Comment on the paper “Bound States in the One-dimensional Hubbard Model”

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We comment on the preprint cond-mat/9805103 by D. Braak and N. Andrei. We point out that the “new” Bethe Ansatz equations presented in [1] are identical to the Bethe equations for strings introduced by M. Takahashi for the description of thermodynamics in 1972 [2]. Some physics suggested in [1] is incorrect. In particular, all former conclusions made on the basis of the string Bethe equations remain valid.

(I) First let us correct a typo in [1]. The bare S-matrix for the scattering of an unbound particle with an m-complex (formula (17) in [1]) should be

\[ S_{u(m)}^{k\phi_{0}^{(m)}} = \frac{\sin k - \phi_{0}^{(m)} - mi\nu/4}{\sin k - \phi_{0}^{(m)} + mi\nu/4} \]  

(see eq. (6.59) of [3]). Let us also mention that by convention the S-matrix in [1] gets inverted upon horizontal transposition of the indices. A straightforward calculation shows that the “new” Bethe equations of [1] ((BAC), eqs. (20)-(22) in [1]) coincide with the standard string Bethe equations [2]. This calculation involves adjustment of notations and making explicit the so-called Λ-strings, which are hidden in the notation of [1] (see appendix A). [The Λ strings are configurations of the spin-rapidities \( \lambda_i \) involving complex \( \gamma_j \). The Λ-strings have to be distinguished from the so-called \( k \)-Λ-strings, which are the subject of criticism in [1] (\( k \)-Λ-strings involve spin-rapidities \( \lambda_i \) as well as charge rapidities \( k_j \)).] Since the “new” Bethe equations in [1] coincide with the string Bethe equations, they can not lead to new physics. It is no surprise, in particular, that the counting of states mentioned in [1] gives the correct number \( 4^L \), since this counting is actually identical to the calculation of [1]. The Algebraic Bethe Ansatz expression at the start of page 2 of [1] is incorrect: the Hamiltonian does not act in this way but needs to be replaced by the inhomogeneous transfer matrix of the spin problem.

(II) Let us now demonstrate that the “new physics” discussed at the end of [1] is partially incorrect and partially known. Let us start with an incorrect statement of [1]: in the last part the holon-antiholon excitation for the half-filled repulsive Hubbard model is considered. This excitation is known in the literature as “charge singlet” (we will follow the discussion given in [5,6]). It is described by two holes in the \( k \)-Fermi sea \( (k_1^h \text{ and } k_2^h) \) and one \( k \)-Λ string. The spin parameter of the \( k \)-Λ string is denoted by \( \phi(q) \) in [1] and by \( \Lambda' \) in [3,4]. The authors of [1] state that the relation

\[ 2\Lambda' = \sin k_1^h + \sin k_2^h \]  

(2)

is not valid (remember that \( \Lambda' = \phi(q) \)). This statement is incorrect: It was shown in [3,4] that (2) follows from the string Bethe equations (which are identical to (20)-(22) alias (BAC) in [1]). Independent confirmation can be found in [4]. We repeat the derivation of (2) in detail in appendix A below: Eq. (2) holds at half-filling! This confirms the expression for the S-matrix obtained in [3,4].

Let us comment on the “new” excitation below half-filling, which is discussed in the last part of [1]. Our comments are: (i) the existence of this excitation was well known [3] and (ii) is of no particular physical significance, since it (a) has a gap and (b) is only one of an infinite number of gapful excitations below half-filling. It is correct that eq. (2) is not satisfied below half-filling and that as a result the \( k \)-Λ string is an independent excitation. However the same holds for the infinite number of longer \( k \)-Λ strings and the one-particle excitations! These excitations also exist at zero density (zero filling) [3]. It is the generic situation for Bethe Ansatz solvable models that the structure of the excitations persists, when going from a trivial to a non-trivial ground state by changing the density (more generally, an order parameter). This means in case of the Hubbard model, that all strings as well as the one-particle excitations are present at finite density below half-filling. Only their dispersion curves get modified. Similarly, for the XXX spin chain the string excitations over the ferromagnetic ground state survive, when the magnetization is diminished. Only the chain with zero magnetization has a different excitation structure. Let us emphasize that the \( k \)-Λ string below half filling has a gap! Hence, the \( k \)-Λ string does not influence the conformal dimensions [5] and does not lead to new low energy physics. The same statement holds true for all longer \( k \)-Λ strings.

(III) Next, we want to comment on eq. (7) of [1]. The authors of [1] do not require (7). They claim that this is the reason for “new” physics. All new physics they present is related to states containing one \( k \)-Λ string (and several real
$k_j$ and real $\lambda_j$). The authors of [3] do not argue the validity of their eqs. (5), (6) and (8). We show in appendix B that for one $k$-A string (and several real $k_j$ and real $\lambda_j$) eq. (7) follows from (5), (6) and (8). This shows an inconsistency in the arguments of [3] and removes the basis for the modification of the physics of the one-dimensional Hubbard model.

(IV) Finally, let us comment on the continuity argument on the bottom of the left column of the page 3 of [3]. The defect of the argument is the following: in the limit $u \to \infty$ [when the bound state (9) satisfy periodic boundary condition] the energy of the bound state is infinite, and it drops out of the Hilbert space. Continuity to finite $u$ cannot be employed to count such states.

We would like to thank H. Frahm for helpful communications.

APPENDIX A: DERIVATION OF EQUATION (2)

In this appendix we show that eqs. (20)-(22) in [3] are identical with the string Bethe equations of Takahashi and that (2) follows from these equations.

Let us first cite the main results of [3]: The $n$-complex is parameterized by an $n$-string of the form

$$\phi_{a,j}^{(n)} = \phi_a^{(n)} + (n + 1 - 2j) \frac{iu}{4}, \quad j = 1, \ldots, n. \quad (A1)$$

[Note the slight change of notation compared to [3]: We made the replacement $\phi_0^{(n)} \to \phi_a^{(n)}$ (and $\phi_j^{(n)} \to \phi_{a,j}^{(n)}$) which allows us to include scattering of two different $n$ complexes of the same length. $a$ enumerates the $n$-strings.] The S-matrix of an unbound particle with an $n$-complex is

$$S_{k \phi_a^{(n)}}^{u(n)} = \frac{\sin k - \phi_a^{(n)} - n \frac{iu}{4}}{\sin k - \phi_a^{(n)} + n \frac{iu}{4}}, \quad (A2)$$

and the S-matrix of an $m$-complex with an $n$-complex is

$$S_{\phi_b^{(m)} \phi_a^{(n)}}^{(m)(n)} = \frac{\phi_b^{(m)} - \phi_a^{(n)} - |n - m| \frac{iu}{4}}{\phi_b^{(m)} - \phi_a^{(n)} + |n - m| \frac{iu}{4}} \prod_{l=1}^{\min(m, n)-1} \left( \frac{\phi_b^{(m)} - \phi_a^{(n)} - |n - m| + 2l \frac{iu}{4}}{\phi_b^{(m)} - \phi_a^{(n)} + |n - m| + 2l \frac{iu}{4}} \right)^2. \quad (A3)$$

Diagonalizing the transfer matrix leads to the following set of equations

$$e^{ik_j L} = \prod_{\delta = 1}^{M^u} \lambda_\delta - \sin k_j - \frac{iu}{4} \prod_{(n,a)} S_{\phi_a^{(n)}}^{(n)u}, \quad (A4)$$

$$\prod_{\delta \neq \lambda}^{M^u} \lambda_\gamma - \lambda_\delta - \frac{iu}{2} = \prod_{j=1}^{N^u} \lambda_\gamma - \sin k_j - \frac{iu}{4} \quad (A5)$$

$$e^{i\mathbf{q}_{(n)}^{(n)} L} = \prod_{(m,b) \neq (n,a)} S_{\phi_b^{(m)} \phi_a^{(n)}}^{(m)(n)} \prod_{j=1}^{N^u} S_{\phi_a^{(n)}}^{u(n)} \prod_{k_j \phi_a^{(n)}} S_{\phi_a^{(n)}}^{(n)u}, \quad (A6)$$

where

$$\mathbf{q}_{(n)}^{(n)}(\phi) = -2\text{Re} \arcsin(\phi + niu/4) + (n + 1)\pi = -(\arcsin(\phi + niu/4) + \arcsin(\phi - niu/4)) + (n + 1)\pi \quad (A7)$$

and

$$S_{\phi_a^{(n)} k_j}^{(n)u} = \left( S_{k_j \phi_a^{(n)}}^{u(n)} \right)^{-1}. \quad (A8)$$

Eqs. (A4)-(A6) are the “new” equations of Braak and Andrei ((20)-(22) or (BAC) in [3]). In order to show that (A4)-(A6) agree with Takahashi’s string Bethe equations let us now adjust notations. Let

$$U = u/4, \quad \Lambda_{\alpha \beta}^{(m)} = \phi_{\alpha \beta}^{(m)}, \quad \alpha = a, \quad \beta = b. \quad (A9)$$
Let us further introduce the functions \( e(x) = \frac{x + 1}{x - 1} \)

\[
E_{nm}(x) = \begin{cases} 
  e \left( \frac{x}{|n - m|} \right) e^2 \left( \frac{x}{|n - m| + 2} \right) \cdots e^2 \left( \frac{x}{n + m - 2} \right) e \left( \frac{x}{n + m} \right) & \text{for } n \neq m, \\
  e^2 \left( \frac{x}{2} \right) e^2 \left( \frac{x}{4} \right) \cdots e^2 \left( \frac{x}{2n - 2} \right) e \left( \frac{x}{2n} \right) & \text{for } n = m. 
\end{cases}
\]  

(A10)

(A11)

Then eqs. (A4)-(A6) turn into

\[
e^{ik_jL} = \prod_{\delta=1}^{M^n} e \left( \frac{\sin k_j - \lambda_\delta}{U} \right) \prod_{(n,\alpha)} e \left( \frac{\sin k_j - \Lambda^n_{\alpha}}{nU} \right),
\]

(A12)

\[
\prod_{j=1}^{N^\nu} e \left( \frac{\lambda_j - \sin k_j}{U} \right) = -\prod_{\delta=1}^{M^n} e \left( \frac{\lambda_\delta - \lambda_\delta}{2U} \right),
\]

(A13)

\[
\exp\{-iL[\arcsin(\Lambda^n_{\alpha} + niU) + \arcsin(\Lambda^n_{\beta} - niU) + (n + 1)\pi]\}
\]

\[
= -\prod_{j=1}^{N^\nu} e \left( \frac{\Lambda^n_{\alpha} - \sin k_j}{nU} \right) \prod_{(m,\beta)} E_{nm} \left( \frac{\Lambda^n_{\alpha} - \Lambda^n_{\beta}}{U} \right).
\]

(A14)

The spin rapidities \( \lambda_j \) in (A12) and (A13) may generally be complex. In order to obtain a set of equations which contains only real unknowns and which transforms into a set of linear integral equations in the thermodynamic limit we have to employ Takahashi’s string hypothesis for \( \Lambda \) strings: As the number \( N \) of electrons becomes large the spin rapidities are driven to string positions characterized by their length \( n \) and their real center \( \Lambda^n_{\alpha} \). Following Takahashi \( ^2 \) we will use the notation \( \Lambda^n_{\alpha,j} \) instead of \( \lambda_j \). \( \Lambda^n_{\alpha,j} \) is the \( j \)-th spin rapidity involved in an \( n \)-\( \Lambda \) string with center \( \Lambda^n_{\alpha} \),

\[
\Lambda^n_{\alpha,j} = \Lambda^n_{\alpha} + (n + 1 - 2j)iU.
\]

(A15)

Following again Takahashi let us assume that in the thermodynamic limit all \( \lambda_j \) are grouped into strings with an accuracy of \( O(\exp(-\delta N)) \), where \( \delta \) is some positive number. Then eqs. (A12)-(A14) lead to

\[
e^{ik_jL} = \prod_{(n,\alpha)} e \left( \frac{\sin k_j - \Lambda^n_{\alpha}}{nU} \right) \prod_{(n,\alpha)} e \left( \frac{\sin k_j - \Lambda^n_{\alpha}}{nU} \right),
\]

(A16)

\[
\prod_{j=1}^{N^\nu} e \left( \frac{\Lambda^n_{\alpha} - \sin k_j}{nU} \right) = -\prod_{(m,\beta)} E_{nm} \left( \frac{\Lambda^n_{\alpha} - \Lambda^n_{\beta}}{U} \right),
\]

(A17)

\[
\exp\{-iL[\arcsin(\Lambda^n_{\alpha} + niU) + \arcsin(\Lambda^n_{\beta} - niU) + (n + 1)\pi]\}
\]

\[
= -\prod_{j=1}^{N^\nu} e \left( \frac{\Lambda^n_{\alpha} - \sin k_j}{nU} \right) \prod_{(m,\beta)} E_{nm} \left( \frac{\Lambda^n_{\alpha} - \Lambda^n_{\beta}}{U} \right).
\]

(A18)

Taking logarithms we arrive at the following form of the string Bethe equations, which is suitable for considering the thermodynamic limit,

\[
k_jL = 2\pi I_j - \sum_{n=1}^{M^n} \sum_{\alpha=1}^{M^n} \theta \left( \frac{\sin k_j - \Lambda^n_{\alpha}}{nU} \right) - \sum_{n=1}^{M^n} \sum_{\alpha=1}^{M^n} \theta \left( \frac{\sin k_j - \Lambda^n_{\alpha}}{nU} \right),
\]

(A19)

\[
\sum_{j=1}^{N^\nu-2M'} \theta \left( \frac{\Lambda^n_{\alpha} - \sin k_j}{nU} \right) = 2\pi J^n_{\alpha} + \sum_{m=1}^{M^n} \sum_{\beta=1}^{M^n} \Theta_{nm} \left( \frac{\Lambda^n_{\alpha} - \Lambda^n_{\beta}}{U} \right),
\]

(A20)

\[
L[\arcsin(\Lambda^n_{\alpha} + niU) + \arcsin(\Lambda^n_{\beta} - niU)] = 2\pi J^n_{\alpha} + \sum_{j=1}^{N^\nu-2M'} \theta \left( \frac{\Lambda^n_{\alpha} - \sin k_j}{nU} \right) + \sum_{m=1}^{M^n} \sum_{\beta=1}^{M^n} \Theta_{nm} \left( \frac{\Lambda^n_{\alpha} - \Lambda^n_{\beta}}{U} \right).
\]

(A21)
Eqs. (A19) and (A20) reduce to quantum numbers \( I \) characterized by \( N \) numbers in (A19)-(A21) having ranges then taking the thermodynamic limit.

Thus the ground state and all excited states can be constructed by specifying a set of numbers rapidities \( \rho \) unbound electrons is \( L \)

Here we assumed \( L = 2 \times \text{odd to be even} \). \( I_j, J_{\alpha}^n, \) and \( J_{\alpha}^{n'} \) are integer or half-odd integer numbers, \( N \) is the total number of electrons, \( M' = \sum_{n=1}^{\infty} nM'_n, \) \( \theta(x) = 2\arctan(x) \), and

\[
\Theta_{nm}(x) = \left\{ \begin{array}{ll}
\theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + \cdots + 2\theta\left(\frac{x}{n+m-2}\right) + \theta\left(\frac{x}{n+m}\right), & \text{if } n \neq m,

2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \cdots + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right), & \text{if } n = m.
\end{array} \right.
\]

(A22)

The branch of arcsin\((x)\) in (A21) is fixed as \(-\pi/2 \leq \text{Re}(\arcsin(x)) \leq \pi/2 \). \( M' \) and \( M'_n \) are the numbers of \( \Lambda \) strings of length \( n \), and \( \Lambda' \) strings of length \( n \) in a specific solution of the system (A19)-(A21). The integer (half-odd integer) numbers in (A19)-(A21) have ranges

\[
\frac{L-1}{2} \leq I_j \leq \frac{L-1}{2}, \tag{A23}
\]

\[
|J_{\alpha}^n| \leq \frac{1}{2}\left(N - 2M' - \sum_{m=1}^{\infty} t_{nm}M_m - 1\right), \tag{A24}
\]

\[
|J_{\alpha}^{n'}| \leq \frac{1}{2}\left(L - N + 2M' - \sum_{m=1}^{\infty} t_{nm}M'_m - 1\right), \tag{A25}
\]

where \( t_{nm} = 2 \min(m,n) - \delta_{mn} \). Each set of numbers \( \{I_j\}, \{J_{\alpha}^n\}, \{J_{\alpha}^{n'}\} \) is in one-to-one correspondence with a set of rapidities \( \{k_j\}, \{\Lambda_{\alpha}^n\}, \{\Lambda_{\alpha}^{n'}\} \), which in turn unambiguously specifies one Bethe eigenstate of the Hubbard Hamiltonian. Thus the ground state and all excited states can be constructed by specifying a set of numbers \( \{I_j\}, \{J_{\alpha}^n\}, \{J_{\alpha}^{n'}\} \) and then taking the thermodynamic limit.

It is our aim to prove eq. (2) for the charge singlet excitation over the half-filled ground state. The ground state is characterized by \( N = L, M_1 = L/2 \). In this case the inequalities (A23) and (A24) lead to unique distributions of the quantum numbers \( I_j, J_1^1, \)

\[
I_j = -(L+1)/2 + j, \quad j = 1,\ldots,L, \quad J_1^1 = -(L+2)/4 + \alpha, \quad \alpha = 1,\ldots,L/2. \tag{A26}
\]

Eqs. (A19) and (A20) reduce to

\[
Lk_j = 2\pi I_j - \sum_{\alpha=1}^{L/2} \theta\left(\frac{\sin k_j - \Lambda_{\alpha}}{U}\right), \quad j = 1,\ldots,L, \tag{A27}
\]

\[
\sum_{j=1}^{L} \theta\left(\frac{\Lambda_{\alpha} - \sin k_j}{U}\right) = 2\pi J_{\alpha}^1 + \sum_{\beta=1}^{L/2} \theta\left(\frac{\Lambda_{\alpha} - \Lambda_{\beta}}{2U}\right), \quad \alpha = 1,\ldots,L/2. \tag{A28}
\]

In the thermodynamic limit eqs. (A27) and (A28) turn into the well known integral equations

\[
\rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \cos k \int_{-\infty}^{\infty} d\Lambda \frac{U}{U^2 + (\sin k - \Lambda)^2} \sigma(\Lambda), \tag{A29}
\]

\[
\sigma(\Lambda) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} dk \frac{U}{U^2 + (\sin k - \Lambda)^2} - \frac{1}{\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{2U}{4U^2 + (\Lambda - \Lambda')^2} \sigma(\Lambda'), \tag{A30}
\]

for the densities \( \rho(k_j) = 1/(L(k_{j+1} - k_j)) \) and \( \sigma(\Lambda_{\alpha}) = 1/(L(\Lambda_{\alpha+1} - \Lambda_{\alpha})) \).

The charge singlet is characterized by \( N = L, M_1 = L/2 - 1 \) and \( M'_1 = 1 \). Thus \( M' = 1 \), and the number of unbound electrons is \( L - 2M' = L - 2 \). We will denote quantities which describe the charge singlet by a tilde. Eqs. (A24) and (A25) uniquely determine the set \( \{\tilde{J}_{\alpha}^1\} \) and the number \( \tilde{J}^1 \),

\[
\tilde{J}_{\alpha}^1 = -L/4 + \alpha, \quad \alpha = 1,\ldots,L/2 - 1, \quad \tilde{J}^1 = 0. \tag{A31}
\]

The set \( \{\hat{I}_j\} \), however, is not uniquely determined by the inequality (A25). There are \( \hat{I}_j \) inequivalent such sets, which are parameterized by two vacancies \( \hat{I}_1^1 \) and \( \hat{I}_2^1 \) in the distribution (A20) of the numbers \( I_j \), which characterize the ground state. These two vacancies determine two charge rapidities \( k_1^h \) and \( k_2^h \) via eqs. (A27) and (A28). In the thermodynamic limit the charge rapidities densely fill the interval \([-\pi, \pi]\), and \( k_1^h \) and \( k_2^h \) become the two free
parameters of the charge singlet excitation. We see already at this stage, that there can not be a third free parameter, since \( \hat{J}^1 = 0 \) is fixed. For the charge singlet excitation eqs. (A19)-(A21) reduce to

\[
\begin{align*}
L \hat{k}_j &= 2\pi \hat{I}_j - \sum_{\alpha=1}^{L/2-1} \theta \left( \sin \frac{k_j - \hat{\Lambda}_\alpha}{U} \right) - \theta \left( \sin \frac{k_j - \Lambda'_j}{U} \right), \quad j = 1, \ldots, L - 2, \\
\sum_{j=1}^{L-2} \theta \left( \frac{\Lambda'_j - \sin k_j}{U} \right) &= 2\pi \hat{I}_j + \sum_{\beta=1}^{L/2-1} \theta \left( \frac{\hat{\Lambda}_\alpha - \Lambda'_\beta}{2U} \right), \quad \alpha = 1, \ldots, L/2 - 1, \\
L(\arcsin(\Lambda' + iU) + \arcsin(\Lambda' - iU)) &= \sum_{j=1}^{L-2} \theta \left( \frac{\Lambda' - \sin k_j}{U} \right).
\end{align*}
\]  

(A32)  

(A33)  

(A34)

Let us subtract (A27) from (A32) for \( \hat{I}_j = \hat{I}_j, \ j = 1, \ldots, L - 2 \), and (A28) from (A33) for \( \alpha = 1, \ldots, L/2 - 1 \). Then, taking the thermodynamic limit, we obtain a pair of integral equations for the shift functions

\[
F_c(k_j) = \frac{\hat{k}_j - k_j}{k_{j+1} - k_j}, \quad F^s(\Lambda_\alpha) = \frac{\hat{\Lambda}_\alpha - \Lambda_\alpha}{\Lambda_{\alpha+1} - \Lambda_\alpha}.
\]  

(A35)

These integral equations are

\[
\begin{align*}
F_{cS}^c(k) &= -\frac{1}{2} - \frac{1}{2\pi} \theta \left( \frac{\sin k - \Lambda'}{U} \right) + \frac{1}{\pi} \int_{-\infty}^{\infty} d\Lambda \frac{U}{U^2 + (\sin k - \Lambda')^2} F_{cS}^s(\Lambda), \\
F_{cS}^s(\Lambda) &= 1 + \frac{1}{2\pi} \sum_{l=1}^{2} \theta \left( \frac{\Lambda - \sin k_l}{U} \right) - \frac{1}{\pi} \int_{-\infty}^{\infty} d\Lambda' \frac{2U}{4U^2 + (\Lambda - \Lambda')^2} F_{cS}^c(\Lambda').
\end{align*}
\]  

(A36)  

(A37)

Here we have supplied an index “CS” to emphasize that we are dealing with the charge singlet excitation. The solution \( F_{cS}^c(k) \) of (A36) can be found on page 526 of [6]. It is a single valued function of \( \sin k \). In the derivation of (A30) and (A37) we made use of the following elementary lemma,

\[
\int_{-\pi}^{\pi} dk \ f(\sin k) \cos k = 0,
\]  

(A38)

which holds for arbitrary single valued functions \( f(x) \). This lemma follows from the identities \( \sin(\pi - k) = \sin k \) and \( \cos(\pi - k) = -\cos k \).

Note that there are still three free parameters, \( k_1^b, k_2^b \) and \( \Lambda' \), in (A36) and (A37). \( \Lambda' \) becomes fixed as a function of \( k_1^b \) and \( k_2^b \) by considering eq. (A34), which in the thermodynamic limit turns into

\[
L(\arcsin(\Lambda' + iU) + \arcsin(\Lambda' - iU)) = L \int_{-\pi}^{\pi} dk \ \theta \left( \frac{\Lambda' - \sin k}{U} \right) \rho(k) = 2 L \int_{-\pi}^{\pi} dk \ \frac{U}{U^2 + (\Lambda' - \sin k)^2} F_{cS}^c(k) \cos k - 2 \sum_{l=1}^{2} \theta \left( \frac{\Lambda' - \sin k_l}{U} \right) + O \left( \frac{1}{L} \right).
\]  

(A39)

In order to obtain the second of the above equalities we have used (A29), (A36) and the lemma (A38). Let us calculate the integral

\[
I(\Lambda') = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \ \theta \left( \frac{\Lambda' - \sin k}{U} \right),
\]  

(A40)

on the right hand side of (A39). First note that \( I(0) = 0 \). The derivative of \( I(\Lambda') \) can be represented as

\[
I'(\Lambda') = \frac{1}{\pi} \int_{-\pi}^{\pi} dk \ \frac{U}{U^2 + (\Lambda' - \sin k)^2} = \text{Re} \left\{ \frac{1}{2\pi i} \oint dz \frac{4}{z^2 + 2z(U - i\Lambda') - 1} \right\},
\]  

(A41)

where the contour of integration is the unit circle. Let
\[ p(z) = z^2 + 2z(U - i\Lambda') - 1 = (z - z_1)(z - z_2). \]  

(A42)

We see that the poles of the integrand in (A41) are related as
\[ z_1 = -1/z_2. \]  

(A43)

Thus, only one of these poles, say \( z_1 \), lies inside the unit circle. Using (A43) we obtain \( I'(\Lambda') \) as a function of \( z_1 \),
\[ I'(\Lambda') = 2\text{Re} \left\{ \frac{2}{z_1 + z_1'} \right\}. \]  

(A44)

Let us parameterize the pole at \( z_1 \) as \( z_1 = e^{i\alpha} \). Since \( z_1 \) is located inside the unit circle, \( \text{Im}(\alpha) > 0 \). Using \( p(z_1) = 0 \) we find that
\[ \sin \alpha = \Lambda' + iU, \quad \text{Im}(\alpha) > 0. \]  

(A45)

This equation fixes \( \alpha \) modulo 2\( \pi \). Now \( U > 0 \) and \( \text{Im}(\alpha) > 0 \), and therefore
\[ \cos \text{Re}(\alpha) = \frac{U}{\sin \text{Im}(\alpha)} > 0. \]  

(A46)

We conclude that \(-\pi/2 < \text{Re}(\alpha) < \pi/2 \). Thus (see the definition below (A22)) \( \alpha = \arcsin(\Lambda' + iU) \). Integrating (A44) with respect to \( \Lambda' \) and using \( I(0) = 0 \) to fix the integration constant we arrive at
\[ I(\Lambda') = 2\text{Re}(\alpha) = 2\text{Re}(\arcsin(\Lambda' + iU)) = \arcsin(\Lambda' + iU) + \arcsin(\Lambda' - iU). \]  

(A47)

We may now insert this result into eq. (A37). This yields
\[ \theta \left( \frac{\Lambda' - \sin k_1^b}{U} \right) = -\theta \left( \frac{\Lambda' - \sin k_2^b}{U} \right). \]  

(A48)

Dividing by 2 and taking tan gives
\[ 2\Lambda' = \sin k_1^b + \sin k_2^b. \]  

(A49)

So we have accomplished our task to show that (2) follows from eqs. (20)-(22) of \[ ]

APPENDIX B:

Let us show that for one \( k-\Lambda \) string (and several real \( k \) and real \( \lambda \) ) eq. (7) of \[ ] follows from eqs. (5), (6), (8) of \[ ]. In case of a single \( k-\Lambda \)-two string the numbers \( N^u \) and \( M^u \) of \[ ] are \( N^u = N - 2, M^u = M - 1 \), where \( N \) is the total number of electrons and \( M \) is the total number of down spins. There are \( N - 2 \) real charge rapidities \( k_1, \ldots, k_{N-2} \), which correspond to unbound particles. The string is characterized by two charge rapidities \( k^+ \) and \( k^- \) and by one spin rapidity \( \Lambda \). \( k^+ \) and \( \Lambda \) satisfy
\[ \sin k^+ = \Lambda - iu/4 + \varepsilon^+ \]
\[ \sin k^- = \Lambda + iu/4 - \varepsilon^- \]  

(B1)

(B2)

where \( \varepsilon^+ \) and \( \varepsilon^- \) become small as the length \( L \) of the lattice becomes large. Eqs. (5)-(8) of \[ ] are
\[ e^{ik_i L} = \frac{\Lambda - \sin k_j - iu/4}{\Lambda - \sin k_j + iu/4} \prod_{\delta=1}^{M-1} \frac{\lambda_\gamma - \Lambda - iu/2}{\lambda_\gamma - \Lambda + iu/2}, \quad j = 1, \ldots, N - 2, \]  

(B3)

\[ \prod_{\delta=1}^{M-1} \frac{\lambda_\gamma - \Lambda - iu/2}{\lambda_\gamma - \Lambda + iu/2} = \prod_{j=1}^{N-2} \frac{\lambda_\gamma - \sin k_j - iu/4}{\lambda_\gamma - \sin k_j + iu/4}, \quad \gamma = 1, \ldots, M - 1, \]  

(B4)

\[ \prod_{\delta=1}^{M-1} \frac{\Lambda - \lambda_\delta - iu/2}{\Lambda - \lambda_\delta + iu/2} = \frac{\varepsilon^+}{\varepsilon^-} \prod_{j=1}^{N-2} \frac{\Lambda - \sin k_j - iu/4}{\Lambda - \sin k_j + iu/4}, \]  

(B5)

\[ e^{i(k^+ + k^-) L} = \frac{\varepsilon^+}{\varepsilon^-} \prod_{\delta=1}^{M-1} \frac{\lambda_\delta - \Lambda - iu/2}{\lambda_\delta - \Lambda + iu/2}. \]  

(B6)
The ratio

$$\frac{\varepsilon^+}{\varepsilon^-} = \frac{\Lambda - \sin k^+ - iu/4}{\Lambda - \sin k^- + iu/4}$$

(B7)

is denoted by $e^{i\varphi}$ in [1]. We want to show that (B5) algebraically follows from (B3), (B4) and (B6). First note that momentum conservation implies that

$$\exp\left(iL \sum_{j=1}^{N-2} k_j + k^+ + k^-\right) = 1.$$  

(B8)

Multiplying all eqs. (B4) we get

$$\prod_{j=1}^{N-2} \prod_{\delta=1}^{M-1} \frac{\lambda_\delta - \sin k_j - iu/4}{\lambda_\delta - \sin k_j + iu/4} = \prod_{\gamma=1}^{M-1} \prod_{\delta=1}^{M-1} \frac{\lambda_\delta - \lambda_\gamma - iu/2}{\lambda_\delta - \lambda_\gamma + iu/2} = 1.$$  

(B9)

Now let us multiply all eqs. (B3). Taking into account (B9) we get

$$e^{iL \sum_{j=1}^{N-2} k_j} = \prod_{j=1}^{N-2} \frac{\Lambda - \sin k_j - iu/4}{\Lambda - \sin k_j + iu/4}.$$  

(B10)

Let us multiply this by (B6) and use (B8). Then

$$1 = \exp\left(iL \sum_{j=1}^{N-2} k_j + k^+ + k^-\right) = \frac{\varepsilon^+}{\varepsilon^-} \prod_{\delta=1}^{M-1} \frac{\lambda_\delta - \Lambda - iu/2}{\lambda_\delta - \Lambda + iu/2} \prod_{j=1}^{N-2} \frac{\Lambda - \sin k_j - iu/4}{\Lambda - \sin k_j + iu/4}.$$  

(B11)

which is equivalent to (B5) (or (7) in [1]).

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