Deformations of the Heisenberg Lie algebra

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Abstract. In this note we compute all deformations of the 3-dimensional Heisenberg Lie algebra $h_3$. This shows that $h_3$ deforms to almost all Lie algebras of dimension 3.

1. Introduction

The study of deformations (and their counterparts degenerations) of Lie algebras is a very interesting topic both in mathematics and physics. Deformation theory have proved to be a very useful tool to obtain different algebraic classifications. For instance, in [1], the classification of kinematical Lie algebras in dimension 2 + 1 have been obtained by studying deformations of the static kinematical Lie algebra.

The obtaining of deformations of a Lie algebra $g$ depends on the computation of the cohomology group $H^2(g, g)$, which can be a very difficult task as the dimension of $g$ grows. In the case of Heisenberg Lie algebras, one has that (see [3]) $\dim H^2(h_3, h_3) = 5$ and $\dim H^2(h_{2n+1}, h_{2n+1}) = 2^n(2n+1)$ for $n > 1$, so already for $h_5$ the infinitesimal deformations are parametrized by a 20-dimensional space.

In this note, we compute all deformations of the Heisenberg Lie algebra of dimension 3. We obtain that $h_3$ deforms to every non-abelian Lie algebra of dimension 3, except for $r_{3,1}$. This resembles the result of [2] which states that every non-abelian 3-dimensional Lie algebra, except for $r_{3,1}$, degenerates to the Heisenberg Lie algebra $h_3$.

2. Preliminaries

A Lie algebra is a pair $g = (V, [\cdot, \cdot])$, where $V$ is a vector space over a field $\mathbb{F}$ and $[\cdot, \cdot] : V \times V \to V$ is a bilinear product satisfying:

- (skew-symmetry) $[x, y] = -[y, x]$,
- (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

for every $x, y, z \in V$.

Consider a complex 3-dimensional vector space with fixed basis $\{e_1, e_2, e_3\}$. We recall the classification of complex 3-dimensional Lie algebras:
Table 1. 3-dimensional Lie algebras.

| Lie algebra | Lie product |
|-------------|-------------|
| $\mathbb{C}^3$ | $[,] = 0$, |
| $\mathfrak{h}_3$ | $[e_1, e_2] = e_3,$ |
| $\mathfrak{v}_3 \oplus \mathbb{C}$ | $[e_1, e_2] = e_2,$ |
| $\mathfrak{v}_3$ | $[e_1, e_2] = e_2,$ $[e_1, e_3] = e_2 + e_3,$ |
| $\mathfrak{v}_3,\lambda$ | $[e_1, e_2] = e_2,$ $[e_1, e_3] = \lambda e_3,$ $0 < |\lambda| \leq 1,$ |
| $\mathfrak{sl}(2, \mathbb{C})$ | $[e_1, e_2] = e_3,$ $[e_1, e_3] = -2e_1,$ $[e_2, e_3] = 2e_2.$ |

2.1. Cohomology and deformations of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra and let $M$ be a $\mathfrak{g}$-module. The space of $p$-cochains with coefficients in $M$ is $C^p(\mathfrak{g}, M) = \text{Hom}(\Lambda^p \mathfrak{g}, M) \simeq \Lambda^p \mathfrak{g}^* \otimes M$, i.e., the space of $p$-alternating maps from $\mathfrak{g}$ to $M$.

Let $f \in C^p(\mathfrak{g}, M)$. Define $\partial(f) \in C^{p+1}(\mathfrak{g}, M)$ by $\partial^p(f) = \partial_0^p(f) + \partial_1^p(f)$, where

$$
\partial_0^p(f)(v_1, \ldots, v_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} f([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1}),
$$

$$
\partial_1^p(f)(v_1, \ldots, v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} v_i \cdot f(v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}),
$$

where $\hat{v}_i$ means that we omitted $v_i$.

Note that if $M = \mathbb{C}$, $\partial^0(f) = \partial_0^0(f)$, and if $M = \mathfrak{g}$ with the adjoint action then

$$
\partial_1^p(f)(v_1, \ldots, v_p) = \sum_{i=1}^{p} (-1)^{i+1} [v_i, f(v_1, \ldots, \hat{v}_i, \ldots, v_p)].
$$

Define the $p$-th cohomology space of $\mathfrak{g}$ with coefficients in $M$ as $H^p(\mathfrak{g}, M) = \frac{Z^p(\mathfrak{g}, M)}{B^p(\mathfrak{g}, M)}$, where, as usual,

- $Z^p(\mathfrak{g}, M) = \{f \in C^p(\mathfrak{g}, M) : \partial^p(f) = 0\}$ is the space of $p$-cocycles, and
- $B^p(\mathfrak{g}, M) = \{f \in C^p(\mathfrak{g}, M) \mid f = \partial^{p-1}(f')$ for some $f' \in C^{p-1}(\mathfrak{g}, M)\}$ is the space of $p$-coboundaries.

We define the following trilinear operation $\circ : C^2(\mathfrak{g}, \mathfrak{g}) \times C^2(\mathfrak{g}, \mathfrak{g}) \to C^3(\mathfrak{g}, \mathfrak{g})$:

$$
\varphi \circ \psi(x, y, z) = \varphi(x, \psi(y, z)) + \varphi(y, \psi(z, x)) + \varphi(z, \psi(x, y)).
$$

A formal deformation of a Lie algebra $\mathfrak{g} = (V, [,]_0)$, or of its Lie product, is a Lie algebra $\mathfrak{g}_t$ with product given by a formal series with parameter $t$:

$$
[,]_t = [,]_0 + \sum_{i=1}^{\infty} t^i \varphi_i,
$$

where the $\varphi_i$'s are skew-symmetric bilinear maps such that $[,]_t$ verifies the formal Jacobi identity: $[,]_t \circ [,]_t = 0$. 

2
From the formal Jacobi identity one obtains that $\varphi_1$ must be a 2-cocycle. Moreover, we have the following relations:

\[
\begin{align*}
\varphi_1 \circ \varphi_1 &= -\partial \varphi_2, \\
\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 &= -\partial \varphi_3, \\
&\vdots \\
\sum_{i=1}^{k-1} \varphi_i \circ \varphi_{k-i} &= -\partial \varphi_k.
\end{align*}
\]

The elements of $Z^2(\mathfrak{g}, \mathfrak{g})$ are called **infinitesimal deformations** of $\mathfrak{g}$. An infinitesimal deformation $\varphi$ is called **integrable** if there exists a formal deformation $[\cdot, \cdot]_t$ with $\varphi_1 = \varphi$; or equivalently, if the system of equations (1) has a solution.

Two deformations of $[\cdot, \cdot]_0$: $[\cdot, \cdot]_t = [\cdot, \cdot]_0 + \sum_{i=1}^{\infty} t^i \varphi_i$ and $[\cdot, \cdot]'_0 = [\cdot, \cdot]'_0 + \sum_{i=1}^{\infty} t^i \psi_i$, are called equivalent if there exists a linear automorphism of the space $V$, $\Phi_t = \text{Id} + \sum_{i=1}^{\infty} t^i f_i$, where $f_i \in \text{End}(V)$ and $[x, y]'_t = \Phi_t([\Phi_t^{-1}(x), \Phi_t^{-1}(y)]_t)$.

A deformation $[\cdot, \cdot]_t$ of $[\cdot, \cdot]_0$ is called **trivial** if it is equivalent to the deformation $[\cdot, \cdot]_0$. It is easy to see that $\varphi_1 - \psi_1 \in B^2(\mathfrak{g}, \mathfrak{g})$ (the space of 2-coboundaries) for equivalent deformations $[\cdot, \cdot]_t$ and $[\cdot, \cdot]'_t$. Therefore we can parametrize the set of infinitesimal deformations of $[\cdot, \cdot]_0$, by the cohomology group $H^2(\mathfrak{g}, \mathfrak{g})$.

### 3. Deformations of $\mathfrak{h}_3$.

Let $\mathfrak{h}_3$ be the Heisenberg Lie algebra of dimension 3, with basis $\{e_1, e_2, e_3\}$ and product given by $[e_1, e_2] = e_3$. We denote the dual basis of $\mathfrak{h}_3$ by $\{e^1, e^2, e^3\}$, i.e., $e^i(e_j) = \delta_{i,j}$.

Then we obtain

**Lemma 3.1.** A basis for the second cohomology group $H^2(\mathfrak{h}_3, \mathfrak{h}_3)$ is given by:

\[
B = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}
\]

where

\[
\begin{align*}
\psi_1 &= e^1 \wedge e^2 \otimes e_1, & \psi_2 &= e^1 \wedge e^2 \otimes e_2, & \psi_3 &= e^1 \wedge e^3 \otimes e_1 - e^2 \wedge e^3 \otimes e_2, \\
\psi_4 &= e^1 \wedge e^3 \otimes e_2, & \psi_5 &= e^2 \wedge e^3 \otimes e_1.
\end{align*}
\]

**Proof.** It is easy to check that

\[
Z^2(\mathfrak{h}_3, \mathfrak{h}_3) = \{e^1 \wedge e^2 \otimes e_1, e^1 \wedge e^2 \otimes e_2, e^1 \wedge e^3 \otimes e_1 - e^2 \wedge e^3 \otimes e_2, e^1 \wedge e^3 \otimes e_2, e^2 \wedge e^3 \otimes e_1, e^1 \wedge e^2 \otimes e_3, e^1 \wedge e^3 \otimes e_1 - e^2 \wedge e^3 \otimes e_3, e^1 \wedge e^3 \otimes e_2 + e^1 \wedge e^2 \otimes e_3\}
\]

and

\[
B^2(\mathfrak{h}_3, \mathfrak{h}_3) = \{e^1 \wedge e^2 \otimes e_3, e^1 \wedge e^2 \otimes e_1 - e^2 \wedge e^3 \otimes e_3, e^1 \wedge e^2 \otimes e_2 + e^1 \wedge e^3 \otimes e_3\}.
\]

\[\square\]
Now consider a 2-cocycle \( \varphi_1 = \sum_{i=1}^{5} \alpha_i \psi_i \). Then

\[
\varphi_1 \circ \varphi_1(e_1, e_2, e_3) = \varphi_1(e_1, \varphi_1(e_2, e_3)) + \varphi_1(e_2, \varphi_1(e_3, e_1)) + \varphi_1(e_3, \varphi_1(e_1, e_2)) = - (\alpha_3 \alpha_1 + \alpha_5 \alpha_2) e^1 \wedge e^2 \wedge e^3 \otimes e_1 + (\alpha_3 \alpha_2 - \alpha_4 \alpha_1) e^1 \wedge e^2 \wedge e^3 \otimes e_2.
\]

From the previous proof one can see that the only non-trivial 3-coboundary is \( \partial^2(e^2 \wedge e^3 \otimes e_2) = e^1 \wedge e^2 \wedge e^3 \otimes e_3 \), therefore \( \varphi_1 \) is integrable if and only if

\[
\begin{align*}
\alpha_3 \alpha_1 + \alpha_5 \alpha_2 &= 0, \\
\alpha_3 \alpha_2 - \alpha_4 \alpha_1 &= 0.
\end{align*}
\]

(2)

This also implies that the deformation is linear.

Solving the system of equations (2), we have the following cases:

(i) If \( \alpha_1 \neq 0 \), \( \alpha_3 = - \frac{\alpha_5 \alpha_2}{\alpha_1} \) and \( \alpha_4 = - \frac{\alpha_5 \alpha_2}{\alpha_1^2} \), we obtain the following deformation:

\[
\begin{align*}
[e_1, e_2]_t &= e_3 + t \alpha_1 \left( e_1 + \frac{\alpha_2}{\alpha_1} e_2 \right), \\
[e_1, e_3]_t &= -t \left( \frac{\alpha_5 \alpha_2}{\alpha_1} \right) \left( e_1 + \frac{\alpha_2}{\alpha_1} e_2 \right), \\
[e_2, e_3]_t &= t \alpha_5 \left( e_1 + \frac{\alpha_2}{\alpha_1} e_2 \right).
\end{align*}
\]

We list every possibility in Table 2:

| #   | Restriction | Base change | New product | 
|-----|-------------|-------------|-------------|
| 1   | \( \alpha_2 \neq 0 \) | \( \alpha_5 \neq -\frac{\alpha_1^2}{4} \) | \( \left( \begin{array}{ccc}
\frac{\alpha_5 \alpha_2}{\alpha_1^2} & \beta & 0 \\
1 & 0 & -(\beta + \alpha_1 t)
\end{array} \right) \) | \( [e_1, e_2] = e_2, \) \\
|     | 0 = \beta^2 + \beta \alpha_1 t + \alpha_5 t & \beta \neq 0 | \( [e_1, e_3] = - \left( \frac{\beta + \alpha_1 t}{\beta} \right) e_3. \) |
| 2   | \( \alpha_2 = 0 \) | \( \alpha_5 \neq -\frac{\alpha_1^2}{4} \) | \( \left( \begin{array}{ccc}
0 & \beta & 0 \\
1 & 0 & -\alpha_1 t - \beta \\
1 & 0 & \beta
\end{array} \right) \) | \( [e_1, e_2] = e_2, \) \\
|     | 0 = \beta^2 + \beta \alpha_1 t + \alpha_5 t & \beta \neq 0 | \( [e_1, e_3] = - \left( \frac{\beta + \alpha_1 t}{\beta} \right) e_3. \) |
| 3   | \( \alpha_5 = -\frac{\alpha_1^2}{4} \) | \( \left( \begin{array}{ccc}
\frac{\alpha_5 \alpha_2}{\alpha_1^2} & \frac{\alpha_2}{\alpha_1} & 0 \\
1 & 0 & 0 \\
1 & 0 & -\alpha_1 t
\end{array} \right) \) | \( [e_1, e_2] = e_2, \) \\
|     | \( [e_1, e_3] = e_2 + e_3. \) | | |

(ii) If \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \), we obtain the following deformation:

\[
[e_1, e_2]_t = e_3, \quad [e_1, e_3]_t = t \left( \alpha_3 e_1 + \alpha_4 e_2 \right), \quad [e_2, e_3]_t = t \left( \alpha_5 e_1 - \alpha_3 e_2 \right),
\]

and the cases are listed in Table 3.
(iii) If \( \alpha_1 = 0, \alpha_3 = 0 \) and \( \alpha_5 = 0 \), we obtain the following deformation:

\[
[e_1, e_2] = e_3 + t\alpha_2 e_2, \quad [e_1, e_3] = t\alpha_4 e_2,
\]

with \( \alpha_2 \neq 0 \) or \( \alpha_4 \neq 0 \). The cases are listed in Table 4.

| 4 | \( \alpha_5 \neq 0 \) | \( 0 \neq 4^2 + \alpha_4 \alpha_5 \) | \( 0 = 16\beta^2 + t\alpha_5 \) | \( 0 = \beta \gamma + \alpha_3 t - 4\sqrt{2\beta} \) |
|---|---|---|---|---|
| 0 | \( \alpha_5 = 0 \) | \( \alpha_3 \neq 0 \) | \( \alpha_3 = 0 \) | \( \alpha_4 = 0 \) | \( \alpha_5 = 0 \) | \( 0 = \beta^2 - \alpha_4 \) |

| # | Restriction | Base change | New product |
|---|---|---|---|
| 6 | \( \alpha_3 \neq 0 \) | \( 0 = \alpha_3^2 + \alpha_4 \alpha_5 \) | \( \alpha_3 = 0 \) | \( \alpha_4 = 0 \) | \( \alpha_5 = 0 \) | \( 0 = \beta^2 + t\alpha_5 \) | \( 0 = \beta^2 - \alpha_4 \) |
| 7 | \( \alpha_3 = 0 \) | \( \alpha_4 = 0 \) | \( \alpha_5 = 0 \) | \( \beta \gamma \neq 0 \) | \( \beta \neq 0 \) | \( \beta^2 - 4\sqrt{2\beta} \) |

From Tables 1, 2, 3 and 4 we obtain

(a) The Lie algebra from cases 1 and 2 is \( \mathfrak{r}_3 \lambda \) with \( \lambda = -\left( \frac{\beta + \alpha_4 t}{\beta} \right) \). Notice that in this case:

- \( \lambda \neq -1 \) because \( \alpha_1 \neq 0 \),
- \( \lambda \neq 1 \) since this would imply \( \alpha_5 = \frac{ta^2}{t-1} \),
- \( \lambda = 0 \) if and only if \( \alpha_5 = 0 \) and in this case we obtain the Lie algebra \( \mathfrak{r}_2 \oplus \mathbb{C} \).
(b) The Lie algebra from cases 6, 7 and 8 is $r_{3,-1}$.
(c) The Lie algebra from cases 4 and 5 is $\mathfrak{sl}(2, \mathbb{C})$.
(d) The Lie algebra from cases 3 and 10 is $r_3$.

Finally, with all this we obtain:

**Theorem 3.2.** The Heisenberg Lie algebra of dimension 3 deforms to every non-abelian 3-dimensional Lie algebra except for $r_{3,1}$.

**Acknowledgments**
The author is supported by Fondo Puente de Investigación de excelencia FPI-18-02 from Universidad de Antofagasta.

4. References

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