Non-linear Gradient Algorithm for Parameter Estimation: Extended version

Juan G. Rueda-Escobedo and Jaime A. Moreno

Abstract—Gradient algorithms are classical in adaptive control and parameter estimation. For instantaneous quadratic cost functions they lead to a linear time-varying dynamic system that converges exponentially under persistence of excitation conditions. In this paper we consider (instantaneous) non-quadratic cost functions, for which the gradient algorithm leads to non-linear (and non Lipschitz) time-varying dynamics, which are homogeneous in the state. We show that under persistence of excitation conditions they also converge globally, uniformly and asymptotically. Compared to the linear counterpart, they accelerate the convergence and can provide for finite-time or fixed-time stability.

I. INTRODUCTION

This work is an extended version of [1]. In this paper the proof of the claims in [1] are presented in the appendices. This proof are omitted in the first work non-linear estimation algorithms, with non-linear correction terms in e, that lead to time-varying dynamical systems. The proposed schemes can be obtained as the negative gradient of an instantaneous convex non smooth function of the output estimation error. The resulting (error) systems are homogeneous (or homogeneous in the bi-limit) in the estimation error x [9], [11], but time-varying. Unlike time invariant homogeneous systems, time-varying systems do not possess such strong properties. For example, neither local asymptotic stability implies global asymptotic stability, nor negative homogeneity degree implies finite time convergence [11]. Some local asymptotic stability results for time-varying systems homogeneous in the state have been obtained, when the homogeneity degree is zero [12], and for positive homogeneity degree [11], using averaging techniques.

Motivated by these results we propose in this work non-linear estimation algorithms, with non-linear correction terms in e, that lead to time-varying dynamical systems. The proposed schemes can be obtained as the negative gradient of an instantaneous convex non smooth function of the output estimation error. The resulting (error) systems are homogeneous (or homogeneous in the bi-limit) in the estimation error x [9], [11], but time-varying. Unlike time invariant homogeneous systems, time-varying systems do not possess such strong properties. For example, neither local asymptotic stability implies global asymptotic stability, nor negative homogeneity degree implies finite time convergence [11]. Some local asymptotic stability results for time-varying systems homogeneous in the state have been obtained, when the homogeneity degree is zero [12], and for positive homogeneity degree [11], using averaging techniques.

We show in this paper, in straightforward a direct manner, that the proposed algorithms converge globally and asymptotically under the PE condition of the regressor. When n = 1, the algorithm is able to converge in finite-time and it is also able to estimate a time-varying parameter. Moreover, adding homogeneous terms of positive degree the estimation error converges in fixed-time, i.e. the convergence time is upper bounded by a constant independent of the initial estimation error. For n > 1 global, uniform and asymptotic stability can be assured and acceleration for large initial conditions can be obtained.
II. MOTIVATION AND PROBLEM STATEMENT

The estimation problem may be regarded as a minimization problem. In this context a common cornerstone is to set a convex cost function of the output estimation error. In this work we choose the following structure for the cost function

\[ J(\hat{\theta})_p := \frac{1}{p+1} |u^T(t)\hat{\theta}(t) - y(t)|^{p+1}. \]

Here the exponent \( p > 0 \) is a parameter to be chosen. The term \( -\frac{\partial}{\partial \theta} J(\hat{\theta})_p \) shows the direction in which the parameter estimated needs to change. With this idea in mind we propose the following algorithm

\[ \dot{\hat{\theta}}(t) := -\frac{\partial}{\partial \theta} J(\hat{\theta})_p \]

\[ = -|u^T(t)\hat{\theta}(t) - y(t)|^p \text{sign}(u^T(t)\hat{\theta}(t) - y(t)) u(t). \]

For the sake of readability let us define \( |w|^p := |w|^p \text{sign}(w) \). With this convention the algorithm is rewritten as

\[ \dot{\hat{\theta}}(t) = -|u^T(t)\hat{\theta}(t) - y(t)|^p u(t). \] (1)

We denote as composite algorithm an algorithm that results from adding vector fields of the form in (1) to avail of the dynamics traits associated to each \( p_i \). This leads to

\[ \dot{\hat{\theta}}(t) = -\left( \sum_{i=1}^h |u^T(t)\hat{\theta}(t) - y(t)|^{p_i} \right) u(t). \] (2)

To analyse the convergence of the algorithms to \( \theta_0 \) the dynamics of the estimation error \( x(t) = \hat{\theta}(t) - \theta_0 \) is needed. Hence it is necessary to compute the time derivative of \( x(t) \). In the case of the single algorithm in (1) this error dynamics is as follow

\[ \dot{x}(t) = -|u^T(t)x(t)|^p u(t). \] (3)

Considering \( f(t,x) = -|u^T(t)x|^p u(t) \) and replacing the argument \( x \) by \( \epsilon x \), with \( \epsilon > 0 \), we have the following relation \( f(t,\epsilon x) = \epsilon^{p-1} \epsilon f(t,x) \). Taking this into consideration and following Definition 1 in [11], we can said that the system (3) is homogeneous with homogeneity degree \( p - 1 \). When \( 0 < p < 1 \) the homogeneity degree is negative; for \( p = 1 \) it is zero; and for \( p > 1 \) the homogeneity degree is positive. Now, the convergence analysis for this class of algorithms reduces to establish the stability and the attractivity of the origin of a time-variant homogeneous system.

Repeating the same analysis for the composite algorithm the error dynamics is

\[ \dot{x}(t) = -\left( \sum_{i=1}^h |u^T(t)x(t)|^{p_i} \right) u(t). \] (4)

Please, notice that this system is no longer homogeneous.

A quick stability check with a quadratic Lyapunov function \( V(x) := \frac{1}{2} x^T x \) yields to global uniform stability (GUS) of the origin of (3) and (4). The time derivative of \( V \) is presented for both cases in order to show its negative semi-definitiveness:

\[ \dot{V}(t) = -|u^T(t)x(t)|^{p+1} \leq 0, \] (5a)

\[ \dot{V}(t) = -\sum_{i=1}^h |u^T(t)x(t)|^{p_i+1} \leq 0. \] (5b)

Notice that the term \( u^T(t)x(t) \) can vanish outside the set \( \{t|u(t) = 0\} \cup \{t|x(t) = 0\} \). To assert the uniform asymptotic stability (UAS) of the origin of (3) and (4) \( u(t) \) needs to be PE. In [13] a convenient description of the Persistent Excitation is given. This description is expressed as a lower bound of the integral of the regressor and is found convenient because it fits well in the study of the estimation-error convergence. For convenience the definition is reproduced below

Definition 1: Let \( u(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^n \) be a piecewise continuous function. It is said that \( u \) is of PE if there exist \( T > 0 \) and \( \epsilon > 0 \) such that

\[ \frac{1}{T} \int_t^{t+T} |u^T(s)| w(s) ds \geq \epsilon, \]

for all \( w \in \mathbb{R}^n \) with \( \|w\| = 1 \).

To conclude this section, we present some definitions of time-varying systems.

Definition 2: Let a time-varying system be represented by \( \dot{x}(t) = f(x(t),t) \) where \( f(0,t) = 0 \) for all \( t \). Let \( \Omega \) be a connected open subset of \( \mathbb{R}^n \), such that \( 0 \in \Omega \). The point \( x = 0 \) is

- Uniformly finite-time stable (UFTS) if it is uniformly stable and for any \( x_0 \in \Omega \) there exist \( 0 \leq T(x_0) < +\infty \) such that \( x(t,x_0) = 0 \) for all \( t \geq t_0 + T(x_0) \). Also, if \( \Omega = \mathbb{R}^n \) then \( x = 0 \) is said to be globally uniformly finite-time stable (GUFTS).
- Uniformly fixed-time stable (UFxTS) if it is GUFTS and exist \( T \geq +\infty \) such that \( T \geq T(x_0) \) for every \( x_0 \in \mathbb{R}^n \).

III. MAIN RESULT

In this section the stability of (3) and (4) is presented. Two cases are recognized: (i) when only one parameter needs to be estimated (scalar case) and (ii) when are more than one (vector case). This division is done because the results in the vector case do not reflect certain phenomena that occurred in the scalar one. The proof of the theorems can be found in the Appendices.
A. Scalar Case

The results in the scalar case are stronger than in the vector one due to the fact that the product $u^T(t)x(t)$ can only be zero if one or both of the variables are zero. Persistent excitation prevents $u$ to stay in zero or to exhibit a growing dwelling time in zero. This implies that, for staying in zero, $x$ needs to be zero. When the exponent $p$ is chosen in the interval $[0,1)$ the algorithm converges in uniform finite-time and a bound of the convergence time is given as an integer multiple of the persistent excitation period $T$. The next statement summarizes this discussion.

**Theorem 1:** Let $u(t)$ be a piece-wise continuous function of $t$ and of PE. Let $0 \leq p < 1$ and $n = 1$, then the origin of (4) is globally uniformly finite-time stable. An upper bound of the convergence time is

$$T(x_0) \leq \left[ \frac{\|x(t_0)\|^{1-p}}{(1-p)T \epsilon^{p+1}} \right] \cdot T$$

where $x(t_0)$ is the estimation error at the initial time; $T$ and $\epsilon$ are as in Definition 1 and $\lceil \cdot \rceil$ denotes the ceiling function.

When $p = 1$ the linear case is obtained and its properties are well known [2], [3], for this is left out of the discussion. For $p > 1$ only global uniform asymptotic stability (GUAS) can be asserted but other interesting property arise. No matter how large the initial error is, the time $\tau(c)$ need to reach a smaller level set $V(x) = c$ is only function of $c$. This is referred to as escape from infinite in finite time uniformly in $t$. We gather these results in the following theorem.

**Theorem 2:** Let $u(t)$ be a piece-wise continuous function of $t$ and of PE, $p > 1$ and $n = 1$, then the origin of (4) is globally uniformly asymptotically stable. Furthermore, the time needed to escape from infinity to a compact region $V(x) \leq c$ is bounded by

$$\tau(c) \leq \left[ \frac{1}{2 \frac{\epsilon}{1} (p-1)T \epsilon^{p+1}} \cdot \frac{1}{e^{\frac{p+1}{2}}} \right] \cdot T$$

The study of the convergence of the composite algorithm can be done using the previous results. For the scalar case the stability can be asserted via a Comparison Lemma for differential inequalities [14]. In short, the trajectories of the system (4) are below the trajectories corresponding to each of the single algorithms with every isolated exponent $p_i$. This can be seen from (5b) which can be rewritten as

$$\dot{V}(t) = - \sum_{i=1}^{h} 2^{p_{i+1}} |u(t)|^{p_{i+1}} V^{p_{i+1}+1}(t) \leq -2^{p_{i+1}} |u(t)|^{p_{i+1}} V^{p_{i+1}+1}(t), \ i = 1, 2, \cdots, h.$$  

An important case occurs when at least one exponent is in $[0,1)$ and another one is greater than one. For this case the time needed for the algorithm to converge to $\theta_0$ is independent of the initial estimation error and the initial time if $u(t)$ is of PE. This is summarized in the following theorem.

**Theorem 3:** Let $u(t)$ be a piece-wise continuous function of $t$ and of PE, also $n = 1$. Consider the system (4) and let $P := \{p_1, p_2, \cdots, p_h\}$ be the set of the exponents. Denote $p_m$ the minimum element in $P$ and $p_M$ the maximum. Assume that $0 \leq p_m < 1$ and $p_M > 1$. Let $P_m$ be the subset of the exponent smaller than one and $P_M$ the subset of the exponents greater than one. Define $\bar{p} \in P_M$ as the exponent which maximizes $(p_i-1)\epsilon^{p_i+1}$ and $p \in P_m$ the exponent which maximizes $(1-p_i)\epsilon^{p_i+1}$, then an upper bound of the time needed to converge to zero is

$$T \leq \left( \frac{1}{(1-p)T \epsilon^{p+1}} + \frac{1}{2 \frac{\epsilon}{1} (\bar{p}-1)T \epsilon^{\bar{p}+1}} \right) \cdot T.$$  

Last but no least, a discontinuous algorithm capable of estimating one varying parameter is presented. This algorithm makes use of the regressor sign and the sign of the output estimation error

$$\hat{\theta}(t) = -L \cdot \text{sign}(u(t)\hat{\theta}(t) - y(t)) \cdot \text{sign}(u(t)), \quad (8)$$

with $L > 0$. Now assume that the parameter variation is bounded, i.e. $\hat{\theta} \in [-\gamma, \gamma], \gamma \geq 0$. Also it is considered that $u(t)$ is of PE and cannot stay in zero for time intervals, but can cross it. The error dynamics induced by (8) is

$$\dot{x}(t) \in -L \cdot \text{sign}(x(t)) + \dot{\theta}(t). \quad (9)$$

which is a differential inclusion [9]. By employing $V(x) = \frac{1}{2}|x|^2$ as Lyapunov Function its derivative along the trajectories of (9) is

$$\dot{V}(t) = -L|x(t)| - \dot{\theta}x(t) = -(L + \dot{\theta}(t)) |x(t)|,$$

which is negative if $L > \gamma$ and $u(t) \neq 0$. If $u(t)$ stay in zero the track of $\theta(t)$ is lost but can be recovered later if $u$ is of PE. To guarantee exact tracking, $u(t)$ cannot stay in zero.

B. Vector Case

When $\theta_0$ is a vector the set where $u^T(t)x(t)$ is zero grows. This changes the general behaviour of our algorithms. Only GUAS can be guaranteed in general with persistent excitation. Also, the discontinuous algorithm which result of selecting $p = 0$ does not converge. In the latter case, a signal of PE can be constructed for which the output estimation error $e(t)$ becomes zero in finite
time but \( x(t) \) does not reach the origin. The stability properties of the origin of (4) are summarized in the following theorem.

**Theorem 4:** Let \( u(t) \) be a piece-wise continuous function of \( t \) and of PE, uniformly bounded by \( u_M \), then the origin of (4) is globally uniformly asymptotically stable for any \( p > 0 \). \( \triangle \)

Although only GUAS can be asserted with PE in general, in the following section two classes of signals of PE are presented. One of them guarantees global uniform exponential stability (GUES) and the other GUFTS; in both cases for \( 0 < p < 1 \). This means that the signals \( u(t) \) which can provide UFTS are in a subset of those of PE.

The composite algorithm still works in the vector case but nothing more than GUAS can be claimed. In contrast to the scalar case the stability of the composite vector algorithm cannot be obtained via the Comparison Lemma. The following theorem synthesizes this discussion.

**Theorem 5:** Let \( u(t) \) be a piece-wise continuous function of \( t \) and of PE, uniformly bounded by \( u_M \), then the origin of (4) is globally uniformly asymptotically stable if \( p_M \leq p_m + 1 \) where \( p_m \) is the minimum exponent in the set \( \{p_1, p_2, \ldots, p_k\} \) and \( p_M \) is the maximum. \( \triangle \)

The extra condition regarding the exponents appears due to the way the proof was done and we think it is not intrinsic to the stability.

In the scalar case the Comparison Lemma gives information about the relationship between the trajectories of the single algorithm but in this case nothing can be concluded. In the next section simulation examples are presented for the composite algorithm. In the Figure 2 and Figure 3 it can be seen that the Lyapunov function is below of the corresponding one for the single algorithms but this does not need to be true in general.

IV. Examples

In this section examples are presented. In the scalar case, simulations of a parameter estimation process are shown, whereas for the vector case also the behaviour of the error is studied for a specific class of signals.

A. Scalar case

Numerical simulation is performed to illustrate the difference between the classic gradient algorithm and the family presented in this work. For this aim we choose:

\[
\theta_0 = 5, \quad \dot{\theta}(t_0) = 0, \quad u(t) = 2 \cos(2t).
\]

Four algorithms were simulated: one with \( p_1 = 3/4 \), other with \( p_2 = 1 \), a third one with \( p_3 = 3/2 \) and a composite one with \( p_1 \) and \( p_3 \). Figure 1 and Figure 2 show the behaviour of the Lyapunov function \( V(t) = \frac{1}{2} |\dot{\theta}(t) - \theta_0|^2 \). In the first figure the initial value of \( V(t) \) and its decaying behaviour are presented for all the algorithms. As can be seen there is an ordering in the decay: Composite \( > p = \frac{3}{4} > p = 1 > p = \frac{3}{2} \), and this is due to the initial value of \( V(t_0) \) \( > 0 \). When the estimation error becomes smaller the order changes as in Figure 3.

In Figure 2 the behaviour is shown when \( V(t) < 0.014 \).

![Fig. 1. Behaviour of \( V(t) \) in the Scalar case (a).](image)

![Fig. 2. Behaviour of \( V(t) \) in the Scalar case (b).](image)

In this region the decay order is: Composite \( > p = \frac{3}{4} > p = 1 > p = \frac{3}{2} \) as is expected from the kind of convergence of every algorithm: u. fixed-time > u. finite-time > u. exponential > u. asymptotic. This order does not change for any future time.

Now a estimation of a time-varying parameter is done via (8). The simulation parameters are:

\[
\theta(t) = \cos(3t) - 4, \quad \dot{\theta}(t_0) = 0, \quad u(t) = \cos(t) + 1.5, \quad L = 3.3.
\]

Since the regressor is non-zero for any time, an exact tracking of the parameter is achieved as shown in Figure 3.

B. Vector case

First simulation results are presented for the estimation process of three parameters with (1) and (2). Four algorithms were simulated: the first one with \( p_1 = 3/4 \), the second with \( p_2 = 1 \), the third one with \( p_3 = 3/2 \) and the last one with \( p_1 \) and \( p_3 \). The simulation results are shown in Figure 4 and Figure 5 to compare the behaviour of the algorithms, when different exponents are chosen. Later, the response of the algorithm for \( 0 < p < 1 \) is found for
piecewise constant signal and the class of convergence is established for it.

For the simulation example the next conditions were selected:

\[
\begin{align*}
    u(t) &= \begin{bmatrix} 2 \cos(2t) - \cos(3t) & 5 \cos(5t) \end{bmatrix}^T, \\
    \theta_0 &= \begin{bmatrix} -3 & \sqrt{2} & 4 \end{bmatrix}^T, \\
    \hat{\theta}(t_0) &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T.
\end{align*}
\]

As in the last section only the plot of \( V(t) = \frac{1}{2}(\hat{\theta}(t) - \theta_0)^T(\hat{\theta}(t) - \theta_0) \) is presented. In Figure 4 the decay of \( V(t) \) is shown at the beginning of the process and the decay order is preserved. Also in Figure 5 the change of order that was found for the scalar case is obtained. However we do not see, in general, a change of order for arbitrary parameter values.

Now the response of the system (3) when \( u(t) \) is piecewise constant is analyzed. Let us define a new variable as follows \( z(t) := u^T(t)x(t) \), its dynamics is \( \dot{z}(t) = u^T(t)\dot{x}(t) = -\|u(t)\|^2 z(t) \) since \( u(t) = 0 \) almost everywhere, where \( \| \cdot \| \) denotes the Euclidean norm. For simplicity we assume that the length of the intervals where \( u \) remains constant has a constant value \( \tau > 0 \). The solution of \( z(t) \) in the interval \( t \in [t_1, t_1 + \tau] \) with \( u(t) = \mu \), \( u_M \geq \|\mu\| > 0 \) and \( p > 0, p \neq 1 \) is

\[
z(t) = \left(|z(t_1)|^{1-p} - (1-p)\|\mu\|^2(t-t_1)\right)^{1/p} \text{sign}(z(t_1)).
\]

For \( 0 < p < 1 \) the expression above holds if \( |z(t_1)|^{1-p} \geq (1-p)\|u\|^2(t-t_1) \) and \( z(t) = 0 \) otherwise. The solution of \( z(t) \) can be used to find the solution of \( x(t) \) by noticing that \( \dot{x}(t) = -[z(t)]^p u(t) \). This is given in the forthcoming expression

\[
x(t) = x(t_1) - \frac{1}{\|\mu\|^2} \mu \left( \frac{1}{1-p} \|\mu\|^2(x(t_1) - \text{sign}(\mu^T x(t_1))) \timesight.
\]

\[
\times \left( (\mu^T x(t_1))^{1-p} - (1-p)\|\mu\|^2(t-t_1) \right)^{1/p} \right).
\]

Restricting the analysis for \( 0 < p < 1 \) it is clear that, if \( \tau \) is large enough then \( x(t_1 + \tau) \) is orthogonal to \( \mu(t_1) \)

\[
x(t_1 + \tau) = x(t_1) - \frac{1}{\|\mu\|^2} \mu^T x(t_1) = \left( I_n - \frac{1}{\|\mu\|^2} \mu \mu^T \right)x(t_1).
\]

Fix \( \tau \), if \( \|x(t_1)\|^{1-p} \leq (1-p)\|\mu\|^2 \tau \) then \( x(t_1 + \tau) \) becomes orthogonal to \( \mu \). This means that there exists a ball centered in zero for which \( \tau \) is always large enough to make \( x \) orthogonal to any \( \mu \). If the sequence \( U = \{\mu_i\}_{i=1}^\infty \) is chosen to fulfill the notion of persistent excitation for discrete systems in [15], then the origin is GUAS by Theorem 4 and \( x(t) \) can reach any ball centered in zero in finite time, i.e. always reach the ball in where \( \tau \) guarantees that \( x \) becomes orthogonal to \( \mu_1 \), and this yields a discrete system which is described by the following difference equation

\[
x_{k+1} = \left( \frac{1}{\|\mu_k\|^2} \mu_{k+1}^T \right)x_k.
\]

Only assuming PE of the sequence \( U \), GUES of the origin can be concluded making an analogue analysis to the one shown in the proof for Theorem 4 in Appendix II-D but taking \( V_k(x) = x_k^T \Delta V = V_{k+1} - V_k(x) \) instead of \( V(x) \) and \( V(t) \) respectively. Now take \( n \) mutual orthogonal vectors \( \{v_1, v_2, \ldots, v_n\} \), \( v_i^Tv_j = 0, i \neq j \), and excite the system with them, then \( x_{k+n} = 0 \)

\[
x_{k+n} = \sum_{i=1}^n \left( I_n - \frac{1}{\|v_i\|^2} v_i v_i^T \right)x_k = 0.
\]

A sequence constructed with these vectors is of PE and also makes the system GUES. This shows that persistent excitation cannot guarantee finite-time convergence in the vector case but it does not forbid it.
V. CONCLUSIONS

In this work a parameter estimation technique is presented. With the proposed algorithms we obtained finite-time and fixed-time convergence to the true parameters. However, this property cannot be guaranteed in general. A deep study of the signals that can assert such important properties is still needed. Even though the algorithms were selected to make the error dynamics homogeneous in the state, this does not help in the analysis. However, the homogeneous nonlinearities can enhance the robustness properties of our algorithms w.r.t additive perturbations in comparison with classical approaches.

ACKNOWLEDGEMENT

The authors thank the financial support from PAPIIT-UNAM (Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica), project IN113614; Fondo de Colaboración II-FI UNAM, Project HSGBS-122-2014; CONACyT (Consejo Nacional de Ciencia y Tecnología), project 241171; and CONACyT CVU: 491701.

REFERENCES

[1] J. G. Rueda-Escobedo and J. A. Moreno, “Non-linear gradient algorithm for parameter estimation,” in Decision and Control (CDC), 2015 IEEE 54th Annual Conference on, 2015, p. to appear.
[2] B. D. O. Anderson, “Exponential stability of linear equations arising in adaptive identification,” IEEE Trans. Automat. Contr., vol. 22, no. 1, pp. 83–88, 1977.
[3] A. P. Morgan and K. S. Narendra, “On the uniform asymptotic stability of certain linear nonautonomous differential equations,” SIAM Journal on Control and Optimization, vol. 15, no. 1, pp. 5–24, Jan. 1977.
[4] S. Bhat and D. Bernstein, “Geometric homogeneity with applications to finite-time stability,” Mathematics of Control, Signals, and Systems, vol. 17, no. 2, pp. 101–127, 2005.
[5] V. Andrieu, L. Praly, and A. Astolfi, “Homogeneous approximation, recursive observer design and output feedback,” SIAM J. Control Optim., vol. 47, no. 4, pp. 1814–1850, 2008.
[6] A. Levant, “Homogeneity approach to high-order sliding mode design,” Automatica, vol. 41, pp. 823–830, 2005.
[7] J. Moreno, “Lyapunov function for analysis and design of second order sliding mode algorithms,” in Sliding Modes after the first decade of the 21st Century, ser. LNCIS, 412, L. Fridman, J. Moreno, and R. Iriarte, Eds. Berlin - Heidelberg: Springer-Verlag, 2011, pp. 113–150.
[8] E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov, “On homogeneity and its application in sliding mode design,” International Journal of Franklin Institute, vol. 351, no. 4, pp. 1866–1901, 2014.
[9] A. Bacciotti and L. Rosier, Liapunov Functions and Stability in Control Theory, ser. Communication and Control Engineering. Berlin - Heidelberg: Springer-Verlag, 2005, ch. 5, pp. 180–186.
[10] E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov, “On an extension of homogeneity notion for sliding intervals,” in European Control Conference, Zurich, Switzerland, Jul. 2013.
[11] J. Peuteman and D. Aeyels, “Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time-varying,” SIAM Journal on Control and Optimization, vol. 37, no. 4, pp. 997–1010, 1999.
[12] R. T. McClosey, “An averaging theorem for time-periodic zero homogeneous differential equations,” Systems & Control Letters, vol. 32, pp. 179–182, 1997.
[13] K. S. Narendra and A. M. Annaswamy, “Persistent excitation in adaptive systems,” International Journal of Control, vol. 45, no. 1, pp. 127–160, 1987.
[14] H. K. Khalil, Nonlinear Systems. Upper Saddle River, New Jersey 07458: Prentice Hall, 2002, ch. 3, pp. 102–104.
[15] T.-H. Lee and K. S. Narendra, “Robust adaptive control of discrete-time systems using persistent excitation,” Automatica, vol. 24, no. 6, pp. 781–788, 1988.
[16] K. S. Narendra and A. M. Annaswamy, Stable Adaptive Systems, Englewood Cliffs, New Jersey 07632: Prentice Hall, 1989, ch. 2, pp. 71–74.

APPENDIX I

PERSISTENT EXCITATION

To prove uniform asymptotic stability rather than uniform stability the persistent excitation in the regressor is needed. For using the property adequately the next proposition is developed.

Proposition 1: If $u(t)$ is of PE, then the following inequality holds for $T$, $\epsilon$ and $w$ as in Definition [1] and $p \geq 0$:

$$\int_{t_0}^{t+T} |u^T(s)w|^{p+1} ds \geq \frac{T \epsilon^{p+1}}{p \epsilon^p}.$$

Proof: Applying the Hölder inequality to $f(t) = |u^T(t)w|$ and $g(t) = 1$ on the interval $[t, t+T]$ and using $\frac{p}{p+1}$ as Hölder conjugates we obtain

$$T^p \int_{t}^{t+T} |u^T(s)w|^{p+1} ds \geq \left( \int_{t}^{t+T} |u^T(s)w| ds \right)^{p+1}$$

using Definition [1] leads to

$$\int_{t}^{t+T} |u^T(s)w|^{p+1} ds \geq \frac{T^{p+1}}{T^p} \epsilon^{p+1} = T \epsilon^{p+1}.$$

This derivation from the PE is done in the aim of easily present the proof of the theorems.

APPENDIX II

PROOF OF THEOREMS [1] TO [5]

A. Proof of Theorem [7]

For $n = 1$ the term $|u(t)^T x(t)|$ can be rewritten as $|u(t)| \cdot |x(t)| = \sqrt{2} |u(t)| V^{\frac{1}{2}}(t)$ replacing this in (5a) yields

$$V(t) = -2^{\frac{p+1}{2}} |u(t)|^{p+1} V^{\frac{p+1}{2}}(t).$$

Solving the differential equation we have

$$V(t) = \left( V^{\frac{1}{2}}(t_0) - 2^{\frac{p+1}{2}} (1 - p) \int_{t_0}^{t} |u(s)|^{p+1} ds \right)^{2}.$$

(10)

This solution is valid for $p \geq 0$ and $p \neq 1$. For $0 \leq p < 1$ the solution exists if $V^{\frac{1}{2}}(t_0) \geq 2^{\frac{p+1}{2}} (1 - p) \int_{t_0}^{t} |u(s)|^{p+1} ds$ after that $V(t) = 0$. The PE guarantees that there exists $t_1 \geq t_0$ when the inequality no longer holds. To estimate this time it is sufficient to find an integer $k$ such that the integral of $|u(t)|^{p+1}$ from...
\[ t_0 + kT \text{ is greater than } 2^{1-p}/(1-p)V^{1-p}(t_0). \]

Using Proposition 1,
\[ \int_{t_0}^{t_0+kT} |u(s)|^{p+1} ds \geq kTE^{p+1} \geq \frac{2^{1-p}}{1-p} V^{1-p}(t_0). \]

Solving for \( k \) and taking the least integer that fulfills the inequality yields
\[ k = \left\lceil \frac{2^{1-p}}{(1-p)TE^{p+1} V^{1-p}(t_0)} \right\rceil, \quad (11) \]
then the time that guarantees that \( V(t) \) reaches zero is
\[ t_1 = k \cdot T + t_0. \]
Notice that \( V(t_0) \) can be replaced by \( \frac{1}{2} |x(t_0)|^2 \) in (11) to obtain (6).

**B. Proof of Theorem 2**

Consider again the solution for \( V(t) \) in (10). Since \( p > 1 \), Equation (10) can be rewritten as
\[ V(t) = \frac{1}{\left( V^{1-p}(t_0) + 2^{\frac{1-p}{2}} (p-1) \int_{t_0}^t |u(s)|^{p+1} ds \right)^{\frac{2}{p-1}}}. \]

As \( u(t) \) is of PE the denominator grows unbounded making \( V(t) \to 0 \) as \( t \to \infty \). Now an estimate of the time needed for \( V(t) \) to decreases from \( V(t_0) \) to a value equal or smaller than \( c \) is calculated. Substituting \( V(t) \) for \( c \) in (10) and evaluating the integral from \( t_0 \) to \( t_1 \) it is clear that the value of the integral needs to be larger enough to satisfy the last inequality
\[ \int_{t_0}^{t_1} |u(s)|^{p+1} ds \geq \frac{1}{2^{\frac{1-p}{2}} (p-1) \int_{t_0}^{t_1} |u(s)|^{p+1} ds} - \frac{1}{V^{1-p}(t_0)}. \]

Fixing \( t_1 - t_0 \) as an integer multiple of \( T \), i.e. \( t_1 - t_0 = kT \), from Proposition 1 we know that the integral is greater or equal to \( kTE^{p+1} \). Forcing the RHS of the last inequality to be less than \( kTE^{p+1} \) we get
\[ kTE^{p+1} \geq \frac{1}{2^{\frac{1-p}{2}} (p-1) \int_{t_0}^{t_1} |u(s)|^{p+1} ds} - \frac{1}{V^{1-p}(t_0)}. \]

Now solving for \( k \) and taking the smallest integer that fulfill the inequality yields
\[ k \geq \left\lceil \frac{1}{2^{\frac{1-p}{2}} (p-1)TE^{p+1}} \left( \frac{1}{\int_{t_0}^{t_1} |u(s)|^{p+1} ds} - \frac{1}{V^{1-p}(t_0)} \right) \right\rceil, \]
\[ k = \left\lceil \frac{1}{2^{\frac{1-p}{2}} (p-1)TE^{p+1}} \left( \frac{1}{\int_{t_0}^{t_1} |u(s)|^{p+1} ds} - \frac{1}{V^{1-p}(t_0)} \right) \right\rceil. \]

Taking the limit when \( V(t_0) \to \infty \) the bound (7) is found
\[ k = \left\lceil \frac{1}{2^{\frac{1-p}{2}} (p-1)TE^{p+1}} \cdot \frac{1}{\int_{t_0}^{t_1} |u(s)|^{p+1} ds} \right\rceil. \]

**C. Proof of Theorem 3**

From (5b) it follows that \( \dot{V}(t) \leq -|u(t)^{T}|x(t)|^{p+1} = 2 \frac{p+1}{2} |u(t)| |V^{\frac{1}{p+1}}(t) \), which define a different differential inequality for each term in the sum. From the Comparison Lemma we know that the solution of \( V(t) \) is for below of each solution of \( V_p(t) \), where \( V_p(t) = -|u(t)| |V^{\frac{1}{p+1}}(t) \). Let \( p_M \) and \( p_m \) be as in the theorem. For each \( p_i \in P_M \) we can assert that \( V_{p_i}(t) \) can escape from infinity to a compact set in finite time and so \( V(t) \). Fix the level set as \( V(x) = 1 \) and estimate the time needed to reach it
\[ \tau(1) \leq \left[ \frac{1}{2^{\frac{p-1}{2}} (p_i - 1)TE^{p+1}} \right] T, \quad p_i \in P_M. \]

The smallest time that the algorithm can guarantee is obtained when \( (p_i - 1)TE^{p+1} \) is maximum. Take that quantity as the estimate. Now, with \( p_j \in P_m \), we can estimate the time needed for each \( V_{p_j}(t) \) to converge from the level set \( V(x) = 1 \) to zero
\[ \mathbf{T}(x_0) \leq \left[ \frac{1}{(1 - p_j)TE^{p+1}} \right] T, \quad p_j \in P_m, x_0 |V(x_0) = 1. \]

Again, the smallest time that the algorithm can guarantee is when \( (1 - p_j)TE^{p+1} \) is maximized. Then the time needed by the algorithm to converge is, at most, the sum of the two estimates.

**D. Proof of Theorem 4**

Following the idea in [16] a lower bound of the integral of \( |u^{T} (x) t^{p+1} | \) is needed. Take the term \( |u^{T} (x(t_1)) | \) and add a zero in the form \( u^{T} (x(t) - u^{T} (x(t_1)) \) inside the absolute value, by means of the triangle inequality the next partition is obtained
\[ |u^{T} (x(t)) | \geq |u^{T} (x(t_1)) | - |u^{T} (x(t_1)) |. \]

Rising both sides to \( p+1 \) and using the Jensen inequality after that, the inequality yields
\[ |u^{T} (x(t)) |^{p+1} \geq \frac{1}{2^{p+1}} |u^{T} (x(t_1)) |^{p+1} \]
\[ - |u^{T} (x(t_1)) |^{p+1}. \]

Integrating both sides from \( t_1 \) to \( t_1 + T \) and using Proposition 1 to bound the first term in the RHS becomes
\[ \int_{t_1}^{t_1 + T} |u^{T} (x(t)) |^{p+1} dt \geq \frac{T^{p+1}}{2^{p+1}} \| x(t_1) \|^{p+1} \]
\[ - \int_{t_1}^{t_1 + T} |u^{T} (x(t_1) - x(t)) |^{p+1} dt. \quad (12) \]

To bound the magnitude of the second term in the RHS of the last inequality, the next procedure may be used.
assuming \( u(t) \) is uniformly bounded, i.e. \( \| u(t) \| \leq u_M \):

\[
\int_{t_1}^{t_1 + T} |u^T(s)(x(t_1) - x(s))|^{p+1} ds \\
\leq u_M^{p+1} \int_{t_1}^{t_1 + T} \|x(t_1) - x(s)\|^{p+1} ds \\
\leq u_M^{p+1} T \sup_{s \in [t_1, t_1 + T]} \|x(t_1) - x(s)\|^{p+1} \\
\leq u_M^{p+1} T \left( \int_{t_1}^{t_1 + T} |\dot{x}(s)| ds \right)^{p+1}.
\]

The norm of \( \dot{x} \) in this case is \( |u^T(t)x(t)|^p \| u(t) \| \) which is less than \( |u^T(t)x(t)|^p u_M \). Hölder inequality can be used with \( f(t) = |u^T(t)x(t)|^p \) and \( g(t) = 1 \) in the interval \([t_1, t_1 + T]\) with \( \frac{p+1}{p} \) and \( p + 1 \) as Hölder conjugates, then

\[
T \left( \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p+1} ds \right)^p \\
\geq \left( \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^p ds \right)^{p+1}.
\]

Using this in (13) and then the result in (12) one gets

\[
\int_{t_1}^{t_1 + T} |u^T(t)x(t)|^{p+1} dt \\
= \frac{T e^{p+1}}{2p} \|x(t_1)\|^{p+1} \\
- T^2 u_M^{2(p+1)} \left( \int_{t_1}^{t_1 + T} |u^T(t)x(t)|^{p+1} dt \right)^p.
\]

This can be rewritten as

\[
\frac{T e^{p+1}}{2p} \|x(t_1)\|^{p+1} \leq \int_{t_1}^{t_1 + T} |u^T(t)x(t)|^{p+1} dt \\
+ T^2 u_M^{2(p+1)} \left( \int_{t_1}^{t_1 + T} |u^T(t)x(t)|^{p+1} dt \right)^p.
\]

By algebraic manipulation and defining \( z := \int_{t_1}^{t_1 + T} |u^T(t)x(t)|^{p+1} dt \) the notation can be simplified:

\[
z + T^2 u_M^{2(p+1)} z^p \geq \frac{T e^{p+1}}{2p} \|x(t_1)\|^{p+1}.
\]

Notice that the polynomial \( P(z) := z + T^2 u_M^{2(p+1)} z^p \) is a strict monotonically increasing function for \( z \geq 0 \) and its inverse exists. By denoting this as \( P^{-1}(z) \) and using in the inequality above, one gets

\[
z = \int_{t_1}^{t_1 + T} |u^T(t)x(t)|^{p+1} dt \geq P^{-1} \left( \frac{T e^{p+1}}{2p} \|x(t_1)\|^{p+1} \right).
\]

Recalling \( \dot{V}(t) \) from (5a) and integrating it for the same time interval it is obvious that

\[
\int_{t_1}^{t_1 + T} \dot{V}(t) dt \leq -P^{-1} \left( \frac{T e^{p+1}}{2p} \|x(t_1)\|^{p+1} \right) < 0.
\]

Since \( P^{-1}(\cdot) \) is also a strict monotonically increasing function and from Theorem 5 in [13] the global uniform asymptotic stability of the origin of (3) is established for any \( p > 0 \).

\[\text{E. Proof of Theorem 5}\]

In (5b) several terms of the form \( |u^T(t)x(t)|^{p+1} \) appear; an analysis of each term separately is done before study the full derivative of \( V \). By taking the term \( |u^T(t)x(t)| \) and adding a zero in the form \( u^T(t)x(t) - u^T(t)x(t) \) inside the absolute sign and then applying the triangle inequality the next partition is found

\[
|u^T(t)x(t)| \leq |u^T(t)(x(t_1) - x(t))| + |u^T(t)x(t)|,
\]

rising to \( p_i + 1 \) and using the Jensen Inequality

\[
\frac{1}{2p_i} |u^T(t)x(t)|^{p_i + 1} \leq |u^T(t)(x(t_1) - x(t))|^{p_i + 1} + |u^T(t)x(t)|^{p_i + 1}.
\]

Solving for \( |u^T(t)x(t)|^{p_i + 1} \) and integrating between \([t_1, t_1 + T]\)

\[
\int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i + 1} ds \geq \frac{1}{2p_i} \int_{t_1}^{t_1 + T} |u^T(s)x(t_1)|^{p_i + 1} ds \\
- \int_{t_1}^{t_1 + T} |u^T(s)(x(t_1) - x(s))|^{p_i + 1} ds.
\]

A lower bound of the first term in the RHS of the inequality can be found using Proposition 1. The following inequality is obtained

\[
\int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i + 1} ds \geq \frac{T e^{p+1}}{2p} \|x(t_1)\|^{p+1} \\
- \int_{t_1}^{t_1 + T} |u^T(s)(x(t_1) - x(s))|^{p_i + 1} ds.
\]

(14)

A upper bound of the magnitude of the second term is quite more difficult. Assuming \( u(t) \) is uniformly bounded by \( u_M \), i.e. \( \| u(t) \| \leq u_M \), the integral can be bounded as

\[
\int_{t_1}^{t_1 + T} |u^T(s)(x(t_1) - x(s))|^{p_i + 1} ds \\
\leq u_M^{p_i + 1} \int_{t_1}^{t_1 + T} |x(t_1) - x(s)|^{p_i + 1} ds \\
\leq u_M^{p_i + 1} T \sup_{s \in [t_1, t_1 + T]} \|x(t_1) - x(s)\|^{p_i + 1} \\
\leq u_M^{p_i + 1} T \left( \int_{t_2}^{t_1 + T} |\dot{x}(s)| ds \right)^{p_i + 1}.
\]

(15)

Now an analysis of the integral of \( |\dot{x}(t)| \) is required. First the norm of \( \dot{x} \) is estimated

\[
|\dot{x}| \leq \sum_{j=1}^{h} \|u^T(t)x(t)|^p \| u(t) \| \leq u_M \sum_{j=1}^{h} |u^T(t)x(t)|^p.
\]

Integrating and using the Jensen inequality for the function \( (\cdot)^{p_i + 1} \)

\[
\left( \int_{t_1}^{t_1 + T} |\dot{x}(s)| ds \right)^{p_i + 1} \leq u_M^{p_i + 1} h^{p_i} \sum_{j=1}^{h} \left( \int_{t_2}^{t_1 + T} |u^T(s)x(s)|^{p_i} ds \right)^{p_i + 1}.
\]

(16)
The restriction $p_M \leq p_{m} + 1$ guarantees that $\frac{p_i + 1}{p_j} \geq 1$ for every $(i, j)$ and then $\frac{p_i + 1}{p_j}$, $\frac{p_i + 1}{p_{i_j}}$ are Hölder conjugates. Now the Hölder inequality is used with $f(t) = |u^T(t)x(t)|^{p_j}$ and $g(t) = 1$ in the interval $[t_1, t_1 + T]$ and Hölder conjugates as described before

\[
T^{p_i - p_j + 1} \left( \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_j + 1} ds \right)^{p_j} \geq \left( \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i} ds \right)^{p_i + 1}.
\]  

(17)

Using (15) in (16), applying (17) in the sum, one has

\[
\sum_{j=1}^{b} \frac{1}{T^{p_j}} \left( \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_j + 1} ds \right)^{p_j} \geq u_3^{p_i + 1} T \left( \int_{t_1}^{t_1 + T} \|\dot{x}(s)\|^{p_i + 1} ds \right)^{p_i + 1}.
\]  

(18)

From (14), (15) and (18) we obtain

\[
\int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i + 1} ds \geq \frac{T^{p_i + 1}}{2p_i} \|x(t_1)\|^{p_i + 1}
\]

\[
- u_3^{2(p_i + 1)} h p_{i} T^{p_{i} + 2} \sum_{j=1}^{b} \frac{1}{T^{p_j}} \left( \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_j + 1} ds \right)^{p_j}.
\]

Defining $z_i := \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i + 1} ds$ the last inequality can be rewritten as

\[
z_i + u_3^{2(p_i + 1)} h p_{i} T^{p_{i} + 2} \sum_{j=1}^{b} \frac{1}{T^{p_j}} z_j^{p_j} \geq \frac{T^{p_i + 1}}{2p_i} \|x(t_1)\|^{p_i + 1}.
\]

The polynomial in the LHS is a strict monotonically increasing function and zero when $z_i = 0$, then its inverse exists and also is a strict monotonically increasing function. By denoting the polynomial as $P_i$ and its inverse as $P_i^{-1}$ one gets

\[
\int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i + 1} ds \geq \frac{T^{p_i + 1}}{2p_i} \|x(t_1)\|^{p_i + 1}.
\]  

(19)

Now integrating (5b) from $t_1$ to $t_1 + T$ and using (19) the next inequality for $V$ is obtained

\[
\int_{t_1}^{t_1 + T} V(s) ds = - \sum_{i=1}^{b} \int_{t_1}^{t_1 + T} |u^T(s)x(s)|^{p_i + 1} ds
\]

\[
\leq - \sum_{i=1}^{b} P_i^{-1} \left( \frac{T^{p_i + 1}}{2p_i} \|x(t_1)\|^{p_i + 1} \right).
\]

This satisfies the assumptions of the Theorem 5 in [13] and guarantees the global uniform asymptotic stability of the origin of (4).