Vortex-line condensation in three dimensions: A physical mechanism for bosonic topological insulators

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Bosonic topological insulators (BTI) in three spatial dimensions are symmetry protected topological (SPT) phases with $U(1) \times \mathbb{Z}_2^T$ symmetry, where $U(1)$ is boson particle number conservation, and $\mathbb{Z}_2^T$ is time-reversal symmetry with $\mathcal{T}^2 = 1$. BTI were first proposed based on the group cohomology theory which suggests two distinct root states, each carrying a $\mathbb{Z}_2$ index. Soon after, surface anomalous topological orders were proposed to identify different root states of BTI, leading to a new BTI root state beyond the group cohomology classification. Nevertheless, it is still unclear what is the universal physical mechanism for BTI phases and what kinds of microscopic Hamiltonians can realize them. In this paper, we answer the first question by proposing a universal physical mechanism via vortex-line condensation in a superfluid, which can potentially be realized in realistic systems, e.g., helium-4 or cold atoms in optical lattices. Using such a simple physical picture, we find three root phases, of which two of them are classified by group cohomology theory while the other is beyond group cohomology classification. The physical picture also leads to a “natural” bulk dynamic topological quantum field theory (TQFT) description for BTI phases and gives rise to a possible physical pathway towards experimental realizations. Finally, we generalize the vortex-line condensation picture to other symmetries and find that in three dimensions, even for a unitary $\mathbb{Z}_2$ symmetry, there could be a nontrivial $\mathbb{Z}_2$ SPT phase beyond the group cohomology classification.

I. INTRODUCTION

Fermionic topological insulators (FTI)[1–7] in free and weakly interacting fermion systems have gained a lot of attention recently. Realistic material candidates for FTIs like Bi$_2$Se$_3$ have also been successfully synthesized and characterized through a remarkable series of experiments.[1–7] In a 3d (three-dimensional) FTI state, the bulk excitation spectrum looks like a conventional band insulator. However, its 2d (two-dimensional) surface exhibits a very anomalous low-energy physics: odd number of Dirac cones. In the language of topological classification, FTIs in free fermions are classified by a $\mathbb{Z}_2$ index signalling the odd/even number of surface Dirac cones. The anomalous physics of odd number of Dirac cones is protected by $U(1) \times \mathbb{Z}_2^T$ symmetry and cannot be achieved in any local lattice fermionic model on a 2d plane unless symmetry is broken. Here, $U(1)$ symmetry corresponds to the fermion particle number conservation, and $\mathbb{Z}_2^T$ is the time-reversal symmetry that acts electrons as $\mathcal{T}^2 = -1$.

The above-mentioned notion of FTIs is just a paradigmatic example of a much more general class of symmetry-protected topological (SPT) phases[8]. SPT states in strongly interacting boson/spin systems have been intensely studied recently.[9–12] By definition, the bulk of a SPT state always only support gapped bosonic excitations but its boundary may exhibit anomalous quantum phenomena in the presence of global symmetry. As such, the aforementioned FTI states can be literally viewed as a fermionic SPT state.[13] Haldane spin chain, which was proposed decades ago, is a typical example of SPT states in 1d.[8, 14–18] Mathematically, given both spatial dimension and symmetry group $G$ as input data, one can apply the “group cohomology theory with $\mathbb{R}/\mathbb{Z}$ coefficient” to systematically classify SPT states.[9] Another mathematical tool “cobordism” has also been applied and some nontrivial SPT states beyond group cohomology theory have been proposed.[19, 20] In addition to the above classification frameworks where advanced mathematics is involved, a lot of progress has also been made from more physical perspectives. In both 2d and 3d, a surge of interest has been shown, such as bosonic lattice model proposal, SPT – topological gauge theory duality, fermionic projective construction, Chern-Simons field theory classification, boundary anomaly, topological response theory, non-linear sigma model approach, “decorated” domain approach, Dijkgraaf-Witten gauge theory obtained by gauging global symmetry of SPT, and so on.[22–54]

In particular, as a bosonic analog of FTIs, the so-called “bosonic topological insulators” (BTI) were proposed first based on the group cohomology theory[9]. By definition, a BTI state is a nontrivial SPT state protected by $U(1) \times \mathbb{Z}_2^T$ symmetry in three dimensions. Here, $U(1)$ symmetry denotes the conservation of boson number, while, time-reversal symmetry $\mathbb{Z}_2^T$ acts on bosons as $\mathcal{T}^2 = 1$ in the bulk. In the framework of group cohomology theory, such SPT states are classified by $\mathbb{Z}_2 \times \mathbb{Z}_2$[9, 37, 55]. In other words, in comparison with the single $\mathbb{Z}_2$ index in FTIs of free fermions, there are two independent $\mathbb{Z}_2$ indices to label distinct BTI states and each index allows us to define a so-called “BTI root state”. The nontrivial phenomena of the first BTI root state can be characterized by its surface $\mathbb{Z}_2$ topological order where both $e$ and $m$ quasiparticles carry half-charge. In addition, if $\mathbb{Z}_2^T$ is explicitly broken on such a surface (e.g. by uniformly adding a layer of ferromagnetic thin film on top of the surface) and the bulk is fabricated in a slab geometry, one may expect a nontrivial electromagnetic response featured by quantum Hall effect with odd-quantized Hall conductance on the surface.
and bulk Witten effect with $\Theta = 2\pi \text{ mod } 4\pi$ [47, 49, 56–58], which is different from $\Theta = \pi \text{ mod } 2\pi$ in FTI states of free fermions. [59] BTI labeled by this $Z_2$ index has been studied in details via fermionic projective construction and dyon condensation. [49] The physical signature of the second BTI root state is characterized by its surface $Z_2$ topological order where both $e$ and $m$ quasiparticles are Kramers’ doublets. Surprisingly, it has been recently known that there is a new $Z_2$ index that is beyond group cohomology classification. [19, 20, 47, 48] As the third BTI root state, it supports a nontrivial surface with the so-called “all-fermion” $Z_2$ topological order where all three nontrivial quasiparticles are self-fermionic and mutual-semionic. Remarkably, an exactly solvable lattice model for this BTI has been proposed via the so-called Walker-Wang approach, [45, 60] which confirms the existence of the third $Z_2$ index. Despite of much progress in diagnosing surface phenomena of BTI states, throughout the paper, we stress that a well-defined bulk theory and bulk definition of symmetry are very crucial towards a controllable understanding of the surface quantum states, which is also highlighted in the conclusion section of [58]. If the bulk theory is unknown, the uniqueness of a proposed surface state is generically unclear. One may also understand the importance of a bulk definition through the following two aspects. Firstly, when a surface phase transition occurs, the bulk doesn’t necessarily experience a bulk phase transition, implying that a many-to-one correspondence between boundary and bulk is generically possible. Incidentally, a many-to-one correspondence was studied in quantum Hall states with high Landau levels. [61, 62] Actually, in the present paper, it will also be shown that the first BTI root state (Sec. V) exhibits a many-to-one correspondence. It generalizes the aforementioned descriptions of surface topological order where only $Z_2$ topological order is allowed. Second, given a 2d state that cannot be symmetrically realized in any 2d lattice model, it does not necessarily mean that the state can be realized on the surface of a 3d SPT phase.

One effective way to derive bulk topological field theory is the so-called “hydrodynamical approach” that we will apply in this paper. Let us first briefly review its application in fractional quantum Hall states (FQHE) which is a 2d problem. In FQHE, the bulk Chern-Simons theory is known to play an important role of encoding “topological data”, such as modular $S_m$ and $T_m$ matrices, chiral central charge $c^-$, and other properties of edge conformal field theory (CFT). [63–70] While it is almost impossible to exactly perform renormalization group analysis directly starting from an intricate microscopic Hamiltonian of interacting electrons gas under strong magnetic field, one can still apply the hydrodynamical approach to effectively handle the problem. The main principle is to study collective modes of Hall system at low energies and long wavelengths, where it is sufficient to only take quantum fluctuations of density $\rho$ and current $j$ into account when the bulk is fully gapped. From this approach, one can obtain various types of bulk topological field theories in a physical and intuitive way. Recently, symmetry implementation in topological field theories has been drawn attention to. Lu and Vishwanath [24] imposed global symmetry to Chern-Simons theory and successfully classified some 2d SPT states with Abelian symmetry.

The success of the hydrodynamical approach in 2d SPT motivates us to develop a “universal hydrodynamical approach” for SPT phases in 3d. However, tackling bulk dynamical topological quantum field theory (TQFT) of a general SPT state in 3d is hard and far beyond the scope of the present work. In this paper, we restrict our attention to investigating the dynamic topological quantum field theory of aforementioned BTI states through considering exotic “vortex-line condensations”, which is pictorially illustrated in Fig. 1. Here vortex-lines mean the configuration of topological line defects in 3d superfluid states, e.g., helium-4. Such a vortex-line condensate state is shown to be described by a topological action in the form of

$$S_{\text{top}} = i \frac{K^{ij}}{2\pi} \int b^i \wedge da^j + i \frac{\Lambda^{ij}}{4\pi} \int b^i \wedge b^j,$$

(1)

($K$ and $\Lambda$ are some $N \times N$ integer matrices that will be elaborated in details in main texts. $I, J = 1, 2, \cdots N$) where $a^I$ are usual 1-form U(1) gauge fields and $b^I$ are 2-form U(1) gauge fields. Surprisingly, we find such a simple physical picture is sufficient to produce all three root states of 3d BTI. More concretely, we find that the first two BTI root states that are within group cohomology classification can be achieved through a pure $b \wedge da$ type term whose physical meaning is illustrated in Fig. 2.

Our above physical approach to 3d BTIs avoids the complications of advanced mathematical topics like cohomology theory and cobordism theory. And this physical picture will shed light on a more challenging question in the future: how to design microscopic interaction terms that can lead to those proposed BTI states. In addition to BTI, the vortex-line condensation picture can be generalized to other symmetry group, e.g., unitary $Z_2$ group and it turns out that there is potentially a nontrivial $b \wedge b$ type term whose physical meaning is illustrated in Fig. 2.

The remainder of the paper is organized as follows. Sec. II is devoted to understand the microscopic origins of bulk dynamical TQFT for 3d BTI. In this section, we will start with a superfluid state in 3d and derive the TQFT description of the vortex-line condensate. In Sec. III, some useful properties of the TQFT are stud-
FIG. 1. Phases obtained by vortex-line condensation. In the Phase transition-1, U(1) symmetry (i.e. boson number conservation) is restored from superfluid to a trivial Mott insulator by condensing strings (i.e. 2π-vortex-lines). Thus, the trivial Mott insulator phase is formed by vortex-line condensation with b ∧ da bulk field theory description. In the Phase transition-2, strings are also condensed and the bulk field theory is also b ∧ da (see Sec. VII). But the resultant Mott phase is a nontrivial SPT state (i.e. bosonic topological insulators, BTI) since either U(1) or Z_2 symmetry transformations is defined in an unusual way. Thus, we end up with two BTI states. In the Phase transition-3, strings are condensed in the presence of a nontrivial Berry phase term, namely, an nontrivial multicomponent b ∧ b term. The nontrivial Mott phase is a BTI phase obtained in Sec. IV, which is a SPT state beyond group cohomology classification and supports “all-fermion” Z_2 surface topological order. Here, Z_2 denotes time-reversal symmetry.

where ρ is the superfluid density. θ is the U(1) phase angle of the superfluid. The spatial gradient of θ costs energy such that a spatially uniform value of θ is picked up in the ground state, rendering a spontaneous symmetry breaking of global U(1) symmetry group (i.e. the particle number conservation of bosons). Since θ physically denotes an angle with periodicity 2π, we may express θ in terms of smooth part and singular part: θ = θ^s + θ^v Substitute this θ decomposition into the Lagrangian of 4d XY theory and introduce a Hubbard-Stratonovich auxiliary field \( J^\mu \): [72–74]

\[
\mathcal{L} = \frac{1}{2\rho} (J_\mu)^2 + i J^\mu (\partial_\mu \theta^s + \partial_\mu \theta^v)
\]

which goes back to the usual Lagrangian of 4d XY theory after \( J^\mu \) is integrated out. It is obvious that \( J^\mu \) can be interpreted as supercurrent of the 3d SF state. Integrating out \( \theta^s \) leads to a constraint \( \delta(\partial_\mu J^\mu) \) in the path integral measure. This constraint can be resolved by introducing a 2-form non-compact U(1) gauge field \( b_{\mu\nu} \): \( J^\mu \overset{\text{def.}}{=} \frac{1}{\pi} \epsilon^\mu\nu\lambda\rho \partial_\nu b_{\lambda\rho} \). Both the physical quantity \( J^\mu \) and the Lagrangian (3) are invariant under the usual smooth gauge transformation:

\[
b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_\nu \xi_\mu,
\]

where \( \xi_\mu \) is a smooth 4-vector. “\( \partial_\nu \Sigma_{\mu\nu} \)” stands for “\( \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \)”. Eventually, the Lagrangian (3) is transformed to the following gauge theory:

\[
\mathcal{L} = \frac{1}{48\pi^2 \rho} h^{\mu\nu\lambda\rho} h_{\mu\nu\lambda} + \frac{i}{2} b_{\mu\nu} \Sigma^{\mu\nu},
\]

where the field strength \( h_{\mu\nu\lambda} \) is a rank-3 antisymmetric tensor: \( h_{\mu\nu\lambda} \overset{\text{def.}}{=} \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu} \). In order to simplify notation, “\( \mathcal{L}_h \)” is introduced via

\[
\mathcal{L}_h \overset{\text{def.}}{=} \frac{1}{48\pi^2 \rho} h^{\mu\nu\lambda\rho} h_{\mu\nu\lambda}
\]

which is the Maxwell term of the two-form U(1) gauge field \( b_{\mu\nu} \). The vortex-line (i.e. string) current operator \( \Sigma_{\mu\nu} \) which is antisymmetric is defined through the singular \( \theta^v \):

\[
\Sigma^{\mu\nu} \overset{\text{def.}}{=} \frac{1}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda \theta^\rho
\]

which is generically nonzero for nontrivial homotopy mapping. The gauge transformation shown above automatically ensures that there is a continuity equation for \( \Sigma_{\mu\nu} \), i.e. \( \partial_\nu \Sigma_{\mu\nu} = 0 \). Hereafter, the nouns “strings”, “vortex-lines”, and “closed loops” are used interchangeably. The vortex-line configuration is very dilute in superfluid. The factor \( \frac{1}{2} \) in the coupling term \( \frac{1}{2} b_{\mu\nu} \Sigma^{\mu\nu} \) naturally arises as a standard convention for the antisymmetric tensor field coupling in 3+1d space-time.

II. MICROSCOPIC ORIGINS OF TOPOLOGICAL FIELD THEORY

A. 3d superfluid state and its dual description

The exotic states to be discussed in this paper are built up from a well known parent state: “3d superfluid (SF) state” which can be described by the following 4d XY continuum field theory at low energies:

\[
\mathcal{L} = \frac{\rho}{2} (\partial_\mu \theta)^2,
\]

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B. Trivial Mott insulators realized by condensing vortex-lines (strings)

Consider strong correlation effects (like Hubbard interactions), we expect that passing through a critical point where the tension of vortex-lines decreases to zero, the string configuration (denoted by the path integral measure $\mathcal{D}\Sigma$) will be proliferated energetically. In other words, “vortex-line condensation” sets in.

The path-integral formalism of vortex-line condensation was ever given in Refs. [75–77]. Here, let us briefly review the basic method. A single fundamental string can be described by reparametrization-invariant Nambu-Goto action. A wave function $\Psi$ can be introduced in quantum theory of strings. Similar to the usual quantum theory of particles, after promoting the quantum mechanics of single-string to field theory of many-strings, $\Psi$ will be viewed as the creation operator (in operator formalism) or the quantum amplitude (in path-integral formalism) of a given string configuration.

In condensate of bosons, the ground state is formed by equal-weight superposition of all kinds of boson configurations in real space, which leads to a macroscopic wave-function, say, $\phi$. The fluctuation of amplitude $|\phi|$ is gapped but the phase fluctuation $\Theta$ is gapless and governed by Eq. (2). Likewise, once vortex-line condensation sets in, all vortex-line configurations have the same quantum amplitude $\Psi$. In contrast to condensate of bosons, the $U(1)$ phase of $\Psi$ of vortex-line condensate is given by a Wilson line $e^{i\int d\Theta^\mu}$ and governed by the Lagrangian given below:

$$L = \frac{1}{2} \phi_0^2 (\partial_\mu \Theta_\nu - b_{\mu\nu})^2 + \mathcal{L}_h,$$  \hspace{1cm} (6)

where the antisymmetrization symbol is defined as usual: $\partial_\mu \Theta_\nu \varepsilon_{\mu\nu} = \partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu$, $|\phi_0|^2$ is the “superfluid density” of the vortex-line condensate. The presence of dynamical gauge field $b_{\mu\nu}$ gaps out the gapless phase fluctuation from $\Theta_\mu$. One may split the phase vector into smooth part $\Theta^S_{\mu}$ and singular part $\Theta^Y_{\mu}$: $\Theta_\mu = \Theta^S_{\mu} + \Theta^Y_{\mu}$, where $\Theta^S_{\mu}$ introduces a solitonic current: $j^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\rho\sigma} \partial_\nu \Theta^S_{\rho} \partial_\sigma \Theta^S_{\mu}$, Its zero component after integrating over 3d space is quantized at integer, i.e. $\int d^3r \frac{1}{2\pi} \nabla \cdot \nabla \times \Theta^Y \in \mathbb{Z}$. Therefore, the gauge group of $b_{\mu\nu}$ is compactified by absorbing $\Theta^Y_\mu$. Note that, in the dual Lagrangian (4) in SF phase, $b_{\mu\nu}$ is non-compact.

Based on Eq. (6), we may formally perform duality transformation in this vortex-line condensation to obtain a $b \wedge da$-term where $a$ is 1-form gauge field. For this purpose, let us introduce a Hubbard-Stratonovich auxiliary tensor field $\Sigma_{\mu\nu}$:

$$L = \frac{1}{2} \Sigma_{\mu\nu} (\partial_\mu \Theta^\nu - b_{\mu\nu})^2 + \mathcal{L}_h + \frac{1}{8\phi_0^2} \Sigma_{\mu\nu} \Sigma_{\mu\nu},$$ \hspace{1cm} (7)

where the physical interpretation of $\Sigma_{\mu\nu}$ is same as the one defined in Eqs. (4,5).

Integrating over $\Theta^S_{\mu}$ in Eq. (7) yields a constraint $\delta (\partial_\mu \Sigma_{\mu\nu})$ in the path integral measure. This constraint can be resolved by introducing a 1-form non-compact $U(1)$ gauge field $a_\mu$:

$$\Sigma_{\mu\nu} \overset{\text{def.}}{=} -\frac{1}{2\pi} \epsilon_{\mu\nu\lambda\rho} \partial_\lambda a_\rho,$$ \hspace{1cm} (8)

indicating that $a_\mu$ field strength is physically identified as the “supercurrent” $\Sigma_{\mu\nu}$ of vortex-lines. The vector field $a_\mu$ is a gauge field since under the gauge transformation:

$$a_\mu \rightarrow a_\mu + \partial_\mu \eta, \hspace{1cm} \text{\textit{9}}$$

the physical observable $\Sigma_{\mu\nu}$ is invariant. $a_\mu$ takes values on whole real axis so that $a_\mu$ is non-compact. Then, the dual formalism of Lagrangian (6) is given by:

$$L = \frac{i}{4\pi} a_\mu \epsilon_{\mu\nu\lambda\rho} \partial_\lambda b_{\rho\nu} + ia_\mu j^\mu + \mathcal{L}_h + \frac{1}{64\pi^2 \phi_0^2} f_{\mu\nu} f^{\mu\nu},$$ \hspace{1cm} (10)

where $b_{\mu\nu}$ is compact and $a_\mu$ is non-compact. The field strength tensor $f_{\mu\nu} \overset{\text{def.}}{=} \partial_\mu a_\nu - \partial_\nu a_\mu$ as usual. $j^\mu$ denotes the excitation of boson particles. Once removing the irrelevant Maxwell terms $\mathcal{L}_h$ and $f_{\mu\nu} f^{\mu\nu}$ at low energies, we end up with a new Lagrangian:

$$L = \frac{i}{4\pi} \epsilon_{\mu\nu\lambda\rho} b_{\mu\nu} \partial_\lambda a_\rho + ia_\mu j^\mu.$$ \hspace{1cm} (11)

As expected, the coefficient $\frac{1}{4\pi}$ indicates that there is no ground state degeneracy (GSD)[80–86] on a 3-torus $\mathbb{T}^3$. In this sense, the bulk state has no intrinsic topological order[80–86]. In terms of exterior products, the term can be rewritten as $\frac{1}{2} b \wedge da$ where $da \equiv f$ is a 2-form strength tensor. In summary, we reach the following remark:

Remark 1 Owing to the microscopic origins of the field theory, in vortex-line condensation phase, $b_{\mu\nu}$ is compact while $a_\mu$ is non-compact.

C. Adding a vortex-line (string) linking Berry phase term into the trivial Mott insulator

In the following, we attempt to explore the possibility of nontrivial Mott insulators. To begin with, we add a topological Berry phase term into Eq. (6) to describe a potential nontrivial “topological vortex-line condensate”:

$$L = \frac{1}{2} \phi_0^2 \left( \partial_\mu \Theta^S_\mu - b_{\mu\nu} \right)^2 + \mathcal{L}_h - \frac{i}{16\pi} \epsilon_{\mu\nu\lambda\rho} \left( \partial_\mu \Theta^Y_\nu - b_{\mu\nu} \right) \left( \partial_\lambda \Theta^S_\rho - b_{\mu\rho} \right),$$ \hspace{1cm} (12)

where $\Theta^Y_\mu$ has been absorbed by $b_{\mu\nu}$ such that $b_{\mu\nu}$ is compactified. The term $\sim d\Theta^S \wedge d\Theta^S$ is a total derivative term.
FIG. 2. Physical meaning of $b \wedge b$ topological term. The larger loop denotes a vortex-line that is static and located on $xy$-plane. The smaller loop is perpendicular to $xy$-plane, parallel to $yz$-plane, and moves toward $z$-direction. There are four slap-shots shown in this figure from left to right. The blue dot in the third slap-shot denotes the intersection of two loops. In the fourth slap-shot, two loops are eventually linked to each other. $b \wedge b$ term will contribute a phase at the third slap-shot of the unlinking-linking process.

and can be neglected. Then, by introducing a Hubbard-Stratonovich auxiliary tensor field $\Xi^{\mu\nu}$ (antisymmetric), Eq. (12) is transformed to:

$$
\mathcal{L} = i\frac{1}{2} \Xi^{\mu\nu}(\partial_\mu \Theta_\nu^a - b^{\mu\nu}) + \frac{1}{8\delta_0^2} \Xi^{\mu\nu} \Xi^{\nu\sigma} + i \frac{\Lambda}{8\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \Theta_\nu^a \delta_{\lambda\rho} \\
- i \frac{\Lambda}{16\pi} \epsilon^{\mu\nu\lambda\rho} b^{\mu\nu} b^{\lambda\rho} + \mathcal{L}_h \\
= i \Theta_\mu^a \partial_\nu \left( \Xi^{\mu\nu} + \frac{\Lambda}{4\pi} \epsilon^{\mu\nu\lambda\rho} b^{\lambda\rho} \right) - i \frac{1}{2} \Xi^{\mu\nu} b^{\mu\nu} \\
- i \frac{\Lambda}{16\pi} \epsilon^{\mu\nu\lambda\rho} b^{\mu\nu} b^{\lambda\rho} + \frac{1}{8\delta_0^2} \Xi^{\mu\nu} \Xi^{\nu\sigma} + \mathcal{L}_h, 
$$

Integrating out $\Theta_\mu^a$ leads to the conservation constraint: $\partial_\nu \left( \Xi^{\mu\nu} + \frac{\Lambda}{4\pi} \epsilon^{\mu\nu\lambda\rho} b^{\lambda\rho} \right) = 0$ which can be resolved by introducing a $1$-form non-compact $U(1)$ gauge field $a_\mu$:

$$
\Xi^{\mu\nu} \overset{\text{def.}}{=} - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda a_\rho - \frac{\Lambda}{4\pi} \epsilon^{\mu\nu\lambda\rho} b^{\lambda\rho}.
$$

This is a ‘modified’ version of Eq. (8) where $b \wedge b$-term is absent. The aforementioned statements (see Remark 1) on non-compactness of $a_\mu$ and compactness of $b^{\mu\nu}$ still hold here. Plugging this expression into the second term in Eq. (13) yields the following topological invariant Lagrangian:

$$
\mathcal{L}_{\text{top}} = \frac{i}{4\pi} \epsilon^{\mu\nu\lambda\rho} b^{\mu\nu} \partial_\lambda a_\rho + i \frac{\Lambda}{16\pi} \epsilon^{\mu\nu\lambda\rho} b^{\mu\nu} b^{\lambda\rho},
$$

where the higher order terms $\mathcal{L}_h$ and $\frac{1}{8\delta_0^2} \Xi^{\mu\nu} \Xi^{\nu\sigma}$ are dropped. We note that the $b \wedge b$ term was previously introduced in mathematical physics\cite{79} and also applied to theory of loop quantum gravity\cite{78} with cosmological constant. Fig. 2 shows that two world-sheets formed by two loops intersect each other will contribute a topological Berry phase to the path integral.

Since the original Lagrangian Eq. (12) is gauge invariant, it is not a surprise that the above topological invariant Lagrangian is also gauge invariant under the following modified gauge transformations:

$$
b^{\mu\nu} \to b^{\mu\nu} + \partial_\mu \xi_\nu, \quad a_\mu \to a_\mu - \Lambda_\xi_\mu = \xi_\mu. \tag{15}
$$

The gauge transformation of $a_\mu$ generalizes the familiar one (9) by incorporating transverse components of $\xi_\mu$. More precisely, if $\xi_\mu$ is purely longitudinal (i.e. $\xi_\mu = \partial_\mu \eta$ where $\eta$ is a scalar function), $a_\mu$ is transformed as usual while $b^{\mu\nu}$ keeps unchanged. But a general vector field $\xi_\mu$ contains both longitudinal and transverse components such that a universal gauge transformation is given by Eq. (15).

**Remark 2** We stress that in Eq. (12) the addition of $b \wedge b$-term is very different from other topological terms (e.g. $da \wedge da \equiv f \wedge f$). The key reason is that a bulk dynamic quantum field theory should be well-defined in a closed manifold as well. However, the Lagrangian density $da \wedge da$ is essentially zero in the bulk due to the non-compactness of $a_\mu$ in Remark 1.

The above single-component theory can be generalized into a multi-component theory:

$$
\mathcal{L}_{\text{top}} = \frac{i}{4\pi} \epsilon^{\mu\nu\lambda\rho} b^{\mu\nu}_a \partial_\lambda a_\rho^J + i \frac{\Lambda}{16\pi} \epsilon^{\mu\nu\lambda\rho} b^{\mu\nu}_a b^{\lambda\rho}_b, \tag{16}
$$

where all Maxwell terms are removed. In terms of exterior products, the action (1) is obtained. Without loss of generality, it is sufficient to consider symmetric matrix $A^{IJ}$ and assume $K^{IJ}$ to be an identity matrix of rank-$N$, i.e.

$$
K = \text{diag}(1, 1, \ldots, 1)_{N \times N} = \mathbb{1}
$$

with $I, J = 1, 2, \ldots, N$. $\{a_\mu^I\}$ are non-compact $1$-form U(1) gauge fields and $\{b^{\mu\nu}_a\}$ are compact $2$-form U(1) gauge fields, respectively, as a straightforward generalization of Remark 1. The above topologically invariant Lagrangian with the particular $K$ matrix Eq.(17) is the central result of this paper (its abstract form in terms of exterior products is given by Eq. (1)), and we will use it to describe all BFI phases as well as some new $Z_N$ SPT phases. Physically, such a multi-component theory can be viewed as a collection of many 3d trivial Mott insulators mutually entangled via $\Lambda$ term.

In general, we can add an excitation term into (16), namely, $\mathcal{L}_{\text{exc}} = L^{IJ}_a a^{I}_\mu a^{J}_\mu + L^{IJ}_b b^{I}_\mu b^{J}_\mu$, and Eq.(16) can also describe intrinsic topological phases with nontrivial particle and string excitations in the bulk if $|\det K| > 1$, parameterized by: $\{K, \Lambda, L_a, L_b\}$. The additional two integer vectors $L_a = (L^1_a, L^2_a, \ldots, L^N_a)^T$ and $L_b = (L^1_b, L^2_b, \ldots, L^N_b)^T$ are excitation vectors that label particles and strings respectively. In the following, we will also call indices $I, J, \ldots$ flavor indices.

### III. GENERAL PROPERTIES OF THE TOPOLOGICAL QUANTUM FIELD THEORY

In this section, we will study the multi-component topological quantum field theory defined by Eq. (16) in a general setting without any global symmetry implementation. In this section as well as Sec. IV, all analyses are
done by implicitly assuming $K$ takes the form in Eq. (17) unless otherwise stated. (e.g. the gauge transformation in Sec. IIIA and $\mathbb{GL}(N,\mathbb{Z})$ transformation in Sec. IIIB are valid for general $K$.)

A. Gauge transformation

As aforementioned, the presence of $b \wedge b$-term drastically changes the gauge structures. The gauge transformation of the multi-component theory Eq. (16) is given by:

$$b_{\mu\nu}^I \rightarrow b_{\mu\nu}^I + \partial_{[\mu}a_{\nu]}^I, \quad a_{\mu}^I \rightarrow a_{\mu}^I - (K^{-1}(A)^I_{\prime\mu}) \xi^I_{\prime}$$  

(18)

which can be understood as a generalized version of Eq. (15).

To see the gauge transformation more clearly, we may reexpress the first two topological terms (denoted by $L_{\text{top}}$) in Lagrangian (16) as:

$$L_{\text{top}} = \frac{iA^I_{\mu\nu}}{32\pi} \epsilon^{\mu\nu\lambda\rho} \left((b_{\mu\nu}^I + (A^{-1}K)^{I}_{\mu\nu}) \partial_{[\mu}a_{\nu]}^I\right) \cdot \left((b_{\lambda\rho}^I + (A^{-1}K)^{I}_{\lambda\rho}) \partial_{[\lambda}a_{\rho]}^I\right).$$  

(19)

From this expression, one may examine the correctness of Eq. (18) easily. To obtain this equivalent expression, we have applied the following two facts: (i) closed space-time manifold is taken. (ii) $a_\mu$ is non-compact such that the term $\sim da \wedge da$ is a total derivative.

B. Bulk $\mathbb{GL}(N,\mathbb{Z})$ transformation

In analog to 2+1d Chern-Simons theory, we can perform two independent general linear ($\mathbb{GL}$) transformations represented by matrices $W, M \in \mathbb{GL}(N,\mathbb{Z})$, on gauge fields $b_{\mu\nu}$, field and $a_{\mu}^I$, respectively. $\mathbb{GL}$ transformations redefine coordinate systems of the charge lattices formed by vortex-line excitation vector $L_b$ and particle excitation vector $L_a$ but leaves geometry of charge lattices unaffected:

$$b_{\mu\nu}^{I} \rightarrow (W^{-1})^{I}_{J}b_{\mu\nu}^{J}, \quad a_{\mu}^{I} \rightarrow (M^{-1})^{I}_{I}a_{\mu}^{I},$$

(20)

where $W, M$ are two $N \times N$ matrices with integer-valued entries and $|\det W| = |\det M| = 1$. These transformations are a “relabeling” of the same low energy physics. After the transformation, in order to formally express the new Lagrangian still in the form of Eq. (16), a new set of parameters $(K, A, L_b, L_a)$ can be introduced via:

$$K = WTKM, \quad A = WTA, \quad L_b = MTL_b, \quad L_a = WTb.$$  

(21)

Regarding these $\mathbb{GL}$ transformations, there is an important remark:

Remark 3 $W$ and $M$ are two independent $\mathbb{GL}$ transformations. In any basis, $|\det K|$ rather than $\det K$ is invariant. Therefore, our choice Eq. (17) is universal once bulk topological order is absent.

C. Flat-connection condition

At classical level, $A$ matrices are arbitrary but at quantum level, we show that there will be constraints from “flat-connection condition”. For simplicity, let us first consider one-component theory:

$$S = \frac{i}{4\pi} \int d^4x b_{\mu\nu} \partial_{\lambda}a_{\rho} \epsilon^{\mu\nu\lambda\rho} + \frac{\Lambda}{16\pi} \int d^4x b_{\mu\nu} b_{\lambda\rho} \epsilon^{\mu\nu\lambda\rho}.$$  

(22)

Equation of motion of $a_{\mu}$ in Eq. (14) leads to the following “flat-connection condition” (Stokes’ theorem is applied):

$$\int_{S} b_{\mu\nu} dS^{\mu\nu} = 2\pi \times \text{integer},$$

(23)

where $S$ is any closed two-dimensional manifold embedded in the four dimensional space-time. Nonzero $2\pi$ contributions arise from the compactness of $b_{\mu\nu}$, $dS^{\mu\nu}$ is the area element on the 2d manifold. It should be noted that, temporarily, there is no Einstein summation over the indices $\mu, \nu, \cdots$ in the above equation. Physically, this condition can be understood as a consequence of (i) the condensation of $2\pi$-vortex-line ($b_{\mu\nu}$ is “higged”) and (ii) the compactness of $b_{\mu\nu}$. For readers who are unfamiliar with 2-form gauge field, one can compare Eq. (23) with a higged 1-form compact gauge field say, $\int_{\gamma} A_{\mu} d\mu = 2\pi \times \text{integer}$ where $L$ is a closed loop. Due to Stokes’ theorem, Eq. (23) leads to quantized “cubic flux density” $\frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \partial_{\mu}b_{\nu\rho}$ of the 2-form gauge field, which is defined in an oriented three-dimensional volume whose boundary is $S$.

More specifically, let us consider the following two tori $\mathbb{T}_{zx}$ and $\mathbb{T}_{yz}$. The former is formed by imaginary time direction and $x$-direction. The latter is formed by $y$- and $z$- directions. In both tori, we have the following flat-connection conditions:

$$\int_{\mathbb{T}_{zx}} b_{0z} d\tau dx = 2\pi \times N_{0z}, \quad \int_{\mathbb{T}_{yz}} b_{0y} dy dz = 2\pi \times N_{0y},$$

where $\tau$ is imaginary time. Note that, above two integrals are done at given $y, z$ and given $\tau, x$ respectively. Since the integrals in both L.H.S. are smooth functions of space-time, the integers $N_{0y}$ ($N_{0y}$) should be independent on the coordinates $y$ and $z$ ($\tau$ and $x$).

Then, one may reformulate $b \wedge b$ term as:

$$i \frac{\Lambda}{16\pi} \int d^4x x \epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} b_{\lambda\rho} =$$

$$i \frac{\Lambda}{16\pi} \int d\tau dx b_{0x} \int_{\mathbb{T}_{yz}} dy dz b_{yz} \epsilon^{xyz} + (x0yz) + (0xyz)$$

$$+ (x0yz) + (y0x0) + (yz0x) + (zp0x) + (zy0)$$

$$= i \frac{8\Lambda}{16\pi} \int d\tau dx b_{0x} \int_{\mathbb{T}_{yz}} dy dz b_{yz} = i \frac{\Lambda}{2\pi} 2\pi N_{0z} 2\pi N_{0z}$$

$$= 2\pi \Lambda N_{0z} N_{0y}.$$  

(24)
By noting that the partition function is invariant if $\int \mathcal{L}$ is shifted by $2\pi$ in a quantized theory, we end up with the following periodicity shift:

$$\Lambda \rightarrow \Lambda + 1. \quad (25)$$

Thus, effectively, $\Lambda$ continuously takes values in $[0,1)$. In the absence of additional quantization conditions, any nonzero $\Lambda$ can be continuously tuned to zero.

More generally, the above periodic shift of the single-component theory is also applicable to the diagonal entries $\Lambda^{IJ}$ of a generic multi-component theory. For off-diagonal entries, the results are still unchanged. For example, since $\Lambda^{13}$ term and $\Lambda^{31}$ term are equal to each other ($\Lambda^{13} = \Lambda^{31}$), the total actions of the mixture of $b_{\mu}^{1}$ and $b_{\mu}^{3}$ are actually:

$$2 \times \frac{\Lambda^{13}}{16\pi} \int d^4 x \epsilon^{\mu \nu \lambda \rho} b_{\mu}^{1} b_{\rho}^{3}. \quad (26)$$

We note that there are eight equivalent copies in Eq. (24) where the two $b$ fields are the same. However, in $\Lambda^{13}$ term where the two $b$ fields are different, there are only four equivalent copies. Overall, loss and gain are balanced such that the periodicity of $\Lambda^{13}$ is still 1.

In summary, all entries of $\Lambda$ can be continuously tuned to zero and thus $\Lambda$ term doesn’t exist for U(1) SPT in three dimensions (space). In this sense, the addition of time-reversal transformations can be consistently defined in the following usual way (the spatial directions are denoted by $i = 1, 2, 3$):

$$\mathcal{T} a_{0}^{\dagger} T^{-1} = a_{0}^{\dagger}, \mathcal{T} a_{1}^{\dagger} T^{-1} = -a_{1}^{\dagger},$$
$$\mathcal{T} j_{0} T^{-1} = j_{0}, \mathcal{T} j_{i} T^{-1} = -j_{i}, \quad (27)$$
$$\mathcal{T} b_{0}^{I} T^{-1} = -b_{0}^{I}, \mathcal{T} b_{i,j}^{I} T^{-1} = b_{i,j}^{I},$$
$$\mathcal{T} \Sigma_{0,i} T^{-1} = -\Sigma_{0,i}, \mathcal{T} \Sigma_{i,j} T^{-1} = \Sigma_{i,j}, \quad (28)$$

where every flavor transformations in the same way. We stress that, if the microscopic origin (i.e. the derivation of field theory from condensation of vortex-lines) is not given, it is impossible to physically define the above symmetry transformations.

The pure $b \wedge f$-term (i.e. $b \wedge b$ term is absent) in Eq. (16) is invariant under these transformation rules. The definition of time-reversal transformation in Eqs. (27) and (28) implies that a pure $b \wedge da$ term necessarily leads to a trivial SPT state for the reason that bulk topological order is absent and symmetry transformation is defined in a usual way. The possibility of nontrivial SPT states arising from pure $b \wedge da$ term will be discussed in Secs. V, VI where either U(1) or time-reversal symmetry has to be modified exotically, as shown in Fig. 1. Therefore, in the present Sec. IV, $\Lambda = 0$ always results in a trivial SPT state by default.

### B. Conditions on $\Lambda$ in the presence of time-reversal symmetry

Under the time-reversal symmetry transformation (28), $b \wedge b$-term is transformed to: $-\frac{\Lambda^{13}}{16\pi} \epsilon^{\mu \nu \lambda \rho} b_{\mu}^{1} b_{\rho}^{3}$. Superficially, this sign change implies that the ground state of topological field theory labeled by $\Lambda$ always “breaks” time-reversal symmetry. However, we will show that the periodicity shift of $\Lambda^{13}$ provides a chance for restoring time-reversal symmetry. We have derived a periodic shift (25) on $\Lambda$ where merely U(1) symmetry is considered. The problem is: are these results still valid in the presence of time-reversal symmetry?

Let us reconsider a simple one-component theory shown in Eq. (22). In the presence of time-reversal symmetry, the space-time manifold becomes non-orientable. Thus, a $\pi$ cubic flux of 2-form gauge field $b_{\mu \nu}$ becomes the minimally allowed value. To have a physical picture for the flux quantization condition on non-orientable manifold, we can consider the simplest case – a flux insertion process for a Mobiouss strip. Very different from a cylinder, where the inserted flux must be in unit of $2\pi$, the Mobius strip allows the inserted flux to be in unit of $\pi$. This is because a Mobius strip must pick up an even winding number to travel back to its origin. In this sense, by using the same notations in Sec. III C, $N_{0x}$ and $N_{0y}$ are now half-integers instead of integers. The last line in Eq. (24) now equals to $\frac{1}{2}\Lambda \pi \times$ integer such that the periodicity of $\Lambda$ now is enhanced to:

$$\Lambda \rightarrow \Lambda + 4. \quad (29)$$
Let us move on to the off-diagonal entires, e.g., $\Lambda^{13}$ in a multi-component theory. At present, there are $N$ components of “topological vortex-line condensations” which, superficially, implies that there are $N\ U(1)$ charge conservation symmetries. However, in our physical system, only one $U(1)$ should be taken into account. Then, when we evaluate the sum of $\Lambda^{13}$ and $\Lambda^{31}$ terms, either $b^1$ gauge group or $b^2$ forms a $\pi$ cubic flux, not both. Therefore, the periodicity of $\Lambda^{13}$ is enhanced from 1 to 2: i.e. $\Lambda^{13} \rightarrow \Lambda^{13} + 2$. In summary, all the above results indicate that $\Lambda^{IJ}$ in the presence of time-reversal symmetry take the following values:

$$\Lambda^{II} = 0, \pm 2, \Lambda^{IJ} = 0, \pm 1 \text{ (for } I \neq J).$$

This quantization condition is protected by time-reversal symmetry.

We note that SPT states (including both trivial and nontrivial states) are defined by the following two common conditions: (i) bulk has no intrinsic topological order; (ii) bulk state respects symmetry. The condition-(i) is always satisfied in our construction since GSD=1 as shown in Sec. III A where $K = I$. If $\Lambda$ entries are defined under the requirement of Eq. (30), the condition-(ii) is also satisfied. Thus, there are infinite number of $\Lambda$ matrices that satisfy Eq. (30) and can be viewed as SPT states with $U(1) \times \mathbb{Z}_2$ symmetry. But which are trivial and which are nontrivial? For example, is $\Lambda = 2$ a trivial or nontrivial SPT? In subsequent discussions, we will aim to answer this question.

### C. Trivial SPT states

As we know that all $\Lambda$ matrices ($K = I$ is implicit all the time) that satisfy the quantization conditions (30) are SPT states. Generically, with an open boundary condition, the surface phenomena of SPT states are expected to capture information of triviality and non-triviality. Therefore, one may wonder what are nontrivial signatures of surface phenomena? Conceptually, one should first understand the set of physical observables that describe the surface physics:

**Definition 1** Physical observables of the surface theory Eq. (26) are composed by: ground state degeneracy, self-statistics and mutual statistics of gapped quasi-particles. All these information can be read out from modular $S_m$ and $T_m$ matrices. Since there are no further 1d boundary, chiral central charge $c^-$ is not an observable on the surface.

Now our question is changed to: how can we use these physical observables to tell a nontrivial SPT from a trivial SPT? The essential physics is the so-called “obstruction” or “anomaly”. More precisely, we define it as:

**Definition 2** By “obstruction”, we mean that the set of physical observables (defined in Definition 1) of the surface theory cannot be reproduced on a 2d plane by any local bosonic lattice model with symmetry. Otherwise, the obstruction is free. Nontriviality of a SPT state corresponds to the presence of obstruction. Distinct kinds of obstruction lead to distinct nontrivial SPT states.

Now, we are at the position to distinguish trivial and nontrivial SPT states.

1. $|\det \Lambda| = 1$

First of all, we consider a subset of $\Lambda$ matrices that satisfy Eq. (30):

$$|\det \Lambda| = 1.$$  \hfill (31)

Mathematically, any $\Lambda$ matrix in this subset can be expressed in terms of two “fundamental blocks”, namely, $\Lambda_{11}$, and $\Lambda_{12}$ given by Cartan matrix of $E_8$ group:[62, 90–92]

$$\Lambda_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda_{12} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$  \hfill (32)

The subscript “t” in “$\Lambda_{11}$” and “$\Lambda_{12}$” stands for “trivial” (to be explained below). One can show that all $\Lambda$ matrices that satisfy Eq. (30) and (31) can be expressed as a direct sum of several $\Lambda_{11}$ and $\pm \Lambda_{12}$ up to an arbitrary $\mathbb{G}/\mathbb{L}$ transformation:

$$\Lambda = W^T(\Lambda_{11} \oplus \Lambda_{11} \oplus \cdots \oplus \pm \Lambda_{12} \oplus \pm \Lambda_{12} \oplus \cdots)W.$$  \hfill (33)

Since $\mathbb{G}/\mathbb{L}$ transformation $W$ doesn’t affect physical observables, whether SPT states in this subset are trivial or not essentially depends on the properties of the two fundamental blocks.

First, $\Lambda_{11}$ gives a trivial SPT state (i.e. a trivial Mott insulator) since its surface physical observables are trivial gapped boson excitations and nothing else. Such surface state can be realized on a 2d lattice model with symmetry. According to Definition 2, the obstruction is free and thus the 3d bulk state is trivial.

Second, in contrast to $\Lambda_{11}$ whose chiral central charge $c_- = 0$, the very feature of $\Lambda_{12}$ is that it has an “irreducible” value of $c_- = 8$ which plays a role of the “generator” of all trivial $\Lambda$ matrices that admit nonzero $c_-$. We call $\pm \Lambda_{12}$ “c-generators”. By “irreducible”, we mean that one can prove that $c_- = 8$ is the minimal absolute value of all $\Lambda$ matrices that satisfy Eq. (30) and (31). Since $c_- \neq 0$, one may wonder if $\Lambda_{12}$ surface state breaks time-reversal symmetry once it is laid on a 2d plane alone. To solve this puzzle, we should, again, focus attentions to physical observables on the surface rather than the formal Lagrangian in Eq. (26). On the grounds that a surface is a 2d closed manifold by definition, there is no further 1d edge so that $c_-$ is not detectable on the
surface, which is also summarized in Definition 1. Thus, \( \Lambda_{12} \) still gives a trivial SPT state.

**Stacking recipe.**—Technically, the triviality of \( \Lambda_{12} \) can be understood by performing the following “stacking recipe”. Its surface state, before laying it on a 2d plane, can be stacked with a \(-\Lambda_{12}\), which leaves the properties of physical observables in Definition 1 unaffected because

\[
\det(\Lambda_{12} \oplus (-\Lambda_{12})) = 1.
\]

This state is essentially formed by direct sum of eight \( \Lambda_{11} \) matrices after a GL transformation, *i.e.* \( W^T (\Lambda_{12} \oplus (-\Lambda_{12})) W = \sum_{t_1} \oplus_{\Lambda_{11}} \). In the end, the resultant state, which merely supports gapped boson excitations (identity particles) and has vanishing chiral central charge, can be realized on a 2d plane without breaking symmetry.

In summary, both fundamental blocks, *i.e.* \( \Lambda_{11} \) and \( \Lambda_{12} \) give trivial SPT states, and therefore, all \( \Lambda \) matrices defined by Eq. (30) and Eq. (31) give trivial SPT states.

2. \( |\det \Lambda| > 1 \)

The above discussion leads to a set of trivial SPT states defined by Eq. (30) and Eq. (31). All of these trivial states do not admit topological order on the surface. How about topologically ordered surface (i.e. \( |\det \Lambda| > 1 \))? While the quantization conditions (30) guarantee that the bulk is symmetric, surface might break symmetry. If symmetry is manifestly broken on the surface, such a surface state can be realized on a 2d plane, which is free of obstruction from symmetry requirement. The corresponding bulk state is a trivial SPT state.

Therefore, hereafter we will merely focus on the symmetry preserving surface state. But a question arises: how can we judge the symmetry is preserved on the surface?

A symmetric surface governed by Eq. (26) with \( \Lambda \) matrix must describe the same set of physical observables as those on the surface with \(-\Lambda\)-matrix. In order to leave the surface physical observables unaffected under time-reversal (i.e. \( \Lambda \rightarrow -\Lambda \)), one obvious way is to consider the following equivalence under GL transformation, namely, “GL-equivalence”:

\[
W^T \Lambda W = -\Lambda, \quad \forall W \in \text{GL}(N, Z) \tag{33}
\]

which does not change physical observables in Definition 1. We introduce the following symbol

\[
\Lambda \overset{\text{GL}}{=} -\Lambda \tag{34}
\]

to denote this equivalence relation. However, such a topological order has no obstruction if it is defined on a 2d plane with symmetry since such a GL transformation can also be regularly performed on a 2d plane. Thus, the only way toward nontrivial SPT states is to find a new method that can transform \( \Lambda \) to \(-\Lambda\) leaving all physical observables unaffected. And this new method is forbidden to be regularly done on a 2d plane. This is what we shall do in Sec. IV D where an “extended GL transformation” Eq. (35) is defined.

Before proceeding further, let us give some examples. The simplest one is \( \Lambda = 2 \) which can be neither connected to \( \Lambda = -2 \) via Eq. (33) nor Eq. (35), so that the corresponding bulk state is a trivial SPT with symmetry-breaking surface. Another example is \( \Lambda = \left( \begin{array}{cc} 0 & -2 \\ 2 & 0 \end{array} \right) \) which can be connected to \( \Lambda = \left( \begin{array}{cc} -2 & 0 \\ 0 & 2 \end{array} \right) \) via Eq. (33), so that the corresponding bulk state is a trivial SPT with symmetry-preserving surface.

### D. Extended GL transformation and nontrivial SPT states

In order to obtain nontrivial SPT states, again, we resort to the fact that the chiral central charge on the surface is not a physical observable as discussed in Sec. IV C. We may relax the GL transformation by arbitrarily adding fundamental blocks like \( \Lambda_{11} \) and \( \pm \Lambda_{12} \) defined in Sec. IV C along the diagonal entries of \( \Lambda \) matrix. This stacking is legitimate since all fundamental blocks correspond to trivial SPT states which do not induce phase transitions.

Technically, we perform a so-called “extended GL transformation” via the following equivalence relation:

\[
W^T (\Lambda \oplus \Lambda_{11} \oplus \Lambda_{11} \cdots) W = (-\Lambda) \oplus \pm \Lambda_{12} \oplus \pm \Lambda_{12} \cdots, \quad \forall W \in \text{GL}(N', Z) \tag{35}
\]

which also leaves physical observables in Definition 1 unaffected. Here, the left-hand-side contains \( (N'-N)/2 \) \( \Lambda_{11} \) matrices, while, the right-hand-side contains \( (N'-N)/8 \) “\( \pm \Lambda_{12} \)” matrices. Here, \( \Lambda \) is \( N \times N \) as usual. The extended GL transformation for \( \Lambda \) means that adding several fundamental blocks both to \( \Lambda \) and \(-\Lambda\) results in two new matrices, *i.e.* \( \Lambda \oplus \Lambda_{11} \oplus \Lambda_{11} \cdots \) and \(-\Lambda \oplus \pm \Lambda_{12} \oplus \pm \Lambda_{12} \cdots \), by performing a GL transformation \( W \).

Since \( (\Lambda_{12} \oplus (-\Lambda_{12})) \overset{\text{GL}}{=} \sum_{t_1} \oplus_{\Lambda_{11}} \), the \( \pm \) signs reduce to an overall sign, *i.e.*

\[
W^T(\Lambda \oplus \Lambda_{11} \oplus \Lambda_{11} \cdots)W = (-\Lambda) \oplus \left[ \pm (\Lambda_{12} \oplus \Lambda_{12} \cdots) \right], \quad \forall W \in \text{GL}(N', Z). \tag{36}
\]

If \( \Lambda \) and \(-\Lambda\) are connected to each other via an extended GL transformation, we define the symbol:

\[
\Lambda \overset{\text{enGL}}{=} -\Lambda
\]

to denote their equivalence relation.

We find that there is a *unique solution* that supports nontrivial SPT. It is the Cartan matrix of SO(8) group, denoted by \( \Lambda_{\text{so8}} \):

\[
\Lambda_{\text{so8}} = \left( \begin{array}{cccc} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array} \right). \tag{37}
\]
More precisely, the following transformation exists:
\[ W^T (\Lambda_{\text{os8}} \sum_4^{i=1} \Lambda_{4i}) W = (-\Lambda_{\text{os8}}) \oplus \Lambda_{12}, \]
\[ \exists W \in GL(12, \mathbb{Z}). \]

Instead of directly looking for the explicit matrix form of \( W \), the existence of \( W \) can be justified by checking the equivalence of the physical observables (see Definition 1) between \( (\Lambda_{\text{os8}} \sum_4^{i=1} \Lambda_{4i}) \)-surface-Chern-Simons theory and \( (-\Lambda_{\text{os8}} \oplus \Lambda_{12}) \)-surface-Chern-Simons theory. Indeed, both share the same excitation spectrum that are formed by four distinct gapped quasiparticles, and, they share the same real \( S_m \) and \( T_m \) matrices. In addition, the chiral central charge \( c \) is obtained and examined rigorously from bulk to boundary step-by-step.

Furthermore, it is a nontrivial SPT, i.e. bosonic topological insulator (BTI). To see its non-triviality clearly, we note that the chiral central charge \( c \) breaks time-reversal symmetry. This obstruction directly indicates that the ground state of topological field theory is a nontrivial SPT state. However these solutions are “reducible” in a sense that all such solutions describe the same surface physical observables as \( \Lambda_{\text{os8}} \) does and thereby can reduce back to \( \Lambda_{\text{os8}} \).

In summary, we derive a BTI state from our bulk field theory where a nontrivial multi-component \( b \wedge b \) term plays an essential role. We stress that this BTI state is obtained and examined rigorously from bulk to boundary step-by-step.

**V. BOSONIC TOPOLOGICAL INSULATORS FROM PURE \( b \wedge da \) TERM—(I)**

In the above discussions, we consider the time-reversal transformation defined by Eqs. (27) and (28) such that the pure \( b \wedge da \) term always describes trivial SPT states, i.e. “trivial Mott insulators” in Fig. 1. In the following Sections V, VI, we will show that two BTI states that are within group cohomology can be obtained by pure \( b \wedge da \) topological term where either U(1) or time-reversal symmetry is defined in an unusual way.

**A. \( \mathbb{Z}_2 \) nature of exotic bulk U(1) symmetry definition**

The unusual U(1) symmetry transformation can be directly characterized by a \( \Theta \)-term \( F \wedge F \) in the response theory, where \( F_{\mu \nu} \) is the field strength of external electromagnetic field \( A_{\mu} \). Technically, one may start with a generic \( b \wedge da \) theory with \( N \) components and then add an electromagnetic coupling terms like \( \sum_{\mu \nu} \frac{q_1}{4\pi} F_{\mu \nu} \partial_\lambda a_\rho^l \epsilon^{\mu \nu \lambda \rho} + \sum_{\mu \nu} \frac{q_2}{4\pi} F_{\mu \nu} \partial_\rho b_\lambda^l \epsilon^{\mu \nu \lambda \rho} \) where \( \{q_1\}, \{q_2\} \) are two integral charge vectors. The first term is the coupling between face variable \( F_{\mu \nu} \) and vortex-line current \( \frac{1}{2\pi} \partial_\lambda a_\rho \epsilon^{\mu \nu \lambda \rho} \), where the additional \( \frac{1}{2\pi} \) is due to the double counting of the pair indices \( \mu, \nu \). The second term is the coupling between link variable \( A_\mu \) and boson particle current \( \frac{1}{2\pi} \partial_\lambda b_\rho \epsilon^{\mu \nu \lambda \rho} \).

By integrating out all \( a^l \) and \( b^l \) gauge fields, we may expect a response action with bulk \( \Theta \)-term in addition to the usual Maxwell terms:

\[ S_{\text{EM Response}} = \int d^4 x \frac{\Theta}{32\pi^2} F_{\mu \rho} F_{\lambda \rho} \epsilon^{\mu \nu \lambda \rho} + \cdots, \]

where \( \cdots \) denotes Maxwell terms. For \( \det K = \pm 1 \) that we consider here, there are only two possibilities: \( \Theta = 0 \mod 4\pi \) or \( 2\pi \mod 4\pi \). This \( \mathbb{Z}_2 \) classification can be understood through various insights, such as the charge lattice of bulk quasiparticles\[49\], statistical Witten effect\[58\], both of which rely on the pioneering studies in \[57\]. As a side note, if all gauge fields (i.e. \( a, b \)) are non-compact, one may directly integrate them in real axis and always end up with a trivial electromagnetic response: \( \Theta = 0 \mod 4\pi \).

Therefore, nontrivial electromagnetic response, if exists, must be realized when some gauge fields are compact. We will again give a \( N = 2 \) simple example in Sec. VB.

Furthermore, the bulk \( \Theta \) term is essentially reduced to a surface Hall response action \( \int d^3 x \frac{\Theta}{8\pi^2} A_\mu \partial_\nu A_\lambda \epsilon^{\mu \nu \lambda} \) if \( A_\mu \) is a smooth configuration.

**B. An example with \( N=2 \)**

Let us start with a \( b \wedge da \) theory with the following \( K \) matrix:

\[ K = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}. \]

More explicitly, the total Lagrangian is given by:

\[ \mathcal{L}_0 = i \frac{1}{4\pi} b^l_{\mu \nu} \partial_\lambda a^2_{\rho \sigma} \epsilon^{\mu \nu \lambda \rho} + i \frac{1}{4\pi} b^2_{\mu \nu} \partial_\lambda a^l_{\rho \sigma} \epsilon^{\mu \nu \lambda \rho} + i \frac{2}{4\pi} b^2_{\mu \nu} \partial_\lambda a^2_{\rho \sigma} \epsilon^{\mu \nu \lambda \rho}. \]


Here, we assume that both $a_\mu^2$ and $b_{\mu\nu}$ are compact, and both $a_\mu^1$ and $b_{\mu\nu}$ are non-compact. Time-reversal symmetry is defined in the usual way shown in Eqs. (27,28).

Let us move on to the surface theory. Starting with Eq. (40), one can integrate out the non-compact gauge field $a_\mu^1$, and $b_{\mu\nu}$ can be resolved by introducing a 1-form $U(1)$ compact gauge field $\tilde{a}_\mu^2$:

$$b_{\mu\nu} \overset{\text{def}}{=} \partial_\mu a_\nu^2.$$  \hspace{1cm} (41)

Thus, by means of Eqs. (28), $\tilde{a}_\mu^2$ is transformed as a pseudo-vector under time-reversal symmetry: $\tilde{a}_0^2 \rightarrow -\tilde{a}_0^2$, $\tilde{a}_i^2 \rightarrow \tilde{a}_i^2$, $i = x, y, z$.

The term $\sim b^2 \wedge da^2$ in Eq. (40) provides a surface Chern-Simons term:

$$L_{\partial} = i \frac{1}{\pi} \epsilon^{\mu\nu\lambda} \tilde{a}_\mu^2 \partial_\nu a_\lambda^2$$  \hspace{1cm} (42)

which can also be reformulated by introducing a matrix $K_\partial \overset{\text{def}}{=} (\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix})$ in the standard convention of $K$-matrix Chern-Simons theory. \cite{63} $a_\mu^2$ and $\tilde{a}_\mu^2$ form a 2-dimensional vector $(a_\mu^2, \tilde{a}_\mu^2)^T$. The ground state of Eq. (42) supports a $Z_2$ topological order \cite{93} associated with four gapped quasiparticle excitations $(1, e, m, \varepsilon)$. Here, “1” denotes identical particles. Both $e$ and $m$ are bosonic, while $\varepsilon$ is a fermion. The particle $e$ carries +1 gauge charge of $a_\mu^2$, while the particle $m$ carries +1 gauge charge of $\tilde{a}_\mu^2$. $\varepsilon$ carries +1 gauge charges of both gauge fields. These three nontrivial quasiparticles all have mutual semionic statistics, i.e. full braiding one particle around another distinct particle leads to a $\pi$ Aharonov-Bohm phase.

To obtain BTI states, we expect the U(1) symmetry transformation may be performed in an unusual way. For the purpose, we may add the following electromagnetic coupling term to $L_\partial$ in Eq. (40):

$$\frac{q_1}{4\pi} F_{\mu\nu} \partial_\lambda a_\mu^2 \epsilon^{\mu\nu\lambda\rho} + \frac{q_2}{4\pi} A_\mu \partial_\nu b_{\mu\nu}^2 \epsilon^{\mu\nu\lambda\rho}.$$  \hspace{1cm} (43)

Let us consider that a surface is located on $z = 0$ plane. The surface theory is described by Eq. (42). The coupling term contributes the following surface electromagnetic coupling terms:

$$\frac{q_1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda^2 + \frac{q_2}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \tilde{a}_\lambda^2,$$  \hspace{1cm} (44)

where Eq. (41) is applied. Based on the Chern-Simons term in Eq. (42), one may calculate the electric charge carried by each quasiparticle:

$$Q_e = (q_2, q_1) (K_\partial)^{-1} (1, 0)^T = \frac{q_1}{2},$$

$$Q_m = (q_2, q_1) (K_\partial)^{-1} (0, 1)^T = \frac{q_2}{2}.$$  \hspace{1cm} (45)

Physically, both $e$ and $m$ quasiparticles can always attach trivial identity particles to change their charges by arbitrary integer so that $q_1$ and $q_2$ are integers mod 2, namely,

$$q_1 \sim q_1 + 2, q_2 \sim q_2 + 2.$$  \hspace{1cm} (46)

For this reason, it is sufficient to merely consider the following four choices: $(q_2, q_1) = (0, 0), (0, 1), (1, 0), (1, 1)$. The first three choices can be realized on a 2d plane without breaking time-reversal symmetry since the Hall conductance $\sigma_{xy} = \frac{1}{2\pi} q^T (K_\partial)^{-1} q = \frac{q_1 + q_2}{2\pi}$ is zero on a 2d plane. However, $(1, 1)$ necessarily breaks time-reversal symmetry on a 2d plane since its Hall conductance $\sigma_{xy} = \frac{1}{2\pi} q^T (K_\partial)^{-1} q = \frac{1}{2\pi} q_1 q_2 = \frac{1}{2\pi}$ is nonzero. Thus, there is a nonvanishing chiral charge flow on the 1d edge of the 2d plane. However, it doesn't break time-reversal symmetry on the surface since chiral charge flow is not a physical observable on the surface. Therefore, the charge assignment $(1, 1)$ faces obstruction in being realized on a 2d plane with $U(1) \times \mathbb{Z}_2^4$ symmetry and this obstruction leads to a BTI in which both $e$ and $m$ carry half-charge on the surface. Note that, the $q_1$-coupling terms in Eq. (43) and Eq. (44) change sign under time-reversal transformation. However, the sign can be removed through shifting $q_1$ to $-q_1$ following the identification in Eq. (45).

In a real transport experiment, one may explicitly break time-reversal symmetry along the normal direction on the surface with charge assignment $(1, 1)$ and put the 3d system in a slab geometry. The surface quantum Hall conductance is quantized at $\frac{1}{2\pi}$, which corresponds to a surface response action in the form of Chern-Simons term: $\frac{1}{2\pi} A_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda\rho}$. It can be formally extended to a bulk $\Theta$-term, i.e. Eq. (38) with $\Theta$ angle given by $\Theta = 2\pi \mod 4\pi$ and the following generic relation:

$$\Theta = 4\pi^2 \sigma_{xy}.$$  \hspace{1cm} (46)

The $4\pi$ periodicity corresponds to even-quantized Hall conductance $\sigma_{xy} = \frac{1}{2\pi} \times 2k$, which can be realized in purely 2D bosonic systems. A projective construction on such a BTI with $\Theta$ response has been made in details in Ref. \cite{49}.

Physically, $e$ and $m$ particles can be regarded as ends of condensed vortex-lines here, and we may think these invisible vortex-lines carry integer charge and form a nontrivial 1D BTI phase. Thus, it is also not a surprise that the end of these vortex-lines will carry half-charge.

\section{C. Many-to-one correspondence between surface and bulk}

The above discussion illustrates how the surface $Z_2$ topological order arise with unusual U(1) symmetry transformations. As a matter of fact, the surface may be of different kinds while all share the same bulk that only supports trivial boson excitations. In other words, the surface topological order may be much richer.

One may replace the $K$-matrix in Eq. (39) by

$$K = \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix},$$  \hspace{1cm} (47)

where $p$ is a nonzero positive integer. The surface term
Eq. (42) now becomes:

\[ \mathcal{L}_\beta = i \frac{p}{2\pi} \epsilon^{\mu\nu\lambda} \hat{a}_\mu^2 \partial_\nu a_\lambda^2 \]  

(48)

which corresponds to a \(Z_p\) topological order on the surface labeled by \(K_\beta = (\frac{0}{0} 1)\).

There are \(p^2\) types of quasiparticles (including the trivial particle), which are labeled by quasiparticle vector \(l = (l_1, l_2)^T\) with \(l_1, l_2 = 0, 1, \cdots, p - 1\). The \(l\)th-quasiparticle carries the electric charge

\[ Q_l = q^T K_\beta^{-1} l = \frac{1}{p}(q_1 l_2 + q_2 l_1). \]  

(49)

Physically all quasiparticles can attach trivial identity particles such that their charges can be changed by any integer. Therefore, the following identification conditions exist:

\[ q_1 \sim q_1 + p, \quad q_2 \sim q_2 + p. \]  

(50)

We note that \(q_1\)-coupling terms in Eq. (43) and Eq. (44) change sign under time-reversal transformation. However, the sign can be removed through shifting \(q_1\) to \(-q_1\) following the identification in Eq. (50). Therefore, bulk time-reversal symmetry requires that \(p = \text{even}\) and only two choices for the integer \(q_1\) are legitimate: \(q_1 = 0 \text{mod} p, \frac{p}{2} \text{mod} p\).

To determine the bulk is trivial or not, one may again examine the surface Hall conductance which is quantized at odd for nontrivial bulk. All even-quantized parts can be removed by stacking several U(1) SPT states on the surface. At present, the Hall conductance is given by \(\sigma_{xy} = \frac{1}{2\pi} q^T K_\beta^{-1} q = \frac{1}{2\pi} 2q^2\). Combined with the relation in Eq. (46), we end up with the following results:

1. \(p = \text{even}\), \(q_1 = \frac{p}{2} \text{mod} p, q_2 = \text{odd}\): BTI states.
2. \(p = \text{even}\), \(q_1 = \frac{p}{2} \text{mod} p, q_2 = \text{even}\): trivial SPT states.
3. \(p = \text{even}\), \(q_1 = 0 \text{mod} p\): trivial SPT states.
4. \(p = \text{even}\), \(q_1 \neq 0 \text{mod} p, q_1 \neq \frac{p}{2} \text{mod} p\): Bulk time-reversal symmetry is explicitly broken.
5. \(p = \text{odd}\): Bulk time-reversal symmetry is explicitly broken.

As a side note, in the first two cases, by means of the identification in Eq. (50), \(q_2\) may be shifted by \(p\). Since \(p\) is even in these two cases, the even / odd property of \(q_2\) is unchanged. Thus, these two cases are consistent with the identification conditions.

When \(p = 2, q_1 = q_2 = 1\), the theory goes back to Sec. VIB. It is clear that the surface \(Z_2\) topological order obtained in Sec. VIB is just one possible surface of BTI states, which manifests the physics of many-to-one correspondence between surface and bulk. For example, we may choose \(p = 4, q_1 = 2, q_2 = 1\) such that the bulk is a BTI state with nontrivial Witten effect. The electric charge carried by totally \(4^2 - 1 = 15\) nontrivial quasiparticles can be calculated by Eq. (49): \(Q_l = \frac{q_1}{2} + \frac{q_2}{4}\). Such an assignment of fractional charge on quasiparticles of \(Z_4\) topological order cannot be realized on a 2d plane unless breaking time-reversal symmetry.

VI. BOSONIC TOPOLOGICAL INSULATORs FROM PURE \(b \wedge da\) TERM—(II)

A. \(Z_2\) nature of bulk time-reversal symmetry definition

Let us consider \(K\) matrix in the form of Eq. (17). A time reversal transformation acting on gauge fields \(a_\mu^i, b_\mu^i\) can be formally expressed as:

\[ T a_\mu^i T^{-1} = T \delta a_\mu^i T \delta a_\mu^i, \quad T b_\mu^i T^{-1} = T b_\mu^i b_\mu^i, \]  

(51)

\[ T a_\mu^i T^{-1} = -T a_\mu^i T \delta b_\mu^i, \quad T b_\mu^i T^{-1} = -T b_\mu^i b_\mu^i, \]  

(52)

where \(T^a\) and \(T^b\) are two integer-valued matrices. In the following, we will simply call \(T^a\) and \(T^b\) “\(T\)-matrices”.

After transforming twice, all gauge variables are unchanged. So we have the constraint \((T^a)^2 = (T^b)^2 = 1\). It indicates that \(|\text{det} T^a| = |\text{det} T^b| = 1\) and both matrices belong to a subset of \(O(N, Z)\) group.

After \(GL\) transformations and time-reversal transformation, \(K\) matrix is transformed to a new one but \(|\text{det} K| = 1\) is still valid such that the bulk still merely supports trivial gapped boson excitations as before. From this perspective, the bulk always keeps time-reversal symmetry although the formal expression of Lagrangian is given by a new \(K\) matrix.

On the other hand, one may apply arbitrary \(GL(N, Z)\) transformations on both sides of all equations in Eqs. (51,52). Using the notation in Eq. (20), we obtain the following equations:

\[ T a_\mu^i T^{-1} = T a_\mu^i T \delta a_\mu^i, \quad T b_\mu^i T^{-1} = T b_\mu^i b_\mu^i, \]  

(53)

\[ T a_\mu^i T^{-1} = -T a_\mu^i T \delta b_\mu^i, \quad T b_\mu^i T^{-1} = -T b_\mu^i b_\mu^i, \]  

(54)

where two \(T\) matrices are transformed to two new ones:

\[ T^a \overset{\text{def.}}{=} M^{-1} T^a M, \quad T^b \overset{\text{def.}}{=} W^{-1} T^b W. \]  

(55)

By keeping \(|\text{det} W| = |\text{det} M| = 1\) in mind, it is clear that the \(\pm\) sign of determinant of \(T\) matrices is manifestly invariant under arbitrary sequence of formal \(GL\) transformations. For the sake of convenience, let us introduce a notation \((a, b)\) that denotes such \(\pm\) signs of determinants:

\[ (a, b) \overset{\text{def.}}{=} (\text{sign of det} T^a, \text{sign of det} T^b). \]  

(56)

Due to the presence of this invariant, we are able to understand usual time-reversal transformation defined in Eqs. (27) and (28) in a much more general background. In terms of \(T\) matrices, the usual time-reversal transformation defined in Eqs. (27) and (28) is denoted by:

\[ T^a = -T^b = \text{diag}(1, 1, \cdots, 1)_{N \times N}, \]  

(56)
where $K$ matrix is fixed as a unit matrix shown in Eq. (17). This specific form of usual time-reversal transformation in a given basis (i.e. $K = 1$) can be generalized and replaced by the following invariant:

$$(a, b) = (1, (-1)^N)$$

(57)

which is a universal property of all specific forms of usual time-reversal transformations.

To sum up, let us consider a 3d bulk state described by a $N$-component $b \land da$ term labeled by $K$ with $|\det K| = 1$. There are $T^a$ and $T^b$ two matrices that define time-reversal transformations. As discussed before, for the purpose of exploring nontrivial SPT states, we consider the cases that have symmetry-preserving surface. Then, if $(a, b) = (1, (-1)^N)$, the state admits a usual time-reversal transformation and thereby a trivial SPT state. If $(a, b) \neq (1, (-1)^N)$, the state is a nontrivial SPT state. From this point of view, a $Z_2$ classification is obtained by attempting to change the definition of time-reversal transformations. Along this line of thinking, in Sec. VI B, we will study $N = 2$ as a simple example which reproduces the BTI state labeled by the first $Z_2$ index introduced in Sec. I.

**B. An example with $N = 2$**

Let us still start with a $b \land da$ theory with $K$ matrix given by (39). The total Lagrangian is given by (40). It is also assumed that both $a_\mu^2$ and $b_\mu^2$ are compact, and, both $a_\mu^1$ and $b_\mu^1$ are non-compact. The microscopic origin of this field theory and the assignment of compactness is given by Appendix A. At present, the time-reversal transformation in each flavor is independent:

$$T a_\mu^2 T^{-1} = a_\mu^2, \quad T a_\mu^1 T^{-1} = -a_\mu^1, \quad T b_{\mu i}^1 T^{-1} = b_{\mu i}^1, \quad T a_{\mu i}^2 T^{-1} = -a_{\mu i}^2, \quad T b_{\mu i}^2 T^{-1} = -b_{\mu i}^2.$$  

(58-61)

Thus, $T$ matrices are given by:

$$T^a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(62)

This definition of time-reversal symmetry transformation is labeled by $(a, b) = (-1, -1)$ which is nontrivial according to Eq. (57). Under the time-reversal transformation, the $b \land da$ term labeled by $K$ is transformed to the term labeled by $K' = (\frac{1}{2}, -\frac{1}{2})$. At first glance, time-reversal symmetry is broken in the bulk. However, two $b \land da$ theories labeled by $K'$ and $K$ are $G\ell$-equivalent by using

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\quad \text{defined in Eq. (21).}$$

More explicitly, these $G\ell$ transformations lead to a sign change in both $b_\mu^2$ and $a_\mu^2$, while $b_\mu^1$ and $a_\mu^1$ are invariant:

$$b_\mu^2 \rightarrow -b_\mu^2, \quad a_\mu^1 \rightarrow -a_\mu^1, \quad b_\mu^1 \rightarrow b_\mu^1, \quad a_\mu^2 \rightarrow a_\mu^2.$$  

(63-64)

Let us move on to the surface theory (42). According to the transformation of $b_\mu^2$ in Eq. (61), the time-reversal transformation rule of $a_\mu^2$ defined in Eq. (41) is automatically fixed:

$$T a_\mu^2 T^{-1} = -a_\mu^2, \quad T a_\mu^2 T^{-1} = a_\mu^2.$$  

(65)

Due to the time-reversal transformations in Eqs. (65,60), both $m$ and $e$, which carry unit gauge charges of $a_\mu^2$ and $a_\mu^1$ respectively, are pseudo-like particles on the surface. Under these time-reversal transformations, a minus sign appears in the Chern-Simons term in Eq. (42). Despite that, the surface state doesn’t break time-reversal. More precisely, the appearance of this minus sign leaves the set of surface physical observables in Definition 1 unaffected by noting that there is a $G\ell$ equivalence relation: $P^T(-K_0)P = K_0$, where $P = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. Therefore, the surface is symmetric under time-reversal transformation.

It is also beneficial to investigate the equations of motion (EOM) of gauge fields under time-reversal symmetry. By adding two quasiparticle currents $j_\mu^c a_\mu^2 + j_\mu^e a_\mu^1$ where $j_\mu^c$ and $j_\mu^e$ are $c$-particle and $m$-particle currents respectively, the EOMs of gauge fields are given by (only zero-component is shown here without loss of generality): $\pi_\mu^c = \nabla \times \tilde{a}_2 = \tilde{b}$, $\pi_\mu^m = \nabla \times a_2 = B$, where $\rho^c = j_\mu^c$ and $\rho^m = j_\mu^e$ are density variables of $c$ and $m$ respectively. $\tilde{B}$ and $B$ are magnetic flux strength of $\tilde{a}_2$ and $a_2$ respectively. Under time-reversal, $\rho_c$ and $\rho_m$ change sign since they are pseudo-like: $\rho_c \rightarrow -\rho_c$, $\rho_m \rightarrow -\rho_m$. On the other hand, both $\tilde{B}$ and $B$ are unchanged under time-reversal transformations Eqs. (65,60). It looks like EOMs break time-reversal. However time-reversal symmetry is still invariant due to the very existence of the 3d bulk. More concretely, the 3d bulk can source a trivial particle that carries two units of $a_\mu^2$ gauge charge. One may attach this trivial particle to a time-reversal partner of $e$ particles on the surface, rendering $-\rho_e + 2\rho_c = \rho_c$. Likewise, the 3d bulk can source a trivial particle that carries two units of $a_\mu^1$ gauge charge. One may attach this trivial particle to a time-reversal partner of $m$ particles, rendering $-\rho_m + 2\rho_m = \rho_m$. As a result, both EOMs respect time-reversal symmetry.

However, this $Z_2$ topological order state on a 2d plane (i.e. no 3d bulk) necessarily breaks time-reversal symmetry. In the absence of 3d bulk, all trivial particles that are used to change sign of $\rho_c$ and $\rho_m$ can only come from 2d state itself. Consequently the magnetic fluxes generated by these trivial particles will change $\int dx dy \tilde{B}$ and $\int dx dy B$ to $\int dx dy \tilde{B} + 2\pi$ and $\int dx dy B + 2\pi$ respectively. Thus, EOMs always break time-reversal symmetry.

In summary, following the definition of obstruction in Definition 2, the 3d bulk is a nontrivial SPT state, i.e. a
BTI state. Physically, the unique way to realize such a time-reversal symmetry on gauge fields, i.e. Eqs. (65,60), is to consider that both $e$ and $m$ are Kramers’ doublets. Other possibilities of Kramers’ degeneracy assignment (e.g. $e$ is Kramers’ doublet while $m$ is Kramers’ singlet) can be realized on a 2d plane without breaking time-reversal symmetry.[48, 71] This obstruction provides us a physical way to understand the nontrivial BTI root phase generated by exotic time reversal symmetry. Indeed, both $e$ and $m$ particles can be regarded as ends of vortex-lines that are condensed and invisible in the bulk. In terms of simple physical picture, we may think these invisible vortex-lines carry integer spin and form a non-trivial 1D SPT phase, e.g., the Haldane phase. Therefore, it is not a surprise that the end of these vortex-lines carry half-integer spins which form Kramers’ doublets under time reversal symmetry.

VII. Z_N SPT IN THREE DIMENSIONS: BEYOND GROUP COHOMOLOGY THEORY

In this section, we use the one component action Eq.(14) to discuss possible $Z_N$ symmetry protected phases beyond group cohomology class. Let us assume a generic $Z_N$ SPT in 3d can be described by:

$$\mathcal{L} = i \frac{1}{4\pi} a_\mu \partial_\nu b_{\lambda \rho} \epsilon^{\mu \nu \lambda \rho} + i \frac{\Lambda}{16\pi} b_{\lambda \nu} b_{\lambda \rho} \epsilon^{\mu \nu \lambda \rho},$$

where only one-component is taken into account for convenience. Here, $a_\mu$ and $b_{\lambda \nu}$ are still non-compact and compact respectively. In contrast to the previous discussion of $U(1)$×$Z_2^T$ where non-vanishing quantized $\Lambda$ needs the help of time-reversal symmetry, $\Lambda$ in $Z_N$ SPT is supposed to be quantized even without the help of time-reversal symmetry. Following Cheng-Gu recipe[34], let us consider the following gauge coupling to “probe the $Z_N$ SPT order”:

$$\mathcal{L}_{coupling} = i \frac{1}{4\pi} B_{\mu \nu} \partial_\lambda a_\rho \epsilon^{\mu \nu \lambda \rho} + i \frac{N}{4\pi} B_{\mu \nu} \partial_\lambda A_\rho \epsilon^{\mu \nu \lambda \rho}.$$

Several explanations are in order. First, the $B_{\mu \nu}$ gauge field in $B \wedge dA$ term is a “2-form compact probe field” that minimally couples to strings (2π-vortex-lines in the ground state). It is this type of coupling that is missed in Dijkgraaf-Witten gauge theory[94] since $H^4[\mathbb{Z}_N, U(1)] = \mathbb{Z}_1$. The term $B \wedge dA$ can be viewed as a Higgs condensate term[34] where $A_\rho$ is a 1-form compact gauge field introduced by Hubbard-Stratonovich transformation. By means of this term, the probe field $B_{\mu \nu}$ is naturally higgsed to $Z_N$ discrete gauge field.

Now, we are in a position to integrate out all SPT degrees of freedom. Integrating the non-compact field $a_\mu$ renders $b_{\mu \nu} = B_{\mu \nu}$. Consequently, we end up with an action in the background fields $B$ and $A$:

$$S = i \int d^4x \frac{N}{4\pi} B_{\mu \nu} \partial_\lambda A_\rho \epsilon^{\mu \nu \lambda \rho} + i \int d^4x \frac{\Lambda}{16\pi} B_{\mu \nu} B_{\rho \lambda} \epsilon^{\mu \nu \lambda \rho}$$

$$= i \int d^4x \frac{N}{8\pi} B_{\mu \nu} F_{\rho \lambda} \epsilon^{\mu \nu \lambda \rho} + i \int d^4x \frac{\Lambda}{16\pi} B_{\mu \nu} B_{\rho \lambda} \epsilon^{\mu \nu \lambda \rho}$$

$$= i \frac{N}{2\pi} \int B \wedge dA + \frac{\Lambda}{4\pi} \int B \wedge B,$$

(66)

where the differential 2-form $dA = F$, the field strength tensor of 1-form gauge field $A$.

All possible values of $\Lambda$ can be found by using the procedures in Sec. III C. The flat-connection condition leads to $N_{0x}$ and $N_{yz}$ quantized at $1/N$ such that the periodicity of $\Lambda$ is:

$$\Lambda \rightarrow \Lambda + N^2.$$

(67)

On the other hand, as a discrete gauge theory, $b \wedge b$ term should also be invariant up to $2\pi$ under large gauge transformation: $b_{\mu \nu} \rightarrow b_{\mu \nu} + \delta b_{\mu \nu}$, where $\delta b_{\mu \nu}$ satisfies:

$$\int_S \delta b_{\mu \nu} \epsilon^{\mu \nu \lambda \rho} = 2\pi \times \text{integer} \ (\text{there is no implicit summation over indices } \mu, \nu \text{ here}).$$

Without loss of generality, let us still consider the pair of tori $T_{0x}$ and $T_{yz}$:

$$\int_{T_{0x}} \delta b_{0x} d\tau dx = 2\pi \times \tilde{N}_{0x}, \quad \int_{T_{yz}} \delta b_{yz} dy dz = 2\pi \times \tilde{N}_{yz},$$

The additional terms $\delta S$ arising from the large gauge transformation are collected as follows:

$$S + \delta S =$$

$$i \frac{\Lambda}{16\pi} \int_{T_{0x}} d\tau dx (b_{0x} + \delta b_{0x}) \int_{T_{yz}} dy dz (b_{yz} + \delta b_{yz}) \epsilon^{0yz}$$

$$+ (x0yz) + (0xzy) + (x0zy) + (yz0x) + (y0xz) + (yzx0)$$

$$= S + \frac{8\Lambda}{16\pi} \left( \frac{2\pi}{N} N_{0x} 2\pi \tilde{N}_{yz} + 2\pi \tilde{N}_{0x} \frac{2\pi}{N} N_{yz} + 2\pi \tilde{N}_{0x} 2\pi \tilde{N}_{yz} \right)$$

$$= S + 2\pi \Lambda \left( \frac{1}{N} \tilde{N}_{0x} \tilde{N}_{yz} + \frac{1}{N} \tilde{N}_{0x} \tilde{N}_{yz} + \tilde{N}_{0x} \tilde{N}_{yz} \right).$$

(68)

To keep the quantum theory invariant under the large gauge transformation, $\delta S$ must equal to integer×$2\pi$, leading to the following quantization condition:

$$\Lambda/N \in \mathbb{Z}.$$  

(69)

The above constraints (67) and (69) suggest $Z_N$ different SPT states protected by $Z_N$ symmetries in three dimensions.

For $Z_2$ case, the above analysis renders $Z_2$ classification, i.e. $\Lambda = 0$ is trivial while $\Lambda = 2$ is nontrivial. To understand the nontrivial bulk response of such a $Z_2$ SPT phase, we can promote the topological response theory Eq.(66) into a dynamical $Z_2$ gauge theory, and according to Ref. [95], the $Z_2$ charge in such a theory is fermion instead of a boson in the usual $Z_2$ gauge theory.

VIII. CONCLUSIONS

In this paper, based on a physical process called “condensation of vortex-lines” of three-dimensional superflu-
ids, we have constructed a bulk dynamical TQFT description Eq. (1) of all three bosonic topological insulator states (BTI). The schematic phase diagram is shown in Fig. 1. Such a physical way of thinking allows us to understand the physical meanings of each gauge field variable, and, most importantly, their symmetry transformations in the bulk. Our method will further shed light on a more challenging question: how to design microscopic interactions to realize BTI states in solid state materials or ultra-cold-atom experiments? Especially, it is quite interesting to explore the possible interaction terms to realize the unlinking-linking Berry phase contributed by \( b \wedge b \) as shown in Fig. 2.

We have shown that one of the three BTI states requires a nontrivial existence of \( b \wedge b \) term, which is beyond group cohomology theory. The remaining two states are within group cohomology theory and can be constructed from pure \( b \wedge da \) term. In contrast to previous works of BTI where the surface topological order is always \( Z_2 \) topological order, now, the surface topological order of the BTI state within group cohomology classification can be a generic \( Z_2 \) topological order with even \( p \). This many-to-one correspondence between surface and bulk explicitly reflects the importance of bulk dynamic TQFT. Our construction also suggests a nontrivial \( Z_2 \) SPT state in three dimensions beyond the group cohomology classification.

In the future, it will be interesting to apply such a physical derivation of bulk dynamical TQFT to other SPT states, even including fermionic SPT states where a spin manifold is required. A challenging problem is the bulk dynamical TQFT (not response theory) description of FTI both in free-fermionic[1–7] and interacting cases[96–99]. There are many previous important efforts, such as [100, 101]. In [102], the functional bosonization techniques are applied and \( b \wedge b \) is also realized. We believe that the basic methodology presented in our work combined with the previous efforts will shed light on this hard problem.

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**Appendix A: Basis change**

Let us start with \( b \wedge da \) term and \( K = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \). The first flavor can be viewed as a Higgs phase where the emergent gauge field \( a_1^1 \) is compact and \( b_{\mu \nu}^1 \) is non-compact. The second flavor can be viewed as a vortex-line condensed phase where \( b_{\mu \nu}^2 \) is compact and \( a_1^2 \) is non-compact. The new time-reversal transformation is defined as follows:

\[
\begin{align*}
\mathcal{T} a_1^0 T^{-1} &= -a_1^1, \\
\mathcal{T} b_{\mu i}^0 T^{-1} &= b_{\mu i}^1, \\
\mathcal{T} a_2^0 T^{-1} &= a_2^0 - 4a_1^2, \\
\mathcal{T} b_{\mu i}^1 T^{-1} &= -b_{\mu i}^0.
\end{align*}
\]

The corresponding \( T \) matrices are given by:

\[
T^a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T^b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The time-reversal transformation in the second flavor is unusual since there is a nontrivial shift in \( a_2^\mu \) gauge field. This shift indicates that, after time-reversal symmetry is implemented, there is a nontrivial entanglement between two flavors despite that the \( K \) matrix is diagonal. To remove such a shift event, we need to perform a \( \mathbb{CSI} \) transformation in order to work in a new basis where time-reversal symmetry in each flavor is independent to each other. For this purpose, we may choose the following matrices defined in Eq. (21):

\[
W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}
\]

which leads to a new \( K \) matrix given by Eq. (39). And time-reversal transformation rules are changed to (eqs. (58) to (61)) where time-reversal transformation in each flavor is independent with each other. Correspondingly, \( \mathbb{CSI} \) transformation rule in Eq. (20) leads to the change in gauge field variables:

\[
\begin{align*}
b_{\mu \nu}^1 \to -b_{\mu \nu}^1, \\
b_{\mu \nu}^2 \to -2a_1^2 + a_2^2, \\
a_1^1 \to a_1^2, \\
a_1^2 \to a_1^1.
\end{align*}
\]

In this new basis, both \( a_2^\mu \) and \( b_{\mu \nu}^1 \) are compact. Both \( a_1^1 \) and \( b_{\mu \nu}^1 \) are non-compact.

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