2013

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Recommended Citation
Zhao, Jianqiang. 2013. "On q-Analogs of Wostenholme Type Congruences for Multiple Harmonic Sums." Integers, 13: 358-368. doi: 10.1515/9783110298161.358 source: http://arxiv.org/abs/1303.3060
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On $q$-Analog of Wolstenholme Type Congruences for Multiple Harmonic Sums

Jianqiang Zhao

Abstract. Multiple harmonic sums are iterated generalizations of harmonic sums. Recently Dilcher has considered congruences involving $q$-analogs of these sums in depth one. In this paper we shall study the homogeneous case for arbitrary depth by using generating functions and shuffle relations of the $q$-analog of multiple harmonic sums. At the end, we also consider some non-homogeneous cases.

Keywords. Multiple harmonic sums, $q$-multiple harmonic sums, shuffle relations.

1 Introduction.

In Shi and Pan extended Andrews’ result on the $q$-analog of Wolstenholme Theorem to the following two cases: for all prime $p \geq 5$

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2[p]_q \pmod{[p]_q^2}, \quad (1)$$

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(1-q)^2 \pmod{[p]_q}, \quad (2)$$

where $[n]_q = (1-q^n)/(1-q)$ for any $n \in \mathbb{N}$ and $q \neq 1$. This type of congruences is considered in the polynomial ring $\mathbb{Z}[q]$ throughout this paper. Notice that the modulus $[p]_q$ is an irreducible polynomial in $q$ when $p$ is a prime. In Dilcher generalized the above two congruences further to sums of the form $\sum_{j=1}^{p-1} \frac{1}{[j]_q}$ and $\sum_{j=1}^{p-1} \frac{q^n}{[j]_q}$ for all positive integers $n$ in terms
of certain determinants of binomial coefficients. However, his modulus is always $[p]_q$. He also expressed these congruences using Bernoulli numbers, Bernoulli numbers of the second kind, and Stirling numbers of the first kind, which we briefly recall now.

The well-known Bernoulli numbers are defined by the following generating series:
\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = 1 - \frac{x}{2!} + \frac{x^2}{6!} - \frac{x^4}{30!} + \cdots.
\]

On the other hand, the Bernoulli numbers of the second kind are defined by the power series (cf. [7, p. 114]).
\[
\frac{x}{\log(1 + x)} = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = 1 + \frac{1}{2!} - \frac{1}{6!} + \frac{1}{4!} - \frac{19}{24!} x^4 + \cdots.
\]

This is a little different from the definition of $\tilde{b}_n$ in [3], which is changed to $b_n$ later in the same paper. Finally, the Stirling numbers of the first kind $s(n, j)$ are defined by
\[
x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{j=0}^{n} s(n, j)x^j.
\]

Define
\[
K_n(p) := (-1)^{n-1} \frac{b_n}{n!} - \frac{(-1)^n}{(n - 1)!} \sum_{j=1}^{[n/2]} \frac{B_{2j}}{2j} s(n - 1, 2j - 1)p^{2j}.
\]

By [3] Thm. 1, (6.5) and Thm. 4 and [4] Thm. 3.1 one gets:

**Theorem 1.1.** If $p > 3$ is a prime, then for all integers $n > 1$ we have
\[
\sum_{j=1}^{p-1} \frac{q^j}{[j]_q^n} \equiv K_n(p)(1 - q)^n \pmod{[p]_q}.
\]

We will need the following easy generalization of this theorem.

**Theorem 1.2.** If $p > 3$ is a prime, then for all integers $n > t \geq 1$ we have
\[
\sum_{j=1}^{p-1} \frac{q^{ij}}{[j]_q^n} \equiv (1 - q)^n \sum_{i=0}^{t-1} \binom{t-1}{i} (-1)^i K_{n-i}(p) \pmod{[p]_q}.
\]
Moreover,
\[
\sum_{j=1}^{p-1} \frac{1}{|j|_q} \equiv (1 - q^n) \left( \frac{p-1}{2} + \sum_{j=2}^{n} K_j(p) \right) \pmod{|p|_q}.
\] (5)

**Proof.** If \( t > 1 \) it is clear that
\[
q^j t^j (1 - (1 - q^j))^{t-1} = q^j \sum_{i=0}^{t-1} \binom{t-1}{i} (-1)^i (1 - q^j)^i.
\]
So (4) follows from Theorem 1.1 immediately. Congruence (5) is a variation of [3, (5.11)].

All of the sums in Theorem 1.1 and 1.2 are special cases of the \( q \)-analog of multiple harmonic sums. The congruence properties of the classical multiple harmonic sums (MHS for short) are systematically investigated in [10]. In this paper we shall study their \( q \)-analogs which are natural generalizations of the congruences obtained by Shi and Pan [8] and Dilcher [3].

Similar to its classical case (compare [10]) a \( q \)-analog of multiple harmonic sum (\( q \)-MHS for short) is defined as follows. For \( s := (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell \), \( t := (t_1, \ldots, t_\ell) \in \mathbb{N}^\ell \) and \( n \in \mathbb{Z}_{\geq 0} \) set
\[
H_q^{(t)}(s; n) := \sum_{1 \leq k_1 < \cdots < k_\ell \leq n} \frac{q^{k_1 t_1 + \cdots + k_\ell t_\ell}}{|k_1|_q^{s_1} \cdots |k_\ell|_q^{s_\ell}}, \quad H_q^{*(t)}(s; n) = H_q^{(t)}(s; n)/(1-q)^{w(s)},
\] (6)
where \( w(s) := s_1 + \cdots + s_\ell \) is the weight, \( \ell \) the depth and \( t \) the modifier. For trivial modifier we set
\[
H_q(s; n) := H_q^{(0, \ldots, 0)}(s; n), \quad H_q^*(s; n) = H_q(s; n)/(1-q)^{w(s)}.
\]
Note that in [3] \( \tilde{H}_q(s; p-1) := H_q^{(1)}(s; p-1) \) are studied in some detail and are related to \( H_q(s; p-1) \). Also note that \( H_q^{(s_1, \ldots, s_\ell-1)}(s; n) \) are the partial sums of the most convenient form of \( q \)-multiple zeta functions (see [10]).

In this paper we mainly consider \( q \)-MHS with the trivial modifier. By convention we set \( H_q^{(t)}(s; 0) = 0 \) for \( r = 0, \ldots, \ell - 1 \), and \( H_q^{(t)}(\emptyset; n) = 1 \). To save space, for an ordered set \( (e_1, \ldots, e_\ell) \) we denote by \( \{e_1, \ldots, e_\ell\}^d \) the ordered set formed by repeating \( (e_1, \ldots, e_\ell) \) \( d \) times. For example \( H_q^{(\{s\}^\ell; n)} \) will be called a homogeneous sum.

Throughout the paper, we use short-hand \( H_q(s) \) to denote \( H_q(s; p-1) \) for some fixed prime \( p \).
2 Homogeneous $q$-MHS.

It is extremely beneficial to study the so-called stuffle (or quasi-shuffle) relations among MHS (see, for e.g., [10]). The same mechanism works equally well for $q$-MHS.

Recall that for any two ordered sets $(r_1, \ldots, r_t)$ and $(r_{t+1}, \ldots, r_n)$ the stuffle operation is defined by

$$\text{Shfl} \left( (r_1, \ldots, r_t), (r_{t+1}, \ldots, r_n) \right) := \bigcup_{\sigma \text{ permutes } \{1, \ldots, n\}, \sigma^{-1}(1) < \cdots < \sigma^{-1}(t), \sigma^{-1}(t+1) < \cdots < \sigma^{-1}(n)} (r_{\sigma(1)}, \ldots, r_{\sigma(n)}).$$

Fix a positive integer $s$. For any $k = 1, \ldots, \ell - 1$, we have by stuffle relation

$$H^*_{q}\left((\ell-k)s\right) \cdot H^*_{q}(\{s\}^k) = \sum_{s \in \text{Shfl}(\{(\ell-k)s\}, \{s\}^k)} H^*_{q}(s) + \sum_{s \in \text{Shfl}(\{(\ell-k+1)s\}, \{s\}^{k-1})} H^*_{q}(s).$$

Applying $\sum_{k=1}^{\ell-1}(-1)^{\ell-k-1}$ on both sides we get

$$H^*_{q}(\{s\}^\ell) = \frac{1}{\ell} \sum_{k=0}^{\ell-1} (-1)^{\ell-k-1} H^*_{q}((\ell-k)s) \cdot H^*_{q}(\{s\}^k). \quad (7)$$

**Theorem 2.1.** Let $s$ be a positive integer and let $\eta_s = \exp(2\pi i / s)$ be the $s$th primitive root of unity. Then

$$\sum_{\ell=0}^\infty H^*_{q}(\{s\}^\ell)x^\ell \equiv \frac{(-1)^s}{p^sx} \prod_{n=0}^{s-1} \left(1 - (1 - \eta_s^n(-x)^{1/s})^p\right) \pmod{[p]_q}.$$  

**Proof.** Let $\zeta = \exp(2\pi i / p)$ be the primitive $p$th root of unity and set

$$P_n = \sum_{j=1}^{p-1} \frac{1}{(1 - \zeta^j)^n}. \quad (8)$$

It is easy to see that $H^*_{q}(n) \equiv P_n \pmod{[p]_q}$. By using partial fractions Dilcher [4] (4.2) obtained essentially the following generating function of $P_n$:

$$g(x) := \sum_{n=0}^\infty P_n x^n = \frac{px(x - 1)^{p-1}}{1 - (1 - x)^p}. \quad (9)$$
Let \( a_\ell = H_q^s \left( \left\{ s \right\}_\ell \right) \) for all \( \ell \geq 0 \). Let \( w(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell \) be its the generating function. By (7) we get

\[
w(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell \equiv 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{k=0}^{\ell-1} (-1)^{\ell-k-1} P_{(\ell-k)s} a_k x^\ell \pmod{[p]_q}.
\]

Differentiating both sides and changing index \( \ell \to \ell + 1 \) we get modulo \([p]_q\):

\[
w'(x) \equiv \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} (-1)^{\ell-k} P_{(\ell-k+1)s} a_k x^\ell
\]

\[
\equiv \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} (-1)^{\ell-k} P_{(\ell-k+1)s} a_k x^\ell
\]

\[
\equiv w(x) \sum_{\ell=0}^{\infty} P_{(\ell+1)s} (-x)^\ell
\]

\[
\equiv \frac{w(x)}{-x} \left( \sum_{\ell=0}^{\infty} P_{\ell s} (-x)^\ell + 1 \right)
\]

\[
\equiv \frac{w(x)}{-sx} \left( s + \sum_{n=0}^{s-1} \sum_{\ell=0}^{\infty} P_{\ell} (\eta^n_s (-x)^{1/s})^\ell \right)
\]

\[
\equiv \frac{w(x)}{-sx} \left( s + \sum_{n=0}^{s-1} g(\eta^n_s (-x)^{1/s}) \right)
\]

\[
\equiv \frac{w(x)}{-sx} \left( s - \sum_{n=0}^{s-1} p\eta^n_s (-x)^{1/s} (\eta^n_s (-x)^{1/s} - 1)^{p-1} \right).
\]

Here \( \eta_s = \exp(2\pi i / s) \) is the \( s \)th primitive root of unity. Thus

\[
\left( \ln w(x) \right)' = \left( -\ln x \right)' + \sum_{n=0}^{s-1} \frac{(1 - (1 - \eta^n_s (-x)^{1/s})^p)'}{1 - (1 - \eta^n_s (-x)^{1/s})^p}.
\]

Therefore by comparing the constant term we get

\[
w(x) \equiv \frac{(-1)^s}{p^sx} \prod_{n=0}^{s-1} \left( 1 - (1 - \eta^n_s (-x)^{1/s})^p \right) \pmod{[p]_q}
\]

as desired. \( \square \)
Corollary 2.2. For all positive integer \( \ell < p \) we have

\[
H_q(\{1\}^\ell) \equiv \frac{1}{\ell + 1} \left( \frac{p - 1}{\ell} \right) \cdot (1 - q)^\ell \pmod{[p]_q}.
\]

Proof. By the theorem we get

\[
\sum_{\ell=0}^{\infty} H_q^*(\{1\}^\ell) x^\ell \equiv \frac{(1 + x)^p - 1}{px} \equiv \frac{1}{px} \sum_{\ell=0}^{\infty} \binom{p}{\ell + 1} x^{\ell + 1} \equiv \sum_{\ell=0}^{\infty} \frac{1}{\ell + 1} \binom{p-1}{\ell} x^\ell \pmod{[p]_q}.
\]

The corollary follows immediately. \(\square\)

Corollary 2.3. For every positive integer \( \ell < p \) we have

\[
H_q(\{2\}^\ell) \equiv (-1)^\ell \cdot \frac{2 \cdot \ell!}{(2\ell + 2)!} \left( \frac{p - 1}{\ell} \right) \cdot F_{2,\ell}(p) \cdot (1 - q)^{2\ell} \pmod{[p]_q},
\]

where \( F_{2,\ell}(p) \) is a monic polynomial in \( p \) of degree \( \ell \).

Proof. By Theorem 2.1 we have modulo \([p]_q\)

\[
\sum_{\ell=0}^{\infty} H_q^*(\{2\}^\ell) x^\ell \equiv \frac{1}{p^2 x} \left( 1 - (1 - i\sqrt{x})^p \right) \left( 1 - (1 + i\sqrt{x})^p \right) \equiv \frac{1}{p^2 x} \left| \sum_{j=1}^{(p-1)/2} \binom{p}{2j} (-1)^j x^j + i\sqrt{x} \sum_{j=0}^{(p-1)/2} \binom{p}{2j+1} (-1)^j x^j \right|^2,
\]

which easily yields

\[
H_q^*(\{2\}^\ell) \equiv \frac{(-1)^\ell}{p^2} \left\{ \sum_{j+k=\ell, 0 \leq j,k < p/2} \binom{p}{2j+1} \binom{p}{2k+1} - \sum_{j+k=\ell+1, 1 \leq j,k < p/2} \binom{p}{2j} \binom{p}{2k} \right\}.
\]

In the first sum above if \( j + k = \ell + 1 \) and \( 1 \leq j,k < p/2 \) then we may assume \( j > \ell/2 \). Then \( (\ell + 1)! \binom{p}{\ell+1} \) is a factor of \( (2j+1)! \binom{p}{2j+1} \) as a polynomial of \( p \), so is \( \ell! \binom{p-1}{\ell} \). Similarly we can see that \( \ell! \binom{p-1}{\ell} \) is a factor of the second sum.
In order to determine the leading coefficient we set
\[ C_1(x) = \sum_{j=0}^{\ell} \frac{(2\ell + 2)! x^{2j + 1}}{(2j + 1)!(2\ell - 2j + 1)!} = \frac{(x + 1)^{2\ell + 2} - (x - 1)^{2\ell + 2}}{2}, \]
\[ C_2(x) = \sum_{j=0}^{\ell + 1} \frac{(2\ell + 2)! x^{2j}}{(2j)!(2\ell - 2j + 2)!} = \frac{(x + 1)^{2\ell + 2} + (x - 1)^{2\ell + 2}}{2}. \]
Hence
\[ \sum_{j+k=\ell} 1 \cdot \frac{1}{(2j + 1)!(2k + 1)!} - \sum_{j+k=\ell+1} 1 \cdot \frac{1}{(2j)!(2k)!} = \frac{C_1(1) - (C_2(1) - 2)}{(2\ell + 2)!} = \frac{2}{(2\ell + 2)!}. \]
This finishes the proof of the corollary.

**Corollary 2.4.** Let \( \ell \) be a positive integer. Set \( \delta_\ell = (1 + (-1)^\ell) \) and \( L = 3\ell + 3 \). Then for every prime \( p \geq L \) we have modulo \([p]\)
\[
H_q(\{3\}^\ell) = \begin{cases} 
-3 \cdot \ell! \cdot \frac{(p-1)}{(3\ell+1)!} \cdot F_{3,\ell}(p) \cdot (1-q)^{3\ell} & \text{if } \ell \text{ is odd}, \\
6 \cdot \ell! \cdot \frac{(p-1)}{(3\ell+3)!} \cdot F_{3,\ell}(p) \cdot (1-q)^{3\ell} & \text{if } \ell \text{ is even}, 
\end{cases}
\]
where \( F_{3,\ell}(p) \) is a monic polynomial in \( p \) of degree \( 2\ell - 1 \) if \( \ell \) is odd and of degree \( 2\ell \) if \( \ell \) is even.

**Proof.** Let \( \eta = \exp(2\pi i/3) \). Then \( \eta^2 + \eta + 1 = 0 \). By Theorem 2.1 we have
\[ \sum_{\ell=0}^{\infty} H_q^*(\{3\}^\ell) x^\ell \equiv \frac{-1}{p^3} \prod_{a=0}^{2} \left(1 - (1 - \eta^a \sqrt[3]{-x})^p\right). \]
We now use two ways to expand this. Set \( y = \sqrt[3]{-x} \). First, the product on
the right hand side of (11) can be expressed as

\[
1 - \sum_{a=0}^{2} (1 - \eta^a y)^p + \sum_{a=0}^{2} (1 - \eta^a y)(1 - \eta^{a+1} y)^p - \prod_{a=0}^{2} (1 - \eta^a y)^p
\]

\[
= 1 - \sum_{j=0}^{p} \left( \frac{p}{j} \right)^2 \eta^{aj} y^j + \sum_{a=0}^{2} (1 + \eta^a y + \eta^{a+1} y^2)^p - (1 + x)^p
\]

\[
= 1 - 3 \sum_{j=0}^{[p/3]} \left( \frac{p}{3j} \right) x^j + 3 \sum_{j,k \geq 0, j+k<p, 2j+k \equiv 0(3)} \frac{p! (-x)^{(j+2k)/3}}{j!k!(p-j-k)!} - (1 + x)^p.
\]

Thus for \( \ell > 0 \) we get

\[
H_q^* \left( \{3\}^\ell \right) \equiv \frac{1}{p^3} \left[ 3\delta_{\ell} \left( \frac{p}{L} \right) + (-1)^\ell \cdot 3 \sum_{k \geq 1} \left( \frac{p}{L-k} \right) \left( \frac{L-k}{k} \right) + \left( \frac{p}{\ell + 1} \right) \right]
\]

Note that if \( \ell \) is odd then the degree of the polynomial is reduced to \( 3\ell - 1 \) with leading coefficient given by

\[
(-1)^\ell \cdot 3 \frac{1}{(L-1)!} \begin{pmatrix} L-1 \\ 1 \end{pmatrix} = \frac{-3}{(L-2)!} = \frac{-3}{(3\ell + 1)!}
\]

as we wanted.

Now to prove \( \ell!(\frac{p}{\ell}) \) is a factor we use the following expansion of (11):

\[
\sum_{\ell=0}^{\infty} \frac{1}{p^3x} \sum_{j,k,n \geq 1} (-1)^{j+k+n} \binom{p}{j} \binom{p}{k} \binom{p}{n} x^{(j+k+n)/3} \eta^{k+2n}.
\]

Thus

\[
H_q^* \left( \{3\}^\ell \right) \equiv \frac{1}{p^3} \sum_{1 \leq j,k,n \leq p, j+k+n=3\ell+3} (-1)^{j+k+n} \binom{p}{j} \binom{p}{k} \binom{p}{n} \eta^{k+2n} \pmod{[p]_q}.
\]

Notice that \( j + k + n = 3\ell + 3 \) implies one of the indices, say \( j \), is at least \( \ell + 1 \). Then clearly \( \binom{p}{\ell} \) contains \( \ell!(\frac{p}{\ell}) \) as a factor, therefore so does \( H_q^* \left( \{3\}^\ell \right) \pmod{[p]_q} \). This completes the proof of the corollary.
3 Some non-homogeneous $q$-MHS congruences.

In this section we consider some non-homogeneous $q$-MHS of depth two with modifiers of special type.

**Theorem 3.1.** Let $m, n$ be two positive integers. For every prime $p$ we have

$$H_q^{(m,n)}(2m, 2n) \equiv \frac{1}{2} \left\{ f(m; p) f(n; p) - f(m + n; p) \right\} \pmod{[p]_q}.$$

where

$$f(N; p) = (1 - q)^2 N \sum_{i=0}^{N-1} \binom{N-1}{i} (-1)^i K_{2N-i}(p)$$

**Proof.** By definition and substitution $i \to p - i$ and $j \to p - j$ we have

$$H_q^{(m,n)}(2m, 2n) = \sum_{1 \leq i < j < p} q^{mi + nj} \frac{(1 - q)^2 m (1 - q)^2 n}{(1 - q^i)^2 m (1 - q^j)^2 n}$$

$$= \sum_{1 \leq j < i < p} q^{pm + pn - mi - nj} \frac{(1 - q^{p-i})^2 m (1 - q^{p-j})^2 n}{(1 - q^j)^2 m (1 - q^p-j)^2 n} \pmod{[p]_q}$$

$$= \sum_{1 \leq j < i < p} q^{mi + nj} \frac{(q^i - q^j)^2 m (q^j - q^p)^2 n}{(1 - p^j)^2 m (1 - p^i)^2 n} \pmod{[p]_q}$$

$$\equiv H_q^{(n,m)}(2n, 2m) \pmod{[p]_q} \quad (12)$$

By shuffle relation we have

$$H_q^{(m)}(2m)H_q^{(n)}(2n) = H_q^{(m,n)}(2m, 2n) + H_q^{(n,m)}(2m, 2n) + H_q^{(m+n)}(2m+2n).$$

Together with (12) this yields

$$2H_q^{(m,n)}(2m, 2n) \equiv H_q^{(m)}(2m)H_q^{(n)}(2n) - H_q^{(m+n)}(2m + 2n) \pmod{[p]_q}.$$

Our theorem follows from (11) quickly. \hfill \Box

In the study of $q$-multiple zeta functions the following function appears naturally (see [9 (47)] or [2 Theorem 1]):

$$\varphi_q(n) = \sum_{k=1}^{\infty} (k - 1) \frac{q^{(n-1)k}}{[k]^n_q} = \sum_{k=1}^{\infty} k q^{(n-1)k} - \zeta_q(n),$$
where \( \zeta_q(n) = \sum_{k=1}^{\infty} \frac{q^{(n-1)k}}{[k]_q} \) is the \( q \)-Riemann zeta value defined by Kaneko et al. in [5]. Using the results we have obtained so far in this paper we discover a congruence related to the partial sums of \( \varphi_q(2) \).

**Proposition 3.2.** For every prime \( p \) we have

\[
\sum_{k=1}^{p-1} \frac{kq^k}{[k]_q^2} \equiv -\frac{p(p-1)(p+1)}{24} (1-q)^2 \pmod{[p]_q}.
\]

**Proof.** We can check the congruence for \( p = 2 \) and \( p = 3 \) easily by hand. Now we assume \( p \geq 5 \). By definition we have

\[
H_q^*(2,1) = \sum_{1 \leq i < j < p} 1 \pmod{(1-q^i)(1-q^j)}.
\]

With substitution \( i \to p-i \) and \( j \to p-j \) we get modulo \([p]_q\)

\[
-H_q^*(2,1) = -\sum_{1 \leq j < i < p} \frac{q^{2i} \cdot q^j}{(q^i-q^j)^2(q^i-q^j)}
\]

\[
\equiv -\sum_{1 \leq j < i < p} \frac{q^{2i} \cdot q^j}{(q^i-1)^2(q^j-1)}
\]

\[
\equiv -\sum_{1 \leq j < i < p} \frac{(q^i-1)^2 + 2(q^i-1) + 1}{(q^i-1)^2} \cdot \frac{1-q^j-1}{1-q^j}
\]

\[
\equiv H_q^*(1,2) - 2H_q^*(1,1) + \sum_{k=1}^{p-1} \frac{p - 3 + k}{1-q^k} - \sum_{k=1}^{p-1} \frac{k - 1}{(1-q^k)^2} - \binom{p-1}{2}
\]

\[
\equiv H_q^*(1,2) - 2H_q^*(1,1) + (p-3)H_q^*(1) + H_q^*(2) - \binom{p-1}{2} - \sum_{k=1}^{p-1} \frac{kq^k}{(1-q^k)^2}.
\]

Notice that we have the stuffle relations

\[
H_q^*(2,1) + H_q^*(1,2) = H_q^*(1)H_q^*(2) - H_q^*(3), \quad 2H_q^*(1,1) = H_q^*(1)^2 - H_q^*(2).
\]

Hence modulo \([p]_q\)

\[
\sum_{k=1}^{p-1} \frac{kq^k}{(1-q^k)^2} \equiv (H_q^*(1)+2)H_q^*(2) - H_q^*(3) - H_q^*(1)^2 + (p-3)H_q^*(1) - \binom{p-1}{2}.
\]
Notice that by [3, Theorem 2]

\[ H_q^*(3) \equiv -\frac{(p-1)(p-3)}{8} \pmod{[p]_q}. \]  

(13)

The proposition now follows from (1) and (2) immediately. \(\square\)

4 A congruence of Lehmer type

Instead of the harmonic sums up to \((p-1)\)-st term Lehmer also studied the following type of congruence (see [6]): for every odd prime \(p\)

\[ \sum_{j=1}^{(p-1)/2} \frac{1}{j} \equiv -2q_p(2) + q_p(2)^2p \pmod{p^2}, \]

where \(q_p(2) = \frac{(2p-1)-1}{p}\) is the Fermat quotient. It is also easy to see that for every positive integer \(n\) and prime \(p > 2n + 1\)

\[ \sum_{j=1}^{(p-1)/2} \frac{1}{j^{2n}} \equiv 0 \pmod{p}. \]

As a \(q\)-analog of the above we have

**Theorem 4.1.** Let \(n\) be a positive integer. For every odd prime \(p\) we have

\[ H_q^{(n)}(2n; (p-1)/2) \equiv \frac{1}{2}(1-q)^{2n} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j K_{2n-j}(p) \pmod{[p]_q}. \]

**Proof.** By definition and substitution \(i \to p-i\) we have

\[ H_q^*(n)(2n) = H_q^*(n)(2n; (p-1)/2) + \sum_{1 \leq i \leq (p-1)/2} \frac{q^{n(p-i)}}{(1-q^{p-i})^{2n}} \]

\[ \equiv 2H_q^*(n)(2n; (p-1)/2) \pmod{[p]_q} \]

By (4) this yields the theorem quickly. \(\square\)

To conclude the paper we remark that the congruence for general \(q\)-MHS should involve some type of \(q\)-analog of Bernoulli numbers and Euler numbers similar to the classical cases treated in [10]. We hope to return to this theme in the future.

**Acknowledgement.** This work is partly supported by NSF grant DMS1162116.
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