Realizing the Box-counting Method for Calculating Fractal Dimension of Urban Form Based on Remote Sensing Image

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Abstract  In the research of fractal cities, the fractal dimension is very important. It is used to describe the fractal character of the city. The authors have designed two approaches to calculate the fractal dimension by the box-counting method through an example of Beijing, which are called the vector method and the grid method, respectively. The former calculates the fractal dimension through an intersecting analysis in ArcView; and the latter is carried out by programming in Matlab. They are compared from three aspects: the calculating process, the limits in use, and the results. As a result, the conclusion is made that there are merits and faults on both methods, and they should be chosen to use properly in practical situation.

Keywords  fractal dimension; box-counting method; urban form; remote sensing; GIS; Matlab

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Introduction

A lot of phenomena have fractal characteristics\cite{1}, including the city we human beings build\cite{2}. In the research of fractal cities, the fractal dimension is very important, describing the fractal characteristic of the city\cite{3}. Through the process of calculating the fractal dimension, we also can determine whether a city is a fractal object or not.

The fractal research of the urban form starts from the analog thinking of the urban boundary line and the coastal line. Next, Batty et al. calculated the fractal dimension with the geometry measure relationship and studied the form and structure of city\cite{3}. In 2000, Benguigui et al. studied the fractal dimension of urban form in Tel Aviv, Israel, with the box-counting method\cite{4}. In 2003, De Keersmaecker used a similar way to study the fractal urban form in Brussels, Belgium\cite{5}. The Chinese scholar Feng calculated the fractal dimension of urban form in Hangzhou using the box-counting method and discussed the fractal structure of the urban land use\cite{6}; Jiang studied the fractal urban form in Beijing, and he calculated not only the capacity dimension but also the radius dimension and information dimension\cite{7, 8}. Scholars often calculate the fractal dimension of the city when they do the city research. Moreover, they will introduce methods of calculating the fractal dimension. However, few scholars will introduce the steps on the realization of calculating the fractal dimension in detail. In fact, since the amount of data used to calculate the fractal dimension is very large, it is very meaningful to work out some kinds of quick and easy ways to calculate the fractal dimension.

There are three methods for the calculation of the fractal dimension of urban form, i.e., geometry measure relationship method,\cite{3, 9} turning radius
method,[10] and the box-counting method[5,6,10]. In this paper, the box-counting method will be used to calculate the fractal dimension, and two ways to realize it will be introduced. According to the file formats used in operation, we name them as the vector method and the grid method. The steps and the key techniques will be introduced in detail, and these two ways will be compared in a certain degree.

1 Principle of box-counting method

The box-counting method was used in the fractal research of the urban system at the earliest[11]. The basic idea is as follows: select a rectangle region (including square), and then, segment it into $4^n$ equal parts step by step; correspondingly, the side is segmented into $2^n$ equal parts. The fractal dimension counted out by this way is called the box-counting method[12]. The principle of the box-counting method is defined as follows: suppose that $F$ is a limited fractal figure in a plane. Cover $F$ with square grids that are built by boxes whose side length is $\delta$ (a grid is namely a box). The number of boxes intersecting on $F$ is $N$. If $N$ is satisfied with the power law

$$N(\delta) = \delta^{-D}$$

Then, as $\delta \to 0$, the ratio of logarithm $D = \ln N(\delta) / \ln(1/\delta)$. $D$ is defined as the box dimension of $F$[13].

The general process of the box-counting method is as follows. First, cover the research object by a rectangle. Certainly, the rectangle is non-empty. The number of non-empty grids is 1. Then, divide the side of the rectangle into two parts, and get 4 grids. The size of the grid is 1/2. Cover them on the object, and the number of the non-empty grids we obtain is $N(1/2)$. Then, quarter the side of the rectangle, and get 16 grids. The size of the grid is $1/2^2$. Cover them on the object, and the number of the non-empty grids we obtain is $N(1/2^2)$. Continue this process until the side of the rectangle is divided into $2^n$ parts. Moreover, we will get $2^n \times 2^n$ grids. Cover them on the object, and the number of the non-empty grids we obtain will be $N(1/2^n)$. Here, $1/2, 1/2^2, \ldots, 1/2^n$ is the size of grid scale $\delta$, and $n$ is seen as the rank of division. Generally, we choose $n=9$. Thus, we can get nine pairs of data about this size and the corresponding number of the non-empty grids.

Then, observe these data to find whether they satisfy the negative power law $N(\delta) \propto \delta^{-D}$. If this condition is satisfied, it can be concluded that the form of object is fractal. As the negative power law is equal to the logarithm linear relationship[8], the problem can be solved through observing whether their distribution in the logarithm coordinate system tend to be linear. If it is linear, we can say the research object is fractal. Next, fit these points with the least square method. The absolute value of the slope of the fit line is just the fractal dimension of the research object.

2 Realizing of the box-counting method

Based on the above principle, we design two ways to realize the box-counting method: the vector method and the grid method. We will introduce them by the example of calculating the fractal dimension of the urban form of Beijing. First it is needed to get the data of the research region. There are two data sources: one is to directly use the land use map from the department of City Planning, and the other is to interpret the remote sensing image and get the urban profile. In this study, we have used the latter.

2.1 Getting the urban area

The data used is from the landsat-7 ETM over the breadth of Beijing, and the date obtaining the image is April 30, 2000. The related auxiliary materials are the 1: 250 000 topographic map of Beijing (1985), traffic and tourist map of Beijing (2003), and the historical atlas of Beijing edited by Hou Renzhi in chief (1988). The remote sensing image processing software ENVI 4.0 is used to do image classification.

The ETM data of Beijing includes the small plain of Beijing and a large piece of hill areas in the North. First, cut out the small plain of Beijing, getting rid of the large amount of the hill areas. The small plain area includes the city group centered in the built-up area of Beijing. Then, reduce the research area further. Taking into account the following grid method, the size of the image should be $2^n$ pixels, so we reduce the research area to a square of 2 048 pixels (see Fig. 1). Because of the uncertain factor in the
fractal development process, the fractal dimensions has a certain degree of dependence on the measuring scope and the size of the city\textsuperscript{[14]}. Our intercepting principle is just including the urban area, which is not too big.

![The remote sensing image of Beijing](image1)

**Fig. 1** The remote sensing image of Beijing

Give 4, 5, and 3 bands with RGB color to synthesize the pseudocolor image\textsuperscript{[15]}. Through visual interpretation, it is determined that there are six land-use types: urban built-up areas, crop 1, crop 2, grass, water, and hills in this urban region. Training samples for these objects are chosen on the pseudo color image, and the maximum likelihood method of the supervised classification is used to classify. The other five ground objects are combined into one type except the built-up area, and we obtained a classification map only having two types of ground objects.

Output the built-up profile as a vector layer (shape file format), which is prepared to be used in the vector method. Then save the two-value classification map as an 8-bit gray image (BMP format) that is prepared to be used in the grid method.

Things that need to be pointed out are that we better do the postprocessing after the classification, getting rid of the tiny polygons. It can reduce the workload, especially the amount of calculation by the vector method.

**2.2 Vector method**

The vector method is a way that the urban profile and the grid are intersected in ArcGIS, counting out the number of the non-empty grids by an arithmetic that is designed by the function of ArcGIS. Here, the urban profile and the grid are all vector formats, so it is called the vector method. The calculation process of the vector method is shown in Fig.2, and the calculating steps are described as follows:

1. Generate the grids in AutoCAD. The size of the first rectangle is equal to that of the research region. Therefore, the urban profile layer obtained from classification is needed to be inputed into AutoCAD, which is used as the reference to draw the rectangle. Here, not only the size of the rectangle should be equal to the research region but also the coordinate of the rectangle should be the same as the research region. Only in this way can the intersect operation be done properly. Then, use this rectangle as the reference of the scale transform and the coordinate and draw the rest of the grids.

2. Input the urban profile layer and the grids in a certain scale into ArcGIS and do the intersect analysis. Observe the attribute tables of the urban profile layer and the grid. The former has five fields: shape, Class_id, parts, length, and area. The shape field indicates that the urban profile consists of polygons. The Class_id field is the indicator of type when the urban profile layer is transformed from the classification map. It indicates that these polygons are of the same type. The Parts field indicates that a certain polygon is constituted by how many parts (including holes). The length and area field record the perimeter and the area of these polygon. The attribute table of the grid has two fields: shape and ID. The shape field also indicates that the grids are polygons. The ID field identifies the different grids. This field is very important for the calculation of the grids.

To illustrate this process of the intersecting operation, we suppose to do the intersect operation of the following plot that has a hole in one of its polygons with a 2×2 grid (Fig. 3). After intersecting, we get their common parts. Two polygons of the plot change into 4. The attribute table of the intersected plot is the sum of those of the plot and the grid. The table’s field value inherits those of the two objects, see Table 1. Among them, the ID field value is inherited from the grid’s attribute table. It indicates the grids that are intersected. For example, in Table 1,
one ID field value is 4. It indicates that the fourth grid has been intersected. Therefore, when we count out the unique value’s number, we get the number of grids intersected with the polygons, namely, get the number of the non-empty grids. In Table 1, the number of the unique value of the ID field is 3. Therefore, the non-empty grids’ number is 3.

![Fig. 3 Intersecting operation of the plot and 2×2 grid](image)

Table 1  Attribute table of plot after intersection

| Shape     | Class_id | Parts | Length | Area | ID |
|-----------|----------|-------|--------|------|----|
| Polygon   | 5        | 2     | 100    | 400  | 1  |
| Polygon   | 5        | 2     | 100    | 400  | 3  |
| Polygon   | 5        | 1     | 10     | 40   | 4  |
| Polygon   | 5        | 2     | 100    | 400  | 4  |

Then, use this way to calculate the number of the non-empty grids after the urban profile layer of Beijing intersects with the grids in different scales. Because the number of polygons constituting the urban profile is large, we design a small arithmetic to count out the number of the unique values. Moreover, we attach this small arithmetic to a button and put it on the toolbar in ArcGIS. Use this button to calculate the numbers of the non-empty grids in nine scales in turn, see Table 2.

Table 2  Numbers of the non-empty grids in nine scales by the vector method

| Rank | δ  | N(δ) |
|------|----|------|
| 1    | 1/2| 4    |
| 2    | 1/4| 16   |
| 3    | 1/8| 64   |
| 4    | 1/16| 256 |
| 5    | 1/32| 1 020 |
| 6    | 1/64| 4 024 |
| 7    | 1/128| 15 784 |
| 8    | 1/256| 59 587 |
| 9    | 1/512| 213 898 |

(3) Observe these nine pairs of data to know whether they satisfy the negative power law. According to the principle that determines whether a figure is fractal, the negative power law equals the logarithm linear relationship, so get the logarithm values (base e) of these data and figure them out in the coordinate system, see Fig. 4. These nine points show an obvious linear tendency, which indicates that the urban form of Beijing is fractal. Finally, fit the points by the least square method and get a fitting line \( y = -1.971 \times 0.059 \times 6 \). Therefore, the fractal dimension of Beijing is 1.971 4.

![Fig. 4 The ln-ln plot on fractal dimension of the urban profile of Beijing by the vector method](image)

2.3 Grid method

The grid method is a way that gets the numbers of the non-empty grids by judging the numbers of the non-zero submatrixes after image segmentation\(^{[16]}\). The calculating process will be illustrated by an example of 4×4 gray image and realized by a program in Matlab.

In the above process of classification, we have taken a two-value classification map, which is saved as an 8-bit gray image. Input this grid image into Matlab and do preprocessing. There are two types of ground objects, the urban area and the other type. Process the image into a 1-0 image, using 1 to represent the urban area and 0 to represent the other type.

We take a 4×4 gray image that has been preprocessed as an example, see Fig. 5. Divide this image into four subimages and number them in turn as A, B, C, and D. Then, judge them if they are non-zero submatrixes or not. In these four subimages, A, B, and C are all non-zero submatrixes, while D is a zero submatrix. Therefore, the numbers of the non-zero submatrixes of this gray image is 3, namely, when the image is covered by a 2×2 grid, the number of the non-empty grids is 3.

![Fig. 5 The 4×4 preprocessed gray image](image)
The same way can be used to figure out the number of the non-zero submatrixes when this image is divided into 16 equal parts, and the result is 5. In fact, we can get the same result by dividing the image into four equal-parts and get four subimages first. Secondly, obtaining the number of the non-empty submatrixes of each subimage by the above process. At last, the sum of these four subimages is just the original image divided into 16 parts.

Therefore, we infer that the calculation of the number of the non-zero submatrixes is a recursive algorithm, which is described as follows. To get the numbers of the non-zero submatrixes of an image divided by $2^n$, we first divide this image into four equal subimages and calculate the numbers of the non-zero submatrixes of each subimage divided by $2^{n-1}$. Their sum is the number of the superior image. Then, calculate the number of the non-zero submatrixes divided by $2^{n-1}$. The execution of the recursive function will stop until when $n=1$, namely, when there is no submatrix.

| Rank | $\delta$ | $M(\delta)$ |
|------|---------|-------------|
| 1    | 1/2     | 4           |
| 2    | 1/4     | 16          |
| 3    | 1/8     | 64          |
| 4    | 1/16    | 256         |
| 5    | 1/32    | 1 020       |
| 6    | 1/64    | 4 022       |
| 7    | 1/128   | 15 732      |
| 8    | 1/256   | 58 563      |
| 9    | 1/512   | 202 781     |

The recursive algorithm is realized in Matlab, and the numbers of the non-empty grids in nine scales are obtained in turn, see Table 3. It can be found that the numbers obtained through this method is not larger than those through the vector method.

The same as the vector method, the logarithm values (base $e$) of these data are figured out in the coordinate system (Fig. 6). The points in the figure show a clear linear tendency that indicates that the urban form of Beijing has the characteristic of fractal. This conclusion is consistent with that of the vector method. Finally, use the least square method to fit the points and get a fitting line $y=-1.964 \times x+0.074 \times 0$. Therefore, the fractal dimension of the urban form of Beijing is 1.964 8 in this way.

3 Comparison of the vector method and the grid method

From the aspect of realization, the vector method is more complex than the grid method. The calculating amount of the vector method is larger than that of the grid method. The vector method involves the generation of the grids and the intersecting analysis, which occupies a large amount of the calculating time, especially the intersection. In the process of the vector method, it also involves a lot of format transforms, from evf to shp and from dxf to shp. However, there is no need to generate the grids and do the intersecting analysis in the grid method, because it operates the matrix of the image directly. Therefore, the process of the grid method is easy, and it executes quickly. The recursive algorithm is the key of the grid method.

From the aspect of results, the numbers of the non-empty grids obtained by the grid method are smaller than those by the grid method. The smaller the grid is, the clearer this phenomena is. Because the vector method is realized by the intersecting operation of the urban profile layer and the grid layer, when the size of the grid is reduced, errors must be created, whereas the grid method segments the matrix of the image directly. Therefore, there is no existence of intersecting errors in the grid method, and its result is more correct.

From the aspect of limits in use, the limit of the vector method toward the size of the image is smaller than that of the grid method. There are not so many limits toward the size of the research region in the vector method; the research region even can be a rectangle. However, in the grid method, the demand
on the size of the image is rather strict. The size of the image is asked to be the \( n \)th power of 2 in order to make the division of \( 2^n \) easy, and \( n \) is asked not to be smaller than 9. In this paper, the image can be divided by \( 2^9 \) at most. When the size of the image is \( 2^n \), it is just divided to pixels. Therefore, the side of the image should not be smaller than \( 2^9 \) pixels. However, we can reduce the limit on the size of the image through improving the algorithm.

4 Conclusion

1) We have designed two approaches of realizing the box-counting method successfully, i.e., the vector method and the grid method. The fractal dimension of the urban form of Beijing is calculated by these two methods, and the conclusion that it is fractal is obtained. These two methods make the fractal judgment of the urban form and the fractal dimension calculation easy, which provides the data for the future research on the fractal cities.

2) Comparing these two methods, the grid method is better than the vector method as a whole. The calculating amount of the former is smaller than that of the latter; moreover, the grid method executes faster and its result is more correct. However, the limit of the grid method on the size of the image is too strict. We can solve this problem by improving the algorithm.

3) The key for realizing the grid method is the recursive function. By means of the recursive approach, the number of the non-empty grids can also be counted out by a division of \( 3^n \) and \( 5^n \). In addition, the fractal dimensions obtained by different divisions can be compared with each other.

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