ABSTRACT. The main aim of this paper is to extend Bochner’s technique to statistical structures. Other topics related to this technique are also introduced to the theory of statistical structures. It deals, in particular, with Hodge’s theory, Bochner-Weitzenböck and Simon’s type formulas. Moreover, a few global and local theorems on the geometry of statistical structures are proved, for instance, theorems saying that under some topological and geometrical conditions a statistical structure must be trivial. We also introduce a new concept of sectional curvature depending on statistical connections. On the base of this notion we study the curvature operator and prove some analogues of well-known theorems from Riemannian geometry.

1. INTRODUCTION

The main tool of the Bochner technique is the Levi-Civita connection. Our purpose is to show that the technique can be extended to the class of statistical connections.

We shall study the following four cases:

i) A torsion-free connection $\nabla$ is statistical for a metric tensor field $g$, that is, $\nabla g$ is symmetric. A statistical structure, that is, a pair $(g, \nabla)$, where $\nabla$ is statistical for $g$ is also called a Codazzi pair.

ii) A statistical structure $(g, \nabla)$ is equiaffine, that is, there is a volume form $\nu$ such that $\nabla \nu = 0$.

iii) A statistical structure $(g, \nabla)$ is equiaffine relative to the volume form $\nu_g$ determined by $g$. It is equivalent to the condition $\text{tr}_g \nabla g(X, \cdot, \cdot) = 0$ for every $X$. We shall call such structures trace-free.

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iv) For a statistical structure \((g, \nabla)\) the curvature tensors for \(\nabla\) and its conjugate connection \(\nabla^*\) are the same.

The oldest examples of statistical structures are the induced structures (consisting of the second fundamental form and the induced connection) on locally strongly convex hypersurfaces in \(\mathbb{R}^{n+1}\) endowed with an equiaffine transversal vector field (in other words – with relative normalization). Within the theory of equiaffine hypersurfaces the case iii) corresponds to Blaschke hypersurfaces, the case iv) – to equiaffine spheres.

But the majority of statistical structures is outside the class of hypersurfaces. Even using the structures obtained on hypersurfaces one can easily modify them and get structures which are not realizable on hypersurfaces. For instance, the product of equiaffine ovaloids is equipped with the product statistical structure but it cannot be realized as a locally strongly convex hypersurface in any \(\mathbb{R}^{n+1}\). It is also easy to find examples of statistical structures which are non-realizable on hypersurfaces even locally.

In Section 2 we provide preliminary information on divergences for volume forms and connections and establish few integral formulas useful in proving classical Bochner’s theorems and their generalizations to statistical structures (e.g. in Section 10).

Basic notions for statistical structures and their subclasses listed above are introduced and discussed in Section 3. Examples of various types of statistical structures are given in Section 4.

In Section 5 we prove a few global theorems saying that under some topological and geometrical assumptions two statistical structures with the same metric must be identical or a statistical structure must be trivial, that is, the statistical connection is the Levi-Civita connection.

On a statistical manifold one can define various Laplacians (acting on differential forms). First we have the Laplacian for the underlying Riemannian structure. One can also define the codifferential \(\delta^*\) relative to a statistical connection \(\nabla\) and then set

\[
\Delta = \delta^* d + d\delta^*,
\]

where \(\nabla^*\) is the conjugate connection for \(\nabla\). For this Laplacian we prove basic properties for compact manifolds and Hodge-type theorems (Sections 8, 9). A differential form \(\omega\) will be called \(\nabla\)-harmonic if \(\Delta \omega = 0\). Bochner’s technique for vector fields and harmonic 1-forms is developed in Sections 10 and 11. Bochner-Weitzenböck formulas for Laplacians acting on differential forms are computed in Section 11. There we also compute Simons’ type formulas for the Laplacians of the square of the length of any tensor field. The Bochner-Weitzenböck curvature operator can be also applied to other tensor fields, in particular, to the metric tensor field of statistical structures.

Another aim of the paper is to introduce a notion of sectional curvature for statistical structures. The curvature tensor of \(\nabla\) does not have, in general, as good symmetries as the curvature tensor of the Levi-Civita connection. But one can modify it and get a tensor field with the same symmetries as the Riemannian curvature tensor. Using the modified tensor one can define an appropriate notion of sectional curvature and the corresponding curvature operator. After the modification there is still a problem with the second Bianchi identity, which plays an essential role in many theorems, e.g. Schur’s lemma or Tachibana’s theorem. In the general case of
statistical structures, Schur’s lemma does not hold. By restricting considerations to the class iv) we get appropriate analogues of such theorems.

2. DIVERGENCES AND INTEGRAL FORMULAS

All the objects considered in this paper are of class $C^\infty$. All connections are linear and torsion-free. Let $M$ be an $n$-dimensional manifold with a fixed volume form $\nu$. For any vector field $X$ on $M$ its Lie derivative $L_X \nu$ is an $n$-form, hence

$$L_X \nu = (\text{div} \nu) X.$$

The function $\text{div} \nu X$ is the divergence relative to the volume form $\nu$. A divergence can also be defined relative to a connection. Namely, if $\nabla$ is a connection, then

$$\text{div} \nabla X = \text{tr} \{Y \rightarrow \nabla_Y X\}$$

for a vector field $X$. More generally, for any tensor field $s$ of type $(1, k)$ we have

$$\text{div} \nabla s(X_1, \ldots, X_k) = \text{tr} \{Y \rightarrow (\nabla_Y s)(X_1, \ldots, X_k)\}.$$

**Lemma 2.1.** For any connection $\nabla$ on $M$ and a tensor field $S$ of type $(1, 1)$ we have

$$X \text{tr} S = \text{tr} \nabla_X S.$$

**Proof.** In the equality (4) both sides depend on $X$ in a tensorial way. Let $x \in M$ and $X \in T_x M$. Take a local frame $e_1, \ldots, e_n$ and its dual frame $\theta_1, \ldots, \theta_n$ defined around $x$ such that $\nabla e_i = 0$, $\nabla \theta^i = 0$ at $x$ for $i = 1, \ldots, n$. Since at $x$

$$0 = (\nabla_X \theta^i)(Se_i) = X(\theta^i(Se_i)) - \theta^i(\nabla_X (Se_i)),$$

we have (at $x$)

$$X \text{tr} S = \sum_{i=1}^n X(\theta^i(Se_i)) = \sum_{i=1}^n \theta^i(\nabla_X (Se_i))$$

$$= \sum_{i=1}^n \theta^i((\nabla_X S)e_i) = \text{tr} \nabla_X S. \quad \square$$

The equality (4) can be written as

$$\nabla_X (\text{tr} S) = \text{tr} (\nabla_X S).$$

Denote by $R$ the curvature tensor of $\nabla$ and by $\text{Ric}$ its Ricci tensor.

**Lemma 2.2.** Let $\nabla$ be a connection on $M$. For a vector field $X$ on $M$ we set

$$S_X Y = \nabla_Y X.$$

$S_X$ is a $(1, 1)$-tensor field and $\text{div} \nabla X = \text{tr} S_X$. We have

$$L_X = \nabla_X - S_X.$$

For any vector fields $X, Y$ on $M$ the following formula holds

$$\text{div} \nabla (\nabla_X Y) = \text{Ric}(X, Y) + \text{tr} (\nabla_X S_Y) + \text{tr} (S_Y \circ S_X)$$

$$= \text{Ric}(X, Y) + X(\text{tr} S_Y) + \text{tr} (S_Y \circ S_X)$$

$$= \text{Ric}(X, Y) + X(\text{div} \nabla Y) + \text{tr} (S_Y \circ S_X).$$
Proof. The equality (7) is well known and it immediately follows from the fact that $L_X$ and $\nabla_X$ are differentiations. For any vector fields $X, Y, Z$ on $M$ the following equalities hold

$$\nabla_Z \nabla_X Y = R(Z, X)Y + \nabla_X \nabla_Z Y + \nabla_{[Z, X]} Y$$

$$= R(Z, X)Y + (\nabla_X \nabla_Z Y - \nabla_{\nabla_X Z} Y) + \nabla_{\nabla_Z X} Y$$

$$= R(Z, X)Y + (\nabla_X S_Y)Z + (S_Y \circ S_X)Z.$$  

Taking the trace relative to $Z$ on both sides we obtain the required equality. 

\[ \square \]

**Lemma 2.3.** Let $\nabla$ be a connection and $\nu$ be a volume form on $M$. Then

$$\text{div}\nabla X = \text{div}^\nu X + \tau(X)$$

for any $X \in \mathcal{X}(M)$, where $\nabla_X \nu = -\tau(X)\nu$.

Proof. It follows from the equality $\nabla_X \nu = L_X \nu + S_X \nu$. 

According to [3], by an equiaffine structure we mean a pair $(\nabla, \nu)$ consisting of a connection $\nabla$ and a volume form $\nu$ such that $\nabla \nu = 0$. Thus for an equiaffine structure $(\nabla, \nu)$ we have $\text{div} \nabla = \text{div}^\nu$.

**Lemma 2.4.** If $M$ is a compact manifold with a volume form $\nu$ and $\nabla$ is a connection on $M$, then for any vector fields $X, Y$ on $M$ we have

$$\int_M \text{Ric}(X, Y)\nu = \int_M \text{tr} S_Y (\text{div}^\nu X)\nu - \int_M \text{tr} (S_Y \circ S_X)\nu + \int_M \tau(\nabla X Y)\nu.$$  

In particular, if $\nabla \nu = 0$ then

$$\int_M \text{Ric}(X, Y)\nu = \int_M \text{tr} S_Y \text{tr} S_Y \nu - \int_M \text{tr} (S_Y \circ S_X)\nu.$$  

Proof. Let $X, Y$ be vector fields on $M$. Set $\varphi = \text{tr} S_Y$. We have

$$L_X (\varphi \nu) = (X \varphi)\nu + \varphi L_X \nu = (X (\text{tr} S_Y))\nu + (\text{tr} S_Y \text{div}^\nu X)\nu.$$  

and by Stokes’ theorem we get

$$\int_M X (\text{tr} S_Y)\nu = -\int_M \text{tr} S_Y (\text{div}^\nu X)\nu.$$  

Using this equality, Lemmas 2.2, 2.3 and the divergence theorem we obtain the result. 

In the 2-dimensional case we get

**Proposition 2.5.** Let $M$ be a compact 2-dimensional manifold with an equiaffine structure $(\nabla, \nu)$. For each vector field $X$ on $M$ we have

$$\int_M \text{Ric}(X, X)\nu = 2 \int_M (\text{det} S_X)\nu.$$  

Proof. For any endomorphism $A$ of a 2-dimensional vector space we have $(\text{tr} A)^2 - \text{tr} A^2 = 2 \text{det} A$ 

If an endomorphism $A$ of a real vector space is diagonalizable, then $\text{tr}(A^2) \geq 0$. Therefore, by (11), we get
**Proposition 2.6.** If $M$ is a compact manifold with an equiaffine structure $(\nabla, \nu)$, a vector field $X \in X(M)$ is without divergence and $S_X$ is diagonalizable at each point $M$, then

$$
\int_M \text{Ric}(X, X)\nu \leq 0.
$$

3. Statistical structures

For a tensor field $s$ and a connection $\nabla$ the notation $\nabla s(X, ...)$ will stand for $(\nabla_X s)(...)$.

Let $g$ be a positive definite Riemannian tensor field on a manifold $M$. We assume that $M$ is oriented. Denote by $\nabla$ the Levi-Civita connection for $g$ and by $\nu_g$ the volume form determined by $g$. We shall study (torsion-free) connections $\nabla$ satisfying the following Codazzi condition:

$$
(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)
$$

for all $X, Y, Z \in T_x M$, $x \in M$. A structure $(g, \nabla)$ satisfying (14) is called a statistical structure. We shall call a connection $\nabla$ satisfying (14) a statistical connection for $g$. Since

$$
2\nabla_X \nu_g = \text{tr}_g(\nabla_X g)(\cdot, \cdot)\nu_g,
$$

the condition $\nabla \nu_g = 0$ is equivalent to the condition

$$
\text{tr}_g(\nabla_X g)(\cdot, \cdot) = 0
$$

for every $X \in TM$. If $\nabla$ is statistical for $g$ and (16) is satisfied, we shall say that the statistical structure is trace-free.

For any connection $\nabla$ one defines its conjugate $\overline{\nabla}$ relative to $g$ by the formula

$$
g(\nabla_X Y, Z) + g(Y, \overline{\nabla}_X Z) = X g(Y, Z).
$$

It is known that if $(g, \nabla)$ is trace-free then so is $(g, \overline{\nabla})$, if $(g, \nabla)$ is a statistical structure then so is $(g, \overline{\nabla})$, see e.g. [3]. Hence statistical structures go in pairs.

If $R$ is the curvature tensor for $\nabla$ and $\overline{R}$ is the curvature tensor for $\overline{\nabla}$, then we have,

$$
g(R(X, Y)Z, W) = -g(\overline{R}(X, Y)W, Z).
$$

It follows that

$$
\overline{\text{Ric}}(Y, W) = -\text{tr}_g g(R(\cdot, Y), W),
$$

where $\overline{\text{Ric}}$ is the Ricci tensor of $\overline{\nabla}$. The function

$$
\rho = \text{tr}_g R(\cdot, \cdot)
$$

will be called the scalar curvature of $(g, \nabla)$. Similarly we define the scalar curvature $\overline{\rho}$ for $(g, \overline{\nabla})$ and we have the usual scalar curvature $\hat{\rho}$ for $(g, \hat{\nabla})$. By (19) we have

$$
\rho = \overline{\rho}.
$$

From now on in this section we assume that $\nabla$ is statistical for $g$. If $K$ is the difference tensor between $\nabla$ and $\hat{\nabla}$, that is,

$$
\nabla_X Y = \hat{\nabla}_X Y + K_X Y,
$$

then

$$
\text{Ric}(X, X)\nu \leq 0.
$$
Comparing this equality with (15) we see that $g$ with Lemma 2.3. We have $(27)$ tr $\tau$ for any $X$ defined by $S_X Y = \nabla_X Y = \hat{S}_X Y = \nabla_X Y$. It is clear that the symmetry of $\nabla g$ and $K$ implies the symmetry of $K_X$ relative to $g$ for each $X$. The converse also holds. Namely, if $K_X$ is symmetric relative to $g$ then we have $(25)$ $\nabla g(X, Y, Z) = -2g(K_XY, Z)$. Set $(26)$ $E = \text{tr}_g K(\cdot, \cdot)$. If $\tau(X) := \text{tr} K_X$ then $\tau(X) = g(E, X)$. By $(25)$ we have $(27)$ $\text{tr}_g \nabla g(\cdot, \cdot, Z) = -2g(E, Z) = -2\tau(Z)$. Comparing this equality with (15) we see that $\nabla_Z \nu_g = -\tau(Z)\nu_g$ (compare also with Lemma 2.3). We have $g(\nabla_X g, \nabla_X g) = 4g(K_X, K_X)$ and since $\nabla g(X, Y, Z) = 2g(K_XY, Z)$, we also have $(28)$ $g(\nabla_X g, \nabla_X g) = 4g(K_X, K_X) = g(\nabla_X g, \nabla_X g)$. Consequently $(29)$ $g(\nabla g, \nabla g) = 4g(K, K) = g(\nabla g, \nabla g)$. Observe also that $(30)$ $g(\nabla_X, \nabla X) = g(\nabla_X, \nabla X) - g(K_X, K_X)$ for any $X \in \mathcal{X}(M)$. Indeed, one has

$$g(\nabla_Y X, \nabla_Y X) = g(\hat{\nabla}_Y X + K_Y X, \hat{\nabla}_Y X - K_Y X) = g(\nabla_Y X, \nabla_Y X) - g(K_Y X, K_Y X).$$

Similarly $(31)$ $g(\nabla_X, \nabla X) + g(\nabla_Y, \nabla X) = 2\{g(\nabla_X, \nabla X) + g(K_X, K_X)\}$. For a vector field $X$ we have three $(1, 1)$-tensor fields $S_X, \hat{S}_X$ and $\nabla_S$ defined by $S_X Y = \nabla_X Y$, $\hat{S}_X Y = \nabla_X Y$, $\nabla_S = \nabla_X Y$. It is known that

$$R(X, Y) = \hat{R}(X, Y) + (\nabla_X K)_Y - (\nabla_Y K)_X + [K_X, K_Y].$$

Writing the same equality for $\nabla$ and adding both equalities we get

$$R(X, Y) + \hat{R}(X, Y) = 2\hat{R}(X, Y) + 2[K_X, K_Y].$$
We also have
\[
\text{tr} \{ X \to [K_X, K_Y] Z \} = \sum_i^n g(e_i, [K_{e_i}, K_Y] Z)
\]
\[
= \sum_i^n (g(e_i, K_{[e_i, K_Y] Z}) - g(e_i, K_Y K_{e_i} Z))
\]
\[
= \sum_i^n (g(K_{e_i} e_i, K_Y Z) - g(K_Y e_i, K_Z e_i))
\]
\[
= -g(K_Y, K_Z) + \tau(K(Y, Z)).
\]

Choose now a point \( x_0 \) and an orthonormal frame \( e_1, ..., e_n \) around \( x_0 \) such that \( \nabla e_i = 0 \) for \( i = 1, ..., n \) at \( x_0 \). Having \( Y, Z \in T_{x_0}M \) we extend the vectors to vector fields, say \( Y, Z \), around \( x_0 \) in such a way that \( \nabla Y = \nabla Z = 0 \) at \( x_0 \). In particular, \( [Y, Z] = 0 \) at \( x_0 \). We obtain at \( x_0 \)
\[
\sum_{i=1}^n \left[ g((\nabla_{e_i} K)(Y, Z), e_i) - g((\nabla_Y K)(e_i, Z), e_i) \right]
\]
\[
= (\text{div}^\nabla K)(Y, Z) - \sum_{i=1}^n Y g(K(e_i, Z), e_i)
\]
\[
= (\text{div}^\nabla K)(Y, Z) - \sum_{i=1}^n Y g(K(e_i, e_i), Z)
\]
\[
= (\text{div}^\nabla K)(Y, Z) - Y g(E, Z)
\]
\[
= (\text{div}^\nabla K)(Y, Z) - Y \tau(Z)
\]
\[
= (\text{div}^\nabla K)(Y, Z) - \nabla \tau(Y, Z).
\]

Therefore
\[
(35) \quad \text{Ric}(Y, Z) = \text{Ric}^\nabla(Y, Z) + (\text{div}^\nabla K)(Y, Z) - \nabla \tau(Y, Z) + \tau(K(Y, Z)) - g(K_Y, K_Z).
\]

It follows that
\[
(36) \quad \text{Ric}(Y, Z) + \text{Ric}^\nabla(Y, Z) = 2\text{Ric}^\nabla(Y, Z) - 2g(K_Y, K_Z) + 2\tau(K(Y, Z)).
\]

In particular, if \( (g, \nabla) \) is trace-free then
\[
(37) \quad 2\text{Ric}^\nabla(X, X) \geq \text{Ric}(X, X) + \text{Ric}^\nabla(X, X).
\]

The above formulas also yield
\[
(38) \quad \text{Ric}(Y, Z) - \text{Ric}(Z, Y) = -g((\nabla K(Y, e_i, Z), e_i) + g((\nabla K(Z, e_i, Y), e_i) = -d\tau(Y, Z).
\]

Hence \( \nabla \) is Ricci-symmetric if and only if \( d\tau = 0 \). The following lemma follows from formulas (18), (33) and (34).

**Lemma 3.1.** Let \( (g, \nabla) \) be a statistical structure. The following conditions are equivalent:
1) \( R = \overline{R} \),
2) \( \nabla K \) is symmetric,
3) \( g(R(X, Y)Z, W) \) is skew-symmetric relative to \( Z, W \).
From 2) and (38) we see that the condition $R = T R$ implies the symmetry of $R i c$. We have proved

**Proposition 3.2.** Let $(g, \nabla)$ be a statistical structure. $R i c$ is symmetric if and only if $d \tau = 0$. If $R = T R$ then $R i c = T R i c$ is symmetric.

Taking now the trace relative to $g$ on both sides of (36) and taking into account that $\rho = \bar{\rho}$, we get

$$\hat{\rho} = \rho + |K|^2 - |E|^2.$$  

In the case where $\nabla$ is the induced connection on a Blaschke hypersurface in $R^{n+1}$ and $g$ is the Blaschke metric, the equality (39) (with $E = 0$) is known as the affine theorema egregium. Indeed, if $H$ is the affine mean curvature then $H = \frac{n}{2} (\frac{n}{2} - 1) \rho$ and $|K|^2 = 4n(n-1)J$, where $J$ is the Pick invariant.

For an orthonormal frame $e_1, \ldots, e_n$ we have

$$|K|^2 = g(K, K) = \sum_{i,j,k} g(K_{ei} e_j, e_k)^2,$$

$$|E|^2 = g(E, E) = \sum_{j,k} g(K_{ej} e_j, e_k)^2.$$  

Thus $|K|^2 - |E|^2 \geq 0$ on $M$. If $|K| = |E|$ then $0 = g(K_{ei} e_j, e_k) = g(K_{ek} e_i, e_j)$ for every $k$ and $i \neq j$. It follows that $K_X$ is a multiple of the identity for each $X$, which is possible only for $K = 0$. Thus we have

**Proposition 3.3.** The functional $\text{scal} : \{\text{statistical connections for } g\} \ni \nabla \rightarrow \text{tr}_g R i c \in C^\infty(M)$ attains its maximum for the Levi-Civita connection at each point of $M$. Conversely, if $\nabla$ is a statistical connection for $g$ and $\text{scal}$ attains its maximum for $\nabla$ at each point on $M$, then $\nabla$ is the Levi-Civita connection for $g$.

**Corollary 3.4.** Let $(g, \nabla)$ be a statistical structure on $M$ and $\rho \geq \hat{\rho}$ on $M$. Then $\nabla$ is the Levi-Civita connection for $g$.

We shall also study equiaffine statistical structures. By an equiaffine statistical structure on $M$ we mean a triple $(g, \nabla, \nu)$, where $(g, \nabla)$ is a statistical structure and $\nu$ is a volume form on $M$ such that $\nabla \nu = 0$. Let us emphasize that $\nu$ is not necessarily the volume form $\nu_g$.

4. Examples

The theory of affine hypersurfaces in $R^{n+1}$ is a natural source of statistical structures. For the theory we refer to [1] or [3]. We recall here only some basic facts.

Let $f : M \rightarrow R^{n+1}$ be a locally strongly convex hypersurface. For simplicity assume that $M$ is connected and oriented. Let $\xi$ be a transversal vector field on $M$. We define the induced volume form $\nu_\xi$ on $M$ (compatible with the given orientation) as follows

$$\nu_\xi (X_1, \ldots, X_n) = \det (f_\ast X_1, \ldots, f_\ast X_n, \xi).$$  

We also have the induced connection $\nabla$ and the second fundamental form $g$ defined by the Gauss formula:

$$D_X f_\ast Y = f_\ast \nabla_X Y + g(X, Y) \xi,$$
where $D$ is the standard flat connection on $\mathbb{R}^{n+1}$. Since the hypersurface is locally strongly convex, $g$ is definite. By multiplying $\xi$ by $-1$, if necessary, we can assume that $g$ is positive definite. A transversal vector field is called equiaffine if $\nabla \nu_\xi = 0$. This condition is equivalent to the fact that $\nabla g$ is symmetric, i.e. $(g, \nabla)$ is a statistical structure. It means, in particular, that for a statistical structure obtained on a hypersurface by a choice of a transversal vector field, the Ricci tensor of $\nabla$ is automatically symmetric.

For later use recall the notion of the shape operator and the Gauss equations. Having a chosen equiaffine transversal vector field and differentiating it we get the Weingarten formula

$$D_X \xi = -\mathfrak{f}_\xi S X.$$

The tensor field $S$ is called the shape operator for $\xi$. If $R$ is the curvature tensor for the induced connection $\nabla$ then

$$(40) \quad R(X, Y)Z = g(Y, Z)S X - g(X, Z)S Y.$$

This is the Gauss equation for $R$. The Gauss equation for $\overline{R}$ is the following

$$(41) \quad \overline{R}(X, Y)Z = g(Y, SZ)Y - g(X, SZ)X.$$

In particular, the dual connection is projectively flat. Recall also that the form $g(SX, Y)$ is symmetric for any equiaffine transversal vector field.

For a locally strongly convex hypersurface there are infinitely many equiaffine transversal vector fields. In fact, if $\xi$ is any equiaffine transversal vector field for $\mathbf{f}$ (for instance a metric normal vector field) and $\phi$ is a nowhere vanishing function on $M$, then $\tilde{\xi} = \mathfrak{f}_\xi Z + \phi \xi$ is equiaffine, where $g(Z, X) = X \phi$. We also have the volume form determined by $g$ on $M$. In general, this volume form is not covariant constant relative to $\nabla$. It can be proved that there is a unique equiaffine transversal vector field $\xi$ such that $\nu_\xi = \nu_g$. This unique transversal vector field is called the affine normal vector field. The second fundamental form for the affine normal is called the Blaschke metric. If the affine lines determined by the affine normal vector field meet at one point or are parallel then the hypersurface is called an affine sphere. In the first case the sphere is called proper in the second one improper. The class of affine spheres is very large. There exist a lot of conditions characterizing affine spheres. For instance, a hypersurface is an affine sphere if and only if $R = \overline{R}$, see Lemma 12.5 below.

As we have already observed, if $\nabla$ is a connection on a hypersurface induced by an equiaffine transversal vector field then the conjugate connection $\overline{\nabla}$ is projectively flat. Therefore the projective flatness of the conjugate connection is a necessary condition for $(g, \nabla)$ to be realizable as the induced structure on a hypersurface.

We will now make few remarks on statistical structures in general, that is, possibly non-realizable on hypersurfaces.

As we have mentioned in the introduction, the cartesian product of statistical manifolds is a statistical manifold which cannot be realized as a locally strictly convex hypersurface.

We shall now produce other statistical structures which are non-realizable on hypersurfaces.

Observe that if $(g, \nabla)$ is a statistical structure with the difference tensor $K$ and $\phi$ is any smooth function on $M$ then

$$\tilde{\nabla} = \nabla + \phi K = \tilde{\nabla} + (1 + \phi) K$$
is a statistical connection for \( g \). Moreover, if \((g, \nabla)\) is trace-free then so is \((g, \tilde{\nabla})\). If \( \phi \) is constant and \( R = \overline{R} \) then \( \tilde{R} = \hat{R} \). Indeed, in this case we have \( \tilde{\nabla}((1 + \phi)K) = (1 + \phi)(\nabla K) \) and we can now use the above lemma. We now have

\textbf{Proposition 4.1.} Assume that \( f : M \to \mathbb{R}^{n+1} \), where \( n > 2 \), be a locally strongly convex affine sphere equipped with the statistical structure \((g, \nabla)\) described above. Assume that the sectional curvature for \( g \) is not constant on \( M \). There is no \( t \in \mathbb{R} \setminus \{0, -2\} \) such that \((g, \tilde{\nabla})\) is realizable on a hypersurface \( \tilde{f} : M \to \mathbb{R}^{n+1} \), where \( \tilde{\nabla} = \nabla + tK \).

Proof. The connection \( \nabla \) is projectively flat, hence

\[ R(X, Y)Z = \overline{R}(X, Y)Z = \gamma(Y, Z)X - \gamma(X, Z)Y \]

for some \((0, 2)\)-tensor field \((\text{the normalized Ricci tensor for } \nabla)\). Since \( \nabla K \) is symmetric, we have

\[ R(X, Y) = \hat{R}(X, Y) + [K_X, K_Y]. \]

We now have

\[ \tilde{R}(X, Y) = \hat{R}(X, Y) + (1 + t^2)[K_X, K_Y] = R(X, Y) + t(2 + t)[K_X, K_Y]. \]

Suppose that \( \overline{R} (= \hat{R}) \) is projectively flat. If \( t \neq 0 \) and \( t \neq -2 \) then

\[ [K_X, K_Y]Z = \gamma_1(Y, Z)X - \gamma_1(X, Z)Y \]

for some \((0, 2)\)-tensor field \( \gamma_1 \). But it means that \( \tilde{\nabla} \) is projectively flat, which contradicts the assumption that the sectional curvature of \( g \) is not constant. \( \square \)

Note that all affine spheres whose Blaschke metric has constant sectional curvature are known. These are quadrics (for which \( \nabla = \hat{\nabla} \)) or hypersurfaces given by the equations

\[ x_1 \cdot \ldots \cdot x_{n+1} = c, \]

where \( x_1, \ldots, x_{n+1} \) are the canonical coordinates in \( \mathbb{R}^{n+1} \) and \( c = \text{const} \neq 0 \), see Theorem 2.2.3.18 in [1].

Come back to the observation that for a statistical structure realizable on a hypersurface in \( \mathbb{R}^{n+1} \) the Ricci tensor of its connection must be symmetric. Assume we have a locally strongly convex hypersurface equipped with an equiaffine transversal vector field and the induced statistical structure on it. Assume that it is not trace-free. We have the non-zero vector field \( E \) and its dual form \( \tau \). Let \( \phi \) be a function on \( M \) such that \( d\phi \neq \tau \). Consider the connection \( \tilde{\nabla} = \nabla + \phi K \). Then \( \tilde{\tau} = (1 + \phi)\tau \). Since the Ricci tensor for the statistical structure is symmetric if and only if \( d\tau = 0 \) and \( d(\phi \tau) = d\phi \wedge \tau \), we see that \( d\tilde{\tau} \neq 0 \). Hence \( \tilde{\nabla} \) is not Ricci symmetric and consequently \((g, \tilde{\nabla})\) cannot be realized on a hypersurface.

5. Further properties of statistical structures

For a given metric tensor \( g \) one has, in general, many statistical connections. Given a connection one also has, in general, many metric tensor fields constituting with the connection a statistical structure. But if we impose additional conditions on the structures and manifolds, the situation might change drastically.

Consider, for instance, the following problem. Let \((g_1, \nabla)\) be a trace-free statistical structure on \( M \). Does there exist another metric tensor \( g_2 \) (non-homothetic to
\(g_1\) on \(M\) for which \((g_2, \nabla)\) is a trace-free statistical structure. For structures realizable on hypersurfaces at least 3-dimensional the answer is negative. The answer is also negative for 2-dimensional ovaloids in \(\mathbb{R}^3\). In the last case, the theorem is, in fact, true for abstract compact 2-dimensional manifolds of genus 0. To illustrate this type of consideration we give a proof of this fact.

Assume that \(M\) is 2-dimensional connected and oriented. Having a statistical structure on \(M\) and, in particular, a positive definite metric tensor field, we also have the underlying complex structure on \(M\). Theorem 5.3. Let \(M\) be a connected compact surface of genus 0. To illustrate this type of consideration we give a proof of this fact.

\[ \text{tr}_g s(\cdot, X_3, \ldots, X_k) = 0. \]

Let \(z = x + iy\) be an isothermal coordinate on \(M\) and \(X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}\). Denote by \(\mathcal{G}_k(M)\) the space of all complex symmetric \(k\)-forms on \(M\). By a straightforward purely algebraic computation one gets

**Lemma 5.1.** For \(s \in \mathcal{G}_k(M)\) the symmetric complex form

\[ \Phi(s) = [s(X, \ldots, X) - i s(Y, X, \ldots, X)] dz^k \]

is well-defined on the whole of \(M\), i.e. it is independent of a choice of isothermal coordinates. The mapping

\[ \Phi : \mathcal{G}_k(M) \ni s \to \Phi(s) \in \mathcal{G}_k^c(M) \]

is a linear isomorphism (over \(C^\infty(M)\)).

A symmetric tensor \(s\) is called a Codazzi tensor for a connection \(\nabla\) if \(\nabla s\) is symmetric. By computing the Cauchy-Riemann or, in a general version, Vekua-Carleman equations one gets

**Lemma 5.2.** If a symmetric tensor field \(s \in \mathcal{G}_k(M)\) is a Codazzi tensor for the Levi-Civita connection \(\tilde{\nabla}\) then the form \(\Phi(s) = [s(X, \ldots, X) - i s(Y, X, \ldots, X)] dz^k\) is holomorphic. If \(s\) is Codazzi for any torsion-free connection then the form \(\Phi(s)\) is pseudo-holomorphic.

By the Riemann-Roch theorem (or the index method) one knows that if \(M\) is compact then a pseudo-holomorphic symmetric complex \(k\)-form is either constantly zero or its zeros are isolated and their number (counted with multiplicities) is equal to \(-2\chi(M)\), where \(\chi(M)\) is the Euler characteristic of \(M\).

We can now prove the following rigidity result due to U. Simon, [7].

**Theorem 5.3.** Let \(M\) be a connected compact surface of genus 0. If \((g_1, \nabla), (g_2, \nabla)\) are two trace-free structures on \(M\) then \(g_1 = cg_2\) on \(M\) for some constant number \(c\).

Proof. We can assume that \(M\) is oriented. Since \(\nabla \nu_{g_1} = \nabla \nu_{g_2} = 0\), by multiplying \(g_2\) by a constant we can assume that \(\nu_{g_1} = \nu_{g_2}\).

Define \(g = g_1 + g_2\) and \(h = g_1 - g_2\). Since both statistical structures are trace-free, we have that the cubic forms \(\nabla g\) and \(\nabla h\) are symmetric. Observe that \(\text{tr}_g h(\cdot, \cdot) = 0\).

There is a basis \(e_1, e_2\) of \(T_p M\) which is \(g_1\)-orthogonal and such that \(g_2(e_1, e_2) = \lambda_1 \delta_{ij}\). By the assumption \(\nu_{g_1} = \nu_{g_2}\) we have \(\lambda_1 \lambda_2 = 1\). The vectors \(\frac{e_1}{\sqrt{\lambda_1 + \lambda_2}}, \frac{e_2}{\sqrt{\lambda_1 + \lambda_2}}\) form a \(g\)-orthonormal basis. We now have

\[ \text{tr}_g h = \frac{1 - \lambda_1}{1 + \lambda_1} + \frac{1 - \lambda_2}{1 + \lambda_2} = 0. \]
Using Lemma 5.2 for $h$ and then the Riemann-Roch theorem finishes the proof.

The same consideration as in the above proof can be applied to surfaces of other topological types. For instance, we have

**Theorem 5.4.** Let $M$ be a connected compact surface of genus 1. If $(g_1, \nabla), (g_2, \nabla)$ are two trace-free structures on $M$ and $g_1 = cg_2$ at one point of $M$ then $g_1 = cg_2$ on the whole $M$.

Other results typical for affine differential geometry are those saying when the induced connection must be the Levi-Civita connection for the second fundamental form. A similar problem is interesting in the case of abstract statistical structures. For instance, using the above considerations we obtain

**Theorem 5.5.** Let $M$ be a connected compact surface and $(g, \nabla)$ be a trace-free statistical structure on $M$. Let $R = \overline{R}$. If $M$ is of genus 0 then $\nabla = \overline{\nabla}$ on $M$. If $M$ is of genus 1 and $K = 0$ at one point of $M$ then $\nabla = \overline{\nabla}$ on $M$.

Proof. It is sufficient to consider the symmetric cubic form $C(X_1, X_2, X_3) = g(K(X_1, X_2, X_3))$. Since $R = \overline{R}$ implies that $\nabla C$ is symmetric, we have that the complex form

$$[C(X, X, X) - iC(Y, X, X)]ds^2$$

is holomorphic. Using the Riemann-Roch theorem finishes the proof.

In the higher-dimensional case we have the following theorem. If a metric tensor field $g$ is given, $k(X \wedge Y)$ will denote the sectional curvature by the plane spanned by $X, Y$ if these vectors are linearly independent.

**Theorem 5.6.** Let $M$ be a compact manifold equipped with a trace-free statistical structure $(g, \nabla)$ such that $R = \overline{R}$. If the sectional curvature $k$ for $g$ is positive then $\nabla = \overline{\nabla}$.

Proof. Since $R = \overline{R}$, $\nabla K$ is symmetric. Consider the function on the unit sphere bundle $UM$

$$\alpha : UM \ni V \rightarrow g(K(V, V), V) \in \mathbb{R}. $$

Let $e_1$ be a point on $UM$, where $\alpha$ attains its maximum. Denote the maximal value by $\lambda_1$. Let $u \in U_pM$ be orthogonal to $e_1$, that is, $u$ is tangent to $U_pM$ at $e_1$. Take the curve $\beta(t) = \cos t e_1 + \sin tu$. Since $\alpha$ attains a maximum at $e_1$, by differentiating $\alpha \circ \beta$ at $t = 0$ we obtain

$$g(K(e_1, e_1), u) = 0, \quad 2g(K(e_1, u), u) - g(K(e_1, e_1), e_1) \leq 0. $$

The first formula yields $K(e_1, e_1) = \lambda_1 e_1$ for some $\lambda_1$, that is, $e_1$ is an eigenvector for $K_{e_1}$. Let $e_1, \ldots, e_n$ be an orthonormal eigenbasis for $K_{e_1}$ and let $\lambda_2, \ldots, \lambda_n$ be eigenvalues corresponding to $e_2, \ldots, e_n$ respectively. By the above inequality we have

$$\lambda_1 - 2\lambda_i \geq 0 \quad \text{for } i = 2, \ldots, n. $$

For any $u \in U_pM$, take the $\nabla$-geodesic $\gamma$ in $M$ with $\gamma(0) = p$ and $\gamma'(0) = u$. Let $e_1(t)$ be the vector field along $\gamma$ obtained by the parallel displacement relative
to \( \hat{\nabla} \) of the vector \( e_1 \). We get a vector field \( e_1(t) \). Since \( \alpha \) attains the maximum at \( e_1 \), by differentiating \( \alpha \circ \gamma \) we obtain

\[
0 = \frac{d}{dt} \bigg|_{t=0} g(K(e_1(t),e_1(t)),e_1(t)) = g(\nabla K(u,e_1,e_1),e_1)
\]

and

\[
0 \geq \frac{d^2}{dt^2} \bigg|_{t=0} g(K(e_1(t),e_1(t)),e_1(t)) = g(\nabla^2 K(u,u,e_1,e_1),e_1).
\]

We have

\[
(\nabla^2 K)(X,Y,Z,W) - (\nabla^2 K)(Y,X,Z,W) = (\hat{\nabla}(X,Y) \cdot K)(Z,W) = \hat{\nabla}(X,Y)(K(Z,W)) - K(\hat{\nabla}(X,Y)Z,W) - K(Z,\hat{\nabla}(X,Y)W)
\]

for every vectors \( X,Y,Z,W \). Therefore

\[
g((\nabla^2 K)(U,V,U,V),V) - g((\nabla^2 K)(V,U,U,V),V) = g(\hat{\nabla}(U,V)(K(U,V)),V) - g(K(\hat{\nabla}(U,U)V),V) - g(K(U,\hat{\nabla}(U,V)V),V)
\]

for any \( U,V \). Using also the symmetry of \( \nabla K \) we get

\[
g((\nabla^2 K)(\epsilon_i,\epsilon_i,e_1,e_1),e_1) = g((\nabla^2 K)(e_1,\epsilon_i,\epsilon_i,e_1),e_1)
\]

\[
+ g(\hat{\nabla}(\epsilon_i,e_1)(K(\epsilon_i,e_1)),e_1) - g(K(\hat{\nabla}(\epsilon_i,e_1)e_i,e_1) - g(K(\epsilon_i,\hat{\nabla}(\epsilon_i,e_1)e_1),e_1)
\]

for every \( i = 1,\ldots,n \). Using now \((44)\), the symmetries of \( K \) and the assumption that \( tr g K = 0 \) we see that

\[
0 \geq \sum_{i=1}^{n} g((\nabla^2 K)(\epsilon_i,\epsilon_i,e_1,e_1),e_1)
\]

\[
= \sum_{i=2}^{n} -\lambda_ig(\epsilon_i,\hat{\nabla}(\epsilon_i,e_1)e_1) + \lambda_1\hat{k}(\epsilon_i \wedge e_1) - \lambda_i\hat{k}(\epsilon_i \wedge e_1)
\]

\[
= \sum_{i=2}^{n} \hat{k}(\epsilon_i \wedge e_i)(\lambda_1 - 2\lambda_i)
\]

Since the curvature \( \hat{k} > 0 \) and \( \lambda_1 - 2\lambda_i \geq 0 \) for \( i = 2,\ldots,n \), we have \( \lambda_1 - 2\lambda_i = 0 \) for \( i = 2,\ldots,n \). Using now the assumption that \( tr K e_1 = 0 \) we see that \( \lambda_1 = 0 \). Hence \( \alpha = 0 \) on \( M \) and consequently \( K \equiv 0 \) on \( M \). \qed

If \((M,g)\) is a compact oriented Riemannian manifold and \( UM \) denotes the unit sphere bundle then for every tensor field \( s \) of type \((0,k)\) we have

\[
\int_{UM} (\nabla s)(U,U)dU = 0
\]

and

\[
\int_{UM} tr g (\nabla s)(\cdot,\cdot,U,U)dU = 0,
\]

where

\[
\int_{UM} f dU = \int_{x\in M} \left( \int_{U_x M} f^g \nu_g \right) \nu_g
\]

for any continuous function \( f : U M \to \mathbb{R} \). These Ros' integral formulas can be found in \([6]\) and \([2]\).
The formula (46) adapted to $(1,k)$-tensor fields says the following. If $s$ is a tensor field of type $(1,k)$ then
\[(47) \quad \int_{UM} (\text{div}^\ast s)(U,...,U)dU = 0.\]

We can now prove

**Theorem 5.7.** Let $M$ be a compact oriented manifold and $(g, \nabla)$ be a statistical structure on it. Then
\[
\int_{UM} \text{Ric}(U,U)dU = \int_{UM} \text{Ric}(U,U)dU
\]
and
\[
\int_{UM} \text{Ric}(U,U)dU = \int_{UM} \text{Ric}(U,U)dU - \int_{UM} g(K_U, K_U)dU + \int_{UM} \tau(K(U,U))dU.
\]

In particular, if $(g, \nabla)$ is trace-free then the following equality of two numbers
\[
\int_{UM} \text{Ric}(U,U)dU = \int_{UM} \text{Ric}(U,U)dU
\]
implies that $\nabla = \hat{\nabla}$ on $M$.

Proof. The first two formulas follow from (35), (36) and Ros’ integral formulas applied to $K$ and $\tau$. To prove the last assertion it is now sufficient to use (49). \[\blacksquare\]

**Corollary 5.8.** Let $M$ be an ovaloid in $\mathbb{R}^{n+1}$ equipped with an equiaffine transversal vector field and $g$ be the corresponding second fundamental form, $\nabla$ the induced connection and $S$ the shape operator. Then
\[
(50) \quad n \int_{UM} g(SU, U)dU = \text{vol}(S^{n-1}) \int_{M} \text{tr} S,
\]
where $\text{vol}(S^{n-1})$ is the volume of the unit sphere in the standard Euclidean space $\mathbb{R}^n$.

Proof. By formulas (40) and (41) one has
\[
\text{Ric}(Y, Z) = g(Y, Z)\text{tr} S - g(Z, SY),
\]
\[
\text{Ric}(Y, Z) = (n - 1)g(SZ, Y).
\]

Using the above theorem we get the assertion. \[\blacksquare\]

In particular, if the third fundamental form $g(S\cdot, \cdot)$ is positive definite (like in the case where the transversal vector field is a metric normal for a locally strictly convex hypersurface) then $\int_M \text{tr} S > 0$.

It is known that if $g$ is the second fundamental form on a hypersurface in $\mathbb{R}^{n+1}$ corresponding to a transversal vector field and $\nabla^2 g = 0$ then $\nabla = \hat{\nabla}$. In the compact case one has a stronger result.

**Proposition 5.9.** Let $M$ be compact oriented and $(g, \nabla)$ be a statistical structure on $M$. Then
\[
(51) \quad \int_{UM} \nabla^2 g(U, U, U, U)dU = 6 \int_{UM} \|K(U, U)\|^2dU
\]

In particular, if $\int_{UM} \nabla^2 g(U, U, U, U)dU = 0$ then $\nabla = \hat{\nabla}$. 
Proof. Denote by $C$ the symmetric cubic form $\nabla g$. We have

$$\nabla^2 g(U, U, U, U) = \hat{\nabla} C(U, U, U, U) + (K_U C)(U, U, U)$$

and

$$(K_U C)(U, U, U) = -3 C(K_U U, U, U)$$
$$= -3 \sum_{i=1}^n g(K(U, U), e_i) C(e_i, U, U) = 6 \sum_{i=1}^n g(K(U, U), e_i)^2.$$ 

Using the integral formula finishes the proof. 

6. LAPLACIANS FOR STATISTICAL STRUCTURES

We adopt the following convention

$$d\omega(X_0, ..., X_k) = \sum_{i=0}^k (-1)^i X_i(\omega(X_0, ..., \hat{X}_i, ..., X_k))$$
$$- \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k)$$

and consequently

$$d\omega(X_0, ..., X_k) = \sum_{i=0}^k (-1)^i (\nabla_{X_i} \omega)(X_0, ..., \hat{X}_i, ..., X_k)$$

for any torsion-free connection $\nabla$.

Let $M$ be an oriented manifold and $g$ a positive definite metric tensor field on $M$. We have the standard Hodge Laplacian

$$\Delta = \delta d + d\delta.$$ 

If $f \in C^\infty(M)$ then

$$\Delta f = -\text{div}^\nabla g \text{grad} f.$$ 

If $(g, \nabla)$ is a trace-free statistical structure and $f$ is a function then

$$\Delta f = -\text{tr} \{ Y \mapsto \nabla_Y (\text{grad} f) \}.$$ 

For any connection $\nabla$ we can define the Hessian of a function $f$

$$\text{Hess}_\nabla f(X, Y) = \nabla^2 f(X, Y) = \nabla_X (\nabla_Y f) = X(df(Y)) - df(\nabla_X Y).$$ 

The Hessian is a tensor field of type $(0, 2)$. If the connection $\nabla$ is torsion-free, the Hessian $\text{Hess}_\nabla f$ is symmetric. We have

**Lemma 6.1.** For any statistical structure $(g, \nabla)$ and any function $f \in C^\infty(M)$ we have

$$\Delta f = -\text{tr}_g \text{Hess}_\nabla f(\cdot, \cdot) - df(E).$$
Proof. We have

\[ \text{Hess}^\nabla f(X, Y) = X(df(Y)) - df(\nabla_X Y) - df(K(X, Y)). \]

\[ \text{Hess}^\nabla f(X, Y) = X(df(Y)) - df(\nabla_X Y) - df(K(X, Y)). \]

From now on we assume that \((g, \nabla)\) is a statistical structure and we shall use the notions introduced in Section 3. For a statistical structure \((g, \nabla)\) we shall study a Laplacian relative to the connection \(\nabla\). Note that the Laplacian we propose is different than the Laplacian relative to a connection (called also the Lichnerowicz Laplacian) defined as \(\text{tr}_g(\nabla^2 s)(\cdot, \cdot)\) for a tensor field \(s\).

If \(f\) is a function then one sets

\[ (55) \quad \Delta^\nabla f = -\text{div}^\nabla \text{grad} f. \]

It is clear that

\[ \Delta^\nabla f = -\text{tr}_g(\nabla df)(\cdot) = -\text{tr}_g(\nabla^2 f). \]

This Laplacian acting on functions was introduced and studied in [8]. We shall extend it to the operator acting on differential forms. We define the codifferential relative to \(\nabla\) acting on differential forms as follows

\[ (56) \quad \delta^\nabla \omega = -\text{tr}_g(\nabla \omega(\cdot, \cdot, \ldots)). \]

A differential form \(\omega\) will be called \(\nabla\)-cocylosed if \(\delta^\nabla \omega = 0\). We shall use

**Lemma 6.2.** For any \(k\)-form \(\omega\) and any orthonormal frame \(e_1, \ldots, e_n\) we have

\[ \sum_i (K_{e_i} \omega)(e_i, \ldots) = -\iota_E \omega, \]

where \(\iota\) stands for the interior product.

Proof. We have

\[ \sum_{i=1}^n (K_{e_i} \omega)(e_i, Y_2, \ldots, Y_k) = -\sum_{i=1}^n \omega(K_{e_i} e_i, Y_2, \ldots, Y_k) + \sum_{i=1}^n \sum_{l=2}^k \omega(e_i, Y_2, \ldots, K_{e_i} Y_l, \ldots, Y_k). \]
If $\omega$ is a 1-form the second term on the right hand side does not appear. If $\omega$ is of degree at least 2, we fix $2 \leq l \leq k$. Using the symmetries of $K$ we compute

$$
\sum_{i=1}^{n} \omega(e_i, Y_2, ..., K_e, Y_l, ..., Y_k) = \sum_{i,j=1}^{n} \omega(K_e, Y_l, e_j) \omega(e_i, Y_2, ..., Y_k)
$$

$$
= \sum_{i,j=1}^{n} g(K_e, Y_l, e_j) \omega(e_i, Y_2, ..., Y_k)
$$

$$
= \sum_{i<j}^{n} g(K_e, Y_l, e_j) \omega(e_i, Y_2, ..., Y_k) + \sum_{j<i}^{n} g(K_e, Y_l, e_j) \omega(e_i, Y_2, ..., Y_k)
$$

$$
= \sum_{i<j}^{n} g(K_e, Y_l, e_j) \omega(e_i, Y_2, ..., e_j, ..., Y_k) - \sum_{i>j}^{n} g(K_e, Y_l, e_j) \omega(e_j, Y_2, ..., e_i, ..., Y_k)
$$

$$
= 0.
$$

From the above proof we also have

**Lemma 6.3.** Let $\alpha$ be an $(m+1)$-form, $m \geq 1$, and an index $l$, $1 \leq l \leq m$, be fixed. We have

$$(57) \quad \sum_{j=1}^{n} \alpha(e_j, X_1, ..., K_e, X_l, ..., X_m) = 0.$$ 

Extending the definition (55) we set

$$\Delta \nabla \omega = \delta \nabla d\omega + d\delta \nabla \omega$$

for any differential form $\omega$. Immediate consequences of Lemma [6.2](#) and the classical Weitzenböck formula: $\delta \omega = -\text{tr}_g \hat{\nabla} \omega(\cdot, \cdot, ..., \cdot)$ are the following relations

$$(58) \quad \delta = \delta \nabla - \iota_E \delta \nabla = \delta \nabla + \iota_E,$$

$$(59) \quad \Delta \nabla = \Delta - \mathcal{L}_E.$$ 

**Lemma 6.4.** For any statistical connection $\nabla$ we have $\delta \nabla \delta \nabla = 0$.

Proof. By (58) it is sufficient to observe that $\iota_E \delta + \delta \iota_E = 0$. The equality trivially holds for 0- and 1-forms. Let $\omega$ be a $k$-form, where $k \geq 2$. Take an orthonormal frame $e_1, ..., e_n$ around a fixed point $x_0 \in M$ such that $\hat{\nabla} e_i = 0$ at $x_0$. Extend vectors $X_1, ..., X_{k-2} \in T_{x_0} M$ to local vector fields around $x_0$ in such a way that $\hat{\nabla} X_j = 0$ at $x_0$ for $j = 1, ..., k-2$. Using the standard Weitzenböck formula we get at $x_0$

$$
(\iota_E \delta \omega)(X_1, ..., X_{k-2}) = - \sum_{i=1}^{n} e_i(\omega(e_i, E, X_1, ..., X_{k-2})),
$$

$$
(\delta \iota_E \omega)(X_1, ..., X_{k-2}) = - \sum_{i=1}^{n} e_i(\omega(E, e_i, X_1, ..., X_{k-2})).
$$
Denote by $\mathcal{F}^k(M)$ the space of all (smooth) differential forms of degree $k$ and by $\mathcal{F}(M)$ the algebra of all differential forms on $M$. The metric tensor $g$ extended to the bundle of tensors on $TM$ will be denoted by $g$.

By Lemma 6.4 the operator $\delta \nabla$ determines the exact sequence

$$0 \leftarrow \mathcal{F}^0(M) \leftarrow \mathcal{F}^1(M) \leftarrow \ldots$$

We shall say that a differential form $\omega$ is $\nabla$-harmonic if $\Delta \nabla \omega = 0$.

Assume now that $(g, \nabla, \nu)$ is an equiaffine statistical structure. Let $\nu = \phi \nu_g$.

We have

$$\hat{\nabla}_X \nu = d\phi(X)\nu_g, \quad \hat{\nabla}_X \nu = -K_X \nu = -K_X(\phi \nu_g) = \phi(tr K_X)\nu_g.$$ 

Thus

$$\tau = d\log \phi,$$

where $\tau(X) = tr K_X$. For any orthonormal frame $e_1, \ldots, e_n$ and any form $\omega$ we now have

$$\delta(\phi \omega) = -\text{tr}_g \hat{\nabla}(\phi \omega)(\cdot, \cdot, \ldots) = \phi \delta \omega - \sum_{i=1}^n \phi \tau(e_i) \omega(e_i, \ldots)$$

Thus

$$\delta(\phi \omega) = \phi \delta \nabla \omega.$$

**Lemma 6.5.** Let $M$ be a compact manifold equipped with an equiaffine statistical structure $(g, \nabla, \nu)$. For any differential forms $\omega, \eta$ we have

$$(61) \quad \int_M g(\omega, d\eta) \nu = \int_M g(\delta \nabla \omega, \eta) \nu.$$ 

**Proof.** Using (60) we obtain

$$\int_M g(\omega, d\eta) \nu = \int_M g(\omega, d\eta) \phi \nu_g = \int_M g(\phi \omega, d\eta) \nu_g$$

$$= \int_M g(\delta(\phi \omega), \eta) \nu_g = \phi g(\delta \nabla \omega, \eta) \nu_g = \int_M g(\delta \nabla \omega, \eta) \nu.$$ 

**Corollary 6.6.** For an equiaffine statistical structure $(g, \nabla, \nu)$ on a compact manifold $M$

$$(62) \quad \int_M g(\Delta \nabla \omega, \eta) \nu = \int_M g(\omega, \Delta \nabla \eta) \nu = \int_M g(d\omega, d\eta) \nu + \int_M g(\delta \nabla \omega, \delta \nabla \eta) \nu.$$ 

**Corollary 6.7.** Let $M$ be a compact manifold equipped with an equiaffine statistical structure $(g, \nabla, \nu)$. A differential form is $\nabla$-harmonic if and only if it is closed and $\nabla$-coclosed.

If a differential form $\omega$ is $\nabla$-parallel then $\omega$ is $\nabla$-coclosed. It is also closed, because $\nabla$ is torsion-free. Therefore we have
Proposition 6.8. For any statistical structure \((g, \nabla)\) \(\nabla\)-parallel forms are \(\nabla\)-harmonic.

7. Formal adjoint operators for statistical connections on compact manifolds

Assume that \((g, \nabla)\) is a statistical structure. We shall construct an appropriate formal adjoint operators for \(\nabla\) using a standard procedure.

We shall use the musical isomorphism notation, that is, \(X^\flat (Y) = g(X, Y)\), \(g(\alpha^\sharp, Y) = \alpha(Y)\) for \(\alpha \in T_x M^*\), \(X, Y \in T_x M\), \(x \in M\).

We restrict our consideration to sections of tensor bundles. Let \(E\) be a vector subbundle of the tensor bundle over \(M\). The set of all smooth sections of this bundle will be denoted by \(C^\infty(M \leftarrow E)\). The metric tensor and the connections \(\nabla\), \(\tilde{\nabla}\) are extended to the bundle \(E\). If \(s_1, s_2\) are sections of \(E\) then

\[ X g(s_1, s_2) = g(\nabla_X s_1, s_2) + g(s_1, \tilde{\nabla}_X s_2) \]

for any \(X \in TM\). We now take the bundle

\[ HOM(TM, E) = \sum_{x \in M} HOM(T_x M, E_x), \]

where \(HOM(T_x M, E_x)\) is the space of all linear mappings from \(T_x M\) to \(E_x\).

A section of the bundle \(HOM(TM, E)\) can be treated as a mapping which sends a vector field \(X \in \mathcal{X}(M)\) to a section of \(E\). The metric tensor \(g\) is again extended to this bundle and as usual denoted by the same letter \(g\). The same deals with the extended connections \(\nabla\) and \(\tilde{\nabla}\). If \(S\) is a section of \(HOM(TM, E)\) then for any \(X, Y \in \mathcal{X}(M)\) we have

\[ (\nabla_X S)Y = \nabla_X (SY) - S(\nabla_X Y). \]

We now regard \(\nabla\) as a mapping

\[ \nabla : C^\infty(M \leftarrow E) \rightarrow C^\infty(M \leftarrow HOM(TM, E)), \]

where

\[ \nabla s = \{ \mathcal{X}(M) \ni X \rightarrow \nabla_X s \in C^\infty(M \leftarrow E) \}, \]

that is,

\[ (\nabla s)(X) = \nabla_X s. \]

If \(\tilde{\nabla}\) is a connection on \(M\) (possibly different than \(\nabla\)) then

\[ (\tilde{\nabla}_Y (\nabla s))(X) = \tilde{\nabla}_Y ((\nabla s)(X)) - (\nabla s)(\tilde{\nabla}_Y X) \]

\[ = \tilde{\nabla}_Y (\nabla_X s) - \tilde{\nabla}_{\tilde{\nabla}_Y X} s. \]

In particular, if \(\tilde{\nabla} = \nabla\) then \(\nabla^2 = \nabla (\nabla s)\). We denote \((\nabla Y (\nabla s))(X)\) by \(\nabla^2_{Y,X} s\).

If \(s\) is a section of \(E\) and \(S\) is a section of \(HOM(TM, E)\) then \(g(\nabla s, S)\) is a function on \(M\). In order to compute it take a local orthonormal frame \(e_1, \ldots, e_n\) in a neighborhood of a fixed point \(x_0 \in M\) such that \(\nabla e_j = 0\) at \(x_0\). In particular, \(\sum_{i=1}^n \nabla e_i = E\) and \(\sum_{i=1}^n \nabla e_i = -E\) at \(x_0\). We obtain at \(x_0\)

\[ g(\nabla s, S) = \sum_{i=1}^n g(\nabla e_i, Se_i) \]

\[ = \sum_{i=1}^n e_i g(s, Se_i) - \sum_{i=1}^n g(s, \nabla e_i(Se_i)) \]

\[ = (\text{div}_s \alpha^\sharp) - \sum_{i=1}^n g(s, \nabla(S(e_i, e_i)) + g(s, SE), \]

where \(\alpha^\sharp = (\nabla s)(X)\).
where $\alpha$ is a 1-form on $M$ given by $\alpha(X) = g(s, SX)$. Indeed, we have the following equalities at $x_0$

\[
\text{div} \nu \cdot \alpha^g = \sum_{i=1}^{n} g(\nabla_{e_i} \alpha^g, e_i) = \sum_{i=1}^{n} (e_i g(\alpha^g, e_i) - g(\alpha^g, \nabla_{e_i} e_i)) = \sum_{i=1}^{n} e_i(\alpha(e_i))
\]

and

\[
\sum_{i=1}^{n} \nabla_{e_i}(Se_i) = \sum_{i=1}^{n} \nabla S(e_i, e_i) + \sum_{i=1}^{n} S(\nabla_{e_i} e_i) = \sum_{i=1}^{n} \nabla S(e_i, e_i) - SE.
\]

Similarly we have at $x_0$

\[
\text{div} \nabla \alpha^g = \sum_{i=1}^{n} (e_i g(\alpha^g, e_i) - g(\alpha^g, \nabla_{e_i} e_i)) = \sum_{i=1}^{n} e_i(\alpha(e_i)) + \alpha(E)
\]

\[
= \sum_{i=1}^{n} e_i(g(s, Se_i)) + g(s, SE)
\]

and consequently

\[
g(\nabla s, S) = \text{div} \nabla \alpha^g - \sum_{i=1}^{n} g(s, \nabla S(e_i, e_i)).
\]

Assume that $M$ is compact and oriented. We first consider the scalar products (both denoted by $\langle \cdot, \cdot \rangle$) on the infinite dimensional vector spaces $C^\infty(M \leftarrow E)$, $C^\infty(TM \leftarrow HOM(TM, \mathcal{E}))$ given by

\[
\langle s_1, s_2 \rangle = \int_M g(s_1, s_2) \nu_g, \quad \langle S_1, S_2 \rangle = \int_M g(S_1, S_2) \nu_g
\]

for sections $s_1, s_2$ of $\mathcal{E}$ and $S_1, S_2$ – sections of $HOM(TM, \mathcal{E})$. Let $\nabla^*$ denote the operator adjoint to $\nabla$ relative to $\langle \cdot, \cdot \rangle$, that is,

\[
\langle \nabla s, S \rangle = \langle s, \nabla^* S \rangle
\]

for each section $s$ of $\mathcal{E}$ and each section $S$ of $HOM(TM, \mathcal{E})$. By the formula (64) and the divergence theorem we obtain

\[
\int_M g(\nabla s, S) \nu_g = \int_M \left( -\sum_{i=1}^{n} g(s, (\nabla_{e_i} S)e_i) + g(s, SE) \right) \nu_g.
\]

It is now justified to set

\[
\nabla^* S = -\text{tr}_g (\nabla S)(\cdot) + SE.
\]

The definition makes sense also in the case where $M$ is neither compact nor oriented. If $S = \nabla s$, where $s \in C^\infty(M \leftarrow \mathcal{E})$ then we have

\[
\nabla^* \nabla s = -\text{tr}_g (\nabla (\nabla s))(\cdot) + \nabla ES.
\]

Observe that if for a fixed point $x_0$ we take a local orthonormal frame $e_1, ..., e_n$ around $x_0$ such that $\nabla_{e_i} = 0$ at $x_0$ then at $x_0$ we have

\[
\nabla^* \nabla s = -\text{tr}_g (\nabla (\nabla s)).
\]

Assume now that $(g, \nabla, \nu)$ is an equiaffine statistical structure. We can consider a scalar product on tensor fields determined by a volume form $\nu$. Namely, we set

\[
\langle s_1, s_2 \rangle_\nu = \int_M g(s_1, s_2) \nu, \quad \langle S_1, S_2 \rangle_\nu = \int_M g(S_1, S_2) \nu
\]
for sections $s_1, s_2$ of $E$ and $S_1, S_2$ – sections of $\text{HOM}(TM, E)$. Denote by $\nabla^{\ast\nu}$ the operator adjoint to $\nabla$ relative to the scalar product $\langle \cdot, \cdot \rangle_{\nu}$, that is,

$$\langle \nabla s, S \rangle_{\nu} = \langle s, \nabla^{\ast\nu} S \rangle_{\nu}.$$ 

By (65) and the divergence theorem for equiaffine structures we get

$$\nabla^{\ast\nu} S = -\text{tr}_g (\nabla.S)(\cdot)$$

Therefore

$$\nabla^{\ast} \nabla s = \nabla^{\ast\nu} \nabla s + \nabla_E s.$$ 

8. Hodge-type theorems for statistical structures

As in the case of Riemannian manifolds we have a Hodge-type decomposition theorem for compact equiaffine statistical manifolds. Let $M$ be a compact manifold endowed with an equiaffine statistical structure $(g, \nabla, \nu)$. In order to prove a decomposition theorem observe first that we can use the classical Fredholm alternative for the product $\langle \cdot, \cdot \rangle_{\nu}$. Namely, we take the bundle $E = \Lambda^k TM^*$ with the Euclidean metric on each $E_x$ given by $g_{\nu}(\omega, \eta) = g(\psi \omega, \psi \eta)$, where $\nu = \psi^2 g$. Then

$$\langle \omega, \eta \rangle_{\nu} = \int_M g_{\nu}(\omega, \eta) \nu_g.$$ 

By (59) it is clear that the smooth linear differential operator $\Delta^{\nabla}$ of order 2 is elliptic. By Corollary 6.6 it is self-adjoint relative to $\langle \cdot, \cdot \rangle_{\nu}$. Therefore, by the Fredholm alternative, (see suitable formulation, for instance, in [?]), we know that if $\eta$ is $\langle \cdot, \cdot \rangle_{\nu}$-orthogonal to $\ker \Delta^{\nabla}(k)(M)$ then there is a unique smooth differential form $\omega$ such that $\Delta^{\nabla} \omega = \eta$.

Denote by $\mathcal{H}^{k, \nabla}(M)$ the space of all smooth $\nabla$-harmonic forms of degree $k$ on $M$. The infinite dimensional space $\mathcal{F}^k(M)$ is equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle_{\nu}$. In this section the orthogonality will mean the orthogonality with respect to this product.

Although one can just say that the following decomposition theorem follows from the theory of elliptic differential operators we give a proof of this theorem for making considerations of this paper complete. The theorem says as follows

**Theorem 8.1.** Let $M$ be a compact manifold equipped with an equiaffine structure $(g, \nabla, \nu)$. We have the following decomposition of the space $\mathcal{F}^k(M)$ into the mutually orthogonal (relative to $\langle \cdot, \cdot \rangle_{\nu}$) subspaces for every $k = 0, 1, \ldots$

$$\mathcal{F}^k(M) = d(\mathcal{F}^{k-1}(M)) \oplus \mathcal{H}^{k, \nabla}(M) \oplus \delta^{\nabla}(\mathcal{F}^{k+1}(M)).$$

**Proof.** We have the mapping

$$h : \mathcal{H}^{k, \nabla}(M) \ni \omega \rightarrow [\omega] \in H^{k}_{\text{DR}}(M),$$

where $H^{k}_{\text{DR}}(M)$ stands for the $k$-th de Rham cohomology group of $M$. Let $\omega$ be $\nabla$-harmonic and exact, i.e. $\omega = d\eta$. Since the operators of the exterior differential and $\nabla$-codifferential are adjoint and $\omega$ is $\nabla$-coclosed we obtain

$$\langle \omega, \omega \rangle_{\nu} = \langle d\eta, \omega \rangle_{\nu} = \langle \eta, \delta^{\nabla} \omega \rangle_{\nu} = 0.$$
Therefore $\omega = 0$ and, consequently, the mapping $h$ is injective. It follows, in particular, that the space $\mathcal{H}^k,\nabla(M)$ is finite dimensional. Let $\beta_1, ..., \beta_r$ be an orthonormal basis of this space. Define the mapping

$$H : \mathcal{F}^k(M) \to \mathcal{H}^k,\nabla(M)$$

given by $H\omega = \sum_{i=1}^r \langle \omega, \beta_i \rangle \beta_i$. For every $\omega \in \mathcal{F}^k(M)$ we have $\omega = H\omega + (\omega - H\omega)$ and it is easily seen that the form $\omega - H\omega$ belongs to $(\mathcal{H}^k,\nabla(M))^\perp$. Hence

$$\mathcal{F}^k(M) = \mathcal{H}^k,\nabla(M) \oplus (\mathcal{H}^k,\nabla(M))^\perp.$$

We now observe that the following subspaces of $\mathcal{F}^k(M)$ are mutually orthogonal

$$\mathcal{H}^k,\nabla(M), \quad d(\mathcal{F}^{k-1}(M)), \quad \delta^\nabla(\mathcal{F}^{k+1}(M)).$$

Namely, let $\omega \in \mathcal{H}^k,\nabla(M)$, $\eta \in \mathcal{F}^{k-1}(M)$, $\mu \in \mathcal{F}^{k+1}(M)$. We have the following obvious equalities

$$\langle \omega, d\eta \rangle = \langle \delta^\nabla \omega, \eta \rangle = 0, \quad \langle \omega, \delta^\nabla \mu \rangle = \langle d\omega, \mu \rangle = 0, \quad \langle d\eta, \delta^\nabla \mu \rangle = \langle d^2 \eta, \mu \rangle = 0.$$

Thus we have the direct orthogonal sum

$$d(\mathcal{F}^{k-1}(M)) \oplus \mathcal{H}^k,\nabla(M) \oplus \delta^\nabla(\mathcal{F}^{k+1}(M)) \subset \mathcal{F}^k(M).$$

Finally we observe that for every form $\omega' \in (\mathcal{H}^k,\nabla(M))^\perp$ the form belongs to $(\mathcal{F}^{k-1}(M)) \oplus \delta^\nabla(\mathcal{F}^{k+1}(M))$. Since $\omega'$ is orthogonal to ker $\Delta^\nabla_{\mathcal{F}^k(M)} = \mathcal{H}^k,\nabla(M)$, by the remarks made before this theorem we know that there is a smooth $k$-form $\eta$ such that $\Delta^\nabla \eta = \omega'$. We now have

$$\omega' = d(\delta^\nabla \eta) + \delta^\nabla (d\eta) \in d(\mathcal{F}^{k-1}(M)) \oplus \delta^\nabla(\mathcal{F}^{k-1}(M)),$$

which finishes the proof. $\square$

Observe now that the mapping $h$ given by (75) is also surjective. Indeed, let $[\omega] \in H^k_{DR}(M)$. There exist $\eta \in \mathcal{F}^{k-1}(M), \omega_H \in \mathcal{H}^k,\nabla(M)$ and $\mu \in \mathcal{F}^{k+1}(M)$ such that

$$\omega = d\eta + \omega_H + \delta^\nabla \mu.$$

Since $0 = d\omega = d\delta^\nabla \mu$, we have $0 = \langle d\delta^\nabla \mu, \mu \rangle = \langle \delta^\nabla \mu, \delta^\nabla \mu \rangle$. Hence $\delta^\nabla \mu = 0$ and consequently $\omega = d\eta + \omega_H$, that is, $[\omega] = [\omega_H]$. Hence the following representation theorem holds

**Theorem 8.2.** Let $M$ be a compact manifold equipped with an equiaffine statistical structure $(g, \nabla, \nu)$. The mapping $h$ given by (75) is an isomorphism. In particular, $\dim \mathcal{H}^k,\nabla(M) = b_k(M)$, where $b_k(M)$ is the $k$-th Betti number of $M$.

9. **Bochner’s technique for vector fields and harmonic 1-forms.**

We shall first collect basic information concerning vector fields and their dual 1-forms on statistical manifolds. In this section we assume that $(g, \nabla)$ is a statistical structure on $M$. As in Section 2 we set $S_X = \nabla X$. Analogously $\tilde{S}_X = \tilde{\nabla} X$ and $\overline{S}_X = \overline{\nabla} X$.

**Lemma 9.1.** Let $(g, \nabla)$ be a statistical structure on $M$. For a vector field $X \in \mathfrak{X}(M)$ the $(1, 1)$-tensor field $S_X$ is symmetric relative to $g$ if and only if $dX^\nu = 0$. 

Proof. Let $\eta = X^\flat$. Using also the assumption that $\nabla$ is torsion-free, we get

\[
d\eta(U, V) = U(\eta(V)) - V(\eta(U)) - \eta([U, V]) = U(g(V, X)) - V(g(U, X)) - g(\nabla_U V, X) + g(\nabla_V U, X).
\]

Let $X \in \mathcal{X}(M)$. Since $\delta X^\flat = -\text{tr}_g \hat{\nabla} X^\flat(\cdot, \cdot) = -\text{tr} \hat{S}_X$, we have $\text{tr} S_X = -\text{tr} \hat{S}_X - \tau(X) = \delta X^\flat - \tau(X)$. Since $X^\flat(E) = \tau(X)$ and $\delta^\nabla = \delta - \iota_E$, one gets

\[
\eta = \delta X^\flat.
\]

Lemma 9.2. Let $(g, \nabla)$ be a statistical structure on $M$. For any $X \in \mathcal{X}(M)$ we have

\[
- \text{tr} S_X = \delta X^\flat - \iota_E X^\flat = \delta^\nabla X^\flat.
\]

Proposition 9.3. Let $(g, \nabla)$ be a statistical structure on $M$ and $X \in \mathcal{X}(M)$. The 1-form $X^\flat$ is closed if and only if $S_X$ is symmetric relative to $g$. The 1-form $X^\flat$ is coclosed if and only if $\text{tr} S_X = \tau(X)$. The 1-form is $\nabla$-coclosed if and only if $\text{tr} S_X = 0$.

Lemma 9.4. Let $(g, \nabla)$ be a statistical structure on $M$. For any $X \in \mathcal{X}(M)$ we have: $g(\nabla X, \nabla X) = g(\nabla X^\flat, \nabla X^\flat)$. In particular, $\nabla X = 0$ if and only if $\nabla X^\flat = 0$.

Proof. Let $\eta = X^\flat$. It is sufficient to observe that

\[
\nabla \eta(U, V) = U(\eta(V)) - \eta(\nabla_U V) = U(g(X, V)) - \eta(X, \nabla_U V) = g(\nabla_U X, V).
\]

By duality $\nabla \eta = 0$ if and only if $\nabla \eta^\flat = 0$ for any 1-form $\eta$.

Corollary 9.5. If $(g, \nabla)$ is a trace-free statistical structure on $M$ and $\nabla X = 0$ for $X \in \mathcal{X}(M)$ then $X^\flat$ is closed and coclosed. If for a 1-form $\eta$ we have $\nabla \eta = 0$ then $\eta$ is closed and coclosed.

In some situations harmonic forms are parallel. A well-known Bochner theorem says that harmonic 1-forms are parallel relative to the Levi-Civita connection on a Ricci non-negative Riemannian manifold. We shall now prove some generalizations of this theorem.

Theorem 9.6. Let $M$ be a connected compact oriented manifold with an equiaffine statistical structure $(g, \nabla, \nu)$. If the Ricci tensor $\text{Ric}$ for $\nabla$ is non-negative on $M$ then every $\nabla$-harmonic 1-form on $M$ is $\nabla$-parallel. In particular, the first Betti number $b_1(M)$ is not greater than $\dim M$. If additionally $\text{Ric} > 0$ at some point of $M$ then $b_1(M) = 0$.

Proof. Let $\eta$ be a $\nabla$-harmonic 1-form and $X = \eta^\flat$. Since $M$ is compact, $\eta$ is closed and $\nabla$-coclosed. Proposition yields that $\text{tr} S_X = 0$ and $S_X$ is symmetric relative to $g$. In particular, $S_X$ is diagonalizable. Therefore $\text{tr} S_X^2 \geq 0$ and the equality holds if and only if $S_X = 0$. By Lemma we have

\[
\int_M \text{Ric}(X, X) \nu = - \int_M \text{tr} S_X^2 \nu \leq 0.
\]
If $\text{Ric} \geq 0$ on $M$ then $\text{tr} S_X^2 = 0$ on $M$ and consequently $S_X$ vanishes on $M$. By Lemma 9.4 we get $\nabla \eta = 0$.

Let $x$ be any point of $M$ and consider the mapping sending each $\nabla$-harmonic 1-form $\eta$ to $\eta_x \in T_xM^*$. The mapping is linear and, since each $\nabla$-harmonic 1-form is covariant constant, the mapping is also a monomorphism. By Theorem 8.2 $b_1(M) \leq \dim M$.

We also have $\text{Ric}(X,X) = 0$ on $M$. Hence, if $\text{Ric} > 0$ at some point then $X = 0$ at this point and consequently, since $X$ is covariant constant, $X = 0$ on $M$. Consequently $\eta = 0$ on $M$. □

In particular, if $(g, \nabla, \nu_g)$ is equiaffine we get

**Corollary 9.7.** Let $M$ be a connected compact oriented manifold with a trace-free statistical structure $(g, \nabla)$. If the Ricci tensor $\text{Ric}$ for $\nabla$ is non-negative on $M$ then every harmonic 1-form on $M$ is $\nabla$-parallel. In particular, the first Betti number $b_1(M)$ is not greater than $\dim M$. If additionally $\text{Ric} > 0$ at some point of $M$ then $b_1(M) = 0$.

If $(g, \nabla)$ is a trace-free statistical structure and $X^\flat$ is harmonic then $\text{tr} S_X = 0$ and $\text{tr} S_X^2 = g(\nabla X, \nabla X)$. Therefore, if $M$ is compact and oriented, by Lemma 2.4 we have

\[
\int \text{Ric}(X,X)\nu_g = -\int g(\nabla X, \nabla X)\nu_g.
\]

The same formula holds for the conjugate connection $\bar{\nabla}$.

In the same way as Theorem 9.6 one gets

**Theorem 9.8.** Let $M$ be a connected compact oriented manifold. Let $(g, \nabla)$ be a trace-free statistical structure on $M$. If $\text{Ric} + \text{Ric}_{\bar{\nabla}} \geq 0$ on $M$ then each harmonic 1-form on $M$ is parallel relative to the connections $\nabla$, $\bar{\nabla}$ and $\nabla_f$. In particular, $b_1(M) \leq \dim M$. If moreover $\text{Ric} + \text{Ric}_{\bar{\nabla}} > 0$ at some point then $b_1(M) = 0$.

Note that the assumption $\text{Ric} + \text{Ric}_{\bar{\nabla}} \geq 0$ implies that $\text{Ric}_{\bar{\nabla}} \geq 0$ and the classical Bochner theorem implies that harmonic 1-forms are $\nabla$-parallel but it does not imply that they are $\nabla$ or $\nabla_f$-parallel.

Consider now the case where $(g, \nabla, \nu)$ is equiaffine and a vector field $X$ equals to $X = \text{grad} f$ for some function $f \in C^\infty(M)$. Let $\eta = df$. We have

\[
\nabla \eta = \nabla^2 f = \text{Hess}^\nabla f.
\]

Since $\text{Hess}^\nabla$ is a symmetric $(0,2)$-tensor field (consequently diagonalizable at each point of $M$) and $\Delta^\nabla f = -\text{tr}_g \text{Hess}^\nabla f$, by the Schwarz inequality we have

\[
n|\text{Hess}^\nabla f|^2 \geq |\Delta^\nabla f|^2.
\]

Since $df$ is closed, $S_X$ is symmetric (by Proposition 9.4) and therefore $\text{tr} S_X^2 = g(S_X, S_X) = g(\nabla X, \nabla X)$. By Lemma 9.4 and the formula (79) we now get

\[
\text{tr} S_X^2 = |\text{Hess}^\nabla f|^2.
\]

We can now prove the following generalization of a Bochner-Lichnerowicz formula and Lichnerowicz’s theorem.
Theorem 9.9. Let $M$ be a compact manifold. If $(g, \nabla, \nu)$ is an equiaffine statistical structure on $M$ then for every function $f \in C^\infty(M)$ we have

\begin{equation}
\int_M \text{Ric}(df^\sharp, df^\sharp)\nu = \int_M |\Delta^\nabla f|^2\nu - \int_M |\text{Hess}^\nabla f|^2\nu
\end{equation}

for any function $f \in C^\infty(M)$. If for some real number $k$

$$\text{Ric} \geq kg,$$

then the first eigenvalue $\lambda_1$ of the Laplacian $\Delta^\nabla$ satisfies the inequality

\begin{equation}
\lambda_1 \geq \frac{n}{n-1} k.
\end{equation}

Proof. The equality (82) immediately follows from (11) and the fact that $\text{tr } S_X = -\Delta^\nabla f$ (by (77)), where $X = df^\sharp$. If $\lambda$ is an eigenvalue of the Laplacian $\Delta^\nabla$ and $f$ is the corresponding eigenfunction then (by Lemma 6.5)

$$\int_M g(\Delta^\nabla f, \Delta^\nabla f)\nu = \int_M \lambda g(f, \Delta^\nabla f)\nu = \int_M \lambda g(df, df)\nu.$$

The second statement now follows from the formulas (82), (80) and the assumed inequality. \qed

The above theorem was proved in [8] by using a different method.

Corollary 9.10. Let $M$ be a compact oriented manifold. If $(g, \nabla)$ is a trace-free statistical structure on $M$ then for every function $f \in C^\infty(M)$ we have

\begin{equation}
\int_M \text{Ric}(df^\sharp, df^\sharp)\nu_g = \int_M |\Delta f|^2\nu_g - \int_M |\text{Hess}^\nabla f|^2\nu_g
\end{equation}

for any function $f \in C^\infty(M)$. If for some real number $k$

$$\text{Ric} \geq kg,$$

then the first eigenvalue $\lambda_1$ of the Laplacian $\Delta$ satisfies the inequality

$$\lambda_1 \geq \frac{n}{n-1} k.$$

Remark 9.11. In the Riemannian case it is known that the equality in (83) holds if and only if $(M, g)$ is isometric to an ordinary sphere. In the case of statistical structures we have the following

Example 9.12. Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex immersed hypersurface. Equipping it with the transversal vector field $\xi = -f$ we get the induced statistical structure on $M$. We say that $\{ : M \to \mathbb{R}^{n+1}$ is a centroaffine hypersurface. If $\alpha$ is a 1-form on $\mathbb{R}^{n+1}$ we define the function $f$ on $M$ by the formula $f = \alpha(\xi) = -\alpha \circ f$. Denote by $\nabla$ the induced connection. We have

\begin{align}
(\nabla df)(X, Y) &= X(df(Y)) - df(\nabla_X Y) = -\alpha(X(Yf)) + \alpha(f(\nabla_X Y)) \\
&= -\alpha(f(\nabla_X Y) - g(X, Y)f) + \alpha(f(\nabla_X Y)) \\
&= -fg(X, Y).
\end{align}
Hence $\nabla df + fg = 0$. Apply this to the case where $f$ is the conormal map of some centroaffine immersion and $g$ is the common second fundamental form for the immersion and its conormal. Then for functions $f$ defined as above we have

$$\nabla df + fg = 0, \tag{85}$$

that is, $\Delta \nabla f = f$. We propose the following conjecture: Let $M$ be a compact manifold equipped with a statistical structure $(g, \nabla)$. Assume that the conjugate connection $\nabla$ is complete. If there is a non-constant function $f$ such that the equation (85) is satisfied then $(g, \nabla)$ can be realized as the induced structure on a centroaffine ovaloid.

We continue considerations of the section assuming that a statistical structure $(g, \nabla)$ is trace-free. Recall the classical Bochner formula for the Laplacian of the square of the length of a vector field $X$ for which $dX^\ast = 0$ (equivalently $S_X, \overline{S}_X, \overline{S}_X$ are symmetric relative to $X$). Specifically, if $\varphi \equiv g(X, X)$ then

$$\Delta \varphi + 2X(\nabla^\ast X) = -2\overline{Ric}(X, X) - 2g(\nabla X, \nabla X). \tag{86}$$

On the other hand, by formula (8) applied for $\nabla$ we have

$$2\nabla^\ast(\nabla_X X) = 2\overline{Ric}(X, X) + 2X(\nabla^\ast X) + 2g(\nabla X, \nabla X).$$

Adding these two equalities we get

$$\Delta \varphi = -2\nabla^\ast \nabla_X X. \tag{87}$$

Since $S_X$ is symmetric, $\text{tr} S_X^2 = g(\nabla X, \nabla X)$. Similarly $\text{tr} \overline{S}_X^2 = g(\nabla X, \nabla X)$. By (8) applied for $\nabla$, $\overline{\nabla}$ and $\nabla$ and the fact that $\text{tr} S_X = \text{tr} \overline{S}_X = \text{tr} \overline{S}_X = \nabla^\ast X$ we get

$$\nabla^\ast (\nabla_X X) = \overline{Ric}(X, X) + X\nabla^\ast X + g(\nabla X, \nabla X),$$

$$\nabla^\ast (\nabla_X X) = \overline{Ric}(X, X) + X\nabla^\ast X + g(\nabla X, \nabla X).$$

Since $2\nabla_X X = \nabla_X X + \nabla_X X$, we have

$$\Delta \varphi + 2X\nabla^\ast X = -\overline{Ric}(X, X) - \nabla^\ast \nabla_X X - g(\nabla X, \nabla X) - g(\nabla X, \nabla X) \tag{88}$$

for any vector field $X$ such that $dX^\ast = 0$. By (31) the last formula can be equivalently written as

$$\Delta \varphi + 2X\nabla^\ast X = -\overline{Ric}(X, X) - \nabla^\ast \overline{Ric}(X, X) - 2g(\nabla X, \nabla X) - 2g(K_X, K_X). \tag{89}$$

**Theorem 9.13.** Let $(g, \nabla)$ be a trace-free statistical structure on a connected manifold $M$ and $\overline{Ric} + \overline{\nabla^\ast} \geq 0$ on $M$. Let $\eta$ be a closed harmonic 1-form on $M$.

1. If $\varphi = |\eta|^2$ attains a local maximum at some point $x_o$ of $M$ then $\nabla \eta = \nabla \eta = \overline{\nabla} \eta = 0$ at $x_o$. If moreover $\overline{Ric} + \overline{\nabla^\ast} > 0$ at $x_o$ then $\eta = 0$ in a neighborhood of $x_o$.

2. If $\overline{Ric} + \overline{\nabla^\ast} > 0$ on $M$ and $\varphi$ attains a global maximum at some point of $M$ then $\eta = 0$ on $M$.

3. If $\overline{Ric} + \overline{\nabla^\ast} > 0$ on $M$, $\varphi$ attains a local maximum at some point and $g$ is analytic then $\eta = 0$ on $M$.

Proof. Let $X = \eta^2$. Of course $\varphi = g(X, X)$. If $\varphi$ attains a local maximum at $x_o$ then $(\Delta \varphi)_{x_o} \geq 0$. Since $\eta$ is closed and harmonic, we have that $\delta \eta$ is constant. By Lemma (22) it follows that $X(\nabla^\ast X) = 0$. Using now (88) we obtain the first assertion in 1). Moreover $Ric(X, X) + \overline{Ric}(X, X) = 0$ at $x_0$. If $Ric + \overline{Ric} > 0$ at $x_0$ then $X_{x_o} = 0$, i.e., $\eta_{x_o} = 0$. Since $\varphi$ attains a local maximum at $x_o$, we have
that \( \eta = 0 \) around \( x_o \). If the maximum is global then \( \eta \) vanishes on \( M \). Since a harmonic form on an analytic Riemannian manifold is analytic, we have 3). \( \square \)

10. BOCHNER–WEITZENBÖCK FORMULAS FOR DIFFERENTIAL FORMS

Let \((g, \nabla)\) be a statistical structure on a manifold \( M \). The Weitzenböck curvature operator for the curvature tensor \( R \) for \( \nabla \) will be denoted by \( \mathcal{W}^R \). Let \( s \) be a tensor field of type \((l, k)\), where \( k > 0 \), on \( M \). One defines a tensor field \( \mathcal{W}^R s \) of type \((l, k)\) as follows

\[
(\mathcal{W}^R s)(X_1, \ldots, X_k) = \sum_{i=1}^k \sum_{j=1}^n (R(e_j, X_i)s)(X_1, \ldots, e_j, \ldots, X_k),
\]

where \( e_1, \ldots, e_n \) is an arbitrary orthonormal frame, \( R(e_j, X_i)s \) means that \( R(e_j, X_i) \) acts as a differentiation on \( s \), and \( e_j \) in the last parenthesis is at the \( i \)-th place. The definition is independent of the choice of an orthonormal basis.

Observe what the \( \mathcal{W}^R \omega \) is in the case where \( \omega \) is a 1-form. In this case we have

\[
\sum_{j=1}^n (R(e_j, X)\omega)(e_j) = -\sum_{j=1}^n \omega(R(e_j, X)e_j) = -\sum_{j=1}^n g(R(e_j, X)e_j, \omega^j) = \sum_{j=1}^n g(R(e_j, X)\omega^j, e_j) = \text{Ric}(X, \omega^j).
\]

Thus for 1-forms

\[
\mathcal{W}^R \omega(X) = \text{Ric}(X, \omega^j).
\]

We shall now prove some generalizations of the Bochner-Weitzenböck formula for the Laplacians acting on differential forms on statistical manifolds.

**Theorem 10.1.** i) For any statistical structure \((g, \nabla)\) we have

\[
\Delta = \nabla^* \nabla + \mathcal{W}^R + \nabla E - L_E = \nabla^* \nabla + \mathcal{W}^R + S_E,
\]

\[
\Delta^\nabla = \nabla^* \nabla + \mathcal{W}^R + 2S_E - \nabla E.
\]

ii) If \((g, \nabla, \nu)\) is an equiaffine statistical structure then

\[
\Delta^\nabla = \nabla^* \nabla + \mathcal{W}^R + 2\nabla E - 2L_E = \nabla^* \nabla + \mathcal{W}^R + 2S_E.
\]

iii) If \((g, \nabla)\) is trace-free then

\[
\Delta = \nabla^* \nabla + \mathcal{W}^R.
\]

**Proof.** i) Let \( \omega \) be a \( k \)-form on \( M \). Let \( x_o \) be a fixed point of \( M \) and \( e_1, \ldots, e_n \) be a local orthonormal frame around \( x_o \) such that \( \hat{\nabla} e_i = 0 \) at \( x_o \). As before, we shall use the fact that \( \sum_{i=1}^n \nabla e_i e_i = E \) at \( x_o \). Let \( X_1, \ldots, X_k \) be arbitrary vectors from \( T_{x_o} M \). We extend them to local vector fields around \( x_o \) in such a way that \( \nabla X_i = 0 \) at \( x_o \) for \( i = 1, \ldots, k \). Then \( (\nabla Y X_i)_{x_o} = (K_Y X_i)_{x_o} \) and \([e_j, X_i]_{x_o} = 0\) for each \( Y \) and every \( i = 1, \ldots, k \) and \( j = 1, \ldots, n \).
We shall now compute at $x_0$

$$d\delta \nabla \omega (X_1, ..., X_k) = \sum_{i=1}^{k} (-1)^{i-1}(\nabla X_i (\delta \nabla \omega))(X_1, ..., X_k)$$

$$= \sum_{i=1}^{k} (-1)^{i-1}(X_i((\delta \nabla \omega)(X_1, ..., X_k))$$

$$- \delta \nabla \omega(\nabla X_i X_1, ..., \hat{X}_i, ..., X_k)$$

$$........ - \delta \nabla \omega(X_1, ..., \hat{X}_i, ..., \nabla X_i X_k)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} (-1)^{i} X_i((\nabla e_j \omega)(e_j, X_1, ..., \hat{X}_i, ..., X_k))$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{k} (-1)^{i-1}(\nabla e_j \omega)(e_j, \nabla X_i X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ ....... + \sum_{j=1}^{n} \sum_{i=1}^{k} (-1)^{i-1}(\nabla e_j \omega)(e_j, ..., \hat{X}_i, ..., \nabla X_i X_k)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} (-1)^{i}(\nabla e_j (\nabla e_j \omega))(e_j, X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{k} (-1)^{i}(\nabla e_j \omega)(\nabla X_i e_j, X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{n} (-1)^{i}(\nabla e_j \omega)(e_j, \nabla X_i X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ ....... + \sum_{j=1}^{n} \sum_{i=1}^{k} (-1)^{i-1}(\nabla e_j \omega)(e_j, ..., \hat{X}_i, ..., \nabla X_i X_k)$$

$$= - \sum_{i=1}^{k} \sum_{j=1}^{n} (\nabla X_i (\nabla e_j \omega))(X_1, ..., e_j, ..., X_k)$$

$$- \sum_{i=1}^{k} \sum_{j=1}^{n} (\nabla e_j \omega)(X_1, ..., K e_j X_i, ..., X_k),$$

where in the last parenthesis in the last two lines $e_j$ and $K e_j X_i$ are at the $i$-th place. Using Lemma 6.3 for $\alpha = d\omega$ and $\alpha = (\nabla X_i \omega)(\cdot, ..., \hat{X}_i, ..., \cdot)$ (if $\omega$ is a 1-form
some lines in the following formula do not appear) we continue computations at $x_o$.

$$\delta d\omega(X_1, ..., X_k) = -\sum_{j=1}^n (\nabla_{e_j} (d\omega)) (e_j, X_1, ..., X_k) - (i_E d\omega)(X_1, ..., X_k)$$

$$= -\sum_{j=1}^n e_j ((d\omega)(e_j, X_1, ..., X_k))$$

$$+ \sum_{j=1}^n d\omega(\nabla_{e_j} e_j, X_1, ..., X_k)$$

$$+ \sum_{j=1}^n d\omega(e_j, \nabla_{e_j} X_1, ..., X_k)$$

$$+ ........ + \sum_{j=1}^n d\omega(e_j, ..., \nabla_{e_j} X_k) - (i_E d\omega)(X_1, ..., X_k)$$

$$= -\sum_{j=1}^n e_j ((\nabla_{e_j} \omega)(X_1, ..., X_k))$$

$$- \sum_{i=1}^k \sum_{j=1}^n (-1)^i e_j ((\nabla_{X_i} \omega)(e_j, X_1, ..., \hat{X}_i, ..., X_k))$$

$$= -\sum_{j=1}^n (\nabla_{e_j} (\nabla_{e_j} \omega))(X_1, ..., X_k)$$

$$- \sum_{j=1}^n (\nabla_{e_j} \omega)(\nabla_{e_j} X_1, ..., X_k)$$

$$- ........ - \sum_{j=1}^n (\nabla_{e_j} \omega)(X_1, ..., \nabla_{e_j} X_k)$$

$$- \sum_{i=1}^k \sum_{j=1}^n (-1)^i \{(\nabla_{e_j} (\nabla_{X_i} \omega))(e_j, X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ (\nabla_{X_i} \omega)(\nabla_{e_j} e_j, X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ (\nabla_{X_i} \omega)(e_j, \nabla_{e_j} X_1, ..., \hat{X}_i, ..., X_k)$$

$$+ ........ + (\nabla_{X_i} \omega)(e_j, X_1, ..., \hat{X}_i, ..., \nabla_{e_j} X_k)\}$$

$$= -\sum_{j=1}^n (\nabla_{e_j} (\nabla_{e_j} \omega))(X_1, ..., X_k)$$

$$- \sum_{j=1}^n \sum_{i=1}^k (\nabla_{e_j} \omega)(X_1, ..., K_{e_j} X_i, ..., X_k)$$

$$+ \sum_{i=1}^k \sum_{j=1}^n (\nabla_{e_j} (\nabla_{X_i} \omega))(X_1, ..., e_j, ..., X_k)$$

$$+ d(i_E \omega)(X_1, ..., X_k) + (S_E \omega)(X_1, ..., X_k),$$
where in the last line $e_j$ is at the $i$-th place and in the last but one line $K_{e_j}X_i$ is at the $i$-th place. Finally we observe that at $x_0$ we have

$$
\sum_{j=1}^n (\nabla e_j(\nabla e_j \omega))(X_1, ..., X_k) = \sum_{j=1}^n ((\nabla e_j - 2K_{e_j})(\nabla e_j \omega))(X_1, ..., X_k)
$$

$$= \sum_{j=1}^n (\nabla e_j (\nabla e_j \omega))(X_1, ..., X_k)
$$

$$+ 2 \sum_{j=1}^n \sum_{i=1}^k (\nabla e_j \omega)(X_1, ..., K_{e_j}X_i, ..., X_k),
$$

where in the last line $K_{e_j}X_i$ is at the $i$-th place. Composing the last three formulas and using (70) we obtain the equality

(96) $\Delta \omega = \nabla^* \nabla \omega + W_R \omega + S_E \omega$

Using (71) and (99) completes the proof of i). To prove ii) it is sufficient to use i), (59) and (73). The last statement is a particular case of i).

Using (91), the last theorem and the formula (25) we get

**Corollary 10.2.** Let $(g, \nabla)$ be a statistical structure. For any 1-form $\omega$ we have

(97) $\Delta \omega = \nabla^* \nabla \omega + \overline{Ric}(\cdot, \omega^\sharp) + \nabla E \omega - L E \omega$,

(98) $(\Delta \omega)(X) = - \text{tr}_g(\nabla^2 \omega)(X) + \overline{Ric}(X, \omega^\sharp) + g(\nabla X g, \nabla E \omega) + \nabla E \omega - L E \omega$

for any vector $X \in T_xM$, $x \in M$. If $(g, \nabla, \nu)$ is an equiaffine statistical structure then

(99) $\Delta \nabla \omega = \nabla^* \nabla \omega + \overline{Ric}(\cdot, \omega^\sharp) + 2 \nabla E \omega - 2 L E \omega$.

The formula (92) can also be written for the connection $\nabla$. Namely, we have

(100) $\Delta = \nabla^* \nabla + W_R + S_E = \nabla^* \nabla + W_R - S_E + 2K_E$.

If $(g, \nabla, \nu)$ is an equiaffine statistical structure then, in general, $(g, \nabla, \nu)$ is not equiaffine. It is equiaffine if an only if $(g, \nabla)$ is trace-free.

The structure $(g, \nabla, \nu)$ is equiaffine if and only if $\nu = e^{e\tau}$ where $\tau = d\rho$. In particular, $\tau$ must be exact. In such a case we define

(101) $\nabla^\nu S = - \text{tr}_g(\nabla S)(\cdot)$

for a section $S$ of $HOM(TM, E)$. If $M$ is compact then $\nabla^\nu$ is the adjoint operator for $\nabla$ relative to $\langle \cdot, \cdot \rangle_\nu$. For the same reasons as (73) one gets

(102) $\nabla^\nu \nabla = \nabla^\nu \nabla - \nabla_E$

We can now compute

$$\nabla^\nu = \Delta - L E = \nabla^* \nabla + W_R - S_E - L E = \nabla^\nu \nabla + W_R - \nabla E - S_E - L E = \nabla^\nu \nabla + W_R - 2 \nabla_E.
$$

We have proved
Proposition 10.3. If \((g, \nabla, \nu)\) is an equiaffine statistical structure and \(\tau\) is exact and equal to \(dp\) then

\[
\Delta \nabla = \nabla^\tau \nabla + \nabla^R - 2\nabla_E,
\]

where \(\tau = e^{-\rho} \nu\).

Consider again a tensor vector bundle \(E\) over \(M\). If \(s\) is a section of \(E\) and \(\tilde{\nabla}\) is a connection then we have

\[
\tilde{\nabla}^2_{Y,X}|s|^2 = Y(d|s|^2(X)) - d|s|^2(\tilde{\nabla}_Y X) = Y(|X|^2) - d|s|^2(\tilde{\nabla}_Y X) = Y\{g(\nabla_X s, s) + g(s, \nabla_X s)\} - d|s|^2(\tilde{\nabla}_Y X) = g(\nabla_Y (\nabla_X s), s) + g(\nabla_X s, \nabla_Y s) + g(\nabla_Y s, \nabla_X s) + g(s, \nabla_Y (\nabla_X s)) - d|s|^2(\tilde{\nabla}_Y X).
\]

Applying this formula to the connections \(\nabla\) and \(\tilde{\nabla}\) one gets

\[
\tilde{\nabla}^2_{Y,X}|s|^2 = g(\nabla_Y (\nabla_X s), s) + g(\nabla_X s, \nabla_Y s) + g(\nabla_Y s, \nabla_X s) + g(s, \nabla_Y (\nabla_X s)) - d|s|^2(\tilde{\nabla}_Y X)
\]

and

\[
\nabla^2_{Y,X}|s|^2 = g(\nabla_Y (\nabla_X s), s) + g(\nabla_X s, \nabla_Y s) + g(\nabla_Y s, \nabla_X s) + g(s, \nabla_Y (\nabla_X s)) - d|s|^2(\nabla_Y X).
\]

For an orthonormal frame \(e_1, \ldots, e_n\) such that \(\nabla e_j = 0\) at a fixed point \(x_0\) we obtain at this point

\[
(104) \quad \Delta|s|^2 = -\sum_{i=1}^n g(\nabla_{e_i}(\nabla_{e_i} s), s) - \sum_{i=1}^n g(\nabla_{e_i}(\nabla_{e_i} s), s) - g(\nabla s, \nabla s) - g(\nabla s, \nabla s)
\]

\[
(105) \quad \Delta|\nabla s|^2 = -\sum_{i=1}^n g(\nabla_{e_i}(\nabla_{e_i} s), s) - \sum_{i=1}^n g(\nabla_{e_i}(\nabla_{e_i} s), s) - g(\nabla s, \nabla s) - g(\nabla s, \nabla s) - E g(s, s).
\]

Therefore, using (104), (105) and (70) for \(\nabla\) and \(\tilde{\nabla}\), we obtain

**Theorem 10.4.** For any statistical structure and any tensor field \(s\) we have

\[
(106) \quad \Delta|s|^2 = g(\nabla^* \nabla s, s) + g(\nabla^* \nabla s, s) - g(\nabla s, \nabla s) - g(\nabla s, \nabla s)
\]

\[
(107) \quad \Delta|\nabla s|^2 = g(\nabla^* \nabla s, s) + g(\nabla^* \nabla s, s) - g(\nabla s, \nabla s) - g(\nabla s, \nabla s) - E g(s, s).
\]

In particular, if \(\omega\) is a differential form then

\[
(108) \quad \Delta|\omega|^2 = 2g(\Delta \omega, \omega) - g(\nabla^R + \nabla^R, \omega) - g(\nabla \omega, \nabla \omega) - g(\nabla \omega, \nabla \omega) - 2g(K_{E \omega}, \omega).
\]

\[
(109) \quad \Delta|\nabla \omega|^2 = 2g(\Delta \nabla \omega, \omega) - g(\nabla^R + \nabla^R, \omega) - g(\nabla \omega, \nabla \omega) - g(\nabla \omega, \nabla \omega) - 2g(\nabla_{E \omega}, \omega).
\]
Proof. To prove (108) it is sufficient to use (106), (92) and (100). Formula (109) follows from (107) and (93).

From (106) and the maximum principle applied to the standard Laplacian we immediately get

**Proposition 10.5.** Let \( M \) be a connected manifold equipped with a statistical structure \((g, \nabla)\). If \( s \) is a tensor field on \( M \) such that

\[
g(\nabla^* \nabla s + \nabla \nabla s, s) \leq 0
\]

on \( M \) and \(|s|\) attains a maximum then \( \nabla s = 0 \) and \( \nabla^* s = 0 \) on \( M \).

11. **Bochner-Weitzenböck formulas for the metric tensor field**

If \((g, \nabla)\) is a statistical structure then we can apply the Weitzenböck curvature operator \( W^R \) to \( g \). One easily sees that

\[
(W^R g)(X,Y) = \overline{\text{Ric}}(X,Y) + \text{Ric}(Y,X) - \text{Ric}(X,Y) - \text{Ric}(Y,X).
\]

For any tensor field \( s \) on \( M \) we have \( R(X,Y)s = \nabla^2_{X,Y} s - \nabla^2_{Y,X} s \). Since \( \nabla g \) is symmetric, we have \( (\nabla^2_{X,Y} g)(Z,W) = (\nabla^2_{X,Z} g)(Y,W) \). Therefore

\[
\sum_{i=1}^n (R(e_i, X)g)(e_i, Y) = -\sum_{i=1}^n (\nabla^2_{X,e_i} g)(e_i, Y) + \text{tr}_g(\nabla^2_{e_i} g)(X,Y).
\]

As usual we choose an orthonormal frame \( e_1, \ldots, e_n \) around a fixed point \( x_0 \) such that \( \nabla e_i = 0 \) at \( x_0 \), that is, \( \nabla_X e_i = K_X e_i \) at \( x_0 \) for any \( X \). Using now formulas (25) and (27) we get at \( x_0 \)

\[
- \sum_{i=1}^n (\nabla^2_{X,e_i} g)(e_i, Y) = -\sum_{i=1}^n (\nabla_X (\nabla g))(e_i, e_i, Y)
\]

\[
= \sum_{i=1}^n 2\nabla g(\nabla X e_i, e_i, Y) - X(\nabla g(e_i, e_i, Y)) + \nabla g(e_i, e_i, \nabla_X Y)
\]

\[
= 2 \sum_{i,j=1}^n g(\nabla X e_i, e_j) \nabla g(e_j, e_i, Y) + 2\nabla \tau(X, Y)
\]

\[
= 2 \sum_{i,j=1}^n g(K_X e_i, e_j) \nabla g(e_j, e_i, Y) + 2\nabla \tau(X, Y)
\]

\[
= -\sum_{i,j=1}^n (\nabla X g)(e_i, e_j) (\nabla Y g)(e_i, e_j) + 2\nabla \tau(X, Y)
\]

\[
= -g(\nabla X g, \nabla Y g) + 2\nabla \tau(X, Y).
\]

On the other hand we have

\[
\sum_{i=1}^n (R(e_i, X)g)(e_i, Y) = \overline{\text{Ric}}(X,Y) - \text{Ric}(X,Y).
\]

We have proved
Lemma 11.1. For any statistical connection $\nabla$ for $g$ we have

\[(112) \quad \text{tr}_g \nabla^2 g(X,Y) - g(\nabla_X g, \nabla_Y g) + 2\nabla \tau(X,Y) = -\text{Ric}(X,Y) + \text{Ric}(X,Y),\]

\[(113) \quad 2\text{tr}_g \nabla^2 g(X,Y) - 2g(\nabla_X g, \nabla_Y g) + 2\nabla \tau(Y,X) + 2\nabla \tau(Y,X) = (\mathcal{W}^R g)(X,Y).\]

Since the scalar curvature of $\nabla$ and the one for $\nabla$ are the same we obtain

Proposition 11.2. For any trace-free statistical structure $(g, \nabla)$ we have

\[\sum_{i,j=1}^n \nabla^2 g(e_i, e_i, e_j, e_j) = g(\nabla g, \nabla g).\]

An example of usage of this theorem is the following

Corollary 11.3. If for a locally strongly convex Blaschke hypersurface for its Blaschke metric $g$ and its induced connection $\nabla$

\[\sum_{i,j=1}^n \nabla^2 g(e_i, e_i, e_j, e_j) = 0,\]

then the hypersurface is a quadric.

Proof. It follows from Berwald’s theorem and the above proposition. \(\square\)

12. The sectional curvature and the curvature operator for statistical structures

12.1. Algebraic preliminaries. Let $V$ be an $n$-dimensional Euclidean vector space with the scalar product $\langle \cdot, \cdot \rangle$. Let $T$ be any $(1,3)$-tensor on $V$. We also set $T(X,Y,Z) := T(X, Y, Z)$. If the tensor is skew-symmetric relative to $X,Y$ and the Bianchi identity: $T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y) = 0$ holds, we call $T$ a curvature-like tensor of type $(1,3)$. For $T$ we define a $(0,4)$-tensor $\tilde{T}$ as follows

\[\tilde{T}(U,Z,X,Y) = \langle T(X,Y)Z, U \rangle.\]

If $T$ is a curvature-like tensor of type $(1,3)$ and the tensor $T$ of type $(0,4)$ is skew-symmetric relative to $U,Z$ then the both tensors $T$ will be called Riemann-curvature-like tensors. For a Riemann-curvature-like tensor we have

\[T(U, Z, X, Y) = T(X, Y, U, Z).\]

The easiest Riemann-curvature-like tensor is $R_0$ defined as follows

\[R_0(X,Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y.\]

The scalar product extended to the exterior products of $V$ gives the isometric identification of $\Lambda^2 V$ and $\Lambda^2 V^*$. The last space will be also isometrically identified with the space $\mathfrak{so}(V)$ of all skew-symmetric endomorphisms of $V$. In particular, if $T(X,Y)Z$ is skew-symmetric in $X,Y$, then $T(X \wedge Y)$ is well defined and consequently $T(\Theta)$ is well-defined for any $\Theta \in \mathfrak{so}(V)$. If $T$ is a Riemann-curvature-like tensor then it defines the curvature operator $\mathcal{F}$ sending 2-vectors into 2-vectors, that is,

\[(\mathcal{F}(X \wedge Y), Z \wedge U) = T(X,Y,Z,U).\]
Because of the properties of $T$, the above formula defines a linear, symmetric relative to the given scalar product, operator $\mathfrak{T} : \Lambda^2 V \to \Lambda^2 V$. In particular, $\mathfrak{T}$ is diagonalizable.

We shall now adapt the material of section 3 from [5] to the case we study. Let $\Theta_\alpha$ be an orthonormal basis of $\Lambda^2 V$. For any Riemann-curvature-like tensor $T$ we get (using the identifications $\Lambda^2 V = \mathfrak{T}(V)$)

$$T(X,Y) = \langle T(X,Y), \Theta_\alpha \rangle \Theta_\alpha = \langle \mathfrak{T}(\Theta_\alpha), X \wedge Y \rangle \Theta_\alpha = -\langle T(\Theta_\alpha), X, Y \rangle \Theta_\alpha.$$

The Weizenböck operator is a purely algebraic notion and can be defined for $T$:

$$(W^T s)(X_1, \ldots, X_k) = \sum_{i,j} (T(e_j, X_i) s)(X_1, \ldots, e_j, \ldots, X_k)$$

for any tensor $s$ of type $(l,k)$, $k > 0$. If $k = 0$, we set $W^T s = 0$.

Lemma 12.1. For any Riemann-curvature-like tensor $T$ and any $(0,k)$-tensor $s$ we have

$$W^T s = -\sum_\alpha T(\Theta_\alpha)(\Theta_\alpha s),$$

where $T(\Theta_\alpha)$ acts on $(\Theta_\alpha s)$ (and $\Theta_\alpha$ acts on $s$) as a differentiation.

Proof. Using the above formula we obtain

$$(W^T s)(X_1, \ldots, X_k) = \sum_{i,j} (T(e_j, X_i) s)(X_1, \ldots, e_j, \ldots, X_k)
= -\sum_{i,j,\alpha} \langle T(\Theta_\alpha), e_j, X_i \rangle \langle (\Theta_\alpha s)(X_1, \ldots, e_j, \ldots, X_k)
= -\sum_{i,j,\alpha} (\Theta_\alpha s)(X_1, \ldots, T(\Theta_\alpha)(e_j) e_j, \ldots, X_k)
= -\sum_{i,\alpha} (\Theta_\alpha s)(X_1, \ldots, T(\Theta_\alpha)X_i, \ldots, X_k)
= -\sum_\alpha (T(\Theta_\alpha)(\Theta_\alpha s))(X_1, \ldots, X_k).$$

Lemma 12.2. If $T$ is a Riemann-curvature like tensor and $\mathfrak{T}$ is the curvature operator for $T$ then

$$(W^T s, s) = \lambda_\alpha |\Theta_\alpha s|^2$$

for any $(0,k)$-tensor $s$, where $\Theta_\alpha$ is an orthonormal eigenbasis for $\mathfrak{T}$ and $\lambda_\alpha$ are corresponding eigenvalues. In particular, if $\mathfrak{T} \geq 0$, then $\langle W^T s, s \rangle \geq 0$.

Proof. Observe first that if $A : V \to V$ is a skew-symmetric endomorphism, then $A$ acting on tensors as a differentiation is also skew-symmetric. Using the above lemma we now get

$$\langle W^T s, s \rangle = -\sum_\alpha (T(\Theta_\alpha)(\Theta_\alpha s), s)
= \sum_\alpha \langle (\Theta_\alpha s), T(\Theta_\alpha) s \rangle = \sum_\alpha \lambda_\alpha |\Theta_\alpha s|^2$$
Lemma 12.3. a) If for a k-form $\omega$, where $0 < k < n$, and for every $A \in \mathfrak{so}(V)$ we have $A\omega = 0$, then $\omega = 0$.

b) If for a Riemann-curvature-like tensor $T$ and for every $A \in \mathfrak{so}(V)$ we have $AT = 0$ then $T$ is a multiple of $R_0$.

Proof. a) Suppose that $\omega \neq 0$. Let $e_1, ..., e_n$ be an orthonormal basis such that $\omega(e_1, ..., e_k) \neq 0$. Take $A \in \mathfrak{so}(V)$ such that $Ae_k = 0$ and $Ae_{k+1} = e_k$.

Then we get the following contradiction

$$0 = (A\omega)(e_1, ..., e_{k-1}, e_{k+1}) = -\omega(e_1, ..., e_k) \neq 0.$$ 

b) It is sufficient to observe that if $X, V, W, Z$ are mutually orthogonal then $T(X, V, W, Z) = 0$. First we take three orthogonal vectors $X, Y, Z$ and $A \in \mathfrak{so}(V)$ such that $AY = 0$ and $AX = Z$. We have $0 = (AT)(X, Y, Y, X) = -T(X, Y, Y, X) - T(X, Y, Y, X) = -2T(X, Y, Y, Z)$. Take now $X, Z, W, V$ orthogonal. Then the vectors $Y = V + W, X, Z$ are orthogonal and from the above formula we get $T(X, V, W, Z) = -T(X, W, V, Z)$. Finally we obtain

$$T(X, V, W, Z) = T(W, V, X, Z) = T(W, X, Z) = 0.$$ 

which implies that $T(X, V, W, Z) = 0$.

12.2. The sectional $\nabla$-curvature. Let $(g, \nabla)$ be a statistical structure on an $n$-dimensional manifold $M$. In general, $g(R(X, Y)U, Z)$ is not skew-symmetric for $U, Z$. Define the following tensor field of type $(0, 4)$

$$(117) \quad R(U, Z, X, Y) = \frac{1}{2}(g(R(X, Y)Z, U) - g(R(X, Y)U, Z))$$

Of course it is skew-symmetric relative to the both pairs of arguments $X, Y$ and $U, Z$. Since $g(R(X, Y)Z, U) = -g(R(X, Y)U, Z)$ we have

$$(118) \quad R(U, Z, X, Y) = \frac{1}{2}g(R(X, Y)Z + R(X, Y)Z, U).$$

It follows that the first Bianchi identity holds:

$$\Xi_{Z,X,Y} R(U, Z, X, Y) = 0,$$

where $\Xi_{Z,X,Y}$ stands for the cyclic permutation sum relative to $Z, X, Y$. Consequently $R$ is a Riemann-curvature-like tensor. We can now define the sectional $\nabla$-curvature of a vector plane $\pi$ spanned by the orthogonal vectors $e_1, e_2$ by the formula

$$k(\pi) = k(e_1 \wedge e_2) = R(e_1, e_2, e_1, e_2) = \frac{1}{2}g(R(e_1, e_2)e_2 + R(e_1, e_2)e_2, e_1).$$

We also have

$$(119) \quad k(X \wedge Y) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

We shall say that the sectional $\nabla$-curvature is point-wise constant if for each point $x \in M$ the sectional $\nabla$-curvature is independent of a plane in $T_x M$ and it is equal to $k(x)$. Of course, in such a case, $k(x)$ is a smooth function and, as in the case of Levi-Civita connections, one has
\( R(X, Y)Z + \overline{R}(X, Y)Z = 2k(x)\{g(Z, Y)X - g(Z, X)Y\} =: 2k(x)R_0(X, Y)Z. \)

Examples of manifolds with constant sectional \( \nabla \)-curvature are locally strongly convex equiaffine spheres (in other terminology relative spheres), where \( \nabla \) is the induced connection and \( g \) is the affine second fundamental form. If the corresponding shape operator is equal to \( \lambda \text{id} \), where \( \lambda \in \mathbb{R} \), then, by the Gauss equation,

\[ R(X, Y)Z = \overline{R}(X, Y)Z = \lambda(g(Y, Z)X - g(X, Z)Y). \]

Hence for such a sphere the sectional \( \nabla \)-curvature equals to \( \lambda \).

In contrast with the classical sectional curvature, the fact that the sectional \( \nabla \)-curvature is point-wise constant does not imply that it is constant on a connected manifold if the dimension of the manifold is greater than 2. To see this let us consider the following example.

**Example 12.4.** Take the hypersurface \( M \) in \( \mathbb{R}^{n+1} \) given by the equation

\[ x_1 \cdot \ldots \cdot x_{n+1} = 1 \]

for \( x_1 > 0, \ldots, x_{n+1} > 0 \). It is a locally strongly convex proper affine sphere (the affine shape operator equals to \( \lambda \text{id} \), where \( \lambda \neq 0 \)) and its Blaschke metric \( g \) is flat, i.e. \( \hat{R} = 0 \). Since

\[ R(X, Y)Z + \overline{R}(X, Y)Z = 2\hat{R}(X, Y)Z + 2[K_X, K_Y], \]

we have

\[ \lambda = g([K_X, K_Y]Y, X) \]

for any orthonormal pair of vectors \( X, Y \). Take

\[ \hat{\nabla} = \hat{\nabla} + \varphi K. \]

Denote by \( \hat{R} \) the curvature tensor of \( \hat{\nabla} \) and by \( \overline{R} \) the curvature tensor of the conjugate connection \( \overline{\nabla} \). We have

\[ \hat{R}(X, Y)Z + \overline{R}(X, Y)Z = 2\hat{R}(X, Y)Z + 2\varphi^2[K_X, K_Y]Z \]

and consequently

\[ \frac{1}{2}[g(\hat{R}(X, Y)Y, X) + g(\overline{R}(X, Y)Y, X)] = \varphi^2 \lambda \]

for orthonormal vectors \( X, Y \). If we take a non-constant function \( \varphi \), we get a structure of point-wise constant but non-constant \( \hat{\nabla} \)-sectional curvature on \( M \).

Let now \( M \) be a locally strongly convex hypersurface with an equiaffine transversal vector field \( \xi \). Let \( g \) be the second fundamental form, \( S \) – the corresponding shape operator and \( \nabla \) – the induced connection. By the Gauss equation we have for orthonormal \( X, Y \)

\[ g(\hat{R}(X, Y)Y + \overline{R}(X, Y)Y, X) = g(SX, X) + g(SY, Y). \]

For equiaffine hypersurfaces we can define the sectional mean curvature. Namely, if \( \pi \) is a plane in the tangent space, then

\[ k(\pi) = g(Se_1, e_1) + g(Se_2, e_2), \]

where \( e_1, e_2 \) is an orthonormal basis of \( \pi \). The above considerations show that the definition is independent of the choice of an orthonormal basis. In particular, an equiaffine surface in \( \mathbb{R}^3 \) of constant sectional \( \nabla \)-curvature is exactly a surface of
constant equiaffine mean curvature. Assume now that the sectional mean curvature for an equiaffine hypersurface is point-wise constant. Then we have

\[ g(Y, Z)SX - g(X, Z)SY + g(SY, Z)X - g(SX, Z)Y = R(X, Y)Z + \overline{R}(X, Y)Z \]

\[ = 2k(g(Y, Z)X - g(X, Z)Y). \]

If \( \dim M > 2 \), then for any \( X \) we can take \( Z \neq 0 \) such that \( g(Z, X) = 0 \), \( g(Z, SX) = 0 \) and \( Y = Z \). We see that \( SX \) is a multiple of \( X \). By the second Codazzi equation (\( \nabla S \) is symmetric) we obtain that \( S = \lambda \text{id} \), where \( \lambda \) is constant if \( M \) is connected. Hence the sectional mean curvature is constant. Roughly speaking, in the case of equiaffine hypersurfaces Schur’s lemma holds.

Schur’s lemma also holds for connections satisfying the condition \( R = \overline{R} \). In the category of Blaschke hypersurfaces the condition describes affine spheres.

More generally, we have

**Lemma 12.5.** Let \( M \) be a connected locally strongly convex hypersurface equipped with a transversal vector field whose induced second fundamental form is \( g \), the induced connection is \( \nabla \) and the induced shape operator is \( S \). The hypersurface is an equiaffine sphere, that is, \( S = \lambda \text{id} \) if and only if \( R = \overline{R} \).

Proof. Assume that \( R = \overline{R} \). The shape operator is diagonalizable. Let \( \dim M = 2 \) and \( e_1, e_2 \) be an orthonormal basis of \( T_xM \) such that \( Se_1 = \lambda_1 e_1 \), \( Se_2 = \lambda_2 e_2 \).

By the Gauss equation we have

\[ (120) \quad g(Y, Z)SX - g(X, Z)SY = g(Y, SZ)X - g(X, SZ)Y. \]

Setting \( X = e_1 \), \( Y = Z = e_2 \) we get \( \lambda_1 = \lambda_2 \).

Assume now that \( n > 2 \). Take any \( T_xM \ni X \neq 0 \) and its orthogonal complementary space \( X^\perp \) in \( T_xM \). The mapping

\[ X^\perp \ni W \rightarrow g(X, SW) \in \mathbb{R} \]

has kernel of dimension at least 1. Take \( Z = Y \neq 0 \) from this kernel. Using (120) we obtain that \( SX \) is proportional to \( X \), which finishes the proof.

We have the following second Bianchi identity for the curvature tensor \( R + \overline{R} \)

**Lemma 12.6.** For any statistical structure \( (g, \nabla) \) we have

\[ \Xi_{U,X,Y}(\hat{\nabla}_U(R + \overline{R}))(X,Y) = \Xi_{U,X,Y}(K_U(\overline{R} - R))(X,Y). \]

Proof. We have

\[ \Xi_{U,X,Y}(\hat{\nabla}_U(R + \overline{R}))(X,Y) \]

\[ = \Xi_{U,X,Y}(\nabla - K_U R)(X,Y) + \Xi_{U,X,Y}(\nabla + K_U \overline{R})(X,Y) \]

\[ = \Xi_{U,X,Y}(K_U(\overline{R} - R))(X,Y). \]

Thus, if \( R = \overline{R} \) then

\[ \Xi_{U,X,Y}(\nabla U R)(X,Y) = \Xi_{U,X,Y}(\nabla U R)(X,Y) = 0. \]

Using the second Bianchi identity for \( \hat{\nabla} R \) one easily gets
Proposition 12.7. Let $M$ be a connected manifold of dimension greater than 2. If $(g, \nabla)$ be a statistical structure on $M$ with $R = \mathcal{R}$. If the sectional $\nabla$-curvature is point-wise constant then it is constant.

12.3. The curvature operator for statistical structures. If $(g, \nabla)$ is a statistical structure then $T = R + \mathcal{R}$ is a Riemann-curvature-like tensor field and we can apply the algebraic results of section 8.1 to this tensor field.

Because of the Bianchi identity proved in the last section exactly in the same way as Theorem 1.2 in [5] one can prove

Theorem 12.8. Assume that for a statistical structure $(g, \nabla)$ we have $R = \mathcal{R}$. The following formula holds

$$(\hat{\nabla}^* \hat{\nabla} R)(X, Y, Z, W) + \frac{1}{2}(W^R R)(X, Y, Z, W)$$

$$= \frac{1}{2}(\hat{\nabla}_X \hat{\nabla}^* R)(Y, Z, W) - \frac{1}{2}(\hat{\nabla}_Y \hat{\nabla}^* R)(X, Z, W)$$

$$+ \frac{1}{2}(\hat{\nabla}_Z \hat{\nabla}^* R)(W, X, Y) - \frac{1}{2}(\hat{\nabla}_W \hat{\nabla}^* R)(Z, X, Y).$$

We can now formulate the following version of Tachibana’s theorem

Theorem 12.9. Let $M$ be a connected compact oriented manifold and $(g, \nabla)$ be a statistical structure on $M$ such that $R = \mathcal{R}$. If the curvature operator $\hat{\mathcal{R}}$ for $\hat{\mathcal{R}}$ is non-negative and $\text{div} \hat{\nabla} R = 0$ then $\hat{\nabla} R = 0$. If additionally $\mathcal{R} > 0$ at some point of $M$ then the sectional $\nabla$-curvature is constant.

Proof. It is clear that $\text{div} \hat{\nabla} R = \nabla^* R$. By Theorem 12.8 we now have

$$\hat{\nabla}^* \hat{\nabla} R + \frac{1}{2} W^R R = 0.$$ 

Consequently

$$0 = \int_M (g(\hat{\nabla}^* \hat{\nabla} R, R) + \frac{1}{2} g(W^R R, R)) \nu_g$$

$$= \int_M g(\hat{\nabla} R, \hat{\nabla} R) \nu_g + \frac{1}{2} \int_M g(W^R R, R) \nu_g$$

By Lemma 12.2 we obtain $\hat{\nabla} R = 0$ and $\sum_{\alpha} \lambda_{\alpha} |\Theta_{\alpha} R|^2 = 0$ at each point of $M$. Therefore, if at some point $x \in M$ the curvature operator $\mathcal{R}$ is positive, then $\Theta_{\alpha} R = 0$ at this point for all $\alpha$ and consequently for any $A \in \mathfrak{so}(T_x M)$ we have $AR = 0$. By Lemma 12.3 b) we get that $R = \lambda R_0$ at $x$. Since $\nabla R = 0$, the same equality holds at each point of $M$.

Finally we observe that a theorem of Meyer-Gallot holds for trace-free statistical structures

Theorem 12.10. Let $M$ be a connected compact oriented manifold and $(g, \nabla)$ be a trace-free statistical structure on $M$. If the curvature operator for $R + \mathcal{R}$ is non-negative on $M$ then each harmonic form is parallel relative to $\nabla$, $\nabla$ and $\hat{\nabla}$. If moreover the curvature operator is positive at some point of $M$, then the Betti numbers $b_1(M) = ... = b_{n-1}(M) = 0$. 
Proof. If $\omega$ is a harmonic form, then by (108) and Lemma 12.2 we obtain

$$0 = \int_M (g(W^R + \bar{R}, \omega) + g(\nabla \omega, \nabla \omega) + g(\nabla \omega, \nabla \omega)) \nu_g$$

$$= \int_M \sum_\alpha \lambda_\alpha |\Theta_\alpha \omega|^2 \nu_g + \int_M g(\nabla \omega, \nabla \omega) \nu_g + \int_M g(\nabla \omega, \nabla \omega) \nu_g.$$ 

This yields the first assertion. If at some point $x \in M$ all $\lambda_\alpha > 0$ then we additionally have $\Theta_\alpha \omega = 0$ at this point. Since the $\Theta_\alpha$ form a basis for the space of $\mathfrak{so}(T_xM)$, we have that $A\omega = 0$ for all $A \in \mathfrak{so}(T_xM)$. Using now Lemma 12.3 we see that $\omega_x = 0$ if the degree of $\omega$ is between 1 and $n - 1$. Since $\omega$ is parallel relative to a connection, it must vanish on the whole of $M$. □

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