Simulating bulk viscosity in neutron stars. II. Evolution in spherical symmetry

Giovanni Camelio,1 Lorenzo Gavassino,1,2 Marco Antonelli,3,1 Sebastiano Bernuzzi,4 and Brynmor Haskell1
1Nicolaus Copernicus Astronomical Center, Polish Academy of Science, Bartycka 18, 00-716 Warsaw, Poland
2Department of Mathematics, Vanderbilt University, Nashville, TN, USA
3CNRS/IN2P3, ENSICAEN, Laboratoire de Physique Corpusculaire, 14000 Caen, France
4Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, 07743, Jena, Germany
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Out-of-equilibrium reactions between different particle species are the main processes contributing to bulk viscosity in neutron stars. In this work, we numerically compare three different approaches to the modeling of bulk viscosity: the multi-component fluid with reacting particle species and two bulk stress formalism based on the Müller-Israel-Stewart theory, namely the Hiscock-Lindblom and the Maxwell-Cattaneo models, whose flux-conservative formulation in radial gauge-polar slicing coordinates and spherical symmetry is derived in a companion paper. To our knowledge, this is the first time that a neutron star is simulated with the complete Hiscock-Lindblom model of bulk viscosity. We find that the Hiscock-Lindblom and Maxwell-Cattaneo models are good approximations of the multi-component fluid for small perturbations and when the non-equilibrium equation of state of the fluid depends on only one independent particle fraction. For more than one independent particle fraction and for large perturbations, the bulk stress approximation is still valid but less accurate. In addition, we include the energy loss due to the luminosity of the reactions in the bulk stress formulation. We find that the energy loss due to bulk viscosity has a larger effect on the dynamics than the bulk stress or the variation in particle composition per se. The new one-dimensional, general-relativistic hydrodynamic code developed for this work, hydro-bulk-1D, is publicly available.

I. INTRODUCTION

Recent advances in fields as diverse as gravitational wave astronomy and heavy ion collisions have lead to a renewed interest in relativistic formulations of dissipative hydrodynamics, as experiments and observations become sensitive enough to probe these effects [1–3].

In neutron star simulations the effects of shear viscosity, which is expected to play a strong role as magneto-hydrodynamical turbulence develops after a neutron star merger, have been considered by few authors [4–10], while bulk viscosity has received less attention. Nevertheless, recent studies suggest that the thermodynamic conditions in a neutron star merger may be such that bulk viscosity could be dynamically significant [11–14], even if no clear evidence of this effect has been found in simulations [15–18], see for example the introductory discussion in our companion paper [19].

Bulk viscosity is an out-of-equilibrium dissipative process, which in neutron stars is mostly due to particle reactions, and in particular to the so-called Urca processes [11]. Bulk viscosity in neutron stars may impact on several aspects of their dynamics. For example, it determines the damping time-scale of radial oscillations which have been excited at the time of the proto-neutron star formation from supernovae. Viscosity also defines the window for the gravitational wave instabilities in rotating neutron stars [20–22] and thereby the theoretical upper limits on their rotation rates [23].

Many calculations of the bulk-viscosity coefficients arising from different kinds of reactions have been reported in literature, e.g. [11,24–29]. Recently, there has been a renewed interest in this problem also because bulk viscosity can in principle leave an observable imprint on the gravitational wave signal emitted during neutron star mergers, even if preliminary studies have found that any effect is probably minor [15–18,30].

In this work, we numerically compare three different approaches to the modeling of bulk viscosity in neutron stars: the multi-component fluid with tracking of the reacting particle species [31], and two bulk stress formalism based on the Müller-Israel-Stewart theory [32], namely the Hiscock-Lindblom [33] and the Maxwell-Cattaneo [34] models, whose flux-conservative formulation in radial gauge-polar slicing coordinates and spherical symmetry is discussed in the companion paper [19]. To our knowledge, this is the first implementation of Müller-Israel-Stewart theories in radial gauge-polar slicing coordinates in spherical symmetry and the first implementation of the complete Hiscock-Lindblom model in the context of neutron stars. We find that the bulk stress formalism (i.e., Hiscock-Lindblom and Maxwell-Cattaneo) is a good approximation of the multi-component fluid for small perturbations and for one independent particle fraction (namely, when we can select the electron fraction as the only independent fraction variable of the equation of state). For more than one independent fraction variable in the equation of state, the bulk stress approximation is still valid but loses accuracy [1]. In addition, we include for the first time the energy loss due to the relaxation-time approximation may no longer be valid [35]. Sometimes, this can lead to exotic dynamics, which cannot be described within the Müller-Israel-Stewart formalism [36].
reaction luminosity in the bulk stress formulations [19], finding that its inclusion influences the dynamics more than the bulk stress alone.

In order to perform this study, we developed ‘hydro-bulk-1D’, a new one dimensional general relativistic hydrodynamic code. We publicly release [37] hydro-bulk-1D on zenodo under the MIT license to allow the community to check and improve our results.

This paper is organized as follows. In Sec. II we describe the bulk viscous models and the hydrodynamic equations. In Sec. III we describe the equation of state and the reaction rates. In Sec. IV we introduce the code. In Sec. V we compare the different approaches to bulk viscosity. We draw our conclusions in Sec. VI. We describe the bulk viscous models and the hydrodynamic equations in radial gauge-polar slicing coordinates and numerical simulations. More details are given in the Appendix A, in detail the numerical implementation of the code, and we derive a formula for the relativistic viscous timescale, and where otherwise specified, we set $c = G = M_\odot = k_B = 1$, which are our code units. In these units, the rest mass saturation density is $\rho_0 \approx 4.34 \times 10^{-4}$, one kilometer is km $\simeq 0.677$, and one millisecond is ms $\simeq 203$.

II. HYDRODYNAMIC MODEL

We briefly recall some defining properties of the three approaches to bulk viscosity that we implement in our numerical simulations. More details are given in the companion paper [19]. Then, we specify their hydrodynamic equations in radial gauge-polar slicing coordinates and spherical symmetry.

A. Bulk viscosity: three approaches

We consider 3 different approaches to bulk viscosity: (i) the multi-component fluid with the tracking of the chemical reactions and particle abundances, and two Müller-Israel-Stewart theories, namely (ii) the Hiscock-Lindblom theory, and (iii) the Maxwell-Cattaneo theory, see Camelo et al. [19].

As long as the fluid elements are isotropic (e.g., in the absence of heat, superfluid or electric currents), the stress-energy tensor for the three models (i)-(iii) can be written as:

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu},$$

where $u^\mu$ is the 4-velocity of the fluid and $g^{\mu\nu}$ is the metric. The physical meaning of the pressure $p$ depends on the model considered: for the multi-component fluid (i) it is $p = p(\rho, \epsilon, \{Y_i\}_i)$, while for the Müller-Israel-Stewart theories (ii)-(iii) it is $p = p^\text{eq}(\rho, \epsilon) + \Pi$. Above, we call $\rho$ the rest mass density, $Y_i = \rho_i/\rho$ the particle fraction of species $i$ ($\rho_i$ is the rest mass density of particle $i$), $\Pi$ the bulk stress, and the ‘eq’ superscript refers to quantities in beta-equilibrium.

Note that $Y_i$ and $\Pi$ are independent variables that have their own dynamics in models (i) and (ii)-(iii), respectively, and in order to evolve the system we need to provide an equation for them. For the multi-component fluid the evolution of each chemical fractions $Y_i$ is given by the corresponding continuity equation:

$$\nabla_\mu (\rho Y_i u^\mu) = m_n R_i,$$

where $m_n$ is the neutron mass and $R_i$ is the net reaction rate of particle species $i$. In the Hiscock-Lindblom model, the additional equation for the bulk stress is:

$$\nabla_\mu (\Pi u^\mu) = -\frac{\Pi}{\tau} - \left( \frac{1}{\chi} - \frac{1}{2} \right) \nabla_\mu u^\mu - \frac{\Pi}{2} u^\mu \nabla_\mu \left( \log \frac{\chi}{T_{\text{eq}}} \right),$$

where $\chi$ is the bulk viscous parameter, $\tau$ is the bulk viscous timescale, and $T_{\text{eq}}$ is the temperature in beta-equilibrium. Finally, Maxwell-Cattaneo is a linearization of Hiscock-Lindblom, and for it the additional equation that describes the evolution of the bulk stress is:

$$\nabla_\mu (\Pi u^\mu) = -\frac{\Pi}{\tau} - \left( \frac{1}{\chi} - \frac{2}{\Pi} \right) \nabla_\mu u^\mu.$$
we can extend this formulation to more than one independent species requiring that the speed of sound of the multi-component fluid coincides with that of the Müller-Israel-Stewart theories for small perturbation from thermodynamic equilibrium [19]:

\[
\tau = \frac{n_\alpha \frac{\partial Y_{\alpha}^{eq}}{\partial \rho} \frac{\partial Y_{\alpha}^{eq}}{\partial \rho}}{m_{\alpha} \frac{\partial^2 u}{\partial \rho d} \frac{\partial Y_{\alpha}^{eq}}{\partial \rho} \frac{\partial Y_{\alpha}^{eq}}{\partial \rho}}.
\]  

(10)

C. Including luminosity

For all the three models (i)-(iii), the total rest mass density satisfies the continuity equation

\[
\nabla_\mu (\rho u^\mu) = 0,
\]

(11)

and the stress-energy tensor evolution is given by the usual momentum and energy conservation equations:

\[
\nabla_\mu (T^{\mu \nu}) = -Q u^\nu,
\]

(12)

where \( Q \) is the total luminosity. For the multi-component fluid, \( Q = \sum_j Q_j \), where \( Q_j \) is the luminosity of reaction \( j \). In Müller-Israel-Stewart theories, the energy luminosity is usually neglected, \( Q = 0 \). However, in the companion paper [19] we show that it is possible to account for the energy luminosity also in this kind of theories by performing an expansion around equilibrium:

\[
Q(\rho, s, X) = Q^{eq}(\rho, s^{eq}) + \frac{\partial Q}{\partial \Pi} \Pi + O(\Pi^2),
\]

(13)

\[
\frac{\partial Q}{\partial \Pi} = \frac{1}{n^2} \left( \frac{\partial Y_{\alpha}^{eq}}{\partial \rho} \right) \frac{1}{n^2} \left( \frac{\partial Y_{\alpha}^{eq}}{\partial \rho} \right) \bigg|_{\rho, s},
\]

(14)

where, for simplicity, the above formula refers to the particular case in which there is a single independent fraction \( Y \) in the non-equilibrium equation of state of the corresponding multi-component fluid, \( \epsilon = \epsilon(\rho, s, Y) \).

D. Hydrodynamic equations in radial gauge, polar slicing coordinates in spherical symmetry

We adopt the radial gauge, polar slicing coordinates in spherical symmetry (i.e. Schwarzschild), whose metric is:

\[
dl^2 = -a^2(r, t)dt^2 + X^2(r, t)dr^2 + r^2d\Omega^2,
\]

(15)

where \( l \) is the proper time, \( t \) and \( r \) are respectively the time and radial coordinates, \( d\Omega \) is the angular element, \( \alpha \) is the lapse function:

\[
\alpha(r, t) = \exp (\phi(r, t)),
\]

(16)

the metric function \( X \) is given by:

\[
X(r, t) = \left( 1 - \frac{2m(r, t)}{r} \right)^{-1/2},
\]

(17)

\( m \) is the gravitational mass:

\[
m(r, t) = 4\pi \int_0^r ((\epsilon + p)W^2 - p) x^2 dx
\]

(18)

\( \phi \) is the general relativistic equivalent of the Newtonian gravitational field:

\[
\phi(r > R, t) = \frac{1}{2} \log \left( 1 - \frac{2m(R, t)}{r} \right),
\]

(19)

where \( R \) is the stellar radius. Finally, the total baryon mass of the star is:

\[
M_b = 4\pi \int_0^R X\rho W x^2 dx.
\]

(22)

In radial gauge, polar slicing coordinates in spherical symmetry, the continuity equations reads [10, II]:

\[
\partial_t D + \frac{1}{r^2} \partial_r \left( \frac{\alpha r^2}{X} Dv \right) = 0,
\]

(23)

\[
D = X\rho W,
\]

(24)

the momentum conservation equation reads [10, II]:

\[
\partial_t S^r + \frac{1}{r^2} \partial_r \left( \frac{\alpha r^2}{X} (S^r v + p) \right) = -\alpha W v Q
\]

\[
- \epsilon \alpha X \left( 8\pi r p + \frac{m}{r^2} \right) + \alpha X p m \frac{2\alpha p}{X r^2} + \frac{2\alpha p}{X r^2}
\]

(25)

\[
S^r = (\epsilon + p)W^2 v,
\]

(26)

the energy conservation equation reads [10, II]:

\[
\partial_t \tau_e + \frac{1}{r^2} \partial_r \left( \frac{\alpha r^2}{X} (S^r - Dv) \right) = -\alpha W Q,
\]

(27)

\[
\tau_e = (\epsilon + p)W^2 - p - D,
\]

(28)

the particle continuity equation reads [10]:

\[
\partial_t (DY_i) + \frac{1}{r^2} \partial_r \left( \frac{\alpha r^2}{X} D Y_i v \right) = \alpha X m R_i,
\]

(29)

\[\text{2} \] Here we assume that neutrinos are emitted isotropically in the fluid rest frame and immediately leave the star. See Appendix A of O'Connor and Ott [10] for a more general setting.
the bulk stress equation in the Hiscock-Lindblom theory reads [19]:

$$\partial_t (XW) + \frac{1}{r^2} \partial_r (r^2 \alpha W v r) = -\frac{\alpha X \Pi}{\tau} - \left(1 - \frac{\Pi}{2}\right) \left(\partial_t (XW) + \frac{1}{r^2} \partial_r (r^2 \alpha W v r)\right) - \frac{\Pi \Pi}{2} \left(\frac{\alpha v_t}{T}\right) \left(\partial_t (XW) + \frac{1}{r^2} \partial_r (r^2 \alpha W v r)\right), \quad (30)$$

and the bulk stress equation in the Maxwell-Cattaneo theory reads [19]:

$$\partial_t (XW) + \frac{1}{r^2} \partial_r (r^2 \alpha W v r) = -\frac{\alpha X \Pi}{\tau} - \left(1 - \frac{\Pi}{2}\right) \left(\partial_t (XW) + \frac{1}{r^2} \partial_r (r^2 \alpha W v r)\right), \quad (31)$$

where as above it is $p = p(\rho, \epsilon, \{Y_i\})$ for the multi-component fluid and $p = p^{\text{eq}}(\rho, \epsilon) + \Pi$ for Hiscock-Lindblom and Maxwell-Cattaneo.

III. MICROPHYSICS

Here we summarize the microscopic input needed for our numerical simulations. For a more detailed discussion, see the companion paper [19].

A. Reaction rates and luminosity

We consider a neutrino-less fluid of neutrons ‘n’, protons ‘p’, electrons ‘e’, and muons ‘µ’, that undergoes direct beta reactions (i.e., direct Urca reactions):

$$\beta^-_e: \quad n \rightarrow p + e^- + \bar{\nu}_e, \quad (32)$$
$$\beta^+_e: \quad p + e^- \rightarrow n + \nu_e, \quad (33)$$
$$\beta^-_\mu: \quad n \rightarrow p + \mu^- + \bar{\nu}_\mu, \quad (34)$$
$$\beta^+_\mu: \quad p + \mu^- \rightarrow n + \nu_\mu. \quad (35)$$

The net number reaction rate for the particle $i = \{e, \mu\}$, which is responsible for the change in the particle fraction [see Eq. (29)], due to neutron decay [Eqs. (32) and (34)] and lepton capture [Eqs. (33) and (35)] and linearized around equilibrium in the affinity $\Delta^i$, is [19]:

$$\mathcal{R}_i = 8.86 \times 10^{31} \text{cm}^{-3} \text{s}^{-1} \sqrt{\frac{Y^\text{eq}_{\nu e}}{\rho_n}} \left(\frac{T^{\text{eq}}}{10^9 \text{K}}\right)^5 \frac{17\pi^4}{30} \frac{\Delta^i}{k_B T}. \quad (36)$$

We chose to linearize the reaction rates in order to simplify the implementation of the implicit step (See Appendix A). The reaction luminosity linearized around equilibrium in $\Delta Y_i = Y_i - Y^\text{eq}_i$ [which in our case is linear in the affinity $\Delta^i$, see Eq. (48)], which is responsible for the change in the energy and momentum of the fluid [see Eqs. (25) and (27)], is [19]:

$$Q_i = 1.22 \times 10^{25} \text{erg}^{-1} \text{cm}^3 \text{s}^{-1} \left(\frac{Y^\text{eq}_{\nu e}}{\rho_n} \left(\frac{T^{\text{eq}}}{10^9 \text{K}}\right)^6 \frac{457 \pi^6}{1260} \left(1 + \frac{\Delta Y_i}{3 Y^{\text{eq}}_i}\right)\right), \quad (37)$$

where $\rho_n$ is the rest mass density at nuclear saturation.

Note that a more realistic set of reactions for the stellar conditions of the models considered in this paper would have included the modified beta reactions (i.e. modified Urca, see Sec. 2.2 of Haensel [42]). Since we are not interested in performing a quantitative study of the neutron star evolution, but rather in comparing the multi-fluid formulation and the bulk stress approximations for bulk viscosity, we focus on the results of simulations including only un-suppressed direct Urca reactions. However, we perform some simulations using the modified Urca number ($\mathcal{R}^m_i$) and energy ($Q^m_i$) rates [19] instead of the direct Urca ones,

$$\mathcal{R}^m_i \approx 5.91 \times 10^{23} \text{cm}^{-3} \text{s}^{-1} \sqrt{\frac{Y^\text{eq}_{\nu e}}{\rho_n}} \left(\frac{T^{\text{eq}}}{10^9 \text{K}}\right)^7 \frac{367 \pi^6}{63} \frac{\Delta^i}{k_B T^{\text{eq}}}, \quad (38)$$
$$Q^m_i \approx 8.15 \times 10^{16} \text{erg}^{-1} \text{cm}^3 \text{s}^{-1} \sqrt{\frac{Y^\text{eq}_{\nu e}}{\rho_n}} \left(\frac{T^{\text{eq}}}{10^9 \text{K}}\right)^8 \frac{11513 \pi^6}{2520} \left(1 + \frac{\Delta Y_i}{3 Y^{\text{eq}}_i}\right). \quad (39)$$

We comment on these in Sec. V D.

B. Analytic non-equilibrium equation of state

We assume an analytic equation of state (EOS) for matter out of beta-equilibrium, defined by:

$$u = k_0 \rho + k_{th} s^2 \rho \Gamma_{th} - 1 + \sum_i k_i \Delta Y_i^2, \quad (40)$$
$$\Delta Y_i = Y_i - Y^\text{eq}_i \rho_n. \quad (41)$$

where $i$ is the index of the independent particle fractions (e.g., $i = e, \mu$ in out-of-equilibrium $npe\mu$ matter). In the above expression, $u = u(\rho, s, \{Y_i\}) = \epsilon/\rho - 1$ is the specific (per unit mass) internal energy, $k_0, k_{th}, k_i$ are constant positive coefficients, $\Gamma_{th}$ is the thermal polytropic exponent, and $Y^\text{eq}_i$ the equilibrium net particle fraction at nuclear saturation. This EOS is a simple extension of a cold $\Gamma = 2$ polytropic EOS: the thermal component was already introduced in Camell et al. [43]. The dependence on $\rho$ of the cold and particle components of the EOS was chosen [19] such that the pressure and the speed of sound are positive if $Y_i \in [0, 1]$ and

$$k_0 > 2 \sum_i k_i Y^\text{eq}_i / \rho_n. \quad (42)$$

Our EOS has the advantages to be simple and analytic, and at the same time it reproduces some behaviors
of a realistic EOS, and in particular: (i) the cold EOS at equilibrium is a polytrope, (ii) the temperature goes to zero as the entropy goes to zero, and (iii) the free parameters $\Gamma_{th}, k_{th}, k_0, Y_i^0$ have been chosen in order to fit the GM3 EOS \cite{43,44}, see Table I and plots (a) and (b) of Fig. 1.

However, we are not trying to accurately reproduce an existing EOS, as for example the polytropic parameters for the cold GM3 EOS are $\Gamma = 2.88, k_0 = 2.78 \times 10^4$ while we are setting $\Gamma = 2, k_0 = 200$ in order for Eq. (42) to be true.

Given our analytical model for the EOS in (40), the equilibrium value of the particle fractions (obtained in the limit $\Lambda^s = \Lambda^0 = 0$) is:

$$Y_i^{eq}(\rho) = Y_i^0 \frac{\rho}{\rho_0} \quad (i = e, \mu).$$

Unfortunately, this expression does not reproduce the correct trend, see plot (c) of Fig. 1. In particular, the equilibrium fraction of muons $Y_\mu^{eq}$ deviates noticeably from the correct trend (see plot (c) of Fig. 1), and as a consequence the contribution of the muons to the dynamics is negligible. Since we are not aiming for an accurate physical description but rather to clarify the role of bulk viscosity in simulations, we decided to artificially enhance the role of muons in the dynamics by modifying their parameters, such that condition (42) is respected. In order to do so, we substitute the muons $\mu$ with an artificially modified muon particle that we call ‘x’ to avoid confusion, such that:

$$Y_x^0 = Y_\mu^0, \quad Y_x^0 = Y_x^0. \quad (45)$$

In Table I we report the EOSs used in this paper for the different models described in Sec. IV. The EOS for the model ‘PF’ (perfect fluid) has both electrons and modified muons at equilibrium, the EOS for model ‘MF’ (multi-component fluid) has electrons out of equilibrium and modified muons at equilibrium, and the EOS for model ‘MF-x’ has both electrons and modified muons out of equilibrium.

The other thermodynamic quantities can be derived from Eq. (40) and the first law of thermodynamics \cite{19}:

$$p = k_0 \rho^2 + (\Gamma_{th} - 1)k_{th} s^2 \rho_{th}$$

$$T = 2m_n k_{th} s \rho_{th}^{-1},$$

$$\Lambda^s = -2m_n k_0 Y_i^0,$$

$$c_{s,uv}^2 = \frac{\partial p}{\partial \rho_{s}} + \sum_i k_i Y_i^0 (2\Delta Y_i - Y_i^0 \frac{\rho}{\rho_0}),$$

$$c_{s,ir}^2 = \frac{\partial p}{\partial \rho_{s}} + \sum_i k_i \Delta Y_i (\Delta Y_i - 2Y_i^0 \frac{\rho}{\rho_0}),$$

$$\frac{\partial \rho^{eq}}{\partial \rho_{s}} \Big|_{s} = 2k_0 \rho + \Gamma_{th}(\Gamma_{th} - 1)k_{th} s^2 \rho_{th}^{-1},$$

$$\frac{\partial \epsilon^{eq}}{\partial \rho_{s}} \Big|_{s} = 1 + 2k_0 \rho + \Gamma_{th} k_{th} s^2 \rho_{th}^{-1},$$

where $c_{s,ir}$ and $c_{s,uv}$ are respectively the ‘infrared’ (where the chemical composition is always at equilibrium) and ‘ultraviolet’ speeds of sound \cite{19} (where the thermodynamic derivatives are performed at constant composition \cite{19}), and the ‘eq’ superscript means that the quantity is taken at equilibrium ($\{Y_i = Y_i^{eq}\}_i$).

For our choice of EOS and reaction rates, we have:

$$\Xi_{ab} = \text{diag}(\Xi_e, \Xi_x),$$

$$\Xi_e = \frac{8.86 \times 10^{11}}{\text{cm}^3 \text{s}} \sqrt{\frac{Y_i^0 \rho^2}{\rho_0^2}} \left(\frac{T}{10^8 \text{K}}\right)^5 \frac{17 \pi^4}{30 k_B T},$$

$$\zeta = n \frac{\rho^2}{\rho_0^2} \sum_i \frac{(Y_i^{eq})^2}{\Xi_i},$$

$$\tau = \frac{n}{2m_n} \times \sum_i \frac{(Y_i^{eq})^2}{\Xi_i} + \sum_i k_i (Y_i^{eq})^2,$$

$$c_{s,uv}^{eq2} = c_{s,ir}^2 + \frac{1}{(\epsilon + \rho \rho^{eq})},$$

$$\Pi = -2 \rho \sum_i k_i \rho^{eq} \Delta Y_i,$$

where $i = \{e, x\}.$
FIG. 1: EOS used in this paper and GM3 EOS. From top to bottom: (a) thermal contribution to the equilibrium pressure (for $s = 4$), (b) dependence of the particle affinity ($A_i = \mu_n - \mu_p - \mu_i$, where $\mu_{(n,p,i)}$ are the chemical potentials) on the particle fraction $Y_i$, (c) dependence of the equilibrium particle fraction on the rest mass density. The value of the muon parameters obtained from the fit are $k_{\mu} = 0.974$, $Y_{0\mu} = 0.00860$, while the other parameters are reported in Table I.

IV. NUMERICAL METHOD

We implemented a new Eulerian, one-dimensional, general-relativistic hydrodynamic code called hydro-bulk-1D in the C programming language and release it on zenodo under the MIT license, in such a way that the community can check our results and improve on them.

The code uses finite-differences with the method of lines on an evenly spaced grid. The time evolution is performed with a 3rd order IMplicit-EXplicit Runge-Kutta (IMEX-RK) solver and a Courant-Friedrichs-Lewy factor CFL = 0.5. The spatial fluxes are obtained with the LLF (Local Lax-Friedrichs) approximate Riemann solver with 2nd order (piecewise linear) spatial reconstruction of the primitive variables and the MINMOD slope limiter. The gravitational mass is computed by integrating Eqs. (18) and (19) with the trapezoid method. We use as primitive variables $\rho$, $Wv$, $u$ with the addition of $Y_e$ and eventually $Y_\alpha$ (in the case of the multi-component fluid), or $\Pi$ (in the case of Hiscock-Lindblom and Maxwell-Cattaneo). We perform the conservative to primitive inversion using the Brent method to find the root of the conserved momentum, using $Wv$ as independent variable. When the density is smaller than the threshold value $\rho_{thr} = 100\rho_{min}$, we set it as an atmosphere at $\rho_{min} = 10^{-20}$. In order to avoid division by zero, we set a floor for the temperature at $T_{min} = 10^{-61}$ (in code units) in the derivative in the source of the Hiscock-Lindblom equation [Eq. (30)]. It is also possible to set in the code a lower order time integration (2nd or 1st) and spatial reconstruction (1st, i.e. piecewise constant), the MCLIM slope limiter [48], and the Harten-Lax-vanLeer (HLL) Riemann-Solver [49]. We adopt an implicit scheme to evolve the equations, as done also by Most and Noronha [50], because the timescale of the reactions can become shorter than the timestep, making an explicit method unstable, see also [51] for a recent discussion. Note that bulk stress formulations (namely Hiscock-Lindblom or Maxwell-Cattaneo) do not solve this issue since the bulk viscous timescale $\tau$ is related to the reaction timescale.

We describe in more detail the code in Appendix A and we test it in Appendix B.

V. RESULTS

In this section, we compare the bulk viscous evolution performed with different models. For each model, we consider both small deviations (neutron star oscillations) and large deviations (neutron star migration from unstable to stable branch) from hydrostatic equilibrium. For the oscillation case, we choose as initial condition the stable and isentropic Tolman-Oppenheimer-Volkoff (TOV) model in beta-equilibrium with central density $\rho_0 = 2\rho_n$. The initial uniform entropy per baryon $s = 0.04$ is such that the dissipative timescale in the center is approximately of the order of 1 ms (see discussion in Appendix C), which is also the order of magnitude of the oscillation period. For this model, $R = 18.7\,\text{km}$, $M = 2.16\,M_\odot$, and $M_b = 2.35\,M_\odot$. The other settings specific for the oscillation case are $R_{\text{max}} = 13$, $N = 800,$
and \( v_{\text{pert}} = 5 \times 10^{-3} \) (\( v_{\text{pert}} \) determines the amplitude of the initial velocity perturbation), see Appendix A for details. For the migration case, we choose as initial condition the unstable cold (\( s = 0 \)) TOV model in beta-equilibrium with central density \( \rho_0 = 4 \times 10^{-3} \approx 9.22 \rho_{\odot} \), which gives a star with \( R = 12.2 \text{ km}, M = 2.05 M_{\odot} \), and \( M_b = 2.17 M_{\odot} \). The other settings specific for the migration model are \( R_{\text{max}} = 80, N = 8001 \) (\( dr = 0.01 \)), and \( v_{\text{pert}} = 0 \). The other settings common to both models are described in Sec. V D. In Fig. 2 we plot the dependence of the total baryon and gravitational masses on the central density of the cold TOV model and we mark the central densities of the oscillation and migration models.

We remark that, contrary to the oscillation and migration models considered in the tests (see Appendix B), the two configurations do not have the same baryon number, namely the end result of the migration is not equivalent to the oscillation model.

For each case (oscillations and migration), we will consider 8 different models:

- **PF**: perfect fluid with all reactions at equilibrium (namely no viscosity and no independent particle fractions),
- **MF**: multi-component fluid with only electron beta reactions out of equilibrium (namely the only independent particle fraction are the electrons) and no luminosity (\( Q = 0 \)),
- **HL**: Hiscock-Lindblom with only electron beta reactions out of equilibrium (as MF) and no luminosity (\( Q = 0 \)),
- **MC**: Maxwell-Cattaneo with only electron beta reactions out of equilibrium (as MF) and no luminosity (\( Q = 0 \)),
- **MF-Q**: multi-component fluid with luminosity and only electron beta reactions out of equilibrium (as MF),
- **MC-Q**: Maxwell-Cattaneo with luminosity and only electron beta reactions out of equilibrium (as MF),
- **MF-x**: multi-component fluid with electrons and modified muons ‘x’ beta reactions out of equilibrium and no luminosity (\( Q = 0 \)),
- **MC-x**: Maxwell-Cattaneo with electrons and modified muons ‘x’ beta reactions out of equilibrium (as MF-x) and no luminosity (\( Q = 0 \)).

In Fig. 3 we show the central rest mass density evolution of the oscillation (left) and migration (right) models. In order to enhance the differences between the models, we show the central rest mass density of models PF, MF, MF-Q, and MF in plots (a) and (b) and the difference between the central rest mass densities of the other models and that of models MF and MF-Q in plots (c)-(f).

### A. Comparison of the three approaches to bulk viscosity

In this subsection, we compare the three dissipative models MF, HL, and MC, together with the additional model PF (no viscosity) that can be used as a benchmark reference. In all cases, we will assume that the modified muons ‘x’ are in equilibrium, so that the only independent fraction that needs to be evolved is \( Y_e \). We also assume that no energy is emitted during the simulation (\( Q = 0 \)).

In plots (a) and (c) of Fig. 3 we show the central rest mass density for each model for the oscillating case (i.e., small perturbations). The oscillation of the central density for the models with bulk viscosity (MF, MC, HL) is damped with time, while the oscillation of the non-viscous model (PF) has no visible damping on the timescale of the simulation. All three viscous models evolve very similarly, which is a corroboration for the chemical reactions-bulk viscosity correspondence and for the code. However, the HL model is more noisy than MC and MF, due to the additional derivative in the source, since close to the surface both \( \chi \) and \( T^\text{eq} \) reach small values and the quantity \( \log(\chi/T^\text{eq}) \) can become problematic. In order to check how the temperature floor \( T_{\text{tiny}} \) in the derivative in the source of Eq. (50) influences the evolution, we made some tests varying its...
FIG. 3: Evolution of the central rest mass density (direct Urca reactions). On the left [plots (a), (c), and (e)] we show the oscillation case, on the right [plots (b), (d), (f)] we show the migration case. In the first row [plots (a) and (b)] we plot the central rest mass density for the PF, MF, and MF-Q models. In the second and third row, we plot the difference of the central rest mass density between different models. Plot (c) shows a detail of the total evolution, up to 5 ms.

value. In Fig. 4 we show the last 5 ms evolution of the central rest mass density (top) and the final temperature profile (bottom) for the oscillating HL model with $T_{\text{tiny}} = \{10^{-65}, 10^{-63}, 10^{-61}\}$. Increasing the value of $T_{\text{tiny}}$ reduces the noise in the evolution of the central density because in this way the relative error in the temperature profile is reduced too (note that the truncation error on the temperature is of the order of $10^{-66}$). On the other hand, $T_{\text{tiny}}$ should be small enough not to overcome the temperature profile. We found that $T_{\text{tiny}} = 10^{-61}$ is a good compromise between reducing the noise and being able to resolve the thermal profile.

In Fig. 5 we plot the value of the bulk stress $\Pi$ directly evolved with the MC model and that obtained with Eq. (8) from the evolution of the MF model, for the oscillation case. The similarity of the bulk stress obtained
FIG. 4: Dependence of the simulation on the parameter $T_{\text{tiny}}$ (direct Urca reactions). On the top, central rest mass density from 15 ms to 20 ms. On the bottom, thermal profile at 20 ms.

with two completely different models is an additional corroboration of the correspondence between the bulk stress theories and the multi-component fluid. As a comparison, the central pressure oscillates around $p \approx 1.5 \times 10^{-3}$. In the final profile, the pressure is at least 9000 times larger than the absolute value of the bulk stress $|\Pi|$ in most of the star, justifying the bulk stress approximation. However, close to the surface, the bulk stress approaches the pressure and as a consequence the results here lose accuracy.

For large perturbations (migration), the relative contribution due to bulk viscosity to the dynamics is smaller, see plots (b) and (d) of Fig. 3. The Müller-Israel-Stewart theories reproduce quite well the results of the multi-component fluid, but the approximation is less accurate than in the oscillation case (as expected for larger deviations from equilibrium). In Fig. 6 we plot the absolute value of bulk stress $|\Pi|$ directly evolved with the MC and HL models and that obtained with Eq. (8) from the evolution of the MF model, for the migration case. As in the oscillation case, also for the migration the evolution with the Hiscock-Lindblom model is more noisy than that with the Maxwell-Cattaneo model, and the bulk stress derived from the multi-component fluid with Eq. (8) is very similar to the one evolved with the Müller-Israel-Stewart theories. In the final profile, the pressure is at least 800 times larger than the absolute value of the bulk stress $|\Pi|$ in most of the star, and hence the bulk stress approximation is still valid.

B. Switching on the luminosity

We now include the luminosity in the evolution. As in the previous subsection, the modified muons ‘x’ are always in beta-equilibrium, so that we only have to evolve $Y_e$. In addition, the luminosity is only due to electron neutrinos.

As can be seen in plots (e) and (f) of Fig. 3 the central rest mass evolution of the bulk stress model (MC-Q) reproduces very well the multi-component fluid (MF-Q), valid.
both for the oscillations and the migration. We also notice that, non-intuitively, the oscillations of the models with luminosity (MF-Q) are larger than those without (MF), see plots (a) and (b) of Fig. 3. In order to understand this behavior, in Fig. 3 we plotted the central temperature (top) and the time evolution of the temperature profile of the MF-Q model (bottom), for the oscillation (left) and migration (right) cases. The energy extraction occurring in the MF-Q model reduces its temperature with respect to the MF model, which in turn implies that also the number rates [which depend strongly on the temperature, cf. Eq. (38)] are reduced. As a consequence, the particle fractions are allowed to oscillate farther from equilibrium than in the case without luminosity, implying larger deviation from equilibrium of the other thermodynamic quantities.

The PF model has no physical viscosity, but due to the always present numerical viscosity there is anyway an increase in the temperature of the star, which is however smaller than that due to the physical bulk viscosity of the MF model (cf. top plots of Fig. 3). We remark that, while in a fluid in equilibrium (as in the PF model) the net number reaction rate vanishes, there can still be luminosity due for example to neutrino emission (see Eq. (37); \( \dot{Q} \neq 0 \) even when \( \Delta Y_i = 0 \). This is the reason why in the oscillating case the dissipating MF-Q model has a temperature lower than that of the non-dissipating PF model: in the latter we do not include the (physical) luminosity (which is present even at chemical equilibrium), and therefore the heating due to the (nonphysical) numerical viscosity cannot be dissipated. On the other hand, the initial entropy of the oscillating MF-Q model was chosen such that the viscosity and hydrodynamic timescales are of the same order, and hence a more efficient emission is expected. Compare this with the migrating case, which happens at temperatures far from resonance between viscosity and hydrodynamics (top right plot of Fig. 7): as expected, the cooling due to density oscillation is smaller than the dissipative heating, cf. MF and MF-Q models.

Due to the many approximations made, and in particular to the use of direct Urca instead of modified Urca reactions, the results presented should not be taken as a quantitative description of what happens in a neutron star, but rather as a qualitative indication of the importance of including luminosity in the simulations.

C. Two independent chemical fractions

Finally, we relax the assumption that there is only one independent particle fraction and we assume that there is no luminosity (\( Q = 0 \)). As mentioned in Sec. 11, we modified the parameters of the muons in the EOS in order to enhance their effects on the dynamics. We refer to the modified muons as particle ‘x’.

The difference between the evolution with independent \( Y_e, Y_\nu \) and the evolution with only one independent fraction (\( Y_\nu \)) for the oscillations is shown in plot (e) of Fig. 3. We note that the addition of another degree of freedom (the fraction \( Y_\nu \)) has a limited effect on the dynamics (cf. models MF-x, MF, and PF), probably due to the particular choice of the parameters of the EOS. More importantly, the bulk stress approximation for two independent particle fractions is less accurate in describing the bulk viscosity than for one independent particle fraction, cf. plots (c) and (e) of Fig. 3. This is due to the fact that the bulk stress correspondence with the multi-component fluid is valid for only one additional independent particle fraction and small perturbations, while in the MC-x model we are considering 2 species out of equilibrium.

For the migration, the results are similar (see plot (f) of Fig. 3), but the bulk stress approximation MC-x of the multi-component fluid MF-x with 2 independent particle species is not appreciably worse than that for one particle species out of equilibrium, cf. with models MC and MF in plots (d) and (f) of Fig. 3, probably due to the fact that the loss in accuracy due to the larger deviations from equilibrium overcome that of considering more than one independent particle fraction.

D. Using modified Urca reactions

In low-density, low-temperature nuclear matter, direct Urca reactions are kinematically suppressed and modified Urca reactions are the main channel for dissipation. For neutron stars in the conditions considered in this paper, direct Urca reactions might not be the relevant rates; for this reason we performed some simulations using the modified Urca reactions instead of the direct Urca ones.

A direct Urca reaction involves less particles (3 plus the escaping neutrino, see Eqs. (1) and (2) of Haensel [12]) than a modified Urca reaction (which involves 5 particles plus the escaping neutrino, see Eqs. (10) and (11) of Haensel [12]). As a consequence, direct Urca rates have a larger prefactor and depend on a smaller power of the temperature than the modified ones, cf. Eqs. (36) and (39). Since for the modified Urca reactions the prefactor is smaller, we need to set a larger initial entropy (i.e., \( s = 0.31 \)) in the oscillation case in order for the reaction timescale at the center to be approximately equal to the oscillation period over \( 2\pi \) and hence maximize the dissipation (see discussion in Appendix C). Apart from this, we use the same initial configurations of the direct Urca case. Note that, in order to be able to compare equivalent configurations, we performed for the oscillation case another perfect fluid (PF) simulation with an initial uniform entropy of \( s = 0.31 \), even if no direct and/or modified Urca reaction is included in the PF model. In Fig. 8 we show the central rest mass density evolution for the oscillation and migration cases, in Fig. 9 we show the central bulk stress evolution for the oscillation case, and in Fig. 10 we show the central temperature evolution for the oscillation and migration cases.

The perturbation in the oscillation case with modi-
FIG. 7: On the top, central temperature evolution for the PF, MF, and MF-Q models (direct Urca reactions). On the bottom, temperature profile evolution for the MF-Q model (direct Urca reactions). On the left, oscillation case, on the right, migration case.

The model with luminosity (MF-Q) with modified Urca reactions has larger oscillations (both in the oscillation and the migration cases) than the model without (MF), see Plots (a) and (b) of Fig. 8. This means that also for the case with modified Urca reactions, the main effect of bulk viscosity on the evolution is indirect and caused by the extraction of energy due to neutrinos leaving the star.

The temperature evolution is qualitatively similar between the direct and modified Urca cases, cf. top row of Fig. 7 with Fig. 10. However, for the oscillation simulations the increase of entropy in the modified Urca case is much smaller due to the fact that the entropy production rate is inversely proportional to the temperature (Eq. (8) of Camelio et al. [19]) and the initial temperature is larger than in the case with direct Urca reactions (as discussed above).

The adopted reaction rates do not qualitatively influence the comparison between different bulk viscosity formulations. In other words, in the regime in which the Müller-Israel-Stewart theories are valid (i.e., for small deviation from equilibrium), both the direct and the modified Urca reaction rates are equivalent to the multi-fluid
In this paper we numerically implement the hydrodynamic equations derived in a companion paper [19]. To do so, we have developed a new one-dimensional, general relativistic, hydrodynamic code called hydro-bulk-1D [37]. We use this code to compare different approaches to the inclusion of bulk viscosity in the context of neutron stars, both for small perturbations (oscillations) and large perturbations (migration from unstable to stable branch). In particular, we consider the bulk viscosity caused by beta reactions and we adopt the following approaches: (i) the multi-component fluid, namely the direct tracking of the chemical reactions and the particle species, and two approximations based on the Müller-Israel-Stewart theory, namely (ii) Hiscock-Lindblom, and (iii) Maxwell-Cattaneo, which is a linearization of Hiscock-Lindblom. In our knowledge, this is the first time that the Hiscock-Lindblom formulation has been implemented in the context of neutron stars, and the first time that Müller-Israel-Stewart theories are implemented in radial gauge, polar slicing coordinates in spherical symmetry. We evolve our model with and without the energy loss due to the luminosity of the reactions and with one (only electrons) or two (both electrons and ‘modified’ muons) species out of equilibrium. The inclusion of the luminosity in the Müller-Israel-Stewart theory is a novel result of the companion paper [19], and this is the first implementation of such extension. Finally, in Appendix C we derive a formula for the relativistic linearized damping time of an oscillation valid in any regime (i.e., in the frozen, quasi-stationary, and intermediate regimes) and discuss its implications on our simulations.

We find that the correspondence between the multi-component description of the fluid and the bulk stress one identified in Gavassino et al. [38] is accurate, both in the case of direct and modified Urca reactions. The accuracy of the Müller-Israel-Stewart models decreases for more than one independent particle species and for large perturbations (because the mathematical correspondence is no longer rigorous), but they still reproduce correctly the qualitative behavior of the system. However, the Hiscock-Lindblom model is more noisy than the Maxwell-Cattaneo one due to the presence of an additional derivative in the source of Hiscock-Lindblom that involves quantities that are vanishing at the surface.

Our simulations show that the largest effect of bulk viscosity in violent (i.e., far from equilibrium, e.g. in the migration case) neutron star dynamics is indirect and due to the energy sink rather than to particle reactions per se, because the inclusion of luminosity reduces the temperature of the star, which in turn reduces the reaction rate, and as a consequence the fluid is allowed to oscillate farther from equilibrium (we find this behavior both in simulation with direct and modified Urca reactions). In other words, it is important to correctly include reaction luminosity in the simulations. This can be done also within the bulk stress approach, as shown in this paper and its companion [19].

When the dominant contribution to bulk viscosity in neutron stars comes from chemical reactions, we find that the implementation of the multi-component fluid is more convenient than the bulk stress models. In fact, it is directly and naturally linked to the physics of the system in consideration, but it is also capable to accurately describe dissipation in systems with more than one independent chemical fraction. In addition, its numerical implementation is simpler than that of the Müller-Israel-Stewart models (it has no derivatives in the source). Moreover, contrary to the Müller-Israel-Stewart models that are perturbative, the chemical framework is robust when the system is driven far from thermodynamic equilibrium [38]. Finally, numerical implementations of a multi-component fluid in the context of neutron stars are much more common and better understood (e.g., supernova simulations) than the ones adopting bulk stress theories.

An alternative approach and possible future line of research is to use the correspondence between the multi-component fluid and the bulk stress [38] to derive the equivalent reaction rates from bulk stress parameters, in order to treat the case when bulk viscosity is not due to the presence of non-equilibrium reactions, still using a multi-component fluid. This approach is the inverse of the one adopted in this paper, and it is discussed in Camelo et al. [17].

As mentioned above, the aim of this paper and its companion [19] is to compare the multi-component fluid description and the bulk stress approximation of bulk viscosity. Other than that, the approximations made allow only for a qualitative description of the neutron star evolution. For a more quantitative description it would be necessary to abandon spherical symmetry, to use a tabulated EOS and accurate reaction rates (including modified Urca and the gradual opening of the direct Urca reactions with density and temperature), and to include trapped neutrinos, which are present at high temperatures.
FIG. 8: Evolution with modified Urca reactions instead of direct Urca reactions. On the left [Plots (a) and (c)] we show the oscillation case, on the right [Plots (b) and (d)] we show the migration case. In the first row [Plots (a) and (b)] we plot the central rest mass density for the PF, MF, and MF-Q models. In the second row [Plots (c) and (d)], we plot the difference of the central rest mass density between different models. Plot (c) shows a detail of the total evolution, up to 5 ms. The damping timescale of the oscillations due to modified Urca reactions is comparable to that due to direct Urca ones (cf. Fig. 3) because we have artificially tuned the temperature, cf. Figs. 10 and 7 and see discussion in Sec. V D.

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FIG. 9: Central bulk stress $\Pi$ for the oscillating MF and MC models with modified Urca reactions instead of direct Urca reactions. The bulk stress for the MF model is obtained with Eq. (8). The HL (not shown) and MC models are very similar.

FIG. 10: Central temperature for the PF, MF, and MF-Q models with modified Urca reactions instead of direct Urca reactions, for the oscillation (top) and the migration (bottom) cases.
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Appendix A: Numerical implementation

In this appendix we discuss the hydro-bulk-1D code in detail.

1. Grid

We use a 1D, evenly spaced, staggered grid with 2 (for the piecewise linear reconstruction) or 1 (for the piecewise constant reconstruction) ghost cells on each side. The position of each grid point is defined by

\[ r_i = \left(i - \frac{1}{2}\right) dr, \quad (A1) \]

\[ dr = \frac{R_{\text{max}}}{N - 1}, \quad (A2) \]

where \( R_{\text{max}} \) is a parameter setting the physical dimension of the grid, \( i = 1 \ldots N \), and \( N \) is the number of grid points (without ghost cells). The ghost cells are filled with symmetry conditions at the center and by linear extrapolation of the primitive quantities at the external boundary.

2. Spatial reconstruction

According to the order of the spatial reconstruction, the primitive variable \( Q \) is obtained at the left and right interfaces of the central position between grid points as:

1st: piecewise constant:

\[ Q_{i-1/2}^R = Q_i + O(dr^2), \quad (A3) \]

\[ Q_{i+1/2}^L = Q_i + O(dr^2), \quad (A4) \]

2nd: piecewise linear:

\[ Q_{i-1/2}^R = Q_i - \frac{r_i - r_{i-1}}{2} Q'_i + O(dr^2) \quad (A5) \]

\[ Q_{i+1/2}^L = Q_i + \frac{r_{i+1} - r_i}{2} Q'_i + O(dr^2) \quad (A6) \]
where \( Q_i' \) is the limited slope. It is necessary to limit the slope to avoid oscillatory solutions. The slope limiter can be set to:

- MINMOD limiter:
  \[
  Q_i' = \begin{cases} 
  Q_-', & |Q_-'| \leq |Q_+'| \text{ and } Q_-' \cdot Q_+ > 0, \\
  0, & \text{otherwise},
  \end{cases}
  \tag{A7}
  \]

- MCLIM limiter (also called MC limiter) [48]:
  \[
  Q_i' = \begin{cases} 
  \min(|Q'|, |2Q_-|, |2Q_+|) \sgn(Q') \quad Q_-' \cdot Q_+ > 0, \\
  0, & \text{otherwise},
  \end{cases}
  \tag{A8}
  \]

where

\[
\begin{align*}
Q_- &= Q_{i+1} - Q_{i-1}, \\
Q_+ &= Q_i - Q_{i-1}, \\
Q_0 &= Q_{i+1} - Q_i
\end{align*}
\tag{A9-A11}
\]

The reconstructed state is used to solve the Riemann problem and to compute the spatial derivatives in the source of Eqs. [30] and [31].

### 3. Riemann solver

The code can use either the HLL or LLF approximate Riemann solvers, which compute the fluxes at the grid interfaces. The HLL Riemann solver is the following:

\[
F_{i-1/2} = \begin{cases} 
  F_L, & \lambda_+ > 0, \\
  F_R, & \lambda_- < 0 < \lambda_+, \\
  \frac{(F_L \lambda_+ - F_R \lambda_- + \lambda_+ (U_R - U_L))}{\lambda_+ - \lambda_-}, & \lambda_- < 0 \text{ and } \lambda_+ < 0,
  \end{cases}
  \tag{A12}
  \]

where \( F \) are the fluxes, \( U \) the conservative variables, the subscripts \( L, R \) refer to the left and right state, respectively, at the center of the cell (namely at \( i - 1/2 \)), and the signal propagation speeds are:

\[
\lambda^+ = \max\left(0, \frac{v_L + c_L}{1 + v_L c_L}, \frac{v_R + c_R}{1 + v_R c_R}\right),
\tag{A13}
\]

\[
\lambda^- = \min\left(0, \frac{v_L - c_L}{1 - v_L c_L}, \frac{v_R - c_R}{1 - v_R c_R}\right),
\tag{A14}
\]

where \( c \) is the 'ultraviolet' speed of sound [Eqs. [49] and [57]].

The LLF Riemann solver is:

\[
F_{i-1/2} = \frac{1}{2} \left( F_L + F_R - \max(|\lambda^+|, |\lambda^-|)(U_R - U_L) \right).
\tag{A15}
\]

### 4. Time evolution

The hydrodynamic equations can be written as:

\[
\frac{dU}{dt} = \mathcal{E}(U, t) + \mathcal{I}(U, t),
\tag{A16}
\]

where \( \mathcal{E}(U, t) \) includes the contribution from the fluxes and the explicit sources, while \( \mathcal{I}(U, t) \) includes the contribution from the implicit sources.

The timestep \( dt \) is determined by:

\[
dt = \text{CFL} \times \min_i (\max(|a_i|, |\lambda_i^+|, |\lambda_i^-|)),
\tag{A17}
\]

where CFL is the Courant-Friedrichs-Lewy factor.

To numerically integrate Eq. [A16] at a given order, we use Implicit-Explicit Runge-Kutta (IMEX RK):

\[
U^i = U(t) + dt \sum_{j=1}^{i-1} a_{ij}^E \mathcal{E}(U^j, t + c_j^E dt)
\]

\[
\quad + dt \sum_{j=1}^{i} a_{ij}^I \mathcal{I}(U^j, t + c_j^I dt),
\tag{A18}
\]

\[
U(t + dt) = U(t) + dt \sum_{j=1}^{n} b_{ij}^E \mathcal{E}(U^j, t + c_j^E dt)
\]

\[
\quad + dt \sum_{j=1}^{n} b_{ij}^I \mathcal{I}(U^j, t + c_j^I dt),
\tag{A19}
\]

where \( a_{ij}, b_{ij}, c_j \) are parameters given by the Butcher tableau, see Table [I]. We report the adopted parameters of the 2nd order IMEX RK in Table [II] and the 3rd order IMEX RK in Table [IV]. When first order integration in time is set, we use an implicit-explicit Euler:

\[
U(t + dt) = U(t) + dt \mathcal{E}(U(t), t)
\]

\[
\quad + dt \mathcal{I}(U(t + dt), t + dt).
\tag{A20}
\]

### 5. Conservative-to-primitive inversion

We choose the quantities \( \rho, W, v, u \) as primitive variables, with the addition of \( \{Y_j\} \) in the case of the multi-component fluid and of \( \Pi \) in the case of Müller-Israel-Stewart theories. For each grid point, we determine the

| Table II: General form of the Butcher tableau. |
|-----------------------------------------------|
| \( c_i^E \) | \( b_i^E \) | \( c_i^I \) | \( b_i^I \) |
| 0 | 0 | 0 | 0 |
| 1/2 | 1/2 | 1/2 | 1/2 |

| Table III: Butcher tableau for IMEX RK at 2nd order, with \( \gamma = 1 - 1/\sqrt{2} \). Compare with Table 2 of Pareschi and Russo [46]. |
|-----------------|-----------------|-----------------|-----------------|
| \( \mathcal{E} \) | \( \mathcal{I} \) | \( \mathcal{E} \) | \( \mathcal{I} \) |
| 0 | 0 | 1 | 0 |
| \( 1 - \gamma \) | \( \gamma \) | \( 1 - 2\gamma \) | \( \gamma \) |
| \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) |
value of the function $X$ from the conservative variables, noting that the integrand of Eq. (18) is equal to $\tau + D$. We then use the Brent method to find the root of the function

$$f(z) = S' - (\hat{\epsilon} + \hat{p}) z \hat{W},$$

where $z$ is the independent variable and

$$\hat{W} = \sqrt{1 + z^2},$$

$$\hat{\rho} = \max\left(\frac{\rho_{\text{min}} - D}{X W}\right),$$

$$\hat{\epsilon} = \max\left(\epsilon_{\text{min}}, \tau + D - \frac{S' z}{W}\right),$$

$$\hat{\rho} = \begin{cases} p(\hat{\rho}, \hat{\epsilon}) & \text{PF,} \\ p(\hat{\rho}, \hat{\epsilon}, \hat{Y}_i) & \text{MF,} \\ p^{\text{eq}}(\hat{\rho}, \hat{\epsilon}) + \Pi & \text{HL and MC,} \end{cases}$$

where $\hat{Y}_i$ and $\hat{\Pi}$ are computed implicitly in the conservative-to-primitive inversion. At the root of $f$, the quantities in Eqs. (A21)-(A25) correspond to the primitives, namely we can drop the hat, and $z = W v$.

The minimal rest mass density is set to $\rho_{\text{min}} = 10^{-20}$ and the minimal energy density $\epsilon_{\text{min}}$ is obtained from $\rho_{\text{min}}$ and the cold EOS. Moreover, when the rest mass density obtained from the conservative-to-primitive inversion or from the reconstruction is smaller than the threshold value $\rho_{\text{thr}} = 100 \rho_{\text{min}}$, we set the primitive variables to their minimal values $\rho = \rho_{\text{min}}, W = 0, u = \epsilon_{\text{min}}/\rho_{\text{min}} - 1, Y_i = 0, \Pi = 0$ (i.e., we set them to their atmospheric values).

The time derivatives in the source of Eqs. (30) and (31) are performed implicitly by including the advanced time quantity in the conservative-to-primitive inversion. We computed implicitly also the terms containing the number reaction rates $R_i$ in Eq. (29) and the viscous timescale $\tau$ and the bulk stress $\Pi$ in the right-hand side of Eqs. (30) and (31). Since the implicit terms are linear, a simple inversion is enough to recover the conservative variables when solving for Eq. (A18), while the implicit term is already known when solving for Eq. (A19).

Appendix B: Tests

In this section we consider a set of standard tests for hydrodynamic codes in special and general relativity. Unless otherwise specified, we use the same code settings described in Sec. IV. In order to make a quantitative evaluation, we define the residual $\Delta f$ of a quantity $f$ as

$$\Delta f = \frac{\sum_{i=1}^{N} |f_i - \bar{f}_i|}{N},$$

where the index $i$ identifies the point in the grid of dimension $N$ and $\bar{f}_i$ is either the exact value or the value at the highest available resolution, according to the test.

In the following, we use 3rd order IMEX-RK with CFL $= 0.5$, 2nd order spatial reconstruction with MIN-MOD limiter, and the LLF Riemann solver.

1. Special-relativistic shocktube

The shocktube is a one dimensional problem where two different thermodynamic states of a fluid are initially separated by a wall, which is lifted at $t = 0$. The hydrodynamic equations are:

$$\partial_t D + \partial_x (D v) = 0, \quad (B2)$$

$$\partial_t S^x + \partial_x (S^x v + p) = 0, \quad (B3)$$

$$\partial_t \tau_v + \partial_x (S^x - D v) = 0, \quad (B4)$$

where $D = \rho W, S^x = (\epsilon + p) W^2 v, \text{ and } \tau_v = (\epsilon + p) W^2 - p - D$.

We use an evenly spaced Cartesian grid from $x = 0$ to $x = 1$ with varying size $N = \{100, 200, 400, 800\}$, constant values of the primitives in the ghost cells, and the ideal EOS:

$$p = (\Gamma - 1) \rho u,$$

with $\Gamma = 4/3$. The fluid is initialized in a rarefaction-shock configuration: $p(x < 0.5) = 0.9, p(x > 0.5) = 0.1, p(x < 0.5) = 1, p(x > 0.5) = 0.001$, and $v = 0$. In Fig. 11 we plot the result of the test at $t = 0.35$. The residuals with respect to the exact solution decrease increasing the grid size, as expected, which proves the convergence of the code. The order of convergence is smaller than the nominal order of our code (which is second order), because of the presence of the shock.

3 We also enforce $0 \leq \hat{Y}_1 \leq 1$ and $-p^{\text{eq}} \leq \hat{\Pi} \leq p^{\text{eq}}$.

4 Note that Chabanov et al. [53] performed an equivalent time derivative in the source term with a two-step predictor-corrector time evolution (Eq. (106) of Chabanov et al. [53]).
The Bjorken flow describes the quark-gluon plasma in the center-of-mass of an ultra-relativistic heavy-ion collision [54]. This is a standard test for viscous codes describing heavy-ion collisions [55], and it has been used also in an astrophysical context by Chabanov et al. [53]. Only in this section, we use the Milne coordinates:

\[ \tilde{t} = t \cosh x \quad \tilde{x} = t \sinh x \]

where \( \tilde{t} \) and \( \tilde{x} \) are the time and spatial coordinates in the Minkowski spacetime. The determinant of the metric is \( \sqrt{-g} = t \) and the only nonzero connection coefficients are \( \Gamma_{xx}^{x} = t \) and \( \Gamma_{xt}^{x} = \Gamma_{tx}^{x} = t^{-1} \). In the boosted system the velocity is null and therefore we have \( v = 0 \), \( W = 1 \), \( u^{\mu} = (1, 0) \), and \( T^{\mu\nu} = \text{diag}(\epsilon, (p^{\rho\kappa}+\Pi)/t^{2}) \). Also, there are no spatial fluxes; the hydrodynamic equations therefore read (see also Sec. 3.4 of Del Zanna et al. [55]):

- continuity equation:
  \[ \partial_{t}(\rho t) = 0, \quad (B7) \]

- since there are no fluxes the momentum conservation equation is trivial,

- energy conservation equation:
  \[ \partial_{t}\epsilon = -\frac{\epsilon + p^{\text{eq}}(\rho, \epsilon) + \Pi}{t}, \quad (B8) \]

- bulk stress evolution in Hiscock-Lindblom theory:
  \[ \partial_{t}(t\Pi) = -\frac{t\Pi}{2} \left( \frac{2}{\tau} - \frac{1}{t} + \partial_{t} \log \left( \frac{\epsilon}{T_{\text{eq}}} \right) \right) - \frac{1}{\chi}, \quad (B9) \]

- bulk stress evolution in Maxwell-Cattaneo theory:
  \[ \partial_{t}(t\Pi) = -t\Pi \left( \frac{1}{\tau} - \frac{1}{t} \right) - \frac{1}{\chi}, \quad (B10) \]

For constant \( \chi \) and \( \tau \) the Maxwell-Cattaneo equation \( (B10) \) has the analytical solution [55]:

\[ \Pi(t) = \Pi(t_0) e^{-\frac{t-t_0}{\tau}} + \frac{\chi}{\tau} e^{-t/\tau} \left( \text{Ei}(t_0/\tau) - \text{Ei}(t/\tau) \right), \quad (B11) \]

where \( t_0 \) is the initial time and \( \text{Ei} \) the exponential integral function.

We use the following EOS [43]:

\[ u(\rho, s) = k_0 \epsilon^{-1} + k_{\text{th}} s^2 \rho^{\Gamma_{\text{th}}-1}, \quad (B12) \]

with \( \Gamma = 2, \Gamma_{\text{th}} = 1.75, k_0 = 100, k_{\text{th}} = 1.5 \). The initial time is \( t_0 = 1 \), the initial rest mass density \( \rho(t_0) = \rho_0 \), the initial entropy \( s(t_0) = 10 \), the initial bulk stress \( \Pi(t_0) = 10^{-8} \), the bulk parameter and bulk timescales are the constant values \( \chi = 100/\rho_0 \) and \( \tau = 1 \), and the timestep is \( dt = \text{CFL} \times 0.1 \). We evolve implicitly the terms containing the viscous timescale \( \tau \), the bulk stress \( \Pi \), and the time derivative in the right-hand side of Eqs. \( (B9) \) and \( (B10) \). In Fig. 12 we plot the results of the simulations. The Maxwell-Cattaneo equation [Eq. \( (B10) \)] reproduces very well the exact solution [Eq. \( (B11) \)], while the Hiscock-Lindblom equation [Eq. \( (B9) \)] has the same trend of the Maxwell-Cattaneo case but differs from it, because the second order effects are important in the Bjorken flow. The solution for the Hiscock-Lindblom equation is closer to local thermodynamic equilibrium (i.e., the absolute value of the bulk stress is smaller) than the Maxwell-Cattaneo equation. This is similar to what happens in viscous cosmological models (see Eq. 5 of Maartens [56]). During the evolution, the pressure remains greater of the magnitude of the bulk stress, but due to the expansion of the Bjorken flow it decreases with time and \( |\Pi| \rightarrow p \), at which point the viscous approximation is no more valid.
is \( dt = \text{CFL} \times dr \). The simulation follows the exact solution for most of the evolution, but (i) there is a buildup of material close to the origin, as noticed also by O’Connor and Ott [40], (ii) the surface of the collapsing cloud is smeared out (note that the velocity tail corresponds to a vanishing density), as noticed also by Romero et al. [41], and (iii) at late times the simulation starts diverging from the exact solution. In any case, the code converges to the exact solution increasing the resolution (see plots (c) and (d) of Fig. 13), even if the order of convergence is lower than 1. These inaccuracies are due to the steep spatial and temporal gradients that develop in the simulation.

4. Neutron star oscillations

The hydrodynamic equation for this test are Eqs. (23)–(28) with \( Q = 0 \). We use the ideal EOS [Eq. (B5)] with \( \Gamma = 2 \). The initial configuration is a cold spherical solution of the TOV equations with central density \( \rho_0 = 1.28 \times 10^{-3} \), initialized with the cold EOS:

\[
p(\rho) = k_0 \rho^\Gamma,
\]

with \( k_0 = 100 \), and we set an initial velocity perturbation \( v(r) = v_{\text{pert}} \times \sin(\pi r/R) \), where \( v_{\text{pert}} = 5 \times 10^{-3} \) (unless otherwise specified) and \( R \) is the stellar radius.

We consider 4 different resolutions for the grid: \( N = \{100, 200, 400, 800\} \), we set \( R_{\text{max}} = 11 \), and we end the evolution at \( t_{\text{end}} = 20 \text{ ms} \).

We show the results in Table V and Fig. 14. We obtain the spectrum of the radial oscillations by Fourier-transforming the cosine-tapered (with a 5\% taper ratio) signal obtained by interpolating (after subtracting a linear fit) the central density on an evenly spaced time grid with the same number of points of the original one. As expected, (i) the frequencies of the first 3 radial oscillation modes converge within the uncertainty to the results from perturbation theory [58] with increasing resolution, (ii) the gravitational mass is the same expected from the spherical solution, (iii) the central rest mass density oscillates around the initial value during the simulation, and (iv) the maximal deviation from the initial total rest mass decreases with increasing resolution. In order to study how the residual changes with the resolution, we have set \( v_{\text{pert}} = 0 \) and we have computed the residual of the rest mass density at \( t = 20 \text{ ms} \) with respect to the initial configuration. The convergence of the residual is closer to the expected 2nd order than in the other test cases, since the oscillation is a smooth problem.

5. Migration from unstable to stable branch

For this test we use the same hydrodynamic equation and EOS of Sec. 5.4, but the initial central rest mass density is \( \rho_0 = 7.993 \times 10^{-3} \), corresponding to a gravitational mass \( M \simeq 1.45 \) and an initial circumferential radius \( R \simeq 5.8 \) [59, 61].
FIG. 13: Pressure-less dust collapse at times $t = 30, 40, 50$. From top to the bottom: density (a) and velocity (b) profiles at grid resolution $N = 800$ against the exact solution (gray line); residual of the rest mass density with respect to the exact solution (c), and the total rest mass variation during the evolution (d).

TABLE V: Frequencies of the first 3 radial modes, gravitational mass, and maximal variation of the absolute value of the baryonic mass for the different resolutions, compared with the result from perturbation theory (last row, see Tables I and II of Baiotti et al. [58]). Since we are evolving the star for $t = 20$ ms, the Fourier spectrum has a resolution of $\delta f = 50$ Hz.

| N  | $f_0$ [Hz] | $f_1$ [Hz] | $f_2$ [Hz] | $M_\text{g} (M_\odot)$ | max $|\Delta M_b/M_b (t = 0)|$ |
|----|------------|------------|------------|------------------|------------------|
| 100| 1450       | 3850       | 5750       | 1.40             | $4.3 \times 10^{-6}$ |
| 200| 1450       | 3900       | 5850       | 1.40             | $4.8 \times 10^{-5}$ |
| 400| 1450       | 3950       | 5900       | 1.40             | $5.7 \times 10^{-10}$ |
| 800| 1450       | 3950       | 5900       | 1.40             | $6.9 \times 10^{-11}$ |

FIG. 14: Neutron star oscillations. On the top, the evolution of the central rest mass density. On the bottom, the residuals at the end of the evolution (20 ms) for different resolutions and for $v_{pert} = 0$.

We set $R_{\text{max}} = 80$ and consider different grid resolutions $dr = \{0.08, 0.04, 0.02, 0.01\}$, which are equivalent to $N = \{1001, 2001, 4001, 8001\}$. We show the results in Fig. 15. The migration causes a series of shock waves at the surface of the neutron star that eject material that eventually reach the computational grid at a time greater than $\sim 1.6$ ms. For this reason, in Fig. 15 we show the profile and the residuals at 1.5 ms, and the mass conser-
vation is considered only up to 1.5 ms, while the central density oscillations are shown up to 5 ms since they are less affected by what happens close to the external grid border (at 5 ms the density at the external grid point is of the order $\sim 10^{-8}$). The final result of the oscillation is the configuration considered in Appendix B 4, but (i) part of the gravitational mass is converted into kinetic energy and (ii) due to the shocks at the surface part of the material is ejected. The code converges with the resolution with an order smaller than 1 because there are strong shocks at the surface of the neutron star.

**Appendix C: Damping timescale**

In this appendix we derive the relativistic linearized damping time of an oscillation without assuming a specific regime (i.e., without assuming that the reaction timescale is much larger or smaller than the perturbation timescale) and discuss its implications on our simulations with direct and modified Urca reactions. Our derivation follows Sec. 3.3 of Alford et al. [62], but it has been refined to properly take into account relativistic corrections. The resulting formula can be seen as a generalization of Eq. (4.9) of Kovtun [63] beyond the Navier-Stokes regime.

In the linear regime, the second law of thermodynamics can be expressed in terms of the information current $E^\mu$ as follows: $\nabla_\mu E^\mu = -\sigma$, where $\sigma$ is the entropy production rate. Working at a scale where the background state can be approximated as homogeneous (in the equilibrium local rest frame [64]), we can integrate the second law in space, obtaining

$$\frac{d}{dt} \int E^0 d^3x = - \int \sigma d^3x. \quad (C1)$$

Treating viscous effects as small corrections, the information density $E^0$ (averaged over the oscillations in space) can be approximated to be the perfect fluid one, which for a sound wave traveling, say, in the $x^1$ direction can be expressed as [65]:

$$\langle E^0 \rangle \approx \epsilon + \rho^{eq}_{\epsilon} \langle \delta u^1 \rangle^2, \quad (C2)$$

where $\delta u^1(t)$ is the amplitude of the velocity perturbation, i.e. $\delta u^1 = \delta u^1(t) e^{ikx^1-\omega t}$ ($\delta u^1$ coincides with the velocity in the equilibrium rest frame), where $k, \omega \in \mathbb{R}$ are the perturbation wavelength and angular velocity, respectively. The entropy production rate is (Eq. (8) of Cameliou et al. [19]) $\sigma = \delta \Pi^2 / \langle T^{eq} \rangle$, where $\delta \Pi$ is the perturbation to the bulk-viscous stress. From the Maxwell-Cattaneo equation (Eq. (14) of Cameliou et al. [19]), we find that $\delta \Pi = - (1 - i\omega \tau)^{-1} \xi k \delta u^1$, where $\tau$ is the reaction timescale. Hence, averaging over space, we obtain (assuming $\omega = c_s k$)

$$\langle \sigma \rangle \approx \frac{\xi \omega^2 \langle \delta u^1 \rangle^2}{2 T^{eq} c_s^2 (1 + \omega^2 \tau^2)}. \quad (C3)$$

**FIG. 15:** Migration from unstable to stable branch. From top to bottom: (a) profile of the rest mass density at 1.5 ms, (b) central rest mass density evolution up to 5 ms, (c) residual at 1.5 ms with respect to the maximum resolution, and (d) baryon mass conservation up to 1.5 ms.
Inserting (C2) and (C3) into (C1), we obtain an ordinary differential equation for $\tilde{\delta u}(t)$, whose solution is in the form $\tilde{\delta u}(t) = \delta \tilde{u}(0)e^{-t/t_{\text{damp}}}$, where the damping timescale is

$$t_{\text{damp}} = \frac{2c_s^2(\epsilon + p_{\text{eq}})(1 + \omega^2\tau^2)}{\zeta\omega^2}. \quad (C4)$$

Note that:

- the quantity $c_s$ is the phase velocity of the wave that is being damped. Hence, it should be either $c_s,\text{ir}$ or $c_s,\text{uv}$ (or something in between) depending on whether the fluid is in the Navier-Stokes (i.e., quasi-stationary or ‘parabolic’ regime, see Sec. II of Gavassino et al. [38] or Sec. II of Camelio et al. [19]) or in the frozen regime.

- in the limit of $\omega \ll \tau$, namely in the Navier-Stokes regime, Eq. (C4) tends, as expected, to Eq. (4.9) of Kovtun [63].

- although in the derivation we made use of the Maxwell-Cattaneo equation, the result still holds unchanged within the Hiscock-Lindblom theory, because the latter is equivalent to the Maxwell-Cattaneo theory in the linear regime.

In Fig. 16 we compare the evolution of the linearized damping time in the center of the star for the oscillation case, for direct and modified Urca reactions, for the MF model. From Plots (a) of Figs. 3 and 8, we estimate an oscillation period of $t_{\text{pert}} = 2\pi/\omega \approx 1\text{ ms}$ in both cases. As can be seen in Fig. 16, the initial central damping time for the direct Urca reaction is much larger than that of the modified Urca one. This is due to the fact that we set (indirectly, by setting the initial uniform entropy in the star) an initial central reaction timescale of $\tau \simeq 1.68\text{ ms} \approx t_{\text{pert}}$ for the direct Urca reactions (i.e., $\omega^2\tau^2 \simeq 111$), while we set $\tau \approx 0.16\text{ ms} \approx 1/\omega$ for the modified Urca ones. However, while the modified Urca damping time remains of the order of 15 ms (which is very similar to the duration of the simulation, i.e. 20 ms) during the whole simulation, the direct Urca one starts at approximately 80 ms and then reaches the same damping timescale of the modified Urca case. This large decrease in the damping time for the direct Urca case is due to the fact that if $\omega\tau \gg 1$, then $t_{\text{damp}} \propto \tau^2/\zeta \propto \Xi^{-1} \propto T^{-4}$, where the last proportionality holds for the direct Urca reactions [Eq. (36)]. Moreover, the entropy production rate is generally inversely proportional to the temperature (Eq. (8) of Camelio et al. [19]). In our case, the initial temperature for the direct Urca reactions is small enough for causing a huge increase of the entropy and as a consequence also of the temperature, see Plot (a) of Fig. 4 which in turn causes a huge decrease of the initial damping time. On the other hand, for the modified Urca case, the initial temperature is much larger and therefore, even accounting for the different order in the dependence on the temperature, it is not enough to cause a significant increase of the entropy and as a consequence also of the temperature, see top plot in Fig. 10. This is the reason why, in order to see the damping of the oscillation during the simulation with modified Urca reactions, we had to choose an initial entropy such that $\tau\omega \approx 1$ instead of $\tau \approx t_{\text{pert}}$ as with the direct Urca reactions.

![FIG. 16: Central linearized damping time for the oscillating MF model with direct and modified reactions. We use Eq. (C4) assuming a constant hydrodynamic timescale of $t_{\text{pert}} = 1\text{ ms}$ and $c_s = c_s,\text{uv}$ (the difference between the ‘ultraviolet’ and the ‘infrared’ speed of sound is of the order of 2% and does not affect our conclusions).](image-url)