Structure of Multi-meron Knot Action

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Abstract

We consider the structure of multi-meron knot action in the Yang-Mills theory and in the \( \text{CP}^1 \) Ginzburg-Landau (GL) model. Self-dual equations have been obtained without identifying orientations in the space-time and in the color space. The dependence of the energy bounds on topological parameters of coherent states in planar systems is also discussed. In particular, it is shown that a characteristic size of a knot in the Faddeev-Niemi model is determined by the Hopf invariant.

1 Introduction

Faddeev's conjecture that the energy of the ground state and properties of low-energy excitations in the non-Abelian field theory are determined by topological invariants of knots and links has been recently developed using the n-field model. One of these topological parameters is the Hopf invariant, which determines the knotting degree of filamental manifolds, where the unit vector \( \mathbf{n} \) is defined.

The study of the behavior of a vortex filament tangle is an active area of research in the field theory and attracts attention due to several reasons. The topological order associated with linking at short distances exists against the background of disorder due to free motion of the tangle filaments. Therefore, such systems of entangled filaments contain data on their behavior in the ultraviolet, as well as infrared limit, while systems of point particles do not contain.

In the present paper we discuss some properties of field configurations within the \( SU(2) \) Yang-Mills theory and within the \( CP^1 \) GL model from this point of view.

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Let us consider the Langrangian density,

\[ L = \frac{1}{4g^2} Tr F_{\mu\nu}^2, \]

\[ F_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu], \quad \hat{A}_\mu = A^a_\mu \frac{g^a}{2i}. \]

We will use special anzatz [4, 5] for the potential \( A^a_\mu \):

\[ A^a_\mu = \varepsilon^{abc} \partial_\mu n^b n^c (K + 1). \]

Here \( g \) is the bare coupling constant, \( K(r,t) \) is the unknown scalar function, \( n \) is the unit vector field, and \( \sigma^a \) denotes Pauli matrices. Using Eq.(3), one can get

\[ L = \frac{1}{2g^2} \left\{ \delta_{\mu\nu} (\partial_\lambda n^2)^2 - (\partial_\mu n)(\partial_\nu n) \right\} \partial^\mu K \partial^\nu K + \]

\[ \frac{1}{4g^2} \{ n \cdot [\partial_\mu n, \partial_\nu n] \}^2 (K^2 - 1)^2. \]

The Lagrangian density in the complete parametrization of the potential \( A^a_\mu \) was computed in Refs.[5]. It is seen from Eq.(4) that multipliers in curly brackets of the kinetic and potential parts of Eq.(4) play the role of the coupling constants for the \( K \)-field and vice versa. This takes place when \( n \neq r/r \) and the foregoing soft version is applicable. Let us fix the \( n \)-field dynamics by the condition \( n = r/r \). Then we will get a significantly simplified problem. Indeed, in this case \( A^a_\mu = \varepsilon^{abc} n^c (K + 1)/r \) and

\[ \{ \delta_{\mu\nu} (\partial_\lambda n^2)^2 - (\partial_\mu n)(\partial_\nu n) \} = \frac{\delta_{\mu\nu}}{r^2}, \quad (n \cdot [\partial_\mu n, \partial_\nu n])^2 = \frac{1}{r^4}. \]

The Lagrangian density and the equation of motion for the scalar field \( K(r,t) \) takes the form

\[ L = \frac{1}{2g^2r^2} (\partial_\mu K)^2 - \frac{1}{4g^2r^4} (K^2 - 1)^2, \]

\[ r^2 \Box K = K(K^2 - 1), \quad \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}. \]

The simplest solution [6] is the meron configuration \( K = t/\sqrt{r^2 - r^2} \). Multimeron configurations of the \( K \)-field were studied in detail in Ref.7. Common property of the solutions of Eq.(6) is existence of singularities of the field \( K \) on closed surfaces located at a finite distance from the point \( r = 0 \). Their existence makes it impossible to consider the infrared limit in the foregoing case, when degrees of freedom of the \( n \)-field are frozen.

A fixed scalar degree of freedom, which is connected with the \( K \)-field can be obtained by integrating over this variable in the Feynman integral. This
problem was investigated in Refs.\cite{5, 8, 9, 10}. As a result, the main contribution to the model in Ref.\cite{1} is characterized by the Lagrangian density

\[ L = c_1 (\partial_\mu n)^2 + c_2 (n \cdot [\partial_\mu n, \partial_\nu n])^2, \]  

(7)

where \( c_1 \) and \( c_2 \) are effective coupling constants. The first term in Eq.(8) describes the infrared limit of the \( n \)-field dynamics, while the second determines the behavior at a short distance, where topological effects of links are important.

Now let us proceed to the general case. This can be done by considering the self-dual equations using the anzatz in Eq.(3). Note that in this case of low symmetry \cite{11} these equations differ from those obtained for spherically-symmetrical \cite{12, 13} or axially-symmetrical \cite{14} cases. Thus, we have

\[ [\partial_\tau n, \partial_k n] = \frac{1}{2} \varepsilon_{kms} [\partial_m n, \partial_s n], \]  

(8)

\[ \partial_\tau K \partial_k n - \partial_k K \partial_\tau n = \frac{1}{2} \varepsilon_{kms} \{ \partial_m K \partial_s n - \partial_s K \partial_m n \}, \]  

(9)

where \( \tau = it \) is the Euclidean time. Since the homotopic group \( \pi_4(SU(2)) = \mathbb{Z}_2 \) is non-trivial \cite{15}, we hope that there exists at least one non-trivial class of configurations of fields \( K \) and \( n \).

3 The \( CP^1 \) Ginsburg-Landau model

We will use the GL functional,

\[ F = \int d^3x \left[ \sum_\alpha \left( \frac{1}{2m} \left| \hbar \partial_k + i \frac{2e}{c} A_k \right| \Phi_\alpha \right)^2 + \sum_\alpha \left( -b_\alpha |\Phi_\alpha|^2 + \frac{c_\alpha}{2} |\Phi_\alpha|^4 + \frac{B^2}{8\pi} \right) \right], \]  

(10)

with a two-component order parameter \( \Phi_\alpha = \sqrt{2m} \rho \chi_\alpha, \quad \chi_\alpha = |\chi_\alpha| e^{i\phi_\alpha} \), which satisfies the \( CP^1 \) condition, \( |\chi_1|^2 + |\chi_2|^2 = 1 \). In the \( (3+0) \)-dimensional case, Eq.(10) has the meaning of free energy and \( \Phi_\alpha \) is a two-component order parameter, which is used either in the context of two-gap superconductivity \cite{16} or in the non-Abelian field theory as a Higgs doublet \cite{17}. In \( (2+1) \)D systems, Eq.(10) has the meaning of action, and the order parameter \( \Phi_\alpha \) gives a two-dimensional non-Abelian representation of the braid group, which is a specific group of symmetry for planar systems at permutations of particles inside them.

As was shown in Ref.\cite{16}, there exists exact mapping of model (10) into the following version of the \( n \)-field model:

\[ F = \int d^3x \left[ \frac{1}{4} \rho^2 (\partial_k n)^2 + (\partial_k \rho)^2 + \frac{1}{16} \rho^2 c^2 + (F_{ik} - H_{ik})^2 + V(\rho, n_3) \right]. \]  

(11)
The equation (11) was obtained with the use of gauge invariant order parameter fields of the unit vector \( n^a = \tilde{\chi} \sigma^a \chi \), where \( \tilde{\chi} = (\chi_1^*, \chi_2^*) \) and the velocity \( \mathbf{c} = \mathbf{J}/\rho^2 \). The total current, \( \mathbf{J} = 2\rho^2(\mathbf{j} - 4\mathbf{A}) \), has paramagnetic (\( j = i[\chi_1 \nabla \chi_1^* - \text{c.c.} + (1 \to 2)] \)) and diamagnetic (-4\( \mathbf{A} \)) parts. Besides, we used in Eq.(11) the following notations: \( F_{ik} = \partial_i c_k - \partial_k c_i \), and \( H_{ik} = \mathbf{n} \cdot [\partial_i \mathbf{n} \times \partial_k \mathbf{n}] := \partial_i a_k - \partial_k a_i \), and the dimensionless units: \( L = (\xi_1 + \xi_2)/2 \) as the unit of length, \( \xi_0 = \hbar/\sqrt{2mb_0} \) as the coherence length; \( \hbar/L \) as the unit of \( c \); \( |\mathbf{c}|^2/(512\pi e^2 L^2) \) as the unit of \( \rho^2 \), and finally, \( \gamma/L \) with \( \gamma = (\hbar c/e)^2/512\pi \) as the unit of energy.

Let us now enumerate some non-trivial (\( \mathbf{n} \neq \text{const} \)) situations that follow from Eq.(11) as a result of competition of the order parameters \( \rho, \mathbf{n} \), and \( \mathbf{c} \):

(i) \( \mathbf{c} = 0, \rho = \text{const}; \)
(ii) \( \mathbf{c} = 0, \rho \neq \text{const}; \)
(iii) \( \mathbf{c} \neq 0, \rho = \text{const}; \)
(iv) \( \mathbf{c} \neq 0, \rho \neq \text{const}. \)

In the case when \( \mathbf{c} \neq 0 \), \( \rho \neq \text{const} \), and \( \mathbf{n} = \text{const} \) we have the problem of a vortex state structure in the classical GL model.

In the limit (i) we get Eq.(7) \( \square \). A numerical study of knotted configurations of the \( \mathbf{n} \)-field in this model was performed in Refs. \( \square \). The lower energy bound in this case, \( F \geq 32\pi^2 |Q|^{3/4} \), is determined \( \square \square \) by the Hopf invariant,

\[
Q = \frac{1}{16\pi^2} \int d^3 x \varepsilon_{ikl} a_i \partial_k a_l .
\]

If the compactification \( \mathbb{R}^3 \to S^3 \) is used and \( \mathbf{n} \in S^2 \), then the integer \( Q \in \pi_3(S^2) = \mathbb{Z} \) shows the degree of linking or knotting of filamental manifolds, where the vector field \( \mathbf{n}(x, y, z) \) is defined. In particular, for two linked rings (Hopf linking) \( Q = 1 \), for the trefoil knot \( Q = 6 \), etc. It is important to note that \( \pi_3(CP^M) = 0 \) at \( M > 1 \) and \( \pi_3(CP^1) = \pi_3(S^2) = \mathbb{Z} \). In the latter case, the order parameter is two-component \( \square \), and linked or knotted soliton configurations are labeled by the Hopf invariant in Eq.(13).

The characteristic size \( R_Q \) of a knot can be found using Eq.(12) and the minimum value of the free energy \( F_{\text{min}} = 2\sqrt{c_1 c_2} \) for the radius \( R_{\text{min}} = \sqrt{c_2/c_1} \). Therefore,

\[
R_Q = \frac{c_2}{16\pi^2 |Q|^{3/4}} .
\]

We see that the characteristic size of such a knot is determined by a combination of dynamical \( c_2 \) and topological \( Q \) features of the system. In case (ii), we have the problem of a soft version of the \( \mathbf{n} \)-field model \( \square \). Case (iv) is most general.

We will focus now on limit (iii). Let us assume that \( \rho \) can be found from the minimum value of the potential \( V(\rho) \) and the velocity \( \mathbf{c} \) is not equal to zero. Eq.(11) in this case takes the following form:

\[
F = F_n + F_v - F_{\text{int}} = \int d^3 x \left[ (\partial_k \mathbf{n})^2 + H_{ik}^2 \right] + \left( \frac{1}{4} \mathbf{c}^2 + F_{ik}^2 \right) - 2F_{ik} H_{ik} .
\]
It is seen from Eq.(15) that the superconducting state with $c \neq 0$ has the energy, which is less than the minimum in the case (i) due to renormalization of the coefficient (= 1) in the second term of the functional $F_n$. In order to find the lower free energy bound in the superconducting state with $c \neq 0$, we will use the following inequality [25]:

$$F_n^{5/6} F_c^{1/2} \geq (32 \pi^2)^{4/3} |L|,$$

where

$$L = \frac{1}{16 \pi^2} \int d^3 x \varepsilon_{ikl} c_i \partial_k a_l$$

(17)

is the degree of mutual linking [20, 27] of the velocity $c$ lines and of the magnetic field $H = [\nabla \times a]$ lines. It is also an integral of motion. [27, 28]

Using the Schwartz-Cauchy-Bunyakovsky inequality, one will find [25] that

$$F \geq F_{\text{min}} = (F_n^{1/2} - F_c^{1/2})^2.$$  

Using Eq.(16) and the condition $Q \neq 0$ the last inequality can be rewritten in the equivalent form

$$F \geq 32 \pi^2 |Q|^{3/4} (1 - |L|/|Q|)^2.$$  

(18)

The minimum value of free energy in Eqs.(18) corresponds to the field configurations with $L = Q \neq 0$ or $a = c$. They satisfy the self-dual condition $F_n = F_c$.

It is clear from Eq.(18) that for all numbers $L < Q$ the energy of the ground state is less than that described in the model [2], for which the inequality (12) is valid. The origin of the energy decrease can be easily understood. Even under the conditions of existence of the paramagnetic part $j$ of the current $J$, the diamagnetic interaction in the superconducting state cancels the own current energy and a part of the energy related to the $n$-field dynamics for all state classes with $L < Q$. It is also important that the current (total momentum of a superconducting pair) is non-zero in the superconducting state. In this regard, the considered inhomogeneous state with the current is similar to the state [22] investigated in Refs. [30, 31].

4 Discussion

Up to now the vector $A$ has characterized the internal charge $U(1)$ gauge symmetry. If we apply the external electromagnetic field, then the vector potential $A$ will be equal to the sum of internal and external gauge potentials. As a result, the velocity $c$ decreases due to the diamagnetism of the superconducting state. This leads to suppression of the superconducting gap. The answer to the question on existence of full or partial Meisner screening in these states depends on the result of the competition between contributions from neutral $j$ and charged $-4A$ parts to the total current $J$.

In the $(3 + 0)D$ case of free energy (10), Hopf invariant (13) is analogous to the Chern-Simons action $(k/4 \pi) \int dt d^2 x \varepsilon_{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda$, which determines strong correlations of $(2 + 1)D$ states [32] for $k \simeq 2$. In planar systems, this coefficient has the sense of braiding degree of excitation world lines. In particular, for the
semion $k = 2$. Keeping in mind the relation of spatial dimensionality of the systems in their quantum and statistical descriptions, we note that the $(2+1)D$ dynamical case $k = 2$ of open world line ends of excitations is equivalent to the compact $(3+0)D$ statistical example of the Hopf linking $Q = 1$.

In conclusion, we have considered the structure of the multi-meron knot action and found the self-dual equation. Using the topological invariants of knots for the analysis of coherent phase states, we have also determined the conditions of applying the theory significantly based on the dynamics and topology of the $n$-field model.

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