Unified Description of Tunneling Transport in Ultracold Atomic Gases

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We derive tunneling Hamiltonians from a microscopic model of ultracold atoms separated by an external potential barrier. Not only usual one-body tunneling but also pair- and spin-tunneling processes naturally arise from a well-known microscopic model without any empirical parameters, where the tunneling process occurs via local transfer in the overlapped region of the many-body wave functions around the potential barrier. Applying the Schwinger-Keldysh formalism to the effective model in the momentum space, we demonstrate how anomalous tunneling currents occur via the tunneling Hamiltonians in strongly interacting systems. Our formulation is useful for various junction systems as well as sheds light on a clear connection between atomtronics and spintronics.

Introduction— Ultracold atomic gases provide tremendous opportunities to study quantum many-body phenomena in a systematic way, called atomtronics [1]. Various transport phenomena have been observed in state-of-the-art experiments in such systems [2–11]. One of the hottest recent topics in cold atoms is anomalous transport in strongly interacting Fermi gases where non-equilibrium many-body physics plays a crucial role [11–13].

Considerable theoretical effort has been made to understand the anomalous tunneling transport observed in these experiments [14–21]. In particular, the effects of pairing fluctuations induced by strong attractive interactions associated with the Feshbach resonance have been discussed extensively for the Bardeen-Cooper-Schrieffer (BCS) to Bose–Einstein-condensation (BEC) crossover [22]. However, the detailed barrier structure for realizing the tunneling junction prevents the development of a microscopic tunneling Hamiltonian. Accordingly, while a single-particle tunneling Hamiltonian with empirical parameters has been introduced in previous theoretical studies [14–21], because of the model ambiguities, it is difficult to determine how many-body effects appear in the tunneling transport.

To understand the interplay between strong fluctuation effects and non-equilibrium properties in cold atomic systems, it is important to develop a versatile framework for tunneling currents from microscopic arguments. Cold-atomic tunneling junctions have been achieved by using an external trap potential, as opposed to the usual sharp wall barrier at the interface considered in condensed-matter systems. Therefore, the overlap of the many-body wave functions in the two reservoirs becomes important, particularly, near unitarity, where the two-body wave function $\psi_2(r) \sim e^{-r/a} / (r \sqrt{a})$ is significantly broadened as a result of the divergent scattering length $a \rightarrow +\infty$ (where $r$ is the relative distance between two atoms) [22]. Moreover, this overlap is crucial to clarify a clear connection with spintronics in condensed-matter systems, where spin-exchange tunneling has been widely discussed [24–38] instead of the single-particle tunneling.

In this study, we derive the tunneling Hamiltonian from a generic model consisting of a potential barrier with an arbitrary form (e.g., a soft wall, as shown in Fig. 1) and a two-body interaction, which is relevant for cold atoms. Not only the usual one-body tunneling term but also the interaction-induced pair- and spin-exchange tunneling terms naturally arise from a conventional microscopic model. This is in sharp contrast to previous studies, where empirical tunneling Hamiltonians were adopted [14–21]. Moreover, our formulation reveals...
a clear connection between cold atoms and condensed-matter junction systems in the context of spintronics. For example, our effective model derived from the usual Hamiltonian can be applied to magnonic spin transport in normal-metal-ferromagnet junctions, where particles are localized in the latter system 23.

Using the momentum-space effective Hamiltonian combined with the Schwinger-Keldysh formalism 23, we demonstrate the conduction properties in strongly interacting spin-1/2 Fermi gases with a two-terminal setup and show that a non-trivial pair-tunneling current occurs even in the linear-response regime. In particular, we demonstrate the temperature dependence of the ratio between the mass conductance and the Seebeck coefficient. One can formally rewrite \( \hat{H} \) as

\[
\hat{H} = \int d^3r \sum_{\sigma} \hat{\psi}_{\sigma}^\dagger(r) \hat{h}_{\sigma}(r) \hat{\psi}_{\sigma}(r) + g \int d^3r \hat{\psi}_{\uparrow}^\dagger(r) \hat{\psi}_{\downarrow}^\dagger(r) \hat{\psi}_{\downarrow}(r) \hat{\psi}_{\uparrow}(r), \tag{1}
\]

where

\[
\hat{h}_{\sigma}(r) = -\frac{\hbar^2}{2m_{\sigma}} + \tilde{V}_{\sigma}(r) \tag{2}
\]

is the one-body local Hamiltonian with the external potential barrier \( \tilde{V}_{\sigma}(r) \). In Eq. (1), \( g \) is the two-body coupling constant. One can formally rewrite \( \hat{\psi}_{\sigma}(r) \) as

\[
\hat{\psi}_{\sigma}(r) = \hat{\psi}_{\sigma,L}(r) + \hat{\psi}_{\sigma,R}(r). \tag{3}
\]

Because we are interested in the low-energy tunneling current between two reservoirs L and R in thermal equilibrium, we assume that \( \hat{\psi}_{\sigma,L(R)}(r) \) denotes the field operator for wave functions with biased probability distributions to the L (and R) reservoirs (noting that these wave functions have an overlap around the potential barrier).

Using \( \hat{\psi}_{\sigma,L,R}(r) \), we rewrite \( \hat{H} \) as

\[
\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_{1L} + \hat{H}_{1R} + \hat{H}_{2L} + \hat{H}_{\text{ind.}}, \tag{4}
\]

where the reservoir Hamiltonian is

\[
\hat{H}_{i=L,R} = \int d^3r \sum_{\sigma} \hat{\psi}_{\sigma,i}^\dagger(r) \hat{h}_{\sigma}(r) \hat{\psi}_{\sigma,i}(r) + g \int d^3r \hat{\psi}_{\downarrow,i}^\dagger(r) \hat{\psi}_{\uparrow,i}^\dagger(r) \hat{\psi}_{\downarrow,i}(r) \hat{\psi}_{\uparrow,i}(r), \tag{5}
\]

the one-body reflection term is

\[
\hat{H}_{1L} = \int d^3r \sum_{\sigma,i} \hat{\psi}_{\sigma,i}^\dagger(r) \hat{\tau}_{\sigma}(r) \hat{\psi}_{\sigma,i}(r), \tag{6}
\]

and the one-body tunneling term is

\[
\hat{H}_{1L} = \int d^3r \sum_{\sigma} \left[ \hat{\psi}_{\sigma,L}^\dagger(r) \hat{\tau}_{\sigma}(r) \hat{\psi}_{\sigma,R}(r) + \text{h.c.} \right]. \tag{7}
\]

In Eq. (7), we introduced \( \hat{\tau}_{\sigma}(r) = \hat{h}_{\sigma}(r) + g \sum_i \hat{N}_{\sigma,i}(r) \) with the density operator \( \hat{N}_{\sigma,i}(r) = \hat{\psi}_{\sigma,i}^\dagger(r) \hat{\psi}_{\sigma,i}(r) \) (\( \tilde{\sigma} \) denotes the opposite spin of \( \sigma \)). Note that Eq. (7) indicates the tunneling and reflection processes occur locally. This fact reflects that the two wave functions in the L and R states are not completely separated but overlap as a result of the proximity effect. Therefore, tunneling occurs via the overlap of the many-body wave functions \( \Psi_L \) and \( \Psi_R \) near the potential barrier in our model (see Fig. 1).

Similarly, we obtain the interaction-induced tunneling term \( \hat{H}_{2L} = \hat{H}_{\text{pair}} + \hat{H}_{\text{spin}} \) consisting of the pair tunneling

\[
\hat{H}_{\text{pair}} = g \int d^3r \left[ \hat{P}_{\uparrow L}^\dagger(r) \hat{P}_{\uparrow R}(r) + \text{h.c.} \right] \tag{8}
\]

with the pair creation operator \( \hat{P}_{\uparrow L}^\dagger(r) = \hat{\psi}_{\uparrow,L}^\dagger(r) \hat{\psi}_{\uparrow,L}^\dagger(r) \) and the spin-exchange tunneling

\[
\hat{H}_{\text{spin}} = g \int d^3r \left[ \hat{S}_{\uparrow L}^\dagger(r) \hat{S}_{\uparrow R}(r) + \text{h.c.} \right]. \tag{9}
\]

with the spin ladder operators \( \hat{S}_{\uparrow L}^\dagger(r) = \hat{\psi}_{\uparrow,L}^\dagger(r) \hat{\psi}_{\uparrow,L}^\dagger(r) \) and \( \hat{S}_{\uparrow R}(r) = \hat{\psi}_{\uparrow,R}^\dagger(r) \hat{\psi}_{\uparrow,R}^\dagger(r) \). Note that a similar two-body tunneling has been discussed in the numerical simulation of a few-body system with a double-well trap potential [40]. Also, the induced interface interaction \( \hat{H}_{\text{ind.}} \) between two systems reads

\[
\hat{H}_{\text{ind.}} = g \int d^3r \sum_{\sigma} \hat{N}_{\sigma,L}(r) \hat{N}_{\sigma,R}(r). \tag{10}
\]

We emphasize that the derivation starting from the model with the contact-type interaction does not involve any approximations. We note that the two-body reflection corresponds to the local two-body interaction in Eq. 5.

**Momentum-space effective Hamiltonian**— Because the potential barrier peaks around the center of the system (\( x = 0 \)) and vanishes at \( x \to \pm \infty \), there are free particles in the far region. Here, we approximately evaluate Eqs. (3)-(6) by substituting the asymptotic form of the wave functions given by,

\[
\psi_{\sigma,L}(r) = \sum_k \tilde{c}_{k,\sigma,L} \left\{ e^{ikr} + A_{k,\sigma} e^{-ikr} \right\} (x < 0), \tag{11}
\]

\[
\psi_{\sigma,R}(r) = \sum_k \tilde{c}_{k,\sigma,R} \left\{ B_{k,\sigma} e^{-ikr} + A_{k,\sigma} e^{ikr} \right\} (x > 0), \tag{12}
\]

where \( \tilde{c}_{k,\sigma,i=L,R} \) is the amplitude of the asymptotic wave function. Note that \( A_{k,\sigma} \) and \( B_{k,\sigma} \) are c-numbers corresponding to the reflection and transmission coefficients.
Assuming that the tunneling effect through the one-body potential is weak, we substitute Eqs. (11) and (12) into Eq. (5) to obtain the effective reservoir Hamiltonian:

\[
H_{i=1,R} = \sum_{p,k,\sigma} \mathcal{R}_{k,p,\sigma,i} \epsilon^{\dagger}_{p,k,\sigma,i} \epsilon^{\dagger}_{c,k,\sigma,i} c_{k,\sigma,i} + g \sum_{k,k',q} c_{k+q,\uparrow}^{\dagger} c_{k',\downarrow}^{\dagger} c_{k',\downarrow,i} c_{k+q,\uparrow,i},
\]

where the amplitude \(\epsilon^{\dagger}_{k,\sigma,i}\) is replaced with a fermionic annihilation operator \(c_{k,\sigma,i}\) and the kinetic energy is defined as \(\epsilon_{p,\sigma} = p^2/(2m_\sigma)\). Similarly, substituting Eqs. (11) and (12) into Eqs. (5) and (7), we obtain the reflection and tunneling Hamiltonians in the momentum space:

\[
H_{1t} = \sum_{p,k,\sigma} \mathcal{R}_{k,p,\sigma,i} c^{\dagger}_{k,\sigma,i} c^{\dagger}_{p,\sigma,i} c_{p,\sigma,i} c_{k,\sigma,i},
\]

\[
H_{1t} = \sum_{p,k,\sigma} \mathcal{T}_{k,p,\sigma} \left[ c^{\dagger}_{k,\sigma,i} c^{\dagger}_{p,\sigma,i} c_{p,\sigma,i} c_{k,\sigma,i} \right],
\]

where terms up to the first-order have been retained in the transmission coefficient. The one-body tunneling amplitude is defined as \(T_{k,p,\sigma} = Z_{k,p,\sigma,1,R} \delta[k,p,\sigma + V_1(k - p)] + \sum_{i} N_{k-p,i}\sigma,i\) and the reflection amplitude is defined as \(R_{k,p,\sigma,i} = Z_{k,p,\sigma,i} V_1(k - p)\). Note that we have defined the overlap integral between the two reservoirs \(Z_{k,p,\sigma,i} = \int \psi^{\dagger}_{k,\sigma,i}(r) \psi_{p,\sigma,i}(r)\), symbolically writing Eqs. (11) and (12) as \(\psi_{\sigma,i}(r) = \sum_{k} \epsilon_{k,\sigma,i} f_{k,\sigma,i}(r)\). For the two-body term, we use the relationship \(\int \psi^{\dagger}_{Q_1}(r) \psi^{\dagger}_{Q_2}(r) \psi_{Q_3}(r) \psi_{Q_4}(r) =: \sum_{Q_1...4} g_{Q_1...4} c^{\dagger}_{Q_1} c^{\dagger}_{Q_2} c_{Q_3} c_{Q_4}\), where \(Q_l\) denotes the state label of each \(\psi\). At the leading order with respect to \(A_{k,\sigma}\) and \(B_{k,\sigma}\), one can obtain \(\mathcal{g}_{Q_1...4} = g \left( B_{k,\sigma}^* B_{k,\sigma} + B_{k,\sigma} B_{k,\sigma}^* \right)\). Assuming the long-wavelength limit for the transmitted waves, namely, \(B_{k,\sigma} \to B_{k,\sigma}\), and therefore \(\mathcal{g} \simeq 2 \Re\left( B_{k,\sigma} B_{k,\sigma}^* \right)\), we obtain the interaction-induced tunneling terms as

\[
H_{\text{pair}} = \sum_{Q} g \left[ P_{Q,L}^{\dagger} P_{-Q,R} + h.c. \right],
\]

\[
H_{\text{spin}} = \sum_{Q} g \left[ S_{Q,L}^{\dagger} S_{-Q,R} + S_{-Q,R}^{\dagger} S_{Q,L} \right],
\]

\[
H_{\text{ind.}} = \sum_{Q,\sigma} g_{Q} N_{Q,\sigma,L} N_{-Q,\sigma,R},
\]

where we keep only the leading-order terms. In Eqs. (13)–(15), \(X_q = \int dr \hat{X}(r) e^{i q \cdot r}\) is the usual Fourier component of \(\hat{X}(r)\). We emphasize that our formulation covers the tunneling properties of not only single-particle and Cooper-pair transfers in the entire BCS–BEC crossover regime but also spin and magnon transfers in the Mott insulators and magnets where one-body tunneling is suppressed by the localization.

\section*{Tunneling currents}

Let us derive the tunneling current formulas based on the momentum-space effective Hamiltonian. The mass current operator is defined by \(I_M = i [N_{\text{tot},L}, H]\), where \(N_{\text{tot},L} = \sum_{\sigma} N_{0,\sigma,L} = \sum_{k,\sigma} \epsilon^{\dagger}_{k,\sigma,i} c_{k,\sigma,i}\). Its explicit form is

\[
I_M = i \sum_{p,k,\sigma} \mathcal{T}_{k,p,\sigma} \left[ c^{\dagger}_{k,\sigma,i} c_{p,\sigma,i} c^{\dagger}_{p,\sigma,i} c_{k,\sigma,i} \right] + 2i \sum_{q} \left[ P_{Q,L}^{\dagger} P_{Q,R} - P_{-Q,R}^{\dagger} P_{Q,L} \right].
\]

We are interested in the statistical average \(\langle I_M \rangle = \text{Tr}[\rho I_M]\), where \(\rho\) is the density matrix. Using the Landau rule with the truncation with respect to the tunnel couplings up to the second order \(I_M\), we obtain

\[
\langle I_M \rangle = 8 g^2 \sum_{Q} \int \frac{d\omega}{2\pi} \text{Im} \mathcal{G}_{\text{ret.}}^{\sigma,\omega} \text{Im} \mathcal{G}_{\text{ret.}}^{\sigma,\omega} \delta_{\text{neq}} q^{\text{neq}} q^{\text{neq}} q^{\text{neq}},
\]

where \(\mathcal{G}_{\text{ret.}}^{\sigma,\omega}\) and \(\Gamma_{\text{ret.}}^{\sigma,\omega}\) are the retarded components of the single-particle Green’s function and pair susceptibility, respectively. The biases between the two reservoirs are incorporated into the non-equilibrium distribution differences \(\delta f_{\text{neq}}^{\sigma,\omega} = f_{\text{ret.}}^{\sigma,\omega} - f_{\text{neq}}^{\sigma,\omega}\) and \(\delta \Gamma_{\text{neq}}^{\sigma,\omega} = b_{\text{ret.}}^{\sigma,\omega} - b_{\text{neq}}^{\sigma,\omega}\) with respect to the one- and two-particle states. Note that these distribution functions involving the local chemical potentials \(\mu_i\) and temperatures \(T_i\) can be obtained from the lesser propagators as \(f_{\text{ret.}}^{\sigma,\omega} = G_{\text{ret.}}^{\sigma,\omega} / [2 \text{Im} G_{\text{ret.}}^{\sigma,\omega} + \delta \Gamma_{\text{neq}}^{\sigma,\omega} / (2 \text{Im} G_{\text{ret.}}^{\sigma,\omega})]\). In the strong-attraction limit \((g < 0)\) of the normal phase \((a^{-1} \to +\infty)\), one can assume an approximate form of the pair susceptibility as \(\Gamma_{\text{ret.}}^{\sigma,\omega} \propto \left( \omega + i \delta - \frac{2}{4m} - \Sigma_{\text{q},\omega} \right)^{-1}\), which is nothing more than the retarded Green’s function of a tightly bound molecule (where \(\Sigma_{\text{q},\omega}\) is the bosonic self-energy). This fact combined with Eq. (21) indicates that the pair-tunneling current naturally appears in the strong-coupling regime within the linear response approach; meanwhile, such a pair transport proportional to \(g^2\) does not appear in the non-interacting case as expected. This is in sharp contrast with the previous studies, where non-linear tunneling currents \(I_M\) and additional tunneling amplitudes with respect to closed-channel molecules \(I_M\) are considered to explain the anomalous transport induced by strong pairing fluctuations. Moreover, the pair-tunneling effect is relevant to nuclear systems.

To see the importance of \(H_{\text{pair}}\), in Fig. 2, we plot the ratio between the pair-tunneling mass conductance...
Indeed, near Tg excitation [47]. Because the shift of temperature. This result indicates that the condensate pair would also play an important role [48].

Consider the case with T → 0 where the single-particle state is localized, ⟨I⟩spin driven by magnon excitations becomes dominant. Even in the itinerant (delocalized) case, the spin current in the repulsive Fermi gas (g > 0) would also be dominated by ⟨I⟩spin because of the divergent spin susceptibility [24,25], which is known as Stoner ferromagnetism [52] and has been observed in recent experiments [53].

Conclusion — In this study, we derived the tunneling Hamiltonian in a two-terminal atomic system with the two-body interaction separated by a potential barrier. The one-body tunneling and reflection, and the pair- and spin-tunneling terms naturally arise from a microscopic model when taking the appropriate separation of the wave functions. We demonstrated that the anomalous current induced by pair- and spin-tunneling arises

FIG. 2: Ratio between the pair-tunneling-induced mass conductivity G and Seebeck coefficient S as a function of temperature T in the strong-coupling limit (a−1 → ∞). Tc is the superfluid critical temperature.

G = limΔµ→0 (Iµ)/Δµ and Seebeck coefficient S = limΔT→0 (Iµ)/ΔT in the strong-coupling limit (a−1 → ∞) at low temperatures, where we defined Δµ = µL − µR and ΔT = TL − TR; however, we eventually take TL = TR ≡ T and µL = µR ≡ µ, leading to Ntot = Ntot,R ≡ Ntot.. One can find a dramatic enhancement of G/S near the molecular BEC temperature Tc ≈ 0.218TF [22] where

T F = k2 B /m is the Fermi energy of an ideal Fermi gas at zero temperature. This result indicates that G is remarkably sensitive to changes in the molecular distribution δb_{q,ω}^{eq}. Indeed, near T = Tc, the bosonic distribution function exhibits an infrared divergence as a result of the gapless excitation [47]. Because the shift of µ is directly relevant to the emergence of the gapless excitation, G tends to be larger than S. On the other hand, S becomes larger than G at high temperatures because the change of T involves a large number of thermal excitations. Even though we consider the strong-coupling BEC limit, the enhancement of pair-induced conduction can manifest anomalous transport in a unitary Fermi gas [2]. A quantitative comparison with existing experiments is left for future work. Note that the single-particle tunneling current ⟨I⟩1t is strongly suppressed in the strong-coupling BEC regime at low temperatures, where most fermions in the system form tightly bound molecules (i.e., δf_{k,p,σ,ω} → 0). Because the system can be described by molecular bosons with one-boson tunneling, our approach below Tc would be consistent with the bosonic superfluid transport discussed in Ref. [21]. In such a case, the conduction of the condensed pair would also play an important role [48]. At high temperature, ⟨I⟩1t becomes large as a result of the enhancement of TK,p,σ associated with thermally excited fermions. In other words, a sufficiently low temperature (e.g., typically T ≪ max[V_σ(r)]) is suitable to see effects of interaction-induced many-body tunneling.

Note that, in practical calculations, µ is obtained by the strong-coupling ansatz Ntot z 2 ∑q(e^2/4π−2µ+Ep−1)−1, where E_b = 1/(ma^2) is the two-body binding energy [49]. For simplicity, we have employed a phenomenological self-energy Σq,σ,ω ≈ −iγ_θ(ω), where the damping γ_σ = 0.1E_F is associated with multiple boson–boson scatterings. We numerically confirmed that a value of γ_σ does not qualitatively change our result. While the value of a−1 modifies the coefficients of G and S associated with g−1, the ratio G/S shown in Fig. 2 does not explicitly depend on a−1 in the limit of a−1 → ∞.

Moreover, H_{spin} induces the spin current operator IS = i [N_{0↑,L} − N_{0↓,L}, H], such that

\[ IS = i \sum_{p,k,σ} \eta_σ^L \bar{c}_{k,σ,L} c_{p,σ,R} - \epsilon_p^L c_{k,σ,R}^\dagger c_{p,σ,L} \]

\[ + 2i \eta_σ^R \sum_q \left[ S_q^+, S_{q,R} - S_{q,L}^+ S_{q,R} \right], \]

where we define η_σ = δ_σ↑ − δ_σ↓. In the presence of a spin-dependent bias, we obtain the statistical average of \langle I_S \rangle that is, \langle I_S \rangle_{1t} + \langle I_S \rangle_{spin}, such that

\[ \langle I_S \rangle_{1t} = 4 \sum_{k,p,σ} \int \frac{dω}{2π} T_{k,p,σ}^2 \text{Im} \chi_{k,p,σ,L,ω} \text{Im} \chi_{k,p,σ,R,ω} \times \eta_σ \delta \epsilon_{eq,k,p,σ,ω}. \]

\[ \langle I_S \rangle_{spin} = 8 \eta_σ \sum_q \int \frac{dω}{2π} \text{Im} \chi_{q,L,ω} \text{Im} \chi_{q,R,ω} \delta G_{q,ω}, \]

where \chi_{q,i,ω} is the retarded component of the dynamical spin susceptibility. \delta G_{q,ω} = \bar{b}_{q,i,ω} - \bar{b}_{q,i,ω} is the non-equilibrium distribution difference of magnon-like excitations. Using the lesser spin susceptibility \chi_{q,i,ω}^<, we define \bar{b}_{q,i,ω} = \chi_{q,i,ω}^< / [2i \text{Im} \chi_{q,i,ω}]. While \langle I_S \rangle_{1t} has been discussed in the context of cold atoms [19], \langle I_S \rangle_{spin} is a crucial term for spin transport extensively discussed in the field of spintronics [23,25]. In particular, if we consider the case with TK,p,σ → 0 where the single-particle state is localized, \langle I_S \rangle_{spin} driven by magnon excitations becomes dominant. Even in the itinerant (delocalized) case, the spin current in the repulsive Fermi gas (g > 0) would also be dominated by \langle I_S \rangle_{spin} because of the divergent spin susceptibility [24,25], which is known as Stoner ferromagnetism [52] and has been observed in recent experiments [53].
even in the linear-response regime. This result is natural in the sense that the strong-coupling BEC limit should be dominated by the tunneling processes of molecular bosons. We discussed how the pair-tunneling-induced mass conductance $G$ and Seebeck coefficient $S$ behave under the strong-coupling ansatz and demonstrated the anomalous enhancement of $G/S$ as a result of the Bose–Einstein statistics of molecules. In addition, we derived the spin current formula for repulsive Fermi gases in terms of Einstein statics of molecules. In addition, we derived the spin susceptibility. Our systematic framework will be useful for understanding transport phenomena in ultracold atoms and will be applicable to other systems such as condensed-matter and nuclear systems.

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