Algorithms for Weighted Pushdown Automata

Alexandra Butoi¹  Brian DuSell²  Tim Vieira³  Ryan Cotterell¹  David Chiang²

¹ETH Zürich   ²University of Notre Dame   ³Johns Hopkins University
alexandra.butoi@inf.ethz.ch  {bdusell1,dchiang}@nd.edu  
{tim.f.vieira,ryan.cotterell}@gmail.com

Abstract

Weighted pushdown automata (WPDA) are at the core of many natural language processing tasks, like syntax-based statistical machine translation and transition-based dependency parsing. As most existing dynamic programming algorithms are designed for context-free grammars (CFGs), algorithms for PDAs often resort to a PDA-to-CFG conversion. In this paper, we develop novel algorithms that operate directly on WPDA. Our algorithms are inspired by Lang’s algorithm, but use a more general definition of pushdown automaton and either reduce the space requirements by a factor of $|\Gamma|$ (the size of the stack alphabet) or reduce the runtime by a factor of more than $|Q|$ (the number of states). When run on the same class of PDAs as Lang’s algorithm, our algorithm is both more space-efficient by a factor of $|\Gamma|$ and more time-efficient by a factor of $|Q| \cdot |\Gamma|$.

https://github.com/rycolab/wpda

1 Introduction

Weighted pushdown automata (WPDA) are widespread in natural language processing (NLP), primarily in syntactic analysis. For instance, WPDA have found use in syntax-based statistical machine translation (Allauzen et al., 2014), and many transition-based dependency parsers (Nivre, 2004; Chen and Manning, 2014; Weiss et al., 2015; Dyer et al., 2015; Andor et al., 2016; Shi et al., 2017; Ma et al., 2018; Fernández-González and Gómez-Rodríguez, 2019) are special cases of WPDA. In addition, PDAs have been used in computational psycholinguistics as models of human sentence processing (Resnik, 1992). Despite their ubiquity, there has been relatively little research on the theory of WPDA themselves. In some ways, WPDA are treated as second-class citizens compared to their equivalent cousins, weighted context-free grammars (WCFG), for which a variety of dynamic programming algorithms exists (Bar-Hillel et al., 1961; Earley, 1970; Stolcke, 1995). To help fill this gap, this paper offers several new and improved algorithms for computing with WPDA.

Figure 1 gives an overview of most of our results. We start by defining a weighted version of the extended PDAs of Aho and Ullman (1972, p. 173) and two special cases: the standard definition (Hopcroft et al., 2006), which we call top-down, and its mirror image, which we call bottom-up. Both top-down and bottom-up WPDA have been used in NLP. Roark’s (2001) generative parser is a top-down PDA as is Dyer et al.’s (2016). Most transition-based dependency parsers, both arc-standard (Nivre, 2004; Huang et al., 2009) and arc-eager (Nivre, 2003; Zhang and Clark, 2008), are bottom-up WPDA.

Next, we give a normal form for WPDA analogous to Chomsky normal form, and we derive new dynamic programming algorithms to compute the weight of a string under top-down and bottom-up WPDA in normal form. We are only aware of one
previous recognition algorithm for PDAs, that of Lang (1974), which we generalize to the weighted case and improve in the following ways:

- On PDAs more general than those Lang considers, our algorithm is more space-efficient by a factor of $|\Gamma|$ (the stack alphabet size);
- We can speed up our algorithm to be more time-efficient by a factor of more than $|Q|$ (the number of states), but without the space-complexity improvement;
- On the same PDAs that Lang considers, which we call simple, our sped-up algorithm is more efficient by a factor of $|\Gamma|$ in space and $|Q| \cdot |\Gamma|$ in time.

Compared with the pipeline of standard procedures for converting a top-down PDA to a CFG, converting to Chomsky normal form, and parsing with CKY, our top-down algorithm is faster by a factor of more than $O(|Q|^3)$.

Finally, we present iterative algorithms for computing the total weight of all runs of a WPDA.

2 Weighted Pushdown Automata

2.1 Preliminaries

Let $[i:j]$ denote the sequence of integers $(i, \ldots, j)$. If $s$ is a string, we write $|s|$ for the length of $s$, $s_i$ for the $i$th symbol of $s$, and $s[i:j]$ for the substring $s_{i+1} \cdots s_j$.

**Definition 1.** A monoid is a tuple $(A, \circ, I)$, where $A$ is a set, $\circ$ is an associative binary operation, and $I \in A$, called the identity element, satisfies $I \circ a = a = a \circ I$ for all $a \in A$. If $a \circ b = b \circ a$ for all $a, b$, we say that the monoid is commutative.

**Definition 2.** A semiring is a tuple $\mathcal{W} = (A, \oplus, \odot, 0, 1)$ such that $(A, \oplus, 0)$ is a commutative monoid and $(A, \odot, 1)$ is a monoid. Additionally, $\odot$ distributes over $\oplus$, that is, $a \odot (b \oplus c) = a \odot b \oplus a \odot c$ and $(a \oplus b) \odot c = a \odot c \oplus b \odot c$, and $0$ is absorbing with respect to $\oplus$, that is, $0 \oplus a = a \oplus 0 = 0$. If $\circ$ is commutative then we say that $\mathcal{W}$ is commutative.

We also sometimes assume $\mathcal{W}$ is continuous; please see the survey by Droste and Kuich (2009) for a definition.

2.2 Definition

Our definition of weighted PDA is more general than usual definitions, in order to accommodate the top-down and bottom-up variants introduced in §2.3. It is roughly a weighted version of extended PDAs of Aho and Ullman (1972, p. 173) and the PDAs of Lewis and Papadimitriou (1997, p. 131).

**Definition 3.** A weighted pushdown automaton (WPDA) over a semiring $\mathcal{W} = (A, \oplus, \odot, 0, 1)$ is a tuple $\mathcal{P} = (Q, \Sigma, \Gamma, \delta, (i, \gamma_1), (f, \gamma_F))$, where:

- $Q$ is a finite set of states;
- $\Sigma$ is a finite set of input symbols, called the input alphabet;
- $\Gamma$ is a finite set of stack symbols, called the stack alphabet;
- $\delta : Q \times \Gamma^* \times (\Sigma \cup \{\varepsilon\}) \times Q \times \Gamma^* \rightarrow A$ is called the transition weighting function;
- $(i, \gamma_1)$ is called the initial configuration and $(f, \gamma_F)$ is called the final configuration, where $i, f \in Q$ and $\gamma_1, \gamma_F \in \Gamma^*$.

Stacks are represented as strings over $\Gamma$, from bottom to top. Thus, in the stack $\gamma = X_1X_2 \cdots X_n$, the symbol $X_1$ is at the bottom of the stack, while $X_n$ is at the top.

**Definition 4.** A configuration of a WPDA is a pair $(q, \gamma)$, where $q \in Q$ is the current state and $\gamma \in \Gamma^*$ is the current contents of the stack.

The initial and final configurations of a WPDA are examples of configurations; it is possible to generalize the initial and final stacks to (say) regular expressions over $\Gamma$, but the above definition suffices for our purposes.

A WPDA moves from configuration to configuration by following transitions of the form $q, \gamma_1 \xrightarrow{a/\omega} r, \gamma_2$, which represents a move from the state $q$ to state $r$, while popping the sequence of symbols $\gamma_1 \in \Gamma^*$ from the top of the stack and pushing the sequence $\gamma_2 \in \Gamma^*$.

**Definition 5.** If $\delta(p, \gamma_1, a, q, \gamma_2) = w$, then we usually write $\delta(p, \gamma_1 \xrightarrow{a} q, \gamma_2) = w$ or that $\delta$ has transition $(q, \gamma_1 \xrightarrow{a/\omega} p, \gamma_2)$. We sometimes let $\tau$ stand for a transition, and we define $\delta(\tau) = w$. We say that $\tau$ scans $a$, and if $a \neq \varepsilon$, we call $\tau$ scanning; otherwise, we call it non-scanning. We say that $\tau$ is k-pop, l-push if $|\gamma_1| = k$ and $|\gamma_2| = l$.

**Definition 6.** If $(q_1, \gamma_1)$ and $(q_2, \gamma_2)$ are configurations, and $\tau$ is a transition $q_1, \gamma_1 \xrightarrow{a/\omega} q_2, \gamma_2$, we write $(q_1, \gamma_1) \Rightarrow \tau (q_2, \gamma_2)$.

**Definition 7.** A run of a WPDA $\mathcal{P}$ is a sequence of configurations and transitions

$$\pi = (q_0, \gamma_0), \tau_1, (q_1, \gamma_1), \ldots, \tau_n, (q_n, \gamma_n)$$

where, for $i = 1, \ldots, n$, we have $(q_{i-1}, \gamma_{i-1}) \Rightarrow \tau_i (q_i, \gamma_i)$. (Sometimes it will be convenient to treat $\pi$ as a sequence of only configurations or only
transitions.) A run is called \textbf{accepting} if \((q_0, \gamma_0)\) is the initial configuration and \((q_n, \gamma_n)\) is the final configuration. If, for \(i = 1, \ldots, n\), \(\tau_i\) scans \(a_i\), then we say that \(\pi\) scans the string \(a_1 \cdots a_n\). We write \(\Pi(\mathcal{P}, s)\) for the set of runs that scan \(s\) and \(\Pi(\mathcal{P})\) for the set of all accepting runs of \(\mathcal{P}\).

2.3 Subclasses of PDAs

Next, we define two special forms for WPDAs, which we call \textbf{top-down} and \textbf{bottom-up}, because they can be used as top-down and bottom-up parsers for CFGs, respectively. The most common definition of PDA (Hopcroft et al., 2006; Autebert et al., 1997) corresponds to top-down PDAs,\(^1\) while the machine used in an LR parser (Knuth, 1965) corresponds to bottom-up PDAs.

\textbf{Definition 8.} A WPDA is called \textbf{bottom-up} if it has only \(1\)-push transitions. Moreover, the initial configuration is \((i, S)\) and the final configuration is \((f, S)\) for some \(i, f \in Q\) and \(S \in \Gamma\).

\textbf{Proposition 1.} Every WPDA is equivalent to some bottom-up WPDA.

\textit{Proof.} Add states \(t', f'\) and a stack symbol \(S'\), and make \((t', e)\) and \((f', S')\) the new initial and final configurations, respectively. Add transitions

\[
\begin{align*}
t', e & \xrightarrow{e/1} t, S' \gamma_1 \\
f, S' \gamma_F & \xrightarrow{e/1} f', S'.
\end{align*}
\]

For each \(k\)-pop, \(l\)-push transition \(p, \gamma \xrightarrow{a/w} r, X_1 \cdots X_l\) where \(l > 1\), create \((l - 1)\) new states \(q_1, \ldots, q_{l-1}\) and replace the transition with

\[
\begin{align*}
p, e & \xrightarrow{e/1} q_1, X_1 \\
q_{i-1}, e & \xrightarrow{e/1} q_i, X_i \quad i = 2, \ldots, l - 1 \\
q_{k-1}, \gamma & \xrightarrow{a/w} r, X_k.
\end{align*}
\]

For each \(k\)-pop, 0-push transition \(q, \gamma \xrightarrow{a/w} p, e\), replace it with the \((k + 1)\)-pop, 1-push transitions \(q, X \gamma \xrightarrow{a/w} p, X\) for every \(X \in \Gamma \cup \{S'\}\). \(\square\)

If the original WPDA had transitions that push at most \(l\) symbols, then the resulting WPDA has \(O(l \cdot |\delta| \cdot |Q|)\) states and \(O((l + |\Gamma|) \cdot |\delta|)\) transitions.

\textbf{Definition 9.} A WPDA is called \textbf{top-down} if it has only \(1\)-pop transitions. Moreover, the initial configuration is \((i, S)\) and the final configuration is \((f, e)\) for some \(i, f \in Q\) and \(S \in \Gamma\).

\textbf{Proposition 2.} Every WPDA is equivalent to some top-down WPDA.

\textit{Proof.} Similar to the bottom-up case. \(\square\)

This conversion crucially makes use of nondeterminism to guess the top \(k\) stack symbols. Aho and Ullman (1972, p. 174) give a different algorithm that uses the state to keep track of the top \(k\) stack symbols. Although this does not require nondeterminism, it creates \(O(|\Gamma|^k \cdot |Q|)\) states.

Finally, Lang (1974) considers a still more restricted subclass of PDAs.\(^2\)

\textbf{Definition 10.} A WPDA is called \textbf{simple} if it only has \(k\)-pop, \(l\)-push transitions for \(k \leq 1\) and \(l \leq 1\).

Because simple PDAs do not condition pushes on the top stack symbol, they can be weighted, but not probabilistic.

2.4 Stringsums and Allsums

\textbf{Definition 11.} The weight \(w(\pi)\) of a run \(\pi \in \Pi(\mathcal{P})\) is the product of the weights of its transitions,

\[
w(\pi) \overset{\text{def}}{=} \bigotimes_{\tau \in \pi} \delta(\tau).
\]

\textbf{Definition 12.} The \textbf{stringsum} \(w(\mathcal{P}, s)\) of a string \(s\) for a WPDA \(\mathcal{P}\) is the total weight of all runs of \(\mathcal{P}\) that scan \(s\),

\[
w(\mathcal{P}, s) \overset{\text{def}}{=} \bigoplus_{\pi \in \Pi(\mathcal{P}, s)} w(\pi).
\]

\textbf{Definition 13.} The \textbf{allsum} \(w(\mathcal{P})\) of a WPDA \(\mathcal{P}\) is the total weight of all runs of \(\mathcal{P}\),

\[
w(\mathcal{P}) \overset{\text{def}}{=} \bigoplus_{\pi \in \Pi(\mathcal{P})} w(\pi).
\]

2.5 Push and Pop Computations

Our algorithms for bottom-up WPDAs make heavy use of \textbf{push computations}. Intuitively, a push computation is a run that pushes exactly one symbol without touching the stack symbols below it.\(^2\)

\(^1\)This definition goes back to Chomsky’s (1963) original definition, which also allows 0-pop transitions.

\(^2\)This definition is also used by Sipser (2012) and seems to go back to Evey’s (1963) original definition of PDAs, which doesn’t allow 1-pop, 1-push transitions. The term “simple PDA” has been used at least twice for two different purposes (Schützenberger, 1963; Lewis and Papadimitriou, 1997); we apologize for introducing a third.
3.1 Binarization

Recall that top-down and bottom-up WPDAs have 1-pop, k-push transitions and k-pop, 1-push transitions, respectively. Since the runtime of our stringsum algorithm depends highly on k, we convert the WPDA into an equivalent one with k ≤ 2. We call this procedure binarization because it is entirely analogous to binarization in CFGs. It is symmetric for top-down and bottom-up WPDAs.

**Proposition 3.** Every bottom-up WPDA is equivalent to a bottom-up WPDA whose transitions are k-pop, 1-push where k ≤ 2.

**Proof.** For each k-pop, 1-push transition

\[ p, Y_1 \cdots Y_k \xrightarrow{a/w} q, X \]

we introduce k − 2 new states \( r_1, \cdots, r_{k-2} \) and we replace the original transition with the following:

\[ p, Y_1 Y_2 \xrightarrow{a/w} r_1, Y_2 \]

\[ r_{i-1}, Y_i Y_{i+1} \xrightarrow{\varepsilon/1} r_i, Y_{i+1} \quad i \in [2:k-2] \]

\[ r_{k-2}, Y_{k-1} Y_k \xrightarrow{\varepsilon/1} q, X. \]

If the original WPDA had transitions that pop at most k symbols, the resulting WPDA has \( O(k \cdot |\delta| \cdot |Q|) \) states and \( O(k \cdot |\delta|) \) transitions.

**Proposition 4.** Every top-down WPDA is equivalent to a top-down WPDA whose transitions are 1-pop, k-push where k ≤ 2.

3.2 Nullary Removal

In this section, we discuss the removal of nullary transitions from WPDAs:

**Definition 16.** In a bottom-up WPDA, a transition is called nullary if it is of the form \( p, \varepsilon \xrightarrow{\varepsilon/1} q, X \).

Although nullary transitions are analogous to nullary productions in a CFG, the standard procedure for removing nullary productions from CFGs does not have an exact analogue for PDAs, and the procedure we describe here is novel.

We assume a bottom-up WPDA, but an identical construction exists for top-down WPDAs. We also assume that the WPDA has been binarized, and semiring \( W \) is commutative and continuous.

The construction consists of three steps: partitioning, precomputation, and removal.

### Partitioning.

For every symbol \( X \in \Gamma \), we replace \( X \) with two stack symbols \( X^e \) and \( X^h \). A push computation that pushes a \( X^e \) scans \( \varepsilon \), and a
push computation that pushes a $X^*$ scans a string that is not $\varepsilon$. To do this, we replace every $k$-pop transition $p, X_1 \cdots X_k \xrightarrow{a/w} q, Y$ with $2^k$ new transitions $p, X_1 Y_1 \cdots X_k Y_k \xrightarrow{a/w} q, Y'$, where $v = \varepsilon$ iff $v_i = \varepsilon$ for all $i$ and $a = \varepsilon$. For instance, we replace transition $p, X, q \xrightarrow{a/w} r, Y$.

We give details on the complexity of this transformation in App. A.

Precomputation. We compute the weight of all non-scanning push computations by solving the quadratic system of equations:

$$N_{pXq} = \delta(p, \varepsilon \xrightarrow{e} q, X)$$

$$\oplus \sum_{Y, r} N_{pYr} \otimes \delta(r, Y \xrightarrow{e} q, X)$$

$$\oplus \sum_{Y, Z, s} N_{pYZs} \otimes \delta(s, YZ \xrightarrow{e} q, X)$$

$N_{pYZs} = \sum_r N_{pYr} \otimes N_{rZs}.$

See §6 for details on solving such systems of equations, which assumes that $W$ is continuous. Then $N_{pXq}$ is the total weight of all push computations of type $[i, p, X, q, i]$ for any $i$.

Removal. First, delete every transition that pushes $X^*$ for each $X \in \Gamma$. If the PDA accepts $\varepsilon$ with weight $w$, add $t, \varepsilon \xrightarrow{e/w} s, S^e$ as the sole nullary transition. (For correctness, we must also ensure that no transition pops $S^e$, no transition enters $s$, and no transition leaves $f$.)

Sometimes an $X^*$ is popped immediately after it is pushed (that is, with no input symbols scanned between the push and the pop). To handle these cases, for the following transitions, we create new versions in which popped $X^*$ symbols are removed, and their corresponding weight multiplied in.

For each:

- Replace with $(\forall s \in Q)$:
  - $t, \varepsilon \xrightarrow{a/N_YZp} q, X^*$.

- Then we modify (again) the 0-pop transitions to first simulate zero or more nullary transitions:

For each transition of the form $p, Y \xrightarrow{e/w} q, X^*$ (where $a \in \Sigma \cup \{\varepsilon\}$), we create transitions for all $r, s, t \in Q$:

$$p, r, Z \xrightarrow{a/N_{XYZp}w} q, r, s, X.$$ (This step is where commutativity is needed.) Finally, add transitions to remove the state annotations, for all $p, X, q$:

$$q, p, X \xrightarrow{e/1} q, X^*.$$
4 Stringsums in Bottom-up WPDA

In this section, we give dynamic programming algorithms for computing the stringsum of an input string $s$ (with $|s| = n$) of bottom-up WPDA in normal form. We give a basic version of the algorithm, which has the same runtime as Lang’s algorithm but improved space requirements, and a fast version that has the same space complexity and runs asymptotically faster. On simple PDAs (for which Lang’s algorithm was designed), the latter version has both improved space and time complexity.

4.1 Basic Algorithm

The algorithm computes stringsums efficiently by exploiting the structural similarities among the WPDA runs. Fig. 3 shows a deductive system (Shieber et al., 1995; Goodman, 1999) for deriving items corresponding to push computations.

The items have the form $[i, p, X, j, q]$ for $p, q \in Q, X \in \Gamma, 0 \leq i \leq j \leq n$. If our algorithm derives this item with weight $w$, then the push computations of type $[i, p, X, j, q]$ have total weight $w$.

We distinguish three categories of push computations, based on their final transition, and we include an inference rule for each. First are those consisting of a single 0-pop, 1-push transition. The other two categories are those ending in a 1-pop transition and a 2-pop transition, respectively. These can be built recursively from shorter push computations.

The goal item is $[0, i, S, n, f]$, which stands for all runs from the initial configuration to the final configuration that scan $s$.

Alg. 1 shows how to compute item weights according to these rules. At termination, the weight of the goal item is the sum of the weights of all accepting runs that scan $s$.

Algorithm 1 Compute the weights of all push computations of a bottom-up WPDA on an input string.

```
1. $w \leftarrow 0$
2. $n \leftarrow |s|$
3. for $i \in [0:n-1]$
4.  $j \leftarrow i + 1$
5. if 0-pop, 1-push
6.  $(p, e \rightarrow s_j/w)^\delta = q, X$
7.  $w[p, i, p, X, j, q] \leftarrow w$
8. for $\ell \in [2:n]$
9.  for $i \in [0:n-\ell+1]$
10.   $j \leftarrow i + \ell$
11.  if 1-pop, 1-push
12.    $(p, e \rightarrow s_j/w)^\delta = q, X$
13.    $w[p, i, p, X, j, q] \leftarrow w$
14.  if 2-pop, 1-push
15.    for $p, r \in Q$
16.      $(s, YZ \rightarrow a/w)^\delta = q, X$
17.      for $k \in [i + 1; j - |a| - 1]$
18.         $w[p, i, p, X, j, q] \leftarrow (w[p, i, p, Y, j, r] \oplus w[k, r, Z, j - |a|, s] \otimes w)$
19.    return $w[0, i, S, n, f]$
```

4.2 Correctness

Theorem 1. Let $P$ be a WPDA and $s \in \Sigma^*$ an input string. The weight $w[i, p, X, j, q]$ is the total weight of all push computations of $P$ of type $[i, p, X, j, q]$.

Proof. By induction on the span length, $\ell = j - i$.

Base Case. Assume that $j - i = 1$. The only push computations from state $p$ to $q$ that push $X$ and scan $s(i, j]$ are ones that have the single transition $\tau = p, e \rightarrow s_j/w^\delta = q, X$. There cannot exist others, because normal form requires that any additional non-scanning transitions would decrease the stack height. So the total weight of all such push computations is $w$, and the algorithm correctly sets $w[i, p, X, j, q] = w$ at line 7.

Inductive Step. Assume that the statement holds for any spans of length at most $(\ell - 1)$ and consider a span of length $\ell$. For such spans, the algorithm computes the total weight of all push computations $\pi$ of type $[i, p, X, j, q]$, for all $X \in \Gamma, p, q \in Q$, and $j - i = \ell$. This weight must be the sum of weights of three types of push computations: those that end with 0-pop transitions, with 1-pop transitions, and with 2-pop transitions.

But ending with a 0-pop transition is impossible, because such push computations must have only
one transition and therefore $j - i \leq 1$. The 1-pop and 2-pop parts of the sum are computed at lines 12–14 and 16–19 of the algorithm, respectively.

**Ending with 1-pop transition.** The algorithm sums over all possible ending transitions $\tau_{\text{end}} = r, Y \xrightarrow{a/w} q, X$. (Normal form requires that this transition be scanning.) Let $\Pi$ be the set of all push computations of type $[i, p, X, j, q]$ ending in $\tau_{\text{end}}$ and let $\Pi'$ be the set of all push computations of type $[i, p, Y, j - 1, r]$. Every push computation in $\Pi$ must be of the form $\pi = \pi' \circ \tau_{\text{end}}$, where $\pi' \in \Pi'$, and conversely, for every $\pi' \in \Pi'$, we have $\pi' \circ \tau_{\text{end}} \in \Pi$. By the induction hypothesis, the total weight of $\Pi'$ was computed in a previous iteration. Then, by distributivity, we have:

$$
\begin{align*}
\left( \bigotimes_{\pi' \in \Pi'} \bigotimes_{\tau' \in \pi'} \delta (\tau') \right) \otimes \delta (\tau_{\text{end}}) &= \left( \bigotimes_{\pi' \in \Pi'} \delta (\tau') \right) \otimes \delta (\tau_{\text{end}}) \\
&= \mathbf{w}[i, p, Y, j - 1, r] \otimes \delta (\tau_{\text{end}}).
\end{align*}
$$

**Ending with 2-pop transition.** The algorithm sums over all possible ending transitions $\tau_{\text{end}} = s, YZ \xrightarrow{a/w} q, X$, $a \in \{s, \varepsilon\}$. Every push computation $\pi$ that ends with $\tau_{\text{end}}$ decomposes uniquely into $\pi' \circ \pi'' \circ \tau_{\text{end}}$, where $\pi'$ and $\pi''$ are push computations of type $[i, p, Y, k, r]$ and $[k, r, Z, j - |a|, s]$, respectively, for some $k \in [i + 1 : j - |a| - 1]$ and $r \in Q$. We call $(k, r)$ the **split point** of $\pi$.

The algorithm sums over all split points $(k, r)$. Let $\Pi$ be the set of all push computations of type $[i, p, X, j, q]$ ending in $\tau_{\text{end}}$ with split point $(k, r)$, and let $\Pi'$ and $\Pi''$ be the sets of all push computations of type $[i, p, Y, k, r]$ and $[k, r, Z, j - |a|, s]$, respectively. Let $\pi \in \Pi$ must be of the form $\pi' \circ \pi'' \circ \tau_{\text{end}}$, where $\pi' \in \Pi'$ and $\pi'' \in \Pi''$, and conversely, for every $\pi' \in \Pi'$ and $\pi'' \in \Pi''$, $\pi' \circ \pi'' \circ \tau_{\text{end}} \in \Pi$. Because $i < k$, we must have $j - |a| - k \leq j - k < j - i$, and because $k < j - |a|$, we must have $k - i < j - |a| - i \leq j - i$. By the induction hypothesis, the total weight of $\Pi'$ and $\Pi''$ were fully computed in a previous iteration. As in the previous case, by distributivity we have

$$
\bigotimes_{\pi \in \Pi} \bigotimes_{\tau \in \pi} \delta (\tau) = \mathbf{w}[i, p, Y, k, r] \otimes \mathbf{w}[k, r, Z, j - |a|, s] \otimes \delta (\tau_{\text{end}}).
$$

### 4.3 Stack Automaton

The distribution over possible configurations that $\mathcal{P}$ can be in after reading $s(0:n)$ can be generated by a weighted finite-state automaton $\mathcal{M}$. The states of $\mathcal{M}$ are of the form $(i, q,)$, with start state $(0, s)$ and accept states $(m, q)$ for all $q \in Q$. There is a transition $(i, q) \xrightarrow{X/w} (j, r)$ for every item $[i, q, X, j, r]$ with weight $w$. Then if an accepting run of $\mathcal{M}$ scans $\gamma$ and ends in state $(m, q)$ with weight $w$, then $\mathcal{P}$ can be in configuration $(q, \gamma)$ with weight $w$.

### 4.4 Complexity Analysis and Speedup

For comparison with our algorithm, we show the original algorithm of Lang (1974) in Fig. 4. It has items of the form $[i, q, XY, j, r]$, which stands for push computations that start with $X$ as the top stack symbol and push a $Y$ on top of it.

Our algorithm stores a weight for each item $[i, p, X, j, q]$, giving a space complexity of $O(n^2 |Q|^2 |\Gamma|^3)$. This is more efficient than Lang’s algorithm, which requires $O(n^3 |Q|^2 |\Gamma|^3)$ space.

Computing the weight of each new item requires, in the worst case (the inference rule for 2-pop transitions), iterating over stack symbols $Y, Z \in \Gamma$, indices $j \in [0:n]$ and states $q, r \in Q$, resulting in a runtime of $O(n |Q|^2 |\Gamma|^3)$ per item. So the algorithm has a runtime of $O(n^3 |Q|^2 |\Gamma|^3)$, the same as Lang’s algorithm.

This runtime can be improved by splitting the

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Figure 4: Deductive system for Lang’s algorithm.
inference rule for 2-pop transitions into two rules:

\[
\begin{align*}
[k, r, Z, j &= |a|, s] & \quad s, YZ \xrightarrow{a/w} q, X \\
(k, r, Y, j, q) & \quad [i, p, X, j, q] \\
[i, p, Y, k, r] & \quad (k, r, Y, X, j, q) \\
[i, p, X, j, q] & \quad [i, p, X, j, q]
\end{align*}
\]

The first rule has \(O(n^2|Q|^3|\Gamma|^3)\) instantiations and the second rule has \(O(n^2|Q|^3|\Gamma|^2)\). So, although we have lost the space-efficiency gain, the total time complexity is now in \(O((n^3|\Gamma|^2 + n^2|\Gamma|^3)|Q|^3)\), a speedup of a factor of more than \(|Q|. We show in App. B an alternative deductive system that achieves a similar speedup.

Furthermore, Lang’s algorithm only works on simple PDAs. To make the algorithms directly comparable, we can assume in the 2-pop, 1-push case that \(X = Y\). This reduces the space complexity by a factor of \(|\Gamma|\) again. Moreover, it reduces the number of instantiations of the inference rules above to \(O(n^3|Q|^3|\Gamma|^2)\) and \(O(n^3|Q|^3|\Gamma|)\), respectively. So the total time complexity is in \(O(n^3|Q|^2|\Gamma|^2)\), which is a speedup over Lang’s algorithm by a factor of \(|Q| \cdot |\Gamma|\).

5 Stringsums in Top-down WPDA

Strings in top-down WPDA can be computed by the left/right mirror image of our bottom-up algorithm. Instead of finding push computations, this algorithm finds pop computations, which decrease (rather than increase) the stack size by exactly one.

**Definition 18** (Pop computation). Let \(P\) be a top-down WPDA and \(s \in \Sigma^*\) an input string. A pop computation of type \([i, p, X, j, q]\), where \(0 \leq i \leq j \leq |s|\), \(p, q \in Q\), and \(X \in \Gamma\), is a run \(\pi = (q_0, \gamma_0), \ldots, (q_m, \gamma_m)\) that scans \(s(i:j)\), where \(q_0 = p\), \(q_m = q\), \(\gamma_0 = \gamma_m X\), and for all \(l < m\), \(|\gamma_l| \geq |\gamma_0|\).

Fig. 5 shows the inference rules used by the dynamic program, which are the mirror image of the rules in Fig. 3. Each item \([i, p, X, j, q]\), which stands for a set of pop computations, is derived using a transition and items corresponding to pop computations that happen later in the run.

---

3 The \() operator, which is just a punctuation mark and does not require any particular interpretation, was chosen to evoke the \(\) in categorial grammar (using Lambek’s “result on top” convention): \(Y/X\) is an \(X\) missing a \(Y\) on the left.
We can use a notion of push computation similar to Def. 14 to derive a space-efficient algorithm for computing allsums in bottom-up PDAs. The item \([p, X, q]\) stands for runs from state \(p\) to state \(q\) that push the symbol \(X\) on top of the stack while leaving the symbols underneath unchanged.

**Definition 19** (Push computation). Let \(\mathcal{P}\) be a bottom-up WPDA. A push computation of type \([p, X, q]\), where \(p, q \in Q, X \in \Gamma\), is a run \(\pi = (q_0, \gamma_0), \ldots, (q_n, \gamma_n)\), where \(q_0 = p, q_n = q\), \(\gamma_n = \gamma_0 X\), and for all \(i > 0\), \(|\gamma_i| \geq |\gamma_n|\).

These items closely resemble those used for computing stringsums, but discard the two variables \(i, j\) that we used for indexing substrings of the input, as we are interested in computing the total weight of runs that scan any string.

**Definition 20.** Let \(\Pi (p, X, q)\) be the set containing all push computations from state \(p\) to state \(q\) that push \(X\). The allsum \(w[p, X, q]\) is defined as

\[
 w[p, X, q] = \bigoplus_{\pi \in \Pi(p, X, q)} w(\pi).
\]

The allsum of a set of push computations can be expressed in terms of other allsums:

\[
 w[p, X, q] = \bigoplus_{a \in \Sigma \cup \{\varepsilon\}} \delta(p, \varepsilon \overset{a}{\rightarrow} q, X) \\
 \bigoplus_{Y \in \Gamma} \bigoplus_{r \in Q} \bigoplus_{a \in \Sigma \cup \{\varepsilon\}} \bigoplus_{\pi \in \Pi(r, Y, s)} \delta(r, Y \overset{a}{\rightarrow} q, X)
\]

In general, allsums cannot be computed recursively, as \(w[p, X, q]\) may rely on allsums that are yet to be computed. Instead, we assume that \(W\) is continuous and solve the system of nonlinear equations using fixed-point iteration or the semiring generalization of Newton’s method (Esparza et al., 2007).

This algorithm computes \(O(|Q|^3|\Gamma|^2)\) items. An allsum algorithm based on Lang’s algorithm would have computed \(O(|Q|^5|\Gamma|^2)\) items; thus we have reduced the space complexity by a factor of \(|\Gamma|^2\).

### 6 Allsums in Bottom-up WPDA

We can use a notion of push computation similar to Def. 14 to derive a space-efficient algorithm for computing allsums in bottom-up PDAs. The item \([p, X, q]\) stands for runs from state \(p\) to state \(q\) that push the symbol \(X\) on top of the stack while leaving the symbols underneath unchanged.

**Definition 20.** Let \(\Pi (p, X, q)\) be the set containing all push computations from state \(p\) to state \(q\) that push \(X\). The allsum \(w[p, X, q]\) is defined as

\[
 w[p, X, q] = \bigoplus_{\pi \in \Pi(p, X, q)} w(\pi).
\]

The allsum of a set of push computations can be expressed in terms of other allsums:

\[
 w[p, X, q] = \bigoplus_{a \in \Sigma \cup \{\varepsilon\}} \delta(p, \varepsilon \overset{a}{\rightarrow} q, X) \\
 \bigoplus_{Y \in \Gamma} \bigoplus_{r \in Q} \bigoplus_{a \in \Sigma \cup \{\varepsilon\}} \bigoplus_{\pi \in \Pi(r, Y, s)} \delta(r, Y \overset{a}{\rightarrow} q, X)
\]

In general, allsums cannot be computed recursively, as \(w[p, X, q]\) may rely on allsums that are yet to be computed. Instead, we assume that \(W\) is continuous and solve the system of nonlinear equations using fixed-point iteration or the semiring generalization of Newton’s method (Esparza et al., 2007).

This algorithm computes \(O(|Q|^3|\Gamma|^2)\) items. An allsum algorithm based on Lang’s algorithm would have computed \(O(|Q|^5|\Gamma|^2)\) items; thus we have reduced the space complexity by a factor of \(|\Gamma|^2\).

### 7 Conclusion

Our study has contributed several results and algorithms whose weighted CFG analogues have long been known, but have previously been missing for weighted PDAs—a normal form analogous to Chomsky normal form and a stringsum algorithm analogous to weighted CKY. But it has also revealed some important differences, confirming that the study of weighted PDAs is of interest in its own right. Most notably, we identified two different normal forms and two corresponding stringsum algorithms (and two allsum algorithms). Since the only existing PDA stringsum algorithm we are aware of, Lang’s algorithm, is better suited to bottom-up PDAs, whereas the more standard definition of PDAs is of top-down PDAs, our algorithm for top-down WPDA fills a significant gap.

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Limitations

Removal of nullary transitions, while similar to removal of nullary rules from a WCFG, is conceptually more complicated, and while it has the same asymptotic complexity, it's currently unknown how the two would compare in practice. Additionally, our nullary removal construction requires a commutative semiring, while removal of nullary productions from a WCFG does not.

Our algorithm for top-down WPDTAs processes a string from right to left, which may be undesirable in some NLP applications and in models of human sentence processing.

Ethics Statement

The authors foresee no ethical concerns with the research presented in this paper.

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A Details of Unary Removal

Since $U$ is a $|Q| \times |\Gamma|$ matrix, computing its transitive closure takes $O((|Q||\Gamma|)^3)$ time. However, if we perform nullary removal first, the stack alphabet could grow by a factor of $|Q|^2$ because of the stack symbols $r_sZ$, which would seem to make the transitive closure take $O((|Q|^3|\Gamma|)^3)$ time.

For comparison, if we converted the PDA to a CFG, it would have $O(|Q|^2|\Gamma|)$ nonterminals, so computing the transitive closure of the unary rules would take $O((|Q|^2|\Gamma|)^3)$ time.

But the graph formed by the unary transitions can be decomposed into several strongly connected components (SCCs), many of which are identical, so the transitive closure can be sped up considerably. Define three matrices for three different forms of unary transitions:

$$U_{ptZ,qX}^1 = \delta(p, r, Z \xrightarrow{\varepsilon} q, r_sX)$$
$$U_{q_pX,qX}^2 = \delta(q, p, X \xrightarrow{\varepsilon} q, X^\#)$$
$$U_{pY,qX}^3 = \delta(p, Y^\# \xrightarrow{\varepsilon} q, X^\#).$$

There are no transitions of the form $p, Y^\# \xrightarrow{a/w} q, r_sX$. Note that in the first equation, the transition weight does not depend on $r$, so $r$ does not occur on the left-hand side. Then let

$$V = U^1 U^2 U^3$$

so that $V_{psY,qX}$ is the total weight of runs from configurations of the form $(p, r, Y)$ to configurations of the form $(q, X^\#)$.

Finally, we remove the unary transitions and modify the non-unary transitions as follows:

For each non-unary: Replace with:

$$p, Y \xrightarrow{a/w} q, r_sX \quad p, Y \xrightarrow{a/w@U_{psX,taw}^1} t_{rs}Y$$
$$p, Y \xrightarrow{a/w@V_{psX,aw}^2} t, Y^\#$$
$$p, Y \xrightarrow{a/w} q, X^\# \quad p, Y \xrightarrow{a/w@U_{psX,taw}^3} r, Y^\#$$

Since $V$ can be computed in $O((|Q|^2|\Gamma|)^3)$ time, the asymptotic time complexity of removing nullary and unary transitions is the same when performed directly on the WPDA as compared with converting to a WCFG and removing nullary and unary rules.

B Fast Bottom-up Stringsum Algorithm

Fig. 7 shows an alternative deductive system for parsing in bottom-up WPDAAS. The algorithm that can be derived from this deductive system achieves a runtime improvement by a factor of $|Q|$ and has the same space complexity as Lang’s algorithm. This algorithm, however, does not achieve further time and space complexity improvements on the special type of automaton used by Lang.