Motivated by the wide range of modern applications of the Erlang-B blocking model beyond communication networks and call centers to sizing and pricing in design production systems, messaging systems, and app-based parking systems, we study admission control for such a system but with unknown arrival and service rates. In our model, at every job arrival, a dispatcher decides to assign the job to an available server or block it. Every served job yields a fixed reward for the dispatcher, but it also results in a cost per unit time of service. Our goal is to design a dispatching policy that maximizes the long-term average reward for the dispatcher based on observing only the arrival times and the state of the system at each arrival that reflects a realistic sampling of such systems. Critically, the dispatcher observes neither the service times nor departure times so that standard reinforcement learning-based approaches that use reward signals do not apply. Hence, we develop our learning-based dispatch scheme as a parametric learning problem 

\textit{a'la} self-tuning adaptive control. In our problem, certainty equivalent control switches between an \textit{always admit if room} policy (explore infinitely often) and a \textit{never admit} policy (immediately terminate learning), which is distinct from the adaptive control literature. Hence, our learning scheme judiciously uses the always admit if room policy so that learning doesn’t stall. We prove that for all service rates, the proposed policy asymptotically learns to take the optimal action and present finite-time regret guarantees. The extreme contrast in the certainty equivalent optimal control policies leads to difficulties in learning that show up in our regret bounds for different parameter regimes: constant regret in one regime versus regret growing logarithmically in the other.

**Key words:** Queueing theory, maximum likelihood estimation, resource allocation, adaptive control.

1. **Introduction**

Queueing systems are widely applicable models used to study resource allocation problems in communication networks, distributed computing systems, supply chains, semiconductor manufacturing, and many other dynamic systems. Queueing models are analyzed under various system information settings, but a common assumption is that the core system parameters like arrival rates, service rates and distributions are available to the system designer (e.g., Srikant and Ying (2013), Harchol-Balter (2013)). However, there are various applications where these parameters are
unknown, and the designer needs to learn them to be able to optimally assign jobs to the servers or block them. For example, the service rate of every server in large-scale server farms may be unknown, or the treatment times in hospitals may be unpredictable and time-varying.

The focus of this paper is the Erlang-B system (Kelly (2011), Srikant and Ying (2013)). The traditional use of this system has been for sizing and analyzing voice and circuit-switched systems, i.e., loss systems. It is also used for sizing and analyzing call-center systems (Gans et al. (2003)) with no waiting room and no reneging. Furthermore, it has been employed to study multiple-access schemes in wireless networks (Marbach et al. (2011)). More recently, these systems have been used to design and size production systems, for instance, by Amazon for its SimpleDB database service, Facebook for the back-end of its chat service, WhatsApp for its messaging servers, Motorola in call processing products used for public safety, etc. These applications motivate us to study a learning problem for an Erlang-B queuing system. Specifically, our problem formulation aligns with the call-center systems mentioned above, assuming the call center is operated by a third-party entity and the servers are homogeneous. It also extends to applications such as the pricing of parking lots in app-based parking systems or messaging systems implemented using third-party cloud servers.

Motivated by such modern applications and to highlight challenges in learning-based optimal control, we study optimal admission control in an Erlang-B queueing system with unknown arrival and service rates, denoted by $\lambda$ and $\mu$, with the goal of designing an optimal learning-based dispatching policy. At every arrival, the dispatcher can accept or block the arrival. Accepted jobs incur a service cost $c$ per unit time, and yield a fixed reward $R$. Assuming that the service rate is known, the dispatcher can maximize its expected reward using a threshold policy: if the service rate exceeds $c/R$, all arrivals are admitted subject to availability; otherwise, all arrivals are rejected. When the service rate equals $c/R$, the dispatcher is indifferent between admitting or rejecting arrivals.

A key aspect of our problem setting is that the information available to the dispatcher consists only of the inter-arrival times and the number of busy servers at each arrival, as the system is sampled at arrivals. Contrarily, the service rate, departure times, and service times are not known to the dispatcher. Hence, the dispatcher cannot form a direct estimate of the service rate (e.g., by taking an empirical average of the observed service times) to then choose its policy and instead has to use the queueing dynamics to estimate the service time for policy determination. This facet of the problem brings it closer to practice but also complicates the analysis. Based on this information structure, our focus is to design an optimal policy that maximizes the long-term average reward.

We study the problem of learning the service rate in the framework of parametric learning of a stochastic dynamic system. Specifically, consider a stochastic system governed by parameter $\theta$:

$$X_{t+1} = \mathcal{F}_t(X_t, U_t, W_t; \theta), \quad t = 0, 1, \ldots$$ (1)
where \( X_t \in \mathcal{X}, U_t \in \mathcal{U}, W_t \in \mathcal{W} \) are the state of the system, control input, and noise at time \( t \) and \( \mathcal{F}_t \) is any measurable function. Further, \( \theta \in \Theta \) is a fixed but unknown parameter, and the initial state and the noise process are mutually independent. In keeping with the bulk of the literature, we focus on a system in which our controller *perfectly observes* the state \( X_t \) and uses the history of its observations to choose the control \( U_t \). For a specified reward function \( r_t(x, u) \) for \( (x, u) \in \mathcal{X} \times \mathcal{U} \), the objective is to maximize the long-term reward. We also assume that the optimal policy \( \mathcal{G}^*(\cdot; \theta) \) is known for every \( \theta \in \Theta \). Section [A](#) provides details for finding optimal policies for general rewards.

To achieve the optimization objective whilst “learning” the unknown parameter \( \theta \), an adaptive control law is applied: using past observations \( X_{1:t} \), an estimate \( \hat{\theta}_{t+1} \) is formed, and then \( \mathcal{G}^*(\cdot; \hat{\theta}_{t+1}) \), the optimal policy according to \( \hat{\theta}_{t+1} \), is applied; note the use of the certainty equivalent control law. One approach to form the estimate \( \hat{\theta}_{t+1} \) is to use the maximum likelihood estimate (MLE). [Mandl (1974)](#) proved that under identifiability, the MLE converges to the true parameter. When these conditions do not hold, to guarantee convergence, [Kumar and Becker (1982)](#) and [Kumar and Lin (1982)](#) utilized reward bias-based exploration schemes to ensure asymptotic optimality.

Our problem fits the above paradigm: the system state \( X_t \) is the number of busy servers at time \( t \) with the dispatcher observing the (continuous-time) system state perfectly at arrivals. Given the policy characterization described earlier, the unknown parameter is the service rate \( \mu \), so \( \Theta = \mathbb{R}_+ \). Using an adaptive control law with forced exploration, we propose a dispatching policy to maximize the long-term average reward. Our main analysis-related contributions are:

1. **Asymptotic optimality.** We prove the convergence of our proposed learning-based policy to the optimal policy. We first focus on a single-server Erlang-B queueing system; see Section [4.1](#). An underlying independence structure lets us establish optimality using the strong law of large numbers. For the multi-server setting, the independence structure does not hold anymore, and in Section [5.1](#) a more intricate argument based on martingale sequences is used to prove convergence.

2. **Finite-time performance analysis.** In Section [4.2](#) we characterize the finite-time regret for the single-server system in two distinct service rate regimes. In the first regime, we show finite regret using independence and concentration inequalities. However, in the other regime, the exploration done by our policy leads to a regret upper bound that scales as \( \log(n) \), where \( n \) is the number of arrivals. We also generalize our results to the multi-server setting to observe that the regret exhibits similar behavior as in the single-server setting; see Section [5.2](#) The analysis for the multi-server setting is based on Doob’s decomposition and concentration inequalities for martingale sequences.

We end by contrasting our problem with the broader literature on learning in stochastic dynamic systems. In our work, we study an example of a parametric learning problem for which we do not

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1 More generally, we can take both the arrival and service rates, \( \lambda \) and \( \mu \), to be the unknown parameters; see Section [A](#).
We discuss the above point in Figure 1, which demonstrates the regret performance of our algorithm for functions $f(n) \in \{n^{2.5}, \exp(n^{0.6}), \exp(n)\}$ where $1/f(n)$ is proportional to the (forced) exploration probability. For $f(n) = n^{2.5}$, exploration is employed aggressively, which results in better performance when the service rate $\mu \in (c/R, +\infty)$, and higher regret in the other regime. On the other hand, when $f(n) = \exp(n)$, aggressive exploitation is enforced, which leads to the opposite behavior. For $\mu \in (c/R, +\infty)$, we show finite regret for $f(n) \in \{n^{2.5}, \exp(n^{0.6})\}$ in Section 5.2, but finite regret is not guaranteed for $f(n) = \exp(n)$ in our analysis. In Section 5.2, for $\mu \in (0, c/R)$, we establish an upper bound that is a polynomial in $\log(n)$ for $f(n) = \exp(n^{0.6})$. Similar arguments lead to a logarithmic upper bound for the regret of $f(n) = \exp(n)$ in the same regime. Based on this discussion, we anticipate that the performance of any proposed algorithm will depend on the parameter regime, and good performance in one regime will lead to worse regret in the other. Further, we also conjecture that when $\mu \in (0, c/R)$, there is an $\Omega(\log(n))$ regret lower bound. We expect this to be true based on our experimental results, and as it is consistent with the lower bound on the asymptotic growth of regret from the literature on learning in unknown stochastic dynamic systems under the assumption that the transition kernels of the underlying controlled Markov chains are strictly bounded away from 0; see Agrawal et al. (1989), Graves and Lai (1997).

Our simulation results in Section 6 investigate other aspects that highlight the subtleties in designing learning schemes. For example, they provide evidence that depending on the relationship between the arrival rate and the service rate, sampling our continuous-time system at a faster rate than the arrivals could reduce the regret. We also show that subtle differences in variable updates
in the learning scheme have a substantial impact on the regret achieved. Thus, the choice of the trade-off of regret between the different parameter regimes determines the learning scheme.

1.1. Related Work

Adaptive control. The self-tuning adaptive control literature studies asymptotic learning in the parametric or non-parametric version of the problem described in (1), and the study was initiated by Mandl. Mandl (1974) showed that the MLE converges to the true parameter under an identifiability condition. Since then, the adaptive control problem has been vastly studied in great generality; see Borkar and Varaiya (1979), Kumar and Becker (1982), Kumar and Lin (1982), Agrawal et al. (1989), Graves and Lai (1997), Gopalan and Mannor (2015). Learning in queueing systems is one of the applications in this literature; see Lai and Yakowitz (1995), Kumar and Varaiya (2015).

A core assumption in the above literature is that the transition kernels of the underlying controlled Markov chains are strictly bounded away from 0 and 1, with the bound uniform in the parameter and the class of (optimal) policies. This core assumption does not hold in our problem: the controlled Markov chain found by sampling the queueing system at arrivals has drastically different behavior under the available class of policies—admit if room or never admit—, and thus the conclusions of this literature do not apply. Furthermore, in the above literature, most of the results are on asymptotic learning, and only recently, finite-time regret guarantees have been obtained. The existing finite-time regret guarantees are largely for certain discrete-time queueing systems with geometrically distributed service times and unknown parameters, which we will discuss below.

Queueing systems. There is a growing body of work on learning-based control in discrete-time queueing systems; see Walton and Xu (2021) for a recent survey. In Krishnasamy et al. (2018), when stability holds, finite regret is shown in a discrete-time multi-class, multi-server queueing problem using a forced exploration-based scheme; regret is defined with respect to the $c\mu$ rule in a system with known service rates. The service rates and the arrival rates of the system are chosen such that the system is in the stability region. Moreover, in the multi-server setting, more restrictive conditions are imposed on the problem parameters to ensure geometric ergodicity and stability. Krishnasamy et al. (2021) also studies learning-based resource allocation in a discrete-time multi-class multi-server queue with unknown class-based service rates, and uses modified UCB and Thompson sampling algorithms to establish polylogarithmic regret bound. Both of these works form empirical service rate estimates by observing and averaging service completions.

In Choudhury et al. (2021), an upper bound on expected regret is proved using a randomized routing-based algorithm in a multi-server discrete-time queueing system. This work assumes no knowledge of service rates and queue lengths, but the dispatcher observes the service times, which are utilized to propose a policy with $\tilde{O}(\sqrt{T})$ finite-time regret, where $T$ is the time horizon. Stahlbuhk et al. (2021) studies the problem of finding the optimum server for service in a
discrete-time multi-server system with unknown service rates and a single queue. Finite regret is achieved for this problem by sampling service rates during idle periods. Ojeda et al. (2021) employ generative adversarial networks to numerically learn the unknown service time distributions in a $G/G/\infty$ queuing system. Zhong et al. (2022) studies the scheduling problem in a multi-class queue with abandonment, where the arrival, service, and abandonment rates are all unknown. By observing the service and patience times, an empirical estimate of unknown service and abandonment rates is formed, and then, using an exploration-exploitation based algorithm, logarithmic regret is established compared to the $c\mu/\theta$ rule. In the context of single-server queuing systems, Zhang et al. (2022) consider a social welfare maximizing admission control problem in an $M/M/1$ queuing system with unknown service and arrival rates for which a specific threshold-based admission control scheme based on the arrival and services rate is known to be optimal (see Naor (1969)). Working with an information structure where the queue is observed at all times, their proposed dispatching algorithm achieves constant regret for one set of parameters, and $O(\log^{1+\epsilon}(n))$ regret for any $\epsilon > 0$ for another set of parameters, where $n$ is the number of arrivals.

In all these works, completed service times are available as the queueing process is observed continuously. This aspect is particularly relevant during the forced exploration time-periods used for parameter estimation and learning. Observing the queue continuously or storing service times may not be feasible in real-world queueing systems due to greatly increased computation and memory requirements: see Stidham (1985), Harchol-Balter (2013). Multiple server settings introduce yet another complication: to correctly identify completed service times, the assignments of the servers also needs to be tracked from the entire process history (even with a homogeneous set of servers). In our work, observations are the (minimal) Markov state of the system at each arrival, which despite being a nonlinear function of service times, is closer to what is possible in real systems. The continuous-time setting of our problem is also novel as it brings up sampling-related issues, which we discuss in our simulations in Section 6.

Learning-based decision-making has also been studied in inventory control and dynamic pricing. Agrawal and Jia (2022) study an inventory control problem with unknown demand distribution. The goal is to minimize the total cost associated with inventory holding and lost sales penalties over $T$ periods by observing the minimum of demand and inventory. A learning algorithm is proposed based on the convexity of the average cost function under the benchmark base-stock policies, and a $O(\sqrt{T})$ regret is established. Chen et al. (2023) study a dynamic pricing problem in a $GI/GI/1$ queue with the objective of determining the optimal service fee and service capacity that maximize the cumulative expected profit. A gradient-based online-learning algorithm is proposed that estimates the gradient of the objective function from the history of arrivals, waiting times, and the server’s busy times and a logarithmic regret bound in the total number of served customers is
established. In another work, Jia et al. (2022) study a price-based revenue management problem with finite reusable resources under price-dependent unknown arrival and service rates. The goal is to find the optimal pricing policy that maximizes the total expected revenue by observing the inter-arrival and service times. Two different online algorithms based on Thompson Sampling and Upper Confidence Bound are proposed, and a cumulative regret upper bound of $\tilde{O}(\sqrt{T})$ is proved, where $T$ is the time-horizon.

Another related line of work focuses on the use of pricing strategies to regulate queue sizes and studies differences between individually optimal and socially optimal strategies (with model parameters known). Naor (1969) studied regulating an $M/M/1$ queue with fixed reward and linear holding cost, which was then generalized in Knudsen (1972) to an $M/M/k$ queuing model with fixed reward and nonlinear holding cost. Similarly, Lippman and Stidham Jr (1977) investigated a stochastic congestion system with random reward and linear holding cost and argued that individuals acting in self-interest over-congest a system relative to the socially optimal rule. In these works, to ensure social optimality, customers are subject to a toll upon joining the queue to counteract the increased congestion when agents selfishly optimize. In Johansen and Stidham (1980), the authors study a general stochastic system with random rewards and non-linear waiting costs, in which each arriving customer can be accepted or rejected. This work again concludes that an individually optimal admission policy accepts more input than a socially optimal rule.

**Reinforcement learning (RL).** Recently, RL methods have been applied to queueing problems with the objective of finding the average cost optimal routing policy. Dai and Gluzman (2022) considers learning optimal parameterized policies in queueing networks with known parameters (for model and costs). When the system parameters are unknown but rewards for actions in all states are available, Massaro et al. (2019) utilizes policy-gradient methods to learn the optimal admission policy in a multiclass queue with a finite buffer. These methods do not apply to our setting as we neither observe the reward sequence directly nor know the expected rewards: the (random) reward is a linear function of the service time of each accepted job which is not observed, and the expected reward for state-action pairs is a function of the arrival and service rates which are unknown; see Section A. We only observe the state of the system, which is a nonlinear and complex function of the received reward. In contrast to the model-agnostic viewpoint in RL, in our problem, the knowledge of the queueing dynamics is exploited to design an algorithm matched to our setting. Although RL methods do not apply to our setting, in Section 6 we consider a fictitious setup wherein the service times are directly observed ahead of the time. In this setup, we implement an average reward RL algorithm, R-learning (Sutton and Barto (2018)), to learn the optimal policy. Despite not observing the service times, our proposed policy outperforms R-learning, which provides evidence that model-class knowledge can be as effective as (in fact, even
better than) observing the reward signal in learning the unknown parameter in stochastic dynamic systems; see Figure [4]. In Section [6] we also compare our algorithm to a Thompson sampling based algorithm Gopalan and Mannor (2015). Using this we can demonstrate that our algorithm that uses model-class knowledge is again as effective as Thompson sampling; again, see Figure [4].

1.2. Organization

The paper is organized as follows. In Section [2] we introduce the problem and the learning objective. Section [3] discusses our learning-based dispatching policy. In Section [4] we analyze the asymptotic behavior of our proposed policy for a single-server Erlang-B queueing system and prove convergence to the optimal policy. Moreover, we characterize the regret of our proposed policy compared to the system with knowledge of the service rate. Section [5] extends the results of Section [2] to the multi-server setting. In Section [6] we study the performance of our proposed policy through experiments and verify our theoretical analysis, and finally, we conclude in Section [7].

2. Problem Formulation

We consider an $M/M/k/k$ queueing system with $k$ identical servers. Arrivals to the system are according to a Poisson process with rate $\lambda$, and at each arrival, a dispatcher decides between admitting the arrival or blocking it. If admitted, the arrival is dispatched to the first available server and serviced with exponentially distributed service times with parameter $\mu$. Otherwise, if blocked, it leaves the system. Each time an arrival is accepted, the dispatcher receives a fixed reward $R$ but incurs a cost of $c$ per unit time service. For simplicity, we suppose that rejecting an arrival does not have any penalty. In our setting, we assume that the dispatcher knows the parameters $R$ and $c$ but has no information about the service rate $\mu$. We also assume that the dispatcher observes the arrival times to the system (but not the arrival rate $\lambda$) and the system state upon arrivals. In contrast to the inter-arrival times, the service times of previous arrivals are unknown.

Consider the queueing system sampled at arrival $i$ for $i \in \{0, 1, \ldots\}$, and let $A_i$ denote the action of the dispatcher to admit or block arrival $i$. If arrival $i$ is blocked, $A_i = 0$; otherwise, if arrival $i$ is admitted, $A_i = 1$ (where this action is only taken if there is an available server). We define $N_i$ as the number of busy servers just before arrival $i$, and let $N_0 = 0$. Let $T_i$ be the inter-arrival time between arrival $i - 1$ and $i$, and $M_i$ be the number of departures during $T_i$. Notice that $N_{i-1} + A_{i-1} = M_i + N_i$ and the value of $M_i$ can be found with the knowledge of $\{N_{i-1}, N_i, A_{i-1}\}$. The dispatcher chooses $A_i$ based on past observations up to arrival $i$, i.e., $\mathcal{H}_i = \{T_1, \ldots, T_i, A_0, A_1, \ldots, A_{i-1}, N_0, N_1, \ldots, N_i\}$.

Using these observations, the dispatcher’s goal is to choose action sequence $\{A_n\}_{n=0}^{\infty}$ to maximize the expected average reward per unit time, which by PASTA (Srikant and Ying (2013)) is

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[K(A_i, \sigma_i)]$$

where $\sigma_i$ is the service time of arrival $i$, and the reward function $K(\cdot, \cdot)$ depends only on arrival $i$ with $K(a, s) = a(R - cs)$; Section [A] describes the general cost problem.
Consider the system where the dispatcher knows the parameter $\mu$. Then, the optimal policy of the dispatcher is to accept all arrivals if $\mu > c/R$ (subject to the availability of the server) and block all arrivals if $\mu < c/R$. We evaluate the performance of a candidate policy with respect to the optimal policy, denoted by $\Pi^\star$. In Section 3, we propose a dispatching policy that uses past observations to estimate the service rate $\mu$, and in Sections 4.1 and 5.1, we show the asymptotic optimality of our proposed policy by proving its convergence to $\Pi^\star$. Further, in Sections 4.2 and 5.2, the finite-time performance of our policy is evaluated using the following definition of the expected regret.

**Definition 1.** Set $A_i^\Pi$ as the action taken at arrival $i$ in a system that follows policy $\Pi$. The expected regret of a policy $\Pi$ with respect to the optimal policy $\Pi^\star$ after $n$ arrivals is defined as

$$
E \left[ R(n); \Pi \right] = \left| E \left[ \sum_{i=0}^{n-1} (A_i^\Pi - A_i^{\Pi^\star}) \right] \right|.
$$

**3. Proposed Maximum Likelihood Estimate-based Dispatching Policy**

In our problem setting, both the arrival and service rates, $\lambda$ and $\mu$, respectively, are unknown, but for the optimal dispatching policy it is sufficient to estimate the service rate. We would like a dispatching policy that (asymptotically) performs optimally, and further, (if possible) we want to minimize the regret of this system with respect to the system in which $\mu$ is known. As mentioned in the introduction, we will take a self-tuning adaptive control viewpoint for this learning problem. Hence, we will consider the system as being driven by parameter $\mu$, and the learning problem as a parameter estimation problem using system measurements given by the sequence of policies chosen. Specifically, we will use maximum likelihood (ML) estimation to estimate the parameter $\mu$, and then we will select the certainty equivalent control but with forced exploration.

**3.1. Maximum Likelihood Estimate Derivation**

In this section, we derive the log-likelihood function and the corresponding MLE. The probability of $m_i$ departures and $n_i$ incomplete services at inter-arrival $t_i$ given $m_i + n_i = N_{i-1} + A_{i-1}$ is

$$
p(m_i, n_i, t_i; \mu) = \binom{n_i + m_i}{n_i} \left(1 - \exp(-\mu t_i)\right)^{m_i} \left(\exp(-\mu t_i)\right)^{n_i}.
$$

From (2), the conditional probability of observing sequences $\{m_i\}_{i=1}^n$ and $\{n_i\}_{i=1}^n$ for a fixed $\mu$ given the inter-arrival sequence $\{t_i\}_{i=1}^n$ is given by

$$
P(M_1 = m_1, \ldots, M_n = m_n, N_1 = n_1, \ldots, N_n = n_n \mid \mu, \{t_i\}_{i=1}^n) = \prod_{i=1}^{n} p(m_i, n_i, t_i; \mu).
$$
In our problem formulation, no prior distribution is assumed for \( \mu \), and thus, the posterior probability of a fixed \( \mu \) given observations of \( \{m_i\}_{i=1}^n, \{n_i\}_{i=1}^n \) and \( \{t_i\}_{i=1}^n \) is proportional to (3). From (2) and (3), we form the likelihood function of the past observations \( \mathcal{H}_n \) under parameter \( \mu \) as
\[
L(\mathcal{H}_n; \mu) := c_b \prod_{i=1}^n (1 - \exp(-\mu t_i))^{M_i} (\exp(-\mu t_i))^{N_i},
\]
where \( c_b \) is the product of the binomial coefficients found in (2) and independent of \( \mu \). Maximization of function \( L(\mathcal{H}_n; \mu) \) is equivalent to maximization of log-likelihood function \( l(\mathcal{H}_n; \mu) \) defined as
\[
l(\mathcal{H}_n; \mu) := \log L(\mathcal{H}_n; \mu) = \log c_b + \sum_{i=1}^n M_i \log (1 - \exp(-\mu t_i)) - \mu \sum_{i=1}^n N_i t_i.
\]
If \( M_i = 0 \) for all \( i \), the maximum of \( l(\mathcal{H}_n; \mu) \) in \( [0, +\infty) \) is obtained for \( \mu = 0 \), and if \( N_i = 0 \) for all \( i \), the maximum is reached at \(+\infty\). Otherwise, from differentiability and strict concavity of the log-likelihood function, it has at most one maximizer, and as \( \lim_{\mu \to 0} l(\mathcal{H}_n; \mu) = \lim_{\mu \to +\infty} l(\mathcal{H}_n; \mu) = -\infty \), there exists a unique \( \hat{\mu}_n > 0 \) that maximizes \( l(\mathcal{H}_n; \mu) \), which can be found by taking the derivative with respect to \( \mu \) and setting it equal to 0. The derivative of \( l(\mathcal{H}_n; \mu) \) is given by
\[
l'(\mathcal{H}_n; \mu) = \sum_{i=1}^n M_i T_i \exp(-\mu t_i) \frac{1 - \exp(-\mu t_i)}{1 - \exp(-\mu t_i)} - \sum_{i=1}^n N_i T_i.
\]
From (6), the maximum likelihood estimate \( \hat{\mu}_n \) is the solution to the following equation:
\[
\sum_{i=1}^n g(T_i, M_i, \hat{\mu}_n) = \sum_{i=1}^n h(T_i, N_i, \hat{\mu}_n),
\]
where \( g(t, m, \mu) := \frac{mt \exp(-\mu t)}{1 - \exp(-\mu t)} \) and \( h(t, n, \mu) := nt \). It is easy to verify that \( \sum_{i=1}^n g(T_i, M_i, \mu) \) is a positive and decreasing function of \( \mu \). Moreover, \( \lim_{\mu \to 0} \sum_{i=1}^n g(T_i, M_i, \mu) = +\infty \) and \( \lim_{\mu \to +\infty} \sum_{i=1}^n g(T_i, M_i, \mu) = 0 \). Since \( \sum_{i=1}^n h(T_i, N_i, \mu) \) is a positive constant independent of \( \mu \), Equation (7) has a unique positive solution \( \hat{\mu}_n \). However, given the simple set of optimal policies for our problem, we do not need to solve this equation to determine our policy. For a given estimate \( \hat{\mu}_n \), the optimal policy only requires a comparison of \( \hat{\mu}_n \) and \( c/R \), and, based on the properties of \( g \) and \( h \), to compare \( \hat{\mu}_n \) with \( c/R \), it suffices to compare \( \sum_{i=1}^n g(T_i, M_i, c/R) \) with \( \sum_{i=1}^n h(T_i, N_i, c/R) \).

### 3.2. The Learning Algorithm

The discussion at the end of the previous subsection leads to the following two cases:

1. \( \sum_{i=1}^n g(T_i, M_i, c/R) > \sum_{i=1}^n h(T_i, N_i, c/R) \) implies that \( \hat{\mu}_n > c/R \).
2. \( \sum_{i=1}^n g(T_i, M_i, c/R) \leq \sum_{i=1}^n h(T_i, N_i, c/R) \) implies that \( \hat{\mu}_n \leq c/R \).

In Case 1 the MLE indicates that the **always admit if room** policy is optimal. In our proposed policy, we follow the MLE whenever Case 1 applies and admit the arrival (if there is a free server). In contrast to Case 1, the MLE in Case 2 suggests blocking all arrivals. However, if we follow the
Algorithm 1 Proposed ML estimate-based Policy for Learning the Optimal Dispatching Policy

1: **Input:** $f : \mathbb{N} \cup \{0\} \rightarrow [1, \infty)$, increasing, and $\lim_{n \to +\infty} f(n) = +\infty$.

2: Initialize $N_0 = 0, \alpha_0 = 0$.

3: At arrival $n \geq 0$, do

4: Update $\alpha_n$ using (8), and find $S(n) = \max \{0 \leq i \leq n : N_i = 0\}$.

5: if $N_n = k$ then

6: Block the arrival.

7: else if $N_n < k$ and $\sum_{i=1}^{S(n)} g(T_i,M_i,c/R) > \sum_{i=1}^{S(n)} h(T_i,N_i,c/R)$ then

8: Admit the arrival.

9: else if $N_n < k$ and $\sum_{i=1}^{S(n)} g(T_i,M_i,c/R) \leq \sum_{i=1}^{S(n)} h(T_i,N_i,c/R)$ then

10: Admit the arrival with probability $p_{\alpha_n} = 1/f(\alpha_n)$.

11: end if

MLE in both cases, we may falsely identify the service rate and incur linear regret. In particular, using the optimal policy in Case 2 results in no arrivals and new samples of the system. Thus, to ensure that learning continues in Case 2 our proposed policy will not use the certainty equivalent control with some small probability that converges to zero. Based on this discussion, we introduce Algorithm 1 for optimal dispatch in an Erlang-B queueing system with unknown service rate.

We label the policy in Algorithm 1 as $\Pi_{Alg1}$. Then $S(n)$ is defined as the last arrival instance before or at arrival $n$ when the system is empty. The probability of using the sub-optimal policy in Case 2 is equal to $p_{\alpha_n} = 1/f(\alpha_n)$, where a valid function $f : \mathbb{N} \cup \{0\} \rightarrow [1, \infty)$ is increasing and converges to infinity as $\alpha_n$ goes to infinity. Further, $\alpha_0 = 0$ and $\alpha_n$ is defined as below for $n \geq 1$

$$\alpha_n = \begin{cases} \alpha_n-1 + 1, & \text{if } \sum_{i=1}^{n-1} g(T_i,M_i,c/R) \leq \sum_{i=1}^{n-1} h(T_i,N_i,c/R), A_{n-1} = 1, N_{n-1} = 0, \\ \alpha_{n-1}, & \text{otherwise}. \end{cases} \quad (8)$$

In other words, $\alpha_n$ is the number of accepted arrivals $0 \leq l < n$ such that $\sum_{i=1}^{l} g_i(c/R) \leq \sum_{i=1}^{l} h_i$ and the system is empty right before arrival $l$. To clarify the effect of function $f$ on the performance of $\Pi_{Alg1}$, we assert that any choice of $f : \mathbb{N} \cup \{0\} \rightarrow [1, \infty)$ that increases to infinity leads to asymptotic optimality of $\Pi_{Alg1}$, as proved in Sections 4.1 and 5.1. However, the class of admissible functions is restricted in Sections 4.2 and 5.2 to provide finite-time performance guarantees.

The parameters of policy $\Pi_{Alg1}$ are only updated when the system becomes empty, i.e., at busy period boundaries, rather than at all arrivals. The reason for this modification is that the busy period boundary is a regenerative epoch that provides sufficient independence for the analysis to proceed without unnecessary complications, whereas the regret of the policy with its parameters updated at all arrivals is hard to analyze theoretically. However, this alternate policy, called $\Pi_{Alg2}$, is also asymptotically optimal, and we empirically compare its performance to $\Pi_{Alg1}$ in Section 6.
Remark 1. When our queueing system consists of a single server, the two policies Π_{Alg1} and Π_{Alg2} are one and the same. In Sections 4 and 5 to provide intuition for the more general setting of the multi-server system, we will initially discuss the case of a single-server system, and then extend our results to the multi-server setting.

4. Single-server Queueing Model

We initially focus on the single-server Erlang-B queueing system to provide a simpler pathway to the multi-server setting. In Section 4.1, we prove that asymptotic learning holds for the proposed policy Π_{Alg1} for any \( \mu \in (0, +\infty) \) and valid function \( f \). In Section 4.2, we evaluate the finite-time performance of our proposed policy in terms of the expected regret defined in Definition 1.

4.1. Asymptotic Optimality

In the single-server setting, the policy Π_{Alg1} is equivalent to the policy Π_{Alg2} that updates at each arrival, so \( S(n) = n \) in Algorithm 1. With this in place, we describe a stochastic process whose limiting behavior will determine the performance of our learning scheme. Define \( \tilde{X}^n = (X^n, N^n, \alpha^n) = \left( \sum_{i=1}^{n} (g(T_i, M_i, c/R) - h(T_i, N_i, c/R)), N^n, \alpha^n \right) \). (9)

We can see that the action at arrival \( n \) defined by Π_{Alg1} is uniquely determined by \( \tilde{X}^n \). Specifically, if the server is available and \( X^n \) is positive, the arrival will be accepted. On the contrary, if \( X^n \) is negative, the arrival will be admitted with probability \( p_{\alpha^n} \). To prove asymptotic optimality, the main idea is to show that eventually, \( X^n \) will always be positive for \( \mu > c/R \), and negative for \( \mu < c/R \). Note that in the stochastic process defined above, \( X^n \) is updated as below

\[
X^n - X^{n-1} = g(T_n, M_n, c/R) - h(T_n, N_n, c/R).
\] (10)

In (10), random variables \( N_n \) and \( M_n \) only depend on the history through the previous state \( \tilde{X}^{n-1} \). In addition, \( \alpha_n \) is updated from \( \tilde{X}^{n-1} \) by \( \tilde{X}^n \). Thus, the stochastic process \( \{\tilde{X}^n\}_{n=0}^{\infty} \) forms a Markov process. Random variables \( \{X^n - X^{n-1}\}_{n=1}^{\infty} \) are not independent since values of \( N_n \) and \( M_n \) depend on \( \tilde{X}^{n-1} \). Hence, it is not straightforward to analyze the asymptotic behavior of the Markov process \( \{\tilde{X}^n\}_{n=0}^{\infty} \). We will define a new stochastic process that will address this issue and establish convergence results for this process. Define \( \{\beta_n\}_{n=0}^{\infty} \) as the sequence of the indices of accepted arrivals, and down-sample \( \{\tilde{X}^n\}_{n=0}^{\infty} \) using sequence \( \{\beta_n\}_{n=0}^{\infty} \) to get the process \( \{\tilde{Y}^n\}_{n=0}^{\infty} \) given by

\[
\tilde{Y}_n = \tilde{X}_{\beta_n} = (X_{\beta_n}, N_{\beta_n}, \alpha_{\beta_n}) =: (Y_n, 0, \alpha_{\beta_n}),
\] (11)

where we define \( Y_n = X_{\beta_n} \) and note that \( N_{\beta_n} = 0 \) as the server is empty just before an arrival is accepted. To ensure this process is well-defined, in Lemma 11, we prove that the number of accepted arrivals following Π_{Alg1} is almost surely infinite; see Section B.1.
Let $E_n$ be the potential service time of arrival $n$ and define $l_n$ as the first arrival after $\beta_n$ that the server is available, i.e., $l_n = \min_m \{ m \geq 1 : \sum_{j=1}^{m} T_{\beta_n+j} \geq E_{\beta_n} \}$. In the following lemma, using the memoryless property of the exponentially-distributed service times and inter-arrival times, we investigate the behavior of $l_n$ by constructing an alternate representation of the process $\{ Y_{n} \}_{n=0}^{\infty}$.

**LEMMA 1.** Random variables $\{ l_n \}_{n=0}^{\infty}$ are geometric, independent, and identically distributed.

The proof of Lemma 1 is given in Section B.2. From the above observation, in Lemma 2, we prove that random variables $\{ Y_n - Y_{n-1} \}_{n=1}^{\infty}$ are independent and identically distributed.

**LEMMA 2.** Random variables $\{ Y_n - Y_{n-1} \}_{n=1}^{\infty}$ are independent and identically distributed.

**Proof of Lemma 2.** We can write $Y_n - Y_{n-1}$ as follows

$$
Y_n - Y_{n-1} = \sum_{j=1}^{\beta_n - \beta_{n-1}} (X_{\beta_n-1+j} - X_{\beta_{n-1}+j-1}) = - \sum_{j=1}^{l_{n-1}-1} T_{\beta_{n-1}+j} + g(T_{\beta_{n-1}+l_{n-1}},1,c/R). \tag{12}
$$

The above equation holds from (10) and the fact that for $j < l_{n-1}$, $N_{\beta_{n-1}+j} = 1$ (the server is busy); otherwise, $N_{\beta_{n-1}+j} = 0$. Also, $M_{\beta_{n-1}+j} = 1$ for $j = l_{n-1}$, and 0 otherwise, which means the arrival has departed in inter-arrival $T_{\beta_{n-1}+l_{n-1}}$. Note that $M_{\beta_{n-1}+j}$ and $N_{\beta_{n-1}+j}$ are both equal to 0 for $l_{n-1} + 1 \leq j \leq \beta_n - \beta_{n-1}$, as the server remains empty until an arrival is accepted. Moreover, $\{ T_n \}_{n=0}^{\infty}$ are independent and identically distributed for different values of $n$. From the fact that $\{ l_n \}_{n=0}^{\infty}$ are also independent and identically distributed geometric random variables, Lemma 2 follows.

Notice that $Y_n$ is the partial sums of independent and identically distributed random variables. As a result, by the strong law of large numbers, we observe that $Y_n$ converges to infinity with the sign depending on $E[Y_n - Y_{n-1}]$. We now present the main result of this subsection in Theorem 1 which proves the asymptotic optimality of policy $\Pi_{\text{Alg1}}$ for any $\mu > 0$ in the single-server setting and argues that convergence of $Y_n$ results in the convergence of $\Pi_{\text{Alg1}}$ to the optimal policy.

**THEOREM 1.** Consider a single-server Erlang-B queueing system with service rate $\mu$. For any $\mu \in (0, +\infty)$, policy $\Pi_{\text{Alg1}}$ converges to the true optimal policy $\Pi^*$. Specifically, for $\mu \in (c/R, +\infty)$, random variable $Y_n$ converges to $+\infty$ almost surely and the proposed policy admits all arrivals after a random finite time subject to the availability of a free server. Similarly, for $\mu \in (0, c/R)$, the random variable $Y_n$ converges to $-\infty$ almost surely and after a random finite time, an arrival is only accepted with a probability that converges to 0 as $n \to +\infty$.

**Proof of Theorem 1.** We assume that $\mu > c/R$, and prove Theorem 1. The proof of case $\mu < c/R$ follows similarly. We first find $E[Y_{i+1} - Y_i]$ using (12) as below

$$
E[Y_{i+1} - Y_i] = \sum_{m=1}^{\infty} \mathbb{P}(l_i = m) E[- \sum_{j=1}^{l_{i-1}} T_{\beta_i+j} + g(T_{\beta_i+l_i},1,c/R) \mid l_i = m]. \tag{13}
$$

We have

$$
E[T_{\beta_i+j} \mid l_i = m, j < l_i] = \frac{\mu + \lambda}{\lambda} \int_{t=0}^{+\infty} \int_{x=t}^{+\infty} t \mu \exp(-\mu x) \lambda \exp(-\lambda t) \, dx \, dt = \frac{1}{\lambda + \mu},
$$

but
To study the finite-time performance of $\Pi_{\text{Alg1}}$, which the system is empty for the first time after $\beta$ (random) finite time, the arrival is accepted whenever the server is available. \(\text{arrivals is made based on the sign of } Y\) binds the expected number of times

Specifically, for $X_{\beta_n+1} = M_{\beta_n+1} = 0$ and in the inter-arrivals after the departure of arrival $\beta_n$, the server remains empty. Hence, for $l_n + 1 \leq j \leq \beta_n + 1 - \beta_n$, \(\text{X}_{\beta_n+1} = M_{\beta_n+1} = 0\), meaning that from arrival $\beta_n + l_n$ at which the system is empty for the first time after $\beta_n$, the decision to accept or reject the following arrivals is made based on the sign of $Y_{n+1}$, which is eventually always positive. Thus, after a (random) finite time, the arrival is accepted whenever the server is available.

\[X_{\beta_n+j} = X_{\beta_n+l_n} + \sum_{i=1}^{j} (g(T_{\beta_n+j}, 0, \frac{c}{R}) - h(T_{\beta_n+j}, 0, \frac{c}{R})) = X_{\beta_n+l_n} \]

Specifically, for $j = \beta_n+1 - \beta_n$, we have $X_{\beta_n+1} = X_{\beta_n+l_n} = Y_{n+1}$, meaning that from arrival $\beta_n + l_n$ at which the system is empty for the first time after $\beta_n$, the decision to accept or reject the following arrivals is made based on the sign of $Y_{n+1}$, which is eventually always positive. Thus, after a (random) finite time, the arrival is accepted whenever the server is available.

\[4.2. \text{ Finite-time Performance Analysis}\]

To study the finite-time performance of $\Pi_{\text{Alg1}}$, we characterize the regret in terms of the processes $\{\hat{X}_n\}_{n=0}^{\infty}$ and $\{\hat{Y}_n\}_{n=0}^{\infty}$. As the sign of $\{Y_n\}_{n=0}^{\infty}$ determines the acceptance law, we would like to upper bound the expected number of times $Y_n$ has an undesirable sign, or it is non-positive when $\mu > c/R$ and positive in the other regime. In Lemma 2, we showed that $\{Y_{n+1} - Y_n\}_{n=0}^{\infty}$ are independent and identically distributed. We further show $Y_{n+1} - Y_n$ is sub-exponentially distributed in Lemma 3.

Lemma 3. Random variables $\{Y_{n+1} - Y_n\}_{n=0}^{\infty}$ are sub-exponentially distributed.
The intuition behind Lemma 3 is: from (12), random variable $Y_{n+1} - Y_n$ can be written as the sum of exponential random variables and a bounded random variable. We formalize the above argument in Section B.3. The results of Lemmas 2 and 4 allow us to use Bernstein’s concentration inequality for the sum of independent sub-exponential random variables and establish an exponentially decaying upper bound for the probability of $Y_n$ resulting in a suboptimal action.

**Lemma 4.** Consider a single-server Erlang-B queueing system with service rate $\mu$ following policy $\Pi_{\text{Alg1}}$. For $\mu \in (c/R, +\infty)$, there exists a positive problem-dependent constant $c_1$ such that

$$\mathbb{P}(Y_n \leq 0) \leq \exp(-c_1 n),$$

and for any $\mu \in (0, c/R)$, for a positive problem-dependent constant $c_2$, the following holds

$$\mathbb{P}(Y_n \geq 0) \leq \exp(-c_2 n).$$

Lemma 4 is proved in Section B.4. We first give an upper bound for the expected regret when $\mu > c/R$. In this regime, when $Y_n$ is positive, $\Pi_{\text{Alg1}}$ follows the optimal policy $\Pi^*$. However, for non-positive $Y_n$, the arrival is only admitted with a given probability. We quantify the impact of the arrivals for which $Y_n$ is non-positive using the exponentially decaying probability established in Lemma 4. Finally, in Theorem 2 we prove that for $\mu \in (c/R, +\infty)$, the expected regret is finite.

**Theorem 2.** Consider a single-server Erlang-B queueing system with service rate $\mu$. For any $\mu \in (c/R, +\infty)$ and (valid) function $f$ such that $\log(f) = o(n)$, the expected regret $\mathbb{E}[R(n); \Pi_{\text{Alg1}}]$ under policy $\Pi_{\text{Alg1}}$ is upper bounded by a constant independent of $n$.

**Proof of Theorem 2.** We define $H_n$ as the number of times an arrival is rejected between arrival $\beta_n$ and $\beta_{n+1}$ when the server is available. Consider the system that accepts all arrivals subject to availability, or follows the optimal policy for $\mu > c/R$; call this system $Q^{(0)}$. We couple $Q^{(0)}$ with our system from the first arrival so that we can ensure whenever our system is busy, $Q^{(0)}$ is also busy and rejects the arrival. Thus, we have the following upper bound for the expected regret:

$$\mathbb{E}[R(n); \Pi_{\text{Alg1}}] \leq \mathbb{E}\left[\sum_{i=0}^{\infty} H_i\right] = \sum_{i=0}^{\infty} \mathbb{E}\left[H_i \mid Y_{i+1} < 0\right] \mathbb{P}(Y_{i+1} \leq 0),$$

where the equality follows from the fact that when $Y_{i+1}$ is positive, the number of rejected arrivals, $H_i$, is zero. Conditioned on the event $\{Y_{i+1} \leq 0\}$, $H_i$ is geometric with parameter $1/f(\alpha_{\beta_i + l_i})$, where $\alpha_{\beta_i + l_i}$ is less than or equal to the number of admitted arrivals up to $\beta_i + l_i$, which is equal to $i + 1$. Consequently, using Lemma 4 we find an upper bound for the expected regret as follows

$$\mathbb{E}[R(n); \Pi_{\text{Alg1}}] \leq \sum_{i=0}^{\infty} f(i+1) \mathbb{P}(Y_{i+1} \leq 0) \leq \sum_{i=0}^{\infty} f(i+1) \exp(-c_1 (i+1)).$$

The above summation converges if $f$ grows slower than the exponential function. \(\square\)
Next, we present the finite-time performance guarantee when \( \mu < c/R \). In this regime, the expected regret consists of two terms. The first term arises from the arrivals for which \( Y_n > 0 \), and we use the exponentially decaying probability of \( \text{Lemma } 5 \) to bound this term. The second term results from the arrivals accepted with a given probability when \( Y_n \) is non-positive. We will use \( \text{Lemma } 5 \) presented below to address this term; proof is given in \( \text{Section B.5} \). In conclusion, \( \text{Theorem } 3 \) proves a polynomial in \( \log n \) upper bound for the expected regret in the case of \( \mu \in (0, c/R) \).

**Lemma 5.** Let \( f(n) = \exp(n^{-\epsilon}) \) and \( d = \lceil 3(\log(n+1))^{1/\epsilon} \rceil \) for a fixed \( \epsilon \in (0, 1) \). Then, for independent geometric random variables \( \{y_i\}_{i=1}^n \) with corresponding success probabilities \( \{f(i)^{-1}\}_{i=1}^n \), the sum \( \sum_{i=d}^{n-1} i \mathbb{P}(y_1 + \cdots + y_i < n, y_1 + \cdots + y_{i+1} \geq n) \) is bounded by a constant determined by \( \epsilon \).

**Theorem 3.** Consider a single-server Erlang-B queueing system with service rate \( \mu \in (0, c/R) \). For \( f(n) = \exp(n^{-\epsilon}) \), the expected regret under policy \( \Pi_{\text{Alg1}} \) is \( \mathbb{E} [\mathcal{R}(n); \Pi_{\text{Alg1}}] = O(\log^{1-\epsilon}(n)) \).

**Proof of Theorem 3.** For \( \mu \in (0, c/R) \), the optimal policy rejects all arrivals, and thus, the expected regret is equal to the expected number of accepted arrivals up to arrival \( n \), or

\[
\mathbb{E} [\mathcal{R}(n); \Pi_{\text{Alg1}}] = \mathbb{E} \left[ \sum_{i=0}^{n-1} \mathbb{1} \{A_i = 1\} \right] = \mathbb{E} \left[ \sum_{i=0}^{n-1} \mathbb{1} \{A_i = 1, X_i > 0\} \right] + \mathbb{E} \left[ \sum_{i=0}^{n-1} \mathbb{1} \{A_i = 1, X_i \leq 0\} \right],
\]

where \( X_i \) is the first component of the state of the Markov chain defined in \( (9) \). Using the sampled process \( \{Y_n\}_{n=0}^\infty \) and \( \text{Lemma } 4 \) we simplify the first term on the RHS of the above equation as,

\[
\mathbb{E} \left[ \sum_{i=0}^{n-1} \mathbb{1} \{A_i = 1, X_i > 0\} \right] \leq \sum_{i=0}^{+\infty} \mathbb{P} (Y_i > 0) \leq \sum_{i=1}^{+\infty} \exp (-c_2 i) < \infty. \tag{16}
\]

We next upper bound the expected number of arrivals accepted when \( X_i \leq 0 \). We consider a system that has infinite servers (to avoid rejecting arrivals) and regardless of the sign of \( X_i \), accepts with probability \( 1/f(i) \), if \( i \) arrivals have already been accepted (the acceptance rule is compatible with the original system when \( X_i \leq 0 \)). By coupling this system with the system following \( \text{Algorithm } \Pi \) taking \( d = \lceil 3(\log n)^{1/\epsilon} \rceil \) and \( \{y_i\}_{i=1}^n \) as defined in \( \text{Lemma } 5 \)

\[
\mathbb{E} \left[ \sum_{i=0}^{n-1} \mathbb{1} \{A_i = 1, X_i \leq 0\} \right] \leq \mathbb{E} \left[ \sum_{i=d}^{d-1} \mathbb{1} \{A_i = 1, X_i \leq 0\} \right] + \mathbb{E} \left[ \sum_{i=d}^{n-1} \mathbb{1} \{A_i = 1, X_i \leq 0\} \right] \\
\leq d + \sum_{i=d}^{n-1} i \mathbb{P} (y_1 + \cdots + y_i < n, y_1 + \cdots + y_{i+1} \geq n).
\]

Finally, By \( \text{Lemma } 3 \) and \( (16) \), the expected regret is bounded by a polylogarithmic function. □

**Remark 2.** There is an exploration-exploitation trade-off in selecting \( f(n) \) on the two sides of \( \mu = c/R \). When admitting is optimal, we want \( f(n) \) to increase to infinity as slow as possible. Also, based on the proof of \( \text{Theorem } 2 \) for our current bound, we cannot take \( f(n) \) to grow exponentially fast since its exponent needs to depend on unknown \( \mu \) to ensure constant regret. Conversely, when
blocking all arrivals is optimal, we need \( f(n) \) to converge to infinity as fast as possible. As the learning algorithm needs to be agnostic about the parameter regime, \( f(n) = \exp(n^{-1-\epsilon}) \) is a good choice: it ensures constant regret in one regime and polynomial regret in \( \log n \) in the other.

We next consider a decreasing sequence of \( \epsilon \) values by choosing \( \epsilon_n = \frac{\epsilon}{\sqrt{1+\log(n+1)}} \) for all \( n \geq 1 \), where \( \epsilon \in (0, 1) \). The algorithm corresponding to the exploration function \( f(n) = \exp(n^{-1-\epsilon_n}) \) is asymptotically optimal from Theorem 1. To determine the regret when \( \mu > c/R \), we observe that \( \log(f) = o(n) \) and the regret in this regime remains finite. For the case of \( \mu < c/R \), we are able to reduce the order of regret further to \( \log n \), as shown in Corollary 1 with proof in Section B.6.

**Corollary 1.** Consider a single-server Erlang-B queueing system with service rate \( \mu \in (0, c/R) \). For \( f(n) = \exp(n^{-1-\epsilon_n}) \) where \( \epsilon_n = \frac{\epsilon}{\sqrt{1+\log(n+1)}} \) for all \( n \geq 1 \) and \( \epsilon \in (0, 1) \), the expected regret under policy \( \Pi_{\text{Alg1}} \) is \( E[R(n); \Pi_{\text{Alg1}}] = O(\log(n)) \).

**Remark 3.** For some parameters our problem setting overlaps with the setting studied in Zhang et al. (2022): when \( \mu \leq c/R \), our setting can be viewed as learning in an \( M/M/1 \) system with the optimal admission threshold being 0, and when \( c/R < \mu \leq h(\lambda, c/R) < +\infty \) (for a function \( h(\cdot, \cdot) \)), our setting corresponds to an \( M/M/1 \) system with an optimal admission threshold of 1. However, our work samples the system only at arrivals, in contrast to Zhang et al. (2022) which samples the system at all times (so that service times of departed customers are known). Despite the information structure in our problem providing less information, our proposed policy exhibits the same regret behavior as Zhang et al. (2022) as shown in Corollary 1 and Theorem 2.

## 5. Multi-server Queueing Model

In this section, we extend the results of Section 4 to the multi-server Erlang-B queueing system. In Section 5.1, the convergence of \( \Pi_{\text{Alg1}} \) to the optimal policy is established by a martingale-based analysis coupled with the strong law of large numbers for martingale sequences. Moreover, in Section 5.2, we prove that the regret bounds of a single-server system hold in the context of a multi-server setting using concentration inequalities for martingale sequences.

### 5.1. Asymptotic Optimality

First, we extend the processes defined in Section 4.1 to the multi-server setting. Define \( \{\tilde{X}_n\}_{n=0}^{\infty} \) as

\[
\tilde{X}_n = (X_n, N_n, \alpha_n) = \left( \sum_{i=1}^{n} \left( g(T_i, M_i, c/R) - h(T_i, N_i, c/R) \right), N_n, \alpha_n \right).
\]

As in the single-server case, \( \{\tilde{X}_n\}_{n=0}^{\infty} \) forms a Markov process. Our goal is to down-sample the Markov process \( \{\tilde{X}_n\}_{n=0}^{\infty} \) at arrival acceptances for which the system is empty and establish convergence results for the resulting process. Similar to the single-server case, we first argue that these instances happen infinitely often (almost surely) in Section C.1. Let \( \{\beta_n\}_{n=0}^{\infty} \) be the sequence of
the indices of accepted arrivals when the system is empty. We down-sample the Markov process \( \{\tilde{X}_n\}_{n=0}^{\infty} \) using the sequence \( \{\beta_n\}_{n=0}^{\infty} \) to get process \( \{\tilde{Y}_n\}_{n=0}^{\infty} \) where \( Y_n \) is defined as \( Y_n := X_{\beta_n} \), and

\[
\tilde{Y}_n := \tilde{X}_{\beta_n} := (X_{\beta_n}, N_{\beta_n}, \alpha_{\beta_n}) = (Y_n, 0, \alpha_{\beta_n}).
\] (18)

In the single-server setting, the processes depicted in (17) and (18) are equivalent to the processes defined in (9) and (11), as the single-server system is empty whenever an arrival is accepted.

In contrast to the single-server case, random variables \( \{Y_n - Y_{n-1}\}_{n=1}^{\infty} \) are not independent in the multi-server setting, as here, unlike the single-server setting, \( Y_n - Y_{n-1} \) depends on the acceptance probabilities. We will argue that process \( \{Y_n\}_{n=0}^{\infty} \) is a submartingale (or supermartingale), and using this result, we will analyze its convergence. We define random variable \( D_i \) as the change in \( X_i \) at inter-arrival \( T_i \), i.e., \( D_i := X_i - X_{i-1} \). Next, for any \( n \geq 0 \), we define process \( \{W_{n,m}\}_{m=0}^{\infty} \) as

\[
W_{n,m} = Y_n + \sum_{i=1}^{m} D_{\beta_n+i} = X_{\beta_n+m}.
\] (19)

We define \( \tau_n \) as the index of the first arrival after \( \beta_n \) that finds the system empty, i.e., \( \tau_n = \min \{i \geq 1 : N_{\beta_n+i} = 0\} \). Note that by (19), \( W_{n,\tau_n} = X_{\beta_n+\tau_n} \). We claim that process \( \{X_n\}_{n=0}^{\infty} \) at the first arrival acceptance after \( \tau_n \), i.e., \( X_{\beta_n+1} \), is equal to \( W_{n,\tau_n} \). Indeed, process \( \{X_n\}_{n=0}^{\infty} \) does not change when there are no departures or ongoing services. Hence, \( W_{n,0} = Y_n \) and \( W_{n,\tau_n} = X_{\beta_n+1} = Y_{n+1} \). Thus, to analyze the convergence of \( \{Y_n\}_{n=0}^{\infty} \), we study the properties of process \( \{W_{n,m}\}_{m=0}^{\infty} \) and random variable \( \tau_n \) for \( n \geq 1 \). We determine the behavior of \( \tau_n \) by coupling the system that runs Algorithm 1 with a system that accepts all arrivals (subject to availability) as follows.

Let \( Q^{(n)} \) denote the system that accepts all arrivals as long as it has at least one available server. We also define random variable \( \zeta_n \) as the first arrival after arrival \( \beta_n \) that finds \( Q^{(n)} \) empty, starting from an empty state. Starting from arrival \( \beta_n \), we couple this system with the system that follows Algorithm 1 such that at each arrival, the number of busy servers in \( Q^{(n)} \) is greater than or equal to our system. We couple the arrival sequences in both systems such that the inter-arrival times are equal. Moreover, when an arrival is accepted in both systems, we assume that its service time is identical in both. System \( Q^{(n)} \) will accept all arrivals unless none of its servers are available. Suppose all of the servers of \( Q^{(n)} \) are busy, and our system accepts an arrival. In this case, we assume that the service time of the accepted arrival in our system equals the remaining service time of the \( k \)th server in \( Q^{(n)} \), which has an exponential distribution with parameter \( \mu \) due to the memoryless property. Using this coupling, we verify that all moments of \( \tau_n \) are finite in Lemma 6.

**Lemma 6.** All moments of random variable \( \tau_n \) are bounded by a constant independent of \( n \).

**Proof of Lemma 6.** By the above coupling of \( Q^{(n)} \) with the system that follows our proposed policy, we ensure that at each arrival, the number of busy servers in \( Q^{(n)} \) is greater than or equal
to our system. Hence, the moments of \( \tau_n \) are bounded by the moments of \( \zeta_n \). In system \( Q^{(n)} \), the number of busy servers just before each arrival forms a finite-state irreducible Markov chain, and random variable \( \zeta_n \) is the first passage time of the state zero starting from zero, and has moments bounded by a constant which only depends on \( \lambda, \mu \) and the number of servers. \( \square \)

After characterizing the behavior of \( \tau_n \), in Lemma 7 we show that the process \( \{W_{n,m}\}_{m=0}^\infty \) is a submartingale or supermartingale depending on the sign of \( \mu - c/R \).

**Lemma 7.** Fix \( n \geq 0 \). For \( \mu \in (c/R, c/R + \infty) \), the stochastic process \( \{W_{n,m}\}_{m=0}^\infty \) forms a submartingale sequence with respect to the filtration \( \{\mathcal{G}_{n,m}\}_{m=0}^\infty \), wherein the \( \sigma \)-algebra \( \mathcal{G}_{n,m} \) is defined as

\[
\mathcal{G}_{n,m} = \sigma(T_{\beta_{n+1}}, \ldots, T_{\beta_{n+1}}, N_{\beta_{n+1}}, \ldots, N_{\beta_{n+1}}, \alpha_{\beta_{n+1}}, \ldots, \alpha_{\beta_{n+1}}, A_{\beta_{n+1}}, \ldots, A_{\beta_{n+1}}, Y_n).
\]

For \( \mu \in (0, c/R) \), the process \( \{W_{n,m}\}_{m=0}^\infty \) is a supermartingale with respect to filtration \( \{\mathcal{G}_{n,m}\}_{m=0}^\infty \).

**Proof of Lemma 7.** We show the proof for the case of \( \mu > c/R \). The other region follows similarly. To prove \( \{W_{n,m}\}_{m=0}^\infty \) is a submartingale sequence, we first show \( \mathbb{E} |W_{n,m}| < \infty \). From (19),

\[
\mathbb{E} |W_{n,m}| \leq \mathbb{E} |Y_n| + \sum_{i=1}^m |D_{\beta_{n+i}}| \leq \mathbb{E} |Y_n| + \sum_{i=1}^m \left| g(T_{\beta_{n+i}}, M_{\beta_{n+i}}, c/R) - h(T_{\beta_{n+i}}, N_{\beta_{n+i}}, c/R) \right|
\]

\[
\leq \mathbb{E} |Y_n| + k \sum_{i=1}^m \left( \mathbb{E} \left| g(T_{\beta_{n+i}}, 1, c/R) \right| + \mathbb{E} |T_{\beta_{n+i}}| \right),
\]

where (20) holds as \( 0 \leq M_{\beta_{n+i}}, N_{\beta_{n+i}} \leq k \). For \( t > 0 \), we have \( g(t, 1, x) \leq \frac{1}{x} \), and thus, the summation in (20) is finite. To show that \( \mathbb{E} |Y_n| < \infty \), it suffices to show \( \mathbb{E} |Y_{n+1} - Y_n| \) is finite for all \( n \):

\[
\mathbb{E} |Y_{n+1} - Y_n| = \mathbb{E} |W_{n,\tau_n} - Y_n| = \mathbb{E} \left( \sum_{i=1}^{\tau_n} D_{\beta_{n+i}} \right) \leq k \mathbb{E} \left( \sum_{i=1}^{\tau_n} \left( T_{\beta_{n+i}} + g(T_{\beta_{n+i}}, 1, c/R) \right) \right)
\]

\[
\leq k \mathbb{E} \left( \sum_{i=1}^{\zeta_n} \left( T_{\beta_{n+i}} + g(T_{\beta_{n+i}}, 1, c/R) \right) \right) = k \mathbb{E} [\zeta_n \mathbb{E} \left( T_{\beta_{n+1}} + g(T_{\beta_{n+1}}, 1, c/R) \right)],
\]

where (21) is derived similar to (20) and (22) follows from the coupling of system \( Q^{(n)} \) and the system that runs Algorithm 1. Specifically, the hitting time \( \zeta_n \) is a stopping time for the finite-state irreducible Markov chain found by sampling \( Q^{(n)} \) just before an arrival and \( \mathbb{E}[\zeta_n] < \infty \). Hence, (22) follows from Wald’s equation (Durrett 2019, Theorem 4.8.6), and \( \mathbb{E} |Y_{n+1} - Y_n| < \infty \), which implies that \( \mathbb{E} |Y_n| < \infty \), and by (20), \( \mathbb{E} |W_{n,m}| < \infty \). We next verify the submartingale property of \( \{W_{n,m}\}_{m=0}^\infty \). From the Markov property of \( \{X_n\}_{n=0}^\infty \),

\[
\mathbb{E} [W_{n,m+1} - W_{n,m} | \mathcal{G}_{n,m}] = \mathbb{E} [X_{\beta_{n+m+1}} - X_{\beta_{n+m}} | X_{\beta_{n+m}}, N_{\beta_{n+m}}, \alpha_{\beta_{n+m}}, A_{\beta_{n+m}}],
\]

which is equal to the expected change in \( X_t \) during inter-arrival \( T_{\beta_{n+m+1}} \). To show \( \mathbb{E}[W_{n,m+1} - W_{n,m} \mathcal{G}_{n,m}] \geq 0 \), we argue that \( \mathbb{E}[X_{i+1} - X_i | X_i, N_i, \alpha_i, A_i] \) is non-negative for all \( i \) as follows,

\[
\mathbb{E} [X_{i+1} - X_i | X_i, N_i, \alpha_i, A_i] \]
where \( \tilde{\sigma} \) is the event that a fixed server from the \( N_i + A_i \) busy servers remains busy during inter-arrival \( T_{i+1} \). We derive \( \mathbb{E}[g(T_{i+1}, 1, c/R)] \) using the same calculations as in (14),

\[
\mathbb{E}[g(T_{i+1}, 1, c/R)] = \int_{t=0}^{\infty} \int_{x=t}^{\infty} t \exp\left(-\frac{t c}{R}\right) \lambda \exp\left(-\lambda t \right) \exp\left(-\mu x \right) dx dt = (N_i + A_i) \frac{\lambda}{(\lambda + \mu)^2}.
\]  

(25)

The second term of (24) can be simplified as follows

\[
\mathbb{E}[N_{i+1}g(T_{i+1}, 1, c/R) \mid N_i, A_i] = (N_i + A_i) \mathbb{E}[g(T_{i+1}, 1, c/R) \mathbb{1}_A]
\]

\[
= (N_i + A_i) \int_{t=0}^{\infty} \int_{x=t}^{\infty} t \mu \exp\left(-\mu x \right) \lambda \exp\left(-\lambda t \right) dx dt = (N_i + A_i) \frac{\lambda}{(\lambda + \mu)^2}.
\]  

(26)

Next, we simplify the third term of (24):

\[
(N_i + A_i) \mathbb{E}[T_{i+1} \mathbb{1}_A] = (N_i + A_i) \int_{t=0}^{\infty} \int_{x=t}^{\infty} t \mu \exp\left(-\mu x \right) \lambda \exp\left(-\lambda t \right) dx dt = (N_i + A_i) \frac{\lambda}{(\lambda + \mu)^2}.
\]

Substituting the terms found in the above equation, (25), and (26), in Equation (24), we get

\[
\mathbb{E}[X_{i+1} - X_i \mid X_i, N_i, \alpha_i, A_i] = \tilde{\delta} (N_i + A_i),
\]  

(27)

where \( \tilde{\delta} := \frac{\lambda}{(\lambda + \mu)^2} + \sum_{j=0}^{\infty} \frac{\lambda}{(\lambda + \mu + (j+1) \frac{c}{R})^2} \) and is positive for \( \mu \in (c/R, +\infty) \). Hence, from (23) and (27),

\[
\mathbb{E}[W_{n,m+1} - W_{n,m} \mid \mathcal{G}_{n,m}] = \tilde{\delta} (N_{\beta_n+m} + A_{\beta_n+m}) \geq 0,
\]  

(28)

and we conclude that \( \{W_{n,m}\}_{m=0}^{\infty} \) is a submartingale sequence with respect to \( \{\mathcal{G}_{n,m}\}_{m=0}^{\infty} \). \( \square \)

Next, in Proposition 1 we argue that the stopped sequence \( \{W_{n,\tau_n}\}_{n=0}^{\infty} \) or \( \{Y_n\}_{n=0}^{\infty} \) also forms a submartingale or supermartingale sequence depending on the problem parameters.

**Proposition 1.** \( \{Y_n\}_{n=0}^{\infty} \) forms a submartingale or supermartingale sequence (depending on the sign of \( \mu - c/R \)) with respect to filtration \( \{\mathcal{F}_n\}_{n=0}^{\infty} \) defined as \( \mathcal{F}_n = \sigma(Y_0, \ldots, Y_n, \alpha_{\beta_0}, \ldots, \alpha_{\beta_n}) \). Specifically, \( \{Y_n\}_{n=0}^{\infty} \) is a submartingale sequence if \( \mu > c/R \) and a supermartingale otherwise.

**Proof of Proposition 1.** We show the proof for the case of \( \mu > \frac{c}{R} \), and the other regime follows similarly. Note that \( Y_{n+1} \) is equal to the submartingale \( \{W_{n,m}\}_{m=0}^{\infty} \) stopped at \( \tau_n \); in other words, \( Y_{n+1} = W_{n,\tau_n} = Y_n + \sum_{i=1}^{\tau_n} D_{\beta_n+i} \). In Lemma 6 we argued that \( \mathbb{E}[\tau_n] < \infty \). Moreover,

\[
\mathbb{E}[|W_{n,m+1} - W_{n,m}| \mid \mathcal{G}_{n,m}] = \mathbb{E}[|D_{\beta_n+m+1} | \mid \mathcal{G}_{n,m}] \leq k \mathbb{E}[g(T_{\beta_n+1}, 1, c/R)] + k \mathbb{E}[T_{\beta_n+1}].
\]  

(29)
As $g$ is bounded, the RHS of (29) is also finite. Hence, we can use Doob’s optional stopping theorem (Durrett 2019, Theorem 4.8.5) for submartingale $\{W_{n,m}\}_{m=0}^{\infty}$ and stopping time $\tau_n$ to get

$$E[Y_{n+1} | \mathcal{G}_{n,0}] = E[W_{n,\tau_n} | \mathcal{G}_{n,0}] \geq E[W_{n,0} | \mathcal{G}_{n,0}] = Y_n.$$ 

Thus, we have

$$E[Y_{n+1} - Y_n | \mathcal{G}_{n,0}] = E[Y_{n+1} - Y_n | \mathcal{F}_n] \geq 0.$$ 

As $E[|Y_n|]$ is finite, $\{Y_n\}_{n=0}^{\infty}$ is a submartingale sequence with respect to $\mathcal{F}_n$. □

Now that we proved the submartingale (or supermartingale) property of $\{Y_n\}_{n=0}^{\infty}$, we can examine the convergence of this process. From Proposition 1 and Doob’s decomposition of $\{Y_n\}_{n=0}^{\infty}$, we have $Y_n = Y_n^A + Y_n^M$, where $Y_n^M$ is a martingale sequence, and $Y_n^A$ is a predictable and almost surely increasing (or decreasing) sequence with $Y_0^A = 0$. In Lemmas 8 and 9, we examine the limiting behavior of sequences $\{Y_n^A\}_{n=0}^{\infty}$ and $\{Y_n^M\}_{n=0}^{\infty}$. The basic idea is to show that $\{Y_n^A\}_{n=0}^{\infty}$ converges to infinity, and $\{Y_n^M\}_{n=0}^{\infty}$ is well-behaved in a way that their sum, $\{Y_n\}_{n=0}^{\infty}$, converges to infinity.

**Lemma 8.** For $\mu \in (c/R, +\infty)$, there exists a positive problem-dependent constant $\delta_1$ such that the process $\{Y_n^A\}_{n=0}^{\infty}$ found using Doob’s decomposition of $\{Y_n\}_{n=0}^{\infty}$ satisfies

$$Y_n^A \geq \delta_1 n \quad (a.s.)$$

Similarly, for $\mu \in (0, c/R)$, there exists a negative constant $\tilde{\delta}_2$ such that the process $\{Y_n^A\}_{n=0}^{\infty}$ satisfies

$$Y_n^A \leq \tilde{\delta}_2 n \quad (a.s.)$$

**Proof of Lemma 8** WLOG, we assume $\mu \in (c/R, +\infty)$. By Proposition 1, sequence $\{Y_n\}_{n=0}^{\infty}$ is a submartingale with respect to filtration $\mathcal{F}_n$. Hence, the increasing sequence is given as below

$$Y_n^A = \sum_{m=0}^{n-1} E \left[ Y_{m+1} - Y_m | \mathcal{F}_m \right] = \sum_{m=0}^{n-1} \left( E \left[ W_{m,\tau_m} | \mathcal{F}_m \right] - Y_m \right). \quad (30)$$

In Lemma 7, we argued $\{W_{n,m}\}_{m=0}^{\infty}$ is a submartingale with respect to $\mathcal{G}_{n,m}$. From Doob’s decomposition, we get $W_{n,m} = W_{n,m}^A + W_{n,m}^M$. For the predictable process $\{W_{n,m}^A\}_{m=0}^{\infty}$, from (28),

$$W_{n,m}^A = \sum_{i=0}^{m-1} E \left[ W_{n,i+1} - W_{n,i} | \mathcal{G}_{n,i} \right] = \sum_{i=0}^{m-1} \delta \left( N_{\beta_{n,i+1}} + A_{\beta_{n,i+1}} \right). \quad (31)$$

Next, we use Doob’s optional stopping theorem for the martingale sequence $\{W_{n,m}^M\}_{m=0}^{\infty}$ to find $E \left[ W_{n,\tau_n}^M | \mathcal{F}_n \right]$. The stopping time $\tau_n$ has finite expectation as argued in Lemma 6, and

$$E \left[ \left| W_{n,i+1}^M - W_{n,i}^M \right| | \mathcal{G}_{n,i} \right] = E \left[ \left| W_{n,i+1} - W_{n,i} - (W_{n,i+1}^A - W_{n,i}^A) \right| | \mathcal{G}_{n,i} \right]$$

$$= E \left[ D_{\beta_{n,i+1}} - E \left[ D_{\beta_{n,i+1}} | \mathcal{G}_{n,i} \right] \right] \leq E \left[ |2D_{\beta_{n,i+1}}| | \mathcal{G}_{n,i} \right], \quad (32)$$
where (32) is bounded by a constant, as argued in (29). After verifying the conditions of the optional stopping theorem, we are able to use this theorem to get

$$\mathbb{E}[W_{n, \tau_n}^M | \mathcal{F}_n] = \mathbb{E}[W_{n, 0}^M | \mathcal{F}_n] = Y_n.$$  \hfill (33)

Inserting (31) and (33) back to (30), we find $Y_n^A$ as follows

$$Y_n^A = \delta \sum_{m=0}^{n-1} \mathbb{E} \left[ \sum_{i=0}^{\tau_n-1} (N_{\beta_n+i} + A_{\beta_n+i}) \right] \mathcal{F}_n.] \hfill (34)$$

Note that $A_{\beta_n} = 1$, as arrival $\beta_n$ is accepted by the definition of the sampling times $\{\beta_n\}_{n=0}^\infty$. Hence, $\mathbb{E} \left[ \sum_{i=0}^{\tau_n-1} (N_{\beta_n+i} + A_{\beta_n+i}) \right] \mathcal{F}_n] \geq 1$, which gives $Y_n^A \geq \tilde{\delta} n$. \hfill \Box

We next state the strong law of large numbers for martingale sequences in Theorem 4 and then, using this result, prove Lemma 9.

**Theorem 4.** (Shiryaev 1996, Corollary 7.3.2) Let $\{M_n\}_{n=0}^\infty$ be a martingale sequence with $M_0 = 0$ and $\mathbb{E}[|M_n|^{2r}] < \infty$ for some $r \geq 1$, and it satisfies $\sum_{n=1}^\infty n^{-(1+r)} \mathbb{E} [\|M_n - M_{n-1}\|^{2r}] < \infty$. Then,

$$\lim_{n \to \infty} \frac{M_n}{n} = 0. \quad \text{(a.s.)}$$

**Lemma 9.** The martingale process $\{Y_n^M\}_{n=0}^\infty$ found by Doob’s decomposition of $\{Y_n\}_{n=0}^\infty$ satisfies

$$\lim_{n \to \infty} \frac{Y_n^M}{n} = 0. \quad \text{(a.s.)}$$

**Proof of Lemma 9** We prove Lemma 9 for $\mu > c/R$. We first derive upper and lower bounds for the martingale difference sequence $Y_{n+1}^M - Y_n^M$. We have

$$Y_{n+1}^M - Y_n^M = Y_{n+1} - Y_n - (Y_{n+1}^A - Y_n^A) = \sum_{i=1}^{\tau_n} D_{\beta_n+i} - \mathbb{E} \left[ \sum_{i=0}^{\tau_n-1} (N_{\beta_n+i} + A_{\beta_n+i}) \right] \mathcal{F}_n \right] \hfill (35)$$

$$= \sum_{i=1}^{\tau_n} \left( g \left( T_{\beta_n+i}, M_{\beta_n+i}, \frac{c}{R} \right) - h \left( T_{\beta_n+i}, N_{\beta_n+i}, \frac{c}{R} \right) \right) - \mathbb{E} \left[ \sum_{i=0}^{\tau_n-1} (N_{\beta_n+i} + A_{\beta_n+i}) \right] \mathcal{F}_n, \right] \hfill (36)$$

where (35) is true by (34), and (36) follows from the definition of $D_i$. To derive an upper bound for the martingale difference sequence, we only consider the non-negative terms in (36) as below

$$Y_{n+1}^M - Y_n^M \leq \sum_{i=1}^{\tau_n} g \left( T_{\beta_n+i}, M_{\beta_n+i}, \frac{c}{R} \right) \leq k \frac{R}{c} \tau_n, \hfill (37)$$

which holds as for $t > 0$, we have $g \left( t, 1, x \right) \leq \frac{1}{x}$. To find a lower bound, using the non-positive terms,

$$Y_{n+1}^M - Y_n^M \geq - \sum_{i=1}^{\tau_n} h \left( T_{\beta_n+i}, N_{\beta_n+i}, \frac{c}{R} \right) - \mathbb{E} \left[ \sum_{i=0}^{\tau_n-1} (N_{\beta_n+i} + A_{\beta_n+i}) \right] \mathcal{F}_n \right] \hfill (38)$$
where we have used the definition of function $h$. From Lemma 6, $\delta k \mathbb{E} [\tau_n | \mathcal{F}_n]$ is bounded by a constant, which we call $c_3$. By (37) and (38), we have

$$-k \sum_{i=1}^{\tau_n} T_{\beta_n+i} - c_3 \leq Y_n^M - Y_{n-1}^M \leq k \frac{R}{c} \tau_n. \tag{39}$$

We next verify the conditions of Theorem 4 for the martingale sequence $Y_n^M$ with $r = 1$. From (39),

$$\mathbb{E} \left[ (Y_n^M - Y_{n-1}^M)^2 \right] \leq k^2 \frac{R^2}{c^2} \mathbb{E} [\tau_n^2] + k^2 \mathbb{E} \left[ \left( \sum_{i=1}^{\tau_n} T_{\beta_n+i} \right)^2 \right] + 2kc_3 \mathbb{E} \left[ \sum_{i=1}^{\tau_n} T_{\beta_n+i} \right] + c_3^2. \tag{40}$$

We aim to show the right-hand side of (40) is bounded by a constant independent of $n$. From Wald’s equation (Durrett 2019, Theorem 4.8.6), we have that $\mathbb{E} \left[ \sum_{i=1}^{\tau_n} T_{\beta_n+i} \right]$ is bounded by a constant. For the second term, we use Wald’s second equation (Durrett 2019, Exercise 4.8.4) for independent and identically distributed random variables $\{\tilde{T}_i\}_{i=1}^{n}$ defined as $\tilde{T}_i := T_{\beta_n+i} - \frac{1}{\lambda}$, with $\mathbb{E}[\tilde{T}_i] = 0$ for all $i$. We take $\tilde{S}_n := \sum_{i=1}^{n} \tilde{T}_i$. From Wald’s second equation, for stopping time $\tau_n$ with finite expectation, $\mathbb{E}[\tilde{S}_{\tau_n}] = \frac{1}{\lambda^2} \mathbb{E} [\tau_n]$. In addition, from the definition of $\tilde{S}_n$, we have $\mathbb{E} [\tilde{S}_{\tau_n}^2] = \mathbb{E} \left[ (\sum_{i=1}^{\tau_n} T_{\beta_n+i} - \frac{1}{\lambda})^2 \right].$

Finally, we bound the second term on the right-hand side of (40) with a constant as below

$$\mathbb{E} \left[ \left( \sum_{i=1}^{\tau_n} T_{\beta_n+i} \right)^2 \right] = \frac{1}{\lambda^2} \mathbb{E} [\tau_n] + \frac{2}{\lambda} \mathbb{E} [\tau_n \sum_{i=1}^{\tau_n} T_{\beta_n+i}] - \frac{1}{\lambda^2} \mathbb{E} [\tau_n^2] \leq \frac{1}{\lambda^2} \mathbb{E} [\tau_n] + \frac{2}{\lambda} \mathbb{E} [\sum_{i=1}^{\tau_n} 2\tau_n T_{\beta_n+i}] \leq \frac{1}{\lambda^2} \mathbb{E} [\tau_n] + \frac{1}{\lambda} \mathbb{E} \left[ \sum_{i=1}^{\tau_n} T_{\beta_n+i}^2 \right] + \frac{1}{\lambda} \mathbb{E} [\tau_n^2]. \tag{41}$$

The last line uses inequality $2xy \leq x^2 + y^2$. We argued that the moments of $\tau_n$ are bounded by the moments of first hitting time of a finite-state irreducible Markov chain found by sampling system $Q^{(n)}$, or $\zeta_n$, and thus, are finite. Hence, the first and third terms of (41) are bounded by a constant. By Wald’s equation, the second term is also bounded by a constant. In conclusion, (41) is bounded by a constant independent of $n$. Similarly, the first term on the right-hand side of (40) is also bounded by a constant. Now, we verify the condition of Theorem 4 as follows

$$\sum_{n=1}^{\infty} \mathbb{E} \left[ \left( \frac{Y_n^M - Y_{n-1}^M}{n^2} \right)^2 \right] \leq c_3 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is finite, and thus the conditions of Theorem 4 are satisfied. Consequently, by Theorem 4

$$\lim_{n \to +\infty} \frac{Y_n^M}{n} = 0, \quad (a.s.)$$

which completes the proof. \(\square\)

We now state Theorem 5, which generalizes Theorem 4 to the multi-server setting and proves the asymptotic optimality of our proposed policy for the multi-server queueing system. The proof of this theorem is based on the submartingale (or supermartingale) property of the sequence $\{Y_n\}_{n=0}^{\infty}$. 

Theorem 5. Consider the multi-server Erlang-B queueing system with \( k \) servers and service rate \( \mu \). For any \( \mu \in (0, +\infty) \), policy \( \Pi_{\text{Alg1}} \) converges to the true optimal policy \( \Pi^* \). Specifically, for \( \mu \in (c/R, +\infty) \), \( Y_n \) converges to \(+\infty\) a.s. and the proposed policy admits all arrivals after a random finite time subject to availability. Similarly, for \( \mu \in (0, c/R) \), \( Y_n \) converges to \(-\infty\) a.s. and after a random finite time, an arrival is only accepted with a probability that converges to 0 as \( n \to +\infty \).

Proof of Theorem 5. For \( \mu \in (c/R, +\infty) \), by Doob’s decomposition for submartingale \( \{Y_n\}_{n=0}^\infty \) and Lemmas 8 and 9, \( \lim_{n \to +\infty} Y_n = +\infty \) a.s. In Algorithm 1, \( X_{S(n)} \) determines the acceptance rule, and between arrival \( \beta_n \) and \( \beta_{n+1} \), \( X_{S(n)} \) is either equal to \( X_{\beta_n} = Y_n \) or \( X_{\beta_{n+1}} = Y_{n+1} \). Hence, the sign of \( Y_n \) and \( Y_{n+1} \) determines the acceptance rule between arrival \( \beta_n \) and \( \beta_{n+1} \). Thus, after a finite time, as long as there is an available server, the arrival is accepted, and the proposed policy converges to the optimal policy \( \Pi^* \). The same arguments apply for the regime of \( \mu \in (0, c/R) \).

5.2. Finite-time Performance Analysis

To extend the finite-time results of the single-server queueing system to the more general setting of the multi-server system, we characterize the regret in terms of the submartingale (or supermartingale) sequence \( \{Y_n\}_{n=0}^\infty \) and processes \( \{Y^A_n\}_{n=0}^\infty \) and \( \{Y^M_n\}_{n=0}^\infty \) found from Doob’s decomposition. As the sign of \( \{Y_n\}_{n=0}^\infty \) determines the acceptance rule, we provide an upper bound for the probability of the event that \( Y_n \) has an undesirable sign. Without loss of generality, in describing the methodology we assume that \( \mu \in (c/R, +\infty) \) and from Doob’s decomposition and Lemma 8,

\[
P(Y_n \leq 0) = P(Y^A_n + Y^M_n \leq 0) \leq P(Y^M_n \leq -\tilde{\delta}_1 n) \text{ for some } \tilde{\delta}_1 > 0.
\]

Thus, it suffices to bound \( P(Y^M_n \leq -\tilde{\delta}_1 n) \), as done in Lemma 10. The proof of Lemma 10 given in Section C.2 verifies a conditional sub-exponential property for the martingale difference sequence \( \{Y^M_{n+1} - Y^M_n\}_{n=0}^\infty \), and utilizes a Bernstein-type bound for martingale difference sequences.

Lemma 10. Consider a multi-server Erlang-B queueing system with service rate \( \mu \) following policy \( \Pi_{\text{Alg1}} \). For \( \mu \in (c/R, +\infty) \), there exists a problem-dependent constant \( c_3 \) such that

\[
P(Y^M_n \leq -\tilde{\delta}_1 n) \leq \exp(-c_3 n),
\]

and for any \( \mu \in (0, c/R) \), there exists a positive problem-dependent constant \( c_4 \) such that

\[
P(Y^M_n \geq -\tilde{\delta}_2 n) \leq \exp(-c_4 n).
\]

We begin with stating Theorem 6 that extends Theorem 2 to the multi-server setting, and argues that for the multi-server queueing system with \( \mu \in (c/R, +\infty) \) and function \( f(n) \) such that \( \log(f) = o(n) \), finite regret is achieved. The proof is similar to Theorem 2 and bounds the expected number of arrivals for which \( Y_n \leq 0 \) using the exponentially decaying probability shown in Lemma 10.
Theorem 6. Consider the multi-server Erlang-B queueing system with \( k \) servers and service rate \( \mu \). For any \( \mu \in (c/R, +\infty) \) and (valid) function \( f \) such that \( \log(f) = o(n) \), the expected regret \( \mathbb{E}[\mathcal{R}(n); \Pi_{\text{Alg1}}] \) under policy \( \Pi_{\text{Alg1}} \) is upper bounded by a constant independent of \( n \).

Proof of Theorem 6. Let \( K_n \) be the number of arrivals rejected after or at \( \beta_n + \tau_n \) and before the first acceptance, \( \beta_{n+1}, \) i.e., \( K_n = \min \{ i \geq 0 : A_{\beta_n + \tau_n + i} = 1 \} = \beta_{n+1} - \beta_n - \tau_n \). Note that if \( Y_n > 0 \), the proposed policy will accept all arrivals from \( \beta_{n-1} + \tau_{n-1} \) up to \( \beta_n + \tau_n \) (subject to availability). In this case, \( \beta_{n-1} + \tau_{n-1} = \beta_n \). But, if \( Y_n \leq 0 \), the arrivals are accepted with a certain probability and can contribute to the expected regret. Thus, we upper bound the expected regret as below

\[
\mathbb{E}[\mathcal{R}(n); \Pi_{\text{Alg1}}] \leq \mathbb{E}[\tau_0] + \mathbb{E}\left[ \sum_{i=1}^{\infty} (\tau_i + K_{i-1}) \mathbb{1} \{ Y_i \leq 0 \} \right]
= \sum_{i=0}^{\infty} \mathbb{E}[\tau_i \mathbb{1} \{ Y_i \leq 0 \}] + \sum_{i=1}^{\infty} \mathbb{E}[K_{i-1} \mathbb{1} \{ Y_i \leq 0 \}]
\leq \sum_{i=0}^{\infty} \mathbb{E}[\tau_i | Y_i \leq 0] \mathbb{P}(Y_i \leq 0) + \sum_{i=1}^{\infty} f(i) \mathbb{P}(Y_i \leq 0)
\leq \sum_{i=0}^{\infty} \mathbb{E}[\tau_i | Y_i \leq 0] \exp(-c_3i) + \sum_{i=1}^{\infty} f(i) \exp(-c_3i).
\]

In the third line, we used the fact that given \( Y_i \leq 0, K_i \) is geometric with \( \mathbb{E}[K_i] = f(i). \) The last inequality follows from (42) and Lemma 10. In Lemma 6, we argued that \( \mathbb{E}[\tau_i | Y_{i-1} \leq 0] \) is bounded by a constant. Hence, for any function \( f \) with \( \log(f) = o(n) \), the expected regret is finite. \( \square \)

Lastly, we argue that the expected regret for a multi-server queueing system with \( \mu \in (0, c/R) \) grows polylogarithmically in \( n \). Analogous to Theorem 3, we bound the expected number of arrivals wherein \( Y_n > 0 \) using Lemma 10. Moreover, we capture the effect of the arrivals being accepted with a given probability by Lemma 5 leading to a polynomial bound in \( \log n \). Further, in Corollary 1 following the same ideas as Corollary 1 we improve the regret to achieve a \( O(\log n) \) regret.

Theorem 7. Consider the multi-server Erlang-B queueing system with \( k \) servers and service rate \( \mu \in (0, c/R) \). For \( f(n) = \exp(n^{1-\epsilon}) \), the expected regret under policy \( \Pi_{\text{Alg1}} \) is \( \mathbb{E}[\mathcal{R}(n); \Pi_{\text{Alg1}}] = O(\log^{1/\epsilon}(n)). \)

Proof of Theorem 7. In this case, the expected regret up to arrival \( n \) equals the expected number of arrivals accepted from the first \( n \) arrivals. Hence, we have

\[
\mathbb{E}[\mathcal{R}(n); \Pi_{\text{Alg1}}] = \mathbb{E}\left[ \sum_{i=0}^{n-1} \mathbb{1} \{ A_i = 1 \} \right]
= \mathbb{E}\left[ \sum_{i=0}^{n-1} \mathbb{1} \{ A_i = 1, X_{S(i)} > 0 \} \right] + \mathbb{E}\left[ \sum_{i=0}^{n-1} \mathbb{1} \{ A_i = 1, X_{S(i)} \leq 0 \} \right].
\]

(43)

We first upper bound the first term using (42) and Lemma 10 as follows

\[
\mathbb{E}\left[ \sum_{i=0}^{n-1} \mathbb{1} \{ A_i = 1, X_{S(i)} > 0 \} \right] \leq \sum_{i=0}^{\infty} \mathbb{E}\left[ \mathbb{1} \{ Y_i > 0 \} \tau_i \right] \leq \sum_{i=0}^{\infty} \mathbb{E}[\tau_i | Y_i > 0] \exp(-c_4i).
\]

(44)
By Lemma 1, the above summation is bounded by a constant \( c_p \). Next, we upper bound the second term of (43). As defined before, \( \tau_i \) is the first \( j > \beta_i \) such that \( N_{\beta_i+j} = 0 \) and \( K_i \) is equal to \( \beta_i+1 - \beta_i - \tau_i \), i.e., the number of rejected arrivals before arrival \( \beta_i+1 \) and after or at \( \beta_i + \tau_i \). If \( X_{\beta_i+\tau_i} \leq 0 \), then \( K_i \) is geometric with parameter \( 1/\alpha_{\beta_i+\tau_i} \). We define \( G(i) \) as the index of the first accepted arrival after \( i-1 \) arrivals, or \( G(i) := \min_m \{ m \geq 0 : \sum_{j=0}^m (\tau_j + K_j) \geq i \} \). We also take \( F(i) \) to be the smallest \( m \) such that the sum of the first \( m+1 \) geometric trials exceeds \( i-1 \), i.e., \( F(i) := \min_m \{ m \geq 0 : \sum_{j=0}^m (K_j + 1) \geq i \} \), where \( B_m = \{ j : 0 \leq j \leq m, X_{\beta_i+j} \leq 0 \} \). From these definitions, it follows that \( G(i) \leq F(i) \). The second term of (43) is less than or equal to the expected number of times an arrival \( i < n \) with \( X_{S(i)} \leq 0 \) is accepted until arrival \( \beta_{G(n)+1} \). Therefore, we have

\[
E \left[ \sum_{i=0}^{n-1} \mathbb{1} \{ A_i = 1, X_{S(i)} \leq 0 \} \right] \leq E \left[ \sum_{i=0}^{G(n)} \tau_i \mathbb{1} \{ X_{\beta_i} \leq 0 \} \right] \leq E \left[ \sum_{i=0}^{F(n)} \tau_i \mathbb{1} \{ X_{\beta_i} \leq 0 \} \right] \\
\leq \sum_{j=0}^{n-1} \left[ \sum_{i=0}^{F(n)} \tau_i \mathbb{1} \{ F(n) = j \} \right] \mathbb{P}(F(n) = j) \\
\leq c_r \sum_{j=0}^d (j+1) \mathbb{P}(F(n) = j) + c_r \sum_{j=d+1}^{n-1} (j+1) \mathbb{P} \left( \sum_{i=1}^{j-1} y_i < n, \sum_{i=1}^j y_i \geq n \right) \quad (45) \\
\leq c_r \mathbb{E} \left[ (F(n) + 1) \mathbb{1} \{ F(n) \leq d \} \right] + c_r \sum_{j=d+1}^{n-2} (j+2) \mathbb{P} \left( \sum_{i=1}^{j-1} y_i < n, \sum_{i=1}^{j+1} y_i \geq n \right), \quad (46)
\]

where \( \{ y_i \}_{i=1}^n \) are defined in Lemma 5, \( d = \lceil 3(\log(n+1))^{1/3} \rceil \), \( c_r \) is found using Lemma 6 and is proportional to \( \sum_{j=0}^k \frac{j}{\mu^{j+1}} \). Furthermore, (45) follows from the fact that the event \( \{ F(n) = j \} \) is equivalent to the event \( \{ \sum_{i=1}^{j-1} y_i < n, \sum_{i=1}^j y_i \geq n \} \). From Lemma 5, (46) is bounded by \( c_r (d+3+c_r) \), where \( c_r \) is a constant determined by \( \epsilon \). Finally, from (44) and (46), Theorem 7 follows.

**Corollary 2.** Consider the multi-server Erlang-B queuing system with \( k \) servers and service rate \( \mu \in (0, c/R) \). For \( f(n) = \exp(n^{1-\epsilon_n}) \) where \( \epsilon_n = \frac{\epsilon}{\sqrt{1+\log(n+1)}} \) for all \( n \geq 1 \) and \( \epsilon \in (0, 1) \), the expected regret under policy \( \Pi_{\text{Alg1}} \) is \( \mathbb{E}[R(n) ; \Pi_{\text{Alg1}}] = O\left( \log(n) \right) \).

We conclude by noting that the finite-time performance guarantees of the single-server setting hold for the general setting of multiple servers. Particularly, for \( \mu \in (0, c/R) \), the expected regret is upper bounded as shown in Theorem 7 and Corollary 2 whereas, in the case of \( \mu \in (c/R, +\infty) \), a finite regret bound is achieved following our proposed policy in Algorithm 1 as argued in Theorem 4.

### 6. Simulation-based Numerical Results

In this section, we empirically evaluate the performance of policy \( \Pi_{\text{Alg1}} \). We calculate the regret by finding the difference in the number of sub-optimal actions taken by Algorithm 1 compared to the optimal policy with the knowledge of the true service rate. The regret is averaged over 2500 simulation runs and plotted versus the number of incoming jobs to the system. From our
Figure 2 Variations of regret for different service rates in a 5 server system with $\lambda = 5$, $c/R = 1.3$, $\epsilon = 0.4$, $\frac{1}{1-\epsilon} = 5/3$, and $f(n) = \exp\left(n^{1-\epsilon}\right)$ following Algorithm 1.

In Figure 3, we compare the performance of Algorithm 1 with an algorithm that updates the policy parameters at every arrival, which we call Algorithm 2. The problem parameters $\lambda, k, c, R$ are the same as the setting of Figure 2. In Algorithm 2, the admission probability decays faster than Algorithm 1, which results in less exploration and a better regret performance when $\mu < c/R$. From Figure 3a for $\mu \in (c/R, +\infty)$, Algorithm 1 has better regret performance resulting from the greater number of arrivals accepted due to the slower decaying admission probability. Another intuitive justification is that using Algorithm 1, the policy is updated after observing a collection
of arrivals, not prematurely after only one sample, and the resulting averaging (and consequent variance reduction) is useful in this regime.

In Figure 4, we compare the performance of Algorithm 1 with two other algorithms: R-learning (Sutton and Barto 2018) and Thompson sampling (Gopalan and Mannor 2015). We consider a system with $k = 5$ servers, arrival rate $\lambda = 5$, and $c/R = 1.3$. We also assume $f(n) = \exp(n^{1-\epsilon})$ with $\epsilon = 0.2$. As noted in Section 1, the R-learning algorithm assumes that the service times are known ahead of the time when an arrival is accepted. Despite not observing the service times, Figure 4 demonstrates that Algorithm 1 outperforms R-learning in both regimes. Furthermore, empirically R-learning seems to have growing regret in both parameter regimes. To implement the Thompson sampling algorithm, we use a uniform prior distribution defined on the two-point support $\{\mu_1, \mu_2\}$, where $\mu_1 = \frac{c}{2R} < \frac{c}{R}$ and $\mu_2 = \frac{3c}{2R} > \frac{c}{R}$, and update the posterior according to (4) upon every arrival. As shown in Figure 4a when $\mu > c/R$, the Thompson sampling algorithm
Figure 5: Comparison of regret performance of Algorithm $\Pi$ for different functions $f(n)$ in a 5 server system with $\lambda = 5$, $c/R = 1.3$, and $\epsilon = \varepsilon = 0.55$. 

has a better final regret value compared to our algorithm, but both algorithms have constant regret. However, in the other regime, i.e., when $\mu < c/R$, Algorithm $\Pi$ significantly outperforms Thompson sampling; empirically, the asymptotic behavior of regret of both the Thompson sampling and R-learning algorithms seem similar. We end by noting that theoretical analysis that would characterize our observed regret performance for both R-learning and Thompson sampling algorithms is not available in the literature.

In Figure 5, we compare the performance of Algorithm $\Pi$ in a 5 server system with $\lambda = 5$ and $c/R = 1.3$ for two different exploration functions $f(n) = \exp(n^{1-\epsilon})$ and $f(n) = \exp(n^{1-\epsilon_n})$, where $\epsilon_n = \frac{\epsilon}{\sqrt{1+\log(n+1)}}$ and $\epsilon = \varepsilon = 0.55$. In Corollary 2, employing $f(n) = \exp(n^{1-\epsilon_n})$ allows us to improve the order of the expected regret from $O(\log^{1-\epsilon}(n))$ to $O(\log(n))$. This improvement is further demonstrated in the numerical results presented in Figure 5b. Since $\epsilon_n$ is decreasing with $n$, the arrival acceptance resulting from the exploration decreases faster, leading to slightly inferior performance when $\mu > c/R$, as shown in Figure 5a.

We next discuss a variant of our setting in which we are able to sample the system at other time instances rather than only at the arrivals. One feasible approach is to modify the learning process as follows. Fix a deterministic sampling duration $d$. At each sampling time $t$, update functions $g$ and $h$ and the admittance probability accordingly. From any sampling time $t$, if an arrival occurs before $d$ units of time, sample the system at the arrival and decide admission according to updated parameters. Otherwise, if $d$ units of time pass without an arrival, then sample the system at $t + d$. After a new sampling is done, repeat the previous steps. Note that (as a rule of thumb) in order for sampling to contribute to the learning, the chosen sampling duration $d$ should be less than $1/\lambda$; setting $d = +\infty$ corresponds to policy $\Pi_{\text{Alg1}}$. In Figure 6, for a system with $k = 2$ servers, arrival rate $\lambda = 2$, $c/R = 1.5$, $\epsilon = 0.4$, and $f(n) = \exp(n^{1-\epsilon})$, we depict the performance of the sampling
scheme. When the service rate is higher than the arrival rate, the performance of Algorithm 1 can be improved by sampling; see Figure 6a. However, as shown in Figure 6b when sampling according to the arrival rate is fast enough, performance does not improve much with this additional sampling. Moreover, Figure 6 suggests that an adaptive sampling scheme might achieve the best trade-off.

7. Conclusions and Future Work

In conclusion, we studied the problem of learning-based optimal admission control of an Erlang-B blocking system where the service rate was not known. We showed how the extreme contrast in the optimal control schemes in different parameter regimes—quickly converging to always admitting arrivals if room in one versus quickly rejecting all arrivals in another—makes learning challenging. With the system being sampled only at arrivals, we designed a dispatching policy based on the maximum likelihood estimate of the unknown service rate followed by the certainty equivalent law coupled with a forced exploration scheme. We proved the asymptotic optimality of our proposed policy, and established finite-time performance guarantees for specific parameter settings: constant regret when $\mu > c/R$ and logarithmic regret when $\mu < c/R$. Through extensive simulations we also showed that our proposed policy achieves a good trade-off of the regret over all parameter regimes.

We plan to explore the following in future work. First, in the regime where $\mu \in (0, c/R)$, we proved a $\log n$ upper bound for the regret, where $n$ is the number of arrivals. One direction is to explore lower bounds in the regime where the never admit policy is optimal; we conjecture that it is $\Omega(\log n)$. Another direction is to allow for different sampling and update schemes (including by an independent Poisson process), and analyzing the regret theoretically. Yet another direction is whether we can generalize our results to other service-time distributions as the optimal admission control policy is unchanged due to insensitivity [Kelly (2011), Srikant and Ying (2013)] of the Erlang-B system. Lastly, a broader theory is needed to study problems like ours where the problem structure changes non-smoothly across parameter choices.
Appendix A: Erlang-B Queueing Model with a General Reward Function

In this section, we generalize the control problem discussed in Section 2 and provide a general framework for studying the extended problem setting. We consider an $M/M/k/k$ queueing system with $k$ identical servers Poisson arrival process (with rate $\lambda$), and exponentially distributed service times (with rate $\mu$). We consider the system sampled at the arrivals. Since PASTA (Srikant and Ying (2013)) ensures that arrival averages are also time-averages, many ergodic (time-average) reward problems where the reward is a function of the state (of the continuous-time Markov chain) or has a renewal-reward formulation can be considered via this approach. At arrival $i$, a controller takes action $A_i \in \{0, 1\}$ and decides between admitting the arrival, $A_i = 1$; or blocking it, $A_i = 0$. As mentioned before, the set of observations of the dispatcher at arrival $n$ is equal to $\mathcal{H}_n = \{T_1, \ldots, T_n, A_0, A_1, \ldots, A_n, N_0, N_1, \ldots, N_n\}$, where $T_i$ is the inter-arrival time between arrival $i - 1$ and $i$, $N_i$ is the state of the system sampled at arrival $i$. Particularly, service rate $\mu$ and arrival rate $\lambda$ are unknown to the dispatcher. Moreover, the dispatcher does not observe the service times of the accepted jobs.

We extend the reward function defined in Section 2 to a general one-step reward function $r : \{0, 1, \ldots, k\} \times \{0, 1\} \to \mathbb{R}$; explicitly, $r(s, a)$ is equal to the expected reward received in state $s$ when action $a$ is taken. In general, although the received (random) reward can depend on the unknown arrival and service rate and is not observed, we assume that the functional form of the expected reward function is known. The goal of the dispatcher is then to choose the action sequence $\{A_n\}_{n=0}^{\infty}$ to maximize the average expected reward.

Arapostathis et al. (1993) argues that there exists a stationary deterministic policy that achieves the optimal average reward. In our model, for every stationary deterministic policy $d : \{0, 1, \ldots, k\} \to \{0, 1\}$ such that $d(k) = 0$, the discrete-time Markov chain attained by sampling the queueing system at job arrivals forms a unichain process (Puterman (1990)); i.e., it consists of a single recurrent class and a possibly non-empty set of transient states. Denote $i^*$ as the smallest state such that $d(i^*) = 0$, or at state $i^*$, the action taken according to policy $d$ is to reject the arrival. Each $0 \leq i^* \leq k$ corresponds to a different class of stationary deterministic policies in which states $\{0, 1, \ldots, i^*\}$ form the single recurrent class and states $\{i^* + 1, \ldots, k\}$ are transient. Denote the class of stationary deterministic policies corresponding to threshold $i^*$ by $\Pi^{i^*}$. In each of the $k + 1$ different classes of the stationary deterministic policies, the underlying Markov process has a unique stationary distribution. Let $\pi^{i^*}$ be the corresponding unique stationary distribution of a Markov chain found by following a stationary deterministic policy in class $i^*$. Then, $\pi^{i^*}$ can be found as

$$
\pi^{i^*}(i) = \begin{cases} 
\frac{\frac{\lambda^i}{\mu} \prod_{j=0}^{i-1} \frac{\mu^j}{\mu^j + \lambda^j}}{\sum_{j=0}^{i-1} \frac{\mu^j}{\mu^j + \lambda^j}}, & 0 \leq i \leq i^* \\
0, & i^* + 1 \leq i \leq k
\end{cases}
$$

(47)

As the state and action space are finite and the Markov process is unichain, for every deterministic stationary policy $d \in \Pi^{i^*}$, the below limit exists, is independent of the initial state, and is given as

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}_d^i \left[ r(N_i, A_i) \right] = \sum_{i=0}^{i^*-1} r(i, 1) \pi^{i^*}(i) + r(i^*, 0) \pi^{i^*}(i^*),
$$

(48)

and the problem of finding the optimal stationary deterministic policy is equivalent to finding the optimal threshold $i^*$ such that the right-hand side of (48) is maximized, or

$$
\max_{d \in \Pi^{i^*}} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}_d^i \left[ r(N_i, A_i) \right] = \max_{0 \leq i^* \leq k} \left\{ \sum_{i=0}^{i^*-1} r(i, 1) \pi^{i^*}(i) + r(i^*, 0) \pi^{i^*}(i^*) \right\}.
$$

(49)
where $\Pi_{SD}$ is the set of all stationary deterministic policies. Note that in our problem setting, the received rewards are not observed directly. Further, as $\lambda$ and $\mu$ are unknown, the rewards cannot be found by observing the state of the system. Hence, RL methods or a direct approach to find the optimal $i^*$ in (49) are not applicable. Thus, we generalize the approach given in Section 3 to get an algorithm, presented below, for learning the optimal policy. In the algorithm, $\{p_n\}_{n=0}^\infty$ is any sequence of probabilities decaying to 0.

1. At arrival $n$, find the MLE vector $(\hat{\lambda}_n, \hat{\mu}_n)$ from the set of observations up to arrival $n$, $\mathcal{H}_n$. From the queueing dynamics, given the inter-arrival times, the conditional probability density of observing the inter-arrival sequence $\{t_i\}_{i=1}^n$ for a fixed $\lambda$ and the conditional probability of the queue evolution depending on the service times factor, so that the MLE $(\hat{\lambda}_n, \hat{\mu}_n)$ can be obtained by maximizing each term separately.

The MLE for the service rate, $\hat{\mu}_n$, is the solution to the following equation: (for derivation, see Section 3.1)

$$\sum_{i=1}^n q(T_i, M_i, \hat{\mu}_n) = \sum_{i=1}^n h(T_i, N_i, \hat{\mu}_n).$$

(50)

Furthermore, the MLE $\hat{\lambda}_n$ can be found by maximizing the log-likelihood function as $\hat{\lambda}_n = \frac{\sum_{i=1}^n T_i}{\sum_{i=1}^n t_i}$.

2. Using the MLE $(\hat{\lambda}_n, \hat{\mu}_n)$ and the functional form of the reward function, find the reward vector $r(\ldots)$.

3. Find the optimal $i^*$ in (49) and the corresponding optimal class of policies $\Pi^*$. With probability $1 - p_n$, take action $A_n$ according to this class of policies as below:

$$A_n = \begin{cases} 1, & \text{for } N_n < i^* \\ 0, & \text{for } N_n = i^* \\ \text{arbitrary}, & \text{for } N_n > i^* \end{cases}$$

Otherwise, with probability $p_n$, implement forced exploration by using the policy with threshold $k$.

As an instance, consider the original problem setting of Section 2. Here, the time spent serving a job before the next job arrival equals the minimum of the inter-arrival time and service time, which is exponential with parameter $\lambda + \mu$. Therefore, the expected reward function for every state $0 \leq i \leq k - 1$ can be found as

$$r(i, 1) = R - \frac{c(i + 1)}{\lambda + \mu}, \quad r(i, 0) = -\frac{ci}{\lambda + \mu},$$

and for $i = k$, we have $r(k, 0) = -\frac{ck}{\lambda + \mu}$. From (47), we have

$$r(i, 1)\pi^*(i) = \left(R - \frac{c}{\lambda + \mu}\right)\pi^*(i) - \frac{c}{\mu} \frac{\lambda}{\lambda + \mu} \pi^*(i - 1), \quad \text{for } i = 1, 2, \ldots, i^* - 1$$

and

$$r(i^*, 0)\pi^*(i^*) = -\frac{c}{\mu} \frac{\lambda}{\lambda + \mu} \pi^*(i^* - 1),$$

and from (48), we find the average expected reward for a Markov process following policy $d \in \Pi^*$ as

$$\sum_{i=0}^{i^*-1} r(i, 1)\pi^*(i) + r(i^*, 0)\pi^*(i^*) = \left(R - \frac{c}{\lambda + \mu} - \frac{c}{\mu} \frac{\lambda}{\lambda + \mu}\right) \sum_{i=0}^{i^*-1} \pi^*(i) = \left(R - \frac{c}{\mu}\right) \left(1 - \pi^*(i^*)\right).$$

The expression above is intuitive and follows a different interpretation of the expected reward—the total expected reward of each accepted arrival is $R - \frac{c}{\mu}$, and as we accept until $i^* - 1$ (for $i^* > 1$) but reject at $i^* - 1$,
and then, the expression easily follows. To find the optimal threshold $i^*$, notice that the Erlang-B blocking probability $\pi_i(i^*)$ is a decreasing function of $i^*$ for $i^* > 1$. As a result, the optimal policy characterization of our problem setting easily follows and depends only on the sign of $R - \frac{\alpha}{\mu}$.

Finally, we point out the difficulties that arise in the problem setting with a general reward function. First, to find the MLE for the service rate, $\hat{\mu}_n$, Equation (50) will need to be solved directly. Secondly, the optimal threshold $i^*$ may not belong to set $\{0, k\}$, which will require proving convergence of the MLE to a set for which the optimal threshold $i^*$ results (again using martingale methods). Finally, characterizing the regret is potentially the hardest step as the queueing dynamics will need to be factored in further.

Appendix B: Analysis of the Single-server Erlang-B Queueing System

B.1. Lemma [11]

Lemma 11. In the single-server Erlang-B queueing system, the number of accepted arrivals following policy $\Pi_{\text{Alg1}}$ is almost surely infinite.

Proof of Lemma [11] Let $A$ be the event that the system stops accepting new arrivals after some finite arrival, $A_1$ the event that the server is always busy after some finite arrival, $A_2$ the event that the server is available after some finite arrival but rejects all subsequent arrivals according to Line 10 of Algorithm 1 and $A_{2,m}$ as the event that for the first time at arrival $m$, the server is available but rejects all arrivals. We have

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2) = \mathbb{P}(A_2) = \sum_{m=0}^{\infty} \mathbb{P}(A_{2,m}) \leq \sum_{m=0}^{\infty} \lim_{n \to +\infty} \left(1 - \frac{1}{f(m)}\right)^n = 0,$$

where the inequality follows from the fact that for $n \geq m$, we have $\alpha_n = \alpha_m \leq m$, which means the acceptance probability is fixed after arrival $m$, as no other arrivals are accepted. From (51), we conclude that almost surely an infinite number of arrivals are accepted following Algorithm 1.

B.2. Proof of Lemma [1]

Proof of Lemma [1] Consider the queueing system sampled at sequence $\{\beta_n\}_{n=0}^{\infty}$. In the original representation, at state $\bar{Y}_n$, the exponentially distributed service time $E_{\beta_n}$ is realized when arrival $\beta_n$ is accepted. Sequence $\{T_{\beta_n+j}\}_{j=1}^{\beta_n+1-\beta_n}$ is also realized until the next accepted arrival $\beta_{n+1}$. Based on the definition of $l_n$, arrival $\beta_n$ departs during $T_{\beta_n+l_n}$. However, in the alternate process, instead of realizing $E_{\beta_n}$ all at once, at each arrival, we generate two independent exponential random variables $T'_{\beta_n+j}$ and $E'_{\beta_n+j}$, with parameters $\lambda$ and $\mu$. Let $l'_n$ be the first arrival such that $l'_n = \min\{m \geq 1 : T'_{\beta_n+m} \geq E'_{\beta_n+m}\}$. For $j < l'_n$, the minimum of two random variables $T'_{\beta_n+j}$ and $E'_{\beta_n+j}$ equals $T'_{\beta_n+j}$, and we assume the server is busy. This event occurs with probability $\lambda/(\lambda + \mu)$, and $T'_{\beta_n+j}$ indicates the inter-arrival time between arrival $\beta_n + j - 1$ and $\beta_n + j$. At $j = l'_n$, random variable $E'_{\beta_n+j}$ is less than $T'_{\beta_n+j}$ for the first time, and we assume the service of arrival $\beta_n$ is complete. For the rest of the process, we only generate the inter-arrivals $T'_{\beta_n+j}$ until an arrival is accepted. Note that $l'_n$ is geometric with parameter $\mu/(\lambda + \mu)$. The equivalence of the process defined using $T'_{\beta_n+j}$ and $E'_{\beta_n+j}$, and the original process follows from the memoryless property of the exponential distribution. Finally, we deduce that $\{l_n\}_{n=0}^{\infty}$ are independent and identically distributed geometric random variables. □
B.3. Proof of Lemma 3

Proof of Lemma 3 Instead of directly verifying that the tail decay of random variables \( \{Y_{n+1} - Y_n\}_{n=0}^\infty \) is at least as fast as an exponential distribution, we argue that an equivalent condition holds (Vershynin 2018 Proposition 2.7.1): there exists a constant \( b > 0 \) such that \( \mathbb{E} \left[ \exp \left( b |Y_{n+1} - Y_n| \right) \right] \leq 2. \) From (12),

\[
\mathbb{E} \left[ \exp \left( b |Y_{n+1} - Y_n| \right) \right] \leq \mathbb{E} \left[ \exp \left( \sum_{j=1}^{l_n-1} T_{\beta_n+j} + bg \left( T_{\beta_n+l_n}, 1, \frac{c}{R} \right) \right) \right] = \sum_{s=1}^{\infty} \mathbb{P}(l_n = s) \mathbb{E} \left[ \exp \left( \sum_{j=1}^{l_n-1} T_{\beta_n+j} + bg \left( T_{\beta_n+l_n}, 1, \frac{c}{R} \right) \right) \bigg| l_n = s \right] = \sum_{s=1}^{\infty} \mathbb{P}(l_n = s) \left( \mathbb{E} \left[ \exp \left( bT_{\beta_n+1} \right) \bigg| l_n = s \right] \right)^{s-1} \mathbb{E} \left[ \exp \left( bg \left( T_{\beta_n+l_n}, 1, \frac{c}{R} \right) \right) \bigg| l_n = s \right].
\]

(52)

For \( s > 1 \) and \( b < \lambda + \mu \), we simplify the first expectation to get

\[
\mathbb{E} \left[ \exp \left( bT_{\beta_n+1} \right) \bigg| l_n = s \right] = \frac{\mu + \lambda}{\lambda} \int_0^{\infty} \int_0^{\infty} \exp(b t) \mu \exp(-\mu x) \lambda \exp(-\lambda t) \, dx \, dt = \frac{\lambda + \mu}{\lambda + \mu - b}.
\]

As \( g(t, 1, c/R) \leq R/c \) for all \( t > 0 \), the second expectation term in (52) is bounded by \( \exp(b R/c) \). Thus,

\[
\mathbb{E} \left[ \exp \left( b |Y_{n+1} - Y_n| \right) \right] \leq \sum_{s=1}^{\infty} \mathbb{P}(l_n = s) \left( \frac{\lambda + \mu}{\lambda + \mu - b} \right)^{s-1} \exp \left( \frac{b R}{c} \right) \leq \sum_{s=1}^{\infty} \left( \frac{\lambda + \mu}{\lambda + \mu - b} \right)^{s-1} \exp \left( \frac{b R}{c} \right) = \frac{\mu \exp \left( \frac{b R}{c} \right)}{\lambda + \mu} \sum_{s=1}^{\infty} \left( \frac{\lambda + \mu}{\lambda + \mu - b} \right)^{s-1}
\]

For \( b < \mu \), the above sum converges, and \( \mathbb{E} \left[ \exp \left( b |Y_{n+1} - Y_n| \right) \right] \leq \frac{\mu}{\lambda + \mu - b} \exp \left( \frac{b R}{c} \right) \), which is less than 2 for small enough \( b \). Thus, the sub-exponential property is proved. \( \square \)

B.4. Proof of Lemma 4

Proof of Lemma 4 We state the proof for \( \mu \in (c/R, +\infty) \); the other regime follows similarly. Note that \( \mathbb{P}(Y_n \leq 0) = \mathbb{P} \left( \sum_{i=0}^{n-1} (Y_{i+1} - Y_i) \leq 0 \right) \). In Lemmas 2 and 4, we showed that \( \{Y_i - Y_{i-1}\}_{i=0}^{n-1} \) are independent and identically distributed sub-exponential random variables, and thus, the centered random variables \( \{Y_{i+1} - Y_i - \mathbb{E}[Y_{i+1} - Y_i]\}_{i=0}^{n-1} \) are also sub-exponential. We showed \( \mathbb{E}[Y_{i+1} - Y_i] > 0 \) in (15); define \( \mathbb{E}[Y_{i+1} - Y_i] = \delta \). Now, we apply Bernstein’s concentration inequality (Vershynin 2018 Theorem 2.8.2) to get

\[
\mathbb{P} \left( \sum_{i=0}^{n-1} (Y_{i+1} - Y_i) < 0 \right) = \mathbb{P} \left( \sum_{i=0}^{n-1} (Y_{i+1} - Y_i) - n \delta < -n \delta \right) \leq \exp \left( -c_B \min \left( \frac{n^2 \delta^2}{n}, n \delta \right) \right) = \exp(-c_1 n).
\]

\( \square \)

B.5. Proof of Lemma 5

Proof of Lemma 5 We first bound the probability term \( \mathbb{P} \left( \sum_{j=1}^i y_j < n, \sum_{j=i+1}^{i+1} y_j \geq n \right) \) using the probability of the first event. We take \( p_i = 1 - q_i = \exp(-i^{1-\epsilon}) \) and then use the Chernoff bound to get

\[
\mathbb{P} \left( y_1 + \cdots + y_i < n, y_1 + \cdots + y_{i+1} \geq n \right) \leq \mathbb{P} \left( y_1 + \cdots + y_i \leq n \right) \leq \min_{t \geq 0} e^{tn} \prod_{j=1}^i e^{p_j} - (1 - p_j).
\]

(53)

Take \( b = \left[ (\log(n + 1))^{1/\epsilon} \right] \) and \( t \geq 0 \) such that \( e^t = \frac{n + 1}{n} q_i \). From (53), for \( i \geq d \geq b \) we have

\[
\mathbb{P} \left( y_1 + \cdots + y_i \leq n \right) \leq \left( \frac{n + 1}{n} \right)^n q_i^{i-1} \prod_{j=1}^i \frac{p_j}{(1 - p_j) (p_j - p_i)} \leq \left( \frac{n + 1}{n} \right)^n q_i^{i-1} \prod_{j=1}^i p_j \prod_{j=b+1}^i \frac{1}{1 - p_i} \prod_{j=1}^b \frac{1}{1 - p_j} \left( \frac{n}{\prod_{j=b+1}^i} \right) \prod_{j=1}^i \frac{1}{1 - \exp \left( -j^{1-\epsilon} \right)}.
\]

(54)
Since \( q_i \leq 1 \) and \( n \geq i - b \), we have \( \left( \frac{n+1}{n} \right)^n q_i^{n-(i-b)} \leq e \). By concavity and gradient inequality, for \( 1 \leq j \leq i \), we have \( i^{1-\epsilon} - j^{1-\epsilon} \geq \frac{1-\epsilon}{\epsilon} (i - j) \). Using this inequality and setting \( \kappa := \lceil i^\epsilon/(1-\epsilon) \rceil \), we have

\[
\prod_{j=1}^{b} \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} (i - j)\right)} \leq \prod_{j=1}^{b} \frac{1}{1 - \exp\left(-\frac{i}{\epsilon} \right)} \leq \prod_{t=1}^{\infty} \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} t\right)}
\]

\[
\leq \prod_{t=1}^{\infty} \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} t\right)} \prod_{t=\kappa}^{\infty} \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} t\right)}
\]

\[
\leq \prod_{t=1}^{\infty} \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} t\right)} \prod_{j=1}^{\kappa} \left( \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} j\right)} \right) \leq (c_\kappa)^{\kappa-1} \prod_{t=1}^{\infty} \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} t\right)}.
\]

The last inequality is true as follows. For \( a_j = (\exp(j) - 1)^{-1} \), using the fact that \( 1 + x \leq \exp(x) \), we have

\[
\prod_{j=1}^{\infty} \frac{1}{1 - \exp(-j)} = \prod_{j=1}^{\infty} (1 + a_j) \leq \exp \left( \sum_{j=1}^{\infty} a_j \right) = e c_u.
\]

For \( 1 \leq t \leq \kappa - 1 \), we have \( \frac{\kappa}{\epsilon} \leq \frac{1}{\epsilon} (\kappa - 1) < 1 \), and \( 1 - \exp(-x) \geq x/2 \) for \( x \leq 1 \). Therefore, we can write \( 1 - \exp\left(-\frac{1}{\epsilon} t\right) \geq \frac{1}{2} \frac{1-\epsilon}{\epsilon} t \). As a result, we can further simplify the second product term in (54) as follows,

\[
\prod_{t=1}^{\kappa} \left( \frac{1}{1 - \exp\left(-\frac{1}{\epsilon} t\right)} \right) \leq (c_\kappa)^{\kappa-1} \frac{1}{(\kappa-1)!} \left( \frac{\kappa}{\epsilon} \right)^{\kappa-1}.
\]

Using (55), we simplify the upper bound found in (54) as below

\[
P(y_1 + \cdots + y_i \leq n) \leq e n^{i-b} (c_u)^{2^{\kappa-1}} \frac{1}{(\kappa-1)!} \left( \frac{\kappa}{\epsilon} \right)^{\kappa-1} \prod_{j=b+1}^{i} p_j.
\]

Using integral lower bound to derive inequality below, we find an upper bound for the term \( \prod_{j=b+1}^{i} p_j \):

\[
(b+1)^{1-\epsilon} + \cdots + i^{1-\epsilon} \geq \frac{1}{2-\epsilon} \left( i^{2-\epsilon} - b^{2-\epsilon} \right).
\]

For \( x > 0 \) and \( k \in \mathbb{N} \), \( x^k/k! \leq \exp(x) \). Thus, \( \frac{c_c^{2^{\kappa-1}}}{(\kappa-1)!} \leq c_c \exp(\frac{2 c_u}{b^{2-\epsilon}}) =: c_b \), which is an \( \epsilon \)-dependent constant. Thus, using (57) we simplify the bound of (56) to get

\[
P(y_1 + \cdots + y_i \leq n) \leq c_b \exp \left( -\frac{1}{2-\epsilon} \left( i^{2-\epsilon} - b^{2-\epsilon} \right) \right) n^{i-b} i^{(\epsilon-1)}.
\]

We upper bound the summation given in the statement of Lemma 5. From (58) and using the fact that \( d \geq b \),

\[
\sum_{i=d}^{n} i P(y_1 + \cdots + y_i \leq n) \leq c_b \sum_{i=d}^{n} i \exp \left( -\frac{1}{2-\epsilon} \left( i^{2-\epsilon} - b^{2-\epsilon} \right) \right) (n+1)^{i-b} i^{(\kappa-1)}
\]

\[
\leq c_b (n+1)^{1-b} \sum_{i=d}^{\infty} i \exp \left( -\frac{i^{2-\epsilon}}{2-\epsilon} + i \log(n+1) + \frac{\epsilon}{1-\epsilon} \log(i)^\epsilon \right)
\]

\[
\leq c_b \exp \left( -b \log(n+1) + \frac{b^{2-\epsilon}}{2-\epsilon} \right)
\]

\[
\leq c_b \exp \left( -b(b-1)^{1-\epsilon} + \frac{b^{2-\epsilon}}{2-\epsilon} \right) = c_b \exp \left( -\frac{b^{2-\epsilon}}{2-\epsilon} \left( \left( \frac{1}{b} \right)^{1-\epsilon} - \frac{1}{2-\epsilon} \right) \right),
\]

where we have used \( b = \lceil (\log(n+1))^{\frac{1}{1-\epsilon}} \rceil \) in the last line. The third inequality holds as for \( i \geq d \), the negative term inside the second exponential function is dominating. Further, as \( n \) grows, \( b \) converges to infinity; hence, in the final term, the exponential term converges to zero. Thus, we can bound the sum with a constant.
B.6. Proof of Corollary 1

Proof. We will follow the same arguments as in Theorem 7 to show a $O(\log n)$ regret. As a parallel to Lemma 5, we upper bound $\sum_{i=d}^{n-1} i \mathbb{P}(y_1 + \cdots + y_i < n, y_1 + \cdots + y_{i+1} \geq n)$ for independent geometric random variables $\{y_i\}_{i=1}^{n}$, with success probability $\{f(i)^{-1}\}_{i=1}^{n}$ following similar arguments to Lemma 5. In addition, denote the smallest $i$ that satisfies $i^{1-\epsilon_i} \geq \log(n+1)$ as $b$ and let $\hat{d}$ be the smallest integer $i$ such that $\log(n+1) \leq \frac{1}{2}i^{1-\epsilon_i}$. We note that $i^{1-\epsilon_i}$ is increasing for $i \geq 1$ as $\epsilon_i$ is a decreasing sequence. Take $p_i = \exp(-i^{1-\epsilon_i})$ and $t \geq 0$ such that $e^t = \frac{n+1}{n}(1-p_i)$, which exists for $i > b$. From (54), for $i > b$,

$$
\mathbb{P}(y_1 + \cdots + y_i \leq n) \leq \epsilon_n 1-b \prod_{j=b+1}^{i} p_j \prod_{j=1}^{b} \frac{1}{1-\exp\left(-\left(i^{1-\epsilon_j} - j^{1-\epsilon_j}\right)\right)}.
$$

Moreover, for $1 \leq j \leq i$, by concavity and gradient inequality, we have $\epsilon_j \geq \epsilon_i$ and

$$i^{1-\epsilon_i} - j^{1-\epsilon_j} \geq i^{1-\epsilon_i} - j^{1-\epsilon_j} \geq \frac{1}{\epsilon_i}(i-j) .
$$

We define $\kappa = \lfloor i^{1-\epsilon_j} / (1-\epsilon_i) \rfloor$ and using (55), simplify the second product term in the RHS of (59) to get

$$
\prod_{j=1}^{b} \frac{1}{1-\exp\left(-\left(i^{1-\epsilon_j} - j^{1-\epsilon_j}\right)\right)} \leq \prod_{j=1}^{b} \frac{1}{1-\exp\left(-\frac{1}{\epsilon_i}(i-j)\right)} \leq \epsilon_n \frac{2^{n-1}}{(n-1)!} \leq \epsilon_n \frac{2^{n-1}}{(n-1)!} (i^{\epsilon_i} - 1 - \epsilon_i) .
$$

Furthermore, using an integral lower bound, we find an upper bound for the term $\prod_{j=b+1}^{i} p_j$:

$$
(b+1)^{1-\epsilon_{b+1}} + \cdots + i^{1-\epsilon_i} \geq (b+1)^{1-\epsilon_{b+1}} + \cdots + i^{1-\epsilon_{b+1}} \geq \frac{1}{2-\epsilon_{b+1}} (i^{2-\epsilon_{b+1}} - b^{2-\epsilon_{b+1}}) .
$$

Using (61), (61), and the fact that $\frac{\epsilon_n 2^{n-1}}{(n-1)!} \leq \epsilon_n \exp(\frac{2\epsilon_n}{1-\epsilon_i})$, we simplify (59) to get

$$
\mathbb{P}(y_1 + \cdots + y_i \leq n) \leq \epsilon_n \exp\left(-\frac{1}{2-\epsilon_{b+1}} (i^{2-\epsilon_{b+1}} - b^{2-\epsilon_{b+1}}) \right) .
$$

Finally, we can bound $\sum_{i=d}^{n-1} i \mathbb{P}(y_1 + \cdots + y_i < n, y_1 + \cdots + y_{i+1} \geq n)$ using (63) as follows

$$
\sum_{i=d}^{n} i \mathbb{P}(y_1 + \cdots + y_i \leq n) \leq c_n (n+1)^{-b} \exp\left(\frac{b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}}\right) \sum_{i=d}^{n} \exp\left(\frac{-i^{2-\epsilon_{b+1}} + \log(n+1) + \frac{\epsilon_i}{1-\epsilon_i} \log(i)^{\epsilon_i}}{2-\epsilon_{b+1}}\right)
$$

$$
\leq c_n (n+1)^{-b} \exp\left(\frac{b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}}\right) \sum_{i=d}^{n} \exp\left(\frac{-i^{2-\epsilon_{b+1}} + \frac{\epsilon_i}{1-\epsilon_i} \log(i)^{\epsilon_i}}{2-\epsilon_{b+1}}\right)
$$

$$
\leq c_n (n+1)^{-b} \exp\left(\frac{b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}}\right),
$$

where the second line follows from $\log(n+1) \leq \frac{1}{2}(d)^{1-\epsilon_b} \leq \frac{1}{2}i^{1-\epsilon_b}$ for $i \geq \hat{d}$. As the negative term inside the second exponential function is the dominating term, we can bound the summation with a constant independent of $n$. From the definition of $b$, we have $(b-1)^{1-\epsilon_b} \leq \log(n+1) \leq b^{1-\epsilon_b}$. Thus

$$
(n+1)^{-b} \exp\left(\frac{b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}}\right) = \exp\left(\frac{b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}} - \log(n+1)\right) \leq \exp\left(\frac{b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}} - (b-1)^{1-\epsilon_b}\right)
$$

$$
= \exp\left(\frac{-b^{2-\epsilon_{b+1}}}{2-\epsilon_{b+1}} (b^{\epsilon\epsilon_{b+1}} - 1) - \frac{1}{2-\epsilon_{b+1}}\right) .
$$

We note that as $b$ grows to infinity, the term $\frac{1}{2-\epsilon_{b+1}}$ converges to 0, and the term $b^{2-\epsilon_{b+1}}$ converges to $\infty$. Since $\epsilon_{b+1} < \epsilon_{b-1}$, the term $b^{\epsilon\epsilon_{b+1}} - 1$ is less than 1. However, we also note that for large enough $b$,

$$
1 > b^{\epsilon\epsilon_{b+1}} - 1 = b^{\frac{\epsilon}{\sqrt{1 + \log(b + 2)}} - \frac{\epsilon}{\sqrt{1 + \log(b + 2)}}} = \exp\left(\frac{\epsilon \log(b)}{\sqrt{1 + \log(b + 2)}} - \frac{\epsilon \log(b)}{\sqrt{1 + \log(b + 2)}}\right)
$$

$$
> \exp(\sqrt{\log(b + 2)} - 1 - \sqrt{\log(b + 2)}) ,
$$
which follows from \( \varepsilon < 1 \) and \((\log(b))^2 > (\log(b+2))^2 - 1\) for sufficiently large \( b \) (since \((\log(b+2))^2 - (\log(b))^2\) converges to 0 as \( b \) grows). Thus, \( b^{a+1-a_b-1} \) converges to 1 as \( b \) increases without bound. Using all of these, we can assert that the RHS of (53) goes to 0 as \( b \) increases to infinity, and so we can bound it by a constant independent of \( n \). Finally, by repeating the arguments of Theorem 3 the expected regret is upper bounded by a linear function of \( \tilde{d} \) and we conclude that the expected regret is of the order \( O(\log n) \). \( \square \)

**Appendix C: Analysis of the Multi-server Erlang-B Queueing System**

**C.1. Lemma 12**

In the multi-server Erlang-B queueing system following policy \( \Pi_{\text{Alg1}} \), the number of accepted arrivals that find the system empty is almost surely infinite.

**Proof.** By observing Markov process \( \{\tilde{X}_n\}_{n=0}^\infty \), we first argue that the system becomes empty infinitely often following our proposed policy. By coupling the two systems, we get

\[
P(\text{returns to state 0 at a finite time} \mid N_n = 0, X_n = x, \alpha_n = \alpha) \geq P(\text{returns to state 0 at a finite time in a system that accepts all arrivals} \mid N_n = 0) = 1.
\]

Thus, state 0 is visited infinitely often. Let \( A \) be the event that the system admits a finite number of arrivals at instances when the server is empty, \( A_1 \) be the event that the system admits a finite number of arrivals, and \( A_2 \) be the event that the system gets empty a finite number of times. We have \( P(A) \leq P(A_1) + P(A_2) = 0 \), wherein \( P(A_1) = 0 \) follows from the same arguments as Lemma 11. \( \square \)

**C.2. Lemma 10**

We first present the following lemma, which is used in the proof of Lemma 10.

**Lemma 13.** (Wainwright 2019, Theorem 2.19) let \( \{ (D_i, F_i) \}_{i=1}^\infty \) be a martingale difference sequence such that for \( \nu_i, \alpha_i > 0 \), we have \( E[\exp(\tilde{\lambda} D_i) \mid F_{i-1}] \leq \exp \left( \frac{\tilde{\lambda} \nu_i^2}{2} \right) \) a.s. for any \( |\tilde{\lambda}| < 1/\alpha_i \). Then the sum \( \sum_{i=1}^n D_i \) satisfies the concentration inequality

\[
P \left( \left| \sum_{i=1}^n D_i \right| \geq t \right) \leq 2 \exp \left( - \min \left( \frac{t^2}{2 \sum_{i=1}^n \nu_i^2}, \frac{t}{2 \max_{i=1,\ldots,n} \alpha_i} \right) \right).
\]

**Proof of Lemma 10.** We state the proof for \( \mu > c/R \). The proof for the other regime follows similarly. Notice that \( \tilde{\delta}_1 \) and \( \tilde{\delta}_2 \) are as defined in Lemma 8. We define the martingale difference sequence \( \{ Y_t^D \}_{n=0}^\infty \) as \( Y_n^D = Y_{n+1}^M - Y_n^M \). To verify the conditions of Lemma 13, we argue that \( E[\exp(\tilde{\lambda} Y_t^D) \mid F_{i-1}] \) is bounded for some positive \( \tilde{\lambda} \). We show this by proving \( E[\exp(\tilde{\lambda} Y_t^D) \mid F_{i-1}] \) and \( E[\exp(-\tilde{\lambda} Y_t^D) \mid F_{i-1}] \) are bounded for some positive \( \tilde{\lambda} \). From the upper bound of \( Y_t^D \) found in (39), we have

\[
E[\exp(\tilde{\lambda} Y_t^D) \mid F_{i-1}] \leq E[\exp(\tilde{\lambda} k \frac{R}{c} \tau_i) \mid F_{i-1}] \leq E[\exp(\tilde{\lambda} k \frac{R}{c} \zeta_i)], \tag{66}
\]

where \( \zeta_i \) is the first passage time of state zero starting from zero in a finite-state irreducible Markov chain, and thus, sub-exponential. From Vershynin 2018, Theorem 2.8.2, the moment generating function of \( \zeta_i \) is bounded at some \( \tilde{\lambda}_i \) independent of \( i \), which leads to a finite bound. For \( E[\exp(-\tilde{\lambda} Y_t^D) \mid F_{i-1}] \), using (39),

\[
E[\exp(-\tilde{\lambda} Y_t^D) \mid F_{i-1}] \leq E \left[ \exp \left( \tilde{\lambda} \left( k \sum_{j=1}^{T_{\beta_i+j} + c_3} \right) \right) \mid F_{i-1} \right] \leq E \left[ \exp \left( \tilde{\lambda} \left( k \sum_{j=1}^{T_{\beta_i+j} + c_3} \right) \right) \right].
\]
From the above inequality, it suffices to show $\sum_{j=1}^{\zeta} T_{i+j}$ is sub-exponential. From Vershynin (2018, Theorem 2.8.2), we need to argue that for some positive $\tilde{\lambda}$, $\mathbb{E}\left[\exp\left(\tilde{\lambda} \sum_{j=1}^{\zeta} T_{i+j}\right)\right] \leq 2$. For $\tilde{\lambda} < \lambda$, we define the martingale sequence $\{M_{i,m}\}_{m=0}^{\infty}$ with respect to filtration $\{\mathcal{G}_{i,m}\}_{m=0}^{\infty}$ as

$$M_{i,m} = \frac{\exp\left(\tilde{\lambda} \sum_{j=1}^{m} T_{i+j}\right)}{\mathbb{E}\left[\exp\left(\tilde{\lambda} \sum_{j=1}^{m} T_{i+j}\right)\right]} = \exp\left(\tilde{\lambda} \frac{\sum_{j=1}^{m} T_{i+j}}{\lambda - \tilde{\lambda}}\right)^m.$$  

The passage time $\zeta_i$ is a finite-mean stopping time for the martingale sequence $\{M_{i,m}\}_{m=0}^{\infty}$. Therefore, using the optional stopping theorem for non-negative supermartingale sequences, we have

$$\mathbb{E}\left[M_{i,\zeta_i}\right] \leq \mathbb{E}\left[M_{i,0}\right],$$

or

$$\mathbb{E}\left[\exp\left(\frac{\tilde{\lambda}}{2} \sum_{j=1}^{\zeta_i} T_{i+j}\right)\right] \leq \sqrt{\mathbb{E}\left[\left(\frac{\lambda}{\lambda - \tilde{\lambda}}\right)^{\zeta_i}\right]} = \sqrt{\mathbb{E}\left[\exp\left(\log\left(\frac{\lambda}{\lambda - \tilde{\lambda}}\right)\zeta_i\right)\right]}.$$  

As $\zeta_i$ is a sub-exponential random variable, we can choose $\tilde{\lambda}$ such that the RHS of (67) is less than or equal to 2 and the conditions of Lemma 13 are verified. Consequently, we apply Lemma 13 to conclude that

$$\mathbb{P}\left(Y_n^M \leq -\tilde{\delta}_1 n\right) = \mathbb{P}\left(\sum_{i=0}^{n-1} (Y_{i+1}^M - Y_i^M) \leq -\tilde{\delta}_1 n\right) \leq \exp\left(-\min\left(\frac{\tilde{\delta}_1^2 n^2}{2\nu}, \frac{\tilde{\delta}_1 n}{2\alpha}\right)\right) = \exp\left(-c_3 n\right),$$

where $\nu$ and $\alpha$ are positive constants independent of $n$. □

References

Agrawal R, Teneketzis D, Anantharam V (1989) Asymptotically efficient adaptive allocation schemes for controlled Markov chains: Finite parameter space. IEEE Transactions on Automatic Control 34(12):1249–1259.

Agrawal S, Jia R (2022) Learning in structured MDPs with convex cost functions: Improved regret bounds for inventory management. Operations Research 70(3):1646–1664.

Arapostathis A, Borkar VS, Fernández-Gaucherand E, Ghosh MK, Marcus SI (1993) Discrete-time controlled Markov processes with average cost criterion: A survey. SIAM Journal on Control and Optimization 31(2):282–344.

Borkar V, Varaiya P (1979) Adaptive control of Markov chains, I: Finite parameter set. IEEE Transactions on Automatic Control 24(6):953–957.

Chen X, Liu Y, Hong G (2023) An online learning approach to dynamic pricing and capacity sizing in service systems. Operations Research .

Choudhury T, Joshi G, Wang W, Shakkottai S (2021) Job dispatching policies for queueing systems with unknown service rates. arXiv preprint arXiv:2106.04707 .

Dai JG, Gluzman M (2022) Queueing network controls via deep reinforcement learning. Stochastic Systems 12(1):30–67.

Durrett R (2019) Probability: Theory and examples, volume 49 (Cambridge university press).
Gans N, Koole G, Mandelbaum A (2003) Telephone call centers: Tutorial, review, and research prospects. *Manufacturing & Service Operations Management* 5(2):79–141.

Gopalan A, Mannor S (2015) Thompson sampling for learning parameterized Markov decision processes. *Conference on Learning Theory*, 861–898 (PMLR).

Graves TL, Lai TL (1997) Asymptotically efficient adaptive choice of control laws in controlled Markov chains. *SIAM journal on control and optimization* 35(3):715–743.

Harchol-Balter M (2013) *Performance modeling and design of computer systems: Queueing theory in action* (Cambridge University Press).

Jia H, Shi C, Shen S (2022) Online learning and pricing for service systems with reusable resources. *Operations Research* .

Johansen SG, Stidham S (1980) Control of arrivals to a stochastic input–output system. *Advances in Applied Probability* 12(4):972–999.

Kelly FP (2011) *Reversibility and stochastic networks* (Cambridge University Press).

Knudsen N (1972) Individual and social optimization in a multiserver queue with a general cost-benefit structure. *Econometrica: Journal of the Econometric Society* 515–528.

Krishnasamy S, Arapostathis A, Johari R, Shakkottai S (2018) On learning the $c\mu$ rule in single and parallel server networks. *2018 56th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 153–154 (IEEE).

Krishnasamy S, Sen R, Johari R, Shakkottai S (2021) Learning unknown service rates in queues: A multi-armed bandit approach. *Operations Research* 69(1):315–330.

Kumar PR, Becker A (1982) A new family of optimal adaptive controllers for Markov chains. *IEEE Transactions on Automatic Control* 27(1):137–146.

Kumar PR, Lin W (1982) Optimal adaptive controllers for unknown Markov chains. *IEEE Transactions on Automatic Control* 27(4):765–774.

Kumar PR, Varaiya P (2015) *Stochastic systems: Estimation, identification, and adaptive control* (SIAM).

Lai TL, Yakowitz S (1995) Machine learning and nonparametric bandit theory. *IEEE Transactions on Automatic Control* 40(7):1199–1209.

Lippman SA, Stidham Jr S (1977) Individual versus social optimization in exponential congestion systems. *Operations Research* 25(2):233–247.

Mandl P (1974) Estimation and control in Markov chains. *Advances in Applied Probability* 6(1):40–60.

Marbach P, Eryilmaz A, Ozdaglar A (2011) Asynchronous CSMA policies in multihop wireless networks with primary interference constraints. *IEEE Transactions on Information Theory* 57(6):3644–3676.

Massaro A, De Pellegrini F, Maggi L (2019) Optimal trunk-reservation by policy learning. *IEEE INFOCOM 2019-IEEE Conference on Computer Communications*, 127–135 (IEEE).
Naor P (1969) The regulation of queue size by levying tolls. *Econometrica* 37(1):15–24, ISSN 00129682, 14680262, URL [http://www.jstor.org/stable/1909200](http://www.jstor.org/stable/1909200).

Ojeda C, Cvejoski K, Georgiev B, Bauckhage C, Schuecker J, Sánchez RJ (2021) Learning deep generative models for queuing systems. *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, 9214–9222.

Puterman ML (1990) Markov decision processes. *Handbooks in operations research and management science* 2:331–434.

Shiryaev AN (1996) *Probability* (Springer).

Srikant R, Ying L (2013) *Communication networks: An optimization, control, and stochastic networks perspective* (Cambridge University Press).

Stahlbuhk T, Shrader B, Modiano E (2021) Learning algorithms for minimizing queue length regret. *IEEE Transactions on Information Theory* 67(3):1759–1781.

Stidham S (1985) Optimal control of admission to a queueing system. *IEEE Transactions on Automatic Control* 30(8):705–713.

Sutton RS, Barto AG (2018) *Reinforcement learning: An introduction* (MIT press).

Vershynin R (2018) *High-dimensional probability: An introduction with applications in data science*, volume 47 (Cambridge university press).

Wainwright MJ (2019) *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48 (Cambridge University Press).

Walton N, Xu K (2021) Learning and information in stochastic networks and queues. *Tutorials in Operations Research: Emerging Optimization Methods and Modeling Techniques with Applications*, 161–198 (INFORMS).

Zhang Y, Cohen A, Subramanian VG (2022) Learning-based optimal admission control in a single server queuing system. *2022 58th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 1–2 (IEEE).

Zhong Y, Birge JR, Ward A (2022) Learning the scheduling policy in time-varying multiclass many server queues with abandonment. *Available at SSRN*. 