THE STRUCTURE OF NORMAL ALGEBRAIC MONOIDS

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Abstract. We show that any normal algebraic monoid is an extension of an abelian variety by a normal affine algebraic monoid. This extends (and builds on) Chevalley’s structure theorem for algebraic groups.

1. Introduction

A classical theorem of Chevalley asserts that any connected algebraic group is an extension of an abelian variety by a connected affine algebraic group. In this note, we obtain an analogous result for normal algebraic monoids. This reduces their structure to that of more familiar objects: abelian varieties, and affine (equivalently, linear) algebraic monoids. The latter have been extensively investigated, see the expositions [7, 8].

To state Chevalley’s theorem and our analogue in a precise way, we introduce some notation. We consider algebraic varieties and algebraic groups over an algebraically closed field $k$ of arbitrary characteristic. By a variety, we mean a separated integral scheme of finite type $X$ over $k$; by a point of $X$, we mean a closed point. An algebraic group is a smooth group scheme of finite type over $k$.

Let $G$ be a connected algebraic group, then there exists a unique connected normal affine algebraic subgroup $G_{\text{aff}}$ such that the quotient $G/G_{\text{aff}}$ is an abelian variety. In other words, we have an exact sequence of connected algebraic groups

$$1 \longrightarrow G_{\text{aff}} \longrightarrow G \xrightarrow{\alpha_G} \mathcal{A}(G) \longrightarrow 0$$

where $G_{\text{aff}}$ is affine and $\mathcal{A}(G)$ is projective (since the group $\mathcal{A}(G)$ is commutative, its law will be denoted additively). It follows that the morphism $\alpha_G$ is affine; hence the variety $G$ is quasi-projective (see [2] for these developments and for a modern proof of Chevalley’s theorem).

Next, let $M$ be an irreducible algebraic monoid, i.e., an algebraic variety over $k$ equipped with a morphism $M \times M \to M$ (the product, denoted simply by $(x, y) \mapsto xy$) which is associative and admits an identity element 1. Denote by $G = G(M)$ the group of invertible elements of $M$. The unit group $G$ is known to be a connected algebraic group, open in $M$ (see [9, Thm. 1]).
Let $G_{\text{aff}} \subseteq G$ be the associated affine group, and $M_{\text{aff}}$ the closure of $G_{\text{aff}}$ in $M$. Clearly, $M_{\text{aff}}$ is an irreducible algebraic monoid with unit group $G_{\text{aff}}$. By [10, Thm. 2], it follows that $M_{\text{aff}}$ is affine. Also, note that

\[(1.2) \quad M = G M_{\text{aff}} = M_{\text{aff}} G\]

as follows from the completeness of $G / G_{\text{aff}} = A(G)$ (see Lemma 3.5 for details).

We may now state our main result, which answers a question raised by D. A. Timashev (see the comments after Thm. 17.3 in [12]):

**Theorem 1.1.** Let $M$ be an irreducible algebraic monoid with unit group $G$. If the variety $M$ is normal, then $\alpha_G : G \to A(G)$ extends to a morphism of algebraic monoids $\alpha_M : M \to A(G)$. Moreover, the morphism $\alpha_M$ is affine, and its scheme-theoretic fibers are normal varieties; the fiber at 1 equals $M_{\text{aff}}$.

In loose words, any normal algebraic monoid is an extension of an abelian variety by a normal affine algebraic monoid.

For nonsingular monoids, Theorem 1.1 follows immediately from Weil’s extension theorem: any rational map from a nonsingular variety to an abelian variety is a morphism. However, this general result no longer holds for singular varieties. Also, the normality assumption in Theorem 1.1 cannot be omitted, as shown by Example 2.7.

Some developments and applications of the above theorem are presented in Section 2. The next section gathers a number of auxiliary results to be used in the proof of that theorem, given in Section 4. The final Section 5 contains further applications of our structure theorem to the classification of normal algebraic monoids, and to their faithful representations as endomorphisms of homogeneous vector bundles on abelian varieties.

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2. SOME APPLICATIONS

With the notation and assumptions of Theorem 1.1, observe that $\alpha_M$ is equivariant with respect to the action of the group $G \times G$ on $M$ via

$$(g_1, g_2) \cdot m = g_1 m g_2^{-1},$$

and its action on $A(G)$ via

$$(g_1, g_2) \cdot a = \alpha(g_1) - \alpha(g_2) + a.$$ 

Since the latter action is transitive, $\alpha_M$ is a $G \times G$-homogeneous fibration. In particular, all fibers are isomorphic, and $\alpha_M$ is faithfully flat.
Also, each irreducible component of the closed subset $M_{\text{aff}} \setminus G_{\text{aff}} \subset M_{\text{aff}}$ is of codimension 1, since $G_{\text{aff}}$ is affine. Together with the above observation, it follows that the same holds for the set $M \setminus G$ of non-units in $M$:

**Corollary 2.1.** Each irreducible component of $M \setminus G$ has codimension 1 in $M$.

Next, we obtain an intrinsic characterization of the morphism $\alpha_M$. To state it, recall from [11] that any variety $X$ admits an Albanese morphism, i.e., a universal morphism to an abelian variety.

**Corollary 2.2.** $\alpha_M$ is the Albanese morphism of the variety $M$.

**Proof.** Let $f : M \to A$ be a morphism (of varieties) to an abelian variety. Composing $f$ with a translation of $A$, we may assume that $f(1) = 0$. Then the restriction $f|_G : G \to A$ is a morphism of algebraic groups by [2, Lem. 2.2]. So $f(G_{\text{aff}})$ equals 0 by [2, Lem. 2.3]. It follows that $f|_G = \varphi \circ \alpha_G$, where $\varphi : A(G) \to A$ is a morphism of algebraic groups. Hence $f$ equals $\varphi \circ \alpha_M$, since both morphisms have the same restriction to the open subset $G$.

Also, since the morphism $\alpha_M : M \to A(G)$ is affine and the variety $A(G)$ is projective (see e.g. [4, Thm. 7.1]), we obtain the following:

**Corollary 2.3.** $M$ is quasi-projective.

Another application of Theorem 1.1 concerns the set

$E(M) = \{ e \in M \mid e^2 = e \}$

of idempotents. Indeed, since $\alpha_M$ is a morphism of monoids and the unique idempotent of $A(G)$ is the origin, we obtain:

**Corollary 2.4.** $E(M) = E(M_{\text{aff}})$.

Next, recall that a monoid $N$ is said to be regular if given any $x \in N$, there exists $y \in N$ such that $x = xyx$.

**Corollary 2.5.** $M$ is regular if and only if $M_{\text{aff}}$ is regular.

**Proof.** If $M_{\text{aff}}$ is regular, then so is $M$ by (1.2). Conversely, assume that $M$ is regular. Let $x \in M_{\text{aff}}$ and write $x = xyx$, where $y \in M$. Then we obtain: $\alpha_M(y) = 0$, so that $y \in M_{\text{aff}}$.

By [3, Thm. 13], every regular irreducible affine algebraic monoid $N$ is unit regular, i.e., given any $x \in N$, there exists $y \in G(N)$ such that $x = xyx$; equivalently, $N = G(N)E(N)$. Together with Corollary 2.5 this implies:

**Corollary 2.6.** If $M$ is regular, then it is unit regular.

Finally, we show that Theorem 1.1 does not extend to arbitrary irreducible algebraic monoids:
Example 2.7. Let $A$ be an abelian variety. Then

$$M := A \times \mathbb{A}^1$$

is a commutative nonsingular algebraic monoid via the product

$$(x, y) (x', y') = (x + x', yy'),$$

with unit group

$$G := A \times \mathbb{G}_m$$

and kernel $A \times \{0\}$. The morphism $\alpha_G$ is the first projection $A \times \mathbb{G}_m \to A$; likewise, the Albanese morphism of the variety $M$ is just the first projection

$$p : A \times \mathbb{A}^1 \to A.$$

Next, let $F \subset A$ be a non-trivial finite subgroup. Let $M'$ be the topological space obtained from $M$ by replacing the closed subset $A \times \{0\}$ with the quotient $A/F \times \{0\}$; in other words, each point $(x + f, 0)$ (where $x \in A$ and $f \in F$) is identified with the point $(x, 0)$. Denote by

$$q : M \to M'$$

the natural map. We claim that $M'$ has a structure of a irreducible, non-normal, commutative algebraic monoid with unit group $G$, such that $q$ is a morphism of monoids; furthermore, $\alpha_G : G \to A$ does not extend to a morphism $M' \to A$.

Indeed, one readily checks that $M'$ carries a unique product such that $q$ is a morphism of monoids. Moreover, the restriction $q|_G$ is an isomorphism onto $G(M')$, and we have a commutative square

$$
\begin{array}{ccc}
M & \xrightarrow{q} & M' \\
p & & \downarrow \alpha \\
A & \xrightarrow{\alpha} & A/F
\end{array}
$$

where $\alpha$ is equivariant with respect to the action of $A$ on $M'$ via the product of $M'$, and the natural action of $A$ on $A/F$. Let

$$N := \alpha^{-1}(0),$$

then the set $N$ is the image under $q$ of the subset $F \times \mathbb{A}^1 \subset A \times \mathbb{A}^1$. So $N$ is a union of copies of the affine line, indexed by the finite set $F$, and glued along the origin. Hence $N$ is a reduced affine scheme, and its product (induced by the product of $M'$) is a morphism: $N$ is a connected, reducible affine algebraic monoid. Furthermore, the natural map

$$A \times^F N \to M'$$

is clearly an isomorphism of monoids, and the left-hand side is also an algebraic monoid. This yields the desired structure of algebraic monoid on $M'$. The map $q$ is induced from the natural map $F \times \mathbb{A}^1 \to N$, which is a morphism; hence so is $q$. Finally, the projection $p : A \times \mathbb{G}_m \to A$ cannot extend to a morphism $M' \to A$: such a morphism would
be $A$-equivariant, and hence restrict to an $A$-equivariant morphism $A/F \times \{0\} \cong A/F \to A$, which is impossible. This completes the proof of the claim.

Alternatively, this claim follows from a general result concerning the existence of pinched schemes (see [3, Thm. 5.4]). Indeed, $M'$ is obtained by pinching the quasi-projective variety $M$ along its closed subset $A \times \{0\} \cong A$ via the finite morphism $A \to A/F$, in the terminology of [3].

One easily checks that $\alpha$ is the Albanese morphism of the variety $M'$, and $q$ is its normalization. Moreover,

$$M'_\text{aff} = q(\{0\} \times A^1) \cong A^1$$

is strictly contained in $N$, and is nonsingular whereas $M'$ is non-normal.

Note finally that $M'$ is weakly normal, i.e., any finite bijective birational map from a variety to $M'$ is an isomorphism. Thus, Theorem 1.1 does not extend to weakly normal monoids.

3. Auxiliary results

We consider a connected algebraic group $G$ and denote by $Z^0$ its connected center regarded as a closed reduced subscheme of $G$, and hence as a connected algebraic subgroup.

**Lemma 3.1.** (i) The scheme-theoretic intersection $Z^0 \cap G_{\text{aff}}$ contains $Z^0_{\text{aff}}$ as a normal subgroup, and the quotient $(Z^0 \cap G_{\text{aff}})/Z^0_{\text{aff}}$ is a finite group scheme.

(ii) The product map $Z^0 \times G_{\text{aff}} \to G$ factors through an isomorphism

$$(Z^0 \times G_{\text{aff}})/(Z^0 \cap G_{\text{aff}}) \cong G,$$

where $Z^0 \cap G_{\text{aff}}$ is embedded in $Z^0 \times G_{\text{aff}}$ as a normal subgroup scheme via the identity map on the first factor and the inverse map on the second factor.

(iii) The natural map $Z^0/(Z^0 \cap G_{\text{aff}}) \to G/G_{\text{aff}} = \mathcal{A}(G)$ is an isomorphism of algebraic groups.

This easy result is proved in [1, Sec. 1.1] under the assumption that $\mathbb{k}$ has characteristic zero; the general case follows by similar arguments. We will also need the following result, see e.g. [1, Sec. 1.2]:

**Lemma 3.2.** Let $G$ act faithfully on an algebraic variety $X$. Then the isotropy subgroup scheme of any point of $X$ is affine.

Next we consider an irreducible algebraic monoid $M$ with unit group $G$. If $M$ admits a zero element, then this point is fixed by $G$ acting by left multiplication, and this action is faithful. Thus, $G$ is affine by Lemma 3.2. Together with [10, Thm. 2], this yields:

**Corollary 3.3.** Any irreducible algebraic monoid having a zero element is affine.
Returning to an arbitrary irreducible algebraic monoid $M$, recall that an ideal of $M$ is a subset $I$ such that $MIM \subseteq I$.

**Lemma 3.4.** (i) $M$ contains a unique closed $G \times G$-orbit, which is also the unique minimal ideal: the kernel $\text{Ker}(M)$.
(ii) If $M$ is affine, then $\text{Ker}(M)$ contains an idempotent.

**Proof.** (i) is part of [9, Thm. 1]. For (ii), see e.g. [8, p. 35]. □

**Lemma 3.5.** (i) Let $Z^0 \cap G_{\text{aff}}$ act on $Z^0 \times M_{\text{aff}}$ by multiplication on the first factor, and the inverse map composed with left multiplication on the second factor. Then the quotient

$$Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}} := (Z^0 \times M_{\text{aff}})/(Z^0 \cap G_{\text{aff}})$$

has a unique structure of an irreducible algebraic monoid such that the quotient map is a morphism of algebraic monoids. Moreover,

$$G(Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}}) = Z^0 \times Z^0 \cap G_{\text{aff}} G_{\text{aff}} \cong G.$$ 

Regarded as a $G$-variety via left multiplication, $Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}}$ is naturally isomorphic to the quotient

$$G \times G_{\text{aff}} M_{\text{aff}} := (G \times M_{\text{aff}})/G_{\text{aff}},$$

where the action of $G_{\text{aff}}$ on $G \times M_{\text{aff}}$ is defined as above.

(ii) The product map $Z^0 \times M_{\text{aff}} \to M$ factors uniquely through a morphism of algebraic monoids

$$\pi : Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}} \to M.$$ 

Moreover, $\pi$ is birational and proper.

(iii) $M = Z^0 M_{\text{aff}}$ and $\text{Ker}(M) = Z^0 \text{Ker}(M_{\text{aff}})$.

**Proof.** (i) and the first assertion of (ii) are straightforward. Also, the restriction of $\pi$ to the unit group is an isomorphism, and hence $\pi$ is birational. To show the properness, observe that $\pi$ factors as a closed immersion

$$Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}} \to Z^0 \times Z^0 \cap G_{\text{aff}} M$$

(induced by the inclusion map $M_{\text{aff}} \to M$), followed by an isomorphism

$$Z^0 \times Z^0 \cap G_{\text{aff}} M \to (Z^0/Z^0 \cap G_{\text{aff}}) \times M \cong A(G) \times M$$

given by $(z, m) \mapsto (z(Z^0 \cap G_{\text{aff}}), zm)$, followed in turn by the projection

$$A(G) \times M \to M$$

which is proper, since $A(G)$ is projective.

(iii) By (ii), $\pi$ is surjective, i.e., the first equality holds. For the second equality, note that $Z^0 \text{Ker}(M_{\text{aff}})$ is closed in $M$ since $\pi$ is proper, and is a unique orbit of $G \times G$ by Lemma [3.1] □
4. Proof of the main result

We begin by showing the following result of independent interest:

**Theorem 4.1.** The morphism
\[ \pi : Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}} = G \times G_{\text{aff}} M \to M \]
is an isomorphism for any normal irreducible monoid \( M \).

*Proof.* We proceed through a succession of reduction steps.

1) It suffices to show that \( \pi \) is finite for any irreducible algebraic monoid \( M \) (possibly non-normal). Indeed, the desired statement follows from this, in view of Lemma 3.5(ii) and Zariski’s Main Theorem.

2) Since \( \pi \) is proper, it suffices to show that its fibers are finite. But the points of \( M \) where the fiber of \( \pi \) is finite form an open subset (by semicontinuity), which is stable under the action of \( G \times G \). Thus, it suffices to check the finiteness of the fiber at some point of the unique closed \( G \times G \)-orbit, \( \text{Ker}(M) \). By Lemmas 3.4 (ii) and 3.5 (iii), \( \text{Ker}(M) \) contains an idempotent \( e \in \text{Ker}(M_{\text{aff}}) \). So we are reduced to showing that the set \( \pi^{-1}(e) \) is finite.

3) Consider the \( Z^0 \)-orbit \( Z^0 e \) and its inverse image under \( \pi \),
\[ \pi^{-1}(Z^0 e) \cong Z^0 \times Z^0 \cap G_{\text{aff}} (Z^0 e \cap M_{\text{aff}}). \]
It suffices to check that the map
\[ p : Z^0 \times Z^0 \cap G_{\text{aff}} (Z^0 e \cap M_{\text{aff}}) \to Z^0 e \]
(the restriction of \( \pi \)) has finite fibers. Since \( p \) is surjective, it suffices in turn to show that \( Z^0 \times Z^0 \cap G_{\text{aff}} (Z^0 e \cap M_{\text{aff}}) \) and \( Z^0 e \) are algebraic groups of the same dimension, and \( p \) is a morphism of algebraic groups.

4) Since \( e \) is idempotent and \( Z^0 \) is a central subgroup of \( G \), the orbit \( Z^0 e \) (regarded as a locally closed, reduced subscheme of \( M \)) is a connected algebraic group under the product of \( M \), with identity element \( e \). Moreover, the intersection \( Z^0 e \cap M_{\text{aff}} \) (also regarded as a locally closed, reduced subscheme of \( M \)) is a closed submonoid of \( Z^0 e \), with the same identity element \( e \). By [S Exer. 3.5.1.2], it follows that \( Z^0 e \cap M_{\text{aff}} \) is a subgroup of \( Z^0 e \). Hence \( Z^0 \times Z^0 \cap G_{\text{aff}} (Z^0 e \cap M_{\text{aff}}) \) is an algebraic group as well, and clearly \( p \) is a morphism of algebraic groups.

5) It remains to show that
\[
\text{dim} Z^0 \times Z^0 \cap G_{\text{aff}} (Z^0 e \cap M_{\text{aff}}) = \text{dim}(Z^0 e).
\]

We first analyze the left-hand side. Since \( Z^0 e \cap M_{\text{aff}} \) is a quasi-affine algebraic group, it is affine. But the maximal connected affine subgroup of \( Z^0 e \) is \( Z^0_{\text{aff}} e \), since \( Z^0 e \cong Z^0 / \text{Stab}_{Z^0}(e) \) as groups. It follows that \( Z^0_{\text{aff}} e \subseteq Z^0 e \cap M_{\text{aff}} \) is the connected component of the identity. Hence
\[
\text{dim} Z^0 \times Z^0 \cap G_{\text{aff}} (Z^0 e \cap M_{\text{aff}}) = \text{dim}(Z^0) - \text{dim}(Z^0 \cap G_{\text{aff}}) + \text{dim}(Z^0_{\text{aff}} e).
\]
But \(\dim(Z_0 \cap G_{\text{aff}}) = \dim(Z_{\text{aff}}^0)\) by Lemma 3.1 and
\[
\dim(Z_{\text{aff}}^0 e) = \dim(Z_{\text{aff}}^0) - \dim \text{Stab}_{Z_{\text{aff}}^0}(e)
\]
so that
\[
\dim Z_0 \times Z_0 \cap G_{\text{aff}} (Z_0 e \cap M_{\text{aff}}) = \dim(Z^0) - \dim \text{Stab}_{Z_{\text{aff}}^0}(e).
\]
On the other hand,
\[
\dim(Z_0 e) = \dim(Z_0) - \dim \text{Stab}_{Z_0}(e),
\]
and \(\text{Stab}_{Z_0}(e)\) is affine by Lemma 3.2. Hence
\[
\dim \text{Stab}_{Z_0}(e) = \dim \text{Stab}_{Z_{\text{aff}}^0}(e).
\]
This completes the proof of Equation (4.1) and, in turn, of the finiteness of \(\pi\).

We may now prove Theorem 1.1. Observe that the projection
\[
M \simeq Z_0 \times Z_0 \cap G_{\text{aff}} M_{\text{aff}} \to Z_0^0 / (Z_0 \cap G_{\text{aff}}) \simeq A(G)
\]
yields the desired extension \(\alpha_M\) of \(\alpha_G\). Clearly, the scheme-theoretic fiber of \(\alpha_M\) at 1 equals \(M_{\text{aff}}\).

To show that the morphism \(\alpha_M\) is affine, observe that \(M_{\text{aff}}\) is an affine variety equipped with an action of the affine group scheme \(Z_0 \cap G_{\text{aff}}\). Thus, \(M_{\text{aff}}\) admits a closed equivariant immersion into a \((Z_0 \cap G_{\text{aff}})\)-module \(V\). Then \(Z_0 \times Z_0 \cap G_{\text{aff}} M_{\text{aff}}\) (regarded as a variety over \(A(G)\)) admits a closed \(Z_0\)-equivariant immersion into \(Z_0 \times Z_0 \cap G_{\text{aff}} V\), the total space of a vector bundle over \(A(G)\).

Finally, to show that the variety \(M_{\text{aff}}\) is normal, consider its normalization \(\widetilde{M_{\text{aff}}}\), an affine variety where \(Z_0 \cap G_{\text{aff}}\) acts such that the normalization map \(f : \widetilde{M_{\text{aff}}} \to M_{\text{aff}}\) is equivariant. This defines a morphism
\[
\tilde{\pi} : Z_0 \times Z_0 \cap G_{\text{aff}} \widetilde{M_{\text{aff}}} \to M
\]
which is still birational and finite. So \(\tilde{\pi}\) is an isomorphism by Zariski’s Main Theorem; it follows that \(f\) is an isomorphism as well.

5. Classification and faithful representation

We begin by reformulating our main results as a classification theorem for normal algebraic monoids:

**Theorem 5.1.** The category of normal algebraic monoids is equivalent to the category having as objects the pairs \((G, N)\), where \(G\) is a connected algebraic group and \(N\) is a normal affine algebraic monoid with unit group \(G_{\text{aff}}\).

The morphisms from such a pair \((G, N)\) to a pair \((G', N')\) are the pairs \((\varphi, \psi)\), where \(\varphi : G \to G'\) is a morphism of algebraic groups and \(\psi : N \to N'\) is a morphism of algebraic monoids such that \(\varphi|_{G_{\text{aff}}} = \psi|_{G_{\text{aff}}}\).
Proof. By Theorem 4.1 any normal irreducible algebraic $M$ with unit group $G$ is determined by the pair $(G, M_{\text{aff}})$ up to isomorphism. Conversely, any pair $(G, N)$ as in the above statement yields a normal algebraic monoid

$$M := G \times^{G_{\text{aff}}} N$$

together with isomorphisms $G \to G(M)$ and $N \to M_{\text{aff}}$, as follows from Lemmas 3.1 and 3.5.

Next, consider a morphism of normal algebraic monoids

$$f : M \to M'.$$

Clearly, $f$ restricts to a morphism of algebraic groups

$$\varphi : G(M) \to G(M').$$

Moreover, the universal property of the Albanese maps $\alpha_M$, $\alpha_{M'}$ yields a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
\downarrow{\alpha_M} & & \downarrow{\alpha_{M'}} \\
A(G(M)) & \xrightarrow{\alpha_f} & A(G(M')),
\end{array}$$

where $\alpha_f$ is a morphism of varieties such that $\alpha_f(0) = 0$, and hence a morphism of abelian varieties (see e.g. [4, Cor. 3.6]). In turn, this yields a morphism of algebraic monoids

$$\psi : M_{\text{aff}} = \alpha_M^{-1}(0) \to \alpha_{M'}^{-1}(0) = M'_{\text{aff}}$$

such that $\varphi|_{G(M)_{\text{aff}}} = \psi|_{G(M')_{\text{aff}}}$. Conversely, any such pair $(\varphi, \psi)$ defines a morphism $f$, as follows from Theorem 4.1 again. □

Remarks 5.2. (i) The irreducible affine algebraic monoids having a prescribed unit group $G$ are exactly the affine equivariant embeddings of the homogeneous space $(G \times G)/\text{diag}(G)$, by [9, Prop. 1]. In the case where $G$ is reductive, such embeddings admit a combinatorial classification, see [9] and [12].

(ii) The normality assumption in Theorem 5.1 cannot be omitted: with the notation of Example 2.7, the monoids $M$ and $M'$ yield the same pair $(A \times \mathbb{G}_m, \mathbb{A}^1)$; but they are not isomorphic as varieties, since the image of their Albanese map is $A$, resp. $A/F$.

Next, we obtain faithful representations of normal algebraic monoids as endomorphisms of vector bundles over abelian varieties. For this, we need additional notation and some preliminary observations.

Let $A$ be an abelian variety, and

$$p : E \to A$$
a vector bundle. Observe that $p$ is the Albanese morphism of the variety $E$ (as follows e.g. from [4, Cor. 3.9]). Thus, any morphism of varieties $f : E \to E$ fits into a commutative square

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E \\
p & \downarrow & p \\
A & \xrightarrow{\alpha(f)} & A,
\end{array}
\]

where $\alpha(f)$ is a morphism of varieties as well. By [4, Cor. 3.9] again, $\alpha(f)$ is the composition of a translation of $A$ with an endomorphism of the abelian variety $A$.

We say that $f$ is an endomorphism (resp. an automorphism) of $E$, if $\alpha(f)$ is the translation $t_a : A \to A, \ x \mapsto a + x$

for some $a = a(f) \in A$, and the induced maps on fibers $f_x : E_x \to E_{a+x} \ (x \in A)$

are all linear (resp. linear isomorphisms).

Clearly, the endomorphisms of $E$ form a monoid under composition, denoted by $\text{End}(E)$; its unit group $\text{Aut}(E)$ consists of the automorphisms. The map

\[(5.1) \quad \alpha : \text{End}(E) \to A, \quad f \mapsto a(f)\]

is a morphism of monoids, and its fiber at a point $a \in A$ is isomorphic to the set of morphisms of vector bundles from $E$ to $t_a^*E$ (a finite-dimensional $k$-vector space).

In particular, the fiber at 0 is the monoid $\text{End}_A(E)$ of endomorphisms of $E$ regarded as a vector bundle over $A$. Moreover, $\text{End}_A(E)$ is a finite-dimensional $k$-algebra; in particular, an irreducible affine algebraic monoid. Its unit group $\text{Aut}_A(E)$ is the kernel of the restriction of $\alpha$ to $\text{Aut}(E)$.

The vector bundle $E$ is called homogeneous if the restriction map $\text{Aut}(E) \to A$ is surjective; equivalently, $E \cong t_a^*E$ for any $a \in A$.

For example, a line bundle is homogeneous if and only if it is algebraically equivalent to 0 (see [4, Sect. 9]). More generally, the homogeneous vector bundles are exactly the direct sums of vector bundles of the form $L \otimes F$, where $L$ is an algebraically trivial line bundle, and $F$ admits a filtration by sub-vector bundles such that the associated graded bundle is trivial (see [5, Thm. 4.17]).

We are now in a position to state:

**Theorem 5.3.** (i) Let $p : E \to A$ be a homogeneous vector bundle over an abelian variety. Then $\text{End}(E)$ has a structure of a nonsingular irreducible algebraic monoid such that its action on $E$ is algebraic.
Moreover, the Albanese morphism of \( \text{End}(E) \) is the map of \([5,7]\), so that \( \text{End}(E)_{\text{aff}} = \text{End}_A(E) \).

(ii) Any normal irreducible algebraic monoid \( M \) is isomorphic to a closed submonoid of \( \text{End}(E) \), where \( E \) is a homogeneous vector bundle over the Albanese variety of \( M \).

Proof. (i) We claim that \( \text{Aut}(E) \) has a structure of a connected algebraic group such that its action on \( E \) is algebraic and the map \( \text{Aut}(E) \to A \) is a (surjective) morphism of algebraic groups.

Indeed, any \( f \in \text{Aut}(E) \) extends uniquely to an automorphism of the projective completion \( \mathbb{P}(E \oplus O_A) \) (regarded as an algebraic variety), where \( O_A \) denotes the trivial bundle of rank 1 over \( A \). Moreover, every automorphism of \( \mathbb{P}(E \oplus O_A) \) induces an automorphism of \( A \), the Albanese variety of \( \mathbb{P}(E \oplus O_A) \). It follows that \( \text{Aut}(E) \) may be identified to the group \( G \) of automorphisms of \( \mathbb{P}(E \oplus O_A) \) that induce translations of \( A \), and commute with the action of the multiplicative group \( \mathbb{G}_m \) by multiplication on fibers of \( E \). Clearly, \( G \) is contained in the connected automorphism group \( \text{Aut}^0 \mathbb{P}(E \oplus O_A) \) (a connected algebraic group) as a closed subgroup; hence, \( G \) is an algebraic group. Moreover, the exact sequence

\[ 1 \to \text{Aut}_A(E) \to \text{Aut}(E) \to A \to 0 \]

implies that \( G \) is connected; this completes the proof of our claim.

This exact sequence also implies that \( \text{Aut}(E)_{\text{aff}} = \text{Aut}_A(E) \). Hence the natural map

\[ \pi : \text{Aut}(E) \times^{\text{Aut}(E)} \text{End}_A(E) \to \text{End}(E) \]

is bijective, since the map \( \alpha : \text{End}(E) \to A \) is a \( \text{Aut}(E) \)-homogeneous fibration. This yields a structure of algebraic monoid on \( \text{End}(E) \), which clearly satisfies our assertions.

(ii) By \([7\), Thm. 3.15], the associated monoid \( M_{\text{aff}} \) is isomorphic to a closed submonoid of \( \text{End}(V) \), where \( V \) is a vector space of finite dimension over \( \mathbb{k} \). In particular, \( V \) is a rational \( G_{\text{aff}} \)-module. We may thus form the associated vector bundle

\[ p : E := G \times^{G_{\text{aff}}} V = Z^0 \times^{Z^0 \cap G_{\text{aff}}} V \to A(G) \]

Since the action of \( A(G) \) on itself by translations lifts to the action of \( G \) on \( E \), then \( E \) is homogeneous. Moreover, one easily checks that the product action of \( Z^0 \times M_{\text{aff}} \) on \( Z^0 \times V \) yields a faithful action of \( M = Z^0 \times Z^0 \cap G_{\text{aff}} M_{\text{aff}} \) on \( E \). \( \square \)

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