On \( p \)-nilpotency of finite group with normally embedded maximal subgroups of some Sylow subgroups

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Abstract. Let \( G \) be a finite group and \( P \) be a \( p \)-subgroup of \( G \). If \( P \) is a Sylow subgroup of some normal subgroup of \( G \), then we say that \( P \) is normally embedded in \( G \). Groups with normally embedded maximal subgroups of Sylow \( p \)-subgroup, where \( (|G|, p - 1) = 1 \), are studied. In particular, the \( p \)-nilpotency of such groups is proved.

Introduction

All groups considered in this paper will be finite. Our notation is standard and taken mainly from [1], [2].

Let \( \mathcal{M}(G) \) be the set of all maximal subgroups of Sylow subgroups of a group \( G \). One of the first results related to the study of the structure of a group with given restrictions on \( \mathcal{M}(G) \) belongs to Srinivasan, see [3]. In particular, in [3] proved that a group \( G \) is supersolvable, if every subgroup of \( \mathcal{M}(G) \) is normal in \( G \). Subsequently, groups with restrictions on subgroups of \( \mathcal{M}(G) \) have been studied in the works of many authors, see the literature in [4].

A subgroup \( H \) of \( G \) is said to be \( S \)-embedded in \( G \), see [5], if \( G \) has a normal subgroup \( N \) such that \( HN \) is \( S \)-permutable in \( G \) and \( H \cap N \leq H_{sG} \), where \( H_{sG} \) is the largest \( S \)-permutable subgroup of \( G \) contained

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On $p$-nilpotency of finite group

in $H$. In the paper [5] the structure of the groups depending on $S$-embedded subgroups is studied. In particular, by Theorem 2.3 [5], follows the $p$-nilpotency of a group $G$ for which every subgroup of $M(P)$ is $S$-embedded in $G$, where $P$ is a Sylow $p$-subgroup of $G$ and $p \in \pi(G)$ such that $(|G|, p - 1) = 1$.

In the present paper, we study another generalization of normality.

**Definition.** A subgroup $H$ of a group $G$ is said normally embedded in $G$, if for every Sylow subgroup $P$ of $H$, there is a normal subgroup $K$ of $G$ such that $P$ is Sylow subgroup of $K$, see [6, I.7.1].

A series of results related to the structure of a group with normally embedded subgroups is presented in [6].

The following examples show that $S$-embedded and normally embedded are different concepts.

In the symmetric group $S_5$ of degree 5 some maximal subgroup $H$ of a Sylow 2-subgroup is a Sylow 2-subgroup in the normal alternating subgroup $A_5$ of degree 5, i.e. $H$ is normally embedded in $S_5$. But, $H$ is not $S$-embedded. In the alternating group $A_4$ of degree 4 some maximal subgroup $M$ of a Sylow 2-subgroup is not normally embedded in $A_4$. But, $M$ is $S$-embedded.

In this paper, the structure of a group $G$ under the condition that every subgroup of $M(P)$ is normally embedded in $G$ is studied, where $P$ is a Sylow $p$-subgroup of $G$ and $p \in \pi(G)$ such that $(|G|, p - 1) = 1$.

The following theorem is proved.

**Theorem.** Let $G$ be a group, $H$ be a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent and $P$ be a Sylow $p$-subgroup of $H$, where $p \in \pi(G)$ with $(|G|, p - 1) = 1$. If every subgroup of $M(P)$ is normally embedded in $G$, then $G$ is $p$-nilpotent.

1. **Preliminaries**

In this section we collect lemmas used in the proof of the main theorem presented in Section 2.

The Fitting subgroup and the Frattini subgroup of $G$ are denoted by $F(G)$ and $\Phi(G)$, respectively; we write $\mathbb{Z}_m$ for a cyclic groups of orders $m$; $O_p(G)$ and $O'_p(G)$ denote the greatest normal $p$-subgroup of $G$ and the greatest normal $p'$-subgroup of $G$, respectively. By $\pi(G)$ denote the set of all prime divisors of the order of $G$; by $H^G$ denote the normal closure of a subgroup $H$ in a group $G$, i.e. the smallest normal subgroup of $G$ containing $H$. We write $H \unlhd G$ for normally embedded subgroup $H$ of $G$ and $G = [A]B$ for the semidirect product of some subgroups $A$ and $B$ with the normal subgroup $A$. 
If the orders of chief factors of $G$ are either equal to $p$ or not divisible on $p$ then $G$ is called $p$-supersolvable. We denote by $pA$ the class of all $p$-supersolvable groups. A group that has a normal Sylow $p$-subgroup is called $p$-closed and a group that has a normal $p'$-Hall subgroup is called $p$-nilpotent.

Let $G$ be a group of order $p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 > p_2 > \ldots > p_k$. We say that $G$ has an ordered Sylow tower of supersolvable type if there exists a series

$$1 = G_0 < G_1 < G_2 < \ldots < G_{k-1} < G_k = G$$

of normal subgroups of $G$ such that $G_i/G_{i-1}$ is isomorphic to a Sylow $p_i$-subgroup of $G$ for each $i = 1, 2, \ldots, k$.

**Lemma 1.** ([6, I.7.3]) Let $U$ be a normally embedded $p$-subgroup of a group $G$, $K$ a normal subgroup of $G$. Then:

1. if $U \leq H \leq G$, then $U \neq H$;
2. $UK/K \neq G/K$;
3. $U \cap K \neq G$;
4. if $K$ is a $p$-group, then $UK \neq G$ and $U \cap K$ is normal in $G$;
5. $U^g \neq H$ for all $g \in G$.

**Lemma 2.** Let $H$ be a normal subgroup of $G$ and every maximal subgroup of Sylow $p$-subgroup of $H$ is normally embedded in $G$. If $N$ is normal in $G$, then every maximal subgroup of every Sylow $p$-subgroup of $HN/N$ is normally embedded in $G/N$. In particular, if $N$ is normal in $G$ and every maximal subgroup of Sylow $p$-subgroup of $G$ is normally embedded in $G$, then every maximal subgroup of every Sylow $p$-subgroup of $G/N$ is normally embedded in $G/N$.

**Proof.** By Lemma 1 (5), follows that $X_1$ is normally embedded in $G$ for any Sylow $p$-subgroup $X$ of $H$ and any maximal subgroup $X_1$ of $X$. Let $T_1 = X/N$ is a maximal subgroup of Sylow $p$-subgroup $T$ of $HN/N$. Then $N \leq X \leq HN$ and there exists a Sylow $p$-subgroup $P$ in $HN$ such that $T = P/N$. By [1, VI.4.6], there exist the Sylow $p$-subgroups $H_p$ in $H$ and $N_p$ in $N$ such that $P = H_pN_p$. Hence $T = H_pN/N$. Further, $N \leq X < PN \leq H_pN$ and $X = (X \cap H_p)N$ by Dedekind’s identity. Since $H_p \cap N = X \cap H_p \cap N$, we have

$$p = |P : T_1| = |H_pN/N : X/N| = |H_pN : X| =$$

$$= |H_pN : (X \cap H_p)N| = \frac{|H_p||N||X \cap H_p \cap N|}{|H_p \cap N||X \cap H_p||N|} = |H_p : X \cap H_p|.$$
So, \( X \cap H_p \) is a maximal subgroup in \( H_p \). By hypothesis, \( X \cap H_p \) is normally embedded in \( G \). By Lemma 1(2), \( (X \cap H_p)N/N = X/N \) is normally embedded in \( G/N \).

For \( H = G \) we obtain the second part of the lemma.

**Lemma 3.** ([7, Lemma 5]) Let \( G \) be a \( p \)-solvable group. Assume that \( G \) does not belong to \( p\mathfrak{A} \), but \( G/K \in p\mathfrak{A} \) for all non-trivial normal subgroups \( K \) of \( G \). Then:

1. \( Z(G) = O_{p'}(G) = \Phi(G) = 1 \);
2. \( G \) contains a unique minimal normal subgroup \( N, N = F(G) = O_p(G) = C_G(N) \);
3. \( G \) is primitive; \( G = [N]M, \) where \( M \) is maximal in \( G \) with trivial core;
4. \( N \) is an elementary Abelian subgroup of order \( p^n, n > 1 \);
5. if \( M \) is Abelian, then \( M \) is cyclic of order dividing \( p^n - 1 \), and \( n \) is the smallest natural number such that \( p^n \equiv 1 \pmod{|M|} \).

A non-nilpotent group whose proper subgroups are all nilpotent is called a Schmidt group.

**Lemma 4.** ([8] Let \( S \) be a Schmidt group. Then:

1. \( S = [P]Q \), where \( P \) is a normal Sylow \( p \)-subgroup, \( Q \) is a non-normal Sylow \( q \)-subgroup, \( p \) and \( q \) are distinct primes;
2. \( Q = < y > \) is cyclic and \( y^q \in Z(S) \);
3. \( |P/P'| = p^m \), where \( m \) is the order of \( p \) modulo \( q \);
4. the chief series of \( S \) has the following system of indexes: \( p, p, ..., p, p^m, q, ..., q \); number of indexes equal to \( p \) coincides with \( n \), where \( p^n = |P'| \); number of indexes equal to \( q \) coincides with \( b \), where \( q^b = |Q| \).

**Lemma 5.** Let \( p \in \pi(G) \) and \( (|G|, p - 1) = 1 \). Then \( G \) is \( p \)-supersolvable if and only if \( G \) is \( p \)-nilpotent. In particular, if a Sylow \( p \)-subgroup is cyclic, then \( G \) is \( p \)-nilpotent.

**Proof.** It is clear that every \( p \)-nilpotent group is \( p \)-supersolvable. Conversely. Let \( G \) be a group of the smallest order such that \( G \) is \( p \)-supersolvable, but is not \( p \)-nilpotent. Let \( H \) be an arbitrary proper subgroup of \( G \). Then \( H \) is \( p \)-supersolvable and \( (|H|, p - 1) = 1 \). Therefore in view of the choice \( G \), the subgroup \( H \) is \( p \)-nilpotent and \( G \) is a minimal non-\( p \)-nilpotent group. By [9, Theorem 10.3.3], \( G \) is a Schmidt group. By Lemma 4(1), \( G = [P]Q \), where \( P \) is a Sylow \( p \)-subgroup and \( Q \) is a cyclic Sylow \( q \)-subgroup. Since \( G \) is \( p \)-supersolvable, then by Lemma 4(4), the order of \( p \) modulo \( q \) is equal 1, i.e. \( m = 1 \). Hence \( q \) divides \( p - 1 \). This is a contradiction.
In particular, if a Sylow \( p \)-subgroup is cyclic, then \( G \) is \( p \)-supersolvable. Then \( G \) is \( p \)-nilpotent by what has been proved above. The lemma is proved.

**Corollary 1.** Let \( p \) be the smallest prime of \( \pi(G) \). Then \( G \) is \( p \)-supersolvable if and only if \( G \) is \( p \)-nilpotent.

**Example 1.** The symmetric group \( G = S_3 \) of degree 3 is 3-supersolvable, but is not 3-nilpotent. Hence, the condition \( (|G|, p - 1) = 1 \) in Lemma 5 can not be removed.

**Example 2.** A group \( G = \mathbb{Z}_5 \times ([\mathbb{Z}_7] \mathbb{Z}_3) \) is 5-supersolvable and is 5-nilpotent. In addition, \( (|G|, 5 - 1) = 1 \), and the prime divisor 5 of \( |G| \) is not the smallest.

Evidently, if a \( p \)-subgroup \( P \) of \( G \) is normally embedded in \( G \), then \( P \) is a Sylow subgroup of \( P^G \).

**Lemma 6.** Let \( G \) be a group, \( \Phi(G) = 1 \), \( P \) be a Sylow subgroup of \( G \) with unprimary order and \( N \) be a unique minimal normal subgroup of \( G \). If every subgroup of \( M(P) \) is normally embedded in \( G \) and \( N \) is Abelian, then \( N \) is not contained in \( P \).

**Proof.** Suppose that \( N \leq P \). If \( N = P \), then by hypothesis, every maximal subgroup \( S \) of \( P \) is normally embedded in \( G \). Then by Lemma 1 (4), \( S \) is normal in \( G \). Since the order of \( P \) is not equal to a prime, we have a contradiction with the fact that \( N \) is a minimal normal subgroup in \( G \).

In the following we assume that \( N < P \). Since \( \Phi(G) = 1 \), it follows that there exists a maximal subgroup \( M \) of \( G \) such that \( N \) is not contained in \( M \). Hence \( G = NM \). By [2, Lemma 2.36], \( N \cap M = 1 \) and \( G = [N]M \). Then by Dedekind’s identity, \( P = P \cap [N]M = [N](P \cap M) \), where \( P \cap M \neq 1 \). Let \( T \) be a maximal subgroup of \( P \) such that \( P \cap M \leq T \). Since \( N \) is a unique minimal normal subgroup of \( G \), it follows that \( N \leq T^G \). Now, \( P = NT \leq T^G \), but by hypothesis, \( T \) is a Sylow subgroup of \( T^G \), a contradiction.

**Lemma 7.** Let \( P \) be a Sylow \( p \)-subgroup of \( G \). If every subgroup of \( M(P) \) is normally embedded in \( G \) and \( (|G|, p - 1) = 1 \), then \( G \) is \( p \)-nilpotent.

**Proof.** We use induction on the order of \( G \). Since \( (|G/N|, p - 1) = 1 \) and by Lemma 2, every maximal subgroup of every Sylow \( p \)-subgroup of \( G/N \) is normally embedded in \( G/N \) for any normal subgroup \( N \neq 1 \) of \( G \), then all quotients of \( G \) satisfy the hypotheses of the lemma.
By the inductive hypothesis, $O_{p'}(G) = 1$. Since the class of all $p$-nilpotent groups is a saturated formation, then $\Phi(G) = 1$ and $N = F(G) = O_p(G)$ is a unique minimal normal subgroup $G$. Hence there is a Sylow $p$-subgroup $R$ of $G$ such that $N \subseteq R$. Since $R$ and $P$ are conjugate in $G$, then by Lemma 1 (5), follows that every maximal subgroup of $R$ is normally embedded in $G$. If $|R| = p$, then $G$ is $p$-nilpotent by Lemma 5. Therefore, we further assume that $|R| > p$. By Lemma 6, $N$ is not contained in $R$. This is a contradiction. The lemma is proved.

2. Proof of Theorem

**Theorem.** Let $G$ be a group, $H$ be a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent and $P$ be a Sylow $p$-subgroup of $H$, where $p \in \pi(G)$ with $|G|, p - 1) = 1$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in $G$, then $G$ is $p$-nilpotent.

**Proof.** In view of Lemma 5, we prove that $G$ is $p$-supersolvable.

By Lemma 1 (1), every maximal subgroup of Sylow $p$-subgroup $P$ of $H$ is normally embedded in $H$ and $(|H|, p - 1) = 1$. By Lemma 7, $H$ is $p$-nilpotent. Since by hypothesis, $G/H$ is $p$-nilpotent, then $G$ is $p$-solvable.

We use induction on the order of $G$. Let $N$ be an arbitrary non-trivial normal subgroup of $G$. Clearly, $HN/N$ is normal in $G/N$ and

$$(G/N)/(HN/N) \cong G/(HN) \cong (G/H)/(HN/H)$$

is $p$-nilpotent. Besides, by Lemma 2, every maximal subgroup of every Sylow $p$-subgroup of $HN/N$ is normally embedded in $G/N$ and $(|G/N|, p - 1) = 1$. Hence the quotients $G/N$ satisfy the hypotheses of the theorem.

By the inductive hypothesis, $G/N$ is $p$-supersolvable. By Lemma 3, $Z(G) = O_{p'}(G) = \Phi(G) = 1$, $G$ contains a unique minimal normal subgroup

$$N = F(G) = O_p(G) = C_G(N), \ G = [N]M,$$

$N$ is an elementary Abelian subgroup of order $p^n$, $n > 1$, $M$ is a maximal subgroup of $G$.

Since $N \leq H$, then $N$ is contained in every Sylow $p$-subgroup $P$ of $H$. By Lemma 6, we have a contradiction. The theorem is proved.

**Corollary 2.** Let $G$ be a group, $H$ be a normal subgroup of group $G$ such that $G/H$ is $p$-nilpotent and $P$ be a Sylow $p$-subgroup of $H$, where $p$ is
the smallest in $\pi(G)$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in $G$, then $G$ is $p$-nilpotent.

**Corollary 3.** Let $G$ be a group and $P$ be a Sylow $p$-subgroup of $G$, where $p \in \pi(G)$ with $|G|, p - 1 = 1$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in $G$, then $G$ is $p$-nilpotent.

**Corollary 4.** Let $G$ be a group and $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest in $\pi(G)$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in $G$, then $G$ is $p$-nilpotent.

**Corollary 5.** Let $G$ be a group. If every subgroup of $\mathcal{M}(G)$ is normally embedded in $G$, then $G$ possesses an ordered Sylow tower of supersolvable type.

**Proof.** Let $p$ be the smallest prime of $\pi(G)$ and $P$ be a Sylow $p$-subgroup of $G$. Then by hypothesis, every subgroup of $\mathcal{M}(P)$ is normally embedded in $G$. By Corollary 4, $G$ is $p$-nilpotent. By Lemma 1 (1) and by the inductive hypothesis, a Hall $p'$-subgroup of $G$ has an ordered Sylow tower of supersolvable type. Consequently, $G$ has an ordered Sylow tower of supersolvable type. □

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