On asymptotic properties of the generalized Dirichlet \( L \)-functions

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Abstract

Let \( q \geq 3 \) be an integer, \( \chi \) denote a Dirichlet character modulo \( q \), for any real number \( a \geq 0 \), we define the generalized Dirichlet L-functions

\[
L(s, \chi, a) = \sum_{n=1}^{\infty} \frac{\chi(n)}{(n+a)^s},
\]

where \( s = \sigma + it \) with \( \sigma > 1 \) and \( t \) both real. It can be extended to all \( s \) by analytic continuation. In this paper, we study the mean value properties of the generalized Dirichlet L-functions, and obtain several sharp asymptotic formulae by using analytic method.

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Key words: generalized Dirichlet L-functions; Dirichlet character; generalized trigonometric sums; mean value properties; asymptotic formulae.

1. Introduction

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Let $q \geq 3$ be an integer, $\chi$ denote a Dirichlet character modulo $q$, Dirichlet $L$-functions $L(s, \chi)$ defined by

$$ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, $$

where $s = \sigma + it$ with $\sigma > 1$ and $t$ both real. It is very important in analytic number theory, and many studies have been done in all directions of Dirichlet $L$-functions. One of the most significant aspects about Dirichlet $L$-functions are the mean value properties. D.R. Heath-Brown, W. Zhang, R. Balasubramanian (see Ref. [1-3]) studied the square mean value properties on Dirichlet $L$-functions on the line $\sigma = \frac{1}{2}$. For example, R. Balasubramanian (see Ref. [3]) got the asymptotic formula

$$ \sum_{\chi \mod q} |L\left(\frac{1}{2} + it, \chi\right)|^2 = \frac{\phi(q)}{q} \log(qt) + O(q(\log \log q)^2) + O\left(te^{\sqrt{\log q}}\right) + O\left(q^\frac{3}{10}t^2e^{10\sqrt{\log q}}\right), $$

which is satisfied for $t \geq 3$ and for all $q$.

W. Zhang, Y. Yi (see Ref. [4] and [5]) got different kinds of the mean value of Dirichlet $L$-functions with weight or not. For instance, Y. Yi and W. Zhang (see Ref. [5]) gave the asymptotic formula of Dirichlet $L$-functions with the weight of $\tau(\chi)$

$$ \sum_{\chi \neq \chi_0} |\tau(\chi)|^n |L(1, \chi)|^{2k} = N^{m-n} \phi(N)^{2k-1}(2) \prod_{p\mid q}(1 - \frac{1}{p^2})^{2k-1} \prod_{p\mid q}(1 - \frac{1 - C_{2k-2}}{p^2}) \prod_{p\mid M}(p^{m+n} - 2p^m + 1) + O(q^{m+n+\epsilon}), $$

where $q$ is an integer $\geq 3$ and $q = MN, (M, N) = 1, M = \prod_{p\mid q} p$.

The first and third authors (see Ref. [6]) also studied the mean value of Dirichlet $L$-functions with trigonometric sums, and gave the asymptotic formula as follow

$$ \sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-1} \chi(a)e\left(\frac{f(a)}{p}\right) \right|^2 |L(1, \chi)|^{2m} = p^2 \zeta^{2m-1}(2) \prod_{p\mid q}(1 - \frac{1 - C_{2m-2}}{p^2}) + O(p^{2-\frac{k+\epsilon}{2}}), $$

where $p$ is prime, $\epsilon$ is any small positive real number, $f(x) = \sum_{i=0}^{k} a_i x^i$ is a polynomial such that $\deg(f(x)) = k$ and $p|(a_0, a_1, \ldots, a_k)$, and $m$ and $k$ are any positive integers, $\prod_{p\mid q}$ denotes the product over all primes different from $p$, $C_m = \frac{m!}{n!(m-n)!}$, and the $O$ constant depends only on $k$ and $\epsilon$. Obviously, let $f(a) = a$, we have the mean value of Dirichlet $L$-functions with the weight of $\tau(\chi)$.

Now let $a \geq 0$ be an integer, generalized Dirichlet $L$-functions $L(s, \chi, a)$ defined by

$$ L(s, \chi, a) = \sum_{n=1}^{\infty} \frac{\chi(n)}{(n + a)^s}, $$

where $s = \sigma + it$ with $\sigma > 1$ and $t$ both real.

About the generalized Dirichlet series, B. C. Berndt (see Ref. [7]-[9]) studied many identical properties satisfying restrictive conditions. It is well known that for $\chi$ a nonprincipal,
The generalized Dirichlet $L$-functions with Dirichlet character and the generalized trigonometric sums have received considerable attention in analytic number theory. Let $\chi$ be a primitive Dirichlet character modulo $q$, for $\sigma > \frac{1}{2} - m$ with $m$ a positive integer, Prof. Berndt (see Ref. [9]) derived

$$L(s, \chi, a) = \frac{a^{-s}}{\Gamma(s)} \left( \sum_{j=0}^{m-1} \frac{(-1)^j \Gamma(s + j)L(-j, \chi)}{j! a^j} + G(s) \right),$$

where $G(s)$ is an analytic function. When $n$ is a nonpositive integer, we can easily calculate $L(n, \chi, a)$, in particular, $L(0, \chi, a) = L(0, \chi)$.

The first and third authors (see Ref. [10]) also got the following asymptotic formula about the generalized Dirichlet $L$-functions

$$\sum_{\chi \neq \chi_0} |L(1, \chi, a)|^2 = \phi(q) \sum_{d|q} \frac{\mu(d)}{d^2} \zeta \left( 2, \frac{a}{d} \right) - \frac{4\phi(q)}{a} \sum_{d|q} \frac{\mu(d)}{d} \sum_{k=1}^{\left\lfloor \frac{q}{d} \right\rfloor} \frac{1}{k} + O \left( \frac{\phi(q) \log q}{\sqrt{q}} \right),$$

(1)

where $\zeta(s, \alpha)(s = \sigma + it, \alpha > 0)$ is the Hurwitz zeta function defined for $\sigma > 1$ by the series

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s},$$

and $\phi$ is the Euler function, $\mu$ is the Möbius function, the $O$ constant only depends on $a$.

On the other hand, trigonometric sums are also the most important research topic in analytic number theory. Let $p$ be a prime, $f(x) = a_0 + a_1 x + \ldots + a_k x^k$ is a $k$-degree polynomial with integral coefficients such that $(p, a_0, a_1, \ldots, a_k) = 1$, trigonometric sums are defined by

$$\sum_{a=1}^{p} \chi(a) e \left( \frac{f(a)}{p} \right),$$

where $\chi$ denotes a Dirichlet Character modulo $p$ and $p|\chi(a_0, a_1, \ldots, a_k)$. When $\chi = \chi_0$, we can see the trigonometric sums enjoy many good properties (see Ref. [11-15]).

If communicated with the generalized trigonometric sums or with Dirichlet character, whether the generalized Dirichlet $L$-functions still show good properties? The authors are very interested in the problems. But there are few references to be referred to about these problems. In this paper, we will study the mean value properties of the generalized Dirichlet $L$-functions with Dirichlet character and the generalized trigonometric sums,

$$\sum_{\chi \neq \chi_0} \chi(k) |L(1, \chi, a)|^2,$$

$$\sum_{\chi \neq \chi_0} \left| \sum_{x=1}^{p-1} \chi(x) e \left( \frac{f(x)}{p} \right) \right|^2 |L(1, \chi, a)|^2,$$

where $p \geq 3$ is odd prime, $\chi$ is a Dirichlet character modulo $p$, $\chi_0$ is the non-principal character modulo $p$, $a \geq 0$ is any real number with $(a, p) = 1$, $e(y) = e^{2\pi i y}$. It could tell us some relationship between the Dirichlet character and the generalized trigonometric sums. More precisely, we prove the following theorems:

**Theorem 1.** Let $k, q$ be two integers with $q \geq 3, k \neq 1, (k, q) = 1$ and $\chi$ denote a Dirichlet character modulo $q$. Then for any positive real number $a \geq 1$, we have the asymptotic formula

$$\sum_{\chi \neq \chi_0} \chi(k) |L(1, \chi, a)|^2 = \frac{\phi(q)}{a(k - 1)} \sum_{d|q} \frac{\mu(d)}{d} \left( \sum_{l=1}^{\left\lfloor \frac{q}{d} \right\rfloor} \frac{1}{l} \sum_{\chi} \frac{(\chi | l)^k}{\chi^{-1}(l)} - \frac{\phi(q) \log q}{\sqrt{q}} \right),$$

where $(\chi | l)^k$ denotes a Dirichlet Character modulo $q$. When $k = 1$, the above theorem becomes the Linnik’s theorem. When $k = 2$, the above theorem becomes the generalization of the Lindelöf hypothesis.
where $\phi$ is the Euler function, $\mu$ is the Möbius function, and the $O$ constant depends only on $a, k$.

**Theorem 2.** Let $p \geq 3$ be an odd prime and $\chi$ denote a Dirichlet character modulo $p$, $f(x) = a_0 + a_1x + \ldots + a_kx^k$ is a $k$-degree polynomial with integral coefficients such that $(p, a_0, a_1, \ldots, a_k) = 1$, $e(y) = e^{2\pi iy}$. Then for any positive real number $a \geq 1$ with $(a, p) = 1$, we have the asymptotic formula

$$
\sum_{\chi \neq \chi_0} \left| \sum_{x=1}^{p-1} \chi(x)e\left(\frac{f(x)}{p}\right) \right|^2 |L(1, \chi, a)|^2
$$

$$
= p^2 \sum_{d|q} \frac{\mu(d)}{d^2} \zeta\left(2, \frac{a}{d}\right) - \frac{4p^2}{a} \sum_{d|q} \frac{\mu(d)}{d} \sum_{l=1}^{\lfloor \frac{q}{d} \rfloor} \frac{1}{l} + O(p^{2-\frac{1}{2}+\epsilon}),
$$

where $\mu$ is the Möbius function and $\zeta(s, \alpha)(s = \sigma + it, \alpha > 0)$ is the Hurwitz zeta function. The $O$ constant is depending on $k, a$ and $\epsilon$.

**Note 1.** For the general case of $2l$-th ($l \geq 2$) power mean value of the generalized Dirichlet $L$-functions and the Dirichlet character

$$
\left| \sum_{\chi \neq \chi_0} \chi(k)|L(1, \chi, a)|^{2l} \right|
$$

it is still an open problem.

**Note 2.** For the general case of $2m$-th ($m \geq 2$) power of the generalized trigonometric sums and $2l$-th ($l \geq 2$) power mean value of the generalized Dirichlet $L$-functions

$$
\left| \sum_{\chi \neq \chi_0} \chi(x)e\left(\frac{f(x)}{p}\right) \right|^{2m} |L(1, \chi, a)|^{2l},
$$

it is still an open problem.

## 2. Some lemmas

To complete the proofs of both of the Theorems, we need the following several lemmas. First, we make an identity of the Dirichlet $L$-functions and the generalized form.

**Lemma 1.** Let $q \geq 3$ be an integer, and $\chi$ denote a nonprincipal Dirichlet character modulo $q$. Let $L(s, \chi)$ denote the Dirichlet $L$-functions corresponding to $\chi$, and $L(s, \chi, a)$ denote the generalized Dirichlet $L$-functions. Then for any real number $a \geq 0$, we have

$$
L(1, \chi, a) = L(1, \chi) - a \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)}.
$$

**Proof.** See Lemma 1, and let $m = 1$ (Ref. [10]).

**Lemma 2.** Let $f(x)$ be a polynomial with integer coefficients as $f(x) = a_0 + a_1x + \ldots + a_kx^k$, and $\chi$ be a Dirichlet character modulo $p$. Then we have

$$
\left| \sum_{x=1}^{p-1} \chi(x)e\left(\frac{f(x)}{p}\right) \right|^2 = p - 1 + \sum_{x=2}^{p-1} \chi(x) \sum_{y=1}^{p-1} e\left(\frac{g(y, x)}{p}\right),
$$

where $g(y, x)$ is the polynomial.
where $g(y, x) = f(xy) - f(y) = \sum_{i=0}^{k} a_i (x^i - 1)y^i$.

**Proof.** Note that for $1 \leq y \leq p - 1$ (p is prime), we have $(y, p) = 1$. According to the properties of characters, we have

$$
\left| \sum_{x=1}^{p-1} \chi(x) \left( \frac{f(x)}{p} \right) \right|^2 = \sum_{x, y=1}^{p-1} \chi(x) \bar{\chi}(y) \left( \frac{f(x) - f(y)}{p} \right)
= \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} \chi(xy) \bar{\chi}(y) \left( \frac{f(xy) - f(y)}{p} \right).
$$

Let

$$
g(y, x) = f(xy) - f(y) = \sum_{i=0}^{k} a_i (x^i - 1)y^i,
$$

we get

$$
\left| \sum_{x=1}^{p-1} \chi(x) \left( \frac{f(x)}{p} \right) \right|^2 = \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} \chi(x) \left( \frac{g(y, x)}{p} \right)
= \sum_{y=1}^{p-1} \chi(1) \left( \frac{g(y, 1)}{p} \right) + \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} \chi(x) \left( \frac{g(y, x)}{p} \right)
= p - 1 + \sum_{x=2}^{p-1} \chi(x) \sum_{y=1}^{p-1} \left( \frac{g(y, x)}{p} \right).
$$

This proves Lemma 2.

**Lemma 3.** Let $f(x)$ satisfy the conditions of Lemma 2. And let $g(z) = g(z, x) = f(xz) - f(z) = \sum_{i=0}^{k} a_i (x^i - 1)z^i$, then we have the following estimate

$$
\left| \sum_{y=1}^{p-1} \left( \frac{g(y, x)}{p} \right) \right| \ll p^{-\frac{1}{2}}, \quad p \nmid (b_0, b_1, \ldots, b_k)
= p - 1, \quad p \mid (b_0, b_1, \ldots, b_k)
$$

where $b_i = a_i (x^i - 1), i = 0, 1, \ldots, k$ and $k$ is the degree of the polynomial $f(x)$.

**Proof.** The result is apparent if $p \mid (b_0, b_1, \ldots, b_k)$. If $p \nmid (b_0, b_1, \ldots, b_k)$, according to the definition of $g(x, a)$, we have (see Ref. [12]),

$$
\left| \sum_{b=1}^{p-1} \left( \frac{g(b, a)}{p} \right) \right| \ll p^{-\frac{1}{2}}.
$$

This proves Lemma 3.

**Lemma 4.** Let $q \geq 3$ be an integer and $\chi$ be the Dirichlet character modulo $q$. Then for any positive integer $a \geq 2$ with $(a, q) = 1$, we have

$$
\sum_{\chi \neq \chi^0} \chi(a)|L(1, \chi)|^2 = \frac{\phi(q)}{a} \zeta(2) \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O(\log^2 q).
$$
Proof. For convenience, we put

\[ A(\chi, y) = \sum_{\frac{q}{a} \leq n \leq y} \chi(n), \quad B(\chi, y) = \sum_{q \leq n \leq y} \chi(n). \]

Then for \( s > 1 \), the series \( L(s, \chi) \) is absolutely convergent, so applying Abel’s identity we have

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} + s \int_{\frac{q}{a}}^{+\infty} \frac{A(\chi, y)}{y^{s+1}} dy
\]

\[
= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} + s \int_{\frac{q}{a}}^{+\infty} \frac{B(\chi, y)}{y^{s+1}} dy.
\]

It is clear that the above formula also holds for \( s = 1 \) and \( \chi \neq \chi_0 \). Hence according to the definition of the Dirichlet \( L \)-function, for any positive integer \( a \neq 1 \) and \( (a, q) = 1 \), we have

\[
\sum_{\chi \neq \chi_0} \chi(a)|L(1, \chi)|^2
= \sum_{\chi \neq \chi_0} \chi(a) \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \right|^2
= \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \right) \left( \sum_{l=1}^{\infty} \frac{\bar{\chi}(l)}{l} \right)
= \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{n=1}^{\frac{q}{a}} \frac{\chi(n)}{n} + \int_{\frac{q}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \left( \sum_{l=1}^{q} \frac{\bar{\chi}(l)}{l} + \int_{q}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right)
= \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{n=1}^{\frac{q}{a}} \frac{\chi(n)}{n} \right) \left( \sum_{l=1}^{q} \frac{\bar{\chi}(l)}{l} \right)
+ \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{n=1}^{\frac{q}{a}} \frac{\chi(n)}{n} \right) \left( \int_{\frac{q}{a}}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right)
+ \sum_{\chi \neq \chi_0} \chi(a) \left( \sum_{l=1}^{q} \frac{\bar{\chi}(l)}{l} \right) \left( \int_{q}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right)
+ \sum_{\chi \neq \chi_0} \chi(a) \left( \int_{\frac{q}{a}}^{+\infty} \frac{A(\chi, y)}{y^2} dy \right) \left( \int_{q}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right)
\equiv A_1 + A_2 + A_3 + A_4.

Now we will estimate each of the above term.

From the orthogonality relation for character sums modulo \( q \), we know that for \( (q, n) = 1 \), we have the identity

\[
\sum_{\chi \mod q} \chi(n)\bar{\chi}(l) = \begin{cases} 
\phi(q), & \text{if } n \equiv l \mod q ; \\
n, & \text{otherwise}.
\end{cases}
\]
Then we can easily get

\[ A_1 = \sum_{\chi \not\equiv \chi_0} \chi(a) \left( \sum_{n=1}^{q/\eta} \frac{\chi(n)}{n} \right) \left( \sum_{l=1}^{q/\eta} \frac{\bar{\chi}(l)}{l} \right) \]

\[ = \sum_{\chi \not\equiv \chi_0} \sum_{n=1}^{q/\eta} \sum_{l=1}^{q/\eta} \frac{\chi(an)\bar{\chi}(l)}{nl} - \sum_{n=1}^{q/\eta} \frac{1}{nl} \]

\[ = \phi(q) \sum_{n=1}^{q/\eta} \frac{1}{an^2} + O(\log^2 q) \]

\[ = \phi(q) \sum_{n=1}^{q/\eta} \frac{1}{an^2} \sum_{d|(n,q)} \mu(d) + O(\log^2 q) \]

\[ = \phi(q) \sum_{d|q} \mu(d) \sum_{n=1}^{q/\eta} \frac{1}{an^2} + O(\log^2 q) \]

\[ = \phi(q) \sum_{d|q} \frac{\mu(d)}{d^2} \sum_{n=1}^{q/\eta} \frac{1}{n^2} + O(\log^2 q) \]

\[ = \phi(q) \sum_{d|q} \frac{\mu(d)}{d^2} \sum_{n=1}^{\infty} \frac{1}{n^2} + O \left( \frac{\phi(q)}{a} \sum_{d|q} \frac{\mu(d)}{d^2} \sum_{n=q/ad}^{\infty} \frac{1}{n^2} \right) + O(\log^2 q) \]

\[ = \frac{\phi(q)}{a} \zeta(2) \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) + O(\log^2 q), \quad (2) \]

where \( \sum_{n}' \) indicates that the sum is over those \( n \) relatively prime to \( q \).

According to Cauchy inequality and Polya-Vinogradiv inequality about character sums we can easily get

\[ A_2 = \sum_{\chi \not\equiv \chi_0} \chi(a) \left( \sum_{n=1}^{q/\eta} \frac{\chi(n)}{n} \right) \left( \int_q^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \]

\[ = \sum_{\chi \not\equiv \chi_0} \chi(a) \left( \sum_{n=1}^{q/\eta} \frac{\chi(n)}{n} \right) \left( \int_q^{q^{3/2}} \frac{\sum_{q \leq n \leq y} \bar{\chi}(n)}{y^2} dy \right) \]

\[ + \sum_{\chi \not\equiv \chi_0} \chi(a) \left( \sum_{n=1}^{q/\eta} \frac{\chi(n)}{n} \right) \left( \int_{q^{3/2}}^{+\infty} \frac{\sum_{q \leq n \leq y} \bar{\chi}(n)}{y^2} dy \right) \]

\[ \leq \int_q^{q^{3/2}} \frac{1}{y^2} \left| \sum_{n=1}^{q/\eta} \frac{1}{n} \sum_{\chi \not\equiv \chi_0} \chi(an)\bar{\chi}(l) \right| dy + q^{3/2} \int_{q^{3/2}}^{+\infty} \frac{1}{y^2} \sum_{\chi \not\equiv \chi_0} |B(\bar{\chi}, y)| dy \]

\[ \ll \int_q^{q^{3/2}} \frac{\phi(q)}{y^2} \left| \sum_{n=1}^{q/\eta} \frac{1}{n} \sum_{\chi \not\equiv \chi_0} \chi(an)\bar{\chi}(l) \right| dy + q^{3/2} \int_{q^{3/2}}^{+\infty} \frac{1}{y^2} \left( \sum_{\chi \not\equiv \chi_0} 1^2 \right)^{1/2} \left( \sum_{\chi \not\equiv \chi_0} |B(\bar{\chi}, y)|^2 \right)^{1/2} dy \]
This completes the proof of Lemma 4.

Similarly, we also have

\[
A_3 = \sum_{\chi \not\equiv \chi_0} \chi(a) \left( \sum_{l=1}^{q} \frac{\bar{\chi}(l)}{l} \right) \left( \int_{\frac{q}{2}}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right) = O \left( \frac{(\phi(q)q^4)}{q} \right),
\]

\[
A_4 = \sum_{\chi \not\equiv \chi_0} \chi(a) \left( \int_{\frac{q}{2}}^{+\infty} \frac{A(\chi, y)}{y^2} \, dy \right) \left( \int_{q}^{+\infty} B(\bar{\chi}, y) \, dy \right) = O \left( \frac{(\phi(q)q^4)}{q} \right).
\]

Combining the formulas (2)-(5), we immediately obtain

\[
\sum_{\chi \not\equiv \chi_0} \chi(a)|L(1, \chi)|^{2m} = \frac{\phi(q)}{a} \zeta(2) \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) + O(\log^2 q).
\]

This completes the proof of Lemma 4.

### 3. Proof of Theorem

In this part, we will prove both of the theorems. Firstly, we will prove Theorem 1.

**Proof of Theorem 1.** According to Lemma 1 and Lemma 4, we have

\[
\sum_{\chi \not\equiv \chi_0} \chi(k)|L(1, \chi, a)|^2
\]

\[
= \sum_{\chi \not\equiv \chi_0} \chi(k) \left| L(1, \chi) - a \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2
\]

\[
= \sum_{\chi \not\equiv \chi_0} \chi(k)|L(1, \chi)|^2 - a \sum_{\chi \not\equiv \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} L(1, \bar{\chi}) -
\]

\[-a \sum_{\chi \not\equiv \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n(n+a)} L(1, \chi) + a^2 \sum_{\chi \not\equiv \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \bigg|^2
\]

\[
= \frac{\phi(q)}{k} \zeta(2) \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) + O(q^4) - aM_1 - aM_2 + a^2M_3,
\]

where

\[
M_1 = \sum_{\chi \not\equiv \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} L(1, \bar{\chi}),
\]

\[
M_2 = \sum_{\chi \not\equiv \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} L(1, \chi),
\]

\[
M_3 = \sum_{\chi \not\equiv \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \bigg|^2.
\]
Now we will estimate each term of the above.

(i) Applying Abel’s identity, by analytic continuation we have

\[ L(1, \bar{\chi}) = \sum_{n=1}^{q} \frac{\bar{\chi}(n)}{n} + \int_{q}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy, \]

\[ \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n + a)} = \sum_{n=1}^{N} \frac{\chi(n)}{n(n + a)} + \int_{N}^{+\infty} \frac{(2y + a)C(\chi, y)}{y^2(y + a)^2} dy, \]

where \( B(\chi, y) = \sum_{q \leq n \leq y} \chi(n) \) as defined in the proof of Lemma 4, \( C(\chi, y) = \sum_{N \leq n \leq y} \chi(n) \), and \( N > q \) is an integer.

\[ M_1 \]
\[ = \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n + a)} L(1, \bar{\chi}) \]
\[ = \sum_{\chi \neq \chi_0} \chi(k) \left( \sum_{n=1}^{N} \frac{\chi(n)}{n(n + a)} + \int_{N}^{+\infty} \frac{(2y + a)C(\chi, y)}{y^2(y + a)^2} dy \right) \left( \sum_{n=1}^{q} \frac{\bar{\chi}(n)}{n} + \int_{q}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy \right) \]
\[ = \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{N} \frac{\chi(n)}{n(n + a)} \sum_{m=1}^{q} \frac{\bar{\chi}(m)}{m} + \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{N} \frac{\chi(n)}{n(n + a)} \int_{N}^{+\infty} \frac{B(\bar{\chi}, y)}{y^2} dy + \]
\[ + \sum_{\chi \neq \chi_0} \chi(k) \sum_{m=1}^{q} \frac{\bar{\chi}(m)}{m} \int_{N}^{+\infty} \frac{(2y + a)C(\chi, y)}{y^2(y + a)^2} dy + \]
\[ + \sum_{\chi \neq \chi_0} \chi(k) \int_{N}^{+\infty} \frac{(2y + a)C(\chi, y)}{y^2(y + a)^2} dy \int_{q}^{+\infty} \frac{B(\bar{\chi}, z)}{z^2} dz \]
\[ \equiv B_1 + B_2 + B_3 + B_4. \]

We will estimate each of them. Firstly we estimate \( B_1 \). From the properties of Dirichlet characters and Möbius function, we have

\[ B_1 = \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{N} \frac{\chi(n)}{n(n + a)} \sum_{m=1}^{q} \frac{\bar{\chi}(m)}{m} \]
\[ = \sum_{\chi \equiv q \mod q} \sum_{n=1}^{N} \frac{\chi(k) \bar{\chi}(m) \chi(n)}{mn(n + a)} + O(\log q) \]
\[ = \sum_{n=1}^{N} \sum_{m=1}^{q} \frac{1}{mn(n + a)} \sum_{\chi \equiv q \mod q} \chi(k) \bar{\chi}(m) \chi(n) + O(\log q) \]
\[ = \phi(q) \sum_{n=1}^{N} \frac{1}{mn(n + a)} + O(\log q) \]
\[ = \phi(q) \sum_{n=1}^{q} \frac{1}{n^2k(n + a)} + \phi(q) \sum_{l=1}^{N/q} \sum_{n=1}^{q} \frac{1}{n(nk + lq)(n + a)} + O(\log q) \]
\[ = \phi(q) \sum_{n=1}^{q} \frac{1}{n^2k(n + a)} \sum_{d(n, q)} \mu(d) + \]
\[ + O \left( \phi(q) \sum_{l=1}^{N/q} \sum_{n=1}^{q} \frac{1}{n(nk + lq)(n + a)} + \log q \right) \]
\[
\begin{align*}
\phi(q) \sum_{d|q} \mu(d) \sum_{n=1}^{q} \frac{1}{n^2 k(n + a)} = O(\log q) \\
\phi(q) \sum_{d|q} \mu(d) \sum_{n=1}^{q/d} \frac{1}{n^2 k(n + a/d)} = O(\log q) \\
\frac{\phi(q)}{k} \sum_{d|q} \mu(d) \sum_{n=1}^{\infty} \frac{1}{n^2(n + a/d)} + O(\log q) \\
\frac{\phi(q)}{k} \sum_{d|q} \mu(d) \sum_{n=q/d}^{\infty} \frac{1}{n^2(n + a/d)} + O(\log q) \\
\frac{\phi(q)}{ak} \sum_{d|q} \mu(d) \sum_{l=1}^{a/d} \frac{1}{l} + O(\log q) \\
\phi(q) \frac{\zeta(2)}{ak} \prod_{\nu|q} \left(1 - \frac{1}{p^2}\right) - \phi(q) \frac{\zeta(2)}{ak} \sum_{d|q} \mu(d) \sum_{l=1}^{a/d} \frac{1}{l} + O(\log q).
\end{align*}
\]

In the following we will estimate $B_2$, $B_3$ and $B_4$. According classical estimate of Dirichlet character sums, we have

\[
\begin{align*}
|B_2| &= \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{N} \frac{\chi(n)}{n(n + a)} \int_{q}^{\infty} \frac{B(\chi, y)}{y^2} dy \\
&\leq \sum_{\chi \neq \chi_0} \sum_{n=1}^{N} \frac{1}{n(n + a)} \int_{q}^{\infty} \left| \sum_{q<n<y} \chi(n) \right| \frac{1}{y^2} dy \\
&\ll q^{\frac{1}{2}} \phi(q) \log q \int_{q}^{\infty} \frac{1}{y^2} dy \\
&\ll \frac{\phi(q) \log q}{\sqrt{q}},
\end{align*}
\]

\[
\begin{align*}
|B_3| &= \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{q} \frac{\chi(n)}{n} \int_{N}^{\infty} \frac{(2y + a)C(\chi, y)}{y^2(y + a)^2} dy \\
&\ll \log q \int_{N}^{\infty} \frac{2y + a}{y^2(y + a)^2} \sum_{\chi \neq \chi_0} |C(\chi, y)| dy \\
&\ll \frac{q^{\frac{1}{2}} \phi(q) \log^2 q}{N^2},
\end{align*}
\]

\[
\begin{align*}
|B_4| &= \sum_{\chi \neq \chi_0} \chi(k) \int_{N}^{\infty} \frac{(2y + a)C(\chi, y)}{y^2(y + a)^2} dy \int_{q}^{\infty} \frac{B(\chi, z)}{z^2} dz \\
&\leq \int_{N}^{\infty} \int_{q}^{\infty} \frac{2y + a}{y^2(y + a)^2 z^2} \sum_{\chi \neq \chi_0} |C(\chi, y)| \cdot |B(\chi, z)| dy dz \\
&\ll \frac{\phi(q) \log^2 q}{N^2},
\end{align*}
\]
where we have used the common estimates. Taking $N = q^2$, then we get

$$M_1 = \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} L(1, \chi)$$

$$= \frac{\phi(q)}{ak} \zeta(2) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) - \frac{\phi(q)}{a^2k} \sum_{d|q} \mu(d) \left\lfloor \frac{a/d}{d} \right\rfloor \sum_{l=1}^{\infty} \frac{1}{l} + O \left(\frac{\phi(q) \log q}{\sqrt{q}}\right).$$

This gives the asymptotic formula of $M_1$. Therefore, we have the asymptotic formula of $M_1$.

(ii) Following the similar method in part (i), we could get the asymptotic formula of $M_2$, that is

$$M_2 = \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} L(1, \chi)$$

$$= \frac{\phi(q)}{ak} \zeta(2) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + \frac{\phi(q)}{a^2} \sum_{d|q} \mu(d) \sum_{l=1}^{\infty} \frac{1}{l} + O (\log q).$$

(iii) Lastly we will derive the asymptotic formula of $M_3$. Let $N > q$ be any integer, then we have

$$M_3 = \sum_{\chi \neq \chi_0} \chi(k) \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2$$

$$= \sum_{\chi \neq \chi_0} \chi(k) \left( \sum_{n=1}^{N} \frac{\chi(n)}{n(n+a)} + \int_{N}^{\infty} \frac{(2y+a)C(\chi, y)}{y^2(y+a)^2} dy \right) \times$$

$$\times \left( \sum_{m=1}^{N} \frac{\chi(m)}{m(m+a)} + \int_{N}^{\infty} \frac{(2z+a)C(\chi, z)}{z^2(z+a)^2} dz \right)$$

$$= \sum_{\chi \neq \chi_0} \chi(k) \left( \sum_{n=1}^{N} \frac{\chi(n)}{n(n+a)} \right)^2 \left( \sum_{m=1}^{N} \frac{\chi(m)}{m(m+a)} \right) +$$

$$+ \sum_{\chi \neq \chi_0} \chi(k) \left( \int_{N}^{\infty} \frac{(2y+a)C(\chi, y)}{y^2(y+a)^2} dy \right) \left( \sum_{m=1}^{N} \frac{\chi(m)}{m(m+a)} \right) +$$

$$+ \sum_{\chi \neq \chi_0} \chi(k) \left( \int_{N}^{\infty} \frac{(2y+a)C(\chi, y)}{y^2(y+a)^2} dy \right) \left( \int_{N}^{\infty} \frac{(2z+a)C(\chi, z)}{z^2(z+a)^2} dz \right)$$

$$= \sum_{n=1}^{N} \frac{1}{n(n+a)} \sum_{m=1}^{N} \frac{1}{m(m+a)} \left( \sum_{\chi \neq \chi_0} \chi(nk)\chi(m) \right) +$$

$$+ O \left( \int_{N}^{\infty} \frac{(2y+a)\sum_{\chi \neq \chi_0} |C(\chi, y)|}{y^2(y+a)^2} dy \right)$$

$$= \phi(q) \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{1}{mn(m+a)(n+a)} + O(1) + O \left( \frac{q^2 \phi(q) \log q}{N^2} \right).$$
\[
\phi(q) \sum_{n=1}^{N} \frac{1}{n^2 k(n+a)(nk+a)} + 2\phi(q) \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{1}{mn(m+a)(n+a)} + O(1)
\]

Taking \( N = q^2 \), we have

\[
M_3 = \sum_{\chi \neq \chi_0} \chi(k) \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2
\]

\[
= \frac{\phi(q)}{a^2 k^2} \zeta(2) \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) + \frac{\phi(q)}{a^3 k(k-1)} \sum_{d \mid q} \mu(d) \frac{1}{d} \sum_{l=1}^{\left\lfloor \frac{d}{k}\right\rfloor} \frac{1}{l} +
\]

\[-\frac{k\phi(q)}{a^3(k-1)} \sum_{d \mid q} \mu(d) \frac{1}{d} \sum_{l=1}^{\left\lfloor \frac{d}{k}\right\rfloor} \frac{1}{l} + O(1) \]

Combining the estimates of (i),(ii), (iii) and Lemma 4, we immediately obtain

\[
\sum_{\chi \neq \chi_0} \chi(k) |L(1, \chi, a)|^2
\]

\[
= \sum_{\chi \neq \chi_0} \chi(k) \left| L(1, \chi) - a \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2
\]

\[
= \sum_{\chi \neq \chi_0} \chi(k) |L(1, \chi)|^2 - a \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n(n+a)} L(1, \chi) -
\]

\[
\approx \sum_{\chi \neq \chi_0} \chi(k) \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2 - a \sum_{\chi \neq \chi_0} \chi(k) \left( \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n(n+a)} L(1, \chi) \right)
\]

\[
= \sum_{\chi \neq \chi_0} \chi(k) \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2 - a \sum_{\chi \neq \chi_0} \chi(k) \left( \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n(n+a)} L(1, \chi) \right)
\]
\[-a \sum_{\chi \neq \chi_0} \chi(k) \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} L(1, \chi) + a^2 \sum_{\chi \neq \chi_0} \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n(n+a)} \right|^2 \]

\[= \frac{\phi(q)}{a(k-1)} \sum_{d|q} \frac{\mu_d(d)}{d} \sum_{l=1}^{\frac{q}{d}} \frac{1}{l} - \frac{\phi(q)}{a(k-1)} \sum_{d|q} \frac{\mu_d(d)}{d} \sum_{l=1}^{\frac{q}{d}} \frac{1}{l} + O \left( \frac{\phi(q) \log q}{\sqrt{q}} \right). \]

\[= \frac{\phi(q)}{a(k-1)} \sum_{d|q} \frac{\mu(d)}{d} \sum_{l=\left\lceil \frac{q}{d} \right\rceil}^{\frac{q}{d}} \frac{1}{l} + O \left( \frac{\phi(q) \log q}{\sqrt{q}} \right), \]

where the \(O\) constant depends on \(a, k\). This proves Theorem 1.

Next, we shall complete the proof of Theorem 2. Theorem 1 will be useful in the proof.

**Proof of Theorem 2.** Firstly from Lemma 2, for any \(p \geq 3\), we have

\[\sum_{\chi \neq \chi_0} \left| \sum_{x=1}^{p-1} \chi(x)e \left( \frac{f(x)}{p} \right) \right|^2 |L(1, \chi, a)|^2 \]

\[= \sum_{\chi \neq \chi_0} \left( p - 1 + \sum_{x=2}^{p-1} \chi(x) \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \right) |L(1, \chi, a)|^2 \]

\[= (p - 1) \sum_{\chi \neq \chi_0} |L(1, \chi, a)|^2 + \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \]

\[= (p - 1) \sum_{\chi \neq \chi_0} |L(1, \chi, a)|^2 \]

\[+ \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \]

\[+ \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2, \]

where \(g(y, x) = \sum_{i=0}^{k} a_i (x^i - 1)y^i\), \(b_i = a_i (x^i - 1)\) and \(\sum_{x=2}^{p-1} \sum_{y=1}^{p-1} \) and \(\sum_{x=2}^{p-1} \sum_{y=1}^{p-1} \) means \(p|\) \((b_0, b_1, \ldots, b_k)\) and \(p|\) \((b_0, b_1, \ldots, b_k)\) respectively. Then we will estimate the two sums respectively.

1. When \(p|\) \((b_0, b_1, \ldots, b_k)\), according to Lemma 3 and Theorem 1, we have

\[\sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \]

\[\ll \sum_{x=2}^{p-1} p^{1-\frac{1}{2}+\epsilon} \left| \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \right| \]

\[\ll \sum_{x=2}^{p-1} p^{2-\frac{1}{2}+\epsilon} \frac{\log p}{a(x - 1)} \]

\[\ll p^{2-\frac{1}{2}+\epsilon}. \]

2. When \(p \mid (b_0, b_1, \ldots, b_k)\), i.e. \(p \mid b_0, p \mid b_1, \ldots, p \mid b_k\), since \(p|\) \((a_0, a_1, \ldots, a_k)\), there is at least one \(a_i\) such that \(p|a_i\), then for this \(l\) we must have \(p \mid (x^i - 1)\), i.e. \(x^i \equiv 1(\text{mod}p)\). But
in the set \{2, 3, \ldots, p - 1\}, there are at most \(l - 1\) numbers \(x\) such that \(p \mid (x^l - 1)\). Also \(l - 1 < k, x^l > x^l - 1 \geq p\), so \(x > p^{\frac{1}{k}} \geq p^{\frac{1}{2}}\). Then from Lemma 3 and Theorem 1 we have

\[
\left| \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \right| \\
\leq \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \left| \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \right| \\
\ll \sum_{x=2}^{p-1} \frac{(p-1)p \log p}{a(x-1)} \\
\ll p^2 \log p \left( \max_{p^{1/k} \leq x < p} \frac{1}{x-1} \right) \times |\{x : x \in \{2, 3, \ldots, p - 1\} \text{ with } x^l \equiv 1 \text{ (mod } p)\}| \\
\ll kp^2 - \frac{1}{k} + \epsilon.
\]

Therefore, combining (1), (2), the formula (1) and Theorem 1, we get the asymptotic formula

\[
\sum_{x=1}^{p-1} \sum_{\chi \neq \chi_0} \chi(x) e \left( \frac{f(x)}{p} \right)^2 |L(1, \chi, a)|^2 \\
= (p-1) \sum_{\chi \neq \chi_0} |L(1, \chi, a)|^2 \\
+ O \left( \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \right) \\
+ O \left( \sum_{x=2}^{p-1} \sum_{y=1}^{p-1} e \left( \frac{g(y, x)}{p} \right) \sum_{\chi \neq \chi_0} \chi(x) |L(1, \chi, a)|^2 \right) \\
= p^2 \sum_{d | q} \frac{\mu(d)}{d^2} \zeta \left( 2, \frac{a}{d} \right) - \frac{4p^2}{a} \sum_{d | q} \frac{\mu(d)}{d} \sum_{l=1}^{[\frac{q}{d}]} \frac{1}{l} + O(p^{2-\frac{1}{k}} + \epsilon),
\]

where \(\mu\) is the Möbius function and \(\zeta(s, \alpha)(s = \sigma + it, \alpha > 0)\) is the Hurwitz zeta function, the \(O\) constant is depending on \(k, a\) and \(\epsilon\). This completes the proof of Theorem 2.

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