A Gentle Introduction to a Beautiful Theorem of Molien

Holger Schellwat

holger.schellwat@oru.se
Orebro universitet, Sweden
Universidade Eduardo Mondlane, Moçambique

12 January, 2017

Abstract

The purpose of this note is to give an accessible proof of Molien’s Theorem in Invariant Theory, in the language of today’s Linear Algebra and Group Theory, in order to prevent this beautiful theorem from being forgotten.

Contents

1 Preliminaries 3
2 The Magic Square 6
3 Averaging over the Group 8
4 Eigenvectors and eigenvalues 11
5 Molien’s Theorem 13
6 Symbol table 17
7 Lost and found 17
References 17
Index 18
Introduction

We present some memories of a visit to the ring zoo in 2004. This time we met an animal looking like a unicorn, known by the name of invariant theory. It is rare, old, and very beautiful. The purpose of this note is to give an almost self contained introduction to and clarify the proof of the amazing theorem of Molien, as presented in [Slo77]. An introduction into this area, and much more, is contained in [Stu93]. There are many very short proofs of this theorem, for instance in [Sta79], [Hu90], and [Tam91].

Informally, Molien’s Theorem is a power series generating function formula for counting the dimensions of subrings of homogeneous polynomials of certain degree which are invariant under the action of a finite group acting on the variables. As an appetizer, we display this stunning formula:

$$\Phi_G(\lambda) := \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\text{id} - \lambda T_g)}$$

We can immediately see elements of linear algebra, representation theory, and enumerative combinatorics in it, all linked together. The paper [Slo77] nicely shows how this method can be applied in Coding theory. For Coding Theory in general, see [Bie04].

Before we can formulate the Theorem, we need to set the stage by looking at some Linear Algebra (see [Rom 08]), Group Theory (see [Hu96]), and Representation Theory (see [Sag 91] and [Tam91]).
1 Preliminaries

Let $V \cong \mathbb{C}^n$ be a finite dimensional complex inner product space with orthonormal basis $\mathcal{B} = (e_1, \ldots, e_n)$ and let $\mathbf{x} = (x_1, \ldots, x_n)$ be the orthonormal basis of the algebraic dual space $V^*$ satisfying $\forall 1 \leq i, j \leq n : x_i(e_j) = \delta_{ij}$. Let $G$ be a finite group acting unitarily linearly on $V$ from the left, that is, for every $g \in G$ the mapping $V \rightarrow V, \mathbf{v} \mapsto g.\mathbf{v}$ is a unitary bijective linear transformation. Using coordinates, this can be expressed as $[g, \mathbf{v}]_\mathcal{B} = [\mathcal{G}]_\mathcal{B} [\mathbf{v}]_\mathcal{B}$, where $[\mathcal{G}]_\mathcal{B}$ is unitary. Thus, the action is a unitary representation of $G$, or in other words, a $G$–module. Note that we are using left composition and column vectors, i.e. $\mathbf{v} = (v_1, \ldots, v_n)$ convention $\mathbb{C}^n \cong [v_1 v_2 \ldots v_n]^\top$, c.f. Ant73.

The elements of $V^*$ are linear forms(linear functionals), and the elements $x_1, \ldots, x_n$, looking like variables, are also linear forms, this will be important later.

Thinking of $x_1, \ldots, x_n$ as variables, we may view (see [Tam91]) $S(V^*)$, the symmetric algebra on $V^*$ as the algebra $R := \mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_n]$ of polynomial functions $V \rightarrow \mathbb{C}$ or polynomials in these variables (linear forms). It is naturally graded by degree as $R = \bigoplus_{d \in \mathbb{N}} R_d$, where $R_d$ is the vector space spanned by the polynomials of (total) degree $d$, in particular, $R_0 = \mathbb{C}$, and $R_1 = V^*$.

The action of $G$ on $V$ can be lifted to an action on $R$.

1.1 Proposition. Let $V, G, R$ as above. Then the mapping $\cdot : G \times R \rightarrow R, (g, f) \mapsto g.f$ defined by $(g.f)(\mathbf{v}) := f(g^{-1}.\mathbf{v})$ for $\mathbf{v} \in V$ is a left action.

Proof. For $\mathbf{v} \in V$, $g, h \in G$, and $f \in R$ we check

1. $(1.f)(\mathbf{v}) = f(1^{-1}.\mathbf{v}) = f(1.\mathbf{v}) = f(\mathbf{v})$

2. $((h.g).f)(\mathbf{v}) = f((h.g)^{-1}.\mathbf{v}) = f((g^{-1}h^{-1}).\mathbf{v}) = f(g^{-1}.(h^{-1}.\mathbf{v})) = (g.f)(h^{-1}.\mathbf{v}) = (h.(g.f))(\mathbf{v})$

In fact, we know more.

1.2 Proposition. Let $V, G, R$ as above. For every $g \in G$, the mapping $T_g : R \rightarrow R, f \mapsto g.f$ is an algebra automorphism preserving the grading, i.e. $g.R_d \subset R_d$ (here we do not bother about surjectivity).

Proof. For $\mathbf{v} \in V$, $g \in G$, $c \in \mathbb{C}$, and $f, f' \in R$ we check

1. $(g.(f + f'))(\mathbf{v}) = (f + f')(g^{-1}.\mathbf{v}) = f(g^{-1}.\mathbf{v}) + f'(g^{-1}.\mathbf{v}) = (g.f)(\mathbf{v}) + (g.f')(\mathbf{v}) = (g.f + g.f')(\mathbf{v})$, thus $g.(f + f') = g.f + g.f'$

2. $(g.(f \cdot f'))(\mathbf{v}) = (f \cdot f')(g^{-1}.\mathbf{v}) = f(g^{-1}.\mathbf{v}) \cdot f'(g^{-1}.\mathbf{v}) = (g.f)(\mathbf{v}) \cdot (g.f')(\mathbf{v}) = (g.f \cdot g.f')(\mathbf{v})$, thus $g.(f \cdot f') = g.f \cdot g.f'$
3. \((g.c.f)(v) = c(f)(g^{-1}.v) = c(f(g^{-1}.v)) = c((g.f)(v)) = (c(g.f))(v)\)

By part 2, it is clear that the grading is preserved.

5. To show that \(f \mapsto g.f\) is bijective it is enough to show that this mapping is injective on the finite dimensional homogeneous components \(R_d\). Let us introduce a name for this mapping, say \(T^d_g : R_d \to R_d, f \mapsto g.f\). Now \(f \in \text{ker}(T^d_g)\) implies that \(g.f = 0 \in R_d\), i.e. \(g.f\) is a polynomial mapping from \(V\) to \(\mathbb{C}\) of degree \(d\) vanishing identically, \(\forall v \in V : (g.f)(v) = 0\).

By definition of the extended action we have \(\forall v \in V : f(g^{-1}.v) = 0\). Since \(G\) acts on \(V\) this implies that \(\forall v \in V : f(v) = 0\), so \(f\) is the zero mapping. Since our ground field has characteristic 0, this implies that \(f\) is the zero polynomial, which we may view as an element of every \(R_d\). See for instance [Cox91], proposition 5 in section 1.1.

6. Note that every \(T^d_g\) is also surjective, since all group elements have their inverse in \(G\).

Both propositions together give us a homomorphism from \(G\) into \(\text{Aut}(R)\). They also clarify the rôle of the induced matrices, which are classical in this area, as mentionend in [Slo77]. Since the monomials \(x_1, \ldots, x_n\) of degree one form a basis for \(R_1\), it follows from the proposition that their products \(x_2 := (x_1^2, x_1x_2, x_1x_3, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots)\) form a basis for \(R_2\), and, in general, the monomials of degree \(d\) in the linear forms (!) \(x_1, \ldots, x_n\) form a basis \(x_d\) of \(R_d\). Clearly, they certainly span \(R_d\), and by the last observation in the last proof they are linearly independent.

1.3 Definition. In the context from above, that is \(g \in G, f \in R^d, \text{ and } v \in V\), we define

\[ T_g^d : R_d \to R_d, f \mapsto g.f : R^d \to \mathbb{C}, v \mapsto f(g^{-1}.v) = f(T_{g^{-1}}.v) \]

1.4 Remark. In particular, we have \((T_g^1(f))(v) = f(T_{g^{-1}}.v))\), see proposition 1.6 below.

Keep in mind that a function \(f \in R_d\) maps to \(T_g^d(f) = g.f\). Setting \(A_g := [T_g^1]|_{x.x}\), then \(A_g^{[d]} := [T_g^1]|_{x_2.x.d}\) is the \(d\)-th induced matrix in [Slo77], because \(T_g^d(f \cdot f') = T_g^d(f) \cdot T_g^d(f')\). Also, if \(f, f'\) are eigenvectors of \(T_g^1\) corresponding to the eigenvalues \(\lambda, \lambda'\), then \(f \cdot f'\) is an eigenvector of \(T_g^2\) with eigenvalue \(\lambda \cdot \lambda'\), because \(T_g(f \cdot f') = T_g(f) \cdot T_g(f') = (\lambda f) \cdot (\lambda' f') = (\lambda \lambda') (f \cdot f')\). All this generalizes to \(d > 2\), we will get back to that later.

We end this section by verifying two little facts needed in the next section.

1.5 Proposition. The first induced operator of the inverse of a group element \(g \in G\) is given by \(T_{g^{-1}} = (T_g^1)^{-1}\).

Proof. Since \(\text{dim}(V^*) < \infty\), it is sufficient to prove that \(T_{g^{-1}} \circ T_g^1 = \text{id}_{V^*}\). Keep in mind that \((T_g^1(f))(v) = f(T_{g^{-1}}.v))\). For arbitrary \(f \in V^*\) we see that

\[(T_{g^{-1}} \circ T_g^1)(f) = T_{g^{-1}}(T_g^1(f)) = T_{g^{-1}}(g.f) = g^{-1}.(g.f) = (g^{-1}g).f = f\]

\[\square\]
We will be mixing group action notation and composition freely, depending on the context. The following observation is a translation device.

1.6 Proposition. For $g \in G$ and $f \in V^*$ the following holds:

$$T^1(f) = g.f = f \circ T_g^{-1}.$$

Proof. For $v \in V$ we see $(T^1(f))(v) = (g.f)(v) \overset{\text{def}}{=} f(g^{-1}.v) = f(T_g^{-1}(v))$. □
2 The Magic Square

Remember that we require a unitary representation of $G$, that is the operators $T_g : V \to V$ need to be unitary, i.e. $\forall g \in G : (T_g)^{-1} = (T_g)^*$. The first goal of this sections is to show that this implies that the induced operators $T_d^g : Rd \to Rd, f \mapsto gf$ are also unitary. We saw that $T_d^1 = V^*$, the algebraic dual of $V$. In order to understand the operator duals of $V$ and $V^*$ we need to look on their inner products first. We may assume that the operators $T_g$ are unitary with respect to the standard inner product $\langle u, v \rangle = [u]_B \cdot [v]_B$, where $\cdot$ denotes the dot product.

Before we can speak of unitarity of the induced operators $T_d^g$ we have to make clear which inner product applies on $R^1 = V^*$. Quite naively, for $f, g \in V^*$ we are tempted to define $\langle f, g \rangle = [f]_x \cdot [g]_x$. We will motivate this in a while, but first we take a look at the diagram in [Rom 08], chapter 10, with our objects:

$$
\begin{array}{ccc}
R^1 = V^* & \xrightarrow{T_d^1} & V^* = R^1 \\
\uparrow P & & \uparrow P \\
V & \xrightarrow{T_g} & V \\
\downarrow \tau & & \downarrow \tau \\
T_g^* & & T_g
\end{array}
$$

Here $P$ (“Rho”) denotes the Riesz map, see [Rom 08], Theorem 9.18, where it is called $R$, but $R$ denotes already our big ring. We started by looking at the operator $T_g$, which is unitary, so its inverse is the Hilbert space adjoint $T_g^*$. Omitting the names of the bases we have $[T_g^*] = [T_g]^*$. We also see the operator adjoint $T_g^*$ with matrix $[T_g^*] = [T_g]^\top$, the transpose. However, the arrow for $T_d^1$ is not in the original diagram, but soon we will see it there, too.

Fortunately, the Riesz map $P$ turns a linear form into a vector and its inverse $\tau : V \to V^*$ maps a vector to a linear form, both are conjugate isomorphisms. This is mostly all we need in order to show that $T_d^1$ is unitary. In the following three propositions we use that $V$ has the orthonormal basis $B$ and that $V^*$ has the orthonormal basis $x$.

2.1 Proposition. For every $f \in V^*$ the coordinates of its Riesz vector are given by

$$[P(f)]_x = (f(e_1), \ldots, f(e_n)).$$

Proof. Writing $\tau$ for the inverse of $P$, we need to show that

$$P(f) = \sum_{i=1}^n \overline{f(e_i)} e_i$$

which is equivalent to

$$f = \tau \left( \sum_{i=1}^n \overline{f(e_i)} e_i \right).$$
It is sufficient to show the latter for values of \( f \) on the basis vectors \( e_j, 1 \leq j \leq n \). We obtain
\[
\left( r \left( \sum_{i=1}^{n} f(e_i)e_i \right) \right)(e_j) = \left\langle e_j, \left( \sum_{i=1}^{n} f(e_i)e_i \right) \right\rangle = \sum_{i=1}^{n} \left\langle e_j, f(e_i)e_i \right\rangle = \sum_{i=1}^{n} (e_j, f(e_i)e_i) = f(e_i) \cdot 1.
\]

In particular, this implies that \( P(x_i) = e_i \).

2.2 Proposition. Our makeshift inner product on \( V^* \) satisfies
\[
\langle f, g \rangle = \langle P(f), P(g) \rangle,
\]
where \( f, g \in V^* \).

Proof. By our vague definition we have \( \langle f, g \rangle = \langle f[x, , x], \overline{g[x, , x]} \rangle \). It is enough to show that \( \langle x_i, x_j \rangle = \langle P(x_i), P(x_j) \rangle \). From the comment after the proof of Proposition 2.1 we obtain
\[
\langle P(x_i), P(x_j) \rangle = \langle e_1, e_j \rangle = \delta_{ij} = e_i \cdot e_j = [x_i][x], \overline{x_j}[x, , x].
\]

Hence, our guess for the inner product on \( V^* \) was correct. We will now relate the Riesz vector of \( f \in V^* \) to the Riesz vector of \( f \circ T_g^{-1} \). Recall that the Riesz vector of \( f \in V^* \) is the unique vector \( w = P(f) \) such that \( f(v) = \langle v, w \rangle \) for all \( v \in V \). If \( f \neq 0 \) it can be found by scaling any nonzero vector in the cokernel of \( f \), which is one–dimensional, see [Rom 08], in particular Theorem 9.18.

2.3 Proposition. Let \( T_g : V \to V \) be unitary, \( f \in V^* \), \( w = P(f) \) the vector of \( f \in V^* \). Then \( T_g(w) \) is the Riesz vector of \( f \circ T_g^{-1} \), i.e. the Riesz vector of \( T_g(f) \).

Proof. We may assume that \( f \neq 0 \). Using the notation \( \langle w \rangle \) for the one–dimensional subspace spanned by \( w \), we start with a little diagram:
\[
\langle w \rangle \odot \ker(f) \xrightarrow{T_g} \langle T_g(w) \rangle \odot \ker(f \circ T_g^{-1}),
\]
where \( \odot \) denotes the orthogonal direct sum.

We need to show that \( f \circ T_g^{-1} = \langle \cdot, T_g(w) \rangle \), i.e. that \( (f \circ T_g^{-1})(v) = \langle v, T_g(w) \rangle \) for all \( v \in V \). Since \( w = P(f) \) the vector of \( f \), we have \( f(v) = \langle v, w \rangle \) for all \( v \in V \). We obtain
\[
(f \circ T_g^{-1})(v) = \langle T_g^{-1}(v), w \rangle T_g, \text{unitary} = \langle v, T_g(w) \rangle.
\]

From Remark 1.3 we conclude that \( f \circ T_g^{-1} = T_g^3(f) \).
Observe that proposition 2.3 implies the commutativity of the following two diagrams.

\[
\begin{array}{ccc}
V^* & \xrightarrow{T_g^1} & V^* \\
\downarrow p & & \downarrow p \\
V & \xrightarrow{T_g} & V
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
V^* & \xrightarrow{(T_g^1)^{-1}} & V^* \\
\downarrow p & & \downarrow p \\
V & \xrightarrow{(T_g)^{-1}} & V
\end{array}
\]

Indeed, 2.3 implies

\[
P \circ T_g^1 = T_g \circ P \\
P \circ (T_g^1)^{-1} = (T_g)^{-1} \circ P
\]

2.4 Proposition. The first induced operator \( T_g^1 \) is unitary.

Proof. We may use that \( T_g \) is unitary, that is,

\[
\langle T_g(v), w \rangle = \langle v, (T_g)^{-1}(w) \rangle = \langle v, (T_{g^{-1}})(w) \rangle \quad (\ast).
\]

Let \( f, h \in V^* \) arbitrary, \( w := P(f) \), and \( u := P(h) \). We need to check that

\[
\langle (T_g^1)(f), h \rangle = \langle f, ((T_g^1)^{-1})(h) \rangle.
\]

We see that

\[
\begin{align*}
\langle (T_g^1)(f), h \rangle & \overset{\text{proposition 2.2}}{=} \langle (P \circ T_g^1)(f), P(h) \rangle \overset{(1)}{=} \langle (T_g \circ P)(f), P(h) \rangle \\
& = \langle (T_g(P))(f), P(h) \rangle = \langle T_g(w), u \rangle = \langle w, T_g^{-1}(u) \rangle \\
& = \langle P(f), T_g^{-1}(P(h)) \rangle = \langle P(f), (T_g)^{-1} \circ P(h) \rangle \\
& \overset{(2)}{=} \langle P(f), (P \circ (T_g^1)^{-1})(h) \rangle = \langle P(f), P((T_g^1)^{-1})(h) \rangle \\
& = \langle f, (T_g^1)^{-1}(h) \rangle
\end{align*}
\]

After having looked at eigenvalues we will see that this generalizes to higher degree, that \( T_g^1 \) is diagonalizable for all \( d \in \mathbb{Z}^+ \). But first let us look at the matrix version of proposition 2.4.

2.5 Proposition.

\[
[T_g^1]_{x,x} = [T_g]_{e,e}
\]

Proof. Let \( A := [T_g|_{B,B}] = [A_1|_B \cdots |A_n|_B] = [a_{i,j}] \) and \( B := [T_g^1]_{x,x} = [B_1|_x \cdots |B_i|_x \cdots |B_n|_x] = [b_{i,j}] \). We will use the commutativity of the diagram, i.e. \( P^{-1} \circ T_g \circ P = T_g \), which we will mark as \( \square \). No, the proof is not finished here.

We get \( T_g(e_i) = A_i = \sum_{k=1}^n a_{i,k} e_k \) and

\[
T_g^1(x_i) \square (P^{-1} \circ T_g \circ P)(x_i) = P^{-1}(T_g(P(x_i))
\]

\[
\overset{2.3}{=} P^{-1}(T_g(e_i)) = P^{-1}\left( \sum_{k=1}^n a_{i,k} e_k \right) \overset{\text{conj.}}{=} \sum_{k=1}^n a_{k,i} e_k P^{-1}(e_k)
\]

\[
\overset{2.3}{=} \sum_{k=1}^n \overline{a_{k,i}} x_k
\]

On the other hand, \( [T_g^1(x_i)]_{x,x} = [T_g]_{x,e_i,x} = B_i \) implies \( T_g^1(x_i) = \sum_{k=1}^n b_{k,i} e_k \).

Together we obtain \( b_{k,i} = \overline{a_{k,i}} \), and the proposition follows. \( \square \)
3 Averaging over the Group

Now we apply averaging to obtain self-adjoint operators.

3.1 Definition. We define the following operators:

1. \( \hat{T} : V \to V, v \mapsto \hat{T}(v) := \frac{1}{|G|} \sum_{g \in G} T_g(v) \)

2. \( \hat{T}^1 : V^* \to V^*, f \mapsto \hat{T}^1(f) := \frac{1}{|G|} \sum_{g \in G} T_g^1(f) \)

These are sometimes called the Reynolds operator of \( G \).

3.2 Proposition. The operators \( \hat{T} \) and \( \hat{T}^1 \) are self-adjoint (Hermitian).

Proof. The idea of the averaging trick is that if \( g \in G \) runs through all group element and \( g' \in G \) is fixed, then the products \( g'g \) run also through all group elements. We will make use of the facts that every \( T_g \) and every \( T_g^1 \) is unitary.

1. We need to show that \( \langle \hat{T}(v), w \rangle = \langle v, \hat{T}(w) \rangle \) for arbitrary \( v, w \in V \).

We obtain

\[
\langle \hat{T}(v), w \rangle = \left\langle \frac{1}{|G|} \sum_{g \in G} T_g(v), w \right\rangle = \frac{1}{|G|} \sum_{g \in G} \langle T_g(v), w \rangle 
\]

\[
= \frac{\text{unit}}{|G|} \sum_{g \in G} \langle v, (T_g)^{-1}(w) \rangle = \frac{1}{|G|} \sum_{g \in G} \langle v, (T_g^{-1})(w) \rangle
\]

\[
= \frac{1}{|G|} \sum_{g' \in G} \langle v, (T_{g'})(w) \rangle = \langle v, \hat{T}(w) \rangle
\]

2. The same proof, \textit{mutatis mutandis}, replacing \( \hat{T} \leftrightarrow \hat{T}^1, T_g \leftrightarrow T_g^1, v \leftrightarrow f \), and \( w \leftrightarrow h \) shows that \( \langle \hat{T}^1(f), h \rangle = \langle f, \hat{T}^1(h) \rangle \).

Consequently, \( \hat{T} \) and \( \hat{T}^1 \) are unitarily diagonalizable with real spectrum.

3.3 Proposition. The operators \( \hat{T} \) and \( \hat{T}^1 \) are idempotent, i.e.

1. \( \hat{T} \circ \hat{T} = \hat{T} \)

2. \( \hat{T}^1 \circ \hat{T}^1 = \hat{T}^1 \).

In particular, the eigenvalues of both operators are either 0 or 1.

Proof. Again, we show only one part, the other part is analog. To begin with, let \( s \in G \) be fixed. Then

\[
T_s \circ \hat{T} = T_s \circ \frac{1}{|G|} \sum_{g \in G} T_g = \frac{1}{|G|} \sum_{g \in G} T_s \circ T_g
\]

\[
= \frac{1}{|G|} \sum_{g \in G} T_s g = \frac{1}{|G|} \sum_{g' \in G} T_{g'} = \hat{T}.
\]
From this it follows that
\[ \hat{T} \circ \hat{T} = \left( \frac{1}{|G|} \sum_{g \in G} T_g \right) \circ \hat{T} = \frac{1}{|G|} \sum_{g \in G} T_g \circ \hat{T} = \frac{1}{|G|} \sum_{g \in G} \hat{T} \]
\[ = \frac{1}{|G|} \cdot |G| \cdot \hat{T} = \hat{T}. \]

From \( \hat{T} \circ \hat{T} = \hat{T} \) we conclude that \( \hat{T} \circ (\hat{T} - \text{id}) = 0 \). Thus the minimal polynomial of \( T \) divides the polynomial \( \lambda(\lambda - 1) \), so all eigenvalues are contained in \( \{0, 1\} \).

We will now look at the eigenvalues of \( T_g \) and \( T_g^1 \) and their interrelation. Since both operators are unitary, their eigenvalues have absolute value 1.

**3.4 Proposition.**
1. If \( v \in V \) is an eigenvector of \( T_g \) for the eigenvalue \( \lambda \), then \( v \) is an eigenvector of \( T_g^{-1} \) for the eigenvalue \( \frac{1}{\lambda} \).
2. If \( f \in V^* \) is an eigenvector of \( T_g^1 \) for the eigenvalue \( \lambda \), then \( f \) is an eigenvector of \( T_g^{-1} \) for the eigenvalue \( \frac{1}{\lambda} \).
3. If \( f \in V^* \) is an eigenvector of \( T_g^1 \) for the eigenvalue \( \lambda \), then \( P(f) \in V \) is an eigenvector of \( T_g \) for the eigenvalue \( \frac{1}{\lambda} \).
4. If \( v \in V \) is an eigenvector of \( T_g \) for the eigenvalue \( \lambda \), then \( P^{-1}(v) \in V^* \) is an eigenvector of \( T_g^1 \) for the eigenvalue \( \frac{1}{\lambda} \).

**Proof.** We will make use of the commutativity of Proposition 3.3. Observe that \( g \cdot v = T_g(v) \) and \( g \cdot f = f \circ T_g \).

1. \( T_g(v) = g \cdot v = \lambda v \Rightarrow g^{-1} \cdot g \cdot v = g^{-1} \cdot \lambda v \Rightarrow g^{-1} \cdot g \cdot v = \lambda g^{-1} \cdot v \)
\[ \Rightarrow v = \lambda g^{-1} \cdot v \Rightarrow T_g^{-1}(v) = g^{-1} \cdot v = \frac{1}{\lambda} v \]

2. \( T_g^1(f) = g \cdot f = \lambda f \Rightarrow g^{-1} \cdot g \cdot f = g^{-1} \cdot \lambda f \Rightarrow g^{-1} \cdot g \cdot f = \lambda g^{-1} \cdot f \)
\[ \Rightarrow f = \lambda g^{-1} \cdot f \Rightarrow T_g^{-1}(f) = g^{-1} \cdot f = \frac{1}{\lambda} f \]

3. \( T_g^1(f) = \lambda f \overset{P\circ}{\Rightarrow} P(T_g^1(f)) = P(\lambda f) \overset{(1)}{=} T_g(P(f)) = P(\lambda f) \)
\[ \Rightarrow T_g(P(f)) = \overline{\lambda} P(f) = \frac{1}{\lambda} P(f) \]

4. \( T_g(v) = \lambda v \overset{P^{-1}\circ}{\Rightarrow} P^{-1}(T_g(v)) = P^{-1}(\lambda v) \overset{\square}{\Rightarrow} (T_g^1 \circ P^{-1})(v) = \overline{\lambda} P^{-1}(v) \)
\[ \Rightarrow T_g^1(P^{-1}(v)) = \frac{1}{\lambda} P^{-1}(v) \]

This implies that if we consider the union of the spectra over all \( g \in G \), then we obtain the same (multi)set, no matter if we take \( T_g \) or \( T_g^1 \).
4 Eigenvectors and eigenvalues

Now we continue from where we left at the end of section 1, fixing one group element \( g \in G \) and compare \( T^1_g \) with \( T^d_g \) for \( d > 1 \). By a method called stars and bars it is easy to see that

\[
\tilde{d} := \text{dim}_C(R_d) = \frac{(n + d + 1)!}{(n - 1)!d!}.
\]

Remember that every \( T^1_g \) is unitarily diagonalizable with eigenvalues of absolute value 1. If \( \text{spec}(T^1_g) = (\omega_1, \ldots, \omega_n) \in U(1)^n \), then \( V^* \) has an orthonormal basis \( y^+_1 := (y_1, \ldots, y_n) \), such that \( T^1_g(y_i) = \omega_i \cdot y_i \) for all \( 1 \leq i \leq n \), and \( [T^1_g]_{y^+_1 y^+_1} = \text{diag}(\omega_1, \ldots, \omega_n) \). Moreover,

\[
[T^1_g y^+_i x^+] = [\text{id}]_{y^+_i x^+} \cdot [T^1_g]_{x^+ x^+} \cdot [\text{id}]_{x^+ y^+_i} = \text{diag}(\omega_1, \ldots, \omega_n),
\]

where \( [\text{id}]_{y^+_i x^+} = [\text{id}]_{x^+ y^+_i} \) is unitary.

For \( d > 1 \) put

\[
x^d := (x_1^d, x_2^d, \ldots, x_n^d, x_1^{d-1} x_2, x_1^{d-1} x_3, \ldots, x_1^{d-1} x_n, \ldots) =: (\tilde{x}_1, \ldots, \tilde{x}_d),
\]

all monomials in the \( x_i \) of total degree \( d \), numbered from 1 to \( \tilde{d} \).

These are certainly linear independent, since we have no relations amongst the variables, and span \( R_d \), since every monomial of total degree \( d \) can be written as a linear combination of these. So the form a basis for \( R_d \). We will not require that this can be made into an orthonormal basis, we do not even consider any inner product on \( R_d \) for \( d > 1 \).

We rather want to establish that

\[
y^d := (y_1^d, y_2^d, \ldots, y_n^d, y_1^{d-1} y_2, y_1^{d-1} y_3, \ldots, y_1^{d-1} y_n, \ldots) =: (\tilde{y}_1, \ldots, \tilde{y}_d)
\]

is a basis of eigenvectors of \( T^d_g \) diagonalizing \( T^d_g \), using the same numbering.

Arranging the eigenvalues of \( T^1_g \) in the same way we put

\[
\omega^d := (\omega_1^d, \omega_2^d, \ldots, \omega_n^d, \omega_1^{d-1} \omega_2, \omega_1^{d-1} \omega_3, \ldots, \omega_1^{d-1} \omega_n, \ldots) =: (\tilde{\omega}_1, \ldots, \tilde{\omega}_d).
\]

Now we establish that the \( \tilde{y}_i, 1 \leq i \leq \tilde{d} \) are the eigenvectors for the eigenvalues \( \tilde{\omega}_i \) of \( T^d_g \).

4.1 Proposition. In the context above,

\[
T^d_g(\tilde{y}_i) = \tilde{\omega}_i \cdot \tilde{y}_i
\]

for all \( 1 \leq i \leq \tilde{d} \).

Proof. The key is proposition 1.2 as in the preliminary observations at the end of section 1. Let

\[
\tilde{y}_i = \prod_{j=1}^{n} y_j^{\tilde{y}_i}
\]

and

\[
\tilde{\omega}_i = \prod_{j=1}^{n} \omega_j^{\tilde{y}_i},
\]

11
where $\epsilon_j \in \mathbb{N}$ and the sum of these exponents is $d$. Then

$$T^d_g(\tilde{y}_i) = T^d_g \left( \prod_{j=1}^{n} y_j^{\epsilon_j} \right) = \prod_{j=1}^{n} T^1_g(y_j^{\epsilon_j}) = \prod_{j=1}^{n} \omega_j^{\epsilon_j} \tilde{y}_j^{\epsilon_j} = \tilde{\omega}_i \cdot \tilde{y}_i$$

As a consequence, $R_d$ has a basis of eigenvectors of $T^d_g$ and $T^d_g$ is similar to the diagonal matrix diag($\tilde{\omega}_1, \ldots, \tilde{\omega}_d$). 

\hfill $\Box$
5 Moliens Theorem

We will now make some final preparations and then present the proof of Moliens Theorem.

For \( f \in R \) and \( g \in G \) we say that \( f \) is an invariant of \( g \) if \( g.f = f \) and that \( f \) is a (simple) invariant of \( G \) if \( \forall g \in G : g.f = f \). The method of averaging from section 3 can also be applied to create invariants:

5.1 Proposition. For \( f \in V^* \) put \( \hat{f} := \hat{T}^1(f) \). Then \( \hat{f} \) is an invariant of \( G \).

Proof. Let \( g \in G \) be arbitrary. We will show that \( g.\hat{f} = \hat{f} \). Clearly, from proposition 1.6 we get that \( g.\hat{f} = \hat{T}^1(f) \circ T_{g^{-1}} \).

\[
\hat{T}^1(f) \circ T_{g^{-1}} = \left( \frac{1}{|G|} \sum_{s \in G} T_s^1(f) \right) \circ T_{g^{-1}} = \left( \frac{1}{|G|} \sum_{s \in G} f \circ T_{s^{-1}} \right) \circ T_{g^{-1}} = \frac{1}{|G|} \sum_{s \in G} f \circ T_{s^{-1}} \circ T_{g^{-1}} = \hat{f}.
\]

Now, we call \( R^G := \{ f \in R : \forall g \in G : g.f = f \} \) the algebra of invariants of \( G \).

5.2 Proposition. \( R^G \) is a subalgebra of \( R \).

Proof. Since the mapping \( f \mapsto g.f \) is linear for every \( g \in G \), \( R^G \) is the intersection of subspaces, and hence a subspace. Let us check the subring conditions in more detail. For arbitrary \( g \in G \), \( f, h \in R^G \), and \( v \in V \) we have \( g.f = f \), \( g.h = h \)

1. For the zero \( 0 \in R \) we obtain \((g.0)(v) = 0(g^{-1}.v) = 0(v)\), so \( 0 \in R^G \).

2. We see

\[
g.(f + h)(v) = (f + h)(g^{-1}.v) = f(g^{-1}.v) + h(g^{-1}.v) = (g.f)(v) + (g.h)(v) = f(v) + h(v) = (f + h)(v)
\]

3. Likewise,

\[
g.(f \cdot h)(v) = (f \cdot h)(g^{-1}.v) = f(g^{-1}.v) \cdot h(g^{-1}.v) = (g.f)(v) \cdot (g.h)(v) = f(v) \cdot h(v) = (f \cdot h)(v)
\]

Our subalgebra \( R^G \) is graded in the same way as \( R \).

5.3 Proposition. The algebra of invariants of \( G \) is naturally graded as

\[
R^G = \bigoplus_{d \in \mathbb{N}} R^G_d,
\]

where \( R^G_d = \{ f \in R_d : \forall g \in G : g.f = f \} \), called the \( d \)-th homogeneous component of \( R^G \).
Proof. This follows directly from proposition 1.1 and proposition 1.2.

5.4 Definition (Molien series). Viewing $R^G_d$ as a vector space, we define
\[ a_d := \dim_C R^G_d, \]
the number of linearly independent homogeneous invariants of degree $d \in \mathbb{N}$, and
\[ \Phi_G(\lambda) := \sum_{d \in \mathbb{N}} a_d \lambda^d, \]
the Molien series of $G$.

Thus, the Molien series of $G$ is an ordinary power series generating function whose coefficients are the numbers of linearly independent homogeneous invariants of degree $d$. The following beautiful formula gives these numbers, its proof is the aim of this paper.

5.5 Theorem (Molien, 1897).
\[ \Phi_G(\lambda) := \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(id - \lambda T_g)} \]

Following [Slo77], we first look the number $a_1$ of linearly independent homogeneous invariants of degree $d$.

5.6 Theorem (Theorem 13 in [Slo77]).
\[ a_1 = \text{Tr}(\hat{T}) = \text{Tr}(\hat{T}^1) \]

Proof. First, we note that the equation $\text{Tr}(\hat{T}) = \text{Tr}(\hat{T}^1)$ follows from the remark at the end of section 3, since the sum for the trace runs over all group elements. Remember that the trace is independent of the choice of basis. From proposition 5.3 we know that both operators are idempotent hermitian and $V^*$ has an orthonormal basis $f = (f_1, \ldots, f_n)$ of eigenvectors of $\hat{T}^1$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \{0, 1\}$, so
\[ [\hat{T}^1]_{f,f} = \text{diag}(\lambda_1, \ldots, \lambda_n). \]

Let us say that this matrix has $r$ entries 1 and the remaining $n - d$ entries 0. By rearranging the eigenvalues and eigenvectors we may assume that the first $r$ entries are 1 and the remaining $n - d$ are 0, i.e.
\[ ([\hat{T}^1]_{f,f})_{i,i} = \begin{cases} 1 : 1 \leq i \leq r \\ 0 : r + 1 \leq i \leq n. \end{cases} \]

Hence $\hat{T}^1(f_i) = f_i$ for $1 \leq i \leq r$ and $\hat{T}^1(f_i) = 0$ for $r + 1 \leq i \leq n$. Any linear invariant of $G$ is certainly fixed by $\hat{T}^1$, so $a_1 \leq r$. On the other hand, by proposition 5.4 $\hat{T}^1(f_i) = \lambda_i f_i$ is an invariant of $G$ for every $1 \leq i \leq r$, so $a_1 \geq r$. Together, $a_1 = r$. □

Before the final proof, let us introduce a handy notation.
5.7 Definition. Let \( p(\lambda) \in \mathbb{C}[\lambda] \) or \( p(\lambda) \in \mathbb{C}[[\lambda]] \). Then \([\lambda^i] : p(\lambda)\) denotes the coefficient of \(\lambda^i\) in \(p(\lambda)\).

So, for example \([x^2] : 2x^3 + 42x^2 - 6 = 42\) and \([\lambda^d] : \Phi_G(\lambda) = a_d\).

Proof. (Moliens Theorem) We just established the case \(d = 1\), so the reader is probably expecting a proof by induction over \(d\). But this is not the case. Rather, the case \(d = 1\) applies to all \(d > 1\). Note that \(a_d\) is equal to the number of linearly independent invariants of all of the \(T_d^g\). So Theorem 5.6 gives us

\[
a_1 = \text{Tr}(\hat{T}) = \text{Tr}(\hat{T}^1) \quad \text{and} \quad a_d = \text{Tr}(\hat{T}^d),
\]

where the latter includes the first. From definition 3.1 we also have

\[
\hat{T}^1 = \frac{1}{|G|} \sum_{g \in G} T^1_g \quad \text{and in general} \quad \hat{T}^d = \frac{1}{|G|} \sum_{g \in G} T^d_g,
\]

so we already know that

\[
a_d = \frac{1}{|G|} \sum_{g \in G} \text{Tr}(T^d_g).
\]

So all we need to show is

\[
[\lambda^d] : \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(id - \lambda T^d_g)} = \frac{1}{|G|} \sum_{g \in G} \text{Tr}(T^d_g).
\]

We will show that for every summand (group element) the equation

\[
[\lambda^d] : \frac{1}{\det(id - \lambda T^d_g)} = \text{Tr}(T^d_g)
\]

holds. From proposition 4.1 we get for every \(g \in G\) that

\[
\text{Tr}(T^d_g) = \text{Tr}(\text{diag}(\tilde{\omega}_1, \ldots, \tilde{\omega}_n)) = \tilde{\omega}_1 + \cdots + \tilde{\omega}_d =
\]

sum of the products of the \(\omega_1, \omega_2, \ldots, \omega_n\), taken \(d\) of them at a time. On the other hand, for the same \(g \in G\) we obtain from section 4 that \([T^d_g]_{\omega_1, \ldots, \omega_n} = \text{diag}(\omega_1, \ldots, \omega_n)\) so that

\[
\det(id - \lambda T^d_g) = \det(id - \lambda \cdot \text{diag}(\omega_1, \ldots, \omega_n)) = (1 - \lambda \omega_1)(1 - \lambda \omega_2) \ldots (1 - \lambda \omega_n),
\]

so

\[
\frac{1}{\det(id - \lambda T^d_g)} = \frac{1}{(1 - \lambda \omega_1)(1 - \lambda \omega_2) \ldots (1 - \lambda \omega_n)}
\]

\[
= \frac{1}{(1 - \lambda \omega_1)} \cdot \frac{1}{(1 - \lambda \omega_2)} \cdots \frac{1}{(1 - \lambda \omega_n)}
\]

\[
= (1 + \lambda \omega_1 + \lambda^2 \omega_1^2 + \ldots)(1 + \lambda \omega_2 + \lambda^2 \omega_2^2 + \ldots) \ldots (1 + \lambda \omega_n + \lambda^2 \omega_n^2 + \ldots)
\]

15
and here the coefficient of $\lambda^d$ is also sum of the products of $\omega_1, \omega_2, \ldots, \omega_n$, taken $d$ of them at a time.

Again, the last claim

$$\frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(id - \lambda T_g)} = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(id - \lambda T_g^1)}$$

follows from the remark at the end of section 3.2 since the sum runs over all group elements.
6 Symbol table

| $\alpha_d$ | number of linearly independent homogeneous invariants of degree $d$ |
| $\tilde{d}$ | Dimension of $R_d$ |
| $\mathcal{B}$ | ON basis for $V$ |
| $G$ | Finite group |
| $\omega_i$ | eigenvalue of $T_g$ ($[\text{Slo}77] = \omega_i$) |
| $P(f)$ | “Rho” Riesz vector of $f$. |
| $\rho$ | Unitary representation $\rho : G \to U(V), g \mapsto T_g$ |

$R$ Big algebra, direct sum of $R_d$ Direct summand of degree $d$
$R^G$ Ring of invariants of $d$
$R^G_d$ Degree $d$ summand
$T_g$ representation of $g$ on $V$, ($[\text{Slo}77]$

$A_\alpha = [T_{g\alpha}]_{\mathcal{B},\mathcal{B}}$

$V$ Complex inner product space
$V^*$ Algebraic dual of $V$

7 Lost and found

Some things to explore from here:

- If we know the conjugacy classes of $G$, we may be able to say more, since every unitary representation splits into irreducible components.
- There seems to be a link to Pólya enumeration.
- We have GAP code, see [GAP].
- An example would be nice.
- Relations on the generators in $S$ of the Cayley graph $\Gamma(G,S)$ should lead to conditions of the minimal polynomial of its adjacency operator $Q(\Gamma(G,S))$.
- Also, Cayley graphs of some finite reflection groups [Hu90] should become accessible.
- Check some more applications, as mentioned in [Slo77].
- For finding invariants, check also [Cox91], Gröbner bases.

References

[Ant73] Howard Anton, *Elementary Linear Algebra*, 6th ed., John Wiley and Sons, New York, 1973.

[Bie04] Jürgen Bierbrauer, *Introduction to Coding Theory*, Discrete Mathematics and Its Applications, Volume: 28, CRC Press Inc, Boca Raton, 2004.

[Cox91] D. Cox, J. Little, D. O’Shea, *Ideals, Varieties, and Algorithms*, Springer-Verlag, New York, 1991.
The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4; 2004, [http://www.gap-system.org](http://www.gap-system.org).

[Hu96] John F. Humphreys, *A Course in Group Theory*, Oxford University Press, Oxford, 1994.

[Hu90] James E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.

[Rom 08] Steven Roman, *Advanced linear algebra, 3rd Edition*, Springer-Verlag, New York, 2008.

[Sag 91] Bruce E. Sagan, *The Symmetric Group*, Wadsworth & Brooks, Pacific Grove, 1991.

[Slo77] Neil J. A. Sloane, ”Error Correcting Codes and Invariant Theory: New Applications of a Nineteenth–Century Technique”, *American Mathematical Monthly*, 84, (1977), 82–107.

[Sta79] Richard P. Stanley, ”Invariants of Finite Groups and their Applications to Combinatorics”, *Bulletin (New Series) of the American Mathematical Society*, 1, No. 3 (1979), 475–511.

[Stu93] B. Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, Wien, New York, 1993.

[Tam91] Torbjörn Tambour, *Introduction to Finite Groups and their Representations*, Lecture notes, Lund, 1991.