Infinite horizon optimal control of forward-backward stochastic differential equations with delay.

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15 May 2013

Abstract

We consider a problem of optimal control of an infinite horizon system governed by forward-backward stochastic differential equations with delay. Sufficient and necessary maximum principles for optimal control under partial information in infinite horizon are derived. We illustrate our results by an application to a problem of optimal consumption with respect to recursive utility from a cash flow with delay.

Keywords: Infinite horizon; Optimal control; Stochastic delay equation; Stochastic differential utility; Lévy processes; Maximum principle; Hamiltonian; Adjoint processes; Partial information.

[2010]MSC: 93EXX; 93E20; 60J75; 60H10; 60H20; 34K50

1 Introduction

One of the problems posed recently and which has got a lot of attention is the optimal control of forward-backward stochastic differential equations (FBSDEs). This theory was first developed in the early 90s by [3], [10], [20] and others.

The paper [20] established the maximum principle of FBSDE in the convex setting and later it was studied by many authors such as [2], [12], [14], [17], [23]. For the existence of an optimal control of FBSDEs, see [4].
The optimal control problem of FBSDE has interesting applications especially in finance like in option pricing and recursive utility problems. The latter was introduced by [7] and for more details about the recursive utility maximization problems, we refer to [6], [20].

The recursive utility is a solution of the backward stochastic differential equation (BSDE) which is not necessarily linear. The BSDE was studied by [18], [19] etc.

All the papers above where dealing with finite horizon FBSDEs. Other related stochastic control publications dealing with finite horizon only are [5], [13] and [22].

Related papers dealing with infinite horizon control, but either without FB systems or without delay, are [1], [9], [11], [21] and [24].

We will study this problem by using a version of the maximum principle which is a combination of the infinite horizon maximum principle in [1] and the finite horizon maximum principle for FBSDEs in [14] and [12]. We extend an application in [12] to infinite horizon and in [15] for the FBSDE.

We emphasize that although the current paper has similarities with [1], the fact that we are considering forward-backward systems and not just forward systems creates a new situation. In particular, we now get additional transversality conditions involving the additional adjoint process \( \lambda \). See Theorem 2.1 and Theorem 3.1.

In this paper we obtain a sufficient and a necessary maximum principle for infinite horizon control of FBSDEs with delay. As an illustration we solve explicitly an infinite horizon optimal consumption problem with recursive utility.

The partial results mentioned above indicate that it should be possible to prove a general existence and uniqueness theorem for controlled infinite horizon FBSDEs with delay. However, this is difficult problem and we leave this for future research.

## 2 Setting of the problem

Let \( (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P) \) be a complete filtered probability space on which a one-dimensional standard Brownian motion \( B(t) \) and an independent compensated Poisson random measure \( \tilde{N}(dt, da) = N(dt, da) - \nu(da)dt \) are defined. We assume that \( \mathbb{F} \) is the natural filtration, made right continuous generated by the processes \( B \) and \( N \).

We study the following infinite horizon coupled forward-backward stochastic differential equations control system with delay:

**FORWARD EQUATION** in the unknown measurable process \( X^u(t) \):

\[
\begin{align*}
    dX(t) &= dX^u(t) = b(t, X^u(t), u(t)) \, dt + \sigma(t, X^u(t), u(t)) \, dB(t) \\
    &\quad + \int_{\mathbb{R}_0} \theta(t, X^u(t), u(t), a) \, \tilde{N}(dt, da); \, t \in [0, \infty), \\
    X(t) &= X_0(t); \quad t \in [-\delta, 0],
\end{align*}
\]

(2.1)

where

\[
X^u(t) = (X^u(t), X_1^u(t), X_2^u(t)),
\]
with
\[ X_1^u(t) = X^u(t - \delta), \quad X_2^u(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X^u(r) dr, \]
and \( X_0 \) is a given continuous (and deterministic) function on \([-\delta, 0]\).

**BACKWARD EQUATION** in the unknown measurable processes \( Y^u(t), Z^u(t), K^u(t, \cdot) \):
\[
dY^u(t) = -g(t, \mathbf{X}^u(t), Y^u(t), Z^u(t), u(t)) dt + Z^u(t) dB(t) + \int_{\mathbb{R}_0} K^u(t, a) N(da); t \in [0, \infty). \tag{2.2}
\]
We interpret the infinite horizon BSDE (2.2) in the sense of Pardoux [18] i.e. for all \( T < \infty \), the triple \( (Y^u(t), Z^u(t), K^u(t, \cdot)) \) solves the equation
\[
Y^u(t) = Y(T) + \int_{T}^{t} \left[ g(s, \mathbf{X}^u(s), Y^u(s), Z^u(s), u(s)) ds - \int_{\mathbb{R}_0} Z^u(s) N(ds, da) + \int_{\mathbb{R}_0} K^u(s, a) N(ds, da) \right]; 0 \leq t \leq T. \tag{2.3}
\]
We call the process \( (Y^u(t), Z^u(t), K^u(t, \cdot)) \) the solution of (2.3) if it also satisfies
\[
E \left[ \sup_{t \geq 0} e^{\kappa t} (Y^u)^2(t) + \int_{0}^{\infty} e^{\kappa t} ((Z^u)^2(t) + \int_{\mathbb{R}_0} (K^u)^2(t, a) \nu(da)) dt \right] < \infty \tag{2.4}
\]
for all constants \( \kappa > 0 \). We refer the reader to Section 4 in [18] for assumptions of the coefficients that insure the existence and uniqueness of the solution of the FBSDE system. Note that (2.4) implies in particular that \( \lim_{t \to \infty} Y^u(t) = 0 \).

Throughout this paper, we introduce the following notations:
\[
\delta > 0, \rho > 0 \text{ are given constants,} \quad b : [0, \infty) \times \mathbb{R}^3 \times \mathcal{U} \times \Omega \to \mathbb{R}, \quad 
\sigma : [0, \infty) \times \mathbb{R}^3 \times \mathcal{U} \times \Omega \to \mathbb{R}, \quad 
g : [0, \infty) \times \mathbb{R}^5 \times \mathcal{U} \times \Omega \to \mathbb{R}, \quad 
\mathbb{R}_0 := \mathbb{R} - \{0\}, \quad 
\theta, K : [0, \infty) \times \mathbb{R}^3 \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}, \quad 
f : [0, \infty) \times \mathbb{R}^5 \times \mathcal{R} \times \mathcal{U} \times \Omega \to \mathbb{R}, \quad 
h : \mathbb{R} \to \mathbb{R}, \quad 
\]
where the coefficients \( b, \sigma, \theta \) and \( g \) are Fréchet differentiable \((C^1)\) with respect to the variables \( (x, y, z, u) \). Here \( \mathcal{R} \) is the set of all functions \( k : \mathbb{R}_0 \to \mathbb{R} \). In the following, we will for simplicity suppress the dependence on \( \omega \in \Omega \) in the notation.

Note that if \( g \) does not depend on \( Y^u(s) \) and \( Z^u(s) \) then the Itô representation theorem for Lévy processes (see [8]), implies that equation (2.3) is equivalent to the equation
\[
Y^u(t) = E[Y(T) + \int_{t}^{T} g(s, \mathbf{X}^u(s), u(s)) ds | \mathcal{F}_t]; t \leq T, \text{ for all } T < \infty. \tag{2.5}
\]
Let $E = \{E_t\}_{t \geq 0}$ with $E_t \subseteq F_t$ for all $t \geq 0$ be a given subfiltration, representing the information available to the controller at time $t$.

Let $U$ be a non-empty convex subset of $\mathbb{R}$. We let $A = A_E$ denote a given locally convex family of admissible $E$-predictable control processes $u$ with values in $U$, such that the corresponding solution $(X^u, Y^u, Z^u, K^u)$ of (2.1) exist and

$$E\left[\int_0^\infty |X^u(t)|^2 dt\right] < \infty.$$

The corresponding performance functional is

$$J(u) = E\left[\int_0^\infty f(t, X(t)) \, dt + h(Y(0))\right],$$

where $f(t, X(t))$ is a short-hand notation for $f(t, X^u(t), Y^u(t), Z^u(t), K^u(t, \cdot), u(t))$. We assume that the functions $f$ and $h$ are Fréchet differentiable ($C^1$) with respect to the variables $(x, y, z, k(\cdot), u)$ and $Y(0)$, respectively, and $f$ satisfies

$$E\left[\int_0^\infty |f(t, X(t))| \, dt\right] < \infty, \text{ for all } u \in A. \quad (2.7)$$

The optimal control problem is to find an optimal control $u^* \in A$ and the value function $\Phi : C([0, \delta]) \to \mathbb{R}$ such that

$$\Phi(X_0) = \sup_{u \in A} J(u) = J(u^*). \quad (2.8)$$

We will study this problem by using a version of the maximum principle which is a combination of the infinite horizon maximum principle in [1] and the finite horizon maximum principle for FBSDEs in [17] and [14].

The Hamiltonian

$$H : [0, \infty) \times \mathbb{R}^5 \times L^2(\nu) \times U \times \mathbb{R}^3 \times L^2(\nu) \to \mathbb{R}$$

is defined by

$$H(t, x, y, z, k(\cdot), u, \lambda, p, q, r(\cdot)) = f(t, x, y, z, k, u) + g(t, x, y, z, u)\lambda + b(t, x, u)p + \sigma(t, x, u)q + \int \theta(t, x, u, a)p + \int r(t, x, a)q \, \nu(da). \quad (2.9)$$

We assume that the Hamiltonian $H$ is Fréchet differentiable ($C^1$) in the variables $x, y, z, k$ and $u$.

We also assume that for all $t$ the Fréchet derivative of $H(t, X^u(t), Y^u(t), Z^u(t), k, u(t), p(t), q(t), r(t, \cdot))$ with respect to $k$, denoted by $\nabla_k H(t, \cdot)$, as a random measure is absolutely continuous with respect to $\nu$, with Radon-Nikodym derivative $\frac{d\nabla_k H}{d\nu}$ satisfying

$$E\left[\int_0^T \int |\frac{d\nabla_k H}{d\nu}(t, a)|^2 \nu(da) dt\right] < \infty, \text{ for all } T < \infty.$$
See Appendix A in [17] for details.

We associate to the problem (2.8) the following pair of forward-backward SDEs in the adjoint processes \( \lambda(t) \), \( (p(t), q(t), r(t, \cdot)) \):

**ADJOINT FORWARD EQUATION:**

\[
\begin{align*}
\frac{d\lambda(t)}{dt} &= \frac{\partial H}{\partial y}(t) \, dt + \frac{\partial H}{\partial z}(t) \, dB(t) + \int_{\mathbb{R}_0} \frac{\partial^2 H}{\partial y \partial z}(t, a) \tilde{N}(dt, da) \\
\lambda(0) &= \mathcal{h}'(Y(0))
\end{align*}
\]

where we have used the short hand notation

\[
\frac{\partial H}{\partial y}(t) = \frac{\partial}{\partial y} H(t, X^u(t), y, Z^u(t), K^u(t, \cdot), u(t), \lambda(t), p(t), q(t), r(t, \cdot)) \big|_{y=Y(t)}
\]

and similarly with \( \frac{\partial H}{\partial z}(t), \frac{\partial H}{\partial x}(t), \ldots \)

**ADJOINT BACKWARD EQUATION:**

\[
dp(t) = E[\mu(t) \mid \mathcal{F}_t] dt + q(t) dB(t) + \int_{\mathbb{R}_0} r(t, a) \tilde{N}(dt, da); t \in [0, \infty)
\]

where

\[
\mu(t) = -\frac{\partial H}{\partial x_1}(t) - \frac{\partial H}{\partial x_2}(t + \delta) - e^{\rho t} \left( \int_t^{t+\delta} \frac{\partial H}{\partial x_2}(s) e^{-\rho s} ds \right).
\]

with terminal condition as in (2.4), i.e.

\[
E[\sup_{t \geq 0} e^{\kappa t} p^2(t) + \int_0^\infty e^{\kappa s} (q^2(s) + \int_{\mathbb{R}_0} r^2(s, a) \nu(da)) ds] < \infty,
\]

for all constants \( \kappa > 0 \).

The unknown process \( \lambda(t) \) is the adjoint process corresponding to the backward system \((Y(t), Z(t), K(t, \cdot))\) and the triple unknown \((p(t), q(t), r(t, \cdot))\) is the adjoint process corresponding to the forward system \(X(t)\).

We show that in this infinite horizon setting the appropriate terminal conditions for the BSDEs for \((Y(t), Z(t), K(t, \cdot))\) and \((p(t), q(t), r(t, \cdot))\) should be replaced by asymptotic transversality conditions. See \((H_3)\) and \((H_6)\) below.

3 Sufficient maximum principle for partial information

We will prove in this section that under some assumptions the maximization of the Hamiltonian leads to an optimal control.

**Theorem 3.1** Let \( \hat{u} \in A \) with corresponding solutions \( \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t, \cdot), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot) \) and \( \hat{\lambda}(t) \) of equations (2.1), (2.2), (2.10) and (2.11). Suppose that
(H₁) (Concavity)

The functions \( x \to h(x) \) and

\[
(x, y, z, k(\cdot), u) \to H(t, x, y, z, k(\cdot), u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))
\]

are concave, for all \( t \in [0, \infty) \).

(H₂) (The conditional maximum principle)

\[
\max_{v \in \mathcal{U}} E[H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t, \cdot), v, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) | \mathcal{E}_t]
= E[H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t, \cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) | \mathcal{E}_t].
\]

Moreover, suppose that for any \( u \in \mathcal{A} \) with corresponding solutions \( X(t), Y(t), Z(t), K(t, \cdot), p(t), q(t), r(t, \cdot) \) and \( \lambda(t) \) we have:

(H₃) (Transversality conditions)

\[
\lim_{T \to \infty} E[\hat{p}(T) \triangle \hat{X}(T)] \leq 0
\]

and

\[
\lim_{T \to \infty} E[\hat{\lambda}(T) \triangle \hat{Y}(T)] \geq 0.
\]

where \( \triangle \hat{X}(T) = \hat{X}(T) - X(T), \triangle \hat{Y}(T) = \hat{Y}(T) - Y(T) \).

(H₄) (Growth conditions I) Suppose that for all \( T < \infty \) the following holds:

\[
E \left[ \int_0^T \left\{ (\triangle \hat{Y}(t))^2 \left\{ (\frac{\partial H}{\partial y})(t) \right\}^2 + \int_{\mathbb{R}_0} \left\| \nabla_k \hat{H}(t, a) \right\|^2 \nu(da) \right\} \right. \\
+ \hat{\lambda}^2(t) \left\{ (\triangle \hat{Z}(t))^2 + \int_{\mathbb{R}_0} (\triangle \hat{K}(t, a))^2 \nu(da) \right\} \\
+ (\triangle \hat{X}(t))^2 \left\{ \hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t, a) \nu(da) \right\} \\
+ \hat{p}^2(t) \left\{ (\triangle \hat{\sigma}(t))^2 + \int_{\mathbb{R}_0} (\triangle \hat{a}(t, a))^2 \nu(da) \right\} dt \right] < \infty.
\] (3.1)

(H₅) (Growth conditions II) Suppose that

\[
E \left[ \int_0^T \left\{ \hat{\lambda}(t) \triangle \hat{g}(t) \right\} + \left| \triangle \hat{Y}(t) \frac{\partial H}{\partial y}(t) \right| + \left| \triangle \hat{Z}(t) \frac{\partial H}{\partial z}(t) \right| \\
+ \int_{\mathbb{R}_0} \nabla_k \hat{H}(t, a) \triangle \hat{K}(t, a) \nu(da) + \left| \triangle \hat{H}(t) \right| + \left| \triangle \hat{b}(t) \hat{p}(t) \right| \\
+ \left| \triangle \hat{\sigma}(t) \hat{q}(t) \right| + \int_{\mathbb{R}_0} \left| \triangle \hat{a}(t, a) \hat{r}(t, a) \right| \nu(da) \\
+ \left| \triangle \hat{X}(t) \frac{\partial H}{\partial x}(t) \right| + \left| \triangle \hat{u}(t) \frac{\partial H}{\partial u}(t) \right| \right] dt < \infty.
\] (3.2)
where \( \sigma(t) = \sigma(t, X(t), u(t)) \), \( \hat{\sigma}(t) = \sigma(t, \hat{X}(t), \hat{u}(t)) \) etc.

Then \( \hat{u} \) is an optimal control for (2.8), i.e.

\[
J(\hat{u}) = \sup_{u \in A} J(u).
\]

**Proof.** Assume that \( u \in A \). We want to prove that \( J(\hat{u}) - J(u) \geq 0 \), i.e. \( \hat{u} \) is an optimal control.

We put

\[
J(\hat{u}) - J(u) = I_1 + I_2,
\]

where

\[
I_1 = E[\int_0^\infty \{ \hat{f}(t) - f(t) \} \, dt],
\]

and

\[
I_2 = E[h(\hat{Y}(0)) - h(Y(0))].
\]

By the definition of \( H \), we have

\[
I_1 = E[\int_0^\infty \{ \Delta \hat{H}(t) - \Delta \hat{g}(t) \hat{\lambda}(t) - \Delta \hat{b}(t) \hat{p}(t) - \Delta \hat{\sigma}(t) \hat{q}(t) - \int_{R_0} \Delta \hat{\vartheta}(t, a) \delta(t, a) \nu(da) \} \, dt],
\]

where we have used the simplified notation

\[
\hat{H}(t) = H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t, \cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))
\]

\[
H(t) = H(t, X(t), Y(t), Z(t), K(t, \cdot), u(t), \lambda(t), p(t), q(t), r(t, \cdot)) \text{ etc.}
\]

Since \( h \) is concave, we have

\[
h(\hat{Y}(0)) - h(Y(0)) \geq h'(\hat{Y}(0)) \Delta \hat{Y}(0) = \hat{\lambda}(0) \Delta \hat{Y}(0).
\]

By It\'s formula, (\( H_4 \)), (2.2) and (2.10), we have for all \( T \)

\[
E[\hat{\lambda}(0) \Delta \hat{Y}(0)] = E[\hat{\lambda}(T) \Delta \hat{Y}(T)] - \int_0^T \hat{\lambda}(t) d(\Delta \hat{Y}(t)) - \int_0^T \Delta \hat{Y}(t) d\hat{\lambda}(t)
\]

\[
- \int_0^T \Delta \hat{z}(t) \frac{\partial \hat{H}}{\partial z}(t) \, dt - \int_0^T \int_{R_0} \nabla_k \hat{H}(t, a) \Delta \hat{K}(t, a) \nu(da) \, dt] .
\]

(3.5)

By (\( H_4 \)) all the local martingales involved in (3.5) are martingales up to time \( T \), for all \( T < \infty \).

Therefore, letting \( T \to \infty \), we obtain by (3.2)

\[
E[\hat{\lambda}(0) \Delta \hat{Y}(0)] = \lim_{T \to \infty} E[\hat{\lambda}(T) \Delta \hat{Y}(T)] - E[\int_0^\infty \{ -\hat{\lambda}(t) \Delta \hat{g}(t) + \Delta \hat{Y}(t) \frac{\partial \hat{H}}{\partial y}(t) + \Delta \hat{z}(t) \frac{\partial \hat{H}}{\partial z}(t) + \int_{R_0} \nabla_k \hat{H}(t, a) \Delta \hat{K}(t, a) \nu(da) \} \, dt].
\]

(3.6)
Combining (3.4) – (3.6), we obtain

\[ J(\hat{u}) - J(u) \geq \lim_{T \to \infty} E \left[ \hat{\lambda}(T) \triangle \hat{Y}(T) \right] + E \int_0^{\infty} \left\{ \triangle \hat{H}(t) - \triangle \hat{b}(t)\hat{p}(t) - \triangle \hat{\sigma}(t)\hat{q}(t) \right\} dt - \int_{R_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da) - \triangle \hat{Y}(t) \frac{\partial H}{\partial y}(t) - \triangle \hat{Z}(t) \frac{\partial H}{\partial z}(t) - \int_{R_0} \nabla_k \hat{H}(t,a) \triangle \hat{K}(t,a)\nu(da) \right\} dt. \]

Since \( H \) is concave, we have

\[ J(\hat{u}) - J(u) \geq \lim_{T \to \infty} E \left[ \hat{\lambda}(T) \triangle \hat{Y}(T) \right] + E \left[ \int_0^{\infty} \left\{ \triangle \hat{\lambda}(t) + \triangle \hat{X}(t) \hat{\mu}(t) + \triangle \hat{\sigma}(t)\hat{q}(t) + \int_{R_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da) \right\} dt \right]. \]

Applying now (H1), (H4) and (H3) together with the It\'{U} formula to \( \hat{p}(t) \triangle \hat{X}(t) \), we get

\[ 0 \geq \lim_{T \to \infty} E \left[ \hat{p}(T) \triangle \hat{X}(T) \right] = E \left[ \int_0^{\infty} \left\{ \triangle \hat{b}(t)\hat{p}(t) - \triangle \hat{\lambda}(t)\hat{\mu}(t) + \triangle \hat{\sigma}(t)\hat{q}(t) + \int_{R_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da) \right\} dt \right]. \]

By the definition (2.12) of \( \hat{\mu} \), we have

\[ E \left[ \int_0^{T+\delta} \triangle \hat{X}(t)\hat{\mu}(t)dt \right] = \lim_{T \to \infty} E \left[ \int_0^{T+\delta} \triangle \hat{X}(t - \delta)\hat{\mu}(t - \delta)dt \right] = \lim_{T \to \infty} E \left[ \int_0^{T+\delta} \frac{\partial \hat{H}}{\partial x_2}(t, \delta) \triangle \hat{X}(t - \delta)dt \right] \]

\[ - \int_0^{T+\delta} \left( \int_0^{t} \frac{\partial \hat{H}}{\partial x_2}(s, \delta)e^{-\rho s}ds \right) e^{\rho(t-\delta)} \delta \hat{X}(t - \delta)dt. \]

Using Fubini’s theorem and the definition of \( \hat{X}_2 \), we obtain

\[ \int_0^{T} \frac{\partial \hat{H}}{\partial x_2}(s) \delta \hat{X}_2(s)ds = \int_0^{T+\delta} \left( \int_0^{t} \frac{\partial \hat{H}}{\partial x_2}(s)e^{-\rho s}ds \right) e^{\rho(t-\delta)} \delta \hat{X}(t - \delta)dt. \]

Combining (3.7) with (3.8) – (3.10), we deduce that

\[ J(\hat{u}) - J(u) \geq \lim_{T \to \infty} E \left[ \hat{\lambda}(T) \triangle \hat{Y}(T) \right] - \lim_{T \to \infty} E \left[ \hat{p}(T) \triangle \hat{X}(T) \right] + E \left[ \int_0^{\infty} \left\{ \triangle \hat{\mu}(t, \delta)\frac{\partial \hat{H}}{\partial u}(t) \right\} dt \right] = \lim_{T \to \infty} E \left[ \hat{\lambda}(T) \triangle \hat{Y}(T) \right] - \lim_{T \to \infty} E \left[ \hat{p}(T) \triangle \hat{X}(T) \right] + E \left[ \int_0^{\infty} \left\{ \triangle \hat{\mu}(t, \delta)\frac{\partial \hat{H}}{\partial u}(t) \right\} E \right] dt. \]

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Then
\[ J(\hat{u}) - J(u) \geq \lim_{T \to \infty} E[\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] - \lim_{T \to \infty} E[\hat{\theta}(T) \triangle \hat{X}(T)] + E[\int_{0}^{\infty} E\{\frac{\partial \hat{H}}{\partial u}(t) | E_{t}\} \triangle \hat{u}(t)dt]. \]

By assumptions \((H_1)\) and \((H_3)\), we conclude \( J(\hat{u}) - J(u) \geq 0 \), i.e. \( \hat{u} \) is an optimal control. ■

4 Necessary conditions of optimality for partial information

A drawback of the previous section is that the concavity condition is not always satisfied in applications. In view of this, it is of interest to obtain conditions for the existence of an optimal control with partial information where concavity is not needed. We assume the following:

\((A_{1})\) For all \( u \in \mathcal{A} \) and all \( \beta \in \mathcal{A} \) bounded, there exists \( \epsilon > 0 \) such that

\[ u + s\beta \in \mathcal{A} \quad \text{for all } s \in (-\epsilon, \epsilon). \]

This implies in particular that the corresponding solution \( X^{u+s\beta}(t) \) of \((2.1) - (2.5)\) exists.

\((A_{2})\) For all \( t_0 > 0, h > 0 \) and all bounded \( \mathcal{E}_{t_0} \)-measurable random variables \( \alpha \), the control process \( \beta(t) \) defined by

\[ \beta(t) = \alpha 1_{[t_0,t_0+h]}(t) \] (4.1)

belongs to \( \mathcal{A} \).

\((A_{3})\) The following derivative processes exist

\[ \xi(t) := \frac{d}{ds}X^{u+s\beta}(t) \big|_{s=0} \] (4.2)

\[ \phi(t) := \frac{d}{ds}Y^{u+s\beta}(t) \big|_{s=0} \] (4.3)

\[ \eta(t) := \frac{d}{ds}Z^{u+s\beta}(t) \big|_{s=0} \] (4.4)

\[ \psi(t, a) := \frac{d}{ds}K^{u+s\beta}(t, a) \big|_{s=0} \] (4.5)
\[(A_4) \quad \text{We also assume that}
\]
\[
\begin{align*}
E[\int_0^\infty & \left\{ \left| \frac{\partial f}{\partial x}(t) \xi(t) \right| + \left| \frac{\partial f}{\partial y}(t) \xi(t) \right| + \left| \frac{\partial f}{\partial z}(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) \, dr \right| + \left| \frac{\partial f}{\partial y}(t) \phi(t) \right| \\
&+ \left| \frac{\partial f}{\partial z}(t) \eta(t) \right| + \left| \frac{\partial f}{\partial y}(t) \beta(t) \right| + \int_{\mathbb{R}_0} \nabla f(t,a) \psi(t,a) \nu(da) \, dt \right] < \infty.
\end{align*}
\]

We can see that
\[
\frac{d}{ds} X_1^{u+s\beta}(t) \big|_{s=0} = \xi(t - \delta)
\]
and
\[
\frac{d}{ds} X_2^{u+s\beta}(t) \big|_{s=0} = \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) \, dr.
\]

Note that
\[
\xi(t) = 0 \quad \text{for} \quad t \in [-\delta, 0].
\]

**Theorem 4.1** Assume that \((A_1) - (A_4)\) hold. Suppose that \(\hat{u} \in \mathcal{A}\) with corresponding solutions \(\hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t, \cdot), \lambda(t), \hat{p}(t), \hat{q}(t)\) and \(\hat{r}(t, \cdot)\) of equations (2.1), (2.2), (2.10) and (2.11).

Assume that \((3.1)\) and the following transversality conditions hold:

\[(H_6)\]
\[
\lim_{T \to \infty} E[\hat{p}(T)\xi(T)] = 0,
\]
\[
\lim_{T \to \infty} E[\hat{\lambda}(T)\phi(T)] = 0.
\]

\[(H_7)\] Moreover, assume that the following growth condition holds
\[
E[\int_0^T \left\{ \hat{\lambda}_2(t)(\eta^2(t) + \int_{\mathbb{R}_0} \psi^2(t,a) \nu(da)) + \phi^2(t)((\hat{\partial H}/\partial z)^2(t) + \int_{\mathbb{R}_0} \nabla_k \hat{H}^2(t,a) \nu(da)) \\
+ \hat{p}^2(t)(\hat{\partial \lambda}/\partial x)(t)\xi(t) + \hat{\partial \lambda}/\partial x_1(t)\xi(t - \delta) + \hat{\partial \lambda}/\partial x_2(t) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) \, dr + (\hat{\partial \lambda}/\partial u)(t)\beta(t))^2 \\
+ \hat{q}^2(t)(\int_{\mathbb{R}_0} \hat{\partial u}(t,a)\xi(t) + \hat{\partial q}/\partial x_1(t,a)\xi(t - \delta) + \hat{\partial q}/\partial x_2(t,a) \int_{t-\delta}^t e^{-\rho(t-r)} \xi(r) \, dr \\
+ \hat{\partial q}/\partial u(t,a)\beta(t))^2 \nu(da)) \, dt \right] < \infty, \quad \text{for all} \quad T < \infty.
\]

Then the following assertions are equivalent.
(i) For all bounded \( \beta \in \mathcal{A} \),
\[
\frac{d}{ds} J(\hat{u} + s \beta) \big|_{s=0} = 0.
\]

(ii) For all \( t \in [0, \infty) \),
\[
E[\frac{\partial}{\partial u} H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t, \cdot), u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) \mid \mathcal{E}_t]_{u=\hat{u}(t)} = 0.
\]

**Proof.** (i) \( \implies \) (ii):

In the following we use the short-hand notation \( \frac{\partial}{\partial x_i}(t) = \frac{\partial}{\partial x_i} b(t, x, u(t))_{x=x(t)} \) etc; \( i = 1, 2, 3 \).

It follows from (2.1) that
\[
\frac{d \xi}{d t}(t) = \{ \frac{\partial}{\partial x}(t) \xi(t) + \frac{\partial}{\partial x_1}(t) \xi(t - \delta) + \frac{\partial}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r) dr + \frac{\partial}{\partial u}(t) \beta(t) \}
\]
\[
+ \{ \frac{\partial}{\partial x}(t) \xi(t) + \frac{\partial}{\partial x_1}(t) \xi(t - \delta) + \frac{\partial}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r) dr + \frac{\partial}{\partial u}(t) \beta(t) \}
\]
\[
+ \int_{\mathbb{R}_0} \{ \frac{\partial}{\partial z}(t, a) \xi(t) + \frac{\partial^2}{\partial z^2}(t, a) \xi(t - \delta) + \frac{\partial}{\partial u}(t, a) \beta(t) \} \hat{N}(dt, da),
\]
and
\[
\frac{d \phi}{d t}(t) = \{- \frac{\partial}{\partial x}(t) \xi(t) - \frac{\partial}{\partial x_1}(t) \xi(t - \delta) - \frac{\partial}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r) dr - \frac{\partial}{\partial u}(t) \phi(t)
\]
\[
- \frac{\partial}{\partial u}(t) \beta(t) - \frac{\partial}{\partial z}(t) \eta(t) \}
\]
\[
+ \int_{\mathbb{R}_0} \{ \frac{\partial}{\partial x}(t) \xi(t) + \frac{\partial}{\partial x_1}(t) \xi(t - \delta) + \frac{\partial}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r) dr + \frac{\partial}{\partial u}(t) \beta(t) \}
\]
\[
+ \frac{\partial}{\partial x}(t) \eta(t) + \frac{\partial}{\partial u}(t) \phi(t) + \int_{\mathbb{R}_0} \nabla_k f(t, a) \psi(t, a) \mu(da) \}
\]
\[
+ h'(\hat{Y}(0)) \phi(0)].
\]

Suppose that assertion (i) holds. Then by (A4) and dominated convergence
\[
0 = \frac{d}{ds} J(\hat{u} + s \beta) \big|_{s=0}
\]
\[
= E\left[ \int_{0}^{\infty} \left\{ \frac{\partial f}{\partial x}(t) \xi(t) + \frac{\partial f}{\partial x_1}(t) \xi(t - \delta) + \frac{\partial f}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r) dr + \frac{\partial f}{\partial u}(t) \phi(t)
\]
\[
+ \frac{\partial f}{\partial z}(t) \eta(t) + \frac{\partial f}{\partial u}(t) \beta(t) + \int_{\mathbb{R}_0} \nabla_k f(t, a) \psi(t, a) \nu(da) \}
\]
\[
+ h'(\hat{Y}(0)) \phi(0) \right] \] (4.7)

We know by the definition of \( H \) that
\[
\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial g}{\partial x}(t) \lambda(t) - \frac{\partial b}{\partial x}(t) p(t) - \frac{\partial \sigma}{\partial x}(t) q(t) - \int_{\mathbb{R}_0} \frac{\partial h}{\partial x}(t, a) r(t, a) \nu(da)
\]
and similarly for \( \frac{\partial f}{\partial x_1}(t), \frac{\partial f}{\partial x_2}(t), \frac{\partial f}{\partial u}(t), \frac{\partial f}{\partial y}(t), \frac{\partial f}{\partial z}(t) \) and \( \nabla_k f(t, a) \).
By the Itô formula and \((H_7)\), the local martingales which appear after integration by parts of the process \(\hat{\lambda}(t)\phi(t)\) are martingales, and we get

\[
E[h'(\hat{Y}(0)\phi(0))] = E[\hat{\lambda}(0)\phi(0)]
= \lim_{T \to \infty} E[\hat{\lambda}(T)\phi(T)]
= \lim_{T \to \infty} E\left[\int_0^T \left\{ \hat{\lambda}(t)(-\frac{\partial g}{\partial x_1}(t)\xi(t) - \frac{\partial g}{\partial x_2}(t)\xi(t) - \frac{\partial g}{\partial y}(t)) \int_0^t e^{-\rho(t-r)}\xi(r)dr - \frac{\partial g}{\partial y}(t)\phi(t)
- \frac{\partial g}{\partial z}(t)\eta(t) - \frac{\partial g}{\partial a}(t)\beta(t)) + \phi(t)\frac{\partial H}{\partial y}(t) + \eta(t)\frac{\partial H}{\partial z}(t) + \int \nabla_k H(t,a)\psi(t,a)\nu(da)\right\} dt\right].
\]

Substituting (4.8) into (4.7) we get

\[
0 = \frac{d}{ds} J(\hat{u} + s\beta) \mid_{s=0}
= E\left[\int_0^T \left\{ \frac{\partial f}{\partial x}(t)\xi(t) + \frac{\partial f}{\partial x_1}(t)\xi(t) - \frac{\partial f}{\partial x_2}(t)\int_0^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial f}{\partial y}(t)\phi(t)
+ \frac{\partial f}{\partial t}(t)\eta(t) + \frac{\partial f}{\partial a}(t)\beta(t) + \int \nabla_k f(t,a)\psi(t,a)\nu(da)\right\} dt\right].
\]

Applying the Itô formula to the process \(\hat{p}(t)\xi(t)\) and using \((H_7)\), we get

\[
0 = \lim_{T \to \infty} E[\hat{p}(T)\xi(T)]
= E\left[\int_0^T \left\{ \frac{\partial h}{\partial x}(t)\xi(t) + \frac{\partial h}{\partial x_1}(t)\xi(t) - \frac{\partial h}{\partial x_2}(t)\int_0^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial h}{\partial a}(t)\beta(t)\right\} dt\right] + \int_0^T \xi(t) E[\mu(t) \mid \mathcal{F}_t]dt + \int_0^T \hat{q}(t)\left\{ \frac{\partial h}{\partial x_2}(t)\xi(t) + \frac{\partial h}{\partial x_1}(t)\xi(t) - \frac{\partial h}{\partial y}(t)\int_0^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial h}{\partial u}(t)\beta(t)\right\} dt
+ \int_0^T \int \hat{r}(t,a)\left\{ \frac{\partial h}{\partial x_2}(t,a)\xi(t) + \frac{\partial h}{\partial x_1}(t,a)\xi(t) - \frac{\partial h}{\partial y}(t,a)\int_0^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial h}{\partial u}(t,a)\beta(t)\right\} \nu(da)dt\right] \mid_{s=0} + E\left[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt\right].
\]

Adding (4.9) and (4.10) we obtain

\[
E[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt] = 0.
\]

Now apply this to

\[
\beta(t) = \alpha 1_{[s,s+h]}(t),
\]

\[12\]
where $\alpha$ is bounded and $\mathcal{E}_{t_0}$-measurable, $s \geq t_0$. Then we get

$$E\left[ \int_{s}^{s+h} \frac{\partial H}{\partial u}(s) ds \alpha \right] = 0.$$  

Differentiating with respect to $h$ at $h = 0$ we obtain

$$E\left[ \frac{\partial H}{\partial u}(s) \alpha \right] = 0.$$  

Since this holds for all $s \geq t_0$ and all $\alpha$, we conclude

$$E\left[ \frac{\partial H}{\partial u}(t_0) \mid \mathcal{E}_{t_0} \right] = 0.$$  

This proves that (i) implies (ii).

(ii) $\implies$ (i):

The argument above shows that

$$\frac{d}{ds} J(u + s\beta) \bigg|_{s=0} = E\left[ \int_{0}^{\infty} \frac{\partial H}{\partial u}(t)\beta(t)dt \right],$$  

for all $u$, $\beta \in \mathcal{A}$ with $\beta$ bounded. So to complete the proof we use that every bounded $\beta \in \mathcal{A}$ can be approximated by linear combinations of controls $\beta$ of the form (4.1). We omit the details. \[\blacksquare\]

## 5 Application to optimal consumption with respect to recursive utility

### 5.1 A general optimal recursive utility problem

Let $X(t) = X^{(c)}(t)$ be a cash flow modeled by

$$\begin{cases}
    dX(t) = X(t - \delta)b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}} \gamma(t,a)\tilde{N}(dt,da) - c(t)dt; t \geq 0, \\
    X(0) = x > 0,
\end{cases}$$  

(5.1)

where $b_0(t)$, $\sigma_0(t)$ and $\gamma(t,a)$ are given bounded $\mathbb{F}$-predictable processes, $\delta \geq 0$ is a fixed delay and $\gamma(t,a) > -1$ for all $(t,a) \in [0,\infty) \times \mathbb{R}$.

The process $u(t) = c(t) \geq 0$ is our control process, interpreted as our relative consumption rate such that $X^{(c)}(t) > 0$ for all $t \geq 0$. We let $\mathcal{A}$ denote the family of all $\mathbb{E}$-predictable
relative consumption rates. To every \( c \in \mathcal{A} \) we associate a recursive utility process \( Y^{(c)}(t) = Y(t) \) defined as the solution of the infinite horizon BSDE

\[
Y(t) = E[Y(T) + \int_t^T g(s, Y(s), c(s)) \, ds \mid \mathcal{F}_t] \text{ for all } t \leq T, \tag{5.2}
\]

valid for all deterministic \( T < \infty \). The number \( Y^{(c)}(0) \) is called the recursive utility of consumption process \( c(t); t \geq 0 \) (See e.g. Duffie & Epstein (1992), [7]).

Suppose the solution \((Y, Z, K)\) of the infinite horizon BSDE \((5.2)\) satisfies the condition \((2.4)\) and let \( c(s); s \geq 0 \) be the consumption rate.

We assume that the function \( g(t, y, c) : \mathbb{R}_+^3 \to \mathbb{R} \) satisfies the following conditions:

1. \( g(t, y, c) \) is concave with respect to \( y \) and \( c \)
2. \[
\int_0^T E[|g(s, Y(s), c(s))|] \, ds < \infty, \text{ for all } c \in \mathcal{A}, T < \infty. \tag{5.3}
\]
3. \( \frac{\partial}{\partial c} g(t, y, c) \) has an inverse:

\[
I(t, v, y) = \begin{cases} 
0 & \text{if } v \geq v_0(t, y), \\
\left( \frac{\partial}{\partial c} g(t, y, c) \right)^{-1}(v) & \text{if } 0 \leq v \leq v_0(t, y),
\end{cases}
\]

where \( v_0(t, y) = \frac{\partial}{\partial c} g(t, y, 0) \).

We want to maximize the recursive utility \( Y^{(c)}(0) \), i.e. we want to find \( c^* \in \mathcal{A} \) such that

\[
\sup_{c \in \mathcal{A}} Y^{(c)}(0) = Y^{(c^*)}(0). \tag{5.4}
\]

We call such a process \( c^* \) an optimal recursive utility consumption rate.

We see that the problem \((5.4)\) is a special case of problem \((2.8)\) with

\[
J(u) = Y(0),
\]

\[
f = 0, \quad h(y) = y, \quad u = c \quad \text{and}
\]

\[
b(t, x, c) = x_1 b_0(t) - c, \\
\sigma(t, x, u) = x_1 \sigma_0(t), \\
\theta(t, x, u, a) = x_1 \gamma(t, a).
\]

In this case the Hamiltonian defined in \((2.9)\) takes the form
\[ H(t, x, y, z, c, p, q, r) = \lambda g(t, y, c) + (x_1 b_0(t) - c) p + x_1 \sigma_0(t) q + x_1 \int_{\mathbb{R}_0} \gamma(t, a) r(a) d\nu(da). \quad (5.5) \]

Maximizing \( E[H \mid \mathcal{E}_t] \) as a function of \( c \) gives the first order condition

\[ E[\lambda(t) \frac{\partial g}{\partial c}(t, Y(t), c(t)) \mid \mathcal{E}_t] = E[p(t) \mid \mathcal{E}_t], \quad (5.6) \]

for an optimal \( c(t) \).

The pair of adjoint processes \((2.10) - (2.11)\) is given by

\[
\begin{cases}
    d\lambda(t) = \lambda(t) \frac{\partial g}{\partial y}(t, Y(t), c(t)) dt, \\
    \lambda(0) = 1,
\end{cases}
\quad (5.7)
\]

and

\[ dp(t) = E[\mu(t) \mid \mathcal{F}_t] dt + q(t) dB(t) + \int_{\mathbb{R}_0} r(t, a) \tilde{N}(dt, da); t \in [0, \infty), \quad (5.8) \]

where

\[ \mu(t) = -[b_0(t + \delta)p(t + \delta) + \sigma_0(t + \delta)q(t + \delta) + \int_{\mathbb{R}_0} \gamma(t + \delta, a)r(t + \delta, a) d\nu(da)]. \quad (5.9) \]

with terminal condition as in \((2.4)\), i.e.

\[ E[\sup_{t \geq 0} e^{\kappa t} p^2(t) + \int_0^\infty e^{\kappa s} (q^2(s) + \int_{\mathbb{R}_0} r^2(s, a) d\nu(da)) ds] < \infty, \]

for all constants \( \kappa > 0 \).

Equation \((5.7)\) has the solution

\[ \lambda(t) = \exp \left( \int_0^t \frac{\partial g}{\partial y}(s, Y(s), c(s)) ds \right); t \geq 0 \quad (5.10) \]

which substituted into \((5.6)\) gives

\[ E[\frac{\partial g}{\partial c}(t, Y(t), c(t)) \exp \left( \int_0^t \frac{\partial g}{\partial y}(s, Y(s), c(s)) ds \right) \mid \mathcal{E}_t] = E[p(t) \mid \mathcal{E}_t]. \quad (5.11) \]

We refer to Theorem 5.1 in \cite{1} for a proof of the existence of the solution of the ABSDE \((5.8)\).
5.2 A solvable special case

In order to get a solvable case we choose the driver \( g \) in (5.2) to be of the form

\[
g(t, y, c) = -\alpha(t)y + \ln c,
\]

where \( \alpha(t) \geq \alpha > 0 \) is an \( \mathbb{F} \)-adapted process.
We also choose

\[
\delta = 0 \quad \text{and} \quad \mathcal{E}_t = \mathcal{F}_t; t \geq 0,
\]

and we represent the consumption rate \( c(t) \) as

\[
c(t) = \rho(t)X(t),
\]

where \( \rho(t) = \frac{c(t)}{X(t)} \geq 0 \) is the relative consumption rate.
We restrict our attention to processes \( c \) such that the wealth process, solution of (5.1), is strictly positive and \( \rho \) is bounded away from 0. This set of controls \( \rho \) is denoted by \( \mathcal{A} \).

The FBSDE system now has the form

\[
\begin{aligned}
dX(t) &= X(t^-)[(b_0(t) - \rho(t))dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,a)\tilde{N}(dt,da)]; t \geq 0, \\
X(0) &= x > 0,
\end{aligned}
\]

and

\[
Y(t) = Y^{(\rho)}(t) = E[Y(T) + \int_t^T (-\alpha(s)Y(s) + \ln \rho(s)X(s)) ds | \mathcal{F}_t],
\]

i.e.

\[
dY(t) = -(-\alpha(t)Y(t) + \ln \rho(t) + \ln(X(t)) dt + Z(t)dB(t); t \geq 0.
\]

We want to find \( \rho^* \in \mathcal{A} \) such that

\[
\sup_{\rho \in \mathcal{A}} Y^{(\rho)}(0) = Y^{(\rho^*)}(0).
\]

In this case the Hamiltonian (2.9) gets the form

\[
H(t, x, y, \rho, \lambda, p, q, r) = \lambda(-\alpha(t)y + \ln \rho + \ln x) + x(b_0(t) - \rho)p \\
+ x\sigma_0(t)q + x \int_{\mathbb{R}_0} \gamma(t,a)r(a)\nu(da).
\]

Maximizing \( H \) with respect to \( \rho \) gives the first order equation

\[
\lambda(t) \frac{1}{\rho(t)} = p(t)X(t),
\]
where, by (1.10) – (1.11) \( \lambda(t) \) and \((p(t), q(t), r(t, \cdot))\) satisfy the FBSDEs
\[
\begin{aligned}
\left\{ \begin{array}{l}
d\lambda(t) = -\alpha(t)\lambda(t)dt, \\
\lambda(0) = 1,
\end{array} \right.
\end{aligned}
\tag{5.21}
\]
and
\[
\begin{aligned}
dp(t) &= -\left[\lambda(t)\frac{1}{X(t)} + (b(t) - \rho(t)) p(t) + \sigma(t)q(t) + \int_{\mathbb{R}_0} \gamma(t, a)r(a)\nu(da)\right]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t, a)\tilde{N}(dt, da),
\end{aligned}
\tag{5.22}
\]
with terminal condition as in (2.4), i.e.
\[
\begin{aligned}
E[\sup_{t \geq 0} e^{\kappa t}p^2(t) + \int_{0}^{\infty} e^{\kappa s}q^2(s) + \int_{\mathbb{R}_0} r^2(s, a)\nu(da)ds] < \infty,
\end{aligned}
\tag{5.23}
\]
for all constants \( \kappa > 0 \).
The infinite horizon BSDE (5.22) – (5.23) has a unique solution, (see e.g. Theorem 3.1 in [9]).
Then, the solutions of (5.21) – (5.22) are respectively,
\[
\begin{aligned}
\lambda(t) &= \exp(-\int_{0}^{t} \alpha(s)ds),
\end{aligned}
\tag{5.24}
\]
and, for all \( 0 \leq t \leq T \) and all \( T < \infty \),
\[
\begin{aligned}
p(t)\Gamma(t) &= E[p(T)\Gamma(T) + \int_{t}^{T} \lambda(s)\frac{\Gamma(s)}{X(s)}ds | \mathcal{F}_t],
\end{aligned}
\tag{5.25}
\]
where \( \Gamma(t) \) is given by
\[
\begin{aligned}
\left\{ \begin{array}{l}
d\Gamma(t) = \Gamma(t^-)[(b(t) - \rho(t))]dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, a)\tilde{N}(dt, da); t \geq 0,
\\
\Gamma(0) = 1.
\end{array} \right.
\end{aligned}
\tag{5.26}
\]
(See e.g. [16]).
This gives
\[
\begin{aligned}
\Gamma(t) &= \exp\left(-\int_{0}^{t} \sigma(s)dB(s) + \int_{0}^{t} \{b_0(s) - \rho(s) - \frac{1}{2}\sigma^2(s)\}ds \right. \\
&\left. + \int_{0}^{t} \int_{\mathbb{R}_0} \{\ln(1 + \gamma(s, a)) - \gamma(s, a)\}\nu(da)ds \\
&\left. + \int_{0}^{t} \int_{\mathbb{R}_0} \ln(1 + \gamma(s, a))\tilde{N}(ds, da); t \geq 0. \right)
\end{aligned}
\tag{5.27}
\]
Comparing with (5.15) we see that
\[
\begin{aligned}
X(t) &= x\Gamma(t); t \geq 0.
\end{aligned}
\tag{5.28}
\]
Substituting this into (5.25) we obtain
\[ p(t)X(t) = E[p(T)X(T) + \int_t^T \exp(-\int_0^s \alpha(r) dr) ds | \mathcal{F}_t]. \] (5.29)

Since \( \rho \) is bounded away from 0 we deduce from (5.20) that
\[ p(T)X(T) = \frac{\lambda(T)}{\rho(T)} = \frac{1}{\rho(T)} \exp\left(-\int_0^T \alpha(r) dr\right) \to 0 \text{ dominatedly as } T \to \infty. \] (5.30)

Hence, by letting \( T \to \infty \) in (5.29) we get
\[ p(t)X(t) = E[\int_t^\infty \exp(-\int_0^s \alpha(r) dr) ds | \mathcal{F}_t]. \] (5.31)

This implies that \( p(t) > 0 \) and hence \( \rho(t) \) given by (5.20) is indeed a maximum point of \( H \).

By (4.20) we therefore get the following candidate for the optimal relative consumption rate
\[ \rho(t) = \rho^*(t) = \frac{\exp\left(-\int_0^t \alpha(r) dr\right)}{E[\int_t^\infty \exp(-\int_0^s \alpha(r) dr) ds | \mathcal{F}_t]} ; t \geq 0, \] (5.32)

If \( \alpha \) is such that this expression for \( \rho^*(t) \) is bounded away from 0, then \( \rho^* \) is optimal. Note that the corresponding optimal net cash flow \( X^*(t) \) is given by
\[ X^*(t) = x \exp\left(\int_0^t \sigma_0(s) dB(s) + \int_0^t \{b_0(s) - \rho(s) - \frac{1}{2}\sigma_0^2(s)\} ds\right); t \geq 0. \] (5.33)

In particular, \( X^*(t) > 0 \) for all \( t \geq 0 \), as required.
In particular, if \( \alpha(r) = \alpha > 0 \) (constant) for all \( r \), then
\[ \rho^*(t) = \alpha; t \geq 0. \] (5.34)

With this choice of \( \rho^* \) we see by (5.31), (5.24) and condition (2.4) for \( Y(t) \) that the transversality conditions (\( H_3 \)) and (\( H_6 \)) hold, and we have proved:

**Theorem 5.1** The optimal relative consumption rate \( \rho^*(t) \) for problem (5.12) \( – \) (5.18) is given by (5.32), provided that \( \rho^*(t) \) is bounded away from 0.
In particular, if \( \alpha(r) = \alpha > 0 \) (constant) for all \( r \), then \( \rho^*(t) = \alpha \) for all \( t \).

**Acknowledgment** We want to thank Brahim Mezerdi for helpful discussions.
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