FORM GEOMETRY
AND THE ’tHOOFT-PLEBANSKI ACTION

Ingemar Bengtsson
Fysikum
University of Stockholm
Box 6730, S-113 85 Stockholm, Sweden

Abstract

Riemannian geometry in four dimensions, including Einstein’s equations, can be described by means of a connection that annihilates a triad of two-forms (rather than a tetrad of vector fields). Our treatment of the conformal factor of the metric differs from the original presentation of this result, due to ’tHooft. In the action the conformal factor now appears as a field to be varied.

Email address: ingemar@vana.physto.se
1. INTRODUCTION.

It is a fact that, in four dimensions, the set of all two-forms can be divided into self-dual and anti-self-dual ones. At first sight our fact seems to be a boring algebraic one, but it is by no means so. Indeed this fact somehow manages to serve as a cornerstone of more than one imposing edifice - Penrose’s twistor theory, Ashtekar’s variables for gravity, and Donaldson’s theory of four manifolds may serve as examples. Less grandly, but nevertheless interestingly, ’tHooft pointed out that it allows a peculiar variation of one of the fundamental theorems of Riemannian geometry, and it is this observation which concerns us here.

The theorem (for \(D\) dimensional spaces) is that one can use a \(D\)-ad of vector fields to relate the Riemann tensor and the curvature tensor of an \(\text{SO}(D)\) connection that obeys

\[
D_{[\alpha} e_{\beta]} I = 0 .
\]

(1)

More precisely, the Riemann tensor that one produces is the Riemann tensor of the metric

\[
g_{\alpha\beta} = e_{\alpha I} e_{\beta I} .
\]

(2)

’tHooft’s observation is that (for \(D = 4\)) we can use a triad of two-forms to relate the Riemann tensor and the \(\text{SO}(3)\) curvature tensor of a connection that obeys

\[
\nabla_{[\mu} \Sigma_{\beta\gamma]\iota} = 0 .
\]

(3)

What is now the analogue of eq. (2)? The answer to this question is known, and will be reviewed in section 2.

In section 3 we make some observations about eq. (3) which are of interest in Yang-Mills theory; after all both the Bianchi identities and the four dimensional Yang-Mills equations can be written in this form.

In section 4 we redo ’tHooft’s analysis. Our treatment differs from his in the way that we handle the conformal factor of the metric.

In section 5 we modify ’tHooft’s formulation of the action principle for Einstein’s equations, and bring it into line with the previous section.

We expect that the formalism discussed here can be useful in various contexts (indeed it can be seen as a natural outgrowth of earlier alternatives to the Newman-Penrose formalism). We also harbour a suspicion that the natural “split” of the metric into conformal factor and conformal structure which happens here may turn out to be of considerable physical interest. However, this is only a suspicion. Some comments on the formalism are to be found in section 6.

2. THE SPACE OF TWO-FORMS.

Before we start, let me tell the reader that all my \(\varepsilon\)-tensors take the values \(\pm 1\) in every coordinate system, hence they have non-zero density weights. To define them, all one
needs is an orientation of the space they live on. I never use a metric to raise or lower indices on an $\epsilon$-tensor.

Now let $V$ be a four dimensional vector space and let $W$ be the six dimensional vector space of two-forms on $V$. We introduce the following operations on $W$:

$$\tilde{\Sigma}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \Sigma_{\gamma\delta}$$  \hspace{1cm} (4)

$$\ast \Sigma^{\alpha\beta} = \frac{1}{2 \sqrt{\pm g}} g_{\alpha\mu} g_{\beta\nu} \epsilon^{\mu\nu\gamma\delta} \Sigma_{\gamma\delta}.$$  \hspace{1cm} (5)

The second definition requires a metric $g_{\alpha\beta}$ on $V$. (Actually a conformal structure is enough, since the definition is conformally invariant.) The minus sign in $\sqrt{\pm g}$ is to be used when this metric has Lorentzian signature. The eigenvectors of the star operation are said to be self-dual or anti-self dual, depending on the sign. Depending on the signature, a self-dual form obeys

\((EK) \quad \ast \Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}\) \hspace{1cm} (6)

\((L) \quad \ast \Sigma_{\alpha\beta} = i \Sigma_{\alpha\beta}\) \hspace{1cm} (7)

where $(EK)$ in front of a formula means that is valid for Euclidean and Kleinian signatures of $g_{\alpha\beta}$, while $(L)$ stands for Lorentzian signature.

The space $W$ admits a natural metric, which is defined by

$$<\Sigma, \Sigma> = \Sigma_{\alpha\beta} \tilde{\Sigma}^{\alpha\beta}.$$  \hspace{1cm} (8)

We observe that this metric has Kleinian signature, and also that the twiddle operation can be regarded as using this metric to raise indices in $W$.

The subspace of self-dual two-forms is three dimensional and will be denoted $W^+$, while the subspace of anti-self dual forms is denoted $W^-$. We assume that we have a basis $\Sigma_{\alpha\beta i}$ of $W^+$ available, where the index $i$ runs from one to three. A useful definition is

$$m_{ij} = <\Sigma_i, \Sigma_j>.$$ \hspace{1cm} (9)

This is a metric on $W^+$. (But note carefully that it is a scalar density of weight one under GL(4) transformations of $V$.) Its inverse will be denoted by $m^{ij}$;

$$m_{ik} m^{kj} = \delta_i^j.$$  \hspace{1cm} (10)

A particularly useful fact is that

\footnote{Kleinian signature means that the number of plus and minus signs are equal. The name is due to Gibbons.}
\[ \Sigma_{\alpha\gamma i} \tilde{\Sigma}^{\gamma\beta j} + \Sigma_{\alpha\gamma j} \tilde{\Sigma}^{\gamma\beta i} = -\frac{1}{2} m_{ij} \delta^\beta_\alpha. \]  

(11)

So far this is very well known; first we introduce a metric on \( V \), then the star operation is defined as a map from \( W \) to \( W \), and finally this map is used to define the self-dual subspace \( W^+ \). What is less well known is that the story can be told in reverse. We simply select any three dimensional subspace of \( W \), define a star operator which turns this subspace into the self-dual subspace of \( W \), and at the end use this star operator to define a metric on \( V \). This metric will be determined by the star operation uniquely up to conformal transformations; the procedure provides an alternative way to characterize metrics on four dimensional spaces. There are several points of view on this, and we refer the reader to the literature for the details [3] [4].

The central theorem in the subject is due to Urbantke:

**Theorem (Urbantke):** The subspace \( W^+ \) of \( W \) is the space of self-dual two-forms with respect to the metric

\[ g_{\alpha\beta} = \frac{8}{3} \sigma \epsilon^{ijk} \Sigma_{\alpha\gamma i} \tilde{\Sigma}^{j\delta} \delta_{\delta\beta k}, \]  

(12)

where the conformal factor \( \sigma \) is arbitrary.

The signature of \( g_{\alpha\beta} \) is

- Euclidean if \( \Sigma_{\alpha\beta i} \) are real and \( m_{ij} \) has definite signature,
- Kleinian if \( \Sigma_{\alpha\beta i} \) are real and \( m_{ij} \) has indefinite signature,
- Lorentzian if \( \Sigma_{\alpha\beta i} \) are complex and \( < \Sigma_i, \Sigma_j' > = 0 \).

With regard to the notation used here: Occasionally we use a basis for the anti-self dual subspace \( W^- \), which is then denoted by \( \Sigma_{\iota} \). In the Lorentzian case these two-forms are related by complex conjugation to the \( \Sigma_i \)'s, otherwise they are unrelated objects.

The conformal factor \( \sigma \) of Urbantke’s metric is arbitrary (the factor \( 8/3 \) has been inserted for convenience), but it is needed to ensure that the metric is invariant under \( GL(3) \) rotations in \( W^+ \). Indeed \( \sigma \) transforms as a scalar density of weight minus one under both \( GL(3) \) and \( GL(4) \). Finally it is useful to know that

\[ (E K) \quad \sqrt{g} = 4 \sigma^2 m \]  

(13)

\[ (L) \quad \sqrt{-g} = 4i \sigma^2 m, \]  

(14)

where \( m \) denotes the determinant of \( m_{ij} \). From now on we adopt Urbantke’s metric in \( V \), so that \( g_{\alpha\beta} \) is in fact defined by our choice of the \( \Sigma_i \)'s, or exactly: The conformal structure represented by \( g_{\alpha\beta} \) is defined by our choice of subspace \( W^+ \).

A number of useful relations may now be derived, such as
\[
(EK) \quad \Sigma_{\alpha \gamma} \tilde{\Sigma}_{\beta j}^\gamma = -\frac{1}{4} m_{ij} g_{\alpha \beta} - \sigma m \epsilon_{ijk} \Sigma_{\alpha \beta}^k \\
(L) \quad \Sigma_{\alpha \gamma} \tilde{\Sigma}_{\beta j}^\gamma = -\frac{1}{4} m_{ij} g_{\alpha \beta} + \sigma m \epsilon_{ijk} \Sigma_{\alpha \beta}^k
\]
\[
\Sigma_{[\alpha \gamma} \tilde{\Sigma}_{\beta j]}^\gamma = 0
\]
\[
(EK) \quad \Sigma_{\alpha \beta i} \tilde{\Sigma}_{\gamma \delta i} = \frac{1}{2} (1 + *)_{\alpha \beta}^\gamma^\delta
\]
\[
(L) \quad \Sigma_{\alpha \beta i} \tilde{\Sigma}_{\gamma \delta i} = \frac{1}{2} (1 - i*)_{\alpha \beta}^\gamma^\delta,
\]
where we have used \( m_{ij} \) and \( g_{\alpha \beta} \) to raise and lower Latin and Greek indices on the two-forms, respectively. (We will do this without comment in the sequel.)

Under certain conditions, the formalism that we are developing reduces to the familiar tetrad formalism based on vectors in \( \mathbf{V} \). A precise statement \[5\] is the following:

**Theorem** (Capovilla et al.): The condition

\[
m_{ij} \propto \delta_{ij}
\]

guarantees the existence of a tetrad of vectors \( e_{\alpha I} \) such that

\[
\Sigma_{\alpha \beta i} = e_{\alpha i}, \quad g_{\alpha \beta} = e_{\alpha I} e_{\beta I},
\]
where \( e_{\alpha i} \) denotes the self-dual part of the two-form

\[
e_{\alpha}^{IJ} \equiv e_{[\alpha} e_{\beta]}^I .
\]

As an application \[2\] of the form formalism we consider the Riemann tensor (or any tensor with the same index symmetries). It is elementary to show that the Riemann tensor can be expressed as

\[
R_{\alpha \beta \gamma \delta} = \Sigma_{\alpha \beta i} r^{ij} \Sigma_{\gamma \delta j} + \Sigma_{\alpha \beta i} \tilde{r}^{ij} \tilde{\Sigma}_{\gamma \delta j} + \Sigma_{\alpha \beta i} s^{ij} \Sigma_{\gamma \delta j} + \Sigma_{\alpha \beta i} \tilde{s}^{ij} \tilde{\Sigma}_{\gamma \delta j},
\]
where

\[
r^{ij} = r^{ji}, \quad \tilde{r}^{ij} = \tilde{r}^{ji}, \quad s^{ij} = s^{ji}, \quad \tilde{s}^{ij} = \tilde{s}^{ji}
\]
and

\[
m_{ij} r^{ij} + \tilde{m}_{ij} \tilde{r}^{ij} = 0 .
\]
The last condition comes from the cyclic property of the Riemann tensor. In the Lorentzian case, the bar denotes complex conjugation. For the traceless Ricci tensor and the curvature scalar we find

\[ R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} = -s^{ij'} (\Sigma_{\alpha\gamma i} \Sigma_{\beta\gamma j'} + \Sigma_{\beta\gamma i} \Sigma_{\alpha\gamma j'}) \]  

\[ (EK) \quad \sqrt{g} R = 2 m_{ij} r^{ij} \]  

\[ (L) \quad \sqrt{-g} R = -2 i m_{ij} r^{ij} . \]

These formulæ will be used below.

3. YANG-MILLS GEOMETRY.

Our first exercise is to solve the equation

\[ D_{[\alpha \Sigma_{\beta\gamma}]} i = 0 , \]  

where

\[ D_{\alpha \Sigma_{\beta\gamma i}} = \partial_{\alpha \Sigma_{\beta\gamma}} + \epsilon_{ijk} A_{\alpha j} \Sigma_{\beta\gamma k} , \]  

and the GL(3) invariance is broken down to SO(3) since we have chosen the Kronecker delta to raise and lower the internal indices of the two-forms. This is twelve equations for twelve unknowns.

The exercise is quite straightforward. For definiteness we choose the conventions that lead to a Euclidean signature for the metric. We find the result

\[ A_{\alpha i} = -2 t_{\alpha\beta\gamma ij} \tilde{\Sigma}^{\beta\gamma} k m^{jk} - 2 \Sigma_{\alpha\beta i} \tilde{\Sigma}^{\beta\gamma} j t_{\gamma\delta\sigma k m} \tilde{\Sigma}^{\delta\sigma} m_{mn} m^{jk} , \]  

where

\[ t_{\alpha\beta\gamma ij} = \epsilon_{ijk} (\partial_{\alpha \Sigma_{\beta\gamma k}} + \partial_{\gamma \Sigma_{\alpha\beta k}} + \partial_{\beta \Sigma_{\gamma\alpha k}}) . \]

The result is of some interest for Yang-Mills theory, in two ways. First we choose

\[ \Sigma_{\alpha\beta i} = F_{\alpha i} , \]  

Then what we have done is that we have solved the Bianchi identities for the connection in terms of the field strength, under the assumption that the latter is non-degenerate in the sense that \( \det m_{ij} \neq 0 \). Hence the Wu-Yang ambiguity is quite “mild” in four dimensions.

Alternatively we can choose

\[ \Sigma_{\alpha i} = * F_{\alpha i} , \]  

5
where the $\star$ denotes the star operator defined using the physical space-time metric, which is not the metric that we can construct from the two-forms. Then what we have done is to solve the Yang-Mills equations for the connection as a function of the field strength and the physical metric, which we denote as $h_{\alpha\beta}$, again under a non-degeneracy condition on the field strength. Where did the dynamics go? Actually it is still there, in the equation

$$F_{\alpha\beta i} = F_{\alpha\beta i}(A(F, h)) . \tag{35}$$

This equation is not a pleasant one to analyze, but if we are prepared to disregard the Wu-Yang ambiguity it may at least be regarded as an interesting curiosity, and perhaps more.

In three dimensions one can not use the Bianchi identities in the same way - in fact there is then only one Bianchi identity, and the Wu-Yang ambiguity becomes more serious. If this is disregarded one can solve the Yang-Mills equations in a manner which is analogous to the above. Moreover, in three dimensions the analogue of eq. (35) can be written as an equation for the Ricci tensor of the “Yang-Mills metric” that one can form from the field strength $F$. I do not know whether a similar interpretation can be made for eq. (35).

4. RIEMANNIAN GEOMETRY.

It is crucial to have a direct relation between the curvature tensor of a connection acting on self-dual two-forms, on the one hand, and the Riemann tensor of the Urbantke metric on the other. Provided that we make some slight changes in the definitions of the previous section, such a relation can be indeed be found. Specifically, we will allow the connection to take values in the Lie algebra of GL(3), and not just in any preassigned SO(3) subspace.

Since the metric will eventually be identified with the physical metric, we choose the conventions appropriate to Lorentzian space-times. We also adopt the convention that GL(3) indices are raised and lowered with the metric $g_{ij}$. And so we impose the condition

$$D_{[\alpha \Sigma_{\beta\gamma}]} = \partial_{[\alpha} \Sigma_{\beta\gamma]} + \mathcal{A}_{[\alpha} \Sigma_{\beta\gamma]} = 0 . \tag{36}$$

These are only twelve equations, so it is hard to see how we can solve them for a GL(3) valued connection. Let us therefore postpone this question and go to the next step, which is to introduce an affine connection through the equation

$$\Gamma_{\alpha\beta}^\delta \Sigma_{\delta\gamma i} - \Gamma_{\alpha\gamma}^\delta \Sigma_{\delta\beta i} = D_{\alpha} \Sigma_{\beta\gamma i} . \tag{37}$$

The condition (36) implies that the affine connection is symmetric. If we then count components, we have 40 unknowns. Since a projection to the self-dual subspace is involved - this is not quite self evident, but it follows because the projection operator is a GL(3) scalar - the number of equations is only 36, so that eq. (37) underdetermines the affine connection.

At this point we recall the arbitrary factor in Urbantke’s metric. We can raise the number of equations for the affine connection to 40 by introducing a field $\sigma(x)$ which
transforms as a scalar density of weight minus one under both GL(3) and GL(4). Our final claim is that, once eq. (36) is imposed, an affine connection is defined by the equations

\[ \nabla_\alpha \Sigma_{\beta\gamma i} = 0 \quad (38) \]

\[ \nabla_\alpha \sigma = 0 \ , \quad (39) \]

where

\[ \nabla_\alpha \Sigma_{\beta\gamma i} = D_\alpha \Sigma_{\beta\gamma i} + A_{\alpha i} \Sigma_{\beta\gamma j} = \partial_\alpha \Sigma_{\beta\gamma i} - \Gamma_{\alpha\beta}^{\delta} \Sigma_{\delta\gamma i} - \Gamma_{\alpha\gamma}^{\delta} \Sigma_{\beta\delta i} + A_{\alpha i} \Sigma_{\beta\gamma j} \ , \quad (40) \]

\[ \nabla_\alpha \sigma = \partial_\alpha \sigma + \Gamma_{\alpha\gamma}^{\delta} \sigma - A_{\alpha i}^j \sigma \ . \quad (41) \]

Note that there are by now three covariant derivatives in the game, \( D_\alpha \), \( D_\alpha \) and \( \nabla_\alpha \).

These are the form compatibility conditions, and we must now verify that we can solve them for the affine connection. To do this we introduce Urbantke’s expression for the metric \( g_{\alpha\beta} \), and use the fact that it is a GL(3) scalar to deduce that

\[ D_\gamma g_{\alpha\beta} = \nabla_\gamma g_{\alpha\beta}(\sigma, \Sigma) = 0 \ . \quad (42) \]

Hence the affine connection is metric compatible, and can be expressed as Christoffel symbols in the usual way.

With eq. (38) and the solution for the affine connection in hand, it is of course straightforward to solve for the GL(3) valued connection as a function of the two-forms and \( \sigma \). We obtain

\[ A_{\alpha i}^j = -\tilde{\Sigma}_{\beta\gamma j} D_\alpha \Sigma_{\beta\gamma i} \ . \quad (43) \]

So this problem is solved.

Our next goal, and our main goal, is to relate the curvature tensors. From

\[ 0 = [\nabla_\alpha, \nabla_\beta] \Sigma_{\gamma\delta i} = R_{\alpha\beta\gamma}^\sigma \Sigma_{\sigma\delta i} - R_{\alpha\beta\delta}^\sigma \Sigma_{\sigma\gamma i} + \mathcal{F}_{\alpha\beta i}^j \Sigma_{\gamma\delta j} \quad (44) \]

we may deduce that

\[ \mathcal{F}_{\alpha\beta i} = -2\sigma m \epsilon_{ijk} R_{\alpha \beta} \Sigma_{\gamma i} k \]

\[ \quad \quad = \frac{1}{2\sigma} \epsilon_{ijk} (r^{km} \Sigma_{\alpha\beta m} + s^{km'} \bar{\Sigma}_{\alpha\beta m'}) \ . \quad (45) \]

(In the second step we made use of the notation for the Riemann tensor that was introduced in section 2.) So we see that the GL(3) curvature lies in an SO(3) subalgebra, whatever the choice of \( \Sigma_i \)'s, and moreover that it is simply related to the self-dual part of the Riemann tensor. Conversely, we may express the latter in terms of the former, namely through the equations
\[ r^{ij} = \sigma \epsilon^{imn} \tilde{\Sigma}^{\alpha \beta j} F_{\alpha \beta mn} \] (46)

\[ s^{ij'} = \sigma \epsilon^{imn} \tilde{\Sigma}^{\alpha \beta j'} F_{\alpha \beta mn} \] (47)

\[ \sqrt{-g} R = -2i \sigma \epsilon^{ijk} \tilde{\Sigma}^{\alpha \beta} i F_{\alpha \beta jk} . \] (48)

With this, our proof of 'tHooft’s form version of the fundamental theorem of Riemannian geometry is complete. A few remarks suggest themselves; first of all the part played by the conformal factor \( \sigma \) in the proof is worth watching. Second, in the end it is not surprising that we were able to solve the twelve equations (36) for the connection, because it turned out to be an SO(3) connection after all; specifically we see that

\[ D_\alpha (\sigma m_{ij}) = \nabla_\alpha (\sigma m_{ij}) = 0 . \] (49)

This defines the metric which selects the relevant SO(3) subspace of GL(3). Third, if we refer back to eq. (23) we see that we have not been able to express the entire Riemann tensor as a function of the curvature tensor \( F_{\alpha \beta} \); we are missing the traceless part of the matrix \( r^{ij'} \), which is the same thing as the anti-self dual part of the Weyl tensor. This happened because a self-dual projection was built into eq. (37); having solved for the affine connection we can of course use the resulting expression to act on anti-self dual forms as well, but this does involve a choice. In the real Lorentzian case the anti-self dual Weyl tensor can be reached from the self-dual Weyl tensor through complex conjugation, but otherwise they are algebraically independent objects. (And the Lorentzian reality conditions are awkward to impose.)

5. GENERAL RELATIVITY.

'tHooft’s paper goes on to show how Einstein’s equations can be derived from an action that is a functional of a connection and a triad of two-forms. We will repeat his construction here, with the minor changes caused by the deviations from his treatment that we have already made.

It should occasion no surprise that the action is

\[ S[A, \Sigma, \sigma] = \int \sqrt{-g} (R - 2\lambda) = -2i \int (\sigma \epsilon^{ijk} \tilde{\Sigma}^{\alpha \beta} i F_{\alpha \beta jk} + 4\lambda \sigma^2 m) . \] (50)

We observe that the trace of the connection drops out of the action, so that this is a functional of an SL(3) valued connection only, quite in accordance with ref. [1].

Varying the action with respect to the connection, we find (after minor manipulations) the equation

\[ \epsilon^{ijk} \sigma \hat{D}_\beta \tilde{\Sigma}^{\alpha \beta} k = -\epsilon^{imn} \tilde{\Sigma}^{\alpha \beta} n m^{jk} \hat{D}_\beta (\sigma m_{km}) . \] (51)
where $\hat{D}_\alpha$ is an SL(3) covariant derivative. It will require some effort to extract the content of this equation; we begin with the observation that the equation remains true if the SL(3) covariant derivative is replaced by a GL(3) covariant derivative $D_\alpha$. The trace of the connection is then at our disposal, and we are free to define it through the equation

$$m^{ij} D_\alpha(\sigma_{mij}) = 0 .$$

(52)

From now on we assume that this has been done. Then the next step is to write the symmetric part of eq. (51) in the form

$$M_{\alpha j}^{\beta kl} D_\beta(\sigma_{mkl}) = 0 ,$$

(53)

where we regard $M$ as a matrix acting on vectors that are symmetric and traceless in their Latin indices. We need to show that this matrix is invertible, but we do not necessarily have to invert it. Now it is not difficult - using eq. (16) - to show that

$$M^2 + \frac{1}{2\sigma} M = \frac{3i}{2\sigma^2} 1 .$$

(54)

By going to Jordan’s canonical form, we see that this matrix equation does not allow $M$ to have any zero eigenvalues, therefore it is indeed invertible, and we may conclude that

$$D_\alpha(\sigma_{mij}) = 0 .$$

(55)

Then the anti-symmetric part of eq. (51) gives

$$D_\beta \tilde{\Sigma}_{\alpha \beta}^i = 0 .$$

(56)

This is precisely eq. (36), and when taken together these equations are equivalent to the form compatibility conditions (38 - 39). Therefore the content of these field equations is given by the formula (45), the one that relates the curvature tensor to the self-dual part of the Riemann tensor of Urbantke’s metric.

When we vary the action with respect to $\tilde{\Sigma}_{\alpha \beta}^i$, and use eq. (45) for the curvature tensor in the resulting equation, we obtain

$$(r_{j}^j + 8\lambda \sigma^2 m)\Sigma_{\alpha \beta}^i + s_{ij} \Sigma_{\alpha \beta j'} = 0$$

$$\Leftrightarrow$$

$$R_{\alpha \beta} = \lambda g_{\alpha \beta} .$$

(57)

(58)

This is Einstein’s equations for an arbitrary cosmological constant $\lambda$.

Finally, variation with respect to $\sigma$ gives nothing new. This concludes the demonstration that ‘tHooft’s action gives Einstein’s equations.

Coupling to electromagnetism is straightforward; the only trick employed is the addition of a surface term to the matter action, which is
\[ S[A, \Sigma, \sigma, A] = \int \sqrt{-g}(R - 2\lambda - \frac{1}{4}F^{\alpha\beta}(1 - i\sigma)F_{\alpha\beta}) = \]

\[ = -2i \int (\sigma\epsilon^{ijk}\tilde{\Sigma}_{\alpha\beta i}F_{\alpha\beta jk} + 4\lambda\sigma^2 m - \frac{1}{4}F_{\alpha\beta}\tilde{\Sigma}_{\alpha\beta i}\tilde{\Sigma}_{\gamma\delta i}F_{\gamma\delta}) . \] (59)

Variation with respect to the vector potential gives Maxwell’s equations, and variation with respect to the two-form triad gives the remaining Einstein-Maxwell equations in the form

\[ (\sigma_{\alpha\beta}^i + 8\lambda\sigma^2 m)\Sigma_{\alpha\beta}^i + (\sigma_{ij}^i - \frac{1}{2}F^i\tilde{F}^i)\tilde{\Sigma}_{\alpha\beta}^i = 0 , \] (60)

where we made the obvious definition

\[ F_{\alpha\beta} = F^i\Sigma_{\alpha\beta i} + \tilde{F}^i\tilde{\Sigma}_{\alpha\beta i} . \] (61)

Coupling to spinorial matter requires more elaborate measures, since SL(3) does not have spinorial representations.

As a matter of fact ‘tHooft’s action is closely related to an action first studied in the seventies by Plebanski [7]. Plebanski’s action differs from ‘tHooft’s only in that Plebanski adds the constraint

\[ \sigma m_{ij} = \delta_{ij} \] (62)

to the action by means of a Lagrange multiplier. Because of the theorem by Capovilla et al. that we quoted earlier, the formalism then rapidly collapses to the familiar tetrad formalism. Moreover it is known that Ashtekar’s Hamiltonian formulation of Einstein’s equations can be obtained from the Plebanski action in a few easy steps [8]. With Plebanski’s constraint added, we are obviously dealing with an SO(3) connection only. This is actually the case also in the more general setting considered by ‘tHooft, with the interesting difference that the relevant SO(3) subspace of GL(3) is then determined dynamically by the field equations, rather than imposed from the outside.

6. DISCUSSION.

Having introduced four different covariant derivatives I may have lost the reader, so let me summarize the results before discussing them. We start with an action that depends on an SL(3) connection:

\[ S = -2i \int \sigma\epsilon^{ijn}m_{mk}\tilde{\Sigma}_{\alpha\beta i}^j F_{\alpha\beta k}^i . \] (63)

Varying the action with respect to the connection, one finds that the field equations can be rewritten using a GL(3) covariant derivative \( D_\alpha \), with the trace of the connection chosen in a particular way, so that the equations take the form
\[ D_\beta \tilde{\Sigma}^{\alpha \beta}_i = 0 \quad D_\alpha (\sigma_{mij}) = 0. \] (64)

These conditions guarantee that the covariant derivative can be further extended to a GL(4) covariant derivative \( \nabla_\alpha \), defined using a symmetric affine connection, such that
\[
\nabla_\alpha \Sigma_{\beta \gamma i} = 0 \quad \nabla_\alpha \sigma = 0. \] (65)

In their turn these conditions guarantee that the derivative \( \nabla_\alpha \) is compatible with the metric
\[
g_{\alpha \beta} = \frac{8}{3} \sigma \epsilon^{ijk} \sigma_{\alpha \gamma i} \tilde{\Sigma}^{\gamma \delta}_j \Sigma_{\delta \beta k} \] (66)

- which is, up to a factor, the unique metric tensor with respect to which the \( \Sigma_i \)'s are self-dual. Now eq. (65) allows us to express the curvature tensor of the original connection in terms of the self-dual Riemann tensor of the metric (66). If we go back to the action, vary with respect to the two-form, and insert the expression for the curvature tensor that we just obtained, we find Einstein’s equations. Variation with respect to \( \sigma \) gives nothing new. A cosmological constant and couplings to matter can be added without any ado.

An obvious drawback which our formalism shares with all “chiral” formalisms for gravity (such as Ashtekar’s variables) is that the Lorentzian reality conditions are awkward to impose. There are also some obvious strengths; formalisms that are related to ours have been extensively used in many problems such as classifying geometries and the like (see ref. [2] for a review). However, the feature that we wish to stress is the clean separation of the metric into conformal structure and conformal factor which is achieved here, through a peculiarly four dimensional mechanism.

The way this happens is somewhat analogous to the appearance of the “complexion” in Rainich’s formulation of electrodynamics [3]; there is a factor left undetermined in the algebraic part of the discussion, which then turns into a field in the differential part. In our case it is essential that the field \( \sigma \) carries non-zero density weight with respect to both GL(3) and GL(4). A certain scalar density plays a similar role also in the CDJ action for gravity [4], which is a functional of this field and the self-dual spin connection alone. However, the CDJ action suffers from two drawbacks which are not shared by our action; it breaks down for certain algebraically special field configurations, and it is hopelessly complicated for almost anything except pure gravity with vanishing cosmological constant [10].

A feature peculiar to the present formalism is that the variation of the action with respect to the field \( \sigma \) - to which we are loosely referring as “the conformal factor” - does not add further content to the field equations. It might therefore seem to be a very innocent bystander in the theory. Nevertheless I suspect that the present formalism can be used to illuminate a wide range of problems in relativity where conformal transformations are being made.
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References

[1] G. ’tHooft, Nucl. Phys. B357 (1991) 211.

[2] W. Israel, Differential Forms in General Relativity, Commun. Dublin Inst. Adv. Stud. Ser. A No 26, 1979.

[3] H. Urbantke, J. Math. Phys. 25 (1984) 2321.
   G. Harnett, J. Phys. A25 (1992) 5649.

[4] S.K. Donaldson and D.P. Sullivan, Acta. Math. 163 (1989) 181.
   T. Dray, R. Kulkarni and J. Samuel, J. Math. Phys. 30 (1989) 1306.
   G. Harnett, J. Math. Phys. 32 (1990) 84.

[5] R. Capovilla, J. Dell, T. Jacobson and L. Mason, Class. Quant. Grav. 8 (1991) 41.

[6] F.A. Lunev, Phys. Lett. B295 99.

[7] J.F. Plebanski, J. Math. Phys. 18 (1977) 2511.

[8] G.Y. Rainich, Trans. Am. Math. Soc. 27 (1925) 106.
   C.W. Misner and J.A. Wheeler, Ann. of Phys. 2 (1957) 525.

[9] R. Capovilla, J. Dell and T. Jacobson, Phys. Rev. Lett. 63 (1989) 2325.

[10] P. Peldán, Class. Quant. Grav. 8 (1991) 1765.