Helmholtz Theorem for Differential Forms in 3-D Euclidean Space∗

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Abstract

There are significant differences between Helmholtz and Hodge’s decomposition theorems, but both share a common flavor. This paper is a first step to bring them together.

We here produce Helmholtz theorems for differential 1–forms and 2–forms in 3-D Euclidean space, \(E_3\). We emphasize their common structure in order to facilitate the understanding of another paper, soon to be made public, where a Helmholtz theorem for arbitrary differential forms in arbitrary Euclidean space is presented and which allows one to connect (actually to derive from it) an improvement of Hodge’s decomposition theorem.

1 Context of this Paper

This paper is a first step in connecting the Helmholtz and Hodge decomposition theorems. The first of these pertains to vector fields (objects of grade one) in 3-D Euclidean space, \(E_3\). It involves the action of the del operator through vector product. This product is specific to three dimensions. It does not exist in arbitrary dimension. Hence the present Helmholtz theorem is a peculiarity of 3-D.

∗To Dr. Howard Brandt, for his contribution to the improvement of my papers and books.
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The second theorem pertains to the vast subject of arbitrary differential forms (in compact oriented Riemannian manifolds) of arbitrary dimension. The first one is of interest mainly for physicists in general, and can be found in the first chapter of a book for them on mathematical methods [1]. The second one is of interest primordially for mathematicians and mathematical physicists, specially those concerned with any of the overlapping subjects of de Rham[2]-Hodge[3] theory, elliptic operators, cohomology, etc. They appear in the thick of books where at last one of those subjects is considered.

This author has made progress in relating those two theorems. The cuspid of his results is constituted by the following:

(a) A decomposition theorem for differential forms in Euclidean spaces, $E_n$, of arbitrary dimension in Helmholtz format. This terminology is justified for making a difference with the Hodge format, where there are only two terms in the decomposition (a would-be-third term goes to zero if and when relevant quantities decay sufficiently fast at infinity). That terminology is also justified because, as in the original theorem and unlike in Hodge’s case, the terms of the decomposition are specified.

(b) An application of (a) through the use of the embedding theorem of Schlafli[4] -Janet[5] -Cartan[6] to improve on the original Hodge theorem [3]. For modern treatments, see [7], [8], [9] and [10]. Let us recall that the original theorem states that a differential form in an oriented compact Riemannian manifolds can be decomposed into three components that are respectively closed, co-closed and harmonic, but it does not state what those components are for given differential form. Our version of the theorem does.

For the proof, we resort to Kähler’s calculus (KC) [11], [12], which generalizes Cartan’s.

None of these results is presented in this paper (I), but in a paper intended to follow (II), which requires more mathematical sophistication. But I may have enough contents to satisfy the interest of most physicists. A second reason is that this paper seeks to pique the interest of a potential sponsor of II for the mathematics-analysis section of the arXiv, where such paper would presumably be assigned if sponsored. It is important to realize that an important indirect goal of both papers is to call attention to the fact that KC is a formidable tool that is being overlooked. It provides opportunities to obtain substantial results in dealing with issues at which one need not be an expert. That is precisely what makes it so important.
2 Introduction

Cartan’s calculus enriched by the Hodge duality operation may suffice to follow this paper, but KC provides a much richer context for the arguments and computations. Consider, for instance, the co-differential. In the so enriched Cartan calculus, this concept involves the metric through Hodge duality. In KC, it is a more general concept introduced through the connection. Both definitions coincide for the Levi-Civita connection, but not in general.

The present author is not aware of whether every particular result he uses is known by practitioners of Cartan’s calculus enriched by the Hodge duality operation. Hence, in section 3, we have introduced well known results (and some which may not be as well known) from a KC perspective. Our source is any of the papers [11] and [12]. Their author owes much to De Rham and Hodge, as he recognized in the first paragraph of [11].

In section 4, we speak of the Helmholtz format. These considerations, later enriched as we make progress in the paper, will allow us to see very early what the Helmholtz format is for general differential forms in euclidean spaces.

In section 5, we derive Helmholtz’ theorem for differential 2−forms. The proof mimics the one for vector fields. In section 6, we derive Helmholtz’ theorem for differential 2−forms, using the fact that any such differential form has a differential 1−form as dual. Hence, from the theorem for the last ones, we obtain the theorem for the first ones. This is instructive in the following specific regard.

Recall that, in the derivation of Helmholtz theorem for vector fields, the treatment of one of the two terms is more complicated than the derivation of the other term. This asymmetry carries to the corresponding theorem for differential 1−forms. The complication is reversed for differential 2−forms. The reason behind it is that the co-differential of a 1−form is a 0−form, and the exterior differential of a 2−form is a 3−form, which, in dimension 3, is almost like a 0−form for certain purposes. In higher dimension both terms of the decomposition will present similar difficulty. Which brings us already to state (without proof) the following qualitative feature of the improvement of our Hodge theorem. The harmonic term is in turn the sum of two terms, each of them harmonic and respectively related to the closed and co-closed terms.

Finally, in section 7, we anticipate how the generalization of Helmholtz theorem of which we spoke above will look like.
How to read this paper? It will be a function of a reader’s knowledge. If familiar with KC, one can comfortably jump to section 5. If not, it may help to read every section.

3 Calculus of Differential Forms

Our differential forms are integrands (known as currents in some of the literature), not skew-symmetric multilinear functions of vectors. Stokes generalized theorem is directly about integrals, and only indirectly about skew-symmetric multilinear functions of vectors. Its underlying algebra, when dealing only scalar-valued differential forms, is Clifford algebra, but is known as Kähler’s algebra when it refers to differential forms. Thus, not all calculi based on Clifford algebra are equivalent. Alternatives are the calculi by Dirac [13] and by Hestenes [14], which are both based on tangent algebra. Worth mentioning is that the problem with negative energies does not arise in KC [12].

From what has been said, Kähler’s algebra is built upon the module of differential forms spanned by \((dx^i)\) and defined by

\[
dx^i dx^j + dx^j dx^i = 2g^{ij}.
\]

When at least one of two factors in a Clifford product is a differential 1-form, \(\alpha\), the identity

\[
\alpha \Gamma \equiv \frac{1}{2}(\alpha \Gamma + \Gamma \alpha) + \frac{1}{2}(\alpha \Gamma - \Gamma \alpha)
\]

allows one to define the two terms on the right of (2) as the interior and exterior product. Which one is which depends on the grade of \(\Gamma\). If \(\Gamma\) is another 1-form, \(\beta\), we have

\[
\alpha \beta = \alpha \wedge \beta + \alpha \wedge \beta
\]

where

\[
\alpha \wedge \beta \equiv \frac{1}{2}(\alpha \beta - \beta \alpha),
\]

\[
\alpha \cdot \beta \equiv \frac{1}{2}(\alpha \beta + \beta \alpha).
\]

Those specific “\(\wedge\)” and “\(\cdot\)” products are of respective grades two and zero. These equations apply in particular to when \(\alpha\) and \(\beta\) are the differentials of the coordinate functions.
Let \( w \) denote the unit differential of highest possible grade. In Cartesian coordinates in 3-D Euclidean space, \( E_3 \), it can be given as

\[
w = dx dy dz.
\] (5)

Since the exterior product raises the grade, the product \( "w \wedge ..." \) by the algebra (except scalars) yields zero. Thus, we have

\[
\Gamma w = \Gamma \wedge w + \Gamma \cdot w = \Gamma \cdot w,
\] (6)

for non-scalars. Multiplication by \( w \) is called the obtaining of the Hodge dual, which corresponds in the tensor calculus to contracting with the Levi-Civita tensor. In \( E_3 \), \( w \) commutes with the whole algebra for that space, and \( w^2 = -1 \).

The Kähler operator, which we shall represent with the symbol \( \partial \), is the sum of a part, \( d \), which raises the grade by one, and a part, \( \delta \), which lowers the grade by one:

\[
\partial = d + \delta.
\] (7)

Readers may refer to it as Dirac’s operator, but we prefer to keep our distance from anything Dirac’s since it may occasionally induce one into error due to different contexts.

When the connection of the manifold is the Levi-Civita connection (LCC), \( \delta \) of a scalar-valued differential form is called co-differential. The operation \( d \) is exterior differentiation —exterior covariant differentiation if applied to tensor-valued or Clifford-valued differential forms. Neither \( \partial \) nor \( \delta \) satisfy the standard Leibniz rule. And yet, Kähler refers to \( \partial \) (for which he uses the symbol \( \delta \)) as interior differentiation. There is strong reason for using the term differentiation in spite of the issue about the Leibniz rule. But Kähler’s better theory should not always be constrained by how terms are used in other theories.

Under the LCC, differential forms \( \Gamma \) of arbitrary grade satisfy

\[
\delta \Gamma = (-1)^\binom{n}{2} d(\Gamma w)w.
\] (8)

In dimension three, we readily get

\[
\delta \Gamma = -wd(\Gamma w),
\] (9)

which is not a definition but a theorem in KC. In arbitrary dimension,

\[
(\delta \Gamma)w = d(\Gamma w)
\] (10)
since \( w^2 = (-1)^{n-n} \).

The Laplacian of differential forms is defined as

\[
\partial \partial \Gamma = (d + \delta)(d + \delta)\Gamma = (dd + d\delta + \delta d + \delta\delta)\Gamma. \tag{11}
\]

For scalar-valued differential forms, \( dd\Gamma \) is zero. If, in addition, the connection is Levi-Civita’s, \( \delta\delta \Gamma = 0 \). Thus,

\[
\partial \partial \Gamma = d\delta \Gamma + \delta d \Gamma, \tag{12}
\]

and further,

\[
\partial \partial f = \delta df \tag{13}
\]

for 0-forms, \( f \).

All the above is well known. What follows may rarely be known. Kähler defines covariant derivatives of \((p, q)\)-valued differential \( r \)-forms, inhomogeneous in general. Their components have three series of indices, \( p, q \) and \( r \), the \( q \) and \( r \) series being of subscripts; \( q \) is for multilinear functions of vectors and \( r \) for functions of hypersurfaces, i.e. integrands. Thus curvature is \((1,1)\)-tensor valued differential 2-form. As a differential 2-form, it is a function of surfaces, surfaces defined by pairs of curves with the same origin and end along which we transport a vector. The second number one in \((1,1)\) refers to the vector being transported. Curvature is then a function of a pair of vector field and surface. This function is vector-valued, meaning the following. It is evaluated on the vector field and then evaluated (read integrated) on the surface. As already mentioned by Cartan, integration surfaces must be infinitesimal, unless the affine curvature is zero.

We only need here scalar-valuedness \((p = q = 0)\). The definition of covariant derivative then simply is

\[
d_h u \equiv \frac{\partial u}{\partial x^h} - \omega^k_h \wedge e_k u, \tag{14}
\]

where \( e_k u \) is \( \omega_k \cdot u \). Kähler then defines \( \partial u \) as

\[
\partial u \equiv dx^h \vee d_h u = du + \delta u \tag{15}
\]

where

\[
 du \equiv dx^h \wedge d_h u, \quad \delta u \equiv dx^h \cdot d_h u. \tag{16}
\]
Consider Euclidean space. It is clear that \( du = dx^h (\partial u / \partial x^h) \) since \( dx^h \wedge \omega^k_h = ddx^k = 0 \). Notice that, in Cartesian coordinates (which presupposes Euclidean space), \( du = dx^h \cdot (\partial u / \partial x^h) \), since \( \omega^k_h \) is then 0. If \( u \) were a differential 1-form, the resulting expression would be the same as the divergence of a vector field that, in terms of a constant orthonormal frame field, has the same components as the differential form in Cartesian coordinates.

Kähler proves that \( \delta u \) so defined coincides with the co-differential. \( dh^u \) satisfies the Leibniz rule, but \( \partial \) and \( \delta \) do not. They rather satisfy

\[
\partial(u \wedge v) = \partial u \wedge v + \eta u \wedge \partial v + e^h u \wedge dh^v + \eta dh^u \wedge e^h v,
\]

(17)

and

\[
\delta(u \wedge v) = \delta u \wedge v + \eta u \wedge \delta v + e^h u \wedge dh^v + \eta dh^u \wedge e^h v.
\]

(18)

In view, however, of how \( \partial, \partial \) and \( \delta \) emerge from the covariant derivative, it seems natural to refer to all of them as differentiations, specially since we also need a name for \( \delta u \) whenever the connection is not the LCC. Hence, we shall refer to \( \partial, \partial \) and \( \delta \) as, respectively, the Kähler, exterior and interior differentials.

A differential form such that \( dh^u \) is zero is called constant differential. It will be denoted as \( c \). They have the property that

\[
\partial(\Gamma c) = (\partial \Gamma)c,
\]

(19)

and, in particular

\[
\partial(\Gamma w) = (\partial \Gamma)w.
\]

(20)

Of great importance is that all polynomials in \( (dx, dy, dz) \) with constant coefficients are constant differentials, examples being \( w \) and the \( dx, dy \) and \( dz \) themselves.

4 Perspective on Helmholtz Theorem

Helmholtz theorem states that smooth vector fields decaying sufficiently fast at infinity can be written as a sum

\[
v = -\nabla \phi + \nabla \times A
\]

(21)

with

\[
\phi(r) = \frac{1}{4\pi} \int \frac{\nabla' \cdot v(r')}{r_{12}} dV'
\]

(22a)
\[ A(r) = \frac{1}{4\pi} \int \frac{\nabla' \times v(r')}{r_{12}} dV' \quad (22b) \]

where \( \nabla' \) refers to differentiation with respect to primed coordinates, where \( V' \) is the volume element in primed coordinates and where

\[ r_{12} \equiv [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}. \quad (23) \]

We shall denote \( v(r') \) as \( v' \). If we substitute (22) in (21), we get

\[ v = \frac{1}{4\pi} \left[ -\nabla \int \frac{1}{r_{12}} (\nabla' \cdot v') dV' + \nabla \times \int \frac{1}{r_{12}} \nabla' \times v' dV' \right], \quad (24) \]

whose format vis-a-vis the del is

\[ -\text{grad} \ldots \text{div}' v' + \text{curl} \ldots \text{curl}' v'. \quad (25) \]

Recalling that the Laplacian, \( \Delta \), satisfies

\[ -\Delta = -\text{grad} \text{ div} + \text{curl} \text{ curl}, \quad (26) \]

we are led to consider

\[ \partial \partial = -d\delta \alpha - \delta d\alpha \quad (27) \]

as replacement for (26) [strict parallelism between differentiations of vector fields and differential 1-forms would lead us to blindly write \( -\Delta v \) equal to \( -\text{grad div} v - \text{div} \text{ grad} v \), which is nonsense since vector fields do not have gradients]. We thus consider the format

\[ -d \ldots \delta' \alpha' - \delta \ldots d' \alpha' \quad (28) \]

for Helmholtz theorem for differential 1-forms, \( \alpha' \) being to \( \alpha \) what \( v(r') \) is to \( v(r) \).

Those observations could be used to make the corresponding changes in (24) to obtain Helmholtz theorem for differential 1-forms. But, in order to get confidence with computations with them beyond the most trivial, we shall formulate the theorem and proceed to prove it as one does in the vector calculus. We shall thus need a uniqueness theorem like the one according to which a vector field (respectively a differential 1-form, \( \alpha \)) is uniquely defined in a region by its divergence and curl and its normal component over the boundary (respectively its \( \delta \) and \( d \) "derivatives" and its components at the boundary...
boundary). The proof resorts to showing that the difference between two hypothetical solutions, $\alpha_1$ and $\alpha_2$, have zero $d$ and $\delta$ derivatives. By the first of these annulments $\alpha_1 - \alpha_2$ is closed and, therefore, locally (meaning “not necessarily globally”) exact, i.e. $\alpha_1 - \alpha_2 = df$. We then have

$$0 = \delta(\alpha_1 - \alpha_2) = \delta df = \partial^2 f. \tag{29}$$

We are now at a point similar to when, in the proof of the uniqueness theorem in the vector calculus, one resorts to Green’s theorem. The namesake theorem in KC’s is far more comprehensive, but it implies in particular that

$$\int_R w f \delta^2 g + \partial f \cdot \partial g = \int_{\partial R} (f dg) w. \tag{30}$$

We specialize this equation to $f = g$. We assume vanishing $f$ (i.e. $\alpha_1 - \alpha_2 = 0$) at the boundary. Equation (30) then yields, using (29),

$$0 = (\partial f)^2 = \partial f = df, \tag{31}$$

all over the region considered. Hence $\alpha_1 = \alpha_2$. The uniqueness theorem has been proved.

## 5 Helmholtz Theorem for Differential 1-Forms in 3-D Euclidean Space

Helmholtz theorem: In Cartesian coordinates in $E_3$, differential 1-forms that are smooth and vanish sufficiently fast at infinity can be written as

$$\alpha(r) = -\frac{1}{4\pi} dI^0 - \frac{1}{4\pi} \delta(dx^j dx^k I^i), \tag{32}$$

$$I^0 \equiv \int \frac{1}{r_{12}} (\delta' \alpha') w', \quad I^i \equiv \int \frac{1}{r_{12}} d' \alpha' \wedge dx^i, \tag{33}$$

with summation over the three cyclic permutations of 1,2,3.

**Proof:** By the uniqueness theorem and the annulment of $dd$ and $\delta \delta$, the proof reduces to showing that $\delta$ and $d$ of respective first and second terms on the right hand side of (32) yield $d \alpha$ and $\delta \alpha$.

Since $\delta dI^0 = \partial \partial I^0$, we write $-(1/4\pi) \delta dI^0$ as

$$-\frac{1}{4\pi} \partial \partial I^0 = -\frac{1}{4\pi} \int_{E_3} (\partial \partial \frac{1}{r_{12}}) (\delta' \alpha') w' = -\frac{1}{4\pi} \int_{E_3} (\delta' \partial' \frac{1}{r_{12}}) (\delta' \alpha') w' = \delta \alpha, \tag{34}$$
after using the relation of \( \partial\partial \) to the Dirac distribution.

For the second term, we use that \( d\delta = \partial\partial - \delta d \) when acting on \( dx^j dx^k \). Let \( \alpha \) be given as \( \alpha_j(x)dx^j \) in terms of the same coordinate system. We get \( d\alpha_i^j \wedge dx^i = (a'_{k,j} - a'_{j,k})w' \). The same property of \( \partial\partial \) now allows us to obtain \( d\alpha \).

For the second part of the second term, we apply \( \delta d \) to \( dx^j dx^k \):

\[
\delta d(dx^j dx^k) = \delta \left( w \frac{\partial I^i}{\partial x^j} \right) = \delta w \left( \frac{\partial I^i}{\partial x^j} \right) = \delta dx^l \frac{\partial^2 I^i}{\partial x^l \partial x^j} = \]

\[
= \delta dx^l \int_{E_3} \left[ \frac{\partial^2}{\partial x'^n \partial x'^m} \left( \frac{1}{r_{12}} \right) \right] (a'_{k,j} - a'_{j,k})w'. \tag{35}
\]

We integrate by parts with respect to \( x'^i \). One of the two resulting terms is:

\[
wdx^l \int_{E'_3} \frac{\partial}{\partial x'^n} \left[ \frac{\partial}{\partial x'^m} \left( \frac{1}{r_{12}} \right) \right] (a'_{k,j} - a'_{j,k})w'. \tag{36}
\]

Application to this of Stokes theorem yields

\[
wdx^l \int_{\partial E'_3} \frac{\partial}{\partial x'^n} \left( \frac{1}{r_{12}} \right) (a'_{k,j} - a'_{j,k})dx'^j dx'^k. \tag{37}
\]

It vanishes for sufficiently fast decay at infinity.

The other term resulting from the integration by parts is

\[
- wdx^l \int_{E'_3} \frac{\partial}{\partial x'^m} \frac{\partial}{\partial x'^n} (a'_{k,j} - a'_{j,k})w', \tag{38}
\]

which vanishes identically (perform the \( \frac{\partial}{\partial x'^m} \) differentiation and sum over cyclic permutations). The theorem has been proved.

### 6 Helmholtz theorem for differential 2–forms in \( E_3 \)

The theorem obtained for differential 1–forms, here denoted as \( \alpha \), can be adapted to differential 2–forms, \( \beta \), by defining \( \alpha \) for given \( \beta \) as

\[
\alpha \equiv w\beta, \quad \beta = -w\alpha. \tag{39}
\]
Then, clearly,
\[ w\delta(w\beta) = -d\beta, \quad wd\beta = \delta(w\beta). \] (40)
Helmholtz theorem for differential 1–forms can then be written as
\[ w\beta = -\frac{1}{4\pi} d \left( \int_{E_3} \frac{\delta'(w'\beta')}{r_{12}} w' \right) - \frac{1}{4\pi} \delta \left( \int_{E_3} \frac{d'(w'\beta') \wedge dx^i}{r_{12}} \right), \] (41)
and, therefore,
\[ \beta = \frac{1}{4\pi} wd \left( \int_{E_3} \frac{\delta'(w'\beta')}{r_{12}} w' \right) + \frac{1}{4\pi} w\delta \left( \int_{E_3} \frac{d'(w'\beta') \wedge dx^i}{r_{12}} \right). \] (42)
The integrals are scalar functions of coordinates \( x \). We shall use the symbol \( f \) to refer to them in any specific calculation. In this way, steps taken are more easily identified.

The first term in the decomposition of \( \beta \), we transform as follows:
\[ wdf = (\partial f)w = \partial(fw) = \delta(fw), \] (43)
where we have used that \( w \) is a constant differential.

For the second term, we have:
\[ w\delta(dx^{jk}f) = w\partial[wdx^if] - wd[fdx^{jk}]. \] (44)
The first term on the right is further transformed as
\[ w\partial(wdx^i f) = wwdx^i \partial f = -dx^i df, \] (45)
where we have used that \( wdx^i \) is a constant differential, which can be taken out of the \( \partial \) differentiation. For the other term, we have
\[ -wd(fdx^{jk}) = -wdf \wedge dx^{jk} = -wf, \quad w = f; i = dx^i \cdot df. \] (46)
From the last three equations, we get
\[ w\delta(dx^{jk}f) = -dx^i df + dx^i \cdot df = -dx^i \wedge df = d(dx^i f). \] (47)

In order to complete the computation, we have to show that \( d(w\beta) \wedge dx^i \) can be written as \( \delta\beta \wedge dx^i \). This can be shown easily by direct calculation. Let \( \alpha \) be given as \( a_idx^i \). Then \( d(w\beta) \wedge dx^1 = d\alpha \wedge dx^1 = (a_{3,2} - a_{2,3})w \). On the other hand, \( \beta = -a_idx^i \) and
\[ \delta\beta = (a_{3,2} - a_{2,3})dx^1 + \text{cyclic permutations}. \] (48)
Hence \( \delta\beta \wedge dx^{23} = (a_{3,2} - a_{2,3})w \) and, therefore,
\[ d(w\beta) \wedge dx^1 = d\alpha \wedge dx^1 = \delta\beta \wedge dx^{23}, \] (49)
and similarly for the cyclic permutations of the indices.
7 Anticipation of Helmholtz theorem for differential $r$–forms in $E_n$

The discussion of section 4 together with the development of the proofs of sections 5 and 6 makes it obvious what Helmholtz theorem for differential $r$–forms will look like, namely

$$\alpha_r(x) = \mu d\left[ dx^{i_1 \ldots i_{r-1}} I^{\delta} \right] + \mu \delta \left[ dx^{k_1 \ldots k_{r+1}} I^{d} \right],$$

(50)

$$I^{\delta} \equiv \int_{E_n} \frac{1}{r!_{12}} (\delta' \alpha') \wedge dx^{j_1 \ldots j_{n-r+1}},$$

(51)

$$I^{d} \equiv \int_{E_n} \frac{1}{r!_{12}} (d' \alpha') \wedge dx^{l_1 \ldots l_{n-r+1}},$$

(52)

$n$ being the dimension of the Euclidean space where $\alpha_r(x)$ would have been defined, or the dimension of a still larger Euclidean space. The integrations are performed over the chosen $E_n$ space. Summation over the basis made by $n!/(r-1)!(n-r+1)!$ independent basis elements $dx^{i_1 \ldots i_{r-1}}$, and the basis made by $n!/(r+1)!(n+r-1)!$ independent basis elements $dx^{k_1 \ldots k_{r+1}}$ is understood. Both $(i_1, \ldots, i_{r-1}, j_1, \ldots, j_{n-r+1})$ and $(k_1, \ldots, k_{r+1}, l_1, \ldots, l_{n-r-1})$ constitute even permutations of $(1, \ldots, n)$. In paper II, the constants $\mu_1$ and $\mu_2$ will be determined, $r!_{12}$ will be specified, and proof of this theorem will be provided.

By the linear nature of the theorem, it is clear that the theorem extends to differential forms of inhomogeneous grade. The summations would then apply, in addition, to all possible values of $r$. The preceding equations would then take the still simpler form

$$\alpha(x) = \mu d\left[ dx^A I^{\delta(A)} \right] + \mu \delta \left[ dx^A I^{d(A)} \right],$$

(53)

$$I^{\delta(A)} \equiv \int_{E'_n} \frac{1}{r!_{12}} (\delta' \alpha') \wedge dx^{tA},$$

(54)

$$I^{d(A)} \equiv \int_{E'_n} \frac{1}{r!_{12}} (d' \alpha') \wedge dx^{tA},$$

(55)

where $A$ labels the basis in the Kähler algebra, and where $dx^{tA}$ is meant to be the element of that basis such that $dx^{tA} \wedge dx^{tA}$ is the basis unit element of grade $n$. Of course, these equations reduce to the previous ones if $\alpha$ is of homogeneous grade.
One can then use that theorem to produce the announced major improvement on Hodge’s decomposition theorem with the help of embedding and through the use of the KC. Retrospectively, the result obtained amounts to the integration of the system constituted by the specification of the exterior differential and co-differential of an arbitrary differential form on Riemannian manifolds to which Stokes theorem can be applied (of course, one would have to actually perform the integrations in each case, as is the case also with Helmholtz theorem). A preprint “Helmholtz Theorem for Differential Forms in Arbitrary Euclidean Space, and a Hodge Theorem that Explicitly Exhibits the Decomposition Terms” will soon be ready for posting in arXiv math-analysis if a reader qualified in that area volunteers to sponsor it.

What I have just stated may perhaps seem a little bit unbelievable. It should not be so. Unbelievable is that KC has so long being overlooked. It is a superb calculus. Using it, I have published several applications in different fields of mathematics and physics. The last one was a paper in high energy physics [15] which would be followed by posting of “U(1) × SU(2) × SU(3) from the tangent bundle” if I found a sponsor for the HEP-theory section, sponsorship which I hereby request to those qualified and who may have consulted that reference.

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