Lipschitz interpolative nonlinear ideal procedure

M. A. S. SALEH
Department of Mathematics and Computer Applications,
College of Science, Al-Nahrain University, Baghdad, Iraq.
E-mail: mas@sc.nahrainuniv.edu.iq

Abstract

We treat the general theory of nonlinear ideals and extend as many notions as possible from
the linear theory to the nonlinear theory. We define nonlinear ideals with special properties
which associate new non-linear ideals to given ones and establish several properties and character-
izations of them. Building upon the results of U. Matter we define a Lipschitz interpolative
nonlinear ideal procedure between metric spaces and Banach spaces and establish this class
of Lipschitz operators is an injective Banach nonlinear ideal and show several standard basic
properties for such class. Extending the work of J. A. López Molina and E. A. Sánchez Pérez we
define a Lipschitz \((p, \theta, q, \nu)\)-dominated operators for \(1 \leq p, q < \infty; 0 \leq \theta, \nu < 1\) and establish
several characterizations. Afterwards we generalize a notion of Lipschitz interpolative nonlinear
ideal procedure between arbitrary metric spaces and prove its a nonlinear ideal. Finally,
we present certain basic counter examples of Lipschitz interpolative nonlinear ideal procedure
between arbitrary metric spaces.

2010 AMS Subject Classification. Primary 47L20; Secondary 26A16, 47A57.

1 Notations and Preliminaries

We introduce concepts and notations that will be used in this article. The letters \(E\), \(F\) and \(G\)
will denote Banach spaces. The closed unit ball of a Banach space \(E\) is denoted by \(B_E\). The dual
space of \(E\) is denoted by \(E^*\). The class of all bounded linear operators between arbitrary Banach
spaces will be denoted by \(\mathcal{L}\). The symbols \(\mathbb{K}\) and \(\mathbb{N}\) stand for the set of all scalar field and the set
of all natural numbers, respectively. The symbols \(W(B_{E^*})\) and \(W(B_{X^*})\) stand for the set of all
Borel probability measures defined on \(B_{E^*}\) and \(B_{X^*}\), respectively. The value of \(a\) at the element
\(x\) is denoted by \(\langle x, a \rangle\). We put \(E^{\text{inj}} := \ell_\infty(B_{E^*})\) and \(J_{E}x := (\langle x, a \rangle)\) for \(x \in E\). Clearly \(J_{E}\) is a
metric injection from \(E\) into \(E^{\text{inj}}\). Let \(0 < p < \infty\). The Banach space of all absolutely \(p\)-summable
sequences \(x = (x_j)_{j \in \mathbb{N}}, \) where \(x_j \in E\), is denoted by \(\ell_p(E)\). We put

\[
\|x\|_{\ell_p(E)} = \left[\sum_{j=1}^{\infty} \|x_j\|^p\right]^{\frac{1}{p}} < \infty.
\]

The Banach space of all weakly absolutely \(p\)-summable sequences \(x \subset E\), is denoted by \(\ell_p^w(E)\).
We put
\[ \|x\|_{L_p^w(E)} = \sup_{a \in B_{E^*}} \left[ \sum_{j=1}^{\infty} |\langle x_j, a \rangle|^p \right]^{\frac{1}{p}}. \tag{1} \]

For the triple sequence \((\sigma, x', x'') \subset \mathbb{R} \times X \times X\). We put
\[ \|\langle \sigma, x', x'' \rangle\|_{\ell_p^w(\mathbb{R} \times X \times X)} = \sup_{f \in B_{X^*}} \left[ \sum_{j=1}^{\infty} |\sigma_j|^p d_X(x'_j, x''_j)^p \right]^{\frac{1}{p}}. \]

And
\[ \|\langle \sigma, x', x'' \rangle\|_{\ell_{p,w}^w(\mathbb{R} \times X \times X)} = \sup_{f \in B_{X^*}} \left[ \sum_{j=1}^{\infty} |\sigma_j|^p |\langle f, x'_j \rangle - \langle f, x''_j \rangle|^p \right]^{\frac{1}{p}}. \]

For \(0 \leq \theta < 1\) and \(1 \leq p < \infty\) we define
\[ \|x\|_{\delta_{p,\theta}(E)} = \sup_{\xi \in B_{E^*}} \left( \sum_{j=1}^{\infty} (|\langle x_j, \xi \rangle|^{1-\theta} \|x_j\|^\theta \right)^{\frac{1}{1-\theta}}. \]

Also for all sequences \((\sigma, x', x'') \subset \mathbb{R} \times X \times X\), we define
\[ \|\langle \sigma, x', x'' \rangle\|_{\delta_{p,\theta}^{Lw}(\mathbb{R} \times X \times X)} = \sup_{f \in B_{X^*}} \left[ \sum_{j=1}^{\infty} (|\sigma_j|^{1-\theta} d_X(x'_j, x''_j)^\theta \right]^{\frac{1}{1-\theta}}. \]

Recall that the definition of an operator ideal between arbitrary Banach spaces of A. Pietsch [13] and [11] is as follows. Suppose that, for every pair of Banach spaces \(E\) and \(F\), we are given a subset \(\mathfrak{A}(E, F)\) of \(\mathfrak{L}(E, F)\). The class
\[ \mathfrak{A} := \bigcup_{E,F} \mathfrak{A}(E, F) \]
is said to be an operator ideal, or just an ideal, if the following conditions are satisfied:

\((\text{OI}_0)\) \(a^* \otimes e \in \mathfrak{A}(E, F)\) for \(a^* \in E^*\) and \(e \in F\).
\((\text{OI}_1)\) \(S + T \in \mathfrak{A}(E, F)\) for \(S, T \in \mathfrak{A}(E, F)\).
\((\text{OI}_2)\) \(BTA \in \mathfrak{A}(E_0, F_0)\) for \(A \in \mathfrak{L}(E_0, E)\), \(T \in \mathfrak{A}(E, F)\), and \(B \in \mathfrak{L}(F, F_0)\).

Condition \((\text{OI}_0)\) implies that \(\mathfrak{A}\) contains nonzero operators.

**Remark 1.** The normed (Banach) operator ideal is designated by \([\mathfrak{A}, \mathfrak{A}]\).
2 Introduction

One important example of operator ideals is the class of $p$-summing operators defined by A. Pietsch [12] as follows: A bounded operator $T$ from $E$ into $F$ is called $p$-summing if and only if there is a constant $C \geq 0$ such that

$$\left\| \left( T x_j \right)_{j=1}^{m} \right\|_{\ell_p(F)} \leq C \cdot \left\| \left( x_j \right)_{j=1}^{m} \right\|_{\ell_p^w(E)}$$

for arbitrary sequence $\left( x_j \right)_{j=1}^{m}$ in $E$ and $m \in \mathbb{N}$. Let us denote by $\Pi_p(E, F)$ the class of all $p$-summing operators from $E$ into $F$ with $\pi_p(T)$ summing norm of $T$ is the infimum of such constants $C$.

J. D. Farmer and W. B. Johnson [6] defined a true extension of the linear concept of $p$-summing operators as follows: a Lipschitz operator $T \in \text{Lip}(X, Y)$ is called Lipschitz $p$-summing map if there is a nonnegative constants $C$ such that for all $m \in \mathbb{N}$, any sequences $x', x''$ in $X$ and $\lambda$ in $\mathbb{R}^+$, the inequality

$$\left\| (\lambda, T x', T x'') \right\|_{\ell_p(\mathbb{R} \times X \times X)} \leq C \cdot \left\| (\lambda, x', x'') \right\|_{\ell_p^w(\mathbb{R} \times X \times X)}$$

holds. Let us denote by $\Pi_p^L(X, Y)$ the class of all Lipschitz $p$-summing maps from $X$ into $Y$ with $\pi^L_p(T)$ Lipschitz summing norm of $T$ is the infimum of such constants $C$.

Jarchow and Matter [8] defined a general interpolation procedure for creating a new operator ideal between arbitrary Banach spaces. Also U. Matter defined in his seminal paper [10] a new class of interpolative ideal procedure as follows: let $0 \leq \theta < 1$ and $[\mathcal{A}, \mathcal{A}]$ be a normed operator ideal. A bounded operator $T$ from $E$ into $F$ belongs to $\mathcal{A}_{\theta}(E, F)$ if there exist a Banach space $G$ and a bounded operator $S \in \mathcal{A}(E, G)$ such that

$$\|Tx|F\| \leq \|Sx|G\|^{1-\theta} \cdot \|x\|^\theta, \ \forall \ x \in E.$$  (3)

For each $T \in \mathcal{A}_{\theta}(E, F)$, we set

$$\mathcal{A}_{\theta}(T) := \inf \mathcal{A}(S)^{1-\theta}$$

where the infimum is taken over all bounded operators $S$ admitted in [3].

**Proposition 1.** [10] $[\mathcal{A}, \mathcal{A}]$ is an injective complete quasinormed operator ideal.

U. Matter [10] applied Inequality [3] to the ideal $[\Pi_p, \pi_p]$ of absolutely $p$-summing operators and obtained the injective operator ideal $[\Pi_p]_\theta$, which is complete with respect to the ideal norm $\pi_p(T)$ and established the fundamental theorem of $(p, \theta)$-summing operators for $1 \leq p < \infty$ and $0 \leq \theta < 1$ as follows:

**Theorem 1.** [10] Let $T$ be a bounded operator from $E$ into $F$ and $C \geq 0$. The following are equivalent:

1. $T \in (\Pi_p)_{\theta}(E, F)$. 
2. $T \in (\Pi_p)_{\theta}^L(E, F)$.
2. There exist a constant $C$ and a probability measure $\mu$ on $B_{E^*}$ such that

$$
\|Tx|F\| \leq C \cdot \left( \int_{B_{E^*}} \left( |(x, x^*)|^{1-\theta} \|x\|^\theta \right) \frac{d\mu(x^*)}{x^*} \right)^{\frac{1-\theta}{p}} , \forall \, x \in E.
$$

3. There exists a constant $C \geq 0$ such that for any $(x_j)_{j=1}^m \in E$, and $m \in \mathbb{N}$ we have

$$
\left\| \left( T x_j \right)_{j=1}^m \left| \ell_p^w(F) \right| \right\| \leq C \cdot \left\| (x_j)_{j=1}^m \left| \ell_p^w(E) \right| \right\| \left\| (b_j)_{j=1}^m \left| \ell_q^w(F^*) \right| \right\|.
$$

In addition, $(\pi_p)_q(T)$ is the smallest number $C$ for which, respectively, (2) and (3) hold.

Another example of operator ideals is the class of $(r, p, q)$-summing operators defined by A. Pietsch [12, Sec. 17.1.1] as follows: Let $0 < r, p, q \leq \infty$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$. A bounded operator $T$ from $E$ into $F$ is called $(r, p, q)$-summing if there is a constant $C \geq 0$ such that

$$
\left\| \left( (Tx_j, b_j) \right)_{j=1}^m \left| f_r \right| \right\| \leq C \cdot \left\| (x_j)_{j=1}^m \left| \ell_p^w(E) \right| \right\| \left\| (b_j)_{j=1}^m \left| \ell_q^w(F^*) \right| \right\|
$$

for arbitrary sequence $(x_j)_{j=1}^m$ in $E$, $(b_j)_{j=1}^m$ in $F^*$ and $m \in \mathbb{N}$. Let us denote by $\mathcal{P}(r, p, q)(E, F)$ the class of all $(r, p, q)$-summing operators from $E$ into $F$ with $\mathcal{P}(r, p, q)(T)$ summing norm of $T$ is the infimum of such constants $C$.

**Proposition 2.** [12, Sec. 17.1.2] $\mathcal{P}(r, p, q), P(r, p, q)$ is a normed operator ideal.

Let $0 < p, q \leq \infty$. A. Pietsch [12] is also defined $(p, q)$-dominated operator as follows: A bounded operator $T$ from $E$ into $F$ is called $(p, q)$-dominated if it belongs to the quasi-normed ideal

$$
\left[ \mathcal{D}(p, q), \mathcal{D}(p, q) \right] := \left[ \mathcal{P}(r, p, q), \mathcal{P}(r, p, q) \right],
$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For a special case, if $q = \infty$, then $\left[ \mathcal{D}(p, \infty), \mathcal{D}(p, \infty) \right] := [\Pi_p, \pi_p]$.

J.A. López Molina and E. A. Sánchez Pérez [9] established the important characteristic of $(p, q)$-dominated operator as follows.

**Proposition 3.** [9] Let $E$ and $F$ be Banach spaces and $T \in \mathfrak{L}(E, F)$. The following are equivalent:

1. $T \in \mathcal{D}(p, q)(E, F)$.

2. There exist a Banach spaces $G$ and $H$, bounded operators $S_1 \in \Pi_p(E, G)$ and $S_2 \in \Pi_q(F^*, H)$ and $C > 0$ such that

$$
|\langle Tx, b \rangle| \leq C \|S_1x\| \|S_2x\|, \quad \forall x \in E, \forall \, b \in F^*.
$$

A general example of $(p, q)$-dominated operators is also defined by J.A. López Molina and E. A. Sánchez Pérez [9] as follows: Let $1 \leq p, q < \infty$ and $0 \leq \theta, \nu < 1$ such that $\frac{1}{p} + \frac{1-\theta}{p} + \frac{1-\nu}{q} = 1$ with $1 \leq r < \infty$. A bounded operator $T$ from $E$ to $F$ is called $(p, \theta, q, \nu)$-dominated if there exist
a Banach spaces $G$ and $H$, a bounded operator $S \in \Pi_p(E, G)$, a bounded operator $R \in \Pi_q(F^*, H)$ and a positive constant $C$ such that
\[
|\langle Tx, b^* \rangle| \leq C \cdot \|x\|^\theta \|Sx|G\|^{1-\theta} \|b^*\|^\nu \|R(b^*)|H\|^{1-\nu}
\] (7)
for arbitrary finite sequences $x$ in $X$ and $b^* \in F^*$.

Let us denote by $D_{(p,\theta,q,\nu)}(E, F)$ the class of all $(p, \theta, q, \nu)$-dominated operators from $E$ to $F$ with
\[
D_{(p,\theta,q,\nu)}(T) = \inf \left\{ C \cdot \pi_p(S)^{1-\theta} \cdot \pi_q(R)^{1-\nu} \right\},
\]
where the infimum is taken over all bounded operators $S$ and $R$ and constant $C$ admitted in (7). They also established an important characteristic of $(p, \theta, q, \nu)$-dominated operator as follows.

**Theorem 2.** Let $E$ and $F$ be Banach spaces and $T \in \mathfrak{L}(E, F)$. The following are equivalent:

1. $T \in D_{(p,\theta,q,\nu)}(E, F)$.
2. There is a constant $C \geq 0$ and regular probabilities $\mu$ and $\tau$ on $B_{F^*}$ and $B_{F^{***}}$, respectively such that for every $x$ in $X$ and $b^* \in F^*$ the following inequality holds
\[
|\langle Tx, b^* \rangle| \leq C \cdot \int_{B_{F^*}} \left( |\langle x, a \rangle|^{1-\theta} \|x\|^\theta \right)^{\frac{1-\theta}{p}} \, d\mu(a) \cdot \int_{B_{F^{***}}} \left( |\langle b^*, b^{**} \rangle|^{1-\nu} \|b^*\|^\nu \right)^{\frac{1-\nu}{q}} \, d\tau(b^{**}).
\]
3. There exists a constant $C \geq 0$ such that for every finite sequences $x$ in $X$ and $b^* \in F^*$ the inequality
\[
\|\langle Tx, b^* \rangle \|_{\ell^\nu} \leq C \cdot \|x\|_{\ell^{p,\theta}(E)} \cdot \|b^*\|_{\ell^{q,\nu}(F^*)}
\] (8)
holds.
4. There are a Banach space $G$, a bounded operator $A \in (\Pi_p)_\theta(X, G)$ and a bounded operator $B \in \mathfrak{L}(E, F)$ such that $B^* \in (\Pi_q)_\nu(F^*, G^*)$ and $T = BA$.

In this case, $D_{(p,\theta,q,\nu)}$ is equal to the infimum of such constants $C$ in either (2), or (3).

We now describe the contents of this paper. In Section 1 we introduce notations and preliminaries that will be used in this article. In Section 2 we first present preliminaries of special cases of those operators that map weakly (Lipschitz) $p$-summable sequences in arbitrary Banach (metric) space into strongly (Lipschitz) $p$-summable ones in Banach (metric) space these operators are called (Lipschitz) $p$-summing operators defined by A. Pietsch [12], J. D. Farmer and W. B. Johnson [6], respectively. Jarchow and Matter [8] defined a general interpolation procedure to create a new ideal from given ideals and U. Matter defined a new class of interpolative ideal procedure in his seminal paper [10]. He established the fundamental characterize result of $(p, \theta)$-summing operators for $1 \leq p < \infty$ and $0 \leq \theta < 1$. For $0 < p, q \leq \infty$. A. Pietsch [12] defined $(p, q)$-dominated operator between arbitrary Banach spaces. J. A. López Molina and E. A. Sánchez Pérez [9] established the fundamental characterize of $(p, q)$-dominated operator. Afterwards a general example
of \((p, q)\)-dominated operators is also defined by J.A. López Molina and E. A. Sánchez Pérez [9]. This class of operators is called \((p, \theta, q, \nu)\)-dominated for \(1 \leq p, q < \infty\) and \(0 \leq \theta, \nu < 1\) such that \(\frac{1}{r} + \frac{1-\theta}{p} + \frac{1-\nu}{q} = 1\) with \(1 \leq r < \infty\). They proved an important characterize of \((p, \theta, q, \nu)\)-dominated operator. In Section 3, we treat the general theory of nonlinear operator ideals. The basic idea here is to extend as many notions as possible from the linear theory to the nonlinear theory. Therefore, we start by recalling the fundamental concepts of an operator ideal defined by A. Pietsch [12], see also [13]. Then, we introduce the corresponding definitions for nonlinear operator ideals in the version close to that of A. Jiménez-Vargas, J. M. Sepulcre, and Moisés Villegas-Vallecillos [3]. Afterwards, we define nonlinear ideals with special properties which associate new non-linear ideals to given ones. Again, this is parallel to the linear theory. For \(0 < p \leq 1\) we also define a Lipschitz \(p\)-norm on nonlinear ideal and prove that the injective hull \(\mathbb{N}^L_{inj}\) is a \(p\)-normed nonlinear ideal. We generalize U. Matter’s interpolative ideal procedure for its nonlinear (Lipschitz) version between metric spaces and Banach spaces and establish several characterizations analogous to linear case of [9] and prove that the class of Lipschitz \((p, \theta, q, \nu)\)-dominated operators is a Banach nonlinear ideal under the Lipschitz \((p, \theta, q, \nu)\)-norm. In Section 4, we define nonlinear operator ideal concept between arbitrary metric spaces. It is also in the version close to that defined in [3]. We generalize a notion of Lipschitz interpolative nonlinear ideal procedure between arbitrary metric spaces and prove its a nonlinear ideal. Finally, we present certain basic counter examples of Lipschitz interpolative nonlinear ideal procedure between arbitrary metric spaces.

3 Nonlinear ideals between arbitrary metric spaces and Banach spaces

Definition 1. Suppose that, for every pair of metric spaces \(X\) and Banach spaces \(F\), we are given a subset \(\mathbb{N}^L(X, F)\) of \(\text{Lip}(X, F)\). The class
\[
\mathbb{N}^L := \bigcup_{X, F} \mathbb{N}^L(X, F)
\]
is said to be a complete \(p\)-normed (Banach) nonlinear ideal \((0 < p \leq 1)\), if the following conditions are satisfied:

\((\text{PNOI}_0)\) \(g \square e \in \mathbb{N}^L(X, F)\) and \(A^L(g \square e) = \text{Lip}(g) \cdot \|e\|\) for \(g \in X^\#\) and \(e \in F\).

\((\text{PNOI}_1)\) \(S + T \in \mathbb{N}^L(X, F)\) and the \(p\)-triangle inequality holds:
\[
A^L(S + T)^p \leq A^L(S)^p + A^L(T)^p \quad \text{for} \quad S, T \in \mathbb{N}^L(X, F).
\]

\((\text{PNOI}_2)\) \(BTA \in \mathbb{N}^L(X_0, F_0)\) and \(A^L(BTA) \leq \|B\| A^L(T) \text{Lip}(A)\) for \(A \in \text{Lip}(X_0, X), \ T \in \mathbb{N}^L(X, F), \) and \(B \in \mathcal{L}(F, F_0)\).
(PNOI$_3$) All linear spaces $\mathbb{A}^L(X,F)$ are complete, where $A^L$ is called a Lipschitz $p$-norm from $\mathbb{A}^L$ to $\mathbb{R}^+$. 

Remark 2. (1) If $p = 1$, then $A^L$ is simply called a Lipschitz norm and $[\mathbb{A}^L, A^L]$ is said to be a Banach nonlinear ideal.

(2) If $[\mathbb{A}^L, A^L]$ be a normed nonlinear ideal, then $\mathbb{A}^L(X,\mathbb{R}) = X^#$ with $\text{Lip}(g) = A^L(g), \forall \ g \in X^#$. 

Proposition 4. Let $\mathbb{A}^L$ be a nonlinear ideal. Then all components $\mathbb{A}^L(X,F)$ are linear spaces.

Proof. By the condition of (PNOI$_1$) it remains to show that $T \in \mathbb{A}^L(X,F)$ and $\lambda \in \mathbb{K}$ imply $\lambda \cdot T \in \mathbb{A}^L(X,F)$. This follows from $\lambda \cdot T = (\lambda \cdot I_F) \circ T \circ I_X$ and (PNOI$_2$). $
$

Proposition 5. If $[\mathbb{A}^L, A^L]$ be a normed nonlinear ideal, then $\text{Lip}(T) \leq A^L(T)$ for all $T \in \mathbb{A}^L$.

Proof. Let $T$ be an arbitrary Lipschitz operator in $\mathbb{A}^L(X,F)$.

$$\text{Lip}(T) = \|T^#_{F^*}\| = \sup \left\{ \text{Lip}(T^# b^*) : b^* \in B_{F^*} \right\}$$

$$= \sup \left\{ \text{Lip} (b^* \circ T) : b^* \in B_{F^*} \right\}$$

Now from Remark 2 we have $\text{Lip}(b^* \circ T) = A^L(b^* \circ T)$ for $b^* \in F^*$. It follows

$$\text{Lip}(T) = \sup \left\{ A^L(b^* \circ T) : b^* \in B_{F^*} \right\} \leq A^L(T).$$

3.1 Nonlinear Ideals with Special Properties

3.1.1 Lipschitz Procedures

A rule

$$\text{new} : \mathbb{A} \rightarrow \mathbb{A}_\text{new}^L$$

which defines a new nonlinear ideal $\mathbb{A}_\text{new}^L$ for every ideal $\mathbb{A}$ is called a Lipschitz semi-procedure. 

A rule

$$\text{new} : \mathbb{A}^L \rightarrow \mathbb{A}_\text{new}^L$$

which defines a new nonlinear ideal $\mathbb{A}_\text{new}^L$ for every nonlinear ideal $\mathbb{A}^L$ is called a Lipschitz procedure.

Remark 3. We now define the following special properties:

(M') If $\mathbb{A}^L \subseteq \mathbb{B}^L$, then $\mathbb{A}_\text{new}^L \subseteq \mathbb{B}_\text{new}^L$ (strong monotony).

(M'') If $\mathbb{A} \subseteq \mathbb{B}$, then $\mathbb{A}_\text{new}^L \subseteq \mathbb{B}_\text{new}^L$ (monotony).

(I) $(\mathbb{A}_\text{new}^L)_{\text{new}} = \mathbb{A}_\text{new}^L$ for all $\mathbb{A}^L$ (idempotence).

A strong monotone and idempotent Lipschitz procedure is called a Lipschitz hull procedure if $\mathbb{A}^L \subseteq \mathbb{A}_\text{new}^L$ for all nonlinear ideals.
3.1.2 Closed Nonlinear Ideals

Let $\mathfrak{A}^L$ be a nonlinear ideal. A Lipschitz operator $T \in \text{Lip}(X,F)$ belongs to the closure $\mathfrak{A}^L_{\text{clos}}$ if there are $T_1, T_2, T_3, \cdots \in \mathfrak{A}^L(X,F)$ with $\lim_n \text{Lip}(T - T_n) = 0$. It is not difficult to prove the following result.

**Proposition 6.** $\mathfrak{A}^L_{\text{clos}}$ is a nonlinear ideal.

The following statement is evident.

**Proposition 7.** The rule $\text{clos}: \mathfrak{A}^L \rightarrow \mathfrak{A}^L_{\text{clos}}$ is a hull Lipschitz procedure.

**Definition 2.** The nonlinear ideal $\mathfrak{A}^L$ is called closed if $\mathfrak{A}^L = \mathfrak{A}^L_{\text{clos}}$.

**Proposition 8.** Let $\mathfrak{G}^L$ be a Lipschitz approximable nonlinear ideal. Then $\mathfrak{G}^L$ is the smallest closed nonlinear ideal.

**Proof.** By the definition of Lipschitz approximable operators in [7] we have $\mathfrak{G}^L = \mathfrak{G}^L_{\text{clos}}$. Hence $\mathfrak{G}^L$ is closed. Let $\mathfrak{A}^L$ be a closed nonlinear ideal. Since $\mathfrak{F}^L$ is the smallest nonlinear ideal, we obtain from the monotonicity of the closure procedure $\mathfrak{G}^L = \mathfrak{F}^L_{\text{clos}} \subseteq \mathfrak{A}^L_{\text{clos}} = \mathfrak{A}^L$.

3.1.3 Dual Nonlinear Ideals

Let $\mathfrak{A}$ be an ideal. A Lipschitz operator $T \in \text{Lip}(X,F)$ belongs to the Lipschitz dual ideal $\mathfrak{A}^L_{\text{dual}}$ if $T^\#_{|F^*} \in \mathfrak{A}(F^*, X^\#)$.

**Lemma 1.** Let $T$ in $\mathfrak{F}^L(X,F)$ with $T = \sum_{j=1}^m g_j \sqcup e_j$. Then $T^\#_{|F^*} = \sum_{j=1}^m \hat{e}_j \otimes g_j$, where $e \mapsto \hat{e}$ is the natural embedding of the space $F$ into its second dual $F^{**}$.

**Proof.** We have $Tx = \sum_{j=1}^m g_j(x)e_j$ for $x \in X$. So for $b^* \in F^*$,

$$\left\langle T^\#_{|F^*} b^*, x \right\rangle_{(X^\#,X)} = \left\langle b^*, Tx \right\rangle_{(F^*,F)} = \sum_{j=1}^m g_j(x)b^*(e_j).$$

Hence $T^\#_{|F^*} b^* = \sum_{j=1}^m b^*(e_j)g_j$. This proves the statement for $T^\#_{|F^*}$. $\blacksquare$

**Lemma 2.** Let $T, S \in \text{Lip}(X,F)$, $A \in \text{Lip}(X_0, X)$, and $B \in \mathfrak{L}(F,F_0)$. Then

1. $(T + S)^\#_{|F^*} = T^\#_{|F^*} + S^\#_{|F^*}$.  

8
2. \((BT A)^\#|_{F_0^*} = A^#T^#|_{F_0^*} B^*\).

**Proof.** For \(b^* \in F^*\) and \(x \in X\), we have

\[
\langle (T+S)|_{F_0^*} b^*, x \rangle_{(X^#, X)} = \langle b^*, (T+S)x \rangle_{(F^*, F)} = \langle b^*, Tx + Sx \rangle_{(F^*, F)}
\]

\[
= \langle b^*, Tx \rangle_{(F^*, F)} + \langle b^*, Sx \rangle_{(F^*, F)}
\]

\[
= \langle T^# b^*, x \rangle_{(F_0^#, X)} + \langle S^# b^*, x \rangle_{(X^#, X)}.
\]

Hence \((T+S)|_{F_0^*} = T^#|_{F_0^*} + S^#|_{F_0^*}\). For \(b_0^* \in F^*\) and \(x_0 \in X_0\), we have

\[
\langle b_0^*, BTA(x_0) \rangle_{(F_0^*, F_0)} = \langle b_0^*, B(TAx_0) \rangle_{(F_0^*, F_0)} = \langle B^*b_0^*, T(Ax_0) \rangle_{(F^*, F)}
\]

\[
= \langle T^# B^*b_0^*, Ax_0 \rangle_{(X^#, X)} = \langle A^#T^# B^*b_0^*, x_0 \rangle_{(X_0^#, X_0)}.
\]

But also \(\langle b_0^*, BTA(x) \rangle_{(F_0^*, F_0)} = \langle BTA)^\#|_{F_0^*} b_0^*, x_0 \rangle_{(X_0^#, X_0)}\). Therefore \((BTA)^\#|_{F_0^*} = A^#T^#|_{F_0^*} B^*\). ■

**Proposition 9.** \(\mathfrak{A}_dual^L\) is a nonlinear ideal.

**Proof.** The algebraic condition \((\mathfrak{PNOI}_0)\) is satisfied, from Lemma 1 we obtain \((g \Box e)^\#|_{F_0^*} = \hat{e} \otimes g \in \mathfrak{A}(F^*, X^#)\). To prove the algebraic condition \((\mathfrak{PNOI}_1)\), let \(T\) and \(S\) in \(\mathfrak{A}_dual^L(X, F)\). Let \(T^#|_{F_0^*}\) and \(S^#|_{F_0^*}\) in \(\mathfrak{A}(F^*, X^#)\), from Lemma 2 we have \((T+S)|_{F_0^*} = T^#|_{F_0^*} + S^#|_{F_0^*} \in \mathfrak{A}(F^*, X^#)\). Let \(A \in \text{Lip}(X_0, X)\), \(T \in \mathfrak{A}_dual^L(X, F)\), and \(B \in \mathcal{L}(F, F_0)\). Also from Lemma 2 we have \((BTA)^\#|_{F_0^*} = A^#T^#|_{F_0^*} B^* \in \mathfrak{A}(F_0^*, X_0^#)\), hence the algebraic condition \((\mathfrak{PNOI}_2)\) is satisfied.

The following proposition is obvious.

**Proposition 10.** The rule

\[
\text{dual} : \mathfrak{A} \longrightarrow (\mathfrak{A})^L_dual
\]

is a monotone Lipschitz procedure.

**Proposition 11.** Let \(\mathfrak{F}^L\) be a nonlinear ideal of Lipschitz finite rank operators, \(\mathfrak{F}\) be an ideal of finite rank operators and \((\mathfrak{F})^L_dual\) be a semi-Lipschitz procedure. Then \(\mathfrak{F}^L = (\mathfrak{F})^L_dual\).

**Proof.** Let \(T \in \mathfrak{F}^L(X, F)\), then \(T\) can be represented in the form \(\sum_{j=1}^m g_j \Box e_j\). From Lemma 1 and \(E^* \otimes F \equiv \mathfrak{F}(E, F)\) we have \(T^#|_{F_0^*} = \sum_{j=1}^m \hat{e}_j \otimes g_j \in F^{**} \otimes X^# \equiv \mathfrak{F}(X^#, X^#)\). Hence \(T \in \mathfrak{F}^L_dual(X, F)\).
Let $T \in \mathcal{G}^L_{dual}(X, F)$ then $T^\# \in \mathcal{G}(F^*, X^\#)$ hence $T^\#_F$ can be represented in the form $\sum_{j=1}^m \hat{e}_j \otimes g_j$. 

For $b^* \in F^*$ and $x \in X$, we have

$$\langle b^*, Tx \rangle_{(F^*, F)} = \left\langle \sum_{j=1}^m \hat{e}_j \otimes g_j (b^*), x \right\rangle_{(X^#, X)} = \left\langle \sum_{j=1}^m \hat{e}_j (b^*) \cdot g_j, x \right\rangle_{(X^#, X)}.$$

Hence $T = \sum_{j=1}^m g_j \boxdot e_j \in \mathcal{G}^L(X, F)$.

### 3.1.4 Injective Nonlinear Ideals

Let $\mathcal{A}^L$ be a nonlinear ideal. A Lipschitz operator $T \in \text{Lip}(X, F)$ belongs to the injective hull $\mathcal{A}^L_{inj}$ if $J_F T \in \mathcal{A}^L(X, F_{inj})$.

**Proposition 12.** $\mathcal{A}^L_{inj}$ is a nonlinear ideal.

**Proof.** The algebraic condition $\text{(PNOI}_0\text{)}$ is satisfied, since $g \boxdot e \in \mathcal{A}^L(X, F)$ and using nonlinear composition ideal property we have $J_F(g \boxdot e) \in \mathcal{A}^L(X, F_{inj})$. To prove the algebraic condition $\text{(PNOI}_1\text{)}$, let $T$ and $S$ in $\mathcal{A}^L_{inj}(X, F)$. Then $J_F T$ and $J_F S$ in $\mathcal{A}^L(X, F_{inj})$, we have $J_F(T + S) = J_F T + J_F S \in \mathcal{A}^L(X, F_{inj})$. Let $A \in \text{Lip}(X_0, X)$, $T \in \mathcal{A}^L_{inj}(X, F)$, and $B \in \mathcal{L}(F, F_0)$. Since $F_{inj}^0$ has the extension property, there exists $B_{inj} \in \mathcal{L}(F_{inj}, F_{inj}^0)$ such that

$$\xymatrix{ X \ar[r]^T & F \ar[d]^{J_F} & \ar[r]^{J_F} & F_{inj} \\
X_0 \ar[u]^A & \ar[r]^{BTA} & F_0 & \ar[r]^{J_{F_0}} & F_{inj} }$$

Consequently $J_{F_0}(BTA) = B_{inj}(J_F T) A \in \mathcal{A}^L$, hence the algebraic condition $\text{(PNOI}_2\text{)}$ is satisfied.

**Proposition 13.** The rule

$$\text{inj} : \mathcal{A}^L \longrightarrow \mathcal{A}^L_{inj}$$

is a hull Lipschitz procedure.

**Proof.** The property $\text{(M}_1\text{)}$ is obvious. To show the idempotence, let $T \in \text{Lip}(X, F)$ belong to $(\mathcal{A}^L_{inj})_{inj}$. Then $J_F T \in \mathcal{A}^L_{inj}(X, F_{inj})$, and the preceding lemma implies $J_F T \in \mathcal{A}^L(X, F_{inj})$. Consequently $T \in \mathcal{A}^L_{inj}(X, F)$. Thus $(\mathcal{A}^L_{inj})_{inj} \subseteq \mathcal{A}^L_{inj}$. The converse inclusion is trivial.
Lemma 3. Let $F$ be a Banach space possessing the extension property. Then $\mathcal{A}^L(X, F) = \mathcal{A}^{L}_{inj}(X, F)$.

Proof. By hypothesis there exists $B \in \mathcal{L}(F^{inj}, F)$ such that $BJ_F = T_F$. Therefore $T \in \mathcal{A}^{L}_{inj}(X, F)$ implies that $T = B(J_F T) \in \mathcal{A}^{L}(X, F)$. This proves that $\mathcal{A}^{L}_{inj} \subseteq \mathcal{A}^{L}$. The converse inclusion is obvious.

Proposition 14. The rule

$$inj : \mathcal{A}^L \longrightarrow \mathcal{A}^{L}_{inj}$$

is a hull Lipschitz procedure.

Proof. The property $(\mathcal{M}')$ is obvious. To show the idempotence, let $T \in \text{Lip}(X, F)$ belong to $(\mathcal{A}^{L}_{inj})_{inj}$. Then $J_F T \in \mathcal{A}^{L}_{inj}(X, F^{inj})$, and the preceding lemma implies $J_F T \in \mathcal{A}^{L}(X, F^{inj})$. Consequently $T \in \mathcal{A}^{L}_{inj}(X, F)$. Thus $(\mathcal{A}^{L}_{inj})_{inj} \subseteq \mathcal{A}^{L}_{inj}$. The converse inclusion is trivial.

3.2 Minimal Nonlinear Ideals

Let $\mathfrak{A}$ be an ideal. A Lipschitz operator $T \in \text{Lip}(X, F)$ belongs to the associated minimal ideal $(\mathfrak{A})^L_{min}$ if $T = B T_0 A$, where $B \in \mathcal{S}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $A \in \mathcal{O}^L(X, G_0)$. In other words $(\mathfrak{A})^L_{min} := \mathcal{S} \circ \mathfrak{A} \circ \mathcal{O}^L$, where $\mathcal{S}$ be an ideal of approximable operators between arbitrary Banach spaces.

Proposition 15. $(\mathfrak{A})^L_{min}$ is a nonlinear ideal.

Proof. The algebraic condition $(\text{PNOI}_0)$ is satisfied, since the elementary Lipschitz tensor $g \boxdot e$ admits a factorization

$$g \boxdot e : X \overset{g \boxdot 1}{\longrightarrow} K \overset{1 \circ 1}{\longrightarrow} K \overset{1 \boxdot e}{\longrightarrow} F,$$

where $1 \circ e \in \mathcal{S}(K, F)$, $1 \boxdot 1 \in \mathfrak{A}(K, K)$, and $g \boxdot 1 \in \mathcal{O}^L(C, X)$. To prove the algebraic condition $(\text{PNOI}_1)$, let $T_i \in \mathcal{S} \circ \mathfrak{A} \circ \mathcal{O}^L(X, F)$. Then $T_i = B_i T_0 A_i$, where $B_i \in \mathcal{S}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $A_i \in \mathcal{O}^L(X, G_0)$. Put $B := B_1 \circ Q_1 + B_2 \circ Q_2$, $T_0 := T_1 \circ T_0 \circ Q_1 + T_2 \circ T_0 \circ Q_2$, and $A := A_1 \circ A_1 + A_2 \circ A_2$. Now $T_0 + T_2 = B \circ T_0 \circ A$, $B \in \mathcal{S}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $A \in \mathcal{O}^L(X, G_0)$ imply $T_1 + T_2 \in \mathcal{S} \circ \mathfrak{A} \circ \mathcal{O}^L(X, F)$. Let $A \in \text{Lip}(X, 0)$, $T \in \mathcal{S} \circ \mathfrak{A} \circ \mathcal{O}^L(X, F)$, and $B \in \mathcal{L}(F, R_0)$. Then $T$ admits a factorization

$$T : X \overset{\tilde{A}}{\longrightarrow} G_0 \overset{T_0}{\longrightarrow} F_0 \overset{\tilde{B}}{\longrightarrow} F,$$

where $\tilde{B} \in \mathcal{S}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $\tilde{A} \in \mathcal{O}^L(X, G_0)$. To show that $BTA \in \mathcal{S} \circ \mathfrak{A} \circ \mathcal{O}^L(X, G_0)$. By using the linear and nonlinear composition ideal properties, we obtain $B \circ \tilde{B} \in \mathcal{S}(F_0, R_0)$ and $\tilde{A} \circ A \in \mathcal{O}^L(X, G_0)$. Hence the Lipschitz operator $BTA$ admits a factorization

$$BTA : X \overset{\tilde{A}}{\longrightarrow} G_0 \overset{T_0}{\longrightarrow} F_0 \overset{\tilde{B}}{\longrightarrow} R_0,$$

where $\tilde{B} = B \circ \tilde{B}$ and $\tilde{A} = \tilde{A} \circ A$, hence the algebraic condition $(\text{PNOI}_2)$ is satisfied.

Proposition 16. The rule

$$\text{min} : \mathfrak{A} \longrightarrow (\mathfrak{A})^L_{min}$$

is a monotone Lipschitz procedure.
Remark 4.  
(1) It is evident \((\mathfrak{A})_{\min}^L \subseteq \Theta^L\).

(2) If \(\mathfrak{A}^L\) is a closed nonlinear ideal, then \((\mathfrak{A})_{\min}^L \subseteq \mathfrak{A}^L\).

Proposition 17. Let \(\mathfrak{A}\) be a closed ideal. Then \((\mathfrak{A})_{\min}^L = \Theta^L\). In particular, the linear and nonlinear ideals of approximable operators are related by \((\Theta)_{\min}^L = \Theta^L\).

The prove of the counterpart of this proposition for ideals of linear operators requires the notion of idempotence of ideals, see \([12, \text{Prop. 4.8.4}]\). In particular, the equalities
\[ F \circ F = F \quad \text{and} \quad G \circ G = G \]  
are needed. Since idempotence does not make sense for nonlinear ideals, we instead use the following equalities.

Proposition 18. Any Lipschitz finite operator can be written as a product of a linear operator with finite rank and a Lipschitz finite operator. Any Lipschitz approximable operator can be written as a product of a linear approximable operator and a Lipschitz approximable operator. That is, \(\mathfrak{F} \circ \mathfrak{F}^L = \mathfrak{F}^L\) and \(\Theta \circ \Theta^L = \Theta^L\).

Proof. Let \(T = \sum_{j=1}^{m} g_j \otimes e_j\) with \(g_1, \cdots, g_m\) in \(X^\#\) and \(e_1, \cdots, e_m\) in \(F\) be a Lipschitz finite operator. Let \(F_0\) be the finite dimensional subspace of \(F\) spanned by \(e_1, \cdots, e_m\) and let \(J : F_0 \to F\) be the embedding. Obviously, \(J\) is a linear operator with finite rank. Moreover, let \(T_0\) be the operator \(T\) considered as an operator from \(X\) to \(F_0\). Then \(T = JT_0\) is the required factorization. Observe that we also have \(\text{Lip}(T_0) \|J\| = \text{Lip}(T)\). The inclusion \(\mathfrak{F} \circ \mathfrak{F}^L \subseteq \mathfrak{F}^L\) is obvious.

Now let \(T \in \Theta^L(X, F)\). Since \(T\) can be approximated by Lipschitz finite operators, we can also find Lipschitz finite operators \(T_n \in \mathfrak{F}^L(X,F)\) such that the sum \(T = \sum_{n=1}^{\infty} T_n\) converges absolutely in \(\text{Lip}(X,F)\), i.e. \(\sum_{n=1}^{\infty} \text{Lip}(T_n) < \infty\). Now each \(T_n\) can be factored as \(T_n = V_n U_n\) with \(V_n \in \mathfrak{F}(F_i,F)\) and \(U_n \in \mathfrak{F}^L(X,F_i)\) such that \(F_i\) is a suitable Banach space and \(\|V_n\| \text{Lip}(U_n) = \text{Lip}(T)\). By homogeneity, we may assume that \(\|V_n\|^2 = \text{Lip}(U_n)^2 = \text{Lip}(T_n)\). Let \(M := \ell_2(F_n)\) and put
\[ V := \sum_{n=1}^{\infty} J_n V_n \quad \text{and} \quad U := \sum_{n=1}^{\infty} U_n Q_n. \]
Then
\[ \text{Lip}(U)^2 \leq \sum_{n=1}^{\infty} \text{Lip}(U_n)^2 < \infty \quad \text{and} \quad \|V\|^2 \leq \sum_{n=1}^{\infty} \|V_n\|^2 < \infty. \]
This shows that the series defining \(U\) and \(V\) are absolutely convergent in \(\text{Lip}(X,M)\) and \(\mathfrak{L}(M, F)\), respectively. Hence \(U \in \Theta^L(X,M)\) and \(V \in \Theta(M,F)\) and \(T = VU\) is the required factorization. Again, \(\Theta \circ \Theta^L \subseteq \Theta^L\) is obvious.

We can now prove Proposition 17.

12
Proof of Proposition 17. By (9) and Proposition 18, we have
\[(G)^L_{\text{min}} = G \circ G \circ G = G \circ G = G.\]
If now \(\mathfrak{A}\) is a closed ideal, then \(G \subseteq \mathfrak{A}\) implies
\[G^L = (G)_{\text{min}}^L = \mathfrak{G} \circ \mathfrak{G} \circ \mathfrak{G} \subseteq G \circ \mathfrak{A} \circ G = (\mathfrak{A})^L_{\text{min}}.\]
The reverse implication was already observed in Remark 4. ■

3.3 Lipschitz interpolative ideal procedure between metric spaces and Banach spaces

Proposition 19. \([\mathfrak{A}^L_{\text{inj}}, \mathfrak{A}^L_{\text{inj}}]\) is a \(p\)-normed nonlinear ideal.

Proof. By Proposition 12 the algebraic conditions of Definition 1 are hold. Then the injective hull \(\mathfrak{A}^L_{\text{inj}}\) is a nonlinear ideal. To prove the norm condition (PNOI\(_1\)), let \(T \) and \(S \) in \(\mathfrak{A}^L_{\text{inj}}(X, F)\). Then
\[\mathfrak{A}^L_{\text{inj}}(S + T)^p := \mathfrak{A}^L([J_F(S + T)]^p) = \mathfrak{A}^L([J_F S + J_F T]^p) \leq \mathfrak{A}^L(J_F S)^p + \mathfrak{A}^L(J_F T)^p = \mathfrak{A}^L_{\text{inj}}(S)^p + \mathfrak{A}^L_{\text{inj}}(T)^p.\]

Let \(A \in \text{Lip}(X_0, X)\), \(T \in \mathfrak{A}^L_{\text{inj}}(X, F)\), and \(B \in \mathfrak{A}^L_{\text{inj}}(F, F_0)\). Then
\[\mathfrak{A}^L_{\text{inj}}(BTA) := \mathfrak{A}^L(J_F(BTA)) = \mathfrak{A}^L(B_{\text{inj}}(J_F T) A) \leq \|B_{\text{inj}}\| \mathfrak{A}^L(J_F T) \text{Lip}(A) = \|B\| \mathfrak{A}^L_{\text{inj}}(T) \text{Lip}(A).\]
Hence the norm condition (PNOI\(_2\)) is satisfied. ■

Definition 3. Let \(0 \leq \theta < 1\) and \([\mathfrak{A}^L, \mathfrak{A}^L]\) be a normed nonlinear ideal. A Lipschitz operator \(T\) from \(X\) into \(F\) belongs to \(\mathfrak{A}^L_{\theta}(X, F)\) if there exist a Banach space \(G\) and a Lipschitz operator \(S \in \mathfrak{A}^L_{\text{inj}}(X, G)\) such that
\[\|Tx' - Tx''|F\| \leq \|Sx' - Sx''|G\|^{1-\theta} \cdot d_X(x', x'')^\theta, \ \forall x', x'' \in X.\] (10)
For each \(T \in \mathfrak{A}^L_{\theta}(X, F)\), we set
\[\mathfrak{A}^L_{\theta}(T) := \inf \mathfrak{A}^L(S)^{1-\theta}\] (11)
where the infimum is taken over all Lipschitz operators \(S\) admitted in (10).
Note that \(\text{Lip}(T) \leq \mathfrak{A}^L_{\theta}(T)\), by definition.

Proposition 20. \([\mathfrak{A}^L_{\theta}, \mathfrak{A}^L_{\theta}]\) is an injective Banach nonlinear ideal.
Proof. The condition of injective hull and the algebraic conditions of Definition [1] are not difficult to prove it. Let $x', x'' \in X$ and $e \in F$ the norm condition $(\text{PNOI}_0)$ is satisfied. Indeed

$$\| (g \square e)x' - (g \square e)x'' | F \| \leq \| (\text{Lip}(g) \cdot |e|)^{\frac{\theta}{1 - \theta}} (g \square e)x' - (g \square e)x'' | F \|^\theta \cdot d_X(x', x''). \quad (12)$$

Since a Lipschitz operator $S := (\text{Lip}(g) \cdot |e|)^{\frac{\theta}{1 - \theta}} (g \square e) \in \mathfrak{A}_L(X, F)$, hence $g \square e \in \mathfrak{A}_L^L(X, F)$. From (11) we have

$$A^L_\theta (g \square e) := \inf A^L((\text{Lip}(g) \cdot |e|)^{\frac{\theta}{1 - \theta}} g \square e)^{1 - \theta} \leq (\text{Lip}(g) \cdot |e|)^{\frac{\theta}{1 - \theta}} A^L(g \square e)^{1 - \theta} = (\text{Lip}(g) \cdot |e|)^{\frac{\theta}{1 - \theta}} \cdot \text{Lip}(g) \cdot |e|)^{1 - \theta} = \text{Lip}(g) \cdot |e|.$$ 

The converse inequality is obvious. To prove the norm condition $(\text{PNOI}_1)$, let $T_1$ and $T_2$ in $\mathfrak{A}_L^L(X, F)$. Given $\epsilon > 0$, there is a Banach space $G_i$ and a Lipschitz operator $S_i \in \mathfrak{A}_L^L(X, G_i)$, $i = 1, 2$ such that

$$\| T_i x' - T_i x'' | F \| \leq \| A^L(S_i)^{-\theta} (S_i x' - S_i x'') | G_i \|^\theta \cdot d_X(x', x'')^\theta, \forall x', x'' \in X \quad (13)$$

and $A^L(S_i)^{-\theta} \leq (1 + \epsilon) \cdot A^L_\theta (T_i)$ ($i = 1, 2$). Introducing the $\ell_1$-sum $G := G_1 \oplus G_2$ and the Lipschitz operator $S := A^L(S_1)^{-\theta} J_1 S_1 + A^L(S_2)^{-\theta} J_2 S_2 \in \mathfrak{A}_L^L(X, G)$ ($J_1, J_2$ the canonical injections) and applying the Hölder inequality, we get for all $x', x'' \in X$

$$\| (T_1 + T_2)x' - (T_1 + T_2)x'' | F \| \leq \| T_1 x' + T_1 x'' | F \| + \| T_1 x' + T_1 x'' | F \| \leq \sum_{i=1}^2 \| A^L(S_i)^{-\theta} (S_i x' - S_i x'') | G_i \|^{\theta} \cdot d_X(x', x'')^\theta \leq \sum_{i=1}^2 A^L(S_i)^{-\theta} \sum_{i=1}^2 A^L(S_i)^{-\theta} \cdot d_X(x', x'')^\theta.$$

Hence $T_1 + T_2 \in \mathfrak{A}_L^L(X, F)$ and furthermore, for $p = 1$ we have

$$A^L_\theta (T_1 + T_2) \leq \left[ A^L(S_1)^{1 - \theta} + A^L(S_2)^{1 - \theta} \right]^\theta A^L(S)^{1 - \theta} \leq A^L(S_1)^{1 - \theta} + A^L(S_2)^{1 - \theta} \leq (1 + \epsilon) \cdot (A^L_\theta (T_1) + A^L_\theta (T_2)).$$

To prove the norm condition $(\text{PNOI}_2)$, let $A \in \mathcal{L}(X_0, X)$, $T \in \mathfrak{A}_L^L(X, F)$, and $B \in \mathcal{L}(F, F_0)$
and $x'_0, x''_0$ in $X$. Then

$$
\|BTAx'_0 - BTAx''_0|F_0\| \leq \|B\| \cdot \|T Ax'_0 - T Ax''_0|F\|
\leq \|B\| \cdot \|S(Ax'_0) - S(Ax''_0)|G\|^{1-\theta} \cdot d_X(Ax'_0, Ax''_0)^\theta
\leq \|B\| \cdot \text{Lip}(A)^\theta \cdot \|S \circ A(x'_0) - S \circ A(x''_0)|G\|^{1-\theta} \cdot d_X(x'_0, x''_0)^\theta
\leq \left\| \|B\|^{1-\theta} \cdot \text{Lip}(A)^{1-\theta} \right\| \left( S \circ A \right)^{1-\theta}
\leq \|B\| \cdot \text{Lip}(A) \cdot \mathcal{A}_\theta(BT A)^{1-\theta}.
$$

(14)

Since a Lipschitz map $\tilde{S} := \left( \|B\|^{1-\theta} \cdot \text{Lip}(A)^{1-\theta} \right) S \circ A \in \mathfrak{N}^L(X_0, G)$, hence $BTA \in \mathfrak{N}_\theta^L(X_0, F_0)$. Moreover, from (11) we have

$$
\mathcal{A}_\theta^L(BT A) := \inf \mathcal{A}_\theta^L(\tilde{S})^{1-\theta}
\leq \mathcal{A}_\theta^L \left( \|B\|^{1-\theta} \cdot \text{Lip}(A)^{1-\theta} \right) S \circ A
\leq \|B\| \cdot \text{Lip}(A) \cdot \mathcal{A}_\theta^L(S \circ A)^{1-\theta}
= \|B\| \cdot \text{Lip}(A) \cdot \mathcal{A}_\theta^L(S)^{1-\theta}.
$$

(15)

Taking the infimum over all such $S \in \mathfrak{N}^L(X, G)$ on the right side of (15), we have

$$
\mathcal{A}_\theta^L(BT A) \leq \|B\| \cdot \mathcal{A}_\theta^L(T) \cdot \text{Lip}(A).
$$

To prove the completeness, condition (PNOI3), let $(T_n)_{n \in \mathbb{N}}$ be a sequence of Lipschitz operator in $\mathfrak{N}_\theta^L(X, F)$ such that $\sum_{n=1}^{\infty} \mathcal{A}_\theta^L(T_n) < \infty$. Since $\text{Lip}(T) \leq \mathcal{A}_\theta^L(T)$ and $\text{Lip}(X, F)$ is a Banach space, there exists $T = \sum_{n=1}^{\infty} T_n \in \text{Lip}(X, F)$. Let $S_n \in \mathfrak{N}^L(X, G_n)$ such that

$$
\|T_n x' - T_n x''|F_n\| \leq \|S_n x' - S_n x''|G_n\|^{1-\theta} \cdot d_X(x', x'')^\theta, \quad \forall x', x'' \in X,
$$

and $\mathcal{A}_\theta^L(S_n)^{1-\theta} \leq \mathcal{A}_\theta^L(T_n) + \frac{\epsilon}{2^\theta}$. Then

$$
\left( \sum_{n=1}^{\infty} \mathcal{A}_\theta^L(S_n) \right)^{1-\theta} \leq \sum_{n=1}^{\infty} \mathcal{A}_\theta^L(S_n)^{1-\theta} \leq \sum_{n=1}^{\infty} \mathcal{A}_\theta^L(T_n) + \frac{\epsilon}{2^\theta}.
$$

Let $S = \sum_{n=1}^{\infty} S_n \in \mathfrak{N}_\theta^L(X, G)$, where $G$ is the $\ell_1$-sum of all $G_n$. Hence

$$
\|Tx' - Tx''|F\| \leq \sum_{n=1}^{\infty} \|T_n x' - T_n x''|F_n\| \leq \sum_{n=1}^{\infty} \|S_n x' - S_n x''|G_n\|^{1-\theta} \cdot d_X(x', x'')^\theta
\leq \|S x' - S x''|G\|^{1-\theta} \left( \sum_{n=1}^{\infty} \mathcal{A}_\theta^L(S_n)^{1-\theta} \right)^\theta \cdot d_X(x', x'')^\theta.
$$
This implies that $T \in \mathfrak{K}_0^L(X, F)$ and

$$A_0^L(T) \leq \sum_{n=1}^\infty A^L(S_n)^{1-\theta} \leq \sum_{n=1}^\infty A_0^L(T_n) + \epsilon < \infty.$$ 

We have

$$A_0^L(T - \sum_{j=1}^n T_j) = A_0^L\left(\sum_{k=n+1}^\infty T_k\right) \leq \sum_{k=n+1}^\infty A_0^L(S_k)^{1-\theta}.$$ 

Thus, $T = \sum_{n=1}^\infty T_n.$ \hfill \Box

**Remark 5.** If $\theta = 0$, then the nonlinear ideal $\mathfrak{N}_0^L$ is just the injective hull of nonlinear ideal $\mathfrak{N}^L$ and Lipschitz norms are equal. Further properties are given in.

**Proposition 21.** Let $0 \leq \theta, \theta_1, \theta_2 < 1$. Then the following holds.

(a) $\mathfrak{N}_{\theta_1}^L \subseteq \mathfrak{N}_{\theta_2}^L$ if $\theta_1 \leq \theta_2$.

(b) $\mathfrak{N}_{\text{inj}}^L \subseteq \mathfrak{N}_{\theta}^L$.

(c) $(\mathfrak{N}_{\theta_2}^{L})_{\theta_2} \subseteq \mathfrak{N}_{\theta_1 + \theta_2 - \theta_1 \theta_2}^L$.

**Proof.** To verify (a), let $T \in \mathfrak{N}_{\theta_1}^L(X, F)$ and $\epsilon > 0$. Then

$$\|T x' - T x''|F\| \leq \|S x' - S x''|G\|^{1-\theta_1} \cdot d_X(x', x'')^{\theta_1}, \quad \forall x', x'' \in X,$$

holds for a suitable Banach space $G$ and a Lipschitz operator $S \in \mathfrak{N}^L(X, G)$ with $A^L(S)^{1-\theta_1} \leq (1 + \epsilon) \cdot A_0^L(T)$. Since

$$\|T x' - T x''|F\| \leq \text{Lip}(S)^{\theta_2-\theta_1} \|S x' - S x''|G\|^{1-\theta_2} \cdot d_X(x', x'')^{\theta_2}, \quad \forall x', x'' \in X,$$

Since a Lipschitz map $\tilde{S} := \text{Lip}(S)^{\theta_2-\theta_1} S \in \mathfrak{N}^L(X, G)$, hence $T \in \mathfrak{N}_{\theta_2}^L(X, F)$ and

$$A_0^L(T) \leq A^L(\tilde{S})^{1-\theta_2} \leq \text{Lip}(S)^{\theta_2-\theta_1} A^L(S)^{1-\theta_2} \leq A^L(S)^{1-\theta_1} \leq (1 + \epsilon) \cdot A_0^L(T).$$

To verify (b), let $T \in \mathfrak{N}_{\text{inj}}^L(X, F)$. Then $J_F T \in \mathfrak{N}^L(X, F_{\text{inj}})$ and

$$\|T x' - T x''|F\| = \|J_F(T x') - J_F(T x'')|F_{\text{inj}}\|$$

$$\leq \text{Lip}(T)^{\theta} \cdot \|J_F \circ T(x') - J_F \circ T(x'')|F_{\text{inj}}\|^{1-\theta} \cdot d_X(x', x'')^{\theta}.$$ 

Since $G := F_{\text{inj}}$ and a Lipschitz map $S := \text{Lip}(T)^{\theta} J_F \circ T \in \mathfrak{N}^L(X, G)$, hence $T \in \mathfrak{N}_{\theta}^L(X, F)$. Moreover,

$$A_0^L(T) := \inf A^L(S)^{1-\theta} \leq A^L(\text{Lip}(T)^{\theta} J_F \circ T)^{1-\theta}$$

$$\leq \text{Lip}(T)^{\theta} \cdot A^L(J_F \circ T)^{1-\theta}$$

$$:= \text{Lip}(T)^{\theta} \cdot A_{\text{inj}}^L(T)^{1-\theta}$$

$$\leq A_{\text{inj}}^L(T)^{\theta} \cdot A^L(T)^{1-\theta}$$

$$= A_{\text{inj}}^L(T).$$
To verify (c), let \( T \in (\mathcal{N}^L_{\theta_1})_{\theta_2} (X, F) \) and \( \epsilon > 0 \). Then
\[
\|Tx' - Tx''\| F \leq \|Sx' - Sx''\| G^{1-\theta_2} \cdot d_X(x', x'')^{\theta_2}, \quad \forall x', x'' \in X,
\]}
holds for a suitable Banach space \( G \) and a Lipschitz operator \( S \in \mathcal{N}^L_{\theta_1}(X, G) \) with \( \mathcal{A}^L_{\theta_1}(S)^{1-\theta_2} \leq (1 + \epsilon) \cdot (\mathcal{A}^L_{\theta_1}(T))_{\theta_2} \) and
\[
\|Sx' - Sx''\| G \leq \|Rx' - Rx''\| G^{1-\theta_1} \cdot d_X(x', x'')^{\theta_1}, \quad \forall x', x'' \in X,
\]}
holds for a suitable Banach space \( \tilde{G} \) and a Lipschitz operator \( R \in \mathcal{N}^L(X, \tilde{G}) \) with \( \mathcal{A}^L(R)^{1-\theta_1} \leq (1 + \epsilon) \cdot \mathcal{A}^L_{\theta_1}(S) \). From (16) and (17) we have
\[
\|Tx' - Tx''\| F \leq \|Rx' - Rx''\| \tilde{G}^{(1-\theta_1)(1-\theta_2)} \cdot d_X(x', x'')^{\theta_1(1-\theta_2)} \cdot d_X(x', x'')^{\theta_2}
\leq \|Rx' - Rx''\| \tilde{G}^{1-\theta_2-\theta_1+\theta_2} \cdot d_X(x', x'')^{\theta_2+\theta_1-\theta_2},
\]}

hence \( T \in \mathcal{N}^L_{\theta_1+\theta_2-\theta_1\theta_2}(X, F) \). Moreover,
\[
\mathcal{A}^L_{\theta_1+\theta_2-\theta_1\theta_2}(T) \leq \mathcal{A}^L(R)^{1-\theta_1-\theta_2+\theta_1\theta_2}
\leq (1 + \epsilon) \frac{1-\theta_2}{1-\theta_1} \cdot \mathcal{A}^L_{\theta_1}(S)^{1-\theta_2+\theta_1\theta_2}
\leq (1 + \epsilon) \frac{1-\theta_2}{1-\theta_1} \cdot (1 + \epsilon) \frac{1-\theta_1-\theta_2+\theta_1\theta_2}{1-\theta_1} \cdot (\mathcal{A}^L_{\theta_1}(T))_{\theta_2},
\]

3.4 Lipschitz \((p, \theta, q, \nu)\)-dominated operators

1 \leq p, q < \infty and 0 \leq \theta, \nu \leq 1 such that \( \frac{1}{r} + \frac{1-\theta}{p} + \frac{1-\nu}{q} = 1 \) with \( 1 \leq r < \infty \). We then introduce the following definition.

**Definition 4.** A Lipschitz operator \( T \) from \( X \) to \( F \) is called Lipschitz \((p, \theta, q, \nu)\)-dominated if there exists a Banach spaces \( G \) and \( H \), a Lipschitz operator \( S \in \Pi_p^L(X, G) \), a bounded operator \( R \in \Pi_q(F^*, H) \) and a positive constant \( C \) such that
\[
\|Tx' - Tx''\| b^* \leq C \cdot d_X(x', x'')^{\theta} \|Sx' - Sx''\| G^{1-\theta} \|b^*\| \nu \|R(b^*)\| H^{1-\nu}
\]}
for arbitrary finite sequences \( x', x'' \) in \( X \), and \( b^* \in F^* \).

Let us denote by \( D^L_{(p, \theta, q, \nu)}(X, F) \) the class of all Lipschitz \((p, \theta, q, \nu)\)-dominated operators from \( X \) to \( F \) with
\[
D^L_{(p, \theta, q, \nu)}(T) = \inf \left\{ C \cdot \pi_p^L(S)^{1-\theta} \cdot \pi_q(R)^{1-\nu} \right\},
\]
where the infimum is taken over all Lipschitz operator \( S \), bounded operator \( R \), and constant \( C \) admitted in (18).
Proposition 22. The ordered pair \( \left( D^L_{(p,\theta,q,\nu)}(X,F), D^L_{(p,\theta,q,\nu)} \right) \) is a normed space.

Proof. We prove the triangle inequality. Let \( i = 1,2 \) and \( T_i \in D^L_{(p,\theta,q,\nu)}(X,F) \). For each \( \epsilon > 0 \), there exists a Banach spaces \( G_i \) and \( H_i \), a Lipschitz operators \( S_i \in \Pi^L_p(X,G_i) \), a bounded operators \( R_i \in \Pi_q(F^*,H_i) \) and a positive constants \( C_i \) such that

\[
\left| \langle T_i x' - T_i x'', b^* \rangle \right| \leq C_i d_X(x', x'')^\theta \left\| S_i x' - S_i x'' \| G_i \right\|^{1-\theta} \left\| b^* \right\|^\nu \left\| R_i(b^*) \right\|_{H_i}^{1-\nu} \forall x', x'' \in X \forall b^* \in F^*
\]

and

\[
C_i \cdot \pi_p^L(S_i)^{1-\theta} \cdot \pi_q(R_i)^{1-\nu} \leq D^L_{(p,\theta,q,\nu)}(T_i) + \epsilon.
\]

For \( x', x'' \in X \) and \( b^* \) we have

\[
\left| \langle T_i x' - T_i x'', b^* \rangle \right| \leq C_i d_X(x', x'')^\theta \left\| S_i x' - S_i x'' \| G_i \right\|^{1-\theta} \left\| b^* \right\|^\nu \left\| R_i(b^*) \right\|_{H_i}^{1-\nu}
\]

where \( \tilde{C}_i = C_i^\frac{\theta}{\nu} \cdot \pi_p^L(S_i)^{1-\theta} \cdot \pi_q(R_i)^{1-\nu} \cdot \pi_q(R_i)^{1-\nu} \cdot \pi_q(R_i)^{1-\nu} \cdot S_i \pi_p^L(S_i) \), and \( \tilde{R}_i = C_i^\frac{\theta}{\nu} \cdot \pi_p^L(S_i)^{1-\theta} \cdot \pi_q(R_i)^{1-\nu} \cdot \pi_q(R_i)^{1-\nu} \cdot \pi_q(R_i)^{1-\nu} \cdot \pi_q(R_i)^{1-\nu} \).

From (19) and (20) we have

\[
C_i \leq \left( D^L_{(p,\theta,q,\nu)}(T_i) + \epsilon \right)^\frac{1}{\theta}.
\]

\[
\pi_p^L(S_i) \leq \left( D^L_{(p,\theta,q,\nu)}(T_i) + \epsilon \right)^\frac{1}{\theta} \quad \text{and} \quad \pi_q(R_i) \leq \left( D^L_{(p,\theta,q,\nu)}(T_i) + \epsilon \right)^\frac{1}{\theta}.
\]

Let \( G \) and \( H \) be a Banach spaces obtained as a direct sum of \( \ell_p \) and \( \ell_q \) by \( G_1 \) and \( G_2 \) and \( H_1 \) and \( H_2 \) respectively. Let \( S \) be a Lipschitz operator from \( X \) into \( G \) such that \( S(x) = (S_i(x))_{i=1}^2 \) for \( x \in X \) and \( R \) be a bounded operator from \( F^* \) into \( H \) such that \( R(b) = (R_i(b))_{i=1}^2 \) for \( b \in F^* \). For each finite sequence \( x', x'' \) in \( X \) we have

\[
\left\| (S(x_j') - S(x_j''))_{j=1}^n \| \ell_p(G) \right\| = \left[ \sum_{j=1}^n \left\| S(x_j') - S(x_j'') \| G \right\|^p \right]^\frac{1}{p} \leq \left[ \sum_{j=1}^n \frac{2}{p} \sum_{i=1}^2 \left\| S_i(x_j') - S_i(x_j'') \right\|^p \right]^\frac{1}{p}
\]

\[
\leq \left[ \sum_{i=1}^2 \pi_p^L(S_i)^{\frac{2}{p}} \sup_{f \in B_{X^*}} \left\| f \right\|_p \left\| f \right\|_p \left\| f \right\|_p \left\| f \right\|_p \right]^\frac{1}{p} \leq \left[ \sum_{j=1}^n \left\| f \right\|_p \left\| f \right\|_p \right]^\frac{1}{p} \leq \left( \sum_{i=1}^2 \pi_p^L(S_i)^{\frac{2}{p}} \right)^\frac{1}{p}.
\]
Remark 6. If $\theta = 0$, then the class of all Lipschitz $(p, q, \nu)$-dominated operators from $X$ to $F$ are the class of all Lipschitz $(p, q)$-dominated operators from $X$ to $F$ considered in [4] with $D^L_{(p, q, \nu)}(X, F) = D^L_{(p, q)}(X, F)$.

Theorem 3. Let $X$ be a metric space, $F$ be a Banach space and $T \in \text{Lip}(X, F)$. The following conditions are equivalent.

1. $T \in D^L_{(p, \theta, q, \nu)}(X, F)$.

2. There is a constant $C \geq 0$ and regular probabilities $\mu$ and $\tau$ on $B_{X^\#}$ and $B_{F^{**}}$, respectively such that for every $x', x''$ in $X$ and $b^*$ in $F^*$ the following inequality holds

$$
\langle Tx' - Tx'', b^* \rangle \leq C \cdot \left[ \int_{B_{X^\#}} \left( |f(x') - f(x'')|^{1-\theta} \ d_X(x', x'')^\theta \right)^{\frac{p}{\theta}} d\mu(f) \right]^{\frac{1-\theta}{\theta}} \cdot \left[ \int_{B_{F^{**}}} \left( |(b^*, b^{**})|^{1-\nu} \ ||b^*||^{\nu} \right)^{\frac{q}{\nu}} d\tau(b^{**}) \right]^{\frac{1-\nu}{\nu}}.
$$
(3) There exists a constant $C \geq 0$ such that for every finite sequences $x', x''$ in $X$; $\sigma$ in $\mathbb{R}$ and $y^* \in F^*$ the inequality
\[
\|\sigma \cdot \langle Tx' - Tx'', b^* \rangle \|_{L_2} \leq C \cdot \|\sigma, x', x''\| \left\|\delta_{p,\theta}(\mathbb{R} \times X \times X)\right\| \|b^*\|_{\delta_{q,\nu}(F^*)}
\] (24)
holds.

In this case, $D^{L}_{(p,\theta,q,\nu)}$ is equal to the infimum of such constants $C$ in either (2), or (3).

Proof. (1) $\Longrightarrow$ (2) If $T \in D^{L}_{(p,\theta,q,\nu)}(X,F)$, then there exists a Banach spaces $G$ and $H$, a Lipschitz operator $S_1 \in \Pi^L_p(X,G)$, a bounded operator $S_2 \in \Pi_q(F^*,H)$ and a positive constant $C$ such that
\[
\|\langle Tx' - Tx'', b^* \rangle\| \leq C \cdot d_X(x', x'')^{\theta} \left\|S_1 x' - S_1 x''\|G\right\|^{1-\theta} \|b^*\|^{\nu} \|S_2(b^*)\|H^{1-\nu}
\] (25)
for arbitrary finite sequences $x', x''$ in $X$ and $b^* \in F^*$. Since $S_1$ is Lipschitz $p$-summing operator and $S_2$ is $q$-summing operator then there exists regular probabilities $\mu$ and $\tau$ on $B_{X^*}$ and $B_{F^*}$, respectively such that

\[
\|\langle Tx' - Tx'', b^* \rangle\| \leq C \cdot \pi_p(S_1)^{1-\theta} \cdot \pi_q(S_2)^{1-\nu} \left[\int_{B_{X^*}} \left( |f(x') - f(x'')|^{1-\theta} \cdot d_X(x', x'')^{\theta} \right)^{\frac{1}{1-\theta}} d\mu(f)\right] \cdot \left[\int_{B_{F^*}} \left( |(b^*, b^{**})|^{1-\nu} \cdot \|b^*\|^{\nu} \right)^{\frac{1}{1-\nu}} d\tau(b^{**})\right]^{\frac{1}{1-\nu}}.
\]

(2) $\Longrightarrow$ (1) Let $x', x''$ be finite sequences in $X$ and $y^*$ be a finite sequence in $F^*$. Let $\varphi_{b^*} := \langle b^*, b^{**} \rangle$. For each $x \in X$, let $\delta_{(x,0)} : X^* \longrightarrow \mathbb{R}$ be the linear map defined by
\[
\delta_{(x,0)}(f) = f(x) \quad (f \in X^*).
\]

By setting $S_1 x := \delta_{(x,0)}$, $S_2 b^* := \varphi_{b^*}$, $G := L_p(B_{X^*}, \mu)$, and $H := L_q(B_{F^*}, \tau)$ we obtain a Lipschitz operator $S_1 \in \Pi^L_p(X,G)$ with $\pi_p(S_1) \leq 1$ and an operator $S_2 \in \Pi_q(F^*,H)$ with $\pi_q(S_2) \leq 1$ such that

\[
\|\langle Tx' - Tx'', b^* \rangle\| \leq C \cdot \left[\int_{B_{X^*}} |f(x') - f(x'')|^p \cdot d\mu(f)\right]^{\frac{1}{1-\theta}} \cdot \left[\int_{B_{F^*}} |(b^*, b^{**})|^q \cdot d\tau(b^{**})\right]^{\frac{1}{1-\nu}} \cdot d_X(x', x'')^{\theta} \|b^*\|^{\nu}
\]

(2) $\Longrightarrow$ (3) Let $x', x''$ be finite sequences in $X$; $\sigma$ in $\mathbb{R}$ and $y^*$ be a finite sequence in $F^*$. By (2)
and using the Hölder inequality with exponents of $1 = \frac{1}{r} + \frac{1-\theta}{p} + \frac{1-\nu}{q}$ we have

$$\left[ \sum_{j=1}^{m} |\sigma_j|^{r'} \left| \langle T x'_j - T x''_{j'}, b'_j \rangle \right|^{r'} \right]^{\frac{1}{r'}} \leq C \cdot \left[ \sum_{j=1}^{m} \left( \int_{B_{X^*}^*} \left| \sigma_j \right| \left| f(x'_j) - f(x''_{j'}) \right|^{1-\theta} d_X(x'_j, x''_{j'})^{\frac{p}{1-\theta}} \ d\mu(f) \right)^{\frac{1-\theta}{p}} \right]^{\frac{1}{r'}} \cdot \left[ \sum_{j=1}^{m} \left( \int_{B_{F^*}^*} \left| \langle b'_j, b^{**} \rangle \right|^{1-\nu} \left| b'_j \right|^{\nu} \right)^{\frac{1}{1-\nu}} \ d\tau(b^{**}) \right]^{\frac{1-\nu}{q}}$$

$$\leq C \cdot \sup_{f \in B_{X^*}^*} \left[ \sum_{j=1}^{m} \left[ \left| \sigma_j \right| \left| f(x'_j) - f(x''_{j'}) \right|^{1-\theta} d_X(x'_j, x''_{j'})^{\frac{p}{1-\theta}} \right]^{\frac{1-\theta}{p}} \right]^{\frac{1}{r'}} \cdot \sup_{b^{**} \in B_{F^*}^*} \left[ \sum_{j=1}^{m} \left[ \left| \langle b'_j, b^{**} \rangle \right|^{1-\nu} \left| b'_j \right|^{\nu} \right]^{\frac{1}{1-\nu}} \right]^{\frac{1-\nu}{q}}$$

$$= C \cdot \| (\sigma, x', x'') \|_{\delta_{p, \theta}^L(\mathbb{R} \times X \times X)} \| b^* \|_{\delta_{p, \nu}^L(F^*)}$$

(3) $\implies$ (2) Take $[C(B_{X^*}^*) \times C(B_{F^*}^*)]^*$ equipped with the weak $C(B_{X^*}^*) \times C(B_{F^*}^*)$-topology. Then $W(B_{X^*}^*) \times W(B_{F^*}^*)$ is a compact convex subset. For any finite sequences $\sigma$ in $\mathbb{R}$, $x'$, $x''$ in $X$ and $y^*$ in $F^*$ the equation

$$\Psi(\mu, \tau) = \sum_{j=1}^{n} \left( \frac{1}{r'} \left| \sigma_j \langle T x'_j - T x''_{j'}, b'_j \rangle \right|^{r'} \right) - \frac{C r'}{1-\theta} \int_{B_{X^*}^*} \left| \sigma_j \right|^{\frac{p}{1-\theta}} d_X(x'_j, x''_{j'})^{\frac{p}{1-\theta}} \left| f(x'_j) - f(x''_{j'}) \right|^p \ d\mu(f)$$

$$- \frac{C r'}{1-\nu} \int_{B_{F^*}^*} \left| b'_j \right|^{\frac{\nu}{1-\nu}} \left| \langle b'_j, b^{**} \rangle \right|^q \ d\tau(b^{**})$$

defines a continuous convex function $\Psi$ on $W(B_{X^*}^*) \times W(B_{F^*}^*)$. From the compactness of $B_{X^*}^*$ and $B_{F^*}^*$, there exists $f_0 \in B_{X^*}^*$ and $y^*_0 \in B_{F^*}^*$ such that

$$\zeta = \left[ \sum_{j=1}^{n} \left| \sigma_j \right| \left| f_0 x'_j - f_0 x''_{j'} \right|^{1-\theta} d_X(x'_j, x''_{j'})^{\theta} \right]^{\frac{1-\theta}{p}}$$

and

$$\beta = \left[ \sum_{j=1}^{n} \left| \langle b'_j, b^{**} \rangle \right|^{1-\nu} \left| b'_j \right|^{\nu} \right]^{\frac{1-\nu}{q}}$$

If $\delta(f_0)$ and $\delta(y^{**}_0)$ denotes the Dirac measure at the point $f_0$ and in $y^{**}_0$, respectively, then we have
\[
\Psi(\delta(f_0), \delta(b_0^*)) = \frac{1}{r'} \sum_{j=1}^{n} \left( |\sigma_j \langle Tx_j' - Tx_j'', b_j^* \rangle|^{r'} - \frac{Cr'}{1-p} |\sigma_j|^{\frac{p}{1-p}} d_X(x_j', x_j'')^{\frac{p}{1-p}} |f_0(x_j') - f_0(x_j'')|^p \right)
\]
\[
- \frac{Cr'}{1-q} \|b_j^*\|^{\frac{q}{1-q}} |\langle b_j^*, b_0^* \rangle|^{q'}
\]
\[
= \frac{1}{r'} \sum_{j=1}^{n} |\sigma_j|^{r'} |\langle Tx_j' - Tx_j'', b_j^* \rangle|^{r'} - \frac{Cr'}{r'} (\zeta \cdot \beta)^{r'}
\]
\[
\leq \frac{1}{r'} \left( \sum_{j=1}^{n} |\sigma_j|^{r'} |\langle Tx_j' - Tx_j'', b_j^* \rangle|^{r'} \right)^{r'} - \frac{Cr'}{r'} (\zeta \cdot \beta)^{r'}
\]
\[
= \frac{1}{r'} \left[ \left( \sum_{j=1}^{n} |\sigma_j|^{r'} |\langle Tx_j' - Tx_j'', b_j^* \rangle|^{r'} \right)^{r'} - (C \cdot \zeta \cdot \beta)^{r'} \right]
\]
\[
\leq 0.
\]

Since the collection \( \Omega \) of all functions \( \Psi \) obtained in this way is concave, by [12, E.4.2] there are \( \mu_0 \in W(B_{X^*}) \) and \( \tau_0 \in W(B_{F^{***}}) \) such that \( \Psi(\mu_0, \tau_0) \leq 0 \) for all \( \Psi \in \Omega \). In particular, if \( \Psi \) is generated by single finite sequences \( \sigma \) in \( \mathbb{R} \), \( x', x'' \) in \( X \) and \( y^* \) in \( F^* \), it follows that

\[
\frac{1}{r'} |\sigma \langle Tx' - Tx'', b^* \rangle|^{r'} - \frac{Cr'}{1-q} \int_{B_{X^*}} |\sigma|^{\frac{p}{1-p}} d_X(x', x'')^{\frac{p}{1-p}} |f x' - f x''|^p d\mu_0(f)
\]
\[
- \frac{Cr'}{1-q} \int_{B_{F^{***}}} \|b^*\|^{\frac{q}{1-q}} |\langle b^*, b^{***} \rangle|^{q'} d\tau_0(b^{***}) \leq 0.
\]

Finally, we put

\[
s_1 := \left[ \int_{B_{X^*}} |\sigma|^{\frac{p}{1-p}} d_X(x', x'')^{\frac{p}{1-p}} |f x' - f x''|^p d\mu_0(f) \right]^{1 - \frac{p}{p}}
\]
and

\[
s_2 := \left[ \int_{B_{F^{***}}} \|b^*\|^{\frac{q}{1-q}} |\langle b^*, b^{***} \rangle|^{q'} d\tau_0(b^{***}) \right]^{1 - \frac{q}{q}}.
\]
Then
\[ \left| \sigma \langle T x' - T x'', b^* \rangle \right| = s_1 s_2 |(s_1^{-1} \sigma) \langle T x' - T x'', s_2^{-1} b^* \rangle| \]
\[ \leq C s_1 s_2 \left[ \frac{r'}{p} \int_{B_X^#} |s_1^{-1} \sigma| \frac{1}{|x'|} d\mu_0(f) \right] \left[ \frac{r'}{q} \int_{B_{F^*}} \|s_2^{-1} b^*\| \frac{1}{|x''|} \left| \langle s_2^{-1} b^*, b^{**} \rangle \right|^q d\tau_0(b^{**}) \right] \frac{1}{p} \]
\[ \leq C s_1 s_2. \]

Hence
\[ \left| \langle T x' - T x'', b^* \rangle \right| \leq C \cdot \left[ \int_{B_X^#} \left( \left| f(x') - f(x'') \right| \frac{1}{|x'|} d\mu(f) \right)^{\frac{1}{p}} \right] \cdot \left[ \int_{B_{F^*}} \left( \left| \langle b^*, b^{**} \rangle \right| \frac{1}{\|b^*\|} \frac{1}{\|b^{**}\|} d\tau(b^{**}) \right)^{\frac{1}{q}} \right]. \]

The aforementioned Theorem 3 will be used to prove the next result.

**Corollary 1.** The linear space \( \mathcal{D}_L^{(p, \theta, \nu)}(X, F) \) is a Banach space under the norm \( \mathcal{D}_L^{(p, \theta, \nu)} \).

**Proof.** To prove that \( \mathcal{D}_L^{(p, \theta, \nu)}(X, F) \) is complete space. We consider an arbitrary Cauchy sequence \((T_n)_{n \in \mathbb{N}} \) in \( \mathcal{D}_L^{(p, \theta, \nu)}(X, F) \) and show that \((T_n)_{n \in \mathbb{N}} \) converges to \( T \in \mathcal{D}_L^{(p, \theta, \nu)}(X, F) \). Since \((T_n)_{n \in \mathbb{N}} \) is Cauchy, for every \( \epsilon > 0 \) there is an \( n_0 \) such that
\[ D_L^{(p, \theta, \nu)}(T_m - T_n) \leq \epsilon \text{ for } m, n \geq n_0. \]  \( \quad (26) \)

Since \( \text{Lip}(T_m - T_n) \leq D_L^{(p, \theta, \nu)}(T_m - T_n) \) then \((T_n)_{n \in \mathbb{N}} \) is also a Cauchy sequence in the Banach space \( \text{Lip}(X, F) \), and there is a Lipschitz map \( T \) with
\[ \lim_{n \to \infty} \text{Lip}(T - T_n) = 0. \]

From (2) of Theorem 3 given \( \epsilon > 0 \) there is \( n_0 \in \mathbb{N} \) such that, for each \( n, m \in \mathbb{N}, n, m \geq n_0 \), there exist probabilities \( \mu_{nm} \) on \( B_X^# \) and \( \tau_{nm} \) on \( B_{F^{**}} \) such that for every \( x', x'' \) in \( X \) and \( b^* \) in
for every $\alpha$

$$F^*$$ the following inequality holds

$$\left| \left( (T_m - T_n) x' - (T_m - T_n) x'', b^* \right) \right| \leq \epsilon \, d_X(x', x'')^\theta \, \|b^*\|^\nu \cdot \left[ \int_{B_{X^*}} \left| f(x') - f(x'') \right|^p \, d\mu_{nm}(f) \right]^{1-\theta} \cdot \left[ \int_{B_{F^*}} \left| \langle b^*, b^* \rangle \right|^q \, d\tau_{nm}(b^*) \right]^{1-\nu}.$$  

Fixed $n \geq n_0$, by the weak compactness of $W(B_X^*)$ and $W(B_{F^*})$, there is a sub-net $(\mu_{nm}(\alpha), \tau_{nm}(\alpha))_{\alpha \in A}$ convergent to $(\mu_n, \tau_n) \in W(B_X^*) \times W(B_{F^*})$ in the topology $\sigma (\{C(B_X^* \times C(B_{F^*}))^*, C(B_X^*) \times C(B_{F^*})\})$. Then, there is $\alpha_0 \in A$ such that for each $x', x''$ in $X$, $b^*$ in $F^*$ and $\alpha \in A$ with $\alpha \geq \alpha_0$ we have

$$\left| \left( (T_{m(\alpha)} - T_n) x' - (T_{m(\alpha)} - T_n) x'', b^* \right) \right| \leq \epsilon \, d_X(x', x'')^\theta \, \|b^*\|^\nu \cdot \left[ \int_{B_{X^*}} \left| f(x') - f(x'') \right|^p \, d(\mu_{nm(\alpha)} - \mu_n)(f) + \int_{B_{X^*}} \left| f(x') - f(x'') \right|^p \, d\mu_n(f) \right]^{1-\theta} \cdot \left[ \int_{B_{F^*}} \left| \langle b^*, b^* \rangle \right|^q \, d(\tau_{nm(\alpha)} - \tau_n)(b^*) + \int_{B_{F^*}} \left| \langle b^*, b^* \rangle \right|^q \, d\tau_n(b^*) \right]^{1-\nu}.$$  

and taking limits when $\alpha \in A$ we have

$$\left| \left( (T_n - T_n) x' - (T_n - T_n) x'', b^* \right) \right| \leq \epsilon \, d_X(x', x'')^\theta \, \|b^*\|^\nu \cdot \left[ \int_{B_{X^*}} \left| f(x') - f(x'') \right|^p \, d\mu_n(f) \right]^{1-\theta} \cdot \left[ \int_{B_{F^*}} \left| \langle b^*, b^* \rangle \right|^q \, d\tau_n(b^*) \right]^{1-\nu}$$  

for every $x', x''$ in $X$ and $b^*$ in $F^*$. It follows that $T = T_n \in D^L_{(p,\theta,q,\nu)}(X,F)$ and therefore $T \in D^L_{(p,\theta,q,\nu)}(X,F)$. From the last inequality it follows that $D^L_{(p,\theta,q,\nu)}(T - T_n) \leq \epsilon$ if $n \geq n_0$ and hence $D^L_{(p,\theta,q,\nu)}(X,F)$ is a Banach space.  

By Proposition 22, Theorem 3 and Corollary 11 we obtain the following result.

**Proposition 23.** $[D^L_{(p,\theta,q,\nu)}, D^L_{(p,\theta,q,\nu)}]$ is a Banach nonlinear ideal.

**Remark 7.** Definition 3 can be generalized as follows. Let $0 \leq \theta < 1$ and $[\mathfrak{A}^L, \mathfrak{A}^L]$ be a normed nonlinear ideal. A Lipschitz operator $T$ from $X$ into $F$ belongs to $(\mathfrak{A}^L, \mathfrak{A}^L)^\theta$ if there exist a
Banach spaces $G_1$, $G_2$ and a Lipschitz operators $S_1 \in \mathcal{L}(X, G_1)$ and $S_2 \in \mathcal{L}(X, G_2)$ such that

$$
\|T x' - T x''\| F \leq \|S_1 x' - S_1 x''\| G_1^{1-\theta} \cdot \|S_2 x' - S_2 x''\| G_2^{\theta}, \ \forall x', x'' \in X.
$$

(27)

For each $T \in (\mathcal{L}, \mathcal{L})_\theta (X, F)$, we set

$$(A^L, B^L)_\theta (T) := \inf A^L(S_1)^{1-\theta} \cdot B^L(S_2)\theta
$$

where the infimum is taken over all Lipschitz operators $S_1, S_2$ admitted in (27).

Note that $\text{Lip}(T) \leq (A^L, B^L)_\theta (T)$, by definition. The nonlinear ideal $[\mathcal{L}^L, A^L_\theta]$ now appear as $[(\mathcal{L}, \text{Lip})_\theta , (A^L, \text{Lip}())_\theta ]$.

4 Nonlinear operator ideals between metric spaces

Definition 5. Suppose that, for every pair of metric spaces $X$ and $Y$, we are given a subset $\mathcal{A}^L(X, Y)$ of $\mathcal{L}(X, Y)$. The class

$$
\mathcal{A}^L := \bigcup_{X, Y} \mathcal{A}^L(X, Y)
$$

is said to be a nonlinear operator ideal, or just a nonlinear ideal, if the following conditions are satisfied:

\begin{itemize}
  \item [(NOI_0)] If $Y = F$ is a Banach space, then $g \boxplus e \in \mathcal{A}^L(X, F)$ for $g \in X^\#$ and $e \in F$.
  \item [(NOI_1)] $BTA \in \mathcal{A}^L(X_0, Y_0)$ for $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{A}^L(X, Y)$, and $B \in \mathcal{L}(Y, Y_0)$.
\end{itemize}

Condition (NOI_0) implies that $\mathcal{A}^L$ contains nonzero Lipschitz operators.

4.1 Lipschitz interpolative ideal procedure between metric spaces

Definition 6. Let $0 \leq \theta < 1$. A Lipschitz map $T$ from $X$ into $Y$ belongs to $\mathcal{A}^L_\theta (X, Y)$ if there exist a constant $C \geq 0$, a metric space $Z$ and a Lipschitz map $S \in \mathcal{A}^L(X, Z)$ such that

$$
d_Y(T x', T x'') \leq C \cdot d_Z(S x', S x'')^{1-\theta} \cdot d_X(x', x'')^\theta, \ \forall x', x'' \in X.
$$

(29)

For each $T \in \mathcal{A}^L_\theta (X, Y)$, we set

$$
A^L_\theta (T) := \inf CA^L(S)^{1-\theta}
$$

(30)

where the infimum is taken over all Lipschitz operators $S$ admitted in (29).

Proposition 24. $\mathcal{A}^L_\theta$ is a nonlinear ideal with $A^L_\theta (BTA) \leq \text{Lip}(B) \cdot A^L_\theta (T) \cdot \text{Lip}(A)$ for $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{A}^L(X, Y)$, and $B \in \mathcal{L}(Y, Y_0)$.
4.2 Basic Examples of Lipschitz interpolative ideal procedure

The proof of condition \( \widetilde{\text{NOI}_0} \) is similar to the proof of algebraic condition \( \widetilde{\text{PNOI}_0} \) in Proposition 20. To prove the condition \( \text{NOI}_1 \), let \( A \in \mathcal{L}(X_0, X) \), \( T \in \mathcal{L}_p(X, Y) \), and \( B \in \mathcal{L}(Y, Y_0) \) and \( x_0', x_0'' \) in \( X \). Then

\[
d_{Y_0}(BTx_0', BTx_0'') \leq \text{Lip}(B) \cdot d_Y(TAx_0', TAx_0'')
\]

\[
\leq \text{Lip}(B) \cdot C \cdot d_Z(S \circ A(x'), S \circ A(x''))^{1-\theta} \cdot d_X(Ax', Ax'')^\theta
\]

\[
\leq C \cdot \text{Lip}(B) \cdot \text{Lip}(A)^\theta \cdot d_Z(S \circ A(x'), S \circ A(x''))^{1-\theta} \cdot d_X(x', x'')^\theta.
\]

Since a Lipschitz map \( \tilde{S} := S \circ A \in \mathcal{L}^L(X_0, Z) \) and \( \tilde{C} := C \cdot \text{Lip}(B) \cdot \text{Lip}(A)^\theta \), hence \( BT \in \mathcal{L}^L(X_0, Y_0) \). Moreover, from (30) we have

\[
\mathcal{A}_{\theta}^L(BTA) := \inf \tilde{C} \cdot \mathcal{A}^L(\tilde{S})^{1-\theta}
\]

\[
\leq C \cdot \text{Lip}(B) \cdot \text{Lip}(A)^\theta \cdot \mathcal{A}^L(S \circ A)^{1-\theta}
\]

\[
= C \cdot \text{Lip}(B) \cdot \text{Lip}(A) \cdot \mathcal{A}^L(S)^{1-\theta}.
\]

Taking the infimum over all such \( S \in \mathcal{L}^L(X, Z) \) on the right side of (31), we have

\[
\mathcal{A}_{\theta}^L(BTA) \leq \text{Lip}(B) \cdot \mathcal{A}_{\theta}^L(T) \cdot \text{Lip}(A).
\]

\[\text{Proof.}\]

\[\text{Remark 8.}\]

(1) If \( \theta = 0 \), then the class \( \Pi^L_{(p, s, \theta)}(X, Y) \) coincides with the class \( \Pi^L_{(p, s)}(X, Y) \) which considered in [14] for \( \infty \geq p \geq q > 0 \) and [2] for \( 1 \leq q < p \).
(2) For a special case, if \( p = s \), a Lipschitz \((p, \theta)\)-summing map defined in [2] if there is a constant \( C \geq 0 \) such that
\[
\| (\lambda, Tx', Tx'') \|_{\ell_p^{(\mathbb{R} \times Y \times Y)}} \leq C \cdot \| (\lambda, x', x'') \|_{\delta_{p,\theta}(\mathbb{R} \times X \times X)}.
\] (33)
for arbitrary finite sequences \( x', x'' \) in \( X \) and \( \lambda \) in \( \mathbb{R}^+ \). Let us denote by \( \Pi_{L}^{(p, \theta)}(X, Y) \) the class of all Lipschitz \((p, \theta)\)-summing maps from \( X \) to \( Y \) with
\[
\pi_{L(p, \theta)}(T) = \inf C.
\]
where the infimum is taken over all constant \( C \) satisfying (33). The next result is a consequence of Proposition 25.

**Corollary 2.** \( \left[ \Pi_{P(p, \theta)}, \pi_{L(p, \theta)} \right] \) is a nonlinear ideal.

As a consequence of a general definition of Lipschitz interpolative ideal procedure between metric spaces the Lipschitz \((p, \theta)\)-summing map has following characterize result.

**Theorem 4.** [2] Let \( 1 \leq p < \infty \), \( 0 \leq \theta < 1 \) and \( T \in \text{Lip}(X, Y) \). The following statements are equivalent.

(i) \( T \in \Pi_{L_{p, \theta}}^{(p, \theta)}(X, Y) \).

(ii) There is a constant \( C \geq 0 \) and a regular Borel probability measure \( \mu \) on \( B_X^\# \) such that
\[
d_Y(Tx', Tx'') \leq C \left( \int_{B_X^\#} \left( |f(x') - f(x'')[1-\theta d_X(x', x'')^\theta] \right)^{1-\theta} \frac{d\mu(f)}{p} \right)^{\frac{1}{1-\theta}}
\]
for all \( x', x'' \in X \).

(iii) There is a constant \( C \geq 0 \) such that for all \( (x'_j)_{j=1}^m, (x''_j)_{j=1}^m \) in \( X \) and all \( (a_j)_{j=1}^m \subset \mathbb{R}^+ \) we have
\[
\left( \sum_{j=1}^m a_j d_Y(T(x'_j), T(x''_j))^{\frac{1-\theta}{p}} \right)^{\frac{1}{1-\theta}} \leq C \sup_{f \in B_X^\#} \left( \sum_{j=1}^m a_j (|f(x'_j) - f(x''_j)|^{1-\theta} d_X(x'_j, x''_j)^\theta)^{\frac{1-\theta}{p}} \right)^{\frac{1}{1-\theta}}.
\]

(iv) There exists a regular Borel probability measure \( \mu \) on \( B_X^\# \) and a Lipschitz operator \( v : X_{\mu, \theta}^\mu \rightarrow Y \) such that the following diagram commutes
\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow_{\delta_X} & & \uparrow_v \\
\delta_X(X) & \xrightarrow{\phi_{oi}} & X_{\mu, \theta}^\mu
\end{array}
\]
Furthermore, the infimum of the constants \( C \geq 0 \) in (2) and (3) is \( \pi_{\mu, \theta}^{(p, \theta)}(T) \).
(2) Lipschitz \((s; q, \theta)\)-mixing operators

D. Achour, E. Dahia and M. A. S. Saleh \cite{Achour2018} defined a Lipschitz \((s; q, \theta)\)-mixing operator if there is a constant \(C \geq 0\) such that

\[
m_{(s; q)}^{L, \theta}(\sigma, T x', T x'') \leq C \cdot \delta_{q \theta}^{L}(\sigma, x', x'')
\]

for arbitrary finite sequences \(x', x''\) in \(X\) and \(\sigma\) in \(\mathbb{R}\). Let us denote by \(M_{(s; q)}^{L, \theta}(X, Y)\) the class of all Lipschitz \((s; q, \theta)\)-mixing maps from \(X\) to \(Y\). In such case, we put

\[
m_{(s; q)}^{L, \theta}(T) = \inf C,
\]

where the infimum is taken over all constant \(C\) satisfying (34). The proof of the following result is not difficult to prove it.

**Proposition 26.** \([M_{(s; q)}^{L, \theta}], m_{(s; q)}^{L, \theta}\) is a nonlinear ideal.

**Remark 9.** Definition (2) can be generalized as follows. Let \(0 \leq \theta < 1\). A Lipschitz map \(T\) from \(X\) into \(Y\) belongs to \((A^{L}, B^{L})_{\theta}(X, Y)\) if there exist a constant \(C \geq 0\), a metric spaces \(Z_{1}, Z_{2}\) and Lipschitz maps \(S_{1} \in A^{L}(X, Z_{1})\) and \(S_{2} \in B^{L}(X, Z_{2})\) such that

\[
d_{Y}(Tx', Tx'') \leq C \cdot d_{Z_{1}}(S_{1}x', S_{1}x'')^{1-\theta} \cdot d_{Z_{2}}(S_{2}x', S_{2}x'')^{\theta}, \quad \forall x', x'' \in X.
\]

For each \(T \in (A^{L}, B^{L})_{\theta}(X, Y)\), we set

\[
(A^{L}, B^{L})_{\theta}(T) := \inf C \cdot A^{L}(S_{1})^{1-\theta} \cdot B^{L}(S_{2})^{\theta}
\]

where the infimum is taken over all Lipschitz operators \(S_{1}, S_{2}\) admitted in (50). The nonlinear ideal \([A^{L}, A^{L}]_{\theta}\) now appear as \(\left((A^{L}, L^{\theta})_{\theta}, (A^{L}, \Lip(\cdot))_{\theta}\right)\).

**References**

[1] D. Achour, E. Dahia and M. A. S. Saleh, *Multilinear mixing operators and Lipschitz mixing operator ideals*, OaM, Oper. Matrices (2018).
[2] D. Achour, P. Rueda, and R. Yahi, \((p, \sigma)\)-absolutely Lipschitz operators, Ann. Funct. Anal. 8 (2017), 38–50.
[3] M. G. Cabrera-Padilla, J. A. Chávez-Domínguez, A. Jimenez-Vargas and M. Villegas-Vallecillos, *Maximal Banach ideals of Lipschitz maps*, Annals of Functional Analysis 7 (2016), 593–608.
[4] J. A. Chávez-Domínguez, *Duality for Lipschitz \(p\)-summing operators*, J. Funct. Anal. 261, (2011), 387–407.
[5] J. A. Chávez-Domínguez, *Lipschitz \((q; p)\)-mixing operators*, Proc. Amer. Math. Soc. 140 (2012), 3101–3115.
[6] J. D. Farmer and W. B. Johnson, *Lipschitz \(p\)-summing operators*, Proc. Amer. Math. Soc. 137 (2009), 2989–2995.
[7] A. Jiménez-Vargas, J. M. Sepulcre, and Moisés Villegas-Vallecillos, *Lipschitz compact operators*, J. Math. Anal. Appl., 415 (2014), 889–901.

[8] H. Jarchow, U. Matter, *Interpolative constructions for operator ideals*, Note. Mat. (1)8(1988), 45–56.

[9] J. A. López Molina and E. A. Sánchez Pérez, *Ideales de operadores absolutamente continuos*, Revista de la Real Academia de Ciencias Exactas Físicas y Naturales 87 (1993), 349–378.

[10] U. Matter, *Absolutely continuous operators and super–reflexivity*, Math. Nachr. 130 (1987), 193–216.

[11] A. Pietsch, *Eigenvalues and s-Numbers*, Geest & Portig, Leipzig, and Cambridge Univ. Press, 1987.

[12] A. Pietsch, *Operator ideals*, Deutsch. Verlag Wiss., Berlin, 1978; North–Holland, Amsterdam–London–New York–Tokyo, 1980.

[13] A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhaüser Boston, 2007; North–Holland, Amsterdam–London–New York–Tokyo, 1980.

[14] M. A. S. Saleh, *New types of Lipschitz summing maps between metric spaces*, Math. Nachr. 290 (2017), 1347–1373.