Locally finite groups with bounded centralizer chains

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Abstract. The $c$-dimension of a group $G$ is the maximal length of a chain of nested centralizers in $G$. We prove that a locally finite group of finite $c$-dimension $k$ has less than $5k$ nonabelian composition factors.

Keywords: locally finite group, nonabelian simple group, lattice of centralizers, $c$-dimension.

Introduction

Let $G$ be a group and $C_G(X)$ be the centralizer of a subset $X$ of $G$. Since $C_G(X) < C_G(Y)$ if and only if $C_G(C_G(X)) > C_G(C_G(Y))$, it follows that the minimal and the maximal conditions for centralizers are equivalent. Thus the length of every chain of nested centralizers in a group with the minimal condition for centralizers is finite. If a uniform bound for the lengths of chains of centralizers of a group $G$ exists, then we refer to maximal such length as $c$-dimension of $G$ following [1]. The same notion is also known as the height of the lattice of centralizers. It is worth to observe that the class of groups of finite $c$-dimension includes abelian groups, torsion-free hyperbolic groups, linear groups over fields and so on. In addition, it is closed under taking subgroups and finite direct products, but the $c$-dimension of a homomorphic image of a group from this class is not necessary finite.

In 1979 R. Bryant and B. Hartley [2] proved that a periodic locally soluble group with the minimal condition for centralizers is soluble. In 2009 E. I. Khukhro published the paper [3], where, in particular, he proved that a periodic locally soluble group of finite $c$-dimension $k$ has the derived length bounded in terms of $k$. The same paper contains the conjecture attributed to A. V. Borovik, which asserts that the number of nonabelian composition factors of a locally finite group of finite $c$-dimension $k$ is bounded in terms of $k$. The purpose of our work is to prove this conjecture.

Theorem. Let $G$ be a locally finite group of $c$-dimension $k$. Then the number of nonabelian composition factors of $G$ is less than $5k$.

§ 1. Preliminaries

Given a locally finite group $G$, denote by $\eta(G)$ the number of nonabelian composition factors of $G$.

The following well-known fact (see, for example, [1 Corollary 3.5]) helps us to derive the theorem from the corresponding statement for finite groups.

Lemma 1.1. If $G$ is a locally finite locally soluble simple group, then $G$ is cyclic.

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Recall that the factor group of a finite group \( G \) by its soluble radical \( R \) is an automorphism group of a direct product of nonabelian simple groups. Thus, if the socle \( Soc(G/R) \) is a direct product of nonabelian simple groups \( S_1, S_2, \ldots, S_n \), then \( G/R \) is a subgroup of the semidirect product \( \langle \text{Aut}(S_1) \times \text{Aut}(S_2) \times \cdots \times \text{Aut}(S_n) \rangle \rtimes \text{Sym}_n \), where \( \text{Sym}_n \) permutes \( S_1, S_2, \ldots, S_n \). By the classification of the finite simple groups, the group of outer automorphisms of a finite simple group is soluble. Therefore, every nonabelian composition factor of \( G \) is either a composition factor of \( Soc(G/R) \), or a composition factor of the corresponding subgroup of \( \text{Sym}_n \).

Next three lemmas give an upper bound for the number of nonabelian composition factors of a subgroup of \( \text{Sym}_n \). We denote by \( \mu(G) \) the degree of the minimal faithful permutation representation of a finite group \( G \).

**Lemma 1.2** ([6], Theorem 2). Let \( G \) be a finite group. Let \( \mathcal{L} \) be a class of finite groups closed under taking subgroups, homomorphic images and extensions. If \( N \) is the maximal normal \( \mathcal{L} \)-subgroup of \( G \), then \( \mu(G) \geq \mu(G/N) \).

**Lemma 1.3** ([6], Theorem 3.1). Let \( S_1, S_2, \ldots, S_r \) be simple groups. Then \( \mu(S_1 \times S_2 \times \cdots \times S_r) = \mu(S_1) + \mu(S_2) + \cdots + \mu(S_r) \).

**Lemma 1.4.** If \( G \) is a subgroup of a symmetric group \( \text{Sym}_n \), then \( \eta(G) \leq (n - 1)/4 \).

**Proof.** We proceed by induction on \( n \). If \( R \) is the soluble radical of \( G \), then Lemma 1.2 implies that \( \mu(G/R) \) does not exceed \( \mu(G) \). Hence, we may assume that the soluble radical of \( G \) is trivial. Let the socle \( Soc(G) \) of \( G \) be the direct product of nonabelian simple groups \( S_1, S_2, \ldots, S_l \). It follows from Lemma 1.3 that \( l \leq n/5 \). Again \( G \) is a subgroup of the semidirect product \( \langle \text{Aut}(S_1) \times \text{Aut}(S_2) \times \cdots \times \text{Aut}(S_l) \rangle \rtimes \text{Sym}_l \). By inductive hypothesis, \( \eta(G) \leq n/5 + (n/5 - 1)/4 = (n - 1)/4 \).

**Remark.** The group \( \text{Sym}_n \), where \( n = 5^k \) with \( k \geq 1 \), contains a subgroup \( G \) isomorphic to the permutation wreath product \( (\cdots ((\text{Alt}_5 \wr \text{Alt}_5) \wr \text{Alt}_5) \cdots) \), where the wreath product is applied \( k - 1 \) times. We have \( \eta(G) = \frac{5^{k-1} - 1}{5 - 1} = \frac{2^{k-1} - 1}{4} \).

The following lemma is a key for bounding the number of composition factors of \( Soc(G/R) \) for a finite group \( G \).

**Lemma 1.5** ([3], Lemma 3). If an elementary abelian \( p \)-group \( E \) of order \( p^n \) acts faithfully on a finite nilpotent \( p' \)-group \( Q \), then there exists a series of subgroups \( E = E_0 > E_1 > E_2 > \cdots > E_n = 1 \) such that all inclusions \( C_Q(E_0) < C_Q(E_1) < \cdots < C_Q(E_n) \) are strict.

As usual, \( O_p(G) \) stands for the largest normal \( p \)-subgroup of a finite group \( G \), while \( O_{p'}(G) \) denotes the largest normal \( p' \)-subgroup of \( G \). If a series of commutator subgroups of a group \( G \) stabilizes, then we denote by \( G^{(\infty)} \) the last subgroup of this series. A quasisimple group is a perfect central extension of a nonabelian simple group. The layer \( E(G) \) is the subgroup of \( G \) generated by all subnormal quasisimple subgroups of \( G \), the latter are called components of \( G \). Recall that the layer is a central product of components of \( G \).
§ 2. Proofs

Proposition 2.1. Let $G$ be a finite group of c-dimension $k$. Then $\eta(G) < 5k$.

Proof. Let $R$ be the soluble radical of $G$. If $P$ is a Sylow subgroup of $R$, then $G/R \simeq N_G(P)/(R \cap N_G(P))$, so nonabelian composition factors of $N_G(P)$ and $G$ coincide. On the other hand, c-dimension of $N_G(P)$ as a subgroup of $G$ is at most $k$. Therefore, we may assume that $N_G(P) = G$ for every Sylow subgroup $P$ of $R$, i.e. that $R$ is nilpotent.

Obviously, we may suppose that $R \neq G$. Put $\overline{G} = G/R$. The socle $\overline{L}$ of $\overline{G}$ is the direct product of nonabelian simple groups $S_1, S_2, \ldots, S_n$. As observed in preliminaries, the group $\overline{G}/\overline{L}$ is an extension of a normal soluble subgroup by a subgroup of the symmetric group $Sym_n$. By Lemma [1.3] an arbitrary subgroup of $Sym_n$ has less than $n/4$ nonabelian composition factors. Thus, it is sufficient to show that $\eta(\overline{L}) = n \leq 4k$. In particular, we may assume that $G$ coincides with $L$, the preimage of $\overline{L}$ in $G$, and nonabelian composition factor of $G$ are the groups $S_1, S_2, \ldots, S_n$.

Let $K = C_G(R)$. The normal subgroup $\overline{K} = KR/R$ of $\overline{G}$ is a direct product of nonabelian simple group. Without loss of generality, we may suppose that $\overline{K} = S_1 \times S_2 \times \ldots \times S_l$ for some $1 \leq l \leq n$. For $i = 1, \ldots, l$ denote by $K_i$ the preimage of $S_i$ in $K$. Then subgroup $H_i = K_i^{(\infty)}$ is normal in $K$ and is a perfect central extension of $S_i$, so it is a component of $K$. Therefore, if $E(K)$ is the layer of $K$, then $KR = E(K)R$ and $E(K)$ is a central product of $H_1, H_2, \ldots, H_l$. Hence $\eta(K) = \eta(E(K)) = l$. Since $[H_i, H_j] = 1$ for $i \neq j$, all inclusions $C_{E(K)}(H_i) < C_{E(K)}(H_iH_j) < \cdots < C_{E(K)}(H_iH_j \cdots H_l)$ are strict. Thus, $l \leq k$.

Let $P$ be a Sylow $p$-subgroup of $G$ and $\overline{P}$ be the image of $P$ in $\overline{G}$. Since $O_p(R) \leq C_G(O_p'(R))$, the action of $P$ on $O_p'(R)$ by conjugation induces the action of $\overline{P}$ on $O_p'(R)$. Given a prime $p$, define the set $\mathcal{F}_p$ as follows: a subgroup $S_i$ of $\overline{G}$ lies in $\mathcal{F}_p$ whenever there is an element $g$ of order $p$ in $S_i$ acting faithfully on $O_p'(R)$. Lemma [1.5] yields that $|\mathcal{F}_p| \leq k$ for every prime $p$. On the other hand, if $S_i$ does not lie in $\mathcal{F}_p$, then $S_i$ is a subgroup of $C_G(O_p'(R))/R/R$. It follows from the classification of finite simple groups that the order of every nonabelian finite simple group is an even number which is a multiple of 3 or 5. Since $R = O_2(R) \times O_2(R)$, every $S_i$ either belongs to $\mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_5$, or is a subgroup of $\overline{K} = C_G(R)/R/R$. Thus, $\eta(G) \leq |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_5| + \eta(K) \leq 4k$, as required.

Proof of the theorem. Now $G$ is locally finite group. Assume $\eta(G) \geq 5k$. Let $\{G_i\}_{i \in I}$ be a composition series of $G$, where $G_i$ is a proper subgroup of $G_j$ for $i < j$. Let $S_1, S_2, \ldots, S_{5k}$ be pairwise distinct nonabelian composition factors of $G$. By Lemma [1.1] every locally finite nonabelian simple group contains a finite insoluble subgroup. Thus, we may choose finite subsets $X_1, X_2, \ldots, X_{5k}$ of $G$ such that the image of $X_i$ in $S_i$ generates an insoluble group. Suppose that $H$ is the finite subgroup of $G$ generated by the union of the sets $X_1, X_2, \ldots, X_{5k}$. Then $\{G_i \cap H\}_{i \in I}$ is a subnormal series of $H$ having at least $5k$ insoluble factors. This contradicts Proposition 2.1. The theorem is proved.

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