CLOSED-CONSTRUCTIBLE FUNCTIONS ARE PIECE-WISE CLOSED

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Abstract. A subset $B \subset Y$ is constructible if it is an element of the smallest family that contains all open sets and is stable under finite intersections and complements. A function $f : X \to Y$ is said to be piece-wise closed if $X$ can be written as a countable union of closed sets $Z_n$ such that $f$ is closed on every $Z_n$. We prove that if a continuous function $f$ takes each closed set into a constructible subset of $Y$, then $f$ is piece-wise closed.

All spaces in this paper are supposed to be separable and metrizable, and all the functions are supposed to be continuous and onto.

A subset $B \subset Y$ is constructible if it is an element of the smallest family that contains all open sets and is stable under finite intersections and complements.

In topology, a constructible set is a finite union of locally closed sets (a set is locally closed or is an LC$_1$-set if it is the intersection of an open set and a closed set).

In algebraic geometry, a constructible set is any zero set of a system of polynomial equations and inequations. A function $f$ is said to be closed-constructible (resp., closed-$F_\sigma$) if $f$ takes closed sets into constructible (resp., $F_\sigma$) ones. It is clear that every closed-constructible function is closed-$F_\sigma$.

A function $f : X \to Y$ is said to be piece-wise closed if $X$ can be written as a countable union of closed sets $Z_n$ such that $f$ is closed on every $Z_n$.

Hansell, Rogers and Jayne gave a corrected form of their previous result [6, Theorem 1], [7] using additional hypotheses (a) – (d) [4, Theorem 3].

Under hypothesis (a) (=Fleissner’s axiom, which is consistent with the usual axiom ZFC), they established the correctness of their first conclusion [6, Lemma 2]: each continuous, closed-$F_\sigma$ function between absolute Souslin sets $X$ and $Y$ is piece-wise closed.

Unfortunately, the theory above is not sufficient for important applications such as the case of closed-LC$_1$ functions between non-Souslin subsets of the real line $\mathbb{R}$.

Motivated by this observation, we study the extensions of the theory of closed-Borel functions, whose major case is closed-LC$_2$ functions (a simple case for such non-continuous functions was recently considered in [10]).

We obtain the following main theorem:

Theorem 1. Every closed-constructible function is piece-wise closed.

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Theorem 1 immediately implies the following:

**Corollary 1.** Each constructible-measurable, closed (or open), one-to-one function is piece-wise continuous.

Note that this result is different from that of Banakh and Bokalo [2], which states that a constructible-measurable function of hereditary Baire space $X$ is piece-wise continuous.

Finally, we give some simple observations to the hypothesis $(d)$ mentioned in the beginning: each preimage $f^{-1}(y)$ of points from $Y$ is compact.

This hypothesis looks fairly strong: as we demonstrated in simple Proposition 3, a function $f$ under hypothesis $(d)$ becomes a closed–$F_\sigma$ function. However, in the same situation (Example 3), such a conclusion becomes false if the requirement $(d)$ is weakened to the following: each preimage $f^{-1}(y)$ of points from $Y$ is completely metrizable.

**Proposition 1.** Let $f : X \rightarrow Y$ be a continuous function with compact fibers and $f$ take elements of a clopen base $B$ into closed sets. Then $f$ is a closed and, hence, closed–constructible function.

Note that in following Example 1 all preimages $f^{-1}(y)$ of points from $Y$ are completely metrizable, but not compete under the given metric.

**Example 1.** There exists a continuous and open function $f : I_X \rightarrow I_Y$ from Polish spaces $I_X, I_Y \subset C$ that takes elements of some clopen base of $I_X$ into clopen sets, but $f$ is not closed–$F_\sigma$.

**References**

[1] Adjan, S. I., Novikov, P. S.: On a semicontinuous function. Moskov. Gos. Ped. Inst. Uchen. Zap. 138, 3–10, (1958) (in Russian)
[2] Banakh, T., Bokalo, B.: On scatteredly continuous maps between topological spaces. Topology and its Applications 157, 108–122 (2010)
[3] Davies, R. O., Tricot, C.: A theorem about countable decomposability. Mathematical Proceedings of the Cambridge Philosophical Society, (03) 91, 457–458 (1982)
[4] Hansell, R. W., Jayne, J. E., Rogers, C. A.: Piece-wise closed functions and almost discretely $\sigma$-decomposable families. Mathematika 32, 229–247 (1985)
[5] Holicky, P., Spurny, J.: $F_\sigma$-mappings and the invariance of absolute Borel classes. Fund. Math. 182, 193–204 (2004)
[6] Jayne, J. E., Rogers, C. A.: Piece-Wise Closed Functions. Math.Ann. 255, 499-518 (1981)
[7] Jayne, J. E., Rogers, C. A.: Piece-Wise Closed Functions: Corrigendum. Math.Ann. 267, 143 (1984)
[8] Keldiš, L.: Sur les fonctions premi'Tres measurables B. Dokl. Akad. Nauk. SSSR, 4, 192–197 (1934)
[9] Ostrovsky, A.: An alternative approach to the decomposition of functions. Topology and its Applications,159, 200–2008 (2012)
[10] Ostrovsky, A.: The structure of $LC$-continuous functions. Acta Mathematica Hungarica, 133, 372-375 (2011)

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