xi_{11} + \xi_{21} \\
  &\quad + y_{n_1}d_{21p} + \beta_1y_{n_1}l_{11p} + o_p(1) \\
  &= p\left[-\frac{2}{(1 - y_{N_2})\tau_p} + \frac{y_{n_1}}{(1 - y_{N_2})^2\tau_p} + \frac{1}{(1 - y_{N_2})^3\tau_p^2}\right] + p \\
  &\quad - \frac{2y_{n_2}}{(1 - y_{N_2})^2} + \frac{y_{n_1} + 2y_{n_1}y_{n_2} + y_{n_2}^2 + 3y_{n_2}}{(1 - y_{N_2})^3} - \frac{2\beta_2y_{n_2}}{(1 - y_{N_2})^4} \\
  &\quad + \frac{2\beta_2y_{n_1}y_{n_2} + \beta_3y_{n_3} + \beta_2y_{n_2}}{(1 - y_{N_2})^2} + \frac{2\beta_2y_{n_2}}{(1 - y_{N_2})^3} + o_p(1).
\end{align*}
\]

Letting \(m_{11} = \int xL_1(x)\) and \(m_{21} = \int x^2L_1(x)\), based on (A.24), (A.25), (A.26), (A.47), (A.48), (A.50) and (A.49), we get

\[
\sigma_{111p}^2 = 4[(y_{N_1} + y_{N_2}\tau_p)^2)p^{-1}\text{tr} \Sigma_1^2 + o_p(1) \rightarrow 4(y_1 + y_2)^2m_{21}^2 = \sigma_{111}
\]

and

\[
\begin{align*}
  \sigma_{121p}^1 &= \left[8(y_{N_1} + y_{N_2}\tau_p)^2y_{N_2}^2 + 4y_{N_2}^2
\right. \\
  &\quad + \left. \frac{8g_{21}^2 - 8y_{N_2}^2\tau_p}{(1 - y_{N_2})^2} + \frac{8g_{22}^2}{(1 - y_{N_2})^3}\right](p^{-1}\text{tr} \Sigma_1)^2 + o_p(1), \\
  \sigma_{121p}^2 &= \left[8(y_{N_1} + y_{N_2}\tau_p)y_{N_1}y_{N_2} + 8y_{N_1}y_{N_2} - 8y_{N_1}y_{N_2}\tau_p
\right. \\
  &\quad + \left. \frac{4y_{N_1}^2}{(1 - y_{N_2})^2\tau_p} + \frac{8y_{N_1}y_{N_2}}{(1 - y_{N_2})^3\tau_p}\right](p^{-1}\text{tr} \Sigma_1)^2 + o_p(1),
\end{align*}
\]
which yields that

\[ \sigma_{121p} = \sigma_{121}^1 + \sigma_{121}^2, \]

\[ \rightarrow \left[ \frac{8y_2(y_1 + y_2)^2 + 4y_1^2 + 4y_2^2}{(1 - y_2)^2} + \frac{8y_1y_2 + 8y_2^3}{(1 - y_2)^3} \right] = \sigma_{121}. \]

As for \( \sigma_{221} \), from (A.27), (A.28), (A.31), (A.34), (A.35), (A.36), (A.48) and (A.49)-(A.55), we get

\[ \sigma_{221p} = \sigma_{221}^1 + \sigma_{221}^2, \]

\[ \rightarrow 4(y_1d_{10} - 1)^2(\eta_{110} + 2y_1d_{20} + n_1y_1l_{110}) + 4(y_1d_{10} - 1)(\eta_{20} + 4y_1d_{30} + 2\beta_1y_1l_{20}) + \eta_{220} + 4y_1(2d_{40} + \beta_1l_{30}) + 4y_1^2d_{20}^2 \]

\[ = \frac{8y_1^2 + 16y_1y_2 + 4y_1^2 + 40y_1^2y_2 + 64y_1y_2}{(1 - y_2)^5} + \frac{8y_1y_2^2 + 56y_1y_2^3 + 48y_2^3 + 8y_1y_2}{(1 - y_2)^7} + \frac{8y_1^2 + 24y_1^2 + 4y_2^2}{(1 - y_2)^8} \]

\[ + 4(\beta_1y_1 + \beta_2y_2) \left[ \frac{(y_1 + y_2)^2}{(1 - y_2)^4} + \frac{2y_2(y_1 + y_2)}{(1 - y_2)^5} + \frac{y_2^2}{(1 - y_2)^6} \right] = \sigma_{220}. \]

Therefore, under the conditions of Theorem 3.1, based on the central limit theorem of martingale difference sequences, we have

\[ T_A + T_B \overset{d}{\longrightarrow} N(0, \sigma_{111} + 2\omega_1\omega_2\sigma_{121} + \omega_2^2\sigma_{220}). \]

Thus, we complete the proof of Theorem 3.1.

The proof of Proposition 3.1. Under Assumptions A-B, from (A.3) and (A.76), we have

\[ p^{-1}\text{tr}\mathbf{S} = p^{-1}\text{tr}\mathbf{S}_w + o_p(1), \quad (A.56) \]

\[ p^{-1}\text{tr}\mathbf{S}^2 = p^{-1}\text{tr}\mathbf{S}_w^2 + \frac{n_1}{p(n_1 + n_2)^2}\text{tr}^2\mathbf{S}_1 + o_p(1), \quad (A.57) \]

\[ + \frac{n_2}{p(n_1 + n_2)^2}\text{tr}^2\mathbf{S}_2 + o_p(1), \]

\[ p^{-1}\text{tr}(\mathbf{S} \circ \mathbf{S}) = p^{-1}\text{tr}(\mathbf{S}_w \circ \mathbf{S}_w) + o_p(1), \quad (A.58) \]

where \( \mathbf{S}_w = n_1/(n_1 + n_2)\mathbf{S}_1 + n_2/(n_1 + n_2)\mathbf{S}_2 \). When \( (\mathbf{S}_1, \mathbf{S}_2) \in \Pi_1 \), we obtain

\[ \hat{\mu}_{10} = \mu_1 + o_p(1), \quad \hat{\sigma}_{110} = \sigma_{111} + o_p(1), \]

\[ \hat{\sigma}_{120} = \sigma_{121} + o_p(1), \quad \hat{\sigma}_{220} = \sigma_{220} + o(1) \]

and

\[ \mu_{21} - \mu_{20} = 2a_1 \left[ \frac{y_{n_1} + y_{n_2}}{(1 - y_{n_2})^2} + \frac{y_{n_2}}{(1 - y_{n_2})^3} \right] + o(1). \]
Therefore, based on Theorem 3.1 and Slutsky’s theorem, we have
\[
P\left( \frac{T_d - \mu_0 - \hat{\mu}_{10}}{\sqrt{\sigma_{110}}} > z_{1-\alpha/2} \right) \\
= P\left( \frac{\sqrt{\sigma_{111}}}{\sqrt{\sigma_{110}}} \cdot \left| \frac{T_d - \mu_0 - \mu_{11} + \mu_{11} - \hat{\mu}_{10}}{\sqrt{\sigma_{111}}} \right| > z_{1-\alpha/2} \right) \\
\rightarrow 1 - \left[ \Phi(z_{1-\alpha/2}) - \Phi(-z_{1-\alpha/2}) \right] = \alpha
\]
and
\[
P\left( \frac{|T_r - \mu_{20}|}{\sqrt{\sigma_{220}}} > z_{1-\alpha/2} \right) \\
= P\left( \frac{\sqrt{\sigma_{220}}}{\sqrt{\sigma_{220}}} \cdot \left| \frac{T_r - \mu_{21} + \mu_{21} - \mu_{20}}{\sqrt{\sigma_{220}}} \right| > z_{1-\alpha/2} \right) \\
\rightarrow 1 - \left[ \Phi(z_{1-\alpha/2} - \Delta_1) - \Phi(-z_{1-\alpha/2} - \Delta_1) \right] > \alpha,
\]
where \( \Phi(\cdot) \) is the distribution function of \( N(0, 1) \) and
\[
\Delta_1 = \frac{2a_1}{\sqrt{\sigma_{220}}} \left[ \frac{y_1 + y_2}{(1 - y_2)^2} + \frac{y_2}{(1 - y_2)^3} \right].
\]
Under the conditions of Theorem 3.1, based on the definition of \( t_\alpha \), we have
\[
P(T_{dr} > t_\alpha) = 1 - P\left( \frac{|T_d - \mu_0 - \hat{\mu}_{10}|}{\sqrt{\sigma_{110}}} \leq t_\alpha, \frac{|T_r - \mu_{20}|}{\sqrt{\sigma_{220}}} \leq t_\alpha \right) \\
= 1 - P\left( \frac{\sqrt{\sigma_{111}}}{\sqrt{\sigma_{110}}} \cdot \left| \frac{T_d - \mu_0 - \mu_{11} + \mu_{11} - \hat{\mu}_{10}}{\sqrt{\sigma_{111}}} \right| \leq t_\alpha, \frac{\sqrt{\sigma_{220}}}{\sqrt{\sigma_{220}}} \cdot \left| \frac{T_r - \mu_{21} + \mu_{21} - \mu_{20}}{\sqrt{\sigma_{220}}} \right| \leq t_\alpha \right) \\
\rightarrow 1 - \int_{-t_\alpha - \Delta_1}^{t_\alpha} \int_{-t_\alpha}^{t_\alpha} f'(x_d, x_r) dx_d dx_r > \alpha,
\]
where \( f'(x_d, x_r) \) is the density of \( N\left( 0_2, \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \right) \) with \( \rho_1 = \sigma_{121} / \sqrt{\sigma_{111} \sigma_{220}} \) and \( t_\alpha' \) satisfies
\[
\alpha = 1 - \int_{-t_\alpha'}^{t_\alpha'} \int_{-t_\alpha'}^{t_\alpha'} f'(x_d, x_r) dx_d dx_r.
\]
According to the definition of \( T_{drx_1} \), we know \( T_{drx_1} \geq T_{dr} \), then we have
\[
P(T_{drx_1} > t_\alpha) \geq P(T_{dr} > t_\alpha),
\]
which yeilds that
\[
\lim_{p \to \infty} P(T_{drx_1} > t_\alpha) \geq \lim_{p \to \infty} P(T_{dr} > t_\alpha) > \alpha.
\]
Based on the definition of $T_{\text{drx}_2}$, we have

$$P(\max\{T_{\text{dr}}, c_\alpha T_X\} > t_{\alpha/2})$$

$$= 1 - P(T_{\text{dr}} \leq t_{\alpha/2}, T_X \leq q_{\alpha/2})$$

$$\geq 1 - P(T_{\text{dr}} \leq t_{\alpha/2}) = P(T_{\text{dr}} > t_{\alpha/2}),$$

then we obtain

$$\lim_{p \to \infty} P(T_{\text{drx}_2} > t_{\alpha/2}) \geq 1 - \int_{-t_{\alpha/2}}^{t_{\alpha/2}} f'(x_d, x_t) dx_d dx_t > \alpha/2,$$

where $t_{1\alpha/2}$ satisfies the following equation

$$\alpha/2 = 1 - \int_{-t_{\alpha/2}}^{t_{\alpha/2}} f'(x_d, x_t) dx_d dx_t.$$ 

**The Proof of Theorem 3.2.** In this subsection, we prove the Theorem 3.2. Similar with the proof of Theorem 2.1, because all quantities are computed under $(\Sigma_1, \Sigma_2) \in \Pi_2$, we add 2 to the subscripts of these quantities. When $(\Sigma_1, \Sigma_2) \in \Pi_2$, we have

$$\Sigma_1 = I_p, \quad \Sigma_2 - \Sigma_1 = a_2/pJ_p, \quad \Sigma_2^{-1} = I_p - a_2/[p(1 + a_2)]J_p. \quad (A.59)$$

Then we have $\text{tr}(\Sigma_1 - \Sigma_2)^2 = a_2^2$, when $\beta_2 = 0$, from (A.12), we obtain

$$\mu_{12} = a_2^2 + y_{n_1} + y_{n_2} + \beta_1 y_{n_1} + o(1).$$

When $\Sigma_1 = I_p$ and $\Sigma_2 = \Sigma_1 + a_2/pJ_p$, we have $\Gamma_2 = \Sigma_1^{-1/2} \Sigma_2^{1/2} = I_p + (\sqrt{1 + a_2} - 1)/pJ_p$ and $T_{2p} = \Gamma_2^T \Sigma_2^{1/2} = \Sigma_2$, which satisfy the Assumptions c-d when we view $\tilde{B}_2$ as $B_n$ in section A.3.3. The eigenvalues of $T_{2p}$ is $1 + a_2, 1, \ldots, 1$, which is coincident with the spiked model considered in [17], based on the Theorem 2 in [17], for $i = 1, 2$, we have

$$c_{i2} = \frac{1}{2\pi p} \oint_{C_1} f_i(-\frac{1}{z} + \frac{y_{n_2}}{1 + z}) \left(\frac{1}{y_{n_2} z} - \frac{(1 + a_2)^2 z}{(1 + (1 + a_2) z)^2}\right) dz$$

$$- \frac{1}{2\pi p} \oint_{C_1} f_i\left(-\frac{1}{z} + \frac{y_{n_2}}{1 + z}\right) \frac{a_2}{(1 + (1 + a_2) z)(1 + z)} \left(\frac{1}{1 + z} - \frac{y_{n_2} z}{(1 + z)^2}\right) dz$$

$$+ \left(1 - \frac{1}{p}\right)c_0 + \frac{1}{p} f_i(\phi_p(a_2)) I(a_2 > \sqrt{y_{n_2}}) + o\left(\frac{1}{p}\right),$$

where $f_1(x) = 1/x$, $f_2(x) = 1/x^2$, $\phi_p(a) = 1 + a_2 + y_{n_2}(1 + a_2)/a_2$ and $C_1$ is a contour counterclockwise, when restricted to the real axes, encloses the interval $[-1/(1 - \sqrt{y_{n_2}}), -1/(1 + \sqrt{y_{n_2}})]$. After calculation, we get

$$c_{i2} = \frac{1}{1 - y_{n_2}} - \frac{a_2}{p(1 - y_{n_2})(1 + a_2)} + o\left(\frac{1}{p}\right) \quad (A.60)$$
Since $B_{\text{a}}B_{\text{d}}\text{e}_\text{a}_{\text{n}}(\text{A.24}), (\text{A.25}), (\text{A.26}), (\text{A.59}), (\text{A.60})$ and $(\text{A.64})$, we get

\[ \delta = \frac{y_{n_2}}{(1 - y_{n_2})^2}, \quad \xi_{12} = \frac{y_{n_2}^2 + 3y_{n_2}}{(1 - y_{n_2})^4}, \quad \xi_{22} = \frac{y_{n_2}^2}{(1 - y_{n_2})^4} \]  

and

\[ \eta_{112} = \frac{2y_2}{(1 - y_2)^4}, \quad \eta_{222} = \frac{4y_2(1 + y_2)}{(1 - y_2)^6}, \quad \eta_{222} = \frac{4y_2(2y_2^2 + 5y_2 + 2)}{(1 - y_2)^8}. \]  

From $(\text{A.59})$ and $(\text{A.17})-(\text{A.23})$, we have

\[ d_{12p} = \frac{1}{(1 - y_{N_2})} + o(1) \to \frac{1}{(1 - y_2)} = d_{10}, \]  

\[ d_{22p} = \frac{1}{(1 - y_{N_2})^3} + o(1) \to \frac{1}{(1 - y_2)^3} = d_{20}, \]  

\[ d_{32p} = \frac{1 + y_{N_2}}{(1 - y_{N_2})^5} + o(1) \to \frac{1 + y_2}{(1 - y_2)^5} = d_{30}, \]  

\[ d_{42p} = \frac{y_{N_2}^2 + 3y_{N_2} + 1}{(1 - y_{N_2})^7} + o(1) \to \frac{y_2^2 + 3y_2 + 1}{(1 - y_2)^7} = d_{40}. \]  

and

\[ l_{12p} = \frac{1}{(1 - y_{N_2})^2} + o(1) \to \frac{1}{(1 - y_2)^2} = l_{10}, \]  

\[ l_{22p} = \frac{1}{(1 - y_{N_2})^4} + o(1) \to \frac{1}{(1 - y_2)^4} = l_{20}, \]  

\[ l_{32p} = \frac{1}{(1 - y_{N_2})^6} + o(1) \to \frac{1}{(1 - y_2)^6} = l_{30}. \]  

Therefore, from $(\text{A.29}), (\text{A.60}), (\text{A.61}), (\text{A.62}), (\text{A.65})$ and $(\text{A.68})$, we have

\[ \mu_{22} = p(y_n, c^2_{12} - 2c_{12} + c_{22}) + p + 2y_n, c_{12} \xi_{12} - 2\xi_{12} + \xi_{22} + y_n, d_{22p} + \beta_1 y_n, l_{12p} + o_p(1) \]

\[ = p \left[ \frac{2}{(1 - y_{n_2})} + \frac{y_{n_1}}{(1 - y_{n_2})^2} + \frac{1}{(1 - y_{n_2})^3} \right] + p \]

\[ + \frac{2a_2^2 - 2a_2(1 + a_2)(y_{n_1} + y_{n_2})}{(1 - y_{n_2})^2} - \frac{2a_2 y_{n_2}}{(1 - y_{n_2})^3(1 + a_2)} \]

\[ - \frac{2y_{n_2}}{(1 - y_{n_2})^4} + \frac{y_{n_1} + 2y_{n_2} y_{n_2}}{(1 - y_{n_2})^4} + \frac{y_{n_2}^2 + 3y_{n_2}}{(1 - y_{n_2})^4} + \frac{\beta_1 y_{n_1}}{(1 - y_{n_2})^2} + o_p(1). \]

Based on $(\text{A.24}), (\text{A.25}), (\text{A.26}), (\text{A.59}), (\text{A.60})$ and $(\text{A.64})$, we get

\[ \sigma_{112p} = 4(y_n, y_{N_2})^2 + o_p(1) \to 4[(y_1 + y_2)^2] = \sigma_{112}. \]
and

\[
\begin{align*}
\sigma_{122p}^1 &= \frac{8(y_{N_1} + y_{N_2})y_{N_2}^2 + 8y_{N_2}^3}{1 - y_{N_2}^2} + \frac{8y_{N_2}^3}{1 - y_{N_2}^2} + o_p(1), \\
\sigma_{122p}^2 &= \frac{8(y_{N_1} + y_{N_2})y_{N_1}y_{N_2} + 8y_{N_1}y_{N_2}}{(1 - y_{N_2})^2} + \frac{8y_{N_1}y_{N_2}}{(1 - y_{N_2})^2} + o_p(1),
\end{align*}
\]

which yields that

\[
\sigma_{122p} = \sigma_{122p}^1 + \sigma_{122p}^2 = \frac{8y_{N_2}(y_{N_1} + y_{N_2}) + 4y_{N_1}^2 + 4y_{N_2}^2}{(1 - y_{N_2})^2} + \frac{8y_{N_1}y_{N_2} + 8y_{N_2}^3}{(1 - y_{N_2})^2} + o_p(1)
\]

\[
\rightarrow \frac{8y_2(y_1 + y_2)^2 + 4y_1^2 + 4y_2^2}{(1 - y_2)^2} + \frac{8y_1y_2 + 8y_2^3}{(1 - y_2)^2} = \sigma_{122}.
\]

From (A.27), (A.28), (A.60), (A.63) and (A.64)-(A.70), we get

\[
\begin{align*}
\sigma_{222p} &= \sigma_{222p}^1 + \sigma_{222p}^2, \\
&\rightarrow 4(y_1d_{10} - 1)^2(\eta_{112} + 2y_1d_{20} + \beta_1y_1l_{10}) \\
&+ 4(y_1d_{10} - 1)(\eta_{122} + 4y_1d_{30} + 2\beta_1y_1l_{20}) \\
&+ \eta_{222} + 4y_1(2d_{40} + \beta_1l_{30}) + 4y_1^2d_{20} \\
&= \frac{8y_1^3 + 16y_1^2y_2}{(1 - y_2)^5} + \frac{4y_1^2 + 40y_1^2y_2 + 64y_1y_2^2}{(1 - y_2)^6} \\
&+ \frac{8y_1y_2^2 + 56y_1y_2^2 + 48y_1^2 + 8y_1y_2}{(1 - y_2)^7} + \frac{8y_2^5 + 24y_3^2 + 4y_2^2}{(1 - y_2)^8} \\
&+ 4\beta_1y_1 \left[ \frac{(y_1 + y_2)^2}{(1 - y_2)^4} + \frac{2y_2(y_1 + y_2)}{(1 - y_2)^5} + \frac{y_1^2}{(1 - y_2)^5} \right] = \sigma_{222}.
\end{align*}
\]

Therefore, under the conditions of Theorem 3.2, based on the central limit theorem of martingale difference sequences, we have

\[
T_A + T_B \xrightarrow{d} N(0, \omega_1^2\sigma_{112} + 2\omega_1\omega_2\sigma_{122} + \omega_2^2\sigma_{222}).
\]

Thus, we complete the proof of Theorem 3.2.

The proof of Proposition 3.2. Under Assumptions A-B, when \((\Sigma_1, \Sigma_2) \in \Pi_2\) and \(\beta_2 = 0\), from (A.56), (A.57) and (A.58), we have

\[
\begin{align*}
\hat{\sigma}_{110} &= \sigma_{112} + o_p(1), \\
\hat{\sigma}_{120} &= \sigma_{122} + o_p(1), \\
\hat{\sigma}_{220} &= \sigma_{222} + o(1), \\
\mu_{12} - \hat{\mu}_{10} &= a_2^2 + o_p(1)
\end{align*}
\]

and

\[
\begin{align*}
\mu_{22} - \hat{\mu}_{20} &= \frac{a_2^2 - 2a_2(1 + a_2)(y_{n_1} + y_{n_2})}{(1 - y_{n_2})^2(1 + a_2)^2} - \frac{2a_2y_{n_2}}{(1 - y_{n_2})^3(1 + a_2)}.
\end{align*}
\]
Therefore, based on Theorem 3.2 and Slutsky's theorem, we have

\[
P\left( \frac{|T_d - \mu_0 - \mu_{10}|}{\sqrt{\sigma_{110}}} > z_{1-\alpha/2} \right) = P\left( \frac{\sqrt{\sigma_{112}}}{\sqrt{\sigma_{110}}} \frac{|T_d - \mu_0 - \mu_{12} + \mu_{12} - \mu_{10}|}{\sqrt{\sigma_{112}}} > z_{1-\alpha/2} \right) \\
\rightarrow 1 - \left[ \Phi(z_{1-\alpha/2} - \Delta_2) - \Phi(-z_{1-\alpha/2} - \Delta_2) \right] > \alpha
\]

and

\[
P\left( \frac{|T_r - \mu_{20}|}{\sqrt{\sigma_{220}}} > z_{1-\alpha/2} \right) = P\left( \frac{\sqrt{\sigma_{222}}}{\sqrt{\sigma_{220}}} \frac{|T_r - \mu_{22} + \mu_{22} - \mu_{20}|}{\sqrt{\sigma_{222}}} > z_{1-\alpha/2} \right) \\
\rightarrow 1 - \left[ \Phi(z_{1-\alpha/2} - \Delta_3) - \Phi(-z_{1-\alpha/2} - \Delta_3) \right] \geq \alpha,
\]

where \(\Delta_2 = \frac{a_2^2}{2(y_1 + y_2)}\) and

\[
\Delta_3 = \frac{1}{\sqrt{\sigma_{222}}} \left[ \frac{a_2^2 - 2a_2(1 + a_2)(y_1 + y_2)}{(1 - y_2)^2(1 + a_2)^2} - \frac{2a_2y_2}{(1 - y_2)^3(1 + a_2)} \right].
\]

Under the conditions of Theorem 3.2, according to the definition of \( \alpha \), we have

\[
P(T_{dr} > t_\alpha) = 1 - P\left( \frac{|T_d - \mu_0 - \mu_{10}|}{\sqrt{\sigma_{110}}} \leq t_\alpha, \frac{|T_r - \mu_{20}|}{\sqrt{\sigma_{220}}} \leq t_\alpha \right) \\
= 1 - P\left( \frac{\sqrt{\sigma_{112}}}{\sqrt{\sigma_{110}}} \frac{|T_d - \mu_0 - \mu_{12} + \mu_{12} - \mu_{10}|}{\sqrt{\sigma_{112}}} \leq t_\alpha, \frac{\sqrt{\sigma_{222}}}{\sqrt{\sigma_{220}}} \frac{|T_r - \mu_{22} + \mu_{22} - \mu_{20}|}{\sqrt{\sigma_{222}}} \leq t_\alpha \right) \\
\rightarrow 1 - \int_{-t_\alpha}^{t_\alpha} \int_{-t_\alpha}^{t_\alpha} f^*(x_d, x_r) dx_d dx_r > \alpha,
\]

where \(f^*(x_d, x_r)\) is the density of \(N\left(0_2, \begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix} \right)\) with \(\rho_2 = \sigma_{122}/\sqrt{\sigma_{112}\sigma_{222}}\) and \(t_\alpha^*\) satisfies

\[
\alpha = 1 - \int_{-t_\alpha^*}^{t_\alpha^*} \int_{-t_\alpha^*}^{t_\alpha^*} f^*(x_d, x_r) dx_d dx_r.
\]

Similar with the proof of Proposition 3.1, we can obtain (IV) and (V). Thus, we complete the proof of Proposition 3.2.

The proof of Proposition 3.3. First, we prove (I). Under the conditions given in the conclusion (I), from (A.56), (A.57) and (A.58), we get

\[
\hat{I}_{10} = p^{-1} \text{tr} \Sigma_w + o_p(1), \\
\hat{I}_{20} = p^{-1} \text{tr} \Sigma^2_w + \frac{y_{n_1}^2 y_{n_2}^2}{(y_{n_1} + y_{n_2})^3} (p^{-1} \text{tr} \Sigma_d)^2 + o_p(1),
\]
where \( \Sigma_d = \Sigma_1 - \Sigma_2 \), which implies the critical value \( t_\alpha \) tends to a constant. Based on (32) and (33) in [5], we get

\[
T_x \overset{a.s.}{\geq} 0.5 \max_{1 \leq i \leq l \leq p} \frac{(\sigma_{1i} \sigma_{2i} - \sigma_{2i} \sigma_{1i})}{\theta_{1i} \theta_{2i} / n_1 + \theta_{2i} \theta_{1i} / n_2} - 4 \log p + 0.5 \log \log p
\]

\[
\overset{a.s.}{\geq} 0.5 \max_{1 \leq i \leq l \leq p} \frac{(\sigma_{1i} \sigma_{2i} - \sigma_{2i} \sigma_{1i})}{\theta_{1i} \theta_{2i} / n_1 + \theta_{2i} \theta_{1i} / n_2} - 4 \log p + 0.5 \log \log p.
\]

Therefore, when \( p \) is large enough, we have

\[
P(T_{dx1} > t_\alpha) = P(T_{dr} + p^2 I(T_x > s(N_1, N_2, p)) > t_\alpha)
\]

\[
\geq P(p^2 I(T_x > s(N_1, N_2, p))) = P(T_x > s(N_1, N_2, p)).
\]

Hence, as \( p \to \infty \), we have \( P(T_{dx1} > t_\alpha) \to 1 \).

Next, we prove (II). According to the definition of \( T_{dx2} \), we get

\[
P(T_{dx2} > t_{\alpha/2}) = P(\max(T_{dr}, c_\alpha T_x) > t_{\alpha/2})
\]

\[
= 1 - P(T_{dr} \leq t_{\alpha/2}, T_x \leq q_{\alpha/2})
\]

\[
\geq 1 - P(T_x \leq q_{\alpha/2}) = P(T_x > q_{\alpha/2}),
\]

when \((\Sigma_1, \Sigma_2) \in \Pi_3\), based on the Theorem 2 in [5], we have \( P(T_{dx2} > t_{\alpha/2}) \to 1 \). Thus, we complete the proof of Proposition 3.3.

### A.3.3. Technical lemmas

We provide some useful lemmas in this subsection. In accordance with the notations in [3], in this subsection, the dimension and sample size are represented by \( n \) and \( N \), respectively. The sample \( x_1, x_2, \ldots, x_N \) is from an \( n \)-dimensional population \( x \). Let \( x_i = (x_{i1}, \ldots, x_{in})^T \) and \( X_n = (x_1, \ldots, x_N) \), define

\[
B_n = N^{-1} \Gamma X_n X_n^\Gamma \Gamma^T,
\]

\[
m_n(z) = m_{F_{B_n}}(z) = \int \frac{1}{\lambda - z} dF_{B_n}(\lambda), \quad \Im(z) \neq 0,
\]

where \( X_n^\Gamma \) denotes the transpose of \( X_n \) and \( \Gamma \) is an invertible \( n \times n \) matrix, \( \Im(z) \) denotes the imaginary part of the complex number \( z \), \( m_n(z) \) denotes the Stieltjes transform of \( F_{B_n} \).

Let \( B_n = N^{-1} X_n^\Gamma \Gamma^T X_n \) (the spectra of which differs from that of \( B_n \) by \(|n - N| \) zeros). Its Stieltjes transform is defined as follow

\[
m_n(z) = m_{F_{B_n}}(z) = \frac{-1 - c_n}{z} + c_n m_n(z),
\]

where \( c_n = n/N \).

- **Assumption a.** The elements \( \{x_{ji}, j = 1, \ldots, n; i = 1, \ldots, N\} \) are \( i.i.d. \) with \( \text{Ex}_{11} = 0, \text{Ex}^2_{11} = 1 \) and \( \beta_x = \text{Ex}^4_{11} - 3 \).
• **Assumption b.** \( N = N(n) \) with \( c_n = n/N \to c > 0 \) as \( n \to \infty \).

• **Assumption c.** The spectral norm of \( T_n = \Gamma \Gamma^T \) is bounded and the ESD \( H_n \) of \( T_n \) converges weakly to a LSD \( H \) as \( n \to \infty \).

• **Assumption d.** The spectral norm of \( T_n^{-1} \), the inverse of \( T_n \), is bounded.

• **Assumption f.** \( \Gamma^T \Gamma \) is diagonal or \( \beta_x = 0 \).

From [20], under Assumptions a-b-c, we known the sample covariance matrix \( B_n \) has the LSD \( F^{c,H} \), namely the Marˇcenko-Pastur distribution of index \((c,H)\), which has support

\[
[a, b] = [(1 - \sqrt{c})^2 I_{0 < c < 1}] \liminf_n \lambda_{\min}^T_n, (1 + \sqrt{c})^2 \limsup_n \lambda_{\max}^T_n.
\]

Moreover, the LSD \( F^{c,H} \) has a Dirac mass \( 1 - 1/c \) at the origin when \( c > 1 \).

Define \( m_c \) to be the Stieltjes transform of the companion LSD \( F^{c,H} = (1 - c)\delta_0 + c F^{c,H} \), where \( \delta_0 \) is the point distribution at zero. Then \( m_c \) is the unique solution in

\[
1 - z = m_c(z) + c \int \frac{t \, dH(t)}{1 + tm_c(z)}, \quad z \in \mathbb{C}^+ = \{ z : \Im z > 0 \}
\]

of the equation

\[
z = -\frac{1}{m_c(z)} + c \int \frac{tdH(t)}{1 + tm_c(z)}, \quad z \in \mathbb{C}^+ = \{ z : \Im z > 0 \}.
\]

Let \( \gamma_j = (1/\sqrt{N}) \Gamma x_j \), \( E_0(\cdot) \) denote expectation and \( E_j(\cdot) \) denote the conditional expectation with respect to the \( \sigma \)-field generated by \( \gamma_1, \cdots, \gamma_j \).

**Lemma A.1.** Let \( M_1, M_2, M_3 \) and \( M_4 \) be symmetric and nonnegative definite matrices with bounded spectral norms, then under Assumptions a-b-c, we have

\[
\sum_{j=1}^N E_{j-1} \{ (E_j - E_{j-1}) \text{tr}(B_n M_1)^2 \} \tag{A.71}
\]

\[
= \frac{1}{N} [2 \text{tr}(T_n^M M_1)^2 + \beta_x \text{tr}(\Gamma^T M_1 \Gamma \circ \Gamma^T M_1 \Gamma)], \tag{A.72}
\]

\[
\sum_{j=1}^N E_{j-1} \{ (E_j - E_{j-1}) \text{tr}(B_n M_3)^2 \}^2 \tag{A.73}
\]

\[
= \frac{4}{N^3} [2 \text{tr}(T_n M_3)^2 + \beta_x \text{tr}(\Gamma^T M_3 \Gamma \circ \Gamma^T M_3 \Gamma)]
\]

\[
+ \frac{8}{N^2} [2 \text{tr}(T_n M_3)^3 + \beta_x \text{tr}(\Gamma^T M_3 T_n M_3 \Gamma \circ \Gamma^T M_3 \Gamma)]
\]

\[
+ \frac{4}{N} [2 \text{tr}(T_n M_3)^4 + \beta_x \text{tr}(\Gamma^T M_3 T_n M_3 \Gamma \circ \Gamma^T M_3 T_n M_3 \Gamma)]
\]

\[
+ \frac{4}{N^2} \text{tr}^2(T_n M_3)^2 + o_p(1),
\]
for \( n \)

For the following of this subsection we will assume \( \nu > 0 \).

Besides, it is worth noting that the identity holds [see (1.15) of [3]]

\[
\sum_{j=1}^{N} \{ (E_j - E_{j-1}) \text{tr}(B_nM_1) [(E_j - E_{j-1}) \text{tr}(B_nM_2)] \} = \frac{1}{N} [2\text{tr}(T_nM_1T_nM_2) + \beta_x \text{tr}(\Gamma^T M_1 \Gamma \circ \Gamma^T M_2 \Gamma)],
\]

\[
\sum_{j=1}^{N} \{ (E_j - E_{j-1}) \text{tr}(B_nM_1) [(E_j - E_{j-1}) \text{tr}(B_nM_3)^2] \} = \frac{2}{N^2} \text{tr}(T_nM_3)[2\text{tr}(T_nM_1T_nM_3) + \beta_x \text{tr}(\Gamma^T M_1 \Gamma \circ \Gamma^T M_3 \Gamma)] + o_p(1)
\]

and

\[
\sum_{j=1}^{N} \{ (E_j - E_{j-1}) \text{tr}(B_nM_3)^2 [(E_j - E_{j-1}) \text{tr}(B_nM_4)^2] \} = \frac{4}{N^3} \text{tr}(T_nM_3)\text{tr}(T_nM_4)[2\text{tr}(T_nM_1T_nM_4) + \beta_x \text{tr}(\Gamma^T M_3 \Gamma \circ \Gamma^T M_4 \Gamma)] + o_p(1).
\]

The proof of Lemma A.1 is similar to that of Lemma 1 given in the Supplementary Material of [21]. Besides, it is worth noting that the identity holds [see (1.15) of [3]]

\[
E[(x_i^T A x_i - \text{tr}A)(x_i^T B x_i - \text{tr}B)] = \beta_x \sum_{i=1}^{n} a_{ii} b_{ii} + \text{tr}AB^T + \text{tr}AB,
\]

for \( n \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \). From (A.76), we have

\[
\text{Etr}(B_nM_1) = \text{tr}(T_nM_1),
\]

\[
\text{Etr}(B_nM_3)^2 = \frac{1}{N} \text{tr}^2(T_nM_3) + \text{tr}(T_nM_3)^2 + \frac{1}{N} [\text{tr}(T_nM_3)^2 + \beta_x \text{tr}(\Gamma^T M_3 \Gamma \circ \Gamma^T M_3 \Gamma)].
\]

Before giving Lemma A.2, some preparatory work is needed. Let \( \nu = \Im(z) \).

For the following of this subsection we will assume \( \nu > 0 \). Without loss of
generality, we assume \(|\|T_n\| \leq 1|\) (\(|\| \cdot \|\) denotes the spectral norm of \(T_n\)) for all \(n\). Let \(D(z) = B_n - zI\), \(D_j(z) = D(z) - \gamma_j^T\),

\[
\varepsilon_j(z) = \gamma_j^T D_j^{-1}(z) \gamma_j - \frac{1}{N} \text{tr} T_n D_j^{-1}(z), \\
\delta_j(z) = \gamma_j^T D_j^{-2}(z) \gamma_j - \frac{1}{N} \text{tr} T_n D_j^{-2}(z) = \frac{d}{dz} \varepsilon_j(z), \\
\beta_j(z) = \frac{1}{1 + \gamma_j^T D_j^{-1}(z) \gamma_j}, \\
\bar{\beta}_j(z) = \frac{1}{1 + N^{-1} \text{tr} T_n D_j^{-1}(z)}, \\
b_n(z) = \frac{1}{1 + N^{-1} \text{tr} T_n D_j^{-1}(z)}.
\]

All of the three latter quantities are bounded in absolute by \(|z|/\nu\) [see (3.4) of [2]]. We have

\[
D^{-1}(z) - D_j^{-1}(z) = -D_j^{-1}(z) \gamma_j^T D_j^{-1}(z) \beta_j(z). \quad (A.79)
\]

For convenience, constants appearing in inequalities will be denoted by \(K\) and may be taken as different values from one expression to the next. Assume that matrix \(A = (a_{ij})\) is a \(n \times n\) matrix with bounded spectral norm, from (A.76) we get

\[
E[(N^{-1} x_1^T A x_1 - N^{-1} \text{tr} A)^2] = \frac{1}{N^2} \left[ \beta_x \sum_{i=1}^n a_{ii}^2 + \text{tr} AA^T + \text{tr} A^2 \right],
\]

which implies that

\[
N^{-1} x_1^T A x_1 = N^{-1} \text{tr} A + o_p(1). \quad (A.80)
\]

**Lemma A.2.** Let \(M_1, M_2, M_3\) and \(M_4\) be \(n \times n\) matrices with bounded spectral norms. Under Assumptions a-b-c-d, when \(B_n\) is invertible, we have

\[
N^{-1} \text{tr}(B_n^{-1} M_1) = \frac{1}{1 - c_n} N^{-1} \text{tr}(T_n^{-1} M_1) + o_p(1), \quad (A.81)
\]

\[
N^{-1} \text{tr}(B_n^{-1} M_1 B_n^{-1} M_2) = \frac{1}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-1} M_1 T_n^{-1} M_2) \]

\[
+ \frac{1}{(1 - c_n)^3} N^{-1} \text{tr}(T_n^{-1} M_1) N^{-1} \text{tr}(T_n^{-1} M_2) + o_p(1), \quad (A.82)
\]
Two-sample tests for high dimensional covariance matrices

\[ N^{-1} \text{tr}(B_n^{-1}M_1B_n^{-1}M_2B_n^{-1}M_3) \quad (A.83) \]

\[ = \frac{1}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2T_n^{-1}M_3) \]

\[ + \frac{1}{(1 - c_n)^4} N^{-1} \text{tr}(T_n^{-1}M_1)^2 N^{-1} \text{tr}(T_n^{-1}M_2T_n^{-1}M_3) \]

\[ + \frac{1}{(1 - c_n)^4} N^{-1} \text{tr}(T_n^{-1}M_2)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_3) \]

\[ + \frac{1}{(1 - c_n)^4} N^{-1} \text{tr}(T_n^{-1}M_3)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2) \]

\[ + \frac{2}{(1 - c_n)^5} N^{-1} \text{tr}(T_n^{-1}M_1)^3 N^{-1} \text{tr}(T_n^{-1}M_2) N^{-1} \text{tr}(T_n^{-1}M_1) + o_p(1) \]

\[ = \frac{1}{(1 - c_n)^4} N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2T_n^{-1}M_3T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_1)^2 N^{-1} \text{tr}(T_n^{-1}M_2T_n^{-1}M_3T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_2)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_3T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_3)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_4)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2T_n^{-1}M_3) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2)^2 N^{-1} \text{tr}(T_n^{-1}M_3T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_3)^2 N^{-1} \text{tr}(T_n^{-1}M_2T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_4)^2 N^{-1} \text{tr}(T_n^{-1}M_2T_n^{-1}M_3) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_2T_n^{-1}M_3)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_4) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_2T_n^{-1}M_4)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_3) \]

\[ + \frac{1}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_3T_n^{-1}M_4)^2 N^{-1} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2) \]

\[ + \frac{2}{(1 - c_n)^6} N^{-1} \text{tr}(T_n^{-1}M_1)^3 N^{-1} \text{tr}(T_n^{-1}M_2) N^{-1} \text{tr}(T_n^{-1}M_1) + o_p(1) \]
\[ D^{-1}(z) = -(zI - b_n(z)T_n)^{-1} + b_n(z)A(z) + B(z) + C(z), \quad \text{(A.85)} \]

where

\[ A(z) = N\sum_{j=1}^{N} (zI - b_n(z)T_n)^{-1}(\gamma_j\gamma_j^T - N^{-1}T_n)D_j^{-1}(z), \]

\[ B(z) = N\sum_{j=1}^{N} (\beta_j(z) - b_n(z))(zI - b_n(z)T_n)^{-1}\gamma_j\gamma_j^T D_j^{-1}(z) \]

and

\[ C(z) = N^{-1}b_n(z)(zI - b_n(z)T_n)^{-1}T_n \sum_{j=1}^{N} (D_j^{-1}(z) - D^{-1}(z)) \]

Under Assumptions a-b-c-d, we know that \( \|(zI - b_n(z)T_n)^{-1}\) and \( \|D_j^{-1}(z)\| \) are bounded. Suppose \( M \) is an \( n \times n \) nonrandom matrix and \( \|M\| \) is bounded, similar with (4.15) and (4.16) in [3], we have

\[ N^{-1}\text{tr}A(z)M = o_p(1), \]

\[ N^{-1}\text{tr}B(z)M = o_p(1) \]

and

\[ N^{-1}\text{tr}C(z)M = o_p(1), \]

therefore

\[ N^{-1}\text{tr}D^{-1}(z)M_1 = -N^{-1}\text{tr}(zI - b_n(z)T_n)^{-1}M_1 + o_p(1). \]

Based on \( b_n(z) = E\beta_1(z) + o(1) \) [see (4.3) of [2]] and

\[ E\beta_1(z) = -zEm_n(z) = 1 - c_n - zc_nEm_n(z), \]

when \( z = 0 \), we have \( b_n(z) = 1 - c_n + o(1) \). Therefore, we get

\[ N^{-1}\text{tr}(B_n^{-1}M_1) = \frac{1}{1 - c_n}N^{-1}\text{tr}(T_n^{-1}M_1) + o_p(1), \]
which is (A.81). Next we will prove (A.82). From (A.79), (A.80) and (A.85), we have

\[
N^{-1} \text{tr} D^{-1}(z)M_1 D^{-1}(z)M_2 \\
= -N^{-1} \text{tr}(zI - b_n(z)T_n)^{-1}M_1 D^{-1}(z)M_2 \\
+ b_n(z) N^{-1} \text{tr} A(z)M_1 D^{-1}(z)M_2 + o_p(1) \\
= -N^{-1} \text{tr} D^{-1}(z)M_2 (zI - b_n(z)T_n)^{-1}M_1 \\
+ b_n(z) N^{-1} \sum_{j=1}^{N} \text{tr}(zI - b_n(z)T_n)^{-1} \\
\times \gamma_j \gamma_j^T D_j^{-1}(z)M_1 (D^{-1}(z) - D_j^{-1}(z))M_2 + o_p(1) \\
= -N^{-1} \text{tr} D^{-1}(z)M_2 (zI - b_n(z)T_n)^{-1}M_1 \\
- b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(z)M_2 (zI - b_n(z)T_n)^{-1} \gamma_j \\
\times \gamma_j D_j^{-1}(z)M_1 D_j^{-1}(z) \gamma_j + o_p(1) \\
= -N^{-1} \text{tr} D^{-1}(z)M_2 (zI - b_n(z)T_n)^{-1}M_1 \\
- b_n(z) N^{-1} \text{tr} D^{-1}(z)M_2 (zI - b_n(z)T_n)^{-1}T_n \\
\times N^{-1} \text{tr} D^{-1}(z)M_1 D^{-1}(z)T_n + o_p(1).
\]

Therefore, when \(z = 0\), we get

\[
N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} M_2 \\
= (1 - c_n)^{-1} N^{-1} \text{tr} B_n^{-1} M_2 T_n^{-1} M_1 \\
+ (1 - c_n)^{-1} N^{-1} \text{tr} B_n^{-1} M_2 N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} T_n + o_p(1).
\]

Let \(M_2 = T_n\), from (A.81), we have

\[
N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} T_n = \frac{1}{(1 - c_n)^3} N^{-1} \text{tr} T_n^{-1} M_1 + o_p(1).
\]

Hence, we have

\[
N^{-1} \text{tr}(B_n^{-1} M_1 B_n^{-1} M_2) \\
= \frac{1}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-1} M_1 T_n^{-1} M_2) \\
+ \frac{1}{(1 - c_n)^3} N^{-1} \text{tr}(T_n^{-1} M_1) N^{-1} \text{tr}(T_n^{-1} M_2) + o_p(1),
\]

which is (A.82). Next we will prove (A.83). Again from (A.79), (A.80) and (A.85),
we get

\[
N^{-1} \text{tr} D^{-1}(z) M_1 D^{-1}(z) M_2 D^{-1}(z) M_3 \\
= -N^{-1} \text{tr} (z I - b_n(z) T_n)^{-1} M_1 D^{-1}(z) M_2 D^{-1}(z) M_3 \\
+ b_n(z) N^{-1} \text{tr} A(z) M_1 D^{-1}(z) M_2 D^{-1}(z) M_3 + o_p(1) \\
= -N^{-1} \text{tr} D^{-1}(z) M_2 D^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} M_1 \\
+ b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j^2(z) \gamma_j^T D_j^{-1}(z) M_1 D_j^{-1}(z) \gamma_j \\
\times \gamma_j^T D_j^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} \gamma_j \\
- b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(z) M_1 D_j^{-1}(z) \gamma_j \\
\times \gamma_j^T D_j^{-1}(z) M_2 D_j^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} \gamma_j \\
- b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(z) M_1 D_j^{-1}(z) M_2 D_j^{-1}(z) \gamma_j \\
\times \gamma_j^T D_j^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} \gamma_j + o_p(1) \\
= -N^{-1} \text{tr} D^{-1}(z) M_2 D^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} M_1 \\
+ b_n(z) N^{-1} \text{tr} D^{-1}(z) M_1 D^{-1}(z) T_n N^{-1} \text{tr} D^{-1}(z) M_2 D^{-1}(z) T_n \\
\times N^{-1} \text{tr} D^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} T_n \\
- b_n^2(z) N^{-1} \text{tr} D^{-1}(z) M_1 D^{-1}(z) T_n \\
\times N^{-1} \text{tr} D^{-1}(z) M_2 D^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} T_n \\
- b_n^2(z) N^{-1} \text{tr} D^{-1}(z) M_1 D^{-1}(z) M_2 D^{-1}(z) T_n \\
\times N^{-1} \text{tr} D^{-1}(z) M_3 (z I - b_n(z) T_n)^{-1} T_n + o_p(1).
\]

Therefore, when \( z = 0 \), we get

\[
N^{-1} \text{tr} (B_n^{-1} M_1 B_n^{-1} M_2 B_n^{-1} M_3) \\
= (1 - c_n)^{-1} N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} M_2 B_n^{-1} M_3 T_n^{-1} M_1 \\
- (1 - c_n)^2 N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} T_n N^{-1} \text{tr} B_n^{-1} M_2 B_n^{-1} T_n N^{-1} \text{tr} B_n^{-1} M_3 \\
+ (1 - c_n) N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} T_n N^{-1} \text{tr} B_n^{-1} M_2 B_n^{-1} M_3 \\
+ (1 - c_n) N^{-1} \text{tr} B_n^{-1} M_1 B_n^{-1} M_2 B_n^{-1} T_n N^{-1} \text{tr} B_n^{-1} M_3 + o_p(1).
\]

Let \( M_3 = T_n \), from (A.81) and (A.82), we have

\[
N^{-1} \text{tr} (B_n^{-1} M_1 B_n^{-1} M_2 B_n^{-1} T_n) \\
= \frac{1}{(1 - c_n)^4} N^{-1} \text{tr} T_n^{-1} M_1 T_n^{-1} M_2 \\
+ \frac{2}{(1 - c_n)^5} N^{-1} \text{tr} T_n^{-1} M_1 N^{-1} \text{tr} T_n^{-1} M_2 + o_p(1).
\]
Hence, we can obtain (A.83). Finally, we will prove (A.84). From (A.79), (A.80)

\[ N^{-1} \text{tr}(D^{-1}(\mathbf{z})M_1 D^{-1}(\mathbf{z})M_2 D^{-1}(\mathbf{z})M_3 D^{-1}(\mathbf{z})M_4) \]

\[ = -N^{-1} \text{tr}((\mathbf{I} - b_n(z)T_n)^{-1}M_1 D^{-1}(\mathbf{z})M_2 D^{-1}(\mathbf{z})M_3 D^{-1}(\mathbf{z})M_4) \]

\[ + b_n(z) N^{-1} \text{tr}(A(z)M_1 D^{-1}(\mathbf{z})M_2 D^{-1}(\mathbf{z})M_3 D^{-1}(\mathbf{z})M_4) + o_p(1) \]

\[ = -N^{-1} \text{tr}(D^{-1}(\mathbf{z})M_2 D^{-1}(\mathbf{z})M_3 D^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1}M_1) \]

\[ - b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j^3(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ + b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j^2(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ + b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ + b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ - b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ + b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ - b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ + b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j^2(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]

\[ - b_n(z) N^{-1} \sum_{j=1}^{N} \beta_j^3(z) \gamma_j^T D_j^{-1}(\mathbf{z})M_1 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_2 D_j^{-1}(\mathbf{z}) \gamma_j \]

\[ \times \gamma_j^T D_j^{-1}(\mathbf{z})M_3 D_j^{-1}(\mathbf{z}) \gamma_j \gamma_j^T D_j^{-1}(\mathbf{z})M_4(z\mathbf{I} - b_n(z)T_n)^{-1} \gamma_j \]
Hence, we can obtain (A.84). Thus, we complete the proof of Lemma A.2.

Let

\[ \left( b_n^2 + (z)N^{-1}trD^{-1}(z)M_1D^{-1}(z)T_nN^{-1}trD^{-1}(z)M_2D^{-1}(z) \times M_3D^{-1}(z)T_nN^{-1}trD^{-1}(z)M_4(zI - b_n(z)T_n)^{-1}T_n \right. \]

Therefore, when \( z = 0 \), we get

\[
N^{-1}tr(B_n^{-1}M_1B_n^{-1}M_2B_n^{-1}M_3B_n^{-1}M_4)
= \left(1 - c_n\right)^{-1}N^{-1}tr(B_n^{-1}M_2B_n^{-1}M_3B_n^{-1}M_4T_n^{-1}M_1)
+ (1 - c_n)^2N^{-1}trB_n^{-1}M_1B_n^{-1}T_nN^{-1}trB_n^{-1}M_2B_n^{-1}T_n
\times N^{-1}trB_n^{-1}M_3B_n^{-1}T_nN^{-1}trB_n^{-1}M_4
\]

\[
+ (1 - c_n)^3N^{-1}trB_n^{-1}M_1B_n^{-1}T_nN^{-1}trB_n^{-1}M_2B_n^{-1}T_n
\times N^{-1}trB_n^{-1}M_3B_n^{-1}M_4
\]

Let \( M_4 = T_n \), from (A.81), (A.82) and (A.83), we have

\[
N^{-1}tr(B_n^{-1}M_1B_n^{-1}M_2B_n^{-1}M_3B_n^{-1}T_n)
= \left(1 - c_n\right)^{-1}N^{-1}trT_n^{-1}M_1T_n^{-1}M_2T_n^{-1}M_3
\]

\[
+ \left(1 - c_n\right)^2N^{-1}trT_n^{-1}M_2N^{-1}trT_n^{-1}M_1T_n^{-1}M_3
\]

\[
+ \left(1 - c_n\right)^3N^{-1}trT_n^{-1}M_3N^{-1}trT_n^{-1}M_2T_n^{-1}M_3
\]

\[
+ \left(1 - c_n\right)^4N^{-1}trT_n^{-1}M_4N^{-1}trT_n^{-1}M_3T_n^{-1}M_2
\]

\[
+ \left(1 - c_n\right)^5N^{-1}trT_n^{-1}M_5N^{-1}trT_n^{-1}M_4T_n^{-1}M_3.
\]

Hence, we can obtain (A.84). Thus, we complete the proof of Lemma A.2. \( \square \)
Lemma A.3. Let \( N' \) be an integer satisfying \( c'_n = n/N' \to c' > 0 \) and \( M_1, M_2 \) and \( M_3 \) be \( n \times n \) matrices with bounded spectral norm. Under Assumptions a-b-c-d, when \( B_n \) is invertible, we have

\[
\frac{1}{N'} \text{tr}(B_n^{-1}M_1B_nM_2) = \frac{1}{N'(1 - c_n)} \text{tr}(T_n^{-1}M_1T_nM_2) \tag{A.86}
\]

\[
- \frac{1}{N'N(1 - c_n)} \text{tr}M_1\text{tr}M_2 + o_p(1)
\]

and

\[
\frac{1}{N'} \text{tr}(B_n^{-1}M_1B_n^{-1}M_2B_nM_3) = \frac{1}{N'(1 - c_n)^2} \text{tr}(T_n^{-1}M_1T_n^{-1}M_2T_nM_3) \tag{A.87}
\]

\[
+ \frac{1}{N'N(1 - c_n)^3} \text{tr}(T_n^{-1}M_1)\text{tr}(T_n^{-1}M_2T_nM_3)
\]

\[
- \frac{1}{N'N(1 - c_n)^2} \text{tr}(M_1T_n^{-1}M_2)\text{tr}M_3
\]

\[
- \frac{1}{N'N(1 - c_n)^2} \text{tr}(T_n^{-1}M_1M_3)\text{tr}M_2
\]

\[
- \frac{1}{N'N^2(1 - c_n)^3} \text{tr}(T_n^{-1}M_1)\text{tr}M_2\text{tr}M_3 + o_p(1).
\]

Proof. Rewrite

\[
B_n = N^{-1}TX_nX_n^T = \sum_{j=1}^{N} \gamma_j\gamma_j^T,
\]

letting \( B_{nj} = B_n - \gamma_j\gamma_j^T \) and \( \eta_j = (1 + \gamma_jB_n^{-1}\gamma_j)^{-1} \), from (A.79), (A.80) and (A.81), we have

\[
\frac{1}{N'} \text{tr}(B_n^{-1}M_1B_nM_2) = \frac{1}{N'} \sum_{j=1}^{N} \gamma_j^T M_2B_n^{-1}M_1\gamma_j
\]

\[
= \frac{1}{N'} \sum_{j=1}^{N} \gamma_j^T M_2(B_{nj}^{-1} - \eta_jB_{nj}^{-1}\gamma_j\gamma_j^T B_{nj}^{-1})M_1\gamma_j
\]

\[
= \frac{1}{N'} \sum_{j=1}^{N} \left[ \gamma_j^T M_2B_{nj}^{-1}M_1\gamma_j - \eta_j\gamma_j^T M_2B_{nj}^{-1}\gamma_j\gamma_j^T B_{nj}^{-1}M_1\gamma_j \right]
\]

\[
= \frac{1}{N'} \left[ \text{tr}(B_n^{-1}M_1T_nM_2) - \eta_j \frac{1}{N'} \text{tr}(B_n^{-1}M_1T_n)\text{tr}(B_n^{-1}T_nM_2) \right] + o_p(1)
\]

\[
= \frac{1}{N'(1 - c_n)} \text{tr}(T_n^{-1}M_1T_nM_2) - \frac{1}{N'N(1 - c_n)} \text{tr}M_1\text{tr}M_2 + o_p(1).
\]
Similarly, from (A.79), (A.80), (A.81) and (A.82), we have

\[
\frac{1}{N} \text{tr}(B_n^{-1}M_1 B_n^{-1}M_2 B_n M_3)
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \gamma_j^T M_2 B_n^{-1}M_1 B_n^{-1}M_2 \gamma_j
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \gamma_j^T M_3 (B_n^{-1} - \eta_j B_n^{-1} \gamma_j \gamma_j^T B_n^{-1}) M_1
\]

\[
\times (B_n^{-1} - \eta_j B_n^{-1} \gamma_j \gamma_j^T B_n^{-1}) M_2 \gamma_j
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \left( \gamma_j^T M_3 B_n^{-1}M_1 B_n^{-1}M_2 \gamma_j
\]

\[
- \eta_j \gamma_j^T M_3 B_n^{-1} \gamma_j \gamma_j^T B_n^{-1}M_1 B_n^{-1}M_2 \gamma_j
\]

\[
- \eta_j \gamma_j^T M_3 B_n^{-1}M_1 B_n^{-1} \gamma_j \gamma_j^T B_n^{-1}M_2 \gamma_j
\]

\[
+ \eta_j^2 \gamma_j^T M_3 B_n^{-1} \gamma_j \gamma_j^T B_n^{-1}M_1 B_n^{-1} \gamma_j \gamma_j^T B_n^{-1}M_2 \gamma_j
\]

\[
= \frac{N}{N'} \left[ N^{-1} \text{tr}(B_n^{-1}M_1 B_n^{-1}M_2 T_n M_3)
\]

\[
- \eta_1 N^{-1} \text{tr}(B_n^{-1} T_n M_3) N^{-1} \text{tr}(B_n^{-1}M_1 B_n^{-1}M_2 T_n)
\]

\[
- \eta_1 N^{-1} \text{tr}(B_n^{-1}M_1 B_n^{-1} T_n M_3) N^{-1} \text{tr}(B_n^{-1} M_2 T_n)
\]

\[
+ \eta_1^2 N^{-1} \text{tr}(B_n^{-1} T_n M_3) N^{-1} \text{tr}(B_n^{-1} M_1 B_n^{-1} T_n)
\]

\[
\times N^{-1} \text{tr}(B_n^{-1} M_2 T_n) \right] + o_p(1)
\]

\[
= \frac{1}{N' (1 - \epsilon_n)^2} \text{tr}(T_n^{-1} M_1 T_n^{-1} M_2 T_n M_3)
\]

\[
+ \frac{1}{N' (1 - \epsilon_n)^3} \text{tr}(T_n^{-1} M_1) \text{tr}(T_n^{-1} M_2 T_n M_3)
\]

\[
- \frac{1}{N' (1 - \epsilon_n)^2} \text{tr}(M_1 T_n^{-1} M_2) \text{tr} M_3
\]

\[
- \frac{1}{N' (1 - \epsilon_n)^2} \text{tr}(T_n^{-1} M_1 M_3) \text{tr} M_2
\]

\[
- \frac{1}{N' (1 - \epsilon_n)^3} \text{tr}(T_n^{-1} M_1) \text{tr} M_2 \text{tr} M_3 + o_p(1). \quad \square
\]

**Lemma A.4.** Let $M_1$, $M_2$ be symmetric and nonnegative definite matrices with bounded spectral norms. Under Assumptions a-b-c-d, when $B_n$ is invertible, we have

\[
\sum_{j=1}^{N} E_j \{ [E_j - E_{j-1}] \text{tr}(B_n^{-1})] [E_j - E_{j-1}] \text{tr}(B_n M_1) \} \tag{A.88}
\]

\[
= - \frac{2}{(1 - \epsilon_n)} N^{-1} \text{tr} M_1 - \frac{\beta_x}{(1 - \epsilon_n)} N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_1 \Gamma) + o_p(1),
\]
\[
\sum_{j=1}^{N} E_{j-1}\left\{(E_j - E_{j-1})\text{tr}(B_n^{-2})\right\}\left\{(E_j - E_{j-1})\text{tr}(B_nM_1)\right\} \quad (A.89)
\]

\[
= -\frac{4}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-1}M_1) - \frac{4}{(1-c_n)^2} N^{-1} \text{tr}T_n^{-1}N^{-1} \text{tr}M_1
\]

\[
- \frac{2\beta_x}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T M_1 \Gamma)
\]

\[
- \frac{2\beta_x}{(1-c_n)^3} N^{-1} \text{tr}T_n^{-1}N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_1 \Gamma) + o_p(1),
\]

\[
\sum_{j=1}^{N} E_{j-1}\left\{(E_j - E_{j-1})\text{tr}(B_n^{-2})\right\}\left\{(E_j - E_{j-1})\text{tr}(B_nM_2^2)\right\} \quad (A.90)
\]

\[
= -\frac{4}{(1-c_n)^2} N^{-1} \text{tr}(T_nM_2^2) - \frac{2\beta_x}{(1-c_n)} N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 T_nM_2 \Gamma)
\]

\[
- \frac{2\beta_x}{(1-c_n)} N^{-1} \text{tr}(T_nM_2)N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 \Gamma) + o_p(1)
\]

and

\[
\sum_{j=1}^{N} E_{j-1}\left\{(E_j - E_{j-1})\text{tr}(B_n^{-2})\right\}\left\{(E_j - E_{j-1})\text{tr}(B_nM_2^2)\right\} \quad (A.91)
\]

\[
= -\frac{8}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-1}M_2 T_nM_2) - \frac{8}{(1-c_n)^3} N^{-1} \text{tr}T_n^{-1}N^{-1} \text{tr}(T_nM_2^2)
\]

\[
+ \frac{4}{(1-c_n)^2} (N^{-1} \text{tr}M_2)^2 - \frac{4\beta_x}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T M_2 T_nM_2 \Gamma)
\]

\[
- \frac{4\beta_x}{(1-c_n)^3} N^{-1} \text{tr}T_n^{-1}N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 T_nM_2 \Gamma)
\]

\[
- \frac{4\beta_x}{(1-c_n)^2} N^{-1} \text{tr}(T_nM_2)N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T M_2 \Gamma)
\]

\[
- \frac{4\beta_x}{(1-c_n)^3} N^{-1} \text{tr}(T_nM_2)N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 \Gamma) + o_p(1).
\]

**Proof.** Based on (A.79), we have

\[
(E_j - E_{j-1})\text{tr}D^{-1}(z)
\]

\[
= \text{tr}\left[(E_j - E_{j-1})(D_j^{-1}(z) - \beta_j(z)D_j^{-1}(z)\gamma_j^T D_j^{-1}(z))\right]
\]

\[
= -(E_j - E_{j-1})\beta_j(z)\gamma_j^T D_j^{-2}(z)\gamma_j.
\]

Since

\[
\beta_j(z) = \beta_j(z) - \beta_j(z)\beta_j(z)\varepsilon_j(z)
\]

\[
= \beta_j(z)\varepsilon_j(z) + \beta_j^2(z)\beta_j(z)\varepsilon_j^2(z),
\]
we obtain
\[-(E_j - E_{j-1})\beta_j(z)\gamma_j^T D_j^{-2}(z)\gamma_j\]
\[= -(E_j - E_{j-1})\left(\bar{\beta}_j(z)\delta_j(z) - \bar{\beta}_j^2(z)\varepsilon_j(z)\delta_j(z)\right.\]
\[\left. - \bar{\beta}_j^2(z)\varepsilon_j(z)N^{-1}\text{tr}(D_j^{-2}(z)T_n) + \bar{\beta}_j^2(z)\beta_j(z)\varepsilon_j^2(z)\gamma_j^T D_j^{-2}(z)\gamma_j\right)\]
\[= -E_j\left(\bar{\beta}_j(z)\delta_j(z) - \bar{\beta}_j^2(z)\varepsilon_j(z)N^{-1}\text{tr}(D_j^{-2}(z)T_n)\right)\]
\[+ (E_j - E_{j-1})\beta_j^2(z)\left(\varepsilon_j(z)\delta_j(z) - \beta_j(z)\gamma_j^T D_j^{-2}(z)\gamma_j\varepsilon_j^2(z)\right).\]

Using (2.3) of [3], we have
\[E\left|\sum_{j=1}^{N} (E_j - E_{j-1})\bar{\beta}_j^2(z)\varepsilon_j(z)\delta_j(z)\right|^2 = \sum_{j=1}^{N} E\left|[E_j - E_{j-1}]\bar{\beta}_j^2(z)\varepsilon_j(z)\delta_j(z)\right|^2\]
\[\leq 4\sum_{j=1}^{N} E\bar{\beta}_j^2(z)\varepsilon_j(z)\delta_j(z)^2 = o(1).\]

This implies that
\[\sum_{j=1}^{N} (E_j - E_{j-1})\bar{\beta}_j^2(z)\varepsilon_j(z)\delta_j(z) \xrightarrow{i.p.} 0.\]

By the same argument, we have
\[\sum_{j=1}^{N} (E_j - E_{j-1})\beta_j(z)\gamma_j^T D_j^{-2}(z)\gamma_j\varepsilon_j^2(z) \xrightarrow{i.p.} 0.\]

Therefore
\[\text{tr}[D^{-1}(z) - ED^{-1}(z)] = \sum_{j=1}^{N} (E_j - E_{j-1})\text{tr}D^{-1}(z)\]
\[= -\sum_{j=1}^{N} (E_j - E_{j-1})\beta_j(z)\gamma_j^T D_j^{-2}(z)\gamma_j\]
\[= -\sum_{j=1}^{N} E_j\left(\bar{\beta}_j(z)\delta_j(z) - \bar{\beta}_j^2(z)\varepsilon_j(z)N^{-1}\text{tr}(D_j^{-2}(z)T_n)\right) + o_p(1).\]

Also because of
\[-E_j \frac{d}{dz} \bar{\beta}_j(z)\varepsilon_j(z) = -E_j \left(\bar{\beta}_j(z)\delta_j(z) - \bar{\beta}_j^2(z)\varepsilon_j(z)N^{-1}\text{tr}(D_j^{-2}(z)T_n)\right),\]
\[(E_j - E_{j-1})\text{tr}(B_n M_1) = \gamma_j^T M_1 \gamma_j - N^{-1}\text{tr}(T_n M_1),\]
based on the central limit theorem of martingale difference sequences, we consider the following sum

\[
\sum_{j=1}^{N} E_{j-1} \left[ - \mathcal{B}_j(z) \varepsilon_j(z) (\gamma_j^T M_1 \gamma_j - N^{-1} \text{tr}(T_n M_1)) \right]
\]

\[
= - \frac{1}{N^2} \sum_{j=1}^{N} E_{j-1} \left[ \mathcal{B}_j(z) (x_j^T \Gamma^T D_j^{-1}(z) \Gamma x_j \right.
\]

\[
- \text{tr}(T_n D_j^{-1}(z)) (x_j^T \Gamma^T M_1 \Gamma x_j - \text{tr}(T_n M_1)) \left. \right] \right]
\]

\[
= - \frac{1}{N^2} \sum_{j=1}^{N} E_{j-1} \left[ \mathcal{B}_j(z) (2 \text{tr}(D_j^{-1}(z) T_n M_1 T_n) \right.
\]

\[
+ \beta_x \text{tr}(\Gamma^T D_j^{-1}(z) \Gamma) + o_p(1) \left. \right]
\]

\[
= - \frac{1}{N^2} \sum_{j=1}^{N} E_{j-1} \left[ b_n(z) (2 \text{tr}(D_j^{-1}(z) T_n M_1 T_n) \right.
\]

\[
+ \beta_x \text{tr}(\Gamma^T D_j^{-1}(z) \Gamma) \left. \right] + o_p(1),
\]

where the last two equalities follow from (2.3) of [3] and

\[
E|\mathcal{B}_j(z) - b_n(z)|^2 \leq K \frac{|z|^4}{N} N^{-1}.
\]

Based on the dominated convergence theorem, we only need to calculate the limits of first derivative and second derivative of \(b_n(z) N^{-1} \text{tr}(D^{-1}(z) T_n M_1 T_n)\) and \(b_n(z) N^{-1} \text{tr}(\Gamma^T D^{-1}(z) \Gamma \circ \Gamma^T M_1 \Gamma)\). Since

\[
b_n'(z) = -b_n(z) N^{-1} \text{tr}(D^{-2}(z) T_n) + o_p(1),
\]

by Lemma A.2, when \(z \to 0\), we have

\[
\frac{d}{dz} \left[ b_n(z) N^{-1} \text{tr}(D^{-1}(z) T_n M_1 T_n) \right]
\]

\[
= -b_n(z) N^{-1} \text{tr}(D^{-2}(z) T_n) N^{-1} \text{tr}(D^{-1}(z) T_n M_1 T_n)
\]

\[
+ b_n(z) N^{-1} \text{tr}(D^{-2}(z) T_n M_1 T_n) + o_p(1)
\]

\[
= \frac{1}{(1 - \epsilon_n)} N^{-1} \text{tr} M_1 + o_p(1)
\]
Therefore, we have

\[
\frac{d^2}{dz^2} \left[ b_n(z)N^{-1} \text{tr}(D^{-1}(z)T_nM_1T_n) \right] = 2\delta_n(z)\left(N^{-1}\text{tr}(D^{-2}(z)T_n)\right)^2N^{-1}\text{tr}(D^{-1}(z)T_nM_1T_n) - 2\delta_n(z)N^{-1}\text{tr}(D^{-3}(z)T_n)N^{-1}\text{tr}(D^{-1}(z)T_nM_1T_n) - 2\delta_n(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(D^{-1}(z)T_nM_1T_n) + 2b_n(z)\text{tr}(D^{-3}(z)T_nM_1T_n) + o_p(1)
\]

\[
= \frac{2}{(1-c_n)^2} N^{-1}\text{tr}(T_n^{-1}M_1) + \frac{2}{(1-c_n)^3} N^{-1}\text{tr}(T_n^{-1}N^{-1}\text{tr}M_1 + o_p(1).
\]

Similarly, we have

\[
\frac{d}{dz} \left[ b_n(z)N^{-1} \text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_1\Gamma) \right] = -\delta_n(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_1\Gamma) + b_n(z)N^{-1}\text{tr}(\Gamma^TD^{-2}(z)\Gamma \circ \Gamma^TM_1\Gamma) + o_p(1)
\]

\[
= \frac{1}{(1-c_n)^2} N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_1\Gamma) + o_p(1)
\]

and

\[
\frac{d^2}{dz^2} \left[ b_n(z)N^{-1} \text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_1\Gamma) \right] = 2\delta_n(z)\left(N^{-1}\text{tr}(D^{-2}(z)T_n)\right)^2N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_1\Gamma) - 2\delta_n(z)N^{-1}\text{tr}(D^{-3}(z)T_n)N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_1\Gamma) - 2\delta_n(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(\Gamma^TD^{-2}(z)\Gamma \circ \Gamma^TM_1\Gamma) + 2\delta_n(z)N^{-1}\text{tr}(\Gamma^TD^{-3}(z)\Gamma \circ \Gamma^TM_1\Gamma)
\]

\[
= \frac{2}{(1-c_n)^2} N^{-1}\text{tr}(T_n^{-2} \circ \Gamma^TM_1\Gamma)
\]

\[
+ \frac{2}{(1-c_n)^3} N^{-1}\text{tr}(T_n^{-1}N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_1\Gamma) + o_p(1).
\]

Therefore

\[
\sum_{j=1}^{N} E_{j-1} \{ [(E_j - E_{j-1})\text{tr}(B_n^{-1})] [(E_j - E_{j-1})\text{tr}(B_nM_1)] \}
\]

\[
= -\frac{1}{N} \sum_{j=1}^{N} E_{j-1} \left[ \frac{2}{(1-c_n)} N^{-1}\text{tr}M_1 + \frac{\beta_x}{(1-c_n)} N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_1\Gamma) \right] + o_p(1)
\]

\[
= -\frac{2}{(1-c_n)} N^{-1}\text{tr}M_1 - \frac{\beta_x}{(1-c_n)} N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_1\Gamma) + o_p(1),
\]

\[
\sum_{j=1}^{N} E_{j-1} \{ [(E_j - E_{j-1})\text{tr}(B_n^{-2})] [(E_j - E_{j-1})\text{tr}(B_nM_1)] \}
\]
Two-sample tests for high dimensional covariance matrices

\[ N \sum_{j=1}^N E_{j-1} \left[ \frac{4}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-1}M_1) \right] \]

\[ + \frac{4}{(1-c_n)^3} N^{-1} \text{tr}(T_n^{-1}N^{-1}trM_1) \]

\[ + \frac{2\beta_x}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^TM_1\Gamma) \]

\[ + \frac{2\beta_x}{(1-c_n)^3} N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^TM_1\Gamma) \] \( + o_p(1) \)

\[ = - \frac{1}{N^3} \sum_{j=1}^N E_{j-1} \left[ \frac{4}{(1-c_n)^2} N^{-1} \text{tr}(T_n^{-1}M_1) \right] \]

For the martingale difference of \( \text{tr}[(B_nM_2)^2] \), we have

\[ (E_j - E_{j-1})\text{tr}[(B_nM_2)^2] \]

\[ = \frac{1}{N^2} \left[ 2(x_j^T\Gamma^TM_2 \sum_{i=1}^{j-1} \Gamma x_i \Gamma^T)M_2 \Gamma x_j - \text{tr}(\Gamma^TM_2 \sum_{i=1}^{j-1} \Gamma x_i \Gamma^T)M_2 \Gamma) \right] \]

\[ + 2\text{tr}(T_nM_2)(x_j^T\Gamma^TM_2 \Gamma x_j - \text{tr}(T_nM_2)) \]

\[ + 2(N - j)(x_j^T\Gamma^TM_2 T_n M_2 \Gamma x_j - \text{tr}(T_nM_2)^2) \] \( + o_p(1) \).

Still based on the central limit theorem of martingale difference sequences, we consider the following sum

\[ \sum_{j=1}^N E_{j-1} \left[ (E_j - E_{j-1})\text{tr}(B_nM_2)^2 (-\beta_j(z)\epsilon_j(z)) \right] \] \( \text{(A.92)} \)

\[ = - \frac{1}{N^3} \sum_{j=1}^N E_{j-1} \left[ 2\beta_j(z)(2\text{tr}(T_nM_2(\sum_{i=1}^{j-1} \Gamma x_i \Gamma^T)M_2 T_n D_j^{-1}(z)) \right] \]

\[ + \beta_x \text{tr}(\Gamma^TM_2 \sum_{i=1}^{j-1} \Gamma x_i \Gamma^T)M_2 \Gamma \circ \Gamma^T D_j^{-1}(z)\Gamma) \]

\[ + 2\beta_j(z)\text{tr}(T_nM_2)(2\text{tr}(T_nM_2 T_n D_j^{-1}(z)) \right) \]

\[ + \beta_x \text{tr}(\Gamma^TM_2 \circ \Gamma^T D_j^{-1}(z)\Gamma) \]

\[ + 2\beta_j(z)(N - j)(2\text{tr}(T_nM_2 T_n M_2 T_n D_j^{-1}(z)) \right) \]

\[ + \beta_x \text{tr}(\Gamma^TM_2 T_n M_2 \Gamma \circ \Gamma^T D_j^{-1}(z)\Gamma) \]

\[ = - \frac{1}{N^3} \sum_{j=1}^N E_{j-1} \left[ 2\beta_j(z)(2 \sum_{i=1}^{j-1} x_i^T \Gamma^TM_2 T_n D_j^{-1}(z)T_n M_2 \Gamma x_i \right] \]
$$+ \beta_x \sum_{i=1}^{j-1} \text{tr}(\Gamma^T \Gamma x_i x_i^T \Gamma \Gamma^T D_{ji}^{-1}(z))$$
$$+ 2\beta_j(z) \text{tr}(T_n \Gamma) (2 \text{tr}(T_n T_n \Gamma^T D_{ji}^{-1}(z)))$$
$$+ \beta_x \text{tr}(\Gamma^T \Gamma^T D_{ji}^{-1}(z))$$
$$+ 2\beta_j(z)(N - j) (2 \text{tr}(T_n \Gamma T_n T_n \Gamma^T D_{ji}^{-1}(z)))$$
$$+ \beta_x \text{tr}(\Gamma^T \Gamma T_n \Gamma \Gamma^T D_{ji}^{-1}(z)) \right].$$

Let

$$D_{ij}(z) = D(z) - \gamma_i \gamma_i^T - \gamma_j \gamma_j^T,$$
$$\beta_{ij}(z) = \frac{1}{1 + \gamma_i^T D_{ij}^{-1}(z) \gamma_i},$$
$$b_i(z) = \frac{1}{1 + N^{-1} \text{etr} T_n D_{ij}^{-1}(z)}.$$

Writing

$$\sum_{i=1}^{j-1} x_i^T \Gamma^T M_2 T_n D_{ji}^{-1}(z) T_n M_2 \Gamma x_i$$
$$= \sum_{i=1}^{j-1} x_i^T \Gamma^T M_2 T_n (D_{ji}^{-1}(z) - \beta_{ji}(z) D_{ji}^{-1}(z) \gamma_i \gamma_i^T D_{ji}^{-1}(z)) T_n M_2 \Gamma x_i$$
$$= \sum_{i=1}^{j-1} \left( x_i^T \Gamma^T M_2 T_n D_{ji}^{-1}(z) T_n M_2 \Gamma x_i$$
$$- \beta_{ji}(z) N^{-1} (x_i^T \Gamma^T M_2 T_n D_{ji}^{-1}(z) \Gamma x_i)^2 \right),$$

and

$$\sum_{i=1}^{j-1} \text{tr}(\Gamma^T \Gamma x_i x_i^T \Gamma \Gamma^T D_{ji}^{-1}(z))$$
$$= \sum_{i=1}^{j-1} \left( \text{tr}(\Gamma^T D_{ji}^{-1}(z)) \Gamma \Gamma^T M_2 \Gamma x_i x_i^T \Gamma \Gamma^T M_2 \Gamma$$
$$- \beta_{ji}(z) N^{-1} \text{tr}(\Gamma^T D_{ji}^{-1}(z) \Gamma x_i x_i^T \Gamma \Gamma^T M_2 \Gamma) \right),$$

we get

$$\text{(A.92)} \quad = - \frac{1}{N^2} \sum_{j=1}^{N} E_{j-1} \left[ 2 b_n(z) (2(j - 1) N^{-1} \text{tr}(D^{-1}(z) T_n M_2 T_n M_2 T_n)$$
$$- 2(j - 1) b_n(z) (N^{-1} \text{tr}(D^{-1}(z) T_n M_2 T_n))^2 \right.$$}
Two-sample tests for high dimensional covariance matrices

\[ +2\text{tr}(T_n M_2) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \]
\[ +2(N - j)N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \]
\[ +\beta_2 b_n(z)(2(j - 1)N^{-1} \text{tr}(\Gamma^T D^{-1}(z)\Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma) \]
\[ +2\text{tr}(T_n M_2) N^{-1} \text{tr}(\Gamma^T D^{-1}(z)\Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma) \]
\[ +2(N - j)N^{-1} \text{tr}(\Gamma^T D^{-1}(z)\Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma)) \right] + o_p(1) \]

\[ = -\frac{1}{N} \sum_{j=1}^{N} E_{j-1} \left[ 2b_n(z) \left( 2N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n M_2 T_n) \right. \right. \]
\[ \left. \left. \left. -2(j - 1)N^{-1} b_n(z)(N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n)) \right)^2 \right. \right. \]
\[ \left. \left. +2N^{-1} \text{tr}(T_n M_2) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n)) \right. \right. \]
\[ \left. \left. +\beta_2 b_n(z)(2N^{-1} \text{tr}(\Gamma^T D^{-1}(z)\Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma) \right. \right. \]
\[ \left. \left. +2N^{-1} \text{tr}(T_n M_2) N^{-1} \text{tr}(\Gamma^T D^{-1}(z)\Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma)) \right. \right. \]
\[ \left. \left. \right) + o_p(1) \right]. \]

Mimicking the discussion above, we need to calculate the limits of the first derivative and second derivative of \( b_n(z) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n M_2 T_n) \), \( b_n(z) \left( N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \right)^2 \), and \( b_n(z) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \). By Lemma A.2, when \( z \to 0 \), we have

\[ \frac{d}{dz} \left[ b_n(z) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n M_2 T_n) \right] \]
\[ = -b_n(z) N^{-1} \text{tr}(D^{-2}(z)T_n) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n M_2 T_n) \]
\[ + b_n(z) N^{-1} \text{tr}(D^{-2}(z)T_n M_2 T_n M_2 T_n) + o_p(1) \]
\[ = \frac{1}{(1 - c_n)} N^{-1} \text{tr}(T_n M_2^2) + o_p(1), \]
\[ \frac{d}{dz} \left[ b_n(z) \left( N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n)) \right)^2 \right] \]
\[ = -2b_n(z) N^{-1} \text{tr}(D^{-2}(z)T_n) \left( N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \right)^2 \]
\[ + 2b_n(z) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) N^{-1} \text{tr}(D^{-2}(z)T_n M_2 T_n) + o_p(1) \]
\[ = \frac{2}{(1 - c_n)} N^{-1} \text{tr}(T_n M_2) N^{-1} \text{tr}(M_2) + o_p(1), \]
\[ \frac{d}{dz} \left[ b_n(z) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \right] \]
\[ = -b_n(z) N^{-1} \text{tr}(D^{-2}(z)T_n) N^{-1} \text{tr}(D^{-1}(z)T_n M_2 T_n) \]
\[ + b_n(z) N^{-1} \text{tr}(D^{-2}(z)T_n M_2 T_n) + o_p(1) \]
\[ = \frac{1}{(1 - c_n)} N^{-1} \text{tr}(M_2) + o_p(1), \]
\[ \frac{d}{dz} \left[ b_n(z) N^{-1} \text{tr}(\Gamma^T D^{-1}(z)\Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma) \right] \]
\[
\begin{align*}
&= -b_n^2(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_2T_nM_2) \\
&\quad + b_n(z)N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_2T_nM_2) + o_p(1) \\
&= \frac{1}{(1 - c_n)}N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_2T_nM_2) + o_p(1)
\end{align*}
\]

and

\[
\begin{align*}
\frac{d}{dz} \left[ b_n(z)N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_2) \right] \\
&= -b_n^2(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(\Gamma^TD^{-1}(z)\Gamma \circ \Gamma^TM_2) \\
&\quad + b_n(z)N^{-1}\text{tr}(\Gamma^TD^{-2}(z)\Gamma \circ \Gamma^TM_2) + o_p(1) \\
&= \frac{1}{(1 - c_n)}N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_2) + o_p(1).
\end{align*}
\]

Therefore

\[
\sum_{j=1}^{N} E_{j-1} \{ [(E_j - E_{j-1})\text{tr}(B_n^{-1})][(E_j - E_{j-1})\text{tr}(B_nM_2)^2] \} \\
= -\frac{4}{(1 - c_n)}N^{-1}\text{tr}(T_nM_2^2) - \frac{2\beta_x}{(1 - c_n)}N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_2T_nM_2) \\
&\quad - \frac{2\beta_x}{(1 - c_n)}N^{-1}\text{tr}(T_nM_2)N^{-1}\text{tr}(T_n^{-1} \circ \Gamma^TM_2) + o_p(1).
\]

Next, the second derivative of those terms are given as follows

\[
\begin{align*}
\frac{d^2}{dz^2} \left[ b_n(z)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_nM_2) \right] \\
&= 2b_n^2(z)(N^{-1}\text{tr}(D^{-2}(z)T_n))^2N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_nM_2T_n) \\
&\quad - 2b_n^2(z)N^{-1}\text{tr}(D^{-3}(z)T_n)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_nM_2) \\
&\quad - 2b_n^2(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(D^{-2}(z)T_nM_2T_nM_2T_n) \\
&\quad + 2b_n(z)N^{-1}\text{tr}(D^{-3}(z)T_nM_2T_nM_2) \\
&\quad + \frac{2}{(1 - c_n)^2}N^{-1}\text{tr}(T_n^{-1}M_2T_nM_2) \\
&\quad + \frac{2}{(1 - c_n)^3}N^{-1}\text{tr}(T_n^{-1}N^{-1}\text{tr}(T_nM_2^2) + o_p(1),
\end{align*}
\]

\[
\begin{align*}
\frac{d^2}{dz^2} \left[ b_n^2(z)(N^{-1}\text{tr}(D^{-1}(z)T_nM_2)) \right] \\
&= 6b_n^4(z)(N^{-1}\text{tr}(D^{-2}(z)T_n))^2(N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n))^2 \\
&\quad - 4b_n^4(z)N^{-1}\text{tr}(D^{-3}(z)T_n)(N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n))^2 \\
&\quad - 8b_n^2(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n) \\
&\quad \times N^{-1}\text{tr}(D^{-2}(z)T_nM_2T_n) \\
&\quad + 2b_n^2(z)(N^{-1}\text{tr}(D^{-2}(z)T_nM_2T_n))^2
\end{align*}
\]
Comparison of empirical powers (in percentages) of the proposed tests with the tests LC and CLX under Scenario 1 when the observations are from the Gaussian and Gamma population.

| No. | p   | (N1, N2) | T_d | T_t | T_dT | LC  | CLX | T_drx1 | T_drx2 |
|-----|-----|----------|-----|-----|------|-----|-----|--------|--------|
|     |     |          |     |     |      |     |     |        |        |
| Gaussian population |
| 1   | 40  | (80,120) | 46.49 | 20.29 | 42.94 | 58.77 | 10.25 | 43.45 | 36.36 |
| 2   | 40  | (120,120) | 65.63 | 23.24 | 61.25 | 75.68 | 10.82 | 61.61 | 53.9  |
| 3   | 40  | (120,160) | 74.78 | 34.25 | 71.4  | 83.46 | 12.03 | 71.71 | 63.92 |
| 4   | 80  | (120,160) | 54.51 | 21.82 | 50.26 | 66.3  | 7.43  | 50.64 | 42.19 |
| 5   | 80  | (160,240) | 82.41 | 46.57 | 80.12 | 89.3  | 9.09  | 80.24 | 73.08 |
| 6   | 80  | (240,240) | 94.85 | 55.07 | 93.14 | 97.24 | 9.61  | 93.19 | 89.26 |
| 7   | 160 | (160,240) | 64.41 | 15.3  | 57.14 | 75.14 | 6.43  | 57.31 | 48.36 |
| 8   | 160 | (240,240) | 82.63 | 15.97 | 76.66 | 89.31 | 6.42  | 76.69 | 68.85 |
| 9   | 160 | (240,320) | 90.96 | 33.43 | 87.73 | 94.77 | 7.23  | 87.73 | 81.55 |
| 10  | 320 | (240,480) | 85.59 | 20.12 | 80.2  | 91.77 | 5.65  | 80.26 | 72.7  |
| 11  | 320 | (320,480) | 95.26 | 20.68 | 92.64 | 97.75 | 5.93  | 92.66 | 88.41 |
| 12  | 320 | (480,480) | 99.45 | 22.55 | 98.96 | 99.81 | 6.41  | 98.96 | 98.13 |
| Gamma population |
| 1   | 40  | (80,120) | 44.14 | 19.89 | 41.78 | 58.19 | 8.01  | 42.18 | 35.46 |
| 2   | 40  | (120,120) | 63.87 | 22.94 | 59.84 | 75.09 | 7.66  | 60.01 | 52.1  |
| 3   | 40  | (120,160) | 73.06 | 32.68 | 70.14 | 82.33 | 9.34  | 70.33 | 62.54 |
| 4   | 80  | (120,160) | 52.32 | 21.11 | 48.4  | 65.04 | 5.63  | 48.56 | 39.91 |
| 5   | 80  | (160,240) | 80.15 | 43.33 | 77.84 | 88.17 | 6.75  | 77.89 | 70.61 |
| 6   | 80  | (240,240) | 93.9  | 50.18 | 92.39 | 92.64 | 7.48  | 92.41 | 88.14 |
| 7   | 160 | (160,240) | 63.43 | 16.03 | 56.4  | 75.11 | 4.5   | 56.54 | 46.82 |
| 8   | 160 | (240,240) | 81.81 | 16.62 | 76.07 | 88.91 | 4.64  | 76.13 | 67.77 |
| 9   | 160 | (240,320) | 90.12 | 32.14 | 86.5  | 94.67 | 5.11  | 86.52 | 80.43 |
| 10  | 320 | (240,480) | 84.51 | 20.03 | 78.71 | 91.6  | 4.52  | 78.73 | 70.39 |
| 11  | 320 | (320,480) | 94.97 | 21.75 | 91.77 | 97.49 | 4.63  | 91.78 | 87.43 |
| 12  | 320 | (480,480) | 99.51 | 22.43 | 99.79 | 99.79 | 4.96  | 99.03 | 98.18 |

\[
+4b_n^2(z)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n)N^{-1}\text{tr}(D^{-3}(z)T_nM_2T_n)
\]

\[
= \frac{4}{(1-c_n)^2}N^{-1}\text{tr}(T_n^2M_2)N^{-1}\text{tr}(T_n^{-1}M_2) + o_p(1),
\]

\[
\frac{d^2}{dz^2}[b_n(z)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n)]
\]

\[
= 2b_n^2(z)(N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n))
\]

\[
-2b_n^2(z)N^{-1}\text{tr}(D^{-2}(z)T_n)N^{-1}\text{tr}(D^{-1}(z)T_nM_2T_n)
\]

\[
-2b_n^2(z)N^{-1}\text{tr}(D^{-3}(z)T_n)N^{-1}\text{tr}(D^{-2}(z)T_nM_2T_n)
\]

\[
+2b_n^2(z)N^{-1}\text{tr}(D^{-3}(z)T_nM_2T_n)
\]
Comparison of empirical powers (in percentages) of the proposed tests with the tests LC and CLX under Scenario 2 when the observations are from the Gaussian and Gamma population.

| No. | \( p \) (\( N_1, N_2 \)) | \( T_{d1} \) | \( T_{d2} \) | \( T_{dr} \) | LC | CLX | \( T_{drx1} \) | \( T_{drx2} \) |
|-----|-----------------|---------|---------|---------|-----|-----|-----------|-----------|
| 1   | 40 (80,120)     | 6.59    | 28.64   | 25.07   | 9.24| 6.2 | 25.64     | 21.32     |
| 2   | 40 (120,120)    | 8.32    | 30.77   | 27.52   | 11.01| 5.11| 27.87     | 22.87     |
| 3   | 40 (120,160)    | 7.83    | 39.17   | 34.6    | 11.14| 6.16| 34.87     | 28.93     |
| 4   | 80 (120,160)    | 7.32    | 37.44   | 32.54   | 10.67| 5.63| 32.91     | 27.12     |
| 5   | 80 (160,240)    | 8.08    | 68.49   | 61.18   | 12.4 | 6.08| 61.3      | 53.34     |
| 6   | 80 (240,240)    | 10.79   | 74.19   | 68.1    | 15.87| 5.75| 68.16     | 60.83     |
| 7   | 160 (160,240)   | 7.66    | 46.33   | 40.3    | 12.15| 5.84| 40.6      | 33.73     |
| 8   | 160 (240,240)   | 9.63    | 48.79   | 43.37   | 15.26| 5.37| 43.6      | 36.19     |
| 9   | 160 (240,320)   | 10.01   | 87.06   | 81.78   | 15.99| 5.56| 81.88     | 75.25     |
| 10  | 320 (240,480)   | 10.48   | 93.17   | 90.38   | 17.8 | 6.72| 90.39     | 85.86     |
| 11  | 320 (320,480)   | 14.85   | 94.25   | 91.58   | 23.66| 6.31| 91.6      | 87.74     |
| 12  | 320 (480,480)   | 21.36   | 95.88   | 93.67   | 31.03| 6.02| 93.69     | 89.92     |

\[
= \frac{2}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-1}M_2) + \frac{2}{(1 - c_n)^3} N^{-1} \text{tr}T_n^{-1}N^{-1} \text{tr}M_2 + o_p(1),
\]

\[
\frac{d^2}{dz^2} \left[ b_n(z) N^{-1} \text{tr}(\Gamma^T D^{-1}(z) \Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma) \right]
\]

\[
= 2b_n^2(z)\left( N^{-1} \text{tr}(D^{-2}(z) T_n) \right)^2 N^{-1} \text{tr}(\Gamma^T D^{-1}(z) \Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma)
-2b_n^2(z)N^{-1} \text{tr}(D^{-3}(z) T_n)N^{-1} \text{tr}(\Gamma^T D^{-1}(z) \Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma)
-2b_n^2(z)N^{-1} \text{tr}(D^{-2}(z) T_n)N^{-1} \text{tr}(\Gamma^T D^{-2}(z) \Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma)
+2b_n(z)N^{-1} \text{tr}(\Gamma D^{-3}(z) \Gamma \circ \Gamma^T M_2 T_n M_2 \Gamma)
\]

\[
= \frac{2}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T M_2 T_n M_2 \Gamma)
- \frac{2}{(1 - c_n)^3} N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 T_n M_2 \Gamma) + o_p(1)
\]
Table A.3

Comparison of empirical powers (in percentages) of the proposed tests with the tests LC and CLX under Scenario 3 when the observations are from the Gaussian and Gamma population.

| No. | p     | (N1, N2) | Td1 | Td2 | TR | LC   | TRx1 | TRx2 |
|-----|-------|----------|-----|-----|----|------|------|------|
| 1   | 40    | (80, 120)| 10.33 | 16.73 | 16.5  | 16.26 | 29.81 | 24.2  | 29.21 |
| 2   | 40    | (120, 120)| 13.86 | 19    | 20.31 | 20.98 | 39.05 | 30.9  | 37.74 |
| 3   | 40    | (120, 160)| 16.48 | 24.98 | 25.04 | 24.66 | 52.75 | 40.68 | 50.93 |
| 4   | 80    | (120, 160)| 7.55  | 9.57  | 9.93  | 11.78 | 32.01 | 19.29 | 27.8  |
| 5   | 80    | (160, 240)| 9.75  | 13.74 | 13.59 | 15.43 | 62.21 | 38.98 | 55.7  |
| 6   | 80    | (240, 240)| 12.94 | 15.89 | 16.66 | 19.85 | 79.95 | 54.76 | 73.89 |
| 7   | 160   | (160, 240)| 5.9   | 7.73  | 7.69  | 9.03  | 46.12 | 24.15 | 39.45 |
| 8   | 160   | (240, 240)| 6.7   | 7.6   | 7.97  | 10.38 | 65.57 | 37.22 | 58.22 |
| 9   | 160   | (240, 320)| 7     | 8.38  | 8.83  | 11.43 | 80.26 | 52.32 | 73.94 |
| 10  | 320   | (240, 480)| 5.99  | 6.83  | 6.62  | 9.36  | 95.2  | 74.93 | 92.28 |
| 11  | 320   | (320, 480)| 6.58  | 6.64  | 6.62  | 9.36  | 95.2  | 74.93 | 92.28 |
| 12  | 320   | (480, 480)| 7.36  | 6.77  | 7.79  | 11.22 | 99.45 | 92.89 | 98.96 |

Gaussian population

| No. | p     | (N1, N2) | Td1 | Td2 | TR | LC   | TRx1 | TRx2 |
|-----|-------|----------|-----|-----|----|------|------|------|
| 1   | 40    | (80, 120)| 11.49 | 17.04 | 17.86 | 18.13 | 28.68 | 24.88 | 29.29 |
| 2   | 40    | (120, 120)| 14.48 | 18.61 | 20.72 | 22.12 | 36.89 | 30.22 | 37.12 |
| 3   | 40    | (120, 160)| 17.07 | 24.65 | 25.64 | 25.83 | 52.63 | 41.4  | 51.19 |
| 4   | 80    | (120, 160)| 8     | 10.4  | 11    | 12.82 | 30.57 | 19.77 | 27.77 |
| 5   | 80    | (160, 240)| 9.87  | 13.19 | 13.65 | 15.44 | 59.92 | 38.61 | 54.54 |
| 6   | 80    | (240, 240)| 12.96 | 14.35 | 15.86 | 19.69 | 77.09 | 52.16 | 71.18 |
| 7   | 160   | (160, 240)| 6.36  | 8.13  | 8.21  | 9.54  | 45.46 | 24.12 | 39.39 |
| 8   | 160   | (240, 240)| 7.15  | 8.14  | 8.95  | 10.44 | 61.49 | 34.69 | 55.13 |
| 9   | 160   | (240, 320)| 7.52  | 8.32  | 9.18  | 12    | 78.66 | 51.86 | 72.31 |
| 10  | 320   | (240, 480)| 6.04  | 6.28  | 6.4   | 8.75  | 83.94 | 56.99 | 78.14 |
| 11  | 320   | (320, 480)| 6.27  | 6.57  | 7.01  | 9.13  | 93.99 | 73.52 | 91.04 |
| 12  | 320   | (480, 480)| 7.36  | 6.77  | 7.79  | 11.22 | 99.27 | 90.62 | 98.51 |

and

\[
\frac{d^2}{dz^2} \left[ b_n(z) N^{-1} \text{tr}(T^{-1}(z) \Gamma \circ \Gamma^T \Sigma \Gamma) \right] \\
= 2b_n(z) (N^{-1} \text{tr}(D^{-2}(z) T_n))^2 N^{-1} \text{tr}(T^T D^{-1}(z) \Gamma \circ \Gamma^T \Sigma \Gamma) \\
- 2b_n^2(z) N^{-1} \text{tr}(D^{-3}(z) T_n) N^{-1} \text{tr}(T^T D^{-1}(z) \Gamma \circ \Gamma^T \Sigma \Gamma) \\
- 2b_n^2(z) N^{-1} \text{tr}(D^{-2}(z) T_n) N^{-1} \text{tr}(T^T D^{-2}(z) \Gamma \circ \Gamma^T \Sigma \Gamma) \\
+ 2b_n^2(z) N^{-1} \text{tr}(T^T D^{-3}(z) \Gamma \circ \Gamma^T \Sigma \Gamma) \\
= \frac{2}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T \Sigma \Gamma) \\
+ \frac{2}{(1 - c_n)^3} N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T \Sigma \Gamma) + o_p(1).
\]
Table A.4
Comparison of empirical powers (in percentages) of the proposed tests with the tests LC and CLX under Scenario 4 when the observations are from the Gaussian and Gamma population.

| No. | p   | (N1, N2)    | Td  | Tr  | Tdr | LC  | CLX   | Td|x1 | Td|x2 |
|-----|-----|-------------|-----|-----|-----|-----|-------|------|------|
|     |     | Gaussian population |     |     |     |     |       |      |      |
| 1   | 40  | (80,120)    | 6.37| 9.95| 9.74| 8.32| 7.79  | 10.89| 10.48|
| 2   | 40  | (120,120)   | 6.41| 10.48|10.74| 8.8 | 8.54  | 12.07| 11.5 |
| 3   | 40  | (120,160)   | 6.65| 11.82|11.33| 9.31| 10.83 | 13.04| 13.21|
| 4   | 80  | (120,160)   | 6.8 | 13.82|12.91| 9.94| 15.04 | 16.27| 17.58|
| 5   | 80  | (160,240)   | 8.45| 23.84|21.64|12.7 | 28.51 | 30.02| 33.08|
| 6   | 80  | (240,240)   | 10.02|27.07|24.07|15.02| 42.55| 38.00| 45.05|
| 7   | 160 | (160,240)   | 12.94|23.77|23.41|20.24| 33.38| 34.02| 38.55|
| 8   | 160 | (240,240)   | 17.53|24.68|27.29|26.22| 48.96| 45.36| 53.45|
| 9   | 160 | (240,320)   | 20.77|53.66|50.06|30.21| 63.57| 68.04| 73.47|
| 10  | 320 | (240,480)   | 76.75|99.98|99.95|85.39| 94.43| 100  | 100  |
| 11  | 320 | (320,480)   | 89.17|100  |99.99|94.12| 98.54| 100  | 100  |
| 12  | 320 | (480,480)   | 97.78|100  |100  |98.87| 99.91| 100  | 100  |

|     |     | Gamma population |     |     |     |     |       |      |      |
| 1   | 40  | (80,120)    | 7.54| 10.86|11.24| 9.31| 6.48  | 12.1 | 10.54|
| 2   | 40  | (120,120)   | 7.71| 11.69|12.31| 9.6  | 7.08  | 13.26| 11.6 |
| 3   | 40  | (120,160)   | 8.35| 12.62|13.17|10.4  | 9.65  | 14.81| 13.56|
| 4   | 80  | (120,160)   | 7.54| 13.82|13.43|10.86| 12.96 | 16.15| 17.01|
| 5   | 80  | (160,240)   | 8.72| 22.18|19.94|13.18| 26.94 | 27.86| 31.66|
| 6   | 80  | (240,240)   | 10.6|24.77|22.8 |15.52| 38.99 | 35.5 | 42.24|
| 7   | 160 | (160,240)   | 12.85|23.98|24.16|19.87| 30.69 | 33.45| 37.37|
| 8   | 160 | (240,240)   | 17.2 |25.3 |27.62|26.39| 45.11 | 42.28| 49.51|
| 9   | 160 | (240,320)   | 21.31|48.88|30.98|59.54| 65.66 | 70.76|
| 10  | 320 | (240,480)   | 76.13|99.96|99.92|84.79| 90.68 | 99.98| 99.96|
| 11  | 320 | (320,480)   | 89.1 |99.97|100  |93.75| 96.66 | 100  | 100  |
| 12  | 320 | (480,480)   | 97.18|100  |100  |98.93| 99.55 | 100  | 100  |

Therefore

\[
\sum_{j=1}^{N} E_{j-1} \{[(E_j - E_{j-1}) \text{tr}(B_n^{-2})][(E_j - E_{j-1}) \text{tr}(B_n M_2)^2]\} \\
= - \frac{8}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-1} M_2 T_n M_2) - \frac{8}{(1 - c_n)^3} N^{-1} \text{tr}(T_n^{-1} N^{-1} \text{tr}(T_n M_2^2)) \\
+ \frac{4}{(1 - c_n)^2} \left( N^{-1} \text{tr}(M_2)^2 \right) - \frac{4\beta_x}{(1 - c_n)^2} N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T M_2 T_n M_2 \Gamma) \\
- \frac{4\beta_x}{(1 - c_n)^3} N^{-1} \text{tr}(T_n M_2 N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 T_n M_2 \Gamma)) \\
- \frac{4\beta_x}{(1 - c_n)^2} N^{-1} \text{tr}(T_n M_2) N^{-1} \text{tr}(T_n^{-2} \circ \Gamma^T M_2 \Gamma)
\]
Two-sample tests for high dimensional covariance matrices

\[-\frac{4\beta_x}{(1-c_n)^3} N^{-1} \text{tr}(T_n M_2) N^{-1} \text{tr}(T_n^{-1} N^{-1} \text{tr}(T_n^{-1} \circ \Gamma^T M_2 \Gamma)) + o_p(1).\]

Thus, the proof of Lemma A.4 is completed. \qed

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