Sticky Brownian Motions and a Probabilistic Solution to a Two-Point Boundary Value Problem

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Abstract
In this paper, we study a two-point boundary value problem consisting of the heat equation on the open interval (0, 1) with boundary conditions which relate first and second spatial derivatives at the boundary points. Moreover, the unique solution to this problem can be represented probabilistically in terms of a sticky Brownian motion. This probabilistic representation is attained from the stochastic differential equation for a sticky Brownian motion on the bounded interval [0, 1].

Keywords Two-point boundary value problem · Sticky Brownian motions

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1 Introduction
Let $u_0 : [0, 1] \rightarrow [0, 1]$. We try to look for a bounded solution $u \in C^{2,1}([0, 1] \times (0, \infty))$ solving the following problem

$$u_t = \frac{1}{2}u_{rr}, \quad \lim_{t \downarrow 0} u(r, t) = u_0(r), \text{ for } r \in (0, 1), \quad (1.1)$$

$$\frac{d}{dt} u(0, t) = \frac{1}{2}u_r(0, t), \quad \frac{d}{dt} u(1, t) = -\frac{1}{2}u_r(1, t). \quad (1.2)$$

Since it is shown in Chapter 6 and Chapter 7 of [7] that there exists a Markov process $\mathbf{B}$ having a generator $A = \frac{1}{2}d^2/dx^2$ on $(0, 1)$ with extension by continuity to the points 0 and 1 and restriction to the domain

$$\mathcal{D}(A) = \left\{ f \in C^2([0, 1]), f''(x) + (-1)^{1-x}f'(x) = 0 \text{ for } x \in \{0, 1\} \right\},$$

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then if \( u_0 \in C^2([0, 1]) \), one can verify that the unique bounded solution to the above problem is given probabilistically, for \((r, t) \in [0, 1] \times (0, \infty)\), by

\[
u(r, t) = E_r [u_0(B(t))],
\]

where \( E_r \) stands for the expectation with respect to the process \( B \) starting from \( r \).

However, when the continuity at the boundary points 0 and 1 of the initial datum \( u_0 \) is relaxed, let us consider the heat equation along with the initial datum (1.1) and the Dirichlet boundary conditions

\[
u(0, t) = v_{0,-}, \quad u(1, t) = v_{0,+} \text{ for } t > 0, \tag{1.3}
\]

where \( v_{0, \pm} \in [0, 1] \). This boundary value problem has a unique solution which can be represented by means of a Brownian motion \( B \) absorbed at 0 and 1 as follows

\[
u(r, t) = E_r [u_0(B(t))1_{\tau > t}] + v_{0,-} P_r (\tau = \tau_0 \leq t) + v_{0,+} P_r (\tau = \tau_1 \leq t), \tag{1.4}
\]

where \( \tau_a \) is the first time when the Brownian motion \( B \) hits \( a \) and \( \tau = \tau_0 \wedge \tau_1 \). Since \( B \) is absorbed whenever it reaches 0 and 1, we are just interested in the boundary conditions of the form (1.3).

In [11], Pang and Stroock investigate the existence of the solution to the heat equation with the initial datum (1.1) and the boundary conditions (1.2) under the assumption on the discontinuity of the initial datum \( u_0 \) at the boundaries. For the boundary conditions (1.2), we are concerned about the mass flux at each boundary. The solution in \( C^2([0, 1] \times (0, \infty)) \) established through a Brownian motion sticky at 0 and 1 is unique if it satisfies further that

\[
\lim_{t \downarrow 0} u(0, t) = v_{0,-}, \quad \lim_{t \downarrow 0} u(1, t) = v_{0,+}. \tag{1.5}
\]

The reason for the requirement (1.5) is that the sticky Brownian motion spends a positive amount of time at 0 and 1 with positive probability.

Moreover, suppose that at each boundary point 0 and 1, there is a reservoir of mass \( v_{\pm}(t) \in C[0, \infty) \) changing in time. The boundary conditions (1.2) describe that the mass flux at 0 and 1 equals the mass change in the left and right reservoir respectively. If we identify \( v_-(t) = u(0, t) \) and \( v_+(t) = u(1, t) \) then it can be checked that

\[
\int_0^1 u(r, t) dr + v_-(t) + v_+(t) = \int_0^1 u_0(r) dr + v_{0,-} + v_{0,+}, \forall t > 0.
\]

This implies the conservation of mass.

Now in the current paper, we will examine an analogous boundary value problem but depending on a parameter \( \epsilon > 0 \). The role of \( \epsilon \) will be explained later. For any fixed \( \epsilon > 0 \), let us consider the heat equation with the initial datum \( u_0^\epsilon \in C(0, 1) \) taking values in \([0, 1]\) as follows

\[
u_\epsilon^\epsilon = \frac{1}{2} \nu_\epsilon^{\epsilon \epsilon}, \quad \lim_{t \downarrow 0} u_\epsilon^\epsilon(r, t) = u_0^\epsilon(r). \tag{1.6}
\]

Let us impose two reservoirs of masses \( \rho^\epsilon_\pm (t) \in C[0, \infty) \) at each boundary. We observe that if \( \rho^\epsilon_\pm (0, \cdot) \) is larger than \( \rho_+^\epsilon (\cdot) \), the rate of the mass change in the left reservoir increases in time. Moreover, the larger this difference is, the faster the rate
increases. We will use the factor $\epsilon^{-1}$ to emphasize this property. Although $\rho_\epsilon(\cdot)$ may be different from $u^\epsilon(0, \cdot)$, the difference between them becomes 0 as $\epsilon$ goes to 0. An analogous phenomenon also happens at the boundary point 1. Then for a parameter $\epsilon > 0$, this fact can be described in a rigorous way

\[
\frac{d}{dt} \rho_\epsilon(t) = \epsilon^{-1} \left[ u^\epsilon(0, t) - \rho_\epsilon(t) \right], \quad (1.7)
\]

\[
\frac{d}{dt} \rho_\pm(t) = \epsilon^{-1} \left[ u^\epsilon(1, t) - \rho_\pm(t) \right]. \quad (1.8)
\]

Furthermore, since we will look for a solution in $C^{2,1}([0, 1] \times (0, \infty))$, the boundary conditions that imply the conservation of mass are expressed in a weak form

\[
\rho_\epsilon(t) = \rho_\epsilon(0) + \lim_{l \to 0^+, t_0 \downarrow 0} \int_{l_0}^t \frac{1}{2} u_\epsilon(l, s) \, ds, \quad (1.9)
\]

\[
\rho_\pm(t) = \rho_\pm(0) - \lim_{l \to 1^-, t_0 \downarrow 0} \int_{l_0}^t \frac{1}{2} u_\epsilon(l, s) \, ds. \quad (1.10)
\]

For a fixed $\epsilon > 0$, if we require that $u_\epsilon(0, \cdot), u_\epsilon(1, \cdot) : [0, \infty) \to [0, 1]$ and $u_\epsilon(0, \cdot), u_\epsilon(1, \cdot) \in C([0, \infty))$, then there exist unique solutions $u^\epsilon \in C^{2,1}([0, 1] \times (0, \infty)), \rho^\epsilon_\pm \in C^1([0, \infty)) \cap C([0, \infty))$ to the boundary value problem (1.6)-(1.10). The first part of the current paper is devoted to arrive at this result. Next, by the suitably chosen initial data, identifying the unique limits of these solutions as $\epsilon \to 0$ and investigating their regularities lead us to the existence and uniqueness of the solution $u \in C^{2,1}([0, 1] \times (0, \infty))$ satisfying (1.1)-(1.5).

Moreover, in the second part of this paper, based on the fact that a sticky Brownian motion on the half line $[0, \infty)$ solves a stochastic differential equation as verified in [5], we will give an analogous characterization for a sticky Brownian motion on the bounded interval $[0, 1]$. This allows us to represent the unique solution to the two-point boundary value problem (1.1)-(1.5) probabilistically in terms of a sticky Brownian motion by applying Ito’s formula.

**Remark 1.1** In [10], we consider a system of particles moving according to the simple symmetric exclusion process in the channel $[1, N]$ with reservoirs at the boundaries. The reservoirs of size $N$ are also particle systems which can be exchanged with the ones in the channel. The hydrodynamic limit equation we obtain for this particle system is the two-point boundary value problem mentioned above. From the physical point of view, the unique solution to this problem is the limit of a sequence of the one-body correlation functions for an appropriately constructed interacting particle system. Furthermore, by duality technique, one can also express the correlation function in terms of a sticky random walk. Since the convergence of a sequence of rescaled sticky random walks to a sticky Brownian motion can be shown based on the arguments presented in [1], it leads us to a probabilistic representation of this unique solution.
2 A Two-Point Boundary Value Problem

Let us set \( U := C^{2,1}((0, 1) \times (0, \infty)) \) and \( H := C^1(0, \infty) \cap C([0, \infty)) \).

**Theorem 2.1** For any fixed \( \varepsilon > 0 \), let \( u_0^\varepsilon \in C(0, 1) \) with values in \( [0, 1] \) and \( \rho_\pm^\varepsilon(0) = v_{0,\pm}^\varepsilon \in [0, 1] \). If \( u^\varepsilon(0, \cdot), u^\varepsilon(1, \cdot) : [0, \infty) \to [0, 1] \) and \( u^\varepsilon(0, \cdot), u^\varepsilon(1, \cdot) \in C(0, \infty) \), then there exists a unique \( (u^\varepsilon, \rho_-^\varepsilon, \rho_+^\varepsilon) \in U \times H \times H \) which satisfies the following problem

\[
\begin{align*}
    u_t^\varepsilon &= \frac{1}{2} u_{rr}^\varepsilon, \quad \lim_{t \downarrow 0} u^\varepsilon(r, t) = r \in (0, 1), \\
    \frac{d}{dt} \rho_-^\varepsilon(t) &= \varepsilon^{-1} [u^\varepsilon(0, t) - \rho_-^\varepsilon(t)], \\
    \frac{d}{dt} \rho_+^\varepsilon(t) &= \varepsilon^{-1} [u^\varepsilon(1, t) - \rho_+^\varepsilon(t)], \\
    \rho_-^\varepsilon(t) &= \rho_-^\varepsilon(0) + \lim_{l \to 0} \lim_{t \downarrow 0} \int_l^t \frac{1}{2} u^\varepsilon(l, s) ds, \\
    \rho_+^\varepsilon(t) &= \rho_+^\varepsilon(0) - \lim_{l \to 1} \lim_{t \downarrow 0} \int_l^t \frac{1}{2} u^\varepsilon(l, s) ds.
\end{align*}
\]  

The existence and uniqueness of \( (u^\varepsilon, \rho_\pm^\varepsilon) \in U \times H \times H \) can be shown by applying the same technique as presented in Proposition 3.10, [10], where we aim to arrive at an integral equation and then construct its unique solution inductively by using the contraction mapping theorem.

**Proof** For \( (r, t) \in [0, 1] \times (0, \infty) \), we denote

\[
\Theta(r, t) = \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(r+2n)^2}{2t}}.
\]

Then as a result of Theorem 6.3.1, [4], for any fixed \( \varepsilon > 0 \), \( (r, t) \in (0, 1) \times (0, \infty) \), the function \( u^\varepsilon \in C^{2,1}((0, 1) \times (0, \infty)) \) defined by the following expression

\[
\begin{align*}
    u^\varepsilon(r, t) &= \int_0^1 u_0^\varepsilon(r') \left[ \Theta(r - r', t) - \Theta(r + r', t) \right] dr' \\
    &\quad - \int_0^t \frac{\partial \Theta}{\partial r}(r, t-s) u^\varepsilon(0, s) ds + \int_0^t \frac{\partial \Theta}{\partial r}(r-1, t-s) u^\varepsilon(1, s) ds
\end{align*}
\]  

satisfies the linear heat (2.1) with boundary values \( u^\varepsilon(0, \cdot), u^\varepsilon(1, \cdot) \) and the initial datum \( u_0^\varepsilon \).

Thus (2.4) and (2.5) give us

\[
\begin{align*}
    \rho_-^\varepsilon(t) &= v_{0, -}^\varepsilon + \lim_{l \to 0} \lim_{t \downarrow 0} \int_l^t \frac{1}{2} \int_0^1 u_0^\varepsilon(r') \left[ \frac{\partial \Theta}{\partial r}(l - r', s) - \frac{\partial \Theta}{\partial r}(l + r', s) \right] dr' ds \\
    &\quad - \int_0^t \Theta(0, t-s) u^\varepsilon(0, s) ds + \int_0^t \Theta(1, t-s) u^\varepsilon(1, s) ds,
\end{align*}
\]  

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and similarly,

\[ \rho^+(t) = v_0^+ + \lim_{l \to 1} \lim_{t_0 \downarrow 0} \int_0^t \frac{1}{2} \int_0^1 u_0^\epsilon(r') \left[ \frac{\partial \Theta}{\partial r} (l - r', s) - \frac{\partial \Theta}{\partial r} (l + r', s) \right] dr'ds \\
+ \int_0^t \Theta(1, t - s) u^\epsilon(0, s) ds - \int_0^t \Theta(0, t - s) u^\epsilon(1, s) ds. \] (2.8)

On the other hand, since it follows from (2.2) and (2.3) that

\[ \rho^-(t) = e^{-\epsilon^{-1}(t-t_0)} \rho^-(t_0) + \int_{t_0}^t e^{-\epsilon^{-1}(t-s)} u^\epsilon(0, s) ds, \] (2.9)
\[ \rho^+(t) = e^{-\epsilon^{-1}(t-t_0)} \rho^+(t_0) + \int_{t_0}^t e^{-\epsilon^{-1}(t-s)} u^\epsilon(1, s) ds. \] (2.10)

then we obtain the following system

\[
\begin{align*}
\int_0^t \left[ \Theta(0, t - s) + \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} \right] u^\epsilon(0, s) ds + \int_0^t - \Theta(1, t - s) u^\epsilon(1, s) ds &= f^\epsilon_x(t) + \left. v_{0,-} \right|_0 \left( 1 - e^{-\epsilon^{-1}t} \right) \\
\int_0^t - \Theta(1, t - s) u^\epsilon(0, s) ds + \int_0^t \left[ \Theta(0, t - s) + \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} \right] u^\epsilon(1, s) ds &= f^\epsilon_x(t) + \left. v_{0,+} \right|_0 \left( 1 - e^{-\epsilon^{-1}t} \right),
\end{align*}
\]

where

\[ f^-_x(t) = \lim_{l \to 0} \lim_{t_0 \downarrow 0} \int_0^t \frac{1}{2} \int_0^1 u_0^\epsilon(r') \left[ \frac{\partial \Theta}{\partial r} (l - r', s) - \frac{\partial \Theta}{\partial r} (l + r', s) \right] dr'ds, \]
\[ f^+_x(t) = - \lim_{l \to 1} \lim_{t_0 \downarrow 0} \int_0^t \frac{1}{2} \int_0^1 u_0^\epsilon(r') \left[ \frac{\partial \Theta}{\partial r} (l - r', s) - \frac{\partial \Theta}{\partial r} (l + r', s) \right] dr'ds. \]

Now multiplying both sides of the first equation of the above system by \((x - t)^{-1/2}\) and integrating with respect to \(t\) from 0 to \(x\) yield

\[
\int_0^x \sqrt{\frac{\pi}{2}} u^\epsilon(0, s) ds \\
+ \int_0^x \left[ \int_0^1 \sum_{n \geq 1} \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{y(1-y)}} e^{-\frac{(2n)^2}{2(y-1)}} dy + \int_0^x \frac{\epsilon}{\sqrt{x-t}} e^{-\epsilon^{-1}(t-s)} dt \right] u^\epsilon(0, s) ds \\
+ \int_0^x \left[ \int_0^1 - \sum_{n \geq 1} \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{y(1-y)}} e^{-\frac{(2n-1)^2}{2(y-1)}} dy \right] u^\epsilon(1, s) ds \\
= \int_0^x \sqrt{\frac{\pi}{x-t}} \left( f^\epsilon_x(t) + \left. v_{0,-} \right|_0 \left( 1 - e^{-\epsilon^{-1}t} \right) \right) dt.
\] (2.12)
If \( u^\varepsilon(0, \cdot), u^\varepsilon(1, \cdot) \in C(0, \infty) \) and take values in \([0, 1]\) on \([0, \infty)\) then this enables us to define for \( x > 0, \)
\[
\psi_-(x) = \int_0^x u^\varepsilon(0, s) \, ds, \quad \psi_+(x) = \int_0^x u^\varepsilon(1, s) \, ds
\]
and thus
\[
\frac{d}{dx} \psi_-(x) = u^\varepsilon(0, x), \quad \frac{d}{dx} \psi_+(x) = u^\varepsilon(1, x).
\]

For \( \alpha := \sqrt{2/\pi}, \) applying integration by parts in (2.12) gives us
\[
\psi_-(x) = \int_0^x \alpha \left[ \sum_{n \geq 1} \frac{-2}{\sqrt{2\pi}} \frac{1}{\sqrt{y(1-y)}} e^{\frac{(2n)^2}{2y^2(x-s)^2}} \left( \frac{2n}{2y(x-s)^2} \right) dy \right. \\
- \int_0^x 2e^{-2} \sqrt{x-s} - \int_s^x 2e^{-3} \sqrt{x-t} e^{-\frac{1}{2}(t-s)} \right] \psi_-(s) \, ds \\
+ \int_0^x \alpha \left[ \sum_{n \geq 1} \frac{2\alpha}{\sqrt{2\pi}} \frac{1}{\sqrt{y(1-y)}} e^{\frac{(2n-1)^2}{2y^2(x-s)^2}} \left( \frac{2n-1}{2y(x-s)^2} \right) dy \right] \psi_+(s) \, ds \\
+ \int_0^x \frac{\alpha}{\sqrt{x-t}} \left[ f^\varepsilon(t) + v_{0,-} \left( 1 - e^{-\frac{1}{2}t} \right) \right] dt.
\]

Making a similar argument as above for the second equation of the system (2.11) leads us to consider the equation
\[
\begin{bmatrix}
\psi_-(x) \\
\psi_+(x)
\end{bmatrix} = \begin{bmatrix}
F_-(x) \\
F_+(x)
\end{bmatrix} + \int_0^x \begin{bmatrix}
K_-(x-s) \\
K_+(x-s)
\end{bmatrix} \begin{bmatrix}
\psi_-(s) \\
\psi_+(s)
\end{bmatrix} ds,
\]
(2.13)
where
\[
K\varepsilon_-(t) = \alpha \left[ \sum_{n \geq 1} \frac{-2}{\sqrt{2\pi}} \frac{1}{\sqrt{y(1-y)}} e^{\frac{(2n)^2}{2y^2t^2}} \left( \frac{2n}{2yt^2} \right) dy \right. \\
- \int_0^t 2e^{-2} \sqrt{t-s} - \int_0^t 2e^{-3} \sqrt{t-\sigma} e^{-\frac{1}{2}\sigma} d\sigma \right],
\]
\[
K\varepsilon_+(t) = \int_0^t \sum_{n \geq 1} \frac{-2\alpha}{\sqrt{2\pi}} \frac{1}{\sqrt{y(1-y)}} e^{\frac{(2n-1)^2}{2y^2t^2}} \left( \frac{2n-1}{2yt^2} \right) dy,
\]
\[
F\varepsilon_+(x) = \int_0^x \frac{\alpha}{\sqrt{x-t}} \left[ f^\varepsilon(t) + v_{0,-} \left( 1 - e^{-\frac{1}{2}t} \right) \right] dt.
\]

Applying the same technique as introduced in Proposition 3.10, [10], one can verify the following result.

**Proposition 2.2** The (2.13) has a unique solution \((\psi_-, \psi_+) \in C(0, T) \times C(0, T)\) for any \(T > 0\).

As a consequence of the above result, there exists a unique solution \((u^\varepsilon(0, \cdot), u^\varepsilon(1, \cdot))\) to the system (2.11) for any \(\varepsilon > 0\) fixed. Then by the expressions
(2.6), (2.9) and (2.10), we obtain the unique existence of the solution \((u^\epsilon, \rho^-^\epsilon, \rho^+^\epsilon) \in U \times H \times H\) to our main problem.

Let us now identify the limits of sequences of functions \(u^\epsilon, \rho^\pm\) as \(\epsilon\) goes to 0 up to subsequences. Since the uniqueness of the limits can be verified, we obtain the identification of the limits \(u, v^\pm\) for the original sequences. Moreover, the boundary conditions can be attained in a strong form in view of the continuous differentiability of \(v^\pm\). More precisely, the limit \(u \in C^{2,1}([0, 1] \times (0, \infty))\) is the unique solution to a two-point boundary value problem.

**Theorem 2.3** Let \(u_0 \in C(0, 1)\) with values in \([0, 1]\) and \(v_{0, \pm} \in [0, 1]\). There exists a unique \(u \in C^{2,1}([0, 1] \times (0, \infty))\) which solves the following problem

\[
\frac{d}{dt} v_-^-(t) = \frac{1}{2} u_r(0, t), \quad \frac{d}{dt} v_+^+(t) = -\frac{1}{2} u_r(1, t), \quad \lim_{t \downarrow 0} v_\pm(t) = v_{0, \pm}. \tag{2.15}
\]

**Proof** For any fixed \(\epsilon > 0\), let \((u^\epsilon, \rho^\pm^\epsilon)\) be the unique solution obtained in Theorem 2.1. At the initial time, the sequence \(u_0^\epsilon\) is chosen such that it converges uniformly to \(u_0\) on any compact set of \((0, 1)\) as \(\epsilon \to 0\). Moreover, let us select the sequences \(v_{0, \pm}^\epsilon\) which converge to \(v_{0, \pm}\), respectively, as \(\epsilon \to 0\).

We observe that from (2.7) and (2.8), for any \(\delta, T > 0\), there exists a constant \(C\) such that for any \(s, t \in [\delta, T]\),

\[
|\rho_-^\epsilon(t) - \rho_-^\epsilon(s)| \leq C|t - s|,
\]

thus the sequences \((\rho^\pm^\epsilon)\) are uniformly equicontinous. Moreover, (2.9) and (2.10) imply the uniform boundedness of these sequences on \([\delta, T]\). Therefore, there exist subsequences \(\rho^\pm^\epsilon\) converging uniformly on \([\delta, T]\) to \(v^\pm\), respectively. Notice that \(v^\pm \in C(0, \infty)\).

On the other hand, by (2.9), we claim that for any \(t > 0\),

\[
\lim_{\epsilon \to 0} \left| \rho_-^\epsilon(t) - u^\epsilon(0, t) \right| = 0.
\]

Indeed, since \(u^\epsilon(0, \cdot)\) is continuous at \(t > 0\) then there exists \(\delta' > 0\) depending on \(\epsilon\) and \(t\) such that \(|u^\epsilon(0, s) - u^\epsilon(0, t)| < \epsilon\) for any \(s \in [t - \delta', t]\). Therefore, our claim follows from the following estimate

\[
\begin{align*}
&\left| \rho_-^\epsilon(t) - u^\epsilon(0, t) \right| \\
&\leq e^{-\epsilon^{-1}t} \left| v_{0, \pm} - u^\epsilon(0, t) \right| + \int_0^t e^{-\epsilon^{-1}(t-s)} \left| u^\epsilon(0, s) - u^\epsilon(0, t) \right| ds \\
&\leq e^{-\epsilon^{-1}t} + \int_{t-\delta'}^t e^{-\epsilon^{-1}(t-s)} ds + e \int_{t-\delta'}^t e^{-\epsilon^{-1}(t-s)} ds.
\end{align*}
\]
Analogously, it can be obtained from (2.10) that for any $t > 0$,

$$\lim_{\epsilon \to 0} \left| \rho^\epsilon_\xi(t) - u^\epsilon(1, t) \right| = 0,$$

then the subsequences $u^{\epsilon_k}(0, \cdot), u^{\epsilon_k}(1, \cdot)$ converge pointwise on $(0, \infty)$ to $v_\pm$, respectively.

Hence, for any $(r, t) \in (0, 1) \times (0, \infty)$, the corresponding subsequence $u^{\epsilon_k}$ converges to $u$ given by

$$u(r, t) = \int_0^1 u_0 \left( r' \right) \left[ \Theta \left( r - r', t \right) - \Theta \left( r + r', t \right) \right] dr'$$

$$- \int_0^t \frac{\partial \Theta}{\partial r} \left( r, t - s \right) v_-(s) ds + \int_0^t \frac{\partial \Theta}{\partial r} \left( r - 1, t - s \right) v_+(s) ds. \quad (2.16)$$

One can easily check that $u \in C^{2,1}((0, 1) \times (0, \infty))$. As already shown in Chapter 6 of [4], the limit $u$ solves the linear heat equation with boundary values $v_\pm$ and the initial datum $u_0$. Moreover, we observe that

$$\lim_{t \downarrow 0} v_\pm(t) = \lim_{t \downarrow 0} \lim_{k \to \infty} \rho^{\epsilon_k}_\pm(t) = \lim_{k \to \infty} \rho^{\epsilon_k}_\pm(t)$$

$$= \lim_{k \to \infty} \rho^{\epsilon_k}_\pm(0) = \lim_{k \to \infty} v^{\epsilon_k}_{0, \pm} = v_{0, \pm}.$$

Taking the limit of both sides of (2.7) and (2.8) along the subsequences $\rho^{\epsilon_k}_\pm$ with our choice of the sequences $u^{\epsilon_k}_0, v^{\epsilon_k}_{0, \pm}$ yields

$$\begin{cases}
  v_-(t) = v_{0,-} + \lim_{l \to 0} \lim_{m \to 0} \int_0^t \frac{1}{2} u_0(r, s) dr d s \\
  v_+(t) = v_{0,+} - \lim_{l \to 1} \lim_{m \to 0} \int_0^t \frac{1}{2} u_0(r, s) dr d s.
\end{cases} \quad (2.17)$$

More precisely, $u$ satisfies (2.14) with $v_\pm(t)$ such that $\lim_{t \downarrow 0} v_\pm(t) = v_{0, \pm}$ and the boundary conditions (2.17).

So far we have just identified the limit of the sequence $u^{\epsilon}$ up to a subsequence. Let us now consider other subsequences $\rho^{\epsilon_m}_\pm$ such that they converge uniformly to $\hat{v}_\pm$, respectively. Then the corresponding limit $\hat{u}$ of the subsequence $u^{\epsilon_m}$ can be given by the same expression as in (2.16), where $v_\pm$ are replaced by $\hat{v}_\pm$. Applying the same argument as before, we deduce that $\hat{u}$ also solves the problem (2.14) with the boundary conditions (2.17), where we replace $v_\pm$ by $\hat{v}_\pm$.

We denote $\bar{v}_\pm = v_\pm - \hat{v}_\pm$. The boundary conditions (2.17) imply that

$$\begin{align*}
  v_-(t) &= v_{0,-} + \lim_{l \to 0} \lim_{m \to 0} \int_0^t \frac{1}{2} \int_0^1 u_0 \left( r' \right) \left[ \frac{\partial \Theta}{\partial r} \left( l - r', s \right) - \frac{\partial \Theta}{\partial r} \left( l + r', s \right) \right] dr' ds \\
  &\quad - \int_0^t \Theta(0, t - s) v_-(s) ds + \int_0^t \Theta(1, t - s) v_+(s) ds,
\end{align*} \quad (2.18)$$

where $\Theta$ is defined in (2.11).
and similarly,
\[ v_+(t) = v_{0,+} - \lim_{l \to t_0} \lim_{l \to 1} \frac{1}{2} \int_0^1 u_0(r') \left[ \frac{\partial \Theta}{\partial r} (l - r', s) - \frac{\partial \Theta}{\partial r} (l + r', s) \right] dr'ds \]
+ \int_0^t \Theta(1, t - s)v_-(s) \, ds - \int_0^t \Theta(0, t - s)v_+(s) \, ds. \]

(2.19)

This implies
\[ \bar{v}_-(t) = -\int_0^t \Theta(0, t - s)\bar{v}_-(s) \, ds + \int_0^t \Theta(1, t - s)\bar{v}_+(s) \, ds, \]
and similarly,
\[ \bar{v}_+(t) = \int_0^t \Theta(1, t - s)\bar{v}_-(s) \, ds - \int_0^t \Theta(0, t - s)\bar{v}_+(s) \, ds. \]

By setting \( V(t) = \bar{v}_-(t) + \bar{v}_+(t) \), the two above expressions allow us to attain that
\[ V(t) = \int_0^t (\Theta(1, t - s) - \Theta(0, t - s))V(s) \, ds. \]

This gives us \( V(t) = 0, \forall t > 0 \), by applying Gronwall’s inequality. Since for any \( t > 0 \), \( \bar{v}_\pm(t) \in [0, 1] \), then \( v_\pm(t) = \bar{v}_\pm(t) \) and \( u(r, t) = \hat{u}(r, t) \) for all \((r, t) \in [0, 1] \times (0, \infty)\). This leads us to the uniqueness of the solution to the problem (2.14) with the boundary conditions (2.17).

Hence, we have verified that \( u \) is the pointwise limit of the sequence \( u^\epsilon \) for \((r, t) \in (0, 1) \times (0, \infty)\) as \( \epsilon \) goes to 0.

Moreover, in Section 3.6, [10], it can be shown that \( v_\pm \in C^1(0, \infty) \). Thus \( u \in C^{2,1}([0, 1] \times (0, \infty)) \) and now one can rewrite the boundary conditions (2.17) in the strong form (2.15). It completes the proof of Theorem 2.3.

3 Sticky Brownian Motion

3.1 Sticky Brownian Motion as a Strong Limit of a Sequence of Rescaled Sticky Random Walks

Sticky random walk \( (X(t))_{t \geq 0} \) moving on \([0, N+1] \cap \mathbb{N}\) is a continuous time random walk with jump rates \( c(x, x \pm 1) = \frac{1}{2}, \forall x \in [1, N] \cap \mathbb{N} \) and \( c(0, 1) = c(N+1, N) = \frac{1}{2N} \).

Let \( Y \) be a simple symmetric random walk on \( \mathbb{Z} \) starting from \( x \). Recall that the sequence of rescaled random walks \( N^{-1}Y(N^2t) \) converges uniformly almost surely on compact intervals of \([0, \infty)\) to a Brownian motion \( B, B_0 = r \in [0, 1]\), defined on some rich enough common probability space \( \tilde{\Omega}, \mathcal{F}, P \), see [8].

We denote by \( Y^{rf} \) the simple random walk \( Y \) reflected at 0 and \( N + 1 \). Let us call
\[ T(0, N + 1; t; Y^{rf}) = \int_0^t (1_{Y^{rf}(s)=0} + 1_{Y^{rf}(s)=N+1}) \, ds \]
the local time spent by $Y^{RF}$ at 0 and $N + 1$. Then it is shown in [10] that the sticky random walk $X$ can be realized by setting

$$X \left( t + (2N - 1)T \left( 0, N + 1; r; Y^{RF} \right) \right) = Y^{RF}(t).$$

**Theorem 3.1** \((2N - 1)N^{-2}T \left( 0, N + 1; N^2t; Y^{RF} \right)\) converges uniformly almost surely on compact intervals of \([0, \infty)\) to $L_t$ which is the local time at 0 and 1 of the reflecting Brownian motion $B^{RF}$ on \([0, 1]\). Moreover, the rescaled sticky random walk $N^{-1}X \left( N^2t \right)$ converges uniformly almost surely on compact intervals of \([0, \infty)\) to the sticky Brownian motion $B^{st}$ on \([0, 1]\) defined as

$$B^{st}(t + L_t) = B^{RF}(t).$$

**Proof** We know that the continuous time random walk $Y$ can be defined by $Y(t) = S_{N(t)}$, where $S$ is a simple symmetric discrete time random walk and $N$ is a Poisson process of parameter 1. Let us denote by $\mu_k^{(m)}$ the number of visits to $m \in \mathbb{Z}$ in the first $k$ steps of the random walk $S$. Then for $\mu_k := \sum_{m \in \mathbb{Z}} \mu_k^{(m(N + 1))}$, we can write

$$\frac{2N - 1}{N^2}T \left( 0, N + 1; N^2t; Y^{RF} \right) = \frac{2N - 1}{N^2} \int_0^{N^2t} \left( 1_{Y^{RF}(s) = 0} + 1_{Y^{RF}(s) = N + 1} \right) ds$$

$$= \frac{2N - 1}{N^2} \int_0^{N^2t} \sum_{(N + 1) - 1 \leq s \leq N + 1} \mathbb{1}_{1\leq m \leq (N + 1)} ds$$

$$= \frac{2N - 1}{N^2} \sum_{k=1}^{\mu_N(N^2t)} G_k,$$

where $G_k$ are independent exponential random variables with parameter 1. We next verify the following result.

**Proposition 3.2** For any $T > 0$ and $m \in \mathbb{Z}$,

$$\sup_{t \in [0, T]} \left| \frac{2N - 1}{N^2} \sum_{k=1}^{\mu_N(N^2t)} G_k - \frac{2}{N} \mu_N(N^2t) \right| \xrightarrow{a.s.} 0.$$

**Proof** As a consequence of the Borel-Cantelli lemma, it is enough to verify that for any $T > 0$ and $\varepsilon > 0$,

$$\sum_{N \to \infty} P \left( \sup_{t \in [0, T]} \left| \frac{2N - 1}{N^2} \sum_{k=1}^{\mu_N(N^2t)} G_k - \frac{2}{N} \mu_N(N^2t) \right| > 2\varepsilon \right) < \infty. \quad (3.3)$$

This follows from applying Doob’s martingale inequality and Markov’s inequality.

$\square$
On the other hand, making use of the same arguments as in [1] leads us to the fact that for any \( T > 0 \) and \( m \in \mathbb{Z} \),

\[
\sup_{t \in [0, T]} \left| \frac{2}{N} \mu_{\mathcal{N}^2} \left[ \frac{m(N+1)}{N^2 t} \right] - 2 L_t^m (B) \right| \underset{a.s.}{\rightarrow} 0,
\]

where \( L_t^m (B) \) stands for the local time at \( m \) of the Brownian motion \( B \). It follows that

\[
\sup_{t \in [0, T]} \left| \frac{2}{N} \mu_{\mathcal{N}^2} (N^2 t) - 2 \sum_{m \in \mathbb{Z}} L_t^m (B) \right| \underset{a.s.}{\rightarrow} 0.
\]

Combining with the above proposition, we can conclude that

\[
\sup_{t \in [0, T]} \left| \frac{2}{N} \sum_{k=1}^{\mu_{\mathcal{N}^2} (N^2 t)} G_k - 2 \sum_{m \in \mathbb{Z}} L_t^m (B) \right| \underset{a.s.}{\rightarrow} 0.
\]

Since it can be checked that

\[
2 \sum_{m \in \mathbb{Z}} L_t^m (B) = 2 \sum_{m \in \mathbb{Z}} L_t^m (B) + 2 \sum_{m \in \mathbb{Z}} L_t^{2m+1} (B) = L_t^0 \left( B^{\text{rf}} \right) + L_t^1 \left( B^{\text{rf}} \right) = L_t,
\]

then for any \( T > 0 \), this implies almost surely that

\[
\lim_{N \to \infty} \sup_{t \in [0, T]} \left| \frac{2}{N} \sum_{m \in \mathbb{Z}} \mathbb{1}_{0 < B(t) < 1} dW(t) + \frac{1}{2} \mathbb{1}_{B(t)=0} dt - \frac{1}{2} \mathbb{1}_{B(t)=1} dt \right| = 0.
\]

For any \( T > 0 \), applying again the same arguments as in [1] yields

\[
P \left( \lim_{N \to \infty} \sup_{t \in [0, T]} \left| N^{-1} X \left( N^2 t \right) - B^{\text{st}} (t) \right| = 0 \right) = 1.
\]

### 3.2 Sticky Brownian Motion as a Solution to a Stochastic Differential Equation

The sticky Brownian motion on the bounded interval \([0, 1]\) also solves a stochastic differential equation similar to the one on the half line \([0, \infty)\) as considered in [5].

**Proposition 3.3** The sticky Brownian motion \( B^{\text{st}} \), \( B^{\text{st}} (0) = r \), defined in (3.2) and the unique solution \( B \) to the following stochastic differential equation

\[
\begin{aligned}
\frac{dB(t)}{dt} &= \mathbb{1}_{0 < B(t) < 1} dW(t) + \frac{1}{2} \mathbb{1}_{B(t)=0} dt - \frac{1}{2} \mathbb{1}_{B(t)=1} dt, \\
B(0) &= r,
\end{aligned}
\]

for some standard Brownian motion \( W \) and \( r \in [0, 1] \), have the same law.

Besides the approach mentioned in Remark 1.1, the above proposition gives us another way to attain the probabilistic representation of the unique solution to the problem (2.14) with the boundary conditions (2.15).
Theorem 3.4 The unique solution \( u \) to the boundary value problem (2.14), (2.15) can be represented probabilistically, for \( r \in [0, 1] \), \( t > 0 \), as
\[
u(r, t) = E_r \left[ u_0(B(t)) \mathbf{1}_{0 < B(t) < 1} + v_0, - \mathbf{1}_{B(t)=0} + v_0, + \mathbf{1}_{B(t)=1} \right], \tag{3.5}
\]
where \( B \) solves the stochastic differential (3.4).

Proposition 3.3 and Theorem 3.4 are verified in the next sections.

3.3 Proof of Proposition 3.3

The existence and uniqueness in law of the solution to the stochastic differential (3.4) are proved in \([13]\).

Therefore, it suffices to verify that the sticky Brownian motion \( B^{st} \) and the unique solution \( B \) have the same law. The idea to show this is looking for a Skorokhod problem that a suitable time change of the process \( B \) and the reflecting Brownian motion satisfy. More precisely, the proof consists of the following steps.

Step 1. First, we show that the stochastic differential (3.4) is equivalent to the following system
\[
\begin{align*}
dB(t) &= \mathbf{1}_{0 < B(t) < 1} dW(t) + \frac{1}{2} dL^0_t(B) - \frac{1}{2} dL^1_{t-}(B), \\
dL^0_t(B) &= \mathbf{1}_{B(t)=0} dt, \\
dL^1_{t-}(B) &= \mathbf{1}_{B(t)=1} dt, \\
B(0) &= r, \\
\end{align*}
\tag{3.6}
\]

where \( L^a_t(B) \) stands for the local time of \( B \) at \( a \).

It is obvious that the system (3.6) implies (3.4). For the converse, we remark that \( 0 \leq B(t) \leq 1 \) almost surely for any \( t \geq 0 \) if \( B \) is a solution of (3.4). This follows from using the Ito - Tanaka formula (see Theorem 1.2 in Chapter 6 in \([12]\)), namely
\[
\begin{align*}
B(t)^- &= -\int_0^t \mathbf{1}_{B(s) < 0} dB(s) + \frac{1}{2} L^0_{t-}(B) = 0, \\
(B(t) - 1)^+ &= \int_0^t \mathbf{1}_{B(s) > 1} dB(s) + \frac{1}{2} L^1_t(B) = 0,
\end{align*}
\]

where we have used that
\[
\begin{align*}
L^0_{t-}(B) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{-\varepsilon \leq B(s) < 0} d[B]_s \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{-\varepsilon \leq B(s) < 0} \mathbf{1}_{0 < B(s) < 1} ds = 0, \\
L^1_t(B) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{1 \leq B(s) < 1 + \varepsilon} d[B]_s \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{1 \leq B(s) < 1 + \varepsilon} \mathbf{1}_{0 < B(s) < 1} ds = 0.
\end{align*}
\]
In view of this remark and using again the Ito - Tanaka formula, we get
\[ B(t) = B(t)^+ = r + \int_0^t \mathbf{1}_{B(s) > 0} dB(s) + \frac{1}{2} L_t^0(B) \]
and this implies \( dL_t^0(B) = \mathbf{1}_{B(t)=0} dt \).

Similarly, we have
\[ -(B(t) - 1) = -(B(t) - 1)^- = -(r - 1) - \int_0^t \mathbf{1}_{B(s)<1} dB(s) + \frac{1}{2} L_t^1(B) \]
and this implies \( dL_t^1(B) = \mathbf{1}_{B(t)=1} dt \). Hence, we obtain the equivalence of (3.4) and (3.6).

Step 2. Next, let us denote
\[ K(t) = \int_0^t \mathbf{1}_{0<B(s)<1} ds, \quad \kappa(t) = \inf\{u \geq 0 : K(u) > t\} \]
Applying the time change \( \kappa \) to the first equation of (3.6) yields
\[ V(t) := B(\kappa(t)) = r + \int_0^{\kappa(t)} \mathbf{1}_{0<B(s)<1} dW(s) + \frac{1}{2} L_0^0(B) - \frac{1}{2} L_{\kappa(t)}^1(B) \]
\[ = r + \int_0^{\kappa(t)} \mathbf{1}_{0<B(s)<1} dW(s) + \frac{1}{2} L_0^0(V) - \frac{1}{2} L_{\kappa(t)}^1(V) \]
Since \( Q(t) := r + \int_0^{\kappa(t)} \mathbf{1}_{0<B(s)<1} dW(s) = r + \int_0^{\kappa(t)} \mathbf{1}_{0<B(\kappa) <1} dW(\kappa(u)) \)
is a continuous local martingale and note that
\[ [Q]_t = \int_0^{\kappa(t)} \mathbf{1}_{0<B(\kappa(u))<1} d\kappa(u) = \int_0^{\kappa(t)} \mathbf{1}_{0<B(s)<1} ds = t, \]
then \( Q \) is a Brownian motion starting from \( r \) by P. Levy’s characterization theorem. Moreover, due to the explicit expression of the solution to the Skorokhod problem
\[ V(t) = Q(t) + \frac{1}{2} L_0^0(V) - \frac{1}{2} L_t^1(V), \]
(see, e.g., [2, 3, 9]), we obtain
\[ V(t) = \tilde{R}_{0,1}(Q(t)), \quad \text{(3.7)} \]
where
\[ \tilde{R}_{0,1}(Q(t)) := Q(t) - (r - 1)^+ \wedge \inf_{u \in [0,t]} Q(u) \vee \sup_{s \in [0,t]} (Q(s) - 1) \wedge \inf_{u \in [s,t]} Q(u). \]

Step 3. Let us denote the Brownian motion \( Q \) reflected at 0 and 1 by
\[ R_{0,1}(Q(t)) := \sum_{m \in \mathbb{Z}} |Q(t) - 2m| \mathbf{1}_{|Q(t) - 2m| \leq 1}. \quad \text{(3.8)} \]
From [6], there is an explicit representation of \( Z := R_{0;1}(Q) \), the Brownian motion \( Q \) starting from \( r \) reflected at two barriers 0 and 1, as follows

\[
Z(t) = r + \int_0^t \left( \mathbf{1}_{Q(s) \in \bigcup_{m \in \mathbb{Z}} (2m,2m+1)} - \mathbf{1}_{Q(s) \in \bigcup_{m \in \mathbb{Z}} (2m+1,2m+2)} \right) dQ(s) + \sum_{m \in \mathbb{Z}} L_{t}^{2m}(Q) - \sum_{m \in \mathbb{Z}} L_{t}^{2m+1}(Q).
\]

Set \( \hat{Q}(t) := r + \int_0^t \left( \mathbf{1}_{Q(s) \in \bigcup_{m \in \mathbb{Z}} (2m,2m+1)} - \mathbf{1}_{Q(s) \in \bigcup_{m \in \mathbb{Z}} (2m+1,2m+2)} \right) dQ(s) \).

We notice that \( \hat{Q} \) is a Brownian motion starting from \( r \) by P. Levy’s characterization theorem. Moreover,

\[
\sum_{m \in \mathbb{Z}} L_{t}^{2m}(Q) = \frac{1}{2} L_{t}^{0}(Z) \quad \text{and} \quad \sum_{m \in \mathbb{Z}} L_{t}^{2m+1}(Q) = \frac{1}{2} L_{t}^{1-}(Z). \tag{3.9}
\]

Hence, we can write

\[
Z(t) = \hat{Q}(t) + \frac{1}{2} L_{t}^{0}(Z) - \frac{1}{2} L_{t}^{1-}(Z).
\]

Using again the explicit representation of the solution to the above Skorokhod problem gives us

\[
Z(t) = \hat{R}_{0;1} \left( \hat{Q}(t) \right) \overset{d}{=} \hat{R}_{0;1}(Q(t)). \tag{3.10}
\]

It follows from (3.7) and (3.10) that

\[
V(t) \overset{d}{=} Z(t) = R_{0;1}(Q(t)). \tag{3.11}
\]

Step 4. Moreover, using the second and the third equation of the system (3.6) gives us

\[
\kappa(t) = \int_0^{\kappa(t)} \mathbf{1}_{0 < \kappa(t) < 1} ds + \int_0^{\kappa(t)} \mathbf{1}_{\kappa(t) = 0} ds + \int_0^{\kappa(t)} \mathbf{1}_{\kappa(t) = 1} ds
\]

\[
= t + L_{\kappa(t)}^{0}(B) + L_{\kappa(t)}^{1-}(B)
\]

\[
= t + L_{t}^{0}(V) + L_{t}^{1-}(V)
\]

\[
\overset{d}{=} t + L_{t}^{0}(Z) + L_{t}^{1-}(Z).
\]

Then in view of (3.11), we deduce that

\[
Z(t) \overset{d}{=} B \left( t + L_{t}^{0}(Z) + L_{t}^{1-}(Z) \right).
\]

Since \( Z(t) \overset{d}{=} B^{H}(t) \), the proof is complete.

### 3.4 Proof of Theorem 3.4

For any \( \delta > 0 \), we fix \( t_0 \geq \delta \). Since the unique solution \( u \in C^{2,1}([0, 1] \times (0, \infty)) \), we apply Ito’s formula to the function \( u(B(t), t_0 - t) \) for \( t \in [0, t_0 - \delta] \) and obtain that

\[
u(B(t), t_0 - t) = u(r, t_0) + \int_0^t u_r(B(s), t_0 - s) dB(s) \]
\[ + \int_{0}^{t} \frac{1}{2} u_{rr}(B(s), t_0 - s) \mathbf{1}_{0 < B(s) < 1} \, ds + \int_{0}^{t} u_s(B(s), t_0 - s) \, ds \]
\[ = u(r, t_0) + \int_{0}^{t} u_r(B(s), t_0 - s) \mathbf{1}_{0 < B(s) < 1} \, dW(s) \]
\[ + \int_{0}^{t} \left[ -\frac{1}{2} u_r(B(s), t_0 - s) + \frac{d}{ds} v_-(t_0 - s) \right] \mathbf{1}_{B(s) = 0} \, ds \]
\[ + \int_{0}^{t} \left[ -\frac{1}{2} u_r(B(s), t_0 - s) + \frac{d}{ds} v_+(t_0 - s) \right] \mathbf{1}_{B(s) = 1} \, ds \]
\[ = u(r, t_0) + \int_{0}^{t} u_r(B(s), t_0 - s) \mathbf{1}_{0 < B(s) < 1} \, dW(s). \]

Let us call \( M(t) := \int_{0}^{t} u_r(B(s), t_0 - s) \mathbf{1}_{0 < B(s) < 1} \, dW(s) \). Then \( M \) is a martingale since \( u \in C^{2,1}([0, 1] \times (0, \infty)) \). So \( E_r[M(t_0 - \delta)] = E_r[M(0)] = 0 \). Hence,
\[ u(r, t_0) = E_r[u(B(t_0 - \delta), \delta)]. \]
Taking the limit \( \delta \downarrow 0 \) of both sides of the above equality gives us
\[ u(r, t_0) = \lim_{\sigma \downarrow 0} E_r[u(B(t_0), \sigma)] \]
\[ = \lim_{\sigma \downarrow 0} E_r[u(B(t_0), \sigma) \mathbf{1}_{0 < B(t_0) < 1} + u(0, \sigma) \mathbf{1}_{B(t_0) = 0} + u(1, \sigma) \mathbf{1}_{B(t_0) = 1}] \]
\[ = E_r[u_0(B(t_0)) \mathbf{1}_{0 < B(t_0) < 1} + v_0_ - \mathbf{1}_{B(t_0) = 0} + v_0_ + \mathbf{1}_{B(t_0) = 1}]. \]
Since \( \delta > 0 \) is arbitrary, we obtain the representation (3.5) for any \( t > 0 \).

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