EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE QUANTUM BOLTZMANN EQUATION FOR SOFT POTENTIALS

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Abstract. In this paper we consider a modified quantum Boltzmann equation with the quantum effect measured by a continuous parameter $\delta$ that can decrease from $\delta = 1$ for the Fermi-Dirac particles to $\delta = 0$ for the classical particles. In case of soft potentials, for the corresponding Cauchy problem in the whole space or in the torus, we establish the global existence and uniqueness of non-negative mild solutions in the function space $L^\infty_t L^\infty_x \cap L^\infty_t L^2_x$ with small defect mass, energy and entropy but allowed to have large amplitude up to the possibly maximum upper bound $F(t, x, v) \leq \frac{1}{\delta}$. The key point is that the obtained estimates are uniform in the quantum parameter $0 < \delta \leq 1$. In particular, as $\delta \to 0$ we can recover the results on the classical Boltzmann equation around global Maxwellians for which solutions may have arbitrarily large oscillations.

1. Introduction

We consider the following Cauchy problem on the modified quantum Boltzmann equation

$$
\partial_t F + v \cdot \nabla_x F = \mathcal{C}_\delta(F), \quad F(0, x, v) = F_0(x, v),
$$

where $F(t, x, v) \geq 0$ is an unknown velocity distribution function of particles with position $x \in \Omega = \mathbb{R}^3$ or $\mathbb{T}^3$ and velocity $v \in \mathbb{R}^3$ at time $t > 0$ and initial data $F_0(x, x)$ is given. The collision operator $\mathcal{C}_\delta$ takes the form of

$$
(\mathcal{C}_\delta(F))(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v-u, \theta) \left[ F(u') F(v') (1 - \delta F(u)) (1 - \delta F(v)) \right. \\
- \left. F(u) F(v) (1 - \delta F(u')) (1 - \delta F(v')) \right] d\omega du,
$$

where $0 \leq \delta \leq 1$ is a continuous parameter denoting the quantum effect. For $\delta = 0$, the equation (1.1) is the classical Boltzmann equation, while for $\delta = 1$, the equation corresponds to the Boltzmann equation for Fermi-Dirac particles. In this paper we consider the continuous parameter $\delta$ taking values in the interval $[0, 1]$. Moreover, we consider only the soft potentials under the Grad’s angular cutoff assumption. Therefore, the collision kernel $B(v-u, \theta)$ satisfies

$$
B(v-u, \theta) = |v-u|^2 b(\theta),
$$

where $-3 < \gamma < 0$, $0 \leq b(\theta) \leq C |\cos \theta|$ and $\cos \theta = \frac{(v-u) \cdot \omega}{|v-u|}$. The post-collision velocities $v'$ and $u'$ are defined by

$$
v' = v - [(v-u) \cdot \omega] \omega, \quad u' = u + [(v-u) \cdot \omega] \omega,
$$

$$
u' + v' = u + v, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2.
$$

Let the equilibrium state $\mu_{\delta, \rho}$ be denoted by

$$
\mu_{\delta, \rho}(v) = \frac{1}{\delta + \rho e^{\frac{|v|^2}{2\delta}}},
$$

where $\rho > 0$ is a constant. Moreover, if $F(t, x, v)$ is a solution to the modified quantum Boltzmann equation (1.1) with the initial datum $F_0(x, v)$, the following conservation laws of the initial defect.
mass, momentum, energy and the defect entropy inequality of \( F(t, x, v) \) hold,

\[
\int_{\Omega} \int_{\mathbb{R}^3} \{ F(t, x, v) - \mu_{\delta, \rho}(v) \} dv dx = \int_{\Omega} \int_{\mathbb{R}^3} \{ F_0(x, v) - \mu_{\delta, \rho}(v) \} dv dx := M_0, \tag{1.4}
\]

\[
\int_{\Omega} \int_{\mathbb{R}^3} v \{ F(t, x, v) - \mu_{\delta, \rho}(v) \} dv dx = \int_{\Omega} \int_{\mathbb{R}^3} v \{ F_0(x, v) - \mu_{\delta, \rho}(v) \} dv dx := J_0,
\]

\[
\int_{\Omega} \int_{\mathbb{R}^3} |v|^2 \{ F(t, x, v) - \mu_{\delta, \rho}(v) \} dv dx = \int_{\Omega} \int_{\mathbb{R}^3} |v|^2 \{ F_0(x, v) - \mu_{\delta, \rho}(v) \} dv dx := E_0, \tag{1.5}
\]

and

\[
\mathcal{H}_{\delta, \rho}(F(t)) := \int_{\Omega} \int_{\mathbb{R}^3} \left\{ F(t, x, v) \log F(t, x, v) + \frac{1}{\delta} (1 - \delta F(t, x, v)) \log (1 - \delta F(t, x, v)) - \mu_{\delta, \rho}(v) \log \mu_{\delta, \rho}(v) - \frac{1}{\delta} (1 - \delta \mu_{\delta, \rho}(v)) \log (1 - \delta \mu_{\delta, \rho}(v)) \right\} dv dx \leq \mathcal{H}_{\delta, \rho}(F(0)). \tag{1.6}
\]

Furthermore, the functional \( \mathcal{E}_{\delta, \rho}(F(t)) \) is given by

\[
\mathcal{E}_{\delta, \rho}(F(t)) := \mathcal{H}_{\delta, \rho}(F(t)) + (\log \rho) M_0 + \frac{1}{2} E_0, \tag{1.7}
\]

with the initial datum \( \mathcal{E}_{\delta, \rho}(F_0) := \mathcal{E}_{\delta, \rho}(F(0)) \).

The quantum Boltzmann equation (1.1) with \( \delta = 1 \) is a kinetic model which describes the behavior of rarefied gas in non-equilibrium state for particles satisfying the Pauli exclusion principle. For the spatially homogeneous case, we mention [23,26]. For the inhomogeneous case, Lu [24,25] obtained the global existence and weak stability of weak solutions for Fermi-Dirac particles with very soft potentials, see also [28,31,30]. Jiang-Xiong-Zhou [19] and Jiang-Zhou [20] studied the incompressible Navier-Stokes-Fourier limit and the compressible Euler and acoustic limits from the quantum Boltzmann equation, respectively. In recent years, more attentions have been paid to study how the equation depends on the continuous quantum parameter. Dolbeault [8] gave the existence and uniqueness results of the renormalized solutions and obtained solutions to the Boltzmann equation by passing the limit with respect to the quantum parameter. He-Lu-Pulvirenti [18] deduced the convergence to the homogeneous Fokker-Planck-Landau equation from the homogeneous quantum Boltzmann equation. Alonso-Bagland-Desvillettes-Lods [11] obtained uniform estimates in the quantum parameter for the entropy dissipation of the Landau-Fermi-Dirac equation. However, there are still many unknowns in the study of the inhomogeneous quantum Boltzmann equation when ones take into account the effect of the quantum parameter.

For the case when \( \delta = 0 \), the quantum Boltzmann equation becomes the classical Boltzmann equation and there have been extensive works on global existence and large time behavior of solutions. For instance, DiPerna-Lions [7] proved the global existence of renormalized solutions for general \( L^p_{x,v} \) initial data with finite mass, energy and entropy. In the perturbation framework near global Maxwellians, Grad [13] and Ukai [32] developed the spatially inhomogeneous well-posedness theory by the spectral analysis and the bootstrap argument, see also [27,29,34]. For more properties of the linearized operator, interested readers may also refer to Ellis-Pinsky [14], Baranger-Mouhot [2] and the references therein. Another important approach using the energy method through the macro-micro decomposition is established by Liu-Yang-Yu [22] and Guo [15] in the 2000s. The case of soft potentials \(-3 < \gamma < 0\) for which the collision frequency \( \nu(v) \sim (1 + |v|^\gamma) \) is degenerate in large velocities are more complicated to treat. For \(-1 < \gamma < 0\), Ukai-Asano [33] and Caflisch [4,5] independently proved the global existence and large-time behavior of the solutions in the whole space and in torus, respectively. When \( \gamma \) is in the full range \((-3, 0)\), Guo [14] constructed the global classical solutions near global Maxwellians and Strain-Guo [39,41] derived the long-time behavior of solutions, see also a recent work [9] for the study of spectral gap formulation in soft potentials.

It is also an interesting topic on how to obtain large amplitude solution with extra smallness assumptions. As \( \delta \to 0 \) corresponding to the limit to the classical Boltzmann equation, the restriction that \( 0 \leq F_0 \leq 1/\delta \) is reduced to the only non-negativity condition that \( F_0 \geq 0 \) and thus the situation is relatively easier than the one in case of \( 0 < \delta \leq 1 \). In case of \( \delta = 0 \), motivated
by the original work Guo [17], Duan-Huang-Wang-Yang [9] developed an $L^\infty_x L^1_t \cap L^\infty_x L^\infty_t$ approach to obtain the global well-posedness of mild solutions under the condition that both $\mathcal{E}(F_0)$ and the $L^1_x L^\infty_t$ norm of $(F_0 - \mu) / \sqrt{\mu}$ are small. The $L^\infty_x$ norm of $\langle v \rangle^\beta (F_0 - \mu) / \sqrt{\mu}$ is only required to be bounded so that the initial data is allowed to have arbitrary large amplitude around the global Maxwellian. See also [21] by the author of this paper for an extension to the large amplitude results in $L^p_x L^q_t L^\infty$ spaces. However, for $0 \leq \delta \leq 1$, it remains unclear whether there exists a large amplitude solution to the quantum Boltzmann equation for soft potentials with estimates that can be uniform in the quantum parameter, in particular, whether or not one can recover the result in [9] by passing the limit $\delta \to 0$.

The perturbation function $f = f(t, x, v)$ is defined by

$$F(t, x, v) = \mu_{\delta, \rho}(v) + \sqrt{\mu_{\delta, \rho}} f(t, x, v), \quad (1.8)$$

where $\mu_{\delta, \rho}(v)$ is given in [13] and $\sqrt{\mu_{\delta, \rho}} f(t, x, v) = \mu_{\delta, \rho}(v) f(t, x, v)$ with

$$\sqrt{\mu_{\delta, \rho}}(v) = \sqrt{\mu_{\delta, \rho}(v)[1 - \delta \mu_{\delta, \rho}(v)]} = \frac{\sqrt{\|v\|^2}}{\delta + \rho e^{\|v\|^2/2}}. \quad (1.9)$$

For simplicity, we use the above notations throughout the paper. Substituting (1.8) into (1.1), the perturbed equation can be written as

$$\partial_t f + v \cdot \nabla_x f + L_\delta f = \Gamma_\delta(f). \quad (1.10)$$

The linear operator $L_\delta$ and the nonlinear term $\Gamma_\delta$ are respectively given by

$$L_\delta = \nu_\delta - K_\delta, \quad (1.11)$$

and

$$\Gamma_\delta(f)(v) = \frac{1}{\sqrt{\mu_{\delta, \rho}(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ \sqrt{\mu_{\delta, \rho}}(u') \sqrt{\mu_{\delta, \rho}} f(v') (1 - \delta \mu_{\delta, \rho}(v) - \delta \mu_{\delta, \rho}(u)) ight.$$

$$- \sqrt{\mu_{\delta, \rho}}(u) \sqrt{\mu_{\delta, \rho}} f(v) (1 - \delta \mu_{\delta, \rho}(v') - \delta \mu_{\delta, \rho}(u')) + \delta \sqrt{\mu_{\delta, \rho}}(u') \sqrt{\mu_{\delta, \rho}} f(v) (\mu_{\delta, \rho}(v) - \mu_{\delta, \rho}(u'))$$

$$+ \delta \sqrt{\mu_{\delta, \rho}}(u) \sqrt{\mu_{\delta, \rho}} f(v) (\mu_{\delta, \rho}(v) - \mu_{\delta, \rho}(u')) + \delta \sqrt{\mu_{\delta, \rho}}(u) \sqrt{\mu_{\delta, \rho}} f(v) \sqrt{\mu_{\delta, \rho}}(v) \sqrt{\mu_{\delta, \rho}}(u')$$

$$+ \delta \sqrt{\mu_{\delta, \rho}}(u) \sqrt{\mu_{\delta, \rho}} f(v) \sqrt{\mu_{\delta, \rho}}(v) \sqrt{\mu_{\delta, \rho}}(v') + \sqrt{\mu_{\delta, \rho}} f(u') \sqrt{\mu_{\delta, \rho}} f(v') \sqrt{\mu_{\delta, \rho}} f(v)$$

$$- \delta \sqrt{\mu_{\delta, \rho}} f(u') \sqrt{\mu_{\delta, \rho}} f(v') \sqrt{\mu_{\delta, \rho}} f(v) \] d\omega du, \quad (1.12)$$

where

$$\nu_\delta(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ \mu_{\delta, \rho}(u) - \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(u) - \delta \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(v') + \delta \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(v) \right] d\omega du, \quad (1.13)$$

$$K_\delta(f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ \frac{\sqrt{\mu_{\delta, \rho}}(u')}{\sqrt{\mu_{\delta, \rho}(v)}} \left[ \mu_{\delta, \rho}(v') - \delta \mu_{\delta, \rho}(v') \mu_{\delta, \rho}(u) - \delta \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(v) + \delta \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(v) \right] f(u') \right.$$

$$\left. \frac{\sqrt{\mu_{\delta, \rho}}(v)}{\sqrt{\mu_{\delta, \rho}(v)}} \left[ \mu_{\delta, \rho}(u) - \delta \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(u') - \delta \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(v') + \delta \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(v) \right] f(v') \right.$$

$$\left. \frac{\sqrt{\mu_{\delta, \rho}}(u)}{\sqrt{\mu_{\delta, \rho}(v)}} \left[ \mu_{\delta, \rho}(v) - \delta \mu_{\delta, \rho}(v) \mu_{\delta, \rho}(u') - \delta \mu_{\delta, \rho}(v) \mu_{\delta, \rho}(v') + \delta \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(v) \right] f(u) \] d\omega du. \quad (1.14)$$
By integrating along the backward trajectory, we yield the mild form of the perturbed equation \((1.10)\) as follows:

\[
f(t, x, v) = e^{-\nu_s(t-\tau)}f_0(x - \nu_s(t-\tau)v) + \int_0^t e^{-\nu_s(t-s)}(K_\delta f)(s, x - v(t-s), v)\,ds
+ \int_0^t e^{-\nu_s(t-s)}\Gamma_\delta(f)(s, x - v(t-s), v)\,ds.
\] (1.15)

For given bounded initial data \(f_0 = f_0(x, v)\) in \(L^\infty_{v,x}\), we first construct a local solution in \(L^\infty_T L^\infty_{v,x}\) space. Then under the smallness assumption for \(\sup_{0\leq \delta \leq 1} \{E_{\delta, \rho}(F_0) + \|f_0\|_{L^1_\delta L^\infty_{v,x}}\}\), we establish the \(L^\infty_T L^\infty_{v,x} \cap L^\infty_T L^\infty_{v,x}^1\) estimates to extend the obtained solution to a unique global solution which can be governed by the \(L^\infty_{v,x}\) bound of the initial datum. We define some notations in different normed spaces for later use. For given functions \(f = f(t, x, v)\) and \(g = g(x, v)\), we define

\[
\|f\|_{L^\infty_T L^\infty_{v,x}} := \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^3} |f(t, x, v)|,
\]

\[
\|f\|_{L^\infty_T L^\infty_{v,x}, L^\infty_{v,x}^1} := \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} |f(t, x, v)|\,dv,
\]

\[
\|g\|_{L^\infty_{v,x}} := \sup_{v \in \mathbb{R}^3} \sup_{x \in \Omega} |g(x, v)|,
\]

\[
\|g\|_{L^1_{v,x}} := \int_{\Omega} \left( \sup_{v \in \mathbb{R}^3} |g(x, v)| \right)\,dx.
\]

If \(T_0 = 0\), we write \(\|f\|_{L^\infty_T L^\infty_{v,x}, L^\infty_{v,x}^1}\) instead of \(\|f\|_{L^\infty_{v,x}, L^\infty_{v,x}^1}\). In this paper, if a constant \(C\) depends on \(\beta_1, \beta_2, \ldots, \beta_n\), we denote \(C = C(\beta_1, \beta_2, \ldots, \beta_n)\) in order to show the dependence clearly. The velocity weight function is given by \(w_\beta(v) = (1 + |v|^\beta)^\gamma\).

Now, two main results of this paper are stated below.

**Theorem 1.1** (Local existence). Let \(-3 < \gamma < 0\) and \(\beta > 6\). Assume \(F_0(x, v) := \mu_\delta \rho(v) + \sqrt{\mu_\delta \rho} f_0(x, v)\) with \(0 \leq F_0 \leq 1/\delta\) and \(\|w_\beta f_0\|_{L^\infty_{v,x}} < \infty\). Then there are constants \(C_1 = C_1(\beta, \gamma) > 0\) independent of \(\rho, C_\rho > 0\) depending continuously on \(\rho \in (0, \infty)\), and a positive time \(T_1 := \frac{C_1}{\tilde{C}_\rho \rho (1 + \|w_\beta f_0\|_{L^\infty_{v,x}} + \|w_\beta f_0\|_{L^2_{v,x}}^2)}\) (1.16)

such that the Cauchy problem on the quantum Boltzmann equation \((1.1)\) has a unique mild solution \(F(t, x, v) = \mu_\delta \rho(v) + \sqrt{\mu_\delta \rho} f(t, x, v), (t, x, v) \in [0, T_1] \times \Omega \times \mathbb{R}^3\), in the sense of \((1.15)\), satisfying \(0 \leq F(t, x, v) \leq 1/\delta\) and

\[
\|w_\beta f\|_{L^\infty_{T, v,x}} \leq 2 \|w_\beta f_0\|_{L^\infty_{v,x}}.
\] (1.17)

**Theorem 1.2** (Global existence). Let all the assumptions in Theorem 1.1 be satisfied. Furthermore, for \(3/(3+\gamma) < p < \infty\), let \(\beta > \max\{6, 16/(5p-1)\}\). There is a constant \(C_2 = C_2(\gamma, \beta) > 0\) such that for any constant \(M \geq 1\), there are constants \(\epsilon = \epsilon(\gamma, \beta, M) > 0\) and \(M_\rho > 0\) such that if it holds that \(\|w_\beta f_0\|_{L^\infty_{v,x}} \leq M\) and

\[
\sup_{0 \leq \delta \leq 1} \{E_{\delta, \rho}(F_0) + \|f_0\|_{L^1_\delta L^\infty_{v,x}}\} \leq \epsilon M_\rho,
\] (1.18)

then the Cauchy problem on the quantum Boltzmann equation \((1.1)\) has a unique global mild solution \(F(t, x, v) = \mu_\delta \rho(v) + \sqrt{\mu_\delta \rho} f(t, x, v), (t, x, v) \in [0, \infty) \times \Omega \times \mathbb{R}^3\), in the sense of \((1.15)\), satisfying \(0 \leq F(t, x, v) \leq 1/\delta\) and

\[
\|w_\beta f\|_{L^\infty_{T, v,x}} \leq C_2 \tilde{C}_\rho \rho M^3,
\] (1.19)

for any \(T \geq 0\). Moreover, both \(\tilde{M}_\rho\) and \(\tilde{C}_\rho\) depend continuously on \(\rho \in (0, \infty)\). \(\tilde{M}_\rho\) is defined in \((1.68)\) and \((1.70)\) for \(\mathbb{R}^3\) and \(T^3\), respectively, and \(\tilde{C}_\rho\) is defined in \((1.68)\).
In what follows we give an example of initial data allowing for the large amplitude in space variable. Indeed, for given \( \rho > 0 \) and \( \beta > 0 \) as in Theorem [1.2], we let \( M \geq C_\beta / \sqrt{\beta} \) be a constant that can be arbitrarily large, where \( C_\beta := \sup_v (1 + |v|)^\beta e^{-|v|^2/4} \) is a finite positive constant. For \( \epsilon > 0 \) and \( 0 < \delta \leq 1 \), we define a set of functions as

\[
A_{\epsilon, \delta} = \{ \phi(x) : 0 < \phi(x) \leq \min\{1 + \frac{\rho}{\delta}, 1 + \frac{M}{C_\beta}\}, \|\phi \ln \phi\|_{L^1_x} + \|\phi - 1\|_{L^1_x} \leq \epsilon \}. \tag{1.20}
\]

We now take

\[
F_0(x, v) = \phi(x) \mu_{\delta, \rho}(v) = \frac{\phi(x)}{\delta + \rho e^{-\frac{|v|^2}{2}}}, \quad \phi(x) \in A_{\epsilon, \delta}. \tag{1.21}
\]

It is straightforward to verify that \( 0 < F_0(x, v) \leq 1/\delta \) and \( \|w_\beta f_0\|_{L^\infty_v} \leq M \) hold true. Moreover, direct computations give that

\[
E_{\delta, \rho}(F_0) = \int \int_{\mathbb{R}^3} \left[ \phi \mu_{\delta, \rho} \log \phi + (\phi - 1) \mu_{\delta, \rho} \log \mu_{\delta, \rho} - (\phi - 1) \mu_{\delta, \rho} \log(1 - \mu_{\delta, \rho}) + \delta (1 - \delta \mu_{\delta, \rho}) \log(1 + \frac{\delta e^{-\frac{|v|^2}{2}}}{\rho} (\phi - 1)) \right] dv dx.
\]

Noticing \( \log(1 + s) \leq s \) for any \( s > -1 \), it then follows that

\[
E_{\delta, \rho}(F_0) + \|f_0\|_{L^1_v L^\infty_x} \leq C_\rho (\|\phi \ln \phi\|_{L^1_x} + \|\phi - 1\|_{L^1_x}) \leq C_\rho \epsilon,
\]

where the upper bound \( C_{\rho, \epsilon} \) can be small as long as \( \epsilon > 0 \) is small enough. Therefore, all the assumptions on \( F_0(x, v) \) in Theorem [1.2] are satisfied. It is obvious to see that such initial data in [1.21] with \( \phi(x) \) in [1.20] allows for the large amplitude in space variable, cf. [9, 10].

For soft potentials, we first consider the structure of the perturbed modified quantum Boltzmann equation around the equilibrium and its relation with the quantum parameter. Due to the lack of spectral gap of the linearized operator, we need to introduce a smooth cut-off function in order to separate the operator into two parts. One part is degenerate and we can let it be small under integral. The other part provides the decay in \( v \), and the decay also depends on the quantum parameter. The proof starts with the mild form of the equation which includes three parts, the initial datum, the term containing the linear operator \( K_\delta \) and the term containing the nonlinear operator \( \Gamma_\delta \). The initial datum will not cause many troubles for us so we only need to take care of the rest two parts. We first deduce an estimate uniformly in \( \delta \) for the nonlinear part so that this part can be controlled by the product of \( \|w_\beta f\|_{L^\infty_v} \) and \( \|f\|_{L^1_v} \). Next we need to treat the second part with \( K_\delta \) in it. Since we divide it into two parts as mentioned above, we can control the degenerate part easily. For the other part, after one time iteration, we can bound it by \( E_{\delta, \rho}(F_0) \) which is defined in [1.40]. Then we can expect that if \( E_{\delta, \rho}(F_0) \) and \( \|f\|_{L^1_v} \) are both small, one can close our a priori assumption. We require \( E_{\delta, \rho}(F_0) \) to be sufficiently small, but we still need to obtain the smallness of \( \|f\|_{L^1_v} \). Actually, \( \|f\|_{L^1_v} \) is able to be small as long as \( \|f_0\|_{L^1_v L^\infty_x} \) and \( E_{\delta, \rho}(F_0) \) are both small uniformly in \( \delta \). Hence, at last we obtain a global solution by closing the a priori estimate. Notice we also track the effect of another parameter \( \rho > 0 \). Then for every smallness or largeness estimate, we need to know exactly how small or how large it is, which needs more accurate calculations.

The paper is organized as follows. In Section 2, we study the linear operator \( L_\delta \) and give an important lemma on the entropy. In Section 3, we construct an approximation sequence under the assumptions in Theorem [1.1] to obtain a local solution by taking the limit. Moreover, the limit function satisfies all the estimates in Theorem [1.1]. In Section 4, we derive the global estimates using the \( L^\infty_T L^\infty_{v,x} \cap L^\infty_T L^\infty_x L^1_v \) approach to prove Theorem [1.2].

2. Preliminaries

It is direct to see that when the quantum parameter \( \delta = 0 \), the equation [1.1] will become the Boltzmann equation in the classical sense. In our proof, one of the key points is to estimate the
linear operator $L_\delta$ defined in [11]. We observe that since $\delta$ is bounded, $\mu_{\delta, \rho}$ is equivalent to the Maxwellian

$$
\mu_0(v) = e^{-\frac{|v|^2}{2}}
$$

in the following sense

$$
\frac{1}{\rho + 1} \mu_0(v) \leq \mu_{\delta, \rho}(v) \leq \frac{1}{\rho} \mu_0(v).
$$

The above inequality provides a clue that we can use some known results for the classic Boltzmann equation around the global Maxwellian to help us estimate the operator $L_\delta$. Thus, we introduce some properties for the classical Boltzmann equation. The operator $L_0$ is given by

$$
L_0 = \nu - K_0,
$$

where

$$
\nu(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \mu_0(u) d\omega du \sim (1 + |v|)^\gamma,
$$

$$
K_0 f(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \sqrt{\mu_0(u)} \left( \sqrt{\mu_0(u')} f(u') + \sqrt{\mu_0(v')} f(u') - \sqrt{\mu_0(v)} f(u) \right) d\omega du.
$$

Then we have the following lemmas. Interested readers may refer to [3, 9, 12, 16] for details.

**Lemma 2.1.** Let $-3 < \gamma < 0$ and $\beta \in \mathbb{R}$. There exists a function $k = k(v, \eta)$ such that $K_0 f(v)$ can be written as

$$
K_0 f(v) = \int_{\mathbb{R}^3} k(v, \eta) f(\eta) d\eta,
$$

with estimates

$$
|k(v, \eta)| \leq C|v - \eta|^{\gamma} e^{-\frac{|v|^2}{4}} e^{-\frac{|\eta|^2}{4}} + \frac{C(\gamma)}{|v - \eta|^2} e^{-\frac{|v|^2}{8}} e^{-\frac{|\eta|^2}{8|v|^2}},
$$

and

$$
\int_{\mathbb{R}^3} |k_m(v, \eta)| d\eta \leq C(\gamma)(1 + |v|)^{-1},
$$

(2.2)

where

$$
k_m(v, \eta) := k(v, \eta) \cdot \frac{w_\beta(v)}{w_\beta(\eta)}.
$$

(2.3)

Furthermore, as in [31], if we introduce a smooth cut-off function $\chi_m = \chi_m(\tau)$ with $0 \leq m \leq 1$, $\chi_m(\tau) = 1$ for $\tau \leq m$, $\chi_m(\tau) = 0$ for $\tau \geq 2m$ and $0 \leq \chi_m \leq 1$ for $m \leq \tau \leq 2m$ and define $K_0^m$ and $K_0^c$ by

$$
K_0^m f(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \chi_m(|v - u|) \left( \frac{\sqrt{\mu_0(v')}}{\sqrt{\mu_0(v)}} \mu_0(u') f(v') + \frac{\sqrt{\mu_0(u')}}{\sqrt{\mu_0(v)}} \mu_0(v') f(u') - \sqrt{\mu_0(u)} \sqrt{\mu_0(v)} f(u) \right) d\omega du
$$

(2.4)

and

$$
K_0^c = K_0 - K_0^m,
$$

(2.5)

then the following lemma holds.
Lemma 2.2. Let \(-3 < \gamma < 0\) and \(\beta \in \mathbb{R}\). For given function \(f = f(v)\), it holds that
\[
|K_0^m f(v)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v - u, \theta) \chi_m(|v - u|) \left( \frac{\sqrt{\mu_0(v')}}{\sqrt{\mu_0(v)}} \mu_0(u') |f(v')| + \frac{\sqrt{\mu_0(v')}}{\sqrt{\mu_0(v)}} \mu_0(v') |f(u')| + \sqrt{\mu_0(v)} \sqrt{\mu_0(v)} |f(u)| \right) d\omega du
\]
\[
\leq C m^{3+\gamma} e^{-\frac{|v|^2}{2\mu}} \|f\|_{L^\infty}. \quad (2.6)
\]
Moreover, there are functions \(l_1(v, \eta), l_2(v, \eta)\), such that \(K_0^\delta f(v)\) can be written as
\[
K_0^\delta f(v) = \int_{\mathbb{R}^3} (l_2(v, \eta) - l_1(v, \eta)) f(\eta) d\eta,
\]
with estimates
\[
|K_0^\delta f(v)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v - u, \theta) (1 - \chi_m(|v - u|)) \left( \frac{\sqrt{\mu_0(v')}}{\sqrt{\mu_0(v)}} \mu_0(u') |f(v')| + \frac{\sqrt{\mu_0(v')}}{\sqrt{\mu_0(v)}} \mu_0(v') |f(u')| + \sqrt{\mu_0(v)} \sqrt{\mu_0(v)} |f(u)| \right) d\omega du
\]
\[
\leq \int_{\mathbb{R}^3} l(v, \eta) |f(\eta)| d\eta, \quad (2.7)
\]
and
\[
l(v, \eta) \leq C |v - \eta|^\gamma e^{-\frac{|v|^2}{\mu}} e^{-\frac{|v|^2}{\mu}} + \frac{C(\gamma)}{|v - \eta|^\gamma} e^{-\frac{|v|^2}{\mu}} e^{-\frac{|v|^2}{\mu}} + \frac{|v|^2 e^{-\frac{|v|^2}{\mu}}}{16e^{-\frac{|v|^2}{\mu}}} , \quad (2.8)
\]
\[
\int_{\mathbb{R}^3} l_w(v, \eta) e^{\frac{|v|^2}{\mu}} d\eta \leq C(\gamma) m^{\gamma-1} \frac{\nu(v)}{(1 + |v|)^2} ,
\]
\[
\int_{\mathbb{R}^3} l_w(v, \eta) e^{\frac{|v|^2}{\mu}} d\eta \leq C(\gamma) (1 + |v|)^{-1},
\]
where
\[
l(v, \eta) := |l_1(v, \eta)| + |l_2(v, \eta)| , \quad (2.9)
\]
\[
and \quad l_w(v, \eta) := \left( |l_1(v, \eta)| + |l_2(v, \eta)| \right) \frac{w_\beta(v)}{w_\beta(\eta)} . \quad (2.10)
\]

Now we can show the linear operator \(L_\delta\) has similar properties as above. First we will simplify the explicit form of \(L_\delta f\). The following lemma is based on direct calculations.

Lemma 2.3. Let \(0 \leq \delta < 1\) and \(\rho > 0\). Let \(\mu_{\delta, \rho}, \bar{\mu}_{\delta, \rho}\) and \(\mu_0\) be defined in \((1.3), (1.9)\) and \((2.1)\), respectively, then we have
\[
C_{1p} \mu_0(u) \leq \mu_{\delta, \rho}(u) - \delta \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(u') \leq C_{2p} \mu_0(u),
\]
\[
\sqrt{\frac{\bar{\mu}_{\delta, \rho}(u)}{\bar{\mu}_{\delta, \rho}(v)}} [\mu_{\delta, \rho}(v) - \delta \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(v')] \leq C_{2p} \sqrt{\mu_0(u) \mu_0(v)},
\]
\[
\sqrt{\frac{\bar{\mu}_{\delta, \rho}(u')}{\bar{\mu}_{\delta, \rho}(v)}} [\mu_{\delta, \rho}(v') - \delta \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(v') \mu_{\delta, \rho}(v)] \leq C_{2p} \sqrt{\mu_0(u) \mu_0(v')},
\]
\[
\sqrt{\frac{\bar{\mu}_{\delta, \rho}(v')}{\bar{\mu}_{\delta, \rho}(v)}} [\mu_{\delta, \rho}(v') - \delta \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(u) \mu_{\delta, \rho}(v') \mu_{\delta, \rho}(v)] \leq C_{2p} \sqrt{\mu_0(u) \mu_0(v')},
\]
where
\[
C_{1p} = \frac{\rho^2}{(\rho + 1)^3}, \quad C_{2p} = \frac{\rho + 1}{\rho^2}. \quad (2.12)
\]
Proof. We prove the first two inequalities and the rest two can be treated similarly. By the definition of $\mu_{\delta,\rho}$ and $\bar{\mu}_{\delta,\rho}$ in (2.3) and (2.4) and the fact that $|u|^2 + |v|^2 = |u|^2 + |v|^2$, we have

$$
\mu_{\delta,\rho}(u) - \delta \mu_{\delta,\rho}(u)\mu_{\delta,\rho}(v) - \delta \mu_{\delta,\rho}(v)\mu_{\delta,\rho}(u) + \delta \mu_{\delta,\rho}(u)\mu_{\delta,\rho}(v)
\leq \frac{\rho e^{-|v|^2} (\delta + \rho e^{-|v|^2})}{(\delta + \rho e^{-|v|^2}) (\delta + \rho e^{-|v|^2})}.
$$

Then the first estimate holds. Then by a similar argument, it follows that

$$
\sqrt{\frac{\bar{\mu}_{\delta,\rho}(u)}{\bar{\mu}_{\delta,\rho}(v)}} [\mu_{\delta,\rho}(v) - \delta \mu_{\delta,\rho}(u)\mu_{\delta,\rho}(v) - \delta \mu_{\delta,\rho}(v)\mu_{\delta,\rho}(u) + \delta \mu_{\delta,\rho}(u)\mu_{\delta,\rho}(v)] = \frac{\rho e^{-|v|^2} e^{-|v|^2}}{\delta (\delta + \rho e^{-|v|^2})} e^{-|v|^2} \leq \frac{1}{\rho} e^{-|v|^2} \leq C_{2\rho} \sqrt{\mu_{\delta,\rho}(u)\mu_{\delta,\rho}(v)}.
$$

This ends the proof of Lemma 2.3. □

We can separate $K_\delta$ into two parts as (2.5) with the same cut-off function $\chi_m = \chi_m(\tau)$ as in (2.4). Let $K_\delta^n$ be given by

$$
K_\delta^n f(v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \chi_m(|v - u|) \left[ \frac{\sqrt{\hat{\mu}_{\delta,\rho}(u') - \delta \hat{\mu}_{\delta,\rho}(u')\hat{\mu}_{\delta,\rho}(v) - \delta \hat{\mu}_{\delta,\rho}(v)\hat{\mu}_{\delta,\rho}(u) + \delta \hat{\mu}_{\delta,\rho}(u)\hat{\mu}_{\delta,\rho}(v)} f(u') \right. \\
\left. \frac{\sqrt{\hat{\mu}_{\delta,\rho}(v') - \delta \hat{\mu}_{\delta,\rho}(v')\hat{\mu}_{\delta,\rho}(u) - \delta \hat{\mu}_{\delta,\rho}(u)\hat{\mu}_{\delta,\rho}(v) + \delta \hat{\mu}_{\delta,\rho}(u)\hat{\mu}_{\delta,\rho}(v)} f(v') \right] d\omega du,
$$

and write

$$
K_0^\gamma = K_0 - K_0^n.
$$

(2.13)

With the above definitions and lemmas, we can control $L_\delta$ by $L_0$ in terms of the lemma below.

Lemma 2.4. Recall that the operators $\nu_\delta$, $K_0^n$ and $K_\delta^\gamma$ are defined in (2.13), (2.13) and (2.14), respectively, and the constants $C_{1\rho}$, $C_{2\rho}$ are defined in (2.12). Let $l = l(v, \eta)$, $k = k(v, \eta)$ be the same functions as in Lemma 2.3 and Lemma 2.4. Then for given function $f = f(v)$, there exists a constant $C > 0$ which is independent of $\rho$ such that

$$
\frac{1}{C} C_{1\rho} (1 + |v|)^\gamma \leq \nu_\delta(v) \leq C C_{2\rho} (1 + |v|)^\gamma,
$$

$$
|K_0^n f(v)| \leq C m^{3+\gamma} C_{2\rho} e^{-|v|^2} \| f \|_{L^\infty},
$$

$$
|K_\gamma^\gamma f(v)| \leq C_{2\rho} \int_{\mathbb{R}^3} |l(v, \eta)| |f(\eta)| d\eta,
$$

and

$$
|K_\delta f(v)| \leq C_{2\rho} \int_{\mathbb{R}^3} |k(v, \eta)| |f(\eta)| d\eta.
$$

Proof. By Lemma 2.3 we have

$$
C_{1\rho} \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \mu_0(u) d\omega du \leq \nu_\delta(v) \leq C_{2\rho} \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \mu_0(u) d\omega du.
$$
and

\[
|K^m_{\delta}f(v)| \leq C_2 \int_{\mathbb{R}^2} \int_{\mathbb{S}^2} B(v-u,\theta)\chi_m(|v-u|) \left( \frac{\mu_0(u')}{\mu_0(v)} |f(u')| + \frac{\mu_0(u')}{\mu_0(v)} |f(u')| + \sqrt{\mu_0(u') \mu_0(v)} |f(u')| \right) d\omega du,
\]

\[
|K^m_{\delta}f(v)| \leq C_2 \int_{\mathbb{R}^2} \int_{\mathbb{S}^2} B(v-u,\theta)(1-\chi_m(|v-u|)) \left( \frac{\mu_0(u')}{\mu_0(v)} |f(u')| + \frac{\mu_0(u')}{\mu_0(v)} |f(u')| + \sqrt{\mu_0(u') \mu_0(v)} |f(u')| \right) d\omega du.
\]

In the last two inequalities above, we have used the fact

\[
\sqrt{\mu_0(u') \mu_0(v)} = \frac{\mu_0(u')}{\mu_0(v)}, \quad \frac{\mu_0(u')}{\mu_0(v)} = \frac{\sqrt{\mu_0(u') \mu_0(v)}}{\mu_0(v)} = \frac{\sqrt{\mu_0(v')}}{\mu_0(v)}.
\]

Then by (2.7) and the fact that \( \frac{1}{(1+|v|)^\gamma} \leq \int_{\mathbb{R}^2} B(v-u,\theta)\mu_0(u)d\omega du \leq C(1+|v|)^\gamma \) for some constant \( C > 0 \) which is independent of \( \rho \), the proof is complete.

Recalling our definition of \( E_{\delta,\rho} \) in (1.3), we have the following lemma which will be used later.

**Lemma 2.5.** Let \( F(t,x,v) \) satisfy (1.4), (1.5) and (1.6), then we have

\[
\int_{\Omega} \int_{\mathbb{R}^2} \left\{ \frac{|F(t,x,v) - \mu_{\delta,\rho}(v)|^2}{4\mu_{\delta,\rho}(v)} \chi(|F(t,x,v) - \mu_{\delta,\rho}(v)| \leq \mu_{\delta,\rho}(v)) \right. \\
\left. + \frac{|F(t,x,v) - \mu_{\delta,\rho}(v)|}{2} \chi(|F(t,x,v) - \mu_{\delta,\rho}(v)| \geq \mu_{\delta,\rho}(v)) \right\} dv dx \leq E_{\delta,\rho}(F_0).
\]

**Proof.** By the entropy inequality (1.6) and Taylor expansion, it follows that

\[
\mathcal{H}_{\delta,\rho}(F(t)) = \int_{\Omega} \int_{\mathbb{R}^2} \left\{ F(t,x,v) \log F(t,x,v) + \frac{1}{\delta} (1 - \delta F(t,x,v)) \log(1 - \delta F(t,x,v)) \right. \\
\left. - \mu_{\delta,\rho}(v) \log \mu_{\delta,\rho}(v) - \frac{1}{\delta} (1 - \delta \mu_{\delta,\rho}(v)) \log(1 - \delta \mu_{\delta,\rho}(v)) \right\} dv dx
\]

\[
= \int_{\Omega} \int_{\mathbb{R}^2} \left\{ \log \frac{\mu_{\delta,\rho}(v)}{1 - \delta \mu_{\delta,\rho}(v)} (F(t,x,v) - \mu_{\delta,\rho}(v)) \\
+ \frac{1}{2} \left( 1 - \frac{\delta}{1 - \delta F} \right) |F(t,x,v) - \mu_{\delta,\rho}(v)|^2 \right\} dv dx \leq \mathcal{H}_{\delta,\rho}(F(0)),
\]

where \( \tilde{F} \) is between \( F \) and \( \mu_{\delta,\rho} \). Since \( 1 - \delta F \geq 0 \) and \( 1 - \delta \mu_{\delta,\rho} \geq 0 \), we have \( 1 - \delta \tilde{F} \geq 0 \). By a direct calculation, we also obtain

\[
\log \frac{\mu_{\delta,\rho}(v)}{1 - \delta \mu_{\delta,\rho}(v)} = -\log \rho - \frac{|v|^2}{2}.
\]

Then it holds that

\[
\int_{\Omega} \int_{\mathbb{R}^2} \frac{1}{2 \tilde{F}} |F(t,x,v) - \mu_{\delta,\rho}(v)|^2 dv dx \leq \int_{\Omega} \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \left( 1 - \frac{\delta}{1 - \delta F} \right) |F(t,x,v) - \mu_{\delta,\rho}(v)|^2 \right\} dv dx
\]

\[
\leq \mathcal{H}_{\delta,\rho}(F(0)) + \log \rho \int_{\Omega} \int_{\mathbb{R}^2} (F(t,x,v) - \mu_{\delta,\rho}(v)) dv dx + \frac{|v|^2}{2} \int_{\Omega} \int_{\mathbb{R}^2} (F(t,x,v) - \mu_{\delta,\rho}(v)) dv dx
\]

\[
= E_{\delta,\rho}(F_0).
\]

In the last equality above, we have used the conservation of mass and energy in (1.4) and (1.5). Noticing that \( F \geq 0 \), then \( |F - \mu_{\delta,\rho}| \geq 0 \) implies \( F \geq 2\mu_{\delta,\rho} \) or \( F = 0 \). Then for \( F \geq 2\mu_{\delta,\rho} \), we have

\[
\frac{|F(t,x,v) - \mu_{\delta,\rho}(v)|}{2F} \geq \frac{F(t,x,v) - \mu_{\delta,\rho}(v)}{2F} = \frac{1}{2} \frac{\mu_{\delta,\rho}}{2F} \geq \frac{1}{4}.
\]
For $F = 0$,
\[ \frac{|F(t, x, v) - \mu_{\delta, \rho}(v)|}{2F} \geq \frac{\mu_{\delta, \rho}(v)}{2\mu_{\delta, \rho}} \geq \frac{1}{4}. \]

Then the lemma follows from the computations above.

**Remark 2.1.** In Lemma 2.8, if we let $\rho = (2\pi)^{\frac{3}{2}}$, $\delta \rightarrow 0$, then it holds that

\[ \mathcal{H}_{\delta, \rho}(F(t)) = \int_{\Omega} \int_{\mathbb{R}^3} \{F(t, x, v) \log F(t, x, v) - F(t, x, v) - \mu_{\delta, \rho}(v) \log \mu_{\delta, \rho}(v) + \mu_{\delta, \rho}(v)\} \, dv \, dx \]

\[ = \int_{\Omega} \int_{\mathbb{R}^3} \{F(t, x, v) \log F(t, x, v) - \mu_{\delta, \rho}(v) \log \mu_{\delta, \rho}(v)\} \, dv \, dx - M_0. \]

$E_{\delta, \rho}(F_0)$ will become

\[ E_{\delta, \rho}(F_0) = \int_{\Omega} \int_{\mathbb{R}^3} \{F(t, x, v) \log F(t, x, v) - \mu_{\delta, \rho}(v) \log \mu_{\delta, \rho}(v)\} \, dv \, dx + (\frac{3}{2} \log 2\pi - 1)M_0 + \frac{1}{2}E_0. \]

This is consistent with Lemma 2.7 in [G].

### 3. Approximation Sequence and Local Existence

In this section we prove Theorem 1.1. We first construct the approximation sequence $\{F^n\}_{n=1}^{\infty}$ from (1.1) as follows:

\[ \partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ F^n(u')F^n(v') \left( 1 - \delta F^n(u) \right) \left( 1 - \delta F^{n+1}(v) \right) \right] \, d\omega \, du, \quad (3.1) \]

with $F^n(0, x, v) = F_0(x, v)$ and $F^0(t, x, v) = 0$. We will prove that under the assumptions in Theorem 1.1, our approximation sequence $F^n = \mu_{\delta, \rho} + \sqrt{\mu_{\delta, \rho}} F^n$ satisfies the pointwise bound $0 \leq F^n \leq \frac{1}{\delta}$. Moreover, for $T_1$ defined in (1.10), it holds that

\[ \|w_\beta f^n\|_{L^\infty_{t,x}} \leq 2 \|w_\beta f_0\|_{L^\infty_{t,x}}, \]

and $\{w_\beta f^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^\infty([0, T_1] \times \Omega \times \mathbb{R}^3)$ with a unique limit. By passing the limit, we can obtain a function $f = \lim_{n \rightarrow \infty} f^n$ which is a mild solution to the quantum Boltzmann equation in the sense of (1.15). We first prove the pointwise bound of $F^n$.

**Lemma 3.1.** The sequence $\{F^n\}_{n=1}^{\infty}$ which is constructed in (3.1) satisfies $0 \leq F^n \leq \frac{1}{\delta}$ for $n = 0, 1, 2, \cdots$.

**Proof.** When $n = 0$, $0 = F^0 \leq 1$. We rewrite (3.1) as

\[ \partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} + g_1 \cdot F^{n+1} = \tilde{C}_1(F^n), \quad (3.2) \]

where

\[ g_1(t, x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ \delta F^n(t, x, u')F^n(t, x, v')(1 - \delta F^n(t, x, u)) \right. \]

\[ \left. + F^n(t, x, u') \left( 1 - \delta F^n(t, x, u') \right) \left( 1 - \delta F^n(t, x, v') \right) \right] \, d\omega \, du, \quad (3.3) \]

and

\[ \tilde{C}_1(F^n) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) F^n(u')F^n(v') \left( 1 - \delta F^n(u) \right) \, d\omega \, du. \quad (3.4) \]

We define $G^n(t, x, v) = 1 - \delta F^n(t, x, v)$. When $n = 0$, $0 \leq G^n(t, x, v) = 1$. By direct calculation we have

\[ \partial_t G^{n+1} + v \cdot \nabla_x G^{n+1} = -\delta \left( \partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} \right) \]

\[ = -\delta \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ F^n(u')F^n(v')G^n(u)G^{n+1}(v) - F^n(u) \frac{1 - G^{n+1}(v)}{\delta} G^n(u')G^n(v') \right] \, d\omega \, du \]

\[ = -g_2 \cdot G^{n+1} + \tilde{C}_2(G^n), \quad (3.5) \]
where
\[ g^n_2(t, x, v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \left[ \delta F^n(t, x, u') F^n(t, x, v') G^n(t, x, u) + F^n(t, x, u) G^n(t, x, u') G^n(t, x, v') \right] d\omega du, \quad (3.6) \]

and
\[ \tilde{C}_2(G^n) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) F^n(u) G^n(u') G^n(v') d\omega du. \quad (3.7) \]

Similarly as above, (3.5) can be written as
\[ \partial_t G^{n+1} + v \cdot \nabla_x G^{n+1} + g^n_2 \cdot G^{n+1} = \tilde{C}_2(G^n). \quad (3.8) \]

For (3.2) and (3.3), we integrate along the characteristics to get
\[ F^{n+1}(t, x, v) = e^{-\tau^n(t,x,v)} F_0(x - vt, v) + \int_0^t e^{-\int_0^\tau \delta^n_2(\tau', x - v(\tau'-t), v) d\tau} \tilde{C}_1(F^n)(s, x - v(t - s), v) ds, \quad (3.9) \]

and
\[ G^{n+1}(t, x, v) = e^{-\tau^n(t,x,v)} G_0(x - vt, v) + \int_0^t e^{-\int_0^\tau \delta^n_2(\tau', x - v(\tau'-t), v) d\tau} \tilde{C}_2(G^n)(s, x - v(t - s), v) ds, \quad (3.10) \]

where
\[ G_0(x, v) = (1 - \delta F_0)(x, v). \]

We prove our lemma by induction. If \( 0 \leq F^n \leq \frac{1}{2} \), then \( 0 \leq G^n \leq \frac{1}{2} \), then \( g^n_1(t, x, v) \geq 0 \), \( g^n_2(t, x, v) \geq 0 \), \( \tilde{C}_1(G^n)(t, x, v) \geq 0 \), \( \tilde{C}_2(G^n)(t, x, v) \geq 0 \) from (3.3), (3.6), (3.4) and (3.7) respectively. Hence, \( F^{n+1}(t, x, v) \geq 0 \) and \( G^{n+1}(t, x, v) \geq 0 \) from (3.9) and (3.10). Then we conclude that \( 0 \leq F^{n+1}(t, x, v) \leq \frac{1}{2} \). Moreover, from our proof, we can also obtain
\[ g^n_1(t, x, v) \geq 0, \quad (3.11) \]

for \( n = 0, 1, 2, \ldots \). \( \square \)

Substituting \( F^n = \mu_{s, \rho} + \sqrt{\mu_{s, \rho}} f^n \) into the mild form (3.9), we get
\[ w_{s}(t, x, v) = e^{-\int_0^t \delta^n_1(\tau, x - v(t - \tau), v) d\tau} w_{s}(v) f_0(x - vt, v) + \int_0^t e^{-\int_0^\tau \delta^n_2(\tau', x - v(\tau'-t), v) d\tau} w_{s}(v) (K_{s}f^n)(s, x - v(t - s), v) ds + \int_0^t e^{-\int_0^\tau \delta^n_2(\tau', x - v(\tau'-t), v) d\tau} w_{s}(v) \Gamma_{s+}(f^n)(s, x - v(t - s), v) ds, \quad (3.12) \]

where \( K_{s} \) is defined in (1.14) and
\[ \Gamma_{s+}(f)(t, x, v) = \frac{1}{\sqrt{\mu_{s, \rho}(v)}} \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \left[ \sqrt{\mu_{s, \rho}(f(t, x, u'))} \sqrt{\mu_{s, \rho}(f(t, x, v'))} (1 - \delta \mu_{s, \rho}(v) - \delta \mu_{s, \rho}(u)) + \delta \sqrt{\mu_{s, \rho}(f(t, x, u'))} \sqrt{\mu_{s, \rho}(f(t, x, v'))} (\mu_{s, \rho}(v) - \mu_{s, \rho}(u')) + \delta \sqrt{\mu_{s, \rho}(f(t, x, u'))} \sqrt{\mu_{s, \rho}(f(t, x, v'))} (\mu_{s, \rho}(v) - \mu_{s, \rho}(v')) - \delta \sqrt{\mu_{s, \rho}(f(t, x, u'))} \sqrt{\mu_{s, \rho}(f(t, x, v'))} (\mu_{s, \rho}(v) - \mu_{s, \rho}(u')) \right] d\omega du. \quad (3.13) \]
By (3.3), we rewrite $g_1^n$ as

$$g_1^n(t, x, v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \left[ \sqrt{\mu_{\delta, \rho} f^n(u)} (1 - \mu_{\delta, \rho}(v') - \mu_{\delta, \rho}(u')) - \delta \sqrt{\mu_{\delta, \rho} f^n(v')} (\mu_{\delta, \rho}(u) - \mu_{\delta, \rho}(u')) - \delta \sqrt{\mu_{\delta, \rho} f^n(u')} (\mu_{\delta, \rho}(v) - \mu_{\delta, \rho}(v')) + \delta \sqrt{\mu_{\delta, \rho} f^n(u')} \sqrt{\mu_{\delta, \rho} f^n(v')} \right] d\omega d\theta + \nu_\delta(v),$$

(3.14)

where $\nu_\delta(v)$ is defined in (1.13). Using properties of $g_1^n$, $K_\delta$ and $\Gamma_{\delta^+}$, we prove the following lemma, which implies the boundedness of $w_\beta f^n$.

**Lemma 3.2.** There exists a constant $C' = C'(\beta, \gamma)$ independent of $\rho$ and a positive time $T_1'$ defined as

$$T_1' = \frac{C'}{C_5 \rho \left( 1 + \|w_\beta f_0\|_{L^\infty_{T, x}} + \|w_\beta f_0\|_{L^\infty_{T, x}}^2 \right)},$$

for some constant $C_5 \rho$, which is given in (3.20), such that we have

$$\|w_\beta f^n\|_{L^\infty_{T, x}} \leq 2 \|w_\beta f_0\|_{L^\infty_{T, x}},$$

(3.15)

for $n = 0, 1, 2, \ldots$.

**Proof.** For $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3$, since we have Lemma 3.1 for the positivity of $g_1^n$, it follows from (3.12) that

$$|w_\beta(v)f^{n+1}(t, x, v)| \leq \|w_\beta f_0\|_{L^\infty_{T, x}} + \int_0^t |w_\beta(v)(K_\delta f^n)(s, x - v(t - s), v)| ds$$

$$+ \int_0^t |w_\beta(v)(\Gamma_{\delta^+}(f^n))(s, x - v(t - s), v)| ds$$

$$= \|w_\beta f_0\|_{L^\infty_{T, x}} + I_1 + I_2.$$

(3.16)

Now we estimate $I_1$. By the definition of $K_\delta$ (1.14) and Lemma 2.3, it holds that

$$I_1 = \int_0^t |w_\beta(v)(K_\delta f^n)(s, x - v(t - s), v)| ds$$

$$\leq C C_2 \rho \int_0^t \int_{\mathbb{R}^3} |w_\beta(v, \eta)| \|w_\beta f^n\|_{L^\infty_{T, x}} |k_w(v, \eta)| |s - v(t - s), \eta)| |ds dq ds$$

$$\leq C C_2 \rho \|w_\beta f^n\|_{L^\infty_{T, x}} \int_0^t \int_{\mathbb{R}^3} |k_w(v, \eta)| ds dq ds$$

$$\leq CTC_2 \rho \|w_\beta f^n\|_{L^\infty_{T, x}}.$$

(17.17)

where $k_w(v, \eta)$ is defined in (2.3), $C_2 \rho$ is given in (2.12). In the last inequality above, we have used the fact that $\int_{\mathbb{R}^3} |k_w(v, \eta)| |dv$ is finite uniformly in $v$ by (2.2).

For $I_2$, denoting $x_1 = x - v(t - s)$, we first observe from (3.13) that

$$|w_\beta(v)(\Gamma_{\delta^+}(f^n))(s, x_1, v)|$$

$$\leq \frac{C}{\sqrt{\mu_{\delta, \rho}(v)}} \int_{\mathbb{R}^3} B(v - u, \theta) \left[ |w_\beta(v)| \sqrt{\mu_{\delta, \rho} f^n(s, x_1, v')} |\sqrt{\mu_{\delta, \rho} f^n(s, x_1, v')}| + |w_\beta(v)| \|\sqrt{\mu_{\delta, \rho} f^n(s, x_1, v')} \| (\mu_{\delta, \rho}(v) + \mu_{\delta, \rho}(u')) \right.$$

$$+ |w_\beta(v)| \|\sqrt{\mu_{\delta, \rho} f^n(s, x_1, u')} \| (\mu_{\delta, \rho}(v) + \mu_{\delta, \rho}(u'))$$

$$+ |w_\beta(v)| \rho^{-\frac{1}{2}} \|\sqrt{\mu_{\delta, \rho} f^n(s, x_1, u')} \| |w_\beta f^n|_{L^\infty_{T, x}} \right] d\omega d\theta$$

$$= I_{21} + I_{22} + I_{23} + I_{24}$$


by the fact that $0 \leq \delta \leq 1$, $0 \leq \delta \mu_{\delta, \mu}(v) = \delta/\left(\delta + \rho e^{\frac{|v|^2}{2}}\right) \leq 1$, and $\sqrt{\mu_{\delta, \mu}(v)} = \sqrt{\rho e^{\frac{|v|^2}{2}}}/(\delta + \rho e^{\frac{|v|^2}{2}}) \leq \rho^{-\frac{1}{2}}$. Moreover, by (12) we have
\[
\frac{\sqrt{\mu_{\delta, \mu}(u')}\sqrt{\mu_{\delta, \mu}(v')}}{\sqrt{\mu_{\delta, \mu}(v)}} = \frac{\sqrt{\rho e^{\frac{|u'|^2}{2}}}\sqrt{\rho e^{\frac{|v'|^2}{2}}} \delta + \rho e^{\frac{|v|^2}{2}}}{\delta + \rho e^{\frac{|v'|^2}{2}}}
\leq C_3 \rho e^{-\frac{|u'|^2}{2}} e^{-\frac{|v'|^2}{2}} e^{\frac{|v|^2}{2}}
= C_3 \rho e^{-\frac{|u|^2}{2}},
\]
where
\[
C_3 \rho = \frac{\sqrt{\rho(\rho + 1)}}{\rho^2}.
\]
Using (12) one more time, since $|v|^2 \leq |u'|^2 + |v'|^2$, we have $w_{\beta}(v) \leq C w_{\beta}(u')w_{\beta}(v')$ for some constant $C$, which yields
\[
I_{21} + I_{24}
= C \left(1 + \rho^{-\frac{3}{2}} \left\|w_{\beta} f^n\right\|_{L^\infty_{v,x}}\right)
\times \frac{\int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(v - u, \theta) e^{-\frac{|u|^2}{2}} \left|w_{\beta}(v)\sqrt{\mu_{\delta, \mu}(v)} f^n(s, x_1, u')\sqrt{\mu_{\delta, \mu}(s, v_1, v')}\right| d\omega du}{C \left(1 + \rho^{-\frac{1}{2}} \left\|w_{\beta} f^n\right\|_{L^\infty_{v,x}}\right) C_3 \rho \left\|w_{\beta} f^n\right\|_{L^\infty_{v,x}}}.
\]
We now turn to $I_{22}$ and $I_{23}$. Recall
\[
I_{22} = C \frac{1}{\sqrt{\mu_{\delta, \mu}(u')}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(v - u, \theta) w_{\beta}(v)
\left|\sqrt{\mu_{\delta, \mu}(v)} f^n(s, x_1, v') \mid \sqrt{\mu_{\delta, \mu}(s, x_1, u)} \left(\mu_{\delta, \mu}(v) + \mu_{\delta, \mu}(u')\right)\right| d\omega du.
\]
Noticing we have $\mu_{\delta, \mu}(v) = \left(\delta + \rho e^{\frac{|v|^2}{2}}\right)^{-1} \leq \rho^{-1} e^{-\frac{|v|^2}{2}}$ and $w_{\beta}(v) = (1 + |v|^2)^{\frac{1}{2}} \leq C (1 + |u'|^2)^{\frac{1}{2}} \leq C (1 + |v'|^2)^{\frac{1}{2}}$, similar calculations as (3.18) and (3.20) show that
\[
w_{\beta}(v)\sqrt{\mu_{\delta, \mu}(v')}\sqrt{\mu_{\delta, \mu}(u)} f^n(v') f^n(u) \mu_{\delta, \mu}(v) \leq C \frac{C_3 \rho}{C} e^{-\frac{|v|^2}{2}} \left\|w_{\beta} f^n\right\|_{L^\infty_{v,x}}
\]
and
\[
w_{\beta}(v)\sqrt{\mu_{\delta, \mu}(v')}\sqrt{\mu_{\delta, \mu}(u)} f^n(s, x_1, v') f^n(s, x_1, u) \left|\mu_{\delta, \mu}(u')\right| \leq C \frac{C_3 \rho}{C} e^{-\frac{|v|^2}{2}} \left\|w_{\beta} f^n\right\|_{L^\infty_{v,x}} \leq C \frac{C_3 \rho}{C} e^{-\frac{|v|^2}{2}} \left\|w_{\beta} f^n\right\|_{L^\infty_{v,x}}^2.
\]
Thus, from the above estimate and (3.16), we have

\[
I_{22} \leq C C_4 \beta \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^\gamma e^{-\frac{|u|^2}{2}} \, d\omega \, du
\]

\[
\leq C C_4 \beta \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2.
\]

where

\[
C_4 \beta = \frac{C_3 \beta}{\rho} = \frac{\sqrt{\rho} (\rho + 1)}{\rho^3}.
\]

Recall

\[
I_{23} = \frac{C}{\sqrt{\mu_\delta,\rho(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) w_f(v) \left| \sqrt{\mu_\delta,\rho} f^n(s, x_1, u') \right| \left( \sqrt{\mu_\delta,\rho} f^n(s, x_1, u) (\mu_\delta,\rho(v) + \mu_\delta,\rho(v')) \right) \, d\omega \, du.
\]

Similarly as (3.22) and (3.23), we have

\[
w_f(v) \sqrt{\mu_\delta,\rho(u')} \sqrt{\mu_\delta,\rho(u)} \left( f^n(s, x_1, u') \right) \left( f^n(s, x_1, u) (\mu_\delta,\rho(v) + \mu_\delta,\rho(v')) \right)
\]

\[
\leq C C_4 \beta \left( e^{-\frac{|v|^2}{4}} \|f^n(s, x_1, u')\| \|f^n(s, x_1, u)\| \left( \mu_\delta,\rho(v) + \mu_\delta,\rho(v') \right) \right)
\]

\[
+ e^{-\frac{|v|^2}{4}} \|f^n(s, x_1, u')\| \|f^n(s, x_1, u)\| \left( \mu_\delta,\rho(v) + \mu_\delta,\rho(v') \right)
\]

\[
\leq C C_4 \beta e^{-\frac{|v|^2}{4}} \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2,
\]

which implies

\[
I_{23} \leq C C_4 \beta \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2.
\]

Combining (3.20), (3.24) and (3.27), we obtain

\[
|w_f(v) \Gamma_{\delta,\rho}(f^n)(s, x_1, v)|
\]

\[
\leq C C_5 \beta \left( 1 + \rho^{-\frac{1}{2}} \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty} \right) \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2 + C C_4 \beta \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2.
\]

Thus, from the above estimate and (3.16) we have

\[
I_2 = \int_0^t \left| w_f(v) \Gamma_{\delta,\rho}(f^n)(s, x - v(t - s), v) \right| \, ds
\]

\[
\leq C T C_5 \beta \left[ 1 + \rho^{-\frac{1}{2}} \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty} \right] \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2 + C T C_4 \beta \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2.
\]

We observe that \( C_3 \rho \rho^{-\frac{1}{2}} = C_2 \rho \). By (3.10), (3.17) and (3.28), it follows that

\[
\|w_f^{n+1}\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty} \leq \|w_f^{0}\|_{\overline{L}_v^\infty \overline{L}_{x}^\infty} + C T C_2 \rho \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty} + C T C_3 \rho \left( 1 + \rho^{-\frac{1}{2}} \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty} \right) \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2 + C T C_4 \rho \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2
\]

\[
\leq \|w_f^{0}\|_{\overline{L}_v^\infty \overline{L}_{x}^\infty} + C T C_5 \rho \left( \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty} + \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^2 + \|w_f^n\|_{\overline{L}_T^\infty \overline{L}_{v,x}^\infty}^3 \right),
\]

where

\[
C_5 \rho = C_2 \rho + C_3 \rho + C_4 \rho.
\]

We choose

\[
T_1 = \frac{1}{16 C_5 \rho \left( 1 + \|w_f^{0}\|_{\overline{L}_v^\infty \overline{L}_{x}^\infty} + \|w_f^{0}\|_{\overline{L}_v^\infty \overline{L}_{x}^\infty}^2 \right)}.
\]

Then (3.18) holds from (3.24) and (3.31).
Based on Lemma 3.2, we can prove that the sequence \(\{w_\beta f^n\}_{n=1}^\infty\) is a Cauchy sequence for \((t, x, v) \in [0, T'_1] \times \Omega \times \mathbb{R}^3\) where \(T'_1\) is defined in the following lemma.

**Lemma 3.3.** There exists a constant \(C'' = C''(\beta, \gamma)\) which is independent of \(\rho\) such that

\[
\left\|w_\beta f^{n+2} - w_\beta f^{n+1}\right\|_{L^\infty_v L^1_{t,x}} \leq \frac{1}{2} \left\|w_\beta f^{n+1} - w_\beta f^n\right\|_{L^\infty_v L^\infty_{t,x}}, \tag{3.32}
\]

for \(n = 0, 1, 2, \ldots\), where the positive time \(T''_1\) is defined by

\[
T''_1 = \min \left\{ T'_1, \frac{C''}{C_{6\rho} \left(1 + \|w_\beta f_0\|_{L^\infty_v L^\infty_{t,x}} + \|w_\beta f_0\|_{L^2_v L^\infty_{t,x}}^2\right)} \right\},
\]

and \(C_{6\rho}\) is given in (4.41).

**Proof.** We first assume that \((t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3\) for \(T \leq T'_1\) where \(T'_1\) is defined in (3.31) so that Lemma 3.2 holds. By the mild form (3.12), we can write \(w_\beta f^{n+2} - w_\beta f^{n+1}\) as

\[
w_\beta(v)(f^{n+2} - f^{n+1})(t, x, v)
= w_\beta(v)f_0(x - vt, v) \left( e^{-\int_0^t g^{n+1}_\beta(\tau, x - v(t - \tau), v) d\tau} - e^{-\int_0^t g^n_\beta(\tau, x - v(t - \tau), v) d\tau} \right)
+ \int_0^t w_\beta(v)(K_\delta f^{n+1})(s, x - v(t - s), v) \left( e^{-\int_s^t g^{n+1}_\beta(\tau, x - v(t - \tau), v) d\tau} - e^{-\int_s^t g^n_\beta(\tau, x - v(t - \tau), v) d\tau} \right) ds
+ \int_0^t w_\beta(v)\Gamma_\delta+(f^{n+1})(s, x - v(t - s), v)
\times \left( e^{-\int_s^t g^{n+1}_\beta(\tau, x - v(t - \tau), v) d\tau} - e^{-\int_s^t g^n_\beta(\tau, x - v(t - \tau), v) d\tau} \right) ds
+ \int_0^t e^{-\int_0^s g^n_\beta(\tau, x - v(t - \tau), v) d\tau} w_\beta(v) \left( K_\delta f^{n+1} - K_\delta f^n \right)(s, x - v(t - s), v) ds
+ \int_0^t e^{-\int_0^s g^n_\beta(\tau, x - v(t - \tau), v) d\tau} w_\beta(v) \left( \Gamma_\delta+(f^{n+1}) - \Gamma_\delta+(f^n) \right)(s, x - v(t - s), v) ds.
\]

By the positivity of \(g^{n}_\beta\) given in (3.11) and the fact that \(|e^{-a} - e^{-b}| \leq |a - b|\) for any \(a, b \geq 0\), it follows that

\[
|w_\beta(v)(f^{n+2} - f^{n+1})(t, x, v)|
\leq |w_\beta(v)f_0(x - vt, v)| \int_0^t \left| (g^{n+1}_\beta - g^n_\beta)(\tau, x - v(t - \tau), v) \right| d\tau
+ \int_0^t \left| w_\beta(v)(K_\delta f^{n+1})(s, x - v(t - s), v) \right| \int_s^t \left| (g^{n+1}_\beta - g^n_\beta)(\tau, x - v(t - \tau), v) \right| d\tau ds
+ \int_0^t \left| w_\beta(v)\Gamma_\delta+(f^{n+1})(s, x - v(t - s), v) \right| \int_s^t \left| (g^{n+1}_\beta - g^n_\beta)(\tau, x - v(t - \tau), v) \right| d\tau ds
+ \int_0^t \left| w_\beta(v) \left( K_\delta f^{n+1} - K_\delta f^n \right)(s, x - v(t - s), v) \right| ds
+ \int_0^t \left| w_\beta(v) \left( \Gamma_\delta+(f^{n+1}) - \Gamma_\delta+(f^n) \right)(s, x - v(t - s), v) \right| ds
= I_3 + I_4 + I_5 + I_6 + I_7. \tag{3.33}
\]
We consider $\int_0^t |(g_1^{n+1} - g_1^n)(\tau, x - v(t - \tau), v)| \, d\tau$ first. From the definition of $g_1^n$ (3.14), we directly obtain that

$$
\int_0^t \int_{\mathbb{R}^2} B(v - u, \theta) \left[ \sqrt{\mu_{\delta, \rho}(u)} \left| (f^{n+1} - f^n)(\tau, u) \right| \right] \, d\omega \, du \, d\tau \\
\leq \int_0^t \int_{\mathbb{R}^2} B(v - u, \theta) \left[ \sqrt{\mu_{\delta, \rho}(u)} \left| (f^{n+1} - f^n)(\tau, u) \right| \right] \, d\omega \, du \, d\tau
$$

Direct calculations show that $0 \leq \delta \leq 1$ and $\sqrt{\mu_{\delta, \rho}(u)}(\leq \rho^{-\frac{1}{2}} e^{-\frac{|u|^2}{4}}$ hold true for $0 \leq \delta \leq 1$. We then have

$$
I_1' \leq C \rho^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}^2} B(v - u, \theta) e^{-\frac{|u|^2}{4}} \sqrt{\mu_{\delta, \rho}(u)} \left| f^{n+1} - f^n \right| \, d\omega \, du \, d\tau
$$

For $I_2'$ and $I_3'$, we can get $e^{-\frac{|u|^2}{4}}$ by the following inequality

$$
\max \left\{ \sqrt{\mu_{\delta, \rho}(v')\mu_{\delta, \rho}(u)}, \sqrt{\mu_{\delta, \rho}(v')\mu_{\delta, \rho}(u)}, \sqrt{\mu_{\delta, \rho}(v')\mu_{\delta, \rho}(u)}, \sqrt{\mu_{\delta, \rho}(v')\mu_{\delta, \rho}(v')} \right\} \leq \rho^{-\frac{1}{2}} e^{-\frac{|u|^2}{4}}.
$$

A similar argument as in (3.35) shows that

$$
I_2' + I_3' \leq CT \rho^{-\frac{1}{2}} \left| f^{n+1} - f^n \right|_{L_T^\infty L^\infty_{v,x}}.
$$

For $I_4'$ and $I_5' + I_6'$, we first split the terms in the $L^\infty$ norm as follows:

$$
\left| f^{n+1}(u')f^{n+1}(v') - f^n(u')f^n(v') \right| \\
\leq \left| f^{n+1}(u') \right| \left| f^{n+1}(u') - f^n(u') \right| + \left| f^n(u') \right| \left| f^{n+1}(v') - f^n(v') \right|.
$$

Then by a similar inequality as (3.36) that

$$
\max \left\{ \sqrt{\mu_{\delta, \rho}(u)}\sqrt{\mu_{\delta, \rho}(v')}, \sqrt{\mu_{\delta, \rho}(u)}\sqrt{\mu_{\delta, \rho}(v')}, \sqrt{\mu_{\delta, \rho}(v')}\sqrt{\mu_{\delta, \rho}(v')} \right\} \leq \rho^{-\frac{1}{2}} e^{-\frac{|u|^2}{4}},
$$

we yield

$$
I_4' + I_5' + I_6' \leq CT \rho^{-\frac{1}{2}} \left( \left\| f^{n+1} \right\|_{L_T^\infty L^\infty_{v,x}} + \left\| f^n \right\|_{L_T^\infty L^\infty_{v,x}} \right) \left\| f^{n+1} - f^n \right\|_{L_T^\infty L^\infty_{v,x}}.
$$

Combining (3.34), (3.35), (3.37) and (3.39), it follows that

$$
\int_0^t \left| (g_1^{n+1} - g_1^n)(\tau, x - v(t - \tau), v) \right| \, d\tau \\
\leq CT \left( \rho^{-\frac{1}{2}} + \rho^{-\frac{1}{2}} + \rho^{-1} \left( \left\| f^{n+1} \right\|_{L_T^\infty L^\infty_{v,x}} + \left\| f^n \right\|_{L_T^\infty L^\infty_{v,x}} \right) \right) \left\| f^{n+1} - f^n \right\|_{L_T^\infty L^\infty_{v,x}}.
$$
Then by (3.32), (3.40) and Lemma 3.29 we have
\[ I_3 + I_4 + I_5 \leq CT \left( \rho^{-\frac{1}{2}} + \rho^{-\frac{3}{2}} + \rho^{-1} \right) \| f_0 \|_{L^{\infty}_{\mathbb{R}^3}} \left\| f^{n+1} - f^n \right\|_{L^{\infty}_{\mathbb{R}^3}} \]
\[ + \int_0^t \left| w_\beta(v) K_\delta f^{n+1}(s, x - v(t - s), v) \right| ds \]
\[ + \int_0^t \left| w_\beta(v) \Gamma_\delta_+(f^{n+1})(s, x - v(t - s), v) \right| ds, \]
where \( \delta \geq 0 \). Recalling our assumption that \( T \leq T' \), similar calculations as in (3.17), (3.28) and (3.29) yield
\[ I_3 + I_4 + I_5 \leq CT \left( \rho^{-\frac{1}{2}} + \rho^{-\frac{3}{2}} + \rho^{-1} \right) \| f_0 \|_{L^{\infty}_{\mathbb{R}^3}} \left\| f^{n+1} - f^n \right\|_{L^{\infty}_{\mathbb{R}^3}} \]
\[ + C_\delta \left( \| w_\beta f_0 \|_{L^{\infty}_{\mathbb{R}^3}} + C T C_\delta \| w_\beta f^{n+1} \|_{L^{\infty}_{\mathbb{R}^3}} + \left\| w_\beta f^{n+1} \right\|_{L^{\infty}_{\mathbb{R}^3}} \right) \]
\[ \leq CT \left( \rho^{-\frac{1}{2}} + \rho^{-\frac{3}{2}} + \rho^{-1} \right) \left( \| w_\beta f_0 \|_{L^{\infty}_{\mathbb{R}^3}} + \| w_\beta f_0 \|_{L^{\infty}_{\mathbb{R}^3}} \right) \left\| f^{n+1} - f^n \right\|_{L^{\infty}_{\mathbb{R}^3}}, \]
(3.41)
where \( C_\delta \) is defined in (3.30).
Recall from (3.32) that
\[ I_6 = \int_0^t \left| w_\beta(v) \left( K_\delta f^{n+1} - K_\delta f^n \right)(s, x - v(t - s), v) \right| ds. \]
Since \( K_\delta \) is linear, we have by Lemma 2.4 that
\[ I_6 \leq C_{2 \rho} \| w_\beta f^{n+1} - w_\beta f^n \|_{L^{\infty}_{\mathbb{R}^3}} \int_0^t \int_{\mathbb{R}^3} \left| k_w(v, \eta) \right| dv ds \]
\[ \leq C T C_\delta \| w_\beta f^{n+1} - w_\beta f^n \|_{L^{\infty}_{\mathbb{R}^3}}, \]
(3.42)
where \( C_{2 \rho} \) and \( k_w \) are defined in (2.12) and (2.3) respectively.

At last we turn to \( I_7 \) where
\[ I_7 = \int_0^t \left| w_\beta(v) \left( \Gamma_\delta_+(f^{n+1}) - \Gamma_\delta_+(f^n) \right)(s, x - v(t - s), v) \right| ds \]
\[ \leq \frac{w_\beta(v)}{\sqrt{\mu_{\delta, \rho}(v)}} \int_0^t \int_{\mathbb{R}^3} \int_{S^2} B(v - u, \theta) \]
\[ \left[ \sqrt{\mu_{\delta, \rho}(v') \mu_{\delta, \rho}(u')} \left\| f^{n+1}(s, u') f^{n+1}(s, v') - f^n(s, u') f^n(s, v') \right\|_{L^{\infty}_{\mathbb{R}^3}} \right] \]
\[ + \delta \sqrt{\mu_{\delta, \rho}(v')} \mu_{\delta, \rho}(u) \left\| f^{n+1}(s, v') f^{n+1}(s, u) - f^n(s, v') f^n(s, u) \right\|_{L^{\infty}_{\mathbb{R}^3}} \mu_{\delta, \rho}(v) + \mu_{\delta, \rho}(u') \]
\[ + \delta \sqrt{\mu_{\delta, \rho}(u') \mu_{\delta, \rho}(u)} \left\| f^{n+1}(s, u') f^{n+1}(s, u) - f^n(s, u') f^n(s, u) \right\|_{L^{\infty}_{\mathbb{R}^3}} \mu_{\delta, \rho}(v) + \mu_{\delta, \rho}(u') \]
\[ + \mu_{\delta, \rho}(u') \mu_{\delta, \rho}(u) \left\| f^{n+1}(s, v') f^{n+1}(s, u') - f^n(s, v') f^n(s, u') \right\|_{L^{\infty}_{\mathbb{R}^3}} \right] dw du ds \]
\[ = I_{71} + I_{72} + I_{73} + I_{74}. \]
(3.43)
As usual, we first obtain the decay \( e^{-\frac{1}{\rho_{\delta, \rho}}} \) from \( \bar{\rho}_{\delta, \rho} \). Then we use the similar split as (3.38) to get the estimate for \( \| w_\beta f^{n+1} - w_\beta f^n \|_{L^{\infty}_{\mathbb{R}^3}} \).

For
\[ I_{71} = \frac{w_\beta(v)}{\sqrt{\mu_{\delta, \rho}(v)}} \int_0^t \int_{\mathbb{R}^3} B(v - u, \theta) \sqrt{\mu_{\delta, \rho}(u')} \sqrt{\mu_{\delta, \rho}(v') \left\| f^{n+1}(s, u') f^{n+1}(s, v') - f^n(s, u') f^n(s, v') \right\|_{L^{\infty}_{\mathbb{R}^3}} 1 + \delta \mu_{\delta, \rho}(v) + \delta \mu_{\delta, \rho}(u) \right) dw du ds, \]
recalling we have \( 0 \leq \delta \mu_\delta, \rho \leq 1 \), and \( w_\beta(v) \leq C w_\beta(u') w_\beta(v') \), it follows that

\[
I_{71} \leq CC_3 \rho \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v-u, \theta) e^{-\frac{|u|^2}{\rho} - \frac{\rho}{4}} \left( \|w_\beta f^{n+1}\|_{L_\infty^T L_\infty^0} + \|w_\beta f^n\|_{L_\infty^T L_\infty^0} \right) \|w_\beta f^{n+1} - w_\beta f^n\|_{L_\infty^T L_\infty^0} \, d\omega \, du \, ds \\
\leq C T C_3 \rho \left( \|w_\beta f^{n+1}\|_{L_\infty^T L_\infty^0} + \|w_\beta f^n\|_{L_\infty^T L_\infty^0} \right) \|w_\beta f^{n+1} - w_\beta f^n\|_{L_\infty^T L_\infty^0},
\]

where \( C_3 \rho \) is defined in (3.19).

\( I_{72} \) can be estimated similarly. The only difference is how we get \( e^{-\frac{|u|^2}{\rho}} \). By direct calculations as (3.25) and (3.29), we have the following inequalities:

\[
w_\beta(v) \frac{\sqrt{\mu_{\delta, \rho}(v') \mu_{\delta, \rho}(u)}}{\sqrt{\mu_{\delta, \rho}(v)}} \mu_{\delta, \rho}(v) = w_\beta(v) \frac{\sqrt{\rho e^{|v'|^2/4}}}{\sqrt{\rho e^{|v|^2/4}}} \frac{\sqrt{\rho e^{|u'|^2/4}}}{\sqrt{\rho e^{|u|^2/4}}} \frac{1}{\delta + \rho e^{\frac{|u|^2}{2}}} \\
\leq C C_4 e^{-\frac{|u|^2}{4}},
\]

\[
w_\beta(v) \frac{\sqrt{\mu_{\delta, \rho}(v') \mu_{\delta, \rho}(u)}}{\sqrt{\mu_{\delta, \rho}(v)}} \mu_{\delta, \rho}(u') = w_\beta(v) \frac{\sqrt{\rho e^{|v'|^2/4}}}{\sqrt{\rho e^{|v|^2/4}}} \frac{\sqrt{\rho e^{|u'|^2/4}}}{\sqrt{\rho e^{|u|^2/4}}} \frac{1}{\delta + \rho e^{\frac{|u'|^2}{2}}} \\
\leq C C_4 w_\beta(u') w_\beta(u') e^{-\frac{|u'|^2}{4}},
\]

where \( C_4 \) is given in (3.26). Then similarly as (3.25), (3.29) and (3.31), we have

\[
I_{72} \leq CC_4 \rho \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v-u, \theta) e^{-\frac{|u|^2}{\rho} - \frac{\rho}{4}} \left( \|w_\beta f^{n+1}\|_{L_\infty^T L_\infty^0} + \|w_\beta f^n\|_{L_\infty^T L_\infty^0} \right) \|w_\beta f^{n+1} - w_\beta f^n\|_{L_\infty^T L_\infty^0} \, d\omega \, du \, ds \\
\leq C T C_4 \rho \left( \|w_\beta f^{n+1}\|_{L_\infty^T L_\infty^0} + \|w_\beta f^n\|_{L_\infty^T L_\infty^0} \right) \|w_\beta f^{n+1} - w_\beta f^n\|_{L_\infty^T L_\infty^0},
\]

Similar the arguments as in (3.29) and (3.40) show that

\[
I_{73} \leq C T C_4 \rho \left( \|w_\beta f^{n+1}\|_{L_\infty^T L_\infty^0} + \|w_\beta f^n\|_{L_\infty^T L_\infty^0} \right) \|w_\beta f^{n+1} - w_\beta f^n\|_{L_\infty^T L_\infty^0}.
\]

Recall

\[
I_{74} = \frac{w_\beta(v)}{\sqrt{\mu_{\delta, \rho}(v)}} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v-u, \theta) \delta \sqrt{\mu_{\delta, \rho}(u') \sqrt{\mu_{\delta, \rho}(u')}} \sqrt{\mu_{\delta, \rho}(u)} \\
\times \|f^{n+1}(s, u') f^{n+1}(s, v') f^{n+1}(s, u) - f^n(s, u') f^n(s, v') f^n(s, u)\|_{L_\infty^T} \, d\omega \, du \, ds.
\]

\( I_{74} \) contains \( |f^{n+1}(u') f^{n+1}(v') f^{n+1}(u) - f^n(u') f^n(v') f^n(u)| \). We divide it into three parts as

\[
|f^{n+1}(u') f^{n+1}(v') f^{n+1}(u) - f^n(u') f^n(v') f^n(u)| \\
\leq |f^{n+1}(u') f^{n+1}(v') f^{n+1}(u) - f^n(u') f^n(u') f^n(u)| \\
+ |f^{n+1}(u') f^{n+1}(u') f^n(u') - f^n(u') f^n(u') f^n(u)| \\
+ |f^n(u') f^n(v') f^n(u)| \left| f^{n+1}(u') - f^n(u') \right|.
\]

Also it is straightforward to see

\[
w_\beta(v) \frac{\sqrt{\mu_{\delta, \rho}(u')} \sqrt{\mu_{\delta, \rho}(v')}}{\sqrt{\mu_{\delta, \rho}(v)}} \mu_{\delta, \rho}(v) = w_\beta(v) \frac{\sqrt{\rho e^{|u'|^2/4}}}{\sqrt{\rho e^{|v'|^2/4}}} \frac{\sqrt{\rho e^{|v|^2/4}}}{\sqrt{\rho e^{|u|^2/4}}} \frac{1}{\delta + \rho e^{\frac{|u|^2}{2}}} \\
\leq C_3 \rho w_\beta(v) e^{-\frac{|u|^2}{4}},
\]

\[
\leq C_2 \rho w_\beta(u') w_\beta(v') e^{-\frac{|u'|^2}{4}}.
\]
Substituting the above two inequalities into (3.47), it holds that

\begin{align}
I_7 \leq CT \left( C_{3p} + C_{4p} \right) \left( \| w_\beta f^{n+1} \|_{L^\infty_T L^\infty_{v,x}} + \| w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \right) \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \\
+ CTC_2 \left( \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \right)^2 \\
+ CT \left( C_{3p} + C_{4p} \right) \left( \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right) \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}},
\end{align}

(3.49)

Finally, (3.33), (3.41), (3.45) and (3.48) yield that

\begin{align}
I_7 \leq CT \left( C_{3p} + C_{4p} \right) \left( \| w_\beta f^{n+1} \|_{L^\infty_T L^\infty_{v,x}} + \| w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \right) \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \\
+ CTC_2 \left( \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \right)^2 \\
+ CT \left( C_{3p} + C_{4p} \right) \left( \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right) \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}},
\end{align}

(3.49)

Then substituting (3.47), (3.48) and (3.49) into (3.33), it follows that

\[ \| w_\beta (f^{n+2} - f^{n+1}) (t, x, v) \| \]

\[ \leq CT \left( \rho^{-\frac{1}{2}} + \rho^{-\frac{1}{2}} + \rho^{-1} \right) \left( \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right) \| f^{n+1} = f^{n} \|_{L^\infty_T L^\infty_{v,x}} \\
+ CTC_2 \left( \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}} \right)^2 \\
+ CT \left( C_{3p} + C_{4p} \right) \left( \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right) \| w_\beta f^{n+1} - w_\beta f^{n} \|_{L^\infty_T L^\infty_{v,x}},
\]

where \( C_{5p} \) is defined in (3.33). Then (3.33) follows if we choose

\[ T_1' = \min \left\{ T_1', \frac{1}{2C C_{5p} \left( 1 + \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right)} \right\}. \]

By Lemma 3.3 for \( (t, x, v) \in [0, T_1'] \times \Omega \times \mathbb{R}^3 \), the sequence \( \{ w_\beta f^n \}_{n=1}^{\infty} \) is a Cauchy sequence. Then we take the limit to obtain a solution \( f = f(t, x, v) \) to the quantum Boltzmann equation (1.1a) satisfying \( 0 \leq F(t, x, v) = \mu_{\delta, \rho}(v) + \sqrt{\mu_{\delta, \rho}} f(t, x, v) \leq 1/\delta \) and

\[ \| w_\beta f \|_{L^\infty_{v,x}} \leq 2 \| w_\beta f_0 \|_{L^\infty_{v,x}}. \]

Similar arguments as how we proved Lemma 3.3 show that if we choose

\[ T_1 \leq \min \left\{ T_1', T_1'', \frac{C'''}{C_{5p} \left( 1 + \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right)} \right\}, \]

(3.50)

for some \( C''' = C'''(\beta, \gamma) \) which is independent of \( \rho \), the solution we obtained is unique for \( (t, x, v) \in [0, T_1] \times \Omega \times \mathbb{R}^3 \). Hence we define

\[ T_1 := \frac{C_1}{C_{s, \rho} \left( 1 + \| w_\beta f_0 \|_{L^\infty_{v,x}} + \| w_\beta f_0 \|_{L^\infty_{v,x}}^2 \right)}, \]

where \( C_1 = \frac{1}{2} \min \{ C', C''', C''' \} \), \( C_{s, \rho} = C_{5p} \). We see \( T_1 \) satisfies (3.50) and (1.17) holds. Thus we finish the proof of Theorem 1.14. \( \square \)
4. Global Existence

In this section we prove Theorem 1.2. First from the mild form (1.15), we can rewrite the equation for $w_\beta f$, which is

$$
w_\beta(v)(t, x, v) = e^{-\nu_\beta(v)t}w_\beta(v)f_0(x - vt, v) + \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)(K^m_\delta f)(s, x - v(t-s), v)\,ds$$
$$+ \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)(K^c_\delta f)(s, x - v(t-s), v)\,ds$$
$$+ \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)\Gamma_\delta(f)(s, x - v(t-s), v)\,ds,$$  \hspace{1cm} (4.1)

where $K^m_\delta$ and $K^c_\delta$ are defined in (2.13) and (2.14) respectively. Throughout this section we assume that $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3$ for positive time $T$. We directly obtain from (4.1) and Lemma 2.4 that

$$|w_\beta(v)(t, x, v)| \leq |w_\beta f_0|_{L^\infty_{v,x}} + Cm^{3+\gamma}C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)}e^{-\frac{|v|^2}{\nu_\beta(v)}}|\beta(v)||f(s)|_{L^\infty_{v,x}}\,ds$$
$$+ C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)}\int_{\mathbb{R}^3} (|l_1(v, \eta)| + |l_2(v, \eta)|) w_\beta(v)|f(s, x - v(t-s), \eta)||\,d\eta\,ds$$
$$+ \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)\Gamma_\delta(f)(s, x - v(t-s), v)\,ds$$
$$\leq |w_\beta f_0|_{L^\infty_{v,x}} + Cm^{3+\gamma}C_{2\rho} e^{-\frac{|v|^2}{\nu_\beta(v)}}|\beta(v)|_L^\infty_{v,x}$$
$$+ C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)}\int_{\mathbb{R}^3} l_w(v, \eta) |w_\beta(\eta)f(s, x - v(t-s), \eta)||\,d\eta\,ds$$
$$+ \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)\Gamma_\delta(f)(s, x - v(t-s), v)\,ds,$$ \hspace{1cm} (4.2)

where $l_w$ is defined in (2.11). Notice that by Lemma 2.4 we have $\nu_\beta(v) \geq \frac{1}{6}C_{1\rho}(1 + |v|)^{\gamma}$. Substituting it into (4.3) we have

$$|w_\beta(v)(t, x, v)| \leq |w_\beta f_0|_{L^\infty_{v,x}} + Cm^{3+\gamma}C_{2\rho} \frac{e^{-\frac{|v|^2}{\nu_\beta(v)}}}{(1 + |v|)^{\gamma}}|\beta(v)|_L^\infty_{v,x}$$
$$+ C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)}\int_{\mathbb{R}^3} l_w(v, \eta) |w_\beta(\eta)f(s, x - v(t-s), \eta)||\,d\eta\,ds$$
$$+ \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)\Gamma_\delta(f)(s, x - v(t-s), v)\,ds$$
$$\leq |w_\beta f_0|_{L^\infty_{v,x}} + Cm^{3+\gamma}C_{2\rho} \frac{e^{-\frac{|v|^2}{\nu_\beta(v)}}}{(1 + |v|)^{\gamma}}|\beta(v)|_L^\infty_{v,x}$$
$$+ C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)}\int_{\mathbb{R}^3} l_w(v, \eta) |w_\beta(\eta)f(s, x - v(t-s), \eta)||\,d\eta\,ds$$
$$+ \int_0^t e^{-\nu_\beta(v)(t-s)}w_\beta(v)\Gamma_\delta(f)(s, x - v(t-s), v)\,ds$$
$$= |w_\beta f_0|_{L^\infty_{v,x}} + Cm^{3+\gamma}C_{2\rho} \frac{e^{-\frac{|v|^2}{\nu_\beta(v)}}}{(1 + |v|)^{\gamma}}|\beta(v)|_L^\infty_{v,x} + J_1 + J_2.$$ \hspace{1cm} (4.3)

Here we can see that in order to obtain the pointwise bound, we need to have good estimate on $w_\beta(v)|\Gamma_\delta(f)(s, x - v(t-s), v)|$. Thus we establish the following Lemma in order to take care of this term.
Lemma 4.1. For any $p > \frac{3}{\gamma}$, let $-3 < \gamma < 0$, $\beta > \max\{6, 16/(5p - 1)\}$, it holds that

$$
|w_\beta(v)\Gamma_\delta(f)(v)| \leq C_{5p} \nu(v) \left(1 + \|w_\beta f\|_{L^\infty}\right)
$$

$$
\times \left\{ \|w_\beta f\|_{L^p}^{\frac{2p-1}{p}} \left( \int_{\mathbb{R}^3} |f(u)| \, du \right)^{\frac{1}{p}} + \|w_\beta f\|_{L^\infty}^{\frac{10p-1}{5p}} \left( \int_{\mathbb{R}^3} |f(u)| \, du \right)^{\frac{1}{5}} \right\},
$$

where $C_{5p}$ is defined in (3.30).

Proof. From the definition of $\Gamma_\delta$ (1.12) and the fact that $0 \leq \max \{\delta, \delta\mu_\delta,p(v)\} \leq 1$ we have

$$
|w_\beta(v)\Gamma_\delta(f)(v)| \leq \frac{C}{\sqrt{\mu_\delta,\rho(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u))} \sqrt{\mu_\delta,\rho(f(v))} \right.
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(u) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
\left. \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{\gamma}} |f(u)| \, du \, d\omega \right|.
$$

There are ten terms above. In order to simplify the calculations, we can replace the terms that include $|\sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} | \mu_\delta,\rho(v) + \mu_\delta,\rho(u')$ and $|\sqrt{\mu_\delta,\rho(f(u')} \sqrt{\mu_\delta,\rho(f(u))} | \mu_\delta,\rho(v) + \mu_\delta,\rho(u')$ in the third row and fourth row by

$$
|\sqrt{\mu_\delta,\rho(f(u')} \sqrt{\mu_\delta,\rho(f(u))} | \mu_\delta,\rho(v) + \mu_\delta,\rho(u')$ and $|\sqrt{\mu_\delta,\rho(f(u')} \sqrt{\mu_\delta,\rho(f(u))} | \mu_\delta,\rho(v) + \mu_\delta,\rho(u')$ respectively since we can exchange $u'$ and $v'$ by a rotation. Then it follows that

$$
|w_\beta(v)\Gamma_\delta(f)(v)| \leq J_{21} + J_{22},
$$

where

$$
J_{21} = \frac{C}{\sqrt{\mu_\delta,\rho(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(v))} \right.
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(u) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
+ w_\beta(v) \sqrt{\mu_\delta,\rho(f(u'))} \sqrt{\mu_\delta,\rho(f(u))} \left| \mu_\delta,\rho(v) + \mu_\delta,\rho(u') \right|
$$

$$
\left. \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{\gamma}} |f(u)| \, du \, d\omega \right|.
$$

It follows by similar arguments as in (3.31), (3.32), (3.30) and (3.34) that

$$
J_{21} \leq C \left( C_{3p} + C_{4p} + C_{2p} \right) \|w_\beta f\|_{L^\infty} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{\gamma}} |f(u)| \, du \, d\omega.
$$

(4.5)

We choose a constant $p$ such that $p > \frac{3}{\delta + \gamma}$, then by Hölder’s inequality it holds that

$$
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{\gamma}} |f(u)| \, du \, d\omega \leq C \left( \int_{\mathbb{R}^3} |u|^{\frac{2p}{p-1}} e^{-\frac{|u|^2}{\gamma}} \, du \right)^\frac{p-1}{p} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |f(u)|^p \, du \, d\omega \right)^{\frac{1}{p}}.
$$

(4.6)
In the last inequality above we have used the fact that
\[
\left( \int_{\mathbb{R}^3} |v - u|^{\frac{2p}{p-1}} e^{-\frac{|v|^2}{4}} du \right)^{\frac{p-1}{p}} \leq (1 + |v|)^p \leq C \nu(v)
\]
since \(-3 < \frac{2p}{p-1} < 0\) by our choice of \(p\). Then it follows from (4.5) and (4.6) that
\[
J_{21} \leq C \nu(v) \left( C_3 \nu + C_4 \nu_c + C_{2p} \| w_\beta f \|_{L^{\frac{2p}{p-1}}_w} \right) \| w_\beta f \|_{L^{\frac{2p}{p-1}}_w} \left( \int_{\mathbb{R}^3} |f(u)| \, du \right)^{\frac{1}{p}}. \tag{4.7}
\]

Notice that since \(|v|^2 \leq |(u')|^2 + |(v')|^2\), either \(|v|^2 \leq 2 |u'|^2\) or \(|v|^2 \leq 2 |v'|^2\). Then there exists a positive constant \(C\) such that \(w_\beta(v) \leq w_\beta(u) \chi_{\{|v|^2 \leq 2 |u'|^2\}} + w_\beta(v) \chi_{\{|v|^2 \leq 2 |v'|^2\}} \leq C (w_\beta(u') + w_\beta(v'))\). Also we can exchange \(u'\) and \(v'\) by a rotation. Then similarly as above we have
\[
J_{22} \leq \frac{C}{\sqrt{\mu_\delta(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ (w_\beta(u') + w_\beta(v')) \sqrt{\mu_\delta(v')} \sqrt{\mu_\delta(u')} \right] \\
+ w_\beta(v) \left[ \sqrt{\mu_\delta(v')} \sqrt{\mu_\delta(u')} \right] |\mu_\delta(v) + \mu_\delta(u')| \\
+ w_\beta(v) \left[ \sqrt{\mu_\delta(v')} \sqrt{\mu_\delta(u')} \right] |\mu_\delta(v) + \mu_\delta(u')| \\
\leq C \left( C_3 \nu + C_4 \nu_c + C_{2p} \| w_\beta f \|_{L^{\frac{2p}{p-1}}_w} \right) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|v|^2}{4}} |f(v')| \, d\omega \, du. \tag{4.8}
\]

It holds from Hölder’s inequality that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|v|^2}{4}} |f(v')| \, d\omega \, du \leq C \left( \int_{\mathbb{R}^3} |v - u|^{\frac{2p}{p-1}} e^{-\frac{|v|^2}{4}} du \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v|^2}{4}} |f(v')|^p \, d\omega \, du \right)^{\frac{1}{p}} \leq C \nu(v) \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v|^2}{4}} |f(v')|^p \, d\omega \, du \right)^{\frac{1}{p}}. \tag{4.9}
\]

Denote \(z = u - v, z_i = u - v, \omega, z_\perp = z - z_i, \eta = v + z_i\). We change variables to yield
\[
\left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v|^2}{4}} |f(v')|^p \, d\omega \, du \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^3} \int_{z_\perp} e^{-\frac{|z|^2}{4} + \frac{|\eta|^2}{4}} \, d\omega \, du \, d\eta \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^3} \frac{(1 + |\eta|)^{-4}}{|\eta - v|^2} \, d\eta \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^3} (1 + |\eta|)^{16} |f(\eta)|^{5p} \, d\eta \right)^{\frac{1}{p}} \leq C \| w_\beta f \|_{L^{\frac{2p}{p-1}}_w} \left( \int_{\mathbb{R}^3} |f(u)| \, du \right)^{\frac{1}{p}}. \tag{4.10}
\]

By choosing \(\beta > \frac{16}{5p-1}\), it follows from (4.8), (4.9) and (4.10) that
\[
J_{22} \leq C \nu(v) \left( C_3 \nu + C_4 \nu_c + C_{2p} \| w_\beta f \|_{L^{\frac{2p}{p-1}}_w} \right) \| w_\beta f \|_{L^{\frac{2p}{p-1}}_w} \left( \int_{\mathbb{R}^3} |f(u)| \, du \right)^{\frac{1}{p}}. \tag{4.11}
\]

Our Lemma follows from (4.7), (4.11) and the definition of \(C_{5p}\) that \(C_{5p} = C_{2p} + C_{3p} + C_{4p}\). \(\square\)

Using the pointwise inequality (4.3), we have the following lemma.
Lemma 4.2. Let \(-3 < \gamma < 0, \beta > \max\{3,3 - \gamma, 16/(5p - 1)\}\), and the constant \(p\) be given in Lemma 4.1. It holds that

\[
\|w_\beta f\|_{L_T^\infty L_x^\infty} \leq C_3 C_{6p} \left( \|w_\beta f_0\|_{L_x^\infty} + \|w_\beta f_0\|_{L_x^\infty}^2 + \|w_\beta f_0\|_{L_x^\infty}^3 \right) \\
+ C_3 C_{6p} \left( 1 + \|w_\beta f\|_{L_T^\infty L_x^\infty} \right) \left\{ \|w_\beta f\|_{L_T^{50p-1} L_x^{50p-1}} \|f\|_{L_T^{3} L_x^{3}} + \|w_\beta f\|_{L_T^{50p-1} L_x^{50p-1}} \|f\|_{L_T^{3} L_x^{3}} \right\} \\
+ C_3 C_{\rho,N,\rho} \left( \lambda_p^{-\frac{2}{3}} \sqrt{E_{\beta,\rho}(F_0)} + \lambda_p^{-\frac{3}{2}} E_{\beta,\rho}(F_0) \right),
\]

where the time \(T_1\) is given in (4.16), the constants \(C_{6p}, C_{\rho,N,\rho}\) and \(\lambda_p\) are defined in (4.11), (4.33), (4.38) and (4.39) respectively, and \(C_3 = C_3(\beta, \gamma)\).

Proof. Now by Lemma 4.1 and (4.3), it is straightforward to see that

\[
J_2 = \int_0^t e^{-\nu_\beta(v)(t-s)} w_\beta(v) |\Gamma_\delta(f)(s, x - v(t-s), v)| \, ds \\
\leq C \left( \int_0^t e^{-\nu_\beta(v)(t-s)} \nu(v) \, ds \right) C_{5\rho} \sup_{0 \leq t \leq T} \left( 1 + \|w_\beta f(t)\|_{L_x^\infty} \right) \\
\times \left\{ \|w_\beta f(t)\|_{L_T^{50p-1} L_x^{50p-1}} \|f(t)\|_{L_T^{3} L_x^{3}} + \|w_\beta f(t)\|_{L_T^{50p-1} L_x^{50p-1}} \|f(t)\|_{L_T^{3} L_x^{3}} \right\} \\
\leq C \frac{C_{5\rho}}{C_{1\rho}} \sup_{0 \leq t \leq T} \left( 1 + \|w_\beta f(t)\|_{L_x^\infty} \right) \\
\times \left\{ \|w_\beta f(t)\|_{L_T^{50p-1} L_x^{50p-1}} \|f(t)\|_{L_T^{3} L_x^{3}} + \|w_\beta f(t)\|_{L_T^{50p-1} L_x^{50p-1}} \|f(t)\|_{L_T^{3} L_x^{3}} \right\},
\]

where in the last inequality above, we have used the fact that

\[
\int_0^t e^{-\nu_\beta(v)(t-s)} \nu(v) \, ds \leq \frac{C}{C_{1\rho}} \nu(v) \leq \frac{C}{C_{1\rho}}.
\]

Next we turn to \(J_1\). Recall

\[
J_1 = C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) |w_\beta(\eta)f(s, x - v(t-s), \eta)| \, d\eta ds.
\]

We substitute (4.3) into \(|w_\beta(\eta)f(s, x - v(t-s), \eta)|\) in \(J_1\) to yield

\[
J_1 \leq C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) \|w_\beta f_0\|_{L_x^\infty} \, d\eta ds \\
+ C m^{3+\gamma} \frac{C_{2\rho}}{C_{1\rho}} \int_0^t e^{-\nu_\beta(v)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) e^{-\frac{|v|^2}{2m}} \|w_\beta f\|_{L_T^\infty L_x^\infty} \, d\eta ds \\
+ C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) \\
\times \int_0^s e^{-\nu_\beta(\eta)(s-s_1)} \int_{\mathbb{R}^3} l_w(\eta, \xi) |w_\beta(\xi)f(s_1, x_1 - \eta(s-s_1), \xi)| \, d\xi d\eta ds \, ds ds_1 \, d\eta ds_1 \\
+ C_{2\rho} \int_0^t e^{-\nu_\beta(v)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) \\
\times \int_0^s e^{-\nu_\beta(\eta)(s-s_1)} w_\beta(\eta) |\Gamma_\delta(f)(s_1, x_1 - \eta(s-s_1), \eta)| \, ds_1 \, d\eta ds_1 \\
= J_{10} + J_{11} + J_{12} + J_{13},
\]

(4.14)
where \( x_1 = x - v(t - s) \). From (2.9) in Lemma 2.2 we have
\[
J_{10} \leq C C_{2\rho} m^{\gamma - 1} \|w_{\beta} f_0\|_{L^\infty_{v,x}} \int_0^t e^{-\nu(s)(t-s)} \frac{\nu(v)}{(1+|v|)^2} ds \\
\leq C \frac{C_{2\rho}}{C_{1\rho}} m^{\gamma - 1} \|w_{\beta} f_0\|_{L^\infty_{v,x}} \int_0^t e^{-\nu(s)(t-s)} \frac{\nu(v)}{(1+|v|)^2} ds \\
\leq C \frac{C_{2\rho}}{C_{1\rho}} m^{\gamma - 1} \|w_{\beta} f_0\|_{L^\infty_{v,x}}.
\]
(4.15)

Since \( e^{-\frac{|v|^2}{2}} \leq C \nu(\eta) \), it follows that
\[
J_{11} \leq C m^{3+\gamma} \frac{C_{2\rho}^2}{C_{1\rho}} \int_0^t e^{-\nu(s)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) \|w_{\beta} f\|_{L^\infty_{v,x}} d\eta ds \\
\leq C m^{3+\gamma} \frac{C_{2\rho}^2}{C_{1\rho}} \|w_{\beta} f\|_{L^\infty_{v,x}}.
\]
(4.16)

Then we estimate \( J_{13} \). Applying Lemma 4.1 again, one can get that
\[
J_{13} \leq C C_{2\rho} C_{5p} \int_0^t e^{-\nu(s)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) \int_0^s e^{-\nu(s)(s-s_1)} \nu(\eta) ds_1 d\eta ds \\
\times \sup_{0 \leq s \leq T} \left( 1 + \|w_{\beta} f(t)\|_{L^\infty_{v,x}} \right) \left\{ \|w_{\beta} f(t)\|_{L^\infty_{v,x}} \left\| f(t) \right\|_{L^1_v L^1_x} \right\} \\
\leq C \frac{C_{2\rho} C_{5p}}{C_{1\rho}} m^{\gamma - 1} \sup_{0 \leq s \leq T} \left( 1 + \|w_{\beta} f(t)\|_{L^\infty_{v,x}} \right) \\
\times \left\{ \|w_{\beta} f(t)\|_{L^\infty_{v,x}} \left\| f(t) \right\|_{L^1_v L^1_x} \right\}.
\]
(4.17)

The estimate of \( J_{12} \) is more delicate. We divide it into four cases. First in order to simplify the integral, recalling \( x_1 = x - v(t - s) \), we rewrite \( J_{12} \) by Fubini’s theorem that
\[
J_{12} = C_{2\rho}^2 \int_0^t e^{-\nu(s)(t-s)} \int_{\mathbb{R}^3} l_w(v, \eta) \\
\times \int_0^s e^{-\nu(s)(s-s_1)} \int_{\mathbb{R}^3} l_w(\eta, \xi) |w_{\beta}(\xi) f(s_1, x_1 - \eta(s-s_1), \xi) | d\xi ds_1 d\eta ds \\
= C_{2\rho}^2 \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu(s)(t-s)} e^{-\nu(s)(s-s_1)} \\
\times l_w(v, \eta) l_w(\eta, \xi) |w_{\beta}(\xi) f(s_1, x_1 - \eta(s-s_1), \xi) | d\xi ds_1 d\eta ds \\
\times \int_0^s \int_{\mathbb{R}^3} e^{-\nu(s)(s-s_1)} \chi_{\{|v| \geq N\}} \\
\times l_w(v, \eta) l_w(\eta, \xi) |w_{\beta}(\xi) f(s_1, x_1 - \eta(s-s_1), \xi) | d\xi ds_1 d\eta ds \\
+ C_{2\rho}^2 \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu(s)(t-s)} e^{-\nu(s)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \geq 3N\}} \\
\times l_w(v, \eta) l_w(\eta, \xi) |w_{\beta}(\xi) f(s_1, x_1 - \eta(s-s_1), \xi) | d\xi ds_1 d\eta ds \\
+ C_{2\rho}^2 \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu(s)(t-s)} e^{-\nu(s)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} \\
\times l_w(v, \eta) l_w(\eta, \xi) |w_{\beta}(\xi) f(s_1, x_1 - \eta(s-s_1), \xi) | d\xi ds_1 d\eta ds \\
+ C_{2\rho}^2 \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu(s)(t-s)} e^{-\nu(s)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \geq 3N\}} \\
\times l_w(v, \eta) l_w(\eta, \xi) |w_{\beta}(\xi) f(s_1, x_1 - \eta(s-s_1), \xi) | d\xi ds_1 d\eta ds.
\]
(4.18)
It is straightforward to see that

\[
J_{121} \leq C_{2p}^2 \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu_3(v)(t-s)} e^{-\nu_4(\eta)(s-s_1)} \chi_{\{|v| \geq N\}} \times l_w(v, \eta)l_w(\eta, \xi) ||w_\beta f||_{L^\infty_{\tau, x}} d\xi d\eta ds_1 ds.
\]

In the last integral above, the only term that contains time variables \(s_1\) and \(s\) is \(e^{-\nu_3(v)(t-s)} e^{-\nu_4(\eta)(s-s_1)}\). Then we can integrate with respect to \(s_1\) and \(s\) first and use (2.8) again to get

\[
J_{121} \leq C_{2p}^2 C_{1p}^2 \int_{\mathbb{R}^3} \chi_{\{|v| \geq N\}} \frac{1}{\nu(v)\nu(\eta)} \times l_w(v, \eta)l_w(\eta, \xi) ||w_\beta f||_{L^\infty_{\tau, x}} d\xi d\eta.
\] (4.19)

Then on the right-hand side of (4.19), using the fact from (2.8) that

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{\nu(v)\nu(\eta)} l_w(v, \eta)l_w(\eta, \xi) d\xi d\eta 
\leq Cm^{\gamma-1} \int_{\mathbb{R}^3} \frac{1}{\nu(v)} l_w(v, \eta) d\eta 
\leq Cm^{2\gamma-2} \frac{1}{(1 + |v|)^2},
\]

we obtain that

\[
J_{121} \leq C_{2p}^2 C_{1p}^2 \int_{\mathbb{R}^3} \chi_{\{|v| \geq N\}} \frac{1}{\nu(v)\nu(\eta)} l_w(v, \eta)l_w(\eta, \xi) ||w_\beta f||_{L^\infty_{\tau, x}} d\xi d\eta \leq Cm^{2\gamma-2} \frac{1}{N} \frac{C_{2p}^2}{C_{1p}^2} ||w_\beta f||_{L^\infty_{\tau, x}}. \tag{4.20}
\]

Notice that if \(|v| \leq N, |\eta| \geq 2N\) or \(|\eta| \leq 2N, |\xi| \geq 3N\), we have either \(|v - \eta| \geq N\) or \(|\eta - \xi| \geq N\). Thus, we have \(e^{-\frac{|v-n|^2}{2\theta}} e^{-\frac{|\eta-x|^2}{2\theta}} \leq \frac{C}{N}\) for some constant \(C\). Then similar arguments as (4.20) show that

\[
J_{122} \leq C_{2p}^2 \int_0^t \int_0^s \int_{\mathbb{R}^3} e^{-\nu_3(v)(t-s)} e^{-\nu_4(\eta)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \geq 2N\}} \chi_{\{|\eta| \leq 2N, |\xi| \geq 3N\}} \times l_w(v, \eta)l_w(\eta, \xi) ||w_\beta f||_{L^\infty_{\tau, x}} d\xi d\eta ds_1 ds
\leq C_{2p}^2 C_{1p}^2 \int_{\mathbb{R}^3} \chi_{\{|v| \leq N, |\eta| \geq 2N\}} \chi_{\{|\eta| \leq 2N, |\xi| \geq 3N\}} \frac{1}{\nu(v)\nu(\eta)} \times l_w(v, \eta) e^{-\frac{|v-n|^2}{2\theta}} e^{-\frac{|\eta-x|^2}{2\theta}} l_w(\eta, \xi) e^{-\frac{|\eta-x|^2}{2\theta}} ||w_\beta f||_{L^\infty_{\tau, x}} d\xi d\eta \leq C \frac{C_{2p}^2}{N} \frac{1}{\nu(v)\nu(\eta)} l_w(v, \eta) e^{-\frac{|v-n|^2}{2\theta}} l_w(\eta, \xi) e^{-\frac{|\eta-x|^2}{2\theta}} ||w_\beta f||_{L^\infty_{\tau, x}} d\xi d\eta \leq Cm^{2\gamma-2} \frac{C_{2p}^2}{N} ||w_\beta f||_{L^\infty_{\tau, x}}. \tag{4.21}
\]
$J_{123}$ can be similarly calculated as above, which yields

$$
J_{123} = C_{2}^{2} \frac{\lambda}{\rho} \int_{0}^{t} \int_{0}^{s} \int_{0}^{s_{1}} \int_{0}^{s_{2}} e^{-\nu_{t}(v)(t-s)} e^{-\nu_{s}(\eta)(s-s_{1})} \times l_{w}(v, \eta) l_{w}(\eta, \xi) w_{3} f(s_{1}, x_{1} - \eta(s - s_{1}), \xi) \mid d\xi d\eta ds_{1} ds
$$

$$
\leq C_{2}^{2} \lambda \int_{0}^{t} \int_{0}^{s} \int_{0}^{s_{1}} \int_{0}^{s_{2}} e^{-\nu_{t}(v)(t-s)} l_{w}(v, \eta) l_{w}(\eta, \xi) w_{3} f \mid \|L_{t}^{\infty} \| \mid \|L_{\infty}^{\infty} \| \mid d\xi d\eta ds_{1} ds
$$

$$
\leq C_{2}^{2} \lambda \int_{0}^{t} \int_{0}^{s} \int_{0}^{s_{1}} \int_{0}^{s_{2}} \frac{1}{\rho} \times l_{w}(v, \eta) l_{w}(\eta, \xi) w_{3} f \mid \|L_{t}^{\infty} \| \mid \|L_{\infty}^{\infty} \| \mid d\xi d\eta
$$

$$
\leq C^{2} m^{2r-2} \lambda \|w_{3} f\| \|L_{t}^{\infty} \| \|L_{\infty}^{\infty} \|.
$$

Now we focus on $J_{124}$ where

$$
J_{124} = C_{2}^{2} \int_{0}^{t} \int_{0}^{s_{2}} \int_{0}^{s_{1}} \int_{0}^{s_{2}} e^{-\nu_{t}(v)(t-s)} e^{-\nu_{s}(\eta)(s-s_{1})} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} \times l_{w}(v, \eta) l_{w}(\eta, \xi) w_{3} f(s_{1}, x_{1} - \eta(s - s_{1}), \xi) \mid d\xi d\eta ds_{1} ds.
$$

Although all the velocity variables are bounded, $|v - \eta|$ or $|\eta - \xi|$ might be small, then it is difficult to have a good pointwise bound of the function $l_{w}$ due to the possible singularity from $\frac{1}{|v - \eta|}$ or $\frac{1}{|\eta - \xi|}$. However, if we take integral, $l_{w}$ is bounded in the following sense,

$$
\sup_{|v| \leq 3N} \int_{|\eta| \leq 3N} l_{w}(v, \eta) d\eta \leq C(\gamma)m^{\gamma-1}.
$$

Then for $0 < \kappa \ll 1$ there exists a smooth function $l_{\kappa} = l_{\kappa}(v, \eta)$ with compact support such that

$$
\sup_{|v| \leq 3N} \int_{|\eta| \leq 3N} |l_{w}(v, \eta) - l_{\kappa}(v, \eta)| d\eta \leq C(\gamma) \kappa^{3+\gamma}.
$$

Moreover we have the pointwise bound of $l_{\kappa}$ which is

$$
\sup_{|v| \leq 3N, |\eta| \leq 3N} |l_{\kappa}(v, \eta)| \leq C(\gamma) N^{\beta} \frac{1}{\kappa^{\frac{\gamma}{2}}}. \tag{4.23}
$$

Such approximation can be directly obtained by removing the singularity using a smooth cut-off function to restrict $l_{w}$ in the region $\{|v| \leq 3N, |\eta| \leq 3N, |v - \eta| \geq \kappa\}$. Now we choose \( \kappa = N^{-\frac{1}{3+2\gamma}} \) and denote $l_{N} = l_{\kappa} \mid_{\kappa = N^{-\frac{1}{3+2\gamma}}}$ to get that

$$
\sup_{|v| \leq 3N} \int_{|\eta| \leq 3N} |l_{w}(v, \eta) - l_{N}(v, \eta)| d\eta \leq C \frac{1}{N^{\gamma}}, \tag{4.23}
$$

and

$$
\sup_{|v| \leq 3N, |\eta| \leq 3N} |l_{N}(v, \eta)| \leq C(\gamma) C_{N}, \tag{4.24}
$$

where

$$
C_{N} = N^{\frac{3(3-2\gamma)}{6+2\gamma} + \beta}. \tag{4.25}
$$

It is direct to see that $l_{N}$ also satisfies $\sup_{|v| \leq 3N} \int_{|\eta| \leq 3N} |l_{N}(v, \eta)| d\eta \leq C(\gamma)$. Since we need to figure out how the constants depend on the parameter $\rho$, then we should calculate exactly how the pointwise and integral bounds of the approximation function $l_{N}$ depend on $N$ for later use. If $\rho$ is a given number instead of a parameter, we will only need (4.23) and (4.24) without the explicit
formula for \( C_N \). Using the approximation function \( l_N \), it follows that

\[
J_{124} \leq C_{2p}^2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |l_w(v, \eta)||l_w(\eta, \xi) - l_N(\eta, \xi)||w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds
\]

\[
+ C_{2p}^2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |l_N(\eta, \xi)||l_w(v, \eta) - l_N(v, \eta)||w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds
\]

\[
\leq C^2_{2p} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds
\]

\[
\times |l_N(v, \eta)||l_w(v, \eta)||w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds ds.
\]

Applying the pointwise bound \( |w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| \leq \|w_\beta f\|_{L^\infty_x L^\infty_{\nu, \eta}} \), then integrating with respect to \( s_1, s, \xi \) and \( \eta \) respectively, one gets that

\[
J_{124} \leq C_{2p}^2 C_{2p}^2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |l_w(v, \eta)||l_w(\eta, \xi) - l_N(\eta, \xi)||w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds
\]

\[
+ C_{2p}^2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |l_N(v, \eta)||l_w(v, \eta) - l_N(v, \eta)||w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds
\]

\[
\leq C_{2p}^2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|v| \leq N, |\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds ds.
\]

Using the inequalities that

\[
\int_{\mathbb{R}^3} |l_w(v, \eta)||l_w(\eta, \xi)\chi_{\{|v| \leq N, |\eta| \leq 2N\}} d\eta \leq C N^6 \int_{\mathbb{R}^3} |l_w(v, \eta)| d\eta \leq C N^6
\]

\[
\int_{\mathbb{R}^3} |l_w(v, \eta) - l_N(v, \eta)|\chi_{\{|v| \leq N, |\eta| \leq 2N\}} d\eta \leq C N^6 \int_{\mathbb{R}^3} |l_w(v, \eta) - l_N(v, \eta)| d\eta \leq \frac{C}{N},
\]

it follows from (4.27) that

\[
J_{124} \leq C_{2p}^2 \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds.
\]

We need to treat the second term on the right-hand side of (4.25) carefully. Recall from Lemma 2.3 that

\[
\nu_\beta(v) \geq C C_{1p} (1 + |v|)^7.
\]

By \( |v| \leq N, |\eta| \leq 2N \) we have

\[
\nu_\beta(v) \geq C C_{1p} N^7.
\]

Then we can bound the second term on the right-hand side of (4.25) by

\[
C C_{2p}^2 C_{2p} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_\alpha(v)(t-s)} e^{-\nu_\eta(v)(s-s_1)} \chi_{\{|\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds
\]

\[
\leq C C_{2p}^2 C_{2p} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-C \nu_\alpha(v)(t-s)} e^{-C \nu_\eta(v)(s-s_1)} \chi_{\{|\eta| \leq 2N, |\xi| \leq 3N\}} d\xi d\eta ds ds ds
\]

\[
\times |w_\beta(\xi)f(s_1, x_1 - \eta(s - s_1), \xi)| d\xi d\eta ds ds ds.
\]
By observing that the only element in the last integral above that contains velocity variables \( \eta, \xi \) is \( w_\beta (\xi) f (s_1, x_1 - \eta (s - s_1), \xi) \), we can consider \( \int \int \chi_{\{ |\eta| \leq 2N, |\xi| \leq 3N \}} |w_\beta (\xi) f (s_1, x_1 - \eta (s - s_1), \xi)| \, d\eta \, d\xi \) first. Applying change of variable \( y = x_1 - \eta (s - s_1) \), we obtain

\[
\int \int |(w_\beta f) (s_1, x_1 - \eta (s - s_1), \xi)| \, d\eta \, d\xi \\
\leq C N^\beta \int \int |(w_\beta f) (s_1, x_1 - \eta (s - s_1), \xi)| \chi_{\{ |\eta| \leq 2N, |\xi| \leq 3N \}} \, d\eta \, d\xi \\
+ C N^\beta \int \int |(F - \mu) (s_1, x_1 - \eta (s - s_1), \xi)| \chi_{\{ |F(s_1, x_1 - \eta (s - s_1), \xi) - \mu (\xi)| \leq \mu (\xi) \}} \, d\eta \, d\xi \\
+ C N^\beta \int \int |(F - \mu) (s_1, y, \xi)| \chi_{\{ |F(s_1, y, \xi) - \mu (\xi)| \geq \mu (\xi) \}} \, d\eta \, d\xi.
\]

Then by (4.14), (4.15), (4.16), (4.35) and (4.17), it holds that

\[
\int \int |(w_\beta f) (s_1, x_1 - \eta (s - s_1), \xi)| \, d\eta \, d\xi \\
\leq C N^\beta \int \int \left\{ \frac{|F - \mu|^2}{\mu} \right\} (s_1, y, \xi) \chi_{\{ F(s_1, y, \xi) - \mu (\xi) \leq \mu (\xi) \}} \, d\eta \, d\xi \\
+ C N^\beta \int \int \left\{ \frac{|F - \mu|^2}{\mu} \right\} (s_1, y, \xi) \chi_{\{ F(s_1, y, \xi) - \mu (\xi) \geq \mu (\xi) \}} \, d\eta \, d\xi.
\]

Substituting (4.35) into (4.30) and using Lemma 2.30, we obtain

\[
C C_2^2 C_N^2 \int_0^t \int_0^{s - \lambda} \int_{\mathbb{R}^3} e^{-\nu_3 (s-t)} e^{-\nu_3 (s-s_1)} \chi_{\{|\eta| \leq 2N, |\xi| \leq 3N \}} \times |w_\beta (\xi) f (s_1, x_1 - \eta (s - s_1), \xi)| \, d\eta \, ds \, ds \\
\leq C \frac{C_2^2}{C_1 N^\gamma} C_N^{2 N^\beta} \left( \frac{1}{C_1 N^\gamma} + \frac{1}{(C_1 N^\gamma)^{3/2}} + \frac{1}{(C_1 N^\gamma)^4} \right) \left( \lambda^{-\frac{3}{2}} \sqrt{E(F_0)} + \lambda^{-3} \mathcal{E}(F_0) \right) \\
= C C_{\rho,N} \left( \lambda^{-\frac{3}{2}} \sqrt{E_{\delta,\rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta,\rho}(F_0) \right),
\]

where

\[
C_{\rho,N} = \frac{C_2^2}{C_1 N^\gamma} C_N^{2 N^\beta} \left( \frac{1}{C_1 N^\gamma} + \frac{1}{(C_1 N^\gamma)^{3/2}} + \frac{1}{(C_1 N^\gamma)^4} \right)
\]

with \( C_N \) defined in (4.25). It follows from (4.25) and (4.32) that

\[
J_{124} \leq C \frac{C_2^2}{C_1 N} \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} + C C_{\rho,N} \left( \lambda^{-\frac{3}{2}} \sqrt{E_{\delta,\rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta,\rho}(F_0) \right).
\]

Combining (4.18), (4.20), (4.21), (4.22) and (4.34), we obtain

\[
J_{12} \leq \frac{C m^{2 \gamma-2} C_2^2}{N} \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} + C m^{2 \gamma-2} \frac{C_2^2}{C_1} \lambda \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} \\
+ C \frac{C_2^2}{C_1 N} \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} + C C_{\rho,N} \left( \lambda^{-\frac{3}{2}} \sqrt{E_{\delta,\rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta,\rho}(F_0) \right).
\]

Then by (4.18), (4.20), (4.21), (4.35) and (4.33), it holds that

\[
J_1 \leq C \frac{C_2^2}{C_1} m^{-1} \| w_\beta f_{0} \|_{L_{\rho,x}^\infty} + C m^{3+\gamma} \frac{C_2^2}{C_1} \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} + C \frac{C m^{2 \gamma-2} C_2^2}{N} \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} \\
+ C m^{2 \gamma-2} \frac{C_2^2}{C_1} \lambda \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} + C \frac{C_2^2}{C_1 N} \| w_\beta f \|_{L_T^\infty L_{\rho,x}^\infty} + C C_{\rho,N} \left( \lambda^{-\frac{3}{2}} \sqrt{E_{\delta,\rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta,\rho}(F_0) \right) \\
+ C \frac{C_2^2 C_5}{C_1 N} m^{-1} \sup_{0 \leq t \leq T} \left( 1 + \| w_\beta f (t) \|_{L_{\rho,x}^\infty} \right) \times \left\{ \| w_\beta f (t) \|_{L_T^\infty L_{\rho,x}^\infty} + \| w_\beta f (t) \|_{L_T^\infty L_{\rho,x}^\infty}^{\frac{1}{2}} \right\}.
\]
Finally, we obtain the $L^\infty$ estimate by (4.33), (4.34), and (4.36) that

$$
|w_\beta(v)f(t, x, v)| \leq \|w_\beta f_0\|_{L^\infty_v} + C m^{3+\frac{2\gamma}{C_2}} \|w_\beta f\|_{L^\infty_v} + C C_{\rho, N} \left( \lambda^{-\frac{2}{3}} \sqrt{E_{\delta, \rho}(F_0)} + \lambda^{-3} E_{\delta, \rho}(F_0) \right)
$$

$$
+ C \frac{C_{2}\rho}{C_1 \rho} m^{-1} \|w_\beta f_0\|_{L^\infty_v} + C m^{3+\frac{2\gamma}{C_2}} \|w_\beta f\|_{L^\infty_v} + \frac{C m^{2\gamma} C_2^2}{N} \|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} + C \frac{C_{2\rho}}{C_1 \rho} \|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} + C \frac{C_{2\rho}}{C_1 \rho} \left( 1 + \frac{C_{2\rho}}{C_1 \rho} m^{-1} \right) \sup_{0 \leq t \leq T} \left( 1 + \|w_\beta f(t)\|_{L^\infty_v} \right)
$$

$$
\times \left\{ \|w_\beta f(t)\|_{L^\infty_v} f(t) \right\}^\frac{1}{2} + \|w_\beta f(t)\|_{L^\infty_v} f(t) \right\}^\frac{10\rho - 1}{L^\infty_v} L^1_{1} \right\}.
$$

We first choose some small $\epsilon_1 = \epsilon_1(\beta, \gamma)$, let

$$
m_\rho = \min \left\{ \left( \frac{C_1 \rho}{C_2 \rho} \right)^{\frac{1}{1+\gamma}}, \left( \frac{C_2 \rho}{C_2 \rho} \right)^{\frac{1}{1+\gamma}} \right\},
$$

and $m = \epsilon_1 m_\rho$ to get

$$
\|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} \leq \|w_\beta f_0\|_{L^\infty_v} + C \frac{C_{2\rho}}{C_1 \rho} m_\rho^{-1} \|w_\beta f_0\|_{L^\infty_v} + C C_{\rho, N} \left( \lambda^{-\frac{2}{3}} \sqrt{E_{\delta, \rho}(F_0)} + \lambda^{-3} E_{\delta, \rho}(F_0) \right)
$$

$$
+ C m^{2\gamma} C_2^2 \frac{C_{2\rho}}{C_1 \rho} \|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} + C m^{2\gamma} C_2^2 \frac{m_\rho^{-1}}{C_1 \rho} \|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} + \frac{C m^{2\gamma} C_2^2}{N} \|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} + C \frac{C_{2\rho}}{C_1 \rho} \left( 1 + \frac{C_{2\rho}}{C_1 \rho} m^{-1} \right) \sup_{0 \leq t \leq T} \left( 1 + \|w_\beta f(t)\|_{L^\infty_v} \right)
$$

$$
\times \left\{ \|w_\beta f(t)\|_{L^\infty_v} f(t) \right\}^\frac{1}{2} + \|w_\beta f(t)\|_{L^\infty_v} f(t) \right\}^\frac{10\rho - 1}{L^\infty_v} L^1_{1} \right\}.
$$

(4.37)

Similarly we choose $\epsilon_2 = \epsilon_2(\beta, \gamma)$ such that if we let

$$
N = N_\rho := \frac{1}{\epsilon_2} \max \left\{ \frac{C_2 \rho}{C_1 \rho}, \frac{m^{2\gamma - 2} C_2^2}{C_2 \rho} \right\},
$$

(4.38)

and

$$
\lambda = \lambda_\rho := \epsilon_2 \frac{C_1 \rho}{m^{2\gamma - 2} C_2 \rho},
$$

(4.39)

it follows from (4.37) that

$$
\|w_\beta f\|_{L^\infty_v} L^\infty_{v,x} \leq \|w_\beta f_0\|_{L^\infty_v} + C \frac{C_{2\rho}}{C_1 \rho} m_\rho^{-1} \|w_\beta f_0\|_{L^\infty_v} + C C_{\rho, N_\rho} \left( \lambda^{-\frac{2}{3}} \sqrt{E_{\delta, \rho}(F_0)} + \lambda^{-3} E_{\delta, \rho}(F_0) \right)
$$

$$
+ C \frac{C_{2\rho}}{C_1 \rho} \left( 1 + \frac{C_{2\rho}}{C_1 \rho} m^{-1} \right) \sup_{0 \leq t \leq T} \left( 1 + \|w_\beta f(t)\|_{L^\infty_v} \right)
$$

$$
\times \left\{ \|w_\beta f(t)\|_{L^\infty_v} f(t) \right\}^\frac{1}{2} + \|w_\beta f(t)\|_{L^\infty_v} f(t) \right\}^\frac{10\rho - 1}{L^\infty_v} L^1_{1} \right\}.
$$
Notice that if we choose $\beta > 3$, we have $\|f(t)\|_{L_T^\infty L_x^\infty} \leq C \|w_\beta f(t)\|_{L_T^\infty L_x^\infty}$. Then by Theorem 1.1 we have
\[
\|w_\beta f\|_{L_T^\infty L_x^\infty} \leq \|w_\beta f_0\|_{L_T^\infty L_x^\infty} + C \frac{C_2\rho}{C_1\rho} \left( 1 + \|w_\beta f_0\|_{L_T^\infty L_x^\infty} \right) \|f(t)\|_{L_T^\infty L_x^\infty} \sup_{t \leq T} \left( 1 + \|w_\beta f(t)\|_{L_T^\infty L_x^\infty} \right)
\]
\[
\times \left\{ \|w_\beta f(t)\|^2_{L_T^\infty L_x^\infty} + \|w_\beta f(t)\|^\frac{3}{2}_{L_T^\infty L_x^\infty} + \frac{10\rho-1}{3} \|f(t)\|_{L_T^\infty L_x^\infty} \right\}.
\]
We simplify the above inequality by writing
\[
\|w_\beta f\|_{L_T^\infty L_x^\infty} \leq C C_6 \rho \left( \|w_\beta f_0\|_{L_T^\infty L_x^\infty} + \|w_\beta f_0\|^2_{L_T^\infty L_x^\infty} + \|w_\beta f_0\|^3_{L_T^\infty L_x^\infty} \right)
\]
\[
+ C \frac{C_5\rho}{C_1\rho} \left( 1 + \|w_\beta f_0\|_{L_T^\infty L_x^\infty} \right) \sup_{t \leq T} \left( 1 + \|w_\beta f(t)\|_{L_T^\infty L_x^\infty} \right)
\]
\[
\times \left\{ \|w_\beta f(t)\|^2_{L_T^\infty L_x^\infty} + \|w_\beta f(t)\|^\frac{3}{2}_{L_T^\infty L_x^\infty} + \frac{10\rho-1}{3} \|f(t)\|_{L_T^\infty L_x^\infty} \right\}.
\]
where
\[
C_6 \rho = 1 + \frac{C_5\rho}{C_1\rho} \left( 1 + \frac{C_2\rho}{C_1\rho} \right) m_\rho^{-\gamma}.
\]

Then (4.12) follows from (4.40).

We can see from (4.12) that if $\|w_\beta f_0\|_{L_T^\infty L_x^\infty}$ and $E_{\delta,\rho}(F_0)$ are uniformly small in $\delta$, then we can close the a priori estimate to obtain a global solution. But we want to remove the smallness assumption on $\|w_\beta f_0\|_{L_T^\infty L_x^\infty}$ so that the initial datum can have arbitrary large $L^\infty$ norm. In the meantime the smallness $\|w_\beta f_0\|_{L_T^\infty L_x^\infty}$ is replaced by the smallness $\|f_0\|_{L_T^1 L_x^\infty}$. In this case, $F_0$ can reach the upper bound $\frac{1}{2}$ and the lower bound $0$. In the view of (4.12), this requires us to prove that the smallness of $\sup_{0 \leq \delta \leq 1} \{ E_{\delta,\rho}(F_0) + \|f_0\|_{L_T^1 L_x^\infty} \}$ implies the smallness of $\|f\|_{L_T^1 L_x^\infty}$.

\section*{Lemma 4.3}

Let $\gamma$, $\beta$ and $p$ satisfy the assumption in Theorem 1.2 and $T_1$ be the constant given in Theorem 1.2. Then for any $T > T_1$ and $(t, x) \in [T_1, T] \times \Omega$, it holds that
\[
\int_{\mathbb{R}^3} |f(t, x, v)| \, dv \leq \int_{\mathbb{R}^3} e^{-\nu_4(t)} |f_0(x - vt, v)| \, dv + C \left( m_3^\gamma + \frac{C_2\rho}{C_1\rho} + \frac{1}{N} \frac{C_2\rho}{C_1\rho} + C_2\rho \lambda \right) \|w_\beta f\|_{L_T^\infty L_x^\infty}
\]
\[
+ C \left( \frac{C_5\rho}{N} \frac{C_2\rho}{C_1\rho} \right) \left( 1 + \|w_\beta f\|_{L_T^\infty L_x^\infty} \right) \|w_\beta f\|^2_{L_T^\infty L_x^\infty}
\]
\[
+ C \|w_\beta f\|^\frac{5p-1}{2p-1}_{L_T^\infty L_x^\infty} C'_p N \left( \lambda^{-\frac{\gamma}{2}} \sqrt{E_{\delta,\rho}(F_0)} + \lambda^{-3} \sqrt{E_{\delta,\rho}(F_0)} \right)
\]
\[
+ C N^3 \|w_\beta f\|^\frac{5p-1}{2p-1}_{L_T^\infty L_x^\infty} C'_p N \left( \lambda^{-\frac{\gamma}{2}} \sqrt{E_{\delta,\rho}(F_0)} + \lambda^{-3} \sqrt{E_{\delta,\rho}(F_0)} \right),
\]
where $C'_p, N$ are given in (4.39) and (4.62) respectively, $m, N, \lambda > 0$ are constants which will be determined later.
Proof. Let \((t, x) \in [T_1, T] \times \Omega\), using the mild form (1.5), we have

\[
\int_{\mathbb{R}^3} |f(t, x, v)| \, dv \leq \int_{\mathbb{R}^3} e^{-\nu_{s}(t)} \| f_0(x - vt, v) \| \, dv + H_1 + H_2 + H_3,
\]

where

\[
H_1 := \int_0^t \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \| (K^s_{\nu}) f(s, x - v(t-s), v) \| \, dvds,
\]

\[
H_2 := \int_0^t \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \| (K^s_{\nu}) f(s, x - v(t-s), v) \| \, dvds,
\]

\[
H_3 := \int_0^t \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \| \Gamma_{\delta}(f)(s, x - v(t-s), v) \| \, dvds.
\]

By Lemma 2.4 one gets that

\[
H_1 \leq C m^{3+\gamma} C_{2 \rho} \| f \|_{L^\infty_T L^\infty_v} \int_0^t \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} e^{-\frac{|v|^2}{2}} \, dvds
\]

\[
\leq C m^{3+\gamma} C_{2 \rho} \| f \|_{L^\infty_T L^\infty_v} \int_{\mathbb{R}^3} \frac{1}{\nu(v)} e^{-\frac{|v|^2}{2}} \, dv
\]

\[
\leq C m^{3+\gamma} C_{2 \rho} \| f \|_{L^\infty_T L^\infty_v},
\]

(4.44)

Once again using Lemma 2.4 and recalling the definition of \(l(v, \eta)\) (2.10), it follows that

\[
H_2 \leq C_{2 \rho} \int_0^t \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \int_{\mathbb{R}^3} l(v, \eta) \| f(s, x - v(t-s), \eta) \| \, d\eta \, dvds.
\]

A similar argument as (4.18) shows that

\[
H_2 \leq C_{2 \rho} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \chi_{\{\nu \geq N\}} l(v, \eta) \frac{1}{w_{\beta}(\eta)} \| f(s, x - v(t-s), \eta) \| \, d\eta \, dvds
\]

\[
+ C_{2 \rho} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \chi_{\{\nu \leq N\}} \frac{1}{w_{\beta}(\eta)} \| f(s, x - v(t-s), \eta) \| \, d\eta \, dvds
\]

\[
+ C_{2 \rho} \int_{t-\lambda}^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \chi_{\{\nu \geq N\}} l(v, \eta) \frac{1}{w_{\beta}(\eta)} \| f(s, x - v(t-s), \eta) \| \, d\eta \, dvds
\]

\[
+ C_{2 \rho} \int_{t-\lambda}^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \chi_{\{\nu \leq N\}} \frac{1}{w_{\beta}(\eta)} \| f(s, x - v(t-s), \eta) \| \, d\eta \, dvds
\]

\[
= H_{21} + H_{22} + H_{23} + H_{24}.
\]

It follows from our assumption \(\beta > 6 > 3 - \gamma\), Lemma 2.2 and similar arguments in (4.20) that

\[
H_{21} \leq C_{2 \rho} \| w_{\beta} f \|_{L^\infty_T L^\infty_v} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_{s}(t-s)} \chi_{\{\nu \geq N\}} l(v, \eta) \frac{1}{w_{\beta}(\eta)} \, d\eta \, dvds
\]

\[
\leq C_{2 \rho} \frac{C_{2 \rho}}{C_{1 \rho}} \| w_{\beta} f \|_{L^\infty_T L^\infty_v} \int_{\mathbb{R}^3} \chi_{\{\nu \geq N\}} \frac{1}{\nu(v)} l(v, \eta) \frac{1}{w_{\beta}(\eta)} \, dv \, d\eta
\]

\[
\leq C_{2 \rho} \frac{C_{2 \rho}}{C_{1 \rho}} \| w_{\beta} f \|_{L^\infty_T L^\infty_v} \int_{\mathbb{R}^3} \chi_{\{\nu \geq N\}} \frac{1}{w_{\beta}(v) \nu(v)(1 + |v|)^2} \, dv
\]

\[
\leq C_{2 \rho} \frac{C_{2 \rho}}{N C_{1 \rho}} \| w_{\beta} f \|_{L^\infty_T L^\infty_v}.
\]

(4.45)
In the last step above we have used the inequality
\[ \int_{\mathbb{R}^3} \chi_{\{|v| \geq N\}} \frac{1}{w_\beta(v) \nu(v)(1 + |v|)^2} dv \]
\[ = \int_{\mathbb{R}^3} \chi_{\{|v| \geq N\}} \frac{1}{(1 + |v|)^{\beta + 3}} dv \]
\[ \leq \frac{C}{N} \int_{\mathbb{R}^3} \frac{1}{(1 + |v|)^{\beta + 3}} dv \leq \frac{C}{N}. \]

By the same approach as (4.21) and (4.45) we also have
\[ H_{22} \leq \frac{C}{N} \frac{C_{2\rho}}{C_1^\rho} \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty}. \quad (4.46) \]

\[ H_{23} \] can be treated in the same way as in (4.22) that
\[ H_{23} \leq C_{2\rho} \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty} \int_{t-L}^t \int_{\mathbb{R}^3} e^{-\nu_5(v)(t-s)} l(v, \eta) \frac{1}{w_\beta(\eta)} d\eta dv ds \]
\[ \leq C_{2\rho} \lambda \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty} \int_{\mathbb{R}^3} l(v, \eta) \frac{1}{w_\beta(\eta)} d\eta dv \]
\[ \leq C_{2\rho} \lambda \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty}. \quad (4.47) \]

Constructing the similar approximation function \( \tilde{I}_N = \tilde{I}_N(v, \eta) \) as in (4.26), we split \( H_{24} \) in the following way
\[ H_{24} \leq C_{2\rho} \int_0^{t-L} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_5(v)(t-s)} \chi_{\{|v| \leq N, |\eta| \leq 2N\}} \times \| l(v, \eta) - \tilde{I}_N(v, \eta) \| \| w_\beta(\eta) f(s, x - v(t-s), \eta) \| d\eta dv ds \]
\[ + C_{2\rho} \int_0^{t-L} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_5(v)(t-s)} \chi_{\{|v| \leq N, |\eta| \leq 2N\}} \times \| \tilde{I}_N(v, \eta) \| \| f(s, x - v(t-s), \eta) \| d\eta dv ds \]
\[ \leq C \frac{C_{2\rho}}{C_1^\rho} \lambda \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty} \]
\[ + C_{2\rho} C_N \int_0^{t-L} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu_5(v)(t-s)} \chi_{\{|v| \leq N, |\eta| \leq 2N\}} \| f(s, x - v(t-s), \eta) \| d\eta dv ds. \]

Using the entropy condition as in (4.31) and (4.32), we obtain that
\[ H_{24} \leq C \frac{C_{2\rho}}{C_1^\rho} \lambda \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty} + C \tilde{C}_{\rho, N} \left( \lambda^{-\frac{5}{2}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right), \quad (4.48) \]
where
\[ \tilde{C}_{\rho, N} = C_{2\rho} C_N \left( \frac{1}{C_1^\rho N^{-\gamma}} + \frac{1}{(C_1^\rho N^{-\gamma})^{3/2}} + \frac{1}{(C_1^\rho N^{-\gamma})^4} \right). \quad (4.49) \]

Combining (4.35), (4.36), (4.37) and (4.48), it holds that
\[ H_2 \leq \frac{C}{N} \frac{C_{2\rho}}{C_1^\rho} \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty} + C_{2\rho} \lambda \| w_\beta f \|_{L_T^\infty L_{v,x}^\infty} \]
\[ + C \tilde{C}_{\rho, N} \left( \lambda^{-\frac{5}{2}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right). \quad (4.50) \]
The last term we need to estimate is $H_3$. Rewrite $H_3$ as follows:

$$H_3 = \int_0^t \int_{\mathbb{R}^3} e^{-\nu_4(v)(t-s)} \frac{1}{w_\beta(v)} |w_\beta(v)\Gamma_\delta(f)(s, x - v(t - s), v)| dvds$$

$$= \int_{1-\lambda}^{t-\lambda} \int_{\mathbb{R}^3} e^{-\nu_4(v)(t-s)} \frac{1}{w_\beta(v)} |w_\beta(v)\Gamma_\delta(f)(s, x - v(t - s), v)| dvds$$

$$+ \int_{0}^{t-\lambda} \chi_{\{v | v \geq N\}} \int_{\mathbb{R}^3} e^{-\nu_4(v)(t-s)} \frac{1}{w_\beta(v)} |w_\beta(v)\Gamma_\delta(f)(s, x - v(t - s), v)| dvds$$

$$+ \int_{0}^{t-\lambda} \chi_{\{v | v \leq N\}} \int_{\mathbb{R}^3} e^{-\nu_4(v)(t-s)} \frac{1}{w_\beta(v)} |w_\beta(v)\Gamma_\delta(f)(s, x - v(t - s), v)| dvds$$

$$= H_{31} + H_{32} + H_{33}.$$ (4.51)

By Lemma 4.11 and similar arguments as in (4.30) and (4.37), we have

$$H_{31} + H_{32} \leq C \left( C_{5p} \lambda + \frac{1}{N} C_{5p} \right) \left( 1 + \| w_\beta f \|_{L^\infty_{\nu} L^\infty_x} \right) \| w_\beta f \|_{L^\infty_{\nu} L^\infty_x}^2.$$ (4.52)

Denoting $x_1 = x - v(t - s)$ and dividing $|w_\beta(v)\Gamma_\delta(f)(s, x - v(t - s), v)|$ into two parts as in (4.32), we obtain

$$|w_\beta(v)\Gamma_\delta(f)(s, x - v(t - s), v)| dvds \leq G_1 + G_2,$$ (4.53)

where

$$G_1 = \frac{C}{\sqrt{\mu_{\delta, \rho}(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ w_\beta(v) \sqrt{|\mu_{\delta, \rho}(s, x_1, u)\sqrt{|\mu_{\delta, \rho}(s, x_1, v)|}} + w_\beta(v) \sqrt{|\mu_{\delta, \rho}(s, x_1, u)\sqrt{|\mu_{\delta, \rho}(s, x_1, v)|}} \right] dvds,$$

and

$$G_2 = \frac{C}{\sqrt{\mu_{\delta, \rho}(v)}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) \left[ w_\beta(v) \sqrt{|\mu_{\delta, \rho}(s, x_1, u')\sqrt{|\mu_{\delta, \rho}(s, x_1, v')|}} + w_\beta(v) \sqrt{|\mu_{\delta, \rho}(s, x_1, u')\sqrt{|\mu_{\delta, \rho}(s, x_1, v')|}} + w_\beta(v) \sqrt{|\mu_{\delta, \rho}(s, x_1, u')\sqrt{|\mu_{\delta, \rho}(s, x_1, v')|}} \right] dvds.$$ (4.54)

Recalling from (5.30) that $C_{5p} = C_{2p} + C_{3p} + C_{4p}$, similar arguments as in (4.33) and (4.38) show that

$$G_1 \leq C C_{5p} \nu(v) \left( 1 + \| w_\beta f \|_{L^\infty_{\nu} L^\infty_x} \right) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{4}} |f(s, x_1, u)| dvdu,$$ (4.55)

and

$$G_2 \leq C C_{5p} \nu(v) \left( 1 + \| w_\beta f \|_{L^\infty_{\nu} L^\infty_x} \right) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{4}} |f(s, x_1, v')| dvdu.$$ (4.56)

Notice that when we estimate $G_1$, we do not write it as in (4.11). This is because the variable $u$ in (4.11) is not the original velocity $u$ defined by (1.2). We first consider the case when $u \geq N$ for the variable $u$ in (4.54) and (4.55). Then we have

$$\int_{u \geq N} |f(s, x_1, u)| du = \int_{u \geq N} \frac{1}{w_{\beta/2}(u)} |w_{\beta/2}(u)f(s, x_1, u)| du,$$ (4.56)

and

$$\int_{u \geq N} e^{-\frac{|u|^2}{4}} |f(s, x_1, u')| du = \int_{u \geq N} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}} e^{-\frac{|u|^2}{4}} |f(s, x_1, u')|^p dvdu.$$ (4.57)
By the discussion above, we take the sum of $G_1$ and $G_2$ and use similar arguments in (4.6), (4.9) and (4.10) to obtain

$$
G_1 + G_2 \leq C \frac{C_5\nu(v)}{N} \left( 1 + \|w_{\beta f}\|_{L_{\infty}^\infty} \right) \|w_{\beta f}\|^2_{L_{\infty}^1 L_{\infty}^\infty} \left( \int_{\mathbb{R}^3} |w_{\beta f/2}(u) f(s, x, u)| \, du \right) \frac{\lambda}{\rho} + C C_5 \nu(v) \left( 1 + \|w_{\beta f}\|_{L_{\infty}^\infty} \right) \int_{u \leq N} \int_{\mathbb{S}^2} B(v - u, \theta) e^{-\frac{|u|^2}{4}} |f(s, x, u)| \, d\omega du du
$$

On the right-hand sides of (4.56) and (4.57), we have either $\frac{1}{w_{\beta f/2}(u)} \leq \frac{\lambda}{\rho}$ or $e^{-\frac{|u|^2}{4}} \leq \frac{\lambda}{\rho}$. Using the argument in (4.10), we conclude that

$$
G_1 + G_2 \leq C \frac{C_5\nu(v)}{N} \left( 1 + \|w_{\beta f}\|_{L_{\infty}^\infty} \right) \|w_{\beta f}\|^2_{L_{\infty}^1 L_{\infty}^\infty} \left( \int_{\mathbb{R}^3} |w_{\beta f/2}(u) f(s, x, u)| \, du \right)
$$

by the fact that

$$
\int_{\mathbb{R}^3} |w_{\beta/2}(u) f(t, x, u)| \, du = \int_{\mathbb{R}^3} \frac{1}{w_{\beta/2}(u)} |w_{\beta}(u) f(s, x, u)| \, du
$$

for $\beta > 6$. Combining (4.53) and (4.58), we have

$$
H_{33} = \int_0^{t-\lambda} \chi_{\{[u]\leq N\}} \int_{\mathbb{R}^3} e^{-\nu_3(v)(t-s)} \frac{1}{w_{\beta}(v)} |w_{\beta}(v) \Gamma_3(f)(s, x - v(t-s), v)| \, dv ds
$$

$$
\leq C \frac{C_5}{C_4 \nu(N)} \left( 1 + \|w_{\beta f}\|_{L_{\infty}^\infty} \right) \|w_{\beta f}\|^2_{L_{\infty}^1 L_{\infty}^\infty} + \int_0^{t-\lambda} \int_{\{[u]\leq N, [u,v]\leq N\}} e^{-\nu_3(v)(t-s)} e^{-\frac{|u|^2}{4w_{\beta}(v)}} B(v - u, \theta) |f(s, x, u)| \, d\omega dv ds
$$

+ \int_0^{t-\lambda} \int_{\{[u]\leq N, [u,v]\leq N\}} e^{-\nu_3(v)(t-s)} e^{-\frac{|u|^2}{4w_{\beta}(v)}} B(v - u, \theta) |f(s, x, u)| \, d\omega dv ds

= C \frac{C_5}{C_4 \nu(N)} \left( 1 + \|w_{\beta f}\|_{L_{\infty}^\infty} \right) \|w_{\beta f}\|^2_{L_{\infty}^1 L_{\infty}^\infty} + H_{331} + H_{332}. \tag{4.59}
$$
For $H_{331}$, we directly apply Hölder’s inequality and (4.29) to obtain

$$H_{331} \leq \int_0^{t-\lambda} \int_{|v| \leq N, |u| \leq N} e^{-C_{1\rho}N^\gamma(t-s)} \frac{|v-u|^2}{w_\beta(v)} B(v-u, \theta) |f(s, x_1, u)| \,dv \,ds \leq C \int_0^{t-\lambda} e^{-C_{1\rho}N^\gamma(t-s)} \left( \int_{|v| \leq N, |u| \leq N} |v-u|^\frac{p}{p-1} e^{-\frac{|w|^2}{w_\beta(v)}} \,dv \right)^{\frac{p}{p-1}} \times \left( \int_{|v| \leq N, |u| \leq N} |f(s, x_1, u)|^p \,dv \right)^{\frac{1}{p}} \,ds \leq C \int_0^{t-\lambda} e^{-C_{1\rho}N^\gamma(t-s)} \left( \int_{|v| \leq N, |u| \leq N} |f(s, x-v(t-s), u)|^p \,dv \right)^{\frac{1}{p}} \,ds. \quad (4.60)$$

It follows from a similar argument as in (4.32) that

$$H_{331} \leq C \|w_\beta f\|_{L^p_TL^\infty_x} \int_0^{t-\lambda} e^{-C_{1\rho}N^\gamma(t-s)} \left( \int_{|v| \leq N, |u| \leq N} |f(s, x-v(t-s), u)| \,dv \right)^{\frac{1}{p}} \,ds \leq C \|w_\beta f\|_{L^p_TL^\infty_x} \left( \frac{1}{C_{1\rho}N^\gamma} + \frac{1}{(C_{1\rho}N^\gamma)^{5/2}} + \frac{1}{(C_{1\rho}N^\gamma)^{4}} \right) \left( \lambda^{-\frac{2}{5}} \sqrt{E(F_0)} + \lambda^{-3} E(F_0) \right)^{\frac{1}{p}} = C \|w_\beta f\|_{L^p_TL^\infty_x} C'_{\rho, N} \left( \lambda^{-\frac{2}{5}} \sqrt{E_{\delta, \rho}(F_0)} + \lambda^{-3} E_{\delta, \rho}(F_0) \right)^{\frac{1}{p}}, \quad (4.61)$$

where $C'_{\rho, N} = \left( \frac{1}{C_{1\rho}N^\gamma} + \frac{1}{(C_{1\rho}N^\gamma)^{5/2}} + \frac{1}{(C_{1\rho}N^\gamma)^{4}} \right)$. \( \quad (4.62) \)

For $H_{332}$, we first apply Hölder’s inequality as (4.60) to get

$$H_{332} \leq C \int_0^{t-\lambda} e^{-C_{1\rho}N^\gamma(t-s)} \left( \int_{|v| \leq N, |u| \leq N} \int_{S^2} e^{-\frac{|w|^2}{w_\beta(v)}} \,d\omega \,dv \right)^{\frac{1}{p}} \,ds \leq C \int_0^{t-\lambda} e^{-C_{1\rho}N^\gamma(t-s)} \left( \int_{|v| \leq N, |u| \leq N} \int_{S^2} |f(s, x-v(t-s), u)|^p \,d\omega \,dv \right)^{\frac{1}{p}} \,ds \leq C \int_0^{t-\lambda} e^{-C_{1\rho}N^\gamma(t-s)} \left( \int_{|v| \leq N, |u| \leq N} \int_{|\eta| \leq 3N} |f(s, x-v(t-s), \eta)|^p \,d\eta \,dv \right)^{\frac{1}{p}} \,ds.$$
By (4.39), (4.61) and (4.63) we have
\[ H_{33} \leq C_{5p} \frac{C_{5p}}{C_{1p}} \left( 1 + \| w_{\beta} f \|_{L_T^p L_x^{\infty}} \right) \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{2} \\
+ C \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{p-1} C_{5p} C_{1p} \left( \lambda^{-\frac{2}{5}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right) \frac{1}{p} \\
+ C N^{3} \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{p-1} C_{5p} C_{1p} \left( \lambda^{-\frac{2}{5}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right) \frac{1}{p}. \]

(4.64)

Then it holds from (4.51), (4.52) and (4.64) that
\[ H_{3} \leq C \left( C_{5p} \lambda + \frac{1}{N} C_{5p} \right) \left( 1 + \| w_{\beta} f \|_{L_T^p L_x^{\infty}} \right) \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{2} \\
+ C \frac{C_{5p}}{C_{1p}} \left( 1 + \| w_{\beta} f \|_{L_T^p L_x^{\infty}} \right) \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{2} \\
+ C \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{p-1} C_{5p} \left( \lambda^{-\frac{2}{5}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right) \frac{1}{p} \\
+ C N^{3} \| w_{\beta} f \|_{L_T^p L_x^{\infty}}^{p-1} C_{5p} \left( \lambda^{-\frac{2}{5}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right) \frac{1}{p}. \]

(4.65)

Hence, (4.62) follows from (4.39), (4.41), (4.43) and (4.64).

With the preparation above, we can prove Theorem 1.2. From Lemma 1.2 we first assume
\[ \| w_{\beta} f \|_{L_T^p L_x^{\infty}} \leq 2 A_{0}, \]
where
\[ A_{0} = 3 C_{5} C_{6_{p}} M^{3} + C_{5} C_{p, N_{p}} \left( \lambda^{-\frac{2}{5}} \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \lambda^{-3} \mathcal{E}_{\delta, \rho}(F_0) \right). \]

Then by (4.12) we have
\[ \| w_{\beta} f \|_{L_T^p L_x^{\infty}} \leq A_{0} \\
+ C_{3} C_{6_{p}} \left( 1 + 2 A_{0} \right) \left\{ \left( 2 A_{0} \right)^{\frac{2p-1}{p}} \| f \|_{L_{T_{1}}^{p} L_{x}^{\infty}} + \left( 2 A_{0} \right)^{\frac{10p-1}{p} - \frac{10}{p}} \| f \|_{L_{T_{1}}^{p} L_{x}^{\infty}} \right\}. \]

(4.66)

A direct calculation shows that
\[ A_{0} \leq 3 C_{5} C_{7_{p}} M^{3} + C_{3} C_{7_{p}} \left( \sqrt{\mathcal{E}_{\delta, \rho}(F_0)} + \mathcal{E}_{\delta, \rho}(F_0) \right), \]
where
\[ C_{7_{p}} = C_{6_{p}} + C_{p, N_{p}} \left( \lambda^{-\frac{2}{5}} + \lambda^{-3} \right). \]

We first require
\[ \mathcal{E}_{\delta, \rho}(F_0) \leq M^{3} \]
to get
\[ A_{0} \leq 5 C_{5} C_{7_{p}} M^{3}. \]

(4.67)

Hence, we see that \( \bar{C}_{5\rho} \) in (1.19) should be defined by
\[ \bar{C}_{5\rho} = C_{7_{p}}. \]

(4.68)
If we can choose \( \|f\|_{L_{x,v}^\infty L_t^1} \) so small that

\[
\|f\|_{L_{x,v}^\infty L_t^1} \leq \min \left\{ \frac{1}{\beta, \gamma, M} \right\} \left( \frac{1}{C_{6p} 4C_{7p}} \right)^{\frac{2p-1}{p}} \left( 1 + 10C_{C_{7p}}M^3 \right) (5C_{C_{7p}}M^3)^{\frac{2p-1}{p}}
\]

we then have from (4.67) that

\[
C_{6p} (1 + 2A_0) (2A_0)^{\frac{2p-1}{p}} \|f\|_{L_{x,v}^\infty L_t^1}^p \leq \frac{3}{4} C_{6p} M^3 \leq \frac{1}{4} A_0.
\]

Similarly as above, we can prove that

\[
C_{6p} (1 + 2A_0) (2A_0)^{\frac{10p-1}{p}} \|f\|_{L_{x,v}^\infty L_t^1} \leq \frac{1}{4} A_0.
\]

We can close our a priori assumption by (4.66), (4.70) and (4.71) that

\[
\|w_{\beta} f\|_{L_{x,v}^\infty L_t^1} \leq A_0 + \frac{A_0}{4} + \frac{A_0}{4} \leq \frac{3}{4} A_0.
\]

We first relax the requirement (4.69) to be

\[
\|f\|_{L_{x,v}^\infty L_t^1} \leq C_4 C_{8p},
\]

where \( C_4 = C_4(\beta, \gamma, M) \) depends only on \( \beta, \gamma \) and \( M \), and

\[
C_{8p} = \frac{1}{C_{7p}^{2p-1} (1 + C_{7p}^p)}.
\]

Then we need to prove that we can actually let \( \|f\|_{L_{x,v}^\infty L_t^1} \) meet the requirement (4.73). From (4.42) in Lemma 4.3 and (4.67), we have

\[
\int_{\mathbb{R}^3} |f(t,v)| dv \leq \int_{\mathbb{R}^3} e^{-\nu(t,v)} |f_0(x - vt,v)| dv + C \left( m_3 + \frac{C_{2p}}{C_{1p}} + \frac{1}{C_{2p}} \lambda \right) 5C_{C_{7p}} M^3 + C \left( C_{5p} + \frac{C_{5p}}{C_{1p}} \right) (1 + 5C_{C_{7p}} M^3) (5C_{C_{7p}} M^3)^2
\]

\[
+ C (5C_{C_{7p}} M^3)^{\frac{3p-1}{p}} C_{p,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\lambda_{\delta,p}(F_0) + \lambda^{-3} \lambda_{\delta,p}(F_0)} \right)^{\frac{1}{p}}
\]

\[
+ C N^3 (5C_{C_{7p}} M^3)^{\frac{3p-1}{p}} \frac{C_{p,N} \left( \lambda^{-\frac{2}{3}} \sqrt{\lambda_{\delta,p}(F_0) + \lambda^{-3} \lambda_{\delta,p}(F_0)} \right)^{\frac{1}{p}}}{\lambda_{\delta,p}(F_0) + \lambda^{-3} \lambda_{\delta,p}(F_0)}.
\]

Similarly as how we define \( m_\rho, N_\rho \), we define

\[
m_\rho = \varepsilon_3 \left( \frac{C_{8p} C_{1p}}{C_{2p} C_{7p}} \right)^{\frac{1}{2p}},
\]

\[
\hat{\lambda}_\rho = \varepsilon_3 \min \left\{ \frac{C_{8p}}{C_{2p} C_{7p}}, \frac{C_{8p}}{C_{5p} C_{7p}}, \frac{C_{8p}}{C_{p,N} C_{7p}} \right\},
\]

\[
\bar{N}_\rho = \frac{1}{\varepsilon_3} \max \left\{ \frac{C_{2p} C_{7p}}{C_{8p} C_{1p}}, \frac{C_{5p} C_{7p}^2}{C_{8p} C_{1p}}, \frac{C_{5p} C_{7p}^2}{C_{8p} C_{1p}} \right\}.
\]
where $\epsilon_3 = \epsilon_3(\beta, \gamma, M)$ is a small constant, such that

$$
\int_{\mathbb{R}^3} |f(t, x, v)| \, dv \leq \int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv + C(5C_3C_7^pM^3)^{\frac{p-1}{p}} \rho \left( \bar{\lambda}_p^{-\frac{2}{3}} \sqrt{E_{\delta, \rho}(F_0)} + \bar{\lambda}_p^{-3} E_{\delta, \rho}(F_0) \right)\left( \lambda_p^{-\frac{2}{3}} \sqrt{E_{\delta, \rho}(F_0)} + \lambda_p^{-3} E_{\delta, \rho}(F_0) \right)^{-\frac{1}{2}}.
$$

We can see if we define

$$
C_{9p} = \min \left\{ C_{8p}^{\frac{2}{p}}, \frac{C_{8p}^{10p}}{(C_{7p})^{2(p-1)}} \lambda_p^{-\frac{2}{3}} + \lambda_p^{-3} \right\} \frac{C_{8p}}{C_{9p}, \bar{N}_p} \left( \lambda_p^{-\frac{2}{3}} + \lambda_p^{-3} \right)^2 \left( \lambda_p^{-\frac{2}{3}} + \lambda_p^{-3} \right)\left( \lambda_p^{-\frac{2}{3}} + \lambda_p^{-3} \right)
$$

then there exists $\epsilon_4 = \epsilon_4(\beta, \gamma, M)$ such that if

$$
E_{\delta, \rho}(F_0) \leq \epsilon_4 C_{9p},
$$

we have

$$
\int_{\mathbb{R}^3} |f(t, x, v)| \, dv \leq \int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv + \frac{C_4 C_{8p}}{2}.
$$

Recall from Theorem 1.1 that $T_1 = \frac{C_1}{C_{9p}}(1 + \|w_{\beta} f_0\|_{L^\infty_v} + \|w_{\beta} f_0\|_{L^\infty_v}^2) > \frac{C}{C_{9p}^3}$. We consider the case that $t \geq T_1$. If $\Omega = \mathbb{R}^3$,

$$
\int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv \leq \frac{1}{T_1} \|f_0\|_{L^1_v L^\infty_t},
$$

Then we choose some small $\epsilon_5(\beta, \gamma, M)$ such that

$$
\|f_0\|_{L^1_v L^\infty_t} \leq \epsilon_5(\beta, \gamma, M) C_{9p} C_{8p}
$$

and $\int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv \leq \frac{1}{2} C_{4} C_{8p}$. Then together with (1.74), (1.73) is satisfied. Hence, (1.71) holds true and we close the a priori estimate by (1.72). Finally, we let $\epsilon = \min \{ \epsilon_4, \epsilon_5 \}$ and

$$
\bar{M}_p = \min \{ 1, C_{9p}, C_{9p} C_{8p} \}
$$

so that

$$
\int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv \leq \frac{4}{M_1} \|f_0(y)\|_{L^\infty} dy,
$$

we obtain

$$
\int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv \leq \int_{\{ |v| \geq M_1 \}} |f_0(x - vt, v)| \, dv + \int_{\{ |v| \leq M_1 \}} |f_0(x - vt, v)| \, dv 
$$

\leq \int_{\{ |v| \geq M_1 \}} |f_0(x - vt, v)| \, dv + \int C \{ M_1^3 \| f_0 \|_{L^1_v L^\infty_t} + M^6 \| f_0 \|_{L^1_v L^\infty_t} \}
$$

\leq M_1^{1-\beta} \| w_{\beta} f_0 \|_{L^\infty_v} + CM_1^3 \| f_0 \|_{L^1_v L^\infty_t} + CM^6 \| f_0 \|_{L^1_v L^\infty_t}.
$$

By choosing $M_1 = \left( \| w_{\beta} f_0 \|_{L^\infty_v} \right)^{-\frac{1}{\beta}}$, we have
Then similarly as in (4.75), we define
\[
\int_{\mathbb{R}^3} |f_0(x - vt, v)| \, dv \leq C \|w_\beta f_0\|_{L_x^\infty}^{\frac{5}{3}} \|f_0\|_{L_x^\infty}^{1-\frac{4}{3}} + \frac{C}{C_{\beta} M^2} M^6 \|f_0\|_{L_x^\infty}^{1-\frac{4}{3}} \leq CM^\frac{5}{3} \|f_0\|_{L_x^\infty}^{1-\frac{4}{3}} + \frac{C}{C_{\beta} M^2} M^6 \|f_0\|_{L_x^\infty}^{1-\frac{4}{3}}.
\]
Then similarly as in (4.75), we define
\[
\bar{M}_\beta = \min\{1, C_{\beta}, C_{\beta} M^2, C_{\beta} M^6\}
\]
(4.76) to obtain (4.18) and (4.19). Therefore we finish the proof of Theorem 1.2. \qed

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