Page Curves for General Interacting Systems

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Abstract

We calculate in detail the Renyi entanglement entropies of cTPQ states as a function of subsystem volume, filling the details of our prior work [Nature Communications 9, 1635 (2018)], where the formulas were first presented. Working in a limit of large total volume, we find universal formulas for the Renyi entanglement entropies in a region where the subsystem volume is comparable to that of the total system. The formulas are applicable to the infinite temperature limit as well as general interacting systems. For example we find that the second Renyi entropy of cTPQ states in terms of subsystem volume is written universally up to two constants, $S_2(\ell) = -\ln K(\beta) + \ell \ln a(\beta) - \ln (1 + a(\beta)^{-L+2\ell})$, where $L$ is the total volume of the system and $a$ and $K$ are two undetermined constants. The uses of the formulas were already presented in our prior work and we mostly concentrate on the theoretical aspect of the formulas themselves. Aside from deriving the formulas for the Renyi Page curves, the expression for the von Neumann Page curve is also derived, which was not presented in our previous work.

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# Contents

1 Introduction 3

2 Notations and properties of Page curves 5
   2.1 Notations .............................................. 5
   2.2 Properties of Page curves .............................. 5

3 Calculation of the entanglement entropy of the random spin system 6
   3.1 Calculation of the \( n \)-th Renyi entropy ................. 6
   3.2 Analytic continuation to \( n = 1 \) .......................... 9
   3.3 Infinite Renyi index limit of the random spin system 11
   3.4 Aside: region where the subsystem Hilbert space dimension is small 11

4 Extension to finite temperature – TPQ state 12
   4.1 TPQ state .................................................. 12
   4.2 Calculation of the \( n \)-th Renyi entropy .................... 12
   4.3 Universality among Renyi entanglement entropies ........... 16
   4.4 von Neumann entanglement entropy in finite temperature systems 18

5 Example: Ising model 20

6 Conclusion and Outlook 22

A Averaging random variables 23

B log of average v.s. average of log 23
   B.1 The idea of the proof .................................... 23
   B.2 Proof ...................................................... 24
1 Introduction

The notion of entanglement has become popular these days as a common language over physicists in the fields of high-energy, condensed matter, and quantum information [1–4]. One useful measure of entanglement is the entanglement entropy, which quantifies the quantum correlation of one subsystem with its compliment. The entanglement entropy of the ground state of locally interacting systems is known to obey an area-law, where the quantity grows as $O(C^{d-1}) + \log$-corrections, where $C$ is a typical length scale of the subsystem and $d$ is the spatial dimension of the system (for a review see [5] and references therein). Incidentally, this has an explanation in terms of the gravity dual of the theory, called Ryu-Takayanagi formula [1, 6, 7]. The entanglement entropy of excited states, however, behaves differently – when the subsystem volume is small compared with the total volume, it follows a volume-law, meaning the entropy grows in proportion to the subsystem volume, i.e., grows as $O(C^d)$ [8]. This is roughly because one needs to specify the field configuration of the whole space, as compared with the entangling surface only for ground states, to determine the entanglement degrees of freedom. In other words, considering excited states, one has to take the thermal entropy of the subsystem itself into account. Therefore, at small subsystem sizes, the thermal effect evades the quantum effect.

How does the entanglement entropy of excited states behave when the subsystem volume is not necessarily small – Can one recover the information of the quantum effect in that way? In particular, what will be the deviation from the volume law when the subsystem volume is almost half the total volume of the system? These questions are very much worth asking as their answer should fully characterise the entanglement entropy for any subsystem sizes, in comparison to the “volume-law”, which is a statement about the entanglement entropy for small subsystem sizes and only teaches us the thermal information about the system. To answer these questions, we have to calculate the entanglement entropy against subsystem volume as a functional form – this graph is called the von Neumann/n-th Renyi Page curve for von Neumann/n-th Renyi entanglement entropy. In the monumental work [9] published in 1993, Don N. Page derived the von Neumann Page curve of random spin-1/2 systems:

$$S(\ell) = \ell \ln 2 - \frac{1}{2} \times \frac{2^\ell}{2^{L-\ell}},$$  \hspace{1cm} (1.1)

where $L$ and $\ell$ is the number of total spins and the number of spins the subsystem contains, respectively. As the Hamiltonian of the random spin system is given by $H = 0$, this gives the form of the Page curve for any systems at infinite temperature ($\beta = 0$). Note that from here one can recover the volume-law for the random spin system, whose coefficient is given by $\ln 2$. This is just the thermal entropy of the system at infinite temperature. Also, you can see that when the volume of the subsystem is comparable to that of the entire system, the entanglement entropy deviates from the above volume law. It even starts decreasing when the subsystem volume is larger than half the total volume, and eventually vanishes when the subsystem is as large as the entire system (See Fig. 3.1.2 for a similar plot for the second Renyi entropy).
result is important as it qualitatively solves the Black Hole information paradox, considering the subsystem as Black Hole radiation and the compliment as the remaining Black Hole \cite{10, 11}. A similar structure for states after a quench was observed in the context of Black Hole formation/evaporation in \cite{12}.

Still, one has to consider finite-temperature systems to sharpen the understanding towards Black Hole information paradox. More importantly, but related, the entanglement entropy of excited states also appears in the context of thermalization \cite{13–18}. A quantum pure state in a scrambled system thermalises using its own subsystem as a thermal bath, and the quantum entanglement substitutes the role of the thermodynamic entropy \cite{19–21} as it reaches the equilibrium. Such an effect and the system’s entanglement/Renyi entropy is already observed in experiments using ultra-cold atoms \cite{22}. This is the reason why the theory is needed which can generally characterize the subsystem volume-dependence of the Page curve in general interacting systems which are fast-scrambled. In general interacting systems, however, it is usually difficult to perform a generic computation applicable to a large class of theories. Even in this case where we restrict attention to fast-scrambled systems, they usually lack simple universal characterizations. The characterization is believed to be made by using out-of-time-order-correlators (OTOCs) \cite{23}, but it still is difficult enough to apply such diagnosis to the problem in question.

Considering the above situation, we set out to compute the calculation of the entanglement entropy using canonical thermal pure quantum (cTPQ) states.\footnote{There are two classes of TPQ states, canonical and micro-canonical type, and our previous paper \cite{24} and this work uses the former, while \cite{25} (appeared on the same day as \cite{24}) used the micro-canonical type.} These states, first introduced in \cite{26}, are known to be a set of typical random states with same macroscopic variables. The expectation values of any local operators distribute around thermally averaged values of those operators, with their variances exponentially small as the total volume of the system grows. One advantage of this method is that it is computationally easy to extract information about physical observables. The expectation value can just be extracted by averaging over random variables, or further, if you pick one random state in a collection of cTPQ states, the value of an observable you get is exponentially close to the one you might have got for the thermal expectation value of the observable.

We will first try to expand \cite{9} and calculate the $n$-th Renyi Page curve of the random spin system, and then compute it for general interacting systems at finite temperature. Especially the second Renyi Page curve and prove that it behaves universally up to two constants (one for the offset of the entropy, and the other for the slope of the volume-law). We also compute the von Neumann Page curve by taking a limit of the Renyi index $n \to 1$. The readers are refereed to our previous work \cite{24} for uses and numerical evidences that back up this result – we conjectured that the precise-ness of the fit of our formula to the actual Page curve constitutes the diagnosis for fast-scrambled systems.

The plan of the paper is as follows. In Section 2, we fix the notation and briefly review some of the properties of Page curves. In Section 3 we calculate the $n$-th Renyi Page curves of the random spin system for any $n$ and prove that the von Neumann
Page curve, obtained by taking \( n \to 1 \) matches with the previous result by Page, (1.1) [9]. In Section 4, we expand the previous section’s result to general interacting systems at finite temperature using TPQ states. We especially focus on the second Renyi Page curve and stress that its form is determined by two constants which can be fitted with numerical data. We also compute the von Neumann Page curve by taking a limit of \( n \to 1 \). In Section 5, we present an example to back up our formula.

2 Notations and properties of Page curves

2.1 Notations

Let us consider a general lattice system \( \Sigma \) with \( L \) spins. We only consider \( s = 1/2 \) spins in this paper for simplicity, so that the dimension of the Hilbert space \( \mathcal{H}_\Sigma \) of \( \Sigma \) is \( d \equiv 2^L \) – extensions of our result to any other spin degrees of freedom is just arithmetic and we do not care to mention. Let us now divide \( \Sigma \) into two parts, \( A \) and \( B \), to evaluate the entanglement of the system. We set the number of spins in \( A \) and \( B \) to be \( \ell \) and \( m \), respectively, and denote the dimension of each Hilbert space associated with \( A \) and \( B \) as \( d_A \equiv 2^\ell \) and \( d_B \equiv 2^m \). Note that \( L = \ell + m \), so that \( d = d_A d_B \).

By using the notations above, the reduced density matrix on subsystem \( A \) constructed from the density matrix \( \rho \) on \( \Sigma \) is defined as

\[
\rho_A \equiv \text{tr}_B \rho,
\]

and by using this, the \( n \)-th Renyi entanglement entropy is defined via

\[
S^A_n \equiv \frac{1}{1-n} \ln \left( \text{tr}_A \rho^n_A \right).
\]

The von Neumann entanglement entropy is defined as

\[
S^A \equiv -\text{tr}_A (\rho_A \ln \rho_A),
\]

and can be calculated by performing an analytic continuation of \( S_n \) and by taking \( n \to 1 \):

\[
S^A \equiv S^A_1 \equiv \lim_{n \to 1} S^A_n
\]

2.2 Properties of Page curves

A Page curve is a graph of entanglement entropy plotted against the subsystem volume, \( \ell \). The entanglement entropy is indeed shape-dependent in general [27], but understand this statement as we have agreed upon one way of choosing the subregion shape. We hereafter call \( S^A_n(\ell) \) as the \( n \)-th Renyi Page curve and \( S^A(\ell) = S^A_1(\ell) \) as the von Neumann Page curve. Additionally, note that \( S^B(\ell) \) denotes the entanglement entropy traced over \( A \) (i.e., \( S^B = -\text{tr}_B (\rho_B \ln \rho_B) \)) when \( B \) contains \( \ell \) spins. Here after we will omit the superscript \( A \) when there are no confusions.

Let us now list a few properties of entanglement entropy and Page curves.
• The Page curve is symmetric under subregion-subregion interchange.
  – If \( \rho \) is pure, it holds that \( S_{n}^{A} = S_{n}^{B} \) because of the triangle inequality.
  – If the system is translational invariant, it follows that \( S_{n}^{A}(\ell) = S_{n}^{B}(\ell) \).
  – Combining the statements above, we have \( S_{n}^{A}(\ell) = S_{n}^{A}(L - \ell) \).

• For any typical excited states at temperature \( \beta \) the Page curve follows a volume law, \( S_{n}(\ell) = s \times \ell + \text{(subleading terms)} \), where \( s \) is a constant which is dependent on \( \beta \). Note that this, modulo subleading terms, should coincide with the Page curve calculated from \( \rho = e^{-\beta H} / \text{tr} e^{-\beta H} \), the Gibbs ensemble in the limit \( L \to \infty \) with fixed \( \ell \), which is a consequence of canonical typicality \([28–30]\).

• For random spin system, the above subleading term is known to be of order \( O(1) \) \([9]\). The extension of this to general interacting systems is the main topic of this paper.

3 Calculation of the entanglement entropy of the random spin system

Calculation of the von Neumann (entanglement) entropy and the second Renyi (entanglement) entropy of the random spin system is already done in \([9]\) and \([31]\). We mainly follow the latter work to expand this calculation to \( n \)-th Renyi entropy. We will also check if this result is consistent with the von Neumann entropy given in the former.

3.1 Calculation of the \( n \)-th Renyi entropy

Although the authors derived them independently, the results of Sec. 3.1 was already derived in the context of Random Matrix Theory. The readers are referred to \([32]\) and references therein for more information. Sec. 3.2 and onwards the content is completely new.

3.1.1 Random pure state

Let us consider the spin system \( \Sigma \) with \( L \) random spins. We divide the system up into two pieces as in Section 2. Following the notations there, general wavefunctions of the system can now be written as

\[
|\psi\rangle = \sum_{a,b} c_{a,b} |a\rangle \otimes |b\rangle,
\]

We call this a random pure state, where we take \( c_{a,s} \) to be uniformly distributed on a unit sphere in \( \mathbb{C}^{d} \).
3.1.2 Calculation of $\text{tr}_A \rho_A^n$

By straightforward calculation, we obtain

$$\text{tr}_A \rho_A^n = \sum_{a_i b_i} c_{a_1 b_1} c_{a_2 b_1} c_{a_3 b_2} \cdots c_{a_n b_n} c_{a_1 b_n},$$  \hspace{1cm} (3.2)

whose cyclicity of the index we represent by the diagram below:

$$\text{(3.3)}$$

We now try to compute $\text{tr}_A \rho_A^n$. Note that $\ln \text{tr}_A \rho_A^n$ is the same as computing $\ln \text{tr}_A \rho_A^n$ at leading order in large-$d$. The complete proof of this fact as well as intuitive explanation is given in Sec. B. Because of the results shown in Appendix A, non-vanishing contributions after averaging are represented by diagrams made by joining $⊙$-• together in (3.3), meaning two pairs of indices, $(a, b)$’s, are the same. We show an example of this contraction for $n = 3$ in Fig. 3.1.2

In a region where $2^\ell$ and $2^m$ are much greater than 1 (note that $2^\ell/2^m$ could be of order 1), at leading order in $d$-scaling only relevant contractions of the graph are such that we contract every link just once and that there is no loop in the resulting diagram. The contribution from one resulting diagram will be equal to $d_A^n d_B^n \times |c|^2 \cdots |c|^2$ when the resulting number of white and blue dots, respectively, is $n_A$ and $n_B$, where $n_A + n_B = n + 1$. We hereafter call those diagrams as diagrams of the order $n_A$.

Now, what is the number of diagrams of the order $n_A$ for general $n$ and $n_A$? This number is the same number as you might have got if you counted the number of non-crossing partitions of $\{1, 2, \ldots, n\}$ of the rank $n_A$, meaning you divide them up into non-crossing $n_A$ pieces. This number is already known as Narayana number [33], denoted $N(n, n_A)$, which is defined by $N(n, n_A) = \frac{1}{n} \binom{n}{n_A} \binom{n}{n - 1}$. We get, by using this notation, the following;

$$\text{tr}_A \rho_A^n = \sum_{\text{All diagrams}} \left( \sum_{k=1}^n N(n, k) \left( \frac{d_A}{d_B} \right)^{k-1} \right), \hspace{1cm} (3.4)$$
Fig. 3.1.2: All the $n = 3$ graphs at leading order in large-$d$.

and the $n$-th Renyi entropy of the random spin system becomes

$$S_n = \ln d_A - \frac{1}{n-1} \ln \left[ \sum_{k=1}^{n} N(n,k) \left( \frac{d_A}{d_B} \right)^{k-1} \right]$$

(3.5)

$$= \ell \ln 2 - \frac{1}{n-1} \ln \left[ \sum_{k=1}^{n} N(n,k) \left( \frac{d_A}{d_B} \right)^{k-1} \right].$$

(3.6)

This means that the $n$-th Renyi entropy of the random spin system approximately follows a volume law ($\ell \ln 2$) when the subsystem $A$ is small, which is then rounded off by the second term as $A$ gets bigger. Especially when $A$ makes up half the volume of the total system, i.e., when $d_A = d_B$, we have

$$S_n(\ell = L/2) = \frac{L}{2} \ln 2 - \frac{1}{n-1} \ln \left[ \sum_{k=1}^{n} N(n,k) \right]$$

(3.7)

$$= \frac{L}{2} \ln 2 - \frac{1}{n-1} \ln [C(n)],$$

(3.8)

where $C(n)$ is the Catalan number [33], defined by $C(n) = \frac{1}{n+1} \binom{2n}{n}$. We show a graph of the second Renyi Page curve in Fig. 3.1.2.
3.2 Analytic continuation to $n = 1$

After getting the results for Renyi entropies for general integer $n$, everyone should be naturally tempted to look into von Neumann entropy by performing an analytic continuation to $n = 1$. We are going to first see the maximal value of the von Neumann entropy for simplicity, and then determine the whole functional form of the entanglement entropy to see if it really matches the result given in [9].

3.2.1 Entanglement entropy at its maximal value

Entanglement entropy, von Neumann or Renyi, takes its maximal value when subsystem $A$ makes up half the volume of the total system. Looking at (3.8) and performing an analytic continuation, we get the maximal value of the von Neumann entropy achieved at $\ell = L/2$:

$$S(\ell = L/2) = \frac{L}{2} \ln 2 - \lim_{n \to 1} \frac{\ln C(n)}{n - 1} = \frac{L}{2} \ln 2 - \frac{1}{2}$$

We can also see with ease that

$$S(\ell = 0) = 0$$

3.2.2 Analytic continuation of the whole function

Analytic continuation of the whole function is a lot more subtle than just calculating its maximal value, and requires knowledge of Narayana polynomial [34]. Narayana

Fig. 3.1.2: The second Renyi Page curve for the random spin system. The curve is convex and symmetric at the centre.
polynomial $N_n(q)$ is defined as

$$N_n(q) = \sum_{k=1}^{n} N(n, k) q^{k-1}$$

and known to be represented in terms of Legendre polynomials as

$$N_n(q) = \frac{(q-1)^{n+1}}{q} \int_0^{\frac{\phi}{\pi}} dx P_n(2x-1)$$

Now let us analytic continue the function $S_n(\ell)$. We work in a region where $0 \ll \ell \leq L/2$, so let us denote

$$d_A/d_B = q,$$

where $0 \leq q \leq 1$. Then we have

$$S_n = \ell \ln 2 - \frac{1}{n-1} \ln N_n(q)$$

For (3.14) to be an analytic continuation of $N_n(q)$, note that $q$ has to satisfy $q \geq 1$, because of the presence of the term like $(q-1)^{n+1}$. This means in a region of interest, $0 \leq q \leq 1$, the expression (3.12), rather than (3.11), must be used alternatively in order to perform an analytic continuation to $n = 1$:

$$\Delta(q) = \ell \ln 2 - \lim_{n \to 1} \frac{1}{n-1} \ln \left[ q^{n-1} N_n(q^{-1}) \right]$$

This expression includes derivatives of Legendre polynomials in terms of their degrees. These are known to be

$$\frac{\partial P_v(z)}{\partial \nu} \bigg|_{\nu=n} = P_n(z) \ln \left( \frac{z+1}{2} \right) + R_n(z),$$

where $R_n(z)$ is a certain polynomial of order $n$ [35]. Specifically, according to the paper above, we have $R_0(z) = 0$ and $R_2(z) = \frac{7}{4} z^2 - \frac{3}{2} z - \frac{1}{4}$. Plugging these into (3.19), we get

$$\Delta(q) = \frac{q}{2}.$$
so that the von Neumann entropy of the random spin system becomes

\[ S = \ell \ln 2 - \frac{1}{2} \frac{d_A}{d_B}, \quad (3.22) \]

which, although derived quite differently, reproduces the result given by Page [9] in 1993 modulo terms that vanish at large-\( d_A \) and \( d_B \). The reason for the difference of order \( 1/d_A \) or \( 1/d_B \) is explained in Sec. 3.4.

### 3.3 Infinite Renyi index limit of the random spin system

Let us also take \( n \to \infty \) in (3.8) to get the first eigenvalue of the reduced density matrix. By using (3.15) and (3.14), we have

\[ \Delta_n(q) = \frac{1}{n-1} \ln \left[ \frac{n}{(4n+2)q} \left( P_{n+1} \left( \frac{1+q}{1-q} \right) - P_{n-1} \left( \frac{1+q}{1-q} \right) \right) \right]. \quad (3.23) \]

Now, for large \( n \), the asymptotic form of the Legendre polynomials can be found in [36, 37]:

\[ P_n(z) = 1 + \frac{\sqrt{1 - z^{-2}}}{\sqrt{2\pi n}} \left( \frac{1 + \sqrt{1 - z^{-2}}}{1 - \sqrt{1 - z^{-2}}} \right)^{n/2} + O(n^{-1}), \quad (3.24) \]

where \( z > 1 \). By using this expression, the finite index limit of \( \Delta_n(q) \) becomes

\[ \lim_{n \to \infty} \Delta_n(q) = \ln(1 - q) + \ln \left[ \frac{1 + \sqrt{q}}{1 - \sqrt{q}} \right] = 2 \ln \left[ 1 + \sqrt{q} \right], \quad (3.25) \]

and the min-entropy of the random spin system becomes

\[ S_{\infty} = \ell \ln 2 - 2 \ln \left[ 1 + \sqrt{\frac{d_A}{d_B}} \right]. \quad (3.26) \]

Incidentally the maximal value of the min-entropy is

\[ \frac{L}{2} \ln 2 - 2 \ln 2, \quad (3.27) \]

which can also be directly checked by taking \( n \to \infty \) in (3.8).

### 3.4 Aside: region where the subsystem Hilbert space dimension is small

In a region where the subsystem Hilbert space dimension is small, or specifically, where \( d = 2^L \gg 1 \) but \( d_A = 2^\ell = O(1) \), we will have to add corrections to the result above. Since we only have \( d_B \)-scaling instead of \( d \)-scaling in that region, we have to take into account terms with the same number of \( d_B \) but with lesser number of \( d_A \). In other words we are forced to add diagrams contracted twice or more to the above
result. The largest contributions of those, large-\(d_B\)-wise, are made by contracting two white dots in graphs of order two. They scale as \(O(d_A^{-n}/d_B)\) in \(\text{tr}_A \rho^n_A\), and hence the \(n\)-th Renyi entropy will be modified like

\[
S_n(\ell) = S^0_n(\ell) + O(1/d_B),
\]

where \(S^0_n\) is the right hand side of (3.6),

\[
S^0_n = \ell \ln 2 - \frac{1}{n-1} \ln \left[ \sum_{k=1}^{n} N(n,k) \left( \frac{d_A}{d_B} \right)^{k-1} \right].
\]

The correction of order \(O(1/d_B)\), therefore, is present for the von Neumann, as well as Renyi, entropy – this explains the \(1/d_B\) discrepancy of (3.22) from the result given in [9].

4 Extension to finite temperature – TPQ state

4.1 TPQ state

Now we are going to consider a general shift-invariant, interacting system with Hamiltonian \(\bar{H}\) at inverse temperature \(\beta\). We are going to prepare a set of states, called TPQ states [26, 38], containing random variables as in (3.1), and calculate various quantities by taking an average. Those TPQ states are defined in terms of Hamiltonian of the system as

\[
|\psi\rangle = \frac{1}{\text{tr}(e^{-\beta H})} \sum_{a,b} c_{a,b} e^{-\beta H/2} |a, b\rangle
\]

(4.1)

Note that these wave functions are not normalised per se – they rather normalise to unity after being averaged over random variables, \(c_{a,b}\). The above two possible normalisations only make a subleading difference in any of the arguments below in terms of large-\(d\) scaling, and hence for the sake of convenience we adopt the latter convention.

4.2 Calculation of the \(n\)-th Renyi entropy

4.2.1 New diagrammatic representation of \(\text{tr}_A \rho^n_A\)

By straightforward calculation, we get

\[
\text{tr}_A \rho^n_A = \frac{1}{\text{tr}(e^{-\beta H})} \sum_{a_0,b_0} \sum_{a_1,b_1} \cdots \sum_{a_n,b_n} c_{a_1,b_1}^* c_{a_2,b_2}^* c_{a_3,b_3}^* \cdots c_{a_n,b_n}^* c_{a_{n-1},b_{n-1}}^* c_{a_n,b_n}^* c_{a_{n-1},b_{n-1}}^* \cdots
\]

\[
\times \langle a_0^0, b_0^0 | e^{-\beta H/2} | a_1^0, b_1^0 \rangle \langle a_1^1, b_1^1 | e^{-\beta H/2} | a_2^0, b_2^0 \rangle \times \cdots
\]

\[
\times \langle a_n^0, b_n^0 | e^{-\beta H/2} | a_n^1, b_n^1 \rangle \langle a_n^2, b_n^2 | e^{-\beta H/2} | a_n^0, b_n^0 \rangle.
\]

(4.2)
We again represent this in terms of diagrams as follows:

Here we represented \( e^{-\beta H/2} |a_1^*, b_1^*\rangle\) and \( \langle a_2^*, b_2^*| e^{-\beta H/2} \) as \( \) and \( \), respectively. These are connected with lines, which represent \( |a_0^*, b_0^*\rangle\) and \( \langle a_0^*, b_0^*| \). By taking an average over random variables, we contract each box only once (Fig. 4.2.1) – again as in Sec. 2.2, contracting twice will only count contributions which is subleading in \( d \)-scaling. Likewise, the contribution which comes from the diagrams which cannot be put on a plane,

are subleading in \( d \)-scaling when \( \beta \) is \( O(1) \) because those graphs would lack the number of traces in the limit \( \beta \to 0 \).

### 4.2.2 Relating new diagrams with the old ones

The new graphs that we invented above have a correspondence with the old ones invented for the random spin system. If we only consider diagrams which are leading in large-\( d \) scaling, the correspondence between the new and the old ones is one-to-one
Fig. 4.2.1: All the \( n = 3 \) graphs at leading order in large-\( d \) and one sub-leading non-planer graph.

and is as follows:

Describing in texts, the boxes are replaced with double lines, while loops are replaced with blue or white dots depending on which subscript they have. Examples of this replacement are shown in the equations below. We are, hereafter, going to resort mostly to the old diagrams when we represent various trace contributions to (4.2) that are leading order in large-\( d \).
4.2.3 Calculation of $\text{tr}_A \rho^n_A$

Calculation of $\text{tr}_A \rho^n_A = \sum_{\text{All}(\text{diagrams})}$ is done in a same manner as in Sec. 3.1.2, but the actual calculation for generic $n$ is much harder, or virtually impossible. Given a concrete value of $n$, however, it is possible to calculate the Renyi entropy with that particular index. We are going to calculate the second and the third Renyi entropies as examples.

(a) Second Renyi entropy  

The second Renyi entropy is

$$S_2 = \begin{array}{c}
\text{diagram 1} \\
\text{diagram 2}
\end{array}$$

(4.6)

$$= \begin{array}{c}
\text{simple diagram 1} \\
\text{simple diagram 2}
\end{array}$$

(4.7)

$$= -\ln \left[ \frac{\text{tr}_A (\text{tr}_B (e^{-\beta H})^2) + \text{tr}_B (\text{tr}_A (e^{-\beta H})^2)}{(\text{tr} e^{-\beta H})^2} \right]$$

(4.8)

(b) Third Renyi entropy  

The third Renyi entropy is

$$S_3 = \frac{1}{3-1} \left( \begin{array}{c}
\text{diagram 1} \\
\text{diagram 2} \\
\text{diagram 3}
\end{array} \right)$$

(4.9)

$$= \frac{1}{2} \left( \begin{array}{c}
\text{simple diagram 1} \\
\text{simple diagram 2} \\
\text{simple diagram 3}
\end{array} \right) + 3$$

(4.10)

$$= -\frac{1}{2} \ln \left[ \frac{\text{tr}_A (\text{tr}_B (e^{-\beta H})^3) + 3 \text{tr} M + \text{tr}_B (\text{tr}_A (e^{-\beta H})^3)}{(\text{tr} e^{-\beta H})^3} \right]$$

(4.11)

where

$$M = e^{-\beta H} \left( \text{tr}_B (e^{-\beta H}) \otimes \text{tr}_A (e^{-\beta H}) \right)$$

(4.12)
4.3 Universality among Renyi entanglement entropies

We are going to try to simplify the above result using the boundedness and the translation-invariance of the Hamiltonian and extensivity of the free energy. This is done in two steps. The implication of the resulting expression is essential – the Renyi entanglement entropy can be determined by finite unknown parameters of order $O(1)$, as promised in the introduction.

4.3.1 First step: rewriting each term with respect to the partition function

As we assume that the interaction of the Hamiltonian is bounded, we can split the Hamiltonian into one in subsystem $A$, one in $B$, and one including interactions in $A$ and $B$:

$$H = H_A + H_B + H_{\text{int}}. \quad (4.13)$$

By using this decomposition, it is possible, at leading order in large-$d_A, d_B$, to replace each of the terms in the $n$-th Renyi entropy using $Z_{A,B}(\beta) = \text{tr}_{A,B}(e^{-\beta H_{A,B}})$ and several unknown $O(1)$ parameters, $P(\beta), Q(\beta)$, etc., coming from the boundary term, $H_{\text{int}}$.

We are listing some of the examples of this type of decomposition below

\begin{align*}
\text{tr}_A \left( \text{tr}_B (e^{-\beta H}) \right)^2 &= P(\beta) \times Z_A(2\beta) \times Z_B(\beta)^2 \quad (4.14) \\
\text{tr}_B \left( \text{tr}_A (e^{-\beta H}) \right)^2 &= P(\beta) \times Z_A(\beta)^2 \times Z_A(2\beta) \quad (4.15) \\
\text{tr} (e^{-\beta H}) &= Q(\beta) \times Z_A(\beta) \times Z_B(\beta). \quad (4.16)
\end{align*}

Now (4.8) becomes

$$S_2 = -\ln R(\beta) - \ln \left( \frac{Z_A(2\beta)}{Z_A(\beta)^2} \frac{Z_B(2\beta)}{Z_B(\beta)^2} \right), \quad (4.17)$$

where $R(\beta) \equiv P/Q$. Likewise, (4.11) becomes

$$S_3 = -\ln R'(\beta) - \ln \left( \frac{Z_A(3\beta)}{Z_A(\beta)^3} + 3 \times \frac{Z_A(2\beta)}{Z_A(\beta)^2} \frac{Z_B(2\beta)}{Z_B(\beta)^2} + \frac{Z_B(3\beta)}{Z_B(\beta)^3} \right), \quad (4.18)$$

where $R'(\beta)$ is again an unknown $O(1)$ parameter coming from the boundary terms. The above procedure is just in the spirit of Suzuki-Trotter decomposition [39,40].

4.3.2 Second step: using extensivity

Extensivity of the free energy lets us even simplify the expression for the Renyi entropies. We here list two examples of extensivity relations that is of use in simplifying

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This fact can be derived from the existence of transfer matrices. Also, be careful about the fact that those parameters are all dependent on $\beta$, although we will refer to them as “parameters”.

16
Note that again these are only true at leading order in large-$d_A$ and $d_B$:

\[
\frac{Z_A(2\beta)}{Z_A(\beta)^2} = S(\beta) \times a(\beta)^{-\ell},
\]

(4.19)

\[
\frac{Z_B(2\beta)}{Z_B(\beta)^2} = S(\beta) \times a(\beta)^{-L+\ell},
\]

(4.20)

where $S(\beta)$ and $a(\beta)$ are, as usual, unknown $O(1)$ parameters coming from the details of the theory. Note that the inequality $1 < a(\beta) \leq 2$ holds because of the concavity and monotonicity of the free energy (the equality holds when $\beta = 0$). By using those relations, (4.8) becomes

\[
S_2 = -\ln K(\beta) + \ell \ln a(\beta) - \ln \left(1 + a(\beta)^{-L+2\ell}\right),
\]

(4.21)

where $K(\beta) \equiv S(\beta)R(\beta)$.\(^3\) This recovers the result for the random spin system at $\beta = 0$. Likewise, (4.11) becomes

\[
S_3 = -\ln K'(\beta) + \ell \ln a'(\beta) - \ln \left(1 + T(\beta)a(\beta)^{-L}a'(\beta)^\ell + a'(\beta)^{-L+2\ell}\right),
\]

(4.22)

again consistent with the already derived expression for the random spin system at $\beta = 0$.

### 4.3.3 Main result and its implication

Let us restate the main result of this section in the following – by using several unknown $O(1)$ coefficients, the $n$-th Renyi entanglement entropy in terms of its subsystem volume can be written as

\[
S_2 = -\ln K(\beta) + \ell \ln a(\beta) - \ln \left(1 + a(\beta)^{-L+2\ell}\right)
\]

(4.23)

\[
S_3 = -\ln K'(\beta) + \ell \ln a'(\beta) - \ln \left(1 + T(\beta)a(\beta)^{-L}a'(\beta)^\ell + a'(\beta)^{-L+2\ell}\right),
\]

(4.24)

with similar looking expressions for $n \geq 4$. The implication of the above statement is clear: the $n$-th Renyi entanglement entropy can universally be decided up to several parameters, which can be fitted with experimental/numerical data afterwards.

The meaning of terms in each expression is also obvious – the first term is an offset, the second means a volume-law (the slope being an effective dimension), and the third is a deviation from it. At $\ell = L/2$, especially, this deviation becomes $\ln 2$ for the second Renyi entropy, independent of the inverse temperature $\beta$ or the Hamiltonian. We would like to stress that this fact is only peculiar to the second Renyi entropy, and generically the deviation at the center does depend on $\beta$ for the Renyi index greater than 2. This can be a favourable fact in actually fitting the second Renyi entropy with the fit function above.

This completes the proof of the formula we presented in [24].

---

3 Note that this expression is symmetric under inversion at $\ell = L/2$ as it should be.
4.3.4 More universality in the thermodynamic limit

As we mentioned in the last subsection, the deviation from a volume-law at the middle is generically dependent on the temperature. This, denoted $\Delta S_n(L/2)$, is schematically written as

$$
\Delta S_n(L/2) = \ln \left( 1 + \sum_i T_i(\beta) \times c_i(\beta)^{-L/2} + (#)^0 \right) = \ln \left( 2 + \sum_i T_i(\beta) \times c_i(\beta)^{-L/2} \right) \tag{4.25}
$$

Again because of the concavity and the monotonicity of the free energy, we have $c_i(\beta) > 1$. Hence, as you approach the thermodynamic limit, or when you take $L$ large, the deviation of the general $n$-th Renyi entropy from a volume-law at $\ell = L/2$ quickly approaches $\ln 2$ for any $\beta > 0$, again independent of the inverse temperature $\beta$ or the details of the model.

4.4 von Neumann entanglement entropy in finite temperature systems

Although it seems as if a very hard task to derive the expression for the $n$-th Renyi entropy and hence the von Neumann entropy at finite $\beta$ as a result of taking a limit of $n \to 1$, it is nevertheless possible to derive the general expression if you wish to expand in terms of $\beta$. We consider the case where $Z_B(\beta) \gg Z_A(\beta)$.

Let us return to Eq. (4.17) and (4.18). $S_2$ is

$$
S_2 = \ln \left( \frac{Z_A(\beta)^2}{Z_A(2\beta)} \right) - \ln \left( 1 + \frac{Z_A(\beta)^2 Z_B(2\beta)}{Z_A(2\beta) Z_B(\beta)^2} \right) - \ln R(\beta), \tag{4.26}
$$

and $S_3$ is simplified as

$$
S_3 \approx \frac{1}{2} \ln \left( \frac{Z_A(\beta)^3}{Z_A(3\beta)} \right) - \frac{1}{2} \ln \left( 1 + 3 \frac{Z_A(\beta)Z_A(2\beta) Z_B(2\beta)}{Z_A(3\beta) Z_B(\beta)^2} \right) - \ln R'(\beta), \tag{4.27}
$$

where $R'(\beta)$ and $R'(\beta)$ are some constants. In the similar manners, we can obtain the simple expression of the Renyi entropies when $Z_B(\beta) \gg Z_A(\beta)$:

$$
S_n \approx \frac{1}{n-1} \ln \left( \frac{Z_A(\beta)^n}{Z_A(n\beta)} \right) - \frac{1}{n-1} \ln \left( 1 + \frac{n Z_B(2\beta)}{2 Z_B(\beta)^2} \sum_{k,m \leq 1,k+m=n} \frac{Z_A(k\beta)Z_A(k\beta)}{Z_A(n\beta)} \right) - \ln R_n(\beta) \tag{4.28}
$$

where $R_n(\beta)$ is a constant of $O(1)$. In order to take the analytic continuation, we expand $Z_A(k\beta)$ and $Z_A(m\beta)$ in terms of $\beta$. Then, the $O(\beta^r)$ terms are

$$
\sum_{q=0}^r Z_A^{(q)}(0) Z_A^{(r-q)}(0) \frac{1}{q!(r-q)!} \sum_{k=1}^{n-1} k^{r-q}(n-k)^q, \tag{4.29}
$$

18
where

\[ Z_A^{(q)}(0) \equiv \frac{\partial Z_A(x)}{\partial x} \bigg|_{x=0}. \quad (4.30) \]

We thus further expand the summation

\[
\sum_{k=1}^{n-1} k^{r-q}(n-k)^q = \sum_{k=1}^{n-1} \sum_{a=0}^{q} (-1)^{q-a} n^a \frac{q}{a} k^{r-a} \\
= \sum_{a=0}^{q} (-1)^{q-a} n^a \frac{q}{a} \frac{(n-1+B)^{r-a+1} - B^{r-a+1}}{r-a+1} \quad (4.31)
\]

where \( B \) is the Bernoulli number in the umbral form. Namely, suppose \( B_j \) is the Bernoulli number, one formally treats the indices \( j \) in a sequence \( B_j \) as if they were exponents. For example, in the umbral form we can write

\[
\sum_{k=1}^{n} \binom{n}{k} B_n = (1 + B)^n 
\]

In Eq. (4.32), it is possible to take \( n \to 1 \) limit and we get

\[
\lim_{n \to 1} \frac{1}{n-1} \sum_{k=1}^{n-1} k^{r-q}(n-k)^q = \sum_{a=0}^{q} (-1)^{q-a} \frac{q}{a} B_{r-a} 
\]

Therefore, we obtain the \( \beta \)-expansion of von Neumann entropy of the TPQ state:

\[
S_{n \to 1} \simeq S_{\text{thermal}} - \frac{1}{2} \sum_{r=1}^{\infty} \frac{\beta^r}{Z_A(\beta)} \sum_{q=0}^{r} \binom{r}{q} Z_A^{(q)}(0) Z_A^{(r-q)}(0) \sum_{a=0}^{q} (-1)^{q-a} \frac{q}{a} B_{r-a} + \ln R_* 
\]  

(4.35)

where

\[
S_{\text{thermal}} \equiv \beta \langle \hat{H}_A \rangle - F_A(\beta), \quad (4.36)
\]

\( F_A(\beta) \equiv \frac{1}{\beta} \ln (Z_A(\beta)) \) is the free energy, \( \langle \hat{H}_A \rangle \equiv \beta \frac{Z_A^{(1)}(\beta)}{Z_A(\beta)} \) is the average energy at the inverse temperature \( \beta \), and

\[
R_* \equiv \lim_{n \to 1} \frac{\ln R_n(\beta)}{n-1}. 
\]

(4.37)

When \( \beta = 0 \), Eq. (4.35) reduces to

\[
S_{n \to 1} \simeq S_{\text{thermal}} - \frac{1}{2} \frac{Z_A(0)}{Z_B(0)}, \quad (4.38)
\]

which reproduces the result given by Page in this limit.
5 Example: Ising model

As an illustration, we apply our formulation to the Ising model, and calculate the second Renyi entropy. The results in this section support the validity of the approximations and our main results in the last section. We consider one-dimensional Ising model

$$\hat{H} = \sum_{i=1}^{L-1} J \sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^{L} h \sigma_i^z,$$  \hspace{1cm} (5.1)

with the open boundary condition for the simplicity. Since the Hamiltonian is diagonal, the reduced density matrix of the TPQ state can be simplified.

$$\rho_A = \frac{1}{Z} \sum_{a_1, a_2, b_1} c_{a_1 b_1} c_{a_2 b_1}^* e^{-\frac{1}{\beta} [E(a_1 b_1) + E(a_2 b_1)] |a_1\rangle \langle a_2|}$$ \hspace{1cm} (5.2)

where $E(a_1 p_1) = \langle a_1 p_1 | \hat{H} | a_1 p_1 \rangle$. Therefore, we get

$$\text{tr}[\rho_A^n] = \frac{1}{Z^n} \sum_{a_1, \ldots, a_p, b_1, \ldots, b_n} c_{a_1 b_1} \ldots c_{a_n b_n} e^{-\frac{1}{\beta} [E(11) + E(21) + \ldots + E(nn) + E(1n)]}.$$ \hspace{1cm} (5.3)

where we use abbreviations $c_{a_i b_j} = c_{ij}$ and $E(a_i b_j) = E(ij)$, and its average is

$$\overline{\text{tr}[\rho_A^n]} = \frac{1}{Z^n} \sum_{a_1, \ldots, a_p, b_1, \ldots, b_n} c_{a_1 b_1} \ldots c_{a_n b_n} e^{-\frac{1}{\beta} [E(11) + E(21) + \ldots + E(nn) + E(1n)]}.$$ \hspace{1cm} (5.4)

When we take the random average, there are many ways to contract $c_{ap}$ and $c_{ap}^*$, and $\overline{\text{tr}[\rho_A^n]}$ is a sum of all the contractions.

At $n = 2$, the r.h.s of Eq. (5.4) consists of two terms:

$$\overline{\text{tr}[\rho_A^2]} = \frac{1}{Z^2} \left[ \sum_{a_1, p_1, p_2} e^{-\beta [E(11) + E(12)]} + \sum_{a_1, a_2, p_1} e^{-\beta [E(11) + E(21)]} \right].$$ \hspace{1cm} (5.5)

Since the r.h.s of Eq.(5.5) is symmetric with respect to the subsystems A and B, we only consider the first term.

$$\sum_{a_1, p_1, p_2} e^{-\beta [E(11) + E(12)]}$$

$$= \sum_{a_1, p_1, p_2} \langle \sigma_1 | T_2^{\ell-1} | \sigma_\ell \rangle \langle \sigma_\ell | T_1 | \tau_{\ell+1} \rangle \langle \tau_{\ell+1} | T_1^{\ell-\ell-2} | \tau_L \rangle \langle \sigma_\ell | T_1 | \tau_{\ell+1} \rangle \langle \tau_{\ell+1} | T_1^{\ell-\ell-2} | \tau_L \rangle$$ \hspace{1cm} (5.6)

where $| \sigma_i \rangle$, $| \tau_i \rangle$, and $| \tau_i' \rangle$ are a set of spins of $a_i$, $p_1$, and $p_2$, respectively, and $T_m$ is a transfer matrix s.t.

$$\langle \sigma_1 | T_m | \sigma_{i+1} \rangle = \exp \left[ m \beta \left( I \sigma_1 \sigma_{i+1} + h \frac{\sigma_i + \sigma_{i+1}}{2} \right) \right].$$ \hspace{1cm} (5.7)
A further analysis is made by using the eigenvalues and the eigenvectors of $T_m$. $\lambda_{\pm}$ and $a_{\pm}|\uparrow\rangle + b_{\pm}|\downarrow\rangle$ are the two eigenvalues (|$\lambda_+|$ > |$\lambda_-$|) and eigenvectors of $T_1$, respectively, and $\chi_{\pm}$ and $c_{\pm}|\uparrow\rangle + d_{\pm}|\downarrow\rangle$ are the two eigenvalues and eigenvectors of $T_2$, respectively. Then, the partition function of this system is

$$Z(l, \beta) \equiv \sum_{\sigma_1, \sigma_\ell = \pm 1} \langle \sigma | T^{\ell - 1} | \sigma_\ell \rangle$$

$$= (a_+^2 \lambda_+^{\ell - 1} + a_-^2 \lambda_-^{\ell - 1}) + (a_+ b_+ \lambda_+^{\ell - 1} + a_- b_- \lambda_-^{\ell - 1})$$

$$+ (b_+ a_+ \lambda_+^{\ell - 1} + b_- a_- \lambda_-^{\ell - 1}) + (b_+^2 \lambda_+^{\ell - 1} + b_-^2 \lambda_-^{\ell - 1})$$

$$= (a_+ + b_+)^2 \lambda_+^{\ell - 1} + (a_- + b_-)^2 \lambda_-^{\ell - 1}$$

Using these results, we calculate Eq. (5.6)

$$- \sum_{a_{1,0}, \ell, \rho_{2}} e^{-\beta [E(11) + E(12)]}$$

$$= (c_+ (c_+ + d_+)^{\ell - 1}) (a_+ (a_+ + b_+) \lambda_+^{\ell - \ell} + a_- (a_- + b_-) \lambda_-^{\ell - \ell})^2$$

$$+ (d_+ (c_+ + d_+)^{\ell - 1}) (b_+ (a_+ + b_+) \lambda_+^{\ell - \ell} + b_- (a_- + b_-) \lambda_-^{\ell - \ell})^2$$

(5.12)

When $L \geq \ell \gg 1$, we drop the terms of $\lambda_-$ and $\chi_-$ in Eq. (5.12) and get

$$- \sum_{a_{1,0}, \ell, \rho_{2}} e^{-\beta [E(11) + E(12)]}$$

$$= (c_+ (c_+ + d_+)^{\ell - 1}) (a_+ (a_+ + b_+) \lambda_+^{\ell - \ell} + a_- (a_- + b_-) \lambda_-^{\ell - \ell})^2$$

$$= (a_+ + b_+)^2 (c_+ + d_+)(a_+^2 c_+ + b_+^2 d_+) \lambda_+^{\ell - 1} \lambda_-^{\ell - \ell}$$

(5.14)

Hence, Eq. (5.5) is

$$\overline{\text{Tr} \rho_{\lambda}^L} \approx (a_+ + b_+)^2 (c_+ + d_+)^2 (a_+^2 c_+ + b_+^2 d_+)(\lambda_+^{\ell - 1} \lambda_-^{L - \ell} + \chi_+^{\ell - 1} \chi_-^{L - \ell})$$

$$= \left( \frac{\lambda_+}{\lambda_+^2} \right)^{\ell - 1} \left( 1 + \frac{\lambda_-}{\lambda_+^2} \right)^{L - 2\ell} \left( c_+ + d_+ \right) \left( a_+^2 c_+ + b_+^2 d_+ \right)$$

$$\left( a_+ + b_+ \right)^2$$

(5.15)

and the second Renyi entropy is

$$\overline{S_2} \approx \ell \ln \alpha - \ln \left( 1 + \frac{1}{a^{2\ell - 2}} \right) + \left( \ln \left( \frac{a_+ + b_+)^2}{(c_+ + d_+)(a_+^2 c_+ + b_+^2 d_+)} \right) - \ln \alpha \right).$$

(5.16)

where $\alpha \equiv \frac{\lambda_+^2}{\lambda_+^2}$. In particular, when $h = 0$

$$\lambda_\pm = e^{\beta l} \pm e^{-\beta l}$$

$$a_\pm = \frac{1}{\sqrt{2}}$$

$$b_\pm = d_\pm = \pm \frac{1}{\sqrt{2}}.$$
Thus, the third term in Eq. (5.16) is simplified.

\[
\ln \left( \frac{(a_+ + b_+)^2}{(c_+ + d_+)(a_+^2 c_+ + b_+^2 d_+)} \right) - \ln \alpha = \ln \frac{2}{\alpha}. 
\]  

(5.20)

The final result (5.16) consist of three terms, the volume-law slope, the deviation from it, and the offset term. The 1st term gives a volume-law contribution. The 2nd term gives the deviation from the volume-law, and it takes a minimum value \(-\ln 2\) at \(\ell = L/2\). The 3rd term is the offset term because it is independent of \(\ell\). The most important observation of this example is that Eq. (5.16) perfectly recovers Eq. (4.23). In contrast to Eq. (4.23), which is obtained by imposing a few assumptions, we do not assume anything to derive Eq. (5.16) in this section. Hence, the results in this section support the validity of the assumptions in Sec. 4.3.

6 Conclusion and Outlook

We have derived the formula for the von Neumann/Renyi Page curves in a finite volume system. We first computed the Renyi Page curves for the infinite temperature systems using a diagrammatic approach, and then analytically continued to get the von Neumann Page curve, reproducing the result of Page [9].

We then expanded the result to general interacting finite-temperature systems by using cTPQ states, and computed the Renyi Page curve using a similar diagrammatic technique. We then explicitly showed the universality of the form of the Page curves using a finite number of thermodynamic constants, from which we infer the effectiveness of the formulas in fitting with numerical or experimental data. We also computed the von Neumann Page curve by using the high-temperature expansion.

There are a number of interesting directions to pursue in the future. As was promoted in our previous work, [24], this formula is conjectured to be a diagnosis for fast-scrambled systems, which might compliment the tedious task of computing the OTOC. It would be interesting to collect evidences in this direction by numerics or experiments. The advantage of this formula is that it works well for fast-scrambled models even at system sizes \(L \sim 15\), and such computations for verifying our formula might be easier to come by than other formulas about entanglement.

It would be also intriguing to derive the von Neumann Page curve for \(\beta = O(1)\). Because the volume-law of Renyi entropies are not actually exact in large total volume limit, and becomes concave rather than convex, our formula surely only applies to the regime where \(L \lesssim 30\). Although by computational or experimental difficulty, this is by no means a practical problem, it would be better to derive a complete formula for the von Neumann Page curve, which is known not to have this issue.

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A Averaging random variables

We rely on the work [41] for averaging random variables in Gaussian unitary ensemble (GUE). Although the work above mostly calculate the average of various random variables in Gaussian orthogonal ensemble (GOE), the generalisation to GUE is straightforward and we will just show the result of the averaging below:

\[
|c_{s,n}|^2 \cdots |c_{s,n}|^2 = \frac{1}{d(d-1) \cdots (d-n+1)} \sim \frac{1}{d^n}
\]  

(A.1)

Other combinations just vanish at leading order in $1/d$.

B log of average v.s. average of log

In this appendix we provide a proof of the following property:

\[
\overline{\log[tr_A(\rho^n_L)]} = \log[tr_A(\rho^n_L)] + O(1/d),
\]  

(B.1)

where $d = \alpha L$, $L$ is the system size, and $1 < \alpha$ is the effective dimension of the system. Note that $\alpha = 2$ at infinite temperature for $S = 1/2$ spin systems. This fact is actually very intuitive, because at large-$d$, the variation for $W[z, \bar{z}] \equiv tr_A(\rho^n_L)$ is suppressed exponentially and one should be able to replace the average of functions with functions of the average.

B.1 The idea of the proof

Let us set up the notations. We denote $W[z, \bar{z}] \equiv tr_A(\rho^n_L)$, where $z$ is the random complex number which we take averages over. We also write $\Omega \equiv W[z, \bar{z}]$, so we are going to prove

\[
\overline{\log W[z, \bar{z}]} = \log \Omega + O(1/d) \iff \overline{\log \frac{W[z, \bar{z}]}{\Omega}} = O(1/d).
\]  

(B.2)
Now we formally expand the log around $\frac{W[z,\bar{z}]}{\Omega} = 1$ and we get the following,

$$\log \left[ \frac{W[z,\bar{z}]}{\Omega} \right] = -\frac{1}{2} \left( \frac{W[z,\bar{z}]}{\Omega} - 1 \right)^2 + \frac{1}{3} \left( \frac{W[z,\bar{z}]}{\Omega} - 1 \right)^3 - \frac{1}{4} \left( \frac{W[z,\bar{z}]}{\Omega} - 1 \right)^4 + \cdots. \quad (B.3)$$

Note that we have used $\frac{W[z,\bar{z}]}{\Omega} - 1 = 0$.

Let us discuss the first term $\left( \frac{W[z,\bar{z}]}{\Omega} - 1 \right)^2$. This gives

$$\left( \frac{W[z,\bar{z}]}{\Omega} - 1 \right)^2 = \frac{W^2 - \Omega^2}{\Omega^2}, \quad (B.4)$$

but $W^2 - \Omega^2$ can be calculated to give $\Omega^2 \times O(1/d)$. Likewise, we can see that the terms like $(W - \Omega)^m$ would only scale as $\Omega^m \times O(1/d^{m/2})$,

$$\frac{(W - \Omega)^m}{\Omega^m} = O(1/d^{m/2}). \quad (B.5)$$

By summing up all the contributions, we will get

$$\log \left[ \frac{W[z,\bar{z}]}{\Omega} \right] = \sum_{l=1}^{\infty} a_l d^{-l}, \quad (B.6)$$

where $a_l$ is independent of $d$ and scales exponentially as $l$ as seen from the direct computation. Therefore, for sufficiently large $d (= O(e^L))$, the right hand side of the above formula converges, which is of order $O(1/d)$.

### B.2 Proof

The rigorous proof of (B.1) can be done using the idea above, but we still have to justify the expansion of the logarithm, because it can include the piece where the argument in the log is greater than 2, which is out of the convergence radius. The rigorous proof, then, only includes the expression using the Taylor expansion up to a finite order and a remaining term.

Let us write the probability distribution of $\Phi = W[z,\bar{z}]/\Omega$ to be $P[\Phi]$, so that we have

$$\log \left[ \frac{W[z,\bar{z}]}{\Omega} \right] = \int_{1/d_A^{-1}}^{d_A^{n-1}} d\Phi P[\Phi] \log \Phi. \quad (B.7)$$

We here take the integration range from $1/d_A^{-1}$ to $d_A^{n-1}$ since by construction $1/d_A^{-1} \leq W[z,\bar{z}] = \text{tr}_A \left\{ \rho_A^p \right\} \leq 1$ and $1/d_A^{n-1} \leq \Phi \leq d_A^{n-1}$, where $d_A$ is the dimension of the subsystem $A$. Here we assume the subsystem $A$ is smaller than the rest of the system, $B = \bar{A}$ (when $A$ is larger than $B$ then the bound is given by $d_B$). Now we expand

$$\log \Phi = (\Phi - 1) - \frac{1}{2} (\Phi - 1)^2 / (2 \xi^2),$$

where $\xi$ is in between 1 and $\Phi$ (the Taylor theorem),

$$\log \Phi = \int_{1/d_A^{-1}}^{d_A^{n-1}} d\Phi P[\Phi] (\Phi - 1) - \frac{1}{2} \int_{1/d_A^{-1}}^{d_A^{n-1}} d\Phi P[\Phi] \frac{(\Phi - 1)^2}{\xi^2}, \quad (B.8)$$

24
but the first term gives zero because $\Phi - 1 = 0$. In the following we divide the range of integration into two parts, $[1/d_{A}^{n-1}, 1/2]$ and $[1/2, d_{A}^{n-1}]$, and evaluate each of them, respectively.

**Integration range $[1/d_{A}^{n-1}, 1/2]$** We would like to evaluate

\[ I_1 \equiv \int_{1/d_{A}^{n-1}}^{1/2} d\Phi P[\Phi] \frac{(\Phi - 1)^2}{\xi^2} \geq 0. \]  
(B.9)

Because $\xi > 1/d_{A}^{n-1}$, we have

\[ I_1 < \int_{1/d_{A}^{n-1}}^{1/2} d\Phi P[\Phi] d_{A}^{2(n-1)}(\Phi - 1)^2, \]  
(B.10)

and also because $(\Phi - 1)^2 < 1$,

\[ I_1 < \int_{1/d_{A}^{n-1}}^{1/2} d\Phi P[\Phi] d_{A}^{2(n-1)}(\Phi - 1)^2 < \int_{1/d_{A}^{n-1}}^{1/2} d\Phi P[\Phi] d_{A}^{2(n-1)}. \]  
(B.11)

This quantity has an upper bound from the Chebyshev inequality for higher moments. The inequality on the $2n$-th moment tells that $\text{Prob}(|\Phi - 1| > 1/2) \leq 2^{2n}(\Phi - 1)^{2n}$ so we obtain

\[ I_1 < 2^{2n} d_{A}^{2(n-1)} \times (\Phi - 1)^{2n} = O(1/d), \]  
(B.12)

where we have used Eq. (B.5) and $d_{A} \leq d^{1/2}$.

**Integration range $[1/2, d_{A}^{n-1}]$** We would then like to evaluate

\[ I_2 \equiv \int_{1/2}^{d_{A}^{n-1}} d\Phi P[\Phi] \frac{(\Phi - 1)^2}{\xi^2} \geq 0. \]  
(B.13)

Because $\xi > 1/2$ we have

\[ I_2 < 4 \times \int_{1/2}^{d_{A}^{n-1}} d\Phi P[\Phi](\Phi - 1)^2. \]  
(B.14)

Also,

\[ I_2 < 4 \times \int_{1/2}^{d_{A}^{n-1}} d\Phi P[\Phi](\Phi - 1)^2 < 4 \times d_{A}^{2(n-1)} \int_{1/2}^{1/2} d\Phi P[\Phi] = O(1/d), \]  
(B.15)

where the last inequality again comes from the result in the previous subsection.

**Sum of the above two terms** Summing up the above two results, we have

\[ \overline{\log \Phi} = O(1/d), \]  
(B.16)

which is the desired result.
Comments on Eq. (B.5)

We have not given any proof of (B.5), since proving this in full generality is too complicated. The proof goes the same as in deriving the result of the average of the Rényi entropy (just contracting the indices in the random number $z$), and when $m = 2$ and $n = 2$ for example we have

$$\frac{(W[z, \bar{z}]-\Omega)^2}{\Omega^2} = \sum_{ijklmnop} Z_{ij} Z_{kl} Z_{mn} Z_{op} + \text{tr}_B \left[ \text{tr}_A Z^2 (\text{tr}_A Z)^2 \right] + (A \leftrightarrow B) \quad (B.17)$$

where $Z = e^{-\beta H}$, taking indices in the subspace $H_A$ (upper) and $H_B$ (lower), respectively. By following the argument in the main text to pull out the extensive contributions, one can see the terms in the right hand side divided by $\Omega^2$ are of the order of $O(1/d)$.

References

[1] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96 (2006) 181602, [hep-th/0603001].

[2] A. Kitaev and J. Preskill, Topological entanglement entropy, Phys. Rev. Lett. 96 (Mar, 2006) 110404.

[3] M. Levin and X.-G. Wen, Detecting topological order in a ground state wave function, Phys. Rev. Lett. 96 (Mar, 2006) 110405.

[4] S. W. Hawking, Breakdown of predictability in gravitational collapse, Phys. Rev. D 14 (Nov, 1976) 2460–2473.

[5] J. Eisert, M. Cramer and M. B. Plenio, Colloquium : Area laws for the entanglement entropy, Rev. Mod. Phys. 82 (Feb, 2010) 277–306.

[6] V. E. Hubeny, M. Rangamani and T. Takayanagi, A Covariant holographic entanglement entropy proposal, JHEP 07 (2007) 062, [0705.0016].

[7] A. Lewkowycz and J. Maldacena, Generalized gravitational entropy, JHEP 08 (2013) 090, [1304.4926].

[8] T. Takayanagi and T. Ugajin, Measuring black hole formations by entanglement entropy via coarse-graining, Journal of High Energy Physics 2010 (2010) 54.

[9] D. N. Page, Average entropy of a subsystem, Phys. Rev. Lett. 71 (1993) 1291–1294.

[10] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, Black Holes: Complementarity or Firewalls?, JHEP 02 (2013) 062, [1207.3123].

[11] D. Harlow, Jerusalem Lectures on Black Holes and Quantum Information, Rev. Mod. Phys. 88 (2016) 015002, [1409.1231].

[12] T. Takayanagi and T. Ugajin, Measuring Black Hole Formations by Entanglement Entropy via Coarse-Graining, JHEP 11 (2010) 054, [1008.3439].
[13] J. M. Deutsch, Quantum statistical mechanics in a closed system, *Phys. Rev. A* **43** (Feb, 1991) 2046–2049.

[14] M. Srednicki, Chaos and quantum thermalization, *Phys. Rev. E* **50** (Aug, 1994) 888–901.

[15] M. Rigol, V. Dunjko and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, *Nature* **452** (Apr, 2008) 854–858.

[16] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, *Rev. Mod. Phys.* **83** (Aug, 2011) 863–883.

[17] G. Biroli, C. Kollath and A. M. Läuchli, Effect of rare fluctuations on the thermalization of isolated quantum systems, *Phys. Rev. Lett.* **105** (Dec, 2010) 250401.

[18] E. Iyoda, K. Kaneko and T. Sagawa, Fluctuation theorem for many-body pure quantum states, *Phys. Rev. Lett.* **119** (Sep, 2017) 100601.

[19] S. Popescu, A. J. Short and A. Winter, Entanglement and the foundations of statistical mechanics, *Nat Phys* **2** (11, 2006) 754–758.

[20] S. Goldstein, J. L. Lebowitz, R. Tumulka and N. Zanghi, Canonical typicality, *Phys. Rev. Lett.* **96** (Feb, 2006) 050403.

[21] R. Fan, P. Zhang, H. Shen and H. Zhai, Out-of-time-order correlation for many-body localization, *Science Bulletin* **62** (2017) 707–711.

[22] A. M. Kaufman, M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss et al., Quantum thermalization through entanglement in an isolated many-body system, *Science* **353** (2016) 794–800.

[23] J. Maldacena, S. H. Shenker and D. Stanford, A bound on chaos, *JHEP* **08** (2016) 106, [1503.01409].

[24] Y. O. Nakagawa, M. Watanabe, S. Sugiura and H. Fujita, Universality in volume-law entanglement of scrambled pure quantum states, *Nature Commun.* **9** (2018) 1635, [1703.02993].

[25] T.-C. Lu and T. Grover, Renyi entropy of chaotic eigenstates, Preprint at http://arxiv.org/abs/1709.08784 (2017).

[26] S. Sugiura and A. Shimizu, Thermal pure quantum states at finite temperature, *Phys. Rev. Lett.* **108** (Jun, 2012) 240401.

[27] T. Faulkner, R. G. Leigh and O. Parrikar, Shape Dependence of Entanglement Entropy in Conformal Field Theories, *JHEP* **04** (2016) 088, [1511.05179].

[28] S. Goldstein, J. L. Lebowitz, R. Tumulka and N. Zanghi, Canonical Typicality, *Phys. Rev. Lett.* **96** (2006) 050403, [cond-mat/0511091].

[29] S. Popescu, A. J. Short and A. Winter, Entanglement and the foundations of statistical mechanics, *Nature Physics* **2** (Nov., 2006) 754–758, [quant-ph/0511225].
[30] H. Tasaki, *From Quantum Dynamics to the Canonical Distribution: General Picture and a Rigorous Example*, *Physical Review Letters* **80** (Feb., 1998) 1373–1376, [cond-mat/9707253].

[31] A. Sugita and A. Shimizu, *Correlations of observables in chaotic states of macroscopic quantum systems*, *Journal of the Physical Society of Japan* **74** (2005) 1883–1886, [https://doi.org/10.1143/JPSJ.74.1883](https://doi.org/10.1143/JPSJ.74.1883).

[32] I. Dumitriu, *Eigenvalue statistics for beta-ensembles*, tech. rep., 2003.

[33] S. A. Blanco and T. K. Petersen, *Counting Dyck paths by area and rank*, *ArXiv e-prints* (June, 2012), [1206.0803].

[34] T. Mansour and Y. Sun, *Identities involving Narayana polynomials and Catalan numbers*, *ArXiv e-prints* (May, 2008), [0805.1274].

[35] R. Szmytkowski, *On the derivative of the legendre function of the first kind with respect to its degree*, *Journal of Physics A: Mathematical and General* **39** (2006) 15147.

[36] F. Olver, D. Lozier, R. Boisvert and C. Clark, *NIST handbook of mathematical functions*, .

[37] N. M. Temme, *Asymptotic Methods for Integrals*, vol. 6 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[38] S. Sugiura and A. Shimizu, *Canonical thermal pure quantum state*, *Phys. Rev. Lett.* **111** (Jul, 2013) 010401.

[39] H. F. Trotter, *On the product of semi-groups of operators*, *Proceedings of the American Mathematical Society* **10** (1959) 545–551.

[40] M. Suzuki, *Generalized trotter’s formula and systematic approximants of exponential operators and inner derivations with applications to many-body problems*, *Communications in Mathematical Physics* **51** (Jun, 1976) 183–190.

[41] N. Ullah, *Invariance hypothesis and higher correlations of hamiltonian matrix elements*, *Nuclear Physics* **58** (1964) 65 – 71.