Fulde-Ferrell-Larkin-Ovchinnikov state of two-dimensional imbalanced Fermi gases

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The ground-state phase diagram of attractively-interacting Fermi gases in two dimensions with a population imbalance is investigated. We find the regime of stability for the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase, in which pairing occurs at finite wavevector, and determine the magnitude of the pairing amplitude $\Delta$ and FFLO wavevector $q$ in the ordered phase, finding that $\Delta$ can be of the order of the two-body binding energy. Our results rely on a careful analysis of the zero temperature gap equation for the FFLO state, which possesses nonanalyticities as a function of $\Delta$ and $q$, invalidating a Ginzburg-Landau expansion in small $\Delta$.

The Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state is a superfluid phase that can occur when two species (or spin-state) of fermion pair and condense in the presence of a density imbalance [1, 2]. Although predicted to occur in a wide range of systems, including electronic materials and high-density quark matter [3, 4], no definitive signature of the FFLO state (such as a spatiallyinhomogeneous pairing amplitude) has been observed.

Experimental advances in atomic physics have recently led to another setting for observing the FFLO state, namely cold atomic gases [5, 6], which exhibit several experimentally tunable parameters, including the fermion densities, the interfermion interactions, and the effective spatial dimension of the gas (controlled by an applied trapping potential).

However, cold atom experiments in the three-dimensional (3D) limit observed no signatures of the FFLO phase [7–10], consistent with theoretical work that found the FFLO to be stable for a very narrow range of density imbalance [11–13]. Imbalanced gases in 1D, studied experimentally at Rice [14], are predicted to have a wide parameter region of stable imbalanced superfluidity [17–23], although again finite-momentum pairing correlations (signifying the FFLO state) have not been seen.

In this paper we investigate Fermi gases in two spatial dimensions, which, in the balanced case, have been studied theoretically in Refs. [24, 25] and experimentally in Refs. [26, 27]. Imbalanced 2D fermion superfluids have been studied theoretically for many years in the condensed matter context [28] and, more recently, by several authors in the present cold-atom context [29–36]. Before presenting our detailed calculations, we first describe our main results on imbalanced 2D Fermi gases that can describe cold atoms with an imposed imbalance but also thin-film superconductors in a Zeeman magnetic field [37].

Using a model Hamiltonian for two species (labeled by $\uparrow, \downarrow$) of fermions in 2D with chemical potentials $\mu_\uparrow = \mu + h$ and $\mu_\downarrow = \mu - h$, we find the ground-state phase diagram (Fig. 1 top panel) as a function of $\mu/\epsilon_b$ and $h/\epsilon_b$, where $\epsilon_b$ is the two-body binding energy characterizing the tunable interactions. Thus, we find a window of FFLO stability that widens with increasing $h$, between a balanced [38] superfluid (SF) and an imbalanced non-superfluid or normal (N) phase. The red curve denotes a continuous N-FFLO transition, and the black curve denotes a first order N-SF transition for $\mu/\epsilon_b < 5/4$ and a first order FFLO-SF transition for $\mu/\epsilon_b > 5/4$. The fixed particle number phase diagram is shown in Fig. 1 (bottom panel), with the first-order transition in the fixed chemical potential ensemble becoming a regime of phase separation (PS) between SF and FFLO for $\epsilon_b/\epsilon_F < 4/5$ and between SF and N for $\epsilon_b/\epsilon_F > 4/5$ (the latter barely visible).

The phase boundary $h_{\text{FFLO}}$ marking the N-FFLO transition is determined by the point where, with decreasing $h$, the system becomes unstable to pairing correlations at a wavevector $q$. As we will show, the function determining this instability has an unusual and singular behavior as a function of the system parameters ($\mu$, $h$, and $q$). Additionally, the function appearing in the gap equation (determining the stable pairing amplitude $\Delta$ in the FFLO state) is a nonanalytic function of $\Delta$ (in the $T \to 0$ limit),
invalidating a Ginzburg-Landau expansion. Despite this nonanalyticity, we find a continuous transition into the FFLO state.

We now proceed to derive these results, starting with the Hamiltonian (where \( \Psi_{\sigma}(r) \) are field operators)

\[
H = \sum_{\sigma=\uparrow,\downarrow} \int d^2r \left[ \hat{p}^2/2m - \mu_{\sigma} \right] \Psi_{\sigma}(r) + \lambda \int d^2r \Psi_{\downarrow}^\dagger(r) \Psi_{\uparrow}^\dagger(r) \Psi_{\downarrow}(r) \Psi_{\uparrow}(r),
\]

where \( \lambda \) parameterizes the short-range attractive interactions. To describe the phases of an imbalanced Fermi gas in 2D, we proceed by making the standard mean-field approximation, assuming an expectation value \( \langle \Psi_{\downarrow}(r) \Psi_{\uparrow}(r) \rangle = \Delta e^{i\mathbf{q} \cdot \mathbf{r}} \), with \( \Delta \) and \( \mathbf{q} \) being variational parameters that must be minimized, at fixed \( \mu_{\uparrow} \) and \( \mu_{\downarrow} \), to determine the equilibrium state. The resulting mean-field ground-state energy is [13]

\[
E_G = -\frac{1}{\lambda} \sum_{\mathbf{p}} (E_p - \tilde{\epsilon}_p) + \sum_{\mathbf{p},\alpha=\uparrow,\downarrow} E_{p\alpha} \Theta(-E_{p\alpha}),
\]

where \( \Theta(x) \) is the Heaviside step function and

\[
E_{p\pm} = E_p \pm (h + \frac{\mathbf{p} \cdot \mathbf{q}}{2m}),
\]

where \( E_p = \sqrt{\tilde{\epsilon}_p^2 + |\Delta|^2} \) and \( \tilde{\epsilon}_p = \epsilon_p - \tilde{\mu} \) with \( \tilde{\mu} = \mu - q^2/8m \) and \( \epsilon_p = \frac{\tilde{\epsilon}_p^2}{2m} \) (below we often take \( m = 1 \), and \( \Delta \) real and positive).

Our model Hamiltonian must include a ultraviolet cutoff \( D \) cutting off all momentum sums. As is standard [24], we express \( \lambda \) in terms of the two-body binding energy \( \epsilon_b \), satisfying

\[
\frac{1}{\lambda} = \sum_{\mathbf{p}} \frac{1}{2\epsilon_p + \epsilon_b},
\]

where the sum is over \( \epsilon_p < D \), leading to \( \epsilon_b \approx 2D \text{exp}[m/\pi\lambda] \). Upon inserting Eq. (4) into Eq. (2), we can take \( D \to \infty \) in the momentum sums, with all dependence on \( D \) absorbed into \( \epsilon_b \).

Our general strategy is to minimize \( E_G \) with respect to \( q \) and \( \Delta \) (as done to obtain the results in Fig. 2), with the latter minimization yielding a function \( S(h, q, \Delta) \), defined by \( \frac{\partial E_G}{\partial \Delta} = -2\Delta S(h, q, \Delta) \). We start by neglecting the FFLO state, setting \( q = 0 \). Within this approximation, Eq. (2) exhibits, with increasing \( h \), a first-order transition from a fully-paired balanced phase to an imbalanced normal phase. The balanced paired phase [24] can be studied by setting \( h = 0 \), yielding the gap equation \( 0 = -2\Delta S(0, 0, \Delta) \) with (converting the momentum sum to an integration, with the system size set to unity)

\[
S(0, 0, \Delta) = \int \frac{d^2p}{(2\pi)^2} \left[ \frac{1}{2\sqrt{\xi_p^2 + \Delta^2}} - \frac{1}{2\epsilon_p + \epsilon_b} \right].
\]

Evaluation of the integral in Eq. (5) leads to the stationary pairing amplitude \( \Delta \) (a minimum of \( E_G \)) at \( \Delta = \sqrt{\epsilon_b^2 + 2\mu + \epsilon_b} \) (so that we must have \( \mu > -\frac{1}{2}\epsilon_b \) for a stable SF phase). Imposing fixed total particle number, \( N = -\frac{\partial E_G}{\partial \mu} \), yields the relation [24] \( n = \frac{1}{\pi^2} \left( \mu + \sqrt{\Delta^2 + \mu^2} \right) \) for the number density. We define the Fermi energy \( \epsilon_F = \pi n \) by the value of \( \mu \) in the normal phase (\( \Delta = 0 \)).

Having briefly reviewed the balanced case, we now turn on nonzero \( h \) to consider the imbalanced case (but still with \( q = 0 \)). We find that the location of the minimum is unchanged with increasing \( h \), although a second minimum of \( E_G \), at \( \Delta = 0 \), appears describing the imbalanced normal phase. The location, \( h_c \), of the first order transition between the SF and N phases is obtained by equating the energies, \( E_{G,SF} = E_{G,N} \), with the latter obtained by setting \( \Delta = 0 \) in Eq. (2). We find

\[
h_c(\mu) = \begin{cases} \epsilon_b \left[ \frac{1}{\sqrt{2}} + (\sqrt{2} - 1)\mu/\epsilon_b \right] & \text{for } \mu < \mu_c \\ \frac{1}{2} \epsilon_b \sqrt{1 + 4\mu/\epsilon_b} & \text{for } \mu > \mu_c \end{cases},
\]

where \( \mu_c = \frac{3}{2}(1+\sqrt{2})\epsilon_b \) separates the cases of a transition into a phase with one (for \( \mu < \mu_c \)) or two (for \( \mu > \mu_c \)) Fermi surfaces. This determines the black curve of Fig. 1 (top panel). As discussed above, for \( \mu/\epsilon_b > 5/4 \), the actual first order transition is between the FFLO and SF phases. However, we find (numerically) that the FFLO and N phases are almost equal in energy, so that the true FFLO-N first order phase boundary is only slightly lower than Eq. (6).

To study the FFLO phase, we first assume a continuous transition, with decreasing \( h \), from the N phase to the FFLO state, occurring when the curvature of \( E_G \) vs. \( \Delta \),
at $\Delta = 0$, becomes zero for some nonzero $q$. This is equivalent to $0 = S(h, q, 0)$ with
\[
S(h, q, 0) = \int \frac{d^2p}{(2\pi)^2} \frac{N(p)}{\xi_{p-\frac{1}{2}q} + \xi_{p+\frac{1}{2}q} - \frac{1}{2c_p + \epsilon_b}},
\tag{7}
\]
where we defined the numerator $N(p) = 1 - n_F(\xi_{p-\frac{1}{2}q}) - n_F(\xi_{p+\frac{1}{2}q})$ with the Fermi function $n_F(x) = \frac{1}{e^{x/T} + 1}$ (for the moment generalizing to nonzero temperature $T$).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3}
\caption{(Color Online) Plots of the curves $p_\pm(\theta)$. Eq. (8), for $\mu = 1.4$ and $h = 0.4$ with $q = 0.25$ (so that $q < q_c$) in the left panel and $q = 0.4$ (so that $q > q_c$). The yellow dashed curve is $0 = \xi_{p-\frac{1}{2}q} + \xi_{p+\frac{1}{2}q}$, or $p^2 = 2\mu$, the curve along which the denominator of the first term of Eq. (7) vanishes.}
\end{figure}

In the limit $T \to 0$, $n_F(x) = \Theta(-x)$, so that $N(p)$ exhibits discontinuities whenever an argument of one of the Fermi functions vanishes. We proceed by choosing the FFLO wavevector to be along the $\hat{x}$ axis, $q = q\hat{x}$ (valid since $S$ is independent of this choice), and define $p_+(\theta)$ and $p_-(\theta)$ to be the solutions to $\xi_{p-\frac{1}{2}q} = 0$ and $\xi_{p+\frac{1}{2}q} = 0$, with $\theta$ the angle between $p$ and $q$. We find:
\[
p_\pm(\theta) = \frac{1}{2} \left( \sqrt{S(\mu \pm h) + q^2 \cos^2 \theta \pm q \cos \theta} \right). \tag{8}
\]
The behavior of Eq. (7) depends crucially on whether the circles $p_+(\theta)$ and $p_-(\theta)$ intersect. The two panels in Fig. 3 show these circles for parameters such that $q < q_c$ (left panel) and $q > q_c$ (right panel) where $q_c \equiv \sqrt{2(\mu + h)} - \sqrt{2(\mu - h)}$ is the difference in Fermi wavevectors of the two species.

As we now discuss, the FFLO phase is only stable as a ground state for $q > q_c$. To show this, we must evaluate the integral in Eq. (7). After evaluating the radial momentum $p$ integral, the remaining integral over angle $\theta$ can be expressed as:
\[
S(h, q, 0) = \frac{1}{8\pi} \int_0^{2\pi} d\theta \ln |f(e^{i\theta})|, \tag{9}
\]
where $f(z) \equiv z^2(2\mu - p_+^2(z))(p_+^2(z) - 2\mu)$, and $p_\pm(z)$ is obtained from replacing $\cos \theta \to \frac{1}{2}(z + \frac{1}{z})$ in Eq. (8). Angle integrals of the form of Eq. (9) can be evaluated using the Jensen formula from complex analysis [39], which states that, for $f(z)$ analytic,
\[
\frac{1}{2\pi} \int_0^{2\pi} d\theta \ln |f(e^{i\theta})| = \ln |f(0)| + \sum_{i=1}^n \frac{1}{|a_i|}, \tag{10}
\]
with $|a_i|$ the zeroes of $f(z)$ inside the unit circle $|z| < 1$. For $q < q_c$, it is easily shown that $f(z)$ only has zeros on the unit circle $|z| = 1$, so they do not contribute and the result only involves the first term on the right side of Eq. (10). For $q > q_c$, $f(z)$ possesses two zeros inside the unit circle at $z_{+} = (\sqrt{h^2 - h_c} - h)/h_c$, where $h_c = \frac{1}{\sqrt{2}} \sqrt{\mu q}$ is the solution to $q = q_c(\mu, h)$. Including these zeros leads to:
\[
S(h, q, 0) = \begin{cases} 
\frac{1}{4\pi} \ln \frac{\sqrt{h_c^2 - 2\mu q}}{h_c} & \text{for } q < q_c, \\
\frac{1}{4\pi} \ln \frac{4\mu h}{q^2} & \text{for } q > q_c, 
\end{cases} \tag{11}
\]
where we note that, for $q > q_c$, $S(h, q, 0)$ is independent of $\mu$ and $h$. Below, we show that this simple result also holds for the full integral $S(h, q, \Delta)$ for sufficiently small $\Delta$. Before doing this, we first use Eq. (11) to find the location of the FFLO phase boundary.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{(Color Online) The top panel shows $S(h, q, \Delta)$ at $\Delta = 0$ for $\mu = 1.4$, $\epsilon_b = 1$, and $h = 3.5$ (green long-dashed), $h = 3$ (blue short-dashed), and $h = 3\sqrt{2} \approx 2.65$ (red solid), with the latter showing a finite $q$ pairing instability where $S(h, q, 0) = 0$. The bottom panel shows $S(h, q, \Delta)$ as a function of $\Delta$ for $\mu = 1$, $q = 1.8$ and $h = 0.5$ (dashed red) and $h = 0.28$ (solid black), with the stable FFLO phase occurring at $S(h, q, \Delta) = 0$. The kinks in $S(h, q, \Delta)$ occur at $\Delta = 1$ and $\Delta = 2$.}
\end{figure}

In Fig. 4 (top panel), we plot $S(h, q, 0)$ for $\mu = 4$ and for three values of $h$, using units such that $\epsilon_b = 1$. The kink in each curve occurs at $q = q_c$ and, consistent with Eq. (11), the curves overlap for $q > q_c$. To interpret these physically, we note that the normal phase is stable against a continuous transition if $S(h, q, 0) < 0$. Therefore, since the maximum of this curve is the kink location (at $q_c$) it is clear the FFLO phase occurs at
$q = q_c (\mu, \hbar_{\text{FFLO}})$ and $S(h, q, 0) = 0$, the simultaneous solution of which is $h_{\text{FFLO}} = \sqrt{2\mu \epsilon_b - \epsilon_q}$ and $q_c = 2\sqrt{\epsilon_b}$, the FFLO-N phase boundary plotted in Fig. 1 (top panel), which crosses $h_c (\mu)$ of Eq. (6) at $\mu = 5\epsilon_b/4$.

To obtain the FFLO-N phase boundary at fixed particle number, we simply need to use the relations for the density $n = n_+ + n_- = \mu/\pi$ and magnetization $M = n_+ + n_- = h/\pi$ in the imbalanced N phase, leading to $P_{\text{FFLO}} = \frac{\alpha}{c_p} \sqrt{2\mu \epsilon_b - 1}$. To get $P_{\text{c2}}$ in Fig. 1 (bottom panel) we must compute $n$ and $M$ in the FFLO state. We have done this using the approximate ground-state energy derived below; however, we find $n$ and $M$ to be well approximated by their values in the $N$ phase, leading to the simple expression $P_{\text{c2}} \approx \frac{1}{2} \frac{\alpha}{c_p} \sqrt{1 + \frac{4}{\epsilon_b}}$ for the FFLO-PS phase boundary [40].

We emphasize that $\Delta$ for $\Delta > \Delta_{c2}$ is easy to obtain, since $S_\pm = 0$. As shown in Ref. [41], for $\Delta < \Delta_{c1}$, $S(h, \mu, \Delta)$ can be written in the form of Eq. (9) with a different analytic function $f(z)$. Applying the Jensen formula in this regime (requiring use of the solution to the quartic equation [42]) leads to the first line of Eq. (13). At $\Delta = \Delta_{c1}$, a branch cut appears in the function $f(z)$, invalidating the use of the Jensen formula.

To conclude, we find that even the simplest 2D FFLO phase exists for $\Delta_{c1} < \Delta < \Delta_{c2}$, where we have no analytic expression for $S$. However, we can derive an approximate result for $S$ valid for $\Delta > \Delta_{c1}$ using the Jensen formula result. To directly evaluate $S_\pm$, Eq. (12), analytically, we need to know the regions where $E_{p\pm} < 0$. For $\Delta < \Delta_{c1}$, the region where $E_{p+} < 0$ is small, allowing an analytic approximation to $S_+$, while the region where $E_{p-} < 0$ is large (precluding a simple analytic approximation to $S_-$). However, since the sum of the three terms adds to the first line of Eq. (13), knowledge of $S_+$ tells us $S_-$. If we assume that $S_-$ is continuous near $\Delta_{c1}$, then this result must also apply for $\Delta > \Delta_{c1}$. With details given in Ref. [41], we take the resulting expression for $S$ and integrate it to obtain the ground state energy (up to a $\Delta$ and $q$ independent constant, needed to correctly obtain the particle number and magnetization via $S = -\partial E_G/\partial h$ and $n = -\partial E_G/\partial \mu$):

$$E_G \approx \frac{1}{4\pi} \Delta^2 \ln \frac{q^2}{4\epsilon_b} + \frac{1}{12\pi h} \sqrt{\frac{q_c}{q - q_c}} (\Delta - \Delta_{c1})^2 (2\Delta + \Delta_{c1}),$$

(14)

where we also approximated $\Delta_{c1} \approx \frac{1}{2} (\mu^2 - h^2) (q - q_c)$, valid for $q - q_c \rightarrow 0$ [41]. Minimizing this with respect to $\Delta$ and $q$ then yields the final results [41]

$$\Delta \approx \frac{\sqrt{2}}{3} \frac{\hbar_{\text{FFLO}}}{\sqrt{\epsilon_b (\mu - \epsilon_b)}} (h_{\text{FFLO}} - h),$$

(15a)

and

$$q \approx 2\sqrt{\epsilon_b} - \frac{1}{3} \frac{\hbar_{\text{FFLO}}}{\sqrt{\epsilon_b (\mu - \epsilon_b)}} (h_{\text{FFLO}} - h),$$

(15b)

showing a continuous onset of FFLO order for $h < h_{\text{FFLO}}$. As shown in Fig. 2, these results agree quite well close to the phase transition.

To conclude, we find that even the simplest 2D FFLO phase has an extremely rich structure in which the gap equation, determining the location of the phase transition and the strength of pairing $\Delta$ in the FFLO state possesses nonanalyticities as a function of $\Delta$ and the FFLO wavevector $q$. We find approximate analytic formulas for the phase boundaries and also find that the equilibrium $\Delta$ can be of the order of the two-body binding energy,
making it plausible to find the FFLO state in 2D imbalanced Fermi gases. Possible future extensions of this work including applying a similar analysis to more complex FFLO-type phases (e.g., of the Larkin-Ovchinnikov type) and analyzing fluctuation effects in 2D [31].

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[1] P. Fulde and R.A. Ferrell, Phys. Rev. 135, A550 (1964).
[2] A.I. Larkin and Yu.N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 47, 1136 (1964) [Sov. Phys. JETP 20, 762 (1965)].
[3] R. Casalbuoni and G. Nardulli, Rev. Mod. Phys. 76, 263 (2004).
[4] M.G. Alford, K. Rajagopal, T. Schaefer, and A. Schmitt, Rev. Mod. Phys. 80, 1455 (2008).
[5] J. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[6] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 80, 1215 (2008).
[7] M.W. Zwierlein, A. Schirotzek, C.H. Schunck, and W. Ketterle, Science 311, 492 (2006).
[8] G.B. Partridge, W. Li, R.I. Kamar, Y.-A. Liao, R.G. Hulet, Science 311, 503 (2006).
[9] Y. Shin, M.W. Zwierlein, C.H. Schunck, A. Schirotzek, W. Ketterle, Phys. Rev. Lett. 97, 030401 (2006).
[10] G.B. Partridge, W. Li, Y. A. Liao, R. G. Hulet, M. Haque, and H.T.C. Stoof, Phys. Rev. Lett. 97, 190407 (2006).
[11] D.E. Sheehy and L. Radzihovsky, Phys. Rev. Lett. 96, 060401 (2006).
[12] M.M. Parish, F.M. Marchetti, A. Lamacraft, and B.D. Simons, Nat. Phys. 3, 124 (2007).
[13] D.E. Sheehy and L. Radzihovsky, Ann. Phys. 322, 1790 (2007).
[14] Y. Liao, A.S.C. Rittner, T. Paprotta, W. Li, G.B. Partridge, R.G. Hulet, S.K. Baur, and E.J. Mueller, Nature (London) 467, 567-569 (2010).
[15] A.T. Sommer, L.W. Cheuk, M.J.H. Ku, W.S. Bakr, and M.W. Zwierlein, Phys. Rev. Lett. 108, 045302 (2012).
[16] Y. Zhang, W. Ong, I. Arakelyan, and J.E. Thomas, Phys. Rev. Lett. 108, 235302 (2012).
[17] G. Orso, Phys. Rev. Lett. 98, 070402 (2007).
[18] H. Hu, X.-J. Liu, and P.D. Drummond, Phys. Rev. Lett. 98, 070403 (2007).
[19] A.E. Feiguin and F. Heidrich-Meisner, Phys. Rev. B 76, 220508 (2007).
[20] X.-J. Liu, H. Hu, and P. D. Drummond, Phys. Rev. A 76, 043605 (2007).
[21] G. G. Batrouni, M.H. Huntley, V.G. Rousseau, and R.T. Scalettar, Phys. Rev. Lett. 100, 116405 (2008).
[22] F. Heidrich-Meisner, A.E. Feiguin, U. Schollwöck, and W. Zwerger, Phys. Rev. A 81, 023629 (2010).
[23] K. Sun and C.J. Boile, Phys. Rev. A 85, 051607 (2012).
[24] M. Randeria, J.-M. Duan, and L.-Y. Shieh, Phys. Rev. Lett. 62, 981 (1989).
[25] D.S. Petrov, M.A. Baranov, and G.V. Shlyapnikov, Phys. Rev. A 67, 031601(R) (2003).
[26] A.T. Sommer, L.W. Cheuk, M.J.H. Ku, W.S. Bakr, and M.W. Zwierlein, Phys. Rev. Lett. 108, 045302 (2012).
[27] Y. Zhang, W. Ong, I. Arakelyan, and J.E. Thomas, Phys. Rev. Lett. 108, 235302 (2012).
[28] H. Shihara, Phys. Rev. B 50, 12760 (1994).
[29] G.J. Conduit, P.H. Conlon, and B.D. Simons, Phys. Rev. A 77, 053617 (2008).
[30] L. Radzihovsky and A. Vishwanath, Phys. Rev. Lett. 103, 010404 (2009).
[31] L. Radzihovsky, Phys. Rev. A 84, 023611 (2011).
[32] M.J. Wolak, B. Grémaud, R.T. Scalettar, and G.G. Batrouni, Phys. Rev. A 86, 023630 (2012).
[33] J. Levinse, Supplementary Material to this manuscript.
[34] H. Caldas, A.L. Mota, R.L.S. Farias, and L.A. Souza, J. Stat. Mech.: Theory and Experiment, P10019 (2012).
[35] M.M. Parish and J. Levinse, Phys. Rev. A 87, 033616 (2013).
[36] S. Yin, J.-P. Martikainen, and P. Törnä, Phys. Rev. B 89, 014507 (2014).
[37] Y.V. Loh, N. Trivedi, Y.M. Xiong, P.W. Adams, and G. Catelani, Phys. Rev. Lett. 107, 067003 (2011).
[38] Here and elsewhere, the term “balanced” means the equilibrium numbers are the same, $N_1 = N_2$, while imbalanced means the equilibrium numbers are different ($N_1 > N_2$).
[39] L.V. Ahlfors, Complex Analysis, Third Edition, McGraw-Hill, New York, 1979.
[40] We note that relaxing this approximation and using $M$ and $n$ in the FFLO phase leads to a slightly lower location for the $P_{c2}$, slightly widening the FFLO window in Fig. 1 (bottom panel).
[41] D.E. Sheehy, Supplementary Material to this manuscript.
[42] Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
Supplementary Material

The full integral we need to evaluate, in the zero temperature limit, is:

\[
S(h, q, \Delta) = \int \frac{d^2p}{(2\pi)^2} \left[ \frac{1 - n_F(E_{p+}) - n_F(E_{p-})}{E_{p+} + E_{p-}} - \frac{1}{2\epsilon_p + \epsilon_b} \right],
\]

where the Fermi function is defined by:

\[
n_F(x) = \frac{1}{e^{x/T} + 1} = \frac{1}{2} \left(1 - \tanh \frac{x}{2T}\right),
\]

equal to a step function \(\Theta(-x)\) in the limit \(T \to 0\). The energies are:

\[
E_{p\pm} = E_p \pm \left(\frac{p \cdot q}{2m}\right),
\]

\[
E_p = \sqrt{\tilde{\xi}_p^2 + \Delta^2},
\]

\[
\tilde{\xi}_p = \frac{p^2}{2m} - \mu + \frac{q^2}{8m} = \frac{p^2}{2m} - \tilde{\mu}.
\]

Our goal is to use the Jensen formula to demonstrate the result

\[
S(h, q, \Delta) = \begin{cases} 
\frac{1}{16} \ln \frac{4\epsilon_b}{q^2} & \text{for } \Delta < \Delta_{c1}, \\
\frac{1}{16} \ln \frac{\epsilon_b}{\sqrt{\Delta^2 + \mu^2 - \tilde{\mu}}} & \text{for } \Delta > \Delta_{c2}, 
\end{cases}
\]

and to derive an approximate formula for \(S(h, q, \Delta)\), and the ground-state energy, that is valid for \(\Delta \gtrsim \Delta_{c1}\).

We first determine the curves \(E_{p\pm} = 0\), since the Fermi functions are only nonzero for \(E_{p\pm} < 0\).

![FIG. 6: Each panel shows the curves \(p_1(\theta)\) and \(p_2(\theta)\) for \(\mu = 1.4\), \(h = 0.3\), and \(q = 1\) (all rescaled in units such that \(\epsilon_b = 1\)). For these parameters, \(\Delta_{c1} = 0.523135\). The red curves are the real parts of \(p_1(\theta)\) and \(p_2(\theta)\). When they are complex, Re[\(p_1(\theta)\)] = Re[\(p_2(\theta)\)]. When they are real, \(p_2(\theta) \geq p_1(\theta)\) and they define the boundaries of blue shaded regions. In the leftmost blue region of each panel, \(E_{p+} < 0\), while in the rightmost blue region of each panel, \(E_{p-} < 0\). The left panel is for \(\Delta = 0.25\), well below \(\Delta_{c1}\). The right panel is for \(\Delta = 0.50\), just below \(\Delta_{c1}\), so that the region where \(E_{p+} < 0\) is about to vanish. We have:

\[
0 = E_p \pm \left(\frac{p \cdot q}{2m}\right) = \sqrt{\left(\frac{p^2}{2m} - \tilde{\mu}\right)^2 + |\Delta|^2} \pm \left(\frac{p \cdot q}{2m}\right).
\]

To solve this, we move the square root to one side and square both sides:

\[
\left(\frac{p^2}{2m} - \tilde{\mu}\right)^2 + |\Delta|^2 = \left(h + \frac{p \cdot q}{2m}\right)^2.
\]
which, after setting $m = 1$, expanding, and multiplying by an overall factor of 4, can be written in terms of a quartic equation:

$$p^4 - (4\tilde{\mu} + q^2 \cos^2 \theta)p^2 - 4hq \cos \theta p + 4(\Delta^2 - h^2 + \tilde{\mu}^2) = 0. \quad (24)$$

When real, the solutions to this equation define the curves $p_1$ and $p_2$ (with $p_2 > p_1$) surrounding the blue shaded regions of Fig. 6.

### Quartic equation

In this section we recall the solution to the quartic equation \[42\]. A general quartic equation is of the form

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad (25)$$

with $a = 1$, $b = 0$ and

$$c = -(4\tilde{\mu} + q^2 \cos^2 \theta), \quad (26)$$
$$d = -4hq \cos \theta, \quad (27)$$
$$e = 4(\Delta^2 - h^2 + \tilde{\mu}^2). \quad (28)$$

The general solution to the quartic equation is:

$$x_{1,2} = -\frac{b}{4a} - S \pm \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{4}{S}, \quad (29)$$
$$x_{3,4} = -\frac{b}{4a} + S \pm \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{4}{S}, \quad (30)$$

where $x_2$ and $x_4$ take the $-$ in each line.

Here we defined:

$$p = \frac{8ac - 3b^2}{8a^2} = -4\tilde{\mu} - q^2 \cos^2 \theta, \quad (31)$$
$$\hat{q} = \frac{b^3 - 4abc + 8a^2d}{8a^3} = -4hq \cos \theta, \quad (32)$$

where the final equalities apply to the present case.

We also define

$$S = \frac{1}{2} \sqrt{-\frac{2}{3}p + \frac{1}{3a} (Q + \frac{\Delta_0}{Q})}, \quad (33)$$
$$Q = \frac{\sqrt{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^2}}}{2}, \quad (34)$$

We note here that the function $S$ needed for the quartic equation has the same symbol as the function $S(h, q, \Delta)$ that we are attempting to compute, and hope that the distinction will be understood from context. An alternate expression for $S$ is:

$$S = \frac{1}{2} \sqrt{-\frac{2}{3}p + \frac{2}{3a} \sqrt{\Delta_0} \cos \frac{\phi}{3}}, \quad (35)$$
$$\phi = \cos^{-1} \frac{\Delta_1}{2\sqrt{\Delta_0^3}}. \quad (36)$$

Here, $\Delta_0$ and $\Delta_1$ are:

$$\Delta_0 = c^2 - 3bd + 12ae, \quad (37)$$
$$\Delta_1 = 48(\Delta^2 - h^2 + \tilde{\mu}^2) + (-4\tilde{\mu} - q^2 \cos^2 \theta)^2, \quad (38)$$

$$\Delta_1 = 2c^3 - 9bcd + 27b^2e + 27ad^2 - 72ace, \quad (39)$$
$$\Delta_1 = 432h^2q^2 \cos^2 \theta + 288(\Delta^2 - h^2 + \tilde{\mu}^2)(4\tilde{\mu} + q^2 \cos^2 \theta) - 2(4\tilde{\mu} + q^2 \cos^2 \theta^3). \quad (40)$$
Note that $S$ and $Q$ are too complex to write out explicitly for the present case. In terms of $\Delta_0$ and $\Delta_1$, the discriminant $\Delta_d$ is
\[
\Delta_d = -\frac{1}{27} (\Delta_1^2 - 4\Delta_0^3).
\] (41)

We also define
\[
P = 8ac - 3b^2 = -8(4\mu + q^2 \cos^2 \theta),
\]
\[
D = 64a^3c - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4 = 256(\Delta_2 - h^2 + \tilde{\mu}^2) - 16(4\mu + q^2 \cos^2 \theta)^2,
\]
which determine the nature of the roots in various regimes. Our $p_1$ and $p_2$ are given by $x_3$ and $x_4$ since $x_{1,2}$ are negative for our parameters (when they are real). Therefore we define:
\[
p_2(\theta) = S + \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}},
\]
\[
p_1(\theta) = S - \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}.
\]
For angles $\theta$ such that $p_1$ and $p_2$ are real, these will define the boundaries of the regions where $E_{p\pm} < 0$. However, in the following we’ll need these solutions even when they are complex, which occurs for angle $\theta$ that do not intersect (for any radial $p$) a region where $E_{p\pm} < 0$.

Then, our integral will be given by
\[
S(h, q, \Delta) = \frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^\infty dp \frac{[1 - n_F(E_{p+}) - n_F(E_{p-})]}{E_{p+} + E_{p-}} - \frac{1}{2\epsilon_p + \epsilon_b}.
\]
\[
= \frac{1}{4\pi} \ln \frac{\epsilon_b}{\sqrt{\Delta^2 + \tilde{\mu}^2 - \tilde{\mu}} - \frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^\infty dp \frac{n_F(E_{p+}) + n_F(E_{p-})}{2E_p}.
\]
To evaluate the final integral, we note that the numerator is zero (for $T \to 0$) for angles $\theta$ that do not intersect regions satisfying $E_{p\pm} < 0$. For angles $\theta$ that do intersect such regions, the numerator is unity for $p_1 < p < p_2$. This allow the final integral to be written as:
\[
S_1 = -\frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^\infty dp \frac{n_F(E_{p+}) + n_F(E_{p-})}{2E_p} = \frac{1}{8\pi^2} \int_0^{2\pi} d\theta \ln \left| \frac{p_2^2 - 2\tilde{\mu} + 2E_{p1}}{p_1^2 - 2\tilde{\mu} + 2E_{p1}} \right|.
\]
To see that this is correct, we note that, when $p_1$ and $p_2$ are real, the final integrand comes from simply evaluating the radial $p$ integral in regions where $E_{p\pm} < 0$. When $p_1$ and $p_2$ are complex, the absolute value bars inside the logarithm ensure that the argument of the logarithm is unity, giving zero for the integrand. This is also correct, since, as noted, $p_1$ and $p_2$ are complex for angles that do not intersect the regions of $E_{p\pm} < 0$.

**Critical pairing amplitudes**

The function $S(h, q, \Delta)$ exhibits slope discontinuities as a function of $\Delta$ at $\Delta_{c1}$ and $\Delta_{c2}$. To define these, we consider the regions of $p$ where $E_{p+} < 0$ and $E_{p-} < 0$, which are the left and right blue regions of the left panel of Fig. 6. At $\Delta_{c1}$, the lefthemost region vanishes and at $\Delta_{c2}$, the rightmost region vanishes.

As a function of angle, the edges of the blue regions are defined by $p_1(\theta) = p_2(\theta)$. Therefore, we simply need to solve this equation for the conditions $\theta = 0$ and $\theta = \pi$. Setting $p_1(\theta) = p_2(\theta)$ yields the condition
\[
0 = -4S^2 - 2p - \frac{q}{S}.
\]
(48)

In the following section, we show directly that the vanishing of this quantity implies that the discriminant $\Delta_d$ vanishes at $\theta = 0$ and $\theta = \pi$. In this section we argue it directly based on the structure of the quartic equation solutions. Therefore we define
\[
R_{\pm}(\theta) = -4S^2 - 2p \pm \frac{q}{S} = -4S^2 + 2q^2 \cos^2 \theta + 8\tilde{\mu} + \frac{4\cos \theta h q}{S},
\]
(49)
with $R_-$ appearing in the quartic equation solutions that we’re interested in (our $p_1(\theta)$ and $p_2(\theta)$), and $R_+$ appearing in the other solutions [Eq. (29)].

We emphasize that $S$ also depends on $\theta$. Although the explicit expression for the discriminant $\Delta_d(\theta)$ is rather complicated, it is easy to verify that $\Delta_d(\theta)$ depends only on $\cos^2\theta$ (so that $\Delta_d(0) = \Delta_d(\pi)$). Additionally, $\Delta_d(\pi) > 0$ for $\Delta \to 0$ and sufficiently small $h$. Since $P < 0$ [Eq. (42)] and $\Delta_d > 0$, from the general theory of the quartic equation we know that the solutions are either all real or all complex; since we know that $p_1(\theta)$ and $p_2(\theta)$ are real near $\theta = 0$ and $\theta = \pi$, we must have the former.

With increasing $\Delta$, as long as $\Delta_d(0) = \Delta_d(\pi) > 0$, we will continue to have $R_+(\theta) > 0$. When the discriminant becomes negative, $\Delta_d(0) = \Delta_d(\pi) < 0$, the general theory of the quartic equation dictates that there are two real solutions and two complex conjugate solutions. Since $R_+(\pi) > R_-(\pi)$, this implies that, at $\theta = \pi$, the two real solutions are Eq. (29) and the two complex conjugate solutions are Eq. (44), with the two complex conjugate solutions coming from $R_-(\pi)$ becoming negative (so that the square root in Eq. (44) is negative while the square root in Eq. (29) is positive). Additionally, since $R_-(0) > R_+(0)$, we have that, at $\theta = 0$, the two real solutions are Eq. (44), while the two complex conjugate solutions are Eqs. (29).

The preceding argument shows that, with increasing pairing amplitude, when the discriminant $\Delta_d(0) = \Delta_d(\pi)$ first vanishes with increasing $\Delta$ (at $\Delta_{c1}$), then $p_1(\pi)$ and $p_2(\pi)$ merge (so that the region where $E_{p_+} < 0$ vanishes) while $p_1(0)$ and $p_2(0)$ remain distinct (so that the region where $E_{p-} < 0$ still exists).

It is easy to check that $\Delta_d(0) = \Delta_d(\pi) > 0$ for large $\Delta$, so that there must be a second crossing, which we call $\Delta_{c2}$. In this large pairing amplitude regime, once again from the theory of the quartic equation, the solutions are all real or all complex. However, from examining the original quartic equation, Eq. (24), it is clear that the solutions are all complex for large $\Delta$. Therefore, the solutions $p_1(0)$ and $p_2(0)$ must merge at $\Delta = \Delta_{c2}$, eliminating the region where $E_{p-} < 0$.

**Direct demonstration that $R_{\pm}$ vanishes when the discriminant vanishes**

We can also directly verify that the vanishing of $R_{\pm}$, Eq. (49), implies a vanishing of the discriminant. $R_{\pm} = 0$ implies:

$$0 = 16S^6 + 4p^2S^2 + 16pS^4 - \tilde{q}^2.$$  \hspace{1cm} (50)

Let’s show that the right side, which we call $F(S) = 16S^6 + 4p^2S^2 + 16pS^4 - \tilde{q}^2$, vanishes when the discriminant vanishes. From Eq. (41), we see that, when the discriminant vanishes, $\Delta_1 = 2\Delta_0^{3/2}$ which furthermore implies that $Q = 2\Delta_0^{3/2}$. Using Eq. (33), we obtain (noting that $\Delta_0 > 0$)

$$S = \frac{1}{2} \sqrt{-\frac{2}{3} p + \frac{2}{3} \sqrt{\Delta_0}},$$ \hspace{1cm} (51)

which, when inserted into $F(S)$, gives

$$F = \frac{1}{27} \left( 2\Delta_0^{3/2} + 6\Delta_0p - 8p^3 - 27\tilde{q}^2 \right).$$ \hspace{1cm} (52)

Using the definitions of $\tilde{q}$ and $p$, we have

$$\Delta_0 = 48(\Delta^2 - h^2 + \tilde{p}^2) + p^2,$$ \hspace{1cm} (53)

$$\Delta_1 = 27\tilde{q}^2 - 288(\Delta^2 - h^2 + \tilde{p}^2)p + 2p^3,$$ \hspace{1cm} (54)

which can be combined to get

$$\Delta_1 = 27\tilde{q}^2 - 6(\Delta_0 - p^2)p + 2p^3.$$ \hspace{1cm} (55)

This can be further combined with Eq. (52) to get

$$F = \frac{1}{27} \left( 2\Delta_0^{3/2} - \Delta_1 \right),$$ \hspace{1cm} (56)

which clearly vanishes when the discriminant, Eq. (41), vanishes.
Approximate formula for $\Delta_{c1}$

The discriminant is a sixth order polynomial in $\Delta$ in which only even powers of $\Delta$ appear. Therefore, to find $\Delta_{c1}$ and $\Delta_{c2}$ we need to solve a cubic equation and take the square root. This can be done, but the solutions are too complex to display here. In this section, we derive an approximate formula for $\Delta_{c1}$ that is valid for $q \rightarrow q_c^+$. The critical pairing $\Delta_{c1}$, is defined above as the point where, for $\Delta > \Delta_{c1}$, $E_{p+}$ is never negative. For $\Delta \rightarrow 0$, $E_{p+}$ only crosses zero for $q > q_c$, with

$$q_c = \sqrt{2(\mu + h)} - \sqrt{2(\mu - h)},$$  

(57)

or, equivalently,

$$q_c = \frac{\sqrt{2} h}{\sqrt{\mu}}.$$  

(58)

Therefore, $\Delta_{c1}$ must vanish for $q \rightarrow q_c^+$. We proceed by assuming the following form:

$$\Delta_{c1} = a(q - q_c) + b(q - q_c)^2 + \cdots,$$  

(59)

which we insert into the discriminant at angle $\theta = 0$ and proceed to Taylor expand order by order in the small parameter $(q - q_c)$, ensuring the discriminant vanishes at each order in $(q - q_c)$. We find:

$$\Delta_{c1} \approx \frac{1}{\sqrt{2}}(\mu^2 - h^2)^{1/4}(q - q_c) + \frac{\sqrt{2}(5h^2 - 2q_c^2\mu)}{2q_c^3\mu - 8h^2q_c}(\mu^2 - h^2)^{1/4}(q - q_c)^2,$$  

(60)

which, in the following, we’ll only use to linear order in $q - q_c$:

$$\Delta_{c1} \approx \frac{1}{\sqrt{2}}(\mu^2 - h^2)^{1/4}(q - q_c),$$  

(61)

Evaluating $S(h, q, \Delta)$ using the Jensen formula

Jensen’s formula, reviewed by Ahlfors [39], states that, for $f(z)$ analytic and free from zeros in $|z| \leq \rho$, then

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \ln |f(\rho e^{i\theta})| = \ln |f(0)|.$$  

(62)

It is also stated in Ahlfors [39] that this formula holds if there are zeros of $f(z)$ on the circle $|z| = \rho$. To use this formula we need to write $S_1$ in this form where $f(z)$ is analytic (we also have $\rho = 1$). This is possible for $\Delta < \Delta_{c1}$, but not for $\Delta_{c1} < \Delta < \Delta_{c2}$. For $\Delta > \Delta_{c2}$, we do not need the Jensen formula, since we can evaluate the integral directly using the knowledge that the Fermi functions in Eq. (16) vanish for $\Delta > \Delta_{c2}$. For $\Delta_{c1} < \Delta < \Delta_{c2}$, we have not been able to evaluate this integral exactly using any method; however, it is easy to verify via numerical integration that the kinks in $S(h, q, \Delta)$ at $\Delta_{c1}$ and $\Delta_{c2}$ are there.

We start in the regime $\Delta < \Delta_{c1}$. The full integral we need to evaluate using the Jensen formula is

$$S_1 = \frac{1}{8\pi^2} \int_0^{2\pi} d\theta \ln \left| \frac{p_1^2 - 2\mu + 2E_{p1}}{p_2^2 - 2\mu + 2E_{p2}} \right|,$$  

(63)

$$= \frac{1}{8\pi^2} \int_0^{2\pi} d\theta \ln \left| \frac{(p_1^2 - 2\mu + 2E_{p1})(p_2^2 - 2\mu - 2E_{p2})}{4\Delta^2} \right|,$$  

(64)

where the second line used the definition of $E_p$ in Eq. (19).

To use the Jensen formula, we simply need to replace $p_1$ and $p_2$ in this expression by $p_1(z)$ and $p_2(z)$, defined by making the replacement $\cos \theta \rightarrow \frac{1}{2}(z + \frac{1}{z})$, in the coefficients appearing in Eq. (44). Then we have:

$$S_1 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ln |f(e^{i\theta})|,$$  

(65)

$$f(z) = \frac{1}{4\pi} \ln \left| \frac{z^2(p_1^2(z) - 2\mu + 2E_{p1}(z))(p_2^2(z) - 2\mu - 2E_{p2}(z))}{4\Delta^2} \right|.$$  

(66)
where we added a factor of \( z^2 \) in the numerator. Since \( |z^2| = 1 \), this is allowed, and this factor will simplify the use of the Jensen formula.

To obtain \( f(0) \), we need \( p_1(0) \) and \( p_2(0) \), most easily obtained by returning to the original quartic equation in the limit \( z \to 0 \). Making the replacement \( \cos \theta \to \frac{1}{2}(z + \frac{1}{z}) \) in Eq. (24), in the limit \( z \to 0 \) we have solutions of the form \( p \propto az \) or \( p \propto b/z \). Plugging in these solutions, Taylor expanding to leading order in small \( z \), and solving for \( p \) yields for \( a \) and \( b \):

\[
a = \frac{4}{q}(-h \pm \sqrt{\Delta^2 + \tilde{\mu}^2}),
\]

\[
b = \pm \frac{q}{2z}.
\]

For \( z \to 0 \), the discriminant \( \Delta_d > 0 \) and the quantity \( P < 0 \), implying the quartic equation solutions are either all real or all complex, and we see that they are all real in this limit. Since the solutions \( x_{1,2} \) are negative, we claim that the analytical continuation of \( p_1 \) and \( p_2 \) are the two positive cases, and obtain (using \( p_2 > p_2 \))

\[
p_2 \sim \frac{q}{2z},
\]

\[
p_1 \sim \frac{4}{q}(-h + \sqrt{\Delta^2 + \tilde{\mu}^2})z.
\]

Plugging this into \( f(z) \), taking the limit \( z \to 0 \), and using Eq. (62) leads to

\[
S_1 = f(0) = \frac{1}{4\pi} \ln \frac{4(\sqrt{\Delta^2 + \tilde{\mu}^2} - \tilde{\mu})}{q^2}
\]

(71)

According to Eq. (46), \( S(h, q, \Delta) \) is

\[
S(h, q, \Delta) = \frac{1}{4\pi} \ln \frac{\epsilon_b}{\sqrt{\Delta^2 + \tilde{\mu}^2} - \tilde{\mu}} + S_1,
\]

(72)

\[= \frac{1}{4\pi} \ln \frac{4\epsilon_b}{q^2},
\]

(73)

the desired result showing that, the gap equation integral is independent of \( \Delta \) for small \( \Delta \).

**Analyticity of \( f(z) \)**

Our next task is to check when the argument of the logarithm is analytic. We will show that a branch cut appears when \( \Delta > \Delta_{c1} \). To do this, we first simplify the numerator in Eq. (64) by multiplying the two factors using

\[
E_{p_1}^2 = (h + \frac{1}{2}pq\cos \theta)^2,
\]

(74)

holding for \( p_1 \) and \( p_2 \), as they satisfy Eq. (23). Therefore we have:

\[
(p_1^2 - 2\tilde{\mu} + 2E_{p_1})(p_2^2 - 2\tilde{\mu} - 2E_{p_2}) = \left(p_1^2 - 2\tilde{\mu} + 2\sqrt{(h + \frac{1}{2}pq\cos \theta)^2}(p_2^2 - 2\tilde{\mu} - 2\sqrt{(h + \frac{1}{2}pq\cos \theta)^2})\right).
\]

(75)

When multiplying these out, we must keep in mind that \( p_1^2 - 2\tilde{\mu} < 0 \) and \( p_2^2 - 2\tilde{\mu} > 0 \) (when \( p_1 \) and \( p_2 \) are real) and that the quantities \( (h + \frac{1}{2}pq\cos \theta) \) change sign as a function of \( \theta \) (so that we can’t naively take the square root). In multiplying out these four terms, we obtain

\[
\frac{1}{4S^2}(4S^2 - q^2\cos^2 \theta)(4S^4 - 4h^2 - 8S,hq\cos \theta - S^2[3q^2\cos^2 \theta + 8\tilde{\mu}])
\]

\[
+ \frac{1}{\sqrt{2S^3}}(2h + q\cos \theta S)^2(q^2\cos^2 \theta - 4S^2)^2(-2S^3 + 2hq\cos \theta + Sq^2\cos^2 \theta + 4S\tilde{\mu}).
\]

(76)

The last factor in the argument of the square root is:

\[
(-2S^3 + 2hq\cos \theta + Sq^2\cos^2 \theta + 4S\tilde{\mu}) = \frac{S}{2}\left(-4S^2 - 2\mu - \frac{\hat{q}}{S}\right),
\]

(77)
containing $R_-$ from Eq. (49). We can also simplify the other terms using $R_-$ and obtain:

\[
(p_1^2 - 2\mu + 2E_{p_1})(p_2^2 - 2\mu + 2E_{p_2}) = (4S^2 - q^2 \cos^2 \theta) \left[ \frac{1}{4S^2} (-S^2 R_- - (2h + q \cos \theta S)^2) + \frac{1}{2S} \sqrt{(2h + q \cos \theta S)^2 R_-} \right]\]

The last factor of the second line contains the quantity $R_- = -4S^2 - 2p - \frac{q}{2}$ in the argument of a square root. As discussed above, this vanishes at $\theta = \pi$ when $\Delta = \Delta_{c1}$, becoming negative for $\Delta > \Delta_{c1}$; this leads to a branch cut in Eq. (78).

To show this, we now repeat the argument presented above, but generalized to the case of having replaced $\cos \theta \rightarrow \frac{1}{2}(z + \frac{1}{z})$ in the discriminant Eq. (41) (obtaining $\Delta_d(z)$) in $P$, Eq. (42) (obtaining $P(z)$), and in $R_\pm$ (Eq. (49)):

\[
R_{\pm}(z) = -4S^2 - 2p \pm \hat{q} \frac{S}{S} = -4S^2 + \frac{1}{2}q^2(z + \frac{1}{z})^2 + 8\mu \mp \frac{2hq}{S}(z + \frac{1}{z}),
\]

where we emphasize that $S$ also depends on $z$. For $\Delta < \Delta_{c1}$, we know from the previous discussion that $\Delta_d(z)$ is positive at $z = -1$ and $z = 1$, and that the quartic equation has four real solutions at these points. In fact, since $\Delta_d(z)$ has minima only at $z = \pm 1$ (due to the fact that $\Delta_d(1/z) = \Delta_d(z)$), we know that $\Delta_d(z)$ is positive on the real axis $-1 < z < 1$. This implies (since $P(z) < 0$) that the quartic equation has four real solutions on the real axis and, furthermore, that $R_-(z)$, that appears in Eq. (78) is real and positive.

Again following the previous argument, for $\Delta > \Delta_{c1}$, for $z$ in the vicinity of $z = -1$ there will be a region where $\Delta_d(z) < 0$, and two of our quartic solutions must become complex. Since $R_-(z) < R_+(z)$ for $z < 0$ on the real axis, it is clear that it is $p_1(z)$ and $p_2(z)$ that become imaginary and that $R_-(z)$ will possess a branch cut near $z = -1$.

To sum up this section, the nonanalyticity in $S(h, q, \Delta)$ for $q > q_c$ as a function of increasing $\Delta_{c1}$, is due to the appearance of a branch cut in the factor $\sqrt{-4S^2 - 2p - \frac{q}{2}}$ in Eq. (78). For $\Delta > \Delta_{c1}$, we cannot analytically evaluate $S(h, q, \Delta)$. However, we are able to make an analytic approximation that is valid for $\Delta \gtrsim \Delta_{c1}$.

**Expansion near $\Delta_{c1}$**

In the present section, we invoke an approximation to $S(h, q, \Delta)$ that holds close to $\Delta_{c1}$ when $q$ is close to $q_c$ and $\Delta$ is close to $\Delta_{c1}$. The full gap-function integral is a sum of three terms:

\[
S(h, q, \Delta) = S_0 + S_+ + S_-,
\]

\[
S_0 = \int \frac{d^2p}{(2\pi)^2} \left( \frac{1}{2E_p} - \frac{1}{2e_p + \epsilon_b} \right) = \frac{1}{4\pi} \ln \frac{\epsilon_b}{\sqrt{\Delta^2 + \mu^2 - \hat{\mu}}},
\]

\[
S_{\pm} = -\int \frac{d^2p}{(2\pi)^2} \frac{n_F(E_{p\pm})}{2E_p}.
\]

Our approximation takes advantage of the fact that, for $\Delta \lesssim \Delta_{c1}$, the region where $E_{p+} < 0$ is small (allowing an analytic approximation to $S_+$) while the region where $E_{p-} < 0$ is large (precluding a simple analytic approximation to $S_-$). That is, the situation is as depicted as in the right panel of Fig. 6, where $E_{p+} < 0$ in the small blue region on the left and $E_{p-} < 0$ in the large blue region on the right.

Although directly finding $S_-$ in this regime is difficult, due to the Jensen theorem, we know that the full integral integral is simply $S = \frac{1}{4\pi} \ln \frac{4\epsilon_2}{\hat{q}}$, allowing us to extract $S_-$. Since $S_-$ is expected to be smooth near $\Delta_{c1}$, we can use this formula for $\Delta > \Delta_{c1}$ as well (where $S_+$ vanishes).

With this in mind, we now proceed to evaluate $S_+$ in the region $\Delta \lesssim \Delta_{c1}$. We define small parameters $\delta = \Delta - \Delta_{c1}$ and $\hat{q} = q - q_c$, and approximate $\Delta_{c1}$ by its form near $q_c$, given in Eq. (61). This implies

\[
\delta = \Delta - \frac{1}{\sqrt{2}}(\mu^2 - h^2)^{1/4} \hat{q}.
\]

For $\Delta \lesssim \Delta_{c1}$, the angle integral of $S_+$ is restricted to the immediate vicinity of $\theta = \pi$, allowing us to simultaneously expand in the small parameters $y = \cos \theta + 1$, $\hat{q}$, and $\delta$. We do this and then also drop terms of order $y^2$ in calculating $p_{\gamma1}$ and $p_{\gamma2}$.
Our quartic equation, expressed in terms of $\mu$ instead of $\hat{\mu}$, is

$$p^4 + \left(-\frac{1}{2}q^2 - 4\mu + 2q^2y - q^2y^2\right)p^2 + (4hq - 4hqu)p + 4\Delta^2 - 4h^2 + \frac{q^4}{16} - q^2\mu + 4\mu^2,$$  \hspace{1cm} (84)

with solutions given in Eq. (30) above, where $S$ is given by Eq. (35). We first get the quantity $\cos \frac{\phi}{\pi}$.

Using the definition of $\cos \phi$, we obtain (keeping all terms of order smallness squared but then omitting $O(y^2)$ terms)

$$\cos \phi = 1 + \frac{27\delta q^2(\hat{\theta} + \sqrt{2\hat{q}(\mu^2 - h^2)})}{64(\mu^2 - h^2)^{3/4}}.$$  \hspace{1cm} (85)

To get $\cos \frac{\phi}{\pi}$ to the same order, we simply need to multiply the second term (which is negative) by $1/9$. Using the same expansion for the other quantities in $S$, and then expanding the square root, we obtain:

$$S = \frac{1}{4}h\left(\Delta^2 + 8(\mu^2 - h^2) - \frac{(2\hat{q} + q_c)y}{\sqrt{\mu^2 - h^2}}\right),$$  \hspace{1cm} (86)

where we re-expressed it in terms of $\Delta$, although it is valid to leading order in the small parameters $\delta, q, \text{ and } y$.

Next, we analyze the square root of Eq. (30). Using the same approximation, we find for the argument of this square root:

$$-4S^2 - 2p - \frac{\hat{q}}{S} = \hat{q}\left(\hat{q} - \frac{4h^2y}{q_c\sqrt{\mu^2 - h^2}}\right) - \frac{2\Delta^2}{\sqrt{\mu^2 - h^2}},$$  \hspace{1cm} (87)

where simplifications came from the following relations among $h, \mu, \text{ and } q_c$:

$$q_c = \sqrt{2(\sqrt{\mu} + h - \sqrt{\mu} - h)},$$  \hspace{1cm} (88)

$$\sqrt{\mu + h} + \sqrt{\mu - h} = \frac{2\sqrt{2h}}{q_c},$$  \hspace{1cm} (89)

$$\sqrt{\mu + \sqrt{\mu^2 - h^2}} = \frac{2h}{q_c}.$$  \hspace{1cm} (90)

Now we have (recall the parameter $b = 0$ since there is no cubic term in Eq. (84)):

$$p_{1,2} = S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{\hat{q}}{S}},$$  \hspace{1cm} (91)

with $S$ given in Eq. (86) and the argument of the square root given by Eq. (87). In terms of $p_1$ and $p_2$, $S_+$ is:

$$S_+ = \frac{1}{4\pi^2}\int_0^\pi d\theta \ln \frac{p_1^2 - 2\hat{\mu} + 2E_{p_1}}{p_2^2 - 2\hat{\mu} + 2E_{p_2}},$$  \hspace{1cm} (92)

where $\theta_c$ is the point where $p_1$ and $p_2$ cross, occurring when the argument of the square root, Eq. (87), vanishes. In terms of $y_c = 1 + \cos \theta_c$, this is

$$y_c = -\frac{\delta q_c}{2\sqrt{2h^2\hat{q}}}(\sqrt{2\hat{\theta} + 2\hat{q}(\mu^2 - h^2)})^{1/4}.$$  \hspace{1cm} (93)

We can also express the argument of the square root in terms of $y_c$, as

$$-4S^2 - 2p - \frac{\hat{q}}{S} = \frac{4h^2\hat{q}(y_c - y)}{q_c\sqrt{\mu^2 - h^2}},$$  \hspace{1cm} (94)

Owing to the fact that $p_1$ and $p_2$ satisfy the quartic equation, it is easy to see that, for $\theta$ near $\pi$,

$$E_{p_{1,2}} = -(h + \frac{p_{1,2}\hat{q}}{2}\cos \theta).$$  \hspace{1cm} (95)
Plugging this into Eq. (92) makes it easier to Taylor expand the logarithm function to leading order in smallness. We Taylor expand to leading order in the simultaneous small parameters $\delta$, $\bar{q}$ and $y$, and then Taylor expand the logarithm again assuming $y - y_c$ is small. This leads to:

$$S_+ = \frac{1}{4\pi^2} \int_{\theta_c}^{\pi} d\theta \left( -8\sqrt{2}h(\mu^2 - h^2)^{3/4} \sqrt{\frac{2}{2\sqrt{\mu^2 - h^2}} \sqrt{y_c - y}} \right)$$

(96)

where we defined

$$a = 2\sqrt{2}\bar{q}(\mu^2 - h^2),$$

(97)

$$b = h^2 \left( \frac{2\sqrt{2}\mu}{q_c} - \frac{q_c}{\sqrt{2}} \right) = \frac{2\sqrt{2}h^2(\mu^2 - h^2)}{q_c},$$

(98)

which, in the parameter range of interest, are always positive.

The range of the integral is the vicinity of $\pi$. In this limit, we can write

$$y = 1 + \cos \theta \approx \frac{1}{2}(\pi - \theta)^2.$$ 

(99)

Then, defining $x = \pi - \theta$, we have $y = \frac{1}{2}x^2$ and $S_+$ is:

$$S_+ = -\frac{2}{\pi^2} \sqrt{\frac{\bar{q}}{q_c}} h(\mu^2 - h^2)^{3/4} \int_0^{x_c} dx \sqrt{\frac{x^2 - x^2}{a - bx^2}},$$

(100)

where we defined $x_c^2 = 2y_c$. We can evaluate the $x$ integral, assuming the denominator does not vanish:

$$\int_0^{x_c} dx \sqrt{\frac{x^2 - x^2}{a - bx^2}} = \frac{\pi}{2b} \left( 1 - \frac{1}{\sqrt{a}} \sqrt{a - bx_c^2} \right).$$

(101)

After plugging in and substituting the definition of $\delta$ in terms of $\Delta$, we obtain

$$S_+ = -\frac{\sqrt{\bar{q}q_c}(\mu^2 - h^2)^{1/4}}{2\sqrt{2}\pi h} \left( 1 - \sqrt{\frac{1}{2} + \frac{\Delta^2}{q_c^2(\mu^2 - h^2)}} \right),$$

(102)

$$= -\frac{\sqrt{\bar{q}q_c}(\mu^2 - h^2)^{1/4}}{2\sqrt{2}\pi h} \left( 1 - \sqrt{\frac{1}{2} + \frac{\Delta^2}{2\Delta_{c_1}^2}} \right),$$

(103)

where in the second line we used the definition of $\Delta_{c_1}$ to this order, i.e., Eq. (61). This expression is only valid to leading order in the simultaneous small parameters $\bar{q}$ and $\Delta - \Delta_{c_1}$. Clearly, the quantity in parentheses vanishes for $\Delta \to \Delta_{c_1}$; to leading order in small $\Delta - \Delta_{c_1}$ the quantity in parentheses is:

$$\left( 1 - \sqrt{\frac{1}{2} + \frac{\Delta^2}{2\Delta_{c_1}^2}} \right) \simeq \frac{1}{2} \left( 1 - \frac{\Delta}{\Delta_{c_1}} \right).$$

(104)

Using this, and again using Eq. (61), we obtain:

$$S_+ = \frac{1}{4\pi h} \sqrt{\frac{\bar{q}q_c}{\bar{q}}} (\Delta - \Delta_{c_1}).$$

(105)

For $\Delta \gtrsim \Delta_{c_1}$

$$S \text{ for } \Delta \gtrsim \Delta_{c_1}$$

$$S = \frac{1}{4\pi} \ln \frac{4\epsilon h}{q} = S_+ + S_- + S_0,$$ 

(106)
so that, in the vicinity of $\Delta_{c1}$,

$$S_- = \frac{1}{4\pi} \ln \frac{4\epsilon_b}{q^2} - S_0 - S_+,$$

$$= \frac{1}{4\pi} \ln \frac{4\epsilon_b}{q^2} - S_0 - \frac{1}{4\pi h} \sqrt{\frac{q_c}{q}} (\Delta - \Delta_{c1}),$$

where we used Eq. (105). For $\Delta > \Delta_{c1}$, $S_+$ is identically zero. However, there is no reason for $S_-$ to have any change in behavior at $\Delta_{c1}$, and we expect it to still be given by Eq. (108). Including the other terms in $S$ then gives:

$$S(h, q, \Delta) = \frac{1}{4\pi} \ln \frac{4\epsilon_b}{q^2} - \frac{1}{4\pi h} \sqrt{\frac{q_c}{q}} (\Delta - \Delta_{c1}),$$

so that the stationary pairing amplitude, $S(h, q, \Delta) = 0$, is:

$$\Delta = \Delta_{c1} + h \sqrt{\frac{q}{q_c}} \ln \frac{4\epsilon_b}{q^2}.$$

This determines the stationary pairing amplitude close to the phase transition, in terms of the FFLO wavevector $q$. To get $\Delta$ vs. $h$ close to the transition, we need the stationarity condition for the FFLO wavevector.

**Ground state energy**

To obtain the stationarity condition for the FFLO wavevector we need the ground-state energy. We know that

$$E_G = -|\Delta|^2 \frac{1}{\lambda} - \sum_p (E_p - \xi_p) + \sum_{p, \alpha = \pm} E_{p\alpha} \Theta(-E_{p\alpha}),$$

which is independent of the wavevector $q$ when $\Delta = 0$ (as it must be). Upon differentiating with respect to $\Delta$, we obtain:

$$\frac{dE_G}{d\Delta} = -2\Delta \left[ \frac{1}{\lambda} + \int \frac{d^2p}{(2\pi)^2} \left[ \frac{1 - n_F(E_{p+}) - n_F(E_{p-})}{E_{p+} + E_{p-}} \right] \right] = -2\Delta S(\Delta)$$

where it is understood that the Fermi functions are evaluated in the $T = 0$ limit. We can therefore obtain $E_G$, up to a $\Delta$ and $q$-independent constant, by integrating $S$ from 0 to $\Delta$. We have:

$$E_G = -2 \int_0^\Delta d\Delta' \Delta' S(h, q, \Delta'),$$

$$= -2 \int_0^{\Delta_{c1}} d\Delta' \Delta' S(h, q, \Delta') - 2 \int_{\Delta_{c1}}^\Delta d\Delta' \Delta' S(h, q, \Delta'),$$

where we will henceforth assume $\Delta > \Delta_{c1}$. The first integral in Eq. (114) is trivial, because $S = \frac{1}{4\pi} \ln \frac{4\epsilon_b}{q^2}$ for $\Delta < \Delta_{c1}$:

$$-2 \int_0^{\Delta_{c1}} d\Delta' \Delta' S(\Delta') = \frac{1}{4\pi} \Delta_{c1}^2 \ln \frac{q^2}{4\epsilon_b},$$

while for the second integral we’ll use our approximate result Eq. (109) along with

$$\int_{\Delta_{c1}}^\Delta d\Delta' \Delta' (\Delta' - \Delta_{c1}) = \frac{1}{6} (\Delta - \Delta_{c1})^2 (2\Delta + \Delta_{c1}),$$

to get:

$$-2 \int_{\Delta_{c1}}^\Delta d\Delta' \Delta' S(h, q, \Delta') = \frac{1}{4\pi} (\Delta^2 - \Delta_{c1}^2) \ln \frac{q^2}{4\epsilon_b} + \frac{1}{12\pi h} \sqrt{\frac{q_c}{q - q_c}} (\Delta - \Delta_{c1})^2 (2\Delta + \Delta_{c1}),$$
leading to:

\[
E_G = \frac{1}{4\pi} \Delta^2 \ln \frac{q^2}{4\epsilon_b} + \frac{1}{12\pi \hbar} \sqrt{\frac{q_c}{q - q_c}} (\Delta - \Delta_{c1})^2 (2\Delta + \Delta_{c1}) ,
\]

where in the second line we see that \( E_G \) does not contain a term linear in \( \Delta \) (as one might have expected from the way we wrote the polynomial in the first line).

Analytically finding the stationary FFLO wavevector, \( q \), turns out to be somewhat tricky. We’ll obtain an approximate formula for \( q \) that is valid near the continuous FFLO transition. We find it most convenient to differentiate the first line of \( E_G \), keeping \( \Delta \) arbitrary (i.e., not necessarily the stationary solution). This leads to:

\[
\frac{\partial E_G}{\partial q} = \frac{1}{96\pi \hbar q^2(q_c + \tilde{q})} \left[ -8\Delta^3 \sqrt{q_c q(q_c + \tilde{q})} + 5\sqrt{2} \sqrt{q_c q} \tilde{q}^{7/2}(q_c + \tilde{q})(\mu^2 - h^2)^{3/4} 
+ 6\Delta^2 \tilde{q}^{3/2}(8h\sqrt{\tilde{q}} - \sqrt{2} \sqrt{q_c q}(q_c + \tilde{q})(\mu^2 - h^2)^{1/4}) \right]
\]

where we inserted the explicit expression for \( \Delta_{c1} \) and we recall that \( \tilde{q} \equiv q - q_c \). The stable FFLO wavevector satisfies \( \frac{\partial E_G}{\partial q} = 0 \), but solving this generally is an arduous task. In the vicinity of the transition, the stationary \( \Delta \) and \( \tilde{q} \) both vanish linearly with \( h - h_{FFLO} \). Thus it makes sense to simultaneously expand the quantity in square brackets in small \( \Delta \) and \( \tilde{q} \). If we formally replace \( \Delta \rightarrow \lambda \Delta \) and \( \tilde{q} \rightarrow \lambda \tilde{q} \) and then keep only the \( \mathcal{O}(\lambda^{7/2}) \) terms we get

\[
0 = \Delta^3 + \frac{3}{2\sqrt{2}} \Delta^2 (\mu^2 - h^2)^{3/4} \tilde{q} - \frac{5}{4\sqrt{2}} (\mu^2 - h^2)^{3/4} \tilde{q}^3,
\]

a cubic equation that we can rewrite as [using Eq. (61) for \( \Delta_{c1} = \frac{1}{\sqrt{2}} (\mu^2 - h^2)^{1/4} \tilde{q} \)]

\[
0 = \Delta^3 + \frac{3}{2} \Delta^2 \Delta_{c1} - \frac{5}{2} \Delta_{c1}^3,
\]

and that has one real solution, given by

\[
\Delta = \Delta_{c1},
\]

equivalent to

\[
\tilde{q} = \frac{\sqrt{2} \Delta}{(\mu^2 - h^2)^{1/4}}.
\]

Thus a leading order analysis finds that \( q \) is adjusted to keep \( \Delta \) equal to \( \Delta_{c1} \). However, this solution is unsatisfactory, because we know the correct solution for \( \Delta \) is slightly above \( \Delta_{c1} \), and we are interested in this small difference.

To derive this, we simply need to include the next-order term the expansion of the small parameter \( \lambda \) [upon rescaling \( \Delta \rightarrow \lambda \Delta \) and \( \tilde{q} \rightarrow \lambda \tilde{q} \) in the numerator of Eq. (120)]. This term is \( \mathcal{O}(\lambda^3) \), given by \( 48\Delta^2 h \tilde{q} \). After simplifying, we obtain the revised stationarity condition for the FFLO wavevector

\[
0 = \Delta^3 + \frac{3}{2} \Delta^2 \Delta_{c1} - \frac{5}{2} \Delta_{c1}^3 - \frac{6\Delta^2 h}{q_c^3/2} \tilde{q}^{3/2}.
\]

We then proceed by assuming \( \tilde{q} = \frac{\sqrt{2} \Delta}{(\mu^2 - h^2)^{1/4}} + \epsilon \). Plugging this into Eq. (125), keeping only terms up to \( \mathcal{O}(\epsilon) \), solving for \( \epsilon \), and dropping subdominant terms in \( \Delta \), we obtain:

\[
\tilde{q} = \frac{\sqrt{2} \Delta}{(\mu^2 - h^2)^{1/4}} - \frac{25/4 h (\Delta_{c1})^{3/2}}{(\mu^2 - h^2)^{5/8}}.
\]
Solving the remaining equations

We are now left with the following two equations for $\bar{q}$ and $\Delta$:

\[
\bar{q} = \sqrt{2} \Delta \left( \frac{\mu^2 - h^2}{(\mu^2 - h^2)^{1/4}} \right)^{3/2} \left( \frac{h^5}{h^5} \right)^{3/8},
\]

\[
\Delta = \Delta_{c1} + h \sqrt{\frac{\mu}{q_c}} \ln \frac{4\epsilon_b}{q^2},
\]

where $\Delta_{c1}$ is given by Eq. (61).

To simplify our analysis, we define

\[
\Delta = ax + bx^{3/2},
\]

\[
\bar{q} = cx + dx^{3/2},
\]

\[
\tilde{q} = ex + fx^{3/2},
\]

with $x \equiv h_{FFLO} - h$ is the distance to the transition (our small parameter). Recall $h_{FFLO} = \epsilon_b \sqrt{2\mu/\epsilon_b - 1}$. Note that, although we only obtain a result valid to $O(x)$, to obtain this we need to keep terms up to $O(x^{3/2})$.

Here, $\tilde{q} = q_{FFLO} - q > 0$, with $q_{FFLO} = 2\sqrt{\epsilon_b}$. Clearly, $\bar{q}$ and $\tilde{q}$ are not independent since they both contain $q$ and can be related by Taylor expanding $q_{FFLO}$ in small $x$. Doing this leads to the relation

\[
e = \frac{h_{FFLO}}{\sqrt{2} \epsilon_b (\mu - \epsilon_b)} - c.
\]

We now plug Eqs. (129), (130), and (131) into Eqs. (127) and (128), expanding all quantities to $O(x^{3/2})$ and demanding equality order by order in $x$. We find:

\[
c = \frac{\sqrt{2}a}{\sqrt{\mu - \epsilon_b}},
\]

\[
d = \frac{\sqrt{2}b}{\sqrt{\mu - \epsilon_b}} - \frac{a^{3/2} \sqrt{2\mu - \epsilon_b}}{2^{1/4} \epsilon_b^{1/4} (\mu - \epsilon_b)^{1/4}},
\]

\[
b = \frac{1}{\sqrt{2}} d \sqrt{\mu - \epsilon_b} + \frac{1}{\sqrt{2}} \epsilon_b^{1/4} \sqrt{c} \sqrt{\frac{2\mu}{\epsilon_b}} - 1,
\]

that lead to the final results

\[
a = \frac{\sqrt{2}}{3} \frac{h_{FFLO}}{\sqrt{\epsilon_b (\mu - \epsilon_b)}},
\]

\[
e = \frac{1}{3} \frac{h_{FFLO}}{\sqrt{\epsilon_b (\mu - \epsilon_b)}},
\]

implying, for $h \to h_{FFLO}$,

\[
\Delta \simeq \frac{\sqrt{2}}{3} \frac{h_{FFLO}}{\sqrt{\epsilon_b (\mu - \epsilon_b)}} (h_{FFLO} - h),
\]

\[
q - 2\sqrt{\epsilon_b} \simeq -\frac{1}{3} \frac{h_{FFLO}}{\sqrt{\epsilon_b (\mu - \epsilon_b)}} (h_{FFLO} - h),
\]

that are used in the main text.