A calculus for ideal triangulations of three-manifolds with embedded arcs

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March 29, 2022

Abstract: Refining the notion of an ideal triangulation of a compact three-manifold, we provide in this paper a combinatorial presentation of the set of pairs \((M, \alpha)\), where \(M\) is a three-manifold and \(\alpha\) is a collection of properly embedded arcs. We also show that certain well-understood combinatorial moves are sufficient to relate to each other any two refined triangulations representing the same \((M, \alpha)\). Our proof does not assume the Matveev-Pergallini calculus for ideal triangulations, and actually easily implies this calculus.

Keywords: 3-manifold, triangulation, presentation, calculus.

MSC (2000): 57Q15.

Introduction

A combinatorial presentation of a class of topological objects (viewed up to the appropriate equivalence relation) is a set of finite combinatorial objects, such that each combinatorial object defines (say “presents”) a unique topological object, and each topological object is presented by at least one combinatorial object. A calculus for a combinatorial presentation is a finite set of moves on the combinatorial objects, such that two combinatorial objects present the same topological object if and only if they are related to each other by a finite sequence of moves in the given set.

Combinatorial presentations are fundamental tools for studying 3-manifolds and links, and for constructing invariants. They translate a topological problem into a combinatorial and, maybe, a simpler one. For instance, an invariant on the class of topological objects can be defined on the combinatorial objects, checking that it is preserved by the moves.

For 3-manifolds, there are several different types of presentations, e.g. (ideal) triangulations, Heegaard diagrams, surgery (on links), and spines. In the present work we concentrate on the pairs \((M, \alpha)\), where \(M\) is a compact connected 3-manifold with non-empty boundary and \(\alpha = \{\alpha^{(1)}, \ldots, \alpha^{(n)}\}\) is a (possibly empty) collection of disjoint arcs properly embedded in \(M\) (viewed up to simultaneous isotopy). We provide a presentation of such pairs and we describe the corresponding calculus. The objects of the presentation are the marked ideal triangulations of the pair \((M, \alpha)\), that is the ideal triangulations of \(M\) that contain as edges all the arcs in \(\alpha\), and the moves of the calculus are the moves
on ideal triangulations (i.e. Matveev-Piergallini moves) which do not kill edges belonging to $\alpha$ (such moves will be called \textit{admissible}).

The calculus for marked ideal triangulations is not new: in fact it has been used by Baseilhac and Benedetti (see \cite{2,3,4}) in the prove that the so-called \textit{quantum hyperbolic invariants} (QHI) for links in 3-manifolds equipped with flat $PSL(2,\mathbb{C})$-bundles are well defined. They derived this calculus from the Matveev-Piergallini one \cite{8,13}, as refined by Turaev and Viro in \cite{14}. They have also used the generalization to the setting of marked ideal triangulations of a result of Makovetskii \cite{7}. We will give a new proof of the calculus (for marked ideal triangulations), which is instead self-contained, see Section 2. Actually, our proof specializes to a new proof of the Matveev-Piergallini calculus. Although our proof is quite long, it is conceptually very simple: in fact it uses only easy results on triangulations and easy topological arguments. For the sake of completeness, we will also describe a sketch of the derivation of the calculus for marked triangulations from the Matveev-Piergallini one, see Subsection 2.5.

The generalized Makovetskii result states that, if two marked ideal triangulations of a pair $(M, \alpha)$ are given, then they are dominated, as far as some \textit{positive} admissible moves are concerned, by another marked ideal triangulation. An admissible move is positive if it increases the number of tetrahedra. In Section 3 we provide the details of the proof of this refinement, and, in Subsection 4.1 we describe the relationship between marked ideal triangulations and links in 3-manifolds.

The initial motivation of the present paper was the remark, due to Frigerio and Petronio \cite{6}, that marked ideal triangulations naturally arise in the study of \textit{complete finite-volume orientable hyperbolic 3-manifolds with geodesic boundary}. In Subsection 4.2 we will describe how this relationship arises.

1 Definitions

From now on, unless explicitly stated, $M$ will be a compact connected 3-manifold with non-empty boundary, and $\alpha = \{\alpha^{(1)}, \ldots, \alpha^{(n)}\}$ will be a (possibly empty) collection of disjoint arcs properly embedded in $M$, viewed up to simultaneous isotopy.

1.1 Standard spines and moves

In this subsection we recall the definition of spine and we describe some moves.

\textbf{Standard spines} A \textit{quasi-standard} polyhedron $P$ is a finite, connected, and purely 2-dimensional polyhedron with singularities of stable nature (\textit{i.e.} triple lines and points where 6 non-singular components meet). Such a polyhedron is called \textit{standard} if it is cellularized by singularity (depending on dimension, we call the components \textit{vertices}, \textit{edges}, and \textit{regions}). A quasi-standard sub-polyhedron $P$ of $M$ contained in $\text{Int}(M)$ is called a \textit{spine} of $M$ if the manifold $M \setminus P \cong \partial M \times [0,1)$. Each spine of $M$ is always viewed up to isotopy. For the sake of completeness, let us recall that, if $M$ is closed, the boundary is created by puncturing $M$ (\textit{i.e.} by considering $M$ minus a ball).
MP-move  Any two (standard) spines of $M$ can be transformed into each other by certain well-understood moves. Let us start from the move shown in Fig. 1-left, which is called MP-move. Such a move will be called positive if it increases (by one) the number of vertices, and negative otherwise. Note that, if we apply an MP-move to a spine of $M$, the result will be another spine of $M$. It is already known (but it will also follow from our Corollary 2.2, after the work of Matveev [8] and Piergallini [13], that any two standard spines of the same $M$ with at least two vertices can be transformed into each other by MP-moves (see Theorem 2.3).

V-move  If one of the two spines of $M$ (we want to transform into each other) has just one vertex, another move is required. The move shown in Fig. 2-left is called V-move. Note that if we apply such a move to a spine of $M$, the result will be another spine of $M$. As above, we have positive and negative V-moves. Note that 3 different positive V-moves can be applied at each vertex.

If a positive V-move is applied to a spine with at least two vertices, the V-move is a composition of MP-moves. In Fig. 3 we show the three positive and the one negative MP-moves giving the V-move.
**Figure 4:** The L-move on a spine (left) and on the dual ideal triangulation (right).

**Figure 5:** Each positive L-move is a composition of V- and MP-moves (case where $R_1$ has more than one vertex).

**L-move** A generalization of the V-move is the L-move, see Fig. 4 left. As above, we have positive and negative L-moves. As opposed to the V-move, this move is non-local, so it must be described with some care. A positive L-move, which increases by two the number of vertices, is determined by an arc $\gamma$ properly embedded in a region $R$ of $P$. The move acts on $P$ as in Fig. 4 left, but, to define its effect non-ambiguously, we must specify which pairs of regions, out of the four regions incident to $R$ at the endpoints of $\gamma$, will become adjacent to each other after the move. This is achieved by noting that $R$ is a disc, so its regular neighborhood in $M$ is a product, and we can choose for $R$ a transverse orientation. Using it, at each endpoint of $\gamma$ we can tell from each other the two regions incident to $R$ as being an upper and a lower one, and we can stipulate that the two upper regions will become incident after the move (and similarly for the lower ones). Obviously, a positive L-move leads to a (standard) spine $P'$ of $M$.

For the negative case the situation is more complicated. A negative L-move can lead to a non-standard spine. If $R_1$ and $R_2$ are contained in the same region, after the negative L-move, the “region” $R$ would not be a disc. To avoid this loss of standardness, we will call negative L-moves only those preserving standardness. So a negative L-move can be applied only if the regions $R_1$ and $R_2$ are different. With this convention, if we apply an L-move to a spine of $M$, the result will be another spine of $M$.

Each positive L-move is a composition of V- and MP-moves. In Fig. 5 we show the one positive V-move and the pairs of (one positive and one negative) MP-moves giving the L-move. Obviously, to apply such moves, $R_1$ must have at least two vertices. If $R_1$ has only one vertex, then $R_2$ has at least two vertices (because $P'$ is standard); so we can take the symmetric picture. For future reference, we note that, if $R_1$ has only one vertex, we can obtain the L-move also as a composition of only one V- and one pair of MP-moves, as shown in Fig. 6.
Figure 6: Each positive L-move is a composition of V- and MP-moves (case where $R_1$ has only one vertex).

Figure 7: The B-move on a spine (left) and on the dual ideal triangulation (right).

Figure 8: The C-move on a spine (left) and the corresponding arch (right).

**B-move** Now we describe the B-move (shown in Fig. 7 left). As above, we have positive and negative B-moves. This move is quite different from the previous ones, because if we apply a positive B-move to a spine $P$ of $M$, the result will be a spine $P_B$ of $M \setminus B^3$ (where $B^3$ is a 3-ball with closure embedded in $M$). So it is obvious that a B-move cannot be a composition of V- and MP-moves. By definition of spine, we have that $M \setminus P_B$ is the disjoint union of $\partial M \times [0, 1)$ and $B^3 \cup (\partial B^3 \times [0, 1))$. The ball $B = B^3 \cup (\partial B^3 \times [0, 1))$ will be called proper ball.

**C-move** In the end, we describe the C-move, see Fig. 8 left. As above, we have positive and negative C-moves. This move is very similar to the B-move, but, if we apply a positive C-move to a spine of $M$, we obtain another spine of the same $M$. In fact, each positive C-move is a composition of V- and MP-moves: the V-move and the (four) MP-moves are shown in Fig. 9. Note also that 12 different positive C-moves can be applied at each vertex.

We will call arch the configuration shown in Fig. 8 right, created by a C-move. Let us compare the spine $P_C$, obtained from a spine $P$ via a C-move, with the spine $P_B$, obtained from $P$ via a B-move (applied at the same vertex). They are different only for the presence of the arch, which joins two different regions ($R_1$ and $R_2$) of the spine $P_B$. Note also that, after the C-move, the proper ball $B$, created by the B-move, is connected to $\partial M \times [0, 1)$ by the cavity of the arch.
In the rest of the paper we will always regard $M$ as being fixed and we will only consider spines and moves embedded in $M$, without explicit mention.

### 1.2 Ideal triangulations

In this subsection we recall the definition of loose triangulation and ideal triangulation, eventually defining the marked ideal triangulations (and spines) of a pair $(M, \alpha)$ and the moves on them.

**Loose and ideal triangulations** A loose triangulation of a polyhedron $|P|$ is a triangulation $\mathcal{P}$ of $|P|$ in a weak sense, namely self-adjacencies and multiple adjacencies are allowed. For any manifold $M$ (as above), let us denote by $\hat{M}$ the space obtained from $M$ by collapsing to a point each component of $\partial M$. An ideal triangulation of a manifold $M$ (as above) is a partition $\mathcal{T}$ of $\text{Int}(M)$ into open cells of dimensions 1, 2, and 3, induced by a loose triangulation $\hat{\mathcal{T}}$ of the space $\hat{M}$ such that the vertices of $\hat{\mathcal{T}}$ are precisely the points of $\hat{M}$ corresponding to the components of $\partial M$. The quotient of $\partial M$ will be denoted by $\partial \hat{M}$. Note that $\hat{M} \setminus \partial \hat{M}$ can be identified with $\text{Int}(M)$. As for spines, each ideal triangulation of $M$ is always viewed up to isotopy.

**Duality** We show now the well-known fact that ideal triangulations exist for each $M$. It turns out that there exists a natural bijection between standard spines and ideal triangulations of a 3-manifold. Given an ideal triangulation $\mathcal{T}$, the corresponding standard spine $P$ is just the 2-skeleton of the dual cellularization, as illustrated in Fig. 10. The inverse passage is also explicit, but it is a little more difficult; so we omit its description. The ideal triangulation $\mathcal{T}$ and the spine $P$ are said to be dual. As said above, every $M$ has standard spines, so dually it has ideal triangulations.
Figure 10: Portion of spine dual to a tetrahedron of an ideal triangulation.

We show in Figg. 1-right, 2-right, 4-right, and 7-right the MP-, V-, L-, and B-moves, respectively, on a spine in terms of the dual ideal triangulations (we have omitted the dual version of the C-move because of the complexity of the picture). In the sequel we will intermingle the spine and the ideal triangulation viewpoints.

Marked ideal triangulations (and spines) Recall that $\alpha$ is a collection of disjoint arcs properly embedded in a manifold $M$. A marked ideal triangulation of the pair $(M, \alpha)$ is a pair $(T, \beta)$, where $T$ is an ideal triangulation of $M$ and $\beta = \{\beta^{(1)}, \ldots, \beta^{(n)}\}$ is a collection of edges of $T$ (simultaneously) isotopic to $\alpha = \{\alpha^{(1)}, \ldots, \alpha^{(n)}\}$. The quotient of $\beta = \{\beta^{(1)}, \ldots, \beta^{(n)}\}$ in $\hat{T}$ will be denoted by $\hat{\beta} = \{\hat{\beta}^{(1)}, \ldots, \hat{\beta}^{(n)}\}$, and the pair $(\hat{T}, \hat{\beta})$ will be said marked loose triangulation corresponding to $(T, \beta)$. With a little abuse of terminology, in the sequel we will say that the edges in $\beta$ and $\hat{\beta}$ belong to $\alpha$.

Using duality, we can give a natural definition of marked spine of a pair $(M, \alpha)$ as a pair $(P, \tilde{\beta})$, where $P$ is the spine dual to a marked ideal triangulation $(T, \beta)$ of the pair $(M, \alpha)$ and $\tilde{\beta} = \{\tilde{\beta}^{(1)}, \ldots, \tilde{\beta}^{(n)}\}$ is the collection of the regions of $P$ dual to the $\beta^{(i)}$'s. With a little abuse of notation, we will drop the tilde, writing only $\beta^{(i)}$ instead of $\tilde{\beta}^{(i)}$, and we will say that the regions $\beta^{(i)}$ also belong to $\alpha$.

Existence of marked ideal triangulations By duality, to prove that every pair $(M, \alpha)$ has marked ideal triangulations, it is enough to prove that it has marked ideal spines. So we prove that $M$ has a spine such that $\alpha$ is isotopic to the collection of the edges dual to $n$ different regions. Let $N(\alpha) = \bigcup_{i=1}^{n} N(\alpha^{(i)})$ be a regular neighborhood of $\alpha$, let $Q$ be a spine of $M \setminus N(\alpha)$. Note that we have a retraction $\pi$ of $M \setminus N(\alpha)$ onto $Q$. For $i = 1, \ldots, n$, let $D^{(i)}$ be a 2-disc properly embedded in $N(\alpha^{(i)})$, embedded in $\text{Int}(M)$, and intersecting $\alpha^{(i)}$ transversely in one point. Now, we can suppose that, by projecting the $\partial D^{(i)}$'s to $Q$ along $\pi$, we obtain “half-open” annuli $\partial D^{(i)} \times [0, 1)$. Up to isotopy, we can also suppose that each $\pi(\partial D^{(i)})$ intersects the singularity of $P$, and that $\pi(\cup_{i=1}^{n} \partial D^{(i)})$ is transversal to the singularity and to itself. Let us define $P$ as the union of the polyhedron $Q$, the discs $D^{(i)}$, and the annuli $\partial D^{(i)} \times [0, 1)$. Obviously, $P$ is the desired spine: in fact, $P$ is a (standard) spine of $M$ and each $\alpha^{(i)}$ coincides with the edge dual to the region $D^{(i)} \cup (\partial D^{(i)} \times [0, 1))$ of $P$.

Admissible moves We will now discuss an extension of the MP-, V-, L-, and C-moves to the context of marked ideal triangulations. Given a marked ideal triangulation $(T, \beta)$ of $(M, \alpha)$, the idea is to consider a move from $T$ to $T'$
admissible if there is a $\beta'$ such that $(T', \beta')$ is a marked ideal triangulation of $(M, \alpha)$, and $\beta'$ coincides with $\beta$ except “near” the portion of $T$ affected by the move. As it turns out, admissibility depends on $\beta$. Moreover, $\beta'$ is sometimes not unique.

By duality, we describe the moves on spines to refer to simpler pictures, but we invite the reader to figure out the dual ideal triangulation pictures. Let $(P, \beta)$ be the marked spine dual to $(T, \beta)$. We describe the moves one by one.

**MPa-move** A positive MP-move from $P$ to $P'$ is admissible whatever $\beta$, and $\beta'$ consists of the same regions as $\beta$ (i.e. the newborn triangular region does not belong to $\beta'$); the move from $(P, \beta)$ to $(P', \beta')$ is called positive MPa-move. A negative MP-move from $P$ to $P'$ is admissible if it is the inverse of a positive MPa-move: namely, the triangular region disappearing during the move must not belong to $\beta$, and $\beta'$ consists of the same regions as $\beta$; the move from $(P, \beta)$ to $(P', \beta')$ is called negative MPa-move. See Fig. 1.

**Va-move** A positive V-move from $P$ to $P'$ is admissible whatever $\beta$, and $\beta'$ consists of the same regions as $\beta$ (i.e. the two newborn little regions do not belong to $\beta'$); the move from $(P, \beta)$ to $(P', \beta')$ is called positive Va-move. A negative V-move from $P$ to $P'$ is admissible if it is the inverse of a positive Va-move: namely, the two little regions disappearing during the move must not belong to $\beta$, and $\beta'$ consists of the same regions as $\beta$; the move from $(P, \beta)$ to $(P', \beta')$ is called negative Va-move. See Fig. 2.

Now, recall that, if there are at least two vertices, a positive V-move is a composition of MP-moves, see Fig. 3, the “admissible” version of this fact is not so obvious but it is true. Namely, if there are at least two vertices, a positive Va-move is a composition of MPa-moves. To prove this, it is enough to note that a positive Va-move is a composition of MP-moves (see again Fig. 3), that the negative MP-move of the sequence eliminates a region created by a previous positive MPa-move (so the region does not belong to $\beta$), and that the position of the $(\beta')^{(i)}$’s after the MPa-moves is the same as after the Va-move.

**La-move** For the L-moves, the situation is more complicated. A positive L-move from $P$ to $P'$ is admissible whatever $\beta$, but $\beta'$ is not uniquely determined. We follow the notation of Fig. 4. We have two cases depending on whether $R$ belongs to $\beta$ or not. If $R$ does not belong to $\beta$, then $\beta'$ consists of the same regions as $\beta$ (i.e. $R_1$, $R_2$, and the newborn little region $D$ do not belong to $\beta'$). In such a case, the move from $(P, \beta)$ to $(P', \beta')$ is called positive La-move. If $R$ belongs to $\beta$, the situation is a little ambiguous: $R$ is divided in two regions, and both of them “are isotopic to $R$” (i.e. the dual edges of $R_1$ and $R_2$ are both isotopic to the dual edge of $R$). If we define $\beta'_1$ as $(\beta \setminus \{R\}) \cup \{R_1\}$ and $\beta'_2$ as $(\beta \setminus \{R\}) \cup \{R_2\}$, we have two admissible L-moves underlying the original L-move: one from $(P, \beta)$ to $(P', \beta'_1)$ and one from $(P, \beta)$ to $(P', \beta'_2)$. Also both these moves are called positive La-moves. Note that the choice of the region, between $R_1$ and $R_2$, is included in the move.

A negative L-move from $P$ to $P'$ is admissible if it is the inverse of a positive La-move. Necessarily, the little region $D$ disappearing during the move must not belong to $\beta$, and only one region between $R_1$ and $R_2$ can belong to $\beta$. Now, we have two cases: if both $R_1$ and $R_2$ do not belong to $\beta$, then $\beta'$ consists of
the same regions as $\beta$; otherwise, if one region $R_i$ (between $R_1$ and $R_2$) belongs to $\beta$, then $\beta'$ is equal to $(\beta \setminus \{R_i\}) \cup \{R\}$. In both cases, the move from $(P, \beta)$ to $(P', \beta')$ is called negative La-move.

Now, recall that each L-move is a composition of V- and MP-moves (see Figs. 5 and 6). As above, we show that each positive La-move is a composition of Va- and MPa-moves. If $R$ does not belong to $\beta$, the situation is analogous to that of Va-move, so we omit its treatment. On the contrary, we suppose that $R_n$ belongs to $\beta'$ (the case for $R_1$ is symmetric). The V- and MP-moves shown in Figs. 5 and 6 (we have two cases depending on whether $R_1$ has one vertex or more) are all admissible, and the (positive) Va-move leaves just $R_2$ in $\beta'$.

**Ca-move** A positive C-move from $P$ to $P'$ is admissible whatever $\beta$, and $\beta'$ consists of the same regions as $\beta$ (i.e. the four newborn regions do not belong to $\beta'$); the move from $(P, \beta)$ to $(P', \beta')$ is called positive Ca-move. See Fig. 8. Note that $R_1$ is joined to $R_2$, so, if $R$ belongs to $\beta$, then the region containing $R_1$ and $R_2$ belongs to $\beta'$.

A negative C-move from $P$ to $P'$ is admissible if it is the inverse of a positive Ca-move: namely, the four regions (included the disc of the arch) disappearing during the move must not belong to $\beta$, and $\beta'$ consists of the same regions as $\beta$. The move from $(P, \beta)$ to $(P', \beta')$ is called negative Ca-move.

As above, it is easy to see that each Ca-move is a composition of Va- and MPa-moves.

**Ba-move** For the B-moves, the situation is quite different because such moves change the homeomorphism class of the manifold. A B-move from $P$ to $P'$ will be considered admissible both if it is positive, or if it is negative and the four regions disappearing do not belong to $\beta$. In such a case, $\beta'$ consists of the same regions as $\beta$. The move from $(P, \beta)$ to $(P', \beta')$ is called Ba-move (positive or negative, respectively). See Fig. 7.

From now on, since a marked ideal triangulation $(T, \beta)$ is a pair while an ideal triangulation $T$ is not, then, for the sake of shortness, we will omit the word “marked” (also for spines and loose triangulations) unless the difference is not clear.

2 The calculus

The main result of this paper is the following.

**Theorem 2.1.** Two marked ideal triangulations of a pair $(M, \alpha)$ can be obtained from each other via a sequence of Va- and MPa-moves.

Recalling that, if there are at least two tetrahedra, each Va-move is a composition of MPa-moves, we obtain the following corollary of Theorem 2.1.

**Corollary 2.2.** Two marked ideal triangulations of $(M, \alpha)$ with at least two tetrahedra can be obtained from each other via a sequence of MPa-moves only.
As a particular case we obtain the Matveev-Piergallini theorem.

**Theorem 2.3 (Matveev-Piergallini).** Two spines of $M$ can be obtained from each other via a sequence of $V$- and $MP$-moves. If moreover both spines have at least two vertices, then they can be obtained from each other via a sequence of $MP$-moves only.

The idea of the proof of Theorem 2.1 consists of the following steps:

- a “desingularization” of the two marked ideal triangulations, say $(T_1, \beta_1)$ and $(T_2, \beta_2)$, via $Ba$, $Va$, and $MPa$-moves (leading to $(T_1', \beta_1')$ and $(T_2', \beta_2')$, respectively);
- an application of the relative version of the Alexander theorem to relate $(T_1', \beta_1')$ and $(T_2', \beta_2')$ via $Ba$- and $MPa$-moves;
- an elimination of each $Ba$-move by substituting it with a $Ca$-move.

We first recall the relative version of the Alexander theorem, and then we describe each of the three steps.

### 2.1 Alexander’s theorem

As said above, our proof relies on the relative version of Alexander’s theorem, so we recall it (the proof is quite easy and can be found in [14]). Let us consider a simplex $\sigma$ of a polyhedron $|P|$ with a (non-loose) triangulation $\mathcal{P}$. Let us define a move on the triangulation $\mathcal{P}$: the substitution of the closed star, $\text{clst}(\sigma)$, of $\sigma$ with the cone on $\partial \text{clst}(\sigma)$ with respect to a point in the interior of $\sigma$ will be called $A$-move; the inverse of an $A$-move will be also called an $A$-move. Note that an $A$-move does not change the homeomorphism class of $|P|$, but only the triangulation $\mathcal{P}$. The following theorem states that $A$-moves are enough to obtain all the triangulations of $|P|$ from any given one, leaving fixed a sub-polyhedron $|Q|$.

**Theorem 2.4.** Let $|P|$ be a dimensionally homogeneous polyhedron and let $|Q|$ be a sub-polyhedron of $|P|$. Then two triangulations of $|P|$, whose restrictions to $|Q|$ coincide, can be obtained from each other via a sequence of $A$-moves which do not change the triangulation of $|Q|$.

### Reduction to B- and MP-moves

Now we prove a modification of a result due to Pachner (Theorem 4.14 of [11]), that he stated only for manifolds. Let us call *singular manifold with boundary* a finite polyhedron $|P|$ such that the link of every point (of $|P|$) is a surface with (possibly empty) boundary. Such a space is the generalization with boundary of the so called *singular manifolds*. In fact, we have an obvious definition of the *boundary* $\partial |P|$ of $|P|$ as the 2-dimensional sub-polyhedron of $|P|$ made of the closure of the triangles lying in only one tetrahedron. Obviously, in $|P|$ there are only a finite number of points having link different from the 2-sphere or the 2-disk. We have the following corollary of Theorem 2.4.

**Proposition 2.5.** Let $|P|$ be a singular manifold with boundary and let $|Q|$ be a sub-polyhedron of $|P|$ containing $\partial |P|$. Then two triangulations of $|P|$, whose restrictions to $|Q|$ coincide, can be obtained from each other via a sequence of $B$- and $MP$-moves, which do not change the common triangulation of $|Q|$.
Proof of 2.5. By Theorem 2.4, the two triangulations can be obtained from each other via a sequence of A-moves which do not change the triangulation of $|Q|$. To conclude the proof, we show that each A-move in this sequence is a composition of B- and MP-moves which do not change the triangulation of $|Q|$. There are four different types of A-move depending on the dimension of the simplex $\sigma$ the A-move is applied to.

$\dim(\sigma) = 0$. This case is obvious; in fact, $\sigma$ is a vertex, so $\text{clst}(\sigma)$ is already the cone on $\partial \text{clst}(\sigma)$ with respect to $\sigma$, and the A-move is the identity.

$\dim(\sigma) = 1$. Here $\sigma$ is an edge, so the A-move on $\sigma$ “divides” $\sigma$ adding a vertex as shown in Fig. 11. Consider the open star, $\text{star}(\sigma)$, of $\sigma$ shown in Fig. 11-left. Note that $\text{star}(\sigma)$ contains at least three tetrahedra; we describe the case for four tetrahedra, other cases being similar. The A-move is the composition of the moves shown in Fig. 12: one positive B-move, two positive MP-moves, and one negative MP-move.

$\dim(\sigma) = 2$. In Fig. 13 we show that the A-move on a triangle is a composition of one positive B-move and one positive MP-move.

$\dim(\sigma) = 3$. The A-move is already a B-move.

Finally, note that all the B- and MP-moves described above do not change the common triangulation of $|Q|$. 

Figure 11: The A-move on the edge $\sigma$ (with four tetrahedra in star($\sigma$)).

Figure 12: The A-move on an edge is a composition of B- and MP-moves.

Figure 13: The A-move on a triangle is a composition of B- and MP-moves.
2.2 Desingularization

Let \((\mathcal{T}, \beta)\) be an ideal triangulation of a pair \((M, \alpha)\). As said above, the idea is to eliminate the singularities of the loose triangulation \((\hat{\mathcal{T}}, \hat{\beta})\), via Ba-, Va-, and MPa-moves, to be able to apply Proposition 2.5. We will see that we cannot eliminate all the singularities, because we cannot eliminate the edges belonging to \(\hat{\beta}\). Since \(\hat{\mathcal{T}}\) is a loose triangulation (of \(\hat{M}\)), there could be a singular edge of \(\hat{\mathcal{T}}\) with coinciding endpoints; such an edge will be called loop.

Proposition 2.6. Let \((\mathcal{T}, \beta)\) be an ideal triangulation of a pair \((M, \alpha)\). Then there exists an ideal triangulation \((\mathcal{T}', \beta')\) of \((\tilde{M} \cup B_k\alpha)\), where the \(B_k\)'s are 3-balls disjoint from each other and from \(\alpha\), such that the following facts hold.

1. \((\mathcal{T}', \beta')\) is obtained from \((\mathcal{T}, \beta)\) via Ba-, Va-, and MPa-moves.
2. The loose triangulation \((\hat{\mathcal{T}}, \hat{\beta}')\) has only the following types of singularities:
   a. an edge \((\hat{\beta}')^{(i)}\) which is a loop,
   b. a pair of edges \((\hat{\beta}')^{(i)}\) sharing both the endpoints,
   c. a pair of edges (giving a multiple adjacency) in \(\text{clst}((\hat{\beta}')^{(i)})\) if \((\hat{\beta}')^{(i)}\) is a loop.
3. Each \((\hat{\beta}')^{(i)}\) has a neighborhood \(N((\hat{\beta}')^{(i)})\) such that:
   a. if \((\hat{\beta}')^{(i)}\) is not a loop, \(N((\hat{\beta}')^{(i)})\) is made of exactly three tetrahedra;
   b. if \((\hat{\beta}')^{(i)}\) is a loop, \(N((\hat{\beta}')^{(i)})\) is the cone on a triangle \(\theta\), where \(\theta\) is triangulated as shown in Fig. 14, the endpoints of the cone on the barycentre \(b\) of \(\theta\) are identified together, and the loop \((\hat{\beta}')^{(i)}\) is just this edge with identified endpoints.
4. \(N((\hat{\beta}')^{(i)}) \cap N((\hat{\beta}')^{(j)}) = (\hat{\beta}')^{(i)} \cap (\hat{\beta}')^{(j)}\) for each \(i \neq j\), and \(N((\hat{\beta}')^{(i)}) \cap \partial\hat{M} = (\hat{\beta}')^{(i)} \cap \partial\hat{M}\) for \(i = 1, \ldots, n\).

Proof of 2.6. The loose triangulation \((\hat{\mathcal{T}}, \hat{\beta})\) has different types of singularity: we eliminate the singularities type by type, being careful not to create any singularity of the types already eliminated. Note that we need to analyze only the singularities for tetrahedra, because both a singular triangle and a singular edge are contained in a singular tetrahedron. There are 6 different types of singularity for tetrahedra. For the sake of shortness, we continue calling \((\mathcal{T}, \beta)\) also the triangulations obtained during the proof, also if they are actually different from \((\mathcal{T}, \beta)\).
Self-adjacency along triangles

The tetrahedron is shown in Fig. 15 left; the Ba-move eliminating the self-adjacency is shown in Fig. 15 right. Note that no new self-adjacency along triangles has been created.

Self-adjacency along edges

This case is more complicated than the previous one: for each tetrahedron the number of edges which are identified together can vary between 2 and 6. An easy induction on the maximal number of edges identified together in a tetrahedron and on the number of tetrahedra having such a maximal number of identifications reduces the number of cases to two.

1. If two edges which are identified together do not share any vertex (in the unfolded version of the tetrahedron), a positive Ba-move is enough to eliminate the singularity, see Fig. 16.

2. If two edges which are identified together share a vertex (in the unfolded version of the tetrahedron) the situation is slightly more difficult. Let us start by calling $T$ the tetrahedron. Note that the tetrahedron $T'$, attached to $T$ along the triangle containing the two identified edges, is different from $T$, because we have already eliminated the self-adjacencies of tetrahedra along triangles. So a positive Ba-move and a positive MPa-move can be applied to eliminate the self-adjacency, see Fig. 17.

Note that no new self-adjacency along either triangles or edges has been created.

Multiple adjacency along triangles or edges

The situation is analogous to the case of self-adjacencies along triangles or edges, respectively; so it can be treated similarly.
Before continuing desingularization, we modify the loose triangulation obtained after the first part of the process to “isolate” each edge \( \hat{\beta}^{(i)} \) of \( \hat{\beta} \). Namely, we apply Ba- and MPa-moves to obtain point 4 of the statement, i.e., \( \mathcal{N}(\hat{\beta}^{(i)}) \cap \mathcal{N}(\hat{\beta}^{(j)}) = (\hat{\beta}^{(i)}) \cap (\hat{\beta}^{(j)}) \) for each \( i \neq j \), and \( \mathcal{N}(\hat{\beta}^{(i)}) \cap \partial M = (\hat{\beta}^{(i)}) \cap \partial M \) for \( i = 1, \ldots, n \). The situation is similar to desingularization: we eliminate the intersections between two \( \text{clst}(\hat{\beta}^{(i)})'s \) and between each \( \text{clst}(\hat{\beta}^{(i)}) \) and \( \hat{\partial} M \) step by step, being careful not to add any intersection of the types already eliminated. First we eliminate the intersections between each \( \text{clst}(\hat{\beta}^{(i)}) \) and \( \hat{\partial} M \). If \( \text{clst}(\hat{\beta}^{(i)}) \cap \hat{\partial} M \) contains a vertex \( v \) different from the endpoints of the edge \( \hat{\beta}^{(i)} \), then we perform the moves already described to eliminate the self-adjacency of tetrahedra along edges (second case); so \( v \) belongs no more to \( \text{clst}(\hat{\beta}^{(i)}) \cap \hat{\partial} M \).

Let us consider now the intersection between two \( \text{clst}(\hat{\beta}^{(i)})'s \). They may share (out of the intersection between the edges \( \hat{\beta}^{(i)} \) tetrahedra, triangles, edges and vertices (different from the endpoints of the edges \( \hat{\beta}^{(i)} \)).

- **Tetrahedra**: if two \( \text{clst}(\hat{\beta}^{(i)})'s \) share a tetrahedron, we note that the two \( \hat{\beta}^{(i)} \) belong to one tetrahedron, so we perform the moves already described to eliminate self-adjacency of tetrahedra along edges.

- **Triangles**: if the common simplex is a triangle, we perform the move already used to eliminate multiple adjacency of tetrahedra along triangles.

- **Edges**: if the common simplex is an edge, we perform the moves already used to eliminate multiple adjacency of tetrahedra along edges.

- **Vertices**: if two \( \text{clst}(\hat{\beta}^{(i)})'s \) share a vertex (different from the endpoints of the edges \( \hat{\beta}^{(i)} \)), we perform the moves already described to eliminate the self-adjacency of tetrahedra along edges (second case).

Note that all the moves described above are admissible, and that no new singularity of the types already eliminated has been created. Let us continue now with desingularization.

**Self-adjacency along vertices**  If two vertices of a tetrahedron are identified together, let us call \( e \) the edge which is a loop (if there is more than one edge like \( e \), we repeat the procedure). There are two cases depending on whether the edge \( e \) belongs to \( \hat{\beta} \) or not.

**First case**: \( e \not\in \hat{\beta} \)  Consider the unfolded version of \( \text{clst}(e) \): the case for four tetrahedra is shown in Fig. 11-left. We know that \( \text{clst}(e) \) contains at least three tetrahedra, because we have already eliminated self-adjacencies and multiple adjacencies of tetrahedra along triangles. The idea is now to “divide” the edge \( e \) by adding a vertex, as shown in Fig. 11. The situation is analogous to that of the proof of Proposition 2.5 when the case of \( \dim(\sigma) = 1 \) is analyzed; the only difference is that now some boundary faces of \( \text{clst}(e) \) could be glued together, but this does not matter: we can repeat the same B- and MP-moves, “dividing” the edge \( e \), as shown in Fig. 12. We conclude by noting that each move is admissible: the first three are positive and the last one eliminates the edge \( e \) which does not belong to \( \hat{\beta} \). Note also that no new singularity of the types already eliminated has been created.
**Figure 18:** How to simplify \( \text{clst}(\hat{\beta}^{(i)}) \) so to have only three tetrahedra in it (case of four tetrahedra in \( \text{clst}(\hat{\beta}^{(i)}) \)). The endpoints of \( \hat{\beta}^{(i)} \) are identified together.

**Second case:** \( e \in \hat{\beta} \). For the sake of clarity, let us call \( \hat{\beta}^{(i)} \) the edge \( e \).

Note that we cannot eliminate the singularity: in fact we cannot eliminate the edge \( \hat{\beta}^{(i)} \), so each tetrahedron in \( \text{clst}(\hat{\beta}^{(i)}) \) always has \( \hat{\beta}^{(i)} \) as an edge and it is always singular. But we will modify a neighborhood of \( \hat{\beta}^{(i)} \) to obtain point 3b of the statement. Recall that \( \text{clst}(\hat{\beta}^{(i)}) \) \( \setminus (\hat{\beta}^{(i)} \cap \partial M) \) and \( \text{clst}(\hat{\beta}^{(j)}) \) \( \setminus (\hat{\beta}^{(j)} \cap \partial M) \) are disjoint for each \( j \neq i \). First we will modify \( \text{clst}(\hat{\beta}^{(i)}) \) via Ba-, Va-, and MPa-moves to have that \( \text{clst}(\hat{\beta}^{(i)}) \) is made of exactly three tetrahedra; then we will modify these tetrahedra to obtain point 3b of the statement.

Let us describe the first modification of \( \text{clst}(\hat{\beta}^{(i)}) \). Note that \( \text{clst}(\hat{\beta}^{(i)}) \) cannot be made of one or two tetrahedra because we have already eliminated self-adjacencies and multiple adjacencies of tetrahedra along triangles. So let us suppose that \( \text{clst}(\hat{\beta}^{(i)}) \) is made of at least four tetrahedra and let us modify the loose triangulation to have that \( \text{clst}(\hat{\beta}^{(i)}) \) is made of three tetrahedra. For the sake of clarity, in Fig. [18] we have shown only the case of four tetrahedra in \( \text{clst}(\hat{\beta}^{(i)}) \): the other cases are analogous. We apply a positive La-move (which is a composition of Va- and MPa-moves), choosing to leave in \( \hat{\beta} \) the edge whose star is made of three tetrahedra; we eliminate the multiple adjacency created by the La-move with a positive Ba-move; we eliminate the singularity of the edge \( e' \) (“parallel” to \( \hat{\beta}^{(i)} \)) created by the La-move as we have done above (\( e' \notin \hat{\beta} \)).

Let us pass to the second modification of \( \text{clst}(\hat{\beta}^{(i)}) \), which is now made of three tetrahedra. Consider the unfolded version of \( \text{clst}(\hat{\beta}^{(i)}) \): it can be seen as a triangulation, say \( \mathcal{X} \), of the 3-ball, see Fig. [19] left. Let \( \mathcal{X}' \) be another triangulation of the 3-ball such that:

- \( \mathcal{X} \) and \( \mathcal{X}' \) coincide on the boundary of the 3-ball and on the edge \( \hat{\beta}^{(i)} \);
- \( \mathcal{X}' \) appears, near \( \hat{\beta}^{(i)} \), as in Fig. [19] right;
- any two boundary faces of \( \mathcal{X}' \) do not belong to the same tetrahedron.

It is very easy to find such an \( \mathcal{X}' \). Now, \( \mathcal{X} \) and \( \mathcal{X}' \) have in common the boundary and the edge \( \hat{\beta}^{(i)} \), so we can apply Proposition 2.5 to obtain \( \mathcal{X}' \) from \( \mathcal{X} \) via B- and MP-moves not involving both the edge \( \hat{\beta}^{(i)} \) and the boundary. Repeating these moves on the folded version of \( \mathcal{X} \) contained in \( T \), we substitute it with a folded version of \( \mathcal{X}' \) using B- and MP-moves which are admissible because they have support in the folded version of \( \mathcal{X} \) and do not involve the edge \( \hat{\beta}^{(i)} \).

Now a neighborhood of \( \hat{\beta}^{(i)} \), say \( \mathcal{N}(\hat{\beta}^{(i)}) \) appears as in Fig. [19] bottom. Note that \( \mathcal{N}(\hat{\beta}^{(i)}) \) is the cone on the triangle \( \theta \) shown in Fig. [14] where the endpoints of the cone on the barycentre \( b \) are identified together, that \( (\hat{\beta}^{(i)}) \) is just this edge with identified endpoints, and that no new singularity of the types already eliminated has been created.
The old clst(\(\hat{\beta}^{(i)}\)) is modified so that the new \(N(\hat{\beta}^{(i)})\) is contained in the old clst(\(\hat{\beta}^{(i)}\)) (shown transparent). We do not show the whole of \(X'\): we show only how it appears near \(\hat{\beta}^{(i)}\). The endpoints of \(\hat{\beta}^{(i)}\) are identified together.

**Multiple adjacency along vertices** The situation is analogous to the case of self-adjacency along vertices, but there are some differences to point out. The idea is to “divide” one of the edges giving the singularity, so the moves to apply are those applied to eliminate self-adjacency along vertices when \(e \notin \hat{\beta}\). But there are two exceptions.

1. We cannot “divide” the edges belonging to \(\hat{\beta}\) so we cannot eliminate the singularity created by two edges of \(\hat{\beta}\) sharing both the endpoints.

2. If an edge \((\hat{\beta})^{(i)}\) (belonging to \(\hat{\beta}\)) is a loop, then we do not divide any of the edges belonging to the closed star of \(\hat{\beta}^{(i)}\), because such an edge has in its closed star a loop (the edge \(\hat{\beta}^{(i)}\)) and the moves described above would create a new multiple adjacency.

For the other cases we can eliminate the multiple adjacency as we have done for self-adjacencies along vertices with \(e \notin \hat{\beta}\), because both the moves are admissible and we do not add any of the singularities of the types already eliminated.

Finally, let us deal with the two exceptions.

1. For each edge \(\hat{\beta}^{(i)}\) which is not a loop, we modify clst(\(\hat{\beta}^{(i)}\)) to have that it is made of exactly three tetrahedra, as we have done above for the first modification of clst(\(\hat{\beta}^{(i)}\)) for the \(\hat{\beta}^{(i)}\)'s which are loops.

2. We do not operate on the edges belonging to the closed star of the \(\hat{\beta}^{(i)}\)'s which are loops.
Conclusion. Repeating the moves described above on the ideal triangulation $(\mathcal{T}, \beta)$ of the pair $(M, \alpha)$, we obtain, via Ba-, Va-, and MPa-moves, an ideal triangulation $(\mathcal{T}', \beta')$ of $(M \setminus \cup B_k, \alpha)$, where the $B_k$’s are 3-balls disjoint from each other and from $\alpha$. We have eliminated almost all the singularities of $\mathcal{T}$, but there are three types of singularity we cannot eliminate (those due to $\alpha$). These three types of singularity are exactly those described in point 2 of the statement. The check that $(\mathcal{T}', \beta')$ is the desired ideal triangulation is straightforward, so we leave it to the reader.

2.3 Application of the Alexander theorem

Let us state (and prove) now a first result, which is a weak version of Theorem 2.1.

Proposition 2.7. Two marked ideal triangulations of a pair $(M, \alpha)$ can be obtained from each other via a sequence of Ba-, Va-, and MPa-moves, such that the negative Ba-moves do not eliminate the spherical boundary components of $\partial M$.

Proof of 2.7. Let $(T_1, \beta_1)$ and $(T_2, \beta_2)$ be two ideal triangulations of $(M, \alpha)$. Let us apply Proposition 2.6 to each $(T_i, \beta_i)$ obtaining $(\hat{T}_i, \hat{\beta}_i)$. Recall that each $(\hat{T}_i, \hat{\beta}_i)$ is obtained from the corresponding $(T_i, \beta_i)$ via Ba-, Va-, and MPa-moves, that each $(\hat{\beta}_i)$ has a particular neighborhood $\mathcal{N}(\hat{\beta}_i)$, and that the loose triangulations $(\hat{T}_i, \hat{\beta}_i)$ are almost desingularized (the singularities are contained in the open neighborhood $\text{Int}(\mathcal{N}(\hat{\beta}_i))$). Moreover, recall that the Ba-move does not involve the spherical boundary components of $\partial M$. Obviously, since we have $\mathcal{N}(\hat{\beta}_j) \cap \mathcal{N}(\hat{\beta}_i) = (\hat{\beta}_j) \cap (\hat{\beta}_i)$ for each $j \neq k$, and $\mathcal{N}(\hat{\beta}_j) \cap \partial M = (\hat{\beta}_j) \cap \partial M$ for $j = 1, \ldots, n$, we can suppose, up to isotopy, that $\mathcal{N}(\hat{\beta}_1)$ and $\mathcal{N}(\hat{\beta}_2)$ coincide.

The strategy will now be to prove that $(T_1', \beta_1')$ and $(T_2', \beta_2')$ are obtained from each other via Ba- and MPa-moves. To do this, we will apply Proposition 2.6 to $\hat{T}_1 \setminus (\cup \text{Int}(\mathcal{N}(\hat{\beta}_1))) = \hat{T}_2 \setminus (\cup \text{Int}(\mathcal{N}(\hat{\beta}_2)))$. Since the singularities of the loose triangulations $\hat{T}_i$ are contained in the $\text{Int}(\mathcal{N}(\hat{\beta}_i))$’s (see Proposition 2.4), the triangulations $\hat{T}_i \setminus (\cup \text{Int}(\mathcal{N}(\hat{\beta}_i)))$ are actually non-loose. Moreover, the two $\hat{T}_i \setminus (\cup \text{Int}(\mathcal{N}(\hat{\beta}_i)))$’s coincide on the boundary and on $\partial \hat{M}$. Then, we can apply Proposition 2.6 to transform $\hat{T}_1 \setminus (\cup \text{Int}(\mathcal{N}(\hat{\beta}_1)))$ into $\hat{T}_2 \setminus (\cup \text{Int}(\mathcal{N}(\hat{\beta}_2)))$ via B- and MP-moves having support out of $\partial M$. Obviously, these moves can be applied on the loose triangulation $\hat{T}_1$ transforming it into $\hat{T}_2$, they are all admissible, and they transform the loose triangulation $(\hat{T}_1, \hat{\beta}_1)$ into $(\hat{T}_2, \hat{\beta}_2)$; moreover, the negative Ba-moves do not eliminate the points belonging to $\partial M$. The desired sequence is obtained by repeating the moves on the ideal triangulations $(T_1', \beta_1')$ of $(M, \alpha)$.

2.4 Elimination of Ba-moves

To deduce Theorem 2.1 from Proposition 2.7, we generalize an idea of Matveev [8] to the setting of marked spines.
Proof of 2.1. Let \((T_1, \beta_1)\) and \((T_2, \beta_2)\) be two ideal triangulations of \((M, \alpha)\). By Proposition 2.7 we have that \((T_2, \beta_2)\) is obtained from \((T_1, \beta_1)\) via Ba-, Va-, and MPa-moves, such that the negative Ba-moves do not eliminate the spherical boundary components of \(\partial M\). The idea of the proof consists in replacing each Ba-move with a Ca-move, and each Va- or MPa-move with suitable sequences of La-, Va-, and MPa-moves. Let us pass to the dual spine viewpoint: for \(i = 1, 2\), let \((P_i, \beta_i)\) be the spine dual to \((T_i, \beta_i)\).

First of all, note that in the passages along the sequence of Ba-, Va-, and MPa-moves we get (standard) spines \(P_i\) of \(M\); minus some balls; so each \(M \setminus P_i\) is a disjoint union of \(\partial M \times [0, 1]\), and some balls. When a positive Ba-move is applied, a proper ball \(B\) appears. Let us continue calling proper ball (and continue indicating it by \(B\)) its transformations after the others Ba-, Va-, and MPa-moves, until it disappears because of a negative Ba-move (each proper ball has to disappear). Note that, conversely, the negative Ba-moves eliminate only the proper balls. Note also that each \(B\) is an open ball with boundary contained in \(P_i\), and it is not touched by the edges belonging to \(\alpha\).

We will not replace all the Ba-moves (with Ca-moves) at the same time; instead, we will concentrate on one positive Ba-move and on the negative Ba-move eliminating the proper ball created by the positive Ba-move. The strategy will be to replace two Ba-moves with two Ca-moves, any other Ba-move with a suitable sequence of only one Ba-move and La-, Va-, and MPa-moves, and each Va- or MPa-move with a suitable sequence of La-, Va-, and MPa-moves only. In such a way we will decrease, by two, the number of Ba-moves in the sequence. By repeating this procedure we can eliminate all the Ba-moves and we can complete the proof.

Let us describe the procedure in details. Let \(s\) be the following sequence of Ba-, Va-, and MPa-moves transforming \((P_1, \beta_1)\) into \((P_2, \beta_2)\):

\[
(P_1, \beta_1) \xrightarrow{s_1} (Q_0, \eta_0) \xrightarrow{Ba^+} (Q_1, \eta_1) \xrightarrow{m_1} (Q_2, \eta_2) \xrightarrow{m_2} \ldots \xrightarrow{m_{r-1}} (Q_r, \eta_r) \xrightarrow{Ba^-} (Q_{r+1}, \eta_{r+1}) \xrightarrow{s_2} (P_2, \beta_2),
\]

where \(s_1\) and \(s_2\) are sequences of moves we will not replace, \(Ba^+\) (respectively, \(Ba^-\)) is the positive (respectively, negative) move we will replace with a positive (respectively, negative) Ca-move, and the \(m_j\’s\) are the other moves we will replace. From now on, we will denote by \(B\) both the proper ball created by \(Ba^+\) (and eliminated by \(Ba^-\)) and its transformations after the \(m_j\’s\). To decrease by two the number of Ba-moves, we will find a sequence \(s’\) transforming \((P_1, \beta_1)\) into \((P_2, \beta_2)\) and appearing as follows:

\[
(P_1, \beta_1) \xrightarrow{s_1} (Q_0, \eta_0) \xrightarrow{Ca^+} (Q_1, \eta_1) \xrightarrow{m_1} (Q_2, \eta_2) \xrightarrow{m_2} \ldots \xrightarrow{m_{r-1}} (Q_r, \eta_r) \xrightarrow{Ca^-} (Q_{r+1}, \eta_{r+1}) \xrightarrow{s_2} (P_2, \beta_2),
\]

where \(s_1\) and \(s_2\) are the same sequences as above, \(Ca^+\) (respectively, \(Ca^-\)) is the positive (respectively, negative) move replacing \(Ba^+\) (respectively, \(Ba^-\)), and the \(m_j\’s\) are sequences of moves (composed either by only one Ba-move and some La-, Va-, and MPa-moves if \(m_j\) is a Ba-move, or by only La-, Va-, and MPa-moves otherwise) replacing the \(m_j\’s\).

Let us start by replacing \(Ba^+\) with a positive Ca-move \(Ca^+\) (the position of the arch can be random). After applying \(Ca^+\) to \((Q_0, \eta_0)\) we obtain a spine.
\((\bar{Q}_1, \bar{\eta}_1)\) which differs from \((Q_1, \eta_1)\) only for the presence of an arch connecting the proper ball \(B\) to \(M \setminus (Q_1 \cup B)\), see Fig. \(\text{S}^{}\) right. Note that the arch joins a region \(R_1\) of \(\partial B\) with another one, \(R_2\), of \(Q_1\); if \(R_2\) belongs to \(\eta_1\), then \(R_1\) is a part of \(\partial B\) belonging to \(\bar{\eta}_1\). Note also that \(R_1\) is the only part of \(\partial B\) which can belong to \(\bar{\eta}_1\), and that \(R_0\) does not belong to \(\bar{\eta}_1\). Now the sequence \(s'\) appears as follows:

\[
(P_1, \beta_1) \xrightarrow{s_1} (Q_0, \eta_0) \xrightarrow{\text{Ca}^+} (\bar{Q}_1, \bar{\eta}_1).
\]

The aim is now to replace the moves \(m_j\). If we try to apply \(m_1\) also on \((\bar{Q}_1, \bar{\eta}_1)\), we could fail because of the presence of the arch created by the move \(\text{Ca}^+\). So the idea is either to apply the move \(m_j\) if the arch is not involved in the move, or to move the arch before applying the move otherwise. To do this, we will use a recursive procedure. Let \((Q_j, \eta_j)\) be a spine (of the sequence \(s\)) of \((M, \alpha)\) minus some balls (let us call \(k\) the number of such balls). Let \(B\) be the connected component of \(M \setminus Q_j\) containing one of such balls. Note that \(B\) is an open ball embedded in \(\text{Int}(M)\), but its closure \(\overline{B}\) may not be a closed ball embedded in \(\text{Int}(M)\). In our recursive procedure \(B\) is the proper ball created by the move \(\text{Ba}^+\) and modified by the moves \(m_i\), with \(i < j\). Let \((Q_j, \bar{\eta}_j)\) be a spine of \((M, \alpha)\) minus \(k - 1\) balls, which differs from \((Q_j, \eta_j)\) only for the presence of an arch connecting the proper ball \(B\) to another connected component of \(M \setminus Q_j\). Let moreover \(m_j\) be an admissible move from \((Q_j, \eta_j)\) to \((Q_{j+1}, \eta_{j+1})\), which does not eliminate the proper ball \(B\). Note that \((Q_{j+1}, \eta_{j+1})\) is a spine of \((M, \alpha)\) minus \(h\) balls, where \(h = k - 1, k, k + 1\) depending on \(m_j\). Let us continue calling \(B\) the transformation of \(B\) under \(m_j\).

The recursive pass consists in describing a sequence \(\tilde{m}_j\) of admissible moves (composed either by only one \(\text{Ba}\)-move and some \(\text{La-}\), \(\text{Va-}\), and \(\text{MPa}\)-moves if \(m_j\) is a \(\text{Ba}\)-move, or by only \(\text{La-}\), \(\text{Va-}\), and \(\text{MPa}\)-moves otherwise) from \((\bar{Q}_j, \bar{\eta}_j)\) to \((\bar{Q}_{j+1}, \bar{\eta}_{j+1})\), where \((\bar{Q}_{j+1}, \bar{\eta}_{j+1})\) is a spine of \((M, \alpha)\) minus \(h - 1\) ball, which differs from \((Q_{j+1}, \eta_{j+1})\) only for the presence of an arch connecting the ball \(B\) to another connected component of \(M \setminus Q_{j+1}\). If \(m_j\) can be applied (i.e. the arch is far from the support of \(m_j\)), then we apply \(m_j\) to \((\bar{Q}_j, \bar{\eta}_j)\) obtaining \((\bar{Q}_{j+1}, \bar{\eta}_{j+1})\), which obviously has all the properties described above. Note that there are some types of moves which can always be applied because the arch is never involved, up to isotopy, in the move: such moves are the positive \(\text{Ba}\)-moves and the positive \(\text{Va}\)-moves. To replace the other \(\text{Ba-}\), \(\text{Va-}\), and \(\text{MPa}\)-moves, maybe we need to move the arch so to be able to apply the move. If \(m_j\) cannot be applied (because of the presence of the arch), then we move the arch before applying \(m_j\). Let us describe how to move the arch; afterwards we will continue the substitution of \(m_j\) with \(\tilde{m}_j\).

**Arch-move** Let \((\bar{Q}_j, \bar{\eta}_j)\) be the spine of \((M, \alpha)\) minus \(k - 1\) balls, which has an arch we want to move. Recall that \(B\) is the proper ball connected to another connected component of \(M \setminus Q_j\) by the arch. Moreover recall that the proper ball \(B\) is an open ball embedded in \(M\), but (because of the moves \(m_i\) with \(i < j\)) its closure \(\overline{B}\) could be not a closed ball embedded in \(M\).

We now define a spine \((\bar{Q}_j', \bar{\eta}_j')\) of \((M, \alpha)\) minus \(k - 1\) balls. Let \(\bar{Q}_j'\) be the spine obtained from \(\bar{Q}_j\) by taking away the arch we want to move and by placing it in another point, so that \(\bar{Q}_j'\) is again a spine of \(M\) minus \(k - 1\) balls and the ball

\[
\text{Arch-move} \quad \text{Let } (\bar{Q}_j, \bar{\eta}_j) \text{ be the spine of } (M, \alpha) \text{ minus } k - 1 \text{ balls, which has an arch we want to move. Recall that } B \text{ is the proper ball connected to another connected component of } M \setminus Q_j \text{ by the arch. Moreover recall that the proper ball } B \text{ is an open ball embedded in } M, \text{ but (because of the moves } m_i \text{ with } i < j) \text{ its closure } \overline{B} \text{ could be not a closed ball embedded in } M.
\]

We now define a spine \((\bar{Q}_j', \bar{\eta}_j')\) of \((M, \alpha)\) minus \(k - 1\) balls. Let \(\bar{Q}_j'\) be the spine obtained from \(\bar{Q}_j\) by taking away the arch we want to move and by placing it in another point, so that \(\bar{Q}_j'\) is again a spine of \(M\) minus \(k - 1\) balls and the ball
Figure 20: The arch-move. We show on the left the situation near the arch we want to remove and on the right the situation near the point where we want to place the arch. 

$\mathcal{B}$ is connected by the new arch to another connected component of $M \setminus Q_j$, see Fig. 20. The two conditions on $\tilde{Q}_j'$ imply that the arch, after the move, should be placed “near” $\partial \tilde{\mathcal{B}}$. To define $\tilde{\eta}'_j$, let us analyze the regions affected by the move. The region $R_1$ of $\tilde{Q}_j$ (intersecting $\partial \mathcal{B}$) is divided (in $\tilde{Q}_j'$) in two regions, $R_1'$ and $R_1''$. Note that these two regions belong also to the spine $(Q_j, \eta_j)$ and that only $R_1'$ can belong to $\eta_j$; if it belongs to $\eta_j$, we impose to leave itself in $\tilde{\eta}_j$. The little region $R_0$, which is eliminated by the arch-move, does not belong to $\tilde{\eta}_j$. The other regions which are modified are the regions $R_2$ and $R_2'$, which unite. Note that these two regions belong also to $(Q_j, \eta_j)$ and that only $R_2$ can belong to $\eta_j$ (because $R_1''$ is contained in $\partial \mathcal{B}$); if $R_2$ belongs to $\eta_j$, we impose to leave the region $E$ in $\tilde{\eta}_j$. The other regions are not modified, so we leave in $\tilde{\eta}_j'$ those belonging to $\tilde{\eta}_j$. Finally, note that $(\tilde{Q}_j', \tilde{\eta}_j')$ differs from $(Q_j, \eta_j)$ only for the presence of the arch (connecting the proper ball $\mathcal{B}$ to another connected component of $M \setminus Q_j$). The transformation of $(\tilde{Q}_j, \tilde{\eta}_j)$ into $(\tilde{Q}_j', \tilde{\eta}_j')$ will be called arch-move.

Now we prove that each arch-move is a composition of La- and MPa-moves. In Fig. 21 we have shown the La- and MPa-moves transforming $(\tilde{Q}_j, \tilde{\eta}_j)$ into $(\tilde{Q}_j', \tilde{\eta}_j')$: let us describe these moves. Note that the only region which can both intersect $\partial \mathcal{B}$ and belong to $\tilde{\eta}_j$ is $R_1$. For the first positive L-move, if $R_1$ belongs to $\alpha$, we choose to leave $R_3$ in $\alpha$. Note that now no region in $\alpha$ intersects $\partial \mathcal{B}$. For the second positive L-move, if $R_2$ belongs to $\alpha$, we choose to leave $R_4$ in $\alpha$. The region $R_5$ does not belong to $\alpha$, because it intersects $\partial \mathcal{B}$, so the third positive L-move is admissible. Let us now describe the move indicated by a dashed arrow. Note that the proper ball $\mathcal{B}$ can be seen as a tube $D^2 \times [0, 1]$, where $D^2 \times \{0\} = D$ and $D^2 \times \{1\} = D'$. Obviously, we can move the disc $D$ through the tube from $D^2 \times \{0\}$ to $D^2 \times \{1\}$ via an isotopy. The move indicated by the dashed arrow consists exactly of this isotopy of the little disc $D$ through the proper ball $\mathcal{B}$: more precisely, if one of the two arches (or both of them) are inside $\mathcal{B}$ (namely, the little discs $R_0$ and $R_0'$ are contained in $\mathcal{B}$), the isotopy is through $\mathcal{B}$ minus both the arch and the tube inside it. At the end of the
Figure 21: The arch-move is a composition of La- and MPa-moves. In each step, we show on the left the situation near the arch we want to remove and on the right the situation near the point where we want to place the arch. The dashed arrow denotes an isotopy of the little disc $D$. 
isotopy the little disc $D$ coincides with $D'$, so it lays near the arch we want to remove. A simple general position argument tells us that the isotopy can be substituted with L- and MP-moves, see Lemma 1.2.16 of [9] for a precise proof. All these moves are admissible because $F_1$, $F_2$, and the regions intersecting $\partial B$ do not belong to $\alpha$. For the same reason (and since $D'$ does not belong to $\alpha$) the last three negative L-moves are admissible (obviously, the regions united in each move are different). To conclude, we note that the position of the regions in $\alpha$ after these moves is the same as after the arch-move.

Continuing substitution  Recall that we want to replace the move $m_j$ which cannot be applied to $(Q_j, \tilde{\eta}_j)$, because of the presence of the arch. We apply first an arch-move to $(Q_j, \tilde{\eta}_j)$ obtaining $(Q'_j, \tilde{\eta}'_j)$ and then the move $m_j$. Let us call $(Q_{j+1}, \tilde{\eta}_{j+1})$ the spine just obtained. Note that, to apply the arch-move, we need to find a place where placing the arch, but it is very easy to find such a place near $\partial B$ and far from the move $m_j$. Note also that, by construction, $(Q_{j+1}, \tilde{\eta}_{j+1})$ differs from $(Q_{j+1}, \eta_{j+1})$ only for the presence of the arch (connecting the proper ball $B$ to another connected component of $M \setminus Q_{j+1}$).

With these substitutions, we have extended the sequence $s'$ obtaining:

$$(P_1, \beta_1) \xrightarrow{s_1} (Q_0, \eta_0) \xrightarrow{C_{\alpha}^+} (Q_1, \tilde{\eta}_1) \xrightarrow{m_1} (Q_2, \tilde{\eta}_2) \xrightarrow{m_2} \ldots \xrightarrow{m_r} (Q_r, \tilde{\eta}_r).$$

Let us consider now the move $Ba^-$. We have noted above that the spine $(Q_r, \tilde{\eta}_r)$ differs from $(Q_r, \eta_r)$ only for the presence of the arch (connecting the proper ball $B$ to another connected component of $M \setminus Q_r$), so $Q_r$ near $B$ appears exactly as in Fig.8 centre. Moreover, $R_1$ is the only part of $\partial B$ which can belong to $\tilde{\eta}_r$ and $R_0$ does not belong to $\tilde{\eta}_r$. Obviously, a negative Ca-move (which we call $Ca^-$) can be applied and the result is just $(Q_{r+1}, \eta_{r+1})$. Now the sequence $s'$ appears as follows:

$$(P_1, \beta_1) \xrightarrow{s_1} (Q_0, \eta_0) \xrightarrow{C_{\alpha}^+} (Q_1, \tilde{\eta}_1) \xrightarrow{m_1} (Q_2, \tilde{\eta}_2) \xrightarrow{m_2} \ldots \xrightarrow{m_{r-1}} (Q_r, \tilde{\eta}_r) \xrightarrow{Ca^-} (Q_{r+1}, \eta_{r+1}).$$

To obtain the desired sequence, it is enough to complete the sequence just obtained by composing it with the sequence $s_2$. This proves the theorem.  

2.5 Another proof

In this subsection we describe how Basehilac and Benedetti have deduced Theorem 2.1 from a result (due to Turaev and Viro) which relies on the Matveev-Piergallini theorem. For the sake of clarity, we describe the ideas of the proof, instead of only stating Theorem 3.4.B of [14]. We restrict ourselves only to a sketch of the proof of Theorem 2.4.1

Sketch of the proof of 2.4.1 For $i = 1, 2$, let $(T_i, \beta_i)$ be the spine dual to an ideal triangulation $(\mathcal{T}_i, \beta_i)$. Let $N(\alpha)$ be a little open regular neighborhood of $\alpha$ and $M_\alpha = M \setminus N(\alpha)$. Note that, up to choosing $N(\alpha)$ small with respect to $P_1$ and $P_2$, we can suppose that $N(\alpha) \cap P_i = \bigcup_{r=1}^p D_i^{(j)}$, where $D_i^{(j)}$ is an open disc.
with closure contained in the (open) region \( \beta_i^{(j)} \), for \( i = 1, 2 \) and \( j = 1, \ldots, n \). Now, for \( i = 1, 2 \), we define two new polyhedra \( Q_i \) and \( R_i \) with \( Q_i \subset R_i \subset P_i \). To get \( Q_i \), we remove from \( P_i \) all the (open) regions \( \beta_i^{(j)} \), and, to get \( R_i \), we remove from \( P_i \) all the open discs \( D_i^{(j)} \). Note that we have a retraction \( \pi_i \) of \( M_n \) onto \( Q_i \). Moreover, we have on \( \partial M_n \) a family \( \lambda_i = \{ \lambda_i^{(1)}, \ldots, \lambda_i^{(n)} \} \) of disjoint simple circles such that \( \lambda_i^{(j)} = \partial D_i^{(j)} \subset \beta_i^{(j)} \) and, up to isotopy, \( R_i \setminus Q_i \) consists precisely of the “half-open” annuli \( \lambda_i^{(j)} \times [0, 1) \) obtained by projecting \( \lambda_i^{(j)} \) to \( Q_i \), along \( \pi_i \). We have already described the “inverse” construction in Subsection 1.2 when we have proved existence of marked ideal triangulations. Of course any move on \( R_i \) not affecting the \( \lambda_i^{(j)} \)’s readily translates into an admissible move on \((P_i, \beta_i)\), and conversely. Obviously, up to isotopy, we can suppose that each \( \lambda_i^{(j)} \) coincides with \( \lambda_2^{(j)} \) and that each \( D_i^{(j)} \) coincides with \( D_2^{(j)} \): let us call simply \( \lambda^{(j)} \) the curve \( \lambda_1^{(j)} = \lambda_2^{(j)} \), \( \lambda \) the collection \( \{ \lambda^{(1)}, \ldots, \lambda^{(n)} \} \), and \( D^{(j)} \) the disc \( D_1^{(j)} = D_2^{(j)} \).

Now, \( Q_1 \) needs not to be standard, but one readily sees that standardness can be achieved using C- and L-moves on \( R_1 \) not affecting the \( \lambda^{(j)} \)’s. Now, \( Q_1 \) and \( Q_2 \) are standard spines of \( M \setminus N(\alpha) \), so, by Matveev-Piergallini theorem (see Theorem 2.23), we can transform \( Q_1 \) into \( Q_2 \) via a deformation \( Q_t \) (with \( t \in [1, 2] \)) with elementary accidents which are L- and MP-moves. Obviously, we can suppose that the elementary accidents occur at different times. Note that the \( Q_t \)’s are all quasi-standard spines, except for a finite number of times when elementary accidents occur so quasi-standardness is lost.

Parallely, we have a deformation \( \pi_t \) of \( \pi_1 \) into \( \pi_2 \), where each \( \pi_t \) is a retraction of \( M \setminus N(\alpha) \) onto \( Q_t \). Obviously, the annuli \( [\lambda^{(j)}, \pi_t(\lambda^{(j)})] \) are transformed into \( [\lambda^{(j)}, \pi_2(\lambda^{(j)})] \) via annuli \( [\lambda^{(j)}, \pi_1(\lambda^{(j)})] \). By a general position argument, we can suppose that the accidents occurring to \( [\lambda, \pi_t(\lambda)] \cup Q_t \) are L-, MP-, and false L-moves not affecting the \( \lambda^{(j)} \)’s, where a false L-move is a negative L-move not preserving standardness (actually it is not an L-move).

Now, we have obtained a sequence of L-, MP-, and false L-moves not affecting the \( \lambda^{(j)} \)’s transforming \( R_1 \) into \( R_2 \). To eliminate the false L-moves, we can use the same technique used in Theorem 1.2.30 of [2], which states the following (we use our notation): "Two standard spines of a 3-manifold \( W \) related by a sequence of L-, MP-, and false L-moves are related by a sequence of L- and MP-moves only." By obviously generalizing this proposition to our setting, we obtain a sequence of L- and MP-moves only, transforming \( R_1 \) into \( R_2 \). By adding the discs \( D^{(j)} \), we obviously obtain the desired sequence of La- and MPa-moves transforming \( (P_1, \beta_1) \) into \( (P_2, \beta_2) \).

### 3 Existence of dominating marked spines

In this section we generalize, to the setting of marked spines, a result of Makovetskii [7] on the existence of a spine which dominates, as far as the positive L-moves and positive MP-moves are concerned, any two given spines of \( M \). Namely, we prove the following.

**Theorem 3.1.** Let \((T_1, \beta_1)\) and \((T_2, \beta_2)\) be two marked ideal triangulations of a pair \((M, \alpha)\). Then there exists a marked ideal triangulation \((T, \beta)\) of \((M, \alpha)\)
obtained from both \((T_1, \beta_1)\) and \((T_2, \beta_2)\) via a sequence of positive La-moves and positive MPa-moves.

For the proof, we follow the ideas of [7].

3.1 Divided spines and moves

Let us give some definitions useful for the proof.

**Dividing strips and divided spines** Let \((P, \beta)\), with \(\beta = \{\beta^{(1)}, \ldots, \beta^{(n)}\}\), be a spine of a pair \((M, \alpha)\). Let \(\gamma : [0, 1] \to P\) be a simple curve such that:

- the endpoints belong to edges (maybe, to the same edge) of \(P\);
- \(\gamma\) intersects the singularities of \(P\) transversely;
- \(\gamma\) contains no vertex of \(P\);
- there exists a strip \(S = \gamma \times [0, 1] \subset M\) intersecting \(P\) exactly in \(\gamma = \gamma \times \{0\}\) and \(\{\gamma(0), \gamma(1)\} \times [0, 1]\).

Such a curve \(\gamma\) divides some regions of \(P\) (those it touches) into discs: for each region \(R\), we will call sub-regions (of \(R\)) such discs if \(R\) is divided by \(\gamma\), or \(R\) itself if it is untouched by \(\gamma\). Let \(\overline{\beta} = \{\overline{\beta}^{(1)}, \ldots, \overline{\beta}^{(n)}\}\) be a collection of sub-regions such that each \(\overline{\beta}^{(i)}\) is a sub-region of \(\beta^{(i)}\). The pair \((S, \overline{\beta})\), where \(S = \gamma \times [0, 1]\), will be said dividing strip of \((P, \beta)\), and the triplet \((P, S, \overline{\beta})\) will be said a divided spine of \((M, \alpha)\).

**Moves on divided spines** Let \((P, S, \overline{\beta})\) be a divided spine of \((M, \alpha)\). We start by defining the obvious generalizations of the positive La- and MPa-moves and then we define two new moves to take into account the strip \(S\). As for admissible moves on marked spines, we will say that an admissible move from \((P, \beta)\) to \((P', \beta')\) gives rise to a divided-admissible move if there is a dividing strip \((S', \overline{\beta}')\) of \((P', \beta')\) such that \((P', S', \overline{\beta}')\) is a divided spine of \((M, \alpha)\), and \((S', \overline{\beta}')\) coincides with \((S, \overline{\beta})\) except “near” the portion of \(P\) affected by the move. As it turns out, divided-admissibility depends on \(S\). Moreover, \(\beta'\) is sometimes not unique.

**MPd-move** Let us consider a positive MPa-move \(m\) from \((P, \beta)\) to another spine \((P', \beta')\) of \((M, \alpha)\), such that the strip \(S\) is not involved in the move (namely, \(S\) does not intersect the part of \(P\) affected by \(m\)). Then, we will say that \(m\) gives rise to an MPd-move from \((P, S, \overline{\beta})\) to \((P', S', \overline{\beta}')\) whatever \(\overline{\beta}\), where \(S'\) coincides with \(S\) and \(\overline{\beta}'\) consists of the same sub-regions as \(\overline{\beta}\) (recall that the newborn triangular region does not belong to \(\beta'\)). Note that an MPd-move always increases (by one) the number of vertices of \(P\).

**Ld-move** For the La-moves, the situation is more complicated. Let us consider a positive La-move \(m\) from \((P, \beta)\) to another spine \((P', \beta')\) of \((M, \alpha)\), such that the strip \(S\) is not involved in the move (namely, \(S\) does not intersect the part of \(P\) affected by \(m\)). As above, we will say that \(m\) gives rise to an Ld-move from \((P, S, \overline{\beta})\) to \((P', S', \overline{\beta}')\) whatever \(\overline{\beta}\), where \(S'\) coincides with \(S\)
and $\overline{\beta'}$ is uniquely determined by $\overline{\beta}$ and $\beta'$. Let us describe $\overline{\beta'}$. Recall that $m$ divides a region $R$ of $P$ in two regions $R_1$ and $R_2$, see Fig. 22 left. Since the strip $S$ is not involved in the move $m$, then the Ld-move divides a sub-region $\overline{R}$ of $(P, S, \overline{\beta})$ in two sub-regions $\overline{R}_1$ and $\overline{R}_2$ (where $\overline{R}_i$ is a sub-region of $R_i$, for $i = 1, 2$). Now, we have two cases depending on whether $\overline{R}$ belongs to $\overline{\beta}$ or not. If $\overline{R}$ does not belong to $\overline{\beta}$, then $\overline{\beta'}$ consists of the same sub-regions as $\overline{\beta}$ (i.e. $\overline{R}_1$, $\overline{R}_2$, and the newborn little region $D$ do not belong to $\beta'$). If $\overline{R}$ belongs to $\overline{\beta}$, then we define $\overline{\beta'}$ as $(\overline{\beta} \setminus \{\overline{R}\}) \cup \{\overline{R}_1\}$ or $(\overline{\beta} \setminus \{\overline{R}\}) \cup \{\overline{R}_2\}$ depending on which region, between $R_1$ and $R_2$, belongs to $\beta'$. Note that an Ld-move always increases (by two) the number of vertices of $P$.

**Md-move** We call Md-move any move from a divided spine $(P, S, \overline{\beta})$ of $(M, \alpha)$ to another divided spine $(P', S', \overline{\beta'})$ of $(M, \alpha)$, where:

- $P'$ coincides with $P$;
- $S'$ is obtained from $S$ as in Fig. 22 (we have two cases depending on whether the endpoints of $\gamma$ are involved in the move or not);
- $\overline{\beta'}$ coincides with $\overline{\beta}$ except that the sub-region $R$, if it lies in $\overline{\beta}$, gets replaced by the sub-region $R_1$.

Note that an Md-move increases (by one) the number of intersections between $\gamma$ and the singularity of $P$, so it can be considered as being “positive”.

**Nd-move** We call Nd-move any move from a divided spine $(P, S, \overline{\beta})$ of $(M, \alpha)$ to another divided spine $(P', S', \overline{\beta'})$ of $(M, \alpha)$, where:

- $P'$ coincides with $P$;
- $S'$ is obtained from $S$ as in Fig. 23;
- $\overline{\beta'}$ coincides with $\overline{\beta}$ except that the sub-regions $R$ and $R'$, if they lie in $\overline{\beta}$, get replaced respectively by either the sub-region $R_1$ or $R_2$, and by $R'_1$.

Note that the choice of which sub-region, between $R_1$ and $R_2$, belongs to $\overline{\beta'}$ is included in the move. Finally, note that an Nd-move increases (by two) the number of intersections between $\gamma$ and the singularity of $P$, so it can be considered as being “positive”.

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If a spine \((P_2, \beta_2)\) is obtained from a spine \((P_1, \beta_1)\) via positive La-moves and positive MPa-moves, we will write \((P_1, \beta_1) \overset{\mark}{\succ} (P_2, \beta_2)\). If a divided spine \((P_2, S_2, \beta_2)\) is obtained from a divided spine \((P_1, S_1, \beta_1)\) via Md-, Nd-, Ld-, and MPd-moves, we will write \((P_1, S_1, \beta_1) \overset{\mark}{\succ} (P_2, S_2, \beta_2)\).

**Swelling** Now we define another move which, taking into account the dividing strip, transforms a divided spine into a (marked) spine. Let \((P, S, \beta)\) be a divided spine of a pair \((M, \alpha)\), where \(S = \gamma \times [0, 1]\). If we apply \(m\) positive L-moves to \(P\) along the curve \(\gamma\) (following the orientation of \(\gamma\)), we obtain a spine, say \(P'\), of \(M\), see Fig. 24. Note that \(m\) is one less than the number of intersections between \(\gamma\) and the singularities of \(P\). Noting that the collection \(\beta\) allows us to choose what regions of \(P'\) remain in \(\alpha\) after the L-moves (with a little abuse of notation, we continue calling \(\beta\) the collection of such regions), it turns out that the L-moves are admissible and that the pair \((P', \beta')\) is a marked spine of \((M, \alpha)\). The spine \((P', \beta')\) will be called swelling of \((P, \beta)\) along \((S, \beta)\) and will be denoted by \(\text{sw}(P, S, \beta)\).

**Remark 3.2.** For future reference, we underline the fact that \((P, \beta) \overset{\mark}{\succ} \text{sw}(P, S, \beta)\).

### 3.2 Existence of dominating marked spines

Let us start with two preliminary results.

**Lemma 3.3.** Let \((P_1, S_1, \beta_1)\) be a divided spine of a pair \((M, \alpha)\) and let \((P_2, \beta_2)\) be a spine of the pair \((M, \alpha)\) such that \((P_1, \beta_1) \overset{\mark}{\succ} (P_2, \beta_2)\). Then there exists a dividing strip \((S_2, \beta_2)\) of \((P_2, \beta_2)\) such that \((P_1, S_1, \beta_1) \overset{\mark}{\succ} (P_2, S_2, \beta_2)\).

**Proof of 3.3.** An easy induction on the number of positive moves transforming \((P_1, \beta_1)\) into \((P_2, \beta_2)\) allows us to analyze only the case of only one positive move between \((P_1, \beta_1)\) and \((P_2, \beta_2)\). There are two moves to analyze: the positive La-move and the positive MPa-moves. We concentrate on the first one (the second one being simpler). If necessary, we first apply Nd-moves to take the strip \(S_1\) away from the part of \(P_1\) affected by the La-move, see Fig. 24 left. Let us call \(S'\) the strip just obtained. We impose that the collection \(\beta'\) consists of the same sub-regions as \(\beta_1\), unless a sub-region divided by one of these Nd-moves belongs to \(\beta_1\), in which case we choose which of the two new sub-regions belongs to \(\beta'\).
Figure 26: If \((P_1, S_1, \beta_1) \tr (P_2, S_2, \beta_2)\) via an Md-move, then \(\text{sw}(P_1, S_1, \beta_1) \tr \text{sw}(P_2, S_2, \beta_2)\) via an La-move (case 1).

Figure 27: If \((P_1, S_1, \beta_1) \tr (P_2, S_2, \beta_2)\) via an Nd-move, then \(\text{sw}(P_1, S_1, \beta_1) \tr \text{sw}(P_2, S_2, \beta_2)\) via two La-moves.

following the choice given by the positive La-move. So \((P_1, S_1, \beta_1)\) is a divided spine of \((M, \alpha)\). Now we are able to apply an Ld-move to \((P_1, S_1, \beta_1)\) to obtain a divided spine \((P_2, S_2, \beta_2)\), see Fig. 25-right. The pair \((S_2, \beta_2)\) is the dividing strip we are searching for.

Lemma 3.4. If \((P_1, S_1, \beta_1) \tr (P_2, S_2, \beta_2)\), then \(\text{sw}(P_1, S_1, \beta_1) \tr \text{sw}(P_2, S_2, \beta_2)\).

Proof of 3.4. An easy induction on the number of moves transforming \((P_1, S_1, \beta_1)\) into \((P_2, S_2, \beta_2)\) allows us to analyze only the case of only one move between \((P_1, S_1, \beta_1)\) and \((P_2, S_2, \beta_2)\). There are four possible moves. If the move is an Ld- or an MPd-move, then obviously \(\text{sw}(P_1, S_1, \beta_1) \tr \text{sw}(P_2, S_2, \beta_2)\) because \(S_1\) is “far” from the move. If the move is an Md-move, we have three cases:

1. If \(\gamma(0)\) is involved in the move (see Fig. 22-right), then \(\text{sw}(P_2, S_2, \beta_2)\) is obtained from \(\text{sw}(P_1, S_1, \beta_1)\) via a positive La-move, as shown in Fig. 26. Note that, if the region \(R\) belongs to \(\beta_1\), we choose to leave in \(\beta_2\) the region \(R_1\); so the spine obtained is exactly the swelling of \((P_2, \beta_2)\) along \((S_2, \beta_2)\).

2. If neither \(\gamma(0)\) nor \(\gamma(1)\) is involved in the move (see Fig. 22-left), then \(\text{sw}(P_2, S_2, \beta_2)\) is obtained from \(\text{sw}(P_1, S_1, \beta_1)\) via two positive MPa-moves.

3. If \(\gamma(1)\) is involved in the move (see again Fig. 22-right), then \(\text{sw}(P_2, S_2, \beta_2)\) is obtained from \(\text{sw}(P_1, S_1, \beta_1)\) via two positive MPa-moves.

If the move is an Nd-move (see Fig. 23), then \(\text{sw}(P_2, S_2, \beta_2)\) is obtained from \(\text{sw}(P_1, S_1, \beta_1)\) via two positive La-moves, as shown in Fig. 27. For the first La-move, if the region \(R\) belongs to \(\beta_1\), we have to choose a region, between \(R_1\) and \(R_2\), to leave in \(\beta_2\); we choose the region depending on which sub-region, between \(R_1\) and \(R_2\), belongs to \(\beta_2\) after the Nd-move. For the second La-move, if the region \(R'\) belongs to \(\beta_1\), we choose to leave in \(\beta_2\) the “nearest” region (between \(R'_1\) and \(R'_2\)) to \(\gamma(0)\). The spine obtained is exactly the swelling of \((P_2, \beta_2)\) along \((S_2, \beta_2)\). This concludes the proof.

Now we are able to prove Theorem 3.4.
Proof of \(3.1\) Let \((P_i, \beta_i)\) the dual spine of \((T_i, \beta_i)\), for \(i = 1, 2\). By applying Theorem 2.1 and by noting that each Va-move is actually an La-move, we obtain a sequence \(s\) of La- and MPa-moves transforming \((T_1, \beta_1)\) into \((T_2, \beta_2)\). The sequence \(s\) can be divided in (sub-)sequences \(s_i\), with \(i = 1, \ldots, 2l\), where the sequences \(s_{2k+1}\) are composed by positive moves while the sequences \(s_{2k}\) are composed by negative moves, and only \(s_1\) and \(s_{2l}\) could be empty. Let us call \(|s_i|\) the number of moves of the sequence \(s_i\). An easy induction on \(S = \sum_{k=1}^{l-1} |s_{2k+1}|\) allows us to prove only the following statement.

If \((P_2, \beta_2)\) is obtained from \((P_1, \beta_1)\) via a sequence \(s\) such that \(l = 2\), \(|s_1| = 0\), \(|s_2| > 0\), \(|s_3| = 1\) and \(|s_4| = 0\), then there exists another sequence \(s'\), transforming \((P_1, \beta_1)\) into \((P_2, \beta_2)\), such that \(l = 1\).

The proof of this statement concludes the proof of the theorem. We have to prove that there exists a spine \((P, \beta)\) such that \((P_1, \beta_1) \nsucceq (P, \beta) \setminus (P_2, \beta_2)\). If we call \((Q, \beta')\) the spine before the positive move \(m\) of the sequence \(s_3\), we have that \((P_1, \beta_1) \setminus (Q, \beta') \nsucceq (P_2, \beta_2)\). Let us start by choosing a dividing strip \((S', \beta')\) for \((Q, \beta')\) (we have two cases depending on \(m\)).

- If \(m\) is a positive La-move, we choose as \(\gamma'\) the curve determining \(m\). Note that there are two different strips \(S' = \gamma' \times [0, 1]\) (up to isotopy): we choose one of them (the choice is immaterial). If the region of \(Q\) divided by \(\gamma'\) is one of the \((\beta')^{(i)}\)'s, we choose the \((\beta')^{(i)}\) following the choice given by \(m\).

- If \(m\) is a positive MPa-move, we choose as \(\gamma'\) a curve parallel to the edge \(e\) of \(Q\) disappearing during \(m\). As above there are two different strips: we choose one of them. If the region of \(Q\) divided by \(\gamma'\) is one of the \((\beta')^{(i)}\)'s, we choose as \((\beta')^{(i)}\) the sub-region which is not adjacent (locally) to \(e\).

By Lemma 3.3 there exists a dividing strip \((S_1, \beta_1)\) for \((P_1, \beta_1)\) such that \((P_1, S_1, \beta_1) \setminus (Q, S', \beta')\); so, by Lemma 3.3, \(\text{sw}(P_1, S_1, \beta_1) \setminus \text{sw}(Q, S', \beta')\). By Remark 3.2, we have that \((P_1, \beta_1) \nsucceq \text{sw}(P_1, S_1, \beta_1)\). Finally, we have two cases depending on \(m\).

- If \(m\) is a positive La-move, then \(\text{sw}(Q, S', \beta') = (P_2, \beta_2)\); so we have that \((P_1, \beta_1) \nsucceq \text{sw}(P_1, S_1, \beta_1) \setminus \text{sw}(Q, S', \beta') = (P_2, \beta_2)\).

- If \(m\) is a positive MPa-move, then \(\text{sw}(Q, S', \beta')\) can be obtained from \((P_2, \beta_2)\) via a positive MPa-move, see Fig. 28 so we have that \((P_1, \beta_1) \nsucceq \text{sw}(P_1, S_1, \beta_1) \setminus \text{sw}(Q, S', \beta') \setminus (P_2, \beta_2)\).

Figure 28: If \(m\) is a positive MPa-move, then \((P_2, \beta_2) \nsucceq \text{sw}(Q, S', \beta')\).
4 Applications

In this section we describe two applications of the previous results. The first one is due to Basehilac and Benedetti [2, 3, 4]. The second one is a natural question arisen in a work of Frigerio and Petronio [6].

4.1 Links in 3-manifolds

Let $M$ be a closed 3-manifold and $L$ a link in $M$. A pair $(T, L)$ is said to be a distinguished triangulation of the pair $(M, L)$ if $T$ is a loose triangulation of $M$, the link $L$ is triangulated by $L$ and $L$ is a Hamiltonian sub-complex of $T$ (i.e. each vertex of $T$ is an endpoint of exactly two germs of edges of $L$). As we have done for marked ideal triangulations, we can define (positive and negative) admissible MP- and L-moves between distinguished triangulations. We need another move allowing us to change the number of vertices of $T$. We will say that the distinguished triangulation $(T', L')$ is obtained from the distinguished triangulation $(T, L)$ via a positive admissible B-move if

- $T'$ is obtained from $T$ via a positive B-move,
- one edge $e$ of the tetrahedron $T$ involved in the move belongs to $L$,
- $L'$ coincides with $L$ except for the edge $e$ which is substituted with the other two edges of the only triangle of $T'$ created by the B-move and containing $e$.

See Fig. 29. Obviously, a negative admissible B-move between distinguished triangulations is defined as the inverse of a positive admissible B-move.

Now we are able to prove the calculus for distinguished triangulations.

**Corollary 4.1.** Two distinguished triangulations of a pair $(M, L)$ can be obtained from each other via a sequence of admissible B- and MP-moves.

**Proof of 4.1** Let $(T_1, L_1)$ and $(T_2, L_2)$ be two distinguished triangulations of $(M, L)$. Obviously, up to applying suitable admissible B-moves, we can suppose that $(T_1, L_1)$ and $(T_2, L_2)$ have the same number of vertices on each component of $L$. Moreover, up to isotopy, we can suppose that the links $L_i$ coincide with $L$, and that the vertices of $T_1$ and the vertices of $T_2$ coincide with each other.

Now, we remove a little star of each vertex of $T_i$: let us call $\overline{M}$ the manifold just obtained. Obviously, after removing the balls, the link $L$ becomes a collection of arcs, say $\overline{L}$, and, for $i = 1, 2$, the pair $(\overline{T_i}, \overline{L_i})$ is a marked loose triangulation corresponding to a marked ideal triangulation of $(\overline{M}, \overline{L})$. So, by applying Corollary 2.2, we obtain that $(\overline{T_2}, \overline{L_2})$ can be obtained from $(\overline{T_1}, \overline{L_1})$ via admissible MP-moves. This concludes the proof.
Using the same technique (and Theorem 3.1), the following result on dominating distinguished triangulations can be proved.

**Corollary 4.2.** Let \((T_1, L_1)\) and \((T_2, L_2)\) be two distinguished triangulations of a pair \((M, L)\). Then there exists a distinguished triangulation \((T, L)\) of \((M, L)\) obtained from both \((T_1, L_1)\) and \((T_2, L_2)\) via a sequence of admissible positive B-, L-, and MP-moves.

### 4.2 Partially truncated triangulations

In this subsection we briefly describe a generalization of ideal triangulations which is useful to study complete finite-volume orientable hyperbolic 3-manifolds with geodesic boundary [6]. (For the sake of shortness, in the rest of the subsection we will just say hyperbolic.) For a complete description see [6].

Let \(N\) be such a hyperbolic manifold. It is a fact that \(N\) consists of a compact portion together with some cusps based either on tori or on annuli. This fact implies that \(N\) has a natural compactification \(\overline{N}\) obtained from \(N\) by adding some tori and some annuli. Let us call \(C\) and \(A\) the collection of such tori and such annuli, respectively. It is a fact that \(N\) can be obtained in a non-ambiguous way from the pair \((\overline{N}, A)\) by removing from \(\overline{N}\) both \(A\) and all the toric components of \(\partial \overline{N}\). Moreover, there is no sphere in \(\partial \overline{N}\) and there is no annulus in \(A\) which is contained in a torus of \(C\).

Let us describe now a generalization of ideal triangulations, which takes into account the annuli. Let us start by defining the pieces substituting ideal tetrahedra. A **partially truncated tetrahedron** is a triple \((T, I, Z)\) where \(T\) is a tetrahedron, \(I\) is a set of vertices of \(T\) (called **ideal vertices**), and \(Z\) is a set of edges of \(T\) (called **length-0 edges**) such that neither of the two endpoints of an edge in \(Z\) belongs to \(I\). Now we define the **topological realization** of a partially truncated tetrahedron \((T, I, Z)\) as the space \(T^*\) obtained by removing from the tetrahedron \(T\) the ideal vertices, the length-0 edges, and small open stars of the non-ideal vertices. We call **lateral hexagon** and **truncation triangle** the intersection of \(T^*\) respectively with a face of \(T\) and with the link in \(T\) of a non-ideal vertex. We note that, if \((T, I, Z)\) has a length-0 edge, some vertices of a truncation triangle of \(T^*\) may be missing and, if \((T, I, Z)\) has ideal vertices or length-0 edges, a lateral hexagon of \(T^*\) may not be a hexagon, because some of its edges may be missing. See Fig. 30.

Let us consider now a manifold \(N\) which is a candidate to be hyperbolic. Namely, let \(\overline{N}\) be a compact orientable manifold, having no sphere in the boundary, and let \(A \subset \partial \overline{N}\) be a family of disjoint annuli not lying on the toric components of \(\partial \overline{N}\); let \(N\) be obtained from \(\overline{N}\) by removing \(A\) and the toric components. Finally, we define a **partially truncated triangulation** of \(N\) as a realization of \(N\).
as the gluing of some $T^*$’s along a pairing of the lateral hexagons induced by a simplicial pairing of the faces of the $T$’s. Note that the truncation triangles of the $T^*$’s give a triangulation of $\partial N$ with some genuine and some ideal vertices, the links of the ideal vertices of the $T$’s give a triangulation of the toric components of $\partial N$, and the links of the length-0 edges of the $T$’s give a decomposition into rectangles of the annuli in $A$.

Let us now translate the theory of partially truncated triangulations into the language of marked ideal triangulations. Let us consider $N$ as above and let us collapse every annulus $[-1, 1] \times S^1 \in A$ to an arc $[-1, 1] \times \{*\}$. It turns out that the space just obtained, say $N'$, is a compact 3-manifold and each $[-1, 1] \times \{*\}$ is an arc properly embedded in $N'$. Let us call $\alpha_N$ the family of the arcs $[-1, 1] \times \{*\}$ in $N'$. It is a fact that partially truncated triangulations of $N$ bijectively correspond to marked ideal triangulations of the pair $(N', \alpha_N)$; under this correspondence, the length-0 edges and the ideal vertices correspond respectively to the edges in $\alpha_N$ and to the vertices on the tori of $\partial N'$ on which there are no ends of arcs in $\alpha_N$.

Obviously, the admissible MP- and V-moves between marked ideal triangulations of $(N, \alpha_N)$ translate into moves between partially truncated triangulations of $N$. Let us call admissible MP- and V-moves also such moves between partially truncated triangulations. Now, Theorem 2.1 and Corollary 2.2 imply the following.

**Corollary 4.3.** Two partially truncated triangulations of $N$ can be obtained from each other via a sequence of admissible V- and MP-moves. If moreover the two partially truncated triangulations have at least two tetrahedra, then they can be obtained from each other via a sequence of admissible MP-moves only.

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