METRIC FOLIATIONS ON THE EUCLIDEAN SPACE

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Abstract. We complete a minor gap in Gromoll and Walschap classification of metric fibrations from the Euclidean space, thus completing the classification of Riemannian foliations on Euclidean spaces.

1. Introduction

A (non-singular) Riemannian foliation is a foliation whose leaves are locally equidistant. A Riemannian submersion is a submersion whose fibers are locally equidistant. Metric foliations and submersions on specific Riemannian manifolds have been studied and classified. For instance, Lytchak–Wilking [LW16] complete the classification of Riemannian foliations of the Euclidean sphere; Gromoll–Walshap [GW01] propose a classification of Riemannian submersions of the Euclidean space and Florit–Goertsches–Lytchak–Töben [FGLT15] prove that any Riemannian foliation $\mathcal{F}$ of the Euclidean space $\mathbb{R}^{n+k}$ is defined by a submersion $\pi : \mathbb{R}^{n+k} \to M^n$ whose fibers coincide with the leaves of $\mathcal{F}$.

However, two gaps in [GW01] were pointed out in [Wei], thus reopening the question of the classification of Riemannian submersions/foliations on the Euclidean space. More specifically, [FGLT15] questions:

Question 1.10. Is any Riemannian foliation on the Euclidean space homogeneous?

The purpose of this note is to complete the gaps in [GW01], answering the question above affirmatively:

Theorem. Every Riemannian foliation with connected fibers on the Euclidean space is homogeneous.

In the next sections, we briefly discuss Gromoll–Walschap’s proof and present a workaround for the gaps pointed out in [Wei]. The new argument happens to be quite elementary and starts just before the first gap, making it easy to be put together for a complete proof.

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2. Original proof and gap

Gromoll–Walschap [GW01] stated the following theorem:

**Theorem 2.1** ([GW01], page 234). Let \( \pi : \mathbb{R}^{n+k} \to M^n \) be a Riemannian submersion of the Euclidean space with connected fibers. Then

1. there is a fiber \( F \) (over a soul of \( M \)) which is an affine subspace of the Euclidean space, that, up to congruence, may be taken to be \( F = \mathbb{R}^k \times \{0\} \).
2. there is a representation \( \phi : \mathbb{R}^k \to \text{SO}(n) \) such that \( \pi \) is the orbit fibration of the free isometric group action \( \psi \) of \( \mathbb{R}^k \) on \( \mathbb{R}^{n+k} = \mathbb{R}^k \times \mathbb{R}^n \) given by

\[
\psi(v)(u,x) = (u + v, \phi(v)x), \quad u, v \in \mathbb{R}^k, \ x \in \mathbb{R}^n.
\]

As a first step, [GW97] proves that the fiber \( \pi^{-1}(b) = F \) over the soul \( \{b\} \) of \( M \) is totally geodesic, concluding item (1) in Theorem 2.1.

Recall that the integrability tensor \( A \) is the vertical restriction \( A_XY = \nabla^v_XY \) of the Levi-Civita connection \( \nabla \), where \( X, Y \) are horizontal vector fields on \( \mathbb{R}^{n+k} \). Moreover, a field is called basic if it is both horizontal and projectable.

The aim of [GW01] is to prove Proposition 2.2 below, thus concluding Theorem 2.1 directly from Theorem 2.6 in [GW97] (see also the paragraph preceding Theorem 2.6 in [GW97]). After presenting two gaps in [GW01], our goal is to establish:

**Proposition 2.2.** For every basic fields \( X, Y, A_XY \) is parallel along \( F \)

In the last paragraph, [GW01] show that \( A_XY \) is parallel along \( F \) for all basic \( X, Y \) if \( A_x y \) is parallel along \( F \) for all parallel horizontal \( x, y \). The overall argument in [GW01] is then to prove that \( A_x y \) is parallel.

**Remark 2.3.** We remark that an argument similar to the one presented in section 3 indeed shows that \( A_x y \) is parallel, achieving Gromoll–Walschap’s aim with a different approach.

Let \( x, y \) be parallel horizontal fields along \( F \). In [GW01, section 3], a very interesting argument using the fiber volume form shows that \( \nabla_v(A_x y) = 0 \) for all \( v \in \text{im}(A_x) + \text{im}(A_y) \). It follows that \( \text{im}(A_x) \) defines an integrable distribution with totally geodesic leaves on \( F \) (at least in the open and dense subset where the rank of \( \text{im}(A_x) \) is constant). The remainder of the proof deals with \( \nabla_u(A_x y) \) for \( u \in (\text{im}(A_x) + \text{im}(A_y))^\perp \) and can be divided in three steps:

**Step 1:** \( \text{im}(A_x) \) defines a foliation by affine subspaces on \( F \)
Step 2: \( \pi \) is the composition of a linear projection \( pr : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k-l} \) followed by a Riemannian submersion \( \pi : \mathbb{R}^{n+k-l} \to \mathbb{R}^n \), such that \( \pi' \) is ‘fully twisted’. Specifically, \( TF' = pr(\bigoplus_{x \in \mathcal{H}} im(A_x)) \), for \( F' = pr(F) \), where \( \mathcal{H} \) denotes the horizontal distribution along \( F \).

Step 3: The integrability tensor of \( \pi' \) is parallel along \( F' \).

The gaps appear in Steps 1 and 3. In Step 1, a gap appears in arguing that \( im(A_x) \) defines a Riemannian foliation on \( F \). In Step 3, it seems to be implicitly assumed that \( \bigoplus_{x \in \mathcal{H}} im(A_x) = im(A_x) + im(A_y) \) for a dense subset \((x,y) \in \mathcal{H} \times \mathcal{H} \) along \( F \). Although this statement is true for the homogeneous submersions in Theorem \[GW01\], one may believe that it generically does not hold if \( \dim(TF) > 2 \dim(\mathcal{H}) \).

2.1. First gap. For \( p \in F \) let \( T_p \mathbb{R}^{n+k} = \mathcal{H}_p + T_p F \) denote the orthogonal decomposition into the horizontal and the vertical space at \( p \). For \( x \in \mathcal{H}_p \), denote the adjoint of \( A_x : \mathcal{H}_p \to T_p F \) by \( A_x^* : T_p F \to \mathcal{H}_p \), noting that \( im(A_x)^\perp = ker(A_x^*) \).

The next step in \[GW01\] was to prove that \( im(A_x) \) defines a foliation by parallel affine subspaces. This could be achieved by proving that, if \( \gamma \) is a geodesic on \( F \) satisfying \( \gamma'(0) \in (im A_x)^\perp = ker(A_x^*) \), then \( \gamma'(t) \in ker(A_x^*) \) for all \( t \). The first gap lies in the following claim (see \[GW01\], section 3).

Claim 2.4. For \( a \in F \), let \( x \in \mathcal{H}_a \) and \( u \in ker(A_x^*) \). Then \( A_x^* \gamma_u(t) = 0 \) for all \( t \), where \( \gamma_u(t) := a + tu \) is a line in \( F \).

Discussion of the proof of Claim 2.4. For \( u \in ker(A_x^*) \) we consider the variation \( V \) on \([0,1] \times (-1,1)\), \( V(t,s) := \exp_{\gamma_u(s)}(tx) \) by horizontal geodesics which projects to the variation \( W = \pi \circ V \) by geodesics on \( M \). Likewise, since \( V \) is by horizontal geodesics, its variational field \( V_x D_s(t,0) \) is a Jacobi field that projects to the Jacobi field \( Y(t) := (\pi \circ V)D_s(t,0) = W_s D_s(t,0) \) on \( M \) induced by \( W \). \( Y \) satisfies \( Y(0) = 0 \) and

\[
Y'(0) = \nabla_{D_t(0,0)}((\pi \circ V)_x D_s) = \pi_* \nabla_{D_t(0,0)}(V_x D_s)^h = -\pi_* \nabla^h_{D_t(0,0)}(V_x D_s)^u = \pi_* A_x^* u = 0,
\]

since \( V_x D_s(0,0) = u \). The second equality follows since the fields \( V_x D_t(t,0) \) and \( (V_x D_s(t,0))^h \) are horizontal fields along \( t \mapsto \exp_{\pi}(tx) \).

The third equality is due to the identity

\[
\pi_* \nabla^h_{D_t(0,0)}(V_x D_s)^h + \pi_* \nabla^h_{D_t(0,0)}(V_x D_s)^u = \pi_* \nabla_{D_t(0,0)}(V_x D_s) = \pi_* \nabla_{\pi u} x = 0.
\]

We follow \( Y \equiv 0 \) along \( t \mapsto W(t,0) \).

At this point, \( x \) is stated to be basic along \( \gamma_u \). However, even though \( W \) is a variation by geodesics emanating from a single point and the variational field \( Y \) is trivial along the geodesic \( t \mapsto W(t,0) \), this is not sufficient to imply that \( x \) is a basic field along \( \gamma_u \). Indeed, one needs to show that \( W_x D_s(t, s) = 0 \) for all \((t, s) \in [0,1] \times (-1,1)\). Then, \( x \) is mapped to a single
vector in $M$ since $t \mapsto W(t, s)$ corresponds to the geodesic $t \mapsto W(t, 0)$ for all $s$. Hence, further arguments are required. The underlying issues can be seen in:

**Example 2.5.** Let $M = \mathbb{R}^2$ and $e_1$ and $e_2$ be the standard basis. Consider the variation $W : [0, 1] \times (-1, 1) \to \mathbb{R}^2$ given by $W(t, s) = te_1 + s^2 e_2$ of the geodesic $t \mapsto te_1$. Then its variational field $Y$ is trivial. But for $s \neq 0$, $t \mapsto W(t, s)$ does not coincide with the geodesic $t \mapsto te_1$.

On the other hand, if we assume that the field $x$ is indeed basic along $\gamma_u$, then

$$A_x^{\ast} \gamma_u = -\nabla_{\gamma_u}^h x = -\nabla_u^h (x \circ \gamma_u) = 0,$$

and Claim 2.4 follows. \hfill \Box

### 2.2. Second gap.

Define the sets

$$A_p = \text{span}\{A_{xy} \mid x, y \in \mathcal{H}_p\}, \quad \text{im}(A) := \bigcup_{p \in F} A_p.$$  

According to Claim 2.4, the distributions $\text{im}(A)$ and $\text{im}(A)^\perp$ define an isometric splitting $F \cong \mathbb{R}^l \times \mathbb{R}^{k-l}$, which extends to the whole ambient space, $\mathbb{R}^{n+k}$, via holonomy transport. These properties result into a factorization of the projection $\pi$:

**Proposition 2.6.** Assume that Claim 2.4 is true. Then $\pi$ factors as an orthogonal projection $\mathbb{R}^{n+k-l} \times \mathbb{R}^l \to \mathbb{R}^{n+k-l} \times \{0\}$ followed by a Riemannian submersion $\pi' : \mathbb{R}^{n+k-l} \to M$. In particular, the fiber $F' := \text{pr}(F)$ is an affine subspace satisfying $TF' = \text{im}(A)$.

$$\begin{array}{ccc}
F & \xrightarrow{\text{pr}} & \mathbb{R}^{n+k} \\
\downarrow \text{pr} & & \downarrow \pi \\
F' = \text{im}(A) \times \{0\} & \xrightarrow{\pi'} & \mathbb{R}^{n+k-l} \times \{0\} & \xrightarrow{\text{pr}} & M
\end{array}$$

What we therefore obtain is a Riemannian submersion $\pi'$ which only contains the 'twisting part' of the former submersion $\pi$. Although $F'$ is spanned by integrability fields, the induced metric foliation $\mathcal{F}'$ of $\pi'$ is not necessarily substantial along $F'$. That is, one can not guarantee that there is a single horizontal $x \in (T_p F')$ such that $\text{im}(A_x) = T_p F'$.

This observation is relevant since the concluding argument in [GW01], in the proof that $A_{xy}$ is parallel, seems to be based on the substantiality of $\mathcal{F}'$; recall that $\nabla_v (A_{xy}) = 0$ for all $v \in \text{im}(A_x) + \text{im}(A_y)$. If $\mathcal{F}'$ is substantial, there is an open and dense set of horizontal vectors $z \in \mathcal{H}_p$, $p \in F'$, such that $\text{im}(A_z) = T_p F'$. In particular, it would follow that $\nabla_v (A_{xy}) = 0$ for all $x, y \in \mathcal{H}$ and $v \in TF' = \text{im}(A_x) = \text{im}(A)$. Otherwise, $A_{xy}$ could be only parallel on $\text{im}(A_x) + \text{im}(A_y)$, but not on the whole $\text{im}(A)$. More specifically, the following statement in [GW01] lacks a proof:
Lemma 3.1. just before the first gap by recalling from [GW01] that:

\[ \nabla_v(A_x y) = 0 \text{ for } v \in \text{im}(A_x) + \text{im}(A_y) \] 

(it is restated in the next section).

3. \( A_X Y \) is parallel

From now on, we fix basic fields \( X, Y \) along \( F \). We directly prove that \( A_X Y \) is parallel along \( F \), avoiding Claim 2.4 and Proposition 2.6. We start just before the first gap by recalling from [GW01] that:

Lemma 3.1 ([GW01], Lemma 2.4). Let \( p \in F \). Then, \( (\nabla_v A)_X Y = 0 \) for all \( X, Y \in \mathcal{H}_p \) and \( v \in \text{im}(A_X) + \text{im}(A_Y) \).

In order to prove that \( A_X Y \) is parallel, we follow the proof of Theorem 2.6 in [GW97] and show that \( A_X Y \) is the gradient of a function \( f : F \to \mathbb{R} \). As in [GW97], we recall that constant length gradients in affine spaces are parallel and that \( ||A_X Y|| \) is constant (as it follows from O'Neill’s equation 3\( ||A_X Y||^2 = R_M(\pi_s X, \pi_t Y, \pi_v Y, \pi_s X) \)).

Denote \( \mathcal{I} = \text{im}(A_X) + \text{im}(A_Y) \), thus \( \mathcal{I}^\perp = \ker(A_X) \cap \ker(A_Y) \).

Lemma 3.2. For every \( u, u' \in \mathcal{I}^\perp \) and \( v \in \mathcal{I} \):

(i) \( \nabla_v(A_X Y) = (\nabla^v S)_X v - (\nabla^v S)_Y v \)
(ii) \( \langle \nabla_u(A_X Y), v \rangle = 0 \)
(iii) \( \langle \nabla_v(A_X Y), u \rangle = 0 \)
(iv) \( \langle \nabla_u(A_X Y), u' \rangle = -\langle (\nabla^v S)_Y u, u' \rangle \)

Proof. O’Neill’s equation (see, e.g., [GW99] page 43) gives

(1) \( (\nabla^v w)_X Y = -A_Y A^*_X w - (\nabla^v X)_Y w \)

for all \( w \in TF \). Therefore, Lemma 3.1 gives

(2) \( A_Y A^*_X v = - (\nabla^v X)_Y v \)

for all \( v \in \mathcal{I} \). Item (i) now follows from a straightforward computation:

(3) \[
\nabla_v(A_X Y) = (\nabla^v A)_X Y + A_Y A^*_X v - A_X A^*_Y v = (\nabla^v S)_Y v - A_X A^*_Y v = (\nabla^v S)_Y v + (\nabla^v X)_Y v,
\]

where equation (3) is valid for all \( w \in TF \) and the last equality follows from (2).

For item (ii), we get

\[
\langle \nabla_u(A_X Y), v \rangle = -\langle A_X A^*_Y u + (\nabla^v X)_Y u, v \rangle = -\langle (\nabla^v S)_Y u, v \rangle = -\langle u, (\nabla^v X)_Y v \rangle = \langle u, A_Y A^*_X v \rangle = 0.
\]
The first and fourth equalities follow from equations (3) and (2), respectively, and the last since $u \perp \text{im}(AX)$. Item (iii) follows from item (i) and equation (2), since $u \in \mathcal{T}$. Item (iv) follows from equation (3) and since $u \in \mathcal{T}$. □

**Proposition 3.3.** There is a function $f : F \to \mathbb{R}$ whose gradient is $AXY$.

**Proof.** Consider the 1-form $\alpha : TF \to \mathbb{R}, \alpha(u) = \langle AXY, u \rangle$. Then $\alpha = df$ for some $f$ if and only if

$$d\alpha(u, v) = \langle \nabla_u(AXY), v \rangle - \langle \nabla_v(AXY), u \rangle = 0$$

for all $u, v \in TF$. But the latter holds by a straightforward computation by distinction of cases for $u, v \in TF = \mathcal{T} + \mathcal{T}^\perp$, using Lemma 3.2 and by observing that $(\nabla_X S)_Y : TF \to TF$ is a symmetric operator since $S_Y$ is symmetric. □

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