PAC fields over finitely generated fields

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Abstract We prove the following theorem for a finitely generated field $K$: Let $M$ be a Galois extension of $K$ which is not separably closed. Then $M$ is not PAC over $K$.

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1 Introduction

A central concept in Field Arithmetic is “pseudo algebraically closed (abbreviated PAC) field”. If $K$ is a countable Hilbertian field, then $K_s(\sigma)$ is PAC for almost all $\sigma \in \text{Gal}(K)^e$ [1, Theorem 18.6.1]. Moreover, if $K$ is the quotient field of a countable Hilbertian ring $R$ (e.g. $R = \mathbb{Z}$), then $K_s(\sigma)$ is PAC over $R$ [3, Proposition 3.1], hence also over $K$.

Here $K_s$ is a fixed separable closure of $K$ and $\text{Gal}(K) = \text{Gal}(K_s/K)$ is the absolute Galois group of $K$. This group is equipped with a Haar measure and “almost all” means “for all but a set of measure zero”. If $\sigma = (\sigma_1, \ldots, \sigma_e) \in \text{Gal}(K)^e$, then $K_s(\sigma)$ denotes the fixed field in $K_s$ of $\sigma_1, \ldots, \sigma_e$.

Recall that a field $M$ is said to be PAC if every nonempty absolutely irreducible variety $V$ defined over $M$ has an $M$-rational point. One says that $M$ is PAC over a subring $R$ if for every absolutely irreducible variety $V$ defined over $M$ of dimension $r \geq 1$ and every dominating separable rational map $\varphi : V \to K_M^r$ there exists an $a \in V(M)$ with $\varphi(a) \in R^r$.
When \( K \) is a number field, the stronger property of the fields \( \bar{K}(\sigma) \) (namely, being PAC over the ring of integers \( O \) of \( K \)) has far reaching arithmetical consequences. For example, \( \bar{O}(\sigma) \) (= the integral closure of \( O \) in \( \bar{K}(\sigma) \)) satisfies Rumely’s local–global principle [4, special case of Corollary 1.9]: If \( V \) is an absolutely irreducible variety defined over \( \bar{K}(\sigma) \) with \( V(\bar{O}) \neq \emptyset \), then \( V \) has an \( O(\sigma) \)-rational point. Here \( \bar{K} \) denotes the algebraic closure of \( K \) and \( \bar{K}(\sigma) \) is, as before, the fixed field of \( \sigma_1, \ldots, \sigma_e \) in \( \bar{K} \).

The article [3] gives several distinguished Galois extensions of \( \mathbb{Q} \) which are not PAC over any number field and notes that no Galois extension of a number field \( K \) (except \( \bar{K} \)) is known to be PAC over \( K \). This lack of knowledge has come to an end in [5], where Neukirch’s characterization of the \( p \)-adically closed fields among all algebraic extensions of \( \mathbb{Q} \) is used in order to prove the following theorem:

**Theorem A** If \( M \) is a Galois extension of a number field \( K \) and \( M \) is not algebraically closed, then \( M \) is not PAC over \( K \).

The goal of the present note is to generalize Theorem A to an arbitrary finitely generated field (over its prime field):

**Theorem B** Let \( K \) be a finitely generated field and \( M \) a Galois extension of \( K \) which is not separably closed. Then \( M \) is not PAC over \( K \).

The proof of Theorem B is based on Proposition 5.4 of [3] which combines Faltings’ theorem in characteristic 0 and the Grauert-Manin theorem in positive characteristic. The latter theorems are much deeper than the result of Neukirch used in the proof of Theorem A.

## 2 Accessible extensions

The proof of Theorem B actually gives a stronger theorem: No accessible extension (see definition prior to Theorem 4) of a finitely generated field \( K \) except \( K_s \) is PAC over \( K \). Technical tools in the proof are the “field crossing argument” and “ring covers”:

An extension \( S/R \) of integral domains with the corresponding extension \( F/E \) of quotient fields is said to be a **cover of rings** if \( S = R[z] \) and \( \text{discr}(\text{irr}(z, E)) \in R^\times \) [1, Definition 6.1.3]. We say that \( S/R \) is a **Galois cover of rings** if \( S/R \) is a cover of rings and \( F/E \) is a Galois extension of fields. Every epimorphism \( \varphi_0 \) of \( R \) onto a field \( E \) extends to an epimorphism \( \varphi \) of \( S \) onto a Galois extension \( F \) of \( E \) and \( \varphi \) induces an isomorphism of the decomposition group \( D_\varphi = \{ \sigma \in \text{Gal}(F/E) | \sigma(\text{Ker}(\varphi)) = \text{Ker}(\varphi) \} \) onto \( \text{Gal}(\bar{F}/\bar{E}) \) [1, Lemma 6.1.4]. In particular, \( \text{Gal}(F/E) \cong \text{Gal}(\bar{F}/\bar{E}) \) if and only if \([F:E] = [\bar{F}:\bar{E}]\).

As in the proof of [1, Lemma 24.1.1], the field crossing argument is the basic ingredient of the construction included in the proof of the following lemma.

**Lemma 1** Let \( K \) be a field, \( M \) an extension of \( K \), \( n \) a positive integer, \( N \) a Galois extension of \( M \) with Galois group \( A \) of order at most \( n \), and \( t \) an indeterminate. Then there exist fields \( D, F_0, F, \hat{F} \) as in diagram (1) such that the following holds:

(a) \( F_0 \) is regular over \( K \), \( F \) and \( D \) are regular over \( M \), and \( \hat{F} \) is regular over \( N \).
(b) \( FD = DN = \hat{F} \).

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