Classification on convex sets in the presence of missing covariates

Levon Demirdjian∗, Majid Mojirsheibani†

Abstract
A number of results related to statistical classification on convex sets are presented. In particular, the focus is on the case where some of the covariates in the data and observation being classified can be missing. The form of the optimal classifier is derived when the class-conditional densities are uniform over convex regions. In practice, the underlying convex sets are often unknown and must be estimated with a set of data. In this case, the convex hull of a set of points is shown to be a consistent estimator of the underlying convex set. The problem of estimation is further complicated since the number of points in each convex hull is itself a random variable. The corresponding plug-in version of the optimal classifier is derived and shown to be Bayes consistent.

Keywords: Classification; Convex hull; Missing covariate; Consistency

1 Introduction
Consider the following two-group classification problem. Let \((X, Y)\) be a random pair with underlying distribution \(F_{X,Y}\), where the observed vector \(X \in \mathbb{R}^d\) is used to predict the unknown class membership of \(Y \in \{0, 1\}\). In classification, one seeks to find a function (classifier) \(\phi : \mathbb{R}^d \rightarrow \{0, 1\}\) whose probability of incorrect prediction, denoted

\[
L(\phi) = P\{\phi(X) \neq Y\},
\]

∗Corresponding author
Email: levondem@ucla.edu
Department of Statistics, University of California Los Angeles, USA
†Department of Mathematics, California State University Northridge, USA
is as small as possible. The rule that minimizes this error is known as the Bayes classifier, and is given by

$$
\phi_B(x) = \begin{cases} 
1 & \text{if } P\{Y = 1 | X = x\} > P\{Y = 0 | X = x\} \\
0 & \text{otherwise},
\end{cases}
$$

(1.1)

with corresponding probability of misclassification

$$
L(\phi_B) = P\{\phi_B(X) \neq Y\};
$$

that is, $L(\phi_B) \leq L(\phi)$ for all classifiers $\phi$. For more on the general problem of statistical classification, one can refer to Devroye et al. (1996) and their list of references.

There are several open questions related to classification when the class-conditional densities of $X$ given $Y$ are assumed to be known. For example, Devroye et al. (1996) examine classification when the class-conditional densities are uniform over the $d$-dimensional hyper-rectangles $R_1$ and $R_0$; that is, $X|Y = 1 \sim \text{Unif}(R_1)$ and $X|Y = 0 \sim \text{Unif}(R_0)$. If $x$ falls in $R_1 - R_0$, for example, there is no probability of incorrect prediction; simply classify $x$ as belonging to class 1. When $x \in R_1 \cap R_0$, however, the problem becomes more challenging.

In this setup, the authors show that the optimal rule in (1.1) becomes

$$
\phi_B(x) = \begin{cases} 
1 & \text{if } x \in R_1 - R_0 \\
0 & \text{if } x \in R_0 - R_1 \\
1 & \text{if } x \in R_1 \cap R_0, \frac{p}{\mu(R_1)} > \frac{1-p}{\mu(R_0)} \\
0 & \text{if } x \in R_1 \cap R_0, \frac{p}{\mu(R_1)} \leq \frac{1-p}{\mu(R_0)},
\end{cases}
$$

(1.2)

where $p = P\{Y = 1\}$ and $\mu(\cdot)$ denotes the Lebesgue measure. In practice, however, the assumption that the conditional densities are uniform over hyper-rectangles is far too restrictive and limits the use of the above classifier. Moreover, the regions $R_1$ and $R_0$ are almost always unknown and must therefore be estimated using a set of iid data. To complicate matters further, there is no mention of how to deal with the more realistic case where there may be missing covariates in both the data and new observation $X$ being classified.

This paper aims to address all of these concerns.

Accurate estimation of the unknown regions $R_1$ and $R_0$ is far more than just a theoretical concern and has widespread applications in many diverse disciplines. In conservation
biology, for example, Kerley et al. (2002) use location data collected from radio-collared tigers in parts of eastern Russia to estimate their home ranges. In a related work, Burgman and Fox (2003) propose using a generalization of the convex hull of observed locations to predict the shape of a species’ true habitat. The cited works empirically verify that species’ habitats almost always form complex geometrical regions. To conduct reliable inference, therefore, there is clearly a need to both generalize the underlying assumptions of the classifier in (1.2) and find an optimal method of estimation.

In order to construct a sample-based version of the optimal rule in (1.2) using estimates of the unknown regions $R_1$ and $R_0$, one typically uses an iid set of data $D_n = \{(X_1, Y_1), ..., (X_n, Y_n)\}$, where $(X_i, Y_i) \sim F_{X,Y}$ for $i = 1, ..., n$. Denoting the resulting data-based classifier as $\phi_n$, the hope is that the performance of $\phi_n$ is similar in some sense to that of the optimal classifier $\phi_B$. In general, the quality of a sample-based classifier is usually assessed by checking whether or not it is consistent; a data-based rule $\phi_n$ is said to be consistent (weakly) if

$$P\{\phi_n(X) \neq Y \mid D_n\} \xrightarrow{p} P\{\phi_B(X) \neq Y\}.$$ 

The rule is said to be strongly consistent if the convergence holds with probability one (Devroye et al. (1996)).

In practice, the situation is more complicated when some of the covariates used to predict the class membership of $X$ are missing. Prior to 2007, most of the existing results in the literature on classification with missing covariates deal with the case where covariates can be missing in the data (i.e., in $X_i$, $i = 1, ..., n$) but not in the new observation $X$. See, for example, Chung and Han (2000) for the parametric case, and Pawlak (1993) for the nonparametric case. Here, however, we consider the more general case where there may be missing covariates in both the data and the new observation being classified. To see why missing values can cause difficulty, first note that if there are missing covariates in the data, the number of regions to be estimated (one corresponding to each class-conditional density) will typically increase (a more detailed explanation will be given in Section 2). Yet another source of difficulty is that in the case with missing covariates, the rule $\phi_B$ given in (1.1) will not necessarily be the optimal classifier any more. In fact, Mojirsheibani and Montazeri (2007) derived the optimal classifier in the presence of missing covariates, which is in general different from (1.1). For more recent results on classification with missing covariates, one may also refer to Mojirsheibani (2012) and Mojirsheibani and Chenouri (2011).
In this paper, we expand upon the above problem of classification when the class-conditional densities are known, and extend our results to the more realistic case where covariates can be missing in both the data and new observation being classified. In Section 2, we present the form of the classifier that is optimal not only in the restrictive case where \( R_1 \) and \( R_0 \) are hyper-rectangles, but in the more general situation where \( R_1 \) and \( R_0 \) can be any two convex sets. In Section 3, we propose a data-based version of the optimal rule which is constructed with sample-based approximations of the two convex regions. The proposed classifier is then shown to be strongly consistent. A formal justification for our method of estimating convex regions can be found in the appendix.

2 Estimation

Our discussion and results for classification on convex sets are based on the following setup. Let \( C \subset \mathbb{R}^d \) be convex and let \( D_n = \{X_1, X_2, \ldots, X_n\} \) denote an iid set of \( d \)-dimensional random vectors, where \( X_i \) is uniformly distributed on \( C \) for \( i = 1, \ldots, n \). Here and throughout the paper, we will be assuming that \( C \) is closed (this is not as restrictive as it seems; if \( C \) is not closed, one can instead consider the closure of \( C \), denoted \( \bar{C} \), since \( P\{X \in C\} = P\{X \in \bar{C}\} \)). Given \( D_n \), how can we estimate \( C \) in a consistent way? To motivate our choice of using the convex hull of \( D_n \) as an estimator of \( C \), we consider the method of maximum likelihood. Let \( \theta \) be the (unknown) volume of \( C \) and note that the likelihood function can be written as

\[
L(\theta|D_n) = \prod_{i=1}^{n} \frac{I\{x_i \in C\}}{\theta} = \frac{I\{x_1, x_2, \ldots, x_n \in C\}}{\theta^n},
\]

where \( I\{\cdot\} \) denotes the indicator function \((I\{A\} = 1 \text{ if } x \in A \text{ and } I\{A\} = 0 \text{ if } x \notin A \text{ for any set } A \subset \mathbb{R}^d)\). Clearly, this expression is maximized when our estimate of \( \theta \) is minimized. With this in mind we take the convex hull of \( D_n \), which we shall denote by \( \hat{C}_n \), to be our estimate of \( C \) (the convex hull of \( D_n \) is, by definition, the smallest convex set containing the points in \( D_n \)). How good is \( \hat{C}_n \) at estimating \( C \)? First, let us show that \( \hat{C}_n \xrightarrow{a.s.} C \) in some sense.

For a fixed point \( t \in C \), define the “distance” between \( t \) and \( \hat{C}_n \) to be \( \inf_{y \in \hat{C}_n} \|t - y\| \), where \( \|\cdot\| \) denotes the Euclidean norm in \( \mathbb{R}^d \). To establish the convergence of \( \hat{C}_n \) to \( C \), we will
consider the Hausdorff distance
\[
d(\hat{C}_n, C) = \max \left\{ \sup_{t \in C} \inf_{\hat{y} \in \hat{C}_n} \| t - \hat{y} \|, \sup_{\hat{y} \in \hat{C}_n} \inf_{t \in C} \| t - \hat{y} \| \right\}.
\]

Since \( \hat{C}_n \subset C \), however, \( d(\hat{C}_n, C) \) can be written equivalently as \( \sup_{t \in C} \inf_{\hat{y} \in \hat{C}_n} \| t - \hat{y} \| \). Therefore, to show that \( \hat{C}_n \xrightarrow{a.s.} C \), we will show that \( \sup_{t \in C} \inf_{\hat{y} \in \hat{C}_n} \| t - \hat{y} \| \xrightarrow{a.s.} 0 \) as \( n \to \infty \).

**Lemma 1** Let \( C \subset \mathbb{R}^d \) be convex and let \( D_n = \{ X_1, X_2, \ldots, X_n \} \) denote a set of iid random vectors, uniformly distributed on \( C \). Also, denote the convex hull of \( D_n \) by \( \hat{C}_n \). Then for all \( t \in C \) and \( \epsilon > 0 \),
\[
P \left\{ \min_{i=1}^{n} \| t - X_i \| > \epsilon \right\} = r^n
\]
for some \( r := r(\epsilon) \in [0, 1) \), where \( r \) does not depend on \( n \).

**Proof of Lemma 1**

Let \( t \in C \) and \( \epsilon > 0 \) be given. Denote the \( d \)-dimensional ball with center \( t \) and radius \( \epsilon \) by \( B(t, \epsilon) \). If \( \mu(A) \) represents the Lebesgue measure of a set \( A \), we get
\[
P \left\{ \min_{i=1}^{n} \| t - X_i \| > \epsilon \right\} = P \{ X_1, \ldots, X_n \notin B(t, \epsilon) \} = \prod_{i=1}^{n} P \{ X_i \notin B(t, \epsilon) \} = [P \{ X_1 \notin B(t, \epsilon) \}]^n = \left[ 1 - \frac{\mu[B(t, \epsilon) \cap C]}{\mu(C)} \right]^n := r^n.
\]
Since \( t \in C \), \( \mu[B(t, \epsilon) \cap C] > 0 \), and so \( 0 \leq r < 1 \). \( \square \)

**Lemma 2** Let \( C \subset \mathbb{R}^d \) be convex and let \( N \sim \text{binomial}(n, p) \). For all \( k \in \{1, 2, \ldots, n\} \), let \( D_k = \{ X_1, X_2, \ldots, X_k \} \) be a set of iid random vectors uniformly distributed on \( C \) and denote the convex hull of \( D_k \) by \( \hat{C}_k \). Then for all \( t \in C \) and \( \epsilon > 0 \),
\[
\sum_{k=1}^{n} P \left\{ \inf_{y \in \hat{C}_k} \| t - y \| > \epsilon \right\} P \{ N = k \} \leq c_1^n + c_2^n,
\]
where \( c_1 := c_1(\epsilon) \in (0, 1) \), \( c_2 := c_2(\epsilon) \in (0, 1) \), and neither \( c_1 \) nor \( c_2 \) depends on \( n \).
Proof of Lemma 2

Let \( t \in \mathcal{C} \) and \( \epsilon > 0 \) be given. Then

\[
\sum_{k=1}^{n} P \left\{ \inf_{y \in \hat{C}_k} \| t - y \| > \epsilon \right\} P \{ N = k \} \leq \sum_{k=1}^{n} P \left\{ \min_{i=1}^{k} \| t - X_i \| > \epsilon \right\} P \{ N = k \}
\]

\[
= \sum_{k=1}^{n} r^k \binom{n}{k} p^k (1 - p)^{n-k} \quad (0 \leq r < 1; \text{by Lemma 1})
\]

\[
= (1 - p)^n \sum_{k=1}^{n} \binom{n}{k} \left( \frac{rp}{1 - p} \right)^k 1^{n-k}
\]

\[
= (1 - p)^n \left[ \left( 1 + \frac{rp}{1 - p} \right)^n - 1 \right]
\]

\[
= [1 - p + rp]^n - (1 - p)^n
\]

\[
:= c_1^n + c_2^n,
\]

where \( 0 < c_1 < 1, 0 < c_2 < 1 \), and neither \( c_1 \) nor \( c_2 \) depends on \( n \). \( \square \)

Lemma 3 If \( \mathcal{C} \subset \mathbb{R}^d \) is convex and closed with finite Lebesgue measure, then \( \mathcal{C} \) is compact.

Proof of Lemma 3

Since \( \mathcal{C} \) is convex and has probability measure 1, it is bounded. By assumption, \( \mathcal{C} \) is also closed, and therefore compact. \( \square \)

Up to this point, we have been assuming that the size of our sample \( D_n \) was fixed. When we discuss classification in the next section, however, we will need to consider the more general case where the sample size is itself a random variable. To see why, note that we can always partition our data into two sets - those \( X_i \)'s for which \( Y_i = 1 \) and those \( X_i \)'s for which \( Y_i = 0 \). Since there is no way of knowing how many points will lie in each set, the sample sizes of both sets will be random.

Before stating our main result, we must first define the convex hull of a set of points containing a random number of elements. Let \( N \) be a non-negative integer valued random variable and let \( D_N = \{ X_1, X_2, ..., X_N \} \). If \( N > 0 \), define \( \hat{C}_N \) to be the convex hull of \( D_N \); if \( N = 0 \), define \( \hat{C}_N \) to be a ball of arbitrary radius \( \delta > 0 \) about the origin.
Theorem 1 Let $C \subset \mathbb{R}^d$ be convex and closed and let $N \sim \text{binomial}(n, p)$. Define $D_N = \{X_1, X_2, \ldots, X_N\}$ to be a set of iid random vectors, uniformly distributed on $C$. Also, denote the convex hull of $D_N$ by $\hat{C}_N$, as defined above. Then

$$\hat{C}_N \xrightarrow{a.s.} C$$

in the Hausdorff metric.

Proof of Theorem 1 Let $\epsilon > 0$ and observe that

$$P \left\{ \sup_{t \in C} \inf_{y \in \hat{C}_N} \|t - y\| > \epsilon \right\} = P \left\{ \sup_{t \in C} \inf_{y \in \hat{C}_N} \|t - y\| > \epsilon, N > 0 \right\}$$

$$+ P \left\{ \sup_{t \in C} \inf_{y \in \hat{C}_N} \|t - y\| > \epsilon, N = 0 \right\}$$

$$\leq P \left\{ \sup_{t \in C} \inf_{y \in \hat{C}_N} \|t - y\| > \epsilon, N > 0 \right\} + P \{N = 0\}$$

$$= \sum_{\ell = 1}^{n} P \left\{ \sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| > \epsilon \mid N = \ell \right\} \cdot P \{N = \ell\} + P \{N = 0\}$$

$$= \sum_{\ell = 1}^{n} P_{\ell, C} \left( \sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| > \epsilon \right) P\{N = \ell\} + P\{N = 0\}$$

$$:= I + II, \quad (2.1)$$

where $P_{\ell, C}\{A\}$ denotes the probability of the event $A$ under the joint uniform distribution of $(X_1, X_2, \ldots, X_\ell)$ on $C$. To deal with term I, consider the collection $\{B(t, \frac{\epsilon}{2}) \mid t \in C\}$, i.e. the set of all open balls of radius $\frac{\epsilon}{2}$ with centers in $C$. Clearly, $C \subset \bigcup_{t \in C} B(t, \frac{\epsilon}{2})$. Furthermore, since $C$ is compact (see Lemma 3), there exists a finite subcover of $C$, say $\bigcup_{i=1}^{k} B_i(t_i, \frac{\epsilon}{2})$. For any $\ell \in \{1, 2, \ldots, n\}$,

$$\sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| \leq \max_{i=1}^{k} \sup_{t \in B(t_i, \frac{\epsilon}{2})} \inf_{y \in \hat{C}_\ell} \|t - y\|. \quad (2.2)$$

By the triangle inequality,

$$\sup_{t \in B(t_i, \frac{\epsilon}{2})} \inf_{y \in \hat{C}_\ell} \|t - y\| \leq \frac{\epsilon}{2} + \inf_{y \in \hat{C}_\ell} \|t_i - y\|$$
for all $i \in \{1, 2, \ldots, k\}$. Therefore,

$$
\max_{i=1}^{k} \sup_{t \in B(t_i, \frac{\varepsilon}{2})} \inf_{y \in \hat{C}_\ell} \|t - y\| \leq \frac{\varepsilon}{2} + \max_{i=1}^{k} \inf_{y \in \hat{C}_\ell} \|t_i - y\| \leq \frac{\varepsilon}{2} + \sum_{i=1}^{k} \inf_{y \in \hat{C}_\ell} \|t_i - y\|. \quad (2.3)
$$

Putting expressions (2.2) and (2.3) together yields

$$
P_{\ell,C} \{\sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| > \epsilon\} \leq P_{\ell,C} \left\{\frac{\varepsilon}{2} + \sum_{i=1}^{k} \inf_{y \in \hat{C}_\ell} \|t_i - y\| > \epsilon\right\}
\leq \sum_{i=1}^{k} P_{\ell,C} \left\{\inf_{y \in \hat{C}_\ell} \|t_i - y\| > \frac{\epsilon}{2k}\right\}
$$

for all $\ell \in \{1, 2, \ldots, n\}$. We can now bound term I in expression (2.1) via

$$
\sum_{\ell=1}^{n} P_{\ell,C} \left\{\sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| > \epsilon\right\} P\{N = \ell\} \leq \sum_{\ell=1}^{k} \sum_{i=1}^{n} P_{\ell,C} \left\{\inf_{y \in \hat{C}_\ell} \|t_i - y\| > \frac{\epsilon}{2k}\right\} P\{N = \ell\}
\leq \sum_{i=1}^{k} (c_1^n + c_2^n) \quad \text{(by Lemma 2)}
$$

where $0 < c_1 < 1$, $0 < c_2 < 1$, and neither $c_1$ nor $c_2$ depends on $n$. To deal with term II in expression (2.1), simply note that

$$
P\{N = 0\} = (1 - p)^n = e^{n \log(1 - p)} = e^{-c_3 n},
$$

where $c_3 > 0$. Putting all of the above together, we have

$$
P\left\{\sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| > \epsilon\right\} \leq k c_1^n + k c_2^n + e^{-c_3 n}.
$$

Therefore

$$
\sum_{n=1}^{\infty} P\left\{\sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| > \epsilon\right\} \leq k \sum_{n=1}^{\infty} c_1^n + k \sum_{n=1}^{\infty} c_2^n + \sum_{n=1}^{\infty} e^{-c_3 n} < \infty,
$$

and by the Borel-Cantelli lemma, $\sup_{t \in C} \inf_{y \in \hat{C}_\ell} \|t - y\| \xrightarrow{a.s.} 0$ as $n \to \infty$. \hfill \Box

3 Classification on Convex Sets

Now that we have shown that $\hat{C}_n$ is a consistent estimator of $C$, we can carefully examine classification on convex sets. Before considering our main results, i.e. those involving classification with missing covariates, we deal with the simpler case where every observation
is available. The ideas in this section will make our main results in the following section more presentable.

Let \( X \in \mathbb{R}^d \) be a vector of covariates to be used to predict the class membership \( Y \in \{0, 1\} \). Here, \( X|Y = 1 \sim \text{Unif}(C_1) \) and \( X|Y = 0 \sim \text{Unif}(C_0) \), where \( C_1 \) and \( C_0 \) are taken to be convex sets in \( \mathbb{R}^d \). In this setup, the optimal rule \( \phi_B \) is given by (1.2), with \( \mathcal{R}_1 \) and \( \mathcal{R}_0 \) replaced by \( C_1 \) and \( C_0 \) respectively. To see this, rewrite expression (1.1) as

\[
\phi_B(x) = I \{ P\{Y = 1|X = x\} > P\{Y = 0|X = x\} \}
= I \{ x \in C_1 \cap C_0, P\{Y = 1|X = x\} > P\{Y = 0|X = x\} \}
+ I \{ x \not\in C_1 \cap C_0, P\{Y = 1|X = x\} > P\{Y = 0|X = x\} \}
:= I + II.
\]

Expand the second term to get

\[
II = I \{ x \in C_1 - C_0, P\{Y = 1|X = x\} > P\{Y = 0|X = x\} \}
+ I \{ x \in C_0 - C_1, P\{Y = 1|X = x\} > P\{Y = 0|X = x\} \}
= I \{ x \in C_1 - C_0 \} + 0
= I \{ x \in C_1 - C_0 \}. \tag{3.1}
\]

Therefore we see that

\[
\phi_B(x) = I \{ x \in C_1 \cap C_0, P\{Y = 1|X = x\} > P\{Y = 0|X = x\} \}
+ I \{ x \in C_1 - C_0 \}. \tag{3.1}
\]

Denote the probability density function of \( X \) conditioned on \( Y = y \) by \( f_{X|Y=y} \), and let \( f_X \) be the marginal density function of \( X \). Next, let \( p = P\{Y = 1\} \) and let \( \mu(\cdot) \) denote the Lebesgue measure of a set and use the fact that whenever \( x \in C_1 \cap C_0 \),

\[
P\{Y = 1|X = x\} = \frac{f_{X|Y=1}(x)P\{Y = 1\}}{f_X(x)} = \frac{p}{\mu(C_1)} + \frac{1-p}{\mu(C_0)} \tag{3.2}
\]

and

\[
P\{Y = 0|X = x\} = \frac{f_{X|Y=0}(x)P\{Y = 0\}}{f_X(x)} = \frac{(1-p)}{\mu(C_0)} \frac{p}{\mu(C_1)} + \frac{1-p}{\mu(C_0)} \tag{3.3}
\]
to rewrite (3.1) as

\[
\phi_B(x) = \begin{cases} 
1 & \text{if } x \in C_1 - C_0 \\
0 & \text{if } x \in C_0 - C_1 \\
1 & \text{if } x \in C_1 \cap C_0, \frac{p}{\mu(C_1)} > \frac{1 - p}{\mu(C_0)} \\
0 & \text{if } x \in C_1 \cap C_0, \frac{p}{\mu(C_1)} \leq \frac{1 - p}{\mu(C_0)} 
\end{cases} 
\]  

(3.4)

Thus, we have proved the following result.

**Theorem 2** Let \( X \in \mathbb{R}^d \) be a vector of covariates to be used to predict the class membership \( Y \in \{0, 1\} \), where \( X|Y = 1 \sim \text{Unif}(C_1) \) and \( X|Y = 0 \sim \text{Unif}(C_0) \) for convex sets \( C_1 \) and \( C_0 \) in \( \mathbb{R}^d \). Then the classifier which minimizes the probability of incorrect prediction is given by \( \phi_B \) in (3.4).

Since \( C_1 \) and \( C_0 \) are generally unknown, we may consider a data-based version of the classifier above. Let \( D_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \) be a set of iid data, distributed as \( (X, Y) \), where \( X_i \in \mathbb{R}^d \) and \( Y_i \in \{0, 1\} \) for \( i = 1, \ldots, n \). Here, \( X_i|Y_i = 1 \sim \text{Unif}(C_1) \) and \( X_i|Y_i = 0 \sim \text{Unif}(C_0) \) for \( i = 1, \ldots, n \). Next, let \( N_1 \sim \text{binomial}(n, p) \) and \( N_0 = (n - N_1) \sim \text{binomial}(n, 1 - p) \) be the number of class 1 and class 0 points respectively (i.e. \( N_k \) is the number of \( (X_i, Y_i) \) pairs for which \( Y_i = k \), \( k = 0, 1 \)). Also, let \( \hat{C}_{1,N_1} := \hat{C}_1 \) denote the convex hull of the class 1 points, i.e. \( \hat{C}_1 \) is the convex hull of those \( X_i \)'s for which \( Y_i = 1 \). Similarly, let \( \hat{C}_{0,N_0} := \hat{C}_0 \) denote the convex hull of those \( X_i \)'s for which \( Y_i = 0 \). Consider the plug in version of the rule in (3.4):

\[
\phi_n(x) = \begin{cases} 
1 & \text{if } x \in \hat{C}_1 - \hat{C}_0 \\
0 & \text{if } x \in \hat{C}_0 - \hat{C}_1 \\
1 & \text{if } x \in \hat{C}_1 \cap \hat{C}_0, \frac{\hat{p}}{\mu(\hat{C}_1)} > \frac{1 - \hat{p}}{\mu(\hat{C}_0)} \\
0 & \text{if } x \in \hat{C}_1 \cap \hat{C}_0, \frac{\hat{p}}{\mu(\hat{C}_1)} \leq \frac{1 - \hat{p}}{\mu(\hat{C}_0)} \\
I \left\{ \frac{\hat{p}}{\mu(\hat{C}_1)} \cdot \inf_{y \in \hat{C}_1} \|x - y\| < \left(1 - \frac{\hat{p}}{\mu(\hat{C}_0)} \right) \cdot \inf_{y \in \hat{C}_0} \|x - y\| \right\} & \text{if } x \notin \hat{C}_1 \cup \hat{C}_0, 
\end{cases} 
\]  

(3.5)

where \( \hat{p} = \frac{1}{n} \sum_{i=1}^{n} Y_i \).
How good is the classifier $\phi_n$ for predicting $Y$? To answer this question, let

$$L_n(\phi_n) = P\{\phi_n(X) \neq Y \mid D_n\} \quad \text{and} \quad L(\phi_B) = P\{\phi_B(X) \neq Y\}$$

be the misclassification probabilities of $\phi_n$ and $\phi_B$ respectively. The following theorem establishes the strong consistency of the plug-in rule $\phi_n$.

**Theorem 3** Let $\phi_n$ be the classifier in equation (3.5). Then $L_n(\phi_n) \xrightarrow{a.s.} L(\phi_B)$ as $n \to \infty$, where $\phi_B$ is the optimal rule in (3.4).

A few lemmas will provide the tools that are necessary to prove Theorem 3.

**Lemma 4** Let $\{\hat{C}_N\}, C$ be as in Theorem 1. Then

$$\mu(\hat{C}_N \Delta C) \xrightarrow{a.s.} 0$$

as $n \to \infty$, where $\hat{C}_N \Delta C$ denotes the symmetric difference of $\hat{C}_N$ and $C$ and where $\mu(\cdot)$ denotes the Lebesgue measure.

**Proof of Lemma 4**

By Theorem 1, $\hat{C}_N \xrightarrow{a.s.} C$ in the Hausdorff metric. Furthermore, a result of Beer (1974) implies that for each $\omega$ in the underlying sample space $\Omega$, if $\hat{C}_N(\omega) \to C$ (in the Hausdorff metric), then $\mu(\hat{C}_N(\omega) \Delta C) \to 0$. Let $\omega \in \{\omega \in \Omega \mid \hat{C}_N(\omega) \to C\}$. Then

$$\omega \in \{\omega \in \Omega \mid \mu(\hat{C}_N(\omega) \Delta C) \to 0\}.$$

The lemma now follows since $P\{\omega \in \Omega \mid \hat{C}_N(\omega) \to C\} = 1$. □

**Corollary 1** Let $\{\hat{C}_N\}, C$ be as in Theorem 1. Then

$$P\left\{X \in \hat{C}_N \Delta C \mid D_n\right\} \xrightarrow{a.s.} 0$$

as $n \to \infty$.

**Proof of Corollary 7**

$$P\left\{X \in \hat{C}_N \Delta C \mid D_n\right\} = P\left\{X \in \hat{C}_N \Delta C, N > 0 \mid D_n\right\} + P\left\{X \in \hat{C}_N \Delta C, N = 0 \mid D_n\right\}$$

11
\[ \leq I\{N > 0\} P \left\{ \mathbf{X} \in \hat{C}_N \Delta C \left| D_n \right. \right\} + I\{N = 0\}. \] (3.6)

Since \( \hat{C}_N \subset C \) for \( N > 0 \) and since \( \mathbf{X} \) is uniformly distributed on \( C \), the inequality in (3.6) becomes

\[ P \left\{ \mathbf{X} \in \hat{C}_N \Delta C \left| D_n \right. \right\} \leq I\{N > 0\} \frac{\mu(\hat{C}_N \Delta C)}{\mu(C)} + I\{N = 0\}. \]

By Lemma 4, \( \frac{\mu(\hat{C}_N \Delta C)}{\mu(C)} \xrightarrow{a.s.} 0 \) as \( n \to \infty \). Finally, \( I\{N = 0\} \xrightarrow{a.s.} 0 \) as \( n \to \infty \) by Markov’s inequality in conjunction with the Borel-Cantelli lemma. \( \square \)

**Lemma 5** Let \( \{\hat{C}_N\}, C \) be as in Theorem 1. Then

\[ \mu(\hat{C}_N) \xrightarrow{a.s.} \mu(C) \]

as \( n \to \infty \).

**Proof of Lemma 5**

\[ \left| \mu(C) - \mu(\hat{C}_N) \right| = \left| \mu(C) - \mu(\hat{C}_N) \right| I\{N > 0\} + \left| \mu(C) - \kappa_0 \right| I\{N = 0\}, \]

where \( \kappa_0 = \mu(B(0, \delta)) \) for some \( 0 < \delta < \infty \) (recall we defined \( \hat{C}_N \) for \( N = 0 \); see the definition before the statement of Theorem 1). Therefore,

\[ \left| \mu(C) - \mu(\hat{C}_N) \right| \leq \left[ \mu(\hat{C}_N \Delta C) + \left| \mu(C) - \kappa_0 \right| I\{N = 0\} \right] \xrightarrow{a.s.} 0. \] \( \square \)

**Proof of Theorem 3**

In what follows, we will be using the well-known inequality

\[ L_n(\phi_n) - L(\phi_B) \leq E \left[ \left| \eta(\mathbf{X}) - \eta_n(\mathbf{X}) \right| \left| D_n \right. \right] \]

to bound the error difference \( L_n(\phi_n) - L(\phi_B) \) (see, for example, Devroye et al. (1996)). Here, \( \eta(\mathbf{X}) = E(Y|\mathbf{X}) \) is given by

\[ \eta(\mathbf{X}) = \frac{p}{\mu(C_1)} I\{\mathbf{X} \in C_1\} + \frac{1-p}{\mu(C_0)} I\{\mathbf{X} \in C_0\}, \]

12
whereas the sample-based approximation $\eta_n(X, D_n)$ is

$$
\eta_n(X, D_n) := \eta_n(X) = \frac{\hat{p}}{\mu(\mathcal{C}_1)} I\{X \in \mathcal{C}_1\} + \frac{(1 - \hat{p})}{\mu(\mathcal{C}_0)} I\{X \in \mathcal{C}_0\}.
$$

First, note that

$$
\left| \eta(X) - \eta_n(X) \right| = \left| \frac{\frac{p}{\mu(\mathcal{C}_1)} I\{X \in \mathcal{C}_1\}}{\mu(\mathcal{C}_1)} - \frac{\frac{(1-p)}{\mu(\mathcal{C}_0)} I\{X \in \mathcal{C}_0\}}{\mu(\mathcal{C}_0)} \right|
$$

$$
\leq \frac{A}{B+C} - \frac{A'}{B' + C'}
$$

$$
= \frac{-A'/[(B' + C') - (B + C)] + A - A'}{(B + C)}
$$

$$
\leq c_0 \left| A - A' \right| + c_1 \left| C - C' \right|,
$$

(3.7)

where $c_0 = 2 \left[ \min \left\{ \frac{p}{\mu(\mathcal{C}_1)}, \frac{(1-p)}{\mu(\mathcal{C}_0)} \right\} \right]^{-1}$ and $c_1 = \frac{1}{2} c_0$. Using a similar manipulation as above, we have

$$
\left| A - A' \right| \leq \left| \frac{\hat{p}}{\mu(\mathcal{C}_1)} - \frac{p}{\mu(\mathcal{C}_1)} \right| + \frac{1}{\mu(\mathcal{C}_1)} I\{X \in \mathcal{C}_1\} - I\{X \in \mathcal{C}_1\}.
$$

But since

$$
\left| I\{X \in \mathcal{C}_1\} - I\{X \in \mathcal{C}_1\} \right| = \left| I\{X \in \mathcal{C}_1\} - I\{X \in (\mathcal{C}_1 \cup (\mathcal{C}_1 \Delta \mathcal{C}_0))\} \right| = I\{X \in (\mathcal{C}_1 \Delta \mathcal{C}_1)\},
$$

we find

$$
\left| A - A' \right| \leq \left| \frac{\hat{p}}{\mu(\mathcal{C}_1)} - \frac{p}{\mu(\mathcal{C}_1)} \right| + \frac{1}{\mu(\mathcal{C}_1)} I\{X \in (\mathcal{C}_1 \Delta \mathcal{C}_1)\}.
$$

(3.8)

Similarly, it can be shown that

$$
\left| C - C' \right| \leq \left| \frac{1 - \hat{p}}{\mu(\mathcal{C}_0)} - \frac{(1-p)}{\mu(\mathcal{C}_0)} \right| + \frac{1}{\mu(\mathcal{C}_0)} I\{X \in (\mathcal{C}_0 \Delta \mathcal{C}_0)\}.
$$

(3.9)
Plugging the inequalities in (3.8) and (3.9) into (3.7) gives

\[ \left| \eta(X) - \eta_n(X) \right| \leq c_0 \left| \frac{\hat{p}}{\mu(\hat{C}_1)} - \frac{p}{\mu(C_1)} \right| + \frac{c_0}{\mu(\hat{C}_1)} I\{X \in (C_1 \Delta \hat{C}_1)\} \\
+ c_1 \left| \frac{(1 - \hat{p})}{\mu(\hat{C}_0)} - \frac{(1 - p)}{\mu(C_0)} \right| + \frac{c_1}{\mu(\hat{C}_0)} I\{X \in (C_0 \Delta \hat{C}_0)\}. \]

Therefore, conditioning on \( D_n \) and taking expectation, we have

\[ E \left[ \left| \eta(X) - \eta_n(X) \right| \big| D_n \right] \leq c_0 \left| \frac{\hat{p}}{\mu(\hat{C}_1)} - \frac{p}{\mu(C_1)} \right| + c_1 \left| \frac{(1 - \hat{p})}{\mu(\hat{C}_0)} - \frac{(1 - p)}{\mu(C_0)} \right| \\
+ \frac{c_0}{\mu(\hat{C}_1)} P \left\{ X \in (C_1 \Delta \hat{C}_1) \big| D_n \right\} + \frac{c_1}{\mu(\hat{C}_0)} P \left\{ X \in (C_0 \Delta \hat{C}_0) \big| D_n \right\} \\
:= A_1 + A_2 + A_3 + A_4. \]

Lemma 5, in conjunction with the strong law of large numbers, implies that both \( A_1 \) and \( A_2 \) converge to 0 almost surely as \( n \to \infty \). Corollary 1 establishes the same result for the terms \( A_3 \) and \( A_4 \), completing the proof. □

4 Classification with Missing Covariates

We now consider the more difficult case where some of the covariates can be missing in both the data and observation to be classified. Let \( Z = (X', V')' \in \mathbb{R}^{d+s} \) be the vector of covariates to be used to predict the class membership \( Y \in \{0, 1\} \), where \( X \in \mathbb{R}^d, d \geq 1 \) is always observable but \( V \in \mathbb{R}^s, s \geq 1 \) can be missing. Also, let \( \delta \) be a \( \{0, 1\} \)-valued random variable defined by \( \delta = 0 \) if \( V \) is missing and \( \delta = 1 \) otherwise.

We will be assuming that \( Z|(\delta = 1, Y = 1) \sim \text{Unif}(C_1) \) and \( Z|(\delta = 1, Y = 0) \sim \text{Unif}(C_0) \) where \( C_1 \) and \( C_0 \) are convex subsets of \( \mathbb{R}^{d+s} \). Similarly, let \( Z|(\delta = 0, Y = 1) \sim \text{Unif}(C'_1) \) and \( Z|(\delta = 0, Y = 0) \sim \text{Unif}(C'_0) \) where \( C'_1 \) and \( C'_0 \) are the projections of \( C_1 \) and \( C_0 \) on \( \mathbb{R}^d \); that is \( C'_1 = \text{proj}_{\mathbb{R}^d}(C_1) \) and \( C'_0 = \text{proj}_{\mathbb{R}^d}(C_0) \). Note that both \( C'_1 \) and \( C'_0 \) are convex subsets of \( \mathbb{R}^d \) (see Rockafeller (1996)).
Now consider a new observation \((Z,Y,\delta)\) that needs to be classified. Mojirsheibani and Montazeri (2007) show that the best classifier (in the sense of having the lowest misclassification error rate) is

\[
\phi_B(Z, \delta) = \delta \tilde{\phi}_1(Z) + (1 - \delta) \tilde{\phi}_0(X),
\]

(4.1)

where

\[
\tilde{\phi}_1(Z) = \begin{cases} 
1 \text{ if } p(Z, 1)P\{Y = 1|Z\} > p(Z, 0)P\{Y = 0|Z\} \\
0 \text{ otherwise,}
\end{cases}
\]

(4.2)

\[
\tilde{\phi}_0(X) = \begin{cases} 
1 \text{ if } (1 - q(X, 1))P\{Y = 1|X\} > (1 - q(X, 0))P\{Y = 0|X\} \\
0 \text{ otherwise,}
\end{cases}
\]

(4.3)

and where

\[p(Z, i) = P\{\delta = 1|Z, Y = i\} \quad \text{and} \quad q(X, i) = P\{\delta = 1|X, Y = i\}, i = 0, 1.\]

The terms \(p(Z, i)\) and \(q(X, i)\) are often referred to as the missingness probability mechanisms. For more recent results on classification with missing covariates, one may also refer to Mojirsheibani (2012) and Mojirsheibani and Chenouri (2011).

To get a more suitable form for the optimal classifier in (4.1), use the arguments in the proof of Theorem 2 to rewrite \(\tilde{\phi}_1(z)\) in expression (4.2) as

\[
\tilde{\phi}_1(z) = I \{z \in C_1 \cap C_0, p(z, 1)P\{Y = 1|Z = z\} > p(z, 0)P\{Y = 0|Z = z\}\}
\]

\[+ I \{z \in C_1 - C_0, p(z, 1) > 0\}.
\]

An analogous argument yields

\[
\tilde{\phi}_0(x) = I \{x \in C_1' \cap C_0', (1 - q(x, 1))P\{Y = 1|X = x\} > (1 - q(x, 0))P\{Y = 0|X = x\}\}
\]

\[+ I \{x \in C_1' - C_0', q(x, 1) < 1\}.
\]

Next, let \(P\{Y = 1\} = p\) and \(P\{Y = 0\} = 1 - p\), where \(0 \leq p \leq 1\). Also recall that \(Z|\delta = 1, Y = 1 \sim \text{Unif}(C_1)\) and \(Z|\delta = 1, Y = 0 \sim \text{Unif}(C_0)\). Using the definition in
we find
\[
\tilde{\phi}_1(z) = I \left\{ z \in C_1 \cap C_0, p(z, 1) \frac{p}{\mu(C_1)} > p(z, 0) \frac{1-p}{\mu(C_0)} \right\}
+ I \left\{ z \in C_1 - C_0, p(z, 1) > 0 \right\}.
\] (4.4)

A similar approach can be used to find \( \tilde{\phi}_0(x) \) in (4.3):
\[
\tilde{\phi}_0(x) = I \left\{ x \in C'_1 \cap C'_0, (1 - q(x, 1)) \frac{p}{\mu(C'_1)} > (1 - q(x, 0)) \frac{1-p}{\mu(C'_0)} \right\}
+ I \left\{ x \in C'_1 - C'_0, q(x, 1) > 0 \right\}.
\] (4.5)

Combining the classifiers in (4.4) and (4.5) gives an expression for the optimal classifier, as expressed in the following theorem.

**Theorem 4** Let \( Z|_{\delta = 1, Y = 1} \sim \text{Unif}(C_1) \) and \( Z|_{\delta = 1, Y = 0} \sim \text{Unif}(C_0) \) where \( C_1 \) and \( C_0 \) are convex subsets of \( \mathbb{R}^{d+s} \). Similarly, let \( Z|_{\delta = 0, Y = 1} \sim \text{Unif}(C'_1) \) and \( Z|_{\delta = 0, Y = 0} \sim \text{Unif}(C'_0) \) where \( C'_1 = \text{proj}_{\mathbb{R}^d}(C_1) \) and \( C'_0 = \text{proj}_{\mathbb{R}^d}(C_0) \). Also, assume that
\[
\inf_{z \in \mathbb{R}^{d+s}} P\{\delta = 1|Z = z, Y = 1\} > 0 \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} P\{\delta = 1|X = x, Y = 1\} < 1.
\]

Then the optimal classifier is given by
\[
\phi_B(Z, \delta) = \delta \tilde{\phi}_1(z) + (1 - \delta) \tilde{\phi}_0(x)
\] (4.6)

where
\[
\tilde{\phi}_1(z) = I \left\{ z \in C_1 \cap C_0, p(z, 1) \frac{p}{\mu(C_1)} > p(z, 0) \frac{1-p}{\mu(C_0)} \right\}
+ I \left\{ z \in C_1 - C_0 \right\}
\]
and
\[
\tilde{\phi}_0(x) = I \left\{ x \in C'_1 \cap C'_0, (1 - q(x, 1)) \frac{p}{\mu(C'_1)} > (1 - q(x, 0)) \frac{1-p}{\mu(C'_0)} \right\}
+ I \left\{ x \in C'_1 - C'_0 \right\}.
\]

Since it is unrealistic to assume that the four convex sets and the functional forms of the missingness probability mechanisms \( p(z, y) \) and \( q(x, y) \) are known, we must proceed by finding a suitable plug-in version of the rule in (4.6). Let \( \hat{C}_1, \hat{C}_0, \hat{C}'_1 \) and \( \hat{C}'_0 \) be the sample based counterparts of \( C_1, C_0, C'_1 \) and \( C'_0 \) respectively; that is, let
\[
\hat{C}_1 = \text{convex hull of } \{(X_i, V_i)|\delta_i = 1, Y_i = 1\}_{i=1}^n,
\]
\[ \hat{C}_0 = \text{convex hull of } \{(X_i, V_i) | \delta_i = 1, Y_i = 0\}_{i=1}^n, \]
\[ \hat{C}_1' = \text{convex hull of } \{X_i | \delta_i = 0, Y_i = 1\}_{i=1}^n, \]
\[ \hat{C}_0' = \text{convex hull of } \{X_i | \delta_i = 0, Y_i = 0\}_{i=1}^n. \]

We propose a data-based version of the classifier in (4.6) given by
\[ \phi_n(Z, \delta) = \delta \phi_{n,1}(Z) + (1 - \delta) \phi_{n,0}(X) \tag{4.7} \]

where
\[ \phi_{n,1}(Z) = \begin{cases} 
1 & \text{if } z \in \hat{C}_1 - \hat{C}_0 \\
0 & \text{if } z \in \hat{C}_0 - \hat{C}_1 \\
1 & \text{if } z \in \hat{C}_1 \cap \hat{C}_0, \ p(z, 1) \frac{\hat{p}}{\mu(C_1)} > \hat{p}(z, 0) \frac{1 - \hat{p}}{\mu(C_0)} \\
0 & \text{if } z \in \hat{C}_1 \cap \hat{C}_0, \ p(z, 1) \frac{\hat{p}}{\mu(C_1)} \leq \hat{p}(z, 0) \frac{1 - \hat{p}}{\mu(C_0)} \\
I \left\{ \hat{p} \cdot \inf_{y \in \hat{C}_1} \|z - y\| < (1 - \hat{p}) \cdot \inf_{y \in \hat{C}_0} \|z - y\| \right\} & \text{if } z \notin \hat{C}_1 \cup \hat{C}_0, \\
\end{cases} \tag{4.8} \]

and
\[ \phi_{n,0}(X) = \begin{cases} 
1 & \text{if } x \in \hat{C}_1' - \hat{C}_0' \\
0 & \text{if } x \in \hat{C}_0' - \hat{C}_1' \\
1 & \text{if } x \in \hat{C}_1' \cap \hat{C}_0', (1 - \hat{q}(x, 1)) \frac{\hat{p}}{\mu(C_1')} > (1 - \hat{q}(x, 0)) \frac{1 - \hat{p}}{\mu(C_0')} \\
0 & \text{if } x \in \hat{C}_1' \cap \hat{C}_0', (1 - \hat{q}(x, 1)) \frac{\hat{p}}{\mu(C_1')} \leq (1 - \hat{q}(x, 0)) \frac{1 - \hat{p}}{\mu(C_0')} \\
I \left\{ \hat{p} \cdot \inf_{y \in \hat{C}_1'} \|x - y\| < (1 - \hat{p}) \cdot \inf_{y \in \hat{C}_0'} \|x - y\| \right\} & \text{if } x \notin \hat{C}_1' \cup \hat{C}_0'. \\
\end{cases} \tag{4.9} \]

Here, \( \hat{p} = n^{-1} \sum_{i=1}^n Y_i \). As for the terms \( \hat{p}(z, y) \) and \( \hat{q}(x, y) \), we consider a method of estimation based on kernel regression.
Recall the forms of the missingness probability mechanisms \( p_y(z) := p(z, y) \) and \( q_y(x) := q(x, y) \):

\[
p_y(z) = P\{\delta = 1|Z = z, Y = y\} = E(\delta|Z = z, Y = y),
q_y(x) = P\{\delta = 1|X = x, Y = y\} = E(\delta|X = x, Y = y).
\]

Under the commonly used assumption of data missing at random (MAR), one assumes that the probability that \( V \) is missing does not depend on \( V \) itself. That is,

\[
p_y(z) = P\{\delta = 1|Z = z, Y = y\} = P\{\delta = 1|X = x, Y = y\} = q_y(x).
\]

Under the MAR assumption, the kernel regression estimates of \( q_y(x), y = 0, 1 \), are

\[
\hat{q}_y(x) = \sum_{j: Y_j = y} \delta_j K\left(\frac{X_j - x}{h_n}\right) / \sum_{j: Y_j = y} K\left(\frac{X_j - x}{h_n}\right),
\]

where \( 0/0 \) is defined to be 0. Here, the kernel \( K \) is a map of the form \( K: \mathbb{R}^d \to \mathbb{R}^+ \) with smoothing parameter \( h_n (h_n \to 0 \text{ as } n \to \infty) \).

To evaluate the performance of the rule \( \phi_n \) in (4.7), let

\[
L(\phi_B) = P\{\phi_B(Z, \delta) \neq Y\} \quad \text{and} \quad L_n(\phi_n) = P\{\phi_n(Z, \delta) \neq Y|D_n\}.
\]

In what follows, we shall assume that the chosen kernel in (4.10) is regular: a nonnegative kernel \( K \) is said to be regular if there are positive constants \( b > 0 \) and \( r > 0 \) for which

\[
K(x) \geq b I\{x \in B(0, r)\} \quad \text{and} \quad \int \sup_{y \in x + B(0, r)} K(y) dx < \infty,
\]

where \( B(0, r) \) is the ball of radius \( r \) centered at the origin (for more on this see, for example, Győrfi et al. (2002)).

**Theorem 5** Let \( \phi_n \) be the classifier in (4.7) where the kernel \( K \) is regular and the bandwidth \( h_n \) satisfies \( h_n \to 0 \) and \( nh_n^d \to \infty \). Then \( L_n(\phi_n) \xrightarrow{a.s.} L(\phi_B) \) as \( n \to \infty \), where \( \phi_B \) is the optimal rule in (4.6).

To prove this result, we need the following lemma.

**Lemma 6** (Mojirsheibani and Montazeri (2007)) Let \( \phi_B \) be the optimal rule given by (4.6). For \( j = 0, 1 \), let \( f_j: \mathbb{R}^{d+s} \to [0, 1] \) and \( g_j: \mathbb{R}^d \to [0, 1] \) be any given functions. Furthermore, let \( \phi_1(z) = I\{f_1(z) > f_0(z)\} \) and \( \phi_0(x) = I\{g_1(x) > g_0(x)\} \). Also, put

\[
\phi(Z, Y) = \delta \phi_1(Z) + (1 - \delta) \phi_0(X).
\]
Then

\[
L(\phi) - L(\phi_B) \leq 2 \sum_{i=0}^{1} E \left[ \left| p(Z, i) P(Y = i | Z) - f_i(Z) \right| D_n \right] \\
+ 2 \sum_{i=0}^{1} E \left[ \left| (1 - q(X, i)) P(Y = i | X) - g_i(X) \right| \right].
\]

**Proof of Theorem 5**

We begin by introducing some notation. Let

\[
f_1(Z) = \hat{p}(Z, 1) \frac{\hat{p}}{\mu(\hat{C}_1)} I\{Z \in \hat{C}_1\} + \frac{(1 - \hat{p})}{\mu(\hat{C}_0)} I\{Z \in \hat{C}_0\},
\]

\[
f_0(Z) = \hat{p}(Z, 0) \frac{\hat{p}}{\mu(\hat{C}_1)} I\{Z \in \hat{C}_1\} + \frac{(1 - \hat{p})}{\mu(\hat{C}_0)} I\{Z \in \hat{C}_0\},
\]

\[
g_1(X) = (1 - \hat{q}(X, 1)) \frac{\hat{p}}{\mu(\hat{C}_1')} I\{X \in \hat{C}_1\} + \frac{(1 - \hat{p})}{\mu(\hat{C}_0')} I\{X \in \hat{C}_0\},
\]

\[
g_0(X) = (1 - \hat{q}(X, 0)) \frac{\hat{p}}{\mu(\hat{C}_1')} I\{X \in \hat{C}_1\} + \frac{(1 - \hat{p})}{\mu(\hat{C}_0')} I\{X \in \hat{C}_0\}.
\]

A simple extension of Lemma 6 yields

\[
L_n(\phi_n) - L(\phi_B) \leq 2 \sum_{i=0}^{1} E \left[ \left| p(Z, i) P(Y = i | Z) - f_i(Z) \right| D_n \right] \\
+ 2 \sum_{i=0}^{1} E \left[ \left| (1 - q(X, i)) P(Y = i | X) - g_i(X) \right| \right]. \tag{4.11}
\]

To prove the theorem, we will show that
\[
E \left[ p(Z, 1)P(Y = 1|Z) - f_1(Z) \right| D_n \right] \xrightarrow{a.s.} 0
\]
as \(n \to \infty\); the other terms in (4.11) can be dealt with analogously. Following the proof of Theorem 3, we find
\[
\left| p(Z, 1)P(Y = 1|Z) - f_1(Z) \right| \leq c_0 \left\{ |B - B'| + |C - C'| + |D - D'| \right\},
\]
where
\[
c_0 = \left[ \min \left\{ \frac{p}{\mu(C_1)}, \frac{(1-p)}{\mu(C_0)} \right\} \right]^{-1},
\]
and where \(B, B', C,\) and \(C'\) are defined as before, with \(Z\) playing the role of \(X\). Here, as before,
\[
|B - B'| \leq \left| \frac{\hat{p}}{\mu(C_1)} - \frac{p}{\mu(C_1)} \right| + \frac{1}{\mu(C_1)}I\{Z \in (C_1 \Delta \hat{C}_1)\}
\]
and
\[
|C - C'| \leq \left| \frac{(1 - \hat{p})}{\mu(C_0)} - \frac{(1 - p)}{\mu(C_0)} \right| + \frac{1}{\mu(C_0)}I\{Z \in (C_0 \Delta \hat{C}_0)\}.
\]
As for the term \(|D - D'|\), it is easy (but tedious) to show that
\[
|D - D'| = \left| \frac{p(Z, 1)}{\mu(C_1)/p \cdot I\{Z \in C_1\}} - \frac{\hat{p}(Z, 1)}{\mu(C_1)/\hat{p} \cdot I\{Z \in \hat{C}_1\}} \right| \\
\leq \left| \frac{\hat{p}}{\mu(C_1)} - \frac{p}{\mu(C_1)} \right| + \frac{1}{\mu(C_1)}I\{Z \in (C_1 \Delta \hat{C}_1)\} + \frac{1}{\mu(C_1)}|\hat{p}(Z, 1) - p(Z, 1)|.
\]
It follows by (4) that
\[
|p(Z, 1)P(Y = 1|Z) - f_1(Z)| \leq 2c_0 \left| \frac{\hat{p}}{\mu(C_1)} - \frac{p}{\mu(C_1)} \right| + \frac{2c_0}{\mu(C_1)}I\{Z \in (C_1 \Delta \hat{C}_1)\} \\
+ c_0 \left| \frac{(1 - \hat{p})}{\mu(C_0)} - \frac{(1 - p)}{\mu(C_0)} \right| + \frac{c_0}{\mu(C_0)}I\{Z \in (C_0 \Delta \hat{C}_0)\} \\
+ \frac{c_0}{\mu(C_1)}|\hat{p}(Z, 1) - p(Z, 1)|.
\]
Therefore, conditioning on $D_n$ and taking expectation, we have

\[
E \left[ \left| p(Z,1)P(Y = 1|Z) - f_1(Z) \right| D_n \right] \leq 2c_0 \left| \frac{\hat{p}}{\mu(C_1)} - \frac{p}{\mu(C_1)} \right| + c_0 \left| \frac{1 - \hat{p}}{\mu(\hat{C}_0)} - \frac{1 - p}{\mu(C_0)} \right| \\
+ \frac{2c_0}{\mu(C_1)} \cdot P \left\{ Z \in (C_1 \Delta \hat{C}_1) \left| D_n \right. \right\} \\
+ \frac{c_0}{\mu(C_0)} \cdot P \left\{ Z \in (C_0 \Delta \hat{C}_0) \left| D_n \right. \right\} \\
+ \frac{c_0}{\mu(C_1)} \cdot E \left[ \left| \hat{p}(Z,1) - p(Z,1) \right| D_n \right].
\]

The strong consistency of the kernel regression estimate $\hat{p}(Z,1)$ implies that

\[
E \left[ \left| \hat{p}(Z,1) - p(Z,1) \right| D_n \right] \overset{a.s.}{\longrightarrow} 0
\]
as $n \to \infty$ (for more on the properties of kernel regression estimates, see, for example, Györfi et al. (2002)); the remaining terms were dealt with in the proof of Theorem 3.

□

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