Abstract

We briefly review why the non-linear realisation of the semi-direct product of a group with one of its representations leads to a field theory defined on a generalised space-time equipped with a generalised vielbein. We give formulae, which only involve matrix multiplication, for the generalised vielbein, the Cartan forms and their transformations. We consider the generalised space-time introduced in 2003 in the context of the non-linear realisation of the semi-direct product of $E_{11}$ and its first fundamental representation. For this latter theory we give explicit expressions for the generalised vielbein up to and including the levels associated with the dual graviton in four, five and eleven dimensions and for the IIB theory in ten dimensions. We also compute the generalised vielbein, up to the analogous level, for the non-linear realisation of the semi-direct product of very extended $SL(2)$ with its first fundamental representation, which is a theory associated with gravity in four dimensions.
1. Introduction

We begin by giving a very brief review of the general theory of non-linear realisations. While some aspects of this are very well known, the non-linear realisations that involve a group whose generators are associated with space-time are less well known. In particular we will make it clear why the non-linear realisations which lead to space-times automatically encode a generalised vielbein.

A non-linear realisation of a group \( G \) with local subgroup \( H \) is constructed from a group element \( g \in G \) which is subject to the transformations

\[
g \to g_0 g, \quad \text{for } g_0 \in G \quad \text{and also} \quad g \to gh, \quad \text{for } h \in H
\]

(1.1)

where \( g_0 \) is a rigid transformation and \( h \) a local transformation. The meaning of rigid and local will be discussed below. The non-linear realisation is an action, or set of equations of motion, that are invariant under the transformations of equation (1.1). The dynamics is usually constructed from the Cartan forms \( \mathcal{V} \) which are inert under the rigid \( g_0 \) transformations but transform under the \( h \) transformations as

\[
\mathcal{V} \to \mathcal{V}' = h^{-1}\mathcal{V}h + h^{-1}dh
\]

(1.2)

Before explaining the particular type of non-linear realisation that will be discussed in this paper it will be instructive to briefly discuss the three types of non-linear realisation.

1.1 Type 1

Non-linear realisations were first introduced to understand the scattering of pions and it was through this work that it became understood that symmetry was to play a crucial role in particle physics [1]. The theoretical underpinning of this method was set out in the classic papers of reference [2]. This work involved a group \( G \) which contained generators that were internal, that is, not associated with space-time. Space-time was introduced in an ad hoc manner by taking the group element \( g \), and so the parameters it contains, to depend on the chosen space-time, which in the application at that time, was just four dimensional Minkowski space-time. As a result, the parameters of the group element \( g \) became the fields of the theory defined on the chosen space-time. The rigid transformation \( g_0 \) is a constant group element, while the local transformation \( h \) is taken to depend on the chosen space-time and so can be used to gauge away parts of \( g \). The Cartan forms can be written as

\[
\mathcal{V} = g^{-1}dg = P_i T^I + Q_i H^i
\]

(1.3)

where \( H^i \) are the generators of \( H \) and \( T^I \) the remaining generators of \( G \). When the group is such that the commutators between generators of \( H \) with \( T^I \) lead again to the generators \( T^I \) (the reductive case), the forms \( P \equiv P_i T^I \) transform homogeneously and can be used to construct an invariant action which is just the space-time integral of \( TrP^2 \). The dynamics of the pions were found to be very well described, in the limit of small pion mass, by the non-linear realisation of \( SU(2) \otimes SU(2) \) with respect to its diagonal subgroup \( SU(2) \).

1.2 Type 2
A non-linear realisation at the other extreme is one where all the generators of the group $G$ are associated with "space-time". In this case we have a simple coset space, often written as $G/H$, which has been studied for a very long time, at least in the mathematics literature. In this case the $H$ transformations enforce the usual equivalence relation that ensures that the group elements of $G$ are regarded as equivalent if they belong to the same coset. Modulo this relation the parameters in the group element label the points in the coset, which for the application that physicists have in mind are the points in space-time. Thus in this case we have no fields.

For these non-linear realisations the Cartan forms can be written as

$$ V = g^{-1}dg = dx^\Pi E_\Pi^A l_A + dx^\Pi \omega_{\Pi i} H^i $$

(1.4)

where $H^i$ are the generators of $H$ and $l_A$ the remaining generators of $G$. The objects $E_\Pi^A$ define a preferred basis of one forms $dx^\Pi E_\Pi^A$ at every point of the coset which are just those swept out, out, using the natural action of the group on the coset, from a basis of one forms at the origin of the coset. As a result we can interpret the objects $E_\Pi^A$ as vielbeins on the coset space. The objects $\omega_{\Pi i}$ can be thought of as the spin-connection on the coset. One can easily verify, using equation (1.1) that both of these objects transform as vielbeins and spin connections should on their $A$ and $i$ indices respectively under the local $H$ transformations.

The classic example of such a non-linear realisation is to take $G$ to be the Poincare group, which can be written as the semi-direct product of the Lorentz group, $SO(1, D-1)$, with a set of generators in its vector representation, denoted $l^{SO(1,D-1)}$, with the local subgroup $H$ being the Lorentz group. We denote this semi-direct product by $SO(1,D-1) \otimes_s l^{SO(1,D-1)}$. Another example is superspace which is the non-linear realisation of the super Poincare group with the local subgroup being the Lorentz group [3].

1.1 Type 3

The final type of non-linear realisation is built from a group $G$ that has some generators that are associated with space-time and some that are not. For simplicity, and as it is the case we wish to consider in this paper, we will take the group to be of a semi-direct product structure, that is, of the form $G = G_1 \otimes_s l_1$ where $G_1$ is a Lie group and $l_1$ is one of its representations. We denote the generators of $G_1$ by $R_\alpha^\delta$ and those of the $l_1$ representation by $l_A$. The Lie algebra for this group can be written in the form

$$ [R_\alpha^\beta, R_\delta^\gamma] = f_\alpha^\beta\gamma R_\delta^\gamma $$

(1.5)

and

$$ [R_\alpha^\beta, l_A] = -(D_\alpha)_A^B l_B $$

(1.6)

The Jacobi identity, implies that the generators $l_A$ belong to representation of $G_1$ and so the matrices in the above equation obey the matrix identity

$$ [D_\alpha^\beta, D_\delta^\gamma] = f_\alpha^\beta\gamma D_\delta^\gamma $$

(1.7)

The commutators of the $l_A$ generators must be consistent with the Jacobi identities and we will take them, for simplicity, to commute.
The group element \( g \in G \) is constructed from the generators \( l_A \) and \( R^\alpha_\beta \) and can be written in the form
\[
g = g_l g_A \equiv e^{x^A l_A} e^{A_\alpha(x) R^\alpha_\beta}
\] (1.8)

The parameters of the \( l_A \) generators can be interpreted as the coordinates of a generalised space-time while the parameters of the \( R^\alpha_\beta \) generators are taken to depend on the coordinates of space-time and are fields defined on the generalised space-time. Rigid in this case means that the group element \( g_0 \) does not depend on the generalised space-time and so its parameters are constants. The local subalgebra \( H \) of \( G \) used in the non-linear realisation is a subalgebra of \( G_1 \) and local \( H \) transformations have a group element \( h \) which does depend on space-time. This transformation can be used to gauge away some of the fields in \( g \). However, once this has been done we have to carry out compensating \( H \) transformation to preserve the form of the group element \( g \) under a rigid \( g_0 \) transformation. This last type of non-linear realisation is a hybrid of the first two types; if we take no \( l_1 \) generators then it is of type one while if we take no generators of the type \( R^\alpha_\beta \) then it is of type two.

The Cartan forms belong to the Lie algebra of \( G \) and so can be written as
\[
\mathcal{V} = \mathcal{V}_l + \mathcal{V}_A
\] (1.9)

were \( \mathcal{V}_l \) contains the generators of \( l_1 \) and \( \mathcal{V}_A \) the generators of \( G_1 \) and as such we can write them in the form
\[
\mathcal{V}_l \equiv dx^{\Pi} E_{\Pi A} l_A = g_A^{-1} dx^{\Pi} l_{\Pi} g_A, \quad \text{and} \quad \mathcal{V}_A \equiv dx^{\Pi} G_{\Pi, \alpha} R^\alpha_\beta = g_A^{-1} dg_A
\] (1.10)

We can interpret the objects \( E_{\Pi A} \) as the vielbein on the generalised space-time.

One of the early examples of this type of non-linear realisation was to take \( G = GL(D) \otimes_{s} l^{GL(D)} \) where \( l^{GL(D)} \) is the vector representation of \( SL(D) \), or equivalently its first fundamental representation \([4,5]\). This non-linear realisation gives, with a judicious choice of a few undetermined constants, Einstein’s theory of gravity \([4,5]\). A more recent example, and the one of interest to us here, is to take \( G \) to be the semi-direct product of \( E_{11} \) and its first fundamental representation \( l_1 \), denoted \( E_{11} \otimes_{s} l_1 \) \([6]\). This is a special case of non-linear realisations constructed from the groups \( G = G^{+++} \otimes_{s} l_1 \) where \( G^{+++} \) is the very extension of any finite dimensional semi-simple Lie algebra and \( l_1 \) is the first fundamental representation of \( G^{+++} \). We note that \( E_{11} = E^{+++}_{8} \). The non-linear realisations \( A_{D-3}^{+++} \otimes_{s} l_1 \) \([6]\) and \( D_{D-2}^{+++} \otimes_{s} l_1 \) \([8]\) are conjectured to be the low energy effective actions for gravity and the closed bosonic string in \( D \) dimensions respectively. A more detailed review of non-linear realisation can be found in \([9]\).

In this paper we will consider non-linear realisation of the last type that is the non-linear realisation of \( G = G_1 \otimes_{s} l_1 \). In section two we derive expressions for the generalised vielbein, Cartan forms and their transformations that require no more than matrix multiplication. In section three we consider the non-linear realisation \( E_{11} \otimes_{s} l_1 \) and compute the generalised vielbeins in eleven, five and four dimensions and the IIB theory in ten dimensions up to levels three, four, two and five respectively. In section four we give the initial steps in the construction of the non-linear realisation of the \( A_{1}^{+++} \otimes_{s} l_1 \) and compute the generalised vielbein up to level two. This later theory is conjectured to be the complete
low energy effective action for four dimensional gravity. In appendix A we recall, up to the level associated with the dual graviton, the $E_{11} \otimes_s l_1$ algebra in the decompositions appropriate to eleven and four dimensions and for five dimensions and the IIB theory in ten dimensions we give these algebras for the first time.

2 Formulae for the generalised vielbein and Cartan forms

In this section we consider the non-linear realisation of the semi-direct product of a group $G_1$ with one of its representations $l_1$ which we denote by $G_1 \otimes_s l_1$ and so we are discussing the case of type three of the section one. In this section the $l_1$ representation can be any representation and not just the first fundamental representation. We will take the generators of the $l_1$ representation to commute. It is straightforward to modify the discussion to the case when the generators of the $l_1$ representation do not commute, but form a group.

The generators of the group $G_1$ in the non-linear realisation are usually taken to be abstract objects, but if we take them to be in the $l_1$ representation then it is straightforward to derive expressions, that involve no more than matrix multiplication, for the generalised vielbein, their transformations, and the Cartan forms. These are well known for the non-linear realisation of $GL(D) \otimes_s l^{GL(D)}$ [4] and were recently given [10] for the generalised vielbein for $E_{11} \otimes_s l_1$.

The generalised vielbein is defined in equation (1.10) and it is straightforward to evaluate using equation (1.6) to find that it is given by

$$E_{\Pi}^A = (e^A)_{\Pi}^A$$ (2.1)

where $(A)_{\Pi}^A \equiv A_\underline{\alpha}(D\underline{\alpha})_{\Pi}^A$ and the expression on the right-hand side is evaluated by expanding the exponential and using matrix multiplication. Taking the generators of the $G_1$ algebra to be in the $l_1$ representations in the expression for the Cartan forms of equation (1.10) we find that

$$-\{\mathcal{V}_A, l_A\} = dx^{\Pi}G_{\Pi,\underline{\alpha}}(D\underline{\alpha})_{\Pi}^A B B = -[g^{-1}_A dg_A, l_A]$$

$$= -g^{-1}_A d(g_A l_A g^{-1}_A)g_A = (E^{-1})_{A} \Delta dE_{\Delta} B l_B$$ (2.2)

and so

$$G_{\Pi, A}^B \equiv G_{\Pi, \underline{\alpha}}(D\underline{\alpha})_A^B = (E^{-1})_{A} \Delta \partial_{\Pi} E_{\Delta}^B$$ (2.3)

Using the expression for the vielbein of equation (2.1) we find that

$$G_{\Pi, A}^B = (\frac{(1 - e^{-A})}{A} \wedge \partial_{\Pi} A)_A^B = (\partial_{\Pi} A - \frac{1}{2}[A, \partial_{\Pi} A] + \frac{1}{3!}[A, [A, \partial_{\Pi} A]] + \ldots)_A^B$$ (2.4)

where we have used the identity

$$e^{-D} de^D = \frac{(1 - e^{-D})}{D} \wedge dD$$ (2.5)

valid for any operator, or matrix, $D$ and where $D^n \wedge dD \equiv [D, [D, [D, \ldots [D, dD]]]] \ldots$. 

5
The action of the rigid transformation \( g_0 \in G^{+++} \), which can be written in the form \( g_0 = e^{\alpha A R^A} \), can also be given in explicit form. As the generators \( l_A \) form a representation of \( G^{+++} \) under this transformation, equation (1.1) implies that

\[
g_l \rightarrow g'_l = g_0 g_l g_0^{-1}, \quad \text{and} \quad g_A \rightarrow g'_A = g_0 g_A g_0^{-1} \tag{2.6}
\]

Using equation (1.6) the first equation is found to imply the coordinate change

\[
x^\Delta \rightarrow x'^\Delta = x^\Pi (e^{-a \cdot D})^\Pi \Delta \tag{2.7}
\]

where \( (a \cdot D)^\Pi^\Delta = a_\beta (D^\beta)^\Pi^\Delta \). While the change in the vielbein can be found by considering

\[
(g^{-1}_A)^l \Pi g'_A = (g^{-1}_A)g_0^{-1} l \Pi g_0 g_A = (e^{a \cdot D})^\Pi^\Delta (g^{-1}_A) l \Delta g_A = (e^{a \cdot D})^\Pi^\Delta E^C C l_C \tag{2.8}
\]

and as a result

\[
E^A \rightarrow E'^A = (e^{a \cdot D})^\Pi^\Delta E^C C l_C \tag{2.9}
\]

We note that \( dx^\Pi E^A \) is inert under rigid \( g_0 \) transformations as it should be.

It is often useful not to parameterise the group element \( g_A \) by a single exponential but by a product of exponentials. In this case one just replaces the above matrix expressions by the corresponding products, for example, if let set \( g_A = e^{A_1 \cdot R} \ldots e^{A_n \cdot R} \) then the vielbein takes the form

\[
E^A = (e^{A_1} \ldots e^{A_n})^\Pi A \tag{2.10}
\]

where \( A_1 = A_1 \cdot (D)^\Pi A \) and there are analogous expressions for the above formulae.

To proceed further we will need the Cartan Involution \( I_c \) which can be taken to act on the generators of \( E_{11} \) as \( I_c(R^B) = -R^{-B} \). In fact we have in previous papers taken a plus sign for some of the involutions, but this can be undone by redefining the negative generators. The Cartan involution acts on the \( l_1 \) representation to give another representation denoted by \( \bar{l}^A \) as \( I_c(l_A) = -J_{AB} \bar{l}^B \) for a suitable matrix \( J_{AB} \). Acting on the commutator of equation (1.6) with the Cartan involution we find that

\[
[R^B, \bar{l}^A] = \bar{l}^B (D^\alpha)_{B^A} \tag{2.11}
\]

where

\[
(D^\alpha)_{B^A} = (JD^{-\alpha}J^{-1})_{B^A}, \quad \text{or in matrix form} \quad D^\alpha = (JD^{-\alpha}J^{-1})^T \tag{2.12}
\]

For the case of \( E_{11} \otimes s l_1 \), the \( l_1 \) representation is a lowest weight representation with lowest weight state \( P_1 \) while \( \bar{l}_1 \) is a highest weight representation with highest weight state \( \bar{P}_1 \) where \( P_a, a = 1, \ldots, D \) are the usual space-time translation generators and \( \bar{P}_a, a = 1, \ldots, D \).
We take the local subalgebra in the $G_1 \otimes l_1$ non-linear realisation to be the Cartan involution subgroup of $G_1$ which consists of group elements which obey $I_c(h) = h$. Following similar arguments one finds that the local $h = e^{b_\alpha (R_\alpha - R_{-\alpha})}$ transformation of the generalised vielbein is given by

$$E_{\Pi}^A \rightarrow E_{\Pi}^{A'} = e^{A'} = (e^A e^{b_\beta (D_\beta - D_{-\beta})})_{\Pi} \equiv E_{\Pi}^B (e^{b_\beta (D_\beta - D_{-\beta})})_{B}^A$$

(2.11)

It is sometimes useful to construct the dynamics not from the Cartan forms, but from the object $M = g_A I_c(g_A^{-1})$ which transforms as $M \rightarrow M' = g_0 M I_c(g_0^{-1})$. We note that $I_c(M^{-1}) = M$ and so $M$ can be written in the form $M = e^{\phi_\alpha (R_\alpha + R_{-\alpha})}$ which confirms that $M$ is a group element that belongs to the coset. The matrix representation of $M$ is given by

$$M_A^B l_B \equiv M^{-1} l_A M = (e^{\phi_\alpha (D_\alpha + D_{-\alpha})})_{A}^B l_B = I_c(g) g^{-1} l_A g I_c(g^{-1})$$

$$= (e^A)_{A}^B I_c(g) l_A I_c(g^{-1}) = (e^A)_{A}^B J_{BC}^1 I_c(g) g^{-1} I_c(g) = (e^A e^{\tilde{A}})_{A}^B l_B$$

(2.13)

where the matrix $\tilde{A} = A_\alpha D_{-\alpha}$. The transformation of $M$ can be written, in matrix form, as $M \rightarrow M' = e^{a_\alpha D_\alpha} M e^{a_\alpha D_{-\alpha}}$

3 Explicit computation of the $E_{11}$ generalised vielbein at low levels

In this section we will consider the non-linear realisation of $E_{11} \otimes s_1$ with the local subgroup being the Cartan involution invariant subalgebra of $E_{11}$; the analogue of the maximal compact subalgebra. This non-linear realisation has been conjectured to be the low energy effective action describing strings and branes [6,11]. The representations of $E_{11}$ can be studied by decomposing them into representations of a finite-dimensional Lie algebras, obtained by removing one node from the Dynkin diagram of $E_{11}$. The Dynkin diagram of $E_{11}$ is given by

```
• 11
|   
• – • – … – • – • – • –  
1  2  7  8  9  10
```

The theories with different number of space-time dimensions emerge when computes the non-linear realisation of the $E_{11} \otimes s_1$ algebra when decomposed into the algebras that follow by removing the different possible nodes [12-14]. In this paper we are interested in four particular cases: removing node 11 leads to $GL(11)$ algebra that corresponds to 11-dimensional theory, removing node 9 results in 10-dimensional type IIB theory with $GL(10) \times SL(2, R)$ algebra, removing node 5 leads to $GL(5) \times E_6$ algebra that describes 5-dimensional theory, and, finally, removing node 4 leads to $GL(4) \times E_7$ algebra that corresponds to the 4-dimensional theory. The fields and coordinates in $D$ dimensions can be classified by a level that is given by the number of down minus up $SL(D)$ indices except that one adds one for the coordinates and divides the results by three in eleven dimensions and two for the ten dimensional IIB theory.

7
The $l_1$ representation decomposed in the way suitable to $D$ dimensions leads to a generalised space-time that contains at level zero the usual coordinates $x^a$ and at level one coordinates that are scalars under the Lorentz group but transform as the $10, \overline{16}, \overline{27}, 56$ and $248 \oplus 1$ of SL(5), SO(5,5), $E_6$, $E_7$ and $E_8$ for $D$ equal to seven, six, five, four and three dimensions respectively [16,17]. The corresponding generalised vielbeins have been partially constructed at low levels for these generalised space-times using the $E_{11} \otimes s l_1$ non-linear realisation. One of the first examples was the construction of the generalised vielbein for the five dimensional theory up to level two [15] which, in conjunction with the corresponding generalised space-time, was used to find all maximally supersymmetric gauged supergravities. In the four dimensional theory reference [18] computed the 56 by 56 vielbein that arises in the space of the level one coordinates [18]. The full generalised vielbein up to and including level one in the four dimensional theory was given in [19]. The generalised vielbein, but restricted to the space of the level one coordinates, was also subsequently computed in [20] in dimensions four up to seven inclusive. The eleven dimensional generalised vielbein was computed up to level two in [21]. A metric that appeared in the duality invariant first quantised actions studied in reference [26] was used in reference [25] to discuss theories formulated on a seven dimensional space-time. However, we note that this generalised space-time is just the part of $l_1$ representation of $E_{11}$ at level one in seven dimensions [6,16,17] and the vielbein, or equivalently the metric, is a truncation of the vielbeins found earlier in the context of $E_{11}$ papers.

Siegel theory [22], sometimes called doubled field theory, was developed in 1993. This was motivated by string theory and it consists of a theory with the same massless fields as appear in the NS-NS sector of the superstring, but defined in a 20-dimensional space-time that transformed in the vector representation of O(10,10). A generalised vielbein defined on this space-time, was found in reference [22], it played an important part in the construction of Siegel theory. The Virasoro operators appeared in construction and they were to contain a corresponding metric which agreed with that found when reducing string theory on a torus. This theory was subsequently formulated as the non-linear realisation of $E_{11} \otimes s l_1$ in ten dimensions at level zero [23]. The extension of this theory to include the R-R sector is just the level one contribution and it was first found in reference [24]. The generalised vielbein computed from this later viewpoint agrees with that found earlier.

In this section we calculate the generalised vielbein in eleven, five and four dimensions and also the for the ten dimensional IIB theory at much higher levels using the $E_{11} \otimes s l_1$ non-linear realisation.

### 3.1 $D = 11$

The eleven dimensional theory is obtained by deleting node 11 from the Dynkin diagram of $E_{11}$.

\[
\otimes 11
\]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

and decomposing the $E_{11} \otimes s l_1$ into representations of $GL(11)$ [11]. In this section we will restrict ourselves with level 3 calculations. The non-negative level generators of the $E_{11}$
where

\[ K^{a b}, R^{a_1 a_2 a_3}, R^{a_1 \ldots a_6}, R^{a_1 \ldots a_8, b}. \]  

(3.1.1)

The negative level generators are

\[ R_{a_1 a_2 a_3}; R_{a_1 \ldots a_6}; R_{a_1 \ldots a_8, b}. \]  

(3.1.2)

The \( l_1 \) representation contains the generators \([6]\)

\[ P^{a}, Z^{a_1 a_2}, Z^{a_1 \ldots a_5}, Z^{a_1 \ldots a_8, b}. \]  

(3.1.3)

The group element \( g = g_l g_A \) can be parametrised in the following way:

\[ g_l = \exp \left( x^a P_a + x_{a_1 a_2} Z^{a_1 a_2} + x_{a_1 \ldots a_5} Z^{a_1 \ldots a_5} + x_{a_1 \ldots a_8} Z^{a_1 \ldots a_8} + x_{a_1 \ldots a_7, b} Z^{a_1 \ldots a_7, b} \right), \]

\[ g_A = \exp \left( h^a_b K^{a b} \right) \exp \left( A_{a_1 \ldots a_8, b} R^{a_1 \ldots a_8, b} \right) \exp \left( A_{a_1 \ldots a_8} R^{a_1 \ldots a_8} \right) \exp \left( A_{a_1 a_2 a_3} R^{a_1 a_2 a_3} \right), \]

where we have introduced the generalised coordinates \([6]\)

\[ x^a; x_{a_1 a_2}; x_{a_1 \ldots a_5}; x_{a_1 \ldots a_8}; x_{a_1 \ldots a_7, b}. \]  

(3.1.4)

(3.1.5)

We have used the local subalgebra to gauge away part of the \( g_A \) group element and we have the, by now well known, fields of the \( E_{11} \otimes_{S} l_1 \) non-linear realisation up to level three, namely, the graviton, the three and six form gauge fields and the dual graviton \([11]\). The corresponding generalised tangent space structure is obvious and the tangent space group is \( I_c(E_{11}) \) which at lowest level is just the Lorentz group and at higher levels has an algebra can be found in reference \([6]\) and also the book of reference \([8]\).

The generalised vielbein is defined in equation (1.10) and, while one can straightforwardly compute it using the commutators of appendix A.1, we will find it using the matrix expression of equation (2.10), which in eleven dimensions takes the form

\[ E^{A}_{\Pi} = c^{A_0} e^{A_3} c^{A_2} e^{A_1} \]  

(3.1.6)

where

\[ A_0 = h^a_b D^a_b, \ A_1 = A_{a_1 a_2 a_3} D^{a_1 a_2 a_3}, \ A_2 = A_{a_1 \ldots a_6} D^{a_1 \ldots a_6}, \ A_3 = A_{a_1 \ldots a_8, b} D^{a_1 \ldots a_8, b} \]  

(3.1.7)

We begin with the level zero matrix which is given by the expression

\[ dx \cdot (A_0) \cdot l = -[h^a_b K_a^b, dx^c P_c] + dx_{c_1 c_2} Z^{c_1 c_2}, \]

\[ +dx_{c_1 \ldots c_5} Z^{c_1 \ldots c_5} + dx_{c_1 \ldots c_8} Z^{c_1 \ldots c_8} + dx_{c_1 \ldots c_7, c} Z^{c_1 \ldots c_7, c} \]  

(3.1.8)

from which we conclude that

\[
(A_0) = 
\begin{pmatrix}
  h^a_b & 0 & -2\delta^{[b_1}_{a_1} h_{a_2]} b_2] & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & -5\delta^{[b_1}_{a_1 \ldots a_4} h_{a_5]} b_5] & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -8\delta^{[b_1 \ldots h_7}_{a_1 \ldots a_7} h_{a_8]} b_8] & 0 \\
  0 & 0 & 0 & 0 & 0 & k^{c_1 \ldots c_7, d}_{a_1 \ldots a_7, c} \\
  -\frac{1}{2} h^c_c I
\end{pmatrix}
\]  

(3.1.9)
We now compute $A$ and $I$ where

$$\delta^{a_1\ldots a_7} b_{a_7} - \delta^{a_1\ldots a_7} b_{a_7} + 8 \delta^{a_1\ldots a_7} b_{a_7}$$

and $I$ is the identity matrix.

It then follows that

$$e^{A_0} = (\det e)^{-\frac{1}{2}}$$

where $e = (e^h)^b_c$ and

$$\left( e^{-1} \right)_{a_1\ldots a_n}^{\mu_1\ldots \mu_n} = \left( e^{-1} \right)_{a_1\ldots a_n}^{\mu_1\ldots \mu_n},$$

$$\left( e^{-1} \right)_{a_1\ldots a_7, b}^{\mu_1\ldots \mu_7, \nu} = \left( e^{-1} \right)_{a_1\ldots a_7}^{\mu_1\ldots \mu_7} (e^{-1})_b^{\nu} - \left( e^{-1} \right)_{a_1\ldots a_7}^{\mu_1} (e^{-1})_b^{\nu}. \quad (3.1.11)$$

We now compute $A_1$ in a similar way by considering

$$dx \cdot (A_1) \cdot l = -[A_{a_1a_2a_3} R^{a_1a_2a_3}, dx^c P_c + x_{c_1c_2} Z^{c_1c_2} + dx_{c_1\ldots c_5} Z^{c_1\ldots c_5} + dx_{c_1\ldots c_7, c} Z^{c_1\ldots c_7, c}] \quad (3.1.12)$$

from which we conclude, using the commutators of appendix A.1, that

$$\left( A_1 \right) = \left( \begin{array}{cccccc} 0 & -3 A_{ab_1b_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (3.1.13)$$

where $A^{a_1\ldots a_5} b_{a_7} = -\delta^{a_1\ldots a_5} A_{b_7} b_{a_7} + \delta^{a_1\ldots a_5} A_{b_7} b_{a_7}$. Proceeding in a similar way we find that

$$\left( A_2 \right) = \left( \begin{array}{cccccc} 0 & 0 & 3 A_{ab_1b_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad (3.1.14)$$

and

$$\left( A_3 \right) = \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (3.1.15)$$
To compute the generalised vielbein we just need to evaluate the matrix expression of equation (3.1.6), being careful to evaluate the unusual index sets, we find that

$$E_1^A = (\det e)^{-\frac{1}{2}}$$

$$\begin{pmatrix}
  e_\mu^a & e_\mu^b & \alpha_b |_{a_1 a_2} & e_\mu^b & \alpha_b |_{a_1 \ldots a_5} & e_\mu^b & \alpha_b |_{a_1 \ldots a_8} & e_\mu^b & \alpha_b |_{a_1 \ldots a_7, a} \\
  0 & (e^{-1})_{a_1 a_2} & \beta^{b_1 b_2} & (e^{-1})_{b_1 b_2} & \beta^{b_1 b_2} & (e^{-1})_{b_1 b_2} & \beta^{b_1 b_2} & (e^{-1})_{b_1 b_2} & \beta^{b_1 b_2} \\
  0 & 0 & 0 & (e^{-1})_{a_1 \ldots a_5} & (e^{-1})_{a_1 \ldots a_5} & (e^{-1})_{a_1 \ldots a_5} & (e^{-1})_{a_1 \ldots a_5} & (e^{-1})_{a_1 \ldots a_5} & (e^{-1})_{a_1 \ldots a_5} \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \end{pmatrix}$$

where the symbols in the first line of this matrix are given by

$$\alpha_a |_{a_1 a_2} = -3 A_{a_1 a_2}, \quad \alpha_a |_{a_1 \ldots a_5} = 3 A_{a_1 \ldots a_5} + \frac{3}{2} A_{a_1 a_2} A_{a_3 a_4 a_5},$$

$$\alpha_a |_{a_1 \ldots a_8} = \frac{3}{2} A_{a_1 \ldots a_8}, a - 3 A_{a_1 \ldots a_5} A_{a_6 a_7 a_8},$$

$$\alpha_a |_{a_1 \ldots a_7, b} = \frac{4}{3} A_{a_1 \ldots a_7}, b + 3 A_{a_1 \ldots a_5} A_{a_6 a_7 b} - \frac{4}{3} A_{a_1 \ldots a_7}, b$$

$$- 3 A_{a_1 \ldots a_5} A_{a_6 a_7 b} - \frac{1}{2} A_{a_1 a_2} A_{a_3 a_4 a_5} A_{a_6 a_7 b},$$

the symbols in the second line are given by

$$\beta^{b_1 b_2} |_{a_1 \ldots a_5} = - \delta_{a_1 a_2}^{b_1 b_2} A_{a_3 a_4 a_5}, \quad \beta^{b_1 b_2} |_{a_1 \ldots a_5} = \delta_{a_1 a_2}^{b_1 b_2} A_{a_3 \ldots a_8},$$

$$\beta^{b_1 b_2} |_{a_1 \ldots a_7, b} = \delta_{a_1 a_2}^{b_1 b_2} A_{a_3 \ldots a_7}, b + \frac{1}{2} \delta_{a_1 a_2}^{b_1 b_2} A_{a_3 a_4 a_5} A_{a_6 a_7 b} - \delta_{a_1 a_2}^{b_1 b_2} A_{a_3 \ldots a_7}, b,$$

and, finally, the symbols in the third line are given by

$$\gamma^{b_1 \ldots b_5} |_{a_1 \ldots a_8} = - \delta_{a_1 \ldots a_5}^{b_1 \ldots b_5} A_{a_6 a_7 a_8}, \quad \gamma^{b_1 \ldots b_5} |_{a_1 \ldots a_7, b} = \delta_{a_1 \ldots a_5}^{b_1 \ldots b_5} A_{a_6 a_7 b} - \delta_{a_1 \ldots a_5}^{b_1 \ldots b_5} A_{a_6 a_7 b}.$$

### 3.2 $D = 10$

There are two ways of obtaining a ten-dimensional theory: removing node 10\ leads to type IIA supergravity theory, while removing node 9 leads to type IIB supergravity [12]. We will be interested in the latter. The corresponding Dynkin diagram is

```
  . 10
  |   |
  \|   |
   1  2  3  4  5  6  7  8  9  11
```
In this case all the generators fall into representations of $GL(10) \times SL(2, R)$. The non-negative level generators of adjoint representation up to level 5 are $[12, 27]

\begin{align*}
K^a_{\beta}, R_{\alpha \beta}, R^{a_1 a_2}_{\alpha}, R^{a_1 a_4}_{\alpha}, R^{a_1 a_6}_{\alpha}, R^{a_1 a_8}_{\alpha}, K^{a_1 a_7, b}, \\
R^{a_1 a_{10}}_{\alpha \beta \gamma}, R^{a_1 a_8 b_1}_{\alpha}, R^{a_1 a_9 b}_{\alpha}.
\end{align*}

The negative level generators are

\begin{align*}
R^{a}_{a_1 a_2}, R^{a}_{a_1 a_4}, R^{a}_{a_1 a_6}, R^{a}_{a_1 a_8}, K^{a_1 a_7, b} \\
R^{a \beta \gamma}_{a_1 a_{10}}, R^{a \alpha}_{a_1 a_8 b_1}, R^{a}_{a_1 a_9 b}.
\end{align*}

The $l_1$ representation generators up to level five are

\begin{align*}
P_a; Z^a_{\alpha}, Z^{a_1 a_2 a_3}_{\alpha}, Z^{a_1 a_5}_{\alpha}, Z^{a_1 a_7}_{\alpha}, Z^{a_1 a_6 b}_{\alpha}, \\
Z^{a_1 a_9}_{\alpha \beta \gamma}, Z^{a_1 a_9}_{\alpha}, Z^{a_1 a_9 b}_{\alpha}, Z^{a_1 a_8 b}_{\alpha}, Z^{a_1 a_7 b_1 b_2}_{\alpha}.
\end{align*}

We note that some of the $l_1$ generators at level five have multiplicity two. Although we have listed the generators up to level five we will only use the generators up to level four, that is, we will work just up to level four in what follows.

Lower case Greek indexes correspond to the fundamental representation of $SL(2, R)$ ($\alpha, \beta, \gamma, \ldots = 1, 2$). Tensors that have multiple Greek indexes are assumed to be symmetric in these indices. The general group element $g = g_A g_A$, up to level 4, can be written as

\begin{align*}
g_l = \exp \left( x^a P_a + x^a \alpha \beta Z^a_{\alpha} \beta + x^{a_1 a_2 a_3} Z^{a_1 a_2 a_3} + x^{a_1 a_2 a_5} Z^{a_1 a_5}_{\alpha} \beta + x^{a_1 a_7} Z^{a_1 a_7}_{\alpha} \beta \right) \\
+ x^{a_1 a_7} Z^{a_1 a_7}_{\alpha} \beta + x^{a_1 a_8 b} Z^{a_1 a_8 b}_{\alpha} \beta, \\
g_A = \exp \left( h^a_{\alpha} K^a_{\alpha} \right) \exp \left( \varphi^{a \beta} R^{a \beta} \right) \exp \left( A^{a_1 a_7, b} K^{a_1 a_7, b} \right) \exp \left( A^{a_1 a_8} R^{a_1 a_8}_{\alpha \beta} \right) \\
\times \exp \left( A^{a_1 a_6} R^{a_1 a_6} \right) \exp \left( A^{a_1 a_4} R^{a_1 a_4} \right) \exp \left( A^{a_1 a_2} R^{a_1 a_2} \right).
\end{align*}

Where we have introduced the generalised coordinates

\begin{align*}
x^a ; x^a_{\alpha} ; x^{a_1 a_2 a_3} ; x^{a_1 a_5} ; x^{a_1 a_7} ; x^{a_1 a_8} ; x^{a_1 a_9} ; x^{a_1 a_6 b}.
\end{align*}

The tangent space group is $I_{\epsilon}(E_{11})$ which at level zero is $SO(1, 9) \times SO(2)$. It is very straightforward to compute at higher levels.

In this section we are going to calculate the generalised vielbein using its definition in equation (1.10) rather than the matrix method of section two. In this approach the generalised vielbein is computed by conjugating the $l_1$ generators with the $E_{11}$ group element. We recall that

\begin{align*}
E_{a_1 a_2} l_A = g_A^{-1} l_{a_1 a_2} g_A.
\end{align*}
Using the algebra from Appendix A.2 we can perform this conjugation for the $D = 10$ case. Conjugation with level 0 group element gives

\[
\exp (-\varphi^{\alpha\beta} R_{\alpha\beta}) \exp (-h_a^b K^a b) \left\{ P_\mu, Z_\alpha^\mu, Z^\mu_1 \mu_2 \mu_3, Z_{\alpha_1 \alpha_2}^\mu, Z_{\bar{\alpha}_1 \bar{\alpha}_2}^\mu, Z^{\mu_1 \cdots \mu_7}, Z^{\mu_1 \cdots \mu_7}, Z^{\mu_1 \cdots \mu_6, \nu} \right\} \\
\times \exp (h_a^b K^a b) \exp (\varphi^{\alpha\beta} R_{\alpha\beta}) =
\]

\[
= (\det e)^{-\frac{1}{2}} \left\{ e^\mu_a P_a, (e^{-1})_a^\mu g_\alpha^\beta Z_\beta^a, (e^{-1})_{\alpha_1 a_2 a_3}^{\mu_1 \mu_2 \mu_3} Z_a^{1 a_2 a_3}, (e^{-1})_{a_1 a_5}^{\mu_1 \cdots \mu_5} g_\alpha^\beta Z_{\beta 1}^a, (e^{-1})_{a_1 a_7}^{\mu_1 \cdots \mu_7} g_\alpha^\beta Z_{1 a_7}, (e^{-1})_{a_1 a_6, b}^{\mu_1 \cdots \mu_6, \nu} Z_{a_1 a_6, b}^a \right\},
\] (3.2.7)

where $\epsilon^b_\mu = (e^h)^b_\mu$, $g_\alpha^\beta = (\epsilon^\gamma \gamma \varphi^* \gamma^* \gamma^*)_\alpha^\beta$ and

\[
(e^{-1})_{a_1 \cdots a_n}^{\mu_1 \cdots \mu_n} = (e^{-1})_{a_1}^{\mu_1} \cdots (e^{-1})_{a_n}^{\mu_n}, \quad g_{\alpha_1 \cdots \alpha_n} = g_{[\alpha_1 \cdots \alpha_n]}^\gamma \gamma^n,
\]

\[
(e^{-1})_{a_1 \cdots a_6, b}^{\mu_1 \cdots \mu_6, \nu} = (e^{-1})_{a_1}^{\mu_1} \cdots (e^{-1})_{a_6}^{\mu_6} (e^{-1})_b^\nu - (e^{-1})_{a_1}^{\mu_1} \cdots (e^{-1})_{a_6}^{\mu_6} (e^{-1})_b^\nu \nu.
\] (3.2.8)

In the above equation and what follows we denote world, rather than tangent, $SL(2)$ indices with a dot, that is $\dot{\alpha}, \ldots$. Conjugating with positive level generators can be obtained by Taylor-expanding the exponents and truncating the series by level 4. For level one $E_{11}$ generator we have

\[
\exp (-A^a_{b_1 b_2} R^b_{a_1 a_2}) \left\{ P_a, Z_\alpha^a, Z_{a_1 a_2 a_3}, Z_{a_1 a_2 ... a_5} \right\} \exp (A^a_{b_1 b_2} R^b_{a_1 a_2}) =
\]

\[
= P_a - A^a_{b_1 b_2} Z_b^a + \frac{1}{2} \varepsilon_{\alpha \beta} A^a_{b_1 a_1} A^b_{a_2 a_3} Z_{a_1 a_2 a_3} - \frac{1}{6} \varepsilon_{\alpha \beta} A^a_{b_1 a_1} A^b_{a_2 a_3} A^\gamma_{a_4 a_5} Z_{\gamma}^{a_1 \cdots a_5} +
\]

\[
\frac{1}{24} \varepsilon_{\alpha \beta} A^a_{b_1 a_1} A^b_{a_2 a_3} A^\alpha_{a_4 a_5} A^\beta_{a_6 a_7} Z_{a_1 a_2 a_3} - \frac{1}{60} \varepsilon_{\alpha \beta} \varepsilon_{\sigma \lambda} A^a_{b_1 a_1} A^b_{a_2 a_3} A^\gamma_{a_4 a_5} A^\delta_{a_6 a_7} Z_{\gamma}^{a_1 \cdots a_7} +
\]

\[
Z_{a_1}^{a_1} - \varepsilon_{\alpha \beta} A^a_{a_2 a_3} Z_{a_1 a_2 a_3} + \frac{1}{2} \varepsilon_{\alpha \beta} A^a_{a_2 a_3} A^\gamma_{a_4 a_5} Z_{\gamma}^{a_1 a_2 a_3} - \frac{1}{6} \varepsilon_{\alpha \beta} A^a_{a_2 a_3} A^\gamma_{a_4 a_5} A^\delta_{a_6 a_7} Z_{\gamma}^{a_1 a_2 a_3} +
\]

\[
\frac{1}{15} \varepsilon_{\alpha \beta} \varepsilon_{\sigma \lambda} A^a_{a_2 a_3} A^\sigma_{a_4 a_5} A^\lambda_{a_6 a_7} Z_{a_1 a_2 a_3} - A^a_{a_4 a_5} Z_{a_1 a_2 a_3} - \frac{1}{2} A^a_{a_4 a_5} A^b_{a_6 a_7} Z_{a_1 a_2 a_3} -
\]

\[
\frac{1}{5} \varepsilon_{\alpha \beta} A^a_{a_4 a_5} A^\beta_{a_6 a_7} Z_{a_1 a_2 a_3} - Z_{a_1 a_2 a_3} - \frac{2}{5} \varepsilon_{\alpha \beta} A^a_{a_4 a_5} Z_{a_1 a_2 a_3}.
\] (3.2.9)

For level 2 $E_{11}$ generator:

\[
\exp (-A_{b_1 \cdots b_4} R^{b_1 \cdots b_4}) \left\{ P_a, Z_{a_1}^{a_1}, Z_{a_1 a_2 a_3} \right\} \exp (A_{b_1 \cdots b_4} R^{b_1 \cdots b_4}) =
\]

\[
= P_a - 2 A_{a a_1 a_2 a_3} Z_{a_1 a_2 a_3} + 2 A_{a a_1 a_2 a_3} A_{a_4 a_7} Z_{a_1 a_2 a_3} - \frac{4}{5} A_{a a_1 a_2 a_3} A_{a_4 a_5 a_6} Z_{a_1 a_2 a_3}.
\]
\[ Z^a_1 + A_{a_2...a_5} Z^a_1...a_5, Z^{a_1a_2a_3} - 2 A_{a_4...a_7} Z^{a_1...a_7} + \frac{4}{5} A_{a_4a_5a_6b} Z^{a_1...a_6,b} \]  

(3.2.10)

For level 3 \( E_{11} \) generator:

\[
\exp \left( -A^\beta_{b_1...b_6} R_{b_1...b_6}^{\beta} \right) \left\{ P_a, Z^a_1 \right\} \exp \left( A^\beta_{b_1...b_6} R_{b_1...b_6}^{\beta} \right) = 
\]

\[
P_a - \frac{3}{4} A^\alpha_{a_1...a_5} Z^{a_1...a_5}_\alpha, Z^a_1 + \frac{1}{4} A^\beta_{a_2...a_7} Z^{a_1...a_7} + \frac{3}{4} \varepsilon_{\alpha\beta} A^\beta_{a_2...a_7} Z^{a_1...a_7} + \frac{1}{20} \varepsilon_{\alpha\beta} A^\beta_{a_2...a_7} Z^{a_2...a_7, a_1}. 
\]

(3.2.11)

and, finally, for level \( E_{11} \) 4 generators:

\[
\exp \left( -A^\beta_{b_1...b_8} R_{b_1...b_8}^{\beta} \right) P_a \exp \left( A^\beta_{b_1...b_8} R_{b_1...b_8}^{\beta} \right) = P_a + A^\alpha_{a_1...a_7} Z^{a_1...a_7}, 
\]

(3.2.12)

\[
\exp \left( -A_{b_1...b_7, b} R_{b_1...b_7, b} \right) P_a \exp \left( A_{b_1...b_7, b} R_{b_1...b_7, b} \right) = P_a + 3 A_{a_1...a_7, a} Z^{a_1...a_7} - \frac{21}{20} A_{a_1...a_6, b} Z^{a_1...a_6, b}. 
\]

Using all these results we find, from equation (3.2.5), that the generalised vielbein is given by

\[ E_{\Pi^A} = (\det e)^{-\frac{1}{2}} \]

(3.2.13)

In the above world indices, that is, \( \mu, \ldots \) or \( \dot{\alpha}, \ldots \) arise, as vielbeins acting on objects with tangent indices, for example

\[
\alpha^\beta_\mu | a = e_\mu^b \alpha_\beta_{b | a}, \quad \beta^\mu_\alpha | a_1a_2a_3 = (e^{-1})^\mu_{b} \gamma_\alpha^{\beta}_{b} \gamma_\gamma^{\beta}_{a_1a_2a_3}, 
\]

\[
\gamma^{\mu_1\mu_2\mu_3}_\beta | a_1...a_5 = (e^{-1})^{\mu_1\mu_2\mu_3}_{b_1b_2b_3} \gamma_\beta^{b_1b_2b_3}_a | a_1...a_5; 
\]

\[
\chi^{\mu_1...\mu_5}_\dot{\alpha} | a_1...a_7 = (e^{-1})^{\mu_1...\mu_5}_{b_1...b_5} g_{\dot{\alpha}}^{\gamma} \gamma^{b_1...b_5}_{\beta} | a_1...a_7; \ldots 
\]

\[
\chi^{\mu_1...\mu_5}_\dot{\alpha} | a_1...a_7 = (e^{-1})^{\mu_1...\mu_5}_{b_1...b_5} \gamma_\dot{\alpha}^{b_1...b_5} | a_1...a_7, 
\]
\[
\chi^{\mu_1...\mu_5}_{|a_1...a_6,b} = (e^{-1})^{\mu_1...\mu_5}_{b_1...b_5} g_{\alpha\gamma} \chi^{b_1...b_5}_{|a_1...a_6,b} \ldots
\]

The symbols in the first line of the above matrix are given by

\[
\begin{align*}
\alpha_{a|b} &= -A^\alpha_{a b}, \quad \alpha_{a|a_1 a_2 a_3} = -2 A_{a a_1 a_2 a_3} + \frac{1}{2} \varepsilon_{\alpha \beta} A^\alpha_{a [a_1} A^\beta_{a_2 a_3]}, \\
\alpha_{a|a_1 a_2 a_3 a_5} &= -\frac{3}{4} A^\alpha_{a a_1 ... a_5} + 2 A_{a [a_1 a_2 a_3} A^\alpha_{a_4 a_5]} - \frac{1}{6} \varepsilon_{\beta \gamma} A^\beta_{a [a_1} A^\gamma_{a_2 a_3} A^\alpha_{a_4 a_5]}, \\
\alpha_{a|a_1 a_2 a_7} &= \alpha_{a_1 a_2 ... a_7} + \frac{3}{4} A^{(a_1}_{a[a_1 ... a_5} A^{a_2)}_{a_6 a_7]} - A_{a[a_1 a_2 a_3} A^\alpha_{a_4 a_5} A^\alpha_{a_6 a_7]} \\
&\quad + \frac{1}{24} \varepsilon_{\beta \gamma} A^\beta_{a [a_1} A^\gamma_{a_2 a_3} A^\alpha_{a_4 a_5} A^\alpha_{a_6 a_7]}, \\
\alpha_{a|a_1 a_2 a_3 a_4 a_5 a_6 a_7} &= 3 A_{a a_4 a_5 a_6 a_7} + \frac{3}{10} \varepsilon_{\alpha \beta} A^\alpha_{a [a_1 ... a_5} A^\beta_{a_6 a_7]} + 2 A_{a[a_1 a_2 a_3} A^\alpha_{a_4 a_5 a_6} A^\beta_{a_6 a_7]} \\
&\quad - \frac{4}{5} A_{a[a_1 a_2 a_3} A^\alpha_{a_4 a_5 a_6} b + \frac{4}{5} A_{a[a_1 a_2 a_3} A^\alpha_{a_4 a_5 a_6} b + \frac{2}{5} \varepsilon_{\alpha \beta} A^\alpha_{a[a_1 a_2 a_3} A^\beta_{a_4 a_5 a_6} A^\beta_{a_6 a_7]} \\
&\quad - \frac{1}{60} \varepsilon_{\alpha \beta} \varepsilon_{\sigma \lambda} A^\sigma_{a [a_1} A^\beta_{a_2 a_3} A^\sigma_{a_4 a_5} A^\lambda_{a_6]} b,
\end{align*}
\]

in the second line are

\[
\begin{align*}
\beta^a_{a[a_1 a_2 a_3} &= -\varepsilon_{\alpha \beta} \delta^a_{[a_1} A^\beta_{a_2 a_3]}, \quad \beta^a_{a[a_1 ... a_5} &= \delta^a_{[a_1} A^\beta_{a_2 a_3} a_5 + \frac{1}{2} \varepsilon_{\alpha \gamma} \delta^a_{[a_1} A^\beta_{a_2 a_3} A^\alpha_{a_4 a_5]}, \\
\beta^a_{a[a_1 a_2 a_7} &= \frac{1}{4} \delta^a_{[a_1} \delta^a_{a_2 a_7} A^{a_3}_{a_2 a_7]} - \delta^a_{[a_1} \delta^a_{a_2 ... a_5} A^{a_2}_{a_6 a_7]} - \frac{1}{6} \varepsilon_{\alpha \gamma} \delta^a_{[a_1} A^\beta_{a_2 a_3} A^\beta_{a_4 a_5} A^\alpha_{a_6 a_7]} \\
&\quad + \frac{3}{4} \varepsilon_{\alpha \beta} \delta^a_{[a_1} A^\beta_{a_2 a_7]} - \varepsilon_{\alpha \beta} \delta^a_{[a_1} A^\beta_{a_2 a_5} A^\beta_{a_6 a_7]}, \quad \beta^a_{a[a_1 ... a_6, b} = \frac{1}{20} \varepsilon_{\alpha \beta} \delta^a_{b} A^\beta_{a[a_1 ... a_6} b \\
&\quad - \frac{2}{5} \varepsilon_{\alpha \beta} \delta^a_{b} A^\beta_{a[a_1 ... a_5} A^\beta_{a_6] b} + \frac{1}{15} \varepsilon_{\alpha \beta} \varepsilon_{\sigma \lambda} \delta^a_{[a_1} A^\beta_{a_2 a_3} A^\lambda_{a_4 a_5} A^\beta_{a_6] b},
\end{align*}
\]

in the third line are

\[
\begin{align*}
\gamma^b_{b_1 b_2 b_3}_{|a_1 ... a_5} &= -\delta^b_{b_1 b_2 b_3} A^\beta_{a_4 a_5]}, \quad \gamma^b_{b_1 b_2 b_3}_{|a_1 ... a_7} = \frac{1}{2} \delta^b_{b_1 b_2 b_3} A^\beta_{a_4 a_5} A^\beta_{a_6 a_7]}, \\
\gamma^b_{b_1 b_2 b_3}_{|a_1 ... a_7} &= -2 \delta^b_{b_1 b_2 b_3} A^\beta_{a_4 a_5}], \quad \gamma^b_{b_1 b_2 b_3}_{|a_1 ... a_6, b} = \frac{4}{5} \delta^b_{b_1 b_2 b_3} A^\beta_{a_4 a_5 a_6] b} \\
&\quad - \frac{4}{5} \delta^b_{b_1 b_2 b_3} A^\beta_{a_4 a_5 a_6} b - \frac{1}{5} \varepsilon_{\alpha \beta} \delta^b_{b_1 b_2 b_3} A^\alpha_{a_4 a_5} A^\beta_{a_6] b}.
\end{align*}
\]

(3.2.16)
and finally in the fourth line are

\[
\chi^{b_1\ldots b_5}_{\alpha} = -\delta^{(a_1 \delta b_1\ldots b_5}_{[a_1\ldots a_5} A^a_{a_6 a_7]} A^\beta_{a_6 a_7]}, \quad \chi^{a_1\ldots a_7} = -\varepsilon_{\alpha\beta} \delta^{b_1\ldots b_5}_{[a_1\ldots a_5} A^\beta_{a_6 a_7]}.
\]

(3.2.17)

\[\chi^{b_1\ldots b_5}_{\alpha} = -\delta^{(a_1 \delta b_1\ldots b_5}_{[a_1\ldots a_5} A^a_{a_6 a_7]} A^\beta_{a_6 a_7]}, \quad \chi^{a_1\ldots a_7} = -\varepsilon_{\alpha\beta} \delta^{b_1\ldots b_5}_{[a_1\ldots a_5} A^\beta_{a_6 a_7]}.
\]

3.3 \( D = 5 \)

The five dimensional theory is obtained by deleting node 5 from the \( E_{11} \) Dynkin diagram, given below, to find the algebra \( GL(5) \times E_6 \) and decomposing the \( E_{11} \otimes_s I_1 \) algebra into representations of this algebra [15].

\[
\begin{array}{cccccccccc}
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

In this decomposition the positive, including zero, level generators of the \( E_{11} \) algebra are

\[
K^a_b, \quad R^a, \quad R^a_M, \quad R^{a_1 a_2}_{N}, \quad R^{a_1 a_2 a_3.\alpha},
\]

\[
R^{a_1 a_2.\alpha}, \quad R^{a_1... a_i N_{i_1} N_{i_2}}, \quad R^{a_1 b_1 b_2 b_3 N}, \ldots
\]

where \( R^{[a_1 a_2.\alpha]} = 0 \) and \( R^{[a_1 b_1 b_2 b_3]N} = 0 \), while those with negative level are given by

\[
R_{a_M}, \quad R^{a_1 a_2 N}, \quad R^{a_1 a_2 a_3 \alpha}, \quad R^{a_1 a_2 b}, \quad R^{a_1... a_i N_{i_1} N_{i_2}}, \quad R^{a_1 b_1 b_2 b_3 N}, \ldots
\]

(3.3.2)

The \( l_1 \) representation decomposes to give the generators [15]

\[
P_a, \quad Z^N, \quad Z^a_N, \quad Z^{a_1 a_2.\alpha}, \quad Z^{a_1 a_2 b}, \quad Z^{a_1 a_2 a_3 b N}, \quad Z^{a_1 a_2 a_3 N_{i_1} N_{i_2}}, \ldots
\]

(3.3.3)

The fifth generator does not obey \( Z^{[a_1 a_2.\alpha]} = 0 \) and the third generator \( Z^{a_1 a_2} \) has no symmetries on its two indices. For these objects the lower case Latin indexes correspond to 5-dimensional representation of \( GL(5) \) (\( a, b, c, \ldots = 1, \ldots, 5 \)). Greek indexes correspond to 78-dimensional adjoint representation of \( E_6 \) (\( \alpha, \beta, \gamma, \ldots = 1, \ldots, 78 \)). Upper and lower case Latin indexes correspond to 27-dimensional and 27-dimensional representations respectively of \( E_6 \) (\( N, M, P, \ldots = 1, \ldots, 27 \)). The 351-dimensional representation can be written as two antisymmetrised indices ie \( X_{N M} \).

An arbitrary group element can be parametrised in the following way:

\[
g_I = \exp (x^a P_a + x_N Z^N + x_{a_1} Z^{a_1 a_2.\alpha} Z^{a_1 a_2} + x_{a b} Z^{a b}),
\]

\[
g_A = \exp (h_a^b K_{a b}) \exp (\varphi_a R^a) \exp (A_{a_1 a_2 a_3,\alpha} R^{a_1 a_2 a_3,\alpha}) \times
\]

\[
\times \exp (A_{a_1 a_2, b} R^{a_1 a_2, b}) \exp (A_{a_1 a_2 N} R^{a_1 a_2} N) \exp (A_{a N} R^a N).
\]

(3.3.4)
We find that the five dimensional theory has a generalised space-time that has the coordinates
\[ x^a, x_N, x_a^N, x_{a_1a_2,\alpha}, x_{ab}, \ldots \] (3.3.5)
and the fields
\[ h_a^b, \varphi_\alpha, A_{aM}, A_{a_1a_2}^N, A_{a_1a_2a_3,\alpha}, A_{a_1a_2,b}, \ldots \] (3.3.6)
which depend on the generalised space-time. The tangent space structure is obvious from the presence of the coordinates and the tangent space group is \( I_c(E_{11}) \) which are lowest level is \( SO(1,4) \otimes Usp(8) \). The generalised vierbein is defined in equation (1.10) and it is straight forward, using the commutators in appendix A.3, to find the generalised vielbein. However, one can also the matrix expression of equation (2.1), or more appropriately equation (2.10), which in the five dimensional case takes the form
\[ E_{\Pi}^A = e^{A_0} e^{A_3} e^{A_3} e^{A_2} e^{A_{3}}, \] (3.3.7)
where
\[ A_0 \equiv h_a^b D_a^b, \quad \tilde{A}_0 \equiv \varphi_\alpha D^\alpha, \quad A_1 \equiv A_{aN} D_a^N, \quad A_2 \equiv A_{a_1a_2}^N D_{a_1a_2}^N, \quad A_3 \equiv A_{a_1a_2a_3}, D_{a_1a_2a_3,\alpha}, \quad \tilde{A}_3 \equiv A_{a_1a_2,b} D_{a_1a_2,b}. \] (3.3.8)

We will compute the generalised vielbein up to level three. We begin by considering the level zero part and noting that
\[ dx \cdot (A_0) \cdot l = - [h_a^b K_a^b, dx^a P_a + dx_N Z_a^N + dx_a^N Z_a^N + dx_{a_1a_2,\alpha} Z_{a_1a_2,\alpha} + dx_{ab} Z_{ab}], \] (3.3.9)
and
\[ dx \cdot (\tilde{A}_0) \cdot l = - [\varphi_\alpha R^\alpha, dx^a P_a + dx_N Z_a^N + dx_a^N Z_a^N + dx_{a_1a_2,\alpha} Z_{a_1a_2,\alpha} + dx_{ab} Z_{ab}], \] (3.3.10)
from which we conclude that
\[ A_0 = \begin{pmatrix} h_a^b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_a^b \delta_M^N & 0 & 0 \\ 0 & 0 & 0 & -2h_{b_1}^{[a_1} \delta_{b_2]}^N \delta_{\beta_1}^\alpha & 0 \\ 0 & 0 & 0 & -h_c^a \delta_d^b - \delta_c^a h_d^b \end{pmatrix} - \frac{1}{2} h_e^a I, \] (3.3.11)
and
\[ \tilde{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (D^\alpha)_M^N & 0 & 0 \\ 0 & 0 & \delta_{b_1}^a \varphi_\alpha (D^\alpha)_N^M & 0 & 0 \\ 0 & 0 & 0 & \delta_{b_1b_2}^{a_1a_2} \varphi_\gamma f^{\gamma\alpha\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \] (3.3.12)
It then follows that
\[ e^{A_0} e^{\tilde{A}_0} = (\det e)^{-\frac{1}{2}} \times \]
\[
\begin{pmatrix}
\varepsilon_{\mu}^a & 0 & 0 & 0 & 0 \\
0 & (d^{-1})_M \dot{N} & 0 & 0 & 0 \\
0 & 0 & d_{\bar{N}}^M (e^{-1})_{a}^\mu & 0 & 0 \\
0 & 0 & 0 & (e^{-1})_{a_1a_2}^{\mu_1\mu_2} (f^{-1})_{\bar{\beta} \bar{\alpha}} & 0 \\
0 & 0 & 0 & 0 & (e^{-1})_{a}^\mu (e^{-1})_{b}^\nu
\end{pmatrix},
\]

where
\[
\varepsilon_{\mu}^a = (e^h)^\mu_a, \quad d_{\dot{N}}^M = \left(e^{\varphi_a D_\alpha}\right)_{\dot{N}}^M, \quad f_{\bar{\alpha} \bar{\beta}} = \left(e^{\varphi_f \gamma \bullet \bullet}\right)_{\bar{\alpha}}^\beta,
\]

and
\[
(e^{-1})_{a_1...a_n}^{\mu_1...\mu_n} = (e^{-1})_{[a_1}^{\mu_1} ... (e^{-1})_{a_n]}^{\mu_n}, \quad d_{\dot{N}_1...\dot{N}_n}^{M_1...M_n} = d_{[\dot{N}_1}^{M_1} ... d_{\dot{N}_n]}^{M_n}.
\]

A dot over an index means that it is a world rather than a tangent index.

We now compute \(A_1\) in a similar way by considering
\[
dx(A_1) \cdot l = -\left[A_{aN} R^a N, \ dx^a P_a + dx_N Z^N + dx_a^N Z^a_N + dx_{a_1a_2, \alpha} Z^{a_1a_2, \alpha} + dx_{ab} Z^{ab}\right],
\]
from which we conclude, using the commutators of appendix A.3, that
\[
A_1 = \begin{pmatrix}
0 & -A_{aM} & 0 & 0 & 0 \\
0 & 0 & d^{NMP} A_{bP} & 0 & 0 \\
0 & 0 & 0 & - (D_{\beta})_{N}^{M} \delta_{[a_1}^{b_1} A_{b_2]M} A_{cN} \delta_{d]}^{a} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Proceeding in a similar way we find that
\[
A_2 = \begin{pmatrix}
0 & 0 & \frac{1}{2} A_{ab}^{M} & 0 & 0 \\
0 & 0 & 0 & (D_{\beta})_{P}^{N} A_{b_1b_2}^{P} & -2 A_{cd}^{N} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
0 & 0 & 0 & -3 A_{ab_1b_2, \beta} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and
\[
\tilde{A}_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & -4 A_{d(a,c)} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
It is now just a matter of matrix multiplication, albeit with unusual index sets, to find the generalised vielbein using equation (3.3.7), the result is

\[ E_\Pi^A = (\det e)^{-\frac{1}{2}} \]

\[
\begin{pmatrix}
  e_{\mu a}^b & e_{\mu b}^a M_{\alpha b | M} & e_{\mu b}^a M_{\alpha b | a_1 a_2, \alpha} & e_{\mu b}^a M_{\alpha b | c d}
  \\
  0 & (d^{-1})_M N & (d^{-1})_P N \beta_P^P & (d^{-1})_P N \beta_P^P
  \\
  0 & 0 & (e^{-1})_a^\mu & (e^{-1})_a^\mu \gamma_P^P
  \\
  0 & 0 & 0 & (e^{-1})_c^\mu (e^{-1})_d^\nu
\end{pmatrix}
\]

(3.3.21)

where in the first line

\[
\alpha_{a | N} = - A_{a N}, \quad \alpha_{a b | N} = - 2 A_{a b}^N - \frac{1}{2} d^{NMP} A_{a M} A_{b P},
\]

\[
\alpha_{a | a_1 a_2, \alpha} = - 3 A_{a a_1 a_2} + 2 A_{a [a_1} N A_{a_2] M} (D_\alpha)_{N M} + \frac{1}{6} A_{a N} A_{a_1 M} A_{a_2] P} d^{NMS} (D_\alpha)_{S P},
\]

\[
\alpha_{a | c d} = - 4 A_{d (a, c)} - 2 A_{a d}^N A_{c N} - \frac{1}{6} A_{a N} A_{b M} A_{c P} d^{NMP},
\]

(3.3.22)

in the second line

\[
\beta_{a M}^N = A_{a P} d^{NMP}, \quad \beta_{a_1 a_2, \alpha}^N = A_{a_1 a_2}^M (D_\alpha)_{M N} - \frac{1}{2} A_{a_1 M} A_{a_2] R} d^{NMP} (D_\alpha)_{P R},
\]

\[
\beta_{a b}^N = - 2 A_{a b}^N + \frac{1}{2} A_{a M} A_{b P} d^{NMP},
\]

(3.3.23)

and in the third line

\[
\gamma_{a N | a_1 a_2, \alpha} = - \delta_{a_1}^a A_{a_2] M} (D_\alpha)_{N M}, \quad \gamma_{a N | c d} = \delta_d^a A_{c N}.
\]

(3.3.24)

\[ 3.4 \quad D = 4 \]

The four dimensional theory is obtained by deleting node 4 from the Dynkin diagram and so decomposing the $E_{11}$ algebra into representations of $GL(4) \times E_7$ [19]. However, it is easier to work with $SL(8)$ subalgebra of $E_7$, instead of $E_7$ itself; the $E_7$ representations can be reconstructed if needed. In this case all the generators belong to different representations of $GL(4) \times SL(8)$.
In this section we are going to calculate the Cartan form up to level 2. The positive (and zero) level generators of $E_{11}$ are

$$K^a_b, \ R^I J, \ R^{I_1\ldots I_4}; \ R^{aI_1 I_2}, \ R^a_{I_1 I_2}; \ \hat{K}^{(ab)}, \ R^{a_1 a_2 I J}, \ R^{a_1 a_2 I_1\ldots I_4}. \quad (3.4.1)$$

The negative level generators are

$$R_{a I_1 I_2}, \ R^a_{I_1 I_2}; \ \hat{K}^{(ab)}, \ R^{a_1 a_2 I J}, \ R^{a_1 a_2 I_1\ldots I_4}. \quad (3.4.2)$$

The $l_1$ representation generators are

$$P_a; \ Z^{I_1 I_2}, \ Z_{I_1 I_2}; \ Z^a, \ Z^{a I}, \ Z^{a I_1\ldots I_4}. \quad (3.4.3)$$

The parametrisation of an arbitrary level 2 group element is of the form

$$g_l = \exp \left( x^a P_a + x_{I_1 I_2} Z^{I_1 I_2} + x_{I_1 I_2} Z_{I_1 I_2} + \hat{x}_a Z^a + x_a^J Z^{a I} + x_{aI_1\ldots I_4} Z^{a I_1\ldots I_4} \right),$$

$$g_A = \exp \left( h^{ab}_a K^a_b \right) \exp \left( \varphi^J J R^I J \right) \exp \left( \varphi_{I_1\ldots I_4} R^{I_1\ldots I_4} \right) \exp \left( \hat{h}^{(ab)} \hat{K}^{(ab)} \right) \times \exp \left( A_{a_1 a_2 J} R^{a_1 a_2 I J} \right) \exp \left( A_{a_1 a_2 I_1\ldots I_4} R^{a_1 a_2 I_1\ldots I_4} \right) \exp \left( A_{a I_1 I_2} R^{a I_1 I_2} + A^{a I_1 I_2} R_{a I_1 I_2} \right),$$

where we have introduced the generalised coordinates

$$x^a; \ x_{I_1 I_2}; \ \hat{x}_a, \ x_a^I J, \ x_{aI_1\ldots I_4}. \quad (3.4.5)$$

and the fields

$$h^{ab}_a, \ \varphi^J J, \ \varphi_{I_1\ldots I_4}; \ A_{a I_1 I_2}, \ A_{a I_1 I_2}; \ \hat{h}^{(ab)}, \ A_{a_1 a_2 J}, \ A_{a_1 a_2 I_1\ldots I_4}. \quad (3.4.6)$$

To calculate the generalised vielbein we used the definition of equation (1.10), which was the same technique as was used in section (3.2) for the ten dimensional IIB theory. Conjugation of any $l_1$ generator with group element that contains the $K^a_b$ and $R^I J$ generators gives the following:

$$\exp \left( - \varphi^J J R^I J \right) \exp \left( - h^{ab}_a K^a_b \right) \left\{ P_\mu, \ Z^{I_1 I_2}, \ Z_{I_1 I_2}, \ Z^a, \ Z^{a I}, \ Z^{a I_1\ldots I_4} \right\}$$

$$= (\det e)^{-\frac{1}{2}} \left\{ e_\mu^a P_a, \ (f^{-1})^{I_1 I_2}_{J_1 J_2}, \ Z^{J_1 J_2}, \ Z_{J_1 J_2}, \ (e^{-1})_\mu^a Z^a, \ (e^{-1})^a_\mu Z^{a I}, \ (e^{-1})^{I_1 I_2}_{J_1 J_2}, \ Z^{a I_1 I_2}, \ (e^{-1})^{I_1 I_2}_{J_1 J_2} Z^{a I_1 I_2}, \ (e^{-1})^{I_1 I_2}_{J_1 J_2} Z^{a I_1 I_2} \right\},$$

where $e_\mu^a = (e^h)_\mu^a, \ f^I J = (e^\varphi)^I J, \ and \ \n$, and

$$\left( e^{-1} \right)^{\mu_1\ldots\mu_n}_{a_1\ldots a_n} = \left( e^{-1} \right)^{\mu_1\ldots\mu_n}_{a_1\ldots a_n}, \ (f^{-1})^{I_1 I_2}_{J_1 J_2} = f^{I_1 [J_1\ldots J_n]}.$$
We place a dot on a SL(8) index to denote that it is a world, rather than a tangent, index. Conjugation with $R_{I_1\ldots I_4}$ generator gives

$$\exp \left( -\phi_{I_1\ldots I_4} R_{I_1\ldots I_4} \right) \left\{ P_a, Z^{I_1 I_2}, Z_{I_1 I_2}, Z^a, Z^{a I_1 I_2}, Z^{a I_1\ldots I_4} \right\} \exp \left( \phi_{I_1\ldots I_4} R_{I_1\ldots I_4} \right) =$$

$$= \left\{ P_a, \beta_{I_1 I_2} Z^{I_1 I_2} + \beta_{I_1 I_2|J_1 J_2} Z_{I_1, J_1, J_2}, \beta_{I_1 I_2} Z_{I_1 I_2} + \beta_{I_1 I_2|J_1 J_2} Z^{J_1 J_2}, Z^a, \gamma_{I_1 K} Z^a K_L + \gamma_{I_1 I_2 J_1 J_2} Z^{a J_1 J_2}, \gamma_{I_1 I_2 J_1 J_2} Z^{a J_1 J_2} + \gamma_{I_1 I_2 K} Z^a K_L \right\}, \tag{3.4.9}$$

where the $\beta$-matrices that mix level 1 elements are defined as

$$\beta_{I_1 I_2} = \left( 1 + \frac{1}{2} P + \frac{1}{4!} P^2 + \frac{1}{6!} P^3 + \ldots \right)_{I_1 I_2}, \quad P_{I_1 I_2} = \frac{1}{24} \varepsilon_{I_1 I_2 I_3 I_4} \phi_{I_3 I_4 I_5 I_6 I_7 I_8 J_1 J_2},$$

$$\beta_{I_1 I_2|J_1 J_2} = -\frac{1}{24} \varepsilon_{J_1 J_2} \left( 1 + \frac{1}{3!} P + \frac{1}{5!} P^2 + \frac{1}{7!} P^3 + \ldots \right)_{I_1 I_2} \phi_{J_3 J_4 \ldots J_8},$$

$$\beta_{I_1 I_2|J_1 J_2} = -\varepsilon_{J_1} \left( 1 + \frac{1}{3!} P + \frac{1}{5!} P^2 + \frac{1}{7!} P^3 + \ldots \right)_{I_1 I_2} \phi_{J_3 J_4 \ldots J_8}, \tag{3.4.10}$$

while the $\gamma$-matrices, responsible for mixing of level 2 elements, are given by

$$\gamma_{I_1 I_2} = \left( 1 + \frac{1}{2} Q + \frac{1}{4!} Q^2 + \frac{1}{6!} Q^3 + \ldots \right)_{I_1 I_2},$$

$$Q_{I_1 I_2} = \left( \frac{1}{72} \delta_{I_1} \phi_{I_3 \ldots I_4} - \frac{1}{9} \delta_{I_2} \phi_{I_1 I_2 I_3 I_4} \right) \varepsilon_{I_1 I_2 I_3 I_4} \phi_{J_1 J_2 J_3 J_4},$$

$$\gamma_{I_1 I_2 J_1 J_2} = \left( 1 + \frac{1}{2} R + \frac{1}{4!} R^2 + \ldots \right)_{I_1 I_2 J_1 J_2},$$

$$R_{I_1 I_2 J_1 J_2} = \varepsilon_{I_1 I_2 J_1 J_2 K_1 K_2 K_3 J} \phi_{K_1 K_2 K_3 J} \left( \frac{1}{72} \delta_{I_1} \phi_{I_3 \ldots I_4} - \frac{1}{9} \delta_{I_2} \phi_{I_1 I_2 I_3 I_4} \right),$$

$$\gamma_{I_1 I_2 J_1 J_2} = \left( 1 + \frac{1}{3!} Q + \frac{1}{5!} Q^2 + \frac{1}{7!} Q^3 + \ldots \right)_{I_1 I_2 J_1 J_2} \left( \frac{4}{3} \delta_{I_1 L} \phi_{L J_2 J_3 J_4} - \frac{1}{6} \delta_{I_2 L} \phi_{J_1 J_2 J_3 J_4} \right),$$

$$\gamma_{I_1 I_2 J_1 J_2} = -\frac{1}{12} \left( 1 + \frac{1}{3!} R + \frac{1}{5!} R^2 + \frac{1}{7!} R^3 + \ldots \right)_{I_1 I_2 J_1 J_2} \varepsilon_{I_1 I_2 J_1 J_2 K_1 K_2 K_3 J} \phi_{K_1 K_2 K_3 J}. \tag{3.4.11}$$

Conjugation with level 1 and level 2 elements is performed by Taylor-expanding the exponents. The generalised vielbein is

$$E_\Pi A = (\det e)^{-\frac{1}{2}} \times$$

21
The quantities in the above matrix which have world indices are given in terms of quantities with all tangent indices by

\[ \alpha_{\mu|J_1J_2} = e_{\mu} e_{b} \alpha_{b|J_1J_2}, \quad \alpha_{\mu|J_1J_2} = e_{\mu} e_{b} \alpha_{b|J_1J_2}, \text{ etc} \]

as well as

\[ \beta^{I_1I_2}_{J_1J_2} = \left( f^{-1} \right)^{I_1I_2}_{K_1K_2} \beta^{K_1K_2}_{J_1J_2}, \quad \beta^{I_1I_2}_{J_1J_2} = f^{K_1K_2}_{I_1I_2} \beta^{K_1K_2}_{J_1J_2}, \]

which form the generalised vielbein on the coset space of the non-linear realisation of \( E_{7} \otimes s L^{56} \) with local subgroup SU(8), and in addition

\[ \beta^{I_1I_2}_{aK} = \left( f^{-1} \right)^{I_1I_2}_{K_1K_2} \beta^{K_1K_2}_{aK}, \quad \beta^{I_1I_2}_{aK} = f^{K_1K_2}_{I_1I_2} \beta^{K_1K_2}_{aK} \]

\[ \gamma^{i}_{jK} = \left( f^{-1} \right)^{i}_{M} f^{N}_{j} \gamma^{M}_{|N|K}, \quad \gamma^{i}_{j,K_1...J_4} = \left( f^{-1} \right)^{i}_{M} f^{N}_{J} \gamma^{M}_{N|J_1...J_4} \]

With these definitions the symbols in the first line of the matrix are given by

\[ \alpha_{a|I_1I_2} = - A_{aI_1I_2}, \quad \alpha_{a}^{I_1I_2} = - A_{a}^{I_1I_2}, \quad \alpha_{a|b} = - \hat{h}_{(ab)} - \frac{1}{2} A_{[aI_1I_2} A_{b]}^{I_1I_2}, \]

\[ \alpha_{a|bI} = \frac{1}{2} A_{aI}^{I} + \frac{1}{2} A_{(aKI} A_{b)}^{KJ}, \quad \alpha_{a|bI...I_4} = \frac{1}{6} A_{abI...I_4}, \]

\[ -\frac{1}{2} A_{a|I_1I_2} A_{bI_3I_4} + \frac{1}{48} \varepsilon_{I_1...I_8} A_{a}^{I_5I_6} A_{b}^{I_7I_8}, \]

and the second line by

\[ \beta^{I_1I_2}_{a} = \beta^{I_1I_2}_{J_1J_2} A_{a}^{aJ_1J_2} - \beta^{I_1I_2|J_1J_2} A_{aJ_1J_2}, \quad \beta^{I_1I_2}_{aI} = - \beta^{I_1I_2}_{K^I A_a} - \beta^{I_1I_2}_{K} A_{aK^I}, \]

\[ \beta^{I_1I_2}_{aJ_1...J_4} = \beta^{I_1I_2}_{[J_1J_2} A_{aJ_3J_4]} - \frac{1}{24} \varepsilon_{J_1...J_8} \beta^{I_1I_2}_{J_5J_6} A_{a}^{J_7J_8}, \]

(3.4.13)
\[ \beta_{l_1 l_2|a} = -\beta_{J_1 J_2}^{l_1 l_2} A_a J_1 J_2 + \beta_{l_1 l_2|J_1 J_2}^{J_1 J_2} A_a, \quad \beta_{l_1 l_2|a}^J = -\beta_{J_1 J_2}^{l_1 l_2} A_a K J - \beta_{l_1 l_2|K J}^{J_1 J_2} A_a, \quad \beta_{l_1 l_2|J_1 J_2}^{J_1 J_2} A_a J_1 J_2 + \beta_{l_1 l_2|J_1 J_2}^{J_1 J_2} A_a J_1 J_2 + \beta_{l_1 l_2|J_1 J_2}^{J_1 J_2} A_a J_1 J_2. \]

(3.4.14)

4 The non-linear realisation of \(A_1^{+++}\) and its generalised vielbein

As we have mentioned the non-linear realisations of the semi-direct product of very extended \(A_1\), denoted \(A_1^{+++}\) with its their first fundamental representation, denoted \(l_1\) is conjectured to lead to the complete low energy effective action for four dimensional gravity \([7]\). The Dynkin diagram for the Kac-Moody algebra \(A_1^{+++}\) is

\[
\bullet \quad - \quad \bullet \quad - \quad \bullet = \bigotimes
\]

which corresponds to the Cartan matrix

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -2 \\
0 & 0 & -2 & 2
\end{pmatrix}.
\] (4.1)

The four dimensional theory appears when we delete node four, as indicated in the above diagram, to leave the algebra \(GL(4)\). Decomposing \(A_1^{+++}\) into this subalgebra we find that the positive level generators of to level 2 are given by

\[ K^a_{\ b}; \ R^{ab}; \ R^{ab,cd}, \]

(4.2)

where the generators obey the conditions \(R^{ab} = R^{ab}\) and \(R^{ab,cd} = R^{[ab],cd}\), while the negative level generators are

\[ R_{ab}; \ R_{ab,cd} \]

(4.3)

and satisfy similar constraints. The level \(\pm 2\) generators satisfy the conditions

\[ R^{[ab,c]d} = R_{[ab,c]d} = 0. \]

(4.4)

For this Kac-Moody algebra the level is the number of up minus down \(GL(4)\) indices on the generator divided by two.

The \(A_1^{+++}\) algebra can be constructed in the usual way, see reference \([8]\) for a review of this process in the context of \(E_{11}\). The commutators for the listed generators preserve the level and must obey the Jacobi identities, as such one proceeds level by level writing down the most general right-hand side for each commutator and then tests the Jacobi identities level by level. The generators belong to representations of \(GL(4)\) and so their commutators with the generators \(K^a_{\ b}\) are

\[ [K^a_{\ b}, K^c_{\ d}] = \delta^c_{\ b} K^a_{\ d} - \delta^a_{\ d} K^c_{\ b}, \]

23
\[ [K^a_b, R^{a_1 a_2}] = 2 \delta^{(a_1}_{(b} R^{a)(a_2)}, \quad [K^a_b, R_{a_1 a_2}] = -2 \delta^{a}_{(a_1} R^{b)(a_2)}, \]
\[ [K^a_b, R^{c d, e f}] = \delta^c_b R^{d, e f} + \delta^a_{a_2} R^{c a, e f} + \delta^e_b R^{c d, a f} + \delta^f_b R^{c d, e a}, \]
\[ [K^a_b, R_{c d, e f}] = -\delta^a_d R_{b d, e f} - \delta^a_{a_2} R_{e b, e f} - \delta^e_{c} R_{c d, b f} - \delta^e_{d} R_{c d, e b}. \quad (4.5) \]

The level 2 \((-2)\) two commutators must give on the right-hand side the unique level 2 \((-2)\) generators and can be written in the form
\[ [R^{ab}, R^{cd}] = R^{ac, bd} + R^{bd, ac}, \quad [R_{ab}, R_{cd}] = R_{ac, bd} + R_{bd, ac}. \quad (4.6) \]

where the normalisation of the level 2 \((-2)\) generators are fixed by these relations. The reader may verify that the right-hand side of these commutators do indeed have the symmetries of the generators which occur in the left-hand side using the constraints on the generators given below equation (4.2). The commutators between the positive and negative level generators are given by
\[ [R^{ab}, R_{cd}] = 2 \delta^{(a}_{(c} K^{e d)} - \delta^{(ad}_{cd} \sum_e K^{e, e}, \]
\[ [R^{a b, c d}, R_{e f}] = \delta^{(bd}_{e f} R^{ac} + \delta^{(b c}_{e f} R^{a d} - \delta^{(a c}_{e f} R^{b d} - \delta^{(a d}_{e f} R^{b c}, \]
\[ [R_{a b, c d}, R^{e f}] = \delta^{e f}_{b d} R^{a c} + \delta^{(e f}_{b c} R_{a d} - \delta^{(e f}_{a c} R_{b d} - \delta^{e f}_{a d} R_{b c}. \quad (4.7) \]

where \(\delta^{(ab}_{cd} = \delta^{(a}_{d} \delta^{b}_{c}\).

The relation of the above generators to the Chevalley generators of \(A_1^{+++}\) is given by
\[ H_1 = K^{1}_{1} - K^{2}_{2}, \quad H_2 = K^{2}_{2} - K^{3}_{3}, \quad H_3 = K^{3}_{3} - K^{4}_{4}, \]
\[ H_4 = -K^{1}_{1} - K^{2}_{2} - K^{3}_{3} + K^{4}_{4}. \quad (4.8) \]
\[ E_1 = K^{1}_{2}, \quad E_2 = K^{2}_{3}, \quad E_3 = K^{3}_{4}, \quad E_4 = R^{44}, \quad (4.9) \]
\[ F_1 = K^{2}_{1}, \quad F_2 = K^{3}_{2}, \quad F_3 = K^{4}_{3}, \quad F_4 = R_{44}. \quad (4.10) \]

One can verify that the satisfy the defining relations
\[ [H_a, E_b] = A_{a b} E_j, \quad [E_a, F_b] = \delta_{a b} H_a, \quad [H_a, F_b] = -A_{a b} F_b \quad (4.11) \]

were \(A_{a b}\) is the Cartan matrix of \(A_1^{+++}\) given in equation (4.1).

The Cartan involution acts on the generators of \(A_1^{+++}\) as follows
\[ I_c (K^{a}_{b}) = - K^{b}_{a}, \quad I_c (R_{a b}) = - R^{a b}, \quad I_c (R^{a b, c d}) = R_{a b, c d}, \quad (4.12) \]

The reader may verify that it leaves invariant the above commutators.

We pause here to review how the above construction of the \(A_1^{+++}\) algebra was carried out as this can act as an illustration of how to construct any Kac-Moody algebra from a knowledge of the generators. We have first written down the commutators of the known generators of equations (4.2) and (4.3) which are consistent with the level, \(\text{SL}(4)\) algebra,
the Cartan involution and the symmetries of the indices on the generators. Strictly we should have included arbitrary constants on the right-hand sides of these commutators, that is, two constants in first of equations (4.7) and one constant in the last two of the equations (4.7), which are related by the action of the Cartan involution. The Jacobi identity \[[R^{ab}, R^{cd}], R_{ef}] + \ldots = 0\] then gives one relation between these three constants.

We have then consider the Chevalley relations which by definition must satisfy the relations of equation (4.11). Those for the first three nodes, that is, \(E_a, F_a\) and \(H_a\), \(a = 1, 2, 3\) are just those for the subalgebra \(A_3\) and are given in equation (4.8-10). The Chevalley generators \(E_4\) must be constructed out of the level one generators \(R^{ab}\). It must also commute with \(F_1, F_2\) and \(F_3\) and as a result it must, up to scale, be \(R^{44}\). We can choose it to be \(E_4 = R^{44}\). Similarly, or using the Cartan involution, we find that \(F_4 = R_{44}\). The Chevalley generator \(H_4\) must be a sum of the \(K^a_a\) generators and finding the correct relations with \(E_1, E_2, E_4\) we find it is as given in equation (4.8). Finally, we impose that \([E_4, F_a] = 2H_4 = [R^{44}, R_{44}]\) which using the first of equations (4.7) fixes the two constants we should have introduced in this relations to be as they are given.

The \(l_1\) representation generators up to level two are given by

\[
P_a, \ Z^a, \ Z^{abc}, \ Z^{ab,c}, (4.13)
\]

where \(Z^{abc} = Z^{(abc)}\), \(Z^{ab,c} = Z^{[ab],c}\) and \(Z^{[ab],c} = 0\). Their commutators with the level 0 generators of \(GL(4)\) are given by

\[
[K^a_b, P_c] = -\delta^a_c P_b + \frac{1}{2} \delta^a_b P_c, \ \ [K^a_b, Z^c] = \delta^a_b Z^a + \frac{1}{2} \delta^a_b Z^c,
\]

\[
[K^a_b, Z^{cde}] = \delta^c_b Z^{ade} + \delta^d_b Z^{cae} + \delta^e_b Z^{cda} + \frac{1}{2} \delta^a_b Z^{cde},
\]

\[
[K^a_b, Z^{cd,e}] = \delta^c_b Z^{ad,e} + \delta^d_b Z^{ca,e} + \delta^e_b Z^{cd,a} + \frac{1}{2} \delta^a_b Z^{cd,e}. \quad (4.14)
\]

The commutators of the level one \(A_1^{++}++\) generators with the \(l_1\) generators must increase their level by one and they can be chosen to be of the form

\[
[R^{ab}, P_c] = \delta^{(a}_c Z^{b)], \ \ [R^{ab}, Z^c] = Z^{abc} + Z^{c(a,b)}. \quad (4.15)
\]

Using the Jacobi identities, the commutator of \(P_a\) with the level 2 generator of \(A_1^{++}++\) is found to be

\[
[R^{ab,cd}, P_e] = -\delta^{[a}_e Z^{b]cd} + \frac{1}{4} \left( \delta^a_e Z^{b(c,d)} - \delta^b_e Z^{a(c,d)} \right) - \frac{3}{8} \left( \delta_c^e Z^{ab,d} + \delta^d_e Z^{ab,c} \right). \quad (4.16)
\]

The commutators with level-lowering generators are given by

\[
[R_{ab}, P_c] = 0, \ \ [R_{ab}, Z^c] = 2 \delta^c_{(a} P_{b)},
\]

\[
[R_{ab}, Z^{cde}] = \frac{2}{3} \left( \delta^c_{(ab)} Z^{e} + \delta^e_{(ab)} Z^{c} + \delta^c_{(ab)} Z^{d} \right).
\]
\[ [R_{ab}, Z^{cd,e}] = \frac{4}{3} \left( \delta^{de}_{(ab)} Z^c - \delta^{ce}_{(ab)} Z^d \right). \quad (4.17) \]

The very first relation reflects the fact that the \( l_1 \) representation is a lowest weight representation.

Having constructed the \( A_1^{++} \otimes_s l_1 \) algebra up to level two we can construct its non-linear realisation. The group element \( g = g_t g_A \) can, up to level two, be written in the form

\[
g_t = \exp \left( x^a P_a + y_a Z^a + x_{abc} Z^{abc} + x_{ab,c} Z^{ab,c} \right),
\]

\[
g_A = \exp \left( h^a_b K^a_b \right) \exp \left( A_{ab,cd} R^{ab,cd} \right) \exp \left( A_{ab} R^{ab} \right), \quad (4.18)\]

We find that we have introduced the fields

\[
h^a_b; A_{ab}; A_{ab,cd} \quad (4.19)
\]

where \( A_{ab} = A_{(ab)}; A_{ab,cd} = A_{[ab],cd} = A_{ab,(cd)} \), and the coordinates

\[
x^a; y_a; x_{abc}, x_{ab,c}, \quad (4.20)
\]

where \( x_{abc} = x_{(abc)} \), \( x_{ab,c} = x_{[ab],c} \). The field \( h^a_b \) is the usual graviton while the field \( A_{ab} \) is the dual graviton. Analogously the coordinates \( x^a \) are the usual coordinates of space-time while the coordinates \( y_a \) are the dual coordinates.

This non-linear realisation is a good arena in which to discuss the dual graviton and the resulting dynamics will be discussed elsewhere. Here we will content ourselves with calculating the generalised vielbein up to level two. We will use the definition of equation (1.10) which involves conjugating the \( l_1 \) generators with \( g_A \) using the above algebra. Conjugation with level 0 group element gives

\[
\exp \left( - h^a_b K^a_b \right) \left\{ P_\mu, Z^\mu, Z^\mu_1 \mu_2 \mu_3, Z^\mu_1 \mu_2 \mu_3 \right\} \exp \left( h^a_b K^a_b \right) =
\]

\[
= \left( \det e \right)^{-\frac{1}{18}} \left\{ e^a_\mu P_a, \left( e^{-1} \right)^{\mu}_a Z^\mu, \left( e^{-1} \right)^{\mu_1 \mu_2 \mu_3}_{(a_1 a_2 a_3)} Z^{a_1 a_2 a_3}, \left( e^{-1} \right)^{[\mu_1 \mu_2]}_{a_1 a_2 a_3} Z^{a_1 a_2 a_3} \right\}, \quad (4.21)
\]

where \( e^a_b = \left( e^h \right)^b_a \) and

\[
\left( e^{-1} \right)^{\mu_1 \ldots \mu_n}_{a_1 \ldots a_n} = \left( e^{-1} \right)^{\mu_1}_{a_1} \ldots \left( e^{-1} \right)^{\mu_n}_{a_n},
\]

\[
\left( e^{-1} \right)^{[\mu_1 \mu_2]}_{a_1 a_2 a_3} = \left( e^{-1} \right)^{\mu_1}_{a_1} \left( e^{-1} \right)^{\mu_2}_{a_2} \left( e^{-1} \right)^{\mu_3}_{a_3} - \left( e^{-1} \right)^{\mu_1}_{a_1} \left( e^{-1} \right)^{\mu_2}_{a_2} \left( e^{-1} \right)^{\mu_3}_{a_3}.
\quad (4.22)
\]

Conjugating with positive level generators can be obtained by Taylor-expanding the exponents and truncating the series by level 2. For the \( E_{11} \) level one generators we have

\[
\exp \left( - A_{bc} R^{bc} \right) \left\{ P_a, Z^a \right\} \exp \left( A_{bc} R^{bc} \right) =
\]

\[
= \left\{ P_a - A_{ab} Z^b + \frac{1}{2} A_{ab} A_{cd} Z^{bcd} + \frac{1}{2} A_{ab} A_{cd} Z^{bc,d}, Z^a - A_{bc} Z^{abc} - A_{bc} Z^{ab,c} \right\}, \quad (4.23)
\]
while for the $E_{11}$ level 2 generator:

$$\exp \left( A_{bc,de} R^{bc,de} \right) P_a \exp \left( A_{bc,de} R^{bc,de} \right)$$

\[ = P_a + A_{ab,cd} Z^{bcd} + \left( \frac{3}{4} A_{bc,ad} - \frac{1}{2} A_{ab,cd} \right) Z^{bc,d}. \]  

(4.24)

As we are only computing up to level two, that is, up to the $l_1$ elements $Z^{abc}$ and $Z^{ab,c}$ the order in which we calculate the action of the group elements on the $l_1$ generators is irrelevant. Combining these results together we find that the generalised vielbein up to level two is given by

$$E_{\Pi}^a = (\det e)^{-\frac{1}{2}}$$

\[ = \left( \begin{array}{cccc}
    e_{\mu}^a & e_{\mu}^b \alpha_{b|a} & e_{\mu}^b \alpha_{b|12a3} & e_{\mu}^b \alpha_{b|12a3} \\
    0 & (e^{-1})_{a|\mu} & (e^{-1})_{b|\mu} \beta_{a12a3} & (e^{-1})_{b|\mu} \beta_{a12a3} \\
    0 & 0 & (e^{-1})_{(1a2a3)} & 0 \\
    0 & 0 & 0 & (e^{-1})_{[\mu12],\mu3} \\
  \end{array} \right), \]  

(4.25)

where the symbols in the first line are given by

$$\alpha_a|b = -A_{ab}, \quad \alpha_a|_{a12a3} = \alpha_a|_{(a12a3)} = A_a(a1,2a3) + \frac{1}{2} A_a(a1, A_{a2a3}),$$

$$\alpha_a|_{a1a2a3} = \frac{3}{4} A_{a1a2a3} - \frac{1}{2} A_{a|a12a3} + \frac{1}{2} A_{a|a12a3} + \frac{1}{2} A_{a|a12a3}, \]  

(4.26)

while the symbols in the second line are given by

$$\beta_{a12a3} = \beta_{(a12a3)} = -\delta_{(a12a3)}, \quad \beta_{a1a2,a3} = \beta_{[a1a2],a3} = -\delta_{[a12a3]}. \]  

(4.27)

**5 Conclusion**

In this paper we have reviewed how to construct the generalised vielbein associated with the generalised space-time that arises in the non-linear realisation of $E_{11} \otimes s l_1$. We find the generalised vielbein up to, and including, the level containing the dual graviton in eleven, five and four dimensions as well as for the ten dimensional IIB theory. To find these results one requires $E_{11} \otimes s l_1$ algebra up to the level concerned. These algebras were previously known in eleven and four dimensions and in this paper we have also found them in five dimensions and for the ten dimensional IIB theory, the explicit formulae being given in appendix A.

In a recent paper the gauge transformations of the fields in the $E_{11} \otimes s l_1$ non-linear realisation were proposed [29]. These are formulated in terms of the generalised vielbein and the results for this object given in this paper will prove useful for finding the explicit gauge transformations.
Acknowledgment

We would like to thank Nikolay Gromov for discussions and the SFTC for support from Consolidated grant number ST/J002798/1.

Appendix A

For convenience we give in this appendix the $E_{11} \otimes_s l_1$ algebra appropriate to four, five and eleven dimensions and also for the IIB ten dimensional theory.

A.1 $D = 11$ algebra

In this appendix we repeat, for convenience, the $E_{11} \otimes_s l_1$ algebra decomposed into representations of $GL(11)$ [11]. The commutators of the $E_{11}$ generators with the generators of $K^a_b$ are given by

$$[K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b, \quad [K^a_b, R^{a_1a_2a_3}] = 3 \delta^{[a_1}_b R^{a][a_2a_3]},$$

$$[K^a_b, R_{a_1a_2a_3}] = -3 \delta^{a}_{[a_1} R^{b]}_{|a_2a_3]},$$

$$[K^a_b, R^{a_1...a_5}] = 5 \delta^{[a_1}_b R^{a][a_2...a_5]}, \quad [K^a_b, R_{a_1...a_5}] = -5 \delta^{a}_{[a_1} R^{b]}_{|a_2...a_5]},$$

$$[K^a_b, R^{a_1...a_8}] = 8 \delta^{[a_1}_b R^{a][a_2...a_8]}, \quad [K^a_b, R_{a_1...a_8}] = -8 \delta^{a}_{[a_1} R^{b]}_{|a_2...a_8]}, \quad (A.1.1)$$

$$[K^a_b, R_{a_1...a_7,c}] = 7 \delta^{[a_1}_b R^{a][a_2...a_7],c} + \delta^c_b R^{a_1...a_7,a},$$

$$[K^a_b, R_{a_1...a_7,c}] = -7 \delta^{a}_{[a_1} R^{b]}_{|a_2...a_7],c} - \delta^c_b R_{a_1...a_7,b}.$$

The positive level commutators are given by

$$[R^{a_1a_2a_3}, R^{a_4a_5a_6}] = 2 R^{a_1...a_6}, \quad [R^{a_1a_2a_3}, R^{b_1...b_6}] = 6 R^{a_1a_2a_3|b_1...b_5,b_6},$$

while the negative level commutators are given by

$$[R_{a_1a_2a_3}, R_{a_4a_5a_6}] = 2 R_{a_1...a_6}, \quad [R_{a_1a_2a_3}, R_{b_1...b_6}] = 6 R_{a_1a_2a_3|b_1...b_5,b_6}. \quad (A.1.2)$$

The commutators between the positive and negative level generators are given by

$$[R^{a_1a_2a_3}, R_{b_1b_2b_3}] = 18 \delta^{[a_1}_b R^{a_2a_3]}_{[b_1b_2b_3]} - 2 \delta^{a_1a_2a_3}_{b_1b_2b_3} K^{a_1},$$

$$[R^{a_1a_2a_3}, R_{b_1...b_6}] = 60 \delta^{a_1a_2a_3}_{b_1b_2b_3} R^{b_4b_5b_6},$$

$$[R^{a_1a_2a_3}, R_{b_1...b_8,b}] = 112 \delta^{a_1a_2a_3}_{b_1b_2b_3} R^{b_4...b_8} - 112 \delta^{a_1a_2a_3}_{b_1b_2b_3} R_{b_3...b_8},$$

$$[R_{a_1a_2a_3}, R^{b_1...b_6}] = 60 \delta^{a_1a_2a_3}_{b_1b_2b_3} R^{b_4b_5b_6},$$

$$[R_{a_1a_2a_3}, R^{b_1...b_8,b}] = 112 \delta^{a_1a_2a_3}_{b_1b_2b_3} R^{b_4...b_8} - 112 \delta^{a_1a_2a_3}_{b_1b_2b_3} R_{b_3...b_8},$$

$$[R^{a_1...a_6}, R_{b_1...b_6}] = \delta^{[a_1...a_6}_b R^{a[a_6]}_{b_1...b_5} - 120 \delta^{a_1...a_6}_{b_1b_2b_3} K^{a_1},$$

$$[R_{a_1...a_6}, R^{b_1...b_8,b}] = \delta^{[a_1...a_6}_b R^{b_1b_2b_3} - 120 \delta^{a_1...a_6}_{b_1b_2b_3} R^{b_4b_5b_6},$$

$$[R_{a_1...a_6}, R^{b_1...b_8,b}] = -3360 \delta^{[a_1...a_6}_b R^{b_1b_2b_3} - 3360 \delta^{a_1...a_6}_{b_1b_2b_3} R^{b_4b_5b_6},$$

28
\[ [R^{a_1 \ldots a_6}, R_{b_1 \ldots b_8, b}] = -3360 \delta^{a_1 \ldots a_6}_{[b_1 \ldots b_6} R_{b_7 b_8]} b - 3360 \delta^{a_1 \ldots a_6}_{[b_1 \ldots b_5} R_{b_6 b_7 b_8]}, \quad (A.1.3) \]

The commutators of the GL(11) generators with those of the \( l_1 \) representation are given by [6]

\[ [K^a_b, P_c] = -\delta^a_c P_b + \frac{1}{2} \delta^a_b P_c, \quad [K^a_b, Z^{a_1 a_2}] = 2 \delta^a_b Z^{a_1 a_2} + \frac{1}{2} \delta^a_b Z^{a_1 a_2}, \]

\[ [K^a_b, Z^{a_1 \ldots a_5}] = 5 \delta^a_b Z^{a_1 \ldots a_5} + \frac{1}{2} \delta^a_b Z^{a_1 \ldots a_5}, \]

\[ [K^a_b, Z^{a_1 \ldots a_8}] = 8 \delta^a_b Z^{a_1 \ldots a_8} + \frac{1}{2} \delta^a_b Z^{a_1 \ldots a_8}, \]

\[ [K^a_b, Z^{a_1 \ldots a_7, c}] = 7 \delta^a_b Z^{a_1 \ldots a_7, c} + \delta^c_b Z^{a_1 \ldots a_7, a} + \frac{1}{2} \delta^a_b Z^{a_1 \ldots a_7, c}, \quad (A.1.4) \]

The commutators of the positive root generators of \( E_{11} \) with \( l_1 \) generators are given by

\[ [R^{a_1 a_2 a_3}, P_a] = 3 \delta^{a_1}_a Z^{a_2 a_3}, \quad [R^{a_1 a_2 a_3}, Z^{a_4 a_5}] = Z^{a_1 \ldots a_5}, \]

\[ [R^{a_1 a_2 a_3}, Z^{b_1 \ldots b_5}] = Z^{b_1 \ldots b_5 a_1 a_2 a_3} + Z^{b_1 \ldots b_5 [a_1 a_2, a_3]} \]

\[ [R^{a_1 \ldots a_6}, P_a] = -3 \delta^{a_1}_a Z^{a_2 \ldots a_6}, \quad [R^{a_1 \ldots a_6}, Z^{b_1 b_2}] = -Z^{b_1 b_2 a_1 \ldots a_6} - Z^{b_1 b_2 [a_1 \ldots a_5, a_6]}, \]

\[ [R^{a_1 \ldots a_8, a}, P_b] = -\frac{4}{3} \delta^a_b Z^{a_1 \ldots a_8} + \frac{4}{3} \delta^c_b Z^{a_1 \ldots a_8} + \frac{4}{3} \delta^d_b Z^{a_1 \ldots a_8} Z^{a_1 \ldots a_8}. \quad (A.1.5) \]

While the commutators of the \( l_1 \) generators with the level minus one \( E_{11} \) generators are given by

\[ [R_{a_1 a_2 a_3}, P_a] = 0, \quad [R_{a_1 a_2 a_3}, Z^{b_1 b_2}] = 6 \delta^{b_1 b_2}_{a_1 a_2} P_{a_3}, \]

\[ [R_{a_1 a_2 a_3}, Z^{b_1 \ldots b_5}] = 60 \delta^{b_1 b_2 b_3}_{a_1 a_2 a_3} Z^{b_4 \ldots b_8}, \quad [R_{a_1 a_2 a_3}, Z^{b_1 \ldots b_8}] = -42 \delta^{b_1 b_2 b_3}_{a_1 a_2 a_3} Z^{b_4 \ldots b_8}, \]

\[ [R_{a_1 a_2 a_3}, Z^{b_1 \ldots b_7, b}] = \frac{945}{4} \delta^{b_1 b_2 b_3}_{a_1 a_2 a_3} Z^{b_4 \ldots b_7} + \frac{945}{4} \delta^{b_1 b_2 b_3}_{a_1 a_2 a_3} Z^{b_4 \ldots b_7}. \quad (A.1.6) \]

**A.2 D = 10 algebra**

In this appendix we give the commutators of \( E_{11} \otimes_{s} l_1 \) algebra, decomposed into representations of \( GL(10) \otimes SL(2, R) \). Parts of this algebra for the form generators were given in references [12] and [27]. The \( l_1 \) multiplet and their commutators with the \( E_{11} \) generators are given for the first time in this paper as are many of the commutators of the \( E_{11} \) algebra that involve the negative level generators. The commutators of the \( E_{11} \) generators with the SL(10) generators \( K^a_b \) are

\[ [K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b, \quad [K^a_b, R_{a\beta}] = 0, \]

\[ [K^a_b, R^{a_1 a_2}_a] = 2 \delta^{a_1}_b R^{a_2}_{[a_2]} + \frac{1}{2} \delta^a_b R^{a_2}_{[a_2]} - 2 \delta^{a_1}_b R^{a_2}_{[a_2]}, \]

\[ [K^a_b, R^{a_1 \ldots a_4}_a] = 4 \delta^{a_1}_b R^{a_2 a_3 a_4}_{[a_1]} - 4 \delta^{a_1}_b R^{a_2 a_3 a_4}_{[a_1]}. \]
The commutators of the $E_{11}$ generators with the $SL(2, R)$ generators $R_{\alpha \beta}$ are

$$ [R_{\alpha \beta}, R_{\gamma \delta}] = \delta^\sigma_\alpha \epsilon_\beta \gamma R_{\sigma \delta} + \delta^\sigma_\alpha \epsilon_\beta \delta R_{\gamma \sigma}, $$

$$ [R_{\alpha \beta}, R_{\gamma} a_2] = \delta^\delta_\alpha \epsilon_\beta \gamma R_{\sigma a_2}, \quad [R_{\alpha \beta}, R_{\gamma} a_1 a_2] = -\delta^\gamma_\alpha \epsilon_\beta \delta R_{\gamma a_1 a_2}, $$

$$ [R_{\alpha \beta}, R^{a_1 \ldots a_4}] = 0, \quad [R_{\alpha \beta}, R_{a_1 \ldots a_4}] = 0, $$

$$ [R_{\alpha \beta}, R_{\gamma} a_1 \ldots a_6] = \delta^\delta_\alpha \epsilon_\beta \gamma R_{\gamma a_1 \ldots a_6}, \quad [R_{\alpha \beta}, R_{\gamma} a_1 a_2] = \delta^\gamma_\alpha \epsilon_\beta R_{\gamma a_1 a_2}, $$

$$ [R_{\alpha \beta}, R_{\gamma} a_1 \ldots a_8] = \delta^\delta_\alpha \epsilon_\beta \gamma R_{\gamma a_1 \ldots a_8}, $$

$$ [R_{\alpha \beta}, R_{a_1 \ldots a_8}] = \delta^\gamma_\alpha \epsilon_\beta \delta R_{a_1 \ldots a_8}, $$

$$ [R_{\alpha \beta}, R_{a_1 \ldots a_7}, b] = 0, \quad [R_{\alpha \beta}, R_{a_1 \ldots a_7}, b] = 0. \quad (A.2.2) $$

The commutators of the positive level $E_{11}$ generators are given by

$$ [R^a_{\alpha a_2}, R^b_{\beta a_4}] = -\epsilon_{\alpha \beta} R^{a_1 \ldots a_4}, \quad [R^a_{\alpha a_2}, R^b_{a_3 a_4}] = -\epsilon_{\alpha \beta} R_{a_1 \ldots a_4}, $$

$$ [R^a_{\alpha a_2}, R^b_{\alpha a_3 \ldots a_6}] = 4 R^a_{\alpha a_1 \ldots a_6}, \quad [R^a_{\alpha a_2}, R^b_{a_1 a_2}, R^c_{a_3 \ldots a_6}] = 4 R^a_{\alpha a_1 \ldots a_6}, $$

$$ [R^a_{\alpha a_2}, R^b_{a_1 a_2}, R^c_{a_3 \ldots a_8}] = -R^a_{\alpha a_1 \ldots a_8} - \epsilon_{\alpha \beta} R_{a_1 a_2 a_3 \ldots a_7, a_8}, $$

$$ [R^a_{\alpha a_2}, R^b_{a_3 \ldots a_8}] = -R^a_{\alpha a_1 \ldots a_8} - \epsilon_{\alpha \beta} R_{a_1 a_2 a_3 \ldots a_7, a_8}, \quad (A.2.3) $$

while

$$ [R^a_{a_1 \ldots a_4}, R^b_{a_5 \ldots a_8}] = \frac{8}{3} R^a_{a_1 \ldots a_4} [a_5 a_6 a_7, a_8], \quad [R_{a_1 \ldots a_4}, R_{a_5 \ldots a_8}] = \frac{8}{3} R_{a_1 \ldots a_4} [a_5 a_6 a_7, a_8]. \quad (A.2.4) $$

To find the commutators between positive and negative level generators we need to utilize Jacobi identities. These commutators up to level 3 are given by

$$ [R^a_{\alpha a_2}, R^b_{b_1 b_2}] = 4 \delta^\beta_\alpha [b_1 K_{a_2 b_2}] - \frac{1}{2} \delta^\beta_\alpha \delta_{b_1 b_2} K_d^d - 2 \delta_{b_1 b_2} \epsilon_{\alpha \beta} R_{\alpha \gamma}, $$

$$ [R^a_{\alpha a_2}, R_{b_1 \ldots b_4}] = -12 \epsilon_{\alpha \beta} \delta_{b_1 b_2} R^\beta_{b_3 b_4}, \quad [R^a_{\alpha a_2}, R^b_{b_1 \ldots b_4}] = -12 \epsilon_{\alpha \beta} \delta_{a_1 a_2} R^a_{b_1 b_2}, $$

$$ [R^a_{a_1 \ldots a_4}, R_{b_1 \ldots b_4}] = 12 \delta_{b_1 \ldots b_4} K_d^d - 96 \delta_{b_1 b_2 b_3} K_{a_4 b_4}. $$
\[
\begin{align*}
\left[ R_{\alpha}^{a_1a_2}, R_{\beta b_1 \ldots b_6}^\beta \right] &= \frac{15}{2} \delta_{\alpha}^\beta \delta_{[b_1b_2} a_{a_1a_2} R_{b_3 \ldots b_6]}, \\
\left[ R_{\alpha}^{a_1 \ldots a_4}, R_{\beta b_1 \ldots b_6}^\beta \right] &= 90 \delta_{\alpha}^{a_1 \ldots a_4} \delta_{[b_1b_2} R_{b_3 \ldots b_6]}^\beta, \\
\left[ R_{\alpha}^{a_1 \ldots a_6}, R_{\beta b_1 \ldots b_6}^\beta \right] &= 270 \delta_{\alpha}^{a_1 \ldots a_6} \delta_{[b_1b_2 \ldots b_5} K_{a_6]}^{a_6} b_6 - \frac{135}{4} \delta_{\alpha}^\beta \delta_{b_1 \ldots b_6} \delta_{[a_1 \ldots a_6} K_d^{a_6} b_6 - 45 \delta_{b_1 \ldots b_6} \epsilon^{\gamma \gamma} R_{\alpha \gamma}.
\end{align*}
\]

The action of the Cartan involution on the adjoint generators is given by

\[
\begin{align*}
\left[ R_{\alpha}^{\alpha a_1a_2}, R_{\beta b_1 \ldots b_8}^\gamma \right] &= -56 \delta_{\alpha}^\beta \delta_{[b_1b_2} a_{a_1a_2} R_{b_3 \ldots b_8]}, \\
\left[ R_{\alpha}^{a_1a_2}, R_{\beta b_1 \ldots b_7}^\beta \right] &= -252 \epsilon_{\alpha \beta} \delta_{[b_1b_2} R_{b_3 \ldots b_7]}^\beta + 252 \epsilon_{\alpha \beta} \delta_{[b_1b_2} R_{b_3 \ldots b_7}^\beta], \\
\left[ R_{\alpha}^{a_1a_2}, R_{\beta b_1 \ldots b_7}^\alpha \right] &= -252 \epsilon_{\alpha \beta} \delta_{[b_1b_2 a_1a_2} R_{b_3 \ldots b_7]}^\beta + 252 \epsilon_{\alpha \beta} \delta_{[b_1b_2 a_1a_2} R_{b_3 \ldots b_7}^\beta], \\
\left[ R_{\alpha}^{a_1 \ldots a_4}, R_{\beta b_1 \ldots b_6}^\beta \right] &= 0, \\
\left[ R_{\alpha}^{a_1 \ldots a_4}, R_{\beta b_1 \ldots b_7}^\beta \right] &= -1260 \delta_{[b_1 \ldots b_4} a_{a_1 \ldots a_4} R_{b_5 \ldots b_7]}^\beta + 1260 \delta_{[b_1 \ldots b_4} a_{a_1 \ldots a_4} R_{b_5 \ldots b_7}^\beta], \\
\left[ R_{\alpha}^{a_1 \ldots a_6}, R_{\beta b_1 \ldots b_7}^\beta \right] &= -1260 \delta_{[b_1 \ldots b_4} a_{a_1 \ldots a_4} R_{b_5 \ldots b_7]}^\beta + 1260 \delta_{[b_1 \ldots b_4} a_{a_1 \ldots a_4} R_{b_5 \ldots b_7}^\beta],
\end{align*}
\]

with levels \(\pm 2\):

\[
\begin{align*}
\left[ R_{\alpha}^{a_1 \ldots a_6}, R_{\beta b_1 \ldots b_8}^\gamma \right] &= 1260 \delta_{\alpha}^\beta \delta_{b_1 \ldots b_6} a_{a_1 \ldots a_6} R_{b_7 b_8]}, \\
\left[ R_{\alpha}^{a_1 \ldots a_4}, R_{\beta b_1 \ldots b_8}^\beta \right] &= 1260 \delta_{\alpha}^\beta \delta_{b_1 \ldots b_6} a_{a_1 \ldots a_6} R_{b_7 b_8}], \\
\left[ R_{\alpha}^{a_1 \ldots a_6}, R_{\beta b_1 \ldots b_7}^\beta \right] &= 1890 \epsilon_{\alpha \beta} \delta_{b_1 \ldots b_4} a_{a_1 \ldots a_6} R_{b_5 \ldots b_7}]^\beta - 1890 \epsilon_{\alpha \beta} \delta_{b_1 \ldots b_4} a_{a_1 \ldots a_6} R_{b_5 \ldots b_7}^\beta], \\
\left[ R_{\alpha}^{a_1 \ldots a_6}, R_{\beta b_1 \ldots b_7}^\beta \right] &= 1890 \epsilon_{\alpha \beta} \delta_{b_1 \ldots b_4} a_{a_1 \ldots a_6} R_{b_5 \ldots b_7}]^\beta - 1890 \epsilon_{\alpha \beta} \delta_{b_1 \ldots b_4} a_{a_1 \ldots a_6} R_{b_5 \ldots b_7}^\beta],
\end{align*}
\]

and, finally, the commutators of level \(\pm 4\) generators between themselves are

\[
\begin{align*}
\left[ R_{\alpha}^{a_1 \ldots a_8}, R_{\beta b_1 \ldots b_8}^{\beta \beta} \right] &= -20160 \delta_{[b_1 \ldots b_7} a_{a_1 \ldots a_8} K_{b_8]}^{a_8}, \\
&+ 2520 \delta_{[b_1 \ldots b_7} a_{a_1 \ldots a_8} K_d^{a_8} d + 5040 \delta_{b_1 \ldots b_8} a_{a_1 \ldots a_8} K_{d}^{a_8} e_{(a_1} \epsilon^e_{a_2)} R_{a_2} e_{),} \\
\left[ R_{\alpha}^{a_1 \ldots a_8}, R_{\beta b_1 \ldots b_7}^{\beta} \right] &= 0, \\
\left[ R_{\alpha}^{a_1 \ldots a_7}, R_{\beta b_1 \ldots b_7}^{\beta} \right] &= -11340 \delta_{b_1 \ldots b_7} a_{a_1 \ldots a_7} K_{d}^{a_7} b + 11340 \delta_{b_1 \ldots b_7} a_{a_1 \ldots a_7} K_{b}^{a_7} d - 11340 \delta_{b_1 \ldots b_7} a_{a_1 \ldots a_7} K_{d}^{a_7} b + 11340 \delta_{b_1 \ldots b_7} a_{a_1 \ldots a_7} K_{b}^{a_7} d.
\end{align*}
\]

The action of the Cartan involution on the adjoint generators is given by

\[
I_c (K_a b) = -K_b a, \quad I_c (R_{\alpha \beta}) = \epsilon_{\alpha \gamma} \epsilon_{\beta \delta} R_{\gamma \delta}, \quad I_c (R_{\alpha}^{a_1a_2}) = -R_{a_1a_2}^\alpha, \quad I_c (R_{\alpha}^{a_1 \ldots a_4}) = R_{a_1 \ldots a_4}.
\]
\[ I_c (R^{a_1 \ldots a_6}_a) = - R^{a_1 \ldots a_6}_a, \quad I_c (R^{a_1 a_2}_{a_1 a_2}) = R^{a_1 a_2}_{a_1 a_2}, \quad I_c (R^{a_1 \ldots a_7, b}) = R_{a_1 \ldots a_7, b}. \] (A.2.10)

One can verify that the above commutators are preserved by this involution.

We now consider the commutators of the \( E^{11} \) generators with those of the \( l_1 \) representation. The members of the \( l_1 \) representation are most easily found using the Nutma programme Simple [30]. The commutators of the \( l_1 \) representation generators with the \( l_0 \) \( E^{11} \) generators, that is the \( SL(11) \) generators, are given by

\[
\begin{align*}
[K^a_b, P_c] &= - \delta^a_c P_b + \frac{1}{2} \delta^a_b P_c, \quad [K^a_b, Z^c_{\alpha}] = \delta^c_{\alpha} Z^a_{\alpha} + \frac{1}{2} \delta^a_b Z^c_{\alpha}, \\
[K^a_{b'}, Z^{a_1 a_2 a_3}] &= 3 \delta^a_b Z^{a_1 a_2 a_3}_{\alpha} + \frac{1}{2} \delta^a_b Z^{a_1 a_2 a_3}, \quad [K^a_{b'}, Z^{a_1 \ldots a_5}] \\
&= 5 \delta^a_b Z^{a_1 a_2 a_3}_{\alpha} + \frac{1}{2} \delta^a_b Z^{a_1 \ldots a_5}, \\
[K^a_{b'}, Z^{a_1 \ldots a_7}] &= 7 \delta^a_b Z^{a_1 a_2 a_3}_{\alpha} + \frac{1}{2} \delta^a_b Z^{a_1 \ldots a_7}, \\
[K^a_{b'}, Z^{a_1 a_2 a_3}] &= 6 \delta^a_b Z^{a_1 a_2 a_3}_{\alpha} + \frac{1}{2} \delta^a_b Z^{a_1 \ldots a_7}, \quad (A.2.11)
\end{align*}
\]

The commutators with the \( SL(2) \) generators \( R_{\alpha \beta} \) are

\[
\begin{align*}
[R_{\alpha \beta}, P_a] &= 0, \quad [R_{\alpha \beta}, Z^a_{\alpha \gamma}] = \delta^a_{\alpha \gamma} Z^a_{\beta \gamma}, \\
[R_{\alpha \beta}, Z^{a_1 a_2 a_3}] &= 0, \quad [R_{\alpha \beta}, Z^{a_1 \ldots a_5}] = \delta^a_{\alpha \gamma} Z^{a_1 \ldots a_5}_{\beta \gamma}, \\
[R_{\alpha \beta}, Z^{a_1 \ldots a_7}] &= \delta^a_{\alpha \gamma} Z^{a_1 \ldots a_7}_{\beta \gamma} + \delta^a_{\beta \gamma} Z^{a_1 \ldots a_7}_{\gamma \gamma}, \\
[R_{\alpha \beta}, Z^{a_1 \ldots a_9}] &= 0, \quad [R_{\alpha \beta}, Z^{a_1 \ldots a_9, b}] = 0. \quad (A.2.12)
\end{align*}
\]

The commutators with level one \( E^{11} \) generators can be taken as

\[
\begin{align*}
[R^{a_1 a_2}_{\alpha}, P_a] &= \delta^{a_1}_{\alpha} Z^{a_2}_{\beta}, \quad [R^{a_1 a_2}_{\alpha}, Z^{a_3}_{\beta}] = - \varepsilon_{\alpha \beta} Z^{a_1 a_2 a_3}_{\beta}, \quad [R^{a_1 a_2}_{\alpha}, Z^{a_3 a_4 a_5}_{\alpha}] = Z^{a_1 a_2 a_3}_{\alpha}, \\
[R^{a_1 a_2}_{\alpha}, Z^{a_3 \ldots a_7}_{\beta}] &= Z^{a_1 \ldots a_7}_{a_3 a_7} - \varepsilon_{\alpha \beta} Z^{a_1 \ldots a_7} - \varepsilon_{\alpha \beta} Z^{a_1 a_2 a_3 a_4 a_5 a_6 a_7}. \quad (A.2.13)
\end{align*}
\]

The commutators with other positive-level generators can be found using the Jacobi identities to be given by

\[
\begin{align*}
[R^{a_1 \ldots a_4}_{\alpha}, P_a] &= 2 \delta^{a_1}_{\alpha} Z^{a_2 a_3 a_4}_{\alpha}, \quad [R^{a_1 \ldots a_4}_{\alpha}, Z^{a_5}_{\alpha}] = - Z^{a_1 \ldots a_5}_{\alpha}, \\
[R^{a_1 \ldots a_4}_{\alpha}, Z^{a_5 a_6 a_7}_{\alpha}] &= 2 Z^{a_1 \ldots a_7} + \frac{3}{5} Z^{a_1 \ldots a_4 a_5 a_6 a_7}_{\alpha}, \quad [R^{a_1 \ldots a_4}_{\alpha}, P_a] = \frac{3}{4} \delta^{a_1}_{\alpha} Z^{a_2 a_3 a_4}_{\alpha},
\end{align*}
\]

32
\[
[R_{\alpha_1 \ldots a_5}, Z_{\beta}^{a_7}] = -\frac{1}{4} Z_{\alpha_\beta}^{a_1 \ldots a_7} + \frac{3}{4} \epsilon_{\alpha_\beta} Z_{a_1 \ldots a_7}^{a_5} + \frac{1}{20} \epsilon_{\alpha_\beta} Z_{a_1 \ldots a_6, a_7},
\]
\[
[R_{\alpha_\beta}^{a_1 \ldots a_8}, P_a] = -\delta_{[a_1}^{a_2} Z_{a_\beta}^{a_2 \ldots a_8}],
\]
\[
[R_{\alpha_\beta}^{a_1 \ldots a_7}, P_a] = -3 \delta_{a_1}^{b} Z_{a_1 \ldots a_7} + 3 \delta_{a_1}^{[b} Z_{a_1 \ldots a_7} + \frac{21}{20} \delta_{a_1}^{[a_2} Z_{a_2 \ldots a_7]}^{b}. 
\]  
(A.2.14)

The commutators with level \(-1\) \(E_{11}\) generators are given by
\[
[R_{a_1 a_2}, P_a] = 0, \quad [R_{a_1 a_2}, Z_{\beta}^{b}] = -4 \delta_{\beta}^{a} \delta_{[a_1}^{a_2} P_{a_2}], \quad [R_{a_1 a_2}, Z_{b_1 b_2 b_3}] = -6 \epsilon_{\alpha_\beta} \delta_{a_1 a_2}^{b_2} Z_{\beta}^{b_3},
\]
\[
[R_{a_1 a_2}, Z_{\beta}^{b_1 \ldots b_5}] = 20 \delta_{\beta}^{a} \delta_{a_1 a_2}^{[b_1 b_2} Z_{b_3 b_4 b_5]}, \quad [R_{a_1 a_2}, Z_{\alpha_1 \alpha_2}^{b_1 \ldots b_7}] = 42 \delta_{(a_1}^{a_2} \delta_{a_1 a_2}^{b_1 b_2} Z_{b_3 \ldots b_7]}, \quad (A.2.15)
\]
\[
[R_{a_1 a_2}, Z_{\beta}^{b_1 \ldots b_7}] = -3 \epsilon_{\alpha_\beta} \delta_{a_1 a_2}^{b_2} Z_{\beta}^{b_3 \ldots b_7},
\]
\[
[R_{a_1 a_2}, Z_{\beta}^{b_{1 \ldots b_6}, b}] = -150 \epsilon_{\alpha_\beta} \delta_{a_1 a_2}^{b_1 b_2} Z_{\beta}^{b_3 \ldots b_6} + 150 \epsilon_{\alpha_\beta} \delta_{a_1 a_2}^{b_1 b_2} Z_{\beta}^{b_3 \ldots b_6},
\]

while the commutators with level \(-2\) generators are
\[
[R_{a_1 \ldots a_4}, P_a] = 0, \quad [R_{a_1 \ldots a_4}, Z_{\beta}^{b}] = 0, \quad [R_{a_1 \ldots a_4}, Z_{b_1 b_2 b_3}] = 48 \delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} P_{b_4},
\]
\[
[R_{a_1 \ldots a_4}, Z_{\beta}^{b_1 \ldots b_5}] = 120 \delta_{a_1 a_4}^{[b_1 b_4} Z_{\beta}^{b_5]}, \quad [R_{a_1 \ldots a_4}, Z_{\alpha_1 \alpha_2}^{b_1 \ldots b_7}] = 0,
\]
\[
[R_{a_1 \ldots a_4}, Z_{\beta}^{b_1 \ldots b_7}] = -120 \delta_{a_1 a_4}^{[b_1 b_4} Z_{\beta}^{b_5 b_6 b_7},
\]
\[
[R_{a_1 \ldots a_4}, Z_{\beta}^{b_{1 \ldots b_6}, b}] = -1800 \delta_{a_1 \ldots a_4}^{b_1 \ldots b_4} Z_{\beta}^{b_5 b_6} + 1800 \delta_{a_1 \ldots a_4}^{b_1 \ldots b_4} Z_{\beta}^{b_5 b_6}, \quad (A.2.16)
\]

with level \(-3\) generators
\[
[R_{a_1 \ldots a_6}, P_a] = 0, \quad [R_{a_1 \ldots a_6}, Z_{\beta}^{b}] = 0, \quad [R_{a_1 \ldots a_6}, Z_{b_1 b_2 b_3}] = 0,
\]
\[
[R_{a_1 \ldots a_6}, Z_{\beta}^{b_1 \ldots b_5}] = -360 \delta_{\beta}^{a} \delta_{[a_1}^{a_2} b_{a_3} P_{a_6}], \quad [R_{a_1 \ldots a_6}, Z_{\alpha_1 \alpha_2}^{b_1 \ldots b_7}] = 1260 \delta_{(a_1}^{a_2} \delta_{[a_1 a_6}^{b_1 b_2} Z_{\beta}^{b_3 \ldots b_7]},
\]
\[
[R_{a_1 \ldots a_6}, Z_{\beta}^{b_1 \ldots b_7}] = 270 \epsilon_{\alpha_\beta} \delta_{a_1 a_6}^{b_1 b_2} Z_{\beta}^{b_3 \ldots b_7},
\]
\[
[R_{a_1 \ldots a_6}, Z_{\beta}^{b_{1 \ldots b_6}, b}] = 900 \epsilon_{\alpha_\beta} \delta_{a_1 a_6}^{b_1 b_6} Z_{\beta}^{b} - 900 \epsilon_{\alpha_\beta} \delta_{a_1 a_6}^{b_1 b_6} Z_{\beta}^{b}, \quad (A.2.17)
\]

and, finally, with level \(-4\) generators
\[
[R_{a_1 \ldots a_8}, P_a] = 0, \quad [R_{a_1 \ldots a_8}, Z_{\beta}^{b}] = 0, \quad [R_{a_1 \ldots a_8}, Z_{b_1 b_2 b_3}] = 0, \quad [R_{a_1 \ldots a_8}, Z_{\beta}^{b_1 \ldots b_5}] = 0,
\]
\[
[R_{a_1 \ldots a_8}, Z_{\beta}^{b_1 \ldots b_7}] = -20160 \delta_{\beta}^{a} \delta_{[a_1 a_7}^{a_2} \delta_{a_1 a_7}^{b_1 b_7} P_{a_8}], \quad [R_{a_1 \ldots a_8}, Z_{\beta}^{b_1 \ldots b_7}] = 0,
\]
\[
[R_{a_1 \ldots a_8}, Z_{\beta}^{b_{1 \ldots b_6}, b}] = 0, \quad [R_{a_1 \ldots a_7, a}, P_a] = 0, \quad [R_{a_1 \ldots a_7, a}, Z_{\beta}^{b}] = 0,
\]
\[
[R_{a_1 \ldots a_7, a}, Z_{\beta}^{b_1 b_2 b_3}] = 0, \quad [R_{a_1 \ldots a_7, a}, Z_{\beta}^{b_1 \ldots b_5}] = 0,
\]

33
\[ [R_{a_1...a_7}, Z^{b_1...b_7}] = 0, \quad [R_{a_1...a_7}, Z^{b_1...b_7}] = 4320 \delta_{a_1...a_7}^{b_1...b_7} P_a - 4320 \delta_{[a_1...a_7}^{b_1...b_7} P_a], \]

\[ [R_{a_1...a_7}, Z^{b_1...b_6}] = -75600 \delta_a \delta_{[a_1...a_6}^{b_1...b_6} P_{a_7}] + 75600 \delta_{a[a_1...a_6}^{b_1...b_6} P_{a_7}], \quad (A.2.18) \]

\section*{A.3 $D = 5$ algebra}

The $E_{11}$ algebra for the generators decomposed into representations of $GL(5) \otimes E_6$ is given below. This algebra for $SL(5)$ and the form generators, up to level four, can be found in references \cite{15,28} and \cite{10}, which also include some useful identities. Here we compute the full $E_{11}$ algebra up to level four and its commutators with the $l_1$ representation up to level 3. These include, in particular, the generators associated with the dual graviton and were given in equation (3.3.1) and (3.3.2). By construction the generators of $E_{11}$ are in representations of SL(5) and this determined their commutators with $K^a_b$ to be given by

\[ [K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_b K^c_d, \quad [K^a_b, R^\alpha] = 0, \]

\[ [K^a_b, R^{aN}] = \delta^a_b R^{aN}, \quad [K^a_b, R_{cN}] = -\delta^a_c R_{bN}, \]

\[ [K^a_b, R^{a_1a_2}_N] = 2 \delta^a_b [R^{a_1a_2}_N], \quad [K^a_b, R_{a_1a_2}^N] = -2 \delta^a_{[a_1} R_{b]a_2}^N, \]

\[ [K^a_b, R^{a_1a_2a_3, \alpha}] = 3 \delta^a_{[a_1} R^{a][a_2a_3], \alpha}], \quad [K^a_b, R_{a_1a_2a_3}^\alpha] = -3 \delta^a_{[a_1} R_{b]a_2a_3}^\alpha, \quad (A.3.1) \]

\[ [K^a_b, R^{a_1a_2, c}] = 2 \delta^a_b [R^{a][a_2, c}], \quad [K^a_b, R_{a_1a_2, c}] = -2 \delta^a_{[a_1} R_{b]a_2, c}, \quad (A.3.2) \]

\[ [K^a_b, R^{a_1a_2a_3, cN}] = 3 \delta^a_{[a_1} R^{a][a_2a_3], cN} + \delta^c_b R^{a_1a_2a_3, aN}, \]

\[ [K^a_b, R_{a_1a_2a_3, cN}] = -3 \delta^a_{[a_1} R_{b]a_2a_3, cN} - \delta^a_c R_{a_1a_2a_3, bN}. \]

The commutation relation of any generator with $R^\alpha$ is determined by the representation of $E_6$ that this generator belongs to:

\[ [R^\alpha, R^\beta] = f^{\alpha\beta\gamma} R^\gamma, \quad [R^\alpha, R^{aM}] = (D^\alpha)_N^M R^{aN}, \quad [R^\alpha, R_{aM}] = -(D^\alpha)_M^N R_{aN}, \]

\[ [R^\alpha, R^{a_1a_2}_M] = -(D^\alpha)_M^N R^{a_1a_2}_N, \quad [R^\alpha, R_{a_1a_2}^N] = (D^\alpha)_M^N R_{a_1a_2}^\alpha, \]

\[ [R^\alpha, R^{a_1a_2a_3, \beta}] = f^{\alpha\beta\gamma} R^{a_1a_2a_3, \gamma}, \quad [R^\alpha, R_{a_1a_2a_3}^\beta] = f^{\alpha\beta\gamma} R_{a_1a_2a_3}^\gamma, \]

\[ [R^\alpha, R^{a_1a_2, b}] = 0, \quad [R^\alpha, R_{a_1a_2, b}] = 0, \]

\[ [R^\alpha, R^{abcd}_M] = -(D^\alpha)_M^N R^{abcd}_P - (D^\alpha)_N^P R^{abcd}_MP, \]

\[ [R^\alpha, R_{abcd}^MN] = (D^\alpha)_P^M R^{abcd}PN + (D^\alpha)_P^N R^{abcd}MP, \]

\[ [R^\alpha, R^{a_1a_2a_3, bM}] = (D^\alpha)_N^M R^{a_1a_2a_3, bN}, \quad [R^\alpha, R_{a_1a_2a_3, bM}] = -(D^\alpha)_M^N R_{a_1a_2a_3, bN}. \quad (A.3.2) \]

where $f^{\alpha\beta\gamma}$ are the structure constants of $E_6$, normalised by $f^{\alpha\beta\gamma} f^{\alpha\beta\delta} = -4 \delta^{\beta\gamma} (D^\alpha)_N^M$ are the generators of $E_6$ in 27 representation. We lower and raise indices with the Killing metric $g_{\alpha\beta}$.
The commutation relations of the positive level $E_{11}$ generators are given by
\[
[R^{aM}, R^{bN}] = d^{MNP} R^{ab}_P, \quad [R^{aN}, R^{bc}_M] = (D_\alpha)_M^N R^{abc,\alpha}_N + \delta_M^N R^{bc,a},
\]
\[
[R^{ab}_M, R^{cd}_N] = R^{abcd}_{MN} - 20 d^{MNP} R^{abc,d}_P, \quad [R^{aN}, R^{bc}_d] = R^{abc,dN} - \frac{1}{3} R^{bcd,aN},
\]
\[
[R^{aN}, R^{bcd,\alpha}] = 3 d^{NMP} (D_\alpha)^P R^{abcd}_{MR} + 6 (D_\alpha)_M^N R^{bcd,aM}.
\] (A.3.3)

where $d^{MNP}$ is the completely symmetric invariant tensor of $E_6$, normalised by $d^{NPR} d^{MPR} = \delta_N^M$.

The commutators of negative-level $E_{11}$ generators are given by
\[
[R_{aN}, R_{bM}] = d_{NMP} R_{ab}^P, \quad [R_{aN}, R_{bc}^M] = (D_\alpha)^M_N R^{abc}_N + \delta_N^M R_{bc,a}.
\]
\[
[R_{ab}^M, R_{cd}^N] = R_{abcd}^{MN} - 20 d^{MNP} R_{ab,c,d}^P, \quad [R_{aN}, R_{bc,d}] = R_{abc,dN} - \frac{1}{3} R_{bcd,aN},
\]
\[
[R_{aN}, R_{bcd,\alpha}] = 3 d_{NMP} (D_\alpha)^P R_{abcd}^{MR} + 6 (D_\alpha)^M_N R_{bcd,aM}.
\] (A.3.4)

The commutators between the positive and negative level generators of $E_{11}$ up to level 4 are given by
\[
[R^{aN}, R_{bM}] = 6 \delta^b_a (D_\alpha)^N_{M} R^a + \delta^N_M K^a b - \frac{1}{3} \delta^N_M \delta^a_d K^c d,
\]
\[
[R_{aN}, R^{bc}_M] = 20 d_{NMP} \delta^b_{[a} R^c_{P]}, \quad [R^{aN}, R_{bc}^M] = 20 d^{NMP} \delta^a_{[b} R_{c]}^P,
\]
\[
[R_{aN}, R^{a_1 a_2 a_3, \alpha}] = 18 (D_\alpha)^N_M \delta_{[a_1}^{M} R^{a_2 a_3]}_{M}, \quad [R^{aN}, R_{a_1 a_2 a_3}^\alpha] = 18 (D_\alpha)^N_M \delta^a_{[a_1} R^{a_2 a_3]}_{M},
\]
\[
[R_{aN}, R^{a_1 a_2}_N] = \delta^b_a R^{a_1 a_2}_N - \delta^b_a R^{a_1 a_2}_{N}, \quad [R^{aN}, R_{a_1 a_2,b}] = \delta^b_a R^{a_1 a_2, N} - \delta^a_{[b} R_{a_1 a_2]}^N,
\]
\[
[R_{aN}, R^{a_1 a_2 a_3 a_4}_N] = 40 d_{N[N_1M]} (D_\alpha)^{N_2}_{N_2} \delta^a_{[a_1} R^{a_2 a_3 a_4]}_{N_2},
\]
\[
[R^{aN}, R_{a_1 a_2 a_3 a_4}^N] = 40 d_{N[N_1M]} (D_\alpha)^{N_2}_{M} \delta^a_{[a_1} R^{a_2 a_3 a_4]}_{N_2},
\]
\[
[R_{aN}, R^{a_1 a_2 a_3,bM}] = (D_\alpha)^N_M \delta^b_a R^{a_1 a_2 a_3, \alpha} - (D_\alpha)^N_M \delta^b_{[a} R^{a_1 a_2 a_3]}_{N} + 3 \delta^M_N \delta^a_{[a_1} R^{a_2 a_3],b},
\]
\[
[R^{aN}, R_{a_1 a_2 a_3,bM}] = (D_\alpha)^N_M \delta^b_a R^{a_1 a_2 a_3} - (D_\alpha)^N_M \delta^a_{[b} R^{a_1 a_2 a_3]_{N}} + 3 \delta^N_M \delta^a_{[a_1} R^{a_2 a_3],b}.
\] (A.3.5)

The Cartan involution acts on the generators of $E_{11}$ as follows
\[
I_c (K^a_b) = - K^b_a, \quad I_c (R^\alpha) = - R^{-\alpha}, \quad I_c (R^{aN}) = - J^{MN} R_{aM},
\]
\[
I_c (R^{ab}_M) = J_{MN}^{-1} R_{ab}^N, \quad I_c (R^{abc,a}) = - R_{abc,-a}, \quad I_c (R^{a_1 a_2,c}) = - R_{a_1 a_2,c},
\]
\[
I_c (R^{abcd}_{MN}) = J_{MP}^{-1} J_{NQ}^{-1} R_{abcd}^{PQ}, \quad I_c (R^{abc,dN}) = J^{MN} R_{abc,dM}.
\] (A.3.6)
We now give the commutators between the generators of $E_{11}$ and those of the $l_1$ representation given in (3.3.3) up to level 3. The commutation relations between the later and the generators of $GL(5)$ are given by

$$[K^a_b, P_c] = - \delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \quad [K^a_b, Z^N] = \frac{1}{2} \delta_b^a Z^N, \quad [K^a_b, Z^c_N] = \delta_b^c Z^a_N + \frac{1}{2} \delta_b^a Z^c_N,$$

$$[K^a_b, Z^{a_1 a_2 \alpha}] = 2 \delta^a_b Z^{[a_1|a_2], \alpha} + \frac{1}{2} \delta_b^a Z^{a_1 a_2, \alpha}, \quad [K^a_b, Z^{c d}] = \delta_b^c Z^{a d} + \delta_b^d Z^{a c} + \frac{1}{2} \delta_b^a Z^{c d}. \quad (A.3.7)$$

while with the generators of $E_6$ we have

$$[R^\alpha_a, P_a] = 0, \quad [R^\alpha_a, Z^M] = (D^\alpha)_N^M Z^N, \quad [R^\alpha_a, Z^a_N] = - (D^\alpha)_N^M Z^a_M,$$

$$[R^\alpha_a, Z^{a_1 a_2, \beta}] = f^{\alpha \beta \gamma} Z^{a_1 a_2, \gamma}, \quad [R^\alpha_a, Z^{a b}] = 0. \quad (A.3.8)$$

The elements of the $l_1$ representation at a given level can be introduced into the algebra by taking the commutators of suitable $E_{11}$ generators of the same level with $P_a$, namely

$$[R^{a N}_a, P_b] = \delta_b^a Z^N, \quad [R^{a_1 a_2}_a, P_a] = 2 \delta_a^{[a_1} Z^{a_2]}_N,$$

$$[R^{a_1 a_2 a_3}_a, P_a] = 3 \delta_a^{[a_1} Z^{a_2 a_3]}_a, \quad [R^{a_1 a_2, b}_a, P_a] = - 2 \delta_b^a Z^{[a_1 a_2] - 2 \delta_a^{[a_1} Z^{b]}_N,}$$

The commutators of the remaining positive level generators of $E_{11}$ with the $l_1$ generators is determined by the Jacobi identities and they are found to be given by

$$[R^{a M}_a, Z^N] = - d^{M N P} Z^P_a, \quad [R^{a N}_a, Z^b_M] = - (D^\alpha)_N^M Z^{a b, \alpha} - \delta^{N}_{M} Z^{a b},$$

$$[R^{a_1 a_2}_a, Z^M] = - (D^\alpha)_N^M Z^{a_1 a_2, \alpha} + 2 \delta^{N}_{M} Z^{[a_1 a_2]}_a. \quad (A.3.9)$$

Commutators between the level $-1$ generators of $E_{11}$ and those of the $l_1$ representation are also determined by the Jacobi identities to be given by

$$[R_{a N}, P_b] = 0, \quad [R_{a N}, Z^M] = \delta^M_N P_a, \quad [R_{a N}, Z^b_M] = - 10 d_{N M P} \delta^b_a Z^P,$$

$$[R_{a N}, P_b] = 0, \quad [R_{a N}, Z^M] = \delta^M_N P_a, \quad [R_{a N}, Z^b_M] = - 10 d_{N M P} \delta^b_a Z^P,$$

$$[R_{a N}, Z^{a_1 a_2}] = - 12 (D^\alpha)_N^M \delta^{[a_1}_a Z^{a_2]}_M, \quad [R_{a N}, Z^{b c}] = - \frac{2}{3} \delta^b_a Z^c_N - \frac{1}{3} \delta^c_a Z^b_N. \quad (A.3.10)$$

**A.4 D = 4 algebra**

In this appendix we give the $E_{11} \otimes l_1$ algebra decomposed into representations of $GL(4) \times SL(8)$ that corresponds to four-dimensional theory [19]. This latter reference contains a few typographical errors in the commutators which are corrected here. We will
first give the commutation relations of level 0 generators with the rest of $E_{11}$ algebra. The commutation relations of any generator with $K^a b$ are

$$[K^a b, K^c d] = \delta^c_b K^a d - \delta^a_d K^c b, \quad [K^a b, R^I] = 0, \quad [K^a b, R^{I_1 \ldots I_4}] = 0,$$

$$[K^a b, R^{I_1 I_2}] = \delta^a_b R^{a I_1 I_2}, \quad [K^a b, R^c I_2] = \delta^c_b R^a I_1 I_2,$$

$$[K^a b, \tilde{R}_{c I_1 I_2}] = -\delta^a_c \tilde{R}_{b I_1 I_2}, \quad [K^a b, \tilde{R}_c I_1 I_2] = -\delta^a_c \tilde{R}_b I_1 I_2,$$

$$[K^a b, \tilde{K}^{c d}] = 2 \delta^c_b \tilde{K}^{a|d}, \quad [K^a b, \tilde{K}_{c d}] = -2 \delta^a_c \tilde{K}_{b|d},$$

$$[K^a b, R^{a_1 a_2 I J}] = 2 \delta^a_{[a_1} R^{a|a_2] I J}, \quad [K^a b, \tilde{R}_{a_1 a_2 I J}] = -2 \delta^a_{[a_1} \tilde{R}_{b|a_2] I J},$$

$$[K^a b, R^{a_1 a_2 I_1 \ldots I_4}] = 2 \delta^a_{[a_1} R^{a|a_2] I_1 \ldots I_4}, \quad [K^a b, \tilde{R}_{a_1 a_2 I_1 \ldots I_4}] = -2 \delta^a_{[a_1} \tilde{R}_{b|a_2] I_1 \ldots I_4}. \quad (A.4.1)$$

The commutators with $SL(8)$ generator $R^I J$ are given by

$$[R^I J, R^K L] = \delta^K_J R^I L - \delta^J_K R^K J, \quad [R^I J, R^{I_1 \ldots I_4}] = 4 \delta^{|I_1|} R^{I I_1 I_2 I_3 I_4} - \frac{1}{2} \delta^K_J R^{I_1 \ldots I_4},$$

$$[R^I J, R^{a I_1 I_2}] = 2 \delta^{|I_1|} R^{a I I_2} - \frac{1}{4} \delta^K_J R^{a I_1 I_2}, \quad [R^I J, R^{a I_1 I_2}] = -2 \delta^{|I_1|} R^{a |I I_2} + \frac{1}{4} \delta^K_J R^{a I_1 I_2},$$

$$[R^I J, \tilde{R}_{a I_1 I_2}] = -2 \delta^{|I_1|} \tilde{R}_{a |I I_2} + \frac{1}{4} \delta^K_J \tilde{R}_{a I_1 I_2}, \quad [R^I J, \tilde{R}_a I_1 I_2] = 2 \delta^{|I_1|} \tilde{R}_a |I I_2| - \frac{1}{4} \delta^K_J \tilde{R}_a I_1 I_2,$$

$$[R^I J, \tilde{K}^{(ab)}] = 0, \quad [R^I J, \tilde{K}_{(ab)}] = 0,$$

$$[R^I J, R^{a_1 a_2 K L}] = \delta^K_J R^{a_1 a_2 I L} - \delta^L_J R^{a_1 a_2 K J},$$

$$[R^I J, \tilde{R}_{a_1 a_2 K L}] = \delta^K_J \tilde{R}_{a_1 a_2 I L} - \delta^L_J \tilde{R}_{a_1 a_2 K J},$$

$$[R^I J, R^{a_1 a_2 I_1 \ldots I_4}] = 4 \delta^{|I_1|} R^{a_1 a_2 |I I_2 I_3 I_4|} - \frac{1}{2} \delta^K_J R^{a_1 a_2 I_1 \ldots I_4},$$

$$[R^I J, \tilde{R}_{a_1 a_2 I_1 \ldots I_4}] = -4 \delta^{|I_1|} \tilde{R}_{a_1 a_2 |I I_2 I_3 I_4|} + \frac{1}{2} \delta^K_J \tilde{R}_{a_1 a_2 I_1 \ldots I_4}. \quad (A.4.2)$$

The commutators with the other $E_7$ generators $R^{I_1 \ldots I_4}$ generators are given by

$$[R^{I_1 \ldots I_4}, R^J_{\ldots J k}] = -\frac{1}{36} \varepsilon^{I_1 \ldots I_4 | J_1 J_2 J_3 J_4 | L} R^{L J k},$$

$$[R^{I_1 \ldots I_4}, R^{a J_1 J_2}] = \frac{1}{24} \varepsilon^{I_1 \ldots I_4 J_1 \ldots J_4} R^{a J_3 J_4}, \quad [R^{I_1 \ldots I_4}, R^{a J_1 J_2}] = \delta^{I_1 |I_2} R^{a J_3 J_4},$$

$$[R^{I_1 \ldots I_4}, \tilde{R}_{a J_1 J_2}] = \delta^{I_1 |I_2} \tilde{R}_{a I_3 I_4}, \quad [R^{I_1 \ldots I_4}, \tilde{R}_a J_1 J_2] = \frac{1}{24} \varepsilon^{I_1 \ldots I_4 J_1 \ldots J_4} R_{a J_3 J_4},$$

37
\[
\left[ R^{I_1 \ldots I_4}, R^{a_1 a_2 I_4} \right] = -4 \delta^{[I_4}_{I_1} R^{a_1 a_2|I_2 I_3 I_4} + \frac{1}{2} \delta^{|I_1}_{J_2} R^{a_1 a_2 I_3 I_4},
\]
\[
\left[ R^{I_1 \ldots I_4}, \tilde{R}_{a_1 a_2} I_4 \right] = - \frac{1}{6} \varepsilon^{I_1 \ldots I_4 J_2 J_3 J_4} \tilde{R}_{a_1 a_2 J_1 J_2 J_3 J_4} \frac{1}{48} \delta^{I_1}_{J_2} \varepsilon^{I_4 J_1 \ldots J_4} \tilde{R}_{a_1 a_2 I_1 \ldots I_4},
\]
\[
\left[ R^{I_1 \ldots I_4}, \tilde{K}^{(ab)} \right] = 0, \quad \left[ R^{I_1 \ldots I_4}, \tilde{K}_{(ab)} \right] = 0,
\]
\[
\left[ R^{I_1 \ldots I_4}, R^{a_1 a_2 J_1 \ldots J_4} \right] = \frac{1}{36} \varepsilon^{I_1 \ldots I_4 [J_1 J_2 J_3 J_4 | L | R^{a_1 a_2} J_4]},
\]
\[
\left[ R^{I_1 \ldots I_4}, \tilde{R}_{a_1 a_2 J_1 \ldots J_4} \right] = -\frac{2}{3} \delta^{I_1 [I_2 I_3} \tilde{R}_{a_1 a_2 J_4].}
\]

(A.4.3)

The commutators of the positive level one $E_{11}$ generators with each other are given by

\[
\left[ R^{a_1} I_2, R^{b} I_4 \right] = -12 R^{ab} I_1 \ldots I_4, \quad \left[ R^{a} I_1 I_2, R^{b} I_3 I_4 \right] = \frac{1}{2} \varepsilon^{I_1 \ldots I_4 J_1 \ldots J_4} R^{ab} I_1 \ldots J_4,
\]
\[
\left[ R^{a_1} I_2, \tilde{R}_{b} I_1 J_2 \right] = 4 \delta_{[I_1} R^{ab I_2]} J_2] + 2 \delta_{i_1 \ldots j_2} \tilde{K}^{(ab)}. \quad \text{(A.4.4)}
\]

The equivalent commutators for the negative level $E_{11}$ generators are

\[
\left[ \tilde{R}_{a_1} I_2, \tilde{R}_{b} I_3 I_4 \right] = -12 \tilde{R}_{ab} I_1 \ldots I_4, \quad \left[ \tilde{R}_{a_1} I_1 I_2, \tilde{R}_{b} I_3 I_4 \right] = \frac{1}{2} \varepsilon^{I_1 \ldots I_4 J_1 \ldots J_4} \tilde{R}_{a_1 a_2 J_1 \ldots J_4},
\]
\[
\left[ \tilde{R}_{a_1} I_2, \tilde{R}_{b} I_1 J_2 \right] = 4 \delta_{[I_1} \tilde{R}_{ab J_2]} J_2] + 2 \delta_{i_1 \ldots j_2} \tilde{K}_{(ab)}. \quad \text{(A.4.5)}
\]

The commutators between the level 1 and $-1$ $E_{11}$ generators are given by

\[
\left[ R^{a_1} I_2, \tilde{R}_{b} J_1 J_2 \right] = 2 \delta_{J_1 J_2} K^a b + 4 \delta^a b \delta_{I_1} K_{I_2} J_2 - \delta^a b \delta_{I_1} J_2 K^c c,
\]
\[
\left[ R^a I_1 I_2, \tilde{R}_{b} J_1 J_2 \right] = -2 \delta_{J_1 J_2} K^a b + 4 \delta^a b \delta_{I_1} K_{I_2} J_2 + \delta^a b \delta_{I_1} J_2 K^c c,
\]
\[
\left[ R^{a_1} I_2, \tilde{R}_{b} I_3 I_4 \right] = -12 \delta^a I_1 \ldots I_4, \quad \left[ R^a I_1 I_2, \tilde{R}_{b} I_3 I_4 \right] = \frac{1}{2} \delta^a \varepsilon^{I_1 \ldots I_4 J_1 \ldots J_4} R^{J_1 \ldots J_4}. \quad \text{(A.4.6)}
\]

The commutators with the level 2 and level $-1 E_{11}$ generators are given by

\[
\left[ R^{ab I}, \tilde{R}_{c} I_1 I_2 \right] = -4 \delta^{[a}_{C} \delta^{|I_1} R^{b]} I_2 | I_2] + \frac{1}{2} \delta^{[a_{c}} \delta^{|I_1} R^{b]} I_1 I_2,
\]
\[
\left[ R^{ab I}, \tilde{R}_{c} I_1 I_2 \right] = 4 \delta^{[a}_{C} \delta^{|I_1} R^{b]} I_2 | I_2] - \frac{1}{2} \delta^{[a_{c}} \delta^{|I_1} R^{b]} I_1 I_2,
\]
\[
\left[ R^{ab I_1 \ldots I_4}, \tilde{R}_{c} I_1 I_2 \right] = 2 \delta^{[a}_{C} \delta^{[I_2} R^{b]} I_3 I_4], \quad \left[ R^{ab I_1 \ldots I_4}, \tilde{R}_{c} I_5 I_6 \right] = \frac{1}{12} \varepsilon^{I_1 \ldots I_8} \delta^{[a}_{C} R^{b]} I_7 I_8,
\]
\[
\left[ \tilde{K}^{ab}, \tilde{R}_{c} I_1 I_2 \right] = -\delta^{(a}_{C} R^{b)} I_1 I_2, \quad \left[ \tilde{K}^{ab}, \tilde{R}_{c} I_1 I_2 \right] = -\delta^{(a}_{C} R^{b)} I_1 I_2. \quad \text{(A.4.7)}
\]
Finally, the commutators of level $-2$ with the level 1 $E_{11}$ generators are

$$\left[ \tilde{R}_{ab}^I, R^{cI_1I_2} \right] = -4 \delta^c_a \delta^I_I \tilde{R}_{b[I} \tilde{R}_{I]2} + \frac{1}{2} \delta^c_a \delta^I_I \tilde{R}_{b[I} \tilde{R}_{I]2},$$

$$\left[ \tilde{R}_{ab}^I, R^{cI_1I_2} \right] = 4 \delta^c_a \delta^I_I \tilde{R}_{b[I} \tilde{R}_{I]2} - \frac{1}{2} \delta^c_a \delta^I_I \tilde{R}_{b[I} \tilde{R}_{I]2},$$

$$\left[ \tilde{R}_{ab}^{I_1...I_4}, R^{cI_1I_2} \right] = 2 \delta^c_a \delta^I_I \tilde{R}_{b[I} \tilde{R}_{I]4}, \quad \left[ \tilde{R}_{ab}^{I_1...I_4}, R^{cI_5I_6} \right] = \frac{1}{12} \varepsilon^{I_1...I_8} \delta^c_a \tilde{R}_{b[I_7I_8},$$

$$\left[ \tilde{R}_{ab}^{I_1...I_4}, R^{cI_1I_2} \right] = -\delta^c_a \tilde{R}_{bI_1I_2}, \quad \left[ \tilde{R}_{ab}^{I_1...I_4}, R^{cI_1I_2} \right] = -\delta^c_a \tilde{R}_{bI_1I_2}. \quad (A.4.8)$$

The Cartan involution preserves the above commutators and is given by

$$I_c(K^a_b) = -K^b_a, \quad I_c(R^I_j) = -R^j_I, \quad I_c(R^{I_1...I_4}) = -\varepsilon^{I_1...I_4} \equiv -\frac{1}{4!} \varepsilon^{I_1...I_4J_1...J_4} R^{J_1...J_4},$$

$$I_c(R^{aI_1I_2}) = -\tilde{R}_{aI_1I_2}, \quad I_c(R^{aI_1I_2}) = \tilde{R}_{aI_1I_2}$$

$$I_c(R^{a_1a_2I}) = -\tilde{R}_{a_1a_2I}, \quad I_c(R^{a_1a_2I_1...I_4}) = \tilde{R}_{a_1a_2I_1...I_4}, \quad I_c(\tilde{K}^{ab}) = -\tilde{K}^{ab}$$

We now give the action of $E_{11}$ on the $l_1$ representation generators whose elements were given in equation (3.4.3). The commutation relations of the $l_1$ representation with level 0 generators of $E_{11}$ are given by

$$[K^a_b, P_c] = -\delta^a_c P_b + \frac{1}{2} \delta^a_b P_c, \quad [K^a_b, Z^{I_1I_2}] = \frac{1}{2} \delta^a_b Z^{I_1I_2}, \quad [K^a_b, Z_{I_1I_2}] = \frac{1}{2} \delta^a_b Z_{I_1I_2},$$

$$[K^a_b, Z^c] = \delta^a_c Z^a + \frac{1}{2} \delta^a_b Z^c, \quad [K^a_b, Z_{cI}] = \delta^a_c Z_{cI} + \frac{1}{2} \delta^a_b Z_{cI},$$

$$[K^a_b, Z_{cI_1...I_4}] = \delta^a_c Z_{cI_1...I_4} + \frac{1}{2} \delta^a_b Z_{cI_1...I_4},$$

$$[R^I_j, P_c] = 0, \quad [R^I_j, Z^{I_1I_2}] = 2 \delta^I_j Z^{I_1I_2} - \frac{1}{4} \delta^I_j Z^{I_1I_2},$$

$$[R^I_j, Z_{I_1I_2}] = -2 \delta^I_j Z_{I_1I_2} + \frac{1}{4} \delta^I_j Z_{I_1I_2},$$

$$[R^I_j, Z^{aK}] = 0, \quad [R^I_j, Z^{aK}L] = \delta^K_j Z^{aK} - \delta^L_j Z^{aK},$$

$$[R^I_j, Z^{aI_1...I_4}] = 4 \delta^I_j Z^{aI_1...I_4} - \frac{1}{2} \delta^I_j Z^{aI_1...I_4},$$

$$[R^{I_1...I_4}, P_a] = 0, \quad [R^{I_1...I_4}, Z^{I_1I_2}] = \frac{1}{24} \varepsilon^{I_1...I_4J_1...J_4} Z_{J_1J_2},$$

$$[R^{I_1...I_4}, Z_{J_1J_2}] = \delta^{I_1I_2} Z_{I_1I_2},$$

39
\[ [R^{I_1 \ldots I_4}, Z^a] = 0, \quad [R^{I_1 \ldots I_4}, Z^{aI}] = -\frac{4}{3} \delta^{[I_1}_{[I_2} Z^{aI_2 I_3 I_4]} + \frac{1}{6} \delta^I_{I_4} Z^{aI_1 \ldots I_4}, \]
\[ [R^{I_1 \ldots I_4}, Z^{aJ_1 \ldots J_4}] = \frac{1}{12} \varepsilon^{J_1 \ldots J_4 I_1 I_2 I_3 I_4 K} Z^{aI_4}]_K. \] (A.4.9)

The commutators with the \( E_{11} \) level 1 generators are given by
\[ [R^{aI_1 I_2}, P_b] = \delta^a_b Z^{I_1 I_2}, \quad [R^{aI_1 I_2}, P_b] = \delta^a_b Z^{I_1 I_2}, \]
\[ [R^{aI_1 I_2}, Z^{J_1 J_2}] = -Z^{aI_1 I_2 J_1 J_2}, \quad [R^{aI_1 I_2}, Z^{J_1 J_2}] = \delta^a_{[I_1} Z^{a] J_2 I_2] + \delta^a_{J_1} Z^a, \]
\[ [R^{aI_1 I_2}, Z^{J_1 J_2}] = \delta^a_{[J_1} Z^{a] J_2]} - \delta^a_{J_1 J_2} Z^a, \quad [R^{aI_1 I_2}, Z^{J_1 J_2}] = \frac{1}{24} \varepsilon^{I_1 I_2 J_1 J_2 K_1 \ldots K_4} Z^{aK_1 \ldots K_4}. \] (A.4.10)

The commutators with the \( E_{11} \) level 2 generators are given by
\[ [\dot{K}^{(a_1 a_2)}, P_a] = \delta^{(a_1}_{a} Z^{a_2)], \quad [R^{a_1 a_2 I} J, P_a] = -\frac{1}{2} \delta^{a_1}_{a} Z^{a_2 I J}, \]
\[ [R^{a_1 a_2 I_1 \ldots I_4}, P_a] = -\frac{1}{6} \delta^{a_1}_{a} Z^{a_2 I_1 \ldots I_4}. \] (A.4.11)

The commutators with the \( E_{11} \) level \(-1\) generators are
\[ [\bar{R}_{aI_1 I_2}, P_b] = 0, \quad [\bar{R}_{aI_1 I_2}, Z^{J_1 J_2}] = 2 \delta^{[I_1}_{I_2} P_a, \quad [\bar{R}_{aI_1 I_2}, Z^{J_1 J_2}] = 0, \]
\[ [\bar{R}_{aI_1 I_2}, P_b] = 0, \quad [\bar{R}_{aI_1 I_2}, Z^{J_1 J_2}] = 0, \quad [\bar{R}_{aI_1 I_2}, Z^{J_1 J_2}] = -2 \delta^a_{I_1 I_2} P_a, \]
\[ [\bar{R}_{aI_1 I_2}, Z^b] = -2 \delta^b_a Z_{I_1 I_2}, \quad [\bar{R}_{aI_1 I_2}, Z^{bl} J] = -8 \delta^b_a \delta^l_{[I_1} Z_{I_2]} J, \]
\[ [\bar{R}_{aI_1 I_2}, Z^{bJ_1 \ldots J_4}] = -12 \delta^b_a \delta^{[I_1}_{I_2} Z^{J_3 J_4]}, \quad [\bar{R}_{aI_1 I_2}, Z^b] = -2 \delta^b_a Z^{l I_1 I_2}, \]
\[ [\bar{R}_{aI_1 I_2}, Z^{bl} J] = 8 \delta^b_a \delta^{[l}_{J} Z^{I_2] J}, \quad [\bar{R}_{aI_1 I_2}, Z^{bJ_1 \ldots J_4}] = -\frac{1}{2} \delta^b_a \varepsilon^{J_1 \ldots J_2 I_1 \ldots I_4} Z_{I_3 I_4}. \] (A.4.12)

The last three commutators in equation (A.4.9), the last four in equation (A.4.10), all in equation (A.4.11) and the last six in equation (A.4.12) are not contained in reference [19] and are taken from a forthcoming publication with Nikolay Gromov.

References

[1] S. Weinberg, Phys. Rev. Lett. 16 (1966) 63; Phys. Rev. Lett. 18 (1967) 188; Phys. Rev. 166 (1968) 1568; Physica 96A (1979) 327.

[2] S. Coleman, J. Wess and B. Zumino, Structure of Phenomenological Lagrangians. 1, Phys.Rev. 177 (1969) 2239; C. Callan, S. Coleman, J. Wess and B. Zumino, Structure of phenomenological Lagrangians. 2, Phys. Rev. 177 (1969) 2247.

[3] A. Salam and J. Strathdee, Superfields and Fermi-Bose symmetry, Phys. Rev. D 11 (1975) 1521; Feynman rules for superfields, Nucl. Phys. B 86 (1975) 142.
[4] A. Borisov and V. Ogievetsky, *Theory of dynamical affine and conformal symmetries as the theory of the gravitational field*, Teor. Mat. Fiz. 21 (1974) 32.

[5] P. West, *Hidden superconformal symmetries of M-theory*, JHEP 0008 (2000) 007, arXiv:hep-th/0005270.

[6] P. West, *E_{11}, SL(32) and Central Charges*, Phys. Lett. B 575 (2003) 333-342, hep-th/0307098.

[7] N. Lambert and P. West, *Coset Symmetries in Dimensionally Reduced Bosonic String Theory*, Nucl. Phys. B615 (2001) 117-132, hep-th/0107209.

[8] F. Englert, L. Houart, A. Taormina and P. West, *The Symmetry of M-Theories*, JHEP 0309 (2003) 020, hep-th/0304206.

[9] P. West, *Introduction to Strings and Branes*, Cambridge University Press, June 2012.

[10] P. West, *Generalised Space-time and Gauge Transformations*, arXiv:1403.6395.

[11] P. West, *E_{11} and M Theory*, Class. Quant. Grav. 18 (2001) 4443, arXiv:hep-th/0044081.

[12] I. Schnakenburg and P. West, *Kac-Moody symmetries of IIB supergravity*, Phys. Lett. B517 (2001) 421, arXiv:hep-th/0107181.

[13] P. West, *The IIA, IIB and eleven dimensional theories and their common E_{11} origin*, Nucl. Phys. B693 (2004) 76-102, hep-th/0402140.

[14] F. Riccioni and P. West, *The E_{11} origin of all maximal supergravities*, JHEP 0707 (2007) 063; arXiv:0705.0752.

[15] F. Riccioni and P. West, *E(11)-extended spacetime and gauged supergravities*, JHEP 0802 (2008) 039, arXiv:0712.1795.

[16] P. West, *E_{11} origin of Brane charges and U-duality multiplets*, JHEP 0408 (2004) 052, hep-th/0406150.

[17] P. West, *Brane dynamics, central charges and E_{11}*, hep-th/0412336.

[18] C. Hillmann, *Generalized E(7(7)) coset dynamics and D=11 supergravity*, JHEP 0903, 135 (2009), hep-th/0901.1581 ; *E(7(7)) and d=11 supergravity*, PhD thesis, arXiv:0902.1509.

[19] P. West, *E11, Generalised space-time and equations of motion in four dimensions*, JHEP 1212 (2012) 068, arXiv:1206.7045.

[20] D. Berman, H. Godazgar, M. Perry and P. West, *Duality Invariant Actions and Generalised Geometry*, JHEP 1202 (2012) 108, arXiv:1111.0459.

[21] P. West, *Generalised Geometry, eleven dimensions and E_{11}*, JHEP 1202 (2012) 018, arXiv:1111.1642.

[22] W. Siegel, *Two vielbein formalism for string inspired axionic gravity*, Phys.Rev. D47 (1993) 5453, hep-th/9302036; *Superspace duality in low-energy superstrings*, Phys.Rev. D48 (1993) 2826-2837, hep-th/9305073; *Manifest duality in low-energy superstrings*, In *Berkeley 1993, Proceedings, Strings '93* 353, hep-th/9308133.

[23] P. West, *E11, generalised space-time and IIA string theory*, Phys.Lett.B696 (2011) 403-409, arXiv:1009.2624.

[24] A. Rocen and P. West, *E11, generalised space-time and IIA string theory; the R-R sector*, arXiv:1012.2744.

[25] D. Berman, M. J. Perry, Generalized Geometry and M theory, JHEP 1106 (2011) 74 arXiv:1008.1763; D. Berman, H. Godazgar and M. J. Perry, SO(5,5) duality in
M-theory and generalized geometry, Phys. Lett. B700 (2011) 65-67. arXiv:1103.5733.

[26] M. Duff, *Duality Rotations In String Theory*, Nucl. Phys. B 335 (1990) 610; M. Duff and J. Lu, Duality rotations in membrane theory, Nucl. Phys. B347 (1990) 394.

[27] P. West, *E_{11}, ten forms and supergravity*, JHEP 0603 (2006) 072, hep-th/0511153.

[28] F. Riccioni, D. Steele and P. West, *The E(11) origin of all maximal supergravities - the hierarchy of field-strengths* JHEP 0909 (2009) 095, arXiv:0906.1177.

[29] P. West, *Generalised Space-time and Gauge Transformations*, arXiv:1403.6395.

[30] T. Nutma, SimpLie, a simple program for Lie algebras, https://code.google.com/p/simplie/.