Elliptic operators and their symbols

Abstract: We consider special elliptic operators in functional spaces on manifolds with a boundary which has some singular points. Such an operator can be represented by a sum of operators, and for a Fredholm property of an initial operator one needs a Fredholm property for each operator from this sum.

Keywords: elliptic operator, local representative, enveloping operator

MSC: 47A05; 58J05

1 Introduction

This paper is devoted to describing the structure of a special class of linear bounded operators on a manifold with non-smooth boundary. Our description is based on Simonenko's theory of envelopes \cite{1} and explains why we obtain distinct theories for pseudo-differential equations and boundary value problems and distinct index theorems for such operators.

1.1 Operators of a local type

In this section we will give some preliminary ideas and definitions from \cite{1}.

Let $B_1, B_2$ be Banach spaces consisting of functions defined on compact $m$-dimensional manifold $M$, $A : B_1 \to B_2$ be a linear bounded operator, $W \subset M$, and $P_W$ be a projector on $W$, i.e.

$$(P_W u)(x) = \begin{cases} u(x), & \text{if } x \in W; \\ 0, & \text{if } x \notin W. \end{cases}$$

Definition 1. An operator $A$ is called an operator of local type if the operator

$$P_U A P_V$$

is a compact operator for arbitrary non-intersecting compact sets $U, V \subset M$.

1.2 Simple examples

These are two of the simplest examples for illustration.

Example 1. If $A$ is a differential operator of the type

$$(Au)(x) = \sum_{|k|=0}^{n} a_k(x) D^k u(x), \quad D^k u = \frac{\partial^k u}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}},$$

then $A$ is an operator of local type.

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Example 2. If $A$ is a Calderon–Zygmund operator with variable kernel $K(x, y) \in C^1(\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ of the following type

$$(Au)(x) = \nu.p. \int_{\mathbb{R}^m} K(x, x - y)u(y)dy,$$

then $A$ is an operator of a local type.

Everywhere below we say “an operator” instead of “an operator of local type”.

1.3 Functional spaces on a manifold

1.3.1 Spaces $H^s(\mathbb{R}^m)$, $L^p(\mathbb{R}^m)$, $C^\alpha(\mathbb{R}^m)$

It is possible to work with distinct functional spaces [2, 3].

Definition 2. [4] The space $H^s(\mathbb{R}^m)$, $s \in \mathbb{R}$, is a Hilbert space of functions with the finite norm

$$||u||_s = \left( \int_{\mathbb{R}^m} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2},$$

where the sign $\sim$ over a function means its Fourier transform.

Definition 3. [2] The space $L^p(\mathbb{R}^m)$, $1 < p < +\infty$, is a Banach space of measurable functions with the finite norm

$$||u||_p = \left( \int_{\mathbb{R}^m} |u(x)|^p dx \right)^{1/p}.$$

Definition 4. [2] The space $C^\alpha(\mathbb{R}^m)$, $0 < \alpha \leq 1$, is a space of continuous functions $u$ on $\mathbb{R}^m$ satisfying the H"older condition

$$|u(x) - u(y)| \leq c|x - y|^\alpha, \quad \forall x, y \in \mathbb{R}^m,$$

with the finite norm

$$||u||_\alpha = \inf\{c\},$$

where infimum is taken over all constants $c$ from the above inequality.

1.3.2 Partition of unity and spaces $H^s(M)$, $L^p(M)$, $C^\alpha(M)$

If $M$ is a compact manifold then there is a partition of unity [5]. It means the following. For every finite open covering $\{U_j\}_{j=1}^k$ of the manifold $M$ there exists a system of functions $\{\varphi_j(x)\}_{j=1}^k$, $\varphi_j(x) \in C^\infty(M)$, such that

- $0 \leq \varphi_j(x) \leq 1$,
- $\text{supp } \varphi_j \subset U_j$,
- $\sum_{j=1}^k \varphi_j(x) = 1$.

So we have

$$f(x) = \sum_{j=1}^k \varphi_j(x)f(x)$$

for an arbitrary function $f$ defined on $M$.

Since every set $U_j$ is diffeomorphic to an open set $D_j \subset \mathbb{R}^m$ we have corresponding diffeomorphisms $\omega_j : U_j \rightarrow D_j$. Further, for a function $f$ defined on $M$ we compose mappings $f_j = f \cdot \varphi_j$ and as long as
supp \( f_j \subset U_j \) we put \( \tilde{f}_j = f_j \circ \omega_j^{-1} \) so that \( \tilde{f}_j : D_j \to \mathbb{R} \) is a function defined on a domain of \( m \)-dimensional space \( \mathbb{R}^m \). We can consider, for example, the following functional spaces [2–4].

**Definition 5.** A function \( f \in H^s(M) \) if the following norm

\[
||f||_{H^s(M)} = \sum_{j=1}^{k} ||\tilde{f}_j||_s
\]

is finite.

A function \( f \in L_p(M) \) if the following norm

\[
||f||_{L_p(M)} = \sum_{j=1}^{k} ||\tilde{f}_j||_p
\]

is finite.

A function \( f \in C^\alpha(M) \) if the following norm

\[
||f||_{C^\alpha(M)} = \sum_{j=1}^{k} ||\tilde{f}_j||_\alpha
\]

is finite.

**2 Operators on a compact manifold**

On the manifold \( M \) we fix a finite open covering and a partition of unity corresponding to this covering \( \{U_j, f_j\}_{j=1}^n \). We then choose smooth functions \( \{g_j\}_{j=1}^n \) so that \( supp g_j \subset V_j, \overline{U_j} \subset V_j, \) and \( g_j(x) \equiv 1 \) for \( x \in supp f_j, supp f_j \cap (1-g_j) = \emptyset \).

**Proposition 1.** The operator \( A \) on the manifold \( M \) can be represented in the form

\[
A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,
\]

where \( T : B_1 \to B_2 \) is a compact operator.

**Proof.** The proof is straightforward. Since

\[
\sum_{j=1}^{n} f_j(x) \equiv 1, \quad \forall x \in M,
\]

then we have

\[
A = \sum_{j=1}^{n} f_j \cdot A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + \sum_{j=1}^{n} f_j \cdot A \cdot (1-g_j),
\]

and the proof is completed. \( \square \)

**Remark 1.** Obviously such an operator is defined uniquely up to a compact operators which have no influence on an index.

By definition, for an arbitrary operator \( A : B_1 \to B_2 \)

\[
|||A||| = \inf ||A + T||,
\]

where infimum is taken over all compact operators \( T : B_1 \to B_2 \).

Let \( B'_1, B'_2 \) be Banach spaces consisting of functions defined on \( \mathbb{R}^m \), and let \( \tilde{A} : B'_1 \to B'_2 \) be a linear bounded operator.
Since $M$ is a compact manifold, then for every point $x \in M$ there exists a neighborhood $U \ni x$ and a diffeomorphism $\omega : U \to D \subset \mathbb{R}^m$, $\omega(x) \equiv y$. We denote by $S_\omega$ the following operator acting from $B_k$ to $B'_{k'}$, $k, k' = 1, 2$. For every function $u \in B_k$ vanishing out of $U$

\[(S_\omega u)(y) = u(\omega^{-1}(y)), \quad y \in D, \quad (S_\omega u)(y) = 0, \quad y \notin D.\]

**Definition 6.** A local representative of the operator $A : B_1 \to B_2$ at the point $x \in M$ is called the operator $\tilde{A} : B'_1 \to B'_2$ such that for all $\varepsilon > 0$ there exists the neighborhood $U_j$ of the point $x \in U_j \subset M$ with the property

\[
|||g_j Af_j - S_\omega^{-1} g_j \tilde{A} f_j S_\omega||| < \varepsilon.
\]

**3 Algebra of symbols**

**Definition 7.** Symbol of an operator $A$ is called the family of its local representatives $\{A_x\}$ at each point $x \in M$.

One can show like [1] this definition of an operator symbol conserves all properties of a symbolic calculus. Namely, up to compact summands we have the following:

- the product and the sum of two operators corresponds to the product and the sum of their local representatives;
- the adjoint operator corresponds to its adjoint local representative;
- a Fredholm property of an operator corresponds to a Fredholm property of its local representative.

**4 Operators with symbols. Examples of operators**

It seems not every operator has a symbol, and we give some examples for operators with symbols.

**Example 3.** Let $A$ be the differential operator from Example 1, and functions $a_k(x)$ be continuous functions on $\mathbb{R}^m$. Then its symbol is an operator family consisting of multiplication operators on the function

\[
\sum_{|k|=0}^n a_k(x)\xi^k,
\]

where $\xi^k = \xi_1^{k_1} \cdots \xi_m^{k_m}$.

**Example 4.** Let $A$ be the Calderon–Zygmund operator from Example 2 and $\sigma(x, \xi)$ be its symbol in the sense of [2], then its symbol is an operator family consisting of multiplication operators on the function $\sigma(x, \xi)$.

The more important point is that the symbol of an operator is simpler than general operator, and it permits to verify its Fredholm properties. For the two above examples a Fredholm property of an operator symbol is equivalent to its invertibility.

**5 Stratification of manifolds and operators**

**5.1 Sub-manifolds**

The above definition of an operator on a manifold supposes that all neighborhoods $\{U_j\}$ have the same type. But even if a manifold has a smooth boundary then there are two types of neighborhoods related to a placement of neighborhood, namely inner neighborhoods and boundary ones. For an inner neighborhood $U$ such that $\overline{U} \subset \overline{M}$ we have the diffeomorphism $\omega : U \to D$, where $D \subset \mathbb{R}^m$ is an open set. For a boundary neighborhood such that $U \cap \partial M \neq \emptyset$ we have another diffeomorphism $\omega_1 : U \to D \cap \mathbb{R}^m_+$, where

\[
\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > 0\}.
\]
Maybe this boundary $\partial M$ has some singularities like conical points and wedges. The conical point at the boundary is such a point, for which its neighborhood is diffeomorphic to the cone

$$C^a_n = \{ x \in \mathbb{R}^m : x_m > a|x'|, \ x' = (x_1, \ldots, x_{m-1}), \ a > 0 \}.$$  

The wedge point of codimension $k$, $1 \leq k \leq m-1$, is such a point for which its neighborhood is diffeomorphic to the set $\{ x \in \mathbb{R}^m : x = (x', x''), x'' \in \mathbb{R}^{m-k}, x' = (x_1, \ldots, x_{m-k-1}), x_{m-k-1} > a|x'''|, x''' = (x_1, \ldots, x_{m-k-2}), \ a > 0 \}$. So if the manifold $M$ has such singularities we suppose that we can extract certain $k$-dimensional submanifolds, namely an $(m - 1)$-dimensional boundary $\partial M$, and $k$-dimensional wedges $M_k, k = 0, \ldots, m - 2$; $M_0$ are a collection of conical points.

### 5.2 Enveloping operators

If the family $\{A_x\}_{x \in M}$ is continuous in the operator topology, then according to Simonenko’s theory there is an enveloping operator, i.e. such an operator $A$ for which every operator $A_x$ is the local representative for the operator $A$ in the point $x \in M$.

**Example 5.** If $\{A_x\}_{x \in M}$ consists of Calderon–Zygmund operators in $\mathbb{R}^m$ [2] with symbols $\sigma(x, \xi)$ parametrized by points $x \in M$ and this family smoothly depends on $x \in M$ then the Calderon–Zygmund operator with variable kernel and symbol $\sigma(x, \xi)$ will be an enveloping operator for this family.

**Example 6.** If $\{A_x\}_{x \in M}$ consists of null operators then an enveloping operator is a compact operator [1].

**Theorem 1.** The operator $A$ has a Fredholm property if and only if its all local representatives $\{A_x\}_{x \in M}$ have the same property.

This property was proved in [1], but we will give the proof (see Lemma 2) including some new constructions because it will be used below for a decomposition of the operator.

### 5.3 Hierarchy of operators

We will remind the reader here of the following definition and Fredholm criteria for operators [6].

**Definition 8.** Let $B_1, B_2$ be Banach spaces, and $A : B_1 \to B_2$ be a linear bounded operator. The operator $R : B_2 \to B_1$ is called a regularizer for the operator $A$ if the following properties

$$RA = I_1 + T_1, \quad AR = I_2 + T_2$$

hold, where $I_k : B_k \to B_k$ is an identity operator, $T_k : B_k \to B_k$ is a compact operator, $k = 1, 2$.

**Proposition 2.** The operator $A : B_1 \to B_2$ has a Fredholm property if and only if there exists a linear bounded regularizer $R : B_2 \to B_1$.

**Lemma 1.** Let $f$ be a smooth function on the manifold $M$, $U \subset M$ be an open set, and $\text{supp} f \subset U$. Then the operator $f \cdot A - A \cdot f$ is a compact operator.

**Proof.** Let $g$ be a smooth function on $M$, $\text{supp} g \subset V \subset M$, moreover $\overline{U} \subset V$, $g(x) \equiv 1$ for $x \in \text{supp} f$. Then we have

$$f \cdot A = f \cdot A \cdot g + f \cdot A \cdot (1 - g) = f \cdot A \cdot g + T_1,$$

$$A \cdot f = g \cdot A \cdot f + (1 - g) \cdot A \cdot f = g \cdot A \cdot f + T_2,$$

where $T_1, T_2$ are compact operators. Let us denote $g \cdot A \cdot g \equiv h$ and write

$$f \cdot A \cdot g = f \cdot g \cdot A \cdot g = f \cdot h, \quad g \cdot A \cdot f = g \cdot A \cdot g \cdot f = h \cdot f,$$

and we obtain the required property. □
**Definition 9.** The operator $A$ is called an elliptic operator if its operator symbol $\{A_x\}_{x \in M}$ consists of Fredholm operators.

Now we will show that each elliptic operator really has a Fredholm property. Our proof in general follows the book [1], but our constructions are more stratified and we need such constructions below.

**Lemma 2.** Let $A$ be an elliptic operator. Then the operator $A$ has a Fredholm property.

**Proof.** To obtain the proof we will construct the regularizer for the operator $A$. For this purpose we choose two coverings like Proposition 1 and write the operator $A$ in the form

$$A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,$$

where $T$ is a compact operator. Without loss of generality we can assume that there are $n$ points $x_k \in U_k \subset V_k$, $k = 1, 2, \ldots, n$. Moreover, we can construct such coverings by balls in the following way. Let $\varepsilon > 0$ be a small enough number. First, for every point $x \in M_0$ we take two balls $U_x$, $V_x$ with the center at $x$ of radius $\varepsilon$ and construct two open coverings for $M_0$ namely $\mathcal{U}_0 = \bigcup_{x \in M_0} U_x$ and $\mathcal{V}_0 = \bigcup_{x \in M_0} V_x$. Second, we consider the set $L_1 = \overline{M}_{\mathcal{U}_0}$ and construct two coverings $\mathcal{U}_1 = \bigcup_{x \in L_1 \cap M_1} U_x$ and $\mathcal{V}_1 = \bigcup_{x \in L_1 \cap M_1} V_x$. Further, we introduce the set $L_2 = \overline{M}_{\mathcal{U}_1}$ and two coverings $\mathcal{U}_2 = \bigcup_{x \in L_2 \cap M_2} U_x$ and $\mathcal{V}_2 = \bigcup_{x \in L_2 \cap M_2} V_x$. Continuing these actions we will come to the set $L_{m-1} = \overline{M}_{\bigcup_{k=0} \mathcal{V}_k}$ which consists of smoothness points of $\partial M$ and inner points of $M$. We then construct two covering $\mathcal{U}_{m-1} = \bigcup_{x \in L_{m-1} \cap \partial M} U_x$ and $\mathcal{V}_{m-1} = \bigcup_{x \in L_{m-1} \cap \partial M} V_x$. Finally, the set $L_m = \bigcup_{k=0} \mathcal{V}_k$ consists of inner points of the manifold $M$ only. We finish this process by choosing the covering $\mathcal{U}_m$ for the latter set $L_m$. So, the covering $\bigcup_{k=0} \mathcal{V}_k$ will be a covering for the whole manifold $M$.

Now we will rewrite the formula (1) in the following way

$$A = \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} f_{jk} \cdot A \cdot g_{jk} \right) + T,$$

where the coverings and partitions of unity $\{f_{jk}\}$ and $\{g_{jk}\}$ are chosen as mentioned above. In other words the operator

$$\sum_{j=1}^{n_k} f_{jk} \cdot A \cdot g_{jk}$$

is related to some neighborhood of the sub-manifold $M_k$; this neighborhood is generated by covering the sub-manifold $M_k$ by balls with centers at points $x_{jk} \in M_k$. Since $A_{x_{jk}}$ is a local representative for the operator $A$ at point $x_{jk}$ we can rewrite the formula (2) as follows

$$A = \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \right) + T.$$

Let us denote $S_{w_j} \tilde{g}_j \equiv \tilde{g}_j$ and $S_{w_j} \equiv \tilde{f}_j$. Further, we can assert that the operator

$$R = \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} f_{jk} \right)$$

will be the regularizer for the operator $A'$; here $A_{x_{jk}}^{-1}$ is a regularizer for the operator $A_{x_{jk}}$.

Indeed,

$$RA = \left( \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} g_{jk} A_{x_{jk}}^{-1} f_{jk} \right) \right) \cdot \left( \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} \cdot (A - A_{x_{jk}} + A_{x_{jk}}) \cdot f_{jk} + T_1 \right).$$
is verified analogously.

\[ \theta_k = \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} \cdot (A - A_{x_{jk}}) \cdot f_{jk} + \sum_{k=0}^{m} f_{jk} \]

because \( f_{jk} \cdot A_{x_{jk}} = A_{x_{jk}} \cdot f_{jk} + \) compact summand, and \( f_{jk} \cdot g_{jk} = f_{jk} \), and

\[ \sum_{k=0}^{m} \sum_{j=1}^{n_k} f_{jk} \equiv 1 \]

as the partition of unity. The same property

\[ AR = I_2 + T_2 + \theta_2, \]

\[ \theta_2 = \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot (A - A_{x_{jk}}) \cdot A_{x_{jk}}^{-1} \cdot f_{jk}, \]

is verified analogously.

\[ \square \]

### 6 Piece-wise continuous operator families

Given an operator \( A \) with the symbol \( \{ A_x \}_{x \in \mathbb{M}} \) which generates a few operators in dependence on a quantity of singular manifolds; we consider this situation in the following way. We will assume additionally some smoothness properties for the symbol \( \{ A_x \}_{x \in \mathbb{M}} \).

**Theorem 2.** If the symbol \( \{ A_x \}_{x \in \mathbb{M}} \) is a piece-wise continuous operator function then there are \( m + 1 \) operators \( A^{(k)} \), \( k = 0, 1, \ldots, m \) such that the operator \( A \) and the operator

\[ A' = \sum_{k=0}^{m} A^{(k)} + T \]  

(4)

have the same symbols, where the operator \( A^{(k)} \) is an enveloping operator for the family \( \{ A_x \}_{x \in \mathbb{M}_k} \), and \( T \) is a compact operator.

**Proof.** We will use the constructions from the proof of Lemma 2, namely the formula (3). We will extract the operator

\[ \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \]

which “serves” the sub-manifold \( M_k \) and consider it in detail. This operator is related to neighborhoods \( \{ U_{jk} \} \) and the partition of unity \( \{ f_{jk} \} \). Really, \( U_{jk} \) is the ball with the center at \( x_{jk} \in M_k \) of radius \( \varepsilon > 0 \), and therefore \( f_{jk}, g_{jk}, n_k \) depend on \( \varepsilon \).

According to Simonenko’s ideas [1] we will construct the component \( A^{(k)} \) in the following way. Let \( \{ \varepsilon_n \}_{n=0}^{\infty} \) be a sequence such that \( \varepsilon_n > 0 \) for all \( n \in \mathbb{N} \), \( \lim_{n \to \infty} \varepsilon_n = 0 \). Given \( \varepsilon_n \) we choose coverings \( \{ U_{jk} \}_{j=1}^{n_k} \) and \( \{ V_{jk} \}_{j=1}^{n_k} \) as above with partition of unity \( \{ f_{jk} \} \) and corresponding functions \( \{ g_{jk} \} \) such that

\[ ||| f_{jk} \cdot (A_x - A_{x_{jk}}) \cdot g_{jk} ||| < \varepsilon_n, \quad \forall x \in V_{jk}; \]

we remind that \( U_{jk}, V_{jk} \) are balls with centers at \( x_{jk} \in \mathbb{M}_k \) of radius \( \varepsilon \) and \( 2 \varepsilon \). This requirement is possible according to continuity of family \( \{ A_x \} \) on the sub-manifold \( \mathbb{M}_k \). Now we will introduce such a constructed operator

\[ A_n = \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \]
and will show that the sequence \( \{ A_n \} \) is a Cauchy sequence with respect to a norm \( ||| \cdot ||| \). We have

\[
A_I = \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik},
\]

where the operator \( A_I \) is constructed for a given \( \varepsilon_I \) with corresponding coverings \( \{ u_{ik} \}_{i=1}^{l_k} \) and \( \{ v_{ik} \}_{j=1}^{l_k} \) with partition of unity \( \{ F_{ik} \} \) and corresponding functions \( \{ G_{ik} \} \) so that

\[
||| F_{ik} \cdot (A_x - A_{y_{ik}}) \cdot G_{ik} ||| < \varepsilon_I, \quad \forall x \in v_{ik};
\]

here \( u_{ik}, v_{ik} \) are balls with centers at \( y_{ik} \in M_k \) of radius \( \tau \) and \( 2\tau \).

We can write

\[
A_n = \sum_{i=1}^{l_k} f_{jk} \cdot A_{y_{jk}} \cdot g_{jk} = \sum_{i=1}^{l_k} F_{ik} \cdot \sum_{i=1}^{l_k} f_{jk} \cdot A_{y_{ik}} \cdot g_{jk}
\]

\[
= \sum_{i=1}^{l_k} \sum_{j=1}^{l_k} F_{ik} \cdot f_{jk} \cdot A_{y_{ik}} \cdot g_{jk} = \sum_{i=1}^{l_k} \sum_{j=1}^{l_k} F_{ik} \cdot f_{jk} \cdot A_{y_{ik}} \cdot g_{jk} + T_1,
\]

and the same can be done for \( A_I \)

\[
A_I = \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} = \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} - \sum_{j=1}^{l_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik}
\]

\[
= \sum_{j=1}^{l_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} + T_2.
\]

Let us consider the difference

\[
||| A_n - A_I ||| = \left| \sum_{j=1}^{l_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk} \right|.
\]

Obviously, summands with non-vanishing supplements to the formula (5) are those for which \( U_{jk} \cap U_{ik} \neq \emptyset \). A number of such neighborhoods are finite always for arbitrary finite coverings, hence we obtain

\[
||| A_n - A_I ||| \leq \sum_{j=1}^{l_k} \sum_{i=1}^{l_k} ||| f_{jk} \cdot F_{ik} \cdot (A_{y_{jk}} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk} |||.
\]

\[
\leq \sum_{x \in U_{jk} \cap U_{ik} \neq \emptyset} ||| f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_{x_{ik}}) \cdot G_{ik} \cdot g_{jk} ||| + \sum_{x \in U_{jk} \cap U_{ik} \neq \emptyset} ||| f_{jk} \cdot F_{ik} \cdot (A_x - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk} |||
\]

\[
\leq 2K \max(e_n, \varepsilon_I),
\]

where \( K \) is a universal constant.

Thus, we have proved that the sequence \( \{ A_n \} \) is a Cauchy sequence, hence there exists \( \lim_{n \to \infty} A_n = A^{(k)} \). \( \square \)

**Corollary 1.** The operator \( A \) has a Fredholm property if and only if all operators \( A^{(k)}, k = 0, 1, \ldots, m \) have the same property.

**Remark 2.** The constructed operator \( A' \) generally speaking does not coincide with the initial operator \( A \) because they act in different spaces. But for some cases they may be the same.

### 7 Conclusion

This paper is a general concept of my vision to the theory of pseudo-differential equations and boundary value problems on manifolds with a non-smooth boundary. The second part will be devoted to applying these abstract results to index theory for such operator families and then to concrete classes of pseudo-differential equations.
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