Learning in Markov Decision Processes under Constraints

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Abstract

We consider reinforcement learning (RL) in Markov Decision Processes (MDPs) in which at each time step the agent, in addition to earning a reward, also incurs an $M$ dimensional vector of costs. The objective is to design a learning rule that maximizes the cumulative reward earned over a finite time horizon of $T$ steps, while simultaneously ensuring that the cumulative cost expenditures are bounded appropriately. The considerations on the cumulative cost expenditures is in departure from the existing RL literature, in that the agent now additionally needs to balance the cost expenses in an online manner, while simultaneously performing optimally the exploration-exploitation trade-off typically encountered in RL tasks. This is challenging since either of the duo objectives of exploration and exploitation necessarily require the agent to expend resources.

When the constraints are placed on the average costs, we present a version of UCB algorithm and prove that its reward as well as cost regrets are upper-bounded as $O\left( T_M S \sqrt{AT \log(T)} \right)$, where $T_M$ is the mixing time of the MDP, $S$ is the number of states, $A$ is the number of actions, and $T$ is the time horizon. We further show how to modify the algorithm in order to reduce regrets of a desired subset of the $M$ costs, at the expense of increasing the regrets of rewards and the remaining costs. We then consider RL under the constraint that the vector comprising of the cumulative cost expenditures until each time $t$ must be less than $c^\text{ub} t$. We propose a “finite (B)-state” algorithm and show that its average reward is within $O\left( e^{-B} \right)$ of $r^*$, the latter being the optimal average reward under average cost constraints. Moreover, the “shifted” regret scales as $O\left( D^{(\ell)} S \sqrt{AT \log(T)} \right)$, where $D^{(\ell)}$ is the diameter of a certain “lifted” process.

1. Introduction

Reinforcement Learning (RL) involves an agent repeatedly interacting with an unknown environment modelled by a Markov Decision Process (MDP), which is characterized by its
states, actions and controlled transition probabilities. In a typical RL task, the goal of the agent is to make control decisions so as to maximize its cumulative rewards over a finite time horizon of $T$ steps, which is taken to be a measure of its performance. In many applications, at each time, the agent also incurs a cost that depends upon the current state and the choice of its action. Hence, it needs to balance the reward earnings with the cost accretion while simultaneously learning the choice of optimal decisions, all in an online manner.

As a motivating example, consider a wireless node that has a queue which contains information packets, and can attempt packet transmissions at varying power levels to a receiver. The channel reliability, i.e., the probability that a transmission at time $t$ is successful depends upon the instantaneous channel state $c_{st}$ and the transmission power $a_t$. The instantaneous reward is equal to the number of packets in queue denoted by $Q_t$, while the cost is equal to $a_t$. Since the node is battery-operated, the total energy consumption has to be less than that available in the storage battery. The channel reliabilities are unknown to the node. The state of the “environment” in this example is given by $(Q_t, c_{st})$. The goal of transmitter is to choose $a_t$ judiciously so as to minimize the average transmission delay while satisfying battery energy constraints.

We consider the following two kinds of constraints on cost expenses:

a) **Average Cost Constraints**: The $M$ dimensional vector comprising of the expectations of the time-average of the total cost expenditures must be lower than the vector comprising of $M$ thresholds $\{c_{ub}^i\}_{i=1,M}$ specified by the agent.

b) **Hard constraints**: This is motivated by a setup in which the $M$ resources of interest are supplied to the agent by an external process. Hence, it is required that the cumulative cost incurred until any time $t$ must be less than the resources supplied until $t$.

### 1.1 Previous Works

For RL problems without constraints, works such as Brafman and Tennenholtz (2002); Bartlett and Tewari (2009); Auer and Ortner (2007); Jaksch et al. (2010) have utilized the principle of “optimism under uncertainty”. The algorithms we propose in this work are based on the UCRL2 algorithm of Jaksch et al. (2010). Below we discuss the existing works on learning in MDPs with average and hard constraints.

**Average Constraints**: Altman and Schwartz (1991) is an early work on optimally controlling unknown CMDPs. It shows that the certainty equivalence approach to learning, i.e., implementing the policy that is optimal for the empirical estimate $\hat{p}_t$ of $p$, does not yield optimal performance. This occurs because there may not exist any feasible policy for $\hat{p}_t$. The authors propose a scheme and prove that it yields asymptotically the optimal average reward in case the CMDP is strictly feasible. However, our analysis is done under the assumption that the CMDP is feasible; infact the use of upper-confidence bounds allows us to work under the feasibility assumption rather than strict feasibility. Borkar (2005) derives a learning scheme based on multi time-scale stochastic approximation Borkar (1997), in which the task of learning optimal policy for the CMDP is decomposed into that of learning the optimal value of dual variables, and that of learning the optimal policy for an unconstrained MDP that is parameterized by the dual variables. However, the proposed scheme lacks finite-time regret analysis, and might suffer from a large regret. Prima facie,
this layered decomposition might not be optimal with respect to sample-complexity of the online learning problem.

**Hard Constraints:** The works Ross and Varadarajan (1989, 1991) initiated studies on the existence and design of policies that are pathwise optimal with respect to the limiting value of average reward. They exclusively cover the case when the MDP is known and the costs are absent, and show that for communicating MDPs, the sample path optimality is equivalent to expected average reward maximization. To the best of authors’ knowledge, ours is the first work that considers the setup of hard sample-path constraints for MDPs. A somewhat related problem called the MAB under budget constraints has been studied in the multi-armed bandit (MAB) literature Badanidiyuru et al. (2018); Badanidiyuru et al. (2013); Ding et al. (2013); Combes et al. (2015). Herein, each pull of a bandit arm yields a random reward and also incurs a random cost, where the probability distributions governing the rewards are arm dependent. The agent can spend a maximum of $B$ units of cost resource, and has to pull the arms so as to maximize the cumulative rewards earned until the resources are exhausted. Analysis of jump diffusion processes constrained to lie in a polyhedral cone has been done under assumptions on the Skorohod map which maps the “system randomness” to the process trajectories Atar et al. (2002). However, such an assumption seems difficult to verify. Our setup is different from these works on the following two accounts. Firstly, unlike the bandit setup, the underlying MDP introduces temporal correlations in the reward and cost process, so that the problem is more challenging. Secondly, in the MAB problem, the agent is provided the $B$ units of resource before the experiment begins, and hence the issue of “burstiness” in the resource consumption does not arise. However, in our case, upon exhausting a resource, the agent needs to wait until the arrival of “resource replenishments”, thereby wasting the opportunity to collect (state-dependent) rewards.

### 1.2 Contributions

For both the problems discussed above, we present learning algorithms based on the upper confidence bounds (UCB) strategy that follow the principle of optimism in the face of uncertainty (OFU). The algorithms are called UCB-CMDP and UCRL-HARD respectively. We summarize our contributions to the two problems, and in the process also state some problems which call for further investigation.

#### Learning Under Average Cost Constraints

1. The use of confidence intervals/UCB principle allows us to overcome the shortcomings of certainty equivalence based approach that was pointed out in Altman and Schwartz (1991). Moreover, unlike Altman and Schwartz (1991), we do not need to resort to “forced explorations”.

2. We show that for UCB-CMDP, the total regret of reward, as well as $M$ costs\(^1\) can be upper-bounded as $O\left( T_M S \sqrt{AT \log(TM)} \right)$, where $T_M$ is the mixing time of the true

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\(^1\) The total $i$-th cost regret is defined as the difference between expected value of total cost incurred until $T$ steps, minus $c_i^T$. 

3
MDP $p$. It remains to be seen if the dependence on the structure of the MDP can be improved to its diameter $D(p)$ as in the UCRL algorithm of Jaksch et al. (2010).

3. We provide a partial characterization of the set of those regret vectors\(^2\) which cannot be attained by any learning algorithm. More specifically, a weighted sum of these $M+1$ regrets is necessarily greater than $O\left(DS\sqrt{AT}\log(T)\right)$ under any algorithm. Since this lower bound on weighted regrets does not depend upon $M$, this raises the question whether it is possible to remove this dependence in the upper-bound discussed in 2.).

4. We also introduce a modification to UCB-CMDP which allows us to tune the $M+1$ dimensional regret vector in case it is more sensitive to cost expenditures of some resources than the others. Such a procedure reduces some components of the vector, at the expense of increased regrets in other components.

### Learning Under Hard Constraints

1. When the MDP is known, we design a class of feasible non-stationary policies parametrized by buffer-size $B$ which choose the action $a_t$ at step $t$ as a function of a certain “lifted process” $s^{(t)}_i$ that contains (in addition to $s_t$) relevant information about the past history of the controlled process, i.e., $\{(s_t, a_t)\}_{t=1}^{t-1}$. This allows the policy to obtain reward within $O(e^{-B})$ of $r^*$, i.e., the optimal reward of CMDP.

2. UCRL-HARD combines the policy of 1. with the OFU principle. Its total regret is shown to be $O(DS\sqrt{AT})$.

3. We conjecture that for the problem of learning under hard constraints, the optimal average reward is equal to that under average cost constraints, $r^*$, and can be attained by an algorithm that uses the entire past information $\{(s_t, a_t)\}_{t=1}^{t-1}$ in order to make control decisions $a_t$.

### 2. Preliminaries

Consider a controlled Markov process $s_t$, $t \in \mathbb{N}$ with finite state and action spaces denoted $S, A$. The controlled transition probabilities are described by $p := \{p(s, a, s')\}_{s, s' \in S, a \in A}$ and are unknown. The reward and $M$ cost functions are denoted $r := \{r(s, a)\}_{(s, a) \in S \times A}$, $c_i := c_i(s, a)_{(s, a) \in S \times A}$, $i \in [1, M]$, respectively, and are assumed to be known to the agent.

A stationary policy $\pi : S \mapsto \Delta(A)^3$ prescribes randomized controls on the basis of the current state $s_t$; i.e., $a_t$, is distributed according to $\pi(s_t)$.

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2. The $M+1$ dimensional vector which contains total reward regret and $M$ cost regrets.
3. $\Delta(A)$ is the $|A|$-simplex
For a stationary policy \( \pi \), let \( \mathbb{E}_\pi(T_{s,s'}) \) denote the expected time taken to hit the state \( s' \) when starting in state \( s \) and evolving under the application of \( \pi \). The MDP \( p \) is called unichain if \( \mathbb{E}_\pi(T_{s,s'}) \) is finite for all \( \pi \), and we let

\[
T_M := \max_{\pi} \left( \max_{s,s' \in \mathcal{S}} \mathbb{E}_\pi(T_{s,s'}) \right),
\]

(1)

denote the mixing time of the MDP Auer and Ortner (2007). The UCRL algorithm of Auer and Ortner (2007) has a regret that scales linearly with \( T_M \).

Assuming that the MDP is unichain might be too strong an assumption. A somewhat weaker assumption is that the MDP is communicating, i.e., for each pair \((s, s')\) of states, there is a policy \( \pi \) (which can depend upon \((s, s')\)) under which \( \mathbb{E}_\pi(T_{s,s'}) \) is finite. In this case, the “diameter” of the MDP is defined as follows,

\[
D(p) := \max_{s,s' \in \mathcal{S}} \left( \min_{\pi} \mathbb{E}_\pi(T_{s,s'}) \right),
\]

(2)

and is finite.

3. Problems Studied

We derive learning algorithms that maximize the average reward under the following two kinds of constraints on the cost expenditures.

3.1 Expected Average Costs

A constrained Markov Decision Process (CMDP) requires designing a policy in order to maximize the infinite-horizon average reward, subject to constraints on average costs, i.e.,

\[
\max \lim_{T \to \infty} \inf \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} r(s_t, a_t) \right]
\]

(3)

\[
\text{s.t. } \lim_{T \to \infty} \sup \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} c_i(s_t, a_t) \right] \leq c_{i}^{ub}, \quad i \in [1, M].
\]

(4)

When the MDP is known, the CMDP can be solved by a stationary policy that necessarily requires randomization, and moreover the average reward and costs are independent of the initial starting state \( s_0 \) Altman (1999). Denote by \( r^* \) the optimal reward of (3)-(4) in this case. The first problem that we study is to solve (3)-(4) when the MDP \( p \) is unknown to the learner. Section 4 derives a learning algorithm to solve this problem.

3.2 Hard Sample-Path Constraints

Any policy that satisfies (4) will only provide probabilistic guarantees on \( \sum_{t<T} c(s_t, a_t) - c_{i}^{ub} T \), i.e., the cost exceedance at the end of the time-horizon \( T \). These guarantees could be tightened in the following two directions:

1. Sample Path constraints: The bounds on the cumulative cost expenditure must now be satisfied in an almost sure sense, similar to MDPs with sample path constraints Ross and Varadarajan (1989, 1991).
2. Hard Constraints: In many applications, we might want the “hard” constraint \( \sum_{t < \ell} c(s_t, a_t) \leq c_i^{ub} \ell \) to hold for each time \( \ell \in [0, T] \). Such is the case when the agent is fed resources by a generator which generates \( c_{i,t} \) units of resource \( i \) at time \( t \). \( c_{i,t} \) could be allowed to be random, as long as the sample path averages tend to \( c_i^{ub} \). However, we exclusively consider the case \( c_{i,t} = c_i^{ub}, \forall t \) pathwise.

Motivated by 1. and 2. above, we propose learning algorithms for the following problem in Section 5,

\[
\max_{\pi} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \sum_{t=1}^{T} r(s_t, a_t) \\
\text{s.t. } \sum_{t < \ell} c(s_t, a_t) \leq c_i^{ub} \ell, \forall \ell \in [0, T], i \in [1, M] \text{ a.s.}
\]  

(5)

(6)

3.3 List of Assumptions Made

The first three assumptions are made while studying both the problems.

**Assumption 1** We will assume that the MDP is unichain.

**Assumption 2** The CMDP (3)-(4) is feasible, i.e., there exists a policy under which the average cost constraints are satisfied.

**Assumption 3** Without loss of generality, we assume that the magnitude of rewards and costs are upper-bounded by 1, i.e.,

\[ |r(\cdot, \cdot)|, |c_i(\cdot, \cdot)| < 1. \]

Hence,

\[ r^*, \{c_i^*\}_{i \in [1, M]} < 1. \]

Hence \( r^*, \{c_i^*\}_{i \in [1, M]} < 1. \)

In addition to the above assumptions, in order to ensure that the set of policies which satisfy (6) is non-empty, we make the following assumption on the cost functions \( c_i \) in Section 5. It is easily verified that if this assumption is violated, then one can construct an MDP \( p \) in which under any policy \( \phi \), the cost expenses incurred during any finite time-interval exceed the quantity of resources generated in the same duration, with a non-zero probability.

**Assumption 4** For each state \( s \in \mathcal{S} \), there is an action \( a_{\text{cheap}}(s) \) (possibly state dependent) under which the instantaneous cost \( c(s, a_{\text{cheap}}(s)) \) is less than \( c_i^{ub} \). We denote by \( \pi_{\text{cheap}} \), a policy that implements such an action in each state \( s \in \mathcal{S} \).
Algorithm 1 UCB-CMDP

**Input:** State-space $\mathcal{S}$, Action-space $\mathcal{A}$, Confidence parameter $\delta$, Time horizon $T$

**Initialize:** Set $t := 1$, and observe the initial state $s_1$.

for Episodes $k = 1, 2, \ldots$ do

Initialize Episode $k$:

1. Set the start time of episode $k$, $\tau_k := t$. For all state-action tuples $(s, a) \in \mathcal{S} \times \mathcal{A}$, initialize the number of visits within episode $k$, $n_k(s, a) = 0$.

2. For all $(s, a) \in \mathcal{S} \times \mathcal{A}$ set $N_{\tau_k}(s, a)$, i.e., the number of visits to $(s, a)$ prior to episode $k$. Also set the transition counts $N_{\tau_k}(s, a, s')$ for all $(s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}$.

3. Compute the empirical estimate $\hat{p}_t$ of the MDP as in (9).

**Compute Policy $\tilde{\pi}_k$:**

1. Let $\mathcal{M}_{\tau_k}$ be the set of plausible MDPs as in (10).

2. Solve (12)-(15) for $\tilde{\pi}_k$ by using the iterative algorithm of Section 5.2.

3. In case (12)-(15) is infeasible, choose $\tilde{\pi}_k$ to be some pre-determined policy (chosen at time $t = 0$).

**Implement $\tilde{\pi}_k$:**

while $n_k(s_t, a_t) < N_k(s_t, a_t)$ do

1. Sample $a_t$ according to the distribution $\tilde{\pi}_k(\cdot | s_t)$. Observe reward $r(s_t, a_t)$, and observe next state $s_{t+1}$.

2. Update $n_k(s_t, a_t) = n_k(s_t, a_t) + 1$.

3. Set $t := t + 1$.

end while

end for
4. UCB-CMDP: A Learning Algorithm for CMDPs

For each \( s, s' \in \mathcal{S} \) and \( a \in \mathcal{A} \), define
\[
N_t(s,a) = \sum_{l < t} 1 \{s_l = s, a_l = a\},
\]
\[
N_t(s,a,s') = \sum_{l < t-1} 1 \{s_l = s, a_l = a, s_{l+1} = s'\},
\]
\[
\hat{p}_t(s,a,s') = \frac{N_t(s,a,s')}{N_t(s,a)} \lor 1.
\]

We now construct the confidence interval \( \mathcal{M}_t \) associated with the estimate \( \hat{p}_t \) as follows. The set \( \mathcal{M}_t \) is composed of MDPs \( p' \) for which
\[
\mathcal{M}_t := \{ p' : \|p'(s,a,\cdot) - \hat{p}_t(s,a,\cdot)\|_1 \leq \epsilon_t(s,a), \forall (s,a) \in \mathcal{S} \times \mathcal{A} \},
\]
where the distance
\[
\epsilon_t(s,a) := \sqrt{\frac{\alpha \log(t/\delta)}{N_t(s,a)}},
\]
and the parameter \( \alpha > 0 \) will be specified shortly.

**Episode**: A new episode begins each time the number of visits to a state-action pair \((s,a)\) doubles. Let \( \tau_k \) denote the start time of episode \( k \), and \( k \)-th episode comprises of the set of time-slots \( \mathcal{E}_k := [\tau_k, \tau_{k+1} - 1] \). Let \( n_k(s,a) \) be the number of times the pair \((s,a)\) is visited during the \( k \)-th episode.

At the beginning of \( \mathcal{E}_k \), the agent solves the following constrained optimization problem, in which the decision variables are the occupation measure \( \{\mu(s,a)\}_{(s,a) \in \mathcal{S} \times \mathcal{A}} \) of the controlled process, and the “candidate” MDP \( p' \),
\[
\max_{\mu, p'} \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \mu(s,a)r(s,a),
\]
\[
s.t. \sum_{a \in \mathcal{A}} \mu(s,a) = \sum_{(s',b) \in \mathcal{S} \times \mathcal{A}} \mu(s',b)p'(s',b,s),
\]
\[
\sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \mu(s,a)c_i(s,a) \leq c_i^{ab}, i \in [1,M]
\]
\[
\|p'(s,a,\cdot) - \hat{p}_{\tau_k}(s,a,\cdot)\|_1 \leq \epsilon_{\tau_k}(s,a),
\]
\[
\mu(s,a) \geq 0 \ \forall (s,a) \in \mathcal{S} \times \mathcal{A}, \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \mu(s,a) = 1.
\]

The maximization w.r.t. \( p' \) denotes that the agent is optimistic regarding the belief of the “true” (but unknown) MDP \( p \), while that w.r.t. \( \mu \) means that it is optimizing its control strategy within the current episode \( k \) for this optimal MDP.

Let \((\tilde{\mu}_k, \tilde{p}_k)\) denote an optimal solution of (12)-(15). Within the \( k \)-th episode, the agent implements the following stationary randomized policy, which is denoted \( \tilde{\pi}_k \). When the state \( s_t \) is equal to \( s \), it chooses the action \( a \) with a probability equal to \( \tilde{\mu}_k(s,a) / \sum_{a' \in \mathcal{A}} \tilde{\mu}_k(s,a') \). In case the problem (12)-(15) is infeasible, the agent implements some pre-specified stationary policy.
4.1 Performance of UCB-CMDP

Let \( r^\star, c^i, i \in [1, M] \) denote the average reward and costs of an optimal policy for (3)-(4).

The “cumulative reward regret” until time \( T \), denoted \( \Delta^{(R)} \) is given by

\[
\Delta^{(R)} \triangleq r^\star T - \sum_{t=1}^{T} r(s_t, a_t). \tag{17}
\]

Similarly, “cumulative cost regret” for the \( i \)-th cost, incurred until \( T \) is given by,

\[
\Delta^{(i)} \triangleq \sum_{t=1}^{T} c_i(s_t, a_t) - c_i^{ub} T. \tag{18}
\]

In the conventional regret analysis of reinforcement learning algorithms, the objective is to bound solely the regret \( \Delta^{(R)} \) associated with the cumulative rewards. However, in our set-up, it is natural to also consider the regrets \( \Delta^{(i)} \). Indeed, as shown in the next section, we can force \( \Delta^{(R)} \) to be arbitrary small at the expense of increased \( \Delta^{(i)} \), and vice versa. The following result is proved in Appendix.

**Theorem 1** With a probability greater than \( 1 - \delta \), we have that,

\[
\Delta^{(R)} \leq 34 \cdot T M S \sqrt{AT \log \left( \frac{T}{\delta} \right)},
\]

\[
\Delta^{(i)} \leq 34 \cdot T M S \sqrt{AT \log \left( \frac{T}{\delta} \right)}, \forall i \in [1, M]. \tag{20}
\]

4.2 Learning Under Bounds on Cost Regret

It was shown that under the UCB-CMDP rule, the reward and cost regrets can be upper-bounded by a term which is \( O(T M S \sqrt{AT}) \). In fact, the upper-bounds on all the \( M + 1 \) regrets were the same. We will now derive algorithms which enable the agent to tune the upper-bounds on the regrets of different costs. This is appealing since the agent might be more sensitive to over-utilizing certain specific costs, as compared to the other costs.

Let \( b = (b_1, b_2, \ldots, b_M) \in \mathbb{R}^M_+ \) be given. We now slightly modify UCB-CMDP in order that the upper-bound (20) on \( \Delta^{(i)} \) becomes \( \Delta^{(i)} \leq b_i \sqrt{T \log \left( \frac{T}{\delta} \right)} \), and also derive an upper-bound on the reward regret \( \Delta^{(R)} \). Within this section, we assume that the true MDP \( p \) satisfies the following.

**Assumption 5** Let \( \epsilon > 0 \) be a parameter. The CMDP (3)-(4) is strictly feasible if the upper bound on the costs is set equal to \( c^{ub} - \epsilon \mathbf{1} \). Let \( \pi_{feas.} \) be a policy under which the vector of average costs \( \bar{c}(\pi_{feas.}) \) is less than \( c^{ub} - \epsilon \mathbf{1} \), i.e., \( \bar{c}_i(\pi_{feas.}) \leq c^{ub}_i - \epsilon, \forall i \in [1, M] \). Let

\[
\eta := \min_{i=1,2,\ldots,M} c^{ub}_i - \epsilon - \bar{c}_i(\pi_{feas.}). \tag{21}
\]
**Modified UCB-CMDP**: The only difference from UCB-CMDP is that at the beginning of episode $k$, instead of solving (12)-(15), the agent now solves the following, slightly different optimization problem in order to obtain the policy $\tilde{\pi}_k$ that has to be implemented during $E_k$.

\[
\max_{\mu, p'} \sum_{(s, a) \in S \times A} \mu(s, a)r(s, a),
\]
\[
\text{s.t. } \sum_{a \in A} \mu(s, a) = \sum_{(s', b) \in S \times A} \mu(s', b)p'(s', b, s),
\]
\[
\sum_{(s, a) \in S \times A} \mu(s, a)c_i(s, a) \leq c_{i}^{ub}, i \in [1, M]
\]
\[
\|p'(s, a, \cdot) - \tilde{p}_k(s, a, \cdot)\| \leq \epsilon_t(s, a),
\]
\[
\mu(s, a) \geq 0 \quad \forall (s, a) \in S \times A, \quad \sum_{(s, a) \in S \times A} \mu(s, a) = 1.
\]

where

\[
d_i := \frac{(34 - b_i) \cdot T_M S \sqrt{A T \log \left(\frac{T}{\delta}\right)}}{T}, i \in [1, M].
\]

Let $\tilde{\mu}_k, \tilde{p}_k$ denote an optimal solution of the above problem. Within the $k$-th episode, the agent implements the following stationary randomized policy, which is denoted $\tilde{\pi}_k$. When the state is equal to $s$, the action $a$ is chosen with a probability $\tilde{\mu}_k(s, a) / \sum_a \tilde{\mu}_k(s, a')$. In case the above problem is infeasible, then the agent implements some pre-specified stationary policy.

**Remark 2** The differing values of $d_i$ emphasize the relative importance placed on various cost constraints. Thus, a smaller $d_i$, or equivalently a larger $b_i$, means that the agent is willing to face a higher amount of regret corresponding to violating the constraint for the corresponding cost.

**Theorem 3** With a probability greater than $1 - \delta$, the cumulative regrets under the modified UCB-CMDP rule can be bounded as follows

\[
\Delta^{(R)} \leq \left(34 + \left(34 - \min_i b_i\right) \cdot \left(\frac{\hat{\eta}}{\eta}\right)\right) \cdot T_M S \sqrt{AT \log \left(\frac{T}{\delta}\right)}, \quad i \in [1, M].
\]

4.3 Achievable Regrets

For the CMDP (3),(4), a problem instance is described by the transition probability $p$ of the underlying MDP.
Definition 4 Let $b = \{b_i\}_{i=1}^{M+1} \in \mathbb{R}_+^{M+1}$, and $f : \mathbb{N} \mapsto \mathbb{R}$ be an increasing function. We say that the tuple $(f, b)$ is achievable if there exists a rule $\phi$ under which the regrets satisfy
\[
\mathbb{E}_\phi \Delta^{(R)} \leq f(T) b_1, \quad \mathbb{E}_\phi \Delta^{(i)} \leq f(T) b_{i+1}, \quad i \in [1, M],
\]
for all time horizon $T \in \mathbb{N}$, and all problem instances $p$. We denote the set of all such tuples by $\Psi$. Similarly, $(f, b)$ is "unachievable", i.e. $(f, b) \in \Psi^c$, if there exists a problem instance $p$ under which for all allowable $\phi$, one of the inequalities in (30) fail to hold for some $T \in \mathbb{N}$.

Remark 5 Note that, unlike the case of unconstrained MDPs (or multi-armed bandit problems) the function $f$ in the above definition is not required to be necessarily increasing in $T$. This is the case because a rule $\phi$ could "temporarily" decrease some components of regrets, at the expense of increasing others.

We now provide a characterization of $\Psi^c$. Let $\lambda = \{\lambda_i\}_{i=1}^M$ be a vector satisfying $\lambda \geq 0$. Consider the Lagrangian relaxation of (3), (4), associated dual function Bertsekas (1997), and the dual problem
\[
\mathcal{L}(\lambda; \pi) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \sum_{t=1}^{T} \left\{ r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \left( c^w_i - c_i(s_t, a_t) \right) \right\},
\]
\[
D(\lambda) := \max_{\pi} \mathcal{L}(\lambda; \pi),
\]
\[
\min_{\lambda \geq 0} D(\lambda).
\]
We then have that.

Theorem 6 Define
\[
\mathcal{R}(\lambda) := \left\{ (\sqrt{T}, x) : x_1 + \sum_{i=1}^{M} x_{i+1} \lambda_i < .015 \cdot \sqrt{DSA} \right\}.
\]
We have
\[
\mathcal{R}(\lambda^*) \subseteq \Psi^c,
\]
where $\lambda^*$ is an optimal solution of the dual problem (33).

5. "Hard" Sample-Path Constraints on Costs
We begin by discussing the case when the MDP $p$ is known to the agent.

5.1 Control with Known MDP
Within this section, we are interested in solving (5), (6) under the assumption that the MDP $p$ is known to the agent. We introduce a class of feasible non-stationary policies, i.e. one for which the current action $a_t$ is not solely a function of $s_t$. A policy belonging to this class is parameterized by "buffer size" $B$. We also prove that the average reward of the policy with the parameter equal to $B$ is within $O(e^{-B})$ of $r^*$. 

11
Definition 7 (The Class \( B \) of Buffer Based Policies) Let \( \epsilon > 0 \), \( L \in \mathbb{N} \), \( B \in \mathbb{N} \) be parameters. The agent solves the following CMDP at time \( t = 0 \),

\[
\max_{\pi} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_{\pi} \sum_{t=1}^{T} r(s_t, a_t)
\]

s.t.

\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_{\pi} \sum_{t=1}^{T} c(s_t, a_t) \leq c^{ub} - \epsilon,
\]

in which there are no hard sample path constraints. Denote the stationary policy obtained above by \( \pi_{soft} \), and its average reward by \( \bar{r}(\pi_{soft}) \).

In the following discussion, the time \( \tau_i^{(m)} \) corresponds to the beginning of the \( i \)-th “mini-episode”, and \( \mathcal{E}_i^{(m)} \) denotes the time-slots belonging to the \( i \)-th mini-episode. The agent maintains a “resource-buffer” which it updates at the beginning of a new mini-episode as follows

\[
B_{\tau_i^{(m)} + 1} = \left( B_{\tau_i^{(m)}} - \sum_{t \in \mathcal{E}_i^{(m)}} c(s_t, a_t) + |\mathcal{E}_i^{(m)}| c^{ub} \right) \wedge B,
\]

where for \( x, y \in \mathbb{R} \), we let \( x \wedge y := \min \{x, y\} \). The policy to be followed during \( \mathcal{E}_i^{(m)} \) is decided on the basis of the level \( B_{\tau_i^{(m)}} \) as follows.

a) If \( B_{\tau_i^{(m)}} > L \), the agent implements \( \pi_{soft} \) for the first \( L \) time-slots. Then, it switches to implementing \( \pi_{cheap} \) until \( s_t \) hits the state 0, which also marks the beginning of new mini-episode.

b) If \( B_{\tau_i^{(m)}} \in [0, L] \), it implements \( \pi_{cheap} \) for the least number of \( 0 \to 0 \) cycles after which the buffer level reaches a value greater than \( L \) units. After this, it follows the same rule as a), i.e., implements \( \pi_{soft} \) for \( L \) time-slots and then \( \pi_{cheap} \) until state 0 is hit.

We denote the policy discussed above by \( \pi_{hard} \). We note that it is parameterized by \( (\epsilon, B, L, p) \); however we will often omit this dependency. We will occasionally also depict the dependence on only a subset of these parameters.

We now show that \( \pi_{hard} \) is feasible for (6), and moreover, its average reward approaches \( r^* \) as \( B \uparrow \infty \).

Theorem 8 The policy \( \pi_{hard} \) satisfies the hard sample-path constraints (6). Its average reward is independent of initial state \( s_0 \), and can be lower-bounded as follows

\[
\liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\pi_{hard}} r(s_t, a_t) \geq r^* - \text{Const.} \left( \frac{e^{-\frac{\epsilon^2}{2L^2}}}{1 - e^{-\frac{\epsilon^2}{2L^2}}} \cdot e^{-\frac{B\epsilon}{L^2}} \right).
\]

Thus, the average reward approaches \( r^* \) as the buffer size \( B \uparrow \infty \).

Remark 9 Denote by \( r^*_{hard} \) the optimal value of the above problem. We note that for any \( \phi \) that satisfies (6), the cost regret vector \( \{\Delta^{(i)}\}_{i=1}^{M} \) defined in (18) is less than or equal to
pathwise, and hence also in expectation. However, it is easily verified that the converse is not necessarily true, i.e., there are $\phi$ for which the limiting value of average expected cost regrets is upper-bounded by $0$, but $\phi$ do not satisfy (6). Hence we have $r_{\text{hard}}^* \leq r_*$. Since the average reward of $\pi_{\text{hard}}$ can be made to approach $r_*$ by letting $B \uparrow \infty$, we have

**Lemma 10** We have that

$$r_{\text{hard}}^* = r_*.$$  \hspace{1cm} (40)

where $r_*$ is the optimal value of CMDP (3), (4).

**The Lifted Process $s_t^{(\ell_i)}$** Note that in order to implement $\pi_{\text{hard}}$, in addition to the value of current state $s_t$, the type of the current mini-episode, time-elapsed in current mini-episode (if mini-episode is of type a)), the agent needs to keep track of the quantity

$$B_{\tau_{k_t}^{(m)}} - \sum_{t \in E_{k_t}^{(m)}} c(s_t, a_t) + |E_{k_t}^{(m)}| \cdot c^{ab}. \quad (41)$$

Since this quantity can assume arbitrarily large values, it might seem that $\pi_{\text{hard}}$ is not a finite-state controller. However, it can be shown that nothing is “lost” by replacing it by the process $x_t$ which evolves as follows. Recall that $k_t$ denotes the index of the current episode.

Case A) If $E_{k_t}^{(m)}$ is of type a), then $x_t$ evolves as follows,

$$x_{t+1} = \begin{cases} x_t + c^{ub} - c(s_t, a_t) & \text{if } t \leq \tau_{k_t}^{(m)} + L, \\ (x_t + c^{ub} - c(s_t, a_t)) \land B & \text{if } t > \tau_{k_t}^{(m)} + L. \end{cases}$$

Case B) If $E_{k_t}^{(m)}$ is of type b), then $x_t$ evolves as follows,

$$x_{t+1} = \left( B_{\tau_{k_t}^{(m)}} + c^{ub} - c(s_t, a_t) \right) \land B.$$

Now define the process $s_t^{(\ell_i)}$ as consisting of the current state $s_t$, the type of the current mini-episode $E_{k_t}^{(m)}$, and $x_t$.

We note that $s_t^{(\ell_i)}$ assumes only finitely many values, and that under the application of $\pi_{\text{hard}}$, it evolves as a finite-state Markov process with transition probabilities denoted by

$$p_t^{(\ell_i)} := \{ p_t^{(\ell_i)}(s^{(\ell_i)}, a, s^{(\ell_i)l}) \}_{s, s' \in S, a \in A}.$$  \hspace{1cm} (42)

**5.2 UCRL-HARD**

We now propose a learning rule inspired by the policy $\pi_{\text{hard}}$, and the upper-confidence bounds based algorithms. For learning rules that satisfy the hard constraints (6), the cumulative cost expenses are bounded pathwise, so that the constraints (4) on average costs become redundant. Hence, we perform a regret analysis of the cumulative rewards only. Throughout this section, we assume that the parameters $(\epsilon, B, L)$ are fixed, so we let $\pi_{\text{hard}}(\epsilon, B, L, p)$ be denoted by $\pi_{\text{hard}}(p)$.

Even if the true MDP $p$ were known, so that the agent could compute and implement $\pi_{\text{hard}}(p)$, the resulting average reward would be $\bar{r}(\pi_{\text{hard}}(p))$. Since $\bar{r}(\pi_{\text{hard}}(p))$ is guaranteed


**Algorithm 2** Policy \( \pi_{\text{hard}} \)

**Input:** State-space \( \mathcal{S} \), Action-space \( \mathcal{A} \), Parameters \( \epsilon, L, B \), and true MDP \( p \)

**Initialize:** Set \( t := 1 \), and observe the initial state \( s_1 \).

for Mini-Episodes \( i = 1, 2, \ldots \) do

**Initialize Mini-episode \( \mathcal{E}_i^{(m)} \):**

1. Set the start time of \( \mathcal{E}_i^{(m)} \), i.e., \( \tau_{i}^{(m)} := t \).

2. Set the type of \( \mathcal{E}_i^{(m)} \), i.e., \( Type(\mathcal{E}_i^{(m)}) = a \) if \( B_{\tau_i^{(m)}} > L \), while \( Type(\mathcal{E}_i^{(m)}) = b \) if \( B_{\tau_i^{(m)}} \leq L \).

3. Update the buffer-value \( B_{\tau_i^{(m)}} = x_t \).

**Compute Policy for \( \mathcal{E}_i^{(m)} \):**

1. If \( Type(\mathcal{E}_i^{(m)}) = a \) then policy follows \( \pi_{\text{soft}} \) for first \( L \) steps, and then \( \pi_{\text{cheap}} \) until the state 0 is hit.

2. If \( Type(\mathcal{E}_i^{(m)}) = b \) then follow \( \pi_{\text{cheap}} \) until the least time \( t \) when \( s_t = 0 \), and \( x_t > L \).

**Implement Policy for \( \mathcal{E}_i^{(m)} \):** If \( Type(\mathcal{E}_i^{(m)}) \) is a) then

while \( t - \tau_i^{(m)} \leq L \) do

1. Sample \( a_t \) according to \( \pi_{\text{soft}}(\cdot|s_t) \). Observe reward \( r(s_t, a_t) \), cost \( c(s_t, a_t) \) and next state \( s_{t+1} \).

2. Update \( x_t \) as \( x_{t+1} = x_t + c^{ab} - c(s_t, a_t) \).

end while

while \( s_t \neq 0 \) do

1. Use control \( a_t = \pi_{\text{cheap}}(s_t) \). Observe reward \( r(s_t, a_t) \), cost \( c(s_t, a_t) \) and next state \( s_{t+1} \).

2. Update \( x_t \) as \( x_{t+1} = (x_t + c^{ub} - c(s_t, a_t)) \land B \).

end while

Else If \( Type(\mathcal{E}_i^{(m)}) \) is b) then

while \( x_t < L \) or \( s_t \neq 0 \) do

1. Choose \( a_t \) according to \( a_t = \pi_{\text{cheap}}(s_t) \). Observe reward \( r(s_t, a_t) \), cost \( c(s_t, a_t) \) and next state \( s_{t+1} \).

2. Update \( x_t \) as \( x_{t+1} = x_t + c^{ab} - c(s_t, a_t) \).

end while

end for

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Learning in Markov Decision Processes under Constraints

to be only within $O(e^{-B\epsilon})$ of $r^*$, we modify the definition (17) of the cumulative regret as follows,

$$
\Delta^{(R)}(\hat{R}) \triangleq \hat{r}(\pi_{\text{hard}}(\epsilon, B, L; p)) \cdot T - \sum_{t=1}^{T} r(s_t, a_t). \tag{42}
$$

**UCRL-HARD**: The agent maintains an estimate $\hat{p}_t$ of the true MDP $p$, and also the confidence intervals $M_t$ in the same manner as in UCB-CMDP, i.e., (7)-(11). The reinforcement learning episodes $E_k$ are defined similar to that of UCB-CMDP, i.e., a new episode begins when the number of visits to a state-action pair $(s, a)$ (corresponding to the original process $s_t$, and not the lifted process $\tilde{s}_t$) doubles. An RL episode should not be confused with the mini-episodes $E_i^{(m)}$ that were used in previous section in order to analyze the lifted process. The agent computes the policy to be implemented within $E_k$ as follows.

For a history-dependent policy $\pi$, and an MDP $p$, we let $\bar{r}(\pi, p)$ denote the average reward earned when the policy $\pi$ is employed on the MDP $p$. At the beginning of $E_k$, the agent solves the following,

$$
\max_{p' \in \mathcal{M}_{\tau_k}} \bar{r}(\pi_{\text{hard}}(p'; \tilde{p}_k)). \tag{43}
$$

Let $\tilde{p}_k$ denote an optimal solution of the above problem. Within the $k$-th episode, the agent implements the policy $\pi_{\text{hard}}(\epsilon, B, L; \tilde{p}_k)$. The following result is proved in Appendix.

**Theorem 11** UCRL-Hard satisfies the following. With probability greater than $1 - \delta$,

$$
\Delta^{(R)}(\hat{R}) \leq 34 \cdot D^{(i)} \cdot \sqrt{AT \log \left( \frac{T}{\delta} \right)}, \tag{44}
$$

where the regret $\Delta^{(R)}(\hat{R})$ is defined as in (42), and $D^{(i)}$ is the diameter of the lifted MDP.

**Appendix A. Proof of Theorem 1**

The proof follows closely the proof of Theorem 2 of Jaksch et al. (2010) with some subtle differences.

**Regret Decomposition**: Let $n_k(s, a)$ be the number of visits to $(s, a)$ during episode $k$, i.e.,

$$
n_k(s, a) := \sum_{t \in E_k} 1 \{s_t = s, a_t = a\}, s \in S, a \in A. \tag{45}
$$

Consider the regrets incurred during $k$-th episode, i.e., let,

$$
\Delta^{(R)}_k \triangleq \sum_{s, a} n_k(s, a) (r^* - r(s, a)), \tag{46}
$$

$$
\Delta^{(i)}_k \triangleq \sum_{s, a} n_k(s, a) \left( c_i(s, a) - c_i^{ab} \right), \quad i \in [1, M]. \tag{47}
$$
Algorithm 3 UCRL-HARD

**Input:** State-space $S$, Action-space $A$, Confidence parameter $\delta$, Time horizon $T$, Parameters $(\epsilon, L, B)$

**Initialize:** Set $t := 1$, and observe the initial state $s_1$.

for Episodes $k = 1, 2, \ldots$ do

**Initialize Episode $k$:**

1. Set the start time of episode $k$, $\tau_k := t$. For all state-action tuples $(s, a) \in S \times A$, initialize the number of visits within episode $k$, $n_k(s, a) = 0$.

2. For all $(s, a) \in S \times A$ set $N_{\tau_k}(s, a)$, i.e., the number of visits to $(s, a)$ prior to episode $k$. Also set the transition counts $N_{\tau_k}(s, a, s')$ for all $(s, a, s') \in S \times A \times S$.

3. Compute the empirical estimate $\hat{p}_t$ of the MDP as in (9).

**Compute Policy $\tilde{\pi}_k$:**

1. Let $M_{\tau_k}$ be the set of plausible MDPs as in (10).

2. Solve $\max_{p' \in M_{\tau_k}} \bar{r}(\pi_{\text{hard}}(p'); p')$ for $\tilde{p}_k$. $\tilde{\pi}_k$ is then given by $\pi_{\text{hard}}(\tilde{p}_k)$.

**Implement $\tilde{\pi}_k$:**

while $n_k(s_t, a_t) < N_k(s_t, a_t)$ do

1. Sample $a_t$ according to the distribution $\tilde{\pi}_k(:, s_t)$. Observe reward $r(s_t, a_t)$, and observe next state $s_{t+1}$.

2. Update $n_k(s_t, a_t) = n_k(s_t, a_t) + 1$.

3. Set $t := t + 1$.

end while

end for
The cumulative regrets $\Delta^{(R)} \{\Delta^{(i)}\}_{i=1}^M$ can be decomposed into the sum of “episodic regrets”, i.e.,

$$
\Delta^{(R)}(R) = \sum_{k=1}^{K} \Delta^{(R)}_k,
$$

(48)

$$
\Delta^{(i)}(i) = \sum_{k=1}^{K} \Delta^{(i)}_k, \ i \in [1, M],
$$

(49)

where $K$ denotes the number of episodes incurred until $T$.

Furthermore, we decompose the episodic regrets as follows,

$$
\Delta^{(R)}_k = \Delta^{(R)}_k \mathbb{1} \{p \notin M_{t_k}\} + \Delta^{(R)}_k \mathbb{1} \{p \in M_{t_k}\},
$$

(50)

and similarly for $\{\Delta^{(i)}\}_{i=1}^M$. Thus, the cumulative regret $\Delta^{(R)}$ (48) can be written as

$$
\Delta^{(R)}(R) = \sum_{k=1}^{K} \Delta^{(R)}_k \mathbb{1} \{p \notin M_{t_k}\} + \sum_{k=1}^{K} \Delta^{(R)}_k \mathbb{1} \{p \in M_{t_k}\},
$$

(51)

and similarly for cost regrets. The two summations in the above are now bounded separately.

**Regret When Confidence Intervals Fail**

We can upper-bound the first summation in (51) as follows.

**Lemma 12** We have that with a probability at least $1 - \frac{\delta}{12T^{5/4}}$,

$$
\sum_{k=1}^{K} \Delta^{(R)}_k \mathbb{1} \{p \notin M_{t_k}\} \leq \sqrt{T}, \text{ and also } \sum_{k=1}^{K} \Delta^{(i)}_k \mathbb{1} \{p \notin M_{t_k}\} \leq \sqrt{T}, \forall i \in [1, M].
$$

**Proof** The proof is similar to Section 4.2 of Jaksch et al. (2010); the difference being that we need to upper-bound multiple regrets simultaneously. The proof carries over to our set-up since the following inequalities hold

$$
\sum_{k=1}^{K} \Delta^{(R)}_k \mathbb{1} \{p \notin M_{t_k}\} \leq \sqrt{T} + \sum_{t=T^{1/4}}^{T} \mathbb{1} \{p \notin M_t\},
$$

$$
\sum_{k=1}^{K} \Delta^{(R)}_k \mathbb{1} \{p \notin M_{t_k}\} \leq \sqrt{T} + \sum_{t=T^{1/4}}^{T} \mathbb{1} \{p \notin M_t\}, \forall i \in [1, M].
$$

The statement then follows by observing that $\mathbb{P} (\exists t : T^{1/4} < t < T : p \notin M_t) < \frac{\delta}{12T^{5/4}}$. ■
Regret When Confidence Intervals are True \( (p \in \mathcal{M}_k) \)

This section derives upper-bounds on the second term in the expression on the r.h.s. of (51). In the event \( \{p \in \mathcal{M}_k\} \), the problem (12)-(15) that needs to be solved by the agent is feasible because under Assumption 5, the CMDP (3)-(4) with controlled transition probabilities equal to the true MDP \( p \) is feasible. Denote by \( r^*_k, c^*_{i,k} \) the reward and costs associated with a solution to (12)-(15). Since \( p \in \mathcal{M}_k \), we have,

\[
r^*_k \geq r^*.
\]

Also,

\[
c^*_{i,k} \leq c^a_{i,k}, i \in [1, M],
\]

holds true trivially. Combining these with (46), (47), we obtain

\[
\Delta_k^{(R)} \{p \in \mathcal{M}_k\} \leq \sum_{s,a} n_k(s,a) (r^*_k - r(s, a)),
\]

(54)

\[
\Delta_k^{(i)} \{p \in \mathcal{M}_k\} \leq \sum_{s,a} n_k(s,a) (c_i(s,a) - c^*_{i,k}), i \in [1, M].
\]

(55)

Using Lemma 21, inequalities (54), (55) can be written as,

\[
\Delta_k^{(R)} \{p \in \mathcal{M}_k\} \leq \sum_{s,a} n_k(s,a) \left( \sum_{s'} \tilde{p}_k(s,a,s') \left( \sum_b \tilde{\pi}_k(b|s') v_k^{(R)}(s') - v_k^{(R)}(s,a) \right) \right),
\]

(56)

\[
\Delta_k^{(i)} \{p \in \mathcal{M}_k\} \leq \sum_{s,a} n_k(s,a) \left( v_k^{(i)}(s,a) - \sum_{s'} \tilde{p}_k(s,a,s') \left( \sum_b \tilde{\pi}_k(b|s') v_k^{(i)}(s',b) \right) \right),
\]

(57)

\[ i \in [1, M].\]

The above inequalities can equivalently be written as follows

\[
\Delta_k^{(R)} \{p \in \mathcal{M}_k\} \leq \sum_{s,a} n_k(s,a) \left( \sum_{s'} \tilde{p}_k(s,a,s') - p(s,a,s') \right) \left( \sum_b \tilde{\pi}_k(b|s') v_k^{(R)}(s',b) \right)
+ \sum_{s,a} n_k(s,a) \left( \sum_{s'} p(s,a,s') \left( \sum_b \tilde{\pi}_k(b|s') v_k^{(R)}(s',b) \right) - v_k^{(R)}(s,a) \right),
\]

(58)

\[
\Delta_k^{(i)} \{p \in \mathcal{M}_k\} \leq \sum_{s,a} n_k(s,a) \left( \sum_{s'} \tilde{p}_k(s,a,s') - p(s,a,s') \right) \left( \sum_b \tilde{\pi}_k(b|s') v_k^{(i)}(s',b) \right)
+ \sum_{s,a} n_k(s,a) \left( \sum_{s'} p(s,a,s') \left( \sum_b \tilde{\pi}_k(b|s') v_k^{(i)}(s',b) \right) - v_k^{(i)}(s,a) \right),
\]

(59)

\[ i \in [1, M].\]
We denote the two expressions in the r.h.s. of inequality (58) by $\Delta^{(R)}_{k,1}$ and $\Delta^{(R)}_{k,2}$ and bound them separately. Similarly define $\{\Delta^{(i)}_{k,1}, \Delta^{(i)}_{k,2}\}_{i=1}^M$.

Bounding $\Delta^{(R)}_{k,1}, \Delta^{(i)}_{k,1}$: We have

$$\Delta^{(R)}_{k,1} \leq \sum_{(s,a) \in S \times A} n_k(s,a)\|\tilde{\pi}_k(s,a,\cdot) - p(s,a,\cdot)\|_1 T_M$$

$$\leq T_M \sum_{(s,a) \in S \times A} n_k(s,a)\epsilon_{\tau_k}(s,a)$$

$$= T_M \sum_{(s,a) \in S \times A} n_k(s,a)\sqrt{\frac{\alpha \log(\tau_k/\delta)}{N_{\tau_k}(s,a)}}, \quad (60)$$

where the first inequality follows from Lemma 21 and the second from UCB confidence interval (10), (11). Similarly we obtain,

$$\Delta^{(i)}_{k,1} \leq T_M \sum_{(s,a) \in S \times A} n_k(s,a)\sqrt{\frac{\alpha \log(\tau_k/\delta)}{N_{\tau_k}(s,a)}}. \quad (61)$$

Summing the above over episodes, we obtain (see (20) of Jaksch et al. (2010) for more details):

**Lemma 13**

$$\sum_{k=1}^K \Delta^{(R)}_{k,1} \leq T_M \left(\sqrt{2} + 1\right) \sqrt{SAT \log\left(\frac{T}{\delta}\right)}, \text{ and similarly}$$

$$\sum_{k=1}^K \Delta^{(i)}_{k,1} \leq T_M \left(\sqrt{2} + 1\right) \sqrt{SAT \log\left(\frac{T}{\delta}\right)}, \forall i \in [1,M]. \quad (62)$$

Bounding $\Delta^{(R)}_{k,2}, \Delta^{(i)}_{k,2}$: Let $k_t$ denote the index of the ongoing episode at time $t$, i.e.,

$$k_t := \max\{k : \tau_k \leq t\}.$$ 

Consider the martingale difference sequence $m^{(R)}_t$ defined as follows

$$m^{(R)}_t := \mathbb{1}_{p \in \mathcal{M}_{\tau_{k_t}}} \left( v^{(R)}_{k_t}(s_{t+1}, a_{t+1}) - \sum_{(s',b)} p(s_t, a_t, s') \tilde{\pi}_{k_t}(b|s') v^{(R)}_{k_t}(s', b) \right).$$

We have

$$\Delta^{(R)}_{k,2} = \sum_{t=\tau_k}^{\tau_{k+1}-1} m^{(R)}_t + v^{(R)}(s_{\tau_{k+1}}, a_{\tau_{k+1}}) - v^{(R)}(s_{\tau_k}, a_{\tau_k})$$

$$\leq \sum_{t=\tau_k}^{\tau_{k+1}-1} m^{(R)}_t + TM, \quad i \in [1,M], \quad (63)$$
where, the inequality follows from Lemma 21. Summing the above over episodes $k$, we obtain

$$\sum_{k=1}^{K} \Delta_{k,2}^{(R)} \leq \sum_{t=1}^{T} m_t^{(R)} + K T_M, \text{ and similarly}$$

$$\sum_{k=1}^{K} \Delta_{k,2}^{(i)} \leq \sum_{t=1}^{T} m_t^{(i)} + K T_M. \tag{65}$$

It follows from Lemma 21 that $|m_t^{(R)}|, |m_t^{(i)}| \leq 2 T_M$. We can use the Azuma-Hoeffding inequality in order to obtain the following bound

$$\Pr\left( \sum_{t=1}^{T} m_t^{(R)} \geq \epsilon \right) \leq \exp\left( -\frac{\epsilon^2}{2T M^2} \right),$$

where $\epsilon > 0$. Similarly, we can upper-bound $\sum_{t=1}^{T} m_t^{(i)}$. We then use union bound in order to obtain

$$\Pr\left( \sum_{t=1}^{T} m_t^{(R)} < \epsilon; \sum_{t=1}^{T} m_t^{(i)} < \epsilon, \forall i \in [1, M] \right) \geq 1 - (M + 1) \exp\left( -\frac{\epsilon^2}{2T M^2} \right).$$

Letting $\epsilon = T_M \sqrt{2T \cdot \frac{5}{4} \log \frac{8T(M+1)}{\delta}}$, we deduce that $\sum_{t=1}^{T} m_t^{(R)}$ and $\left\{ \sum_{t=1}^{T} m_t^{(i)} \right\}_{i\in[1,M]}$ are all upper-bounded by $T_M \sqrt{2T \cdot \frac{5}{4} \log \frac{8T(M+1)}{\delta}}$ w.p. greater than $1 - \frac{\delta}{12T^{5/4}}$.

The quantity $K$, i.e. the total number of episodes incurred during $T$ steps, can be upper-bounded similar to Jaksch et al. (2010) Appendix C.2 as follows,

$$K \leq S A \log \left( \frac{8T}{SA} \right). \tag{66}$$

Combining (64), (65), (66) with the probabilistic bounds on $\sum_{t=1}^{T} m_t^{(R)}$, $\left\{ \sum_{t=1}^{T} m_t^{(i)} \right\}_{i\in[1,M]}$, $K$ discussed above, we obtain,

$$\Pr\left( \sum_{k=1}^{K} \Delta_{k,2}^{(R)} \leq x; \sum_{k=1}^{K} \Delta_{k,2}^{(i)} \leq x, \forall i \in [1, M] \right) \geq 1 - \frac{(M + 1)\delta}{12T^{5/4}}, \tag{67}$$

where $x = T_M \sqrt{2T \cdot \frac{5}{4} \log \frac{8T(M+1)}{\delta}} + T_M S A \log \left( \frac{8T}{SA} \right)$. Combining the bound (67) with Lemma 13 yields the following result.

\[20\]
Lemma 14  With a probability greater than $1 - \frac{\delta}{12T^{5/4}}$, we have that

$$\sum_{k=1}^{K} \mathbb{1}_{\{p \in \mathcal{M}_{\tau_k}\}} \Delta_{k}^{(R)} \leq T MS \left(\sqrt{2} + 1\right) \sqrt{AT \log \left(\frac{T}{\delta}\right)} + T M \sqrt{2T \cdot \frac{5}{4} \log \left(\frac{8T(M + 1)}{\delta}\right)}$$

$$+ T M S A \log \left(\frac{8T}{SA}\right),$$

and

$$\sum_{k=1}^{K} \mathbb{1}_{\{p \in \mathcal{M}_{\tau_k}\}} \Delta_{k}^{(i)} \leq T MS \left(\sqrt{2} + 1\right) \sqrt{AT \log \left(\frac{T}{\delta}\right)} + T M \sqrt{2T \cdot \frac{5}{4} \log \left(\frac{8T(M + 1)}{\delta}\right)}$$

$$+ T M S A \log \left(\frac{8T}{SA}\right), \forall i \in [1, M].$$

Proof  Easily obtained by using the definitions of $\Delta_{k,1}^{(R)}$, $\Delta_{k,2}^{(R)}$, $\{\Delta_{k,1}^{(i)}, \Delta_{k,2}^{(i)}\}_{i=1}^{M}$, the bound (67), and Lemma 13. ■

Completing the Proof

Recall the regret decompositions (51)

$$\Delta_{k}^{(R)} = \Delta_{k}^{(R)} \mathbb{1}_{\{p \notin \mathcal{M}_{\tau_k}\}} + \Delta_{k}^{(R)} \mathbb{1}_{\{p \in \mathcal{M}_{\tau_k}\}}, \quad (68)$$

$$\Delta_{k}^{(i)} = \Delta_{k}^{(i)} \mathbb{1}_{\{p \notin \mathcal{M}_{\tau_k}\}} + \Delta_{k}^{(i)} \mathbb{1}_{\{p \in \mathcal{M}_{\tau_k}\}}, \forall i \in [1, M]. \quad (69)$$

Substituting the upper-bounds derived in Lemma 12, Lemma 14, and utilizing the union bound, we infer that the following holds with a probability greater than $1 - \frac{2\delta}{12T^{5/4}}$,

$$\Delta^{(R)} \leq \sqrt{T} + T M S \left(\sqrt{2} + 1\right) \sqrt{AT \log \left(\frac{T}{\delta}\right)} + T M \sqrt{2T \cdot \frac{5}{4} \log \left(\frac{8T(M + 1)}{\delta}\right)} + T M S A \log \left(\frac{8T}{SA}\right),$$

and

$$\Delta^{(i)} \leq \sqrt{T} + T M S \left(\sqrt{2} + 1\right) \sqrt{AT \log \left(\frac{T}{\delta}\right)} + T M \sqrt{2T \cdot \frac{5}{4} \log \left(\frac{8T(M + 1)}{\delta}\right)} + T M S A \log \left(\frac{8T}{SA}\right),$$

$\forall i \in [1, M]$.

The third term in the r.h.s. can be written as

$$T M S \sqrt{AT} \left(\frac{5}{2A} \log \left(\frac{T}{\delta} \cdot 8(M + 1)\right)\right) \leq T M S \sqrt{AT} \left(\frac{5}{2} \log \left(\frac{T}{\delta} \cdot 8(M + 1)\right)\right),$$

which, for $T > 8\delta(M + 1)$ can be upper-bounded by $T M S \sqrt{AT} \left(\frac{5}{2} \log \left(\frac{T}{\delta}\right)\right)$

The last term in the above upper-bound can be written as

$$T M S A \log \left(\frac{8T}{SA}\right) = T M S \sqrt{AT} \left(\sqrt{\frac{A}{T}} \log \left(\frac{T}{\delta} \cdot \frac{8\delta}{SA}\right)\right),$$
which for $T > \frac{8\delta^2}{S\delta}$ can be upper-bounded by $T_MS\sqrt{AT\log \frac{T}{\delta}}$. Combining these bounds completes the proof.

### Appendix B. Proof of Theorem 3

Throughout this section, we let $\langle x, y \rangle$ denote the dot product between two vectors $x$ and $y$.

#### Preliminary Results

We begin by deriving some results on the variations in the value of the CMDP (3), (4) as a function of the vector $c_{ub} := \{c_{ub}^i\}_{i=1}^M$.

Let $\hat{c}_{ub} = \{\hat{c}_{ub}^i\}_{i=1}^M$ satisfy

$$c_{ub} \geq \hat{c}_{ub} \geq c_{ub}^i - \epsilon, \forall i \in [1, M].$$

(70)

Now consider the following CMDP in which the upper-bounds on the average costs are equal to $\{\hat{c}_{ub}^i\}_{i=1}^M$.

$$\max_{\pi} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \sum_{t=1}^T r(s_t, a_t)$$

s.t. $\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi \sum_{t=1}^T c_i(s_t, a_t) \leq \hat{c}_{ub}^i, i \in [1, M].$

(71) (72)

**Lemma 15** Let $\lambda^*$ be an optimal dual variable/Lagrange multiplier associated with (71), (72). Then, the vector $\lambda^*$ satisfies

$$\langle \lambda^*, 1 \rangle \leq \frac{\hat{\eta}}{\eta},$$

where the constant $\eta$ is as in (21), $1$ is the vector comprising of all ones, and $\hat{\eta}$ is defined as

$$\hat{\eta} := \max_{(s, a) \in S \times A} r(s, a) - \min_{(s, a) \in S \times A} r(s, a).$$

(73)

**Proof** Within this proof, we let $\pi^*(\hat{c}_{ub})$ be an optimal policy for (71), (72). Let $\bar{r}(\pi^*(\hat{c}_{ub})), \bar{c}(\pi^*(\hat{c}_{ub}))$ denote its average reward, and the vector comprising of average costs respectively. Recall that the policy $\pi_{feas.}$ satisfies Assumption 5. We have

$$\max_{(s, a) \in S \times A} r(s, a) \geq \bar{r}(\pi^*(\hat{c}_{ub}))$$

$$= \bar{r}(\pi^*(\hat{c}_{ub})) + \langle \lambda^*, \hat{c}_{ub} - \bar{c}(\pi^*(\hat{c}_{ub})) \rangle$$

$$\geq \bar{r}(\pi_{feas.}) + \langle \lambda^*, \hat{c}_{ub} - \bar{c}(\pi_{feas.}) \rangle$$

$$\geq \min_{(s, a) \in S \times A} r(s, a) + \langle \lambda^*, \hat{c}_{ub} - \bar{c}(\pi_{feas.}) \rangle$$

$$\geq \min_{(s, a) \in S \times A} r(s, a) + \eta \langle \lambda^*, 1 \rangle,$$

22
where the second inequality follows since a policy that is optimal for (71), (72) maximizes the Lagrangian when the multiplier is set equal to $\lambda^*$. Rearranging the above inequality yields the desired result.

**Lemma 16** If $r^*(\hat{c}^{ub})$ denotes optimal reward value of (71), (72), then we have that

$$r^* - r^*(\hat{c}^{ub}) \leq \max_i \left\{ c_i^{ub} - \hat{c}_i^{ub} \right\} \times \frac{\hat{\eta}}{\eta},$$

where $\eta$ is as in (73), $\eta$ is as in (21), and $r^*$ is the optimal reward of the CMDP (3), (4).

**Proof** Since the CMDPs (3) - (4) and (71), (72) are strictly feasible under Assumption 5, strong duality Bertsekas (1997) holds true. Thus,

$$r^* = \sup_{\pi} \inf_{\lambda} \bar{r}(\pi) + \langle \lambda, c^{ub} - \bar{c}(\pi) \rangle,$$

$$r^*(\hat{c}^{ub}) = \sup_{\pi} \inf_{\lambda} \bar{r}(\pi) + \langle \lambda, \hat{c}^{ub} - \bar{c}(\pi) \rangle.$$

Let $\pi_1, \pi_2$ and $\lambda_1, \lambda_2$ denote optimal policies and dual variables for the above two relations. Then,

$$r^* \leq \bar{r}(\pi_1) + \langle \lambda_2, c^{ub} - \bar{c}(\pi_1) \rangle,$$

$$r^*(\hat{c}^{ub}) \geq \bar{r}(\pi_1) + \langle \lambda_2, \hat{c}^{ub} - \bar{c}(\pi_1) \rangle.$$

Subtracting the second inequality from the first yields

$$r^* - r^*(c^{ub}) \leq \langle \lambda_2, c^{ub} - \hat{c}^{ub} \rangle$$

$$\leq \max_i \left\{ c_i^{ub} - \hat{c}_i^{ub} \right\} \langle \lambda_2, 1 \rangle$$

$$\leq \max_i \left\{ c_i^{ub} - \hat{c}_i^{ub} \right\} \frac{\hat{\eta}}{\eta},$$

where the last inequality in the above follows from Lemma 15.

**Proof** [Theorem 3]

The proof is similar to that of Theorem 1, and hence we only discuss the modifications required in the proof of Theorem 1.

Since the construction of the confidence bounds remains the same as in UCB-CMDP, the analysis for the case when the confidence intervals fail remains the same as in proof of Theorem 1. Thus, the quantities $\sum_{k=1}^{K} \Delta_k^{(R)} \{ p \notin \mathcal{M}_{\tau_k} \}, \{ \sum_{k=1}^{K} \Delta_k^{(i)} \{ p \notin \mathcal{M}_{\tau_k} \} \}_{i=1}^{M}$ can be upper-bounded as in Lemma 12.

The regret analysis on the set $\{ p \in \mathcal{M}_{\tau_k} \}$ undergoes the following changes. Using Lemma 16, the inequalities (52), (53) are modified as follows,

$$r_k^* \geq r^* - \max_{i \in [1,M]} \{ d_i \} \left( \frac{\hat{\eta}}{\eta} \right),$$

and $c_{i,k}^* \leq c_i^{ub} - d_i, \forall i \in [1, M]$ (74)
Thus, in comparison with (60), the bound on $\Delta_{k,1}^{(R)}$ now also constitutes an extra additive term of $\sum_{s,a} n_k(s,a) \max_i \{d_i\} \left( \frac{2}{\eta} \right)$, while that on $\Delta_{k,1}^{(i)}$ contains $-d_i \sum_{s,a} n_k(s,a)$ additionally. Summing these bounds over episodes $k$ yields

$$
\sum_{k=1}^{K} \Delta_{k,1}^{(R)} \leq (\sqrt{2} + 1)\sqrt{SAT} + \max_i (34 - b_i) \cdot \left( \frac{\eta}{\eta} \right) \cdot T_M S \sqrt{A T \log \left( \frac{T}{\delta} \right)} ,
$$

and

$$
\sum_{k=1}^{K} \Delta_{k,1}^{(i)} \leq (\sqrt{2} + 1)\sqrt{SAT} - (34 - b_i) \cdot T_M S \sqrt{A T \log \left( \frac{T}{\delta} \right)} , \forall i \in [1,M].
$$

The upper-bound (67) on $\Delta_{k,2}^{(R)}, \Delta_{k,2}^{(i)}$ remain unchanged. Substituting the modified bounds (75) into the proof of Theorem 1 yields the desired result.

**Appendix C. Proof of Theorem 6**

We begin by stating the following result, which will be utilized in the proof of theorem.

**Lemma 17** Under Assumption 5, the CMDP (3), (4) is strictly feasible, so that Slater’s constraint (Boyd and Vandenberghe, 2004) is satisfied, and consequently strong duality holds true. Thus, if $\lambda^*$ solves the dual problem (33), we then have that

$$D(\lambda^*) = r^*.$$  \hspace{1cm} (76)

Let $\lambda = \{\lambda_i\}_{i=1}^{M}$ satisfy $\lambda \geq 0$ and $\phi$ be a rule. Consider

$$
\mathbb{E}_\phi \sum_{t=1}^{T} \left\{ r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \left( c_i^{ub} - c_i(s_t, a_t) \right) \right\} = \mathbb{E}_\phi \sum_{t=1}^{T} r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \mathbb{E}_\phi \sum_{t=1}^{T} \left( c_i^{ub} - c_i(s_t, a_t) \right)
$$

$$= r^* T - \left( r^* T - \mathbb{E}_\phi \sum_{t=1}^{T} r(s_t, a_t) \right)$$

$$- \sum_{i=1}^{M} \lambda_i \mathbb{E}_\phi \sum_{t=1}^{T} \left( c_i(s_t, a_t) - c_i^{ub} \right)$$

$$= r^* T - \mathbb{E}_\phi \Delta^{(R)} - \sum_{i=1}^{M} \lambda_i \mathbb{E}_\phi \Delta^{(i)}. \hspace{1cm} (77)$$

Now, we consider an auxiliary reward maximization problem that involves the same MDP $p$, but in which the reward received at time $t$ by the agent is equal to $r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \left( c_i(s_t, a_t) - c_i^{ub} \right)$. However, this auxiliary problem does not impose constraints on the cost expenditures. Denote its optimal reward by $r^*(\lambda)$ and let $\phi'$ be a policy for this auxiliary problem. The regret for cumulative rewards collected by $\phi'$ in the auxiliary problem is given by

$$r^*(\lambda) T - \mathbb{E}_{\phi'} \left[ \sum_{t=1}^{T} r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \left( c_i(s_t, a_t) - c_i^{ub} \right) \right].$$
It follows from Theorem 5 of Jaksch et al. (2010) that the parameter \( p \) can be chosen so that the regret is greater than \( 0.015 \cdot \sqrt{DSAT} \), i.e.,

\[
r^*(\lambda) \cdot T - \mathbb{E}_{\phi'} \left[ \sum_{t=1}^{T} r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \left( c_i(s_t, a_t) - c_i^{ub} \right) \right] \geq 0.015 \cdot \sqrt{DSAT}. \tag{78}
\]

We observe that since \( \phi \) is a valid rule for the original constrained problem, it is also a valid rule for the auxiliary problem. Thus, we let \( \phi' = \phi \) in the above to obtain

\[
r^*(\lambda) \cdot T - \mathbb{E}_{\phi'} \left[ \sum_{t=1}^{T} r(s_t, a_t) + \sum_{i=1}^{M} \lambda_i \left( c_i(s_t, a_t) - c_i^{ub} \right) \right] \geq 0.015 \cdot \sqrt{DSAT}. \tag{79}
\]

We now substitute (77) in the above to obtain

\[
\mathbb{E}_{\phi} \Delta(R) + \sum_{i=1}^{M} \lambda_i \mathbb{E}_{\phi} \Delta^{(i)} \geq 0.015 \cdot \sqrt{DSAT} + r^*T - r^*(\lambda) \cdot T. \tag{80}
\]

Since the expression in the r.h.s. is maximized for values of \( \lambda \) which are optimal for the dual problem, we set it equal to \( \lambda^* \), and then use Lemma 17 in order to obtain

\[
\mathbb{E}_{\phi} \Delta(R) + \sum_{i=1}^{M} \lambda_i \mathbb{E}_{\phi} \Delta^{(i)} \geq 0.015 \cdot \sqrt{DSAT}. \tag{81}
\]

This completes the proof.

**Appendix D. Proof of Theorem 8**

We divide the proof into two parts. First we show that \( \pi_{hard} \) satisfies the hard constraints. Then we provide guarantees on its average reward.

**Feasibility**

We will first show that \( \pi_{hard} \) satisfies (6). We begin by showing that

\[
B_{\tau_t^{(m)}} \geq 0, \forall i \in \mathbb{N}. \tag{82}
\]

It is easily verified that the total cost incurred under any mini-episode of type a) is upper-bounded by \( L \) units. Since any mini-episode of type a) begins only when buffer level is larger than \( L \), (82) holds true at the end of mini-episodes of type a). Next, since the buffer value at the end of the 0 → 0 cycles is greater than \( L \), (82) is also true at the end of mini-episodes of type b). This proves (82).

To see this, construct an auxiliary process \( \hat{B}_{\tau_t^{(m)}} \) on the same probability space as follows. Let \( \hat{B}_{\tau_t^{(m)}} = B_{\tau_t^{(m)}} \), and update it as

\[
\hat{B}_{\tau_t^{(m)}} = \left( \hat{B}_{\tau_t^{(m)}} - \sum_{t \in E_{\tau_t^{(m)}}} c(s_t, a_t) + |E_{\tau_t^{(m)}}| c^{ub} \right)
\]
instead of (38). Then,

$$- \sum_{t \leq t_i^{(m)}} c(s_t, a_t) + \left( \sum_{j \leq i} |E_{j}^{(m)}| \right) c^{ub} = \sum_{j \leq i} \left( \sum_{t \in E_{j}^{(m)}} -c(s_t, a_t) + |E_{j}^{(m)}| c^{ub} \right) = \hat{B}_{\tau_i^{(m)}}$$

$$\geq B_{\tau_i^{(m)}}$$

$$> 0,$$  \hspace{1cm} (83)

where the first inequality follows by comparing (38) with the update equation for $\hat{B}_{\tau_i^{(m)}}$, while the second inequality follows from (82). This shows that (6) holds for those times $\ell$ which mark beginning of a new mini-episode.

For times $\ell \in [\tau_i^{(m)}, \tau_{i+1}^{(m)})$, we have

$$- \sum_{t < \ell} c(s_t, a_t) + c^{ub} \ell = \hat{B}_{\tau_i^{(m)}} + c^{ub} (\ell - \tau_i^{(m)}) - \sum_{t = \tau_i^{(m)}}^{\ell} c(s_t, a_t).$$  \hspace{1cm} (84)

We consider two cases separately:

Case i) $E_i^{(m)}$ is of type a): Since $B_{\tau_i^{(m)}} > L$, we have from (83) that $\hat{B}_{\tau_i^{(m)}} > L$. During the first $L$ steps, $\sum_{t = \tau_i^{(m)}}^{\ell} c(s_t, a_t)$ is upper-bounded by $L$, so that the r.h.s. of (84) is greater than 0. After $L$ steps, $\pi_{cheap}$ gets implemented. Since under $\pi_{cheap}$, the quantity $c^{ub} - c(s_t, a_t)$ is positive, the r.h.s. of (84) continues to be greater than 0.

Case ii) $E_i^{(m)}$ is of type b): We have $B_{\tau_i^{(m)}} > 0$, so that we have $\hat{B}_{\tau_i^{(m)}} > 0$ from (83). During phase i) $c^{ub} - c(s_t, a_t)$ is always positive, so that the r.h.s. of (84) is greater than 0. The proof during phase ii) is similar to that of Case i) and hence omitted.

Thus, we have shown that $\pi_{hard}$ satisfies (6).

**Lower-Bound on Average Reward**

We begin by deriving some preliminary results.

**Lemma 18**

$$\mathbb{P} \left( B_{\tau_i^{(m)}} \leq L \right) < \frac{e^{-\frac{2}{2L^2}}}{1 - e^{-\frac{2}{2L^2}}} \cdot e^{-\frac{B_L}{L^2}}, \forall i \in \mathbb{N}.$$  \hspace{1cm} (85)

**Proof** We construct an auxiliary controlled process $(\tilde{s}_t, \tilde{a}_t)$, $t \in [1, T]$ by applying a controller which differs from $\pi_{hard}$ in that the episodes are always of type a), i.e., it does not switch to implementing $\pi_{cheap}$ when buffer level is less than $L$. Let $\hat{B}_{\tau_i^{(m)}}$ be the corresponding buffer-level process that is updated similar to (38).

The auxiliary processes are constructed on the same probability space so that they share the elementary random variables. We show the following.
Claim 1

\[ B_{\tau_i}^{(m)} \geq \tilde{B}_{\tau_i}^{(m)}, \forall i \in \mathbb{N}. \]  

Proof We will show that if (86) holds true for some \( j \in \mathbb{N} \), then it also holds true for \( j + 1 \). A simple induction argument then completes the proof since the relation holds true with equality for \( j = 1 \). Thus, we assume that (86) is true for some \( j \). Since the quantity

\[-\sum_{t \in E_j^{(m)}} c(s_t, a_t) + |E_j^{(m)}| c^{ub} \text{ is greater (pathwise) when mini-episode is of type b)},

we have

\[-\sum_{t \in \tilde{E}_j^{(m)}} c(s_t, a_t) + |\tilde{E}_j^{(m)}| c^{ub} \geq -\sum_{t \in \tilde{E}_j^{(m)}} c(s_t, \tilde{a}_t) + |\tilde{E}_j^{(m)}| c^{ub}.\]

Since the buffer-update equation is the same for \( B_{\tau_j}^{(m)}, \tilde{B}_{\tau_j}^{(m)} \), we use the above inequality to conclude that

\[ B_{\tau_j+1}^{(m)} \geq \tilde{B}_{\tau_j+1}^{(m)}. \]

This completes the proof. \( \blacksquare \)

It follows from (86) that

\[ P\left(B_{\tau_i}^{(m)} \leq L\right) \leq P\left(\tilde{B}_{\tau_i}^{(m)} \leq L\right). \]  

(87)

In the remainder of this proof, we will derive upper-bound on the term in r.h.s.

Consider the “netput process” defined as follows

\[ \delta \tilde{B}_i := \sum_{t \in \tilde{E}_i^{(m)}} c(s_t, a_t) - |\tilde{E}_i^{(m)}| c^{ub}. \]

The \( \delta \tilde{B}_i \) are i.i.d. with

\[ E\left(\delta \tilde{B}_i\right) \geq L \epsilon. \]  

(88)

We have

\[ \left\{ \tilde{B}_{\tau_i} \leq L\right\} \subset \bigcup_{n=1}^{i} \left\{ \sum_{j=1}^{n} \delta \tilde{B}_i > B \right\}. \]  

(89)

Next, we bound the probability of \( \left\{ \sum_{j=1}^{n} \delta \tilde{B}_j > B \right\} \).

Let \( \theta > 0 \), and \( F_i := \sigma\left(\{s_t, \tilde{a}_t\}_{t \in \{j=1}^{(m)}}\right) \) and define \( X_j := E\left(\sum_{i=1}^{n} \delta \tilde{B}_i|F_j\right) \). We then have that

\[ \sum_{i=1}^{n} \delta \tilde{B}_i - E \sum_{i=1}^{n} \delta \tilde{B}_i = E\left(\sum_{i=1}^{n} \delta \tilde{B}_i|F_n\right) - E\left(\sum_{i=1}^{n} \delta \tilde{B}_i|F_0\right) = \sum_{j=1}^{n} X_j - X_{j-1}, \]  

(90)
Let $Y_j := X_j - X_{j-1}$ so that it follows from (90) that
\[
\mathbb{P} \left( \sum_{i=1}^{n} \delta \tilde{B}_i - \mathbb{E} \sum_{i=1}^{n} \delta \tilde{B}_i > x \right) = \mathbb{P} \left( \sum_{j=1}^{n} Y_j > x \right) = \mathbb{P} \left( e^{\theta \sum_{j=1}^{n} Y_j} > e^{\theta x} \right) \leq e^{-\theta x} e^{\theta \sum_{j=1}^{n} Y_j}. \tag{91}
\]

Now,
\[
\mathbb{E} e^{\theta \sum_{j=1}^{n} Y_j} = \mathbb{E} \left( e^{\theta \sum_{j=1}^{n-1} Y_j} \cdot \mathbb{E} \left[ e^{\theta Y_n | \mathcal{F}_{n-1}} \right] \right) \leq \mathbb{E} \left( e^{\theta \sum_{j=1}^{n-1} Y_j} e^{\frac{\theta^2 n L^2}{2}} \right). \tag{92}
\]

Continuing similarly, we obtain
\[
\mathbb{E} e^{\theta \sum_{j=1}^{n} Y_j} \leq e^{\frac{\theta^2 n L^2}{2}}. \tag{93}
\]

Substituting the above into inequality (91), we obtain
\[
\mathbb{P} \left( \sum_{i=1}^{n} \delta \tilde{B}_i - \mathbb{E} \sum_{i=1}^{n} \delta \tilde{B}_i > x \right) \leq e^{-\theta x} e^{\frac{\theta^2 n L^2}{2}}. \tag{94}
\]

The r.h.s. in the above bound is minimized at $\theta = x/nL^2$. Setting $\theta = x/nL^2$ in the above inequality yields
\[
\mathbb{P} \left( \sum_{i=1}^{n} \delta \tilde{B}_i - \mathbb{E} \sum_{i=1}^{n} \delta \tilde{B}_i > x \right) \leq e^{-\frac{x^2}{2nL^2}}. \tag{95}
\]

After substituting the value of $x$ equal to $n\epsilon + B$, we obtain
\[
\mathbb{P} \left( \sum_{j=1}^{n} \delta \tilde{B}_j > B \right) \leq e^{-\frac{(n^2 \epsilon^2 + B^2 + 2nB \epsilon)}{2nL^2}}. \tag{96}
\]

Substituting (96) into (89), and utilizing the union bound gives us
\[
\mathbb{P} \left( \tilde{B}_i \leq L \right) \leq \sum_{n=1}^{i} e^{-\frac{(n^2 \epsilon^2 + B^2 + 2nB \epsilon)}{2nL^2}} \leq \sum_{n=1}^{i} e^{-\frac{n^2 \epsilon^2}{2L^2}} \cdot e^{-\frac{B \epsilon}{L^2}} \leq \frac{e^{-\frac{2}{2L^2}}}{1 - e^{-\frac{2}{2L^2}}} \cdot e^{-\frac{B \epsilon}{L^2}}. \tag{97}
\]

The proof then follows by combining the above with inequality (87). \hfill \blacksquare
Lemma 19 We have

\[
\mathbb{E}_{\pi_{\text{hard}}} \left( \sum_{t \in \mathcal{E}_i^{(m)}} r(s_t, a_t) \bigg| B_{\tau_i^{(m)}} > L \right) \geq L \left( r^* - \epsilon \hat{\eta}/\eta \right) + \text{Const.}, \tag{98}
\]

\[
\mathbb{E}_{\pi_{\text{hard}}} \left( \sum_{t \in \mathcal{E}_i^{(m)}} c(s_t, a_t) \bigg| B_{\tau_i^{(m)}} > L \right) \leq L \left( c_{\text{ub}} - \epsilon \right) + \text{Const.} \tag{99}
\]

Let \( \bar{T}^{(a)} \), \( \bar{r}^{(a)} \) denote the expected duration and average reward of a mini-episode of type \( a \). We have

\[
\bar{T}^{(a)} \leq L + T_M. \tag{100}
\]

From (98), (100) we obtain that for values of \( L \) much larger than \( T_M \), we have

\[
\bar{r}^{(a)} \geq r^* - \epsilon \hat{\eta}/\eta + o(r^*). \tag{101}
\]

Proof In the discussion within this proof, we omit the conditioning on \( \{B_{\tau_i^{(m)}} > L\} \) in order to ease the notation. We have,

\[
\mathbb{E}_{\pi_{\text{hard}}} \left( \sum_{t \in \mathcal{E}_i^{(m)}} r(s_t, a_t) \right) \geq \mathbb{E}_{\pi_{\text{hard}}} \left( \sum_{t = \tau_i^{(m)}}^{\tau_i^{(m)} + L} r(s_t, a_t) \right)
\]

\[
= \mathbb{E}_{\pi_{\text{hard}}} \left( \sum_{t = \tau_i^{(m)}}^{\tau_i^{(m)} + L} \sum_{(s, a) \in \mathcal{S} \times \mathcal{A}} r(s, a) \mathbb{1}\{s_t = s, a_t = a\} \right)
\]

\[
= \mathbb{E}_{\pi_{\text{soft}}} \left( \sum_{t = \tau_i^{(m)}}^{\tau_i^{(m)} + L} \sum_{(s, a) \in \mathcal{S} \times \mathcal{A}} r(s, a) \mathbb{1}\{s_t = s, a_t = a\} \right)
\]

\[
= \sum_{t = \tau_i^{(m)}}^{\tau_i^{(m)} + L} \sum_{(s, a) \in \mathcal{S} \times \mathcal{A}} r(s, a) \mathbb{P}(s_t = s, a_t = a)
\]

\[
= \sum_{t = \tau_i^{(m)}}^{\tau_i^{(m)} + L} \left[ \sum_{(s, a) \in \mathcal{S} \times \mathcal{A}} r(s, a) \left( \mathbb{P}(s_t = s, a_t = a) - \mu_{\pi_{\text{soft}}}(s, a) \right) \right]
\]

\[
+ \sum_{t = \tau_i^{(m)}}^{\tau_i^{(m)} + L} \left[ \sum_{(s, a) \in \mathcal{S} \times \mathcal{A}} r(s, a) \mu_{\pi_{\text{soft}}}(s, a) \right], \tag{102}
\]

where the second equality follows since \( \pi_{\text{hard}} \) implements \( \pi_{\text{soft}} \) during the first \( L \) steps of a mini-episode of type \( a \). Since the probability distribution of a stationary Markov chain
converges geometrically to its steady state distribution, we have that

\[
\tau^{(m)}_{t_i} + \sum_{t=\tau^{(m)}_{t_i}}^{\tau^{(m)}_{t_i}+L} \sum_{(s,a) \in S \times A} r(s, a) \left( \mathbb{P}(s_t = s, a_t = a) - \mu_{\pi_{sof t}}(s, a) \right) \leq \sum_{t=\tau^{(m)}_{t_i}}^{\tau^{(m)}_{t_i}+L} \text{Const.} \rho^t - \tau^{(m)}_{t_i} + \sum_{t=\tau^{(m)}_{t_i}}^{\tau^{(m)}_{t_i}+L} \text{Const.} \rho^t < \infty,
\]

where \( \rho \in [0, 1) \). Substituting this and the bound on average reward of \( \pi_{sof t} \) from Lemma 16 into (102) completes proof of (98).

In order to bound the cost, we note that

\[
\mathbb{E}_{\pi_{\text{hard}}} \left( \sum_{t \in E^{(m)}_{t_i}} c(s_t, a_t) \right) \leq \mathbb{E}_{\pi_{\text{sof t}}} \left( \sum_{t=\tau^{(m)}_{t_i}}^{\tau^{(m)}_{t_i}+L} c(s_t, a_t) \right) + \max_{(s,a) \in S \times A} c(s,a) \max_{s,s' \in S} \mathbb{E}_{\pi_{sof t}} (T_{s,s'}) 
\]

\[
\leq L (c(\pi_{sof t}) + \text{Const.} + \max_{(s,a) \in S \times A} c(s,a) T_M 
\]

\[
\leq L \left( c^{ub} - \epsilon \right) + \text{Const.} + \max_{(s,a) \in S \times A} c(s,a) T_M,
\]

where \( T_M \) is as in (1).

**Lemma 20** The quantities \( \frac{k_T^{(m)}}{\tau^{(m)}_{t_i}}, \frac{k_T^{(m,b)}}{k_T^{(m)}} \) converge as \( T \to \infty \). Moreover,

\[
\lim_{T \to \infty} \frac{k_T^{(m,b)}}{k_T^{(m)}} \leq \frac{e^{-\frac{2}{2L^2}}}{1 - e^{-\frac{\epsilon}{2L^2}}} \cdot e^{-\frac{B\epsilon}{L^2}}. \tag{103}
\]

**Proof** Under the application of \( \pi_{\text{hard}} \), the process \( s_t^{(\ell_i)} \) evolves as a finite state Markov process, and hence is necessarily ergodic, so that \( \tau^{(m)}_{t_i} \) is convergent. Since the state 0 is recurrent under any policy, it follows that each mini-episode is of finite duration, and moreover \( \tau^{(m)}_{t_i} \) converges to a positive quantity. The convergence of \( \frac{k_T^{(m,b)}}{k_T^{(m)}} \) also follows from the ergodicity of \( s_t^{(\ell_i)} \). Since \( \frac{k_T^{(m,b)}}{k_T^{(m)}} = \frac{k_T^{(m,b)}}{k_T^{(m)}} / T \), we infer convergence of \( \frac{k_T^{(m,b)}}{k_T^{(m)}} \). Next, we derive the bound (103).

Consider the embedded Markov process \( B_{\tau^{(m)}_{t_i}}, i \in \mathbb{N} \) that evolves as a finite-state Markov process and assumes values in the set \( [0, B] \). Since the process is ergodic, the probabilities \( \mathbb{P}(B_{\tau^{(m)}_{t_i}} < L), \mathbb{P}(B_{\tau^{(m)}_{t_i}} \geq L) \) converge as \( i \to \infty \), and also that

\[
\lim_{i \to \infty} \frac{k_T^{(m,b)}}{k_T^{(m)}} / \tau^{(m)}_{t_i} = \lim_{i \to \infty} \mathbb{P}(B_{\tau^{(m)}_{t_i}} < L).
\]
Since $k_t^{(m,b)}/k_t^{(m)} = k_t^{(m,b)}/k_t^{(m)}, \forall t \in \mathcal{E}_i^{(m)},$ and $|\mathcal{E}_i^{(m)}| < \infty, \forall i \in \mathbb{N},$ we have $\lim_{t \to \infty} k_t^{(m,b)}/k_t^{(m)} = \lim_{i \to \infty} k_t^{(m,b)}/k_t^{(m)},$ and hence

$$\lim_{t \to \infty} k_t^{(m,b)}/k_t^{(m)} = \lim_{i \to \infty} \mathbb{P}(B_{\tau_i^{(m)}} < L).$$

The inequality (103) then follows from Lemma 18.

We are now in a position to prove the second statement of Theorem 8 which lower-bounds the average reward of $\pi_{\text{hard}}.$ Let $k_t^{(m)}$ denote the number of mini-episodes until time $t,$ i.e.,

$$k_t^{(m)} := \max \left\{ i \in \mathbb{N} : \tau_i^{(m)} \leq t \right\}.$$ 

Also let $k_t^{(m,a)}, k_t^{(m,b)}$ be the number of type a), and type b) mini-episodes until $t,$ and let $\mathcal{E}_i^{(m,a)}, \mathcal{E}_i^{(m,b)}$ denote the $i$-th mini-episode of type a) and type b) respectively.

Also, let $\bar{r}^{(a)}, \bar{R}^{(a)}$ be the average and cumulative rewards, and $\bar{T}^{(a)}$ the time spent in a mini-episode of type a); similarly for mini-episode of type b), so that it follows by using elementary results from embedded Markov chains Asmussen (2008) that

$$\bar{r}^{(a)} = \frac{\bar{R}^{(a)}}{\bar{T}^{(a)}}, \quad \bar{r}^{(b)} = \frac{\bar{R}^{(b)}}{\bar{T}^{(b)}}.$$ (104)
In the discussion below, we omit the sub-script \( \pi_{\text{hard}} \). The average reward of \( \pi_{\text{hard}} \) is given by

\[
\liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \sum_{t=1}^{T} r(s_t, a_t) \right) = \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \sum_{i=1}^{k_{T}^{(m,a)}} \sum_{t \in E_{i}^{(m,a)}} r(s_t, a_t) \right) \\
= \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \sum_{i=1}^{k_{T}^{(m,a)}} \sum_{t \in E_{i}^{(m,a)}} r(s_t, a_t) + \sum_{i=1}^{k_{T}^{(m,b)}} \sum_{t \in E_{i}^{(m,b)}} r(s_t, a_t) \right) \\
= \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left( k_{T}^{(m,a)} \bar{R}(a) + k_{T}^{(m,b)} \bar{R}(b) \right) \\
\geq \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left( k_{T}^{(m,b)} \bar{R}(b) \right) \\
\geq \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left( k_{T}^{(m,a)} \bar{R}(a) + k_{T}^{(m,b)} \bar{R}(b) \right) \\
= \bar{r}(a) \lim_{T \to \infty} \frac{\mathbb{E} \left( k_{T}^{(m,a)} \bar{R}(a) + k_{T}^{(m,b)} \bar{R}(b) \right)}{T} \\
= \bar{r}(a) \lim_{T \to \infty} \frac{\mathbb{E} \left( k_{T}^{(m,a)} \bar{R}(a) + k_{T}^{(m,b)} \bar{R}(b) \right)}{T} \\
\geq \bar{r}(a) - \bar{r}(a) \bar{T}(b) \lim_{T \to \infty} \frac{k_{T}^{(m,a)} + k_{T}^{(m,b)}}{T} \cdot \lim_{T \to \infty} \frac{k_{T}^{(m,b)}}{k_{T}^{(m,b)} + k_{T}^{(m,a)}} \\
\geq \bar{r}(a) - \bar{r}(a) \bar{T}(b) \bar{e}(m) \frac{e^{-\frac{2}{2\epsilon^2}}}{1 - e^{-\frac{2}{2\epsilon^2}}} \cdot e^{-\frac{B\epsilon}{4}}.
\]

The proof is completed by substituting the lower bound on \( \bar{r}(a) \) from Lemma 19.

Appendix E. Proof of Theorem 11

**Proof** The proof is similar to the regret analysis of the UCRL algorithm of Jaksch et al. (2010), and hence we only point out the differences.

**Notation:** Within this proof, a generic state-action tuple for the original MDP will be denoted by \((s, a)\), while that for the lifted MDP by \((s^{(\ell_i)}, a)\). Note that, as discussed in Section 5.1, policies from the class \( \mathcal{B} \) choose control actions on the basis of \( s^{(\ell_i)} \). Hence, we let \( \pi_{\text{hard}}(\epsilon, B, L; \tilde{p}_k) \) be denoted \( \tilde{\pi}_k := \{ \tilde{\pi}_k(s^{(\ell_i)}) \}_{s^{(\ell_i)} \in \mathcal{S}(\ell_i)} \). We also let \( r^{*}_k \) denote the average reward of \( \pi_{\text{hard}}(\tilde{p}_k) \) when the true MDP is \( \tilde{p}_k \). \( D^{(\ell_i)} \) is used to denote the diameter of the lifted MDP, and \( \tilde{p}_k^{(\ell_i)} \) to denote the transition probabilities of the lifted MDP associated with the (non-lifted) MDP \( \tilde{p}_k \). We continue to use \( r \) as the reward function for the lifted process, i.e., reward earned at time \( t \) is equal to \( r(s_t, a_t) \), or equivalently \( r(s_t^{(\ell_i)}, a_t) \).
Regret analysis for case when confidence intervals fail remains same as in proof of Theorem 1, and hence it can be upper-bounded by \( \sqrt{T} \) with probability larger than \( 1 - \frac{\delta}{12 \cdot T^{1/4}} \); hence we shift focus to analyzing \( \Delta_k \).

**Regret during \( \mathcal{E}_k \):**

Note that

\[
\tau_k^* = \bar{r}(\pi_{\text{hard}}(\tilde{p}_k); \tilde{p}_k) \geq \bar{r}(\pi_{\text{hard}}(\bar{p}); \bar{p}).
\]

Hence, it follows from (42) and the inequality above that,

\[
\Delta_{(R)} > \{ p \in \mathcal{M}_{\tau_k} \} \leq \sum_{(s, a) \in S \times A} n_k(s, a) (\tau_k^* - r(s, a))
= \sum_{(s(\ell_i), a) \in S(\ell_i) \times A} n_k(s(\ell_i), a) (\tau_k^* - r(s(\ell_i), a)), \tag{105}
\]

We obtain the following using the discussion in Section 4.3 of Jaksch et al. (2010),

\[
\tau_k^* + v_k(\pi_k(s(\ell_i), a) = r(s(\ell_i), a) + \sum_{s(\ell_i) \in S(\ell_i)} \tilde{p}_k(s(\ell_i), a, s(\ell_i)) \left( \sum_b \tilde{\pi}_k(b|s(\ell_i), a) v_k(R)(s(\ell_i), b) \right),
\]

where \( \|v_k\|_\infty \leq \frac{D(\ell_i)}{2} \). Thus,

\[
\Delta_{(R)} > \{ p \in \mathcal{M}_{\tau_k} \} \leq \sum_{(s(\ell_i), a) \in S(\ell_i) \times A} n_k(s(\ell_i), a) (\tau_k^* - r(s(\ell_i), a))
= \sum_{(s(\ell_i), a) \in S(\ell_i) \times A} n_k(s(\ell_i), a) \left( \sum_{s(\ell_i), \ell_i} \tilde{p}_k(s(\ell_i), a, s(\ell_i)) \left( \sum_b \tilde{\pi}_k(b|s(\ell_i), a) v_k(R)(s(\ell_i), b) \right) - v_k(R)(s(\ell_i), a) \right)
\leq \sum_{(s(\ell_i), a)} n_k(s(\ell_i), a) \left( \sum_{s(\ell_i), \ell_i} \tilde{p}_k(s(\ell_i), a, s(\ell_i)) - p(\pi_k(s(\ell_i), a, s(\ell_i))) \right) \left( \sum_b \tilde{\pi}_k(b|s(\ell_i), a) v_k(R)(s(\ell_i), b) \right)
+ \sum_{(s(\ell_i), a)} n_k(s(\ell_i), a) \left( \sum_{s(\ell_i), \ell_i} p(\pi_k(s(\ell_i), a, s(\ell_i))) \left( \sum_b \tilde{\pi}_k(b|s(\ell_i), a) v_k(R)(s(\ell_i), b) \right) - v_k(R)(s(\ell_i), a) \right).
\]

Denote the two expressions in above bound as \( \Delta_{k,1}, \Delta_{k,2} \). We will now bound them separately.

**Bounding \( \Delta_{k,1} \)**
\[
\Delta_{k,1}^{(R)} \leq \sum_{(s,a)} n_k(s,a) \| \tilde{p}_k(s,a) - p(s,a) \|_1 D^{(R)} K \sum_{k=1}^T \Delta_{k,2}^{(R)} \leq T m_t + K \sum_{t=1}^T D^{(R)}.
\]

where the first inequality follows from (106), and the last from the form of the UCB confidence intervals (10), (11).

Bounding \( \Delta_{k,2}^{(R)} \)

Utilizing the modified bounds on \( \Delta_{k,1}^{(R)}, \Delta_{k,2}^{(R)} \), and performing the same analysis as that of UCRL, we obtain the desired result.

### Appendix F. Computing an Optimistic Optimal Policy

Consider the problem (12)- (16) that involves solving for an optimistic MDP, and the corresponding optimal policy. We repeat it below for the convenience of readers,

\[
\max_{\mu, p'} \sum_{(s,a) \in S \times A} \mu(s,a) r(s,a), \\
\text{s.t. } \sum_{a \in A} \mu(s,a) = \sum_{(s',b) \in S \times A} \mu(s',b) p'(s',b,s), \\
\sum_{(s,a) \in S \times A} \mu(s,a) c_i(s,a) \leq c_i^{ub}, i \in [1,M] \\
\| p'(s,a,\cdot) - \hat{p}_k(s,a,\cdot) \|_1 \leq \epsilon_{\tau_k}(s,a), \\
\mu(s,a) \geq 0 \ \forall (s,a) \in S \times A, \sum_{(s,a) \in S \times A} \mu(s,a) = 1.
\]

Let \( M_{\tau_k} \) denote the confidence interval associated with the estimate \( \hat{p}_k \). Now consider an equivalent “augmented” CMDP in which the decision \( a_t^{(+)} \) at each time \( t \) comprises of (i)
the action $a_t$ corresponding to the control input to the original MDP, (ii) the MDP $p_t$, i.e., the controlled transition probabilities $p_t \in \mathcal{M}_{\tau_k}$. The state and action-spaces are given by $\mathcal{S}$ and $\mathcal{A} \times \mathcal{M}_{\tau_k}$ respectively. We denote by $\pi^{(+)}$ a policy that makes decisions for this augmented problem. Similarly, we will also let $\mu^{(+)} : \mathcal{S} \times \mathcal{M}_{\tau_k} \mapsto \mathbb{R}_+$ denote an occupation measure for the augmented problem. The augmented CMDP is described as follows,

$$\max \lim \inf \frac{1}{T} T \mathbb{E}_{\pi^{(+)}} \sum_{t=1}^{T} r(s_t, a_t^{(+)})$$

s.t. $\lim \sup \frac{1}{T} T \mathbb{E}_{\pi^{(+)}} \sum_{t=1}^{T} c_i(s_t, a_t^{(+)}) \leq c_i^{ub}, i \in [1, M], \tag{109}$

and an optimal policy can be obtained by solving an infinite LP. Approximation schemes for an infinite LP can be found in Hernández-Lerma and Lasserre (1998); Hernández-Lerma et al. (2003).

Appendix G. Some Properties of MDPs

Lemma 21 Consider a unichain MDP $p$ operating under a stationary policy $\pi$ that chooses action $a$ with a probability $\pi(a|s)$ when the system state is equal to $s$. Then, there exists functions $v_\pi^{(R)}, \{v_\pi^{(i)}\}_{i=1}^{M} : \mathcal{S} \mapsto \mathbb{R}$ which satisfy the following,

$$\tilde{r}(\pi) + v_\pi^{(R)}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left( r(s, a) + \sum_{s' \in \mathcal{S}} p(s, a, s') v_\pi^{(R)}(s') \right), s \in \mathcal{S},$$

$$\tilde{c}_i(\pi) + v_\pi^{(i)}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left( c_i(s, a) + \sum_{s' \in \mathcal{S}} p(s, a, s') v_\pi^{(i)}(s') \right), s \in \mathcal{S}, i \in [1, M], \tag{111}$$

where $\tilde{r}(\pi)$ denotes the average reward under $\pi$, while $\tilde{c}_i(\pi)$ denotes the average $i$-th cost incurred under $\pi$. Moreover,

$$|v_\pi^{(R)}(s) - v_\pi^{(R)}(s')| \leq \max_{(s, a)} T_M, \forall s, s' \in \mathcal{S}$$

$$|v_\pi^{(i)}(s) - v_\pi^{(i)}(s')| \leq \max_{(s, a)} T_M, \forall s, s' \in \mathcal{S}, i \in [1, M]. \tag{112}$$

A similar result also holds true for the augmented CMDPs.

Proof We will only prove the result for rewards, since the proofs for costs are similarly derived. Let $J_n(s)$ denote the reward collected in first $n$ steps after starting in state $s$ and following policy $\pi$. We then have that

$$J_n(s) = \sum_{a \in \mathcal{A}} \pi(a|s) r(s, a) + \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} p(s, a, s') J_{n-1}(s'), \forall s \in \mathcal{S}. \tag{113}$$

Consider the following,

$$J_n(s) = n \tilde{r}(\pi) + j_n(s), s \in \mathcal{S},$$
where

\[ j_n(s) = \mathbb{E}_\pi \left( \sum_{t=1}^n r(s_t, a_t) - \bar{r}(\pi) \right| s_0 = s ). \]

Substituting the above into (113), we obtain the following recursions,

\[ \bar{r}(\pi) + j_n(s) = \sum_{a \in A} \pi(a|s) r(s, a) + \sum_{a \in A} \pi(a|s) \sum_{s' \in S} p(s, a, s') j_{n-1}(s'), \forall s \in \mathcal{S}. \] (114)

Also,

\[ j_n(s) \leq \text{Const.} \ \forall s \in \mathcal{S} \text{ and } , \]

\[ j_n(s) - j_n(s') = \mathbb{E}_\pi \left( \sum_{t=1}^n r(s_t, a_t) \right| s_0 = s ) - \mathbb{E}_\pi \left( \sum_{t=1}^n r(s_t, a_t) \right| s_0 = s') \]

\[ \leq \max_{(s,a)} r(s,a) T_M, \forall s, s' \in \mathcal{S}. \]

Since \( \{j_n(s)\}_{s \in \mathcal{S}, n \in \mathbb{N}} \) is compact, we pass on to a subsequence and deduce the existence of functions satisfying (86).

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