**b-Symbol Distance of Constacyclic Codes of Length** \( p^5 \) **Over** \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \)**

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**ABSTRACT** In this research paper, the repeated-root constacyclic codes over the chain ring \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \) are considered, where \( p \) is a prime and \( m > 0 \) is any integer. The \( b \)-symbol distance for prime power length, i.e. \( p^b \), is also studied for any integer \( b > 0 \). The Hamming and symbol-pair distances of all \( \delta \)-constacyclic codes have been thoroughly studied in [18], where \( \delta \) is an unit in the ring \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \) which is of the form \( \zeta \) and \( \phi + u\varphi \), where \( 0 \neq \phi, \varphi, \zeta \in \mathcal{F}_{p^m} \). In this paper, the \( b \)-symbol distance of all such \( \delta \)-constacyclic codes of prime power length is computed for \( 1 \leq b \leq \lfloor \frac{p}{2} \rfloor \). Furthermore, as an application, all MDS \( b \)-symbol constacyclic codes of length \( p^b \) over \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \) are established.

**INDEX TERMS** Constacyclic codes, repeated-root, MDS codes, \( b \)-symbol distance.

**I. INTRODUCTION**

The theory of error correcting codes attracted the attention of many researchers around the globe since its eruption. At the beginning, the message in the noisy channel was split up into information units called symbols. Researchers used to perform the operations of writing and reading on these individual symbols, which created a lot of disruptions. These conditions were sorted out with the introduction of symbols that can be read and written in possible overlapping groups. Cassuto and Blaum [1] were the first to propose this method in which the outputs of the channel is overlapping pair of symbols. The model proposed by Cassuto and Blaum [1] and later by Cassuto and Litsyn [2] deals with the possibly corrupted outputs generated by a string of read operations having overlapping pairs of adjoining symbols, called pair-read symbols. Later, Kai et al. [20] developed the theory given by Cassuto and Litsyn [3, Th. 10] for simple-root constacyclic codes. Many researchers scrutinized symbol-pair distances over constacyclic codes since then in [11], [14], [16]–[18], [21] over many years.

As a result of rich algebraic structure and practical implementations, constacyclic codes play a remarkable role in coding theory. Repeated-root constacyclic codes were inducted by Castagnoli [4] and Van Lint [22], where they established that the repeated-root constacyclic codes have a sequential structure. The researchers like Cao further established some significant results over repeated-root constacyclic codes in [5]–[10]. But the existence of optimal repeated-root constacyclic codes influenced the researchers to put these codes under further scrutiny.

The results established for symbol-pair read channels were further generalized to \( b \)-symbol read channels by Yaakobi et al. [23], in which the read operation is performed on overlapping \( b \)-pairs of adjacent symbols, where \( b \geq 3 \).

In [12], Ding et al. expressed the \( b \)-symbol read vector of \( x \) as

\[
\pi_b(x) = [(x_0, x_1, \ldots, x_{b-1}), (x_1, x_2, \ldots, x_b), \ldots, (x_{n-1}, x_0, \ldots, x_{b-2})] \in \mathbb{Z}^{b\cdot n},
\]

where \( \mathbb{Z} \) is an alphabet consisting of symbols of size \( q \) and \( x = (x_0, x_1, \ldots, x_{n-1}) \) is a vector in \( \mathbb{Z}^n \). The \( b \)-symbol distance between vectors \( y \) and \( x \) in \( \mathbb{Z}^n \) is denoted by \( d_b(y, x) \) and defined as

\[
d_b(y, x) = d_H(\pi_b(y), \pi_b(x)).
\]
A Singleton Bound for $b$-symbol codes over $\mathbb{F}_q$ was established in [12] as follows: Let $q$ be a prime power and $b \leq d_0 \leq n$, for any $b$-symbol code $C$ of length $n$ with size $M$ and minimum $b$-distance $d_0$ over $\mathbb{F}_q$, $M \leq q^{n-d_0+b}$. The $b$-symbol code $C$ is called an optimal code or maximum distance separable (MDS) $b$-symbol code, if the equality holds, with respect to Singleton bound. Some families of linear MDS $b$-symbol codes over finite field for some special $b$ are also constructed.

The Hamming and symbol-pair distances over $F_{p^m} + uF_{p^m}$ and $b$-distances of over $F_{p^m}$ of repeated root constacyclic codes of prime power lengths have been computed by Dinh et al. [18], [19], recently. These works motivate us to study $b$-symbol distances of repeated root constacyclic codes of $p^k$ length over $F_{p^m} + uF_{p^m}$, $u^2 = 0$. We determine $b$-distance of all the constacyclic codes of length $p^k$ over $F_{p^m} + uF_{p^m}$ for $1 \leq b \leq \lfloor \frac{p}{2} \rfloor$. As an application, we establish all MDS $b$-symbol constacyclic codes of length $p^k$ over the ring $F_{p^m} + uF_{p^m}$.

The organization of this paper is as follows. Some preliminary results are discussed in Section 2. In Section 3, the $b$-distance of constacyclic codes of length $p^k$ is established over the ring $F_{p^m} + uF_{p^m}$. All the MDS $b$-symbol constacyclic codes of length $p^k$ are identified in Section 4 and in Section 5, we conclude the paper.

II. PRELIMINARIES

Let $\mathcal{R} = F_{p^m} + uF_{p^m}$, $u^2 = 0$ be a finite commutative ring. Let $\delta$ be an unit of $\mathcal{R}$. The $\delta$-constacyclic shift $\Omega_\delta$ on $\mathcal{R}^n$ is given by

$$\Omega_\delta(c_0, c_1, c_2, \ldots, c_{n-1}) = (\delta c_{n-1}, c_0, c_1, \ldots, c_{n-2})$$

If $\Omega_\delta(C) = C$, then $C$ is called $\delta$-constacyclic code.

Consider the polynomial $c(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$ in the ring $\mathcal{R}[x]/(x^n - \delta)$. The polynomial $c(x)$ can be used to express the codeword $c = (c_0, c_1, c_2, \ldots, c_{n-1})$ of the code $C$. And $\chi_C(x)$ corresponds to the $\delta$-constacyclic shift of $c(x)$.

We have a well-known result about $\delta$-constacyclic codes.

**Proposition 2.1:** A linear code $C$ is an ideal of $\mathcal{R}[x]/(x^n - \delta)$ if and only if $C$ is a $\delta$-constacyclic code of length $n$ over $\mathcal{R}$.

So, for any unit $\delta$ of $F_{p^m}$, the linear $\delta$-constacyclic code of length $p^k$ over $F_{p^m}$ are precisely the ideals in $F_{p^m}[x]/(x^{p^k} - \delta)$.

From Division Algorithm, there exists non-negative integers $k_0$, $k_1$, such that $s = k_0 m + k_1$, and $0 \leq k_1 \leq m - 1$. Let $d_0 = \delta^{s/p^k + 1}$. Then $d_0^{p^k} = \delta^{sk_0 + 1} = \delta$.

**Theorem 2.2:** $x - \delta_0$ is a nilpotent element in $F_{p^m}[x]/(x^{p^k} - \delta)$ with nilpotency index $p^k$. Moreover, the ring $F_{p^m}[x]/(x^{p^k} - \delta)$ is a chain ring with the maximal ideal of $(x - \delta_0)$. $\delta$-constacyclic codes of length $p^k$ over $F_{p^m}$ are the ideals $(x - \delta_0)^y \subseteq F_{p^m}[x]/(x^{p^k} - \delta)$, where $0 \leq y \leq p^k$, which forms the strictly inclusion chain $F_{p^m}[x]/(x^{p^k} - \delta) = \{0\} \supseteq (x - \delta_0) \supseteq (x - \delta_0)^2 \supseteq \cdots \supseteq (x - \delta_0)^{p^k-1} \supseteq (x - \delta_0)^{p^k} = \{0\}$.

Each $\delta$-constacyclic code $\langle(x - \delta_0)^y \rangle \subseteq F_{p^m}[x]/(x^{p^k} - \delta)$ has $p^{m(p^k-1)}$ codewords.

**Proof:** We have

$$(x - \delta_0)^{p^k} = x^{p^k} - \delta_0^{p^k} + \sum_{i=1}^{p^k-1} (-1)^i \binom{p^k}{i} x^i \delta_0^{p^k-i}.$$ Since, $p$ divides $\binom{p^k}{i}$ for all $1 \leq i \leq p^k - 1$. Hence,

$$0 \leq p^k - 1 \leq \binom{p^k}{i} = 0 \text{ in } F_{p^m}[x]/(x^{p^k} - \delta).$$ Thus,

$$(x - \delta_0)^{p^k} = x^{p^k} - \delta_0^{p^k} = x^{p^k} - \delta \text{ in } F_{p^m}[x]/(x^{p^k} - \delta).$$ Hence, $(x - \delta_0)$ is a nilpotent element in $F_{p^m}[x]/(x^{p^k} - \delta)$ with nilpotency index $p^k$.

Let $f(x) = a_0 + a_1 x + \ldots + a_{p^k-1} x^{p^k-1} \in F_{p^m}[x]/(x^{p^k} - \delta)$, where $a_0, a_1, \ldots, a_{p^k-1} \in F_{p^m}$. Suppose, there exists $b_0, b_1, \ldots, b_{p^k-1}$ such that $f(x)$ can be represented as

$$f(x) = b_0 + b_1(x - \delta_0) + b_2(x - \delta_0)^2 + \cdots + b_{p^k-1}(x - \delta_0)^{p^k-1}.$$ If $b_0 = 0$, then $f(x) = (x - \delta_0)g(x)$, which implies $f(x) \in (x - \delta_0)$. And if $b_0 \neq 0$, then $f(x) = b_0 + (x - \delta_0)g(x)$. Since, $x - \delta_0$ is a nilpotent element in $F_{p^m}[x]/(x^{p^k} - \delta)$, this implies $f(x)$ is an invertible element in $F_{p^m}[x]/(x^{p^k} - \delta)$. Hence, we have shown that $f(x)$ is either a unit or $f(x) \in (x - \delta_0)$. That means, $F_{p^m}[x]/(x^{p^k} - \delta)$ is a local ring with the maximum ideal $(x - \delta_0)$, hence, $F_{p^m}[x]/(x^{p^k} - \delta)$ is a chain ring. Since the nilpotency index of $x - \delta_0$ is $p^k$, the ideals of $F_{p^m}[x]/(x^{p^k} - \delta)$ form the desired strictly inclusive chain. Clearly, for $t = 0, 1, \ldots, p^k$, the cardinality of each code $(x - \delta_0)^t \subseteq F_{p^m}[x]/(x^{p^k} - \delta)$ is $p^{m(p^k-t)}$.

**Theorem 2.3:** The dual of the $\delta$-constacyclic codes is the $\delta^{-1}$-constacyclic codes over $F_{p^m}$ and is given by $\langle x - \delta^{-1} \rangle^{p^k} \subseteq F_{p^m}[x]/(x^{p^k} - \delta^{-1})$, which contains $p^m$ codewords.

Hamming weight of a codeword $a = (a_0, a_1, \ldots, a_{n-1}) \in C$ is the non-zero entries in $a$ and is denoted by $wt(q)$.

The no. of non-zero entries in the codeword $a = b = (a_0 - b_0, a_1 - b_1, \ldots, a_{n-1} - b_{n-1})$ is the Hamming distance between two codewords $a = (a_0, a_1, \ldots, a_{n-1})$ and $b = (b_0, b_1, \ldots, b_{n-1}) \in \mathcal{C}$ and is denoted by $d_H(a, b)$. The Hamming distance of all $\delta$-constacyclic code depends on the characteristic of $F_{p^m}$ and length of the code.

**Theorem 2.4:** Hamming distance of the constacyclic code $C = (x - \delta_0)^y \subseteq F_{p^m}[x]/(x^{p^k} - \delta)$ for $0 \leq y \leq p^k$ is given by the following:

Case 1: If $t = 0$, then $d_H(C) = 1$.

Case 2: If $p^k - p^k - 1 + 1 \leq t \leq p^k - p^k - 1 + 1$, where $1 \leq e \leq p - 1$ and $0 \leq e \leq s - 1$, then $d_H(C) = (e + 1)p^k$.

Case 3: If $t = p^k$, then $d_H(C) = 0$.

Consider the ring $\mathcal{R} = F_{p^m}[u]/(u^2) = F_{p^m} + uF_{p^m}$ ($u^2 = 0$). The elements of $\mathcal{R}$ can be written as $a + ub$, where $a, b \in F_{p^m}$.

The construction of all constacyclic codes of prime power length over $\mathcal{R}$ is provided by Dinh in [13] as follows.
Theorem 2.5: Let \( \delta \) be a unit of the ring \( R \), i.e. \( \delta \) is of the form \( \phi + \omega \tau \) or \( \zeta \), where \( \phi \neq \psi \), \( \phi, \tau, \zeta \in \mathbb{F}_{p^m}^* \).

- If \( \delta = \phi + \omega \psi \), then the ring \( R[x]/(\phi x - \delta) \) is a finite chain ring with maximal ideal \( \langle \phi x - 1 \rangle \), and \( (x - \delta) \mathbb{F}_{p^m} \) is a unit in \( \mathbb{F}_{p^m} \).

- If \( \delta = \zeta \), then the ring \( R[x]/(x - \delta) \) is a finite chain ring with maximal ideal \( \langle x - \delta \rangle \), but it is not a chain ring. The \( \zeta \)-constacyclic codes of \( p^s \) length over \( R \), i.e., ideals of the ring \( R[x]/(x - \delta) \), are
  - Type 1 (trivial ideals): \( \langle \delta \rangle \).
  - Type 2 (principal ideals generated by nonmonic polynomials): \( \langle u(x - \delta) \rangle \), where \( 0 \leq u \leq p^s - 1 \).
  - Type 3 (principal ideals generated by monic polynomials): \( \langle (x - \delta) \rangle \), with \( x \leq p^s - 1 \).

When the unit \( \delta \) is of the form \( \phi + \omega \tau \), \( d_H(\mathbb{C}) \) of each \( \phi \) and \( \omega \tau \)-constacyclic codes of \( p^s \) length over \( R \) were provided in [13].

Theorem 2.6: Let \( \mathcal{C} \subseteq R_{\phi, \psi} = R[x]/(x^{p^s} - (\phi + \psi)) \), then \( \mathcal{C} \subseteq \langle (\phi x - 1) \rangle \), for \( \tau \in \{0, 1, 2 \} \), and the Hamming distance \( d_H(\mathcal{C}) \) is completely determined by

Case 1: If \( 0 \leq u \leq p^s \), then \( d_H(\mathcal{C}) = 1 \).

Case 2: If \( 2 p^s - pr + (e - 1) \leq u \leq 2 p^s - pr + er \), where \( r = p^{s-1} \), \( 1 \leq e \leq p - 1 \), \( 0 \leq \epsilon \leq s - 1 \), then \( d_H(\mathcal{C}) = (e + 1)p^s \).

Case 3: If \( t = 2 p^s \), then \( d_H(\mathcal{C}) = 0 \).

If the unit \( \delta \) is of the form \( \zeta \), then \( d_H(\mathbb{C}) \) of each \( \zeta \)-constacyclic codes is given in [18]. We note that \( \mathcal{F}_{p^m} \) is a subring of \( R \), for a code \( \mathbb{C} \) over \( R \), we denote \( d_H(\mathbb{C}) \) as the Hamming distance of \( \mathbb{C} \) over \( \mathcal{F}_{p^m} \).

Theorem 2.7: Let \( \mathcal{C} = \langle u(x - \delta) \rangle \), \( 0 \leq u \leq p^s - 1 \) be a \( \zeta \)-constacyclic codes of length \( p^s \) over \( R \) of Type 2. Then \( d_H(\mathbb{C}) = d_H((x - \delta)^\mathbb{C}) \), and is determined by

Case 1: If \( t = 0 \), then \( d_H(\mathbb{C}) = 1 \).

Case 2: If \( p^s - pr + (e - 1) \leq u \leq p^s - pr + er \), where \( r = p^{s-1} \), \( 1 \leq e \leq p - 1 \), \( 0 \leq \epsilon \leq s - 1 \), then \( d_H(\mathbb{C}) = (e + 1)p^s \).

For Type 3 ideals, i.e., ideals of the form \( \langle x - \delta \rangle + u(x - \delta)^\mathbb{C} \), where \( 1 \leq u \leq p^s - 1 \), \( 0 \leq \epsilon \leq s - 1 \), and either \( h(x) \) or \( h(x) = 0 \) is a unit in \( \mathcal{F}_{p^m} \), Hamming distance is given by the following theorem.

Theorem 2.8: Let \( \mathcal{C} \) be a \( \zeta \)-constacyclic codes of length \( p^s \) over \( R \) of Type 3, i.e., \( \mathcal{C} = \langle (x - \delta) + u(x - \delta)^\mathbb{C} \rangle \), where \( 1 \leq u \leq p^s - 1 \), \( 0 \leq \epsilon \leq s - 1 \), and either \( h(x) \) or \( h(x) = 0 \) is a unit in \( \mathcal{F}_{p^m} \). Then \( d_H(\mathbb{C}) = d_H((x - \delta)^\mathbb{C}) \), and is determined by \( d_H(\mathbb{C}) = (e + 1)p^s \), where \( p^s - pr + (e - 1) \leq u \leq p^s - pr + er \), \( r = p^{s-1} \), \( 1 \leq e \leq p - 1 \), and \( 0 \leq \epsilon \leq s - 1 \).

For \( \zeta \)-constacyclic codes of the form \( \langle (x - \delta) + u(x - \delta)^\mathbb{C} \rangle \), \( h(x) \) as in Type 3, \( \deg h(x) \leq \kappa - 1 \), and \( \kappa < T \), \( T \) is the smallest integer such that \( u(x - \delta)^T \in \langle (x - \delta) + u(x - \delta)^\mathbb{C} \rangle \); and \( T = 1 \), if \( h(x) = 0 \), otherwise \( T = \min(t, p^s + 1 - t) \). Hamming distance is given by the following theorem.

Theorem 2.9: Let \( \mathcal{C} \) be a \( \zeta \)-constacyclic codes of length \( p^s \) over \( R \) of Type 4, i.e., \( \mathcal{C} = \langle (x - \delta) + u(x - \delta)^\mathbb{C} \rangle \), \( h(x) \) as in Type 3, \( \deg h(x) \leq \kappa - 1 \), and \( \kappa < T \), \( T \) is the smallest integer such that \( u(x - \delta)^T \in \langle (x - \delta) + u(x - \delta)^\mathbb{C} \rangle \); \( T = 1 \), if \( h(x) = 0 \), otherwise \( T = \min(t, p^s - 1 + 1) \). Then \( d_H(\mathbb{C}) = d_H((x - \delta)^\mathbb{C}) \), and is determined by \( d_H(\mathbb{C}) = (e + 1)p^s \), where \( p^s - pr + (e - 1) \leq u \leq p^s - pr + er \), \( r = p^{s-1} \), \( 1 \leq e \leq p - 1 \), and \( 0 \leq \epsilon \leq s - 1 \).

III. B-SYMBOL DISTANCE

Recently, the b-symbol distance distribution of all constacyclic code of length \( p^s \) over \( \mathcal{F}_{p^m} \) has been discussed by Dinh et al. in [19].

Theorem 3.1: Let \( 1 \leq b \leq \left[ \frac{s}{2} \right] \) and \( 1 \leq \theta \leq p - 1 \). Given a \( \delta \)-constacyclic code of length \( p^s \) over \( \mathcal{F}_{p^m} \), then it has the form \( \mathcal{C} = \langle (x - \delta)^\mathbb{C} \rangle \). The \( b \)-distance \( d_b(\mathcal{C}) \) of \( \mathcal{C} \) is completely determined by

\[
d_b(\mathbb{C}) = \begin{cases} 
 b, & \text{if } t = 0 \\
 (\theta + b)(\psi + 1)p^s, & \text{if } t = p^s - p^s - \theta - \psi p^{s-1} + \theta, \\
 0 \leq \psi < p - 2, \\
 \theta(\psi + 1) \leq b \text{ or } \theta + b \leq p \\
 0 < \psi \leq p - 2, \\
 \theta(\psi + 1) > b, \text{ or } \theta + b > p \\
 \psi \leq p - b, \text{ or } \theta + b \leq p \\
 0, & \text{if } t = p^s - \theta + b \\
 \end{cases}
\]
The $b$-symbol distances of all $\delta$-constacyclic codes of length $p^s$ over the ring $\mathfrak{R}$ is computed in this section. Let $\delta = \phi + u\psi$ be the unit, where $\phi, \psi \in F_{p^m}[x]$, i.e., $(\phi + u\psi)$-constacyclic codes of the length $p^s$ over $\mathfrak{R}$. Now for $(\phi \circ \chi - 1)^{p^s} \in \langle u \rangle$ in $F_{p^m}[x]^{\frac{n}{p^s} - (\phi + u\psi)}$. We have two cases:

Case 1: $1 \leq t \leq p^s$. Then $u \in ((\phi \circ \chi - 1)^{t})$, whence $(\phi \circ \chi - 1)^{t}$ has a $b$-symbol distance of $b$.

Case 2: $p^s + 1 \leq t \leq 2p^s - 1$. Then $(\phi \circ \chi - 1)^{t} = (\phi \circ \chi - 1)^{t-2p^s}$. Thus, the codewords of the code $(\phi \circ \chi - 1)^{t}$ in $F_{p^m}[x]^{\frac{n}{p^s} - (\phi + u\psi)}$ are precisely the codewords of the code $(\phi \circ \chi - 1)^{t-2p^s}$ in $F_{p^m}[x]^{\frac{n}{p^s} - (\phi + u\psi)}$, multiplied by $u$, which have the same $b$-symbol weights. Furthermore, the codes $(\phi \circ \chi - 1)^{t-2p^s}$ in $F_{p^m}[x]^{\frac{n}{p^s} - (\phi + u\psi)}$ are $\phi$-constacyclic codes of length $p^s$ over $\mathfrak{R}$, whose $b$-symbol distances are computed in Theorem 3.1.

Thus, following the methodology of [18] we obtain the $b$-symbol distances of each $(\phi + u\psi)$-constacyclic codes of length $p^s$ over $\mathfrak{R}$ are given below:

**Theorem 3.2:** Let $1 \leq b \leq \left[ \frac{p^s}{2} \right]$ and $1 \leq \theta \leq p - 1$. Let $\mathcal{C}$ be a $(\phi + u\psi)$-constacyclic code of length $p^s$ over $\mathfrak{R}$, then it has the form $\mathcal{C} = (\phi \circ \chi - 1)^{t} \leq \frac{\mathfrak{R}[x]}{\langle x^p - \phi \circ \chi \rangle}$, for $t \in \{ 0, 1, \ldots, 2p^s \}$. The $b$-distance $d_b(\mathcal{C})$ of $\mathcal{C}$ is completely determined by:

$$d_b(\mathcal{C}) = \begin{cases} b, & \text{if } 0 \leq t \leq p^s \\ (\theta + b)(\psi + 1)p^s, & \text{if } t = 2p^s - p^s - \epsilon + 1 + \theta, \\ & \text{if } 0 \leq \epsilon \leq s - 2, \\ & \text{if } 0 \leq \psi \leq p - 2, \\ & \text{if } \theta(\psi + 1) \leq b \text{ and } \theta + b \leq p \\ b(\psi + 2)p^s, & \text{if } 2p^s - p^s - \epsilon + 1 + \theta \\ & \leq t \leq 2p^s - p^s - \epsilon + 1 + \theta, \\ & \text{if } 0 \leq \epsilon \leq s - 2, \\ & \text{if } 0 \leq \psi \leq p - 2, \\ & \text{if } \theta(\psi + 1) > b \text{ or } \theta + b > p \\ (\phi + b)p^{s-1}, & \text{if } t = 2p^s - p + \epsilon, \\ & \text{if } 0 \leq \epsilon \leq p - b \text{ or } \theta \geq 0 \leq \psi \leq p - b \\ p^s, & \text{if } t = 2p^s. \end{cases}$$

Now, we consider the case where the unit in the form $\delta = \zeta \in F_{p^m}[x]$. We observe that if $c(x) = f(x) + ug(x)$, where $f(x), g(x) \in F_{p^m}[x]$, then $w_T(c(x)) \geq \max\{w_T(f(x)), w_T(g(x))\}$. Thus, we provide the $b$-symbol distances of all $\zeta$-constacyclic codes of length $p^s$ over $\mathfrak{R}$, for $\zeta \in F_{p^m}[x]$ by following the methodology given in [18].

**Theorem 3.3:** Let $\delta = \zeta \in F_{p^m}[x]$ be a unit in $\mathfrak{R}$. The $\zeta$-constacyclic codes of length $p^s$ over $\mathfrak{R}$ have their $b$-symbol distances completely determined as follows.

- **Type 1 trivial ideals:** $\langle 0 \rangle, \langle 1 \rangle$: $d_b(\langle 0 \rangle) = 0, d_b(\langle 1 \rangle) = b$.

- **Type 2 principal ideals generated by non-monic polynomial:** $\mathcal{C} = (u(x - \zeta)^t)$, where $0 \leq t \leq p^s - 1$.

$$d_b(\mathcal{C}) = \begin{cases} b, & \text{if } t = 0 \\ (\theta + b)(\psi + 1)p^s, & \text{if } t = 2p^s - p^s - \epsilon + 1 + \theta, \\ & \text{if } 0 \leq \epsilon \leq s - 2, \\ & \text{if } 0 \leq \psi \leq p - 2, \\ & \text{if } \theta(\psi + 1) \leq b \text{ and } \theta + b \leq p \\ b(\psi + 2)p^s, & \text{if } 2p^s - p^s - \epsilon + 1 + \theta \\ & \leq t \leq 2p^s - p^s - \epsilon + 1 + \theta, \\ & \text{if } 0 \leq \epsilon \leq s - 2, \\ & \text{if } 0 \leq \psi \leq p - 2, \\ & \text{if } \theta(\psi + 1) > b \text{ or } \theta + b > p \\ (\phi + b)p^{s-1}, & \text{if } t = 2p^s - p + \epsilon, \\ & \text{if } 0 \leq \epsilon \leq p - b \text{ or } \theta \geq 0 \leq \psi \leq p - b \\ p^s, & \text{if } t = 2p^s. \end{cases}$$

- **Type 3 principal ideals generated by monic polynomial:** $\mathcal{C} = (u(x - \zeta)^t + u(x - \zeta)^t h(x))$, where $0 \leq t \leq p^s - 1, 0 \leq \tau < t$, and either $\theta(h(x)) = 0$ or $\theta(h(x))$ is a unit in $F_{p^m}[x]^{\frac{n}{p^s} - (\phi + u\psi)}$.

$$d_b(\mathcal{C}) = \begin{cases} (\theta + b)(\psi + 1)p^s, & \text{if } t = 2p^s - p^s - \epsilon + 1 + \theta, \\ & \text{if } 0 \leq \epsilon \leq s - 2, \\ & \text{if } 0 \leq \psi \leq p - 2, \\ & \text{if } \theta(\psi + 1) \leq b \text{ and } \theta + b \leq p \\ b(\psi + 2)p^s, & \text{if } 2p^s - p^s - \epsilon + 1 + \theta \\ & \leq t \leq 2p^s - p^s - \epsilon + 1 + \theta, \\ & \text{if } 0 \leq \epsilon \leq s - 2, \\ & \text{if } 0 \leq \psi \leq p - 2, \\ & \text{if } \theta(\psi + 1) > b \text{ or } \theta + b > p \\ (\phi + b)p^{s-1}, & \text{if } t = 2p^s - p + \epsilon, \\ & \text{if } 0 \leq \epsilon \leq p - b \text{ or } \theta \geq 0 \leq \psi \leq p - b \\ p^s, & \text{if } t = 2p^s. \end{cases}$$

- **Type 4 non-principal ideals:** $\mathcal{C} = (u(x - \zeta)^t + u(x - \zeta)^t h(x), u(x - \zeta)^t h(x))$, with $h(x)$ as in Type 3, $\theta(h(x)) \geq \kappa - \tau - 1$, and $\kappa < T$, where $T$ is the smallest integer such that $u(x - \zeta)^t \in ((x - \zeta)^t + u(x - \zeta)^t h(x))$; that is $T$ can be determined as:

$$T = \begin{cases} t, & \text{if } h(x) = 0 \\ \min(t, p^s - t + \tau), & \text{if } h(x) \neq 0. \end{cases}$$
Then
\[
d_b(\mathcal{C}) = \begin{cases} 
(\theta + b)(\psi + 1)p^s, & \text{if } \kappa = p^b - p^{s-\epsilon} \\
\psi p^{s-\epsilon - 1} + \theta, & \text{where } 0 \leq \epsilon \leq s - 2, \\
0 \leq \psi \leq p - 2, \\
\theta(\psi + 1) \leq b \text{ and } \theta + b \leq p \\
\psi^{-p} - p^{s-\epsilon} + \psi p^{s-\epsilon - 1} + \theta, & \text{if } \kappa = p^b - p^{s-\epsilon} \\
\psi \leq \kappa, \\
\psi p^{s-\epsilon} + (\psi + 1)p^{s-\epsilon - 1}, & \text{where } 0 \leq \epsilon \leq s - 2, \\
0 \leq \psi \leq p - 2, \\
\theta(\psi + 1) > b, \text{ or } \theta + b > p \\
\phi + b)^{s-\epsilon - 1}, & \text{if } \kappa = p^b - p + \phi, \\
\{ \phi + b \}^{s-\epsilon - 1}, & \text{if } \kappa = p^b - p + \phi, \\
p^s, & \text{if } \kappa = p^b - p + 1 \leq \kappa \leq p^b - 1.
\end{cases}
\]

It can be noted that in Theorem 3.2 and 3.3, when \( b = 1 \) and \( b = 2 \), the \( b \)-distance gives the Hamming distance and the symbol-pair distance of all the \( b \)-constacyclic codes of length \( p^n \) over \( \mathcal{R} = F_{p^n} + u F_{p^n} \), respectively. It can be observed that the resultant Hamming and symbol-pair distances are similar to the distances deduced in [18]. It can also be inferred that the Hamming distances can be obtained for \( p \geq 2 \), whereas the symbol-pair distances can be acquired for \( p \geq 5 \).

Now, let us consider an example of \( b \)-symbol constacyclic codes of length \( p^n \) over \( F_{p^n} + u F_{p^n} \).

Example 3.4: Consider the ring \( \mathcal{R} = F_{7} + u F_{7} \), where \( p = 7 \), \( m = 1 \). The units in the ring \( F_{7} + u F_{7} \) are of the form \( \phi + w \psi \) and \( \xi \) where \( \phi, \psi, \xi \in F_{7} \). Let \( b = 3 \) and \( s = 2 \). Thus, we will determine the 3-symbol distance of all \( b \)-constacyclic codes of length \( n = p^s = 49 \), where \( \delta \) is in the form \( \phi + w \psi \) and \( \xi \).

For \((\phi + w \psi)\)-constacyclic code of length 49 over \( \mathcal{R} \) the generators are of the form \( \mathcal{C}_i = \langle (\phi \delta x - 1)^{\tau} \rangle \), for \( i \in [0, 1, \ldots, 98] \). Then the 3-symbol distance is determined in Table 1.

| Range of \( i \) | \( d_{3}(\mathcal{C}_s) \) |
|-----------------|----------------|
| \( 0 \leq i \leq 49 \) | 3 |
| \( i = 50 \) | 4 |
| \( i = 51 \) | 5 |
| \( 52 \leq i \leq 56 \) | 6 |
| \( i = 57 \) | 8 |
| \( 58 \leq i \leq 63 \) | 9 |
| \( 64 \leq i \leq 70 \) | 12 |
| \( 71 \leq i \leq 77 \) | 15 |

Now, \( \xi \)-constacyclic codes of length 49 over \( \mathcal{R} \) has four types of generator. The distances corresponding to different generators are given as follows:

- **Type 1 (trivial ideals)**: \( \langle 0 \rangle, \langle 1 \rangle \);
  \[ d_3(\langle 0 \rangle) = 0, \quad d_3(\langle 1 \rangle) = 3. \]

| \( i \) | \( d_{3}(\mathcal{C}_s) \) |
|-----|----------------|
| \( 0 \leq i \leq 49 \) | 3 |
| \( i = 50 \) | 4 |
| \( i = 51 \) | 5 |
| \( 52 \leq i \leq 56 \) | 6 |
| \( i = 57 \) | 8 |
| \( 58 \leq i \leq 63 \) | 9 |
| \( 64 \leq i \leq 70 \) | 12 |
| \( 71 \leq i \leq 77 \) | 15 |

**TABLE 2.** 3-symbol distance of \( \xi \)-constacyclic codes of Type 2 over \( F_{7} + u F_{7} \).

| Range of \( i \) | \( d_{3}(\mathcal{C}_s) \) |
|-----------------|----------------|
| \( i = 0 \) | 3 |
| \( i = 1 \) | 4 |
| \( i = 2 \) | 5 |
| \( 3 \leq i \leq 7 \) | 6 |
| \( i = 8 \) | 8 |
| \( 9 \leq i \leq 14 \) | 9 |
| \( 15 \leq i \leq 21 \) | 12 |
| \( 22 \leq i \leq 28 \) | 15 |
| \( 29 \leq i \leq 35 \) | 18 |
| \( 36 \leq i \leq 42 \) | 21 |
| \( 43 \leq i \leq 49 \) | 24 |

**TABLE 3.** 3-symbol distance of \( \xi \)-constacyclic codes of Type 2 over \( F_{7} + u F_{7} \).

| Range of \( i \) | \( d_{3}(\mathcal{C}_s) \) |
|-----------------|----------------|
| \( i = 1 \) | 4 |
| \( i = 2 \) | 5 |
| \( 3 \leq i \leq 7 \) | 6 |
| \( i = 8 \) | 8 |
| \( 9 \leq i \leq 14 \) | 9 |
| \( 15 \leq i \leq 21 \) | 12 |
| \( 22 \leq i \leq 28 \) | 15 |
| \( 29 \leq i \leq 35 \) | 18 |
| \( 36 \leq i \leq 42 \) | 21 |
| \( 43 \leq i \leq 49 \) | 24 |

**IV. MDS \( b \)-SYMBOL CONSTACYCLIC CODES OF LENGTH \( p^n \) OVER \( F_{p^m} + u F_{p^m} \)**

All the MDS \( b \)-symbol constacyclic codes are identified in this section by utilizing the \( b \)-symbol distance obtained in Section III of all \( b \)-symbol constacyclic codes of prime power length over \( F_{p^m} + u F_{p^m} \).

**A. \((\phi + w \psi)\)-CONSTACYCLIC CODES**

**Theorem 4.1:** Let \( b \leq \lfloor \frac{q}{2} \rfloor \). Let \( \mathcal{C} = \langle (\phi \delta x - 1)^{\tau} \rangle \subseteq F_{p^m} \langle (\phi^{p^m} - (\phi + w \psi))^{\tau} \rangle \) be a \((\phi + w \psi)\)-constacyclic code of length \( p^n \) over \( \mathcal{R} \) for \( i \in [0, 1, \ldots, 2p^n] \). Then \( \mathcal{C} \) is a MDS \( b \)-symbol constacyclic code if and only if \( i = 0, d_3(\mathcal{C}_s) = b \).
TABLE 4. 3-symbol distance of $\varsigma$-constacyclic codes of Type 4 over $\mathbb{F}_2 + u\mathbb{F}_2$.

| Range of $\kappa$ | $d_3(C_{\kappa})$ |
|-------------------|-------------------|
| $\kappa = 1$      | 2                 |
| $\kappa = 2$      | 3                 |
| $3 \leq \kappa \leq 7$ | 4                 |
| $\kappa = 8$      | 5                 |
| $9 \leq \kappa \leq 14$ | 6                 |
| $15 \leq \kappa \leq 21$ | 7                 |
| $22 \leq \kappa \leq 28$ | 8                 |
| $29 \leq \kappa \leq 35$ | 9                 |
| $36 \leq \kappa \leq 42$ | 10                |
| $\kappa = 43$     | 11                |
| $\kappa = 44$     | 12                |
| $\kappa = 45$     | 13                |
| $46 \leq \kappa \leq 48$ | 14                |

Proof: For $(\phi + u\psi)$-constacyclic codes, we have $|\mathcal{C}| = p^m(2p^s - 1)$. By the Singleton bound, $\mathcal{C}$ is a $b$-symbol MDS code if and only if $2p^s - t = 2(p^s - d_b(\mathcal{C}) + b)$, i.e., $t = 2d_b(\mathcal{C}) - 2b$. The $b$-symbol distance $d_b(\mathcal{C})$ for all $t \in \{0, 1, \ldots, 2p^s\}$ is established in Theorem 3.2. We consider cases according to the range of $t$.

Case 1: $0 \leq t \leq p^s$. Then $d_b(\mathcal{C}) = b$, so obviously, MDS $b$-symbol code can be obtained when $t = 0$.

Case 2: $t = 2p^s - p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \psi \leq p - 2$, $\theta(\psi + 1) \leq b$ and $\theta + b \leq p$. Then $d_b(\mathcal{C}) = (\theta + b)(\psi + 1)p^s$, and

$$t = 2p^s - p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta = p^s - (p^s - 1) + \psi p^s - \epsilon - 1 + \theta$$

$$\geq p^s - (p^s - 1) + \psi p^s + \theta$$

(when equality when $\epsilon = s - 2$ or $\epsilon = s$)

$$\geq (\psi + 2)p^s - (p^s - 1) + \psi p^s + \theta$$

(when equality when $\psi = p - 2$)

$$= 2(\psi + 1)p^s + 2p^s - \epsilon - 1 - 2p + \theta$$

(when equality when $\theta = b$)

$$\geq 2(\psi + 1)(\theta + b)p^s + 2p^s - \epsilon - 1 - 2p + \theta$$

(when equality when $\theta = b$)

$$\geq 2(\psi + 1)(\theta + b)p^s + \theta$$

(when equality when $\epsilon = 0$)

$$= 2d_b(\mathcal{C}) + (\theta + 2b) - 2b$$

(when equality when $\theta = b$)

(when equality when $\theta + b > 0$).

Hence, no MDS code can be obtained here.

Case 3: $2p^s - p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta \leq t \leq 2p^s - p^s - \epsilon + (\psi + 1)p^s - \epsilon - 1$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \psi \leq p - 2$, $\theta(\psi + 1) > b$, or $\theta + b > p$. Then $d_b(\mathcal{C}) = b(\psi + 2)p^s$, and

$$t = 2p^s - p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta$$

$$= p^s - (p^s - 1) + \psi p^s - \epsilon - 1 + \theta$$

$$\geq p^s(\psi + 1)p^s - \epsilon - 1 + \theta$$

(when equality when $\epsilon = s - 2$ or $\epsilon = s$)

$$\geq (\psi + 2)p^s(2p^s - 1) + \psi p^s + \theta$$

(when equality when $\psi = p - 2$)

$$\geq (\psi + 2)p^s(2p^s - 1) + \psi p^s + \theta$$

(when equality when $\psi = p - 2$)

$$= 2(\psi + 2)p^s - (p^s - 1) + \psi p^s + \theta$$

(when equality when $\psi = p - 2$)

Thus, in this case there is no MDS code.

Case 4: $t = 2p^s - p + \varrho$, where $0 \leq \varrho \leq p - b$. Then $d_b(\mathcal{C}) = (\varrho + b)p^s - 1$, and

$$t = 2p^s - p + \varrho = p(2p^s - 1) + \theta$$

(when equality when $\varrho = p - b$, or $\varrho = 1$)

$$= 2(\varrho + b)p^s - 1 - b$$

$$= 2d_b(\mathcal{C}) - b$$

(when $\theta + 2b > 0$).

Therefore, in this case there is no MDS $b$-symbol constacyclic code.

Case 5: $2p^s - b + 1 \leq t \leq 2p^s - 1$. Then $d_b(\mathcal{C}) = p^b$, and

$$t = 2p^s - b + 1 \geq 2d_b(\mathcal{C}) - b + 1 > 2d_b(\mathcal{C}) - 2b$$

Since, $t > p^s + d_b(\mathcal{C}) - b$, no MDS $b$-symbol code can be obtained in this case.

Case 6: $t = 2p^s$. Then $d_b(\mathcal{C}) = 0$, and $t = 2p^s > 2d_b(\mathcal{C}) - 2b$.

Since, $t > 2d_b(\mathcal{C}) - 2b$, no MDS $b$-symbol code exists in this case.

Thus, the only MDS $b$-symbol $(\phi + u\psi)$-constacyclic codes of length $p^s$ over $\mathbb{F}_2$ is the trivial code (1).

Now, the case where the unit $\delta \in \mathbb{F}_2$ is considered. From [13], it is acquired that for a $\varsigma$-constacyclic code, four types of ideals can be obtained and the dimension of the code $\mathcal{C}$ varies with each ideal. Here, the $b$-symbol MDS codes for each type of ideal is determined, whose $b$-symbol distance has been discussed in Theorem 3.3.

B. $\varsigma$-CONSTACYCLIC CODES

1) TYPE 1 (TRIVIAL IDEALS)

If $\mathcal{C} = \{0\}$, then $|\mathcal{C}| = 1$ and $d_b(\mathcal{C}) = 0$. Thus by Singleton bound, $\mathcal{C}$ is a $b$-symbol MDS code if and only if $0 = 2(p^s - d_b(\mathcal{C}) + b)$, i.e., $p^s = b$, which is not possible.

Again, if $\mathcal{C} = \{1\}$, then $|\mathcal{C}| = p^{2mp^s}$ and $d_b(\mathcal{C}) = b$. Thus by Singleton bound, $\mathcal{C}$ is a $b$-symbol MDS code if and only if $2p^s = 2(p^s - d_b(\mathcal{C}) + b)$, i.e., $d_b(\mathcal{C}) = b$. 

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Thus, there is only one $b$-symbol code for trivial ideals, i.e., $(1)$.

2) TYPE 2 (PRINCIPAL IDEALS GENERATED BY NONMONIC POLYNOMIAL)

Here, we have $\mathcal{C} = \langle u(x - \zeta_0) \rangle$, where $0 \leq t \leq p^s - 1$ and $|\mathcal{C}| = p^{m(p^s - t)}$. Thus by Singleton bound, $\mathcal{C}$ is a $b$-symbol MDS code if and only if $p^s - t = 2(p^s - d_b(\mathcal{C}) + b)$, i.e., $t = 2d_b(\mathcal{C}) - p^s - 2b$. Hence, we have the following theorem.

**Theorem 4.2:** Let $\mathcal{C} = \langle u(x - \zeta_0) \rangle \subseteq \mathbb{F}_{pm} / \langle x^s - \zeta \rangle$ be a $\zeta$-constacyclic code of length $p^s$ over $\mathcal{R}$, for $i \in 0, 1, \ldots, p^s - 1$. Then $\mathcal{C}$ is not a MDS $b$-symbol constacyclic code.

**Proof:** We get MDS code for $i = 2d_b(\mathcal{C}) - p^s - 2b$.

Now, we consider the cases according to the range of $i$.

Case 1: $i = 0$, then $d_b(\mathcal{C}) = b$, and the Singleton bound is satisfied when $p^s = 0$, which is a contradiction. Thus, no MDS $b$-symbol constacyclic code can be obtained in this case.

Case 2: $i = p^s - p^s - s + 1 + \psi p^s - s - 1 + \theta$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \psi \leq p - 2$, $\theta(\psi + 1) \leq b$ and $b + \theta \leq p$. Then $d_b(\mathcal{C}) = (\theta + b)(\psi + 1)p^s$, and

\[
i = p^s - p^s - s + 1 + \theta + \psi p^s - s - 1 + \psi p^s - s - 1 + \theta = p^s - (2p^s - 1) - p^s + \psi p^s - s - 1 + \theta \geq p^s - (2p^s - 1) - p^s + \psi p^s - s - 1 + \theta (equality when \epsilon = s - 2 or s = 2) \geq (\psi + 2)p(2p^s - 1) - p^s + \psi p^s - s - 1 + \theta (equality when \psi = p - 2) = 2(\psi + 1)p^{s+1} - p^s + 2p^{s+1} - 2p + \theta \geq 2(\theta + b)(\psi + 1)p^s - p^s + 2p^{s+1} - 2p + \theta (equality when \theta + b = p) \geq 2d_b(\mathcal{C}) - p^s + \theta \geq 2d_b(\mathcal{C}) - p^s - 2b. \]

Since, $i > 2d_b(\mathcal{C}) - p^s - 2b$, in this case no MDS $b$-symbol constacyclic code exist.

Case 3: $p^s - p^s - s + 1 + \psi p^s - s - 1 + \theta \leq t \leq p^s - p^s - s + 1 + \psi p^s - s - 1 + \theta$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \psi \leq p - 2$, $\theta(\psi + 1) > b$, or $\theta + b > p$. Then $d_b(\mathcal{C}) = b(\psi + 2)p^s$, and

\[
i \geq p^s - p^s - s + 1 + \psi p^s - s - 1 + \theta = p^s - (2p^s - 1) - p^s + \psi p^s - s - 1 + \theta \geq p^s - (2p^s - 1) - p^s + \psi p^s - s - 1 + \theta (equality when \epsilon = s - 2 or s = 2) \geq (\psi + 2)p(2p^s - 1) - p^s + \psi p^s - s - 1 + \theta (equality when \psi = p - 2) = (\psi + 2)p^{s+1} - p^s + \theta (equality when \psi = 0 and \epsilon = 0) \geq 2b(p + 2)p^s - p^s + \theta (equality when \theta = 0) = 2d_b(\mathcal{C}) - p^s + \theta > 2d_b(\mathcal{C}) - p^s - 2b. \]

Since, $i > 2d_b(\mathcal{C}) - p^s - 2b$, in this case no MDS $b$-symbol constacyclic code exist.

Case 4: $i = p^s - p + \varrho$, where $0 \leq \varrho \leq p - b$. Then $d_b(\mathcal{C}) = (\varrho + b)p^s - 1$, and

\[
i = p^s - p + \varrho p(2p^s - 1) - p^s + \varrho p^s + \varrho + \theta (equality when \varrho = p - b) = 2(\varrho + b)p^s - 1 - p^s - b \geq 2d_b(\mathcal{C}) - p^s - b \geq 2d_b(\mathcal{C}) - p^s - 2b. \]

Since, $i > 2d_b(\mathcal{C}) - p^s - 2b$, in this case no MDS $b$-symbol constacyclic code exist.

Case 5: $p^s - b + 1 \leq i \leq p^s - 1$, then $d_b(\mathcal{C}) = p^s$ and

\[
i = p^s - b + 1 \geq 2p^s - p^s + b \geq 2d_b(\mathcal{C}) - p^s - b + 1 \geq 2d_b(\mathcal{C}) - p^s - 2b. \]

Since, $i > 2d_b(\mathcal{C}) - p^s - 2b$, in this case no MDS $b$-symbol constacyclic code can be obtained.

This completes the proof.

square

3) TYPE 3 (PRINCIPAL IDEALS GENERATED BY MONIC POLYNOMIAL)

Here, $\mathcal{C} = \langle (x - \zeta_0)^i + u(x - \zeta_0)^i h(x) \rangle$, where $1 \leq i \leq p^s - 1$. Thus, the following two cases can be obtained:

Case 1: When $h(x) = 0$ then, $|\mathcal{C}| = p^{2m(p^s - i)}$. Thus by Singleton bound, $\mathcal{C}$ is a $b$-symbol MDS code if and only if $2(p^s - t) = 2(p^s - d_b(\mathcal{C}) + b)$, i.e., $i = d_b(\mathcal{C}) - b$. Hence, the MDS codes for these ideals are similar to the MDS codes obtained for $\delta$-constacyclic $b$-symbol codes over $\mathcal{F}_{pm}$. Hence, we have the following theorem:

**Theorem 4.3:** Let $\mathcal{C} = \langle (x - \zeta_0)^i \rangle \subseteq \mathbb{F}_{pm} / \langle x^s - \zeta \rangle$ be a $\zeta$-constacyclic code of length $p^s$ over $\mathcal{R}$, for $i \in \{1, \ldots, p^s - 1\}$. Then $\mathcal{C}$ is a MDS $b$-symbol code if and only if one of the following conditions holds:

- If $s = 1$, then $i = \varrho$, for $1 \leq \varrho \leq p - b$, then $d_b(\mathcal{C}) = \varrho + b$.
- If $s \geq 2$, then
  - $i = \varrho$ for $1 \leq \varrho \leq b$, then $d_b(\mathcal{C}) = \varrho + b$.
  - $i = p^s - b$, $d_b(\mathcal{C}) = p^s$.

Case 2: When $h(x)$ is a unit $[13]$ then,

\[
|\mathcal{C}| = \begin{cases} p^{2m(p^s - i)}, & \text{if } 1 \leq i \leq p^s - 1 + \lfloor \frac{i}{2} \rfloor \\ p^{m(p^s - i)}, & \text{if } p^s - 1 + \lfloor \frac{i}{2} \rfloor < i \leq p^s - 1. \end{cases}
\]
Therefore, when $1 \leq \tau \leq p^s - 1 + \lfloor \frac{1}{p^s} \rfloor$, MDS $b$-symbol constacyclic codes can be obtained when $\tau = d_b(\mathcal{C}) - b$. Hence, the MDS codes for these ideals are similar to the MDS codes obtained in the above case. But, we have $2\tau - 2p^{s-1} \leq \tau$ and $0 \leq \tau < n$, i.e., $p^{s-1} \leq \tau < 2p^{s-1}$. Thus, when $s = 1$, $\tau = 1$ and when $s \geq 2$, $2 \leq \tau < 2p^{s-1}$. Again, $p^s - b < 2p^{s-1}$ if and only if $p = 2$. Hence, we conclude the theorem as follows.

**Theorem 4.4:** Let $\mathcal{C} = \langle (x - \zeta^s) \rangle \subseteq \langle (x - \zeta^s) \rangle$ be a $\zeta$-constacyclic code of length $p^s$ over $\mathbb{F}_q$, for $1 \leq \tau \leq p^{s-1} + \lfloor \frac{1}{p^s} \rfloor$. Then $\mathcal{C}$ is a MDS $b$-symbol code if and only if one of the following conditions holds:

- If $s = 1$, then $\tau = \varrho$, for $1 \leq \varrho \leq p - b$, then $d_b(\mathcal{C}) = \varrho + b$.
- If $s \geq 2$, then
  - $\tau = \theta$ for $2 \leq \theta \leq b$, then $d_b(\mathcal{C}) = \theta + b$,
  - $\tau = 2s - b$, then $d_b(\mathcal{C}) = 2s$.

When $p^{s-1} + \lfloor \frac{1}{p^s} \rfloor < \tau \leq p^s - 1$, i.e., when $0 \leq \tau < 2s - 2p^{s-1}$, MDS $b$-symbol constacyclic codes can be obtained when $\tau = 2d_b(\mathcal{C}) - p^s - 2b$, i.e., when $2s - 2p^{s-1} > 2d_b(\mathcal{C}) - p^s - 2b$. In the following theorem we discuss the case when $2s - 2d_b(\mathcal{C}) - p^{s-1}(p - 2) - 2b$ and $\tau \geq 0$.

**Theorem 4.5:** Let $\mathcal{C} = \langle (x - \zeta^s) \rangle \subseteq \langle (x - \zeta^s) \rangle$ be a $\zeta$-constacyclic code of length $p^s$ over $\mathbb{F}_q$, for $p^{s-1} + \lfloor \frac{1}{p^s} \rfloor < \tau \leq p^s - 1$. Then, $\mathcal{C}$ is a MDS $b$-symbol code if and only if one of the following conditions:

- For $p \geq 3$, $\tau = p^s - b$, then $d_b(\mathcal{C}) = p^s$.
- $p^s - b + 1 \leq \tau \leq p^s - 1$ then $d_b(\mathcal{C}) = p^s$.

**Proof:** We get MDS codes for $2s - 2d_b(\mathcal{C}) - p^{s-1}(p - 2) - 2b$, where $p^{s-1} + \lfloor \frac{1}{p^s} \rfloor < \tau \leq p^s - 1$ and $\tau \geq 0$. For $p = 2$, the condition for MDS code is modified to $2s - 2d_b(\mathcal{C}) - 2b$ and $\tau \geq 0$. Now, by considering $d_b(\mathcal{C})$ for different ranges of $\tau$ from Theorem 3.3 we have the following cases:

**Case 1:** For $\tau = p^s - p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \psi \leq p - 2$, and $\theta = p + 1) \leq b$ and $\theta + b \leq p$. Then $d_b(\mathcal{C}) = \theta + b(\psi + 1)p^s$, and

$$2\tau = 2p^s - 2p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta$$

$$= 2p^{s-1}(p^{s-1} - 2) + 2b + 2(\psi + 1)p^s - 2$$

Now, we consider two sub-cases:

**Subcase 1.1:** For $p = 2$, we have $2\tau \geq 2d_b(\mathcal{C}) - 2b + 4(p^{s-2} - 2) + 2\theta + 2b$.

Now, for $\epsilon = 0$, we have $2\tau \geq 2d_b(\mathcal{C}) - 2b - 4 + 2\theta + 2b$. Thus, MDS $b$-symbol constacyclic code can be obtained when $\theta > 2 - b$, which is a contradiction, since we have the equality $\theta + b = 2$.

Again for $\epsilon \geq 1$, we have $2\tau \geq 2d_b(\mathcal{C}) - 2b + 2\theta + 2b > 2d_b(\mathcal{C}) - 2b$. Thus, MDS $b$-symbol constacyclic code can be obtained with equality $\epsilon = s - 2$, $\psi = 0$ and $\theta + b = 2$, i.e., when $\tau = 2s - 3$ and $d_b(\mathcal{C}) = 2s - 1$. Thus, $\tau = 2d_b(\mathcal{C}) - p^s - 2b = -2b < 0$, which is a contradiction.

Thus, MDS $b$-symbol constacyclic code can not be obtained in this case.

**Subcase 1.2:** For $p \geq 3$, we have

$$2\tau \geq 2d_b(\mathcal{C}) - p^s - 1(p - 2) + 2b + 2p(p^{s-2} - 2)$$

$$+ p^s - 1(p - 2) + 2b + 2\theta$$

$$> 2d_b(\mathcal{C}) - p^s - 1(p - 2) - 2b.$$

Thus, MDS $b$-symbol constacyclic code can be obtained with equality $\epsilon = s - 2$, $\psi = 0$ and $\theta + b = 2$. But $\theta(\psi + 1) \leq p^s - 2$ gives $p^s - 2 \leq 1$, which is a contradiction. Thus, MDS $b$-symbol constacyclic code can not be obtained in this case.

**Case 2:** $p^s - p^s - \epsilon + \psi p^s - \epsilon - 1 + \theta \geq p^s - p^s - \epsilon + (\psi + 1)p^s - \epsilon - 1$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \psi \leq p - 2$, $\theta(\psi + 1) > b$, or $\theta + b > p$. Then $d_b(\mathcal{C}) = b(\psi + 2)p^s$, and

$$2\tau \geq 2p^s - 2p^s - \epsilon + \psi p^s - \epsilon - 1 + 2\theta$$

$$= 2p^{s-1}(p^{s-1} - 2) + 2b + 2\theta$$

$$(\text{equality when } \epsilon = s - 2 \text{ or } s = 2)$$

$$\geq 2(\psi + 2)p^s - 1 + 2\psi p + 2\theta$$

$$(\text{equality when } \psi = 0)$$

$$\geq 2(\psi + 2)p^s - 1 + 2\psi p + 2\theta$$

$$(\text{equality when } \psi = 0)$$

$$\geq 2(\psi + 2)p^s - 1 + 2\psi p + 2\theta$$

$$(\text{equality when } \epsilon = 0)$$

$$= 2d_b(\mathcal{C}) - p^s - 1(p - 2) - 2b + 2p^s - 1(p - 2) + (\psi - 2)p + 2\theta$$

$$(\text{since } \theta + b > p)$$

$$> 2d_b(\mathcal{C}) - p^s - 1(p - 2) - 2b + 2p^s - 1(p - 2) + \psi p$$

$$(\text{since } \theta + b > p)$$

$$> 2d_b(\mathcal{C}) - p^s - 1(p - 2) - 2b.$$

Hence, $2\tau \geq 2d_b(\mathcal{C}) - p^s - 1(p - 2) - 2b$ with the equality $\epsilon = 0$, $s = 2$, $\psi = p - 2$ and $\theta = 2b$, i.e., when $\tau = p^s - 2p + \theta$ and $d_b(\mathcal{C}) = b.p^s - 1$. Thus, $\tau = 2d_b(\mathcal{C}) - p^s - 2b = 2b - p^s - 2b = -2b < 0$, which is a contradiction. Therefore, no MDS $b$-symbol constacyclic code can be obtained in this case.

**Case 3:** $\tau = p^s - p + 2$, where $0 \leq \varrho \leq p - b$. Then $d_b(\mathcal{C}) = (\varrho + b)p^s - 1$, and

$$2\tau = 2p^s - 2p + 2\varrho$$

$$= 2p(p^s - 1) + 2\varrho$$
4) TYPE 4 (NONPRINCIPAL IDEALS)

We have, \(\mathcal{C} = \{(x - \zeta_0)^i + u(x - \zeta_0)^j \mid h(x), u(x - \zeta_0)^k\}\), where \(1 \leq i \leq p^s - 1\), \(0 \leq \tau < i\), and either \(h(x)\) is either 0 or a unit in \(\frac{\mathbb{F}_{p^m}[x]}{(x - \zeta_0^s)}\). Thus, \(\mathcal{C}\) is a b-symbol MDS code if and only if \(2p^s - i - \kappa = 2(p^s - d_{\mathcal{C}} - b)\), i.e., \(\kappa = 2d_{\mathcal{C}}(C) - 2b - \tau\). Let \(\iota = p^s - \eta\), where \(1 \leq \eta \leq p^s - 1\). Thus, the condition for \(\mathcal{C}\) to be a b-symbol MDS constacyclic code becomes \(\kappa \geq 2d_{\mathcal{C}} - 2b - p^s + \eta\). Hence, we can conclude the following theorem:

**Theorem 4.6:** Let \(\mathcal{C} = \{(x - \zeta_0)^i + u(x - \zeta_0)^j \mid h(x), u(x - \zeta_0)^k\}\) be a \(\zeta\)-constacyclic code of length \(p^s\) over \(\mathbb{F}_p\), where \(i \leq 1, \ldots, p^s - 1\), \(0 \leq \tau < i\), and either \(h(x)\) is either 0 or a unit in \(\frac{\mathbb{F}_{p^m}[x]}{(x - \zeta_0^s)}\), \(\deg(h(x)) \leq \kappa - \tau - 1\) and \(\kappa < T\), where

\[
T = \begin{cases} 
\iota, & \text{if } h(x) = 0 \\
\min[\iota, p^s - i + \tau], & \text{if } h(x) \neq 0.
\end{cases}
\]

Then \(\mathcal{C}\) is not a MDS b-symbol constacyclic code.

**Proof:** We get MDS code for \(\kappa = 2d_{\mathcal{C}} - p^s - 2b + \eta\), where \(1 \leq \eta \leq p^s - 1\) and \(\kappa < \iota\). Now, we consider the cases according to the range of \(\kappa\).

**Case 1:** \(\kappa = p^s - p^s - \eta + \psi \geq 0\), where \(0 \leq \eta \leq p^s - 2\), \(\theta(\psi + 1) \leq b\) and \(\theta + b \leq p\). Then \(d_{\mathcal{C}}(C) = (\theta + b)(\psi + 1)p^s\), and

\[
\kappa = p^s - p^s - \eta + \psi \geq p^s - 2p^s - 1 - p^s + \psi p^s - \theta \geq 0.
\]

**Case 2:** \(\kappa = p^s - p^s - \eta + \psi < 0\), where \(0 \leq \eta \leq p^s - 2\), \(\theta(\psi + 1) > b\) and \(\theta + b > p\). Then \(d_{\mathcal{C}}(C) = (\theta + b)(\psi + 1)p^s\), and

\[
\kappa = p^s - p^s - \eta + \psi < 0.
\]

This completes the proof. \(\square\)

\(\eta \geq 2d_{\mathcal{C}} - p^s - 2b + \eta\) if and only if \(\eta \geq 2b\), i.e., \(\eta \leq \theta + 2b\). Thus, \(\mathcal{C}\) is a b-symbol constacyclic code if and only if \(\eta \geq 2b\), i.e., \(\eta \leq \theta + 2b\). Hence, \(\mathcal{C}\) is a b-symbol constacyclic code can obtained in this case.

**Case 2:** \(\kappa = p^s - p^s - \eta + \psi \leq 0\), where \(0 \leq \eta \leq p^s - 2\), \(\theta(\psi + 1) > b\) and \(\theta + b > p\). Then \(d_{\mathcal{C}}(C) = (\theta + b)(\psi + 1)p^s\), and

\[
\kappa = p^s - p^s - \eta + \psi \leq 0.
\]

This completes the proof. \(\square\)
Case 4: $p^5 - b + 1 \leq \kappa \leq p^5 - 1$, then $d_b(\mathcal{C}) = p^5$ and

$$\kappa = p^5 - b + 1 = 2p^5 - p^5 - b + 1.$$

$\kappa \geq 2d_b(\mathcal{C}) - p^5 - 2b + 1$ if and only if $-b + 1 \geq -2b + 1$. i.e., $\eta \leq b + 1$, i.e., $t = p^5 - b + 1$, $\kappa = p^5 - b + 1$. Clearly, $t < \kappa$, which is a contradiction. Hence, no MDS $b$-symbol constacyclic code can be obtained in this case.

This completes the proof. \(\square\)

Consequently, we have the list of all MDS $b$-symbol constacyclic codes of length $p^5$ over $\mathbb{F}_p$.

**Theorem 4.7:** Let $1 \leq b \leq \lfloor \frac{p^5}{5} \rfloor$ and $1 \leq \theta \leq p - 1$. Then all MDS $b$-symbol constacyclic codes of length $p^5$ over $\mathbb{F}_p$ are determined as follows:

- \((\phi + u\psi)\)-constacyclic codes: $\mathcal{C} = \langle (\phi_0 x - 1)^t \rangle \subseteq \mathbb{F}_p[x]$, where $0 \leq t \leq 2p^5$. Then $\mathcal{C}$ is a MDS $b$-symbol constacyclic code if and only if $t = 0$, i.e. (1), in such case $d_b(\mathcal{C}) = b$.

- For $\zeta$-constacyclic codes, there are four types of ideals:
  - Type 1 (trivial ideals): (1) is the only $b$-symbol constacyclic code with $d_b(\mathcal{C}) = b$.
  - Type 2 (principal ideals generated by nonmonic polynomial): $\mathcal{C} = \langle u(x - \zeta_0)^t \rangle$, where $0 \leq t \leq p^5 - 1$. No MDS $b$-symbol constacyclic codes can be obtained in this case.
  - Type 3 (principal ideals generated by monic polynomial): $\mathcal{C} = \langle (x - \zeta_0)^t + u(x - \zeta_0)^t \theta(x) \rangle$, where $1 \leq t \leq p^5 - 1$, $0 \leq \tau < t$, and either $\theta(x)$ is $0$ or a unit in $\mathbb{F}_p[x]$. Then $\mathcal{C}$ is a MDS $b$-symbol code if and only if $t = 0$, i.e. (1), in such case $d_b(\mathcal{C}) = b$.

- Type 4 (nonprincipal ideals): $\mathcal{C} = \langle (x - \zeta_0)^t + u(x - \zeta_0)^t \theta(x) \rangle$, where $1 \leq t \leq p^5 - 1$, $0 \leq \tau < t$, and either $\theta(x)$ is either $0$ or a unit in $\mathbb{F}_p[x]$.

No MDS $b$-symbol constacyclic codes can be obtained in this case.

We conclude this section by some examples of MDS $b$-symbol constacyclic codes.

**Example 4.8:** Let us consider the example 3.4. Here, we consider $b = 3$, i.e., 3-symbol $\delta$-constacyclic codes of length 49 over $\mathbb{F}_7 + u\mathbb{F}_7$, where $p = 7$, $m = 1$, $s = 2$ and $\delta$ is a unit in $\mathbb{F}_7 + u\mathbb{F}_7$ of the form $\phi + u\psi$ and $\zeta$, with $\phi$, $\psi$, $\zeta \in \mathbb{F}_7$. In example 3.4, we determined all the 3-symbol distances for all the ranges of $\tau$ and $\kappa$. From the values of $\tau$, $\kappa$ and $d_3(\mathcal{C})$, we have the following list of MDS $3$-symbol $\delta$-constacyclic codes.

- \((\phi + u\psi)\)-constacyclic codes: $\mathcal{C} = \langle (\phi_0 x - 1)^t \rangle$, where $0 \leq t \leq 98$. Here, the condition for MDS code is given by $t = 2d_3(\mathcal{C}) - 6$ and the only MDS code obtained is $t = d_3((1)) = 3$.

- For $\zeta$-constacyclic codes, there are four types of ideals:
  - Type 1 (trivial ideals): (1) is the only MDS code with $d_3((1)) = 3$.
  - Type 2 (principal ideals generated by the non monic polynomial): $\mathcal{C} = \langle (x - \zeta_0)^t \rangle$, where $0 \leq t \leq 48$. MDS $3$-symbol codes for these codes are obtained by the condition $t = 2d_3(\mathcal{C}) - 55$, which is not satisfied by any value of $t$ and $d_3(\mathcal{C})$. Thus, no MDS code is obtained in this case.
  - Type 3 (principal ideals generated by the monic polynomial): $\mathcal{C} = \langle (x - x_0)^t + u(x - \zeta_0)^t \theta(x) \rangle$, where $1 \leq t \leq 48$, $0 \leq \tau < t$, and either $\theta(x)$ is 0 or a unit in $\mathbb{F}_7[x]$. Then $\mathcal{C}$ is a MDS $b$-symbol code if and only if $t = 0$, i.e. (1), in such case $d_b(\mathcal{C}) = b$.

For $\theta(x)$ is 0, the MDS code condition is $t = d_3(\mathcal{C}) - 3$ and the MDS codes are:

- $t = 1$, then $d_3(\mathcal{C}) = 4$.
- $t = 2$, then $d_3(\mathcal{C}) = 5$.
- $t = 3$, then $d_3(\mathcal{C}) = 6$.
- $t = 46$, then $d_3(\mathcal{C}) = 49$.

For $\theta(x)$ is a unit, when $1 \leq t \leq 7 + \lfloor \frac{7}{3} \rfloor$, the MDS code condition is $t = d_3(\mathcal{C}) - 3$ and the MDS codes are:

- $t = 2$, then $d_3(\mathcal{C}) = 5$.
- $t = 3$, then $d_3(\mathcal{C}) = 6$.

For $\theta(x)$ is a unit, when $7 + \lfloor \frac{7}{3} \rfloor < t \leq 48$, the MDS code condition is $t = d_3(\mathcal{C}) - 41$ and $t \geq 0$ and the MDS codes are:

- $t = 46$, then $d_3(\mathcal{C}) = 49$.
- $47 \leq t \leq 48$, then $d_3(\mathcal{C}) = 49$.

- Type 4 (nonprincipal ideals): $\mathcal{C} = \langle (x - \zeta_0)^t + u(x - \zeta_0)^t \theta(x) \rangle$, where $1 \leq t \leq 48$, $0 \leq \tau < t$, and either $\theta(x)$ is either 0 or a unit in $\mathbb{F}_7[x]$.
deg(h(x)) \leq \kappa - \tau - 1, \text{ and }
\kappa < T = \begin{cases} 
\iota, & \text{if } h(x) = 0 \\
\min\{\iota, 49 - \iota + \tau\}, & \text{if } h(x) \neq 0.
\end{cases}

MDS 3-symbol codes in this case depend on \( \kappa \) and given by \( \kappa = 2d_b(\mathbb{C}) - \iota - 6 \). But, no values of \( \kappa, \iota \) or \( d_s(\mathbb{C}) \) satisfy the condition. So, no MDS 3-symbol constacyclic code is obtained in this case.

V. CONCLUSION

In this research article, all b-symbol distances of repeated-root constacyclic codes having length \( p^s \) have been determined over \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \), influenced by the notion of Dinh et al. \[18\], \[19\]. The b-symbol distances over \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \) are quite similar to those over \( \mathcal{F}_{p^m} \), with variable range of i for each type of ideal. All MDS b-symbol codes are determined as an application among repeated-root constacyclic codes of \( p^s \) length over the ring \( \mathcal{F}_{p^m} + u\mathcal{F}_{p^m} \) and consequently on \( \mathbb{R} \). It will be interesting to obtain some more optimal codes.

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