Two Greedy Consequences for Maximum Induced Matchings

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Abstract

We prove that, for every integer $d$ with $d \geq 3$, there is an approximation algorithm for the maximum induced matching problem restricted to $\{C_3, C_5\}$-free $d$-regular graphs with performance ratio $0.7083d + 0.425$, which answers a question posed by Dabrowski et al. (Theor. Comput. Sci. 478 (2013) 33-40). Furthermore, we show that every graph with $m$ edges that is $k$-degenerate and of maximum degree at most $d$ with $k < d$, has an induced matching with at least $m/((3k - 1)d - k(k + 1) + 1)$ edges.

Keywords: induced matching; greedy algorithm; approximation algorithm; strong chromatic index; degenerate graph
1 Introduction

A set \( M \) of edges of a graph \( G \) is an induced matching of \( G \) if the set of vertices of \( G \) that are incident with the edges in \( M \) induces a 1-regular subgraph of \( G \), or, equivalently, if \( M \) is an independent set of the square of the line graph of \( G \). The induced matching number \( \nu_2(G) \) of \( G \) is the maximum cardinality of an induced matching of \( G \). Induced matchings were introduced by Stockmeyer and Vazirani [15] as a variant of classical matchings [11]. While classical matchings are structurally and algorithmically well understood [11], it is hard to find a maximum induced matching [2,15] and efficient algorithms are only known for special graph classes [1,3,5]. The problem to determine a maximum induced matching in a given graph, called Maximum Induced Matching for short, is even APX-complete for bipartite \( d \)-regular graphs for every \( d \geq 3 \) [5,7].

On the positive side, a natural greedy strategy applied to a \( d \)-regular graph \( G \), which mimics the well-known greedy algorithm for the maximum independent set problem applied to the square of the line graph of \( G \), produces an induced matching with at least \( \frac{m(G)}{2d^2 - 2d + 1} \) edges. Since every induced matching of a \( d \)-regular graph contains at most \( \frac{m(G)}{2d} \) edges, this already yields an approximation algorithm for Maximum Induced Matching in \( d \)-regular graphs with performance ratio \( d - \frac{1}{2} + \frac{1}{4d^2} \) as observed by Zito [18]. This was improved slightly by Duckworth et al. [7] who describe an approximation algorithm with asymptotic performance ratio \( d - 1 \). The best known approximation algorithm for Maximum Induced Matching restricted to \( d \)-regular graphs for general \( d \) is due to Gotthilf and Lewenstein [9] who elegantly combine a greedy strategy with a local search algorithm to obtain a performance ratio of \( 0.75d + 0.15 \). In [10] Joos et al. describe a linear time algorithm that finds an induced matching with at least \( \frac{m(G)}{2d^2} \) edges for a given 3-regular graph \( G \), which yields an approximation algorithm for cubic graphs with performance ratio \( \frac{3}{2} \).

At the end of [5] Dabrowski et al. propose to study approximation algorithms for regular bipartite graphs, and to determine whether the above performance ratios can be improved in the bipartite case. As our main result we show that this is indeed possible.

**Theorem 1** For every integer \( d \) with \( d \geq 3 \), there is an approximation algorithm for Maximum Induced Matching restricted to \( \{C_3, C_5\} \)-free \( d \)-regular graphs with performance ratio \( \frac{17}{24}d + \frac{17d}{4d^2 - 24} \leq 0.7083d + 0.425 \).

Our proof of Theorem 1 builds on the approach of Gotthilf and Lewenstein [9], and all proofs are postponed to Section 2.

Our second result, which also relies on a greedy strategy, is a lower bound on the induced matching number of degenerate graphs. This result is related to recent bounds on the strong chromatic index \( \chi'_s(G) \) of a graph \( G \) [8], which is defined as the minimum number of induced matchings into which the edge set of \( G \) can be partitioned. The most prominent conjecture concerning this notion was made by Erdős and Nešetřil in 1985 and states that the strong chromatic index of a graph \( G \) of maximum degree at most \( d \) is at most \( \frac{2}{3}d^2 \). The most significant progress towards this conjecture is due to Molloy and Reed [12] who proved \( \chi'_s(G) \leq 1.998d^2 \) provided that \( d \) is sufficiently large. Again a natural greedy edge coloring implies \( \chi'_s(G) \leq 2d^2 - 2d + 1 \).

Recall that a graph \( G \) is \( k \)-degenerate for some integer \( k \), if every non-empty subgraph of \( G \) has a vertex of degree at most \( k \). Recently, Chang and Narayanan [4] studied the strong chromatic index of 2-degenerate graphs and inspired the following results about the strong chromatic index of a
Theorem 2

If the proofs of (1) all rely in some way on greedy colorings and (1) immediately implies that \( \nu_F(G) \geq \frac{m(G)}{\chi'_s(G)} \geq \frac{m(G)}{4kd + O(kd + d)} \) for the considered graphs. We show that the factor 4 can be reduced to 3.

Theorem 2 If \( G \) is a \( k \)-degenerate graph of maximum degree at most \( d \) with \( k < d \), then \( \nu_F(G) \geq \frac{m(G)}{(3k-1)d - k(k+1)+1} \).

Before we proceed to the proofs of Theorems 1 and 2 we collect some notation and terminology.

We consider finite, simple, and undirected graphs, and use standard terminology and notation. For a graph \( G \), we denote the vertex set, edge set, order, and size by \( V(G) \), \( E(G) \), \( n(G) \), and \( m(G) \), respectively. If \( G \) has no cycle of length 3 or 5, then \( G \in \{C_3, C_5\} \)-free. Let \( L(G) \) denote the line graph of \( G \) and let \( G^2 \) denote the square of \( G \). For an edge \( e \) of \( G \), let \( C_G(e) = \{e\} \cup N_{L(G)}(e) = \{f \in E(G) : \text{dist}_{L(G)}(e, f) \leq 2\} \) and let \( c_G(e) = |C_G(e)| \). Note that a set of edges of \( G \) is an induced matching if and only if it does not contain two distinct edges \( e \) and \( f \) with \( f \in C_G(e) \), or, equivalently, \( e \in C_G(f) \). In a maximal induced matching \( M \) of a graph \( G \), for every edge \( f \in E(G) \setminus M \), there is some edge \( e \) in \( M \) with \( f \in C_G(e) \). For some edges \( f \), the choice of \( e \) might be unique, which motivates the following definition. For a set \( M \) of edges of \( G \), let \( PC_G(M, e) = C_G(e) \setminus \bigcup_{f \in M \setminus \{e\}} C_G(f) \) and let \( pc_G(M, e) = |PC_G(M, e)| \). For two disjoint sets \( X \) and \( Y \) of vertices of \( G \), let \( E_G(X, Y) \) be the set of edges \( uv \) of \( G \) with \( u \in X \) and \( v \in Y \), and let \( m_G(X, Y) = |E_G(X, Y)| \). For a set \( E \) of edges of \( G \), let \( G - E = (V(G), E(G) \setminus E) \). For a positive integer \( k \), let \( [k] = \{1, 2, \ldots, k\} \).

2 Proofs

The greedy strategy for maximum induced matching relies on the following lemma.

Lemma 3 Let \( G \) be a graph. If \( G_0, \ldots, G_k \) are such that

- \( G_0 = G \), and
- for \( i \in [k] \), there is an edge \( e_i \) of \( G_{i-1} \) such that \( G_i = G_{i-1} - C_{G_{i-1}}(e_i) \),

then,

(i) If \( e \) and \( f \) are edges of \( G_i \) for some \( i \in [k] \), then \( f \in C_{G_i}(e) \) if and only if \( f \in C_G(e) \).

(ii) The set \( \{e_1, \ldots, e_k\} \) is an induced matching.

Proof: (i) Let \( e = uv \) and \( f = xy \) be edges of \( G_i \) for some \( i \in [k] \). Since \( G_i \) is a subgraph of \( G \), it follows that \( f \in C_{G_i}(e) \) immediately implies \( f \in C_G(e) \). Hence, for a contradiction, we may assume that \( f \notin C_{G_i}(e) \), the two edges \( e \) and \( f \) do not share a vertex. Since \( f \notin C_G(e) \), we may assume that the graph \( G \) contains the edge \( ux \). Since \( f \notin C_{G_i}(e) \), the edge \( ux \) does not belong to \( G_i \), which implies \( ux \in C_{G_{i-1}}(e_j) \) for some \( j \leq i \). Note that \( e \) and \( f \) are edges of
If \( u \varepsilon \) is incident with \( e_j \), then one of \( e \) and \( f \) is incident with \( e_j \), and belongs to \( C_{G_j-1}(e_j) \). If \( u \varepsilon \) is not incident with \( e_j \), then, by symmetry, we may assume that \( G_j-1 \) contains an edge between \( u \) and a vertex incident with \( e_j \), which implies that \( e \) belongs to \( C_{G_j-1}(e_j) \). In both cases we obtain the contradiction that one of the two edges \( e \) and \( f \) does not belong to \( G_j \) and hence also not to \( G_i \).

(ii) If \( e_j \in C_G(e_i) \) for some \( i, j \in [k] \) with \( i < k \), then, by (i), we have \( e_j \in C_{G_{i-1}}(e_i) \), which implies the contradiction that \( e_j \) does not belong to \( G_i \) and hence also not to \( G_{j-1} \). \( \square \)

Algorithm 1 called \texttt{Greedy}(\( f \)), corresponds to the greedy algorithm used by Gotthilf and Lewenstein.

\begin{algorithm}
\textbf{Greedy} (\( f \))
\begin{algorithmic}
\State \textbf{Input}: A \textit{d}-regular graph \( G \).
\State \textbf{Output}: A pair \((M, G')\) such that \( M \) is an induced matching of \( G \) and \( G' \) is a subgraph of \( G \).
\State \( G_0 \leftarrow G \);
\State \( G' \leftarrow G_0 \);
\State \( i \leftarrow 1 \);
\State \( M \leftarrow \emptyset \);
\While {\( \min\{c_{G_{i-1}}(e) : e \in E(G_{i-1})\} \leq f \)}
\State Choose an edge \( e_i \) of \( G_{i-1} \) with \( c_{G_{i-1}}(e_i) \leq f \);
\State \( M \leftarrow M \cup \{e_i\} \);
\State \( G_i \leftarrow G_{i-1} - c_{G_{i-1}}(e_i) \);
\State \( G' \leftarrow G_i \);
\State \( i \leftarrow i + 1 \);
\EndWhile
\State \textbf{return} \((M, G')\);
\end{algorithmic}
\end{algorithm}

Algorithm 1: The greedy algorithm of Gotthilf and Lewenstein depending on \( f \).

Obviously, \texttt{Greedy}(\( f \)) can be performed in polynomial time, and, by Lemma 3, it works correctly. The intuitive idea behind \texttt{Greedy}(\( f \)) is to restrict the greedy choices to edges that are not too expensive in the sense that their inclusion in the induced matching does not eliminate too many of the remaining edges. In fact, \texttt{Greedy}(\( f \)) adds a fraction of at least \( \frac{1}{2d^2 - 2d + 1} \) of the edges in \( E(G) \setminus E(G') \) to \( M \), which is better than the trivial fraction \( \frac{1}{2d^2 - 2d + 1} \) for \( f < 2d^2 - 2d + 1 \). The drawback of \texttt{Greedy}(\( f \)) is that it might not consume all edges of \( G \), that is, \( E(G) \setminus E(G') \) might be small compared to \( E(G) \).

By Lemma 3, the union of \( M \) with any induced matching of \( G' \) is an induced matching of \( G \). Therefore Gotthilf and Lewenstein combine \texttt{Greedy}(\( f \)) applied to \( G \) with Algorithm 2 called \texttt{Local Search}, applied to \( G' \). \texttt{Local Search} starts with an empty augmentation operations as long as possible:

- Add an edge from \( E(G') \setminus M' \) to \( M' \) if this results in an induced matching.
- Replace one edge in \( M' \) with two edges from \( E(G') \setminus M' \) if this results in an induced matching.

Clearly, \texttt{Local Search} can be performed in polynomial time, and, as observed above, the union of \( M \) with its output \( M' \) is an induced matching of \( G \). The crucial observation is that the graph \( G' \) produced by \texttt{Greedy}(\( f \)) has an additional structural property. As a subgraph of \( G \), it is trivially a graph of maximum degree at most \( d \) but additionally each of its edges \( e \) satisfies \( c_{G'}(e) > f \), which allows the improved analysis of \texttt{Local Search} in the following two Lemmas 4 and 5.
Local Search

Input: A graph $G$.
Output: An induced matching $M$ of $G$.

$M \leftarrow \emptyset$;
repeat
  if $M \cup \{e\}$ is an induced matching of $G$ for some edge $e \in E(G) \setminus M$ then
    $M \leftarrow M \cup \{e\}$;
  end
  if $(M \setminus \{e\}) \cup \{e', e''\}$ is an induced matching of $G$ for some three distinct edges $e \in M$ and $e', e'' \in E(G) \setminus M$ then
    $M \leftarrow (M \setminus \{e\}) \cup \{e', e''\}$;
  end
until $M$ does not increase during one iteration;
return $M$;

Algorithm 2: The local search algorithm of Gotthilf and Lewenstein.

Lemma 4 Let $G$ be a $\{C_3, C_5\}$-free graph of maximum degree at most $d$ for some $d \geq 3$ such that
$\min\{c_G(e) : e \in E(G)\} > \frac{17}{15}d^2$. If $M$ is an induced matching of $G$ produced by Local Search applied to $G$, then, for every edge $e \in M$, $pc_G(M, e) \leq \frac{5}{3}d^2 + 1$.

Proof: Since $M$ is produced by Local Search, it has the following properties.

(a) For every edge $e$ of $G$, there is some edge $e'$ in $M$ with $e \in C_G(e')$; because otherwise $e \notin M$ and $M \cup \{e\}$ is an induced matching of $G$.

(b) If $e$ is in $M$ and $e'$ and $e''$ are two distinct edges in $PC_G(M, e)$, then $e'' \in C_G(e')$; because otherwise $e', e'' \notin M$ and $(M \setminus \{e\}) \cup \{e', e''\}$ is an induced matching of $G$.

Let $e = xy \in M$.

Let $X$ be the set of neighbors $u$ of $x$ distinct from $y$ such that $xu \in PC_G(M, e)$. Let $Y$ be the set of neighbors $v'$ of $y$ distinct from $x$ such that $yu' \in PC_G(M, e)$. Since $G$ is $\{C_3, C_5\}$-free, the sets $X$ and $Y$ are disjoint. Let $X_2$ be the set of vertices $v$ in $V(G) \setminus (\{x, y\} \cup X \cup Y)$ such that there is some vertex $u$ in $X$ with $uv \in PC_G(M, e)$. Let $Y_2$ be the set of vertices $v'$ in $V(G) \setminus (\{x, y\} \cup X \cup Y)$ such that there is some vertex $u'$ in $Y$ with $vu' \in PC_G(M, e)$. Since $G$ is $\{C_3, C_5\}$-free, the sets $X_2$ and $Y_2$ are disjoint. Let $X_1$ be the set of vertices in $X$ that have a neighbor in $X_2$. Let $Y_1$ be the set of vertices in $Y$ that have a neighbor in $Y_2$. Since $G$ is $\{C_3, C_5\}$-free, the sets $X_1$, $Y_1$, $X_2$, and $Y_2$ are independent, and there are no edges between $X_2$ and $Y$ as well as between $Y_2$ and $X$.

By definition, $\{x, y\} \cup E_G(\{x\}, X) \cup E_G(\{y\}, Y)$ is the set of edges in $PC_G(M, e)$ that are identical or adjacent with $e$.

Note that is $f \in PC_G(M, e)$ is not identical or adjacent with $e$, then $G$ contains an edge $g$ that is adjacent with $e$ and $f$. If $g \notin PC_G(M, e)$, then $g \in C_G(e')$ for some edge $e'$ in $M$ distinct from $e$, which implies the contradiction $f \in C_G(e')$. Hence $g \in PC_G(M, e)$. This implies that all edges in $PC_G(M, e) \setminus \{e\}$ are incident with a vertex in $X \cup Y$.

If $uv$ is an edge such that $u \in X_1$ and $v \in X_2$, then, by definition, $xu \in PC_G(M, e)$ and $u'v \in PC_G(M, e)$ for some $u' \in X$. Clearly, $uv \in C_G(e)$. If $uv \notin PC_G(M, e)$, then $uv \in C_G(e')$ for some edge $e'$ in $M$ distinct from $e$, which implies the contradiction that one of the two edges $xu$ and $u'v$
belongs to $C_G(e')$. Hence, by symmetry, all edges in $E_{G}(X_1, X_2) \cup E_{G}(Y_1, Y_2)$ belong to $PC_G(M, e)$. Similarly, it follows that all edges in $E_{G}(X, Y)$ belong to $PC_G(M, e)$, and hence

$$PC_G(M, e) = \{x, y\} \cup E_{G}(\{x\}, X) \cup E_{G}(\{y\}, Y)$$

$$\cup E_{G}(X_1, X_2) \cup E_{G}(Y_1, Y_2) \cup E_{G}(X, Y).$$

(2)

If $u \in X_1$ and $u' \in Y$, then there is some $v \in X_2$ such that $uv \in PC_G(M, e)$ and, by definition, $yu' \in PC_G(M, e)$. Since $G$ is $\{C_3, C_5\}$-free, property (b) implies that $u$ and $u'$ are adjacent, that is, every vertex in $X_1$ is adjacent to every vertex in $Y$, and, by symmetry, every vertex in $Y_1$ is adjacent to every vertex in $X$.

If $u_1, u_2 \in X_1$ and $v_1, v_2 \in X_2$ are four distinct vertices such that $u_1v_1$ and $u_2v_2$ are edges of $G$, then, by (2), $u_1v_1, u_2v_2 \in PC_G(M, e)$ and, since $G$ is $\{C_3, C_5\}$-free, property (b) implies that $u_2$ is a neighbor of $u_1$ or $v_1$ is a neighbor of $u_2$, that is, the bipartite graph between $X_1$ and $X_2$ is $2K_2$-free. This implies that the sets $N_G(u) \cap X_2$ for $u$ in $X_1$ are ordered by inclusion. Hence, if $X_1$ is non-empty, then $X_1$ contains a vertex $u_x$ that is adjacent to all vertices in $X_2$. By symmetry, if $Y_1$ is non-empty, then the set $Y_1$ contains a vertex $u_y$ that is adjacent to all vertices in $Y_2$.

We consider three cases.

**Case 1 $X_1$ and $Y_1$ are both non-empty.**

Since $u_x$ is adjacent to each vertex in $\{x\} \cup X_2 \cup Y$, we obtain $|X_2| + |Y| \leq d - 1$, and, by symmetry, $|Y_2| + |X| \leq d - 1$. Now (2) implies

$$pc_G(M, e) = 1 + m_N(x, X) + m_N(y, Y) + (m_G(\{u_x\}, X_2) + m_G(X_1 \setminus \{u_x\}, X_2))$$

$$+ (m_G(y, Y_2) + m_G(Y_1 \setminus \{y\}, Y_2)) + m_G(X, Y)$$

$$= 1 + |X| + |Y| + (|X_2| + m_G(X_1 \setminus \{u_x\}, X_2))$$

$$+ (|Y_2| + m_G(Y_1 \setminus \{y\}, Y_2)) + m_G(X, Y)$$

$$\leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(Y_1 \setminus \{u_y\}, Y_2) + m_G(X, Y)$$

(3)

Note that $u_x$ is adjacent to all vertices in $X_2 \cup Y$ and that $x$ is adjacent to all vertices in $\{y\} \cup (X \setminus \{u_x\})$. There are $|Y|$ edges between $y$ and $Y$, $m_G(X, Y) - |Y|$ edges between $X \setminus \{u_x\}$ and $Y$, and $m_G(X_1 \setminus \{u_x\}, X_2)$ edges between $X_2$ and $X \setminus \{u_x\}$. This implies

$$c_G(xu_x) \leq 2d^2 - 2d + 1 - m_G(N_G(x) \setminus \{u_x\}, N_G(u_x) \setminus \{x\})$$

$$\leq 2d^2 - 2d + 1 - |Y| - (m_G(X, Y) - |Y|) - m_G(X_1 \setminus \{u_x\}, X_2)$$

$$= 2d^2 - 2d + 1 - (m_G(X, Y) + m_G(X_1 \setminus \{u_x\}, X_2)).$$

Since, by assumption, $c_G(xu_x) > \frac{17}{12}d^2$, we obtain

$$m_G(X, Y) + m_G(X_1 \setminus \{u_x\}, X_2) \leq \frac{7}{12}d^2 - 2d + 1, \text{ and, by symmetry,}$$

(4)

$$m_G(X, Y) + m_G(Y_1 \setminus \{u_y\}, Y_2) \leq \frac{7}{12}d^2 - 2d + 1.$$

By symmetry, we may assume that $|X_1| \geq |Y_1|$.

Since every vertex in $Y_1$ is adjacent to $y$ and to all vertices in $X_1$, it has at most $d - 1 - |X_1|$ neighbors in $Y_2$, which implies $m_G(Y_1 \setminus \{u_y\}, Y_2) \leq (|Y_1| - 1)(d - 1 - |X_1|) \leq (|Y_1| - 1)(d - 1 - |Y_1|)$. 

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Note that regardless of the value of $|Y_1|$, we have $-|Y_1|^2 + d|Y_1| \leq \frac{d^2}{4}$. Together with (3) and (4) we obtain

$$pc_G(M, e) \leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(Y_1 \setminus \{u_y\}, Y_2) + m_G(X, Y)$$

$$\leq 2d - 1 + \frac{7}{12}d^2 - 2d + 1 + (|Y_1| - 1)(d - 1 - |Y_1|)$$

$$= \frac{7}{12}d^2 - |Y_1|^2 + d|Y_1| - d + 1$$

$$\leq \frac{7}{12}d^2 + \frac{1}{4}d^2 - d + 1$$

$$= \frac{5}{6}d^2 - d + 1.$$ 

**Case 2** $X_1$ is non-empty but $Y_1$ is empty.

As in Case 1, we have $|X_2| + |Y| \leq d - 1$. Clearly, $|Y| \leq d - 1$. Now (2) implies

$$pc_G(M, e) = 1 + m_G(\{x\}, X) + m_G(\{y\}, Y) + m_G(\{u_x\}, X_2) + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y)$$

$$= 1 + |X| + |Y| + |X_2| + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y)$$

$$\leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y).$$

Exactly as Case 1, we obtain

$$m_G(X, Y) + m_G(X_1 \setminus \{u_x\}, X_2) \leq \frac{7}{12}d^2 - 2d + 1,$$

and hence

$$pc_G(M, e) \leq 2d - 1 + m_G(X_1 \setminus \{u_x\}, X_2) + m_G(X, Y)$$

$$\leq 2d - 1 + \frac{7}{12}d^2 - 2d + 1$$

$$\leq \frac{7}{12}d^2.$$

Since $d \geq 3$, it follows that $pc_G(M, e) \leq \frac{5}{6}d^2 - d + 1$.

**Case 3** $X_1$ and $Y_1$ are both empty.

Note that in this case also both sets $X_2$ and $Y_2$ are empty. Now (2) implies

$$pc_G(M, e) = 1 + m_G(\{x\}, X) + m_G(\{y\}, Y) + m_G(X, Y)$$

$$\leq 2d - 1 + m_G(X, Y).$$

If $m_G(X, Y) \leq \frac{5}{6}d^2 - 2d + 2$, then $pc_G(M, e) \leq \frac{5}{6}d^2 + 1$. Hence, we may assume that $m_G(X, Y) > \frac{5}{6}d^2 - 2d + 2$. Since $|X| \leq d - 1$, this implies the existence of a vertex $\tilde{u}_x$ in $X$ with at least

$$\frac{\frac{5}{6}d^2 - 2d + 2}{d - 1} = \frac{5}{6}d - \frac{7}{6} + \frac{5}{6(d - 1)} \geq \frac{5}{6}d - \frac{7}{6}.$$
neighbors in \( Y \). Let \( \tilde{Y} \) be the set of neighbors of \( \tilde{u}_x \) in \( Y \). Since \( |Y| \leq d - 1 \), we have \( |Y \setminus \tilde{Y}| \leq (d - 1) - \left( \frac{5}{6}d - \frac{5}{6} \right) = \frac{1}{6}d + \frac{1}{6} \) and thus

\[
m_G(X \setminus \{\tilde{u}_x\}, \tilde{Y}) = m_G(X,Y) - m_G(\{\tilde{u}_x\}, Y) - m_G(X \setminus \{\tilde{u}_x\}, Y \setminus \tilde{Y}) \\
\geq \left( \frac{5}{6}d^2 - 2d + 2 \right) - (d - 1) - (d - 2) \left( \frac{1}{6}d + \frac{1}{6} \right) \\
= \frac{2}{3}d^2 - \frac{17}{6}d + \frac{10}{3}.
\]

Note that \( \tilde{u}_x \) is adjacent to all vertices in \( \tilde{Y} \) and that \( x \) is adjacent to all vertices in \( \{y\} \cup (X \setminus \{\tilde{u}_x\}) \). There are \( |\tilde{Y}| \) edges between \( y \) and \( \tilde{Y} \), and \( m_G(X \setminus \{\tilde{u}_x\}, \tilde{Y}) \) edges between \( X \setminus \{\tilde{u}_x\} \) and \( \tilde{Y} \). This implies

\[
c_G(x\tilde{u}_x) \leq 2d^2 - 2d + 1 - m_G(N_G(x) \setminus \{\tilde{u}_x\}, N_G(\tilde{u}_x) \setminus \{x\}) \\
\leq 2d^2 - 2d + 1 - |\tilde{Y}| - m_G(X \setminus \{\tilde{u}_x\}, \tilde{Y}) \\
\leq 2d^2 - 2d + 1 - \left( \frac{5}{6}d - \frac{7}{6} \right) - \left( \frac{2}{3}d^2 - \frac{17}{6}d + \frac{10}{3} \right) \\
\leq \frac{4}{3}d^2 - \frac{7}{6},
\]

contradicting the assumption \( c_G(xu_x) > \frac{17}{12}d^2 \).

This completes the proof. \( \square \)

**Lemma 5** Let \( G \) be a \( \{C_3,C_5\}\)-free graph of maximum degree at most \( d \) for some \( d \geq 3 \) such that \( \min\{c_G(e) : e \in E(G)\} > \frac{17}{12}d^2 \). If \( M \) is an induced matching of \( G \) produced by Local Search applied to \( G \), then \( |M| \geq \frac{m(G)}{1.44d^2-d+1} \).

**Proof:** We consider the number \( p \) of pairs \((e,f)\) where \( e \in M \) and \( f \in C_G(e) \). Since \( c_G(e) \leq 2d^2 - 2d + 1 \), we have \( p \leq (2d^2 - 2d + 1)|M| \). In order to obtain a lower bound on \( p \), we observe that the edges \( f \) of \( G \) for which there is exactly one edge \( e \) in \( M \) with \( f \in C_G(e) \) are exactly those in \( \bigcup_{e \in M} PC_G(M,e) \). Hence, by Lemma 4, \( p \geq 2m(G) - \sum_{e \in M} PC_G(M,e) \geq 2m(G) - \left( \frac{5}{6}d^2 + 1 \right)|M| \). Both estimates for \( p \) together yield the desired bound. \( \square \)

**Algorithm 3:** An approximation algorithm for \( \{C_3,C_5\}\)-free \( d\)-regular graphs.

**APPBip(d)**

**Input:** A \( \{C_3,C_5\}\)-free \( d\)-regular graph \( G \).

**Output:** An induced matching \( M \cup M' \) of \( G \).

Apply GREEDY(\( \frac{17}{12}d^2 \)) to \( G \) and denote the output by \( (M,G') \);

Apply LOCAL SEARCH to \( G' \) and denote the output by \( M' \);

return \( M \cup M' \).

**Lemma 6** Let \( G \) be a \( \{C_3,C_5\}\)-free graph of maximum degree at most \( d \) for some \( d \geq 3 \). The algorithm \textsc{APPBip}(d) (cf. Algorithm 3) applied to \( G \) produces an induced matching \( M \cup M' \) of \( G \) with \( |M \cup M'| \geq \frac{12m(G)}{17d^2} = \frac{m(G)}{1.44d^2} \).

**Proof:** By Lemma 4, the set \( M \cup M' \) produced by \textsc{APPBip}(d) applied to \( G \) is an induced matching of \( G \). If \( m = m(G) - m(G') \), then \( |M| \geq \frac{m}{12d^2} \). By Lemma 5, \( |M'| \geq \frac{m(G')}{12d^2} \geq \frac{m(G) - m}{12d^2} \). Altogether, we obtain \( |M \cup M'| \geq \frac{m(G)}{14d^2} \). \( \square \)
We are now in a position to prove Theorem 1.

Proof of Theorem 1: For the sake of completeness, we give the argument for the upper bound. If $N$ is some induced matching of the $d$-regular graph $G$, then, for each edge $e$ in $N$, the two vertices incident with $e$ are incident with exactly $2d - 1$ edges of $G$. Since $N$ is induced, these sets of $2d - 1$ edges are disjoint, which implies $|N| \leq \frac{m(G)}{2d-1}$. By Lemma 6, this implies that the performance ratio of APPROXBip$(d)$ applied to a $\{C_3, C_5\}$-free $d$-regular graph is at most $\frac{17d^2}{12(2d-1)} = \frac{17}{24}d + \frac{17}{48}d - \frac{24}{24}$, which completes the proof. □

For the proof of Theorem 2, we need the following lemma.

Lemma 7 If $G$ is a non-empty $k$-degenerate graph of maximum degree at most $d$ with $k < d$, then $G$ has an edge $e$ with $c_G(e) \leq (3k - 1)d - k(k + 1) + 1$.

Proof: Let $X$ denote the set of vertices of $G$ of degree at most $k$ and let $Y = V(G) \setminus X$. If two vertices $u$ and $v$ in $X$ are adjacent, then $c_G(e) \leq 1 + 2(k - 1)d$. If $X$ is an independent set, then, since the graph $G[Y]$ has a vertex $v$ of degree at most $k$ in $G[Y]$ but degree more than $k$ in $G$, some vertex $u$ in $X$ is adjacent to this vertex $v$ in $Y$. Each of the at most $k$ neighbors of $u$ has degree at most $d$. Since $v$ has degree at most $k$ in $G[Y]$, it has $r$ neighbors in $Y$ for some $r \leq k$, which have degree at most $d$. The remaining at most $d - r$ neighbors of $v$ are all in $X$ and hence have degree at most $k$. Altogether this implies that

\[
c_G(uv) \leq 1 + (k - 1)d + rd + (d - 1 - r)k = 1 + 2kd + r(d - k) - d - k \leq 1 + 2kd + k(d - k) - d - k = 1 + (3k - 1)d - k(k + 1).
\]

Since $2(k - 1)d \leq (3k - 1)d - k(k + 1)$, the proof is complete. □

Proof of Theorem 2: By definition, every subgraph of $G$ is $k$-degenerate and of maximum degree at most $d$. Therefore, by Lemma 7, the algorithm Greedy($(3k - 1)d - k(k + 1) + 1)$ applied to the graph $G$ outputs a pair $(M, G')$ where $M$ is an induced matching of $G$ and the graph $G'$ has no edges. This implies that $|M| \geq \frac{m(G)}{(3k-1)d-k(k+1)+1}$, which completes the proof. □

References

[1] A. Brandstät and R. Mosca, On distance-3 matchings and induced matchings, Discrete Appl. Math. 159 (2011) 509-520.
[2] K. Cameron, Induced matchings, Discrete Appl. Math. 24 (1989) 97-102.
[3] K. Cameron, Induced matchings in intersection graphs, Discrete Math. 278 (2004) 1-9.
[4] G.J. Chang and N. Narayanan, Strong chromatic index of 2-degenerate graphs, J. Graph Theory 73 (2013) 119-126.
[5] K.K. Dabrowski, M. Demange, and V.V. Lozin, New results on maximum induced matchings in bipartite graphs and beyond, Theor. Comput. Sci. 478 (2013) 33-40.
[6] M. Dębski, J. Grytczuk and M. Śleszyńska-Nowak, Strong chromatic index of sparse graphs, \[\text{arXiv:1301.1992}\] (9 January 2013).

[7] W. Duckworth, D.F. Manlove, and M. Zito, On the approximability of the maximum induced matching problem, J. Discrete Algorithms 3 (2005) 79-91.

[8] J.L. Fouquet and J.L. Jolivet, Strong edge-colouring of graphs and applications to multi-\(k\)-gons, Ars Combin. 16A (1983) 141-150.

[9] Z. Gotthilf and M. Lewenstein, Tighter Approximations for Maximum Induced Matchings in Regular Graphs, Lecture Notes in Computer Science 3879 (2006) 270-281.

[10] F. Joos, D. Rautenbach, and T. Sasse, Induced Matchings in Subcubic Graphs, to appear in SIAM J. Discrete Math.

[11] L. Lovász and M.D. Plummer, Matching Theory, vol. 29, Annals of Discrete Mathematics, North-Holland, Amsterdam, 1986.

[12] V.V. Lozin, On maximum induced matchings in bipartite graphs, Inf. Process. Lett. 81 (2002) 7-11.

[13] R. Luo and G. Yu, A note on strong edge-colorings of 2-degenerate graphs, \[\text{arXiv:1212.6092}\] (25 December 2012).

[14] M. Molloy and B. Reed, A bound on the strong chromatic index of a graph, J. Combin. Theory Ser. B 69 (1997) 103-109.

[15] L.J. Stockmeyer and V.V. Vazirani, NP-completeness of some generalizations of the maximum matching problem, Inf. Process. Lett. 15 (1982) 14-19.

[16] G. Yu, Strong edge-colorings for \(k\)-degenerate graphs, \[\text{arXiv:1212.6093}\] (31 March 2013).

[17] T. Wang, Strong chromatic index of \(k\)-degenerate graphs, \[\text{arXiv:1304.0285}\] (19 July 2013).

[18] M. Zito, Induced Matchings in Regular Graphs and Trees, Lecture Notes in Computer Science 1665 (1999) 89-100.