On the Uniqueness of Inverse Problems with Fourier-domain Measurements and Generalized TV Regularization

Julien Fageot, Thomas Debarre, and Quentin Denoyelle

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Abstract

We study the super-resolution problem of recovering a periodic continuous-domain function from its low-frequency information. This means that we only have access to possibly corrupted versions of its Fourier samples up to a maximum cut-off frequency \( K_c \). The reconstruction task is specified as an optimization problem with generalized total-variation regularization involving a pseudo-differential operator. Our special emphasis is on the uniqueness of solutions. We show that, for elliptic regularization operators (e.g., the derivatives of any order), uniqueness is always guaranteed. To achieve this goal, we provide a new analysis of constrained optimization problems over Radon measures. We demonstrate that either the solutions are always made of Radon measures of constant sign, or the solution is unique. Doing so, we identify a general sufficient condition for the uniqueness of the solution of a constrained optimization problem with TV-regularization, expressed in terms of the Fourier samples.

1 Introduction

In recent years, total-variation regularization techniques for continuous-domain inverse problems have shown to be very fruitful, with rapidly-growing theoretical developments [1, 2, 3, 4], algorithmic progress [5, 6, 7], and data science applications [8, 9, 10]. As is well known for its discrete-domain counterpart (i.e., \( \ell_1 \) optimization), this leads to variational problems whose solutions are not necessarily unique. Our goal in this paper is to provide a systematic study of the uniqueness of TV-based optimization problems for the special case of Fourier sampling measurements.

The torus is the interval \( T = \mathbb{R}/2\pi \mathbb{Z} = [0, 2\pi] \) where the two points 0 and \( 2\pi \) are identified. We study the reconstruction of an unknown periodic real function \( f_0 : T \to \mathbb{R} \) from the knowledge of its possibly corrupted low-frequency Fourier series coefficients. Let \( K_c \geq 0 \) be the maximum frequency, we therefore have access to

\[
y = (y_0, y_1, \ldots, y_{K_c}) \in \mathbb{R} \times \mathbb{C}^{K_c}
\]

such that \( y_k \) is approximately the \( k \)th Fourier series coefficient \( \hat{f}_0[k] \) of \( f_0 \). Note that, since \( f_0 \) is a real function, \( y_0 \in \mathbb{R} \) is the (approximated) mean \( \hat{f}_0[0] = \langle f_0, 1 \rangle \) of \( f_0 \), while \( y_k \in \mathbb{C} \) for \( k \neq 0 \). Moreover, the Fourier series of \( f_0 \) is Hermitian symmetric, meaning that \( \hat{f}_0[-k] = \overline{\hat{f}_0[k]} \in \mathbb{C} \) for every \( k \in \mathbb{Z} \). The observation \( y \) in (1) therefore has \( 2K_c + 1 \) (real) degrees of freedom: one for the real mean \( y_0 \) and two for each other complex Fourier series coefficients in \( \mathbb{C} \). Finally, we will always assume that \( y_0 \geq 0 \) (otherwise, we can consider the reconstruction of \(-f_0\) instead of \( f_0 \)).

1.1 Reconstruction via TV-based Optimization

The recovery of a periodic function from finitely many observations is of course ill-posed. We therefore formulate the reconstruction task as a regularized optimization problem. More precisely,
the reconstruction \( f^* \) of \( f_0 \) is defined as a solution of

\[
f^* \in \text{arg min}_f E(y, \nu(f)) + \lambda \|Lf\|_M,
\]

where \( y \in \mathbb{R}^+ \times \mathbb{C}^{K_c} \) is the observation vector, \( \nu(f) \) is the measurement vector

\[
\nu(f) = (\hat{f}[0], \hat{f}[1], \ldots, \hat{f}[K_c]) \in \mathbb{R} \times \mathbb{C}^{K_c},
\]

\( E(\cdot, \cdot) : (\mathbb{R} \times \mathbb{C}^{K_c})^2 \to \mathbb{R}^+ \cup \{\infty\} \) is a data-fidelity functional, strictly convex over its domain\(^{1}\), lower semi-continuous (lsc), and coercive with respect to its second argument, \( \|\cdot\|_M \) is the total-variation norm on periodic Radon measures, and \( L \) is a regularization operator acting on periodic functions.

The data-fidelity term encourages the measurement vector \( \nu(f) \) to be close to the observation \( y \). A typical example is the quadratic functional

\[
E(y, \nu(f)) = \frac{1}{2} \|y - \nu(f)\|_2^2 = \frac{1}{2} \sum_{k=0}^{K_c} |y_k - \hat{f}[k]|^2.
\]

The data fidelity (4) corresponds to an additive noise model where the measurements \( y \) are generated via the model \( y = \nu(f_0) + n \) with \( n \) a complex Gaussian vector (see [12, Section IV-B] for more details). We will also be interested in constrained optimization problems of the form

\[
\text{arg min}_f \|Lf\|_M,
\]

which corresponds to \( E(y, \nu(f)) = 0 \) if \( y = \nu(f) \) and \( \infty \) otherwise\(^{2}\). Other classical data-fidelity functionals can be found in [9, Section 7.5].

The choice of the total-variation norm promotes sparse and adaptive continuous-domain reconstruction, and has recently received a lot of attention (see Section 1.3). The operator \( L \) controls the transform domain in which sparsity is enforced together with the regularity properties of the recovery: Dirac recovery corresponds to \( L = \text{Id} \) [1], and higher-order operators induce smoother reconstructions [4].

### 1.2 Contributions

The existence of solutions for problems such as (2) is well established, and simply follows from the convexity of the cost functional. However, the solution is in general not unique (the simplest case of non-uniqueness is with \( K_c = 0 \) and \( L = \text{Id} \), see Section 3.1). It is actually well known that even finite-dimensional \( \ell_1 \)-regularization, of which the total-variation norm for Radon measures is the continuous-domain generalization, can lead to non-unique solutions [13, 14]. In this paper, we focus on characterizing the cases of uniqueness for (2). Our contributions can be detailed as follows.

(i) **Optimization over Radon measures: positivity versus uniqueness.** We first consider the constrained problem

\[
\mathcal{V}(y) = \text{arg min}_{\nu(w) = y} \|w\|_M
\]

over periodic Radon measures \( w \in \mathcal{M}(\mathbb{T}) \), with \( y \in \mathbb{R}^+ \times \mathbb{C}^{K_c} \) and \( \nu(w) = (\hat{w}[0], \hat{w}[1], \ldots, \hat{w}[K_c]) \).

We show that two (not necessarily mutually exclusive) scenarios are possible: either (i) the solution

\(^{1}\)The domain of a convex function \( g : X \to \mathbb{R}^+ \cap \{\infty\} \) is the set \( \{x \in X \mid g(x) < \infty\} \) [11]

\(^{2}\)In this case, the value of the regularization parameter \( \lambda > 0 \) plays no role.
is unique, or (ii) the solution set consists of all positive Radon measures satisfying the constraints. The two scenarios are characterized by a simple positive-definiteness condition on the vector \( y \) (see Theorem 2). We then link (2) with regularized optimization problems of the form (6), leading to our second main contribution.

(ii) **Uniqueness of the solution for elliptic operators.** Our main theoretical result focuses on the case where \( L \) is a periodic elliptic operator, meaning that its null space consists of constant functions (see Definition 1 thereafter). An \( L \)-spline is a function \( f \) such that

\[
Lf = \sum_{k=1}^{K} a_k \delta(\cdot - x_k)
\]

is a finite sum of Dirac combs, the distinct Dirac locations \( x_k \) being the knots of the spline (see Section 2.1).

**Theorem 1.** Let \( L \) be an elliptic periodic operator, \( K_c \geq 0 \), and \( y \in \mathbb{R}^+ \times \mathbb{C}^{K_c} \). Then, there exists a unique solution to (2). Moreover, this solution is a \( L \)-spline whose number of knots is bounded by \( 2K_c \).

To the best of our knowledge, Theorem 1 is the first systematic uniqueness result for the analysis of a variational problem of the form (2).

**1.3 Related Works**

**Optimization over Radon measures:** The historical motivation to consider total-variation as a regularization norm was to extend discrete \( \ell_1 \) regularization techniques, used in the theory of compressed sensing to recover sparse vectors \([15, 16]\), for continuous-domain Dirac recovery. The goal is to recover point sources, modelled as a sum of Dirac masses, from finitely many measurements. This has received a considerable attention in the 21st century, including FRI (finite rate of innovation) techniques \([17, 18]\) and Prony’s methods \([19]\). Several data-science problems can indeed be formulated as a Dirac recovery problem, including radio-astronomy \([20]\), super-resolution microscopy \([6]\), or 3D image deconvolution \([10]\).

After the development of the theory of compressed sensing \([15, 21]\), inherently discrete in its seminal formulation, several infinite-dimensional extensions of the compressed sensing framework have been proposed \([22, 23, 24, 25, 26]\). In this context, in the early 2010’s, De Castro and Gamboa \([27]\), Candès and Fernandez-Granda \([1, 28]\), and Bredies and Pikkarainen \([2]\) all considered optimization tasks of the form (6) (or its penalized version), with both theoretical analyses and novel algorithmic approaches to recover a sparse measure solution.

**From sparse measures to splines and beyond:** The study of optimization problems of the form (6) can be traced back to the pioneering works of Beurling \([29]\), where Fourier-sampling measurements were also considered. The existence of sparse measure solutions, i.e., solutions of the form \( \sum_{k=1}^{K} a_k \delta(\cdot - x_k) \), seems to have been proven for the first time in \([30]\) and was later improved by Fisher and Jerome in \([31]\). Since then, a remarkable revival around TV optimization has occurred recently \([32, 33, 34, 35, 7]\).

These works have revealed that the signal model induced by the total-variation norm is a stream of Dirac. Several authors extended this framework to smoother continuous-domain signals by considering generalized total-variation regularization. In \([4]\), Unser et al. revealed the connection between
constrained problems (2) (in a non-periodic setting) and spline theory for general measurement functionals: the extreme-point solutions are necessarily L-splines. This result was revisited, extended, and refined by several authors [9, 36, 37, 38, 39, 40, 41, 42, 43, 44]. This manuscript will strongly rely on the periodic theory of TV-based optimization problem recently developed in [44].

Uniqueness results for generalized TV optimization: It is well-known that replacing the total-variation norm by the $L_2$ norm in (2) (Tikhonov regularization, also known as ridge regression) leads to optimization problems whose solution is unique (see [12] for a detailed study of the quadratic case in the periodic setting with general operators $L$ and measurements $\nu$). This is no longer true for TV regularization, and it becomes important to understand the cases of uniqueness. Many uniqueness results for constrained or penalized TV-based optimization problems have been given in the literature, but from different perspectives than the one studied in this paper. In [27], de Castro and Gamboa introduced the concept of extrema Jordan type measure (see [27, Definition 1]), which gives sufficient conditions on a given signed sparse measure to be the unique solution of a TV-based optimization problem. Candès and Fernandez-Granda also studied the super-resolution problem of recovering a ground-truth sparse Radon measure $w_0$ from its low-frequency measurements. They have shown that if the minimal distance between the spikes of $w_0$ is large enough, then the optimization problem has a unique solution, which is $w_0$ itself [1, Theorem 1.2]. Duval and Peyré identified the so-called non-degenerate source condition [3, Definition 5], under which the uniqueness of the reconstruction together with the support recovery of the underlying ground-truth sparse measure are shown. These results are based on the key notion of dual certificates, will play an important role in our work. This notion has been introduced for discrete compressed sensing problems in [45] and connected to TV-based optimization problems in [27].

All these works are clearly related to this paper. However, the perspective we propose is different: we aim at characterizing the cases of uniqueness directly over the measurement vector $y$, and are oblivious to the ground-truth signal that generated it. The closest work in this direction is our recent publication [46], where we provide a full description of the solution set of TV optimization problems with a regularization operator $L = D^2$ where $D$ is the derivative operator (which leads to piecewise-linear reconstructions), and spatial sampling measurements, which includes the characterization of the cases of uniqueness [46, Proposition 6 and Theorem 2].

1.4 Outline

The paper is organized as follows. Section 2 introduces the mathematical material used in this paper. Section 3.1 is dedicated to an illustrative toy optimization problem with $K_c = 1$, highlighting interesting phenomena which also occurs in the general case. In the rest of Section 3, we fully characterize the solution set of (6) when there exists a non-negative measure solution (Section 3.2), and show that the solution is unique when no non-negative measure is a solution (Section 3.3). Section 4 uses the results of the previous section to study (2) for invertible (Section 4.1) and elliptic operators (Section 4.2). We conclude in Section 5.

2 Mathematical Preliminaries

We first introduce some notations and recall some basic facts concerning periodic functions and their Fourier series. More details can be found in [44, Section 2]. The space of infinitely smooth periodic function is denoted by $\mathcal{S}(\mathbb{T})$, endowed with its usual Fréchet topology. Its topological dual is the space of periodic generalized functions $\mathcal{S}'(\mathbb{T})$. 
For \( k \in \mathbb{Z} \), let \( e_k : \mathbb{T} \to \mathbb{C} \) be the complex exponential function \( e_k(x) = \exp(2\pi kx) \), which is clearly in \( \mathcal{S}(\mathbb{T}) \). The Fourier series coefficients of \( f \in \mathcal{S}'(\mathbb{T}) \) are given by \( \hat{f}[k] = (f, e_k) \in \mathbb{C} \). For a real function \( f \), these coefficients are Hermitian symmetric, i.e., such that \( \hat{f}[-k] = \bar{\hat{f}}[k] \) for all \( k \in \mathbb{Z} \), which implies in particular that \( \hat{f}[0] \in \mathbb{R} \). We then have that \( \hat{f} = \sum_{k \in \mathbb{Z}} \hat{f}[k] e_k \) for any \( f \in \mathcal{S}'(\mathbb{T}) \), where the convergence is in \( \mathcal{S}'(\mathbb{T}) \). The Dirac stream is defined as \( \mathbb{V} = \sum_{n \in \mathbb{Z}} \delta(\cdot - 2\pi n) \). Its Fourier coefficients are \( \mathbb{V}[k] = 1 \) for each \( k \in \mathbb{Z} \).

### 2.1 Operators, Green’s Functions, and Splines on the Torus

We consider periodic, linear, and shift-invariant operators \( L : \mathcal{S}'(\mathbb{T}) \to \mathcal{S}'(\mathbb{T}) \). As is well known, \( L \) is fully characterized by its Fourier sequence \( \{\hat{L}[k]\}_{k \in \mathbb{Z}} \) such that \( L[f] = \hat{L}[k] \hat{f}[k] e_k \), in the sense that

$$ Lf = \sum_{k \in \mathbb{Z}} \hat{L}[k] \hat{f}[k] e_k. \quad (8) $$

In this paper, we will only consider operators \( L \) that are invertible or elliptic, as defined thereafter. The space of periodic linear and shift-invariant operators is denoted by \( \mathcal{L}_{SI}(\mathcal{S}'(\mathbb{T})) \).

**Definition 1.** An operator \( L \in \mathcal{L}_{SI}(\mathcal{S}'(\mathbb{T})) \) is invertible if there exists an operator \( L^{-1} \in \mathcal{L}_{SI}(\mathcal{S}'(\mathbb{T})) \) such that

$$ LL^{-1} f = L^{-1} L f = f, \quad \forall f \in \mathcal{S}'(\mathbb{T}). \quad (9) $$

A periodic operator is elliptic \(^3\) if \( \hat{L}[0] = 0 \) and if \( L + 1 \) is invertible.

The operator \( K = L + 1 \) in Definition 1 has the same Fourier sequence as \( L \) except that \( \hat{K}[0] = \hat{L}[0] + 1 \). It is such that \( Kf = Lf + \hat{f}[0] \). Adding this constant allows the operator to be invertible, which is excluded for \( L \) by the condition \( \hat{L}[0] = 0 \). Invertibility and ellipticity can be characterized by the Fourier sequence of \( L \) as follows. The proof is given in Appendix A.

**Proposition 1.** A periodic operator is invertible if and only if there exist \( A, B > 0 \) and \( \alpha, \beta \in \mathbb{R} \) such that

$$ A(1 + |k|)^\alpha \leq |\hat{L}[k]| \leq B(1 + |k|)^\beta, \quad \forall k \in \mathbb{Z}. \quad (10) $$

In this case, the Fourier sequence of \( L^{-1} \) is \( (1/\hat{L}[k])_{k \in \mathbb{Z}} \).

A periodic operator is elliptic if and only if there exist \( A, B > 0 \) and \( \alpha, \beta \in \mathbb{R} \) such that

$$ A|k|^\alpha \leq |\hat{L}[k]| \leq B|k|^\beta, \quad \forall k \in \mathbb{Z}. \quad (11) $$

Then, the null space of \( L \) is the one-dimensional space of constant functions.

**Definition 2.** We say that \( L^\dagger \in \mathcal{L}_{SI}(\mathcal{S}'(\mathbb{T})) \) is a pseudoinverse of \( L \in \mathcal{L}_{SI}(\mathcal{S}'(\mathbb{T})) \) if we have\(^4\)

$$ LL^\dagger L = L, \quad \text{and} \quad L^\dagger LL^\dagger = L^\dagger. \quad (12) $$

Any elliptic operator has a unique pseudoinverse whose Fourier sequence is given by \( \hat{L}^\dagger [0] = 0 \) and \( \hat{L}^\dagger[k] = 1/\hat{L}[k] \) for \( k \neq 0 \). This is a particular case of [44, Proposition 2.4] but it can easily be proved in this case.

The pseudoinverse operator allows to define the Green’s function of \( L \in \mathcal{L}_{SI}(\mathcal{S}'(\mathbb{T}^d)) \). This notion is here adapted to the periodic setting, as discussed after [44, Definition 4].

\(^3\)We borrow the terminology for pseudo-differential operators in partial differential equations [47] whose Fourier symbols are assumed to be non-vanishing except at the origin. Elliptic operators are generalizations of the Laplace operator.

\(^4\)A pseudoinverse should also satisfy the self-adjoint relations \( (LL^\dagger)^* = LL^\dagger \) and \( (L^\dagger L)^* = L^\dagger L \), but they are automatically satisfied in this case, as shown in [44, Section 2.2].
Definition 3. We fix $L \in \mathcal{L}_{SI}(S'(\mathbb{T}))$ and we assume that it admits a pseudoinverse $L^\dagger$. Then, the Green’s function of $L$ is $\rho_L = L^\dagger III$.

The Green’s function of an invertible periodic operator is simply $\rho_L = L^{-1}III$. For an elliptic operator, it is the generalized function such that $\hat{\rho}_L[0] = 0$ and $\hat{\rho}_L[k] = 1/\hat{L}[k]$ for $k \neq 0$.

Definition 4. Let $L$ be an elliptic or invertible periodic operator. We say that $f$ is a periodic $L$-spline (or simply an $L$-spline) if

$$Lf = w = \sum_{n=1}^{N} a_n III(\cdot - x_n)$$

where $N \geq 0$, $a_n \in \mathbb{R}\setminus\{0\}$, and distinct knots $x_n \in \mathbb{T}$. We call $w$ the innovation of the $L$-spline $f$.

For invertible operators, (13) is equivalent to $f = \sum_{n=1}^{N} a_n \rho_L(\cdot - x_n)$ where $\rho_L = L^{-1}III$ the Green’s function of $L$. When $L$ is elliptic, $f$ satisfies (13) if and only if $f = a_0 + \sum_{n=1}^{N} a_n \rho_L(\cdot - x_n)$ where $\rho_L = L^\dagger III$ and $a_0 \in \mathbb{R}$. In this case, we necessarily have that $\sum_{n=1}^{N} a_n = 0$. This is a particular case of [44, Proposition 2.8] and simply follows from taking the mean (or 0th Fourier coefficient) in (13), giving $0 = \tilde{L}[0] \hat{f}[0] = \sum_{n=1}^{N} \hat{a}_n$. It is worth noting that the Green’s function of an elliptic operator is not a $L$-spline. However, $\rho_L - \rho_L(\cdot - 1/2)$ is a periodic $L$-spline.

Examples. Invertible operators include differential operators $D + \alpha I$ with $\alpha \in \mathbb{R}\setminus\{0\}$ and Sobolev operators $L = (\alpha^2 I - \Delta)^{\gamma/2}$ for $\gamma > 0$ and $\alpha^2 > 0$ (whose Fourier sequence is $\hat{L}[k] = (\alpha^2 + k^2)^{\gamma/2}$). The fractional derivatives $D^{\gamma}$ and fractional Laplacians $(-\Delta)^{\gamma/2}$ with $\gamma > 0$ are examples of elliptic operators. More examples can be found in [44, Section 5.1], together with the representation of the corresponding Green’s functions and splines.

2.2 Periodic Radon Measures, Native Spaces, and Representer Theorem

Let $\mathcal{M}(\mathbb{T})$ be the space of periodic Radon measures. By the Riesz-Markov theorem [48], it is the continuous dual of the space $\mathcal{C}(\mathbb{T})$ of continuous periodic functions endowed with the supremum norm. The total-variation norm on $\mathcal{M}(\mathbb{T})$, for which it forms a Banach space, is given by

$$\|w\|_\mathcal{M} = \sup_{f \in \mathcal{C}(\mathbb{T}), \|f\|_\infty \leq 1} \langle w, f \rangle.$$ (14)

We denote by $\mathcal{M}_0(\mathbb{T})$ the set of Radon measures with zero mean, i.e., $\mathcal{M}_0(\mathbb{T}) = \{w \in \mathcal{M}(\mathbb{T}), \hat{w}[0] = 0\}$. It is the continuous dual of the space $\mathcal{C}_0(\mathbb{T}) = \{f \in \mathcal{C}(\mathbb{T}), \hat{f}[0] = 0\}$ of continuous functions with zero mean. We also consider the set of positive Radon measures $\mathcal{M}_+(\mathbb{T})$ which are Radon measures $w$ such that $\langle w, \varphi \rangle \geq 0$ for any positive continuous function $\varphi$. The set of probability measures, i.e., non-negative measures $w$ with total-variation $\|w\|_\mathcal{M} = 1$, is denoted by $\mathcal{P}(\mathbb{T})$.

The total-variation norm upper-bounds the Fourier coefficients of a Radon measure, as stated in Proposition 2 which also provides elementary characterizations for non-negative Radon measures. The proof is provided in Appendix A.

Proposition 2. Let $w \in \mathcal{M}(\mathbb{T})$. Then,

1. For any $k \in \mathbb{Z}$, we have $|\hat{w}[k]| \leq \|w\|_\mathcal{M}$.
2. $w \in \mathcal{M}_+(\mathbb{T})$ if and only if $\|w\|_\mathcal{M} = \hat{w}[0]$.
3. $w \in \mathcal{P}(\mathbb{T})$ if and only if $\|w\|_\mathcal{M} = \hat{w}[0] = 1$. 

Let $L$ be a periodic operator. We define the native space associated to $L$ as

$$\mathcal{M}_L(\mathbb{T}) = \{ f \in \mathcal{S}'(\mathbb{T}), \ Lf \in \mathcal{M}(\mathbb{T}) \}. \quad (15)$$

Periodic native spaces have been studied for general spline-admissible operators (i.e., periodic operators with finite-dimensional null space and which admit a pseudoinverse) in [44, Section 3].

**Proposition 3** (Theorem 3.2 in [44]). Let $L$ be a periodic operator. If $L$ is invertible, then $\mathcal{M}_L(\mathbb{T}) = L^{-1}(\mathcal{M}(\mathbb{T}))$ inherits the Banach structure of $\mathcal{M}(\mathbb{T})$ for the norm

$$\|f\|_{\mathcal{M}_L} = \|Lf\|_{\mathcal{M}}. \quad (16)$$

If $L$ is elliptic, then we have that the direct sum relation

$$\mathcal{M}_L(\mathbb{T}) = L^\dagger \mathcal{M}_0(\mathbb{T}) \oplus \text{Span}\{1\} \quad (17)$$

and any $f \in \mathcal{M}_L(\mathbb{T})$ has a unique decomposition as

$$f = L^\dagger w + a \quad (18)$$

where $w \in \mathcal{M}_0(\mathbb{T})$ and $a \in \mathbb{R}$ are given by $w = Lf$ and $a = \hat{f}[0]$. Then, $\mathcal{M}_L(\mathbb{T})$ is a Banach space for the norm

$$\|f\|_{\mathcal{M}_L} = \|w\|_{\mathcal{M}} + |a|. \quad (19)$$

The case of invertible operators is particularly simple: the bijection induces an isometry between $\mathcal{M}(\mathbb{T})$ and the native space $\mathcal{M}_L(\mathbb{T})$. Native spaces of elliptic operators require to deal with constants from the null space of $L$. Note that the measurement functional $\nu$ in (3) is well defined over $\mathcal{M}_L(\mathbb{T})$, and more generally over $\mathcal{S}'(\mathbb{T})$, since the complex exponentials are infinitely smooth.

It is known that TV-based optimization problems with regularization operators lead to splines solutions [4]. This is both an existence result and a representer theorem, which provides the form of the (extreme point) solutions of the optimization task. We now recall the representer theorem associated to (2). This follows the same line as many recent works on representer theorems for TV optimization [4, 36, 40, 39, 38], that has been recently adapted to the periodic setting in [44].

**Proposition 4** (Theorem 4 in [44]). Let $L$ be an invertible or elliptic periodic operator, $K_e \geq 0$ be the cut-off frequency of the low-pass filter $\nu : \mathcal{M}_L(\mathbb{T}) \to \mathbb{R}^+ \times \mathbb{C}^{K_e}$ defined in (3), $y \in \mathbb{R}^+ \times \mathbb{C}^{K_e}$, $E(\cdot, \cdot) : (\mathbb{R} \times \mathbb{C}^{K_e})^2 \to \mathbb{R}^+$ be a functional which is strictly convex over its domain, lsc, and coercive with respect to its second argument, and $\lambda > 0$. Then, the solution set

$$\nu(y) = \arg \min_{f \in \mathcal{M}_L(\mathbb{T})} E(y, \nu(f)) + \lambda \|Lf\|_{\mathcal{M}} \quad (20)$$

is non-empty, convex, weak* compact, and its extreme points are periodic $L$-splines whose number of knots is upper-bounded by $2K_e + 1$.

**Proof.** Proposition 4 is a direction application of [44, Theorem 4] to the case $d = 1$ and for Fourier sampling linear functionals $\nu_m = \epsilon_m$. The latter, being in $\mathcal{S}(\mathbb{T})$, fulfill the hypotheses of the theorem as justified in [44, Section 6.1]. The number of real measurements is given by $M = 2K_e + 1$: $y_0$ is one measurement, while each $y_k \in \mathbb{C}$ for $k = 1, \ldots, K_e$ provides two real measurements via their real and imaginary parts. \qed

The main result of our paper can be seen as the uniqueness counterpart of Proposition 4. We shall identify general sufficient conditions ensuring that the problem only admits one solution. Proposition 4 then implies that this unique solution is necessarily a periodic $L$-spline.
3 TV-based Constrained Problems over Radon Measures

As we have seen in Section 1.3, many works deal with the reconstruction of Dirac streams from Fourier-domain measurements. This is a super-resolution problem, because one aims to recover a Dirac stream from its low-frequency information. In this section, we focus on the constrained optimization problem

\[ V(y) = \arg \min_{w \in \mathcal{M}(\mathbb{T}), \nu(w)=y} \|w\|_\mathcal{M}, \tag{21} \]

where \( K_c \geq 0, \ y = (y_0, y_1, \ldots, y_{K_c}) \in \mathbb{R}^+ \times \mathbb{C}^{K_c}, \) and \( \nu(w) = (\hat{w}[0], \hat{w}[1], \ldots, \hat{w}[K_c]) \). Our goal is to derive new results on the solution set \( V(y) \). We first provide a useful lower bound on the minimal value of (21).

**Lemma 1.** For any \( K_c \geq 0 \) and \( y \in \mathbb{R}^+ \times \mathbb{C}^{K_c}, \)

\[
\min_{w \in \mathcal{M}(\mathbb{T}), \nu(w)=y} \|w\|_\mathcal{M} \geq \max_{0 \leq k \leq K_c} |y_k|. \tag{22}
\]

**Proof.** We know that \( V(y) \) is non-empty, so the minimum value is reached by at least one measure \( w_0 \). Then, we have \( \|w_0\|_\mathcal{M} \geq |\hat{w}[k]| = |y_k| \) for all \( 0 \leq k \leq K_c \) according to Proposition 2, which gives (22). \( \square \)

### 3.1 A Toy Problem: The Case \( K_c = 1 \)

We start our investigation on constrained problems over Radon measures (i.e., no regularization operator \( L \)), as in (6). First of all, there exist cases of non-uniqueness, as exemplified with the moderately interesting problem of reconstructing a Radon measure \( w \) uniquely from its mean \( \hat{w}[0] = y_0 > 0 \). In our framework, this corresponds to solving

\[ V(y_0) = \arg \min_{w \in \mathcal{M}(\mathbb{T}), \hat{w}[0]=y_0} \|w\|_\mathcal{M}. \tag{23} \]

The solution set is \( V(y_0) = \{ w \in \mathcal{M}_+(\mathbb{T}), \hat{w}[0] = y_0 \} = y_0 \cdot \mathcal{P}(\mathbb{T}). \) Of course, this set is infinite, and its extreme points are of the form \( y_0 \cdot \text{III}(\cdot - x_0) \) for \( x_0 \in \mathbb{T} \), which is itself uncountably infinite. We now consider the case \( K_c = 1 \) in the following result, whose proof is given in Appendix B.

**Proposition 5.** Let \( y_0 \in \mathbb{R}^+ \) and \( y_1 = \text{re}^{\text{i}\alpha} \in \mathbb{C} \) such that \( (y_0, y_1) \neq (0, 0) \). Consider the optimization problem

\[ V(y) = V(y_0, y_1) = \arg \min_{w \in \mathcal{M}(\mathbb{T}), \hat{w}[0]=y_0 \& \hat{w}[1]=y_1} \|w\|_\mathcal{M}. \tag{24} \]

Then, the measure

\[
w^* = \frac{y_0 + r}{2} \text{III}(\cdot + \alpha) + \frac{y_0 - r}{2} \text{III}(\cdot + \alpha + \pi) \tag{25}\]

is always a solution to (24). Moreover, we have the following scenarios.

- If \( y_0 \leq r \), then \( w^* \) is the unique solution. Note that \( w^* = y_0 \text{III}(\cdot + \alpha) \) is a single Dirac mass for \( y_0 = r \) and is a non-negative measure in this case.
- If \( y_0 > r \), then the problem has uncountably infinitely many extreme point solutions, which are necessarily non-negative measures.
Proposition 5 is somehow representative of the general case (see Theorem 2). Indeed, we shall see that, for general cut-off frequency $K_C \geq 1$ and $y_0 \geq 0$, either the solution set consists of non-negative measures, or the solution is unique. Note moreover that the two situations are not exclusive: with $y_0 = r$, the unique solution is a non-negative measure. For $y_0 > r$, our proof of Proposition 5 provides more than what is stated, with a full characterization of the extreme point solutions with 2 Dirac masses (see Appendix B).

3.2 On the Existence of Non-negative Measure Solutions

We first study the case where the optimization problem (21) admits a solution that is a non-negative measure. Theorem 2 reveals that this has strong implications on the solution set.

**Theorem 2.** Let $K_C \geq 0$ and $y = (y_0, y_1, \ldots, y_{K_c}) \in \mathbb{R}^+ \times \mathbb{C}^{K_c}$. We also set, for $1 \leq k \leq K_c$, $y_{-k} = \overline{y_k}$. Then, the following conditions are equivalent:

1. There exists $w_0 \in \mathcal{M}_+(\mathbb{T})$ such that $\nu(w_0) = y$.

2. We have the equality

$$y_0 = \min_{w \in \mathcal{M}(\mathbb{T}), \nu(w) = y} \|w\|_{\mathcal{M}}. \quad (26)$$

3. The solution set (21) is $\mathcal{V}(y) = \{w \in \mathcal{M}_+(\mathbb{T}), \nu(w) = y\}$.

4. For any complex numbers $z_0, z_1, \ldots, z_{K_c} \in \mathbb{C}$,

$$\sum_{k, \ell = 0}^{K_c} y_{k-\ell} z_k \overline{z_\ell} \geq 0. \quad (27)$$

5. The Toeplitz Hermitian matrix $T_y \in \mathbb{C}^{(K_c+1) \times (K_c+1)}$ given by

$$T_y = \begin{bmatrix}
y_0 & y_1 & y_2 & \ldots & y_{K_c} \\
y_{-1} & y_0 & y_1 & \ldots & y_{K_c-1} \\
y_{-2} & y_{-1} & y_0 & \ldots & y_{K_c-2} \\
\vdots & & & \ddots & \\
y_{-K_c} & y_{-K_c+1} & y_{-K_c+2} & \ldots & y_0
\end{bmatrix} = [y_{k-\ell}]_{0 \leq k, \ell \leq K_c} \quad (28)$$

is positive semi-definite.

**Proof.** Set $m = \min_{w \in \mathcal{M}(\mathbb{T}), \nu(w) = y} \|w\|_{\mathcal{M}}$. We have the equivalence, for any $w \in \mathcal{M}(\mathbb{T})$:

$$w \in \mathcal{M}_+(\mathbb{T}) \iff \|w\|_{\mathcal{M}} = \hat{w}[0]. \quad (29)$$

1. $\Rightarrow$ 2. The existence of $w_0$ ensures that $m \leq \|w_0\|_{\mathcal{M}} = \hat{w}[0] = y_0$ according to (29). From Lemma 1, we know moreover that $m \geq y_0$, hence $m = y_0$.

2. $\Rightarrow$ 3. Any $w \in \mathcal{V}(y)$ is such that $\hat{w}[0] = y_0 = \|w\|_{\mathcal{M}}$, hence, by Proposition 2, $w$ is a non-negative measure satisfying the constraints $\nu(w) = y$. Conversely, non-negative measures $w$ satisfying the constraints are such that $\|w\|_{\mathcal{V}} = \hat{w}[0] = y_0 = m$, and are therefore solutions.

3. $\Rightarrow$ 1. This is obvious since we know (for instance using Proposition 4) that $\mathcal{V}(y)$ is non-empty.

1. $\Rightarrow$ 4. Let $w_0$ be a non-negative measure solution. According to the Herglotz theorem (see Proposition 8 above), a measure in $\mathcal{M}(\mathbb{T})$ is positive if and only if its Fourier sequence is positive-definite in the sense of (56); that is, if and only if for any sequence $(z_k)_{k \in \mathbb{Z}}$ with finitely many
non-zero terms, \( \sum_{k,\ell \in \mathbb{Z}} \hat{w}|k-\ell| z_k \bar{z}_\ell \geq 0 \). In particular, restricting to sequences such that \( z_k = 0 \) for \( k < 0 \) and \( k > K_c \), we have that
\[
\sum_{k,\ell = 0}^{K_c} \hat{w}|k-\ell| z_k \bar{z}_\ell \geq 0. \tag{30}
\]

Applying this relation to \( w_0 \), we remark that \( |k-\ell| \leq K_c \) for \( 0 \leq k, \ell \leq K_c \) so that \( \hat{w}|k-\ell| = y_{k-\ell} \), hence (30) is equivalent to (27).

4. \( \Rightarrow \) 1. Recall that \( \mathcal{P}_K(\mathbb{R}) \) is the space of real trigonometric polynomials of degree at most \( K \). Consider the mapping \( \Phi : \mathcal{P}_K(\mathbb{R}) \to \mathbb{R} \) such that
\[
\Phi(p) = \Phi \left( \sum_{|k| \leq K_c} c_k e_k \right) = \sum_{|k| \leq K_c} c_k y_k. \tag{31}
\]

Then, \( \Phi \) is linear and positive. Indeed, let \( p \in \mathcal{P}_K(\mathbb{R}) \) such that \( p \geq 0 \). According to Proposition 9, \( p \) can be written as \( p = |q|^2 \) for some complex trigonometric \( q = \sum_{k=0}^{K_c} z_k e_k \) with \( q_k \in \mathbb{C} \). This implies that \( p = |q|^2 = \sum_{k,\ell=0}^{K_c} z_k \bar{z}_\ell e_{k-\ell} \) and therefore
\[
\Phi(p) = \sum_{k,\ell=0}^{K_c} y_{k-\ell} z_k \bar{z}_\ell \geq 0 \tag{32}
\]
due to (27). According to Proposition 10, \( \Phi \) can be extended as a positive, linear, and continuous functional from \( \mathcal{C}(\mathbb{R}) \) to \( \mathbb{R} \). Hence, \( \Phi \in (\mathcal{C}(\mathbb{R}))' = \mathcal{M}(\mathbb{R}) \) specifies a positive Radon measure \( \Phi = w_0 \) due to the Riesz-Markov theorem [48]. Then, \( w_0 \in \mathcal{M}(\mathbb{R}) \) satisfies \( \nu(w_0) = \mathbf{y} \) and 1. is proved.

4. \( \Leftarrow \) 5. The relation (27) can be written in matrix form as \( \sum_{k,\ell=0}^{K_c} y_{k-\ell} z_k \bar{z}_\ell = \langle z, T \mathbf{y} \rangle \), with \( z = (z_0, \cdots, z_{K_c}) \in \mathbb{C}^{K_c+1} \), with \( \langle \cdot, \cdot \rangle \) the canonical inner product over \( \mathbb{C}^{K_c+1} \). Hence, (27) is equivalent to \( \langle z, T \mathbf{y} \rangle \geq 0 \) for any \( z \in \mathbb{C}^{K_c+1} \), which is equivalent to being positive semi-definite.

Theorem 2 characterizes the situations where (21) admits a non-negative measure as solution. Conditions 1. to 3. are expressed in term of the solution set \( \mathcal{V}(\mathbf{y}) \), while 4. and 5. give direct characterizations on the observation vector \( \mathbf{y} \). The condition 3. implies that the solution set \( \mathcal{V}(\mathbf{y}) \) is completely understood when there exists a non-negative measure solution. The next section considers the case where the conditions of Theorem 2 do not occur.

3.3 Sufficient Condition for Uniqueness

Remarkably, if the conditions of Theorem 2 do not occur, the optimization problem (21) has a unique solution. This is shown in the next theorem. This section also highlights some interesting consequences of this fact.

**Theorem 3.** Let \( K_c \geq 1 \) and \( \mathbf{y} \in \mathbb{R}^+ \times \mathbb{C}^{K_c} \) such that \( \mathbf{y} \neq 0 \). Assume that the equivalent conditions of Theorem 2 are not satisfied. Then, the optimization problem (21) has a unique solution, which is a sum of at most \( 2K_c \) periodic Dirac impulses.

**Proof.** By assumption, there is no non-negative measure solution. Let \( \eta \) be a dual certificate of the optimization problem (21) (it always exists due to Proposition 11 in Appendix D). Then, we have that \( \|w^*\|_\mathcal{M} = \langle w^*, \eta \rangle \) for any \( w^* \in \mathcal{V}(\mathbf{y}) \) (see again Proposition 11).

Thanks to Proposition 12, we know that either the dual certificate is constant, or the solution of (21) is unique. We assume by contradiction that \( \eta \) is constant. We therefore have that
\[
\langle w^*, \eta \rangle = \|w^*\|_\mathcal{M} = c \hat{w}[0] = \epsilon y_0. \tag{33}
\]
This means in particular that $\epsilon y_0 > 0$. Yet, $y_0 \geq 0$, which implies $\epsilon = 1$. This shows that $\hat{w}[0] = \|w^*\|_M$, which together with Proposition 2 implies that $w^* \in \mathcal{M}_+(\mathbb{T})$. This contradicts our initial assumption.

This shows by contradiction that $\eta$ is necessarily non-constant, and therefore, due to Proposition 12, that the solution is unique and is a sum of at most $2K_c$ Dirac masses, as expected.

**Remark.** When the solution is unique, Theorem 1 provides a slight refinement of existing results, summarized in Proposition 4. Indeed, it is known that extreme points of $\mathcal{V}(y)$ consists of at most $(2K_c + 1)$ Dirac masses, which is the dimension of the observation space. In this case, the unique solution has at most $2K_c$ Dirac masses.

**Corollary 1.** Let $K_c \geq 0$ and $y \in \mathbb{R}^+ \times \mathbb{C}^{K_c}$ be such that $y_0 < |y_{k_0}|$ for some $k_0 \neq 0$. Then, (21) has a unique solution, which is a sum of at most $2K_c$ periodic Dirac impulses.

**Proof.** According to Lemma 1, we have $\min_{\nu(\omega) \neq y} \|w\|_M \geq |y_{k_0}| > y_0$. This means in particular that (26) does not hold, and therefore, by the equivalence between (26) and (27) in Proposition 2, we are in a case of uniqueness described by Proposition 3.

Corollary 1 gives a deceptively simple sufficient condition ensuring that the solution of (21) is unique. It will play a crucial role in our proof of Theorem 1.

## 4 TV-based Penalized Problems with Regularization Operators

We explore the consequences of the previous section for penalized problems of the form

$$
\mathcal{V}_\lambda(y) = \arg\min_{f \in \mathcal{M}_L(\mathbb{T})} E(y, \nu(f)) + \lambda \|Lf\|_M
$$

(34)

with an observation vector $y \in \mathbb{R}^+ \times \mathbb{C}^{K_c}$, a measurement vector $\nu(f) \in \mathbb{R} \times \mathbb{C}^{K_c}$, a data-fidelity functional $E(\cdot, \cdot) : (\mathbb{R} \times \mathbb{C}^{K_c})^2 \rightarrow \mathbb{R}^+ \cup \{\infty\}$ which is strictly convex over its domain, lsc, and coercive with respect to its second argument, $\lambda > 0$, and a regularization operator $L$. We consider invertible operators in Section 4.1 and elliptic operators (with the proof of Theorem 1) in Section 4.2.

### 4.1 The Case of Invertible Regularization Operators

**Proposition 6.** Let $L$ be an invertible periodic operator, $y \in \mathbb{R}^+ \times \mathbb{C}^{K_c}$, and $\lambda > 0$. Then, (34) is non-empty and all the solutions $f^* \in \mathcal{V}_\lambda(y)$ share an identical measurement vector $y_\lambda = \nu(f^*)$. We set $z_\lambda \in \mathbb{R} \times \mathbb{C}^{K_c}$ such that $z_{\lambda,0} = \hat{L}[k]y_{\lambda,k}$ for all $0 \leq k \leq K_c$. Then, the following statements hold, assuming first that $z_{\lambda,0} \geq 0$.

1. The problem (34) admits a non-negative measure solution if and only if the Toeplitz Hermitian matrix $T_{z_\lambda}$ is positive semi-definite, where $T_{z_\lambda}$ is given by (28) with $y = z_\lambda$. In this case,

$$
\mathcal{V}_{\lambda}(y) = \{f \in \mathcal{M}_L(\mathbb{T}), \ Lf \in \mathcal{M}_+(\mathbb{T}) \text{ and } \nu(f) = y\}
$$

(35)

2. If $\mathcal{V}(z_\lambda) \cap \mathcal{M}_+(\mathbb{T}) = \emptyset$, then (34) has a unique solution, which is a periodic $L$-spline with at most $2K_c$ knots.

If $z_{\lambda,0} \leq 0$, then the same conclusions remain except that non-negative measures are substituted by non-positive measures and that the positive semi-definiteness of $T_{z_\lambda}$ is replaced by the negative semi-definiteness.
Proof. We treat the case $z_{\lambda,0} \geq 0$. The case $z_{\lambda,0} \leq 0$ is deduced by remarking that $\mathcal{V}(\mathbf{y}) = -\mathcal{V}(\mathbf{y})$. The existence of a common $\mathbf{y}_\lambda$ is classic and uses the strict convexity of the data fidelity $E(\mathbf{y}, \cdot)$; see for instance [46, Proposition 7]. It implies in particular that the optimization problem (34) has exactly the same solution set than

$$\arg\min_{f \in \mathcal{M}_L(T), \nu(f) = y_\lambda} \|Lf\|_{\mathcal{M}}.$$  

(36)

Using the invertibility of $L$, we then remark that $\mathcal{V}_\lambda(\mathbf{y}) = L^{-1}(\mathcal{V}(z_\lambda))$, where

$$\mathcal{V}(z_\lambda) = \arg\min_{w \in \mathcal{M}(\mathbb{T}), \nu(w) = z_\lambda} \|w\|_{\mathcal{M}},$$

(37)

as in (21). Then, the case (1) follows by applying Theorem 2 and the case (2) is a direct consequence of Theorem 3.

Remark. Once we have recognized the link between (21) and (34), Proposition 6 is a reformulation of the results of Section 3. However, one cannot easily deduce the modified vector $\mathbf{y}_\lambda$ from $\mathbf{y}$, and thus adjudicate on uniqueness using Theorem 2. In order to compute $\mathbf{y}_\lambda$, we need to solve (34), find one solution $f^*$, compute $\nu(f) = y_\lambda$, and then apply the criterion on $y_\lambda$.

4.2 The Case of Elliptic Regularization Operators

This section is dedicated to the proof of Theorem 1. We therefore consider Problem (34) for elliptic operators $L$, such as the (fractional) derivative or the (fractional) Laplacian. We start with the following preparatory result. We recall that $\mathcal{M}_0(\mathbb{T})$ is the space of Radon measure with zero mean.

Lemma 2. Let $L$ be an elliptic periodic operator and $\mathbf{y} \in \mathbb{R}^+ \times \mathbb{C}^{K_e}$. Let $f^* = L^*w^* + a^* \in \mathcal{M}_L(T)$ such as the (fractional) derivative or the (fractional) Laplacian. We start with the following preparatory result. We recall that $\mathcal{M}_0(\mathbb{T})$ is the space of Radon measure with zero mean.

Lemma 2. Let $L$ be an elliptic periodic operator and $\mathbf{y} \in \mathbb{R}^+ \times \mathbb{C}^{K_e}$. Let $f^* = L^*w^* + a^* \in \mathcal{M}_L(T)$ where $(w^*, a^*) \in \mathcal{M}_0(\mathbb{T}) \times \mathbb{R}$ are uniquely determined as in (18). Then, we have the equivalence:

$$f^* \in \arg\min_{f \in \mathcal{M}_L(T), \nu(f) = y} \|Lf\|_{\mathcal{M}} \iff a^* = y_0 \text{ and } w^* \in \arg\min_{w \in \mathcal{M}(\mathbb{T}), \nu(w) = z} \|w\|_{\mathcal{M}},$$

(38)

with $z = (z_0, z_1, \ldots, z_{K_e}) \in \mathbb{R} \times \mathbb{C}^{K_e}$ such that $z_0 = 0$ and $z_k = \hat{L}[k]y_k$ for $k \neq 0$.

Proof. The uniqueness of the decomposition $f^* = L^*w^* + a^* \in \mathcal{M}_L(T)$ with $w^* \in \mathcal{M}_0(\mathbb{T})$ and $a^* \in \mathbb{R}$ implies that

$$f^* \in \arg\min_{f \in \mathcal{M}_L(T), \nu(f) = y} \|Lf\|_{\mathcal{M}} \iff (w^*, a^*) \in \arg\min_{(w,a) \in \mathcal{M}_0(\mathbb{T}) \times \mathbb{R}} \|w\|_{\mathcal{M}}.$$ 

(39)

Moreover, $\nu(L^*w^* + a^*) = (a^*, \hat{w}^*[1]/\hat{L}[1], \ldots, \hat{w}^*[K_e]/\hat{L}[K_e])$, hence

$$\nu(f^*) = y \iff a^* = y_0 \text{ and } \hat{w}^*[k] = \hat{L}[k]y_k, \forall 1 \leq k \leq K_e \iff a^* = y_0 \text{ and } \nu(w^*) = z.$$ 

(40)

Coupling (39) and (40) implies the equivalence of Lemma 2.

Proof of Theorem 1. As was briefly discussed in the proof of Proposition 6, the solutions of (34) share a common value $\nu(f^*) = y_\lambda$. Hence, the optimization problem is equivalent to

$$\mathcal{V}_\lambda(y) = \arg\min_{f \in \mathcal{M}_L(T), \nu(f) = y_\lambda} \|Lf\|_{\mathcal{M}}.$$ 

(41)
Moreover, the optimization problem
\[
\arg \min_{w \in \mathcal{M}(\mathbb{T}), \nu(w) = z} \|w\|_{\mathcal{M}}
\]
has a unique solution. Indeed, this is obvious for \( z = 0 \) (the unique solution being \( w^* = 0 \)). For \( z \neq 0 \), we are in the conditions of Corollary 1 since \( z_0 = 0 < |z_k| \) for some \( k \), which implies uniqueness. Due to Lemma 2, the optimization problem on the right in (38) therefore also admits a unique solution. Hence, (41) also admits a unique solution, as expected.

Theorem 1 guarantees the uniqueness of the spline solution for (34). For many classic data-fidelity, one can actually be slightly more precise and show that the mean of the solution is known.

**Proposition 7.** We assume that we are under the conditions of Theorem 1 and moreover that the data fidelity \( E \) is such that \( y_0 = \arg \min_{z_0 \in \mathbb{R}} E(y, z) \) (43)
where \( y = (y_0, y_1, \ldots, y_{K_c}) \in \mathbb{R}^+ \times \mathbb{C}^{K_c} \) and \( z = (z_0, z_1, \ldots, z_{K_c}) \in \mathbb{C}^{K_c} \). Then, the unique solution \( f^* \) to (34) is such that \( f^*[0] = y_0 \).

**Proof.** We use the unique decomposition \( f = a + L^\dagger w \in \mathcal{M}_L(\mathbb{T}) \) with \( a \in \mathbb{R} \) and \( w \in \mathcal{M}_0(\mathbb{T}) \). This decomposition implies that \( f^* = a^* + L^\dagger w^* \) where
\[
(a^*, w^*) = \arg \min_{a \in \mathbb{R}, \, w \in \mathcal{M}_0(\mathbb{T})} E(y, \nu(a + L^\dagger w)) + \|w\|_{\text{TV}}.
\]
Then, we have that \( \nu(a + L^\dagger w) = (a, \widehat{L^\dagger w}[1], \ldots, \widehat{L^\dagger w}[K_c]) \). Moreover, \( w \) being fixed, (43) implies that \( E(y, \nu(a + L^\dagger w)) \geq E(y, \nu(y_0 + L^\dagger w)) \), with equality if and only if \( a = y_0 \). The constant \( a \) does not impact the regularization hence we always have that \( a^* = f^*[0] = y_0 \) as expected.

**Remark.** The relation (43) is typically satisfied by the quadratic data-fidelity (4), and actually by most of the data-fidelity encountered in practice. Proposition 7 hence ensures that the mean of the solution \( f^* \) is given by the observation \( y_0 \). It then suffices to focus on the measure \( w^* \in \mathcal{M}_0(\mathbb{T}) \) in the unique decomposition \( f^* = L^\dagger w^* + a^* \) (see Proposition 3).

## 5 Conclusion

This paper deals with continuous-domain inverse problems, where the goal is to recover a periodic function from its low-pass measurements. The reconstruction task is formalized as an optimization problem with a TV-based regularization involving a pseudo-differential operator. It was known that spline solutions always exist (representer theorem), and our goal was to investigate the uniqueness issue. We have shown that two scenarios occur in general: either the solution is unique, or the innovation of any solution is a measure of constant sign. Once applied to elliptic regularization operator, we were able to exclude the second scenario, implying that the solution is always unique in this case.
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A  Deferred Proofs from Section 2

Proof of Proposition 1. The Fourier sequence of $L$ is also the one of $L\{\text{III}\} \in S'(\mathbb{T})$, which is bounded by a polynomial (as any Fourier sequence of a periodic generalized function; see [49, Chapter VII]). This implies the existence of $B$ and $\beta$ in (10). For an invertible $L$, the Fourier sequence of $L^{-1}$ is $(1/L[k])_{k \in \mathbb{Z}}$ and is also bounded by a polynomial, giving $A$ and $\alpha$ in (10). The relation (11) is then obvious for $k = 0$ (all the quantities are vanishing) and easily deduced from (10) applied to $L + 1$ for $k \neq 0$. Finally, $L f = 0$ if and only if $\hat{f}(k) = 0$ for any $k \in \mathbb{Z}$. For $L$ elliptic, due to (11), this is equivalent to $\hat{f}(k) = 0$ for any $k \neq 0$. Hence, the null space of $L$ is $\text{Span}\{1\}$. 

Proof of Proposition 2. The first relation is obvious for $k = 0$ since

$$|\hat{w}(0)| = |\langle w, 1 \rangle| \leq \sup_{\|\varphi\|_{\infty} \leq 1} \langle w, \varphi \rangle = \|w\|_{M}$$

(45)

by picking both $\varphi \equiv 1$ and $\varphi \equiv -1$. It is worth noting that this argument does not work for $k \neq 0$ because $\hat{w}(k) = \langle w, e_k \rangle$ with $e_k$ a complex continuous function. Let us fix $k \neq 0$ and assume that $\hat{w}(k) = r e^{i\alpha} \neq 0$ with $r > 0$ and $\alpha \in [0, 2\pi)$ (the case $\hat{w}(k) = 0$ is obvious). Let $\psi = e^{-i\alpha} e_k$.

Then, we have $\langle w, \psi \rangle = e^{-i\alpha} \hat{w}(k) = r$ and $\langle w, \overline{\psi} \rangle = \langle \overline{w}, \psi \rangle = \langle w, \psi \rangle = r$, where we used the fact that $w$ is real on the second equality. Hence, setting $\varphi = \Re(\psi)$, we have that $\varphi \in \mathcal{C}(\mathbb{T})$ with $\|\varphi\|_{\infty} \leq \sup_{x \in \mathbb{T}} |\psi(x)| = 1$ and

$$|\hat{w}(k)| = r = \frac{\langle w, \psi \rangle + \langle w, \overline{\psi} \rangle}{2} = \langle w, \varphi \rangle \leq \sup_{\|\varphi\|_{\infty} \leq 1} \langle w, \varphi \rangle = \|w\|_{M},$$

(46)

as expected.

Any $w \in M(\mathbb{T})$ can be uniquely decomposed as $w = w_+ - w_-$ where $w_+$ and $w_- \in M_+(\mathbb{T})$ (Jordan decomposition). Moreover, $\|w\|_{M} = \|w_+\|_{M} + \|w_-\|_{M} = \langle w_+, 1 \rangle + \langle w_-, 1 \rangle = \hat{w}_+[0] + \hat{w}_-[0]$. Then,

$$w \in M_+(\mathbb{T}) \iff w_- = 0 \iff \hat{w}_-[0] = 0 \iff \|w\|_{M} = \hat{w}_+[0] = \hat{w}[0],$$

(47)

as expected. Finally, $w \in M_+(\mathbb{T})$ is a probability measure if and only if $\|w\|_{M} = 1$, which concludes the proof. 

B  Proof of Proposition 5

We first observe that

$$\mathcal{V}(y_0, e^{i\theta} y_1) = \{w \in M(\mathbb{T}), w(\cdot - \theta) \in \mathcal{V}(y_0, y_1)\}.$$  

(48)

One can therefore assume that $y_1 = r e^{i\alpha}$ is such that $\alpha = 0$, the solution set with a general $y_1$ then being obtained by a translation of $(-\alpha)$. 

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According to Lemma 1, we have that \( \min \{ \tilde{w}[0] = y_0, \tilde{w}[1] = r \} \|w\|_\mathcal{M} \geq \max(y_0, r) \). In particular, if \( w \) satisfies \( \tilde{w}[0] = y_0, \tilde{w}[1] = r, \) and \( \|w\|_\mathcal{M} = \max(y_0, r) \), then \( w \in \mathcal{V}(y_0, r) \). We then verify that this is the case for \( w^* \) given by (25). Indeed, we have

\[
\tilde{w}[0] = \frac{y_0 + r}{2} + \frac{y_0 - r}{2} = y_0, \\
\tilde{w}[1] = \frac{y_0 + r}{2} + \frac{y_0 - r}{2} e^{i\pi} = \frac{y_0 + r}{2} - \frac{y_0 - r}{2} = r, \quad \text{and} \\
\|w\|_\mathcal{M} = \max(y_0, r),
\]

which shows that \( w^* \) is solution and that \( \min \{ \tilde{w}[0] = y_0, \tilde{w}[1] = r \} \|w\|_\mathcal{M} = \max(y_0, r) \). Moreover, this implies that

\[
\mathcal{V}(y_0, r) = \{w \in \mathcal{M}(\mathbb{T}), \tilde{w}[0] = y_0, \tilde{w}[1] = r, \text{ and } \|w\|_\mathcal{M} = \max(y_0, r) \}. 
\]

The case \( y_0 < r \). According to Corollary 1\(^5\), this implies that the solution to (24) is unique, and is therefore given by \( w^* \) in (25).

The case \( y_0 = r \). According to Proposition 4, the extreme point solutions are sum of at most \( 2K_r + 1 = 3 \) Dirac masses. Moreover, (24) has a unique solution if and only if it has a unique extreme point (since \( \mathcal{V}(y) \) is the weak-* closure of the convex hull of its extreme points). Let \( w = a_1 \mathbb{I}(\cdot - \alpha_1) + a_2 \mathbb{I}(\cdot - \alpha_2) + a_3 \mathbb{I}(\cdot - \alpha_3) \) be an extreme point solution with distinct \( \alpha_p \). Any solution is such that \( \|w\|_\mathcal{M} = y_0 = \tilde{w}[0] \), which is only possible for non-negative measures (see Proposition 2). This implies that \( a_1, a_2, a_3 \) are non-negative. Without loss of generality, we assume that \( a_1 \geq a_2 \geq a_3 \geq 0 \), with \( a_1 \neq 0 \). Then,

\[
y_0 = |a_1 e^{i\alpha_1} + a_2 e^{i\alpha_2} + a_3 e^{i\alpha_3}| \leq |a_1| + |a_2| + |a_3| = a_1 + a_2 + a_3 = y_0. 
\]

Hence, \( |a_1 e^{i\alpha_1} + a_2 e^{i\alpha_2} + a_3 e^{i\alpha_3}| = a_1 + a_2 + a_3 \), which is only possible if the four points \( a_p e^{i\alpha_p}, \) \( p = 1, 2, 3 \) and \( z = 1 \) of the 2d plan \( \mathbb{C} \) are positively colinear. Because the \( \alpha_p \) are distinct, this implies that \( \alpha_1 = 0 \) and \( a_2 = a_3 = 0 \), and then \( a_1 = y_0 \). Hence, \( w^* = y_0 \mathbb{I} \): the solution is unique and we recover \( w^* \) in (25).

The case \( y_0 > r \). Our goal is to characterize the solutions of the form

\[
w = a_1 \mathbb{I}(\cdot - \alpha_1) + a_2 \mathbb{I}(\cdot - \alpha_2). 
\]

One easily sees that there is no solution with only one Dirac mass \( (w = a \mathbb{I}(\cdot - \alpha) \) is such that \( |\tilde{w}[0]| = |a| = |\tilde{w}[1]|) \), hence the solutions with 2 Dirac masses are extreme points. As for \( y_0 = r \), the solutions are non-negative measures, hence \( a_1, a_2 > 0 \). Without loss of generality, one can assume that \( a_1 \geq a_2 \).

Then, if \( a_2 < \frac{y_0 - r}{2} \), \( w \) cannot be solution. Indeed, we have that \( r = a_1 e^{i\alpha_1} + a_2 e^{i\alpha_2} \) with \( a_2 = y_0 - a_1 \). Hence, \( r - y_0 e^{i\alpha_1} = a_2 (e^{i\alpha_2} - e^{i\alpha_1}) \) and therefore

\[
y_0 - r \leq |r - y_0 e^{i\alpha_1}| = a_2 |e^{i\alpha_2} - e^{i\alpha_1}| \leq 2a_2 
\]

We therefore fix \( a_2 \in \left[ \frac{y_0 - r}{2}, \frac{y_0 - r}{2} \right] \), meaning that \( \alpha_1 = y_0 - a_2 \in \left[ \frac{y_0}{2}, \frac{y_0 + r}{2} \right] \). Then, denoting by \( C(z, r) \) the circle of center \( z \in \mathbb{C} \) and radius \( r > 0 \), the point \( a_1 e^{i\alpha_1} \) is at the intersection between \( C(0, a_1) \) and \( C(r, a_2) \). Due to \( a_2 \geq \frac{y_0 - r}{2} \), this intersection is non-empty: it contains 1 element if \( a_2 = \frac{y_0 - r}{2} \) and 2 if \( a_2 > \frac{y_0 - r}{2} \). This gives one or two possibilities for \( \alpha_1 \), and once \( \alpha_1 \) is determined, \( \alpha_2 \) has only one possibility. This gives a complete geometric description of the solutions with 2 Dirac masses. This construction shows in particular that there are uncountably infinitely many extreme points in \( \mathcal{V}(y_0, r) \), as expected.

\(^5\) It is possible to give an elementary demonstration of this fact without using the machinery behind Corollary 1.
C Trigonometric Toolbox

In this section is dedicated to known theoretical results (or easily deducible from known ones) that play a crucial role for the main result of this paper.

A sequence \((a_k)_{k \in \mathbb{Z}}\) of complex numbers is positive-definite if \(a_0 \in \mathbb{R}^+, a_{-k} = \overline{a_k}\) for any \(k \geq 0\), and for any sequence \((z_k)_{k \in \mathbb{Z}}\) of complex numbers with finitely many non-zero terms, we have

\[
\sum_{k, \ell \in \mathbb{Z}} a_{k-\ell} z_k \overline{z_\ell} \geq 0.
\]  

(56)

**Proposition 8** (Herglotz Theorem). A sequence \((a_k)_{k \in \mathbb{Z}}\) is positive-definite if and only if there exists a non-negative measure \(w \in \mathcal{M}_+(\mathbb{T})\) such that \(\hat{w}[k] = a_k\) for all \(k \in \mathbb{Z}\).

This theorem was obtained by Herglotz in [50]. For a modern exposition, we refer to [51, Theorem 7.6].

For \(K \geq 0\), we denote by \(\mathcal{P}_K(\mathbb{T})\) the set of real trigonometric polynomial of degree at most \(K\); i.e., functions of the form \(p = \sum_{|k| \leq K} c_k e_k\) with \(z_k\) such that \(c_0 \in \mathbb{R}\) and \(c_{-k} = \overline{c_k} \in \mathbb{C}\) for any \(1 \leq k \leq K\).

**Proposition 9** (Fejér–Riesz Theorem). Let \(p = \sum_{|k| \leq K} c_k e_k \in \mathcal{P}_K(\mathbb{T})\) be a positive trigonometric polynomial of degree \(K \geq 0\). Then, there exists a complex trigonometric polynomial \(q = \sum_{k=0}^K z_k e_k\) such that \(p = |q|^2\).

The Fejér–Riesz theorem was conjectured by Fejér [52] and shown by Riesz [53]. See [54, p. 26] for a recent exposition of this classic result. The next proposition deals with the extension of positive linear functionals from trigonometric polynomials to the space of continuous functions.

**Proposition 10.** Let \(K \geq 0\). Let \(\Phi : \mathcal{P}_K(\mathbb{T}) \to \mathbb{R}\) be a linear and positive functional (i.e., \(\Phi(p) \geq 0\) for any \(p \geq 0\)). Then, there exists an extension \(\hat{\Phi} : \mathcal{C}(\mathbb{T}) \to \mathbb{R}\) which is still linear and positive. Moreover, any such extension is continuous on \(\mathcal{C}(\mathbb{T}), ||\cdot||_\infty\).

**Proof.** Let \(E\) be an ordered topological vector space, \(C\) its positive cone, and \(M \subset E\). Then, according to [55, Corollary 2 p. 227], if \(C \cap M\) contains an interior point of \(C\), then any continuous, positive, and linear form over \(M\) can be extended as a continuous, positive, and linear form over \(E\). We apply this result to \(E = \mathcal{C}(\mathbb{T})\), whose positive cone is the space of positive continuous functions \(\mathcal{C}_+(\mathbb{T})\), and to \(M = \mathcal{P}_K(\mathbb{T})\). Then, \(C \cap M = \mathcal{C}_+(\mathbb{T}) \cap \mathcal{P}_K(\mathbb{T})\) contains the constant function \(p = 1\), which is an interior point of \(\mathcal{C}_+(\mathbb{T})\) due to \(\{f \in \mathcal{C}(\mathbb{T}), ||f - 1||_\infty \leq \frac{1}{2}\} \subset \mathcal{C}_+(\mathbb{T})\).

In our case, \(\Phi\) is continuous over \((\mathcal{P}_K(\mathbb{T}), ||\cdot||_\infty)\), since it is a linear functional over a finite-dimensional space. Hence, \(\Phi\) is continuous, positive, and linear, and admits the desired extension. \(\square\)

D Duality and Certificates for Convex Optimization on Measure Spaces

The analysis of (2) and (6) benefits from the theory of duality for infinite dimensional convex optimization, as exposed for instance by Ekeland and Temam in [56]. This line of research has proven to be extremely fruitful for optimization on measure spaces [3, 33, 57, 46]. We focus on the following problem for fixed \(K_c \geq 0\) and \(y \in \mathbb{R}^+ \times \mathbb{C}^{K_c}:\)

\[
\mathcal{V}(y) = \arg \min_{w \in \mathcal{M}(\mathbb{T}), \nu(w) = y} ||w||_\mathcal{M}
\]

(57)
We will mostly rely on the concepts and results exposed in [3, 46]. We recall that a Radon measure \(w\) can be uniquely decompose as \(w = w_+ - w_\) where \(w_+\) and \(w_-\) are non-negative measures (Jordan decomposition).

**Definition 5.** Let \(w \in M(\mathbb{T})\). We define the signed support of \(w\) by

\[
supp_\pm(w) = supp(w_+) \times \{1\} \cup supp(w_-) \times \{-1\}.
\]

Let \(\eta \in C(\mathbb{T})\) be such that \(\|\eta\|_\infty \leq 1\). The positive and negative saturation sets of \(\eta\) are given by

\[
sat_+(\eta) = \eta^{-1}(\{1\}) \quad \text{and} \quad sat_-(\eta) = \eta^{-1}(\{-1\}),
\]

respectively. Finally, we define the signed saturation set of \(\eta\) by

\[
sat_\pm(\eta) = sat_+(\eta) \times \{1\} \cup sat_-(\eta) \times \{-1\}.
\]

We summarize the main results which help us to characterize the cases of uniqueness for (57).

**Proposition 11.** Let \(K_c \geq 0\) and \(y \in \mathbb{R}^+ \times \mathbb{C}^{K_c}\) and \(w \in M(\mathbb{T})\). The following statements are equivalent.

- The measure \(w\) is a solution of (57).
- There exists a real trigonometric polynomial \(\eta \in C(\mathbb{T})\) of degree at most \(K_c\) such that \(\|\eta\|_\infty \leq 1\) and \(\|w\|_M = \langle w, \eta \rangle\).
- There exists a real trigonometric polynomial \(\eta \in C(\mathbb{T})\) of degree at most \(K_c\) such that \(\|\eta\|_\infty \leq 1\) and \(supp_\pm(w) \subset sat_\pm(\eta)\).

In this case, for any other solution \(w^* \in V(y)\), we have that

\[
supp_\pm(w^*) \subset sat_\pm(\eta).
\]

The function \(\eta\) in Proposition 11 is called a dual certificate of (57). Note that a dual certificate always exist for Fourier sampling [1]. Proposition 11 is an application of the main results of [3], where dual certificates of optimization problems of the form (57) are studied. A key role is played by the adjoint operator \(\nu^*: \mathbb{R}^+ \times \mathbb{C}^{K_c} \to C(\mathbb{T})\) (denoted by \(\Phi^*\) in [3]). For the Fourier sampling scenario, we have that

\[
\nu^*(c_0, c_1, \ldots, c_{K_c}) = \sum_{|k| \leq K_c} c_k \hat{c}_k
\]

with the convention that \(c_{-k} = c_k\) for \(1 \leq k \leq K_c\). This explains the role of trigonometric polynomials in Proposition 11. We do not provide a detailed proof of Proposition 11 since it has already been exposed elsewhere. It is for instance done in [46, Propositions 1 & 2] in a different setting, but the arguments can be readily adapted.

An important consequence for the uniqueness is the following proposition, that can also be deduced from [3]; see also [27]. We provide a proof for the sake of completeness.

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6Duval and Peyré consider more general measurement operators that can even be infinite dimensional and exemplify their results for low-frequency measurements.

7Strictly speaking, the fact that we use complex Fourier series measurements implies that the dual certificates of Problem (57) are complex trigonometric polynomials of the form \(\sum_{k=0}^{K_c} c_k \hat{c}_k\), which differs from (62). However, one can modify the initial problem into an equivalent one by imposing that \(\tilde{w}[k] = y_k\) for \(-K_c \leq k \leq K_c\) with the convention \(\tilde{y}_{-k} = \tilde{y}_k\), leading to (62). It is indeed more convenient to work with real dual certificates when dealing with real Radon measures.
Proposition 12. If there exists a non-constant dual certificate for the optimization problem (57), then it has a unique solution of the form \( w = \sum_{k=1}^{K} a_k \Pi(\cdot - x_k) \) with \( K \leq 2K_c \).

Proof. We start with three observations:

1. With Proposition 11, we know that for any certificate \( \eta \) and any solution \( w^* \) to (57), we have that \( \text{supp}_\pm(w^*) \subset \text{sat}_\pm(\eta) \).

2. Assume that a non-constant certificate \( \eta \) exists. Firstly, \( \eta \) is a trigonometric polynomial of degree at most \( K_c \) and so is its derivative \( \eta' \). Each saturation point \( x_0 \in \mathbb{T} \) is a local optimum of \( \eta \) and is therefore a root of \( \eta' \). The non-degeneracy of \( \eta \) implies that \( \eta' \) has finitely many roots, the number of which is then bounded by \( 2K_c \) [58, p. 150].

3. Let \( \tau = (\tau_1, \ldots, \tau_P) \in \mathbb{T}^P \) be the distinct roots of \( \eta' \), with \( P \leq 2K_c \). For \( a \in \mathbb{R}^P \), we introduce \( w_{a,\tau} = \sum_{p=1}^P a_p \Pi(\cdot - \tau_p) \). Consider the mapping \( \Phi : \mathbb{R}^P \to \mathbb{R} \times \mathbb{C}^{K_c} \) such that

\[
\Phi(a) = \nu(w_{a,\tau}).
\]

Then, \( \Phi \) is injective. Indeed, let \( a \) such that \( \Phi(a) = 0 \). Then, for each \( -K_c \leq k \leq K_c \), we have that \( \hat{w}_{a,\tau}(k) = 0 \) (this is due to (63) for \( k \geq 0 \) and to Hermitian symmetry, \( w \) being real by assumption, for \( k < 0 \)). This provides \( 2K_c + 1 \) linear equations over \( a \) that can be written in matrix form as \( Ma = 0 \) with

\[
M = \begin{bmatrix}
e^{-iK_c\tau_1} & \ldots & e^{-iK_c\tau_P} \\
\vdots & \ddots & \vdots \\
e^{-i\tau_1} & \ldots & e^{-i\tau_P} \\
1 & \ldots & 1 \\
e^{+i\tau_1} & \ldots & e^{+i\tau_P} \\
\vdots & \ddots & \vdots \\
e^{+iK_c\tau_1} & \ldots & e^{+iK_c\tau_P}
\end{bmatrix} = \begin{bmatrix} e^{ik\tau_p} \end{bmatrix}_{1 \leq p \leq P, -K_c \leq k \leq K_c} \in \mathbb{C}^{(2K_c+1) \times P}.
\]

One recognizes a Vandermonde-type matrix, which is therefore of full rank \( P \) since the \( e^{i\tau_p} \) are distinct and \( P < 2K_c + 1 \). This implies that \( a = 0 \) and thus that \( \Phi \) is injective.

Let \( \eta \) be a non-constant certificate. Due to 2., it has at most \( 2K_c \) saturation points in \( \text{sat}_\pm(\eta) \), that we denote by \( \tau_1, \ldots, \tau_P \) with \( P \leq 2K_c \). According to 1., any solution is a sparse measure whose support is included in \( \text{sat}_\pm(\eta) \). In particular, it is of the form \( w_{a,\tau} \) for some \( a \in \mathbb{R}^K \). However, due to 3., there is at most one measure \( w_{a,\tau} \) such that \( \nu(w_{a,\tau}) = y \). Hence, the solution, if it exists, is unique. Finally, we already know from Proposition 4 that solutions exist, which concludes the proof.

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*In the more general case studied in [3], this corresponds to the non-degeneracy condition of the dual certificate.*
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