M-best solutions for a class of fuzzy constraint satisfaction problems

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Abstract—The article considers one of the possible generalizations of constraint satisfaction problems where relations are replaced by multivalued membership functions. In this case operations of disjunction and conjunction are replaced by maximum and minimum, and consistency of a solution becomes multivalued rather than binary. The article studies the problem of finding $d$ most admissible solutions for a given $d$. A tractable subclass of these problems is defined by the concepts of invariants and polymorphisms similar to the classic constraint satisfaction approach. These concepts are adapted in two ways. Firstly, the correspondence of “invariant-polymorphism” is generalized to (min,max) semirings. Secondly, we consider non-uniform polymorphisms, where each variable has its own operator, in contrast to the case of one operator common for all variables. The article describes an algorithm that finds $d$ most admissible solutions in polynomial time, provided that the problem is invariant with respect to some non-uniform majority operator. It is essential that this operator needs not to be known for the algorithm to work. Moreover, even a guarantee for the existence of such an operator is not necessary. The algorithm either finds the solution or discards the problem. The latter is possible only if the problem has no majority polymorphism.

Index Terms—constraint satisfaction, discrete optimization, labeling, invariants, polymorphisms.

1 INTRODUCTION.

The constraint satisfaction problem (CSP) \cite{8} is one of the paradigms of machine intelligence. The problem is to find values for variables satisfying a given set of constraints or to determine inconsistency of the constraints. The set of all possible constraint satisfaction problems forms an NP-complete class. However, three tractable subclasses are known. Each of these subclasses is defined in terms of polymorphisms \cite{4}, \cite{5}, i.e. operators under which the problem is invariant. The article considers sets of constraints invariant under majority operators.

A stronger version is the counting CSP, where the goal is to count the number of solutions of a CSP rather than merely to decide if a solution exists. The complexity of counting CSPs has been analyzed in papers \cite{2}, \cite{3}. Evidently, this problem is stronger than the consistency problem, because any algorithm that solves the counting problem can be used to determine consistency. Unfortunately, the counting problem turns out to be much harder. The three known tractable subclasses of constraint satisfaction problems become NP-complete for the counting problem. Under some additional conditions only problems invariant under Mal'tsev operators are tractable \cite{2}, \cite{3}. This essential difference between consistency and counting problems makes it worthwhile to state and analyze intermediate problems.

We are interested in a problem that is weaker than the counting problem but still stronger than the consistency problem. The problem is to determine whether a given set of constraints has more than $d$ solutions where $d$ is a given number. To the best of our knowledge this problem has not been stated yet, let alone analyzed.

One of our results is an algorithm which proves whether a given set of constraints has more than $d$ solutions, provided that the constraints are invariant under a majority operator. The task is solved in polynomial time, avoiding an NP-complete counting problem. In case of a positive answer and $d > 0$, the algorithm returns $d$ possible solutions for the given set of constraints. This particular result is closest to traditional constraint satisfaction theory. The article as a whole covers a more general set of questions.

We consider one of the possible modifications of constraint satisfaction problems with multilevel constraints. Instead of simply categorizing solutions into consistent and inconsistent ones, they rather define a level of consistency. This modification can be interpreted as fuzzy constraint satisfaction problem \cite{6}, \cite{9}, where the problem is to find the solution with highest level of consistency or, for the sake of brevity, the maximum admissible solution. The search of the maximum admissible solution can be reduced to discrete optimization tasks for special functions and proves to be tractable if the problem has a majority polymorphism \cite{10}. The main novelty of the present article is to show that $d$ best solutions (and not only the most admissible one) can be found in polynomial time under the same assumptions. The equivalent task in the context of standard constraint satisfaction problems is to determine whether the number of solutions is greater than $d$. The exact formulation of the result is given in Section 2 after the main definitions. Section 3 explains relations to known results.
2 Problem definition and main result.

The article uses the denotation \( \text{arg}(d) \min_{i \in I} f(i) \) similar to the commonly used denotation \( \text{argmin}_{i \in I} f(i) \).

**Definition 1.** For a finite set \( I \), an ordered set \( W \), a function \( f : I \rightarrow W \) and an integer \( 0 < d < |I| \), the expression \( I' = \text{arg}(d) \min_{i \in I} f(i) \) means that \( I' \) is a subset of \( I \) such that \( |I'| = d \) and \( f(i) \leq f(j) \) holds for any pair \( i \in I' \), \( j \notin I' \). For \( d \geq |I| \) the expression \( I' = \text{arg}(d) \min_{i \in I} f(i) \) means that \( I' = I \).

The subset \( I' \subset I \) specified by this definition is not necessarily unique, in the same way as the element \( i^* \in I \) defined by the expression \( i^* = \text{argmin}_{i \in I} f(i) \) is not necessarily unique. The set \( I' \) is equivalently defined by the inequalities

\[
\max_{i \in I'} f(i) \leq \max_{i \in I} f(i),
\]

which must be fulfilled for any subset \( I' \subset I \) with \( d \) elements.

Let \( T \) and \( K \) be two finite sets called the set of objects and the set of labels. A function \( \bar{x} : T \rightarrow K \) will be called a labeling. Let \( x_i \) denote the value of a labeling \( \bar{x} : T \rightarrow K \) for an object \( i \in T \) and let \( x_S \) denote its restriction to a subset \( S \subset T \). Let \( K^S \) denote the set of all possible labelings \( x_S : S \rightarrow K \) for any \( S \subset T \). Whenever we want to stress that the domain of a labeling is a union of a pair of disjoint subsets \( A, B \subset T \), the labeling will be denoted by \( \langle x_A, x_B \rangle \), and not by \( x_{A \cup B} \). Let \( 2^T \) denote the set of all possible subsets \( S \subset T \). A set \( S \subset 2^T \) will be called a structure of the set \( T \), the number \( \max_{S \in \mathbb{S}} |S| \) being the order of a structure.

Let \( W \) be a totally ordered set, \( \mathbb{S} \) be a structure and let \( \varphi_S : K^S \rightarrow W \) be a function given for each structure element \( S \in \mathbb{S} \). We assume that each of these functions is defined by a table \( \text{Tab}(S) = \{ (x, \varphi_S(x)) \mid x \in K^S \} \).

**Definition 2.** The input data of a minimax labeling problem or, simply a problem, is a quintuple

\[
\Phi = \langle T, K, W, \mathbb{S} \subset 2^T, (\varphi_S : K^S \rightarrow W \mid S \in \mathbb{S}) \rangle.
\]

The order of the problem \( \Phi \) is defined as the order of the structure \( \mathbb{S} \).

The article considers arbitrary but fixed sets \( K \) and \( W \). Therefore, we refer to problems also in form of a triple \( \Phi = \langle T, S, (\varphi_S : S \in \mathbb{S}) \rangle \) and not by a quintuple \( \langle T, S, W, \mathbb{S} \rangle \).

The input data of a problem \( \Phi \) define its objective function \( \varphi : K^T \rightarrow W \) with values \( \varphi(\bar{x}) = \max_{S \in \mathbb{S}} \varphi_S(x_S) \), \( \bar{x} \in K^T \), where \( x_S \) is the restriction of \( \bar{x} \) to \( S \).

**Definition 3.** For a given positive integer \( d \) the solution of a problem \( \Phi \) is a subset \( \text{Sol}(\Phi) = \text{arg}(d) \min_{\bar{x} \in K^T} \varphi(\bar{x}) \).

The set of problems \( \langle \rangle \) forms an NP-complete class, because any constraint satisfaction problem can be expressed in this format. We formulate a tractable subclass of such problems based on the concepts of polymorphisms and invariants, which are the main tools for tractability analysis of constraint satisfaction problems \([4], [5]\). We generalize these concepts in order to analyze problems \( \langle \rangle \), which are more general than constraint satisfaction problems.

Let \( p_i : K \times K \times K \rightarrow K \) be a ternary operator defined for each \( i \in T \). A collection \( P = \{ p_i \mid i \in T \} \) of such operators is understood as an operator \( K^S \times K^S \times K^S \rightarrow K^S \), defined for each \( S \subset T \). Applying it to a triple \( x, y, z \in K^S \) gives the labeling \( P(x, y, z) = v \in K^S \) defined by \( v_i = p_i(x_i, y_i, z_i), i \in S \).

**Definition 4.** A function \( \varphi_S : K^S \rightarrow W \) is invariant under the operator \( P = \{ p_i \mid i \in T \} \) and an operator \( P \) is a polymorphism of the function \( \varphi_S \) if the inequality

\[
\max \{ \varphi_S(x), \varphi_S(y), \varphi_S(z) \} \geq \varphi_S(P(x, y, z)) \text{ holds for each triple } x, y, z \in K^S.
\]

This definition was first introduced in \([1]\) and is more general than the polymorphism-invariant correspondence commonly used in constraint satisfaction theory. Definition \([4]\) assumes that each object \( i \in T \) gets assigned its own operator \( p_i : K \times K \times K \rightarrow K \), instead of assigning a single operator to all variables. If it is necessary to emphasize that the components \( p_i \) of an operator \( P = \{ p_i \mid i \in T \} \) depend on \( i \) and, may differ from each other, we call the operator non-uniform.

**Definition 5.** An operator \( P = \{ p_i \mid i \in T \} \) is a polymorphism of the problem \( \Phi = \langle T, S, (\varphi_S \mid S \in \mathbb{S}) \rangle \), and a problem \( \Phi \) is invariant under the operator \( P \) if \( P \) is a polymorphism of all functions \( \varphi_S, S \in \mathbb{S} \).

**Definition 6.** An operator \( P = \{ p_i \mid i \in T \} \) is a majority operator if the equalities

\[
p_i(y, x, x) = p_i(x, y, x) = p_i(x, x, y) = x
\]

hold for all \( i \in T \) and for all \( x, y \in K \).

The result of this paper is an algorithm that solves problems \( \langle \rangle \) if they have a majority polymorphism. Its time complexity depends on parameters \( |T|, |K|, |\mathbb{S}| \), the number of required labelings \( d \) and the total size \( \sum_{S \in \mathbb{S}} |\text{Tab}(S)| \) of the tables, which represent the functions \( \varphi_S, S \in \mathbb{S} \). The main idea is to transform a problem of arbitrary order into an equivalent problem of order 2, and then to solve the second order problem by sequentially excluding variables. The order reduction procedure is described in Section \([4]\) the approach for solving second order problems is described in Section \([5]\).

The set of problems \( \langle \rangle \) solvable by the algorithm for \( d = 1 \) includes a well known subclass of constraint satisfaction problems and its fuzzy modifications. Example \([1]\) illustrates the likely less known fact that certain clustering problems can be expressed in the form \( \langle \rangle \). Example \([2]\) shows, how solving \( \langle \rangle \) for \( d > 1 \) can improve a certain workaround for solving problems with additional global constraints.

**Example 1. Clustering.** Consider a finite set \( T \) and a function \( r : T \times T \rightarrow W \) defining a dissimilarity \( r(s, t) \).
for each pair \(s, t \in T\). A partition of the set \(T\) into two subsets is a pair \((T_1, T_2)\) such that \(T_1 \cup T_2 = T\), \(T_1 \cap T_2 = \emptyset\) and its quality is defined by the value

\[
F(T_1, T_2) = \max\left\{ \max_{s,t \in T_1} r(s, t), \max_{s,t \in T_2} r(s, t) \right\}.
\]

One possible definition of a clustering problem is to find the best partition

\[
(T_1^*, T_2^*) = \arg\min_{(T_1, T_2)} \max\left\{ \max_{s,t \in T_1} r(s, t), \max_{s,t \in T_2} r(s, t) \right\}.
\]  

This problem is reduced to a minimax problem (2) by

\[
T_1^*, T_2^* = \arg\min_{(T_1, T_2)} \max\left\{ \max_{s,t \in T_1} r(s, t), \max_{s,t \in T_2} r(s, t) \right\}.
\]  

Example 2. Constraint relaxation. Suppose that the task is to find the best labeling for given data (2) under some additional constraints. Formally put, the labeling must belong to some given set \(K\) of labellings. This might be a condition which is easy to verify. For example, it might be required that a certain label \(k_0 \in K\) appears in a labeling \(\vec{x}\) at most \(l\) times. However, seeking the best labeling

\[
\vec{x}^* = \arg\min_{\vec{x} \in K} \left[ \max_{S \subseteq \bar{S}} \varphi_S(x_S) \right]
\]  

under such additional constraints may turn out to be much harder than seeking the best labeling

\[
\vec{x}^* = \arg\min_{\vec{x} \in K^T} \left[ \max_{S \subseteq \bar{S}} \varphi_S(x_S) \right].
\]

without such constraints. Moreover, it might happen that the additional constraints are hard to formalize. The set \(K\) may represent, for example, a user who rejects labellings based on informal personal preferences.

A workaround is to find the best labeling (5) and to check condition \(\vec{x}^* \in K\) afterwards. Obviously, if the condition holds, then \(\vec{x}^*\) is a solution of (4). However, this requirement is rather too strong. It can be weakened by finding \(d\) best labellings. The approach for solving (4) is to consider labellings one by one, from best to worst. The first labeling in the sequence which fulfills \(\vec{x}^* \in K\) is a solution of the task \(\arg\min_{\vec{x} \in K} \left[ \max_{S \subseteq \bar{S}} \varphi_S(x_S) \right]\). Of course, this labeling may appear late in the sequence and the problem (4) will remain unsolved. However, an incorrect solution is excluded in any case.

3 Relations to known results.

The closest counterpart to problem (2) are constraint satisfaction problems. It is known that constraint satisfaction problems with a majority polymorphism form a tractable subclass [4]. This result can be easily generalized to problems (2) for \(d = 1\). Solving \(\arg\min_{\vec{x} \in K^T} \varphi(\vec{x})\) is tractable because it can be reduced to solving \(\log(|K| \times |T|)\) constraint satisfaction problems.

We solve the task for arbitrary \(d\). For constraint satisfaction problems this means to prove existence of at least \(d\) solutions satisfying the constraints. As far as we know, this question has not yet been studied for constraint satisfaction problems.

The article presents an algorithm that solves a certain subclass of an \(NP\)-complete class of problems. This subclass is defined in terms of existence of a non-uniform majority polymorphism. For practical application of the algorithm, it is necessary to either know its behavior on problems instances without such a polymorphism or to have a method for proving existence of a non-uniform majority polymorphism for a given problem instance. There are known methods for proving whether a problem has a uniform majority polymorphism [5], however, we are not aware of such a method for non-uniform majority polymorphisms. We conjecture this to be a nontrivial task. The advantage of the presented algorithm is that such a prior control of input data is not required. For any problem (2) from an \(NP\)-complete class given on its input, the algorithm stops in polynomial time either returning a set of \(d\) best labellings or discarding the problem. The latter is possible only if the problem has no majority polymorphism. Therefore, the algorithm solves any problem (2) that is invariant under some non-uniform majority operator and avoids to answer the potentially hard question of existence of such an operator let alone to find it.

4 Transforming problems of arbitrary order to problems of second order.

Let \(T\) be a set of objects, \(S \subseteq T\) and \(R = T \setminus S\).

Definition 7. The projection of a function \(\varphi : K^T \rightarrow W\) onto the subset \(S\) is the function \(\varphi_S : K^S \rightarrow W\), obtained by minimizing over all variables not in \(S\), i.e.

\[
\varphi_S(x_S) = \min_{x_R \in K^R} \varphi(x_S, x_R).
\]

The following property immediately follows from this definition.

Lemma 1. Let \(S \subseteq R \subseteq T\), \(\varphi_S\) and \(\varphi_R\) be the projections of a function \(\varphi : K^T \rightarrow W\) onto \(S\) and \(R\) respectively and \(\varphi_*\) be the projection of \(\varphi_R\) onto \(S\). Then \(\varphi_* = \varphi_S\).

The next two lemmas express properties of functions invariant under some operator.

Lemma 2. If a function \(\varphi : K^T \rightarrow W\) is invariant under an operator \(P = (p_i)_{i \in T}\), then its projection \(\varphi_S\) is invariant under the same operator.
Proof: Let us denote $R = T \setminus S$. Let $x_S, y_S, z_S$ be three labellings of the form $S \to K$. Since $\varphi_S$ is the projection of $\varphi$ onto $S$, there exist three labellings

$$\bar{x} = (x_S, x_R), \ \bar{y} = (y_S, y_R), \ \bar{z} = (z_S, z_R)$$

of the form $T \to K$, such that

$$\varphi(\bar{x}) = \varphi_S(x_S), \ \varphi(\bar{y}) = \varphi_S(y_S), \ \varphi(\bar{z}) = \varphi_S(z_S).$$

Let us denote

$$\bar{u} = P(\bar{x}, \bar{y}, \bar{z}), \ \bar{u}_S = P(x_S, y_S, z_S), \ u_R = P(x_R, y_R, z_R).$$

Because $\varphi_S$ is the projection of $\varphi$ onto $S$ and $\varphi$ is invariant under $P$, it follows that

$$\varphi_S(P(x_S, y_S, z_S)) = \varphi_S(u_S) = \min_{x_R \in K^R} \varphi(u_S, x_R) \leq \varphi(u_S, u_R) = \varphi(\bar{u}) \leq \max \{ \varphi(\bar{x}), \varphi(\bar{y}), \varphi(\bar{z}) \} = \max \{ \varphi_S(x_S), \varphi_S(y_S), \varphi_S(z_S) \}.$$ 

$\square$

Lemma 3. If two functions $\varphi, \psi: K^T \to W$ are invariant under an operator $P = \{ p_i \mid i \in T \}$, then their element-wise maximum, i.e. the function $\omega: K^T \to W$ with values $\omega(\bar{x}) = \max \{ \varphi(\bar{x}), \psi(\bar{x}) \}$, $\bar{x} \in K^T$, is invariant under the same operator.

Proof: Let $x_i, i = 1, 2, 3$ be three labellings and $\bar{y} = P(x_1, x_2, x_3).$ The fact that $\varphi$ and $\psi$ are invariant under $P$ means that $\max_i \varphi(\bar{x}_i) \geq \varphi(\bar{y})$ and $\max_i \psi(\bar{x}_i) \geq \psi(\bar{y}).$ It follows that

$$\max \max_i \{ \varphi(\bar{x}_i), \psi(\bar{x}_i) \} \geq \max \{ \varphi(\bar{y}), \psi(\bar{y}) \},$$

and, equivalently,

$$\max_i \omega(\bar{x}_i) \geq \omega(\bar{y}) = \omega(P(\bar{x}_1, \bar{x}_2, \bar{x}_3)).$$

$\square$

If a function $\varphi: K^T \to W$ has a majority operator then it has an important additional property.

Lemma 4. Let $\varphi: K^T \to W$ be a function which has a majority polymorphism and let $Q$, $R$, $S$ be pairwise disjoint subsets of $T$ such that $Q \cup R \cup S = T$. Denote by $\varphi_{QR}$, $\varphi_{QS}$, $\varphi_{RS}$ the projections of $\varphi$ onto the subsets $Q \cup R$, $Q \cup S$, $R \cup S$ respectively. Then the equality

$$\varphi(\bar{x}) = \max \{ \varphi_{QR}(x_Q, x_R), \varphi_{QS}(x_Q, x_S), \varphi_{RS}(x_R, x_S) \}$$

holds for any labeling $\bar{x} = (x_Q, x_R, x_S) \in K^T$.

Proof: Let us pick an arbitrary labeling $\bar{x}$ for the following considerations. Let $x_Q$, $x_R$, $x_S$ denote the restrictions of the labeling $\bar{x}$ onto the subsets $Q$, $R$, $S$ respectively. By definition of projection the inequalities

$$\varphi(\bar{x}) \geq \varphi_{QR}(x_Q, x_R),$$

$$\varphi(\bar{x}) \geq \varphi_{QS}(x_Q, x_S),$$

$$\varphi(\bar{x}) \geq \varphi_{RS}(x_R, x_S),$$

are valid, and, consequently we have

$$\varphi(\bar{x}) \geq \max \{ \varphi_{QR}(x_Q, x_R), \varphi_{QS}(x_Q, x_S), \varphi_{RS}(x_R, x_S) \}.$$  

Let us prove the converse inequality

$$\varphi(\bar{x}) \leq \max \{ \varphi_{QR}(x_Q, x_R), \varphi_{QS}(x_Q, x_S), \varphi_{RS}(x_R, x_S) \}.$$ 

Because $\varphi_{QR}, \varphi_{QS}, \varphi_{RS}$ are the projections of the function $\varphi$ onto the subsets $Q \cup R$, $Q \cup S$, $R \cup S$, there exist three labellings $y_S: S \to K$, $y_R: R \to K$ and $y_Q: Q \to K$ such that

$$\varphi(x_Q, x_R, y_S) = \varphi_{QR}(x_Q, x_R),$$

$$\varphi(x_Q, y_R, x_S) = \varphi_{QS}(x_Q, x_S),$$

$$\varphi(y_Q, x_R, x_S) = \varphi_{RS}(x_R, x_S).$$

The function $\varphi$ has some majority polymorphism $P$, therefore, (6) implies the chain

$$\varphi(\bar{x}) = \varphi(x_Q, x_R, x_S) = \varphi(P(x_Q, y_Q, y_R, x_R, x_S), y_S, x_R, x_S)) \leq \max \{ \varphi(x_Q, x_R, y_S), \varphi(x_Q, y_R, x_S), \varphi(y_Q, x_R, x_S) \} = \max \{ \varphi_{QR}(x_Q, x_R), \varphi_{QS}(x_Q, x_S), \varphi_{RS}(x_R, x_S) \}.$$ 

$\square$ 

Lemma 4 shows that any function $\varphi: K^T \to W$ of $|T|$ arguments that has a majority polymorphism, can be expressed in terms of three projections $\varphi_{A}$, $\varphi_{B}$, $\varphi_{C}$ onto subsets $A, B, C \subset T$, provided that the union of their pairwise intersections coincides with $T$. Each of these functions depends on less variables than $\varphi$ and, according to Lemma[2], they are invariant under the same operator as $\varphi$. Therefore, each of the functions $\varphi_{A}$, $\varphi_{B}$, $\varphi_{C}$ can in turn be expressed in terms of functions of less variables. Moreover, there will be no collisions when projecting some functions $\varphi_{A}$ and $\varphi_{B}$ onto $D \subset A \cap B$. According to Lemma[4] both projections are equal to the projection of $\varphi$ onto $D$. Therefore, any function $\varphi: K^T \to W$ that has a majority polymorphism can be expressed in form of $\varphi(\bar{x}) = \max_{i,j \in T} \varphi_{ij}(x_i, x_j)$, $\bar{x} \in K^T$, where $\varphi_{ij}$ are the projections of $\varphi$ onto $\{i,j\}$ and have the same majority polymorphism as $\varphi$.

The stated properties allow us to transform any problem with a majority polymorphism into an equivalent problem of second order, even if the polymorphism itself is not known. Instead of denoting the second order problem by a triple $(T, S, (\varphi_S \mid S \in S))$, we will denote it by a tuple $(T, (\varphi_{ij} \mid i, j \in T))$, where $\varphi_{ij}$, $i, j \in T$, are functions $K \times K \to W$ such that $\varphi_{ij}(k, k') = \varphi_{ij}(k', k)$ for all $i, j \in T$, $k, k' \in K$ and $\varphi_{ij}(k, k') = \min W$ for all $i \in T$. The value of the objective function for a labeling $\bar{x} \in K^T$ is $\max_{i,j \in T} \varphi_{ij}(x_i, x_j)$

Theorem 1. Any problem $\Phi = \langle T, S, (\varphi_S \mid S \in S) \rangle$ that has a majority polymorphism can be transformed to a second order problem $\Psi = \langle T, (\psi_{ij} \mid i, j \in T) \rangle$ such that

$$\max_{S \in S} \psi_{S}(x_S) = \max_{i,j \in T} \psi_{ij}(x_i, x_j), \ \bar{x} \in K^T,$$

where $x_S$ is the restriction of labeling $\bar{x}$ onto $S \in S$ and $x_i$ are its values for $i \in T$. The second order problem $\Psi$ is invariant under the same majority operator as $\Phi$. 

Proof: Let us denote the projection of \( \varphi_S \) onto \( \{i, j\} \subset S \) by \( \varphi_S^i_j \). We assume \( \varphi_S^i_j(x_i, x_j) = \min W \) for \( \{i, j\} \not\subset S \). The functions \( \varphi_S^i_j \) are invariant under the same operator as the function \( \varphi_S \). Let us define functions \( \psi_{ij} \) of the problem \( \Psi \) as \( \psi_{ij} = \max_{S \in S} \varphi^i_j \). According to Lemma 3 these functions are invariant under the same operator as \( \varphi^i_j \). Therefore, both problems \( \Phi \) and \( \Psi \) are invariant under the same majority operator. And, the following chain

\[
\max_{S \in S} \varphi_S(x_S) = \max_{S \in S} \max_{i, j \in S} \varphi^i_j(x_i, x_j) = \max_{i, j \in T} \max_{S \in S} \varphi^i_j(x_i, x_j) = \max_{i, j \in T} \psi_{ij}(x_i, x_j)
\]

holds for each labeling \( \bar{x} \in K^T \).

The proof of the theorem implicitly contains an algorithm for transforming the problem \( \Phi = \langle T, S, \varphi_S \mid S \in \mathbb{S} \rangle \) into the problem \( \Psi = \langle T, \psi_{ij} \mid i, j \in T \rangle \). Assuming that the functions \( \varphi_S, S \in \mathbb{S} \), which define the problem \( \Phi \), are given in form of a tables \( \text{Tab}(S) = \{(x, \varphi_S(x)) \mid x \in K^S\} \), this algorithm reads as follows.

**Algorithm 1. Reducing the problem’s order.**

**Input:** problem \( \Phi = \langle T, S, \varphi_S \mid S \in \mathbb{S} \rangle \).

**Output:** problem \( \Psi = \langle T, \psi_{ij} \mid i, j \in T \rangle \).

1. For each \( i, j \in T, k, k' \in K \)
   \( \psi_{ij}(k, k') = \min W \);
2. For each \( S \in \mathbb{S} \)
   1. For each \( i, j \in S, k, k' \in K \)
      \( \varphi^i_j(k, k') = \max W \);
   2. For each \( (\bar{x}, w) \in \text{Tab}(S) \) and each \( i, j \in S \)
      \( \varphi^i_j(x_i, x_j) = \min \{\varphi^i_j(x_i, x_j), w\} \);
   3. For each \( (\bar{x}, w) \in \text{Tab}(S) \)
      if \( w \neq \max_{i, j \in S} \varphi^i_j(x_i, x_j) \),
      then return "discard";
   4. For each \( i, j \in S, k, k' \in K \)
      \( \psi_{ij}(k, k') = \max \{\psi_{ij}(k, k'), \varphi^i_j(k, k')\} \).

If the input problem \( \Phi \) has a majority polymorphism then Algorithm 1 is guaranteed to transform the problem into an equivalent problem \( \Psi \) of order two. Testing conditions in p.1.2 is redundant in this case. None of these holds. However, testing these conditions extends the scope of the algorithm to cover any problem, and not only those which have a majority polymorphism.

The absence of a “discard” message guarantees that the algorithm has successfully converted the input problem into a second order problem. This is true regardless of presence or absence of a majority polymorphism. Notice however, that the resulting problem has no majority polymorphism if the input problem lacks one. As will be shown in section 5.3, this does not violate the applicability of algorithms given there for solving problems of second order. The “discard” message means that the problem \( \Phi \) is not in the applicability range of the algorithm. This is possible only if the problem has no majority polymorphism.

The complexity of Algorithm 1 as well as of all other presented algorithms is measured by the number of max\( \{w, w'\} \) and min\( \{w, w'\} \) operations. The complexity of Algorithm 1 depends polynomially on the parameters of the problem: the numbers \(|T|, |K|, |S|\), the order \( n \) and the size \( l = \sum_{S \in \mathbb{S}} |\text{Tab}(S)| \) of the input data.

The total complexity of p.0 is of order \(|T|^2 \times |K|^2\). The total complexity for all \( S \in \mathbb{S} \) of p.1.0 and 1.3 is of order \(|S| \times |K|^2 \times n^2\). The total complexity for all \( S \in \mathbb{S} \) of p.1.1 and 1.2 is of order \( l \times n^2\).

5 Second order problems.

5.1 A general approach for excluding variables.

Let us define two problems \( \Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle \) and \( \Phi^* = \langle S, (\varphi_{ij} \mid i, j \in S) \rangle \), for a set \( T \) and the set \( S = T \setminus \{t\} \) obtained from \( T \) by removing an arbitrary element \( t \in T \). Consider an algorithm that “reduces” the search of a solution \( \text{Sol}(\Phi) \) to the search of \( \text{Sol}(\Phi^*) \) in a greedy way. Algorithms of this type minimize a function of \( n \) variables by minimizing an auxiliary function of \( (n - 1) \) variables first, and then find the optimal value of the remaining \( n \)-th variable by keeping the other variables fixed to the previously found minimizer of the auxiliary function. The auxiliary function itself is minimized by the same greedy algorithm, so that the optimization over \( n \) variables is eventually “reduced” to \( n \) optimizations over a single variable.

We modify this idea in two ways. Firstly, a greedy algorithm is used to find \( d \) best solutions, and not only the best solution. Secondly, we include an essential test that allows to detect situations in which the result of the algorithm differs from \( \text{Sol}(\Phi) \).

**Algorithm 2. Greedy algorithm.**

**Input:** a problem \( \Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle \).

**Output:** \( \text{Sol}(\Phi) \) or a message “discard”.

1. If \(|\mathcal{T}| = 2, T = \{a, b\}\)
   then \( \text{Sol}(\Phi) = \arg\min \varphi_{ab}(x_a, x_b); (x_a, x_b) \in K^2 \)
   else
2. using Algorithm 1 find \( \text{Sol}(\Phi^*) \),
3. if there is at least one labeling \( x \in \text{Sol}(\Phi^*) \)
   fulfilling the inequality
   \( \max_{i, j \in S} \psi_{ij}(x_i, x_j) < \min_{k \in K} \varphi_{ik}(x_i, k) \),
   then return “discard”;
4. construct the auxiliary set
   \( \mathcal{W} = \{(x_i, x_j) \in K^T \mid x \in \text{Sol}(\Phi^*), x_i \in K\} \);
5. find \( \text{Sol}' = \arg\min \max_{(x_i, x_j) \in \mathcal{W}} \psi_{ij}(x_i, x_j) \).

The subset \( \text{Sol}' \) returned by the algorithm is not necessarily the solution of the problem. However, testing conditions in p.3 allows to detect situations in which \( \text{Sol}' \) is a solution. The next lemma proves that \( \text{Sol}' \) is the required labeling subset if the algorithm does not output "discard".
Lemma 5. Let $\Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle$ and $\Phi^* = \langle S, (\varphi_{ij} \mid i, j \in S) \rangle$ be two problems such that $S = T \setminus \{t\}$ and $t \in T$. Let
\[
\text{WORK} = \{(x, x_t) \in K^T \mid x \in \text{Sol}(\Phi^*), x_t \in K\}
\]
denote all possible extensions of labellings $x \in \text{Sol}(\Phi^*)$ and let
\[
\text{Sol}' = \arg\min_{x \in \text{WORK}} \max_{i,j \in T} \varphi_{ij}(x_i, x_j).
\]
be a set of $d$ best labellings in WORK. If the inequality
\[
\max_{i,j \in S} \varphi_{ij}(x_i, x_j) \geq \min_{k \in K} \max_{i \in S} \varphi_{it}(k, x_i)
\]
holds for each labeling $x \in \text{Sol}(\Phi^*)$, then Sol' is a solution of $\Phi$, i.e. $\text{Sol}' = \arg\min_{x \in K^T} \max_{i,j \in T} \varphi_{ij}(x_i, x_j)$.

Proof: Denote
\[
\varphi_S(x) = \max_{i,j \in S} \varphi_{ij}(x_i, x_j), \quad \theta^* = \max\{\varphi_S(x) \mid x \in \text{Sol}(\Phi^*)\},
\]
\[
\varphi(\bar{x}) = \max_{i,j \in T} \varphi_{ij}(x_i, x_j), \quad \theta' = \max\{\varphi(\bar{x}) \mid \bar{x} \in \text{Sol}'\}
\]
for $x \in K^S$ and $\bar{x} \in \text{Sol}'$.

Let us first prove that $\theta' \leq \theta^*$. We enumerate labellings from $\text{Sol}(\Phi^*)$ by numbers $l \in \{1, 2, \ldots, d\}$ and denote by $x(l) \in \text{Sol}(\Phi^*)$ the labeling with number $l$, so that $\text{Sol}(\Phi^*) = \{x(l) \mid l = 1, 2, \ldots, d\}$. Let us define the label $\varphi_S(x(l)) = \arg\min_{x \in K^S} \varphi_{it}(x_i, l, k)$ for each labeling $x(l) \in \text{Sol}(\Phi^*)$. This gives $d$ labellings $(x(l), x_t(l)) \in \text{WORK}, l \in \{1, 2, \ldots, d\}$. Because of assumption (7) they fulfill the equalities
\[
\varphi(x(l), x_t(l)) = \varphi_S(x(l)), \quad l \in \{1, 2, \ldots, d\},
\]
which leads to the chain
\[
\theta' = \max_{\bar{x} \in \text{Sol}'} \varphi(\bar{x}) \leq \max_{1 \leq l \leq d} \varphi(x(l), x_t(l)) = \max_{x \in \text{Sol}(\Phi^*)} \varphi_S(x) = \theta^*.
\]
The inequality in this chain follows from the property (1) of the set $\text{Sol}' = \arg\min_{x \in \text{WORK}} \varphi(\bar{x})$. The next equality is valid due to (8).

Let us prove that the inequality $\varphi(\bar{x}) \geq \theta'$ holds for all $\bar{x} \in K^T \setminus \text{Sol}'$. For labelings $\bar{x} \in \text{WORK} \setminus \text{Sol}'$ this inequality follows directly from the definition of Sol'. It is also true for labelings $\bar{x} \in K^T \setminus \text{WORK}$, because
\[
\varphi(\bar{x}) = \max_{i,j \in T} \varphi_{ij}(x_i, x_j) \geq \max_{i \in S} \varphi_{it}(x_i, x_t) = \varphi_S(x) \geq \theta^* \geq \theta',
\]
where $x_S$ is the restriction of $\bar{x}$ to $S$. The second inequality of this chain follows from the fact that $x_S \notin \text{Sol}(\Phi^*)$ if $\bar{x} \notin \text{WORK}$. We obtain, that $\varphi(\bar{x}) \geq \theta'$ holds for all $\bar{x} \notin \text{Sol}'$ and $\varphi(\bar{x}) \leq \theta'$ holds for all $\bar{x} \in \text{Sol}'$. According to Definition (1) this means that Sol' is applicable for the whole NP-complete class of problems.

Its output is either “discard” or a correct solution. Unfortunately, this correct solution is obtained only for a very limited set of simple problems. We expand this set by replacing the problem $\Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle$ in step 2 of the algorithm by an equivalent problem $\Omega = \langle T, (\omega_{ij} \mid i, j \in T) \rangle$. Equivalence means here that both problems have the same objective function, i.e. that $\max_{x \in T} \omega_{ij}(x_i, x_j) = \max_{x \in T} \varphi_{ij}(x_i, x_j)$ holds for any labeling $x \in K^T$. Section 5.2 shows how to construct this equivalent problem in such a way, that the extension of Algorithm 2 solves all problems with a majority polymorphism.

5.2 Equivalent transformation of a problem.

Definition 8. A structure $\{(t, i) \mid i \in T \setminus \{t\}\}$ defined for a set $T$ is called a star with center $t$; a structure $\{(i, j) \mid i, j \in S, i \neq j\}$ defined for a set $S$ is called a simplex.

The objective function of a problem defined on a star structure $\{(t, i) \mid i \in T \setminus \{t\}\}$ is
\[
\varphi(\bar{x}) = \max_{i \in T \setminus \{t\}} \varphi_{ti}(x_i, x_t), \quad \bar{x} \in K^T.
\]

Definition 9. A transformation of a star into a simplex is the transformation of a problem $\Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle$, defined on a star structure, into a problem $\Psi = \langle T \setminus \{t\}, (\psi_{ij} \mid i, j \in T \setminus \{t\}) \rangle$, defined on a simplex structure, such that $\psi_{ij}$ are the projections of the objective function of $\Phi$ onto $\{i, j\}$.

The starting point for the following construction is to represent the objective function of the problem $\Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle$, defined on a simplex, in form of a maximum of two functions
\[
\varphi(\bar{x}) = \max_{i,j \in T} \varphi_{ij}(x_i, x_j) = \max_{i \in S} \varphi_{ti}(x_i, x_t), \quad \bar{x} \in K^T.
\]
The first of these functions is the objective function of a problem defined on a star. The second one is the objective function of a problem defined on a smaller simplex.

Lemma 6. Let $\Phi = \langle T, (\varphi_{ij} \mid i, j \in T) \rangle$ be a problem, $t \in T, S = T \setminus \{t\}$ and let $\psi_{ij} : K \times K \to W$ be the projections of the function $\max_{x \in S} \varphi_{it}$ onto $\{i, j\}$. Denote by $\omega_{ij} = \max_{x \in S} \psi_{ij}$ the point-wise maximum of the functions $\varphi_{ij}$ and $\psi_{ij}$ for $i, j \in S$. Then the equality
\[
\varphi(\bar{x}) = \max_{i,j \in S} \max_{x \in S} \omega_{ij}(x_i, x_j), \quad \bar{x} \in K^T.
\]

Proof: The functions $\psi_{ij}$ are the projections of the function $\max_{x \in S} \varphi_{it}$ onto $\{i, j\}$. Therefore, the inequality $\psi_{ij}(x_i, x_j) \leq \max_{x \in S} \varphi_{it}(x_t, x_i)$ holds for any $\bar{x} \in K^T$. The right-hand side of this inequality does not depend on $(i, j)$, and $\max_{i,j \in S} \psi_{ij}(x_i, x_j) \leq \max_{x \in S} \varphi_{it}(x_t, x_i)$
follows as a consequence. Hence,
\[ \varphi(\bar{x}) = \max_{i,j \in S} \max_{i \in S} \varphi_{ij}(x_i, x_j), \max_{i \in S} \varphi_{ti}(x_i, x_i) = \]
\[ = \max_{i,j \in S} \max_{i \in S} \psi_{ij}(x_i, x_j), \max_{i \in S} \varphi_{ij}(x_i, x_j), \max_{i \in S} \varphi_{ti}(x_t, x_i), \]
and (10) follows immediately.

The function on the right-hand side of (10) can be thought of as the objective function of the problem
\[ \Omega = \langle T, (\omega_{ij} | i, j \in T) \rangle \] where \( \omega_{ij} = \max_{\omega, \Omega, k \in \Omega} \varphi_{ij}(\omega, \Omega, k) \) for \( i, j \in S \) and \( \omega_{ij} = \varphi_{ij} \) for \( j \in S \). The problems \( \Phi \) and \( \Omega \) are equivalent because they have the same objective function. Notice that this equivalence is not conditioned upon existence of a majority polymorphism.

The following additional property holds in the event that the problem \( \Phi \) is invariant under some majority operator.

Lemma 7. Let \( \Phi = \langle T, (\varphi_{ij} | i, j \in T) \rangle \) be a problem with a majority polymorphism. Let \( t \in T \) and let \( \psi_{ij} : K \times K \to W \) be the projections of the function \( \max_{\omega \in \Omega} \varphi_{t\omega} \) onto \( \{i, j\} \). Then the function \( \omega : K^S \to W \), with values defined by
\[ \omega(x_S) = \max_{i,j \in S} \max_{i \in S} \varphi_{ij}(x_i, x_j), \max_{i \in S} \psi_{ij}(x_i, x_j), \]
(11) is the projection of the objective function of \( \Phi \) onto \( S \).

Proof: The following chain holds for the projection of the objective function \( \varphi : K^T \to W \) of \( \Phi \) onto the subset \( S \) and for any labeling \( x_S \in K^S \)
\[ \min_{x_t \in K} \varphi(x_S, x_t) = \]
\[ = \min_{x_t \in K} \max_{i,j \in S} \max_{i \in S} \varphi_{ij}(x_i, x_j), x_t, \max_{i \in S} \varphi_{ti}(x_t, x_i) = \]
\[ = \max_{i,j \in S} \max_{i \in S} \varphi_{ij}(x_i, x_j), \max_{i \in S} \varphi_{ti}(x_t, x_i) = \]
\[ = \max_{i,j \in S} \max_{i \in S} \varphi_{ij}(x_i, x_j), \max_{i \in S} \psi_{ij}(x_i, x_j) \]
The first equality repeats (9), the second equality holds because \( \max_{i,j \in S} \varphi_{ij}(x_i, x_j) \) does not depend on \( x_t \) and the third equality holds because the function \( \max_{i \in S} \psi_{ij}(x_i, x_j) \) is the projection of \( \max_{i \in S} \varphi_{t\omega} \) onto \( S \), as shown in Lemma 4.

In summary, given a problem \( \Phi = \langle T, (\varphi_{ij} | i, j \in T) \rangle \), an arbitrary element \( t \in T \) and the subset \( S = T \setminus \{t\} \), we have constructed the problem \( \Omega = \langle T, (\omega_{ij} | i, j \in T) \rangle \), defined by
\[ \omega_{ij} = \max_{\psi_{ij}, \Omega} \varphi_{ij}(\psi_{ij}, \Omega), \]
(12)
\[ \omega_{ji} = \varphi_{ij} \]
for \( i, j \in S \), (12)
for \( j \in S \).

According to Lemma 4, the objective functions of both problems are the same, therefore so are their projections onto \( S \). Moreover, if the problem \( \Phi \) has a majority polymorphism then, according to Lemma 7, the projection of the function \( \omega(\bar{x}) = \max_{i,j \in T} \omega_{ij}(x_i, x_j), \bar{x} \in K^T \), onto \( S \) is simply the function \( \omega(x_S) = \max_{i,j \in S} \omega_{ij}(x_i, x_j), x_S \in K^S \).

Since the problem \( \Omega \) is defined in (12) in terms of functions \( \psi_{ij} \), the procedure of transforming \( \Phi \) into \( \Omega \) is defined up to the procedure of transforming a star into a simplex. This transformation can be expressed explicitly due to the following equivalences for expressions composed of operations \( \max \) and \( \min \).

Let \( X \) and \( Y \) be some finite sets and let \( f : X \to W \) and \( g : Y \to W \) be two functions. Then
\[ \min_{x \in X} \min_{y \in Y} \max_{x \in X} \min_{y \in Y} \{ f(x), g(y) \} = \max_{x \in X} \min_{y \in Y} \{ f(x), g(y) \} \]
(13)
and for any \( x \in X \)
\[ \min_{y \in Y} \max_{x \in X} \{ f(x), g(y) \} = \max_{x \in X} \min_{y \in Y} \{ f(x), g(y) \} \]
(14)
Let \( I \) be some finite set, let \( X_i, i \in I \), be finite sets and let \( f_i : X_i \to W, i \in I \cup \{0\} \) be some functions. Then
\[ \max_{i \in I} \min_{x_i \in X_i} \max_{i \in I} \min_{x_i \in X_i} f_i(x_i) = \max_{i \in I} \min_{x_i \in X_i} f_i(x_i) \]
(15)
for any \( x_0 \in X_0 \).

Using rules (13) - (15), the star-to-simplex transformation is constructed in the following way. Let us pick two objects \( mn \in S \) and fix them for the following considerations. Denote \( R = S \setminus \{m, n\} = T \setminus \{t, m, n\} \). The following chain of equalities holds for the values \( \psi_{mn}(x_m, x_n) \) of the projection of \( \varphi(\bar{x}) = \max_{i \in S} \varphi_{t\omega}(x_t, x_i) \) onto \( \{m, n\} \):
\[ \psi_{mn}(x_m, x_n) = \]
\[ \min_{x_t \in K} \max_{x_t \in K} \min_{x_t \in K} \max_{x_t \in K} \{ \varphi_{tm}(x_m, x_t), \varphi_{tn}(x_n, x_t) \} = \]
\[ = \max_{x_t \in K} \min_{x_t \in K} \max_{x_t \in K} \min_{x_t \in K} \{ \varphi_{tm}(x_m, x_t), \varphi_{tn}(x_n, x_t) \} = \]
\[ = \max_{x_t \in K} \min_{x_t \in K} \max_{x_t \in K} \min_{x_t \in K} \{ \varphi_{tm}(x_m, x_t), \varphi_{tn}(x_n, x_t) \} \]
The first two equalities are valid by definition. The third one is valid according to rule (14), the fourth one is valid according to (13) and the fifth one is valid according to (15). The following explicit expression for the star-to-simplex transformation
\[ \psi_{ij}(x_i, x_j) = \]
\[ = \max_{x_t \in K} \min_{x_t \in K} \max_{x_t \in K} \min_{x_t \in K} \{ \varphi_{ti}(x_i, x_t), \varphi_{tj}(x_j, x_t) \} \]
(16)
is obtained as a result.

Algorithm 3 implements the transformation of the problem \( \Phi \) into the problem \( \Omega \) based on expressions (12) and (16).

Algorithm 3. Equivalent transformation of problems.
Input: a problem \( \Phi = \langle T, (\varphi_{ij} | i, j \in T) \rangle \) and \( t \in T \).
Output: a problem \( \Omega = \langle T, (\omega_{ij} | i, j \in T) \rangle \).

1. Let \( S = T \setminus \{t\} \);
2. for all \( k \in K \) compute \( \varphi_{tk}(k, x) \);
\[
\psi_{ij}(x, y) = \min_{k \in K} \max \{ \varphi_{ik}(k, x), \varphi_{kj}(k, y), q(k) \};
\]
\[
\omega_{ij}(x, y) = \max \{ \varphi_{ij}(x, y), \psi_{ij}(x, y) \};
\]
3. for all \( i \in S, k, x \in K \) let \( \omega_{iti}(k, x) = \varphi_{ti}(k, x) \).

The complexity of Algorithm [3] is of order \(|K|^3 \times |T|^2\).

5.3 Solving problems of order two.

We include Algorithm [3] for equivalent transformation into the general Algorithm [2] for exclusion of variables.

Algorithm 4. Solving a problem of order two.

**Input:** a problem \( \Phi = \langle T, (\varphi_{ij}, i, j \in T) \rangle \).

**Output:** either \( \text{Sol}(\Phi) \) or a message “discard”.

1. If \(|T| = 2, T = \{a, b\} \)
   
   then \( \text{Sol}(\Phi) = \text{arg}(a) \min_{(x_a, x_b) \in K^2} \varphi_{ab}(x_a, x_b) \);

2. if at least one labeling \( x \in \text{Sol}(\Omega^*) \)
   
   fulfills the inequality
   \[
   \max_{i,j \in S} \omega_{ij}(x_i, x_j) < \min_{k \in K} \max_{i,j \in S} \omega_{iti}(k, x_i),
   \]
   
   then return the message “discard”;

3. **Proof:** Let \( \Omega = \langle T, (\omega_{ij}, i, j \in T) \rangle \) and its restriction \( \Omega^* = \langle S, (\omega_{ij}, i, j \in S) \rangle \) onto \( S \) denote the auxiliary problems constructed in steps 2 and 3 of the algorithm.

4. **Proof:** Let us prove the first statement of the theorem. If the algorithm has not returned a “discard” message, then the inequality
   \[
   \max_{i,j \in S} \omega_{ij}(x_i, x_j) \geq \min_{k \in K} \max_{i,j \in S} \omega_{iti}(k, x_i)
   \]  
   holds for each labeling \( x \in \text{Sol}(\Omega^*) \). It follows from Lemma 5 and condition (18) that

5. **Proof:** Let us prove the second statement. Since \( \Phi \) is assumed to have a majority polymorphism, it follows from Lemma 7 that the objective function of the problem \( \Omega^* \) is the projection of the objective function of the problem \( \Phi \) onto \( S \). This means that inequalities (18) are valid for all \( x \in K^S \) including \( x \in \text{Sol}(\Omega^*) \). Consequently, the proof can be completed by repeating from hereon the proof of the first statement.

6. **Conclusion.**

We have analyzed the problem of finding \( d \) best labellings \( \bar{x} : T \rightarrow K \) where \( T \) and \( K \) are finite sets and the quality \( \varphi : K^T \rightarrow W \) of a labeling is given in a format similar to constraint satisfaction theory. This addresses the search of \( d \) smallest numbers in a set of \( |K|^T \) numbers. If the function \( \varphi \) is invariant under a majority operator, then this problem is reduced to a sequence of \((|T| - 2)\) essentially easier problems. Each of them seeks \( d \) smallest numbers in a set of \( |K| \times d \) numbers. In particular, if \( d = 1 \) then the \(|T|\)-variate minimization is reduced to \((|T| - 2)\) univariate minimizations.

This strength would be severely weakened, if the behavior of the algorithm on problems with no majority polymorphism was not known. This would require an additional algorithm for testing the existence of a majority polymorphism for the input problem. We do not know such an algorithm and expect it to be quite complex. The advantage of the proposed algorithm is that it does not require such control. It copes with the whole NP-complete class of minimax problems of a certain format. For any such problem the algorithm returns either the solution or a “discard” message. The latter is possible only if the problem has no majority polymorphism.
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