SPlicing Matroids

Joseph E. Bonin and William R. Schmitt

Abstract. We introduce and study a natural variant of matroid amalgams. For matroids $M(A)$ and $N(B)$ with $M.(A \cap B) = N.(A \cap B)$, we define a splice of $M$ and $N$ to be a matroid $L$ on $A \cup B$ with $L|A = M$ and $L.B = N$. We show that splices exist for each such pair of matroids $M$ and $N$; furthermore, there is a freest splice of $M$ and $N$, which we call the free splice. We characterize when a matroid $L(A \cup B)$ is the free splice of $L|A$ and $L.B$. We study minors of free splices and the interaction between free splice and several other matroid operations. Although free splice is not an associative operation, we prove a weakened counterpart of associativity that holds in general and we characterize the triples for which associativity holds. We also study free splice as it relates to various classes of matroids.

In memory of Tom Brylawski

1. Introduction

Considering a variation of the well-known problem of matroid amalgams led us to the new matroid construction that we develop in this paper. This construction turns out to have many attractive properties that have no counterparts for amalgams.

We first briefly recall matroid amalgams; for an excellent account of this topic, see [11, Section 12.4]. Given matroids $M(A)$ and $N(B)$ with $M.(A \cap B) = N.(A \cap B)$, an amalgam of $M$ and $N$ is a matroid $L$ on $A \cup B$ with $L|A = M$ and $L.B = N$. The most familiar special case is the generalized parallel connection, which glues two matroids together, in the freest possible way, along what is essentially a modular flat of one of the matroids [3, 11]. Apart from this atypically nice case, though, two matroids may have incompatible structure and thus have no amalgam. Also, pairs of matroids that have amalgams may have no freest amalgam.

Now consider a variation on this theme. Since deletion and contraction commute, if we are given a matroid $L(A \cup B)$, then the restriction $M = L|A$ and contraction $N = L.B$ have the property that $M.(A \cap B) = N.(A \cap B)$. Hence, starting with matroids $M(A)$ and $N(B)$ satisfying $M.(A \cap B) = N.(A \cap B)$, it is natural to ask if there is a matroid $L(A \cup B)$ such that $L|A = M$ and $L.B = N$; we call such a matroid $L$ a splice of $M$ and $N$. We show that, in contrast to the situation for amalgams, splices of such $M$ and $N$ always exist and, furthermore, there is always a freest splice; we call this matroid the free splice of $M$ and $N$ and denote it by $M \boxtimes N$. If $A \cap B = \emptyset$, then the free splice is the free product $M \boxtimes N$.

In this paper we introduce matroid splices and study the free splice operation in depth. Consistent with a common phenomenon in matroid theory, there are many ways that one may approach the free splice; we use the Higgs lift to provide an
efficient framework for developing these various approaches. A key component of Section 2, which treats preliminary matters, is a unified treatment of cryptomorphic formulations of the Higgs lift; this section also treats minors of Higgs lifts and the behavior of the Higgs lift with respect to the weak and strong orders. In Section 3, we define splices in general and show that they form a filter in a certain natural suborder of the weak order. In the main part of this section, we define the free splice as a particular Higgs lift, we show that it is indeed the freest splice, we show that it preserves the weak and strong orders, and we provide cryptomorphic formulations of this operation. One attractive result is the relation between free splice and duality: \((M \otimes N)^* = N^* \otimes M^*\). Using cyclic flats, the basic question of when a matroid \(L(A \cup B)\) is the free splice of \(L|A\) and \(L|B\) is answered in Section 4; a corollary of this work is that a matroid is irreducible with respect to free splice if and only if, for each ordered pair \(x, y\) of distinct elements, there is cyclic flat that contains \(x\) but not \(y\). In Section 5 we study the interaction between free splice and several other matroid operations; we show, for instance, that direct sums and generalized parallel connections of irreducible matroids are irreducible, we study minors of free splices, and we show that the free splice can be realized as the intersection of certain free products. Like the free product operation, free splice is noncommutative but, in contrast to both free product and direct sum, it is also nonassociative. In Section 6 we prove a weakened version of associativity for free splice and characterize triples of matroids for which associativity holds. We also describe the (very special) conditions under which free splice is commutative. In contrast to free product, free splice does not preserve many commonly studied matroid properties (e.g., being representable, transversal, or base-orderable); Section 7 contains examples addressing these points and gives a necessary condition for a minor-closed class of matroids to be closed under free splice.

We assume the reader is familiar with basic matroid theory. Background on particular topics can be found in \([11, 12, 13]\).

2. Preliminaries

The Higgs lift provides an efficient framework for defining the free splice and treating its cryptomorphic formulations. To pave the way, the Higgs lift is reviewed in the main part of this section and a unified account of its cryptomorphic formulations is presented. (Many of these formulations are known; others, such as that using cyclic flats, as well as the unifying perspective itself, may be new.) Similarly, results in this section on minors of Higgs lifts and the fact that the Higgs lift preserves the weak and strong orders will be used when treating corresponding results about free splice. A short section on submodular functions preceeds and prepares the way for the work on the Higgs lift.

2.1. Notation and terminology. A set \(X\) in a matroid \(M(E)\) is cyclic if \(X\) is a union of circuits of \(M\), that is, if \(M|X\) has no isthmuses. We write \(M^*\) for the dual of \(M\) and denote by \(\mathcal{I}(M)\), \(\mathcal{B}(M)\), \(\mathcal{C}(M)\), \(\mathcal{S}(M)\), \(\mathcal{F}(M)\), and \(\mathcal{Z}(M)\), respectively, the collections of independent sets, bases, circuits, spanning sets, flats, and cyclic flats of \(M\). For any family \(\mathcal{U}\) of subsets of \(E\), we write \(\mathcal{U}^c\) for the family \(\{E - X : X \in \mathcal{U}\}\). Since \(E - X\) is a flat of \(M^*\) if and only if \(M^*|X\) has no loops, that is, \(M|X\) has no isthmuses, it follows that \(\mathcal{F}(M^*)^c\) is the collection of cyclic sets of \(M\), so \(\mathcal{Z}(M) = \mathcal{F}(M) \cap \mathcal{F}(M^*)^c\). We denote the rank function of \(M\) by \(r_M\) and write \(r(M)\) for \(r_M(E)\). We write \(\text{Isth}(M)\) and \(\text{Loop}(M)\) for the sets of
Lemma 1. We provide a proof here for the convenience of the reader.

Lemma 2.1. Recall that a submodular function on a set $E$ is a function $f : 2^E \to \mathbb{Z}$ that satisfies $f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq E$; also, a submodular function $f$ is the rank function of a matroid on $E$ if and only if $0 \leq f(X) \leq |X|$, for all $X \subseteq E$, and $f$ is order-preserving (that is, $(f(X) \leq f(Y)$ for all $X \subseteq Y \subseteq E$). The following result was stated without proof in [12]. We provide a proof here for the convenience of the reader.

**Lemma 2.1.** If $f$ and $g$ are submodular functions on $E$ such that $f - g$ is order-preserving, then $h : 2^E \to \mathbb{Z}$ given by $h(X) = \min \{f(X), g(X)\}$ is submodular.

**Proof.** Let $\mathcal{J}_\leq = \{X \subseteq E : f(X) \leq g(X)\}$ and $\mathcal{J}_\geq = \{X \subseteq E : f(X) \geq g(X)\}$. Note that $\mathcal{J}_\leq \cup \mathcal{J}_\geq = 2^E$ and that, since $f - g$ is order-preserving, $\mathcal{J}_\leq$ is an ideal and $\mathcal{J}_\geq$ is a filter in $2^E$. If $X, Y \in \mathcal{J}_\leq$, then

$$h(X) + h(Y) = f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \geq h(X \cup Y) + h(X \cap Y),$$

and similarly for $X, Y \in \mathcal{J}_\geq$. If $X \in \mathcal{J}_\leq$ and $Y \in \mathcal{J}_\geq$, then $X \cup Y \in \mathcal{J}_\leq$ and $X \cap Y \in \mathcal{J}_\geq$. Using the fact that $(f - g)(Y) \geq (f - g)(X \cap Y)$, we thus have

$$h(X) - h(X \cup Y) = g(X) - g(X \cup Y) \geq g(X \cap Y) - g(Y) \geq f(X \cap Y) - f(Y) = h(X \cap Y) - h(Y).$$

Hence $h$ is submodular. \[\square\]

We note that if, in addition, one of the functions $f$ and $g$ is the rank function of a matroid and the other is nonnegative and order-preserving, then $h$ is the rank function of a matroid.

2.2. Submodular functions. Recall that a submodular function on a set $E$ is a function $f : 2^E \to \mathbb{Z}$ that satisfies $f(X) + f(Y) \leq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq E$; also, a submodular function $f$ is the rank function of a matroid on $E$ if and only if $0 \leq f(X) \leq |X|$, for all $X \subseteq E$, and $f$ is order-preserving (that is, $(f(X) \leq f(Y)$ for all $X \subseteq Y \subseteq E$). The following result was stated without proof in [12]. We provide a proof here for the convenience of the reader.

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$$h(X) + h(Y) = f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \geq h(X \cup Y) + h(X \cap Y),$$

and similarly for $X, Y \in \mathcal{J}_\geq$. If $X \in \mathcal{J}_\leq$ and $Y \in \mathcal{J}_\geq$, then $X \cup Y \in \mathcal{J}_\leq$ and $X \cap Y \in \mathcal{J}_\geq$. Using the fact that $(f - g)(Y) \geq (f - g)(X \cap Y)$, we thus have

$$h(X) - h(X \cup Y) = g(X) - g(X \cup Y) \geq g(X \cap Y) - g(Y) \geq f(X \cap Y) - f(Y) = h(X \cap Y) - h(Y).$$

Hence $h$ is submodular. \[\square\]

We note that if, in addition, one of the functions $f$ and $g$ is the rank function of a matroid and the other is nonnegative and order-preserving, then $h$ is the rank function of a matroid.

2.3. The Higgs lift. Suppose that $N \subseteq M$ in $\mathcal{M}(E)$ and $i \geq 0$. By Lemma 2.1 and the observation after it, the function $\min \{r_M, i + r_N\}$ is the rank function of a matroid, which we denote by $L^i_{N,M}$ and which is known as the $i$th Higgs lift of $N$ towards $M$. It is immediate that $L^0_{N,M} = N$, that $L^i_{N,M} = M$ for all $i \geq r(M) - r(N)$, and that $L^i_{N,M}$ is an elementary quotient of $L^{i+1}_{N,M}$ for all $i \geq 0$. We extend the definition of $L^i_{N,M}$ to all $i \in \mathbb{Z}$ by setting $L^i_{N,M} = N$ for $i < 0$. We define the family $\mathcal{J}_<(M, N) \subseteq 2^E$ by

$$\mathcal{J}^<(M, N) = \{X \subseteq E : r_M(X) - r_N(X) < i\},$$

all isthmuses and loops of $M$ and denote by $I(E)$ the free matroid on $E$; thus, $\text{Isth}(I(E)) = \text{Loop}(I^+(E)) = E.$

For any finite set $E$ we denote by $\mathcal{M}(E)$ the set of all matroids on $E$. The weak order on $\mathcal{M}(E)$, denoted by $\leq$, is given by $N \leq M$ if and only if $\mathcal{I}(N) \subseteq \mathcal{I}(M)$ or, equivalently, $r_N(X) \leq r_M(X)$ for all $X \subseteq E$. We say that $M$ is freeer than $N$ if $N \leq M$. Note that if $r(M) = r(N)$, then $N \leq M$ if and only if $\mathcal{B}(N) \subseteq \mathcal{B}(M)$; thus in this case, $N \leq M$ if and only if $N^* \leq M^*$. The strong order on the set of matroids $\mathcal{M}(E)$ is defined by setting $N \preceq M$ if and only if $\mathcal{F}(N) \subseteq \mathcal{F}(M)$. This relation is also described by saying that $N$ is a quotient of $M$, or that $M$ is a lift of $N$. It is well known that $N \leq M$ if and only if $r_N(Y) - r_N(X) \leq r_M(Y) - r_M(X)$ for all $X \subseteq Y \subseteq E$; it follows immediately that $N \leq M$ if and only if $M^* \leq N^*$ and that $N \leq M$ implies $N \leq M$. 


similarly define $\mathcal{J}_c, \mathcal{J}_s, \mathcal{J}_2, \mathcal{J}_c$, and let $\mathcal{J}_c = \{X \subseteq E : r_M(X) - r_N(X) = i + 1\}$. The rank function of $L = L^1_{N,M}$ is thus given, for $i \geq 0$, by

$$r_L(X) = \begin{cases} r_M(X) & \text{if } X \in \mathcal{J}_c, \\ r_N(X) + i & \text{if } X \in \mathcal{J}_s, \end{cases}$$

for all $X \subseteq E$. Note that, since $N$ is a quotient of $M$ and thus $r_M - r_N$ is order-preserving, $\mathcal{J}_< \text{ and } \mathcal{J}_\leq$ are order ideals, and $\mathcal{J}_>$ and $\mathcal{J}_\geq$ are order filters, in the Boolean algebra $2^E$.

The following result about the dual of a Higgs lift is used extensively throughout this paper.

**Proposition 2.2.** If $N \leq M$ and $i + j = r(M) - r(N)$, then $(L^1_{N,M})^* = L^1_{M^*,N^*}$ and $\mathcal{J}^2_c(M,N)^c = \mathcal{J}^2_c(N^*,M^*)$.

**Proof.** The claim is trivially true unless $0 \leq i \leq r(M) - r(N)$, so assume this inequality holds. Let $L = L^1_{N,M}$ and $L' = L^1_{M^*,N^*}$. For all $X \subseteq E$, we have

$$r_{L'}(X) = \min\{r_{M^*}(X), r_{M^*}(X) + j\} = |X| - r(N) - i + \min\{r_M(E - X), r_N(E - X) + i\} = |X| - r(L) + r_L(E - X) = r_{L^*}(X),$$

and so $L' = L^*$. A straightforward computation gives

$$r_{M^*}(X) - r_{M^*}(X) - j = -(r_M(E - X) - r_N(E - X) - i)$$

for all $X \subseteq E$. Hence $X \in \mathcal{J}^2_c(N^*,M^*)$ if and only if $E - X \in \mathcal{J}^2_c(M,N)$, that is, if and only if $X \in \mathcal{J}^2_c(M,N)^c$. \hfill \square

The same argument also gives $\mathcal{J}^2_c(M,N)^c = \mathcal{J}^2_c(N^*,M^*)$ and $\mathcal{J}^2_c(M,N)^c = \mathcal{J}^2_c(N^*,M^*)$.

In the next section we define, in terms of Higgs lifts, the free splice operation on matroids. Suitably interpreting the next result then gives the various equivalent formulations of this new operation.

**Theorem 2.3.** If $N \leq M$ and $L = L^1_{N,M}$ with $i \geq 0$, then

1. $\mathcal{I}(L) = \mathcal{I}(M) \cap \mathcal{J}_c$,
2. $\mathcal{S}(L) = \mathcal{S}(N) \cap \mathcal{J}_c$,
3. $\mathcal{B}(L) = \mathcal{B}(M) \cap \mathcal{S}(N) \cap \mathcal{J}_c$,
4. $\mathcal{C}(L) = (\mathcal{C}(M) \cap \mathcal{J}_c) \cup (\mathcal{I}(M) \cap \mathcal{F}(N^*)^c \cap \mathcal{J}_c)$,
5. $\mathcal{cl}_L(X) = \begin{cases} \mathcal{cl}_M(X) & \text{if } X \in \mathcal{J}_c, \\ \mathcal{cl}_N(X) & \text{if } X \in \mathcal{J}_s, \end{cases}$

for all $X \subseteq E$,
6. $\mathcal{F}(L) = (\mathcal{F}(M) \cap \mathcal{J}_c) \cup \mathcal{F}(N)$,
7. $\mathcal{Z}(L) = (\mathcal{Z}(M) \cup \mathcal{J}_c) \cap (\mathcal{Z}(N) \cup \mathcal{J}_c) = (\mathcal{Z}(M) \cap \mathcal{J}_c) \cup (\mathcal{Z}(N) \cap \mathcal{J}_c) \cup (\mathcal{Z}(M) \cap \mathcal{Z}(N))$.

**Proof.** A set $X \subseteq E$ belongs to $\mathcal{I}(L)$ if and only if $|X| = \min\{r_M(X), r_N(X) + i\}$. Since $r_M(X) \leq |X|$, this is the case if and only if $|X| = r_M(X) \leq r_N(X) + i$, that is, if and only if $X \in \mathcal{I}(M) \cap \mathcal{J}_c$, so (1) holds. Statement (2) is dual to (1), following
from Proposition 2.2 and the fact that $S(L) = I(L^*)^c$, and (3) is immediate from (1) and (2).

It is immediate from (1) that if $X \in J_c$, then $X \in C(L)$ if and only if $X \in C(M)$. For $X \in J_c$, we have $X \in C(L)$ if and only if $X \in T(M)$ and $r_M(X) \leq r_N(X - x) + i$ for all $x \in X$. Since $r_M(X) > r_N(X) + i$, this is equivalent to $X \in T(M)$ and $r_N(X) = r_N(X - x)$ for all $x \in X$. Hence (4) follows.

Suppose $X \subseteq E$ and $Y = X \cup x$ for some $x \in E$. If $X \in J_c$, then $Y \in J_c$; thus $r_L(Y) - r_L(X) = r_M(Y) - r_M(X)$, so $x \in \text{cl}_L(X)$ if and only if $x \in \text{cl}_M(X)$. If $X \in J_c$, then $Y \in J_c$, since $J_c$ is an order filter; thus $r_L(Y) - r_L(X) = r_N(Y) - r_N(X)$, and so $x \in \text{cl}_L(X)$ if and only if $x \in \text{cl}_M(X)$. Thus (5) holds. It is immediate from (5) that $F(L^*) = (F(M) \cap J_c) \cup (F(N) \cap J_c)$. The fact that $N$ is a quotient of $M$ gives $F(N) \subseteq F(M)$, so (6) follows.

By Proposition 2.2, we have $F(L^*) = F(L^*_{r - r(N)})$, where $j = r(M) - r(N) - i$, and so $F(L^*) = (F(N^*) \cap J_c(N^*, M^*)) \cup F(M^*) = (F(N^*) \cap J_c(M^*, N^*) \cup F(M^*)$. Hence

$$Z(L) = F(L) \cap F(L^*)^c$$

$$= (F(M) \cap J_c) \cup F(N)) \cap ((F(N^*)^c \cap J_c) \cup F(M^*)^c)$$

$$= (F(M) \cap J_c \cap F(M^*)^c) \cup (F(N) \cap F(N^*)^c \cap J_c) \cup (U(N) \cap F(M^*)^c).$$

Since $N \leq M$, and thus $M^* \leq N^*$, we have $F(N) \subseteq F(M)$ and $F(M^*) \subseteq F(N^*)$. Hence $F(N) \cap F(M^*)^c = F(M) \cap F(M^*)^c \cap F(N) \cap F(N^*)^c = Z(M) \cap Z(N)$, and thus

$$Z(L) = (Z(M) \cap J_c) \cup (Z(N) \cap J_c) \cup (Z(M) \cap Z(N)).$$

□

For $N \leq M$, the interval from $N$ to $M$ in the strong order, which we denote by $[N, M]_{\downarrow}$, is graded of rank $r(M) - r(N)$. The following characterization of the Higgs lift was given in [10].

**Proposition 2.4.** For $N(E) \leq M(E)$ and $0 \leq i \leq r(M) - r(N)$, the Higgs lift $L^i_{N, M}$ is the maximum element in the weak order on \{ $L \in [N, M]_{\downarrow}$ : $r(L) - r(N) = i$ \}.

**Proof.** Suppose $L \in [N, M]_{\downarrow}$ with $r(L) - r(N) = i$. Let $X \subseteq E$. Since $N \leq L$, the function $r_L - r_N$ is order preserving, so $r_L(X) - r_N(X) \leq r_L(E) - r_N(E) = i$, that is, $r_L(X) \leq r_N(X) + i$. Since $L \leq M$, we have $L \leq M$, so $r_L(X) \leq r_M(X)$. Hence $r_L(X) \leq \min\{r_M(X), r_N(X) + i\} = r_{L^i_{N, M}}(X)$, so $L \leq L^i_{N, M}$. □

We next show that the Higgs lift preserves the strong and weak orders.

**Proposition 2.5.** Suppose that $N(E) \leq M(E)$ and $N'(E) \leq M'(E)$.

(1) If $N \leq N'$ and $M \leq M'$, then $L^i_{N, M} \leq L^i_{N', M'}$.

(2) If $N \leq N'$ and $M \leq M'$, then $L^i_{N, M} \leq L^k_{N', M'}$ where $k = i + r(M') - r(M)$.

**Proof.** Suppose $N \leq N'$ and $M \leq M'$. By part (1) of Theorem 2.3, it suffices to show that $I(M) \cap J_c(M, N) \leq J_c(M', N')$ in order to prove that $L^i_{N, M} \leq L^k_{N', M'}$. To see the inclusion, note that if $X \in I(M) \cap J_c(M, N)$, then $|X| = r_M(X) = r_M(X)$ and $i \geq |X| - r_N(X) \geq |X| - r_N(X)$, and thus $X \in J_c(M', N')$.

It is immediate from part (6) of Theorem 2.3 that $F(N) \subseteq F(N')$ implies $F(L^i_{N, M}) \subseteq F(L^i_{N', M'})$, that is, $N \leq N'$ implies $L^i_{N, M} \leq L^i_{N', M'}$. Now suppose that
Let $j = r(M) - r(N) - i$. Using this result, together with Proposition 2.2 applied twice, we have

$$L^i_{N,M} = (L^j_{M,N^*})^* \leq (L^j_{(M')^*,N^*})^* = L^k_{N,M'},$$

where $k = r(N^*) - r((M')^*) - j = i + r(M') - r(M)$.

We end this section with a result that expresses minors of Higgs lifts as Higgs lifts of corresponding minors. Recall that, by convention, $L^i_{N,M} = N$, for $i < 0$.

**Proposition 2.6.** If $N$ and $M$ are matroids on $E$ with $N \leq M$, then for all $A \subseteq E$ and $i \in \mathbb{Z}$,

$$L^i_{N,M}|A = L^i_{N|A,M|A} \quad \text{and} \quad L^i_{N,M}/A = L^i_{N|A,M/A} - k,$$

where $k = r_M(A) - r_N(A)$.

**Proof.** The result obviously holds for $i < 0$ so assume $i \geq 0$. Let $L = L^i_{N,M}$. If $X \subseteq A$, then

$$r_{L|A}(X) = r_L(X) = \min\{r_M(X), r_N(X) + i\} = \min\{r_{M|A}(X), r_{N|A}(X) + i\} = r_{L_{N|A,M|A}}(X),$$

and so the first equality in the theorem holds. The second equality follows from the first by duality, as follows: using Proposition 2.2 twice, we have

$$L^i_{N,M}/A = ((L^j_{N,M})^* \setminus A)^* = (L^j_{M,N^* \setminus A})^* = (L^j_{M,N^* \setminus A, N^* \setminus A})^* = L^j_{N^* \setminus A, (M^* \setminus A)^*} = L^j_{N/A,M/A},$$

where $j = r(M) - r(N) - i$ and

$$\ell = r(N^* \setminus A) - r(M^* \setminus A) - j = i - r_M(A) + r_N(A).$$

Thus the second equality in the theorem holds.

3. **Splices**

In the first subsection below we introduce splices of matroids and show that all splices of $M(A)$ and $N(B)$ lie in a certain interval $[N_0, M_1]_{\preceq}$ in the strong order on $\mathcal{M}(A \cup B)$; indeed, all splices have the same rank, and, if we order the matroids in $[N_0, M_1]_{\preceq}$ of this rank by the weak order, then the splices are a filter. In the second subsection we show that this filter is nonempty and hence $M$ and $N$ have a freest splice, given by a Higgs lift of $N_0$ towards $M_1$; this is the free splice $M \otimes N$. This subsection also treats duals of free splices, the effect of the free splice operation on the weak and strong orders, and a variety of equivalent formulations of the free splice.
3.1. Matched matroids and splices. We say that matroids \( M(A) \) and \( N(B) \) are matched or, more precisely, that the ordered pair \((M, N)\) is matched, if \( M \) and \( N \) agree on \( A \cap B \) in the sense that \( M.(A \cap B) = N|(A \cap B) \). In particular, any pair of matroids on disjoint sets is matched; also, the pair \((M, N)\) is matched if and only if \((N^*, M^*)\) is matched. If \( M(A) \) and \( N(B) \) are matched, then a matroid \( L \) on \( A \cup B \) is called a splice of \( M \) and \( N \) if \( M = L|A \) and \( N = L.B \). We write \( \text{Sp}(M, N) \) for the set of all splices of \( M \) and \( N \). Note that, for any matroid \( L(A \cup B) \), the matroids \( L|A \) and \( L.B \) are matched and \( L \in \text{Sp}(L|A, L.B) \). Also note that \( L \in \text{Sp}(M, N) \) if and only if \( L^* \in \text{Sp}(N^*, M^*) \), and that the rank of any \( L \in \text{Sp}(M, N) \) is equal to \( r_M(A - B) + r_N = r(M) + r(N) - r_N(A \cap B) \).

Figure 1 shows matched matroids \( M \) and \( N \), together with the set \( \text{Sp}(M, N) \), ordered by the weak order. The freest of these splices is the 3-whirl, \( W^3 \). Section 7 contains additional examples of splices.

If \( M(A) \) and \( N(B) \) are matched, we denote by \( M_1 \) the matroid \( M \oplus I(B - A) \), obtained by adjoining the elements of \( B - A \) to \( M \) as isthmuses, and by \( N_0 \) the matroid \( N \oplus I^*(A - B) \), obtained by adjoining the elements of \( A - B \) to \( N \) as loops. Note that \( r_{M_1}(X) = r_M(X \cap A) + |X - A| \) and \( r_{N_0}(X) = r_N(X \cap B) \) for all \( X \subseteq A \cup B \).

**Proposition 3.1.** If \( M(A) \) and \( N(B) \) are matched, then \( N_0 \preceq M_1 \).

**Proof.** If \( F \in \mathcal{F}(N) \), then \( F \cap A \in \mathcal{F}(N|(A \cap B)) = \mathcal{F}(M.(A \cap B)) \); therefore \((A - B) \cup (F \cap A) \in \mathcal{F}(M) \). The inclusion \( B - A \subseteq \text{Isth}(M_1) \) now gives \((A - B) \cup F \in \mathcal{F}(M_1) \) for any \( F \in \mathcal{F}(N) \). Since any flat of \( N_0 \) has the form \((A - B) \cup F \) for some \( F \in \mathcal{F}(N) \), we get \( \mathcal{F}(N_0) \subseteq \mathcal{F}(M_1) \). \( \Box \)

The next result identifies some order-theoretic structure of the set of splices of two matched matroids.

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**Figure 1.** Matched matroids \( M \) and \( N \), along with their splices; the grey lines indicate the weak order.
**Proposition 3.2.** If \( M(A) \) and \( N(B) \) are matched, then \( \text{Sp}(M,N) \) is a filter in the weak order on the set \( L = \{ L \in [N_0, M_1] \subseteq : r(L) - r(N) = r_M(A - B) \} \).

**Proof.** Let \( L \in \text{Sp}(M,N) \). Since \( L|A = M \), we have \( F(M) = \{ X \cap A : X \in F(L) \} \) and hence \( F(L) \subseteq F(M) = \{ X \subseteq A \cup B : X \cap A \in F(M) \} \), that is, \( L \subseteq M_1 \). Now \( L \in \text{Sp}(M,N) \) gives \( L^* \in \text{Sp}(N^*, M^*) \), hence \( L^* \leq (N^*)_1 = (N_0)^* \) or, equivalently, \( N_0 \leq L \). It is clear that \( r(L) - r(N_0) = r_M(A - B) \); hence \( \text{Sp}(M,N) \subseteq L \).

Suppose that \( L \in \text{Sp}(M,N) \) and that \( P \in L \) satisfies \( L \leq P \). Restriction preserves both the weak and strong orders; hence \( M = L|A \leq P|A = M_1|A = M \), so \( P|A = M \). Since duality reverses strong order, and preserves weak order for matroids of the same rank, we have \( L^* \leq P^* \leq (N_0)^* = (N^*)_1 \) and so the same argument gives \( P^*B = N^* \), that is, \( P.B = N \). Hence \( P \in \text{Sp}(M,N) \).  

**3.2. The free splice.** Given matched matroids \( M(A) \) and \( N(B) \), we denote by \( M \times N \) the Higgs lift \( L^i_{N_0,M_1} \), where \( i = r_M(A - B) \). We will show that \( M \times N \) is the freest splice of \( M \) and \( N \). Denote by \( \mathcal{J}_z \) or \( \mathcal{J}_z(M,N) \), the order ideal \( \mathcal{J}_z(M_1,N_0) \), and similarly define \( \mathcal{J}_z, \mathcal{J}_z, \mathcal{J}_z, \) and \( \mathcal{J}_z \). Hence

\[
\mathcal{J}_z = \{ X \subseteq A \cup B : r_M(X) - r_{N_0}(X) < r_M(A - B) \},
\]

that is,

\[
\mathcal{J}_z = \{ X \subseteq A \cup B : r_M(X \cap A) + |X - A| < r_N(X \cap B) + r_M(A - B) \}.
\]

The rank function of \( M \times N \) is thus given by

\[
(3.1) \quad r_{M \times N}(X) = \min\{r_M(X \cap A) + |X - A|, r_N(X \cap B) + r_M(A - B)\}
\]

\[
= \begin{cases} r_M(X \cap A) + |X - A| & \text{if } X \in \mathcal{J}_z, \\ r_N(X \cap B) + r_M(A - B) & \text{if } X \in \mathcal{J}_z, \end{cases}
\]

for all \( X \subseteq A \cup B \).

Note that \( \mathcal{J}_z(N^*, M^*) = \mathcal{J}_z((N^*)_1, (M^*)_0) \), where \( j = r_N((B - A)) \). It follows that \( i + j = r(M_1) - r(N_0) \), and so by Proposition 2.2 we have the following lemma.

**Lemma 3.3.** If \( M \) and \( N \) are matched, then \( \mathcal{J}_z(N^*, M^*) = \mathcal{J}_z(M, N)^c \).

Given sets \( A \subseteq B \subseteq E \), we use the standard notation \([A, B]\) for the interval \( \{X : A \subseteq X \subseteq B\} \) in the Boolean algebra \( 2^E \).

**Lemma 3.4.** If \( M(A) \) and \( N(B) \) are matched, then

1. \( [0, A] \subseteq \mathcal{J}_z \); also \( X \in [0, A] \) belongs to \( \mathcal{J}_z \) if and only if \( A - B \subseteq \text{cl}_M(X) \),
2. \( [A - B, A \cup B] \subseteq \mathcal{J}_z \); also \( X \in [A - B, A \cup B] \) belongs to \( \mathcal{J}_z \) if and only if \( B - A \subseteq \text{cl}_N(B - X) \), that is, if and only if \( X - A \subseteq \text{Isth}(N)(X \cap B) \),
3. \( [A - B, A] \subseteq \mathcal{J}_z \).

**Proof.** If \( X \subseteq A \), then \( r_{M_1}(X) = r_M(X) \) and

\[
r_{N_0}(X) = r_N(X \cap B)
\]

\[
= r_{N|\langle A \cap B \rangle}(X \cap B)
\]

\[
= r_{M|\langle A \cap B \rangle}(X \cap B)
\]

\[
= r_M(X) - r_M(A - B).
\]

Since \( r_M(X) \leq r_M(X \cup (A - B)) \), we thus have \( r_M(X) \leq r_{N_0}(X) + r_M(A - B) \), with equality if and only if \( A - B \subseteq \text{cl}_M(X) \). Hence (1) holds. The second statement is the dual of the first: if \( A - B \subseteq X \), then \( (A \cup B) - X \subseteq B \) and so, by part (1)
together with Lemma 3.3 we have \((A \cup B) - X \in \mathcal{J}_c(N^*, M^*) = \mathcal{J}_c(M, N)^c\), that is, \(X \in \mathcal{J}_c\) with equality if and only if \(B - A \subseteq \text{cl}_{N^*}(B - X)\). The third statement is an obvious consequence of the first two.

**Corollary 3.5.** If \((M, N)\) are matched, then a matroid \(L(A \cup B)\) belongs to \(\text{Sp}(M, N)\) if and only if \(r_L(X) = r_{M \otimes N}(X)\) for all \(X \in [\emptyset, A] \cup [A - B, A \cup B]\).

**Proof.** Lemma 3.4 and Equation 3.1 give \(r_{M \otimes N}(X) = r_M(X)\) for all \(X \subseteq A\) and \(r_{M \otimes N}(X) = r_M(X \cap B) + r_M(A - B)\) for all \(X \subseteq A - B\). However, \(L \in \text{Sp}(M, N)\) if and only if \(L|A = M\) and \(L|B = N\), that is, if and only if \(r_L(X) = r_M(X)\) for all \(X \subseteq A\) and \(r_L(X) = r_M(X \cap B) + r_M(A - B)\) for all \(X \subseteq A - B\).

**Theorem 3.6.** If \(M\) and \(N\) are matched, then \(M \otimes N\) is the freest splice of \(M\) and \(N\), that is, \(M \otimes N\) is the maximum element of \(\text{Sp}(M, N)\) in weak order.

**Proof.** It is immediate from Corollary 3.5 that \(M \otimes N \in \text{Sp}(M, N)\). Hence the result follows directly from Propositions 2.4 and 3.4.

We refer to \(M \otimes N\) as the *free splice* of \(M\) and \(N\). Note that if \(A \subseteq B\), and so \(M = N|A\), then \(M \otimes N = N\) is the unique splice of \(M\) and \(N\). Similarly, if \(B \subseteq A\) then \(M \otimes N = M\) is the unique splice of \(M\) and \(N\). If \(A\) and \(B\) are disjoint, then the free splice \(M \otimes N\) is the *free product* \(M \boxtimes N\) of \(M\) and \(N\), which was introduced in [5]. In this case, the set \(\text{Sp}(M, N)\) has minimum element \(M \oplus N\). The following example shows that, in general, \(\text{Sp}(M, N)\) does not have a unique minimal element.

**Example 3.7.** Let \(M\) be the uniform matroid \(U_{2,3}\) on \(\{a, b, c\}\). Let \(N\) be the rank-2 matroid on \(\{b, c, d, e\}\) in which \(\{b, c\}\) is the only nonspanning circuit. The free splice of \(M\) and \(N\) is a plane on \(\{a, b, c, d, e\}\) in which \(\{a, b, c\}\) is the only nonspanning circuit. There are only two other splices of \(M\) and \(N\); one has \(b, d, e\) collinear and the other has \(c, d, e\) collinear. Thus, \(\text{Sp}(M, N)\) has more than one minimal element.

The next result gives the simple relation between the free splice and duality.

**Proposition 3.8.** If \((M, N)\) are matched, then \((M \otimes N)^* = N^* \otimes M^*\).

**Proof.** The result is immediate from Proposition 2.2 and Lemma 3.3. Alternatively, it follows from Theorem 3.6 and the fact that the duality operator is a poset isomorphism \(\text{Sp}(M, N) \rightarrow \text{Sp}(N^*, M^*)\).

The following proposition shows that the free splice operation preserves the weak order and that the operator defined by taking the free splice with a fixed matroid, on either the left or the right, preserves the strong order.

**Proposition 3.9.** Suppose that \((M, N)\) are matched.

1. If \((M', N')\) is matched, with \(M \leq M'\) and \(N \leq N'\), then \(M \otimes N \leq M' \otimes N'\).
2. If \((M, N')\) is matched and \(N \leq N'\), then \(M \otimes N \leq M \otimes N'\).
3. If \((M', N)\) is matched and \(M \leq M'\), then \(M \otimes N \leq M' \otimes N\).

**Proof.** Suppose \((M', N')\) is matched. If \(M \leq M'\) and \(N \leq N'\), then \(M \otimes N \leq (M')_1\) and \(N \otimes N \leq (N')_1\); also, \(r_M(A - B) \leq r_{M'}(A - B)\). Thus, statement (1) follows immediately from part (1) of Proposition 2.6.
If \((M, N)\) and \((M, N')\) are matched and \(N \subseteq N'\), then \(N_0 \subseteq (N')_0\), so statement (2) follows from part (2) of Proposition 2.5. Now if \((M', N)\) is matched and \(M \subseteq M'\), then \((M')^* \subseteq M^*\), so by part (2) and Proposition 3.8 we have
\[
(M' \bowtie N)^* = N^* \bowtie (M')^*
\]
\[
\subseteq N^* \bowtie M^*
\]
\[
= (M \bowtie N)^* ,
\]
and hence \(M \bowtie N \subseteq M' \bowtie N\).

The equivalent formulations of the Higgs lift in Theorem 2.3 yield the following equivalent formulations of the free splice.

**Theorem 3.10.** Suppose that \(M(A)\) and \(N(B)\) are matched. Let \(L = M \bowtie N\), \(E = A \cup B\), \(S = A - B\), and \(r = r(L) = r_M(S) + r(N)\).

1. \(I(L) = \{X \subseteq E : X \cap A \in I(M) \text{ and } |X| \leq r_m(S) + r_N(X \cap B)\}\).

2. \(S(L) = \{X \subseteq E : X \cap B \in S(N) \text{ and } r_m(X \cap A) + |X - A| \geq r\}\).

3. \(B(L) = \{X \subseteq E : |X| = r, X \cap A \in I(M), \text{ and } X \cap B \in S(N)\}\).

4. \(C(L) = C(M) \cup C, \text{ where } C \text{ is the set of all } X \subseteq E \text{ such that } X \cap A \in I(M) \text{ and } \text{Isth}(N)(X \cap B) = \emptyset, \text{ with } r_N(X \cap B) = |X| - r_m(S) - 1\).

5. \(\text{cl}_L(X) = \begin{cases} \text{cl}_m(X \cap A) \cup X & \text{if } r_m(X \cap A) + |X - A| < r_m(S) + r_N(X \cap B), \\ \text{cl}_m(X \cap B) \cup S & \text{otherwise,} \end{cases}\)

for all \(X \subseteq E\).

6. \(F(L) = F_1 \cup F_2\), where \(F_2 = \{S \cup F : F \in F(N)\}\) and \(F_1\) is the family \(\{X \subseteq E : X \cap A \in F(M) \text{ and } r_m(X \cap A) + |X - A| < r_m(S) + r_N(X \cap B)\}\).

7. \(Z(L) = Z_1 \cup Z_2 \cup Z_3\), where
\[
Z_1 = \{Z \in Z(M) : S \notin Z\},
\]
\[
Z_2 = \{Z \in Z(M) : S \subseteq Z \text{ and } Z \cap B \in Z(N)\},
\]
\[
Z_3 = \{Z \subseteq E : S \subseteq Z \notin A \text{ and } Z \cap B \in Z(N)\}.
\]

**Proof.** Each statement of the theorem follows from its counterpart in Theorem 2.3 with \(M_1\) and \(N_0\) replacing \(M\) and \(N\), together with the observations below.

1. \(I(M_1) = \{X \subseteq E : X \cap A \in I(M)\}\); also, for \(X \in I(M_1)\), we have \(X \in J_\leq\) if and only if \(|X| \leq r_N(X \cap B) + r_m(S)\).

2. \(S(N_0) = \{X \subseteq E : X \cap B \in S(N)\}\), and if \(X \cap B \in S(N)\), then \(r_N(X \cap B) + r_m(S) = r\).

3. The statement is immediate from the previous two.

4. Note that \(C(M_1) = C(M)\) and \(C(M) \subseteq J_\leq\) by part (1) of Lemma 3.4. Also, \(E - X \in F((N_0)^*)\) if and only if \(X\) is a union of circuits of \(N_0\), that is, if and only if \(\text{Isth}(N)(X \cap B) = \emptyset\). If \(X \in I(M_1)\), that is, if \(X \cap A \in I(M)\), then \(X \in J_\leq\) if and only if \(|X| - 1 = r_N(X \cap B) + r_m(S)\).
(5) \( \text{cl}_{M_1}(X) = \text{cl}_M(X \cap A) \cup X \) and \( \text{cl}_{N_0}(X) = \text{cl}_N(X \cap B) \cup S \) for all \( X \subseteq E \).

(6) \( \mathcal{F}(M_1) = \{ X \subseteq E : X \cap A \in \mathcal{F}(M) \} \) and \( \mathcal{F}(N_0) = \mathcal{F}_2 \).

(7) \( \mathcal{Z}(M_1) = \mathcal{Z}(M) \) and \( \mathcal{Z}(N_0) = \{ Z \cup S : Z \in \mathcal{Z}(N) \} \), so \( \mathcal{Z}(M_1) \cap \mathcal{Z}(N_0) = \mathcal{Z}_2 \).

By Lemma 3.3 we have \( \mathcal{Z}(M_1) \subseteq \mathcal{J}_\mathcal{Z} \) and \( \mathcal{Z}(N_0) \subseteq \mathcal{J}_\mathcal{Z} \). Lemma 3.4 also tells us that if \( X \subseteq A \) is closed in \( M \), then \( X \in \mathcal{J}_\mathcal{Z} \) if and only if \( S \not\subseteq X \), and that if \( X \supseteq S \) is such that \( X \cap B \) is cyclic in \( N \), then \( X \in \mathcal{J}_\mathcal{Z} \) if and only if \( X \not\subseteq A \).

Hence \( \mathcal{Z}(M_1) \cap \mathcal{J}_\mathcal{Z} = \mathcal{Z}_1 \) and \( \mathcal{Z}(N_0) \cap \mathcal{J}_\mathcal{Z} = \mathcal{Z}_3 \).

\[ \square \]

The following characterization of loops in free splices is immediate from part (5) of Theorem 3.10 or directly from Equation 3.1; duality gives the characterization of isthmuses.

**Corollary 3.11.** If \( M(A) \) and \( N(B) \) are matched, then
\[
\text{Loop}(M \Join N) = \begin{cases} 
\text{Loop}(M) \cup \text{Loop}(N) & \text{if } A - B \subseteq \text{Loop}(M), \\
\text{Loop}(M) & \text{otherwise}
\end{cases}
\]
and
\[
\text{Isth}(M \Join N) = \begin{cases} 
\text{Isth}(M) \cup \text{Isth}(N) & \text{if } B - A \subseteq \text{Isth}(N), \\
\text{Isth}(N) & \text{otherwise}.
\end{cases}
\]

### 4. Factorization with respect to free splice

If a matroid \( L(E) \) factors with respect to the free splice operation, the factorization is necessarily of the form \( L = L_1 \Join L_2 \) for some ordered pair \((A, B)\) of subsets of \( E \) with \( A \cup B = E \). In this case we call the pair \((A, B)\) a **free separator** of \( L \). A free separator \((A, B)\) is **nontrivial** if both \( A - B \) and \( B - A \) are nonempty. A matroid is **reducible** if it has a nontrivial free separator; otherwise it is **irreducible**.

In this section we present a number of results on the structure of free separators. In particular, Theorem 4.2 characterizes free separators of matroids in terms of their cyclic flats. This result, which will be used extensively throughout the rest of this paper, immediately gives a characterization of irreducible matroids.

We write \( \mathcal{FS}(L) \) for the set of all free separators of \( L \). The set \( \mathcal{FS}(L) \) is partially ordered by setting \((A, B) \leq (A', B')\) if and only if \( A \subseteq A' \) and \( B \subseteq B' \). A free separator of \( L \) is **minimal** if it is minimal with respect to this ordering. Note that the pair \((A, E - A)\) is a free separator of \( L \) if and only if \( A \) is a free separator of \( L \) in the sense of [6], that is, if and only if \( L \) is the free product \( L[A \square L/A] \). Note that \((A, B)\) is a (minimal) free separator of \( L \) if and only if \((B, A)\) is a (minimal) free separator of \( L^* \).

For example, the 3-whirl \( W^3 \) (the maximum element of \( \text{Sp}(M, N) \) in Figure 1) has among its free separators \((\{a, c, d, e\}, \{b, c, d, e, f\})\) and \((\{a, c, d, e, f\}, \{b, c, d, e, f\})\); the first of these is minimal.

The next lemma is used in proving the main result of this section, Theorem 4.2.

**Lemma 4.1.** Suppose \( L \leq P \) in the weak order. If \( r_L(Z) = r_P(Z) \) for all \( Z \in \mathcal{Z}(L) \), then \( L = P \).

**Proof.** If \( L < P \), then there is some \( C \in \mathcal{C}(L) \) with \( C \in \mathcal{I}(P) \); however \( Z = \text{cl}_L(C) \in \mathcal{Z}(L) \) and \( r_L(Z) = |C| - 1 < |C| = r_P(C) \leq r_P(Z) \). \( \square \)
Theorem 4.2. A pair of sets \((A, B)\) is a free separator of a matroid \(L(A \cup B)\) if and only if \(Z(L) \subseteq [\emptyset, A] \cup [A - B, A \cup B]\).

Proof. Theorem 3.6 implies that \(L \leq L|A \otimes L.B\) for any matroid \(L(A \cup B)\). If \(Z(L) \subseteq [\emptyset, A] \cup [A - B, A \cup B]\), then it follows from Corollary 4.5 that \(r_L(Z) = r_{L|A \otimes L.B}(Z)\) for all \(Z \in Z(L)\). Hence Lemma 4.4 implies that \(L = L|A \otimes L.B\), that is, \((A, B)\) is a free separator of \(L\). On the other hand, if \((A, B)\) is a free separator of \(L\), then \(Z(L) \subseteq [\emptyset, A] \cup [A - B, A \cup B]\), by Theorem 3.10.

The following corollary is immediate.

Corollary 4.3. If \((A, B)\) is a free separator of \(L(E)\), and \(A', B' \subseteq E\) satisfy \(A \subseteq A'\) and \(B \subseteq B'\), then \((A', B')\) is a free separator of \(L\); in other words \(FS(L)\) is an order filter in the Boolean algebra \(2^E \times 2^E\).

The following result provides a much more efficient test for irreducibility than direct application of Theorem 4.2.

Corollary 4.4. A matroid \(L(E)\) is irreducible if and only if, for each ordered pair \(x, y\) of distinct elements of \(E\), there is some \(Z \in Z(L)\) with \(x \in Z\) and \(y \notin Z\).

Proof. If there is a pair \(x, y \in E\) such that every \(Z \in Z(L)\) containing \(x\) also contains \(y\), then, by Theorem 4.2, the pair \((E - x, E - y)\) is a free separator of \(L\). On the other hand, if \((A, B)\) is a free separator of \(L\), and \(x \in B - A\) and \(y \in A - B\), then, again by Theorem 4.2, any cyclic flat of \(L\) that contains \(x\) also contains \(y\).

We note the following immediate consequence of Corollary 4.4.

Corollary 4.5. If \(|E| \geq 2\) and \(L(E)\) contains either a loop or an isthmus, then \(L\) is reducible.

Recall that elements \(x\) and \(y\) of a matroid \(L\) are clones if the map switching \(x\) and \(y\) and fixing all other elements is an automorphism of \(M\). Equivalently, \(x\) and \(y\) are clones if they are contained in precisely the same cyclic flats. The following corollary is immediate from Corollary 4.4.

Corollary 4.6. Any matroid having clones is reducible. Any identically self-dual matroid without clones is irreducible.

Corollary 4.7. For a matroid \(L(E)\) and proper subsets \(A\) and \(B\) of \(E\) with \(A \cup B = E\), both \((A, B)\) and \((B, A)\) are free separators of \(L\) if and only if all elements of \((A - B) \cup (B - A)\) are clones in \(L\).

We next address the question of when a pair \((A', B')\) of \(2^E \times 2^E\) below a free separator \((A, B)\) of \(L\) is also a free separator of \(L\).

Proposition 4.8. Suppose \(L(E) = M(A) \otimes N(B)\). If \(A' \subseteq A\) and \(B' \subseteq B\) satisfy \(A' \cup B' = E\), then \((A', B') \in FS(L)\) if and only if \((A', A \cap B') \in FS(M)\) and \((A' \cap B, B') \in FS(N)\).

Proof. If \((A', B') \in FS(L)\), then \(Z(L) \subseteq [\emptyset, A'] \cup [A' - B', E]\), by Theorem 3.10. Since any cyclic flat of \(M = L|A\) is of the form \(Z \cap A\), for some \(Z \in Z(L)\), it follows that \(Z(M) \subseteq [\emptyset, A'] \cup [A' - B', A] = [\emptyset, A'] \cup [A' - (A \cap B')]\) so, by Theorem 4.2, we have \((A', A \cap B') \in FS(M)\). Duality then gives \((A' \cap B, B') \in FS(N)\).

Now suppose \((A', A \cap B') \in FS(M)\) and \((A' \cap B, B') \in FS(N)\). We have (i) \(Z(M) \subseteq [\emptyset, A'] \cup [A' - B', A]\) and (ii) \(Z(N) \subseteq [\emptyset, A' \cap B] \cup [B - B', B]\) by
Proposition 4.11. If $M(A)$ and $N(B)$ are matched, and $i = r_M(A-B)$, then

$$L_{N_0,M_1}^i = (T_{A-B}^{i-j})^i M \otimes L_{A-B}^{j-i} N = \begin{cases} M \otimes L_{A-B}^{j-i} N & \text{if } j \geq i, \\ (T_{A-B}^{i-j})^i M \otimes N & \text{if } j \leq i, \end{cases}$$

for all $j$.

Proof. Since $Z(M_1) \subseteq [0, A]$ and $Z(N_0) \subseteq [A-B, E]$, it follows from Theorem 2.20 that $Z(L_{N_0,M_1}) \subseteq [A-B, E]$, and hence by Theorem 4.2 that $L_{N_0,M_1}^i = L_{N_0,M_1}^i A \otimes L_{N_0,M_1}^i B$. By Proposition 2.3, we have $L_{N_0,M_1}^i A = L_{N_0,M_1}^{i-k} A - B$, and $L_{N_0,M_1}^i B = L_{N_0,M_1}^{i-k} B$, where $k = r_M(A-B) - r_M(A-B) - r_{N_0}(A-B) = i$. It is clear that $M_1 | A = M$ and $N_0,B = N$, and, since $M$ and $N$ are matched, we have
\[ M_1, B = N_1 | (A \cap B) \oplus I(B - A) \] and \[ N_0 | A = M_1 | (A \cap B) \oplus I^*(A - B); \] hence the result follows. \( \square \)

5. Interaction between free splice and other constructions

Among the results proven in this section are that direct sums and generalized parallel connections of irreducible matroids are irreducible. This section also shows how to express minors of free splices as free splices. Also, while the free product is a special case of the free splice, we show how to obtain the free splice as an intersection of certain free products.

5.1. Direct sum.

**Proposition 5.1.** If \( M(A), N(B), P(C) \) are matroids with \((M, N)\) matched and \((A \cup B) \cap C = \emptyset\), then \((M \boxplus N) \oplus P = (M \oplus P) \boxtimes (N \oplus P)\).

**Proof.** Let \( L = (M \boxtimes N) \oplus P \) and \( E = A \cup B \cup C \). Since \( Z(L) \) is given by \( \{ X \cup Y : X \in Z(M \boxtimes N) \text{ and } Y \in Z(P) \} \) and \( Z(M \boxtimes N) \subseteq [\emptyset, A] \cup \{ A - B, A \cup B \} \), we have \( Z(L) \subseteq [\emptyset, A \cup C] \cup \{ A - B, E \} = [\emptyset, A \cup C] \cup \{ A \cup C - (B \cup C), E \} \), and hence \((A \cup C, B \cup C) \in FS(L)\). Therefore \( L = L(A \cup C) \bowtie L(B \cup C) \), that is, \((M \boxtimes N) \oplus P = (M \oplus P) \boxtimes (N \oplus P)\). \( \square \)

**Proposition 5.2.** If \( L = M(A) \oplus N(B) \) and \(|A|, |B| \geq 2\), then \( L \) is irreducible if and only if \( M \) and \( N \) are irreducible.

**Proof.** Suppose \( M \) and \( N \) are irreducible. Let \( x \) and \( y \) be distinct elements of \( A \cup B \). If \( x, y \in A \), then some \( Z \in Z(M) \subseteq Z(L) \) contains \( x \) and not \( y \); a similar argument applies if \( x, y \in B \). Suppose \( x \in A \) and \( y \in B \). Since \( M \) is irreducible and \(|A| \geq 2\), it follows that \( M \) has no isthmuses, so \( A \in Z(M) \). Hence \( A \) is a cyclic flat of \( L \) that contains \( x \) and not \( y \). A similar argument applies if \( x \in B \) and \( y \in A \), so \( L \) is irreducible by Corollary 4.2. The converse is immediate from Proposition 5.1. \( \square \)

5.2. Generalized parallel connection. We start by recalling the generalized parallel connection of matroids \( M(A) \) and \( N(B) \). (For more information on this operation, see [3].) Set \( T = A \cap B \) and \( K = M|T| \). If \( N|T = K \), if \( cl_M(T) \) is a modular flat of \( M \), and if each element of \( cl_M(T) - T \) is either a loop or parallel to an element of \( T \), then there is a freest amalgam of \( M \) and \( N \). This freest amalgam, denoted \( P_K(M, N) \), is the generalized parallel connection of \( M \) and \( N \); its flats are the subsets \( F \) of \( A \cup B \) for which \( F \cap A \) is a flat of \( M \) and \( F \cap B \) is a flat of \( N \). Thus, in particular, \( P_K(M, N) = M \boxplus N \) if \( T = \emptyset \). Note that the closure in \( P_K(M, N) \) of \( X \subseteq A \) is \( cl_M(X) \cup cl_N(cl_M(X) \cap T) \); we will denote this closure by \( cl(X) \). A similar formula for \( cl(X) \) holds if \( X \subseteq B \).

**Proposition 5.3.** Let \( M \) and \( N \) be as above, each with at least two elements. If \( M \) and \( N \) are both irreducible, then so is \( P_K(M, N) \).

**Proof.** Since \(|A|, |B| \geq 2\), the matroids \( M \) and \( N \), being irreducible, have no loops, no isthmuses, and no parallel elements. In particular, \( cl_M(T) = T \). It suffices to show that if \( x \) and \( y \) are distinct elements of \( A \cup B \), then some cyclic flat of \( P_K(M, N) \) contains \( x \) and not \( y \). This property follows easily if \( x \) and \( y \) are both in \( A \) or both in \( B \) since \( M \) and \( N \) are restrictions of \( P_K(M, N) \).

Assume \( x \in B - T \) and \( y \in A - T \). Since \( N \) has no isthmuses, \( x \) is in some circuit \( C \) of \( N \). No element of \( A - T \) is in \( cl(C) = cl_M(C) \cup cl_N(cl_M(C) \cap T) \), so \( cl(C) \) is the required cyclic flat.

\[ \]
the result with a second copy of $M$. If $C \in \mathcal{C}(M)$ with $x \in C$ and $r_M(cl_M(C \cap T) \leq 1$, then the description of $cl(C)$ preceding this proposition and the fact that $N$ has neither loops nor parallel elements give $y \notin cl(C)$, so $cl(C)$ is the required cyclic flat. The following step will therefore complete the proof: given $C \in \mathcal{C}(M)$ with $x \in C$ and $r_M(cl_M(C \cap T) \geq 2$, we construct $C' \in \mathcal{C}(M)$ with $x \in C'$ and $r_M(cl_M(C') \cap T) = 1$.

Let $C$ be as just stated. Set $k = r_M(C \cup T) - r_M(T)$ and $F = cl_M(C)$. The fact that $T$ is a modular flat of $M$ gives $r_M(F) - r_M(F \cap T) = k$. Let $Y \subseteq C - (T \cup x)$ be a basis of $F - T$ in $M/(F \cap T)$. Then $r_M(cl_M(Y)) - r_M(cl_M(Y) \cap T) = k$ since $|Y| = k$ and $cl_M(Y) \cap T = \emptyset$. Modularity then gives $r_M(Y \cup T) - r_M(T) = k$, from which $cl_M(Y \cup T) = cl_M(C \cup T)$ follows. Now $cl_M(Y \cup x)$ is a rank-$(k + 1)$ flat of $M$ with $cl_M(Y \cup x \cup T)$ also being $cl_M(C \cup T)$, so modularity implies that $cl_M(Y \cup x) \cap T$ is a point, say $p$. The fact that $p$ is not in $cl_M(Y)$ implies that $x$ is in the fundamental circuit $C'$ of $p$ with respect to the basis $Y \cup x$ of $cl_M(Y \cup x)$; thus, as needed, $C' \in \mathcal{C}(M)$, $x \in C'$, and $r_M(cl_M(C') \cap T) = 1$. □

The property in Proposition 5.3 does not hold for amalgams in general. For instance, let $M$ be the irreducible matroid $M(K_4)$, the cycle matroid of the complete graph $K_4$. Form $N$ from $M$ by relabelling one element $a$ as $a'$. The only amalgam of $M$ and $N$ has the elements $a$ and $a'$ parallel, so the amalgam is reducible.

In contrast to Proposition 5.2, the converse of Proposition 5.3 can fail, even in the special case of a parallel connection at a point. For example, starting with the matroid $U_{2,3}$ on $\{a, b, p\}$, form $M$ by first taking the parallel connection, at $a$, of this matroid with a copy of $M(K_4)$, and then take the parallel connection, at $b$, of the result with a second copy of $M(K_4)$. The resulting rank-6 matroid $M$ on the set $A$ is reducible since every cyclic flat that contains $p$ also contains $a$. Relabel the elements in $A - p$ to get a matroid $N$ on $B$ isomorphic to $M$ and with $A \cap B = p$. It is easy to see that the parallel connection of the reducible matroids $M$ and $N$ at $p$ is irreducible.

### 5.3. Minors

We now show that minors of free splices are also free splices. From this point on, we adhere to the convention that all unary operations on matroids are performed before binary operations; so, for example, $M \bowtie N \backslash X$ means $M \bowtie (N \backslash X)$ rather than $(M \bowtie N) \backslash X$.

**Theorem 5.4.** If $M(A)$ and $N(B)$ are matched, $i = r_M(A - B)$, and $X \subseteq A \cup B$, then

$$(M \bowtie N)(X) = M|(X \cap A) \bowtie N' \quad \text{and} \quad (M \bowtie N).X = M' \bowtie N.(X \cap B),$$

where

$$N' = L^j_{N \setminus (X \cap B), M \setminus (X \cap B)} \quad \text{and} \quad M' = L^k_{N \setminus (X \cap A), M \setminus (X \cap A)},$$

with $j = i - r_M(X - B)$ and $k = i - r_M(A - X) + r_N(B - X) - |B - (A \cup X)|$.

**Proof.** Let $L = M \bowtie N$. Since $(A, B)$ is a free separator of $L$ and any cyclic flat of $L|X$ is of the form $Z \cap X$ for some $Z \in \mathcal{Z}(L)$, it is immediate from Theorem 4.2 that $(X \cap A, X \cap B)$ is a free separator of $L|X$, so

$$L|X = L|X|(X \cap A) \bowtie L|X|(X \cap B) = M|(X \cap A) \bowtie L|X|(X \cap B).$$
Proposition 2.6 gives $L X = L_{N_0,M_1}^k X = L_{N_0|(X \cap B)}^k X$. Using Proposition 2.6 again, and recalling that the elements of $A - B$ are loops of $N_0$, gives
\[ L_x (X \cap B) = L_{N_0|(X \cap B),M_1}^{x-f} (X \cap B), \]
where $f = r_{M_1|x}(X - (X \cap B)) - r_{N_0|x}(X - (X \cap B)) = r_{M_1}|x(X - B)$. Thus, $L_x (X \cap B) = N'$, and so $L_x = M'(X \cap A) \Join N'$. The formula for $L_x$ follows either by duality or by a similar application of Proposition 2.6. \[ \Box \]

We next characterize the special cases in which restriction and contraction of free splices can be expressed in the simplest way. Mildly extending the definition often seen for flats, we call a pair $(X,Y)$ of subsets of $E$ a modular pair in the matroid $M(E)$ if $r_M(X) + r_M(Y) = r_M(X \cup Y) + r_M(X \cap Y)$, that is, if equality holds in the semimodular inequality.

Corollary 5.5. Assume $M(A)$ and $N(B)$ are matched and $X \subseteq A \cup B$. We have $(M \Join N)|X = M|(X \cap A) \Join N|(X \cap B)$ if and only if either
\[(a) \quad r_M(A - B) = r_M(X - B), \text{ that is, } A - B \subseteq c_M(X - B), \text{ or} \]
\[(b) \quad \text{all elements of } X - A \text{ are isthmuses of } N|(X \cap B) \text{ and } (A - B, X \cap A) \text{ is a modular pair in } M. \]
Likewise, $(M \Join N).X = M.(X \cap A) \Join N.(X \cap B)$ if and only if either
\[(a') \quad \text{all elements of } (B - A) - X \text{ are isthmuses of } N|(B - (X - A)), \text{ or} \]
\[(b') \quad \text{all elements of } X - B \text{ are loops of } M.(X \cap A) \text{ and } (A \cap B, B - X) \text{ is a modular pair in } N. \]

Proof. Let $j = r_M(A - B) - r_M(X - B)$. By Theorem 5.4, the expression for $(M \Join N)|X$ in this corollary is valid if and only if $L_{N|(X \cap B),M_1}^j|X = N|(X \cap B)$. Since $j \geq 0$, this is the case if and only either $j = 0$, which is just statement (a), or $M_1|X(X \cap B) = N|(X \cap B)$. Since all elements of $B - A$ are isthmuses of $M_1$, this last equation holds if and only if all elements of $X - A$ are isthmuses of $N|(X \cap B)$ and
\[ M_1|(X \cap B)(X \cap A \cap B) = N|(X \cap A \cap B). \]
Simplifying and using the matching condition $N|(A \cap B) = M.(A \cap B)$, this equation can be rewritten as
\[ M|(X \cap A),(X \cap A \cap B) = M.(A \cap B)|(X \cap A \cap B), \]
which says that in $M\backslash((A \cap B) - X)/(X - (A \cap B))$, the same minor results whether $A - (B \cup X)$ is deleted or contracted. Using standard results on connectivity and separators (see, e.g., [11] Section 4.2), this statement can be recast as saying that $(A - B, X \cap A)$ is a modular pair of sets in $M$. The assertion about contractions follows by duality, noting that the containment $B - A \subseteq c_{N^*}(X - A)$ holds if and only if $(B - A) - X \subseteq \text{Isth}(N|(B - (X - A))).$ \[ \Box \]

Corollary 5.6. Suppose that $M(A)$ and $N(B)$ are matched and that $X \subseteq A \cup B$.
\[(1) \quad \text{If } A - B \subseteq X, \text{ then } (M \Join N)|X = M|(X \cap A) \Join N|(X \cap B). \]
\[(2) \quad \text{If } B - A \subseteq X, \text{ then } (M \Join N).X = M.(X \cap A) \Join N.(X \cap B). \]
5.4. Free product. As mentioned in Section 5.2 whenever the ground sets of $M(A)$ and $N(B)$ are disjoint, the free splice of $M$ and $N$ is the free product $M \square N$. The various cryptomorphic descriptions of free product given in [6] are obvious specializations of their free splice counterparts, given in Theorem 5.10.

It was shown in [7] that in the case of disjoint ground sets, the set $\text{Sp}(M, N) = \{L : L|A = M \text{ and } L.B = N\}$ is the interval $[M \oplus N, M \square N]$ in the weak order on $\mathcal{M}(A \cup B)$. (Recall that, as we saw in Example 5.7, the set $\text{Sp}(M, N)$ is not in general an interval, since it may have many minimal elements.)

Suppose $M(A)$ and $N(B)$ are given, where $A$ and $B$ need not be disjoint, and denote by $I_1$ and $I_2$ the intervals $I_1 = [M|(A - B) \oplus N, M|(A - B) \sqcap N]$ and $I_2 = [M \oplus N.(B - A), M \sqcap N.(B - A)]$ in $\mathcal{M}(A \cup B)$. We then have

$$I_1 = \{L : L|(A - B) = M|(A - B) \text{ and } L.B = N\}$$

and

$$I_2 = \{L : L|A = M \text{ and } L.(B - A) = N.(B - A)\},$$

and hence it follows that $\text{Sp}(M, N) = I_1 \cap I_2$. Note that when $A$ and $B$ are disjoint we have $I_1 = I_2 = [M \oplus N, M \sqcap N] = \text{Sp}(M, N)$.

Suppose that $L$, $P$, and $Q$ are matroids on the same set. The matroid $L$ is the intersection of $P$ and $Q$ if $\mathcal{I}(L) = \mathcal{I}(P) \cap \mathcal{I}(Q)$. When $P$ and $Q$ have the same rank, this is the case if and only if $\mathcal{B}(L) = \mathcal{B}(P) \cap \mathcal{B}(Q)$. (For arbitrary matroids $P$ and $Q$ on the same set, the intersection $\mathcal{I}(P) \cap \mathcal{I}(Q)$ is not the collection of independent sets of a matroid.) We now show that a free splice is the intersection of two free products.

**Proposition 5.7.** If $M(A)$ and $N(B)$ are matched, then the free splice $M \mathbb{\&} N$ is the intersection of the matroids $M|(A - B) \square N$ and $M \sqcap N.(B - A)$.

**Proof.** Let $r = r_M(A - B) + r(N)$. By Theorem 5.10 the $r$-element subsets $X$ of $A \cup B$ such that $X - B \in \mathcal{I}(M)$ and $X \cap B \in \mathcal{S}(N)$ are the bases of $M|(A - B) \sqcap N$, and those satisfying $X \cap A \in \mathcal{I}(M)$ and $X - A \in \mathcal{S}(N.(B - A))$, that is, $X \cap A \in \mathcal{I}(M)$ and $(X - A) \cup (A \cap B) \in \mathcal{S}(N)$, are the bases of $M \sqcap N.(B - A)$. The containment $\mathcal{B}(M|(A - B) \sqcap N) \cap \mathcal{B}(M \sqcap N.(B - A)) \subseteq \mathcal{B}(M \mathbb{\&} N)$ is thus apparent. For the reverse inclusion, it suffices to note that, for all $X \subseteq A \cup B$, we have $X - B \subseteq X \cap A$ and $X \cap B \subseteq (X - A) \cup (A \cap B)$, and so $X \cap A \in \mathcal{I}(M)$ implies that $X - B \in \mathcal{I}(M)$, and $X \cap B \in \mathcal{S}(N)$ implies that $(X - A) \cup (A \cap B) \in \mathcal{S}(N)$. □

6. Algebraic properties of the free splice operation

Unlike free product and direct sum, free splice is a nonassociative operation. In this section, we characterize the triples of matroids for which associativity holds. Furthermore, in Theorem 6.2 below we show that a weakened version of associativity holds in general. The key to the proof of this result is a basic property of free separators, given in the following lemma.

**Lemma 6.1.** Let $L$ be a matroid on $A \cup B \cup C$.

1. If $(A \cup B, C) \in \mathcal{FS}(L)$ and $(A, B) \in \mathcal{FS}(L|(A \cup B))$, then $(A, B \cup C) \in \mathcal{FS}(L)$.
2. If $(A, B \cup C) \in \mathcal{FS}(L)$ and $(B, C) \in \mathcal{FS}(L|(B \cup C))$, then $(A \cup B, C) \in \mathcal{FS}(L)$.

**Proof.** Suppose that $(A \cup B, C) \in \mathcal{FS}(L)$ and $(A, B) \in \mathcal{FS}(L|(A \cup B))$. By Theorem 5.10 we have

$$Z(L) \subseteq Z(L|(A \cup B)) \cup [(A \cup B) - C, A \cup B \cup C]$$

and
and 

$$Z(L|(A \cup B)) \subseteq [\emptyset, A] \cup [A - B, A \cup B].$$

Since \( A - (B \cup C) \subseteq (A \cup B) - C \) and \( A - (B \cup C) \subseteq A - B \), it follows that 

$$Z(L) \subseteq [\emptyset, A] \cup [A - (B \cup C), A \cup B \cup C],$$

and hence Theorem 6.2 implies that 

$$(A, B \cup C) \in FS(L).$$

Statement (2) follows by duality. \( \square \)

**Theorem 6.2.** Suppose \( M(A), N(B), \) and \( P(C) \) are matroids. Let \( U = A \cap (B \cup C) \) and \( V = (A \cup B) \cap C \).

1. If \((M, N)\) and \((M \otimes N, P)\) are matched, then \((M \otimes N) \otimes P = M \otimes (N' \otimes P)\), where \( N' = M \cup U \otimes N \).
2. If \((N, P)\) and \((M, N \otimes P)\) are matched, then \(M \otimes (N \otimes P) = (M \otimes N') \otimes P\), where \( N' = N \otimes P \mid V \).

**Proof.** Let \( L = (M \otimes N) \otimes P. \) Since \((A, B) \in FS(M \otimes N) = FS(L|(A \cup B))\) and \((A \cup B, C) \in FS(L), \) Lemma 6.1 implies that 

$$(A, B \cup C) \in FS(L),$$

that is, \( L = L|(A \otimes L(B \cup C)). \) Now \( L|A = L|(A \cup B)|A = (M \otimes N)|A = M, \) and Corollary 6.3 gives 

\[
L|(B \cup C) = (M \otimes N),( (A \cup B) \cap (B \cup C)) \otimes P \\
= (M.(A \cap ((A \cup B) \cap (B \cup C))) \otimes N) \otimes P \\
= (M.U \otimes N) \otimes P,
\]

and hence (1) follows. Statement (2) follows by duality. \( \square \)

**Corollary 6.3.** Suppose that \( M(A), N(B), \) and \( P(C) \) are matroids.

1. If \((M, N)\) and \((M \otimes N, P)\) are matched and \( B \subseteq C \), then \((M \otimes N) \otimes P = M \otimes P\).
2. If \((N, P)\) and \((M, N \otimes P)\) are matched and \( B \subseteq A \), then \(M \otimes (N \otimes P) = M \otimes P\).

**Proof.** If \( B \subseteq C \), then the ground set of the matroid \( N' \) in Theorem 6.2 is contained in \( C \). As noted after Theorem 6.3 it follows that \( N' \otimes P = P \), and so (1) follows by Theorem 6.2. Statement (2) follows by duality. \( \square \)

We now consider the simplest situation in which free splice is associative.

**Proposition 6.4.** If \( M(A), N(B), \) and \( P(C) \) are matroids with \( A \cap C \subseteq B \), then the following are equivalent:

1. \((M, N)\) and \((N, P)\) are matched,
2. \((M, N)\) and \((M \otimes N, P)\) are matched,
3. \((N, P)\) and \((M, N \otimes P)\) are matched.

Furthermore, if these conditions are satisfied, then \((M \otimes N) \otimes P = M \otimes (N \otimes P)\).

**Proof.** Suppose \((M, N)\) is matched. Since \( A \cap C \subseteq B \), we have \((A \cup B) \cap C = B \cap C\), and so the pair \((M \otimes N, P)\) being matched means that \((M \otimes N),(B \cap C) = P|(B \cap C)\).

But \((M \otimes N),(B \cap C) = (M \otimes N),B,(B \cap C) = N,(B \cap C)\), and thus \((M \otimes N, P)\) is matched if and only if \(N,(B \cap C) = P|(B \cap C)\). Hence (1) and (2) are equivalent. The equivalence of (1) and (3) follows by duality.

By Theorem 6.2, statement (2) implies that \((M \otimes N) \otimes P = M \otimes (N' \otimes P)\), where \( N' = M.(A \cap (B \cup C)) \otimes P \). The fact that \( A \cap (B \cup C) \subseteq B \) then gives \( N' = N \). \( \square \)

The next proposition is a type of commutativity result, showing when a matroid occurs as both a left and a right factor in a free splice. This result is essential to the proof of Theorem 6.8 which characterizes associative triples of matroids.
Proposition 6.5. Suppose $L(E)$ is a matroid with $E = A \cup B = B \cup C$. Let $S = A - B = C - B$. The pairs $(A, B)$ and $(B, C)$ are free separators of $L$ and $L|B = LB$ if and only if either $S = \emptyset$ or one of the statements (a)–(c) holds:

(a) $S \cup (B - A) \subseteq \text{Isth}(L)$,
(b) $S \cup (B - C) \subseteq \text{Loop}(L)$,
(c) (i) $B \subseteq A \cup C$,
     (ii) $B - A \subseteq \text{Isth}(L)$ and $B - C \subseteq \text{Loop}(L)$, and
     (iii) $(S, A \cap B \cap C)$ is a modular pair in $L$.

Proof. The assertion is obvious if $S = \emptyset$, so assume $S \neq \emptyset$. Suppose $L|B = LB$ and $(A, B), (B, C) \in \mathcal{FS}(L)$. Thus, $L = L|A \otimes LB$. Since $L|B = LB$ and $A \cap B \subseteq B$, we have $L|B = L|(A \cap B) \otimes LB|B$, so, by Corollary 5.5 with $X = B$, one of the following statements holds:

(1) $r_{L|A}(S) = 0$, so all elements of $S$ are loops of $L$, or
(2) all elements of $B - A$ are isthmuses of $LB$ and so of $L$; also, $(S, A \cap B)$ is a modular pair in $L|A$ and so in $L$.

Similarly, $L = L|B \otimes LC$ and $L.B = L|(A \cap B) \otimes LB|B$, so Corollary 5.5 implies that one of the following statements holds:

(1') all elements of $S$ are isthmuses of $LC$ and so of $L$, or
(2') all elements of $B - C$ are loops of $LB$ and so of $L$; also, $(B \cap C, S)$ is a modular pair in $LC$, that is, $(B, S \cup (B - C))$ is a modular pair in $L$.

Since $S$ is nonempty, statements (1) and (1') are incompatible. Statements (2) and (2') can both hold only if $B \subseteq A \cup C$; furthermore, in this case the assertions about loops and isthmuses reduce the modularity conditions to one, namely, that in part (iii) of (c). Note also that the modularity assertion in statement (2') would follow immediately from statement (1); a similar remark applies to statements (1') and (2'). Hence one of the statements (a)–(c) holds.

Conversely, when (a) holds, each cyclic flat of $L$ is contained in $A \cap B$; when (b) holds, each cyclic flat contains $B - C$ and $S$; and when (c) holds, each cyclic flat contains $B - C$ and is contained in $A$. In each case, Theorem 4.2 implies that $(A, B), (B, C) \in \mathcal{FS}(L)$. When (a) or (b) holds, the fact that the elements of $S$ are loops or isthmuses yields $L|B = LB; when (c) holds, (ii)$ and (iii) yield $L|B = LB$.

For convenience, we give the following restatement of Proposition 6.5.

Proposition 6.6. Let $M(A), N(B)$, and $P(C)$ be matroids with $A \cup B = B \cup C$. Let $S = A - B = C - B$ and $T = A \cap B \cap C$. The equality $M \otimes N = N \otimes P$ holds if and only if either $S = \emptyset$ or one of the statements (a)–(c) holds:

(a) $M = I(S) \oplus Q$, $N = Q \oplus I(B - A)$, and $P = I(S) \oplus Q.T \oplus I(C - A)$, for some matroid $Q(A \cap B)$,
(b) $M = I^*(S) \oplus R(T) \oplus I^*(A - C)$, $N = R \oplus I^*(B - C)$, and $P = I^*(S) \oplus R$, for some $R(B \cap C)$, or
(c) $M = Q(S) \oplus R(T) \oplus I^*(A - C)$, $N = R(T) \oplus I^*(A - C) \oplus I(C - A)$, and $P = Q(S) \oplus R(T) \oplus I(C - A)$, for some $Q(S)$ and $R(T)$.

As a special case, we have the following characterization of the pairs for which free splice is commutative.
Corollary 6.7. If \( M(A) \) and \( N(B) \) are matched, then \( M \otimes N = N \otimes M \) if and only if one of the following conditions holds:

(a) \( A - B \subseteq \text{Loop}(M) \) and \( B - A \subseteq \text{Loop}(N) \),
(b) \( A - B \subseteq \text{Isth}(M) \) and \( B - A \subseteq \text{Isth}(N) \),
(c) \( B \subseteq A \) and \( M = M[B \oplus M](A - B) \),
(d) \( A \subseteq B \) and \( N = N[A \oplus N](B - A) \).

We now characterize the triples for which free splice is associative.

Theorem 6.8. Suppose that \( M(A), N(B), \) and \( P(C) \) are matroids and that the pairs \( (M, N) \), \( (M \otimes N, P) \), \( (N, P) \), and \( (M, N \otimes P) \) are matched. The equality \( (M \otimes N) \otimes P = M \otimes (N \otimes P) \) holds if and only if either \( (A \cap C) - B = \emptyset \) or one of statements (a)–(c) holds:

(a) \( (A \cap C) - B \subseteq \text{Isth}(M) \) and \( B - A \subseteq \text{Isth}(N) \),
(b) \( (A \cap C) - B \subseteq \text{Loop}(P) \) and \( B - C \subseteq \text{Loop}(N) \),
(c) \( i) B \subseteq A \cup C \),
   (ii) \( B - A \subseteq \text{Isth}(N) \) and \( B - C \subseteq \text{Loop}(N) \),
   (iii) \( ((A \cap C) - B, A \cap B \cap C) \) is a modular pair in \( P \).

Proof. Proposition 6.4 treats the case \( A \cap C \subseteq B \), so assume \( (A \cap C) - B \neq \emptyset \). Let \( U = A \cap (B \cup C) \) and \( V = (A \cup B) \cap C \). The hypotheses and Theorem 6.2 give

\[ (M \otimes N) \otimes P = M \otimes ((M.U \otimes N) \otimes P) \]

Also, \( N \otimes P = (N \otimes P[V]) \otimes P \) by Corollary 6.3 and the hypotheses, so

\[ M \otimes (N \otimes P) = M \otimes ((N \otimes P[V]) \otimes P) \]

Hence \( (M \otimes N) \otimes P = M \otimes (N \otimes P) \) is equivalent to

\[ M \otimes ((M.U \otimes N) \otimes P) = M \otimes ((N \otimes P[V]) \otimes P) \]

Since the underlying free separators are the same, it follows that this equation is equivalent to \( (M.U \otimes N) \otimes P = (N \otimes P[V]) \otimes P \) and, in turn, to

\[ (6.1) \]

\[ M.U \otimes N = N \otimes P[V] \]

If this equation holds, then one of statements (a)–(c) in Proposition 6.5 holds; in a straightforward manner, these statements imply, respectively, statements (a)–(c) above.

For the converse, first assume that (a) holds. Let \( S = U - B = V - B \) and \( T = U \cap B \cap V \). Since \( B - A \subseteq \text{Isth}(N) \), Corollary 6.11 implies that all isthmuses of \( M \) are isthmuses of \( M \otimes N \). Since, in addition, \( (M \otimes N).V = P[V] \) (because the pair \( (M \otimes N, P) \) is matched), the hypothesis \( S \subseteq \text{Isth}(M) \) gives \( S \subseteq \text{Isth}(P[V]) \). Setting \( Q = M.(A \cap B) = N|(A \cap B) \) now gives

\[ M.U = I(S) \oplus Q, \quad N = Q \oplus I(B - V), \quad \text{and} \quad P[V] = I(S) \oplus Q.T \oplus I(V - U) \]

From this and part (a) of Proposition 6.3, Equation (6.1) follows, as needed. Statement (b) is handled similarly or via duality. Now assume statement (c) holds. Note that \( (M, P) \) is matched since \( (M \otimes N, P) \) is matched and \( B \subseteq A \cup C \). By (iii), the matroid \( M.(A \cap C) = P|(A \cap C) \) is \( Q(S) \oplus R(T) \) for some matroids \( Q \) and \( R \). Since \( (M \otimes N).V = P[V] \), statement (ii) gives \( B - A \subseteq \text{Isth}(P[V]) \), that is, \( V - U \subseteq \text{Isth}(P[V]) \). Similarly, \( U - V \subseteq \text{Loop}(M.U) \). These observations give

\[ M.U = Q(S) \oplus R(T) \oplus I^*(U - V), \quad N = R(T) \oplus I^*(U - V) \oplus I(V - U), \]
and

\[ P|V = Q(S) \oplus R(T) \oplus I(V - U). \]

Thus, part (c) of Proposition 6.6 applies and gives Equation (6.1), thereby completing the proof. □

7. Splices and classes of matroids

Given a class \( C \) of matroids, it is natural to ask whether \( C \) contains all splices, the free splice, or at least one splice of any two matched matroids in \( C \). Much of this section shows that even the weakest of these questions has a positive answer for few of the commonly-studied classes of matroids. Also, we show that even the simplest nontrivial class of matroids that is generated by the free splice — that obtained by starting with loops and isthmuses and iteratively taking the free splice — is huge and has some striking properties, notably its failure to be minor-closed. These results may make one wonder whether any nontrivial minor-closed class of matroids is closed under free splice; to address this question, we identify sufficient conditions for the excluded minors that guarantee that the corresponding minor-closed class of matroids is also closed under the free splice, and we show that, for ranks three and greater, binary projective geometries and cycle matroids of complete graphs, as well as their duals, satisfy these conditions.

7.1. Representable and algebraic matroids. We start by considering representable matroids. Crapo and Schmitt [6] showed that the free product of two matroids that are representable over a given field is representable over every sufficiently large field of the same characteristic; however, the class of matroids that are representable over a given finite field is not closed under free product. In contrast, examples below show that the free splice of matroids that are representable over a given field might not be representable over any field. However, there is a positive result for splices of binary and ternary matroids, the key to which is unique representability. Recall that two matrix representations of a matroid over a field \( F \) are equivalent if one can be obtained from the other by the following operations: interchange two rows; interchange two columns (along with their labels); multiply a row or a column by a nonzero element of \( F \); replace a row by its sum with another row; replace every matrix entry by its image under an automorphism of \( F \); and delete or adjoin rows of zeroes. A matroid that is representable over \( F \) is uniquely \( F \)-representable if all of its matrix representations over \( F \) are equivalent. Brylawski and Lucas, who introduced this important idea in [4], proved that a binary matroid is uniquely representable over every field over which it is representable; also, ternary matroids are uniquely representable over \( GF(3) \). It follows that if a matroid \( M \) on \( A \) is binary (resp., ternary) and if \( P \) is a binary (resp., ternary) matrix that represents a restriction \( M|S \), then there is a binary (resp., ternary) matrix that represents \( M \) and for which the columns corresponding to the elements in \( S \) form \( P \), possibly with some added rows of zeroes.

**Proposition 7.1.** Fix \( F \in \{ GF(2), GF(3) \} \). If the matched matroid \( M(A) \) and \( N(B) \) are representable over \( F \), then so is some splice of \( M \) and \( N \).

**Proof.** Let \( X \) be a basis of \( M|(A - B) \); let \( |X| = k \). Thus, \( M \) has a matrix representation over \( F \) of the form

\[
\begin{pmatrix}
I_k & R & S \\
0 & 0 & T
\end{pmatrix}
\]
where the columns of the identity matrix $I_k$ correspond to the elements of $X$, the columns of $R$ correspond to the elements of $(A - B) - X$, and the zeroes denote the zero matrices of the appropriate sizes. The matrix realization of contraction shows that $T$ is a representation, over $F$, of $M.(A \cap B)$, that is, $N|(B \cap A)$, so unique representability implies that $N$ has a representation over $F$ of the form

$$
\begin{pmatrix}
T & U \\
0 & V
\end{pmatrix}.
$$

It follows that the matroid represented over $F$ by the matrix

$$
\begin{pmatrix}
I_k & R & S & W \\
0 & 0 & T & U \\
0 & 0 & 0 & V
\end{pmatrix},
$$

for any matrix $W$ over $F$ of the appropriate size, is a splice of $M$ and $N$.  

We note that it is possible for the first two displayed matrices in the proof above to be totally unimodular without the third having this property, even if $W = 0$. Thus, if the counterpart of this result is true for regular matroids, a different approach to the proof would be needed.

The following example stands in contrast to the result above. The Vámos cube, $V_8$, is the rank-4 matroid on \{a, a', b, b', c, c', d, d'\} in which the proper, nonempty cyclic flats (all of rank 3) are all sets of the form \{x, x', y, y'\} except \{a, a', d, d'\} (see Figure 2). This matroid is neither representable nor algebraic over any field. Since $b$ and $b'$ are clones, Theorem 4.2 implies that $V_8$ is the free splice of $V_8 \setminus b$ and $V_8 / b'$, both of which are representable over all fields except GF(2) and GF(3); also, both are gammoids and both are algebraic over all fields. Furthermore, the geometric argument that $V_8$ is not representable over any field also shows that no splice of $V_8 \setminus b$ and $V_8 / b'$ is representable over any field; likewise, the argument Ingleton and Main [9] give to show that $V_8$ is not algebraic applies to all splices. Splices other than the free splice have additional cyclic planes; for instance, \{a, b, c, d\} might be a cyclic plane.) It follows that there is no counterpart of Proposition 7.1 for any field other than GF(2) and GF(3); also, no result of that type applies to the class of gammoids or to the class of matroids that are algebraic over any fixed field.

7.2. Matroids with no $U_{2,q+2}$-minor. For an integer $q > 1$, let $\mathcal{U}(q)$ be the class of matroids that have no $U_{2,q+2}$-minor. These classes arise often in extremal matroid theory. Note that $\mathcal{U}(2)$ is the class of binary matroids, but if $q$ is any other prime power, then $\mathcal{U}(q)$ properly contains the class of matroids that are
representable over \( \text{GF}(q) \). We now show that for \( q > 2 \), the counterpart of Proposition 7.1 fails for \( U(q) \). Let \( p \) be the largest prime power less than \( q \). The projective plane \( \text{PG}(2, p) \) is in \( U(q) \); let this be \( M \). Fix a point \( a \) in the ground set \( A \) of \( M \) and an element \( x \notin A \). Let \( N \) be the rank-2 matroid on \( (A - a) \cup x \) whose rank-1 flats are \( x \) and the sets \( \ell - a \) as \( \ell \) runs over the lines of \( M \) with \( a \notin \ell \). Note that \( M \) and \( N \) are matched; also, \( N \in U(q) \). Any splice of \( M \) and \( N \) extends \( M \) by putting \( x \) either freely in \( M \) or freely on (only) one of the lines of \( M \) not containing \( a \); neither type of splice is in \( U(q) \) since the former has \( p^2 + p + 1 \) lines through \( x \) while the latter has \( p^2 + 1 \), and both numbers exceed \( q + 1 \).

### 7.3. Transversal matroids

We now show that the counterpart of Proposition 7.1 fails for the class of transversal matroids and bicircular matroids. First recall the geometric representation of transversal matroids from [2]: a matroid is transversal if and only if it has a geometric representation on a simplex in which each cyclic flat of rank \( k \) spans a \( k \)-vertex face of the simplex. Fundamental transversal matroids have such a representation in which, in addition, there is an element of the matroid at each vertex of the simplex. Bicircular matroids have such a representation in which each element is along an edge or at a vertex of the simplex. Now consider the 3-whirl on \( \{c, c', d, e, f, g\} \) with non-spanning circuits \( \{e, f, g\}, \{e', d, g\} \), and \( \{c, d, e\} \); add an isthmus \( a \) to get the matroid \( M \), which is both a fundamental transversal matroid and a bicircular matroid. In any simplex representation of \( M \), the elements \( d, e, g \) are at vertices of the simplex. Let \( N \) be the rank-3 matroid on \( \{a, b, b', c', d, e, f, g\} \) whose proper nonempty cyclic flats are \( \{a, e\} \) of rank 1, and, of rank 2, \( \{a, b, b', c'\} \) and \( \{c', d, e, f, g\} \). Note that \( N \) is also a fundamental transversal matroid and a bicircular matroid; also, \( M \) and \( N \) are matched. Since \( \{a, b, b', c'\} \) or \( \{a, b, b', c, c'\} \) is a cyclic flat of \( N \), it follows that in any splice of \( M \) and \( N \), either \( \{a, b, b', c'\} \) or \( \{a, b, b', c, c'\} \) is a cyclic flat: however, a cyclic flat in a transversal matroid must intersect the face \( \{c, c', d, e, f, g\} \) of the simplex in some face, not, for instance, at \( c \) and \( c' \). Thus, no splice of \( M \) and \( N \) is transversal.

### 7.4. Base-orderable matroids

We now show that the free splice need not preserve the property of being base-orderable. We first recall some definitions. For bases \( B \) and \( B' \) of a matroid \( M \), elements \( x \in B \) and \( x' \in B' \) are exchangeable if both \( (B - x) \cup x' \) and \( (B' - x') \cup x \) are bases of \( M \). A matroid \( M \) is base-orderable if for each pair of bases \( B \) and \( B' \) of \( M \), there is a bijection \( \phi : B \to B' \) such that for all \( x \in B \), the elements \( x \) and \( \phi(x) \) are exchangeable. It is known that the class of base-orderable matroids is closed under minors, duals, free extension, and matroid union; from this and Joseph Kung’s observation that free products can be expressed as certain matroid unions (see his review of [6] in Mathematical Reviews, MR2177484), it follows that the class of base-orderable matroids is closed under free product. It is also known (and follows from the results just mentioned) that all gammoids are base-orderable. The matroid \( M \) in Figure 8 (which appears in [8] and is used to illustrate the theory developed there) is not base-orderable since, for \( B = \{a, b, c, d\} \) and \( B' = \{s, t, u, v\} \), the elements \( a \) and \( b \) are each exchangeable only with \( u \). Note that \( c \) and \( d \) are clones, so \( M \) is the free splice of \( M \setminus c \) and \( M / d \), which are transversal and so are base-orderable.
Figure 3. A rank-4 matroid $M$ that is not base-orderable, yet is the free splice of its base-orderable minors $M_c$ and $M/d$.

7.5. The smallest nontrivial class that is closed under free splice. We next highlight the complexity of the free splice operation by considering the smallest nontrivial class of matroids that it generates. First, recall that the matroids obtained by starting with loops and isthmuses and repeatedly taking free products are the nested matroids (called freedom matroids in [6]), which can be characterized as the matroids whose cyclic flats form a chain under inclusion. Up to isomorphism, there are $2^n$ nested matroids on $n$ elements; they form a small subclass of the class of fundamental transversal matroids; also, minors of nested matroids are nested. Now consider the counterpart for free splice: let $\mathcal{N}$ be the class of matroids obtained by starting with loops and isthmuses and repeatedly taking free splices. We show that the resulting class is much larger than the class of nested matroids; indeed, it includes all fundamental transversal matroids, and much more. The proof of the next result uses the following sufficient condition for a collection $\mathcal{C}$ of matroids to be contained in $\mathcal{N}$: if each matroid $L(E)$ in $\mathcal{C}$, with $|E| \geq 2$, can be written as $M \bowtie N$ where $M$ and $N$ are proper minors of $L$ that are in $\mathcal{C}$, then $\mathcal{C} \subseteq \mathcal{N}$. (Note that $\mathcal{C}$ does not have to be minor-closed.)

Proposition 7.2. All fundamental transversal matroids are in $\mathcal{N}$. Furthermore, $\mathcal{N}$ is closed under taking duals but not under taking minors.

Proof. A fundamental transversal matroid $L$ of rank $k$ has a geometric realization on a $k$-vertex simplex in which there is at least one element of $L$ at each vertex. Let $v_1, v_2, \ldots, v_k$ be at the $k$ vertices. Easy arguments show that we may assume that $L$ has no loops and that it has elements in addition to $v_1, v_2, \ldots, v_k$. Let $x$ be an element of $L$ not among $v_1, v_2, \ldots, v_k$. Thus, $x$ is freely in some face $F$ of the simplex. It follows that the elements of $L$ in $F$ form a cyclic flat and that this cyclic flat is contained in every cyclic flat that contains $x$. Therefore $L = (L \setminus x) \bowtie (L/v_i)$ for any $v_i$ in $F$. Since $v_i$ is at a vertex of the simplex and $x$ is not, both $L/v_i$ and $L \setminus x$ are fundamental transversal matroids. The assertion about these matroids now follows from the sufficient condition noted above.

The class $\mathcal{N}$ is clearly closed under duality. To see that it is not closed under minors, consider the bicircular matroid $B_n$ that has one element at each vertex of the $n$-vertex simplex and one element freely on each edge of the simplex. Thus, $B_n$ has $n + \binom{n}{2}$ elements. Since $B_n$ is a fundamental transversal matroid, it is in $\mathcal{N}$. Let $B'_n$ be the restriction of $B_n$ to the $\binom{n}{2}$ elements that are freely on the edges of the simplex. Note that if $n \geq 5$, then, for each vertex $v$ of the simplex, the $\left(\begin{array}{c} n-1 \\ 2 \end{array}\right)$ elements that are freely on the edges of the simplex that do not contain the vertex
v form a cyclic flat of $B'_n$. It follows from this observation and Corollary 4.3 that $B'_n$ is irreducible for $n \geq 5$, and so is not in $\mathcal{N}$. □

We also note that since every lattice path matroid contains either a loop, an isthmus, or a pair of clones, and since this class of matroids is closed under minors, it follows that all lattice path matroids are in $\mathcal{N}$. (Although they are transversal matroids, not all lattice path matroids are fundamental transversal matroids.) Not all matroids in $\mathcal{N}$ are transversal: $V_8$ is in $\mathcal{N}$, as is the truncation of $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$ to rank 2.

### 7.6. A sufficient condition for a minor-closed class to be closed under free splice

We now turn to classes of matroids that yield positive answers to a question posed at the start of this section: Theorem 7.4 gives sufficient conditions for the excluded minors of a minor-closed class $\mathcal{C}$ of matroids so that the free splice of any two matched matroids in $\mathcal{C}$ will also be in $\mathcal{C}$. We then show that, for ranks three and greater, binary projective geometries and cycle matroids of complete graphs, as well as their duals, satisfy these sufficient conditions. We start with a lemma.

**Lemma 7.3.** Assume $G$ is a matroid on the ground set $Z \cup a$ where $a \notin Z$ and $a$ is neither a loop nor an isthmus of $G$. If the ground set of a matroid $K$ is the disjoint union of $X$, $Y$, and $Z$, and if $K|Z = G|a$ and $K.Z = G/a$, then either $K/X$ or $K\setminus Y$ has a minor isomorphic to $G$.

**Proof.** That $a$ is neither a loop nor an isthmus of $G$ implies that $G$ is the only extension $G'$ of $G\setminus a$ by the element $a$ with $G'/a = G/a$. Therefore if $t$ is a non-loop in $\text{cl}_K(Z) \cap (X \cup Y)$, then $K|(Z \cup t)$ is isomorphic to $G$, for otherwise $K/t$ could not have $G/a$ as the further contraction $K/(X \cup Y)$. More generally, the same observation shows that in any contraction $K/U$ with $U \subseteq X \cup Y$ for which $r_{K/U}(Z) = r_K(Z)$, if $t$ is a non-loop of $K/U$ in $\text{cl}_{K/U}(Z) \cap (X \cup Y)$, then $K/U|(Z \cup t)$ is isomorphic to $G$. It follows that if $K/X$ does not have a minor isomorphic to $G$, then $r_{K/X}(Z) = r(G/a) = r_K(Z) - 1$. In this case, there is a subset $X'$ of $X$ and element $t \in X - X'$ for which $K/X'|((Z \cup t)$ is isomorphic to $G$, so $K\setminus Y$ has a minor isomorphic to $G$. □

In what follows, for a matroid $G$ and element $a$ of $G$, we let $G_a$ denote the principal truncation of $G$ at $a$, that is, $(G/a) \oplus I^*(a)$; dually, $G^a$ denotes the principal lift of $G$ at $a$, that is, $(G\setminus a) \oplus I(a)$.

**Theorem 7.4.** Let $\mathcal{C}$ be a minor-closed class of matroids that has the following properties: the excluded minors for $\mathcal{C}$ have neither loops nor isthmuses, and whenever an excluded minor $G$ for $\mathcal{C}$ is written as a Higgs lift $E_{G_1,G_2}$ of a proper quotient $G_1$ and a proper lift $G_2$ of $G$, then $j = 1$ and there is an element $a$ in $G$ such that

1. $G_1$ is the principal truncation $G_a$ and $a$ is the only loop of $G_a$, and
2. $G_2$ is the principal lift $G^a$ and $a$ is the only isthmus of $G^a$.

If $M(A)$ and $N(B)$ are matched matroids in $\mathcal{C}$, then $M \otimes N$ is in $\mathcal{C}$.

**Proof.** Let $K = M \otimes N$. Toward getting a contradiction, assume $K \notin \mathcal{C}$; say $K/X/Y = G$ where $G$ is one of the excluded minors for $\mathcal{C}$. By Corollary 5.0 and the assumption that $\mathcal{C}$ is minor closed, we may assume $X \subseteq A - B$ and $Y \subseteq B - A$. These inclusions and Proposition 2.4 give $G = K/X/Y = L^j_{(X/Y),0}(M\setminus X)_t$ for some integer $j$. If $(M\setminus X)_1$ were $G$, then either $M\setminus X = G$ or $G$ would have isthmuses;
neither conclusion is possible, so \((M\setminus X)_{1}\) is a proper lift of \(G\). Similarly, \((N/Y)_{0}\) is a proper quotient of \(G\). By the hypotheses about the excluded minors for \(C\), we have \(j = 1\), \((M\setminus X)_{1} = G\), and \((N/Y)_{0} = G_{a}\) for some element \(a\) in \(G\). Let the ground set of \(G\setminus a\) be \(Z\). Only loops and isthmuses of \((M\setminus X)_{1}\) and \((N/Y)_{0}\) can be outside \(A \cap B\), so the hypotheses give \(Z \subseteq A \cap B\). We now consider which of \(A\) and \(B\) the element \(a\) is in.

First assume \(a \in A - B\). Set \(K' = K/a\). Since \(a \not\in B\), from \((N/Y)_{0} = G_{a}\) we get \(N/Y = G/a\), so

\[
K' \setminus Z = K/(X \cup Y \cup a) = N/Y = G/a.
\]

Note that \(K'[Z \cup a]\) is an extension of \(G\setminus a\) since \((M\setminus X)_{1} = G\). The idea in the proof of Lemma 7.3 shows that if \(a\) is a non-loop of \(K\) that is in \(\text{cl}_{K}(Z)\), then, since \(G/a\) is a minor of \(K/a\), we would have \(K'[Z \cup a] = G\), contrary to \(M\) not having \(G\) as a minor. Therefore \(a\) is either a loop or an isthmus of \((M\setminus X)_{1}\), so

\[
K' \setminus Z = K' \setminus (X \cup Y)/a = M/X/a = G/a.
\]

Therefore, by Lemma 7.3 either \(K'/X\) or \(K'\setminus Y\) has a minor isomorphic to \(G\). Restated, this conclusion is that either \(N\) or \(M/a\) (and so \(M\)) has a minor isomorphic to \(G\). This contradiction completes the argument if \(a \in A - B\). The case \(a \in B - A\) follows by duality. (Using duality is justified by the observation that the class \(C^{*} = \{M^{*} : M \in C\}\) that is dual to \(C\) satisfies the hypotheses of the theorem.)

Finally, assume \(a \in A \cap B\). Thus, \(M/X = G\) and \(N/Y = G_{a}\), so

\[
k \setminus (X \cup Y \cup a) = M \setminus (X \cup a) = G \setminus a
\]

and

\[
k \setminus (X \cup Y \cup a) = N/(Y \cup a) = G/a.
\]

Therefore by Lemma 7.3 either \(K/(X \cup Y)\) or \(K'\setminus Y\) has a minor isomorphic to \(G\), that is, either \(N/a\) (and so \(N\)) or \(M\) has such a minor. This contradiction completes the proof.

Some familiar matroids have the properties that are hypothesized in Theorem 7.3 for the excluded minors of \(C\). Below we show that any binary projective geometry \(\text{PG}(n - 1, 2)\) of rank 3 or more has these properties, as does the cycle matroid of a complete graph, \(M(K_{n+1})\), for \(n \geq 3\). To mildly extend the collection of matroids known to have these properties, note that if \(M\) has these properties, then so does \(M^{*}\). The arguments below use two well-known results about quotients: if \(N(E) \subseteq M(E)\), then there is a matroid \(K\) with \(K|E = M\) and \(K.E = N\); also, if \(N \subseteq M\) and \(r(N) = r(M)\), then \(N = M\). The arguments also use two simple observations: if \(N \subseteq M\), then \(N|A \subseteq M|A\) for any \(A \subseteq E\); also, a cyclic flat of size 3 that has rank at least two is a line.

First consider the binary projective plane, \(\text{PG}(2, 2)\), that is, the Fano plane, \(F_{7}\). Assume \(F_{7} = L^{j}_{G_{1}, G_{2}}\) where \(G_{1}\) and \(G_{2}\) are, respectively, a proper quotient and a proper lift of \(F_{7}\). Thus, \(r(G_{1}) < 3 < r(G_{2})\). By part (7) of Theorem 2.4, each line of \(F_{7}\) must be a cyclic flat of either \(G_{1}\) or \(G_{2}\). It follows that if \(r(G_{1}) < 2\), then at least six of the lines of \(F_{7}\) would be lines of \(G_{2}\), which, by the structure of the lines of \(F_{7}\), would yield the contradiction \(r(G_{2}) = 3\). Thus, \(r(G_{1}) = 2\). Therefore there is a single-element extension \(P'\) of \(F_{7}\) by a non-loop \(x\) such that \(G_{1} = P'/x\). Since \(F_{7}\) is modular, \(P'\) extends \(F_{7}\) by adding \(x\) freely in the plane, on a line, or parallel to a point \(a\); the first two options would force all but at most one line of
$F_7$ to be lines of $G_2$, which would yield the contradiction $r(G_2) = 3$, so $G_1$ is the principal truncation $(F_7)_a$. Therefore, all lines of $F_7$ that do not contain $a$ must be lines of $G_2$; this conclusion and the structure of the lines of $F_7 \backslash a$ imply that $G_2$ is the principal lift, $(F_7)^a$.

Now consider the cycle matroid $M(K_4)$. Assume $M(K_4)$ is $L^{(j)}_{G_1,G_2}$ where $G_1$ and $G_2$ are, respectively, a proper quotient and a proper lift of $M(K_4)$. By part (7) of Theorem 2.3 each cyclic flat of $M(K_4)$ must be a cyclic flat of at least one of $G_1$ and $G_2$. Since $r(G_2) > 3$, at most two of the four cyclic lines of $M(K_4)$ are cyclic flats (necessarily lines) of $G_2$. Since $M(K_4)$ is self-dual, it follows that at most two of the four cyclic lines of $M(K_4)$ are cyclic flats of $G_1$. Thus, two cyclic lines of $M(K_4)$ are cyclic flats of $G_1$ and the other two are lines of $G_2$. From these conclusions, it follows that there is some $a$ in $M(K_4)$ for which $G_1$ is the principal truncation $M(K_4)_a$ and $G_2$ is the principal lift $M(K_4)^a$.

Now consider $n \geq 4$. Let $G$ be either $PG(n - 1,2)$ or $M(K_{n+1})$. Let $A$ be the ground set of $G$. Let $G_1$ and $G_2$ be a proper quotient and a proper lift of $G$ with $G = L^{(j)}_{G_1,G_2}$. Thus, $j > 0$. By the structure of $G$, if all lines of $G$ were lines of $G_2$, then $r(G_2) = n$, contrary to $G_2$ being a proper lift of $G$. Let $\ell$ be a line of $G$ that is not a line of $G_2$ and let $X$ be a plane of $G$ that contains $\ell$; if $G$ is $M(K_{n+1})$, choose $X$ so that $G|X$ is $M(K_4)$. By Proposition 2.6 $G|X = L^{(j)}_{G_1|X,G_2|X}$. Since $\ell$ is a line of $G|X$ but not of $G_2|X$, it follows that $G_2|X$ is a proper lift of $G|X$; this conclusion and the inequality $j > 0$ imply that $G_1|X$ is a proper quotient of $G|X$. By the results in the previous two paragraphs, $G_1|X$ is the principal truncation $(G|X)_a$ for some $a \in X$, so $a$ is a loop of $G_1$. Let $G'$ be an extension of $G$ to the set $A \cup T$ such that $G|T = G$ and $G'/T = G_1$. Thus, $r(G') = r(G)(T \cup a)$. For any line $\ell'$ of $G$ with $a \notin \ell'$, we have $r(G)(T \cup \ell') = r(G)(T \cup \ell' \cup a)$, so $\ell'$ is not a line of $G_1$. Part (7) of Theorem 2.3 then implies that each line $\ell'$ of $G$ with $a \notin \ell'$ is a line of $G_2$. This conclusion and the structure of $G\backslash a$ gives $r(G_2\backslash a) = n$, from which it follows that $G_2\backslash a = G\backslash a$, so $G_2$ is the principal lift $G^a$. Also, $a$ must be the only loop of $G_1$. By part (7) of Theorem 2.3 and the observation that no cyclic flat of $G$ that contains $a$ is a cyclic flat of $G^a$, it follows that all such sets are cyclic flats of $G_1$; by considering a maximal chain of such flats, we see that $G_1$ must have rank $n - 1$. These conclusions and that $a$ is a loop of $G_1$ imply that $G_1$ is the principal truncation $G_a$.

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Department of Mathematics, The George Washington University, Washington, D.C. 20052, USA

E-mail address: jbonin@gwu.edu, wschmitt@gwu.edu