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Some new gradient estimates for two nonlinear parabolic equations under Ricci flow

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Abstract. In this paper, by maximum principle and cutoff function, we investigate gradient estimates for positive solutions to two nonlinear parabolic equations under Ricci flow. The related Harnack inequalities are deduced. As applications, gradient estimates and Harnack inequalities for positive solutions to the heat equation under Ricci flow are derived. These results in the paper can be regard as generalizing the gradient estimates of Li-Yau, J. Y. Li, Hamilton and Li-Xu to the Ricci flow. Our results also improve the estimates of S. P. Liu and J. Sun to the nonlinear parabolic equation under Ricci flow.

1. Introduction

Beginning with the pioneering work of Li and Yau [14], gradient estimates are also known as differential Harnack inequalities, which have tremendous impact in geometric analysis, as shown for example in [14, 15, 16]. Moreover, both have very important applications in singularity analysis. In perelman’s geometrization conjecture [22, 23] on the poincaré conjecture, a differential Harnack inequality played an important role.

Next, we simply introduce research progress associated with this article.

Let \((M^n, g)\) be a complete Riemannian manifold. Li and Yau [14] established a famous gradient estimate for positive solutions to the following heat equation

\[ u_t = \Delta u \]  

on \((M^n, g)\), which is described as

**Theorem A** (Li-Yau [14]) Let \((M^n, g)\) be a complete Riemannian manifold. Suppose that on the ball \(B_{2R}\), \(\text{Ricci}(B_{2R}) \geq -K\). Then for any \(\alpha > 1\),

\[
\sup_{B_{2R}} \left( \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \right) \leq \frac{C\alpha^2}{R^2} \left( \frac{\alpha^2}{\alpha^2 - 1} + \sqrt{KR} \right) + \frac{n\alpha^2K}{\alpha - 1} + \frac{na^2}{2t}. 
\]  

In general, on a complete Riemannian manifold, if \(\text{Ricci}(M) \geq -k\), by letting \(R \to \infty\) in (1.2), one inferred

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n\alpha^2k}{2(\alpha - 1)} + \frac{na^2}{2t}. 
\]  

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In 1991, Li [15] generalized Li and Yau’s estimates to the nonlinear parabolic equation
\[ \left( \Delta - \frac{\partial}{\partial t} \right) u(x, t) + h(x, t)u^\alpha(x, t) = 0 \] (1.4)
on \((M^n, g)\). In 1993, Hamilton in [8] generalized the constant \(\alpha\) of Li and Yau’s result to the function \(\alpha(t) = e^{2Kt}\). In 2006, Sun [27] also obtained a gradient estimate of different coefficient. In 2011, Li and Xu in [17] further promoted Li and Yau’s result, and found two new functions \(\alpha(t)\). Recently, first author and Zhang in [28] further generalized Li and Xu’s results to the nonlinear parabolic equation (1.4). Related results can be found in [5, 11, 32].

In this paper, we investigate the two nonlinear parabolic equations
\[ \partial_t u(x, t) = \Delta u(x, t) + h(x, t)u^l(x, t) \] (1.5)
and
\[ \partial_t u(x, t) = \Delta u(x, t) + au(x, t) \log u(x, t) \] (1.6)
under Ricci flow, where the function \(h(x, t) \geq 0\) is defined on \(M^n \times [0, T]\), which is \(C^2\) in the first variable and \(C^1\) in the second variable, \(T\) is a positive constant and \(l, a \in \mathbb{R}\), respectively.

Recently, there are a number of studies on Ricci flow on manifolds by R. Hamilton [9, 10] and others, because the Ricci flow is a powerful tool in analyzing the structure of manifolds. Assume \(M^n\) is an \(n\)-dimensional manifold without boundary, and let \((M^n, g(t))_{t \in [0, T]}\) be an \(n\)-dimensional complete manifold with metric \(g(t)\) evolving by the Ricci flow
\[ \frac{\partial g(t)}{\partial t} = -2R_{ij}, \quad (x, t) \in M^n \times [0, T]. \] (1.7)

In 2008, Kuang and Zhang [11] proved a gradient estimate for positive solutions to the conjugate heat equation under Ricci flow on a closed manifold. In 2009, Liu [18] derived a gradient estimate for positive solutions to the heat equation under Ricci flow. Afterwards, Sun[26] generalized Liu’s results to general geometric flow. In 2010, Bailesteanu, Cao and Pulemotov [1] established some gradient estimates for positive solutions to the heat equation under Ricci flow. In 2016, Li and Zhu [19] generalized J. Y. Li’s [15] estimates under Ricci flow. Recently, Cao and Zhu [3] derived some Aronson and Bénilan estimates for porous medium equation
\[ u_t = \Delta u^m, \quad m > 1 \]
under Ricci flow. Li, Bai and Zhang [13] studied fast diffusion equation
\[ u_t = \Delta u^m, \quad 0 < m < 1 \]
under the Ricci flow. Zhao and Fang [31] generalized Yang’s result [30] to the Ricci flow.

Firstly, we introduce three \(C^1\) functions \(\alpha(t), \varphi(t)\) and \(\gamma(t)\) : \((0, +\infty) \to (0, +\infty)\). Suppose that three \(C^1\) functions \(\alpha(t), \varphi(t)\) and \(\gamma(t)\) satisfy the following conditions:
\( (C1) \; \alpha(t) > 1, \; \varphi(t) \) and \( \gamma(t) \).
(C2) \( \alpha(t) \) and \( \varphi(t) \) satisfy the following system
\[
\begin{cases}
\frac{2\varphi}{n} - 2\alpha K \geq \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha}, \\
\frac{2\varphi}{n} - \alpha' > 0, \\
\varphi^2 + \alpha \varphi' \geq 0.
\end{cases}
\]

(C3) \( \gamma(t) \) satisfies
\[
\frac{\gamma'}{\gamma} - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} \leq 0.
\]

(C4) \( \gamma(t) \) is non-decreasing, and \( \alpha(t) \) is also non-decreasing or is bounded uniformly.

This paper is organized as follows: We prove gradient estimates for the equation (1.5) in Section 2 and gradient estimates for the equation (1.6) in Section 3. We derive related Harnack inequalities in Section 4. As special case, we deduce gradient estimates and Harnack inequality to the heat equation in section 5. Detailed calculation of some specific functions \( \alpha(t) \), \( \varphi(t) \) and \( \gamma(t) \) are given in section 6.

2. Gradient estimates for the equation (1.5)

In this section, we will derive some new gradient estimates for positive solutions to equation (1.5) under the Ricci flow.

2.1. Main results.

We state our results as follows.

**Theorem 2.1.** Let \( (M^n, g(t))_{t \in [0,T]} \) be a complete solution to the Ricci flow (1.7). Assume that \(|\text{Ric}(x, t)| \leq K \) for some \( K > 0 \) and all \( t \in [0,T] \). Suppose that there exist three functions \( \alpha(t) \), \( \varphi(t) \) and \( \gamma(t) \) satisfy conditions (C1), (C2), (C3) and (C4).

Given \( x_0 \in M^n \) and \( R > 0 \), let \( u \) be a positive solution of the equation (1.5) in the cube \( B_{2R,T} := \{(x, t)|d(x, x_0, t) \leq 2R, 0 \leq t \leq T\} \). Let \( h(x, t) \) be a function defined on \( M^n \times [0,T] \) which is \( C^1 \) in \( t \) and \( C^2 \) in \( x \), satisfying \(|\nabla h|^2 \leq \delta_2 h \) and \( \Delta h \geq -\delta_3 \) on \( B_{2R,T} \) for some positive constants \( \delta_2 \) and \( \delta_3 \).

1. If \( \frac{2\alpha^4}{\alpha^4} \leq C_1 \) for some constant \( C_1 \), then
\[
\frac{|\nabla u|^2}{u^2} - \frac{u}{u} + \alpha h(x, t)u^{l-1} \leq C\alpha^2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^2\alpha^2 K
\]
\[
+ \alpha\sqrt{n\mu_1}\delta_3 + n\alpha^2\mu_1\delta_1 + \sqrt{\frac{2-l}{2}}\alpha^2\sqrt{n\mu_1}\delta_2 + \alpha\varphi.
\]

If \( \frac{\gamma}{\alpha^4} \leq C_2 \) for some constant \( C_2 \), then
\[
\frac{|\nabla u|^2}{u^2} - \frac{u}{u} + \alpha h(x, t)u^{l-1} \leq C\alpha^2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^2\alpha^2 K
\]

If \( \frac{\gamma}{\alpha^4} \leq C_2 \) for some constant \( C_2 \), then
\[
\frac{|\nabla u|^2}{u^2} - \frac{u}{u} + \alpha h(x, t)u^{l-1} \leq C\alpha^2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^2\alpha^2 K
\]

If \( \frac{\gamma}{\alpha^4} \leq C_2 \) for some constant \( C_2 \), then
\[
\frac{|\nabla u|^2}{u^2} - \frac{u}{u} + \alpha h(x, t)u^{l-1} \leq C\alpha^2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^2\alpha^2 K
\]
\[ + \alpha \sqrt{n\overline{\pi}_1} \delta_3 + n\alpha^2 \overline{\pi}_1 \delta_1 + \sqrt{2 - l - \alpha^2 \overline{\pi}_1} \delta_2 + \alpha \varphi, \]

where \( C \) is a positive constant depending only on \( n \) and set
\[ \overline{\pi}_1 := \max_{B_{2R,T}} u^{l-1}, \quad \delta_1 := \max_{B_{2R,T}} h(x,t). \]

(2) \( l > 1 \). If \( \frac{\alpha^4}{\alpha - 1} \leq C_1 \) for some constant \( C_1 \), then
\[
\frac{\nabla u^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leq C\alpha^2 \left( \frac{1}{R^2} + \sqrt{K} + K \right) + Cn^2 \alpha^4 \frac{R^2}{R^{2\gamma}} + n^2 \alpha^2 K + n\alpha^2 (l - 1) \delta_1 \overline{\pi}_2 \\
\quad + \alpha \sqrt{\frac{n(l\alpha - 1)\overline{\pi}_2 \delta_2}{l-1}} + \alpha^2 \sqrt{n(l - 1)\delta_1 \varphi + \alpha^2 \sqrt{n\overline{\delta}_3 \overline{\pi}_2}} + \alpha \varphi.
\]

If \( \frac{\alpha^4}{\alpha - 1} \leq C_2 \) for some constant \( C_2 \), then
\[
\frac{\nabla u^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leq C\alpha^2 \left( \frac{1}{R^2} + \sqrt{K} + K \right) + Cn^2 \alpha^4 \frac{R^2}{R^{2\gamma}} + n^2 \alpha^2 K + n\alpha^2 (l - 1) \delta_1 \overline{\pi}_2 \\
\quad + \alpha \sqrt{\frac{n(l\alpha - 1)\overline{\pi}_2 \delta_2}{l-1}} + \alpha^2 \sqrt{n(l - 1)\delta_1 \varphi + \alpha^2 \sqrt{n\overline{\delta}_3 \overline{\pi}_2}} + \alpha \varphi,
\]

where \( C \) is a positive constant depending only on \( n \) and set
\[ \overline{\pi}_2 := \max_{B_{2R,T}} u^{l-1}, \quad \delta_1 := \max_{B_{2R,T}} h(x,t). \]

Let us list some examples to illustrate the Theorem 2.1 holds for different circumstances and see appendix in section 6 for detailed calculation process.

**Corollary 2.1.** Suppose that \((M^n, g(t))_{t \in [0,T]}\) satisfies the hypotheses of Theorem 2.1. Then the following special estimates are valid.

1. **Li-Yau type:**
   \[
   \alpha(t) = \text{constant}, \quad \varphi(t) = \frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha - 1}, \quad \gamma(t) = t^\theta \quad \text{with} \quad 0 < \theta \leq 2.
   \]
   If \( l \leq 1 \), then
   \[
   \frac{\nabla u^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
   \leq C\alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{K} R) + \frac{\alpha^2}{\alpha - 1} \frac{1}{R^2} + K \right] + \alpha \varphi \\
   + n^2 \alpha^2 K + n\alpha^2 \overline{\delta}_3 + n\alpha^2 \overline{\pi}_1 \delta_1 + \sqrt{\frac{2 - l}{2} \alpha^2 \overline{n\overline{\pi}_1} \overline{\delta}_2}.
   \]
   If \( l > 1 \), then
   \[
   \frac{\nabla u^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
   \leq C\alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{K} R) + \frac{\alpha^2}{\alpha - 1} \frac{1}{R^2} + K \right] + \alpha \varphi
   \]
\[ + n^{\frac{3}{2}} \alpha^2 K + \alpha \sqrt{n(l-1) \delta_2} + \alpha \sqrt{n(l-1) \delta_2 \varphi} + \alpha^2 \sqrt{n\delta_3 \varphi}. \]

2. Hamilton type:
\[ \alpha(t) = e^{2Kt}, \quad \varphi(t) = \frac{n}{t} e^{4Kt}, \quad \gamma(t) = te^{2Kt}. \]

If \( l \leq 1 \), then
\[ \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x, t)u^{l-1} \]
\[ \leq C \alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{KR} + K) \right] + \frac{C}{R^2 t} e^{2Kt} + \alpha \varphi \]
\[ + n^{\frac{3}{2}} \alpha^2 K + \alpha \sqrt{n\delta_3} + \alpha^2 \delta_1 + \sqrt{2 - l} \frac{2 - l}{2} \alpha \sqrt{n\delta_2}. \]

If \( l > 1 \), then
\[ \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x, t)u^{l-1} \]
\[ \leq C \alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{KR} + K) \right] + \frac{C}{R^2 t} \tanh(Kt) + \alpha \varphi \]
\[ + n^{\frac{3}{2}} \alpha^2 K + \alpha \sqrt{n\delta_3} + \alpha^2 \delta_1 + \sqrt{2 - l} \frac{2 - l}{2} \alpha \sqrt{n\delta_2}. \]

3. Li-Xu type:
\[ \alpha(t) = 1 + \frac{\sinh(Kt) \cosh(Kt) - Kt}{\sinh^2(Kt)}, \quad \varphi(t) = 2nK[1 + \coth(Kt)], \]
\[ \gamma(t) = \tanh(Kt). \]

If \( l \leq 1 \), then
\[ \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x, t)u^{l-1} \]
\[ \leq C \alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{KR} + K) \right] + \frac{C}{R^2 \tanh(Kt)} + \alpha \varphi \]
\[ + n^{\frac{3}{2}} \alpha^2 K + \alpha \sqrt{n\delta_3} + \alpha^2 \delta_1 + \sqrt{2 - l} \frac{2 - l}{2} \alpha \sqrt{n\delta_2}. \]

If \( l > 1 \), then
\[ \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x, t)u^{l-1} \]
\[ \leq C \alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{KR} + K) \right] + \frac{C}{R^2 \tanh(Kt)} + \alpha \varphi \]
\[ + n^{\frac{3}{2}} \alpha^2 K + \alpha \sqrt{n\delta_3} + \alpha^2 \delta_1 + \sqrt{2 - l} \frac{2 - l}{2} \alpha \sqrt{n\delta_2}. \]

where \( \alpha(t) \) is bounded uniformly.
4. Linear Li-Xu type:

\[ \alpha(t) = 1 + 2Kt, \varphi(t) = \frac{n}{t} + nK(1 + 2Kt + \mu Kt), \gamma(t) = Kt \quad \text{with} \quad \mu \geq \frac{1}{4}. \]

If \( l \leq 1 \), then

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leq C \alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{K}R) + K \right] + \frac{C \alpha^4}{R^2 K t} + \alpha \varphi \\
+ n^{2} \alpha^2 K + \alpha \sqrt{n \bar{u}_1 \delta_3} + n \alpha^2 \bar{u}_1 \delta_1 + \sqrt{\frac{2-l}{2}} \alpha \sqrt{n \bar{u}_1 \delta_2}. 
\]

If \( l > 1 \), then

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leq C \alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{K}R) + K \right] + \frac{C \alpha^4}{R^2 K t} + \alpha \varphi \\
+ n^{2} \alpha^2 K + n \alpha^2 (l-1) \delta_1 \bar{u}_2 + \alpha \sqrt{\frac{n(l-1) \bar{u}_2 \delta_2}{l-1}} \\
+ \alpha^{\frac{2}{l}} \sqrt{n/l} \bar{u}_2 \delta_1 + \alpha^{\frac{2}{l}} \sqrt{n \bar{u}_3 \delta_2}. 
\]

Remark 2.1. The above results can be regard as generalizing the gradient estimates of Li-Yau [14], J. Y. Li [15], Hamilton [8] and Li-Xu [17] to the Ricci flow. Our results also generalize the estimates of S. P. Liu [18] and J. Sun [26] to the nonlinear parabolic equation under the Ricci flow.

The local estimates in Theorem 2.1 imply global estimates.

Corollary 2.2. Let \((M^n, g(t))_{t \in [0,T]}\) be a complete solution to the Ricci flow (1.7). Assume that \( |\text{Ric}(x,t)| \leq K \) for some \( K > 0 \) and all \( (x,t) \in M^n \times [0,T] \). Let \( u(x,t) \) be a positive solution to equation (1.5) on \( M^n \times [0,T] \). Let \( h(x,t) \) be a function defined on \( M^n \times [0,T] \) which is \( C^1 \) in \( t \) and \( C^2 \) in \( x \), satisfying \( |\nabla h|^2 \leq \delta_2 h \) and \( \Delta h \geq -\delta_3 \) on \( M^n \times [0,T] \) for some positive constants \( \delta_2 \) and \( \delta_3 \).

If \( l \leq 1 \) and for \( (x,t) \in M^n \times (0,T) \), then

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leq \alpha \varphi + C \alpha \left[ \alpha K + \sqrt{n \bar{u}_1 \delta_3} + \alpha \bar{u}_1 \delta_1 + \sqrt{\frac{2-l}{2}} \sqrt{n \bar{u}_1 \delta_2} \right], 
\]

where where \( C \) is a positive constant depending only on \( n \) and set

\[ \bar{u}_1 := \max_{M^n \times [0,T]} u^{l-1}, \quad \delta_1 := \max_{M^n \times [0,T]} h(x,t). \]

If \( l > 1 \) and for \( (x,t) \in M^n \times (0,T) \), then

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \leq \alpha \varphi \\
\leq C \alpha \left[ \alpha K + (l-1) \alpha \bar{u}_2 \delta_1 + \sqrt{\frac{(l-1) \bar{u}_2 \delta_2}{l-1}} + \alpha^{\frac{2}{l}} \sqrt{(l-1) \bar{u}_1 \delta_1} + \alpha^{\frac{2}{l}} \sqrt{n \bar{u}_3 \delta_2} \right], 
\]
where where $C$ is a positive constant depending only on $n$ and set

$$\bar{\omega}_1 := \max_{M^n \times [0,T]} u^{l-1}, \quad \delta_1 := \max_{M^n \times [0,T]} h(x,t).$$

We can derive a gradient estimate for any positive solution to the following nonlinear parabolic equation under the Ricci flow on a closed manifold without any curvature conditions. The method of the proof is inspired by Hamilton [10], Shi [23] and Liu [18].

**Theorem 2.2.** Let $(M^n, g(x,t))_{t \in [0,T]}$ be a solution to the Ricci flow (1.7) on a closed manifold. If $u$ is a positive solution to equation

$$\partial_t u = \Delta u + h(t)u^{l-1},$$

where $h(t)$ is a $C^1$ function and $h(t) \leq 0$. Then for $l \geq 1$, we have

$$|\nabla u(x,t)|^2 \leq \frac{1}{2t} \left( \max_{x \in M^n} u^2(x,0) - u^2(x,t) \right) \text{ for } (x,t) \in M^n \times [0,T]. \quad (2.1)$$

### 2.2. Auxiliary lemma.

To prove main results, we need a lemma. Let $f = \ln u$. Then

$$f_t = \Delta f + |\nabla f|^2 + hu^{l-1}. \quad (2.2)$$

Let $F = |\nabla f|^2 - \alpha f_t + \alpha hu^{l-1} - \alpha \varphi$, where $\alpha = \alpha(t) > 1$ and $\varphi = \varphi(t) > 0$.

**Lemma 2.1.** Suppose that $(M^n, g(t))_{t \in [0,T]}$ satisfies the hypotheses of Theorem 2.1. We also assume that $\alpha(t) > 1$ and $\varphi(t) > 0$ satisfy the following system

$$\begin{cases}
\frac{2\varphi}{n} - 2\alpha K \geq \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha}, \\
\frac{2\varphi}{n} - \alpha' > 0, \\
\varphi^2 + \alpha \varphi' \geq 0,
\end{cases} \quad (2.3)$$

and $\alpha(t)$ is non-decreasing. Then

$$(\Delta - \partial_t)F \geq |f_{ij} + \frac{\varphi}{n} g_{ij}|^2 + \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} F - \alpha^2 n^2 K^2 - 2|\nabla f| \nabla F + 2c(\alpha - 1)(l-1)u^{l-1}|\nabla f|^2 + \alpha(l-1)^2 hu^{l-1} |\nabla f|^2 + \alpha(l-1)hu^{l-1} \Delta f + \alpha u^{l-1} \Delta h + 2(\alpha - 1)u^{l-1} \Delta \nabla f \cdot \nabla h. \quad (2.4)$$

**Proof.** By directly computing, we have

$$\begin{aligned}
\Delta F &= \Delta |\nabla f|^2 - \alpha \Delta (f_t) + \alpha \Delta (hu^{l-1}) \\
&= 2|f_{ij}|^2 + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha \Delta (f_t) + \alpha h \Delta (u^{l-1}) \\
&\quad + \alpha u^{l-1} \Delta h + 2\alpha \nabla h \nabla u^{l-1} \\
&= 2 \left( |f_{ij}|^2 + \alpha R_{ij} f_j \right) + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha (\Delta f)_t \\
&\quad + \alpha h \Delta (u^{l-1}) + \alpha u^{l-1} \Delta h + 2\alpha \nabla h \nabla u^{l-1},
\end{aligned}$$

where we have used Bochner’s formula and

$$\Delta (f_t) = (\Delta f)_t - 2 \sum_{i,j=1}^n R_{ij} f_{ij}.$$
Applying Young’s inequality

\[ R_{ij}f_{ij} \leq |R_{ij}||f_{ij}| \leq \frac{\alpha}{2}|R_{ij}|^2 + \frac{1}{2\alpha}|f_{ij}|^2, \]

we conclude for \(|R_{ij}| \leq K\),

\[ \Delta F \geq |f_{ij}|^2 - \sum \alpha^2|R_{ij}|^2 + 2f_jf_{ij} + 2R_{ij}f_if_j - \alpha(\Delta f)_t + \alpha h\Delta(u^{-1}) + \alpha u^{-1}\Delta h + 2\alpha\nabla h\nabla u^{-1} \]

\[ \geq |f_{ij}|^2 - \alpha^2n^2K^2 + 2f_jf_{ij} + 2R_{ij}f_if_j - \alpha(\Delta f)_t + \alpha h\Delta(u^{-1}) + \alpha u^{-1}\Delta h + 2\alpha\nabla h\nabla u^{-1}. \] (2.5)

On the other hand, we infer

\[ \partial_tF = (|\nabla f|^2)_t - \alpha f_{tt} - \alpha'f_t + \alpha'hu^{-1} + ah(u^{-1})_t \]

\[ + \alpha u^{-1}h_t - \alpha\varphi' - \alpha\varphi + 2\nabla f\nabla(f_t) + 2R_{ij}f_if_j - \alpha f_{tt} - \alpha'f_t + \alpha'hu^{-1} + \alpha u^{-1}h_t \]

\[ + \alpha h(u^{-1})_t - \alpha\varphi' - \alpha\varphi. \] (2.6)

We follow from (2.5) and (2.6),

\[ (\Delta - \partial_t)F \geq |f_{ij}|^2 - \alpha^2n^2K^2 + 2\nabla f\nabla(\Delta f) - \alpha(\Delta f)_t + \alpha h\Delta(u^{-1}) + \alpha u^{-1}\Delta h + 2\alpha\nabla h\nabla u^{-1} - 2\nabla f\nabla(f_t) + \alpha f_{tt} + \alpha'f_t + \alpha'hu^{-1} - \alpha(u^{-1})_t + \alpha u^{-1}h_t + \alpha\varphi' + \alpha\varphi \]

\[ = |f_{ij}|^2 - \alpha^2n^2K^2 + 2\nabla f\nabla(\Delta f) + \alpha(\nabla f)^2 - hu^{-1})_t + \alpha h\Delta(u^{-1}) + \alpha u^{-1}\Delta h \]

\[ + 2\alpha\nabla h\nabla u^{-1} - 2\nabla f\nabla(f_t) + \alpha f_{tt} + \alpha'f_t + \alpha'hu^{-1} - \alpha(u^{-1})_t - \alpha h\Delta(u^{-1}) + \alpha\varphi' + \alpha\varphi \]

By using the formula

\[ (|\nabla f|^2)_t = 2\nabla f \cdot \nabla(f_t) + 2\text{Ric}(\nabla f, \nabla f), \]

we obtain

\[ (\Delta - \partial_t)F \geq |f_{ij}|^2 - \alpha^2n^2K^2 + 2\nabla f\nabla(\Delta f) + 2\alpha\nabla f\nabla(f_t) + 2(\alpha - 1)|\nabla f|^2 + \alpha h\Delta(u^{-1}) + \alpha u^{-1}\Delta h \]

\[ + 2\alpha\nabla h\nabla u^{-1} + \alpha f_{tt} - \alpha'hu^{-1} + \alpha\varphi' + \alpha\varphi. \] (2.7)

Applying the following two equations

\[ \nabla(u^{-1}) = (l - 1)u^{-1}\nabla f, \]

\[ \Delta(u^{-1}) = (l - 1)^2u^{-1}|\nabla f|^2 + (l - 1)u^{-1}\Delta f, \]

to (2.7), we have

\[ (\Delta - \partial_t)F \geq |f_{ij}|^2 + 2\alpha R_{ij}f_if_j - \alpha^2n^2K^2 + 2\nabla f\nabla F + \alpha u^{-1}\Delta h \]
Therefore, (2.4) is derived from (2.3) and (2.9). The proof is complete.

Further applying unit matrix \((\delta_{ij})_{n \times n}\) and (2.8), we derive

\[
(\Delta - \partial_t) F \geq \ |f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 - 2\alpha K|\nabla f|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla F
+ 2h(\alpha - 1)(l - 1)u^{l-1}|\nabla f|^2 + 2[(\alpha - 1) + \alpha(l - 1)]u^{l-1}\nabla f \cdot \nabla h
+ h\alpha(1 - 1)(l - 1)^2u^{l-1}T f - \alpha u^{l-1} \Delta f + \alpha u^{l-1} \Delta h
+ \alpha' f_t - \alpha' cu^{l-1} + \alpha' \varphi' + \alpha' \varphi - \frac{\varphi^2}{n} - 2\frac{\varphi}{n} \Delta f.
\]  

Applying (2.2), we have

\[
(\Delta - \partial_t) F \geq \ |f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 + 2\frac{\varphi}{n} - 2\alpha K|\nabla f|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla F
+ 2h(\alpha - 1)(l - 1)u^{l-1}|\nabla f|^2 + 2[(\alpha - 1) + \alpha(l - 1)]u^{l-1}\nabla f \cdot \nabla h
+ h\alpha(1 - 1)(l - 1)^2u^{l-1}T f - \alpha u^{l-1} \Delta f + \alpha u^{l-1} \Delta h
+ \alpha' f_t - \alpha' cu^{l-1} + \alpha' \varphi' + \alpha' \varphi - \frac{\varphi^2}{n} + 2\frac{\varphi}{n} - \alpha \frac{\alpha' \varphi}{\alpha}.
\]  

(2.9)

Therefore, (2.4) is derived from (2.3) and (2.9). The proof is complete. \(\square\)

2.3. Proof of Theorem 2.1 and 2.2.

In this section, we will prove the Theorem 2.1 and 2.2.

**Proof of Theorem 2.1.** Let \(G = \gamma(t)F\) and \(\gamma(t) > 0\) be non-decreasing. Then

\[
(\Delta - \partial_t) G = \gamma(\Delta - \partial_t) F - \gamma' F
\]

\[
\geq \gamma |f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 + 2\frac{\varphi}{n} - \alpha' \frac{1}{\alpha} G - \alpha^2 n^2 K^2 - 2\nabla f \nabla G
+ 2h\gamma(\alpha - 1)(l - 1)u^{l-1}|\nabla f|^2 + 2[\alpha(1 - 1)u^{l-1}\nabla f \cdot \nabla h
+ h\gamma(1 - 1)(l - 1)^2u^{l-1}T f - \alpha u^{l-1} \Delta f + \alpha u^{l-1} \Delta h - \gamma' F
= \gamma |f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 + 2\frac{\varphi}{n} - \alpha' \frac{1}{\alpha} - \frac{\gamma}{\gamma} G - \alpha^2 n^2 K^2 - 2\nabla f \nabla G
+ 2h\gamma(\alpha - 1)(l - 1)u^{l-1}|\nabla f|^2 + 2[\alpha(1 - 1)u^{l-1}\nabla f \cdot \nabla h
+ \gamma(1 - 1)(l - 1)^2h^2u^{l-1}T f + \gamma(1 - 1)hu^{l-1} \Delta f + \alpha u^{l-1} \Delta h. \quad (2.10)
\]

Now let \(\varphi(r)\) be a \(C^2\) function on \([0, \infty)\) such that

\[
\varphi(r) = \begin{cases} 
1 & \text{if } r \in [0, 1], \\
0 & \text{if } r \in [2, \infty),
\end{cases}
\]

and

\[
0 \leq \varphi(r) \leq 1, \quad \varphi'(r) \leq 0, \quad \varphi''(r) \leq 0, \quad \frac{|\varphi'(r)|}{\varphi(r)} \leq C,
\]

where \(C\) is a constant. Then

\[
(\Delta - \partial_t) \varphi(\Omega) = \gamma(\Delta - \partial_t) \varphi(\Omega) - \gamma' \varphi(\Omega)
\]

\[
\geq \gamma |f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 + 2\frac{\varphi}{n} - \alpha' \frac{1}{\alpha} \varphi(\Omega) - \alpha^2 n^2 K^2 - 2\nabla f \nabla \varphi(\Omega)
+ 2h\gamma(\alpha - 1)(l - 1)u^{l-1}|\nabla f|^2 + 2[\alpha(1 - 1)u^{l-1}\nabla f \cdot \nabla \varphi(\Omega)
+ \gamma(1 - 1)(l - 1)^2h^2u^{l-1}T f + \gamma(1 - 1)hu^{l-1} \Delta f + \alpha u^{l-1} \Delta \varphi(\Omega).
\]

Therefore, (2.5) is derived from (2.4). The proof is complete. \(\square\)
where $C$ is an absolute constant. Let define by

$$\phi(x, t) = \phi(d(x, x_0, t)) = \phi \left( \frac{d(x, x_0, t)}{R} \right) = \phi \left( \frac{\rho(x, t)}{R} \right),$$

where $\rho(x, t) = d(x, x_0, t)$. By using maximum principle, the argument of Calabi [2] allows us to suppose that the function $\phi(x, t)$ with support in $B_{2R,T}$, is $C^2$ at the maximum point. By utilizing the Laplacian theorem, we deduce that

$$\frac{\|\nabla \phi\|^2}{\phi} \leq \frac{C}{R^2}, \quad -\Delta \phi \leq \frac{C}{R^2} (1 + \sqrt{K} R), \quad (2.11)$$

For any $0 \leq T_1 \leq T$, let $H = \phi G$ and $(x_1, t_1)$ be the point in $B_{2R,T_1}$ at which $H$ attains its maximum value. We can suppose that the value is positive, because otherwise the proof is trivial. Then at the point $(x_1, t_1)$, we infer

$$0 = \nabla (\phi G) = G \nabla \phi + \phi \nabla G, \quad \Delta (\phi G) \leq 0, \quad \partial_t (\phi G) \geq 0. \quad (2.12)$$

By the evolution formula of the geodesic length under the Ricci flow [6], we calculate

$$\phi_t G = -G \phi' \left( \frac{\rho}{R} \right) \frac{1}{R} \frac{d\rho}{dt} = G \phi' \left( \frac{\rho}{R} \right) \int_{\gamma_{t_1}} \text{Ric}(S, S) ds \leq G \phi' \left( \frac{\rho}{R} \right) K \rho \leq G \phi' \left( \frac{\rho}{R} \right) K_2 \leq G \sqrt{C} K,$$

where $\gamma_{t_1}$ is the geodesic connecting $x$ and $x_0$ under the metric $g(t_1)$, $S$ is the unite tangent vector to $\gamma_{t_1}$, and $ds$ is the element of the arc length.

All the following computations are at the point $(x_1, t_1)$. It is not difficult to find that

$$|f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 \geq \frac{1}{n} \left( \text{tr} |f_{ij} + \frac{\varphi}{n} \delta_{ij}| \right)^2 = \frac{1}{n} \left( \Delta f + \varphi \right)$$

$$= \frac{1}{n} \left[ -\frac{1}{\alpha} F - \frac{1}{\alpha} (\alpha - 1) |\nabla f|^2 \right] = \frac{1}{\alpha^2 n} \left[ G - (\alpha - 1) |\nabla f|^2 \right]. \quad (2.13)$$

and

$$\Delta f = f_t - |\nabla f|^2 - \alpha^{l-1} \leq - \frac{F}{\alpha} - \frac{\alpha - 1}{\alpha} |\nabla f|^2 - \varphi < 0. \quad (2.14)$$

To obtain main results, two cases will be shown.

**Case 1**  \(l \leq 1\).

From (2.14), we have $\Delta f \leq 0$. Then by substituting it into (2.10), we obtain

$$(\Delta - \partial_t) G = \gamma (\Delta - \partial_t) F - \gamma' F \geq \gamma |f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 + \left[ \frac{2\varphi}{n} - \alpha' \frac{1}{\alpha} - \gamma' \gamma \right] G - \gamma \alpha^2 n^2 K^2 - 2\nabla f \nabla G$$

$$+ 2h \gamma (\alpha - 1) (l - 1) u^{l-1} |\nabla f|^2 + \alpha \gamma u^{l-1} \Delta h$$
Multiply $\phi$ to inequality (2.15), we have
\[
0 \geq \phi G \left[ \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} + \left( \frac{2 \phi}{\alpha} - \alpha' \frac{\phi}{\alpha} - \frac{\gamma'}{\gamma} \phi \right) \phi + \frac{\phi^2 G^2}{\alpha^2 n \gamma} \right] (1) + (a - 1) |\nabla f|^2 + 2 \phi G \nabla f \cdot \nabla h
\]
\[
= G \left( \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} + \phi (\Delta - \partial_t) \alpha G - G \phi_t \right) + \phi G \left[ G \alpha^2 n \gamma \right] (1) + (a - 1) |\nabla f|^2 \right]^2
\]
\[
+ \left[ \frac{2 \phi}{2} - \alpha' \right] \frac{1}{\alpha - \gamma} \phi G - \gamma \phi \alpha^2 n^2 K^2 - 2 \phi \nabla f \cdot \nabla G
\]
\[
+ 2 \gamma \phi (a - 1) (l - 1) u^{l - 1} |\nabla f|^2 + \phi \alpha \gamma u^{l - 1} \Delta h
\]
\[
+ 2 [(a - 1) + \alpha (1 - 1)] \phi \gamma u^{l - 1} \nabla f \cdot \nabla h - G^2 \nabla h.
\]
Using the Cauchy inequality
\[
|\nabla f \cdot \nabla h| \geq - |\nabla f| |\nabla h| \geq - h |\nabla f|^2 - \frac{|\nabla h|^2}{4 h},
\]
\[
|\nabla f \cdot \nabla h| \leq h |\nabla f|^2 + \frac{|\nabla h|^2}{4 h},
\]
we conclude
\[
0 \geq \phi G \left[ \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} + \left( \frac{2 \phi}{\alpha} - \alpha' \frac{\phi}{\alpha} - \frac{\gamma'}{\gamma} \phi \right) \phi + \frac{\phi^2 G^2}{\alpha^2 n \gamma} + \frac{\phi^2 (a - 1)^2 \gamma}{\alpha^2 n^2 \gamma} |\nabla f|^4 \right]
\]
\[
+ \frac{2 \phi^2 (a - 1)}{\alpha n^2} - \gamma \phi \alpha^2 n^2 K^2 + 2 \phi G \nabla f \cdot \nabla f
\]
\[
- 2 \phi \gamma^2 \gamma (a - 1) (1 - l) u^{l - 1} |\nabla f|^2 - 2 [(a - 1) + \alpha (1 - 1)] \phi^2 \gamma u^{l - 1} h |\nabla f|^2
\]
\[
+ \frac{1}{2} [(a - 1) + \alpha (1 - 1)] G \nabla h^2 \nabla h + \phi \alpha \gamma u^{l - 1} \Delta h - \phi G \nabla h
\]
\[
\geq \phi G \left[ \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} + \left( \frac{2 \phi}{\alpha} - \alpha' \frac{\phi}{\alpha} - \frac{\gamma'}{\gamma} \phi \right) \phi + \frac{\phi^2 G^2}{\alpha^2 n \gamma} + \frac{\phi^2 (a - 1)^2 \gamma}{\alpha^2 n^2 \gamma} |\nabla f|^4 \right]
\]
\[
+ \frac{2 \phi^2 (a - 1)}{\alpha n^2} - \gamma \phi \alpha^2 n^2 K^2 + 2 \phi G \nabla f \cdot \nabla f
\]
\[
- 2 \phi \gamma^2 \gamma (a - 1) (1 - l) u^{l - 1} |\nabla f|^2 - 2 [(a - 1) + \alpha (1 - 1)] \phi^2 \gamma u^{l - 1} h |\nabla f|^2
\]
\[
+ \frac{1}{2} [(a - 1) + \alpha (1 - 1)] G \nabla h^2 \nabla h + \phi \alpha \gamma u^{l - 1} \Delta h - \phi G \nabla h.
\]
+ \phi^2 \alpha \gamma u^{l-1} \Delta h - \phi G \sqrt{CK}, \quad \text{(2.16)}

where we use the fact that \((\alpha - 1)(l - 1) + (\alpha - 1) + \alpha(1 - l) \leq \alpha(3 - 2l) - 1\). Further using the inequality \(Ax^2 + Bx \geq -\frac{B^2}{4A}\) with \(A > 0\), we have

\[
\frac{2\phi^2(\alpha - 1)}{n\alpha^2}G|\nabla f|^2 + 2\phi G \nabla \phi \nabla f \geq -\frac{n\alpha^2}{2(\alpha - 1)} \frac{\nabla \phi|^2}{\phi} G,
\]

and

\[
\frac{\phi^2(\alpha - 1)^2}{n\alpha^2} |\nabla f|^4 - 2 \alpha(3 - 2\alpha) - 1 \phi^2 \gamma u^{l-1} h |\nabla f|^2 \\
\geq - \frac{n\alpha^2}{(\alpha - 1)^2} \gamma \phi^2 u^{2(l-1)} h^2 \\
\geq - \frac{n\alpha^2}{(\alpha - 1)^2} \gamma \phi^2 \delta_1^2.
\]

Substituting above two inequalities into (2.16), we deduce that

\[
0 \geq \phi G \left[ \Delta \phi - 2 \frac{\nabla \phi|^2}{\phi} + \left( \frac{2\phi}{n} - \alpha' \right) \phi \frac{\phi'}{\alpha} - \frac{\nabla \phi|^2}{\phi} \right] - \phi G + \frac{\phi^2 G^2}{\alpha \gamma} - \gamma \phi \alpha^2 n^2 K^2 - \frac{n\alpha^2 \gamma}{(\alpha - 1)^2} \phi \alpha^2 \gamma^2 K^2 \\
- \frac{n\alpha^2}{(\alpha - 1)^2} \gamma \phi^2 \delta_1^2 \\
- \frac{1}{2} [(\alpha - 1) + \alpha(1 - l)] \phi^2 \gamma \delta_2 - \phi^2 \alpha \gamma \delta_3. \quad \text{(2.17)}
\]

Applying (2.11), we infer

\[
0 \geq \begin{cases} 
\frac{C}{R^2} (1 + \sqrt{K} R) - \frac{2C}{R^2} + \left( \frac{2\phi}{n} - \alpha' \right) \phi \frac{\phi'}{\alpha} - \frac{\nabla \phi|^2}{\phi} \\
- \frac{n\alpha^2}{2(\alpha - 1)} \frac{C}{R^2} - \sqrt{CK} \right] \phi G + \frac{\phi^2 G^2}{\alpha \gamma} - \gamma \phi \alpha^2 n^2 K^2 \\
- \frac{\alpha^2 n^2}{(\alpha - 1)^2} \phi \alpha^2 \gamma^2 K^2 \\
- \frac{1}{2} [(\alpha - 1) + \alpha(1 - l)] \phi^2 \gamma \delta_2 - \phi^2 \alpha \gamma \delta_3. 
\end{cases}
\]

For the inequality \(Ax^2 + Bx \leq C\), one has \(x \leq \frac{B}{A} + (\frac{C}{A})^{\frac{1}{2}}\), where \(A, B, C > 0\). By using this inequality to (2.17) and then we arrive at

\[
\phi G(x, t_1) \leq (\phi G)(x_1, t_1)
\]

\[
\leq \begin{cases} 
\left\{ n\alpha^2 \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + \frac{n\alpha^2}{2(\alpha - 1)} \frac{C}{R^2} + \sqrt{CK} \right] \\
+ n\gamma^2 \left[ \frac{\gamma'}{\gamma} - \frac{2\phi}{n} - \alpha' \frac{1}{\alpha} \right] + n^2 \gamma \alpha^2 \phi K \\
+ \frac{n\alpha^2}{\alpha - 1} \phi \alpha^2 \gamma \delta_1 + \alpha \phi \gamma \sqrt{n \delta_3} \\
+ \sqrt{\left[ \alpha^2 \gamma \delta_2 \right]}(x_1, t_1). 
\end{cases}
\]
If $\gamma$ is nondecreasing which satisfies the system
\[
\begin{cases}
\frac{\gamma'}{\gamma} - \left(\frac{2 \varphi}{n} - \alpha'\right) \frac{1}{\alpha} \leq 0, \\
\frac{\gamma\alpha^4}{\alpha - 1} \leq C.
\end{cases}
\] (2.18)

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1)
\leq n\gamma(T_1)\alpha^2(T_1) \left[ \frac{C}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{n^2C}{R^2}
+ n^2\gamma(T_1)\alpha^2(T_1)\phi K + \phi\alpha(T_1)\gamma(T_1)\sqrt{n\pi_1\delta_3}
+ \frac{n\alpha^2(T_1)[\alpha(T_1)(3 - 2\alpha(T_1)) - 1]}{\alpha(T_1) - 1} \phi\gamma(T_1)\pi_1\delta_1
+ \sqrt{\left((\alpha(T_1) - 1) + \alpha(T_1)(1 - l)\right)\alpha(T_1)\phi\gamma(T_1)\sqrt{n\pi_1\delta_2}}.
\]

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,
\[
F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2} \left( 1 + \sqrt{KR} \right) + CK \right] + \frac{n^2C}{R^2\gamma(T_1)}
+ n^2\alpha^2(T_1)K + \alpha(T_1)\gamma(T_1)\sqrt{n\pi_1\delta_3}
+ \frac{n\alpha^2(T_1)[\alpha(T_1)(3 - 2\alpha(T_1)) - 1]}{\alpha(T_1) - 1} \phi\gamma(T_1)\pi_1\delta_1
+ \sqrt{\left((\alpha(T_1) - 1) + \alpha(T_1)(1 - l)\right)\alpha(T_1)\phi\gamma(T_1)\sqrt{n\pi_1\delta_2}}.
\]

If $\gamma$ is nondecreasing which satisfies the system
\[
\begin{cases}
\frac{\gamma'}{\gamma} - \left(\frac{2 \varphi}{n} - \alpha'\right) \frac{1}{\alpha} \leq 0, \\
\frac{\gamma\alpha^4}{\alpha - 1} \leq C.
\end{cases}
\] (2.19)

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1)
\leq n\gamma(T_1)\alpha^2(T_1) \left[ \frac{C}{R^2} \left( 1 + \sqrt{KR} \right) + \frac{n\alpha^2}{\alpha - 1} \frac{C}{R^2} + CK \right]
+ n^2\gamma(T_1)\alpha^2(T_1)\phi K + \phi\alpha(T_1)\gamma(T_1)\sqrt{n\pi_1\delta_3}
+ \frac{n\alpha^2(T_1)[\alpha(T_1)(3 - 2\alpha(T_1)) - 1]}{\alpha(T_1) - 1} \phi\gamma(T_1)\pi_1\delta_1
+ \sqrt{\left((\alpha(T_1) - 1) + \alpha(T_1)(1 - l)\right)\alpha(T_1)\phi\gamma(T_1)\sqrt{n\pi_1\delta_2}}.
\]

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,
\[
F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{n^2C\alpha^4}{R^2\gamma(T)}
\]
Using (2.13), we infer

\[ -\frac{\phi}{\alpha^2 n^2} \geq \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G - \gamma \alpha^2 n^2 K^2 \]

\[ -2 \nabla f \nabla G + 2(\alpha - 1)\gamma u^{l-1} \nabla f \cdot \nabla h - h(l-1)u^{l-1} G - h\gamma(l-1)u^{l-1} \Delta h. \]

Because \( T_1 \) is arbitrary in \( 0 < T_1 < T \) and \( \alpha(3-2\alpha) - 1 \leq \alpha - 1 \) and \( \alpha - 1 + \alpha(l-1) \leq \alpha(2-l) \), thus the conclusion is valid.

**Case 2.** \( l > 1 \).

Substituting (2.14) into (2.10), we have

\[ (\Delta - \partial_t)G \geq \gamma |f_{ij} + \frac{\phi}{n} \delta_{ij}|^2 + \left[ \frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G - \gamma \alpha^2 n^2 K^2 \]

\[ - 2 \nabla f \nabla G + 2(\alpha - 1)\gamma u^{l-1} \nabla f \cdot \nabla h - h(l-1)u^{l-1} G - h\gamma(l-1)u^{l-1} \Delta h. \]

Using (2.13), we infer

\[ 0 \geq (\Delta - \partial_t)(\phi G) \]

\[ = G \left( \Delta \phi - 2 \frac{\nabla \phi}{\phi} \right) + \phi (\Delta - \partial_t)G - \gamma G \phi \]

\[ \geq G \left( \Delta \phi - 2 \frac{\nabla \phi}{\phi} \right) + \frac{\phi \gamma}{\alpha^2 n^2} \left[ \frac{G}{\gamma} + (\alpha - 1)|\nabla f|^2 \right]^2 \]

\[ + \left[ \frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G - \gamma \phi \alpha^2 n^2 K^2 - 2\phi \nabla f \nabla G \]

\[ + 2(\alpha - 1)\delta \gamma u^{l-1} \nabla f \cdot \nabla h - h(l-1)u^{l-1} G - h\gamma(l-1)u^{l-1} \phi \alpha \phi \frac{\phi^2}{\alpha^2 n^2} \Delta h + h\phi \gamma(l-1)(l-1)u^{l-1} \phi |\nabla f|^2 - G \sqrt{C}. \]  

(2.20)

Multiply \( \phi \) to (2.20), we have

\[ 0 \geq \phi G \left( \Delta \phi - 2 \frac{\nabla \phi}{\phi} \right) + \left[ \frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G + (\alpha - 1)|\nabla f|^2 \]

\[ - \gamma \phi \alpha^2 n^2 K^2 - 2\phi \nabla \phi \nabla G + 2(\alpha - 1)\phi^2 \gamma u^{l-1} \nabla f \cdot \nabla h \]

\[ + \phi^2 \alpha \gamma u^{l-1} \Delta h - h(l-1)u^{l-1} \phi G - h\gamma(l-1)u^{l-1} \phi^2 \alpha \phi \]

\[ + \phi^2 \gamma(l-1)(l-1)u^{l-1} |\nabla f|^2 - \phi G \sqrt{C} K \]

\[ \geq \phi G \left( \Delta \phi - 2 \frac{\nabla \phi}{\phi} \right) + \left[ \frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G + \frac{\phi^2 G^2}{\alpha^2 n^2} \]

\[ + \frac{2\phi^2 (\alpha - 1)}{n \alpha^2} G |\nabla f|^2 - \gamma \phi^2 \alpha^2 n^2 K^2 + 2\phi G \nabla \phi \nabla f \]

\[ + 2(\alpha - 1)\phi^2 \gamma u^{l-1} \nabla f \cdot \nabla h + \phi^2 \alpha \gamma u^{l-1} \Delta h \]

\[ - h(l-1)u^{l-1} \phi G - h\gamma(l-1)u^{l-1} \phi^2 \alpha \phi \]

\[ + \phi^2 \gamma(l-1)(l-1)u^{l-1} |\nabla f|^2 - \phi G \sqrt{C} K, \]  

(2.21)

where we drop one term \( \frac{\phi^2 (\alpha - 1)^2}{n \alpha^2} |\nabla f|^4 \).
Further using the inequality \( Ax^2 + Bx \geq -\frac{B^2}{4A} \) with \( A > 0 \), we have 
\[
\frac{2\phi^2(\alpha - 1)}{n\alpha^2} G |\nabla f|^2 + 2\phi G \nabla \phi \nabla f \geq -\frac{n\alpha^2}{2(\alpha - 1)} \frac{\nabla \phi^2}{\phi} \phi G,
\]
and
\[
h\phi^2 \gamma (l - 1)(\alpha - 1)u^{l-1} |\nabla f|^2 + 2(\alpha - 1)\phi^2 \gamma u^{l-1} \nabla f \cdot \nabla h \\
\geq h\phi^2 \gamma (l - 1)(\alpha - 1)u^{l-1} |\nabla f|^2 - 2(\alpha - 1)\phi^2 \gamma u^{l-1} |\nabla f| |\nabla h| \\
\geq \frac{l\alpha - 1}{l - 1} \gamma \phi^2 u^{l-1} |\nabla h|^2 \frac{1}{h} \\
\geq -\frac{l\alpha - 1}{l - 1} \gamma \phi^2 \pi_2 \delta_2.
\]

Substituting above two inequalities into (2.21), we deduce that 
\[
0 \geq \phi G \left[ \Delta \phi - 2\frac{\nabla \phi^2}{\phi} + \left( \frac{2\phi}{n} - \alpha' \right) \frac{\phi}{\alpha} - \frac{\gamma'}{\gamma} \phi - \frac{n\alpha^2}{2(\alpha - 1)} \frac{\nabla \phi^2}{\phi} \right] \\
-\delta_1 (l - 1) \pi_2 - \sqrt{C}K \right] + \frac{\phi^2 G^2}{\alpha^2 n\gamma} - \gamma \phi^2 \alpha^2 n^2 K^2 \\
-\frac{l\alpha - 1}{l - 1} \gamma \phi^2 \pi_2 \delta_2 - \gamma (l - 1) \pi_2 \phi^2 \delta_1 \alpha \phi - \phi^2 \gamma \gamma \pi_2 \delta_3.
\]

Applying (2.11), we infer
\[
0 \geq \left[ -\frac{C}{R^2} + \left( \frac{2\phi}{n} - \alpha' \right) \frac{\phi}{\alpha} - \frac{n\alpha^2}{2(\alpha - 1)} \frac{C}{R^2} \right] \\
-\delta_1 (l - 1) \pi_2 - \sqrt{C}K \right] + \frac{\phi^2 G^2}{\alpha^2 n\gamma} - \gamma \phi^2 \alpha^2 n^2 K^2 \\
-\frac{l\alpha - 1}{l - 1} \gamma \phi^2 \pi_2 \delta_2 - \gamma (l - 1) \pi_2 \phi^2 \delta_1 \alpha \phi - \phi^2 \gamma \gamma \pi_2 \delta_3.
\]

For the inequality \( Ax^2 - 2Bx \leq C \), one has \( x \leq \frac{2B}{A} + \left( \frac{C}{A} \right)^{\frac{1}{2}} \), where \( A, B, C > 0 \).

By using this inequality to (2.22) and then we arrive at 
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq \left\{ \begin{array}{l}
n\gamma \alpha^2 \left[ \frac{C}{R^2} + \frac{n\alpha^2}{2(\alpha - 1)} \frac{C}{R^2} + \delta_1 (l - 1) \pi_2 + \sqrt{C}K \right] \\
+ n\gamma \alpha^2 \left[ \frac{\gamma'}{\gamma} - \left( \frac{2\phi}{n} - \alpha' \right) \frac{1}{\alpha} \right] + n^2 \gamma \phi \alpha^2 K + \alpha \phi \sqrt{\frac{n(l\alpha - 1)\pi_2}{l - 1}} \\
+ \alpha^2 \gamma \phi \sqrt{n(l - 1)\pi_2 \delta_1 \phi} + \alpha^2 \gamma \phi \sqrt{n\delta_3 \pi_2} \end{array} \right\} (x_1, t_1).
\]

If \( \gamma \) is nondecreasing which satisfies the system
\[
\left\{ \begin{array}{l}
\frac{\gamma'}{\gamma} - \left( \frac{2\phi}{n} - \alpha' \right) \frac{1}{\alpha} \leq 0, \\
\frac{\gamma \alpha^4}{\alpha - 1} \leq C.
\end{array} \right. \quad (2.23)
\]

Recall that \( \alpha(t) \) and \( \gamma(t) \) are non-decreasing and \( t_1 < T_1 \). Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1)
\]
Because

$$\phi$$

is nondecreasing which satisfies the system

$$\left\{ \begin{array}{l}
\frac{\gamma'}{\gamma} - \frac{2\varphi}{n} - \frac{1}{\alpha} \leq 0,
\frac{\gamma}{\alpha - 1} \leq C.
\end{array} \right. \tag{2.24}$$

Recall that \(\alpha(t)\) and \(\gamma(t)\) are non-decreasing and \(t_1 < T_1\). Hence, we have

$$\phi G(x, T_1) \leq (\phi G)(x_1, t_1)$$

\[
\leq n\gamma(T_1)\alpha^2(T_1) \left[ \frac{C}{R^2} \left(1 + \sqrt{KR} \right) + \frac{Cn\alpha^2(T_1)}{R^2} + CK \right]
+ n\gamma(T_1)\alpha^2(T_1)(l - 1)\delta_1\pi_2 + n\frac{2}{\gamma}\gamma(T_1)\phi\alpha^2(T_1)K
+ \alpha(T_1)\phi\gamma(T_1)\sqrt{\frac{n(l\alpha(T_1) - 1)\pi_2\delta_2}{l - 1}}
+ \alpha^2(T_1)\gamma(T_1)\phi\sqrt{n(l - 1)\delta_1\varphi} + \alpha^2(T_1)\gamma(T_1)\phi\sqrt{n\delta_3\pi_2}.
\]

Hence, we have for \(\phi \equiv 1\) on \(B_{R,T}\),

$$F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2} \left(1 + \sqrt{KR} \right) + \frac{Cn\alpha^2(T_1)}{R^2} + CK \right]
+ n\alpha^2(T_1)(l - 1)\delta_1\pi_2 + n\frac{2}{\gamma}\gamma(T_1)\phi\alpha^2(T_1)K
+ \alpha(T_1)\sqrt{\frac{n(l\alpha(T_1) - 1)\pi_2\delta_2}{l - 1}}
+ \alpha^2(T_1)\gamma(T_1)\phi\sqrt{n(l - 1)\delta_1\varphi} + \alpha^2(T_1)\gamma(T_1)\phi\sqrt{n\delta_3\pi_2}.$$}

Because \(T_1\) is arbitrary in \(0 < T_1 < T\), the conclusion is valid.

\(\square\)

**Proof of Theorem 2.2.** Since \(u_t = \nabla u + h(x, t)u^l\), we have

$$\partial_t(|\nabla u|^2) = 2\text{Ric}(\nabla u, \nabla u) + 2 < \nabla u, \nabla(u_t) >$$

$$= 2\text{Ric}(\nabla u, \nabla u) + 2 < \nabla u, \nabla(\Delta u) > + 2 < \nabla u, \nabla(h(x, t)u^l) >.$$
Applying Bochner's formula, above equation becomes
\[ \partial_t (|\nabla u|^2) = \Delta (|\nabla u|^2) - 2|\nabla^2 u|^2 + 2 < \nabla u, \nabla (h(x,t)u^l) >. \] (2.25)

Besides,
\[ \partial_t (u^2) = \Delta (u^2) - 2|\nabla u|^2 + 2 h(t)u^{l+1}. \] (2.26)

Let \( F = t|\nabla u|^2 + X u^2 \), where \( X \) is a constant to be decided. Then combining (2.25) with (2.27), we obtain
\[ \partial_t F = |\nabla u|^2 + t [\Delta (|\nabla u|^2) - 2|\nabla^2 u|^2 + 2 < \nabla u, \nabla (h(t)u^l) >] \\
+ X [\Delta (u^2) - 2|\nabla u|^2 + 2 h(t)u^{l+1}] \\
= |\nabla u|^2 + t [\Delta (|\nabla u|^2) - 2|\nabla^2 u|^2 + 2 h(t)(l-1)u^{l-2}|\nabla u|^2 \\
+ X [\Delta (u^2) - 2|\nabla u|^2 + 2 h(t)u^{l+1}] \\
\leq \Delta F + (1 - 2X)|\nabla u|^2. \] (2.27)

Selecting \( X = \frac{1}{2} \) and using maximum principle, we infer
\[ F(x,t) \leq \max_{x \in M^n} F(x,0) = \frac{1}{2} \max_{x \in M^n} u^2(x,0), \]
which implies the theorem is valid. \( \square \)

3. Gradient estimates for the equation (1.6)

Recalled that \((M^n, g(t))\) is called a gradient Ricci soliton if there is a smooth function \( f \) on \( M^n \) such that for some constant \( c \in \mathbb{R} \), which satisfies
\[ Rc = cg + D^2 f, \] (3.1)
where \( D^2 f \) is the Hessian of \( f \). Let \( u = e^f \), after some computation applying (3.1) as done in [21], we get
\[ \Delta u + 2cu \log u = (A_0 - nc)u \quad \text{in} \quad M^n \]
for some constant \( A_0 \), where \( n \) is the dimension of \( M^n \). In [21], Ma proved the local gradient estimate of positive solutions to the equation
\[ \Delta u + au \log u + bu = 0 \quad \text{in} \quad M^n, \]
where \( a > 0 \) and \( b \in \mathbb{R} \) are constants for complete noncoompact manifolds with a fixed metric and curvature locally bounded below. In [31], Yang generalized Ma’s result and derived a local gradient estimates for positive solutions to the equation
\[ u_t = \Delta u + au \log u + bu = 0 \quad \text{in} \quad M^n \times (0, T], \]
where \( a, b \in \mathbb{R} \) are constants for complete noncompact manifolds with a fixed metric and curvature locally bounded below. Replacing \( u \) by \( e^{b/a}u \), above equation becomes
\[ u_t = \Delta u + au \log u. \] (3.2)

One can find in [29, 30] some related results for equation (3.2) on manifolds.

In this section, we consider the nonlinear parabolic equation (1.6) under the Ricci flow.
3.1. Main results.

Our main results state as follows.

**Theorem 3.1.** Let $(M^n, g(t))_{t \in [0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\text{Ric}(x, t)| \leq K$ for some $K > 0$ and all $t \in [0,T]$. Suppose that there exist three functions $\alpha(t), \varphi(t)$ and $\gamma(t)$ satisfy the following conditions (C1), (C2), (C3) and (C4).

Given $x_0 \in M$ and $R > 0$, let $u$ be a positive solution of the nonlinear parabolic equation

$$\partial_t u = \Delta u + au \log u$$

in the cube $B_{2R,T} := \{(x,t)|d(x,x_0,t) \leq 2R, 0 \leq t \leq T\}$, where $a$ is a constant.

1. For $a \leq 0$. If $\frac{\alpha^4}{\alpha - 1} \leq C_1$ for some constant $C_1$, then

$$\frac{\nabla u^2}{u^2} - \frac{\alpha u_t}{u} + \alpha a \log u \leq C \alpha^2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{Cn^2}{R^2} + n^2 \alpha^2 K + n|a|\alpha^2 + \alpha \varphi.$$

2. For $a > 0$. If $\frac{\alpha^4}{\alpha - 1} \leq C_2$ for some constant $C_2$, then

$$\frac{\nabla u^2}{u^2} - \frac{\alpha u_t}{u} + \alpha a \log u \leq C \alpha^2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + a + K \right) + \frac{n^2 C}{R^2} + n^2 \alpha^2 K + \alpha \varphi.$$

where $C$ is a constant.

**Corollary 3.1.** Let $(M^n, g(t))_{t \in [0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\text{Ric}(x, t)| \leq K$ for some $K > 0$ and all $t \in [0,T]$. Given $x_0 \in M$ and $R > 0$, let $u$ be a positive solution of the nonlinear parabolic equation (1.6) in the cube $B_{2R,T} := \{(x,t)|d(x,x_0,t) \leq 2R, 0 \leq t \leq T\}$. Then the following special estimates are valid.

1. Li-Yau type: $\alpha(t) = \text{constant, } \varphi(t) = \frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha - 1}, \gamma(t) = t^{\theta} \text{ with } 0 < \theta \leq 2.$
If \( a \leq 0 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} + a \alpha \log u \leq C_\alpha \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + \frac{\alpha^2}{\alpha - 1} \frac{1}{R^2} + K \right) + n^\frac{3}{2} \alpha^2 K + n|a|\alpha^2 + \alpha \varphi.
\]

If \( a > 0 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} + a \alpha \log u \leq C_\alpha \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + \frac{\alpha^2}{\alpha - 1} \frac{1}{R^2} + a + K \right) + n^\frac{3}{2} \alpha^2 K + \alpha \varphi.
\]

2. Hamilton type:
\[
\alpha(t) = e^{2Kt}, \quad \varphi(t) = \frac{n}{t} e^{4Kt}, \quad \gamma(t) = te^{2Kt}.
\]

If \( a \leq 0 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} + a \alpha \log u \leq C_\alpha \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{C_\alpha^4}{R^2 e^{2Kt}} + \alpha \varphi + n^3 \alpha^2 K + n|a|\alpha^2 + \alpha \varphi.
\]

If \( a > 0 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} + a \alpha \log u \leq C_\alpha \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{C_\alpha^4}{R^2 e^{2Kt}} + n^3 \alpha^2 K + \alpha \varphi.
\]

3. Li-Xu type:
\[
\alpha(t) = 1 + \frac{\sinh(Kt) \cosh(Kt) - Kt}{\sinh^2(Kt)}, \quad \varphi(t) = 2nK[1 + \coth(Kt)],
\]
\[
\gamma(t) = \tanh(Kt).
\]

If \( a \leq 0 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} + a \alpha \log u \leq C_\alpha \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{C}{R^2 \tanh(Kt)} + n^\frac{3}{2} \alpha^2 K + n|a|\alpha^2 + \alpha \varphi.
\]

If \( a > 0 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} + a \alpha \log u \leq C_\alpha \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + \frac{C}{R^2 \tanh(Kt)} + n^\frac{3}{2} \alpha^2 K + \alpha \varphi.
\]

4. Linear Li-Xu type:
\[
\alpha(t) = 1 + 2Kt, \varphi(t) = \frac{n}{t} + nK(1 + 2Kt + \mu Kt), \gamma(t) = Kt \quad \text{with} \quad \mu \geq 1.
\]
If $a \leq 0$, then
\[
\frac{\|\nabla u\|^2}{u^2} - \alpha \frac{u_t}{u} + a \alpha \log u \leq C\alpha^2 \left( \frac{1}{R^2} + \sqrt{\frac{K}{R}} + K \right)
\]
\[
+ \frac{C\alpha^4}{R^2 K t} + n^\frac{4}{3} \alpha^2 K + n |a| \alpha^2 + \alpha \varphi.
\]

If $a > 0$, then
\[
\frac{\|\nabla u\|^2}{u^2} - \alpha \frac{u_t}{u} + a \alpha \log u \leq C\alpha^2 \left( \frac{1}{R^2} + \sqrt{\frac{K}{R}} + a + K \right)
\]
\[
+ \frac{C\alpha^4}{R^2 K t} + n^\frac{4}{3} \alpha^2 K + \alpha \varphi.
\]

The local estimates above imply global estimates.

**Corollary 3.2.** Let $(M^n, g(0))$ be a complete noncompact Riemannian manifold without boundary, and assume $g(t)$ evolves by Ricci flow in such a way that $|\text{Ric}| \leq K$ for $t \in [0, T]$. Let $u(x, t)$ be a positive solution to the equation (1.6). If $l \in \mathbb{R}$ and for $(x, t) \in M^n \times (0, T]$, then
\[
\frac{\|\nabla u\|^2}{u^2} - \alpha \frac{u_t}{u} + a \alpha \log u \leq C\alpha^2 (K + |a|) + \alpha \varphi.
\]

**Remark 3.1.** The above results may be regard as generalizing the gradient estimates of Yang [30] to the Ricci flow.

### 3.2. Auxiliary lemma.

To prove the theorem 3.1, the following a lemma is needed.

Let $f = \log u$. Then
\[
(\Delta - \partial_t) f = -\|\nabla f\|^2 + af.
\]

Let $F = \|\nabla f\|^2 - \alpha f_t + a \alpha f - \alpha \varphi$, where $\alpha = \alpha(t)$ and $\varphi = \varphi(t)$. Then
\[
\Delta f = f_t - af - \|\nabla f\|^2
\]
\[
= \frac{F}{\alpha} - (\frac{\alpha - 1}{\alpha}) \|\nabla f\|^2 - \varphi.
\]

**Lemma 3.1.** We assume that $\alpha(t) > 1$ and $\varphi(t) > 0$ satisfy the following system (2.3). Then
\[
(\Delta - \partial_t) F \geq |f_{ij} + \frac{2}{n} \delta_{ij}|^2 + \frac{2\varphi}{n - \alpha'} \frac{1}{\alpha} F - \alpha^2 n^2 K^2 - 2 \nabla f \nabla F
\]
\[
+ 2a(\alpha - 1) \|\nabla f\|^2 + a a \Delta f.
\]

**Proof.** A computation is shown that
\[
\Delta F = \Delta \|\nabla f\|^2 - \alpha \Delta (f_t) + a \alpha \Delta f
\]
\[
= 2|f_{ij}|^2 + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha \Delta (f_t) + a \alpha \Delta f
\]
\[
= 2 \left( |f_{ij}|^2 + \alpha R_{ij} f_i f_j \right) + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha (\Delta f)_t + a \alpha \Delta f
\]
\[
\geq 2|f_{ij}|^2 - 2 \sum \alpha |R_{ij}||f_{ij}| + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha (\Delta f)_t + a \alpha \Delta f
\]
\[
\geq 2|f_{ij}|^2 - \sum (\alpha^2 |R_{ij}|^2 + |f_{ij}|^2) + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha (\Delta f)_t + a \alpha \Delta f
\]
\[
\geq |f_{ij}|^2 - \sum \alpha^2 |R_{ij}|^2 + 2f_j f_{ij} + 2R_{ij} f_i f_j - \alpha (\Delta f)_t + a \alpha \Delta f
\]
Further, by utilizing the unit matrix (\mathbf{I})

\[ \text{Proof of Theorem 3.3} \]

We follow that from (3.6) and (3.7)

\[
(\Delta - \partial_t)F \geq |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla (\Delta f) - \alpha(\Delta f)_t + \alpha \Delta f - 2\nabla f \nabla (f_t)
\]

\[
\text{and}
\]

\[
\partial_t F = (|\nabla f|^2)_t - \alpha f_{tt} - \alpha' f_t + \alpha' \alpha f + \alpha \Delta f - \alpha \varphi' - \alpha' \varphi
\]

\[
= 2\nabla f \nabla (f_t) + 2R_{ij} f_i f_j - \alpha f_{tt} - \alpha' f_t + \alpha' \alpha f
\]

\[ + \alpha \Delta f - \alpha \varphi' - \alpha' \varphi. \tag{3.7} \]

We follow that from (3.6) and (3.7)

\[ (\Delta - \partial_t)F \geq |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla (\Delta f) - \alpha(\Delta f)_t + \alpha \Delta f - 2\nabla f \nabla (f_t)
\]

\[
\tag{3.6} + \alpha f_{tt} + \alpha' f_t - \alpha' \alpha f + \alpha \varphi' + \alpha' \varphi
\]

\[ = |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla (\Delta f) - \alpha(\Delta f)_t + \alpha \Delta f
\]

\[
+ \alpha \Delta f - 2\nabla f \nabla (f_t) + \alpha' f_t - \alpha' \alpha f + \alpha \varphi' + \alpha' \varphi
\]

\[ = |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla (\Delta f) + 2\alpha \nabla f \nabla (f_t) + 2\alpha R_{ij} f_i f_j
\]

\[
+ \alpha \Delta f - 2\nabla f \nabla (f_t) + \alpha' f_t - \alpha' \alpha f + \alpha \varphi' + \alpha' \varphi
\]

\[ = |f_{ij}|^2 + 2\alpha R_{ij} f_i f_j - \alpha \Delta f + 2\nabla f \nabla F + 2a(a - 1)(l - 1)|\nabla f|^2
\]

\[ + \alpha \Delta f + \alpha' f_t - \alpha' \alpha f + \alpha \varphi' + \alpha' \varphi. \tag{3.8} \]

Further, by utilizing the unit matrix (\delta_{ij})_{n\times n} and (3.8), we obtain

\[ (\Delta - \partial_t)F \geq |f_{ij}|^2 + \frac{\varphi}{n} \delta_{ij}^2 - 2\alpha K|\nabla f|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla F
\]

\[
+ 2a(a - 1)|\nabla f|^2 + \alpha \Delta f + \alpha' f_t - \alpha' \alpha f + \alpha \varphi' + \alpha' \varphi
\]

\[ - \frac{\varphi^2}{n} - 2\frac{\varphi}{n} \Delta f
\]

\[ = |f_{ij}|^2 + \frac{\varphi}{n} \delta_{ij}^2 + \left( \frac{2\varphi}{n} - 2\alpha K \right)|\nabla f|^2 - \left( \frac{2\varphi}{n} - \alpha' \right)f_t
\]

\[ + \left( \frac{2\varphi}{n} - \alpha' \right) a \Delta f - \alpha^2 n^2 K^2 - 2\nabla f \nabla F + 2a(a - 1)|\nabla f|^2
\]

\[ + \alpha \Delta f + \alpha \varphi' + \alpha' \varphi - \frac{\varphi^2}{n} + \left( \frac{2\varphi}{n} - \alpha' \right) \frac{\alpha \varphi}{\alpha}. \tag{3.9} \]

\[ \square \]

### 3.3. The proof of Theorem

In this section, we will prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( G = \gamma(t)F \) and \( \gamma(t) > 0 \) be non-decreasing. Then

\[ (\Delta - \partial_t)G = \gamma(\Delta - \partial_t)F - \gamma' F \]

\[ \geq \gamma |f_{ij}|^2 + \frac{\varphi}{n} g_{ij}^2 + \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} G - \gamma \alpha^2 n^2 K^2 - 2\nabla f \nabla G
\]

\[ + 2a \gamma (a - 1)|\nabla f|^2 + a \gamma \alpha \Delta f - \gamma' F
\]

\[ = \gamma |f_{ij}|^2 + \frac{\varphi}{n} g_{ij}^2 + \left[ \frac{2\varphi}{n} - \alpha' \right] 1 - \frac{\gamma'}{\gamma} \left[ G - \gamma \alpha^2 n^2 K^2
\]

\[ - 2\nabla f \nabla G + 2a \gamma (a - 1)|\nabla f|^2 + a \gamma \alpha \Delta f. \tag{3.10} \]
Now, let \( \varphi(r) \) be a \( C^2 \) function on \([0, \infty)\) such that
\[
\varphi(r) = \begin{cases} 
1 & \text{if } r \in [0, 1], \\
0 & \text{if } r \in [2, \infty), 
\end{cases}
\]
and
\[
0 \leq \varphi(r) \leq 1, \quad \varphi'(r) \leq 0, \quad \varphi''(r) \leq 0, \quad \frac{\varphi'(r)}{\varphi(r)} \leq C,
\]
where \( C \) is an absolute constant. Define by
\[
\phi(x, t) = \varphi(d(x, x_0, t)) = \varphi \left( \frac{d(x, x_0, t)}{R} \right) = \varphi \left( \frac{\rho(x, t)}{R} \right),
\]
where \( \rho(x, t) = d(x, x_0, t) \). By using maximum principle, the argument of Calabi [2] allows us to suppose that the function \( \phi(x, t) \) with support in \( B_{2R, T} \), is \( C^2 \) at the maximum point. By utilizing the Laplacian theorem, we deduce that
\[
|\nabla \phi|^2 \leq \frac{C}{R^2}, \quad -\Delta \phi \leq \frac{C}{R^2} (1 + \sqrt{K} R), \tag{3.11}
\]
For any \( 0 \leq T_1 \leq T \), let \( H = \phi G \) and \( (x_1, t_1) \) be the point in \( B_{2R, T_1} \) at which \( H \) attains its maximum value. We can suppose that \( H \) is positive, because otherwise the proof is trivial. Then at the point \( (x_1, t_1) \), we infer
\[
\begin{align*}
0 = \nabla(\phi G) &= G \nabla \phi + \phi \nabla G, \\
\Delta(\phi G) &\leq 0, \\
\partial_t(\phi G) &\geq 0.
\end{align*} \tag{3.12}
\]
By the evolution formula of the geodesic length under the Ricci flow [6], we calculate
\[
\phi_t G = -G \phi' \left( \frac{\rho}{R} \right) \frac{1}{R} \frac{d \rho}{dt} = G \phi' \left( \frac{\rho}{R} \right) \int_{\gamma_{t_1}} \text{Ric}(S, S) ds \\
\leq G \phi' \left( \frac{\rho}{R} \right) \frac{1}{R} K \rho \leq G \phi' \left( \frac{\rho}{R} \right) K_2 \leq G \sqrt{C} K,
\]
where \( \gamma_{t_1} \) is the geodesic connecting \( x \) and \( x_0 \) under the metric \( g(t_1) \), \( S \) is the unite tangent vector to \( \gamma_{t_1} \), and \( ds \) is the element of the arc length.

All the following computations are at the point \( (x_1, t_1) \). Since
\[
|f_{ij} + \frac{\varphi}{n} \delta_{ij}|^2 \geq \frac{1}{n} \left( \text{tr} |f_{ij} + \frac{\varphi}{n} \delta_{ij}| \right)^2 \\
= \frac{1}{n} \left( \Delta f + \varphi \right) \\
= \frac{1}{n} \left[ -\frac{1}{\alpha} F - \frac{1}{\alpha} (\alpha - 1)|\nabla f|^2 \right]^2 \\
= \frac{1}{\alpha^2 n} \left[ \frac{G}{\gamma} + (\alpha - 1)|\nabla f|^2 \right]^2. \tag{3.13}
\]
and
\[
\Delta f = f_t - |\nabla f|^2 - af \\
= -\frac{F}{\alpha} - \frac{\alpha - 1}{\alpha} |\nabla f|^2 - \varphi < 0. \tag{3.14}
\]
Case 1 $a \leq 0$. Combining (3.14) with (3.10), we have
\[
(\Delta - \partial_t)G \geq \gamma |f_{ij} + \frac{\phi}{n} \delta_{ij}|^2 + \left[\frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G - \gamma \alpha^2 n^2 K^2 - 2\nabla f \nabla G + 2a\gamma (a - 1)|\nabla f|^2.
\]
Using (3.12) and (3.13), we infer
\[
0 \geq (\Delta - \partial_t)(\phi G) = G\left(\Delta \phi - 2\frac{\nabla \phi^2}{\phi}\right) + \phi(\Delta - \partial_t)G - \gamma G \phi_t \geq G\left(\Delta \phi - 2\frac{\nabla \phi^2}{\phi}\right) + \phi \frac{\phi_G}{\alpha^2 n} \frac{G}{\gamma} + (a - 1)|\nabla f|^2 \right)^2 + \left[\frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \phi_G - \gamma \alpha^2 n^2 K^2 - 2\phi \nabla f \nabla G + 2a\phi \gamma (a - 1)|\nabla f|^2 - G \sqrt{C} K.
\]
Multiply $\phi$ to (3.15), we have
\[
0 \geq \phi G\left[\Delta \phi - 2\frac{\nabla \phi^2}{\phi}\right] + \left[\frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \phi \frac{\phi_G}{\alpha^2 n} \frac{G}{\gamma} + (a - 1)|\nabla f|^2 \right)^2 - \gamma \phi^2 \alpha^2 n^2 K^2 - 2\phi^2 \nabla f \nabla G + 2a\phi \gamma (a - 1)|\nabla f|^2 - \phi G \sqrt{C} K \geq \phi G\left[\Delta \phi - 2\frac{\nabla \phi^2}{\phi}\right] + \left[\frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \phi \frac{\phi_G^2}{\alpha^2 n^2} + \phi \frac{\phi^2 (a - 1)^2 \gamma}{\alpha^2 n^2} |\nabla f|^4 + \frac{2\phi^2 (a - 1)}{na^2} G|\nabla f|^2 - \gamma \phi^2 \alpha^2 n^2 K^2 + 2\phi G \nabla \phi \nabla f + 2a\phi^2 \gamma (a - 1)|\nabla f|^2 - \phi G \sqrt{C} K.
\]
We use the fact
\[
\frac{2\phi^2 (a - 1)}{na^2} G|\nabla f|^2 + 2\phi G \nabla \phi \nabla f \geq - \frac{na^2}{2(a - 1)} \frac{|\nabla \phi|^2}{\phi} \phi G,
\]
and
\[
\frac{\phi^2 (a - 1)^2 \gamma}{na^2} |\nabla f|^4 + 2a\phi^2 \gamma (a - 1)|\nabla f|^2 \geq - na^2 \alpha^2 \gamma \phi^2,
\]
to (3.16), we deduce that
\[
0 \geq \phi G\left[\Delta \phi - 2\frac{\nabla \phi^2}{\phi}\right] + \left[\frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \phi \frac{\phi_G^2}{\alpha^2 n^2} - \frac{na^2}{2(a - 1)} \frac{|\nabla \phi|^2}{\phi} - \sqrt{C} K + \frac{\phi^2 G^2}{\alpha^2 n^2} - \gamma \phi^2 \alpha^2 n^2 K^2 - na^2 \alpha^2 \gamma \phi^2
\]
\[
\geq \left[\frac{\phi}{R^2} (1 + \sqrt{k} R) - \frac{2C}{R^2} + \left[\frac{2\phi}{n} - \alpha' \right] \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \phi \frac{\phi_G^2}{\alpha^2 n^2} - \frac{na^2}{2(a - 1)} \frac{C}{R^2} - \sqrt{C} K \right] \phi G + \frac{\phi^2 G^2}{\alpha^2 n^2} - \gamma \phi^2 \alpha^2 n^2 K^2 - na^2 \alpha^2 \gamma \phi^2.
\]
For the inequality $Ax^2 - 2Bx \leq C$, one has $x \leq \frac{2B}{2A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}$, where $A, B, C > 0$. Hence, we infer
\[
\phi G(x, t_1) \leq (\phi G)(x_1, t_1)
\]
\[
\begin{align*}
&\leq \left\{ n\gamma^2 \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + \frac{n\alpha^2}{2(\alpha - 1)} \frac{C}{R^2} + \sqrt{C} K \right] \\
&\quad + n\gamma^2 \left[ \frac{\gamma' - \frac{2\varphi}{n} - \alpha'}{\gamma} \right] \frac{1}{\alpha} \\
&\quad + n^2 \gamma \alpha^2 \phi K + n\alpha^2 \gamma \phi \right\}(x_1, t_1).
\end{align*}
\]

If \( \gamma \) is nondecreasing which satisfies the system

\[
\begin{cases}
\gamma' - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} \leq 0, \\
\frac{\gamma^2}{\alpha} \leq C.
\end{cases}
\]

Recall that \( \alpha(t) \) and \( \gamma(t) \) are non-decreasing and \( t_1 < T_1 \). Hence, we have

\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq n\gamma(T_1) \alpha^2(T_1) \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + K \right] + \frac{n^2 C}{R^2} \\
+ n^2 \gamma^2(T_1) K + n\alpha^2(T_1) \gamma(T_1).
\]

Hence, we have for \( \phi \equiv 1 \) on \( B_{R,T} \),

\[
F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + CK \right] + \frac{n^2 C}{R^2 \gamma(T_1)} \\
+ n^2 \gamma^2(T_1) K + n\alpha^2(T_1) \gamma(T_1).
\]

If \( \gamma \) is nondecreasing which satisfies the system

\[
\begin{cases}
\gamma' - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} \leq 0, \\
\frac{\gamma}{\alpha - 1} \leq C.
\end{cases}
\]

Recall that \( \alpha(t) \) and \( \gamma(t) \) are non-decreasing and \( t_1 < T_1 \). Hence, we have

\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq n\gamma(T_1) \alpha^2(T_1) \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + Cn\alpha^4 \right] \\
+ \frac{Cn\alpha^4}{R^2 \gamma(T_1)} \\
+ n^2 \gamma^2(T_1) K + n\alpha^2(T_1) \gamma(T_1).
\]

Hence, we have for \( \phi \equiv 1 \) on \( B_{R,T} \),

\[
F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + K \right] + \frac{n^2 C\alpha^4}{R^2 \gamma(T_1)} \\
+ n^2 \gamma^2(T_1) K + n\alpha^2(T_1).
\]

Because \( T_1 \) is arbitrary in \( 0 < T_1 < T \), the conclusion is valid.

**Case 2** \( a \geq 0 \). It is not difficult to find \( \Delta f \leq -\frac{F}{\alpha} \) form (3.14). Then, we have from (3.10)

\[
(\Delta - \partial_t) G \geq \gamma |f_{ij} + \frac{2\varphi d_{ij}|^2 + \left[ \frac{2\varphi}{n} \gamma \right] - \alpha' \frac{1}{\alpha} - \frac{\gamma'}{\gamma} G} \\
- \gamma \alpha^2 n^2 K^2 2\nabla f \nabla G - a G.
\]
Using (3.13) and (3.13), we infer
\[
0 \geq (\Delta - \partial_t) (\phi G) \\
= G (\Delta \phi - 2 \frac{1}{\phi} \phi \nabla \phi^2) + \phi (\Delta - \partial_t) G - \gamma G \phi_t \\
\geq G (\Delta \phi - 2 \frac{1}{\phi} \phi \nabla \phi^2) + \frac{\phi \gamma}{\alpha^2n} \left[ \frac{G}{\gamma} + (\alpha - 1) |\nabla \phi|^2 \right]^2 \\
+ \left[ (\frac{2}{\alpha} - \alpha') \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \right] \phi G - \gamma \phi \alpha^2 n^2 K^2 - 2 \phi \nabla \phi \nabla G \\
- a \phi G - G \sqrt{C} K.
\]

Multiply \( \phi \), we have
\[
0 \geq \phi G \left[ \Delta \phi - 2 \frac{1}{\phi} \phi \nabla \phi^2 + \left( \frac{2}{\alpha} - \alpha' \right) \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \right] + \frac{\phi^2 \gamma}{\alpha^2 n} \left[ \frac{G}{\gamma} + (\alpha - 1) |\nabla \phi|^2 \right]^2 \\
- \gamma \phi^2 \alpha^2 n^2 K^2 - 2 \phi^2 \nabla \phi \nabla G - a \phi G - \phi G \sqrt{C} K \\
\geq \phi G \left[ \Delta \phi - 2 \frac{1}{\phi} \phi \nabla \phi^2 + \left( \frac{2}{\alpha} - \alpha' \right) \frac{1}{\alpha} - \frac{\gamma'}{\gamma} \right] + \frac{\phi^2 G^2}{\alpha^2 n \gamma} + \frac{2 \phi^2 (\alpha - 1)}{na^2} G |\nabla \phi|^2 \\
- \gamma \phi^2 \alpha^2 n^2 K^2 + 2 \phi G \nabla \phi \nabla f - a \phi^2 G - \phi G \sqrt{C} K, \tag{3.17}
\]

where we drop the term \( \frac{\phi^2 (\alpha - 1)}{na^2} |\nabla \phi|^4 \). We use the fact
\[
\frac{2 \phi^2 (\alpha - 1)}{na^2} G |\nabla \phi|^2 + 2 \phi G \nabla \phi \nabla f \geq - \frac{na^2}{2(\alpha - 1)} \frac{|\nabla \phi|^2}{\phi} \phi G,
\]
to (3.17), we deduce that
\[
0 \geq \phi G \left[ \Delta \phi - 2 \frac{1}{\phi} \phi \nabla \phi^2 + \left( \frac{2}{\alpha} - \alpha' \right) \frac{1}{\alpha} - \frac{\gamma'}{\gamma} - \frac{na^2}{2(\alpha - 1)} |\nabla \phi|^2 \phi - a \phi - \sqrt{C} K \right] \\
+ \frac{\phi^2 G^2}{\alpha^2 n \gamma} - \gamma \phi^2 \alpha^2 n^2 K^2 \\
\geq \left[ - \frac{\gamma}{\alpha} - \frac{2}{\alpha} (\frac{1}{n} - \alpha') \frac{1}{\alpha} - \frac{\gamma'}{\gamma} - \frac{na^2}{2(\alpha - 1)} \frac{C}{R^2} - \sqrt{C} K \right] \phi G \\
+ \frac{\phi^2 G^2}{\alpha^2 n \gamma} - \gamma \phi^2 \alpha^2 n^2 K^2.
\]

For the inequality \( Ax^2 - 2Bx \leq C \), one has \( x \leq \frac{2B}{A} + \left( \frac{C}{A} \right)^{\frac{1}{2}} \), where \( A, B, C > 0 \).
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \\
\leq \left\{ n \gamma \alpha^2 \left[ \frac{C}{R^2} (1 + \sqrt{K} R) + \frac{na^2}{2(\alpha - 1)} \frac{C}{R^2} + a \phi + \sqrt{C} K \right] \right\} \\
\left\{ \gamma' - \left( \frac{2}{\alpha} - \alpha' \right) \frac{1}{\alpha} \right\} + \frac{n^2 \gamma \alpha^2 \phi K}{\alpha - 1} (x_1, t_1).
\]

If \( \gamma \) is nondecreasing which satisfies the system
\[
\begin{cases}
\gamma' - \left( \frac{2}{\alpha} - \alpha' \right) \frac{1}{\alpha} \leq 0, \\
\frac{\gamma \alpha^4}{\alpha - 1} \leq C.
\end{cases}
\]
Recall that \( \alpha(t) \) and \( \gamma(t) \) are non-decreasing and \( t_1 < T_1 \). Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1)
\]
\[
\leq n\gamma(T_1)\alpha^2(T_1) \left[ \frac{C}{R^2}(1 + \sqrt{KR}) + a\phi + K \right] + n^2 C \frac{|\alpha|}{R^2} + n^2 \gamma(T_1)\alpha^2(T_1)K.
\]
Hence, we have for \( \phi \equiv 1 \) on \( B_{R,T} \),
\[
\sup_{B_R} F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2}(1 + \sqrt{KR}) + a + CK \right] + n^2 C \frac{|\alpha|}{R^2 \gamma(T_1)} + n^2 \alpha^2(T_1)K.
\]
If \( \gamma \) is nondecreasing which satisfies the system
\[
\begin{cases}
\frac{\gamma'}{\gamma} - \frac{2\phi}{n} - \frac{\alpha'}{\alpha} & \leq 0, \\
\frac{\gamma}{\alpha} - 1 & \leq C.
\end{cases}
\]
Recall that \( \alpha(t) \) and \( \gamma(t) \) are non-decreasing and \( t_1 < T_1 \). Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1)
\]
\[
\leq n\gamma(T_1)\alpha^2(T_1) \left[ \frac{C}{R^2}(1 + \sqrt{KR}) + \frac{C n^4}{R^2} + a\phi + CK \right] + n^2 \gamma(T_1)\alpha^2(T_1)K.
\]
Hence, we have for \( \phi \equiv 1 \) on \( B_{R,T} \),
\[
F(x, T_1) \leq n\alpha^2(T_1) \left[ \frac{C}{R^2}(1 + \sqrt{KR}) + a + K \right] + n^2 C \frac{n^4}{R^2 \gamma(T_1)} + n^2 \alpha^2(T_1)K.
\]
Because \( T_1 \) is arbitrary in \( 0 < T_1 < T \), the conclusion is valid. This proof is complete.

\[ \square \]

4. Harnack Inequalities

In this section, as application of main theorems, some Harnack inequalities are derived.

**Theorem 4.1.** Let \( (M^n, g(x,t))_{t \in [0,T]} \) be a complete solution to the Ricci flow (1.7). Suppose that \( |\text{Ric}| \leq K \) for some \( K > 0 \), and all \( (x,t) \in M^n \times [0,T] \). Assume that \( u(x,t) \) is a positive solution for (1.6). Let \( h(x,t) \) be a function defined on \( M^n \times [0,T] \) which is \( C^1 \) in \( t \) and \( C^2 \) in \( x \), satisfying \( |\nabla h|^2 \leq \delta_2 h \) and \( \Delta h \geq -\delta_3 \) on \( M^n \times [0,T] \) for some positive constants \( \delta_2 \) and \( \delta_3 \). Then for all \( (x_1, t_1) \in M^n \times (0,T) \) and \( (x_2, t_2) \in M^n \times (0,T) \) such that \( t_1 < t_2 \), we have
\[
u(x_2, t_2) \leq \begin{cases}
u(x_1, t_1) \times \text{exp} (\Gamma(t_1, t_2, \delta_1, \delta_2, \delta_3, \overline{\nu}_1)), & l \leq 1, \\
u(x_1, t_1) \times \text{exp} (\Lambda(t_1, t_2, \delta_1, \delta_2, \delta_3, \overline{\nu}_1)), & l > 1,
\end{cases}
\]
where
\[
\Gamma(t_1, t_2, \delta_1, \delta_2, \delta_3, \overline{\nu}_1)
\]
Proof. Firstly, the estimate in Corollary 2 where \( \mu \) and \( l \) to Theorem 4. Integrating above inequality over \( \gamma \), The proof is complete. \( \square \)

Let \( l(1.7) \)

Now we only prove the conclusion for \( l(1.6) \). The proof is similar \( (1.6) \) and \( l(1) = \log u(0, t_2) \) and \( l(1) = \log u(x, t_1) \). Direct calculation shows

\[
\frac{\partial l(s)}{\partial s} = (t_2 - t_1) \left( \frac{\nabla u}{u} \frac{\gamma'(s)}{u} - \frac{u_t}{u} \right)
\]

\[
\leq (t_2 - t_1) \left[ \frac{\nabla u}{u} \frac{\gamma'(s)}{u} - \frac{1}{\alpha(t)} \frac{|\nabla u|^2}{u^2} - h(x, t) u^{l-1} + \varphi + C \alpha(K + \mu_1) \right]
\]

\[
\leq \alpha(t) \frac{|\gamma'(s)|^2}{4 (t_2 - t_1)} + (t_2 - t_1) [\varphi + C \alpha(K + \mu) + \delta_1 \varphi_1].
\]

Integrating above inequality over \( \gamma(s) \), we obtain

\[
\log \frac{u(x, t_1)}{u(y, t_2)} = \int_0^1 \frac{\partial l(s)}{\partial s} ds
\]

\[
\leq \int_0^1 \left[ \frac{\alpha(t) |\gamma'(s)|^2}{4 (t_2 - t_1)} + (t_2 - t_1) [\varphi + C \alpha(K + \mu_1) + \delta_1 \varphi_1] \right] ds
\]

\[
\leq \int_0^1 \frac{|\gamma'(s)|^4}{2(t_2 - t_1)^2} ds + \int_0^{t_2} \frac{\alpha^2(t)}{32} dt
\]

\[
\leq \int_0^1 \frac{|\gamma'(s)|^4}{32} ds + \int_0^{t_2} \frac{\alpha^2(t)}{32} dt
\]

\[
+ \int_0^{t_2} |\varphi + C \alpha(K + \mu) + \delta_1 \varphi_1| dt.
\]

The proof is complete. \( \square \)

We also derive an Harnack inequality for the equation (1.6). The proof is similar to Theorem 4.1, so we omit it.

**Theorem 4.2.** Let \( (M^n, g(x, t))_{t \in [0, T]} \) be a complete solution to the Ricci flow (1.7). Suppose that \( |\text{Ric}| \leq K \) for some \( K > 0 \), and all \( (x, t) \in M^n \times [0, T] \). Assume
that \( u(x, t) \) is a positive solution for (1.6). Then for all \((x_1, t_1) \in M^n \times (0, T) \) and \((x_2, t_2) \in M^n \times (0, T) \) such that \( t_1 < t_2 \), we have
\[
\begin{align*}
  u(x_2, t_2) & \leq u(x_1, t_1) \\
  & \times \exp \left( \int_0^1 \frac{|\gamma(s)|^4}{2(t_2 - t_1)^2} ds + \int_{t_1}^{t_2} \frac{\alpha_2^2(t)}{32} dt + \int_{t_1}^{t_2} [\varphi + C\alpha(K + \mu) + \delta_1 \overline{\nabla}] dt \right)
\end{align*}
\]

5. Application to heat equation

According to Theorem 2.1 and Theorem 3.1, we derive corresponding gradient estimates and Harnack inequalities to the heat equation under Ricci flow

**Theorem 5.1.** Let \((M^n, g(t))_{t \in [0, T]} \) be a complete solution to the Ricci flow (1.7).

Assume that \(|\text{Ric}(x, t)| \leq K\) for some \( K > 0 \) and all \( t \in [0, T] \). Suppose that there exist three functions \( \alpha(t), \varphi(t) \) and \( \gamma(t) \) satisfy the following conditions (C1), (C2), (C3) and (C4).

Given \( x_0 \in M \) and \( R > 0 \), let \( u(x,t) \) be a positive solution of the heat equation
\[
  u_t = \Delta u,
\]
in the cube \( B_{2R, T} := \{(x, t) | d(x, x_0, t) \leq 2R, 0 \leq t \leq T\} \), where \( c \) is a constant.

If \( \frac{\alpha^2}{R} \leq C_1 \) for some constant \( C_1 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} \leq C_1 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + Cn^2 \frac{\alpha}{R^2 \gamma} + \alpha \varphi.
\]

If \( \frac{\alpha^2}{R^2} \leq C_2 \) for some constant \( C_2 \), then
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} \leq C_2 \left( \frac{1}{R^2} + \frac{\sqrt{K}}{R} + K \right) + Cn^2 \frac{\alpha}{R^2 \gamma} + \alpha \varphi.
\]

where \( C \) is a constant.

**Corollary 5.1.** Let \((M^n, g(t))_{t \in [0, T]} \) be a complete solution to the Ricci flow (1.7).

Assume that \(|\text{Ric}(x, t)| \leq K\) for some \( K > 0 \) and all \( t \in [0, T] \). Given \( x_0 \in M \) and \( R > 0 \), let \( u(x,t) \) be a positive solution of the heat equation (5.1) in the cube \( B_{2R, T} := \{(x, t) | d(x, x_0, t) \leq 2R, 0 \leq t \leq T\} \). Then the following special estimates are valid.

1. Li-Yau type:
\[
\alpha(t) = \text{constant, } \varphi(t) = \frac{n}{t} + \frac{nK\alpha^2}{\alpha - 1}, \gamma(t) = t^\theta \text{ with } 0 < \theta \leq 2.
\]
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} \leq C_1 \left[ \frac{1}{R^2} (1 + K \sqrt{R}) + \frac{\alpha^2}{\alpha - 1} \frac{1}{R^2} + K \right] + \alpha \varphi + n^2 \alpha^2 K.
\]

2. Hamilton type:
\[
\alpha(t) = e^{2Kt}, \varphi(t) = \frac{n}{t} e^{4Kt}, \gamma(t) = te^{2Kt}.
\]
\[
\frac{\nabla u}{u^2} - \frac{u_t}{u} \leq C_2 \left[ \frac{1}{R^2} (1 + \sqrt{K} \sqrt{R}) + K \right] + C_0^4 \frac{\alpha}{R^2 te^{2Kt}}.
\]
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3. Li-Xu type:
\( \alpha(t) = 1 + \frac{\sinh(Kt) \cosh(Kt) - Kt}{\sinh^2(Kt)} \), \( \varphi(t) = 2nK[1 + \coth(Kt)] \), \( \gamma(t) = \tanh(Kt) \).
\[
\frac{\|
abla u\|^2}{u^2} - \frac{\alpha u_t}{u} \leq C\alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{KR}) + K \right] + \frac{C}{R^2 \tanh(Kt)} + \alpha \varphi + n^2 \alpha^2 K.
\]

4. Linear Li-Xu type:
\( \alpha(t) = 1 + 2Kt \), \( \varphi(t) = nK (1 + 2Kt + \mu Kt) \), \( \gamma(t) = Kt \) with \( \mu \geq \frac{1}{4} \).
\[
\frac{\|
abla u\|^2}{u^2} - \frac{\alpha u_t}{u} \leq C\alpha^2 \left[ \frac{1}{R^2} (1 + \sqrt{KR}) + K \right] + \frac{C\alpha^4}{R^2 Kt} + \alpha \varphi + n^2 \alpha^2 K.
\]

Let \( R \to \infty \), a global estimate is derived.

**Corollary 5.2.** Let \((M^n, g(t))_{t \in [0, T]} \) be a complete solution to the Ricci flow (1.7). Assume that \( |\text{Ric}(x, t)| \leq K \) for some \( K > 0 \) and all \( t \in [0, T] \). Suppose that there exist three functions \( \alpha(t), \varphi(t) \) and \( \gamma(t) \) satisfy the following conditions (C1), (C2), (C3) and (C4).

Given \( x_0 \in M \) and \( R > 0 \), let \( u(x, t) \) be a positive solution of the heat equation (5.2) in the cube \( M^n \times [0, T] \). Then
\[
\frac{\|
abla u\|^2}{u^2} - \frac{\alpha u_t}{u} \leq C\alpha^2 K + \alpha \varphi,
\]
where \( C \) is a constant.

Using theorem 4.1, we derive a Harnack inequality.

**Corollary 5.3.** (Harnack Inequality) Let \((M^n, g(t))_{t \in [0, T]} \) be a complete solution to the Ricci flow (1.7). Suppose that \( |\text{Ric}| \leq K \) for some \( K > 0 \), and all \( (x, t) \in M^n \times [0, T] \). Assume that \( u(x, t) \) is a positive solution for (5.1). Then for all \( (x_1, t_1) \in M^n \times (0, T) \) and \( (x_2, t_2) \in M^n \times (0, T) \) such that \( t_1 < t_2 \), we have
\[
u(x_2, t_2) \leq u(x_1, t_1) \times \exp \left( \int_0^1 \frac{|\gamma(s)|^2}{2(t_2 - t_1)^2} ds + \int_{t_1}^{t_2} \frac{\alpha^2(t)}{32} dt + \int_{t_1}^{t_2} [\varphi + C\alpha K] dt \right)
\]

6. Appendix

We will check some special functions \( \alpha(t) > 1, \varphi(t) > 0 \) and \( \gamma(t) > 0 \) satisfy the following two systems
\[
\begin{cases}
\frac{2\varphi}{n} - 2\alpha K \geq \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha}, \\
\frac{2\varphi}{n} - \alpha' > 0, \\
\frac{\varphi^2}{n} + \alpha \varphi' \geq 0.
\end{cases}
\]
(6.1)
and

\[
\begin{aligned}
\gamma' - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha} &\leq 0, \\
\gamma^\alpha - (2\phi - \alpha') \frac{1}{\alpha} &\leq 0, \\
\gamma^\alpha - (2\phi - \alpha') &\leq 0,
\end{aligned}
\]  \hspace{1cm} (6.2)

Besides, \(\alpha(t)\) and \(\gamma(t)\) are non-decreasing.

(1) Let \(\alpha(t) = 1 + 2Kt, \varphi(t) = \frac{n}{t} + nK(1 + 2Kt + \mu Kt) (\mu \geq \frac{1}{4})\) and \(\gamma(t) = Kt\). One can has

(i) \[
\frac{2\varphi}{n} - \alpha' = \frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K > 0,
\]

(ii) \[
\frac{\varphi^2}{n} + \alpha \varphi' = \frac{n}{t^2} + nK^2(1 + 2Kt + \mu Kt)^2 + \frac{2nK}{t}(1 + 2Kt + \mu Kt)
\]

\[+(1 + 2Kt)(- \frac{n}{t^2} + 2nK^2 + n\mu K^2)
\]
\[= nK^2(1 + 2Kt + \mu Kt)^2 + \frac{2nK}{t}(2Kt + \mu Kt)
\]
\[+(1 + 2Kt)(2nK^2 + n\mu K^2) > 0,
\]

(iii) \[
\frac{2\varphi}{n} - 2\alpha K - (\frac{2\varphi}{n} - \alpha') \frac{1}{\alpha}
\]
\[= \frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K(1 + 2Kt)
\]
\[= \left[ \frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K \right] \cdot \frac{1}{1 + 2Kt}
\]
\[= \frac{4Kt(\mu K^2t^2 - Kt + 1)}{t(1 + 2Kt)} \geq 0, \text{ for } \mu \geq \frac{1}{4}.
\]

Hence, \(\alpha(t) = 1 + 2Kt, \varphi(t) = \frac{n}{t} + nK(1 + 2Kt + \mu Kt) (0 < \mu \leq \frac{1}{4})\) satisfy system (6.1).

On the other hand, one has

\[
\frac{\gamma'}{\gamma} - \left( \frac{2\varphi}{n} - \alpha' \right) \frac{1}{\alpha}
\]
\[= \frac{1}{t} \left( \frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K \right) \cdot \frac{1}{1 + 2Kt}
\]
\[= \frac{1}{t(1 + 2Kt)} \left[ -(4K^2 + 2K\mu)t^2 + 2Kt - 1 \right]
\]
\[= \frac{1}{t(1 + 2Kt)} \left[ -(3K^2 + 2K\mu)t^2 - (Kt - 1)^2 \right]
\]
\[\leq 0, \text{ for } t \geq 0.
\]

and \(\frac{\gamma}{\gamma'} = \frac{1}{2}\). So, (6.2) is also satisfied.

(2) \(\alpha(t) = e^{2Kt}, \varphi(t) = \frac{n}{t} e^{4Kt}\) and \(\gamma(t) = te^{2Kt}\), where \(0 < Kt \leq 1\). Direct calculation gives

(i) \[
\frac{2\varphi}{n} - \alpha' = \frac{2}{t} e^{2Kt} (e^{2Kt} - Kt) > 0,
\]

(ii) \[
\frac{\varphi^2}{n} + \alpha \varphi' = \frac{n}{t^2} e^{6Kt} (e^{2Kt} - 1 + 4Kt) > 0,
\]
(iii) \[
\frac{2\varphi}{n} - 2\alpha K - \left(\frac{2\varphi}{n} - \alpha'\right) \frac{1}{\alpha} = \frac{2}{t} e^{4Kt} - 2K e^{2Kt} - \frac{2}{t} e^{2Kt} + 2K
\]
\[
= (e^{2Kt} - 1)(\frac{2}{t} e^{2Kt} - 2K) \geq 0.
\]
Hence, \( \alpha(t) = e^{2Kt} \) and \( \varphi(t) = \frac{n}{n} e^{4Kt} \) satisfy system (6.1).

Besides, we have

\[
\frac{\alpha'}{\gamma} - \left(\frac{2\varphi}{n} - \alpha'\right) \frac{1}{\alpha} = \frac{1 + 2Kt}{t} - \left(\frac{2}{t} e^{2Kt} - 2K\right)
\]
\[
= \frac{1}{t} (1 + 4Kt - 2e^{2Kt})
\]
\[
\leq 0, \quad \text{for } t \geq 0.
\]

and as \( t \to 0^+ \), \( \frac{\alpha'}{\alpha} \to \frac{K^2}{e^{2Kt}} \to \frac{1}{\alpha_K} \). This implies \( \frac{\alpha'}{\alpha} \leq C \). So, (6.2) is also satisfied.

(3) \( \alpha(t) = 1 + \frac{\sinh(Kt) \cosh(Kt) - K}{\sinh^2(Kt)} \), \( \varphi(t) = 2nK[1 + \coth(Kt)] \) and \( \gamma(t) = \tanh(Kt) \). Direct calculation gives

(ii) \[
\frac{2\varphi}{n} - 2\alpha K - \left(\frac{2\varphi}{n} - \alpha'\right) = 4K[1 + \coth(Kt)] - 2K + 2K \coth^2(Kt) - \frac{2K^2t}{\sinh^2(Kt)} \coth(Kt)
\]
\[
= 2K + 2K(1 + \alpha) \coth(Kt) > 0,
\]

(iii) \[
\frac{\varphi^2}{n} + \alpha \varphi' = \frac{2nK^2}{\sinh^2(Kt)} \left[2(1 + \coth(Kt))^2 \sinh^2(Kt) - \alpha\right]
\]
\[
= \frac{2nK^2}{\sinh^2(Kt)} \left[2e^{2Kt} - 1 - \frac{e^{4Kt} - 1 - 4Kte^{2Kt}}{(e^{2Kt} - 1)^2}\right]
\]
\[
= \frac{4nK^2e^{2Kt}}{(e^{2Kt} - 1)^2 \sinh^2(Kt)} \left[e^{4Kt} - 3e^{2Kt} + 2 + 4Kt\right].
\]

Let \( f(x) = e^{4x} - 3e^{2x} + 2 + 4x \) with \( x \leq 0 \). Obviously, \( f(0) = 0 \) and

\[
f'(x) = 4e^{4x} - 6e^{2x} + 4 > 0.
\]

Then we get \( f(x) > 0 \) for \( x > 0 \). Hence, we have

\[
\frac{2\varphi}{n} - \alpha'\varphi + \alpha \varphi' + \alpha' \varphi - \frac{\varphi^2}{n}
\]
\[
= \frac{4nK^2e^{2Kt}}{(e^{2Kt} - 1)^2 \sinh^2(Kt)} \left[e^{4Kt} - 3e^{2Kt} + 2 + 4Kt\right] > 0.
\]
Hence, $\alpha(t) = 1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh(Kt)}$ and $\varphi(t) = 2nK[1 + \coth(Kt)]$ satisfy system (6.1).

On the other hand, as $t \to 0$, we have $\frac{2\alpha^4}{\alpha - 1} \to 2$; $\frac{2\alpha^4}{\alpha - 1} \to 1$ for $t \to \infty$. These imply $\frac{2\alpha^4}{\alpha - 1} \leq C$, here $C$ is a universal constant.

Besides, we have

\[
\gamma' = \left(\frac{2\varphi}{n} - \alpha'\right)\frac{1}{\alpha}
\]

\[
= \frac{1}{\alpha}\left[\frac{K\alpha}{\sinh(Kt)\cosh(Kt)} - 2K - 2K(1 + \alpha)\coth(Kt)\right]
\]

\[
= \frac{1}{\alpha}\left[\frac{K}{\sinh(Kt)\cosh(Kt)}[\alpha - 2(1 + \alpha)\cosh^2(Kt)] - 2K\right]
\]

\[
= \frac{1}{\alpha}\left[\frac{K}{\sinh(Kt)}[\alpha(1 - 2\cosh(Kt)) - 2\cosh(Kt)] - 2K\right]
\]

\[
\leq 0, \quad \text{for} \quad t \geq 0.
\]

So, (6.2) is also satisfied.

(4) $\alpha(t) = \text{constant}$, $\varphi(t) = \frac{an}{t} + \frac{nK\alpha^2}{\alpha - 1}$ and $\gamma(t) = t^\theta$ with $0 < \theta \leq 2$. Direct calculation gives

(i) $\frac{2\varphi}{n} - \alpha' = 2\left[\frac{\alpha n}{t} + \frac{K\alpha^2}{\alpha - 1}\right] > 0$,

(ii) $\frac{\varphi^2}{n} + \alpha\varphi' = \frac{n\alpha^2}{t^2} + \frac{n^2K^2\alpha^4}{(\alpha - 1)t} + \frac{2nK\alpha^2}{(\alpha - 1)t} - \frac{n\alpha^2}{t^2} > 0$.

(iii) $\left(\frac{2\varphi}{n} - 2\alpha K\right) - \left(\frac{2\varphi}{n} - \alpha'\right)\frac{1}{\alpha}$

\[
= \frac{2\varphi}{n\alpha}(\alpha - 1) - 2K\alpha
\]

\[
\geq \frac{2}{n\alpha}(\alpha - 1)\frac{nK\alpha^2}{\alpha - 1} - 2K\alpha = 0.
\]

Hence, $\alpha(t) = \text{constant}$, and $\varphi(t) = \frac{an}{t} + \frac{nK\alpha^2}{\alpha - 1}$ satisfy system (6.1).

On the other hand, we have

\[
\gamma' = \left(\frac{2\varphi}{n} - \alpha'\right)\frac{1}{\alpha}
\]

\[
= \frac{\theta}{t} - \frac{2}{\alpha - 1} - \frac{2K\alpha}{\alpha - 1}
\]

\[
\leq 0, \quad \text{for} \quad t \geq 0 \quad \text{and} \quad 0 < \theta \leq 2.
\]

So, (6.2) is also satisfied.

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References

[1] M. Bailesteanu, X. D. Cao, A. Pulemomov, Gradient estimates for the heat equation under the Ricci flow, J. Funct. Anal., 258 (2010), 3517-3542.

[2] E. Calabi, An extension of E. Hopf’s maximum principle with an application to Riemannian geometry, Duke Math. J. 25 (1958), 45-56. MR19,1056eZbl0079.11801
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[3] H. Cao, M. Zhu, Aronson-Bénilan estimates for the porous medium equation under the Ricci flow, Journal De Mathmatiques Pures Et Appliqus, 2015, 104(4):90-94

[4] X. D. Cao, B. F. Ljungberg, B. W. Liu, Differential Harnack estimates for a nonlinear heat equation, J. Funct. Anal., (2003): 1-19

[5] D. G. Chen, C.W. Xiong, Gradient estimates for doubly nonlinear diffusion equations. Nonlinear Anal. 112 (2015), 156-164.

[6] B. Chow and D. Knopf, The Ricci flow: An introduction, mathematical Surveys and Monographs 110, American Society, Providence, RI, 2004 MR2005e:53101z

[7] C. M. Guenter, The fundamental solution on manifolds with time-dependent metric, J. Geom. Anal. 12:3 (2002), 425-456. MR2003a:58034z

[8] R. S. Hamilton, A matrix Harnack estimates for the heat equation, Comm. Anal. Geom., 1 (1993), 113-126.

[9] R. S. Hamilton, Three manifolds with positive Ricci curvature, J. Differential Geom. 17:2 (1982): 255-306 MR84a:53050

[10] R. S. Hamilton, The formation of singularities in the Ricci flow, pp. 7-136 in Surveys in differential geometry, II (Cambridge, MA, 1993), edited by S. T. Yau, International, Cambridge, MA, 1995. MR97e:53075

[11] G. Y. Huang, Z. J. Huang, H. Z. Li, Gradient estimates and differential Harnack inequalities for a nonlinear parabolic equation on Riemannian manifolds, Ann Glob Anal Geom, (2013) 43: 209-232.

[12] S. Kuang, Q. S. Zhang, A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow, J. Funct. Anal. 255:4 (2008), 1008-1023. MR2439602

[13] H. Li, H. Bai, G. Zhang, Hamilton's gradient estimates for fast diffusion equation under the Ricci flow, J. Math. Anal. Appl., 444 (2016) 1372-1379.

[14] P. Li, S. T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math., 156 (1986), 153-201. MR87f:58156

[15] J. Y. Li, Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds, J. Funct. Anal., 100 (1991), 233-256.

[16] J. Y. Li, Gradient estimate for the heat kernel of a complete Riemannian manifold and its applications, J. Funct. Anal., 97:2 (1991), 293-310. MR92f:58174

[17] J. Li and X. Xu, Defferential Harnack inequalities on Riemannian manifolds I: Linear heat equation, Adv. in Math., 226 (2011),4456-4491.

[18] S. P. Liu, Gradient estimates for solutions of the heat equation under flow, Pacific J. of Math., 243 (1), (2009), 165-179. MR2010g:53122

[19] Y. Li, X. R. Zhu, Harnack estimates for a heat-type equation under the Ricci flow, J. Differential Equations 260 (2016), 3270-3301

[20] P. Lu, L. Ni, J. L. Vázquez and C. Villani, Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds, J. Math. Pures Appl. 91 (2009), 1-19.

[21] L. Ma, Gradient estimate for a sample elliptic equation on non-compact Riemannian manifolds, J. Funct. Anal., 241 (2006), 374-382 MR2264255

[22] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, preprint, 2002, arXiv.math.DG/0211159

[23] G. Perelman, Ricci flow with surgery on three manifolds, preprint, 2003, arXiv.math.DG/0203109

[24] W. X. Shi, Deforming the metric on complete Riemannian manifolds, J. Differential Geom., 30: 1 (1989), 223-301. MR90i:53044

[25] B. H. Song, The Harnack Estimate for a Nonlinear Parabolic Equation under the Ricci Flow, Acta Mathematica Sinica, English Series 27(10),(2011): 1935C1940 DOI:10.1007/s10479-011-0747-z

[26] J. Sun, Gradient estimates for positive solutions of the heat equation under geometric flow, Pacific J. Math. 253, 489-510 (2011)

[27] H. J. Sun, Higher Eigenvalue Estimates on Riemannian Manifolds with Ricci Curvature Bounded Below, Acta Math. Sinica (Chin. Ser.) 49 (2006), 3, 539-548.

[28] W. Wang P. Zhang, Some Gradient Estimates and Harnack Inequalities for Nonlinear Parabolic Equations on Riemannian Manifolds, Mathematische Nachrichten, 1-13 (2016) DOI:10.1002/mana.201500287
[29] J. Y. Wu, Li-Yau type estimates for a nonlinear parabolic equation on complete manifolds, 
J. Math. Anal. Appl., 369 (2010), 400-407.
[30] Y. Y. Yang, Gradient estimate for a nonlinear parabolic equation on Riemannian manifold, 
Proc. Amer. Math. Soc. 136 (2008), 4095-4102.
[31] L. Zhao, S. Fang, Gradient estimates for a nonlinear lichnerowicz equation under generalize 
geometric flow on complete noncompact manifolds, Pacific J. Math. 285 (1) (2016):243-256.
[32] X. B. Zhu, Gradient estimates and Liouville theorems for nonlinear parabolic equations on 
noncompact Riemannian manifolds, Nonlinear Analysis, 74 (2011), 5141-5146.

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