Robin eigenvalues on domains with peaks

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Abstract

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with an outward power-like peak which is assumed not too sharp in a suitable sense. We consider the Laplacian $u \mapsto -\Delta u$ in $\Omega$ with the Robin boundary condition $\partial_n u = \alpha u$ on $\partial \Omega$ with $\partial_n$ being the outward normal derivative and $\alpha > 0$ being a parameter. We show that for large $\alpha$ the associated eigenvalues $E_j(\alpha)$ behave as $E_j(\alpha) \sim -\epsilon_j \alpha^\nu$, where $\nu > 2$ and $\epsilon_j > 0$ depend on the dimension and the peak geometry. This is in contrast with the well-known estimate $E_j(\alpha) = O(\alpha^2)$ for the Lipschitz domains.

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1. Introduction

Given a domain $\Omega \subset \mathbb{R}^N, N \geq 2$, with a suitably regular boundary and a parameter $\alpha > 0$, we consider the self-adjoint operator $Q_\alpha^\Omega$ in $L^2(\Omega)$ generated by the quadratic form

$$ q_\alpha^\Omega(u, u) = \int_{\Omega} |\nabla u|^2 \, dx - \alpha \int_{\partial \Omega} u^2 \, d\sigma, \quad u \in H^1(\Omega), $$

where $d\sigma$ stands for the $(N-1)$-dimensional Hausdorff measure. Informally, the operator $Q_\alpha^\Omega$ can be viewed as the Laplacian with the Robin boundary condition $\partial_n u = \alpha u$, where $\partial_n$ is the outward normal derivative. Various properties of the operator $Q_\alpha^\Omega$ have been analysed in the literature over the last decades, see e.g. the recent paper [3] for a review of results and a collection of open problems. In the present paper we are interested in the asymptotic behaviour of the lowest eigenvalues $E_j(Q_\alpha^\Omega)$ of $Q_\alpha^\Omega$ in the limit $\alpha \to +\infty$. It is a standard result that for Lipschitz domains for large $\alpha$ one has the bound $E_1(Q_\alpha^\Omega) \geq -K\alpha^2$ with $K > 0$, see Subsection 2.1 below. Under additional regularity assumptions, i.e. for
the case when $\Omega$ is a so-called corner domain, the lower bound can be upgraded to the asymptotics $E_1(Q_\alpha^\Omega) \sim -K\alpha^2$ with some $K \geq 1$ depending on the regularity of the boundary as described in [2, 14], in particular, $K = 1$ for the $C^1$ domains, see [15], and for planar domains the value $K$ is determined by the smallest corner at the boundary, see [10, 14]. More precise asymptotic expansions of $E_1(Q_\alpha^\Omega)$ have been obtained, under various geometric assumptions, in [6, 18, 8, 9, 19, 20]. Analogous version of the problem for the $p$-Laplacian was treated in [13] by the authors.

One arrives then at the following natural question: what kind of results can be obtained for non-Lipschitz domains? An easy revision of the above mentioned works shows that inward peaks do not influence the eigenvalue asymptotics (the first terms in the asymptotic expansions are determined by the rest of the boundary). So in the present work we are going to study the operator $Q_\alpha^\Omega$ for domains $\Omega$ with a suitably defined outward peak. It is well-known that if the peak is too sharp, then the quadratic form $q_\alpha^\Omega$ fails to be semibounded, hence, the first eigenvalue of $Q_\alpha^\Omega$ does not exist, see Remark 4 below, so our objective is to describe the asymptotic behavior of the eigenvalues in a more detailed way for the peak of a “moderate” sharpness, see Figure 1. More precisely, we restrict our attention to power-like outward peaks characterized by two parameters

$$p \in (1, 2), \quad m > 0$$

as follows:

**Assumption 1.** There exist $\ell_0 > 0$ and a positive $C^1$ function $\mu$ on $(-\ell_0, \ell_0)^{N-1}$ with $\mu(0) = m$ such that

- for some $\ell \in (0, \ell_0)$ one has
  $$\Omega \cap (-\ell, \ell)^N = \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 \in (0, \ell), \frac{|x'|}{\mu(x')} < x_1^p \right\},$$

- for some $h \in (0, \ell)$ the domain $\Omega \setminus [-h, h]^N$ is Lipschitz.

In order to state the main results we need some notation and an auxiliary one-dimensional operator. It will be convenient to use the shorthand

$$n := N - 1.$$ 

If $H$ is a self-adjoint operator, we denote by $E_j(H)$ its $j$th eigenvalue (when enumerated in the non-decreasing order and counted according to the multiplicities) if it exists. If $E \in \mathbb{R}$, we denote by $N(H, E)$ the number of eigenvalues of $H$ in $(-\infty, E)$. 

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Figure 1: An example of a domain $\Omega$ with a peak in two dimensions. On the left a zoom near the vertex of the peak is the presented.
Given \( \lambda > 0 \) we consider the symmetric differential operator in \( L^2(0, \infty) \) given by

\[
C_0^\infty(0, \infty) \ni f \mapsto -f'' + \left( \frac{(np-1)^2 - 1}{4s^2} - \frac{n}{\lambda s^p} \right)f,
\]

and denote by \( A_\lambda \) its Friedrichs extension in \( L^2(0, \infty) \), then it is standard to see that the essential spectrum of \( A_\lambda \) is \([0, +\infty)\) and that \( A_\lambda \) has infinitely many negative eigenvalues \( E_j(A_\lambda) \) accumulating at zero (see Subsection 3.1 for a more detailed discussion). Our result on the asymptotics of individual eigenvalues of \( Q_{a_\alpha}^\Omega \) is as follows:

**Theorem 2.** For any fixed \( j \in \mathbb{N} \) one has

\[
E_j(Q_{a_\alpha}^\Omega) = \left( \frac{a}{m} \right)^{\frac{2}{2-p}} E_j(A_1) + o(\alpha^{\frac{2}{2-p}})
\]

as \( \alpha \) tends to \(+\infty\).

In addition, we provide an estimate for the number of eigenvalues below a moving threshold:

**Theorem 3.** For some \( B > 0 \) there holds

\[
N(Q_{a_\alpha}^\Omega, -Ba_\alpha^{p+1}) = M_p \alpha^{\frac{2}{2-p}} + o(\alpha^{\frac{2}{2-p}})
\]

as \( \alpha \) tends to \(+\infty\), where

\[
M_p = \frac{1}{2\pi B^{\frac{n-2}{2}} \left( \frac{m}{n} \right)^{\frac{1}{2-p}}} \int_0^1 \sqrt{\frac{1 - s^p}{s^p}} \, ds.
\]

**Remark 4.** Let us explain the restriction \( p \in (1, 2) \) in Assumption 1. If \( p \leq 1 \), then \( \Omega \) is Lipschitz. Hence the condition \( p > 1 \) is necessary for \( \Omega \) to have a peak. On the other hand, for \( p > 2 \) one has \( E_1(Q_{a_\alpha}^\Omega) = -\infty \) for all \( \alpha > 0 \), and for \( p = 2 \) one has \( E_1(Q_{a_\alpha}^\Omega) > -\infty \) only if \( \alpha \leq \alpha_0 \) with some \( \alpha_0 > 0 \), see [4, 16, 17]. As explained in [4], this is equivalent to the fact that for \( p > 2 \) the trace operator \( H^1(\Omega) \rightarrow L^2(\partial\Omega) \) does not exist, and for \( p = 2 \) it exists, but is not compact.

**Remark 5.** An upper bound for \( E_1(Q_{a_\alpha}^\Omega) \) on the planar domain \( \Omega = \{ (x_1, x_2) : |x_2| < x_1^p \} \) was obtained in [14, Example 3.4] using a test function argument: it was shown that \( E_1(Q_{a_\alpha}^\Omega) \leq -C a_\alpha^{\frac{2}{2-p}} \) for large \( \alpha > 0 \) and some \( C > 0 \).

**Remark 6.** Due to the assumption \( p \in (1, 2) \) one has \( \frac{2}{2-p} > 2 \) which indeed shows that the eigenvalues \( E_j(Q_{a_\alpha}^\Omega) \) tend to \(-\infty \) much faster then for the Lipschitz case. Furthermore, the gaps \( G_j := E_{j+1}(Q_{a_\alpha}^\Omega) - E_j(Q_{a_\alpha}^\Omega) \) have the same order in \( \alpha \), which is in contrast to the previously studied cases with more regularity: as shown in [10], for the curvilinear polygons one has \( G_j = O(\alpha^2) \), and for the \( C^k \) smooth domains one has \( G_j = O(\alpha) \) if \( k = 2 \) and \( G_j = O(\sqrt{\alpha}) \) if \( k = 3 \), see [20].

**Remark 7.** Concerning Theorem 3 we remark that if \( \Omega \) is Lipschitz, \( B > 0 \) and \( p > 1 \), then \( N(Q_{a_\alpha}^\Omega, -Ba_\alpha^{p+1}) = 0 \) for \( \alpha \) large enough, see Corollary 9. Therefore, the growing number of eigenvalues below \(-Ba_\alpha^{p+1}\) is purely due to the presence of the peak.
2. Scheme of proof

In order to prove the main results we perform first a number of truncations and
dilations in order to isolate the peak and to reduce the problem to the study of some
models domains.

2.1. Lipschitz domains

Let us recall some known facts about Robin Laplacians on Lipschitz domains. The
following result is quite standard, see e.g. in [7, Theorem 1.5.1.10]:

Proposition 8. Let $U$ be a bounded Lipschitz domain. Then there exists a constant
$K > 0$ such that

\[
\eta \int_U |\nabla u|^2 \, dx + \eta^{-1} \int_U u^2 \, dx \geq K \int_{\partial U} u^2 \, d\sigma
\]

holds for all $u \in H^1(U)$ and all $\eta \in (0, 1)$.

An immediate consequence of Proposition 8 is the following

Corollary 9. Let $U$ be a bounded Lipschitz domain. Then there exist constants $K > 0$
and $\alpha_0 > 0$ such that

\[
E_1(Q^U_\alpha) \geq -K \alpha^2 \quad \forall \alpha \geq \alpha_0. \tag{2.1}
\]

2.2. Isolating the peak

Recall that $\Omega$ satisfies Assumption 1, which implies a choice of strictly positive con-
stants $\ell$ and $h$ appearing in the formulation. For $\delta \in (0, h)$ denote

\[
\Lambda_\delta := \{(x_1, x') : x_1 \in (0, \delta), \frac{|x'|}{\mu(x')} < x_1^p\},
\]

\[
\Omega_\delta := \Omega \setminus \Lambda_\delta,
\]

\[
\partial_0 \Lambda_\delta := \{(x_1, x_2) \in \partial \Lambda_\delta : x_1 < \delta\},
\]

then $\Omega = \Lambda_\delta \cup \Omega_\delta$, while $\Omega_\delta$ is a bounded Lipschitz domain by construction. A standard
application of the min-max principle shows that for any $j \in \mathbb{N}$ one has

\[
E_j(R^{N,\delta}_\alpha) \leq E_j(Q^\Omega_\alpha) \leq E_j(R^{D,\delta}_\alpha) \tag{2.2}
\]

where $R^{N/D,\delta}_\alpha$ are the self-adjoint operators in $L^2(\Lambda_\delta)$ defined by the following quadratic
forms $r^{N/D,\delta}_\alpha$:

\[
r^{N,\delta}_\alpha(u, u) = \int_{\Lambda_\delta} |\nabla u|^2 \, dx - \alpha \int_{\partial_0 \Lambda_\delta} u^2 \, d\sigma, \quad D(r^{N,\delta}_\alpha) = H^1(\Lambda_\delta),
\]

\[
r^{D,\delta}_\alpha(u, u) = r^{N,\delta}_\alpha(u, u), \quad D(r^{D,\delta}_\alpha) = \{u \in H^1(\Lambda_\delta) : u(\delta, \cdot) = 0\},
\]

and $K^{N,\delta}_\alpha$ is the self-adjoint operator in $L^2(\Omega^\delta_\alpha)$ defined by the quadratic form

\[
k^{N,\delta}_\alpha(u, u) = \int_{\Omega^\delta_\alpha} |\nabla u|^2 \, dx - \alpha \int_{\partial \Omega^\delta_\alpha \cap \partial_0 \Omega} u^2 \, d\sigma, \quad D(k^{N,\delta}_\alpha) = H^1(\Omega^\delta_\alpha).
\]

Proposition 10. For any $\delta \in (0, h)$ there exist constants $\alpha_0 > 0$ and $K > 0$ such that
for $\alpha > \alpha_0$ and $j \in \{1, \ldots, N(Q^\Omega_\alpha, -K\alpha^2)\}$ there holds $E_j(R^{N,\delta}_\alpha) \leq E_j(Q^\Omega_\alpha) \leq E_j(R^{D,\delta}_\alpha)$.

Proof. For large $\alpha$ for some $K > 0$ one has $K^{N,\delta}_\alpha \geq -K \alpha^2$ due to Corollary 9
and the min-max principle. Hence for $E_j(Q^\Omega_\alpha) < -K \alpha^2$ one has $E_j(R^{N,\delta}_\alpha \oplus S^{N,\delta}_\alpha) = E_j(R^{N,\delta}_\alpha)$. 

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2.3. Reduction to a model peak

In order to estimate the eigenvalues of $R^{D/N,\delta}_\alpha$ we compare them with Robin Laplacians on some model domains. Namely, for $k > 0$ and $a > 0$ denote

$$V_{k,a} = \{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^n : x_1 \in (0, a), \, |x'| < kx_1^p \} \subset \mathbb{R}^n,$$

$$\partial_0 V_{k,a} = \{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^n : x_1 \in (0, a), \, |x'| = kx_1^p \} \subset \partial V_{k,a},$$

(2.3)

Let $S^{k,a}_\alpha$ and $\tilde{S}^{k,a}_\alpha$ be the self-adjoint operators in $L^2(V_{k,a})$ generated respectively by the quadratic forms $s^{k,a}_\alpha$ and $\tilde{s}^{k,a}_\alpha$ given by

$$s^{k,a}_\alpha(u, u) = \int_{V_{k,a}} |\nabla u|^2 dx - \alpha \int_{\partial_0 V_{k,a}} u^2 ds, \quad \mathcal{D}(s^{k,a}_\alpha) = H^1(V_{k,a}),$$

$$\tilde{s}^{k,a}_\alpha(u, u) = s^{k,a}_\alpha(u, u), \quad \mathcal{D}(\tilde{s}^{k,a}_\alpha) = \tilde{H}^1(V_{k,a}).$$

Proposition 11. There exist $c > 0$ and $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and $\alpha > 0$ there holds

$$E_j(R^{N,\delta}_\alpha) \geq (1 - c\delta)E_j(S^{m,\delta}_{\alpha N}), \quad \alpha_N := \frac{1 + c\delta}{1 - c\delta} \alpha,$$

$$E_j(R^{D,\delta}_\alpha) \leq (1 + c\delta)E_j(S^{m,\delta}_{\alpha D}), \quad \alpha_D := \frac{1 - c\delta}{1 + c\delta} \alpha.$$

Proof. Consider the map $\Phi : \Lambda_\delta \to V_{m,\delta}$ given by

$$\Phi(x_1, x') = \left( x_1, \frac{m}{\mu(x')} x' \right),$$

then $\nabla \Phi(x) = \text{Id} + \mathcal{O}(|x|)$ for $x \to 0$. Hence, for sufficiently small $\delta$ the map $\Phi$ is a diffeomorphism, and its inverse $\Psi : V_{m,\delta} \to \Lambda_\delta$ satisfies $\nabla \Psi(x) = \text{Id} + \mathcal{O}(|x|)$ as well. Therefore, there exist $c_0 > 0$ and $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and all $u \in H^1(\Lambda_\delta)$ one can estimate, with $v := u \circ \Psi \in H^1(V_{m,\delta}),$

$$(1 - c_0\delta) \|v\|^2_{L^2(V_{m,\delta})} \leq \|u\|^2_{L^2(\Lambda_\delta)} \leq (1 + c_0\delta) \|v\|^2_{L^2(V_{m,\delta})},$$

$$(1 - c_0\delta) \int_{V_{m,\delta}} |\nabla v|^2 dx \leq \int_{\Lambda_\delta} |\nabla u|^2 dx \leq (1 + c_0\delta) \int_{V_{m,\delta}} |\nabla v|^2 dx,$$

$$(1 - c_0\delta) \int_{\partial_0 V_{m,\delta}} v^2 ds \leq \int_{\partial_\delta \Lambda_\delta} u^2 ds \leq (1 + c_0\delta) \int_{\partial_0 V_{m,\delta}} v^2 ds.$$

The substitution of these inequalities into the expressions for $s^{k,a}_\alpha$ and $\tilde{s}^{k,a}_\alpha$ and then into the min-max principle gives the result.

2.4. The rescaled peak

Now we need to study the eigenvalues of $S^{m,\delta}_\alpha$ and $\tilde{S}^{m,\delta}_\alpha$ for large $\alpha$. It is easy to see that

$$\alpha V_{m,\delta} = V_{m\alpha^{1-\gamma},\delta\alpha},$$

\[5\]
and the change of variables $x = y/\alpha$ in the above expressions for $s_\alpha^{k,a}$ and $\tilde{s}_\alpha^{k,a}$ leads to the equalities
\[
E_j(S_\alpha^{m,\delta}) = \alpha^2 E_j(S_1^{m\alpha^{-p},\delta\alpha}), \quad E_j(\tilde{S}_\alpha^{m,\delta}) = \alpha^2 E_j(\tilde{S}_1^{m\alpha^{-p},\delta\alpha}). \tag{2.4}
\]
Hence we denote
\[
\varepsilon := m\alpha^{-p}, \quad \delta\alpha = b e^{-\frac{1}{\varepsilon}}, \quad b := m^{\frac{1}{p}} \delta
\]
and study the rescaled operators
\[
Q_{\varepsilon,b} := S_1^{\varepsilon,b} e^{\frac{1}{\varepsilon}}, \quad \tilde{Q}_{\varepsilon,b} := \tilde{S}_1^{\varepsilon,b} e^{\frac{1}{\varepsilon}} \tag{2.6}
\]
as $\varepsilon \to 0$. In Section 5 we prove the following crucial result:

**Proposition 12.** There exist $K_1 > 0$, $k_1 > 0$, $\varepsilon_0 > 0$ such that
\[
E_j(\tilde{Q}_{\varepsilon,b}) \leq (1 - k_1 \varepsilon) \varepsilon^{\frac{2}{p}} E_j(A_1) \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \quad 1 \leq j \leq \mathcal{N}(A_1, -K_1 \varepsilon^{\frac{2}{p}}). \tag{2.7}
\]
Furthermore, one can find $K_2 > 0$, $k_2 > 0$, $B > 0$ with
\[
E_j(Q_{\varepsilon,b}) \geq (1 + k_2 \varepsilon) \varepsilon^{\frac{2}{p}} E_j(A_1) - K_2 \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \quad 1 \leq j \leq \mathcal{N}(Q_{\varepsilon,b}, -B/\varepsilon). \tag{2.8}
\]
Before presenting the proofs of the main results of our paper, let us state two simple but important consequences of Proposition 12.

**Corollary 13.** There exist $K_1 > 0$, $k_1 > 0$, $\alpha_0 > 0$ such that
\[
E_j(S_\alpha^{m,\delta}) \leq (1 - k_1 \alpha^{-p}) \left(\frac{\alpha}{m}\right)^{\frac{2}{p}} E_j(A_1) \quad \text{for all } \alpha > \alpha_0 \text{ and } 1 \leq j \leq \mathcal{N}(A_1, -K_1 \alpha^{-\frac{1}{p}}). \tag{2.9}
\]
Furthermore, there exist $K_2 > 0$, $k_2 > 0$, and $B > 0$ such that
\[
E_j(S_\alpha^{m,\delta}) \geq (1 + k_2 \alpha^{-p}) \left(\frac{\alpha}{m}\right)^{\frac{2}{p}} E_j(A_1) - K_2 \quad \text{for all } \alpha > \alpha_0 \text{ and } 1 \leq j \leq \mathcal{N}(S_\alpha^{m,\delta}, -B\alpha^{p+1}). \tag{2.10}
\]

**Proof.** The inverse passage from $\varepsilon$ to $\alpha$ in Proposition 12 implies the claim.

**Corollary 14.** There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ the following assertions hold true:

(a) there exist $K_1 > 0$, $k_1 > 0$, $\alpha_0 > 0$ such that
\[
E_j(R_\alpha^{D,\delta}) \leq (1 - k_1 \delta^{-p-1}) \left(\frac{\alpha}{m}\right)^{\frac{2}{p}} E_j(A_1) \quad \text{for all } \alpha > \alpha_0 \text{ and } 1 \leq j \leq \mathcal{N}(A_1, -K_1 \alpha^{-\frac{1}{p}}), \tag{2.11}
\]
(b) there exist $K_2 > 0$, $k_2 > 0$, $B > 0$ such that
\[
E_j(R_\alpha^{N,\delta}) \geq (1 + k_2 \delta^{-p-1}) \left(\frac{\alpha}{m}\right)^{\frac{2}{p}} E_j(A_1) - K_2 \quad \text{for all } \alpha > \alpha_0 \text{ and } 1 \leq j \leq \mathcal{N}(R_\alpha^{N,\delta}, -B\alpha^{p+1}). \tag{2.12}
\]

**Proof.** This follows by substituting Corollary 13 into Proposition 11.
2.5. Proof of main results

In order to prove Theorem 2 it suffices to insert the inequalities of Corollary 14 into Proposition 10 and to remark that \( \delta > 0 \) can be chosen arbitrarily small.

Let us now turn to a proof of Theorem 3. To this aim, we need an additional result on the operator \( A_1 \), which is proved in subsection 3.1.

Proposition 15. For \( \varepsilon \to 0^+ \) there holds

\[
N(A_1, -\varepsilon) = J_p \varepsilon^{-\frac{2-p}{2p}} + o(\varepsilon^{-\frac{2-p}{2p}})
\]

with

\[
J_p = \frac{n^\frac{1}{2}}{2\pi} \int_0^1 \sqrt{\frac{1-s^p}{s^p}} \, ds.
\]

Now let us take \( \delta_0 > 0 \) sufficiently small and \( \delta \in (0, \delta_0) \). Let \( B > 0 \) and \( \alpha > 0 \) be sufficiently large. Using the lower bound of Proposition 10 we see that if

\[
E_j(Q_\alpha, -B\alpha^{p+1}) \leq -(1 - k\delta^{p-1})Bm \frac{\alpha^{p-1}}{p-1},
\]

where \( k \) does not depend on \( \delta \). It follows that

\[
N(Q_\alpha, -B\alpha^{p+1}) \leq N(A_1, -(1 - k\delta^{p-1})Bm \frac{\alpha^{p-1}}{p-1}),
\]

and by Proposition (15) one has

\[
N(Q_\alpha, -B\alpha^{p+1}) \leq J_p(1 - k\delta^{p-1}) \frac{2-p}{2p} B^{-\frac{2-p}{2p}} m^{-\frac{1}{2}} \alpha^{\frac{p-1}{p-2}} + o(\alpha^{\frac{p-1}{p-2}}).
\]

As \( \delta > 0 \) can be chosen sufficiently small, we obtain the upper bound of Theorem 3. The lower bound is obtained in an analogous way.

2.6. Outline of the paper

The rest of the paper is dedicated to the proof of the key Proposition 12. This is done in several steps. First we prove some auxiliary results on the one-dimensional operator \( A_1 \) and show that it can be approximated by a truncated operator \( L_{\varepsilon,a} \) acting on an interval \((0,a)\) for small \( \varepsilon \), see Section 3. In Section 3.2 we study the effective contribution from the Robin Laplacian defined on the cross-section of the peak, see Lemma 18. In Section 4 we establish a connection between the truncated one-dimensional operator \( L_{\varepsilon,a} \) and the operator \( \tilde{S}_{\varepsilon,a} \) for a fixed value of \( a \) and small \( \varepsilon \). Finally, the operators \( Q_{\varepsilon,b} \) and \( \tilde{Q}_{\varepsilon,b} \) are studied in Section 5 using an additional truncation, which completes the proof.

3. Auxiliary estimates

In this section we prove a number of estimates for various operators appearing in the proof of Proposition 12. For a scalar product in a Hilbert space \( \mathcal{H} \) we will use the symbol \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \). Given \( r > 0 \), we denote by

\[
\mathcal{B}_r = \{ x \in \mathbb{R}^n : |x| < r \}
\]

the \( n \)-dimensional ball of radius \( r \) entered in the origin. Finally, \( \omega_k \) stands for the surface area of the \( k \)-dimensional unit sphere.
3.1. One-dimensional comparison operators

For \( \lambda > 0 \), consider the symmetric differential operator in \( L^2(0, \infty) \) given by

\[
C_0^\infty(0, \infty) \ni f \mapsto -f'' + \left( \frac{(np - 1)^2 - 1}{4s^2} - \frac{n}{\lambda s^p} \right) f.
\]

Since \( (np - 1)^2 > 0 \), the operator is semi-bounded from below in view of the classical Hardy inequality,

\[
\int_0^\infty |f'|^2 ds \geq \int_0^\infty \frac{f^2}{4s^2} ds \text{ for } f \in C_0^\infty(0, \infty), \quad (3.1)
\]

and we denote by \( A_\lambda \) its Friedrichs extension in \( L^2(0, \infty) \). The potential term is for large enough \( s \) attractive and decays at infinity as \( s^{-p} \). Hence standard spectral theoretical arguments show that the essential spectrum of \( A_\lambda \) is \([0, +\infty)\) and that \( A_\lambda \) has infinitely many negative eigenvalues accumulating at zero. Moreover, a scaling argument shows the equalities

\[
E_j(A_\kappa \lambda) = \kappa^{-\frac{1}{2p}} E_j(A_\lambda), \quad \kappa > 0, \quad (3.2)
\]

in particular, the individual eigenvalues of \( A_\lambda \) are continuous in \( \lambda \).

In what follows we will work with truncated versions of \( A_\lambda \). Namely, given \( \lambda > 0 \) and \( a > 0 \) we denote by \( L_{\lambda, a} \) and \( M_{\lambda, a} \) the Friedrichs extensions in \( L^2(0, a) \) and \( L^2(a, \infty) \) of the operators \( C_0^\infty(0, a) \ni f \mapsto A_\lambda f \) and \( C_0^\infty(a, \infty) \ni f \mapsto A_\lambda f \) respectively.

PROOF OF PROPOSITION 15. Since imposing a Dirichlet boundary at one point represents a perturbation of rank one of the resolvent of \( A_1 \), it follows that

\[
\mathcal{N}(A_1, -\varepsilon) \leq \mathcal{N}(L_{1, a} \oplus M_{1, a}, -\varepsilon) + 1
\]

holds for all \( a > 0 \). As the operator \( L_{1, a} \) has compact resolvent, it follows that \( N_a := \mathcal{N}(L_{1, a}, 0) < +\infty \). Hence the above inequality and the min-max principle show that

\[
\mathcal{N}(M_{1, a}, -\varepsilon) \leq \mathcal{N}(A_1, -\varepsilon) \leq \mathcal{N}(M_{1, a}, -\varepsilon) + N_a + 1. \quad (3.3)
\]

Let \( \delta > 0 \), then the parameter \( a \) can be chosen sufficiently large to have

\[
\frac{(np - 1)^2 - 1}{4s^2} \geq -\frac{\delta}{s^p} \text{ for } s \in (a, \infty).
\]

Hence, if denote by \( K_{k, a} \) the self-adjoint operator in \( L^2(a, \infty) \) obtained as the Friedrichs extension of

\[
C_0^\infty(a, \infty) \ni f \mapsto -f'' - \frac{k}{s^p} f, \quad k > 0,
\]

one has the form inequality \( K_{n+\delta, a} \leq M_{1, a} \leq K_{n, a} \) implying, for any \( \varepsilon > 0 \),

\[
\mathcal{N}(K_{n, a}, -\varepsilon) \leq \mathcal{N}(A_1, -\varepsilon) \leq \mathcal{N}(K_{n+\delta, a}, -\varepsilon) + N_a + 1. \quad (3.4)
\]

At any fixed values of \( k \) and \( a \), the operator \( K_{k, a} \) can be analyzed using standard approaches, in particular, by [21, Theorem XIII.82] we have, for \( \varepsilon \to 0+ \),

\[
\mathcal{N}(K_{k, a}, -\varepsilon) \sim \frac{1}{2\pi} \int_a^\infty \frac{1}{s} \sqrt{\frac{k}{s^p} - \varepsilon} \, ds.
\]
where \( x_+ = x \) for \( x \geq 0 \) and \( x_+ = 0 \) for \( x < 0 \), and an elementary analysis shows that

\[
N(K_{k,a}, -\varepsilon) \sim \frac{k^2}{2\pi} \varepsilon^{-\frac{3+\mu}{2}} \int_0^1 \frac{1 - s^\mu}{\varepsilon s^\mu} ds.
\]

It remains to substitute the last estimate into (3.4) and to use the fact that \( \delta > 0 \) can be chosen arbitrarily small.

We are now going to relate the eigenvalues of \( L_{\lambda,a} \) to those of the comparison operator \( A_\lambda \). First remark that due to the min max-principle one has

\[
E_j(L_{\lambda,a}) \geq E_j(A_\lambda) \text{ for any } a > 0, \lambda > 0, j \in \mathbb{N}.
\]

Let us now obtain an asymptotic upper bound for \( E_j(L_{\lambda,a}) \).

**Lemma 16.** Let \( a > 0 \). Then there exist \( K > 0, k > 0, \varepsilon_0 > 0 \) such that

\[
E_j(L_{\varepsilon,a}) \leq e^{-\frac{\varepsilon}{2\varepsilon_0}} E_j(A_1) + k \text{ for all } \varepsilon \in (0, \varepsilon_0), \quad j \in \{1, \ldots, N(A_1, -K\varepsilon_0)\}.
\]

**Proof.** The proof is quite standard using a so-called IMS partition of unity. Let \( \chi_1 \) and \( \chi_2 \) be two smooth functions on \( \mathbb{R} \) such that \( \chi_1^2 + \chi_2^2 = 1 \), \( \chi_1(s) = 0 \) for \( s > \frac{1}{2} a \), \( \chi_2(s) = 0 \) for \( s < \frac{1}{2} a \). We set \( k := \|\chi_1^2\|_{\infty}^2 + \|\chi_2^2\|_{\infty}^2 \). A direct computation shows that for \( f \in C_0^\infty(0, \infty) \) one has

\[
\int_0^\infty (f')^2 ds = \int_0^\infty |(\chi_1 f)'|^2 ds + \int_0^\infty |(\chi_2 f)'|^2 ds - \int_0^\infty ((\chi_1')^2 + (\chi_2')^2) f^2 ds
\]

\[
\geq \int_0^\infty |(\chi_1 f)'|^2 ds + \int_0^\infty |(\chi_2 f)'|^2 ds - k \|f\|_{L_2(0, \infty)}^2.
\]

Hence

\[
\langle f, A_\varepsilon f \rangle_{L_2(0, \infty)} \geq \langle \chi_1 f, A_\varepsilon (\chi_1 f) \rangle_{L_2(0, \infty)} + \langle \chi_2 f, A_\varepsilon (\chi_2 f) \rangle_{L_2(0, \infty)} - k \|f\|_{L_2(0, \infty)}^2,
\]

which can be rewritten as

\[
\langle f, A_\varepsilon f \rangle_{L_2(0, \infty)} + k \|f\|_{L_2(0, \infty)}^2 \geq \langle \chi_1 f, A_\varepsilon (\chi_1 f) \rangle_{L_2(0, \infty)} + \langle \chi_2 f, A_\varepsilon (\chi_2 f) \rangle_{L_2(0, \infty)}
\]

Using the equality

\[
\|f\|_{L_2(0, \infty)}^2 = \|\chi_1 f\|_{L_2(0, a)}^2 + \|\chi_2 f\|_{L_2(a/4, \infty)}^2, \quad \chi_1 f \in C_0^\infty(0, a), \quad \chi_2 f \in C_0^\infty(a/4, \infty)
\]

and the min-max principle one obtains

\[
E_j(A_\varepsilon) + k \geq \inf_{S \subset C_0^\infty(0, \infty) \atop \dim S = j} \sup_{f \in S, f \neq 0} \frac{\langle \chi_1 f, A_\varepsilon (\chi_1 f) \rangle_{L_2(0, \infty)} + \langle \chi_2 f, A_\varepsilon (\chi_2 f) \rangle_{L_2(0, \infty)}}{\|f\|_{L_2(0, \infty)}^2}
\]

\[
= \inf_{S \subset C_0^\infty(0, \infty) \atop \dim S = j} \sup_{f \in S, f \neq 0} \frac{\langle \chi_1 f, A_\varepsilon (\chi_1 f) \rangle_{L_2(0, \infty)} + \langle \chi_2 f, A_\varepsilon (\chi_2 f) \rangle_{L_2(0, \infty)}}{\|\chi_1 f\|_{L_2(0, a)}^2 + \|\chi_2 f\|_{L_2(a/4, \infty)}^2}
\]

\[
\geq \inf_{S \subset C_0^\infty(0, \infty) \atop \dim S = j} \sup_{(u_1, u_2) \in S} \frac{\langle u_1, A_\varepsilon u_1 \rangle_{L_2(0, a)} + \langle u_2, A_\varepsilon u_2 \rangle_{L_2(a/4, \infty)}}{\|u_1\|_{L_2(0, a)}^2 + \|u_2\|_{L_2(a/4, \infty)}^2}
\]

\[
= E_j(L_{\varepsilon,a} \oplus M_{\varepsilon,a/4}).
\]
With the help of the Hardy inequality (3.1) we conclude that $M_{\varepsilon,a/4} \geq -K_0\varepsilon^{-1}$ for $K_0 := 4^p\bar{a}^{-p}$. Now take any $K > K_0$ and set $\varepsilon_0 := (K - K_0)/k$, then for any $j \leq N(A_1, -K\varepsilon^{\frac{p}{2p-p}})$ and $\varepsilon \in (0, \varepsilon_0)$ one has then, using (3.2),

$$E_j(A_\varepsilon) + k = \varepsilon^{-\frac{2}{p}}E_j(A_1) + k < -K\varepsilon^{-1} + k = -(K-k\varepsilon)\varepsilon^{-1} \leq -K_0\varepsilon^{-1} \leq E_1(M_{\varepsilon,a/4}).$$

This in combination with (3.6) shows that $E_j(L_{\varepsilon,a} \oplus M_{\varepsilon,a/4}) = E_j(L_{\varepsilon,a})$ and the result follows.

**Lemma 17.** Let $a > 0$, then there exist $\varepsilon_0 > 0$ and $K > 0$ such that

$$N(L_{\varepsilon,a}, 0) \geq N(A_1, -K\varepsilon^{\frac{p}{2p-p}}) \quad \forall \varepsilon \in (0, \varepsilon_0).$$

**Proof.** It follows from Lemma 16 that one can find $K > 0$, $k > 0$ and $\varepsilon_0 > 0$ such that

$$E_j(L_{\varepsilon,a}) \geq \frac{K}{\varepsilon} + k$$

for $\varepsilon \in (0, \varepsilon_0)$ and $j \leq N(A_1, -K\varepsilon^{\frac{p}{2p-p}})$. Adjust the value $\varepsilon_0$ to have $-K/\varepsilon_0 + k < 0$, then $E_j(L_{\varepsilon,a}) < 0$ for all $j \leq N(A_1, -K\varepsilon^{\frac{p}{2p-p}})$.

### 3.2. Robin Laplacian on a ball

Let $B_{\varepsilon,r}$ be the self-adjoint operator in $L^2(\mathcal{B}_1)$ generated by the quadratic form

$$b_{\varepsilon,r}(f, f) = \int_{\mathcal{B}_1} |\nabla f|^2 \, dy - r \int_{\partial \mathcal{B}_1} |f|^2 \, ds, \quad f \in H^1(\mathcal{B}_1),$$

(3.7)

where $r \in \mathbb{R}$ and let $E_j(B_{\varepsilon,r})$ denote the eigenvalues of $B_{\varepsilon,r}$. In other words, the operator $B_{\varepsilon,r}$ is the Laplacian $f \mapsto -\Delta f$ with the Robin boundary condition $D_n f = rf$ with $D_n$ being the outward normal derivative.

We have

**Lemma 18.** The following assertions hold true:

(a) $E_j(B_{\varepsilon,r}) = \varepsilon^{-2}E_j(B_{1,\varepsilon r})$.

(b) The mapping $\mathbb{R} \ni x \mapsto E_1(B_{1,x}) \in \mathbb{R}$ is $C^\infty$. Moreover, if $\psi_{\varepsilon,r}$ denotes the positive eigenfunction relative to $E_1(B_{\varepsilon,r})$ and normalised to 1 in $L^2(\mathcal{B}_1)$, then for any $\varepsilon > 0$ the mapping $\mathbb{R} \ni r \mapsto \psi_{\varepsilon,r} \in L^2(\mathcal{B}_1)$ is $C^\infty$.

(c) There exists $\varphi \in L^\infty(0, \infty)$ such that

$$E_1(B_{1,x}) = -nx + x^2\varphi(x) \quad \forall x > 0.$$  

(3.8)

(d) Let $E_2^N > 0$ denote the second eigenvalue of the Neumann Laplacian on $\mathcal{B}_1$. Then

$$E_2(B_{1,x}) = E_2^N + o(1), \quad x \to 0.$$

(e) Let $\psi_{\varepsilon,r}$ be as in part (b). Then for any $r_0 > 0$ there exists $\varepsilon_0 > 0$ and a constant $K > 0$ such that

$$\int_{\mathcal{B}_r} |\partial_r \psi_{\varepsilon,r}(y)|^2 \, dy \leq K\varepsilon^2 \quad \forall \varepsilon \in (0, \varepsilon_0), \forall r \in (0, r_0).$$  

(3.9)
Proof. The property (a) is easily obtained by dilations. From the compactness of the embedding $H^1(B_1) \hookrightarrow L^2(\partial B_1)$ it follows that for any $\eta > 0$ there exists $C_\eta$ such that
\begin{equation}
\int_{\partial B_1} |f|^2 \, d\sigma \leq \eta \int_{B_1} |\nabla f|^2 \, dy + C_\eta \int_{B_1} |f|^2 \, dy
\end{equation}
holds true for all $f \in H^1(B_1)$. Hence the mapping $x \mapsto B_{1,x}$ is a type (B) analytic family, which implies (b) and (d). Moreover, since $B_{1,0}$ is the Neumann Laplacian on $B_1$ whose first eigenvalue is simple and the associated eigenfunction is constant, the analytic perturbation theory gives
\begin{equation}
E_1(B_{1,x}) = -nx + O(x^2), \quad x \to 0.
\end{equation}
where we have used the fact that $|\partial B_1| = n|B_1|$. On the other hand, $E_1(B_{1,x}) = -x^2 + o(x^2)$ as $x \to +\infty$, see e.g. [15]. This together with (3.11) gives (c).

It remains to prove (e), which is done by rather direct computations. For the proof in the case $n = 1$ we refer to [11, Lemma 4.7]. Consider now the case $n \geq 2$ and let
\begin{equation}
\lambda = \lambda(\varepsilon, r) := \sqrt{-E_1(B_{\varepsilon,r})}.
\end{equation}
By symmetry $\psi_{\varepsilon,r}(y) = \phi_{\varepsilon,r}(|y|)$, where $\phi_{\varepsilon,r}$ is a positive solution of
\begin{equation}
-\partial_t^2 \phi_{\varepsilon,r}(t) - \frac{n-1}{t} \partial_t \phi_{\varepsilon,r}(t) = -\lambda^2(\varepsilon, r) \phi_{\varepsilon,r}(t),
\end{equation}
satisfying $\partial_t \phi_{\varepsilon,r}(t)|_{t=\varepsilon} = r \phi_{\varepsilon,r}(\varepsilon)$. Writing
\begin{equation}
\phi_{\varepsilon,r}(t) = (\lambda t)^{-\nu} v(\lambda t), \quad \nu := \frac{n}{2} - 1
\end{equation}
and $s = \lambda t$, we find out that
\begin{equation}
s^2 v''(s) + s v'(s) - (s^2 + \nu^2) v(s) = 0.
\end{equation}
By [1, Sec. 9.6.1] the solutions of the last equation are given by the modified Bessel functions $I_{\nu}$ and $K_{\nu}$. Since $s^{-\nu} K_{\nu}(s) \notin H^1(B_\varepsilon)$, see [1, Eqs. (9.6.8-9) & (9.6.28)], it follows that
\begin{equation}
\phi_{\varepsilon,r}(t) = \beta(\lambda, \varepsilon) u_{\varepsilon,r}(\lambda t) := \beta(\lambda, \varepsilon) (\lambda t)^{-\nu} I_{\nu}(\lambda t),
\end{equation}
where $\beta(\lambda, \varepsilon)$ is chosen so that $\|\psi_{\varepsilon,r}\|_{L^2(B_1)} = 1$. The latter condition implies
\begin{equation}
\omega_n \beta^2(\lambda, \varepsilon) \int_0^\infty I_{\nu}^2(s) s \, ds = \lambda^n.
\end{equation}
To prove estimate (3.9) we use the identity
\begin{align}
\partial_r \psi_{\varepsilon,r}(y) &= \partial_r \phi_{\varepsilon,r}(|y|) \frac{\partial \lambda(\varepsilon, r)}{\partial r} \\
&= -\frac{1}{2\lambda(\varepsilon, r)} \frac{\partial E_1(B_{\varepsilon,r})}{\partial r} \left( \partial_r \beta(\lambda, \varepsilon) u_{\varepsilon,r}(\lambda |y|) + |y| \beta(\lambda, \varepsilon) u'_{\varepsilon,r}(\lambda |y|) \right),
\end{align}
(3.17)
where $u'_{\varepsilon,r}(s) = \partial_s u_{\varepsilon,r}(s)$. Differentiating equation (3.16) with respect to $\lambda$ gives

$$n \lambda^{n-1} - \omega_n \beta^2(\lambda, \varepsilon) \varepsilon^2 \lambda I_\nu^2(\varepsilon \lambda) = 2 \omega_n \beta(\lambda, \varepsilon) \partial_\lambda \beta(\lambda, \varepsilon) \int_0^\lambda I_\nu^2(s) s \, ds. \quad (3.18)$$

By [1, Eq. (9.6.10)] we have

$$I_\nu(s) = 2^{-\nu} s^{\nu} \sum_{k=0}^\infty \frac{4^{-k} s^{2k}}{k! \Gamma(k + \nu + 1)}. \quad (3.19)$$

Keeping in mind that $2 \nu = n - 2$, it follows that

$$I_\nu^2(s) = a_\nu^2 s^{n-2} + 2 a_\nu b_\nu s^n + o(s^n), \quad s \to 0 \quad (3.20)$$

holds true with

$$a_\nu = \frac{1}{2 \nu \Gamma(\nu + 1)}, \quad b_\nu = \frac{1}{2^{2\nu} \Gamma(\nu + 2)}. \quad$$

This together with (3.16) implies

$$\omega_n \beta^2(\lambda, \varepsilon) \varepsilon^2 \lambda I_\nu^2(\varepsilon \lambda) = \frac{\varepsilon^2 \lambda^{n+1} I_\nu^2(\varepsilon \lambda)}{\int_0^\lambda I_\nu^2(s) s \, ds} = \frac{\lambda^{n+1} \varepsilon^2 a_\nu^2 (\varepsilon \lambda)^{n-2} (1 + c_\nu \varepsilon^2 \lambda^2 + o(\varepsilon^2 \lambda^2))}{\frac{1}{2} \varepsilon^n \lambda^2 a_\nu^2 \left(1 + \frac{n}{n+2} c_\nu \varepsilon^2 \lambda^2 + o(\varepsilon^2 \lambda^2)\right)}$$

$$= n \lambda^{n-1} \left(1 + \frac{2}{n+2} c_\nu \varepsilon^2 \lambda^2 + o(\varepsilon^2 \lambda^2)\right), \quad \varepsilon \to 0,$$

where

$$c_\nu = \frac{2 b_\nu}{a_\nu} = \frac{\Gamma(\nu + 1)}{2 \Gamma(\nu + 2)} = \frac{1}{2(\nu + 1)}. \quad$$

Here we have used the identity $\Gamma(z + 1) = z \Gamma(z)$. Hence

$$n \lambda^{n-1} - \omega_n \beta^2(\lambda, \varepsilon) \varepsilon^2 \lambda I_\nu^2(\varepsilon \lambda) = -\frac{n}{n+2} \lambda^{n+1} \varepsilon^2 + o(\lambda^{n+1} \varepsilon^2), \quad \varepsilon \to 0. \quad$$

Using (3.16) and (3.19) again one easily verifies that

$$\omega_n \beta(\lambda, \varepsilon) \int_0^\lambda I_\nu^2(s) s \, ds = \frac{\sqrt{\omega_n}}{2 \nu \sqrt{n}} \varepsilon^{\frac{3}{2} \lambda^n} + o(\varepsilon^{\frac{3}{2} \lambda^n}), \quad \varepsilon \to 0.$$

Equation (3.18) thus gives

$$\partial_\lambda \beta(\lambda, \varepsilon) = -\frac{n^{3/2} 2^{\nu-1} \lambda \varepsilon^{2 - \frac{3}{2}}}{\sqrt{\omega_n} (n + 2)(\nu + 1)} + o(\lambda \varepsilon^{2 - \frac{3}{2}}), \quad \varepsilon \to 0. \quad (3.21)$$

Since

$$u'_{\varepsilon,r}(s) = 2^{-\nu} \sum_{k=1}^\infty \frac{4^{-k} 2 k s^{2k-1}}{k! \Gamma(k + \nu + 1)},$$
see (3.15) and (3.19), the above estimates and a simple calculation show that the upper bound
\[
\int_{\mathbb{R}^n} \left( \partial_\lambda \beta(\lambda, \varepsilon) u_{\varepsilon, r}(\lambda |y|) + |y|\beta(\lambda, \varepsilon) u'_{\varepsilon, r}(\lambda |y|) \right)^2 \, dy \leq C_1 \lambda^2 \varepsilon^4
\]  
holds for all \(\varepsilon \in (0, \varepsilon_0)\) and \(\forall r \in (0, r_0)\) and with a constant \(C_1\) depending only on \(n, r_0\) and \(\varepsilon_0\). On the other hand, part (a) of the Lemma in combination with (3.11) implies
\[
\left| \frac{\partial E_i(B_{\varepsilon, r})}{\partial r} \right| \leq C_2 \varepsilon^{-1} \quad \forall \varepsilon \in (0, \varepsilon_0), \; \forall r \in (0, r_0).
\]
In view of (3.17) and (3.22) this proves estimate (3.9) for \(n \geq 2\).

4. Model peak operator

4.1. Problem setting

Throughout this section, we keep fixed a value of \(a > 0\). Our goal is to study the properties of the operator \(\widetilde{S}_{1, \varepsilon}^a\) for \(\varepsilon \to 0\). In order to simplify the notation we denote
\[
T_{\varepsilon, a} := \widetilde{S}_{1, \varepsilon}^a.
\]  
Then \(T_{\varepsilon, a}\) is the self-adjoint operator in \(L^2(V_{\varepsilon, a})\) generated by the quadratic form
\[
t_{\varepsilon, a}(u, u) = \widetilde{s}_{1, \varepsilon}^a(u, u) = \int_{V_{\varepsilon, a}} |\nabla u|^2 \, dx - \int_{\partial V_{\varepsilon, a}} u^2 \, d\sigma, \quad D(t_{\varepsilon, a}) = \widetilde{H}_{0}^1(V_{\varepsilon, a}),
\]  
see section 2 for the notation. We start with a technical result. Denote
\[
D_0(t_{\varepsilon, a}) = \{ u \in C^\infty(V_{\varepsilon, a}) : \exists b, c \in (0, a) \text{ such that } u(x) = 0 \text{ for } x_1 < b \text{ and for } x_1 > c \}.
\]

**Lemma 19.** The subspace \(D_0(t_{\varepsilon, a})\) is dense in \(\widetilde{H}_{0}^1(V_{\varepsilon, a})\) in the norm of \(H^1(V_{\varepsilon, a})\).

**Proof.** We provide a quite standard proof for sake of completeness. First remark that the subspace \(D^\infty := \widetilde{H}_{0}^1(V_{\varepsilon, a}) \cap L^\infty(V_{\varepsilon, a})\) is dense in \(\widetilde{H}_{0}^1(V_{\varepsilon, a})\) in the norm of \(H^1(V_{\varepsilon, a})\). Indeed, for \(u \in \widetilde{H}_{0}^1(V_{\varepsilon, a})\) and \(k > 0\) set \(u_k := \min \{ \max \{ u, -k \}, k \} \), then \(u_k \in L^\infty(V_{\varepsilon, a})\) and it is standard to check that \(u_k \in \widetilde{H}_{0}^1(V_{\varepsilon, a})\) and that \(u_k\) converges to \(u\) in \(H^1(V_{\varepsilon, a})\) as \(k \to +\infty\). Therefore, it is sufficient to check that any function from \(D^\infty\) is the limit in \(H^1(V_{\varepsilon, a})\) of functions from \(D_0(t_{\varepsilon, a})\).

Let \(u \in D^\infty\). Let \(\chi : \mathbb{R} \to \mathbb{R}\) be an increasing \(C^\infty\)-function with \(\chi(s) = 0\) for \(s < \frac{1}{2}\) and \(\chi(s) = 1\) for \(s > 1\). For \(\delta > 0\) we denote by \(u_\delta\) the function on \(V_{\varepsilon, a}\) given by
\[
u_\delta(x) = u(x) \chi \left( \frac{x_1}{\delta} \right) \chi \left( \frac{a - x_1}{\delta} \right).
\]
Then for some \(C > 0\) and small \(\delta > 0\) one has
\[
\|u - u_\delta\|_{\widetilde{H}_{1}^1(V_{\varepsilon, a})} \leq C \int_{V_{\varepsilon, a}} (|u|^2 + |
abla u|^2) \left( 1 - \chi \left( \frac{x_1}{\delta} \right) \chi \left( \frac{a - x_1}{\delta} \right) \right)^2 \, dx
\]
\[
+ \frac{C}{\delta^2} \int_{V_{\varepsilon, a} : x_1 < \delta} |u|^2 \, dx + \frac{C}{\delta^2} \int_{V_{\varepsilon, a} : x_1 > a - \delta} |u|^2 \, dx =: I_1 + I_2 + I_3.
\]
By the monotone convergence theorem the term $I_1$ tends to zero as $\delta \to 0$. To estimate $I_2$ we remark that

$$
\int_{V_{\varepsilon,a}} |u|^2 \, dx \leq \|u\|_\infty^2 \int_{V_{\varepsilon,a}} 1 \, dx = \|u\|_\infty^2 \int_0^\delta \int_{B_{\varepsilon,a}} dx' \, dx_1 = O(\delta^{np+1}),
$$

and due to $np + 1 > 2$ we see that $I_2$ tends to zero as $\delta \to 0$. To estimate $I_3$ we first remark that almost everywhere one has

$$
u(x_1, x') = \int_a^{x_1} \partial_{x_1} u(t, x') \, dt,$$

hence, using Hölder’s inequality,

$$|u(x_1, x')|^2 \leq (a - x_1) \int_{x_1}^a (\partial_{x_1} u(t, x') \, dx_1)^2 \leq (a - x_1) \int_{x_1}^a |\nabla u(t, x')|^2 \, dt,$$

and then

$$I_3 \leq \frac{C}{\delta^2} \int_{V_{\varepsilon,a} : x_1 > a - \delta} (a - x_1) \int_{x_1}^a |\nabla u(t, x')|^2 \, dt \, dx_1$$

$$= \frac{C}{\delta^2} \int_{a - \delta}^a (a - x_1) \int_{x_1}^a \int_{-\varepsilon x_1^p}^{\varepsilon x_1^p} |\nabla u(t, x')|^2 \, dx' \, dt \, dx_1$$

$$\leq \frac{C}{\delta^2} \int_{a - \delta}^a (a - x_1) \int_{x_1}^a \int_{-\varepsilon t^p}^{\varepsilon t^p} |\nabla u(t, x')|^2 \, dx' \, dt \, dx_1$$

$$\leq \frac{C}{\delta^2} \int_{a - \delta}^a (a - x_1) dx_1 \int_{a - \delta}^a \int_{-\varepsilon t^p}^{\varepsilon t^p} |\nabla u(t, x')|^2 \, dx' \, dt$$

$$= \frac{C}{2} \int_{V_{\varepsilon,a} : x_1 > a - \delta} |\nabla u|^2 \, dx,$$

and the last term tends to 0 for $\delta \to 0$ due to $|\nabla u|^2 \in L^1(V_{\varepsilon,a})$. Therefore, $u_\delta$ converges to $u$ in $H^1(V_{\varepsilon,a})$.

As $u_\delta = 0$ for $x_1 < \delta/2$ and $x_1 > a - \delta/2$, it follows from the preceding constructions that the set

$$D_1(\varepsilon, a) = \{u \in H^1(V_{\varepsilon,a}) : \exists b, c \in (0, a) \text{ such that } u(x) = 0 \text{ for } x_1 < b \text{ and } x_1 > c\}$$

is dense in $\tilde{H}_{0}^1(V_{\varepsilon,a})$ in the norm of $H^1(V_{\varepsilon,a})$. On the other hand, in the same norm $D_0(\varepsilon, a)$ is dense in $D_1(\varepsilon, a)$ using the standard mollifying procedure.

In view of Lemma 19 it follows by the min-max principle that for any $j \in \mathbb{N}$ one has

$$E_j(T_{\varepsilon,a}) = \inf_{S \subset D_0(\varepsilon, a)} \sup_{u \in S} \frac{t_{\varepsilon,a}(u, u)}{\|u\|_{L^2(V_{\varepsilon,a})}^2}. \quad (4.2)$$
4.2. Change of variables

Let \((s, t) = (s, t_1, t_2, \ldots, t_n) \in \Pi_\varepsilon\), where
\[
\Pi_\varepsilon = (0, a) \times \mathcal{B}_\varepsilon, \quad \mathcal{B}_\varepsilon \subset \mathbb{R}^n.
\] (4.3)

Then \(V_{\varepsilon, \alpha} = X(\Pi_\varepsilon)\) for \(X(s, t) = (s, ts^\alpha)\), and the transform
\[
u \mapsto \mathcal{U} \nu, \quad \mathcal{U} \nu(s, t) := \nu(X(s, t))
\] (4.4)
maps \(L^2(\Pi_\varepsilon)\) unitarily on \(L^2(\Pi_\varepsilon, s^{\alpha p} \, ds \, dt)\). We are going to study the quadratic form \(q_{\varepsilon}\) in \(L^2(\Pi_\varepsilon, s^{\alpha p} \, ds \, dt)\) given by
\[
q_{\varepsilon}(\nu, \nu) := t_{\varepsilon, \alpha}(\mathcal{U}^{-1} \nu, \mathcal{U}^{-1} \nu)
\]
with the domain \(\mathcal{D}(q_{\varepsilon}) = \mathcal{U} \mathcal{D}(t_{\varepsilon, \alpha})\). For this purpose, denote
\[
\mathcal{D}_0(q_{\varepsilon}) := \mathcal{U} \mathcal{D}_0(t_{\varepsilon, \alpha})
\]
\[
\equiv \{ \nu \in C^\infty(\overline{\Pi}_\varepsilon) : \exists b, c \in (0, a) \text{ such that } \nu(s, t) = 0 \text{ for } s < b \text{ and for } s > c \}, \quad (4.5)
\]
which is a core of \(q_{\varepsilon}\) by construction. Hence in view of (4.2) one has
\[
E_j(T_{\varepsilon, \alpha}) = \inf_{\mathcal{S} \subset \mathcal{D}_0(q_{\varepsilon})} \sup_{u \in \mathcal{S}} \frac{q_{\varepsilon}(u, u)}{\|u\|_{L^2(\Pi_\varepsilon, s^{\alpha p} \, ds \, dt)}^2}, \quad (4.6)
\]
and a standard calculation then shows that for \(\nu \in \mathcal{D}_0(q_{\varepsilon})\) there holds
\[
q_{\varepsilon}(\nu, \nu) = \int_0^a \int_{\mathcal{B}_\varepsilon} \langle \nabla \nu, G \nabla \nu \rangle_{\mathbb{R}^n} s^{\alpha p} \, ds \, d\tau - \int_0^a \int_{\partial \mathcal{B}_\varepsilon} u^2 s^{(n-1)} \, d\tau \, ds,
\] (4.7)
where \(d\tau\) denotes the \((n-1)\)-dimensional Hausdorff measure, and \(G\) is an \(N \times N\) matrix given by
\[
G = \left( \begin{array}{cc}
1 + p^2 t^2 s^{2p-2} & ps^{2p-1} t \\
p^{2p-1} t^T & s^{2p} I
\end{array} \right)^{-1}.
\]
Here \(I\) stands for the \(n \times n\) identity matrix. One checks directly that
\[
G = \left( \begin{array}{cc}
1 & -\frac{t}{s} t^T \\
-\frac{t}{s} t^T & C
\end{array} \right) \quad \text{with} \quad C_{jk} = \left\{ \begin{array}{ll}
s^{-2p} + p^2 t_j s^{-2} & \text{if } j = k, \\
p^{2p-2} t_j t_k & \text{if } j \neq k.
\end{array} \right.
\] (4.8)

Using the Young inequality and equation (4.8) we find that for \(\varepsilon\) small enough
\[
(1 - n\varepsilon p) |\partial_s u|^2 + \left(1 - \frac{n\varepsilon p^2}{s^2} + \varepsilon p \right) |\nabla_t u|^2 \leq \langle \nabla u, G \nabla u \rangle_{\mathbb{R}^n} \leq \left(1 + n\varepsilon p\right) |\partial_s u|^2 + \left(1 + \frac{n\varepsilon p^2}{s^2} + \varepsilon p \right) |\nabla_t u|^2.
\] (4.9)

In what follows we will also need the transform
\[
u \mapsto (\mathcal{V} \nu)(s, t) = s^{-\frac{\alpha p}{2}} \nu(s, t),
\] (4.10)
which maps \(L^2(\Pi_\varepsilon)\) unitarily onto \(L^2(\Pi_\varepsilon, s^{\alpha p} \, ds \, dt)\).
4.3. Upper bound

We start with a comparison between \( T_{\varepsilon,a} \) and the one-dimensional operator \( L_{\varepsilon,a} \).

**Lemma 20.** There exist \( c > 0, c' > 0, \varepsilon_0 > 0 \) such that

\[
E_j(T_{\varepsilon,a}) \leq (1 + c\varepsilon)E_j(L_{(1+c\varepsilon)\varepsilon,a}) + c' \quad \forall j \in \mathbb{N}, \forall \varepsilon \in (0, \varepsilon_0).
\]  

\[ (4.11) \]

**Proof.** By equations (4.7) and (4.9), for \( u \in D_0(q_\varepsilon) \) one has

\[
q_\varepsilon(u, u) \leq q_\varepsilon^+(u, u) := \int_0^a \int_{\mathbb{R}_s} \left(1 + np\varepsilon\right) |\partial_s u|^2
+ \left(\frac{1}{s^{2p}} + \frac{pe + np^2 \varepsilon^2}{s^2}\right) |\nabla_t u|^2 \right) s^{np} dt ds
- \int_0^a \int_{\partial \mathbb{R}_s} s^{p(n-1)} u^2 d\tau ds.
\]

A simple calculation then shows that for \( u \in D_0(r_\varepsilon^+) := \mathcal{V}^{-1}D_0(q_\varepsilon) \equiv D_0(q_\varepsilon) \) there holds

\[
r_\varepsilon^+(u, u) := q_\varepsilon^+(\mathcal{V} u, \mathcal{V} u)
= \int_0^a \int_{\mathbb{R}_s} \left(1 + np\varepsilon\right) \left(\partial_s u - \frac{nu}{2s}\right)^2
+ \left(\frac{1}{s^{2p}} + \frac{pe + np^2 \varepsilon^2}{s^2}\right) |\nabla_t u|^2 \right) dt ds
- \int_0^a \frac{1}{s^p} \int_{\partial \mathbb{R}_s} u^2 d\tau ds.
\]

Eq. (4.6) implies then

\[
E_j(T_{\varepsilon,a}) \leq \inf_{S \subseteq D_0(r_\varepsilon^+) \atop \dim S = j} \sup_{u \in S \atop u \neq 0} \frac{r_\varepsilon^+(u, u)}{\|u\|_{L^2(S)}}
\]  

\[ (4.12) \]

The integration by parts gives that for \( u \in D_0(r_\varepsilon^+) \) one has

\[
\int_0^a u \partial_s u \frac{ds}{s} = \int_0^a \frac{u^2}{2s^2} ds,
\]  

\[ (4.13) \]

which implies that

\[
r_\varepsilon^+(u, u) = \int_0^a \int_{\mathbb{R}_s} \left(1 + np\varepsilon\right) \left(\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2\right)
+ \left(\frac{1}{s^{2p}} + \frac{pe + np^2 \varepsilon^2}{s^2}\right) |\nabla_t u|^2 \right) dt ds
- \int_0^a \frac{1}{s^p} \int_{\partial \mathbb{R}_s} u^2 d\tau ds.
\]

Having in mind that due to (3.1)

\[
\int_0^a \left(\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2\right) ds \geq \frac{1}{16} \int_0^a \left(\partial_s u|^2 - \frac{u^2}{4s^2}\right) ds \geq 0,
\]
for $\varepsilon$ small enough one can estimate, with a suitable $c > 0$, 
\[
r^+_{\varepsilon}(u, u) \leq (1 + c\varepsilon) \int_0^a \int_{\mathcal{M}_\varepsilon} \left( |\partial_s u|^2 - \frac{n^2 p^2 - 2np}{4s^2} u^2 + \frac{1}{s^{2p}} |\nabla_t u|^2 \right) dt ds \\
- \int_0^a \int_{\partial\mathcal{M}_\varepsilon} u^2 d\tau ds \\
= (1 + c\varepsilon) \int_0^a \int_{\mathcal{M}_\varepsilon} \left( |\partial_s u|^2 - \frac{n^2 p^2 - 2np}{4s^2} u^2 \right) dt ds \\
+ (1 + c\varepsilon) \int_0^a \frac{1}{s^{2p}} \left\{ \int_{\mathcal{M}_\varepsilon} |\nabla_t u|^2 dt - \frac{s^p}{1 + c\varepsilon} \int_{\partial\mathcal{M}_\varepsilon} u^2 d\tau \right\} ds.
\]
Note that the functional in the curly brackets is the quadratic form $b_{\varepsilon, \rho(s, \varepsilon)}$ as defined in subsection 3.2 with 
\[
\rho(s, \varepsilon) = (1 + c\varepsilon)^{-1} s^p.
\]
Denote by $\psi \equiv \psi_{\varepsilon, \rho(s, \varepsilon)}$ the positive normalized eigenfunction of the associated $B_{\varepsilon, \rho(s, \varepsilon)}$ relative to the eigenvalue $E_1(B_{\varepsilon, \rho(s, \varepsilon)})$.

Now let $S \subset C_0^\infty(0, a)$ be a linear subspace with dimension $j$ and define 
\[
\tilde{S} = \{ u : \Pi_\varepsilon \rightarrow \mathbb{R} : u(s, t) = f(s) \psi_{\varepsilon, \rho(s, \varepsilon)}(t), f \in S \}. 
\]
Then $\dim \tilde{S} = j$ and $\tilde{S} \subset D_0(r^+_{\varepsilon})$ due to Lemma 18. Hence for $u \in \tilde{S}$ one has 
\[
\|u\|_{L^2(\Pi_\varepsilon)} = \|f\|_{L^2(0, a)} : \int_{\mathcal{M}_\varepsilon} |\nabla_t u|^2 dt - \frac{s^p}{1 + c\varepsilon} \int_{\partial\mathcal{M}_\varepsilon} u^2 d\tau = E_1(B_{\varepsilon, \rho(s, \varepsilon)}) f(s)^2.
\]
Moreover, 
\[
\int_0^a \int_{\mathcal{M}_\varepsilon} \left( |\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2 \right) dt ds \\
= \int_0^a \left[ |f'|^2 + \left( \frac{n^2 p^2 - 2np}{4s^2} + \int_{\mathcal{M}_\varepsilon} |\partial_s \psi_{\varepsilon, \rho(s, \varepsilon)}|^2 dt \right) f^2 \right] ds.
\]
Using (3.9) we see that there exists $K > 0$ such that for $\varepsilon$ small enough we have 
\[
\int_{\mathcal{M}_\varepsilon} |\partial_s \psi_{\varepsilon, \rho(s, \varepsilon)}|^2 dt = \int_{\mathcal{M}_\varepsilon} \left( \partial_s \psi_{\varepsilon, \rho(s, \varepsilon)} \right) \right|_{\rho(\varepsilon, s)} \frac{\partial \rho(\varepsilon, s)}{\partial s} dt \\
= \frac{p^2 s^{2p - 2}}{(1 + c\varepsilon)^2} \int_{\mathcal{M}_\varepsilon} \left( \partial_s \psi_{\varepsilon, \rho(s, \varepsilon)} \right) \right|_{\rho(\varepsilon, s)}^2 dt \leq K \frac{p^2 s^{2p - 2}}{(1 + c\varepsilon)^2} \varepsilon^2 < \varepsilon \quad \forall \varepsilon \in (0, a).
\]
Hence 
\[
r^+_{\varepsilon}(u, u) \leq (1 + c\varepsilon) \int_0^a \left[ |f'|^2 + \left( \frac{n^2 p^2 - 2np}{4s^2} + \varepsilon + \frac{E_1(B_{\varepsilon, \rho(s, \varepsilon)})}{s^{2p}} \right) f^2 \right] ds.
\]
To continue we apply Lemma 18 which implies that there exists $c_0 > 0$, independent of $\varepsilon$, such that 
\[
\frac{E_1(B_{\varepsilon, \rho(s, \varepsilon)})}{s^{2p}} \leq \frac{n \varepsilon \rho(s, \varepsilon) + c_0 \varepsilon^2 \rho^2(s, \varepsilon)}{\varepsilon^2 s^{2p}} \leq \frac{c_0}{(1 + c\varepsilon) \varepsilon s^p} + \frac{c_0}{(1 + c\varepsilon)^2}.
\]
This implies that the inequality
\[
\frac{r^+_2(u, u)}{\|u\|^2_{L^2(\Pi_\varepsilon)}} \leq (1 + c\varepsilon) \int_0^a \left[ |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} - \frac{n}{(1 + c\varepsilon)} s^p \right) \right] ds + \frac{c_0}{1 + c\varepsilon} + \varepsilon(1 + c\varepsilon) 
\]
holds for each \(u \in \tilde{S}\). Therefore,
\[
\inf_{S \subset D_0(r_2^+)} \sup_{u \in \overline{S}} \frac{r^+_2(u, u)}{\|u\|^2_{L^2(\Pi_\varepsilon)}} \leq \inf_{S \subset C_0^0(0, a)} \sup_{u \notin S} \frac{r^+_2(u, u)}{\|u\|^2_{L^2(\Pi_\varepsilon)}} 
\]
\[
\leq (1 + c\varepsilon) \inf_{S \subset C_0^0(0, a)} \sup_{f \in S} \frac{\langle f, L_{(1+c\varepsilon)} f \rangle_{L^2(0, a)}}{\|f\|^2_{L^2}} + \frac{c_0}{1 + c\varepsilon} + \varepsilon(1 + c\varepsilon) 
\]
\[
= (1 + c\varepsilon) E_j(L_{(1+c\varepsilon)} f, a) + \frac{c_0}{1 + c\varepsilon} + \varepsilon(1 + c\varepsilon),
\]
and the substitution into (4.12) concludes the proof.

A combination of Lemma 20 with Lemma 16 gives then the main result of this subsection:

**Proposition 21.** There exist \(K > 0, k > 0\) and \(\varepsilon_0 > 0\) such that
\[
E_j(T_{\varepsilon, a}) \leq (1 - k\varepsilon) \varepsilon^{\frac{j}{2}} E_j(A_1) \quad \forall \varepsilon \in (0, \varepsilon), \quad 1 \leq j \leq \mathcal{N}(A_1, -K\varepsilon^{\frac{n}{n-2}}).
\]

### 4.4. Lower bound

**Lemma 22.** There exist \(\varepsilon_0 > 0, b > 0\) and \(B > 0\) such that
\[
E_j(T_{\varepsilon, a}) \geq (1 - b\varepsilon) E_j(L_{(1-b\varepsilon)} f, a) - B \quad \text{for all} \quad j \in \{1, \ldots, \mathcal{N}(T_{\varepsilon, a}, -B)\}.
\]

**Proof.** There exist \(c > 0\) and \(\varepsilon_0 > 0\) such that for all \(\varepsilon \in (0, \varepsilon_0)\) and \(s \in (0, a)\) there holds
\[
\frac{1}{s^{2p}} - \frac{pe + np^2 \varepsilon^2}{s^2} \geq 1 - \frac{(pe + np^2 \varepsilon^2) s^{2p-2}}{s^{2p}} \geq 1 - \frac{c\varepsilon}{s^{2p}},
\]
\[
\sqrt{1 + p^2 \varepsilon^2 s^{2p-2}} \leq \sqrt{1 + p^2 \varepsilon^2 s^{2p-2}} \leq \frac{1}{1 - c\varepsilon}.
\]
Combining the two inequalities with (4.9) we estimate the quadratic form \(q_{\varepsilon}\) from below as follows:
\[
q_{\varepsilon}(u, u) \geq q_{\varepsilon}^-(u, u) := (1 - c\varepsilon) \int_0^a \int_{\partial \beta \varepsilon} (|\partial_a u|^2 + s^{-2p} |\nabla u|^2) s^{np} dtds
\]
\[
- \frac{1}{1 - c\varepsilon} \int_0^a \int_{\partial \beta \varepsilon} u^2 s^{p(n-1)} dtds \quad \forall \varepsilon \in D_0(q_\varepsilon).
\]
By construction we then have

$$E_j(T_{\varepsilon,a}) \geq \inf_{S \subseteq D_0(q_c)} \sup_{u \in S \setminus \{0\}} \frac{q_c^{-}(u,u)}{\|u\|^2_{L^2(\Pi_s)}}.$$  

Now consider the quadratic form \( r_c^{-} \) in \( L^2(\Pi_s) \) given by \( r_c^{-}(u,u) = q_c^{-}(\mathcal{V}u,\mathcal{V}u) \) defined on \( D_0(r_c^{-}) = \mathcal{V}^{-1}D_0(q_c) \equiv D_0(q_c) \). Hence

$$E_j(T_{\varepsilon,a}) \geq \inf_{S \subseteq D_0(r_{c}^{-})(\varepsilon)} \sup_{u \not\in S} \frac{r_{c}^{-}(u,u)}{\|u\|^2_{L^2(\Pi_s)}}. \tag{4.18}$$

The direct substitution in combination with (4.13) shows that

$$r_{c_{\varepsilon}}^{-}(u,u) = (1 - \varepsilon) \int_0^a \int_{\mathcal{B}_s} (|\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2 + s^{-2p} |\nabla u|^2) \, dt \, ds \tag{4.19}$$

$$- \frac{1}{1 - \varepsilon} \int_0^a \int_{\mathcal{B}_s} \frac{u^2}{s^p} \, dt \, ds$$

$$= (1 - \varepsilon) \int_0^a \int_{\mathcal{B}_s} (|\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2) \, dt \, ds$$

$$+ (1 - \varepsilon) \int_0^a \frac{1}{s^{2p}} \left\{ \int_{\mathcal{B}_s} |\nabla u|^2 \, dt - \frac{s^p}{(1 - \varepsilon)^2} \int_{\partial \mathcal{B}_s} u^2 \, d\tau \right\} \, ds.$$

The expression in the curly brackets is the quadratic form \( b_{c_{\varepsilon}}(s,\varepsilon) \) with

$$\varrho(s,\varepsilon) = \frac{s^p}{(1 - \varepsilon)^2} \in (0,M), \quad M := \frac{\alpha p}{(1 - \varepsilon_0)^2}, \quad \varepsilon \in (0,\varepsilon_0), \tag{4.20}$$

see section 3.2. Let \( \psi_{c_{\varepsilon}}(s,\varepsilon) \) be the positive normalized eigenfunction of \( B_{c_{\varepsilon}}(s,\varepsilon) \) relative to the eigenvalue \( E_1(B_{c_{\varepsilon}}(s,\varepsilon)) \). We decompose each \( u \in D_0(r_{c}^{-}) \) as

$$u = v + w, \text{ where } v(s,t) = \psi_{c_{\varepsilon}}(s,\varepsilon)(t) f(s), \quad f(s) := \int_{\mathcal{B}_s} u(s,t) \psi_{c_{\varepsilon}}(s,\varepsilon)(t) \, dt. \tag{4.21}$$

Notice that by construction we have \( f \in C_0^\infty(0,a) \). Furthermore,

$$\int_{\mathcal{B}_s} w(s,t) \psi_{c_{\varepsilon}}(s,\varepsilon)(t) \, dt = 0 \quad \forall s \in (0,a), \tag{4.22}$$

$$\|f\|^2_{L^2(0,a)} + \|w\|^2_{L^2(\Pi_s)} = \|u\|^2_{L^2(\Pi_s)}, \tag{4.23}$$

and the spectral theorem implies that

$$\int_{\mathcal{B}_s} |\nabla u|^2 \, dt - \varrho(s,\varepsilon) \int_{\partial \mathcal{B}_s} u^2 \, d\tau \geq E_1(B_{c_{\varepsilon}}(s,\varepsilon)) f(s)^2 + E_2(B_{c_{\varepsilon}}(s,\varepsilon)) \int_{\mathcal{B}_s} u^2 \, dt. \tag{4.24}$$

Recall, see Lemma 18(c), that one can find a constant \( c_1 > 0 \) such that

$$E_1(B_{1,x}) = -nx + O(x^2) \geq \frac{-nx}{1 - c_1 x} \quad \text{for small } x > 0.$$
By Lemma 18(a) we have $E_1(B_{\varepsilon,\varrho}(s,\varepsilon)) = \varepsilon^{-2}E_1(B_{1,\varrho}(s,\varepsilon))$, and $\varrho(s,\varepsilon) \in [0, M\varepsilon]$. By adjusting the value of $\varepsilon_0$ we conclude that there exists $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $s \in (0, a)$ it holds

$$\frac{E_1(B_{\varepsilon,\varrho}(s,\varepsilon))}{\varepsilon^2 s^{2p}} = \frac{E_1(B_{1,\varrho}(s,\varepsilon))}{\varepsilon^2 s^{2p}} \geq \frac{-n \varrho(s,\varepsilon)}{\varepsilon^2 s^{2p}(1 - c_2 \varrho(s,\varepsilon))} \geq -\frac{n}{\varepsilon(1 - c_2 \varepsilon)s^p}.$$

In a similar way, using the fact that $E_2(B_{1,\varrho}) = E_2^N + O(\varepsilon) \geq A_0 > 0$ for small $x$, see Lemma 18(d), we conclude that if $\varepsilon_0$ is sufficiently small, then for all $s \in (0, a)$ and $\varepsilon \in (0, \varepsilon_0)$ there holds

$$\frac{E_2(B_{\varepsilon,\varrho}(s,\varepsilon))}{s^p} = \frac{E_2(B_{1,\varrho}(s,\varepsilon))}{\varepsilon^2 s^{2p}} \geq \frac{A_0}{\varepsilon^2 s^{2p}}.$$

Inserting these eigenvalue estimates into (4.24) we arrive the inequality

$$\int_{\mathcal{M}_s} |\nabla u|^2 dt - \varrho(s, \varepsilon) \int_{\mathcal{M}_s} u^2 d\tau \geq -\frac{n f(s)^2}{\varepsilon(1 - c_2 \varepsilon)s^p} + \frac{1}{\varepsilon^2 s^{2p}} \int_{\mathcal{M}_s} w^2 dt,$$

which is valid for all $u \in D_0(r_e^-)$. The substitution of the last inequality into (4.19) shows that one can find $k > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in D_0(r_e^-)$ there holds

$$r_e^- (u, u) \geq (1 - c\varepsilon) \int_0^a \int_{\mathcal{M}_s} \left(|\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2\right) dt ds + (1 - c\varepsilon) \int_0^a \int_{\mathcal{M}_s} \frac{w^2}{s^{2p}} ds - \int_0^a \frac{n}{(1 - k\varepsilon)s^p} f^2 ds.$$

In the sequel, for the sake of brevity we will adopt the notation $\psi := \psi_{\varepsilon,\varrho(s,\varepsilon)}$ and $\psi_s := \partial_s \psi, \quad v_s := \partial_s v, \quad w_s := \partial_s w$.

Let us study the first term on the right-hand side of (4.25). Using (4.22) we get

$$\int_0^a \int_{\mathcal{M}_s} \left(|\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2\right) dt ds = \int_0^a \int_{\mathcal{M}_s} \left(u_s^2 + \frac{n^2 p^2 - 2np}{4s^2} v_s^2\right) dt ds + \int_0^a \int_{\mathcal{M}_s} \frac{w_s^2}{s^{2p}} ds - \int_0^a \frac{n}{(1 - k\varepsilon)s^p} f^2 ds + 2 \int_0^a \int_{\mathcal{M}_s} v_s w_s dt ds.$$

Since $\psi$ is normalized, one has

$$\int_{\mathcal{M}_s} \psi v_s dt = 0,$$

and the first term on the right-hand side of (4.26) can be bounded from below as follows:

$$\int_0^a \int_{\mathcal{M}_s} \left(v_s^2 + \frac{n^2 p^2 - 2np}{4s^2} v_s^2\right) dt ds = \int_0^a \left[|f'|^2 + \frac{n^2 p^2 - 2np}{4s^2} |f|^2\right] ds \geq \int_0^a \left[|f'|^2 + \frac{n^2 p^2 - 2np}{4s^2} f^2\right] ds.$$

(4.27)
In order to estimate the last two terms in (4.26) we note that
\[ \int_0^a \int_{\partial \mathcal{L}_e} \psi w_s \, dt \, ds = \int_0^a \int_{\partial \mathcal{L}_e} f' \psi w_s \, dt \, ds + \int f \int_{\partial \mathcal{L}_e} \psi w_s \, dt \, ds. \] (4.28)

Then, in view of (4.22)
\[ \int_{\partial \mathcal{L}_e} \psi w_s \, dt = - \int_{\partial \mathcal{L}_e} \psi_s w \, dt. \]

Hence, using the Cauchy-Schwarz inequality,
\[ \left| \int_0^a \int_{\partial \mathcal{L}_e} f' \psi w_s \, dt \, ds \right| = \left| \int_0^a \int_{\partial \mathcal{L}_e} \psi_s w' \, dt \, ds + \| w \|_{L^2(\Omega_e)}^2 \right|. \]

To estimate the last term in (4.28) we use again the Young inequality;
\[ \left| \int_0^a \int_{\partial \mathcal{L}_e} \psi_s w_s \, dt \, ds \right| \leq \frac{1}{\varepsilon} \int_0^a \int_{\partial \mathcal{L}_e} \psi_s^2 \, dt \, ds + \varepsilon \| w \|_{L^2(\Omega_e)}^2. \]

By (3.9) and (4.20) there is \( K > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \) and \( s \in (0, a) \) one has
\[ \int_{\partial \mathcal{L}_e} \psi_s^2 \, dt \leq K \varepsilon^2 \leq \varepsilon. \]

Putting the above estimates together we obtain the upper bound
\[ \left| \int_0^a \int_{\partial \mathcal{L}_e} \psi w_s \, dt \, ds \right| \leq \varepsilon \int_0^a |f'|^2 \, ds + K \varepsilon \int_0^a f^2 \, ds + \| w \|_{L^2(\Omega_e)}^2 + \varepsilon \| w \|_{L^2(\Omega_e)}^2. \]

In view of (4.26) and (4.27) it follows that there exists \( C > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( u \in D_0(\mathcal{L}_e) \) one has
\[ \int_0^a \int_{\partial \mathcal{L}_e} \left( |\partial_s u|^2 + \frac{n^2 p^2 - 2np}{4s^2} u^2 \right) \, dt \, ds \geq \int_0^a \left( 1 - \varepsilon \right) |f'|^2 + \left( \frac{n^2 p^2 - 2np}{4s^2} - C \right) f^2 \right) \, ds \]
\[ + \int_0^a \int_{\partial \mathcal{L}_e} \left( 1 - \varepsilon \right) w_s^2 + \left( \frac{n^2 p^2 - 2np}{4s^2} - C \right) w^2 \, dt \, ds. \]

By (4.25) this in turn gives
\[ r_\varepsilon^- (u, u) \geq (1 - \varepsilon \varepsilon) \int_0^a \left( 1 - \varepsilon \right) |f'|^2 + \left( \frac{n^2 p^2 - 2np}{4s^2} - C \right) f^2 \right) \, ds \]
\[ + (1 - \varepsilon \varepsilon) \int_0^a \int_{\partial \mathcal{L}_e} \left( 1 - \varepsilon \right) w_s^2 + \left( \frac{n^2 p^2 - 2np}{4s^2} - C \right) w^2 \, dt \, ds \]
\[ + (1 - \varepsilon \varepsilon) \int_0^a \frac{A_0 u_s^2}{\varepsilon^2 s^3} \, dt \, ds - \int_0^a \frac{n f^2}{(1 - k \varepsilon)^2 s^3} \, ds, \]
\[ 21 \]
and using the norm equality (4.23) one may rewrite

\[ r_{-}(u, u) + (1 - c\varepsilon) C \| u \|_{L^2_{\mathbb{P}}_{\varepsilon}}^2 \]

\[ \geq (1 - c\varepsilon) \int_0^a \left[ (1 - \varepsilon) |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) f^2 \right] ds \]

\[ + (1 - c\varepsilon) \int_0^a \int_{\mathbb{H}_s} \left[ (1 - \varepsilon) w_s^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) w^2 \right] dt ds \quad (4.29) \]

Next we notice that due to the Hardy inequality (3.1) we have

\[ \int_0^a \frac{g(s)^2}{s^2} ds \leq \frac{4}{(np - 1)^2} \int_0^a \left[ (1 - \varepsilon) |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) f^2 \right] ds \]

Therefore, there exists \( c_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( u \in D_0(r_{-}) \) one has

\[ \int_0^a \left[ (1 - \varepsilon) |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) f^2 \right] ds \]

\[ \geq (1 - c_0\varepsilon) \int_0^a \left[ (1 - \varepsilon) |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) f^2 \right] ds, \]

\[ \int_0^a \int_{\mathbb{H}_s} \left[ (1 - \varepsilon) w_s^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) w^2 \right] dt ds \]

\[ \geq (1 - c_0\varepsilon) \int_0^a \int_{\mathbb{H}_s} \left[ w_s^2 + \left( \frac{n^2p^2 - 2np}{4s^2} \right) w^2 \right] dt ds. \]

We thus conclude that there exist \( b > 0 \) and \( B > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and all \( u \in D_0(r_{-}) \) there holds

\[ r_{-}(u, u) + B \| u \|_{L^2_{\mathbb{P}}_{\varepsilon}}^2 \geq (1 - bc) \int_0^a \left[ |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} - \frac{n}{(1 - bc)\varepsilon s^p} \right) f^2 \right] ds \]

\[ + (1 - bc) \int_0^a \int_{\mathbb{H}_s} \left[ w_s^2 + \left( \frac{n^2p^2 - 2np}{4s^2} + \frac{A_0}{\varepsilon s^p} \right) w^2 \right] dt ds. \]

Assuming \( \varepsilon_0 \) sufficiently small we have

\[ \frac{n^2p^2 - 2np}{4s^2} + \frac{1}{\varepsilon^2 s^{2p}} \geq 0 \quad \forall s \in (0, a). \]

Therefore,

\[ r_{-}(u, u) + B \| u \|_{L^2_{\mathbb{P}}_{\varepsilon}}^2 \]

\[ \geq (1 - bc) \int_0^a \left[ |f'|^2 + \left( \frac{n^2p^2 - 2np}{4s^2} - \frac{n}{(1 - bc)\varepsilon s^p} \right) f^2 \right] ds. \quad (4.30) \]

Note that by construction of \( f \) and \( w \) and the norm equality (4.23) the map \( u \mapsto (f, w) \) extends to a unitary map \( \Psi : L^2(\mathbb{P}_{\varepsilon}) \to L^2(0, a) \times \mathcal{H} \), where \( \mathcal{H} \) is a closed subspace of
Let \( R^{-}_{\varepsilon} \) be the self-adjoint operator in \( L^2(\Pi_{\varepsilon}) \) generated by the closure of \( r^{-}_{\varepsilon} \), then the inequality (4.18) means that
\[
E_j(T_{\varepsilon,a}) \geq E_j(R^{-}_{\varepsilon}) \quad \forall \ j \in \mathbb{N}.
\]
On the other hand, let \( h_\varepsilon \) be the quadratic form in \( L^2(0, a) \times \mathcal{H} \) defined as the closure of the form
\[
C_\infty^0(0, a) \times \mathcal{H} \ni (f, w) \mapsto \int_0^a \left[ |f'|^2 + \frac{n^2 p^2 - 2np}{4s^2} \frac{\varepsilon}{(1 - b\varepsilon)^2 s^p} \right] ds,
\]
then the corresponding self-adjoint operator in \( L^2(0, a) \times \mathcal{H} \) is \( H_\varepsilon = L_{(1 - b\varepsilon)^2, \varepsilon,a} \). The inequality (4.30) reads as
\[
r^{-}_{\varepsilon}(u, u) + B\|u\|^2_{L^2(\Pi_{\varepsilon})} \geq (1 - b\varepsilon) h_\varepsilon(\Psi u, \Psi u), \quad u \in \mathcal{D}_0(r^{-}_{\varepsilon})
\]
which by the min-max principle implies that \( E_j(R^{-}_{\varepsilon}) + B \geq (1 - b\varepsilon) E_j(H_\varepsilon) \). Assume now that \( j \in \{1, \ldots, N(T_{\varepsilon,a}, -B)\} \). Then \( E_j(T_{\varepsilon,a}) < -B \) and \( E_j(R^{-}_{\varepsilon}) + B < 0 \), which shows that for the same \( j \) one has \( E_j(H_\varepsilon) < 0 \), and then \( E_j(T_{\varepsilon,a}) = E_j(L_{(1 - b\varepsilon)^2, \varepsilon,a}) \).

Now we can state the main result of the subsection.

**Proposition 23.** Let \( a > 0 \), then there exist \( K > 0, k > 0 \) and \( \varepsilon_0 > 0 \) such that
\[
E_j(T_{\varepsilon,a}) \geq (1 + k\varepsilon) \varepsilon^{\frac{s}{p-1}} E_j(A_1) - K \quad \forall \ \varepsilon \in (0, \varepsilon_0), \ 1 \leq j \leq N(T_{\varepsilon,a}, -K).
\]

**Proof.** Due to (3.5) and (3.2) one has, for any \( j \in \mathbb{N} \) and a suitably chosen \( k > 0 \),
\[
E_j(L_{(1 - b\varepsilon)^2, \varepsilon,a}) \geq E_j(A_{(1 - b\varepsilon)^2_{\varepsilon}}) = (1 - b\varepsilon)^\frac{s}{p-1} \varepsilon^{\frac{s}{p}} E_j(A_1) \geq (1 + k\varepsilon) \varepsilon^{\frac{s}{p}} E_j(A_1).
\]
The substitution of the above lower bound into the result of Lemma 22 completes the proof.

**5. Proof of Proposition 12**

In order to simplify the notation, for \( b > 0 \) and \( \varepsilon > 0 \), we denote
\[
\ell := b\varepsilon^{\frac{1}{p-1}}.
\]
Then, the operators \( Q_{\varepsilon,b} \) and \( \tilde{Q}_{\varepsilon,b} \) defined in (2.6) are generated by the quadratic forms
\[
q_{\varepsilon,b}(u, u) = \int_{V_{\varepsilon,\ell}} |\nabla u|^2 dx - \int_{\partial V_{\varepsilon,\ell}} u^2 ds, \quad \mathcal{D}(q_{\varepsilon,b}) = H^1(V_{\varepsilon,\ell}), \quad \text{and}
\]
\[
\tilde{q}_{\varepsilon,b}(u, u) = q_{\varepsilon,b}(u, u), \quad \mathcal{D}(\tilde{q}_{\varepsilon,b}) = \tilde{H}^1_0(V_{\varepsilon,\ell})
\]
respectively, with \( V_{\varepsilon,\ell} \) defined in (2.3). Note that the domain inclusions imply the obvious inequalities
\[
E_j(Q_{\varepsilon,b}) \leq E_j(\tilde{Q}_{\varepsilon,b}) \quad \text{for all} \ j \in \mathbb{N},
\]
\[
E_j(\tilde{Q}_{\varepsilon,b}) \leq E_j(T_{\varepsilon,a}) \quad \text{for all} \ a \leq \ell \text{ and } j \in \mathbb{N},
\]
where the operator \( T_{\varepsilon,a} \) is defined in (4.1). Let us give a lower bound for \( Q_{\varepsilon,b} \):
Lemma 24. Let \( a > 0 \). Then there exist \( B > 0, k > 0 \) and \( \varepsilon_0 > 0 \) such that
\[
E_j(Q_{\varepsilon,b}) \geq E_j(T_{\varepsilon,a}) - k \quad \forall \varepsilon \in (0, \varepsilon_0), \quad 1 \leq j \leq N(Q_{\varepsilon,b} - B/\varepsilon).
\]

Proof. One may assume from the very beginning that \( a \leq \ell \). Let \( \phi_1 \) and \( \phi_2 \) be two smooth functions on \( \mathbb{R} \) with the following properties:
\[
\phi_1^2 + \phi_2^2 = 1, \quad \phi_1(s) = 0 \text{ for } s > a, \quad \phi_2(s) = 0 \text{ for } s < a/2.
\]
We set
\[
k := \|\phi_1^2\|_\infty^2 + \|\phi_2^2\|_\infty^2, \quad \chi_j(x) := \phi(x_1), \quad j \in \{1, 2\}.
\]
By a direct computation, for any \( u \in D(q_{\varepsilon,b}) \) there holds
\[
q_{\varepsilon,b}(u,u) = q_{\varepsilon,b}(\chi_1 u, \chi_1 u) + q_{\varepsilon,b}(\chi_2 u, \chi_2 u) - \int_{V_{\varepsilon,\ell}} (|\nabla \chi_1|^2 + |\nabla \chi_2|^2) u^2 \, dx
\geq q_{\varepsilon,b}(\chi_1 u, \chi_1 u) + q_{\varepsilon,b}(\chi_2 u, \chi_2 u) - k\|u\|_{L^2(V_{\varepsilon,\ell})}^2.
\]
Denote
\[
r_{\varepsilon}(u,u) = \int_{W_\varepsilon} |\nabla u|^2 \, dx - \int_{\partial_0 W_\varepsilon} u^2 \, ds, \quad D(r_{\varepsilon}) = \tilde{H}_0^1(W_\varepsilon),
\]
\[
W_\varepsilon = \{ (x_1, x') : x_1 \in (a/2, \ell), |x'| < \varepsilon x_1^p \} \subset \mathbb{R}^N,
\]
\[
\partial_0 W_\varepsilon = \{ (x_1, x') : x_1 \in (a/2, \ell), |x'| = \varepsilon x_1^p \} \subset \partial V_{\varepsilon,\ell},
\]
\[
\tilde{H}_0^1(W_\varepsilon) = \{ u \in H^1(W_\varepsilon) : u(\ell, \cdot) = u(a/2, \cdot) = 0 \},
\]
and let \( R_{\varepsilon} \) be the self-adjoint operator acting in \( L^2(W_\varepsilon) \) and generated by \( r_{\varepsilon} \). Then one has
\[
\|\chi_1 u\|_{L^2(V_{\varepsilon,\ell})} = \|\chi_1 u\|_{L^2(V_{\varepsilon,a})}, \quad \|\chi_2 u\|_{L^2(V_{\varepsilon,\ell})} = \|\chi_2 u\|_{L^2(W_\varepsilon)},
\]
\[
q_{\varepsilon,b}(\chi_1 u, \chi_1 u) = t_{\varepsilon,a}(\chi_1 u, \chi_1 u), \quad q_{\varepsilon,b}(\chi_2 u, \chi_2 u) = r_{\varepsilon}(\chi_2 u, \chi_2 u),
\]
and for any \( j \in \mathbb{N} \) the min-max principle gives
\[
E_j(Q_{\varepsilon,b}) + k \geq \inf_{S \subset D(q_{\varepsilon,b})} \sup_{u \in S, u \neq 0} \frac{q_{\varepsilon,b}(\chi_1 u, \chi_1 u) + q_{\varepsilon,b}(\chi_2 u, \chi_2 u)}{\|\chi_1 u\|_{L^2(V_{\varepsilon,a})}^2 + \|\chi_2 u\|_{L^2(W_\varepsilon)}^2}
= \inf_{S \subset D(q_{\varepsilon,b})} \sup_{u \in S, u \neq 0} \frac{t_{\varepsilon,a}(\chi_1 u, \chi_1 u) + r_{\varepsilon}(\chi_2 u, \chi_2 u)}{\|\chi_1 u\|_{L^2(V_{\varepsilon,a})}^2 + \|\chi_2 u\|_{L^2(W_\varepsilon)}^2}
\geq \inf_{S \subset D(t_{\varepsilon,a}) \oplus D(r_{\varepsilon})} \sup_{(u_1, u_2) \in S} \frac{t_{\varepsilon,a}(u_1, u_1) + r_{\varepsilon}(u_2, u_2)}{\|u_1\|_{L^2(V_{\varepsilon,a})}^2 + \|u_2\|_{L^2(W_\varepsilon)}^2}
= E_j(T_{\varepsilon,a} \oplus R_{\varepsilon}).
\]
Let us now obtain a lower bound for \( R_{\varepsilon} \). Using Fubini's theorem, for \( u \in H^1(W_\varepsilon) \) one
has
\[
\int_{W_\varepsilon} |\nabla u|^2\, dx - \int_{\partial_\varepsilon W_\varepsilon} u^2\, d\sigma = \int_{\varepsilon/2}^\varepsilon \int_{\partial_\varepsilon x_1^p} |\nabla u|^2\, dx' - \sqrt{1 + \varepsilon^2 p^2 x_1^{2p-2}} \int_{\partial_\varepsilon x_1^p} u^2\, d\tau \, dx_1
\]
\[
\geq \int_{\varepsilon/2}^\varepsilon \left[ \int_{\partial_\varepsilon x_1^p} |\nabla u|^2\, dx' - \sqrt{1 + \varepsilon^2 p^2 x_1^{2p-2}} \int_{\partial_\varepsilon x_1^p} u^2\, d\tau \right] \, dx_1
\]
\[
\geq \int_{\varepsilon/2}^\varepsilon \left[ E_1(B_{\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}}) \int_{\partial_\varepsilon x_1^p} u^2\, d\tau \right] \, dx_1 \geq \Lambda \int_{W_\varepsilon} u^2\, dx \tag{5.4}
\]
with
\[
\Lambda := \inf_{x_1 \in (a/2, \ell)} E_1(B_{\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}}).
\]
Due to the parts (a) and (c) of Lemma 18 one has
\[
E_1(B_{\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}}) = \frac{1}{\varepsilon^2 x_1^{2p}} E_1(B_{1,\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}})
\]
\[
= -n \sqrt{1 + \varepsilon^2 p^2 x_1^{2p-2}} \left( \frac{1}{\varepsilon x_1^p} + (1 + \varepsilon^2 p^2 x_1^{2p-2}) \varphi(\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}) \right)
\]
with \( \varphi \in L^\infty(0, \infty) \). Hence,
\[
E_1(B_{\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}}) = -n \sqrt{\frac{1}{x_1^{2p}} + \frac{\varepsilon^2 p^2}{x_1^p}} \frac{1}{\varepsilon}
\]
\[
+ (1 + \varepsilon^2 p^2 x_1^{2p-2}) \varphi(\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2})
\]
Note that under the assumptions \( 0 < \varepsilon < \varepsilon_0 \) and \( a/2 < x_1 < \ell \) one has
\[
B_1 := \sqrt{\frac{1}{a^{2p}} + \frac{\varepsilon^2 p^2}{a^2}} \geq \sqrt{\frac{1}{x_1^{2p}} + \frac{\varepsilon^2 p^2}{x_1^p}},
\]
\[
B_2 := 1 + b^2 p^2 x_1^{2p-2} \geq 1 + \varepsilon^2 p^2 x_1^{2p-2},
\]
implying
\[
E_1(B_{\varepsilon x_1^p,1+\varepsilon^2 p^2 x_1^{2p-2}}) \geq -\frac{n B_1}{\varepsilon} - B_2 \|\varphi\|_\infty.
\]
Hence \( \Lambda \geq -n B_1/\varepsilon - B_2 \|\varphi\|_\infty \geq -B/\varepsilon \) for \( B := n B_1 + B_2 \|\varphi\|_\infty \varepsilon_0 \). Therefore, by (5.4), for \( j \leq \mathcal{N}(T_{\varepsilon,a}, B/\varepsilon) \) one has \( E_j(T_{\varepsilon,a} \oplus R_{\varepsilon}) = E_j(T_{\varepsilon,a}) \), and the substitution into (5.3) gives the result.

**Proof of Proposition 12.** The upper bound (2.7) follows by combining inequality (5.2) with Proposition 21. On the other hand, Lemma 24 together with Proposition 23 imply the lower bound (2.8).
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