MULTIPlicity AND ORBITAL STABILITY OF NORMALIZED SOLUTIONS TO NON-AUTONOMOUS SCHröDINGER EQUATION WITH MIXED NONLINEARITIES

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Abstract. This paper studies the multiplicity of normalized solutions to the Schrödinger equation with mixed nonlinearities
\begin{equation}
\begin{cases}
-\Delta u = \lambda u + h(\epsilon x)|u|^{q-2}u + \eta |u|^{p-2}u, & x \in \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}
\end{equation}
where $a, \epsilon, \eta > 0$, $q$ is $L^2$-subcritical, $p$ is $L^2$-supercritical, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $h$ is a positive and continuous function. It is proved that the numbers of normalized solutions are at least the numbers of global maximum points of $h$ when $\epsilon$ is small enough. Moreover, the orbital stability of the solutions obtained is analyzed as well. In particular, our results cover the Sobolev critical case $p = 2N/(N-2)$.

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1. Introduction and main results

In this paper, we study the multiplicity and orbital stability of normalized solutions to the non-autonomous Schrödinger equation with mixed nonlinearities
\begin{equation}
\begin{cases}
-\Delta u = \lambda u + h(\epsilon x)|u|^{q-2}u + \eta |u|^{p-2}u, & x \in \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |u|^2 dx = a^2,
\end{cases}
\end{equation}
where $N \geq 1$, $a, \epsilon, \eta > 0$, $2 < q < 2 + \frac{4}{N}$, $2^* := \frac{2N}{N-2}$, $N \geq 3$, and $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier. The function $h$ satisfies the following conditions:
\begin{enumerate}
\item[(h\textsubscript{1})] $h \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < h_0 = \inf_{x \in \mathbb{R}^N} h(x) \leq \max_{x \in \mathbb{R}^N} h(x) = h_{\text{max}}$;
\item[(h\textsubscript{2})] $h_{\infty} = \lim_{|x| \to +\infty} h(x) < h_{\text{max}}$;
\item[(h\textsubscript{3})] $h^{-1}(h_{\text{max}}) = \{a_1, a_2, \ldots, a_l\}$ with $a_1 = 0$ and $a_j \neq a_i$ if $i \neq j$.
\end{enumerate}

A solution $u$ to the problem (1.1) corresponds to a critical point of the functional
\begin{equation}
E_\epsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon x)|u|^{q} dx - \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^{p} dx
\end{equation}
restricted to the sphere
\begin{equation}
S(a) := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a^2\}.
\end{equation}

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It is well known that $E_e \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$E'_e(\varphi) = \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx - \int_{\mathbb{R}^N} h(\varepsilon x)|u|^{q-2}u \varphi dx - \eta \int_{\mathbb{R}^N} |u|^{p-2}u \varphi dx$$

for any $\varphi \in H^1(\mathbb{R}^N)$.

One motivation driving the search for normalized solutions to (1.1) is the non-linear Schrödinger equation

$$\frac{\partial \psi}{\partial t} + \Delta \psi + g(|\psi|^2)\psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.3)$$

Since the mass $\int_{\mathbb{R}^N} |\psi|^2 dx$ is preserved along the flow associated with (1.3), it is natural to consider it as prescribed. Searching for standing wave solution $\psi(t, x) = e^{-i\lambda t}u(x)$ of (1.3) leads to (1.1) for $u$ with $g(|s|^2)s = h(\varepsilon x)|s|^{q-2}s + \eta|s|^{p-2}s$. In recent decades, the research of finding normalized solutions to Schrödinger equations has received a special attention. This seems to be particularly meaningful from the physical point of view, as the $L^2$-norm is a preserved quantity of the evolution and the variational characterization of such solutions is often a strong help to analyze their orbital stability, see [7, 9, 28, 29] and the references therein.

In the study of normalized solutions to the Schrödinger equation

$$-\Delta u = \lambda u + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \quad (1.4)$$

the number $\bar{p} := 2 + 4/N$, labeled $L^2$-critical exponent, is a very important number, because in the study of (1.4) using variational methods, the functional

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx, \quad u \in H^1(\mathbb{R}^N)$$

is bounded from below on $S(a)$ for the $L^2$-subcritical problem, i.e., $2 < p < 2 + 4/N$. Thus, a solution of (1.4) can be found as a global minimizer of $J|_{S(a)}$, see [27, 31]. For the purely $L^2$-supercritical problem, i.e., $2 + 4/N < p < 2^*$, $J|_{S(a)}$ is unbounded from below (and from above). Related to this case, a seminar paper due to Jeanjean [15] exploited the mountain pass geometry to get a normalized solution, see [3, 4, 8, 12, 14, 18, 19] for more results. In the purely $L^2$-critical case (i.e., $p = 2 + 4/N$), the result is delicate. Recently, the Schrödinger equation with double power form nonlinearity $\mu|u|^{r-2}u + |u|^{p-2}u$ has been extensively studied due to Soave [28, 29], see [2, 16, 17, 20, 30, 33] for more results. The multiplicity of normalized solutions to the autonomous Schrödinger equation or systems has also been considered extensively at the last years, see [2, 3, 5, 10, 12, 14, 17, 18, 19, 23, 24]. As for the existence of normalized solutions to the non-autonomous Schrödinger equation

$$-\Delta u = \lambda u + f(x, u), \quad x \in \mathbb{R}^N, \quad (1.5)$$

we refer to [6, 13, 21, 22, 25] and the references therein.

Our study is motivated by Alves [1], where they considered the multiplicity of normalized solutions to

$$-\Delta u = \lambda u + h(\varepsilon x)f(u), \quad x \in \mathbb{R}^N \quad (1.6)$$

with $f$ being $L^2$-subcritical. Their arguments depend on the existence of global minimizer and the relative compactness of any minimizing sequence of the functional $\tilde{J}|_{S(a)}$ corresponding to the limit problem

$$-\Delta u = \lambda u + \mu f(u), \quad x \in \mathbb{R}^N. \quad (1.7)$$
While in our problem (1.1), the appearance of the $L^2$-supercritical term $\eta|u|^{p-2}u$ makes the functional to the limit problem
\begin{equation}
-\Delta u = \lambda u + \mu|u|^{q-2}u + \eta|u|^{p-2}u, \quad x \in \mathbb{R}^N
\end{equation}
with $q < 2 + 4/N < p$ is unbounded from below (and from above). But in view of the studies of [17, 28], we know that the functional in this case has a local minimizer. So employing the truncated skill used in [2, 26], we can isolate the local minimizer and obtain the multiplicity of normalized solutions to the problem (1.1). The application of truncated functions and the appearance of the Sobolev critical exponent $p = 2^*$ make more delicate analysis is needed. Furthermore, we also consider the orbital stability of the solutions obtained (see Section 5). We should point out that in [2], the authors studied the multiplicity of normalized solutions to the autonomous Schrödinger equation (1.8) with $q < 2 + 4/N < p = 2^*$ in radial symmetry space $H^1_{rad}(\mathbb{R}^N)$ by using truncated skill and genus theory. Note that our problem (1.1) is non-autonomous and not radially symmetry, so their method is not work in our problem.

The main results of this paper are as follows.

**Theorem 1.1.** Let $N, \epsilon, a, \eta, p, q, h$ be as in (1.1). We further assume that $h_{max}, a$ and $\eta$ satisfy some condition (i.e., (2.3)) holds for $p < 2^*$, (2.3) and (2.6) hold for $p = 2^*$). Then there exists $\epsilon_0 > 0$ such that (1.1) admits at least $l$ couples $(u_j, \lambda_j) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $\epsilon \in (0, \epsilon_0)$ with $\int_{\mathbb{R}^N} |u_j|^2dx = a^2$, $\lambda_j < 0$ and $E_\epsilon(u_j) < 0$ for $j = 1, 2, \cdots, l$.

**Theorem 1.2.** The solutions obtained in Theorem 1.1 is orbitally stable in some sense. To state this theorem, we need some notations used in the proof of Theorem 1.1, so we give the details in Section 5. We point out that we give the stability of $l$ different sets, which is very different from the existing results.

**Remark 1.3.** In [16, 17, 28, 29], the authors considered the normalized solutions to (1.8) with $q < 2 + 4/N < p$. They obtained a ground state solution to (1.8) with negative energy which is local minimizer and orbitally stable and a mountain-pass type solution with positive energy which is strongly instable. In this paper, the solutions obtained in Theorem 1.1 are also local minimizers, but we do not know whether they are ground state solutions. The appearance of the potential $h$ increases the number of the local minimizer and maintains its stability.

This paper is organized as follows. In Section 2, we define the truncated functional used in the study. In Section 3, we study the properties of the truncated autonomous functional. In Section 4, we study the truncated non-autonomous problem and give the proof of Theorem 1.1. In Section 5, we study the orbital stability of the solutions obtained in Theorem 1.1.

**Notation:** For $t \geq 1$, the $L^t$-norm of $u \in L^t(\mathbb{R}^N)$ is denoted by $\|u\|_t$. The usual norm of $u \in H^1(\mathbb{R}^N)$ is denoted by $\|u\|$. $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant and may be change from line to line. $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$. ’$\to$’ denotes strong convergence and ’$\rightharpoonup$’ denotes weak convergence. $B_r(x_0) := \{x \in \mathbb{R}^N: |x - x_0| < r\}$.

2. **Truncated functionals**

In the proof of Theorem 1.1, we will adapt for our case a truncated function found in Peral Alonso ([26], Chapter 2, Theorem 2.4.6).
In what follows, we will consider the functional \( E_{\varepsilon} \) given by (1.2) restricts to \( S(a) \). By the Sobolev embedding and the Gagliardo-Nirenberg inequality (see [34])

\[
\begin{align*}
&\|u\|_t \leq C_{N,t} \|u\|_t^{1-\gamma_t} \|\nabla u\|_t^{\gamma_t}, \quad 2 < t < \infty, \quad N = 1, 2, \\
&\|u\|_t \leq C_{N,2^*} S^{-\frac{4}{2}}, \quad \text{and} \quad S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \frac{\int_{\mathbb{R}^N} |u|^{2^*} \, dx}{2} \right)^{2/2}} \right\},
\end{align*}
\]

we have

\[
E_{\varepsilon}(u) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{q} \|u\|_q dx - \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p \, dx
\]

\[
\geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{q} \|u\|_q^q - \frac{\eta}{p} \|u\|_p^p = g_\alpha(\|\nabla u\|_2)
\]

for any \( u \in S(a) \), where

\[
g_\alpha(r) := \frac{1}{2} r^2 - \frac{1}{q} \|u\|_q^q - \frac{\eta}{p} \|u\|_p^p, \quad r > 0.
\]

Set \( g_\alpha(r) = r^2 w_\alpha(r) \) with

\[
w_\alpha(r) := \frac{1}{2} - \frac{1}{q} \|u\|_q^q - \frac{\eta}{p} \|u\|_p^p, \quad r > 0.
\]

Now we study the properties of \( w_\alpha(r) \). Note that

\[
t\gamma_t \begin{cases} 
< 2, & 2 < t < 2 + 4/N, \\
= 2, & t = 2 + 4/N, \\
> 2, & 2 + 4/N < t \leq 2^*
\end{cases}
\]

and \( \gamma_{2^*} = 1 \).

It is obvious that \( \lim_{t \to -q \gamma_{2^*}} w_\alpha(r) = -\infty \) and \( \lim_{t \to +\infty} w_\alpha(r) = -\infty \). By direct calculations, we obtain that

\[
w_\alpha'(r) = \frac{1}{q} \|u\|_q^q - \frac{\eta}{p} \|u\|_p^p.
\]

Then the equation \( w_\alpha'(r) = 0 \) has a unique solution

\[
r_0 = \left( \frac{(2 - q \gamma_q) \frac{1}{q} \|u\|_q^q - \frac{\eta}{p} \|u\|_p^p}{(2 \gamma_\gamma - 2) \frac{1}{q} \|u\|_q^q - \frac{\eta}{p} \|u\|_p^p} \right)^{\frac{2 - \gamma_q}{2 - q \gamma_q}}
\]

and the maximum of \( w_\alpha(r) \) is

\[
w_\alpha(r_0) = \frac{1}{2} - B \left( \|u\|_q^q \right)^{\frac{2 - \gamma_q}{2 - q \gamma_q}} \left( \frac{\eta}{p} \right)^{\frac{2 - q \gamma_q}{2 - q \gamma_q}},
\]

where

\[
B = \frac{p \gamma_q - q \gamma_q}{2 - q \gamma_q} \left( \frac{2 - q \gamma_q}{2 \gamma_\gamma - q \gamma_q} \right)^{\frac{p \gamma_q - 2}{p \gamma_q - q \gamma_q}} \left( \frac{C_{N,q}}{q} \right)^{\frac{p \gamma_q - 2}{p \gamma_q - q \gamma_q}} \left( \frac{C_{N,p}}{p} \right)^{\frac{2 - q \gamma_q}{2 - q \gamma_q}}.
\]

Thus, if we assume

\[
\left( \|u\|_q^q \right)^{\frac{2 - \gamma_q}{2 - q \gamma_q}} \left( \frac{\eta}{p} \right)^{\frac{2 - q \gamma_q}{2 - q \gamma_q}} < \frac{1}{2B^2},
\]

we have
then the maximum of \( w_a(r) \) is positive and \( w_a(r) \) has exactly two zeros \( 0 < R_0 < R_1 < \infty \), which are also the zeros of \( g_a(r) \). It is obvious that \( g_a(r) \) has the following properties

\[
\begin{align*}
& g_a(0) = g_a(R_0) = g_a(R_1) = 0; \\
& g_a(r) < 0 \text{ for } r > 0 \text{ small}; \\
& \lim_{r \to +\infty} g_a(r) = -\infty; \\
& \text{\( r_1 \in (0, R_0) \) and \( r_2 \in (R_0, R_1) \) with } g_a(r_1) < 0 \text{ and } g_a(r_2) > 0.
\end{align*}
\]

(2.4)

For \( p = 2^* \), we further assume that \( R_0 < \eta^{-\frac{\gamma}{2}} N \frac{2}{p} \), which is satisfied if we assume

\[
r_0 < \eta^{-\frac{\gamma}{2}} N \frac{2}{p},
\]

(2.5)

because \( R_0 < r_0 < R_1 \). By the expression of \( r_0, (2.5) \) is equivalent to

\[
(\max h_a \eta^{1-\gamma})^{\frac{1}{p}} \frac{\eta^{\frac{\gamma}{2}}}{p-\eta q} \leq \left( \frac{(2-q\gamma)C_{N,q}^{*} 2^* N \frac{2}{p} S^N}{q(p-2)} \right)^{\frac{1}{p-\eta q}} S^N.
\]

(2.6)

Now fix \( \tau : (0, +\infty) \to [0, 1] \) as being a non-increasing and \( C^\infty \) function that satisfies

\[
\tau(x) = \begin{cases} 1, & \text{if } x \leq R_0, \\
0, & \text{if } x \geq R_1
\end{cases}
\]

(2.7)

and consider the truncated functional

\[
E_{\tau,T}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} h(\varepsilon x) |u|^q dx - \frac{\eta}{p} \tau(\|\nabla u\|_2) \int_{\mathbb{R}^N} |u|^p dx.
\]

(2.8)

Thus,

\[
E_{\tau,T}(u) \geq \frac{\eta}{2} \|\nabla u\|_2^2 - \frac{1}{q} \max h\eta^{1-\gamma} \|\nabla u\|_2^{\gamma q}
\]

(2.9)

for any \( u \in S(a) \), where

\[
\bar{g}_a(r) := \frac{1}{2} r^2 - \frac{1}{q} \max h \eta^{1-\gamma} q r^{\gamma q} - \frac{\eta}{p} \tau(\|\nabla u\|_2) C_{N,p}^{*} \eta^{1-\gamma} r^{p \gamma p}.
\]

It is easy to see that \( \bar{g}_a(r) \) has the following properties

\[
\begin{align*}
& \bar{g}_a(r) \equiv g_a(r) \text{ for all } r \in [0, R_0]; \\
& \bar{g}_a(r) \text{ is positive and strictly increasing in } (R_0, +\infty).
\end{align*}
\]

(2.10)

Correspondingly, for any \( \mu \in (0, \max h) \), we denote by \( J_\mu \), \( J_{\mu,T} : H^1(\mathbb{R}^N) \to \mathbb{R} \) the following functionals

\[
J_\mu(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx - \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p dx
\]

(2.11)

and

\[
J_{\mu,T}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx - \frac{\eta}{p} \tau(\|\nabla u\|_2) \int_{\mathbb{R}^N} |u|^p dx.
\]

(2.12)

The properties of \( J_{\mu,T} \) and \( E_{\tau,T} \) will be studied in Sections 3 and 4, respectively.
3. The truncated autonomous functional

In this section, we study the properties of the functional $J_{\mu,T}$ defined in (2.12) restricted to $S(a_1)$, where $\mu \in (0, h_{\max})$ and $a_1 \in (0, a]$.

**Lemma 3.1.** Let $N, a, \eta, p, q$ be as in (1.1), (2.3) hold, $\mu \in (0, h_{\max})$, $0 < a_1 \leq a$. Then the functional $J_{\mu,T}$ is bounded from below in $S(a_1)$.

**Proof.** By (2.9) and (2.10), for any $u \in S(a_1)$,

$$J_{\mu,T}(u) \geq J_{\max,T}(u)$$

$$\geq \frac{1}{2}||\nabla u||^2 + \frac{1}{q} h_{\max} C_{N,q} a_1^{q(1-\gamma_q)} ||\nabla u||^{\gamma_q}$$

$$- \frac{\eta}{p} \tau (||\nabla u||^2) C_{N,p} a_1^{p(1-\gamma_p)} ||\nabla u||^{\gamma_p}$$

$$\geq \bar{g}_a (||\nabla u||) \geq \inf_{r \geq 0} \bar{g}_a (r) > -\infty.$$

The proof is complete. \hfill $\square$

**Lemma 3.2.** Let $N, a, \eta, p, q$ be as in (1.1), (2.3) hold, $\mu \in (0, h_{\max})$, $0 < a_1 \leq a$. $\Upsilon_{\mu,T,a_1} := \inf_{u \in S(a_1)} J_{\mu,T}(u) < 0$.

**Proof.** Fix $u \in S(a_1)$. For $t > 0$, we define $u_t(x) = tN^u(tx)$. Then $u_t \in S(a_1)$ for all $t > 0$. By $\tau \geq 0$ and $q\gamma_q < 2$, we obtain that

$$J_{\mu,T}(u_t) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_t|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u_t|^q dx$$

$$= \frac{1}{2} t^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx$$

$$< 0$$

for $t > 0$ small enough. Thus $\Upsilon_{\mu,T,a_1} < 0$. The proof is complete. \hfill $\square$

**Lemma 3.3.** Let $N, a, \eta, p, q$ be as in (1.1), (2.3) hold, $\mu \in (0, h_{\max})$. Then

(1) $J_{\mu,T} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$;

(2) Let $a_1 \in (0, a]$. If $u \in S(a_1)$ such that $J_{\mu,T}(u) < 0$, then $||\nabla u|| < R_0$ and $J_{\mu,T}(v) = J_{\mu}(v)$ for all $v$ satisfying $||v|| < a$ and being in a small neighborhood of $u$ in $H^1(\mathbb{R}^N)$.

**Proof.** (1) is trivial. Now we prove (2). It follows from $J_{\mu,T}(u) < 0$ and

$$J_{\mu,T}(u) \geq \bar{g}_a (||\nabla u||^2) \geq \bar{g}_a (||\nabla u||)$$

that $\bar{g}_a (||\nabla u||^2) < 0$, which implies that $||\nabla u|| < R_0$ by (2.10). By (1) and $J_{\mu,T}(u) < 0$, we obtain that $J_{\mu,T}(v) < 0$ for all $v$ in a small neighborhood of $u$ in $H^1(\mathbb{R}^N)$, which combined with $||v|| < a$ gives that $||\nabla v|| < R_0$ and thus $J_{\mu,T}(v) = J_{\mu}(v)$. The proof is complete. \hfill $\square$

For any $a_1 \in (0, a]$, we define

$$m_\mu(a_1) := \inf_{u \in V(a_1)} J_{\mu}(u), \ V(a_1) := \{ u \in S(a_1) : ||\nabla u|| < R_0 \}.$$

Since $J_{\mu,T}(u) \geq \bar{g}_a (||\nabla u||_2) \geq \bar{g}_a (||\nabla u||)$ for any $u \in S(a_1)$, by Lemmas 3.1-3.3, we obtain that

(3.1) $\Upsilon_{\mu,T,a_1} = \inf_{u \in S(a_1)} J_{\mu,T}(u) = m_\mu(a_1)$.

In ([16], Lemma 2.6 and Theorem 1.2), the authors obtained that
Lemma 3.4. Let $N,a,η,p,q$ be as in (1.1), (2.3) hold, $μ ∈ (0, h_{\text{max}}]$. Then
(1) $a_1 ∈ (0, a] \mapsto m_μ(a_1)$ is continuous;
(2) Let $0 < a_1 < a_2 ≤ a$, then $\frac{a_2^2}{a_1^2} m_μ(a_2) < m_μ(a_1) < 0$.

Consequently, by (3.1) and Lemma 3.4, we obtain that

Lemma 3.5. Let $N,a,η,p,q$ be as in (1.1), (2.3) hold, $μ ∈ (0, h_{\text{max}}]$. Then
(1) $a_1 ∈ (0, a] \mapsto Υ_{μ,T,a_1}^{\pm}$ is continuous;
(2) Let $0 < a_1 < a_2 ≤ a$, then $\frac{a_2^2}{a_1^2} Υ_{μ,T,a_2} < Υ_{μ,T,a_1} < 0$.

The next compactness lemma is useful in the study of the autonomous problem as well as in the non-autonomous problem.

Lemma 3.6. Let $N,a,η,p,q$ be as in (1.1), (2.3) hold, $μ ∈ (0, h_{\text{max}}]$, $a_1 ∈ (0,a]$. Let $\{u_n\} ⊂ S(a_1)$ be a minimizing sequence with respect to $Υ_{μ,T,a_1}$. Then, for some subsequence, either
(i) $\{u_n\}$ is strongly convergent,
or (ii) There exists $\{y_n\} ⊂ \mathbb{R}^N$ with $|y_n| → ∞$ such that the sequence $v_n(x) = u_n(x + y_n)$ is strongly convergent to a function $v ∈ S(a_1)$ with $J_{μ,T}(v) = Υ_{μ,T,a_1}$.

Proof. Noting that $\|∇u_n\|_2 < R_0$ for $n$ large enough, there exists $u ∈ H^1(\mathbb{R}^N)$ such that $u_n → u$ in $H^1(\mathbb{R}^N)$ for some subsequence. Now we consider the following three possibilities.

(1) If $u ≠ 0$ and $||u||_2 = b ≠ a_1$, we must have $b ∈ (0, a_1)$. Setting $v_n = u_n - u$, $d_n = ||v_n||_2$, and by using

$||u_n||_2^2 = ||v_n||_2^2 + ||u||_2^2 + o_n(1)$,

we obtain that $||v_n||_2 → d$, where $a_1^2 = d^2 + b^2$. Noting that $d_n ∈ (0, a_1)$ for $n$ large enough, and using the Brézis-Lieb Lemma (see [35]), Lemma 3.5, $\|∇u_n\|_2^2 = \|∇v_n\|_2^2 + \|∇u_n\|_2^2 + o_n(1)$, $\|∇u\|_2^2 ≤ \lim inf_{n→+∞} \|∇u_n\|_2^2$, $τ$ is continuous and non-increasing, we obtain that

$Υ_{μ,T,a_1} + o_n(1) = J_{μ,T}(u_n) = \frac{1}{2} ||∇v_n||_2^2 - \frac{μ}{q} ||v_n||_q^q - \frac{η}{p} τ(||∇u_n||_2)||v_n||_p^p$

$+ \frac{1}{2} ||∇u||_2^2 - \frac{μ}{q} ||u||_q^q - \frac{η}{p} τ(||∇u||_2)||u||_p^p + o_n(1)$

$≥ J_{μ,T}(v_n) + J_{μ,T}(u) + o_n(1)$

$≥ Υ_{μ,T,d_n} + Υ_{μ,T,b} + o_n(1)$

$≥ \frac{d_n^2}{a_1^2} Υ_{μ,T,a_1} + Υ_{μ,T,b} + o_n(1)$.

Letting $n → +∞$, we find that

$Υ_{μ,T,a_1} ≥ \frac{d_n^2}{a_1^2} Υ_{μ,T,a_1} + Υ_{μ,T,b}$

$> \frac{d_n^2}{a_1^2} Υ_{μ,T,a_1} + \frac{b_1^2}{a_1^2} Υ_{μ,T,a_1} = Υ_{μ,T,a_1}$,

which is a contradiction. So this possibility can not exist.

(2) If $||u||_2 = a_1$, then $u_n → u$ in $L^2(\mathbb{R}^N)$ and thus $u_n → u$ in $L^t(\mathbb{R}^N)$ for all $t ∈ (2, 2^*)$. 


Case $p < 2^*$, then

$$
\Upsilon_{\mu,T,a_1} = \lim_{n \to +\infty} J_{\mu,T}(u_n)
= \lim_{n \to +\infty} \left( \frac{1}{2} \|\nabla u_n\|_2^2 - \frac{\mu}{q} \|u_n\|_q^q - \frac{\eta}{p} \tau(\|\nabla u_n\|_2)\|u_n\|_p^p \right)
\geq J_{\mu,T}(u).
$$

As $u \in S(a_1)$, we infer that $J_{\mu,T}(u) = \Upsilon_{\mu,T,a_1}$, then $\|\nabla u_n\|_2 \to \|\nabla u\|_2$ and thus $u_n \to u$ in $H^1(\mathbb{R}^N)$, which implies that (i) occurs.

Case $p = 2^*$, noting that $\|\nabla v_n\|_2 \leq \|\nabla u_n\|_2 < R_0$ for $n$ large enough, and using the Sobolev inequality, we have

$$
J_{\mu,T}(v_n) = J_{\mu}(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |v_n|^q dx - \frac{\eta}{p} \int_{\mathbb{R}^N} |v_n|^p dx
\geq \frac{1}{2} \|\nabla v_n\|^2_2 - \frac{\eta}{2^*} S^{2^*}_{q\gamma} \|\nabla v_n\|^{2^*-2}_2 + o_n(1)
\geq \|\nabla v_n\|^2_2 \left( \frac{1}{2} - \frac{\eta}{2^*} S^{2^*}_{q\gamma} \right) + o_n(1)
\geq \|\nabla v_n\|^2_2 \left( \frac{1}{2} - \frac{\eta}{2^*} S^{2^*}_{q\gamma} R_0^{2^*-2} \right) + o_n(1)
\geq \|\nabla v_n\|^2_2 \frac{1}{q} h_{\text{max}} C_{N,q}^q \gamma(1 - \gamma) R_0^{\gamma_q - 2} + o_n(1),
$$

because $w_\alpha(R_0) = \frac{1}{2} - \frac{\eta}{2^*} S^{2^*}_{q\gamma} R_0^{2^*-2} - \frac{1}{q} h_{\text{max}} C_{N,q}^q \gamma(1 - \gamma) R_0^{\gamma_q - 2} = 0$. Now we remember that

$$
\Upsilon_{\mu,T,a_1} \leq J_{\mu,T}(u_n) \geq J_{\mu,T}(v_n) + J_{\mu,T}(u) + o_n(1).
$$

Since $u \in S(a_1)$, we have $J_{\mu,T}(u) \geq \Upsilon_{\mu,T,a_1}$, which combined with (3.2) and (3.3) gives that $\|\nabla v_n\|_2 \to 0$ and then $u_n \to u$ in $H^1(\mathbb{R}^N)$. This implies that (i) occurs.

(3) If $u \equiv 0$, that is, $u_n \to 0$ in $H^1(\mathbb{R}^N)$. We claim that there exist $R, \beta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$
\int_{B_R(y_n)} |u_n|^2 dx \geq \beta, \text{ for all } n.
$$

Indeed, otherwise we must have $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2^*)$. Thus, for $p < 2^*$, $J_{\mu,T}(u_n) \geq \frac{1}{2} \|\nabla u_n\|_2^2 + o_n(1)$, which contradicts $J_{\mu,T}(u_n) \to \Upsilon_{\mu,T,a_1} < 0$. For $p = 2^*$, similarly to (3.2), we obtain that

$$
J_{\mu,T}(u_n) \geq \|\nabla u_n\|^2_2 \frac{1}{q} h_{\text{max}} C_{N,q}^q \gamma(1 - \gamma) R_0^{\gamma_q - 2} + o_n(1),
$$

We also get a contradiction in this case. Hence, in all cases, (3.4) holds and $\|y_n\| \to +\infty$ obviously. From this, considering $\bar{u}_n(x) = u_n(x + y_n)$, clearly $\{\bar{u}_n\} \subset S(a_1)$ and it is also a minimizing sequence with respect to $\Upsilon_{\mu,T,a_1}$. Moreover, there exists $\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\bar{u}_n \to \bar{u}$ in $H^1(\mathbb{R}^N)$. Following as in the first two possibilities of the proof, we derive that $\bar{u}_n \to \bar{u}$ in $H^1(\mathbb{R}^N)$, which implies that (ii) occurs. This proves the lemma. \hfill \Box

**Lemma 3.7.** Let $N, a, \eta, p, q$ be as in (1.1), $\mu \in (0, h_{\text{max}}]$, $a_1 \in (0, a]$, (2.3) hold. Then $\Upsilon_{\mu,T,a_1}$ is attained.
Proof. By Lemmas 3.1-3.3, there exists a bounded minimizing sequence \( \{ u_n \} \subset S(a_1) \) satisfying \( J_{\mu,T}(u_n) \to \Upsilon_{\mu,T,a_1} \) as \( n \to +\infty \). Now, applying Lemma 3.6, there exists \( u \in S(a_1) \) such that \( J_{\mu,T}(u) = \Upsilon_{\mu,T,a_1} \). The proof is complete.

An immediate consequence of Lemma 3.7 is the following corollary.

**Corollary 3.8.** Let \( N, a, \eta, p, q \) be as in (1.1), (2.3) hold. Fix \( a_1 \in (0,a) \) and let \( 0 < \mu_1 < \mu_2 \leq \mu_{\text{max}} \). Then \( \Upsilon_{\mu_2,T,a_1} < \Upsilon_{\mu_1,T,a_1} \).

Proof. Let \( u \in S(a_1) \) satisfying \( J_{\mu_1,T}(u) = \Upsilon_{\mu_1,T,a_1} \). Then, \( \Upsilon_{\mu_2,T,a_1} \leq J_{\mu_2,T}(u) < J_{\mu_1,T}(u) = \Upsilon_{\mu_1,T,a_1} \). \( \square \)

4. **Proof of Theorem 1.1**

In this section, we first prove some properties of the functional \( E_{\epsilon,T} \) defined in (2.8) restricted to the sphere \( S(a) \), and then give the proof of Theorem 1.1.

Denote
\[
J_{\text{max},T} := J_{h_{\text{max}},T}, \quad \Upsilon_{\text{max},T,a} := \Upsilon_{h_{\text{max}},T,a},
\]
and
\[
J_{\infty,T} := J_{h_{\infty,T}}, \quad \Upsilon_{\infty,T,a} := \Upsilon_{h_{\infty},T,a}.
\]
It is obvious that \( J_{\infty,T}(u) \geq J_{\text{max},T}(u) \) and \( E_{\epsilon,T}(u) \geq J_{\text{max},T}(u) \) for any \( u \in S(a) \).

By Lemma 3.1, the definition
\[
\Gamma_{\epsilon,T,a} := \inf_{u \in S(a)} E_{\epsilon,T}(u)
\]
is well defined and \( \Gamma_{\epsilon,T,a} \geq \Upsilon_{\text{max},T,a} \).

The next lemma establishes some crucial relations involving the levels \( \Gamma_{\epsilon,T,a}, \Upsilon_{\infty,T,a} \) and \( \Upsilon_{\text{max},T,a} \).

**Lemma 4.1.** Let \( N, a, \eta, p, q, h, \epsilon \) be as in (1.1), (2.3) hold. Then
\[
\limsup_{\epsilon \to 0^+} \Gamma_{\epsilon,T,a} \leq \Upsilon_{\text{max},T,a} < \Upsilon_{\infty,T,a} < 0.
\]
Proof. By Lemma 3.7, choose \( u \in S(a) \) such that \( J_{\text{max},T}(u) = \Upsilon_{\text{max},T,a} \). Then,
\[
\Gamma_{\epsilon,T,a} \leq E_{\epsilon,T}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon x)|u|^q dx - \frac{\eta}{p^*} \tau(\|\nabla u\|_2) \int_{\mathbb{R}^N} |u|^p dx.
\]

Letting \( \epsilon \to 0^+ \), by the Lebesgue dominated convergence theorem, we deduce that
\[
\limsup_{\epsilon \to 0^+} \Gamma_{\epsilon,T,a} \leq \limsup_{\epsilon \to 0^+} E_{\epsilon,T}(u) = J_{h(0),T}(u) = J_{\text{max},T}(u) = \Upsilon_{\text{max},T,a},
\]
which combined with Lemma 3.2 and Corollary 3.8 completes the proof. \( \square \)

By Lemma 4.1, there exists \( \epsilon_1 > 0 \) such that \( \Gamma_{\epsilon,T,a} < \Upsilon_{\infty,T,a} \) for all \( \epsilon \in (0, \epsilon_1) \).

In the following, we always assume that \( \epsilon \in (0, \epsilon_1) \). Similarly to the proof of Lemma 3.3, we have the following result, whose proof is omitted.

**Lemma 4.2.** Let \( N, a, \eta, p, q, h, \epsilon \) be as in (1.1), \( \epsilon \in (0, \epsilon_1) \), (2.3) hold. Then
\begin{enumerate}
  \item \( E_{\epsilon,T} \in C^1(H^1(\mathbb{R}^N), \mathbb{R}) \);
  \item If \( u \in S(a) \) such that \( E_{\epsilon,T}(u) < 0 \), then \( \|\nabla u\|_2 < R_0 \) and \( E_{\epsilon,T}(v) = E_{\epsilon}(v) \) for all \( v \) satisfying \( \|v\|_2 \leq a \) and being in a small neighborhood of \( u \) in \( H^1(\mathbb{R}^N) \).
\end{enumerate}

The next two lemmas will be used to prove the \( (PS) \) condition for \( E_{\epsilon,T} \) restricts to \( S(a) \) at some levels.
Lemma 4.3. Let $N, a, \eta, p, q, h$ be as in (1.1), $\epsilon \in (0, \epsilon_1)$, (2.3) hold. Assume
\( \{u_n\} \subset S(a) \) such that $E_{\epsilon,T}(u_n) \to c$ as $n \to +\infty$ with $c < \Upsilon_{\infty,T,a}$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then $u \neq 0$.

Proof. Assume by contradiction that $u \equiv 0$. Then,
\[
c + o_n(1) = E_{\epsilon,T}(u_n) = J_{\infty,T}(u_n) + \frac{1}{q} \int_{\mathbb{R}^N} (h_{\infty} - h(\epsilon x))|u_n|^q dx.
\]
By $(h_2)$, for any given $\delta > 0$, there exists $R > 0$ such that $h_{\infty} \geq h(x) - \delta$ for all $|x| \geq R$. Hence,
\[
c + o_n(1) = E_{\epsilon,T}(u_n) \geq J_{\infty,T}(u_n) + \frac{1}{q} \int_{B_R(0)} (h_{\infty} - h(\epsilon x))|u_n|^q dx
\]
\[-\frac{\delta}{q} \int_{B_R(0)} |u_n|^q dx.
\]
Recalling that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $u_n \to 0$ in $L^t(B_{R/\epsilon}(0))$ for all $t \in [1, 2^*)$, it follows that
\[
c + o_n(1) = E_{\epsilon,T}(u_n) \geq J_{\infty,T}(u_n) - \delta C + o_n(1)
\]
for some $C > 0$. Since $\delta > 0$ is arbitrary, we deduce that $c \geq \Upsilon_{\infty,T,a}$, which is a contradiction. Thus, $u \neq 0$. \hfill \Box

Lemma 4.4. Let $N, a, \eta, p, q, h$ be as in (1.1), $\epsilon \in (0, \epsilon_1)$, (2.3) hold. If $p = 2^*$, we further assume (2.6) hold. Let $\{u_n\}$ be a $(PS)_{c^*}$ sequence of $E_{\epsilon,T}$ restricted to $S(a)$ with $c < \Upsilon_{\infty,T,a}$ and let $u_n \rightharpoonup u_\epsilon$ in $H^1(\mathbb{R}^N)$. If $u_n \not\to u_\epsilon$ in $H^1(\mathbb{R}^N)$, there exists $\beta > 0$ independent of $\epsilon \in (0, \epsilon_1)$ such that
\[
\limsup_{n \to +\infty} \|u_n - u_\epsilon\|_2 \geq \beta.
\]

Proof. By Lemma 4.2, we must have $\|\nabla u_n\|_2 < R_0$ for $n$ large enough, and so, $\{u_n\}$ is also a $(PS)_{c^*}$ sequence of $E_{\epsilon}$ restricted to $S(a)$. Hence,
\[
E_{\epsilon}(u_n) \to c \text{ and } \|E_{\epsilon}'(S(a))(u_n)\| \to 0 \text{ as } n \to +\infty.
\]
Setting the functional $\Psi : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by
\[
\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 dx,
\]
it follows that $S(a) = \Psi^{-1}(a^2/2)$. Then, by Willem ([35], Proposition 5.12), there exists $\{\lambda_n\} \subset \mathbb{R}$ such that
\[
\|E_{\epsilon}'(u_n) - \lambda_n \Psi'(u_n)\|_{H^{-1}(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty.
\]
By the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$, we know $\{\lambda_n\}$ is bounded and thus, for some subsequence, there exists $\lambda_\epsilon$ such that $\lambda_n \to \lambda_\epsilon$ as $n \to +\infty$. This together with (4.1) leads to
\[
E_{\epsilon}'(u_\epsilon) - \lambda_\epsilon \Psi'(u_\epsilon) = 0 \text{ in } H^{-1}(\mathbb{R}^N)
\]
and then
\[
\|E_{\epsilon}'(v_n) - \lambda_\epsilon \Psi'(v_n)\|_{H^{-1}(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty,
\]
where $v_n := u_n - u_\epsilon$. By direct calculations, we get that

\[
\Upsilon_{\infty, T, a} > \lim_{n \to +\infty} E_\epsilon(u_n)
\]

\[
= \lim_{n \to +\infty} \left( E_\epsilon(u_n) - \frac{1}{2} E'_\epsilon(u_n)u_n + \frac{1}{2} \lambda_n \|u_n\|_2^2 + o_n(1) \right)
\]

\[
= \lim_{n \to +\infty} \left( \left( \frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} h(\epsilon x) |u_n|^q dx + \left( \frac{1}{2} - \frac{1}{p} \right) \eta \int_{\mathbb{R}^N} |u_n|^p dx + \frac{1}{2} \lambda_n a^2 + o_n(1) \right)
\]

\[
\geq \frac{1}{2} \lambda_\epsilon a^2,
\]

which implies that

\[
\lambda_\epsilon \leq \frac{2 \Upsilon_{\infty, T, a}}{a^2} < 0 \quad \text{for all } \epsilon \in (0, \epsilon_1).
\]

By (4.3), we know

\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \lambda_\epsilon \int_{\mathbb{R}^N} |v_n|^2 dx - \int_{\mathbb{R}^N} h(\epsilon x) |v_n|^q dx - \eta \int_{\mathbb{R}^N} |v_n|^p dx = o_n(1),
\]

which combined with (4.4) gives that

\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{2 \Upsilon_{\infty, T, a}}{a^2} \int_{\mathbb{R}^N} |v_n|^2 dx
\]

\[
\leq h_{\text{max}} \int_{\mathbb{R}^N} |v_n|^q dx + \eta \int_{\mathbb{R}^N} |v_n|^p dx + o_n(1).
\]

If $u_n \not\to u_\epsilon$ in $H^1(\mathbb{R}^N)$, that is, $v_n \not\to 0$ in $H^1(\mathbb{R}^N)$, by (4.6) and the Sobolev inequality, we deduce that

\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{2 \Upsilon_{\infty, T, a}}{a^2} \int_{\mathbb{R}^N} |v_n|^2 dx
\]

\[
\leq h_{\text{max}} C_{N, q} \|v_n\|_q^q + \eta C_{N, p} \|v_n\|_p^p + o_n(1).
\]

So there exists $C > 0$ independent of $\epsilon$ such that $\|v_n\| \geq C$ and then by (4.6)

\[
\limsup_{n \to +\infty} \left( h_{\text{max}} \int_{\mathbb{R}^N} |v_n|^q dx + \eta \int_{\mathbb{R}^N} |v_n|^p dx \right) \geq C.
\]

Case $p < 2^*$, by (4.7) and the Gagliardo-Nirenberg inequality (2.1), there exists $\beta > 0$ independent of $\epsilon \in (0, \epsilon_1)$ such that

\[
\limsup_{n \to +\infty} \|v_n\|_2 \geq \beta.
\]

Case $p = 2^*$, if

\[
\limsup_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^q dx \geq C
\]

for some $C > 0$ independent of $\epsilon$, we obtain (4.8) as well. If

\[
\liminf \limsup_{\epsilon \to 0^+} \int_{\mathbb{R}^N} |v_n|^q dx = 0 \quad \text{and} \quad \liminf \limsup_{\epsilon \to 0^+} \|v_n\|_2 = 0,
\]

by (4.7) we have

\[
\liminf \limsup_{\epsilon \to 0^+} \int_{\mathbb{R}^N} |v_n|^p dx \geq C,
\]
and by (4.5) we have
\[ \liminf_{\epsilon \to 0^+} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = \liminf_{\epsilon \to 0^+} \limsup_{n \to +\infty} \eta \int_{\mathbb{R}^N} |v_n|^p \, dx. \]

Applying the Sobolev inequality to the above equality, we obtain that
\[ \liminf_{\epsilon \to 0^+} \limsup_{n \to +\infty} \|v_n\|_2^2 = \liminf_{\epsilon \to 0^+} \limsup_{n \to +\infty} \eta \|v_n\|_2^{2^*} \leq \liminf_{\epsilon \to 0^+} \limsup_{n \to +\infty} \eta S^{-2^*/2} \|\nabla v_n\|_2^{2^*}, \]
which implies that
\[ R_0 \geq \liminf_{\epsilon \to 0^+} \limsup_{n \to +\infty} \|v_n\|_2 \geq \eta^{-\frac{N-2}{2}} S^{N/4}. \]

That contradicts \( R_0 < \eta^{-\frac{N-2}{2}} S^{N/4} \), that is, assumption (2.6). So we must have (4.8) for the case \( p = 2^* \).

The proof is complete. \( \square \)

Now we give the compactness lemma.

**Lemma 4.5.** Let \( N, a, \eta, p, q, b \) be as in (1.1), \( \epsilon \in (0, \epsilon_1) \), \( \beta \) be as in Lemma 4.4,
\[ 0 < \rho_0 \leq \min \left\{ \frac{\max_{T,a} \beta^2}{2} \left( \frac{\max_{T,a}}{\min_{T,a}} \right) \right\}, \]
and (2.3) hold. If \( p = 2^* \), we further assume (2.6) hold. Then \( E_{\epsilon,T} \) satisfies the \((PS)\) condition restricted to \( S(a) \) if \( c < \max_{T,a} \rho_0 + \rho_0 \).

**Proof.** Let \( \{u_n\} \subset S(a) \) be a \((PS)_c\) sequence of \( E_{\epsilon,T} \) restricted to \( S(a) \). Noting that \( c < \max_{T,a} \rho_0 < 0 \), by Lemma 4.2, \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). Let \( u_n \to u_e \) in \( H^1(\mathbb{R}^N) \). By Lemma 4.3, \( u_e \neq 0 \). Set \( v_n := u_n - u_e \). If \( u_n \to u_e \) in \( H^1(\mathbb{R}^N) \), the proof is complete. If \( u_n \not\to u_e \) in \( H^1(\mathbb{R}^N) \) for some \( \epsilon \in (0, \epsilon_1) \), by Lemma 4.4,
\[ \limsup_{n \to +\infty} \|v_n\|_2 \geq \beta. \]

Set \( b = \|u_e\|_2, \) \( d_n = \|v_n\|_2 \) and suppose that \( \|v_n\|_2 \to d \), then we get \( d \geq \beta > 0 \) and \( a^2 = b^2 + d^2 \). From \( d_n \in (0, a) \) for \( n \) large enough, we have
\[ c + o_n(1) = E_{\epsilon,T}(u_n) \geq E_{\epsilon,T}(v_n) + E_{\epsilon,T}(u_e) + o_n(1). \]

Since \( v_n \to 0 \) in \( H^1(\mathbb{R}^N) \), similarly to the proof of Lemma 4.3, we deduce that
\[ E_{\epsilon,T}(v_n) \geq J_{\epsilon,T}(v_n) - \delta C + o_n(1) \]
for any \( \delta > 0 \), where \( C > 0 \) is a constant independent of \( \delta, \epsilon \) and \( n \). By (4.9) and (4.10), we obtain that
\[ c + o_n(1) = E_{\epsilon,T}(u_n) \geq J_{\epsilon,T}(v_n) + E_{\epsilon,T}(u_e) - \delta C + o_n(1) \]
\[ \geq \max_{T,a} \rho_0 + \max_{T,b} \beta C + o_n(1). \]

Letting \( n \to +\infty \), by Lemma 3.5 and the arbitrariness of \( \delta > 0 \), we obtain that
\[ c \geq \max_{T,d} + \max_{T,b} \delta^2 \frac{\max_{T,a}}{\min_{T,a}} \geq \frac{d^2}{a^2} \max_{T,a} + \frac{\max_{T,a}}{\min_{T,a}} \]
\[ \geq \max_{T,a} + \frac{\max_{T,a}}{\min_{T,a}} \]
which contradicts \( c < \max_{T,a} + \frac{\max_{T,a}}{\min_{T,a}} \). Thus, we must have \( u_n \to u_e \) in \( H^1(\mathbb{R}^N) \).

\( \square \)
Lemma 4.6. Let \( N, a, \eta, p, q, h \) be as in (1.1), and (2.3) hold. Then there exist \( \epsilon_2 \in (0, \epsilon_1), \rho_1 \in (0, \rho_0] \) such that if \( \epsilon \in (0, \epsilon_2), u \in S(a) \) and \( E_{\epsilon, T}(u) \leq \Upsilon_{\text{max}, T, a} + \rho_1 \), then

\[
Q_{\epsilon}(u) \in K_{\tilde{\rho}}.
\]

**Proof.** If the lemma does not occur, there must be \( \rho_n \to 0, \epsilon_n \to 0 \) and \( \{u_n\} \subset S(a) \) such that

\[
E_{\epsilon_n, T}(u_n) \leq \Upsilon_{\text{max}, T, a} + \rho_n \text{ and } Q_{\epsilon_n}(u_n) \notin K_{\tilde{\rho}}.
\]

Consequently,

\[
\Upsilon_{\text{max}, T, a} \leq J_{\text{max}, T}(u_n) \leq E_{\epsilon_n, T}(u_n) \leq \Upsilon_{\text{max}, T, a} + \rho_n,
\]

then

\[
\{u_n\} \subset S(a) \text{ and } J_{\text{max}, T}(u_n) \to \Upsilon_{\text{max}, T, a}.
\]

According to Lemma 3.6, we have two cases:

(i) \( u_n \to u \) in \( H^1(\mathbb{R}^N) \) for some \( u \in S(a) \), or

(ii) There exists \( \{y_n\} \subset \mathbb{R}^N \) with \( |y_n| \to +\infty \) such that \( v_n(x) = u_n(x+y_n) \) converges in \( H^1(\mathbb{R}^N) \) to some \( v \in S(a) \).

Analysis of (i): By the Lebesgue dominated convergence theorem,

\[
Q_{\epsilon_n}(u_n) = \left[ \int_{\mathbb{R}^N} \chi(\epsilon_n x) |u_n|^2 dx \right] - \left[ \int_{\mathbb{R}^N} |u_n|^2 dx \right] = 0 \in K_{\tilde{\rho}}.
\]

From this, \( Q_{\epsilon_n}(u_n) \in K_{\tilde{\rho}} \) for \( n \) large enough, which contradicts \( Q_{\epsilon_n}(u_n) \notin K_{\tilde{\rho}} \) in (4.11).

Analysis of (ii): Now we will study two cases: (I) \( |\epsilon_n y_n| \to +\infty \) and (II) \( \epsilon_n y_n \to y \) for some \( y \in \mathbb{R}^N \).

If (I) holds, the limit \( v_n \to v \) in \( H^1(\mathbb{R}^N) \) provides

\[
E_{\epsilon_n, T}(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon_n x + \epsilon_n y_n) |v_n|^q dx
\]

\[
- \frac{\eta}{p} (||v_n||_2^2) \int_{\mathbb{R}^N} |v_n|^p dx
\]

\[
\to J_{\infty, T}(v).
\]
Since $E_{\epsilon_n,T}(u_n) \leq \Upsilon_{\text{max},T,a} + \rho_n$, we deduce that
\[ \Upsilon_{\infty,T,a} \leq J_{\infty,T}(v) \leq \Upsilon_{\text{max},T,a}, \]
which contradicts $\Upsilon_{\infty,T,a} > \Upsilon_{\text{max},T,a}$ in Lemma 4.1.

Now if (II) holds, then
\[
E_{\epsilon_n,T}(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon_n x + \epsilon_n y_n)|v_n|^q dx \\
- \frac{\eta}{p} \tau(\|\nabla v_n\|_2) \int_{\mathbb{R}^N} |v_n|^p dx \\
\rightarrow J_{h(y),T}(v),
\]
which combined with $E_{\epsilon_n,T}(u_n) \leq \Upsilon_{\text{max},T,a} + \rho_n$ gives that
\[ \Upsilon_{h(y),T,a} \leq J_{h(y),T}(v) \leq \Upsilon_{\text{max},T,a}. \]

By Corollary 3.8, we must have $h(y) = h_{\text{max}}$ and $y = a_i$ for some $i = 1, 2, \ldots, l$. Hence,
\[
Q_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x)|u_n|^2 dx}{\int_{\mathbb{R}^N} |u_n|^2 dx} = \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x + \epsilon_n y_n)|u_n|^2 dx}{\int_{\mathbb{R}^N} |v_n|^2 dx} \\
\rightarrow \frac{\int_{\mathbb{R}^N} \chi(y)|v|^2 dx}{\int_{\mathbb{R}^N} |v|^2 dx} = a_i \in K^\perp_2,
\]
which implies that $Q_{\epsilon_n}(u_n) \in K^\perp_2$ for $n$ large enough. That contradicts (4.11). The proof is complete. \qed

From now on, we will use the following notations:
- $\theta^i_\epsilon := \{ u \in S(a) : |Q_{\epsilon}(u) - a_i| \leq \tilde{\rho}\}$;
- $\partial \theta^i_\epsilon := \{ u \in S(a) : |Q_{\epsilon}(u) - a_i| = \tilde{\rho}\}$;
- $\beta^i_\epsilon := \inf_{u \in \theta^i_\epsilon} E_{\epsilon,T}(u)$;
- $\tilde{\beta}^i_\epsilon := \inf_{u \in \partial \theta^i_\epsilon} E_{\epsilon,T}(u)$.

**Lemma 4.7.** Let $N,a,\eta,p,q,h$ be as in (1.1), (2.3) hold, $e_2$ and $p_1$ be obtained in Lemma 4.6. Then there exists $\epsilon_3 \in (0,\epsilon_2)$ such that
\[ \beta^i_\epsilon < \Upsilon_{\text{max},T,a} + \frac{p_1}{2} \text{ and } \tilde{\beta}^i_\epsilon < \tilde{\beta}^i_\epsilon, \text{ for any } \epsilon \in (0,\epsilon_3). \]

**Proof.** Let $u \in S(a)$ be such that
\[ J_{\text{max},T}(u) = \Upsilon_{\text{max},T,a}. \]
For $1 \leq i \leq l$, we define
\[ \hat{u}_i^\epsilon(x) := u \left( x - \frac{a_i}{\epsilon} \right), \quad x \in \mathbb{R}^N. \]
Then $\hat{u}_i^\epsilon \in S(a)$ for all $\epsilon > 0$ and $1 \leq i \leq l$. Direct calculations give that
\[
E_{\epsilon,T}(\hat{u}_i^\epsilon) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon x + a_i)|u|^q dx \nonumber \\
- \frac{\eta}{p} \tau(\|\nabla u\|_2) \int_{\mathbb{R}^N} |u|^p dx,
\]
and then
\[
(4.12) \quad \lim_{\epsilon \to 0^+} E_{\epsilon,T}(\hat{u}_i^\epsilon) = J_{h(a_i),T}(u) = J_{\text{max},T}(u) = \Upsilon_{\text{max},T,a}. \]

Note that
\[
Q_{\epsilon}(\hat{u}_i^\epsilon) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon x + a_i)|u|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} \rightarrow a_i \text{ as } \epsilon \to 0^+. \]
So \( \hat{u}_i^\epsilon \in \theta_i^\epsilon \) for \( \epsilon \) small enough, which combined with (4.12) implies that there exists \( \epsilon_3 \in (0, \epsilon_2) \) such that

\[
\beta_i^\epsilon < \Upsilon_{\text{max}, T, a} + \frac{\rho_1}{2} \quad \text{for any } \epsilon \in (0, \epsilon_3).
\]

For any \( v \in \partial \theta_i^\epsilon \), that is, \( v \in S(a) \) and \( |Q_\epsilon(v) - a_i| = \tilde{\rho} \), we obtain that \( |Q_\epsilon(v) \not\in K_i^\epsilon \). Thus, by Lemma 4.6,

\[
E_{\epsilon, T}(v) > \Upsilon_{\text{max}, T, a} + \rho_1, \quad \text{for all } v \in \partial \theta_i^\epsilon \text{ and } \epsilon \in (0, \epsilon_3),
\]

which implies that

\[
\tilde{\beta}_i^\epsilon = \inf_{v \in \partial \theta_i^\epsilon} E_{\epsilon, T}(v) \geq \Upsilon_{\text{max}, T, a} + \rho_1, \quad \text{for all } \epsilon \in (0, \epsilon_3).
\]

Thus,

\[
\beta_i^\epsilon < \tilde{\beta}_i^\epsilon, \quad \text{for all } \epsilon \in (0, \epsilon_3).
\]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Set \( \epsilon_0 := \epsilon_3 \), where \( \epsilon_3 \) is obtained in Lemma 4.7. Let \( \epsilon \in (0, \epsilon_0) \). By Lemma 4.7, for each \( i \in \{1, 2, \cdots, l\} \), we can use the Ekeland’s variational principle to find a sequence \( \{u_n^i\} \subset \theta_i^\epsilon \) satisfying

\[
E_{\epsilon, T}(u_n^i) \to \beta_i^\epsilon \text{ and } \|E_{\epsilon, T}^{\prime|S(a)}(u_n^i)\| \to 0 \text{ as } n \to +\infty,
\]

that is, \( \{u_n^i\}_n \) is a \((PS)_{\beta_i^\epsilon}\) sequence for \( E_{\epsilon, T} \) restricted to \( S(a) \). Since \( \beta_i^\epsilon < \Upsilon_{\text{max}, T, a} + \rho_0 \), it follows from Lemma 4.5 that there exists \( u^i \) such that \( u_n^i \to u^i \) in \( H^1(\mathbb{R}^N) \). Thus

\[
u^i \in \theta_i^\epsilon, \quad E_{\epsilon, T}(u^i) = \beta_i^\epsilon \quad \text{and} \quad E_{\epsilon, T}^{\prime|S(a)}(u^i) = 0.
\]

As

\[
Q_\epsilon(u^i) \subset B_{\tilde{\rho}}(a_i), \quad Q_\epsilon(u^j) \subset B_{\tilde{\rho}}(a_j),
\]

and

\[
B_{\tilde{\rho}}(a_i) \cap B_{\tilde{\rho}}(a_j) = \emptyset \text{ for } i \neq j,
\]

we conclude that \( u^i \not\equiv u^j \) for \( i \neq j \) while \( 1 \leq i, j \leq l \). Therefore, \( E_{\epsilon, T} \) has at least \( l \) nontrivial critical points for all \( \epsilon \in (0, \epsilon_0) \).

As \( E_{\epsilon, T}(u^i) < 0 \) for any \( i = 1, 2, \cdots, l \), by Lemma 4.2, \( u^i (i = 1, 2, \cdots, l) \) are in fact the critical points of \( E_\epsilon \) on \( S(a) \) with \( E_\epsilon(u^i) < 0 \) and then there exists \( \lambda_i \in \mathbb{R} \) such that

\[
-\Delta u^i = \lambda_i u^i + h(\epsilon x)|u^i|^{q-2}u^i + \eta |u^i|^{p-2}u^i, \quad x \in \mathbb{R}^N.
\]

By using \( E_\epsilon(u^i) = \beta_i^\epsilon < 0 \) and \( E_\epsilon^{\prime}(u^i)u^i = \lambda_i a^2 \), we obtain that

\[
\frac{1}{2} \lambda_i a^2 = E_\epsilon(u^i) + \left( \frac{1}{q - 1} - \frac{1}{2} \right) \int_{\mathbb{R}^N} h(\epsilon x)|u^i|^q dx + \left( \frac{1}{p} - \frac{1}{2} \right) \int_{\mathbb{R}^N} |u^i|^p dx,
\]

which implies that \( \lambda_i < 0 \) for \( i = 1, 2, \cdots, l \). The proof is complete.
5. Orbital stability

In this section we investigate the orbital stability of the solutions obtained in Theorem 1.1. We first give the definition of orbital stability.

**Definition 5.1.** A set $\Omega \subset H^1(\mathbb{R}^N)$ is orbitally stable under the flow associated with the problem

$$
\begin{aligned}
\begin{cases}
  i \frac{\partial \psi}{\partial t} + \Delta \psi + h(\epsilon x)|\psi|^{q-2}\psi + \eta|\psi|^{p-2}\psi = 0, & t > 0, x \in \mathbb{R}^N, \\
  \psi(0, x) = u_0(x)
\end{cases}
\end{aligned}
$$

(5.1)

if for any $\theta > 0$ there exists $\gamma > 0$ such that for any $u_0 \in H^1(\mathbb{R}^N)$ satisfying

$$
\text{dist}_{H^1(\mathbb{R}^N)}(u_0, \Omega) < \gamma,
$$

the solution $\psi(t, \cdot)$ of problem (5.1) with $\psi(0, x) = u_0$ satisfies

$$
\sup_{t \in \mathbb{R}^+} \text{dist}_{H^1(\mathbb{R}^N)}(\psi(t, \cdot), \Omega) < \theta.
$$

For any $i = 1, 2, \cdots, l$, we define

$$
\Omega_i := \{ v \in \theta_i^e : E_{\epsilon, T}(\delta(v)) = 0, E_{\epsilon, T}(v) = \beta_i^e \} = \{ v \in \theta_i^e : E_{\epsilon}(\delta(v)) = 0, E_{\epsilon}(v) = \beta_i^e, \|\nabla v\|_2 \leq R_0 \}.
$$

Next we show the stability of the sets $\Omega_i (i = 1, \cdots, l)$ in two cases $p < 2^*$ or $p = 2^*$.

**Theorem 5.2.** Let $N, a, \eta, q, h, \epsilon_0$ be as in Theorem 1.1, $p < 2^*$, $\epsilon \in (0, \epsilon_0)$, (2.3) hold. Then $\Omega_i (i = 1, \cdots, l)$ is orbitally stable under the flow associated with the problem (5.1).

**Proof.** Letting $r_1$ be such that $\tilde{g}_a(r_1) = \beta_i^e$, and considering (2.9), the definition of $\Omega_i$ and $\beta_i^e < 0$, we know that

$$
\|\nabla v\|_2 \leq r_1 < R_0, \text{ for any } v \in \Omega_i.
$$

Let $a_1 > a$ be such that $\tilde{g}_{a_1}(R_0) = \frac{\beta_i^e}{2}$. There exists $\delta > 0$ such that if

$$
u_0 \in H^1(\mathbb{R}^N) \text{ and } \text{dist}_{H^1(\mathbb{R}^N)}(u_0, \Omega_i) < \delta,
$$

then

$$
\|\nabla u_0\|_2 \leq a_1, \|\nabla u_0\|_2 \leq r_1 + \frac{R_0 - r_1}{2}, E_{\epsilon, T}(u_0) \leq \frac{2}{3} \beta_i^e.
$$

Denoting by $\psi(t, \cdot)$ the solution to (5.1) with initial data $u_0$ and denoting by $[0, T_{\text{max}})$ the maximal existence interval for $\psi(t, \cdot)$, we have classically that either $\psi(t, \cdot)$ is globally defined for positive times, or $\|\nabla \psi(t, \cdot)\|_2 = +\infty$ as $t \to T_{\text{max}}^-$, see ([32], Section 3). Set $\tilde{a} = \|u_0\|_2$. Note that $\|\psi(t, \cdot)\|_2 = \|u_0\|_2$ for all $t \in (0, T_{\text{max}})$ by the conservation of the mass. If there exists $\tilde{t} \in (0, T_{\text{max}})$ such that $\|\nabla \psi(\tilde{t}, \cdot)\|_2 = R_0$, then

$$
E_{\epsilon}(\psi(\tilde{t}, \cdot)) = E_{\epsilon, T}(\psi(\tilde{t}, \cdot)) \geq \tilde{g}_a(R_0) \geq \tilde{g}_{a_1}(R_0) = \frac{\beta_i^e}{2},
$$

which contradicts the conservation of the energy

$$
E_{\epsilon}(\psi(t, \cdot)) = E_{\epsilon}(u_0) \leq \frac{2}{3} \beta_i^e, \text{ for all } t \in (0, T_{\text{max}}).
$$

Thus,

$$
\|\nabla \psi(t, \cdot)\|_2 < R_0 \text{ for all } t \in [0, T_{\text{max}}),
$$

which implies that $\psi(t, \cdot)$ is globally defined for positive times.
Next we prove that $\Omega_i$ is orbitally stable. The validity of Lemma 4.5 for complex valued function can be proved exactly as in Theorem 3.1 in [11]. Thus, the orbital stability of $\Omega_i$ can be proved by modifying the classical Cazenave-Lions argument [9] (see also [21]). For the completeness, we give the proof here. Suppose by contradiction that there exist sequences $\{u_{0,n}\} \subset H^1(\mathbb{R}^N)$ and $\{t_n\} \subset \mathbb{R}^+$ and a constant $\theta_0 > 0$ such that for all $n \geq 1$,

$$\inf_{v \in \Omega_i} \|u_{0,n} - v\| < \frac{1}{n}$$

and

$$\inf_{v \in \Omega_i} \|\psi_n(t_n, \cdot) - v\| \geq \theta_0,$$

where $\psi_n(t, \cdot)$ is the solution to (5.1) with initial data $u_{0,n}$. By (5.2), there exists $n_0$ such that for $n > n_0$ it holds that $\|\nabla \psi_n(t, \cdot)\|_2 < R_0$ for all $t \geq 0$.

By (5.3), there exists $\{v_n\} \subset \Omega_i$ such that

$$\|u_{0,n} - v_n\| < \frac{2}{n}.$$  

That $\{v_n\} \subset \Omega_i$ implies that $\{v_n\} \subset \theta^i$ is a $(PS)_{\beta_i}$ sequence restricted to $S(u)$. From the proof of Theorem 1.1, there exists $v \in \Omega_i$ such that

$$\lim_{n \to +\infty} \|v_n - v\| = 0,$$

which combined with (5.5) gives that

$$\lim_{n \to +\infty} \|u_{0,n} - v\| = 0.$$  

Hence,

$$\lim_{n \to +\infty} \|u_{0,n}\|_2 = \|v\|_2 = a, \quad \lim_{n \to +\infty} E_\epsilon(u_{0,n}) = E_\epsilon(v) = \beta^i_\epsilon < \tilde{\beta}^i_\epsilon.$$  

Then by the conservation of mass and energy, we obtain that

$$\lim_{n \to +\infty} \|\psi_n(t, \cdot)\|_2 = a, \quad \lim_{n \to +\infty} E_\epsilon(\psi_n(t, \cdot)) = \beta^i_\epsilon, \quad \text{for any} \ t \geq 0.$$

Define

$$\varphi_n(t, \cdot) = \frac{\alpha \psi_n(t, \cdot)}{\|\psi_n(t, \cdot)\|_2}, \quad t \geq 0.$$  

Then $\varphi_n(t, \cdot) \in S(u)$ and

$$\|\varphi_n(t, \cdot) - \psi_n(t, \cdot)\| \to 0 \text{ as } n \to +\infty \text{ uniformly in } t \geq 0,$$

which combined with (5.6) gives that

$$\lim_{n \to +\infty} \|\varphi_n(0, \cdot) - v\| = \lim_{n \to +\infty} \left\| \frac{\alpha u_{0,n}}{\|u_{0,n}\|_2} - v \right\| = 0.$$  

Hence, $\varphi_n(0, \cdot) \in \theta^i_\epsilon \setminus \partial \theta^i_\epsilon$ for $n$ large enough because $v \in \theta^i_\epsilon \setminus \partial \theta^i_\epsilon$. Using the method of continuity, $\lim_{n \to +\infty} E_\epsilon(\varphi_n(t, \cdot)) = \beta^i_\epsilon$ for all $t \geq 0$, and $\beta^i_\epsilon < \tilde{\beta}^i_\epsilon$, we obtain that

$$\lim_{n \to +\infty} \varphi_n(t, \cdot) \in \theta^i_\epsilon \setminus \partial \theta^i_\epsilon \text{ for all } t \geq 0.$$  

From (5.7)-(5.9), $\{\varphi_n(t_n, \cdot)\} \subset \theta^i_\epsilon$ is a minimizing sequence of $E_\epsilon, T$ at level $\beta^i_\epsilon$, and from the proof of Theorem 1.1, there exists $\tilde{v} \in \theta^i_\epsilon$ such that

$$\lim_{n \to +\infty} \|\varphi_n(t_n, \cdot) - \tilde{v}\| = 0,$$
which combined with (5.8) gives that
\[
\lim_{n \to +\infty} \| \psi_n(t_n, \cdot) - \tilde{v} \| = 0.
\]
That contradicts (5.4). Hence \( \Omega_i \) is orbitally stable for any \( i = 1, 2, \ldots, l \). \( \square \)

**Theorem 5.3.** Let \( N, a, \eta, q, h, \epsilon_0 \) be as in Theorem 1.1, \( p = 2^* \), \( \epsilon \in (0, \epsilon_0) \), (2.3) and (2.6) hold. Further assume that \( h(x) \in C^1(\mathbb{R}^N) \) and \( h'(x) \in L^\infty(\mathbb{R}^N) \). Then \( \Omega_i (i = 1, \ldots, l) \) is orbitally stable under the flow associated with the problem (5.1).

**Proof.** The proof can be done by modifying the arguments of Sections 3 and 4 in [16] and using the arguments of the proof of Theorem 5.2, so we omit it. \( \square \)

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