Quantum strings in $AdS_5 \times S^5$
and AdS/CFT duality$^1$

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**Abstract**

We review some recent progress in understanding the spectrum of energies/dimensions of strings/operators in $AdS_5 \times S^5$ – planar $\mathcal{N}=4$ super Yang-Mills correspondence. We consider leading strong coupling corrections to the energy of lightest massive string modes in $AdS_5 \times S^5$, which should be dual to members of the Konishi operator multiplet in the SYM theory. This determines the general structure of strong-coupling expansion of the anomalous dimension of the Konishi operator. We use 1-loop results for semiclassical string states to extract information about the leading coefficients in this expansion.

1 Introduction

Two important problems of modern theoretical high energy physics are to understand quantum gauge theories at any coupling (with applications to both perturbative and non-perturbative issues) and to understand string theories in non-trivial backgrounds (e.g. Ramond-Ramond ones relevant for flux compactifications). The AdS/CFT duality relates the two questions suggesting solving them together rather than separately is the best strategy. This is the modern analog of the “harmonic oscillator” or “Ising” problem – to solve two most symmetric non-trivial 4-d gauge theory and 10-d superstring theory – planar $\mathcal{N}=4$ SYM theory and its dual – free superstring in $AdS_5 \times S^5$. They have powerful hidden symmetries allowing to solve problem “in principle” using integrability methods. The $\mathcal{N}=4$ SYM theory has maximal supersymmetry and conformal invariance but it is a priori unclear in which sense it could be integrable. Integrability might be expected in the spectrum of anomalous dimensions (as it was previously observed in YM gluonic sector – emergence of XXX spin chain Hamiltonian as 1-loop anomalous dimension operator [1]) but hidden symmetries should play broader role as they are “inherited” via AdS/CFT from 2-d integrable QFT – string $\sigma$-model. In that sense one may hope to use 2-d integrable QFT to solve 4-d CFT.

Superstring sigma model in $AdS_5 \times S^5$ is integrable in “canonical” sense being an example of a sigma model on symmetric space. Its classical equations admit infinite number of conserved charges. In particular, it is closely related (via Pohlmeyer reduction) to (super) sine-Gordon and

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non-abelian Toda models [2]. For example, special motions of strings are described by integrable 1-d mechanical systems (Neumann, etc.). The integrability of string theory extends to quantum level: we have evidence of that to 2 loops in $\alpha'$ expansion using AdS/CFT “bootstrap” reasoning (see ref. [3] and refs. there).

Quantum integrability of string theory sigma-model should control the spectrum of string energies on $R \times S^1$ (or anomalous dimensions of 2-d primary operators = vertex operators on $R^{1,1}$). It may also have implications for correlation functions of vertex operators (or closed-string scattering amplitudes) but to which extent that might be true is unclear (for example, in flat space the string action is just a collection of free oscillators but space-time amplitudes are rather nontrivial).

The quantum integrability of string theory for all values of string tension $\sqrt{\lambda}$ should then imply integrability on the gauge theory side for any value of 't Hooft coupling $\lambda = g^2_{YM} N_c$. Again, this should be true for the spectrum of anomalous dimensions of single-trace gauge invariant operators, while implications of integrability for correlation functions of such operators a priori seem rather limited (though this may turn out not to be so given remarkable hidden symmetries in the on-shell gluon scattering amplitudes related to cusped Wilson loops discovered recently, see, e.g., [4]).

Regarding the spectrum of states, in the last 7 years there was an impressive progress (for reviews, see, e.g. [5]). The spectrum of “long” operators or “semiclassical” string states with large quantum numbers is now understood to be described by the Asymptotic Bethe Ansatz (ABA) in its final BES [6] form. The ABA was constructed using information from perturbative gauge theory (spin chain for 1-loop anomalous dimensions, ...) and perturbative string theory (classical and 1-loop phase,...), symmetries (magnon S-matrix), and the assumption of exact integrability. The consequences of ABA were checked against all available perturbative gauge and string theory data. The key example is the ABA prediction for the cusped anomalous dimension or dimension $\Delta$ of twist 2 operator $\text{Tr}(\Phi D_S^2 \Phi)$. Namely, $\Delta - 2 =_{s \to \infty} f(\lambda) \ln S$ where $f(\lambda)$ satisfies exact integral equation solution of which is known in principle to any order in small $\lambda$ or large $\lambda$ expansion and agrees with known perturbative results (see, e.g., [7, 3, 8]):

$$f(\lambda \ll 1) = \frac{\lambda}{2\pi^2} \left[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{2^8 \cdot 45} - \left(\frac{73}{630} + \frac{4\zeta^2(3)}{\pi^6}\right)\frac{\lambda^3}{2^7} + \ldots\right],$$

(1)

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi} \left[1 - \frac{3 \ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - \ldots\right],$$

(2)

where $K$ is Catalan’s constant.

More recently, there was a substantial progress towards understanding the spectrum of “short” operators, i.e. the spectrum of all quantum string states. This was achieved solely on the string side by using 2-d integrable field theory methodology generalizing the ABA describing states on $R^{1,1}$ to Thermodynamic Bethe Ansatz (TBA) which should be describing states on $R \times S^1$ [9, 10]. It remains to be understood why inclusion of wrapping diagram contributions to anomalous dimensions should lead to a similar modification from the gauge theory perspective. The construction of TBA is rather non-trivial due to lack of 2-d Lorentz invariance in the standard “BMN-vacuum-adapted” i.e. gauge on string theory side. In few special cases the ABA “improved” by Luscher corrections is enough: 4- and 5-loop Konishi
operator dimension, and 4-loop minimal twist operator dimension were computed [11] in this way. What remains to do is to thoroughly check the TBA predictions against perturbative string and gauge theory data.

The key example of a “short” operator is the Konishi operator [12] $\text{Tr}(\bar{\Phi}^i \Phi^i)$ for which the weak-coupling anomalous dimension is now known up to 5 loop orders[13, 11]:

$$
\gamma(\lambda \ll 1) = \frac{12\lambda}{(4\pi)^2} \left[ 1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} \right] - \frac{\lambda^3}{(4\pi)^6} \left[ 208 - 48\zeta(3) + 120\zeta(5) \right] + 8\left[ 158 + 72\zeta(3) - 54\zeta^2(3) - 90\zeta(5) + 315\zeta(7) \right] \frac{\lambda^4}{(4\pi)^8} + \ldots
$$

(3)

The planar perturbation theory should have a finite radius of convergence; suppose we sum up this series and re-expand at strong coupling $\lambda$ – what should we expect to get? The AdS/CFT correspondence suggests [15] that $\gamma(\lambda \gg 1)$ should start with $\sim 4\sqrt{\lambda}$ term. As was argued in [14] and will be discussed below, in general string-theory arguments imply that

$$
\gamma(\lambda \gg 1) = 2\sqrt{\lambda} + b_0 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \frac{b_3}{(\sqrt{\lambda})^3} + \ldots.
$$

(4)

Here the values of $b_0, b_1$ should be rational, while $b_3$ should be transcendental. The analysis in [14] leads to

$$
b_0 = -4, \quad b_1 = 1, \quad b_2 = 0,
$$

(5)

while $b_3$ should contain $\zeta(3)$. At the same time, the numerical result found from the TBA/Y-system approach [10] gives $b_1$ that is approximately twice as big. The reason for this disagreement remains to be understood but it is very encouraging that the numerical values of $b_0, b_1$ in [10] suggest that they are integer, i.e. rational as it should be according to the string-theory logic.

There are many open questions remaining: (i) which is radius of convergence of weak coupling expansion? (ii) how to find the analytic form of strong-coupling expansion from TBA/Y-system? (iii) how to carry out direct matching of short operators onto string spectrum found in near-flat-space expansion? (iv) is there “level crossing” as one increases $\lambda$? (v) is strong-coupling expansion of dimensions of short operators Borel summable or not as in the case of the cusp anomaly? (vi) are there exponential corrections $e^{-a\sqrt{\lambda}}$ in (4) like in the cusp anomaly case [8]? There are also deeper issues: how to solve string theory from first principles – which are fundamental variables? how to preserve 2-d Lorentz invariance? how to prove quantum integrability? is there a useful lattice version of string “supercoset” sigma model?

Below we shall concentrate on explaining the origin of the expansion (4) of the energy of a string state which is dual to Konishi operator following ref. [14].

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3The 5-loop result of [11] was not so far reproduced directly on the gauge theory side and so remains a string-theory/integrability prediction.

4The BMN dispersion relation $e(p) = \sqrt{1 + \frac{1}{\pi^2} \sin^2 \frac{p}{2}}$ suggests an upper bound on the radius of convergence: $|\lambda| < \frac{1}{4}$. 4
2 Approaches to finding energies of quantum strings in \(AdS_5 \times S^5\)

According to AdS/CFT energies of quantum strings in AdS should be the same as dimensions of the corresponding gauge theory operators. The aim is to compute the leading \(\lambda' \sim \frac{1}{\sqrt{\lambda}}\) corrections to the energy of the “lightest” massive string state which should be dual to the Konishi operator in SYM theory.

The members of the Konishi multiplet are operators related to the lowest canonical dimension singlet by supersymmetry. They correspond to highest-weight states of \(SO(2, 4) \times SO(6)\) labelled by (see, e.g., [16]) \([J_2 - J_3, J_1 - J_2, J_2 + J_3]^{a_0}_{(s_l, s_R)}\) with the singlet being \([0, 0, 0]_{(0,0)}\). Then full dimension is \(\Delta = \Delta_0 + \gamma(\lambda)\), where \(\Delta_0 = 2, \frac{5}{2}, 3, ..., 10\). All operators in supermultiplet have the same anomalous dimension \(\gamma\). Examples of such operators are \(\text{Tr}(\Phi^i \Phi_i)\), \((i = 1, 2, 3)\) with \(\Delta_0 = 2\); \(\text{Tr}(\Phi_1, \Phi_2)^2)\) in \(su(2)\) sector with \(\Delta_0 = 4\); \(\text{Tr}(\Phi_1 D^2 \Phi_1)\) in \(sl(2)\) sector with \(\Delta_0 = 4\).

Assuming no “level crossing”, the Konishi operator having lowest nontrivial dimension at weak coupling should be dual to the “lightest” among massive \(AdS_5 \times S^5\) string states. At large \(\lambda = \frac{g^2}{4\alpha'}\) a “small” string at the “center” of \(AdS_5\) is in nearly flat space. At strong coupling we are dealing with perturbative string theory: string states are built out of “flat-string” oscillators and there is a large degeneracy of mass spectrum which is lifted once the curvature is switched on. In flat space case we have string masses \(m_n = \frac{4(n-1)}{n} \), \(n = \frac{1}{2} (N + \bar{N}) = 1, 2, ..., N = \bar{N}\). The \(n = 1\) is the massless IIB supergravity (BPS) level (l.c. vacuum \(|0 >: (8+8)^2 = 256\) states). The \(n = 2\) is the first massive level; it is highly degenerate: \(|(a_{1-1} + S^2_{11})|0 >|^2 = [(8+8)\times(8+8)]^2\). Switching on \(AdS_5 \times S^5\) background fields lifts degeneracy: states with “lightest” mass are thus at first excited string level and thus should correspond to Konishi multiplet.

The string spectrum in \(AdS_5 \times S^5\) is organised [16] in long multiplets of \(PSU(2, 2|4)\): remarkably, flat-space string spectrum can be re-organized in multiplets of \(SO(2, 4) \times SO(6) \subset PSU(2, 2|4)\). Namely, \(SO(4) \times SO(5) \subset SO(9)\) reps can be lifted to \(SO(4) \times SO(6)\) reps of \(SO(2, 4) \times SO(6)\), etc. Then the Konishi long multiplet \(\tilde{T}_1 = (1 + Q + Q \wedge Q + ...)\) \([0, 0, 0]_{(0,0)}\) determines the Kaluza-Klein “floor” of 1-st excited string level: \(H_1 = \sum_{\gamma=0}^{\infty} [0, J, 0]_{(0,0)} \times \tilde{T}_1\).

States on the first excited level with \(J \neq 0\) are outside Konishi multiplet, i.e. they should have higher anomalous dimension.

For scalar massive state represented by a field in \(AdS_5\) one expects to find \((-\nabla^2 + m^2)\Phi + ... = 0\), i.e. \(\Delta(\Delta - 4) = (mR)^2 + O(\alpha') = 4(n-1)\frac{R^2}{\alpha'} + O(\alpha')\) or \(\Delta = 2 + \sqrt{(mR)^2 + 4} + O(\alpha')\) and thus [15]

\[
\Delta(\lambda \gg 1) = \sqrt{4(n-1)^2}\sqrt{\lambda} + ...
\] (6)

Thus for the first massive level \(n = 2\): \(\Delta = 2\sqrt{\lambda} + ....\) What about subleading corrections? In general, comparison between gauge and string theory states is non-trivial. In gauge theory for \((\lambda \ll 1)\) the operators are built out of free fields, canonical dimension \(\Delta_0\) determines states that can mix. In string theory with \((\lambda \gg 1)\) the near-flat-space string states are built out of

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5That follows from the assumption of exact superconformal algebra in which the dilatation operator commutes with supersymmetry generator \(Q\) on \(Q\) itself, i.e. \(Q\) has exact conformal dimension 1/2. Then members of supermultiplet obtained from “ground state” by acting by \(Q\) will have dimensions differing by (half)integers.
free oscillators, and the level $n$ determines states that can mix. One non-trivial question is then about the meaning of $\Delta_0$ at strong coupling and the meaning of $n$ at weak coupling. A possible strategy is to relate states with same global charges and to assume “non-intersection principle” [17]: no level crossing for states with same quantum numbers as $\lambda$ changes from strong to weak coupling.

There are several approaches to computation of corrections to string energies: (i) vertex operator approach: use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in anomalous dimension of corresponding vertex operator [17, 18]; (ii) space-time effective action approach: use near-flat-space expansion and NSR vertex operators to reconstruct $\alpha' \sim \sqrt{\lambda}$ corrections to corresponding massive string state equation of motion [19]; (iii) “light-cone” quantization approach: start with light-cone gauge $AdS_5 \times S^5$ string action and compute corrections to energy of corresponding flat-space oscillator string state [20]; (iv) semiclassical approach: identify short string state as small-spin limit of semiclassical string state – reproduce the structure of strong-coupling corrections to short operators [21, 14].

To find the spectrum one should solve the marginality (1,1) conditions on vertex operators, i.e. to diagonalize the 2-d anomalous dimension operator. For example, the scalar anomalous dimension operator $\hat{\gamma}$ acts on $T(x) = \sum c_{n...m}x^n...x^m$ or on the coefficients $c_{n...m}$. $\hat{\gamma}$ is a differential operator in target space found from the $\beta$-function for the perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2z \left[ G_{mn}(x) \partial x^m \partial x^n + T(x) \right]$$

$$\beta T = -2T - \frac{\alpha'}{2} \hat{\gamma} T + O(T^2)$$

$$\hat{\gamma} = \Omega^{mn} D_m D_n + \cdots + \Omega^{m...k} D_m ... D_k + \cdots$$

$$\Omega^{mn} = G^{mn} + O(\alpha'^3), \quad \Omega^{...} \sim \alpha'^m R_p^{...}$$

Solving $-\hat{\gamma} T + m^2 T = 0$ is the same as diagonalising $\hat{\gamma}$. Similar approach applies to massless (graviton, ...) and massive states: e.g., $\beta_{mn} = \alpha' R_{mn} + O(\alpha'^3)$ gives the Lichnerowitz operator as anomalous dimension operator

$$(\hat{\gamma} h)_{mn} = -D^2 h_{mn} + 2R_{mknl} h^{kl} - 2R_{k(m} h_{n)} + O(\alpha'^3)$$

For massive string states in a curved background we should get the action

$$\int d^D x \sqrt{g} \left[ \Phi(-D^2 + m^2 + X) \Phi + \cdots \right]$$

$$m^2 = \frac{4}{\alpha'} (n - 1), \quad X = R_{...} + O(\alpha')$$

In the case of $AdS_5 \times S^5$ background

$$R_{mn} - \frac{1}{96} (F_5 F_5)_{mn} = 0, \quad R = 0, \quad F_5^2 = 0$$

so for a 10-d scalar field $\Phi$ the leading term in $X$ should vanish. This apparently implies that the leading $\alpha'$ correction to the scalar string state mass should be zero:

$$\left[ -D^2 + m^2 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right] \Phi = 0,$$

$$\Delta = 2 + \sqrt{4(n - 1) + 4 + O\left(\frac{1}{\sqrt{\lambda}}\right)}.$$
\[ \Delta_{(n=2)} = 2 + 2 \sqrt{\lambda} \left[ 1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right] \] (10)

There are possible subtleties in this argument, so one should try to rederive this result using other approaches. Also, one should compare this to predictions for non-singlet Konishi descendant states – they should have the same anomalous dimension. For example, \( \text{Tr}[\Phi_1, \Phi_2]^2 \) corresponding to \( SO(6) \) \((2,2,0)\) state \( J_1 = J_2 = 2 \) should be described by a string state with a tensor wave function \( \Phi_{mn;kl} \).

To find \( \hat{\gamma} \) for tensor states one may use effective action approach: derive equation of motion for a massive string field in curved background from quadratic effective action reconstructed from flat-space NSR S-matrix. An example is a totally symmetric NS-NS 10-d tensor – a state for a massive string field in curved background from quadratic effective action reconstructed from flat-space NSR S-matrix. We may consider a particular tensor with indices in \( AdS_5 \) so that near-flat-space expansion should be applicable. The resulting expression is [19] \( (a_i \text{ are from } AdS_5 \text{ and } m_k \text{ are from } S^5) \)

\[
L = R - \frac{1}{2 \alpha'} F^2 + \mathcal{O}(\alpha') \n \frac{1}{2} (D_\mu \Phi D^\mu \Phi + m^2 \Phi^2) + \sum_{k \geq 1} (\alpha')^{k-1} \Phi X_k (R, F_5, D) \Phi + ... \] (11)

The assumption is that \( \alpha' n R \ll 1 \), i.e. \( n \ll \sqrt{\lambda} \). That corresponds to a small massive string in the middle of \( AdS_5 \) so that near-flat-space expansion should be applicable. The resulting expression is [19] \( (a_i \text{ are from } AdS_5 \text{ and } m_k \text{ are from } S^5) \)

\[
L = \frac{1}{2} \Phi_{\mu_1...\mu_{2n}} (-D^2 + m^2) \Phi^{\mu_1...\mu_{2n}} 
+ \frac{n^2}{R^2} \left( \Phi_{a_1 a_2 \mu_3...\mu_{2n}} \Phi^{a_1 a_2 \mu_3...\mu_{2n}} - \Phi_{m_1 m_2 \mu_3...\mu_{2n}} \Phi^{m_1 m_2 \mu_3...\mu_{2n}} \right) + ... \] (12)

We may consider a particular tensor with \( S \) indices in \( AdS_5 \) and \( K \) indices in \( S^5 \) and then end up with the anomalous dimension operator

\[
[-D^2 + (m^2 + \frac{K^2 - S^2}{2R^2})] \Phi = 0 , \quad D^2 = D^2_{AdS_5} + D^2_{S^5} \] (13)

This symmetric transverse traceless tensor corresponds the highest-weight state with the Young table labels \( (\Delta, S, 0; K, J, 0) \) where \( J \) is an additional KK momentum. If we extract \( AdS_5 \) radius \( R \) and set \( \sqrt{\lambda} = \frac{R^2}{\alpha'} \) then [19]

\[
(-D^2_{AdS_5} + M^2) \Phi = 0 , \quad M^2 = 2\sqrt{\lambda}(S + K - 2) + \frac{1}{2}(K^2 - S^2) + J(J + 4) - K \] (14)

Equivalently, the marginality condition for the corresponding vertex operator is

\[
0 = -\sqrt{\lambda}(S + K - 2) 
+ \frac{1}{2} \left[ \Delta(\Delta - 4) + \frac{1}{2}S(S - 2) - \frac{1}{2}K(K - 2) - J(J + 4) \right] + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \] (15)

\[ \text{It is assumed here that } \Phi_{\mu_1...\mu_{2n}} \text{ is traceless and } D^\mu \Phi_{\mu_1...\mu_{2n}} = 0. \text{ In general, the quadratic terms may also contain mixing of different types of fields which we ignore here.} \]
The lowest BPS level is $n = \frac{1}{2}(S + K) = 1$ and the first massive level is $n = \frac{1}{2}(S + K) = 2$. A state from the first massive level on leading Regge trajectory is $S = K = 2$, $J = 0$ and then

$$\Delta = 2 + \sqrt{4\sqrt{\lambda} + 4 + \mathcal{O}(\frac{1}{\sqrt{\lambda}})} = 2 + 2\sqrt{\lambda}\left[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})^2\right] \quad (16)$$

This is the same prediction as for a singlet scalar state found above in (10). The constant term 2 here corresponds to a $\Delta_0 = 6$ operator ($\Delta_0 - 4 = 2$) [14].

Let us now review how the above expressions may appear directly in the vertex operator approach. If we would like to calculate the 2d anomalous dimensions from “first principles” we should start with the $AdS_5 \times S^5$ superstring theory action [22]

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \left[ - \partial N_p \bar{\partial} N^p + \partial m_k \bar{\partial} n_k + \text{fermions} \right] \quad (17)$$

$$N_p N^p \equiv N_+ N_- - N_u N_u^* - N_v N_v^* = 1, \quad n_x n_x^* + n_y n_y^* + n_z n_z^* = 1$$

$$N_\pm = N_0 \pm i N_5, \quad N_u = N_1 + i N_2, \ldots, \quad n_x = n_1 + i n_2, \ldots$$

and construct marginal (1,1) operators in terms of $N_p$ and $n_k$. For example, a vertex operator for a dilaton field (highest-weight state) is

$$V_f = (N_+)^{-\Delta}(n_x)^J \left[ - \partial N_p \bar{\partial} N^p + \partial n_k \bar{\partial} n_k + \text{fermions} \right] \quad (18)$$

where $N_+ \equiv N_0 + i N_5 = \frac{1}{z}(z^2 + x_m x_m) \sim e^{it}$, $n_x \equiv n_1 + i n_2 \sim e^{i\varphi}$. The marginality condition gives

$$0 = 2 - 2 + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - J(J + 4)] + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \quad (19)$$

i.e. $\Delta = 4 + J$ as it should be for a BPS state. The vertex operator for a bosonic string state on the leading Regge trajectory in flat space is

$$V_S = e^{-iE(x)\partial x^2}/S, \quad x = x_1 + ix_2, \quad \alpha' E^2 = 2(S - 2)$$

Candidate operators for states on leading Regge trajectory in $AdS_5 \times S^5$ are

$$V_f = (N_+)^{-\Delta}(n_x)^{J/2}, \quad n_x \equiv n_1 + i n_2 \quad (20)$$

$$V_S(\xi) = (N_+)^{-\Delta}(\partial N_u \bar{\partial} N_u)^{S/2}, \quad N_u \equiv N_1 + i N_2 \quad (21)$$

where we ignore fermionic terms and possible $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of 2-d anomalous dimension matrix. A non-trivial question is how such operators mix with operators with same charges and dimension. In general, $(\partial n_x \bar{\partial} n_x)^{J/2}$ mixes with singlets

$$(n_x)^2 n_2 (\partial n_x)^{J/2-2}(\bar{\partial} n_x)^{2}(\partial n_m \bar{\partial} n_m)^p(\partial n_k \bar{\partial} n_k)^q$$

For operators like $O_{f,s} = f_{k_1...k_m} n_{k_1} ... n_{k_l} \partial n_{m_1} \bar{\partial} n_{m_2} ... \partial n_{m_{2s-1}} \bar{\partial} n_{m_{2s}}$ this question was studied, e.g., in ref. [23]. An example of higher-level scalar operator is

$$N_+^{-\Delta}[(\partial n_k \bar{\partial} n_k)^r + \ldots], \quad r = 1, 2, ...$$

7In the simplest case of the operator $f_{k_1...k_l} n_{k_1} ... n_{k_l}$ with traceless $f_{k_1...k_l}$ it has the same anomalous dimension $\gamma$ as its highest-weight representative $V_f = (n_x)^J$, i.e. $\gamma = 2 - \frac{1}{2\sqrt{\lambda}} J(J + 4) + \ldots$. This is the same result as found for a scalar spherical harmonic that solves the Laplace equation on $S^5$. Similarly for the $AdS_5$ or $SO(2,4)$ model: replacing $n_x$ and $\partial n_m \bar{\partial} n_m$ with $N_+^{-\Delta}$ and $\partial N_p \bar{\partial} N_p$, with $J = -\Delta$ and $\frac{1}{\sqrt{\lambda}} \rightarrow -\frac{1}{\sqrt{\lambda}}$. For example, the dimension of $N_+^{-\Delta} \partial N^p \bar{\partial} N_p$ is $\gamma = \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4)] + \mathcal{O}(\frac{1}{\sqrt{\lambda}})^2$. 

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for which [24]

\[
0 = -2(r - 1) + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) + 2r(r - 1)] \\
+ \frac{1}{(\sqrt{\lambda})^2}\left[\frac{2}{3}r(r - 1)(r - \frac{7}{2}) + 4r\right] + ... .
\] (22)

For \( r = 2 \) this is a candidate for a state on the first excited level. However, the contribution of fermions may change the value of the 2-d anomalous dimension.

In general, the 2-d anomalous dimensions are given by a regular series expansion in \( \alpha' = \frac{1}{\sqrt{\lambda}} \), while \( \frac{1}{(\sqrt{\lambda})^k} \) appear as a result of solving quadratic-type equations for \( E = \Delta \) following from the marginality condition. 8

As an illustration, for operators with two spins \( J_1 = K, J_2 = J \) in \( S^5 \):

\[
V_{K,J} = N_+^{-\Delta} \sum_{u,v=0}^{K/2} c_{uv} M_{uv} \\
M_{uv} = n_y^{-u-v} n_x^{u+v} (\partial n_y)^u (\partial n_x)^v (\partial n_y)^{K/2-u} (\partial n_x)^{K/2-v}
\] (23)

the highest and lowest eigen-values of the 1-loop anomalous dimension matrix are [18]

\[
2 - K + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}K(K + 10) - J(J + 4) - 2JK] + O(\frac{1}{(\sqrt{\lambda})^2}) \\
2 - K + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - \frac{1}{2}K(K + 6) - J(J + 4)] + O(\frac{1}{(\sqrt{\lambda})^2})
\] (24)

Again, the fermionic contributions may alter terms linear in \( K \).

The main question on the way to finding energies or \( \Delta \)'s of the corresponding states is thus how to take the \( AdS_5 \times S^5 \) fermionic contributions into account? To this end we shall follow ref. [14] and employ the semiclassical approach based on the full superstring action.

3 Semiclassical approach: small-spin expansion

In semiclassical expansion for an energy of a string state one fixes the “classical” spin parameter \( \mathcal{J} = \frac{J}{\sqrt{\lambda}} \) and expands in large string tension \( \sim \sqrt{\lambda} \),

\[
E = \mathcal{E}(\frac{J}{\sqrt{\lambda}}, \sqrt{\lambda}) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + ...
\] (25)

In the “short” string limit \( \mathcal{J} \ll 1 \)

\[
\mathcal{E}_n = \sqrt{\mathcal{J}} \left( a_{0n} + a_{1n} \mathcal{J} + a_{2n} \mathcal{J}^2 + ... \right)
\] (26)

\(^8\)In particular, there cannot be any log \( \lambda \) terms that appear in the strong-coupling expansion of the anomalous dimensions computed using asymptotic Bethe ansatz equations [26].
Since $\sqrt{\lambda} \gg 1$ and $J = \sqrt{\lambda}$=fixed, here $J \sim \sqrt{\lambda} \gg 1$. If we knew all the terms in this expansion we could express $J$ in terms of $J$, fix $J$ to a finite value and re-expand in $\sqrt{\lambda}$. Then we would get

$$E = \sqrt{\lambda} J \left[ a_{00} + \frac{a_{10} J + a_{01}}{\sqrt{\lambda}} + \frac{a_{20} J^2 + a_{11} J + a_{02}}{(\sqrt{\lambda})^2} + ... \right]$$  \hfill (27)

To trust the coefficient of $\frac{1}{(\sqrt{\lambda})^n}$ one would need to know coefficients of up to $n$-loop terms, e.g., the classical $a_{10}$ and the 1-loop $a_{01}$ coefficients are sufficient in order to fix the $\frac{1}{\sqrt{\lambda}}$ term.

Since here we are interested in a short string probing the near-flat-space limit, we may (i) start with classical string solutions in flat space representing states at 1-st excited string level, and then (ii) embed them into $AdS_5 \times S^5$ and compute the 1-loop correction to their energy. The two basic examples are [14]: (1) circular string with 2 spins in two orthogonal planes, and (2) folded string spinning in one plane.

The rigid circular string rotating in two planes of flat $R^4$ space is described by

$$t = \kappa \tau \ , \quad x_1 \equiv x_1 + i x_2 = a \ e^{i(\tau + \sigma)} \ , \quad x_3 \equiv x_3 + i x_4 = a \ e^{i(\tau - \sigma)} \ , \quad E_{\text{flat}} = \frac{\kappa}{\alpha'} = \sqrt{\frac{4}{\alpha'} J} \ , \quad J_1 = J_2 = J = \frac{a^2}{\alpha'} .$$  \hfill (28)

Identifying oscillator modes that are excited on this solution we may associate it with a quantum string state created by

$$e^{-iEt} \left[ (\partial n_x \bar{n}_{x})^J \left( \partial n_y \bar{n}_{y} \right)^J + ... \right] , \quad \alpha' E^2 = 2(J_1 + J_2 - 2)$$  \hfill (29)

In the $J_1 = J_2$ case to get the quantum-state analog of classical expression one would need to shift $J \rightarrow J - 1$, i.e.

$$E_{\text{flat}} = \frac{\lambda}{\alpha'}(J - 1) .$$

Then $J_1 = J_2 = 2$ corresponds to state on 1-st string level $n = 2$.

A folded string rotating in a plane in flat space is represented by

$$t = \kappa \tau \ , \quad x_1 \equiv x_1 + i x_2 = a \sin \sigma \ e^{i \tau} \ , \quad E_{\text{flat}} = \sqrt{\frac{2}{\alpha'} S} \ , \quad S = \frac{a^2}{2 \alpha'} .$$  \hfill (30)

This is a semiclassical counterpart of a quantum string state on the leading Regge trajectory created by the vertex operator

$$e^{-iEt} \left[ (\partial x_x \bar{x}_x)^J \left( \partial n_y \bar{n}_{y} \right)^J + ... \right] , \quad \alpha' E^2 = 2(S - 2) .$$  \hfill (31)

There are 3 obvious choices of how to embed the circular solution into $AdS_5 \times S^5$: (i) the two 2-planes may belong to $S^5$: $J_1 = J_2$ “small string”; (ii) the two 2-planes may belong to $AdS_5$: $S_1 = S_2$ “small string”; (iii) one plane in $AdS_5$ and the other in $S^5$: $S = J$ “small string”. There are similar 3 choices for the folded string. We may then study each case in $AdS_5 \times S^5$, compute 1-loop correction to energy and interpolate to small values of $S, J$ and try to match
to states in the Konishi multiplet table. Getting the same expressions up to a constant shift would verify the universality of the strong-coupling expansion of the 4-d anomalous dimension of the dual gauge theory operators in the same supermultiplet.

The final result found in [14] is

$$E \equiv \Delta = 2\sqrt{\lambda} + b_0 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \frac{b_3}{(\sqrt{\lambda})^3} \ldots$$  (32)

$$b_0 = \Delta_0 - 4, \quad b_1 = 1$$  (33)

where $\Delta_0 = 4$ for the 3 circular string cases and $\Delta_0 = 6$ for the 3 folded string cases. The value of the coefficient $b_2$ is sensitive to the 2-loop string corrections which were not computed so far but we conjecture that that it is zero due to supersymmetry (see ref. [14]).

The simplest example is circular rotating string in $S^5$ with $J_1 = J_2 = J$ (see refs. [25, 14]) which for $J_1 = J_2 = 2$ should be dual to the Konishi descendant: $\text{Tr}([\Phi_1, \Phi_2]^2)$. The above flat solution can be directly embedded into $R_t \times S^5$ inside of $AdS_5 \times S^5$ – it represents a string on a small sphere inside $S^5$

$$X_1 + iX_2 = a e^{i(\tau + \sigma)/2}, \quad X_3 + iX_4 = a e^{i(\tau - \sigma)/2}, \quad \tilde{t} = \kappa \tau, \quad J = J_1 = J_2 = a^2, \quad E^2 = \kappa^2 = 4J.$$  (34)

Remarkably, the exact classical energy $E_0$ is just as in flat space

$$E_0 = \sqrt{\lambda E} = \sqrt{4\sqrt{\lambda}J}, \quad J = \sqrt{\lambda J}.$$  (35)

The 1-loop quantum string correction to the energy is given by the sum of the bosonic and fermionic fluctuation frequencies

$$E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[ 4\omega_n + 2n + \omega_{n+} + \omega_{n-} - 4(\omega_{n+}^f + \omega_{n-}^f) \right],$$  (36)

where

$$AdS_5: \quad \omega_n^2 = n^2 + 4J$$
$$S^5: \quad \omega_n^{\pm} = n^2 + 4(1 - J) \pm 2\sqrt{4(1 - J)n^2 + 4J^2}$$
Fermions: \quad $(\omega_n^f)^2 = n^2 + 1 + J \pm \sqrt{4(1 - J)n^2 + 4J}$

Expanding in small $J$ and doing the sums one finds (UV divergences cancel)

$$E_1 = \frac{1}{\sqrt{J}} \left[ J - \frac{3}{8}(1 + 8\zeta(3))J^2 + \ldots \right]$$
$$E = E_0 + E_1 = 2\sqrt{\lambda}J \left[ 1 + \frac{1}{2\sqrt{\lambda}} - \frac{3}{16}(1 + 8\zeta(3))\frac{J}{(\sqrt{\lambda})^2} + \ldots \right]$$  (37)
To ensure the correct short string limit in flat space we need to shift $J \to J - 1$. In $J = J_1 = J_2 = 2$ case that would suggest that for a Konishi state $[2, 0, 2]_{(0,0)}$

$$E = 2\sqrt{\lambda}[1 + \frac{1}{2\sqrt{\lambda}} + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)]$$  \hspace{1cm} (38)

The dual state in the Konishi table has $\Delta_0 = 4$. Thus in (32) one gets $b_1 = 1$ and $b_2 = 0$ (at 1-loop order) while $b_3$ is transcendental (1-loop contribution to it contains $\zeta(3)$).

The same results are found for the other (2 circular and 5 folded) string solutions representing 5 other states at the 1-st massive string level dual to 5 particular operators in the Konishi multiplet table. We refer to [14] for the details.

4 Conclusions

Due to an impressive recent progress based on integrability methods we are now at the beginning of understanding the quantum string spectrum in $AdS_5 \times S^5$ or the spectrum of “short” operators in planar $\mathcal{N}=4$ SYM theory for any value of ‘t Hooft coupling. The predictions of the integrability approach are still to be checked against perturbative string results. This requires better understanding of perturbative quantum $AdS_5 \times S^5$ string theory.

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9In general, the correct flat-space limit is consistent also with $J \to J - 1 + \frac{1}{\sqrt{\lambda}} + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)$. 

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