FINITNESS OF PROLONGATIONS OF GRADED LIE ALGEBRAS

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Abstract. We find necessary and sufficient conditions for the finiteness of Tanaka’s maximal prolongation of fundamental graded Lie algebras. In the final part we discuss some examples of simple prolongations.

Contents

Introduction 2
1. Fundamental graded Lie algebras and prolongations 4
2. Tanaka’s finiteness criterion 7
2.1. Left and right $V$-modules 7
2.2. Right $m$-modules 10
2.3. Reduction to first kind 12
3. $\mathfrak{g}$-prolongations of graded Lie algebras of the first kind 13
3.1. Prolongation of irreducible representations 19
4. $\mathfrak{gl}(V)$-prolongations of (FGLA)’s of the second kind 19
5. $\mathfrak{gl}(V)$-prolongations of (FGLA)’s of higher kind 23
6. $\mathfrak{g}$-prolongations of (FGLA)’s of general kind 25
7. Some semisimple examples 26
7.1. Structure algebras of type $A$ 28
7.2. Structure algebras of type $B$ 35
7.3. Structure algebras of type $C$ 36
7.4. Structure algebras of type $D$ 37
7.5. Real Spin representations of real Lie algebras of type $D$ 38
7.6. Complex spin representations of real lie algebras of type $D$ 40
7.7. Quaternionic spin representations of real Lie algebras of type $D$ 41
References 43

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The concept of $G$-structure was introduced to treat various interesting differential geometrical structures in a unified manner (see e.g. \cite{22, 23, 24, 25, 26, 27, 28}). At a chosen point $p_0$ of a manifold $M$, a $G$-structure is described by the datum of a Lie algebra $\mathfrak{L}$ of infinitesimal transformations, acting as linear maps on the tangent space $V=T_{p_0}M$. It is convenient to envision $V\oplus\mathfrak{L}$ as an Abelian extension of $\mathfrak{L}$ and to look for its maximal effective prolongation, to read the differential invariants of the structure. When this turns out to be finite dimensional, Cartan’s method can be used to study the automorphism group and the equivalence problem for the corresponding $G$-structure. This algebraic point of view was taken up systematically in \cite{19, 35} and pursued in \cite{2, 29, 36, 37} to study generalised contact and CR structures: the datum of a smooth distribution defines on $T_{p_0}M$ a structure of $\mathbb{Z}$-graded nilpotent Lie algebra $\mathfrak{m}=\sum_{p=1}^\infty \mathfrak{g}_{-p}$. The condition that the distribution be completely non integrable, or satisfies Hörmander’s condition at $p_0$, translates into the fact that $\mathfrak{m}$ is fundamental, i.e. that the part $\mathfrak{g}_{-1}$, tangent at $p_0$ to the contact distribution, generates $\mathfrak{m}$.

Having recently proved in \cite{30} a finiteness result for the automorphism group of a class of homogeneous CR manifolds by applying a result of N.Tanaka to a suitable filtered object, we got interested in the general preliminary problem of the finiteness of the effective $L$-prolongations of general fundamental $\mathbb{Z}$-graded Lie algebras, starting from rereading Tanaka’s seminal paper \cite{37}.

Let us describe the contents of this work. Being essentially algebraic, all results are formulated for an arbitrary ground field $K$ of characteristic zero.

In §1 we describe fundamental graded Lie algebras as quotients of the free Lie algebra $\mathfrak{f}(V)$ generated by a vector space $V$ by its graded ideals $\mathfrak{K}$. After getting from \cite{9} the maximal effective prolongation $\tilde{\mathfrak{f}}(V)$ of $\mathfrak{f}(V)$, we use this result to characterise, in Theorem 1.8, the maximal effective $\mathfrak{L}$-prolongation of $\mathfrak{m}$ in terms of $\tilde{\mathfrak{f}}(V)$ and $\mathfrak{K}$.

In §2 we review the finiteness theorem proved by N.Tanaka in \cite{37}. Here we make explicit the role of duality, which allows to reduce to commutative algebra. This was hinted in Serre’s appendix to \cite{19} and also in \cite{37}, but, in our opinion, in a way which left part of the arguments rather obscure. We hope that our exposition would make this important theorem more understandable. The final result, Theorem 2.11, reduces the finiteness of the maximal $\mathfrak{L}$-prolongation to the analogous problem for an $\mathfrak{L}'$-prolongation of an Abelian Lie algebra. For the application, one needs on one hand to have a viable criterion for studying prolongations of Abelian Lie algebras and on the other to obtain an explicit description of $\mathfrak{L}'$ after knowing $\mathfrak{L}$ and $\mathfrak{m}$. This is done in the following sections.

In §3 we get an effective criterion for the finiteness of $\mathfrak{L}$-prolongations of fundamental graded Lie algebras of the first kind, using duality to reduce the question to the study the co-primary decomposition of finitely generated
modules over the polynomials: this boils down to computing the rank of a matrix $M_1(\mathcal{L}, z)$ of first degree homogeneous polynomials associated to $\mathcal{L}$ (see Theorem 3.7). This condition is equivalent to ellipticity in the sense of Kobayashi ([22, p.4]) for $\mathbb{F} \otimes \mathcal{L}$, where $\mathbb{F}$ is the algebraic closure of $\mathbb{K}$.

We illustrate this procedure by studying the classical examples of the $G$-structures treated, e.g., in [22, 35].

When $m$ has kind $\mu > 1$, the effect of the terms $g_{-p}$ with $p \geq 2$ is of restricting the structure algebra $L$ to a smaller algebra $L'$. The final criterion is obtained by applying Theorem 3.7 to $L'$. In fact, there is a difference in the way the $g_{-p}$ summands contribute to $L'$ between the cases $p=2$ and $p>2$.

In §4 we study the maximal $\mathfrak{gl}_\mathbb{K}(V)$-prolongation in the case $\mu = 2$. We show in Theorem 4.8 that the condition can be expressed in terms of the rank of the Lie product, considered as an alternate bilinear form on $\mathbb{F} \otimes g_{-1}$, for the algebraic closure $\mathbb{F}$ of the ground field $\mathbb{K}$. We show by an example that this rank condition, that was known to be necessary over $\mathbb{K}$ (see e.g. [22]), becomes necessary and sufficient when stated over $\mathbb{F}$. As in §3 there is an equivalent formulation in terms of the rank a suitable matrix $M_2(\mathcal{K}, z)$.

In §5 we show that one can take into account the non zero summands $g_{-p}$ in $m$, with $p \geq 3$, by adding to $M_2(\mathcal{K}, z)$ a matrix $M_3(\mathcal{K}, z)$, which has rank $n= \dim g_{-1}$ when $z$ does not belong to a subspace $W(\mathcal{K})$ of $\mathbb{F} \otimes V$. Thus the finite dimensionality criterion of §4 has to be checked on a smaller subspace of $z$'s.

In §6 Theorem 6.1 collects the partial results of the previous sections, to state a finiteness criterion for the maximal $\mathcal{L}$-prolongation of an $m$ of any finite kind $\mu$ in terms of the rank of the matrix $(M_1(\mathcal{L}, z), M_2(\mathcal{K}, z), M_3(\mathcal{K}, z))$ which is obtained by putting together the contributions coming from $\mathcal{L}$ (kind one), $g_{-2}$ (kind two) and $\sum_{p \geq -3}^\infty g_p$ (kind $>2$).

Semisimple $\mathbb{Z}$-graded Lie algebras are a source of interesting instances of the structures we study in this paper. We collect in §7 several examples, related to the exceptional Lie algebras, where spin representations play an important role. We believe that some of them could be of some interest in physics, where graded Lie algebras may play a role in understanding the basic symmetries of nature.

Notation.

- We shall indicate by $\mathbb{K}$ a field of characteristic zero and by $\mathbb{F}$ its algebraic closure.
- The acronym (FGLA) stands for fundamental graded Lie algebra.
- The acronym (EPFGLA) stands for effective prolongation of a fundamental graded Lie algebra.
- $T(V) = \sum_{p=0}^\infty T^p(V)$ is the tensor algebra of the vector space $V$.
- $\Lambda(V) = \sum_{p=0}^n \Lambda^p(V)$ is the Grassmann algebra of an $n$-dimensional vector space $V$.
- $\mathfrak{gl}_\mathbb{K}(V)$ is the Lie algebra of $\mathbb{K}$-linear endomorphisms of the $\mathbb{K}$-linear space $V$. 
• $\mathfrak{gl}_n(\mathbb{K})$ is the Lie algebra of $n \times n$ matrices with entries in the field $\mathbb{K}$.

1. Fundamental graded Lie algebras and prolongations

**Definition 1.1** (see [37]). A $\mathbb{Z}$-graded Lie $\mathbb{K}$-algebra

\[(FGLA) \quad \mathfrak{m} = \sum_{p \geq 1} \mathfrak{g}_{-p},\]

is called fundamental if $\mathfrak{g}_{-1}$ is a finite dimensional $\mathbb{K}$-vector space and generates $\mathfrak{m}$ as a Lie $\mathbb{K}$-algebra. We call $\mu = \sup\{p \mid \mathfrak{g}_{-p} \neq \{0\}\}$ the kind of $\mathfrak{m}$.

At variance with [37], we do not require here that $\mathfrak{m}$ be finite dimensional. The only (FGLA) with $\dim(\mathfrak{g}_{-1}) = 1$ is the trivial one-dimensional Lie algebra. We will consider in the following (FGLA)'s $\mathfrak{m}$ with $\dim(\mathfrak{m}) > 1$.

Let $V$ be a vector space of finite dimension $n \geq 2$ over $\mathbb{K}$ and $\mathfrak{f}(V)$ the free Lie algebra over $\mathbb{K}$ generated by $V$ (see e.g. [8, 33]). It is an (FGLA) of infinite kind with the natural $\mathbb{Z}$-gradation

\[(1.1) \quad \mathfrak{f}(V) = \sum_{p=1}^{\infty} \mathfrak{f}_{-p}(V).\]

obtained by setting $\mathfrak{f}_{-1}(V) = V$. The free Lie algebra of $V$ is characterised by the universal property:

**Proposition 1.1.** For any Lie algebra $\mathfrak{L}$ over $\mathbb{K}$ and any $\mathbb{K}$-linear map $\phi: V \to \mathfrak{L}$, there is a unique Lie algebras homomorphism $\tilde{\phi}: \mathfrak{f}(V) \to \mathfrak{L}$ extending $\phi$. □

By Proposition 1.1 we can consider any finitely generated Lie algebra as a quotient of a free Lie algebra.

**Remark 1.2.** For any integer $\mu \geq 2$, the direct sum

\[(1.2) \quad \mathfrak{f}_{[\mu]}(V) = \sum_{p=\mu}^{\infty} \mathfrak{f}_{-p}(V)\]

is a proper $\mathbb{Z}$-graded ideal in $\mathfrak{f}(V)$ and hence the quotient

\[(1.3) \quad \mathfrak{f}_{[0]}(V) = \mathfrak{f}(V)/\mathfrak{f}_{[\mu+1]}(V)\]

is an (FGLA) of kind $\mu$, that is called the free (FGLA) of kind $\mu$ generated by $V$ (see e.g. [39]).

Let $\mathfrak{m}$ be an (FGLA) over $\mathbb{K}$ and set $V = \mathfrak{g}_{-1}$. Since $\mathfrak{m}$ is generated by $V$, there is a surjective homomorphism

\[(1.4) \quad \pi: \mathfrak{f}(V) \longrightarrow \mathfrak{m},\]

which preserves the gradations. Its kernel $\mathcal{K}$ is a $\mathbb{Z}$-graded ideal

\[(1.5) \quad \mathcal{K} = \sum_{p \geq 2} \mathcal{K}_{-p}\]

and $\mathfrak{m}$ has finite kind $\mu$ iff there is a smaller integer $\mu \geq 1$ such that

$\mathcal{K} \supseteq \mathfrak{f}_{[\mu+1]}(V)$.

**Definition 1.2.** An ideal $\mathcal{K}$ of $\mathfrak{f}(V)$ is said to be $\mu$-cofinite if $\mathfrak{f}_{[\mu+1]}(V) \subseteq \mathcal{K}$. 

Lemma 1.5. The elements of a string \((K_{-2}, \ldots, K_{-m}, \ldots)\) of linear subspaces \(K_{-p} \subseteq \mathfrak{f}(V)\) are the homogeneous summands of a \(\mathbb{Z}\)-graded ideal \((1.5)\) of \(\mathfrak{f}(V)\) if and only if they satisfy the compatibility condition
\[
(1.6) \quad [K_{-p}, V] \subseteq K_{-(p+1)}, \quad \forall p \geq 2.
\]

Since \(\mathbb{K}\) has characteristic 0, we can canonically identify \(\mathfrak{f}(V)\) with a \(GL(V)\)-invariant subspace of \(T(V)\).

Proposition 1.4. Every \((\text{FGLA})\) \(m\) is isomorphic to a quotient \(\mathfrak{f}(V)/\mathcal{K}\), for a \(\mathbb{Z}\)-graded ideal \(\mathcal{K}\) of the form \((1.5)\). Two \((\text{FGLA})'s\) \(\mathfrak{f}(V)/\mathcal{K}\) and \(\mathfrak{f}(V)/\mathcal{K}'\) are isomorphic if and only if \(\mathcal{K}\) and \(\mathcal{K}'\) are \(GL(V)\)-congruent. \(\square\)

Definition 1.3. A \(\mathbb{Z}\)-graded Lie algebra over \(\mathbb{K}\)
\[
(1.7) \quad \mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p,
\]
is said to be an effective prolongation of a fundamental graded Lie algebra if
\begin{itemize}
  \item [(i)] \(\mathfrak{g}_{<0} = \sum_{p < 0} \mathfrak{g}_p\) is a \((\text{FGLA})\);
  \item [(ii)] \(\mathfrak{g}_{\geq 0} = \sum_{p \geq 0} \mathfrak{g}_p\) is effective: this means that
  \[
  \{X \in \mathfrak{g}_{\geq 0} \mid [X, \mathfrak{g}_{-1}] = \{0\}\} = \{0\}.
  \]
\end{itemize}
In this case, we say for short that \(\mathfrak{g}\) is an \((\text{EPFGLA})\) of \(\mathfrak{g}_{<0}\).

An \((\text{EPFGLA})\) \((1.7)\) is said to be \textit{of type} \(\mathfrak{l}\), or an \(\mathfrak{l}\)-prolongation, if \(\mathfrak{l}\) is a Lie subalgebra of \(gl(V)\)(\(\mathfrak{g}_{-1}\)) and, for each \(A \in \mathfrak{g}_0\), the map \(v \rightarrow [A, v]\), for \(v \in \mathfrak{g}_{-1}\), is an element of \(\mathfrak{l}\).

Lemma 1.5. If \((1.7)\) is an \((\text{EPFGLA})\), then \(\dim(\mathfrak{g}_p) < \infty\) for all \(p \in \mathbb{Z}\).

Proof. Let \(\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p\) be an \((\text{EPFGLA})\) and set \(V = \mathfrak{g}_{-1}\). For each \(\xi \in \mathfrak{g}_0\), we define by recurrence
\[
\begin{cases}
\xi(v_0) = [\xi, v_0], & \forall v_0 \in V, \\
\xi(v_0, v_1) = [[\xi, v_0], v_1], & \forall v_0, v_1 \in V, \\
\xi(v_0, \ldots, v_p) = [\xi(v_0, \ldots, v_{p-1}), v_p], & \forall v_0, v_1, \ldots, v_p \in V.
\end{cases}
\]
By the effectiveness assumption, for each \(p \geq 0\) the map which associates to \(\xi \in \mathfrak{g}_0\) the \((p+1)\) multilinear map \(\mathfrak{g}^{p+1} \ni (v_0, \ldots, v_p) \rightarrow \xi(v_0, \ldots, v_p) \in V\) is injective from \(\mathfrak{g}_p\) to \(V \otimes T^{p+1}(V)\). \(\square\)

We already noted that, since \(\mathbb{K}\) has characteristic zero, \(\mathfrak{f}(V)\) can be canonically identified with a Lie subalgebra of the Lie algebra of the tensor algebra \(T(V)\); \(gl(V)\) acts as an algebra of zero degree derivations on \(T(V)\), leaving \(\mathfrak{f}(V)\) invariant. Denote by \(T_A\) the derivation of \(T(V)\) associated to \(A \in gl(V)\). With the Lie product defined by
\[
(1.8) \quad [A, X] = T_A(X), \quad \forall A \in gl(V), \ \forall X \in \mathfrak{f}(V),
\]
the direct sum
\[
(1.9) \quad \tilde{\mathfrak{f}}(V) = \sum_{p \geq 0} \tilde{\mathfrak{f}}_{-p}(V), \quad \text{with} \quad \tilde{f}_0(V) = gl(V)
\]
is an (EPFGLA) of \( f(V) \) and, for any Lie subalgebra \( \mathcal{L} \) of \( \mathfrak{gl}_V(V) \), the sum \( \mathcal{L} \oplus f(V) \) is a graded Lie subalgebra of \( \tilde{f}(V) \).

By using [8, Ch.2, §2,Prop.8], we obtain

**Proposition 1.6.** The maximal effective prolongation \( \tilde{f}(V) \) of \( f(V) \) is

\[
\tilde{f}(V) = \sum_{p=-\infty}^{\infty} f_p(V), \quad \text{where} \quad f_p(V) = V \otimes T^{p+1}(V^*), \quad \text{for} \ p \geq 0,
\]

is the space of \((p+1)\)-linear \( V \)-valued maps on \( V \) and the Lie product is defined in such a way that, for \( \xi \in f_p(V) \), with \( p \geq 0 \), we have

\[
[\xi, v_0](v_1, \ldots, v_p) = \xi(v_0, v_1, \ldots, v_p), \quad \forall v_0, v_1, \ldots, v_p \in V. \tag{1.11}
\]

**Proposition 1.7.** Let \( \mathcal{K} \) be a Lie subalgebra of \( \mathfrak{gl}_V(V) \). Then

\[
\tilde{f}(V, \mathcal{K}) = f(V) \oplus \mathcal{K} \oplus \sum_{p \geq 1} f_p(V, \mathcal{K}), \quad \text{with}
\]

\[
f_p(V, \mathcal{K}) = \{ \xi \in f_p(V) \mid \xi(v_1, \ldots, v_p) \in \mathcal{K}, \ \forall v_1, \ldots, v_p \in V \} \tag{1.12}
\]

is the maximal (EPFGLA) prolongation of type \( \mathcal{K} \) of \( f(V) \). \( \square \)

Fix a \( \mathbb{Z} \)-graded ideal \( \mathcal{K} \) of \( f(V) \), contained in \( f_{(2)}(V) \), and denote by

\[
m(\mathcal{K}) = \sum_{p \geq 1} g_{-p}(\mathcal{K}) \tag{1.13}
\]

the (FLGA) defined by the quotient \( f(V)/\mathcal{K} \). Let \( \mathcal{L} \) be a Lie subalgebra of \( \mathfrak{gl}_V(V) \). Note that \( \mathcal{K} \) is a Lie subalgebra of \( \tilde{f}(V, \mathcal{L}) \). We associate to the pair \((\mathcal{K}, \mathcal{L})\) the normalizer

\[
\mathcal{R}(\mathcal{K}, \mathcal{L}) = \{ \xi \in \tilde{f}(V, \mathcal{L}) \mid [\xi, \mathcal{K}] \subseteq \mathcal{K} \} \tag{1.14}
\]

of \( \mathcal{K} \) in \( \tilde{f}(V, \mathcal{L}) \). It is a \( \mathbb{Z} \)-graded Lie subalgebra of \( \tilde{f}(V, \mathcal{L}) \) and the largest which contains \( \mathcal{K} \) as an ideal. The quotient

\[
g(\mathcal{K}, \mathcal{L}) = \mathcal{R}(\mathcal{K}, \mathcal{L})/\mathcal{K} \tag{1.15}
\]

has a natural \( \mathbb{Z} \)-grading for which the natural projection \( \mathcal{R}(\mathcal{K}, \mathcal{L}) \to g(\mathcal{K}, \mathcal{L}) \) is a \( \mathbb{Z} \)-graded epimorphism of Lie algebras.

**Theorem 1.8.** The commutative diagram

\[
0 \longrightarrow f(V) \longrightarrow \mathcal{R}(\mathcal{K}, \mathcal{L}) \quad \downarrow \quad \downarrow
\]

\[
0 \longrightarrow m(\mathcal{K}) \longrightarrow g(\mathcal{K}, \mathcal{L})
\]

defines on \( g(\mathcal{K}, \mathcal{L}) \) the structure of an (EPFGLA) of type \( \mathcal{L} \) of \( \mathfrak{g}(\mathcal{K}, \mathcal{L}) \) and \( g(\mathcal{K}, \mathcal{L}) \) is, modulo isomorphisms, its maximal (EPFGLA) of type \( \mathcal{L} \). \( \square \)

We have

\[
g(\mathcal{K}, \mathcal{L}) = \sum_{p \in \mathbb{Z}} g_p(\mathcal{K}, \mathcal{L}), \quad \text{with} \ g_p(\mathcal{K}, \mathcal{L}) = g_p(\mathcal{K}) \text{ for } p < 0, \text{ and } \ g_p(\mathcal{K}, \mathcal{L}) = \{ \xi \in f_p(V, \mathcal{L}) \mid [\xi, \mathcal{K}] \subseteq \mathcal{K} \}, \text{ for } p \geq 0. \tag{1.16}
\]
Remark 1.9. This theorem was proved in [37, §5] for the case in which $\mathcal{K}$ is a cofinite ideal of $f(V)$. The summands $g_p(\mathcal{K}, \mathcal{V})$ in (1.16) were recursively defined there by setting
\begin{equation}
\begin{aligned}
g_p(\mathcal{K}, \mathcal{V}) &= g_p(\mathcal{K}), \\
g_{<p}(\mathcal{K}, \mathcal{V}) &= \sum_{q<p} g_q(\mathcal{K}, \mathcal{V}), \\
g_p(\mathcal{K}, \mathcal{V}) &= \text{Der}_p(m, g_{<p}(\mathcal{K}, \mathcal{V})),
\end{aligned}
\end{equation}
for $p<0$, $p\geq 0$. Each $g_{<p}(\mathcal{K}, \mathcal{V})$ is a right $m(\mathcal{K})$-module and $\text{Der}_p(m, g_{<p}(\mathcal{K}, \mathcal{V}))$ is the space of the $g_{<p}$-valued degree $p$-derivations of $m$. One can check that this definition is equivalent to the one we gave above.

2. TANAKA’S FINITENESS CRITERION

2.1. Left and right $V$-modules. Let $V$ be an $n$-dimensional $\mathbb{K}$-vector space, that we consider as an abelian Lie algebra. Its universal enveloping algebra is the graded algebra
\begin{equation}
\mathcal{S}(V) = \sum_{p=0}^{\infty} \mathcal{S}_p(V)
\end{equation}
of symmetric elements of its tensor algebra. Since $\mathbb{K}$ has characteristic zero, $\mathcal{S}(V)$ is the ring of polynomials of $V$ with coefficients in $\mathbb{K}$.

A right action of $V$ on a $\mathbb{K}$-vector space $E$ is a bilinear map
\begin{equation}
E \times V \ni (e, v) \mapsto e \cdot v \in E,
\end{equation}
with $(e \cdot v_1) \cdot v_2 = (e \cdot v_2) \cdot v_1$, $\forall e \in E$, $\forall v_1, v_2 \in V$.

The map (2.2) extends to a right action
\begin{equation}
E \times \mathcal{S}(V) \ni (e, a) \mapsto e \cdot a \in E.
\end{equation}
If $E^*$ is a $\mathbb{K}$-vector space which is in duality with $E$ by a pairing
\begin{equation}
E \times E^* \ni (e, \eta) \mapsto \langle e | \eta \rangle \in \mathbb{K},
\end{equation}
a left action
\begin{equation}
V \times E^* \ni (v, \eta) \mapsto v \cdot \eta \in E^*
\end{equation}
of $V$ on $E^*$ is dual of (2.2) if
\begin{equation}
\langle e | v \cdot \eta \rangle = \langle e \cdot v | \eta \rangle, \quad \forall v \in V, \forall e \in E, \forall \eta \in E^*.
\end{equation}
Clearly a dual left action of $V$ extends to a left action of $\mathcal{S}(V)$.

Definition 2.1. A $\mathbb{Z}$-gradation of the right $V$-module $E$ is a direct sum decomposition
\begin{equation}
E = \sum_{p=-\infty}^{\infty} E_p \quad \text{with} \quad E_p \cdot V \subseteq E_{p-1} \quad \text{for} \quad p \in \mathbb{Z}.
\end{equation}

We say that $E$ satisfies condition (C) if
\begin{equation}
\begin{aligned}
\dim(E_p) &< +\infty, \quad \forall p \in \mathbb{Z}, \\
\exists p_0 \in \mathbb{Z} \quad \text{s.t.} \quad E_p = \{0\} \quad \text{for} \quad p < p_0, \\
e &\in \sum_{h \geq 0} E_p \quad \text{and} \quad e \cdot V = \{0\} \Rightarrow e = 0.
\end{aligned}
\end{equation}
Let us consider the \(\mathbb{Z}\)-graded vector space
\[
E^* = \sum_{h=-\infty}^{\infty} E^*_h,
\]
with \(E^*_h\) the dual of \(E_{-h}\), for all \(h \in \mathbb{Z}\).

We call \(E^*\) the \textit{graded dual} of \(E\). When \(E\) is a graded right \(V\)-module, its graded dual \(E^*\) is a graded left \(V\)-module under the action (2.5) which is described on the homogeneous elements by
\[
\langle e \mid v \eta \rangle = \langle e \cdot v \mid \eta \rangle \quad \forall e \in E_{1-h}, \forall \eta \in E^*_h, \forall v \in V.
\]

\textbf{Lemma 2.1.} The following are equivalent:

1. \(E\) satisfies condition (C);
2. \(E^*\) satisfies condition
   \[
   (C^*) \quad \begin{cases}
   \dim(E^*_q) < +\infty, \forall q \in \mathbb{Z}, \\
   \exists q_0 \in \mathbb{Z} \quad \text{s.t.} \quad E^*_q = 0, \forall q > q_0; \\
   \sum_{q \geq 1} E^*_q \text{ generates } E^* \text{ as a left } S(V)-\text{module}.
   \end{cases}
   \]

\textbf{Proof.} We have \(E_p = \{0\} \iff E^*_{-p} = \{0\}\).

Given any basis \(v_1, \ldots, v_n\) of \(V\), the maps
\[
E_p \ni e \rightarrow (e \cdot v_1, \ldots, e \cdot v_n) \in (E_{p-1})^n,
\]
\[
E^*_{-p} \ni (\eta_1, \ldots, \eta_n) \rightarrow v_1 \cdot \eta_1 + \cdots + v_n \cdot \eta_n \in E^*_p
\]
are dual of each other and all vector spaces involved are finite dimensional. Hence the first one is injective if and only if the second one is surjective. These remarks yield the statement. \(\square\)

Let \((V)\) denote the maximal ideal of \(S(V)\), consisting of polynomials vanishing at 0. For the notions of commutative algebra that will be used below we refer to [7, Ch.IV]

\textbf{Proposition 2.2.} Let \(E\) satisfy condition (C). Then the following are equivalent:

1. \(E\) is finite dimensional;
2. \(E^*\) is finite dimensional;
3. \(E^*\) is \((V)\)-coprimary, i.e. all \(v \in V\) are nilpotent on \(E^*\).

\textbf{Proof.} Clearly (i) \(\iff\) (ii). Also (ii) \(\Rightarrow\) (iii) is clear. Indeed, if \(E^*\) is finite dimensional, all \(v \in V\), lowering the degree by one unit, are nilpotent. Vice versa, since \(E^*\) is finitely generated, if the set \(\mathfrak{A}(E^*)\) of its associated primes is \(\{(V)\}\), then all \(E^*_q\) are zero for \(q < q_0\) for some \(q_0 \in \mathbb{Z}\). \(\square\)

\textbf{Corollary 2.3.} A necessary and sufficient condition for a right \(V\)-module \(E\) satisfying condition (C) to be infinite dimensional is that there is \(v \in V\) such that \(E \cdot v = E\).

\textbf{Proof.} By Proposition 2.2 a necessary and sufficient condition for \(E\) to be infinite dimensional is that \(E^*\) is not \((V)\)-coprimary. If \(p_1, \ldots, p_m\) are the associated primes of \(E^*\), since \(V \cap p_j = V_j\) is, for each \(j\), a proper vector subspace of \(V\), it suffices to choose a \(v\) which does not belong to \(V_1 \cup \cdots \cup V_m\).
In fact, for such a \( v \), all maps \( E^* \ni \eta \mapsto v \cdot \eta \in E^*_{q-1} \) are injective. By duality, all maps \( E_p \ni e \mapsto e \cdot v \in E_{p-1} \) are surjective. This gives \( E \cdot v = E \).

Vice versa, if right multiplication by \( v \) on \( E \) is surjective, by duality left multiplication by \( v \) is injective on \( E^* \) and hence \( E^* \) is not \((V)\)-coprimary. \( \Box \)

It will be useful that the thesis of this corollary be verified by a \( v \) which has a special form. To this aim we prove the following lemma.

**Lemma 2.4.** Let \( U, W \) be finite dimensional vector spaces over \( \mathbb{K} \) and \( \omega : U \times W \to V \) a bilinear map such that \( \{ \omega(u, w) \mid u \in U, w \in W \} \) spans \( V \). Then the image of \( \omega \) is not contained in any finite union of proper linear subspaces of \( V \).

**Proof.** It suffices to show that \( \omega(U \times W) \) is not contained in a finite union of hyperplanes. Let \( \xi_1, \ldots, \xi_m \in V^* \) and assume that \( \omega(U \times W) \) is contained in \( \ker(\xi_1) \cup \cdots \cup \ker(\xi_m) \). The condition that \( \omega(U \times W) \) spans \( V \) implies that each quadric \( Q_j = \{(u, w) \in U \times W \mid \xi_j(\omega(u, w)) = 0\} \) is properly contained in \( U \times W \). Hence \( Q_1 \cup \cdots \cup Q_m \not\subseteq U \times W \) and the \( \omega \)-image of each pair which is not contained in \( Q_1 \cup \cdots \cup Q_m \) does not belong to \( \ker(\xi_1) \cup \cdots \cup \ker(\xi_m) \). \( \Box \)

**Proposition 2.5.** Let \( E \) be a right \( V \)-module satisfying condition (C). If \( E \) is infinite dimensional, then we can find linearly independent vectors \( v_1, \ldots, v_m \) in \( V \) such that, with

\[
(2.10) \quad E^{(h)} = \{ e \in E \mid e \cdot v_i = 0, \forall i \leq h \}, \text{ for } 0 \leq h \leq m,
\]

the following conditions are satisfied:

(i) either \( m = n \) and \( v_1, \ldots, v_n \) is a basis of \( V \), or \( 0 < m < n \) and \( E^{(m)} \) is finite dimensional;

(ii) the maps \( T_{h+1} : E^{(h)} \ni e \mapsto e \cdot v_{h+1} \in E^{(h)} \) are surjective for \( 0 \leq h \leq m-1 \).

**Proof.** Note that \( E^{(0)} = E \). We know from Proposition 2.2 that, if \( \dim(E) = \infty \), then \( E^* \) is not \((V)\)-coprimary. Hence we can find \( v_1 \in V \) such that the left multiplication by \( v_1 \) is injective on \( E^* \). By duality this means that the map \( T_1 : E \ni e \mapsto e \cdot v_1 \in E \) is surjective.

Let \( V_1 \) be a hyperplane in \( V \) which does not contain \( v_1 \). We have a natural duality pairing between \( E^{(1)} \) and the quotient \( E^*/v_1 \cdot E^* \), that we can consider as a left \( V_1 \)-module. In case \( E^{(1)} \) is infinite dimensional, \( (V_1) \) is not an associated prime of \( E^*/v_1 \cdot E^* \) and hence we can find \( v_2 \in V_1 \) which is not a zero divisor in \( E^*/v_1 \cdot E^* \). By duality this yields a surjective

\[
T_2 : E^{(1)} \ni e \mapsto e \cdot v_2 \in E^{(1)}.
\]

The recursive argument is now clear and we get the statement after a finite number of steps. \( \Box \)

**Remark 2.6.** In the hypothesis of Lemma 2.4, the vectors \( v_1, \ldots, v_m \) in the statement of Proposition 2.5 can be chosen of the form \( v_i = \omega(u_i, w_i) \), with \( u_i \in U \) and \( w_i \in W \), for \( 1 \leq i \leq m \).
2.2. **Right \( m \)-modules.** We consider now an (FGLA) \( m = \sum_{p=1}^{\mu} g_p \) of finite kind \( \mu \geq 1 \).

Let \( E = \sum_{p \in \mathbb{Z}} E_p \) be a \( \mathbb{Z} \)-graded right \( m \)-module. This means that there is a bilinear action

\[
E \times m \ni (e, X) \mapsto e \cdot X \in E,
\]

which has the properties:

\[
\begin{cases}
(e \cdot X) \cdot Y = e \cdot (X \cdot Y), & \forall e \in E, \forall X, Y \in m, \\
E_q g_{-p} \subseteq E_{q-p}, & \forall q \in \mathbb{Z}, \forall 1 \leq p \leq \mu.
\end{cases}
\]

**Definition 2.2.** We say that the graded right \( m \)-module \( E \) satisfies condition \( (C(m)) \) if

\[
(C(m)) \quad \begin{cases}
\dim(E_p) < \infty, & \forall p \in \mathbb{Z}, \\
\exists p_0 \in \mathbb{Z} \text{ s.t. } E_p = 0, & \forall p < -p_0, \\
e \in \sum_{p \geq 0} E_p \text{ and } e \cdot g_{-1} = \{0\} \implies e = 0.
\end{cases}
\]

The following lemma is similar to [37, Lemma 11.4].

**Lemma 2.7. Assume that**

- \( m \) has kind \( \mu \geq 2 \);
- \( E \) is a graded right \( m \)-module satisfying condition \( (C(m)) \);
- there are \( X \in g_{-1}, Y \in g_{1-\mu} \) and \( p_0 \in \mathbb{Z} \) such that

\[
(2.11) \quad \psi_p : E_p \ni e \mapsto e \cdot [X, Y] \in E_{p-\mu}
\]

is an isomorphism for \( p \geq p_0 \).

Then \( E_p = \{0\} \) for \( p \geq p_0 \).

**Proof.** We consider the right multiplication by \( Z = [X, Y] \) as a linear map \( R_Z \) on \( E \). Note that \( E_{<q_0} = \sum_{p < q_0} E_p \) is a right \( m \)-submodule of \( E \). Thus we can consider the truncation \( \sum_{p \geq 0} E_p \) as the quotient right \( m \)-module \( E/E_{<q_0} \).

By substituting to \( E \) its truncation, if needed, we can as well assume that \( E = \sum_{p \geq 0} E_p \). Then \( R_Z \) has a right inverse \( \Psi : E \ni E \) such that \( \Psi \circ R_Z \) restricts to the identity on each \( E_p \) with \( p \geq p_0 \), i.e. \( \Psi \) is also a left inverse of \( R_Z \) on \( E_p \) for \( p \geq p_0 \). Denote by \( R_X \) and \( R_Y \) the linear maps on \( E \) defined by the right multiplication by \( X \) and \( Y \), respectively. We claim that

\[
\Psi \circ R_X = R_X \circ \Psi \quad \text{and} \quad \Psi \circ R_Y = R_Y \circ \Psi \quad \text{on} \quad E_p, \quad \forall p \geq p_0 - 1.
\]

Indeed, \( \Psi \circ R_X(E_p) \cup R_X \circ \Psi(E_p) \subseteq E_{p+1} \) and \( \Psi \circ R_Y(E_p) \cup R_Y \circ \Psi(E_p) \subseteq E_{p+1} \), so that, since both \( p+1 \) and \( p+\mu-1 \) are \( p \geq p_0 - 1 \), and \( \Psi \) is a right inverse of \( R_Z \) in this range, these equalities are equivalent to

\[
R_X = R_Z \circ R_X \circ \Psi \quad \text{and} \quad R_Y = R_Z \circ R_Y \circ \Psi,
\]

and thus are verified because \( R_Z \) commutes with \( R_X \) and \( R_Y \). Fix \( p \geq p_0 - 1 \) and consider the finite dimensional \( \mathbb{K} \)-vector space \( W = \sum_{h=1}^{n} E_{p+h} \). We define
two endomorphisms \(T_X, T_Y\) on \(W\) by setting
\[
T_X(e) = \begin{cases} R_X \circ \Psi(e) \in E_{p+\mu}, & \text{if } e \in E_{p+1}, \\ R_X(e) \in E_{h-1}, & \text{if } e \in E_h, \text{ with } p+1 < h \leq \mu, \end{cases}
\]
\[
T_Y(e) = \begin{cases} R_Y \circ \Psi(e) \in E_{h+1}, & \text{if } e \in E_h \text{ with } p+1 \leq h < p+\mu, \\ R_Y(e) \in E_{p+1} & \text{if } e \in E_{p+\mu}. \end{cases}
\]

One easily checks that
\[
T_X \circ T_Y - T_Y \circ T_X = (R_X \circ R_Y - R_Y \circ R_X) \circ \Psi = I_W \quad \text{on } W
\]
and hence
\[
\dim(W) = \text{trace}(I_W) = \text{trace}(T_X \circ T_Y - T_Y \circ T_X) = 0.
\]
This proves the lemma. \(\square\)

Let \(E = \sum_{h=-p_0}^\infty E_p\) be a right \(\mathfrak{m}\)-module, satisfying condition \((\mathbb{C}(\mathfrak{m}))\). Set
\[
(2.12) \quad n = \sum_{h \geq 2} g_{-h}
\]
and
\[
(2.13) \quad N(E) = \sum_{h=-p_0}^\infty N_p(E), \quad \text{with } N_p(E) = \{e \in E_p | e \cdot n = \{0\}\}.
\]

**Theorem 2.8.** Let \(E\) be a \(\mathbb{Z}\)-graded right \(\mathbb{Z}\)-module satisfying condition \((\mathbb{C}(\mathfrak{m}))\). Then \(E\) is finite dimensional if and only if \(N(E)\) is finite dimensional.

**Proof.** We argue by recurrence on the kind \(\mu\) of \(\mathfrak{m}\). In fact, when \(\mu = 1\), we have \(n = \{0\}\) and hence \(N(E) = E\) and the statement is trivially true. Moreover, since \(N(E) \subseteq E\), we only need to show that \(N(E)\) is infinite dimensional when \(E\) is infinite dimensional.

Assume that \(\mu > 1\). The subspace \(g_{-\mu}\) is an ideal of \(\mathfrak{m}\) and hence \(g' = \mathfrak{m}/g_{-\mu}\) is an (FGLA) of kind \(\mu - 1\). Then
\[
(2.14) \quad F = \sum_{f=-p_0}^\infty F_f, \quad \text{with } F_f = \{e \in E_p | e \cdot g_{-\mu} = \{0\}\}
\]
can be viewed as a \(\mathbb{Z}\)-graded \(\mathfrak{m}'\)-module which satisfies condition \(C(\mathfrak{m}')\).

Since \(N(F) = N(E)\), by our recursive assumption \(F\) and \(N(E)\) are either both finite, or both infinite dimensional.

Hence it will suffice to prove that, if \(E\) is infinite dimensional, also \(F\) is infinite dimensional. The \(\mathbb{Z}\)-grading
\[
(2.15) \quad M(E) = \sum_{j \in \mathbb{Z}} M_j(E), \quad \text{where } M_j(E) = \sum_{j=0}^{\mu-1} E_{j+p \mu}.
\]
defines on \(E\) the structure of a \(\mathbb{Z}\)-graded left \(V\)-module, for \(V = g_{-\mu}\). Since, by assumption, it is infinite dimensional, by Proposition 2.5 and Lemma 2.4 we can find \(X_1, \ldots, X_m \in g_{-1}\) and \(Y_1, \ldots, Y_m \in g_1 - \mu\) such that
\[
Z_1 = [X_1, Y_1], \ldots, Z_m = [X_m, Y_m]
\]
are linearly independent and have the properties:
(i) either $m < \dim_{\mathbb{E}}(g_{-\mu})$ and $E^{(m)} = \{ e \in E \mid eZ_i = 0, \forall i = 1, \ldots, m \}$ is finite dimensional, or $m = n$ and $Z_1, \ldots, Z_m$ is a basis of $g_{-\mu}$;

(ii) with $E^{(h)} = \{ e \in E \mid eZ_i = 0, \forall i \leq h \}$, the maps

$$T_{h+1} : E^{(h)} \ni e \mapsto eZ_{h+1} \in E^{(h)}$$

are surjective for $0 \leq h < m$.

For all $0 \leq h < m$ we obtain exact sequences

$$
0 \longrightarrow E^{(h+1)} \longrightarrow E^{(h)} \stackrel{T_{h+1}}{\longrightarrow} E^{(h)} \longrightarrow 0.
$$

Note that $E^{(0)} = E$ and that $E^{(n)} = F$ when $m = n$. We want to prove that $m = n$ and that $F$ is infinite dimensional. We argue by contradiction. If our claim is false, then $E^{(m)}$ is finite dimensional and, from the exact sequence (2.16), we obtain that $E^{(m-1)}_{p} \ni e \mapsto eZ_{m} \in E^{(m-1)}_{p-m}$ is an isomorphism for all $p \geq p_m$ for some integer $p_m$, which, by Lemma 2.7, implies that $E^{(m-1)}$ is finite dimensional. Using again the exact sequence (2.16) for $h = m - 2$ and Lemma 2.7 we obtain that also $\dim_{\mathbb{E}}(E^{(m-2)}) < \infty$ and, repeating the argument, we end up getting that $E^{(0)} = E$ is finite dimensional. This yields a contradiction, proving that $F$, and thus also $N(E)$, must be infinite dimensional when $E$ is infinite dimensional. □

2.3. Reduction to first kind. Let us get back to the prolongations defined in §1. We will use Theorem 2.8 to show that the finiteness of the maximal effective $\mathcal{L}$-prolongation of an (FGLA) of finite kind $\mu \geq 2$ is equivalent to that of the $\mathcal{L}'$-prolongation of an (FGLA) of the first kind for a suitable $\mathcal{L}' \subseteq \mathcal{L}$.

Let $\mathfrak{m} = \sum_{p = 1}^{\mu} g_{-p}$ be an (FGLA) of finite kind $\mu$. Set $V = g_{-1}$. We showed in §1 that $\mathfrak{m} = \hat{\mathcal{I}}(V)/\mathcal{K}$ for a $\mathbb{Z}$-graded ideal $\mathcal{K}$ of $\hat{\mathcal{I}}(V)$, contained in $\hat{\mathcal{I}}_{2}(V)$. As in (2.13), we denote by $\mathfrak{n}$ the ideal $\sum_{p < -1} g_{p}$ of $\mathfrak{m}$. Fix a Lie subalgebra $\mathcal{L}$ of $g_{1}(g_{-1})$ and denote by $g(\mathcal{K}, \mathcal{L})$ the maximal (EPFGLA) of type $\mathcal{L}$ of $\mathfrak{m}$, that was characterised in Theorem 1.8. We have the following finiteness criterion:

**Theorem 2.9.** The maximal effective $\mathcal{L}$-prolongation $g(\mathcal{K}, \mathcal{L})$ of $\mathfrak{m}(\mathcal{K})$ is finite dimensional if, and only if,

$$N(g(\mathcal{K}, \mathcal{L})) = \{ \xi \in g(\mathcal{K}, \mathcal{L}) \mid [\xi, n] = \{0\} \}
$$

is finite dimensional.

**Proof.** The statement follows by applying Theorem 2.8 to $g(\mathcal{K}, \mathcal{L})$, considered as a right $\mathfrak{m}$-module. □

Set

$$a = (\mathfrak{m}/\mathfrak{n}) \oplus \sum_{p \geq 0} a_p, \text{ with } a_p = \{ \xi \in g_p(\mathcal{K}, \mathcal{L}) \mid [\xi, n] = \{0\} \}.
$$

We note that $\mathfrak{m} \oplus \sum_{p \geq 0} a_p$ is a Lie subalgebra of $g(\mathcal{K}, \mathcal{L})$, which contains $\mathfrak{n}$ as an ideal. There is a natural isomorphism of $\mathfrak{a}$ with $((\mathfrak{m} \oplus \sum_{p \geq 0} a_p)/\mathfrak{n}$, which defines its Lie algebra structure. Set $a_{-1} = \mathfrak{m}/\mathfrak{n} \simeq V$. 


Lemma 2.10. The summand \(a_0\) in (2.17) is given by
\[
(2.19) \quad a_0 = a_0(\mathcal{K}, \mathcal{L}) = \{ A \in \mathcal{L} | T_A(\bar{\mathfrak{f}}_{[2]}(V)) \subseteq \mathcal{K} \}.
\]

Proof. The statement follows because \(n = \bar{\mathfrak{f}}_{[2]}(V)/\mathcal{K} \). \(\square\)

Theorem 2.11. The \(\mathbb{Z}\)-graded Lie algebra \(a\) of (2.18) is the maximal effective prolongation of type \(a_0(\mathcal{K}, \mathcal{L})\) of \(m(\bar{\mathfrak{f}}_{[2]}(V))\) and the following are equivalent:

(i) \(g(\mathcal{K}, \mathcal{L})\) is finite dimensional;
(ii) \(g(\bar{\mathfrak{f}}_{[2]}(V), a_0(\mathcal{K}, \mathcal{L}))\) is finite dimensional.

Proof. Set \(a_0 = a_0(\mathcal{K}, \mathcal{L})\) and denote by \(b = \sum_{p \geq 0} b_p\) the maximal (EPFGLA) \(g(\bar{\mathfrak{f}}_{[2]}(V), a_0)\) of type \(a_0\) of \(a = \bar{\mathfrak{f}}(V)/\bar{\mathfrak{f}}_{[2]}(V) \cong V\). By construction, \(b_0 = a_0\).

Let \(\pi : m \to (m/n)\) be the canonical projection. Then \(X \in m\) acts to the right on \(b\) by \(\beta \cdot X = [\beta, \pi(X)]\) and
\[
m \oplus g_0(\mathcal{K}, \mathcal{L}) \oplus \sum_{p>0} b_p
\]
is an effective prolongation of type \(\mathcal{L}\) of \(m(\mathcal{K})\). Hence \(b_p \subseteq a_p\) for \(p>0\). On the other hand, since \(a\) is an effective prolongation of type \(a_0\) of \(m(\bar{\mathfrak{f}}_{[2]}(V))\), we also have the opposite inclusion. This yields \(b_p = a_p\) for all \(p>0\), proving the first part of the statement. The equivalence of (i) and (ii) is then a consequence of Theorem 2.9. \(\square\)

3. \(\mathcal{L}\)-prolongations of graded Lie algebras of the first kind

By Theorem 2.11 finiteness of the maximal \(\mathcal{L}\)-prolongation of an (FGLA) \(m\) of any finite kind is equivalent to that of the \(\mathcal{L}'\)-prolongation of its first kind quotient \(V = m/n\), for a Lie subalgebra \(\mathcal{L}'\) of \(\mathcal{L}\) that can be computed in terms of \(m\) and \(\mathcal{L}\). It is therefore a key issue to establish a viable criterion for (FGLA)’s of the first kind.

Using duality, we will translate questions on the maximal effective prolongations of (FGLA)’s of the first kind to questions of commutative algebra for finitely generated modules over polynomial rings.

Let \(V\) be a finite dimensional \(\mathbb{K}\)-vector space, that we will consider as a commutative Lie algebra over \(\mathbb{K}\), and
\[
S(V^*) = \sum_{p=0}^{\infty} S_p(V^*)
\]
the \(\mathbb{Z}\)-graded unitary associative algebra over \(\mathbb{K}\) of symmetric multilinear forms on \(V\). Its product is described on homogeneous forms by
\[
(\xi \cdot \eta)(v_1, \ldots, v_{p+q}) = \frac{1}{(p+q)!} \sum_{\alpha \in S_{p+q}} \xi(v_{\alpha_1}, \ldots, v_{\alpha_p}) \cdot \eta(v_{\alpha_{p+1}}, \ldots, v_{\alpha_{p+q}}),
\]
\(\forall \xi \in S_p(V^*), \forall \eta \in S_q(V^*), \forall v_1, \ldots, v_{p+q} \in V\).

The duality pairing
\[
(3.2) \quad V \times V^* \ni (\nu, \xi) \longrightarrow \langle \nu | \xi \rangle \in \mathbb{K}
\]
extends to a degree-\((-1)\)-derivation \(D_p\) of \(S(V^*)\), with

\[
(D_p\xi)(v_1, \ldots, v_p) = (p+1)\xi(v, v_1, \ldots, v_p),
\]

\((3.3)\)  

\(\forall v, v_1, \ldots, v_p \in V, \ \forall \xi \in S_{p+1}(V^*).\)

The tensor product

\[
\mathcal{X}(V) = S(V^*) \otimes V = \sum_{p \geq 1} \mathcal{X}_p(V), \quad \text{with} \quad \mathcal{X}_p(V) = S_{p+1}(V^*) \otimes V,
\]

(3.4)  
is the maximal \(\mathfrak{gl}_{\mathbb{K}}(V)\)-prolongation of the commutative Lie algebra \(\mathfrak{m} = V\), where each vector in \(V\) is considered as a homogeneous element of degree \((-1)\). We can identify \(\mathcal{X}(V)\) with the space of vector fields with polynomial coefficients on \(V\). The Lie product in \(\mathcal{X}(V)\) is described, on rank one elements, by

\[
[\xi \otimes v, \eta \otimes w] = (\xi(D_v \eta)) \otimes w - (\eta(D_w \xi)) \otimes v, \ \forall \xi \in S_{0}(V^*), \ \forall \eta \in S_{0}(V^*), \ \forall v, w \in V.
\]

If \(a_0\) is any Lie subalgebra of \(\mathfrak{gl}_{\mathbb{K}}(V)\), then the direct sum \(V \oplus a_0\) is a Lie subalgebra of the Abelian extension \(V \oplus \mathfrak{gl}_{\mathbb{K}}(V)\) of \(\mathfrak{gl}_{\mathbb{K}}(V)\). The maximal effective \(a_0\)-prolongation of \(V\) can be described as a Lie subalgebra of \(\mathcal{X}(V)\).

\textbf{Proposition 3.1.} Let \(a_0\) be any Lie subalgebra of \(\mathfrak{gl}_{\mathbb{K}}(V)\) and

\[
a = V \oplus a_0 \oplus \sum_{p \geq 1} a_p\]

(3.5)  

the maximal effective prolongation of type \(a_0\) of \(V\). Its summands of positive degree are

\[
a_p = \{\xi \in \mathcal{X}_p(V) \mid \{V \ni v \rightarrow \xi(v_1, \ldots, v_p, v) \in a_0, \ \forall w \in V\} \}
\]

(3.6)  

\(p\) times

\textbf{Proof.} It is known (see e.g. [35] Ch.VII §3 or [22] Ch.1 §5]) that the elements of \(a_p\), for \(p \geq 1\), are the \(\xi \in S_{p+1}(V^*) \otimes V\) for which

\[
\{V \ni v \rightarrow \xi(v_1, \ldots, v_p, v) \in a_0, \ \forall v_1, \ldots, v_p \in V\}
\]

(3.6)  

\text{Formula (3.6) follows by polarization.} \qed

\textbf{Remark 3.2.} Denote by \(C : S_{p+1}(V^*) \otimes V \rightarrow S_p(V^*)\) the contraction map. The Casimir element \(c\) of \(V^* \otimes V\) is the sum \(\sum_{i=1}^{n} e_i \otimes e_i\), where \(e_1, \ldots, e_n\) is any basis of \(V\) and \(e_1^*, \ldots, e_n^*\) its dual basis in \(V^*\). We note that

\[
C(\xi \cdot c) = (p+1)\xi, \quad \forall \xi \in S_p(V^*),
\]

so that the symmetric right product by \((p+1)^{-1}c\) is a right inverse of the contraction. In particular,

\[
\mathcal{X}_p(V) = \mathcal{X}'_p(V) \oplus \mathcal{X}''_p(V), \quad \text{with} \quad \mathcal{X}'_p(V) = \ker \left( C : S_{p+1}(V^*) \otimes V \rightarrow S_p(V^*) \right),
\]

\[
\mathcal{X}''_p(V) = \{\xi : c \mid \xi \in S_p(V^*)\},
\]

is the decomposition of \(\mathcal{X}_p\) into a direct sum of irreducible \(\mathfrak{gl}_{\mathbb{K}}(V)\)-modules.

Since the operators \(D_v, D_w\) on \(S(V^*)\) commute,

\[
\mathcal{X}'(V) = V \oplus \mathfrak{s}_{\mathbb{K}}(V) \oplus \sum_{p \geq 1} \mathcal{X}'_p(V)
\]

is also the decomposition of \(\mathcal{X}(V)\) into a direct sum of irreducible \(\mathfrak{gl}_{\mathbb{K}}(V)\)-modules.
Example 3.3. Let $V, W$ be finite dimensional vector spaces over $\mathbb{K}$ and $\delta : V \times V \to W$ a non-degenerate symmetric bilinear form.

If $a_0$ is the orthogonal Lie algebra $\frak{o}_0(V)$, consisting of $X \in \frak{gl}_\mathbb{K}(V)$ such that

$$\delta(X(v_1), v_2) + \delta(v_1, X(v_2)) = 0, \text{ for all } v_1, v_2 \in V,$$

then $a_1 = 0$.

Proof. An element $\xi \in a_1$ is a map $\xi : V \to \frak{o}_0(V)$ such that $\xi(v_1)(v_2) = \xi(v_2)(v_1)$ for all $v_1, v_2 \in V$. Then, for $v_1, v_2, v_3 \in V$, we have

$$\delta(\xi(v_1)(v_2), v_3) = -\delta(v_2, \xi(v_1)(v_3)) = -\delta(v_2, \xi(v_3)(v_1)) = \delta(\xi(v_3)(v_2), v_1)$$

$$= \delta(\xi(v_2)(v_3), v_1) = -\delta(v_3, \xi(v_2)(v_1)) = -\delta(\xi(v_1)(v_2), v_3).$$

Since we assumed that $\delta$ is non-degenerate, this implies that $\xi(v_1)(v_2) = 0$ for all $v_2 \in V$ and hence that $\xi(v_1) = 0$ for all $v_1 \in V$, i.e. that $\xi = 0$. $\square$

The action

$$(\xi \otimes v) \cdot w = (D_w \xi) \otimes v, \quad \forall \xi \in S(V^*), \forall v, w \in V$$

defines on $X(V) = S(V^*) \otimes V$ the structure of a $\mathbb{Z}$-graded right $V$-module. Its $\mathbb{Z}$-graded dual

$$(3.7) \quad X^*(V) = \sum_{p=1}^{\infty} X^*_p(V), \quad \text{with} \quad X^*_p(V) = S_{p+1}(V) \otimes V^*$$

has a natural dual structure of left $V$-module (see §2.2). The dual action of $V$ on $X^*(V)$ is left multiplication, which extends to left multiplication by elements of $S(V)$.

By using the right-$V$-module structure of $X(V)$, we can rewrite the summands $a_p$ in (3.5) by

$$a_p = \{ \xi \in X^*_p(V) \mid \xi \cdot v_1, \ldots, v_p \in a_0, \forall v_1, \ldots, v_p \in V \}.$$  

Since $a$ is a right-$V$-submodule of $X(V)$, its graded dual $a^*$ is the quotient of the left $V$-module $X^*(V)$ by the annihilator $\mathcal{M}$ of $a$ in $X^*(V)$.

The duality pairing of $V$ and $V^*$ makes $\frak{gl}_\mathbb{K}(V) = V \otimes V^*$ self-dual, its pairing being defined by the trace form

$$\langle X, Y \rangle = \text{trace}(X \cdot Y), \quad \forall X, Y \in \frak{gl}_\mathbb{K}(V).$$

Let

$$(3.9) \quad a_0^0 = \{ X \in \frak{gl}_\mathbb{K}(V) \mid \text{trace}(X \cdot A) = 0, \forall A \in a_0 \}$$

be the annihilator of $a_0$ in $\frak{gl}_\mathbb{K}(V)$. Then the annihilator $\mathcal{M}$ of $a$ in $X^*(V)$ is the graded left-$V$-module

$$(3.10) \quad \mathcal{M} = S(V^*) \cdot a_0^0 \subseteq X^*(V).$$

Proposition 3.4. The dual of the maximal effective $a_0$-prolongation $a$ of $V$ is the quotient module

$$(3.11) \quad a^* = X^*(V)/\mathcal{M}. \quad \square$$
Proposition 3.5. Let $V$ be a finite dimensional $\mathbb{K}$-vector space and $\mathfrak{a}_0$ a Lie subalgebra of $\mathfrak{g}(\mathbb{K}) \in (V)$. Then the following are equivalent:

1. the maximal effective $\mathfrak{a}_0$-prolongation of $V$ is finite dimensional;
2. the $S(V)$-module $\mathfrak{a}^*$ is $(V)$-coprimary;
3. $S_h(V) \cdot \mathfrak{a}_0^h = S_{h+1}(V) \otimes V^*$, for some $h \geq 0$.

Proof. We recall that $\mathfrak{a}^*$ is said to be coprimary if, for each $s \in S(V)$, the homothety $\{a^* \ni a^* s \in \mathfrak{a}^* \}$ is either nilpotent or injective. If this is the case, the radical $\sqrt{Ann(\mathfrak{a}^*)}$ of the ideal $Ann(\mathfrak{a}^*) = \{s \in S(V) | s \cdot \mathfrak{a}^* = \{0\} \}$ is a prime ideal in $S(V)$. [See e.g. [38, p.8].]

Being $\mathbb{Z}$-graded, $\mathfrak{a}^*$ is finite dimensional if and only if all $v \in V$ define nilpotent homotheties on $\mathfrak{a}_0^*$. This shows that (1) and (2) are equivalent. Finally, (3) is equivalent to the fact that $\mathfrak{a}_0^*$, and hence $\mathfrak{a}$, is finite dimensional. \hfill $\Box$

The advantage of using duality is to reduce the question about the finite dimensionality of the maximal prolongation to an exercise on finitely generated modules over the ring of polynomials with coefficients in $\mathbb{K}$ and eventually to linear algebra.

Having fixed a basis $\xi_1, \ldots, \xi_n$ of $V^*$ we can identify $X^*(V)$ to $S(V)^n$. Each element $X$ of $\mathfrak{a}_0^*$ can be viewed as a column vector $X \nu$ of $S(V)^n$, whose entries are first degree polynomials in $V$. By taking a set $X_1, \ldots, X_m$ of generators of $\mathfrak{a}_0^*$ we obtain a matrix of homogeneous first degree polynomials

\[(3.12) \quad M(\nu) = (X_1 \nu, \ldots, X_m \nu) \in V^{m \times m} \subset S(V)^{m \times m},\]

that we can use to give a finite type presentation of $\mathfrak{a}^*$:

\[(3.13) \quad S(V)^m \xrightarrow{M(\nu)} S(V)^n \longrightarrow \mathfrak{a}^* \longrightarrow 0.\]

Theorem 3.6. Let $\mathfrak{g}(\mathfrak{M})$ be the ideal generated by the order $n$ minor determinants of $M(\nu)$. A necessary and sufficient condition for $\mathfrak{a}$ to be finite dimensional is that $\sqrt{\mathfrak{g}(\mathfrak{M})} = \{0\}$.

Proof. Indeed, the ideals $\mathfrak{g}(\mathfrak{M})$ and $Ann(\mathfrak{a}^*) = \{f \in S(V) | f \cdot \mathfrak{a}^* = \{0\} \}$ have the same radical (see e.g. [38, ch.I, §2]). \hfill $\Box$

Let $\mathbb{F}$ be the algebraic closure of the ground field $\mathbb{K}$ and set $V(\mathbb{F}) = \mathbb{F} \otimes_{\mathbb{K}} V$. By taking the tensor product by $\mathbb{F}$ we deduce from (3.13) the exact sequence

\[(3.14) \quad S(V(\mathbb{F}))^m \xrightarrow{M(\epsilon)} S(V(\mathbb{F})) \longrightarrow \mathbb{F} \otimes_{\mathbb{K}} \mathfrak{a}^* \longrightarrow 0.\]

We observe that $\mathfrak{a}^*$ is $(V)$-coprimary if and only if $\mathbb{F} \otimes_{\mathbb{K}} \mathfrak{a}^*$ is $(V(\mathbb{F}))$-coprimary. Therefore Theorem (3.6) translates into

Theorem 3.7. A necessary and sufficient condition for the maximal prolongation $\mathfrak{a}$ to be finite dimensional is that

\[(3.15) \quad \text{rank}(M(\epsilon)) = n = \dim(V), \quad \forall \epsilon \in V(\mathbb{F}) \setminus \{0\}.\]
Proof. In fact, since \( \mathbb{F} \) is algebraically closed, by the Nullstellensatz (see e.g. [20]) the necessary and sufficient condition for an ideal \( I \) of \( S(V(\mathbb{F})) \) to have \( \sqrt{I}=(V(\mathbb{F})) \) is that \( \{z\in V(\mathbb{F})\mid f(z)=0, \forall f\in I\}=[0] \). \( \square \)

**Example 3.8.** Denote by \( \mathfrak{co}(n, \mathbb{K})=\{A\in gl_n(\mathbb{K})\mid A^\top+A\in \mathbb{K}\cdot I_n\} \) the Lie algebra of conformal transformations of \( \mathbb{K}^n \).

(1). Let \( a_0=\mathfrak{co}(2, \mathbb{K}) \). Its orthogonal in \( gl_2(\mathbb{K}) \) consists of the traceless symmetric matrices. Take the basis consisting of

\[
M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We obtain

\[
M(z) = \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix}, \quad \text{with } \det(M(z)) = z_1^2 + z_2^2.
\]

Since the equation \( z_1^2 + z_2^2 = 0 \) has non zero solutions in \( \mathbb{F}^2 \), by Theorem [3,7], \( \mathbb{K}^2 \) has an infinite dimensional effective \( \mathfrak{co}(2, \mathbb{K}) \)-prolongation.

(2). Let us consider next the case \( n>2 \). The orthogonal \( a_0^0 \) of \( a_0=\mathfrak{co}(n, \mathbb{K}) \) consists of the traceless \( n\times n \) symmetric matrices. As a basis of \( a_0^0 \) we can take the matrices

\[
\Delta_h=(\delta_{1,j}\delta_{1,j}-\delta_{h,j}\delta_{h,j}), \quad (h=2, \ldots, n) \quad \text{and} \quad T_{h,k}=(\delta_{h,j}\delta_{k,j}+\delta_{k,j}\delta_{h,j}), \quad (1\leq h<k\leq n).
\]

Accordingly, we get

\[
M(z) = \begin{pmatrix} z_1 & z_1 & \cdots & z_1 & z_2 & \cdots & z_n & 0 & \cdots \\ -z_2 & 0 & \cdots & 0 & z_1 & 0 & \cdots & 0 & z_2 & \cdots \\ 0 & -z_3 & \cdots & 0 & 0 & z_1 & \cdots & 0 & z_2 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -z_n & 0 & 0 & \cdots & z_1 & 0 & \cdots \end{pmatrix},
\]

We want to show that \( M(z) \) has rank \( n \) when \( z\in \mathbb{F}^n\setminus\{0\} \). The minor of the \((j-1)\)-st, \( n\)-th, \( \ldots, (n-2)\)-nd columns is \((z_j^2+z^j^n)\cdot z_1^{n-2} \), for \( j=2, \ldots, n \). Likewise, we can show that the ideal of order \( n \) minor determinants of \( M(z) \) contains all polynomials \((z_j^2+z^j^n)\cdot z_1^{n-2} \) for \( 1\leq j\neq h\leq n \). Denote by \( \pm i \) the roots of \((-1)\) in \( \mathbb{F} \) and assume by contradiction that \( M(z) \) has rank \( <n \) for some \( z\neq 0 \). We can assume that \( z_1\neq 0 \). This yields \( z_j=\pm i \cdot z_1 \) for \( j=2, \ldots, n \). But then

\[
(z_2^2+z_3^2)\cdot z_1^{n-2} = -2(\pm i)^{n-2}z_1^n \neq 0.
\]

This contradiction proves that \( M(z) \) has rank \( n \) for all \( z\in \mathbb{F}^n\setminus\{0\} \), showing, by Theorem [3,7], that the maximal (EPFGLA) of type \( \mathfrak{co}(n, \mathbb{K}) \) of \( \mathbb{K}^n \) is finite dimensional if \( n\geq 3 \).

**Example 3.9.** Let us fix integers \( 0<p\leq q \) and set \( \mathcal{B} = \left\{ \begin{pmatrix} 0 & B^\top \\ B & 0 \end{pmatrix} \mid B\in \mathbb{K}^{p\times q} \right\} \).

Set \( n=p+q \) and let

\[
a_0 = \{X\in gl_2(n) \mid X^\top B+B X \in \mathcal{B}, \forall B\in \mathcal{B}\}.
\]
be the Lie algebra of $\mathcal{B}$-conformal transformations of $\mathbb{K}^n$. We have

$$a_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \middle| X \in \mathfrak{gl}_\mathbb{K}(q), Y \in \mathfrak{gl}_\mathbb{K}(p) \right\}$$

and hence $a_0^0 = \left\{ \begin{pmatrix} 0 & E \\ F^* & 0 \end{pmatrix} \middle| E, F \in \mathbb{K}^{q \times p} \right\}$.

By taking the canonical basis of $\mathbb{K}^{q \times p}$ we obtain

$$M(z) = \begin{pmatrix} z_{q+1} I_q & \cdots & z_q I_q & 0 & \cdots & 0 \\ 0 & \cdots & 0 & z_1 I_p & \cdots & z_r I_r \end{pmatrix}.$$ 

Clearly both $(z_1, \ldots, z_p)$ and $(z_{p+1}, \ldots, z_n)$ are associated ideals of $J(M)$ and hence the maximal effective $a_0$-prolongation of $V$ is infinite dimensional. Note that, if we take the $\mathcal{B}$-orthogonal Lie algebra

$$a_\mathcal{B} = \left\{ X \in \mathfrak{gl}_\mathbb{K}(n) \mid X^\top B + BX = 0, \forall B \in \mathcal{B} \right\},$$

the maximal $a_\mathcal{B}$-prolongation of $\mathbb{K}^n$ is finite dimensional by Example 3.3, because, if $B_1, \ldots, B_{pq}$ is a basis of the $\mathbb{K}$-vector space $\mathcal{B}$, the symmetric bilinear form $\mathbb{K}^n \times \mathbb{K}^n \ni (v_1, v_2) \mapsto (v_1^\top B_1 v_2)_{1 \leq i \leq pq} \in \mathbb{K}^p$ is nondegenerate.

**Example 3.10.** Let $V = \mathbb{K}^{2n}$ and

$$a_0 = \mathfrak{sp}(n, \mathbb{K}) = \left\{ A \in \mathfrak{gl}_n(\mathbb{K}) \mid A^\top \Omega + \Omega A = 0 \right\}, \quad \Omega = \begin{pmatrix} 0 & \begin{pmatrix} I_n \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -I_n \end{pmatrix} & 0 \end{pmatrix},$$

$$= \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \mid A, B, C \in \mathfrak{gl}_n(\mathbb{K}), \ B^\top = B, \ C^\top = C \right\}.$$ 

Then

$$a_0^0 = \left\{ \begin{pmatrix} A & B \\ C & A^\top \end{pmatrix} \mid A, B, C \in \mathfrak{gl}_n(\mathbb{K}), \ B^\top = -B, \ C^\top = -C \right\}.$$ 

Take any basis $A_1, \ldots, A_h$ of $\mathfrak{gl}_n(\mathbb{K})$ (with $h=n^2$) and $B_1, \ldots, B_k$ of $o(n, \mathbb{K})$ (with $k=\frac{1}{2}n(n-1)$). Then we have, for $z, w \in \mathbb{K}^n$,

$$M(z, w) = \begin{pmatrix} A_1 z & \cdots & A_h z & B_1 w & \cdots & B_k w & 0 & \cdots & 0 \\ A_1^\top w & \cdots & A_h^\top w & 0 & \cdots & 0 & B_1 z & \cdots & B_k z \end{pmatrix}.$$ 

If we take e.g. $w=0$, we see that, for $z_0 \neq 0$, $M(z_0, 0)$ has rank $(2n-1)$, because $B_1 z_0, \ldots, B_k z_0$ span the orthogonal hyperplane $z_0^\perp = \{ z \in \mathbb{K}^n \mid z^\top z_0 = 0 \}$ to $z_0$ in $\mathbb{K}^n$. By Theorem 3.7, this implies that the maximal $\mathfrak{sp}(n, \mathbb{K})$-prolongation of $\mathbb{K}^{2n}$ is infinite dimensional.

Let us state now the main result of this section.

**Theorem 3.11.** Let $V$ a finite dimensional $\mathbb{K}$-vector space, $a_0$ a Lie subalgebra of $\mathfrak{gl}_\mathbb{K}(V)$ and $\mathbb{F}$ the algebraic closure of $\mathbb{K}$.

The maximal (EPFGLA) of type $a_0$ of $V$ is infinite dimensional if and only if $\mathbb{F} \otimes a_0$ contains an element of rank one on $V(\mathbb{F})$.

**Proof.** We can as well assume that $\mathbb{K}$ is algebraically closed.

All rank one elements of $\mathfrak{gl}_\mathbb{K}(V)$ can be written in the form $v \otimes \xi$, with nonzero $v \in V$ and $\xi \in V^*$. If $X \in \mathfrak{gl}_\mathbb{K}(V)$, we obtain

$$(*) \quad \langle X v \mid \xi \rangle = \text{trace}(X(v \otimes \xi)).$$
If \( a_0 \) contains \( \psi \otimes \xi \), then \( X \psi \in \xi = \{ w \in V | \langle w, \xi \rangle = 0 \} \) for all \( X \in a_0 \), showing that \( M(\psi) \) has rank less than \( n = \dim(V) \). Then \( \dim(a) = \infty \) by Theorem 3.7.

Vice versa, if \( M(\psi) \) has rank less than \( n \) for some \( \psi \neq 0 \), we can find a nonzero \( \xi \in V^* \) such that \( X \psi \in \xi \) for all \( X \in a_0 \). Since the trace form is non-degenerate on \( gl_{\mathbb{K}}(V) \), by (3) this implies that \( (\psi \otimes \xi) \) is a rank one element of \( a_0 \).

\[ \square \]

3.1. **Prolongation of irreducible representations.** Assume that \( \mathbb{K} \) is algebraically closed and let \( V \) be a finite dimensional faithful irreducible representation of a reductive Lie algebra \( a_0 \) over \( \mathbb{K} \), having a center \( \mathfrak{z}_0 \) of dimension \( \leq 1 \). Let \( \mathfrak{s} = [a_0, a_0] \) be the semisimple ideal of \( a_0 \), \( \mathfrak{h} \) its Cartan subalgebra and \( \mathfrak{r}_s, \Lambda_s \) its corresponding root system and weight lattice. Fix a lexicographic order on \( \mathfrak{r}_s \) corresponding to the choice of a Borel subalgebra of \( \mathfrak{s} \). Then \( V \) is a faithful irreducible \( \mathfrak{s} \)-module. Let \( \Lambda(V) \subset \Lambda_s \) be the set of its weights and \( \mathfrak{h} \) its dominant weight. If \( \psi \) is minimal in \( \Lambda(V) \), then \( \phi - \psi \) is dominant in \( \Lambda(V \otimes V^*) \). Let \( \nu_\psi \) be a maximal vector in \( V \) and \( \xi_{-\psi} \) a maximal covector in \( V^* \). Then \( \nu_{\psi} \otimes \xi_{-\psi} \) is an element, of rank one on \( V \), generating an irreducible \( \mathfrak{s} \)-sub-module \( L_{\psi-\psi} \) of \( V \otimes V^* \). By a Theorem of Dynkin (see [5], [10] Ch.XIV), [14]) we know that all nonzero elements of \( L_{\psi-\psi}^0 \) have rank larger than one. Hence by using Theorem 3.11 we obtain

**Theorem 3.12.** The maximal effective prolongation of type \( a_0 \) of \( V \) is infinite dimensional if and only if

\[ L_{\psi-\psi} \subset \mathfrak{s}. \]

**Proof.** The Lie algebra \( \mathfrak{s} \) decomposes into a direct sum \( S_1 \oplus \cdots \oplus S_k \) of irreducible \( \mathfrak{s} \)-sub-modules of \( V \otimes V^* \). The summands \( S_i \) which are distinct from \( L_{\psi-\psi} \) are contained in \( L_{\psi-\psi}^0 \). The statement follows from this observation.

\[ \square \]

Under the assumptions above, if \( \mathfrak{s} = [a_0, a_0] \) is simple, then the maximal prolongation \( a \) of type \( a_0 \) of \( V \) is primitive in the sense explained in [16]. Then Theorem 3.12 yields easily a result about the infinite dimensionality of the maximal effective primitive prolongations that has been already proved by several Authors (see [11], [16], [17], [18], [21], [27], [32], [34], [40]).

**Proposition 3.13.** Assume that \( \mathbb{K} \) is algebraically closed, that \( a_0 \) is reductive, with \( [a_0, a_0] \) simple, and that \( V \) is a faithful irreducible \( a_0 \)-module. Then the maximal effective prolongation of type \( a_0 \) of \( V \) is infinite dimensional if and only if one of the following is verified

(i) \( a_0 \) is equal either to \( gl_{\mathbb{K}}(V) \) or \( sl_{\mathbb{K}}(V) \);

(ii) \( V \cong \mathbb{K}^{2n} \) for some integer \( n \geq 2 \) and \( a_0 \) is isomorphic either to \( sp(n, \mathbb{K}) \) or to \( \mathfrak{osp}(n, \mathbb{K}) \).

\[ \square \]

4. \( gl_{\mathbb{K}}(V) \)-prolongations of (FGLA)’s of the second kind

Let \( m = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \) be an (FGLA) of the second kind. Set \( V = \mathfrak{g}_{-1} \). By Proposition 1.4 \( m \) is isomorphic to a quotient \( \mathfrak{t}(V) / \mathfrak{r} \), for a graded ideal \( \mathfrak{r} \) of
f(V) with f_{13}(V) \subseteq \mathcal{K} \subseteq f_{12}(V). We note that \mathcal{K} = \mathcal{K}_{-2} \oplus f_{13}(V) is a graded ideal of f(V) for every proper vector subspace \mathcal{K}_{-2} of f_{-2}(V).

Since f_{-2}(V) = \Lambda^2(V), the subspace \mathcal{K}_{-2} is the kernel of the surjective linear map \lambda : \Lambda^2(V) \rightarrow g_{-2} associated to the bilinear map (v_1, v_2) \rightarrow [v_1, v_2] defined by the Lie product of elements of V:

\[ V \times V \xrightarrow{(v_1, v_2) \rightarrow v_1 \wedge v_2} \Lambda^2(V) \]

\[ \mathfrak{g}_{-2} = \mathcal{K}_{-2} \setminus \mathcal{K}_{-1} \]

\[ \mathfrak{g}_{-2} = \{ \mathfrak{g}_{-1}, \mathfrak{g}_{-1} \} = \langle \frac{\partial}{\partial h_i} | 1 \leq i < j \leq n \rangle \] so that \( \mathfrak{g} \cong \mathbb{R}^n \oplus \Lambda^2(\mathbb{R}^n) \), corresponds to the case \( \mathcal{K}_{-2} = \{ 0 \} \).

We keep the notation (1.13). For the case of (FGLA)'s of the second kind Proposition 1.4 reads

**Proposition 4.2.** Two graded fundamental Lie algebras of the second kind \( \mathfrak{m}(\mathcal{K}) \) and \( \mathfrak{m}(\mathcal{K}') \) are isomorphic if and only if the homogeneous parts of second degree \( \mathcal{K}_{-2} \) and \( \mathcal{K}'_{-2} \) of \( \mathcal{K} \) and \( \mathcal{K}' \) are \( \text{GL}_V(V) \)-congruent.

Fix a proper vector subspace \( \mathcal{K}_{-2} \) of \( f_{-2}(V) = \Lambda^2(V) \) and the corresponding 2-cofinite ideal \( \mathcal{K} = \mathcal{K}_{-2} \oplus f_{13}(V) \). By (1.16), the Lie algebra of zero-degree derivations of \( \mathfrak{m}(\mathcal{K}) \) is characterised by

\[ \mathfrak{g}_0(\mathcal{K}) = \{ A \in \mathfrak{g}(\mathcal{K}) | T_A(\mathcal{K}_{-2}) \subseteq \mathcal{K}_{-2} \}. \]

We recall from (1) that, for \( \mathcal{K} = \mathcal{K}_{-2} \oplus f_{13}(V) \), the maximal effective canonical \( \mathfrak{g}(\mathcal{K}) \)-prolongation of \( \mathfrak{m}(\mathcal{K}) \) is the \( \mathbb{Z} \)-graded Lie algebra

\[ \mathfrak{g}(\mathcal{K}) = \sum_{p \geq -2} \mathfrak{g}_p(\mathcal{K}) \]

whose homogeneous summands are defined by

\[ \begin{cases} 
  \mathfrak{g}_{-2}(\mathcal{K}) = \Lambda^2(V)/\mathcal{K}_{-2}, \\
  \mathfrak{g}_{-1}(\mathcal{K}) = V, \\
  \mathfrak{g}_h(\mathcal{K}) = \text{Der}_h \left( \mathfrak{m}, \sum_{p \geq h} \mathfrak{g}_p(\mathcal{K}) \right), \quad \text{for } h \geq 0.
\end{cases} \]

The spaces \( \mathfrak{g}_0(\mathcal{K}) \) are defined by recurrence and consist of the degree \( h \) homogeneous derivations of \( \mathfrak{m}(\mathcal{K}) \) with values in the \( \mathfrak{m}(\mathcal{K}) \)-module \( \sum_{p \geq h} \mathfrak{g}_p(\mathcal{K}) \):

\[ \mathfrak{g}_0(\mathcal{K}) = \{ A \in \mathfrak{g}(\mathcal{K}) | T_A(\mathcal{K}_{-2}) \subseteq \mathcal{K}_{-2} \}. \]
an element $\alpha$ of $\mathfrak{g}_b(\mathcal{K})$ is identified with a map
\[
\alpha : (V, \Lambda^2(V)) \rightarrow (\mathfrak{g}_{b-1}(\mathcal{K}), \mathfrak{g}_{b-2}(\mathcal{K})), \quad \text{with}
\]
\[
\alpha(v \wedge w) = (\alpha(v))(w) - (\alpha(w))(v), \quad \forall v, w \in V, \quad \text{and} \quad \alpha(0) = 0, \quad \forall 0 \in \mathcal{K}_{-2}.
\]
By Theorem 2.9, $\mathfrak{g}(\mathcal{K})$ is finite dimensional if and only if
\[
\alpha(\mathcal{K}) = \sum_{p=1}^{\infty} a_p(\mathcal{K}), \quad \text{with} \quad a_p(\mathcal{K}) = \{ \alpha \in g_p(\mathcal{K}) | [\alpha, g_{-2}(\mathcal{K})] = 0 \}
\]
is finite dimensional.

**Lemma 4.4.** We have $\mathfrak{a}_0(\mathcal{K}_{-2}) = \{ A \in g_0(R) | T_A(\Lambda^2(R^n)) \subseteq \mathcal{K}_{-2} \}$. □

Fix an identification $V \cong \mathbb{K}^n$, to consider the non degenerate symmetric bilinear form $b(v, w) = v^T \cdot w$ on $V$. It yields an isomorphism
\[
\rho : \Lambda^2(V) \rightarrow \mathfrak{o}(V), \quad \text{with} \quad \rho(v \wedge w) = v^T w - w^T v, \quad \forall v, w \in V,
\]
between $\Lambda^2(V)$ and the orthogonal Lie algebra
\[
\mathfrak{o}(V) = \{ X \in g_0(\mathcal{K}) | X^T + X = 0 \}.
\]
Under this identification, the action of $A \in g_0(\mathcal{K})$ on $\Lambda^2(V)$ can be described by $A \cdot X = A \cdot X + A \cdot X^T$. We use $\mathfrak{p}$ to identify $\mathcal{K}_{-2}$ with a linear subspace of $\mathfrak{o}(V)$. With this notation, we introduce
\[
\mathcal{K}_{-2}^{-1} = \{ X \in \mathfrak{o}(V) | \text{trace}(X K) = 0, \quad \forall K \in \mathcal{K}_{-2} \} \cong g_{-2}(\mathcal{K}).
\]

**Remark 4.5.** For the ideal $\mathcal{K}' = \mathcal{K}_{-2}^{-1} \oplus \mathcal{F}_3(V)$ we have
\[
\mathfrak{g}_0(\mathcal{K}') = \{ A^T | A \in \mathfrak{g}_0(\mathcal{K}) \}.
\]
which is a Lie algebra anti-isomorphic to $\mathfrak{g}_0(\mathcal{K})$, but, of course, $\mathfrak{a}_0(\mathcal{K})$ and $\mathfrak{a}_0(\mathcal{K}')$ may turn out to be quite different.

Note that $\mathcal{K}_{-2}^{-1}$ can be canonically identified with $\mathfrak{g}_{-2}$.

**Lemma 4.6.** We have $\mathfrak{a}_0^0(\mathcal{K}) = \{ XY | X \in \mathfrak{o}(V), \quad Y \in \mathcal{K}_{-2}^{-1} \}$. 

**Proof.** In fact, for $A \in g_0(\mathcal{K})$ and $X, Y \in \mathfrak{o}(V)$, we obtain
\[
\text{trace}((AX + XA^T)Y) = \text{trace}(AXY) + \text{trace}(YXA^T) = 2\text{trace}(A(XY)),
\]
because $(YXA^T)^T = A(XY)$. Thus we obtain
\[
A \in \mathfrak{a}_0 \iff \text{trace}((AX + XA^T)Y) = 2\text{trace}(A(YX)) = 0, \quad \forall X \in \mathfrak{o}(V), \forall Y \in \mathcal{K}_{-2}^{-1},
\]
proving our statement. □

To apply Theorem 3.7 to investigate the finite dimensionality of the maximal $\mathfrak{g}_0(\mathcal{K})(V)$-prolongation of $\mathfrak{m}(\mathcal{K})$, we need to construct the matrix $M(z)$ in (3.14), relative to $\mathfrak{a}_0^0(\mathcal{K})$, that we will denote by $M_2(\mathcal{K}, z)$. As usual, we denote by $\mathbb{F}$ be the algebraic closure of $\mathbb{K}$ and set $V(\mathbb{F}) = \mathbb{F} \otimes \mathbb{K} V$.

We can proceed as follows. Fix generators $Y_1, \ldots, Y_m$ of $\mathcal{K}_{-2}^{-1}$ and, for $z \in V(\mathbb{F})$ and $1 \leq i \leq m$, consider the vectors $Y_i z \in V(\mathbb{F})$. By the identification $V \cong \mathbb{K}^n$,
\[
\Phi_\mathcal{K}(z) = (Y_1 z, \ldots, Y_m z) \in (\mathbb{K}^m) \cong \mathbb{K}^{m \times m},
\]
is a matrix of first order homogeneous polynomials in $\mathbb{K}[z_1, \ldots, z_n]$ and, after choosing generators $X_1, \ldots, X_N$ of $o(V)$ as a $\mathbb{K}$-linear space, we take
\begin{equation}
M_2(\mathcal{K}, z) = (X_1 \Phi_K(z), \ldots, X_N \Phi_K(z)) \in (\mathbb{K}[z_1, \ldots, z_n])^{n \times (mN)}.
\end{equation}

The $o(V)$-orbit of a non zero vector $z$ of $\mathbb{P}^n$ spans the hyperplane
\[ z^\perp = \{ w \in \mathbb{P}^n \mid z^\top w = 0 \}. \]

To check this fact, we can reduce to the case where $\mathbb{F} = \mathbb{K}$. Take any $u \in V$ with $z^\top u = 1$. If $w \in z^\perp$, then the matrix $X = w - uu^\top$ belongs to $o(V)$ and $X z = w$. This shows that $z^\perp \subseteq o(V) z$. The opposite inclusion is obvious, since $z^\top X z = 0$ for all $X \in o(V)$ and $z \in V$.

Hence, if $z_1, z_2 \in V(\mathbb{F})$ are linearly independent, then $o(n) z_1 + o(n) z_2$ spans $V(\mathbb{F})$, so that $M_2(\mathcal{K}, z)$ has rank $n$ for all $z$ for which $\Phi_K(z)$ has rank $\geq 2$. We proved the following

**Proposition 4.7.** Let $\mathcal{K}_{-2} \subset \mathfrak{f}_{-2}(V)$ and $\mathcal{K} = \mathcal{K}_{-2} \oplus \mathfrak{f}_{[3]}(V)$. Then the maximal $\mathfrak{gl}_g(V)$-prolongation $\mathfrak{g}(\mathcal{K})$ of $m(\mathcal{K})$ is finite dimensional if and only if
\begin{equation}
\{ z \in V(\mathbb{F}) \mid \text{rank}(\Phi_K(z)) < 2 \} = \{ 0 \}. \quad \square
\end{equation}

We can give an equivalent formulation of Proposition 4.7 involving the rank of the $\mathbb{F}$-bilinear extension of the alternate bilinear form on $V$ defined by the Lie brackets.

**Definition 4.1.** Let $\mathbb{F}$ be the algebraic closure of $\mathbb{K}$. We call the integer
\begin{equation}
\ell = \inf\{ \text{dim}_\mathbb{F}(\langle z, V(\mathbb{F}) \rangle) \mid 0 \neq z \in V(\mathbb{F}) \}
\end{equation}
the algebraic minimum rank of $m(\mathcal{K})$.

**Theorem 4.8.** Let $\mathcal{K}_{-2} \subset \mathfrak{f}_{-2}(V)$ and $\mathcal{K} = \mathcal{K}_{-2} \oplus \mathfrak{f}_{[3]}(V)$. Then the maximal $\mathfrak{gl}_g(V)$-prolongation $\mathfrak{g}(\mathcal{K})$ of $m(\mathcal{K})$ is finite dimensional if and only if $m(\mathcal{K})$ has algebraic minimum rank $\ell \geq 2$.

**Proof.** If we identify $z \in V(\mathbb{F})$ with the corresponding numerical vector in $\mathbb{F}^n$, then $w^\top Y z$ is the $Y_i$-component of $\llbracket w, z \rrbracket$. The condition that the minimum rank of $m(\mathcal{K})$ is larger or equal to two is then equivalent to the fact that $\Phi_K(z)$ has rank $\geq 2$ for all $z \in V(\mathbb{F})$. \quad $\square$

**Example 4.9.** Let $\mathbb{K} = \mathbb{R}$, $n=4$ and $\mathcal{K} = \langle (\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}) \mid A \in \mathbb{R}^{2 \times 2}, A = A^\top \rangle$.

With $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a basis of $\mathcal{K}^\perp$ is given by the matrices
\[ Y_1 = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}. \]

This yields
\[ \Phi_K(z) = \begin{pmatrix} z_2 & 0 & z_4 \\ -z_1 & 0 & -z_3 \\ 0 & z_4 & z_2 \\ 0 & -z_3 & -z_1 \end{pmatrix}. \]
Writing $\Delta_{hk}^{ij}$ for the minor of the lines $i, j$ and the columns $h, k$, we obtain

$$
\Delta_{1:3}^{2,4} = e_1^2, \; \Delta_{1:3}^{1,3} = e_2^2, \; \Delta_{1:3}^{2,4} = e_3^2, \; \Delta_{2:3}^{1,3} = e_4^2,
$$

showing that $g(\mathcal{K})$ is finite dimensional.

**Example 4.10.** Let $\mathbb{K} = \mathbb{R}$, $n=4$ and $\mathcal{K}^+$ generated by the matrices

$$
Y_1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \; Y_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \; Y_3 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Then

$$
\Phi_{\mathcal{K}}(z) = \begin{pmatrix}
z_3 & z_4 & z_2 \\
z_4 & -z_3 & -z_4 \\
-z_1 & z_2 & 0 \\
-z_2 & -z_1 & 0
\end{pmatrix}.
$$

The matrix $\Phi_{\mathcal{K}}(z)$ has rank $\geq 2$ for all nonzero $z \in \mathbb{R}^4$, but

$$
\Phi_{\mathcal{K}}(0, 0, 1, i) = \begin{pmatrix}
1 & i & 0 \\
i & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

has rank 1. Thus $g(\mathcal{K})$ is infinite dimensional, although $m(\mathcal{K})$ has (real) minimal rank 2.

We note that, if $\Phi_{\mathcal{K}}(z)$ has at most 2 columns, then the set

$$
\{z \in V(\mathbb{P}) | \text{rank}(\Phi_{\mathcal{K}}(z)) < 2\}
$$

is an algebraic affine variety of positive dimension in $V(\mathbb{P})$. In particular, we obtain (cf. e.g. [29])

**Corollary 4.11.** If $\dim(g_{-2}(\mathcal{K})) \leq 2$, then $g(\mathcal{K})$ is infinite dimensional. $\square$

5. $g_{\mathbb{K}}(V)$-prolongations of (FGLA)’s of higher kind

Let $V$ be a $\mathbb{K}$-vector space of finite dimension $n \geq 2$ and, for an integer $\mu \geq 3$, fix a $\mu$-cofinite $\mathbb{Z}$-graded ideal $\mathcal{K} \subset \mathfrak{f}_2(V)$ of $\mathfrak{f}(V)$. Let

$$
\begin{align*}
&\mathfrak{m}(\mathcal{K}) = \sum_{p=-\mu}^{-1} \mathfrak{g}_p(\mathcal{K}) \cong \mathfrak{f}(V)/\mathcal{K}, \quad \mathfrak{n}(\mathcal{K}) = \sum_{p=-\mu}^{-2} \mathfrak{g}_p(\mathcal{K}) \cong \mathfrak{f}_2(\mathcal{K}).
\end{align*}
$$

The canonical maximal $\mathbb{Z}$-graded effective $g_{\mathbb{K}}(V)$-prolongation

$$
\mathcal{G}(\mathcal{K}) = \sum_{p \geq -\mu} \mathfrak{g}_p(\mathcal{K})
$$

of $m(\mathcal{K})$ is the quotient by $\mathcal{K}$ of its normaliser $\mathfrak{N}(\mathcal{K})$ in $\mathfrak{G}(V)$. By Tanaka’s criterion (Theorem 2.11), $g(\mathcal{K})$ is finite dimensional if, and only if,

$$
\alpha(\mathcal{K}) = \sum_{p \geq -1} \alpha_p(\mathcal{K}), \quad \text{with} \quad \begin{cases}
\alpha_{-1} = \mathfrak{m}(\mathcal{K})/\mathfrak{n}(\mathcal{K}) \cong V, \\
\alpha_p(\mathcal{K}) = \{X \in g_p(\mathcal{K}) | [X, n(\mathcal{K})] = \{0\}\} \text{ for } p \geq 0,
\end{cases}
$$

is finite dimensional. As in §4 we get (see also Lemma 2.10):
Proposition 5.1. Let $\mathcal{K}$ be a $\mathbb{Z}$-graded ideal of $\mathfrak{f}(V)$, contained in $\mathfrak{f}_{[2]}$. Then

\begin{align}
(5.2) \quad & \begin{cases}
\mathfrak{g}_0(\mathcal{K}) = \{ A \in \mathfrak{gl}_2(V) | T_A(\mathcal{K}) \subset \mathcal{K} \}, \\
\mathfrak{a}_0(\mathcal{K}) = \{ A \in \mathfrak{gl}_2(V) | T_A(\mathfrak{f}_{[-1]}(V)) \subseteq \mathcal{K}_{-p}, \forall p \geq 2 \}.
\end{cases}
\end{align}

Set (brackets are computed in $\mathfrak{f}(V)$)

\begin{align}
(5.3) \quad W(\mathcal{K}) = \{ v \in V | [v, \mathfrak{f}_{[2]}(V)] \subset \mathcal{K} \}.
\end{align}

Proposition 5.2. We have the following characterisation:

\begin{align}
(5.4) \quad & \mathfrak{a}_0(\mathcal{K}) = \{ A \in \mathfrak{gl}_2(V) | T_A(\mathfrak{f}_{[-2]}(V)) \subseteq \mathcal{K}_{-2}, \mathfrak{A}(V) \subseteq W(\mathcal{K}) \}.
\end{align}

Proof. We use the characterisation in Proposition 5.1 and the fact that $V$ generates $\mathfrak{f}(V)$. Let us denote by $b$ the right hand side of (5.4).

We first show that if $A \in b$, then $T_A(\mathfrak{f}_{[-p]}(V)) \subseteq \mathcal{K}_{-p}$ for $p \geq 2$. If $A \in b$ then $[A(v), X] \in \mathcal{K}_{-p}$ for $p \geq 2$ and hence both the left hand side and the second summand of the right hand side of (5.4) belong to $\mathcal{K}_{-p}$ for all $v \in V$ and $X \in \mathfrak{f}_{[-p]}(V)$, this shows that $T_A(\mathfrak{f}_{[-p]}(V)) \subseteq \mathcal{K}_{-p}$ for all $p \geq 2$ and hence that $A(V) \subseteq W(\mathcal{K})$. Thus $a_0(\mathcal{K}) \subseteq b$, completing the proof of the proposition.

Let us fix a basis $e_1, \ldots, e_n$ whose first $m$ vectors $e_1, \ldots, e_m$ are a basis of $W(\mathcal{K})$. The matrices of the elements $A$ of $\mathfrak{gl}_2(V)$ that map $V$ into $W(\mathcal{K})$ are of the form

$A = \begin{pmatrix} B \\ 0 \end{pmatrix}$, with $B \in \mathbb{R}^{m \times n}$.

Their orthogonal in $\mathfrak{gl}_2(V)$ consists of the matrices of the form

$Y = \begin{pmatrix} 0 \\ C \end{pmatrix}$, with $C \in \mathbb{R}^{n \times (n-m)}$.

Taking the canonical basis $Y_1, \ldots, Y_r$, with $r = n \times (n-m)$, for the linear space of the matrices of this form, we obtain a matrix

\begin{align}
(5.5) \quad & M_3(\mathcal{K}, z) = (z_{m+1}I_n, \ldots, z_nI_n).
\end{align}

Since the orthogonal of an intersection of linear subspaces is the sum of their orthogonal subspaces, the matrix of the form $M_3(\mathcal{K}, z)$ associated to $a_0(\mathcal{K})$ is in this case

\begin{align}
(5.6) \quad & M(z) = (M_2(\mathcal{K}, z), M_3(\mathcal{K}, z)).
\end{align}
Taking into account that $M_3(\mathcal{K}, z)$ is described by (5.5), the matrix (5.6) has rank $n$ whenever $(\varepsilon_{n+1}, \ldots, \varepsilon_{n}) \neq (0, \ldots, 0)$. Let $\Phi_k(z)$ the matrix defined by (4.10). Then Theorem 2.11 yields

**Theorem 5.3.** Assume that $\mathcal{K}$ is $\mu$-cofinite for some integer $\mu \geq 2$ and let $e_1, \ldots, e_m$ be a basis of $W(\mathcal{K})$. Then $\mathfrak{g}(\mathcal{K})$ is finite dimensional if and only if $\Phi_k(z)$ has rank $\geq 2$ for all $0 \neq z \in W(\mathcal{K})$.

6. $\mathfrak{g}$-prolongations of (FGLA)’s of general kind

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}_{\mathbb{K}}(V)$. Fix a set $A_1, \ldots, A_k$ of generators of

$$\mathfrak{g}^\perp = \{ A \in \mathfrak{gl}_{\mathbb{K}}(V) \mid \text{trace}(AX) = 0, \forall X \in \mathfrak{g} \}$$

and form

$$M_1(\mathfrak{g}, z) = (A_1 z, \ldots, A_k z), \quad z \in V(\mathfrak{g}).$$

If we fix a basis $e_1, \ldots, e_n$ of $V$, we may consider $M(z)$ as a matrix of homogeneous first degree polynomials of $z=(z_1, \ldots, z_n)$ with coefficients in $\mathbb{K}$.

Let $m = \sum_{\mu=1}^n \alpha_\mu$ be an (FGLA) of finite kind $\mu \geq 1$ and set $V = \mathfrak{g}_{-1}$. Then $m$ is isomorphic to a quotient $\mathfrak{t}(V)/\mathcal{K}$, for a graded ideal $\mathcal{K}$ of $\mathfrak{t}(V)$ with $\mathfrak{t}(V) \subseteq \mathfrak{g} \subseteq \mathfrak{t}(2)(V)$. Let $\mathfrak{l}$ be a Lie subalgebra of $\mathfrak{gl}_{\mathbb{K}}(\mathfrak{g}_{-1})$. We showed in \S 4 that the maximal effective prolongation of type $\mathfrak{l}$ of $m$ is the quotient by $\mathcal{K}$ of its normaliser $\mathfrak{g}(\mathcal{K}, \mathfrak{l})$ in $\mathfrak{g}(V, \mathfrak{l})$.

By Theorem 2.11 this maximal effective $\mathfrak{l}$-prolongation $\mathfrak{g}(\mathcal{K}, \mathfrak{l})$ of $m$ is finite dimensional if and only if the maximal effective $\mathfrak{a}_0$-prolongation of $V$ is finite dimensional, where $\mathfrak{a}_0$ the Lie subalgebra of $\mathfrak{l}$ defined by

$$\mathfrak{a}_0 = \{ A \in \mathfrak{g} \mid T_A(\mathfrak{t}_2)(V) \subseteq \mathcal{K} \}.$$  

Since the orthogonal of an intersection of linear subspaces is the sum of their orthogonal subspaces, we can use the results (and the notation) of \S 3, 4, 5 to construct the matrix $M(z)$ of Theorem 3.7 obtaining

$$M(z) = (M_1(\mathfrak{g}, z), M_2(\mathcal{K}, z), M_3(\mathcal{K}, z)).$$

Let $W = W(\mathcal{K})$ be the subspace of $V$ defined in \S 5. Then we obtain

**Theorem 6.1.** Fix the basis $e_1, \ldots, e_n$ of $V$ in such a way that the first $m$ vectors $e_1, \ldots, e_m$ form a basis of $W(\mathcal{K})$. Then a necessary and sufficient condition in order that the maximal effective $\mathfrak{l}$-prolongation of $m$ be finite dimensional is that

$$(M_1(\mathfrak{g})(z), M_2(\mathcal{K}, z))$$

has rank $n$ for all $z \in W(\mathfrak{g}) \setminus \{0\} = \{ z \in \mathbb{K}^n \setminus \{0\} \mid \zeta_0 = 0, \forall i > m \}$. 

**Example 6.2.** Let $V$ be a finite dimensional $\mathbb{K}$-vector space of dimension $n \geq 3$ and $b : V \times V \to \mathbb{K}$ a nondegenerate bilinear form on $V$, having nonzero symmetric and antisymmetric components $b_s$ and $b_a$. Let

$$\mathfrak{g} = \{ X \in \mathfrak{gl}_{\mathbb{K}}(V) \mid \exists \epsilon(X) \in \mathbb{K} \text{ s.t. } b(Xv, w) + b(v, Xw) = \epsilon(X)b(v, w), \forall v, w \in B \}.$$
Then it is natural to take $m = g_{-1} \oplus g_{-2}$, with $g_{-1} = V$ and $g_{-2} = \mathbb{K}$, defining the Lie brackets on $V$ by

$$[v, w] = b_s(v, w), \quad \forall v, w \in V.$$ 

Then $m = f(V)/\mathcal{K}$, with $\mathcal{K} = \mathcal{K}_{-2} \oplus f_{[3]}$ for

$$\mathcal{K}_{-2} = \left\{ \sum v_i \wedge w_i \in \Lambda^2(V) \left| \sum b_s(v_i, w_i) = 0 \right. \right\}.$$

The maximal effective $\mathfrak{g}$-prolongation $g = g(\mathcal{K}, \mathfrak{g})$ of $m$ is equal to the maximal effective $\text{co}_{s_t}(V)$-prolongation $g(\mathcal{K}, \text{co}_{s_t}(V))$ of $m$.

In particular, $g$ is finite dimensional by Example 3.8 because $n \geq 3$, when $b_s$ is nondegenerate.

Consider now the case where $b_s$ has rank $0 < p < n$. We can find a basis $e_1, \ldots, e_n$ of $V$ such that $b_s$ is represented by a matrix

$$B = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{with } D = \text{diag}(\lambda_1, \ldots, \lambda_p) \in \mathbb{K}^{p \times p}, \quad \text{with } \lambda_1 \cdots \lambda_p \neq 0.$$

Let $A$ be the antisymmetric $n \times n$ matrix representing $b_s$ in this basis and $A_1, \ldots, A_n$ its rows. Then the matrix $M(z)$ of (6.3) has the form

$$M(z) = (M_1(z), M_2(z)),$$

with $M_1(z) = (z_1 E_1, \ldots, z_p E_p)$ for invertible $n \times n$ matrices $E_1, \ldots, E_q$ and

$$M_2(z) = \begin{pmatrix} A_2 z & \ldots & A_n z & 0 & \ldots & 0 \\ -A_1 z & \ldots & 0 & A_3 z & \ldots & A_n z \\ 0 & \ldots & 0 & -A_2 z & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & -A_1 z & 0 & \ldots & -A_2 z \end{pmatrix}$$

In particular, when $z_i = 0$ for $1 \leq i < n$, the first row of $M(z)$ is zero because $A$ is antisymmetric and therefore $g$ is infinite dimensional by Theorem 6.1 (here $W_{(p)} = V_{(p)}$ because $\mathcal{K}_{-3} = f_{[3]}(V)$).

### 7. Some semisimple examples

Let $g = \sum_{p=1}^n g_p$ be a finite dimensional $\mathbb{Z}_p$-graded semisimple Lie algebra over $\mathbb{K}$. We say that the gradation is not trivial if $g_0 \neq \{0\}$ and $g_0$ does not contain any non-trivial ideal of $g$. Note that $g_0$ is reductive, since the restriction to $g_0$ of the Killing form of $g$ is nondegenerate. Set $m = \sum_{p=1}^{n-1} g_p$ and $V = g_{-1}$. Assuming that $m$ is fundamental, $g$ is an (EPFGA) of $m$ if and only if the action of $g_0$ on $V$ is faithful; in this case $g_0$ can be identified with a Lie subalgebra of $\text{gl}(\mathbb{K})$. The derived algebra $[g_0, g_0]$ of $g_0$ is semisimple and $V$ decomposes into a direct sum $V = V_1 \oplus \cdots \oplus V_k$ of its irreducible representations. For each $1 \leq i \leq k$, the set $A_i$ of $\mathbb{K}$-endomorphisms of $V_i$ which commute with the action of $[g_0, g_0]$ is, by Schur’s lemma, a division $\mathbb{K}$-algebra. The elements of $A_i$ uniquely extend to derivations of $g$ vanishing on $V_j$ for $j \neq i$. In particular, the identity of $A_i$ is a projection $\eta_i : V \to V_i$. Since every
derivation of a semisimple finite dimensional Lie algebra is inner, we can consider the $A_i$’s as subalgebras of $g_0$.

**Lemma 7.1.** Assume that $g$ is finite dimensional and semisimple and that the action of $g_0$ on $V$ is faithful. Then the center of $A_1 \oplus \cdots \oplus A_k$ is the center of $g_0$. \qed

**Remark 7.2.** We note that $A_i$ may contain a simple Lie algebra over $\mathbb{K}$, that will contribute as a summand to $[g_0, g_0]$. For instance, when $\mathbb{K}$ is the field of real numbers, the possible $A_i$ are $\mathbb{R}$ itself, or $\mathbb{C}$, or the non commutative real division algebra $H$ of quaternions, and $\mathbb{H} \approx \mathfrak{o}(3) \oplus \mathbb{R}$. A very easy example of this situation is the simple Lie algebra $\mathfrak{sl}_2(\mathbb{H})$, with $g_0 = \mathfrak{o}(3) \oplus \mathbb{R} \approx \mathfrak{co}(4) \oplus \mathbb{R}$, with the standard action on $V = \mathbb{R}^4 \approx \mathbb{H}$, where $\mathfrak{sl}_2(\mathbb{H})$ can be viewed as a maximal (EPFGLA) of type $\mathfrak{o}(4)$ of $V$. This presentation corresponds to the cross marked Satake diagram (see e.g. [3])

Semisimple and maximal (EPFGLA) prolongations are related by the following proposition.

**Proposition 7.3.** Assume that $g$ is semisimple and that the action of $g_0$ on $V$ is faithful. Then $g$ is maximal among the finite dimensional effective prolongations of type $\mathfrak{gl}_\mathbb{Z}(V)$ of $m$.

**Proof.** Indeed, an effective finite dimensional prolongation $\mathfrak{g}_1 = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $m$ containing $g$ is a $g$-module. Since the finite dimensional linear representations of a semisimple $g$ are completely reducible, $g$ has in $\mathfrak{g}_1$ a complementary $\mathbb{Z}$-graded $g$-module $\mathfrak{g}_1'$. The conditions that $[\mathfrak{g}_1', m] \subseteq \mathfrak{g}_1'$ and that $\mathfrak{g}_1'$ is contained in $\mathfrak{g}_1 = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ and is effective implies that $\mathfrak{g}_1' = \{0\}$. \qed

In the rest of this section, we will exhibit (EPFGLA) structures on semisimple Lie algebras, that we think could be interesting in geometrical and physical applications.

Let us explain the pattern of our constructions. We start from a semisimple Lie algebra $\mathfrak{g}_0$ and its faithful linear representation $V$, so that we identify $\mathfrak{g}_0$ with a Lie subalgebra of $\mathfrak{gl}_\mathbb{C}(V)$. The $\mathfrak{g}_0$-module $V$ decomposes into a direct sum $V = V_1 \oplus \cdots \oplus V_k$ of irreducible $\mathfrak{g}_0$-modules. Denote by $A_i$ the $k$-division algebra of the linear endomorphisms of $V_i$ which commute with the action of $\mathfrak{g}_0$. We consider the elements of $A_i$ as endomorphisms of $V$ by letting them act trivially on $\sum_{j \neq i} V_j$. Then we consider $g_0 = \mathfrak{g}_0 + \mathfrak{A}_1 + \cdots + \mathfrak{A}_k$ as a Lie subalgebra of $\mathfrak{gl}_\mathbb{C}(V)$ and take it to be the linear isotropy subalgebra (corresponding to the structure group of the corresponding differential geometrical object), see e.g. [22]. This is a reductive Lie algebra, with $\mathfrak{g}_0 \subseteq \{g_0, g_0\}$ and center $c$ contained in $A_1 \oplus \cdots \oplus A_k$.

The exterior power $\Lambda^2(V)$ is a $g_0$-module and we choose for $\mathfrak{g}_{-2}$ any $g_0$-submodule of $\Lambda^2(V)$. Next, we can choose $\mathfrak{g}_{-3}$ as a $g_0$-submodule of $(V \otimes \mathfrak{g}_{-2}) \cap \mathfrak{f}_{-3}(V)$, and, by recurrence, $\mathfrak{g}_{-p}$ as a $g_0$-submodule of $(V \otimes \mathfrak{g}_{-p-1}) \cap \mathfrak{f}_{-p-1}(V)$. In this way we build up the (FGLA) $m$. 


Assume that $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers. Then a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the direct sum of $\mathfrak{c}$ and of a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{L}_0$. Let $\mathcal{R}_0$ and $\mathcal{W}_0$ be the root system and the weight lattice associated to $\mathfrak{h}_0$. Each $\mathfrak{g}_{-p}$ decomposes into a direct sum of irreducible $\mathfrak{L}_0$-modules, that, after fixing on $\mathcal{R}_0$ a lexicographic order, can be described by their dominant weight $\omega$. Searching for a semisimple (EPFGLA) of type $\mathfrak{g}_0$ of $\mathfrak{m}$, we need to enlarge the euclidean space $E$ where $\mathcal{W}_0$ sits. This is done by adding linearly independent markers, corresponding to the $\eta_i$'s of $\mathfrak{c}$, in a manner that will be clarified by the examples.

To discuss the case of real Lie algebras, we may observe that the complexification of the real (EPFGLA) of a real (FGLA) $\mathfrak{m}$ is the complex (EPFGLA) of the complexification of $\mathfrak{m}$. In this way we reduce to the complex case. Namely, for semisimple prolongation, we can observe that the gradation of $\mathfrak{g}$ reflects into a gradation of the associated Dynkin/Satake diagrams. On a Satake diagram $\Sigma$ (see e.g. [3, 4]), the conditions for the associated real semisimple Lie algebra being an effective prolongation of a finite dimensional (FGLA) can be read on $\Sigma$ by requiring that

(i) the eigenspaces corresponding to fundamental roots of $\Sigma$ are homogeneous of degree either 0 or 1;
(ii) compact roots have degree 0 and those joined by an arrow have the same degree;
(iii) the degree 0 roots are the nodes of the Satake diagram of $[\mathfrak{g}_0, \mathfrak{g}_0]$.

Crosses can be added under the nodes of a Satake diagram to indicate the roots of positive degree. Complex type representation can be associated to couples of positive roots joined by an arrow; for this we refer e.g. to [31].

As usual, we denote by $(\alpha|\beta)$ the standard scalar product of $\alpha, \beta \in \mathbb{R}^n$ and set $(\alpha|\beta) = 2(\alpha|\beta)/||\beta||^2$.

For a semisimple $\mathbb{Z}$-graded $\mathfrak{g}$, having fixed its Cartan subalgebra $\mathfrak{h}$ and the corresponding root system $\mathcal{R}$, we will write for simplicity $\langle \alpha_1, \ldots, \alpha_r \rangle$ to indicate the direct sum of the complex eigenspaces

$\mathcal{V}_{\alpha_i} = \{ X \in \mathfrak{g} | [H, X] = \alpha_i(H) \cdot X, \forall H \in \mathfrak{h} \}$.

This is convenient, as each $\mathfrak{g}_p$ decomposes into a direct sum of root spaces. We shall also use the notation $\mathcal{W}(\omega)$ to indicate the set of weights of an irreducible representation with dominant weight $\omega$.

Exceptional Lie algebras naturally arise as (EPFGLA) of (FGLA)'s in which the structure algebra is non exceptional. These constructions are related to the investigation of their maximal rank reductive subalgebras (see e.g. [1, 15]).

7.1. Structure algebras of type A. To describe the root system of $\mathfrak{sl}_n(\mathbb{C})$, it is convenient to use an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ and set

$\mathcal{R} = \{ \pm(e_i - e_j) | 1 \leq i < j \leq n \}$. 
The Dynkin diagram is

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1}
\end{array}
\]

with simple roots

\[\alpha_i = e_i - e_{i+1}, \quad \text{for} \ 1 \leq i \leq n-1.\]

Set \(e_0 = e_1 + \cdots + e_n\).

The simple positive weights in \(\langle e_i - e_j \mid 1 \leq i < j \leq n \rangle \approx \mathbb{R}^{n-1}\) are \(\omega_k = \sum_{j=1}^{k} e_j - \frac{k}{n} e_0\), for \(1 \leq i \leq n-1\), with \(\omega_k\) corresponding to the irreducible representation \(\Lambda^k(\mathbb{C}^m)\).

The kind one abelian Lie algebra \(\mathfrak{m} = V_{0,1} \simeq \mathbb{C}^n\) has the simple (EPFGLA) \(\mathfrak{sl}_{n+1}(\mathbb{C})\), but this is contained in its infinite dimensional maximal (EPFGLA) \(\chi(V)\) (see §3). We have

\[\langle \omega_j, \omega_k \rangle = j \left(1 - \frac{k}{n}\right) > 0, \quad \forall 1 \leq j < k < n\]

and hence

\[
\left\| \sum_{j=1}^{n-1} a_j \omega_j \right\|^2 = \sum_{j=1}^{n-1} j a_j \left[ a_j \left(1 - \frac{j}{n}\right) + 2 \sum_{k=j+1}^{n-1} a_k \left(1 - \frac{k}{n}\right) \right] > \sum_{j=1}^{n-1} a_j^2 \cdot j \left(1 - \frac{j}{n}\right).
\]

A marker for an irreducible representation of \(\mathfrak{sl}_n(\mathbb{C})\), with dominant weight \(\omega_0\), could be taken of the form \(\epsilon = c \cdot e_0\). If we aim to obtain (EPFGLA)'s of type \(\mathfrak{g}_n(\mathbb{C})\) which are simple, a necessary condition is that the dominant weight \(\omega'\) of an irreducible representation appearing in the \(\mathfrak{g}_{-2}\) summand of \(\mathfrak{m}\) has an admissible value of \(\|\omega' - p \epsilon\|^2\). This means that we should have

\[
\begin{cases}
\|\omega' - p \epsilon\|^2 = 2, & \text{if } \mathfrak{g} \text{ is of type A, D, E}, \\
\|\omega' - p \epsilon\|^2 \text{ either } \in \{1, 2\}, & \text{or } \in \{2, 4\} \text{ if } \mathfrak{g} \text{ is of type B, C, F}, \\
\|\omega' - p \epsilon\|^2 \text{ either } \in \{2, 6\}, & \text{or } \in \{2, \frac{4}{3}\} \text{ if } \mathfrak{g} \text{ is of type G}_2.
\end{cases}
\]

7.1.1. Construction of \(B_2\) and \(G_2\). For the standard representation \(V_{0,1} \simeq \mathbb{C}^n\), also \(\Lambda^2(V_{0,1})\) is irreducible, with dominant weight \(\omega_2\). Thus the only possible choice for kind two is to take \(\mathfrak{m} = \mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n) \simeq \mathfrak{f}(\mathbb{C}^n)/\mathfrak{f}_{(3)}(\mathbb{C}^n)\). With \(\epsilon = \frac{1}{n} e_0\), we have

\[
\|\omega_1 + \epsilon\|^2 = \|e_1\|^2 = 1, \quad \|\omega_2 + 2\epsilon\|^2 = ||e_1 + 2\epsilon||^2 = 2.
\]

By Corollary [4.11] when \(n \leq 2\), \(\dim(\mathfrak{g}_{-2}) \leq 1\) and therefore maximal (EPFGLA) of \(\mathfrak{m}\) is infinite dimensional. For \(n \geq 3\), the maximal (EPFGLA) is finite dimensional by Theorem [4.3] and is in fact isomorphic to \(\mathfrak{o}(2n+1, \mathbb{C})\) (see
e.g. \cite{[41]}). Indeed, setting

\[
\begin{align*}
\mathcal{R}_{-2} &= \{\omega + 2\epsilon \mid \omega \in \mathcal{W}(\omega_2)\} = \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\}, \\
\mathcal{R}_{-1} &= \{\omega + \epsilon \mid \omega \in \mathcal{W}(\omega_1)\} = \{\epsilon_i \mid 1 \leq i \leq n\}, \\
\mathcal{R}_0 &= \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}, \\
\mathcal{R}_1 &= \{\omega - \epsilon \mid \omega \in \mathcal{W}(\omega_{n-1})\} = \{-\epsilon_i \mid 1 \leq i \leq n\}, \\
\mathcal{R}_2 &= \{\omega - 2\epsilon \mid \omega \in \mathcal{W}(\omega_{n-2})\} = \{-\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\},
\end{align*}
\]

the union \(\mathcal{R} = \bigcup_{p=-2}^{2} \mathcal{R}_p\) is a root system of type \(B_n\) and the prolongation

\[\mathfrak{g} = \sum_{p=-2}^{2} \mathfrak{g}_p \cong \mathfrak{o}(2n+1, \mathbb{C})\]

with \(\mathfrak{h}_n\) a Cartan subalgebra of dimension \(n\), is by Proposition \cite{73} a maximal (EPFGLA) of \(\mathfrak{m}\), because \(\mathfrak{o}(2n+1, \mathbb{C})\) is simple.

We note that indeed we got

\[\mathcal{R} = \{\pm \epsilon_i \mid 1 \leq i \leq n\} \cup \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}\]

for an orthonormal basis \(\epsilon_1, \ldots, \epsilon_n\) of \(\mathbb{R}^n\). The grading of \(\mathfrak{g}\) could also have been obtained form the cross marked Dynkin diagram (see \cite{9})

\[
\begin{array}{ccccccc}
\circ & - & - & - & - & \circ & \circ \\
\alpha_0 & \alpha_1 & \ldots & \alpha_{n-1} & \alpha_{n-1} & \circ & \circ
\end{array}
\]

with

\[
\alpha_n = e_1, \quad \alpha_i = e_i - e_{i-1}, \quad \text{for } 1 \leq i \leq n-1.
\]

by setting \(\deg(\alpha_n) = 1, \deg(\alpha_i) = 0\) for \(1 \leq i \leq n-1\).

The (EPFGLA) of \(\mathfrak{f}(\mathbb{C}^2)/\mathfrak{f}_3(\mathbb{C}^2)\) is infinite dimensional when \(n=2\), because in this case \(\mathfrak{g}_{-2}\) has dimension 1 (see Corollary \cite{41}). For \(n=2\), the summands \(\mathfrak{f}_{-p}(\mathbb{C}^2)\) (for \(p \geq 1\)) are all irreducible and isomorphic either to the trivial one-dimensional representation on \(\Lambda^2(\mathbb{C}^2) \cong \mathbb{C}\), for \(p\) even, or to the two-dimensional standard representation \(\Lambda^1(\mathbb{C}^2) \cong \mathbb{C}^2\) for \(p\) odd. Therefore we will consider the (FGLA) of the third kind \(\mathfrak{m} = \mathfrak{f}(\mathbb{C}^2)/\mathfrak{f}_3(\mathbb{C}^2)\).

Denote by \(\omega = \frac{1}{2}(e_1 - e_2)\) the fundamental weight for \(\mathfrak{s}l_2(\mathbb{C})\) and take the marker \(\epsilon = \frac{1}{2\sqrt{3}}(e_1 + e_2)\). Then

\[\|\omega + \epsilon\|^2 = \frac{2}{3}, \quad \|2\epsilon\|^2 = \frac{2}{3}, \quad \|\omega + 3\epsilon\|^2 = 2\]

and, by setting

\[
\mathcal{R} = \bigcup_{p=-3}^{3} \mathcal{R}_p, \quad \text{with} \quad \mathcal{R}_0 = \{\pm 2\omega,\}, \\
\mathcal{R}_{\pm 1} = \{\pm \epsilon + \omega, \pm \epsilon - \omega\}, \\
\mathcal{R}_{\pm 2} = \{\pm 2\epsilon\}, \\
\mathcal{R}_{\pm 3} = \{\pm 3\epsilon + \omega, \pm 3\epsilon - \omega\},
\]

because in this case \(\mathfrak{g}_{-2}\) has dimension 1 (see Corollary \cite{41}).
we obtain a root system of type $G_2$. With a 2-dimensional Cartan subalgebra $h_2$, the graded Lie algebra

$$\sum_{p=1}^{3} g_p,$$

with $g_0=\mathfrak{h}_2 \oplus \langle R_0 \rangle$ and $g_p=\langle R_p \rangle$ if $p \neq 0$,

is an (EPFGLA) of an (FGLA) of the third kind. It is the maximal one. Indeed, by Theorem 5.3, the (EPFGLA)’s of $m=\sum_{p=0}^{3} g_p$ are finite dimensional, because in this case $W=\{0\}$. Then, since $g$ is simple, it is maximal by Proposition 7.3. The cross marked Dynkin diagram is

$$\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_1=2\omega, \quad \alpha_2=\epsilon-\omega,
\end{array}$$

with $\deg(\alpha_1)=0$, $\deg(\alpha_2)=1$ see e.g. [12, 15, 40]).

This gradation and those introduced for $B_2$ are compatible only with the split real forms of the complex simple Lie algebras considered above: we obtain structures of (EPFGLA) of type $\mathfrak{gl}_n(\mathbb{R})$ and second kind on $\mathfrak{o}(n, n+1)$ for $n \geq 3$ and of the third kind on the split real form of $G_2$ for $n=2$.

7.1.2. Construction of $C_n$ and $G_2$. Let us consider now the irreducible faithful representation of $\mathfrak{gl}_n(\mathbb{C})$ on the space $S_2(\mathbb{C}^n)$ of degree two symmetric tensors. From §3.1 we know that the maximal (EPFGLA) of type $\mathfrak{gl}_n(\mathbb{C})$ of $S_2(\mathbb{C}^n)$ is finite dimensional if $n \geq 3$. In fact, it is isomorphic to a Lie algebra of type $C_n$. The dominant weight of $S_2(\mathbb{C}^n)$ is $2\omega_1$ and we have $\|2\omega_1\|^2 = 4 - \frac{4}{n}$. We take the marker $\epsilon = \frac{2}{n} \omega_0$. Set

$$\begin{align*}
\mathcal{R}_{-1} &= \{ \omega + \epsilon \mid \omega \in W(2\omega_1) \} = \{(e_i+e_j) \mid 1 \leq i \leq j \leq n\}, \\
\mathcal{R}_0 &= \{ e_i-e_j \mid 1 \leq i < j \leq n \}, \\
\mathcal{R}_1 &= \{ \omega - \epsilon \mid \omega \in W(2\omega_{n-1}) \} = \{-(e_i+e_j) \mid 1 \leq i \leq j \leq n\}.
\end{align*}$$

Then $\mathcal{R} = \bigcup_{p=-1}^{1} \mathcal{R}_p$ is a root system of type $C_n$. With a Cartan subalgebra $h_2$ of dimension $n$ we obtain the maximal (EPFGA) of type $\mathfrak{gl}_n(\mathbb{C})$ of $S_2(\mathbb{C}^n)$ in the form

$$\mathfrak{g} = \sum_{p=-1}^{1} g_p \cong \mathfrak{sp}(n, \mathbb{C}), \quad \text{with } g_0=\mathfrak{h}_2 \oplus \langle R_0 \rangle, \quad g_p=\langle R_p \rangle, \quad \text{for } p \neq 0.$$ 

Indeed,

$$\mathcal{R} = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{2e_i \mid 1 \leq i \leq n\}$$

is the set of roots of $\mathfrak{sp}(n, \mathbb{C})$ and we can obtain the gradation above from its cross marked Dynkin diagram

$$\begin{array}{c}
\alpha_n \\
\alpha_1 \\
\ldots \\
\alpha_2 \\
\alpha_1 = 2\omega, \quad \alpha_2 = \epsilon - \omega,
\end{array}$$

with $\alpha_i = e_i - e_{i+1}$, for $1 \leq i \leq n-1$, $\alpha_n = -2\epsilon_1$, by requiring that $\deg(\alpha_1)=0$ for $1 \leq i \leq n-1$ and $\deg(\alpha_n)=1$.

For $n=2$, we consider the representation of $\mathfrak{gl}_2(\mathbb{C})$ on $S_2(\mathbb{C}^2) \cong V_{2\omega}$, where we denoted by $\omega = (e_1-e_2)/2$ the fundamental weight of $\mathfrak{sl}_2(\mathbb{C})$. The exterior
square $\Lambda^2(S_3(\mathbb{C}^2))$ contains the irreducible representation $V_0=\Lambda^2(\mathbb{C}^2) \cong \mathbb{C}$. We can therefore consider the (FGLA) $m=g_{-1} \oplus g_{-2}$ with $g_{-1} = S_3(\mathbb{R}^2)$ and $g_{-2} = \Lambda^2(\mathbb{C}^2)$. One can check, by using §3 §4 §6 that its maximal effective $gl_2(\mathbb{C})$-prolongation $g$ is finite dimensional. It is in fact a simple Lie algebra of type $G_2$.

Set $\epsilon = \frac{\sqrt{3}}{2}e_0$. Then

$$\|3\omega+\epsilon\|^2 = 6, \quad \|\omega+\epsilon\|^2 = 2, \quad \|2\epsilon\|^2 = 6.$$

We can set

$$\mathcal{R}_{-2} = \{2\epsilon\},$$
$$\mathcal{R}_{-1} = \{\epsilon+\omega, \epsilon+3\omega\},$$
$$\mathcal{R}_0 = \{\pm 2\omega\},$$
$$\mathcal{R}_1 = \{-\epsilon+\omega, -\epsilon+3\omega\},$$
$$\mathcal{R}_2 = \{-2\epsilon\}.$$

The union $\mathcal{R} = \bigcup_{p=-2}^2 \mathcal{R}_p$ is a root system of type $G_2$ and the prolongation

$$g = \sum_{p=-2}^2 g_p, \quad \text{with} \quad g_0 = h_2 \oplus (\mathcal{R}_0), \quad g_p = (\mathcal{R}_p),$$

with $h_2$ a Cartan subalgebra of dimension 2, is by Proposition 7.3 a maximal (EPFGLA) of $m$, because $g$ is simple.

The discussion above only applies to the split real form $gl_n(\mathbb{R})$ of $gl_n(\mathbb{C})$, exhibiting $sp(n, \mathbb{R})$ and the real split form of $G_2$ as (EPGFLA)'s of (FGLA) with structure algebra $gl_n(\mathbb{R})$.

7.1.3. Construction of $D_n$. The next example refers to the representation of $sl_n(\mathbb{C})$ on $V_{\omega_2}=\Lambda^2(\mathbb{C}^n)$, for $n\geq 4$. Since the Dynkin diagram of a simple extension would have a ramification node, all its roots would have the same square length 2. Thus we take the marker $\epsilon = \frac{1}{n}e_0$, so that $\|\omega+\epsilon\|^2=2$, for all $\omega \in \Lambda(\omega_2)$. We set $m=g_{-1} = \Lambda^2(\mathbb{C}^n)$ and

$$\mathcal{R}_{-1} = \{\omega+\epsilon \mid \omega \in \mathcal{W}(\omega_2)\} = \{e_i+e_j \mid 1 \leq i < j \leq n\},$$
$$\mathcal{R}_0 = \{e_i-e_j \mid 1 \leq i \neq j \leq n\},$$
$$\mathcal{R}_1 = \{\omega-\epsilon \mid \omega \in \mathcal{W}(\omega_{n-2})\} = \{-(e_i+e_j) \mid 1 \leq i < j \leq n\}.$$

The union $\mathcal{R} = \bigcup_{p=-2}^2 \mathcal{R}_p$ is a root system of type $D_n$ and the prolongation

$$g = \sum_{p=-2}^2 g_p \approx o(2n, \mathbb{C}), \quad \text{with} \quad g_0 = h_2 \oplus (\mathcal{R}_0), \quad g_p = (\mathcal{R}_p),$$

with $h_n$ a Cartan subalgebra of dimension $n$, is, by Proposition 7.3 a maximal (EPFGLA) of $m$, because $g$ is simple. Indeed,

$$\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$
is the root system of $\mathfrak{o}(2n, \mathbb{C})$, with Dynkin diagram

with $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = -(e_1 + e_2)$. By setting $\deg(e_i) = \frac{1}{2}$ for $1 \leq i \leq n$ we obtain $\deg(\alpha_i) = 0$ for $1 \leq i \leq n-1$, $\deg(\alpha_n) = 1$ and hence the gradation above for $\mathfrak{g}$.

The gradation above is compatible both with the Satake diagram of the split form $\mathfrak{gl}_n(\mathbb{R})$, yielding $\mathfrak{o}(n,n)$, and, for $n=2m$ even, also with $\mathfrak{sl}_m(\mathbb{H})$, which has the Satake diagram

In this case the corresponding real representation $V$ is of quaternionic type. The maximal (EPFGLA) of type $\mathfrak{sl}_m(\mathbb{H})$ of the (FGLA) of the first kind $V$ is then isomorphic to $\mathfrak{o}^*(2m)$, having Satake diagram

7.1.4. Construction of $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$. Let us consider (EPFGLA)'s which are constructed on the representation of $\mathfrak{sl}_n(\mathbb{C})$ on $\Lambda^3(\mathbb{C}^n)$, for $n \geq 6$. We know from §3.1 and §6 that they are finite dimensional. The dominant weight for $\Lambda^3(\mathbb{C}^n)$ is $\omega_3 = e_1 + e_2 + e_3 - \frac{1}{n} e_0$ and a necessary condition to embed $\omega_3 + e_n$, for a suitable marker $e_n$, into a larger root system containing $\{e_i - e_j \mid 1 \leq i \neq j \leq n\}$ is that

$$||\omega_3||^2 = 3 - \frac{9}{n} < 2.$$

Thus the only possible choices are $n=6, 7, 8$. Natural choices are

$$\epsilon_6 = \frac{\sum_{i=1}^{6} e_i}{2 \sqrt{3}}, \quad \epsilon_7 = \sqrt{2} \frac{\sum_{i=1}^{7} e_i}{7}, \quad \epsilon_8 = \frac{\sum_{i=1}^{8} e_i}{8}.$$

Let us set

\[
\begin{align*}
\mathcal{R}^{(6)}_{-2} &= \{2\epsilon_6\}, \\
\mathcal{R}^{(6)}_{-1} &= \{\epsilon_6 + \omega \mid \omega \in \mathcal{W}(\omega_3)\} \\
\mathcal{R}^{(6)}_0 &= \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq 6\}, \\
\mathcal{R}^{(6)}_1 &= \{-\epsilon_6 + \omega \mid \omega \in \mathcal{W}(\omega_{n-3})\} \\
\mathcal{R}^{(6)}_2 &= \{-2\epsilon_6\},
\end{align*}
\]
For all cases we have 
\[ R_{\alpha}^\pm = \{ \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq n \} \],

\[ R_{\alpha}^0 = \{ -\epsilon_i \mid \omega \in \mathcal{W}(\omega_1) \}, \]

\[ R_{\alpha}^2 = \{ -2\epsilon_i + \omega \mid \omega \in \mathcal{W}(\omega_1) \}, \]

\[ R_{\alpha}^{(8)} = \{ 2\epsilon_i + \omega \mid \omega \in \mathcal{W}(\omega_6) \}, \]

\[ R_{\alpha}^{(8)} = \{ \epsilon_i + \omega \mid \omega \in \mathcal{W}(\omega_2) \}, \]

\[ R_{\alpha}^{(8)} = \{ -\epsilon_i \mid 1 \leq i \neq j \leq 8 \}, \]

\[ R_{\alpha}^{(8)} = \{ -\epsilon_i + \omega \mid \omega \in \mathcal{W}(\omega_2) \}, \]

\[ R_{\alpha}^{(8)} = \{ -2\epsilon_i + \omega \mid \omega \in \mathcal{W}(\omega_2) \}, \]

Then one can show that, for each \( n = 6, 7, 8 \), the sets \( \mathcal{R}^{(n)} = \bigcup_{p=-2}^{2} \mathcal{R}_{p}^{(n)} \) are root systems of type \( E_n \), with Dynkin diagrams

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \]

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \]

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \]

\[ \alpha_i = \epsilon_i - \epsilon_{i+1} \]

Accordingly, we obtain on the complex Lie algebras of type \( E_6, E_7, E_8 \) structures of (EPFGGLA)'s of type \( g_{\alpha}(\mathbb{C}) \), for \( 6 \leq n \leq 8 \), by setting

\[ g = \sum_{p=-2}^{2} g_p, \quad \text{with} \quad \begin{cases} g_p = \langle g_{\alpha}^{(n)} \rangle, & \text{for } p = \pm 1, \pm 2, \\ g_0 = b_n \oplus \langle g_{\alpha}^{(n)} \rangle \simeq sl_n(\mathbb{C}), \end{cases} \]

where \( b_n \) is an \( n \)-dimensional Cartan subalgebra.

In all cases we have \( g_{-1} \simeq \Lambda^3(\mathbb{C}^n) \) and \( g_{-2} \simeq \Lambda^5(\mathbb{C}^n) \), with the Lie brackets defined by the exterior product. This gradation is compatible with the non compact real forms \( E_I, E_{II}, E_{III} \) of \( E_6 \) (see below). In these cases \( g_{-1} \simeq \Lambda^3(\mathbb{R}^6) \), while \( [g_0, g_0] \) is simple and of type \( sl_6(\mathbb{R}) \), \( su(3, 3) \) and \( su(1, 4) \), respectively. For \( n = 7, 8 \), the gradation is only consistent with the real split forms of \( E_7 \) and \( E_8 \) yielding the real analogue of the examples above.
7.2. **Structure algebras of type B.** The root system of a complex Lie algebra of type $\mathbf{B}_m$ (isomorphic to $\mathfrak{o}(2n+1, \mathbb{C})$) is

$$\mathcal{R} = \{ \pm e_i \mid 1 \leq i \leq m \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq m \}$$

for an orthonormal basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$ and its Dynkin diagram

```
α_1
   α_2
     ...  ...
    α_{m-1}
   α_m
```

has simple roots that can be chosen to be $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < m$ and $\alpha_m = e_m$. Its fundamental weights are

$$\sigma_1 = e_1, \quad \sigma_2 = e_1 + e_2, \quad \ldots, \quad \sigma_{m-1} = e_1 + \cdots + e_{m-1}, \quad \sigma_m = \frac{1}{2}(e_1 + \cdots + e_m).$$

Its irreducible representation with maximal weight $\sigma_m$ is called its *complex spin representation* and indicated by $S^C_{m+1}$. Its weights

$$\mathcal{W}(\sigma_m) = \left\{ \frac{1}{2}(\pm e_1 \pm \cdots \pm e_m) \right\}$$

are all simple, so that $\dim(S^C_{m+1}) = 2^m$.

We know (see Examples 3.3, 3.8) that all the (EPFLGA) of type $\mathfrak{o}(n, \mathbb{C})$, with $n \geq 2$, or $\mathfrak{co}(n, \mathbb{C})$, with $n \geq 3$, of an (FGLA) of the first kind are finite dimensional. Then this holds also for (FGLA)'s of any finite kind.

7.2.1. *A presentation of the exceptional Lie algebra of type $\mathbf{F}_4$.* Let us take an $m$ with $g_{-1}$ equal to the spin representation $S = S^C_{m+1}$ of $\mathfrak{o}(2m+1, \mathbb{C})$.

To obtain a semisimple (EPFLGA) of type $\mathfrak{o}(2m+1, \mathbb{C})$ of an (FGLA) with $g_{-1} = S$, since its dominant weight is attached to a simple root of length 1, it is necessary to produce a *new length 1 root* by adding a *marker* to the dominant weight $\sigma_m$ of $S$. Since $||\sigma_m||^2 = \frac{m}{2}$, this is possible iff $m \leq 3$. Putting aside the cases $m = 1, 2$, in which the prolongations are $\mathfrak{o}(5, \mathbb{C})$ and $\mathfrak{sp}(3, \mathbb{C})$, respectively, we concentrate on the case $m = 3$, to show that the semisimple (EPFLGA) of type $\mathfrak{o}(7, \mathbb{C})$ of an (FGLA) with $g_{-1}$ equal the 8-dimensional spin representation is the simple complex Lie algebra of type $\mathbf{F}_4$.

With a *marker* $\epsilon$ orthogonal to $\langle e_1, e_2, e_3 \rangle$ and of length $\frac{1}{2}$, we obtain

$$||\sigma_3 + \epsilon||^2 = 1, \quad ||\epsilon_1 + 2\epsilon||^2 = 2, \quad ||2\epsilon||^2 = 2.$$

We note that the vector representation $V_{\epsilon_1} \simeq \mathbb{C}^7$ of $\mathfrak{o}(7, \mathbb{C})$ is an irreducible summand of $\Lambda^2(S)$. Then we can take $m = S \oplus V_{\epsilon_1}$. Set

$$\begin{align*}
\mathcal{R}_{-2} &= \{2\epsilon + \omega \mid \omega \in \mathcal{W}(e_1)\} = \{2\epsilon\} \cup \{2\epsilon \pm e_i \mid 1 \leq i \leq 3\}, \\
\mathcal{R}_{-1} &= \{\epsilon + \omega \mid \omega \in \mathcal{W}(\sigma_3)\} = \{\epsilon + \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)\}, \\
\mathcal{R}_0 &= \{\pm e_i \mid 1 \leq i \leq 3\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 3\}, \\
\mathcal{R}_1 &= \{-\epsilon + \omega \mid \omega \in \mathcal{W}(\sigma_3)\} = \{-\epsilon + \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)\}, \\
\mathcal{R}_2 &= \{-2\epsilon + \omega \mid \omega \in \mathcal{W}(e_1)\} = \{-2\epsilon\} \cup \{-2\epsilon \pm e_i \mid 1 \leq i \leq 3\}. 
\end{align*}$$

We observe that $\mathcal{R} = \bigcup_{p=-2}^2 \mathcal{R}_p$ is a root system of type $\mathbf{F}_4$. 

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*PROLONGATIONS OF (FGLA)*

35
With the four dimensional Cartan algebra \( h_4 \) of \( \mathfrak{so}(7, \mathbb{C}) \), we obtain an (EPFGLA) of \( \mathfrak{m} \) in the form

\[
g = \sum_{p=-2}^{2} g_p, \quad \text{with} \quad g_p = \begin{cases} \langle R_p \rangle, & \text{for } p = \pm 1, \pm 2, \\ h_4 \oplus \langle R_0 \rangle \simeq \mathfrak{so}(7, \mathbb{C}), & \text{for } p = 0. \end{cases}
\]

There are two non compact real forms of \( F_4 \), with Satake diagrams

(FI) \hspace{1cm} (FII)

The 0-degree parts are in both case reductive, with semisimple ideal equal to \( \mathfrak{o}(3,4) \) in the first and to \( \mathfrak{o}(7) \) in the second case. Their spin representations in degree \( \pm 1 \) are then real and isomorphic to \( \Lambda(\mathbb{R}^3) \); the vector representations in degree \( \pm 2 \) are also real and isomorphic to \( \mathbb{R}^7 \).

7.3. **Structure algebras of type \( C \).** Let us consider the complex Lie algebra \( \mathfrak{sp}(m, \mathbb{C}) \), of type \( C_m \). Its root system is

\[
\mathcal{R}_c = \{ \pm 2e_i \mid 1 \leq i \leq m \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq m \}
\]

for an orthonormal basis \( e_1, \ldots, e_m \) of \( \mathbb{R}^m \) and its Dynkin diagram

\[
\alpha_1 - \cdots - \alpha_{m-1} - \alpha_m
\]

has simple roots that can be chosen to be \( \alpha_i = e_i - e_{i+1} \) for \( 1 \leq i < m \) and \( \alpha_m = 2e_m \). Its weights lattice is the \( \mathbb{Z} \)-module \( \mathbb{Z}^m \) in \( \mathbb{R}^m \), with fundamental weights \( \gamma_j = \sum_{i=1}^{j} e_i \), for \( j = 1, \ldots, m \). We know that the maximal (EPFGLA) of type \( \mathfrak{sp}(m, \mathbb{C}) \) of the (FGLA) of the first kind \( C^{2m} \) are infinite dimensional (see Example 3.10 or [22, p.10]). On the other hand, if we choose another faithful irreducible representation \( V \) of \( \mathfrak{sp}(m, \mathbb{C}) \), then its maximal (EPFGLA) of type \( \mathfrak{csp}(m, \mathbb{C}) \) is finite dimensional by Proposition 3.13

Let us take for instance \( V = V_{\gamma_m} \subset \Lambda^m(\mathbb{C}^{2m}) \). Then \( \mathfrak{sp}(m, \mathbb{C}) \) acts on \( V \) as an algebra of transformations that keep invariant the bilinear form on \( V \) that can be obtained from the exterior product of elements of \( \Lambda^m(\mathbb{C}^{2m}) \). If \( m \) is even, this is a nondegenerate symmetric bilinear form and the finite dimension of the maximal prolongation can be also checked by using Examples 3.3 and 3.8.

When \( m \) is odd, we can use the exterior product on \( \Lambda^m(\mathbb{C}^{2m}) \) to define a Lie product on \( V \), yielding an (FGLA) \( m = V \oplus \Lambda^2(\mathbb{C}^{2m}) \) of the second kind, which has a finite dimensional maximal (EPFGLA) of type \( \mathfrak{csp}(m, \mathbb{C}) \). The fundamental weight \( \gamma_m \) is attached to the long root \( 2e_m \). In order to be able to find a marker \( \gamma \) to embed \( \gamma_m + \gamma \) into the root system of a simple Lie algebra, we need that \( \| \gamma \|^2 = m < 4 \), i.e. that \( m \leq 3 \). The case \( m = 3 \) leads to another presentation of the exceptional Lie algebra \( F_4 \).
7.3.1. The exceptional Lie algebra of type $\mathbf{F}_4$. We take the marker $\epsilon$ as a unit vector orthogonal to $e_1, e_2, e_3$. We note that

$$
\|\gamma_3+\epsilon\|^2=4, \quad \|e_1+\epsilon\|^2 = 2, \quad \|2\epsilon\|^2 = 4
$$

and set

$$
\mathcal{R} = \{2\epsilon\}, \quad \mathcal{R}_1 = \{\epsilon, \omega \mid \omega \in \mathcal{W}(\gamma_3)\} = \{\epsilon \pm e_1 \pm e_2 \pm e_3\}, \quad \mathcal{R}_0 = \{\pm 2e_i \mid 1 \leq i \leq 3\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 3\},
$$

$$
\mathcal{R}_1 = \{\pm \epsilon, \omega \mid \omega \in \mathcal{W}(\gamma_3)\} = \{\pm \epsilon e_1 e_2 e_3\}, \quad \mathcal{R}_2 = \{-2\epsilon\}.
$$

Then

$$
\mathcal{R} = \bigcup_{p=-2}^{2} \mathcal{R}_p = \{\pm 2e_i \mid 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \{\pm e_1 e_2 e_3 e_4\},
$$

is a root system of type $\mathbf{F}_4$. We have, with a 4-dimensional Cartan subalgebra $\mathfrak{h}$,

$$
\mathfrak{g} = \sum_{p=-2}^{2} \mathfrak{g}_p, \quad \text{with} \quad \mathfrak{g}_p = \begin{cases} \langle \mathcal{R}_p \rangle, & \text{for } p=\pm 1, \pm 2, \\ \mathfrak{h}_4 \oplus \langle \mathcal{R}_0 \rangle \simeq \mathfrak{o}(7, \mathbb{C}) \oplus \mathbb{C}, & \text{for } p=0. \end{cases}
$$

We have $\dim(\mathfrak{g}_0)=22$, $\dim(\mathfrak{g}_1)=14$, $\dim(\mathfrak{g}_2)=1$.

This is the maximal (EPFGLA) of an (FGLA) of the second kind. The cross marked Dynkin diagram associated to $\mathfrak{g}$ is

with simple roots $\alpha_1 = e_1-e_2$, $\alpha_2 = e_2-e_3$, $\alpha_3 = 2e_1$, $\alpha_4 = e_4-e_1-e_2-e_3$. The gradation is obtained by setting $\deg(e_i)=0$ for $1 \leq i \leq 3$ and $\deg(e_4)=1$.

Only the split real form $\mathbf{F}_4$ is compatible with this grading, yielding a real equivalent of the complex case.

7.4. Structure algebras of type $\mathbf{D}$. The diagram $\mathbf{D}_m$ (we assume $m \geq 4$) corresponds to the orthogonal algebra $\mathfrak{o}(2m, \mathbb{C})$. Its root system is defined, in an orthonormal basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$, by

$$
\mathcal{R}(\mathbf{D}_m) = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq m\},
$$

and we consider the corresponding Dynkin diagram

with $\alpha_m = e_m$.
with \( \alpha_i = e_i - e_{i+1} \) for \( 1 \leq i \leq m-1 \) and \( \alpha_m = e_{m-1} + e_m \). The maximal root is \( e_1 + e_m \) and the fundamental weights are

\[
\omega_j = \sum_{i=1}^{j} e_i \text{ for } 1 \leq j \leq m-2, \quad \omega_{m-1} = \frac{1}{2} (e_1 + \cdots + e_{m-1} - e_m), \quad \omega_m = \frac{1}{2} (e_1 + \cdots + e_m).
\]

The last two are the dominant weights of two complex spin representations \( S^C \), with opposite chiralities and simple weights

\[
\mathcal{W}(\omega_{m-1}) = \mathcal{W}'(m) = \left\{ \frac{1}{2} \sum_{i=1}^{m} a_i e_i \mid a_i = \pm 1, \ a_1 \cdots a_m = -1 \right\},
\]

\[
\mathcal{W}(\omega_m) = \mathcal{W}'_s(m) = \left\{ \frac{1}{2} \sum_{i=1}^{m} a_i e_i \mid a_i = \pm 1, \ a_1 \cdots a_m = 1 \right\}.
\]

We call \( V^C_4 = V_{w_1} \cong C^{2m} \) the complex vector representation.

### 7.5. Real Spin representations of real Lie algebras of type D

Let us first consider complex (EPFGLA)’s with structure algebra \( \mathfrak{o}(2m, \mathbb{C}) \) in which \( \mathfrak{g}_- \) is a spin representation. A necessary condition for finding a marker \( \epsilon \) for which \( \omega_{m-1} + \epsilon \) or \( \omega_m + \epsilon \) could be embedded into the root system of a simple Lie algebra is that \( \|\omega_{m-1}\|^2 = \|\omega_m\|^2 = \frac{m}{4} < 2 \), i.e. that \( m = 4, 5, 6, 7 \).

When \( m = 4 \), the spin and the vector representations are isomorphic and the maximal (EPFGLA) of type \( \mathfrak{o}(8, \mathbb{C}) \) of the abelian Lie algebra \( V^C_4 \cong S^C_4 \) is just the orthogonal algebra \( \mathfrak{o}(10, \mathbb{C}) \) (see Example 3.8).

For \( m = 5, 6, 7 \) we obtain the three exceptional Lie algebras of type \( \mathbb{E} \). Denote by \( e_{m+1} \) a vector of \( \mathbb{R}^{m+1} \), orthogonal to \( e_1, \ldots, e_m \) and with \( \|e_{m+1}\|^2 = \frac{8-m}{4} \). We define the sets

\[
\mathcal{R}^{(6)}_{m-1} = \{2\epsilon_1 + w \mid w \in \mathcal{W}'_s(5)\},
\]

\[
\mathcal{R}^{(6)}_{-1} = \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 5\},
\]

\[
\mathcal{R}^{(6)}_{1} = \{\epsilon_1 + w \mid w \in \mathcal{W}'_s(5)\}
\]

\[
\mathcal{R}^{(7)}_{-2} = \{2\epsilon_2\},
\]

\[
\mathcal{R}^{(7)}_{-1} = \{\epsilon_1 + w \mid w \in \mathcal{W}'_s(6)\},
\]

\[
\mathcal{R}^{(7)}_{0} = \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 6\},
\]

\[
\mathcal{R}^{(7)}_{1} = \{-\epsilon_1 + w \mid w \in \mathcal{W}'_s(6)\},
\]

\[
\mathcal{R}^{(7)}_{2} = \{-2\epsilon_2\},
\]

\[
\mathcal{R}^{(8)}_{-2} = \{2\epsilon_2 \pm \epsilon_i \mid 1 \leq i \leq 7\},
\]

\[
\mathcal{R}^{(8)}_{-1} = \{\epsilon_8 + w \mid w \in \mathcal{W}'_s(7)\},
\]

\[
\mathcal{R}^{(8)}_{0} = \{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 7\},
\]

\[
\mathcal{R}^{(8)}_{1} = \{-\epsilon_8 + w \mid w \in \mathcal{W}'_s(7)\},
\]

\[
\mathcal{R}^{(8)}_{2} = \{-2\epsilon_8 \pm \epsilon_i \mid 1 \leq i \leq 7\},
\]

Then \( \mathcal{R}^{(6)} = \bigcup_{p=1}^{5} \mathcal{R}^{(6)}_p, \mathcal{R}^{(7)} = \bigcup_{p=2}^{7} \mathcal{R}^{(7)}_p, \mathcal{R}^{(8)} = \bigcup_{p=2}^{8} \mathcal{R}^{(8)}_p, \) are root systems of type \( \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8 \), respectively.
Then we obtain on the complex Lie algebras of type $E_n$ structures of (EPFGLA) for structure algebras of type $D_{n-1}$ and their spin representations, of the form

$$g = \sum g_p, \text{ with } g_0 = \mathfrak{o}(2n-2, \mathbb{C}), \ g_p = \langle R^{(n)}_p \rangle \text{ for } p \neq 0.$$  

We note that $g_{\pm 1}$ are for $n=6,8$ spin representations with opposite chirality, while for $E_7$ the representations $g_{\pm 1}$ have equal chiralities. For $E_8$ the $g_{\pm 2}$ representations are vectorial.

Since the complexification of the spin representation that we found in these cases are by construction irreducible over $\mathbb{C}$, these complex (EPFGLA) must be complexifications of real spin representations. We recall that the spin representations of $\mathfrak{o}(p, q)$ are (see e.g. [13, p.103])

$$\begin{cases} 
\text{real} & \text{if } q-p \equiv 0, 1, 7 \pmod{8}, \\
\text{complex} & \text{if } q-p \equiv 2, 6 \pmod{8}, \\
\text{quaternionic} & \text{if } q-p \equiv 3, 4, 5 \pmod{8}.
\end{cases}$$

Thus, for the real forms of $\mathfrak{o}(10, \mathbb{C})$, the semisimple ideal of $g_0$ can only be $\mathfrak{o}(5, 5) \mathfrak{o}(1, 9)$. Corresponding, we obtain the real forms of $E_6$ having Satake diagrams

(El) \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \\
| \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \\
\alpha_1 \hspace{1cm} \alpha_3 \hspace{1cm} \alpha_4 \hspace{1cm} \alpha_5 \hspace{1cm} \alpha_6 \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hs
For \( EV \) the semisimple part of the degree zero subalgebra is \( o(6, 6) \) and for \( EV I \) it is \( o(2, 10) \). In both cases the spin representations involved are real.

We recall that the real form \( o^*(12) \) is obtained from \( o(12, \mathbb{C}) \) by the conjugation which is described on the roots by

\[
\bar{e}_{2h-1} = e_{2h}, \text{ for } 1 \leq h \leq 3.
\]

Accordingly, \( so^*(12) \) has both a real and a quaternionic spin representation.

The real forms of \( o(14, \mathbb{C}) \) admitting real spin representations are \( o(7, 7) \) and \( o(3, 11) \) (\( so^*(14) \) has a complex spin representation). In this case, \( g_{\pm 1} \) are spin representations with opposite chiralities and \( g_{\pm 2} \) dual copies of the vector representation of \( o(14, \mathbb{C}) \), which correspond to real vector representations \( V_7^{\pm} = \mathbb{R}^{14} \). Thus we obtain all non compact real Lie algebras of type \( E_8 \) as \((EPFGLA)'s\) of type \( o(p, q) \) for \((FGLA)'s\) \( m \) whose \( g_{-1} \) component is a real spin representation. Their Satake diagrams are

\[\begin{align*}
\text{(EVII)} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
This is possible only for $m=4$. Thus we assume $m=4$ and take markers $\epsilon_\pm$ with

$$||\epsilon_\pm||^2 = 1, \quad (\epsilon_+|\epsilon_-) = -\frac{1}{2}.$$ 

Then we define

$$
\begin{align*}
\mathcal{R}^{(6)}_{-2} &= \{\epsilon_+ + \epsilon_+ \pm \epsilon_i | 1 \leq i \leq 4\}, \\
\mathcal{R}^{(6)}_{-1} &= \{\epsilon_+ + w | w \in \mathcal{W}_+(4)\} \cup \{\epsilon_+ + w | w \in \mathcal{W}_-(4)\}, \\
\mathcal{R}^{(6)}_0 &= \{\pm \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq 4\}, \\
\mathcal{R}^{(6)}_1 &= \{-\epsilon_- + w | w \in \mathcal{W}_+(4)\} \cup \{-\epsilon_- + w | w \in \mathcal{W}_-(4)\}, \\
\mathcal{R}^{(6)}_2 &= \{-\epsilon_- - \epsilon_+ \pm \epsilon_i | 1 \leq i \leq 4\},
\end{align*}
$$

One can check that $\mathcal{R}^{(6)} = \bigcup \mathcal{R}^{(6)}_p$ are root systems of type $E_6$, and

$$\mathfrak{g} = \sum_{p=0}^{2} \mathfrak{g}_p,$$

with

$$
\begin{align*}
\mathfrak{g}_0 &= \mathfrak{h}_0 \oplus \mathcal{R}^{(6)}_0, \quad \dim \mathfrak{h}_0 = 6, \\
\mathfrak{g}_p &= \langle \mathcal{R}^{(6)}_p \rangle, \quad p \neq 0,
\end{align*}
$$

is the maximal (EPFGLA) of type $\mathfrak{g}_0$, with $\mathfrak{g}_0$ reductive and $[\mathfrak{g}_0, \mathfrak{g}_0] \simeq \mathfrak{o}(8, \mathbb{C})$, of a (FGLA) with $\mathfrak{g}_1 \simeq S^x_4(4) \oplus S^y_4(4)$.

The real forms of $\mathfrak{o}(8, \mathbb{C})$ having a complex spin representation are $\mathfrak{o}(3, 5)$ and $\mathfrak{o}(1, 7)$, corresponding to the two real non compact forms of $E_6$ having Satake diagrams

\[
\begin{align*}
\text{(EII)} & \quad \begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{(EIII)} & \quad \begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \\
\end{array}
\end{align*}
\]

(the irreducible complex representation is represented in the diagram by two white nodes joined by an arrow). Note that the summands on degree $\pm 2$ in $\mathfrak{g}$ are the vector representations $V^\mathbb{C}_4 \simeq \mathbb{C}^8$ in the complex and $V^\mathbb{R}_4 \simeq \mathbb{R}^8$ in the real cases.

7.7. **Quaternionic spin representations of real Lie algebras of type D.**

Let $\eta_1, \eta_2$ be an orthonormal basis of $\mathbb{R}^2$. We denote by $\{\pm (\eta_1 - \eta_2)\}$ the root system of $\mathfrak{sl}_2(\mathbb{C})$, and by $\gamma = \frac{1}{2}(\eta_1 - \eta_2)$ its fundamental weight.

Then $\{\pm (\eta_1 - \eta_2)\} \cup \{\pm \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq m\}$ is the root system of $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{o}(2m, \mathbb{C})$ and, taking the usual lexicographic orders, its fundamental weights are $\gamma, \omega_1, \ldots, \omega_m$.

The complexification of an irreducible spin representation of quaternionic type lifts to an irreducible complex representation of $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{o}(2m, \mathbb{C})$ whose dominant weight is either $\gamma + \omega_{m-1}$, or $\gamma + \omega_m$. Let us assume it is $\gamma + \omega_m$. A necessary condition to find a semisimple complex Lie algebra which is an (EPFGLA) of type $\mathfrak{g}_0$, with $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{o}(2m, \mathbb{C})$ of an $m$ with $\mathfrak{g}_1$ equal
to the irreducible $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{o}(2m, \mathbb{C})$-module with dominant weight $\gamma + \omega_m$ is that

$$||\gamma + \omega_m||^2 = \frac{1}{2} + \frac{m}{4} < 2,$$

i.e. that $m=4, 5$. With $\epsilon_4 = \frac{1}{2}(\eta_1 + \eta_2)$, $\epsilon_5 = \frac{1}{\sqrt{6}}(\eta_1 + \eta_2)$, set

$$\mathcal{R}^{(4)}_m = \{2\epsilon_4\},$$
$$\mathcal{R}^{(4)}_{-2} = \{\epsilon_4 \pm \eta + w \mid w \in \mathcal{W}_r(4)\},$$
$$\mathcal{R}^{(4)}_0 = \{\pm(\eta_1 - \eta_2)\} \cup \{\pm \eta_i \pm \epsilon_j \mid 1 \leq i < j \leq 4\},$$
$$\mathcal{R}^{(4)}_{-1} = \{-\epsilon_4 \pm \eta + w \mid w \in \mathcal{W}_r(4)\},$$
$$\mathcal{R}^{(4)}_2 = \{-2\epsilon_4\},$$
$$\mathcal{R}^{(5)}_m = \{2\epsilon_5 \pm \epsilon_1 \mid 1 \leq i \leq 5\},$$
$$\mathcal{R}^{(5)}_{-1} = \{\epsilon_5 \pm \eta + w \mid w \in \mathcal{W}_r(5)\},$$
$$\mathcal{R}^{(5)}_0 = \{\pm(\eta_1 - \eta_2)\} \cup \{\pm \eta_i \pm \epsilon_j \mid 1 \leq i < j \leq 5\},$$
$$\mathcal{R}^{(5)}_{-2} = \{-\epsilon_5 \pm \eta + w \mid w \in \mathcal{W}_r(5)\},$$
$$\mathcal{R}^{(5)}_2 = \{-2\epsilon_5 \pm \epsilon_1 \mid 1 \leq i \leq 5\}.$$

Then $\mathcal{R}^{(4)}_m = \bigcup_{p=-2}^{2} \mathcal{R}_p^{(4)}$ is a root system of type $\mathbf{D}_6$ and $\mathcal{R}^{(5)}_m = \bigcup_{p=-2}^{2} \mathcal{R}_p^{(5)}$ a root system of type $\mathbf{E}_7$. Set

$$\mathfrak{g}^{(m)} = \sum_{p=-2}^{2} \mathfrak{g}_p^{(m)}, \quad \text{with} \quad \begin{cases} 
\mathfrak{g}_0^{(m)} = \mathfrak{b}_m \oplus (\mathcal{R}_0^{(m)}), & \dim \mathfrak{b}_m = m+2, \\
\mathfrak{g}_p^{(m)} = (\mathcal{R}_p^{(m)}), & p \neq 0,
\end{cases}$$

for $m=4, 5$. Then $\mathfrak{g}_0^{(m)} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{o}(2m) \oplus (\sigma)$, where $\sigma$ satisfies $[\sigma, X] = p \cdot X$ for $X \in \mathfrak{g}_p$, and the $\mathfrak{g}^{(m)}$ are maximal (EPFGLA) of type $\mathfrak{g}_0^{(m)}$.

For $m=4$ each of the summands $\mathfrak{g}_4^{(4)}$ consists of two copies of $S^C_+(4)$ and each of $\mathfrak{g}_{\pm 2}^{(4)}$ is the scalar representation.

For $m=5$ each of the summands $\mathfrak{g}_5^{(4)}$ consists of two copies of $S^C_+(5)$ and each of $\mathfrak{g}_{\pm 2}^{(4)}$ is the complex vector representation $V_m^C \simeq \mathbb{C}^{2m}$.

The only real form of $\mathfrak{o}(8, \mathbb{C})$ having a quaternionic spin representation is $\mathfrak{o}(2, 6)$. Its prolongation is the real form of $\mathfrak{g}^{(4)}$, isomorphic to $\mathfrak{o}(4, 8)$, whose graded structure is represented by the cross-marked Satake diagram

```
\begin{tikzpicture}
  \node (a1) at (0,0) {$\alpha_1$};
  \node (a2) at (1,0) {$\alpha_2$};
  \node (a3) at (2,0) {$\alpha_3$};
  \node (a4) at (3,0) {$\alpha_4$};
  \node (a5) at (4,0) {$\alpha_5$};
  \node (a6) at (5,0) {$\alpha_6$};
  \node (x) at (4,-1) {$\times$};

  \draw[thick] (a1) -- (a2) -- (a3) -- (a4) -- (a5);
  \draw[thick] (a1) -- (a6);

  \path (a1) -- (x);
\end{tikzpicture}
```
The only real form of \( \mathfrak{o}(10, \mathbb{C}) \) having a quaternionic spin representation is \( \mathfrak{o}(3, 7) \) and the corresponding real form of \( \mathfrak{g}^{(5)} \) is of type \( \mathbf{EVI} \), with cross-marked Satake diagram

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \bullet & \times & \circ & \bullet \\
\end{array}
\]

\[\alpha_1 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7\]

\[\alpha_2\]

REFERENCES

1. J.F. Adams, Z. Mahmud, and M. Mimura, *Lectures on exceptional Lie groups*, Chicago Lectures in Mathematics, University of Chicago Press, 1996.
2. D. Alekseevsky and L. David, *Prolongation of Tanaka structures: an alternative approach*, Ann. Mat. Pura Appl. (4) 196 (2017), no. 3, 1137–1164.
3. A. Altomani, C. Medori, and M. Nacinovich, *The CR structure of minimal orbits in complex flag manifolds*, J. Lie Theory 16 (2006), no. 3, 483–530.
4. S. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ 13 (1962), 1–34.
5. H. Aslaksen, *Determining summands in tensor products of lie algebra representations*, Journal of Pure and Applied Algebra 93 (1994), no. 2, 135–146.
6. A. O. Barut and A. J. Bracken, *The remarkable algebra \( \mathfrak{so}^*(2n) \), its representations, its Clifford algebra and potential applications*, J. Phys. A23 (1990), 641.
7. N. Bourbaki, *Commutative algebra*, Springer-Verlag, Berlin, 1989.
8. __________, *Lie groups and Lie algebras. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989, Translated from the French, Reprint of the 1975 edition.
9. __________, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.
10. R.N. Cahn, *Semi-simple Lie algebras and their representations*, Frontiers in Physics, vol. 59, Benjamin/Cummings, 1984.
11. E. Cartan, *Les groupes de transformations continus, infinis, simples*, Annales scientifiques de l’École Normale Supérieure 26 (1909), 93–161 (fre).
12. __________, *Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. Sup. (3) 27 (1910), 109–192.
13. Edited Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten, *Quantum fields and strings. a course for mathematicians*, Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, American Mathematical Society, 1999.
14. E. B. Dynkin, *Maximal subgroups of the classical groups*, Trudy Moskov. Mat. Obšč 1 (1952), 39–166 (russian).
15. M. Golubitsky and B. Rothschild, *Primitive subalgebras of exceptional Lie algebras*, Pacific J. Math. 39 (1971), no. 2, 371–393.
16. V.W. Guillemin, *Infinite dimensional primitive Lie algebras*, J. Differential Geom. 4 (1970), no. 3, 257–282.
17. V.W. Guillemin, D. Quillen, and S. Sternberg, *The classification of the complex primitive infinite pseudogroups*, Proceedings of the National Academy of Sciences of the United States of America 55 (1966), no. 4, 687–690.
18. __________, *The classification of the irreducible complex algebras of infinite type*, Journal d’Analyse Mathématique 18 (1967), no. 1, 107–112.
19. V.W. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964), 16–47.
20. R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.
21. V. G. Kac, *Simple graded Lie algebras of finite height*, Funkcional. Anal. i Priložen **1** (1967), no. 4, 82–83.
22. S. Kobayashi, *Transformations groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 70, Springer-Verlag, Berlin, 1972.
23. S. Kobayashi and T. Nagano, *On filtered Lie algebras and geometric structures I*, Journal of Mathematics and Mechanics **13** (1964), no. 5, 875–907.
24. S. Marini, C. Medori, M. Nacinovich, and A. Spiro, *A theorem on filtered Lie algebras and its applications*, Bull. Amer. Math. Soc. **70** (1964), no. 3, 401–403.
25. S. Marini, C. Medori, M. Nacinovich, and A. Spiro, *On a fundamental theorem of Weyl-Cartan on G-structures*, J. Math. Soc. Japan **17** (1965), no. 1, 84–101.
26. S. Marini, C. Medori, M. Nacinovich, *Classification of semisimple Levi-Tanaka algebras*, Ann. Mat. Pura Appl. (4) **174** (1998), 285–349.
27. T. Morimoto and N. Tanaka, *The classification of the real primitive infinite Lie algebras*, J. Math. Kyoto Univ. **10** (1970), no. 2, 207–243.
28. C. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press, Oxford University Press, New York, 1993, Oxford Science Publications.
29. S. Shnider, *The classification of real primitive infinite lie algebras*, J. Differential Geom. **4** (1970), no. 1, 81–89.
30. S. Sternberg, *Lectures on differential geometry*, Chelsea Publ. Co., New York, 1983, First edition: Prentice Hall, Inc. Englewood Cliffs, N.J., 1964.
31. N. Tanaka, *On generalized graded Lie algebras and geometric structures. I*, J. Math. Soc. Japan **19** (1967), 215–254.
32. J.C. Tougeron, *Ideaux de fonctions differentiables*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge, Springer Berlin Heidelberg, 1972.
33. B. Warhurst, *Tanaka prolongation of free Lie algebras*, Geometriae Dedicata **130** (2007), no. 1, 59–69.
34. R.L. Wilson, *Irreducible Lie algebras of infinite type*, Proc. Amer. Math. Soc. (1971), no. 29, 243–249.
35. K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Adv. Stud. Pure Math. **22** (1993), 413–494, Progress in differential geometry.

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