Projective modules for the symmetric group and Young’s seminormal form.

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We study the representation theory of the symmetric group $S_n$ in positive characteristic $p$. Using features of the LLT-algorithm we give a conjectural description of the projective cover $P(\lambda)$ of the simple module $D(\lambda)$ where $\lambda$ is a $p$-restricted partition such that all ladders of the corresponding ladder partition are of order less than $p$. Inspired by the recent theory of Khovanov-Lauda-Rouquier algebras we explain an algorithm that allows us to verify this conjectural description for $n \leq 15$, at least.

1. Introduction

In this paper we continue the investigation from [RH1-3] that seeks to demonstrate the relevance of Young’s seminormal form for the representation theory of the symmetric group $S_n$ in characteristic $p > 0$. Our particular interest is here James’ Conjecture for the decomposition numbers for $S_n$.

Let $p$ be a prime and let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ be the finite field of $p$ elements. Let $\text{Par}_{\text{res},n}$ denote the set of $p$-restricted partitions of $n$. In [RH3] we proved that the simple $\mathbb{F}_p S_n$-module $D(\lambda)$ given by $\lambda \in \text{Par}_{\text{res},n}$, is generated by $a_\lambda E_\lambda$ where $E_\lambda \in \mathbb{Q} S_n$ is the Jucys-Murphy idempotent associated with $\lambda$ and $a_\lambda \in R$ is the least common multiple of the denominators of the coefficients of $E_\lambda$ when it is expanded in the natural basis of $\mathbb{Q} S_n$. We here remark that although $a_\lambda E_\lambda$ is a preidempotent, its reduction modulo $p$, that is $\bar{a}_\lambda \bar{E}_\lambda$, will in general have zero square and will therefore not be a preidempotent. This corresponds to the fact that $D(\lambda)$ is not a projective module for $\mathbb{F}_p S_n$ in general.

This remark leads us to ask if it might be possible to describe the projective cover $P(\lambda)$ of $D(\lambda)$ within the theory of Young’s seminormal form, and indeed the present paper is dedicated to this question.

Let us now briefly explain our results. For all $\lambda \in \text{Par}_{\text{res},n}$, such that all ladders of the corresponding ladder tableau are of length less than $p$, we first construct an idempotent $\bar{e}_\lambda \in \mathbb{F}_p S_n$ that plays an important role throughout the paper. The main ingredients for its construction are Murphy’s tableau class idempotents for the ladder class of $\lambda$, and a symmetrization procedure over the ladder group. We observe that under the condition $n < p^2$ of James’ Conjecture for the adjustment matrix of the decomposition matrix, all ladder lengths as above are automatically less than $p$.

Defining $\widehat{A}(\lambda) := \mathbb{F}_p S_n \bar{e}_\lambda$ we certainly obtain a projective $\mathbb{F}_p S_n$-module, but it is decomposable in general, that is $A(\lambda) \neq P(\lambda)$. On the other hand, we are able

*Supported in part by FONDECYT grant 1051024, by Programa Redes y Simetría and by the MathAmSud project OPECSHA 01-math-10
to show in our Theorem 2 below that there is triangular expansion of the form
\[
\widetilde{A}(\lambda) = P(\lambda) \oplus \bigoplus_{\mu, \mu \triangleright \lambda} P(\mu)^{\oplus \lambda_{\mu}}
\]
for certain nonnegative integers \(m_{\lambda\mu}\) where \(\triangleright\) is the usual dominance order on partitions.

Recall now the Lascoux-Leclerc-Thibon (LLT) algorithm that gives a way of calculating the global crystal basis \(\{G(\lambda) \mid \lambda \in \text{Par}_{\text{res}, n}\}\), for the basic submodule \(\mathcal{M}_q\) of the \(q\)-Fock space \(\mathcal{F}_q\). An important tool for this algorithm is given by certain combinatorially defined elements \(A(\lambda) \in \mathcal{M}_q\) called “the first approximation of the global basis” in [LLT]. Indeed, the LLT-algorithm is a ‘triangular recursion’ based on these elements. Using this we observe that they satisfy the following triangular expansion property with respect to the global crystal basis
\[
A(\lambda) = G(\lambda) + \sum_{\mu, \mu \triangleright \lambda} n_{\lambda\mu}(q) G(\mu)
\]
where \(n_{\lambda\mu}(q) \in \mathbb{Z}[q, q^{-1}]\) satisfy \(\overline{n_{\lambda\mu}(q)} = n_{\lambda\mu}(q)\) for \(q \mapsto q^{-1}\).

Our main point is now to consider \(A(\lambda)\) as an object of interest in itself, and not just a tool for calculating \(G(\lambda)\). In this spirit, for \(n < p^2\) we conjecture that \(A(\lambda)\) should be categorified by \(\widetilde{A}(\lambda)\), or to be more precise that
\[
n_{\lambda\mu}(1) = m_{\lambda\mu}.
\]
This conjectural formula is the main theme of our paper. An important motivation for studying it is contained in our Theorem 5 below, showing that if it is true, then James’ Conjecture is also true. Theorem 5 is proved by inverting equation (1).

In section 5 of the paper we describe a method for verifying formula (2) for \(n\) not too big, which we believe is of independent interest. It is based on the isomorphism between \(\mathbb{F}_p S_n\) and \(\mathcal{R}_n\), the cyclotomic Khovanov-Lauda-Rouquier (KLR) algebra of type \(A\), that was proved by Brundan and Kleshchev in [BK]. In this situation, \(\varepsilon_\lambda\) can be seen as a symmetrization of the KLR-idempotents and so, with \(S(\mu)\) denoting the Specht module, we get that \(\text{Hom}_{\mathbb{F}_p S_n}(\widetilde{A}(\lambda), S(\mu)) = \varepsilon_\lambda S(\mu)\) identifies with a symmetrized generalized eigenspace for the action of the Jucys-Murphy elements in \(S(\mu)\). Let \(\langle \cdot, \cdot \rangle_\mu\) be the symmetric, \(S_n\)-invariant, bilinear form on \(S(\mu)\), given by the cellular algebra structure on \(\mathbb{F}_p S_n\). Then for \(\mu \in \text{Par}_{\text{res}, n}\) we have \(D(\mu) = S(\mu)/\text{rad}(\cdot, \cdot)_\mu\), and so \(\text{dim } D(\mu) = \text{rank}(\langle \cdot, \cdot \rangle_\mu)\), where \(\text{rank}(\langle \cdot, \cdot \rangle_\mu)\) is the \(p\)-rank of the matrix associated with the form. Unfortunately, \(\text{dim } S(\mu)\) grows fast with respect to \(n\). Already for \(n \approx 20\) we have \(\text{dim } S(\mu)\) at magnitudes of several millions, and even fast computers will in general not be able to calculate the rank. On the other hand, the eigenspaces \(\varepsilon_\lambda S(\mu)\) have much smaller dimensions, for example less than 10 for \(n \approx 20\) and \(p = 5\), and are orthogonal with respect to \(\langle \cdot, \cdot \rangle_\mu\).

Still, this is not quite enough to perform calculations. As a matter of fact, in order to do them we need to rely on our recent results from [RH3] on the compatibility of the “intertwining elements” from Brundan and Kleshchev’s work with Young’s seminormal form. They allow us to describe the action of the KLR-generators \(\psi_i\) completely in terms of Young’s seminormal form, and thus to calculate the rank of \(\langle \cdot, \cdot \rangle_\mu\) on the restriction to \(\varepsilon_\lambda S(\mu)\). For \(n \approx 20\) and \(p = 5\), our GAP-implementation
calculates the individual ranks in less than one second. Our partial verification of
the conjectural formula (2) follows from these calculations.

In the final section of the paper, we take the relationship with the KLR-algebra
one step further. Indeed, one of the important aspects of the KLR-algebra is the
fact that it is a Z-graded algebra in a nontrivial way and hence it is possible to
speak of graded modules over it. By comparison with certain idempotents that
occur naturally in the nilHecke algebra, we then show in Theorem 9 that the
idempotent $\tilde{e}_\lambda$ is a homogeneous idempotent of $R_n$, thus making $A(\lambda)$ a graded
module for $R_n$. Summing up, the situation becomes in this way reminiscent of
Soergel’s theory of bimodules over the coinvariant algebra, with $A(\lambda)$ playing the
role of the Bott-Samelson bimodule, see [So], [EW].

2. Basic Notation and a couple of Lemmas

Let $p > 2$ be a prime and let $R$ be the localization of $\mathbb{Z}$ at $p$. Let $S_n$ be the
symmetric group on $n$ letters and write $\sigma_i := (i - 1, i)$. We are interested in the
representation theory of $S_n$ over the finite field $\mathbb{F}_p = R/pR$.

Over $\mathbb{Q}$, the irreducible representations of $S_n$ are parametrized by the set $\text{Par}_n$
of partitions of $n$, that is the set of nonincreasing sequences of positive integers
$\lambda = (\lambda_1, \ldots, \lambda_k)$ with sum $n$. Over $\mathbb{F}_p$, they are parametrized by the set of $p$-
restricted partitions $\text{Par}_{res,n}$, consisting of those $\lambda \in \text{Par}_n$ that satisfy $\lambda_i - \lambda_{i+1} < p$
for all $i$ where by convention $\lambda_1 = 0$ for $i \geq k + 1$. For $\lambda \in \text{Par}_n$ we denote by
$S(\lambda)$ the Specht module for $R_{S_n}$, see below for the precise definition. In general,
for an $R_{S_n}$-module $M$ we denote by $\overline{M} := M \otimes_R \mathbb{F}_p$, the $\mathbb{F}_pS_n$-module obtained
by reduction modulo $p$, but sometimes, when there is no risk of confusion, we also
refer to it simply as $M$. There is a bilinear, symmetric $S_n$-invariant form $\langle \cdot, \cdot \rangle_\lambda$ on
$S(\lambda)$ which is nonzero iff $\lambda \in \text{Par}_{res,n}$ and we obtain the parametrization of the
simple modules for $\mathbb{F}_pS_n$ via $\lambda \in \text{Par}_{res,n} \mapsto D(\lambda) := \overline{S(\lambda)}/\overline{\text{rad}(\lambda)}$.

In the paper we shall be specially interested in the projective covers of the
simple modules. For $\lambda \in \text{Par}_{res,n}$ we denote by $P(\lambda)$ the projective cover of $D(\lambda)$.
By definition, $P(\lambda)$ is the unique indecomposable projective $\mathbb{F}_pS_n$-module such
that $D(\lambda)$ is a homomorphic image of $P(\lambda)$. By general theory, $P(\lambda)$ is of the form
$P(\lambda) = \mathbb{F}_pS_n e_\lambda$ for some idempotent $e_\lambda \in \mathbb{F}_pS_n$. Unfortunately, there is in general
no concrete description of $e_\lambda$.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k) \in \text{Par}_n$ is represented graphically via its “Young
diagram”. It consists of $k$, left aligned, files of boxes, called nodes, in the plane,
with the first file containing $\lambda_1$ nodes, the second file containing $\lambda_2$ nodes and so on.
The nodes are indexed using matrix convention, with the $[i, j]$th node situated in the
de file of the $i$th file. For $\lambda \in \text{Par}_n$, a $\lambda$-tableau $t$ is a filling of the nodes of
$\lambda$ with the numbers $\{1, 2, \ldots, n\}$. We write $t[i, j] = k$ if the $[i, j]$th node of $t$ is filled
with $k$ and $c_t(k) = j - i$ if $t[i, j] = k$. Then $c_t(k)$ is the content of $t$ at $k$, whereas its
image in $\mathbb{F}_p$, denoted $r_t(k)$, is the $p$-residue of $t$ at $k$. For $k \in \{1, 2, \ldots, n\}$ we define
$t(k) := [i, j]$ where $t[i, j] = k$. A tableau $t$ is called standard if $t[i, j] \leq t[i, j + 1]$ and
$t[i, j] \leq t[i + 1, j]$ for all relevant $i, j$. The set of standard tableaux of partitions
of $n$ is denoted $\text{Std}(n)$ and the set of standard tableaux with underlying partition
$\lambda$ is denoted $\text{Std}(\lambda)$. For $\lambda \in \text{Par}_n$ and $t \in \text{Std}(\lambda)$-tableau we write $\text{Shape}(t) := \lambda$.

Let $t$ be a $\lambda$-tableau with node $[i, j]$. The $[i, j]$-hook consists of the nodes of the
Young diagram of $\lambda$ situated to the right and below the $[i, j]$ node and its cardinality
is called the hook-length $h_{ij}$. The product of all hook-lengths is denoted $h_\lambda$. The
hook-quotient of the tableau $t \in \text{Std}(\lambda)$ at $n$ is the number $\gamma_{tn} = \prod_{i=1}^{b_i}^{h_i}$ where the product is taken over all nodes in the row of $\lambda$ that contains $n$, omitting hooks of length one. For a general $i$, we define $\gamma_i$ similarly, by first deleting from $t$ the nodes containing $i + 1, i + 2, \ldots, n$. Finally we define $\gamma_t = \prod_{i=2}^{n} \gamma_{ti}$.

Let us recall the combinatorial concepts of ladders and ladder tableaux that play an important role for the LLT-algorithm, although we shall use conventions that are dual to the ones of [LLT]. Let $\mu$ be a $p$-restricted partition. The 'ladders' of $\mu$ are the straight 'line segments' through the Young diagram of $\mu$ with 'slope' $1/(p-1)$, that is the subsets of the nodes of $\mu$ of the form $L_b := \{ [i,j] | j = b - (p-1)(i-1) \}$. If $\mu \in \text{Par}_{res,n}$ we have that the ladders are 'unbroken', that is $\pi_1(L_a)$ is of the form $\{q, q+1, a+2, \ldots, r \}$ for some $q < r$ where $\pi_1$ is the first projection. We say that $L_b$ is smaller than $L_{b'}$ if $b < b'$. The ladder tableau $\mu_{lad}$ of $\mu$ is defined as the $\mu$-tableau with the numbers $1, 2, \ldots, n$ filled in one ladder at the time, starting with the smallest ladder and continuing successively upwards, the numbers being filled in from top to bottom in each ladder. Note that the residues are constant on each ladder.

For a partition $\mu$, the $p$-residue diagram $res_\mu$ is obtained by writing the residue $r_i(k)$ in the $[i,j]$'th node of the Young diagram of $\mu$. For example, if $\mu = (6,5,3,1)$ and $p = 3$ then the residue diagram and ladder tableau are as follows

$$res_\mu = \begin{array}{cccc}
0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 & \\
0 & & & \\
\end{array} , \quad \mu_{lad} = \begin{array}{cccc}
1 & 2 & 3 & 5 \\
4 & 6 & 8 & 11 \\
9 & 12 & 14 & \\
15 & & & \\
\end{array}$$

with ladders $L_1 = \{1\}$, $L_2 = \{2\}$, $L_3 = \{3, 4\}$, $L_4 = \{5, 6\}$, $L_5 = \{7, 8, 9\}$, $L_6 = \{10, 11, 12\}$ and $L_7 = \{13, 14, 15\}$. We denote by $i_{lad,\mu}$ the residue sequence given by the ladder tableau for $\mu$. In the above example it is $i_{lad,\mu} = (0, 1, 2, 2, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0)$.

The ladders define a sequence of subpartitions $\mu_{lad,\leq 1}, \ldots, \mu_{lad,\leq m}$ of $\mu$ where $\mu_{lad,\leq k}$ is defined as the union of the ladders $L_1, L_2, \ldots, L_k$.

We define positive integers $n_0, \ldots, n_m$ by $n_0 := 0$ and

$$n_k := |L_1| + |L_2| + \ldots + |L_k|. \quad (3)$$

We may then introduce the ladder group $S_{lad,\mu} \leq S_n$ for $\mu$ as $S_{lad,\mu} := \prod_k S_{L_k}$, where $S_{L_k}$ is the symmetric group on the letters $n_{k-1} + 1, \ldots, n_k$.

The dominance order $\preceq$ on partitions is defined by

$$\lambda \preceq \mu \text{ if } \sum_{i=1}^{m} \lambda_i \leq \sum_{i=1}^{m} \mu_i \text{ for } m = 1, 2, \ldots, \min(k, l)$$

for $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$. When $\lambda$ is used as a subscript where a tableau is expected, it refers to the unique maximal $\lambda$-tableau $t^\lambda$, having the numbers $\{1, \ldots, n\}$ filled in along the rows. The dominance order extends to tableaux by considering them as series of partitions.

Following Murphy in [Mu83], we define an equivalence relation on the set of all standard tableaux via $t \sim_p s$ if $r_i(k) = r_i(s) \mod p$ for all $k$. The classes of $\sim_p$ are called tableaux classes. The tableau class containing $t$ is denoted $[t]$. The
tableaux classes are given by residue sequences, that is elements of \( (F_p)^n \), although a given residue sequence \( i \in (F_p)^n \) may give rise to the empty class. The ladder tableaux are ‘minimal’ in their classes in the sense of the following Lemma. Note that throughout we use the convention that \( S_n \) acts on the left on tableaux by place permutations.

**Lemma 1.** Assume that \( \lambda \) is \( p \)-restricted. Then if \( t \in [\lambda_{\text{lad}}] \) we have that either \( \text{Shape}(t) \triangleright \lambda \) or \( \text{Shape}(t) = \lambda \) and \( t = \sigma \lambda_{\text{lad}} \) for \( \sigma \in S_{\text{lad}, \lambda} \).

**Proof:** Omitted. \( \square \)

Suppose that \( \mu \in \text{Par}_n \). A node of \( \mu \) is called removable if it can be removed from \( \mu \) with the result being the diagram of a partition \( \lambda \). Dually, that node is called an addable node of \( \lambda \). It is called an \( i \)-node if its \( p \)-residue is \( i \).

**Lemma 2.** Assume that \( \mu \) is \( p \)-restricted and that \( T_{\mu \lambda} := \{ t \in [\mu_{\text{lad}}] \mid \text{Shape}(t) = \lambda \} \neq \emptyset \).

Then the ladder group \( S_{\text{lad}, \mu} \) acts faithfully on \( T_{\mu \lambda} \).

**Proof:** In general, we may think of the tableau class \([t]\) in an algorithmic way. Indeed, setting \( t' := i_1 i_2 \ldots i_n \in (F_p)^n \) where \( i_k := r_t(k) \) we obtain the tableaux in \([t]\) by starting with the one-node partition, to which we add in all possible ways an addable \( i_2 \)-node. For each arising partition, we add in all possible ways an addable \( i_3 \)-node and so on. The set of tableaux that arises in this way after \( n \) steps is exactly \([t]\). From this, it is clear that \( S_{\text{lad}, \mu} \) acts faithfully on \( T_{\mu \lambda} \). \( \square \)

For \( k = 1, 2, \ldots, n \) the Jucys-Murphy elements \( L_k \in \mathbb{Z}S_n \) are defined by

\[
L_k := (1, k) + (2, k) + \ldots + (k - 1, k)
\]

with the convention that \( L_1 := 0 \). An important application of the \( L_k \) is the construction of orthogonal idempotents \( E_t \in \mathbb{Q}S_n \), the Jucys-Murphy idempotents, indexed by tableaux \( t \), that can be used to derive Young’s seminormal form. Their construction is as follows

\[
E_t := \prod_{\{c \mid -n < c < n\} \setminus \{i \mid c_t(i) \neq c\}} \frac{L_i - c}{c_t(i) - c}.
\]

For \( t \) standard we have \( E_t \neq 0 \), whereas for \( t \) nonstandard either \( E_t = 0 \), or \( E_t = E_s \) for some standard tableau \( s \) related to \( t \). Running over all standard tableaux, the \( E_t \) form a set of primitive and complete idempotents for \( \mathbb{Q}S_n \), that is their sum is 1. Moreover, they are eigenvectors for the action of the Jucys-Murphy operators in \( \mathbb{Q}S_n \), since

\[
(L_k - c_t(k))E_t = 0 \text{ or equivalently } L_k = \sum_{t \in \text{Std}(n)} c_t(k)E_t. \quad (4)
\]

For \( \lambda \in \text{Par}_n \), we let \( \text{Stab}_\lambda \) denote the row stabilizer of \( t^\lambda \) and define \( x_\lambda \) and \( y_\lambda \) as the following elements of \( RS_n \)

\[
x_\lambda = \sum_{\sigma \in \text{Stab}_\lambda} \sigma \quad \text{and} \quad y_\lambda = \sum_{\sigma \in \text{Stab}_\lambda} (-1)^{|\sigma|} \sigma
\]
where $|\sigma|$ is the sign of $\sigma$. For $t \in \text{Std}(\lambda)$, we define the associated element $d(t) \in S_n$ by
\[ d(t)^t = t. \]
Then for pairs of standard $(s, t)$ of $\lambda$-tableaux, Murphy’s standard basis and dual standard basis, mentioned above, consist of the elements
\[ x_{st} = d(s)x_{\lambda}d(t)^{-1} \quad \text{and} \quad y_{st} = d(s)y_{\lambda}d(t)^{-1}. \]
They are bases for $RS_n$ and also for $\mathbb{F}_p S_n$. Set
\[ (RS_n)^{>\lambda} := \text{span}_R \{ x_{st} | \text{Shape}(s) > \lambda \}. \]
Then $(RS_n)^{>\lambda}$ is an ideal of $RS_n$ and the Specht module $S(\lambda)$, mentioned above, is the span of $\{ x_{\lambda} + (RS_n)^{>\lambda} | s \in \text{Std}(\lambda) \}$. These elements form an $R$-basis for $S(\lambda)$. Define $S(\lambda)_Q := S(\lambda) \otimes \mathbb{Q}$ and let $\xi_{st} := E_s x_{st} E_t$. Then $\{ \xi_{st} | (s, t) \in \text{Std}(\lambda)^2, \lambda \in \text{Par}_n \}$ is the seminormal basis for $\mathbb{Q}S_n$ and, moreover, $\{ \xi_{\lambda} | s \in \text{Std}(\lambda) \}$ is a basis for $S(\lambda)_Q$.

The action of $S_n$ on the standard basis $\{ x_{\lambda} \}$ is given by a recursion using the Garnir relations, whereas the action of $S_n$ on $\{ \xi_{\lambda} \}$ is given by the following formulas, that appear for example in Theorem 6.4 of [Mu93] (in the more general context of Hecke algebras, but note the sign error there: the expression for $h$ should be replaced by $-h$).

**Theorem 1.** Let $h = c_s(i - 1) - c_t(i)$ be the radial distance between the $i - 1$ and $i$-nodes of $s \in \text{Std}(\lambda)$. Let $t := \sigma_i s$ where still $\sigma_i = (i - 1, i)$. Then the action of $\sigma_i$ on $\xi_{x_{\lambda}}$ is given by the formulas
\[ \sigma_i \xi_{x_{\lambda}} := \begin{cases} 
\xi_{x_{\lambda}} & \text{if } h = -1 \ (i - 1 \text{ and } i \text{ are in same row}) \\
-\frac{1}{n} \xi_{x_{\lambda}} + \xi_{x_{\lambda}} & \text{if } h = 1 \ (i - 1 \text{ and } i \text{ are in same column}) \\
-\frac{1}{n} \xi_{x_{\lambda}} + \xi_{x_{\lambda}} & \text{if } h > 1 \ (i - 1 \text{ is above } i) \\
-\frac{1}{n} \xi_{x_{\lambda}} + \frac{1}{n} \xi_{x_{\lambda}} & \text{if } h < -1 \ (i - 1 \text{ is below } i). 
\end{cases} \] (5)

3. **Projective modules**

For the results of this section, we first need a description of Robinson’s $i$-induction functor in terms of the Jucys-Murphy idempotents. There are related descriptions available in the literature, see for example [HuMa3], but our description of the 'divided power' functor seems to be new. It relies on a result from our recent paper [RH3].

Recall that the blocks for $\mathbb{F}S_n$ are given by the Nakayama Conjecture (which is a Theorem). Murphy showed in [Mu83] how to describe the corresponding block idempotents in terms of the Jucys-Murphy idempotents $E_i$. Indeed, let $T = [t]$ be the class of $t \in \text{Std}(n)$ under $\sim_p$ and consider for $\lambda \in \text{Par}_n$ the following tableau set
\[ T_{\lambda} := \{ s | \text{ there is a tableau of shape } \lambda \text{ in } [s] \}. \] (6)

Let $[\lambda]$ be the class of $\lambda$ under the equivalence relation on $\text{Par}_n$ given by $\lambda \sim_p \mu$ if $T_{\lambda} = T_{\mu}$. Then Murphy showed in loc. cit. that $E_T := \sum_{t \in T} E_t$ and $E_{[\lambda]} := \sum_{t \in T_{\lambda}} E_t$ lie in $RS_n$ and that $E_{[\lambda]} \in \mathbb{F}_p S_n$ is the block idempotent for the block given by $[\lambda]$. In particular the $E_{[\lambda]}$ are pairwise orthogonal and central in $\mathbb{F}_p S_n$ with sum $1$. 


Let $\mathbb{F}_pS_n\text{-mod}$ denote the category of finite dimensional $\mathbb{F}_pS_n$-modules and let

$$T_{n-1}^n : \mathbb{F}_pS_{n-1}\text{-mod} \to \mathbb{F}_pS_n\text{-mod}, \; M \mapsto \mathbb{F}_pS_n \otimes_{\mathbb{F}_pS_{n-1}} M$$

be the induction functor from $\mathbb{F}_pS_{n-1}\text{-mod}$ to $\mathbb{F}_pS_n\text{-mod}$.

Assume that $\lambda \in \text{Par}_{n-1}$ is a subpartition of $\mu \in \text{Par}_n$ and that $\mu \setminus \lambda$ consists of one node of residue $i$. Then Robinson’s $i$-induction functor $f_i$ is defined as

$$f_i : \mathbb{F}_pS_{n-1}\text{-mod} \to \mathbb{F}_pS_n\text{-mod}, \; M \mapsto \mathbb{F}_pS_n \otimes_{\mathbb{F}_pS_{n-1}} M.$$

Consider the following set $T_{i,n}$ of tableaux classes of $n$-tableaux

$$T_{i,n} := \{ [t] | s[n] = i \mod p \text{ for some (any) } s \in [t] \}$$

and set $E_{i,n} := \sum_{T \in T_{i,n}} E_T$. Then $T_{i,n}$ is a union of tableaux classes and so $E_{i,n}$ is an idempotent in $\mathbb{F}_pS_n$ and moreover $\sum_i E_{i,n} = 1$. We now have the following Lemma.

**Lemma 3.** Suppose that $M$ lies in the $[\lambda]$-block of $\mathbb{F}_pS_{n-1}$. Then there is an isomorphism of $\mathbb{F}_pS_n$-modules

$$f_iM \cong \mathbb{F}_pS_n \overline{E_{i,n}} \otimes_{\mathbb{F}_pS_{n-1}} M.$$

**Proof:** Since the $E_i$’s sum to 1, we have that $E_{[\mu]}$, viewed as an element of $RS_n$, is the sum of all $E_s$ where $s$ is obtained from a tableau in $T_\lambda$ by adding an addable $i$-node. From this we deduce

$$E_{[\mu]} = E_{[\mu]} E_{[\lambda]} = E_{i,n} E_{[\lambda]}.$$

On the other hand $E_{[\mu]}$ is central in $\mathbb{F}_pS_n$ and so we get

$$f_iM \cong \mathbb{F}_pS_n E_{[\mu]} \otimes_{\mathbb{F}_pS_{n-1}} M \cong \mathbb{F}_pS_n E_{i,n} E_{[\lambda]} \otimes_{\mathbb{F}_pS_{n-1}} M \cong \mathbb{F}_pS_n E_{i,n} \otimes_{\mathbb{F}_pS_{n-1}} M$$

as claimed. \qed

We next introduce the notation that allows us to generalize the Lemma to the ‘divided powers’. Assume therefore that $\mu$ is a $p$-restricted partition of $n$. Assume furthermore that all its ladders $\mathcal{L}_k$ are of length $|\mathcal{L}_k|$ strictly less than $p$. The partition $\mu = (6, 5, 3, 1)$ considered above violates this condition, since for example $|\mathcal{L}_5| = 3$, whereas the partition $\nu = (4, 4, 3, 1)$ meets it. Its $p$-residue diagram is

$$\nu_{\text{res}} = \begin{array}{cccc}
      0 & 1 & 2 & 0 \\
      1 & 2 & 3 & 1 \\
      2 & 3 & 4 & 2 \\
      0 & 1 & 2 & 3 \\
   \end{array}$$

and the ladder lengths are 1, 1, 2, 2, 2, 1, 2, all less than 3.

Let $\mathcal{H}_n(q)$ be the integral Hecke algebra of finite type $A$, that is the $\mathbb{Z}[q, q^{-1}]$-algebra on generators $\{ T_i | i = 1, 2, \ldots, n - 1 \}$ subject to the braid relations and

$$(T_i - q)(T_i + 1) = 0 \text{ for all } i.$$

For $\xi \in \mathbb{C}^\times$, the specialized Hecke algebra is defined as

$$\mathcal{H}_n(\xi) := \mathcal{H}_n(q) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}.$$
where \( \mathbb{C} \) is made into a \( \mathbb{Z}[q, q^{-1}] \)-algebra by sending \( q \) to \( \xi \).

In [Ja], James formulated a conjecture concerning the decomposition numbers for \( q \)-Schur algebras. A special case of his conjecture is the statement that for \( n < p^2 \) and for \( \xi \) a primitive \( p' \)th root of unity, the decomposition numbers for \( \mathcal{H}_n(\xi) \) and for \( \mathbb{F}_pS_n \) should coincide, via a modular reduction procedure sending \( \xi \) to 1. We shall refer to this last statement as James’ Conjecture.

**Lemma 4.** Assume that \( n \) is an integer for which James’ conjecture for \( \mathcal{H}_n(\xi) \) and \( \mathbb{F}_pS_n \) should hold, that is \( n < p^2 \). Then for all \( p \)-restricted partitions \( \mu \) of \( n \), all ladders have lengths strictly less than \( p \).

**Proof:** Suppose that \( \mu \in \text{Par}_{\text{res}}(n) \). Let \( \mathcal{L} = \mathcal{L}_k \) be a ladder for \( \mu \) with top node \((a_1, b_1)\) and bottom node \((a_2, b_2)\) and suppose that it has length \( l \), that is \( l := a_2 - a_1 + 1 \). For contradiction we assume that \( l \geq p \). The subdiagram of \( \mu \) consisting of the nodes \((a, b)\) satisfying \( a_1 \leq a \leq a_2 \) and \( b_2 \leq b \leq b_1 \) is a \( p \)-core partition of shape 
\[
\lambda_{\text{core}} = (1 + (p - 1)(l - 1), \ldots, (p - 1) + 1, 1).
\]
 Its parts form an arithmetic series with constant term 1 and difference \( p - 1 \) and so the cardinality of \( \lambda_{\text{core}} \) is
\[
|\lambda_{\text{core}}| = \frac{l(p - 1)(l - 1) + 2}{2} \quad (7)
\]
which is greater than
\[
p \left( \frac{(p - 1)^2 + 2}{2} \right) = \frac{p(p^2 - 2p + 3)}{2}. \quad (8)
\]
On the other hand, we have that
\[
\frac{p(p^2 - 2p + 3)}{2} \geq p^2. \quad (9)
\]
Indeed, \( p(p^2 - 2p + 3) < p^2 \) is equivalent to \( (p - 1)(p - 3) < 0 \) which is impossible since we assume from the beginning that \( p \neq 2 \). We get the desired contradiction by combining (7), (8) and (9) with \( n < p^2 \). \( \square \)

Motivated by the Lemma, we fix for the rest of this section a partition \( \mu \in \text{Par}_{\text{res}, n} \) whose ladders \( \mathcal{L}_k \), \( k = 1, 2, \ldots, m \) are all of cardinality strictly less than \( p \). We denote the residue of the nodes of \( \mathcal{L}_k \) by \( \iota_k \). With respect to \( \mu \) we define numbers \( n_k \) as in (3) and write \( \mathcal{I}_{n_k-1} \) for the induction functor from the category of finite dimensional \( \mathbb{F}_pS_{n_k-1} \)-modules to the category of finite dimensional \( \mathbb{F}_pS_{n_k} \)-modules, that is 
\[
\mathcal{I}_{n_k-1} : M \mapsto \mathbb{F}_pS_{n_k} \otimes_{\mathbb{F}_pS_{n_k-1}} M.
\]
Generalizing \( \mathcal{T}_{i,n} \), we introduce the following set \( \mathcal{T}_{\mathcal{L}_k} \) of tableaux classes for \( S_{n_k} \)
\[
\mathcal{T}_{\mathcal{L}_k} := \{ [T] \mid t[j] = \iota_k \text{ mod } p \text{ for } t \in [T] \text{ and } j = n_{k-1} + 1, \ldots, n_k \}.
\]
This gives rise to the following idempotents
\[
E_{\mathcal{L}_k} := \sum_{T \in \mathcal{T}_{\mathcal{L}_k}} E_T \in RS_{n_k}, \quad E_{\mathcal{L}_k} \in \mathbb{F}_pS_{n_k}.
\]
Now since we are assuming $|\mathcal{L}_k| < p$, we can define another idempotent

$$e_k := \frac{1}{|\mathcal{L}_k|!} \sum_{\sigma \in \mathcal{S}_{\mathcal{L}_k}} \sigma \in \mathbb{F}_p S_{n_k}.$$ 

We combine it with $E^{(\mathcal{L}_k)}$ to define

$$E(\mathcal{L}_k) := E^{(\mathcal{L}_k)} e_k \in \mathbb{F}_p S_{n_k}, \quad \bar{e}_\mu := \prod_k E(\mathcal{L}_k).$$

Note that it is not obvious from the definition that $\bar{e}_\mu$ is nonzero, although each of its factors is it. But the following Lemma follows easily from [RH3].

**Lemma 5.** $E(\mathcal{L}_k)$ and $\bar{e}_\mu$ are idempotents of $\mathbb{F}_p S_{n_k}$ and $\bar{e}_\mu = E(\mu) \prod_k e_k$.

**Proof:** By Lemma 1 of [RH3] the two factors of $E(\mathcal{L}_k)$ commute and so it is indeed an idempotent. Moreover, we have that $E(\mathcal{L}_k)$ commutes with $\mathbb{F}_p S_{n_k-1}$ and so all factors of $\bar{e}_\mu$ commute and it is also an idempotent. The last claim also follows from this. □

We now get our divided power induction functor as

$$f_i^{(\mathcal{L}_k)} : \mathbb{F}_p S_{n_k-1} \text{-mod} \to \mathbb{F}_p S_{n_k} \text{-mod}$$

$$M \mapsto \mathbb{F}_p S_{n_k} E(\mathcal{L}_k) \otimes_{\mathbb{F}_p S_{n_k-1}} M.$$ (10)

By Lemma 3, we have that if $|\mathcal{L}_k| = 1$ then $f_i^{(\mathcal{L}_k)} = f_i$.

With this at hand, we can now formulate the definition of the $\mathbb{F}_p S_n$-module $\widehat{A}(\mu)$, mentioned in the introduction of the paper. It is defined as

$$\widehat{A}(\mu) := f_{i_1}^{(\mathcal{L}_{m_1})} \ldots f_{i_2}^{(\mathcal{L}_{m_2})} f_{i_1}^{(\mathcal{L}_1)} \mathbb{F}_p.$$ (12)

The following Theorem contains the basic properties of $\bar{e}_\mu$ and $\widehat{A}(\mu)$ that shall be used throughout the paper.

**Theorem 2.** Suppose that $\mu \in \text{Par}_{\text{res}, n}$ is chosen as above, that is all its ladders $\mathcal{L}_k$, $k = 1, 2, \ldots, m$ are of cardinality strictly less than $p$. Then the follow hold:

a) There is an isomorphism of $\mathbb{F}_p S_n$-modules $\widehat{A}(\mu) \cong \mathbb{F}_p S_n \bar{e}_\mu$. In particular, $\widehat{A}(\mu)$ is a projective $\mathbb{F}_p S_n$-module.

b) Define $M := E(\mu) S(\lambda)$ where $\lambda \in \text{Par}_n$ and $\mu \in \text{Par}_{\text{res}, n}$. Then $M$ is a free $\mathbb{F}_p S_{\text{lad}, \mu}$-module. In particular, if $M \neq 0$ then $\bar{e}_\mu S(\lambda) \neq 0$.

c) For $\lambda, \mu \in \text{Par}_{\text{res}, n}$ there are nonnegative integers $m_{\lambda \mu}$ and a triangular expansion of the form

$$\widehat{A}(\mu) = P(\mu) \oplus \bigoplus_{\lambda, \lambda > \mu} P(\lambda)^{m_{\lambda \mu}}.$$ 

In particular $\bar{e}_\mu \neq 0$.

**Proof:** By definition $\widehat{A}(\mu)$ is isomorphic to

$$\mathbb{F}_p S_{n_m} E(\mathcal{L}_m) \otimes_{\mathbb{F}_p S_{n_m-1}} \ldots \otimes_{\mathbb{F}_p S_{n_2}} E(\mathcal{L}_2) \otimes_{\mathbb{F}_p S_{n_2}} \mathbb{F}_p S_{n_1} E(\mathcal{L}_1) \otimes_{\mathbb{F}_p S_{n_1}} \mathbb{F}_p.$$
Note that \( n_1 = 1, \mathbb{F}_p S_1 = \mathbb{F}_p \) and \( \mathbb{E}(\mathbb{L}_i) = 1 \). Since \( \mathbb{E}(\mathbb{L}_k) \) commutes with \( \mathbb{F}_p S_{n_j} \) for all \( j < k \), this simplifies to

\[
\mathbb{F}_p S_n \prod_k \mathbb{E}(\mathbb{L}_k) = \mathbb{F}_p S_n \mathbb{e}_\mu
\]

proving a).

In order to show b), we first note that \( \mathbb{M} \) indeed is an \( \mathbb{F}_p S_{lad, \mu} \)-module, since the elements of \( S_{lad, \mu} \) commute with \( \mathbb{E}[\mu_{lad}] \) by Lemma 1 of [RH3]. Consider now the set of tableaux \( T_{\mu, \lambda} \) as in Lemma 2. Let \( t_1, t_2 \ldots, t_k \in T_{\mu, \lambda} \) be the maximal elements of the \( S_{lad, \mu} \) orbits in \( T_{\mu, \lambda} \). By Lemma 2, the orbits \( S_{lad, \mu} t_i \) are all of cardinality \( |S_{lad, \mu}| \). For each \( i \), we now check that the homomorphism

\[
\varphi_i : \mathbb{F}_p S_{lad, \mu} \to \mathbb{M}, \; \sigma \mapsto \mathbb{E}[\mu_{lad}] \sigma x_{t_1, \lambda}
\]

is injective. First of all, for \( \sigma \in S_{lad, \mu} \) we have that

\[
\mathbb{E}[\mu_{lad}] \sigma x_{t_1, \lambda} = \sigma x_{t_1, \lambda} = x_{\sigma t_1, \lambda}
\]

modulo higher terms, that is modulo an \( \mathbb{F}_p \)-linear combination of terms \( x_{s, \lambda} \) satisfying \( s \triangleright \sigma t_1 \) and terms \( x_{s, \mu} \) satisfying that \( \text{Shape}(s) = \text{Shape}(t) \triangleright \lambda \). Indeed, \( x_{\sigma t_1, \lambda} \) is an element of Murphy’s standard basis and so the claim follows from the fact that the \( L_i \)’s act upper triangularly on the standard basis elements by Murphy’s theory, see for example [Ma]. To show injectivity of \( \varphi_i \), we now suppose that \( \sum_{\sigma \in S_{lad, \mu}} \lambda_\sigma \sigma = 0 \), and choose \( \sigma \) with \( \mu_{lad} \) minimal subject to \( \lambda_\sigma \neq 0 \). By the previous remark we find that the coefficient of \( x_{\sigma t_1, \lambda} \) in \( \varphi_i(\sum_{\sigma \in S_{lad, \mu}} \lambda_\sigma \sigma) \) is nonzero, and so \( \varphi_i \) indeed is injective.

On the other hand, the tableaux \( \sigma t_1 \) that appear in (14), where \( \sigma \in S_{lad, \mu} \) and \( i \in \{1, \ldots, k\} \), are precisely those of \( T_{\mu, \lambda} \), and so the elements of (14) form a basis for \( \mathbb{M} \), see equation (2.4) of [Mu83]. We now conclude that \( \mathbb{M} = \oplus \im \varphi_i \) and so b) is proved.

Assume now that \( P(\lambda) \) is a summand of \( \widehat{A}(\mu) \). Then \( \text{Hom}_{\mathbb{F}_p S_n}(\widehat{A}(\mu), D(\lambda)) \neq 0 \) and hence \( \text{Hom}_{\mathbb{F}_p S_n}(\widehat{A}(\mu), \overline{S(\lambda)}) \neq 0 \) since \( D(\lambda) \) is a quotient of \( \overline{S(\lambda)} \) and \( \widehat{A}(\mu) \) is projective. On the other hand, by the definition of \( \widehat{A}(\mu) \) we have that

\[
\text{Hom}_{\mathbb{F}_p S_n}(\widehat{A}(\mu), \overline{S(\lambda)}) = \overline{\mathbb{e}_\mu S(\lambda)} = \prod_k \mathbb{E}[\mu_{lad}] \overline{S(\lambda)}.
\]

We now show that \( \mathbb{E}[\mu_{lad}] \overline{S(\lambda)} \neq 0 \) implies that \( \lambda \geq \mu \). We view \( \mathbb{E}[\mu_{lad}] \) as an element of \( \mathbb{Q} S_n \) and get via Lemma 11 that in the expansion of it as a sum of \( E_l \), only those \( t \) with \( \text{Shape}(t) \preceq \mu \) can appear. On the other hand, over \( \mathbb{Q} \) the standard basis \( \{ s_{x, \lambda}, s \in \text{Std}(\lambda) \} \) for \( S(\lambda) \) may be replaced by the seminormal basis \( \{ s_{x, \lambda}, s \in \text{Std}(\lambda) \} \), as defined in [Mu92] via \( s_{x, \lambda} = E_l s_{x, \lambda} \), and since \( E_l s_{x, \lambda} \neq 0 \) implies \( \text{Shape}(t) = \lambda \) we get the triangularity property of b).

To show that \( P(\mu) \) occurs with multiplicity one in \( \widehat{A}(\mu) \), we set \( \lambda = \mu \) in (15) and verify that \( \overline{\mathbb{e}_\mu S(\lambda)} \) has dimension one over \( \mathbb{F}_p \). We consider once again \( M := \mathbb{E}[\mu_{lad}] S(\mu) \). It is a free \( \mathbb{R} \)-module being a submodule of \( S(\mu) \). Let us determine its rank by extending scalars from \( \mathbb{R} \) to \( \mathbb{Q} \). By Lemma 11 we get that in the expansion of \( \mathbb{E}[\mu_{lad}] \) as a sum of \( E_l \)’s, the occurring \( t \) with \( \text{Shape}(t) = \mu \) are exactly those of the form \( \sigma \mu_{lad} \) where \( \sigma \in S_{lad, \mu} \) and hence, over \( \mathbb{Q} \), we get a basis for \( M \).
consisting of \{\xi_s\mu\} where \(s = \sigma_{\mu_{ad}}\). In other words, \(M\) has dimension \(|S_{ad,\mu}|\overline{Q}\). Hence \(M\) has rank \(|S_{k,\mu}|\overline{R}\) and so \(\overline{M}\) is a free rank one \(\mathbb{F}_pS_{ad,\mu}\)-module, by b) of the Theorem. Finally, we use that \(E_{[\mu_{ad}]}S(\mu) = E_{[\mu_{ad}]}D(\mu)\), as one gets by combining Lemma 3.35 y 3.37 of [Ma], and c) follows. \(\square\)

4. The conjecture and the LLT-algorithm

Let us recall the Fock space \(F_q\) associated with the representation theory of the Hecke algebra \(H_n(\xi)\) at a \(p\)’th root of unity. As a \(\mathbb{C}(q)\)-vector space, we have

\[
F_q := \bigoplus_{\lambda \in \text{Par}} \mathbb{C}(q)\lambda
\]

where \(\text{Par} := \bigcup_{n=0}^{\infty} \text{Par}_n\) with the convention that \(\text{Par}_0 := \{\emptyset\}\). It is an integrable module for the quantum group \(U_q(\widehat{sl}_p)\), where we use the version of \(U_q(\widehat{sl}_p)\) that appears for example in [LLT]. It is the \(\mathbb{C}(q)\)-algebra on generators \(e_i, f_i, i = 0, 1, \ldots, p - 1\) and \(k_h\) for \(h\) belonging to the Cartan subalgebra \(h\) of the associated Kac-Moody algebra, all subject to certain well known relations that we do not detail here. Let us explain the action of \(U_q(\widehat{sl}_p)\) in \(F_q\). Assume that \(\gamma = \mu \setminus \lambda\) is a removable \(i\)-node of \(\mu\). We then define

\[
N_i^l(\gamma) := |\text{addable } i\text{-nodes to the left of } \gamma| - |\text{removable } i\text{-nodes to the left of } \gamma|,
\]

\[
N_i^r(\gamma) := |\text{addable } i\text{-nodes to the right of } \gamma| - |\text{removable } i\text{-nodes to the right of } \gamma|.
\]

The action of \(e_i, f_i, i = 0, 1, \ldots, p - 1\) on \(F_q\) is now given by the following formulas

\[
f_i\lambda = \sum_{\mu \in \text{Par}_n, \gamma = \mu \setminus \lambda} q^{N_i^l(\gamma)}\mu, \quad e_i\mu = \sum_{\lambda \in \text{Par}_{n-1}, \gamma = \mu \setminus \lambda} q^{-N_i^r(\gamma)}\lambda \quad (16)
\]

where \(\gamma\) runs over addable \(\lambda\)-nodes in the first sum, and over removable \(\mu\)-nodes in the second sum. Note that since [LLT] use the duals of our Specht modules, the formulas for the action on \(F_q\) that appear there are slightly different. There are similar formulas for the action of the other generators, but we leave them out.

For \(k \in \mathbb{Z}\) we let \([k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}\) be the usual Gaussian integer, with the convention \([0]_q = 0\), and define \([k]_q! := [k]_q[k-1]_q \ldots [1]_q\) and the divided powers \(f_i^{(k)} := \frac{1}{[k]_q!} f_i^k\) and \(e_i^{(k)} := \frac{1}{[k]_q!} e_i^k\). We then introduce \(U_\mathbb{Q}\) as the \(\mathbb{Q}[q, q^{-1}]\)-subalgebra of \(U_q(\widehat{sl}_p)\) generated by \(e_i^{(k)}, f_i^{(k)}, k = 1, 2, 3, \ldots\). We define \(M_q := U_q(\widehat{sl}_p)\emptyset\) and \(M_\mathbb{Q} := U_\mathbb{Q}\emptyset\).

\(M_q\) is the basic module for \(U_q(\widehat{sl}_p)\). It is irreducible and therefore provided with a canonical basis/global crystal by Lusztig and Kashiwara’s general theory. To be more precise, let \(u \mapsto \overline{u}\) be the usual bar involution of \(U_{\mathbb{Q}}\), satisfying \(\overline{\overline{u}} = q^{-1}\overline{u}\), \(\overline{f_i} = f_i^{(1)}\) and \(\overline{e_i} = e_i^{(1)}\). It induces an involution \(m \mapsto \overline{m}\) of \(M_q\), satisfying \(\overline{\emptyset} = \emptyset\) and \(\overline{uv} = \overline{v}\overline{u}\) for \(u \in U_{\mathbb{Q}}\) and \(v \in M_q\).

Let \(A := \{f(q)/g(q), f(q), g(q) \in \mathbb{Q}[q], g(q) \neq 0\}\). Then \(A\) is a local subring of \(\mathbb{Q}(q)\) with maximal ideal \(qA\) and we define \(L\) as the \(A\)-sublattice of \(F\) generated by all \(\lambda \in \text{Par}\). The following Theorem follows from Kashiwara and Lusztig’s general theory.
Theorem 3. There is a unique $\mathbb{Q}[q,q^{-1}]$-basis $\{G(\lambda) : \lambda \in \bigcup_n \text{Par}_{res,n}\}$ for $\mathcal{M}_q$, called the lower global crystal basis, satisfying

\[
a) G(\lambda) \equiv \lambda \mod qL, \quad \quad \quad b) \overline{G(\lambda)} = G(\lambda).
\]

Recall now that Lascoux, Leclerc and Thibon introduced in [LLT] for $\mu \in \text{Par}_{res,n}$ an element $A(\mu)$ of $\mathcal{M}_q$ called “the first approximation to $G(\lambda)$”. It is defined as

\[
A(\mu) := f_{t_m}^{(1)}(E_m) \cdots f_{t_2}^{(1)}(E_2) f_{t_1}^{(1)}(E_1) \emptyset
\]

where $E_1, E_2, \ldots, E_m$ still are the ladders for $\mu$ with residues $t_1, \ldots, t_m$. Based on this, they explain a recursive algorithm, the LLT-algorithm, that determines $G(\mu)$ in terms of $A(\lambda)$ where $\lambda \in \text{Par}_{res,n}$ and $\lambda \succeq \mu$. The following is an immediate consequence of that algorithm.

Theorem 4. For $\mu, \lambda \in \text{Par}_{res,n}$ there is an expansion of the form

\[
A(\mu) = G(\mu) + \sum_{\lambda, \lambda \triangleright \mu} n_{\lambda\mu}(q)G(\lambda)
\]

for certain $n_{\lambda\mu}(q) \in \mathbb{Z}[q,q^{-1}]$ satisfying $n_{\lambda\mu}(q) = n_{\lambda\mu}(q)$.

The main purpose of our paper is to study the following conjecture.

Conjecture 1. Suppose that $n < p^2$. Then for $\lambda, \mu \in \text{Par}_{res,n}$ we have that

\[
n_{\lambda\mu}(1) = m_{\lambda\mu}
\]

where $n_{\lambda\mu}(q)$ is as in Theorem 4 and $m_{\lambda\mu}$ as in Theorem 2. In particular $n_{\lambda\mu}(1)$ is a nonnegative integer.

Remark. Our main interest in studying the Conjecture comes from Theorem 5 below, which shows that it implies James’s Conjecture. In particular, our Conjecture cannot hold outside the region of validity of James’ Conjecture. In the next section we give strong experimental evidence in favor of the Conjecture, see Theorem 6 below. Still, this evidence does not reach the limits of the region of validity of James’ Conjecture as stated in [Ja] (that is $n < p^2$) and so we must acknowledge that the proposed region of validity of our Conjecture is only loosely founded.

We note at this point that in the cases that are covered by Theorem 5 below, we always have that $n_{\lambda\mu}(q) = n_{\lambda\mu}(1)$, that is $n_{\lambda\mu}(q)$ is a constant polynomial, and so the condition $n_{\lambda\mu} = n_{\lambda\mu}(1)$ may be necessary for the Conjecture to be valid. On the other hand, experimental evidence beyond the cases that are covered by Theorem 5 suggest that, as predicted by the Conjecture, $n_{\lambda\mu}(1)$ is nonnegative even when $n_{\lambda\mu}$ is nonconstant, for instance for $p = 5$ we have checked that $n_{\lambda\mu}(1) \geq 0$ for all $n < 5^2$. This last piece of experimental evidence is our main motivation for keeping $n < p^2$ as the region of validity of our Conjecture.

Let $G(n)$ be the Grothendieck group of finitely generated $F_pS_n$-modules, and let $K(n)$ be the Grothendieck group of finitely generated projective $F_pS_n$-modules. If $M$ is a (projective) $F_pS_n$-module, we denote by $[M]$ its image in $G(n)$ ($K(n)$). We have that $G(n)$ and $K(n)$ are free Abelian groups with bases given by $\{[D(\mu)]\}$ and $\{[P(\mu)]\}$ for $\mu \in \text{Par}_{res,n}$. There is a non-degenerate bilinear pairing $(\cdot, \cdot)$ between $G(n)$ and $K(n)$ which is given by $([P], [M]) = \dim \text{Hom}_{F_pS_n}(P, M)$. Using it, we have the following formula for the decomposition number for $F_pS_n$

\[
d_{\lambda\mu} = ([P(\mu)], [S(\lambda)]).
\]
These constructions and definitions can also be carried out for the Hecke algebra $\mathcal{H}_n(\xi)$, and we shall in general use a superscript 'Hecke' for the corresponding quantities.

Our interest in Conjecture \ref{conj} comes from the following Theorem.

**Theorem 5.** Suppose that Conjecture \ref{conj} is true. Then James’ Conjecture holds.

**Proof:** For any $v \in \mathcal{F}_q$, we define $v_\lambda \in \mathbb{C}(q)$ as the coefficient of $\lambda$ in the expansion of $v$ in the natural basis $\text{Par}$. We first check that

\[
(A(\mu)_\lambda)(1) = \dim_{\mathbb{F}_p}(\tilde{e}_\mu S(\lambda)).
\]  

To calculate $(A(\mu)_\lambda)(1)$ we put $q = 1$ in the formula \ref{eq} to arrive at

\[
(\tilde{f}_{1m}^{(L_m)} \ldots \tilde{f}_{12}^{(L_2)} \tilde{f}_{11}^{(L_1)} \emptyset)_\lambda(1).
\]

But this is exactly the number of tableaux in $[\mu_{lad}]$ of shape $\lambda$, that is the cardinality of $T_{\mu\lambda}$ from Lemma \ref{lem}, as can be seen from the combinatorial description of $[\mu_{lad}]$ given in that Lemma \ref{lem}. On the other hand we have

\[
E_{[\mu_{lad}]} S(\lambda) = \sum_{t \in [\mu_{lad}]} E_t S(\lambda) = \sum_{t \in T_{\mu\lambda}} E_t S(\lambda).
\]

But as mentioned already in the proof part b) of Theorem \ref{thm}, Murphy gave in [Mu83] a basis for this space, from which we deduce that its dimension is the cardinality of $T_{\mu\lambda}$, as well. Finally we obtain \ref{eq}, using that $S_{\text{lad}}$ acts faithfully on $T_{\mu\lambda}$, as shown in Lemma \ref{lem} combined with part b) of Theorem \ref{thm}.

Let us now assume that Conjecture \ref{conj} holds and let $(a_{\lambda\mu}) := (n_{\lambda\mu}(1))^{-1}$. Then we have the following formulas

\[
G(\mu)(1) = A(\mu)(1) + \sum_{\lambda,\lambda \succ \mu} a_{\lambda\mu} A(\lambda)(1), \quad [P(\mu)] = [\widetilde{A(\mu)}] + \sum_{\lambda,\lambda \succ \mu} a_{\lambda\mu} [\widetilde{A(\lambda)}]
\]  

(19)

where the last equality takes place in $\mathcal{K}(n)$. We get from this last formula that

\[
d_{\tau \mu} = ([\widetilde{A(\mu)}],[S(\tau)]) + \sum_{\lambda,\lambda \succ \mu} a_{\lambda\mu} ([\widetilde{A(\lambda)}],[S(\tau)])
\]

which, using equation \ref{eq} and the definition of $(\cdot,\cdot)$, can be rewritten as

\[
d_{\tau \mu} = (A(\mu)_\tau)(1) + \sum_{\lambda,\lambda \succ \mu} a_{\lambda\mu} (A(\lambda)_\tau)(1) = (G(\mu)_\tau)(1)
\]

where we for the last equality used the first equality of (19). Finally, by Ariki’s proof of the main Conjecture of [LLT], we know that $(G(\mu)_\tau)(1) = d_{\text{hecke}}^{\mu\tau}$. The Theorem is proved. \hfill \Box

\section{Partial verification of Conjecture \ref{conj}}

In this section we give a method for verifying Conjecture \ref{conj} for $n$ not too big. It is inspired by the recent theory of KLR-algebra algebras. Let therefore $\mathcal{R}_n$ be the cyclotomic KLR-algebra (Khovanov-Lauda-Rouquier) of type $A$ over $\mathbb{F}_p$ and let

\[
(a_{ij})_{i,j \in \mathbb{F}_p} = \begin{cases} 
2 & \text{if } i = j \mod p \\
-1 & \text{if } i = j \pm 1 \mod p \\
0 & \text{otherwise}
\end{cases}
\]
be the Cartan matrix of affine type $A_{p-1}^{(1)}$. Then $R_n$ is the $\mathbb{F}_p$-algebra on the generators

$$\{e(i) \mid i \in (\mathbb{F}_p)^n\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$$

subject to the following relations

$$y_1e(i) = 0 \text{ if } i_1 = 0 \mod p$$

$$e(i) = 0 \text{ if } i_1 \neq 0 \mod p$$

$$e(i)e(j) = \delta_{i,j}e(i)$$

$$\sum_{i \in (\mathbb{F}_p)^n} e(i) = 1$$

$$y_re(i) = e(i)y_r$$

$$\psi_re(i) = e(\sigma_{r+1})\psi_r$$

$$y_ry_s = y_sy_r$$

$$\psi_ry_s = \psi_s\psi_r$$

$$\psi_ry_{r+1}e(i) = \begin{cases} (y_re(i)+1)e(i) & \text{ if } i_r = i_{r+1} \mod p \\ y_re(i) & \text{ if } i_r \neq i_{r+1} \mod p \end{cases}$$

$$y_{r+1}\psi_re(i) = \begin{cases} (\psi_r)e(i) & \text{ if } i_r = i_{r+1} \mod p \\ (\psi_r)e(i) & \text{ if } i_r \neq i_{r+1} \mod p \end{cases}$$

$$\psi_r^2e(i) = \begin{cases} 0 & \text{ if } i_r = i_{r+1} \mod p \\ e(i) & \text{ if } i_r \neq i_{r+1} \pm 1 \mod p \\ (y_{r+1}e(i)-y_re(i)) & \text{ if } i_{r+1} = i_r + 1 \mod p \\ (y_re(i)-y_{r+1}e(i)) & \text{ if } i_{r+1} = i_r - 1 \mod p \end{cases}$$

$$\psi_r\psi_{r+1}\psi_re(i) = \begin{cases} (\psi_{r+1}\psi_r)e(i) & \text{ if } i_r = i_{r+1} - 1 \mod p \\ (\psi_{r+1}\psi_r)e(i) & \text{ if } i_r = i_{r+1} + 1 \mod p \end{cases}$$

$$\psi_r^2\psi_{r+1}\psi_r^2e(i) = \begin{cases} (\psi_r)e(i) & \text{ if } i_r = i_{r+1} \mod p \\ (\psi_r)e(i) & \text{ if } i_r \neq i_{r+1} \mod p \end{cases}$$

$$\psi_r^3e(i) = \begin{cases} 0 & \text{ if } i_r = i_{r+1} \mod p \\ e(i) & \text{ if } i_r \neq i_{r+1} \pm 1 \mod p \\ (y_{r+1}e(i)-y_re(i)) & \text{ if } i_{r+1} = i_r + 1 \mod p \\ (y_re(i)-y_{r+1}e(i)) & \text{ if } i_{r+1} = i_r - 1 \mod p \end{cases}$$

where $\sigma_{r+1} = (r, r+1)$ acts on $(\mathbb{F}_p)^n$ by permutation of the coordinates $r, r+1$. It is an important point that $R_n$ is a $\mathbb{Z}$-graded algebra. Indeed, the conditions

$$\text{deg } e(i) = 0, \quad \text{deg } y_r = 2, \quad \text{deg } \psi_se(i) = -a_{i_r,i_{r+1}}$$

for $1 \leq r \leq n$, $1 \leq s \leq n-1$ and $i \in (\mathbb{F}_p)^n$ are homogeneous with respect to the relations and therefore define a unique $\mathbb{Z}$-grading on $R_n$ with degree function deg.

In their important paper [BK], Brundan and Kleshchev proved the existence of an $\mathbb{F}_p$-algebra isomorphism $f : R_n \cong \mathbb{F}_pS_n$ and hence we may view $\mathbb{F}_pS_n$ as a $\mathbb{Z}$-graded algebra via $f$.

Let us now return to the situation of the previous section. We still set $(a_{\lambda\mu}) := (n_{\lambda\mu}(1))^{-1}$ and consider for $\mu \in \text{Par}_{r,s,n}$ the element $P(\mu) \in K(n)$ given by

$$P(\mu) := [A(\mu)] + \sum_{\lambda,\lambda\mu} a_{\lambda\mu} [A(\lambda)].$$

Assume from now on that $n < p^2$. In order to prove Conjecture $\Pi$ we must show for all $\lambda, \mu \in \text{Par}_{r,s,n}$ that $(P(\mu), [D(\tau)]) = \delta_{\mu\tau}$ or equivalently

$$\dim_{\mathbb{F}_p}(\bar{e}_\mu D(\tau)) + \sum_{\lambda,\lambda\mu} a_{\lambda\mu} \dim_{\mathbb{F}_p}(\bar{e}_\lambda D(\tau)) = \delta_{\mu\tau}$$

(34)
since we would then have that \( \mathcal{P}(\mu) = [P(\mu)] \). The number of terms in the summation of \( [34] \) is relatively small, so in order to verify these equations, we essentially need a way of determining the dimension of \( \tilde{e}_\lambda D(\tau) \).

Under Brundan and Kleshchev’s isomorphism \( f : \mathcal{R}_n \cong \mathbb{F}_p S_n \), the generators \( \{e(i)\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\} \) are mapped to elements of \( \mathbb{F}_p S_n \) that we denote in the same way. For instance, we know from [BK] that \( e(i) \in \mathbb{F}_p S_n \) is the idempotent projector on the generalized eigenspace for the Jucys-Murphy elements, that is

\[
e(i) \mathbb{F}_p S_n = \{a \in \mathbb{F}_p S_n \mid \text{for all } k \text{ there is } M \text{ such that } (L_k - i_k)^M v = 0\}
\]

and hence we get that

\[
E_{[\lambda_{ad},]} = e(i_{lad,\lambda})
\]

where \( i_{lad,\lambda} \) is the residue sequence of the ladder tableau for \( \lambda \) as in section 2. This is the key Lemma 4.1 of [HuMa1]. Hence via Lemma [35] we get that \( \tilde{e}_\lambda \) is the symmetrized idempotent projector on a generalized weight space for the Jucys-Murphy operators. By orthogonality of the \( e(i) \), we deduce that the dimension of \( \tilde{e}_\lambda D(\tau) \) is equal to the \( p \)-rank of \( \langle \cdot, \cdot \rangle_\tau \) on the restriction to \( \tilde{e}_\lambda S(\tau) \).

Let us now turn to the elements \( \psi_1, \ldots, \psi_{n-1} \) in \( \mathbb{F}_p S_n \). In [BK] they are constructed as suitable adjustments of certain ‘intertwining elements’ \( \phi_1, \ldots, \phi_{n-1} \) and, as a matter of fact, in this section we shall mostly focus on these intertwining elements. In [RH3] we found a natural realization of them, completely within the theory of Young’s seminormal form. Indeed, we have that \( \phi_i = \sigma_i + \frac{1}{\kappa_e} \) where \( \frac{1}{\kappa_e} = \frac{1}{\kappa_{i-1} - \kappa_i} \) is defined in Lemma 5 of [RH3].

Let \( \{\xi_{st} \mid \langle s, t \rangle \in \text{Std}(\lambda)^2, \lambda \in \text{Par}_n\} \) be the seminormal basis for \( \mathbb{Q} S_n \), introduced in section 2. The action of \( S_n \) on it is given by the seminormal form, that is by the formulas of Theorem [36] but these formulas take place in \( S(\lambda) \mathbb{Q} \) and therefore do not immediately help us in the modular setting. But note that by [Mu92] we have that

\[
\xi_{\lambda \lambda} = x_{\lambda \lambda} \mod (\mathbb{Q} S_n)^{>\lambda}
\]

and so, using the seminormal form in a reduced expression \( d(s) = \sigma_1, \ldots, \sigma_iN \) for \( d(s) \), we can express the standard basis element \( x_{s \lambda} \), when viewed as an element of \( S(\lambda) \), as a linear combination of the seminormal basis elements \( \xi_{s \lambda} \), but with coefficients in \( \mathbb{Q} \). The next Theorem is based on this idea.

**Theorem 6.** Let \( T \) be a tableau class and suppose that \( x \in E_T S(\lambda) \). Then \( x \) can be written as \( x = \sum_{t \in T} a_t \xi_{t \lambda} \). The action of the intertwiner \( \phi_i \) is given by \( \phi_i x = \sum_{t \in T} a_t \phi_i^m \xi_{s \lambda} \) where

\[
\phi_i^m \xi_{s \lambda} := \begin{cases} 
0 & \text{if } |h| = 1 \\
\xi_{t \lambda} & \text{if } h > 1 \text{ and } s \not\sim_p t \\
h^2 - 1 \xi_{t \lambda} & \text{if } h < 1 \text{ and } s \not\sim_p t \\
(1 - \frac{1}{p}) \xi_{s \lambda} + \xi_{t \lambda} & \text{if } h > 1 \text{ and } s \not\sim_p t \\
(1 - \frac{1}{p}) \xi_{s \lambda} + \frac{h^2 - 1}{\kappa_{t-1}} \xi_{t \lambda} & \text{if } h < 1 \text{ and } s \not\sim_p t
\end{cases}
\]

for \( t := \sigma_i s \). We say that the first three cases of these formulas are the ‘regular’ ones whereas the last two cases are the ‘singular’ ones.

**Proof:** The first statement is a consequence of the realization of the tableau class idempotent \( E_T = \sum_{t \in T} E_t \) and the fact that \( E_t S(\lambda) \mathbb{Q} = \mathbb{Q} \xi_{t \lambda} \) for \( t \in \text{Std}(\lambda) \).
In order to prove the second statement, we need to recall the construction of \( \phi_i \) from [RH3]. Let \( S := [s], T := [t] \) be as in the announcement of the Theorem and suppose first that \( S \neq T \). Choose arbitrarily \( t \in T \) and define \( c_T(i - 1) := c_T(i) \in R \) and \( c_T(i) := c_T(i) \in R \) and set \( h_T(i) := c_T(i - 1) - c_T(i) \). Although \( h_T(i) \in R \) depends on the choice of \( t \in T \), we showed in [RH3] that for any \( a \in E_T S(\lambda) \), we have that \( (L_{i-1} - L_i - h_T(i))^N a \) belongs to \( pE_T S(\lambda) \) for \( N \) sufficiently big, independently of the choice of \( t \). Then \( \frac{1}{L_{i-1} - L_i} \) is the linear transformation on \( E_T S(\lambda) \) given by the corresponding geometric series. To be precise, for \( a \in E_T S(\lambda) \) it is given by

\[
\frac{1}{L_{i-1} - L_i} a := \frac{1}{h_T(i)} \sum_k (-1)^k \left( \frac{L_{i-1} - L_i - h_T(i)}{h_T(i)} \right)^k a
\]  

(36)

where the sum may be assumed to be finite by the above remark. Finally, \( \phi_i \) is the linear transformation on \( E_T S(\lambda) \) given by \( \phi_i := \sigma_i + \frac{1}{L_{i-1} - L_i} \). Note the slight variation from [RH3], where we used the definition \( \phi_i := \sigma_i - \frac{1}{L_{i-1} - L_i} \). With this convention, \( \phi_i \) coincides exactly with Brundan and Kleshchev’s element \( \phi_i \) and it verifies the following intertwining property

\[
E_S \phi_i = \phi_i E_T
\]  

(37)

in \( \text{Hom}_{p}(E_T(F_p S_n), F_p S_n) \), corresponding to Lemma 7 of [RH3].

Now, recall that the argument in [RH3] to show that \( (L_{i-1} - L_i - h_T(i))^N a \in pS(\lambda) \) for \( N \gg 0 \) used the expansion of \( a \in E_T S(\lambda) \) in the seminormal basis \( \xi_{i\lambda} \). For each term of this expansion we indeed get

\[
(L_{i-1} - L_i - h_T(i))^N \xi_{i\lambda} = (c_T(i - 1) - c_T(i) - h_T(i))^N \xi_{i\lambda} \in pS(\lambda)
\]

for \( N \gg 0 \). We conclude from it that the series \( [36] \) can be calculated by lifting \( a \in E_T S(\lambda) \) to \( a \in E_T S(\lambda) \), then expanding this \( a \) in the \( \xi_{i\lambda} \)'s, next applying the series to each term and finally reducing modulo \( p \). On each of these terms we get

\[
\frac{1}{L_{i-1} - L_i} \xi_{s\lambda} = \frac{1}{h_T(i)} \sum_k (-1)^k \left( \frac{c_{i-1}(t) - c_i(t) - h_T(i)}{h_T(i)} \right)^k \xi_{s\lambda}
\]

(38)

that equals \( \frac{1}{c_{i-1}(t) - c_i(t)} \xi_{s\lambda} \). From this the regular cases \( [35] \) of the Theorem follow by applying the classical formulas for Young’s seminormal form, that is Theorem \( 4 \).

The singular cases are easier to handle, since we then have \( \phi_i = \sigma_i + 1 \) and so we finish by applying Theorem \( 4 \) once again. \( \square \)

**Remark.** In their recent work [HuMa2], independent of ours, Hu and Mathas prove in a systematic way results that are similar to the Theorem. They use them to develop the theory of KLR-algebras, including integral versions of them, completely from the point of view of Young’s seminormal form.

**Theorem 7.** Let \( \lambda \in \text{Par}_n \) and let \( \{ \xi_{s\lambda} \mid s \in \text{Std}(\lambda) \} \) be the seminormal basis for \( S(\lambda)_q \). Then we have that

\[
\langle \xi_{s\lambda}, \xi_{s\lambda} \rangle_\lambda = \gamma_s.
\]

**Proof:** This is contained in Murphy’s papers where it is shown by induction. The induction basis is given by \( \xi_{\lambda\lambda} = x_{\lambda\lambda} \) and the induction step by Young’s seminormal form \( 6 \). \( \square \)
With the above Theorems at our disposal we can now describe an algorithm for calculating $\dim_{R} \tilde{e}_{\lambda} D(\tau)$, or equivalently the rank of $\langle \cdot, \cdot \rangle_{\tau}$ on $\tilde{e}_{\lambda} S(\tau)$. Note that Step 3 of the algorithm depends on the results from our previous work [RH3], giving a cellular basis in terms of the $\phi_{i}$'s.

Algorithm.
Step 1. Determine the set $T_{\lambda\tau} = \{ s \in [\lambda_{lad}] \mid \text{Shape}(s) = \tau \}$. As indicated above, $T_{\lambda\tau}$ can be read off from the calculation of the first approximation of $A(\mu)$ at $q = 1$, since the successive actions of $f_{i}$ may be viewed as producing tableaux rather than partitions.
Step 2. Write the elements of $\{ d(s) \mid s \in T_{\lambda\tau} \} \subset S_{n}$ as reduced products of simple transpositions $\sigma_{i}$. The longest element of $S_{n}$ has length $l(w_{0}) = n(n - 1)/2$, and so each of the reduced products has less than $n(n - 1)/2$ terms.
Step 3. For each $d(s) = \sigma_{i_{k}} \ldots \sigma_{i_{1}}$ from step 3, calculate $\phi_{i_{k}} \ldots \phi_{i_{1}} \xi_{\lambda\lambda}$ using (35). By Theorem 2 and Lemma 9 of [RH3], we get in this way an $R$-basis for $\epsilon_{[\lambda_{lad}]} S(\tau)$. The basis elements are given as linear combinations of seminormal basis elements. The number of terms $\xi_{\lambda\lambda}$ in this expansion will be less than $2^{B}$ where $B$ is the number of indices in the reduced expression for $d(s)$ that involve the singular cases of (35).
Step 4. Symmetrize each basis element from the previous step with respect to the ladder group $S_{lad,\lambda}$, to get a basis for $\tilde{e}_{\lambda} S(\tau)$.
Step 5. Calculate the matrix of the form $\langle \cdot, \cdot \rangle_{\tau}$ on $\tilde{e}_{\lambda} S(\tau)$ with respect to the basis given in the previous step. Since the basis elements are expanded in terms of the seminormal basis, this step now follows easily from the previous Theorem. The matrix will have values in $R$ although the coefficients of the expansions are rational.
Step 6. Reduce the matrix modulo $p$ and determine its rank.

Remark. The algorithm can also be implemented using the classical Young's seminormal form, that is formula (3). On the other hand, that algorithm will be much less efficient with expansions that grow too fast. In fact, the main point of our algorithm, as presented above, is that the indices of the reduced expressions will mostly correspond to the regular cases of (35), thus reducing, as much as possible, the doubling up of terms.

Example. Suppose that $p = 3$. We verify Conjecture 1 for $S_{5}$ using our algorithm. Although the LLT-algorithm does not involve any subtractions in this example, the example is still big enough to illustrate our algorithm.

We have that
$$\text{Par}_{res,5} = \{ [3, 2], [3, 1^{2}], [2^{2}, 1], [2, 1^{3}], [1^{5}] \}.$$ The ladder groups are
$$S_{lad,[3,2]} = S_{lad,[3,1^{2}]} = \{ (3, 4) \}, S_{lad,[2^{2},1]} = S_{lad,[2,1^{3}]} = S_{lad,[1^{5}]} = 1.$$ The first approximations are
$$A([3, 2]) = [3, 2] + q[4, 1], \quad A([3, 1^{2}]) = [3, 1^{2}],$$
$$A([2^{2}, 1]) = [2^{2}, 1] + q[5], \quad A([2, 1^{3}]) = [2, 1^{3}] + q[2^{2}, 1],$$
$$A([1^{5}]) = [1^{5}] + q[3, 2].$$ From this we conclude, as already mentioned above, that $G(\lambda) = A(\lambda)$ for all $\lambda \in \text{Par}_{res,5}$. Thus, Conjecture 1 is in this case the affirmation that $P(\lambda) = A(\lambda)$ for all $\lambda \in \text{Par}_{res,5}$, or by the above that
$$\dim \tilde{e}_{\lambda} D(\tau) = \delta_{\lambda\tau} \quad \text{for all } \lambda, \tau \in \text{Par}_{res,5}.$$
We calculate the rank of $\langle \cdot , \cdot \rangle$ on each symmetrized weight space $\tilde{e}_\lambda S(\tau)$, using our algorithm. Recall that $\dim \tilde{e}_\lambda S(\tau)$ can be read off from the first approximation $A(\lambda)$. For example we have that $\dim \tilde{e}_{[3,2]} S([4,1]) = 1$ since the coefficient $q$ evaluates to 1, although the eigenspace $\tilde{e}_{[3,2]} S([4,1])$ is irrelevant to us since $[4,1] \notin \text{Par}_{\text{res},5}$.

By going through the first approximations, we see that the relevant eigenspaces are $\tilde{e}_{[2,1^3]} S([2^2,1])$ and $\tilde{e}_{[1^5]} S([3,2])$, both of dimension one. We verify that the ranks of the corresponding forms are zero, or equivalently that the forms are zero.

We first consider $\tilde{e}_{[2,1^3]} S([2^2,1])$. The residue diagram $res_{[2,1^3]}$ of $\lambda := [2,1^3]$ is

$$
\begin{array}{c}
\text{res}_{[2,1^3]} :=
\begin{array}{c}
0 \\
2 \\
1 \\
0
\end{array}
\end{array}
\begin{array}{c}
\text{t} :=
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1
\end{array}
\end{array}
\begin{array}{c}
\text{res}_\tau :=
\begin{array}{c}
0 \\
1 \\
2 \\
0
\end{array}
\end{array}
$$

and so we have $i_{\text{ad},[2,1^3]} = (0,1,2,1,0)$. The only tableau of shape $\tau := [2^2,1]$ in the ladder class of $\lambda$ is therefore $t$ as given above. We have $d(t) = (4,5) = \sigma_5$, and so we get from formula (33) that the basis for $\tilde{e}_{[2,1^3]} S([2^2,1])$ is $\{ \phi_5 \xi_{\lambda} \} = \{ \xi_{\lambda} \}$. Finally, by Theorem 4 we get that $\langle \xi_{\lambda}, \xi_{\lambda} \rangle = 3 = 0 \mod 3$, as claimed.

We next consider $\tilde{e}_{[1^5]} S([3,2])$ where we basically proceed as before. The residue diagram $res_{[1^5]}$ is

$$
\begin{array}{c}
\text{res}_{[1^5]} :=
\begin{array}{c}
0 \\
2 \\
1 \\
0
\end{array}
\end{array}
\begin{array}{c}
\text{s} :=
\begin{array}{c}
1 \\
3 \\
5 \\
2 \\
4
\end{array}
\end{array}
\begin{array}{c}
\text{res}_s :=
\begin{array}{c}
0 \\
1 \\
2 \\
0
\end{array}
\end{array}
$$

and we have $i_{\text{ad},[2,1^3]} = (0,2,1,0,2)$. The only tableau of shape $\nu := [3,2]$ in the ladder class of $[1^5]$ is $s$ as given above. We have $d(s) = (2,3)(4,5)(3,4) = \sigma_3 \sigma_5 \sigma_4$, and so we get from formula (33) that the basis for $\tilde{e}_{[1^5]} S([3,2])$ is

$$
\{ \phi_3 \phi_5 \phi_4 \xi_{\lambda} \} = \{ \xi_{\lambda} \}
$$

and then by Theorem 4 we get $\langle \xi_{\lambda}, \xi_{\lambda} \rangle = 3 = 0 \mod 3$, as claimed. This concludes the verification of Conjecture 1 and then by Theorem 5 also of James’s Conjecture, in this case.

We have implemented the algorithm using the GAP-system and have found the following results, that without doubt can be improved on.

**Theorem 8.** If $p = 3$ then Conjecture 2 is true.
If $p = 5$ then Conjecture 1 is true for $n < 16$.
If $p = 7$ then Conjecture 1 is true for $n < 19$.
If $p = 11$ then Conjecture 1 is true for $n < 22$.
If $p = 13$ then Conjecture 1 is true for $n < 22$.

**Remark.** The Theorem provides, via Theorem 5 decomposition numbers for $F_p S_n$. But as pointed out to us by A. Mathas, the partitions involved in the Theorem are all of $p$-weight less than three and hence the corresponding decomposition numbers are already present in the literature, see M. Richards’ paper [Ri] for the weight one and two cases, and M. Fayers’s paper [F] for the weight three case.
6. \( \widetilde{A(\mu)} \) as a Graded Module

In this section we show that \( \widetilde{A(\mu)} \) may be viewed as a graded module for \( \mathbb{F}_p S_n = \mathcal{R}_n \). We do so by showing that \( \tilde{e}_\mu \) is a homogeneous idempotent in \( \mathbb{F}_p S_n \), necessarily of degree zero.

Let \( \mathcal{R}_n \) be the noncycloctotomic KLR-algebra of type \( A \) over \( \mathbb{F}_p \), or more precisely the \( \mathbb{F}_p \)-algebra on the same generators and relations as \( \mathcal{R}_n \), but without the relations (20) and (21). This is the algebra that was first considered in [R] and in [KL] from a diagrammatic point of view. Assume that \( \lambda \in \text{Par}_{\text{res},n} \) has associated ladder residue sequence \( \text{ladder}(\lambda) \) and let \( n_0, \ldots, n_m \) be as in (9). Then for all \( k \), the residues \( i_{n_k-1}, \ldots, i_{n_k} \) are equal, being the residue \( i_k \) of the \( k \)'th ladder \( L_k \) of \( \lambda_{\text{ladder}} \). Let \( \mathcal{R}_{k,n} \) be the subalgebra of \( \mathcal{R}_n \) defined by

\[
\mathcal{R}_{k,n} := \langle e(i_{\text{ladder}},\lambda), y_i e(i_{\text{ladder}},\lambda), \psi_j e(i_{\text{ladder}},\lambda) \mid a_k \leq i \leq b_k, \ a_k \leq j \leq b_k - 1 \rangle
\]

where we write \( L_k = \{ a_k, a_k+1, \ldots, b_k \} \) or just \( \{ a, a+1, \ldots, b \} \) for simplicity. Then \( \mathcal{R}_{k,n} \) is isomorphic to the nilHecke algebra, or to be more precise, setting \( y_i := y_i e(i_{\text{ladder}},\lambda) \) and \( \partial_r := \psi_j e(i_{\text{ladder}},\lambda) \), to the infinite dimensional \( \mathbb{F}_p \)-algebra generated by \( y_r \) and \( \partial_r \) subject to the relations

\[
\begin{align*}
\partial^2_r & = 0 \\
\partial_r \partial_{r+1} \partial_r & = \partial_{r+1} \partial_r \partial_{r+1} \\
\partial_r \partial_s & = \partial_s \partial_r & \text{if } |r-s| > 1 \\
y_r y_s = y_s y_r & \\
\partial_r y_{r+1} - y_r \partial_r & = 1 & \text{if } r \neq r, r+1
\end{align*}
\]

with one-element \( e(i_{\text{ladder}},\lambda) \). Note that to get the nilHecke algebra presentation used in [KL], one should use the isomorphism given by \( \partial_r \mapsto -\partial_r, y_i \mapsto y_i \). For \( w \in S_k \) we define \( \partial_w = \partial_{i_1} \ldots \partial_{i_k} \) where \( w = \sigma_{i_1} \ldots \sigma_{i_k} \) is a reduced expression; this is independent of the chosen reduced expression. Note that \( \prod_k \mathcal{R}_{k,n} \) is a subalgebra of \( \mathcal{R}_n \). For \( w_0,k \in S_{\mathcal{L}_k} \) the longest element we define

\[
e_{KL,\lambda,k,n} := (-1)^{m_k} \frac{m_k}{2} \partial_{w_0,k} y_{a_1} y_{a_2}^{m_2-1} \ldots y_{a_k}^{m_k-1} \in \mathcal{R}_{k,n}
\]

where \( m_k = |L_k| \). Then, by section 2.2 of [KL], \( e_{KL,\lambda,k,n} \) is a homogeneous idempotent of \( \mathcal{R}_{k,n} \) and hence \( e_{KL,\lambda,n} \) is a homogeneous idempotent of \( \mathcal{R}_n \). Let \( g : \mathcal{R}_n \to \mathcal{R}_n \) be the quotient map.

**Theorem 9.** We still assume that \( \lambda \in \text{Par}_{\text{res},n} \) and that all ladders of \( \lambda_{\text{ladder}} \) are of length strictly less than \( p \). Then we have that \( g(e_{KL,\lambda,n}) = \tilde{e}_\lambda \). In particular, \( \widetilde{A(\mu)} \) may be considered a graded module for \( \mathbb{F}_p S_n \).

**Proof:** Clearly, \( g(e_{KL,\lambda,n}) \) is an idempotent, although possibly zero. By the definitions we have \( g(e_{KL,\lambda,n}) = \prod_k g(e_{KL,\lambda,k,n}) \) where

\[
g(e_{KL,\lambda,k,n}) = (-1)^{m_k} \frac{m_k}{2} \partial_{w_0,k} y_{a_1} y_{a_2}^{m_2-1} \ldots y_{a_k}^{m_k-1}
\]

where we use the same notation for \( x \in \mathcal{R}_n \) and its image \( g(x) \in \mathcal{R}_n \).

Let us now recall Brundan and Kleshchev's construction of the element \( g(\partial_r) \), that is \( \psi_j e(i_{\text{ladder}},\lambda) \). As already mentioned it is an adjustment of the intertwining
element \( \phi_j \), that we described in Theorem B. On the other hand, since \( a_k \leq j \leq b-1 \) we are in the singular case of Theorem \( B \) and so we have \( \phi_j = \sigma_j + 1 \) when acting in the generalized eigenspace corresponding to \( e(1_{lad}, \lambda) \). In general, the adjustment element \( q_i(1_{lad}, \lambda) \) satisfying \( \phi_j = \psi_j q_i(1_{lad}, \lambda)^{-1} \) is an invertible power series in the \( y_i \)'s. Let now \( j \) be any number such that \( a_k \leq j \leq b_k - 1 \). Then there is a reduced expression for \( w_{0,k} \) of the form \( w_{0,k} = \sigma_j w' \) for some \( w' \) from which we deduce that

\[
\partial_{w_{0,k}} = (\sigma_j + 1)q_j(1_{lad}, \lambda)^{-1} a
\]

for some \( a \). Hence we conclude that \( g(e_{KLR, \lambda, k, n}) \) is invariant under the action of \( \sigma_j \) for all \( j \) such that \( a_k \leq j \leq b_k - 1 \) and so we have that \( g(e_{KLR, \lambda, n}) = c \tilde{e}_\lambda \) for some scalar \( c \). Since \( g(e_{KLR, \lambda, n}) \) is an idempotent, the only possibilities for \( c \) are now \( c = 1 \) or \( c = 0 \).

It remains to show that \( g(e_{KLR, \lambda, n}) \neq 0 \), or that \( c = 1 \). Let \( x_{lad, \lambda} \in S(\lambda) \) be the Murphy basis element as above. From equation (13) we have that \( \tilde{e}_\lambda x_{lad, \lambda} \neq 0 \), so in order to show that \( c = 1 \) it is enough to check the equality

\[
g(e_{KLR, \lambda, n})\tilde{e}_\lambda x_{lad, \lambda} = \tilde{e}_\lambda x_{lad, \lambda}. \tag{39}
\]

Without loss of generality, it is enough to consider the \( k \)'th ladder \( L_k = L \) and to assume that \( S_{lad, \lambda} = S_L \). Let \( w_{k,0} = w_0 \) be the longest element of \( S_L \). In order to show (39), we first lift both sides to \( \mathbb{Q} \), then expand in terms of the seminormal basis \( \xi_{\sigma, \lambda, \lambda}, \sigma \in S_L \) and finally verify that in both expansions the coefficient of \( \xi_{w_{0}, \lambda, \lambda} \) is the same, up to a unit in \( R^x \).

We first consider the right hand side of (39). It is the reduction modulo \( p \) of the following element, that exists over \( R \), but that shall also be considered over \( \mathbb{Q} \)

\[
LIFT := 1/|S_L| \sum_{\sigma \in S_L} \sigma E_{[\lambda_{lad}]} x_{lad, \lambda}.
\]

Using Young’s seminormal form (1) repeatedly we get that the coefficient of \( \xi_{w_{0}, \lambda, \lambda} \) in \( LIFT \) is \( 1/|S_L| \).

We next work out the left hand side of (39), using the realization in [BK] of the \( y_i \)'s as Jucys-Murphy elements. Indeed, we have from (3.21) of loc. cit. that

\[
y_r = \sum_{i \in (\mathbb{F}_p)^n} (L_r - i_r) e(i). \tag{40}
\]

For \( T \) a tableau class with residue sequence \( \mathbf{i} \), this can be lifted to \( R \) to

\[
y_r = \sum_{i \in T} (c_i(r) - \hat{i}_r) E_t \tag{41}
\]

where \( \hat{i}_r \in \mathbb{Z} \) is chosen such that \( \hat{i}_r \mod p = i_r \). Writing \( LIFT = \sum_{\sigma \in S_L} a_{\sigma} \xi_{\sigma, \lambda_{lad, \lambda}} \) for \( a_{\sigma} \in \mathbb{Q} \), we get

\[
y_a^{m-1}y_{a+1}^{m-2} \cdots y_{b-1} LIFT = y_a^{m-1}y_{a+1}^{m-2} \cdots y_{b-1} \sum_{\sigma \in S_L} a_{\sigma} \xi_{\sigma, \lambda_{lad, \lambda}}. \tag{42}
\]

Now, for the factors of \( y_a^{m-1} \) we now choose for the lift \( \hat{i}_r \) as in (40) all the contents that appear in \( L \) each exactly once, except \( c_{\lambda_{lad}}(a) \). Similarly, for the factors of \( y_{a+1}^{m-2} \) we choose for \( \hat{i}_r \) as in (40) all the contents that appear in \( L \) each once, except
this time \( c_{\lambda\mu}(a) \) and \( c_{\lambda\mu}(a+1) \) and so on. With these choices, only one term survives in the action of the \( y_i \)'s in (12), giving

\[
y_a^{m-1} y_{a+1}^{m-2} \cdots y_b^{-1} \sum_{\sigma \in S_L} a_{\sigma} \xi_{\sigma\lambda\mu,\lambda} = u a_1 p^{m(m-1)/2} \xi_{\lambda\mu,\lambda}
\]

for some \( u \in R^x \) where we use that \(|L| < p\) to get the factor \( p^{m(m-1)/2}\).

We now need to work out \( a_1 \), the coefficient of \( \xi_{\lambda\mu,\lambda} \) in \( LIFT \). The coefficient of \( \xi_{\lambda\mu,\lambda} \) is 1, as already mentioned above, so we need a formula relating the coefficients of the \( \xi_{\lambda\mu,\lambda} \)'s. But using Young’s seminormal form (1) on the equality

\[
\sigma_i \sum_{\sigma \in S_L} a_{\sigma} \xi_{\lambda\mu,\lambda} = \sum_{\sigma \in S_L} a_{\sigma} \xi_{\lambda\mu,\lambda}\quad \text{for all } i
\]

we get that

\[
a_{\sigma} = ((h-1)/h) a_{\sigma}, \sigma \text{ if } \sigma_i \sigma > \sigma
\]

where \( h \) is the radial distance between \( \lambda\mu \) and \( \sigma_i \lambda\mu \). We find from this that

\[
a_1 = u p^m (m-1)/2 \quad \text{for some } u \in R^x, \text{ and so indeed } c = 1. \text{ The Theorem is proved.} \quad \square
\]

Using Brundan and Kleshchev’s isomorphism \( \mathbb{F}_p S_n \cong \mathbb{R}_n \), we may introduce \( \mathbb{F}_p S_n\text{-grmod} \), the category of finite dimensional graded \( \mathbb{F}_p S_n \)-modules. For \( M \) an object of \( \mathbb{F}_p S_n\text{-grmod} \), we have a decomposition of \( \mathbb{F}_pS\text{-spaces} \)

\[
M = \bigoplus_{i \in \mathbb{Z}} M_i
\]

such that \( (\mathbb{F}_p S_n)_i M_j \subset M_{i+j} \) for all \( i, j \).

Let \( v : \mathbb{F}_p S_n\text{-grmod} \to \mathbb{F}_p S_n\text{-mod} \) be the forgetful functor. Using the results from [CF] on the representation theory of general \( \mathbb{Z} \)-graded rings, we have that an object \( M \) in \( \mathbb{F}_p S_n\text{-grmod} \) is indecomposable if and only if \( v(M) \) is indecomposable in \( \mathbb{F}_p S_n\text{-mod} \) and that every indecomposable object in \( \mathbb{F}_p S_n\text{-mod} \) is of the form \( v(M) \) for some \( M \). Moreover, by Lemma 2.5.3 of [BGS] we know that the grading on indecomposable modules is unique up to a shift in degree. That is, if \( M \) and \( N \) are indecomposable objects in \( \mathbb{F}_p S_n\text{-grmod} \) satisfying \( v(M) \cong v(N) \), then \( M \cong N(k) \) for some \( k \in \mathbb{Z} \) where \( N(k) \) is the degree shift of \( N \) of order \( k \), that is

\[
N(k)_i := N_{i+k}.
\]

Let us write \( \widetilde{A(\mu)}^{gr} \) for \( A(\mu) \) considered as a graded module, via the previous Theorem. By applying the above mentioned results from the literature, we now get, corresponding to b) of the Theorem 2, a triangular expansion

\[
\widetilde{A(\mu)}^{gr} = P(\mu)^{gr} \oplus \bigoplus_{\lambda\lambda' \geq \mu} (P(\lambda)^{gr} \langle k_{\lambda\mu}, k_{\lambda'\mu} \rangle)^{\oplus m_{\lambda\mu}}
\]

for certain integers \( m_{\lambda\mu}, k_{\lambda\mu} \) where \( P(\mu)^{gr}, \) satisfying \( v(P(\mu)^{gr}) = P(\mu) \), is chosen as the first term in the expansion for all \( \mu \).

Formally, the relationship between \( \widetilde{A(\mu)}^{gr} \) and \( P(\mu)^{gr} \) is now the same as the one between the Bott-Samelson bimodule and the indecomposable Soergel bimodule in Soergel’s theory of bimodules over the coinvariant ring of a Coxeter group, see [So]. It would therefore be interesting to investigate to what extent the methods of Elias and Williamson’s paper [EW] for solving Soergel’s Conjecture can be applied to our situation.


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