Logarithmic superdiffusivity of the 2-dimensional anisotropic KPZ equation

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We study an anisotropic variant of the two-dimensional Kardar-Parisi-Zhang equation, that is relevant to describe growth of vicinal surfaces and has Gaussian, logarithmically rough, stationary states. While the folklore belief (based on one-loop Renormalization Group) is that the equation has the same scaling behaviour as the (linear) Edwards-Wilkinson equation, we prove that, on the contrary, the non-linearity induces the emergence of a logarithmic super-diffusivity. This phenomenon is similar in flavour to the super-diffusivity for two-dimensional fluids and driven particle systems.

Stochastic growth phenomena are ubiquitous in non-equilibrium statistical physics [1]. Over the last 20 years most of the attention has focused on one-dimensional (1d) growing interfaces (e.g. the boundary of a bacterial colony spreading in a two-dimensional medium). Experimental, theoretical and mathematical results succeeded in unveiling the universal features (most notably, scaling exponents and non-Gaussian limiting distributions) of what is by now known as the 1d KPZ universality class. Also in dimension d ≥ 3 progress was made in both the physics and mathematics literature and recently the prediction [2] of asymptotically Gaussian behaviour for small coupling constant has been rigorously established. Instead, the harder case of 2d growth, on which we focus here, is still to a large extent unexplored. We study an anisotropic version of the 2d KPZ equation for which we determine super-diffusive behaviour, contradicting the claim of diffusivity repeatedly made in the previous literature.

The KPZ equation is the stochastic partial differential equation

$$\partial_t H = \frac{1}{2} \Delta H + \lambda \langle \nabla H, Q \nabla H \rangle + \xi,$$  \hspace{1cm} (1)

where $H = H(t, x)$ depends on time $t \geq 0$ and on a $d$-dimensional space coordinate $x$, $\Delta$ is the $d$-dimensional Laplacian, $\xi$ is the space-time Gaussian white noise, i.e. $\mathbb{E}[\xi(x,t)]=0$, $\mathbb{E}(\xi(x,t)\xi(y,s)) = \delta(x-y)\delta(t-s)$ ($\mathbb{E}(\cdots)$ denoting the average), $Q$ is a fixed $d \times d$ symmetric matrix and $\lambda \geq 0$ tunes the strength of the non-linearity (here, $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$).

The equation was introduced in a seminal paper by Kardar, Parisi and Zhang [2], that focused on the situation in which $Q$ is the identity matrix, thus reducing the non-linearity to $|\nabla H|^2$. In this case (1) is also connected to the partition function $Z$ of a $(d+1)$-dimensional directed polymer in a random potential (the time variable is the $(d+1)$-th space coordinate) via the transformation $Z = \exp(\lambda H)$. More generally, (1) serves as a model for $(d+1)$-dimensional stochastic growth, the non-linear term encoding the slope-dependence of the growth mechanism, and it is presumed to arise as the scaling limit of a large class of interacting particle systems. The phenomenological connection with microscopic growth models is the following: for (1) to correctly describe the height fluctuation process around a macroscopically flat state of slope $\rho \in \mathbb{R}^d$, one should take $Q = D^2 v(\rho)$, where $v(\rho)$ is the average speed of growth and $D^2 v$ is the Hessian of $v$.

A natural problem associated to (1) is to determine whether the non-linearity is relevant or not in a Renormalization Group (RG) sense, i.e. whether large-scale features of the equation, such as roughness and growth exponents $\alpha, \beta$, differ or coincide with those of the linear Edwards-Wilkinson (EW) equation corresponding to (1) with $\lambda = 0$. It has been argued in [2] and confirmed since then in many works [3, 4] that the non-linearity is relevant in dimension $d = 1$ (the growth exponent changes from $\beta_{\text{EW}} = 1/4$ to $\beta_{\text{KPZ}, d = 1} = 1/3$), whereas it is not if $d \geq 3$, provided $\lambda$ is smaller than a critical threshold $\lambda_c(d)$ (the mathematical proofs of this [8, 10] require that $Q = I$). In 2 dimensions, however, the situation is more subtle: the non-linearity is dimensionally marginal and the qualitative behaviour of (1) was predicted in [1, 11] to depend on the sign of the determinant of $Q$. In the case of $\det Q > 0$ the non-linearity changes the growth and roughness exponent (see e.g. [12]) to two universal values $\alpha_{\text{KPZ}, d = 2} \approx 0.39..., \beta_{\text{KPZ}, d = 2} \approx 0.24...$, compatible with the exact scaling relation $\alpha + z = 2$, with $z = \alpha/\beta$ the dynamic critical exponent. Instead, for $\det Q \leq 0$, which is called “anisotropic KPZ” (AKPZ) and includes both the linear equation $Q = 0$, as well as models of growth of vicinal surfaces [11], the exponents should be the same as for the EW equation, i.e. $\alpha_{\text{EW}, d = 2} = \beta_{\text{EW}, d = 2} = 0$, with logarithmic instead of power-like fluctuation growth. This has been conjectured on the basis of one-loop RG computations [1, 11] and supported by numerical simulations [13] of a discretized version of (1). Further, it has been claimed [1, 11, 13] that the large-scale fixed point of (1) is the EW equation. The purpose of the
present work is to disprove the latter claim. Indeed our main result is that if $d = 2$ and $Q = \text{diag}(+1, -1)$ is the diagonal matrix with entries $(+1, -1)$ then, as soon as $\lambda \neq 0$, \((1)\) is \textit{logarithmically super-diffusive}, namely the correlation length $\ell(t)$ behaves like $\sqrt{t \times (\log t)^\delta}$ as time grows, for some $\delta > 0$, while EW has the usual diffusive growth $\ell(t) \sim \sqrt{t}$. Interestingly, the exponent $\delta$ does not continuously go to zero as $\lambda \to 0$ and, in fact, a mode-coupling theory computation suggests that $\delta = 1/2$ for every $\lambda \neq 0$. A more precise statement of the results, together with an idea of the proof, is given below. A full mathematical proof can be found in \cite{14}. Before we proceed, let us remark that, even though in the context of $2d$ growth our findings were unexpected, logarithmic corrections to the diffusive scaling have already been observed \(2\) for the asymmetric simple exclusion process, in which case though the value of $\delta$ is $2/3$) and fluid models \(3\) for other two-dimensional out of equilibrium systems such as driven particle systems \(4\) for the asymmetric simple exclusion process, in which case though the value of $\delta$ is $2/3$) and fluid models \(5\) for other two-dimensional out of equilibrium systems such as driven particle systems \(6\) for the asymmetric simple exclusion process, in which case though the value of $\delta$ is $2/3$) and fluid models \(7\) for other two-dimensional out of equilibrium systems such as driven particle systems for which case though the value of $\delta$ is $2/3$) and fluid models for other two-dimensional out of equilibrium systems such as driven particle systems.

A distinguishing feature \(8\) of the 2d equation \(9\) with $Q = \text{diag}(+1, -1)$, i.e. the AKPZ equation
\[
\partial_t H = \frac{1}{2} \Delta H + \lambda [ (\partial_{x_1} H)^2 - (\partial_{x_2} H)^2 ] + \xi
\]
is that it has a Gaussian log-correlated stationary state $\eta$. More precisely, $\eta$ is a zero-mean Gaussian field (GFF in the mathematical jargon) whose covariance is $\mathbb{E}(\eta(x)\eta(y)) \sim \log |x - y|$ (with $x = (x_1, x_2)$), showing a vanishing roughness exponent. Note that the stationary state is independent of $\lambda$. As remarked in \(10\), \(2\) is the only version of the 2d KPZ equation (up to rotations) whose stationary state is Gaussian.

The equation \(2\) is mathematically ill-posed: the solution at fixed time is a GFF, that is merely a distribution, so that the square $(\partial_{x_1} H)^2$ does not make sense. A usual way out (that was already adopted implicitly in \(2\)) is to regularize the equation. In \(11\), we replaced $(\partial_{x_1} H)^2$ by $\Pi((\Pi\partial_{x_1} H)^2)$, where $\Pi$ is a cut-off in Fourier space, that removes all modes $|k| \geq 1$. The non-linearity then becomes $N(H) = \Pi((\Pi\partial_{x_1} H)^2 - (\Pi\partial_{x_2} H)^2)$. As observed in \(12\), the stationary state of the regularized equation is still the GFF $\eta$ and from now on we work with the stationary process with initial condition $H(0) = \eta$. We expect that our results would hold unchanged if we regularized the noise instead, as is often done. Also, in \(11\) we work on a torus of side length $2\pi N$ instead of the infinite plane, and $N$ is sent to infinity before any other limit is taken. For lightness, we drop the $N$-dependence in the formulas below.

A convenient way of encoding the growth in time of the correlation length is through the \textit{bulk diffusion coefficient} $D_{\text{bulk}}(t)$ \(21\). For the KPZ equation, the usual way to define it is as in \(22\). In our context, we let $U = (-\Delta)^{1/2} H$ (this operation just means that, in Fourier space, each Fourier mode $\hat{U}(t, k)$ is given by $|k|\hat{H}(t, k)$), that solves a 2d stochastic Burgers equation whose stationary state is simply the Gaussian white noise $\rho$, with $\mathbb{E}\rho(x) = 0$, $\mathbb{E}(\rho(x)\rho(y)) = \delta(x - y)$. Then, $D_{\text{bulk}}$ reads
\[
D_{\text{bulk}}(t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^2 S(t, x) \, dx,
\]
with
\[
S(t, x) = \mathbb{E}U(t, x)U(0, 0).
\]
Note that $S(0, x) = \delta(x)$ and $t \times D_{\text{bulk}}(t)$ measures the spread of correlations in time in a mean-square sense. The explicit solution of the EW equation yields that $D_{\text{EW}}(t) = 1$ independently of $t$, corresponding to the usual $\sqrt{t}$ growth of correlation length. Our main result is that in contrast, as soon as $\lambda \neq 0$, there exists $0 < \delta \leq 1/2$ such that
\[
(\log t)^\delta \leq D_{\text{bulk}}(t) \leq (\log t)^{1-\delta}
\]
for $t$ large (to be precise, \(5\) is proven in the sense of Laplace transforms, see \(13\) below). While we do not pin down the precise value of $\delta$, we can prove that $\delta$ does not tend to zero as $\lambda \to 0$, while as mentioned it equals zero for $\lambda = 0$. The result can be reformulated by saying that the dynamic exponent $z$ is still $z = 2$ like for EW, but the effect of non-linearity changes the power-law behaviour by a non-trivial logarithmic factor.

Another natural question for stochastic growth processes is how they behave under rescaling. The $2d$ EW stationary equation is well known to be scale-invariant under diffusive scaling, i.e.
\[
H^z(t, x) := H(t/\varepsilon^2, x/\varepsilon)
\]
has the same law as $H(t, x)$. Our second result shows that, for the non-linear equation \(2\) with $\lambda \neq 0$, this is not true, not even asymptotically for $\varepsilon \to 0$. Namely, we prove that the fields $H^z(t, \cdot)$ and $H^z(0, \cdot)$ already decorrelate at times of order $|\log \varepsilon|^{-\delta} \ll 1$ for $0 < \delta \leq 1/2$ as above. We quantify this by verifying \(14\) Th. 1.2) that given a smooth test function $\phi$ and letting $H^z_\phi(t)$ be the centered random variable $H^z_\phi(t) = \int_{\mathbb{R}^2} \text{d}x \phi(x) H^z(t, x)$, the normalized covariance
\[
\frac{\text{Cov}(H^z_\phi(t), H^z_\phi(0))}{\text{Var}(H^z_\phi(0))}
\]
is strictly smaller than $1$ for $t \approx |\log \varepsilon|^{-\delta} \ll 1$, \textit{uniformly as $\varepsilon \to 0$}. This result again indicates that the large scale behaviour of \(2\) differs from that of EW.

We emphasize that the above does not contradict the numerical findings of \(13\), but only its conclusion that the solution of \(2\) shows a “very rapid, unrelenting and nearly immediate crossover to the EW fixed point”. In fact, \(12\) numerically observes $\sqrt{\log t}$ growth of fluctuations in time for \(2\), which is the same growth as for
by momentum then readily implies that the contractions of logarithmic corrections to the diffusive scaling or to \( D_{\text{bulk}} \), which turns out to be the feature that really distinguishes between the EW and AKPZ equations.

Before explaining how we prove \([13]\), let us briefly give a heuristics, based on a mode-coupling approximation \([13, 21]\) which moreover leads to the conjecture \( \delta = 1/2 \). Let \( \hat{S}(t,k) = (2\pi)^{-2} \mathbb{E} \langle \hat{U}(t,k) \hat{U}(0,-k) \rangle \), \( k = (k_1, k_2) \) be the Fourier transform of \( S \). A direct computation shows that \( \hat{S} \) satisfies the exact identity (see \([14]\), App. B) for details

\[
(\partial_t + \frac{|k|^2}{2}) \hat{S}(t,k) = -\frac{|k|^2 \lambda^2}{(2\pi)^3} \int_0^t ds e^{-\frac{|k|^2}{2}(t-s)} \int dp \int dq K_{p,k-p} K_{q,-k-q} \mathbb{E} \left[ \hat{U}(s,p) \hat{U}(s,k-p) \hat{U}(0,q) \hat{U}(0,-k-q) \right]
\]

(8)

where \( K_{p,q} = \langle p_2q_2 - p_1q_1 \rangle \langle |p| |q| \rangle \) comes from the Fourier representation of the non-linearity \([2]\) and the integration is over the two-dimensional momenta \( p, q \) subject to the conditions \( |p|, |q|, |k-p|, |k+q| \leq 1 \) due to the Fourier regularisation induced by \( \Pi \). To get an (approximate) closed equation for \( \hat{S} \), we perform a Gaussian approximation which allows to replace the average in \([8]\) by a Gaussian one. The conservation of momentum then readily implies that the contractions \( \mathbb{E} \langle \hat{U}(s,k) \hat{U}(s,k-p) \rangle, \mathbb{E} \langle \hat{U}(0,q) \hat{U}(0,-k-q) \rangle \) multiplied by \( |k|^2 \) do not contribute, and we obtain

\[
(\partial_t + \frac{|k|^2}{2}) \hat{S}(t,k) = -\frac{2|k|^2 \lambda^2}{(2\pi)^3} \int_0^t ds e^{-\frac{|k|^2}{2}(t-s)} \int dp |K_{p,k-p}|^2 \hat{S}(s,p) \hat{S}(s,k-p).
\]

(9)

We now make the Ansatz

\[
\hat{S}(t,k) = \hat{S}(0,0) e^{-\frac{|k|^2}{2} t - c |k|^2 (\log t)^{4}},
\]

(10)

for \( k \) small and \( t \) large. Notice that in this regime \( k \approx \rho \), which means \( (k_{k-p})^2 \approx 1 \), and \( \exp(-|k|^2 (t-s)/2) \approx 1 \). Hence, computing the left and right hand side of \([9]\) with \( \hat{S} \) as in \([10]\) and then equating them, we are led to

\[
-|k|^2 (\log t)^{4} \approx -|k|^2 \lambda^2 (\log t)^{1-\delta}
\]

which imposes the choice \( \delta = 1/2 \).

The actual proof of \([13]\) given in \([14]\) starts by rewriting the bulk diffusion coefficient in its Green-Kubo formulation

\[
D_{\text{bulk}}(t) = 1 + \frac{2\lambda^2}{t} \mathbb{E} \left[ (\int_0^t ds \mathcal{N}(U(s)))^2 \right]
\]

(11)

where \( \mathcal{N}(U(s)) \) is the spatial average of \( \mathcal{N}(H(s, \cdot)) = \mathcal{N}((-\Delta)^{-1/2} U(s, \cdot)) \). Now, thanks to \([20]\) and the fact that \( U \) is a stationary Markov process whose law at every fixed time is that of the spatial white noise \( \rho \), the Laplace transform in \( t \times D_{\text{bulk}}(t) \), which we denote by \( D_{\text{bulk}} \), can be written as

\[
D_{\text{bulk}}(\mu) = \frac{1}{\mu^2} + \frac{1}{\mu^2} \mathbb{E}[\mathcal{N}(\rho) (\mu - \mathcal{L})^{-1} \mathcal{N}(\rho)]
\]

(12)

with \( \mathcal{L} \) the generator of the Markov process \( U \) and where the expectation is taken with respect to the law of the stationary state \( \rho \). In the Laplace transform sense, \([14]\) for large \( t \) is equivalent to

\[
\frac{1}{\mu^2} |\log \mu|^\delta \leq D_{\text{bulk}}(\mu) \leq \frac{1}{\mu^2} |\log \mu|^{1-\delta}
\]

(13)

for \( \mu \) small.

To have a better understanding of the expectation in \([12]\), recall that the bosonic Fock space associated to \( \rho \) can be decomposed as \( \oplus_{n \geq 0} \Gamma L_n^2 \), the \( n \)-particle sector \( \Gamma L_n^2 = L_n^2 (\mathbb{R}^{2n}) \) being the space of square integrable functions which are symmetric in their \( n \)-dimensional coordinates, endowed with the usual \( L^2 \)-scalar product \( \langle \cdot, \cdot \rangle \). Let us remark that, denoting by \( \mathcal{N} \) the representation in Fock space of \( \mathcal{N}(\rho) \), that belongs to \( \Gamma L_2^2 \) since \( \mathcal{N}(\rho) \) is quadratic in \( \rho \). Let \( P_n \) be the orthogonal projection onto \( \Gamma L_{\leq n}^2 = \oplus_{j \leq n} \Gamma L_j^2 \) and set \( \mathcal{L}_n = P_n \mathcal{L} P_n \). It turns out (see \([14]\), Lemma 3.1)) that the sequence \( b_j(\mu) = \langle n, (\mu - \mathcal{L})^{-1} n \rangle \), satisfies

\[
b_3(\mu) \leq b_5(\mu) \leq \cdots \leq b_4(\mu) \leq b_2(\mu)
\]

(14)

and

\[
\lim_{j \to \infty} b_j(\mu) = b(\mu) := \langle n, (\mu - \mathcal{L})^{-1} n \rangle,
\]

where the right hand side equals the expectation in \([12]\). Therefore, in order to prove \([13]\), it suffices to determine suitable upper and lower bounds for \( b_{2j}(\mu) \) and \( b_{2j+1}(\mu) \), respectively. To do so, note first that the symmetric part of \( \mathcal{L} \), \( \mathcal{L}_0 \), acts in Fock space as \( -\frac{1}{2} \Delta \) so that in particular it leaves \( \Gamma L_n^2 \) invariant, i.e., it conserves the particle number. On the other hand, the antisymmetric part \( \mathcal{A} \) can be written as the sum of \( \mathcal{A}_+ + \mathcal{A}_- \), such that \( -\mathcal{A}_+ = \mathcal{A}_- \) and the former maps \( \Gamma L_n^2 \) to \( \Gamma L_{n+1}^2 \)
while the latter to $\Gamma L_0^2$. If we recursively define the operators $H_j$’s as

\[ H_3 = -\mathcal{A}_-(\mu - \mathcal{L}_0)^{-1} \mathcal{A}_+ \]
\[ H_j = -\mathcal{A}_-(\mu - \mathcal{L}_0 + H_{j-1})^{-1} \mathcal{A}_+ \]

we obtain the alternative representation

\[ b_j(\mu) = \langle n, (\mu - \mathcal{L}_0 + H_j)^{-1} n \rangle_2. \] (16)

From (15) it is immediate to verify that, for all $j$, $H_j$ leaves each of the $\Gamma L_0^2$’s invariant. In order to treat the inverse of $\mu - \mathcal{L}_0 + H_{j-1}$ and get meaningful estimates for $b_j$, we need to control the $H_j$’s in terms of explicit multiplication operators which act diagonally in momentum space, as $\mathcal{L}_0$ does. Thus, to the structure of (15), we can iteratively bound the $H_j$’s starting from $H_3$ and ultimately attain

\[ H_{2j+1} \lesssim C^{2j+1} (-\mathcal{L}_0) \frac{\log(1 + (\mu - \mathcal{L}_0)^{-1})}{T_{j-1}(\mu - \mathcal{L}_0)} \] (17)
\[ H_{2j+2} \gtrsim \frac{1}{C^{2j+2}} (-\mathcal{L}_0) T_j(\mu - \mathcal{L}_0) \] (18)

(see [14, Theorem 3.3] for the precise statement), where the inequalities above are to be intended in the sense of operators, $C > 1$ is a constant, uniformly bounded from below for $\lambda$ small, arising from the approximations made in each step of the iteration and the function $T_j$ is defined as a Taylor expansion truncated at level $j$, i.e.

\[ T_j(x) = \sum_{\ell=0}^{j} \frac{\ell!}{\ell!} \log(1 + x^{-1})^\ell. \] (19)

Plugging (18) and (17) into (16), choosing $\mu$ sufficiently small and $j$ sufficiently large depending on $\mu$ and $C$ ($j \approx C^{-2} \log \log(1/|\mu|)$), (17) follows with $\delta \approx 1/C^2$.

To give a taste of the computations involved, let us show how to derive (17) for $j = 1$. Testing $H_3$ against a $n$-particle state $\phi = \Gamma L_0^2$ and using the explicit expression for $\mathcal{A}_\pm$, we get

\[ \langle \phi, H_3 \phi \rangle_n \sim \lambda^2 \int dk_{1:n} |k_{1:n}|^2 |\tilde{\phi}(k_{1:n})|^2 \]
\[ \times \int dp \frac{(K_{p,k_{1:p}})^2}{\mu + |p|^2 + |k_{1:p} + k_{2:n}|^2} \] (20)

with $k_{1:n} = (k_1, \ldots, k_n)$ and $|k_{1:n}|^2 = |k_1|^2 + \cdots + |k_n|^2$. Using $(K_{p,k_{1:p}})^2 \leq 1$ and performing the integral on $p$, one obtains an upper bound of the form

\[ \lambda^2 \int dk_{1:n} |k_{1:n}|^2 |\tilde{\phi}(k_{1:n})|^2 \log \left( 1 + (\mu + |k_{1:n}|^2)^{-1} \right) \] (21)

which implies (17) for $j = 1$. Let us remark that already this first bound implies a divergence of the bulk diffusivity, of order at least $\log \log t$ for $t$ large. Indeed, plugging it into (16) with $j = 3$, one can show that

\[ b_3(\mu) \gtrsim \int dp \frac{(K_{p,k_{1:p}})^2}{\mu + |p|^2 (1 + \log(1 + (\mu + |p|^2)^{-1})))}, \] (22)

where $|p| \leq 1$. Carefully evaluating the integral yields $b_3(\mu) \gtrsim \log \log(1/\mu)$, from which the claim follows at once.

Conclusions. We have studied the AKPZ equation, an anisotropic variant of the 2d KPZ equation (1) with $\det Q < 0$, which has a Gaussian, logarithmically rough stationary state. The common folklore belief is that it has the same scaling behaviour as the (linear) EW equation. While indeed our results confirm that both have the same (vanishing) roughness and growth exponents, we prove that non-linearity produces non-trivial logarithmic corrections to the diffusive scaling and to the bulk diffusion coefficient. In fact, we propose these corrections as a distinguishing feature between the EW and the AKPZ universality classes for 2d stochastic growth. It would be extremely interesting to find (numerical and/or analytical) evidence of analogous logarithmic super-diffusivity for discrete growth models as those in [22, 25], that are conjectured to have the same qualitative features as the AKPZ equation, or for the equation with non-linearity given by $[(\partial_x H)^2 - a(\partial_x^2 H)]^2$, $a > 0$. In fact, for $a \neq 1$ the stationary state is not Gaussian [19] but the RG analysis of [14] suggests that the behaviour should be the same as for $a = 1$, in particular the stationary state should be asymptotically Gaussian on large scales, as indicated by the simulations in [18]. Even though log corrections to diffusivity may look too tiny to be observed, we emphasise the importance of analogous logarithmic super-diffusivity in 2d driven diffusive models has been very recently numerically measured [26].

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