OPEN STRINGS, SIMPLE CURRENTS AND FIXED POINTS

A.N. Schellekens

NIKHEF Theory Group
P.O. Box 41882, 1009 DB Amsterdam, The Netherlands

Abstract

Some applications of simple current techniques and fixed point resolution to theories of open strings are discussed. In addition to a brief review of work presented in two recent papers with L. Huiszoon and N. Sousa, some new results concerning uniqueness of crosscap coefficients are presented, as well as a strange sum rule for the modular matrix implied by the existence of crosscaps.
1 Introduction

Open strings have enjoyed a rather varied amount of interest during the past three decades of string theory. In the early days open strings were used to describe mesons, with quarks attached to the endpoints. Indeed, the Chan-Paton labels still used today date back to as early as 1969 [1]. This provided the first method for obtaining gauge groups in string theory. The possible gauge groups were classified much later [2], in a period when string theory in general had fallen into decline. In 1983 Alvarez-Gaumé and Witten [3] showed that open superstrings (type-I) were plagued by chiral anomalies for any gauge group, whereas closed superstrings (type-II) were automatically anomaly free. Although this looked like a fatal blow, there was a brief revival after Green and Schwarz [4] found a novel mechanism to cancel the anomaly for the gauge group \(SO(32)\). But within months the heterotic string was discovered [5]. This theory could in addition to \(SO(32)\) have the gauge group \(E_8 \times E_8\) [6], which at first sight seemed phenomenologically more attractive. During the subsequent ten years open strings were almost completely neglected in favor of heterotic strings, which went through a phase of rapid development. Apart from being phenomenologically disfavored, open strings looked ugly and complicated: their description requires world-sheets with boundaries and crosscaps, and to obtain finite one-loop diagrams one has to cancel tadpoles by hand.

All this changed drastically in 1995, for several reasons. First of all open strings were found to be part of the duality picture, and in particular in ten dimensions the strong coupling limit of the type-I string was conjectured to be the heterotic string [7]. Secondly, the discovery of D-branes swept out by the endpoints of open strings made boundaries more respectable [8]. Furthermore in some cases tadpole cancellation was found to be equivalent to charge cancellation between D-branes and orientifold planes [9], which sounds somewhat less \textit{ad hoc}. Furthermore it was pointed out that the relation between the gauge and gravitational coupling is different in open string theories and in closed ones, which makes it possible to separate the unification scale and the string scale [10]. This has opened new avenues in string phenomenology, involving open strings (see \textit{e.g.} [11]). These developments make it clear that open strings must be considered seriously again.
2 Closed strings, CFT and modular invariance

During the period 1984-1994 there has been a lot of progress in the description of lower-dimensional closed strings in terms of conformal field theory (here and in the following “closed” is an abbreviation for “closed and oriented”). This had led to a very economical formalism based on a few simple, algebraic constraints, from which very general theorems can be derived. It would be nice to have a similar description of open strings. The algebraic constraints in the closed string case are (essentially) Lorentz invariance in the number of dimensions one considers, (super)conformal invariance and modular invariance. If one builds the internal (“compactified”) part of the theory out of some (super)CFT, the constraint of modular invariance is that the integer matrix $Z$ appearing in the one-loop partition function

$$P(\tau, \bar{\tau}) = \sum_{ij} \chi_i(\bar{\tau}) Z_{ij} \chi_j(\tau)$$ (1)

must commute with the generators $S$ and $T$ of the modular group of the torus. Here $\chi_i(\tau)$ are the Virasoro characters of the internal CFT (which may be combined with the space-time CFT in a more intricate way than suggested here, but this is easy to take into account). Modular invariance is a simple and powerful constraint, from which one can for example derive the Green-Schwarz factorization of chiral anomalies [12] or prove the existence of fractional charges in a large class of heterotic string theories [13].

Unfortunately all this is limited to perturbative, closed string theories. There may be non-perturbative states in the spectrum of closed strings that are not controlled by the modular invariant partition function. Nevertheless modular invariance is, in my opinion, too nice a principle to simple give up. I would hope to find some sort of generalization, a principle that governs the presence or absence of states in string theory, M-theory or whatever string theory generalizes to. Such a principle would justify the term “theory of everything”. Although this expression has fallen out of favor because it is usually maliciously misinterpreted by adversaries of string theory in particular and science in general, it is justified in the following precise sense. In field-theoretic descriptions of our world it is always possible to add some new particles to a successful theory (and in particular to the standard
model). There are few theoretical constraints, but if one makes the new
particles sufficiently massive, unstable and weakly coupled, it is not hard
to evade all experimental and cosmological constraints. This implies that
one can never claim to have arrived at a complete description of all physics
in our universe, since experimental and cosmological constraints are always
limited to some subset of any relevant parameter space. This seems like an
inevitable fact: one can never know more than one has measured. However,
string theory provides a potential way out. Adding extra particles to a given
string theory makes it inconsistent. The only thing that one may do is remove
some states, and add others in their place. This is perhaps best known in
the example of orbifold constructions, where one removes states from the
spectrum that are not invariant under a certain symmetry, and replaces them
by “twisted states”. This is not limited to orbifold constructions, but is in
fact a general property of modular invariant, perturbative closed strings.
If it generalizes beyond perturbative closed strings it would in principle be
possible to make a unique choice among the huge amount of string vacua,
based on a finite number of experimental results. If further experiments find
additional particles not predicted by this particular string theory, then they
can only be accommodated at the expense of some particles that were general
found before. In other words, any further experiments would rule out string
theory as a whole, or in still other words, we would have a very strong
prediction for the existence or non-existence of any other particle, no matter
how massive or weakly coupled, in our universe. This would certainly deserve
the name “theory of everything”.

The argument given above is summarized in the above two pictures, with
on the left the string theory way of going from theory A to a different theory
B, and on the right the field theory way.
At present these pictures are only a caricature of reality. There is little hope of finding the right string “vacuum” without additional information, even if the first picture is the right one. But in addition non-perturbative effects, and in particular open and unoriented strings (which are non-perturbative from the point of view of closed strings) pose a serious challenge to this picture.

Open and unoriented strings are usually constructed by starting from a consistent closed, oriented theory. To the one-loop closed string amplitude, the torus, one adds an unoriented closed string diagram, the Klein bottle. This acts as a projection, removing certain states from the spectrum. Furthermore one adds open string diagrams, which at the one-loop level are the annulus and the Moebius strip. These diagrams come with a free parameter, the Chan-Paton multiplicity, for each boundary. Until this point this looks reminiscent of the orbifold procedure, with the Klein bottle projection playing the rôle of the removal of non-invariant states, and with open strings playing the rôle of twisted sectors. However, the presence of “twisted” sectors is in this case not governed by modular invariance, but by a different principle, the cancellation of massless tadpoles that lead to infinities. Unlike modular invariance this is a target space criterion, and hence one loses the nice feature of closed oriented theories that a consistent world sheet theory is sufficient to get a consistent target space theory. Worse yet, there exist unoriented string theories for which the tadpole cancellation conditions require all Chan-Paton labels to vanish \cite{14 \cite{15}. This means that there are no open string states at all, even though the Klein bottle still acts as a projection. Hence in this case the states of the closed, unoriented theory are a subset of those of the closed oriented theory, as in the second picture above. Note that this can never happen in modular invariant partition functions. It is easy to show that if \( Z_{ij} \) yields a modular invariant, then any matrix \( 0 \leq \tilde{Z}_{ij} \leq Z_{ij} \) can only be modular invariant if \( \tilde{Z} = Z \) (this follows from \( Z_{00} = \tilde{Z}_{00} = 1 \) and \( S_{0i} > 0 \)).

This implies that the first picture does not hold for perturbative states. It might still be saved by non-perturbative states, if the unoriented theory has non-perturbative states not present in the oriented theory. On the other hand it is possible that the first picture has to given up, and that one has to allow for a discrete set of consistent truncations of certain string theories. This does not necessarily invalidate the discussion given above. Most people
would probably agree with the statement that in string theory and its generalizations the possibilities for adding or removing states are severely limited by consistency requirements. It would be nice to make that more precise.

3 Simple currents and fixed points

In search of a principle that governs the presence of states in a string theory, I now turn to something much more down-to-earth, namely a tool that plays an essential rôle in dealing with modular invariance in the closed, oriented case: simple currents [16]. Simple currents are primary fields that upon fusion with any other field yield just one field [10] [17]. They can be used to build a non-diagonal partition function. If one has a closed set of integral spin simple currents, these currents can extend the chiral algebra, and one obtains a partition function

\[
\sum_{Q(i)=0} N_i \left| \sum_{j \in \text{Orbit}(i)} \chi_j \right|^2
\]  

Here \( Q \) is a charge (or set of charges) defined for each current, and the orbit is the set of distinct fields generated by the set of currents acting on \( i \). In general some currents may fix \( i \), and then the action of the currents covers the orbit \( N_i \) times. Fractional spin currents also generate modular invariants, but they correspond to automorphisms of the fusion algebra, which pair the left and right representations in an off-diagonal way.

Simple currents are used for a variety of purposes in the construction of closed string theories, such as

- Field identification in coset CFT’s [18] [19]
- World-sheet supersymmetry projections [20]
- Space-time supersymmetry projections [20]
- D-type invariants [21] [22] [10] [17]
- Inverse orbifolds (under conditions discussed in [23])

As we will see, they play a useful rôle in open string constructions as well.
Another concept that comes back in open string theories is the resolution of fixed points \[18\][24]. The presence of the multiplicities \(N_i\) in partition functions implies usually (but not always \[25\]), that the corresponding terms are reducible representations of the extended chiral algebra. To describe the modular properties of the characters of the extended algebra (the orbit sums) one needs a set of matrices that act on the resolved fixed points. There is such a matrix \(S^J_{ij}\) for any current that has fixed points, and it is defined only on the fixed points \(i\) and \(j\) of \(J\). In terms of these matrices, and the original matrix \(S \equiv S^0\), the matrix \(S\) of the extended theory takes the form \[25\]

\[
S_{(i,\mu)(j,\nu)} = \frac{|G|}{\sqrt{|S_i||\mathcal{U}_i||S_j||\mathcal{U}_j|}} \sum_J \Psi^J_{\mu} S^J_{ij} (\Psi^J_{\nu})^* \tag{3}
\]

Here \(\mu, \nu\) label the components into which the orbits of \(i\) and \(j\) are resolved, \(G\) is the simple current group that extends the chiral algebra, and \(S_i\) (the stabilizer) is the subgroup that fixes \(i\). The group \(\mathcal{U}_i\) is a subgroup of \(S_a\) called the untwisted stabilizer; for the precise definition see \[33\]. The factors \(\Psi\) are discrete group characters of \(\mathcal{U}_i\), and \(i\) is resolved into \(|\mathcal{U}_i|\) components.

In general one can derive a list of properties that the matrices \(S^J\) should satisfy in order for \(S_{(i,\mu)(j,\nu)}\) to be a proper modular transformation matrix. In the case of WZW-models one can find a natural set of matrices that satisfy all those properties, and that are therefore obvious candidates for \(S^J\) \[24\][25]. This works as follows \[26\]. To each WZW-model belongs a Dynkin diagram, which is an extended Dynkin diagram of an ordinary Lie algebra. Simple currents are related to symmetries of this extended Dynkin diagram that move the extended root (with one exception for \(E_8\) level 2 \[27\]). Given a Dynkin diagram symmetry one can define a folded diagram in a fairly obvious way (for details see \[26\]), and to that diagram one can associate a Cartan matrix. This defines a new algebra, which we call the orbit Lie algebra. There is such an algebra for any simple current, and the modular transformation matrices of the characters of this orbit Lie algebra are equal to the matrices \(S^J\), up to a calculable phase. The orbit Lie algebras of simple currents are affine Lie algebras, so that their modular transformation matrices are calculable using the Kac-Peterson formula \[28\]. In one case the orbit Lie algebra is a twisted affine Lie algebra, but luckily this is precisely the only twisted affine Lie algebra whose characters have good modular properties.

The generalization of the foregoing to arbitrary CFT’s is not completely understood yet. The concept of an orbit Lie algebra seems restricted to
WZW-models. But in any case most known rational CFT’s are related to WZW-models by the coset construction. Since field identification can be described in terms of simple currents (with a few exceptions), formula (3) applies to those cases. One can apply formula (3) to a combination of field identification and any chiral algebra extension of the coset theory, and read off the matrices $S_J^J$ of the coset theory. In [29] a formula for these matrices was derived.

4 Open string CFT

Here a very brief introduction to open string conformal field theory is given. Only those aspects needed in the rest of the paper are mentioned.

Open string conformal field theory is defined on surfaces with handles, boundaries and crosscaps. Any such surface has a double cover which only has handles, and on which one defines a closed, oriented conformal field theory. This CFT is the starting point for constructions of open (and un-oriented) strings, which are referred to as “open descendants” of the closed string theories [30]. The presence of boundaries and crosscaps is described by boundary and crosscap “states”, which are not really states themselves, but in fact non-normalizable linear combinations of states in the closed string Hilbert space.

The closed string CFT has a chiral algebra which includes the Virasoro algebra. The boundary may preserve all or only part of the closed string chiral algebra, but must at least preserve the Virasoro algebra. Here we will assume that the entire chiral algebra remains unbroken (“trivial gluing”). The condition that a symmetry is not broken by a boundary or a crosscap is

$$ (J_n + (-1)^{h_J} J_n^+) |B\rangle = 0 \; ; \; \; (J_n + (-1)^{h_J+n} J_n^+) |C\rangle = 0 $$

where $J_n$ is a mode of a chiral current, $J_n^+$ a mode of an anti-chiral current, $|B\rangle$ a boundary state and $|C\rangle$ a crosscap state. A basis for the solutions to these conditions is formed by the Ishibashi states [31]

$$ |B_i\rangle = \sum_I |I_i⟩ \otimes U_B |I\rangle^{\bar{c}} \; ; \; \; |C_i\rangle = \sum_I |I_i⟩ \otimes U_C |I\rangle^{\bar{c}} . $$

Here the $i$ labels a representation of the chiral algebra and $\bar{c}$ its charge conjugate. The sum is over all states in the representation, and $U_B$ and $U_C$
are operators satisfying
\[ \tilde{J}_n U_B = (-1)^{h_j} U_B \tilde{J}_n ; \quad \tilde{J}_n U_C = (-1)^{h_j+n} U_C \tilde{J}_n \] (6)

Any boundary state must be a linear combination of these Ishibashi states, \textit{i.e.}
\[ |B_a\rangle = \sum_i B_{ai} |B_i\rangle ; \quad |C\rangle = \sum_i \Gamma_i |C_i\rangle \] (7)

It turns out that in general one can allow for several boundary states, labelled by a boundary label \( a \), but for only one crosscap state. A choice of a set of boundary labels \( a \), and a set of coefficients \( B_{ai} \) and \( \Gamma_i \) form part of the data that define an open string CFT. Although more is required to specify all correlation functions on arbitrary surfaces, this information is sufficient to compute the one-loop diagrams without external lines that contribute to the open and closed string partition functions. Hence we can at least compute the spectrum of the theory. The diagrams are computed in the transverse channel, in which closed strings propagate between two boundaries, a boundary and a crosscap, or two crosscaps.

**Transverse Annnulus:** \[ N_a N_b \langle B^c_a | e^{i\tau H} | B_b \rangle \] (8)

**Transverse Moebius strip:** \[ N_a \left( \langle B^c_a | e^{i\tau H} | C \rangle + \langle C^c | e^{i\tau H} | B_a \rangle \right) \] (9)

**Transverse Klein bottle:** \[ N_a \langle C^c | e^{i\tau H} | C \rangle \] (10)

Here \( H \) is the closed string Hamiltonian: \( H = 2\pi (L_0 + \tilde{L}_0 - c/12) \), and \( \tau \) is a real number representing the length of the cylinder. The subscript \( “c” \) indicates that a CPT conjugate state is to be used. The integers \( N_a \) are the Chan-Paton multiplicities. One can express these amplitudes in terms of characters of the representation \( i \). By means of a transformation of the parameter \( \tau \) one can then compute the corresponding amplitudes in the direct channel (the open and closed string loop channels). In the case of the Klein bottle and the annulus this transformation acts on the characters as the modular transformation matrix \( S \), whereas in the case of the Moebius strip one uses the matrix \( P = \sqrt{T} S T^2 S \sqrt{T} \), with \( \sqrt{T} \) defined as \( \exp i\pi (L_0 - c/24) \).

Then one arrives at the following expressions

**Direct Annnulus:** \[ \frac{1}{2} N_a N_b A^i_{\alpha \beta \chi} \left( \frac{1}{2} \tau \right) \] (11)

**Direct Moebius strip:** \[ \frac{1}{2} N_a M^i_{\alpha \chi} \left( \frac{1}{2} + \frac{1}{2} \tau \right) \] (12)
Direct Klein bottle: \( \frac{1}{2} K_i \chi_i (2 \tau) \)  

Here \( \tilde{\chi}_i \equiv T^{-\frac{1}{2}} \chi_i \), and the parameter \( \tau \) is purely imaginary. The coefficients are

\[
A^i_{ab} = \sum_m S_i^m B_{am} B_{bm} ; \quad M^i_a = \sum_m P_i^m B_{am} \Gamma_m ; \quad K^i = \sum_m S_i^m \Gamma_m \Gamma_m ; \quad (13)
\]

4.1 Constraints

These coefficients are subject to several constraints. Since the direct channel contributions yield the open and closed string partition functions, the coefficients are subject to the requirement that all multiplicities should be positive integers (if one applies the formalism to fermionic strings one would like to see negative integers for space-time fermions, but those signs come out automatically if one takes into account ghosts properly). This yields two important conditions:

\[
\text{Closed sector:} \quad |K^i| = Z_{ii} \quad (15)
\]
\[
\text{Open sector:} \quad M^i_a = A^{i}_{aa} \mod 2 , \quad |M^i_a| \leq A^{i}_{aa} \quad (16)
\]

In writing down the first condition we assume that \( Z_{ii} \leq 1 \). One can write down modular invariant partition functions for which that is not the case, but this always means that some fields must be resolved into irreducible components first. Then the combination \( \frac{1}{2} (Z_{ii} + K^i) \) is either a symmetric or anti-symmetric projection, or vanishes completely. In the second condition it is assumed that \( A^i_{aa} \) is non-negative. Then \( \frac{1}{2} (N_a A^i_{aa} + N_a M^i_a) \) can be interpreted as \( \frac{1}{2} (A^i_{aa} + M^i_a) \) symmetric tensors of the Chan-Paton group, plus \( \frac{1}{2} (A^i_{aa} - M^i_a) \) anti-symmetric tensors.

There are other constraints that are easy to check. First of all the “completeness conditions” \([32]\)

\[
A_{ia}^b A_{jb}^c = N_{ij}^k A_{ka}^c \quad (17)
\]
\[
A_{iab} A_{v_{ad}}^c = A_{iac} A_{v_{bd}}^c \quad (18)
\]

Here the boundary indices are raised and lowered with the “boundary metric” \( A_{0_{ab}} = \sum_n S_{bn} B_{na} B_{nb} \), which must be order-2 permutation. In particular the
matrix $A^0_{ab}$ must have entries 0 or 1, and must be numerically equal to its own inverse $A^{0ab}$. The three allowed Chan-Paton gauge groups correspond precisely to the three allowed combinations of $A$ and $M$: $A^0_{aa} = M^0_a = 1$ gives $Sp(N_a)$ (if $N_a$ is even), $A^0_{aa} = -M^0_a = 1$ gives $SO(N_a)$, $A^0_{ab} = A^0_{ba} = 1, M^0_a = M^0_b = 0, a \neq b$ gives $U(N_a)$ (if $N_a = N_b$).

It is convenient to define the matrices

$$R_{ia} = B_{ia} \sqrt{S_i} \quad (19)$$

A sufficient condition for the completeness conditions as well as the properties of $A^0_{ab}$ is then

$$R_{ia} R^*_{ib} = \delta_{ab}$$
$$R_{ib} R^*_{jb} = \delta_{ij}$$

This may not be the most general solution, but the only one considered here. Given the matrices $R$ it is natural to define also

$$U_i = \Gamma_i \sqrt{S_i} \quad (20)$$

Another easy constraint is the “Klein bottle constraint”

$$K_i K_j K_k \geq 0 \quad \text{if} \quad N_{ijk} \neq 0 \quad (21)$$

This ensures that the Klein bottle defines a consistent truncation on the spectrum. If this were not satisfied two states that are in the projected spectrum can have couplings to a third state that is not in the projected spectrum, so that the latter can appear in the intermediate channels of tree-diagrams. A slightly weaker form of this constraint was conjectured in [36].

There are other constraints (see [37]), but they involve additional quantities (such as OPE coefficients and duality matrices), that are not readily available, except in a few special cases. We will see, however, that the constraints described above are already very restrictive.

5 Open strings and simple currents

In this section various simple current modifications of the basic open descendant construction are discussed. This basic construction is often referred to as the “Cardy case”, and consists of a natural ansatz for the boundaries,
supplemented with an ansatz for the crosscap. We will discuss this case first, and show that the crosscap ansatz follows directly from the boundary ansatz and the positivity requirements.

5.1 Uniqueness of crosscaps in the Cardy case

Cardy [38] conjectured a general ansatz for the annulus

\[ B_{ai} = \frac{S_{ai}}{\sqrt{S_{0i}}} \]

This ansatz satisfies the completeness condition, and yields an annulus coefficient equal to the Verlinde fusion coefficients for \( S \). It follows that all fields propagate in the transverse channel, and hence that all fields \( \phi_{i,j} \) must appear in the bulk theory. Therefore this ansatz requires the torus partition function to be defined in terms of \( Z = C \), the charge conjugation matrix.

To check any of the other conditions we need to know the crosscap coefficients \( \Gamma_i \). They were first determined in an \( SU(2) \) model by Sagnotti et. al. and this result was used as a conjecture for all other cases. Additional support for this conjecture was given in [39], where it was shown that this conjecture satisfies all positivity and integrality constraints; see also [40][41] for related results and [42][43] for a discussion of the Klein bottle constraint.

The results of the latter paper can in fact be turned around: we may impose positivity and integrality and derive the crosscap coefficients. To do this we make use of the fact that the set of boundaries in the Cardy case is in one-to-one correspondence with the bulk labels, and in particular there is a boundary “0”. Then \( A_{i00} = N_{i00} = \delta_{i0} \), and hence \( M_i^0 = \pm 1 \). Hence

\[ \sum_m P_i^m B_{0m} \Gamma_m = \pm \delta_{i0} \]

from which we read off immediately

\[ \Gamma_m = \pm \frac{P_{0m}}{\sqrt{S_{0m}}} \]

where the sign is undetermined, but does not depend on \( m \) (it would ultimately determined by the tadpole cancellation condition). We conclude
that the crosscap coefficients are in fact uniquely determined by the boundary coefficients (up to an overall sign, which is not fixed by any CFT constraint). Recently the crosscap coefficient has also been determined using 3-dimensional TFT \cite{44}, but the argument given here has the advantage of being considerably simpler. It does not necessarily generalize to other annuli, because the boundary “0” need not exist. But it will generalize to the cases considered below.

5.2 Simple current modifications

Various simple current related modifications of the Cardy ansatz have been studied. A possibly incomplete list is

- Extensions of the closed chiral algebra
- Non-trivial Klein bottle projections
- Simple current automorphisms
- Broken bulk symmetries

The first item is dealt with entirely within the closed string theory and requires no further discussion. Examples of the second kind have been around for a while \cite{35,45}, and were studied in general in \cite{39} where also the consistency conditions were shown to hold in general. In the third class the bulk theory is defined by means of a simple current automorphism of the fusion rules. Such examples were first studied in \cite{46}. In \cite{47} automorphisms of spin-$\frac{1}{2}$ simple currents were studied in general. These authors gave an interpretation of the boundary label “a” in terms of the label of representations of a “classifying algebra”, which in the C-diagonal case is just the Verlinde algebra. A remarkable feature of this case is the appearance of the fixed point resolution matrix of the spin-$\frac{1}{2}$ current in the formulas for the boundary coefficient. This matrix does not play a direct rôle in the closed string theory. The fourth case is studied for example in \cite{48,49} and \cite{23,50}. The latter papers reveal an even more interesting appearance of fixed point resolution matrices. I will not discuss any of these results in detail here, but in the rest of this section I will show how in the second and third case one may also derive the crosscap coefficients directly from the positivity constraints.
5.3 A formula for $P$

To derive these results I will need a formula relating matrix elements of the matrix $P$ on simple current orbits. It is analogous to the well-known formula for $S$ \cite{17,51}:

$$S_{a,J}^{b} = e^{2\pi i Q_J(a)} S_{ab}$$ \hspace{1cm} (24)

The corresponding relation for $P$ is more complicated due to the factors $\sqrt{T}$ in the definition of $P$, and only works if the indices are shifted by even powers of the current

$$P_{a,J}^{b} = \rho(\ell) e^{2\pi i \Delta(2l,b)} e^{2\pi i Q_J(a)} P_{ab}$$ \hspace{1cm} (25)

where

$$\Delta(\ell,c) = h_{Jc} - h_{Jt} - h_c + \ell Q(c)$$ \hspace{1cm} (26)

with $0 \leq Q < 1$, and

$$\rho(\ell) = e^{\pi i (r\ell + M_\ell)}$$ \hspace{1cm} (27)

where

$$M_\ell = h_{Jt} - \frac{r\ell(N - \ell)}{2N},$$ \hspace{1cm} (28)

where $r$ is the monodromy parameter of the current. The derivation is straightforward, provided one replaces the ill-defined quantity $\sqrt{T}$ systematically by the well-defined quantity $\exp(i\pi(h - c/24))$. The first two factors in (25) are signs.

5.4 Non-trivial Klein bottle projection

The first simple current modification I will consider was called a “non-trivial Klein bottle projection” in \cite{39}, because in some cases it produces sign changes in the coefficients $K_i$ with respect to the Cardy case. Here I will study it from a different starting point, namely the annulus.

Consider the following set of reflection coefficients, $R_{ma} = S_{ma} \sqrt{S_{mJ}}$, which satisfy the completeness conditions. Then

$$A_{i00}^j = \sum_m \frac{S_{jm} R_{ma} R_{mb}}{S_{mJ}} = N^j_{i00}$$

Hence $M_{0i}^j = \pm \delta_{0i}^j = \pm \delta_{jc}^j$. On the other hand

$$M_{0i}^i = \sum_m P_{m} U_m \sqrt{\frac{S_{m0}}{S_{mJ}}}$$
so that

\[ U_m = \sum_i P_{mi} M^i_0 \sqrt{\frac{S_m J}{S_{m0}}} \]  
\[ = \pm P_{mJc} \sqrt{\frac{S_m J}{S_{m0}}} \]  
\[ = \pm e^{2\pi i Q_{Jc}(m)} P_{ac} \]  
(29)  
(30)  
(31)

Now we use the formula

\[ P_{a,K^2c} = \epsilon(K, c)e^{2\pi i Q_K(a)} P_{ac} \]

where \( \epsilon(K, c) \) is a sign. We choose \( K = J^c \) and \( c = J \). Then

\[ U_m = \pm P_{mJc} \sqrt{\frac{S_m J}{S_{m0}}} \]  
\[ = \pm e^{2\pi i Q_{Jc}(m)} P_{mJ} \sqrt{\frac{S_m J}{S_{m0}}} \]  
\[ = \pm P_{mJ} \sqrt{\frac{S_m J}{S_{mJ}}} \]  
(32)  
(33)  
(34)

This is the formula used in [39], which turns out to be the only possibility, given the annulus coefficients. The proof that the other constraints are satisfied can be found in that paper.

### 5.5 Non-trivial simple current automorphism

Up to now the bulk modular invariant was the charge conjugation invariant, \( Z = C \). A general modular invariant is characterized by a left and right extension of the chiral algebra, the modules of which are paired by an automorphism. For the construction of “open descendants” the left and right extensions must be identical, and the automorphism symmetric. The extension can be dealt with at the closed string level, which leaves the possibility of non-trivial automorphisms. One may distinguish three basic types: the charge conjugation invariant, simple current invariants and anything else, which by definition is “exceptional”.

A solution is known for simple current automorphisms generated by \( Z_{\text{odd}} \) and \( Z_2 \) simple currents. Both cases are described by the following formula for \( R \)

\[ R_{m,ai} = \frac{1}{\sqrt{|G|}} \sum_{J \in S_a} \sum_{K \in G/S_a} \tilde{S}_{m,Ka}^J \psi_i^J \]
Here $G$ is the full simple current group that produces the automorphism, $S_a \subset G$ is the stabilizer of $a$, $\tilde{S}^J$ is the (appropriate generalization of) the orbit Lie algebra $S$-matrix corresponding to $J$ and $\psi^J_i$ is a discrete group character of $S_a$. The boundary labels are taken to correspond to be $G$-orbits, with each orbit split into $|S_a|$ components. The label $m$ ranges over all fields with $Z_{mm^e} = 1$. This formula is in any case correct for $G = \mathbb{Z}_2$ (in which case it summarizes the four expressions given in [47]) and $G = \mathbb{Z}_{\text{odd}}$, and is a good candidate for a general formula, although this has not been investigated yet. From here on we will assume that $G = \mathbb{Z}_N$, with $N$ odd or equal to 2.

In [36] a consistent ansatz was presented for the crosscap coefficient. Here I will show how to derive it, demonstrating that this ansatz is in fact unique (given the reflection coefficients). Consider the “zero-boundary” $a = 0$. It is not hard to show that

$$A^i_{00} = \sum_J \delta^i_J ,$$

so that

$$M^i_0 = \eta(i) \sum_J \delta^i_J ,$$

where $\eta(i)$ is a sign.

Computing $M^i_0$ directly gives

$$M^i_0 = \sum_{m,Q(m)=0} P^i_m \sqrt{|G|} U_m . \quad (35)$$

The restriction to zero charge is due to the fact that $R_{m0}$ vanishes if $m$ has non-zero charge. Using the inverse of $P$ we get, for $Q(m) = 0$

$$U_m = \frac{1}{\sqrt{|G|}} \sum_{J \in G} \eta(J) P_{Jm} , \quad (36)$$

Note that we get no information about the coefficients for non-zero charge. On the other hand, the fact that the terms with $Q(m) \neq 0$ do not contribute to the sum (35) implies that

$$\sum_{J \in G} \eta(J) P_{Jm} = 0 \quad \text{for } Q(m) \neq 0$$

This puts strong constraints on the possible choices for $\eta(J)$. 

16
We can write (36) as a sum over even and odd elements:

\[ V_m = \frac{1}{\sqrt{N}} \left[ \sum_{\ell=0} \eta(2\ell) P_{J2\ell,m} + \sum_{\ell=0} \eta(2\ell + 1) P_{J2\ell+1,m} \right] \]

Here \( V_m = U_m \) if \( Q(m) = 0 \) and \( V_m = 0 \) (with \( U_m \) undetermined) if \( Q(m) \neq 0 \). Using the relation (25) we can express the first terms in terms of \( P_{0m} \) and the second ones in terms of \( P_{Jm} \). If \( N \) is odd we can furthermore express \( P_{Jm} = P_{JN+1,m} \) in terms of \( P_{0m} \). Then the final result can be expressed as

\[ V_m = \frac{1}{\sqrt{N}} P_{0m} \sum_{\ell=0}^{N-1} \sigma(\ell) e^{2\pi i Q(m)} , \]

where \( \sigma(\ell) \) is a sign. This can only vanish for all \( Q(m) \neq 0 \) if \( \sigma(\ell) = \pm 1 \), independent of \( \ell \). Then also \( U_m = V_m \) for \( Q(m) = 0 \) is determined,

\[ U_m = \pm \sqrt{N} P_{0m} \quad (37) \]

For \( N = 2 \) we find

\[ V_m = \frac{1}{\sqrt{2}} [\eta(0) P_{0m} + \eta(J) P_{Jm}] . \]

Since \( P_{0m} \) and \( P_{Jm} \) are unrelated we cannot simplify the result further. In addition \( P_{0m} = P_{Jm} = 0 \) for \( Q(m) = \frac{1}{2} \), so that we get no further constraints. It may seem that there are two solutions now (not counting the overall sign), but closer inspection of the positivity constraints of fixed point boundary labels reveals that for each CFT only one definite relative sign \( \eta(0) / \eta(J) \) is allowed. The other sign is the correct one for a different choice of reflection coefficients (see [36] for more details).

To determine the coefficients \( U_m \) for charged \( m \) we may use closed sector positivity for bulk label \( i = 0 \). This leads to the requirement \( \sum_m U_m^2 = 1 \). For \( N \) odd there are no transverse channel fields with non-zero charge, and it is easy to show that (37) satisfies this condition. In the case \( N = 2 \) there is an additional problem, namely that we do not know \( U_m \) for fixed points. Allowing for an unknown contribution on the fixed points we get

\[ K^i = \sum_{m,Q(m)=0} S^i_m U_m U_m S_{0m} + \sum_{f,Jf=f} S^i_f U_f U_f S_{0f} = K_1^i + K_2^i , \]
where $U_m = V_m$ given above, and $U_f$ is unknown. It can be shown that $K_1$ already satisfies the positivity constraints. Then the only allowed values for $K_2$ are $K_2^i = k_i K_1^i$, where $k_i = -2$ or $0$. But it is easy to show that $K_2^{ji} = -K_2^j$ whereas $K_1^{ji} = K_1^j$. Then we must have $K_2^i = 0$ and hence $U_f = 0$. So also in this case the crosscap is unique.

5.6 Other automorphisms

For automorphisms that are not simple current modifications of the $C$-invariant there is an amusing observation to be made. The coefficients $U_m$ must vanish whenever $m$ is not paired with its charge conjugate. The coefficients $K^i$ must vanish if $i$ is not paired with itself. This leads to the sum rule

$$\sum_{i,Z_{ii}=1} K^i S_{im} = 0, \text{if } Z_{mm^c} = 0$$

For example in the interesting case $Z = 1$ this leads to the sum rule $\sum_i K^i S_{im} = 0$ for complex fields $m$. Empirically this rule is indeed satisfied, with all $K^i$ equal to $1$. Although in some cases (e.g. $A_2$ level 1) this sum rule is satisfied in a trivial way, there are many other examples (e.g. $A_2$ level 3) where it is non-trivial, and implies relations between matrix elements of $S$ that are hard to derive in any other way. This illustrates that by studying open, unoriented CFT one may learn something interesting about the closed, oriented case.

Acknowledgements

I would like to thank Lennaert Huiszoon and Nuno Sousa for discussions and Christoph Schweigert for comments on the manuscript. Special thanks to the organizers for inviting me to give this talk, and for making this a successful conference despite what turned out to be very difficult circumstances.

References

[1] J. E. Paton and H. Chan, Nucl. Phys. B10 (1969) 516.
[2] N. Marcus and A. Sagnotti, Phys. Lett. B119 (1982) 97.
[3] L. Alvarez-Gaume and E. Witten, Nucl. Phys. B234 (1984) 269.
[4] M. B. Green and J. H. Schwarz, Phys. Lett. B149 (1984) 117.
[5] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502.
[6] J. Thierry-Mieg, Phys. Lett. B156 (1985) 199.
[7] J. Polchinski and E. Witten, Nucl. Phys. B460 (1996) 525 [hep-th/9510163].
[8] J. Polchinski, Phys. Rev. D50 (1994) 6041 [hep-th/9407031].
[9] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724 [hep-th/9510017].
[10] E. Witten, Nucl. Phys. B471 (1996) 135 [hep-th/9602070].
[11] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436 (1998) 257 [hep-ph/9804398].
[12] A.N. Schellekens and N.P. Warner, Phys. Lett. B177 (1986) 317; Phys. Lett. B181 (1986) 339; Nucl. Phys. B287 (1987) 317.
[13] A.N. Schellekens, Phys. Lett. B237 (1990) 363.
[14] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y. S. Stanev, Phys. Lett. B387 (1996) 743 [hep-th/9607229].
[15] A. Dabholkar and J. Park, Nucl. Phys. B477 (1996) 701.
[16] A. N. Schellekens and S. Yankielowicz, Nucl. Phys. B327 (1989) 673.
[17] K. Intriligator, Nucl. Phys. B332 (1990) 541.
[18] A. N. Schellekens and S. Yankielowicz, Nucl. Phys. B334 (1990) 67.
[19] J. Fuchs, B. Schellekens and C. Schweigert, Nucl. Phys. B461 (1996) 371 [hep-th/9509105].
[20] A. N. Schellekens and S. Yankielowicz, Nucl. Phys. B330 (1990) 103.
[21] A. Cappelli, C. Itzykson and J. B. Zuber, Commun. Math. Phys. 113 (1987) 1.

[22] D. Bernard, Nucl. Phys. B288 (1987) 628.

[23] J. Fuchs and C. Schweigert, Nucl. Phys. B558 (1999) 419 [hep-th/9902132].

[24] A. N. Schellekens and S. Yankielowicz, Int. J. Mod. Phys. A5 (1990) 2903.

[25] J. Fuchs, A. N. Schellekens and C. Schweigert, Nucl. Phys. B473 (1996) 323 [hep-th/9601078].

[26] J. Fuchs, B. Schellekens and C. Schweigert, Commun. Math. Phys. 180 (1996) 39 [hep-th/9506135].

[27] J. Fuchs, Commun. Math. Phys. 136 (1991) 345.

[28] V.G. Kac and D.H. Peterson, Adv. in Math. 53 (1984) 125.

[29] A. N. Schellekens, Nucl. Phys. B558 (1999) 484 [math.qa/9905153].

[30] A. Sagnotti and Y. S. Stanev, Fortsch. Phys. 44 (1996) 585 [hep-th/9605042].

[31] T. Onogi and N. Ishibashi, Mod. Phys. Lett. A4 (1989) 161; N. Ishibashi, Mod. Phys. Lett. A4 (1989) 251.

[32] G. Pradisi, A. Sagnotti & Ya.S. Stanev,

[33] Ya.S. Stanev, Open descendants of Gepner models in D=6, talk given at workshop ‘Conformal Field Theory of D-branes’ at DESY, Hamburg, Germany. Transparencies on http://www.desy.de/~jfuchs/CftD.html.

[34] R. E. Behrend, P. A. Pearce, V. B. Petkova and J. Zuber, [hep-th/9908036].

[35] A. Sagnotti & Ya.S. Stanev, hep-th/9605042, Fortsch. Phys. 44 (1996) 585; Nucl. Phys. Proc. Suppl. 55B (1997) 200;

[36] L. R. Huiszoon, A. N. Schellekens and N. Sousa, hep-th/9911229.

20
[37] D. Lewellen, Nucl. Phys. B 372 (1992) 654; D. Fioravanti, G. Pradisi & A. Sagnotti, Phys. Lett. B 321 (1994) 349;

[38] J. L. Cardy, Nucl. Phys. B324 (1989) 581.

[39] L. R. Huiszoon, A. N. Schellekens and N. Sousa, Phys. Lett. B470 (1999) 95 [hep-th/9909117].

[40] L. Borisov, M. B. Halpern and C. Schweigert, Int. J. Mod. Phys. A13 (1998) 125 [hep-th/9701061].

[41] T. Gannon, hep-th/9910148.

[42] P. Bantay, Phys. Lett. B394 (1997) 87 [hep-th/9610192].

[43] P. Bantay, hep-th/0001173.

[44] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, hep-th/9912233.

[45] A. Sagnotti, Some properties of open-string constructions, hep-th/9509080; Talk presented at SUSY’95.

[46] G. Pradisi, A. Sagnotti & Ya.S. Stanev, hep-th/9506014; Phys. Lett. B356 (1995) 230;

[47] J. Fuchs and C. Schweigert, Phys. Lett. B414 (1997) 251 [hep-th/9708141].

[48] J. Fuchs and C. Schweigert, Nucl. Phys. B530 (1998) 99 [hep-th/9712257].

[49] A. Recknagel and V. Schomerus, Nucl. Phys. B531 (1998) 185 [hep-th/9712186].

[50] J. Fuchs and C. Schweigert, Phys. Lett. B447 (1999) 266; hep-th/9902132; hep-th/9908023; L. Birke, J. Fuchs and C. Schweigert, Adv. Theor. Math. Phys. 3 (1999) 3 [hep-th/9905038].

[51] A.N. Schellekens and S. Yankielowicz, Phys. Lett. B227 (1989) 387.