Quantization in singular real polarizations: Kähler regularization, Maslov correction and pairings

João N Esteves\textsuperscript{1}, José M Mourão\textsuperscript{2} and João P Nunes\textsuperscript{3,4}

\textsuperscript{1} Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049–001 Lisboa, Portugal
\textsuperscript{2} Lehrstuhl für Theoretische Physik III, FAU Erlangen-Nürnberg and Center for Mathematical Analysis, Geometry and Dynamical Systems, Department of Mathematics, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049–001 Lisboa, Portugal
\textsuperscript{3} Center for Mathematical Analysis, Geometry and Dynamical Systems, Department of Mathematics, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049–001 Lisboa, Portugal

E-mail: joao.n.esteves@tecnico.ulisboa.pt, jmourao@math.tecnico.ulisboa.pt and jpnunes@math.tecnico.ulisboa.pt

Received 19 January 2015, revised 15 April 2015
Accepted for publication 17 April 2015
Published 11 May 2015

Abstract
We study the Maslov correction to semiclassical states by using a Kähler regularized BKS pairing map from the energy representation to the Schrödinger representation. For general semiclassical states, the existence of this regularization is based on recently found families of Kähler polarizations degenerating to singular real polarizations and corresponding to special geodesic rays in the space of Kähler metrics. In the case of the one-dimensional harmonic oscillator, we show that the correct phases associated with caustic points of the projection of the Lagrangian curves to the configuration space are correctly reproduced.

Keywords: Maslov correction, Kähler quantization, geometric quantization

1. Introduction
Consider a classical system with phase space given by a $2\pi$-dimensional symplectic manifold $(M, \omega)$, where $\omega$ is assumed to be quantizable in a sense that we will recall in the next section.
The geometric quantization of \((M, \omega)\) then requires that the classical data be complemented by additional choices including a so-called polarization, \(\mathcal{P}\). The corresponding Hilbert space of quantum states, \(H_{\mathcal{P}}\), is then \(\mathcal{P}\)-dependent and is given by \(\mathcal{P}\)-polarized sections (of an appropriate line bundle) on \(M\) (see (2.1)). Choosing a polarization corresponds, locally on \(M\), to the choice of a maximal set of, possibly complex valued, functionally independent Poisson commuting observables. Global observables which locally are only functions of the observables defining \(\mathcal{P}\) act as multiplication operators on \(H_{\mathcal{P}}\).

When the local observables defining \(\mathcal{P}\) can all be chosen to be real valued, the polarization is said to be real. This includes the case of completely integrable systems in which there are \(n\) globally defined functionally independent Poisson commuting observables. For instance, in the standard Schrödinger quantization of \(\mathbb{R}^{2n}\), with position and momentum coordinates \((q, p)\), these observables are taken to be \(\{q_j\}_{j=1,\ldots,n}\) and the resulting quantum wave functions turn out to be functions of position only (see (4.1)). In the case when \((M, \omega)\) admits an appropriate compatible structure as a complex manifold (making \(M\) a Kähler manifold), the local holomorphic coordinates define a polarization, called Kähler polarization. The corresponding Hilbert space of quantum states is then a space of holomorphic sections on \(M\) obtained in a standard way thus making Kähler quantization mathematically well controlled. By contrast, real polarizations will correspond to (singular, in general) foliations of \(M\) and the quantum wave functions will often be distributional, supported on the so-called Bohr–Sommerfeld leaves of \(\mathcal{P}\). (States supported on Lagrangian Bohr–Sommerfeld leaves correspond to generalizations of the Bohr quantization conditions for the atomic energy levels in old quantum theory.) If the foliation contains singular (i.e. of dimension lower than \(n\)) leaves, the quantization process is more difficult to define \([BFMN, Ham, HM, KMN1, So]\).

Though the process of Kähler quantization is mathematically better defined, the quantization in (possibly singular) real or mixed polarizations is frequently physically more interesting. This is partly due to the fact that the observables preserving mixed polarizations are likely to be physically more interesting than those preserving a Kähler polarization. In the present paper we continue along the lines proposed in \([BFMN, Ki, KMN1, KW2]\), which motivate the definition of the quantization for real or mixed polarizations via degeneration of quantizations on suitable families of Kähler polarizations. We call such process Kähler regularization.

In this work, we use Kähler regularization to study the Maslov phases that appear in the semiclassical approximation to eigensates of the energy operator in the Schrödinger quantization. This approach is conceptually simpler than the standard one where one needs to look for caustic points of the projection of a Lagrangian submanifold \(\mathcal{L}\), in a level set of the energy, to the configuration space and keep track of the corresponding correct phases (see, for instance, Example 4.12 of \([BW]\)). These phases enter in the definition of the Maslov corrected semiclassical state \(\psi_{\mu}^{\text{c}}\) corresponding to \(\mathcal{L}\). In our approach, on the other hand, \(\psi_{\mu}^{\text{c}}\) is defined as the image under the BKS pairing map of the standard energy representation state supported on \(\mathcal{L}\), see (4.18). Kähler regularization plays then a crucial role in making the BKS pairing map well defined despite the non-transversality between the Schrödinger and energy polarizations at the caustic points. This is achieved by defining the pairing as a limit of well defined BKS pairings between two families of Kähler polarizations approaching these two real polarizations, see (4.16).

More concretely, we adopt the following strategy, detailed in section 4.1. We consider cases for which the given Lagrangian submanifold is a leaf of a (possibly singular) Lagrangian fibration corresponding to the level sets of a moment map \(\mu\) of a completely integrable system. Denoting the corresponding real polarization by \(\mathcal{P}_\mu\), the proposal of \([MN2]\)
corresponds then to obtaining the Hilbert space \( \mathcal{H}_\mu \), of quantum states in the polarization \( P_\mu \), as the infinite imaginary time limit \( s \to +\infty \), of the family defined by applying the imaginary time flow of the Hamiltonian vector field of the norm square of the moment map, \( \chi_{(\mu)} \), to the Hilbert space corresponding to a starting Kähler quantization. In the toric case, these families of polarizations were first introduced in [BFMN]. These families were also used in [KMN1] to derive the Maslov shift of levels of the Bohr–Sommerfeld leaves. In order to relate the Schrödinger representation to this Kähler polarization, we consider the Thiemann complexifier method [Th1, Th2] adapted to geometric quantization in [HK1, HK2, KMN2, KMN3, KMN4]. The momentum space quantization \( T^\ast K \), for compact Lie groups \( K \), was also defined in [KW2] through the infinite imaginary time limit of the flow of the Hamiltonian vector field of the norm squared of the (in general non-abelian) moment map of the action of \( K \) on \( T^\ast K \) (see also [KMN2]). In the present paper, we focus on obtaining semiclassical states associated with real polarizations on \( \mathbb{C}^n \), non-invariant under translations. We are particularly interested in polarizations for which only Kähler regularizations of the second type exist (see section 3 below). We propose a general formalism in section 4.1 and apply it to the harmonic oscillator in section 4.2.

We note that, while for translation invariant real or mixed polarizations on a symplectic vector space it is easy to construct families of Kähler polarizations degenerating to the given one, for polarizations with singular leaves it is usually not easy to find such Kähler families explicitly (see the no-go theorem 3.4 and the conjecture 3.5 below). Building up on previous works [BFMN, KMN1, KW2, MN1], in [MN2] a general strategy is proposed to find families of well behaved polarizations degenerating to a wide class of singular real polarizations corresponding to the level sets of completely integrable systems. For integrable systems with singularities of elliptic type, Kähler regularization (see [BFMN, KMN1, KMN4]) improves on other approaches such as the cohomological approach extended from [Sn] to this singular case in [Ham]. If one does not include the half-form correction, Kähler regularization gives, as expected, a quantum Hilbert space where to each connected Bohr–Sommerfeld leaf (including the singular ones) there corresponds a quantum state, while the cohomological approach gets no contribution from singular Bohr–Sommerfeld leaves. Also, for systems with hyperbolic singularities (see [HM]) the cohomological approach produces a Hilbert space with infinite dimensions, even in the compact case. The treatment of hyperbolic singularities within the framework of Kähler regularization would produce a finite-dimensional space of quantum states in the compact case. Singularities of focus–focus type were studied from the cohomological point of view in [So] and would also be interesting to study using Kähler regularization.

The Maslov correction has been extensively studied (see, for example, [BW, EHHL, Go, GS, MF, Wo, Wu] and references therein). In particular, in [Wu] the holonomy of the natural projectively flat connection along geodesic triangles in the space of Kähler polarizations on \( \mathbb{R}^{2n} \) invariant under translations, was shown to yield the triple Maslov index of Kashiwara when the vertices of the triangle approach mutually transverse polarizations at geodesic infinity.

2. Preliminaries

Let \( (M, \omega, I) \) be a connected Kähler manifold such that \( \frac{\omega}{2\pi\hbar} \), \( \frac{1}{2} \iota_\omega(M) \in H^2(M, \mathbb{Z}) \) so that the canonical line bundle \( K_I \) has \( I \)-holomorphic square roots. Let \( \sqrt{K_I} \) denote one such square root with Chern connection \( \nabla^I \) and let us fix a complex line bundle \( L \to M \) with first
Chern class $c_1(L) = \left[ \frac{\omega}{2\pi\hbar} \right]$. We consider on $L$ a connection $\nabla$ with curvature $F_\nabla = -\frac{i}{\hbar} \omega$ and a compatible Hermitian structure $h$. The half-form corrected quantum Hilbert space corresponding to $I$ is then

$$\mathcal{H}_I = \left\{ s \in \Gamma\left( L \otimes \sqrt{K_I^*} \right) : \left( \nabla_h \otimes 1 + 1 \otimes \nabla_h^* \right) s = 0 \right\}, \quad (2.1)$$

where $\mathcal{P}_I$ denotes the polarization generated by $I$-anti-holomorphic vector fields and the bar denotes completion with respect to the inner product defined by

$$\langle \sigma, \sigma' \rangle = \frac{1}{(2\pi\hbar)^n} \int_X h^i(\sigma, \sigma') \frac{\omega^p}{n!}. \quad (2.2)$$

In cases when the canonical bundle $K_I$ is trivial and $\Omega_I$ is a global trivializing section we choose as $\sqrt{K_I}$ the trivial square root and denote by $\sqrt{\Omega_I}$ one of the two trivializing sections of $\sqrt{K_I}$ which square to $\Omega_I$.

The half-form corrected prequantization of a function $f \in \mathcal{C}^\infty(M)$ is given by

$$\hat{f}^{pQ} = i\hbar \nabla_f \otimes 1 + f \otimes 1 + i\hbar 1 \otimes \mathcal{L}_f,$$

where $X_f$ denotes the Hamiltonian vector field corresponding to $f$, or, if a local trivializing section $\sigma$ of $L$ is given, such that

$$\nabla_\sigma = \frac{i}{\hbar} \Theta \sigma, \quad (2.3)$$

where $d\Theta = -\omega$, we obtain, in the local trivialization of $L$ defined by $\sigma$,

$$\hat{f}^{pQ} = i\hbar X_f \otimes 1 - L_f \otimes 1 + i\hbar 1 \otimes \mathcal{L}_f, \quad (2.4)$$

where $L_f = \Theta(X_f) - f$ is called the Lagrangian of $f$.

3. Kähler regularizations

In the present section we study regularizations associated with the imaginary time flow of hamiltonians, $h \in \mathcal{C}^\infty(M)$, which we call regulators. Depending on the mixed polarization $\mathcal{P}$ we will consider regulators of two types.

**Definition 3.1.** $\mathcal{P}$-regulators of the first type or Thiemann (partial) complexifiers are regulators $h$ for which, there exists a $T > 0$ such that the polarization $\mathcal{P}_I^h = e^{i\mathcal{C}_{3h}(P)}$ exists and is Kähler for $t \in (0, T)$.

In interesting families of examples, we can then define also a sensible limit of the corresponding Hilbert spaces of polarized quantum states,

$$\mathcal{H}_P := \lim_{\iota \to 0} \mathcal{H}_I^h, \quad (3.1)$$

In the examples in [KMN1, KMN2, KMN4], the space of $\mathcal{P}$-polarized quantum states was already known and the limit actually recovers the correct Hilbert space. Conjecturally, however, one could possibly start with a badly behaved (and hence difficult to quantize directly) polarization $\mathcal{P}$ and define $\mathcal{H}_P$ as a limit of well-behaved quantizations in Kähler polarizations.
Regulators of the first type were introduced by Thiemann in the context of non-perturbative quantum gravity [Th1, Th2] to transform the SU(2) spin connection to the SL(2, C) Ashtekar connection. The prototypical example in finite dimensions is that of the vertical polarization on a cotangent bundle $M = T^*X$ of a compact manifold $X$. Hall and Kirwin [HK1, HK2] showed, both for the canonical symplectic form $\omega$, and for a symplectic form modified by a magnetic field, $\omega + B$, that the imaginary time flow of the kinetic energy, $h = E$, corresponding to a Riemannian metric $\gamma$ on $X$ defines, at $t = 1$ and on a tubular neighborhood of the zero section, a Kähler structure, which, for $B = 0$, coincides with the adapted Kähler structure introduced by Guillemin–Stenzel [GS1, GS2] and Lempert–Szöke [LS]. In the cases when the Kähler structure extends to $T^*X$, as is the case of compact Lie groups $X$ with bi-invariant metric, $E$ can be used as a regulator of the first type [KMN2].

Remark 3.2. Regulators of the first type however do not allow to obtain Kähler regularizations of many real polarizations as we show below in theorem 3.4. See also the conjecture 3.5.

Definition 3.3. We call $h \in C^\infty(M)$ a $P$-regulator of the second type or (partial) decomplexifier if there exist a polarization $P_0$ such that the polarization $P_t^h = e^{it\omega_0}(P_0)$ exists and is Kähler for $t > 0$ and

$$\lim_{t \to +\infty} P_t^h = P,$$

in an open, dense subset of $M$.

Then, as above, one can look for a sensible definition of the space of $P$-polarized quantum states by considering the limit

$$H_P := \lim_{t \to \infty} H_{P_t^h},$$

(3.2)
in an appropriate sense.

The need for regulators of the second type comes from the difficulty in finding regulators of the first type for example for real polarizations with compact fibers. In fact, we can prove easily the following result concerning the important case of completely integrable systems on compact manifolds.

Theorem 3.4. Let $(M, \omega)$ be a compact real analytic completely integrable system defined by $n$ Hamiltonian functions $H_1, ..., H_n$ in Poisson involution, with $dH_1 \wedge \cdots \wedge dH_n \neq 0$ on an open dense subset of $M$. Let $P$ be the real (necessarily singular) polarization with integral leaves corresponding to the level sets of $\mu = (H_1, ..., H_n)$. Then there can be no real analytic $P$-regulator of the first type.

Proof. Recall that $P$ is pointwise generated by the global Hamiltonian vector fields $X_{H_j}, j = 1, ..., n$. Suppose that there exists a $P$-regulator of the first type, $h \in C^\infty(M)$. From [MN1], it then follows that there exists $\epsilon > 0$ such that $P_t^h = e^{it\omega_0}P$ is, for all $\tau \in \mathbb{C}$ : $|\tau| < \epsilon$, a polarization generated by the (complex) Hamiltonian vector fields of the global functions $H^\tau_j = e^{i\tau H_j}$. Then there exists a $\epsilon' \leq \epsilon$ such that $P_t^{h'}$ is Kähler for all $t > 0$, $0 < |\tau| < \epsilon'$ and $H^\tau_j$ are nonconstant global holomorphic functions which contradicts the compacteness of $M$. □
Conjecture 3.5. We conjecture that do not exist regulators of the first type for singular polarizations $\mathcal{P}$ such that there exist points $x \in M$ for which $P_i$ is an isotropic non-Lagrangian subspace of $T_xM \otimes \mathbb{C}$.

Fortunately, as shown in [KMN4, MN2], there are regulators of the second type for many of the above examples. They are given by strongly convex functions of the Hamiltonians $H_j$.

4. Schrödinger semiclassical states and Maslov phases

4.1. Kähler regularized semiclassical states

Let $\mathcal{L} \subset T^*\mathbb{R}^n$ be a compact closed Lagrangian submanifold and consider the Schrödinger representation, that is the prequantum line bundle $L = T^*\mathbb{R}^n \times \mathbb{C}$ with global trivializing section the constant function, with connection $\nabla : \forall \mathbf{V} = \frac{i}{\hbar}pdq = \frac{i}{\hbar} \sum_{j=1}^{n} p_j dq_j$ and

$$P_{\text{Sch}} = \left( \frac{\partial}{\partial q} \right)_C = \left( \frac{\partial}{\partial q_1}, ..., \frac{\partial}{\partial q_n} \right)_C,$$

$$H_{\text{Sch}}^0 = L^2(\mathbb{R}^n, dq) \otimes \sqrt{dq}.$$  (4.1)

Our goal in the present section will be to use Kähler regularization to construct a semiclassical state $\psi_\varepsilon$ in the Schrödinger representation that is an approximate eigenvector of a quantum Hamiltonian $\hat{h}$ corresponding to the quantization of a function $\mathbf{h} \in C^\infty(T^*\mathbb{R}^n)$ such that

$$\mathcal{L} \subset \{ (q, p) \in \mathbb{R}^{2n} : \mathbf{h}(q, p) = \varepsilon \}. \quad (4.2)$$

$\psi_\varepsilon$ will therefore be an approximate solution of the eigenvalue equation

$$\hat{h}_\varepsilon \psi_\varepsilon = \varepsilon \psi_\varepsilon. \quad (4.3)$$

Such states have been obtained mainly by using the WKB method of constructing approximate solutions of (4.3) and then imposing the Maslov correction to improve the solution (see, for example, [BW, GS, Wo]). The Maslov correction changes the energy levels (and therefore the set of quantizable Lagrangians) by correcting the Bohr–Sommerfeld quantization conditions and introduces phases in the caustic points of the projection $\pi$, where $\pi$ is the canonical projection $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$, $\pi(q, p) = q$.

To construct $\psi_\varepsilon$ with the help of Kähler regularization, we will consider Kähler regularizations of both the energy and Schrödinger representations, such that they are both deformed, through one-parameter families, to Kähler polarizations. We then use the limit of the BKS pairing map between the Kähler polarizations along these families, to define the pairing map $B$ from the energy representation to the Schrödinger representation. We construct $\psi_\varepsilon$ in the energy representation and define $\psi_\varepsilon = B(\psi_\varepsilon)$ (see (4.13), (4.18)).

For the vertical polarization, $P_{\text{Sch}} = \left( \frac{\partial}{\partial q_j}, j = 1, ..., n \right)_C$, there are many Thiemann complexifiers or regulators of the first type. It follows from [KMN2] that any strongly convex function of the momenta is a $P_{\text{ch}}$ regulator of the first type. Functions of both $p$ and $q$ can also be used as, for example, the Hamiltonians of harmonic oscillators which will be studied in [Es]. Let $h_1$ denote such a regulator and assume that, even though $h_1$ does not preserve the vertical polarization (otherwise it could not be a $P_{\text{Sch}}$-regulator of the first type), it has a natural quantization on the Schrödinger representation, $h^{\text{Sch}}_1$. Then let
and define the following Kostant–Souriau–Heisenberg (KSH) regularization map for Schrödinger states
\[ U_1^{\text{tr}} = e^{i h_1^{tQ} tQ} e^{-i h_1^{\text{Sch}} \text{Sch}} : \mathcal{H}_{\text{Sch}} \rightarrow \mathcal{H}_{\text{Sch}}, \] (4.5)
where \( h_1^{tQ} \) is given by (2.4). In the context of studying the equivalence of quantizations for different polarizations, this map was introduced in [KMN2, KMN3] for the case of cotangent bundles of a compact Lie group, while for toric symplectic manifolds it was considered in [KMN4], and, more generally, it is studied in [MN2]. (For \( K \) a compact Lie group, \( M = T^*K \) and \( h_1 \) the Hamiltonian corresponding to geodesic motion on \( K \) for the bi-invariant metric, the KSH map is equivalent to the Hall coherent state transform [Hal1, Hal2, KW1], see [KMN2].)

Next, we will need to assume that \( h \) is an Hamiltonian in a classically completely integrable system with integrals of motion defining a moment map, \( \mu : \mathbb{C}^n \rightarrow \mathfrak{g}^* \), such that \( \mathcal{L} \) is a corrected Bohr–Sommerfeld fiber. I.e., for some \( c_0 \in \mathbb{R}^n \),
\[ \mathcal{L} = \mathcal{L}_{c_0} = \{ (q, p) \in \mathbb{R}^{2n} : \mu(q, p) = c_0 \}, \] (4.6)
and
\[ h = F_{\mu}, \] (4.7)
for some \( F \in C^\infty(\mathbb{R}^n) \) with \( F(c_0) = E \). Let \( \mathcal{P}_{\mu} \) denote the real polarization having the level sets \( \mathcal{L}_c, c \in \mathbb{R}^n \) as leaves.

**Definition 4.1.** We will call the polarization \( \mathcal{P}_{\mu} \) the energy polarization and the corresponding quantization on \( \mathcal{H}_{\mu} \) the energy quantization.

A real polarization on \( \mathbb{R}^{2n} \) having a compact fiber will typically have singular (lower dimensional) fibers and therefore, due to conjecture 3.5, we do not expect a Kähler regulator of the first type to exist for \( \mathcal{P}_{\mu} \) so that we will need to consider a regulator of the second type for \( \mathcal{P}_{\mu} \). This problem was studied in the toric case in [BFMN, KMN1] and in general in [MN2] and we will review now some of the results.

Let us assume that the level sets \( \mathcal{L}_c \) are compact for noncritical values \( c \in \mathbb{R}^n \) and that a function \( G : \mathbb{R}^n \rightarrow \mathbb{R} \) exists such that \( h_2 = G_{\mu} \) is strongly convex as a function of all action variables on equivariant neighborhoods of all regular fibers. This, plus some technical assumptions on the Fourier coefficients of local holomorphic functions for some initial complex structure \( J_0 \), imply that the polarization
\[ \mathcal{P}_{h_2}^{\text{tr}} = e^{i \mathcal{L}_{\text{tr}} \text{tr}} \mathcal{P}_{\mu} \] (4.8)
where \( \mathcal{P}_{\mu} \) is the polarization associated with \( J_0 \), converges to \( \mathcal{P}_{\mu} \) as \( t \rightarrow \infty \),
\[ \lim_{t \rightarrow +\infty} \mathcal{P}_{h_2}^{\text{tr}} = \mathcal{P}_{\mu}, \]
in an appropriate (weak) sense [BFMN, MN2]. We will also assume that there are local \( J_0 \)-holomorphic coordinates, \( \{ u_j \}_{j=1, \ldots, n} \), such that pointwise, on a neighborhood of every point, one has
\[ \lim_{t \rightarrow +\infty} (t \rightarrow +\infty) \mathcal{P}_{h_2}^{\text{tr}} = dH_j \wedge \cdots \wedge dH_n, \] (4.9)
for some sooth, positive function $\beta \in C^\infty((0, \infty))$, where $u^{(\mu)}_j = e^{i\theta_j(u_j)}$. Consider the following modification of the KSH map, introduced in [MN2],

$$U_2^{(\mu)} = e^{-i\hat{h}_2^{(\mu)} \alpha \sigma^{(\mu)}},$$

(4.10)

where $\hat{h}_2^{\rho\sigma}$ is defined in (2.4) and $\hat{h}_2^{\mu}$ is the following self-adjoint operator

$$\hat{h}_2^{\mu}(f \otimes \sqrt{d\nu_1^{(\mu)} \wedge \cdots \wedge d\nu_n^{(\mu)}}) = G\left(\tilde{H}_1^{\rho\sigma}, \tilde{H}_n^{\rho\sigma}\right)(f) \otimes \sqrt{d\nu_1^{(\mu)} \wedge \cdots \wedge d\nu_n^{(\mu)}},$$

(4.11)

densely defined on $L^2(\mathbb{R}^{2n}) \otimes \sqrt{d\nu_1^{(\mu)} \wedge \cdots \wedge d\nu_n^{(\mu)}}$, where $\tilde{H}_j^{\rho\sigma}$ denotes the prequantization of $H_j$ without the half-form correction, $\tilde{H}_j^{\rho\sigma} = i\hbar X_j - L_{H_j}$ (compare with (2.4)).

The operator $U_2^{(\mu)}$ in (4.10) maps, for all $t > 0$, $H_P$ to, in general non-polarized, subspaces of $L^2(\mathbb{R}^{2n}) \otimes \sqrt{d\nu_1^{(\mu)} \wedge \cdots \wedge d\nu_n^{(\mu)}},$

We will further assume that the following limit, for every $\psi \in H_P$,

$$U_2^{(\mu)}(\psi) = \lim_{t \to \infty} U_2^{(\mu)}(\psi)$$

(4.12)

e exists and the resulting map, $U_2^{(\mu)} : H_P \to H_P$, is an isomorphism onto the space of polarized Dirac delta distributions supported on Maslov corrected Bohr–Sommerfeld Lagrangian leaves (see [BFMN, KMN1, MN2]). Then the distributional section

$$\hat{\psi}_{c,\nu}(H, \theta) = \delta(H - c_0) e^{i\theta_0} \otimes \sqrt{d\nu} \tilde{H} \in H_P,$$

(4.13)

where $(H, \theta)$ are local action-angle coordinates, the phase factor $\theta_0 \in \hbar\mathbb{Z}^n$ corresponds to the uncorrected Bohr–Sommerfeld leaf, $H = \tilde{c}$ (compare with (4.31) for the harmonic oscillator), is the image of a uniquely defined section $\hat{\psi}_{c} \in H_P$.

$$\hat{\psi}_{c} = U_2^{(\mu)}(\hat{\psi}_{c}).$$

(4.14)

Summarizing, our proposal to use Kähler regularization to construct semiclassical solutions of (4.3) can be divided in the following steps.

1. Choice of regulators $h_1$, $h_2$ and construction of the KSH maps in $U_1^{(\mu)}$ in (4.5) and $U_2^{(\mu)}$ in (4.10):

   (i) Choose the Thiemann complexifier or regulator of first type, $h_1$, for the Schrödinger polarization and define the one-parameter family of Kähler polarizations (4.4) and Kähler regularizations of the Schrödinger representation (4.5). A standard choice is the free particle Hamiltonian, $h_1(q, p) = \frac{1}{2}||p||^2$, for which $p^{(\mu)}_u = (X_{1^{(\mu)}}^u, ..., X_{n^{(\mu)}}^u),$$

   where $c^{(\mu)}_0 = q + itp$. Another interesting possibility, studied in [Es], consists in using the regulator of second type used for the polarization $p^{(\mu)}_u$ also as Thiemann complexifier, i.e. $h_1 = h_2 = h$. Find $U_1^{(\mu)}$ in (4.5).

   (ii) Choose a $p^{(\mu)}_u$-regulator of second type, $h_2$, for the energy representation of definition 4.1, define the one-parameter family of Kähler polarizations (4.8) and the Kähler regularized Hilbert space as the image, $U_2^{(\mu)}(H_P)$, of (4.10), for large $t$, leading to (4.12), (4.13) and (4.14). As mentioned above, from [BFMN, KMN1, KMN4, MN2], it follows that $h_2$ should be a function on $\mathbb{R}^{2n}$ that is strongly convex in the action coordinates.
(2) **BKS pairing between the Schrödinger representation and the energy representation of definition 4.1:** for the states $\psi_1 \in \mathcal{H}_{\text{Sch}}$ and $\psi_2 \in \mathcal{H}_p$ define their time $(t_1, t_2)$ regularized BKS pairing as

$$\{\psi_1, \psi_2\}^{b_{12}}_{\text{BKS}} = \left\{ U^{12}_{12}(\psi_1), U^{12}_{21}(\psi_2) \right\}_{\text{BKS}},$$

where $\hat{\psi}_2$ is the state in $\mathcal{H}_p$, mapped to $\psi_2$ by $U^{12}_{21}$, that is $\hat{\psi}_2 = (U^{12}_{21})^{-1}(\psi_2)$ and the pairing in the right hand side is the usual half-form corrected BKS pairing of geometric quantization. The BKS pairing between states in $\mathcal{H}_{\text{Sch}}$ and $\mathcal{H}_p$ is then defined by the following limit of Kähler regularized pairings

$$\{\psi_1, \psi_2\}^{b_{12}}_{\text{BKS}} = \lim_{(t_1, t_2) \to (0, \infty)} \left\{ \psi_1, \psi_2 \right\}^{b_{12}},$$

in case the limit exists $\forall \psi_1 \in \mathcal{H}_{\text{Sch}}, \psi_2 \in \mathcal{H}_p$.

(3) **Definition of the semiclassical state $\psi_\mathcal{E}$:** if the limit in (4.16) exists and is continuous on $\mathcal{H}_{\text{Sch}} \times \mathcal{H}_p$, it defines a pairing map

$$B : \mathcal{H}_p \to \mathcal{H}_{\text{Sch}}.$$ (4.17)

The semiclassical state corresponding to $\mathcal{L}$ via the Kähler regularization of the pairing is then defined to be

$$\psi_\mathcal{E} = B\left( \hat{\psi}_\mathcal{E} \right),$$

(4.18)

where $\hat{\psi}_\mathcal{E}$ is the state defined in (4.13). So, $\psi_\mathcal{E}$ is the unique state in $\mathcal{H}_{\text{Sch}}$ such that

$$\{\psi, \psi_\mathcal{E}\}_{\text{Sch}} = \{\psi, \hat{\psi}_\mathcal{E}\}_{\text{BKS}}, \forall \psi \in \mathcal{H}_{\text{Sch}}.$$

**Remark 4.2.** The pairing map $B$ and thus the semiclassical states, could in principle depend on the Kähler regularizations, though we would expect to obtain the same result for generic choices.

### 4.2. Maslov phases for the harmonic oscillator

Let us illustrate the general method of the previous section in the case of the one-dimensional harmonic oscillator. We have $n = 1$, $M = \mathbb{R}^2$, $\omega = dq \wedge dp$, $\Theta = pdq$, $L = \mathbb{R}^2 \times \mathbb{C}$, $\nabla 1 = \frac{i}{\hbar} \Theta$ and $h(q, p) = \mu = H = \frac{1}{2}(p^2 + q^2)$. As in (4.1), we have

$$\mathcal{H}_{\text{Sch}} = L^2(\mathbb{R}, dq) \otimes \sqrt{dq}.$$ For $\mathcal{H}_p$, since $\hbar$ is the action variable for the standard toric structure in $\mathbb{R}^2$, we know from [KMN1] that

$$\mathcal{H}_p = \left\{ \delta \left( \hbar - \hbar \left( m + \frac{1}{2} \right) \right) e^{-i\theta \psi_0} e^{i\omega_0}, m \in \mathbb{N}_0 \right\} \otimes \sqrt{d\hbar}. $$ (4.19)

Let us now follow the three steps described in the previous section, in this example.

(1) **Choice of regulators $h_1$, $h_2$ and construction of $U^{12}_{12}$ in (4.5) and $U^{12}_{21}$ in (4.10):**

(i) Let us choose the standard Thiemann complexifier for the Schrödinger representation, $h_1(q, p) = \frac{1}{2}p^2$. Then
\[ \mathcal{P}_h^{it} = e^{(i\mathcal{L}_\kappa) t} \mathcal{P}_{\text{Sch}} = e^{(i\mathcal{L}_\kappa) t} \{ X_\kappa \} \mathcal{C}, \]
\[ = \{ X_{e^{(i\mathcal{L}_\kappa) t}} \} \mathcal{C} = \{ X_{q+itp} \} \mathcal{C} = \{ X_{z^{(it)}} \} \mathcal{C}, \]

where \( z^{(it)} = q + itp \) defines a Kähler polarization for all \( t > 0 \), confirming that \( h_1 \) is a \( \mathcal{P}_{\text{Sch}} \)-regulator of the first type. This is the simplest example of imaginary time geodesic flows starting at the Schrödinger polarization and leading (at time \( t = -1 \)) to adapted Kähler structures on tubular neighbourhoods (the whole \( T^*\mathbb{R} \) in the present case) of the zero section of cotangent bundles of compact Riemannian manifolds \([HK1]\). The equation for \( \mathcal{P}_h^{it} \)-polarized sections of \( L \) reads,

\[
\begin{pmatrix}
\psi
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
\psi
\end{pmatrix} \times \mathcal{E}
\end{pmatrix}.
\]

We see that, due to the fact that \( p \) preserves the Schrödinger polarization, \( \hat{\mathcal{P}}^Q \) acts on the Schrödinger representation. We can then define

\[
\mathcal{H}_p = \left\{ f(z^{(it)}) e^{-\frac{1}{2}p^2} \otimes \sqrt{\mathcal{E}(z^{(it)})} \right\},
\]

and therefore

\[
\mathcal{H}_p = \left\{ f(z^{(it)}) e^{-\frac{1}{2}p^2} \otimes \sqrt{\mathcal{E}(z^{(it)})} : \int_{\mathbb{R}^2} \left| f(z^{(it)}) \right|^2 \right\},
\]

where \( f \) is \( \mathcal{P}_h^{it} \)-holomorphic.

Let us now obtain the corresponding Kähler regularization maps, \( \mathcal{U}_1^t \). From (2.4) we obtain

\[
\hat{\mathcal{P}}^Q = i\hbar \left( \frac{\partial}{\partial q} \otimes 1 + 1 \otimes \mathcal{L}_\pi \right) \quad \text{and} \quad \hat{\mathcal{P}}^Q = i\hbar \left( p \frac{\partial}{\partial q} \otimes 1 + 1 \otimes \mathcal{L}_p \right) - \frac{p^2}{2} \otimes 1.
\]

We see that, due to the fact that \( p \) preserves the Schrödinger polarization, \( \hat{\mathcal{P}}^Q \) acts on the Schrödinger representation. We can then define

\[
\mathcal{H}_p = \frac{1}{2} \left( \mathcal{P}_h^{it} \right)^2 \left| \mathcal{P}_{\text{Sch}}^{it} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} \otimes 1. \right.
\]

It is convenient to act with \( \mathcal{U}_1^t \) on \( \psi \in \mathcal{H}_{\text{Sch}} \), written in the form

\[
\psi(q) \otimes \sqrt{\mathcal{E}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} p \cdot q} \psi(p_0) \ dp_0 \otimes \sqrt{\mathcal{E}}.
\]

From (4.5), (4.21) and (4.22) we obtain for \( \mathcal{U}_1^t(\psi) \in \mathcal{H}_p^\mathcal{P} \),

\[
\mathcal{U}_1^t(\psi)(q, p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} p \cdot q} e^{-\frac{1}{2}(p+\mu)^2} \psi(p_0) \ dp_0 \otimes \sqrt{\mathcal{E}(p+\mu)}.
\]

(ii) For the choice of \( h_2 \) notice that \( h = H = \mu \) is an action coordinate in \( \mathbb{R}^2 \setminus \{0\} \) generating a global \( S^1 \) action. The angle coordinate is the polar angle \( \theta = \arctan(p/q) \). We can therefore choose as \( h_2 \) a strongly convex function of \( H = h \), e.g.
Let \( w = \varepsilon^{(i)} = q + ip = \sqrt{2/h} \ e^{i\theta} \) and choose as starting polarization, the \( S^1 \)-invariant toric polarization \( P_0 = \langle X_w \rangle_C \). We have \( X_{h_z} = -\hbar \frac{d}{d\theta} \) and therefore the one-parameter family of polarizations obtained by flowing with this vector field in imaginary time is also toric and a simple particular example of those studied in [BFMN] and [KMN1]

\[
P_{\nu^2} = e^{iu(z)} P_0 = e^{iu(z)} \langle X_w \rangle_C,
\]
where

\[
\frac{w(i)}{\sqrt{2}} = \sqrt{2/h} e^{i\theta} = e^{\frac{1}{2} \log(h) + i\theta} = e^{\frac{1}{2} h \theta^2}, \tag{4.24}
\]

and \( g(h) = \frac{1}{2} h \log(h) - \frac{h}{2} + \frac{h^2}{2} \) is the strongly convex toric symplectic potential. (See e.g. [BFMN].) A local \( J_0 \)-holomorphic coordinate satisfying (4.9) is \( u = \log(w/\sqrt{2}) = th + \frac{1}{2} \log(h) + i\theta \) with \( \beta(t) = 1/t \). To find the Hilbert space it will be convenient to use the \( S^1 \)-invariant trivializing section (we will return to the trivialization defined by \( \sigma(q, p) = 1 \) when we calculate the pairing of states of different polarizations) \( \tilde{\sigma} = e^{-\frac{h}{2} \theta^2} \sigma \) so that

\[
V \tilde{\sigma} = -\frac{i}{\hbar} hd\theta \tilde{\sigma}.
\]

The equation for \( P_{\nu^2} \)-polarized sections then reads

\[
V_{\tilde{\sigma}} \varphi = 0 \Leftrightarrow \left( \frac{\partial}{\partial \nu} + i \frac{\partial}{\partial \theta} + \frac{h}{\hbar} \right) \tilde{\Psi}(q, p) = 0. \tag{4.25}
\]

Since \( g \) is strictly convex and \( v = \frac{dg}{dh} \), the inverse Legendre map is \( h = \frac{dk}{dv} \), where \( k(v) = h(v)v - g(h(v)) \) is the Kähler potential. We see that the solutions of (4.25) are given by

\[
\tilde{\Psi}(q, p) = f \left( w(i) \right) e^{-\frac{1}{2} h \theta^2},
\]

where \( f \) is an arbitrary \( P_{\nu^2} \)-holomorphic function. Therefore, we have, in the trivialization defined by \( \tilde{\sigma} \),

\[
\tilde{H}_{\nu^2} = \left\{ f \left( w(i) \right) e^{-\frac{1}{2} h \theta^2} \otimes \sqrt{dw(i)} : \right. \right.
\]

\[
\int_{\mathbb{R}^+ \times S^1} \left| f \left( w(i) \right) \right|^2 e^{-\frac{1}{2} h \theta^2} \sqrt{g^*(h)} \ dhd\theta < \infty \right\}, \tag{4.26}
\]

Let us now obtain \( U_{\nu^2}^{(i)} \) in (4.10). From (2.4) we obtain
where \( a_m = 2^{-\frac{i}{2}}(2\pi \hbar)^{-\frac{1}{2}}(\hbar (m + \frac{1}{2}))^{-\frac{1}{2}}e^{\frac{i}{2}(m+\frac{1}{2})^2} \), form an orthogonal basis of eigensections of \( \hat{h}_2^\mu \) (see (4.11)), with

\[
\hat{h}_2^\mu \left( \varphi_m^{(i)} \right) = \frac{1}{2} \left( h \left( m + \frac{1}{2} \right) \right)^2 \varphi_m^{(i)}. \tag{4.29}
\]

The constants \( a_m \) in (4.28) are chosen to have \( \varphi \psi = \varphi \), \( \varphi \psi = \varphi \), see (4.14). In this case \( \hat{h}_2^\mu \) acts on the space of polarized sections because \( h \) preserves the polarizations \( \mathcal{F}_{m} \) for every \( \in [0, \infty) \). Eventhough \( h_2 \), itself does not preserve the polarization the operator \( \hat{h}_2^\mu \) is defined through \( \hbar \) in (4.11) and therefore has a well defined action on the Hilbert spaces \( \mathcal{H}_{m}^\mu \). Note that (4.28) is a local expression of a global \( \mathcal{H}_{m}^\mu \)-holomorphic section of the half-form corrected prequantum bundle in spite of the factor of \( \theta e^{i2} \). In fact, as explained in Section 3 and in the Appendix of [KMN1], this factor gets canceled against a similar factor arising from the fact that \( u \) effectively carries a factor of \( \theta e^{i2} \) as remarked above, we obtain

\[
\psi_{\mathcal{L}_m} = \lim_{t \to +\infty} U_2^\mu \left( \varphi_m^{(i)} \right) = \delta \left( h - h \left( m + \frac{1}{2} \right) \right) e^{im\theta} \sqrt{\hbar}, \quad m \in \mathbb{N}_0, \tag{4.31}
\]

so that indeed, \( \psi_{\mathcal{L}_m} = \varphi_m^{(i)} \).
(2) **BKS pairing between the Schrödinger representation and the energy representation:**

**Proposition 4.3.** For the holomorphic forms in (4.23) with \( t = t_1 \) and (4.28) with \( t = t_2 \), we obtain

\[
\left( \frac{1}{2} \right) \mathbf{d} \sigma^{(it_1)} \wedge \left( \left( \frac{1}{2} \hbar + t_2 \right) \mathbf{d} \vartheta - \mathbf{i} \mathbf{d} \theta \right) = \left( \frac{1}{2} \right) \frac{(p - iq)(1 + t_1)}{p^2 + q^2} + t_2(p - it_1q) \mathbf{d}w.
\] (4.32)

**Proof.** The result follows directly from \( \zeta^{(it_1)} = q + it_p \) and (4.24) with \( t = t_2 \). □

**Proposition 4.4.** The pairing in (4.16) for the harmonic oscillator is given by

\[
\left\langle \psi, \hat{\psi}_e \right\rangle_{BKS} = \lim_{(t_1, t_2) \to (0, +\infty)} \left\langle U_t^{it_1}(\psi)(q, p), e^{ipq/\hbar} U_t^{it_2}(\varphi_m^{(0)}) \right\rangle_{BKS}
\]

\[
= \frac{i}{2} \int_{\mathbb{R}^2} \psi(q) e^{-ipq/\hbar} e^{-im\theta} \sqrt{\hbar} \delta \left( \hbar - \hbar \left( m + \frac{1}{2} \right) \right) dq dp,
\] (4.33)

where \( \psi \in H_{Sch} \).

**Proof.** From (4.23), (4.30) and (4.32) we obtain the BKS pairing

\[
\left\langle U_t^{it_1}(\psi)(q, p), e^{ipq/\hbar} U_t^{it_2}(\varphi_m^{(0)}) \right\rangle_{BKS} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ipq/\hbar} e^{-m\theta} \left( \frac{p + q}{4} \right)^{\frac{1}{2}} \psi \times (p_0) h^{\frac{1}{2}} \mathbf{e}^{-ipq/\hbar} e^{-m\theta} \left( \hbar - \hbar \left( m + \frac{1}{2} \right) \right) e^{-im\theta},
\]

\[
\left( \frac{(p - iq)(1 + t_1)}{p^2 + q^2} + t_2(p - it_1q) \right)^{\frac{1}{2}} \mathbf{d}p_0 dq dp.
\]

Due to the gaussians, the integrals are convergent and bounded and, therefore, the limit \( (t_1, t_2) \to (0, +\infty) \) exists and can be taken inside the integral. Taking the limit \( t_2 \to +\infty \) gives,
\[
\lim_{t_1 \to +\infty} \left\langle U_{t_1}^{\psi}(\psi(q, p), e^{ipq/2\hbar}U_{t_1}^{(i)}(q_{m}^{(i)}) \right\rangle_{\text{BKS}} \\
= \sqrt{\hbar} a_m \frac{2^{m} + 1}{2} \left( \hbar \left( m + \frac{1}{2} \right) \right)^{2 + \frac{1}{2}} e^{-\left( \frac{m + \frac{1}{2}}{2} \right)}. \\
\cdot \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i \mu_q p_0 q} e^{-\frac{i}{\hbar} \left( p + p_0 \right)^2} \psi(p_0) e^{-ipq/2\hbar} \\
\times e^{-i m \hbar \left( p - i t_1 q \right)^2} \delta \left( \hbar - \hbar \left( m + \frac{1}{2} \right) \right) dp_0 dq dp.
\]

The limit \( t_1 \to 0 \) and the integration in \( dp_0 \) then produces the result. \( \square \)

(3) \text{The semiclassical state } \psi_c: \text{ from above we obtain the following}

\textbf{Proposition 4.5.} Let \( L_m, m \in \mathbb{N}_0 \), be the Lagrangian cycles where \( h = \hbar (m + \frac{1}{2}) \), as above. The pairing map in (4.17) and (4.18) for the harmonic oscillator reads

\[
\psi_{L_m}(q) = B\left( \psi_{L_m}(q) \right) = \psi_{L_m}^+(q) + \psi_{L_m}^-(q),
\]

such that \( \psi_{L_m}^+, \psi_{L_m}^- \) have support in \([-\hbar(2m + 1), \hbar(2m + 1)]\) where they are given by

\[
\psi_{L_m}^+(q) = \sqrt{\frac{1}{2}} \left( \hbar (2m + 1) - q^2 \right)^{-\frac{1}{2}} e^{\pm i \sqrt{\hbar(2m+1)-q^2}} e^{im \arctan \frac{\sqrt{\hbar(2m+1)-q^2}}{q}} \otimes \sqrt{dq}
\]

and

\[
\psi_{L_m}^-(q) = \sqrt{\frac{1}{2}} e^{-i q^2 \frac{1}{2}} \left( \hbar (2m + 1) - q^2 \right)^{-\frac{1}{2}} \\
\times e^{\mp i \sqrt{\hbar(2m+1)-q^2}} e^{-im \arctan \frac{\sqrt{\hbar(2m+1)-q^2}}{q}} \otimes \sqrt{dq}.
\]

\textbf{Proof.} Let \( \psi \) in (4.33) be a continuous function in \( L^2(\mathbb{R}) \). Then, since the inner product in \( \mathcal{H}_{\text{Sch}} \) is given by integration in \( q \), we obtain from (4.17), (4.18) and (4.33),

\[
\psi_{L_m}(q) = B\left( \psi_{L_m}(q) \right) = \psi_{L_m}^+(q) + \psi_{L_m}^-(q),
\]

where

\[
\psi_{L_m}^+(q) = \sqrt{\frac{1}{2}} \int_{0}^{\frac{\hbar (m+1)}{2}} e^{\pm in_{L_m} \arctan \frac{\sqrt{p^2 + q^2}}{2}} \delta \left( \hbar (m + \frac{1}{2}) \right) dp \otimes \sqrt{dq}
\]

(4.37)
\[ \psi_{\mathcal{L}}(q) = \left( \frac{1}{2} \right)^{1/4} e^{\frac{i}{\hbar} \int_{-2\hbar(m+2)}^{0} \frac{p^2 + q^2}{2} - \hbar \left( m + \frac{1}{2} \right) dp} \otimes \sqrt{dq} \] (4.38)

and the result follows.

We observe that \( \psi_{\mathcal{L}} \), which is supported in the ‘classically allowed region’ \([-\hbar(2m+1), \hbar(2m+1)]\), contains two contributions, weighted with a relative (Maslov) phase \( e^{\frac{i}{\hbar} \int} \) which arises at the caustic points \( q = \pm \sqrt{2\hbar (m + \frac{1}{2})} \), \( p = 0 \), for the projection of \( \mathcal{L}_m \) onto the \( q \)-axis. Moreover, by explicitly evaluating \( \int p dq \) with \( (q, p) \in \mathcal{L}_m \), we see that \( \psi_{\mathcal{L}} \) has the form of the usual WKB wave function in the classically allowed region. (See, for example, chapter 7 of [Me].)

Acknowledgments

We would like to thank the referees for very useful suggestions. The authors were partially supported by FCT/Portugal through the projects PEst-OE/EEI/LA009/2013, UID/MAT/04459/2013 EXCL/MAT-GEO/0222/2012, PTDC/MAT/119689/2010, PTDC/MAT/1177762/2010. The first author was supported by the FCT fellowship SFRH/BPD/77123/2011 and by a postdoctoral fellowship of the project PTDC/MAT/120411/2010. The second author thanks S Wu for useful discussions and is thankful for generous support from the Emerging Field Project on Quantum Geometry from Erlangen–Nürnberg University.

References

[BFMN] Baier T, Florentino C, Mourão J M and Nunes J P 2011 Toric Kähler metrics seen from infinity, quantization and compact tropical amoebas J. Differ. Geom. 89 411–54
[BW] Bates S and Weinstein A 1997 Lectures on the Geometry of Quantization (Berkeley Mathematical Lecture Notes) vol 8 (Providence, RI: American Mathematical Society)
[EHHL] Esterlis I, Haggard H M, Hedeman A and Littlejohn R G 2014 Maslov indices, Poisson brackets, and singular differential forms Europhys. Lett. 106 50002
[Es] Esteves J N Generalized Coherent State Transforms on \( \mathbb{F}^n \) in preparation.
[Go] de Gosson M 1997 Maslov Classes, Metaplectic Representation, and Lagrangian Quantization (Berlin: Akademie Verlag)
[GS] Guillemin V and Sternberg S 1990 Geometric Asymptotics Mathematical Survey and Monographs vol 14 (Providence, RI: American Mathematical Society)
[GS1] Guillemin V and Stenzel M 1991 Grauert tubes and the homogeneous Monge–Ampère equation J. Differ. Geom. 34 561–70
[GS2] Guillemin V and Stenzel M 1992 Grauert tubes and the homogeneous Monge–Ampère equation II J. Differ. Geom. 35 627–41
[Hal1] Hall B C 1994 The Segal–Bargmann ‘coherent-state’ transform for Lie groups J. Funct. Anal. 122 103–51
[Hal2] Hall B C 2002 Geometric quantization and the generalized Segal–Bargmann transform for Lie groups of compact type Commun. Math. Phys. 226 233–68
[Ham] Hamilton M 2010 Locally toric manifolds and singular Bohr–Sommerfeld leaves Mem. Am. Math. Soc. 207 971
[HK1] Hall B C and Kirwin W D 2011 Adapted complex structures and the geodesic flow Math. Ann. 350 455–74
[HK2] Hall B C and Kirwin W D 2015 Complex structures adapted to magnetic flows J. Geom. Phys. 90 111–31

[HM] Hamilton M and Miranda E 2010 Geometric quantization of integrable systems with hyperbolic singularities Ann. Inst. Fourier (Grenoble) 60 51–85

[Ki] Kirwin W D 2014 Quantizing the geodesic flow via adapted complex structures arXiv:1408.1527

[KMN1] Kirwin W D, Mourão J M and Nunes J P 2013 Degeneration of Kähler structures and half-form quantization of toric varieties J. Sympl. Geom. 11 603–43

[KMN2] Kirwin W D, Mourão J M and Nunes J P 2013 Complex time evolution in geometric quantization and generalized coherent state transforms J. Funct. Anal. 265 1460–93

[KMN3] Kirwin W D, Mourão J M and Nunes J P 2014 Complex time evolution and the Mackey–Stone–Von Neumann theorem J. Math. Phys. 55 102101

[KMN4] Kirwin W, Mourão J and Nunes J P 2015 Complex symplectomorphisms and pseudo-Kähler islands in the quantization of toric manifolds Math. Ann. doi:10.1007/s00208-015-1205-0

[KW1] Kirwin W D and Wu S 2006 Geometric quantization, parallel transport and the Fourier transform Comm. Math. Phys. 266 577–94

[KW2] Kirwin W D and Wu S 2011 Momentum space for compact Lie groups and the Peter–Weyl theorem in preparation.

[LS] Lempert L and Szőke R 1991 Global solutions of the homogeneous complex Monge–Ampère equation and complex structures on the tangent bundle of Riemannian manifolds Math. Ann. 319 689–712

[Me] Merzbacher E 1970 Quantum Mechanics (New York: Wiley)

[MF] Maslov V P and Fedoriuk M V 1981 Semi-Classical Approximation in Quantum Mechanics (Boston: Reidel)

[MN1] Mourão J and Nunes J P 2015 On complexified Hamiltonian flows and geodesics on the space of Kähler metrics Int. Math. Res. Not. doi:10.1093/imrn/rnv004

[MN2] Mourão J and Nunes J P Decomplexification of integrable systems and quantization in preparation

[So] Solha R 2015 Circle actions in geometric quantization J. Geom. Phys. 87 450–60

[Sn] Sniatycki J 1975 On Cohomology Groups Appearing in Geometric Quantization (Lecture Notes in Mathematics) vol 570 (Berlin/Heidelberg: Springer-Verlag)

[Th1] Thiemann T 1996 Reality conditions inducing transforms for quantum gauge field theory and quantum gravity Class. Quantum Grav. 13 1383–404

[Th2] Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)

[Wo] Woodhouse N 1991 Geometric Quantization (Oxford: Oxford University Press)

[Wu] Wu S 2011 Geometric phases in the quantisation of bosons and fermions J. Aust. Math. Soc. 90 221–35