Existence of efficient and properly efficient solutions to problems of constrained vector optimization

Do Sang Kim1 · Boris S. Mordukhovich2,3 · Tiến Sơn Phạm4 · Nguyen Van Tuyen5,6

Received: 2 May 2018 / Accepted: 8 June 2020 / Published online: 1 July 2020
© Springer-Verlag GmbH Germany, part of Springer Nature and Mathematical Optimization Society 2020

Abstract
The paper is devoted to the existence of global optimal solutions for a general class of nonsmooth problems of constrained vector optimization without boundedness assumptions on constraint set. The main attention is paid to the two major notions of optimality in vector problems: Pareto efficiency and proper efficiency in the sense of Geoffrion. Employing adequate tools of variational analysis and generalized differentiation, we first establish relationships between the notions of properness, $M$-tameness, and the Palais–Smale conditions formulated for the restriction of the vector cost mapping on the constraint set. These results are instrumental to derive verifiable necessary and sufficient conditions for the existence of Pareto efficient solutions in vector optimization. Furthermore, the developed approach allows us to obtain new sufficient conditions for the existence of Geoffrion-properly efficient solutions to such constrained vector problems.

Keywords Existence theorems · Pareto efficient solutions · Geoffrion-properly efficient solutions · $M$-tameness · Palais–Smale conditions · Properness

Mathematics Subject Classification 90C29 · 90C30 · 90C31 · 49J30

The first author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2019R1A2C1008672). Research of second author was supported by the US National Science Foundation under Grants DMS-1512846 and DMS-1808978, by the Air Force Office of Scientific Research under Grant #15RT0462, and by the Australian Research Council under Discovery Project DP-190100555. The third author is partially supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED), Grant 101.04-2019.302. The fourth author was supported by Hanoi Pedagogical University 2 (HPU2) under Grant No. C.2019-18-05.

Extended author information available on the last page of the article
1 Introduction

This paper concerns with some fundamental issues of global vector optimization that are revolved around the existence of efficient and properly efficient solutions under unbounded constraints. Such issues have been addressed in many publications; see, e.g., the books [22,28,34] and the papers [2,3,6–9,13–17,19,21,26,27,29,30] with the references therein. We offer here a new approach to these topics that allows us to derive significantly new existence theorems for a general class of problems in vector optimization. This approach is mainly based on advanced tools of variational analysis and generalized differentiation that provide essential improvements of known results even in the case of problems with smooth data.

The basic problem under consideration is formulated as follows:

\[
\min_{\mathbb{R}^m_+} \left\{ f(x) \mid x \in \Omega \right\},
\]

where \( f: \mathbb{R}^n \to \mathbb{R}^m \) is a locally Lipschitz mapping, \( \Omega \subset \mathbb{R}^n \) is a nonempty and closed (not necessarily bounded) set, \( \mathbb{R}^m_+ \) is the nonnegative orthant of \( \mathbb{R}^m \), i.e.,

\[
\mathbb{R}^m_+ := \left\{ y := (y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_i \geq 0, \ i = 1, \ldots, m \right\},
\]

and where “minimization” is understood in conventional terms of vector optimization that are specified below. Recall that the mapping \( f \) is called locally Lipschitz if for all \( x \in \mathbb{R}^n \) there is a neighborhood \( U \) of \( x \) and a real number \( L > 0 \) such that

\[
\|f(u) - f(v)\| \leq L\|u - v\| \text{ for all } u, v \in U.
\]

In the case of unconstrained problems (VP) with \( \Omega = \mathbb{R}^n \), existence theorems for weak Pareto/weak efficient and the so-called relative Pareto (while not Pareto efficient) solutions were obtained in [2,3,32] by using appropriate set-valued extensions of the Ekeland variational principle under the following major assumptions:

- \( f \) is bounded from below, i.e., there exists a bounded subset \( B \subset \mathbb{R}^m \) such that \( f(\mathbb{R}^n) \subset B + \mathbb{R}^m_+ \).
- \( f \) satisfies a certain Palais–Smale condition.

Somewhat related results for weak Pareto minimizers were obtained in [16] under more restrictive assumptions. As discussed in [25], such assumptions are rather limited. To improve them, powerful methods of semialgebraic geometry and polynomial optimization were invoked in [25]. In this way the equivalence between the following conditions was proved therein when \( \Omega = \mathbb{R}^n \) and \( f \) is polynomial in (VP); see below for the exact definitions:

- \( f \) is proper at the sublevel \( \bar{y} \).
- \( f \) satisfies the Palais–Smale condition at the sublevel \( \bar{y} \).
- \( f \) satisfies the weak Palais–Smale condition at the sublevel \( \bar{y} \).
• $f$ is $M$-tame at the sublevel $\bar{y}$.

As consequences of these results, some sufficient conditions for the existence of Pareto efficient solutions of the unconstrained polynomial problem (VP) were given in [25].

The main contributions of this paper are significantly different from [25]. First of all, we study the constrained problem (VP) with an arbitrary closed constraint set $\Omega$ and without any polynomial requirement on $f$, which is now replaced by local Lipschitz continuity. To proceed, we do not use methods of semialgebraic geometry but employ instead advanced tools of variational analysis and generalized differentiation. To the best of our knowledge, this is a novel approach to the existence theory in vector optimization, which is of its own interest and certainly of further perspectives while not being limited just to the results presented in this paper. Here it allows us to establish existence theorems for Pareto efficient and Geoffrion-properly efficient solutions to nonconvex (and nonsmooth) vector optimization problems, which have been largely underinvestigated in the literature in contrast to their weak counterparts. In particular, in this way we obtain, for the first time in literature, necessary and sufficient conditions for the existence of Pareto efficient solutions that seem to be new even for scalar optimization problems.

Besides the developed variational approach, the major results of this paper are as follows:

(a) Assuming that the image set $f(\Omega)$ has a bounded section at some $\bar{y} \in f(\Omega)$, which is indeed necessary for the existence of Pareto efficient solutions to (VP), we show that the following statements are equivalent:

– the restriction $f|_\Omega$ of $f$ on $\Omega$ is proper at the sublevel $\bar{y}$.
– the restriction $f|_\Omega$ satisfies the Palais–Smale condition at the sublevel $\bar{y}$.
– the restriction $f|_\Omega$ satisfies the weak Palais–Smale condition at the sublevel $\bar{y}$.
– the restriction $f|_\Omega$ is $M$-tame at the sublevel $\bar{y}$.

Here and in the following, by the sublevel of $f|_\Omega$ at $\bar{y}$ we mean the set $\{x \in \Omega \mid f(x) \leq \bar{y}\}$.

(b) Based on these results, we obtain, for the first time in the literature, necessary and sufficient conditions for the existence of Pareto efficient solutions to the problem (VP). As a byproduct of our approach, new sufficient conditions for the existence of Geoffrion-properly efficient solutions to (VP) are also derived.

The rest of the paper is organized as follows. In Sect. 2 we recall some definitions and preliminary results from variational analysis and generalized differentiation. Section 3 is devoted to establish relationships between properness, Palais–Smale conditions, and $M$-tameness. In Sect. 4 we prove the existence of Pareto efficient and Geoffrion-properly efficient solutions to the vector optimization problem (VP). The concluding Section 5 contains discussions of open problems to address in our future research.

2 Preliminaries

Our notation and terminology are standard in variational analysis and vector optimization; see, e.g., the books [22,28,31,33]. Recall that for any number $n \in \mathbb{N} := \{1, 2, \ldots\}$
we denote \( x := (x_1, \ldots, x_n) \) and equip the space \( \mathbb{R}^n \) with the usual scalar product \( \langle \cdot, \cdot \rangle \) and the Euclidean norm \( \| \cdot \| \). The closed unit ball in \( \mathbb{R}^n \) is denoted by \( B^n \).

### 2.1 Definitions of optimal solutions

Let \( a, b \in \mathbb{R}^m \). Then \( a \leq b \) means \( b - a \in \mathbb{R}^m_+ \), \( a \leq b \) means \( b - a \in \mathbb{R}^m_+ \setminus \{0\} \), and \( a < b \) means \( b - a \in \text{int} \mathbb{R}^m_+ \), where \( \text{int} \mathbb{R}^m_+ := \{ y := (y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_i > 0, \; i = 1, \ldots, m \} \).

**Definition 2.1** Consider a map \( f : \mathbb{R}^n \to \mathbb{R}^m \), a set \( \Omega \subset \mathbb{R}^n \) and a point \( \bar{x} \in \Omega \). We say that:

(i) \( \bar{x} \) is a **Pareto efficient solution** to (VP) if there is no \( x \in \Omega \) such that

\[
    f(x) \leq f(\bar{x}) \quad \text{and} \quad f(x) \neq f(\bar{x}).
\]

(ii) \( \bar{x} \) is a **Geoffrion-properly efficient solution** to (VP) if it is a Pareto efficient solution and there is a real number \( C > 0 \) such that whenever \( i \in \{1, \ldots, m\} \) and \( x \in \Omega \) satisfying \( f_i(x) < f_i(\bar{x}) \) there exists an index \( j \in \{1, \ldots, m\} \) with \( f_j(\bar{x}) < f_j(x) \) and

\[
    \frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq C.
\]

**Remark 2.1** (i) By definition, every Geoffrion-properly efficient solution is a Pareto efficient solution to (VP) but not vice versa; see e.g., Example 4.1 below.

(ii) It has been realized in vector optimization (see, e.g., [22]) that in the setting under consideration the concept of Geoffrion-properly efficient solutions agrees with the notions of properly efficient solutions in the senses of Benson [4] and Henig [20], and that every Geoffrion-properly efficient solution is also properly efficient in the sense of Borwein [5].

### 2.2 Normals and subdifferentials

Here we recall the notions of the normal cones to closed sets and the subdifferential of real-valued functions used in this paper. The reader is referred to [31,33] for more details.

**Definition 2.2** Consider a set \( \Omega \subset \mathbb{R}^n \) and a point \( \bar{x} \in \Omega \).

(i) The **regular normal cone** (known also as the prenormal or Fréchet normal cone) \( \hat{N}(\bar{x}; \Omega) \) to \( \Omega \) at \( \bar{x} \) consists of all vectors \( v \in \mathbb{R}^n \) satisfying

\[
    \langle v, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|), \quad x \in \Omega,
\]

where the term \( o(\|x - \bar{x}\|) \) satisfies \( o(\|x - \bar{x}\|)/\|x - \bar{x}\| \to 0 \) whenever \( x \to \bar{x}, \; x \in \Omega \setminus \{\bar{x}\} \).

\[\square\] Springer
(ii) The limiting normal cone (known also as the basic or Mordukhovich normal cone) $N(\bar{x}; \Omega)$ to $\Omega$ at $\bar{x}$ consists of all vectors $v \in \mathbb{R}^n$ such that there are sequences $x^k \to \bar{x}$ with $x^k \in \Omega$ and $u^k \to v$ with $u^k \in \tilde{N}(x^k; \Omega)$ as $k \to \infty$.

**Definition 2.3** Consider a function $\phi : \mathbb{R}^n \to \mathbb{R}$ and a point $\bar{x} \in \mathbb{R}^n$. The (limiting) subdifferential of $\phi$ at $\bar{x}$ is defined by

$$\partial \phi(\bar{x}) := \{ v \in \mathbb{R}^n \mid (v, -1) \in N((\bar{x}, \phi(\bar{x})); \text{epi} \phi) \}$$

via the limiting normal cone to the epigraph epi $\phi$ of $\phi$ given by

$$\text{epi} \phi := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R} \mid \phi(x) \leq y \}.$$ 

In [31–33] the reader can find equivalent analytic descriptions of the subdifferential $\partial \phi(\bar{x})$ and comprehensive studies of it and related constructions. In the case of convex sets and functions the above normal cone and subdifferential notions reduce to the corresponding concepts of convex analysis. Furthermore, we have $\partial \phi(\bar{x}) = \{ \nabla \phi(\bar{x}) \}$ if $\phi$ is strictly differentiable at $\bar{x}$; in particular, when it is smooth around this point.

Next we present several known statements, which play significant roles in the proofs of the main results. The first lemma is a classical subdifferential formula of convex analysis.

**Lemma 2.1** For each $\bar{x} \in \mathbb{R}^n$ we have

$$\partial \left( \| \cdot - \bar{x} \| \right)(x) = \begin{cases} \frac{x - \bar{x}}{\| x - \bar{x} \|} & \text{if } x \neq \bar{x}, \\ B_n & \text{otherwise}. \end{cases}$$

The following major results of subdifferential calculus and necessary optimality conditions for scalar nonsmooth optimization are used below in our derivation of the existence theorems for (VP) even in the case of problems with smooth initial data.

**Lemma 2.2** (see [31, Theorem 3.36]) Let $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be locally Lipschitz functions around $\bar{x} \in \mathbb{R}^n$. Then we have the subdifferential sum rule

$$\partial(\phi_1 + \cdots + \phi_m)(\bar{x}) \subset \partial \phi_1(\bar{x}) + \cdots + \partial \phi_m(\bar{x}).$$

**Lemma 2.3** (see [31, Theorem 3.46]) Let $\phi_1, \ldots, \phi_m : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions around $\bar{x} \in \mathbb{R}^n$. Then the maximum function

$$\phi(x) := \max_{1 \leq i \leq m} \phi_i(x), \quad x \in \mathbb{R}^n,$$

is locally Lipschitz around $\bar{x}$, and we have the inclusion

$$\partial \phi(\bar{x}) \subset \left\{ \sum_{i \in I(\bar{x})} \alpha_i \partial \phi_i(\bar{x}) \mid \alpha_i \geq 0 \quad \text{and} \quad \sum_{i \in I(\bar{x})} \alpha_i = 1 \right\},$$
where the active index set is defined by $I(\bar{x}) := \{ i \mid \phi_i(\bar{x}) = \phi(\bar{x}) \}$.

**Lemma 2.4** (see [32, Corollary 6.6]) Let $\phi_i : \mathbb{R}^n \to \mathbb{R}, \ i = 0, \ldots, m + r$, be locally Lipschitz functions around $\bar{x} \in \Omega$, and let the set $\Omega \subset \mathbb{R}^n$ be locally closed around this point. If $\bar{x}$ is a local minimizer of the function $\phi_0$ on the set

$$\{ x \in \Omega \mid \phi_i(x) \leq 0, \ i = 1, \ldots, m, \ \phi_i(x) = 0, \ i = m + 1, \ldots, m + r \},$$

then there exists a nonzero collection of multipliers $(\lambda_0, \ldots, \lambda_{m+r}) \in \mathbb{R}^{m+r+1}$ satisfying the sign conditions $\lambda_i \geq 0$ for $i = 0, \ldots, m$ and

$$0 \in \sum_{i=0}^{m} \lambda_i \partial \phi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i \partial^0 \phi_i(\bar{x}) + N(\bar{x}; \Omega),$$

$$\lambda_i \phi_i(\bar{x}) = 0, \ i = 1, \ldots, m,$$

where $\partial^0 \phi_i(\bar{x})$ is the symmetric subdifferential of $\phi_i$ at $\bar{x}$ and defined by

$$\partial^0 \phi_i(\bar{x}) := \partial \phi_i(\bar{x}) \cup [-\partial(-\phi_i)(\bar{x})], \ i = m + 1, \ldots, m + r.$$

**Remark 2.2** If the function $\phi_i$ is strictly differentiable at $\bar{x}$, then

$$\partial^0 \phi_i(\bar{x}) = \partial \phi_i(\bar{x}) = \{ \nabla \phi_i(\bar{x}) \}.$$

### 3 Properness, Palais–Smale conditions, and $M$-Tameness

Let $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz mapping, and let $\Omega \subset \mathbb{R}^n$ be a nonempty and closed set. In this section we establish close relationships between the properness, Palais–Smale conditions, and $M$-tameness for the restriction $f|_\Omega$ of $f$ on $\Omega$. To proceed, the following notions are needed.

**Definition 3.1** Let $A$ be a subset of $\mathbb{R}^m$, and let $y \in \mathbb{R}^m$. The set $A \cap (y - \mathbb{R}_+^m)$ is called a section of $A$ at $y$ and is denoted by $[A]_y$. The section $[A]_y$ is said to be **bounded** if there exists a vector $a \in \mathbb{R}^m$ such that

$$[A]_y \subset a + \mathbb{R}_+^m.$$  

It is easy to observe that for a Pareto efficient solution $\bar{x}$ to (VP) we have

$$[f(\Omega)]_{f(\bar{x})} = \{ f(\bar{x}) \}.$$

Thus the condition that $f(\Omega)$ admits at least one nonempty compact section is necessary for the existence of Pareto efficient solutions to (VP). In fact, this necessary condition is also sufficient for the existence of Pareto efficient solutions to (VP); see, e.g., [6, Theorem 1], or, equivalently, [28, Corollary 3.8, p. 48]. In the finite-dimensional setting, the compactness of a set is equivalent to the set being closed and
bounded. Thus, a necessary and sufficient condition for the existence to (VP) is that the set \( f(\Omega) \) admits at least one bounded and closed section. In general, it can be difficult to verify the closedness of a section of \( f(\Omega) \).

Next we introduce the notions of \textit{properness} for the restricted cost mapping, which are instrumental to prove the closedness of a given bounded section of \( f(\Omega) \).

\begin{definition}
We say that:

(i) The restriction \( f|_\Omega \) of \( f \) on \( \Omega \) is \textit{proper at sublevel} \( y \in \mathbb{R}^m \) if

\[
\forall \{x^k\} \subset \Omega, \|x^k\| \to \infty, \quad f(x^k) \leq y \implies \|f(x^k)\| \to \infty \quad \text{as} \quad k \to \infty.
\]

(ii) The restriction \( f|_\Omega \) is \textit{proper} if it is proper at every sublevel \( y \in \mathbb{R}^m \).
\end{definition}

\begin{remark}
The notions of properness were introduced in \cite[Definition 3.4]{25} for the case of \( \Omega = \mathbb{R}^n \). When \( m = 1 \) and \( f \) is bounded from below, it is easy to see that the properness of \( f|_\Omega \) is equivalent to the \textit{coercivity} of \( f|_\Omega \), i.e.,

\[
\lim_{x \in \Omega, \|x\| \to \infty} f(x) = +\infty.
\]

In case of \( m \geq 2 \), the properness of \( f|_\Omega \) is weaker than other well-known coercivity conditions, for example, the \textit{\( \mathbb{R}^m \)-zero-coercivity} of \( f \) on \( \Omega \) introduced in \cite[Definition 3.1]{15}. To see this, recall that the mapping \( f \) is said to be \( \mathbb{R}^m_+ \)-zero-coercive on \( \Omega \) with respect to \( \alpha \in \mathbb{R}^m_+ \setminus \{0\} \) if

\[
\lim_{x \in \Omega, \|x\| \to \infty} \langle \alpha, f(x) \rangle = +\infty.
\]

By definition, it is easy to check that if \( f \) satisfies the \( \mathbb{R}^m_+ \)-zero-coercivity on \( \Omega \) with respect to some \( \alpha \in \mathbb{R}^m_+ \setminus \{0\} \), then \( f|_\Omega \) is proper. However, the converse fails to hold in general. For example, let \( f : \mathbb{R} \to \mathbb{R}^2 \) be the polynomial mapping defined by

\[
f(x) := (x, -x),
\]

and let \( \Omega = \mathbb{R} \). Clearly, \( f \) is proper but not \( \mathbb{R}^2_+ \)-zero-coercive on \( \Omega \) for all \( \alpha \in \mathbb{R}^2_+ \setminus \{0\} \).

For each \( \bar{y} \in (\mathbb{R} \cup \{\infty\})^m \), consider the sets

\[
\tilde{K}_{\infty, \leq \bar{y}}(f, \Omega) := \{ y \in \mathbb{R}^m \mid \exists \{x^k\} \subset \Omega, \quad f(x^k) \leq \bar{y}, \quad \|x^k\| \to \infty, \quad f(x^k) \to y, \quad \text{and} \quad \nu(x^k) \to 0 \text{ as } k \to \infty \},
\]

\[
K_{\infty, \leq \bar{y}}(f, \Omega) := \{ y \in \mathbb{R}^m \mid \exists \{x^k\} \subset \Omega, \quad f(x^k) \leq \bar{y}, \quad \|x^k\| \to \infty, \quad f(x^k) \to y, \quad \text{and} \quad \|x^k\|\nu(x^k) \to 0 \text{ as } k \to \infty \},
\]
where \( \nu : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \) is the (extended) Rabier function defined by
\[
\nu(x) := \inf \left\{ \left\| \sum_{i=1}^{m} \alpha_i v^i + w \right\| \mid v^i \in \partial f_i(x), \ w \in N(x; \Omega), \ \alpha \in \mathbb{R}_+^m, \sum_{i=1}^{m} \alpha_i = 1 \right\}.
\]

From the definitions, it is clear that \( K_{\infty, \leq \bar{y}}(f, \Omega) \subset \tilde{K}_{\infty, \leq \bar{y}}(f, \Omega) \), but the reverse inclusion may not hold (see [25, Remark 3.5(ii)]).

Following [18, Chapter 2], we also consider the set
\[
T_{\infty, \leq \bar{y}}(f, \Omega) := \left\{ y \in \mathbb{R}^m \mid \exists \{x^k\} \subset \Gamma(f, \Omega), \ f(x^k) \leq \bar{y}, \ \|x^k\| \to \infty, \text{ and } f(x^k) \to y \text{ as } k \to \infty \right\},
\]
defined via the tangency variety of \( f \) on \( \Omega \)
\[
\Gamma(f, \Omega) := \left\{ x \in \Omega \mid \exists v^i \in \partial f_i(x) \text{ for } i = 1, \ldots, m, \right. \\
\left. \exists (\alpha, \mu) \in \mathbb{R}_+^m \times \mathbb{R} \text{ with } \sum_{i=1}^{m} \alpha_i + |\mu| = 1 \text{ such that } \right. \\
\left. 0 \in \sum_{i=1}^{m} \alpha_i v^i + \mu x + N(x; \Omega) \right\}.
\]

Geometrically, the tangency variety \( \Gamma(f, \Omega) \) of \( f \) on \( \Omega \) consists of all points \( x \in \Omega \) where the level sets of \( f|_\Omega \) are “tangent to the sphere” in \( \mathbb{R}^n \) centered in the origin with radius \( \|x\| \). We refer the reader to [18] and the bibliography therein for more details on this notion and its applications to polynomial optimization.

If \( \bar{y} = (\infty, \ldots, \infty) \), we simplify the notation by writing \( \tilde{K}_\infty(f, \Omega), \ K_\infty(f, \Omega), \) and \( T_\infty(f, \Omega) \) instead of \( \tilde{K}_{\infty, \leq \bar{y}}(f, \Omega), \ K_{\infty, \leq \bar{y}}(f, \Omega), \) and \( T_{\infty, \leq \bar{y}}(f, \Omega) \), respectively. Clearly,
\[
J_\infty(f, \Omega) = \bigcup_{\bar{y} \in \mathbb{R}^m} J_{\infty, \leq \bar{y}}(f, \Omega), \ J \in \{ \tilde{K}, K, T \}.
\]

**Definition 3.3** Let \( \bar{y} \in (\mathbb{R} \cup \{ \infty \})^m \). We say that:

(i) \( f|_\Omega \) satisfies the **Palais–Smale condition** at the sublevel \( \bar{y} \) if
\[
\tilde{K}_{\infty, \leq \bar{y}}(f, \Omega) = \emptyset.
\]

(ii) \( f|_\Omega \) satisfies the **weak Palais–Smale condition** at the sublevel \( \bar{y} \) if
\[
K_{\infty, \leq \bar{y}}(f, \Omega) = \emptyset.
\]

(iii) \( f|_\Omega \) is \( M \)-**tame** at the sublevel \( \bar{y} \) if \( T_{\infty, \leq \bar{y}}(f, \Omega) = \emptyset \).
When \( \bar{y} = (\infty, \ldots, \infty) \), we say for short that \( f|_{\Omega} \) satisfies the Palais–Smale condition, the weak Palais–Smale condition or \( f|_{\Omega} \) is \( M \)-tame, respectively.

It follows from the definitions that the properness of \( f|_{\Omega} \) at sublevel \( \bar{y} \in \mathbb{R}^m \) yields

\[
T_{\infty, \leq \bar{y}}(f, \Omega) = \tilde{K}_{\infty, \leq \bar{y}}(f, \Omega) = K_{\infty, \leq \bar{y}}(f, \Omega) = \emptyset.
\]

The converse does not hold in general. Indeed, let \( \Omega := \mathbb{R}^2 \), and let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a real-valued function defined by \( f(x_1, x_2) := x_1 + x_2 \). It is easy to check that

\[
T_{\infty, \leq \bar{y}}(f, \Omega) = \tilde{K}_{\infty, \leq \bar{y}}(f, \Omega) = K_{\infty, \leq \bar{y}}(f, \Omega) = \emptyset \quad \text{for all} \quad \bar{y} \in \mathbb{R},
\]

while \( f \) is not proper at every sublevel. Nevertheless, we have the following rather surprising result whose proof is based on variational arguments and subdifferential calculus.

**Theorem 3.1** Assume that there exists \( \bar{y} \in f(\Omega) \) such that the section \( [f(\Omega)]_{\bar{y}} \) is bounded. Then the following statements are equivalent:

(i) \( f|_{\Omega} \) is proper at the sublevel \( \bar{y} \).

(ii) \( f|_{\Omega} \) satisfies the Palais–Smale condition at the sublevel \( \bar{y} \).

(iii) \( f|_{\Omega} \) satisfies the weak Palais–Smale condition at the sublevel \( \bar{y} \).

(iv) \( f|_{\Omega} \) is \( M \)-tame at the sublevel \( \bar{y} \).

Furthermore, the sets \( \{x \in \Omega \mid f(x) \leq \bar{y}\} \) and \( [f(\Omega)]_{\bar{y}} \) are compact provided that one of the above equivalent conditions is satisfied.

**Proof** Note first that implications (i)⇒(ii), (ii)⇒(iii), and (i)⇒(iv) are obvious.

To prove (iii)⇒(i), we argue by contradiction and assume that \( f|_{\Omega} \) is not proper at the sublevel \( \bar{y} \). Consider the nonempty set

\[
X := \{x \in \Omega \mid f_i(x) \leq \bar{y}_i, \ i = 2, \ldots, m\}. \tag{2}
\]

Then it is clear that the set \( X \) is unbounded and the number

\[
c := \liminf_{x \in X, \|x\| \to \infty} f_1(x) \tag{3}
\]

is finite. For each \( R > 0 \) consider the quantity

\[
m(R) := \inf_{x \in X, \|x\| \geq R} f_1(x)
\]

and observe that \( m \) is a nondecreasing function with \( \lim_{R \to \infty} m(R) = c \). Thus for each \( k \in \mathbb{N} \) there exists \( R_k > k \) satisfying

\[
m(R) \geq c - \frac{1}{6k} \quad \text{whenever} \quad R \geq R_k.
\]
On the other hand, by definition, it is easy to check that

$$ m(R) = \inf_{x \in X, \|x\| \geq R} f_1(x) $$

for all $R > 0$. Hence, we can choose $x^k \in X$ with $\|x^k\| > 2R_k$ such that $f(x^k) \leq \bar{y}$ and

$$ f_1(x^k) < m(2R_k) + \frac{1}{6k}. $$

We clearly have the chain of inequalities

$$ m(2R_k) \leq f_1(x^k) < m(2R_k) + \frac{1}{6k} \leq c + \frac{1}{6k} \leq m(R_k) + \frac{1}{3k} = \inf_{x \in X, \|x\| \geq R_k} f_1(x) + \frac{1}{3k}. $$

We are now in a position to apply the Ekeland variational principle [12] (see, e.g., [31, Theorem 2.26]) to the function $f_1$ on the closed set \{ $x \in X \mid \|x\| \geq R_k$ \} with the parameters $\epsilon := \frac{1}{3k}$ and $\lambda := \frac{\|x^k\|}{2}$ therein. Note that in the finite-dimensional setting under consideration this result and other variational principles can be proved easily; see [32, Theorem 2.12]. In this way we find $u^k \in X$ with $\|u^k\| \geq R_k$ satisfying the following conditions:

(a) $m(R_k) \leq f_1(u^k) \leq f_1(x^k)$,
(b) $\|u^k - x^k\| \leq \lambda$, and
(c) $f_1(x) + \frac{\epsilon}{\lambda} \|x - u^k\| \geq f_1(u^k)$ for all $x \in X$ with $\|x\| \geq R_k$.

It follows from (a) that $\{ f(u^k) \} \subset [ f(\Omega) ]_{\bar{y}}$ and $f_1(u^k) \to c$ as $k \to \infty$, while (b) yields

$$ R_k < \frac{\|x^k\|}{2} \leq \|u^k\| \leq \frac{3}{2} \|x^k\| $$

and implies, in particular, that $\|u^k\| \to \infty$ as $k \to \infty$. On the other hand, by the condition (c), $u^k$ is a minimizer of the scalar optimization problem

$$ \min_{x \in X, \|x\| \geq R_k} f_1(x) + \frac{\epsilon}{\lambda} \|x - u^k\|. $$
Applying the necessary optimality conditions from Lemma 2.4 to this problem allows us to find \( \alpha \in \mathbb{R}_+^m \) for which

\[
0 \in \alpha_1 \partial \left[ f_1(\cdot) + \frac{\varepsilon}{\lambda} \cdot - u^k \right] (u^k) + \sum_{i=2}^m \alpha_i \partial f_i(u^k) + N(u^k; \Omega),
\]

(4)

\[
\alpha_i \{ f_i(u^k) - \bar{y}_i \} = 0 \quad \text{for} \quad i = 2, \ldots, m, \quad \text{and} \quad \sum_{i=1}^m \alpha_i = 1.
\]

The subdifferential sum rule from Lemma 2.2 and the calculation of Lemma 2.1 give us

\[
\partial \left[ f_1(\cdot) + \frac{\varepsilon}{\lambda} \cdot - u^k \right] (u^k) \subset \partial f_1(u^k) + \frac{\varepsilon}{\lambda} \mathbb{B}^n.
\]

(5)

From (4) and (5) we have

\[
0 \in \sum_{i=1}^m \alpha_i \partial f_i(u^k) + \alpha_1 \frac{\varepsilon}{\lambda} \mathbb{B}^n.
\]

It follows from the definition of the Rabier function \( \nu \) that

\[
\nu(u^k) \leq \alpha_1 \frac{\varepsilon}{\lambda} \leq \frac{\varepsilon}{\lambda} = \frac{2}{3k\|x^k\|} \leq \frac{1}{k\|u^k\|}.
\]

Consequently, we get the estimate

\[
\|u^k\| \nu(u^k) \leq \frac{1}{k} \quad \text{for each} \quad k \in \mathbb{N},
\]

and therefore \( \|u^k\| \nu(u^k) \to 0 \) as \( k \to \infty \).

On the other hand, it follows from the boundedness of the section \([f(\Omega)]_{\bar{y}}\) and the inclusion \(\{f(u^k)\} \subset [f(\Omega)]_{\bar{y}}\) that the sequence \(\{f(u^k)\}\) has an accumulation point, say \(y \in \mathbb{R}^m\). Thus \(y \in K_{\infty, \leq \bar{y}}(f; \Omega)\), a contradiction that verifies implication (iii) \(\Rightarrow\) (i).

Next we prove (iv) \(\Rightarrow\) (i). Assume on the contrary that \(f|_{\Omega}\) is not proper at the sublevel \(\bar{y}\). Then there exists a sequence \(\{x^k\} \subset \Omega\) such that \(\|x^k\| \to \infty\) as \(k \to \infty\) and \(f(x^k) \leq \bar{y}\).

As above, consider the set \(X\) from (2) and the number \(c\) from (3). For each \(k \in \mathbb{N}\) we form the following scalar nonsmooth optimization problem:

\[
\begin{aligned}
\text{minimize} & \quad f_1(x) \\
\text{subject to} & \quad x \in X \quad \text{and} \quad \|x\|^2 - \|x^k\|^2 = 0.
\end{aligned}
\]

Since the constraint set here is nonempty and compact, this problem admits an optimal solution denoted by \(v^k\). The usage of necessary optimality conditions from Lemma 2.4
give us a pair \((\alpha, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}\) satisfying the relationships

\[
0 \in \sum_{i=1}^{m} \alpha_i \partial f_i(v^k) + \mu v^k + N(v^k; \Omega),
\]

\[
\alpha_i \left( f_i(v^k) - \bar{y}_i \right) = 0 \quad \text{for} \quad i = 2, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i + |\mu| = 1.
\]

We clearly get \(v^k \in \Gamma(f, \Omega)\).

Thus we constructed the sequence \(\{v^k\}\) with the following properties:

(a) \(\{v^k\} \subset \Gamma(f, \Omega)\);
(b) \(\|v^k\| = \|x^k\| \to \infty\) as \(k \to \infty\);
(c) \(f_1(v^k) \leq f_1(x^k)\) for all \(k \in \mathbb{N}\);
(d) \(f(v^k) \leq \bar{y}\) for all \(k \in \mathbb{N}\).

It follows from the boundedness of the section \([f(\Omega)]_{\bar{y}}\) and the inclusion \(\{f(v^k)\} \subset [f(\Omega)]_{\bar{y}}\) that the sequence \(\{f(v^k)\}\) has an accumulation point \(y \in \mathbb{R}^m\). Therefore \(y \in T_{\infty, \leq \bar{y}}(f, \Omega)\), a contradiction. This completes the proof of the equivalence between all the properties (i)–(iv).

Let us finally verify the last statement of the theorem. Suppose that (i) holds and we show that the set \(X' := \{x \in \Omega \mid f(x) \leq \bar{y}\}\) is bounded. To proceed, take an arbitrary sequence \(\{x^k\} \subset \Omega\) such that \(f(x^k) \leq \bar{y}\) for all \(k \in \mathbb{N}\). Since the sequence \(\{f(x^k)\}\) is contained in the bounded set \([f(\Omega)]_{\bar{y}}\), it follows from (i) that the sequence \(\{x^k\}\) is bounded. On the other hand, \(X'\) is closed because the set \(\Omega\) is closed and the mapping \(f\) is continuous. Therefore \(X'\) is compact. The compactness of the section \([f(\Omega)]_{\bar{y}}\) is directly deduced from the compactness of \(X'\) and the continuity of the mapping \(f\). The proof of Theorem 3.1 is completed.

The results of Theorem 3.1 significantly extend the recent ones from [25], where such an equivalence is established in the case of \(\Omega = \mathbb{R}^n\) and polynomial mappings \(f\) by using methods of semialgebraic geometry. The proof of [25] is based on the inclusion

\[
T_{\infty, \leq \bar{y}}(f, \mathbb{R}^n) \subset K_{\infty, \leq \bar{y}}(f, \mathbb{R}^n)
\]  

valid when \(f\) is polynomial. The following example shows that if \(f\) is not polynomial, then (6) fails. Thus the approach of [25] cannot be applied to our general setting, while the new approach of variational analysis allows to treat (VP) in full generality.

**Example 3.1** Let \(\bar{y} = 0\), and let \(f: \mathbb{R} \to \mathbb{R}\) be defined by \(f(x) := \sin x\). We claim that \(0 \in T_{\infty, \leq \bar{y}}(f, \mathbb{R}) \setminus K_{\infty, \leq \bar{y}}(f, \mathbb{R})\) and so \(T_{\infty, \leq \bar{y}}(f, \mathbb{R}) \not\subset K_{\infty, \leq \bar{y}}(f, \mathbb{R})\). Indeed, let \(x^k = 2k\pi\) for all \(k \in \mathbb{N}\). It is easily seen that \(\Gamma'(f, \mathbb{R}) = \mathbb{R}\) and hence \(\{x^k\} \subset \Gamma'(f, \mathbb{R})\). Since \(x^k \to \infty\) and \(f(x^k) \to 0\) as \(k \to \infty\), we have that \(0 \in T_{\infty, \leq \bar{y}}(f, \mathbb{R})\). To show now that \(0 \notin K_{\infty, \leq \bar{y}}(f, \mathbb{R})\), assume the contrary and then find a sequence \(\{u^k\} \subset \mathbb{R}\) such that \(f(u^k) \leq 0\), \(u^k \to \infty\), \(f(u^k) \to 0\), and \(u^k \nabla f(u^k) \to 0\) as
k → ∞. This implies that sin u^k → 0 and u^k cos u^k → 0 as k → ∞. Using u^k → ∞ and u^k cos u^k → 0, we get that cos u^k → 0 as k → ∞. Consequently, sin^2 u^k + cos^2 u^k → 0 as k → ∞, which contradicts the fact that sin^2 u^k + cos^2 u^k = 1 for all k ∈ N and thus verifies the failure of the inclusion (6).

We conclude this section with an immediate consequence of Theorem 3.1; cf. [18, Theorem 2.5] for the case where f is polynomial and Ω = R^n.

**Corollary 3.1** Assume that every section of the set f(Ω) is bounded. Then the following assertions are equivalent:

(i) f|Ω is proper.
(ii) f|Ω satisfies the Palais–Smale condition.
(iii) f|Ω satisfies the weak Palais–Smale condition.
(iv) f|Ω is M-tame.

Furthermore, every section of the set f(Ω) is compact provided that one of the above equivalent conditions is satisfied.

### 4 Existence of optimal solutions

This section contains our main results on the existence of optimal solutions to constrained vector optimization problems in the general nonsmooth setting. We start with deriving verifiable necessary and sufficient conditions for the existence of Pareto efficient solutions.

#### 4.1 Existence of Pareto efficient solutions

Given \( \bar{y} \in (\mathbb{R} \cup \{ \infty \})^m \), denote

\[
K_{0, \leq \bar{y}} (f, \Omega) := \{ f(x) \in \mathbb{R}^m \mid x \in \Omega, \; f(x) \leq \bar{y}, \; \nu(x) = 0 \},
\]

where \( \nu(x) \) is the Rabier function defined in (1). The motivation behind this definition comes from the observation that if \( \bar{x} \) is a Pareto efficient solution to the problem (VP), then \( \nu(\bar{x}) = 0 \) and so \( f(\bar{x}) \in K_{0, \leq \bar{y}} (f, \Omega) \) with \( \bar{y} = f(\bar{x}) \) (see the proof of Theorem 4.1).

**Theorem 4.1** The following assertions are equivalent:

(i) The problem (VP) admits a Pareto efficient solution.
(ii) There exists a vector \( \bar{y} \in f(\Omega) \) such that the section \( [f(\Omega)]_{\bar{y}} \) is bounded and the inclusion \( K_{\infty, \leq \bar{y}} (f, \Omega) \subseteq K_{0, \leq \bar{y}} (f, \Omega) \) holds.
(iii) There exists a vector \( \bar{y} \in f(\Omega) \) such that the section \( [f(\Omega)]_{\bar{y}} \) is bounded and the inclusion \( K_{\infty, \leq \bar{y}} (f, \Omega) \subseteq K_{0, \leq \bar{y}} (f, \Omega) \) holds.
(iv) There exists a vector \( \bar{y} \in f(\Omega) \) such that the section \( [f(\Omega)]_{\bar{y}} \) is bounded and the inclusion \( T_{\infty, \leq \bar{y}} (f, \Omega) \subseteq K_{0, \leq \bar{y}} (f, \Omega) \) holds.
Proof First we justify in parallel implications (i)⇒(ii), (i)⇒(iii), and (i)⇒(iv). To this end, let \( \bar{x} \in \Omega \) be a Pareto efficient solution to \((VP)\), and let \( \tilde{y} := f(\bar{x}) \). As mentioned above, the section \([f(\Omega)]_{\tilde{y}}\) is just \([\tilde{y}]\) while containing in this case the sets \( K_{0,\leq \tilde{y}}(f, \Omega) \), \( \tilde{K}_{\infty,\leq \tilde{y}}(f, \Omega) \), \( K_{\infty,\leq \tilde{y}}(f, \Omega) \), and \( T_{\infty,\leq \tilde{y}}(f, \Omega) \). Furthermore, it is easy to check that

\[
\bar{x} \in \text{argmin}_{\Omega} \max_{1 \leq i \leq m} \{ f_i(x) - f_i(\bar{x}) \}.
\]

Then, the necessary optimality conditions from Lemma 2.4 and the calculation of Lemma 2.3 ensure the existence of \( \alpha \in \mathbb{R}^m_+ \) such that

\[
0 \in \sum_{i=1}^{m} \alpha_i \partial f_i(\bar{x}) + N(\bar{x}; \Omega).
\]

By definition, we have \( \nu(\bar{x}) = 0 \), and therefore \( \tilde{y} \in K_{0,\leq \tilde{y}}(f, \Omega) \). Thus the conditions in (ii), (iii), and (iv) follow immediately from these facts giving us necessary conditions for the existence of Pareto efficient solutions to \((VP)\).

Next we verify implications (ii)⇒(i), (iii)⇒(i), and (iv)⇒(i), which justify the sufficiency of conditions (ii)–(iv) for the existence of Pareto efficient solutions to \((VP)\).

Let \( \tilde{y} \in f(\Omega) \) be such that the section \( Y := [f(\Omega)]_{\tilde{y}} \) is bounded. Then the closure \( \overline{Y} \) of \( Y \) is a nonempty compact set.

Claim 1 For almost every \( \alpha \in \mathbb{R}^m \), the optimization problem \( \min_{y \in \overline{Y}} \langle \alpha, y \rangle \) has a unique minimizer.

To see this, define the function \( \theta : \mathbb{R}^m \rightarrow \mathbb{R} \) by

\[
\theta(\alpha) := \min_{y \in \overline{Y}} \langle \alpha, y \rangle.
\]

It is easy to see that \( \theta \) is locally Lipschitz. By the Rademacher theorem (see, for example, [33, Theorem 9.60]), \( \theta \) is almost everywhere differentiable. Let \( \alpha \in \mathbb{R}^m \) be such that the derivative \( \nabla \theta(\alpha) \) of \( \theta \) at \( \alpha \) exists and let \( \hat{y} \in \overline{Y} \) be a minimizer of the problem \( \min_{y \in \overline{Y}} \langle \alpha, y \rangle \). We can write

\[
\theta(\alpha + h) - \theta(\alpha) - \langle h, \nabla \theta(\alpha) \rangle = o(\|h\|) \quad \text{as} \quad h \to 0.
\]

Since \( \langle \alpha + h, \hat{y} \rangle \geq \theta(\alpha + h) \) and \( \langle \alpha, \hat{y} \rangle = \theta(\alpha) \), it follows that

\[
\langle h, \hat{y} \rangle - \langle h, \nabla \theta(\alpha) \rangle \geq o(\|h\|) \quad \text{as} \quad h \to 0,
\]

which implies easily that \( \hat{y} = \nabla \theta(\alpha) \). This proves Claim 1.

By Claim 1, we can find a vector \( \alpha \in \text{int} \mathbb{R}^m_+ \) such that the scalar optimization problem

\[
\min_{y \in \overline{Y}} \langle \alpha, y \rangle \quad (7)
\]
has the unique optimal solution, say \( \hat{y} \in \overline{Y} \).

**Claim 2** If \( \Omega \cap f^{-1}(\hat{y}) \neq \emptyset \), then every point \( \hat{x} \in \Omega \cap f^{-1}(\hat{y}) \) is a Pareto efficient solution to (VP).

Indeed, by contradiction, suppose that there exists \( x \in \Omega \) such that

\[
f(x) \leq f(\hat{x}) \quad \text{and} \quad f(x) \neq f(\hat{x}).
\]

Componentwise it can be equivalently written as

\[
f_i(x) \leq f_i(\hat{x}) \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad f_j(x) < f_j(\hat{x}) \quad \text{for some} \quad j.
\]

Hence in the case of \( f(x) \in Y \) we arrive at the contradiction by

\[
\langle \alpha, f(x) \rangle < \langle \alpha, f(\hat{x}) \rangle = \langle \alpha, \hat{y} \rangle.
\]

If otherwise \( f(x) \notin Y \), we have that \( f_i(x) > \hat{y}_i \) for some \( i \in \{1, \ldots, m\} \), and so

\[
f_i(\hat{x}) = \hat{y}_i \leq \hat{y}_i < f_i(x) \leq f_i(\hat{x}),
\]

which is also a contradiction and the proof of Claim 2 is complete.

**Claim 3** If \( \Omega \cap f^{-1}(\hat{y}) = \emptyset \), then \( \hat{y} \in K_{\infty, \leq \hat{y}} (f, \Omega) \cap T_{\infty, \leq \hat{y}} (f, \Omega) \).

Indeed, assume that \( \Omega \cap f^{-1}(\hat{y}) = \emptyset \). Note that \( \hat{y} \leq \overline{y} \) because \( \hat{y} \in \overline{Y} \). Let

\[
I := \{i \in \{1, \ldots, m\} \mid \hat{y}_i < \overline{y}_i\}.
\]

Then \( I \) is nonempty because if otherwise, \( \hat{y} = \overline{y} \) and so \( \Omega \cap f^{-1}(\overline{y}) \neq \emptyset \), a contradiction. We first show that

\[
\text{argmin}\left\{ \sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y} \right\} = \{\hat{y}\}.
\]  \(\text{(8)}\)

In fact, by the compactness of \( \overline{Y} \), the set argmin\(\{\sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y}\}\) is nonempty. Let \( y \in \text{argmin}\{\sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y}\} \). By definition, then

\[
\langle \alpha, y \rangle = \sum_{i \in I} \alpha_i y_i + \sum_{i \notin I} \alpha_i y_i \\
\leq \sum_{i \in I} \alpha_i \hat{y}_i + \sum_{i \notin I} \alpha_i \overline{y}_i \\
= \sum_{i \in I} \alpha_i \hat{y}_i + \sum_{i \notin I} \alpha_i \overline{y}_i = \langle \alpha, \hat{y} \rangle.
\]

On the other hand, since \( \hat{y} \) is the unique minimizer of the problem (7), it holds that \( y = \hat{y} \), as required.
Let
\[ X := \{ x \in \Omega \mid f_1(x) \leq \bar{y}_1, \ldots, f_m(x) \leq \bar{y}_m \} \]
and define the (locally Lipschitz) function \( \phi: \mathbb{R}^n \to \mathbb{R} \) by
\[ \phi(x) := \sum_{i \in I} \alpha_i f_i(x). \]

We next show that
\[ \min \left\{ \sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y} \right\} = \inf_{x \in X} \phi(x). \tag{9} \]

Indeed, for any \( x \in X \), one has \( f(x) \in f(X) = Y \) and so
\[ \min \left\{ \sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y} \right\} \leq \sum_{i \in I} \alpha_i f_i(x) = \phi(x). \]

This implies that
\[ \min \left\{ \sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y} \right\} \leq \inf_{x \in X} \phi(x). \]

On the other hand, since \( \hat{y} \in \overline{Y} = f(X) \), there exists a sequence \( \{ x_k \} \subset X \) such that \( f(x_k) \to \hat{y} \). Consequently, for each \( k \in \mathbb{N} \) we have
\[ \inf_{x \in X} \phi(x) \leq \phi(x_k) = \sum_{i \in I} \alpha_i f_i(x_k). \]

By letting \( k \to \infty \), we obtain
\[ \inf_{x \in X} \phi(x) \leq \sum_{i \in I} \alpha_i \hat{y}_i = \min \left\{ \sum_{i \in I} \alpha_i y_i \mid y \in \overline{Y} \right\}, \]

where the second equation follows from (8). Therefore, the Eq. (9) holds.

If \( \phi \) attains its infimum on \( X \), say \( \hat{x} \in X \), then it follows from (8) and (9) that \( f(\hat{x}) = \hat{y} \) and so \( \Omega \cap f^{-1}(\hat{y}) \neq \emptyset \), a contradiction. Therefore, \( \phi \) does not attain its infimum on \( X \). Then, as \( X \) is closed, it has to be unbounded. Now by using arguments similar to those employed to establish implication (iii)\( \Rightarrow \) (i) of Theorem 3.1 we find a sequence \( \{ u^k \} \subset X \) such that
\[ \| u^k \| \to +\infty \quad \text{and} \quad \phi(u^k) \to \inf_{x \in X} \phi(x) \]
and for each \( k \in \mathbb{N} \) there exists a pair \((\kappa, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+^m\) satisfying the following relationships

\[
0 \in \kappa \partial \left[ \phi(\cdot) + \frac{\varepsilon}{\lambda} \cdot \|u^k\| \right](u^k) + \sum_{i=1}^m \beta_i \partial f_i(u^k) + N(u^k; \Omega),
\]

\[
\beta_i(f_i(u^k) - \bar{y}_i) = 0 \quad \text{for } i = 1, \ldots, m,
\]

\[
1 = \kappa + \sum_{i=1}^m \beta_i,
\]

where \( 0 < \frac{\varepsilon}{\lambda} \leq \frac{1}{k\|u^k\|} \). Note that, the subdifferential sum rule from Lemma 2.2 and the calculation of Lemma 2.1 give us

\[
\partial \left[ \phi(\cdot) + \frac{\varepsilon}{\lambda} \cdot \|u^k\| \right](u^k) \subset \partial \phi(u^k) + \frac{\varepsilon}{\lambda} \mathbb{B}^n = \sum_{i \in I} \alpha_i \partial f_i(u^k) + \frac{\varepsilon}{\lambda} \mathbb{B}^n.
\]

On the other hand, it follows from the boundedness of the section \([f(\Omega)]\bar{y}\) and the inclusion \{\{f(u^k)\} \subset [f(\Omega)]\bar{y}\} that the sequence \{f(u^k)\} has an accumulation point, say \( y \in \mathbb{R}^m \). Then we deduce from (8) and (9) that \( y = \bar{y} \). Consequently, the sequence \{f(u^k)\} converges to \( \bar{y} \), taking a subsequence if necessary. By the definition of the index set \( I \), for all \( i \in I \) and all \( k \) sufficiently large, we have \( f_i(u^k) - \bar{y}_i < 0 \), and so \( \beta_i = 0 \). Therefore,

\[
0 \in \sum_{i \in I} \kappa \alpha_i \partial f_i(u^k) + \sum_{i \notin I} \beta_i \partial f_i(u^k) + N(u^k; \Omega) + \kappa \frac{\varepsilon}{\lambda} \mathbb{B}^n.
\]

It follows from the definition of the Rabier function \( \nu \) that

\[
\nu(u^k) \leq \frac{\kappa}{\sum_{i \in I} \kappa \alpha_i + \sum_{i \notin I} \beta_i \lambda} \frac{\varepsilon}{\lambda} \leq \frac{1}{\sum_{i \in I} \alpha_i \lambda} \frac{\varepsilon}{\lambda} \leq \frac{1}{\sum_{i \in I} \alpha_i \kappa k\|u^k\|}.
\]

Consequently, we get the estimate

\[
\|u^k\| \nu(u^k) \leq \frac{1}{\sum_{i \in I} \alpha_i \kappa k},
\]

and so \( \|u^k\| \nu(u^k) \to 0 \) as \( k \to +\infty \). Therefore, \( \bar{y} \in K_{\infty, \leq \bar{y}}(f, \Omega) \).

Finally, by using arguments similar to those used to prove implication \((iv) \Rightarrow (i)\) of Theorem 3.1, we can find a sequence \{\bar{v}^k\} \subset X \cap \Gamma(f, \Omega) satisfying the relationships

\[
\|\bar{v}^k\| \to +\infty \quad \text{and} \quad \phi(\bar{v}^k) \to \inf_{x \in X} \phi(x) \quad \text{as} \quad k \to +\infty.
\]

Again by similar arguments as above, we can see that \( f(\bar{v}^k) \to \bar{y} \) as \( k \to \infty \). This gives us by definition that \( \bar{y} \in T_{\infty, \leq \bar{y}}(f, \Omega) \). The proof of Claim 3 is complete.
We are now in a position to finish the proof of Theorem 4.1. In fact, we can see from Claim 3 that, if $\Omega \cap f^{-1}(\hat{y}) = \emptyset$, then $\hat{y} \in K_{\infty, \leq \hat{y}}(f, \Omega) \cap T_{\infty, \leq \hat{y}}(f, \Omega)$, which yields together with the assumptions made that $\hat{y} \in K_{0, \leq \hat{y}}(f, \Omega)$. Hence $\hat{y} = f(\hat{x})$ for some $\hat{x} \in \Omega$, a contradiction. Therefore, $\Omega \cap f^{-1}(\hat{y}) \neq \emptyset$; by Claim 2, then the theorem follows. □

In this way we arrive at the following verifiable sufficient conditions for the existence of Pareto efficient solutions in constrained vector optimization with nonsmooth data.

**Corollary 4.1** Assume that there is $\bar{y} \in f(\Omega)$ such that the section $[f(\Omega)]_{\bar{y}}$ is bounded. Then the problem (VP) admits a Pareto solution provided that one of the following equivalent conditions holds:

(i) $f|_{\Omega}$ is proper at the sublevel $\bar{y}$.

(ii) $f|_{\Omega}$ satisfies the Palais–Smale condition at the sublevel $\bar{y}$.

(iii) $f|_{\Omega}$ satisfies the weak Palais–Smale condition at the sublevel $\bar{y}$.

(iv) $f|_{\Omega}$ is M-tame at the sublevel $\bar{y}$.

**Proof** This is a direct consequence of Theorems 3.1 and 4.1; see also [6, Theorem 1], or, equivalently, [28, Corollary 3.8, p. 48]. □

**Remark 4.1** (i) The literature on sufficient conditions for the existence of Pareto efficient solutions to (VP) is rich; see, e.g., the books [28,34] and the papers [3,6–8,15,19] with the references therein. However, there are only a few results studying necessary conditions for the existence of Pareto efficient solutions. For instance, in [9, Theorems 3.1, 3.2 and Lemma 3.3] and [15, Theorems 4.1 and 4.2], the authors provided some necessary and sufficient conditions in terms of the so-called *first-order asymptotic directions* for the nonemptiness and boundedness of the Pareto efficient solution set of a convex vector optimization problem. However, such conditions are rather restrictive, because they require the boundedness of the solution set. In [19, Section 4], the authors introduced a new concept of *second-order asymptotic directions* of nonconvex unbounded sets and applied it to derive sufficient conditions for the existence of Pareto efficient points. To the best of our knowledge, Theorem 4.1 is the first result, which gives necessary and sufficient conditions for the existence of Pareto efficient solutions of nonconvex vector optimization problems. The criteria (ii)–(iv) of Theorem 4.1 are new even for the scalar case.

(ii) The equivalent conditions in Corollary 4.1 are sufficient conditions but not necessary ones for the existence of Pareto efficient solutions to (VP). For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be the polynomial mapping defined by

$$f(x_1, x_2) := x_1^2$$

and let $\Omega = \mathbb{R}^2$. Clearly, $f$ is not proper at every sublevel $\hat{y} \in f(\Omega) = [0, \infty)$, and hence Corollary 4.1 cannot be applied. However, $f$ satisfies any of the equivalent conditions (ii)–(iv) of Theorem 4.1 for all $\hat{y} \geq 0$, and so $f$ attains its infimum on $\mathbb{R}^2$. Note that, by easy computations, we can see that [2, Theorem 4],
(iii) We also note here that the sufficient condition (i) for the existence of Pareto solutions to (VP) in Corollary 4.1 does not require the Lipschitz continuity of $f$. In fact, if the functions $f_i, i = 1, \ldots, m$, are lower semicontinuous, $f|_{\Omega}$ is proper at the sublevel $\tilde{y} \in f(\Omega)$, and the section $[f(\Omega)]_{\tilde{y}}$ is bounded, then it is not hard to check that the problem (VP) admits a Pareto solution. However, it is important to note that the problem of testing whether a function is proper (or coercive) is strongly NP-hard even for polynomials of degree 4 (see [1, Theorem 3.1]).

On the other hand, when $f$ is a polynomial mapping and $\Omega = \mathbb{R}^n$, the sets $K_0(f, \Omega), K_\infty(f, \Omega)$, and $T_\infty(f, \Omega)$ can be computed effectively as shown recently in [10,11,23]. We therefore hope that the Palais–Smale conditions and the $M$-tameness will be useful in studying the existence of Pareto solutions of the problem (VP) at least for the polynomial case.

### 4.2 Existence of Geoffrion-properly efficient solutions

The last part of this paper is devoted to the existence of Geoffrion-properly efficient solutions to (VP). First we show that the equivalent conditions of Corollary 4.1 do not guarantee the existence of a Geoffrion-properly solution to this problem.

**Example 4.1** Let $\Omega := \{x \in \mathbb{R} | -x \leq 0\}$, and let $f : \mathbb{R} \to \mathbb{R}^2$ be a polynomial mapping defined by $f(x) = (f_1(x), f_2(x)) := (-x^2, x)$. It is easy to see that every section of $f(\Omega)$ is bounded and that $f|_{\Omega}$ is proper. Corollary 4.1 tells us that the problem (VP) admits a Pareto efficient solution. In fact, it is not hard to check that the whole set $\Omega$ consists of Pareto efficient solutions to (VP).

We claim that the problem (VP) has no Geoffrion-properly efficient solution. Indeed, let $\bar{x}$ be an arbitrary element of $\Omega$. We have to show that for all $C > 0$ there exists an index $i \in \{1, 2\}$ and a point $x \in \Omega$ with $f_i(x) < f_i(\bar{x})$ such that

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} > C$$

whenever $j \in \{1, 2\}$ with $f_j(x) < f_j(x)$. To proceed, pick $x > \max\{C, \bar{x}\}, i = 1$, and $j = 2$. Then we have $f_1(x) < f_1(\bar{x}), f_2(\bar{x}) < f_2(x)$, and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} = x + \bar{x} > C,$$

which verifies the claim.

As shown in [20, Theorem 5.2] (see also [29, Proposition 2.1]), a necessary condition for the existence of Geoffrion-properly efficient solutions to (VP) is

$$[f(\Omega) + \mathbb{R}_+^m] \oplus \cap (-\mathbb{R}_+^m) = \{0\}.$$
By using [34, Lemma 3.2.4], this can be equivalently rewritten as

$$[f(\Omega)]^{\oplus} \cap (-\mathbb{R}^m_+) = \{0\},$$  

where given $Y \subset \mathbb{R}^m$, the symbol $[Y]^{\oplus}$ stands for the recession cone of $Y$, i.e.,

$$[Y]^{\oplus} := \{d \in \mathbb{R}^m \mid \exists \{(t_k, y^k)\} \subset \mathbb{R}_+ \times Y \text{ such that } t_k \to 0 \text{ and } t_k y^k \to d\}.$$ 

Let us emphasize that condition (10) is necessary for the existence of Geoffrion-properly efficient solutions to (VP) but is not sufficient except for convex cases; see [9, Lemma 3.3(d)]. The next result provides sufficient conditions for the existence of Geoffrion-properly efficient solutions to nonconvex constrained vector optimization problems.

**Theorem 4.2** Under the validity of the necessary optimality condition (10), the following equivalent conditions are sufficient for the existence of a Geoffrion-properly efficient solution to the vector optimization problem (VP):

(i) $f|_\Omega$ is proper;
(ii) $f|_\Omega$ satisfies the Palais–Smale condition;
(iii) $f|_\Omega$ satisfies the weak Palais–Smale condition;
(iv) $f|_\Omega$ is $M$-tame.

**Proof** Let us first check that conditions (i)–(iv) are indeed equivalent in the setting under consideration. By using Corollary 3.1, it suffices to show that every section of $f(\Omega)$ is bounded. Supposing the contrary gives us a point $y \in \mathbb{R}^m$ for which the section $[f(\Omega)]_y$ is unbounded. Then we find a sequence $\{x^k\} \subset \Omega$ such that $f(x^k) \leq y$ whenever $k \in \mathbb{N}$ and $\|f(x^k)\| \to \infty$ as $k \to \infty$. Denoting $$t_k := \frac{1}{\|f(x^k)\|} \quad \text{and} \quad y^k := f(x^k), \quad k \in \mathbb{N},$$

we have $t^k \to 0$ as $k \to \infty$ and $\|t_k y^k\| = 1$ for all $k \in \mathbb{N}$. Without loss of generality, assume that the sequence $\{t_k y^k\}$ converges to some $d \in \mathbb{R}^m$ with $\|d\| = 1$. It follows from the definition that $d \in [f(\Omega)]^\oplus$. Since $f(x^k) \leq y$ for all $k \in \mathbb{N}$, we arrive at $d \in -\mathbb{R}_+^m$, a contradiction.

Now we are ready to prove that the set of Geoffrion-properly efficient solutions to (VP) is nonempty. Invoking [4, Theorem 3.2] and [20, Theorems 2.1 and 5.1], it suffices to show that the set $f(\Omega) + \mathbb{R}_+^m$ is closed. To proceed, we deduce from Corollary 3.1 that every section of $f(\Omega)$ is compact. Pick an arbitrary sequence $\{a^k\} \subset f(\Omega) + \mathbb{R}_+^m$ that converges to some $a \in \mathbb{R}^m$ and find sequences $\{y^k\} \subset f(\Omega)$ and $\{d^k\} \subset \mathbb{R}_+^m$ such that $a^k = y^k + d^k$ for all $k \in \mathbb{N}$. Since the sequence $\{a^k\}$ is convergent, there is $\tilde{a} \in \mathbb{R}^m$ with $a^k \leq \tilde{a}$ for all $k \in \mathbb{N}$. It clearly follows that $y^k \leq \tilde{a}$ whenever $k \in \mathbb{N}$, and thus $\{y^k\} \subset [f(\Omega)]_{\tilde{a}}$. The compactness of $[f(\Omega)]_{\tilde{a}}$ gives us a subsequence of $\{y^k\}$, which converges to some $y \in [f(\Omega)]_{\tilde{a}}$. This implies that the corresponding subsequence of $\{d^k\}$ is also convergent to some $d \in \mathbb{R}_+^m$, and therefore

$$a = y + d \in [f(\Omega)]_{\tilde{a}} + \mathbb{R}_+^m \subset f(\Omega) + \mathbb{R}_+^m.$$
which completes the proof of the theorem. □

**Remark 4.2**

(i) The first part of the proof of Theorem 4.2 shows that if condition (10) holds, then every section of \( f(\Omega) \) is bounded. By Corollary 3.1, the equivalent conditions of Theorem 4.2 guarantee that every section of \( f(\Omega) \) is compact. Hence, the existence of a Geoffrion-properly efficient solution to (VP) can also be obtained by applying [28, Corollary 3.15 and Proposition 2.9, Chapter 2]. On the other hand, for the reader's convenience we present a detailed proof here.

(ii) By Remark 2.1(ii), Theorem 4.2 also provides sufficient conditions for the existence of properly efficient solutions in the senses of Benson [4], Borwein [5], and Henig [20].

(iii) It follows from [17, Theorem 5.1] that the problem (VP) admits a Henig-properly efficient solution if \( \Omega = \mathbb{R}^n \) and \( f: \mathbb{R}^n \to \mathbb{R}^m \) is bounded from below and satisfies the Palais–Smale condition. Clearly, if \( f \) is bounded from below, then

\[
\left[ f(\mathbb{R}^n) \right]^\oplus \subset \left[ a + \mathbb{R}_+^m \right]^\oplus = \mathbb{R}_+^m,
\]

and therefore \( \left[ f(\mathbb{R}^n) \right]^\oplus \cap (-\mathbb{R}_+^m) = \{0\} \). The converse is true when \( m = 1 \) but fails in general. Thus our Theorem 4.2 essentially improves and extends [17, Theorem 5.1]. To illustrate that we get a proper improvement even for unconstrained problems on \( \mathbb{R} \) with polynomial objectives, consider the problem (VP) with \( \Omega := \mathbb{R} \) and the cost mapping \( f: \mathbb{R} \to \mathbb{R}^2 \) defined by \( f(x) = (f_1(x), f_2(x)) := (x, x^4 - 3x^3 + 2x^2) \). It is easy to check that

\[
\left[ f(\mathbb{R}) \right]^\oplus \cap (-\mathbb{R}_+^2) = \{0\}
\]

and that \( f \) satisfies the Palais–Smale condition. Theorem 4.2 tells us that this problem admits a Geoffrion-properly efficient solution. However, the mapping \( f \) under consideration is not bounded from below on \( \mathbb{R} \), and hence [17, Theorem 5.1] is not applicable in this case. Since \( f_2 \) is nonconvex, [9, Lemma 3.3] cannot be applied for this example. Note that the set \( f(\mathbb{R}) + \mathbb{R}_+^2 \) is nonconvex and hence, [26, Proposition 3.1] also cannot be employed.

We finish this section by presenting an example that shows that, under the validity of condition (10), the equivalent conditions (ii)–(iv) of Theorem 4.1 are not sufficient for the existence of Geoffrion-properly efficient solutions of (VP). We further observe that the conclusion of Theorem 4.2 does not hold if conditions (i)–(iv) of this theorem are replaced by equivalent conditions (i)–(iv) of Corollary 4.1.

**Example 4.2** Let

\[
\Omega := \left\{ x \in \mathbb{R}^5 \mid (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)(x_2x_3 - 1)^2 + (x_4x_5 - 1)^2 = 0, x_2 \geq 0, x_4 \geq 0 \right\},
\]

and let \( f: \mathbb{R}^5 \to \mathbb{R}^2 \) be the mapping defined by

\[
f(x_1, x_2, x_3, x_4, x_5) := (x_1^2 + x_2, x_1 + x_4).
\]
An easy computation shows that

\[ f(\Omega) = \{(0, 0)\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > -\sqrt{y_1}, y_1 > 0\} \]

and the Pareto efficient solution set of \((VP)\) is given by

\[ (f|_{\Omega})^{-1}(0, 0) = \{(0, 0, 0, 0, 0)\}. \]

Hence, conditions (ii)–(iv) of Theorem 4.1 are satisfied at \(\bar{y} = (0, 0)\). Furthermore, we can check that condition (10) holds. However, it is easy to see that for any convex cone \(\Lambda\) with \(\text{int} \Lambda \supset \mathbb{R}_+^2 \setminus \{(0, 0)\}\), \(\bar{y}\) is not a Pareto efficient point of \(f(\Omega)\) with respect to \(\Lambda\) (see Fig. 1). Hence, by [20, Theorem 2.1], the problem \((VP)\) has no Geoffrion-properly efficient solution.

Furthermore, since \((f|_{\Omega})^{-1}(\bar{y}) = \{(0, 0, 0, 0, 0)\}\), \(f|_{\Omega}\) is proper at sublevel \(\bar{y}\). Hence, the equivalent conditions (i)–(iv) of Corollary 4.1 are satisfied. However, the problem \((VP)\) has no Geoffrion-properly efficient solution.

5 Conclusions

This paper demonstrates that developing a novel approach of variational analysis and generalized differentiation to the existence of global optimal solutions to constrained problems of vector optimization allows us to derive truly new results in this area in both smooth and nonsmooth settings. In this way we show that the developed variational approach leads us to verifiable necessary optimality conditions for the existence of Pareto efficient solutions as well as sufficient conditions for the existence of properly efficient solutions to general constrained problems with locally Lipschitz cost mappings. In particular, the obtained results dramatically improve the very recent existence theorems of Pareto efficient solutions established in [25] for unconstrained problems.
with polynomial cost mappings by using techniques of semialgebraic geometry and polynomial optimization.

We see the following natural directions of future developments of the variational approach to the existence theorems in problems of multiobjective optimization.

1. Avoiding the Lipschitz continuity assumption on cost mappings by considering vector optimization problems with merely continuous and also lower semicontinuous cost mappings that frequently arise in applications. According to the scheme implemented above, this requires further investigations of fundamental issues of generalized differential calculus and necessary optimality conditions dealing with non-Lipschitz mappings.

2. Studying optimization and equilibrium problems with set-valued cost mappings, which are at the core of most recent developments in multiobjective optimization and practical applications to various models in economics, finance, behavioral sciences, etc.; see, e.g., the monographs [24,32] and the references therein.

3. Considering vector and set-valued optimization problems in infinite-dimensional spaces. This would open the gate to cover, in particular, various dynamical equilibrium models arising in macroeconomic, mechanics, and systems control governed by constrained evolution equations, inclusions, variational conditions, etc; see, e.g., [31,32,35]. Variational principles and appropriate tools of generalized differentiation provide powerful machinery to successfully proceed in the theoretical developments in this direction with subsequent applications.

Acknowledgements  The authors are grateful to three anonymous referees and handling editors for careful reading of the original submission and for helpful suggestions and kind remarks.

References

1. Ahmadi, A.A., Zhang, J.: On the complexity of testing attainment of the optimal value in nonlinear optimization. Math. Program. Ser. A 1, 1 (2019). https://doi.org/10.1007/s10107-019-01411-1

2. Bao, T.Q., Mordukhovich, B.S.: Variational principles for set-valued mappings with applications to multiobjective optimization. Control Cybern. 36, 531–562 (2007)

3. Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers for multiobjective problems: existence and optimality conditions. Math. Program. 122, 301–347 (2010)

4. Benson, H.P.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Math. Anal. Appl. 71, 232–241 (1979)

5. Borwein, J.M.: Proper efficient points for maximizations with respect to cones. SIAM J. Control Optim. 15, 57–63 (1977)

6. Borwein, J.M.: On the existence of Pareto efficient points. Math. Oper. Res. 8, 64–73 (1983)

7. Deng, S.: Characterizations of the nonemptiness and compactness of solution sets in convex vector optimization. J. Optim. Theory Appl. 96, 123–131 (1998)

8. Deng, S.: Coercivity properties and well-posedness in vector optimization. RAIRO Oper. Res. 37, 195–208 (2003)

9. Deng, S.: Boundedness and nonemptiness of the efficient solution sets in multiobjective optimization. J. Optim. Theory Appl. 144, 29–42 (2010)

10. Dias, L.R.G., Tibãr, M.: Detecting bifurcation values at infinity of real polynomials. Math. Z. 279, 311–319 (2015)

11. Dias, L.R.G., Tanabé, S., Tibãr, M.: Toward effective detection of the bifurcation locus of real polynomial maps. Found. Comput. Math. 17, 837–849 (2017)

12. Ekeland, I.: Nonconvex minimization problems. Bull. Am. Math. Soc. 1, 443–474 (1979)
13. Flores-Bazán, F.: Ideal, weakly efficient solutions for vector optimization problems. Math. Program. 93, 453–475 (2002)
14. Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. 22, 618–630 (1968)
15. Gutiérrez, C., López, R., Novo, V.: Existence and boundedness of solutions in infinite-dimensional vector optimization problems. J. Optim. Theory Appl. 162, 515–547 (2014)
16. Ha, T.X.D.: Variants of the Ekeland variational principle for a set-valued map involving the Clarke normal cone. J. Math. Anal. Appl. 316, 346–356 (2006)
17. Ha, T.X.D.: The Ekeland variational principle for Henig proper minimizers and super minimizers. J. Math. Anal. Appl. 364, 156–170 (2010)
18. Hà, H.V., Phạm, T.S.: Genericity in Polynomial Optimization. World Scientific Publishing, Singapore (2017)
19. Hadjisavvas, N., Luc, D.T.: Second order asymptotic directions of unbounded sets with application to optimization. J. Convex Anal. 18, 181–202 (2011)
20. Henig, M.I.: Proper efficiency with respect to cones. J. Optim. Theory Appl. 36, 387–407 (1982)
21. Huy, N.Q., Kim, D.S., Tuyen, N.V.: Existence theorems in vector optimization with generalized order. J. Optim. Theory Appl. 174, 728–745 (2017)
22. Jahn, J.: Vector Optimization: Theory, Applications, and Extensions. Springer, Berlin (2004)
23. Jelonek, Z., Kurdyka, K.: Reaching generalized critical values of a polynomial. Math. Z. 276, 557–570 (2014)
24. Khan, A.A., Tammer, C., Zălinescu, C.: Set-Valued Optimization. An Introduction with Applications. Springer, Berlin (2015)
25. Kim, D.S., Phạm, T.S., Tuyen, N.V.: On the existence of Pareto solutions for polynomial vector optimi-
26. mization problems. Math. Program. Ser. A 177, 321–341 (2019)
27. Lara, F.: Generalized asymptotic functions in nonconvex multiobjective optimization problems. Optimization 66, 1259–1272 (2017)
28. Luc, D.T.: An existence theorem in vector optimization. Math. Oper. Res. 14, 693–699 (1989)
29. Luc, D.T.: Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
30. Luc, D.T.: Recession cones and the domination property in vector optimization. Math. Program. 49, 113–122 (1990)
31. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. Vol. 1: Basic Theory, Vol. 2: Applications. Springer, Berlin (2006)
32. Mordukhovich, B.S.: Variational Analysis and Applications. Springer, Cham (2018)
33. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin (1998)
34. Sawaragi, Y., Nakayama, N., Tanino, T.: Theory of Multiobjective Optimization. Academic Press, New York (1985)
35. Xiao, Y.B., Huang, N.J., Cho, Y.J.: A class of generalized evolution variational inequalities in Banach spaces. Appl. Math. Lett. 25, 914–920 (2012)
