Landau and leading singularities in arbitrary space-time dimensions

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Using the decomposition of the $D$-dimensional space-time into parallel and perpendicular subspaces, we study and prove a connection between Landau and leading singularities for $N$-point one-loop Feynman integrals by applying the multi-dimensional theory of residues. We show that if $D = N$ and $D = N + 1$, the leading singularity corresponds to the inverse of the square root of the leading Landau singularity of the first and second type, respectively. We make use of this outcome to systematically provide differential equations of Feynman integrals in canonical forms and the extension of the connection of these singularities at the multi-loop level by exploiting the loop-by-loop approach. Illustrative examples with the calculation of Landau and leading singularities are provided to supplement our results.

I. INTRODUCTION

The study of Feynman integrals as analytic functions has a long history. It is already apparent in the definition of Feynman propagators, which is the integral in the complex plane with the poles moved from the real axis by the $+i\epsilon$ prescription allowing integration over real momenta. However, the real rise of the analytic approach to Feynman integrals, or the scattering matrix in general, began at the turn of the 50s and 60s of the previous century with the works of Landau, Bjorken and Nakanishi, who introduced the notion of what is presently known as Landau equations and the Landau singularity [1–3]. Landau equations give a necessary condition for a function defined by an integral to have singularities. Together with an attempt to formulate nuclear interactions in terms of the scattering matrix in a program called S-matrix theory, there had been a period of intensive studies of the analytic properties of the scattering matrix and Feynman integrals. This program resulted in multifarious methods of studying the analytic structure of Feynman integrals including Hadamard’s lemma, topological methods, differential equations, and series expansion, to mention some [4–7].

After the success of QCD, however, the S-matrix program had been mostly abandoned, leaving one branch which focused on differential equations for generalised functions still going for a few more years in terms of micro-local analysis and the hyper-function theory [8, 9]. In the 90s of the XX century there began a renaissance of analytic studies of scattering amplitudes and Feynman integrals, which started with works on unitarity and cuts [10–12], some other important developments include introduction of amplituhedron whose definition is based on the singularity structure [13, 14]. In this work, we focus on two fundamental concepts, Landau singularities and leading singularities. The first one results from a system of equations, and describes a variety in a space spanned by kinematic variables, which is a singular locus of integrals depending on parameters allowing for a study of their analytic structure. For mathematical definitions of Landau varieties, we refer the Reader to [15, 16]. Recently, Landau singularities again grabbed the attention of the community and interesting works appeared focusing on mathematical aspects of these structures [16–24].

The second notion describes the maximal residue of Feynman integrals [25]. The knowledge of leading singularities plays a crucial role in the analytic calculation of Feynman integrals through differential equations methods [26–28]. The identification of $d\log$ structures has led to a systematic procedure to evaluate Feynman integrals in terms of transcendental numbers and special functions [29]. Therefore, the connection between Landau and leading singularities needs to be provided and mathematically supported. This is, in effect, the main target of this communication, in which we will show that for any scalar one-loop Feynman integral the leading singularities can be expressed by the leading Landau singularities, of the first and second type, by accounting for the appropriated space-time dimension.

In order to elaborate on the connection between Landau and leading singularities, we depth on the study of Feynman integrals in an arbitrary space-time dimension, by performing an analysis at the integrand level. This is carried out through a decomposition of the space-time dimension in terms of two independent and complementary subspaces: parallel and perpendicular. This decomposition has been initially considered in [30–33] and recently applied to simplifications of multi-loop scattering amplitudes within the integrand reduction method framework [34, 35]. We analyse in detail the analytic and singular structure of scalar one-loop Feynman integrals with $N$ external momenta (often referred to as $N$-gons, and as they are interesting objects there has already been effort to understand their properties [19, 36–42]).

Contributions

- By elaborating on the decomposition of the space-time dimension into two independent subspaces (parallel and perpendicular) and an extensive use of the multivariate theory of residues given by Leray, we prove that for one-loop integrals leading
Feynman integrals in arbitrary space-time dimensions

We discuss Feynman integrals in arbitrary space-time dimensions. This paper is organised as follows. In Section II, we introduce an alternative representation of Feynman integrals which splits space-time into parallel and perpendicular components, this will lay a basis for our studies. In Section III, we introduce the notion of Landau singularities, including simple examples. Then we derive an equation for the Landau singularity in the parallel and perpendicular representation, showing that it obeys a very simple form, and all the information about the singularity is included in the perpendicular component. This allows us for an alternative derivation of the Landau singularity based on on-shell conditions. Section IV is devoted to leading singularities, where we give illustrative one-loop examples. Then, we discuss and formulate the two main theorems concerning the dependence of leading singularities on the space-time dimension and its connection to leading Landau singularities. This section ends with an application of the obtained results in the context of the method of differential equations for Feynman integrals. Section V is a prelude to the multiloop investigation of Landau and leading singularities and the connection between them, supported by examples of integrals with leading singularities. The work ends with conclusions and a discussion of possible directions for further studies. The paper is aided by Appendices containing mathematical background on Leray’s theory of residues (A), supplementary results

Having at hand a method to extract leading singularities by simply looking at Landau singularities, we discuss the application of this relation in the analytic calculation of Feynman integrals by the method of differential equations [26, 27]. We show that within our approach the so-called canonical form of the differential equation is straightforwardly obtainable regardless of the space-time dimension [29].

On top of the studies carried out for one-loop Feynman integrals, we employed the multivariate theory of residues given by Leray [43] in the calculation of leading singularities of two-loop planar and non-planar three-point functions with massive external and massless internal momenta. As a by-product of this direct calculation, we elaborated on the loop-by-loop approach and providing the $L$-loop ladder diagrams for three and four-point functions.

Outline

This paper is organised as follows. In Section II, we discuss Feynman integrals in arbitrary space-time dimension $D$. Then, we introduce an alternative representation of Feynman integrals which splits space-time into parallel and perpendicular components, this will lay a basis for our studies. In Section III, we introduce the notion of Landau singularities, including simple examples. Then we derive an equation for the Landau singularity in the parallel and perpendicular representation, showing that it obeys a very simple form, and all the information about the singularity is included in the perpendicular component. This allows us for an alternative derivation of the Landau singularity based on on-shell conditions. Section IV is devoted to leading singularities, where we give illustrative one-loop examples. Then, we discuss and formulate the two main theorems concerning the dependence of leading singularities on the space-time dimension and its connection to leading Landau singularities. This section ends with an application of the obtained results in the context of the method of differential equations for Feynman integrals. Section V is a prelude to the multiloop investigation of Landau and leading singularities and the connection between them, supported by examples of integrals with leading singularities. The work ends with conclusions and a discussion of possible directions for further studies. The paper is aided by Appendices containing mathematical background on Leray’s theory of residues (A), supplementary results

which are used in the derivation of main results (B), more examples of one-loop leading singularities (C), and the proof of the main theorem in the momentum representation (D).

II. FEYNMAN INTEGRALS IN ARBITRARY SPACE-TIME DIMENSIONS

Let us start by recalling that a $D$-dimensionally regularised Feynman integral, where $D$ is a complex number, at $L$ loops with $N$ external momenta can be written as,

$$J_{N}^{(L),D} (1,\ldots,n;n+1,\ldots,m) = \int \frac{L}{(2\pi)^{D/2}} \prod_{i=1}^{L} d^{D} \ell_{i} \prod_{k=n+1}^{m} D_{k}^{\nu_{k}} \prod_{j=1}^{n} D_{j}^{\nu_{j}},$$ (1)$$

where the first $n$ propagators, with integer powers, $\nu_{i} \in \mathbb{Z}$, represent the loop topology and are expressed as

$$D_{i} = q_{i}^{2} - m_{i}^{2} + i\delta.$$ (2)$$

Here, $q_{i}$ contains the dependence on loop ($\ell_{i}$) and external ($p_{i}$) momenta. For instance, in one-loop topologies, we can set, $q_{i} = \ell_{i} + P_{i-1}$, with $P_{i-1} = p_{1} + \ldots + p_{i-1}$ and $P_{0} = 0$, for a combination of external momenta (see Fig. 1). For the $D$-dimensional integral (1), loop and external momenta are elements of infinite dimensional vector space. The mass of the internal propagators and infinitesimal Feynman prescription are, respectively, $m_{i}$ and $+i\delta$. Throughout this paper, we will consider different kinematic configurations, hence, no assumption on internal and external masses is given at this point.

The remaining $(m-n)$ propagators in (1) correspond to irreducible scalar products often named as auxiliary propagators, whose structure can be chosen as in Eq. (2) or as a product between external and internal momenta, $2\ell_{i} \cdot p_{j}$. For this set of propagators, their powers are only positive integer numbers, $\nu_{i} \in \mathbb{Z}_{\geq 0}$.

![FIG. 1. Loop-momentum configuration for one-loop $n$-point Feynman integrals (N-gons).](image-url)
A. Feynman integrals in parallel and perpendicular space

Since Feynman integrals can be characterised by the number of external momenta, vertices, and internal propagators, one can think of a decomposition in terms of two independent and complementary subspaces, which we will refer to parallel, with dimension $D_{||}$, and perpendicular, of dimension $D_{\perp}$ (we will use the same symbols for the dimensions and spaces itself, the meaning will be clear from the context),

$$D = D_{||} + D_{\perp},$$

(3)
in which the first one, say $D_{||} = \min (|D|, E)$, is spanned by the independent external momenta, $E$. The second subspace, on the contrary, is spanned by momenta that are orthogonal to the external momenta, $D_{\perp} = D - D_{||}$.

Throughout this paper we follow decomposition (3) to cast all information of external momenta in $D_{||}$. We will observe in the following discussion that, regardless of the space-time dimension, it is possible to recast the formal $D$-dimensional integral (1) as an ordinary $m$-dimensional integral, where $m$ corresponds to the number of propagators, showing in this way a clear connection with the Baikov representation [44, 45]. We emphasize that to carry out this analysis, we do not consider a particular dimension, instead, we perform this analysis by considering the most general features of Feynman integrals in the context of leading and Landau singularities.

Since we perform an analysis at the integrand level, one can translate the decomposition of the space-time dimension to the parametrization of the $D$-dimensional loop momenta, which, within this framework, we can express as:

$$\ell_{i,[D]}^\alpha = \ell_{i,[D_{||}]}^\alpha + \ell_{i,[D_{\perp}]}^\alpha,$$

(4)

that in the parametrization of parallel and perpendicular directions of the loop momenta, we have, by definition, $\ell_{i,[D_{||}]}^\alpha \ell_{j,[D_{\perp}]}^\alpha = \ell_{i,[D_{||}]}^\alpha \cdot \ell_{j,[D_{\perp}]}^\alpha = 0$.

B. One-loop decomposition

The decomposition into parallel and perpendicular subspaces will help us to make sense of the formal $D$-dimensional integral (1) and transform it to the form of finite (regularised) dimensional integrals which will be suitable for our analysis. We will focus only on scalar Feynman integrals and present arguments for the one-loop case. Multi-loop Feynman integrals within this representation will be discussed in the next section.

The one-loop scalar Feynman integral in the discussed decomposition (4) can be written as,

$$J_N^{(1),D} = \frac{1}{i\pi^{D/2}} \int \frac{d\ell_{\perp}^1}{\prod_{i=1}^{D_{\perp}}} \int d^{D_{||}}\ell_{\parallel} \frac{1}{\prod_{j=1}^{D_{||}} D_j},$$

(5)

where to simplify notation, $\ell_{\|} = \ell_{1,[D_{||}]}$ and $\ell_{\perp} = \ell_{1,[D_{\perp}]}$.

Since scalar Feynman integrals depend only on scalar products between external and loop momenta, we can transform integration over perpendicular components to spherical coordinates and perform integration over angular components,

$$J_N^{(1),D} = \frac{\Omega D_{\perp}^{D_{\perp} - 1}}{i\pi^{D/2}} \int \int_{0}^{\infty} d\ell_{\perp}^1 \int d^{D_{||}}\ell_{\parallel} \frac{1}{\prod_{j=1}^{D_{||}} D_j}.$$

(6)

In effect we obtained finite-dimensional integral of $D_{\perp} + 1$ variables $\ell_{1,[D_{\perp}]}$ which will serve as a definition of the formal $D$-dimensional integral (1). The denominators $D_j$ are functions of the vectors $q_j^\alpha = q_{j,\|}^\alpha + q_{j,\perp}^\alpha$, which in view of the above definition can be defined as the $D_{\perp} + 1$ dimensional vectors (i.e., $\alpha = 1, \ldots, D_{||} + 1$). As we will be mostly concerned with the integrands, this form is enough for our analysis. In general, convergence of (6) depends on $D$, and the convergence region can be determined by analytic continuation in $D$ [47].

To elucidate further the convergence of integral (6), let us consider the particular case of $D = 4 - 2\epsilon$ space-time dimensions in the four-dimensional helicity scheme [48]. Integration measure in one-loop Feynman integrals, as suggested by Ref. [49], can be decomposed into two independent subspaces,

$$d^{D}\ell_{1} = d\ell_{1,[4]} d^{2\epsilon}\ell_{1,[-2\epsilon]}.$$

(7)

This decomposition, after integration over the angular components of spherical coordinates in the additional $-2\epsilon$ dimension and keeping in mind external momenta in $D = 4$, allows to cast the wedge product in $D = 4 - 2\epsilon$ space-time dimensions as,

$$d^{4-2\epsilon}\ell_{1} = d\ell_{1}^\epsilon \wedge \ldots \wedge d\ell_{4}^\epsilon \wedge d\mu^2,$$

(8)

where $\mu^2 = -\ell_{1,[-2\epsilon]}^\alpha \ell_{1,[-2\epsilon]}^\alpha \cdot$ With this decomposition in mind, we continue our analysis in parallel and perpendicular subspaces.

The dimension of the parallel subspace is associated with the number of linearly independent external momenta. Let us then assume that this subspace is spanned by independent external momenta,

$$\ell_{||}^\alpha = \ell_{1,[D_{||}]}^\alpha = \sum_{j=1}^{D_{||}} a_j p_j^\alpha.$$

(9)
We now introduce the following change of variables $\ell^2 = \lambda_{11}$, with the differential $d\ell_\perp = \frac{1}{2} \lambda_{11}^{-\frac{3}{2}} d\lambda_{11}$. In the new variables, integral (6) is given by,

$$J_N^{(1)} = \mathcal{J}_{[D_i]}^{(1)} \Omega_{D\perp - 1} \int P_i d\alpha_i \lambda_{11}^{(D_{\perp} - 2)/2} \frac{1}{\prod_{j=1}^{D_i} D_j},$$

with $\mathcal{J}_{[D_i]}$ being the Jacobian of the transformation of the parallel loop-momentum components $(a_i)$. To find this Jacobian, we write the parallel loop momentum in terms of components,

$$\ell^1_i = \sum_{j=1}^{D_i} p_j^1 a_j,$$

$$\ell^2_i = \sum_{j=1}^{D_i} p_j^2 a_j,$$

$$\vdots$$

$$\ell^{D_i} = \sum_{j=1}^{D_i} p_j^{D_i} a_j.$$

Now, we can express the differential form $d\ell^1_{\parallel} \wedge \ldots \wedge d\ell^{D_i}_{\parallel}$ in the new variables $(a_i)$,

$$d\ell^1_{\parallel} \wedge \ldots \wedge d\ell^{D_i}_{\parallel} = \left( \sum_{j=1}^{D_i} p_j^1 da_j \right) \wedge \ldots \wedge \left( \sum_{j=1}^{D_i} p_j^{D_i} da_j \right),$$

that, because of Lemma B.2, amounts to,

$$d\ell^1_{\parallel} \wedge \ldots \wedge d\ell^{D_i}_{\parallel} = \det(J) da_1 \wedge \ldots \wedge da_{D_i}.$$

Notice that the determinant of the Jacobian matrix $J$ of this transformation in the above form is not very insightful. To put it in a more useful form, we make use of the identity $\det(J^T gJ) = \det(J)^2 \det(g) \Rightarrow \det(J) = \pm \sqrt{\det(J^T gJ)/\det(g)}$, where $g = \text{diag}(1, -1, \ldots, -1)$ is the metric tensor in Minkowski space with, $\det(g) = (-1)^{D_\perp - 1}$. By calculating the product,

$$J^T gJ = \begin{pmatrix} p_1^1 & p_2^1 & \ldots & p_{D_i}^1 \cr p_1^2 & p_2^2 & \ldots & p_{D_i}^2 \cr \vdots & \vdots & \ddots & \vdots \cr p_1^{D_i} & p_2^{D_i} & \ldots & p_{D_i}^{D_i} \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \cr 0 & -1 & \ldots & 0 & 0 \cr \vdots & \vdots & \ddots & \vdots & \vdots \cr 0 & 0 & \ldots & 0 & -1 \end{pmatrix},$$

we get,

$$J^T gJ = \begin{pmatrix} p_1 \cdot p_1 & \ldots & p_1 \cdot p_{D_i} \cr \vdots & \ddots & \vdots \cr p_1 \cdot p_{D_i} & \ldots & p_{D_i} \cdot p_{D_i} \end{pmatrix}.$$ (15)

Therefore, we have,

$$\det(J^T gJ) = \det(p_1 \cdot p_j).$$

Compiling all together, we have the following formula for the Jacobian,

$$\mathcal{J}_{[D_i]} = \det(J) = \pm \sqrt{(-1)^{D_\perp - 1} \det(p_1 \cdot p_j)}$$

$$= \pm \sqrt{(i)^{D_i} - \det(p_1 \cdot p_j)}.$$ (17)

After this preliminary observation on the integrand representation, let us comment on the structure of the integrand. Feynman propagators $D_j$’s, due to parametrization (4), are polynomials quadratic in $a_i$ and linear in $\lambda_{11}$. For instance, in $D_{\parallel} = n - 1 > 0$ and $D_{\perp} = D - n + 1 > 0$, Feynman propagators present in one-loop Feynman integrals, become,

$$D_i = (\ell_1 + p_1)^2 - m_i^2$$

$$= \sum_{j,k=1}^{n-1} (p_j \cdot p_k) a_j a_k$$

$$+ 2 \sum_{j=1}^{n-1} \sum_{k=1}^{i-1} (p_j \cdot p_k) a_j + \sum_{j,k=1}^{i-1} (p_j \cdot p_k) - m_i^2 + \lambda_{11}.$$

In the above decomposition, we notice that the analysis of any one-loop Feynman integral can be carried out by only keeping track of the number of independent external momenta. In effect, regardless of the space-time dimension, one needs to perform $D_{\parallel} + 1$ (or $N$) integrations to elucidate the singular structure of the integral, since all integrations coming from the perpendicular subspace are reduced to a single one ($\lambda_{11}$).
C. Multi-loop decomposition

Let us now continue our analysis with the generic multi-loop Feynman integral, where inspired by the differential form obtained in the integration measure for one-loop Feynman integrals, one can think of an extension at \( L \) loops given by,

\[
\prod_{i=1}^{L} d^D \ell_i = d^{L+\frac{D}{2}} \Lambda d^D \ell_\parallel.
\]

By taking into account the parametrization of loop momenta (4), relations (19) take the explicit form,

\[
d^{L\parallel} \ell_\parallel = \mathcal{J}^{(L)}_\parallel \prod_{i=1}^{\parallel} d\ell_i,
\]

\[
d^{L+\frac{D}{2}} \Lambda = \frac{\Omega^{(L)}_\parallel}{2^L} [G(\lambda_{ij})]^{\frac{D-2}{2}} \prod_{1\leq i\leq j}^L d\lambda_{ij},
\]

with \( G(\lambda_{ij}) \) corresponding to the Gram determinant of the perpendicular directions \( \lambda_{ij} \equiv \ell_i \cdot [\ell_j] \cdot [D_\perp] \) and \( \mathcal{J}^{(L)}_\parallel \) the Jacobian of the transformation given by,

\[
\mathcal{J}^{(L)}_\parallel = \pm (i)^L D_\parallel \prod_{i=1}^L \left\{ -\det \left( \frac{\partial \ell_{\alpha i}^{(L)}}{\partial \alpha_i} \frac{\partial \ell_{\alpha j}^{(L)}}{\partial \alpha_k} \right) \right\},
\]

with \( j, k \leq D_\parallel \),

where the overall sign \( \pm \) in the Jacobian originates from the square roots and, for the purpose of our discussion does not introduce any ambiguity. Likewise, notice that the overall prefactor of \( i \) in the Jacobian does not play any important role. Therefore, in the discussion of the following sections, we will neglect it.

In view that our analysis is performed at the integrand level, we observe that all scalar products between external and internal momenta (as well as propagators) are expressed in terms of the loop-momentum components \( a_{ij} \) and \( \lambda_{ij} \). A more involved structure that arises when considering helicity amplitudes can methodically be integrated out through Gegenbauer polynomials, as elucidated in Ref. [34].

Therefore, similar to one-loop, the integration of spherical coordinates in scalar multi-loop Feynman integrals are cast in,

\[
\Omega^{(L)}_\perp = \prod_{i=1}^{L} \Omega^{L-\parallel}_{D_\perp}.
\]

With the integration measure given by Eq. (19) in the expression of a generic multi-loop Feynman integral (1) at our disposal, an alert reader might notice a similarity with the Baikov representation. In effect, this derivation has been performed in [50, 51], where the decomposition of space-time dimension, in parallel and perpendicular directions, was employed. Said differently, one obtains,

\[
d^{D_\parallel} \ell_\parallel d^{\frac{L+\parallel}{2}} \Lambda \sim \prod_{i=1}^{m} dD_i,
\]

in which the number of integrations in l.h.s. and r.h.s. is the same, \( L(L+1)/2 + EL = m \), as earlier anticipated.

Elaborating on this loop-momentum parametrization, we can proceed in the following to analyze Landau and leading singularities of different loop topologies. We shall recall that the physical information is contained in the parallel direction of the space-time dimension. The remaining integrations are simply cast in \( \lambda_{ij} \). Therefore, we will mostly draw our attention to integrations in the parallel subspace.

III. LANDAU SINGULARITIES

We will focus on the form of the Landau equations in the decomposition of the space-time in terms of parallel and perpendicular components.

Let us first look at what form Landau equations take for our definition of \( D \)-dimensional integral (6). Landau equations can be derived by requiring that polar sets \( D_j \) are in non-general positions, i.e. when differential \( dD_j \) are not linearly independent together with the requirement that we are confined to a singular variety of the integrand (on-shell conditions). Thus, we have,

\[
D_i = 0 \quad \text{for} \quad i = 1, \ldots, N,
\]

\[
\sum_i \alpha_i dD_i =
\]

\[
\sum_i \alpha_i \left( \frac{\partial D_i}{\partial \ell_\parallel} d\ell_\parallel + \cdots + \frac{\partial D_i}{\partial \ell_\perp} d\ell_\perp \right) =
\]

\[
\left( \sum_i \alpha_i \frac{\partial D_i}{\partial \ell_\parallel} \right) d\ell_\parallel + \cdots + \left( \sum_i \alpha_i \frac{\partial D_i}{\partial \ell_\perp} \right) d\ell_\perp = 0.
\]

Taking into account the independence of differential \( d\ell \) the brackets in the last line of (25b) give the well-known form of the Landau equations. In principle one can study singularities of subdiagrams by imposing \( \alpha_i = 0 \) for some \( i \), then Eq. (25a) in general should be stated as \( D_i = 0 \) or \( \alpha_i = 0 \). However, since we are interested in leading Landau singularities, all \( \alpha \)'s are non-zero.

We impose that the Landau equations has nontrivial solution for \( \alpha \)'s which happens when \( \det(\frac{\partial D_i}{\partial \ell}) \) vanishes. By multiplying by the transpose of the matrix \( (\frac{\partial D_i}{\partial \ell}) \), this system can be put into the form,

\[
\sum_i \alpha_i (q_i \cdot q_j) = 0, \quad j = 1, \ldots, N.
\]
There is one such system for each independent loop in the diagram. Together with on-shell conditions and momentum conservation, this gives us the equation for the leading Landau variety which we also call Landau singularity. In practice solving Landau equations for complicated Feynman diagrams can be very difficult. However, for one-loop diagrams, as we have only one equation of the type (26) it simplifies a lot. Thus, here and in the following, we define, for one-loop diagrams,

$$\text{LanS}_n^{(1)} = \det(q_i \cdot q_j), \quad i, j = 1, \ldots, N,$$  \hspace{1cm} (27)

which should be understood that on-shell conditions and momentum conservation are taken into account. Then, the solution LanS$_n^{(1)} = 0$ gives us the Landau singularity for one-loop diagrams. In this respect, Eq. (27) is given by the Gram determinant of internal momenta which can vanish identically if the number of internal momenta is larger than the dimension of the space. We will study the dimensional dependence of Landau singularities for one-loop diagrams.

In the following, we will explore Landau equations in the parametrization of the loop momenta according to (4). To start, we will observe in one-loop Feynman integrals that solutions to these equations, because of the decomposition of the dimension, are cast in $\lambda_{11} = \ell_1^2$, once on-shell conditions are imposed.

**A. One-loop Landau singularities**

Because of the decomposition of the space-time dimension into parallel and perpendicular subspaces that was laid out in Sec. II A, one can work out, within this framework, the general solution of Eq. (26). We will show that this provides the following expression for Landau singularities of a one-loop $n$-point Feynman integral,

$$\text{LanS}_n^{(1)} = \lambda_{11} \det((p_i \cdot p_j)_{(n-1) \times (n-1)}),$$  \hspace{1cm} (28)

which holds only when on-shell conditions are imposed. To achieve that, let us first express the Gram determinant of propagators (26) in terms of parallel and perpendicular components,

**Proposition III.1.**

$$\text{LanS}_n^{(1)} = \det(q_i \cdot q_j) = \det((q_i \cdot q_j) + \lambda_{11})_{n \times n}$$
$$= \det((q_i \cdot q_j))_{n \times n}$$
$$+ \lambda_{11} \sum_{k=1}^{n} \det\left((q_i \cdot q_j)(1 - \delta_{jk}(1 - \frac{1}{q_i \cdot q_j}))\right)_{n \times n}$$
$$= \lambda_{11} \sum_{k=1}^{n} \left((q_i \cdot q_j)(1 - \delta_{jk}(1 - \frac{1}{q_i \cdot q_j}))\right)_{n \times n},$$  \hspace{1cm} (29)

with $q_i^\alpha = \ell_1^\alpha [P_i] + P_i^\alpha$, according to Fig. 1.

Proof. We work in a matrix form and write $q_{i,\parallel} \cdot q_{j,\parallel}$ as $q_{ij}$; we compactly express $\det(q_{ij} + \lambda_{11})$ in a column form as follows,

$$|Q_1 + \lambda_{11}, Q_2 + \lambda_{11}, \ldots, Q_N + \lambda_{11}|,$$  \hspace{1cm} (30)

where $Q_i = (q_{ij}, \ldots, q_{nj})^T$ and $\lambda_{11}$ should be understood as a column vector with all entries equal to $\lambda_{11}$.

We now use multi-linearity of the determinant, starting from the additivity,

$$|Q_1 + \lambda_{11}, Q_2 + \lambda_{11}, \ldots, Q_N + \lambda_{11}|$$
$$= |Q_1, Q_2, \ldots, Q_N| + |\lambda_{11}, Q_2, \ldots, Q_N|$$
$$+ |Q_1, \lambda_{11}, \ldots, Q_N| + \ldots + |Q_1, Q_2, \ldots, \lambda_{11}|.$$

Further, by applying the homogeneity, we get,

$$|Q_1 + \lambda_{11}, Q_2 + \lambda_{11}, \ldots, Q_N + \lambda_{11}|$$
$$= |Q_1, Q_2, \ldots, Q_N| + \lambda_{11}|I, Q_2, \ldots, Q_N|$$
$$+ \lambda_{11}|Q_1, I, \ldots, Q_N| + \ldots + \lambda_{11}|Q_1, Q_2, \ldots, I|,$$

where $I = (1, \ldots, 1)^T$.

Moreover, as there are only $n - 1$ linearly independent vectors $q_i, i = 1, \ldots, n$ Gram determinant $\det((q_i \cdot q_j)_{n \times n}) = |Q_1, Q_2, \ldots, Q_n|$ vanishes. Thus, by factoring out $\lambda_{11}$ in (33) we obtain the assertion.

Now we are in the position to show that the sum in Proposition III.2 is equal to the Gram determinant of independent external momenta.

**Proposition III.2.** $\sum_{k=1}^{n} \det(q_{i,\parallel} \cdot q_{j,\parallel}(1 - \delta_{jk}(1 - \frac{1}{q_i \cdot q_j})))$ is equal to $\det((p_i \cdot p_j)_{(n-1) \times (n-1)})$.

Proof. By making use of Lemma B.1, we can write $\sum_{k=1}^{n} \det(q_{i,\parallel} \cdot q_{j,\parallel}(1 - \delta_{jk}(1 - \frac{1}{q_i \cdot q_j})))$ as $|I|Q_2 - Q_1)(Q_3 - Q_1)\ldots(Q_n - Q_1)$. Let us subtract the last row from all other rows. This gives us $|I'Q_2 - Q_1')(Q_3 - Q_1')\ldots(Q_n - Q_1')$, where, in particular, $I' = (0, 0, \ldots, 0, 1)^T$. By the Laplace expansion along the first column, we get the $(n - 1) \times (n - 1)$ determinant $(-1)^{n-1}|Q_2'' - Q_1''(Q_3'' - Q_1'')\ldots(Q_n'' - Q_1'')$, where $Q_i'' = Q_i - Q_1$ for $j = 2, \ldots, n$. Let us look at the elements of this determinant. The elements of $Q_i - Q_1$, for $j = 2, \ldots, n$, are given by $q_{ij} - q_{1j}$. Then, by subtracting the last row, which has elements $q_{n_j} - q_{1j}$, we get elements of $Q_i'' - Q_1''$ given by $q_{ij} - q_{1j} - (q_{nj} - q_{1j}) = q_{ij} + q_{1j} - q_{nj}$ for $i = 1, \ldots, n - 1$, but each $q_{ij} = \frac{1}{2}(q_{ii} + q_{jj} - f_{ij}(P))$ and thus all $q_{kk}$ cancel in
each element, leaving only functions depending on the external momenta.

The function $f_{ij}(P)$ for each $q_{ij}$ has the following structure,

$$f_{ij}(P) = \left( \sum_{k=i}^{j-1} p_k \right)^2 \equiv f_{ij}, \quad \text{(34)}$$

where we assumed $i \leq j$, which is justified by the fact that $q_{ij} = q_{ji}$. Thus, we get,

$$q_{ij} + q_{n1} - q_{i1} - q_{nj} = \frac{1}{2}(-f_{ij} + f_{n1} - f_{i1} + f_{nj}). \quad \text{(35)}$$

Now, let us subtract $(i+1)$-th row from $i$-th row for each $i = 1, \ldots, n-2$ to get,

$$\frac{1}{2}(-f_{ij} + f_{(i+1)j} + f_{i1} - f_{(i+1)1}). \quad \text{(36)}$$

Next, we subtract $j$-th column from $(j+1)$-th column, for $j = 2, \ldots, n$, which gives,

$$\frac{1}{2}(-f_{(j+1)j} + f_{ij} - f_{(i+1)j} + f_{(i+1)(j+1)}). \quad \text{(37)}$$

Let us make the following change of variables $p_i = x_i - x_{i+1}$ (dual variables). Then, $f_{ij}(P) = (x_i - x_j)^2$ and we have,

$$\frac{1}{2}(- (x_i - x_{j+1})^2 + (x_i - x_j)^2 - (x_i - x_{j+1})^2 + (x_{i+1} - x_{j+1})^2) =$$

$$= (x_i \cdot x_{j+1} - x_i \cdot x_j + x_{i+1} \cdot x_j - x_{i+1} \cdot x_{j+1}) =$$

$$= -(x_i - x_{i+1})(x_j - x_{j+1}) = -p_i \cdot p_j. \quad \text{(38)}$$

Thus, by factoring out $-1$ from each column, we get $(-1)^{n-1}$, and combining it with factors coming from the Laplace expansion, we finally obtain $(-1)^{n+1}(-1)^{n-1} = (-1)^{2n} = 1$, which assures the statement. \hfill \square

From the combination of Propositions III.1 and III.2, we support our claim,

$$\lambda_{11} = \frac{\text{LanS}^{(1)}_n}{\det ((p_i \cdot p_j)_{(n-1) \times (n-1)})}. \quad \text{(39)}$$

Let us emphasize that this relation is true only when on-shell conditions are imposed, as it is the way we defined LanS$_n^{(1)}$ in (27). In other words, one can use on-shell conditions to eliminate parallel loop components and express $\lambda_{11}$ solely in terms of external kinematics, which we present below.

Notice that, due to the way Feynman propagators are expressed in this parametrization (see Eq. (18)), one can determine $\lambda_{11}$ from the remaining Landau equations (i.e., on-shell conditions),

$$D_i - D_{i+1} = 0, \quad D_1 = 0. \quad \text{(40)}$$

The first $n-1$ on-shell conditions $D_i - D_{i+1} = 0$ give $n-1$ equations for the loop-momentum components $a_i$, for $i = 1, \ldots, n-1$, of Eq. (18), which can then be expressed in terms of external kinematics and internal masses.

$$0 = D_i - D_{i+1}, \quad \text{(41)}$$

$$0 = -m_i^2 + m_{i+1}^2 - 2 \sum_{j=1}^{i-1} (p_i \cdot p_j) - 2 \sum_{j=1}^{D_i} (p_i \cdot p_j) a_j. \quad \text{(42)}$$

Furthermore, without loss of generality, we can consider the last equation $D_1 = 0$ (see Eq. (18)) to solve for $\lambda_{11}$, whose explicit form at one-loop is given by,

$$\lambda_{11} = m_i^2 - \sum_{j=1}^{D_i} (p_j \cdot p_k) a_j a_k, \quad \text{(43)}$$

where, by using the solutions (41), we can further express $\lambda_{11}$ as a function depending only on external kinematics and masses. From (39), we also observe that $\lambda_{11} = 0$ exactly corresponds to an equation for Landau variety.

**Corollary III.1.** For one-loop scalar Feynman integrals with equal internal masses, LanS can be split into a term proportional to the mass and a term independent of the mass. The proportionality factor for the mass term is the Gram determinant of independent external momenta, $\det(p_i \cdot p_j)$.

**Proof.** It follows from (39), (41), and (42). \hfill \square

### B. One-loop bubble and triangle Landau singularities

We finalise this section with examples of explicit computations of Landau singularities for one-loop scalar bubble and triangle integrals. For additional examples of up-to six-point Feynman integrals, we refer the Reader to Appendix C.

#### a. One-loop bubble integral

For the one-loop scalar bubble the parametrization of the loop momentum, by the decomposition studied in the paper is the following,

$$\ell_1^a = a_1 \ p_1^a + \lambda_1^a, \quad \text{(43)}$$

with $p_1^a \neq 0$. The Gram determinant of propagators, before imposing the on-shell conditions, becomes,

$$\det (q_i \cdot q_j) = \lambda_{11} p_1^a, \quad \text{(44)}$$

---

3 Notice that $f_{ij}(P)$ can also be written in a more symmetric way $f_{ij}(P) = \left( \sum_{k=1}^{\min(i,j)} p_k - \frac{1}{2} (i+j-\min(i,j)) \right)^2$, where we avoid this assumption.
then, from Eqs. (41) and (42) we obtain,
\[ a_1 = \frac{1}{2p_1^2} \left( -m_1^2 + m_2^2 - p_1^2 \right), \]
\[ \lambda_{11} = m_1^2 - p_1^2 a_1^2, \]
that corresponds to imposing on-shell conditions and allows to find the well-known Landau singularity,
\[ \text{LanS}_2^{(1)} = \frac{1}{4} \lambda_K \left( p_1^2, m_1^2, m_2^2 \right), \]
in terms of the Källén function, \( \lambda_K(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \).

Furthermore, by considering the equal mass case, \( m_1^2 = m_2^2 = m^2 \),
\[ \text{LanS}_2^{(1)} \big|_{m_1^2=m_2^2} = -\frac{1}{4} \left( p_1^2 \right)^2 + m^2 p_1^2. \]

b. One-loop triangle integral

We continue with the explicit calculation of the Landau singularity of the one-loop triangle and different kinematic scales. The loop momentum can be parametrized in terms of two independent momenta,
\[ \ell_i^2 = a_1 p_1^2 + a_2 p_2^2 + \lambda_i^2, \]
with \( p_1^2, p_2^2, p_3^2 \neq 0 \) and, because of the momentum conservation, \( 2p_1 \cdot p_2 = p_3^2 - p_1^2 - p_2^2 \). Thus, the loop-momentum components for this loop topology take the form,
\[ a_1 = \frac{1}{\lambda_K(p_1^2, p_2^2, p_3^2)} \left[ 2p_2^2 \left( m_1^2 - m_2^2 + p_1^2 \right) \right. \]
\[ - \left( p_1^2 + p_2^2 - p_3^2 \right) \left( -m_2^2 + m_3^2 + p_1^2 - p_3^2 \right) \right], \]
\[ a_2 = \frac{1}{\lambda_K(p_1^2, p_2^2, p_3^2)} \left[ p_1^2 \left( m_1^2 + m_2^2 - 2m_1^2 + p_2^2 + p_3^2 \right) \right. \]
\[ + \left( m_1^2 - m_2^2 \right) \left( p_2^2 - p_3^2 \right) - \left( p_1^2 \right)^2 \right], \]
\[ \lambda_{11} = m_1^2 - a_2^2 p_1^2 - a_1 a_2 (p_1^2 + p_2^2 - p_3^2), \]
whose combination amounts to the Landau singularity,
\[ \text{LanS}_3^{(1)} = -\frac{1}{4} \left( \frac{p_1^2 p_2^2 p_3^2}{3} - p_2^2 \left( m_1^2 - m_2^2 \right) \left( m_2^2 - m_3^2 \right) \right. \]
\[ + m_2^2 \left( p_1^2 + p_2^2 - p_3^2 \right) \right] + \text{cycl. perm.}. \]

The Landau singularity for the equal-mass case takes the form,
\[ \text{LanS}_3^{(1)} \big|_{m_1^2=m_2^2} = -\frac{1}{4} \left( p_1^2 p_2^2 p_3^2 + m^2 \lambda_K(p_1^2, p_2^2, p_3^2) \right). \]

Notice that in the equal-mass case in the above examples, the term proportional to \( m^2 \) corresponds to the Gram determinant of the external momenta, in full agreement with corollary III.1. For instance, in the one-loop triangle, we have, \( \text{det}(p_i \cdot p_j) = -1/4 \lambda_K(p_1^2, p_2^2, p_3^2) \).

IV. LEADING SINGULARITIES

By continuing with the same strategy in the decomposition of space-time dimension presented in Sec. II.A, we now focus on leading singularities of one-loop Feynman integrals. By leading singularities we understand residues in Leray’s sense evaluated over all polar sets, i.e.,
\[ LS = \int_{\delta^m \sigma} \phi = (2\pi i)^m \int_{\sigma} \text{res}^m[\phi], \]
where \( \phi \) is a closed form with \( m \) polar sets \( S_1, \ldots, S_m, \sigma \in \mathbb{Z}_{p-m}(S_1 \cap \ldots \cap S_m) \) is a cycle and \( \delta^m \sigma \) is its composed co-boundary (see Appendix A). It can happen that the number of polar sets is larger than the dimension of the space, in this situation Leinartas decomposition [52, 53] allows to write \( \phi \) as a sum of forms, each having at most as many polar sets as the dimension of the space.

To carry out this analysis, we present in this section illustrative computations of selected one-loop topologies, and provide the Reader with additional examples in Appendix C.

Additionally, because of the pattern displayed by explicit calculations of leading singularities in the various examples, we conjecture and prove the structure of these singularities in particular space-time dimensions – a connection between Landau and leading singularities. Similar results to those presented in this section can already be found in [19]. However, whereas authors of the mentioned paper focused on dimensional regularisation and limited the analysis to a class of integrals where the integer dimension of the space-time is always even, omitting cases like the one-loop scalar triangle in the three space-time dimensions, we study the integrals in arbitrary space-time dimensions.

In the last part of this section, we consider differential equations of one-loop Feynman integrals, showing that from the knowledge of their leading singularities the construction of differential equations in canonical form is straightforwardly carried out. We illustrate this with explicit calculations.

A. One-loop leading singularities

In the spirit of showing the simplicity of this integrand representation when extracting leading singularities, we explicitly consider in the following one-loop bubble and triangle Feynman integrals, where for the sake of this analysis, we drop overall factors coming from angular integrations.

a. One-loop bubble integral

The integrand of the one-loop bubble in \( D \geq 2 \) becomes,
\[ J_2^{(1),D}(1, 2) = \frac{1}{2} \int_{|p_1|}^{1} da_1 \int_{0}^{\infty} d\lambda_{11} \frac{\lambda_{11}^{(D_1-2)/2}}{D_1 D_2}. \]
\[ J_2^{(1), D=2} \sim \pm \frac{1}{2\sqrt{\lambda_K (p_1^2, m_1^2, m_2^2)}} \int d \log \frac{\sqrt{\lambda_{11}^2} - \sqrt{-D_2\|}}{\sqrt{-\lambda_{11} + \sqrt{-D_2\|}}}, \]
\[ \times \left( \frac{d \log D_{21} + \sqrt{D_{21} + r_1^+} + \sqrt{D_{21} + r_1^-} + \sqrt{r_1^+} \sqrt{r_1^-}}{D_{21} + \sqrt{D_{21} + r_1^+} + \sqrt{D_{21} + r_1^-} - \sqrt{r_1^+} \sqrt{r_1^-}} - (r_1^+ \rightarrow r_2^+) \right), \] (54a)
\[ J_2^{(1), D=3} \sim \frac{1}{\sqrt{-p_1^2}} \left[ \int_{-\infty}^{\infty} \frac{dD_{12}}{D_{12}} \int_{A_1}^{\infty} \frac{dD_1}{D_1} + \int_{-\infty}^{\infty} \frac{dD_{21}}{D_{21}} \int_{A_2}^{\infty} \frac{dD_2}{D_2} \right], \] (54b)
\[ J_2^{(1), D>4} \rightarrow \text{No \ dlog \ representation}, \] (54c)

where \( p_1^2 \neq 0 \) and, for this loop topology, \( D_\| = 1 \) and \( D_\perp = D - 1 \).

We observe that, by examining the structure of the integrand for different values of \( D_\perp = D - 1 \), and by relying on a partial fractioning along the lines of the approach presented in \[54, 55\], we find,

\[ J_2^{(1), D=2} \sim \pm \frac{1}{2\sqrt{\lambda_K (p_1^2, m_1^2, m_2^2)}} \int d \log \frac{\sqrt{\lambda_{11}^2} - \sqrt{-D_2\|}}{\sqrt{-\lambda_{11} + \sqrt{-D_2\|}}}, \]
\[ \times \left( \frac{d \log D_{21} + \sqrt{D_{21} + r_1^+} + \sqrt{D_{21} + r_1^-} + \sqrt{r_1^+} \sqrt{r_1^-}}{D_{21} + \sqrt{D_{21} + r_1^+} + \sqrt{D_{21} + r_1^-} - \sqrt{r_1^+} \sqrt{r_1^-}} - (r_1^+ \rightarrow r_2^+) \right), \] (54a)
\[ J_2^{(1), D=3} \sim \frac{1}{\sqrt{-p_1^2}} \left[ \int_{-\infty}^{\infty} \frac{dD_{12}}{D_{12}} \int_{A_1}^{\infty} \frac{dD_1}{D_1} + \int_{-\infty}^{\infty} \frac{dD_{21}}{D_{21}} \int_{A_2}^{\infty} \frac{dD_2}{D_2} \right], \] (54b)
\[ J_2^{(1), D>4} \rightarrow \text{No \ dlog \ representation}, \] (54c)

with \( A_1 = \frac{\lambda_K (m_2^2 - D_{12}, m_1^2, p_1^2)}{4p_1^2} \), \( A_2 = \frac{\lambda_K (m_1^2 - D_{21}, m_2^2, p_1^2)}{4p_1^2} \) and \( D_{ij} = D_i - D_j \). The prefactor (given only in terms of kinematic invariants) of the various integrals corresponds to the leading singularity of a Feynman integral in a particular space-time dimension. The no dlog representation in (54c) means that there is a double pole and thus it is not possible to transform it to the logarithmic representation.

Feynman Integrals (54) were obtained with the aid of relations (C14), summarised in Appendix C. Also, to make explicit the parallel component of propagators, we define, \( D_{\|} = D_1 - \lambda_{11} \), that is at most a quadratic polynomial in \( a_1 \) and absent of \( \lambda_{11} \) (see Eq. (18)). To simplify the notation, here and in the following, we use the shorthand notation \( D_{ij} = D_i - D_j \). The values of \( r_i^\pm \), for \( i = 1, 2 \), are,

\[ r_1^\pm = m_2^2 - p_1^2 \pm 2 \sqrt{p_1^2 \sqrt{m_1^2}}, \] (55a)
\[ r_2^\pm = m_1^2 + p_1^2 \pm 2 \sqrt{p_1^2 \sqrt{m_2^2}}, \] (55b)

in which both of them satisfy the relation \( r_i^+ r_i^- = \lambda_K (p_1^2, m_1^2, m_2^2) \).

Let us, however, remark that the integrands (54), expressed as products of dlog forms, are often referred to as integrands in dlog representation and are by no means unique (see e.g. [56]). One can find alternative and more compact representations in terms of single dlog integrands, by accounting for an educated change of variables. For instance,

\[ J_2^{(1), D=2} \sim \pm \frac{1}{2\sqrt{\lambda_K (p_1^2, m_1^2, m_2^2)}} \int d \log \frac{D_1}{D_\|} \wedge d \log \frac{D_2}{D_\perp}, \] (56a)
\[ J_2^{(1), D=3} \sim \frac{1}{4\sqrt{-p_1^2}} \int d \log D_1 \wedge d \log D_2, \] (56b)

where, \( D_\pm = (\ell_1 - \ell_\pm)^2 \), with \( \ell_\pm \) either of the two solutions of the maximal cut conditions, \( D_1 = D_2 = 0 \). The \( \pm \) sign in front of (56a) is related to the solution one has chosen for \( D_\pm \). The domain of integration \( \mathcal{K} \) is a function of external and internal kinematics which we were not able to find explicitly. Its determination remains an open problem. The domain of integration \( \mathcal{K} \) is given by the condition \( \det(q_1 q_{\pm}) \geq 0 \), where \( q_i \) are expressed in terms of variables \( D_i \) as \( q_i^2 = D_i + m_i^2 \), for \( i, j = 1, 2 \).

\[ J_2^{(1), D>3} \] can also be written as an iterated integral as

\[ J_2^{(1), D>3} \sim \int_{-m_2^2}^{\infty} \frac{dD_2}{D_2} \int_{-m_2^2+m_2^2+s-2\sqrt{s(D_2+m_2^2)}}^{\infty} \frac{dD_1}{D_1} \] (57)

Since the leading singularity turns out to be the same, regardless of the dlog representation of a given Feynman integral, in the explicit computations, we draw only our attention to the features of this singularity in different space-time dimensions.

b. One-loop triangle integral

We now continue with the one-loop triangle in \( D \geq 3 \) (i.e., \( D_\| = 2 \) and \( D_\perp = D - 2 \geq 1 \)), whose leading singularities become,

\[ J_3^{(1), D=3} \sim \pm \frac{1}{8\sqrt{-\Delta S_3^{(1)}}} \int d \log \frac{D_1}{D_\|} \wedge d \log \frac{D_2}{D_\|} \wedge d \log \frac{D_3}{D_\|}, \] (58a)
\[ J_3^{(1), D=4} \sim \frac{1}{4\sqrt{\lambda_K (p_1^2, p_2^2, p_3^2)}} \int d \log D_1 \wedge d \log D_2 \wedge d \log D_3, \] (58b)
\[ J_3^{(1), D>5} \rightarrow \text{No \ dlog \ representation}, \] (58c)

where the ratios \( D_i/D_\| \) are meromorphic functions on the variables \( a_i \) (with \( i = 1, 2 \)) and \( \lambda_{11} \), and, similar to Eq. (58a), \( D_\| = (\ell_1 - \ell_\pm)^2 \), is obtained from the on-shell conditions, \( D_1 = D_2 = D_3 = 0 \). The domain of integration \( \mathcal{K} \) is a function of external and internal kinematics which we were not able to find explicitly. Its
determination remains an open problem. The domain of integration \( C \) is given by the condition \( \det(q, q_j) \geq 0 \), where \( q_i \) are expressed in terms of variables \( D_i \) as \( q_i^2 = D_i + m_i^2 \); for \( i, j = 1, 2, 3 \) and \( k, l = 1, 2 \). The leading singularity can exactly be cast in terms of the Landau singularity of a one-loop triangle integral (50). The no \( d \log \) form in (58c), similarly to the bubble integral (54c), means that there is a double pole.

As anticipated in Sec. II, the decomposition of the space-time dimension in terms of parallel and perpendicular subspaces leads to the Baikov representation, which can clearly be appreciated in Eqs. (56b) and (58b). In details, we consider the integration

\[
\int \prod_{i=1}^{n-1} da_i \frac{1}{D_1 \ldots D_n},
\]

where \( D_{ki} = D_k - D_i \). The integration over \( \lambda_{11} \) clearly manifests a \( d \log \) representation, whose residue is \( +1 \). The remaining integrations (and residues) are then calculated according to Lemma B.4, finding,

\[
\int \prod_{i=1}^{n-1} \int \frac{d\lambda_{11}}{D_i} \prod_{k \neq i} \frac{1}{D_{ki}} \sim \frac{1}{2^{n-1} \sqrt{-\det(p_i \cdot p_j)}},
\]

(61)

where we identify the variables \( x \) of Lemma B.4 as, \( x_{ij} = 2p_i \cdot p_j \).

Therefore, by recalling \( J_{[n-1]}^{(1)} \) from Eq. (17), we find,

\[
J_{n, n+1}^{(1)} \sim \frac{\pm 1}{2^n \sqrt{-\det(p_i \cdot p_j)}},
\]

(62)

completing the proof of this theorem.

\[\square\]

**Theorem IV.2.** The leading singularity of an \( n \)-point one-loop Feynman integral in \( D = n \) space-time dimensions is equal to \( \pm 1/(2^n \sqrt{-1)^{D-1} \text{LanS}} \).

**Proof.** The Feynman integrand in the parallel and perpendicular representation has, in \( D = n \), the following form

\[
\omega = \frac{J_{[n]}^{(1)}}{2} \frac{1}{\sqrt{\lambda_{11}}} \frac{d\lambda_{11} \wedge da_1 \wedge \ldots \wedge da_{n-1}}{D_1 \ldots D_n}.
\]

(63)

Let us calculate the Leray residue around polar sets \( D_1, \ldots, D_n \), where each \( D_i \) is given by (18),

\[
\text{res}^n \omega = \frac{J_{[n]}^{(1)}}{2} \frac{1}{\sqrt{\lambda_{11}}} \frac{d\lambda_{11} \wedge da_1 \wedge \ldots \wedge da_{n-1}}{dD_1 \wedge \ldots \wedge dD_n}.
\]

(64)

Each differential \( dD_i \) has the following form

\[
dD_i =
\]

\[
= 2 \sum_{j=1}^{n-1} p_j \cdot p_k \sum_{k=1}^{n-1} a_j \, da_k + 2 \sum_{j=1}^{n-1} p_j \cdot p_k \, da_k + d\lambda_{11}
\]

\[
= \sum_{k=1}^{n-1} \left( 2 \sum_{j=1}^{n-1} p_j \cdot p_k \sum_{k=1}^{n-1} a_j \right) \, da_k + d\lambda_{11}.
\]

(65)

Thus,
\[
\text{res}^n[\omega] = \frac{\mathcal{J}^{(1)}_{[\mathcal{D}_1]}}{2 \sqrt{\lambda_{11}}} \frac{1}{d\lambda_{11} \wedge da_1 \wedge \ldots \wedge da_{n-1}} d\lambda_{11} \wedge \sum_{k=1}^{n-1} \left( 2 \sum_{j=1}^{n-1} p_j \cdot p_k a_j \right) da_k \wedge \ldots \wedge \left[ d\lambda_{11} + \sum_{k=1}^{n-1} \left( 2 \sum_{j=1}^{n-1} p_j \cdot p_k a_j + 2 \sum_{j=1}^{n-1} p_j \cdot p_k \right) da_k \right].
\]

By Lemma B.2 this is equal to,

\[
\text{res}^n[\omega] = \frac{\mathcal{J}^{(1)}_{[\mathcal{D}_0]}}{2 \sqrt{\lambda_{11}}} \frac{1}{d\lambda_{11} \wedge da_1 \wedge \ldots \wedge da_{n-1}} d\lambda_{11} \wedge \det(B) d\lambda_{11} \wedge \ldots \wedge da_{n-1} = \frac{\mathcal{J}^{(1)}_{[\mathcal{D}_0]}}{2 \sqrt{\lambda_{11}}} \frac{1}{\det(B)},
\]

where,

\[
\det(B) = \begin{vmatrix}
1 & 2 \sum_{j=1}^{n-1} p_j \cdot p_1 a_j & \ldots & 2 \sum_{j=1}^{n-1} p_j \cdot p_{n-1} a_j \\
1 & 2 \sum_{j=1}^{n-1} p_j \cdot p_1 a_j + 2p_1 \cdot p_1 & \ldots & 2 \sum_{j=1}^{n-1} p_j \cdot p_{n-1} a_j + 2p_1 \cdot p_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 \sum_{j=1}^{n-1} p_j \cdot p_1 a_j + 2 p_1 \cdot p_j & \ldots & 2 \sum_{j=1}^{n-1} p_j \cdot p_{n-1} a_j + 2 p_j \cdot p_{n-1} \\
1 & 2 \sum_{j=1}^{n-1} p_j \cdot p_1 a_j + 2 \sum_{j=1}^{n-1} p_j \cdot p_k & \ldots & 2 \sum_{j=1}^{n-1} p_j \cdot p_{n-1} a_j + 2 \sum_{j=1}^{n-1} p_j \cdot p_{n-1}
\end{vmatrix}
\]

Let us calculate \( b_{(i+1)k} - b_{ik} \) with \( i, k = 1, \ldots, n - 1 \),

\[
\left( 2 \sum_{j=1}^{n-1} p_j \cdot p_k a_j + 2 \sum_{j=1}^{i-1} p_j \cdot p_k \right) - \left( 2 \sum_{j=1}^{n-1} p_j \cdot p_k a_j + 2 \sum_{j=1}^{i-1} p_j \cdot p_k \right) = 2p_i \cdot p_k.
\]

and by repeating this until we reach the first row we will end up with,
Therefore, the residue of \(\omega\) is equal to,

\[
\text{res}^n[\omega] = \frac{\mathcal{J}^{(1)}_{[D_1]} 1}{2^n \sqrt{\lambda_{11}} \frac{1}{2^{n-1}} \det(p_i \cdot p_j)} .
\]

(73)

However, since we have calculated residues around polar sets \(D_1 = \ldots = D_n = 0\) and we showed in Sec. IIIA that under these conditions,

\[
\lambda_{11} = \frac{\text{LanS}}{\det(p_i \cdot p_j)} ,
\]

we can insert this into our residue formula to obtain,

\[
\text{res}^n[\omega] = \frac{\mathcal{J}^{(1)}_{[D_1]} 1}{2^n} \sqrt{\frac{\det(p_i \cdot p_j)}{\text{LanS}}} .
\]

(75)

Taking into account that \(\mathcal{J}^{(1)}_{[D_1]} = \pm \sqrt{(-1)^{D-1} \det(p_i \cdot p_j)}\), we get,

\[
\text{res}^n[\omega] = \pm \sqrt{(-1)^{D-1} \det(p_i \cdot p_j)} \frac{1}{2^n} \sqrt{\frac{\det(p_i \cdot p_j)}{\text{LanS}}} .
\]

(76)

We notice from the examples listed in this section (and in Appendix C) that one-loop scalar Feynman integrals in \(D = n\) and \(D = n + 1\) space-time dimensions manifest a \(d\log\) representation, where the arguments of \(d\log\)'s have very simple structure. In effect, combining this observation with the above results for the one-loop scalar integrals we make the following conjecture.\(^4\)

**Conjecture 1.** An \(n\)-point Feynman integral can be written in one of the following forms depending on the space-time dimension \(D\),

\[
\begin{align}
J_n^{(1), D = n} &\sim \pm \frac{1}{2^n \sqrt{-\text{LanS}}^{(1)}} \int_{\mathcal{K}} d\log D_1 \wedge \ldots \wedge d\log D_n, \quad (77a) \\
J_n^{(1), D = n + 1} &\sim \frac{1}{2^n \sqrt{-\det(p_i \cdot p_j)}} \int_{\mathcal{C}} d\log D_1 \wedge \ldots \wedge d\log D_n , \\
J_n^{(1), D \geq n + 2} &\sim \text{No } d\log \text{ representation},
\end{align}
\]

(77b)

(77c)

where \(\det(p_i \cdot p_j)\) can be identified with the Landau singularity of the second type, and \(D_\pm\) are calculated by imposing the on-shell conditions \(D_1 = D_2 = \ldots = D_n = 0\). The \(\pm\) sign in front of \((77a)\) is related to the solution one has chosen for \(D_\pm\). The domain of integration \(\mathcal{K}\) is a function of external and internal kinematics which we were not able to find explicitly. Its determination remains an open problem. The domain of integration \(\mathcal{C}\) is given by the condition \(\frac{\det(q_i \cdot q_j)}{\det(p_i \cdot p_j) \geq 0}\), where \(q_i\) are expressed in terms of variables \(D_i\) as \(q_i^2 = D_i + m_i^2\), for \(i, j = 1, \ldots, n\) and \(k, l = 1, \ldots, n - 1\). In other words \(\mathcal{C}\) is given by the condition \(\lambda_{11} \geq 0\), similarly to (39) without imposing on-shell conditions and expressing it in terms of variables \(D_i\). The \(d\log\) form means that there is a double pole and it cannot be transformed to logarithmic representation.

### C. Differential equations of one-loop Feynman integrals

The connection between leading and Landau singularities in one-loop Feynman integrals motivates us to continue our analysis with the calculation of Feynman integrals through differential equations methods. As revealed in Ref. [29], the difficulty in this analytic calculation can be largely reduced once the differential equation is in the canonical form. In the following,

\(^4\) An evidence of a similar result in a dual formulation of planar and non-planar \(\mathcal{N} = 4\) SYM was pointed out in [57, 58].
according to convention (1) and, respectively, displayed in independent master integrals, internal massive propagators with mass of master integrals with off-shell momenta and internal momenta. We choose to reduce the number present in the four-point integrals with massive external equation in canonical form of the set of master integrals this observation with the calculation of the differential D integrals to pass from space-time dimensions (specific space-time dimensions. equations in the canonical form by appropriately in Laporta basis \[59\], one can generate differential of integrals (often referred to as master integrals) we show that from the knowledge of the minimal set of integrals in \(D = 4 - 2\epsilon\). From the analysis of Sec. IVB, we know that not all master integrals (78) manifest a leading singularity in \(D = 4\). For tadpole and bubble integrals, however, we can make use of dimensional recurrence relations to consider these integrals in \(D = 2 - 2\epsilon\),

\[
J^{(1),D-2}_1(1) = \frac{(D - 2)}{2m^2} J^{(1),D}_1(1),
\]

\[
J^{(1),D-2}_2(i, j) = \frac{(D - 2)}{3m^2} J^{(1),D}_1(1) + \frac{2(D - 3)}{3m^2} J^{(1),D}_2(i, j).
\]

Thus, by normalising leading singularities to 1 (in the sense that irrelevant overall numerical factors independent of kinematic scales are dropped), we choose,

\[
g_1 = \epsilon J^{(1),D=2-2\epsilon}_1(1),
\]

\[
g_2 = \epsilon m^2 J^{(1),D=2-2\epsilon}_2(1, 4),
\]

\[
g_3 = \epsilon \sqrt{4m^2 - s} J^{(1),D=2-2\epsilon}_2(1, 3),
\]

\[
g_4 = \epsilon \sqrt{t(4m^2 - t)} J^{(1),D=2-2\epsilon}_2(2, 4),
\]

\[
g_5 = \epsilon^2 \sqrt{4m^2 - s} J^{(1),D=4-2\epsilon}_3(1, 2, 4),
\]

\[
g_6 = \epsilon^2 \sqrt{t(4m^2 - t)} J^{(1),D=4-2\epsilon}_3(2, 1, 3),
\]

\[
g_7 = \epsilon^2 \sqrt{st(12m^4 - 4m^2(s + t) + st)} J^{(1),D=4-2\epsilon}_4(1, 2, 3, 4).
\]

that automatically satisfies the differential equation in the canonical form,

\[
\partial_\eta \bar{g} = \epsilon A_\eta \bar{g},
\]

with \(\eta \in \{s, t, m^2\}\) and \(\bar{g} = \{g_1, \ldots, g_7\}\). For instance, we find for \(g_7\) the differential equation w.r.t. \(s\),

\[
\partial_s g_7 = \frac{\epsilon}{m^2} \left[ - \frac{6R_{22}}{(R_1^2 + 1)R_3R_{11}} g_2 + \frac{R_{22}}{R_1(R_1^2 + 1)R_3} g_3 + \frac{R_2}{R_3R_{11}} g_4 - \frac{2}{R_3(R_1^2 + 1)(R_1^2 + R_2^2 + 4)} \frac{R_3 g_5}{R_3(R_1^2 + 2)(R_3^2 + 2) R_{11}} g_6 - \frac{2R_2(R_2^2 + 2)}{(R_1^2 + R_2^2 + 4)R_3R_{11}} g_6 - \frac{(R_2^2 + 2)^2}{(R_1^2 + R_2^2 + 4)R_3^2} g_7 \right],
\]

with,

\[
R_{11}^2 = s/m^2, \quad R_2^2 = R_{11}^2 - 4, \quad R_{22} = t/m^2, \quad R_3^2 = R_{22}^2 - 4, \quad R_3^2 = R_1^2 R_2^2 - 4.
\]

Similar structures are found for the other derivatives.

---

5 We generated integration-by-parts identities with the aid of the publicly available software Form \[61\], whereas for differential equations w.r.t. \(s, t, \) and \(m^2\) we employed the FormRed framework \[62\].
a. Differential equation in $D = 5 - 2\epsilon$

In view that our method does not introduce any limitation w.r.t. the choice of the space-time dimension, we consider as an illustrative example the construction of differential equation in canonical form of the four-point integral family in $D = 5 - 2\epsilon$.

We begin by selecting the space-time dimension where master integrals (78) are known to manifest leading singularities. Besides relations (79), we consider the following relation for the one-loop scalar triangle,

$$J^{(1),D-2}_3(1,2,k) = \frac{(D-2)(14m^2 - 5Q_k)}{6m^4(3m^2 - Q_k)(4m^2 - Q_k)} J^{(1),D}_1(1)$$

$$+ \frac{(D-3)(2m^2 - Q_k)}{m^2(3m^2 - Q_k)(4m^2 - Q_k)} J^{(1),D}_2(2,k)$$

$$+ \frac{2(D-3)}{3m^2(3m^2 - Q_k)} J^{(1),D}_2(1,k)$$

$$+ \frac{(D-4)(4m^2 - Q_k)}{2m^2(3m^2 - Q_k)} J^{(1),D}_2(1,2,k), \quad (84)$$

for $k = 3, 4$, and $Q_3 = s$ and $Q_4 = t$.

Thus, with relations (79) and (84), we choose,

$$g_1 = \epsilon \sqrt{m^2} J^{(1),D=1-2\epsilon}_1(1),$$

$$g_2 = \epsilon^2 \sqrt{m^2} J^{(1),D=3-2\epsilon}_2(1,4),$$

$$g_3 = \epsilon^2 \sqrt{-s} J^{(1),D=3-2\epsilon}_2(1,3),$$

$$g_4 = \epsilon^2 \sqrt{-t} J^{(1),D=3-2\epsilon}_2(2,4),$$

$$g_5 = \epsilon^2 \sqrt{-sm^2(3m^2 - s)} J^{(1),D=3-2\epsilon}_3(1,2,4),$$

$$g_6 = \epsilon^2 \sqrt{-tm^2(3m^2 - t)} J^{(1),D=3-2\epsilon}_3(1,2,3),$$

$$g_7 = \epsilon^2 \sqrt{-st(4m^2 - s - t)} J^{(1),D=5-2\epsilon}_4(1,2,3,4). \quad (85)$$

finding differential equations w.r.t. $s, t$, and $m^2$ in canonical form (81).

In these illustrative examples, we showed that from the knowledge of master integrals in terms of one-loop scalar integrals, the construction of differential equations in the canonical form is automatic from the clear knowledge of leading singularities in specific space-time dimensions. Allowing, in this way, to focus more on the interesting properties of Feynman integrals given by their singularities.

V. PRELIMINARIES ON MULTI-LOOP FEYNMAN INTEGRALS

To this end, in this section, we explicitly showcase computations of particular two-loop Feynman integrals (with internal massless propagators and external massive momenta) whose leading singularities can be extracted along the lines of the approach carried out for one-loop Feynman integrals. Additionally, as a by-product of the explicit calculations of two-loop planar and non-planar triangles, we make use of the loop-by-loop approach and multi-dimensional residues to provide an alternative proof of analytic expressions for the four-dimensional leading singularities of $L$-loop ladder triangle and box integrals, whose expressions are known for a long time [63].

A. Application of the Leray residue at two loops

a. Two-loop planar triangle

Let us start considering the two-loop topology displayed in Fig. 3(a), whose Feynman integral with internal massless propagators can be described as,

$$J^{(2),D=4}_{P,3-\text{pt}} = \int d^4k_1 d^4k_2 \frac{1}{l_1^2 (l_1 - k_2)^2 (l_1 - p_1)^2 (l_2 - p_1)^2 (l_1 + p_2)^2 (l_2 + p_2)^2}, \quad (86)$$

with the same convention for external kinematics ($p_1^2 \neq 0$) utilised in Secs. III and IV.

Then, by taking into account the decomposition of space-time dimension into parallel ($D_\parallel = 2$) and perpendicular ($D_\perp = 2$) directions, along the lines of
Sec. II C, we parametrize the loop momenta as,
\[ \ell_i^a = a_{i1} p_1^a + a_{i2} p_2^a + \lambda_i^a, \]  
finding for the integrand, \( J_{P;3:pt}^{(2),D=4} \) is found to be
\[
J_{P;3:pt}^{(2),D=4} \sim \pm \frac{1}{16 \lambda_K (p_1^2, p_2^2, p_3^2)} \int \frac{da_{11} da_{12} da_{21} da_{22} d\lambda_{11} d\lambda_{22} d\lambda_{12}}{\sqrt{\lambda_{11} \lambda_{22} - \lambda_{12}^2}} \ell_2 (\ell_1 - \ell_2)^2 \cdots (\ell_2 + p_2)^2, 
\]
where, as in the examples at one-loop, we omit overall terms that arise from angular integrations. Notice that propagators are now expressed in terms of the loop-momentum components \( a_{ij} \) and \( \lambda_{ij} \), e.g.,
\[
(\ell_1 - \ell_2)^2 = p_1^2 (a_{11} - a_{21})(a_{11} - a_{12} - a_{21} + a_{22})
+ p_2^2 (a_{12} - a_{22})(-a_{11} + a_{12} + a_{21} - a_{22})
+ p_3^2 (a_{11} - a_{21})(a_{12} - a_{22}) + \lambda_{11} - 2 \lambda_{12} + \lambda_{22}. 
\]
Then, the residue takes the form, \( J_{P;3:pt}^{(2),D=4} \) is found to be
\[
J_{P;3:pt}^{(2),D=4} \sim \pm \frac{1}{16 \lambda_K (p_1^2, p_2^2, p_3^2)} \frac{1}{\sqrt{\lambda_{11} \lambda_{22} - \lambda_{12}^2}} 
\times \frac{\ell_2 (\ell_1 - \ell_2)^2 \cdots (\ell_2 + p_2)^2}{(\lambda_{12} (a_{11} - a_{12} - 1) + \lambda_{11} (-a_{21} + a_{22} + 1))}, 
\]
where we are left to find the values of the integration variables by accounting from the seven conditions,
\[
\sqrt{\lambda_{11} \lambda_{22} - \lambda_{12}^2} = \ell_2 = (\ell_1 - \ell_2)^2 \cdots = (\ell_2 + p_2)^2 = 0, \]
finding in this way,
\[
J_{P;3:pt}^{(2),D=4} \sim \pm \frac{1}{16 \lambda_K (p_1^2, p_2^2, p_3^2)} \frac{1}{\sqrt{\lambda_{11} \lambda_{22} - \lambda_{12}^2}}, 
\]
that is in complete agreement with the leading singularity delivered by the automated package DLOGBASIS.

b. Two-loop non-planar triangle

Let us now draw our attention to the non-planar triangle of Fig. 3(b), whose Feynman integral with internal massless propagators can be characterized as,
\[
J_{NP;3:pt}^{(2),D=4} \sim \int \frac{d^4 \ell_1}{i \pi^2} \frac{d^4 \ell_2}{i \pi^2} \times \frac{1}{\ell_1^2 (\ell_1 - p_1)^2 (\ell_2 - p_1)^2 (\ell_2 + p_2)^2 (\ell_1 - \ell_2)^2 (\ell_1 - \ell_2 - p_2)^2}. 
\]

By parametrizing the loop momenta according to (88) and following the same procedure carried out in the calculation of the two-loop planar triangle, we find the residue,
\[
J_{NP;3:pt}^{(2),D=4} \sim \pm \frac{1}{16 \lambda_K (p_1^2, p_2^2, p_3^2)} \frac{1}{\sqrt{\lambda_{11} \lambda_{22} - \lambda_{12}^2}} 
\times \frac{\lambda_{11} (a_{21} - a_{22} - 1) - \lambda_{12} (a_{11} - a_{12} - a_{22} - 1) - a_{12} \lambda_{22}}{\lambda_{11} \lambda_{22} - \lambda_{12}^2}, 
\]
in which, the loop-momentum components \( a_{ij} \) and \( \lambda_{ij} \) are fixed from the conditions,
\[
\sqrt{\lambda_{11} \lambda_{22} - \lambda_{12}^2} = \ell_1^2 = (\ell_1 - p_1)^2 
= \cdots = (\ell_1 - \ell_2 - p_2)^2 = 0, \]
amounting to,
\[
J_{NP;3:pt}^{(2),D=4} \sim \pm \frac{1}{16 \lambda_K (p_1^2, p_2^2, p_3^2)}, 
\]
finding complete agreement with the result delivered by DLOGBASIS.

B. Loop-by-loop approach

In view of the connection between leading and Landau singularities considered in Sec. IV, one can exploit this connection by following a loop-by-loop approach based on the former results. To this end, we discuss this approach in the two-loop triangles previously considered and, as a by-product of these results, give the explicit expression of the \( L \)-loop ladder planar triangle and box scalar integrals.

---

6 The calculation of residues through Leray’s theory has been implemented in MATHEMATICA through its built-in function TensorWedge.
a. Two-loop planar triangle and box integrals

Let us then begin with \( J^{(2), D=4}_{P, 3\text{-pt}} \) of Eq. (87), which, by separately performing loop integrations, according to

\[
J^{(2), D=4}_{P, 3\text{-pt}} \bigg|_{\text{L-R}} = \int \frac{d^4 \ell_2}{16 \pi^2} \frac{1}{(\ell_2 - p_1)^2 (\ell_2 + p_2)^2} \int \frac{d^4 \ell_1}{16 \pi^2} \frac{1}{\ell_1^2 (\ell_1 - \ell_2)^2 (\ell_1 - p_1)^2 (\ell_1 + p_2)^2}
\]

\[
\sim \int \frac{d^4 \ell_2}{16 \pi^2} \frac{1}{\ell_2^2 (\ell_2 - p_1)^2 (\ell_2 + p_2)^2} \frac{1}{p_3^2} \lambda_K \left( 1, \frac{\ell_2^2 (\ell_2 + p_2)^2}{p_3^2}, \frac{\ell_2^2 (\ell_2 - p_1)^2}{p_3^2} \right),
\]

(98a)

\[
J^{(2), D=4}_{P, 3\text{-pt}} \bigg|_{\text{R-L}} = \int \frac{d^4 \ell_1}{16 \pi^2} \frac{1}{\ell_1^2 (\ell_1 - p_1)^2 (\ell_1 + p_2)^2} \int \frac{d^4 \ell_2}{16 \pi^2} \frac{1}{\ell_2^2 (\ell_1 - \ell_2)^2 (\ell_1 - p_2)^2 (\ell_1 + p_2)^2}
\]

\[
\sim \int \frac{d^4 \ell_1}{16 \pi^2} \frac{1}{\ell_1^2 (\ell_1 - p_1)^2 (\ell_1 + p_2)^2} \frac{1}{p_3^2} \lambda_K \left( 1, \frac{(\ell_1 - p_1)^2}{p_3^2}, \frac{(\ell_1 + p_2)^2}{p_3^2} \right),
\]

(98b)

where we have taken into account the expressions of leading singularities for one-loop triangle and box in four space-time dimensions provided by theorems IV.1 and IV.2. Thus, obtaining the explicit integration of the integrands containing square roots of Källen functions (whose arguments are propagators of the loop topology). A similar result is obtained for the application of the loop-by-loop approach in the non-planar two-loop triangle of Fig. 3(b).

\[
J^{(2), D=4}_{P, 4\text{-pt}} = \int \frac{d^4 \ell_1}{16 \pi^2} \frac{d^4 \ell_2}{16 \pi^2} \ell_1^2 (\ell_1 - p_1)^2 (\ell_1 + p_2)^2 \ell_2^2 (\ell_2 - p_1)^2 (\ell_2 + p_2 + p_3)^2
\]

\[
= \int \frac{d^4 \ell_2}{16 \pi^2} \frac{1}{\ell_2^2 (\ell_2 - p_1)^2 (\ell_2 + p_2 + p_3)^2} \int \frac{d^4 \ell_1}{16 \pi^2} \frac{1}{\ell_1^2 (\ell_1 - \ell_2)^2 (\ell_1 - p_1)^2 (\ell_1 + p_2)^2}
\]

\[
\sim \int \frac{d^4 \ell_1}{16 \pi^2} \frac{1}{\ell_1^2 (\ell_2 - p_1)^2 (\ell_2 + p_2 + p_3)^2} \ell_2^2 \lambda_K \left( 1, \frac{\ell_1^2 (\ell_2 + p_2)^2}{s \ell_2^2}, \frac{\ell_1^2 (\ell_2 - p_1)^2}{s \ell_2^2} \right),
\]

(99a)

where, with the observation we just made,

\[
J^{(2), D=4}_{P, 4\text{-pt}} \sim \frac{1}{s} \sqrt{\lambda_K \left( 1, \frac{p_1^2 s}{p_3^2}, \frac{p_2^2 s}{p_3^2} \right)}.
\]

(99b)

with \( s = (p_1 + p_2)^2 \) and \( t = (p_2 + p_3)^2 \).

b. L-loop ladder triangle and box integrals

With the results of Eqs. (98) and (99) at our disposal we can provide explicit expressions of leading singularities for ladder L-loop topologies, by exploiting the loop-by-loop approach. Our findings are summarised in the following theorems.

\[\text{Theorem V.1. The leading singularity of the three-point L-loop ladder Feynman integral of Fig. 4(a) in four space-time dimensions with off-shell external momenta (} p_i^2 \neq 0 \text{ for } i = 1, 2, 3) \text{ and massless propagators is equal to}\]

\[\left( p_3^2 \right)^L \sqrt{\lambda_K \left( 1, \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right)} \cdot \\]
Proof. We prove this theorem by induction on \( L \). For \( L = 2 \), it was explicitly shown in Sec. V A through the direct application of Leray’s residues. Let us then consider the \( L = 3 \) case. 

We begin with the three-loop planar integral,

\[
J_{3,\text{pt}}^{(3),D=4} = \int \prod_{i=1}^{3} \frac{d^4\ell_i}{i\pi^2} \frac{1}{(\ell_i - p_1)^2 (\ell_i + p_2)^2 (\ell_{i-1} - \ell_i)^2},
\]

with \( \ell_0 = 0 \) (see Fig. 4(a)).

We then use the result of the leading singularity in \( D = 4 \) of the two-loop \((L = 2)\) planar triangle,

\[
J_{3,\text{pt}}^{(3),D=4} \sim \int \frac{d^4\ell_3}{i\pi^2} \frac{1}{(\ell_3 - p_1)^2 (\ell_3 + p_2)^2} \int \left( \prod_{i=1}^{2} \frac{d^4\ell_i}{i\pi^2} \frac{1}{(\ell_i - p_1)^2 (\ell_i + p_2)^2 (\ell_{i-1} - \ell_i)^2} \right) \frac{1}{(\ell_2 - \ell_3)^2} \sim \int \frac{d^4\ell_3}{i\pi^2} \frac{1}{(\ell_3 - p_1)^2 (\ell_3 + p_2)^2 (\ell_2 - p_1)^2 (\ell_3 - \ell_2)^2} \left( \frac{1}{\lambda_K(1, \frac{p_1^2(\ell_3+p_2)^2}{p_3^2}, \frac{p_2^2(\ell_3-p_1)^2}{\ell_2^2})} \right),
\]

where by applying the results \((98)\), we find,

\[
J_{3,\text{pt}}^{(3),D=4} \sim \frac{1}{(p_3^2)^3} \sqrt{\lambda_K \left( 1, \frac{p_1^2}{p_3}, \frac{p_2^2}{p_3} \right)}, \quad (102)
\]

in agreement with the induction hypothesis.

Let us now assume that the induction hypothesis is true for \( L \) (with \( L \geq 3 \)). Then, with the same procedure we just followed for \( L = 3 \), we find,

\[
J_{3,\text{pt}}^{(L+1),D=4} = \int \left( \prod_{i=1}^{L+1} \frac{d^4\ell_i}{i\pi^2} \frac{1}{(\ell_i - p_1)^2 (\ell_i + p_2)^2 (\ell_{i-1} - \ell_i)^2} \right)
\]

\[
= \int \frac{d^4\ell_{L+1}}{i\pi^2} \frac{1}{(\ell_{L+1} - p_1)^2 (\ell_{L+1} + p_2)^2} \int \left( \prod_{i=1}^{L} \frac{d^4\ell_i}{i\pi^2} \frac{1}{(\ell_i - p_1)^2 (\ell_i + p_2)^2 (\ell_{i-1} - \ell_i)^2} \right) \frac{1}{(\ell_{L+1} - \ell_L)^2} \sim \int \frac{d^4\ell_{L+1}}{i\pi^2} \frac{1}{(\ell_{L+1} - p_1)^2 (\ell_{L+1} + p_2)^2 (\ell_{L+1} - \ell_L)^2} \left( \frac{1}{\lambda_K(1, \frac{p_1^2(\ell_{L+1}+p_2)^2}{p_3^2}, \frac{p_2^2(\ell_{L+1}-p_1)^2}{p_3^2})} \right),
\]

proving in this way our theorem,

\[
J_{3,\text{pt}}^{(L+1),D=4} \sim \frac{1}{(p_3^2)^{L+1}} \sqrt{\lambda_K \left( 1, \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right)}, \quad (103)
\]

Theorem V.2. The leading singularity of the four-point \( L \)-loop ladder Feynman integral of Fig. 4(b) in four space-time dimensions with off-shell external momenta \((p_i^2 \neq 0\) for \( i = 1, 2, 3, 4 \)) and massless propagators is equal to

\[
\left( s^{L/2} \sqrt{\lambda_K \left( 1, \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right)} \right)^{-1}, \quad \text{with} \quad s = (p_1 + p_2)^2 \quad \text{and} \quad t = (p_2 + p_3)^2.
\]
we find agreement with the induction hypothesis.

\[ J_{t,4-p}^{(3), D=4} \sim \frac{1}{s^3 t^4} \sqrt{\lambda_K \left( 1, \frac{p_1^2 p_2^2}{st}, \frac{p_3^2 p_4^2}{st} \right)}. \]  

Thus, completing the proof of this theorem.

Interestingly, from the results of theorems V.1 and V.2 and their extension to higher multiplicity in external momenta and non-planar topologies, it is possible to predict \( d \log \) integrands at multi-loop level without an explicit calculation. This information turns out to be very useful in techniques that aim at the construction of differential equation in canonical form from the knowledge of a single integral, e.g. Ref. [64].

Also, we would like to remark that the results of theorems V.1 and V.2 were found in Ref. [63] by following an approach based on Mellin-Barnes representation of Feynman integrals.

VI. CONCLUSIONS

Elaborating on the decomposition of the space-time dimension into two independent and complementary subspaces, parallel and perpendicular, in this paper, we carried out a study of the analytic properties of Feynman integrals within this framework. We observed that the various features of Feynman integrals can systematically be tracked down to an analysis at the integrand level, in which techniques and results based on integrand reduction methods were crucial for the results presented in this paper.

Since it is known that Feynman integrals display singularities depending on kinematic invariants in given space-time dimensions, we considered Landau and leading singularities as well as the connections between them. For Landau singularities, because of the way how the space-time dimension was decomposed, we provided and proved the algorithm to calculate these singularities by focusing only on a linear system of equations. We observed that Landau singularities can be cast in the single variable (39), present in the loop-momentum parametrization, whose value is determined by imposing on-shell conditions.

Regarding leading singularities, we observed for one-loop scalar Feynman integrals that the number of integrations is lessened to the number of independent external momenta (\( E \)). This feature, obtained as a product of the general decomposition of the space-time dimension in terms of parallel and perpendicular directions, led us to perform an integrand analysis,
finding a connection between leading and Landau singularities. In detail, we noted and proved that leading singularities in $D = E$ and $D = E + 1$, respectively, correspond to the inverse of the square root of the leading Landau singularity of the first and second type. This connection was at first glance noted with the explicit calculation of leading and Landau singularities of up-to six-point one-loop scalar Feynman integrals. Then, to prove at all multiplicities, we use the multi-dimensional theory of residues provided by Leray.

As a by-product of the connection between Landau and leading singularities, we observed that one-loop scalar Feynman integrals can be cast in a very simple product of $d \log$ forms. We checked this for integrals up to six points and stated the conjecture that it holds for all one-loop scalar Feynman integrals. Additionally, we briefly discussed the connection of our work with the method of differential equations in dimensional regularisation. In this matter, the knowledge of leading singularities for an arbitrary one-loop graph makes the problem of finding the canonical form for a differential equation much simpler.

On top of the aforementioned one-loop analysis, we obtained, promising preliminary results for Feynman multi-loop integrals by applying developed methods. We considered particular families of integrals, namely $L$-loop three- and four-point ladder diagrams. Treating them as iterated integrals, we were able to apply Leray’s residues in the loop-by-loop approach to establish their leading singularities.

Our work which was focused mostly on the one-loop Feynman integrals just scratch the surface of possible applications of methods presented here and other ideas coming from mathematics. We were able to identify possible directions for further studies

1. The first natural extension of this work is to investigate the application of multi-dimensional residues to general multi-loop Feynman integrals, starting from two loops, and its connection to the notion of leading singularity. At two loops, one can already observe that the number of poles is rarely equal to the degree of the form, so even by taking the global residue, we cannot reduce this form to the zero form, which means that there is some integration which is left. Here the crucial point will be the determination of cycles of integration. In the one-loop case, this problem was already studied in [19] in the context of the calculation of non-maximal residues.

2. Further investigation of the connection between the differential equation method, and Landau singularities. On the one hand, it would be interesting to see how much we can learn from the knowledge of leading singularities to find the canonical form for differential equations for Feynman integrals as well as using the knowledge of Landau singularities in integrals that are known to be described by elliptic curved and beyond. On the other hand, it would be very interesting to see if one can recover the full alphabet from Landau singularities by considering all possible residues for a given diagram. In that context, it would also be interesting to look at the connection to cluster algebras.

3. Lastly, it would be very interesting to investigate the $d \log$ structure of Feynman integrals. The multi-dimensional residues can shed new light on this issue. It would be ideal to find a set of criteria which determine whether a given differential form has a full $d \log$ structure. In connection to this, it also would be interesting to see if it is possible to determine the geometry necessary for a given Feynman diagram by considering its Landau singularities.

All the above perspectives can be explored within the approach proposed in this work.

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Appendix A: Leray residues

We will give a brief introduction, without any proofs, to the multi-dimensional theory of residues due to Leray. It is one of the possible extensions of the one-dimensional residue theory. It is particularly useful in our application as it introduces a residue operator that acts directly on differential forms. For a detailed presentation and proofs of this theory, we refer the Reader to the original work of Leray [43] or [15, 53].

Let $X$ be a complex analytic manifold and $S$ a closed analytic submanifold of complex codimension 1. If $\phi$ is a closed differential form in $X - S$ with a pole of first order
on $S$, then in a neighbourhood $U_a$ of any point $a \in S$ the form $\phi$ can be represented as,
\[
\phi = \frac{ds}{s} \wedge \psi + \theta , \tag{A1}
\]
where, $s = s_a(z)$, is a defining function for the manifold $S$ in $U_a$, and $\psi, \theta$ are forms which are regular on $U_a$ The restriction $\psi|_S$ is called the residue form of the form $\phi$, $\psi|_S \equiv res[\phi]$.

**Theorem A.1.** For an arbitrary closed form $\phi$ of degree $p$ on $X - S$ and a cycle $\sigma \in \mathbb{Z}_{p-1}(S)$ there is a formula,
\[
\int_{\delta \sigma} \phi = 2\pi i \int_{\sigma} res[\phi] , \tag{A2}
\]
where $\delta \sigma \in \mathbb{Z}_{p}(X - S)$ is the co-boundary of the cycle $\sigma$.

If the form $\phi \in \mathbb{Z}^p(X - (S_1 \cup \ldots \cup S_m))$ has a pole of first order on $S_1, \ldots, S_m$ then one can define an iterated residue form $res^m[\phi] \in \mathbb{Z}^{p-m}(S_1 \cap \ldots \cap S_m)$ and a homomorphism,
\[
res^m : H^p(X - (S_1 \cup \ldots \cup S_m)) \to H^{p-m}(S_1 \cap \ldots \cap S_m), \tag{A3}
\]
as the composition of homomorphisms,
\[
H^p(X - (S_1 \cup \ldots \cup S_m)) \xrightarrow{res} H^{p-1}(S_1 - (S_2 \cup \ldots \cup S_m)) \xrightarrow{res} \ldots \xrightarrow{res} H^{p-m}(S_1 \cap \ldots \cap S_m).
\]
There is also a sequence of co-boundary homomorphisms
\[
H_p(X - (S_1 \cup \ldots \cup S_m)) \xrightarrow{\delta_1} H_{p-1}(S_1 - (S_2 \cup \ldots \cup S_m)) \xrightarrow{\delta_2} \ldots \xrightarrow{\delta_m} H_{p-m}(S_1 \cap \ldots \cap S_m).
\]
Iterating Theorem A.1 we can write the formula for composed residue
\[
\int_{\delta^n \sigma} \phi = (2\pi i)^m \int_{\sigma} res^m[\phi] , \tag{A6}
\]
The composed residue of a form $\phi$ with polar singularities of order $q_1, q_2, \ldots, q_m$ on $S_1, \ldots, S_m$ is written
\[
res^m[\phi] = \frac{1}{(q_1 - 1)! \ldots (q_m - 1)!} \frac{d\omega^{q_1 + \ldots + q_m - m}}{dS_1^{q_1} \wedge \ldots \wedge dS_m^{q_m}} |_{S_1 \cap \ldots \cap S_m}, \tag{A7}
\]
where $\omega = s_1^{q_1} \ldots s_m^{q_m} \phi$.

In our case, we deal with scalar one-loop Feynman integrals that in momentum components can be cast as,
\[
\phi = \frac{dk_1 \wedge \ldots \wedge dk_D}{D_1 \ldots D_m} , \tag{A8}
\]
where $D_i = (k + P_{i-1})^2 - m_i^2$, with $P_{i-1} = p_1 + \ldots + p_{i-1}$ and $P_0 = 0$.

**Proposition A.1.** A one-loop scalar Feynman integrand $\phi = \frac{dk_1 \wedge \ldots \wedge dk_D}{D_1 \ldots D_m}$ is a closed differential form, i.e., $d\phi = 0$.

**Proof.** We are in a $D$-dimensional $k$-space, i.e., $\phi$ is a top holomorphic form. Therefore, $d\phi = (\frac{\partial \phi}{\partial k_1} dk_1 + \ldots + \frac{\partial \phi}{\partial k_D} dk_D) \wedge dk_1 \wedge \ldots \wedge dk_D = 0$. \hfill $\square$

Since we are interested in the calculation of composite residue form for Feynman integrands, we have $\phi = \frac{d\omega^{q_1} \wedge \ldots \wedge d\omega^{q_m}}{D_1 \ldots D_m}$ and $q_1 = \ldots = q_m = 1 \Rightarrow q_1 + \ldots + q_m = m$. Then, we need to consider the highest co-dimensional residue, i.e., all $D_i$ intersect $D_1 = \ldots = D_m = 0$, and the case when $d = m$, thus,
\[
res^m[\phi] = \frac{dz_1 \wedge \ldots \wedge dz_n}{dD_1 \wedge \ldots \wedge dD_n} \in H^0(D_1 \cap \ldots \cap D_n) . \tag{A9}
\]
a. Example: one-loop bubble in $D = 2$
\[
\phi = \frac{dk_1 \wedge dk_2}{(k^2 - m^2)((k + p)^2 - m^2)}, \tag{A10}
\]
\[
D_1 = k^2 - m^2 = k_2^2 - m^2, \quad D_2 = (k + p)^2 - m^2 = (k_1 + p_1)^2 - (k_2 + p_2)^2 - m^2.
\]
The differentials of polar sets $D_1$ and $D_2$ are,
\[
dD_1 = 2k_1 dk_1 - 2k_2 dk_2, \tag{A11}
dD_2 = 2(k_1 + p_1) dk_1 - 2(k_2 + p_2) dk_2,
\]
and their wedge product is
\[
dD_1 \wedge dD_2 =
\]
\[
\begin{align*}
&= -4k_1(k_2 + p_2) dk_1 \wedge dk_2 - 4k_2(k_1 + p_1) dk_2 \wedge dk_1 \\
&= -4(k_1 p_2 - k_2 p_1) dk_1 \wedge dk_2 . \tag{A12}
\end{align*}
\]
Hence the residue form is given by
\[
res^2[\phi] = \frac{dk_1 \wedge dk_2}{-4(k_1 p_2 - k_2 p_1) dk_1 \wedge dk_2} = \frac{1}{-4(k_1 p_2 - k_2 p_1)} . \tag{A13}
\]
By solving $D_1 = D_2 = 0$ for $k_1$ and $k_2$ we get two solutions which give us
\[
res^2[\phi] = \pm \frac{1}{2\sqrt{-s(4m^2 - s)}} . \tag{A14}
\]
which is the expected result.

**Appendix B: Supplementary results**

In this section, we collect results which we use to prove statements in the main text.
Lemma B.1. The sum $\sum_{k=1}^{n} \det(A_1 A_2 \ldots A_k \ldots A_n)$ is equal to $\det(I(A_2 - A_1)(A_3 - A_1) \ldots (A_n - A_1))$, where $A_j = (a_{ij}, \ldots, a_{in})^T$ for $j = 1, \ldots, n$ are columns of the determinants and $I_k = (1, \ldots, 1)^T$ replaces the $k$'th column $A_k$ in $k$'th term of the sum.

Proof.

$$
|IA_2 A_3 \ldots A_n| + |A_1 I A_3 \ldots A_n| + \\
+ \ldots + |A_1 A_2 A_3 \ldots A_{n-1}| = \\
|IA_2 A_3 \ldots A_n| - |A_1 A_3 \ldots A_n| + \\
+ \ldots + |A_1 A_2 A_3 \ldots A_{n-1}| = \\
|I(A_2 - A_1) A_3 \ldots A_n| + |A_1 A_2 I A_n| + \\
+ \ldots + |A_1 A_2 A_3 \ldots A_{n-1}| = \\
|I(A_2 - A_1) A_3 \ldots A_n| - |I A_2 A_1 \ldots A_n| + \\
+ \ldots + |A_1 A_2 A_3 \ldots A_{n-1}| = \\
|I(A_2 - A_1)(A_3 - A_1) \ldots A_n| + \ldots + |A_1 A_2 A_3 \ldots A_{n-1}|
$$

(B1)

By repeating the same procedure for remaining terms we get the result.

Lemma B.2. \(\det(a_{ij}) dx_1 \wedge \ldots \wedge dx_n = (a_{11} dx_1 + \ldots a_{1n} dx_n) \wedge \ldots \wedge (a_{n1} dx_1 + \ldots a_{nn} dx_n)\).

Proof. The statement will be proved by the induction. Let us prove this for $n = 2$,

\[
(a_{11} dx_1 + a_{12} dx_2) \wedge (a_{21} dx_1 + a_{22} dx_2) = \\
= a_{11} a_{21} dx_1 \wedge dx_1 + a_{11} a_{22} dx_1 \wedge dx_2 + \\
+ a_{12} a_{21} dx_2 \wedge dx_1 + a_{12} a_{22} dx_2 \wedge dx_2 = \\
= a_{11} a_{22} - a_{12} a_{21} dx_1 \wedge dx_2 \\
= |a_{11} a_{12}|
\]

Let us assume that the statement holds for $n = m > 2$, i.e.,

\[
|a_{i1} \ldots a_{im}|
\]

\[
\vdots \quad \vdots \quad \vdots \\
|a_{m1} \ldots a_{mm}|
\]

\[
= (a_{i1} dx_i \wedge \ldots \wedge a_{im}) \wedge \ldots \wedge (a_{m1} dx_i \wedge \ldots \wedge a_{mm}),
\]

(B3)

with $i_1 < \ldots < i_m$.

Let us now check the statement for $m + 1$,

\[
|a_{11} \ldots a_{1m} a_{1(m+1)}| \\
\vdots \quad \vdots \quad \vdots \\
|a_{m1} \ldots a_{mm} a_{m(m+1)}| \\
|a_{(m+1)1} \ldots a_{(m+1)m} a_{(m+1)(m+1)}|
\]

(B4)

By expanding this determinant with respect to the last row,

\[
(-1)^{(m+1)+1} a_{(m+1)1} \\
|a_{m2} \ldots a_{mm} a_{m(m+1)}|
\]

(B5)

where by induction hypothesis we get,

\[
(-1)^{2(m+1)} \left[ (a_{12} dx_2 + \ldots a_{1(m+1)} dx_{m+1}) \right] \wedge \left( a_{m2} dx_2 + \ldots + a_{m(m+1)} dx_{m+1} \right) \\
+ \ldots + \\
(-1)^{2(m+1)} \left[ (a_{11} dx_1 + \ldots a_{1m} dx_m) \right] \wedge \left( a_{m1} dx_1 + \ldots + a_{mm} dx_m \right) \\
\wedge a_{(m+1)(m+1)} dx_{m+1}
\]

(B6)

However, because of $dx_i \wedge dx_j = 0$, we can add to each bracket one missing one-form,

\[
[(a_{11} dx_1 + a_{12} dx_2 + \ldots a_{1(m+1)} dx_{m+1}) \wedge \ldots \wedge \\
(a_{m1} dx_1 + a_{m2} dx_2 + \ldots + a_{m(m+1)} dx_{m+1})] \wedge (a_{m+1} dx_1 + \ldots + a_{m+1} dx_{m+1}) \\
+ \ldots + \\
[(a_{11} dx_1 + \ldots a_{1m} dx_m + a_{1(m+1)} dx_{m+1}) \wedge \ldots \wedge \\
(a_{m1} dx_1 + \ldots + a_{mm} dx_m + a_{m(m+1)} dx_{m+1})] \wedge \\
\wedge a_{(m+1)(m+1)} dx_{m+1},
\]

(B7)

that allows us to factor out the square bracket,

\[
[(a_{11} dx_1 + a_{12} dx_2 + \ldots a_{1(m+1)} dx_{m+1}) \wedge \ldots \wedge \\
(a_{m1} dx_1 + a_{m2} dx_2 + \ldots + a_{m(m+1)} dx_{m+1})] \\
\wedge (a_{(m+1)1} dx_1 + \ldots + a_{(m+1)(m+1)} dx_{m+1}) = \\
= (a_{11} dx_1 + a_{12} dx_2 + \ldots a_{1(m+1)} dx_{m+1}) \wedge \ldots \wedge \\
(a_{m1} dx_1 + a_{m2} dx_2 + \ldots + a_{m(m+1)} dx_{m+1}) \wedge \\
\wedge (a_{(m+1)1} dx_1 + \ldots + a_{(m+1)(m+1)} dx_{m+1})
\]

(B8)

completing the proof of this lemma.

Alternatively the statement of the Lemma B.2 can be seen as the definition of the determinant as follows: Let
Lemma B.3. The leading singularity of the integrand \( j \int \prod_{i=1}^{n} \frac{1}{x_{i}} \) is equal to \( \sum_{i=1}^{n} \frac{1}{D_{i}} \prod_{i=1}^{1} D_{j,i} \), where \( D_{j,i} = D_{j} - D_{i} \).

Proof. We prove the result by induction on \( n \). For \( n = 2 \) or 3, the result is evident. Let us now assume that this relation is true for \( n = m \), with \( m \geq 3 \), and corresponds to our induction hypothesis. Then, by working out the product of \( (m+1) \) denominators and keeping in mind the induction hypothesis,

\[
\frac{1}{D_{1} \cdots D_{m} D_{m+1}} = \frac{1}{D_{m+1}} \sum_{i=1}^{m} \frac{1}{D_{i}} \prod_{j=1, j \neq i}^{1} D_{j,i} \]

\[
= \sum_{i=1}^{m+1} \left( \frac{1}{D_{i}} - \frac{1}{D_{m+1}} \right) \prod_{j=1, j \neq i}^{m+1} D_{j,i} ,
\]

(B10)

where we took into account the relation for \( n = 2 \) in the product of denominators \( D_{i} \) and \( D_{m+1} \), we recalled that \( D_{i,m+1} = -D_{m+1,i} \), and added a vanishing term in the sum.

In (B10), we observe that the first term corresponds to the relation for \( (m+1) \) in the product of denominators \( D_{1}, D_{2}, \ldots, D_{m}, D_{m+1} \). This means that we are left to prove,

\[
\sum_{i=1}^{m+1} \frac{1}{\prod_{j=1, j \neq i}^{m+1} D_{j,i}} = 0 .
\]

(B11)

Because of the induction hypothesis, we can write (B11) as,

\[
\sum_{i=1}^{m+1} \frac{1}{\prod_{j=1, j \neq i}^{m+1} D_{j,i}} = \sum_{i=1}^{m} \frac{1}{\prod_{j=1, j \neq i}^{m+1} D_{j,i}} + \sum_{i=1}^{m} \frac{1}{D_{i,m+1}} \prod_{j=1, j \neq i}^{m+1} (D_{j,m+1} - D_{i,m+1})
\]

(B12)

In the last line, we took into account that \( D_{j,m+1} - D_{i,m+1} = D_{j,i} \) and \( D_{i,m+1} = -D_{m+1,i} \). Therefore, the proof is complete by induction. \( \Box \)

Lemma B.4. The leading singularity of the integrand \( j \int \prod_{i=1}^{n} \frac{1}{x_{i}} \) is equal to \( 1/\det(x_{ij}) \), where \( K_{i} = x_{i0} + \sum_{j=1}^{n} x_{ij} a_{j} \).

Proof. Let us calculate a composite Leray residue of degree \( n \),

\[
res^{n} = \frac{da_{1} \wedge \ldots \wedge da_{n}}{dK_{1} \wedge \ldots \wedge dK_{n}} .
\]

(B13)

For any \( i \) we have,

\[
dK_{i} = x_{i0} da_{1} + \ldots + x_{in} da_{n} .
\]

(B14)

Thus the residue is equal to,

\[
res^{n} = \frac{da_{1} \wedge \ldots \wedge da_{n}}{(x_{11} da_{1} + \ldots + x_{in} da_{n}) \wedge \ldots \wedge (x_{n1} da_{1} + \ldots + x_{nn} da_{n})} .
\]

(B15)

By Lemma B.2 this is equal to,

\[
res^{n} = \frac{1}{\det(x_{ij}) da_{1} \wedge \ldots \wedge da_{n}} = \frac{1}{\det(x_{ij})} .
\]

(B16)

Thus, proving our claim. \( \Box \)

Appendix C: Additional examples of one-loop Landau singularities

In this appendix, we supplement the results provided in Secs. III and IV with additional examples of explicit calculations of singularities of up-to six-point one-loop Feynman integrals, and summarise the identities employed in Sec. IV to find \( d \log \) representations of one-loop bubble and triangle integrals in \( D \geq n \) space-time dimension.

To explicitly display the results of scalar one-loop box, pentagon, and hexagon Feynman integrals, we consider massless external momenta, \( p_{i}^{2} = 0 \), and equal internal masses. Lastly, since Landau and leading singularities are related through theorems IV.1 and IV.2 (or equivalently Eqs. (77)), we only focus on the derivation of Landau singularities, by listing explicit expressions for the determinants of propagators and external momenta.
a. One-loop box integral
By taking into account the parametrization of the loop momentum in terms of three independent external momenta, say $p_1, p_2, p_3$, and keeping generic the internal masses of propagators, the loop momentum components become,

$$a_1 = - \frac{m_{13}^2 t + s (m_{24}^2 - m_{34}^2 + 2s + t)}{2s(s + t)},$$

$$a_2 = \frac{t}{2} \left( \frac{m_{13}^2}{s} + \frac{m_{24}^2 + t}{t} \right),$$

$$a_3 = \frac{(m_{12}^2 - m_{34}^2) t - s (m_{24}^2 + t)}{2t(s + t)},$$

$$\lambda_{11} = m_1^2 + a_1 (a_3(s + t) - a_2 s) - a_2 a_3 t,$$  \hspace{1cm} (C1)

with the Landau singularity,

$$\text{LanS}^{(1)}_{5} = - \frac{1}{16} \left[ s^2 t^2 - t^2 \lambda_R \left( m_1^2, m_2^2, s \right) - s^2 \lambda_R \left( m_2^2, m_4^2, t \right) + 2st \left( m_{12}^2 m_{34}^2 - m_{14}^2 m_{23}^2 \right) \right],$$ \hspace{1cm} (C2)

in terms of the kinematic invariants $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$, and, to simplify the notation, we defined $m_i^2 = m_i^2 - m_j^2$.

Thus, the Landau singularity in the equal-mass case configuration becomes,

$$\text{LanS}^{(1)}_{5} \bigg|_{m_i^2 = m} = \frac{s^2 t^2}{16} - \frac{1}{4} m^2 s t(s + t),$$ \hspace{1cm} (C3)

where we recall that the Gram determinant for this kinematic configuration is, $\det (p_1 \cdot p_2) = -1/4 st(s + t)$.

b. One-loop pentagon integral
For the one-loop pentagon, we find for $\lambda_{11}$,

$$\lambda_{11} = m_1^2 - a_1 a_3 a_4 s_34 + a_2 \left( a_4 \left( s_{23} + s_{34} - s_{51} \right) - a_3 s_{23} \right) + a_1 \left( -a_2 s_{12} + a_3 \left( s_{12} + s_{23} - s_{45} \right) \right) + a_4 \left( -a_2 s_{23} + a_4 \left( s_{34} + s_{45} - s_{51} \right) \right),$$ \hspace{1cm} (C4)

in terms of five kinematic invariants, $s_{ij} = (p_i \cdot p_j)^2$, whose explicit expressions for $a_i = \tilde{a}_i / (16 \det (p_1 \cdot p_3))$ in the equal mass case are,

$$\tilde{a}_1 = -s_{12}^2 \left( s_{23} - s_{51} \right)^2 + s_{12} \left( 2 s_{45} s_{51}^2 - \left( s_{23} s_{34} + 2 \left( s_{23} + s_{34} \right) s_{45} \right) s_{51} + s_{23} s_{34} \left( s_{23} - s_{45} \right) \right) + s_{45} \left( s_{34} - s_{51} \right) \left( s_{23} s_{34} + s_{45} \left( s_{34} - s_{34} \right) \right),$$

$$\tilde{a}_2 = -s_{12}^2 \left( s_{23} - s_{51} \right)^2 + s_{12} \left( 2 s_{45} s_{51}^2 - \left( s_{23} s_{34} + 2 \left( s_{34} + s_{45} \right) s_{51} + s_{23} s_{34} \left( s_{23} - s_{45} \right) \right) s_{51} + s_{23} s_{34} \left( s_{23} - s_{45} \right) \right) + s_{45} s_{51} \left( s_{23} s_{34} + s_{45} \left( s_{51} - s_{54} \right) \right),$$

$$\tilde{a}_3 = -s_{12} \left( s_{34} s_{45} + \left( s_{34} + s_{45} \right) s_{51} \right) s_{23} + s_{12} \left( s_{23} - s_{51} \right)^2 - s_{23} s_{34} + s_{45} \left( s_{34} - s_{51} \right) s_{51},$$

$$\tilde{a}_4 = -s_{12} s_{23} \left( -s_{23} s_{34} + s_{12} \left( s_{23} - s_{51} \right) + s_{45} \left( s_{34} + s_{51} \right) \right),$$

$$\tilde{a}_5 = -s_{12} s_{23} \left( -s_{23} s_{34} + s_{12} \left( s_{23} - s_{51} \right) + s_{45} \left( s_{34} + s_{51} \right) \right).$$ \hspace{1cm} (C7)

Thus, the Landau singularity takes the form,

$$\text{LanS}^{(1)}_{6} \bigg|_{m_i^2 = m} = - \frac{1}{16} s_{12} s_{23} s_{34} s_{45} s_{51} + m^2 \det (p_1 \cdot p_2),$$ \hspace{1cm} (C8)

with,

$$\det (p_1 \cdot p_2) =$$

$$\frac{1}{16} \left[ s_{12}^2 \left( s_{23} - s_{51} \right)^2 + \left( s_{23} s_{34} + s_{45} \left( s_{51} - s_{34} \right) \right)^2 + 2 s_{12} \left( s_{34} s_{45} + \left( s_{34} + s_{45} \right) s_{51} \right) s_{23} - s_{23} s_{34}^2 + s_{45} \left( s_{34} - s_{51} \right) s_{51} \right].$$ \hspace{1cm} (C9)

c. One-loop hexagon integral
In order to describe the Landau singularities for the one-loop hexagon, we choose, without loss of generality, the following set of nine kinematic scales,

$$\{ s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345} \},$$ \hspace{1cm} (C10)

with $s_{ijk} = s_{ij} + s_{ik} + s_{jk}$. This choice of variables allows us to express $\lambda_{11}$ as,

$$\lambda_{11} = m_1^2 + a_1 \left( -a_2 s_{12} + a_3 \left( s_{12} + s_{23} - s_{123} \right) + a_4 \left( -a_2 s_{23} + a_4 \left( s_{34} + s_{45} - s_{345} \right) \right) + a_3 \left( a_5 \left( s_{34} + s_{45} - s_{345} \right) - a_4 s_{45} \right) - a_4 a_5 s_{45},$$ \hspace{1cm} (C11)

and, therefore, the Landau singularity in the equal-mass case,

$$\text{LanS}^{(1)}_{6} \bigg|_{m_i^2 = m} =$$

$$\frac{1}{64} \left[ -s_{12}^2 s_{23}^2 s_{34}^2 - \left( s_{23} s_{34} + 2 \left( s_{23} + s_{34} \right) s_{45} \right) s_{51} + s_{23} s_{34} \left( s_{23} - s_{34} \right) \right] + m^2 \det (p_1 \cdot p_2),$$ \hspace{1cm} (C12)

with the Gram determinant calculated in terms of the kinematic scales (C10).

d. Identities used in the derivation of d log bubble and triangle Feynman integrals
The results reported in Eqs. (54) and (58) were obtained through the use of the following identities,

$$\frac{1}{D_1 D_2} = \frac{1}{D_1} - \frac{1}{D_2} \frac{1}{D_2} D_2 - \frac{1}{D_1},$$ \hspace{1cm} (C14a)

$$\frac{dx}{\sqrt{x + y}} = \frac{1}{\sqrt{-y}} d \log \frac{\sqrt{x + y}}{\sqrt{x} + \sqrt{-y}},$$ \hspace{1cm} (C14b)
\[ \frac{dx}{x \sqrt{(x + r_1)(x + r_2)}} = \frac{1}{\sqrt{r_1 r_2}} d \log \left( \frac{x + \sqrt{x + r_1} \sqrt{x + r_2}}{x + \sqrt{x + r_1} \sqrt{x + r_2} - \sqrt{r_1 r_2}} \right) \]

where in (C14b) and (C14c) forms depend only on the \( x \) variable while \( y, r_1 \) and \( r_2 \) are kept fixed.

Appendix D: Proof of Theorem IV.2 in momentum representation

**Proposition D.1.** The leading singularity of an \( n \)-point one-loop Feynman integral in \( D = n \) space-time dimensions is equal to \( \pm 1 / \left( 2^n \sqrt{(-1)^{D-1} \text{LanS}} \right) \)

**Proof.** The one-loop Feynman integrand \( \omega \) has the following form,

\[ \omega = \frac{d^n k}{D_1 \ldots D_n} = \frac{d k_1 \wedge \ldots \wedge d k_n}{D_1 \ldots D_n}, \]  

where \( k = (k_1, \ldots, k_n) \) is an \( n \)-dimensional loop momentum vector and \( D_i = (k + \sum_{j=1}^{i-1} p_j)^2 - m_i^2 \) with \( p_j = (p_{j1}, \ldots, p_{jn}) \) being \( n \)-dimensional external momenta. From the definition, the Leading singularity is the highest co-dimensional residue of the Feynman integral. Thus, let us calculate composite Leray residue of degree \( n \) of the \( \omega \),

\[ \text{res}^n [\omega] = \frac{d k_1 \wedge \ldots \wedge d k_n}{d D_1 \wedge \ldots \wedge d D_n}. \]  

Each \( d D_i \) is given by,

\[ d D_i = 2(k_1 + \sum_{j=1}^{i-1} p_{j1}) dk_1 + (-2)(k_1 + \sum_{j=1}^{i-1} p_{j2}) dk_2 + \ldots + (-2)(k_n + \sum_{j=1}^{i-1} p_{jn}) dk_n. \]

Hence, we get,

\[ \text{res}^n [\omega] = \frac{d k_1 \wedge \ldots \wedge d k_n}{\text{det}(A) d k_1 \wedge \ldots \wedge d k_n} = \frac{1}{\text{det}(A)} \]  

where,

\[ A = \begin{pmatrix} 2k_1 & -2k_2 & \ldots & -2k_n \\ 2(k_1 + p_{11}) & -2(k_2 + p_{12}) & \ldots & -2(k_n + p_{1n}) \\ \vdots & \ddots & \ddots & \vdots \\ 2(k_1 + \sum_{j=1}^{n-1} p_{j1}) & -2(k_2 + \sum_{j=1}^{n-1} p_{j2}) & \ldots & -2(k_n + \sum_{j=1}^{n-1} p_{jn}) \end{pmatrix}. \]

By the linearity of determinants, \( \text{det}(A) \) can be written as,

\[ \text{det}(A) = \begin{vmatrix} 2k_1 & \ldots & -2k_n \\ 2k_1 & \ldots & -2k_n \\ \vdots & \ddots & \vdots \\ 2(k_1 + \sum_{j=1}^{n-1} p_{j1}) & \ldots & -2(k_n + \sum_{j=1}^{n-1} p_{jn}) \end{vmatrix} + \begin{vmatrix} 2k_1 & \ldots & -2k_n \\ 2k_1 & \ldots & -2k_n \\ \vdots & \ddots & \vdots \\ 2k_1 & \ldots & -2k_n \end{vmatrix} \]

\[ \begin{vmatrix} 2p_{11} & \ldots & -2p_{1n} \\ \vdots & \ddots & \vdots \\ 2(k_1 + \sum_{j=1}^{n-1} p_{j1}) & \ldots & -2(k_n + \sum_{j=1}^{n-1} p_{jn}) \end{vmatrix}, \]
where the first term is equal to zero and by repeating this procedure we end up with,

\[
\det(A) = \begin{vmatrix}
2k_1 & \cdots & -2k_n \\
2p_{11} & \cdots & -2p_{1n} \\
\vdots & \ddots & \vdots \\
2p_{(n-1)1} & \cdots & -2p_{(n-1)n}
\end{vmatrix} = (-1)^{n-1}2^n.
\]

Let us denote by \( B \) the following matrix,

\[
B = \begin{pmatrix}
2k_1 & \cdots & -2k_n \\
2p_{11} & \cdots & -2p_{1n} \\
\vdots & \ddots & \vdots \\
2p_{(n-1)1} & \cdots & -2p_{(n-1)n}
\end{pmatrix},
\]

and we have \( \det(B) = \det(A) \). On the other hand, we have,

\[
\det(BgB^T) = \det(B) \det(g) \det(B^T) = \det(B)^2 \det(g) \Rightarrow \det(B) = \pm \sqrt{\det(BgB^T)/\det(g)}.
\]

This gives us

\[
BgB^T = 2^2 \begin{pmatrix}
q_1 \cdot q_1 & q_1 \cdot q_2 - q_1 \cdot q_1 & \cdots & q_1 \cdot q_n - q_1 \cdot q_1 \\
q_1 \cdot q_2 - q_1 \cdot q_1 & q_2 \cdot q_2 + q_1 \cdot q_1 - 2q_1 \cdot q_2 & \cdots & q_2 \cdot q_n + q_1 \cdot q_n - q_2 \cdot q_1 - q_1 \cdot q_2 \\
\vdots & \vdots & \ddots & \vdots \\
q_1 \cdot q_n - q_1 \cdot q_{n-1} & q_2 \cdot q_n + q_1 \cdot q_n - q_2 \cdot q_1 - q_1 \cdot q_2 & \cdots & q_n \cdot q_n + q_{n-1} \cdot q_n - 2q_{n-1} \cdot q_n
\end{pmatrix}.
\]

Let us now transform \( \det(BgB^T) \), firstly by performing operations on rows. Let us add the first row to the second row and then a new second row from the third row and repeat this procedure until we reach the \( n \)-th row. After this, we will have,

\[
\det(BgB^T) = 2^{2n} \begin{vmatrix}
q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_n \\
q_1 \cdot q_2 & q_2 \cdot q_2 & \cdots & q_2 \cdot q_n \\
\vdots & \vdots & \ddots & \vdots \\
q_1 \cdot q_n & q_2 \cdot q_n & \cdots & q_n \cdot q_n
\end{vmatrix} = 2^{2n} \det(q_1 \cdot q_j).
\]

This is exactly the function which together with on-shell conditions provides the Landau variety, but on-shell conditions are satisfied as we calculated the residue around polar sets \( D_1 = \ldots = D_n = 0 \). Combining together all the results we have,

\[
\det A = \det(B) = \pm \sqrt{\det(BgB^T)/\det(g)} = \pm \sqrt{2^{2n}(-1)^{n-1} \det(q_1 \cdot q_j)} = \pm 2^n \sqrt{(-1)^{n-1} \det(q_1 \cdot q_j)},
\]

where \( g = \text{diag}(1, -1, \ldots, -1) \) is a metric tensor and \( \det(g) = (-1)^{n-1} \).

However, \( BgB^T \) is equal to,

\[
BgB^T = 2^2 \begin{pmatrix}
k \cdot k \cdot k \cdot p_1 & k \cdot p_1 & \cdots & k \cdot p_{n-1} \\
k \cdot p_1 & p_1 \cdot p_1 & \cdots & p_{n-1} \cdot p_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
k \cdot p_{n-1} & p_1 \cdot p_{n-1} & \cdots & p_{n-1} \cdot p_{n-1}
\end{pmatrix}.
\]
and finally
\[ \text{res}^n[\omega] = \frac{1}{\text{det}(A)} = \frac{\pm 1}{2^n \sqrt{(-1)^{n-1} \text{det}(q_i \cdot q_j)}}, \quad (D17) \]
that ends our proof.

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