On conserved charges and thermodynamics of the AdS$_4$ dyonic black hole

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Abstract: Four-dimensional gravity in the presence of a dilatonic scalar field and an Abelian gauge field is considered. This theory corresponds to the bosonic sector of a Kaluza-Klein dimensional reduction of eleven-dimensional supergravity which induces a determined self-interacting potential for the scalar field. We compute the conserved charges and carry out the thermodynamics of an anti-de Sitter (AdS) dyonic black hole solution recently proposed. The charges coming from symmetries of the action are computed by using the Regge-Teitelboim Hamiltonian approach. These correspond to the mass, which acquires contributions from the scalar field, and the electric charge. Integrability conditions are introduced because the scalar field leads to non-integrable terms in the variation of the mass. These conditions are generically solved by introducing boundary conditions that arbitrarily relates the leading and subleading terms of the scalar field fall-off. The Hamiltonian Euclidean action, computed in the grand canonical ensemble, is obtained by demanding the action to attain an extremum. Its value is given by a radial boundary term plus an additional polar angle boundary term due to the presence of a magnetic monopole. Remarkably, the magnetic charge can be identified from the variation of the additional polar angle boundary term, confirming that the first law of black hole thermodynamics is a consequence of having a well-defined and finite Hamiltonian action principle, even if the charge does not come from a symmetry of the action. The temperature and electrostatic potential are determined demanding regularity on the black hole solution, whereas the value of the magnetic potential is already identified in the variation of the additional polar angle boundary term. Consequently, the first law of black hole thermodynamics is identically satisfied by construction.
Contents

1 Introduction 2

2 AdS$_4$ dyonic black hole solution 3

3 Hamiltonian and surface integrals 5
   3.1 Conserved charges of the AdS$_4$ dyonic black hole 6

4 Thermodynamics of the AdS$_4$ dyonic black hole 9
   4.1 Hamiltonian action and Euclidean minisuperspace 10
   4.2 Gibbs free energy and first law 12

5 Concluding remarks 16
1 Introduction

It has been recently conjectured about the need of additional terms in the first law of black hole thermodynamics, interpreted as scalar charges for a new class of dyonic black hole introduced in [1], [2]. This black hole is asymptotically AdS provided the system is endowed with a self-interacting potential. The theory is described by the action of the bosonic sector coming from a consistent truncation of a $S^7$ reduction of eleven-dimensional supergravity [3]. In the thermodynamic analysis of [1], the authors claimed that the first law is not satisfied unless one adds a term which is interpreted as a scalar charge [1], [2]. However, this argument conflicts with the fact that there is no gauge symmetry related to the presence of a single scalar field, i.e., there is no a Noether charge associated to this Lagrangian (nor a topological charge as is the case of the magnetic charge). Moreover, it has been clearly identified that the scalar field generically contributes to the mass with non-integrable terms. This has been proved with general asymptotic conditions through Hamiltonian [4] and other methods [5], [6], and even with an explicit black hole example in presence of gauge fields in three dimensions [7]. In this context, in [8] is presented a new class of dyonic AdS black hole solutions of four-dimensional $\mathcal{N} = 8$, SO(8) gauge supergravity where the AdS$_4$ dyonic dilatonic black hole of [1] is included. It is found in fact that the missing term, supposedly related to a scalar charge, is a contribution to the variation of the mass. It is also pointed out in [8] that the non-integrability of this term leads to an ill-defined mass with independent electric and magnetic charges. However, as it will be shown in this manuscript, it is possible to impose general integrability conditions. One can settle some physical criterium that is manifested through suitable boundary conditions which determine the arbitrary functions coming from the integrability conditions. One possible criterium is to demand the preservation of the AdS symmetry of the scalar field fall-off [9], [4], obtaining that the scalar field contribution is cancelled by a term coming from the gravitational contribution to the mass.

Other motivation for studying the black hole solution presented in [1] has to do with the computation of its Gibbs free energy. For doing so, it requires to have a well defined action and also presenting other thermodynamic features such that to exhibit all the charges of the solution with their respective chemical potentials. The Gibbs free energy presented in [1] fails on that, since they cannot recover the magnetic contribution to the Euclidean action, not even including the supposed scalar charge. The value of the action in [1] was obtained through the holographic renormalization method described in [10] by adding counterterms which only considers radial surface terms to get a finite action principle. As we will show bellow, it is necessary an additional polar angle boundary term for obtaining the magnetic contribution to the Euclidean action. For proving the latter, we formulate a well-defined and finite Hamiltonian action principle for the system and we prove that this additional boundary term comes from a total derivative in the polar angle which appears due to the black hole solution is endowed with a magnetic monopole.

The aim of this paper is to compute the conserved charges of the AdS$_4$ dyonic black hole and also to formulate a well-defined and finite Hamiltonian action principle for the described system which allows to obtain the value of the Hamiltonian Euclidean action. By
imposing the grand canonical ensemble, the Gibbs free energy is chosen as our thermodynamic potential which, unlike the free energy computed in [1], exhibits all the conserved charges of the solution, i.e., the mass (with its respective scalar field contribution), the electric charge and the magnetic charge. The plan of this manuscript is the following, in the Section 2 we present the Lagrangian and the AdS$_4$ dyonic dilatonic black hole solution of [1]. Section 3 is focused on the Hamiltonian analysis and the corresponding conserved charges. The mass and the electric charge are computed by using the Regge-Teitelboim Hamiltonian approach. In the variation of the mass there are two contributions, the gravitational and the scalar field part (already identified in [8]). Integrability conditions are needed to be imposed since the presence of the scalar field leads to a non-integrable term. Suitable boundary conditions are chosen in order to preserve the AdS symmetry of the scalar field fall-off. This implies a precise relation on the coefficients of the leading and subleading terms of the scalar field, as was noted for this kind of systems in [4]. In Section 4 we perform the thermodynamic analysis of the solution. In there, the Hamiltonian Euclidean action is introduced, where for simplicity the calculations are done in a suitable Euclidean minisuperspace. For obtaining the Gibbs free energy we compute the value of the Euclidean Hamiltonian action endowed with a suitable radial boundary term and an additional polar angle boundary term. These terms are needed to be added in order to have a well-defined and finite Hamiltonian action principle. From the variation of the boundary term at infinity is possible to identify the variation of the Hamiltonian conserved charges of this system, which are the mass and electric charge. On the other hand, the variation of the magnetic charge comes from the additional polar boundary term. This boundary term has to be considered due to the presence of a magnetic monopole. The chemical potentials associated to Noether charges are Lagrange multipliers of the system at infinity. They are obtained through regularity conditions at the horizon, unlike the magnetic potential which remarkably already appears determined in the variation of the boundary term besides the magnetic charge. It is worth to note that the first law of black hole thermodynamics is satisfied independently of the integrability conditions on the mass, since the relation only involves the variation of the conserved charges. Once the Gibbs free energy is obtained, the value of the mass, electric charge, magnetic charge and entropy are verified by using the known thermodynamic relations. Finally, Section 5 is devoted to some concluding remarks.

2 AdS$_4$ dyonic black hole solution

We consider four-dimensional gravity with negative cosmological constant in presence of an Abelian gauge field and a dilatonic scalar field with a self-interacting potential. The action reads

$$I[g_{\mu\nu}, A_\mu, \phi] = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} e^{-\sqrt{3}\phi} F^{\mu\nu} F_{\mu\nu} - V(\phi) \right).$$

(2.1)
Hereafter the gravitational constant is chosen as $\kappa = 1/2$. The self-interacting potential of the scalar field is given by

$$V(\phi) = -6g^2 \cosh\left(\frac{\phi}{\sqrt{3}}\right),$$

(2.2)

where the coupling constant $g$ determines the AdS radius as $\ell^2 = g^{-2}$. The theory given by (2.1) corresponds to the bosonic sector of two possible dimensional reductions, which depend on the coupling constant $g$. In the case of vanishing $g$, the action is obtained after a $S^1$ reduction of five-dimensional pure gravity. On the other hand if $g \neq 0$ the action can be obtained after a $S^7$ reduction of eleven-dimensional supergravity as was found in [3].

The gravitational field equations for the action (2.1) are

$$G_{\mu\nu} = T^\phi_{\mu\nu} + T^A_{\mu\nu},$$

(2.3)

where the contributions to the energy-momentum tensor of the dilatonic scalar field and the gauge field are given by

$$T^\phi_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} \partial^\lambda \phi \partial^\lambda \phi + \frac{1}{2} g_{\mu\nu} V(\phi),$$

(2.4)

$$T^A_{\mu\nu} = \frac{1}{2} e^{-\sqrt{3}\phi} \left( F_{\mu}^\lambda F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F_{\lambda\rho} \right),$$

(2.5)

respectively. The equation for the scalar field is

$$\Box \phi + \frac{\sqrt{3}}{4} e^{-\sqrt{3}\phi} F^{\mu\nu} F_{\mu\nu} - \frac{dV}{d\phi} = 0,$$

(2.6)

and the equation for the gauge field reads

$$\nabla_\mu \left( e^{-\sqrt{3}\phi} F^{\mu\nu} \right) = 0.$$  

(2.7)

This system admits an AdS dyonic black hole which is static and spherically symmetric [1]. The line element of this configuration can be written as

$$ds^2 = -\left(H_1 H_2\right)^{-1/2} f dt^2 + \frac{dr^2}{\left(H_1 H_2\right)^{-1/2} f} + \left(H_1 H_2\right)^{1/2} r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2\right),$$

(2.8)

where the functions determining the solution are given by

$$f(r) = f_0(r) + g^2 r^2 H_1(r) H_2(r), \quad f_0(r) = 1 - \frac{2\mu}{r},$$

(2.9)

$$H_1(r) = \gamma_1^{-1} \left(1 - 2\beta_1 f_0(r) + \beta_1 \beta_2 f_0(r)^2\right), \quad H_2(r) = \gamma_2^{-1} \left(1 - 2\beta_2 f_0(r) + \beta_1 \beta_2 f_0(r)^2\right),$$

(2.10)

with $\gamma_1 = 1 - 2\beta_1 + \beta_1 \beta_2$, and $\gamma_2 = 1 - 2\beta_2 + \beta_1 \beta_2$. The dilatonic scalar field is given by

$$\phi(r) = \frac{\sqrt{3}}{2} \log \left( \frac{H_2(r)}{H_1(r)} \right),$$

(2.11)

\[1\] The vacuum permeability constant located in front of the Maxwell-like action in (2.1) turns out to be normalized to one after the dimensional reduction.
whereas the one-form gauge field has the following form

\[ A = A_t(r)dt + A_\varphi(\theta)d\varphi. \]  

The time component of (2.12) is

\[ A_t(r) = \frac{\sqrt{2}(1 - H_1(r) - \beta_1(f_0 - H_1(r)))}{\sqrt{\beta_1 \gamma_2 H_1(r)}}, \]  

while the definition of the angular component of the gauge potential changes depending on the hemisphere in order to avoid the Dirac string [11]. Hence

\[ A_\varphi(\theta) = \begin{cases} p(1 + \cos(\theta)) & , \quad 0 \leq \theta < \frac{\pi}{2} - \delta, \\ p(-1 + \cos(\theta)) & , \quad \frac{\pi}{2} + \delta < \theta \leq \pi, \end{cases} \]  

where \( p = 2\sqrt{2}\mu\gamma_1^{-1}\sqrt{\beta_2 \gamma_1} \) and \( \delta \to 0 \) (Wu-Yang monopole [12], [13]). In this solution the coordinates range as \( 0 < r < \infty, -\infty < t < \infty, 0 \leq \theta < \pi \) and \( 0 \leq \varphi < 2\pi \). All the integration constants \( (\mu, \beta_1, \beta_2, \gamma_1, \gamma_2) \) are restricted to be positive.

In the case of \( \beta_1 = \beta_2 \), the dilatonic scalar field is decoupled and the solution turns out to be an AdS dyonic Reissner-Nordström black hole where the electric and magnetic charges have the same value. If \( \beta_1 = 0 \) the solution is purely magnetic and in the case of \( \beta_2 = 0 \) the configuration becomes purely electric. If \( \mu = 0 \) the solution turns out to be the anti-de Sitter spacetime.

3 Hamiltonian and surface integrals

The Hamiltonian generator for the Lagrangian(2.1) reads as

\[ H[\xi, \xi^A] = \int d^3x \left( \xi^A H_\perp + \xi^i H_i - \xi^A G \right) + Q[\xi, \xi^A], \]  

where the boundary term \( Q[\xi, \xi^A] \), which corresponds to the conserved charges in the Regge-Teitelboim approach, ensures that the Hamiltonian generator has well-defined functional derivatives [14]. The bulk term appearing in (3.1) is a linear combination of the constraints \( H_\perp, H_i \), and \( G \), where the first two are the energy and momentum densities, and the last one corresponds to the Gauss constraint associated to the Abelian gauge field. The asymptotic surface deformations of the spacetime are given by the vector \( \xi = (\xi^\perp, \xi^i) \), and \( \xi^A \) in turn is the gauge parameter of the Abelian symmetry. The constraints are explicitly given by

\[ H_\perp = \frac{1}{\sqrt{\gamma}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} (\pi^i_i)^2 \right) - \sqrt{\gamma} R \]

\[ + \frac{\pi^2_{\phi}}{2\sqrt{\gamma}} + \sqrt{\gamma} \left( \frac{1}{2} \partial^i \phi \partial_i \phi + V(\phi) \right) + e^{\sqrt{3} \phi} \frac{\pi^i_i \pi_{ij}}{2\sqrt{\gamma}} + \frac{1}{4} \sqrt{\gamma} e^{-\sqrt{3} \phi} F^{ij} F_{ij}, \]  

\[ H_i = 2\nabla_j \pi^j_i + \pi_\phi \partial_i \phi + \pi^i F_{ij}, \]  

\[ G = \partial_i \pi^i. \]
The dynamical variables of the system are the spatial components of the fields $\{\gamma_{ij}, A_i, \phi\}$, where $\gamma_{ij}$ is the spatial metric of the ADM decomposition. Here $R$ stands for the scalar curvature of the 3-dimensional spatial metric $\gamma_{ij}$ and the self-interacting potential of the scalar field $V(\phi)$ is defined in eq. (2.2). The momentum conjugated to the 3-dimensional metric $\gamma_{ij}$ is

$$\pi^{ij} = -\sqrt{\gamma} \left( K^{ij} - \gamma^{ij} K \right),$$

(3.5)

where the extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N} \left( \nabla_i N_j + \nabla_j N_i - \gamma_{ij} \dot{N} \right).$$

(3.6)

The momentum for the dilatonic field $\phi$ reads

$$\pi_\phi = \frac{\sqrt{\gamma}}{N} \left( \dot{\phi} - N^i \partial_i \phi \right),$$

(3.7)

and for the gauge field $A_i$,

$$\pi^i = -\sqrt{\gamma_e^{-\sqrt{3}} \phi} \left( -\gamma^{ij} F_{ij} + N^j \gamma^{ik} F_{jk} \right).$$

(3.8)

The variation of the surface term exhibits different contributions according to the field content of the theory, such that

$$\delta Q \left[ \xi^A, \xi^A \right] = \delta Q^G \left[ \xi, \xi^A \right] + \delta Q^\phi \left[ \xi, \xi^A \right] + \delta Q^A \left[ \xi, \xi^A \right],$$

(3.9)

where $\delta Q$ was obtained after demanding that $\delta H = 0$ on the constraint surface (vanishing constraints). The explicit expressions for the surface integrals are given by

$$\delta Q^G = \int dS_i G^{ijkl} \left( \xi^k \nabla_l \delta \gamma_{ij} - \partial_k \xi^l \delta \gamma_{ij} \right) + \int dS_i \left[ 2 \xi_k \delta \pi^{kl} + \left( 2 \xi^k \pi^{jl} - \xi^l \pi^{kj} \right) \delta \gamma_{lj} \right],$$

(3.10)

$$\delta Q^\phi = -\int dS_i \left( \xi^l \sqrt{\gamma} \partial_l \phi \delta \phi + \xi^l \pi_\phi \delta \phi \right),$$

(3.11)

$$\delta Q^A = -\int dS_i \left[ \xi^l \sqrt{\gamma} \phi \delta \phi + \xi^l \pi_\phi \delta \phi \right],$$

(3.12)

with

$$G^{ijkl} = \frac{1}{2} \sqrt{\gamma} \left( \gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} - 2 \gamma^{ij} \gamma^{kl} \right).$$

(3.13)

### 3.1 Conserved charges of the AdS$_4$ dyonic black hole

In order to obtain the above surface integrals, let us consider a static and spherically symmetric minisuperspace in which the AdS$_4$ dyonic black hole (2.8) is included. For simplicity in the upcoming analysis we perform the following change of variable in the radial coordinate

$$\rho^2 = \sqrt{H_1(r) H_2(r)} r^2.$$

(3.14)
The line element then reads
\[ ds^2 = -N^\perp (\rho)^2 dt^2 + \frac{d\rho^2}{F(\rho)} + \rho^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right). \] (3.15)

The gauge field ansatz is given by
\[ A = A_t (\rho) dt + A_\varphi (\theta) d\varphi, \] (3.16)
and the scalar field also depends on the radial coordinate \( \phi = \phi(\rho) \). Taking into consideration this minisuperspace the only nonvanishing momentum is the radial component of the electromagnetic one, where \( \pi^\rho = p^\rho (\rho, \theta) \). Therefore, the value of the Hamiltonian charges, computed on the sphere \( S^2 \) of infinite radius, is given by
\[
\delta Q = \left[ -\xi^t \left( \frac{8\pi \rho N^\perp \delta F}{\sqrt{F}} + 4\pi \sqrt{F} N^\perp \rho^2 \partial_\rho \phi \delta \phi + \pi \left( \int \frac{N^\perp e^{-\sqrt{3}\phi}}{\sqrt{F} \rho^2} \, d\rho \right) \csc(\theta) \delta A_\varphi \partial_\theta A_\varphi \right]_{\theta=\pi}^{\theta=0} \\
+ 2\pi \xi^A \int_0^\pi \delta \rho^A \, d\theta \right]_{\rho=\infty}. \] (3.17)

Here, we have applied the definition of the deformation vectors \( \xi^\perp \) and \( \xi^i \) in terms of the Killing vectors \( \xi^t \) and \( \bar{\xi}^i \), which reads
\[
\xi^\perp = N^\perp \xi^t, \\
\xi^i = \bar{\xi}^i + N^i \xi^t. \] (3.18)

In order to compute and perform a proper analysis of the charges, we must give suitable asymptotic conditions that represent the behavior of the fields at infinity. These conditions are specified up to the orders that contribute to the charges, such that
\[
F(\rho) = g^2 \rho^2 + 1 + F_0 + \frac{F_1}{\rho} + O \left( \frac{1}{\rho^2} \right), \] (3.20)
\[
N^\perp (\rho) = g\rho + O \left( \frac{1}{\rho} \right), \] (3.21)
\[
\phi(\rho) = \frac{\phi_1}{\rho} + \frac{\phi_2}{\rho^2} + O \left( \frac{1}{\rho^3} \right), \] (3.22)
\[
p^\rho(\rho, \theta) = p_0 \sin(\theta) + O \left( \frac{1}{\rho} \right), \] (3.23)
\[
\xi^A = \xi^A_0 + O \left( \frac{1}{\rho} \right). \] (3.24)

The coefficients in the expansions given above are parameters that depend on the integration constants of the corresponding solution. The variation of the charge obtained after replacing the proposed asymptotic behavior in (3.17) is given by\(^2\)

\(^2\)It has to be noted that a divergent term appears in the variation of the charge but it vanishes once the solution is replaced. This is because the divergent part of the gravitational contribution is cancelled by the divergent part of the scalar field contribution by virtue of the relation \( \delta F_0 = \frac{\partial^2}{\partial^2_\phi} \phi_1 \delta \phi_1 \).
\[
\delta Q = \xi^t \left[ -8\pi \delta F_1 + 4\pi g^2 \left( 2\phi_2 \delta \phi_1 + \phi_1 \delta \phi_2 \right) \right] + 4\pi \xi^A_0 \delta \rho_0. \tag{3.25}
\]

The mass is the conserved charge associated to the time spacetime translations, which in this approach is obtained from \( \delta M = \delta Q \left[ \xi^t \right] \), while the electric charge is the charge associated to the Abelian gauge transformations, where \( \delta Q_e = \delta Q \left[ \xi^A \right] \). Then, the variation of the mass and the electric charge reads

\[
\delta M = -8\pi \delta F_1 + 4\pi g^2 \left( 2\phi_2 \delta \phi_1 + \phi_1 \delta \phi_2 \right), \tag{3.26}
\]

\[
\delta Q_e = 4\pi \delta \rho_0, \tag{3.27}
\]

respectively. The electric charge can be directly integrated for the AdS_4 dyonic black hole, which in terms of the integration constants of the solution is written as

\[
Q_e = \frac{16\pi \sqrt{2}\mu \sqrt{\beta_1 \gamma_2}}{\gamma_1}. \tag{3.28}
\]

In contrast, the mass is generically non-integrable and its variation is explicitly given by

\[
\delta M = \delta \left( 16\pi \left( 1 + \beta_1 \right) \left( 1 - \beta_2 \right) \left( 1 - \beta_1 \beta_2 \right) \mu \right) + 64\pi g^2 \mu^3 \left( 1 - \beta_1 \beta_2 \right) \left( \beta_1 - \beta_2 \right)^2 \gamma + \Phi, \tag{3.29}
\]

with

\[
\gamma = \beta_1 + \beta_2 - 8\beta_1 \beta_2 + 6\beta_1^2 \beta_2 + 6\beta_1 \beta_2^2 - 8\beta_1^2 \beta_2^2 + \beta_1^3 \beta_2^2 + \beta_1^2 \beta_2^3. \tag{3.30}
\]

Note that the variation of the mass coincides with the one computed in [8], which has the non-integrable term \( \Phi \) that comes from the scalar field part of the energy density. This term is given by

\[
\Phi = 4\pi g^2 \left( 2\phi_2 \delta \phi_1 + \phi_1 \delta \phi_2 \right), \tag{3.31}
\]

where the leading and subleading terms of the scalar field fall-off are respectively

\[
\phi_1 = \frac{2\sqrt{3} \left( \beta_2 \left( 1 + \beta_2 \right) - \beta_1 \left( 1 + \beta_2 \right) \right) \mu}{\gamma_1 \gamma_2}, \tag{3.32}
\]

\[
\phi_2 = \frac{2\sqrt{3} \left( -\beta_2^2 \left( 1 - \beta_1^2 \right) - 2\beta_1 \beta_2 \left( -4 + 3\beta_2 \right) - 2\beta_1^2 \beta_2 \left( -3 + 4\beta_2 \right) - \beta_1^2 \left( -1 + 8\beta_2 - 8\beta_2^2 + \beta_1^2 \right) \right) \mu^2}{\gamma_1^2 \gamma_2^2}. \tag{3.33}
\]

The presence of a non-integrable term \( \Phi \) in the variation of the mass (3.26) demands integrability relations among the fall-off coefficients of the scalar field. The integrability condition \( \delta^2 M = 0 \) implies that \( \delta \Phi = 0 \), this implies in turn that \( \phi_2 = \phi_2 (\phi_1) \). Hence, the mass generically takes the form

\[
M = -8\pi F_1 + 4\pi g^2 \int \left( 2\phi_2 + \phi_1 \frac{d\phi_2}{d\phi_1} \right) d\phi_1. \tag{3.34}
\]

In this point it is necessary to impose a boundary condition that fixes a precise relation between the leading and subleading terms of the scalar field behavior at infinity. One possible criterium is to demand preservation of the AdS symmetry of the scalar field asymptotic.
fall-off, which can be carried out since the AdS$_4$ dyonic dilatonic black hole of [1] is within
the asymptotic conditions for AdS spacetimes analyzed in [9], [4]. In there, they construct
a set of boundary conditions for having a well-defined and finite Hamiltonian generators
for all the elements of the anti-de Sitter algebra in the case of gravity minimally coupled to
scalar fields. Then, we are allowed to impose certain relations on the leading and subleading
terms of the scalar field fall-off provided the scalar field does not break the AdS symme-
try at infinity. These boundary conditions are ($\phi_1 = 0$, $\phi_2 \neq 0$), ($\phi_1 \neq 0$, $\phi_2 = 0$) and
$\phi_2 = c\phi_1^2$, where $c$ is a constant without variation. In terms of the integration constants
the latter relation becomes

$$\phi_2 = c\phi_1^2,$$

From eq. (3.35) we observe three cases, two of them nontrivial. When $\mu = 0$ (3.35) holds,
the mass, the electric charge and magnetic charge vanish giving rise to the vacuum solution
which turns out to be the AdS$_4$ spacetime. The other two cases imply that $\beta_1 = \beta_1 (\beta_2)$ in
such a way that they make the terms proportionals to $g^2$ in (3.29) to vanish. Hence, the
mass becomes the AMD mass [15], [16] obtained in [1],

$$M = \frac{16\pi (1 - \beta_1) (1 - \beta_2) (1 - \beta_1\beta_2) \mu}{\gamma_1\gamma_2}.$$  

This fact is in agreement with [17], where it is pointed out that in hairy spacetimes the
mass does not receive scalar field contributions provided the scalar field respects the AdS
invariance at infinity, making some holographic prescriptions suitable for computing the
mass.

4 Thermodynamics of the AdS$_4$ dyonic black hole

The thermodynamic analysis of the AdS$_4$ dyonic dilatonic black hole is performed in this
section. We define the Euclidean Hamiltonian action of the theory including a surface term
and a additional polar boundary term to have a finite action principle. The presence of
the latter additional term is due to the solution counts with a magnetic monopole. For
simplicity, we take a minisuperspace in which the AdS$_4$ dyonic black hole is included. The
variation of the Euclidean Hamiltonian action is computed in the grand canonical ensemble,
where the chemical potentials are fixed. Remarkably, the magnetic charge emerges from
the additional polar boundary term accompanied by its respective chemical potential. The
value of the temperature and electric potential, on the other hand, are settled imposing
regularity conditions. When the variation of the additional surface and polar boundary
terms is determined, as was mentioned above, integrability conditions are needed to be
impose to determine the value of the Euclidean Hamiltonian action leading to the Gibbs
free energy.
4.1 Hamiltonian action and Euclidean minisuperspace

Let us consider spacetimes with manifold of topology $\mathbb{R}^2 \times S^2$. The plane $\mathbb{R}^2$ is centered at the event horizon $r_+$ and is parametrized by the periodic Euclidean time $\tau$ and the radial coordinate $r$. These plane coordinates range as

\begin{align}
0 &\leq \tau < \beta, \quad (4.1) \\
0 &\leq r < \infty, \quad (4.2)
\end{align}

with $\beta$ the inverse of the Hawking temperature and the 2-sphere $S^2$ stands for the topology of the base manifold. The Hamiltonian Euclidean action for the system is given by

\begin{equation}
I^E = \int_0^\beta d\tau \int d^3x \left[ \dot{x}_{ij} \pi^{ij} + \dot{A}_i \pi^i + \dot{\phi} \pi_\phi - \left( N^\perp \mathcal{H}_\perp + N^i \mathcal{H}_i - A_\tau \mathcal{G} \right) \right] + B, \quad (4.3)
\end{equation}

where $\Sigma = \mathbb{R} \times S^2$ is the spatial section of the manifold. Note that the additional term $B$ in (4.3) needs to be added to the action in order to have a well-defined variational principle, and it is crucial for determining the value of the action for stationary configurations.

The Euclidean continuation of the AdS$_4$ dyonic black hole (2.8) is considered. The line element reads

\begin{equation}
ds^2 = N^\perp (r)^2 d\tau^2 + \frac{dr^2}{F(r)} + H(r) \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right), \quad (4.4)
\end{equation}

where the gauge field ansatz and the scalar field are given by

\begin{align}
A &= A_\tau(r) d\tau + A_\varphi(\theta) d\varphi, \quad (4.5) \\
\phi &= \phi(r). \quad (4.6)
\end{align}

The radial component of the electromagnetic field momentum is $\pi^r = p^r(r, \theta)$ (all the other momenta of the fields vanish). Hence, from (4.3) is possible to obtain the following reduced action

\begin{equation}
I^E = -2\pi\beta \int_{r_+}^{\infty} dr \int_0^\pi d\theta \left[ N^\perp (r) \mathcal{H}_\perp - A_\tau(r) \mathcal{G} \right] + B, \quad (4.7)
\end{equation}

where the reduced constraints take the form

\begin{equation}
\mathcal{H}_\perp = \frac{e^{-\sqrt{3}\phi} \sin(\theta)}{2\sqrt{F}H} \left[ -\csc^2(\theta) (\partial_\theta A_\varphi)^2 - 2e^{\sqrt{3}\phi} H \left( \partial_r F \partial_r H + 2F \partial_r^2 H - 2 \right) \\
+ e^{\sqrt{3}\phi} H^2 \left( 12g^2 \cosh\left( \frac{\phi}{\sqrt{3}} \right) - F \left( \partial_r \phi \right)^2 \right) \right] + e^{\sqrt{3}\phi} F \left( \partial_r H \right)^2 + \csc^2(\theta) e^{2\sqrt{3}\phi} (p^r)^2, \quad (4.8) \\
\mathcal{G} = \partial_r p^r. \quad (4.9)
\end{equation}

The variation of the reduced action (4.7) with respect to the Lagrange multipliers $N^\perp$ and $A_\tau$ indicates that the constraints have to vanish

\begin{equation}
\mathcal{H}_\perp = 0, \quad \mathcal{G} = 0. \quad (4.10)
\end{equation}
These equations define the constraint surface. On the other hand, the variation of (4.7) with respect to the independent functions of the dynamical fields in the minisuperspace leads to the field equations. The field equations related to \( F(r) \) and \( H(r) \) are respectively given by

\[
\frac{e^{-\sqrt{3}\phi} \sin(\theta)}{4F^{3/2}H} \left( N^\perp \left( -\csc^2(\theta) (\partial_\theta A_\phi)^2 - F (\partial_r H)^2 e^{\sqrt{3}\phi} + \csc^2(\theta) e^{2\sqrt{3}\phi} (p^r)^2 \right) \right. \\
+ H^2 N^\perp e^{\sqrt{3}\phi} \left( F (\partial_r \phi)^2 + 12g^2 \cosh \left( \frac{\phi}{\sqrt{3}} \right) \right) + 4He^{\sqrt{3}\phi} \left( N^\perp - F \partial_r H \partial_r N^\perp \right) \right) = 0, \\
\tag{4.11}
\]

\[
\frac{e^{-\sqrt{3}\phi} \sin(\theta)}{2\sqrt{F}H^2} \left( N^\perp \left( -\csc^2(\theta) (\partial_\theta A_\phi)^2 - F (\partial_r H)^2 e^{\sqrt{3}\phi} + \csc^2(\theta) e^{2\sqrt{3}\phi} (p^r)^2 \right) \right. \\
- H^2 e^{\sqrt{3}\phi} \left( N^\perp \left( 12g^2 \cosh \left( \frac{\phi}{\sqrt{3}} \right) - F (\partial_r \phi)^2 \right) - 2 \left( \partial_r F \partial_r N^\perp + 2F \partial_r^2 N^\perp \right) \right) \\
+ He^{\sqrt{3}\phi} \left( N^\perp (\partial_r F \partial_r H + 2F \partial_r^2 H) + 2F \partial_r H \partial_r N^\perp \right) \right) = 0. \\
\tag{4.12}
\]

The field equations associated to \( A_\phi (r, \theta) \) and \( p^r (r, \theta) \) are

\[
\frac{N^\perp e^{-\sqrt{3}\phi} \csc(\theta)}{\sqrt{F(r, s)H(r, s)}} (\partial_\theta A_\phi \cot(\theta) - \partial_\theta^2 A_\phi) = 0, \quad \partial_r A_\tau + \frac{\csc(\theta) N^\perp e^{\sqrt{3}\phi} p^r}{\sqrt{FH}} = 0, \tag{4.13}
\]

and finally the scalar field equation reads

\[
-\frac{e^{-\sqrt{3}\phi} \sin(\theta)}{2\sqrt{F}H} \left( N^\perp \left( \sqrt{3} \csc^2(\theta) (\partial_\theta A_\phi)^2 + 2FH e^{\sqrt{3}\phi} \partial_r H \partial_r \phi \right) \right. \\
+ H^2 e^{\sqrt{3}\phi} \left( \partial_r F \partial_r \phi + 2F \partial_r^2 \phi + 4\sqrt{3}g^2 \sinh \left( \frac{\phi}{\sqrt{3}} \right) \right) \\
+ \sqrt{3} \csc^2(\theta) e^{2\sqrt{3}\phi} (p^r)^2 \right) 2FH^2 e^{\sqrt{3}\phi} \partial_r \phi \partial_r N^\perp \right) = 0. \tag{4.14}
\]

Then, the variation of the reduced action (4.7) on the constraint surface and evaluated on-shell (eqs. from (4.11) to (4.14) have to be satisfied) turns out to be

\[
\delta I^F \bigg|_{\text{on-shell}} = -2\pi \beta \int_0^\pi d\theta \left[ N^\perp \sin(\theta) \left( \frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F} \partial_r H \delta H}{H} \right) \\
+ \frac{\sqrt{F} \partial_r \phi \delta \phi}{\sqrt{F}} + 2\sqrt{F} \partial_r \delta H \right] \left. - \partial_r \left( 2N^\perp \sin(\theta) \sqrt{F} \right) \delta H - A_\tau \delta p^r \right]_r^\infty \\
- 2\pi \beta \int_{r_+}^\infty dr \left[ \frac{N^\perp e^{-\sqrt{3}\phi}}{H \sqrt{F} \sin \theta} \partial_\theta A_\phi \delta A_\phi \right]_0^\pi + \delta B. \tag{4.15}
\]
If we demand the action to attain an extremum, i.e., \( \delta I[E]_{\text{on-shell}} = 0 \), the variation of additional term \( \delta B \) necessarily must be given by

\[
\delta B = 2\pi \beta \int_0^\pi d\theta \left[ N^\perp \sin(\theta) \left( \frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F} \partial_r H \delta H}{H} \right) \right.
\]

\[
+ \frac{\sqrt{F} \partial_r \phi \delta \phi}{2} + 2\sqrt{F} \partial_r \delta H \bigg] \bigg|_{r^+}^{\infty}
\]

\[
+ 2\pi \beta \int_{r^+}^{\infty} dr \left[ N^\perp e^{-\sqrt{3} \phi} \partial_\phi A_\phi \delta A_\phi \right]_0^\pi.
\]

(4.16)

In this expression is possible to recognize two kind of terms, the surface term coming from a total derivative in the radial coordinate and a boundary term that comes from a total derivative in the polar angle, which clearly is not vanishing since the presence of an angular component depending on the polar angle in the gauge field. The analysis of the variation of the term \( B \) and the evaluation on the AdS_4 dyonic black hole (4.16) will be performed in the following subsection.

### 4.2 Gibbs free energy and first law

From (4.16), we can identify different contributions depending whether the term comes from a total derivative in the radial coordinate, or whether the term comes from a total derivative in the polar angle which is identified as a polar boundary term. The surface term evaluated at infinity will be denoted by \( \delta B(\infty) \) and, on the other hand, \( \delta B(r^+) \) will stand for the surface term at the horizon. The polar boundary term in turn will be denoted by \( \delta B_\theta \). Hence, the variation of \( B \) (4.16) can be written as

\[
\delta B = \delta B(\infty) + \delta B(r^+) + \delta B_\theta, \tag{4.17}
\]

where the surface term at infinity \( \delta B(\infty) \) is given by

\[
\delta B(\infty) = 2\pi \beta \int_0^\pi d\theta \left[ N^\perp \sin(\theta) \left( \frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F} \partial_r H \delta H}{H} \right) \right.
\]

\[
+ \frac{\sqrt{F} \partial_r \phi \delta \phi}{2} + 2\sqrt{F} \partial_r \delta H \bigg] \bigg|_{r^+}^{\infty}
\]

(4.18)

the surface term at the horizon \( \delta B(r^+) \) is

\[
\delta B(r^+) = -2\pi \beta \int_0^\pi d\theta \left[ N^\perp \sin(\theta) \left( \frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F} \partial_r H \delta H}{H} \right) \right.
\]

\[
+ \frac{\sqrt{F} \partial_r \phi \delta \phi}{2} + 2\sqrt{F} \partial_r \delta H \bigg] \bigg|_{r^+}^{\infty}
\]

(4.19)
and the polar boundary term $\delta B_\theta$ reads as

$$
\delta B_\theta = 2\pi \beta \int_{r_+}^{\infty} dr \left[ \frac{N_{\perp} e^{-\sqrt{3} \phi}}{H \sqrt{F} \sin(\theta)} \partial_\theta A_\varphi \delta A_\varphi \right]_0^\pi .
$$

(4.20)

Once the different contributions to the variation of $B$ are identified one can analyze their physical content. In the surface term at infinity $\delta B(\infty)$ is possible to find the variation of the charges coming from symmetries of the action together with their respective chemical potentials, which correspond to the Lagrange multipliers at infinity of the respective symmetry (as was shown in Section 3). This is because at the end of the day the term (4.18) is obtained from the boundary term of the Hamiltonian which ensures that the canonical generators have well-defined functional derivatives [14]. Then, from $\delta B(\infty)$ it will be identified the variations of the mass and the electric charge of the AdS$_4$ dyonic dilatonic black hole. From the surface term at the horizon $\delta B(r_+)$ will be obtained the entropy of the black hole which corresponds to the Bekenstein-Hawking entropy. Finally, from the polar boundary term $\delta B_\theta$ can be identified the contribution of the topological charge of the system leading to the variation of the magnetic charge multiplied by the magnetic potential.

Let us to introduce the Euclidean continuation of the AdS$_4$ dyonic dilatonic black hole that satisfies the field equations (4.11)-(4.14) and the constraints (4.10). This is obtained after performing the identifications $t \rightarrow -i\tau$ and $\beta_1 \rightarrow -\beta_1$ in the Lorentzian solution, which make the black hole functions to take the form

$$
H_1(r) = \gamma_1^{-1} \left( 1 + 2\beta_1 f_0(r) - \beta_1 \beta_2 f_0(r)^2 \right), \quad H_2(r) = \gamma_2^{-1} \left( 1 - 2\beta_2 f_0(r) - \beta_1 \beta_2 f_0(r)^2 \right),
$$

(4.21)

where $\gamma_1 = 1 + 2\beta_1 - \beta_1 \beta_2$ and $\gamma_2 = 1 - 2\beta_2 - \beta_1 \beta_2$. The functions in the line element (4.4) are considered to be

$$
F(r) = \frac{f(r)}{\sqrt{H_1(r) H_2(r)}}, \quad H(r) = \sqrt{H_1(r) H_2(r)} r^2,
$$

(4.22)

where the function $f(r)$ is the same as the one given in (2.9). The lapse function is $N_{\perp}(r) = \sqrt{F(r)}$. The scalar field is defined in (2.11), and the temporal component of the gauge field is given by

$$
A_\tau(r) = -\sqrt{2} \frac{(1 - H_1(r) + \beta_1 (f_0(r) - H_1(r)))}{\sqrt{\beta_1 \gamma_2 H_1(r)}} + \Phi_e.
$$

(4.23)

Note that the possibility of adding a constant $\Phi_e$ allows to have a regular gauge field at the horizon. This constant is related to the electrostatic potential of the solution when the regularity conditions on the black hole are established. The angular component of the gauge field takes the same definition given in (2.14).

Replacing the Euclidean continuation of the AdS$_4$ dyonic dilatonic black hole in the surface term at infinity $\delta B(\infty)$ given in eq. (4.18), it is obtained that

$$
\delta B(\infty) = -\beta \delta M - \beta \Phi_e \delta Q_e,
$$

(4.24)
where the variations of the mass and electric charge are respectively given by
\[
\delta M = \delta \left( \frac{16\pi (1 + \beta_1) (1 - \beta_2) (1 + \beta_1\beta_2)\mu}{\gamma_1\gamma_2} \right) + \Theta, \tag{4.25}
\]
\[
\delta Q_e = 4\pi\delta \left( \frac{2\sqrt{2}\mu\sqrt{\beta_1\gamma_2}}{\gamma_1} \right). \tag{4.26}
\]

The above variations coincide with the values computed in (3.29) and (3.28). In the variation of the mass is clearly obtained a contribution
\[
\Theta = \frac{64\pi g^2\mu^3 (1 + \beta_1\beta_2) (\beta_1 + \beta_2)^2\gamma}{\gamma_1^2\gamma_2^2} + \Phi^E
\]
\[
- \frac{32\pi g^2\mu^3 (\beta_1 + \beta_2)}{\gamma_1^2\gamma_2^2} (\beta_2 (1 - \beta_1 - 2\beta_2 + \beta_1\beta_2) \delta\beta_1 - \beta_1 (1 + 2\beta_1 + 2\beta_2 + \beta_1\beta_2) \delta\beta_2), \tag{4.27}
\]
where \(\Phi^E\) is the Euclidean continuation of \(\Phi\). Here \(\Theta\) corresponds to the supposed scalar charge term in the context of [1].

The inverse of the temperature \(\beta\) and the electrostatic potential \(\Phi_e\) are determined through the regularity conditions at the horizon. Indeed,
\[
\beta = \frac{4\pi\sqrt{H_1(r_+) H_2(r_+)}}{f'(r_+)} , \quad \Phi_e = -\sqrt{\frac{2}{\beta_1\gamma_2}} \left( 1 + \beta_1 - \frac{1 + \beta_1 f_0 (r_+)}{H_1(r_+)} \right). \tag{4.28}
\]
The temperature is obtained by demanding the absence of conical singularities around the event horizon, while the electrostatic potential comes from the trivial holonomy condition of the gauge field around a temporal cycle on the plane \(r - \tau\) at the event horizon. Introducing the values of the chemical potentials (4.28) into the surface term at the horizon \(\delta B (r_+)\), we get that this term exactly coincides with the Bekenstein-Hawking entropy
\[
\delta B (r_+) = \delta \left( 16\pi^2 \sqrt{H_1(r_+) H_2(r_+)} r_+^2 \right) = \delta S. \tag{4.29}
\]
The polar boundary term \(\delta B_\theta\) has to be carefully computed by using the definition of the angular component of the gauge field given in (2.14). Then,
\[
\delta B_\theta = 2\pi\beta \left( \int_{r_+}^\infty dr \frac{e^{-\sqrt{3}\phi}}{H} \left[ \frac{\partial_\theta A_\phi \delta A_\phi}{\sin(\theta)} \right]_0^{\pi/2-\delta} + \left[ \frac{\partial_\theta A_\phi \delta A_\phi}{\sin(\theta)} \right]_0^{\pi/2+\delta} \right)_{\delta \to 0}
\]
\[
= -2\pi\beta \left( \int_{r_+}^\infty dr \frac{e^{-\sqrt{3}\phi}}{H} \left[ p\delta p (1 + \cos(\theta))_0^{\pi/2-\delta} + [p\delta p (-1 + \cos(\theta))]_0^{\pi/2+\delta} \right]_{\delta \to 0}
\]
\[
= 4\pi\beta \left( \int_{r_+}^\infty dr \frac{e^{-\sqrt{3}\phi}}{H} p\delta p. \right. \tag{4.30}
\]
This term can be conveniently written as
\[
\delta B_\theta = -\beta\Phi_m\delta Q_m, \tag{4.31}
\]
where it can be identified the magnetic potential
\[
\Phi_m = -\sqrt{\frac{2}{\beta_2 \gamma_1}} \left( 1 - \beta_2 - \frac{1 - \beta_2 f_0(r_+)}{H_2(r_+)} \right), \tag{4.32}
\]
and also the value of variation of the magnetic charge
\[
\delta Q_m = 4\pi \delta \left( \frac{2\sqrt{2} \mu \sqrt{\beta_2 \gamma_1}}{\gamma_2} \right), \tag{4.33}
\]
In consequence, the variation of the boundary term \(B\) is given by
\[
\delta B = \delta S - \beta \delta M - \beta \Phi_e \delta Q_e - \beta \Phi_m \delta Q_m. \tag{4.34}
\]
Note that once this term is integrated, the value of \(B\) corresponds to the Euclidean Hamiltonian action \(I^E\) evaluated on stationary configurations and on the constraints surface. In the grand canonical ensemble \(I^E\) is related to the Gibbs free energy as \(I^E = -\beta G\). It is also worth to point out that since the first law of black hole thermodynamics,
\[
\delta M = T \delta S - \Phi_e \delta Q_e - \Phi_m \delta Q_m, \tag{4.35}
\]
is a consequence that the Euclidean action to attain an extremum, \eqref{4.35} is identically satisfied independently of the boundary conditions on the mass. This is because \eqref{4.35} is a relation that only involves the variation of the conserved charges. This can be explicitly proved by introducing the computed value for the charge variations \eqref{4.25}, \eqref{4.26}, \eqref{4.33} and the chemical potentials obtained by using regularity conditions \eqref{4.28} into \eqref{4.35}.

Once the mass is integrated using arbitrary boundary conditions (see Section 3), it is possible to find the value of the Gibbs free energy which is equivalent Euclidean Hamiltonian action evaluated on-shell,
\[
I^E = S - \beta M - \beta \Phi_e Q_e - \beta \Phi_m Q_m. \tag{4.36}
\]
Recalling that we have chosen the grand canonical ensemble and taking the Euclidean action as our thermodynamic potential, the values of the extensive quantities; mass, electric charge, magnetic charge and the entropy are obtained through the following thermodynamic relations
\[
M = -\left( \frac{\partial I^E}{\partial \beta} \right)_{\Phi_e, \Phi_m} + \frac{\Phi_e}{\beta} \left( \frac{\partial I^E}{\partial \Phi_e} \right)_{\beta, \Phi_m} + \frac{\Phi_m}{\beta} \left( \frac{\partial I^E}{\partial \Phi_m} \right)_{\beta, \Phi_e}, \tag{4.37}
\]
\[
Q_e = -\frac{1}{\beta} \left( \frac{\partial I^E}{\partial \Phi_e} \right)_{\beta, \Phi_m}, \tag{4.38}
\]
\[
Q_m = -\frac{1}{\beta} \left( \frac{\partial I^E}{\partial \Phi_m} \right)_{\beta, \Phi_e}, \tag{4.39}
\]
\[
S = I_E - \beta \left( \frac{\partial I^E}{\partial \beta} \right)_{\Phi_e, \Phi_m}. \tag{4.40}
\]
The values of the charges and the entropy computed above coincide with \eqref{3.34}, \eqref{3.28}, \eqref{4.33} and \eqref{4.29}, respectively.
5 Concluding remarks

We have carried out the thermodynamic analysis of a new class of AdS$_4$ dyonic dilatonic black hole recently proposed in [1], which is a solution of the bosonic sector of a Kaluza-Klein dimensional reduction of eleven-dimensional supergravity. The Noether conserved charges were computed using the Regge-Teitelboim Hamiltonian approach. These correspond to the mass, which acquires contributions from the scalar field and the electric charge. It was also shown that the mass acquires non-integrable contributions from the scalar field, then it was necessary to impose integrability conditions for having a definite mass. These conditions are generically solved by introducing boundary conditions that arbitrarily relate the leading and subleading terms of the scalar field fall-off. A possible physical criterium to establish the arbitrary functions coming from the integrability condition is to preserve the AdS symmetry of the scalar field behavior at infinity as was established in [9], [4].

The Hamiltonian Euclidean action was computed by demanding the action to attain an extremum, where its value was given by the corresponding radial boundary term plus an additional polar angle boundary term, since the presence of a magnetic monopole. The computation was performed in the grand canonical ensemble. From the thermodynamic analysis, the conserved charges were identified. Noether charges, mass and electric charge, were obtained from the radial boundary term at infinity unlike the magnetic charge, that comes from the additional polar angle boundary term. Remarkably, the magnetic potential appeared already determined in the variation of such boundary term, unlike the chemical potentials associated to Noether charges which are Lagrange multipliers of the system at infinity. They are obtained through regularity conditions at the horizon. Considering the above, it is possible to verify that the first law of black hole thermodynamics is identically satisfied. This is a consequence of having a well-defined and finite Hamiltonian action principle.

A different way to deal with the thermodynamics of dyonic black holes is to consider a manifestly duality invariant action that involves two U(1) symmetries, producing the appearance of electric and magnetic Gauss constraints [18]. The dyonic Reissner-Nordström black hole is solution of system proposed in [18], however in this case the magnetic and electric field appear as Coulombian potentials, hence the solution is devoid of stringy singularities. In this case, all the conserved charges appearing in the first law come from symmetries of the action.

It would be interesting to analyze the existence of phase transitions between this dyonic dilatonic black hole solution and the dyonic Reissner-Nordström black hole, and studying the chance that bellow a critical temperature the dyonic Reissner-Nordström black hole undergoes spontaneously to a dressing state with a dilaton scalar field. This kind of results have been reproduced, for instance, in the case of four-dimensional topological black holes dressed with a scalar field in [19].
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