Functional Relations in Solvable Lattice Models II: Applications

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Abstract. Reported are two applications of the functional relations ($T$-system) among a commuting family of row-to-row transfer matrices proposed in the previous paper Part I. For a general simple Lie algebra $X_r$, we determine the correlation lengths of the associated massive vertex models in the anti-ferroelectric regime and central charges of the RSOS models in two critical regimes. The results reproduce known values or even generalize them, demonstrating the efficiency of the $T$-system.
1. Introduction

This is a continuation of our previous paper, hereafter called Part I [1]. There, we have proposed the functional relation (FR), the $T$-system, among a commuting family of row-to-row transfer matrices for a class of solvable lattice models. They are the vertex and RSOS type models associated with any simple Lie algebra $X_r$ in the sense of section 3.3 of Part I. Let $\{T_m^{(a)}(u)\}$ be the family of their row-to-row transfer matrices acting on a common quantum space. Here $u$ denotes the spectral parameter and the $(a, m)$ labels the fusion type, i.e., signifies that the auxiliary space is the $U_q(X_r^{(1)})$-module $W_m^{(a)}(u)$. See section 3.2 and Fig.2 in Part I. Then the $T$-system is the following three term FRs for $a = 1, \ldots, r, m = 1, 2, \ldots$:

$$T_m^{(a)}(u - \frac{1}{2t_a})T_m^{(a)}(u + \frac{1}{2t_a}) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + g_m^{(a)}(u)\prod_{b=1}^r T(a, b, m, u)^{I_{ab}}, \quad (1.1a)$$

where the explicit forms of $T(a, b, m, u)$ are given by

$$T(a, b, m, u) = T^{(b)}_{m/b/m} (u) \quad \text{for } \frac{t_b}{t_a} = 1, 2, 3, \quad (1.1b)$$

$$= T^{(b)}_{m/b/2m} (u - \frac{1}{4})T^{(b)}_{2m/2m} (u + \frac{1}{4}) \times T^{(b)}_{m-1/m-1} (u)T^{(b)}_{m+1/m+1} (u) \quad \text{for } \frac{t_b}{t_a} = \frac{1}{2}, \quad (1.1c)$$

$$= T^{(b)}_{m/b/3m} (u - \frac{1}{3})T^{(b)}_{3m/3m} (u)T^{(b)}_{m+1/m+1} (u + \frac{1}{3}) \times T^{(b)}_{m-1/m-1} (u - \frac{1}{6})T^{(b)}_{m+2/m+2} (u) \quad \text{for } t_b/t_a = \frac{1}{3}, \quad (1.1d)$$

under the conventions $\forall T_0^{(a)}(u) = 1$ and $T_m^{(a)}(u) = 1$ if $m \not\in \mathbb{Z}$. In (1.1a), $g_m^{(a)}(u)$ is a scalar function depending on the quantum space. The integers $t_a \in \{1, 2, 3\}$ and $I_{ab} \in \{0, 1\}$ are defined from the root system of $X_r$ (cf. Part I, eq.(3.1)):

$$I_{ab} = 2\delta_{ab} - B_{ab},$$

$$B_{ab} = C_{ab} \frac{t_b}{t_{ab}}, \quad C_{ab} = \frac{2(\alpha_a|\alpha_b)}{(\alpha_a|\alpha_a)}, \quad (1.2)$$

$$t_a = \frac{2}{(\alpha_a|\alpha_a)}, \quad t_{ab} = \max(t_a, t_b).$$
We call (1.1) the *unrestricted* $T$-system when we let $m$ extend over all positive integers.

On the other hand, if $T_{t_α, t_{α+1}}^{(a)}(u) \equiv 0$ ($1 \leq a \leq r$) is imposed for some integer $ℓ ≥ 1$, (1.1) closes among those $T_{m}^{(a)}(u)$’s with $1 ≤ a ≤ r$, $1 ≤ m ≤ t_α ℓ$. We call it the (level $ℓ$) *restricted* $T$-system. These are to hold for the vertex and the level $ℓ$ RSOS models associated with $U_q(X_r^{(1)})$, respectively.

Applications of the above $T$-system to the calculation of physical quantities are the main purpose in this report Part II. More specifically, we will determine the correlation lengths of massive vertex models in anti-ferroelectric regime (section 2) and central charges of critical RSOS models (section 3). The idea of using the FRs for computing thermodynamic quantities is not itself new but often known to bypass complicated Bethe ansatz analyses (cf. [2,3,4]). Our $T$-system approach indeed possesses such an advantage and generalizes the earlier studies [5-7] for the simplest $X_r = A_1$ case.

As for the correlation lengths of the massive vertex models, the derivation reduces to finding a periodicity of $T_{m}^{(a)}(u)$ that obeys a truncated version (2.6) of (1.1a). We call it the “bulk” $T$-system and prove that the real period is just the dual Coxeter number, thereby giving a unified expression (2.3) for the correlation length. This extends the earlier analysis for the $sl(2)$ case in [5]. As an independent check, we shall also invoke the standard Bethe ansatz method [8] and show the agreement of the results.

To evaluate the central charges, we generalize the treatment of the $sl(2)$ case [6,7] to arbitrary $X_r$. We convert the $T$-system into integral equations for finite size corrections and extract the central charges by the standard argument [9,10]. It leads to the expressions (3.29) and (3.49) in terms of the Rogers dilogarithm, which turn out to be calculable by means of the conjecture in [11,12]. The results reproduce those in [12-14] obtained from the thermodynamic Bethe ansatz (TBA) analyses.

These agreements show the efficiency of our $T$-system approach. It works well even for the models that defy the root density method [15] and leads to scaling dimensions as shown in [6,7] for $sl(2)$. Though we only determine the central charges in section 3, our formulation also covers the scaling dimensions, for which the relevant dilogarithm conjecture is now available in [16]. We hope to further extend the analyses presented in this paper to actually derive the scaling dimensions or even to produce the character itself for the corresponding conformal field theories (CFTs) (cf. [17]).

Appendix A supplements the proof of the periodicity of the bulk $T$-system. Appendix B provides a list of the matrix functions needed in the Bethe ansatz computation in section 2.3. Appendix C is a calculation of a certain ratio of the bulk eigenvalues of
the transfer matrices \([14]\) for the \(A_r^{(1)}\) RSOS models \([18]\). It is an important input to the integral equations in section 3. Appendix D is devoted to an exposition of the generalized dilogarithm conjecture in \([16]\). Throughout the paper, the equations in Part I \([1]\), e.g., (B.3), will be referred to as (IB.3), etc.

2. Correlation Lengths of Massive Vertex Models

As the first application of our \(T\)-system (1.1), we shall compute the correlation lengths of the massive vertex models in anti-ferroelectric regime. The same quantities will be calculated through a Bethe ansatz method as an independent check.

2.1. Massive vertex models in anti-ferroelectric regime

Let us briefly sketch what we mean by massive vertex models in anti-ferroelectric (AF) regime. As the simplest example, consider the 6-vertex model with the Boltzmann weights given by

\[
\begin{align*}
  w(\pm 1, \pm 1, \pm 1, \pm 1) &= \frac{\sinh \lambda (1 - u)}{\sinh \lambda}, \\
  w(\pm 1, \mp 1, \pm 1, \mp 1) &= \frac{\sinh \lambda u}{\sinh \lambda}, \\
  w(\pm 1, \pm 1, \mp 1, \mp 1) &= 1,
\end{align*}
\]

(2.1)

where the variables \(\epsilon_1, \ldots, \epsilon_4\) in \(w(\epsilon_1, \ldots, \epsilon_4)\) is ordered clockwise from the left edge of the vertex. We impose the periodic boundary condition and assume that the system size is even. In (2.1) one observes that

(i) the Boltzmann weights are trigonometric functions of \(u\) and \(\lambda\) and are analytic with respect to \(u\). The partition function with periodic boundaries is invariant under \(u \rightarrow u + \frac{\pi i}{\lambda}\).

The invariance can be seen by noting that the weights \(w(1, 1, -1, -1)\) and \(w(-1, -1, 1, 1)\) always occur in pairs. We shall consider the regime \(0 < u < \frac{1}{2}\), \(\lambda > 0\). Then (2.1) corresponds to \(\Delta < -1\) in the conventional parameter \([2]\) and it is well known that

(ii) the model is critical as \(\lambda \rightarrow 0\) and is in an AF ground state as \(\lambda \rightarrow \infty\).

This is natural in view of that all the weights (2.1) compete when \(\lambda \rightarrow 0\) while only the last one dominates exponentially as \(\lambda \rightarrow \infty\). The ground state configurations are invariant in the SW-NE direction and alternating in the NW-SE direction. Thus the range \(0 < \lambda < \infty\)
corresponds to the AF phase. As we saw in section 2 of Part I, one can form a length \( N \)
row-to-row transfer matrix \( T_1(u) \) acting in the vertical direction and obeying the \( T \)-system
\[
T_1(u - \frac{1}{2})T_1(u + \frac{1}{2}) = T_2(u) + g_2^N(u)\text{Id} \tag{2.2}
\]
with \( g_2(u) = \sinh \lambda (3/2 - u) \sinh \lambda (1/2 + u)/\sinh^2 \lambda \). Here \( T_2(u) \) is the transfer matrix for
the degree 2 fusion model in the auxiliary space. (The spectral parameter in section 2 of
Part I is denoted by \(-u\) here.) When evaluating (2.2) on the common eigenvectors, it is
generally expected in the regime \( 0 < u < \frac{1}{2}, \lambda > 0 \) [3,19] that
(iii) \( \frac{\text{eigenvalue of the 1st term}}{\text{eigenvalue of the 2nd term}} < e^{-\delta N} \) with some \( \delta > 0 \) on the rhs of the \( T \)-system,
which simplifies (2.2) in the thermodynamic limit \( N \to \infty \). Excitations are described by
various eigenstates of the transfer matrices. Then,
(iv) there are finitely many excited states that degenerate to the lowest one as \( N \to \infty \).
We shall call all these finitely many states as the first excited states. In the present 6-vertex
case, there is only one first excited state relevant to the interfacial tension. The lowest
excitations above the first ones are called the second excitations. Then it is known that
(v) the second excited states form an energy band for \( N \to \infty \) and they are higher than
the first excitation energy by a certain gap,
which implies that the model is massive and the correlation length is just the inverse of
the energy gap. Note that this defines the vertical correlation length which should be the
same for all the fusion models whose transfer matrices act on the common quantum space.

Now recall that the 6-vertex model (2.1) is associated to the fundamental representa-
tion of the quantum group \( U_q(A_1^{(1)}) \) with \( q = e^{-\lambda} \) [20,21]. Analogously we consider the
fusion vertex models associated to \( U_q(X_r^{(1)}) \) with the fixed homogeneous quantum space
\( W_s^{(p)\otimes N} \) for some integers \( 1 \leq p \leq r \) and \( s \geq 1 \). See section 3 of Part I. As discussed there,
they are expected to obey the unrestricted \( T \)-system (1.1). Moreover we suppose (i) and
that there exists a certain parameter regime in \( u \) and \( \lambda \) such that all the features (ii)-(v)
are valid. We call them the massive \( U_q(X_r^{(1)}) \) vertex models in AF regime.

In the rest of this section we exclude the case \( X_r = A_r \) with \( r \geq 2 \) and take the AF
regime as \( 0 < u < \frac{1}{2}, \lambda > 0 \). We will show that the correlation length \( \xi \) of these vertex
models is given by a unified formula
\[
\xi = -\frac{1}{\log k}. \tag{2.3a}
\]
Here $0 < k < 1$ is determined by
\[ \frac{K'(k)}{K(k)} = \frac{g\lambda}{\pi}, \] (2.3b)
where $g$ is the dual Coxeter number of $X_r$ and $K(k) (K'(k))$ denotes the complete elliptic integral of the first (second) kind with modulus $k$ (cf. section 15 in [2]). Namely,
\[ k = 4y^{\frac{1}{2}} \prod_{n=1}^{\infty} \left( \frac{1 + y^{2n}}{1 + y^{2n-1}} \right)^4 y = e^{-g\lambda}. \] (2.3c)
Note that (2.3) is consistent with the property (ii): $\xi \to 0(\infty)$ as $\lambda \to \infty(0)$. Actually it reproduces the 6-vertex model result eq.(8.10.12) in [2] for $X_r = A_1 (g = 2)$ up to a trivial overall factor which depends on the normalization of $\xi$. Such factors will not be concerned hereafter. The result (2.3) will be shown by two independent methods. One uses our $T$-systems and the other is a rather conventional Bethe ansatz analysis where we will use a string hypothesis (vi) (see section 2.3) as well. Both methods lead to (2.3), supporting our hypotheses on the $T$-system. (In the Bethe ansatz analysis, we shall actually consider a less general setting as will be noted in the beginning of section 2.3.)

In the working below the AF regime will always be taken as $0 < u < \frac{1}{2}$. This is actually needed in the derivation using the Bethe ansatz in section 2.3. See a remark before (2.42). However, the analyses indicate that (2.3) is valid in a wider range $0 < u < \frac{g}{2}$ ($g$: dual Coxeter number). This has been observed for the 6-vertex model in p155 of [2] and will also be argued in the end of section 2.3.

### 2.2. Correlation length from the $T$-system

#### 2.2.1 Double Periodicity

Consider the $T$-system (1.1) among the length $N$ transfer matrices $T_m^{(a)}(u)$ for the massive vertex models. We denote by $L_m^{(a)}(u)$ the ratio of the eigenvalues of $T_m^{(a)}(u)$ for a second excitation to the ground state. Then $L_m^{(a)}(u)$ turns out to be a doubly periodic function with respect to $u$ in the thermodynamic limit, which is crucial in deriving the correlation lengths [5]. From the property (i), $L_m^{(a)}(u)$ is a meromorphic function of $u$ and has the periodicity in the imaginary direction of $u$,
\[ L_m^{(a)}(u) = L_m^{(a)}(u + \frac{\pi i}{\lambda}). \] (2.4)

We set $N \to \infty$ from now on and seek the periodicity in the real direction. We shall exclusively consider the case $m \in t_a \mathbb{Z}_{\geq 0}$ in the sequel. In this limit, $L_m^{(a)}(u)$ actually
measures the energy gap due to the property (iv). Let us quote the unrestricted $T$-system (1.1a) symbolically as a three term relation

$$T_0 = T_1 + T_{-1},$$

(2.5)

where $T_0 = T_m(a)(u + \frac{1}{2t_a})T_m(a)(u - \frac{1}{2t_a})$, $T_1 = T_{m+1}(a)T_{m-1}(a)$ and the third term $T_{-1}$ is also a product of $T^{(a')}_{m'}$s and the scalar function $g_m(a)(u)$. We assume that $|T_1| \ll |T_{-1}|$ as $N \to \infty$ in AF regime. This is a natural extension of the property (iii) for the sl(2) case [19] where it is consistent with the fact that the ground state is singlet. In this view, we introduce the truncated version of (2.5)

$$T_0 = T_{-1}|_{g_m(a)(u)\equiv 1},$$

(2.6)

which we call the bulk $T$-system. By the definition, $L_m(a)(u)$ satisfies (2.6) because the $g_m(a)(u)$ factor cancels when taking the ratio of the second excitation to the ground state. Then, the point is that the bulk $T$-system itself imposes the periodicity as follows.

**Proposition.** Suppose $T_m(a)(u)$ satisfies the bulk $T$-system (2.6). Then the following periodicity is valid for $m \in t_a\mathbb{Z}_{\geq 0}$.

$$T_m(a)(u) = T_m(a)(u + g) \quad \text{for any } X_r,$$

(2.7)

$$\bar{T}_m(a)(u)\bar{T}_m(a)(u + \frac{g}{2}) = 1 \quad \text{if } X_r \neq A_{r \geq 2},$$

(2.8a)

$$T_m(a)(u) = \begin{cases} T_m(a)(u)T_m(a)(u + 1)T_m(a)(u + 2) & \text{if } X_r = E_6 \text{ or } E_8, \\ T_m(a)(u) & \text{if } X_r \neq A_{r \geq 2}, E_6 \text{ or } E_8, \end{cases}$$

(2.8b)

where $g$ denotes the dual Coxeter number.

Here we have also included the case $X_r = A_{r \geq 2}$ in the statement although it is irrelevant to the correlation length calculation. Note that (2.7) is just a corollary of (2.8) if $X_r \neq A_{r \geq 2}, E_6$ or $E_8$. The proposition claims another period than (2.4) as

$$L_m(a)(u) = L_m(a)(u + g) \quad \text{for } m \in t_a\mathbb{Z}_{\geq 0}.$$

(2.9)

This is consistent with $Y$-system’s period (IB.7) in view of the connection (I3.19) and the fact that the bulk $T$-system formally corresponds to the level 0 restricted $T$-system. In the rest of this subsection, we illustrate the proof of the proposition for $X_r = A_r, B_r$ and leave the other cases to appendix A.
Proof. \( X_r = A_r \): The bulk \( T \)-system is
\[
T_m^{(a)}(u - \frac{1}{2})T_m^{(a)}(u + \frac{1}{2}) = T_m^{(a+1)}(u)T_m^{(a-1)}(u) \quad 1 \leq a \leq r,
\]
where we have put \( T_m^{(0)}(u) = T_m^{(r+1)}(u) = 1 \). By induction, it is easy to show
\[
T_m^{(a)}(u) = \prod_{j=1}^{a} T_m^{(1)}(u + j - \frac{a + 1}{2}),
\]
for \( 1 \leq a \leq r + 1 \). Setting \( a = r + 1 \) and \( u = u' + r/2 \) in the above, we obtain
\[
T_m^{(1)}(u') \cdots T_m^{(1)}(u' + r) = 1,
\]
from which \( T_m^{(1)}(u) = T_m^{(1)}(u + r + 1) \) hence (2.7) follows. Note that eq.(2.8a) holds only for \( r = 1 \) as clearly seen from the above derivation.

\( X_r = B_r \): The relevant bulk \( T \)-system is
\[
T_m^{(a)}(u - \frac{1}{2})T_m^{(a)}(u + \frac{1}{2}) = T_m^{(a+1)}(u)T_m^{(a-1)}(u) \quad \text{for} \quad 1 \leq a \leq r - 2,
\]
\[
T_m^{(r-1)}(u - \frac{1}{2})T_m^{(r-1)}(u + \frac{1}{2}) = T_m^{(r-2)}(u)T_{2m}^{(r)}(u),
\]
\[
T_{2m}^{(r)}(u - \frac{1}{4})T_{2m}^{(r)}(u + \frac{1}{4}) = T_m^{(r-1)}(u - \frac{1}{4})T_m^{(r-1)}(u + \frac{1}{4}),
\]
where \( T_m^{(0)}(u) = 1 \). From (2.13a) we have
\[
T_m^{(a)}(u) = \prod_{j=1}^{a} T_m^{(1)}(u + j - \frac{a + 1}{2}),
\]
for \( 1 \leq a \leq r - 1 \). Using this in (2.13b), we find
\[
T_{2m}^{(r)}(u) = \prod_{j=1}^{r} T_m^{(1)}(u + j - \frac{r + 1}{2}),
\]
Substitute (2.14) and (2.15) into (2.13c) and cancel the common factors. The result reads
\[
T_m^{(1)}(u - \frac{2r - 1}{4})T_m^{(1)}(u + \frac{2r - 1}{4}) = 1.
\]
This establishes (2.8) for all \( 1 \leq a \leq r \) because of (2.14,15) and \( g = 2r - 1 \).
2.2.2 Correlation Lengths from the double periodicity

So far we have seen that $L_m^{(a)}(u)$ is a meromorphic and doubly periodic function of $u$ as specified in (2.4) and (2.9). Next we define $0 < k < 1$ by (2.3b,c) and put

\[ h_1(u, u_0) = \sqrt{k} \text{sn} \left( \frac{2i\lambda K(k)}{\pi}(u - u_0) \right), \]

\[ h_2(u, u_0) = \sqrt{k} \text{sn} \left( \frac{2i\lambda K(k)}{\pi}(u - u_0 + \frac{g}{2}) \right), \]

where Jacobi’s elliptic function sn is of modulus $k$. These are meromorphic, $g$-periodic, $\frac{\pi i}{\lambda}$-anti-periodic functions of $u$ and satisfy

\[ h_j(u, u_0)h_j(u + \frac{g}{2}, u_0) = 1 \quad \text{for} \quad j = 1, 2. \] (2.18)

In the rectangle $[0, \frac{g}{2}) \times [0, \frac{\pi i}{\lambda})$ on the complex $u$-plane, $h_1(u, u_0)$ ($h_2(u, u_0)$) has one simple zero (pole) and no poles (zeros). Put

\[ \bar{L}_m^{(a)}(u) = \begin{cases} L_m^{(a)}(u)L_m^{(a)}(u + 1)L_m^{(a)}(u + 2) & \text{if } X_r = E_6 \text{ or } E_8, \\ L_m^{(a)}(u) & \text{otherwise}, \end{cases} \] (2.19)

and let $\{u_z\}$, $\{u_p\}$ be the set of zeros and poles of $\bar{L}_m^{(a)}(u)$ in the rectangle $u \in [0, \frac{g}{2}) \times [0, \frac{\pi i}{\lambda})$, respectively. Then the ratio

\[ h(u) = \frac{\bar{L}_m^{(a)}(u)}{\prod_{u_z} h_1(u, u_z) \prod_{u_p} h_2(u, u_p)} \] (2.20)

is analytic and non-zero for $0 \leq \text{Re}(u) < \frac{g}{2}$. Furthermore from (2.8) we have

\[ h(u)h(u + \frac{g}{2}) = 1 \] (2.21)

except for $X_r = A_{r \geq 2}$. Therefore $h(u)$ is doubly periodic with periods $g$ and $\frac{\pi i}{\lambda}$ (if $\#\{u_z\} + \#\{u_p\}$ is even) and it is analytic and non-zero on the whole complex $u$-plane. From the Liouville theorem and (2.21) it follows that $h(u) = \pm 1$. Thus we obtain

\[ \bar{L}_m^{(a)}(u) = \pm \prod_{u_z} \sqrt{k} \text{sn} \left( \frac{2i\lambda K(k)}{\pi}(u - u_z) \right) \prod_{u_p} \sqrt{k} \text{sn} \left( \frac{2i\lambda K(k)}{\pi}(u - u_p + \frac{g}{2}) \right), \] (2.22)

where there must be even number of sn’s in total on the rhs for $\bar{L}_m^{(a)}(u)$ to be $\frac{\pi i}{\lambda}$-periodic.
Consider now the correlation function $G(R) = \langle \delta_{\sigma_0 \rho_1} \delta_{\sigma_R \rho_2} \rangle$ of the two vertical edge variables $\sigma_0$, $\sigma_R$ on the same column and separated by $R$ vertices. Here $\rho_1$, $\rho_2$ denote some given edge states. The vertical correlation length $\xi$ is then defined by

$$G(R) - G(\infty) \simeq \text{const} \cdot e^{-\frac{R}{\xi}} \quad \text{as } R \to \infty. \quad (2.23)$$

To extract the $\xi$ from (2.22) and (2.23) we follow [5,8] and need a few more assumptions. Firstly, we assume that $L_m^a(u)$ is free of poles in $0 \leq \text{Re}(u) \leq \frac{2\pi}{T}$. This may be natural since its denominator is the ground state eigenvalue of $T_m^a(u)$. Secondly, the leading part of $G(R) - G(\infty)$ will come from those $L_m^a(u)$ that contain only two zeros. Thus the relevant case of (2.22) has the from

$$L_m^a(u) = \pm k \sin\left(\frac{2i\lambda K(k)}{\pi}(u-u_1)\right) \sin\left(\frac{2i\lambda K(k)}{\pi}(u-u_2)\right). \quad (2.24)$$

Various choices for $u_1$ and $u_2$ here will correspond to the energy band mentioned in the property (v). The correlation function is now evaluated by assembling the contribution from the band as

$$G(R) - G(\infty) \simeq \int du_1 \int du_2 g(u_1, u_2) \left(\tilde{L}_m^a(u; u_1, u_2)\right)^R, \quad (2.25)$$

where $g(u_1, u_2)$ is some weight function and the integration is over a finite range. Finally, we assume that the integral $\int du_1 \int du_2 g(u_1, u_2) \sin^R\left(\frac{2i\lambda K(k)}{\pi}(u-u_1)\right) \sin^R\left(\frac{2i\lambda K(k)}{\pi}(u-u_2)\right)$ does not develop a dominant $R$-dependence in the limit $R \to \infty$ compared with $k^R$. The discrepancy between $\tilde{L}_m^a$ and $L_m^a$ for $X_a = E_6$ and $E_8$ is expected to be irrelevant to extracting the dominant $R$-dependence. Thus substitution of (2.24) into (2.25) leads to

$$G(R) - G(\infty) \simeq \text{const} \cdot k^R.$$

Comparing this with (2.23) we arrive at (2.3).

### 2.3. Correlation length from Bethe ansatz

#### 2.3.1 Explicit forms of the free energy and the Bethe ansatz equation

The Bethe ansatz equation for our $U_q(X_r^{(1)})$ massive vertex model has the form (1 $\leq a \leq r$) [22-24]

$$\left(\frac{\sin\left(\frac{1}{2}v_j^{(a)} + \frac{i}{2l_s} s \lambda \delta_{ap}\right)}{\sin\left(\frac{1}{2}v_j^{(a)} - \frac{i}{2l_s} s \lambda \delta_{ap}\right)}\right)^N = \prod_{b=1}^{r} \prod_{k} \frac{\sin\left(\frac{1}{2}v_j^{(a)} - v_k^{(b)} + i(\alpha_a|\alpha_b)\lambda\right)}{\sin\left(\frac{1}{2}v_j^{(a)} - v_k^{(b)} - i(\alpha_a|\alpha_b)\lambda\right)}. \quad (2.26)$$
Here, the integers $1 \leq p \leq r$ and $s \geq 1$ signify that the transfer matrices $T_m^{(a)}(u)$ act on the fixed quantum space $W_s^{(p)} \otimes W_{N_s}$ as noted before (2.3). Note that

$$X_r \neq A_{r \geq 2}, \quad 0 < u < \frac{1}{2} \quad (2.27a)$$

as in subsection 2.2. In addition, we shall restrict ourselves to the case

$$t_p = 1 \quad (2.27b)$$
in this subsection for a technical reason and exclusively study the fusion vertex model corresponding to the transfer matrix $T_m^{(a)}(u)$ with $(a, m) = (p, s)$. Plainly, it is the model whose horizontal and vertical fusion types are both $W_s^{(p)}$ in the sense of section 3.3 in Part I. In this case, one may deduce [2] the free energy from the energy of the corresponding one dimensional quantum system (cf. eq.(2.19) in [12]) as

$$F(u) = - \sum_{a=1}^{r} \sum_{j} \log \left( \frac{\sin \left( \frac{1}{2} v_j^{(a)} \right) + i \frac{m}{2t_a} s \lambda \delta_{ap} + i \lambda (u - u^*)}{\sin \left( \frac{1}{2} v_j^{(a)} - i \frac{m}{2t_a} s \lambda \delta_{ap} + i \lambda (u - u^*) \right)} \right). \quad (2.28)$$

In the above, $u^*$ signifies the point where the Hamiltonian limit is to be taken. In the case of the $sl(r + 1)$ models [18,14], it is given by $1 - p$ (see appendix C). Below, we set $u^* = 0$ assuming that its effect has been absorbed into the spectral parameter $u$ that obeys (2.27a). Eqs.(2.26,28) are consistent with the analytic Bethe ansatz in [23] and actually reduces to the known 6-vertex model result for $X_r = A_1, s = p = 1$. Notice that the sin function appears here in contrast to the sinh function for the “massless” regime. In the thermodynamic limit $N \to \infty$, the solution to (2.26) is expected to form the pattern

$$v_j^{(a)} = v_{j,m}^{(a)} + i \frac{(m + 1 - 2\alpha) \lambda}{t_a}, \quad \alpha = 1, 2, \ldots, m \quad (2.29)$$

for some $v_{j,m}^{(a)} \in (-\pi, \pi]$ and $m \in \mathbb{Z}_{\geq 1}$, which is called the color $a$ $m$-string with center $v_{j,m}^{(a)}$. For each color $1 \leq a \leq r$ and length $m \in \mathbb{Z}_{\geq 1}$, let $\rho_m^{(a)}(v)$ and $\sigma_m^{(a)}(v)$ denote the densities of $m$-strings and $m$-holes with the center $v \in (-\pi, \pi]$, respectively (cf. [25-28]). Then eq.(2.26) can be rewritten in terms of these densities as we did in appendix B of Part I.

For a $2\pi$-periodic function $f(v) = f(v + 2\pi)$, we define its Fourier components as

$$\hat{f}[n] = \int_{-\pi}^{\pi} f(v) e^{inv} dv \quad \text{for} \quad n \in \mathbb{Z}, \quad f(v) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}[n] e^{-inv}. \quad (2.30)$$
By performing the Fourier transformation with respect to the centers, (2.26) is rewritten as
\[ \delta_{ap} \hat{A}_{pa}^{sm} [n] = \delta_{m}^{(a)} [n] + \sum_{b=1}^{r} \sum_{k \geq 1} \hat{M}_{ab} [n] \hat{A}_{ab}^{mk} [n] \hat{\rho}_{k}^{(b)} [n], \] (2.31)
where
\[ \hat{M}_{ab} [n] = B_{ab} + 2 \delta_{ab} (\cosh \frac{n \lambda}{t_a} - 1), \] (2.32a)
\[ \hat{A}_{ab}^{mk} [n] = \frac{\sinh (\min \left( \frac{m}{t_a}, \frac{k}{t_b} \right) n \lambda)}{\sinh \frac{n \lambda}{t_{ab}}} \exp (-\max \left( \frac{m}{t_a}, \frac{k}{t_b} \right) |n| \lambda). \] (2.32b)

See (1.2) for the definitions of $B_{ab}$ and $t_{ab}$. For given string densities, the free energy (2.28) is evaluated as
\[ \frac{F(u)}{N} = - \sum_{m \geq 1} \sum_{\alpha=1}^{m} \int_{-\pi}^{\pi} dv \rho_{m}^{(p)} (v) \log \left( \sin \left( \frac{\lambda}{2} v + i \frac{(m+1-2\alpha+s) \lambda}{2} + i \lambda u \right) \right) \left( \frac{\sin \left( \frac{\lambda}{2} v + i \frac{(m+1-2\alpha-s) \lambda}{2} + i \lambda u \right)}{\sin \left( \frac{\lambda}{2} v + i \frac{(m+1-2\alpha-s) \lambda}{2} + i \lambda u \right)} \right) = \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} e^{-2\lambda nu} \hat{A}_{pp}^{sm} [n] \hat{\rho}_{m}^{(p)} [-n] \frac{n}{n}, \] (2.33)
for $N \to \infty$.

### 2.3.2 Excitation energy as a function of hole locations

First we seek the ground state by employing the string hypothesis:

(vi) The ground state is given by the Dirac sea of the color $a$ $st_a$-strings, which agrees with many earlier investigations on the special cases [12-14,29,30]. In terms of the density functions, this is equivalent to
\[ \delta_{m}^{(a)} [n] = 0 \text{ for all } 1 \leq a \leq r \text{ and } m, \] (2.34a)
\[ \hat{\rho}_{m}^{(a)} [n] = 0 \text{ unless } m = st_a. \] (2.34b)

Substituting this into (2.31) we find
\[ \hat{\rho}_{st_a}^{(a)} [n]_{gr} = \hat{Z}_{ap} [n] \hat{A}_{pp}^{st_a, st_a} [n], \] (2.35)
where the subscript $gr$ means the ground state and $\hat{Z}_{ap} [n]$ is defined by the matrix relation
\[ (\hat{Z}_{ab} [n])_{1 \leq a, b \leq r} = \text{inverse of } (\hat{M}_{ab} [n] \hat{A}_{ab}^{st_a, st_b} [n])_{1 \leq a, b \leq r} \]
\[ = \frac{e^{s|n| \lambda}}{\sinh (s n \lambda)} \cdot (\hat{D} [n])^{-1}_{1 \leq a, b \leq r}, \] (2.36a)
\[ \hat{D} [n] = \left( \frac{\hat{M}_{ab} [n]}{\sinh \left( \frac{n \lambda}{t_{ab}} \right)} \right)_{1 \leq a, b \leq r}. \] (2.36b)
The explicit form of the inverse matrix $\hat{D}[n]^{-1}$ is listed in appendix B for all the simple Lie algebras $X_r$.

Next, we turn to the excited states that are obtained by making "holes" in the sea.

$$\sigma^{(a)}_{st_a}(v) = \frac{1}{N} \sum_{j} \delta(v - \theta^{(a)}_j) \quad 1 \leq a \leq r, \quad (2.37)$$

where $\theta^{(a)}_j \in (-\pi, \pi]$ specifies the location of a hole. The excitation energy should be given as a function $\Delta F(u; \{\theta^{(a)}_j\})$ of the hole locations $\{\theta^{(a)}_j\}$ and the spectral parameter $u$. To find its explicit form, solve (2.31) in the presence of (2.37) but still keeping (2.34b). The result reads

$$\hat{\rho}^{(a)}_{st_a} = \hat{\rho}^{(a)}_{st_a}[n]_{gr} - \sum_{b=1}^{r} \sum_{j} \frac{\hat{Z}_{ab}[n]}{N} \epsilon^{in\theta^{(a)}_j}, \quad (2.38)$$

giving the deviations of the string densities from their ground state values. To be precise, we have neglected here the contributions from "nearest", "intermediate" and "wide" strings [31], which we suppose will not affect the final result for the excitation energy. This is indeed the case for $X_r = A_1$ as shown in [31-33]. By combining (2.32-34), (2.36) and (2.38), the excitation energy $\Delta F(u; \{\theta^{(a)}_j\})$ is expressed as the infinite series

$$\Delta F(u; \{\theta^{(a)}_j\}) = -\sum_{a=1}^{r} \sum_{j} \sum_{n \in \mathbb{Z}} \frac{\sinh(n(-2\lambda u + i\theta^{(a)}_j))}{n \sinh n\lambda} (\hat{D}[n]^{-1})_{pa}. \quad (2.39)$$

This consists of the contributions from the color $a$ holes (2.37) for each of $1 \leq a \leq r$. However, we find it more natural to modify (2.39) slightly in order to fit the Dynkin diagram symmetry as follows. (The technical reason for doing this has been stated in the end of appendix B.)

$$\Delta F(u; \{\theta^{(a)}_j\}) = -\sum_{a \in \{1, \ldots, r\}/\text{Aut}} \sum_{j} \sum_{n \in \mathbb{Z}} \frac{\sinh(n(-2\lambda u + i\theta^{(a)}_j))}{n \sinh n\lambda} \sum_{\tau \in \text{Aut}} (\hat{D}[n]^{-1})_{p\tau(a)}. \quad (2.40)$$

Here, Aut stands for the set of Dynkin diagram automorphisms of $X_r$. For our case (2.27a), it is trivial, i.e., $\text{Aut} = \{id\}$ except for $X_r = D_r$ and $E_6$ where $\text{Aut} = \{id, \tau_0\}$ and the involution $\tau_0$ is specified by

$$\tau_0(a) = \begin{cases} a & \text{if } 1 \leq a \leq r - 2 \\ r & \text{if } a = r - 1 \\ r - 1 & \text{if } a = r \\ 6 - a & \text{if } 1 \leq a \leq 5 \\ 6 - a & \text{for } X_r = E_6. \end{cases} \quad (2.41)$$
Correspondingly, the range \{1, \ldots, r\}/Aut of the \(a\)-sum in (2.40) is just \{1, \ldots, r\} itself except when \(X_r = D_r\) and \(E_6\). In these cases one can choose, for example, \{1, \ldots, r\}/Aut = \{1, \ldots, r-1\} and \{1, 2, 3, 6\} according as \(X_r = D_r\) and \(E_6\), respectively. As is evident form the expression, (2.40) measures the excitation energy of the holes located symmetrically with respect to Aut. It is real for the distributions of holes invariant under \(\theta_j^{(a)} \rightarrow -\theta_j^{(a)}\).

The \(n\)-sum in (2.40) is actually convergent in our regime \(0 < u < \frac{1}{2}\) (2.27a) owing to (B.11b). \(\Delta F(u; \{\theta_j^{(a)}\})\) can be further evaluated explicitly in terms of Jacobi’s elliptic function by the formula (\(|\omega| < \frac{g\lambda}{2}\))

\[
\sum_{n \in \mathbb{Z}} \frac{\sinh n\omega}{n \cosh \frac{an\lambda}{2}} = 2 \log \left( -i\sqrt{k} \sin \left( \frac{iK(k)}{\pi} (\omega + \frac{g\lambda}{2}) \right) \right),
\]

(2.42)

where the modulus \(k\) of \(\text{sn}\) is specified by (2.3b,c). Substituting (B.12) into (2.40) and performing the \(n\)-sum by (2.42) we arrive at

\[
e^{-\Delta F(u; \{\theta_j^{(a)}\})} = \prod_{a, j, m \in \pm 1} \left( -i\sqrt{k} \sin \left( \frac{2i\lambda K(k)}{\pi} (-u + \frac{i\theta_j^{(a)}}{2\lambda} + \frac{\beta^{(p,a)}_{m}}{2} + \frac{g}{4}) \right) \right)^{z_m^{(p,a)}},
\]

(2.43a)

\[
= \prod_{a, j, m} \left( -k \sin \left( \frac{K(k)}{\pi} \theta_j^{(a)} - i\eta_{m, +}^{(p,a)} \right) \sin \left( \frac{K(k)}{\pi} \theta_j^{(a)} - i\eta_{m, -}^{(p,a)} \right) \right)^{z_m^{(p,a)}},
\]

(2.43b)

\[
\eta_{m, \pm}^{(p,a)} = \frac{2\lambda K(k)}{\pi} (-u \pm \frac{\beta^{(p,a)}_{m}}{2} + \frac{g}{4}).
\]

(2.43c)

Here \(\beta^{(p,a)}_{m} \geq 0\) and \(z_m^{(p,a)} > 0\) are the constants defined by the expansion (B.12). Notice that \(\frac{1}{2} \leq \pm \frac{1}{2} \beta^{(p,a)}_{m} + \frac{g}{4} \leq \frac{g-1}{2}\) from (B.13b). From this and \(0 < u < \frac{1}{2}\) (2.27a), it follows that

\[
0 < \eta_{m, \pm}^{(p,a)} < \frac{(g-1)\lambda K(k)}{\pi} = (1 - \frac{1}{g})K'(k) < K'(k),
\]

(2.44)

where (2.3b) has been used. Thus from (2.43b) and (2.44) we verify that \(e^{-\Delta F(u; \{\theta_j^{(a)}\})}\) is free of poles in the range \(\theta_j^{(a)} \in (-\pi, \pi]\).

2.3.3 Correlation lengths from the excitation energy

Having obtained the excitation energy \(\Delta F(u; \{\theta_j^{(a)}\})\) (2.43), we can express the correlation function \(G(R)\) explicitly as (cf. [8])

\[
G(R) - G(\infty) \simeq \int d\{\theta_j^{(a)}\} \sigma(\{\theta_j^{(a)}\}) e^{-\Delta F(u; \{\theta_j^{(a)}\})} R,
\]

(2.45)
where $\sigma(\{\theta_j^{(a)}\})$ is the hole distribution function. As $g(u_1, u_2)$ in (2.25), its explicit form is not necessary for our purpose of extracting the correlation length $\xi$. In order to determine $\xi$, we consider the simplest case where the only two holes of the same color $a$ are excited at $\pm\theta$, for which (2.43) is certainly real (cf. [8]). Substituting the corresponding $e^{-\Delta F(u; \{\pm\theta\})}$ into (2.45), we get the final result,

$$G(R) - G(\infty) \simeq k^z R \int_{-\pi}^{\pi} d\theta \sigma(\\{\theta, -\theta\}) \prod_m |\text{sn}(\frac{K(k)}{\pi} \theta - i\eta_{m,+}^{(p,a)})\text{sn}(\frac{K(k)}{\pi} \theta - i\eta_{m,-}^{(p,a)})|^2 z_m^{(p,a)} R.$$  

(2.46)

Here $z = 2 \sum_m z_m^{(p,a)}$ is a positive constant thanks to (B.13a). The integral is finite because the integrand is so as we verified from (2.44). For a fixed $0 < k < 1$, the decay of (2.46) as $R \to \infty$ will then be controlled by the prefactor $k^z R$. Since $z$ is an order one constant, it will not be concerned for our definition of the correlation length as mentioned after (2.3c). Therefore from (2.46) we obtain the result (2.3). Notice the similarity of the $u$-dependence in (2.43b) and (2.24).

In the above calculations, we have actually used the condition $0 < u < \frac{1}{2}$ (2.27a) in (2.33) and to assure the convergence of (2.40). However, the result (2.46) seems valid in the wider regime $0 < u < \frac{g}{2}$ as we commented in the end of section 2.1. To support this, we firstly note that (2.43b) is actually free of poles when $0 < u < \frac{g}{2}$ for any $\theta_j^{(a)} \in (-\pi, \pi]$. Secondly, $\Delta F(u; \{\theta_j^{(a)}\}) > 0$ must hold for the excitation energy and this can be proved throughout $0 < u < \frac{g}{2}$ at $\theta_j^{(a)} = 0$ and $\pi$. Similar “extension” of the regime has been observed in p155 of [2]. These arguments imply that eqs.(2.25) and (2.46) actually admit the same parameter range for $u$.

3. Critical RSOS models

3.1. General remarks

By critical RSOS models we mean those sketched in section 3 of Part I [1]. They are specified by the quantum affine algebra $U_q(X_r^{(1)})$ and the three integers $\ell, p$ and $s$ subject to the condition,

$$\ell \geq 1, \quad 1 \leq p \leq r \quad \text{and} \quad 1 \leq s \leq t_p \ell - 1.$$  

(3.1a)
The corresponding one is called the level $\ell U_q(X_r^{(1)})$ RSOS model with fusion type $W_s^{(p)}$. Throughout this section we shall reserve the letters $\ell, p$ and $s$ for this meaning and use the notations

$$\ell_a = t_a \ell \quad \text{for} \quad 1 \leq a \leq r, \quad G = \{(a,m)|1 \leq a \leq r, 1 \leq m \leq \ell_a - 1\} \quad (3.1b)$$

in accordance with (I3.3). The well known Andrews-Baxter-Forrester (ABF) model [34] corresponds to $X_r = A_1, (p, s) = (1, 1)$ and belongs to the hierarchy $X_r = A_r$ with general $(p, s)$ [18]. The cases $X_r = B_r, C_r, D_r$ with $(p, s) = (1, 1)$ and $X_r = G_2$ with $(p, s) = (2, 1)$ have also been constructed in [35] and [36], respectively. See also [37,38].

As shown in section 2 of Part I, the restricted $T$-system is indeed valid for $X_r = A_r$ RSOS models. It is the FRs among the row-to-row transfer matrices $T_{m}(a)(u)$ with various auxiliary spaces $W_m^{(a)}$ but acting on the common quantum space $W_s^{(p)\otimes N}$ in the corresponding vertex model picture. Supposing this for all $X_r$, we shall show how the central charges can be computed from the $T$-system. The results perfectly agree with those in [12-14]. They are also consistent to the 1-point function results as argued in [12].

There are two regimes to consider corresponding to $\epsilon = +1$ and $\epsilon = -1$ in [12-14]. For the ABF model these are the regime I/II boundary and the regime III/IV boundary, respectively. The difference of these two regimes lies in which of the two terms dominates on the rhs of $T$-system (1.1a) in the thermodynamic limit $N \to \infty$. To explain this more concretely, we again quote (1.1a) (restricted case) in the form

$$T_0 = T_1 + T_{-1}, \quad (3.2)$$

as done in (2.5) for the unrestricted case. In the above, $T_i$’s are products of the transfer matrices $T_{m}(a)(u)$’s, which are dependent on the horizontal system size $N$. When evaluating (3.2) on the common eigenvector with the largest eigenvalue of $T_0$, we suppose

$$|T_\epsilon| \gg |T_{-\epsilon}| \quad \text{as} \quad N \to \infty \quad \text{in the regime} \quad \epsilon = \pm 1. \quad (3.3)$$

This is a natural extension from the $sl(2)$ case [6,7]. In particular for $\epsilon = -1$, ABF’s ground state is anti-ferroelectric like [34], hence (3.3) is also consistent with the assumption made after (2.5) for the underlying vertex model. From (3.3), the bulk eigenvalue must be a solution of $T_0 = T_\epsilon$. On the other hand, the ratio $T_{-\epsilon}/T_\epsilon$ will measure the finite size correction, which will yield the central charge in the corresponding CFT [9,10]. It is
through this route that we determine the central charge starting from our restricted $T$-system. The precise formulation will be given below for each regime. We note that such a calculation was firstly done by Klümper and Pearce in [6,7] for $X_r = A_1$. They also reproduced the scaling dimensions predicted earlier in [39]. Thus it is our hope to further extend the treatment of this paper to derive the scaling dimensions for all the $X_r$ cases. We leave it as a future problem but give a formulation which will be of use for that purpose as far as possible. We remark that the dilogarithm conjecture in [16] (cf. appendix D) is an important step toward this direction.

Concerning the case $X_r = A_r$, we have a proof of the $T$-system in section 2 of Part I and the explicit result on the asymptotic form of the bulk eigenvalues in appendix C. Therefore our derivation of the central charge relies only on the assumption of the ANZC property which will be explained after (3.7).

In the working below, we find it convenient to renormalize the spectral parameter in the $T$-system as $\mathcal{T}'_{\text{old}} = \mathcal{T}'_{\text{new}}$. In this convention, the $T$-system (1.1) (or (3.2)) becomes

$$
\mathcal{T}'_0 = \mathcal{T}'_1 + \mathcal{T}'_{-1},
$$

$$
\mathcal{T}'_0 = T_m^{(a)}(u - \frac{i}{t_a})T_m^{(a)}(u + \frac{i}{t_a}), \quad \mathcal{T}'_1 = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u),
$$

$$
\mathcal{T}'_{-1} = g_m^{(a)}(u) \prod_{b=1}^r \mathcal{T}(a, b, m, \frac{u}{2t})^{I_{ab}} |_{\mathcal{T}_m^{(a)}(x) \to \mathcal{T}_m^{(a)}(2ix)},
$$

with an obvious redefinition of the scalar function $g_m^{(a)}(u)$. The same assumption as (3.3) will still be implied for $\mathcal{T}'_{\pm 1}$. We recall that the combination $Y_m^{(a)}(u) = \mathcal{T}'_{-1}/\mathcal{T}'_1$ solves the restricted $Y$-system (IB.6) due to the connection (I3.19).

### 3.2. Regime $\epsilon = +1$

#### 3.2.1 Integral equation for the finite size correction

Consider the restricted $T$-system (3.4), where $T_m^{(a)}(u)$ is to be understood either as a matrix or its eigenvalues. For $(a, m) \in G$ (3.1b), put $Y_m^{(a)}(u) = \mathcal{T}'_{-1}/\mathcal{T}'_1$. Thus from (3.3) we see that $|Y_m^{(a)}(u)| \ll 1$ in the present regime $\epsilon = +1$. In view of this we rewrite the $Y$-system (IB.6) slightly to exclude $Y_m^{(a)}(u)^{-1}$ as

$$
\frac{Y_m^{(a)}(u - \frac{i}{t_a})Y_m^{(a)}(u + \frac{i}{t_a})}{Y_{m-1}^{(a)}(u)Y_{m+1}^{(a)}(u)} = \prod_{b=1}^r \prod_{k=1}^2 F_k(a, m, b; u) \delta_{t_a k, t_{ab} b} |_{Y_m^{(a)}(x) \to Y_m^{(a)}(2ix)}. \quad (3.5)
$$
Note that the rhs only contains the combinations $1 + Y_m^{(a')}(u')$ hence it is close to 1 for $N$ large. To separate the bulk and the finite size correction parts we put

$$Y_m^{(a)}(u) = \frac{YF_m^{(a)}(u)}{YB_m^{(a)}(u)}, \quad (3.6)$$

where the bulk part obeys

$$\frac{YB_m^{(a)}(u - \frac{i}{t_a})YB_m^{(a)}(u + \frac{i}{t_a})}{YB_{m-1}^{(a)}(u)YB_{m+1}^{(a)}(u)} = 1. \quad (3.7)$$

Moreover we assume that for the largest eigenvalue, the finite size correction part $YF_m^{(a)}(u)$ and $1 + Y_m^{(a)}(u)$ are Analytic, Non-Zero in some strip on the complex $u$-plane including the real axis and asymptotically Constant as $Re(u) \to \pm \infty$. This property is referred to as ANZC [7]. Unfortunately this is not derivable solely from the $T$-system but we believe it true based on some arguments in the $X_r = A_1$ case [7]. The significance of a function $f(u)$ for being ANZC is that it allows the Fourier transformation after the logarithmic derivative

$$(\log f)(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} du e^{-iux} \frac{\partial}{\partial u} \log f(u). \quad (3.8)$$

Combining (3.5-7) we get

$$\frac{YF_m^{(a)}(u - \frac{i}{t_a})YF_m^{(a)}(u + \frac{i}{t_a})}{YF_{m-1}^{(a)}(u)YF_{m+1}^{(a)}(u)} = \prod_{b=1}^{r} \prod_{k=1}^{3} F_k(a, m, b; u) I_{ab} \delta_{t_a k, t_ab}, \quad (3.9)$$

Now that the both sides consist of ratios of ANZC functions one can take the $\log$ of them. Solving the resulting equation for the finite size correction part, one finds

$$\log YF_m^{(a)} = \sum_{(b, k) \in G} \hat{\Psi}_{ab}^{mk} \log(1 + Y_k^{(b)}). \quad (3.10)$$

Here $\hat{\Psi}_{ab}^{mk}$ has been defined in (IB.13) and we have used the fact that the restricted $Y$-system (IB.6) can be rewritten in the form (IB.25). By taking the inverse Fourier transformation and integrating over $u$, (3.10) becomes

$$\log Y_m^{(a)} = -\log YB_m^{(a)} + \sum_{(b, k) \in G} \Psi_{ab}^{mk} \log(1 + Y_k^{(b)}) + \pi i D_m^{(a)}, \quad (3.11)$$

where the last term is the integration constant and the symbol $\ast$ is the convolution as specified in (IB.9).
3.2.2 Integral equation in the scaling limit

In (3.11) the dependence on the system size \( N \) enters only through \( \log Y_B^{(a)} \). For \( X_r = A_r \), we have determined the explicit asymptotic behavior of this quantity in appendix C as follows.

\[
\lim \log Y_B^{(a)}(\pm(u + \ell \pi \log N)) \equiv 2\delta_{pa} \frac{\sin \frac{m}{\ell_p} \sin \frac{s}{\ell_p}}{\sin \frac{\pi}{\ell_p}} e^{-\frac{\pi u}{\ell_p}} + O\left(\frac{1}{N}\right) \mod 2\pi iZ, \tag{3.12}
\]

where the limit \( N \to +\infty \) is taken under the condition \( N \in i_0\mathbb{Z} \) with \( i_0 = 2\ell \). See (C.13). Supported by this we assume (3.12) for a general \( X_r \) with some integer \( i_0 \). This is consistent with (3.7). Correspondingly we introduce the functions in the same scaling limit

\[
y^{(a)}_{m,\pm}(u) = \lim \log Y^{(a)}_m(\pm(u + \ell \pi \log N)). \tag{3.13}
\]

Dropping the \( 2\pi i\mathbb{Z} \) part in (3.12) that can be absorbed into the branch choice, one finds from (3.11) and (3.13) that \( y^{(a)}_{m,\pm}(u) \) satisfies the integral equation

\[
\log y^{(a)}_{m,\pm} = -2\delta_{pa} \frac{\sin \frac{m}{\ell_p} \sin \frac{s}{\ell_p}}{\sin \frac{\pi}{\ell_p}} e^{-\frac{\pi u}{\ell_p}} + \sum_{(b,k) \in G} \Psi_{ab}^{mk} \log(1 + y^{(b)}_{k,\pm}) + \pi i D^{(a)}_m. \tag{3.14}
\]

3.2.3 Finite size correction to \( T^{(a)}_m(u) \) and the effective central charge

Let us split \( T^{(a)}_m(u) \) into the bulk and the finite size correction parts as

\[
T^{(a)}_m(u) = TB^{(a)}_m(u)TF^{(a)}_m(u). \tag{3.15}
\]

On the bulk part we impose

\[
TB^{(a)}_m(u - \frac{i}{t_a})TB^{(a)}_m(u + \frac{i}{t_a}) = TB^{(a)}_{m-1}(u)TB^{(a)}_{m+1}(u), \tag{3.16}
\]

which is the same form as (3.7). From the \( T \)-system (3.4) and (3.15-16) it follows that

\[
\frac{TF^{(a)}_m(u - \frac{i}{t_a})TF^{(a)}_m(u + \frac{i}{t_a})}{TF^{(a)}_{m-1}(u)TF^{(a)}_{m+1}(u)} = 1 + Y^{(a)}_m(u). \tag{3.17}
\]

Assuming that \( TF^{(a)}_m(u) \) is ANZC as well, one can solve this for it by following the similar procedure to that from (3.9) to (3.11). The result reads

\[
\log TF^{(a)}_m - C^{(a)}_m = \sum_{n=1}^{\ell_{a-1}} A^{mn}_{aa} \log(1 + Y^{(a)}_n), \tag{3.18}
\]

\[18\]
where $C_m^{(a)}$ is an integration constant and $A_{a a}^{m n}$ has been defined in (IB.11) by its Fourier component. To see the behavior in the limit $N \to \infty$, we rewrite the rhs further as

$$\begin{align*}
\ell_a - \frac{1}{\ell_a} \sum_{n=1}^{\ell_a-1} & \int_{-\frac{\pi}{\ell_a} \log N}^{\infty} dv A_{a a}^{m n} (u + v + \frac{\ell}{\pi} \log N) \log (1 + Y_n^{(a)} (-v - \frac{\ell}{\pi} \log N)) \\
+ \sum_{n=1}^{\ell_a-1} & \int_{-\frac{\pi}{\ell_a} \log N}^{\infty} dv A_{a a}^{m n} (u - v - \frac{\ell}{\pi} \log N) \log (1 + Y_n^{(a)} (v + \frac{\ell}{\pi} \log N)).
\end{align*}$$

(3.19)

By combining (3.13), (3.19) and (IB.15), the following scaling behavior can be derived for $N$ large.

$$\log TF_{m}^{(a)} (u) = \frac{\sin \frac{\pi m}{\ell_a}}{N \sin \frac{\pi}{\ell_a}} \sum_{n=1}^{\ell_a-1} \sum_{\kappa = \pm} \sin \frac{\pi n}{\ell_a} \int_{-\infty}^{\infty} dv e^{-\frac{\pi u}{\ell_a}} \log (1 + y_{n, \kappa}^{(a)} (v)) + C_m^{(a)} + o\left(\frac{1}{N}\right).$$

(3.20)

We are interested in the non-trivial finite size correction proportional to $1/N$ for $(a, m) = (p, s)$ in the above. Following appendix C of [7], the central charge $c$ and the scaling dimensions $\Delta_{\pm}$ may be deduced as

$$\log TF_{s}^{(p)} (u) = \frac{\pi c}{6 N} \cosh \frac{\pi u}{\ell} - \frac{2\pi}{N} \left( (\Delta_{+} + \Delta_{-}) \cosh \frac{\pi u}{\ell} + (\Delta_{+} - \Delta_{-}) \sinh \frac{\pi u}{\ell} \right)$$

$$= \frac{\pi}{12 N} e^{\frac{\pi u}{\ell}} (c - 24 \Delta_{+}) + \frac{\pi}{12 N} e^{-\frac{\pi u}{\ell}} (c - 24 \Delta_{-}),$$

(3.21)

up to $o(1/N)$ terms. Here the combinations $c_{\text{eff}, \pm} = c - 24 \Delta_{\pm}$ are the effective central charges for the two chiral halves. Comparing (3.21) with (3.20) one gets

$$c_{\text{eff}, \pm} = \frac{12 \sin \frac{\pi s}{\ell_p}}{\pi \ell \sin \frac{\pi}{\ell_p}} \sum_{m=1}^{\ell_p-1} \sin \frac{\pi m}{\ell_p} \int_{-\infty}^{\infty} dv e^{-\frac{\pi u}{\ell_p}} \log (1 + y_{m, \pm}^{(p)} (u)).$$

(3.22)

3.2.4 Effective central charge as the Rogers dilogarithm

Let us evaluate the integral in (3.22) by means of the integral equation in the scaling limit (3.14). As it turns out, the result is expressed in terms of the Rogers dilogarithm. Consider the identity

$$\sum_{(a, m) \in G} \int_{-\infty}^{\infty} du \left( \log (1 + y_{m, \pm}^{(a)} (u)) \frac{\partial}{\partial u} \log y_{m, \pm}^{(a)} (u) - \log (y_{m, \pm}^{(a)} (u)) \frac{\partial}{\partial u} \log (1 + y_{m, \pm}^{(a)} (u)) \right)$$

$$= 2 \sum_{(a, m) \in G} \left( L\left( \frac{y_{m, \pm}^{(a)} (\infty)}{1 + y_{m, \pm}^{(a)} (\infty)} \right) - L\left( \frac{y_{m, \pm}^{(a)} (-\infty)}{1 + y_{m, \pm}^{(a)} (-\infty)} \right) \right),$$

(3.23)
which follows directly from the definition (D.2) of the Rogers dilogarithm. In the lhs of (3.23), replace the log \( y_{m,\pm}(u) \) by the rhs of (3.14). All the terms involving \( \Psi_{ab}^{mk} \) vanishes from the property (IB.14). Partially integrating the remaining parts, one can derive

\[
\frac{2\pi \sin \frac{\pi s}{\ell} \sum_{m=1}^{\ell-1} \sin \frac{\pi m}{\ell} \int_{-\infty}^{\infty} du \frac{e^{\pi u}}{2} \log(1 + y_{m,\pm}(u))}
= \sum_{(a,m)\in G} \left[ L\left( \frac{y_{m,\pm}^{(a)}(u)}{1 + y_{m,\pm}^{(a)}(u)} \right) - \pi i D_{m}^{(a)} \log\left( \frac{1}{1 + y_{m,\pm}^{(a)}(u)} \right) \right]_{u=+\infty}^{u=-\infty}. \tag{3.24}
\]

Up to some trivial factor, the lhs of (3.24) is precisely the combination appearing in (3.22). Thus we have

\[
\frac{\pi^2}{6} c_{\text{eff},\pm} = \sum_{(a,m)\in G} \left[ L\left( \frac{y_{m,\pm}^{(a)}(\infty)}{1 + y_{m,\pm}^{(a)}(\infty)} \right) \right. \\
- \left. \pi i D_{m}^{(a)} \log\left( \frac{1}{1 + y_{m,\pm}^{(a)}(\infty)} \right) \right]_{u=+\infty}^{u=-\infty}. \tag{3.25a}
\]

The constant \( D_{m}^{(a)} \) here can be determined from (3.14) as

\[
\pi i D_{m}^{(a)} = \log\left( \frac{y_{m,\pm}^{(a)}(\infty)}{1 + y_{m,\pm}^{(a)}(\infty)} \right) - \sum_{(b,k)\in G} K_{ab}^{mk} \log\left( \frac{1}{1 + y_{k,\pm}^{(b)}(\infty)} \right), \tag{3.25b}
\]

which must be independent of the \( \pm \) suffix. In deriving (3.25b) we have assumed that \( y_{m,\pm}^{(a)}(u) \) approaches its limit rapidly enough when \( u \to \infty \). Since \( \Psi_{ab}^{mk}(u) \) is decaying for \( |u| \) large, \( \sum_{(b,k)\in G} \Psi_{ab}^{mk} \log(1 + y_{k,\pm}^{(b)}) \) could then be replaced by \( \sum_{(b,k)\in G} \hat{\Psi}_{ab}^{mk}(0) \log(1 + y_{k,\pm}^{(b)}(\infty)) \) for which one can use (IB.13) and (IB.21a). Eq.(3.25) and (3.14) are the key ingredients in deriving the central charge and the scaling dimensions.

**3.2.5 Central charges**

As in [16,17], (3.25a) may be interpreted as defining a function that depends on the ways the rhs is analytically continued. To study the resulting spectrum of \( c_{\text{eff},\pm} \) will be an interesting problem and it has been argued in appendix D for some simplified case (formally without \( u = -\infty \) term in (3.25a)) in the light of the dilogarithm conjecture in [16]. Here we concentrate on the simplest case where \( c_{\text{eff},\pm} \) becomes the central charge itself. Such a situation is realized when \( \forall y_{m,\pm}^{(a)}(u) > 0 \) for \( -\infty < u < \infty \) and \( \forall D_{m}^{(a)} = 0 \) as in [6,7]. Then all the formulas (3.22-25) may be treated on an equal footing for both indices \( \pm \). We shall henceforth suppress it and write \( c_{\text{eff},\pm} \) simply as \( c \). Under the condition \( \forall y_{m}^{(a)}(\infty) > 0 \), the eq.(3.25b) with \( \forall D_{m}^{(a)} = 0 \) has the unique solution

\[
0 < \frac{y_{m}^{(a)}(\infty)}{1 + y_{m}^{(a)}(\infty)} = f_{m}^{(a)} < 1, \tag{3.26}
\]

20
where \( f_{m}^{(a)} \) is \( f_{m}^{(a)}(z = 0) \) of (D.4). See (D.5). As detailed in appendices A and B of Part I and [40,12], this quantity is a purely Lie algebraic data specified by the algebra \( X_{r} \) and the level \( \ell \in \mathbb{Z}_{\geq 1} \). To seek the value of \( y_{m}^{(a)}(-\infty) \), consider next (3.14) (with \( D_{m}^{(a)} = 0 \)) in the limit \( u \to -\infty \). The negative divergence from the first term on its rhs will be compensated by the behavior

\[
y_{m}^{(p)}(u) \xrightarrow{u \to -\infty} 0 + 0 \quad \text{for} \quad 1 \leq m \leq \ell_{p} - 1.
\]

Eq.(3.14) is then valid when \( u \to -\infty \) if the other \( y_{m}^{(a)}(-\infty) \)'s obey the reduced equation

\[
0 = \log\left(\frac{y_{m}^{(a)}(-\infty)}{1 + y_{m}^{(a)}(-\infty)}\right) - \sum_{(b,k) \in G, b \neq p} K_{ab}^{mk} \log\left(\frac{1}{1 + y_{k}^{(b)}(-\infty)}\right) \quad \text{for} \quad a \neq p.
\]

Eq.(3.27b) can be analyzed similarly to section 3.1 of [12]. To do so we visualize the set \( G \) (3.1b) by a tableau whose \( a \)-th column \((1 \leq a \leq r)\) consists of the \( \ell_{a} - 1 \) rectangles each having the depth \( 1/t_{a} \) and the width 1. The elementary rectangle at the \( a \)-th column and the \( m \)-th row corresponds to \( y_{m}^{(a)}(-\infty) \) \((\text{or} \frac{y_{m}^{(a)}(-\infty)}{1 + y_{m}^{(a)}(-\infty)})\). We call such a tableau the \( G \)-tableau for \((X_{r}, \ell)\). See fig.1 in [12] for examples. As depicted there, hatch the vertical strip \( G_{+} = \{(p, m) | 1 \leq m \leq \ell_{p} - 1\} \) in the \( G \)-tableau to signify (3.27a). Removing the strip divides the \( G \)-tableau into a few unhatched subdomains. But each such domain is again identical to a \( G \)-tableau for some \((X'_{r}, \ell')\). Let \( \mathcal{H}_{p}(X_{r}, \ell) \) denote the multiple set of the pairs \((X'_{r}, \ell')\) so obtained. For example,

\[
\begin{align*}
\mathcal{H}_{1}(F_{4}, 3) &= \{(C_{3}, 3)\}, & \mathcal{H}_{2}(F_{4}, 3) &= \{(A_{1}, 3), (A_{2}, 6)\}, \\
\mathcal{H}_{3}(F_{4}, 3) &= \{(A_{2}, 3), (A_{1}, 6)\}, & \mathcal{H}_{4}(F_{4}, 3) &= \{(B_{3}, 3)\},
\end{align*}
\]

(3.28a)

corresponding to (3.7a) in [12]. \( \mathcal{H}_{p}(X_{r}, \ell) \) has a multiplicity when the \( X_{r} \) Dynkin diagram has a branch, e.g.,

\[
\begin{align*}
\mathcal{H}_{1}(E_{6}, 2) &= \mathcal{H}_{5}(E_{6}, 2) = \{(D_{5}, 2)\}, & \mathcal{H}_{6}(E_{6}, 2) &= \{(A_{5}, 2)\}, \\
\mathcal{H}_{2}(E_{6}, 2) &= \mathcal{H}_{4}(E_{6}, 2) = \{(A_{1}, 2), (A_{4}, 2)\}, & \mathcal{H}_{3}(E_{6}, 2) &= \{(A_{1}, 2), (A_{2}, 2), (A_{2}, 2)\},
\end{align*}
\]

(3.28b)

Returning to (3.27b), we now see that it decouples into independent systems of equations and each of them again takes the form (D.5b) for the pairs \((X'_{r}, \ell') \in \mathcal{H}_{p}(X_{r}, \ell)\). Therefore \( \frac{y_{m}^{(a)}(-\infty)}{1 + y_{m}^{(a)}(-\infty)} \) contained in the corresponding \( G \)-tableau is identified with some \( f_{m'}^{(a')} \) attached
to the \((X'_r, \ell')\). In this way the values of \(y_m^{(a)}(\infty)\)'s are specified for all \((a, m) \in G\). Substituting them and (3.26) into (3.25a) and using \(D_m^{(a)} = 0\) we find

\[
\frac{\pi^2}{6} c = \sum_{(a,m) \in G} L(f_m^{(a)}) - \sum_{(X'_r, \ell') \in \mathcal{H}_p(X_r, \ell)} L(f_m^{(a)}) \text{ for } (X'_r, \ell').
\]  

(3.29)

With the aid of the dilogarithm conjecture (D.6), (3.29) is computed explicitly as

\[
c = \frac{\ell \dim X_r}{\ell + g} - r - \sum_{(X'_r, \ell') \in \mathcal{H}_p(X_r, \ell)} \left( \frac{\ell' \dim X'_r}{\ell' + g'} - r' \right),
\]

(3.30)

where \(g'\) denotes the dual Coxeter number of \(X'_r\). The result (3.30) completely reproduces the central charge (3.9) of [12]. It depends on the fusion type \(W_s^{(p)}\) only through \(p\).

### 3.3. Regime \(\epsilon = -1\)

Here we shall exclusively deal with the simply laced case \(X_r = A_r, D_r\) and \(E_{6,7,8}\) to avoid a technical complexity. Then the analysis is fairly parallel to the regime \(\epsilon = +1\). We note however that it is quite straightforward to write down the integral equation analogous to (3.11) for any \(X_r\).

#### 3.3.1 Integral equation for the finite size correction

Throughout section 3.3, we set

\[
Y_m^{(a)}(u) = \frac{T_m^{(a)}(u) T_{m-1}^{(a)}(u)}{g_m^{(a)}(u) \prod_{b=1}^{r} (T_m^{(b)}(u)) I_{ab}},
\]

(3.31)

which is the inverse of the definition \(Y_m^{(a)}(u) = T_{-1}^{(a)} / T_{1}^{(a)}\) in the previous section 3.2. Then the level \(\ell\) restricted \(Y\)-system (IB.30) is changed into

\[
Y_m^{(a)}(u + i)Y_m^{(a)}(u - i) = \frac{\prod_{k=1}^{\ell-1} (1 + Y_k^{(a)}(u)) \tilde{I}_{mk}}{\prod_{b=1}^{r} (1 + Y_m^{(b)}(u)^{-1}) I_{ab}},
\]

(3.32)

where \(I\) is the incidence matrix (1.2) and \(\tilde{I}\) denotes the \(I\) for \(A_{\ell-1}\) as in appendix B.5 of Part I. The definition (3.31) corresponds to \(Y_m^{(a)}(u) = T_{-1}^{(a)} T_{1}^{(a)}\) in the notation of (3.4) hence \(|Y_m^{(a)}(u)| \ll 1\) from (3.3) in the regime \(\epsilon = -1\) under consideration. As in (3.5) let us rewrite (3.32) so as to exclude \(Y_m^{(a)}(u)^{-1}\) as

\[
\frac{Y_m^{(a)}(u + i)Y_m^{(a)}(u - i)}{\prod_{b=1}^{r} Y_m^{(b)}(u) I_{ab}} = \frac{\prod_{k=1}^{\ell-1} (1 + Y_k^{(a)}(u)) \tilde{I}_{mk}}{\prod_{b=1}^{r} (1 + Y_m^{(b)}(u)) I_{ab}} \quad \text{for } (a, m) \in G,
\]

(3.33)
in which the both sides are close to 1 for $N$ large. We separate the bulk and the finite size correction parts as

$$Y_m^{(a)}(u) = Y_B^{(a)}(u)Y_F^{(a)}(u),$$  \hfill (3.34)

where the former satisfies

$$\frac{Y_B^{(a)}(u+i)Y_B^{(a)}(u-i)}{\prod_{b=1}^r Y_{B_m}^{(b)}(u)/\epsilon_{ab}} = 1.$$  \hfill (3.35)

The $Y_m^{(a)}$’s on the lhs of (3.33) can be replaced with $Y_F^{(a)}$’s due to (3.34,35). Assuming that the $Y_F^{(a)}(u)$ and $1 + Y_m^{(a)}(u)$ are ANZC, one can solve the resulting equation by taking the log on both sides. This is a similar calculation to (3.8-11) leading to

$$\log Y_m^{(a)} = \log Y_B^{(a)} + \sum_{(b,k) \in G} \Phi_{ab}^{mk} \ast \log(1 + y_k^{(b)}) + \pi i D_m^{(a)},$$  \hfill (3.36)

where $D_m^{(a)}$ is an integration constant. $\Phi_{ab}^{mk}$ is defined by (IB.31) and has the same property as (IB.14). The equation (3.36) is the regime $\epsilon = -1$ analogue of (3.11).

### 3.3.2 Integral equation in the scaling limit

For the bulk part entering (3.36), we assume the following large $N$ behavior

$$\lim \log Y_B^{(a)}(\pm (u + \frac{g}{\pi} \log N)) \equiv -\delta_{sm} \frac{g \chi^{(a)} \chi^{(p)}}{\sin \frac{\pi a}{g}} e^{-\frac{\pi u}{g}} + O\left(\frac{1}{N}\right) \mod 2\pi i \mathbb{Z},$$  \hfill (3.37)

where the limit $N \to \infty$ is taken under $N \in i_0 \mathbb{Z}$ for some $i_0$. $\chi = (\chi^{(a)})_{1 \leq a \leq r}$ is the normalized Perron-Frobenius eigenvector of the incidence matrix $I_{ab}$ as in (IB.35). This has been established actually for $X_r = A_r$ in appendix C where one has $g = r + 1$ and $\chi^{(a)} = \sqrt{\frac{2}{r+1}} \sin \frac{\pi a}{r+1}$. Compare (3.37) and (C.12) with $N \in 2g \mathbb{Z}$. Eq.(3.37) is also consistent with (3.35) due to (IB.35a). Introducing the functions in the same scaling limit

$$y_m^{(a),\pm}(u) = \lim Y_m^{(a)}(\pm (u + \frac{g}{\pi} \log N)),$$  \hfill (3.38)

one derives from (3.36-37) the integral equation

$$\log y_m^{(a),\pm} = -\delta_{sm} \frac{g \chi^{(a)} \chi^{(p)}}{\sin \frac{\pi a}{g}} e^{-\frac{\pi u}{g}} + \sum_{(b,k) \in G} \Phi_{ab}^{mk} \ast \log(1 + y_k^{(b),\pm}) + \pi i D_m^{(a)},$$  \hfill (3.39)

where the $2\pi i \mathbb{Z}$ part in (3.37) has been dropped by the same reason as in (3.14).
3.3.3 Finite size correction to $T^{(a)}_m(u)$ and the effective central charge

Let us separate $T^{(a)}_m(u)$ into the bulk and the finite size correction parts as in (3.15), where the former is to fulfill the regime $\epsilon = -1$ analogue of (3.16) as

$$TB^{(a)}_m(u - i)TB^{(a)}_m(u + i) = g^{(a)}_m(u) \prod_{b=1}^r TB^{(b)}_m(u)^{I_{ab}}. \quad (3.40)$$

In view of (3.15), (3.31) and (3.40), the $T$-system (3.4) is now equivalent to

$$\frac{TF^{(a)}_m(u + i)TF^{(a)}_m(u - i)}{\prod_{b=1}^r TF^{(b)}_m(u)^{I_{ab}}} = 1 + Y^{(a)}_m(u). \quad (3.41)$$

By assuming that $TF^{(a)}_m(u)$ is ANZC, (3.41) can be transformed into an analogous equation to (3.18) as follows.

$$\log TF^{(a)}_m = - C^{(a)}_m = \sum_{b=1}^r \hat{\gamma} (\hat{M}^{-1})_{ab} \ast \log (1 + Y^{(b)}_m), \quad (3.42)$$

where $C^{(a)}_m$ is an integration constant. The symbol $\hat{\gamma} (\hat{M}^{-1})_{ab}$ denotes the inverse Fourier transformation (cf. (IB.8)) of the $(a, b)$-element of the matrix $\hat{M}^{-1}$ inverse to (IB.10). Apply the asymptotic form (IB.34) to (3.42). Then the large $N$ behavior of $\log TF^{(a)}_m$ can be deduced in a similar manner to (3.18-20) as

$$\log TF^{(a)}_m(u) = \frac{1}{2N \sin \frac{\pi}{g}} \sum_{\kappa = \pm 1} \sum_{b=1}^r \chi^{(b)} \int_{-\infty}^{\infty} dv e^{-\frac{\pi u}{g}} \log (1 + y^{(b)}_{m,\kappa}(v)) + C^{(a)}_m + o\left(\frac{1}{N}\right). \quad (3.43)$$

This is the finite size correction proportional to $1/N$ from which the central charge $c$ and the scaling dimensions $\Delta_\pm$ are to be extracted. Guided by appendix B of [7], we postulate the equation (3.21) with $\ell$ replaced by $g$ in the present regime $\epsilon = -1$. Then the effective central charges $c_{\text{eff}, \pm} = c - 24\Delta_\pm$ for the two chiral halves are given by

$$c_{\text{eff}, \pm} = \frac{6\chi^{(p)}}{\pi \sin \frac{\pi}{g}} \sum_{a=1}^r \chi^{(a)} \int_{-\infty}^{\infty} dv e^{-\frac{\pi u}{g}} \log (1 + y^{(a)}_{s,\pm}(u)). \quad (3.44)$$

3.3.4 Effective central charge as the Rogers dilogarithm
The integral in (3.44) can be evaluated in a similar way to section 3.2.4 by considering the identity (3.23). Replacing \( \log y^{(a)}_{m,\pm}(u) \) there by the rhs of (3.39) one can show
\[
\frac{\pi \chi^{(p)}}{\sin \frac{\pi}{g}} \sum_{a=1}^{r} \chi^{(a)} \int_{-\infty}^{\infty} du \left( e^{-\frac{\pi y^{(a)}_{m,\pm}(u)}} \log (1 + y^{(a)}_{s,\pm}(u)) \right)
= - \sum_{(a,m) \in G} \left[ L \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) + \frac{\pi i}{2} D^{(a)}_{m} \log \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) \right]_{u=+\infty} - \sum_{(a,m) \in G} \left[ L \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) + \frac{\pi i}{2} D^{(a)}_{m} \log \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) \right]_{u=-\infty},
\]
which is the regime \( \epsilon = -1 \) analogue of (3.24). Combining (3.44) and (3.45) we find the expression of \( c_{eff,\pm} \) in terms of the Rogers dilogarithm
\[
\frac{\pi^{2}}{6} c_{eff,\pm} = - \sum_{(a,m) \in G} \left[ L \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) + \frac{\pi i}{2} D^{(a)}_{m} \log \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) \right]_{u=+\infty} - \sum_{(a,m) \in G} \left[ L \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) + \frac{\pi i}{2} D^{(a)}_{m} \log \left( \frac{1}{1 + y^{(a)}_{m,\pm}(u)} \right) \right]_{u=-\infty}.
\]

The constant \( D^{(a)}_{m} \) is specified from the \( u \to +\infty \) limit of (3.39) as
\[
\pi i D^{(a)}_{m} = - \sum_{(b,k) \in G} (C^{-1})_{ab} \bar{C}_{mk} \left( \log \left( \frac{1}{1 + y^{(b)}_{k,\pm}(\infty)} \right) - \sum_{(c,j) \in G} K^{kj}_{bc} \log \left( \frac{y^{(c)}_{j,\pm}(\infty)}{1 + y^{(c)}_{j,\pm}(\infty)} \right) \right)
\]
and this should be independent of the suffix \( \pm \). In (3.46b) we have followed a similar procedure to the derivation of (3.25b) and used (IB.32c,d) for simply laced \( X_{r} \). Eq.(3.46) and (3.39) are our basic ingredients in studying the central charge and the scaling dimensions in the regime \( \epsilon = -1 \).

### 3.3.5 Central charges

As in the regime \( \epsilon = +1 \) we shall hereafter restrict our analysis of (3.46) to the simplest case \( \forall y_{m,\pm}^{(a)}(u) > 0 \) for \( -\infty < u < \infty \) and \( \forall D_{m}^{(a)} = 0 \), which corresponds to the central charge \( c_{eff,\pm} = c \). We suppress the \( \pm \) indices and the identical argument is to be implied for both choices. Investigations of the full aspects of (3.46) and (3.39) will be an interesting future problem relevant to the scaling dimensions.

The constraints \( \forall y_{m}^{(a)}(u) > 0 \) and \( \forall D_{m}^{(a)} = 0 \) on (3.46b) uniquely determine the limit \( y_{m}^{(a)}(\infty) \) as
\[
0 < \frac{1}{1 + y_{m}^{(a)}(\infty)} = f_{m}^{(a)} < 1,
\]

25
where \( f_m^{(a)} \) is again \( f_m^{(a)}(z = 0) \) of (D.4). On the other hand, the negative divergence from the \( u \to -\infty \) limit of the \( e^{-\pi u / s} \)-term in (3.39) compels

\[
y_s^{(a)}(u) \xrightarrow{u \to -\infty} +0 \quad \text{for} \quad 1 \leq a \leq r.
\]  

(3.48a)

Then (3.39) (with \( D_m^{(a)} = 0 \)) holds for \( u \to -\infty \) if the other \( y_m^{(a)}(\infty) \)'s satisfy

\[
0 = \log \left( \frac{1}{1 + y_m^{(a)}(\infty)} \right) - \sum_{(b,k) \in G, k \neq s} K_{ab}^m \log \left( \frac{y_k^{(b)}(\infty)}{1 + y_k^{(b)}(\infty)} \right) \quad \text{for} \quad m \neq s.
\]

(3.48b)

which can be analyzed similarly to section 3.2.5 with the \( G \)-tableau for the pair \((X_r, \ell)\).

For the simply laced algebras \( X_r \), it has the simple \((\ell - 1) \times r \) rectangular shape. This time we hatch the horizontal strip \( G_{-1} = \{(a,s)|1 \leq a \leq r\} \) in it to depict (3.48a). Removal of it splits the \( G \)-tableau into those corresponding to \((X_r, s)\) and \((X_r, \ell - s)\). Accordingly, (3.48b) decouples into the independent systems (D.5b) for these pairs within which \( \frac{1}{1 + y_m^{(a)}(\infty)} \) is determined as in (3.47). Thus (3.46a) with \( D_m^{(a)} = 0 \) yields

\[
\frac{\pi^2}{6} c = - \sum_{(a,m) \in G} L(f_m^{(a)}) + rL(1) + \sum_{(a,m) \in G} L(f_m^{(a)}) \quad \text{for} \quad (X_r, s)) \nonumber \]

\[
+ \sum_{(a,m) \in G} L(f_m^{(a)}) \quad \text{for} \quad (X_r, \ell - s)).
\]

(3.49)

By using the dilogarithm conjecture (D.6) and \( L(1) = \frac{\pi^2}{6} \), this is evaluated explicitly as

\[
c = \frac{s \dim X_r}{s + g} + \frac{(\ell - s) \dim X_r}{\ell - s + g} - \frac{\ell \dim X_r}{\ell + g},
\]

(3.50)

which corresponds to the coset pair [41]

\[
X_r^{(1)} \oplus X_r^{(1)} \supset X_r^{(1)}
\]

lev els \( s \quad \ell - s \quad \ell \).

(3.51)

The result (3.50) agrees with that in [12-14]. It depends on the fusion type \( W_s^{(p)} \) only through \( s \).

4. Summary and Discussions
In this paper, we have shown two applications of our $T$-system proposed in Part I [1]. In section 2, it has been used to determine the correlation length of the massive vertex model in AF regime associated with any $X_r \neq A_r \geq 2$. By assuming the dominance $|T_1| \ll |T_{-1}|$ in (2.5) in the thermodynamic limit, the problem essentially reduces to finding the periodicity of the bulk $T$-system (2.6). We have proved that it is just given by the dual Coxeter number and thereby obtained the result (2.3). It generalizes the earlier ones [2,5,8] for the $sl(2)$ case. In section 3, we have used the $T$-system to derive the central charges of the level $\ell$ critical $U_q(X^{(1)}_r)$ RSOS model with fusion type $W^{(p)}_s$. This is an extension of the approach in [6,7] for the $sl(2)$ case. Supposing the ANZC property, we have converted the $T$-system into the integral equations (3.14) and (3.39). Then the effective central charges $c_{\text{eff}, \pm}$ are extracted from the finite size corrections and expressed by the Rogers dilogarithm as in (3.25) and (3.46), respectively. Full investigations of these equations will yield all the scaling dimensions in the underlying CFT. We have actually restricted our analysis to the simplest situation where $c_{\text{eff}, \pm}$ becomes the central charge itself. Then it has been evaluated explicitly in (3.30) and (3.50) by means of the dilogarithm conjecture [11,12]. They perfectly reproduce the results in [12-14].

The central charge calculation in section 3 exhibits a strong resemblance to the TBA analyses in [12-14]. In fact, the both approaches should be equivalent as the means of central charge calculations at least spiritually [9,10]. The TBA treats the infinite length one-dimensional quantum system and seeks its specific heat, which vanishes linearly with the temperature $\beta^{-1} \to 0$. On the other hand, the FR approach in this paper effectively deals with the corresponding two dimensional classical system on an infinite strip of width $N$ and evaluates the $1/N$ correction to the ground state energy. So they are essentially the same thing under the identification $\beta = N \to \infty$. Our analysis not only verifies this natural coincidence but goes beyond. It manifests that there underlies a common mathematical structure between the two approaches, TBA and FR. The basic ingredients in the two methods are the $Y$ and the $T$-systems, respectively and they are intriguingly connected through (I3.19). The connection makes the integral equations (3.14) and (3.39) possess an almost identical structure to the TBA equation (IB.4), hence the analyses thereafter essentially the same in the two methods. It therefore appears ever fundamental to unveil the full content of the key relation (I3.19) between the $Y$ and the $T$-systems. In the technical aspects, the TBA relies crucially on the string hypothesis, while the FR needs the ANZC property instead. These assumptions are yet to be justified in general.
There are several future problems that stem from the present paper. Firstly, one should be able to analyze the full aspects of the effective central charges (3.25) and (3.46). As shown in [7] for the $sl(2)$ case, it will yield the scaling dimensions as well as the central charge. Appendix D is a study in this direction, leading to a generalized dilogarithm conjecture [16]. One may even stumble to reproduce the character itself for the underlying CFT in the spirit of [17]. Secondly, our $T$-system is expected to hold even in off-critical RSOS models. Thus one might hope to derive the correlation lengths and the interfacial tensions in those models by extending the analysis in [3].

Acknowledgements

This work is supported in part by JSPS fellowship, NSF grants PHY-87-14654, PHY-89-57162 and Packard fellowship.

Appendix A. Periodicity of the bulk $T$-system

Let us continue the proof of the proposition in section 2.2.1. $X_r = C_r$: The relevant bulk $T$-system is

\[ T_{2m}^{(a)}(u - \frac{1}{4})T_{2m}^{(a)}(u + \frac{1}{4}) = T_{2m}^{(a+1)}(u)T_{2m}^{(a-1)}(u) \quad \text{for} \quad 1 \leq a \leq r - 2, \quad (A.1a) \]

\[ T_{2m}^{(r-1)}(u - \frac{1}{4})T_{2m}^{(r-1)}(u + \frac{1}{4}) = T_{m}^{(r)}(u - \frac{1}{4})T_{m}^{(r)}(u + \frac{1}{4})T_{2m}^{(r-2)}(u), \quad (A.1b) \]

\[ T_{m}^{(r)}(u - \frac{1}{2})T_{m}^{(r)}(u + \frac{1}{2}) = T_{2m}^{(r-1)}(u), \quad (A.1c) \]

where $T_{2m}^{(0)}(u) = 1$. From (A.1a) we have

\[ T_{2m}^{(a)}(u) = \prod_{j=1}^{a} T_{2m}^{(1)}(u + \frac{j}{2} - \frac{a+1}{4}) \quad (A.2) \]

for $1 \leq a \leq r - 1$. Applying it to (A.1b,c) one gets

\[ T_{m}^{(r)}(u - \frac{1}{4})T_{m}^{(r)}(u + \frac{1}{4}) = \prod_{j=1}^{r} T_{2m}^{(1)}(u + \frac{j-1}{2} - \frac{r-1}{4}), \quad (A.3a) \]

\[ T_{m}^{(r)}(u - \frac{1}{2})T_{m}^{(r)}(u + \frac{1}{2}) = \prod_{j=1}^{r-1} T_{2m}^{(1)}(u + \frac{j}{2} - \frac{r}{4}), \quad (A.3b) \]
from which $T_{2m}^{(1)}(u + r/4) = T_m^{(r)}(u)/T_m^{(r)}(u - 1/2)$ follows. Substituting this back into the rhs of (A.3b) we find $T_m^{(r)}(u)T_m^{(r)}(u + r/4) = 1$. From this and (A.2), (2.8) is easily seen for all $1 \leq a \leq r$.

$X_r = D_r$: The bulk $T$-system is

$$T_m^{(a)}(u - \frac{1}{2})T_m^{(a)}(u + \frac{1}{2}) = T_m^{(a+1)}(u)T_m^{(a-1)}(u) \quad \text{for} \quad 1 \leq a \leq r - 3,$$

(A.4a)

$$T_m^{(r-2)}(u - \frac{1}{2})T_m^{(r-2)}(u + \frac{1}{2}) = T_m^{(r-3)}(u)T_m^{(r-1)}(u)T_m^{(r)}(u),$$

(A.4b)

$$T_m^{(r-1)}(u - \frac{1}{2})T_m^{(r-1)}(u + \frac{1}{2}) = T_m^{(r-2)}(u),$$

(A.4c)

$$T_m^{(r)}(u - \frac{1}{2})T_m^{(r)}(u + \frac{1}{2}) = T_m^{(r-2)}(u),$$

(A.4d)

where $T_m^{(0)}(u) = 1$. By the same argument as the $X_r = A_r$ case, (2.11) holds for $1 \leq a \leq r - 2$. Substituting it into (A.4b), we have

$$T_m^{(r-1)}(u)T_m^{(r)}(u) = \prod_{j=1}^{r-1} T_m^{(1)}(u + j - \frac{r}{2}).$$

(A.5)

Using this in the product of (A.4c) and (A.4d), we deduce $T_m^{(1)}(u)T_m^{(1)}(u + r - 1) = 1$, from which (2.8) can be shown for all the $T_m^{(a)}(u)$’s.

$X_r = E_r, r = 6, 7, 8$: The bulk $T$-system is

$$T_m^{(a)}(u - \frac{1}{2})T_m^{(a)}(u + \frac{1}{2}) = \prod_{b=1}^{r} T_m^{(b)}(u)T_{ab} \quad \text{for} \quad 1 \leq a \leq r.$$

(A.6)

When $r = 6$, all the $T_m^{(a)}(u)$’s are expressible via $T_m^{(1)}(u)$ by combining (A.6). In addition we have one consistency condition

$$T_m^{(1)}(u + 3) = T_m^{(1)}(u)T_m^{(1)}(u + 1)T_m^{(1)}(u + 5)T_m^{(1)}(u + 6).$$

(A.7)

Rewriting $T_m^{(1)}(u + 5)$ on the rhs by the same equation, we have

$$T_m^{(1)}(u)T_m^{(1)}(u + 1)T_m^{(1)}(u + 2) = \frac{1}{T_m^{(1)}(u + 6)T_m^{(1)}(u + 7)T_m^{(1)}(u + 8)},$$

(A.8)

which leads to $T_m^{(1)}(u)T_m^{(1)}(u + 1)\cdots T_m^{(1)}(u + 11) = 1$. Therefore we obtain (2.7) and (2.8) for $a = 1$. All the other cases $a \neq 1$ are assured by this.
When \( r = 7 \), from (A.6) all the \( T_m^{(a)}(u) \)'s are expressible in terms of \( T_m^{(6)}(u) \) which must obey the consistency condition

\[
T_m^{(6)}(u + 3)T_m^{(6)}(u + 4) = T_m^{(6)}(u)T_m^{(6)}(u + 1)T_m^{(6)}(u + 6)T_m^{(6)}(u + 7). \quad (A.9)
\]

One can show \( T_m^{(6)}(u)T_m^{(6)}(u + 9) = 1 \) from this, hence (2.8) is valid.

When \( r = 8 \), from (A.6) all the \( T_m^{(a)}(u) \)'s are expressible by \( T_m^{(1)}(u) \) which must satisfy the consistency condition

\[
T_m^{(1)}(u - 1)T_m^{(1)}(u)T_m^{(1)}(u + 1) = T_m^{(1)}(u - 4)T_m^{(1)}(u - 3)T_m^{(1)}(u + 3)T_m^{(1)}(u + 4). \quad (A.10)
\]

From this one can deduce

\[
T_m^{(1)}(u)T_m^{(1)}(u + 1)T_m^{(1)}(u + 2) = \frac{1}{T_m^{(1)}(u + 15)T_m^{(1)}(u + 16)T_m^{(1)}(u + 17)}, \quad (A.11)
\]

which leads to \( T_m^{(1)}(u)T_m^{(1)}(u + 1) \cdots T_m^{(1)}(u + 29) = 1 \), proving (2.7) and (2.8).

\( X_r = F_4 \): The relevant bulk \( T \)-system is

\[
egin{align*}
T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u + \frac{1}{2}) &= T_m^{(2)}(u), \quad (A.12a) \\
T_m^{(2)}(u - \frac{1}{2})T_m^{(2)}(u + \frac{1}{2}) &= T_m^{(1)}(u)T_2^{(3)}(u), \quad (A.12b) \\
T_2^{(3)}(u - \frac{1}{4})T_2^{(3)}(u + \frac{1}{4}) &= T_m^{(2)}(u - \frac{1}{4})T_m^{(2)}(u + \frac{1}{4})T_2^{(4)}(u), \quad (A.12c) \\
T_2^{(4)}(u - \frac{1}{4})T_2^{(4)}(u + \frac{1}{4}) &= T_2^{(3)}(u). \quad (A.12d)
\end{align*}
\]

All the \( T_m^{(a)}(u) \)'s are expressible by \( T_m^{(1)}(u) \) which must satisfy the consistency condition

\[
T_m^{(1)}(u) = T_m^{(1)}(u - \frac{3}{2})T_m^{(1)}(u + \frac{3}{2}). \quad (A.13)
\]

This leads to \( T_m^{(1)}(u)T_m^{(1)}(u + \frac{9}{2}) = 1 \), proving (2.8).

\( X_r = G_2 \): The relevant \( T \)-system is

\[
egin{align*}
T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u + \frac{1}{2}) &= T_3^{(2)}(u), \quad (A.14a) \\
T_3^{(2)}(u - \frac{1}{6})T_3^{(2)}(u + \frac{1}{6}) &= T_m^{(1)}(u - \frac{1}{3})T_m^{(1)}(u)T_m^{(1)}(u + \frac{1}{3}). \quad (A.14b)
\end{align*}
\]
Substituting the first equation into the second we get
\[ T_m^{(1)}(u) = T_m^{(1)}(u - \frac{2}{3})T_m^{(1)}(u + \frac{2}{3}). \]  

(A.15)

From this it follows that \[ T_m^{(1)}(u)T_m^{(1)}(u + 2) = 1, \]
proving (2.8).

Appendix B. Inverse of the matrix \( \hat{D}[n] \)

Let \( \hat{D} = \hat{D}[n] \) be the symmetric matrix defined by (2.32a) and (2.36b). Below we list the explicit form of the matrix elements of its inverse \((\hat{D}^{-1})_{ab}\) using the notation
\[ s(i_1, \ldots, i_\alpha) = \prod_{k=1}^{\alpha} \sinh(i_k n \lambda), \quad c(i_1, \ldots, i_\alpha) = \prod_{k=1}^{\alpha} \cosh(i_k n \lambda). \]  

(B.1)

We shall omit some matrix elements in view of the symmetry \((\hat{D}^{-1})_{ab} = (\hat{D}^{-1})_{ba}\).

\(X_r = A_r:\)

\[ (\hat{D}^{-1})_{ab} = \frac{s(\min(a, b), r + 1 - \max(a, b))}{s(r + 1)}, \]  

for \( 1 \leq a, b \leq r - 1, \)

(B.2)

\(X_r = B_r:\)

\[ (\hat{D}^{-1})_{ra} = \frac{s(a)}{2c(r - \frac{1}{2})}, \quad (\hat{D}^{-1})_{rr} = \frac{s(r)}{4c(\frac{1}{2}, r - \frac{1}{2})}. \]  

(B.3)

\(X_r = C_r:\)

\[ (\hat{D}^{-1})_{ab} = \frac{s(\frac{1}{2}\min(a, b))c(\frac{1}{2}(r + 1 - \max(a, b)))}{c(\frac{1}{2}(r + 1))}. \]  

(B.4)

\(X_r = D_r:\)

\[ (\hat{D}^{-1})_{ab} = \frac{s(\min(a, b))c(r - 1 - \max(a, b))}{c(r - 1)}, \]  

for \( 1 \leq a, b \leq r - 2, \)

(B.5)
For the remaining exceptional algebras we shall only present the upper half of the \(\hat{D}^{-1}\).

\[ X_r = E_6 : \]

\[
\hat{D}^{-1} = \frac{1}{c(6)} \begin{pmatrix}
\frac{s(1,8)}{2s(3)} & \frac{s(1,5)c(2)}{s(3)} & \frac{1}{2}s(4) & \frac{s(2,4)}{2s(3)} & \frac{s(1,4)}{2s(3)} & \frac{s(1)c(2)}{2s(3)} \\
\frac{s(4,5)}{2s(3)} & \frac{s(4)c(1)}{s(3)} & \frac{2s(3)c(1,2)}{s(3)} & \frac{1}{2}s(4) & \frac{s(2,4)c(1)}{s(3)} & \frac{s(3)c(2)}{2s(3)} \\
\frac{2s(3)c(1,2)}{s(3)} & \frac{s(4)c(1)}{s(3)} & \frac{1}{2}s(4) & \frac{s(2,4)c(1)}{s(3)} & \frac{s(1,5)c(2)}{s(3)} & \frac{s(1)c(2)}{2s(3)} \\
\frac{1}{c(6)} & \frac{s(4)c(3)}{2c(1)} & \frac{s(2,4)c(1)}{s(3)} & \frac{s(1,5)c(2)}{s(3)} & \frac{s(1)c(2)}{2s(3)} & \frac{s(1)c(2)}{2s(3)} \\
\end{pmatrix} \quad (B.62)
\]

\[ X_r = E_7 : \] For a typographical reason we split \(\hat{D}^{-1}\) into two pieces.

\[
(\hat{D}^{-1})_{1 \leq a \leq 7, 1 \leq b \leq 4} = \frac{1}{c(9)} \begin{pmatrix}
2s(1)c(3, 5) & \frac{s(6)c(3)}{2c(1)} & s(4)c(3) & \frac{1}{2}s(6) \\
s(6)c(3) & 2s(4)c(1, 3) & s(6)c(1) & \frac{s(3, 6)}{2s(1)} \\
2s(6)c(1, 2) & 2s(6)c(1, 2) & s(6)c(1) & \frac{s(5, 6)}{2s(2)} \\
\end{pmatrix} \quad (B.7a)
\]

\[
(\hat{D}^{-1})_{1 \leq a \leq 7, 5 \leq b \leq 7} = \frac{1}{c(9)} \begin{pmatrix}
s(2)c(3) & s(1)c(3) & 2s(1)c(2, 3) \\
2s(2)c(1, 3) & s(2)c(3) & s(4)c(3) \\
s(6)c(1) & s(6)c(1) & s(6)c(2) \\
s(5)c(3) & \frac{s(5)c(3)}{2c(1)} & \frac{s(3, 6)}{2s(1)} \\
2s(2)c(3, 4) & 2s(1)c(3, 4) & \frac{1}{2}s(6) \\
\frac{s(12)}{8c(1, 2)} & \frac{s(12)}{8c(1, 2)} & \frac{s(6)}{8c(1, 2)} \\
\end{pmatrix} \quad (B.7b)
\]

\[ X_r = E_8 : \] For a typographical reason we split \(\hat{D}^{-1}\) into two pieces.

\[
(\hat{D}^{-1})_{1 \leq a \leq 8, 1 \leq b \leq 4} = \frac{1}{c(15)} \begin{pmatrix}
2s(1)c(5, 9) & \frac{s(12)c(5)}{4c(1, 2)} & 4s(1)c(3, 4, 5) & \frac{s(10)c(3)}{2c(1)} \\
\frac{s(12)c(5)}{4c(1, 2)} & 2s(6)c(3, 4, 5) & 4s(2)c(3, 4, 5) & \frac{s(10)c(3)}{2c(1)} \\
\frac{s(12)c(5)}{2c(2)} & \frac{s(12)c(5)}{2c(2)} & 2s(6)c(4, 5) & \frac{s(6, 10)}{2s(2)} \\
\frac{s(12)c(5)}{4c(1, 2)} & \frac{s(12)c(5)}{4c(1, 2)} & \frac{s(12)c(5)}{2c(2)} & 2s(10)c(2, 3) \\
\end{pmatrix} \quad (B.8a)
\]
\[(\mathcal{D}^{-1})_{1 \leq a \leq 8, 5 \leq b \leq 8} = \frac{1}{c(15)} \begin{pmatrix}
2s(6)c(5) & 2s(2)c(3, 5) & 2s(1)c(3, 5) & \frac{s(6)c(5)}{2c(1)} \\
2s(6)c(1, 5) & 4s(2)c(1, 3, 5) & 2s(2)c(3, 5) & s(6)c(5) \\
\frac{s(3, 6)c(5)}{s(1)} & 2s(6)c(1, 5) & s(6)c(5) & \frac{s(3, 6)c(5)}{s(2)} \\
\frac{4s(6)c(1, 2, 5)}{s(6, 10)} & \frac{4s(4)c(1, 3, 5)}{2s(1)} & 2s(4)c(3, 5) & 2s(6)c(2, 5) \\
2s(10)c(1, 3) & \frac{s(7)c(3, 5)}{c(1)} & \frac{s(10)c(3)}{2c(1)} & \frac{s(10)c(3)}{s(8)c(3, 5)} \\
2s(7)c(3, 5) & \frac{s(7)c(3, 5)}{c(1)} & \frac{s(10)c(3)}{2c(1)} & \frac{s(10)c(3)}{s(8)c(3, 5)} \\
4s(1)c(3, 5, 6) & \frac{s(7)c(3, 5)}{c(1)} & \frac{s(10)c(3)}{2c(1)} & \frac{s(10)c(3)}{s(8)c(3, 5)} 
\end{pmatrix} \]  \tag{B.8b}

\[X_r = F_4 : \]

\[\hat{\mathcal{D}}^{-1} = \frac{1}{c(\frac{9}{2})} \begin{pmatrix}
2s(1)c(\frac{3}{2}, 2) & s(3)c(\frac{1}{3}) & 2s(1)c(\frac{1}{2}, \frac{3}{2}) & s(1)c(\frac{3}{2}) \\
2s(3)c(\frac{3}{2}, 1) & 2s(2)c(\frac{1}{3}, \frac{3}{2}) & s(2)c(\frac{3}{2}) & \frac{s(3)c(\frac{1}{2})}{2c(\frac{1}{2})} \\
s(3)c(\frac{3}{2}) & \frac{s(3)c(\frac{1}{2})}{2c(\frac{1}{2})} & 2s(\frac{1}{2})c(\frac{3}{2}, \frac{5}{2}) 
\end{pmatrix} \]  \tag{B.9}

\[X_r = G_2 : \]

\[\hat{\mathcal{D}}^{-1} = \frac{1}{c(2)} \begin{pmatrix}
2s(1)c(\frac{4}{3}, \frac{2}{3}) & s(1)c(\frac{2}{3}) \\
s(1)c(\frac{2}{3}) & 2s(\frac{1}{2})c(\frac{2}{3}, 1) 
\end{pmatrix} \]  \tag{B.10}

In general the matrix \(\hat{\mathcal{D}}[n]^{-1}\) has the properties

\[\hat{\mathcal{D}}[n]^{-1} = -\hat{\mathcal{D}}[-n]^{-1}, \quad (\hat{\mathcal{D}}[n]^{-1})_{pa} = O(e^{\gamma|n|\lambda}) \quad \text{for some} \quad \gamma \leq 0 \quad \text{as} \quad |n| \to \infty. \quad \tag{B.11b}\]

When \(t_p = 1\), all the combinations appearing in (2.40) can be expanded into a finite sum of the form

\[ \sum_{\tau \in Aut(\hat{\mathcal{D}}[n]^{-1})_{p, \tau(a)}} \frac{z_m^{(p, a)} \cosh \beta_m^{(p, a)} n \lambda}{\sinh n \lambda} = \sum_m z_m^{(p, a)} \frac{\cosh \beta_m^{(p, a)} n \lambda}{\cosh \frac{ng \lambda}{2}}, \quad \tag{B.12}\]

where \(g\) is the dual Coxeter number. \(z_m^{(p, a)}\) and \(\beta_m^{(p, a)} \geq 0\) are the rational constants uniquely specified by the above expansion. They satisfy

\[z_m^{(p, a)} > 0, \quad \tag{B.13a}\]

\[\pm \beta_m^{(p, a)} - \frac{g}{2} \leq -1. \quad \tag{B.13b}\]

Here, (B.13b) is a direct consequence of (B.11b). The property (B.13a), which is by no means obvious from the definition, is crucial to assure \(z > 0\) etc, in (2.46) in the main text. It can be checked directly by using the explicit forms (B.2-9). We note that the expansion
as (B.12) with the property (B.13a) is impossible in general without “averaging over” Aut. The case \( X_r = E_6, a = p = 1 \) is such an example.

**Appendix C. Asymptotic form of \( \log Y B_m^{(a)}(u) \) for \( A_r \)**

Here we shall derive the asymptotic forms (3.12) and (3.37) directly from the known eigenvalue of the transfer matrix [14] for \( A_r \) RSOS models [18]. We shall calculate it firstly in the regime \( \epsilon = -1 \) in the sense of section 3.1. The behavior in the other regime \( \epsilon = +1 \) can then be deduced from the level-rank duality [42].

Let \( \text{Tab}(\mu) \) denote the set of semi-standard tableaux on the Young diagram \( \mu \) for \( sl(r+1) \). Namely, \( t \in \text{Tab}(\mu) \) is an assignment of an integer \( t(c,q) \) to each box at position \( (c,q) \) in \( \mu \) such that
1. \( 1 \leq t(c,q) \leq r + 1 \),
2. \( t(c,q) < t(c+1,q) \),
3. \( t(c,q) \leq t(c,q+1) \).

Then the eigenvalue of transfer matrix with spectral parameter \( u \) is given by [14]

\[
\Lambda^{\lambda,\mu}(u) = \sum_{t \in \text{Tab}(\mu)} \prod_{(c,q) \in \mu} X^{t(c,q)}(u + \mu_1 + c - q - 1).
\] (C.1)

Here \( \lambda \) is the Young diagram specifying that the transfer matrix of the corresponding vertex model acts on the quantum space \( V_{\lambda} \otimes V_{\mu} \) (\( V_\lambda \): the irreducible \( sl(r+1) \)-module with the highest weight corresponding to \( \lambda \)). Similarly, \( \mu \) signifies that the auxiliary space of the transfer matrix is \( V_{\mu} \). \( \lambda_i (\mu_i) \) denotes the number of boxes in the \( i \)-th row of \( \lambda (\mu) \) and

\[
X^c(u) = \omega_c \frac{Q^{c-1}(u-1)Q^c(u+1)}{Q^{c-1}(u)Q^c(u)} f(u - \lambda_c),
\]

\[
f(u) = \sin^N\left(\frac{\pi}{L}u\right), \quad L = \ell + r + 1,
\]

\[
Q^0(u) = Q^{r+1}(u) = 1,
\] (C.2)

\[
Q^c(u) = \prod_{j=1}^{N_c} \sin^c\left(\frac{\pi}{L}(u - u_j^c)\right) \quad 1 \leq c \leq r.
\]

In the above, the length \( N \) of the transfer matrix is taken as \( N \equiv 0 \mod r+1 \) and \( N_c \) is given by (IB.1). \( \{\omega_c\} \) are some phase factors [14] and \( \{u_j^c\} \) are the solutions to the Bethe ansatz equation (BAE)

\[
\frac{\omega_c f(u_j^c - \lambda_c)}{\omega_{c+1} f(u_j^c - \lambda_{c+1})} = -\frac{Q^{c+1}(u_j^c + 1)Q^c(u_j^c - 1)Q^{c-1}(u_j^c)}{Q^{c+1}(u_j^c)Q^c(u_j^c + 1)Q^{c-1}(u_j^c - 1)} \quad 1 \leq c \leq r.
\] (C.3)
In the remainder of this appendix, we set $\lambda = p \times s$, $\mu = a \times m$ rectangular shapes in accordance with the setting in section 3.

When $N \to \infty$, the dominant contribution in (C.1) is expected to come from the semi-standard tableau such that $t(c, q) = c, \forall q$ as in the case $\lambda = \mu$ [14]. Being concerned with the bulk behavior, we retain only this term and thereby get

$$\log \Lambda^{\lambda, \mu}(u) \simeq \log \frac{Q^a(u + m + a - 1)}{Q^a(u + a - 1)} + \sum_{c=1}^{a} \sum_{q=1}^{m} \log f(u + q + c - 2 - \lambda_c). \quad \text{(C.4)}$$

Let us rewrite (C.4) in the form suitable for the evaluation at the ground state in the regime $\epsilon = -1$. With the change of the variables

$$u_j^c \to iw_j^c/2 + (s + c - p)/2,$$

eq.(C.3) reads $(h(w) = \sinh(\pi w/L))$

$$\frac{\omega_c}{\omega_{c+1}} \left( \frac{h(w_j^c/2 + is\delta_{cp}/2)}{h(w_j^c/2 - is\delta_{cp}/2)} \right)^N = -\prod_{j'} h((w_j^c - w_{j'}^{c+1})/2 - i/2) \prod_{j''} h((w_j^c - w_{j''}^{c})/2 - i) \prod_{j'''} h((w_j^c - w_{j'''}^{c-1})/2 - i/2) \prod_{j''} h((w_j^c - w_{j''}^{c})/2 + i/2), \quad \text{(C.5)}$$

which is nothing but the BAE (IB.2) for $X_r = A_r$. In the ground state of the regime $\epsilon = -1$ at $N \to \infty$, the roots of (C.5) are expected to form the $s$-strings $w_j^c = v_j^c + i(s + 1 - 2\alpha), \alpha = 1, \cdots, s$. Then the distribution function of the color--$a$ $s$-string centers $\{v_j^a\}$ is given by

$$A^{(r+1)}_{a}(v), \quad \text{(C.6)}$$

defined in (IB.32e) by its Fourier component. To see this, note that $\forall \sigma_m^{(a)}, \rho_m^{(a)} \equiv 0$ except for $\rho_s^{(a)}$’s in the Dirac sea of $s$-strings. Due to (IB.12) and (IB.32e), the BAE (IB.3) then becomes $\delta_{pa} A^{(f)}_{sm} = \sum_{b=1}^{r} A^{(f)}_{sm} M_{ab} \rho_s^{(b)}$, which has the solution (C.6) because of (IB.17a) and (IB.18). In terms of the $\{v^a_j\}$, (C.4) is rewritten as

$$\log \Lambda^{\lambda, \mu}(u) \simeq \sum_{c=1}^{a} \sum_{q=1}^{m} \log f(u + q + c - 2 - \lambda_c) + \sum_{j} \sum_{\alpha=1}^{s} \log \frac{h(v_j^\alpha + i(u + m + \frac{a+p-1}{2}) - i\alpha)}{h(v_j^\alpha + i(u + \frac{a+p-1}{2}) - i\alpha)}. \quad \text{(C.7)}$$

35
Replacing $\sum_j \to N \int dv A_{ap}^{(r+1)}(v)$ and passing to the Fourier components, we get the bulk eigenvalue

$$\log \Lambda_{\text{bulk}}^{\lambda,\mu}(u) = \sum_{c=1}^{a} \sum_{q=1}^{m} \log f(u + q + c - 2 - \lambda_c) + N \int dk \frac{\sinh(2u + m - s + a + p - 2k)}{k} \hat{A}_{sm}^{(L)}(k) \hat{A}_{ap}^{(r+1)}(k).$$  \hfill (C.8)

The eigenvalue of the $T_m^{(a)}(u)$ in section 3 is given by $\Lambda^{\lambda,\mu}(\frac{u}{2\pi} + \frac{s-p+r+1}{2} - m - a)$ here. Thus the bulk term for the $Y$-system (3.31) is given by

$$Y B_m^{(a)}(u) = \frac{\Lambda^{\lambda,\mu(1)}(\frac{u}{2\pi} + \frac{s-p+r+1}{2} - m - a)}{\Lambda^{\lambda,\mu(2)}(\frac{u}{2\pi} + \frac{s-p+r+1}{2} - m - a)} \frac{\Lambda^{\lambda,\mu(3)}(\frac{u}{2\pi} + \frac{s-p+r+1}{2} - m - a)}{\Lambda^{\lambda,\mu(4)}(\frac{u}{2\pi} + \frac{s-p+r+1}{2} - m - a)},$$  \hfill (C.9)

where $\mu^{(1)}, \ldots, \mu^{(4)}$ are the Young diagrams with the rectangular shapes $a \times (m+1), a \times (m+1), (a+1) \times m, (a-1) \times m$, respectively. Here we have used the facts $g_m^{(a)}(u) = 1 (1 \leq a \leq r-1)$ and $g_m^{(r)}(u) = \Lambda^{\lambda,\mu'(\frac{u}{2\pi} + \frac{s-p+r+1}{2} - m)} (\mu' = r \times m$ Young diagram) in (3.31). These are derivable from section 2.3 in Part I and eq.(3.12) in [14]. Substitute (C.8) into the logarithm of (C.9) and simplify the result by means of the formulas that follow from the definition (IB.32e)

$$\hat{A}_{a b+1}^{(T)}(k) + \hat{A}_{a b-1}^{(T)}(k) = 2 \cosh k \hat{A}_{ab}^{(T)}(k) - \delta_{ab},$$  \hfill (C.10a)

$$\int dk \frac{\sinh(2u + b - a + 1)k}{k} \hat{A}_{ab}^{(T)}(k) = - \sum_{j=1}^{b} \log \frac{\sin \left( \frac{\pi}{4} (u - a + j) \right)}{\sin \left( \frac{\pi}{4} (u + j) \right)},$$  \hfill (C.10b)

which is valid for any $a, b \in \{1, 2, \ldots, T-1\}, T \in \mathbb{Z}_{\geq 2}$. After some calculations we obtain

$$\log Y B_m^{(a)}(u) = N \delta_{s m} \sum_{c=0}^{p-1} \log \frac{\sin \left( \frac{\pi}{4} (\frac{u}{2\pi} + \frac{s-p+r+1-a-1}{2} + c) \right)}{\sin \left( \frac{\pi}{4} (\frac{u}{2\pi} + \frac{s-p+r+1+a-1}{2} + c) \right)},$$  \hfill (C.11)

which indeed satisfies (3.35). From this one can easily derive the $N \to \infty$ asymptotic form in the regime $\epsilon = -1$

$$\log Y B_m^{(a)}(\pm(u + \frac{r+1}{\pi} \log N)) = \delta_{s m} \left( \frac{a p \pi i}{r+1} - 2 \sin \frac{\pi a}{r+1} \sin \frac{\pi p}{r+1} e^{-\frac{\pi u}{r+1}} + O(\frac{1}{N}) \right).$$  \hfill (C.12)
To get the behavior in the other regime \( \epsilon = +1 \), we invoke the level-rank duality [42]. Noting that the rhs of (C.12) is symmetric under \( a \leftrightarrow p \) and \( m \leftrightarrow s \), we formally exchange \( r + 1 \leftrightarrow \ell \), \( a \leftrightarrow m \) and \( p \leftrightarrow s \) there to get

\[
- \log Y B_m^{(a)}(\pm (u + \frac{\ell}{\pi} \log N)) = \delta_{pa} \left( \mp m s N \pi i \frac{\ell}{\ell} - 2 \frac{\sin \frac{\pi m}{\ell} \sin \frac{\pi s}{\ell}}{\sin \frac{\pi}{\ell}} e^{-\frac{\pi u}{\ell}} + O\left( \frac{1}{N} \right) \right) \tag{C.13}
\]

in the regime \( \epsilon = +1 \). Here the overall sign factor on the lhs has been attached to take the difference of (3.6) and (3.34) into account. When \( r = 1 \), (C.13) indeed agrees, with a suitable adjustment of the variables, with the regime II result eq.(4.32) in [7].

**Appendix D. Rogers dilogarithm and the conjecture in [16]**

This appendix is devoted to a description of the Rogers dilogarithm and the curious conjecture proposed in [16]. The latter relates the parafermion scaling dimensions to a certain sum of the dilogarithm special values. As promised there we present here some interesting calculations that elucidate the origin of the conjecture. In section D.1 we recall the log and the dilogarithm functions and specify our convention on their analytic continuations. Our dilogarithm conjecture will be formulated in D.2 following [16] for any classical simple Lie algebra \( X_r \), integer \( \ell \geq 1 \) and dominant integral weight \( \Lambda \in P_\ell \). (See (I3.4) for the definition of \( P_\ell \).) Its special case \( \Lambda = 0 \) concerns the earlier one in [11,12] and is relevant to our calculation in section 3 of the main text. (Conversely we expect that the content of section 3 could be extended so that the \( \Lambda \)-generic (\( \in P_\ell \)) dilogarithm conjecture becomes relevant.) Section D.3 contains a proof of the important property eq.(16) in [16] based on the congruence index explored in appendix A of Part I. We remark that the connection of the dilogarithm with CFT has attracted much attention recently and several insights have been gained in [17,43-47].

**D.1. \( \log f, L(f) \) and their analytic continuations**

Let \( \log x \) signify the logarithm in the branch \(-\pi < \text{Im}(\log x) \leq \pi \) for \( x \neq 0 \). Namely, \( \log x = \log |x| + i\pi \text{arg}(x) \) with \(-1 < \text{arg}(x) \leq 1 \) for any non-zero \( x = |x|e^{i\pi \text{arg}(x)} \). Under this convention for the arg function, we have

\[
\log(x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}) = r_1 \log x_1 + \cdots + r_k \log x_k - 2\pi i H(r_1 \text{arg}(x_1) + \cdots + r_k \text{arg}(x_k)), \tag{D.1a}
\]
for any nonzero complex numbers \(x_1, \ldots, x_k\) and real numbers \(r_1, \ldots, r_k\). The function \(H(x) \in \mathbb{Z}\) for \(x \in \mathbb{R}\) is given by the rule

\[
H(x) = n \quad \text{for} \quad 2n - 1 < x \leq 2n + 1, \quad n \in \mathbb{Z}.
\] (D.1b)

By the definition, the functions \(\log x\) and \(\log(1-x)\) have the branch cuts \((-\infty, 0]\) and \([1, \infty)\) on the real \(x\)-axis and they actually belong to the upper and lower half plane, respectively. The Rogers dilogarithm is a function of a complex number \(f\) defined by

\[
L(f) = -\frac{1}{2} \int_0^f \left( \frac{\log(1-x)}{x} + \frac{\log x}{1-x} \right) dx,
\] (D.2)

where the integral is along a contour that does not cross the cuts mentioned above.

To discuss the analytic continuations of these functions, we introduce the universal covering space \(\mathcal{R}\) of \(\mathbb{C}\backslash\{0, 1\}\) and the covering map \(\tilde{i}: \mathcal{R} \to \mathbb{C}\backslash\{0, 1\}\). Given \(f \in \mathbb{C}\backslash\{0, 1\}\), a corresponding point \(\tilde{f} \in \mathcal{R}\), \(\tilde{i}(\tilde{f}) = f\) is specified by a contour \(\mathcal{C}\) from an arbitrarily fixed base point to \(f\) in \(\mathbb{C}\backslash\{0, 1\}\). Since \(\tilde{f}\) depends only on the homotopy class of \(\mathcal{C}\), we shall parameterize it by the integers \(\xi_j, \eta_j\) \((j \geq 1)\) as \(\mathcal{C} = \mathcal{C}[f|\xi_1, \xi_2, \ldots |\eta_1, \eta_2, \ldots]\), which signifies the following contour. It firstly goes across the cut \([1, \infty)\) for \(\eta_1\) times then crosses the other cut \((-\infty, 0]\) for \(\xi_1\) times then \([1, \infty)\) again for \(\eta_2\) times, \((-\infty, 0]\) for \(\xi_2\) times and so on before approaching \(f\) finally. Here intersections have been counted as +1 when the contour goes across the cut \((-\infty, 0]\) (resp. \([1, \infty)\)) from the upper (resp. lower) half plane to the lower (resp. upper) and -1 if opposite. We call \(\xi_j\) and \(\eta_j\) the winding numbers and assume that they are all zero for \(j\) sufficiently large. Let \(\tilde{f} \in \mathcal{R}\) be so specified by \(\mathcal{C}\) and let \(\text{Log}\tilde{f}\) and \(\tilde{L}(\tilde{f})\) denote the analytic continuations of \(\log f\) and \(L(f)\) to \(\mathcal{R}\), respectively. From the definitions one deduces the formulas

\[
\log(\tilde{f}) = \log f + 2\pi i \sum_{j \geq 1} \xi_j, \quad \log(1 - \tilde{f}) = \log(1 - f) + 2\pi i \sum_{j \geq 1} \eta_j, \quad (D.3a)
\]

\[
\tilde{L}(\tilde{f}) = L(f) + \pi i \sum_{j \geq 1} \xi_j \log(1 - f) - \pi i \sum_{j \geq 1} \eta_j \log f
- 2\pi^2 \sum_{j \geq 1} \xi_j (\sum_{j \geq 1} \eta_j) + 4\pi^2 \sum_{j \geq 1} \xi_j (\eta_1 + \cdots + \eta_j), \quad (D.3b)
\]

which make the dependences on the contour \(\mathcal{C} = \mathcal{C}[f|\xi_1, \xi_2, \ldots |\eta_1, \eta_2, \ldots]\) explicit. We note that eq.(9b) and (12d) in [16] are erroneous (though it does not affect the conclusion there) and should be corrected as (D.3b) and (D.12d) in this paper, respectively.
D.2. Dilogarithm conjectures

In the remainder of this appendix, we fix an integer $\ell \geq 1$ and use the notation (3.1b). Define a complex valued function $f_m^a(z)$ on the dual space of the Cartan subalgebra $z \in \mathcal{H}^*$ by

$$f_m^a(z) = 1 - \frac{Q_m^{a-1}(z)Q_m^{a+1}(z)}{Q_m^a(z)^2} \quad \text{for } (a, m) \in G,$$  

(D.4)

where the quantity $Q_m^a(z)$ has been detailed in section 3.2 and appendix A of Part I. Among other values of $z$, $Q_m^a(0)$ is real positive for all $(a, m) \in G$ as noted after (I3.8). As shown in (IB.28), it follows from (D.4) that

$$0 < f_m^a(0) < 1 \quad \text{for } (a, m) \in G,$$  

(D.5a)

$$\log f_m^a(0) = \sum_{(b, k) \in G} K_{ab}^{mk} \log (1 - f_k^b(0)),$$  

(D.5b)

where $K_{ab}^{mk}$ is defined in (IB.21) and related to $J_{ba}^{km}$ in (I3.7), (IB.22) via (IB.23a). It has been conjectured in [11,12] that

$$\frac{6}{\pi^2} \sum_{(a, m) \in G} L(f_m^a(0)) = \frac{\ell \dim X_r}{\ell + g} - r.$$  

(D.6)

The rhs is the well known level $\ell X_r^{(1)}$ parafermion central charge $c_{PF}$ [48] while the intricate lhs arises both from the $T$-system analysis in section 3 and the thermodynamic Bethe ansatz [12-14]. So far the proof of (D.6) has been known for $X_r = A_r$ in [11,49] as well as several insights [17,43-47] for related problems. In particular, a new $q$-series formula has been conjectured in [17] for a level $\ell X_r^{(1)}$ string function [50], which may be viewed as a $q$-analogue of (D.6).

Let us proceed to the generalization [16] of (D.6) where the rhs includes the combination $c_{PF} - 24 \times $ (scaling dimensions mod $\mathbb{Z}$). Following [16], we call an element $z \in \mathcal{H}^*$ regular if it satisfies $Q_m^a(z) \neq 0$ in (I3.6) for all $1 \leq a \leq r, 1 \leq m \leq \ell_a$ and singular otherwise. The most typical example of the former is the choice $z = 0$ since all the relevant $Q_m^a(0)$’s are real positive. Suppose $z \in \mathcal{H}^*$ is regular. Then $f_m^a(z)$ (D.4) is finite and $f_m^a(z) \neq 0, 1$ for all $(a, m) \in G$. Take a contour $C_{a,m} = C[f_m^a(z)|\xi_{m,1}^a, \xi_{m,2}^a, \ldots |\eta_{m,1}^a, \eta_{m,2}^a, \ldots]$ and denote by $f_m^a(z)$ the point on $\mathcal{R}$ specified by the $C_{a,m}$ with $\tilde{i}(\tilde{f}_m^a(z)) = f_m^a(z)$. Then the collection $(\tilde{f}_m^a(z))_{(a,m) \in G}$ may be viewed as a point on $\mathcal{R}^{|G|}$ parameterized by a regular
\[ z \in H^* \text{ and the winding numbers } S = (\xi_{m,j}, \eta_{m,j})_{(a,m) \in G, j \geq 1}. \] According to eq.(11) in [16] we introduce the following functions of regular \( z \) and \( S \),

\[ \frac{\pi^2}{6} c(z, S) = \sum_{(a,m) \in G} \left( \tilde{L}(\tilde{f}_m^a(z)) - \frac{\pi i}{2} D_m^a(z, S) \log(1 - \tilde{f}_m^a(z)) \right), \quad \text{(D.7a)} \]

\[ \pi i D_m^a(z, S) = \log(\tilde{f}_m^a(z)) - \sum_{(b,k) \in G} K_{a,b}^{m,k} \log(1 - \tilde{f}_k^b(z)). \quad \text{(D.7b)} \]

Note that for the typical regular element \( z = 0 \in H^* \), (D.7) provides an analytic continuation of the \( u = +\infty \) term in (3.25) due to (3.26). In particular, if all the winding numbers are zero further, \( \log \tilde{f}_m^a(0) \) becomes \( \log \tilde{f}_m^a(0) \) itself and (D.7b) vanishes because of (D.5b). Therefore the simplest case \( c(z = 0, S = \text{trivial}) \) of (D.7a) is nothing but the lhs of (D.6). Thus we expect, in the light of the calculation in section 3.2.4, that the function \( c(z, S) \) will be relevant to the combination \( c_{\text{eff}} = c_{PF} - 24 \times \text{ (scaling dimension)} \) and possibly leads to a generalization of (D.6). This seems indeed the case as will be formulated in our conjecture shortly. We remark that there is yet another route to reach (D.7) from the TBA argument as done in [17]. These are the two sources mentioned in [16] that led to the invention of (D.7).

Given a regular \( z \in H^* \), define an element \( \lambda(z) \in H^* \) by (cf. eq.(13) in [16])

\[ \lambda(z) = \sum_{a=1}^{r} \lambda_a(z) \Lambda_a, \quad \text{(D.8a)} \]

\[ \lambda_a(z) = \frac{1}{2\pi i} \sum_{b=1}^{r} C_{ab} \left( \sum_{k=1}^{\ell_b-1} k \log(1 - f_k^b(z)) \right. \\
+ \ell_b \left( \log Q_{\ell_b-1}^b(z) - \log Q_{\ell_b}^b(z) \right) \right), \quad \text{(D.8b)} \]

where \( \log \) denotes the logarithm specified in section D.1. From now on we shall mainly concern the specialization to the dominant integral weights \( z = \Lambda \in P_t \) (3.4). In case \( \Lambda \) is singular, formulas are to be understood via a fixed way of the limit \( z \in H^*_R \rightarrow \Lambda \). Based on (IB.23) and (IA.11), it will be shown in Proposition 1 of section D.3 that

\[ \lambda(\Lambda) \equiv \Lambda \mod Q, \quad \text{(D.9)} \]

where \( Q \) stands for the root lattice as in section 3.1 of Part I. This is eq.(16a) of [16]. Given the winding numbers \( S = (\xi_{m,j}, \eta_{m,j})_{(a,m) \in G, j \geq 1} \), we put

\[ \beta(S) = \sum_{(a,m) \in G} mN_m^a \alpha_a \in Q, \quad \text{(D.10a)} \]

\[ N_m^a = \sum_{j \geq 1} \eta_{m,j} \in \mathbb{Z}, \quad \text{(D.10b)} \]
in which (D.10b) is finite since only finitely many \(\eta_{m,j}^{(a)}\)'s are non-zero as noted before (D.3).

Now the generalized dilogarithm conjecture can be stated as follows.

**Conjecture (eq.(17) of [16]).** Let \(c(z, S)\) be as defined in (D.7) and take a dominant integral weight \(\Lambda \in P_\ell\). Then,

\[
\begin{align*}
c(\Lambda, S) &= \frac{\ell \dim X_r}{\ell + g} - r - 24(\Delta^\Lambda_{\lambda(\Lambda)} + \beta(S) + \text{integer}), \\
\Delta^\Lambda_y &= \frac{(z|z + 2\rho) - |y|^2}{2(\ell + g)} \quad \text{for } y, z \in H^*,
\end{align*}
\]

where \(\rho\) stands for the half sum of the positive roots of \(X_r\) (I3.1e).

The rhs of (D.11a) with the congruence properties (D.9) and (D.10a) is well known as the central charge \(-24 \times (\text{scaling dimension mod } Z)\) of the level \(\ell X_r^{(1)}\) parafermion CFT [48]. As the winding number \(S\) is chosen variously; \(\beta(S)\) (D.10a) ranges over the root lattice \(Q\). Thus all the spectra in parafermion CFTs seem to come out from the Rogers dilogarithm through the quantity \(c(\Lambda, S)\) (D.7), which was the main observation in [16]. In fact, when \(\Lambda = 0 \in P_\ell\), a more concrete correspondence can be inferred between vacuum parafermion module and the contours represented by \(S\) as argued in [17].

In the rest of this section, we rewrite our conjecture in a simpler form suitable for the discussion in section D.3. Firstly, we extract the \(S\)-dependence of the quantity \(c(\Lambda, S)\) by applying the formula (D.3). A little manipulation leads to

\[
\begin{align*}
c(\Lambda, S) &= c_0(\Lambda) - 24T(\Lambda, S), \\
\frac{\pi^2}{6} c_0(\Lambda) &= \sum_{(a, m) \in G} \left( L(f_m^{(a)}(\Lambda)) - \frac{\pi i}{2} \eta_{m}^{(a)}(\Lambda) \log(1 - f_m^{(a)}(\Lambda)) \right), \\
\pi i d_{m}^{(a)}(\Lambda) &= \log f_m^{(a)}(\Lambda) - \sum_{(b, k) \in G} K_{ab}^{m k} \log(1 - f_k^{(b)}(\Lambda)), \\
T(\Lambda, S) &= \frac{1}{2} \sum_{(a, m) \in G} K_{ab}^{m k} N_m^{(a)} N_k^{(b)} - \frac{1}{2} \sum_{(a, m) \in G} d_{m}^{(a)}(\Lambda) N_m^{(a)} \\
&\quad - \sum_{(a, m) \in G} \sum_{j \geq 1} \xi_{m,j}^{(a)}(\eta_{m,1}^{(a)} + \cdots + \eta_{m,j}^{(a)}),
\end{align*}
\]

where \(N_m^{(a)}\) is specified from \(S = (\xi_{m,j}^{(a)}, \eta_{m,j}^{(a)})_{(a, m) \in G, j \geq 1}\) by (D.10b). Here comes the point which enabled us to find the conjecture (D.11). If one expects the parafermion scaling dimension from \(c(\Lambda, S)\) at all, there must be a square of some vector corresponding to
−\frac{|y|^2}{2\ell} \text{ in (D.11b)} \text{ such that } y \equiv \Lambda \mod Q. \text{ Our idea is to regard } T(\Lambda, S) \text{ (D.12d)} \text{ as a quadratic form of } N_m^{(a)} \in \mathbb{Z} \text{ and try to complete a square out of it mod } \mathbb{Z}. \text{ Actually this postulate was almost enough to find the vector } \lambda(\Lambda) \text{ (D.8)} \text{ for which eq.(16b) in [16], i.e.,}

\[ T(\Lambda, S) \equiv -\frac{1}{2\ell}|\lambda(\Lambda) + \beta(S)|^2 + \frac{1}{2\ell}|\lambda(\Lambda)|^2 \mod \mathbb{Z} \quad \text{(D.13)} \]
certainly holds. See Proposition 2 in section D.3 for the proof. In view of (D.13) and (D.12a) the conjecture (D.11) reduces to a simpler statement (eq.(14) in [16])

\[ c_0(\Lambda) = \frac{\ell \dim X_r}{\ell + g} - r - 24(\Delta^\Lambda_{\lambda(\Lambda)} + \text{integer}) \quad \text{(D.14)} \]

for \( c_0(\Lambda) \) defined by (D.12b,c). We have checked this numerically for all the regular \( \Lambda \in P_\ell \) in \((X_r, \ell) = (A_1, 2 \sim 15), (A_2, 2 \sim 6), (A_3, 2 \sim 4), (A_4, 5, 2 \sim 3), (B_r, 6 - r), (C_r, 6 - r)\) for \( 2 \leq r \leq 5, (B_6,2), (D_4,5,2 \sim 3) \) and \( (D_6,2) \). Singular \( \Lambda \) case has also been confirmed in the above examples under several numerical limits \( z(\in \mathcal{H}_R^*) \rightarrow \Lambda \). We note that when \( \Lambda = 0 \), \( d_m^{(a)}(0) = \lambda(0) = 0 \) holds, therefore the above conjecture (with “integer” = 0) reduces to (D.6).

**D.3. Proof of (D.9) and (D.13)**

Let us show the properties (D.9) and (D.13) based on (IA.11). They have been announced earlier in eqs.(16a) and (16b) of [16], respectively. We shall exclusively consider the specialization \( z = \Lambda \in P_\ell \). As mentioned after (D.8), formulas involving \( \Lambda \) should then be understood via a limit \( z \in \mathcal{H}_R^* \rightarrow \Lambda \) whenever \( \Lambda \) is singular. For simplicity we often abbreviate \( Q_m^{(a)}(\Lambda) \) to \( Q_m^{(a)} \) etc and use the notation \( \theta_m^{(a)} = \arg Q_m^{(a)}(\Lambda) \). Thus we have

\[ Q_m^{(a)}(\Lambda) = |Q_m^{(a)}(\Lambda)|\exp(i\pi \theta_m^{(a)}), \quad -1 < \theta_m^{(a)} \leq 1, \quad \text{(D.15)} \]

for \( 1 \leq a \leq r, 0 \leq m \leq \ell_a \) in accordance with section D.1.

Our first step is to rewrite the \( d_m^{(a)} \) (D.12c) as

**Lemma D.1.** For \((a, m) \in G\) we have

\[
d_m^{(a)} = -\frac{m}{\ell_a \pi i} \sum_{b=1}^{r} C_{ab} \log Q_{\ell_b}^{(b)} + 2 \sum_{(b,k) \in G} K_{ab}^{mk} H(\theta_{k+1}^{(b)} + \theta_{k-1}^{(b)} - 2\theta_k^{(b)})
\]
\[ -2H(-2 \sum_{b=1}^{r} \sum_{k=1}^{\ell_b} J_{ba}^{km} \theta_k^{(b)}), \quad \text{(D.16a)} \]
\[
d_m^{(a)} \equiv md_1^{(a)} \mod 2\mathbb{Z}. \quad \text{(D.16b)}
\]
Proof. Noting \( Q_0^{(b)} = 1 \) but \( Q_{\ell_b}^{(b)} \neq 1 \) in general, the definition (D.4) can be written as

\[
1 - f_k^{(b)} = Q_{\ell_b}^{(b)}\delta_{k,\ell_b-1} \prod_{n=1}^{\ell_b-1} Q_n^{(b)} - \bar{C}_{nk}^b,
\]

(D.17a)

for \((b, k) \in G\), where \( \bar{C}_{nk}^b \) denotes the Cartan matrix of \( A_{\ell_b-1} \) as in (IB.16b). Combining (D.4) with (IA.9a) we get

\[
f_m^{(a)} = \prod_{(b, k)\in G} Q_{k}^{(b)} - 2J_{ba}^{k,m} \prod_{b=1}^{r} Q_{\ell_b}^{(b)} - 2J_{ba}^{\ell_b,m},
\]

(D.17b)

for \((a, m) \in G\), where we have extracted the factor corresponding to \( k = \ell_b \) in (IA.9a) explicitly. Expanding the log of (D.17a) according to the rule (D.1a) gives

\[
\log(1 - f_k^{(b)}) = \delta_{k,\ell_b-1} \log Q_{\ell_b}^{(b)} - \sum_{n=1}^{\ell_b-1} \bar{C}_{nk}^b \log Q_n^{(b)} - 2\pi iH(\theta_{k+1}^{(b)} + \theta_{k-1}^{(b)} - 2\theta_{k}^{(b)}).
\]

(D.18)

Substitute this and the similar expression for \( \log f_m^{(a)} \) obtainable from (D.17b) into (D.12c). The result reads

\[
\pi id_m^{(a)} = -2 \sum_{(b, k)\in G} J_{ba}^{k,m} \log Q_k^{(b)} + \sum_{b=1}^{r} J_{ba}^{\ell_b,m} \log Q_{\ell_b}^{(b)} - 2\pi iH(-2 \sum_{b=1}^{r} \sum_{k=1}^{\ell_b} J_{ba}^{k,m} \theta_{k}^{(b)})
\]

\[
- \sum_{(b, k)\in G} K_{ab}^{m} \left[ \delta_{k,\ell_b-1} \log Q_{\ell_b}^{(b)} - \sum_{n=1}^{\ell_b-1} \bar{C}_{nk}^b \log Q_n^{(b)} - 2\pi iH(\theta_{k+1}^{(b)} + \theta_{k-1}^{(b)} - 2\theta_{k}^{(b)}) \right],
\]

(D.19)

which consists of 6 pieces. The first one cancels the fifth due to (IB.23a). The second one can be combined with the fourth by means of (IB.23c), which proves (D.16a). To show (D.16b), recall from the definition (D.1b) that \( H(\cdot) \)’s in (D.16a) are just integers. Using the explicit forms (IB.21b) and \((\alpha_a | \alpha_b) = C_{ab}/t_a \) further, one finds

\[
d_m^{(a)} \equiv -\frac{m}{\ell_a n b} \sum_{b=1}^{r} C_{ab} \log Q_{\ell_b}^{(b)} - \frac{2m}{\ell_a} \sum_{(b, k)\in G} C_{ab} k H(\theta_{k+1}^{(b)} + \theta_{k-1}^{(b)} - 2\theta_{k}^{(b)}) \mod 2\mathbb{Z},
\]

(D.20)

from which (D.16b) follows.

Similarly \( \lambda_a(\Lambda) \) (D.8b) is expressed as
Lemma D.2. For $1 \leq a \leq r$,
\[
\lambda_a(\Lambda) = -\frac{1}{2\pi i} \sum_{b=1}^{r} C_{ab} \log Q^{(b)}_{\ell_b}(\Lambda) - \sum_{(b,k) \in G} C_{ab} k H(\theta_{k+1}^{(b)} + \theta_{k-1}^{(b)} - 2\theta_k^{(b)}), \tag{D.21a}
\]
\[
d_1^{(a)}(\Lambda) \equiv \frac{2}{\ell_a} \lambda_a \mod 2\mathbb{Z}, \tag{D.21b}
\]
\[
\lambda_a(\Lambda) \in \mathbb{Z}. \tag{D.21c}
\]

Proof. To see (D.21a), replace $\log(1 - f_k^{(b)})$ in (D.8b) with the rhs of (D.18) and reduce the result by $\sum_{\ell_b=1}^{\ell_b \in \mathbb{Z}} k \delta_{\ell_n} \ell_{b,1}$. Comparing (D.21a) and (D.20) we get (D.21b).

Eq.(D.21c) is valid if the first term on the rhs of (D.21a) is an integer. From the data (IA.11), it can be computed explicitly as (when $X_r \neq D_r$ for example)
\[
-\frac{1}{2\pi i} \sum_{b=1}^{r} C_{ab} \left( -\frac{2\pi i \Gamma(\Lambda) \gamma_b}{\kappa} - 2\pi i H\left( -\frac{2\Gamma(\Lambda) \gamma_b}{\kappa} \right) \right) \equiv \sum_{b=1}^{r} C_{ab} \gamma_b \equiv 0 \mod \mathbb{Z}, \tag{D.22}
\]
where we have used (D.1a), (IA.14) and the fact $\Gamma(\Lambda), H(\cdot) \in \mathbb{Z}$. The case $X_r = D_r$ is verified similarly. This completes the proof of (D.21c), hence Lemma D.2.

We remark that
\[
d_m^{(a)}(\Lambda) \equiv \frac{2m \lambda_a(\Lambda)}{\ell_a} \mod 2\mathbb{Z}, \tag{D.23}
\]
due to (D.16b) and (D.21b). From (D.21c) we now know that $\lambda(\Lambda)$ is an integral weight. Moreover it has a remarkable property as follows.

**Proposition 1 (eq.(D.9)).** $\lambda(\Lambda)$ and $\Lambda$ are congruent with respect to the root lattice $Q$.

Proof. We consider the case $X_r \neq D_r$. The case $X_r = D_r$ is similar. In view of (IA.13) it is sufficient to check $\Gamma(\lambda(\Lambda)) \equiv \Gamma(\Lambda) \mod \kappa \mathbb{Z}$. Substitute (D.21a) into the sum $\Gamma(\lambda(\Lambda)) = \sum_{a=1}^{r} \gamma_a \lambda_a(\Lambda)$ (IA.12a). Because of (IA.14) the second term on the rhs of (D.21a) makes no contribution mod $\kappa \mathbb{Z}$. Thus from (D.21a) and (IA.11) we have
\[
\Gamma(\lambda(\Lambda)) \equiv -\frac{1}{2\pi i} \sum_{1 \leq a,b \leq r} \gamma_a C_{ab} \log Q^{(b)}_{\ell_b}(\Lambda) \mod \kappa \mathbb{Z}
\]
\[
\equiv -\frac{1}{2\pi i} \sum_{1 \leq a,b \leq r} \gamma_a C_{ab} \left( -\frac{2\pi i \Gamma(\Lambda) \gamma_b}{\kappa} - 2\pi i H\left( -\frac{2\Gamma(\Lambda) \gamma_b}{\kappa} \right) \right) \mod \kappa \mathbb{Z} \tag{D.24}
\]
\[
\equiv \Gamma(\Lambda) \mod \kappa \mathbb{Z},
\]
where in the last step we used (IA.14) and (IA.15a).

Eq.(D.9) is now proved. The significance of $\lambda(\Lambda)$ is not only the congruence property $\lambda(\Lambda) \equiv \Lambda \mod Q$ established above. It emerges when one tries to complete a square out of the $T(\Lambda, S)$ (D.12d) as a quadratic form of $\{N_m^{(a)} \mid (a, m) \in G\} \mod \mathbb{Z}$. Quoting (D.13) and (D.10a) again we have
Proposition 2 (eq.(D.13)). \( T(\Lambda, S) \) defined in (D.12d) satisfies

\[
T(\Lambda, S) \equiv -\frac{1}{2\ell}|\lambda(\Lambda) + \beta(S)|^2 + \frac{1}{2\ell}|\lambda(\Lambda)|^2 \mod Z,
\]

\[
\beta(S) = \sum_{a=1}^{r} \sum_{m=1}^{\ell_a-1} mN_m^{(a)} \alpha_a \in Q.
\]

Proof. Consider the first term

\[
\frac{1}{2} \sum K_{ab}^{mk} N_m^{(a)} N_k^{(b)} = \frac{1}{2} \sum (\min(t_{bm}, t_{ak}) - \frac{mk}{\ell})(\alpha_a|\alpha_b)N_m^{(a)} N_k^{(b)}
\]

appearing in \( T(\Lambda, S) \) (D.12d) (see (IB.21b)). The \( \min(, ) \) part here can be ignored mod \( Z \). As for the \( -\frac{1}{2} \sum d_m^{(a)} N_m^{(a)} \) term in (D.12d), one can replace the \( d_m^{(a)} \) by \( \frac{2m}{\ell_a} \lambda_a(\Lambda) \mod Z \) thanks to (D.23). The result now reads

\[
T(\Lambda, S) \equiv -\frac{1}{2\ell} \sum_{1\leq a, b \leq r} (\alpha_a|\alpha_b) \left( \sum_{m=1}^{\ell_a-1} mN_m^{(a)} \right) \left( \sum_{k=1}^{\ell_b-1} kN_k^{(b)} \right)
- \frac{1}{\ell} \sum_{a=1}^{r} \frac{\lambda_a(\Lambda)}{t_a} \left( \sum_{m=1}^{\ell_a-1} mN_m^{(a)} \right) \mod Z.
\]

Noting (D.25b) and \( (\lambda(\Lambda)|\alpha_a) = \lambda_a(\Lambda)/t_a \), we find that the rhs of (D.26) equals

\[-\frac{1}{2\ell}|\beta(S)|^2 - \frac{1}{\ell} (\lambda(\Lambda)||\beta(S)), \]

from which (D.25a) follows.
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