Low Energy Consequences of Loop Quantum Gravity

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Abstract

As the loop quantum gravity is based on polymer quantization, we will argue that the polymer length (like string length) can be several orders larger than the Planck length, and this can have low energy consequences. We will demonstrate that a short distance modification of a quantum system by polymer quantization and by string theoretical considerations can produce similar behavior. Moreover, it will be demonstrated that a family of different deformed Heisenberg algebras can produce similar low energy effects. We will analyze such polymer correction to a degenerate Fermi gases in a harmonic trap, and its polymer corrected thermodynamics.

The polymer quantization has been motivated from loop quantum gravity \cite{1,2}. In fact, the polymer quantization of simple matter fields has been studied and the conclusions reached resemble the conclusions obtained in loop quantum gravity \cite{3,4}. It has been demonstrated that the polymer quantization leads to radiation in inertial frames \cite{5}, and a modification of blackbody radiation at high temperatures \cite{6}. The usually polymer length in polymer quantization is assumed to be of the same order as the Planck length. However, it is possible for the polymer length to be larger than Planck length, and this can have low energy consequences (which can be experimentally measured). It may be noted that it has already been argued that the minimal length can be related to the string length $l_s$ as $l_{\text{min}} = g_s^{1/4} l_s$, with $g_s$ being the string coupling constant \cite{7}. In fact, the short distance correction to the quantum mechanics from a string length much greater than Planck length has been analyzed \cite{8,9}. It has been suggested that Landau levels and Lamb shift would be modified by a deformation of quantum mechanics produced from such a minimal length, and this can be measured experimentally \cite{10}. It has also been proposed that an opto-mechanical setup can be used to measure such a deformation, which occurs at several orders of magnitude larger than the Planck scale \cite{11}.

Motivated by the mentioned results above in string theory, we consider the polymer length to be several orders of magnitude larger than Planck length (only bounded by experiments). We will show that this can modify short distance behavior of quantum systems. Indeed, we will analyze the corrections brought by the polymer quantization to the Fermi-Dirac distribution as well as to the thermodynamics of a Fermi gas in harmonic trap.
We start by analyzing the effect polymer quantization would have on the Heisenberg algebra. It is known that in the polymer quantization, the usual phase space \((x, p)\) can be replaced by
\[
x \rightarrow x, \quad p \rightarrow \frac{1}{\mu} \sin(\mu p)
\]
where \(\mu\) is the polymer length scale. This mapping modifies the usual Heisenberg algebra to
\[
[x, p] \rightarrow \frac{1}{\mu} [x, \sin(\mu p)] = i \left(1 - \frac{\mu^2 p^2}{2} + f(p)\right)
\]
where the function of momentum \(f(p) = \mu^4 p^4 + \cdots\). Note that the first order correction to the ordinary Heisenberg algebra is of order \(\mu^2 p^2\), which is the correction that will be observed in any low energy future experiment. These corrections (of the form \(\mu^2 p^2\)) are similar to those obtained from generalized uncertainty principle [12,13]. Such modification of uncertainty principle is motivated from string theory and has similar consequences as polymer quantization [14]. As in string theory, fundamental strings are the smallest probes, then it is not possible to probe spacetime below string length scale, which leads to a string length being the minimal measurable length in spacetime [15]. This modification of spacetime by minimal measurable length of the order of string length \(l_s = \alpha'\) gives rise to a generalized uncertainty principle [16,17]. In perturbative string theory, the minimal measurable length is related to the string length \(l_s\) as \(l_{\text{min}} = g_s^{1/4} l_s\) [7]. Even though non-perturbative effects add point like probes to string theory, such as \(D0\)-branes, it has been demonstrated that even with such non-perturbative objects, a minimal length of the order \(l_{\text{min}} = l_s g_s^{1/3}\) exists in string theory [7,18].

As the winding number \(w\) and the excitation number \(n\) can be used to obtain the total energy of quantized strings, it can be argued that the description of string theory below \(l_s\) is the same as its description above \(l_s\) [7]. This is because under \(T\)-duality, we have the correspondences \(R \rightarrow l_s^2 / R\), and \(n \rightarrow w\). The \(T\)-duality has also been used to study the behavior of effective path integral for string propagating in compactified extra dimensions, and similar conclusions have been obtained form such system [19,20]. The construction of double field theories is motivated from \(T\)-duality [21,22], and it has been observed that a minimal length also exist in double field theories [23]. Thus, such deformation of Heisenberg algebra can be motivated from string theory results.

Now, as polymer quantization, and the modification of quantum mechanics by string theory, lead to the same first order corrections to the Heisenberg algebra, it can be argued that like string length scale, polymer length scale can also be several orders of magnitude larger than the Planck scale. In fact, it is possible that in nature two polymer scales exist, such that one of them is the gravitational polymer scale \(\mu_{gr}\), and the other is the polymer scale for matter fields \(\mu\) (as polymer quantization of matter fields has also been studied [3,4]), such that \(\mu_{gr} = g \mu\), where \(g\) is some coupling in a more fundamental theory. Under this similarity, it is possible to analyze physical consequences of polymer quantization on condensed matter systems in similar way to those of generalized uncertainty principle on such low energy system [8–11]. It may be noted that any deformation of the Heisenberg algebra \([x, p] = i \left(1 - \frac{\mu^2 p^2}{2} + f(p)\right)\), with any function of \(f(p) = \mu^4 p^4 + \cdots\), cannot be distinguishable from the polymer quantization, as any near future experiment would first measure the first order correction of the form \(\mu^2 p^2\). As the non-local deformation of the Heisenberg algebra,
\[
[x, p] = i \sqrt{1 - \mu^2 p^2}
\]
cannot be distinguished by near future experiments from polymer quantization, it has been used to analyze the consequences of polymer quantization [24]. It is also possible to take other modifications of Heisenberg algebra, such as

\[ [x, p] = i \cos \mu p \]  

and all these can be distinguished from polymer quantization by any near future experiments, because all these first order corrections are of the form \( \mu^2 p^2 \).

Now we can study the effect of such short distance corrections on a degenerate Fermi gas in a harmonic trap, as such a system can be physically realized [25–28]. In fact, spin-polarized fermionic alkali atoms, like \(^6\)Li and \(^{40}\)K, have been studied in magnetic traps [29–32]. This has motivated the study of Fermi gas in isotropic or anisotropic harmonic traps [33, 34]. In this study, the thermodynamic properties of such Fermi gas have been analyzed the trapping of ultracold atoms in a harmonic potential. It is expected that such a system will be very sensitive to short distance modification of quantum system. So, a modification of such thermodynamic properties for such a system can be experimentally measured [35, 36]. The Fermi polarons have been studied using spin down impurities in a spin up Fermi sea of ultracold atoms [37]. The polaron energy for various interaction strengths have been studied around a Feshbach resonance, for this system. The investigation of Fermi polarons in a spin-imbalanced Fermi gas has been done using using photoemission spectroscopy [38, 39]. Now as these systems would be modified by polymer quantization, it is important to understand the deformation of such a system by polymer quantization, as this can be measured experimentally.

It is expected that a deformation of the Heisenberg algebra will deform the angular momentum algebra. Now as the first order consequences of polymer quantization, and generalized uncertainty principle have the same mathematical form, we can use the deformation of the angular momentum algebra by generalized uncertainty principle [40] to write the polymer deform algebra of angular momentum as

\[ L^2 |l, m\rangle = \hbar^2 l(l + 1)(1 - \mu^2 p^2)^2 |l, m\rangle, \quad L_z |l, m\rangle = \hbar m (1 - \mu^2 p^2) |l, m\rangle. \]  

As the Fermi gas in isotropic or anisotropic harmonic traps [33, 34] have been studied, and can be measured experimentally, we can analyze the polymer quantization of the following isotropic harmonic potential

\[ V(x, y, z) = \frac{1}{2} M \omega^2 \left( x^2 + y^2 + z^2 \right) \]  

and the polymer corrections to the energy eigenvalues can be expressed as

\[ E_n = \omega \left( n + \frac{3}{2} \right) (1 - \mu^2 p^2) \]  

where the degeneracy of shell with energy \( E_n \) is given by

\[ g_n = \frac{1}{2} (n + 1)(n + 2). \]  

The total number of quantum states \( S_\alpha \) with energy less or equal to \( E_\alpha \) can be written as

\[ \sum_{n=0}^{\alpha} g_n = \frac{1}{6} (\alpha + 1)(\alpha + 2)(\alpha + 3). \]
Now we can obtain the polymer deformation of Fermi-Dirac distribution

\[ f_n = \frac{1}{e^{-\beta \nu + \beta \omega (n + \frac{3}{2}) (1 - \mu^2 p^2)} + 1} \]  

(10)

where \( \nu \) is the chemical potential and \( \beta = \frac{1}{k_B T} \), with \( k_B \) is the Boltzmann constant. The total number of particles in the trap and Fermi energy are given by

\[ N = \sum_n g_n f_n, \quad E_F = \omega \left( n_F + \frac{3}{2} \right) \left( 1 - \mu^2 p^2 \right). \]  

(11)

In the Fig. 1, we plot the deformed Fermi-Dirac distribution function \( f_n \) to observe the corrections due to the polymer quantization. We set \( -\beta \nu \equiv y = 5 \), and \( \beta \omega (n + \frac{3}{2}) \equiv x \), to plot \( f_n \) in terms of \( x \) for several values of \( \mu \) in Fig. 1 (a) and in terms of correction parameter for several values of \( x \) in Fig. 1 (b). In the case of \( \mu = 0 \) (dash dotted red line of Fig. 1 (a)), we have usual Fermi-Dirac distribution. For \( \mu < 1 \), \( f_n \) increases (see Fig. 1 (b)) and the Fermi energy is reduced by \( \mu \). For \( \mu \geq 1 \) as well as \( x = 0 \) (corresponding to high temperature), we can find \( f_n \approx 1 \) as expected.

Figure 1: Fermi-Dirac distribution function for \( p = 1 \).

For high temperature regime, the chemical potential can be written as

\[ \nu(T) = \omega \left( n_F + \frac{3}{2} \right) \left( 1 - \mu^2 p^2 \right) \left( 1 - \frac{\pi^2}{3} \left( \frac{k_B T}{\omega \left( n_F + \frac{3}{2} \right) \left( 1 - \mu^2 p^2 \right)} \right)^2 \right) \]  

(12)

In the Fig. 2, we can observe the typical behavior of the chemical potential for different values of \( \mu p \). We can observe a completely opposite behavior for \( \mu > 1 \) and \( \mu < 1 \). As expected, for the small correction (\( \mu < 1 \)), we the behavior resembles the uncorrected chemical potential. Both of them coincide at low temperature. Hence, the polymer corrections are only important at high temperature.

However, in order to analyze the present system for low temperature regime, we start by first defining the number \( N_>(T) \) of exited atoms above \( E_F \), number \( N_<(T) \) of atoms below \( E_F \) and corrected
short distance,

\[
N_>(T) = \sum_{n=n_F+1}^{\infty} g_n \frac{g_n}{z^{-1}e^{-n\omega\beta(1-\mu^2p^2)} + 1} \\
N_<(T) = \sum_{n=0}^{\infty} g_n \left(1 - \frac{1}{z^{-1}e^{-n\omega\beta(1-\mu^2p^2)} + 1}\right)
\]

where the fugacity is given by

\[
z = e^{\beta(\nu-\frac{1}{2}\hbar\omega(1-\mu^2p^2))}
\]

For low temperature regime, we have two conditions, i.e., \(k_B T \ll (1-\mu^2p^2)E_{n_F+1} - \nu\) and \(k_B T \ll \nu - (1-\mu^2p^2)E_F\). Under these conditions we can write

\[
\Sigma_> = \sum_{n=n_F+1}^{\infty} g_n e^{-(E_n-E_{n_F+1})(1-\mu^2p^2)\beta}
\]

\[
\Sigma_< = \sum_{n=0}^{n_F} g_n e^{-(E_F-E_n)(1-\mu^2p^2)\beta}
\]

Now we can use the following approximation,

\[
N_>(T) \approx \Sigma_> e^{-\beta((1-\mu^2p^2)E_{n_F+1} - \nu)}
\]

\[
N_<(T) \approx \Sigma_< e^{-\beta((\nu-1-\mu^2p^2)E_{n_F})}
\]

The product of \(N_>(T)\) and \(N_<(T)\) can now be written as

\[
N_>(T)N_<(T) \approx \Sigma_> \Sigma_< e^{-\beta((1-\mu^2p^2)\omega)}
\]
So, when \( N \in \{ S_\alpha \} \), the Fermi shell is totally filled at \( T = 0 \), \( N_\uparrow(T) = N_\downarrow(T) \), and hence we obtain

\[
N_\uparrow(T) = \left( \sum_\uparrow \sum_\downarrow e^{-(1-\mu^2 p^2)\beta \omega} \right)^{-\frac{1}{2}}
\]  

Therefore the chemical potential takes the form

\[
\nu(T) = \omega(n_F + 2)(1 - \mu^2 p^2) - \frac{1}{2} k_B T \ln \frac{\sum_\uparrow}{\sum_\downarrow}.
\]  

We can also analyze the low temperature limit for \( N \notin \{ S_\alpha \} \). For very low temperature regime, the Fermi function can be approximated by \( f_n = 1 \) for \( n < n_F \) and \( f_n = 0 \) for \( n > n_F \). The number of occupied states in the Fermi shell \( \Delta N = N - S_{n_f-1} \) can be approximated by

\[
\Delta N = \frac{g_n}{z^{-1}e^{\beta \alpha \omega(1-\mu^2 p^2)} + 1}
\]  

and then the corresponding chemical potential is given by,

\[
\nu(T) = \omega \left( n_f + \frac{3}{2} \right) (1 - \mu^2 p^2) - k_B T \ln \left( \frac{g_{n_f}}{\Delta N} - 1 \right).
\]  

Finally, the energy of the present system can be written as

\[
U(T) = \sum_{n=0}^{\infty} \frac{g_n n \omega (1 - \mu^2 p^2)}{z^{-1}e^{\beta n \omega(1-\mu^2 p^2)} + 1}
\]  

We find that, effect of polymer correction is increasing of the energy. Thus, all the thermodynamic quantities for a harmonic oscillator are deformed because of the polymer quantization. Such modification to various important thermodynamic quantities can change the properties of such Fermi gases, which can be detected experimentally.

In that case the single grand partition function is given by

\[
Z = \text{Tr} e^{-\beta H} = \text{Tr} e^{\beta \nu - \beta \alpha \omega(n + \frac{3}{2})}
\]  

with \( \alpha = (1 - \mu^2 p^2) \). Now the partition function can be written as

\[
Z = \frac{e^{\beta \nu}}{\left( 2 \sinh \frac{\beta \alpha \omega}{\beta} \right)^{\frac{3}{2}}}
\]  

Hence, we can obtain entropy by using the following relation

\[
S = k_B \ln Z + k_B T \left( \frac{\partial \ln Z}{\partial T} \right)_V.
\]  

In the Fig. 3, we observe the effect of correction on the entropy. It is clear that polymer correction increases value of the entropy.

The specific heat in constant volume is obtained using,

\[
C_V = T \left( \frac{\partial S}{\partial T} \right)_V.
\]  

In the plots of Fig. 4, we draw specific heat for some values of correction parameter. For \( \mu \geq 1 \), we find that specific heat is negative. This can be interpreted as instability in the system. For \( \mu < 1 \), the
Figure 3: Typical behavior of the entropy for $p = k_B = \omega = 1$.

Figure 4: Typical behavior of the specific heat in constant volume for $p = k_B = \omega = 1$.

Specific heat is increasing function of the temperature (see Fig. 4). At low temperature the effect of $\mu$ is negligible and the specific heat tends to zero. For the high temperature limit it yields to a constant.
We can also calculate Helmholtz free energy as,

\[ F = -k_B T \ln Z. \]  

We find critical value for the correction parameter where Helmholtz free energy is zero (see long dash cyan line of Fig. 5). Below which Helmholtz free energy is positive at low temperature and yields to negative value at high temperature. Otherwise for the larger value of \( \mu \) value of Helmholtz free energy is totally negative. There is a maximum for the Helmholtz free energy which may be interpreted as unstable point.

![Graph of Helmholtz free energy vs. T](image)

**Figure 5:** Typical behavior of the Helmholtz free energy versus \( T \) for \( p = k_B = \omega = 1 \).

Finally it is interesting to see behavior of the internal energy with correction parameter which is illustrated by the Fig. 6. We observe that the internal energy given by the equation (25) has a minimum for special value of \( \mu \). For the case of \( \mu \leq 1 \) we can see that internal energy is decreasing function of \( \mu \). Comparing this with the experimental value of internal energy could be used to obtain a bound on the value of \( \mu \).

Motivated by the deformation of Heisenberg algebra brought from string theory, and its low energy consequences [8–11], we argued that the polymer length for matter fields can be much greater than Planck length. Such a modification of polymer quantization can have interesting low energy consequences, and these can experimentally be measured. As an example, we analyzed the generate Fermi gas in a harmonic trap using such deformation. We explicitly analyzed the modification to Fermi-Dirac distribution as well as to the thermodynamics of the present system. It has been demonstrated that polymer quantization leads to a loss of manifest Lorentz symmetry in Minkowski space. [4]. So, it would be interesting to analyze the effect of polymer quantization on the Poincare algebra [41, 42]. This can have important consequences for particle physics, if the polymer length is
assumed to be much larger than Planck scale. It would be interesting to construct gauge theories using such a deformed quantum system, and analyze its consequences. In general, we expect that this will lead to new types of vertices between gauge fields and matter, and their coupling constants can be constrained from particle physics experiments. This can in turn lead to new and interesting bounds on the polymer length.

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