FUNDAMENTAL GROUP OF A GEOMETRIC INVARIANT
THEORETIC QUOTIENT

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Abstract. Let $M$ be an irreducible smooth projective variety, defined over an algebraically closed field, equipped with an action of a connected reductive affine algebraic group $G$, and let $L$ be a $G$–equivariant very ample line bundle on $M$. Assume that the GIT quotient $M/G$ is a nonempty set. We prove that the homomorphism of algebraic fundamental groups $\pi_1(M) \to \pi_1(M//G)$, induced by the rational map $M\dashrightarrow M//G$, is an isomorphism.

If $k = \mathbb{C}$, then we show that the above rational map $M\dashrightarrow M//G$ induces an isomorphism between the topological fundamental groups.

1. Introduction

Let $M$ be an irreducible smooth projective variety defined over an algebraically closed field $k$ of arbitrary characteristic. Fix a very ample line bundle $\mathcal{L}$ on $M$. Let $G$ be a connected reductive affine algebraic group over $k$ acting algebraically on both $M$ and $\mathcal{L}$ such that $\mathcal{L}$ is a $G$–equivariant line bundle. The geometric invariant theoretic quotient $M//G$ for this action of $G$ on $(M, \mathcal{L})$ will be denoted by $X$. We assume that $X$ is nonempty. The quotienting produces a rational morphism $M \dashrightarrow X$. This rational morphism produces a homomorphism

$$h : \pi_1(M) \to \pi_1(X)$$

between the algebraic fundamental groups.

When $k = \mathbb{C}$, the above rational morphism $M \dashrightarrow X$ produces a homomorphism

$$h' : \pi_t_1(M) \to \pi_t_1(X)$$

between the topological fundamental groups.

We prove the following (see Theorem 3.3 and Theorem 4.4):

Theorem 1.1. Let $G/k$ be a connected reductive algebraic affine group acting on a smooth connected projective variety $M/k$.

(1) The above homomorphism $h$ between the algebraic fundamental groups is an isomorphism.

(2) If $k = \mathbb{C}$, the homomorphism $h'$ between the topological fundamental groups is an isomorphism.
Hui Li proved that for an Hamiltonian action of a compact group on a compact symplectic manifold $M$, the fundamental group of the symplectic quotient coincides with $\pi_1(M)$ [Li, p. 346, Theorems 1.2, 1.3]. A theorem of Kirwan and Kempf–Ness says that for a linear action of a complex reductive group on a smooth complex projective variety, if the stabilizer of every semistable point is finite, then the GIT quotient is homeomorphic to the symplectic quotient [Ki, Theorem 7.5, Remark 8.14], [KN].

2. Lifting an action

Let $k$ be an algebraically closed field. Let $G$ be a connected reductive affine algebraic group over $k$ and $M/k$ an irreducible smooth projective variety equipped with an algebraic action

$$\theta : M \times G \rightarrow M$$

of $G$. Continuing with the above notation we have the following proposition.

**Proposition 2.1.** Let $\varphi : M' \rightarrow M$ be an étale Galois covering morphism such that $M'$ is connected. Then there is a unique action of $G$ on $M'$

$$\theta' : M' \times G \rightarrow M'$$

that lifts $\theta$, meaning $\theta \circ (\varphi \times \text{Id}_G) = \varphi \circ \theta'$.

**Proof.** Since $M' \times G$ is connected, for any two lifts $\theta_1, \theta_2 : M' \times G \rightarrow M'$ of the morphism $\theta$, there is a deck transformation $\gamma \in \text{Gal}(\varphi)$ such that $\theta_2 = \gamma \circ \theta_1$. On the other hand, for an action $\theta'$ of $G$ on $M'$, we have $\theta'(z, e) = z$ for all $z \in M'$, where $e \in G$ denotes the identity element. Therefore, there can be at most one action $\theta'$ of $G$ satisfying the condition that it lifts $\theta$.

We will now prove the existence of a lifted action. For any closed point $x \in M$, let

$$\varphi^x : G \rightarrow M$$

be the morphism defined by $g \mapsto \theta(x, g)$. Consider the induced homomorphism of algebraic fundamental groups

$$\varphi^x_* : \pi_1(G, e) \rightarrow \pi_1(M, x).$$

We will show that

$$\varphi^x_* = 0.$$  \hspace{2cm} (2.2)

To prove (2.2), first take $x$ to be such that its orbit $\theta(x, G) \subset M$ is of minimal dimension among all the $G$–orbits in $M$. Since the boundary of $\theta(x, G)$

$$\overline{\theta(x, G)} \setminus \theta(x, G) \subset M$$

is preserved by the action of $G$, the condition that the dimension of $\theta(x, G)$ is the minimum one implies that this boundary is empty. In other words, the orbit $\theta(x, G)$ is a complete
subvariety of $M$. Let $G_x \subset G$ be the isotropy group-scheme for the point $x$ (we note that $G_x$ need not be reduced). The reduced group
\begin{equation}
(2.3) \quad P := G_{x,\text{red}} \subset G_x \subset G
\end{equation}
is a parabolic subgroup of $G$ because $\theta(x,G)$ is complete. Consider the natural projection
\begin{equation}
(2.4) \quad \xi : P \backslash G \longrightarrow G_x \backslash G = \theta(x,G)
\end{equation}
given by the first inclusion in (2.3). Any solvable algebraic group defined over $k$ is isomorphic as a variety (not as an algebraic group) to a product of copies of $\mathbb{A}^1_k$ and $\mathbb{G}_m$ (recall that $k$ is algebraically closed). Therefore, from the Bruhat decomposition of $G$ it follows that the variety $G$ is rational. Combining this with the fact that the quotient morphism $G \longrightarrow P \backslash G$ is separable we conclude that $P \backslash G$ is separably rationally connected. This implies that the variety $P \backslash G$ is simply connected [Kol, p. 75, Theorem 13].

Let $\nu : \tilde{C} \longrightarrow \theta(x,G)$
be a finite étale Galois covering. Let
\begin{equation}
(2.5) \quad \begin{array}{ccc}
(P \backslash G) \times_{\theta(x,G)} \tilde{C} & \overset{\delta}{\longrightarrow} & \tilde{C} \\
\downarrow \mu & & \downarrow \\
P \backslash G & \overset{\xi}{\longrightarrow} & \theta(x,G)
\end{array}
\end{equation}
be the pullback of the covering $\nu$ to $P \backslash G$ by the morphism $\xi$ in (2.4). Since $P \backslash G$ is simply connected, the covering $\mu$ in (2.5) admits a section
\begin{equation}
\eta : P \backslash G \longrightarrow (P \backslash G) \times_{\theta(x,G)} \tilde{C}.
\end{equation}
Now consider the composition
\begin{equation}
\delta \circ \eta : P \backslash G \longrightarrow \tilde{C},
\end{equation}
where $\delta$ is the projection in (2.5). The image of $\delta \circ \eta$ is a connected component of $\tilde{C}$ that projects isomorphically to $\theta(x,G)$. Indeed, this follows immediately from the fact that $\xi$ in (2.4) is bijective on closed points. Hence we conclude that
\begin{equation}
\pi_1(\theta(x,G), x) = 0.
\end{equation}
Therefore, (2.2) holds for this point $x$.

For another closed point $y$ of $M$, let $\varphi^y : G \longrightarrow M$ be the morphism defined by $g \longmapsto \theta(y,g)$. Let
\begin{equation}
\psi^x : G \longrightarrow M \times G \quad \text{and} \quad \psi^y : G \longrightarrow M \times G
\end{equation}
be the embeddings defined by $g \longmapsto (x, g)$ and $g \longmapsto (y, g)$ respectively. So, we have $\varphi^x = \theta \circ \psi^x$ and $\varphi^y = \theta \circ \psi^y$. Therefore, $\varphi^x_* = \theta_* \circ \psi^x_*$ and $\varphi^y_* = \theta_* \circ \psi^y_*$, where $\varphi^x_*$ is the homomorphism in (2.1). Since $M$ is projective,
\begin{equation}
\pi_1(M \times G, (y, e)) = \pi_1(M, y) \times \pi_1(G, e)
\end{equation}
[SAG1, Exposé X, §1, Corollaire 5.1]. The two groups $\pi_1(M \times G, (y, e))$ and $\pi_1(M \times G, (x, e))$ are identified up to conjugation, and such an identification takes the image
Take $M'$ in the statement of the proposition. Since $M'$ is projective, we have
\[
\pi_1(M' \times G, (x', e)) = \pi_1(M', x') \times \pi_1(G, e)
\]
SGA1 Exposé X, § 1, Corollaire 5.1. Consider the diagram
\[
\begin{array}{ccc}
M' \times G & \xrightarrow{\theta} & M' \\
\downarrow \varphi \times \text{Id}_G & & \downarrow \varphi \\
M \times G & \xrightarrow{\text{Id}} & M
\end{array}
\]
Take any closed point $x' \in \varphi^{-1}(x)$. Now consider the induced homomorphism of algebraic fundamental groups
\[
(\theta \circ (\varphi \times \text{Id}_G))_* : \pi_1(M' \times G, (x', e)) = \pi_1(M', x') \times \pi_1(G, e) \longrightarrow \pi_1(M, x).
\]
From (2.2) it follows immediately that we have
\[
(\theta \circ (\varphi \times \text{Id}_G))_* (\{e_0\} \times \pi_1(G, e)) = 0,
\]
where $e_0 \in \pi_1(M', x')$ denotes the identity element. Consequently, the image of $(\theta \circ (\varphi \times \text{Id}_G))_*$ coincides with the image of the homomorphism
\[
\varphi_* : \pi_1(M', x') \longrightarrow \pi_1(M, x).
\]
This implies that the pulled back Galois étale covering
\[
(M' \times G) \times_M M' \longrightarrow M' \times G
\]
is identified with the trivial covering $M' \times G \times \varphi^{-1}(x) \longrightarrow M' \times G$. Consequently, there is a unique morphism
\[
(2.6) \quad \theta' : M' \times G \longrightarrow M'
\]
such that the following two conditions hold:
\[
(2.7) \quad \varphi \circ \theta' = \theta \circ (\varphi \times \text{Id}_G)
\]
and $\theta'(x', e) = x'$.

In view of (2.7), the morphism $\theta'_e : M' \longrightarrow M'$ defined by $y \longmapsto \theta'(y, e)$ is a lift of the identity map of $M$ because $\theta(z, e) = z$ for all $z \in M$. Therefore, from the given condition that $\theta'(x', e) = x'$ it follows that $\theta'_e = \text{Id}_{M'}$.

Next consider the two morphisms
\[
a, b : M' \times G \times G \longrightarrow M'
\]
defined by $(z, g_1, g_2) \longmapsto \theta'((z, g_1), g_2)$ and $(z, g_1, g_2) \longmapsto \theta(z, g_1, g_2)$ respectively. The morphism $a$ (respectively, $b$) is a lift of the morphism $M \times G \times G \longrightarrow M$ defined by $(z, g_1, g_2) \longmapsto \theta((z, g_1), g_2)$ (respectively, $(z, g_1, g_2) \longmapsto \theta(z, g_1, g_2)$). These two morphisms from $M \times G \times G$ to $M$ coincide because $\theta$ is an action of $G$ on $M$. Also,
\[
a(z, e, e) = z = b(z, e, e)
\]
for all $z \in M'$. Therefore, we conclude that $a = b$. Consequently, the morphism $\theta'$ defines an action of $G$ on $M'$.

Let $\Gamma = \text{Gal}(\varphi) \subset \text{Aut}(M')$ be the Galois group for the Galois covering $\varphi$.

**Lemma 2.2.** The Galois action of $\Gamma$ on $M'$ commutes with the action of $G$ on $M'$ given by $\theta'$ in Proposition 2.1.

**Proof.** Take any $\gamma \in \Gamma$. The morphism

$$\gamma'': M' \times G \rightarrow M', \quad (z, g) \mapsto \gamma'(\theta'^{-1}(z), g)$$

is an action of $G$ on $M'$ that lifts the action $\theta$ of $G$ on $M$. Now from the uniqueness of $\theta'$ it follows that $\gamma'' = \theta'$. This immediately implies that the actions of $\Gamma$ and $G$ on $M'$ commute. \qed

3. *Fundamental group of the quotient*

Let $L$ be a $G$–equivariant very ample line bundle on $M$. The action of any $g \in G$ on any $v \in L$ will be denoted by $v \cdot g$. Let

$$(3.1) \quad X := M//G$$

be the geometric invariant theoretic (GIT) quotient of $M$ for the action of $G$ on $(M, L)$. We assume that $X$ is nonempty. This $X$ is an irreducible normal projective variety. Let $U \subset M$ be the largest Zariski open subset over which the rational map to the GIT quotient $M \dashrightarrow X$ is defined. Consider the homomorphism

$$(3.2) \quad \pi_1(U, u_0) \rightarrow \pi_1(X, x_0)$$

induced by the quotient map, where $u_0 \in U$ is a point lying over a point $x_0 \in X$. The codimension of the complement $M \setminus U \subset M$ is at least two because this complement is the indeterminacy locus of a rational morphism. Since $M$ is smooth, this codimension condition implies that the homomorphism

$$\pi_1(U, u_0) \rightarrow \pi_1(M, u_0),$$

induced by the inclusion map $U \hookrightarrow M$, is an isomorphism. Using this isomorphism, the homomorphism in (3.2) produces a homomorphism

$$(3.3) \quad h : \pi_1(M, u_0) \rightarrow \pi_1(X, x_0).$$

Take $(M', \varphi)$ as in Proposition 2.1. Consider the ample line bundle $\varphi^*L$ on the covering $M'$ and the action $\theta'$ of $G$ on $M'$ (constructed in Proposition 2.1). Since $\theta'$ is a lift of the action $\theta$, the action of $G$ on $L$ produces an action of $G$ on $\varphi^*L$. The action of any $g \in G$ sends any $v \in (\varphi^*L)_x$ to the element in $(\varphi^*L)_{\theta'(x,g)}$ that corresponds to $v \cdot g$ by the natural identification $L_{\theta(\varphi(x),g)} = (\varphi^*L)_{\theta'(x,g)}$ after we consider $v$ as an element of
\[ \mathcal{L}_{\varphi(x)} \] using the identification \((\varphi^*\mathcal{L})_x = \mathcal{L}_{\varphi(x)}\). This action of \(G\) on \(\varphi^*\mathcal{L}\) evidently lifts the action \(\theta'\). Let

\[
(3.4) \quad \tilde{X}' := M'/\!/G
\]

be the GIT quotient for the action of \(G\) on \((M', \varphi^*\mathcal{L})\).

Let \(M^{ss} \subset M\) be the semistable locus for the action of \(G\); it is an open subscheme of \(M\). Let

\[
\tilde{M} := \varphi^{-1}(M^{ss}) \subset M'
\]

be the inverse image. We note that the subset \(\tilde{M}\) is left invariant under the action of \(G\) on \(M'\) because \(M^{ss}\) is preserved by the action of \(G\) on \(M\) and \(\varphi\) is \(G\)-equivariant.

There is a finite collection of \(G\)-invariant nonzero sections \(\{s_i\}_{i=1}^N\) of \(\mathcal{L}\) such that the collection

\[
U_i := \text{Spec} \ A_i = \{z \in M \mid s_i(z) \neq 0\}
\]

is an affine open cover of \(M^{ss}\). Since \(s_i\) is \(G\)-invariant, the subset \(U_i\) is preserved by the action of \(G\) on \(M\). The GIT quotient \(X = M/\!/G\) is obtained by patching together the affine open subschemes \(V_i = \text{Spec}(A_i^G)\) (see [New, Ch. 3, §3] and [New, Ch. 3, §4] for affine and projective GIT quotients respectively).

Consider the affine open cover of \(\tilde{M}\)

\[
U'_i := \varphi^{-1}(U_i) = \text{Spec} \ B_i = \{z \in \tilde{M} \mid \varphi^* s_i(z) \neq 0\} \subset \tilde{M}.
\]

Note that each \(U'_i\) is preserved by the action of \(G\) on \(M'\). We may patch together the affine open subschemes \(V'_i := \text{Spec}(B_i^G)\) to construct a quotient \(\tilde{X}\) (see the proof of Theorem 1.10 in [Mu, p. 38]). Clearly,

\[
\tilde{X} \subset \tilde{X}'
\]

is an open subscheme, where \(\tilde{X}'\) is the quotient in (3.4).

**Proposition 3.1.** The natural morphism

\[
f : \tilde{X} \longrightarrow X
\]

is an étale Galois covering with Galois group \(\Gamma = \text{Gal}(\varphi)\). Moreover the restriction

\[
\varphi|_{\tilde{M}} : \tilde{M} \longrightarrow M^{ss}
\]

is the pullback of \(f\) via the quotient map \(q : M^{ss} \longrightarrow M/\!/G = X\).

**Proof.** From (2.2) it follows immediately that the restriction of the covering \(\varphi\) to any orbit \(\theta(x, G) \subset M\) is trivial. In other words, the inverse image \(\varphi^{-1}(\theta(x, G))\) is a disjoint union of copies of \(\theta(x, G)\). In view of Lemma 2.2 this implies that the Galois group \(\Gamma\) acts simply transitively on the set of connected components of \(\varphi^{-1}(\theta(x, G))\). In particular, for any \(y \in M'\), the restriction of \(\varphi\) to the orbit \(\theta'(y, G)\) is injective.

Therefore, to prove the proposition, we may replace \(M\) by the spectrum of an integral finite type algebra \(A\), with quotient field \(K\), equipped with an action of \(G\). Similarly, the variety \(\tilde{M}\) and the action of \(G\) on it are replaced by a connected finite étale algebra \(B\), with the quotient field \(L\), over \(A\) with Galois group \(\Gamma\), and a lifting to \(B\) of the action of
G on A that commutes with the action of Γ on B. The quotients \( \tilde{X} \) and \( X \) get replaced by \( \text{Spec}(A^G) \) and \( \text{Spec}(B^G) \) respectively. Since \( M \) is smooth, hence normal, and the map \( \varphi \) restricted to any closed orbit of \( G \) is injective, the following lemma completes the proof of the proposition.

**Lemma 3.2.** Suppose the \( G \)-equivariant finite étale map \( f : \tilde{M} \rightarrow M \) of affine varieties is defined by an inclusion \( A \subset B \) of finite type \( k \) algebras such that

- \( A \) is normal, and
- \( f \) restricted to each closed orbit of \( G \) is an injection.

Then the induced map on the quotients \( \tilde{X} \rightarrow X \) is also finite étale, and \( \tilde{M} \) is the fiber product \( \tilde{X} \times_X M \).

**Proof.** This can be found in [Dr] (see [Dr, Proposition 4.16] and [Dr, Proposition 4.18]). In [Dr] it is assumed that the characteristic of the field \( k \) is zero. However, the proof can be checked to be characteristic free. For the convenience of the reader, we give a brief outline of the proof.

The actions of \( G \) on \( A \) and \( B \) extend to the quotient fields \( K \) and \( L \) respectively. The spaces of invariants for the action of the Galois group \( \Gamma \) on \( B \) and \( L \) are \( A \) and \( K \) respectively.

Since the actions of \( G \) and \( \Gamma \) on \( B \) commute, the natural inclusion \( A^G \subset (B^G)^\Gamma \) is an isomorphism. This implies that \( B^G \) is finite over \( A^G \).

We first observe that \( B^G \) is the integral closure of \( A^G \) in \( L \). Indeed, if an element \( a \in L \) is integral over \( A^G \), then all \( G \) translates of \( a \) are also solutions of the same equation. Therefore, the connectedness of \( G \) implies that \( a \) is fixed by \( G \).

Consequently, \( \text{Spec}(B^G) \) is a normal affine variety equipped with an action of \( \Gamma \) such that the quotient is \( \text{Spec}(A^G) \). So to show that the map \( \tilde{X} \rightarrow X \) in the lemma is étale it is enough to check that this action of \( \Gamma \) is free on the closed points.

Let \( x \in \text{Spec}(B^G) \) be a closed point such that there is an element \( \gamma \in \Gamma \) with \( \gamma \cdot x = x \). Let \( y \) be a closed point in the unique closed orbit in the fiber of the map \( \tilde{M} \rightarrow \tilde{X} \) over \( x \). Since \( \gamma \) commutes with \( G \) (see Lemma 2.2) we get that \( \theta'(\gamma \cdot y, G) = \gamma \cdot \theta'(y, G) \) is also the unique closed orbit projecting to \( x \). Hence, we have \( \theta'(\gamma \cdot y, G) = \theta'(y, G) \). Now from the injectivity of the map from \( \Gamma \) to the permutations of the components of \( \theta'(y, G) \) we conclude that \( \gamma \cdot y = y \). Consequently, we have \( \gamma = e \). This proves that the morphism \( \tilde{X} \rightarrow X \) is étale.

For the isomorphism \( \tilde{M} = \tilde{X} \times_X M \), first note that the \( G \)-equivariant inclusion \( A \rightarrow B \) factors via

\[
A \subset A \otimes_{A^G} B^G.
\]

Therefore, in order to prove that \( \tilde{M} = \tilde{X} \times_X M \) it is enough to prove it under the assumption that the natural \( G \)-equivariant homomorphism

\[
A \otimes_{A^G} B^G \rightarrow B
\]
is an isomorphism.

By using the conclusion of the first part of the lemma that $B^G$ is finite and étale over $A^G$ of cardinality $|\Gamma|$, the base change $A \rightarrow A \otimes_{A^G} B^G$ is also finite and étale of the same cardinality. Since we started with a finite and étale algebra $B$ over $A$ we conclude that $A \otimes_{A^G} B^G \rightarrow B$ is also finite and étale.

Now the isomorphism $\widetilde{M} = \widetilde{X} \times_X M$ follows because both $\text{Spec } B$ and $\text{Spec}(A \otimes_{A^G} B^G)$ are finite and étale over $\text{Spec } A$ of same fiber cardinality $|\Gamma|$. □

The construction in Proposition 3.1 of a covering $\widetilde{X}$ starting from an étale covering $M'$ of $M$ is functorial (compatible with the standard operations on coverings), and defines a homomorphism

\[(3.5) \quad H : \pi_1(X, x_0) \rightarrow \pi_1(M, u_0).\]

Given an étale Galois covering $\phi : Y \rightarrow X$, the étale Galois covering of $M$ corresponding to the homomorphism $h$ in (3.3) is constructed as follows. Consider the pullback of the covering $\phi$ to the open subset of $M$ where the rational map $M \rightarrow X$ is defined. This covering extends to $M$ because the complement of the open subset has codimension at least two and $M$ is smooth.

The above two constructions, namely the construction of a covering of $X$ from a covering of $M$, and the construction of a covering of $M$ from a covering of $X$, are clearly inverses of each other. Therefore, for the two homomorphisms $h$ and $H$ in (3.3) and (3.5), we have $h \circ H = \text{Id}_{\pi_1(X, x_0)}$ and $H \circ h = \text{Id}_{\pi_1(M, u_0)}$. Consequently, the following is proved:

**Theorem 3.3.** The homomorphism $h$ in (3.3) is an isomorphism.

**Remark 3.4.** Let $M = \mathbb{P}_k^1/\{0 = \infty\}$ be the unique nodal curve of arithmetic genus one. The action of $\mathbb{G}_m$ on $\mathbb{P}_k^1$ defined by $t \cdot (x_1, x_2) = (tx_1, x_2/t)$ produces an action of $\mathbb{G}_m$ on $M$. While $M$ is not simply connected, the GIT quotient $M/\mathbb{G}_m = \text{Spec } k$ is so. So the condition in Theorem 3.3 that $M$ is smooth is essential.

4. **The topological fundamental group**

In this section we assume that $k = \mathbb{C}$. The topological fundamental group of any $Y$ with base point $y_0 \in Y$ will be denoted by $\pi_1(Y, y_0)$. We recall that an irreducible smooth complex projective variety is also called a complex projective manifold.

As before, fix a $G$–equivariant algebraic line bundle on the complex projective manifold $M$.

Let $\varphi : M' \rightarrow M$ be a holomorphic étale Galois covering of $M$ such that $M'$ is connected. It should be clarified that the degree of $\varphi$ is now allowed to be infinite.

**Proposition 4.1.**
(1) There is a unique holomorphic action of $G$ on $M'$

$$\theta' : M' \times G \longrightarrow M'$$

that lifts the action $\theta$ of $G$ on $M$.

(2) The actions of $G$ and $\Gamma = \text{Gal}(\varphi)$ on $M'$ commute.

Proof. The proof of the first (respectively, second) part of the proposition is exactly identical to the proof of Proposition 2.1 (respectively, Lemma 2.2). \qed

The geometric invariant theory is not applicable for the action of $G$ on $M'$ because $M'$ is not an algebraic variety in general. However we will construct from $M'$ a covering of the GIT quotient $X = M//G$.

Let $M^{ss} \subset M$ be the semistable locus for the action of $G$ on $M$; it is a Zariski open subset preserved by the action of $G$. Let

$$\beta : M^{ss} \longrightarrow X$$

be the quotient map. This map $\beta$ is surjective.

Take an affine open subset $U \subset X$. The inverse image

$$\beta^{-1}(U) \subset M^{ss}$$

is also an affine open subset. Fix a maximal compact subgroup

$$K_G \subset G.$$ 

There is a $K_G$–invariant subset

$$U_K \subset \beta^{-1}(U)$$

such that

- $\beta^{-1}(U)$ admits a deformation retraction to $U_K$,
- the map $\beta|_{U_K} : U_K \longrightarrow U$ is surjective,
- the quotient map $U_K/K_G \longrightarrow U$ is a homeomorphism, and
- the subset $U_K$ is contained in the union of all closed $G$–orbits satisfying the condition that the intersection with $U_K$ is a $K$–orbit.

(See [Nec, p. 422, Corollary 1.4] and [Nec, p. 424, Theorem 2.1]; also stated in the first paragraph of the introduction in [Nec, p. 419]. See also [KN].)

Let

$$\tilde{U}_K := \varphi^{-1}(U_K) \subset M'$$

be the inverse image. As $\beta^{-1}(U)$ is a nonempty Zariski open subset of $M$, and $M$ is smooth, the homomorphism of topological fundamental groups

$$\pi_1(\beta^{-1}(U)) \longrightarrow \pi_1(M),$$

(4.3)
induced by the inclusion of $\beta^{-1}(U)$ in $M$, is surjective. Since $U_K$ is a deformation retraction of $\beta^{-1}(U)$, from the surjectivity of the homomorphism in (4.3) it follows immediately that the homomorphism

\begin{equation}
\pi_1(U_K) \rightarrow \pi_1(M)
\end{equation}

induced by the inclusion of $U_K$ in $M$ is surjective. Note that surjectivity of the homomorphism induced on fundamental groups induced by a map is equivalent to the condition that the pullback to the domain (of the map) of any connected étale covering of the target space (of the map) remains connected. Consequently, from the surjectivity of the homomorphism in (4.4) it follows that the inverse image $\tilde{U}_K$ in (4.2) is connected (recall that $M'$ is connected).

**Lemma 4.2.** Take a point $x \in M$. Let $Z := \theta(x, K_G) \subset M$ be the $K_G$-orbit of $x$. Then $\varphi^{-1}(Z)$ is a disjoint union of copies of $Z$. More precisely, the restriction of $\varphi$ to each connected component $S$ of $\varphi^{-1}(Z)$ is a homeomorphism from $S$ to $Z$.

**Proof.** Let $\iota^x : Z \hookrightarrow M$ be the inclusion map. To prove the lemma it suffices to show that the homomorphism

\[ \iota^x_\ast : \pi_1(Z, x) \rightarrow \pi_1(M, x), \]

induced by the inclusion $\iota^x$, is trivial.

Let $G_x \subset G$ be the isotropy subgroup for $x$. The $G$–orbit $\theta(x, G)$ is identified with the quotient $G/G_x$. Since $M$ is projective, there is an irreducible smooth complex projective variety $\tilde{Z}$ containing $G/G_x$ as a Zariski open subset such that the inclusion map

\[ G/G_x = \theta(x, G) \hookrightarrow M \]

extends to a morphism

\begin{equation}
\tau : \tilde{Z} \rightarrow M.
\end{equation}

Since $G$ is a rational variety, the quotient $G/G_x$ is unirational. So $\tilde{Z}$ is unirational. Since $\tilde{Z}$ is smooth, this implies that $\tilde{Z}$ is simply connected [Se, p. 483, Proposition 1].

The above inclusion map $\iota^x$ coincides with the composition

\[ Z = K_G/(K_G \cap G_x) \hookrightarrow G/G_x \hookrightarrow \tilde{Z} \xrightarrow{\tau} M, \]

where $\tau$ is the map in (4.5). Since $\tilde{Z}$ is simply connected, this implies that $\iota^x_\ast = 0$. As noted before, the lemma follows from this.

**Corollary 4.3.** The image in $\pi_1(M)$ of the fundamental group of any $G$ orbit in $M$ is trivial.

**Proof.** We saw in the proof of Lemma 4.2 that the inclusion of the orbit $\theta(x, G)$ in $M$ factors through the simply connected variety $\tilde{Z}$.

Define

\begin{equation}
\tilde{U} := \tilde{U}_K/K_G,
\end{equation}
where \( \tilde{U}_K \) is defined in (4.2). We note that \( \tilde{U} \) is connected because \( \tilde{U}_K \) is connected. The natural map
\[
(\beta \circ \varphi)|_{\tilde{U}_K} : \tilde{U}_K \longrightarrow \mathcal{U},
\]
is clearly \( K_G \)–invariant. So it descends to a map
\[
(4.7) \quad \tilde{\varphi}_U : \tilde{U} \longrightarrow \mathcal{U},
\]
where \( \tilde{U} \) is defined in (4.6).

Since the actions of \( G \) and \( \Gamma \) on \( M' \) commute (Proposition 4.1(2)), the action of \( \Gamma \) on \( \tilde{U}_K \) descends to an action of \( \Gamma \) on the quotient \( \tilde{U} \) in (4.6). Note that the quotient map
\[
(4.8) \quad \tilde{U} \longrightarrow \tilde{U}/\Gamma
\]
is an étale Galois covering map with Galois group \( \Gamma \) because the covering
\[
\varphi|_{\tilde{U}_K} : \tilde{U}_K \longrightarrow U_K
\]
is \( K_G \)–equivariant and the map in (4.8) is the quotient for the actions of \( K_G \).

Take a finite collection of affine open subsets \( \{U_i\}_{i=1}^n \) of \( X \) that cover \( X \). Let \( \tilde{U}_i \) be the topological space constructed as in (4.6) from \( U_i \). Let \( X' \) be the topological space constructed by performing the above gluing on the disjoint union \( \bigsqcup_{i=1}^n \tilde{U}_i \). Recall that all \( \tilde{U}_i \) are connected, and for any pair \( \tilde{U}_i \) and \( \tilde{U}_j \), the gluing between them is done over connected open subsets of \( \tilde{U}_i \) and \( \tilde{U}_j \). Therefore, the resulting topological space \( X' \) is connected.

Recall that each \( \tilde{U}_i \) is an étale Galois covering of \( U_i \) with Galois group \( \text{Gal}(\varphi) \) (via the map \( \tilde{\varphi}_U \) in (4.7)). The intersection of \( \tilde{U}_i \) with \( \tilde{U}_j \) inside \( X' \) is an étale Galois covering of \( U_i \cap U_j \) with Galois group \( \text{Gal}(\varphi) \). Therefore, \( X' \) is an étale Galois covering of \( X \) with Galois group \( \text{Gal}(\varphi) \).

Just as Theorem 3.3 was proved, we now have the following:

**Theorem 4.4.** The homomorphism between topological fundamental groups induced by the rational map \( M \longrightarrow M/G \) is an isomorphism.
Remark 4.5. It is easy to construct examples to show that Theorem 1.1 is false if $M$ is not assumed to be projective. Let $G = \mathbb{G}_m$ act on $M = \mathbb{G}_m$ by left translations. Then the quotient $M/\!/G$ coincides with $\text{Spec } k$. Clearly $\pi_1(\mathbb{G}_m) \longrightarrow \pi_1(\text{Spec } k)$ is not an isomorphism.

5. Acknowledgements

Niels Borne pointed out a gap in the proof of Proposition 2.1 given in an earlier version. We are very grateful to him for this. We are very grateful to Jakob Stix for some very helpful correspondences. We thank Michel Brion for a useful correspondence. We are very grateful to the two referees for their comments. We thank the managing editors for pointing out [Li]. The first-named author acknowledges the support of the J. C. Bose Fellowship.

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