Data-Driven Set-Based Estimation using Matrix Zonotopes with Set Containment Guarantees

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Abstract—We propose a method to perform set-based state estimation of an unknown dynamical linear system using a data-driven set propagation function. Our method comes with set-containment guarantees, making it applicable to safety-critical systems. The method consists of two phases: (1) an offline learning phase where we collect noisy input-output data to determine a function to propagate the state-set ahead in time; and (2) an online estimation phase consisting of a time update and a measurement update. It is assumed that known finite sets bound measurement noise and disturbances, but we assume no knowledge of their statistical properties. These sets are described using zonotopes, allowing efficient propagation and intersection operations. We propose a new approach to compute a set of models consistent with the data and noise-bound, given input-output data in the offline phase. The set of models is utilized in replacing the unknown dynamics in the data-driven set propagation function in the online phase. Then, we propose two approaches to perform the measurement update. Simulations show that the proposed estimator yields state sets comparable in volume to the 3σ confidence bounds obtained by a Kalman filter approach, but with the addition of state-containment guarantees. We observe that using constrained zonotopes yields smaller sets but with higher computational costs than unconstrained ones.

I. INTRODUCTION

Set-based estimation involves the computation of a set, which is guaranteed to contain the system’s true state at each time step given bounded uncertainties [1]. Existing set-based observers require a system model to propagate the state set at each time step [2], [3]. We address the problem of propagating the state set using only noisy offline input-output data and merging this with online measurements to obtain a time-varying state set which is guaranteed to contain the true system’s state at each time step. This problem is essential in safety-critical applications [4].

Two popular set-based estimators are interval observers and set-membership observers. Interval-based observers generally generate state estimates by utilizing an observer gain to fuse a model-based time update of the state with current measurements. For example, the authors in [5] propose an exponentially stable interval-based observer for time-invariant linear systems. Set-membership observers generally follow a geometrical approach by intersecting the state-space regions consistent with the model with those from the measurements to obtain the current state set [6]. This approach has been extended to sensor networks with event-based communication in [7] and multi-rate systems in [8]. Various set representations have been used for set-membership observers such as ellipsoids [9], polytopes [10] and zonotopes [11]. Zonotopes are a special class of polytopes for which one can efficiently compute linear maps, and Minkowski sums — both frequent operations performed by set-based observers.

All the aforementioned observers use a model of the underlying system to propagate the state set. However, identifying a system model is often time-consuming, and the identified model is not necessarily well-suited for estimation or control. Recent works based on Willems’ fundamental lemma [12] have shown that system trajectories can be used directly to synthesize controllers. The authors in [13] present an extended Kalman filter and model predictive control (MPC) scheme computed directly from system trajectories. Stability and robustness guarantees for such a data-driven control scheme are presented in [14], and for an MPC scheme in [15]. An alternative approach is to find a set of models that is consistent with data and use this set of models to propagate a state set [16].

Our contribution is a novel method to perform set-based state estimation with set-containment guarantees given bounded, noisy measurements and known inputs. The algorithm, summarized in Fig. I, consists of an offline learning phase to determine a state-propagation function \( f(\cdot) \) directly from data, and an online estimation phase to perform a time update using \( f(\cdot) \) and measurements iteratively to track the system state. A new approach to compute the set of models consistent with the data and noise bound from input-output data is proposed different from input-state data in [16], [17]. Then, we present two approaches to perform the measurement update utilizing either the singular value decomposition (SVD) of the observation matrix or an optimization formulation. We compare the approaches in simulation. Our method is shown to yield set-based state estimates similar in size to 3σ confidence bounds of an approach based on system identification and a Kalman filter, but with the addition of set-containment guarantees. The code to recreate our findings is publicly available.

The rest of this paper is outlined as follows. Sec. II introduces the preliminaries and problem statement. We present our method in Sec. III and evaluate it in Sec. IV. Finally, Sec. V concludes the paper.

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https://github.com/alexberndt/data-driven-set-based-estimation-zonotopes
II. PRELIMINARIES AND PROBLEM STATEMENT

We denote the i-th element of a vector or list A by A(i).

Definition 1. (Zonotope [18]) Given a center c ∈ R^n and a number ξ ∈ N of generator vectors in a generator matrix G = [g(1), ..., g(ξ)] ∈ R^{n×ξ}, a zonotope is a set
\[ Z = \{ x ∈ R^n \mid x = c + \sum_{i=1}^{ξ} \beta(i) g(i), -1 ≤ \beta(i) ≤ 1 \} . \] (1)

We use the shorthand notation Z = (c, G).

Given two zonotopes Z_1 and Z_2, we use the notation + for the Minkowski sum, and Z_1 − Z_2 to denote Z_1 + (−Z_2) not the Minkowski difference.

Definition 2. (Matrix zonotope [4, p.52]) Given a center matrix C ∈ R^{n×k} and ξ ∈ N generator matrices G(i) ∈ R^{n×k} where i = {1, ..., ξ}, a matrix zonotope is the set
\[ M = \{ X ∈ R^{n×k} \mid X = C + \sum_{i=1}^{ξ} \beta(i) G(i), -1 ≤ \beta(i) ≤ 1 \} . \]

We use the notation M = (C, G(i)), where G(i) = [G(1), ..., G(ξ)].

Definition 3. (Interval matrix [4, p.42]) An interval matrix I specifies the interval of all possible values for each matrix element between the left limit L and right limit R:
\[ I = [ L, R ] , \quad L, R ∈ R^{n×c} . \] (2)

We consider estimating the set of all possible system states using an array of q sensors. Our system is described as
\[ x(k+1) = A_n x(k) + B_n u(k) + w(k) , \] (3a)
\[ y_i(k) = C_i x(k) + v_i(k) , \quad i ∈ \{ 1, ..., q \} , \] (3b)
where x(k) ∈ R^n is the system state, u(k) ∈ R^m the input, y_i(k) ∈ R^{p_i} the measurement of sensor i, x(0) ∈ X_0 the initial condition where X_0 is the initial bounding zonotope. Furthermore, the system matrices A_n ∈ R^{n×n} and B_n ∈ R^{n×m} are unknown whereas C_i ∈ R^{p_i×n} is known for all i ∈ {1, ..., q}. The noise w(k) ∈ Z_w and v_i(k) ∈ Z_{v,i} are assumed to belong to the bounding zonotopes Z_w = ⟨c_w, G_w⟩ ∈ R^n and Z_{v,i} = ⟨c_{v,i}, G_{v,i}⟩ ∈ R^{p_i} for i ∈ {1, ..., q}, respectively. We denote the Frobenius norm by ∥∥_F and the null space of a matrix A by ker(A). We compute the pseudoinverse of an interval matrix by adapting [19, Thm 2.40]. The pseudoinverse of an interval matrix is denoted by †.

Let R_k denote a set containing x(k) given the exact system model and bounded, but unknown, process and measurement noise. The problem addressed in this paper is to develop an algorithm that returns a set R_k ⊇ R_k, which is guaranteed to contain the true state x(k) at each time instance i.e., x(k) ∈ R_k for all k, given input-output data and bounds for model uncertainties and measurement noise without knowledge of the model [A_n, B_n].

III. DATA-DRIVEN SET-BASED ESTIMATION

Our proposed data-driven set estimator consists of two phases: an offline learning phase and an online estimation phase. In the offline phase, we compute the function to perform the time update. The online phase consists of iteratively performing a time update and a measurement update.

We denote the time and measurement updated sets at k by R_{k+1} ⊇ R_k and R_{k+1} ⊇ R_k, respectively.

A. Offline Learning Phase

The objective of this phase is to compute a function f : R^n × R^m → R^n, such that R_{k+1} = f(R_k, U_k), i.e., f returns R_{k+1} given a known input zonotope U_k and the measurement updated set R_k at time-step k such that we can guarantee x(k+1) ∈ R_{k+1} for all k. During this phase, we assume that we have offline access to an input sequence u(k) and noisy output z(k) such that
\[ z(k) = C_i x(k) + γ(k) , \] (4)
where the noise γ(k) is bounded by the zonotope Z_{γ} = ⟨c_{γ}, G_{γ}⟩, i.e., γ(k) ∈ Z_{γ}, ∀k. We have for all sensors vertically combined noisy output z(k) = z^{γ}(k) and similarly for γ and C. For the sake of clarity, we differentiate the notation of the offline noisy output z(k) from the online noisy output y(k) and
similarly for the measurement noise. Given an experiment yielding a sequence of noisy data of length $T$, we can construct the following sequences

$$\begin{align*}
Z^+ &= \begin{bmatrix} z(1) & \cdots & z(T) \end{bmatrix}, \\
Z^- &= \begin{bmatrix} z(0) & \cdots & z(T-1) \end{bmatrix}, \\
U^- &= \begin{bmatrix} u(0) & \cdots & u(T-1) \end{bmatrix}.
\end{align*}$$

(5)

We further construct

$$Z = [z(0) \ldots z(T)],$$

and similarly for other signals. The data $D = [U^- \ Z]$ can be from one sensor or multiple sensors. Furthermore, we denote the sequence of unknown and similarly for other signals. The data $D = [U^- \ Z]$ can be from one sensor or multiple sensors. Furthermore, we denote the sequence of unknown

$$\langle c_w, \{g^{(i)}_w\} \rangle$$

as

$$C_{M_w} = \begin{bmatrix} c_{w} & \cdots & c_{w} \end{bmatrix},$$

$$G_{M_w}^{(j+(-1)T)} = \begin{bmatrix} g^{(i)}_w \ 0_{n \times (T-1)} \end{bmatrix},$$

$$G_{M_w}^{(j+(-1)T)} = \begin{bmatrix} 0_{n \times (j-1)} & g^{(i)}_w \ 0_{n \times (T-j)} \end{bmatrix},$$

$$G_{M_w}^{(j+(-1)T)} = \begin{bmatrix} 0 & g^{(i)}_w \ 0_{n \times (T-1)} \end{bmatrix},$$

for all $i = \{1, \ldots, \xi\}, j = \{2, \ldots, T-1\}$ [16]. In a similar fashion, we describe the unknown noise and matrix zonotope of $\gamma(k)$ as $\Gamma^+, \Gamma^- \in M_{\gamma} = (C_{M_{\gamma}}, G_{M_{\gamma}}^{(1:T)})$. We denote all system matrices $[A \ B]$ that are consistent with the data

$$\begin{align*}
N_\Sigma &= \{ [A \ B] \mid X^+ &= AX^- + BU^- + W^-, \\
Z^- &= CX^- + \Gamma^- , W^- \in M_{\gamma} , \Gamma^+ \in M_{\gamma}, \\
\Gamma^- \in M_{\gamma} \},
\end{align*}$$

By definition, $[A_t \ B_t] \in N_\Sigma$ as $[A_t \ B_t]$ is one of the systems that are consistent with the data. The following theorem finds a set of models $M_{\Sigma}$ that over-approximates $N_\Sigma$, i.e., $N_\Sigma \subseteq M_{\Sigma}$, which defines $f(\cdot)$ introduced above.

For this, we aim to determine the mapping of the observation $Z^+$ and $Z^-$ to the corresponding state-space region. Specifically, we construct a zonotope $Z_{x,z}(k) \subseteq M_{\eta}$ that contains all possible $x \in M_{\eta}$ given $z^*(k)$, which defines $f(\cdot)$ for each $i$. This can be written as $Z_{x,z}(k) = \{ x \in M_{\eta} \mid C^i x = z^i(k) - Z_{\gamma,i} \}$. Extending $\tilde{\beta}_{M_{u},w}$ to a matrix zonotope allows to find the mapping of $Z^+$ and $Z^-$ to the state space region.

$$\begin{align*}
M_{\Sigma} &= \{ M_{x,z} \mid \text{consistent with the data} \}.
\end{align*}$$

(7)

contains all matrices $[A \ B]$ that are consistent with the data $D$ and the noise bounds, i.e., $N_\Sigma \subseteq M_{\Sigma}$, with $M_{x,z}^+ = (C_{M,x,z}^{(1:T+1)}, G_{M,x,z}^{(1:T+1)})$ and $M_{x,z}^- = (C_{M,x,z}^{(1:T+1)}, G_{M,x,z}^{(1:T+1)})$.

where

$$\begin{align*}
C_{M,x,z}^{+} &= V_1 \Sigma_{r \times r} P_1^T \left(Z^+ - C_{M,\gamma}\right), \\
C_{M,x,z}^{-} &= V_1 \Sigma_{r \times r} P_1^T \left(Z^- - C_{M,\gamma}\right), \\
G_{M,x,z}^{(i)} &= V_1 \Sigma_{r \times r} P_1^T G_{M,\gamma}^{(i)}, \quad i = \{1, \ldots, \xi T\}, \\
G_{M,x,z}^{(T+1)} &= M V_2 1_{(n-r) \times T},
\end{align*}$$

(8) (9) (10) (11)

for all $M \geq \|x\|_2$, with $P_1$, $V_1$, $\Sigma$ and $V_2$ obtained from the SVD of $C$. Assuming $C$ has rank $r$, then

$$\begin{align*}
C &= \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \Sigma_{r \times r} \ 0_{(r-n-r) \times r} \end{bmatrix} \begin{bmatrix} V_1^T \ V_2^T \end{bmatrix},
\end{align*}$$

(12)

where a matrix with non-positive index is an empty matrix.

**Proof.** From (12), we rewrite $G_{M,x,z}^{(i)}$ as $P_1 V_1^T x = z - \gamma$. Since $\gamma$ is bounded by $Z_{\gamma} = (c_{\gamma}, G_{\gamma})$, we can write

$$\begin{align*}
x &= V_1 \Sigma_{r \times r}^{-1} P_1^T (z - c_{\gamma}) - V_2 \Sigma_{r \times r}^{-1} P_2^T G_{\gamma} \beta, \quad |\beta| \leq 1.
\end{align*}$$

This set corresponds to all possible $x$ values within the range space of $C$ satisfying (4). By definition, if $r = n$, then $V_2 = 0$, $V_1$ spans the domain of $x$, and $\{c_{\gamma}, G_{\gamma}\}$ sufficiently defines all possible $x$ satisfying (4). However, if $r < n$, $V_1$ only spans a subset of the domain of $x$. To ensure $Z_{x,z}$ contains all possible $x$ we include a basis for ker($C$) in $G_{M,x,z}$ by appending the generator $V_2 M$ to $G_{M,x,z}$, ensuring $M \geq \|x\|_2$ such that $V_2 M$ includes all $x$ values in the directions of $V_2$. In both cases for $r$, the generator matrix can be written as

$$G_{M,x,z} = \left[ G_{M,x,z}^{(i)} \ V_2 M \right] = \left[ V_1 \Sigma_{r \times r}^{-1} P_1^T G_{\gamma} \ V_2 M \right],$$

and the set $Z_{x,z} = \{c_{x,z}, G_{x,z}\}$. This result extends to the case when $r < p$ using similar argumentation in the respective cases $r = n$ and $r < n$. Considering the matrix version of $Z_{x,z}$ results in proving $M_{x,z}^+$ and $M_{x,z}^-$. Then, we extend the proof of [17, Lem.1] for input-output data: For any $[A \ B] \in N_\Sigma$, we know that there exists a $W^- \in M_{\eta}$ such that

$$AX^- + BU^- = X^- - W^-.$$

(13)

Every $W^- \in M_{\eta}$ can be represented by a specific choice $\beta_{M_{u},w}^{(i)}$, $-1 \leq \beta_{M_{u},w}^{(i)} \leq 1$, $i = 1, \ldots, \xi M_{\eta}$, that results in a matrix inside the matrix zonotope $M_{w}$:

$$W^- = C_{M_{w}} + \sum_{i=1}^{\xi M_{w}} \beta_{M_{u},w}^{(i)} G_{M_{u},w}^{(i)}.$$

(14)

Rearranging (13) and considering $M_{x,z}^+$ and $M_{x,z}^-$ as an over-approximation of $X^+$ and $X^-$, respectively, yields

$$\begin{align*}
[A \ B] = \left( M_{x,z}^+ - C_{M_{x,z}} - \sum_{i=1}^{\xi M_{w}} \beta_{M_{u},w}^{(i)} G_{M_{u},w}^{(i)} \right) \begin{bmatrix} U^- \ \end{bmatrix}^T
\end{align*}$$

Hence, for all $[A \ B] \in N_\Sigma$, there exists $\beta_{M_{u},w}^{(i)}$, $-1 \leq \beta_{M_{u},w}^{(i)} \leq 1$, $i = 1, \ldots, \xi M_{w}$, such that (14) holds. Therefore, for all $[A \ B] \in N_\Sigma$, it also holds that $[A \ B] \in M_{\Sigma}$ as defined in (7), which concludes the proof. □
Given that we have found a matrix zonotope $\mathcal{M}_Z$ that contains the true system dynamics $[A_t \quad B_t] \in \mathcal{M}_Z$, we can utilize it in computing the time update reachable set $\tilde{R}_k$ in the following theorem.

**Theorem 1.** The set $\tilde{R}_k$ over-approximates the exact reachable set, i.e., $\tilde{R} \supseteq \tilde{R}_k$ where

$$\tilde{R}_{k+1} = \mathcal{M}_Z(\tilde{R}_k \times U_k) + \mathcal{Z}_w,$$

and $\tilde{R}_0 = \chi_0$.

**Proof.** As $[A_t \quad B_t] \in \mathcal{M}_Z$ according to Lemma 1 and starting from the same initial set $\chi_0$, it follows that $\tilde{R}_k \supseteq \tilde{R}_k$.

### B. Online Estimation Phase using Zonotopes

In this subsection, we present the online estimation phase. We are now considering the system (35) with observations (36). This phase consists of a time update and a measurement update. In Sec. III-A, we derived the function $f(\cdot)$ for the time update. We next present two approaches to perform the measurement update.

1) **Approach 1 - Reverse-Mapping:** For this approach, we aim to determine the mapping of an observation $y^i(k)$ to the corresponding state-space region. Similar to Lemma 1, we construct a zonotope $Z_{x|y^i}(k) \subseteq \mathbb{R}^n$ that contains all possible $x \in \mathbb{R}^n$ given $y^i(k)$, $C^i$ and bounded noise $v^i(k) \in Z_{v,i}$ satisfying (3b), for each $i$.

**Proposition 1.** Assume $\|x\|_2 \leq K$. Given a measurement $y^i(k)$ with noise $v^i(k) \in Z_{v,i} = \langle c_{v,i}, G_{v,i} \rangle$ satisfying (3b), the possible states $x$ that correspond to this measurement are contained within the zonotope $Z_{x|y^i} = \langle c_{x|y^i}, G_{x|y^i} \rangle$, where

$$c_{x|y^i} = V_1 \Sigma^{-1} \pi_p, P_1^T (y^i(k) - c_{v,i}),$$

$$G_{x|y^i} = [V_1 \Sigma^{-1} \pi_p, P_1^T G_{v,i}, V_2 M_1],$$

for all $M \geq K$, with $P_1$, $V_1$, $\Sigma$ and $V_2$ obtained from the SVD of $C^i$ as in (12).

**Proof.** The proof follows immediately from Lemma 1.

**Remark 1.** In our case, $Z_{x|y^i}(k)$ will eventually be intersected with $\tilde{R}_k = \langle \hat{c}_k, \tilde{G}_k \rangle$. It is therefore sufficient to set $M \geq \text{radius}(\tilde{R}_k) + \|V_2 \Sigma \pi_p\|_2$ instead of the more conservative $M \geq \|x\|_2$, where radius($\tilde{R}_k$) returns the radius of a minimal hyper-sphere containing $\tilde{R}_k$.

Having determined the sets $Z_{x|y^i}(k)$ for all $i \in \{1, \ldots, q\}$, we can compute the measurement updated set $\tilde{R}_k$ given the predicted set $\tilde{R}_k$ and each measurement set $Z_{x|y^i}(k)$ as

$$\tilde{R}_k = \tilde{R}_k \cap \bigcap_{i=1}^q Z_{x|y^i}(k),$$

which can be performed using the standard intersection operations presented in [11], [20].

2) **Approach 2 - Implicit Intersection:** Contrary to Approach 1, here, we do not explicitly determine the sets $Z_{x|y^i}(k)$. Instead, $\tilde{R}_k$ is determined directly from the set $\tilde{R}_k$, the measurements $y^i(k)$ and some weights $\lambda^i_k$ for $i \in \{1, \ldots, q\}$. We then optimize over the weights to minimize the volume of $\tilde{R}_k$.

**Proposition 2.** The intersection of $\tilde{R}_k = \langle \hat{c}_k, \tilde{G}_k \rangle$ and the $q$ regions for $x$ corresponding to $y^i(k)$ with noise $v^i(k) \in Z_{v,i} = \langle c_{v,i}, G_{v,i} \rangle$ satisfying (3b) can be over-approximated by the zonotope $\tilde{R}_k = \langle \hat{c}_k, \tilde{G}_k \rangle$ with

$$\hat{c}_k = \hat{c}_k + \sum_{i=1}^q \lambda_k^i (y^i(k) - C^i \hat{c}_k - c_{v,i}),$$

$$\tilde{G}_k = \left[ (I - \sum_{i=1}^q \lambda_k^i C^i) \tilde{G}_k - \lambda_k^i G_{v,i} \right],$$

where $\lambda_k^i \in \mathbb{R}^n, \Sigma^i$ for $i \in \{1, \ldots, q\}$ are weights.

**Proof.** The proof is based on [21, Prop.1] but with zonotopes as measurements instead of strips. Let $x \in \tilde{R}_k \cap Z_{x|y^i} \cap \cdots \cap Z_{x|y^q}$. Then there exists a $z$ such that $x = \hat{c}_k + \tilde{G}_k z$. Adding and subtracting $\sum_{i=1}^q \lambda_k^i C^i \tilde{G}_k z$ yields

$$x = \hat{c}_k + \sum_{i=1}^q \lambda_k^i (y^i(k) - C^i \hat{c}_k - c_{v,i} - G_{v,i} d^i) + (I - \sum_{i=1}^q \lambda_k^i C^i) \tilde{G}_k z,$$

which we insert into (20) to obtain

$$x = \hat{c}_k + \sum_{i=1}^q \lambda_k^i (y^i(k) - C^i \hat{c}_k - c_{v,i} - G_{v,i} d^i) + (I - \sum_{i=1}^q \lambda_k^i C^i) \tilde{G}_k z,$$

Note that $z \in [-1, 1]$ since $d^i \in [-1, 1]$ and $z \in [-1, 1]$.

The online estimation phase is illustrated in the block diagram of Fig. 1. The detailed estimation phase is presented in Algorithm 1. The function $\text{measZon}(\cdot)$ executes Proposition 1 and $\text{optZon}(\cdot)$ Proposition 2. The function $\text{reduce}(\tilde{R}_{k+1})$ reduces the order of $\tilde{R}_{k+1}$ using the method proposed in [22], which ensures the number of generators in $\tilde{R}_{k+1}$ remains relatively low, avoiding potential tractability issues after multiple iterations.

### C. Online Estimation Phase using Constrained Zonotopes

When intersecting zonotopes, the result is an over-approximation of the true intersection. However, it is possible
Algorithm 1 Online Estimation Phase

\[ \hat{R}_0 = x_0 \]

\[ k = 1 \]

while True do

\[ \hat{R}_k = f(\hat{R}_{k-1} \cup (u(k-1), 0)) \text{ using } (15) \]

if Approach 1 then

foreach \( i \in \{1, \ldots, q\} \) do

\[ Z_{x_i y_i}(k) = \text{measZon}(y_i(k), Z_{v_i}, C^i) \text{ using } (16) \]

end

\[ \hat{R}_k = \hat{R}_k \cap \bigcap_{i=1}^q Z_{x_i y_i}(k) \]

if Approach 2 then

\[ (\hat{c}_k, \hat{G}_k) = \text{optZon}(\hat{R}_k, y(k), C, Z_v) \]

\[ \hat{G}_k, \lambda^* \leftarrow \text{Solve } (21) \]

\[ \hat{R}_k = \text{reduce}(\hat{R}_k) \text{ using } (22) \]

\[ k \leftarrow k + 1 \]

end

to determine the exact intersection of constrained zonotopes.

Definition 4. (Constrained zonotope [23]) An \( n \)-dimensional constrained zonotope is

\[ C = \{ x \in \mathbb{R}^n \mid x = c_C + G_C \beta, A_C \beta = b_C, \| \beta \| \leq 1 \} \]

where \( c_C \in \mathbb{R}^n \) is the center, \( G_C \in \mathbb{R}^{n \times n_y} \) the generator matrix and \( A_C \in \mathbb{R}^{n_y \times n} \) and \( b_C \in \mathbb{R}^{n_y} \) the constraints. In short, we write \( C = (c_C; G_C, A_C; b_C) \).

When using constrained zonotopes, we replace the time and measurement updated sets \( \hat{R}_k \) and \( \hat{R}_k \) by the constrained zonotopes \( \hat{C}_k \) and \( \hat{C}_k \), respectively.

1) Approach 1 - Reverse-Mapping: This approach works directly with constrained zonotopes. The sets \( Z_{x_i y_i}(k) \) of Proposition 1 are constrained zonotopes with no \( A_C, b_C \) constraints. The intersection in (17) becomes \( \hat{C}_k = \bigcap_{i=1}^q Z_{x_i y_i}(k) \) which can be performed as described in [23].

2) Approach 2 - Implicit Intersection: We adapt Proposition 3 to use constrained zonotopes.

Proposition 3. The intersection of \( \hat{C}_k = (\hat{c}_k, \hat{G}_k, \hat{A}_k, \hat{b}_k) \) and \( q \) regions for \( x \) corresponding to \( y_i(k) \) as in (26) can be described by the constrained zonotope \( \hat{C}_k = (\hat{c}_k, \hat{G}_k, \hat{A}_k, \hat{b}_k) \) with weights \( \lambda^*_k \in \mathbb{R}^{n \times n_y} \) for \( i \in \{1, \ldots, q\} \) where

\[ \hat{c}_k = \hat{c}_k + \sum_{i=1}^q \lambda^*_k(y^i(k) - C^i \hat{c}_k - c_{v,i}), \]

\[ \hat{G}_k = \left( I - \sum_{i=1}^q \lambda^*_k C^i \right) \hat{G}_k - \lambda^*_k G_{v,1} \ldots - \lambda^*_k G_{v,q} \]

\[ \hat{A}_k = \begin{bmatrix} A_k & 0 & \ldots & 0 \\ C^1 \hat{G}_k & G_{v,1} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ C^q \hat{G}_k & 0 & \ldots & G_{v,q} \end{bmatrix} \]

Adding and subtracting \( \sum_{i=1}^q \lambda^*_k C^i \hat{G}_k z_k \) to (26) yields

\[ x_k = \hat{c}_k + \sum_{i=1}^q \lambda^*_k C^i \hat{G}_k z_k + (I - \sum_{i=1}^q \lambda^*_k C^i) \hat{G}_k z_k. \]

If we now insert (28) into (29), we obtain

\[ x = \left( I - \sum_{i=1}^q \lambda^*_k C^i \right) \hat{G}_k - \lambda^*_k G_{v,1} \ldots - \lambda^*_k G_{v,q} \]

Hence, \( x(k) \in \hat{C}_k \) and \( (\hat{C}_k \cap Z_{x_i y_i} \cap \cdots \cap Z_{x_i y_i}) \subseteq \hat{C}_k \).

IV. EVALUATION

We evaluate our method by considering an input-driven variant of the rotating target described in [11]. We set

\[ A_{tr} = \begin{bmatrix} 0.9455 & -0.2426 \\ 0.2486 & 0.9455 \end{bmatrix}, \quad B_{tr} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \]
with \( q = 3 \) measurements parameterized as follows
\[
C^1 = \begin{bmatrix} 1 & 0.4 \end{bmatrix}, C^2 = \begin{bmatrix} 0.9 & -1.2 \end{bmatrix}, C^3 = \begin{bmatrix} -0.8 & 0.2 \\ 0 & 0.7 \end{bmatrix},
\]
\[
Z_{v,1} = \langle 0, 1 \rangle, Z_{v,2} = \langle 0, 1 \rangle, Z_{v,3} = \langle 0 \rangle^T, I_2).
\]
The noise signals are characterized by the zonotopes \( Z_\gamma = \langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T, 0.02I_2 \rangle \) and \( Z_w = \langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T, 0.02I_2 \rangle \). We run the offline learning phase with \( T = 500 \) and inputs sampled uniformly from the set \( \mathcal{U} = \langle 0, 10 \rangle \). The noise signals \( \gamma(k) \), \( w(k) \) and \( \mu(k) \) are sampled uniformly from their respective zonotope sets using the command randPoint\( (Z) \) as described in [20].

After learning \( f(\cdot) \), we run the online estimation phase. The initial state set is \( X_0 = \langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T, 15I_2 \rangle \) and the true initial state is \( x(0) = \begin{bmatrix} -10 \\ 10 \end{bmatrix}^T \). Once again, we sample the inputs uniformly from \( \mathcal{U} \). We evaluate both the zonotope and constrained zonotope methods, each time using either of the two proposed measurement update approaches. Fig. 2a shows the bounds of \( \hat{R}_k \) in the \( x_1 \) state dimension for both approaches. Fig. 2b shows the equivalent results when our method uses constrained zonotopes. As expected, \( x(k) \) is always contained within \( \hat{R}_k \) (or \( \hat{C}_k \)) at each time step. Although both measurement update approaches yield similar set sizes on average, the set evolution of Approach 2 is comparatively smoother.

Furthermore, we compare our results with N4SID subspace identification [25] combined with a Kalman filter (KF). In Fig. 3, we show the sets \( \hat{R}_k \) and \( \hat{C}_k \) using either measurement update approach, using zonotopes or constrained zonotopes. We also show the ellipse corresponding to the \( 3\sigma \) uncertainty bound of the KF estimate, indicating that our estimator provides state sets comparable in size to that of the KF. We should mention that KF bounds come without any guarantees.

Referring to both Fig. 2 and Fig. 3, it is clear that the constrained zonotopes yield smaller state sets at each time step. However, this comes at the cost of increased computational load. Running our simulations on a Dell laptop with an 8-core i5-8365U processor at 1.6GHz, the average computation time per iteration for Approach 1 increased from 0.656sec to 1.267sec when using constrained zonotopes; for Approach 2, the corresponding times were 0.221sec and 0.971sec, respectively. For all our approaches, we observed that reducing the order of the sets to 5, which reduces the number of generators in \( \hat{R} \) (or \( \hat{C} \)), was critical to keep the computational load low.

V. CONCLUSIONS AND RECOMMENDATIONS

In this paper, we introduced a novel zonotope-based method to perform set-based state estimation with set containment guarantees using a data-driven set propagation function. We presented an approach to compute the set of model that is consistent with the data and noise bounds given input-output data. Then, we presented two approaches to perform the measurement update which merges the time updated state set with the observed measurements. We extended our method to use constrained zonotopes, which yielded smaller state sets at the cost of increased computational load. Our results show state sets comparable in size to the \( 3\sigma \) uncertainty bounds obtained when running N4SID subspace identification and a Kalman filter, but with the added feature of set-containment guarantees and without requiring any knowledge of the statistical properties of the noise.

Future work includes evaluating our proposed estimator on real-world examples as well as gaining more insight into the limitations of our method when applied to more complex dynamical systems. Additionally, improving the zonotope intersection operation to lessen the degree of over-approximation of the resultant state set would yield tighter state set estimates at each time step.
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