Predictability in Spatially Extended Systems with Model Uncertainty

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Abstract

Macroscopic models for spatially extended systems under random influences are often described by stochastic partial differential equations (SPDEs). Some techniques for understanding solutions of such equations, such as estimating correlations, Liapunov exponents and impact of noises, are discussed. They are relevant for understanding predictability in spatially extended systems with model uncertainty, for example, in physics, geophysics and biological sciences. The presentation is for a wide audience.

Key Words: Stochastic partial differential equations (SPDEs), correlation, Liapunov exponents, predictability, uncertainty, invariant manifolds, impact of noise

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1 Motivation

Scientific and engineering systems are often subject to uncertainty or random influence. Randomness can have delicate impact on the overall evolution of such systems, for example, stochastic bifurcation [8], stochastic resonance [28], and noise-induced pattern formation [21]. Taking stochastic effects into account is of central importance for the development of mathematical models of complex phenomena in engineering and science.

Macroscopic models for systems with spatial dependence (“spatially extended”) are often in the form of partial differential equations (PDEs). Randomness appears in these models as stochastic forcing, uncertain parameters, random sources or inputs, and random boundary conditions (BCs). These models are usually called stochastic partial differential equations (SPDEs). Note that SPDEs may also serve as intermediate “mesoscopic” models in some multiscale systems. Although we may think that SPDEs could be reduced to large systems of stochastic ordinary differential equations (SODEs) in numerical approaches [37, 1], it is beneficial to work on SPDEs directly when dealing with some dynamical issues [6, 5, 14, 15, 20, 26, 29, 32, 40, 41, 52, 53, 55].

There is a growing recognition of a role for the inclusion of stochastic terms in the modeling of complex systems. For example, there has been increasing interest in mathematical modeling via SPDEs, for the climate system, condensed matter physics, materials sciences, mechanical and electrical engineering, and finance, to name just a few. The inclusion of stochastic effects has led to interesting new mathematical problems at the interface of dynamical systems, partial differential equations, scientific computing, and probability theory. Problems arising in the context of stochastic dynamical modeling have inspired interesting research topics about, for example, the interaction between noise, nonlin-
earity and multiple scales, and about efficient numerical methods for simulating random phenomena.

There has been some promising new development in understanding dynamics of SPDEs via invariant manifolds [16, 17, 38, 59] and stochastic homogenization [57, 58]. But we will not discuss these issues in this paper. For general background on SPDEs, see [13, 51, 61, 10, 48].

Although some progress has been made in SPDEs in the past decade, many challenges remain and new problems arise in modeling basic mechanisms in complex systems under uncertainty. These challenging problems include overall impact of noise, stochastic bifurcation, ergodic theory, invariant manifolds, and predictability of dynamical behavior, to name just a few. Solutions for these problems will greatly enhance our ability in understanding, quantifying, and managing uncertainty and predictability in engineering and science. Breakthroughs in solving these challenging problems are expected to emerge.

This article is organized as follows. After reviewing some basic concepts on probability in Hilbert space in §2, we discuss stochastic analysis and SPDEs in §3. Then we derive correlations of some linear SPDEs, Lyapunov exponents, and the impact of uncertainty in §4, §5 and §6, respectively.

2 Stochastic Tools in Hilbert Space

2.1 Hilbert space
Recall that the Euclidean space $\mathbb{R}^n$ is equipped with the usual metric or distance $d(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$, norm or length $\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2}$, and the usual scalar product $x \cdot y = \langle x, y \rangle = \sum_{j=1}^{n} x_j y_j$. The Borel $\sigma$-field of $\mathbb{R}^n$, i.e., $\mathcal{B}(\mathbb{R}^n)$ is generated by all open balls in $\mathbb{R}^n$.

Hilbert space $H$ is a set with three mathematical operations: scalar multiplication, addition and scalar product $\langle \cdot, \cdot \rangle$, satisfying the usual properties as we are familiar with in elementary mathematics. The scalar product induces a natural norm $\|u\| = \sqrt{\langle u, u \rangle}$. The Borel $\sigma$-field of $H$, i.e., $\mathcal{B}(H)$ is generated by all open balls in $H$.

2.2 Probability in Hilbert space
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with sample space $\Omega$, $\sigma$-field $\mathcal{F}$ and probability measure $\mathbb{P}$. Consider a random variable in Hilbert space $H$ (i.e., taking values in $H$): $X : \Omega \to H$.

Its mean or mathematical expectation is defined in terms of the integral with respect to the probability measure $\mathbb{P}$:

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega).$$
Its variance is:

\[ \text{Var}(X) = \text{E}(X - \text{E}(X), X - \text{E}(X)) = \text{E}\|X - \text{E}(X)\|^2 = \text{E}\|X\|^2 - \|\text{E}(X)\|^2 \]

Especially, if \( \text{E}(X) = 0 \), then \( \text{Var}(X) = \text{E}\|X\|^2 \).

Covariance operator of \( X \) is defined as

\[ \text{Cov}(X) = \text{E}[(X - \text{E}(X)) \otimes (X - \text{E}(X))], \tag{1} \]

where for any \( a, b \in H \), we denote \( a \otimes b \) the linear operator in \( H \) defined by

\[ a \otimes b : H \rightarrow H, \quad (a \otimes b)h = a(b, h), \quad h \in H. \tag{2} \]

Let \( X \) and \( Y \) be two random variables taking values in Hilbert space \( H \). The correlation operator of \( X \) and \( Y \) is defined by

\[ \text{Cor}(X, Y) = \text{E}[(X - \text{E}(X)) \otimes (Y - \text{E}(Y))]. \tag{4} \]

**Remark 1.** \( \text{Cov}(X) \) is a symmetric positive and trace-class linear operator with trace

\[ \text{Tr} \text{Cov}(X) = \text{E}(X - \text{E}(X), X - \text{E}(X)) = \text{E}\|X - \text{E}(X)\|^2. \tag{5} \]

Moreover,

\[ \text{Tr} \text{Cor}(X, Y) = \text{E}(X - \text{E}(X), Y - \text{E}(Y)). \tag{6} \]

### 2.3 Gaussian random variables

Recall that a random variable taking values in \( \mathbb{R}^n \)

\[ X : \Omega \rightarrow \mathbb{R}^n \]

is called Gaussian, if for any \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), \( X \cdot a = a_1X_1 + \cdots + a_nX_n \) is a scalar Gaussian random variable. A Gaussian random variable in \( \mathbb{R}^n \) is denoted as \( X \sim \mathcal{N}(m, Q) \), with mean vector \( m \) and covariance matrix \( Q \). The covariance matrix \( Q \) is symmetric and non-negative (i.e., eigenvalue \( \lambda_j \geq 0 \), \( j = 1, \ldots, n \)). The trace of \( Q \) is written as \( \text{Tr}(Q) = \lambda_1 + \cdots + \lambda_n \). The covariance matrix is defined as

\[ Q = (Q_{ij}) = (\text{E}[(X_i - m_i)(X_j - m_j)]). \]

We use the notations \( E(X) = m \) and \( \text{Cov}(X) = Q \). The probability density function for this Gaussian random variable \( X \) in \( \mathbb{R}^n \) is

\[ f(x) = f(x_1, \ldots, x_n) = \frac{\sqrt{\det(A)}}{(2\pi)^{n/2}} e^{-\frac{1}{2} \sum_{j,k=1}^{n} (x_j - m_j)a_{jk}(x_k - m_k)}, \tag{7} \]

where \( A = Q^{-1} = (a_{jk}) \).
The probability distribution function of $X$ is

$$F(x) = P(\omega: X(\omega) \leq x) = \int_{-\infty}^{x} f(x)dx.$$  

(8)

The probability distribution measure $\mu$ (or law $\mathcal{L}_X$) of $X$ is:

$$\mu(B) = \int_B f(x)dx, \quad B \in \mathcal{B}(\mathbb{R}^n).$$  

(9)

Here are some observations. For $a, b \in \mathbb{R}^n$,

$$\mathbb{E}(X, a) = \mathbb{E}\sum_{i=1}^{n} a_i X_i = \sum_{i=1}^{n} a_i \mathbb{E}(X_i) = \sum_{i=1}^{n} a_i m_i = \langle m, a \rangle$$  

(10)

$$\mathbb{E}((X - m, a)\langle X - m, b \rangle) = \mathbb{E}\left(\sum_{i} a_i (X_i - m_i) \sum_{j} b_j (X_j - m_j)\right) = \sum_{i,j} a_i b_j [E(\langle X_i - m_i, X_j - m_j \rangle)]$$  

(11)

$$= \sum_{i,j} a_i b_j Q_{ij}$$  

(12)

$$= \langle Qa, b \rangle$$  

(13)

In particular, $\langle Qa, a \rangle = \mathbb{E}(X - m, a)^2 \geq 0$, which confirms that $Q$ is non-negative. Also, $\langle Qa, b \rangle = \langle a, Qb \rangle$, which implies that $Q$ is symmetric.

**Definition 1.** A random variable $X : \Omega \rightarrow H$ in Hilbert space $H$ is called a Gaussian random variable and denoted as $X \sim \mathcal{N}(m, Q)$, if for every $a \in H$, the real random variable $(X, a)$ is a scalar Gaussian random variable (i.e., taking values in $\mathbb{R}^1$).

**Remark 2.** If $X$ is a Gaussian random variable taking values in Hilbert space $H$, then for all $a, b \in H$,

(i) Mean vector $\mathbb{E}(X) = m : \mathbb{E}(X, a) = \langle m, a \rangle$;

(ii) Covariance operator $\text{Cov}(X) = Q : \mathbb{E}((X - m, a)\langle X - m, b \rangle) = \langle Qa, b \rangle$

**Remark 3.** The Borel probability measure $\mu$ on $(H, \mathcal{B}(H))$, induced by a Gaussian random variable $X$ taking values in Hilbert space $H$, is called a Gaussian measure. If $\mu$ is a Gaussian measure in $H$, then there exist an element $m \in H$ and a non-negative symmetric continuous linear operator $Q : H \rightarrow H$ such that:

For all $h, h_1, h_2 \in H$,

(i) Mean vector $m : \int_H \langle h, x \rangle d\mu(x) = \langle m, h \rangle$;

(ii) Covariance operator $Q : \int_H \langle h_1, x \rangle \langle h_2, x \rangle d\mu(x) - \langle m, h_1 \rangle \langle m, h_2 \rangle = \langle Qh_1, h_2 \rangle$
Since the covariance operator $Q$ is non-negative and symmetric, the eigenvalues of $Q$ are non-negative and the eigenvectors $e_n$’s form an orthonormal basis for Hilbert space $H$:

$$Qe_n = q_ne_n, \quad n = 1, 2, \cdots.$$  

Moreover, trace $Tr(Q) = \sum_{n=1}^{\infty} q_n$.

Note that

$$X - m = \sum X_ne_n \quad (15)$$

with coefficients $X_n = \langle X - m, e_n \rangle$.

$$\mathbb{E}X_n^2 = \mathbb{E}(\langle X - m, e_n \rangle \langle X - m, e_n \rangle) = \langle Qe_n, e_n \rangle = \langle q_ne_n, e_n \rangle = q_n. \quad (16)$$

Therefore,

$$\|X - m\|^2 = \sum X_n^2 \quad (17)$$

$$\mathbb{E}\|X - m\|^2 = \sum \mathbb{E}X_n^2 = \sum q_n = Tr(Q). \quad (18)$$

We use $L^2(\Omega, H)$, or just $L^2(\Omega)$, to denote the (new) Hilbert space of square-integrable random variables $x : \Omega \rightarrow H$. In Hilbert space $L^2(\Omega, H)$, the scalar product is

$$<x, y> = \mathbb{E}<x(\omega), y(\omega)>,$$

where $\mathbb{E}$ denotes the mathematical expectation (or mean) with respect to probability $\mathbb{P}$. This scalar product induces the usual mean square norm

$$\|x\| := \sqrt{\mathbb{E}\|x(\omega)\|^2},$$

which provides an appropriate convergence concept.

### 2.4 Brownian motion

Recall that a Brownian motion (or Wiener process) $W(t)$, also denoted as $W_t$, in $\mathbb{R}^n$, is a Gaussian stochastic process on a underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a sample space, $\mathcal{F}$ is a $\sigma$–field composed of measurable subsets of $\Omega$ (called “events”), and $\mathbb{P}$ is a probability (also called probability measure). Being a Gaussian process, $W_t$ is characterized by its mean vector (taking to be the zero vector) and its covariance operator, a $n \times n$ symmetric positive definite matrix (taking to be the identity matrix). More specifically, $W_t$ satisfies the following conditions [11]:

(a) $W(0)=0, \quad$ a.s.
(b) $W$ has continuous paths or trajectories, \quad a.s.
(c) $W$ has independent increments,
(d) $W(t)-W(s) \sim N(0, (t-s)I), \quad t \text{ and } s > 0 \text{ and } t \geq s \geq 0$, where $I$ is the $n \times n$ identity matrix. The Brownian motion in $\mathbb{R}^1$ is called a scalar Brownian motion.
Remark 4. (i) The covariance operator here is a constant $n \times n$ identity matrix $I$, i.e., $Q = I$ and $\text{Tr}(Q) = n$.

(ii) $W(t) \sim N(0, tI)$, i.e., $W(t)$ has probability density function $p_t(x) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{x^2}{2t}}$.

(iii) For every $\alpha \in (0, \frac{1}{2})$, for a.e. $\omega \in \Omega$, there exists $C(\omega)$ such that

$$|W(t, \omega) - W(s, \omega)| \leq C(\omega)|t - s|^{\alpha},$$

namely, Brownian paths are Hölder continuous with exponent less than one half.

Note that the generalized time derivative of Brownian motion $W_t$ is a mathematical model for white noise [3].

Now we define Wiener process, or Brownian motion, in Hilbert space $U$. We consider a symmetric nonnegative linear operator $Q$ in $U$. If the trace $\text{Tr}(Q) < +\infty$, we say $Q$ is a trace class (or nuclear) operator. Then there exists a complete orthonormal system (eigenfunctions) $\{e_k\}$ in $U$, and a (bounded) sequence of nonnegative real numbers (eigenvalues) $q_k$ such that

$$Qe_k = q_k e_k, \ k = 1, 2, \ldots.$$

A stochastic process $W(t)$, or $W_t$, taking values in $U$ for $t \geq 0$, is called a Wiener process with covariance operator $Q$ if:

(a) $W(0) = 0$, a.s.
(b) $W$ has continuous trajectories, a.s.
(c) $W$ has independent increments,
(d) $W(t) - W(s) \sim N(0, (t-s)Q)$, $t \geq s$.

Hence, $E W(0) = 0$ and $\text{Cov}(W(t)) = tQ$.

We can think the covariance matrix $Q$ as a $\infty \times \infty$ diagonal matrix, with diagonal elements $q_1, q_2, \ldots, q_n, \ldots$.

For any $a \in H$,

$$a = \sum_n < a, e_n > e_n$$

$$Qa = \sum_n < a, e_n > Qe_n = \sum_n q_n < a, e_n > e_n$$

We define, for $\gamma > 0$, especially for $\gamma \in (0, 1)$,

$$Q^\gamma a = \sum_n q_n^\gamma < a, e_n > e_n, \quad (19)$$

when the right hand side is defined.

Representations of Brownian motion in Hilbert space:

It is known that $W_t$ has an infinite series representation [13]:

$$W_t(\omega) = \sum_{n=1}^{\infty} \sqrt{q_n} W_n(t)e_n, \quad (20)$$
where
\[ W_n(t) := \begin{cases} \frac{(W(t), e_n)}{\sqrt{q_n}}, & q_n > 0, \\ 0, & q_n = 0. \end{cases} \] (21)

are the standard scalar independent Brownian motions. Namely, \( W_n(t) \sim N(0, t) \), \( EW_n(t) = 0 \), \( EW_n(t)^2 = t \) and \( EW_n(t)W_n(s) = \min(t, s) \).

This infinite series converges in \( L^2(\Omega) \), as long as \( Tr(Q) = \sum q_n < \infty \).

**Remark 5.** For example in \( H = L^2(0,1) \), we have an orthonormal basis \( e_n = \sin(n\pi x) \). In the above infinite series representation, taking derivative with respect to \( x \), we get
\[ \partial_x W_t(\omega) = \sum_{n=1}^{\infty} \sqrt{2(n\pi)}\sqrt{q_n}W_n(t) \cos(n\pi x). \] (22)

In order for this series to converge, we need \( \sqrt{2(n\pi)}\sqrt{q_n} \) converges to zero sufficiently fast as \( n \to \infty \). So \( q_n \) being small helps. In this sense, the trace \( Tr(Q) = \sum q_n \) may be seen as a measurement for spatial regularity of white noise \( W_t \): the smaller the trace \( Tr(Q) \), the more regular of the noise.

We do some calculations. For \( a, b \in H \), we have the following identities.
\[ \mathbb{E}(W_2, W_1) = \mathbb{E}\|W_1\|^2 = \mathbb{E}\left( \sum_{n=1}^{\infty} \sqrt{q_n}W_n(t)e_n, \sum_{n=1}^{\infty} \sqrt{q_n}W_n(t)e_n \right) \]
\[ = \sum_{n=1}^{\infty} q_n \mathbb{E}(W_n(t), W_n(t)) \]
\[ = t \sum_{n=1}^{\infty} q_n = t Tr(Q). \] (23)

\[ \mathbb{E}(W_1, a) = \langle 0, a \rangle = 0. \] (24)
\[ \mathbb{E}((W_t, a) \langle W_t, b \rangle) = \mathbb{E}[\sum_{n=1}^{\infty} \sqrt{\eta_n}W_n(t)e_n, a] \langle \sum_{n=1}^{\infty} \sqrt{\eta_n}W_n(t)e_n, b \rangle \]
\[ = \mathbb{E} \sum_{m,n} \sqrt{\eta_m \eta_n}W_m(t)W_n(t) < e_m, a > < e_n, b > \]
\[ = \sum_n t\eta_n < e_n, a > < e_n, b > \]
\[ = t \sum_n < e_n, a > < q_n e_n, b > \]
\[ = t \sum_n < e_n, a > < Q e_n, b > \]
\[ = t < Q \sum_n < e_n, a > e_n, b > \]
\[ = t < Q a, b > , \]

where we have used the fact that \( a = \sum_n < e_n, a > e_n \) in the final step.

In particular, taking \( a = b \), we obtain
\[ \mathbb{E}((W_t, a)^2) = t \langle Qa, a \rangle, \] (25)
\[ \text{Var}((W_t, a)) = t \langle Qa, a \rangle. \] (26)

More generally,
\[ \mathbb{E}(\langle W_t, a \rangle \langle W_s, b \rangle) = \min(t, s) < Qa, b > . \] (27)

Moreover,
\[ \mathbb{E}[W_t(x)W_s(y)] = \mathbb{E}\{\sum_{n=1}^{\infty} \sqrt{\eta_n}W_n(t)e_n(x) \sum_{m=1}^{\infty} \sqrt{\eta_m}W_m(s)e_m(y)\} \]
\[ = \sum_{n,m=1}^{\infty} \sqrt{\eta_n \eta_m} \mathbb{E}[W_n(t)W_m(s)]e_n(x)e_m(y) \]
\[ = \min(t, s) \sum_{n=1}^{\infty} q_n e_n(x)e_n(y) \]
\[ = \min(t, s)q(x, y), \] (28)

where
\[ q(x, y) = \sum_{n=1}^{\infty} q_n e_n(x)e_n(y). \]
On the other hand, the covariance operator may be represented in terms of $q(x, y)$:

$$ Qa = Q \sum_n < e_n, a > e_n = \sum_n < e_n, a > Qe_n $$

$$ = \sum_n < e_n, a > q_n e_n $$

$$ = \sum_n \int_0^1 a(y) e_n(y) dy q_n e_n(x) $$

$$ = \int_0^1 q(x, y) a(y) dy. \quad (29) $$

Sometimes we call the kernel function $q(x, y)$ the spatial correlation. The smoothness of $q(x, y)$ depends on the decaying property of $q_n$’s.

## 3 Stochastic Partial Differential Equations

### 3.1 Stochastic calculus in Hilbert space

We define the Ito stochastic integral:

$$ \int_0^T \Phi(s, \omega) dW_s. $$

Note that since $W_t$ takes values in Hilbert space $U$. The integrand $\Phi(t, \omega)$ is usually a linear operator from $U$ to $H$ (for each time $t$ and each sample $\omega$):

$$ \Phi : U \rightarrow H. $$

It is also possible to take $W_t$ as a scalar, real-valued Brownian motion. For example, in $\int_0^T u(s) dW_s$, if $W_t$ is a scalar Brownian motion, we can interpret the integrand $u$ as a multiplication operator.

For Brownian motion $W_t$ in $U$

$$ W_t(\omega) = \sum_{n=1}^\infty \sqrt{\alpha_n} W_n(t) e_n, \quad (30) $$

we define

$$ \int_0^T \Phi(s, \omega) dW_s(\omega) = \sum_{n=1}^\infty \sqrt{\alpha_n} \int_0^T \Phi(s, \omega) e_n dW_n(s). \quad (31) $$

A property of Ito integrals:

$$ E \int_0^T \Phi(s, \omega) dW_s(\omega) = 0. \quad (32) $$
3.2 Deterministic calculus in Hilbert space

In order to discuss more tools to handle stochastic calculus in Hilbert space, we need to recall some concepts of deterministic calculus.

For calculus in Euclidean space $\mathbb{R}^n$, we have concepts derivative and directional derivative. In Hilbert space, we have the corresponding Fréchet derivative and Gateaux derivative \cite{4, 64}.

Let $H$ and $\hat{H}$ be two Hilbert spaces, and $F : U \subset H \rightarrow \hat{H}$ be a map, whose domain of definition $U$ is an open subset of $H$. Let $L(H, \hat{H})$ be the set of all bounded linear operators $A : H \rightarrow \hat{H}$. In particular, $L(H) := L(H, H)$. We can also introduce a multilinear operator $A_1 : H \times H \rightarrow \hat{H}$. The space of all these multilinear operators is denoted as $L(H \times H, \hat{H})$.

**Definition 2.** The map $F$ is Fréchet differentiable at $u_0 \in U$ if there is a linear bounded operator $A : H \rightarrow \hat{H}$ such that

$$\lim_{h \rightarrow 0} \frac{\|F(u_0 + h) - F(u_0) - Ah\|}{\|h\|} = 0,$$

i.e.,

$$\|F(u_0 + h) - F(u_0) - Ah\| = o(\|h\|),$$

where $\|\cdot\|$ denotes norms in $H$ or $\hat{H}$ as appropriate. The linear bounded operator $A$ is called the Fréchet derivative of $F$ at $u_0$, and is denoted as $F'_u(u_0)$, or sometimes $F'(u_0)$.

If $F$ is linear, its Fréchet derivative is itself.

**Definition 3.** The directional derivative of $F$ at $u_0 \in U$ in the direction $h \in H$ is defined by the limit

$$\delta F(u_0; h) := \lim_{t \rightarrow 0} \frac{F(u_0 + th) - F(u_0)}{t}.$$  
If this limit exists for every $h \in H$, and $F'_u(u_0)h := \delta F(u_0; h)$ is a linear map, then we say that $F$ is Gateaux differentiable at $u_0$. The linear map $F'_G(u_0)$ is called the Gateaux derivative of $F$ at $u_0$.

In fact, if $F$ is Fréchet differentiable at $u_0$, then it is also Gateaux differentiable at $u_0$ and they are equal \cite{4, 64}:

$$F'_u(u_0) = F'_G(u_0).$$

But the converse is not usually true. It is true under suitable conditions; see \cite{4}, p. 68.

For any nonlinear map $F : U \subset H \rightarrow Y$, its Fréchet derivative $F'(u_0)$ is a linear operator, i.e., $F'(u_0) \in L(H, Y)$. Similarly, we can define higher order Fréchet derivatives. Each of these derivatives is a multilinear operator. For example,

$$f''(u_0) : H \times H \rightarrow Y,$$

$$(h, k) \mapsto f''(u_0)(h, k).$$
We denote
\[ f''(u_0)h^2 := f''(u_0)(h, h), \]
\[ f'''(u_0)h^3 := f'''(u_0)(h, h, h), \]
and similarly for higher order derivatives.

Then we have the Taylor expansion in Hilbert space
\[ f(u + h) = f(u) + f'(u)h + \frac{1}{2!}f''(u)h^2 + \cdots + \frac{1}{m!}f^{(m)}(u)h^m + R_{m+1}(u, h), \]
where the remainder
\[ R_{m+1}(u, h) = \frac{1}{(m + 1)!} \int_0^1 (1 - s)^m f^{(m+1)}(u + sh)h^{m+1}ds. \]

Remark 6. It is interesting to relate these two concepts with the classical concept of variational derivative (or functional derivative) that is used in the context of calculus of variations. The variational derivative is usually considered for functionals defined as spatial integrals, such as a Langrange functional in mechanics. For example,
\[ F(u) = \int_0^l G(u(x), u_x(x))dx, \]
where \( u \) is defined on \( x \in [0, l] \) and satisfies zero Dirichlet boundary condition at \( x = 0, l \). Then it is known [27] that
\[ F_u(u)h = \int_0^l \frac{\delta F}{\delta u} h(x)dx, \quad (33) \]
for \( h \) in the Hilbert space \( H^1_0(0, l) \). The quantity \( \frac{\delta F}{\delta u} \) is the classical variational derivative of \( F \). The equation (33) above gives the relation between Fréchet derivative and variational derivative.

### 3.3 Ito’s formula in Hilbert space

We get back to stochastic calculus in Hilbert space \( H \). We first look at the Ito’s formula; see [13] or [48].

**Theorem 1.** Let \( u \) be the solution of the SPDE
\[ du = b(u)dt + \Phi(u)dW_t, \quad u(0) = u_0. \quad (34) \]
Assume that \( F(t, u) \) be a given smooth (deterministic) function:
\[ F : [0, \infty) \times H \to \mathbb{R}^1. \]
Then
(i) Ito’s Formula: Differential form
\begin{equation}
\frac{dF}{dt}(t, u(t)) = F_t(t, u(t)) + \{F_t(t, u(t)) + F_u(t, u(t))(b(u(t)) + \frac{1}{2}Tr[F_{uu}(t, u(t))(\Phi(u(t))Q^\frac{1}{2})) (\Phi(u(t))Q^\frac{1}{2})^*]\}dt,
\end{equation}

where \(F_u\) and \(F_{uu}\) are Fréchet derivatives, \(F_t\) is the usual partial derivative in time, and \(^*\) denotes adjoint operator. This formula is understood with the following symbolic operations in mind:
\begin{align*}
\langle dt, dW_t \rangle &= \langle dt, dW_t \rangle = 0, \\
\langle dW_t, dW_t \rangle &= Tr(Q) dt,
\end{align*}

(ii) Ito’s Formula: Integral form
\begin{equation}
F(t, u(t)) = F(0, u(0)) + \int_0^t F_u(s, u(s))(\Phi(u(s))dW_s) \\
+ \int_0^t \{F_t(s, u(s)) + F_u(s, u(s))(b(u(s)) + \frac{1}{2}Tr[F_{uu}(s, u(s))(\Phi(u(s))Q^\frac{1}{2})) (\Phi(u(s))Q^\frac{1}{2})^*]\}ds,
\end{equation}

where \(F_u\) and \(F_{uu}\) are Fréchet derivatives, and \(F_t\) is the usual partial derivative in time. Moreover,
\begin{align*}
\int_0^t F_u(s, u(s))(\Phi(u(s))dW_s) &= \int_0^t \tilde{\Phi}(u(s))dW_s \\
and for all \(s, v \in H, \omega \in \Omega\), the operator \(\tilde{\Phi}(u(s))\) is defined by \\
\tilde{\Phi}(u(s))(v) &= F_u(s, u(s))(\Phi(u(s))v).
\end{align*}

Also,
\begin{equation}
Tr[F_{uu}(s, u(s))(\Phi(u(s))Q^\frac{1}{2})) (\Phi(u(s))Q^\frac{1}{2})^*] = Tr[(\Phi(u(s))Q^\frac{1}{2})^* F_{uu}(s, u(s)) (\Phi(u(s))Q^\frac{1}{2})].
\end{equation}

Note that for the symmetric non-negative covariance operator \(Q\) with eigenvalues \(q_n \geq 0\) and eigenvector \(e_n\), \(n = 1, 2, \cdots\), we have
\begin{equation}
Qu = \sum_n q_n(u, e_n)e_n, Q^\frac{1}{2}u = \sum_n \sqrt{q_n}(u, e_n)e_n.
\end{equation}

In fact, for a given function \(h : \mathbb{R} \to \mathbb{R}\), we define the operator \(h(Q)\) through the following natural formula \([63], p. 293-294\),
\begin{equation}
h(A)u = \sum_n h(q_n)(u, e_n)e_n,
\end{equation}
when the right hand side is defined.
Example 1. A typical application of Ito’s formula for SPDEs.

\[
du = b(u)dt + \Phi(u)dW_t, \quad u(0) = u_0. \tag{37}
\]

Take Hilbert space \( H = L^2(D), D \subset \mathbb{R}^n \), with the usual scalar product \( \langle u, v \rangle = \int_D udv \).

Energy functional \( F(u) = \frac{1}{2} \int_D u^2 dx = \frac{1}{2}\|u\|^2 \). In this case, \( F_u(h) = \int_D uhdx \) and \( F_{uu}(u)(h, k) = \int_D h(x)k(x)dx \).

\[
\frac{1}{2}d\|u\|^2 = \{\langle u, b(u) \rangle + \frac{1}{2}\text{Tr} \int_D [(\Phi(u)Q^\frac{1}{2})(\Phi(u)Q^\frac{1}{2})^*]dx\}dt + \langle u, \Phi(u)dW_t \rangle.
\]

Integrating and taking mathematical expectation, we obtain

\[
\frac{1}{2}\mathbb{E}\|u\|^2 = \frac{1}{2}\mathbb{E}\|u(0)\|^2 + \mathbb{E} \int_0^t \langle u, b(u) \rangle dt + \frac{1}{2}\mathbb{E} \int_0^t \text{Tr} \int_D [(\Phi(u(r))Q^\frac{1}{2})(\Phi(u(r))Q^\frac{1}{2})^*]dxdr
\]

Note that in this special case, \( F_u \) is a bounded operator in \( L(H, \mathbb{R}) \), which can be identified with \( H \) itself due to the Riesz representation theorem.

Example 2. Energy functional \( F(u) = \int_D |u|^{2p}dx = \int_D (|u|^2)^pdx \) for \( p \in [1, \infty) \). In this case, \( F_u(u_0)(h) = 2p\int_D |u_0|^{2p-2}u_0hdx \) and \( F_{uu}(u_0)(h, k) = 2p\int_D |u_0|^{2p-2}khdx + 4p(p-1)\int_D |u_0|^{2p-4}u_0h(x)u_0k(x)dx \).

Example 3. ([10], p. 153) Let \( H \) be a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \|\cdot\|^2 = \langle \cdot, \cdot \rangle \).

Consider an energy functional \( F(u) = \|u\|^{2p} \) for \( p \in [1, \infty) \). In this case, \( F_u(u_0)(h) = 2p\|u_0\|^{2p-2} < u_0, h > \) and

\[
F_{uu}(u_0)(h, k) = 2p\|u_0\|^{2p-2} < h, k > + 4p(p-1)\|u_0\|^{2p-4} < u_0, h > < u_0, k >
\]

\[
= 2p\|u_0\|^{2p-2} < h, k > + 4p(p-1)\|u_0\|^{2p-4} < (u_0 u_0)h, k >,
\]

where \( (a \otimes b)h := a < b, h > \); see [13] or [13], p25.

Stochastic product rule:
Let \( u \) and \( v \) be solutions of two SPDEs. Then

\[
d(uv) = udv + (du)v + dudv. \tag{38}
\]

Ito isometry:

\[
\mathbb{E}\| \int_0^t \Phi(t, \omega)dW_t \|^2 = \mathbb{E} \int_0^t \text{Tr} [(\Phi(r)Q^\frac{1}{2})(\Phi(r)Q^\frac{1}{2})^*]dr. \tag{39}
\]

Generalized Ito isometry:
\[ E(\int_0^a F(t, \omega) dW_t, \int_0^b G(t, \omega) dW_t) \]
\[ = \ E \int_{a \wedge b} \text{Tr}[(G(r, \omega)Q^\frac{1}{2})(F(r, \omega)Q^\frac{1}{2})^*)dr, \quad (40) \]

where \( a \wedge b = \min(a, b) \).

### 3.4 Stochastic partial differential equations

A general class of SPDEs may be written as

\[ du_t = [Au + f(u)] dt + G(u) dW_t, \quad (41) \]

where \( Au \) is the linear part, \( f(u) \) is the nonlinear part, \( G(u) \) the noise intensity (usually an operator), and \( W_t \) a Brownian motion. When \( G \) depends on \( u \), \( G(u) dW_t \) is called a multiplicative noise, otherwise it is an additive noise.

For general background on SPDEs, such as wellposedness and basic properties of solutions, see [13, 48, 51].

### 4 Correlation

In this section, we discuss correlation of solutions, at different time instants, of some linear SPDEs. We first recall some information about Fourier series in Hilbert space.

#### 4.1 Hilbert-Schmidt theory and Fourier series in Hilbert space

A separable Hilbert space \( H \) has a countable orthonormal basis \( \{e_n\}_{n=1}^\infty \). Namely, \( \langle e_m, e_n \rangle = \delta_{mn} \), where \( \delta_{mn} \) is the Kronecker delta function. Moreover, for any \( h \in H \), we have Fourier series expansion

\[ h = \sum_{n=1}^\infty \langle h, e_n \rangle e_n. \quad (42) \]

In the context of solving stochastic PDEs, we may chose to work on a Hilbert space with an appropriate orthonormal basis. This is naturally possible with the help of the Hilbert-Schmidt theory [63], p.232.

The Hilbert-Schmidt theorem ([63], p.232) says that a linear compact symmetric operator \( A \) on a separable Hilbert space \( H \) has a set of eigenvectors that form a complete orthonormal basis for \( H \). Moreover, all the eigenvalues of \( A \) are real, each non-zero eigenvalue has finite multiplicity, and two eigenvectors that correspond to different eigenvalues are orthogonal.
This theorem applies to a strong (self-adjoint) elliptic differential operator $B$

$$Bu = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u), \quad x \in D \subset \mathbb{R}^n,$$

where the domain of definition of $B$ is an appropriate dense subspace of $H = L^2(D)$, depending on the boundary condition specified for $u(x)$.

In this case, $A := B^{-1}$ is a linear symmetric compact operator in a Hilbert space, e.g., $H = L^2(D)$. We may consider $(L + aI)^{-1}$. This may be necessary in order for the operator to be invertible, i.e., no zero eigenvalue, such as in the case of Laplace operator with zero Neumann boundary condition.

By the Hilbert-Schmidt theorem, eigenvectors (also called eigenfunctions in this context) of $A = B^{-1}$ form an orthonormal basis for $H = L^2(D)$. Note that $A$ and $B$ share the same set of eigenfunctions. So we can claim that the strong elliptic operator $B$'s eigenfunctions form an orthonormal basis for $H = L^2(D)$.

In the case of one spatial variable, the elliptic differential operator is the so-called Sturm-Liouville operator [63], p.245. For example

$$Bu = -(pu')' + qu, \quad x \in (0, l),$$

where $p(x)$, $p'(x)$ and $q(x)$ are continuous on $(0, l)$. This operator arises in the method of separating variables for solving linear (deterministic) partial differential equations in the next section. By the Hilbert-Schmidt theorem, eigenfunctions of the Sturm-Liouville operator form an orthonormal basis for $H = L^2(0, l)$.

### 4.2 The wave equation with additive noise

Consider the stochastic wave equation with additive noise:

$$u_{tt} = c^2 u_{xx} + \epsilon W_t, \quad 0 < x < l, \quad t > 0$$

$$u(0, t) = u(l, t) = 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x),$$

where $\epsilon$ is a real parameter modeling the noise intensity, $c > 0$ is a constant (wave speed), and $W_t$ is a Brownian motion taking values in Hilbert space $H = L^2(0, l)$.

Method of eigenfunction expansion:

$$u = \sum_{n=1}^{\infty} u_n(t)e_n(x),$$

$$W_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} W_n(t)e_n(x),$$

where

$$e_n(x) = \sqrt{2/l} \sin \frac{n\pi x}{l}, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \cdots.$$
Putting these into the SPDE \( u_{tt} = c^2 u_{xx} + \epsilon W_t \) we obtain
\[ u_n(t) + c^2 \lambda_n u_n = \epsilon \sqrt{q_n} \dot{W}_n(t), n = 0, 1, 2, \cdots. \] (48)
For each \( n \), this second order SDE may be solved by converting to a linear system of first order SDEs [41]:
\[
\begin{align*}
    u_n(t) &= [A_n - \epsilon \frac{l}{cn\pi} \sqrt{q_n} \int_0^t \sin \frac{cn\pi}{l} s dW_n(s)] \cos \frac{cn\pi}{l} t \\
    &\quad + [B_n + \epsilon \frac{l}{cn\pi} \sqrt{q_n} \int_0^t \cos \frac{cn\pi}{l} s dW_n(s)] \sin \frac{cn\pi}{l} t
\end{align*}
\] (49)
with \( A_n \) and \( B_n \) constants.

The final solution is
\[
    u(x, t) = \sum_{n=1}^{\infty} \left[ A_n - \epsilon \frac{l}{cn\pi} \sqrt{q_n} \int_0^t \sin \frac{cn\pi}{l} s dW_n(s) \right] \cos \frac{cn\pi}{l} t \\
    + \left[ B_n + \epsilon \frac{l}{cn\pi} \sqrt{q_n} \int_0^t \cos \frac{cn\pi}{l} s dW_n(s) \right] \sin \frac{cn\pi}{l} t.
\] (50)
where the constants \( A_n \) and \( B_n \) are determined by the initial condition as follows
\[
    A_n = \langle f, e_n \rangle, \quad B_n = \frac{l}{cn\pi} \langle g, e_n \rangle.
\]
When the noise is at one mode, say at the first mode \( e_1(x) \) (i.e., \( q_1 > 0 \) but \( q_n = 0, \ n = 2, 3, \cdots \)), we see that the solution contains randomness only at that mode. So for the linear stochastic diffusion system, there is no interactions between modes. In other words, if we randomly force a few fast modes, then there is no impact on slow modes.

**Mean value** for the solution:
\[
    \mathbb{E} u(x, t) = \sum_{n=1}^{\infty} [A_n \cos \left( \frac{cn\pi t}{l} \right) + B_n \sin \left( \frac{cn\pi t}{l} \right)] e_n(x).
\] (51)

**Covariance** for the solution:
Now we calculate the covariance of solution \( u \) at different time instants \( t \) and \( s \), i.e., \( \mathbb{E} < u(x, t) - \mathbb{E} u(x, t), u(x, s) - \mathbb{E} u(x, s) > \).
Using the Itô’s isometry, we get
\[
    \begin{align*}
    \mathbb{E} < u(x, t) - \mathbb{E} u(x, t), u(x, s) - \mathbb{E} u(x, s) > &= \sum_{n=1}^{\infty} \frac{\epsilon^2 l^2 q_n}{c^2 n^2 \pi^2} \left[ \int_0^{t \wedge s} \sin^2 \frac{cn\pi r}{l} dr \cos \frac{cn\pi t}{l} \cos \frac{cn\pi s}{l} \\
    &\quad + \int_0^{t \wedge s} \cos^2 \frac{cn\pi r}{l} dr \sin \frac{cn\pi t}{l} \sin \frac{cn\pi s}{l} \\
    &\quad - \int_0^{t \wedge s} \sin \frac{cn\pi r}{l} \cos \frac{cn\pi r}{l} dr (\cos \frac{cn\pi t}{l} \sin \frac{cn\pi s}{l} + \cos \frac{cn\pi s}{l} \sin \frac{cn\pi t}{l}) \right]
    \end{align*}
\]
After integrations, we get the covariance as

\[
\text{Cov}(u(x,t), u(x,s)) = \mathbb{E} < u(x,t) - \mathbb{E} u(x,t), u(x,s) - \mathbb{E} u(x,s) > \\
= \sum_{n=1}^{\infty} \frac{e^{2i^2q_n}(t \wedge s) \cos \frac{cn\pi(t - s)}{l}}{2c^2u^2n^2\pi^2} \\
- \frac{l}{2cn\pi} \sin \frac{2cn\pi(t \wedge s)}{l} \cos \frac{cn\pi(t + s)}{l} \\
+ \frac{l}{2cn\pi} \cos \frac{2cn\pi(t \wedge s)}{l} \sin \frac{cn\pi(t + s)}{l} \\
- \frac{l}{2cn\pi} \sin \frac{cn\pi(t + s)}{l} \\
= \sum_{n=1}^{\infty} \frac{e^{2i^2q_n}(t \wedge s) \cos \frac{cn\pi(t - s)}{l}}{2c^2u^2n^2\pi^2} \\
+ \frac{l}{2cn\pi} \sin \frac{cn\pi(t + s - 2(t \wedge s))}{l} \\
- \frac{l}{2cn\pi} \sin \frac{cn\pi(t + s)}{l}.
\]

In particular, for \( t = s \) we get the variance.

**Variance** for the solution:

\[
\text{Var}(u(x,t)) = \sum_{n=1}^{\infty} \frac{e^{2i^2q_n}(t - \frac{l}{2cn\pi} \sin \frac{2cn\pi}{l}t)}{2c^2u^2n^2\pi^2}. \tag{52}
\]

**Energy evolution** for the solution:

\[
E(t) = \frac{1}{2} \int_0^l [u_t^2 + c^2u_x^2]dx. \tag{53}
\]

Taking time derivative,

\[
\dot{E}(t) = \int_0^l u_t[u_{tt} - c^2u_{xx}]dx = \epsilon \int_0^l u_t(x,t) \dot{W}_t(x)dx. \tag{54}
\]

Or in integral form,

\[
E(t) = E(0) + \epsilon \int_0^t \int_0^l u_s(x,t) dW_s(x)dx.
\]

Thus

\[
\mathbb{E}E(t) = E(0). \tag{55}
\]
\[
V \text{ar}(E(t)) = \varepsilon^2 \mathbb{E} \left( \int_0^t \int_0^t \partial_t u(x, s) dW_s dx \right)^2,
\] (56)

where \( W_t \) is in the following form
\[
W_t = W(t) = \sum_{n=1}^{\infty} \sqrt{q_n} W_n(t) e_n(x),
\] (57)

and \( \partial_t u \) can be written as

\[
\partial_t u = \sum \{ -A_n \frac{cn \pi}{l} \sin \frac{cn \pi t}{l} + B_n \frac{cn \pi}{l} \cos \frac{cn \pi t}{l} \\
+ \varepsilon \sqrt{q_n} \int_0^t \sin \frac{cn \pi s}{l} dW_n(s) \sin \frac{cn \pi t}{l} \\
+ \int_0^t \cos \frac{cn \pi s}{l} dW_n(s) \cos \frac{cn \pi t}{l} \} e_n(x).
\] (58)

Set \( \frac{cn \pi}{l} = \mu_n \) and rewrite

\[
\partial_t u = \sum \{ F_n(t) + \varepsilon \sqrt{q_n} \int_0^t (\sin \mu_n s \cdot \sin \mu_n t + \cos \mu_n s \cdot \cos \mu_n t) dW_n(s) \} e_n(x)
\]

\[
= \sum \{ F_n(t) + \varepsilon \sqrt{q_n} \int_0^t \cos \mu_n (t - s) dW_n(s) \} e_n(x),
\]

where

\[
F_n(t) := -A_n \mu_n \sin \mu_n t + B_n \mu_n \cos \mu_n t, \quad n = 1, 2, \ldots
\]

For the simplicity of notations, set

\[
G_n(t) := F_n(t) + \varepsilon \sqrt{q_n} \int_0^t \cos \mu_n (t - s) dW_n(s), \quad n = 1, 2, \ldots
\]

then we have

\[
\partial_t u = \sum G_n(t) e_n(x).
\]

Thus

\[
\mathbb{E} \left( \int_0^t \int_0^t \partial_t u(x, s) dW_s dx \right)^2 = \mathbb{E} \left( \int_0^t \int_0^t \sum_{n=1}^{\infty} \sqrt{q_n} e_n(x) \int_0^t u_s dW_n(s) dx \right)^2
\]

\[
= \mathbb{E} \left( \sum_{n=1}^{\infty} \sqrt{q_n} \int_0^t \int_0^t u_s e_n(x) dW_n(s) dx \right)^2
\]

\[
= \mathbb{E} \left( \sum_{n=1}^{\infty} \sqrt{q_n} \int_0^t \int_0^t e_n(x) \sum_{j=1}^{\infty} \int_0^t G_j(s) e_j(x) dW_n(s) \right)^2
\]

\[
= \mathbb{E} \left( \sum_{n=1}^{\infty} \sqrt{q_n} \int_0^t \sum_{j=1}^{\infty} G_j(s) \int_0^t e_n(x) e_j(x) dW_n(s) \right)^2
\]

\[
= \mathbb{E} \left( \sum_{n=1}^{\infty} \sqrt{q_n} \int_0^t G_n(s) dW_n(s) \right)^2
\]

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Putting these into the above SPDE (59), we further obtain the following system of SODEs:

\[ 0 = \sum_{n=1}^{\infty} q_n \mathbb{E} \int_0^t G_n^2(s) \, ds \]
\[ = \sum_{n=1}^{\infty} q_n \mathbb{E} \int_0^t \left[ F_n(s) + \varepsilon \sqrt{q_n} \int_0^s \cos \mu_n (s-r) \, dW_n(r) \right]^2 ds \]
\[ = \sum_{n=1}^{\infty} q_n \int_0^t F_n^2(s) \, ds + \mathbb{E} \sum_{n=1}^{\infty} \varepsilon^2 q_n^2 \int_0^t \left[ \int_0^s \cos \mu_n (s-r) \, dW_n(r) \right]^2 ds \]
\[ = \sum_{n=1}^{\infty} q_n \int_0^t F_n^2(s) \, ds + \sum_{n=1}^{\infty} \varepsilon^2 q_n^2 \int_0^t \left[ \int_0^s \cos^2 \mu_n (s-r) \, dr \right] ds \]
\[ = \sum_{n=1}^{\infty} q_n \left[ A_n^2 \mu_n^2 \left( \frac{t}{2} - \frac{1}{4 \mu_n} \sin 2\mu_n t \right) + B_n^2 \mu_n^2 \left( \frac{t}{2} + \frac{1}{4 \mu_n} \sin 2\mu_n t \right) \right] - \frac{1}{2} A_n B_n \mu_n (1 - \cos 2\mu_n t) + \sum_{n=1}^{\infty} \varepsilon^2 q_n^2 \frac{t^2}{4} + \frac{1}{8 \mu_n^2} (1 - \cos 2\mu_n t). \]

Therefore,

\[ \text{Var}(E(t)) = \sum_{n=1}^{\infty} \varepsilon^2 q_n [A_n^2 \mu_n^2 \left( \frac{t}{2} - \frac{1}{4 \mu_n} \sin 2\mu_n t \right) + B_n^2 \mu_n^2 \left( \frac{t}{2} + \frac{1}{4 \mu_n} \sin 2\mu_n t \right)] - \frac{1}{2} A_n B_n \mu_n (1 - \cos 2\mu_n t) + \sum_{n=1}^{\infty} \varepsilon^4 q_n^2 \frac{t^2}{4} + \frac{1}{8 \mu_n^2} (1 - \cos 2\mu_n t). \]

### 4.3 The diffusion equation with multiplicative noise

Consider the stochastic diffusion equations with zero Dirichlet boundary condition

\[ u_t = u_{xx} + \epsilon u \dot{w}_t, \quad 0 < x < 1, \quad (59) \]
\[ u(x, 0) = f(x), \quad (60) \]

where \( w_t \) is a scalar Brownian motion. We take Hilbert space \( H = L^2(0, 1) \) with an orthonormal basis \( e_n = \sqrt{2} \sin(n\pi x) \). We use the method of eigenfunction expansion:

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(t) e_n(x), \quad (61) \]
\[ u_{xx} = \sum_{n=1}^{\infty} u_n(t) e'_n(x) = \sum_{n=1}^{\infty} -u_n(t)(n\pi)^2 e_n(x). \quad (62) \]

Putting these into the above SPDE (59), with \( \lambda_n = (n\pi)^2 \), we get

\[ \sum_{n=1}^{\infty} \dot{u}_n(t) e_n(x) = \sum_{n=1}^{\infty} u_n(t) (-\lambda_n) e_n + \epsilon \sum_{n=1}^{\infty} u_n(t) e_n(x) \dot{w}_t. \quad (63) \]

We further obtain the following system of SODEs:

\[ du_n(t) = -\lambda_n u_n(t) + \epsilon u_n(t) dw(t), \quad n = 1, 2, 3, \ldots. \quad (64) \]
Thus
\[ u_n(t) = u_n(0) \exp((-\lambda_n - \frac{1}{2} \epsilon^2)t + \epsilon w(t)), \] (65)
where \( u_0 = \sum < u_0, e_n(x) > e_n(x) \). Therefore, the final solution is:
\[ u(x, t) = \sum a_n e_n(x) \exp(b_n t + \epsilon w_t), \] (66)
with \( a_n = < f(x), e_n(x) > \) and \( b_n = (-\lambda_n - \frac{1}{2} \epsilon^2) \).
Note that \( \mathbb{E} \exp(b_n t + \epsilon w_t) = \exp(b_n t) \mathbb{E} \exp(\epsilon w_t) = \exp(b_n t) \exp(\frac{1}{2} \epsilon^2 t) = \exp(-\lambda_n t) \). Therefore, we can find out the mean, variance, covariance and correlation of the solution:
\[ E(u(x, t)) = \sum a_n e_n(x) \exp(-\lambda_n t). \] (67)
\[ Var(u(x, t)) = \mathbb{E}(u(x, t) - E(u(x, t)), u(x, t) - E(u(x, t))) \]
\[ = \sum a_n^2 \exp(-2\lambda_n t) [\exp(\epsilon^2 t) - 1]. \] (68)
For \( \tau \leq t \), we have
\[ \mathbb{E} \exp\{\epsilon(w_t + w_\tau)\} = \mathbb{E} \exp\{\epsilon(w_t - w_\tau) + 2\epsilon w_\tau\} \]
\[ = \mathbb{E} \exp\{\epsilon(w_t - w_\tau)\} \cdot \mathbb{E} \exp\{2\epsilon w_\tau\} \]
\[ = \exp\left(\frac{1}{2} \epsilon^2 (t - \tau)\right) \cdot \exp\left(2\epsilon^2 \tau\right) \]
\[ = \exp\left(\frac{1}{2} \epsilon^2 \tau(t + \tau + 2(t \wedge \tau))\right), \]
therefore, by direct calculation, we can get
\[ Cov(u(x, t), u(x, \tau)) = \sum a_n^2 \exp\{-\lambda_n t + \frac{1}{2} \epsilon^2 ((t + \tau) + 2(t \wedge \tau))\} \]
\[ + \exp(-\lambda_n \tau + b_n t + \frac{1}{2} \epsilon^2 t) - \exp(-\lambda_n t + b_n t + \frac{1}{2} \epsilon^2 \tau) \]
\[ = \sum a_n^2 \exp\{-\lambda_n (t + \tau)\} [\exp(\epsilon^2 (t \wedge \tau)) - 1] \]
and
\[ Corr(u(x, t), u(x, \tau)) = \frac{Cov(u(x, t), u(x, \tau))}{\sqrt{Var(u(x, t))} \sqrt{Var(u(x, \tau))}} \]
\[ = \frac{\sum a_n^2 \exp\{-\lambda_n (t + \tau)\} [\exp(\epsilon^2 (t \wedge \tau)) - 1]}{\sqrt{\sum a_n^2 \exp(-2\lambda_n t) [\exp(\epsilon^2 t) - 1]} \sqrt{\sum a_n^2 \exp(-2\lambda_n \tau) [\exp(\epsilon^2 \tau) - 1]}} \]

5 **Lyapunov Exponents**

Lyapunov exponents are tools for quantifying growth or decay of linear systems (e.g., PDEs or SPDEs). The following discussions are from [7][32].
5.1 A deterministic PDE system

Let us first look at the following deterministic PDE:

\[
\frac{\partial u}{\partial t} = u_{xx} + \alpha u, \tag{69}
\]

\[
u(0, x) = f(x), \tag{70}
\]

\[
u(t, x) = 0, \; x \in \partial D \tag{71}
\]

where \(D = \{x : 0 \leq x \leq 1\}\) and the function \(f \in L^2(0, 1)\). An orthonormal basis for \(L^2(0, 1)\) is \(\{e_n(x)\}, n = 0, 1, 2, \ldots\), \(\partial_{xx}e_j = -\lambda_j e_j\). Note that \(0 \leq \lambda_j \uparrow \infty\).

We then can write:

\[
f = \sum_{j=0}^{\infty} f_j e_j, \text{ where } f_j = \langle f, e_j \rangle. \tag{72}
\]

By using the method of eigenfunction expansion, it is known that the unique solution to the problem is given below:

\[
u(t, x) = \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x), \; t \geq 0. \tag{73}
\]

**Theorem 2.** Let us fix an initial condition \(f, f \neq 0\). Let \(j_0\) be the smallest integer \(j \geq 0\) in the expansion (72) of \(f\) such that \(f_{j_0} \neq 0\). Then the Lyapunov exponent of the system (69) – (71) exists as a limit and is given by

\[
\lambda^u(f) = -\lambda_{j_0} + \alpha. \tag{74}
\]

**Proof.** For a class of initial conditions \(f\) we calculate the Lyapunov exponents, which are defined as

\[
\lambda^u(f) = \limsup_{t \to \infty} \frac{1}{t} \log \|u(t)\|_{L^2}. \tag{75}
\]

By applying (73), we obtain the Lyapunov exponents regarding to PDE system (69) – (71),

\[
\lambda^u(f) = \limsup_{t \to \infty} \frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\|. \tag{76}
\]

On the one hand,

\[
\frac{1}{t} \log \left\| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right\| \leq \frac{1}{t} \log \left( \sum_{j=j_0}^{\infty} |\exp(t(-\lambda_{j_0} + \alpha)) f_j|^2 \right)^{1/2} = -\lambda_{j_0} + \alpha + \frac{1}{t} \log \|f\|. \tag{77}
\]
On the other hand,
\[
\frac{1}{t} \log \left| \sum_{j=0}^{\infty} \exp(t(-\lambda_j + \alpha)) f_j e_j(x) \right| \geq \frac{1}{t} \log |\exp(t(-\lambda_{j_0} + \alpha)) f_{j_0}| \\
= -\lambda_{j_0} + \alpha + \frac{1}{t} \log |f_{j_0}|. \tag{78}
\]

\[\square\]

5.2 A SPDE system

We now consider the following SPDE
\[
dv = (v_{xx} + \beta v) dt + \gamma v dw_t, \tag{79}
\]
\[
v(0, x, \omega) = f(x), x \in D, \tag{80}
\]
\[
v(t, x, \omega) = 0, x \in \partial D, \tag{81}
\]
where \(w_t\) is a scalar Brownian motion. The conditions (80) and (81) hold for a. a. \(\omega \in \Omega\).

We seek the solution with expansion with respect to the basis \(\{e_j\}\) (see the last subsection)
\[
v(t, x) = \sum_{j=0}^{\infty} y_j(t) e_j(x), \tag{82}
\]
where \(y_j(t), j = 0, 1, 2, \ldots\) satisfy the following stochastic ordinary differential equations:
\[
dy_j(t) = (-\lambda_j + \beta) y_j(t) dt + \gamma y_j(t) dw_t, \tag{83}
\]
\[
y_j(0) = f_j. \tag{84}
\]

So
\[
y_j(t) = \exp(\gamma w_t) \exp \left( \left(-\lambda_j + \beta - \frac{1}{2} \gamma^2 \right) t \right) f_j. \tag{85}
\]

Thus from (82), we obtain,
\[
v(t, x) = \sum_{j=0}^{\infty} \exp(\gamma w_t) \exp \left( \left(-\lambda_j + \beta - \frac{1}{2} \gamma^2 \right) t \right) f_j e_j. \tag{86}
\]

Observe that
\[
v(t, x) = \exp(\gamma w_t) \exp \left( \left(\beta - \alpha \frac{1}{2} \gamma^2 \right) t \right) u(t, x), \tag{87}
\]
where \(u(t, x)\) is the solution to the above deterministic PDE (69) - (71).
By \([86]\), we can calculate the Lyapunov exponent of the stochastic system \([79] - [81]\) as a function of the Lyapunov exponent of the deterministic system \([69] - [71]\) as follows:

\[
\lambda^v(f) = \lim_{t \to \infty} \frac{1}{t} \log \|v(t)\| = \lim_{t \to \infty} \frac{1}{t} \log \left\| \exp(\gamma w_t) \exp \left( (\beta - \alpha) - \frac{1}{2} \gamma^2 \right) u(t) \right\| = \lambda^u(f) + (\beta - \alpha) - \frac{1}{2} \gamma^2, \text{ a.s.} \tag{87}
\]

by the strong law of large number.

Let us state the result in the following theorem.

**Theorem 3.** Let \(f \neq 0\). Then the Lyapunov exponent of the SPDE \([79] - [81]\) almost surely exists as a limit, is non-random and is given in the following formula:

\[
\lambda^v(f) = \lambda^u(f) + (\beta - \alpha) - \frac{1}{2} \gamma^2, \text{ a.s.} \tag{88}
\]

**Remark 7.** Let us consider a special case when \(\alpha = \beta\). Then by the above theorem, for a fixed initial condition \(f\), the Lyapunov exponent of the stochastic system \([79] - [81]\) is

\[
\lambda^v(f) = \lambda^u(f) - \frac{1}{2} \gamma^2, \tag{89}
\]

which obviously is smaller than the Lyapunov exponent of the corresponding deterministic system \([69] - [71]\). The result implies that this stochastically perturbed system is more stable than the original deterministic system.

### 6 Impact of Uncertainty

In this section, we first recall some inequalities for estimating solutions of SPDEs, and then we estimate the impact of noises on solutions of the nonlinear Burgers equation.

#### 6.1 Differential and integral inequalities

**Gronwall inequality:** Differential form \([56]\)

Assuming that \(y(t) \geq 0\), \(g(t)\) and \(h(t)\) are integrable, if \(\frac{dy}{dt} \leq g(t)y + h(t)\) for \(t \geq t_0\), then

\[
y(t) \leq y(t_0)e^{\int_{t_0}^t g(\tau)d\tau} + \int_{t_0}^t h(s)[e^{\int_s^t g(\tau)d\tau}]ds, \quad t \geq t_0.
\]

In particular, if \(\frac{dy}{dt} \leq gy + h\) for \(t \geq t_0\) with \(g, h\) being constants and \(t_0 = 0\), we have

\[
y(t) \leq y(0)e^{gt} - \frac{h}{g}(1 - e^{gt}), \quad t \geq 0.
\]
Note that when constant \( g < 0 \), then \( \lim_{t \to \infty} y(t) = -\frac{h}{g} \).

**Gronwall inequality**: Integral form [11, 24]

If \( u(t), v(t) \) and \( c(t) \) are all non-negative, \( c(t) \) is differentiable, and \( v(t) \leq c(t) + \int_{t_0}^{t} u(s)v(s)ds \) for \( t \geq t_0 \), then

\[
v(t) \leq v(t_0)e^{\int_{t_0}^{t} u(\tau)d\tau} + \int_{t_0}^{t} c'(s)[e^{\int_{s}^{t} u(\tau)d\tau}]ds, \quad t \geq t_0.
\]

In particular, assuming that \( y(t) \geq 0 \) and is continuous and \( y(t) \leq C + K \int_{0}^{t} y(s)ds \), with \( C, K \) being positive constants, for \( t > 0 \). Then

\[
y(t) \leq Ce^{Kt}, \quad t \geq 0.
\]

### 6.2 Sobolev inequalities

We first introduce some common Sobolev spaces. For \( k = 1, 2, \cdots \), we define

\[
H^k(0, l) := \{ f : f, f', \cdots, f^{(k)} \in L^2(0, l) \}
\]

Each of these is a Hilbert space with the scalar product

\[
\langle u, v \rangle_k = \int_{0}^{l} [uv + u'v' + \cdots + u^{(k)}v^{(k)}]dx,
\]

and the norm

\[
\| u \|_k = \sqrt{\langle u, u \rangle_k} = \sqrt{\int_{0}^{l} [u^2 + (u')^2 + \cdots + (u^{(k)})^2]dx}.
\]

For \( k = 1, 2, \cdots \) and \( p \geq 1 \), we further define another class of Sobolev spaces

\[
W^{k,p}(D) = \{ u : u, Du, \cdots, D^\alpha u \in L^p(D), |\alpha| \leq k \},
\]

with norm

\[
\| u \|_{k,p} = (\| u \|_{L^p}^p + \| u' \|_{L^p}^p + \cdots + \| u^{(k)} \|_{L^p}^p)^{\frac{1}{p}}.
\]

Moreover, \( H^k_0(0, l) \) denotes the closure of \( C_c^\infty(0, l) \) in \( H^k(0, l) \) (i.e., under the norm \( \| \cdot \|_k \)). It is a sub-Hilbert space in \( H^k(0, l) \). Similarly, \( W^{k,p}_0(0, l) \) denotes the closure of \( C_c^\infty(0, l) \) in \( W^{k,p}(0, l) \) (i.e., under the norm \( \| \cdot \|_{k,p} \)). It is a sub-Hilbert space in \( W^{k,p}(0, l) \).

Standard abbreviations \( L^2 = L^2(D), H^k_0 = H^k_0(D), k = 1, 2, \cdots, \) are used for the common Sobolev spaces in fluid mechanics, with \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denoting the usual (spatial) scalar product and norm, respectively, in \( L^2(D) \):

\[
\langle f, g \rangle := \int_{D} fgdxdy, \quad \| f \| := \sqrt{\langle f, f \rangle} = \sqrt{\int_{D} f(x,y)dxdy}.
\]

**Cauchy-Schwarz inequality**:
In the space $L^2(D)$ of square-integrable functions defined on a domain $D \subset \mathbb{R}^n$:

$$| \int_D f(x)g(x)dx | \leq \sqrt{\int_D f^2(x)dx} \sqrt{\int_D g^2(x)dx}.$$  

**Hölder inequality:**

In the space $L^r(D)$ of functions defined on a domain $D \subset \mathbb{R}^n$:

$$| \int_D f(x)g(x)dx | \leq ( \int_D |f(x)|^r dx )^{\frac{1}{r}} ( \int_D |g(x)|^q dx )^{\frac{1}{q}}.$$  

**Minkowski inequality:**

In the space $L^p(D)$ of functions defined on a domain $D \subset \mathbb{R}^n$:

$$\left( \int_D |f(x) \pm g(x)|^p dx \right)^{\frac{1}{p}} \leq \left( \int_D |f(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_D |g(x)|^p dx \right)^{\frac{1}{p}}.$$  

**Poincaré inequality** [56]:

For $g \in H_0^1(D)$,

$$\|g\|^2 = \int_D g^2(x, y) dxdy \leq \frac{|D|}{\pi} \int_D |\nabla g|^2 dxdy = \frac{|D|}{\pi} \|\nabla g\|^2,$$

where $|D|$ is the Lebesgue measure of the domain $D$.

For $u \in W_0^{1,p}(D)$, $1 \leq p < \infty$ and $D \subset \mathbb{R}^n$ a bounded domain

$$\|u\|_p \leq C \|\nabla u\|_p,$$

where $C$ is a positive constant depending only on the domain $D$.

Let $u \in W^{1,p}(D)$, $1 \leq p < \infty$ and $D \subset \mathbb{R}^n$ a bounded convex domain. Let $S \subset D$ be a measurable subset, and define the spatial average of $u$ over $S$ by $u_S = \frac{1}{|S|} \int_D u dx$ (here $|S|$ is the volume or Lebesgue measure of $S$). Then

$$\|u - u_S\|_p \leq C \|\nabla u\|_p,$$

where $C$ is a positive constant depending only on the domain $D$ and $S$.

**Agmon inequality** [56]:

Let $D \subset \mathbb{R}^n$. There exists a constant $C$ depending only on domain $D$ such that

$$\|u\|_{L^\infty(D)} \leq C \|u\|_{H^{\frac{n+1}{2}}(D)}^\frac{1}{2} \|u\|_{H^{\frac{n+1}{2}}(D)}^\frac{1}{2},$$

for $n$ odd,

$$\|u\|_{L^\infty(D)} \leq \|u\|_{H^{\frac{n-2}{2}}(D)}^{\frac{1}{4}} \|u\|_{H^{\frac{n-2}{2}}(D)}^{\frac{1}{4}},$$

for $n$ even.

In particular, for $n = 1$ and $u \in H^1(0, l)$,

$$\|u\|_{L^\infty(0, l)} \leq C \|u\|_{L^2(0, l)} \|u\|_{H^1(0, l)}.$$

Moreover, for $n = 1$ and $u \in H^1_0(0, l)$,

$$\|u\|_{L^\infty(0, l)} \leq C \|u\|_{L^2(0, l)} \|u_x\|_{L^2(0, l)}.$$
6.3 Stochastic Burgers equation

We now consider the Burgers equation with additive noise forcing as in [5]:

\[ \partial_t u + u \cdot \partial_x u = \nu \partial_x^2 u + \sigma \dot{W}_t \]

\( u(\cdot, 0) = 0, \quad u(\cdot, l) = 0, \quad u(x, 0) = u_0(x), \)

where \( W_t \) is a Brownian motion, with covariance \( Q \), taking values in the Hilbert space \( L^2(0, l) \) with the usual scalar product \( \langle \cdot, \cdot \rangle \). We assume that the trace \( Tr(Q) \) is finite. So \( \dot{W}_t \) is noise colored in space but white in time.

Taking \( F(u) = \frac{1}{2} \int_0^1 u^2 dx = \frac{1}{2} \langle u, u \rangle \) and applying the Ito’s formula, we obtain

\[ \frac{1}{2} d\|u\|^2 = \langle u, \sigma dW_t \rangle + [\langle u, \nu u_{xx} - uu_x \rangle + \frac{1}{2} \sigma^2 \| u \|^2] dt. \]  

Thus

\[ \frac{d}{dt} E\|u\|^2 = 2E\langle u, \nu u_{xx} - uu_x \rangle + \sigma^2 \| u \|^2 \mbox{ } \| \mbox{ } l \mbox{ } Tr(Q) \]

\[ = -2\nu E\|u_x\|^2 + \sigma^2 \| u \|^2 \mbox{ } l \mbox{ } Tr(Q). \]

By the Poincare inequality \( \| u \|^2 \leq c\| u_x \|^2 \) for some positive constant depending only on the interval \( (0, l) \), we have

\[ \frac{d}{dt} E\|u\|^2 \leq -\frac{2\nu}{c} E\|u\|^2 + \sigma^2 \| u \|^2 \mbox{ } l \mbox{ } Tr(Q). \]

Then using the Gronwall inequality, we finally get

\[ E\|u\|^2 \leq E\|u_0\|^2 e^{-\frac{2\nu}{c}t} + \frac{1}{2} c \sigma^2 \| u \|^2 l \mbox{ } Tr(Q) ] \mbox{ } [1 - e^{-\frac{2\nu}{c}t}]. \]  

Note that the first term in this estimate involves the initial data, and the second term involves the noise intensity \( \sigma \) as well as the trace of the noise covariance.

We finally consider the Burgers equation with multiplicative noise forcing.

\[ \partial_t u + u \cdot \partial_x u = \nu \partial_x^2 u + \sigma u \dot{w}_t, \]

with the same boundary condition and initial condition as above, where \( W_t \) is a scalar Brownian motion (e.g., with covariance \( Q = 1 \) and the trace \( Tr(Q) = 1 \)).

So \( \dot{W}_t \) is noise homogeneous in space but white in time.

By the Ito’s formula, we obtain

\[ \frac{1}{2} d\|u\|^2 = \langle u, \sigma ud\dot{w}_t \rangle + [\langle u, \nu u_{xx} - uu_x \rangle + \frac{1}{2} \sigma^2 \| u \|^2] dt. \]

Thus

\[ \frac{d}{dt} E\|u\|^2 = 2E\langle u, \nu u_{xx} - uu_x \rangle + \sigma^2 E\|u\|^2 \]

\[ = -2\nu E\|u_x\|^2 + \sigma^2 E\|u\|^2 \]

\[ \leq (\sigma^2 - \frac{2\nu}{c}) E\|u\|^2. \]  

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Therefore,

\[ E\|u\|^2 \leq E\|u_0\|^2 e^{(\sigma^2 - 2\nu c)t}. \] (99)

Note here that the multiplicative noise affects the mean energy growth or decay rate, while the additive noise affects the mean energy upper bound.

### 6.4 Likelihood for staying bounded

By the Chebyshev inequality, we can estimate the likelihood of solution orbits staying inside or outside a bounded domain in Hilbert space \( H = L^2(0,l) \). Taking the bounded domain as a ball centered at the origin with radius \( \delta > 0 \). For example, for the above Burgers equation with multiplicative noise, we have

\[ \mathbb{P}(\omega : \|u\| \geq \delta) \leq \frac{1}{\delta^2} E\|u\|^2 \leq \frac{E\|u_0\|^2}{\delta^2} e^{(\sigma^2 - 2\nu c)t}. \] (100)

and

\[ \mathbb{P}(\omega : \|u\| < \delta) = 1 - \mathbb{P}(\omega : \|u\| \geq \delta) \geq 1 - \frac{E\|u_0\|^2}{\delta^2} e^{(\sigma^2 - 2\nu c)t}. \] (101)

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