The Levels of Quasiperiodic Functions on the Plane, Hamiltonian Systems and Topology

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Abstract: Topology of levels of the quasiperiodic functions with \( m = n + 2 \) periods on the plane is studied. For the case of functions with \( m = 4 \) periods full description is obtained for the open everywhere dense family of functions. This problem is equivalent to the study of Hamiltonian systems on the \( n + 2 \)-torus with constant rank 2 Poisson bracket. In the cases under investigation we proved that this system is topologically completely integrable in some natural sense where interesting integer-valued locally stable topological characteristics appear. The case of 3 periods has been extensively studied last years by the present author, Zorich, Dynnikov and Maltsev for the needs of solid state physics (“Galvanomagnetic Phenomena in Normal Metals”); The case of 4 periods might be useful for the Quasicrystals.

Let us consider a periodic function \( f(x), x = (x^1, \ldots, x^m), n = m + 2, \) in the space \( \mathbb{R}^{n+2} \) with the group of periods \( \mathbb{Z}^m \subset \mathbb{R}^m, \) i.e. the function on the torus \( f : T^m \to R, T^m = R^m/\mathbb{Z}^m. \) We may think that the lattice \( \mathbb{Z}^m \) is generated by the standard basic vectors \( e_j = (0, \ldots, 0, 1, 0, \ldots). \) For every plane \( \mathbb{R}^2 \subset \mathbb{R}^{n+2} \) with linear coordinates \( (y^1, y^2) \) given by the system of the linear equations \( l_i = b_i, l_i \in \mathbb{R}^{m*}, i = 1, \ldots, n, \) we have a restriction \( g(l)(y) \) of the function \( f(x) \) on the plane \( g_l(y) = f(x(y)). \) Such a function \( g_l(y) \) is called Quasiperiodic with \( m \) periods on the plane.

Problem: What can be said about Topology of the levels \( g_l(y) = \text{const} \)?

The case \( m = 3 \) has been extensively studied in connection with the Galvanomagnetic Phenomena in the single crystal normal metals since late 50s by the school of I.Lifshitz (M.Azbel, M.Kaganov, V.Peschanski and others-see the survey and references in [1]). Topological investigations of this problem have been started after the article [2] in 1982 in the present authors seminar (see [1, 2, 3, 4, 5] for the main topological and physical results of our group).

The space \( T^3 = R^3/\mathbb{Z}^3 \) here is a space of quasimomenta, \( Z^3 \) is a reciprocal lattice, the function \( f(x) \) is a dispersion relation. The standard notation for quasimomenta would be \( k \) or \( p \) but we use the letter \( x \) for them here.

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The most important is Fermi level \( f = \epsilon_F \) which is a Fermi surface in \( R^3/Z^3 \). In the magnetic field \( B \) ”semiclassical” electrons move in the planes \( R^2 \) orthogonal to \( B \) in the space of quasimomenta. This family of planes leads to the quasiperiodic functions with 3 periods in the planes.

The case \( m = 4 \) is the main subject of this work. Our results may be useful for the theory of quasicrystals. By definition, the Rational Plane is one given by two linear equations \( l_1 = b_1, l_2 = b_2 \) where both linear forms \( l_1, l_2 \) have rational coefficients in the standard basis of lattice above (no restrictions for right-hand parts \( b_1, b_2 \)).

**Theorem 1** There exist nonempty open neighborhoods \( U_j \subset RP_3 \) of the rational directions \( l_j \in U_j, j = 1, 2 \) such that:

For every directions \( l'_j \in U_j, j = 1, 2 \) the connectivity component of the level \( g_{l'}(y) = \text{const} \) is either compact or belongs to the strip of finite width between two parallel lines in \( R^2 \). Except codimension one subset of the ”nongeneric levels” this situation is stable under the variation of all parameters involved (including variation of the directions \( l' \) and of the periodic function \( f \) in \( R^4 \)). The direction of the strip is an intersection of the 2-plane \( l' \) with some 3-dimensional hyperplane \( R^3 \subset R^4 \) which is integral in the standard lattice basis above and stable under the small variations of parameters.

We may consider this problem from the different point of view. Let Poisson Bracket is given on the torus \( T^4 \) which is constant and degenerate with Annihilator (Casimir) generated by 2 multivalued functions \( l'_1, l'_2 \). The Hamiltonian function \( f(x) \) generates a flow whose trajectories are equal to our curves—sections of the level \( M^3_a : f = a \) by the family \( l' \) of 2-planes \( l'_1 = b_1, l'_2 = b_2 \). Remove from the level \( M^3_a \) all compact nonsingular trajectories (CNST):

\[
M^3_a = (CNST) \bigcup_i M_i
\]

Now fill in all boundaries of \( M_i \) by the family of 2-discs in the corresponding 2-planes whose boundaries are the separatrix trajectories. We are coming to the piecewise smooth 3-manifolds \( \bar{M}^3_i \subset T^4 \) representing the 3-cycles \( z_i \in H_3(T^4, Z) \). Under the same restrictions as in the previous theorem, we have following

**Theorem 2** All cycles \( z_i \) are nonzero in \( H_3(T^4, Z) \), equal to each other up to the sign, and sum of them is equal to zero. Every manifold \( \bar{M}^3_i \) and
corresponding cycle $z_i$ is represented as an image of the 3-torus $T^3 \to T^4$. These cycles are stable under the small variation of parameters.

The idea of the proof is following. Take any pair of rational directions $l_1, l_2$. Assume that $l_1 = x^4$, and the corresponding levels are tori $x^4 = c$. Take small enough generic variation $l'_2$ of the direction $l_2$. The intersections of hyperplanes $l'_2 = b_2$ with tori $T^3_c : x^4 = c$ leads us to the one-parametric family of problems previously solved [3] about the plane sections of the Fermi surfaces $M^3_a \cap T^3_c$ in the 3-tori $T^3_c$. We may meet also a finite number of the singular sections $M^3_c$. In the nonsingular sections $M^3_c$ there are compact trajectories (i.e. closed in the universal covering space $\mathbb{R}^3$) and open trajectories lying on the 2-tori $T^2_{i,c} \subset T^3_c$ nonhomologous to zero in the homology group $H_2(T^4, \mathbb{Z})$. All other trajectories are the singular limits of these types.

Both these types are topologically stable. So we may have one-parametric family of compact trajectories or one-parametric family of 2-tori. Let this family be generic. The nonsingular compact trajectories remain compact after small perturbations (in particular, after replacing the direction $l_1$ by $l'_1$). They will be removed from our manifold according to the construction above.

One-parametric family of 2-tori $T^2_{i,c}$ may have generic singularities. The pair of 2-tori may meet each other in the nonsingular level $c$. The structure of this process can be extracted from the arguments given in the work [4]: it is the same picture as one when the pair of 2-tori are meeting each other in the generic boundary point of the stability zone. These tori have the opposite homology classes, so they are homotopic in the torus $T^3$. After meeting they leave a tail consisting of the compact orbits. So all our picture is covered by the map $T^2 \times I \to M^3_c$. It looks like 2-torus “turned back” being reflected by this level. Another situation might happen when 2-torus meets a singular level $M^3_c$ (i.e. the closed 1-form $dx_4$ has a Morse critical point). The local and minima play no role here. We concentrate on the case of the Morse index equal to 2 (the case of index 1 corresponds to the inverse process). The vanishing cycle on the torus $T^2_c$ should be homologous to zero in $T^2$ because it is homologous to zero in $T^4$. By that reason we shall come to the union $T^2 \cup S^2$ after surgery. However all plane sections of the topological 2-sphere are compact. So our torus passed critical level homotopically unchanged giving raise to the tail of compact orbits. Let us mention that such tails of compact orbits might appear also from the more trivial reasons. Now we perturb the direction $l_1$ replacing it by the generic
small perturbation \( l'_1 \). Cutting off all compact nonsingular sections and filling in all boundary separatrix compact curves by 2-discs in the corresponding plane, we are coming to the natural images of the 3-tori \( T^3 \to \bar{M}^3 \subset T^4 \). Here the map is monomorphic in homology groups. After that both our theorems follow from the same arguments as before, for the 3-dimensional case.

**Remarks:** 1. It became clear after discussion of the present author with Dynnikov that no generalization of this results is possible for the number of periods more than 4 for the directions not closed to the rational ones. 2. For the case of 4 periods our conjecture is that Theorems 1 and 2 are valid for the measure one family of 2-planes in the grassmanian, but in generic case the Hausdorf codimension of the exceptional set is less than one. For the even (and therefore nongeneric) cases like \( \sum_i \cos(x^i) = 0 \) these theorems are probably not true. 3. Some generalization of our results for the directions closed to the rational one is probably possible for any number of periods.

**References**

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