

# Imprecise Probability for Multiparty Session Types in Process Algebra

## Abstract

In this paper we introduce imprecise probability for session types. More exactly, we use a probabilistic process calculus in which both nondeterministic external choice and probabilistic internal choice are considered. We propose the probabilistic multiparty session types able to codify the structure of the communications by using some imprecise probabilities given in terms of lower and upper probabilities. We prove that this new probabilistic typing system is sound, as well as several other results dealing with both classical and probabilistic properties. The approach is illustrated by a simple example inspired by survey polls.

**Keywords** Imprecise Probability · Nondeterministic and Probabilistic Choices · Multiparty Session Types.

## 1 Introduction

Aiming to represent the available knowledge more accurately, imprecise probability generalizes probability theory to allow for partial probability specifications applicable when a unique probability distribution is difficult to be identified. The idea that probability judgments may be imprecise is not novel. It would be enough to mention the Dempster-Shafer theory for reasoning with uncertainty by using upper and lower probabilities [1]. The idea appeared even earlier in 1921, when J.M. Keynes used explicit intervals for approaching probabilities [2] (B. Russell called this book “undoubtedly the most important work on probability that has appeared for a very long time”). In analytic philosophy, the probability judgements were formulated in terms of interval-valued functions in [3], and in terms of upper and lower probabilities in [4]. The term *imprecise probability* was introduced explicitly in 1991 [5], while in 2000 the theory of interval probability was presented as a unifying concept for uncertainty [6]. Information on how imprecise probability differs from the classic Bayesian approach could be found in [7].

Using imprecise probability in the framework of computer science involving process calculi and session types represents the contribution of this paper. More exactly, imprecise probability are used to represent and quantify uncertainty in the study of concurrent processes involving multiparty session types. We consider a probabilistic process calculus using nondeterministic branching (choices made by an external process) and probabilistic selection (choices made internally by the process). This new probabilistic approach satisfies the main standard axioms for determining the probability of an event [1].

Session types [9] and multiparty session types [10] describe a type discipline for communication-centric systems having their roots in process calculi. Essentially, session types provide a typing discipline ensuring that a message-passing process implements a given (multiparty) session protocol defined as a structured sequence of interactions without errors. Such a structure is abstracted as a type through an intuitive syntax which is used to validate programs.

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1The probability of an event is a real number in the interval $[0, 1]$, and the sum of the probabilities of the elementary events is 1; these axioms represent a simplified version of those introduced in 1933 by A.Kolmogorov in the context of measure theory [8].
Session types are terms of a process calculus that also contains a selection construct (an internal choice among a set of branches), a branching construct (an external choice offered to the environment) and recursion.

To understand deeper the quantitative aspects of uncertainty that might arise in multiparty session types, we use *imprecise probabilities* by considering sets of probabilities (rather than a single probability) in terms of lower and upper probabilities. The imprecise probabilities appear for instance in voting polls, where it is reported an interval given by a percentage and a margin error. The probability interval is determined by subtracting and adding the sampling (margin) error to the sample mean (estimated percentage). This confidence interval may be wider or narrower, depending on the degree of (un)certainty or a required precision. To illustrate our approach, we consider a simple example inspired by probabilistic survey polls (which represent an alternative to the classic survey polls). In such a probabilistic poll the pollsters communicate their own beliefs and receive the beliefs of others in the form of numerical probabilities [11]. One of the advantages of such an approach is that the probability provides a well-defined numerical scale for responses; in [12] it is studied the accuracy of voting probabilities in the US presidential election (reporting the first large-scale application of probabilistic polling). The concept of probabilistic polling is based on the assumption that the opinions of the people accurately represent the distribution of opinions across the entire population; however, this can never be completely true. The margin of error describes the uncertainty that comes from having a small sample size (relative to the size of the polling population).

In the following, Section 2 presents the syntax and semantics of our calculus; the key ideas are explained by using a running example. Section 3 describes both global and local types used in session types, as well as the connection between them. Section 4 presents the new probabilistic typing system, and the main results. The probabilistic extension preserves the classical properties of a typing system; additionally, it satisfies the axioms of the probability theory, and can determine the imprecise probability of certain behaviours. Section 5 concludes and discusses certain related probabilistic approaches involving typing systems.

## 2 Processes in Probabilistic Multiparty Sessions

In this section we describe a probabilistic extension of the multiparty session process calculus presented in [13], by using the same notations as in [13]. This probabilistic extension allows probabilistic selection made internally by the communicating processes, and also branching controlled by an external process.

### 2.1 Syntax

Informally, a session represents a unit of conversation given by a series of interactions between multiple parties. A session is established via a shared name representing a public interaction point, and consists of communication actions performed on fresh session channels. The syntax is presented in Table 1 where the following notations are used: probabilities $p$, channels $c$ and $s$, values $v$, variables $x$, labels $l$, role $r$ and process variables $X$ in order to define recursive behaviours. In what follows, $f_c(P)$ denotes the set of free channels with roles in $P$, while $f_v(P)$ denotes the set of free variables in $P$.

| Processes | $P, Q$ ::= $c[r] \bar{\oplus}_{i \in I} \langle p_i : l_i(v_i) ; P_i \rangle$ (selection towards role $r$, $I \neq \emptyset$) |
|-----------|---------------------------------------------------------------|
|           | $c[r] \& \bar{\in}_{i \in I} \{ l_i(x_i) ; P_i \}$ (branching from role $r$, $I \neq \emptyset$) |
|           | $(\nu s) P$ (restriction) |
|           | $\text{def } D \text{ in } P$ (process definition) |
|           | $X(\bar{x})$ (process call) |
|           | $0$ (inaction) |
|           | $P \mid Q$ (parallel) |
| Declaration | $D ::= X(\bar{x}) = P$ (process declaration) |
| Context | $E ::= [ ] \mid P$ (evaluation context) |
|           | $(\nu s) E$ |
|           | $\text{def } D \text{ in } E$ |
|           | $E \mid E$ |
| Channel | $c ::= x \mid s[r]$ (variable, channel with role $r$) |
| Values | $v ::= s \mid \text{true} \mid \text{false} \mid 3 \ldots$ (base value) |

Table 1: Syntax of the probabilistic multiparty session calculus
In both selection and branching, the labels $l_i$ ($i \in I$) are all different and their order is irrelevant. This ensures that a labelled choice in a selection can be matched uniquely by its corresponding label in a branching. The process $c[v] \oplus_{i \in I} (p_i : l_i(v_i) : P_i)$ performs an internal choice on the channel $c$ towards role $r$, the labelled value $l_i(v_i)$ is sent with the probability $p_i$, and the execution continues as process $P_i$. Depending on which label is selected, there are $|I|$ possibilities of continuation, all unique due to the uniqueness of the labels in each set. When all the sets of probabilistic choices of a process $P$ contain only one element with probability 1, then the process $P$ can be defined using the syntax presented in [13] (by discarding all the probabilities). On the other hand, the process $c[v] \&_{i \in I} \{l_i(x_i) : P_i\}$ waits for an external choice on the channel $c$ from role $r$. If the labelled $l_i$ is used to communicate, then the execution continues as $P_i$ in which all the occurrences of the variable $x_i$ are replaced by the received value. Note that for all $i \in I$, the variables $x_i$ are bound with scope $P_i$. The restriction $(v)sP$ delimits the scope of a session $s$ to $P$. Process definition $\text{def } D$ in $P$ and process call $X(\bar{v})$ describe recursion, with $D$ being a process declaration of the form $X(\bar{x}) = P$. The call invokes $X$ by replacing its formal parameters $\bar{x}$ with the actual ones $\bar{v}$ from the process call. Just like in [13], we consider closed process declarations for which we have $fv(P) \subseteq \bar{x}$ and $fc(P) = \emptyset$ whenever $X(\bar{x}) = P$. The inaction $\emptyset$ represents a terminated process, while the parallel composition $P | Q$ represents two processes that execute concurrently and possibly communicate on a shared channel. As usual, evaluation contexts are processes with some holes [14]. A channel $c$ can be either a variable $x$ or a channel with role $s[r]$, namely a multiparty communication channel for role $r$ in a session $s$. Values $v$ can be integers, Boolean and strings (i.e., base values).

**Example 1.** The following process describes a simple survey poll in which Alice either responds to some questions of Bob or declines the polling. Bob can either stop the polling if it is pleased with the responses received till now, or has some other polling to perform. Just like in [13], Alice and Bob communicate their own choices and receive the choices of others having numerical probabilities attached to them, probabilities providing a well-defined numerical scale for responses.

\[
\text{System} = (\nu s)(\langle \text{Bob} \mid (\nu \text{PhoneNoA}, \text{IMIdA})\text{Alice } \rangle).
\]

The name $s$ indicates the session involving the processes Alice and Bob composed in parallel. Session $s$ has two roles: $r_B$ for the process performing the survey poll, and $r_A$ for the process answering the poll. The processes Alice and Bob communicate in this session by using the channel with role $s[r_A]$ and $s[r_B]$, respectively. In a similar manner, PhoneNoA and IMIdA represent sessions names used to model sessions established between Alice and Bob after the communication performed inside the session $s$ ends. Note that the scopes of PhoneNoA and IMIdA include only the process Alice initially, as the process Bob is informed about which session to join after communication with Alice.

\[
\begin{align*}
\text{Alice} = & \text{def } A(y) = A_1 \text{ in } A_2, \text{ where } A_1 \text{ and } A_2 \text{ are the following processes:} \\
A_1 = & y[r_B] & \& \{\text{talk}(t_1).y[r_B] \oplus (0.6 : \text{yes}(t).A(y) , 0.3 : \text{no}(t).A(y) , 0.1 : \text{quit}(t).0) \}, \text{quit}(t_2).0\} ; \\
A_2 = & s[r_A][r_B] \oplus (0.6 : \text{ComPhone(PhoneNoA)}.A(\text{PhoneNoA}[r_A]) , \\
& 0.35 : \text{ComIM(IMIdA)}.A(\text{IMIdA}[r_A]) , \\
& 0.05 : \text{noComm}(\text{no}).0) .
\end{align*}
\]

\[
\begin{align*}
\text{Bob} = & \text{def } B(y, t) = B_1 \text{ in } B_2, \text{ where } B_1 \text{ and } B_2 \text{ are the following processes:} \\
B_1 = & y[r_A] \oplus (0.95 : \text{talk}(t).y[r_A] & \& \{\text{yes}(t).B(y, next(t)) , \text{no}(t).B(y, next(t)) , \\
& \text{unsure}(t).B(y, next(t)) , \text{quit}(t).B\} , \\
& 0.05 : \text{quit}(\text{no}).0) ; \\
B_2 = & s[r_B][r_A] \& \{\text{ComPhone}(x_A).B(x_A[r_B], Q_1) , \text{ComIM}(x_A').B(x'_A[r_B], Q_1) , \text{noComm}(x''_A).0\} .
\end{align*}
\]

We explain the concurrent evolution of processes Alice and Bob by describing the interaction between them. In the first step, using the channel with role $s[r_A]$, the process Alice decides whether to respond to the poll either by phone, instant messaging, or not answering now. In each case, process Alice has attached the probabilities marking its willingness to take the appropriate branch. Suppose the ComPhone label is used; in this case Alice and Bob communicate by using the session channel PhoneNoA that represents the unique phone number of Alice. The process Bob is implemented by invoking $B(x_A[r_B])$, where $x_A$ becomes PhoneNoA after communication. Here $Q_1$ stands for the first question that process $A$ has to respond by using one of the labels yes, no, unsure or quit. Notice that the process Alice does not provide the answer by using unsure, as does not consider this to be an option for the given question. After receiving an answer to the question, process Bob can continue to ask the next question from its list (illustrated by next). The processes can quit the poll by performing a selection of either the label quit or noComm (when appropriate). Notice that the quit and noComm have low attached probabilities; this means that these options have very low chances of being chosen when the survey poll is conducted.
2.2 Operational Semantics

In what follows, \(s \notin fc(P)\) means that it does not exist an \(r\) such that \(s[r] \in fc(P)\). Also, \(dpv(D)\) denotes the set of process variables declared in \(D\), while \(fpv(P)\) denotes the set of process variables which occur free in \(P\).

The operational semantics is based on the notion of structural congruence:

\[
P_1 = P_2 \quad \text{if} \quad (\forall v) \quad (\forall s) \quad P_1[s/v] = P_2[s/v] = P
\]

where \(P\) is a process term.

\[
(\forall s) P = \text{true} \quad \text{if} \quad (\forall s') P[s'/s] = \text{true}
\]

The operational semantics provides the opportunity of uniquely identifying each individual transition. To do this, we use the transition labels attached to each execution step. The semantics of our probabilistic calculus is presented in Table 2.

| Rule | Description |
|------|-------------|
| \(s[r_1][r_2] \oplus_{j \in J} \langle p_j : l_j(v_j); P_j \rangle \mid s[r_2][r_1] \&_{i \in I} \{l_i(x_i); P_i\} \) | \(\frac{(r_1,r_2,l_k)}{p_k} P_k \mid P'_k \{\tilde{v}_k/\tilde{x}_k\}\) (COMM) |
| \(\text{def } X(\tilde{x}) = P \in (X(\tilde{v}) \mid Q) \) | \(\tilde{x}_1\) \(\text{def } X(\tilde{x}) = P \in (P[\tilde{v}/\tilde{x}] \mid Q)\) (CALL) |
| \(P \xrightarrow{\ell_p} P'\) implies \(E[P] \xrightarrow{\ell} E[P']\) (where \(p' = p \cdot \text{nextProc}(P)\)/\(\text{nextProc}(E(P))\)) | (CTXT) |
| \(P = P'\) and \(P' \xrightarrow{\ell} Q'\) and \(Q \equiv Q'\) implies \(P \xrightarrow{\ell} Q\) | (STRUCT) |

Table 2: Operational semantics of the probabilistic multiparty session calculus

Rule (COMM) models the communication between two roles \(p\) and \(q\) by using a branching process and a probabilistic selection process. If the process \(s[r_1][r_2] \oplus_{j \in J} \langle p_j : l_j(v_j); P_j \rangle\) chooses a branch with label \(l_k\) and probability \(p_k\), then the corresponding branch with label \(l_k\) is chosen also in process \(s[r_2][r_1] \&_{i \in I} \{l_i(x_i); P_i\}\). To identify which branch was selected, we use the transition label \((r_1,r_2,l_k)\). The first process continues as \(P_k\), while the second one as \(P'_k\{\tilde{v}_k/\tilde{x}_k\}\) obtained by using the communicated value \(\tilde{v}_k\) and substituting it for the existing variable \(\tilde{x}_k\). It is worth noting that it is possible that the options offered by the selection process are fewer than the ones offered by the branching process because the last one has to take into account all the possible continuations (depending on different scenarios). However, all the options of the selection set \(J\) should be present in the branching set \(I\) (expressed by the condition \(J \subseteq I\)) in order to avoid choosing a branch with no correspondence.

Rule (CALL) instantiates a process call by using its definition and replacing its formal parameters \(\tilde{x}\) with the actual ones \(\tilde{v}\). The side condition of this rule ensures that the number of variables of the definition is equal with the number of variables appearing in the process call. Since the continuation is unique, we should not mention what it was executed, and so it is used the transition label \(\varepsilon\).

Rule (CTXT) states that the transition relation is closed under structural congruence.

Rule (STRUCT) states that the transition can happen under restriction, process definition and parallel composition; the transition label \(tl\) stands for either a tuple of the form \((r_1,r_2,l_j)\) or \(\varepsilon\). The new value \(p'\) of the probability attached to the reduction arrow is a normalization of the old probability value \(p\) with respect to the number of rules (COMM) and (CALL) that can be applied. To compute the number of possible rules (COMM) and (CALL) that can be applied to a process \(P\), we use the \(\text{nextProc}\) function defined as follows:

\[
\text{nextProc}(P) = \begin{cases} 
1 + \text{nextProc}(P') & \text{if } P = s[r_1][r_2] \oplus_{j \in J} \langle p_j : l_j(v_j); P_j \rangle \\
\text{nextProc}(P) + \text{nextProc}(P') & \text{if } P = ((\forall s') P') \mid P'' \\
1 + \text{nextProc}(\text{def } D \text{ in } P') \mid P'' & \text{if } P = (\text{def } D \text{ in } (X(\tilde{v}) \mid P')) \mid P'' \\
\text{nextProc}(P') \mid P'' & \text{if } P = (\text{def } D \text{ in } P) \mid P'' \\
0 & \text{otherwise}
\end{cases}
\]

where \(D := X(\tilde{x}) = Q\)
3 Global and Local Types

Aiming to represent the available knowledge more accurately, we use imprecise probability given in terms of lower and upper probabilities to represent and quantify uncertainty in multiparty session types. Since in the processes presented in Section 2 the probabilities are static, the global types are used to check whether the probabilities of executing actions are in the intervals indicated by the imprecise probabilities.

The global types $G, G’, \ldots$ presented in Table 3 describe the global behaviour of a probabilistic multiparty session process. We use the imprecise probability $\delta$ having the form $[a_1, a_2]$ in which $a_1, a_2 \in [0, 1]$ and $a_1 \leq a_2$. If $\delta = [a, a]$, we use the shorthand notation $\delta = a$.

| Global              | $G ::= r \rightarrow r_2 : \{\delta_i : l_i(S_i).G_i\}_{i \in I}$ (interaction with $I \neq \emptyset$) |
|---------------------|---------------------------------------------------------------------------------------------------|
|                     | $\mu t.G$ (recursive with $G \neq t$)                                                             |
|                     | $t$ (variable)                                                                                     |
|                     | end (end)                                                                                           |

Table 3: Global types of the session typing

Type $r_1 \rightarrow r_2 : \{\delta_i : l_i(S_i).G_i\}_{i \in I}$ states that a participant with role $r_1$ sends with a probability belonging to the imprecise probability $\delta$, a message of type $S_i$ to a participant with role $r_2$ by using label $l_i$; then we have the interactions described by $G_i$. Each $l_i$ is unique, as it was already assumed when defining the processes. Type $\mu t.G$ is recursive, where type variable $t$ is guarded in the standard way (it appears only under a prefix).

Example 2. The following global type formalizes the PhoneNoA session of Example 1:

$$G_A = \mu t.B \rightarrow r_A : \begin{cases} 
\delta_1 : \text{talk(string)}.r_A \rightarrow r_B : \begin{cases} 
\delta_2 : \text{yes(string)}.t, \\
\delta_3 : \text{no(string)}.t, \\
\delta_4 : \text{unsure(string)}.t, \\
\delta_5 : \text{quit(string)}.\text{end}
\end{cases} \\
\delta_6 : \text{quit(string)}.\text{end}
\end{cases}.$$ 

If for all $i \in \{1, \ldots, 6\}$ it holds that $\delta_i = [0, 1]$, then the probabilities in the processes used in Example 7 fulfilling the global type $G_A$ are useless. This means that any probability used in the PhoneNoA session of Example 7 does not affect the fulfilling of global type $G_A$ (they are irrelevant for the global type). However, if $\delta_6 = [0.95, 1]$, then this implies that Bob (the person in charge of performing the poll) prefers to avoid talking with other persons in order to complete the poll (as required). This would lead to a sampling and coverage errors. This kind of behaviour should not be allowed by imposing $\delta_6 = [0, 0.1]$, meaning that the chance for Bob to stop the polling is minimal. To obtain the imprecise probabilities of the global types, we just need to subtract and add the error to a sample mean; these values can be obtained by using statistical methods.

The local types $T, T', \ldots$ presented in Table 4 describe the local behaviour of processes; they also represent a connection between the global types and processes of our calculus.

| Local               | $T ::= r \oplus_{i \in I} \delta_i : !l_i(S_i).T_i$ (selection towards role $p$, $I \neq \emptyset$) |
|---------------------|---------------------------------------------------------------------------------------------------|
|                     | $r \&_{i \in I} ? l_i(S_i).T_i$ (branching from role $p$, $I \neq \emptyset$)                  |
|                     | $\mu t.T$ (recursive with $T \neq t$)                                                            |
|                     | $t$ (variable)                                                                                     |
|                     | end (end)                                                                                           |

Table 4: Local types of the session typing

The selection $r \oplus_{i \in I} \delta_i : !l_i(S_i).T_i$ describes a channel that can choose a label $l_i$ with a probability belonging to the imprecise probability $\delta_i$ (for any $i \in I$), and send it to $r$ together with a variable of type $S_i$; then the channel must be used as described by $T_i$. The branching type $r \&_{i \in I} ? l_i(S_i).T_i$ describes a channel that can receive a label $l_i$ from role $r$ (for some $i \in I$, chosen by $r$), together with a variable of type $S_i$; then the channel must be used as described in $T_i$. The labels $l_i$ of selection and branching types are all distinct, and their order is irrelevant. The recursive type
When none of the side conditions hold, the projection is undefined. The set \( \{ \}
\) defines now the projection of a global type to a local type for each participant. The type assignment system for processes is given in Table 5, where the typing system uses a map from shared names to their sorts (\( \Delta \)) domains contain disjoint sets of session channels).

**Definition 1.** The projection \( G \upharpoonright r \) for a participant with role \( r \) appearing in a global type \( G \) is inductively defined as:

\[
\begin{align*}
\cdot (r_1 \rightarrow r_2 : \{ \delta_i : l_i(S_i), G_i \}_{i \in I} \upharpoonright r) &= \begin{cases} 
  r_1 \otimes \delta \upharpoonright l_i(S_i) \cdot G_1 \upharpoonright r & \text{if } r = r_1 \neq r_2 \\
  r_2 \& \delta \upharpoonright ?l_i(S_i) \cdot G_i \upharpoonright r & \text{if } r = r_2 \neq r_1 \\
  G_1 \upharpoonright r & \text{if } r \neq r_1 \text{ and } r \neq r_2 \\
  \forall i, j \in J, G_i \upharpoonright r = G_j \upharpoonright r 
\end{cases}
\end{align*}
\]

**Definition 2.** Consider a set of probabilities \( \{ \delta_i \}_{i \in I} \), where the imprecise probabilities have the form \( \delta_i = [d_{i1}; d_{i2}] \). The set \( \{ \delta_i \}_{i \in I} \) is called proper if

\[
\sum_{i \in I} d_{i1} \leq 1 \leq \sum_{i \in I} d_{i2}
\]

while the set \( \{ \delta_i \}_{i \in I} \) is called reachable if

\[
(\sum_{i \neq j} d_{i1} + d_{i2}) \leq 1 \leq (\sum_{i \neq j} d_{i2}) + d_{i1}
\]

By defining a set to be proper and reachable, we avoid cases in which there does not exist \( p_i \in \delta_i \) for all \( i \in I \) such that \( \sum_{i \in I} p_i = 1 \). More details about the imprecise probabilities and operations on them can be found in [15].

From now on we assume that all the global types are well-formed, i.e. \( G \upharpoonright r \) is defined for all roles \( r \) occurring in \( G \), and all the types with imprecise probabilities contain reachable sets of imprecise probabilities.

**Example 3.** Consider the global type \( G_A \) of Example 2 in which for all \( i \in \{1, \ldots, 5\} \) it holds that \( \delta_i = [0, 1] \), and also \( \delta_6 = [0.95, 1] \). According to Definition 2 \( G_A \upharpoonright r \) is defined for all roles. Moreover, according to the above definitions, the set \( \{ \delta_i \}_{i \in \{2, 3, 4, 5\}} \) is proper and reachable, and the set \( \{ \delta_i \}_{i \in \{1, 6\}} \) is proper but not reachable.

## 4 Probabilistic Multiparty Session Types

Here we show how to connect the probabilistic processes of Section 2 to the global and local types using imprecise probabilities presented in Section 3. For this, we introduce sortings and typings with the purpose of defining types for probabilistic behaviours:

\[
\Gamma := \emptyset \mid \Gamma, x : S \mid \Gamma, X : S^T \quad \text{and} \quad \Delta := \emptyset \mid \Delta, \{s[r_i] : T_i\}_{i \in I}
\]

The typing system uses a map from shared names to their sorts \( (S, S', \ldots) \). A sorting \( (\Gamma, \Gamma', \ldots) \) is a finite map from names to sorts, and from process variables to sequences of sorts and types. A typing \( (\Delta, \Delta', \ldots) \) records linear usage of session channels. Given two typings \( \Delta \) and \( \Delta' \), their disjoint union is denoted by \( \Delta \uplus \Delta' \) (by assuming that their domains contain disjoint sets of session channels).

The type assignment system for processes is given in Table 5 where \( pid(G) \) denotes the set of participants in \( G \). We use the judgement \( \Gamma \vdash P \triangleright \Delta \) saying that “under the sorting \( \Gamma \), process \( P \) has typing \( \Delta \)” (TVAR) and (TVAL) are the rules for typing variables and values. (TSELECT) and (TBRANCH) are the rules for typing selection and branching, respectively. As these rules contain probabilities, the rules should check all the possible choices with respect to \( \Gamma \). These two rules state that selection (branching) process is well-typed if \( c[r] \) has a compatible selection (branching) type, and the continuations \( P_i \) (for all \( i \in I \)) are well-typed with respect to the session types. Rule (TCONE) composes two processes if their local types are disjoint. Rule (TEND) is standard: “\( \Delta \text{ end only} \)” means that \( \Delta \) contains only end as session types. (TRES) is the restriction rule for session names and claims that \( (vs)P \) is well-typed in \( \Gamma \); this
We consider that these annotations are natural for our framework. For typing annotated processes, we assume the following substitution and weakening results.

Given an annotated process, we need a normalization of the probability attached to the transition arrow in order to take into account the spawning of subprocesses. We use the same notation $\Gamma \vdash P \triangleright \Delta$ for the typing of $P$ in context $\Gamma$. If there exists a typing $\Delta$ out of $P$ and $\Gamma$, the algorithm aborts when a constraint in a rule is violated. Note that if there exists a derivation for $P$ under $\Gamma$ in the typing system, then the construction of the algorithm is possible. Therefore, the algorithm gives a decidable procedure for the typability of annotated processes.

**Theorem 1.** Given an annotated process $P$ and a sorting $\Gamma$, it is decidable whether there is a typing $\Delta$ such that $\Gamma \vdash P \triangleright \Delta$. If such a typing $\Delta$ exists, there is an algorithm to build one.

**Proof.** The annotated rules obtained from the rules of Table 5 are used to construct the typing of $P$ under $\Gamma$. If the construction does not succeed at some point, the algorithm fails. We use the same notation $\Gamma \vdash P \triangleright \Delta$ to denote the construction of $\Delta$ out of $P$ and $\Gamma$. The algorithm aborts when a constraint in a rule is violated. Note that if there exists a derivation for $P$ under $\Gamma$ in the typing system, then the construction of the algorithm is possible. Therefore, the algorithm gives a decidable procedure for the typability of annotated processes.

Since the processes interact, their dynamics is formalized as in [10] by a labelled type reduction relation $\triangleright$ on typing $\Delta$ given by the following two rules:

- $s[r_1] : r_2 \oplus_{i \in I} d_i : l_i(S_i)T_i, s[r_2] : r_1 \&_{i \in I} ?l_i(S_i)T_i' \xrightarrow{(r_1, r_2, l_k)} \delta_k s[r_1] : T_k, s[r_2] : T'_k$ for $k \in I$;
- $\Delta, \Delta' \xrightarrow{\delta} \Delta, \Delta''$ whenever $\Delta' \nmid \Delta''$.

The first rule corresponds to sending/receiving a value of type $S_k$ by using label $l_k$ on session channel $s$ by the participant $r_2$. The second rule is used to compose typings when only a part of a typing is changing. Just like for process, we need a normalization of the probability attached to the transition arrow in order to take into account the components of the typing.

We are able to prove now that this probabilistic typing system is sound, namely its term checking rules admit only terms that are valid with respect to the structural congruence and operational semantics. **Subject reduction** ensures that the type of an expression is preserved during its evaluation. For the proof of the subject reduction, we need the following substitution and weakening results.

**Lemma 1.**

- (substitution) $\Gamma, x : S \vdash P \triangleright \Delta$ and $\Gamma \vdash v : S$ imply $\Gamma \vdash P[v/x] \triangleright \Delta$. 

Table 5: Typing system for processes in probabilistic multiparty sessions

| Rule | Description |
|------|-------------|
| $\Gamma, x : S \vdash x : S$ | (TVAR), (TVAL) |
| $\forall i, \Gamma \vdash v_i : S_i$ | (TGEN) |
| $\Gamma \vdash e[r] \oplus_{i \in I} (p_i : l_i(v_i); P_i) \triangleright \Delta, c : T_i$ | (TSELECT) |
| $\forall i, \Gamma, x_i : S_i \vdash P_i \triangleright \Delta, c : T_i$ | (TBRANCH) |
| $\Delta \text{ end only}$ | (TEND), (TCONC) |
| $\Gamma \vdash 0 \triangleright \Delta$ | |
| $\Gamma \vdash P \triangleright \Delta$ | |
| $\Gamma \vdash Q \triangleright \Delta'$ | |
| $\Gamma \vdash P | Q \triangleright \Delta, \Delta'$ | |
| $\Gamma \vdash (\nu s)P \triangleright \Delta$ | (TRES) |
| $\Gamma \vdash \text{def } X(\bar{x}_1 \ldots c_n) = P \text{ in } Q \triangleright \Delta$ | (TDEF) |
| $\Gamma \vdash \vec{v} : \bar{S}$ | (TCALL) |
| $\Gamma, X : \bar{S}T_1 \ldots T_n \vdash P \triangleright c_1 : T_1, \ldots, c_n : T_n$ | |

We consider that these annotations are natural for our framework. For typing annotated processes, we assume the obvious updates for the rules of Table 5. For instance, in the annotated rule obtained from rule (TR), happens only if the types $T_i$ of the session $s$ are exactly the projections of the same global type $G$ into all participants of $G$. (TDEF) says that a process definition $\text{def } X(\bar{x}_1 \ldots c_n) = P$ in $Q$ is well-typed if both $P$ and $Q$ are well-typed in their typing contexts. Rule (TCALL) says that a process call $X(\bar{v}_1 \ldots c_n)$ is well-typed if the actual parameters $\bar{v}_1 \ldots c_n$ have compatible types with respect to $X$. 

$$\Gamma, x : S \vdash x : S \quad \Gamma \vdash v : S$$ (TVAR), (TVAL)

$$\forall i, \Gamma \vdash v_i : S_i \quad \Gamma \vdash P_i \triangleright \Delta, c : T_i \quad \sum_{i \in I} p_i = 1 \quad p_i \in \delta_i$$ (TSELECT)

$$\Gamma \vdash e[r] \oplus_{i \in I} (p_i : l_i(v_i); P_i) \triangleright \Delta, c : r \oplus_{i \in I} \delta_i : l_i(S_i)T_i$$ (TBRANCH)

$$\Delta \text{ end only} \quad \Gamma \vdash P \triangleright \Delta \quad \Gamma \vdash Q \triangleright \Delta'$$ (TEND), (TCONC)

$$\Gamma \vdash 0 \triangleright \Delta$$ (TRES)

$$\Gamma \vdash \text{def } X(\bar{x}_1 \ldots c_n) = P \text{ in } Q \triangleright \Delta$$ (TDEF)

$$\Gamma \vdash \vec{v} : \bar{S} \quad \Delta \text{ end only}$$ (TCALL)

Table 5: Typing system for processes in probabilistic multiparty sessions

As in [10], an annotated process $P$ is the result of annotating the bound names of $P$, e.g., $(\nu s : G)P$ and $s?(x : S)P$. We consider that these annotations are natural for our framework. For typing annotated processes, we assume the following updates for the rules of Table 5. For instance, in the annotated rule obtained from rule (TRES) for typing $(\nu s : G)P$, the set $\{s[r_i] : T_i\}_{i \in I}$ is obtained from projecting the type $G$ of $s$ such that $T_i$ is the projection of $G$ onto $r_i$ for all $i \in I$. It is worth mentioning that some rules which do not involve variables are the same as the ones of Table 5.

**Theorem 1.** Given an annotated process $P$ and a sorting $\Gamma$, it is decidable whether there is a typing $\Delta$ such that $\Gamma \vdash P \triangleright \Delta$. If such a typing $\Delta$ exists, there is an algorithm to build one.

**Proof.** The annotated rules obtained from the rules of Table 5 are used to construct the typing of $P$ under $\Gamma$. If the construction does not succeed at some point, the algorithm fails. We use the same notation $\Gamma \vdash P \triangleright \Delta$ to denote the construction of $\Delta$ out of $P$ and $\Gamma$. The algorithm aborts when a constraint in a rule is violated. Note that if there exists a derivation for $P$ under $\Gamma$ in the typing system, then the construction of the algorithm is possible. Therefore, the algorithm gives a decidable procedure for the typability of annotated processes.

Since the processes interact, their dynamics is formalized as in [10] by a labelled type reduction relation $\triangleright$ on typing $\Delta$ given by the following two rules:

- $s[r_1] : r_2 \oplus_{i \in I} d_i : l_i(S_i)T_i, s[r_2] : r_1 \&_{i \in I} ?l_i(S_i)T_i' \xrightarrow{(r_1, r_2, l_k)} \delta_k s[r_1] : T_k, s[r_2] : T'_k$ for $k \in I$;
- $\Delta, \Delta' \xrightarrow{\delta} \Delta, \Delta''$ whenever $\Delta' \nmid \Delta''$.

The first rule corresponds to sending/receiving a value of type $S_k$ by using label $l_k$ on session channel $s$ by the participant $r_2$. The second rule is used to compose typings when only a part of a typing is changing. Just like for process, we need a normalization of the probability attached to the transition arrow in order to take into account the components of the typing.

We are able to prove now that this probabilistic typing system is sound, namely its term checking rules admit only terms that are valid with respect to the structural congruence and operational semantics. **Subject reduction** ensures that the type of an expression is preserved during its evaluation. For the proof of the subject reduction, we need the following substitution and weakening results.

**Lemma 1.**

- (substitution) $\Gamma, x : S \vdash P \triangleright \Delta$ and $\Gamma \vdash v : S$ imply $\Gamma \vdash P[v/x] \triangleright \Delta$. 


Why by inverting a rule, we can describe how the (sub)processes of a well-typed process are typed. This is a basic Theorem 2

Proof. The proofs are rather standard, following those presented in [10].

In our setting, at most one typing rule can be applied for any arbitrary given well-typed process. This is the reason why, by inverting a rule, we can describe how the (sub)processes of a well-typed process are typed. This is a basic property used in several papers when reasoning on induction on the structure of processes (for instance, [10][14][16]).

Theorem 2 (type preservation under equivalence), \( \Gamma \vdash P \triangleright \Delta \) and \( P \equiv P' \) imply \( \Gamma \vdash P' \triangleright \Delta \).

Proof. The proof is by induction on \( \equiv \), showing (in both ways) that if one side has a typing, then the other side has the same typing.

• Case \( P | 0 \equiv P \).

\[ \Rightarrow \text{Assume } \Gamma \vdash P | 0 \triangleright \Delta. \text{ By inverting the rule (TCONC), we obtain } \Gamma \vdash P \triangleright \Delta_1 \text{ and } \Gamma \vdash 0 \triangleright \Delta_2, \text{ where } \Delta_1, \Delta_2 = \Delta. \text{ By inverting the rule (TEND), } \Delta_2 \text{ is only end and } \Delta_2 \text{ is such that } dom(\Delta_1) \cap dom(\Delta_2) = \emptyset. \text{ Then, by type weakening, we get that } \Gamma \vdash P \triangleright \Delta, \text{ where } \Delta = \Delta_1, \Delta_2. \]

\[ \Leftarrow \text{Assume } \Gamma \vdash P \triangleright \Delta. \text{ By rule (TEND), it holds that } \Gamma \vdash 0 \triangleright \Delta', \text{ where } \Delta' \text{ is only end and } dom(\Delta) \cap dom(\Delta') = \emptyset. \text{ By applying the rule (TCONC), we get } \Gamma \vdash P | 0 \triangleright \Delta, \text{ and for } \Delta' = \emptyset \text{ we obtain } \Gamma \vdash P \triangleright \Delta, \text{ as required.} \]

• Case \( (\nu s)0 \equiv 0 \).

\[ \Rightarrow \text{Assume } \Gamma \vdash (\nu s)0 \triangleright \Delta. \text{ By inverting the rule (TRES), we get } \Gamma \vdash 0 \triangleright \Delta, \{s[r_i] : T_i\}_{i \in I} \text{ where } pid(G) = |I| \text{ and } \forall i. G | r_i = T_i. \text{ By inverting the rule (TEND), } \Delta, \{s[r_i] : T_i\}_{i \in I} \text{ is only end, namely } \Delta \text{ is only end. Using the rule (TRES), we get } \Gamma \vdash 0 \triangleright \Delta. \]

\[ \Leftarrow \text{Assume } \Gamma \vdash 0 \triangleright \Delta. \text{ By rule (TRES), it holds that } \Gamma \vdash (\nu s)0 \triangleright \Delta_1 \text{ where } pid(G) = |I|, \forall i. G | r_i = \text{end}, \Delta_2 = \{s[r_i] : \text{end}\}_{i \in I} \text{ and } \Delta = \Delta_1, \Delta_2. \text{ Then, by type weakening, we get that } \Gamma \vdash (\nu s)0 \triangleright \Delta, \text{ as required.} \]

• Case \( P \mid Q \equiv P \mid R \).

\[ \Rightarrow \text{Assume } \Gamma \vdash P \mid Q \triangleright \Delta. \text{ By inverting the rule (TCONC), we obtain } \Gamma \vdash P \triangleright \Delta_1 \text{ and } \Gamma \vdash Q \triangleright \Delta_2, \text{ where } \Delta_1, \Delta_2 = \Delta. \text{ Using the rule (TCONC), we get } \Gamma \vdash Q \mid P \triangleright \Delta. \]

\[ \Leftarrow \text{In a similar way (actually symmetric to } \Rightarrow). \]

• Case \( (P \mid Q) | R \equiv P | (Q | R) \).

\[ \Rightarrow \text{Assume } \Gamma \vdash (P \mid Q) | R \triangleright \Delta. \text{ By inverting the rule (TCONC), we obtain } \Gamma \vdash P \triangleright \Delta_1, \Gamma \vdash Q \triangleright \Delta_2 \text{ and } \Gamma \vdash R \triangleright \Delta_3, \text{ where } \Delta_1, \Delta_2, \Delta_3 = \Delta. \text{ Using the rule (TCONC), we get } \Gamma \vdash P \mid (Q | R) \triangleright \Delta. \]

\[ \Leftarrow \text{In a similar way (actually symmetric to } \Rightarrow). \]

• Case \( (\nu s)(\nu s')P \equiv (\nu s')(\nu s)P \).

\[ \Rightarrow \text{Assume } \Gamma \vdash (\nu s)(\nu s')P \triangleright \Delta. \text{ By inverting the rule (TRES) twice, we obtain } \Gamma \vdash P \triangleright \Delta, \{s[r_i] : T_i\}_{i \in I}, \{s'[r_i'] : T_i'\}_{i \in I'} \text{ where } pid(G) = |I|, \forall i. G | r_i = T_i, \text{ pid}(G') = |I'| \text{ and } \forall i'. G' | r_i' = T_i'. \text{ Using the rule (TRES), we get } \Gamma \vdash (\nu s')(\nu s)P \triangleright \Delta. \]

\[ \Leftarrow \text{In a similar way (actually symmetric to } \Rightarrow). \]

• Case \( (\nu s)P | Q \equiv (\nu s)(P | Q) \text{ (if } s \not\in fc(Q)) \).

\[ \Rightarrow \text{Assume } \Gamma \vdash (\nu s)P | Q \triangleright \Delta. \text{ By inverting the rule (TCONC), we obtain } \Gamma \vdash (\nu s)P \triangleright \Delta_1 \text{ and } \Gamma \vdash Q \triangleright \Delta_2, \text{ where } \Delta_1, \Delta_2 = \Delta. \text{ By inverting the rule (TRES), we obtain } \Gamma \vdash P \triangleright \Delta_1, \{s[r_i] : T_i\}_{i \in I} \text{ where } pid(G) = |I| \text{ and } \forall i. G | r_i = T_i. \text{ Using rule (TCONC), we get } \Gamma \vdash P \mid Q \triangleright \Delta_1, \Delta_2, \{s[r_i] : T_i\}_{i \in I} \text{ where } pid(G) = |I| \text{ and } \forall i. G | r_i = T_i, \text{ by using rule (TRES), we get } \Gamma \vdash (\nu s)(P | Q) \triangleright \Delta, \text{ where } \Delta_1, \Delta_2 = \Delta. \]
The following result is known also as ‘Subject Reduction’. According to this result, if a well-typed process takes a transition step of any kind, the resulting process is also well-typed.
Theorem 3 (type preservation under evolution).

\[ \Gamma \vdash P \triangleright \Delta \text{ and } P \overset{\triangleright}{\rightarrow} P' \text{ imply } \Gamma \vdash P' \triangleright \Delta', \text{ where } \Delta' = \Delta \text{ or } \Delta \overset{\triangleright}{\rightarrow} \Delta' \text{ with } p \cdot \text{nextProc}(P) \in \delta. \]

**Proof.** By induction on the derivation of \( P \overset{\triangleright}{\rightarrow} P' \). There is a case for each operational semantics rule, and for each operational semantics rule we consider the typing system rule generating \( \Gamma \vdash P \triangleright \Delta \).

- **Case (COM):** \( s[1]_{1} \otimes_{j\in J} l_j(v_j); P_j \mid s[2]_{1} \otimes_{k\in K} l_k(x_k); P'_k \mid \overset{r_1 \cdot r_2 \cdot \ldots \cdot r_k}{\triangleright} \quad P_1 \mid P_2 \mid \ldots \mid P_k \mid \overset{\tilde{v}/\tilde{x}}{\triangleright} \Delta' \).

  By assumption, \( \Gamma \vdash s[1]_{1} \otimes_{j\in J} l_j(v_j); P_j \mid s[2]_{1} \otimes_{k\in K} l_k(x_k); P'_k \triangleright \Delta \). By inverting the rule (ACONC), we get \( \Gamma \vdash s[1]_{1} \otimes_{j\in J} l_j(v_j); P_j \mid s[2]_{1} \otimes_{k\in K} l_k(x_k); P'_k \triangleright \Delta_1 \) and \( \Gamma \vdash s[2]_{1} \otimes_{k\in K} l_k(x_k); P'_k \triangleright \Delta_2 \) where \( \Delta = \Delta_1, \Delta_2 \). Since these can be inferred only from (TSELECT) and (BREACH), we know that \( \Delta_1 = \Delta'_1, s[1]_1 \otimes_{j\in J} l_j(S_1)_T \) and \( \Delta_2 = \Delta'_2, s[2]_1 \otimes_{k\in K} l_j(S_1)_T \). By inverting the rules (TSELECT) and (BREACH), we get that \( \forall i, \Gamma \vdash v_i \colon S_i, \forall i, \Gamma \vdash P_i \triangleright \Delta'_1, s[1]_1 \otimes_{j\in J} l_j(S_1)_T \) and \( \forall i, \Gamma \vdash x_i \colon S_i, \forall i, \Gamma \vdash P'_i \triangleright \Delta'_2, s[2]_1 \otimes_{k\in K} l_j(S_1)_T \). From \( \Gamma \vdash P_i \triangleright \Delta'_1, s[1]_1 \otimes_{j\in J} l_j(S_1)_T \), by applying the substitution part of Lemma [1], we get \( \Gamma \vdash P'_i \triangleright \Delta'_2, s[2]_1 \otimes_{k\in K} l_j(S_1)_T \). By applying the rule (TRCONC), we obtain \( \Gamma \vdash P \mid P \mid v_i/x_i \triangleright \Delta_1, s[1]_1 \otimes_{j\in J} l_j(S_1)_T \), \( \Delta_2, s[2]_1 \otimes_{k\in K} l_j(S_1)_T \). By using the type reduction rule, we get that \( \Delta \overset{r_1 \cdot r_2 \cdot \ldots \cdot r_k}{\triangleright} \Delta' \), where \( \Delta' = \Delta'_1, s[1]_1 \otimes_{j\in J} l_j(S_1)_T \), \( \Delta'_2, s[2]_1 \otimes_{k\in K} l_j(S_1)_T \) and \( p_k \in \delta_k \).

- **Case (CALL):** \( \text{def } X(\tilde{x}) = R \text{ in } (X(\tilde{v}) \otimes Q) \overset{\tilde{x}}{\triangleright} \text{def } X(\tilde{x}) = R \text{ in } (R \{ \tilde{v}/\tilde{x} \}) \).

  By assumption, \( \Gamma \vdash X(\tilde{x}) = R \text{ in } (X(\tilde{v}) \otimes Q) \triangleright \Delta \). By inverting rule (TDEF), we obtain \( \Gamma \vdash X(\tilde{x}) = R \text{ in } (X(\tilde{v}) \otimes Q) \triangleright \Delta \). By inverting the rule (TRCONC), we obtain \( \Gamma \vdash X \triangleright \Delta \) where \( \Delta = \Delta_1, s[1]_1 \otimes_{j\in J} l_j(S_1)_T \). By inverting the rule (TCALL), we get \( \Gamma \vdash \tilde{v} \colon \tilde{x}, \Delta_1 \triangleright \Delta_1 \). Applying substitution (Lemma [1]), we get \( \Gamma \vdash \tilde{v} \colon \tilde{x}, \Delta_1 \triangleright \Delta_1 \). As \( \Delta_1 \triangleright \Delta_1 \) is end only, then by type weakening (Lemma [1]) we get that \( \Gamma \vdash X \colon \\tilde{v} \colon \tilde{x}, \Delta_1 \triangleright \Delta_1 \). Applying rule (TRCONC), we get \( \Gamma \vdash X \colon \\tilde{v} \colon \tilde{x}, \Delta_1 \triangleright \Delta_1 \). Applying rule (TDEF), we obtain \( \Gamma \vdash \text{def } X(\tilde{x}) = R \text{ in } (R \{ \tilde{v}/\tilde{x} \}) \triangleright \Delta \), as desired.

- **Case (CTX):** \( P \overset{\triangleright}{\rightarrow} P' \) implies \( E[P] \overset{\triangleright}{\rightarrow} E[P'] \), where \( p' = p \cdot \text{nextProc}(P) \) where \( p' = p \cdot \text{nextProc}(P) \).

  Based on the form of the context we got several cases.

  - **Case \( E = [\nu s] \) in \( [\nu s] \).** By assumption, \( \Gamma \vdash (\nu s)P \triangleright \Delta \). By inverting the rule (TRES), we get that \( \Gamma \vdash (\nu s)P \triangleright \Delta \), \( s[1]_1 \otimes_{j\in J} l_j(S_1)_T \), where \( \Delta = \Delta \triangleright \Delta' \) and \( p \cdot \text{nextProc}(P) \in \delta \). By using the rule (TRES), we obtain \( \Gamma \vdash (\nu s)P \triangleright \Delta \), \( \Delta = \Delta \triangleright \Delta' \) and \( p \cdot \text{nextProc}(P) \in \delta \), as required.

  - **Case \( E = \text{def } D \) in \( [\text{def } D] \).** By assumption, \( \Gamma \vdash \text{def } D \triangleright \Delta \). By inverting rule (TDEF), we obtain \( \Gamma \vdash (\text{def } D) \triangleright \Delta \), \( s[1]_1 \otimes_{j\in J} l_j(S_1)_T \), where \( \Delta = \Delta \triangleright \Delta' \) and \( p \cdot \text{nextProc}(P) \in \delta \). By using the rule (TDEF), we obtain \( \Gamma \vdash (\text{def } D) \triangleright \Delta \), \( \Delta = \Delta \triangleright \Delta' \) and \( p \cdot \text{nextProc}(P) \in \delta \), as required.

- **Case \( E = [\nu s] \) in \( [\nu s] \).** By assumption, \( \Gamma \vdash P \triangleright \Delta \). By inverting rule (TCONC), we get \( \Gamma \vdash (\nu s)P \triangleright \Delta \). By inverting the rule (TCONC), we get \( \Gamma \vdash P \triangleright \Delta \) and \( \Gamma \vdash P \triangleright \Delta \), \( \Delta = \Delta \triangleright \Delta' \) where \( \Delta = \Delta \triangleright \Delta' \) and \( p \cdot \text{nextProc}(P) \in \delta \). By using the rule (TCONC), we get \( \Gamma \vdash P \triangleright \Delta \), \( \Delta = \Delta \triangleright \Delta' \) and \( p \cdot \text{nextProc}(P) \in \delta \), as required.

- **Case (STRUCT):** \( P = P' \) and \( P' \overset{\triangleright}{\rightarrow} Q' \) and \( Q \equiv Q' \) implies \( P \overset{\triangleright}{\rightarrow} Q \).

  It is obtained by using the structural congruence (Theorem [2]).
stuck process). Negative premises are used to denote the fact that the passing to a new step is performed based on the absence of actions; the use of negative premises in this way does not lead to any inconsistency.

**Theorem 4** (deadlock freedom). Let $\emptyset \vdash P \triangleright \emptyset$, where $P \equiv (\nu s : G) |_{i \in I} P_i$, and each $P_i$ interacts only on $s[r_i]$ of type $T_i$, where $T_i = G \upharpoonright r_i$. Then $P$ is deadlock-free: i.e., $P \xrightarrow{\mu \xi_p} P' \not\rightarrow$ implies $P' \equiv 0$.

**Proof.** Since the probabilities do not influence the behaviour, the proof is similar to the approach presented in [13].

Lock-freedom was introduced in [17] as a property stronger than deadlock freedom, requiring that every action that can be executed will eventually be executed. Session types not only ensure deadlock-freedom, but also lock-freedom [18]. Checking if such results still hold in our framework represents future work.

As it can be noticed from the rules of Table 5, the obtained types are not unique. This is due to the fact that we use imprecise probabilities allowing the processes to be considered well-typed within the interval defined by lower and upper probabilities. Having several types for the same process, we can obtain some refinements of these typings. Inspired by [16], we use an erase function that, when applied to a type, removes its probability annotations. Considering $erase(\Delta_1) = erase(\Delta_2)$, let us define the intersection of the imprecise probabilities appearing in the typing of processes as follows:

$$\Delta_1 \cap_r \Delta_2 = \{ s[r_2] : (T_1 \cap_p T_2) | s[r_2] : T_1 \in \Delta_1 \land s[r_2] : T_2 \in \Delta_2 \}$$

such that

$$T_1 \cap_r T_2 = \begin{cases} 
    r_1 \oplus_{i \in I} (\delta_{i1} \cap \delta_{i2}) : !l_i(S_i), (T_{i1} \cap_r T_{2i}) & \text{if } T_j = r_1 \oplus_{i \in I} \delta_{ji} : !l_i(S_i), T_{ji} \\
    r_1 \&_{i \in I} ?l_i(S_i), (T_{i1} \cap_r T_{2i}) & \text{if } T_j = r_1 \&_{i \in I} ?l_i(S_i), T_{ji} \\
    \mu \xi_t(T_{i1} \cap_r T_{2i}) & \text{if } T_1 = \mu \xi_t T_{i1} \land T_2 = \mu \xi_t T_{i2} \\
    T_1 & \text{otherwise.} 
\end{cases}$$

In a similar manner, it can be defined the intersection $\Gamma_1 \cap_p \Gamma_2$ of two sortings.

For the next three results we assume that $erase(\Delta_1) = erase(\Delta_2)$ and $erase(\Gamma_1) = erase(\Gamma_2)$. It is worth noting that a process well-typed with different types still remains well-typed if we consider the intersections of all corresponding imprecise probabilities.

**Theorem 5.** If $\Gamma_1 \vdash P \triangleright \Delta_1$ and $\Gamma_2 \vdash P \triangleright \Delta_2$, then $\Gamma_1 \cap_r \Gamma_2 \vdash P \triangleright \Delta_1 \cap_r \Delta_2$.

**Proof.** The proof follows by induction on the structure of process $P$ and by using the rules of Table 5.

In what follows we associate to any execution path (starting from a process $P$ and leading to a process $Q$) an execution probability of going along this path. This is computed by using the operational semantics presented in Table 2.

**Definition 3** (evolution paths). A sequence of evolution transitions represents an evolution path $ep = P_0 \xrightarrow{tl_1} P_1 \ldots \xrightarrow{tl_k} P_k$. Two evolution paths $ep = P_0 \xrightarrow{tl_1} P_1 \ldots \xrightarrow{tl_k} P_k$ and $ep' = P_0' \xrightarrow{tl_1'} P_1' \ldots \xrightarrow{tl_k'} P_k'$ are identical if $k = k'$, and for all $t \in \{0, \ldots, k\}$ we have $P_t \equiv P'_t$, $p_i = p'_i$ and $tl_i = tl'_i$.

**Definition 4** (evolution probability). The probability to reach a process $P_j$ starting from a process $P_i$ (with $i < j$) along the evolution path $ep = P_0 \xrightarrow{tl_1} P_1 \ldots \xrightarrow{tl_k} P_k$ is denoted by $prob(ep, P_i, P_j) = p_{i+1} \ast \ldots \ast p_j$. The evolution probability to reach $P_j$ starting from a process $P_i$ (with $i < j$) taking all the possible evolution paths (without considering identical paths) is denoted by $prob(P_i, P_j) = \sum_{ep} prob(ep, P_i, P_j)$.

For any process $P$ we can define the sets $Reach_k(P)$ of processes reached in $k$ steps starting from $P$ by using the rules of Table 2.

**Definition 5** (reachable sets). $Reach_1(P) = \{ Q \mid P \xrightarrow{tl_p} Q \}$. For $k \geq 2$,

$$Reach_k(P) = \{ Q \mid Q \in Reach_{k-1}(P) \land Q \not\rightarrow \} \cup \{ R \mid Q \in Reach_{k-1}(P) \land Q \xrightarrow{tl_p} R \}.$$

The following result is based on the fact that for any process there exist various possible transitions, each occurring with a given probability. The approach is solid if for each well-typed process, the sum of these probabilities is 1.
Theorem 6. Given a well-typed process $P$ such that $\text{Reach}_k(P) \neq \emptyset$ for $k \geq 1$, then
\[ \sum_{Q \in \text{Reach}_k(P)} \text{prob}(P, Q) = 1. \]

Proof. By induction on the number $k$ of the evolution steps.

- Case $k = 1$.

  By induction on the structure of $P$, namely on the number of parallel components:
  \begin{itemize}
  \item Case $P = P_1$. We have several subcases:
    \begin{itemize}
    \item $P = \text{end}$ or $P = c[r] \oplus_{i \in I} \langle p_i : l_i(v_i); P_i \rangle$ or $P = c[r] k_{i \in I} \{ l_i(x_i); P_i \}$ or $P = (\nu s) P_1$, meaning that no rule is applicable. Hence $\text{Reach}_1(P) = \emptyset$, and the sum is not computed.
    \item $P = \text{def } X(\bar{x}) = P'$ in $(X(\bar{v}))(Q)$ meaning that the rule (CALL) is applied, and $P$ evolves with probability 1 to def $X(\bar{x}) = P'$ in $(P'(\bar{v}/\bar{v}))(Q)$. This means that $\sum_{Q \in \text{Reach}_1(P)} \text{prob}(P, Q) = 1$ (as desired).
    \item Case $P = P_1 \mid P_2$. We have several subcases:
      \begin{itemize}
      \item $P_1 = s[r_1][r_2] \oplus_{j \in J} \langle p_j : l_j(v_j); P_j \rangle$ and $P_2 = s[r_2][r_1] k_{i \in I} \{ l_i(x_i); P_i \}$ meaning that the rule (COM) is applied, and $P$ evolves with probability $p_k$ (with $k \in J$) to $P_k | P_k'(\bar{v}_k/\bar{x}_k)$. Since the process $P$ is well-typed, it follows that $\sum_{k \in J} p_k = 1$. Thus, $\sum_{Q \in \text{Reach}_1(P)} \text{prob}(P, Q) = \sum_{k \in J} p_k = 1$ (as desired).
    \end{itemize}
  \end{itemize}

The remaining cases are proved in a similar manner.

- Case $k > 1$. Since $\text{Reach}_k(P) \neq \emptyset$, this means that $\text{Reach}_{k-1}(P) \neq \emptyset$. By considering the inductive step, we obtain that $\sum_{Q \in \text{Reach}_{k-1}(P)} \text{prob}(P, Q) = 1$. Let us consider that another evolution step is performed, namely $k$ steps starting from process $P$. This means that $\sum_{Q \in \text{Reach}_k(P)} \text{prob}(P, Q)$ can be broken into two parts: one computing the probability for the first $(k - 1)$ steps, and another one using the probabilities obtained in the additional step. Thus it holds that $\sum_{Q \in \text{Reach}_k(P)} \text{prob}(P, Q) = \sum_{Q \in \text{Reach}_{k-1}(P)} \text{prob}(P, Q_1) \ast \sum_{Q \in \text{Reach}_1(Q_1)} \text{prob}(Q_1, Q)$. Since we have $\sum_{Q \in \text{Reach}_1(Q_1)} \text{prob}(Q_1, Q) = 1$, then $\sum_{Q \in \text{Reach}_k(P)} \text{prob}(P, Q) = \sum_{Q \in \text{Reach}_{k-1}(P)} \text{prob}(P, Q_1).$ Also, since we have $\sum_{Q \in \text{Reach}_{k-1}(P)} \text{prob}(P, Q_1) = 1$, then $\sum_{Q \in \text{Reach}_k(P)} \text{prob}(P, Q) = 1$, as required.

\[ \blacksquare \]

5 Conclusion and Related Work

Aiming to represent and quantify uncertainty, imprecise probability generalizes probability theory to allow for partial probability specifications applicable when a unique probability distribution is difficult to be identified. In this paper we have used imprecise probability by introducing imprecise probabilities for multiparty session types in process algebra. According to our knowledge, there is no work in computer science dealing (explicitly) with imprecise probabilities.

We use a probabilistic extension of the process calculus presented in [13] by allowing both nondeterministic external choices and probabilistic internal choices. We define probabilistic multiparty session types able to codify the structure of the communications by using imprecise probabilities given in terms of lower and upper probabilities. Moreover, we have defined and studied a typing system extending the multiparty session types with imprecise probabilities; this typing system in which the channels have specific roles in a session is inspired by the system presented in [14]. The new typing system has several properties and features. It preserves the classical typing properties; additionally, it is specified in such a way to satisfy the axioms of the standard probability theory and some properties of the imprecise probability theory. To illustrate our calculus, we considered an example inspired by probabilistic survey poll in which the probabilities are provided by a well-defined numerical scale for responses.

In [19] there are proposed two semantics of a probabilistic variant of the $\pi$-calculus. For these, the types are used to identify a class of nondeterministic probabilistic behaviours which can preserve the compositionality of the parallel operator in the framework of event structures. The authors claim to perform an initial step towards a good typing discipline for probabilistic name passing by employing Segala automata [20] and probabilistic event structures. In comparison with them, we simplify the approach and work directly with processes, giving a probabilistic typing in the context of multiparty session types. Moreover, we include the imprecise probabilities in the framework of multiparty session types.
In a different framework presented in [21], the authors study the long-run behaviour of probabilistic models in the presence of uncertainty given by lower and upper bounds.

Several formal tools have been proposed for probabilistic reasoning. Some of these approaches make use of certain probabilistic logics. In [22], terms are assigned probabilistically to types via probabilistic type judgements, and from an intuitionistic typing system is derived a probabilistic logic as a subsystem [23]. However, the existing probabilistic tools do not account for the imprecision of the probabilistic parameters in the model.

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