Topological Entropy and Diffeomorphisms of Surfaces with Wandering Domains*

Ferry Kwakkel † Vladimir Markovic

Abstract

Let $M$ be a closed surface and $f$ a diffeomorphism of $M$. A diffeomorphism is said to permute a dense collection of domains, if the union of the domains are dense and the iterates of any one domain are mutually disjoint. In this note, we show that if $f \in \text{Diff}^{1+\alpha}(M)$, with $\alpha > 0$, and permutes a dense collection of domains with bounded geometry, then $f$ has zero topological entropy.

1 Definitions and statement of results

A result of A. Norton and D. Sullivan [7] states that a diffeomorphism $f \in \text{Diff}^{3}_0(T^2)$ having Denjoy-type can not have a wandering disk whose iterates have the same generic shape. By diffeomorphisms of Denjoy-type are meant diffeomorphisms of the two-torus, isotopic to the identity, that are obtained as an extension of an irrational translation of the torus, for which the semi-conjugacy has countably many non-trivial fibers. If these fibers have non-empty interior, then the corresponding diffeomorphism has a wandering disk. Further, by generic shape is meant that the only elements of $\text{SL}(2, \mathbb{Z})$ preserving the shape are elements of $\text{SO}(2, \mathbb{Z})$, such as round disks and squares. In a similar spirit, C. Bonatti, J.M. Gambaudo, J.M. Lion and C. Tresser in [1] show that certain infinitely renormalizable diffeomorphisms of the two-disk that are sufficiently smooth, can not have wandering domains if these domains have a certain boundedness of geometry.

In this note, we study an analogous problem, namely the interplay between the geometry of iterates of domains under a diffeomorphism and its topological entropy. To state the precise result, we first need some definitions. Let $(M, g)$ be a closed surface, that is, a smooth, closed, oriented Riemannian two-manifold, equipped with the canonical metric $g$ induced from the standard conformal metric of the universal cover $\mathbb{P}^1, \mathbb{C}$ or $\mathbb{D}^2$. We denote by $d(\cdot, \cdot)$ the distance function relative to the metric $g$. Let $\text{Diff}^r(M)$ be the group of diffeomorphisms of $M$, where for

*2000 Mathematics Subject Classification: Primary 30C62, Secondary 28D20
†The first author was supported by Marie Curie grant MRTN-CT-2006-035651 (CODY).
r \geq 0 \text{ finite}, f \text{ is said to be of class } C^r \text{ if } f \text{ is continuously differentiable up to order } [r] \text{ and the } [r]-\text{th derivative is } (r)\text{-Hölder, with } [r] \text{ and } (r) \text{ the integral and fractional part of } r \text{ respectively. We identity } \text{Diff}^0(M) \text{ with } \text{Homeo}(M), \text{ the group of homeomorphisms of } M.

Given } f \in \text{Homeo}(M), \text{ for each } n \geq 1, \text{ define the metric } d_n \text{ on } M \text{ given by } d_n(x, y) = \max_{1 \leq i \leq n} \{d(f^i(x), f^i(y))\}. \text{ Given } \epsilon > 0, \text{ a subset } U \subset M \text{ is said to be } (n, \epsilon) \text{ separated if } d_n(x, y) \geq \epsilon \text{ for every } x, y \in U \text{ with } x \neq y. \text{ Let } N(n, \epsilon) \text{ be the maximum cardinality of an } (n, \epsilon) \text{ separated set. The topological entropy is defined as }

\begin{equation}
    h_{\text{top}}(f) = \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} \sup \frac{1}{n} \log N(n, \epsilon) \right).
\end{equation}

Next, we make precise the notion of a homeomorphism of a surface permuting a dense collection of domains.

**Definition 1.1.** Let } S \subset M \text{ be compact and } D := \{D_k\}_{k \in \mathbb{Z}} \text{ the collection of connected components of the complement of } S, \text{ with the property that } \text{Int}(\text{Cl}(D_k)) = D_k, \text{ where } \text{Cl}(D) \text{ is the closure of } D \text{ in } M. \text{ We say } f \in \text{Homeo}(M) \text{ permutes a dense collection of domains if }

1. } f(S) = S \text{ and } \text{Cl}(D_k) \cap \text{Cl}(D_{k'}) = \emptyset \text{ if } k \neq k',
2. \text{ for every } k \in \mathbb{Z}, \text{ } f^n(D_k) \cap D_k = \emptyset \text{ for all } n \neq 0, \text{ and }
3. \bigcup_{k \in \mathbb{Z}} D_k \text{ is dense in } M.

Note that we do not assume a domain to be recurrent, nor do we assume the orbit of a single domain to be dense. A wandering domain is a domain with mutually disjoint iterates under } f \text{ such that the orbit of the domain is recurrent. Thus a diffeomorphism with a wandering domain with dense orbit is a special case of definition 1.1. Denote } \exp_p : T_p M \to M \text{ the exponential mapping at } p \in M. \text{ The injectivity radius at a point } p \in M \text{ is defined as the largest radius for which } \exp_p \text{ is a diffeomorphism. The injectivity radius } \iota(M) \text{ of } M \text{ is the infimum of the injectivity radii over all points } p \in M. \text{ As } M \text{ is compact, } \iota(M) \text{ is positive.}

**Definition 1.2 (Bounded geometry).** A collection of domains } \{D_k\}_{n \in \mathbb{Z}} \text{ on a surface } M \text{ is said to have bounded geometry if the following holds: } \text{Cl}(D_k) \text{ is contractible in } M \text{ and there exists a constant } \beta \geq 1 \text{ such that for every domain } D_k \text{ in the collection, there exist } p_k \in D_k \text{ and } 0 < r_k \leq R_k \text{ such that }

\begin{equation}
    B(p_k, r_k) \subset D_k \subset B(p_k, R_k), \text{ with } R_k/r_k \leq \beta,
\end{equation}

where } B(p, r) \subset M \text{ is the ball centered at } p \in M \text{ with radius } r > 0. \text{ If no such } \beta \text{ exists, then the collection is said to have unbounded geometry.}
By Cl$(D_k)$ being contractible in $M$ we mean that Cl$(D_k)$ is contained in an embedded topological disk in $M$. Our definition of bounded geometry is equivalent to the notion of bounded geometry in the theory of Kleinian groups and complex dynamics. It is not difficult, given a surface of any genus, to construct homeomorphisms of that surface with positive entropy that permute a dense collection of domains. We show that producing examples that have a certain amount of smoothness is possible only to a limited degree.

**Theorem A** (Topological entropy versus bounded geometry). *Let $M$ be a closed surface and $f \in \text{Diff}^{1+\alpha}(M)$, with $\alpha > 0$. If $f$ permutes a dense collection of domains with bounded geometry, then $f$ has zero topological entropy.*

The outline of the proof of Theorem A is as follows. First we show that the bounded geometry of the permuted domains, combined with their density in the surface, give bounds on the dilatation of $f$ on the complement of the union of the permuted domains. The differentiability assumptions on $f$ allow us to estimate the rate of growth of the dilatation on the whole surface $M$. Using a result by Przytycki [8], we show this rate of growth is slow enough so as to ensure the topological entropy of $f$ is zero.

2 Entropy and diffeomorphisms with wandering domains

First, we study the relation between geometry of domains and the complex dilatation of a diffeomorphism.

2.1 Geometry of domains and complex dilatation

We denote $\lambda$ the measure associated to $g$ and $d\lambda$ the Riemannian volume form. By compactness of $M$, there exists a constant $\kappa > 0$ such that

$$\lambda(B(p,r)) = \int_{B(p,r)} d\lambda \geq \kappa r^2.$$  

(2)

where $B(p,r) \subset M$ is the ball centered at $p$ with radius $r < \iota(M)/2$. A sequence of positive real numbers $x_k$ is called a null-sequence, if for every given $\epsilon > 0$ there exist only finitely elements of the sequence for which $x_k \geq \epsilon$. Henceforth, we denote $\ell_k := \text{diam}(D_k)$, the diameter of $D_k$ measured in $g$, with $D_k \in \mathcal{D}$.

**Lemma 2.1.** Let $(M,g)$ be a closed surface and let $\{D_k\}_{k \in \mathbb{Z}}$ be a collection of mutually disjoint domains with bounded geometry. Then the sequence $\ell_k$ is a null-sequence.

**Proof.** Suppose, to the contrary, that $\{D_k\}_{k \in \mathbb{Z}}$ is not a null-sequence. Then there exist an $\epsilon > 0$ and an infinite subsequence $k_t$ such that $\text{diam}(D_{k_t}) \geq \epsilon$. By the bounded geometry property,
we have that \( \text{diam}(D_k t) \leq \beta r_k t \) and therefore \( r_k t \geq \epsilon/\beta \). Therefore, by (2),

\[
\lambda(D_k t) \geq \kappa r_k t^2 \geq \frac{\kappa \epsilon^2}{\beta^2},
\]

for every \( t \in \mathbb{Z} \). But this yields that

\[
\sum_{t \in \mathbb{Z}} \lambda(D_k t) = \infty,
\]

contradicting the fact that \( \lambda(M) < \infty \) as \( M \) is compact. \qed

Recall that \( S \) is the complement of the union of the permuted domains, i.e. \( S = M \setminus \bigcup_{k \in \mathbb{Z}} D_k \).

**Lemma 2.2.** Let \( f \in \text{Homeo}(M) \) permute a dense collection \( \mathcal{D} \) of domains with bounded geometry. For every \( p \in S \), there exists a sequence of domains \( D_{k_t} \) with \( \text{diam}(D_{k_t}) \to 0 \) for \( t \to \infty \) such that \( D_{k_t} \to p \).

**Proof.** Fix \( p \in S \) and let \( U \subset M \) be an open (connected) neighbourhood of \( p \). First assume that \( p \in S \setminus \bigcup_{k \in \mathbb{Z}} \partial D_k \). This set is non-empty, as otherwise the surface \( M \) is a union of countably many mutually disjoint continua; but this contradicts Sierpiński’s Theorem, which states that no countable union of disjoint continua is connected. We claim that \( U \) intersects infinitely many different elements of \( \mathcal{D} \). Indeed, if \( U \) intersects only finitely many elements \( D_{k_1}, ..., D_{k_m} \), then \( \Omega := \bigcup_{i=1}^m \text{Cl}(D_{k_i}) \) is closed. This implies that \( U \setminus \Omega \) is open and non-empty, as otherwise \( M \) would be a finite union of disjoint continua, which is impossible. However, as the union of the elements of \( \mathcal{D} \) is dense, \( U \setminus \Omega \) can not be open. Thus, there are infinitely many distinct elements \( D_{k_1}, D_{k_2}, ... \) of \( \mathcal{D} \) that intersect \( U \). Taking a sequence of nested open connected neighbourhoods \( U_t \) containing \( p \), we can find elements \( D_{k_t} \subset U_t \setminus U_{t+1} \) for every \( t \geq 1 \). By Lemma 2.1 \( \text{diam}(D_{k_t}) \) is a null-sequence and thus we obtain a sequence of domains \( D_{k_t} \) with \( \text{diam}(D_{k_t}) \to 0 \) for \( t \to \infty \) such that \( D_{k_t} \to p \).

As \( \text{Int}(\text{Cl}(D_k)) = D_k \), given \( p \in \partial D_k \) and given any neighbourhood \( U \ni p \), \( U \) has non-empty intersection with \( M \setminus \text{Cl}(D_k) \). By the same reasoning as above, \( p \) is again a limit point of arbitrarily small domains in the collection \( \mathcal{D} \). Thus we have proved the claim for all points \( p \in S \) and this concludes the proof. \qed

Next, we turn to the *complex dilatation* of a diffeomorphism \( f \in \text{Diff}(M) \) and its behaviour under compositions of diffeomorphisms, see e.g. [4]. We first consider the case where \( f \in \text{Diff}(\mathbb{C}) \). The complex dilatation \( \mu_f \) of \( f \) is defined by

\[
\mu_f: \mathbb{C} \to \mathbb{D}^2, \quad \mu_f(p) = \frac{f(z)}{f'(z)}(p),
\]

and the corresponding differential

\[
\mu_f(p) \frac{dz}{dz}.
\]
is the Beltrami differential of $f$. The dilatation of $f$ is defined by

$$K_f(p) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|},$$

which equals

$$K_f(p) = \frac{\max_v |Df_p(v)|}{\min_v |Df_p(v)|},$$

where $v$ ranges over the unit circle in $T_p\mathbb{C}$ and the norm $|\cdot|$ is induced by the standard (conformal) Euclidean metric $g$ on $\mathbb{C}$. Denote $[\cdot, \cdot]$ be the hyperbolic distance in $D^2$, i.e. the distance induced by the Poincaré metric on $D^2$. When one composes two diffeomorphisms $f, g: \mathbb{C} \to \mathbb{C}$, then

$$\mu_{g \circ f}(p) = \mu_f(p) + \theta_f(p)\mu_g(f(p)),$$

where $\theta_f(p) = \frac{\partial f}{\partial z}(p)$. It follows that

$$\mu_{f^{n+1}}(p) = \frac{\mu_f(p) + \theta_f(p)\mu_f^n(f(p))}{1 + \mu_f(p)\theta_f(p)\mu_f^n(f(p))},$$

We can rewrite (7) as

$$\mu_{g \circ f}(p) = T_{\mu_f(p)}(\theta_f(p)\mu_g(f(p)))$$

where

$$T_a(z) = \frac{a + z}{1 + \overline{a}z} \in \text{Möb}(D^2)$$

is an isometry relative to the Poincaré metric, for a given $a \in D^2$. Further, the following relation holds

$$\log(K_{g \circ f^{-1}}(f(p))) = [\mu_g(p), \mu_f(p)].$$

To define the complex (and maximal) dilatation of a diffeomorphism of a surface $M$, we first lift $f: M \to M$ to the universal cover $\tilde{f}: \widetilde{M} \to M$ and denote $\pi: \widetilde{M} \to M$ be the corresponding canonical projection mapping, where $M = \widetilde{M}/\Gamma$, with $\Gamma$ a Fuchsian group. We assume here that $\widetilde{M}$ is either $\mathbb{C}$ or $D^2$, the trivial case of the sphere $\mathbb{P}^1$ is excluded here. As $\pi$ is an analytic local diffeomorphism, $\tilde{f}$ is a diffeomorphism. Further, as $M$ is compact, $f$ is $K$-quasiconformal on $M$ for some $K \geq 1$ and thus $\tilde{f}$ is $K$-quasiconformal on $\widetilde{M}$. Since $\tilde{f} \circ h \circ \tilde{f}^{-1}$ is conformal for every $h \in \Gamma$, it follows from (7) that

$$\mu_{\tilde{f}}(p) = \mu_{\tilde{f}}(h(p)\frac{h}{h_z}(p)).$$

In other words, $\mu_{\tilde{f}}$ defines a Beltrami differential on $\widetilde{M}$ for the group $\Gamma$, or equivalently, it defines a Beltrami differential for $f$ on the surface $M$. Furthermore, the same formulas (5) and (6),
defined relative to the canonical (conformal) metric defined on \( M \), hold for the dilatation \( K_f \) of \( f \) on \( M \).

The following lemma shows that the bounded geometry assumption of the domains has a strong effect on the dilatation of iterates of \( f \) on \( S \). We say \( f \) has uniformly bounded dilatation on \( S \subset M \), if \( K_{f^n}(p) \) is bounded by a constant independent of \( n \in \mathbb{Z} \) and \( p \in S \).

**Lemma 2.3** (Bounded dilatation). Let \( f \in \text{Diff}^1(M) \) permute a dense collection of domains \( \mathcal{D} \). If the collection \( \mathcal{D} \) has bounded geometry, then \( f \) has uniformly bounded dilatation on \( S \).

**Proof.** Suppose the collection of domains \( \mathcal{D} = \{ D_k \}_{k \in \mathbb{Z}} \) has \( \beta \)-bounded geometry for some \( \beta \geq 1 \). Fix \( N \in \mathbb{Z} \) and \( p \in S \) and take a small open neighbourhood \( U \subset M \) containing \( p \). By Lemma 2.2, there exists a subsequence of domains \( D_{k_t} \), where \( |k_t| \to \infty \) and \( \text{diam}(D_{k_t}) \to 0 \) for \( t \to \infty \) and such that \( D_{k_t} \to p \). Denote \( q = f^N(p) \in S \). We may as well assume that for all \( t \geq 1 \) the domains \( D_{k_t} \) are contained in \( U \). Define \( D'_{k_t} := f^N(D_{k_t}) \). If we denote \( U' = f^N(U) \), then the sequence \( D'_{k_t} \) converges to \( q \) and \( D'_{k_t} \subset U' \). By the bounded geometry assumption, for every \( t \geq 1 \), there exists \( p_t \in D_{k_t} \) and \( 0 < r_t \leq R_t \) such that

\[
B(p_t, r_t) \subseteq D_{k_t} \subseteq B(p_t, R_t)
\]

with \( R_t/r_t \leq \beta \). As \( f \in \text{Diff}^1(M) \), the local behaviour of \( f^N \) around \( q \) converges to the behaviour of the linear map \( Df^N_q \). In particular, if we take \( p_t \in D_{k_t} \), then \( p_t \to p \) and thus \( q_t := f^N(q_t) \to q \), and in order for all \( D'_{k_t} \) to have \( \beta \)-bounded geometry, it is required that

\[
K_{f^N}(p) \leq \frac{R \beta}{r}.
\]

Indeed, this is easily seen to hold if the map acts locally by a linear map and is thus sufficient as \( f \in \text{Diff}^1(M) \) and the increasingly smaller domains approach \( q \). As \( R/r \leq \beta \), we must therefore have \( K_{f^N}(p) \leq \beta^2 \). As this argument holds for every (fixed) \( N \in \mathbb{Z} \) and every \( p \in S \), we find \( \beta^2 \) the uniform bound on the dilatation on \( S \).

Our smoothness assumptions on \( f \) allow us to give bounds on the (complex) dilatation of iterates of \( f \) on \( M \) in terms of the diameters of the permuted domains.

**Lemma 2.4** (Sum of diameters). Let \( f \in \text{Diff}^{1+\alpha}(M) \), with \( \alpha > 0 \), which permutes a collection of domains \( \mathcal{D} = \{ D_k \}_{k \in \mathbb{Z}} \) with \( \beta \)-bounded geometry. Then there exists a constant \( C = C(\beta) > 0 \) such that, if \( p \in D_t \) (for some \( t \in \mathbb{Z} \)) and \( q \in \partial D_t \), then

\[
\left[ \mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q) \right] \leq C \sum_{s=t}^{t+n} \ell_s^{\alpha}, \quad (13)
\]

where the domains are labeled such that \( f^s(D_t) = D_{t+s} \).
To prove Lemma 2.4 we use the following.

**Lemma 2.5.** Let \( f \in \text{Diff}^1(M) \) and \( p_0, q_0 \in M \). Then

\[
[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)] \leq \sum_{s=0}^{n} \left[ T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1})) \right], \tag{14}
\]

where \( p_s = f^s(p_0) \) and \( q_s = f^s(q_0) \).

**Proof.** Using (9), we write

\[
[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)] = [T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^{n}}(p_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^{n}}(q_1))].
\]

By the triangle inequality, we thus have the following inequality

\[
[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)] \leq [T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^{n}}(p_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^{n}}(q_1))]
\]

\[
+ [T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^{n}}(q_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^{n}}(q_1))].
\]

As both \( T_a \) (as defined by (10)) and rotations are isometries in the Poincaré disk, we have that

\[
[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^{n}}(p_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^{n}}(q_1))] = [\mu_{f^n}(p_1), \mu_{f^n}(q_1)].
\]

Inequality (14) now follows by induction. \( \square \)

As \( \partial D_1 \subset S \), by Lemma 2.3 \( \mu_{f^{n-s}}(q_{s+1}) \in B_\delta \), with \( B_\delta \subset \mathbb{D}^2 \) the compact hyperbolic disk centered at 0 \( \in \mathbb{D}^2 \) with radius

\[
\delta = \frac{\beta^2 - 1}{\beta^2 + 1}. \tag{15}
\]

Further, define

\[
\delta' = \sup_{p \in M} |\mu_f(p)| < 1, \tag{16}
\]

and let \( B_{\delta'} \subset \mathbb{D}^2 \) be the compact hyperbolic disk centered at 0 \( \in \mathbb{D}^2 \) and radius \( \delta' \).

**Lemma 2.6.** There exists a constant \( C_1(\delta, \delta') \) such that

\[
[T_a(z), T_b(z)] \leq C_1 [a, b], \tag{17}
\]

for given \( a, b \in B_{\delta'} \) and \( z \in B_\delta \).
Proof. First we observe that there exists a constant $0 < \delta'' < 1$ (depending only on $\delta$ and $\delta'$), such that $[T_a(z), 0] \leq \delta''$, for every $a \in B_{\delta'}$ and every $z \in B_{\delta}$, as the disks $B_{\delta}, B_{\delta'} \subset \mathbb{D}^2$ are compact. Define $\delta = \max\{\delta, \delta', \delta''\}$ and $B_{\delta} \subset \mathbb{D}^2$ the compact disk with center $0 \in \mathbb{D}^2$ and radius $\delta$.

As the Euclidean metric and the hyperbolic metric are equivalent on the compact disk $B_{\delta}$, it suffices to show that there exists a constant $C'_1(\delta)$ such that

$$|T_a(z) - T_b(z)| \leq C'_1|a - b|,$$  

(18)

where $|z - w|$ denotes the Euclidean distance between two points $z, w \in \mathbb{D}^2$. Indeed, if this is shown then (17) follows for a constant $C_1$ which differs from $C'_1$ by a uniform constant depending only on $\delta$. To prove (18), we compute that

$$|T_a(z) - T_b(z)| = \left| \frac{(a - b) + (\bar{a} \bar{b} - \bar{a} \bar{b})z + (\bar{b} - \bar{a})z^2}{(1 + \bar{a}z)(1 + b\bar{z})} \right|. \quad (19)$$

As $a, b \in B_{\delta'}$ and $z \in B_{\delta}$, there exists a constant $Q_1(\delta, \delta') > 0$ so that

$$|(1 + \bar{a}z)(1 + b\bar{z})| \geq Q_1.$$  

Therefore, it holds that

$$|T_a(z) - T_b(z)| \leq Q_1 \left( |a - b| + \delta'|a\bar{b} - \bar{a}b| + (\delta')^2|a - b| \right). \quad (20)$$

In order to prove (18), we show there exists a constant $Q_2(\delta') > 0$ such that

$$|a\bar{b} - \bar{a}b| \leq Q_2|a - b|. \quad (21)$$

To this end, write $a = re^{i\phi}$ and $b = r'e^{i\phi'}$ and $x = ab$, so that $x = rr'e^{i\nu}$ with $\nu = \phi - \phi'$. We may assume that $\nu \in [0, \pi)$. It follows that $a\bar{b} - \bar{a}b = x - \bar{x} = 2irr'e^{i\nu}$. Therefore,

$$|a\bar{b} - \bar{a}b| = |x - \bar{x}| = 2rr' \sin(\nu) \leq 2\delta' r \sin(\nu), \quad (22)$$

as $r' \leq \delta'$. As the angle between the vectors $a, b \in B_{\delta'}$ is $\nu$, it is easily seen that $|a - b| \geq r \sin(\nu)$. Combining this estimate with (22), we obtain that

$$|a\bar{b} - \bar{a}b| \leq 2\delta' r \sin(\nu) \leq 2\delta'|a - b|. \quad (23)$$

Setting $Q_2 = 2\delta'$ yields (21). If we now combine (23) in turn with (20), we obtain a uniform constant

$$C'_1(\delta, \delta') = Q_1(1 + \delta' Q_2 + (\delta')^2) = Q_1(1 + 3(\delta')^2)$$

for which (18) holds, as required. \qed
Proof of Lemma 2.4. As \( f \in \text{Diff}^{1+\alpha}(M) \), we have that \( \mu_f(p) \in C^\alpha(M, \mathbb{D}^2) \) and \( \theta_f \in C^\alpha(M, \mathbb{C}) \), are uniformly Hölder continuous by compactness of \( M \). By the triangle inequality, we can estimate the summand in the right-hand side of (14) of Lemma 2.5 as

\[
\left[ T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^n-s}(q_{s+1})) \right] \leq \]

(24)

\[
\left[ T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})) \right] + \]

(25)

\[
\left[ T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^n-s}(q_{s+1})) \right]. \]

(26)

To estimate (25), define

\[
z_s := \theta_f(p_s)\mu_{f^n-s}(q_{s+1}) \in B_3 \quad \text{and} \quad a_s = \mu_f(p_s), b_s = \mu_f(q_s) \in B_{3r} \subset \mathbb{D}^2.
\]

Then (25) reads

\[
\left[ T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})) \right] = [T_{a_s}(z_s), T_{b_s}(z_s)]. \tag{27}
\]

By Lemma 2.6 there exists a constant \( C_1 > 0 \) such that

\[
[T_{a_s}(z_s), T_{b_s}(z_s)] \leq C_1[a_s, b_s]. \tag{28}
\]

By Hölder continuity of \( \mu_f \), there exists a constant \( \tilde{C}_1 \) such that

\[
a_s, b_s \leq \tilde{C}_1(d(p_s, q_s))^{\alpha}. \tag{29}
\]

Therefore, combining equations (27), (28) and (29), we obtain that

\[
\left[ T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^n-s}(q_{s+1})) \right] \leq \tilde{C}_1\ell_{t+s}^\alpha, \tag{30}
\]

as \( d(p_s, q_s) \leq \ell_{t+s} \), with \( \tilde{C}_1 := C_1\tilde{C}_1 \).

To estimate (26), we note that the hyperbolic distance and the Euclidean distance are equivalent on the compact disk \( B_3 \). Therefore, as the (Euclidean) distance between a point \( z \in B_3 \) and \( e^{i\phi}z \) is bounded from above by a constant (depending only on \( \delta \)) multiplied by the angle \( |\phi| \), by Hölder continuity of \( \theta_f \) there exists a constant \( \tilde{C}_2(\delta) \), such that

\[
[\theta_f(p)z, \theta_f(p')z] \leq \tilde{C}_2(d(p, p'))^{\alpha},
\]

for all \( z \in B_3 \) and \( p, p' \in M \), using the local equivalence of the hyperbolic and Euclidean metric. Hence, (26) reduces to

\[
[\theta_f(p_s)\mu_{f^n-s}(q_{s+1}), \theta_f(q_s)\mu_{f^n-s}(q_{s+1})] \leq \tilde{C}_2d(p_s, q_s)^\alpha \leq \tilde{C}_2\ell_{t+s}^\alpha, \tag{31}
\]

as \( d(p_s, q_s) \leq \ell_{t+s} \). Therefore, if we set \( C := \tilde{C}_1 + \tilde{C}_2 \), then (13) follows. \( \square \)
2.2 Upper bounds on the entropy of a surface diffeomorphism

Next, we relate the topological entropy of a diffeomorphism to its dilatation.

**Lemma 2.7 (Entropy and dilatation).** Let \( f \in \text{Diff}^{1+\alpha}(M) \) with \( \alpha > 0 \). Then

\[
h_{\text{top}}(f) \leq \lim_{n \to \infty} \sup \frac{1}{2n} \log \int_M K_f^n(p) d\lambda(p),
\]

with \( K_f \) the dilatation of \( f \).

To prove this we use a result of F. Przytycki [8]. We need the following notation. Let \( L : \mathbb{R}^m \to \mathbb{R}^m \) be a linear map and \( L^\wedge : \mathbb{R}^{m \wedge k} \to \mathbb{R}^{m \wedge k} \) the induced map on the \( k \)-th exterior algebra of \( \mathbb{R}^m \). \( L^\wedge \) denotes the induced map on the full exterior algebra. The norm \( \|L^\wedge\| \) of \( L^k \) has the following geometrical meaning. Let \( \text{Vol}_k(v_1, \ldots, v_k) \) be the \( k \)-dimensional volume of a parallelepiped spanned by the vectors \( v_1, \ldots, v_k \), where \( v_i \in \mathbb{R}^m \) with \( 1 \leq i \leq k \). Then

\[
\|L^\wedge\| = \sup_{v_i \in \mathbb{R}^m} \frac{\text{Vol}_k(L(v_1), \ldots, L(v_k))}{\text{Vol}_k(v_1, \ldots, v_k)},
\]

\[
\|L\wedge\| = \max_{1 \leq k \leq m} \|L^\wedge\|.
\]

Further, let

\[
\|L\| = \sup_{|v|=1} |L(v)|,
\]

the standard norm on operators, with \( v \in \mathbb{R}^m \) and \( |\cdot| \) induced by the corresponding inner product on \( \mathbb{R}^m \). The following result is due to F. Przytycki [8] (see also [3]).

**Theorem 2.8.** Given a smooth, closed Riemannian manifold \( M \) and \( f \in \text{Diff}^{1+\alpha}(M) \) with \( \alpha > 0 \). Then

\[
h_{\text{top}}(f) \leq \lim_{n \to \infty} \sup \frac{1}{n} \log \int_M \|(Df^n)^\wedge\| d\lambda(p).
\]

where \( h_{\text{top}}(f) \) is the topological entropy of \( f \), \( \lambda \) is a Riemannian measure on \( M \) induced by a given Riemannian metric, \( (Df^n)^\wedge \) is a mapping between exterior algebras of the tangent spaces \( T_pM \) and \( T_{f^{-n}(p)}M \), induced by the \( Df^n_p \) and \( \|\cdot\| \) is the norm on operators, induced from the Riemannian metric.

**Proof of Lemma 2.7.** Fix \( p \in M \) and let \( Df^n_p : T_pM \to T_{f^{-n}(p)}M \). Then

\[
\|Df^n_p\|^2 = K_{f^n}(p)J_{f^n}(p).
\]

Thus

\[
\|(Df^n_p)^\wedge\| = \sqrt{K_{f^n}(p)J_{f^n}(p)}, \text{ and } \|(Df^n_p)^{2\wedge}\| = J_{f^n}(p).
\]
It follows that
\[ \| (Df^n)^\wedge \| = \max \left\{ \sqrt{K_{f^n}(p)J_{f^n}(p)}, J_{f^n}(p) \right\}. \tag{38} \]
As
\[ \max \left\{ \sqrt{K_{f^n}(p)J_{f^n}(p)}, J_{f^n}(p) \right\} \leq \sqrt{K_{f^n}(p)J_{f^n}(p) + J_{f^n}(p)}, \]
we have that
\[ \int_M \| (Df^n)^\wedge \| d\lambda(p) \leq \int_M \left( \sqrt{K_{f^n}J_{f^n} + J_{f^n}} \right) d\lambda \]
\[ = \lambda(M) + \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \]
as \( \lambda(M) = \int_M J_{f^n} d\lambda \), for every \( n \in \mathbb{Z} \). Either \( \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \) is bounded as a sequence in \( n \), in which case (32) holds trivially, or the sequence is unbounded in \( n \), in which case it is readily verified that
\[ \lim_{n \to \infty} \sup_{1 \leq n} \frac{1}{n} \log \left( \lambda(M) + \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \right) = \lim_{n \to \infty} \sup_{1 \leq n} \frac{1}{n} \log \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda. \]
By the Cauchy-Schwartz inequality, we have that
\[ \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \leq \sqrt{\lambda(M)} \cdot \sqrt{\int_M K_{f^n} d\lambda}. \]
and thus,
\[ \log \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \leq \frac{1}{2} \log \lambda(M) + \frac{1}{2} \log \int_M K_{f^n} d\lambda. \]
It now follows that
\[ \lim_{n \to \infty} \sup_{1 \leq n} \frac{1}{n} \log \int_M \| (Df^n)^\wedge \| d\lambda \leq \lim_{n \to \infty} \sup_{1 \leq n} \frac{1}{2n} \log \int_M K_{f^n} d\lambda. \]
and this proves (32).

2.3 Proof of Theorem A

Let us now complete the proof. Let \( f \in \text{Diff}^{1+\alpha}(M) \), with \( \alpha > 0 \), and suppose that \( f \) permutes a dense collection of domains \( \{D_k\}_{k \in \mathbb{Z}} \) with bounded geometry. By Lemma 2.1, the sequence \( \ell_k \) is a null-sequence. Therefore, \( \ell_k^\alpha \) is a null-sequence as well, for every \( \alpha > 0 \). Let \( p \in D_t \) for some \( t \in \mathbb{Z} \) and \( q \in \partial D_t \) and label the domains such that \( f^t(D_t) = D_{t+s} \). By (11),
\[ \log K_{f^n}(f(p)) = [\mu_{f^n+1}(p), \mu_f(p)] \]
and thus, by the triangle inequality,

$$\log K_{f^n}(f(p)) \leq [\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)] + [\mu_{f^{n+1}}(q), \mu_f(p)]$$  \hspace{1cm} (39)$$

As the second term in the right hand side of (39) stays uniformly bounded, we have that

$$\log K_{f^n}(f(p)) \leq [\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)] + C'$$  \hspace{1cm} (40)$$

for some constant $C' > 0$, independent of $p \in M$ and $n \in \mathbb{Z}$. Define

$$\xi(n) = \max \sum_{i=0}^{n} \ell^\alpha_{k_i}$$

where the maximum is taken over all collections of $n + 1$ distinct elements $\{D_{k_0}, \ldots, D_{k_n}\}$ of $\mathcal{D}$. As $\ell^\alpha_k$ is a null-sequence, we have that

$$\lim_{n \to \infty} \sup_n \frac{\xi(n)}{n} = 0.$$  \hspace{1cm} (41)$$

By Lemma 2.4, we have that

$$[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)] \leq C \sum_{s=t}^{t+n} \ell^\alpha_s,$$

for some constant $C > 0$. Combined with (40), we obtain the following uniform estimate

$$\log K_{f^n}(f(p)) \leq C \xi(n) + C',$$  \hspace{1cm} (42)$$

for every $p \in M$ and $n \in \mathbb{Z}$. Therefore

$$\log \int_M K_{f^n} d\lambda \leq \log \int_M \exp(C \xi(n) + C') d\lambda$$  \hspace{1cm} (43)$$

$$= \log \left( \exp(C \xi(n) + C') \lambda(M) \right)$$  \hspace{1cm} (44)$$

$$= C \xi(n) + C' + \log(\lambda(M)).$$  \hspace{1cm} (45)$$

Combining (45) in turn with Lemma 2.7 yields

$$h_{top}(f) \leq \lim_{n \to \infty} \sup_n \frac{1}{2n} \log \int_M K_{f^n} d\lambda \leq C \lim_{n \to \infty} \sup_n \frac{\xi(n)}{2n} = 0,$$  \hspace{1cm} (46)$$

by (41). This proves Theorem A.
3 Concluding remarks

The proof of Theorem A, more precisely condition (41) in section 2.3, fails in the case where the Hölder constant $\alpha = 0$. This leads to the following natural

Question 1 (Differentiable counterexamples). Do there exist diffeomorphisms $f \in \text{Diff}^3(M)$ with positive entropy that permute a dense collection of domains with bounded geometry?

References

[1] C. Bonatti, J.M. Gambaudo, J.M. Lion and C. Tresser, Wandering Domains for Infinitely Renormalizable Diffeomorphisms of the Disk, Proceedings of the AMS 122-4, (1994), 1273-1278.

[2] A. Fletcher and V. Markovic, Quasiconformal Maps and Teichmüller Theory, Oxford Graduate Texts in Mathematics 11, Oxford University Press (2007).

[3] O.S. Kozlovski, An integral formula for topological entropy of $C^\infty$ maps, Ergodic Theory & Dynamical Systems 18, (1998), 405-424.

[4] O. Lehto, Univalent functions and Teichmüller spaces, Graduate Texts in Mathematics 109, Springer-Verlag, (1987).

[5] P. McSwiggen, Diffeomorphisms of the Torus with Wandering Domains, Proceedings of the AMS 117-4, (1993), 1175-1186.

[6] A. Navas, Wandering disks for diffeomorphisms of the $k$-torus: a remark on a theorem by Norton and Sullivan, preprint, (2007).

[7] A. Norton and D. Sullivan, Wandering domains and invariant conformal structures for mappings of the 2-torus, Ann. Acad. Sci. Fenn. Math. 21, (1996), 51-68.

[8] F. Przytycki, An Upper Estimation for Topological Entropy of Diffeomorphisms, Inventiones Mathematicae 59, (1980), 205-213.