A Kind of Incremental Kalman Filter in Fast Implementation

YAN Guangming and SUN Xiaojun*
Electrical Engineering Institute, Heilongjiang University, Harbin, 150080, China
*sxj@hlju.edu.cn

Abstract. Now the observation error is widespread in applications. It is usually caused by the influence of surrounding environment, the error of measuring equipment or the improper selection of model and parameters. When the observation equation of system has not been verified or corrected under certain environmental conditions, applying it will yield to unknown system errors and filtering errors. For the systems under poor observation condition, an incremental Kalman filter in fast implementation is presented in this paper, which can eliminate the unknown system errors and improve the accuracy of state estimators. Moreover, it also has the smaller computation load compared with the general incremental Kalman filter. It is in fast implementation and easy to be applied in engineering practice. Two simulation examples show its effectiveness and feasibility.

1. Introduction
Kalman filtering is an important state estimation method. This time domain method was first proposed in the 60s of last century [1-3]. Its main advantage is that its recursive form is convenient for computer realization and real-time application, and it can handle the filtering problem of time-varying system, non-stationary random signal and multidimensional signal. At present, it has been widely used in many high technology fields such as target tracking, guidance, GPS positioning, communication and signal processing, fault diagnosis, robotics and other [4].

However, the Kalman filtering algorithm requires accurate system model parameters and noise statistics [5]. In the practical application, the observation error will be caused due to the influence of surrounding environment, the error caused by the measuring equipment itself, the improper selection of model and parameters. And the traditional Kalman filtering algorithm is difficult to eliminate the error of this kind of system [6-8]. In order to solve this problem, a series of incremental filtering algorithms have been proposed. For the nonlinear systems under poor observation condition, the extended incremental Kalman filtering algorithm was proposed by [9]. The incremental particle filter was proposed by [10], and the adaptive high order unscented incremental Kalman filtering algorithm was proposed by [11]. In [12] and [13], the incremental Kalman filtering and smoothing algorithms were presented for the systems under poor observation condition, which successfully eliminate the measurement error and improve the filtering accuracy. However, the inverse operation of the matrix with the same dimension yields the effect on the speed.

In this paper, an incremental Kalman filter in fast implementation is presented. This method can effectively eliminate the unknown error of the system under poor observation condition, and greatly improve the estimation accuracy. Compared with the general incremental Kalman filter, it has a faster implementation speed. The simulation example shows its effectiveness and feasibility.
2. Problem formulation

Consider a target tracking system as follows

\[ x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} w_{k-1} \]  

(1)

where \( x_k \in R^n \) is the state at time \( k \), \( \Phi_k \in R^{m \times n} \) is the state transformation matrix, \( \Gamma_k \in R^{m \times m} \) is the system noise distribution matrix, \( w_k \in R^m \) is the system noise vector.

**Assumption 1** \( w_k \) is Gauss white noise with zero mean and variance \( Q_k \). \( x_0 \) is the initial state for the target, and \( \text{cov}[w_k, w_j] = Q_k \delta_{kj} \), \( \text{cov}[x_0, w_k] = 0 \).

The observation equation can be denoted by

\[ z_k = H_k x_k + v_k + a_k \]  

(2)

where \( z_k \in R^m \) is the observation vector at time \( k \), \( H_k \in R^{m \times n} \) is the observation matrix at time \( k \). \( a_k \) is the observation system error at time \( k \). \( v_k \in R^m \) is observation noise vector at time \( k \). \( v_k \) is a Gauss white noise with zero mean and variance \( R_k \).

**Assumption 2.** Suppose the observation noises are uncorrelated with each other at the same time \( k \), as well as the observation noises at different times. i.e. \( \text{cov}[v_k, v_j] = R_k \delta_{kj} \), \( \text{cov}[v_k, v_j] = 0 \) and \( \text{cov}[x_0, v_k] = 0 \).

Based on (2), we can obtain the incremental observation equation as follows

\[ \Delta z_k = H_k x_k - H_{k-1} x_{k-1} + v_k - v_{k-1} \]  

(3)

where \( \Delta z_k \) is the observation vector increment at time \( k \), \( \Delta z_k = z_k - z_{k-1} \). Suppose that the stochastic vectors \( v_k \) and \( v_{k-1} \) are Gauss white noises uncorrelated with each other, then \( V_k = v_k - v_{k-1} \) is also a Gauss white noise with zero mean and variance \( 2R_k \).

Furthermore, applying (1) and (3) yields

\[ \Delta z_k = \overline{H}_k x_{k-1} + \overline{V}_k \]  

(4)

where \( \overline{H}_k = H_k \Phi_{k-1} - H_{k-1} \), \( \overline{V}_k = H_k \Gamma_{k-1} w_{k-1} + V_k \). In (4), the unknown observation error is eliminated. Eq. (1) and Eq. (4) contribute to the incremental filtering fusion equation. The objectives are to find a incremental Kalman filter in fast implementation \( \hat{x}_{k|k} \) for the state \( x_k \), based on the measurement \( (\Delta z_1, \Delta z_2, \ldots, \Delta z_k) \) at time \( k \).

**Remark 1.** In engineering practice, the observation system errors between two adjacent observations \( z_k \) and \( z_{k-1} \) are close to size, so that the system error of \( \Delta z_k \) is relatively small and can be ignored [12]. In addition, \( \Delta z_k \) and \( \Delta z_{k-1} \) are better to satisfy the independence requirement than that of \( z_k \) and \( z_{k-1} \) according to the principle of independent increment stochastic process [14].

3. A incremental Kalman filter in fast implementation

**Lemma 1** [15]. For the known non-singular matrices \( P \in R^{n \times n} \), \( R \in R^{m \times m} \), and \( H \in R^{m \times n} \) , if the inverses of \( P^{-1} + H^T R^{-1} H \) and \( HPH^T + R \) exist, we have the following matrix equation

\[ [I + PH^T R^{-1} H]^{-1} P = [P^{-1} + H^T R^{-1} H]^{-1} = P - PH^T (HPH^T + R)^{-1} HP \]  

(5)

Applying Lemma 1 and [13] yields a new incremental Kalman filter as follows.

**Theorem 1.** The system (1) and (4) with Assumptions 1 and 2 has the following Kalman filter in fast implementation

\[ \hat{x}_{k+1|k} = \Phi_k \hat{x}_{k|k} \]  

(6)
\[ P_{k+1|k} = \Phi_k P_{k|k} \Phi_k^T + \Gamma_k Q_k \Gamma_k^T \]  

\[ \tilde{R}_{k+1}^{-1} = \frac{1}{2} R_{k+1}^{-1} - \frac{1}{2} R_{k+1}^{-1} H_k \Gamma_k \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1}) + Q_k^{-1} \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1}) \]  

\[ K_{k+1} = \Phi_k P_{k|k} \tilde{H}_{k+1}^{-1} [\tilde{R}_{k+1}^{-1} - \tilde{R}_{k+1}^{-1} \tilde{H}_{k+1}^{-1} (P_{k|k} + \tilde{H}_{k+1}^{-1} \tilde{R}_{k+1}^{-1} H_k) \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1})]^{-1} \tilde{H}_{k+1}^{-1} \]  

\[ \hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} (\hat{z}_{k+1} - \tilde{H}_{k+1} \hat{x}_{k+1|k}) \]  

\[ P_{k+1|k+1} = P_{k+1|k} + K_{k+1} \tilde{R}_{k+1}^{-1} K_{k+1}^T + K_{k+1} (H_k \Gamma_k Q_k \Gamma_k^T H_k^T + 2R_{k+1}^{-1}) K_{k+1}^T \]  

Proof. From [13], we have (6), (7), (10) and the filtering error variance matrix as follows  

\[ K_{k+1} = \Phi_k P_{k|k} \tilde{H}_{k+1}^{-1} [\tilde{R}_{k+1}^{-1} - \tilde{R}_{k+1}^{-1} \tilde{H}_{k+1}^{-1} (P_{k|k} + \tilde{H}_{k+1}^{-1} \tilde{R}_{k+1}^{-1} H_k) \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1})]^{-1} \tilde{H}_{k+1}^{-1} \]  

Setting \( R_{k+1} = H_k \Gamma_k Q_k \Gamma_k^T H_k^T + 2R_{k+1}^{-1} \), applying Lemma 1 yields  

\[ K_{k+1} = \Phi_k P_{k|k} \tilde{H}_{k+1}^{-1} [\tilde{R}_{k+1}^{-1} - \tilde{R}_{k+1}^{-1} \tilde{H}_{k+1}^{-1} (P_{k|k} + \tilde{H}_{k+1}^{-1} \tilde{R}_{k+1}^{-1} H_k) \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1})]^{-1} \tilde{H}_{k+1}^{-1} \]  

Furthermore, we have  

\[ \tilde{R}_{k+1}^{-1} = \frac{1}{2} R_{k+1}^{-1} - \frac{1}{2} R_{k+1}^{-1} H_k \Gamma_k \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1}) + Q_k^{-1} \Gamma_k^T H_k^T (\frac{1}{2} R_{k+1}^{-1}) \]  

In summary, we can obtain an incremental Kalman filter in fast implementation. The proof is completed.

Remark 2. In the multi-sensor information fusion Kalman filtering problem [15], it will encounter the measurement matrix \( H_k \) in \( m \times n \) dimension. Here \( m \) is far greater than \( n \) such as that in simulation example 2. In [13], it is necessary to calculate the inverse matrix of the high-dimensional matrix [\( H_k \Gamma_k Q_k \Gamma_k^T H_k^T + 2R_{k+1}^{-1} \) or \( (H_k \Gamma_k Q_k \Gamma_k^T H_k^T + 2R_{k+1}^{-1}) \) online. This problem can be solved applying (8) and (9), which can reduce the amount of incremental Kalman computation and speed up the implementation.

4. Simulation model and result analysis

Many simulation experiments show that the system errors can not be eliminated by the traditional Kalman filtering and self-adaptive Kalman filtering when the observation data has unknown errors due to the environments. In this paper, the presented incremental Kalman filter in fast implementation can eliminate the system error and greatly improve the filtering accuracy.

Example 1. Consider the one dimension discrete system

\[ x_k = 0.8 x_{k-1} + w_k \]  

\[ z_k = x_k + a_k + v_k \]  

where \( w_k \) and \( v_k \) are independent Gauss white noises, the system noise \( w_k \) has the mean \( q = 0 \) and variance \( Q = 0.1 \), the observation noise \( v_k \) has the mean \( r = 0 \) and variance \( R = 1 \). They are all known. \( a_k \) is the unknown observation error. Suppose \( a_k = 3 \).

Applying the method in [13] and Theorem 1 yields two kinds of incremental Kalman filters respectively. The simulation results are shown in Figure 1, Figure 2 and Table 1. Figure 1 shows the contrast curve between the real value and filter \( \hat{x}_{k|k} \) for the state, where the solid line represents the true value and the dashed line represents the estimated value. 100 Monte Carlo runs are carried out and the mean square error (MSE) curves are shown by Figure 2. Table 1 gives a comparison of calculation between the method in [13] and Theorem 1 in 200 steps. It can be seen that the proposed incremental
Kalman filtering algorithm can realize the same accuracy for state estimation at a faster speed, and the method is effective and feasible.

![Figure 1. $x_k$ and $\hat{x}_{4k}$](image1.png)

**Figure 1.** $x_k$ and $\hat{x}_{4k}$

![Figure 2. MSE curves for an incremental Kalman filter in fast implementation](image2.png)

**Figure 2.** MSE curves for an incremental Kalman filter in fast implementation

**Table 1.** The comparison of calculation time between the proposed and existed incremental Kalman filters for example 1

| Estimator                        | The calculation time used to in 200 steps (second) |
|----------------------------------|--------------------------------------------------|
| Existed incremental Kalman filter| 0.125                                             |
| Proposed incremental Kalman filter| 0.116                                             |

**Example 2.** Consider the one dimension discrete system

\[
x_k = 0.5x_{k-1} + w_k
\]

\[
z_k = H_kx_k + a_k + v_k
\]

(18)  

(19)

where the system noise $w_k$ is a Gauss white noise with zero mean and variance $Q = 10$, the observation noise $v_k$ is a Gauss white noise with zero mean and variance $R = 1$, they are all known. The observation matrix $H_k = [1 1 0 1]^T$. $a_k = [3 3 3 3]^T$ is unknown observation error.
Applying the method in [13] and Theorem 1 yields the two kinds of incremental Kalman filters respectively. The simulation result is shown in Table 2. Table 2 gives a comparison of calculation between the method in [13] and Theorem 1 in 100 steps. It can be seen that the proposed incremental Kalman filtering algorithm is effective and feasible.

Table 2. The comparison of calculation time between the proposed and existed incremental Kalman filters for example 2

| Estimator                      | The calculation time used to in 100 steps (second) |
|-------------------------------|-----------------------------------------------|
| Existed incremental Kalman filter | 0.136                                           |
| Proposed incremental Kalman filter | 0.094                                          |

5. Conclusion
Considering the systems under poor observation condition, an incremental Kalman filter in fast implementation is proposed in this paper. On the one hand, the error of the system is eliminated, and the accuracy of state estimation is improved. On the other hand, compared with the existed incremental Kalman filter, it has a faster implementation speed. Two simulation examples show its effectiveness.

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