Recurrent Neural Networks as Weighted Language Recognizers

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Abstract

We investigate computational complexity of questions of various problems for simple recurrent neural networks (RNNs) as formal models for recognizing weighted languages. We focus on the single-layer, ReLU-activation, rational-weight RNNs with softmax, which are commonly used in natural language processing applications. We show that most problems for such RNNs are undecidable, including consistency, equivalence, minimization, and finding the highest-weighted string. However, for consistent RNNs the last problem becomes decidable, although the solution can be exponentially long. If additionally the string is limited to polynomial length, the problem becomes NP-complete and APX-hard. In summary, this shows that approximations and heuristic algorithms are necessary in practical applications of those RNNs. We also consider RNNs as unweighted language recognizers and situate RNNs between Turing Machines and Random-Access Machines regarding their real-time recognition powers.

1 Introduction

Recurrent neural networks (RNN) are an attractive apparatus for probabilistic language modeling (Mikolov and Zweig, 2012). Recent experiments show that RNNs significantly outperform other methods in assigning high probability to held-out English text [Jozefowicz et al., 2016].

Roughly speaking, an RNN works as follows. At each time step, it consumes one input token, updates its hidden state vector, and predicts the next token by generating a probability distribution over all permissible tokens. The probability of an input string is simply obtained as the product of the predictions of the tokens constituting the string followed by a terminating token. In this manner, each RNN defines a weighted language, i.e. a total function from strings to weights. [Siegelmann and Sontag (1995)] showed that single-layer rational-weight RNNs with saturated linear activation can compute any computable function. To this end, a specific architecture with 886 hidden units can simulate any Turing machine in real-time (i.e., simulates each Turing machine step in a single time step). However, their RNN encodes the whole input in its internal state, performs the actual computation of the Turing machine when reading the terminating token, and then encodes the output (provided an output is produced) in a particular hidden unit. In this way, their RNN allows “thinking” time after the input has been encoded (this is equivalent to the computation time of the Turing machine).

We consider a different variant of RNNs that is commonly used in natural language processing applications. It uses ReLU activations, consumes an input token at each time step, and produces softmax predictions for the next token. It thus immediately halts after reading the last input token and the weight assigned to the input is simply the product of the input token predictions in each step.

Other formal models that are currently used to implement probabilistic language models such as probabilistic finite-state automata (PFSA) and probabilistic context-free grammars (PCFG) are by now well-understood. A fair share of their utility di-
rectly derives from their nice algorithmic properties. For example, the weighted languages computed by weighted finite-state automata are closed under intersection, union, difference, complementation, and \( \epsilon \)-removal (Droste et al., 2013). Moreover, toolkits like OpenFST (Allauzen et al., 2007) and Carmel\(^1\) implement efficient algorithms on automata like minimization, intersection, finding the highest-weighted path and the highest-weighted string for WFSA.

RNN practitioners naturally face many of these same problems. For example, an RNN-based machine translation systems should extract the highest-weighted output string (i.e., the most likely translation) generated by an RNN, (Sutskever et al., 2014; Bahdanau et al., 2014). Currently this task is solved by approximation techniques like heuristic greedy and beam searches. To facilitate the deployment of large RNNs onto limited memory devices (like mobile phones) minimization techniques would be beneficial. Again currently only heuristic approaches like knowledge distillation (Kim and Rush, 2016) are available. It is even unclear whether we can determine if the computed weighted language is consistent; i.e., if it is a probability distribution on the set of all strings. Without a determination of the overall probability mass assigned to all finite strings, a fair comparison of language models with regard to perplexity is simply impossible.

The goal of this paper is to study some of the mentioned properties for the variant of RNNs already introduced. More specifically, we ask and answer the following questions:

1. Consistency: Are all RNNs consistent? Is consistency of an RNN decidable?
2. Highest-weighted string: Can we (efficiently) find the highest-weighted string for a given RNN?
3. Equivalence: Can we decide whether two given RNNs compute the same weighted language?
4. Minimization: Can we minimize the number of neurons for a given RNN?

Another question of great interest to natural language researchers is: What is the class of languages that RNNs can recognize? In the last section we define RNNs as unweighted language recognizers so as to compare them with Turing Machines (TMs) and Random-Access Machines with unit cost (RAMs) regarding their real-time recognition powers. This is a first step towards understanding the theoretical limits of RNNs’ expressiveness.

2 Definitions and Notations

Before we introduce our RNN model formally, we recall some basic notions and notation. An alphabet \( \Sigma \) is a finite set of symbols and we write \( |\Sigma| \) for the number of symbols in \( \Sigma \). A string \( s \) over alphabet \( \Sigma \) is a finite sequence of zero or more symbols drawn from \( \Sigma \) and we write \( \Sigma^* \) for the set of all strings over \( \Sigma \), of which \( \epsilon \) is the empty string. A weighted language \( L \) is a total function from strings to arbitrary rational-valued weights: \( L : \Sigma^* \rightarrow \mathbb{R} \). For example, \( L(a^n) = \frac{1}{2^n} \), for \( n \geq 0 \) is a weighted language.

We restrict the weights in our RNNs to rational weights \( \mathbb{Q} \). A single-layer RNN \( R \) is a 9-tuple \( \langle \Sigma, B, N, h_0, U, W, W_b, E, E_b \rangle \), where:

- \( \Sigma \) is an alphabet that does not include the reserved string boundary symbol \( \$ \).
- \( B \) is an alphabetization which is a one-to-one mapping from symbols to integers \( B : \Sigma \cup \$ \rightarrow \{1, 2, \ldots, |\Sigma| + 1\} \), with \( \$ \) mapped onto \(|\Sigma| + 1\).
- Suppose \( c \in \Sigma \cup \$, \( p \) is a vector of length \( |\Sigma| + 1 \), then \( p[B(c)] \) represents the \( B(c) \)-th dimension of \( p \).
- \( N \) is the finite number of neurons (or states),
- \( h_0 \) is an initial activation of the \( N \) neurons,
- \( U \) is a rational-valued input matrix of size \( N \times (|\Sigma| + 1) \).
- \( W \) is a rational-valued transition matrix of size \( N \times N \).
- \( W_b \) is a rational-valued bias vector of size \( N \).
- \( E \) is a rational-valued output matrix of size \( (|\Sigma| + 1) \times N \).
- \( E_b \) is a rational-valued bias vector of size \(|\Sigma| + 1\).

For a string \( s = s_1s_2\ldots s_n \in \Sigma^n \), \( n \geq 0 \), let \( s' \) be the string \( \$s\$s \), indices of which range between 1 and \( n + 2 \). Our RNNs use ReLUs (Rectified Linear Units), so for every vector \( v \), \( \sigma(v) \) (the ReLU function) is applied element-wise such that for each element \( x \) in \( v \), \( \sigma(x) = \max\{0, x\} \). Given an RNN \( R \) as above, we define the following vectors for every \( 0 \leq t \leq n \):

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\(^1\)https://www.isi.edu/licensed-sw/carmel/
the hidden state \( h_t = \sigma(W \cdot h_{t-1} + U[s_t] + W_b) \) where \( U[s_t] \) denotes the \( B[s_t] \)-th column of matrix \( U \) using the standard matrix product and component-wise vector addition,

- the next-token prediction, \( y_t = E \cdot h_t + E_b \),
- the normalized next-token probabilities \( p_t = \text{softmax}(y_t) \) where for any vector \( x \) of length \( m \), \( \text{softmax}(x)_i = \frac{e^{x_i}}{\sum_{j=1}^{m} e^{x_j}} \).

Then the RNN \( R \) assigns the weight \( P(R, s) = \prod_{t=1}^{n+1} p_t[B(s(t+1))] \) to the input \( s \). An RNN clearly assigns a weight in the interval \((0, 1)\) to each input string. Similar to a PFSA or PCFG each RNN is a compact, finite way to specify a weighted language as an infinite set of \(<\text{string}, \text{weight}>\) pairs. In addition, the softmax-operation enforces that the probability 0 is impossible as a string weight.

The hidden state \((h_t)\) of an RNN can be used as scratch space for computation. It can count input symbols via:

\[
h_t = \sigma(1 + h_{t-1}).
\]

Similarly, for an arbitrary alphabet \( \Sigma = \{a_1, \ldots, a_m\} \) we can utilize the method of Siegelmann and Sontag (1995) to encode the complete input string in base \( m + 1 \) using:

\[
h_t = \sigma(B[s_t] + (|\Sigma| + 1)h_{t-1})
\]

for every \( 1 \leq t \leq n \). In principle, we can thus store the entire input string (of unbounded length) in the hidden state vector, but our RNN model outputs weights at each step and terminates immediately once the final delimiter \( $ \) is read. It must assign a probability to a string incrementally as \( P(s_1) \cdot P(s_2|s_1) \ldots P($|s_1 \ldots s_n) \).

Let us illustrate our notion of RNNs on some examples. All example RNNs use the alphabet \( \Sigma = \{a\} \) and are illustrated and formally specified in Figure 1. The first column shows an RNN \( R_1 \) that assigns \( P(R_1, a) = \frac{1}{2}, P(R_1, a) = \frac{1}{4}, P(R_1, aa) = \frac{1}{8} \), and so forth. The \( E \) matrix ensures equal output probabilities for \( a \) and \( $ \) at every time step. Similarly, the second column shows an RNN \( R_2 \) that assigns \( P(R_2, e) \approx 0 \) and \( P(R_2, a^n) = \frac{1}{2^n} \). In the beginning, \( E_b \) heavily biases the next symbol prediction toward \( a \). This bias is countered starting at \( t = 2 \), via \( h_t = \sigma([t, t-1, t-2]) \) which counts time steps. The third RNN \( R_3 \) uses a similar counting mechanism with \( h_t = \sigma([t - 100, t - 101, t]) \). The first two components are RELU-thresholded to zero until \( t > 101 \), at which point they overwhelm the \( E_b \) bias toward \( a \), turning all future predictions to \( $ \).

- RNNs computing languages \( L_1 = \{a^n b^n : n \geq 0, P \approx 1/2^{n+1} \} \) and \( L_2 = \{a^n b^n e^n : n \geq 0, P \approx 1/2^{n+1} \} \) are given in Appendix A.

3 Consistency

We first analyze the consistency of RNNs, which is the question of whether the weighted language recognized by an RNN is a probability distribution\(^3\). An RNN \( R \) is consistent provided that \( \sum_{s \in \Sigma^*} P(R, s) = 1 \). We first show that there are non-consistent RNNs, which together with our examples shows that consistency is a nontrivial property of RNNs.

**Lemma 3.1.** Let \( S \) be the sequence \( \{S_i = \Sigma^{i+1} \}_{i=1}^{\infty} \). The following limit exists and is lower-bounded by

\[
\lim_{n \to \infty} \prod_{i=1}^{n} S_i > \frac{1}{2}.
\]

The proof is given in Appendix B.

**Theorem 3.2.** Not all RNNs are consistent.

**Proof sketch.** We construct an RNN whose termination probability shrinks rapidly as it reads each input token. This RNN wastes probability mass on the non-terminating sequence.

1. For comparison, all PFSAs are consistent, provided no transitions exit final states. Not all PCFGs are consistent; necessary and sufficient conditions for a PCFG to be consistent are given by Booth and Thompson (1973). PCFGs obtained by training on a finite corpus using popular methods (such as expectation-maximization) are guaranteed to be consistent (Nederhof and Satta, 2006).
\[
P(a^n) = \frac{1}{n+1} (n \geq 0) \quad P(a^n) \approx \frac{1}{n} (n \geq 1) \quad P(a^{100}) \approx 1
\]

\[
P(\epsilon) \approx 0 \quad P(a^n) \approx 0 (n \neq 100)
\]

| $\Sigma$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
|-------|-------|-------|-------|
| $B$   | $\{a \rightarrow 1, \$ \rightarrow 2\}$ | $\{a \rightarrow 1, \$ \rightarrow 2\}$ | $\{a \rightarrow 1, \$ \rightarrow 2\}$ |
| $N$   | 1     | 3     | 3     |
| $h_0$ | 0     | 0     | 0     |
| $U$   | $[0 \ 0]$ | $[1 \ 1]$ | $[0 \ 0]$ |
| $W$   | 0     | $[1 \ 0 \ 0]$ | $[0 \ 0 \ 1]$ |
| $W_b$ | 1     | 0     | $-99$ |
| $E$   | $1 \ 1$ | $0 \ \infty + 1 \ -1$ | $-\infty \ \infty \ 0$ |
| $E_b$ | $0 \ 0$ | $-\infty$ | $-\infty$ |

$y_t = [h_t + \ln 2, 0]$

\[
p_t = \begin{bmatrix} 2t+1 + 1 \ 2t+1 + 1 \end{bmatrix}
\]

$E_b$ is as given in Lemma [3.1]. This language assigns the following string weights: $P(\epsilon) = \frac{1}{5}$, $P(a) = \frac{4}{5} \times \frac{1}{5}$, $P(aa) = \frac{4}{5} \times \frac{8}{5} \times \frac{1}{17}$, etc. Since the infinite product $\prod_{i=1}^{\infty} S_t$ converges to a constant greater than $\frac{1}{2}$, the weights assigned to all finite strings sum to less than $\frac{1}{2}$, so this RNN is inconsistent. The following RNN generates this language:

\[
h_0 = 0 \quad U = [0, 0]
\]

\[
W = 1 \quad W_b = \ln 2
\]

\[
E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad E_b = \begin{bmatrix} \ln 2 \\ 0 \end{bmatrix}
\]

by implementing the following updates:

\[
h_t = \sigma(h_{t-1} + \ln 2)
\]

Next, we look at the consistency problem for RNN:

Consistency for RNN (CONSISTENCY)

Given: RNN $R$

Return: Yes if $R$ is consistent, otherwise no.

We first prove the following lemma, a corollary of Theorem 2 in [Siegelmann and Sontag, 1995]:

**Lemma 3.3.** Let $M$ be a $p$-stack Turing machine computing $\phi : \{0,1\}^+ \rightarrow \{0,1\}^+$ in time $T$. There exists an RNN of our definition (with ReLU activation) with alphabet $\{a\}$, alphabetization $\{a \rightarrow 1, \$ \rightarrow 2\}$, and 4 pre-designated neurons $H^1, G^1, H^2, G^2$, so that property below hold:

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From now on, we will save space by giving update equations rather than full RNN descriptions.
Suppose we input $aaa$... (input $a$ for all $t \geq 2$). For each $\omega \in \{0, 1\}^+$, if $\phi(\omega)$ is undefined, $H_1^n - G_1^n = 0$ for all $t \geq 0$. If instead $\phi(\omega)$ is defined and computed in time $T$, then

$$\forall t : 0 \leq t \leq T - 1 : H_1^n - G_1^n = 0$$

where $\delta[s]$ is the unique encoding of any finite binary string to a rational number between 0 and 1: $\phi : \{0, 1\}^+ \rightarrow [0, 1]$, as defined in equation (5) in [Siegelmann and Sontag, 1995].

Proof: In proving this lemma we disregard the output part of our RNN (i.e., vector $y$ is never used); the “output” of the Turing machine is encoded in the difference between 2 hidden units, $H_1^n - G_1^n$.

For any given $\phi$ and $\omega$, construct an RNN $R'$ of the type defined in Theorem 2 of [Siegelmann and Sontag, 1995]. Suppose $R'$ has $N'$ neurons: $h_1^n, h_2^n, \ldots, h_{N'}^n$. $R'$ is a single-layer rational-weight RNN with:

1. Initialization: $h_1^n_0 = \delta[\omega]$, $h_2^n_0 = h_3^n = \cdots = h_{N'}^n_0 = 0$.
2. Saturated-linear activation

$$\sigma'(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

3. Output neurons: If $\phi(\omega)$ is undefined, $h_1^n = 0$ for all $t \geq 0$. If $\phi(\omega)$ is defined and computed in time $T$, then

$$\forall t : 0 \leq t \leq T - 1 : h_2^n = 0$$

$$h_3^n = \delta[\phi(\omega)].$$

We now construct an RNN of our definition $R$ with $N = 2N'$ neurons: $h_1^n, g_1^n, h_2^n, g_2^n, \ldots, h_{N'}^n, g_{N'}^n$.

1. Initialization: $h_1^n_0 = \delta[\omega]$; all other neurons initialize to 0.
2. Suppose in $R'$:

$$\forall j \in [N'], h_2^n_0 = \sigma'(\sum_{i=1}^{N'} w_{ij}^{h_1} h_{t-1}^{h_1} + W_b[j]).$$

In $R$, for all $j \in [N']$, define

$$h_2^n = \sigma'(\sum_{i=1}^{N'} w_{ij}^{h_1} (h_{t-1}^{h_1} - g_1^{t-1}) + W_b[j])$$

$$g_2^n = \sigma'(\sum_{i=1}^{N'} w_{ij}^{h_1} (h_{t-1}^{h_1} - g_1^{t-1}) + W_b[j] - 1)$$

3. Output neurons: Designate $h_2^n, g_2^n, h_3^n, g_3^n$ as $H_1^n, G_1^n, H_2^n, G_2^n$, respectively. Now we only have to show that for all $t \geq 0$,

$$H_1^n - G_1^n = h_2^n, H_2^n - G_2^n = h_3^n.$$

In fact, we will prove through induction on $t$ that:

$$\forall t \geq 0, \forall j \in [N']: h_2^n - g_2^n = h_3^n.$$

When $t = 0$ we clearly have $h_2^n_0 - g_2^n_0 = h_3^n_0$. When $t \geq 1$, according to our definition,

$$h_2^n - g_2^n = \sigma'(\sum_{i=1}^{N'} w_{ij}^{h_1^n} h_{t-1}^{h_1^n} + W_b[j])$$

$$- \sigma'(\sum_{i=1}^{N'} w_{ij}^{h_1^n} h_{t-1}^{h_1^n} + W_b[j] - 1)$$

Notice that $\sigma'(x) = \sigma(x) - \sigma(x - 1)$ for any rational number $x$, so $h_2^n - g_2^n = h_3^n$.

Siegelmann and Sontag (1995) constructed an RNN which simulates any multi-stack Turing machine. The above lemma shows how our RNNs relate to theirs and means our RNN can simulate any multi-stack Turing machine in real time; it can be used to show that RNN with ReLU activation is Turing-complete. Previously only RNNs with sigmoidal activations were proven to be Turing-complete [Kilian and Siegelmann, 1993].

Theorem 3.4. CONSISTENCY is undecidable.

Proof sketch. We reduce from the Halting Problem to CONSISTENCY: given any Turing machine, we construct an RNN that simulates the Turing machine while producing the same output as the inconsistent RNN in Theorem 3.2 until the simulation halts, at which point our RNN will turn consistent. Therefore our RNN is consistent if and only if the simulation halts.

Proof. We prove by contradiction. Suppose Turing Machine $M$ takes any RNN as its input and decides...
whether the RNN is consistent. For any Turing Machine $M'$ and input $x$, construct an RNN $R$ as described in Lemma 3.3 that simulates $M'$ on $x$. Recall that this RNN has a pair of data output neurons $H_t^1, G_t^1$ and a pair of validation output neuron $H_t^1, G_t^1$. If $M'$ never halts, $H_t^1 - G_t^1 = 0$ at all times. If $M'$ halts, $H_t^1 - G_t^1 = 0$ at all times except for one moment $t = T$ when $H_t^1 - G_t^1 = 1$ and $H_t^2 - G_t^2$ encodes the final (halting) output of $M'$. Add 3 additional hidden units $h_t^1, h_t^II$ and $h_t^III$ to $R$ as following:

$$h_t^I = h_t^II = h_t^III = 0$$

$$h_t^I = \sigma(h_{t-1}^I + H_t^I - G_t^I)$$

$$h_t^II = \sigma(h_{t-1}^II + \ln 2)$$

$$h_t^III = \sigma(h_{t-1}^III + 5 \ln 2 h_{t-1}^I)$$

$$y_t = [h_t^II + \ln 2, h_t^III]$$

If $M'$ never halts, $h_t^I, h_t^III$ remains 0, we have the same output as the RNN in Theorem 3.2:

$$\forall t \geq 1, p_t = \begin{bmatrix} 2^{t+1} & 1 \\ 2^{t+1} + 1 & 2^{t+1} + 1 \end{bmatrix}.$$ 

So our RNN $R$ is inconsistent.

If $M'$ halts, suppose $H_T - G_T = 1$, then $h_{T+1}^I = 1, h_{T+1}^III = 5 \ln 2(t - T - 1)$ for all $t \geq T + 1$. So for all $1 \leq t \leq T + 1$,

$$y_t = [(t + 1) \ln 2, 0]$$

$$p_t = \begin{bmatrix} 2^{t+1} & 1 \\ 2^{t+1} + 1 & 2^{t+1} + 1 \end{bmatrix}$$

For all $t \geq T + 2$,

$$y_t = [(t + 1) \ln 2, 5(t - T - 1) \ln 2]$$

$$p_t = \begin{bmatrix} 1 \\ 1 + 2^{t-5T-6} \end{bmatrix}$$

$$= \begin{bmatrix} 2^{t-5T-6} \\ 1 + 2^{t-5T-6} \end{bmatrix}$$

In the latter case the prediction for $a$ (the first dimension of $p_t$) approaches 0 as $t$ goes to infinity, so $R$ is consistent. Therefore we can run $M$ on input $R$. If $R$ is consistent, $M'$ halts on $x$, else it does not halt. Therefore, the Halting Problem would be decidable if CONSISTENCY were decidable, so CONSISTENCY is undecidable.

As mentioned in footnote $3$ PCFGs obtained after training on a finite corpus using the most popular methods are guaranteed to be consistent. At least for 2-layer RNNs with ReLU activation, this does not hold.

**Theorem 3.5.** A two-layer RNN trained to local optimum using Back-propagation-through-time (BPTT) on a finite corpus is not necessarily consistent.

**Proof.** The first layer of the RNN has the following behavior,

$$h_0^I = 0$$

$$h_t^I = \sigma(h_{t-1}^I + 1)$$

The second layer takes $h_t^I$ as input at time $t$ and:

$$y_t = (h_t^I \ln 2 - 2 \ln 2)$$

$$p_t = \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{2^{t-2}}, \frac{1}{2^{t-2}} \frac{1}{2^{t-2}} \end{bmatrix}$$

Let the training data be a single string ‘a’. Then the objective we wish to maximize is simply $P(a)$. The derivative of this objective with respect to each parameter is 0. Therefore, applying gradient descent updates does not change any of the parameters and we have converged to an inconsistent RNN.

It is an open question whether there is a single-layer RNN that also exhibits this behavior.

**4 Highest-Weighted String**

Finding the highest-weighted string in RNN as a language model is of particular interest, since it is the same as finding the most-likely translation for a decoder RNN in machine translation.

For deterministic PFSAs, deterministic PCFGs, and RNNs, only one path or derivation exists for any string, so finding the highest weighted string is the same task as finding the most probable path or derivation; for nondeterministic PFSAs and PCFGs, finding the highest weighted string is harder, since the weight of a string is the sum of the probabilities all possible paths or derivations. A comparison of the hardness of finding the most probable derivation and the highest-weighted string for various models
is summarized in Table 1 (results of our paper are marked with *).

In this section we present various results concerning the hardness of finding one string most highly weighted by an RNN. We also summarize some algorithms available. First, we list formulations of the various optimization and decision problems we consider.

1. Optimization Problems with Unbounded Solution Length

Highest Weighted String for General RNN (MaxPS)

Given: RNN \( R \)
Return: \( \arg\max_{s \in \Sigma^*} P(R, s) \)

Highest Weighted String for Consistent RNN (MaxPS-CST)

Given: Consistent RNN \( R \)
Return: \( \arg\max_{s \in \Sigma^*} P(R, s) \)

2. Optimization Problems Restricted to Solutions of Polynomial Length

We use \( Q : \mathbb{N}^+ \to \mathbb{N}^+ \) to represent any function in \( \{ O(n^c) : c \text{ is a constant} \} \) such that \( Q(n) \geq n, \forall n \in \mathbb{N}^+ \). So below are two families of optimization problems.

Highest Weighted String for Consistent RNN (MaxPS-CST-P)

Given: Consistent RNN \( R \)
Return: \( \arg\max_{s \in \Sigma^*, |s| \leq Q(|R|)} P(R, s) \)

3. Decision Problems with Unbounded Solution Length

Highest Weighted String for General RNN (MPS)

Given: RNN \( R \), Probability \( p \in (0, 1) \)
Return: Is the strict cut-point set \( \{ s | P(R, s) > p \} \) empty?

Highest Weighted String for Consistent RNN (MPS-CST)

Given: Consistent RNN \( R \), Probability \( p \in (0, 1) \)
Return: Is there a string \( s \in \Sigma^* \) with \( P(R, s) > p \)?

4. Decision Problems Restricted to Solutions of Polynomial Length

Restricted Highest Weighted String for Consistent RNN (RMPS-CST-P)

Given: Consistent RNN \( R \), d \( \in \mathbb{N} \), \( d \leq Q(|R|) \)

Return: Is there a string \( s \in \Sigma^d \) with \( P(R, s) > p \)?

4.1 Undecidability of highest-weighted string for general RNN

**Theorem 4.1.** MPS is undecidable.

*Proof sketch.* We still reduce from the Halting Problem: given any Turing machine, we construct an RNN that simulates the Turing machine while producing the same output as the inconsistent RNN in Theorem 3.2, so the weight of longer strings would monotonically decrease, until the simulation

| General RNN | Undecidable* |
|-------------|--------------|
| Consistent RNN | NP-complete |
| Deterministic PFSA | \( \leq O(|Q|^2) \)|
| Nondeterministic PFSA | NP-complete |
| Deterministic PCFG | \( \leq O(|Q|^2) \)|
| Nondeterministic PCFG | NP-complete |

**Table 1:** Comparison of the hardness of finding the most probable derivation (Best-path) and the highest-weighted string (Best-string) for various models
halts, at which point our RNN will rapidly increase termination probability. Our design ensures that the weight of strings longer than halting time would eventually surpass that of \( \epsilon \), before dropping to close to 0. Therefore the highest-weighted string would be \( \epsilon \) if and only if the simulation does not halt.

Proof. We prove by contradiction. Suppose Turing Machine \( M \) computes MPS. For any Turing Machine \( M' \) and input \( x \) (assume w.l.o.g that the starting state is not the halting state), construct the same RNN \( R \) as in Theorem 3.4. If \( M' \) never halts, the weight of string \( a^n \) monotonically decreases as \( n \) increases. If \( M' \) halts at some time \( T \), the weight would suddenly shoot up (see graph below for an example where \( T = 24 \)).

![Image](https://via.placeholder.com/150)

We now prove that if the simulation halts, some longer string has weight higher than \( \epsilon \), therefore \( \epsilon \) cannot be the highest-weighted string. Formally:

If \( H_T^1 - G_T^1 = 1 \), \( T = 4k + q (k + q \geq 1, q \in \{0, 1, 2, 3\}) \), then \( P(R, a^{5k+q+1}) > P(R, \epsilon) = \frac{1}{5} \).

We will only prove the case when the halting time is a multiple of 4 (\( q = 0 \)):

\[
P(R, a^{5k+1}) = \prod_{i=1}^{5k+1} \frac{2^{i+1}}{2^{i+1} + 1} \prod_{i=4k+2}^{5k+1} \frac{2-4i+20k+6}{2-4i+20k+6 + 1} > \lim_{n \to \infty} \prod_{i=1}^{n} S_i > \frac{1}{2}.
\]

Therefore

\[
P(R, a^{5k+1}) > \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{5} = P(R, \epsilon).
\]

The proof for other cases where \( q \in \{1, 2, 3\} \) is similar.

Therefore we can run \( M \) on input \( \langle R, \frac{1}{5} \rangle \). If \( M \) accepts this input, \( M' \) halts on \( x \), else it does not halt. Therefore the Halting Problem would be decidable if MPS is decidable. Therefore MPS is undecidable.

\[\square\]

4.2 Highest-weighted string for consistent RNN is decidable, but the solution can be exponentially long

Theorem 4.2. MPS-CST is decidable.

Proof. Since the set \( \Sigma^* \) is countable, we give each string \( s \in \Sigma^* \) a unique positive-integer index. We examine the strings in the order of \( i = 1, 2, 3, \ldots \). Store the sum of the weights of strings \( s_1, s_2, \ldots, s_i \) in variable \( A_i \). We stop when \( A_i > 1 - p \), and output Yes iff. the highest weight examined so far is greater than \( p \).

Since \( R \) is consistent, \( \lim_{i=1}^{\infty} A_i = 1 \). So \( \forall \delta > 0, \exists N_0 \text{ such that } \forall N \geq N_0, A_N > 1 - \delta \). In particular, we pick \( \delta = p \). So this algorithm is guaranteed to terminate. Therefore MPS-CST is decidable. \[\square\]

Theorem 4.3. The highest-weighted string for a consistent RNN can be exponentially long in not only the length of the RNN input encoded in binary, but also the sum of the absolute values of all the parameters of the RNN.

Proof. Consider the following family of RNNs indexed by a positive integer \( K \) (inspired by the construction for exponential-length most probable string of PFSA (de la Higuera and Oncina, 2013a)): Let hidden unit \( l_i^t \) be 1 if and only if \( t \) is a multiple of \( p_i \), and 0 otherwise. We need \( 3K - 2 \) neurons total to produce this behavior:

1. The design for \( p_1 = 2 \) is trivial:

\[
l_0^1 = 1, \quad l_i^1 = \sigma(1 - l_{i-1}^0)
\]
2. For any \( i \in \{2, 3, \ldots, K\} \):
\[
x_i^0 = z_i^0 = t_i^0 = 0
\]
\[
x_i^t = \sigma((z_{i-1} + (4 - p_i))
\]
\[
z_i^t = \sigma(2 - p_i)z_{i-1}^t + z_{i-1}^t + 1
\]
\[
l_i^t = \sigma(x_i^t - 1)
\]

Add neurons \( h^{II}, h^{III} \) as defined in Theorem 3.4

Define \( y \) in the same way as in Theorem 3.4

Clearly \( q_t = 1 \) only when \( t = \prod_{i=1}^{K} p_i + 1; \) \( q_t = 0 \) for all other \( t \).

Let \( L_1, L_2, L_3 \) denote the length of the highest-weighted string, the length of the RNN input encoded in binary, and the total sum of the absolute values of the parameters of the RNN, respectively. Then \( L_1 \geq \prod_{i=1}^{K} p_i \). On the other hand, the RNN has \( 3K - 2 + 4 = 3K + 2 \) neurons, so \( L_2 \) is linear in \( \sum_{i=1}^{K} \log p_i \); \( L_3 \) is linear in \( \sum_{i=1}^{K} p_i \approx \frac{1}{2} K^2 \log K \) (Bach and Shallit, 1996). So \( L_1 \) is exponential in both \( L_2 \) and \( L_3 \).

\( \text{MPS-CST-P} \) and \( \text{RMPS-CST-P} \) are variations of \( \text{MPS-CST} \) restricted to solutions of polynomial length. In \( \text{RMPS-CST-P} \), we look for solutions of a certain fixed length.

4.3 Limiting to solutions of polynomial length, finding the highest-weighted string for a consistent RNN is NP-complete and APX-hard.

Theorem 4.4. \( \text{MPS-CST-P} \) and \( \text{RMPS-CST-P} \) are NP-Complete and APX-Hard.

Proof. Clearly, given an input string of polynomial length, we can run the RNN and verify whether its weight \( \geq p \) in polynomial time. Therefore \( \text{MPS-CST-P} \in \text{NP} \) and \( \text{RMPS-CST-P} \in \text{NP} \).

We now reduce from 0-1 Integer Linear Program Feasibility Problem to \( \text{MPS-CST-P} \) and \( \text{RMPS-CST-P} \).

0-1 Integer Linear Program Feasibility
Given: \( n \) variables \( x_1, x_2, \ldots, x_n \) which can only take values in \( \{0, 1\} \), and \( k \) constraints \( (k \geq 1) \):
\[
\sum_{j=1}^{n} A_{ij} x_j - B_i \leq 0, \forall i \in [k].
\]
\( A \in \mathbb{Z}^{k \times n}, B \in \mathbb{Z}^k \).

Return: Yes iff. there is a feasible solution \( x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n \) that satisfies all \( k \) constraints.

Suppose we are given an instance of the above problem. Construct an instance of \( \text{MPS-CST-P} \) with input \( (R, p) \) and an instance of \( \text{RMPS-CST-P} \) with input \( (R, d, p) \), where:
1. \( R \) is an RNN as follows:
\[
\Sigma = \{0, 1\}
\]
\[
\forall j \in [n], 0 \leq t \leq j - 1 : h_i^t = 0
\]
\[
\forall j \in [n], t \geq j : h_i^t = x_j
\]
\[
\forall i \in [k], 0 \leq t \leq n : g_i^t = l_i^t = 0
\]
\[
\forall i \in [k], t \geq n + 1 : g_i^t = \sigma(1 - \sum_{j=1}^{n} A_{ij} h_{i-1}^j + B_i)
\]
\[
\forall i \in [k], t \geq n + 1 : l_i^t = \sigma(-\sum_{j=1}^{n} A_{ij} h_{i-1}^j + B_i)
\]

Let \( c = \sum_{i=1}^{k} (g_i^t - l_i^t), \) \( \delta_2 = \frac{1}{k+2} \). We pick a small enough positive constant \( \delta_1 \) so that
\[
\delta_1 < \left(\frac{1 - \delta_1}{2}\right)^n \delta_2
\]
Let
\[
\beta = \ln \frac{1}{\delta_1} - 1
\]
When \( t \neq n + 1 \), set
\[
y_t = [\beta, \beta, 0]
\]
Therefore
\[
p_t = \left[\frac{1 - \delta_1}{2}, \frac{1 - \delta_1}{2}, \delta_1\right]
\]
When \( t = n + 1 \), one can verify that we can set
\[
y_t = [\gamma_c, \gamma_c, 0]
\]
where
\[
\gamma_c = \ln \frac{1}{(c+1)\delta_2} - 1
\]
\[
\delta_1 < \left(\frac{1 - \delta_1}{2}\right)^n \delta_2
\]
so that
\[ p_t = \frac{1}{2} (1-(c+1)\delta_2), \frac{1}{2} (1-(c+1)\delta_2), (c+1)\delta_2, \]
since the range of \( c \) is a finite set of values \( \{0, 1, 2, \ldots, k\} \).

2. \( d = n \)
3. \( p = (\frac{1-\delta_1}{2})^n (k\delta_2) \)

From equation [1] we get
\[ -\ln \delta_1 > -n \ln \frac{1-\delta_1}{2} + \ln (k+2), \]
so we can pick a \( \delta_1 \) such that the length of \( \beta \) written in binary
\[ \log_2 \beta = \log_2 \left( \frac{1}{\delta_1} \right) - 2 \]
is logarithmic in \( n \) and \( k \). So the weights in matrices \( E, E_0 \) that produce \( \beta \) are polynomial in \( n \) and \( k \). Same is true for the weights that produce \( \gamma_c \). \( p \) written in binary has length
\[ -\log_2 p = -n \log_2 \frac{1-\delta_1}{2} - \log_2 (k\delta_2) \]
\[ = n - n \log_2 (1-\delta_1) - \log_2 \frac{k}{k+2} \]
which is polynomial in \( n \) and \( k \). So our construction is polynomial.

We now prove that if we can solve either the \( \langle R, p \rangle \)-instance of MPS-CST-P or the \( \langle R, d, p \rangle \)-instance of RMPS-CST-P in polynomial time, we can also solve the given instance of 0-1 Integer Linear Program Feasibility in polynomial time.

By our design, at time \( 1 \leq t \leq n \), \( R \) reads a binary string \( x \in \{0, 1\}^n \) into neurons \( h^1, h^2, \ldots, h^n \) while predicting almost half-half probability for either 0 or 1 and infinitesimal probability \( \delta_1 \) for termination. Therefore no string with length less than \( n \) has weight greater than \( p \).

At time \( t = n+1 \), since \( \sum_{j=1}^n A_{ij} h_{t-1}^j - B_i \) is an integer, \( g_t^i - l_t^i \) is the indicator for whether the \( i \)-th constraint is satisfied:
\[ g_t^i - l_t^i = \begin{cases} 0 & \sum_{j=1}^n A_{ij} h_{t-1}^j - B_i \geq 1 \\ 1 & \sum_{j=1}^n A_{ij} h_{t-1}^j - B_i \leq 0 \end{cases} \]
Therefore \( c \) is the total number of clauses satisfied by a given setting of \( x = (x_1, x_2, \ldots, x_n) \) \( 0 \leq c \leq k \). The termination probability at \( t = n+1 \) is \((c+1)\delta_2 = \frac{c+1}{k+2} \). If all \( k \) clauses are satisfied, this setting of \( x \) would have termination probability \( 1 - \delta_2 \) and therefore weight \((\frac{1-\delta_1}{2})^n (1-\delta_2) > p \). If fewer than \( k \) clauses are satisfied, \( x \) would have weight at most \( p \).

When \( t \geq n+2 \), \( R \) continues to assign almost half-half probability for either 0 or 1 and infinitesimal probability for termination. Therefore any string of length greater than \( n+1 \) has a weight smaller than \( c \). From that point on the output vector is constant, so the RNN is consistent. Notice that the weights of strings monotonically decrease with length except for at length \( n \).

Therefore our construction ensures that the only length at which a string can have weight greater than \( p \) is \( n \). Thus, if there is any string whose weight is greater than \( p \), the given instance of 0-1 Integer Linear Program is feasible; otherwise it is not.

Define the maximum number of clauses satisfied by all assignments of \( x \in \{0, 1\}^n \):
\[ c_{\text{max}} = \max_{x \in \{0, 1\}^n} c(x). \]

By our construction, when \( c_{\text{max}} \geq 1 \), the highest-weighted string will occur at length \( n \), and has weight \((\frac{1-\delta_1}{2})^n (c_{\text{max}} + 1)\delta_2 = (\frac{1-\delta_1}{2})^n c_{\text{max}} + 1 \)
which is proportional to \( c_{\text{max}} + 1 \). The empty string has the highest weight among all strings of length not equal to \( n \). Its weight is \( \delta_1 < (\frac{1-\delta_1}{2})^n \delta_2 \) which will always be less than the weight of any length-\( n \) string corresponding to a setting of variables satisfying at least 1 constraint (\( \geq (\frac{1-\delta_1}{2})^n (2\delta_2) \)).

Therefore, given any rational number \( \zeta = \frac{\eta_1}{\eta_2} > 0 \) \((\eta_1, \eta_2 \in \mathbb{N}^+) \), define \( \delta(\zeta) = \frac{\eta_1}{\eta_2+1} \). If binary string \( s \) is a \((1+\delta(\zeta))\)-approximation to MaxPS-CST-P or RMaxPS-CST-P, then reading \( s \) as a vector of \( n \) variables \( x = (x_1, x_2, \ldots, x_n) \), \( x \) would be a \((1+\zeta)\)-approximation to 0-1 Integer Linear Program Maximum Satisfiability (the optimization version of the problem finding a setting of variables to satisfy the greatest number of constraints).

Thus our reduction is PTAS-reduction and preserves approximability. Since 0-1 Integer Linear Program Feasibility is NP-complete and APX-complete, MPS-CST-P and RMPS-CST-P are NP-complete and APX-hard, meaning there is no Polynomial-Time Approximation Scheme (i.e. the
best we can hope for in polynomial time is a constant-factor approximation algorithm) unless \( P = \text{NP} \).

### 4.4 Randomized and Deterministic Algorithms for MPS-CST

Suppose the solution length is bounded by \( b \) (which we have shown can be exponential in sum of all parameters and the total input length of the RNN), and suppose calculating the forward pass of an RNN with \( N \) hidden neurons for one time step takes time \( O(N + |A|) \), we can directly convert the algorithm in [15] for most probable string in PFSA to get an \( O \left( \frac{b(N + |\Sigma|)}{p} \right) \) randomized algorithm based on sampling and an \( O \left( \frac{b(N + |\Sigma|)}{p} \right) \) deterministic algorithm based on keeping the \( \frac{1}{p} \)-best prefixes (akin to beam search [16] used most widely in practice).

We can also use their sampling method to find \( b \): \( \forall p > 0, \delta > 0 \), if we draw a sample of size at least \( \frac{1}{p} \ln \frac{1}{\delta} \), the following holds with probability at least \( 1 - \delta \): the probability of sampling a string \( x \) longer than any string we have seen in \( S \) is less than \( p \).

### 5 Equivalence

We prove that equivalence of two RNNs is undecidable. For comparison, equivalence of two deterministic WFSA can be tested in time \( O(|\Sigma|(|Q_A| + |Q_B|)^3) \), where \(|Q_A|, |Q_B|\) are the number of states of the two WFSA and \(|\Sigma|\) is the size of the alphabet [17]; equivalence of nondeterministic WFSA are undecidable [18]. The decidability of language equivalence for deterministic probabilistic push-down automata (PPDA) is still open [19], although equivalence for deterministic unweighted push-down automata (PDA) is decidable [20].

The equivalence problem is formulated as follows:

**Equivalence of two RNNs (EQ)**

Given: RNNs \( R \) and \( R' \)

Return: Yes iff \( \forall s \in \Sigma^*, P(R, s) = P(R', s) \).

**Theorem 5.1.** \( \text{EQ} \) is undecidable.

**Proof sketch.** Theorem 5.1 and Theorem 6.1 are both corollaries of Lemma 3.3. Again we reduce from the Halting Problem: given any Turing machine, we construct an RNN that simulates the Turing machine while producing the same output as a second RNN until the simulation halts, at which point our RNNs will differ. Therefore those two RNNs are different if and only if the simulation halts.

**Proof.** We prove by contradiction. Suppose Turing machine \( M \) takes two RNNs as its input and decides whether they produce the same weighted language. For any Turing Machine \( M' \) and input \( x \) (assume w.l.o.g that the starting state is not the halting state), construct the RNN \( R \) that simulates \( M' \) on input \( x \) as described in Lemma 3.3. Let \( y_t = [0, H^1_t - G^1_t] \). When \( H^1_t - G^1_t = 0, y_t = [0, 0] \) so \( p_t = \left[ \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right] \); when \( H^1_t - G^1_t = 1, y_t = [0, 1] \) so \( p_t = \left[ \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right] \).

Let \( R' \) be the simple RNN that outputs \( \{a^n : P(a^n) = \frac{1}{2^n}, n \geq 0 \} \). We can run \( M \) on input \( (R, R') \). If \( R \) and \( R' \) are not equal, \( M' \) halts on \( x \); else it does not halt. Therefore Halting Problem would be decidable if \( \text{EQ} \) is decidable. Therefore \( \text{EQ} \) is undecidable.

### 6 Minimization

We look next at minimization of RNNs. For comparison, state-minimization of a deterministic PFSA is \( O(|E| \log |Q|) \) where \(|E|\) is the number of transitions and \(|Q|\) is the number of states [21]. Minimization of a non-deterministic PFSA is \( \text{PSPACE} \)-complete [22].

We focus on minimizing the number of hidden neurons \( (N) \) in RNNs:

**Minimizing an RNN (MIN)**

Given: RNN \( R \) and non-negative integer \( n \)

Return: Yes iff \( \exists \) RNN \( R' \) with hidden unit size \( N' \leq n \) such that \( \forall s \in \Sigma^*, P(R, s) = P(R', s) \).

**Theorem 6.1.** \( \text{MIN} \) is undecidable.

**Proof.** We reduce from the Halting Problem to MIN. Suppose Turing Machine \( M \) decides \( \text{MIN} \). For any Turing Machine \( M' \) and input \( x \) (assume w.l.o.g that the starting state is not the halting state), construct the same RNN \( R \) as in Theorem 5.1. We run \( M \) on input \( (R, 0) \). Note that an RNN with no hidden unit can only output constant \( p_t \) for all \( t \). Therefore the number of hidden units in \( R \) can be minimized to 0 if and only if it always outputs
\[ p_t = \left[ \frac{1}{2}, \frac{1}{2} \right] \]. If \( M \) outputs yes, \( M' \) does not halt on \( x \), else it halts. Therefore the Halting Problem would be decidable if \( \text{MIN} \) is decidable. Therefore \( \text{MIN} \) is undecidable. \( \square \)

7 Real-Time Recognition by RNN

We now prove a theorem regarding the theoretical limits of RNNs’ expressive power. We shall define the class of unweighted languages recognized by RNN and locate this class within the “Chomsky hierarchy” of real-time computation.

Following the concept of real-time computation introduced in (Rabin, 1963), a machine is on-line if it produces an output for each input symbol it reads and produces the \( t \)-th output just before reading the \((t+1)\)st input symbol for \( t \geq 1 \); An on-line machine is real-time if it makes only a constant number of steps between two readings (Galil, 1981). RNN is a real-time machine since it consumes one input token and predicts the next token in one time step.

Previous works have investigated unweighted languages recognizable by real-time Turing Machines (TMs) and real-time Random-Access Machines with unit cost (RAMs). For example, Galil proved that given an alphabet \( \Sigma \), \( L = \{ x | x \in \Sigma^{+}, x = x^{R} \} \), the language of all palindromes over \( \Sigma \), is TM-recognizable in real-time, i.e., \( \exists \) a multi-tape, multi-head-per-tape TM such that at time \( t \), it reads \( s_t \in \Sigma \), and, within a constant number of steps, outputs 1 if \( s_1 s_2 \ldots s_t \) is a palindrome, 0 if it is not (Galil, 1978). In order to compare RNN with TM and RAM in terms of real-time recognition power, we need to define RNN as a real-time unweighted language recognizer, the acceptance of which can be manifested as, say, predicting high ending probability for the next token.

We define an unweighted language \( L \subseteq \Sigma^{*} \) as RNN-recognizable if and only if \( \exists \) RNN \( R \) with two designated neurons \( H, G \) such that \( \forall t \geq 1, R \) consumes \( s'_t \) as input (recall that \( s' \) is \( s \) with padded boundary \( \$ \) so \( \forall t > 1, s'_t = s_{t-1} \)), and

\[
H_t - G_t = \begin{cases} 
0 & s_1 s_2 \ldots s_{t-1} \notin L \\
1 & s_1 s_2 \ldots s_{t-1} \in L 
\end{cases}
\]

Given \( H_t, G_t \), it is easy to set \( E_a, E_b \) so that the RNN predicts high ending probability (say, \( p_t[B($)] > c \) for some constant \( 0 < c < 1 \)) if and only if the length-\((t - 1)\) string read so far is in \( L \). We first prove that RNN can simulate any \( p \)-stack real-time Turing Machine with one additional one-way read-only input tape and one additional one-way write-only output tape that takes only 1 step between two readings. The input alphabet is \( \Sigma (|\Sigma| \geq 2) \), the stack alphabet is \( \Gamma (|\Gamma| \geq 2) \), and the output alphabet is \{0, 1\} (0 signifies rejection and 1 signifies acceptance).

Lemma 7.1. Let \( M \) be a real-time Turing Machine as described above recognizing \( \phi : \Sigma^{*} \rightarrow \{0, 1\} \). Then \( L = \{ s : s \in \Sigma^{*}, \phi(s) = 1 \} \) is RNN-recognizable.

Proof. Reading from a one-way input tape is the same as reading from a pop-only “stack” with un-bounded storage. Writing to a one-way output tape is the same as writing to a push-only stack. Therefore, this lemma is Theorem 2(b) in (Siegelmann and Sontag, 1995) with only three modifications: changing the stack alphabet from binary to \(|\Gamma| \geq 2\), changing the method of input from popping from the input stack to feeding from \( U_{B[\delta]} \), and changing the activation function from saturated linear to ReLU. We provide a sketch below on how to modify their proof into a proof of our lemma.\[ \begin{align*}
1. & \text{For each working stack } i, \text{ for each stack symbol } k \in \Gamma, \text{ and for } j \in \{1, 2, 3, 4\}, \text{ use a noisy sub-top indicator neuron } \tau_{ijk} \text{ which is 1 if the top of stack } i \text{ is } k, \text{ and 0 if the top is not } k \text{ or the stack is empty, instead of the original } \tau_{ij}. \text{ For input stack } u \text{ and output stack } v \text{ there is no need to check for empty stacks, so for each } k \in \Gamma, \text{ use noisy top indicator neurons } \tilde{\tau}_{uk} \text{ which is 1 iff. the current token is } k \text{ and } \tilde{\tau}_{vk} \text{ which is 1 iff. the current output is } k. \\
2. & \text{Use an encoding which guarantees that any negative value of the noisy sub-top and sub-non-empty indicators has an absolute value of at least } (|\Gamma| + 1)(p + 1) - 1 \text{ times any positive value of them, instead of } 2p - 1. \\
3. & \text{Suppose Siegelmann and Sontag’s construction gives an update equation } h_t = \sigma'(W : h_{t-1} + W_b). \text{ We now eliminate neurons } \tau_{uk} \text{ for } k \in \Gamma \text{ by deleting the column in } W \text{ corresponding to the weights of } \tilde{\tau}_{uk} \text{ and moving this column to } U_{B[k]}. \text{ This step changes the input method to feeding from } U. \\
4. & \text{Replace each of the saturated linear activation function } \sigma' \text{ with } \text{ReLU.}
\end{align*} \]
neuron with the difference of 2 ReLU neurons, as we did in the proof for Lemma 3.3. In particular, the two neurons replacing $\tilde{v}_1$ are designated $H, G$.

Now we prove the following theorem:

**Theorem 7.2.** Let $\mathcal{R}_{TM} = \{ L : L$ is recognizable by some real-time TM $\}$, $\mathcal{R}_{RNN} = \{ L : L$ is recognizable by some RNN $\}$, $\mathcal{R}_{RAM} = \{ L : L$ is recognizable by some real-time RAM $\}$, then $\mathcal{R}_{TM} \subseteq \mathcal{R}_{RNN} \subseteq \mathcal{R}_{RAM}$.

**Proof.** We first show that $\mathcal{R}_{TM} \subseteq \mathcal{R}_{RNN}$, i.e., any unweighted language $L$ recognizable by some real-time Turing Machine is recognizable by some single-layer, ReLU RNN.

Suppose $L$ is recognizable by a multi-tape, multi-head-per-tape real-time TM $M$ with a one-way, single-head, read-only input tape and one-way, single-head, write-only output tape. By the simulation of (Fischer et al., 1972), we can transform $M$ into a multi-tape TM $M'$ with one head per tape that takes a constant number of steps between two readings. By the constant speedup theorem (Hartmanis and Stearns, 1965), we can turn $M'$ into TM $M''$ recognizing the same language taking only 1 step between two readings. Suppose $M''$ has $p$ working tapes (not including input and output). We then replace each tape with two stacks, which does not affect the time behavior.

We now construct a single-layer ReLU RNN simulating this 2p-stack real-time Turing Machine (Lemma 7.1). Since this simulation is also real-time, one time step in this RNN equals one step of Turing computation between two readings. Therefore this RNN can recognize $L$ in real-time.

On the other hand, each time step of RNN (disregarding the part calculating $y_t, p_t$) consists of a constant number of algebraic operations $+, -, \times, /, <$, so RNN can be simulated by a real-time, unit-cost algebraic RAM. Therefore $\mathcal{R}_{RNN} \subseteq \mathcal{R}_{RAM}$. □

As a consequence, all unweighted languages recognizable by a real-time Turing Machine are also recognizable by RNN. Examples include palindrome ($xx^R$) (Galil, 1978), copy ($xx$), and string matching (3 flavors: (a) The pattern $x$ is preprocessed and the input is $y$; (b) The input is $x^*y$; (c) The input is $[x_0x_1\ldots x_n y_0 y_1 \ldots y_n y_{n+1} \ldots]$) (Galil, 1981).

All unweighted languages unrecognizable by a real-time unit-cost algebraic RAM are also unrecognizable by RNN. For example, if $P$ is a decision problem with super-linear time lower bound on RAM, and we input instances of $P$ as a binary stream into RNN, no RNN can “output” accept / reject for the whole string read so far immediately after reading each new token.

Since there are languages unrecognizable by real-time Turing Machine but recognizable by real-time RAM (Galil, 1981), whether $\mathcal{R}_{TM}$ is a proper subset of $\mathcal{R}_{RNN}$ and whether $\mathcal{R}_{RNN}$ is a proper subset of $\mathcal{R}_{RAM}$ are open questions.

8 Conclusion

We proved the following hardness results regarding RNN as a recognizer of weighted languages:

1. Consistency:
   (a) Inconsistent RNNs exist.
   (b) Consistency of RNN is undecidable.

2. Highest-weighted string:
   (a) Finding the highest-weighted string for arbitrary RNN is undecidable.
   (b) Finding the highest-weighted string for a consistent RNN is decidable, but the solution can be exponentially long in the sum of the absolute values of all the parameters of the RNN.
   (c) Restricting to solutions of polynomial length, finding the highest-weighted string is NP-complete and APX-hard.

3. Testing equivalence of RNNs and minimizing the number of neurons in an RNN are both undecidable.

We also considered RNNs as unweighted language recognizers and situated RNNs between Tur-
ing Machines and Random-Access Machines regarding their real-time recognition powers.

Although our undecidability results are upshots of the Turing-completeness of RNN (Siegelmann and Sontag, 1995), our NP-completeness and APX-hardness results are original, and surprising, since the analogous hardness results in PFSA relies on the fact that there are multiple derivations for a single string (Casacuberta and de la Higuera, 2000).

Our results show the non-existence of (efficient) algorithms for interesting problems that researchers using RNN in natural language processing tasks may have hoped to find. On the other hand, the non-existence of such efficient or exact algorithms gives evidence for the necessity of approximation, greedy or heuristic algorithms to solve those problems in practice. In particular, since finding the highest-weighted string in RNN is the same as finding the most-likely translation in a sequence-to-sequence RNN decoder, our NP-completeness and APX-hardness results provide some justification for employing greedy and beam search algorithms in practice.

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A RNNs expressing

$L_1 = \{a^nb^n : n \geq 0, P \approx 1/2^{n+1}\}$ and
$L_2 = \{a^nb^n c^n : n \geq 0, P \approx 1/2^{n+1}\}$

The following RNN represents $L_1$:

\[
\begin{align*}
h_0 &= g_0 = 0 \\
h_1^t &= \sigma(h_{t-1}^1 + u_a) \\
g_1^t &= \sigma(h_{t-1}^1 + u_a - 1) \\
h_2^t &= \sigma(h_{t-1}^2 + u_b) \\
g_2^t &= \sigma(h_{t-1}^2 + u_b - 1) \\
h_3^t &= \sigma(h_{t-1}^1 + u_a - h_{t-1}^2 - u_b) \\
g_3^t &= \sigma(h_{t-1}^1 + u_a - h_{t-1}^2 - u_b - 1)
\end{align*}
\]

where $u_a = 1$ iff the current letter is $a$ and $u_b = 1$ iff the current letter is $b$.

The following RNN represents $L_2$:

\[
\begin{align*}
h_0 &= g_0 = 0 \\
h_1^t &= \sigma(h_{t-1}^1 + u_a) \\
g_1^t &= \sigma(h_{t-1}^1 + u_a - 1) \\
h_2^t &= \sigma(h_{t-1}^2 + u_b) \\
g_2^t &= \sigma(h_{t-1}^2 + u_b - 1) \\
h_3^t &= \sigma(h_{t-1}^1 + u_a - h_{t-1}^2 - u_b) \\
g_3^t &= \sigma(h_{t-1}^1 + u_a - h_{t-1}^2 - u_b - 1) \\
y_t &= [\ln \epsilon(h_t^2 - g_t^2), \ln \epsilon(1 - h_t^3 + g_t^3), \ln \epsilon(h_t^3 - g_t^3)]
\end{align*}
\]

Proof: Since $\lim_{k \to \infty} 2^{-k} = 0$,

\[
\lim_{k \to \infty} \frac{\ln (1 + 2^{-k})}{2^{-k}} = \lim_{x \to 0} \frac{\ln (1 + x)}{x} = 1.
\]

By the limit comparison test, since

\[
\lim_{n \to \infty} \sum_{k=2}^{n} 2^{-k}
\]

converges,

\[
\lim_{n \to \infty} \sum_{k=2}^{n} \ln (1 + 2^{-k})
\]

also converges.
Since
\[\forall k \geq 2, \ln (1 + 2^{-k}) \leq 2^{-k},\]
\[
\lim_{n \to \infty} \sum_{k=2}^{n} \ln (1 + 2^{-k}) \leq \lim_{n \to \infty} \sum_{k=2}^{n} 2^{-k} = \frac{1}{2} < \ln 2,
\]
so
\[
\lim_{n \to \infty} \prod_{i=1}^{n} \frac{2^{i+1}}{2^{i+1} + 1} = \lim_{n \to \infty} \prod_{k=2}^{n} \frac{1}{1 + 2^{-k}} > \frac{1}{2}.
\]