Higher-order approximation for uncertainty quantification in time series analysis

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Abstract

For time series with high temporal correlation, the empirical process converges rather slowly to its limiting distribution. Many statistics in change-point analysis, goodness-of-fit testing and uncertainty quantification admit a representation as functionals of the empirical process and therefore inherit its slow convergence. Inference based on the asymptotic distribution of those quantities becomes highly impacted by relatively small sample sizes. We assess the quality of higher-order approximations of the empirical process by deriving the asymptotic distribution of the corresponding error terms. Based on the limiting distribution of the higher-order terms, we propose a novel approach to calculate confidence regions for statistical quantities such as the median. In a simulation study, we compare coverage rate and size of our confidence regions with those based on the asymptotic distribution of the empirical process and highlight some of our method’s benefits.

Keywords: uncertainty quantification; confidence intervals; empirical process; quantiles; long-range dependence.

AMS subject classification: Primary: 62G15, 60F17. Secondary: 62G20.

1 Introduction

Let \( X_n, n = 1, \ldots, N, \) be a time series stemming from a stochastic process \( X_n, n \in \mathbb{N}, \) with marginal distribution function \( F. \) We study the empirical distribution function \( F_N(x) := \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{X_n \leq x\}}. \) The rate of convergence, i.e. the increase of the sequence \( a_N, N \in \mathbb{N}, \) which ensures weak convergence of the empirical process

\[
a_N^{-1} N(F_N(x) - F(x))
\]

to a non-degenerate limit, crucially depends on the behavior of the process’ autocorrelation function \( \gamma(k) := \text{Cov}(X_1, X_{k+1}). \) For short-range dependent time series, i.e. for stochastic processes with summable autocorrelations, \( a_N = \sqrt{N}. \) In contrast, for long-range dependent time series, i.e. for \( \gamma(k) = k^{-(2-2H)}L(k) \) with \( L \) some slowly varying function and \( H \in (1/2, 1) \) the so-called Hurst parameter, we have \( a_N = N^H L_+(N). \) In fact, under long-range dependence, the distribution of the empirical process converges much slower to its limit than under short-range dependence.

To illustrate the slow convergence of the empirical process under strong temporal correlation, we would like to draw the reader’s attention to Figure 1. The figure depicts the empirical

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Figure 1: The empirical distribution of the centered empirical distribution $F_N(x) - F(x)$ evaluated at zero ($x = 0$) under the assumption of Gaussian long-range dependent data with different Hurst parameters and for sample sizes $m = 100, 200, 1000$. The red graph depicts the standard Gaussian density function.

distribution of the centered empirical distribution $F_N(x) - F(x)$ evaluated at zero for different sample sizes. For a small Hurst parameter ($H = 0.55$), one can observe how the empirical distribution approaches the Gaussian density function (left-hand side). In contrast, for a large Hurst parameter ($H = 0.95$) and implied stronger temporal correlation, the empirical distribution converges much slower (right-hand side). This phenomenon is specific to long-range dependent time series and focus of this work.

The empirical process is a powerful tool to infer the asymptotic behavior of a variety of test statistics used in change-point analysis and goodness-of-fit testing (Wilcoxon, Kolmogorov-Smirnov and Cramér-von Mises statistics); see Beran (1992), Dehling, Rooch, and Taqqu (2013), Betken (2016, 2017), Tewes (2018). When testing the hypothesis of stationarity against the alternative hypothesis of a structural change in a time series, the phenomenon illustrated in Figure 1 results in a high number of false positives; see Dehling et al. (2013).

Our contributions in this paper are twofold. On the one hand, we address a statistical issue arising in the calculation of confidence regions under strong temporal correlation. On the other hand, we push forward the theoretical investigation of the empirical process by proving a novel limit theorem:

- We study the construction of confidence bands for the marginal distribution and confidence intervals for its quantiles in long-range dependent time series. We propose a novel approach to calculate confidence regions based on a higher-order approximation of the empirical distribution function. In the long-range dependence (LRD) case, an asymptotic expansion of the empirical distribution that is similar in spirit to a Taylor expansion can be derived. This expansion can be used to obtain a higher-order approximation of certain statistical functionals of the empirical process.

- We establish the theoretical validity of our method for statistics that are functionals of the empirical process. For statistical applications beyond confidence regions, e.g. change-point and goodness-of-fit tests, uniform convergence of the one-parameter empirical process (1.1) does not suffice in order to derive limit distributions of corresponding statistics. These
typically require consideration of the two-parameter (or sequential) empirical process
\[ a_N \{ N \} \{ F_{N} \} (x) - F(x), \quad t \in [0, 1], \quad x \in \mathbb{R}. \]

We derive the asymptotic distribution of a higher-order approximation of the sequential empirical process by proposing a new chaining technique.

Constructing confidence regions for unknown quantities in time series is a problem of substantial interest in statistics. In the statistical literature, the main focus has been on approximating the limiting distribution through finite sample procedures like subsampling and bootstrap; see Bühlmann (2002), Shao (2010), Nordman, Bunzel, and Lahiri (2013), Kim, Zhao, and Shao (2015), Huang and Shao (2016). From an entirely theoretical perspective, Youndjé and Vieu (2006) investigate consistency properties of kernel-type estimators of quantiles under long-range dependence. The interest in confidence regions is also due to their relevance for uncertainty quantification in other sciences where they are used in a variety of fields including climate science, economics, finance, industrial engineering and machine learning; see Massah and Kantz (2016), Fang, Xu, and Yang (2018), Hoga (2019), Purwanto and Sudargini (2021).

Empirical process theory became one of the major themes in the historical progress of non-parametric statistics; see Donsker (1952), Dudley (1978), Doukhan and Surgailis (1998), Shorack and Wellner (2009), Wellner and van der Vaart (2013). The applications are manifold, especially since many statistics have a representation as functionals of the empirical process, such that statistical inference can be based on the properties of the empirical process itself. In the empirical sciences, confidence regions for unknown parameters or critical values for hypothesis tests are derived from the distributional properties of the empirical process.

For stationary Gaussian processes Koul and Surgailis (2002) derived the asymptotic distribution of the higher-order terms of the empirical process. We extend their results substantially by considering the sequential empirical process and by allowing the underlying time series to be driven by subordinated Gaussian processes. Subordination extends the models flexibility by allowing for a large class of marginal distributions. Furthermore, we are the first to propose how to utilize the higher-order approximation of the empirical process to calculate confidence regions in time series which are robust to high temporal correlation.

Although long-range dependent processes are a popular modeling tool in a variety of domains (Rust, Kallache, Schellnhuber, and Kropp (2011), Weron (2002)), the construction of confidence intervals under long-range dependence has not gotten much attention. We provide an empirical study comparing confidence regions derived from the asymptotic distribution of the empirical process with confidence regions based on our procedure.

For the population mean, Hall, Jing, and Lahiri (1998) proposes a sampling window method to set confidence intervals under long-range dependence. Nordman, Sibbertsen, and Lahiri (2007) consider the empirical likelihood for confidence intervals. For mean functions, Bagchi, Banerjee, and Stoev (2016) study a monotone function plus noise model with potential long-range dependence in the noise term and derive confidence intervals for the monotone functions. In contrast, we will deal with a different, rank-based class of statistics.

The literature review, as well as our motivation illustrated in Figure 1 show the strong influence of high temporal correlation on the performance of statistics derived from the empirical process. In this paper, we aim to address this issue by introducing a procedure based on a higher-order approximation of the empirical process to construct confidence regions for statistics of long-range dependent time series robust to high temporal correlation. Our theoretical contribution is of independent interest and potentially has further applications in change-point analysis and goodness-of-fit testing. Furthermore, it allows to use existing finite sample procedures to approximate the limiting distribution for short-range dependent time series under the assumption of long-range dependence.
The rest of the paper is organized as follows. In Section 2, we introduce our setting in all details. Section 3 motivates the consideration of the higher-order approximation of the empirical process. Section 4 focuses on our theoretical contribution which manifests the theoretical validity of our method. In Section 5, we discuss how to calculate confidence regions based on the asymptotic distribution of the empirical process and propose our approach based on a higher-order approximation. The numerical study in Section 6 provides a comparison between the two methods. We conclude with Section 7. The proofs of our theoretical results can be found in Appendices A, B, C, and D.

2 Preliminaries

While Section 1 provides insight into the motivation for considering higher-order approximations of statistics, this section introduces model assumptions which allow for this type of approximations (Section 2.1) and gives some technical details necessary for our analysis (Section 2.2).

2.1 Setting

For future reference, we subsume assumptions on the data-generating process under the following model specification:

**Model 2.1.** Let \( X_n, n \in \mathbb{N} \), be a subordinated Gaussian process, i.e. \( X_n = G(\xi_n) \) for some measurable function \( G : \mathbb{R} \rightarrow \mathbb{R} \) and with \( \xi_n, n \in \mathbb{N} \), denoting a (standardized) long-range dependent Gaussian process, i.e. \( \mathbb{E} \xi_n = 0, \var{\xi_n} = 1 \), and autocovariance function

\[
\gamma(k) = \text{Cov}(\xi_1, \xi_{k+1}) = \mathbb{E}(\xi_1 \xi_{k+1}) = k^{-D}L(k),
\]

where \( D \in (0,1) \) (the so-called long-range dependence (LRD) parameter) and \( L \) a slowly varying function.

Relation (2.1) corresponds to one of multiple different ways to define long-range dependence. A more general definition characterizes long-range dependent time series by the non-summability of the absolute values of its autocovariance function; see (2.1.6) in Pipiras and Taqqu (2017). In fact, the relation (2.1) implies that the series of the autocovariances diverges. We refer to Chapter 2.1 in Pipiras and Taqqu (2017) for a detailed representation of the different ways to define long-range dependence and how they relate to each other.

Relation (2.1) is satisfied whenever the slowly varying function \( L \) is quasi-monotone; see Definition 2.2.11 in Pipiras and Taqqu (2017) for a definition of quasi-monotonicity and the discussion therein.

For any particular distribution function \( F \), an appropriate choice of the transformation \( G \) yields subordinated Gaussian processes with marginal distribution \( F \). Moreover, there exist algorithms for generating Gaussian processes that, after suitable transformation, yield subordinated Gaussian processes with marginal distribution \( F \) and a predefined covariance structure; see Pipiras and Taqqu (2017).

The following example presents a process which satisfies Model 2.1.

**Example 2.2** (Definition 2.8.3 in Pipiras and Taqqu (2017)). Let \( B_H(t), t \in \mathbb{R} \), be a fractional Brownian motion. Then, the process \( \xi_H(k), k \in \mathbb{Z} \), defined by

\[
\xi_H(k) := B_H(k+1) - B_H(k)
\]

is called fractional Gaussian noise with Hurst parameter \( H \).
2.2 Functions of Gaussian Random Variables

In the study of functionals of Gaussian processes, Hermite polynomials play a fundamental role. In particular, they form a basis for the space of finite-variance functions of Gaussian random variables. Since they are an inevitable tool in our analysis, we provide a detailed review.

Let $L^2(\mathbb{R}, \varphi(x)dx)$ be a separable space of real-valued square integrable functions with respect to the measure $\varphi(x)dx$ associated with the standard normal density function $\varphi$ with seminorm

$$\|g\|_2^2 := \int_{-\infty}^{\infty} g^2(x)\varphi(x)dx, \ g \in L^2(\mathbb{R}, \varphi(x)dx).$$

Then, $L^2(\mathbb{R}, \varphi(x)dx)$ can be defined as the corresponding Hilbert space of equivalence classes of $\varphi(x)dx$-almost everywhere equal functions with inner product

$$\langle g_1, g_2 \rangle_2 := \int_{-\infty}^{\infty} g_1(x)g_2(x)\varphi(x)dx = E g_1(\xi)g_2(\xi)$$

with $\xi$ denoting a standard normally distributed random variable. Subordinated Gaussian random variables $X_n = g(\xi_n)$, $n \in \mathbb{N}$, can be considered as elements of the Hilbert space $L^2(\mathbb{R}, \varphi(x)dx)$. A collection of orthogonal elements in $L^2(\mathbb{R}, \varphi(x)dx)$ is given by the sequence of Hermite polynomials; see Proposition 5.1.3 in Pipiras and Taqqu (2017).

**Definition 2.3.** For $n \geq 0$, the Hermite polynomial of order $n$ is defined by

$$H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}, \ x \in \mathbb{R}.$$  

Orthogonality of the sequence $H_n$, $n \geq 0$, in $L^2(\mathbb{R}, \varphi(x)dx)$ follows from

$$\langle H_n, H_m \rangle_2 = \begin{cases} n! & \text{if } n = m, \\ 0 & \text{if } n \neq m; \end{cases}$$

see Proposition 5.1.1 in Pipiras and Taqqu (2017). Moreover, it can be shown that the Hermite polynomials form an orthogonal basis of $L^2(\mathbb{R}, \varphi(x)dx)$. As a result, every $g \in L^2(\mathbb{R}, \varphi(x)dx)$ has an expansion in Hermite polynomials, i.e. for $g \in L^2(\mathbb{R}, \varphi(x)dx)$ and $\xi$ standard normally distributed, we have

$$g(\xi) = \sum_{r=0}^{\infty} \frac{J_r(g)}{r!} H_r(\xi), \ J_r(g) := \langle g, H_r \rangle_2 = E g(\xi)H_r(\xi), \quad (2.2)$$

where $J_r(g), r \geq 0$, are the so-called Hermite coefficients. Equation (2.2) holds in an $L^2$-sense, meaning

$$\lim_{n \to \infty} \left\| g(\xi) - \sum_{r=0}^{n} \frac{J_r(g)}{r!} H_r(\xi) \right\|_2 = 0,$$

where $\| \cdot \|_2$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_2$.

Given the Hermite expansion (2.2), it is possible to characterize the dependence structure of subordinated Gaussian time series $g(\xi_n)$, $n \in \mathbb{N}$. In fact, it holds that

$$\text{Cov}(g(\xi_1), g(\xi_{k+1})) = \sum_{r=1}^{\infty} \frac{J_r^2(g)}{r!} (\gamma(k))^r, \quad (2.3)$$
where $\gamma$ denotes the autocovariance function of $\xi_n$, $n \in \mathbb{N}$; see Proposition 5.1.4 in Pipiras and Taqqu (2017). Under the assumption that, as $k$ tends to $\infty$, $\gamma(k)$ converges to 0 with a certain rate, the asymptotically dominating term in the series (2.3) is the summand corresponding to the smallest integer $r$ for which the Hermite coefficient $J_r(g)$ is non-zero. This index, which decisively depends on $g$, is called Hermite rank.

**Definition 2.4** (Definition 5.2.1 in Pipiras and Taqqu (2017)). Let $g \in L^2(\mathbb{R}, \varphi(x)dx)$ with $\mathbb{E}g(\xi) = 0$ for standard normally distributed $X$ and let $J_r(g)$, $r \geq 0$, be the Hermite coefficients in the Hermite expansion of $g$. The smallest index $k \geq 1$ for which $J_k(g) \neq 0$ is called the Hermite rank of $g$, i.e.

$$r := \min \{k \geq 1 : J_k(g) \neq 0\}.$$ 

### 3 Higher-order approximation

We utilize our model assumptions and give details on a characterization of the empirical process as a sum of first- and higher-order terms.

Given time series data $X_1, \ldots, X_N$ stemming from a subordinated Gaussian process $X_n$, $n \in \mathbb{N}$, according to Model 2.1 with marginal distribution function $F$, we are interested in characterizing higher-order approximations of the sequential empirical process

$$e_N(t, x) := \sum_{n=1}^{[Nt]} \left(1_{\{X_n \leq x\}} - F(x)\right), \quad t \in [0, 1], \quad x \in \mathbb{R}. \quad (3.1)$$

Higher-order approximations can be derived through the Hermite expansion

$$1_{\{X_n \leq x\}} - F(x) = \sum_{l=r}^{\infty} \frac{\tilde{c}_l(x)}{l!} H_l(\xi_n),$$

where $\tilde{c}_l(x) = \mathbb{E}\left(1_{\{G(\xi_0) \leq x\}} H_l(\xi_0)\right)$ and $r$ denotes the corresponding Hermite rank

$$r := \min_{x \in \mathbb{R}} r(x) \quad \text{with} \quad r(x) := \min\{q \geq 1 : \tilde{c}_q(x) \neq 0\}.$$ 

Dehling and Taqqu (1989) show that the first summand of this expansion determines the asymptotic distribution of the empirical process through the reduction principle

$$\frac{1}{d_{N,r}} \sum_{n=1}^{N} \left(1_{\{X_n \leq x\}} - F(x)\right) = \frac{\tilde{c}_r(x)}{r!} \frac{1}{d_{N,r}} \sum_{n=1}^{N} H_r(\xi_n) + o_P(1); \quad (3.2)$$

see Taqqu (1975).

In order to study higher-order terms, we utilize the following simple observation

$$\sum_{n \in \mathbb{N}} |\text{Cov}(H_l(\xi_1), H_l(\xi_{n+1}))| = l! \sum_{n \in \mathbb{N}} |\gamma(n)|^l \begin{cases} = \infty, & lD < 1, \\ < \infty, & lD > 1; \end{cases} \quad (3.3)$$

see equation (5.1.1) in Pipiras and Taqqu (2017) for the first equality in (3.3). Then, distinguishing the two cases in (3.3), the last relation is a consequence of (2.1) and our assumption $\frac{1}{lD} \notin \mathbb{N}$. The absolute convergence of the series of autocovariances is another way to differentiate short- and long-range dependence. While convergence indicates exponential decay of the autocovariance function, divergence is a consequence of power-like decaying autocovariances.
a result, the sequence $H_l(\xi_n), n \in \mathbb{N}$, can be considered as long-range dependent when $lD < 1$, while as short-range dependent when $lD > 1$. Moreover, it holds that

$$\frac{\widetilde{c}_l(x)}{l!} \sum_{n=1}^{N} H_l(\xi_n) = \mathcal{O}_P(N^H L^{\frac{1}{2}}(N)) \text{ for } l < \frac{1}{D}, \text{ while } \frac{\widetilde{c}_l(x)}{l!} \sum_{n=1}^{N} H_l(\xi_n) = \mathcal{O}_P(\sqrt{N}) \text{ for } l > \frac{1}{D},$$

(3.4)

where the memory parameter $D$ corresponds to the Hurst parameter $H$ through the relation $H = 1 - \frac{D}{2}$.

Motivated by the behavior of the series over the autocovariances in (3.3) and the different convergence rates in (3.4), we separate

$$\sum_{l=r}^{\infty} \frac{\widetilde{c}_l(x)}{l!} H_l(\xi_n) = L_n(x) + S_n(x),$$

where

$$L_n(x) = \sum_{l=\lceil \frac{r}{D} \rceil}^{\infty} \frac{\widetilde{c}_l(x)}{l!} H_l(\xi_n) \text{ and } S_n(x) = \sum_{l=0}^{\infty} \frac{\widetilde{c}_l(x)}{l!} H_l(\xi_n).$$

(3.5)

Based on (3.3), the series over autocovariances of $L_n(x)$ diverges, while $S_n(x)$ has an absolutely summable autocovariance function. We refer to $L_n(x)$ in (3.5) as “lower-order term” and to $S_n(x)$ as “higher-order term”.

For the empirical process (3.1), higher-order approximations result from

$$\frac{1}{N} e_N(t, x) = \frac{1}{N} \sum_{n=1}^{[Nt]} L_n(x) + \frac{1}{N} \sum_{n=1}^{[Nt]} S_n(x).$$

Based on the previous considerations, the two summands are expected to converge with different convergence rates. For our purpose, we aim at proving the convergence of $\frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} S_n(x)$ parameterized in $t$ and $x$.

To illustrate the observations made in this section, we turn to Figure 2. Figure 2 depicts the Hurst parameter $H \in (\frac{1}{4}, 1)$ on the $x$-axis and the corresponding number of summands $\lceil \frac{r}{D} \rceil$ with $D = 2 - 2H$ which contribute to the long-range dependent part $L_n(x)$ in (3.5). Note that the number of summands contributing to the lower-order term increases exponentially with the value of the Hurst parameter, while the interval length, i.e. the length of the subintervals of $(\frac{1}{4}, 1)$ which correspond to a certain number of summands, decreases.

4 Main result

In this section, we present our main technical contributions. Our main result is stated in Section 4.1, followed by a layout of the proof ideas in Section 4.2.
4.1 Statement

We establish a limit theorem for the higher-order term in the decomposition of the sequential empirical process in two parameters. For this, recall that

$$\frac{\lfloor Nt \rfloor}{\sqrt{N}} (F_{\lfloor Nt \rfloor}(x) - F(x)) = \sum_{l=r}^{\lfloor Nt \rfloor} (-1)^{l} N^{-\frac{1}{p}} L_{l}^{\frac{1}{p}}(N) Z_{N}^{(l)}(t, x) + \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} S_{n}(x)$$

with $Z_{N}^{(l)}(t, x) = N^{\frac{1}{2p}} \sum_{n=1}^{[Nt]} \tilde{c}_{n}(x) \frac{H_{l}(\xi_{n})}{l!}$. According to Theorem 5.3.1 in Pipiras and Taqqu (2017), if suitably standardized, each of the first $[\frac{N}{2}] - r + 1$ summands converges to a Hermite process of order $l$. More precisely, it holds that

$$Z_{N}^{(l)}(t, x) = N^{\frac{1}{2p}} \sum_{n=1}^{[Nt]} \tilde{c}_{n}(x) \frac{H_{l}(\xi_{n})}{l!} \xrightarrow{D} \frac{\ell_{l,H}}{l!} \beta_{l,H} Z_{H}^{(l)}(t)$$

in $D([-\infty, \infty] \times [0,1])$, where $\beta_{l,H}$ is a constant and $Z_{H}^{(l)}(t)$, $t \in [0,1]$, an Hermite process of order $l$ with self-similarity parameter $H = 1 - \frac{1}{2p}$. The limit of the higher-order term $\frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} S_{n}(x)$ is characterized by the following theorem.

**Theorem 4.1.** Suppose $X_{n}, n \in \mathbb{N}$, satisfies Model 2.1 and $X_{n}$ has a strictly monotone distribution function $F$ with density $f \in L^{p}$ for some $p > 1$ and $\frac{1}{p} \notin \mathbb{N}$. Then, as $N \to \infty$,

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} S_{n}(x) \xrightarrow{D} S(x, t)$$

in $D([-\infty, \infty] \times [0,1])$, where $S(x, t)$ is a mean zero Gaussian process with cross-covariances

$$\text{Cov}(S(x, t), S(y, u)) = \min\{t, u\} \sum_{n \in \mathbb{Z}} \text{Cov}(S_{0}(x), S_{n}(y)).$$  (4.1)

The proof of Theorem 4.1 can be found in Appendix A.

**Remark 4.2.** Note that we exclude the case $\frac{1}{p} \in \mathbb{N}$. That excludes in particular the case $D = \frac{1}{2}$, that is when the underlying time series is short-range dependent. Therefore, our result as it is stated, cannot recover existing results for short-range dependent time series. Under short-range dependence, the empirical process is known to converge to the so-called Kiefer-Müller process; see Müller (1970), Kiefer (1972).

4.2 Proof

While the detailed proof of Theorem 4.1 is given in Appendices A, B, and C, we aim here to provide a roadmap of our proofs and to emphasize some of the main technical challenges.

Not surprisingly, the proof requires convergence of the finite-dimensional distributions and tightness; see Sections A.1 and A.2, respectively. While one does not face any major issues in proving convergence of the finite-dimensional distributions, the main technical challenges arise in the proof of tightness. Those challenges are subject of this section.

The partial sum of the higher-order terms in (3.5) is denoted as

$$\hat{m}_{N}(x, t) := \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} S_{n}(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} \sum_{l=[\frac{1}{p}]}^{\infty} \frac{\tilde{c}_{n}(x)}{l!} H_{l}(\xi_{n})$$  (4.2)
and is a process in two parameters, which is one reason why proving tightness becomes particularly challenging. Another challenge is the structure of the higher-order terms. In contrast to the empirical process, the higher-order terms are no longer bounded. While the transformed variables $H_l(\xi_n), n \in \mathbb{N}$, for $l \geq \lfloor \frac{\theta}{2} \rfloor$ are short-range dependent, the underlying process $\xi_n, n \in \mathbb{N}$, is still long-range dependent with non-summable autocovariance function. The dependence on the memory parameter $D$ appears in the summation, determining the number of summands going into the higher-order terms.

The articles Dehling and Taqqu (1989), Koul and Surgailis (2002) and El Ktaibi and Ivanoff (2016) are closest to our work. In the following layout of our proof, we will emphasize the differences to those works.

1. It is necessary to prove tightness in two parameters, more precisely, in the space $D([-\infty, \infty] \times [0, 1])$. Furthermore, we allow the underlying process to be subordinated Gaussian. This makes our proofs decisively different from the one in Koul and Surgailis (2002), who only considered (4.2) for fixed $t = 1$ and did not allow for subordinated transformations of the underlying Gaussian process.

2. The first step of our proof is to reduce tightness in $D([-\infty, \infty] \times [0, 1])$ to proving tightness in $D([0, 1] \times [0, 1])$. The corresponding object in $D([0, 1] \times [0, 1])$ can be written as

$$m_N(x, t) := \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} \sum_{l=\lfloor \frac{\theta}{2} \rfloor}^{\infty} c_l(x) H_l(\xi_n) \quad \text{with} \quad c_l(x) = E \left( \mathbb{1}_{\{F(\xi_0) \leq x\}} H_l(\xi_0) \right). \quad (4.3)$$

3. We use a tightness criterion introduced in Ivanoff (1980) and later utilized in El Ktaibi and Ivanoff (2016) to prove tightness of the sequential empirical process under short-range dependence. El Ktaibi and Ivanoff (2016) take advantage of the boundedness of the empirical process. Those techniques fail for (4.2) since the higher-order terms of the empirical process can no longer be represented as an indicator function.

4. In the main part of our proof, we reduce the tightness criterion in El Ktaibi and Ivanoff (2016) to bounding the probability

$$P \left( \bar{m}_{N,b}(y, x) > \lambda \right) \quad \text{with} \quad \bar{m}_{N,b}(x, y) := \sup_{t \in [0, b]} \left| m_N(y, t) - m_N(x, t) \right|$$

for some $b > 0$ and $x, y \in [0, 1]$ with $m_N(y, t)$ as in (4.3). Typically, such bounds are derived through chaining techniques. Dehling and Taqqu (1989) establish a corresponding argument for proving tightness of the empirical process of long-range dependent observations. For this, they take advantage of the reduction principle as stated in (3.2). The reduction principle reduces the problem to proving convergence of the partial sums of the dominating Hermite polynomial. Since none of the summands of the infinite series (4.2) is asymptotically negligible, the chaining technique of Dehling and Taqqu (1989) does not apply to the considered situation. Betken, Giraudo, and Kulik (2023) establish a chaining technique for proving tightness of the tail empirical process of Long Memory Stochastic Volatility (LMSV) time series. The major difference to our argument results from a martingale structure of the tail empirical process of LMSV time series. This allows to apply Freedman’s inequality, i.e. a Bernstein-type inequality for martingale difference sequences which, as well, does not apply to the situation in this paper.

5. A crucial part of the proof and second main technical contribution is to find a bound of the form

$$P \left( \bar{m}_{N,b}(y, x) > \lambda \right) \leq C_1 \frac{1}{\lambda^4} b^{2p-\theta} \frac{1}{N^{\theta}} (y - x) + C_2 \frac{1}{\lambda^4} b^2 (y - x)^{\frac{3}{2}}$$
for some $\theta > 0$, any $b > 0$ and all $x, y \in [0, 1]$. The result is formally stated in Lemma B.1. Our proof consists of two major parts. The first one is to extend Theorem 12.2 in Billingsley (1968). Theorem 12.2 in Billingsley (1968) provides a probabilistic bound for maxima over partial sums. In Lemma C.1 we provide a similar result, allowing the bound to take a more general form. The second main part of the proof is to verify the assumptions of Lemma C.1. Both, Lemma C.1 and Theorem 12.2 in Billingsley (1968) are written in great generality and do not impose any assumptions on the dependence structure of the underlying process. However, both are based on a probabilistic bound on the distances between partial sums. Given strong temporal dependence, as in our setting, verifying this condition becomes particularly challenging.

5 Confidence regions

In this section, we focus on how to utilize the higher-order approximation of the empirical process for the construction of confidence regions. To begin with, we determine confidence bands for values of the marginal distribution $F(x)$ of a time series $X_1, \ldots, X_N$ following Model 2.1. The confidence bands are based on the empirical analogue $F_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{X_i \leq x\}}$ of $F(x)$; see Section 5.1. Following this, we derive confidence intervals for quantiles of the marginal distribution; see Section 5.2. Section 5.3 provides a comparative discussion.

First and foremost, we are interested in how well these confidence regions approximate optimal confidence region. For this, note that the goodness of confidence regions can be assessed on the basis of the following two criteria:

C1: A high coverage probability, i.e. the probability that the true value of the estimated quantity lies in the considered confidence region should be high.

C2: A small area covered by the confidence region.

Based on these criteria we aim to compare the confidence regions derived from higher-order approximations of the empirical process to confidence regions that result from the asymptotic distribution of $F_N(x)$. Therefore, we will first rephrase how to compute the asymptotic confidence regions and then move on to introducing our approach to derive confidence regions.

For ease of computations, we base all analysis on the assumption that we are given a subordinated Gaussian time series $X_n = G(\xi_n)$, $n = 1, \ldots, N$, resulting from a strictly monotone function $G$. In this case, the Hermite rank $r$ equals 1 and the Hermite coefficients $\tilde{c}_1(x)$ can be determined analytically. In particular, it holds that

$$
\tilde{c}_1(x) = \begin{cases} 
-H_{-1}(G^{-1}(x))\varphi\left(G^{-1}(x)\right) & \text{if } G \text{ is increasing}, \\
H_{-1}(G^{-1}(x))\varphi\left(G^{-1}(x)\right) & \text{if } G \text{ is decreasing};
\end{cases}
$$

see Lemma D.2.

5.1 Confidence bands for the marginal distribution

Asymptotic confidence bands: For a construction of confidence bands based on the asymptotic distribution of the empirical process, note that

$$
\frac{N}{d_N} (F_N(x) - F(x)) \overset{D}{\to} \tilde{c}_1(x) Z,
$$

(5.1)
where $Z$ is a standard normally distributed random variable, $\tilde{c}_1(x) = E\left(\mathbf{1}_{\{G(\xi_0) \leq x\}}\xi_0\right)$ and $d_N^2 := \text{Var}\left(\sum_{i=1}^{N} \xi_i\right) \sim N^{2H^2} L(N)$; see Dehling and Taqqu (1989). Due to the fact that convergence in (5.1) holds in $D[-\infty, \infty]$, we have

$$1 - \alpha = \text{Pr}\left(\left|\tilde{c}_1(x)\right|^{\frac{-1}{N}} d_N(F_N(x) - F(x)) \in \left[\frac{z_1 - \alpha}{\sqrt{\frac{N}{2}}}, \frac{z_1 + \alpha}{\sqrt{\frac{N}{2}}}\right] \quad \forall x \in \mathbb{R}\right) + o(1),$$

where $z_\alpha := \Phi^{-1}(\alpha)$ and $\Phi$ denotes the standard normal distribution function. Therefore, an approximate $1 - \alpha$ confidence interval for $F(x)$ based on the asymptotic distribution of the empirical process is given by

$$\left(F_N(x) - \frac{d_N}{N}|\tilde{c}_1(x)|^{\frac{-1}{2}} z_1 - \frac{\sigma}{\sqrt{\frac{N}{2}}}, F_N(x) - \frac{d_N}{N}|\tilde{c}_1(x)|^{\frac{-1}{2}} z_1 + \frac{\sigma}{\sqrt{\frac{N}{2}}}\right).$$  (5.2)

Circling back to Example 2.2, the following example establishes these confidence bands (5.2) for fractional Gaussian noise.

**Example 5.1.** For fractional Gaussian noise time series with Hurst parameter $H$, $d_N \sim \sqrt{H(2H - 1)} N^{H}$ and $|\tilde{c}_1(x)| = \varphi(x)$, such that the band in (5.2) equals

$$\left(F_N(x) - \sqrt{H(2H - 1)} N^{H-\frac{1}{2}} \varphi(x) z_1 - \frac{\sigma}{\sqrt{\frac{N}{2}}}, F_N(x) - \sqrt{H(2H - 1)} N^{H-\frac{1}{2}} \varphi(x) z_1 + \frac{\sigma}{\sqrt{\frac{N}{2}}}\right).$$

Confidence bands based on the higher-order approximation: For a construction of confidence bands based on the higher-order approximation, note that according to Theorem 4.1

$$\sqrt{N} (F_N(x) - F(x)) - \frac{1}{\sqrt{N}} \sum_{n=1}^{N} L_n(x) \xrightarrow{D} Z(x),$$

where $Z(x)$ is normally distributed with mean zero and variance $\sigma^2(x) := \sum_{n \in \mathbb{Z}} \text{Cov}(S_0(x), S_n(x))$ and convergence hold in $D[-\infty, \infty]$. As a result, we have

$$1 - \alpha = \text{Pr}\left(\left(\sigma(x)\right)^{-1} \left(\sqrt{N} (F_N(x) - F(x)) - \frac{1}{\sqrt{N}} \sum_{n=1}^{N} L_n(x)\right) \in \left[\frac{z_1 - \alpha}{\sqrt{\frac{N}{2}}}, \frac{z_1 + \alpha}{\sqrt{\frac{N}{2}}}\right] \quad \forall x \in \mathbb{R}\right) + o(1).$$

Therefore, an approximate $1 - \alpha$ confidence band for $F(x)$ based on higher-order approximations of the empirical process is given by

$$\left(F_N(x) - \frac{1}{N} \sum_{n=1}^{N} L_n(x) - \frac{\sigma(x)}{\sqrt{\frac{N}{2}}} z_1 - \frac{\sigma}{\sqrt{\frac{N}{2}}}, F_N(x) - \frac{1}{N} \sum_{n=1}^{N} L_n(x) - \frac{\sigma(x)}{\sqrt{\frac{N}{2}}} z_1 + \frac{\sigma}{\sqrt{\frac{N}{2}}}\right).$$  (5.3)

Recall that

$$L_n(x) = \sum_{l=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \frac{1}{l!} \tilde{c}_l(x) H_l(\xi_n),$$

where the Hermite coefficients can be explicitly calculated as

$$\tilde{c}_l(x) = \begin{cases} -H_{l-1}(G^{-1}(x)) \varphi\left(G^{-1}(x)\right) & \text{if } G \text{ is increasing}, \\ H_{l-1}(G^{-1}(x)) \varphi\left(G^{-1}(x)\right) & \text{if } G \text{ is decreasing}; \end{cases}$$

see Lemma D.2.
5.2 Confidence intervals for quantiles

In this section, we establish confidence intervals for quantiles of the marginal distribution of long-range dependent time series. Initially, we describe the construction of confidence intervals for quantiles based on the convergence of the empirical process. Subsequently, we discuss the construction of confidence intervals for quantiles based on higher-order approximations of the empirical process.

Asymptotic confidence intervals: The asymptotic distribution of empirical quantiles can be derived from the asymptotic behavior of the empirical process (5.1) and an application of the delta method. In fact, Hössjer and Mielniczuk (1995) showed that for a functional \( \phi : (D[-\infty, \infty], \| \cdot \|_\infty) \to \mathbb{R} \), compactly differentiable at \( F \),

\[
\frac{N}{d_N} (\phi(F_N) - \phi(F)) \overset{D}{\to} Z \phi'(F)(\phi(F)) \tilde{c}_1(\phi(F)),
\]

where \( Z \) is a standard normally distributed random variable, \( \tilde{c}_1(x) = E(1_{[\xi_0 \leq x]} \xi_0) \) and \( d_N^2 := \text{Var} \left( \sum_{i=1}^{N} \xi_i \right) \sim N^{2H} L(N) \); see Theorem 1 in Hössjer and Mielniczuk (1995). Since our goal is to establish confidence intervals for \( \tau \)-quantiles \( q_\tau = \inf \{ x \mid F(x) \geq \tau \} \), we consider \( \phi : (D[-\infty, \infty], \| \cdot \|_\infty) \to \mathbb{R}, \phi(F) = F^{-1}(p) \). Given that \( r = 1 \), (5.4) corresponds to

\[
\frac{N}{d_N} (F_N^{-1}(p) - F^{-1}(p)) \overset{D}{\to} -Z \frac{1}{F'(F^{-1}(p))} \tilde{c}_1(F^{-1}(p)).
\]

As a result, we have

\[
1 - \alpha = \text{Pr} \left( -F'(F^{-1}(p))|\tilde{c}_1(F^{-1}(p))|^{-1} \frac{N}{d_N} (F_N^{-1}(p) - F^{-1}(p)) \in \left( z_{1-\alpha}, z_{1-\alpha} \right) \right) + o(1).
\]

Therefore, an approximate \( 1 - \alpha \) confidence interval for \( F^{-1}(p) \) based on the asymptotic distribution of the empirical process is given by

\[
\left( F_N^{-1}(p) - \frac{d_N}{N} \frac{1}{F'(F^{-1}(p))} |\tilde{c}_1(F^{-1}(p))| z_{1-\alpha}, F_N^{-1}(p) - \frac{d_N}{N} \frac{1}{F'(F^{-1}(p))} |\tilde{c}_1(F^{-1}(p))| z_{1-\alpha} \right).
\]

Example 5.2. For fractional Gaussian noise time series with Hurst parameter \( H \), \( d_N \sim \sqrt{H(2H-1)N^H} \) and \( \tilde{c}_1(x) = \varphi(x) \), such that the interval in (5.5) equals

\[
\left( F_N^{-1}(p) + \sqrt{H(2H-1)N^{H-1}z_{1-\alpha}}, F_N^{-1}(p) + \sqrt{H(2H-1)N^{H-1}z_{1-\alpha}} \right).
\]

Confidence intervals based on the higher-order approximation: We propose an alternative way to derive confidence intervals for the quantiles of the marginal distribution of long-range dependent time series based on higher-order approximations of the empirical process. Recall that quantiles can be written as a functional of the distribution \( F \) as well as their estimated counterparts. Based on Taylor approximation of the functional \( \phi \), we can then write

\[
\phi(F_N) - \phi(F) = \phi'(F)(F_N - F) + o_P(1);
\]

see Theorem 2.8 in Van der Vaart (2000). The right-hand side can be further simplified by

\[
\phi'(F)(F_N - F) = \frac{p - F_N(F^{-1}(p))}{F'(F^{-1}(p))} = -\frac{(F_N - F)(F^{-1}(p))}{F'(F^{-1}(p))};
\]
see p. 294 in Van der Vaart (2000). Under the assumption that the underlying time series has Gaussian marginals and for \( p = \frac{1}{2} \), which corresponds to the median, we get \( \phi(F_N - F) = -\Phi(F_N - F)^{10} \). Then, an approximate \( 1 - \alpha \) confidence interval of \( F^{-1}(p) \) can be written as

\[
\left( \phi(F_N) + \frac{1}{\varphi(F^{-1}(p))} \left( \frac{1}{N} \sum_{n=1}^{N} L_n(F^{-1}(p)) + \frac{1}{\sqrt{N}} \sigma(F^{-1}(p)) z_{1-\frac{\alpha}{2}} \right) \right),
\]

\[
\phi(F_N) + \frac{1}{\varphi(F^{-1}(p))} \left( \frac{1}{N} \sum_{n=1}^{N} L_n(F^{-1}(p)) + \frac{1}{\sqrt{N}} \sigma(F^{-1}(p)) z_{1-\frac{\alpha}{2}} \right); \tag{5.8}
\]

see Lemma D.1 and its proof for more details on the calculations.

5.3 Discussion

We would like to point out how the confidence regions based on the asymptotic distributions (5.5) differ from those we propose based on the higher-order approximation of the empirical process in (5.8). The major difference is that (5.8) only depends on the Hurst parameter \( H \) through \( L_n \) as in (5.3). In particular, the ceiling function applied to \( H \) determines the number of summands included in estimation. In other words, the confidence regions are less sensitive to small estimation errors. Given that inference on long-range dependent time series relies on how well one estimates the corresponding Hurst parameter, we expect that our confidence regions are more robust with respect to estimation errors in estimating the Hurst parameter.

In Section 5.2, we focused on deriving confidence intervals for the quantiles of the marginal distribution based on a higher-order approximation of the empirical process. The proposed procedure takes advantage of the Taylor expansion (5.6) of a general functional \( \phi \). Due to the generality of the results, we believe that similar results can be achieved for other estimators that are functionals of the empirical process like Huber’s estimator and M-functionals.

6 Numerical Studies

For our numerical studies, we consider the procedures proposed in Section 5. We compare the coverage rate as well as the length of asymptotic confidence region with those based on higher-order approximations. To assess the performance of the proposed procedure, we assume for simplicity that the underlying time series follows Model 2.1 with \( G = \id \), i.e. the time series is assumed to be long-range dependent with Gaussian marginals. We focus on confidence bands for the marginal distribution and confidence intervals for the median (Sections 6.2 and 6.3). Section 6.1 discusses the estimation of the long-run variance and the Hurst parameter.

6.1 Estimation

In order to compute the confidence regions discussed in Section 5, we need to estimate the long-run variance. Furthermore, we provide simulation results under the assumptions that the Hurst parameter \( H \) is known and unknown.

The long-run variance \( \sigma^2(x) := \sum_{n \in \mathbb{Z}} \text{Cov}(S_0(x), S_n(x)) \) cannot be computed analytically. In order to make our results applicable, we therefore need to estimate \( \sigma^2(x) \). For this, we use the kernel smoothing long-run variance estimator

\[
\hat{\sigma}^2(x) = \sum_{j=-(N-1)}^{N-1} K \left( \frac{j}{b_N} \right) \hat{\gamma}(j),
\]
where \( K(x) = (1 - |x|) \mathbf{1}_{(|x| \leq 1)} \) is the Bartlett kernel function, \( b_N \) denotes a bandwidth parameter and \( \hat{\gamma}_N(j) \) is the sample autocovariance at lag \( j \). For our simulation study, we use the command `hurstexp` in the R package `cointReg`. To determine the bandwidth, we use the command `getBandwidth`. For an estimation of the Hurst parameter \( H \) we used the R/S procedure following the description in Section 2.1 in Weron (2002). The estimator is implemented in `getLongRunVar` in the R package `pracma`.

6.2 Confidence bands for the marginal distribution

We construct confidence bands for the marginal distribution \( F \) based on the asymptotic distribution and based on higher-order approximations of the empirical process of long-range dependent time series. For a visual comparison of the two different methods see Figure 3. To numerically assess the quality of the computed bands, we report their coverage rate and width evaluated at different \( x \). In our simulation study, we consider different scenarios ranging from small to large sample sizes (\( N = 200 \) and \( N = 1000 \)) as well as from small to large Hurst parameters (\( H = 0.55 \) and \( H = 0.95 \)). The numerical studies show that even a significant increase of the sample size (from 200 to 1000) does not seem to have much impact on either coverage rate, nor width; see Figures 4, 5, 6, and 7. That said, we fix the sample size to \( N = 200 \) and compare Figures 4 and 6. Focusing on \( x = 0 \), one can observe that for larger Hurst parameters, the width increases significantly (from 0.016 to 0.96) for the asymptotic method. Naturally, the increase in width results in a higher coverage rate. Nonetheless, for the asymptotic method, the coverage rate does not exceed the value 0.3 at a significance level of 0.95. Confidence bands based on the proposed higher-order approximation method (HOA), however, are robust with respect to the value of the Hurst parameter and outperform the traditional construction of confidence bands with respect to the coverage rate; see Figures 4 and 6.

6.3 Confidence intervals for the median

In this section, we consider confidence intervals for the median based on long-range dependent time series characterized by different Hurst parameters. Again, we consider different scenarios
Figure 4: The coverage rate and bandwidth of confidence bands for the marginal distribution $F(x)$ evaluated at different $x$. The two displayed methods to calculate the confidence bands are based on the asymptotic distribution (asymp) and the higher-order approximation (HOA). The simulations are based on 2000 repetitions for Gaussian time series of length $N = 200$ with Hurst parameter $H = 0.55$. The dashed gray line depicts the significance level of 95%.

Figure 5: The coverage rate and length of confidence intervals for the marginal distribution $F$ evaluated at different $x$. The two displayed methods to calculate the confidence intervals are based on the asymptotic distribution (asymp) and the higher-order approximation (HOA). The simulations are based on 2000 repetitions for Gaussian time series of length $N = 1000$ with Hurst parameter $H = 0.55$. The dashed gray line depicts the significance level of 95%.
Figure 6: The coverage rate and length of confidence intervals for the marginal distribution $F$ evaluated at different $x$. The two displayed methods to calculate the confidence intervals are based on the asymptotic distribution (asymp) and the higher-order approximation (HOA). The simulations are based on 2000 repetitions for Gaussian time series of length $N = 200$ with Hurst parameter $H = 0.95$. The dashed gray line depicts the significance level of 95%.

Figure 7: The coverage rate and length of confidence intervals for the marginal distribution $F$ evaluated at different $x$. The two displayed methods to calculate the confidence intervals are based on the asymptotic distribution (asymp) and the higher-order approximation (HOA). The simulations are based on 2000 repetitions for Gaussian time series of length $N = 1000$ with Hurst parameter $H = 0.95$. The dashed gray line depicts the significance level of 95%.
ranging from small to large sample sizes ($N = 200$ and $N = 1000$) as well as from small to large Hurst parameters (from $H = 0.55$ to $H = 0.95$), and we assess the quality of the confidence intervals through interval length and coverage rate. Pointing towards Figures 8 and 9, which are based on sample sizes $N = 200$ and $N = 1000$, the sample size does not have any impact on coverage rate or interval length. Therefore, we focus on how varying the Hurst parameter influences the coverage rate and interval length. In this regard, Figures 8 and 9 clearly demonstrate that the length of a confidence interval constructed on the basis of the asymptotic distribution of the empirical process increases almost exponentially with increasing value of $H$. This may be attributed to the exponential increase of the number of summands needed to calculate the lower-order terms of the empirical process; see Figure 2. As a result, the coverage rate of confidence intervals increases when $H$ takes higher values. Nonetheless, it barely reaches the 0.95 significance level. In contrast to basing confidence intervals on the asymptotic distribution of the empirical process, Figures 8 and 9 illustrate robustness of the confidence interval lengths to different values of the Hurst parameter if the construction of confidence intervals is based on higher-order approximations of the empirical process. For Hurst parameters bigger than $H = 0.9$ a significant drop of the coverage rate can be observed. Again, this may be due to the exponential increase of the number of summands needed to calculate the lower-order terms of the empirical process; see Figure 2.

6.4 Estimated Hurst parameter

To make the construction of confidence regions based on higher-order approximations of the empirical process feasible for practical purposes, we need to consider the case where the Hurst parameter is unknown. As discussed in Section 6.1, we base estimation of the Hurst parameter on the so-called $R/S$-method. In this section, we focus on studying confidence intervals for the median since the consideration of different Hurst parameters illustrates how an estimated Hurst parameter changes the empirical coverage rates and lengths; see Figure 10.

Next, we compare the numerical results based on estimation of the Hurst parameter (Figure
10) with the numerical results that are based on the assumption that the Hurst parameter is known (Figures 8 and 9). It is notable that the lengths of confidence intervals resulting from approximation of the empirical process by its asymptotic distribution tend to be shorter when the Hurst parameter is estimated while their coverage rate is lower. Although the coverage rate of confidence intervals that are based on higher-order approximations of the empirical process and estimation of the Hurst parameter declines for Hurst parameters larger than 0.9, this effect is not as pronounced as for the simulations that assumed knowledge of the Hurst parameter. The reason for this lies in the fact that R/S estimation tends to underestimate the Hurst parameter and that the higher the value of the Hurst parameter, the bigger the estimation bias; see Taqqu, Teverovsky, and Willinger (1995).

7 Conclusions

In this work, we studied higher-order approximations of the empirical process as an approach to improving statistical inference for long-range dependent time series. More precisely, we study confidence regions for values of the empirical process and for quantiles of the marginal distribution of stationary time series that are based on an approximation of the empirical process through higher-order terms in its Hermite expansion.

The main theoretical contribution of this article is a proof for the convergence of higher-order terms in the Hermite expansion of the sequential empirical process for long-range dependent time series. This result is of general interest for empirical process theory and paves the way for novel approaches with respect to statistical inference for long-range dependent time series. First numerical approaches using the established theory illustrate an alternative way of constructing confidence regions based on long-range dependent observations.

In comparison to the construction of confidence regions based on the asymptotic distribution of the empirical process, the proposed procedure improves on the quality of confidence regions for the empirical process and quantiles of the marginal distribution. We envision to use similar
Figure 10: Coverage rate and interval length of confidence intervals for the median $F^{-1}(1/2)$ based on long-range dependent time series characterized by different Hurst parameters. For this, the distribution of the median is approximated by the asymptotic distribution (asymp) and by higher-order approximation (HOA) of the empirical process. The Hurst parameter is replaced by its $R/S$-estimator. Simulations are based on 2000 repetitions for Gaussian time series of length $N = 200$ (upper row) and $N = 1000$ (lower row). The dashed gray line depicts the significance level of 95%.

![Coverage rate and interval length of confidence intervals for the median](image)
techniques for change-point analysis and goodness-of-fit testing under long-range dependence. Generally speaking, our results provide the sufficient theoretical groundwork for utilization of higher-order approximations for statistical inference on long-range dependent time series.

A Proof of Theorem 4.1

In order to prove Theorem 4.1, we first investigate the convergence of the finite-dimensional distributions and then tightness in $D([-\infty, \infty] \times [0, 1])$; see Sections A.1 and A.2, respectively.

For the proof we will make use of the following notation

$$m_N (x, t) := \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} S_n(x)$$  \hspace{1cm} (A.1)

with $S_n(x)$ as in (3.5).

A.1 Convergence of the finite-dimensional distributions

We need to show convergence of the finite-dimensional distributions, i.e.

$$\tilde{m}_N (x, t) \overset{f.d.d.}{\rightarrow} S(x, t),$$

where $\{S(x, t)\}$ is the limiting process with cross-covariances given in (4.1). It then suffices to show that for all $q \in \mathbb{N}, a_1, \ldots, a_q \in \mathbb{R},$ and $(x_i, t_i) \in [-\infty, \infty] \times [0, 1], i=1, \ldots, q,$

$$\sum_{i=1}^{q} a_i \tilde{m}_N (x_i, t_i) \overset{D}{\rightarrow} \sum_{i=1}^{q} a_i S(x_i, t_i);$$

see Theorem 2 in Bickel and Wichura (1971). Note that this is equivalent to showing convergence of

$$\sum_{i=1}^{q} a_i m_N (x_i, t_i),$$

where $m_N (x, t) := \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} \sum_{l=\lfloor\frac{n}{T}\rfloor}^{\infty} \frac{c_l(x)}{l!} H_l(\xi_n),$  \hspace{1cm} (A.2)

for all $q \in \mathbb{N}, a_1, \ldots, a_q \in \mathbb{R},$ and $(x_i, t_i) \in [0, 1] \times [0, 1], i=1, \ldots, q.$ This can be shown by an application of Theorem 1 (ii) in Bardet and Surgailis (2013). Assume, without loss of generality, that $t_1 \leq t_2 \leq \ldots \leq t_q.$ We get

$$\sum_{i=1}^{q} a_i m_N (x_i, t_i) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt_0]} \sum_{i=1}^{q} f_{n,N}(\xi_n),$$

where, setting $t_0 = 0,$

$$f_{n,N}(\xi_n) := \sum_{i=1}^{q} \sum_{l=\lfloor\frac{n}{T}\rfloor}^{\infty} \frac{c_l(x_k)}{l!} H_l(\xi_n).$$

Let $\xi$ denote a standard normally distributed random variables. Define

$$\phi_r(\xi) := \sum_{i=1}^{q} h_i(\xi) 1_{\{t_{i-1} < r < t_i}\},$$
It then holds that
\[
\lim_{N \to \infty} \sup_{\tau \in (0,1)} E \left[ (f_{[N\tau],N}(\xi) - \phi_{\tau}(\xi))^2 \right] = 0.
\]
Moreover, we have
\[
\sigma^2_N := E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{[NT_2]} f_{n,N}(\xi_n) \right)^2 \right] = E \left[ \left( \sum_{i=1}^{q} a_i m_N(x_i, t_i) \right)^2 \right]
\]
\[
= \frac{1}{N} \sum_{i_1=1}^{q} \sum_{i_2=1}^{q} a_{i_1} a_{i_2} \sum_{n_1=1}^{[NT_{i_1}]} \sum_{n_2=1}^{[NT_{i_2}]} \sum_{l_1, l_2=\lfloor \frac{1}{N} \rfloor}^{\infty} c_{l_1}(x_{i_1}) c_{l_2}(x_{i_2}) \frac{1}{l_1! l_2!} E(H_{l_1}(\xi_{n_1}) H_{l_2}(\xi_{n_2}))
\]
\[
= \frac{1}{N} \sum_{i_1=1}^{q} \sum_{i_2=1}^{q} a_{i_1} a_{i_2} \sum_{n_1=1}^{[NT_{i_1}]} \sum_{n_2=1}^{[NT_{i_2}]} \sum_{l=\lfloor \frac{1}{N} \rfloor}^{\infty} c_l(x_{i_1}) c_l(x_{i_2}) \frac{1}{l!} \gamma^l(n_1 - n_2).
\]
Since \( r \) denotes the Hermite rank of \( I_{\{G(\xi) \leq x\}} \) and consequently, due to strict monotonicity of \( F \), also that of \( I_{\{F(G(\xi)) \leq x\}} \), and \( r \leq \lfloor \frac{1}{N} \rfloor \),
\[
\sum_{l=\lfloor \frac{1}{N} \rfloor}^{\infty} \frac{c_l^2(x)}{l!} \leq E \left[ \left( \sum_{l=r}^{\infty} \frac{c_l(x)}{l!} H_l(\xi) \right)^2 \right] = E \left( I_{\{F(G(\xi)) \leq x\}} \right) = x.
\]
For \( l \geq \lfloor \frac{1}{N} \rfloor \)
\[
\sum_{n \in \mathbb{Z}} |\gamma(n)|^l < \infty
\]
due to (3.3). As a result, we obtain
\[
\frac{1}{N} \sum_{n_1=1}^{[NT_{i_1}]} \sum_{n_2=1}^{[NT_{i_2}]} \gamma^l(n_1 - n_2) \to \min(t_{i_1}, t_{i_2}) \sum_{n \in \mathbb{Z}} \gamma^l(n)
\]
for all \( 1 \leq i_1, i_2 \leq q \). Consequently, \( \sigma^2_N \) converges to
\[
\sigma^2 = \sum_{i_1=1}^{q} \sum_{i_2=1}^{q} a_{i_1} a_{i_2} \min(t_{i_1}, t_{i_2}) \sum_{l=\lfloor \frac{1}{N} \rfloor}^{\infty} \frac{c_l(x_{i_1}) c_l(x_{i_2})}{l!} \sum_{n=-\infty}^{\infty} \gamma^l(n).
\]
According to Theorem 1 in Bardet and Surgailis (2013) it then follows that
\[
\sum_{i=1}^{q} a_i m_N(x_i, t_i) \xrightarrow{D} N(0, \sigma^2).
\]

### A.2 Tightness

Since the object of interest \( \hat{m}_N(x, t) \) in (A.1) is a process in two parameters, proving tightness becomes particularly challenging. We will first give a tightness criterion in \( D([-\infty, \infty] \times [0, 1]) \) and then argue that it suffices to prove tightness in \( D([0, 1] \times [0, 1]) \).

In order to prove tightness of \( \hat{m}_N(x, t) \) in \( D([-\infty, \infty] \times [0, 1]) \), we validate the following tightness criterion: for all \( \varepsilon > 0 \)
\[
\lim_{N \to \infty} \lim_{\delta \to 0} P \left( \sup_{|x_2-x_1| < \delta} \sup_{|t_2-t_1| < \delta} \sup_{x_1, x_2 \in \mathbb{R}} \sup_{0 \leq t_1, t_2 \leq 1} |\hat{m}_N(x_2, t_2) - \hat{m}_N(x_1, t_1)| > \varepsilon \right) = 0;
\]
see formula (26) in El Ktaibi and Ivanoff (2016). In a more general setting, the criterion was introduced in Ivanoff (1980). We further write

$$\tilde{m}_N(x_2, t_2) - \tilde{m}_N(x_1, t_1) = \tilde{m}_N(x_2, t_2) - \tilde{m}_N(x_1, t_2) + \tilde{m}_N(x_1, t_2) - \tilde{m}_N(x_1, t_1).$$

Then, it suffices to show

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup_{x_2, x_1 \in R} \left( \sup_{x_2-x_1 < \delta} \sup_{t \in [0, 1]} |\tilde{m}_N(x_2, t) - \tilde{m}_N(x_1, t)| > \epsilon \right) = 0, \tag{A.3}$$

and

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup_{x_2, x_1 \in R} \left( \sup_{|x_2-x_1| < \delta} \sup_{0 \leq t_1, t_2 \leq 1} |\tilde{m}_N(x_2, t_2) - \tilde{m}_N(x_1, t_1)| > \epsilon \right) = 0. \tag{A.4}$$

In order to prove (A.3) and (A.4), note that since $F$ is strictly monotone

$$c_l(x) = E \left( \mathbb{1}_{\{X_0 \leq x\}} H_l(\xi_0) \right) = E \left( \mathbb{1}_{\{F(X_0) \leq F(x)\}} H_l(\xi_0) \right) = c_l(F(x)).$$

where $c_l(x) = E \left( \mathbb{1}_{\{F(X_0) \leq x\}} H_l(\xi_0) \right)$. Recall from (A.2), that

$$m_N(x, t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \sum_{l=1}^{\infty} \frac{c_l(x)}{l!} H_l(\xi_n), \tag{A.5}$$

Since the density $f \in L^p$, $p > 1$, $F$ is Hölder continuous and it follows that the criteria (A.3) and (A.4) can be reformulated as

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup_{x_2, x_1 \in R} \left( \sup_{x_2-x_1 < \delta} \sup_{t \in [0, 1]} |m_N(x_2, t) - m_N(x_1, t)| > \epsilon \right) = 0, \tag{A.6}$$

and

$$\lim_{\delta \to 0} \lim_{N \to \infty} \sup_{x \in R} \left( \sup_{|t_2-t_1| < \delta} \sup_{0 \leq t_1, t_2 \leq 1} |m_N(x, t_2) - m_N(x, t_1)| > \epsilon \right) = 0. \tag{A.7}$$

with $m_N(x, t)$ as in (A.5).

We consider (A.6) and (A.7) separately. Both proofs are based on chaining techniques following the ideas in Dehling and Taqqu (1989, p. 1778) and Betken et al. (2023, Section 5.1.4).
A.2.1 Proof of (A.6).

In order to prove (A.6), we apply a chaining technique. For this, we define the intervals

\[ I_{1,p} := [2p\delta, 2(p+1)\delta] \quad \text{and} \quad I_{2,p} := [(2p+1)\delta, (2(p+1))\delta] \]

for \( p = 0, \ldots, L_\delta := \left[\frac{1}{2} - \frac{3}{2}\right]. \) Then, the expression inside \( P \) in (A.6) can be bounded as

\[
\sup_{0 \leq x_1, x_2 \leq 1} \sup_{t \in [0,1]} |m_N(x_2, t) - m_N(x_1, t)|
\leq \max_{0 \leq p \leq L_\delta} \sup_{x_1, x_2 \in I_{1,p}, t \in [0,1]} |m_N(x_2, t) - m_N(x_1, t)|
+ \max_{0 \leq p \leq L_\delta} \sup_{x_1, x_2 \in I_{2,p}, t \in [0,1]} |m_N(x_2, t) - m_N(x_1, t)|. \tag{A.8}
\]

In the following, we consider only the first summand in (A.8), since for the second summand analogous considerations hold. For this reason, it remains to show that

\[
\lim_{\delta \to 0} \limsup_{N \to \infty} \sup_{x \in [0,2\delta]} \left( \max_{0 \leq p \leq L_\delta} \sup_{x_1, x_2 \in I_{1,p}, t \in [0,1]} |m_N(x_2, t) - m_N(x_1, t)| > \varepsilon \right) = 0.
\]

For this, it suffices to show that

\[
\lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{\delta} \max_{0 \leq p \leq L_\delta} \sup_{x_1, x_2 \in I_{1,p}, t \in [0,1]} |m_N(x_2, t) - m_N(x_1, t)| > \varepsilon = 0.
\]

We write \( I_{1,p} = [a_p, a_{p+1}] \), i.e., \( a_p := 2p\delta \) and \( a_{p+1} := 2(p+1)\delta \). Note that

\[
\sup_{x_1, x_2 \in I_{1,p}, t \in [0,1]} |m_N(x_2, t) - m_N(x_1, t)| \leq 2 \sup_{x \in [0,2\delta]} \sup_{t \in [0,1]} |m_N(a_p, t) - m_N(a_p + x, t)|. \tag{A.9}
\]

Define refining partitions \( x_i(k) \) for \( k = 0, \ldots, K_N \) with \( K_N \to \infty \), for \( N \to \infty \), and

\[
x_i(k) := a_p + \frac{i}{2^k} 2\delta, \quad i = 0, \ldots, 2^k, \tag{A.10}
\]

and choose \( i_k(x) \) such that

\[
a_p + x \in (x_{i_k(x)}(k), x_{i_k(x)}+1(k)).
\]

We write

\[
\bar{m}_N(x, y) := \sup_{t \in [0,b]} |m_N(y, t) - m_N(x, t)|, \quad \bar{m}_N(x, y) := \bar{m}_N,1(x, y). \tag{A.11}
\]

Then, with help of the introduced partition (A.10), (A.9) can be bounded as

\[
\sup_{t \in [0,1]} |m_N(a_p, t) - m_N(a_p + x, t)| \\
\leq \sum_{k=1}^{K_N} \bar{m}_N(x_{i_k(x)}(k), x_{i_{k-1}(x)}(k-1)) + \bar{m}_N(x_{i_{K_N}(x)}(K_N), a_p + x). \tag{A.12}
\]

Consequently, (A.12) can be used to infer (A.13) below

\[
P \left( \sup_{x \in [0,2\delta]} \sup_{t \in [0,1]} |m_N(a_p, t) - m_N(a_p + x, t)| > \varepsilon \right)
\]
\begin{align*}
\sum_{k=1}^{K_N} \sum_{i=0}^{2^k-1} \Pr \left( \bar{m}_N(x_{i+1}(k), x_i(k)) > \frac{\varepsilon}{(k+3)^2} \right)
& \leq C \sum_{k=1}^{K_N} \sum_{i=0}^{2^k-1} \frac{(k+3)^8}{\varepsilon^4} \left( \frac{1}{N^{\theta}} (x_{i+1}(k) - x_i(k)) + (x_{i+1}(k) - x_i(k))^{\frac{2}{3}} \right) \\
& \leq C \sum_{k=1}^{K_N} \sum_{i=0}^{2^k-1} \frac{(k+3)^8}{\varepsilon^4} \left( \frac{1}{N^{\theta} 2^k} + \left( \frac{2\delta}{2^k} \right)^{\frac{2}{3}} \right) \\
& \leq C\delta \sum_{k=1}^{K_N} \frac{(k+3)^8}{\varepsilon^4} \left( \frac{1}{N^{\theta}} + C\delta \sum_{k=1}^{K_N} \frac{(k+3)^8}{\varepsilon^4} \right) \\
& \leq C\delta \frac{1}{\varepsilon^4} \left( \frac{1}{N^{\theta} 2^k} + \left( \frac{2\delta}{2^k} \right)^{\frac{2}{3}} \right) \\
& \leq C\delta \frac{1}{\varepsilon^4} \left( \frac{1}{N^{\theta}} + 2\delta \right) \\
\end{align*}

for sufficiently large \( N \), where (A.15) follows from Lemma B.1 with \( b = 1 \) and (A.16) is a simple consequence of the choice of our partition in (A.10). The last inequality (A.17) is then satisfied for large enough \( N \) since \( \sum_{k=1}^{\infty} \frac{(k+3)^8}{\varepsilon^4} \left( \frac{1}{N^{\theta} 2^k} + \left( \frac{2\delta}{2^k} \right)^{\frac{2}{3}} \right) < \infty \) by the ratio test for the convergence of series and \( \sum_{k=1}^{K_N} \frac{(k+3)^8}{\varepsilon^4} \frac{1}{N^{\theta}} \sim K_N \frac{1}{N^{\theta}} \to 0 \) choosing \( K_N \) such that \( K_N = o(N^\theta) \).

Now, we consider the second summand in (A.14). Choosing \( K_N \) such that \( K_N = o(N^\theta) \), but \( \frac{K_N}{\log_2(N)} \to \infty \), we get

\[
\lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^{\theta}} \left( \sup_{x \in [0,2^\delta]} \bar{m}_N(x_{i+1}(x)(K_N), a_p + x) > \frac{\varepsilon}{2} \right) \leq \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^{\theta}} \max \frac{1}{N^{\theta} 2^k} \left( \frac{N}{2K_N^5} \right) = 0
\]

for all \( N \geq N_\varepsilon \) by applying Lemma B.2 below with \( a = 2\delta, b = 1 \) and \( c = a \).

**Proof of (A.7):** In order to prove (A.7), we first split the interval over \( t_1, t_2 \) in (A.7) into subintervals. This allows to bound the quantity of interest in terms of a supremum over a single parameter \( t \) in a specific interval. We then apply a similar chaining technique as in the proof of (A.6). Note that here the chaining is applied to \( x \in [0,1] \).

To deal with the supremum over \( t_1, t_2 \) in (A.7), define

\[
I_{1,p} := [2p\delta, 2(p+1)\delta] \quad \text{and} \quad I_{2,p} := [(2p+1)\delta, (2p+1)+1)\delta]
\]

for all \( p \in \mathbb{Z} \).
for \( p = 0, \ldots, L_\delta := \lceil \frac{1}{25} - \frac{3}{7} \rceil \). We first note that the expression in \( P \) in (A.7) can be bounded through

\[
\sup_{0 \leq x \leq 1} \sup_{0 \leq t_1, t_2 \leq 1, |t_2 - t_1| \leq \delta} |m_N(x, t_2) - m_N(x, t_1)| \leq \sup_{0 \leq x \leq 1} \max_{0 \leq p \leq L_\delta} \sup_{t_1, t_2 \in I_{1,p}} |m_N(x, t_2) - m_N(x, t_1)| + \sup_{0 \leq x \leq 1} \max_{0 \leq p \leq L_\delta} \sup_{t_1, t_2 \in I_{2,p}} |m_N(x, t_2) - m_N(x, t_1)|.
\]

(A.18)

In the following, we consider only the first summand in (A.18), since for the second summand analogous considerations hold. For this reason, it remains to show that

\[
\lim_{\delta \to 0} \lim_{N \to \infty} P \left( \sup_{0 \leq x \leq 1} \max_{0 \leq p \leq L_\delta} \sup_{t_1, t_2 \in I_{1,p}} |m_N(x, t_2) - m_N(x, t_1)| > \varepsilon \right) = 0.
\]

We write \( I_{1,p} = [a_p, a_{p+1}] \), i.e. \( a_p := 2p\delta \) and \( a_{p+1} := 2(p + 1)\delta \). Note that

\[
\sup_{t_1, t_2 \in I_{1,p}} |m_N(x, t_2) - m_N(x, t_1)| \leq \sup_{t_2 \in I_{1,p}} |m_N(x, t_2) - m_N(x, a_p)| + \sup_{t_1 \in I_{1,p}} |m_N(x, a_p) - m_N(x, t_1)| \\
\leq 2 \sup_{t \in [0,2\delta]} |m_N(x, a_p) - m_N(x, a_p + t)|.
\]

(A.19)

For the supremum over \( x \in [0, 1] \), we apply a similar chaining technique as in the proof of (A.6). Define refining partitions \( x_i(k) \) for \( k = 0, \ldots, K_N \) with \( K_N \to \infty \), for \( N \to \infty \), and

\[
x_i(k) = \frac{i}{2^k}, \quad i = 0, \ldots, 2^k,
\]

(A.20)

and choose \( i_k(x) \) such that

\[
x \in \left( x_{i_k(x)}(k), x_{i_k(x)}+1(k) \right).
\]

Moreover, define

\[
m_N(x, t, p) := m_N(x, a_p) - m_N(x, a_p + t) \\
m_N(x, y, t, p) := m_N(y, t, p) - m_N(x, t, p)
\]

(A.21)

Then, continuing with (A.19), it follows that

\[
\max_{0 \leq p \leq L_\delta} \sup_{t \in [0,2\delta]} |m_N(x, a_p) - m_N(x, a_p + t)| \\
= \max_{0 \leq p \leq L_\delta} \sup_{t \in [0,2\delta]} |m_N(x, t, p)| \\
= \max_{0 \leq p \leq L_\delta} \sup_{t \in [0,2\delta]} |m_N(0, t, p) - m_N(x, t, p)|,
\]

since \( c_1(0) = E(1_{F(X_0) \leq 0} H_1(c_0)) = 0 \). We have

\[
\max_{0 \leq p \leq L_\delta} \sup_{t \in [0,2\delta]} |m_N(0, t, p) - m_N(x, t, p)| \\
\leq \sum_{k=1}^{K_N} \max_{0 \leq p \leq L_\delta} \sup_{t \in [0,2\delta]} |m_N(x_{i_k(x)}(k), x_{i_{k-1}(x)}(k-1), t, p)|
\]

25
Combining \((\text{A.24})\) below

\[
A.22 \quad \text{where } \bar{m}_N(x, y) := \max_{0 \leq p \leq L} \sup_{t \in [0, 2\delta]} |m_N(x, y, t, p)|. \text{ Consequently, (A.22) can be used to infer (A.23) below}
\]

\[
\begin{align*}
\mathbb{P} \left( \sup_{x \in [0, 1]} \max_{0 \leq p \leq L} \sup_{t \in [0, 2\delta]} |m_N(0, t, p) - m_N(x, t, p)| > \varepsilon \right) \\
&\leq \sum_{k=1}^{K} \mathbb{P} \left( \sup_{x \in [0, 1]} \bar{m}_N(x_0(x), x_{k-1}(x)(k-1)) > \frac{\varepsilon}{(k+3)^2} \right) \\
&\quad + \mathbb{P} \left( \sup_{x \in [0, 1]} \bar{m}_N(x_{iK}(x))(K_N), x > \varepsilon - \sum_{k=0}^{\infty} \frac{\varepsilon}{(k+3)^2} \right) \\
&\leq \sum_{k=1}^{K} \sum_{i=0}^{2^k} \mathbb{P} \left( \bar{m}_N(x_{i+1}(k), x_i(k)) > \frac{\varepsilon}{(k+3)^2} \right) \\
&\quad + \mathbb{P} \left( \sup_{x \in [0, 1]} \bar{m}_N(x_{iK}(x))(K_N), x > \frac{\varepsilon}{2} \right),
\end{align*}
\]

since \(\sum_{k=0}^{\infty} \frac{\varepsilon}{(k+3)^2} \leq \frac{\varepsilon}{2}\). We consider the two summands in (A.24) separately. For the first summand in (A.24) we need some preliminary results. Note that for any \(\eta > 0\),

\[
\begin{align*}
\mathbb{P} \left( \bar{m}_N(x_{i+1}(k), x_i(k)) > \eta \right) &= \mathbb{P} \left( \max_{0 \leq p \leq L} \sup_{t \in [0, 2\delta]} m_N(x_{i+1}(k), x_i(k), t, p) > \eta \right) \\
&\leq \sum_{p=0}^{L} \mathbb{P} \left( \sup_{t \in [0, 2\delta]} m_N(x_{i+1}(k), x_i(k), t, 0) > \eta \right). \quad \text{(A.25)}
\end{align*}
\]

Due to stationarity it follows that

\[
\begin{align*}
\mathbb{P} \left( \sup_{t \in [0, 2\delta]} m_N(x_{i+1}(k), x_i(k), t, p) > \eta \right) &= \mathbb{P} \left( \sup_{t \in [0, 2\delta]} m_N(x_{i+1}(k), x_i(k), t, 0) > \eta \right). \quad \text{(A.26)}
\end{align*}
\]

Combining (A.25) and (A.26), we get

\[
\begin{align*}
\mathbb{P} \left( \bar{m}_N(x_{i+1}(k), x_i(k)) > \eta \right) &\leq \frac{1}{\delta} \mathbb{P} \left( \sup_{t \in [0, 2\delta]} m_N(x_{i+1}(k), x_i(k), t, 0) > \eta \right). \quad \text{(A.27)}
\end{align*}
\]

Due to the notation in (A.21), we have

\[
\begin{align*}
m_N(x_{i+1}(k), x_i(k), t, 0) &= m_N(x_i(k), t, 0) - m_N(x_{i+1}(k), t, 0) \\
&= m_N(x_i(k), 0) - m_N(x_i(k), t) - (m_N(x_{i+1}(k), 0) - m_N(x_{i+1}(k), t))
\end{align*}
\]

such that

\[
\begin{align*}
\sup_{t \in [0, 2\delta]} |m_N(x_{i+1}(k), x_i(k), t, 0)| &\leq 2 \sup_{t \in [0, 2\delta]} |m_N(x_{i+1}(k), t) - m_N(x_i(k), t)|. 
\end{align*}
\]
We can then bound the first summand in (A.24), with further explanations given below, as follows
\[
\sum_{k=1}^{K_N} \sum_{i=0}^{2^k} P \left( \bar{m}_N(x_{i+1}(k), x_i(k)) > \frac{\varepsilon}{(k+3)^2} \right)
\leq \sum_{k=1}^{K_N} \sum_{i=0}^{2^k} \frac{1}{\delta} P \left( 2 \sup_{t \in [0,26]} |m_N(x_{i+1}(k), t) - m_N(x_i(k), t)| > \frac{\varepsilon}{(k+3)^2} \right) \tag{A.28}
\leq C \frac{1}{\delta} \sum_{k=1}^{K_N} \sum_{i=0}^{2^k} \frac{(k+3)^8}{\varepsilon^4} \left( \delta^{2-\theta} \frac{1}{N^\theta} (x_{i+1}(k) - x_i(k)) + \delta^2 (x_{i+1}(k) - x_i(k))^2 \right) \tag{A.29}
\leq \frac{1}{\delta} C \sum_{k=1}^{K_N} \sum_{i=0}^{2^k} \frac{(k+3)^8}{\varepsilon^4} + C \frac{1}{\delta^2} \sum_{k=1}^{K_N} \frac{(k+3)^8}{\varepsilon^4} \left( \frac{1}{2^k} \right)^2 \tag{A.30}
\leq C \max\{\delta^{1-\theta}, \delta\} \tag{A.31}
\]
for sufficiently large \(N\), where (A.28) is due to (A.27) and (A.29) follows from Lemma B.1 with \(b = \delta\), (A.30) is a simple consequence of the choice of our partition in (A.20). The last inequality (A.31) is then satisfied for large enough \(N\) since \(\sum_{k=1}^{\infty} \frac{(k+3)^8}{\varepsilon^4} \left( \frac{1}{N^\theta} \right)^2 \frac{1}{N^\theta} \sim K_N^9 = 0\) choosing \(K_N\) such that \(K_N^9 = o(N^\theta)\).

Now, we consider the second summand in (A.24).
\[
P \left( \sup_{x \in [0,1]} \bar{m}_{N,\delta}(x|K_N, x) > \frac{\varepsilon}{2} \right)
= P \left( \sup_{x \in [0,1]} \max_{0 \leq p \leq L_3} \sup_{t \in [0,26]} |m_N(x_{iK_N}(x), x, t, p)| > \frac{\varepsilon}{2} \right)
\leq \frac{1}{\delta} P \left( \sup_{x \in [0,1]} \sup_{t \in [0,26]} |m_N(x_{iK_N}(x), x, t)| > \frac{\varepsilon}{2} \right)
\leq \frac{1}{\delta} P \left( 2 \sup_{x \in [0,1]} \sup_{t \in [0,26]} |m_N(x_{iK_N}(x), x, t) - m_N(x, t)| > \frac{\varepsilon}{2} \right).
\]
Choosing \(K_N\) such that \(K_N^9 = o(N^\theta)\), but \(\frac{K_N}{\log_2(N)} \to \infty\), we get
\[
\frac{1}{\delta} P \left( 2 \sup_{x \in [0,1]} \bar{m}_{N,\delta}(x|K_N, x) > \frac{\varepsilon}{2} \right) \leq C \frac{1}{\varepsilon^4} \delta^\frac{1}{3} \max \left\{ \frac{1}{N^\theta}, \frac{N}{2^{K_N^9}} \right\} \to 0 \quad \text{as} \quad \delta \to 0,
\]
for all \(N \geq N_c\) by applying Lemma B.2 below with \(a = 1\), \(b = \delta\) and \(c = 0\). \qed

B Technical results and their proofs

In this section, we provide some technical results and their proofs.

Lemma B.1. Let \(\bar{m}_{N,\delta}\) be as in (A.11). Then, there are constants \(C_1, C_2 > 0\) and a \(\theta \in (0, \frac{1}{3}]\) such that for any \(\lambda > 0\),
\[
P (\bar{m}_{N,\delta}(x, y) > \lambda) \leq C_1 \frac{1}{\lambda^2} b^{2-\theta} \frac{1}{N^\theta} (y - x) + C_2 \frac{1}{\lambda^2} b^2 (y - x)^2 \tag{B.1}
\]
for any $b > 0$ and all $x, y \in [0, 1]$ with $y > x$.

Proof: In order to bound the probability in (B.1), we use arguments from Billingsley (1968). For this, we express $\bar{m}_{N,b}$ in (A.11) as

$$
\bar{m}_{N,b}(x,y) = \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \left| \sum_{n=1}^{[Nt]} \sum_{l=1}^{\lfloor \frac{1}{y} \rfloor} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_n) \right|
$$

$$
= \max_{1 \leq k \leq \lceil Nb \rceil} \frac{1}{\sqrt{N}} \left| \sum_{n=1}^{k} \sum_{l=\left\lfloor \frac{1}{y} \right\rfloor}^{\infty} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_n) \right|
= \max_{1 \leq k \leq \lceil Nb \rceil} |s_k|.
$$

Note that

$$
s_j - s_i = \frac{1}{\sqrt{N}} \sum_{n=1}^{j} \sum_{l=\left\lfloor \frac{1}{y} \right\rfloor}^{\infty} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_n)
$$

and define

$$
h_{x,y}(\xi_n) := \sum_{l=\left\lfloor \frac{1}{y} \right\rfloor}^{\infty} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_n) = \mathbb{1}_{\{ x < F(G(\xi_n)) \leq y \}} - \sum_{l=r}^{1} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_n).
$$

Then,

$$
E |s_j - s_i|^4 = E \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{j} h_{x,y}(\xi_n) \right)^4
$$

$$
= \frac{1}{N^2} E (\Sigma_1 + 4\Sigma_{21} + 3\Sigma_{22} + 6\Sigma_3 + \Sigma_4)
$$

with

$$
\Sigma_1 := \sum_{n=1}^{j} h_{x,y}^4(\xi_n),
$$

$$
\Sigma_{21} := \sum_{n=1}^{j} h_{x,y}(\xi_n) h_{x,y}(\xi_{n+1}),
$$

$$
\Sigma_{22} := \sum_{n=1}^{j} h_{x,y}^2(\xi_n) h_{x,y}^2(\xi_{n+1}),
$$

$$
\Sigma_3 := \sum_{n=1}^{j} h_{x,y}(\xi_n) h_{x,y}(\xi_{n+1}) h_{x,y}(\xi_{n+2}),
$$

$$
\Sigma_4 := \sum_{n=1}^{j} h_{x,y}(\xi_n) h_{x,y}(\xi_{n+1}) h_{x,y}(\xi_{n+2}) h_{x,y}(\xi_{n+3}),
$$

where $\sum'$ extends over all different indices $i + 1 \leq n_1, \ldots, n_p \leq j$, $n_r \neq n_s$, $r \neq s$, $p = 1, \ldots, 4$.

Note that for any even integer $p \geq 2$, there is a constant $C > 0$ such that

$$
E (h_{x,y}^p(\xi_0)) \leq C (y - x)
$$

(B.3)

since

$$
E (h_{x,y}^p(\xi_0)) \leq C \left( E \mathbb{1}_{\{ x < F(G(\xi_0)) \leq y \}} + E \left( \sum_{l=r}^{1} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_0) \right)^p \right)
$$

$$
\leq C \left( E \mathbb{1}_{\{ x < F(G(\xi_0)) \leq y \}} + \sum_{l=r}^{1} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_0) \right)^p E (|H_l(\xi_0)|^p)
$$

(B.4)
It then holds that
\[
E \left( \mathbf{1}_{\{x < F(G(x)) \leq y\}} \right) \leq C (y - x), \tag{B.6}
\]
where \( C \) is a generic constant that can change upon each appearance. Inequality (B.4) follows from
\[
\left( \sum_{k=1}^{n} |x_k| \right)^p \leq n^{p-1} \sum_{k=1}^{n} |x_k|^p
\]
which holds for any \( p \geq 1 \) and is a direct consequence of Hölder’s inequality. Inequality (B.5) follows by Nelson’s inequality; see Nourdin and Rosiński (2014) Lemma 2.1. Finally, (B.6) is a consequence of applying the Cauchy-Schwarz inequality
\[
(c_l(y) - c_l(x))^2 = E^2 \left( \mathbf{1}_{\{x < F(G(x)) \leq y\}} H_l(\xi_0) \right) \leq (y - x) E \left( H_l^2(\xi_0) \right) = (y - x)!
\]
and by noticing that \( (y - x)^{\frac{p}{2}} \leq y - x \) for \( p \geq 2 \) and \( x, y \in (0, 1) \).

We now consider the summands on the right-hand side of formula (B.2) separately. Starting with \( \Sigma_1 \), note that (B.3) gives
\[
E |\Sigma_1| \leq C(j - i)(y - x).
\]
In order to estimate the remaining quantities, we make use of Lemma 2.2 in Koul and Surgailis (2002). This together with (B.3) above, immediately yields
\[
E |\Sigma_2| \leq C(j - i)^{\frac{3}{2}} \left( E \left( h_{x,y}^2(\xi_0) \right) \right)^{\frac{1}{2}} \left( E \left( h_{x,y}^6(\xi_0) \right) \right)^{\frac{1}{2}} \leq C(j - i)^{\frac{3}{2}}(y - x),
E |\Sigma_3| \leq C(j - i)^2 E \left( h_{x,y}^2(\xi_0) \right) \left( E \left( h_{x,y}^4(\xi_0) \right) \right)^{\frac{1}{2}} \leq C(j - i)^2(y - x)^{\frac{3}{2}},
E |\Sigma_4| \leq C(j - i)^2 E^2 \left( h_{x,y}^2(\xi_0) \right) \leq C(j - i)^2(y - x)^2.
\]
It remains to find an upper bound for \( E \Sigma_{22} \). For this, define
\[
L_{x,y}(\xi_n) := \frac{1}{l!} \sum_{i=r}^{l} \frac{c_i(y) - c_i(x)}{l!} H_l(\xi_n) \quad \text{and} \quad \tilde{h}_{x,y}(z) = \mathbf{1}_{\{x < F(G(z)) \leq y\}} - (y - x).
\]
It then holds that
\[
E \left( h_{x,y}^2(\xi_n_1) h_{x,y}^2(\xi_n_2) \right) = E \left( \tilde{h}_{x,y}(\xi_n_1) - L_{x,y}(\xi_n_1) \right)^2 \left( \tilde{h}_{x,y}(\xi_n_2) - L_{x,y}(\xi_n_2) \right)^2 \leq 4 \left( E \left( \tilde{h}_{x,y}^2(\xi_n_1) + L_{x,y}^2(\xi_n_1) \right) \left( \tilde{h}_{x,y}^2(\xi_n_2) + L_{x,y}^2(\xi_n_2) \right) \right)^{\frac{1}{2}}
\leq 4 \left( E \left( \tilde{h}_{x,y}^2(\xi_n_1) \tilde{h}_{x,y}^2(\xi_n_2) \right) + E \left( \tilde{h}_{x,y}^2(\xi_n_1) L_{x,y}^2(\xi_n_2) \right) + E \left( \tilde{h}_{x,y}^2(\xi_n_2) L_{x,y}^2(\xi_n_1) \right) + E \left( L_{x,y}^2(\xi_n_1) L_{x,y}^2(\xi_n_2) \right) \right).
\]
Before we consider the four summands in (B.8) separately, we make the following observation. With arguments given below,
\[
E \left( L_{x,y}^4(\xi_n) \right) = E \left( \sum_{i=r}^{l} \frac{c_i(y) - c_i(x)}{l!} H_l(\xi_n) \right)^4
\]
\[ C \sum_{l=r}^{\infty} \frac{(c_l(y) - c_l(x))^4}{l!} 3^{2l}, \quad (B.10) \]

where (B.9) follows by H"older’s inequality and (B.10) by Nelson’s inequality; see Nourdin and Rosiński (2014) Lemma 2.1. Then, combining (B.10) with (B.7),

\[ E(L^4_{x,y}(\xi_n)) \leq C(y - x)^2. \quad (B.11) \]

In the following, we consider the summands in (B.8) separately. For the last one, the Cauchy-Schwarz inequality and (B.11) yield

\[ E(L^2_{x,y}(\xi_n) L^2_{x,y}(\xi_n)) \leq (E(L^4_{x,y}(\xi_n)))^{\frac{1}{2}} (E(L^4_{x,y}(\xi_n)))^{\frac{1}{2}} \leq C(y - x)^2. \quad (B.12) \]

Since \( E(\tilde{h}^2_{x,y}(\xi_n)L^2_{x,y}(\xi_n)) \) and \( E(L^2_{x,y}(\xi_n)\tilde{h}^2_{x,y}(\xi_n)) \) in (B.8) can be treated analogously, we only consider \( E(\tilde{h}^2_{x,y}(\xi_n)L^2_{x,y}(\xi_n)) \). Given the definition of \( \tilde{h}^2_{x,y} \) in (B) and with further explanations provided below, we get

\[ E(\tilde{h}^2_{x,y}(\xi_n)L^2_{x,y}(\xi_n)) = E((\mathbb{1}_{\{x < F(\xi_n)\} \leq y} - (y - x))^2 L^2_{x,y}(\xi_n)) \\
= E(\mathbb{1}_{\{x < F(\xi_n)\} \leq y} L^2_{x,y}(\xi_n)) + (y - x)^2 E(L^2_{x,y}(\xi_n)) \\
+ 2(y - x)E(\mathbb{1}_{\{x < F(\xi_n)\} \leq y} L^2_{x,y}(\xi_n)) \leq C(y - x)^{\frac{3}{2}} + C(y - x)^3 + C(y - x)^{\frac{5}{2}} \leq C(y - x)^{\frac{3}{2}}. \quad (B.13) \]

For the first and third summand in (B.13), Cauchy-Schwarz inequality and (B.11) yield

\[ E(\mathbb{1}_{\{x < F(\xi_n)\} \leq y} L^2_{x,y}(\xi_n)) \leq \left( E\left( \mathbb{1}_{\{x < F(\xi_n)\} \leq y}\right) \right)^{\frac{1}{2}} \left( E(L^4_{x,y}(\xi_n)) \right)^{\frac{1}{2}} \leq C(y - x)^{\frac{1}{2}}. \]

Moreover, we have, by orthogonality of the Hermite polynomials, that the second summand in (B.13) can be bounded as

\[ E(L^2_{x,y}(\xi_n)) = E \left( \sum_{l=r}^{\infty} \frac{c_l(y) - c_l(x)}{l!} H_l(\xi_n) \right)^2 = \sum_{l=r}^{\infty} \frac{(c_l(y) - c_l(x))^2}{(l!)^2} E H_l^2(\xi_n) \leq C(y - x). \]

The first summand in (B.8) requires some more calculations, leading to

\[ E(\tilde{h}^2_{x,y}(\xi_n)\tilde{h}^2_{x,y}(\xi_n)) \\
= E \left( \left( \mathbb{1}_{\{x < F(X_n)\} \leq y} - (y - x) \right)^2 \left( \mathbb{1}_{\{x < F(X_n)\} \leq y} - (y - x) \right)^2 \right) \\
= E \left( \mathbb{1}_{\{x < F(X_n)\} \leq y} + (y - x)^2 - 2(y - x) \mathbb{1}_{\{x < F(X_n)\} \leq y} \right) \\
\times \left( \mathbb{1}_{\{x < F(X_n)\} \leq y} + (y - x)^2 - 2(y - x) \mathbb{1}_{\{x < F(X_n)\} \leq y} \right) \]

\[ \leq \sum_{l=r}^{\infty} \frac{(c_l(y) - c_l(x))^2}{(l!)^2} E H_l^2(\xi_n) \leq C(y - x). \]
\[
\begin{align*}
&= \mathbb{E}\left( \mathbb{1}_{x<F(X_{n_1})\leq y} \mathbb{1}_{x<F(X_{n_2})\leq y} \right) (1 - 2(y - x))^2 + 2(y - x)^3 - 3(y - x)^4 \\
&= \mathbb{E}\left( \tilde{h}_{x,y}(\xi_{n_1})\tilde{h}_{x,y}(\xi_{n_2}) \right) (1 - 2(y - x))^2 + (y - x)^2(1 - (y - x))^2 \\
&\leq \left| \mathbb{E}\left( \tilde{h}_{x,y}(\xi_{n_1})\tilde{h}_{x,y}(\xi_{n_2}) \right) \right| + (y - x)^2.
\end{align*}
\]

By orthogonality of the Hermite expansion

\[
\begin{align*}
\mathbb{E}\left( \tilde{h}_{x,y}(\xi_{n_1})\tilde{h}_{x,y}(\xi_{n_2}) \right) &= \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) + \sum_{l=r}^{\left\lfloor \frac{n_1}{2} \right\rfloor} \left( \frac{c_l(y) - c_l(x)}{l!} \right)^2 \mathbb{E}(H_l(\xi_{n_1})H_l(\xi_{n_2})) \\
&= \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) + \sum_{l=r}^{\left\lfloor \frac{n_1}{2} \right\rfloor} \left( \frac{c_l(y) - c_l(x)}{l!} \right)^2 \gamma^l(n_1 - n_2) \\
&\leq \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) + (y - x) \sum_{l=r}^{\left\lfloor \frac{n_1}{2} \right\rfloor} \gamma^l(n_1 - n_2) \\
&\leq \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) + (y - x)C\gamma^r(n_1 - n_2).
\end{align*}
\]

Then,

\[
\left| \mathbb{E}\left( \tilde{h}_{x,y}(\xi_{n_1})\tilde{h}_{x,y}(\xi_{n_2}) \right) \right| \leq \left| \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) \right| + C(y - x)\gamma^r(n_1 - n_2),
\]

such that

\[
\mathbb{E}\left( \tilde{h}_{x,y}(\xi_{n_1})\tilde{h}_{x,y}(\xi_{n_2}) \right) \leq \left| \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) \right| + C(y - x)\gamma^r(n_1 - n_2). \tag{B.15}
\]

Combining (B.8), (B.12), (B.14) and (B.15) finally gives

\[
\begin{align*}
\mathbb{E}\left( h_{x,y}^2(\xi_{n_1})h_{x,y}^2(\xi_{n_2}) \right) &\leq C\left| \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) \right| \\
&+ C(y - x)^2 + C(y - x)^{\frac{2}{r}} + C(y - x)\gamma^r(n_1 - n_2).
\end{align*}
\]

Lemma 2.2 in Koul and Surgailis (2002) yields

\[
\sum' \left| \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) \right| \leq C(j - i)(y - x).
\]

Furthermore, we get

\[
\sum_{i+1 \leq n_1 \neq n_2 \leq j} \gamma^r(n_1 - n_2) = (j - i)^{2-rD}L^r(j - i) \leq C(j - i)^{2-rD+\eta}
\]

for all \(\eta > 0\) due to the proof of Lemma 3.1 (p. 1777) in Dehling and Taqqu (1989). Therefore, we get

\[
\begin{align*}
\mathbb{E}\left| \Sigma_{22} \right| &\leq C\sum' \left| \mathbb{E}(h_{x,y}(\xi_{n_1})h_{x,y}(\xi_{n_2})) \right| + C(j - i)^2(y - x)^2 \\
&+ C(j - i)^2(y - x)^{\frac{2}{r}} + C(y - x)(j - i)^{2-rD+\eta} \\
&\leq C(j - i)^2(y - x)^{\frac{2}{r}} + C(j - i)(y - x) + C(j - i)^{2-rD+\eta}(y - x).
\end{align*}
\]

Finally, we can use the bounds on \(\mathbb{E}\Sigma_1\) to \(\mathbb{E}\Sigma_4\) to continue bounding (B.2) as follows

\[
\mathbb{E}\left| s_j - s_i \right|^4 = \frac{1}{N^2} \mathbb{E}\left( \Sigma_1 + 4\Sigma_{21} + 3\Sigma_{22} + 6\Sigma_3 + \Sigma_4 \right)
\]
Let Lemma B.2. Note first that for all \( n \in \mathbb{N} \) for all of the interval \( r \) there is an \( \hat{p} \) such that

\[
\hat{p} \leq C \frac{1}{N^2} (j-i)(y-x) + (j-i)^2 y - x) + (j-i)^2 (y-x)^2 \leq C \frac{1}{N^2} (j-i)^2 (y-x) + (j-i)^2 (y-x)^2 \leq C \frac{1}{N^2} (j-i)^2 - \theta (y-x) + (j-i)^2 (y-x)^2
\]

with \( \theta = \min \{ \frac{1}{2}, r D - \eta \} \). We obtain,

\[
E |s_j - s_i|^4 \leq C \frac{1}{N^4} \left( \frac{j-i}{N} \right)^2 - \theta (y-x) + C \left( \frac{j-i}{N} \right)^2 (y-x)^2 \leq C \left( (y-x)^2 \sum_{q=i+1}^{j} \frac{1}{N^{2-\theta}} \right) + C \left( (y-x)^2 \sum_{q=i+1}^{j} \frac{1}{N} \right)^2.
\]

Then, applying Lemma C.1 with \( \gamma = 4 \) and \( \alpha_1 = 2 - \theta, \alpha_2 = 2 \) yields

\[
P \left( \max_{1 \leq k \leq \lfloor Nb \rfloor} |s_k| > \lambda \right) \leq C_{\alpha_1, \gamma} \frac{1}{\lambda^4} \left( (y-x)^2 \sum_{n=1}^{\lfloor Nb \rfloor} \frac{1}{N^{2-\theta}} \right)^2 + C_{\alpha_2, \gamma} \frac{1}{\lambda^4} \left( (y-x)^2 \sum_{n=1}^{\lfloor Nb \rfloor} \frac{1}{N^2} \right)^2 \leq C_{\alpha_1, \gamma} \frac{1}{\lambda^4} b^2 - \theta (y-x) + C_{\alpha_2, \gamma} \frac{1}{\lambda^4} b^2 (y-x)^2
\]

The subsequent lemmas are used in Appendix A and are all formulated in terms of a generic sequence of refining partitions which covers the two sequences of partitions (A.10) and (A.20) in Appendix A. For \( k = 0, \ldots, K_N \) define refining partitions

\[
x_i(k) := \tilde{a}_p + \frac{i}{2^k} 2\delta, \ i = 0, \ldots, 2^k,
\]

of the interval \( [\tilde{a}_p, \tilde{a}_p + a] \) and for \( x \in [0, a] \) choose \( i_k(x) \) such that

\[
\tilde{a}_p + x \in (x_{i_k(x)}(k), x_{i_k(x)+1}(k)].
\]

Note that for the partitions defined in (A.10) and (A.20), we consider \( \tilde{a}_p = a_p, a = 2\delta \) and \( \tilde{a}_p = 0, a = 1 \), respectively. All following lemmas in this section refer to these partitions.

Lemma B.2. Let \( m_N \) be defined as in (A.5) and let \( \sqrt{N}/2^K \to 0 \). Then, for all \( \lambda, b \in (0, 1) \) there is an \( N_\lambda \) such that

\[
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \left| m_N(x, t) - m_N(x+c, t) \right| > \lambda \right) \leq C \frac{1}{\lambda^2} b^2 \max \left\{ \frac{1}{N^2}, \frac{N}{2^{KN/2}} \right\}
\]

for all \( N > N_\lambda \) and for all \( c \geq 0 \) such that \( x + c \leq x_{iK_N(x)+1}(K_N) \).

Proof: Note first that for all \( x, y \),

\[
|m_N(y, t) - m_N(x, t)| = \left| \sum_{|l|=\frac{|N|}{2}} \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{\sqrt{N}} H_l(\xi_n) \right|
\]

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\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \mathbb{1}_{\{x < F(G(\xi_n)) \leq y\}} - \sum_{l=r}^{\frac{[N]}{l!}} \frac{c_l(y) - c_l(x)}{l!} \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} H_l(\xi_n) \leq \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \mathbb{1}_{\{x < F(G(\xi_n)) \leq y\}} + \sum_{l=r}^{\frac{[N]}{l!}} \frac{c_l(y) - c_l(x)}{l!} \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} H_l(\xi_n) \quad \text{. (B.17)}
\]

Then,

\[
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \left| m_N(x + c, t) - m_N(x_{iK_N}(x)(K_N), t) \right| > \lambda \right) \leq P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \mathbb{1}_{\{x_{iK_N}(x)(K_N) < F(G(\xi_n)) \leq x + c\}} \right| > \frac{\lambda}{2} \right) 
+ P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \left| \sum_{l=r}^{\frac{[N]}{l!}} \frac{c_l(x + c) - c_l(x_{iK_N}(x)(K_N))}{l!} \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} H_l(\xi_n) \right| > \frac{\lambda}{2} \right) \quad \text{. (B.18)}
\]

\[
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \mathbb{1}_{\{x_{iK_N}(x)(K_N) < F(G(\xi_n)) \leq x + c\}} \right| > \frac{\lambda}{2} \right) 
+ P \left( \sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nb]} \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c\}} \right| \right) 
\leq P \left( \sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nb]} \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c\}} \right| \right) 
\leq P \left( \sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[ Nb]} \left( \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c\}} - \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c + 1(K_N)\}} \right) \right) \quad \text{. (B.19)}
\]

where (B.18) follows by (B.17) and (B.19) is a consequence of applying Lemma B.3 below. It remains to bound the probability in (B.19). Therefore, we write

\[
P \left( \sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nb]} \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c\}} \right| > \frac{\lambda}{2} \right) 
\leq P \left( \sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nb]} \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c\}} \right| \right) 
\leq P \left( \sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[ Nb]} \left( \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c\}} - \mathbb{1}_{\{x_{iK_N}(x)(K_N) \leq x + c + 1(K_N)\}} \right) \right) \quad \text{. (B.20)}
\]

The second summand in (B.20) is deterministic and can be bounded by

\[
\sup_{x \in [0,a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nb]} \left( x_{iK_N}(x+1(K_N)) - x_{iK_N}(x)(K_N) \right) \right| \leq \frac{\sqrt{N}}{2K_N} b \leq \frac{\sqrt{N}}{2K_N}
\]

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Then, choose $N_\lambda \geq 1$ such that $\frac{x}{2KN} < \frac{1}{\lambda}$ for all $N \geq N_\lambda$. As a result, we get

$$
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \mathbb{I} \left\{ x_{iK_N}(x) < F(G(\xi_n)) \leq x+c \right\} > \frac{\lambda}{2} \right) \leq \frac{\lambda}{4}$$

for all $N \geq N_\lambda$. Lemma B.4 gives an upper bound on the probability in (B.21):

$$
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} \mathbb{I} \left\{ x_{iK_N}(x) < F(G(\xi_n)) \leq x+c \right\} > \frac{\lambda}{2} \right) \leq \frac{C}{\lambda^4} b^2 \max \left\{ \frac{1}{N^2}, \frac{N}{2KN^2} \right\}.$$  

\[ \square \]

**Lemma B.3.** Let $c_1$ be defined as in (4.3). For all $\lambda > 0$ it holds that

$$
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \sum_{l=r}^{\infty} c_1 (x+c) - c_1 (x_{iK_N}(x)(K_N)) \frac{1}{l!} \sum_{n=1}^{[N]} H_l(\xi_n) > \lambda \right) \leq \frac{C}{\lambda^4} b^2 \frac{N}{2KN^2}$$

for $a > 0$ and $c \geq 0$, such that $x + c \leq x_{iK_N}(x+1)(K_N)$.

**Proof:** In order to bound the probability of interest, we use (B.7) in (B.22) below

$$
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \sum_{l=r}^{\infty} \frac{c_1 (x+c) - c_1 (x_{iK_N}(x)(K_N))}{l!} \sum_{n=1}^{[N]} H_l(\xi_n) > \lambda \right) \leq \frac{\lambda}{4}$$

$$
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \sum_{l=r}^{\infty} \frac{(x+c-x_{iK_N}(x)(K_N))^{1/2}}{l!} \sum_{n=1}^{[N]} H_l(\xi_n) > \lambda \right)$$

$$
P \left( \sup_{x \in [0,a]} \sup_{t \in [0,b]} \frac{1}{\sqrt{N}} \sum_{l=r}^{\infty} \frac{(x_{iK_N}(x)+1(K_N)-x_{iK_N}(x)(K_N))^{1/2}}{l!} \sum_{n=1}^{[N]} H_l(\xi_n) > \lambda \right)$$

$$
P \left( \sum_{l=r}^{\infty} \frac{1}{\sqrt{2KN l!}} \sum_{n=1}^{[N]} |H_l(\xi_n)| > \lambda \right)$$

$$
\leq \frac{1}{2^{KN}} \left[ \sum_{l=r}^{\infty} \frac{1}{\sqrt{l!}} \sum_{n=1}^{[N]} |H_l(\xi_n)| \right]^2 \left( \frac{1}{\lambda} \right)^2$$

(B.23)
\[
\begin{align*}
&\leq \frac{1}{2KN} \sum_{i_1, i_2 = r}^{\lfloor \frac{1}{\lambda} \rfloor} \frac{1}{\sqrt{1/2N}} \sum_{n_1, n_2 = 1}^{\lfloor Nb \rfloor} E(\|H_{i_1}(\xi_{n_1})\|H_{i_2}(\xi_{n_2})) \left( \frac{1}{\lambda} \right)^2 \\
&\leq \frac{1}{2KN} \sum_{i_1, i_2 = r}^{\lfloor \frac{1}{\lambda} \rfloor} \frac{1}{N} \sum_{n_1, n_2 = 1}^{\lfloor Nb \rfloor} \left( \frac{1}{\lambda} \right)^2 \\
&\leq \frac{N}{2KN} b^2 \left( \frac{1}{\alpha} \right)^2 \left( \frac{1}{\lambda} \right)^2
\end{align*}
\]  

where (B.23) follows by Markov’s inequality. We then used Cauchy-Schwarz inequality to get (B.24).

\[\Box\]

**Lemma B.4.** Let \( F \) denote the marginal distribution function of \( X_n, n \in \mathbb{N} \). Then, it holds that

\[
P \left( \sup_{x \in [0, a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor Nb \rfloor} \left( 1 \{ x_{i_{K_N}(x)}(K_N) < F(x_n) \} \right) - \left( x_{i_{K_N}(x)+1}(K_N) - x_{i_{K_N}(x)}(K_N) \right) \right| > \lambda \right) \leq \frac{C}{K^\frac{3}{2}} \max \left\{ \frac{1}{N^\frac{1}{2}}, \frac{N}{2KN^\frac{1}{2}} \right\}
\]

for \( a > 0 \) and \( b, \lambda \in [0, 1] \).

**Proof:** With further explanations given below, we can infer the following bounds

\[
P \left( \sup_{x \in [0, a]} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor Nb \rfloor} \left( 1 \{ x_{i_{K_N}(x)}(K_N) < F(x_n) \} \right) - \left( x_{i_{K_N}(x)+1}(K_N) - x_{i_{K_N}(x)}(K_N) \right) \right| > \lambda \right) \leq \frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor Nb \rfloor} \left| H_i(\xi_n) \right| > \frac{\lambda}{2}
\]

\[
\leq \frac{2KN}{\lambda^2} \sum_{i=0}^{2KN} (x_{i+1}(K_N) - x_i(K_N)) + \frac{C}{K^\frac{3}{2}} b^2 \sum_{i=0}^{2KN} (x_{i+1}(K_N) - x_i(K_N))^2 + \frac{C}{K^2} \sum_{i=0}^{2KN} b^2 \frac{N}{2K} \]

(B.25)

(B.26)

where we used the representation (A.5) in the first summand of (B.25). The first probability in (B.26) can be bounded by Lemma B.1 and the second one by Lemma B.3. We deduce the last inequality by using that \( b, \lambda \in (0, 1] \).

\[\Box\]
C  A complementary result and its proof

In order to prove Lemma B.1, we use a slightly modified version of Theorem 12.2 in Billingsley (1968). We recall some notation from Chapter 12 in Billingsley (1968). Let \( \xi_1, \ldots, \xi_N \) be independent or identically distributed random variables and \( s_k = \sum_{j=1}^{k} \xi_j \) with \( s_0 = 0 \) and set

\[
M_N = \max_{0 \leq k \leq N} |s_k|.
\]

**Lemma C.1.** Suppose there are \( \gamma > 0, \alpha_1, \alpha_2 > 1, v_1, v_2 > 0 \) and a positive sequence \( (u_\ell)_{1 \leq \ell \leq N} \), such that for all \( \lambda > 0 \),

\[
P(|s_j - s_i| > \lambda) \leq \frac{1}{\lambda^\gamma} \left( \left( v_1 \sum_{\ell = i+1}^{j} u_\ell \right)^{\alpha_1} + \left( v_2 \sum_{\ell = i+1}^{j} u_\ell \right)^{\alpha_2} \right), \quad 0 \leq i \leq j \leq N. \tag{C.1}
\]

Then, there are constants \( C_{\alpha_1,\gamma}, C_{\alpha_2,\gamma} > 0 \) only depending on \( \alpha_i, i = 1, 2 \) and \( \gamma \), such that

\[
P(M_N > \lambda) \leq \frac{C_{\alpha_1,\gamma}}{\lambda^\gamma} \left( v_1 \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \frac{C_{\alpha_2,\gamma}}{\lambda^\gamma} \left( v_2 \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2}.
\]

**Remark C.2.** Note that Lemma C.1 reduces to Theorem 12.2 in Billingsley (1968) by setting \( v_2 = 0 \). In particular, our statement remains true if either \( v_1 \) or \( v_2 \) are zero. Our proof below reveals that the generalization only works when the two summation in (C.1) depend on the same sequence \( (u_\ell)_{1 \leq \ell \leq N} \).

**Proof:** The proof follows the proofs of Theorems 12.1 and 12.2 in Billingsley (1968) and requires only slight modifications of the arguments. First, note that

\[
P(M_N > \lambda) \leq P(M'_N > \lambda/2) + P(s_N > \lambda/2) \tag{C.2}
\]

with \( M'_N = \max_{0 \leq \ell \leq N} \min(|s_\ell|, |s_N - s_\ell|) \). We consider the two probabilities in (C.2) separately. Using assumption (C.1) with \( j = N \) and \( i = 0 \), the second probability can be bounded as

\[
P(s_N > \lambda/2) \leq \frac{2^\gamma}{\lambda^\gamma} \left( \left( v_1 \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \left( v_2 \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} \right).
\]

We prove a bound for the first probability in (C.2) via induction over \( N \). Our induction hypothesis is, for \( \mu = \lambda/2 \),

\[
P(M'_N > \mu) \leq \frac{C_{\alpha_1,\gamma}}{\mu^\gamma} \left( v_1 \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \frac{C_{\alpha_2,\gamma}}{\mu^\gamma} \left( v_2 \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2}. \tag{C.3}
\]

Beginning the induction with the base case \( N = 2 \), we get

\[
P(M'_2 > \mu) = P(\min(|s_1|, |s_2 - s_1|) > \mu) \leq \frac{1}{\mu^\gamma} \left( \left( v_1 \sum_{\ell = 1}^{2} u_\ell \right)^{\alpha_1} + \left( v_2 \sum_{\ell = 1}^{2} u_\ell \right)^{\alpha_2} \right). \tag{C.4}
\]

by applying Lemma C.3 with \( i = 0, j = 1, k = 2 \) in (C.4).

For the inductive step, we assume that the induction hypothesis (C.3) is satisfied for all integers smaller and equal to \( N - 1 \) and move towards \( N \) during the inductive step. Note that there is an \( h \) such that

\[
\sum_{\ell = 1}^{h-1} u_\ell \leq \frac{1}{2} \sum_{\ell = 1}^{N} u_\ell \leq \sum_{\ell = 1}^{h} u_\ell, \tag{C.5}
\]
where the sum on the left is zero if \( h = 1 \). By some algebra one can infer

\[
\sum_{\ell = h+1}^{N} u_\ell \leq \frac{1}{2} \sum_{\ell = 1}^{N} u_\ell \leq \sum_{\ell = h}^{N} u_\ell.
\]  

(C.6)

By (12.36) in Billingsley (1968),

\[ M_N' \leq \max\{U_1 + D_1, U_2 + D_2\} \]

and therefore

\[ P(M_N' > \mu) \leq P(U_1 + D_1 > \mu) + P(U_2 + D_2 > \mu) \]  

(C.7)

with

\[
U_1 = \max_{0 \leq i \leq h-1} \min\{|s_i|, |s_{h-1} - s_i|\}, \quad U_2 = \max_{h \leq i \leq N} \min\{|s_j - s_h|, |s_N - s_j|\},
\]

(C.8)

\[
D_1 = \min\{|s_{h-1}|, |s_N - s_{h-1}|\}, \quad D_2 = \min\{|s_h|, |s_N - s_h|\}.
\]

(C.9)

The tail probabilities of the random variables (C.8) and (C.9) can be bounded by using the inductive hypothesis (C.3) and Lemma C.3, respectively. Exemplarily, we consider \( U_1 \) and \( D_1 \). For \( U_1 \), we get the following bounds

\[
P(U_1 > \mu) \leq \frac{C_{\alpha_1, \gamma}}{\mu^\gamma} \left( \sum_{\ell = 1}^{h-1} u_\ell \right)^{\alpha_1} + \frac{C_{\alpha_2, \gamma}}{\mu^\gamma} \left( \sum_{\ell = 1}^{h-1} u_\ell \right)^{\alpha_2}
\]

\[
\leq \frac{C_{\alpha_1, \gamma}}{\mu^\gamma} \left( \frac{1}{2^{\alpha_1}} \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} \right) + \frac{C_{\alpha_2, \gamma}}{\mu^\gamma} \left( \frac{1}{2^{\alpha_2}} \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} \right)
\]  

(C.10)

by applying the inductive hypothesis (C.3) in (C.10) and the inequality (C.5) in (C.11). The tail probability of \( U_2 \) can be dealt with analogously by applying (C.6). For \( D_1 \), we get

\[
P(D_1 > \mu) \leq \frac{1}{\mu^\gamma} \left( \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} \right)
\]  

(C.12)

by Lemma C.3 with \( i = 0, j = h - 1 \) and \( k = N \). The tail probability of \( D_2 \) can be handled analogously by applying Lemma C.3 with \( i = 0, j = h \) and \( k = N \).

We now continue with bounding (C.7) with focus on the first summand since the second summand can be bounded by analogous arguments. With explanations given below, for some positive \( \mu_0, \mu_1 \) with \( \mu_0 + \mu_1 = \mu \),

\[
P(U_1 + D_1 > \mu) \leq P(U_1 > \mu_0) + P(D_1 > \mu_1)
\]

\[
\leq \frac{C_{\alpha_1, \gamma}}{\mu_0^{\alpha_1}} \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \frac{C_{\alpha_2, \gamma}}{\mu_0^{\alpha_2}} \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} + \frac{1}{\mu_1} \left( \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} \right)
\]

\[
= \frac{1}{\mu^\gamma} \left( \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \frac{C_{\alpha_2, \gamma}}{\mu_0^{\alpha_2}} \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} \right) + \left( \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_1} + \left( \sum_{\ell = 1}^{N} u_\ell \right)^{\alpha_2} \right)^{\frac{1}{2}}
\]  

(C.13)
\[
\frac{1}{\mu^\gamma} \left( \left(C_{\alpha_1,\gamma} \left( v_1 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_1} + C_{\alpha_2,\gamma} \left( v_2 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_2} \right) \delta \left[ \left( \frac{1}{2\alpha_1} + \frac{1}{2\alpha_2} \right)^{\frac{1}{2}} \delta \left( \frac{1}{C_{\alpha_1,\gamma}} + \frac{1}{C_{\alpha_2,\gamma}} \right) \right]^{\frac{1}{2}} \right) \leq \frac{1}{\mu^\gamma} \left( C_{\alpha_1,\gamma} \left( v_1 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_1} + C_{\alpha_2,\gamma} \left( v_2 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_2} \right),
\]

(C.15)

where (C.13) follows by (C.11) and (C.12). For (C.14) we recall (12.39) in Billingsley (1968). It states that for positive numbers \( A, B, \lambda, \)

\[
\min_{\frac{\lambda_0, \lambda_1 > 0}{\lambda_0 + \lambda_1 = \lambda}} \left( \frac{A}{\lambda_0^\gamma} + \frac{B}{\lambda_1^\gamma} \right) = \frac{1}{\lambda^\gamma} (A^\delta + B^\delta)^{\frac{1}{2}}
\]

with \( \delta = \frac{1}{\gamma + 1} \). The inequality (C.15) follows since

\[
\left( v_1 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_1} + \left( v_2 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_2} \leq \left( C_{\alpha_1,\gamma} \left( v_1 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_1} + C_{\alpha_2,\gamma} \left( v_2 \sum_{\ell=1}^{N} u_{\ell} \right)^{\alpha_2} \right) \left( \frac{1}{C_{\alpha_1,\gamma}} + \frac{1}{C_{\alpha_2,\gamma}} \right).
\]

Finally, we get (C.16) by choosing the constants \( C_{\alpha_1,\gamma}, C_{\alpha_2,\gamma} \) large enough to get

\[
\left[ \left( \frac{1}{2\alpha_1} + \frac{1}{2\alpha_2} \right)^{\frac{1}{2}} \delta \left( \frac{1}{C_{\alpha_1,\gamma}} + \frac{1}{C_{\alpha_2,\gamma}} \right) \right]^{\frac{1}{2}} \leq 1
\]

(C.17)

which is possible since (C.17) is equivalent to

\[
\left( \frac{1}{2\alpha_1} + \frac{1}{2\alpha_2} \right) \delta \left( \frac{1}{C_{\alpha_1,\gamma}} + \frac{1}{C_{\alpha_2,\gamma}} \right) \leq 1
\]

and due to our assumption that \( \alpha_1, \alpha_2 > 1 \).

\[ \blacksquare \]

**Lemma C.3.** Suppose there are \( \gamma > 0, \alpha_1, \alpha_2 > 1, \) (C.1) is satisfied with \( v_1, v_2 > 0 \) and a positive sequence \( (u_{\ell})_{1 \leq \ell \leq N} \). Then, for all \( \lambda > 0, \)

\[
P \left( |s_j - s_i| > \lambda, |s_k - s_j| > \lambda \right) \leq \frac{1}{\lambda^\gamma} \left( \left( v_1 \sum_{\ell=i+1}^{k} u_{\ell} \right)^{\alpha_1} + \left( v_2 \sum_{\ell=j+1}^{k} u_{\ell} \right)^{\alpha_2} \right)^{\frac{1}{2}}, \quad 0 \leq i \leq j \leq k \leq N.
\]

**Proof:** We follow the arguments in the proof of Theorem 12.1 in Billingsley (1968). That is,

\[
P \left( |s_j - s_i| > \lambda, |s_k - s_j| > \lambda \right)
\]

\[
\leq P \frac{1}{\lambda^\gamma} \left( \left( v_1 \sum_{\ell=i+1}^{j} u_{\ell} \right)^{\alpha_1} + \left( v_2 \sum_{\ell=i+1}^{j} u_{\ell} \right)^{\alpha_2} \right)^{\frac{1}{2}} \left( \left( v_1 \sum_{\ell=j+1}^{k} u_{\ell} \right)^{\alpha_1} + \left( v_2 \sum_{\ell=j+1}^{k} u_{\ell} \right)^{\alpha_2} \right)^{\frac{1}{2}} \leq \frac{1}{\lambda^\gamma} \left( \left( v_1 \sum_{\ell=i+1}^{j} u_{\ell} \right)^{\alpha_1} + \left( v_2 \sum_{\ell=i+1}^{j} u_{\ell} \right)^{\alpha_2} \right)^{\frac{1}{2}}, \quad \lambda > 0
\]

(C.18)
An approximate Lemma D.1.

Note that Proof:

ř

For shortness’ sake we set we consider the asymptotic behavior of the higher-order approximation of the empirical process.

D Additional results and their proofs

For shortness’ sake we set \( \tilde{L}_n(x) = \frac{1}{N} \sum_{n=1}^{N} L_n(x) \) in this section.

Lemma D.1. An approximate \( 1 - \alpha \) confidence interval of \( F^{-1}(p) \) can be written as

\[
\left( \phi(F_N) + \frac{1}{\varphi(F^{-1}(p))} \left( \tilde{L}_N(F^{-1}(p)) + \frac{1}{\sqrt{N}} \sigma(F^{-1}(p)) z_{1 - \frac{\alpha}{2}} \right) \right)
\]

\[\cdot \phi(F_N) + \frac{1}{\varphi(F^{-1}(p))} \left( \tilde{L}_N(F^{-1}(p)) + \frac{1}{\sqrt{N}} \sigma(F^{-1}(p)) z_{\frac{\alpha}{2}} \right) \).

Proof: Note that

\[
P \left( c_1 \leq \frac{N}{d_N} (\phi(F_N) - \phi(F)) \leq c_2 \right) = P \left( \frac{d_N}{N} c_1 \leq \phi(F_N) - \phi(F) \leq \frac{d_N}{N} c_2 \right)
\]

\[= P \left( \phi(F_N) - \frac{d_N}{N} c_2 \leq \phi(F) \leq \phi(F_N) - \frac{d_N}{N} c_1 \right) \quad (D.1)
\]

In order to find an approximate \( 1 - \alpha \) confidence interval, one has to determine the critical values \( c_1, c_2 \) in (D.1). Instead of utilizing the asymptotic distribution of the empirical process, we consider the asymptotic behavior of the higher-order approximation of the empirical process.

Then, with explanations given below,

\[
P \left( c_1 \leq \frac{N}{d_N} (\phi(F_N) - \phi(F)) \leq c_2 \right)
\]

\[\approx P \left( \frac{d_N}{N} c_1 \leq \phi_F(F_N - F) \leq \frac{d_N}{N} c_2 \right) \quad (D.2)
\]

\[= P \left( \frac{d_N}{N} c_1 \leq \frac{-1}{\varphi(F^{-1}(p))} (F_N(F^{-1}(p) - F(F^{-1}(p)))) \leq \frac{d_N}{N} c_2 \right) \quad (D.3)
\]

\[= P \left( -\varphi(F^{-1}(p)) \frac{d_N}{N} c_2 \leq F_N(F^{-1}(p)) - F(F^{-1}(p)) \leq -\varphi(F^{-1}(p)) \frac{d_N}{N} c_1 \right)
\]

\[= P \left( -\varphi(F^{-1}(p)) \frac{d_N}{\sqrt{N}} c_2 \leq \sqrt{N} \tilde{L}_N(F^{-1}(p)) \right)
\]

\[\approx \sqrt{N} S_N(F^{-1}(p)) \approx -\varphi(F^{-1}(p)) \frac{d_N}{\sqrt{N}} c_1 - \sqrt{N} \tilde{L}_N(F^{-1}(p)) \]

\[\approx P \left( \sigma(F^{-1}(p)) z_{1 - \frac{\alpha}{2}} \leq S(F^{-1}(p), 1) \leq \sigma(F^{-1}(p)) z_{\frac{\alpha}{2}} \right), \quad (D.4)
\]
D.2

where (D.2) is due to the Taylor approximation (5.6), (D.3) follows by the relation (5.7) and (D.4) with $\sigma^2(x) = \sum_{n \in \mathbb{Z}} \text{Cov}(S_n(x), S_n(x))$.

Based on the last approximation (D.4), we can infer the following relation between $c_1, c_2$ and the quantiles of the normal distribution

$$\sigma(F^{-1}(p))z_{1-\frac{p}{2}} = -\varphi(F^{-1}(p))\frac{d_N}{\sqrt{N}}c_1 - \sqrt{N}L_N(F^{-1}(p)),$$

$$\sigma(F^{-1}(p))z_{\frac{p}{2}} = -\varphi(F^{-1}(p))\frac{d_N}{\sqrt{N}}c_2 - \sqrt{N}L_N(F^{-1}(p)).$$  \hfill (D.5)

Combining (D.1) and (D.5), we can then infer the statement of the lemma since for example

$$\phi(F_N) - \frac{d_N}{N}c_1$$

$$= \phi(F_N) - \frac{1}{\varphi(F^{-1}(p))} \frac{d_N}{N} \left( -\sqrt{N}L_N(F^{-1}(p)) - \sigma(F^{-1}(p))z_{1-\frac{p}{2}} \right)$$

$$= \phi(F_N) + \frac{1}{\varphi(F^{-1}(p))} \left( \tilde{L}_N(F^{-1}(p)) + \frac{1}{\sqrt{N}}\sigma(F^{-1}(p))z_{1-\frac{p}{2}} \right).$$

\[\square\]

**Lemma D.2.** Suppose $G: \mathbb{R} \to \mathbb{R}$ is a monotonically increasing (decreasing), bijective function. Then,

$$\tilde{c}_l(x) = \begin{cases} -H_{l-1}(G^{-1}(x))\varphi(G^{-1}(x)) & \text{if } G \text{ is increasing} \\ H_{l-1}(G^{-1}(x))\varphi(G^{-1}(x)) & \text{if } G \text{ is decreasing} \end{cases}.$$  

**Proof of Lemma D.2:** For a strictly monotonically increasing, bijective function $G$, it holds that

$$\tilde{c}_l(x) = E\left(\mathbb{1}_{\{G(x_0) \leq x\}} H_l(x_0)\right)$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y \leq x\}} H_l(y)\varphi(y)dy$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y \in G^{-1}(x)\}} H_l(y)\varphi(y)dy$$

$$= \int_{G^{-1}(x)} H_l(y)\varphi(y)dy = -H_{l-1}(G^{-1}(x))\varphi(G^{-1}(x)),$$

where the last equality follows from the definition of the Hermite polynomial $H_l$ as

$$H_l(x) = (-1)^l \frac{1}{\varphi(x)} \frac{\partial^l}{\partial x^l} \varphi(x);$$

see formula (4.1.1) in Pipiras and Taqqu (2017). Analogously, it follow that for a strictly monotonically decreasing, bijective function $G$, it holds that

$$\tilde{c}_l(x) = E\left(\mathbb{1}_{\{G(x_0) \geq x\}} H_l(x_0)\right)$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y \geq x\}} H_l(y)\varphi(y)dy$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y \notin G^{-1}(x)\}} H_l(y)\varphi(y)dy$$

$$= \int_{G^{-1}(x)}^\infty H_l(y)\varphi(y)dy = H_{l-1}(G^{-1}(x))\varphi(G^{-1}(x)).$$

\[\square\]
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