THE VALIDITY OF PERTURBATION THEORY FOR THE $O(N)$ NON-LINEAR SIGMA MODELS

James M. Cline

California Institute of Technology, Pasadena, CA 91125, USA

Received 14 March 1986

Abstract

Recently it has been claimed that ordinary perturbation theory (OPT) gives incorrect weak coupling expansions for lattice $O(N)$ non-linear sigma models in the infinite volume limit, and in particular that the two-dimensional non-abelian models are not asymptotically free, contrary to previous findings. Here it is argued that the problem occurs only for one-dimensional infinite lattices, and that in general, OPT gives correct expansions if physical quantities are first computed on a finite lattice, and the infinite volume limit is taken at the end. In one dimension the expansion is sensitive to boundary conditions because of the severe infrared behavior, but this is not expected to happen in higher dimensions. It is concluded that spin configurations which are far from the perturbative vacuum have too small a measure in the path integral to invalidate OPT, even though they are energetically allowed for non-zero values of the coupling.

Two-dimensional spin systems have been important field theoretic laboratories because of their similarities to four-dimensional gauge theories. For example, when $N > 2$ the $O(N)$ symmetry is non-abelian and the theory has been shown to be asymptotically free [1,2]. The sigma model could also be regarded as a testing ground for perturbation theory. As for gauge theories, the perturbative vacuum, with all spins aligned, is quite different from the true vacuum, since there is no magnetization for $d \leq 2$ [3]. A further similarity is that the expansion proceeds in powers of the coupling and the fields, although there is no mass to damp out configurations far away from the perturbative vacuum. Therefore one might worry that very long wavelength excitations, where the fields eventually get large, will contribute significantly to the path integral, and that ordinary perturbation theory (OPT) may not account for them correctly.

Thusly Patrasciou has argued that OPT gives incorrect low-temperature expansions for the free energy and spin correlations of the $O(N)$ non-linear sigma model, for $N > 2$ and infinite volume lattices [4]. (For finite lattices there always exist temperatures low enough so that the spins are all relatively aligned, and OPT should have no problem.) To remedy this, he formulates a new perturbation expansion which treats the gradients of the fields, rather than the fields themselves, as small quantities. Despite the method’s computational difficulty (it is very nonlocal), he extracts a value for the one-oop spin-spin correlation function in two dimensions, which yields an opposite sign for the Callan-Symanzik $\beta$ function, relative to the OPT result.

It is important to investigate this claim, for if it is true, then it could conceivably be that the $\beta$ function for QCD or other gauge theories is different—in magnitude, if not in sign—from what is presently accepted. In this letter I show that OPT gives the correct result in one dimension, where the theory is exactly soluble, if it is formulated on a finite lattice with the correct boundary condition. Since this result is independent of the lattice size, $L$, it is trivially correct in the infinite volume limit. This procedure should give correct results in higher dimensions, where the IR behavior is milder. It is shown that starting with an infinite lattice gives results consistent with the exact solution for $N \to \infty$ (as well as the infinite volume limit of the finite lattice calculations) except in the one-dimensional models, apparently because of their more severe IR divergences.

The controversy over OPT’s validity stems from the difference between the OPT result for the free energy
density on a $d$-dimensional, infinite lattice [5],
\[
\frac{\ln Z}{L^d} = \frac{d}{g} + \frac{1}{2}(N-1)\ln g + \frac{(N-2)}{8d}g + O(g^2)
\]  
(1)

versus the expansion of the exact result for $d = 1$ [6]
\[
\frac{\ln Z}{L} = \frac{1}{g} + \frac{1}{2}(N-1)\ln g - \frac{1}{4}(N-1)(N-3)g + O(g^2)
\]  
(2)

where the lattice partition function is given by
\[
Z = \int \prod_{x} d\delta(S_x^2 - 1) \exp \left\{ \frac{1}{g} \sum_{\mu \neq 1}^{d} S_x \cdot S_{x+\hat{\mu}} \right\}.
\]  
(3)

OPT proceeds by rewriting $S$ as $(\pi, \sigma)$, where $\pi$ is $(N-1)$-dimensional, and solving the constraint, so that $\sigma = \pm (1 - \pi^2)^{1/2}$. The vacuum state is chosen to be $\sigma = 1$, and configurations with $\sigma < 0$ are discarded, as they give, naively, contributions of $O(\exp(-1/g))$. After rescaling $\pi^2$ to $g\pi^2$, $Z$ becomes
\[
Z = \exp(dL^d/g) \int \prod_{x} \frac{d\pi_x}{(1 - g\pi_x^2)^{1/2}} \times \exp\left\{ -\sum_{x,u} \frac{1}{2}(\Delta_u\pi_x)^2 - \frac{1}{2g} \sum_{x} \left[ \Delta_u(1 - g\pi_x^2)^{1/2} \right]^2 \right\}.
\]  
(4)

To do perturbation theory, the integration region must be extended from $\pi^2 \leq 1/g$ to $\pi^2 \leq \infty$ and the radicals Taylor-expanded.

More crucially, the zero modes of the $\pi$ field must be removed. This could be accomplished by introducing a magnetic field in the $e_N$ direction (i.e., $\pi = 0$ and $\sigma = 1$), which is equivalent to a mass for the $\pi$, and removing it at the end of the calculation. However, this method is known to give wrong results even for a two-spin system, beyond the tree level [7]. It is easy to see why this happens, heuristically: when the factor $(1 - g\pi^2)^{1/2}$ in the measure is reexpressed as a term in the action, it contributes a mass of the wrong sign, $m^2 = -g$. Then, as the magnetic field becomes smaller than $g$, the action becomes unbounded from below, and the gaussian integrations no longer make sense. A better way is to use the global $O(N)$ invariance to fix one of the spins, say $S_0$, to be in the $e_N$ direction. More generally, one can use the Faddeev-Popov procedure to fix any linear combination of the $S_x$, thus removing the zero modes, and this gives agreement with exact solutions in every case that has been checked [7]. However, fixing spins is not sufficient for an infinite lattice, because it provides a low-momentum cutoff of $O(1/L)$ ($L$ being the lattice size) which vanishes as $L \to \infty$ causing the $\pi$ propagators to become undefined. Thus one is forced to bring in a magnetic field again. Since this is a bad procedure for finite systems, there is no a priori reason for it to work on an infinite lattice. But this is exactly how (1) was obtained; hence the discrepancy with (2).

Patrascioiu has a different explanation for why OPT goes wrong as $L \to \infty$ in 1 dimension, however. Note that the two-point function is given by
\[
\langle S_0 \cdot S_{\sigma} \rangle = g^2 \left( \langle \pi_0 \cdot \pi_\sigma \rangle + (1 - g\pi_0^2)^{1/2}(1 - g\pi_\sigma^2)^{1/2} \right) = 1 + g(N-1)D(x) + O(g^2)
\]  
(5)

at tree level, where $D(x) \equiv G(x) - G(0)$ and $G(x)$ is the massless scalar propagator in $d$ dimensions. The large-$x$ dependence of $D(x)$ is $|x|, -\ln |x|$ and $|x|^{-1}$ for $d = 1, 2, 3$, respectively. Therefore we have configurations in which the spins wander far away from $e_N$, the perturbative vacuum, over a distance $|x| \sim g^{-1}$ ($|x| \sim \exp(g^{-1})$) for $d = 1$ ($d = 2$), whereas these are energetically suppressed for $d \geq 3$ (magnetization). Of course, OPT is not designed to account for spins pointing far away from $e_N$, since $\sigma < 0$ was discounted. The argument is that for finite $L$ there should be no problem, because there always exists a $g$ small enough so that $gL$ or $g\ln L$ is $\ll 1$: then no large excursions away from $e_N$ are allowed. For $L = \infty, d \leq 2$, the phase transition occurs exactly at $g = 0$, so that for finite $g$ these long spin waves are unsuppressed, and OPT may fail.

In order to clarify why OPT fails in one dimension, I compute the correlation function $\langle S_0 \cdot S_{\sigma} \rangle$ in $d$ dimensions, using a somewhat different method, which is to obtain the non-linear $\sigma$ model from the infinite mass limit of the linear model. This has the advantage of avoiding the OPT approximations which made the long spin waves impossible to represent correctly. It is instructive to do the calculation on both finite and infinite lattices, for it will become apparent that only in one dimension does any discrepancy appear.

First consider the finite lattice. We start by noticing that the $\delta$ functions in (3) can be rewritten using the identity $\delta(x) = \lim_{M \to \infty}(M/\sqrt{\pi}) \exp(-M^2 x^2)$. After the change of variables $S \to (\sqrt{\pi}g, 1 + \sqrt{\pi}g)$ and "gauge
\[ Z = \lim_{M \to \infty} \left( \frac{M}{\sqrt{8\pi}} \right)^L \exp \left\{ L^d \left( \frac{d}{g} + \frac{1}{2}(N-1) \ln g \right) \right\} \times \int \text{D} \pi \text{D} \sigma [\Delta_F(\pi) \delta(F(\pi))] \]  

\[ \times \exp \left\{ -\frac{1}{4} \sum_{\mu} \left[ (\Delta_{\mu} \pi_\mu)^2 + (\Delta_{\mu} \sigma_\mu)^2 + M^2 \sigma^2_\mu \right] \right\} \]  

\[ - M^2 \sum_\mu \left[ \frac{2}{g} (\pi_\mu^2 + \sigma_\mu^2) + \frac{1}{2} \sqrt{8g} \sigma_\mu (\pi_\mu^2 + \sigma_\mu^2) \right] \]  

where \( \delta(F) \) removes the zero modes of \( \pi \), and \( \Delta_F \) is the associated determinant. A convenient choice which preserves the \( \pi \) propagator’s translational invariance is \( F = \sum_\mu \pi_\mu \), in which case \( \Delta_F \) turns out to be \( (1 + \sqrt{8L} \Delta_{\mu} \sum_\sigma \sigma_\mu)^{N-1} \) (see ref. [7]), and the bare propagator is

\[ \langle \pi_\mu(0) \pi_\nu(0) \rangle = (N-1) \Gamma(x) \]  

\[ = (N-1) L^d \sum_{\rho \nu} \exp \left\{ i2\pi (\rho \cdot x/L) \right\} P(p) \]  

\[ P(p) = 4 \sum_{\mu} \sin^2 \pi p_\mu/L. \]  

The propagators and vertices of this theory are shown in fig. 1, including the lowest order vertex due to \( \Delta_F \). To compute the two-point function

\[ \langle S_0 \cdot S_\mu \rangle = 1 + \sqrt{8}(\langle \sigma_\mu \rangle + \langle \sigma_\nu \rangle) + g(\langle \sigma_\nu \sigma_\mu \rangle + \langle \pi_\mu \cdot \pi_\nu \rangle) \]  

leading \( M^2 \)-dependences are nonvanishing. Graphs 3 gives examples of graphs which vanish as \( M \to \infty \) because they have more \( \sigma \) propagators than non-\( \Delta_F \) vertices. Although some of the graphs in fig. 1 diverge like \( M^2 \), their sums are finite, as expected, and the factor \( M^{d/2} \) in \( Z \) is exactly cancelled by \( \det(\Delta_F + M^2) \) coming from the \( \sigma \) integrations. After much algebra, the one-loop contribution to \( \langle S_0 \cdot S_\mu \rangle \) obtained is

\[ \langle S_0 \cdot S_\mu \rangle^{(2)} = g^2 \left( - (N-1)(N-2) + \frac{1}{L^{d-2}} \sum_{\rho \nu} \exp(2\pi \rho \cdot x/L) - 1 \right) \]  

\[ + \frac{1}{d} \sum_{\rho \nu} \exp(2\pi \rho \cdot x/L) \]  

\[ \frac{1}{d} \left( 1 - \frac{1}{L} + D(x) \right) \]  

in exact agreement with the OPT result of Hasenfratz [7]. This just shows that the OPT approximations—ignoring \( \sigma < 0 \) configurations, and enlarging the integration region beyond the radius of convergence of \( (1 - g\pi^2)^{1/2} \)’s Taylor series—are justified for \( L < \infty \) as expected.

Before carrying out the finite lattice calculation of \( \langle S_0 \cdot S_\mu \rangle \), I will show that the procedure leading to (9) is correct for \( d = 1 \) and any \( L \) (and thus, as \( L \to \infty \)). This case is important because: (1) only here is the exact solution known (unless \( N = \infty \); (2) the low-energy spin waves should affect OPT most severely when \( d = 1 \), by Patrascioiu’s argument; and (3) the only known failure of OPT occurs when \( d = 1 \). For simplicity, let us only consider the average interaction energy per spin,

\[ f = \frac{1}{L} \sum_n \langle S_n \cdot S_{n+1} \rangle = -g^2 \frac{d \ln Z}{d \beta} \]  

From (9) we find that

\[ \lim_{L \to \infty} f_{\text{OPT}} = 1 - \frac{1}{d} g(N-1) \]  

\[ + \frac{1}{4g} g^2 (N-1)(N-5) + O(g^3) \]
whereas the expansion of the exact result is [6]

\[ f_{\text{exact}} = 1 - \frac{1}{8}g(N - 1) + \frac{1}{8}g^2(N - 1)(N - 3) + O(g^3) \] (12)

independent of \( L \). In comparing these it must be remembered that the exact result is obtained for an open chain of spins, in which the ends do not interact with each other, whereas OPT is normally done on a periodic lattice, which is the case in (11). To repeat the OPT calculation (4) for the open chain, one must use the open chain lattice propagator, which in \( \sum x \pi_x = 0 \) gauge and at \( O(g^0) \) is exactly

\[ G_{\text{open}}(x,y) = G_{xy} + L(G_{1y} - G_{1x})(G_{1x} - G_{1y}) \] (13)

Here \( G_{xy} \) is the periodic lattice propagator in (7), which for \( d = 1 \) has the closed form

\[ G_{xy} = \frac{1}{11}(L - L^{-1}) - \frac{1}{2}|x - y \mod L| + \frac{1}{2L}(x - y \mod L)^2 \] (14)

Therefore the momentum sums in loop graphs can be done explicitly. The result is (12), including the exact \( L \)-independence which distinguished it. Thus, OPT agrees with the exact result in \( d = 1 \) as long as the same boundary conditions as in the exact solution are used. This does not mean that (11) is incorrect. To prove that the difference between (11) and (12) is due to the physical difference made by the extra link interaction, and not some fluke of OPT, I applied Patrascioiu’s method to the periodic spin chain with \( L \) spins and confirmed (11). (This letter has no complaint with the method of ref. [4], but only its result.)

Of course, the exact expressions for \( f \) must agree for the two boundary conditions as \( L \to \infty \). The only plausible explanation for the difference in the perturbative expansions is that the exact expressions differ by a function like

\[ f_{\text{open}} - f_{\text{closed}} \sim L^{-1} \exp[-g^2L(N - 1)(N - 3)/12] \] (15)

for example. Even though this vanishes as \( L \to \infty \), its expansion in powers of \( g \) does not, and in fact it predicts that the weak-coupling expansion of \( f_{\text{closed}} \) is IR-divergent, starting in \( O(g^3) \). From the point of view of the perturbative calculation, the reason for sensitivity to the boundary condition, which is a \( 1/L \) effect, is that the propagator is linear in distance and so acquires values of \( O(L) \) in the loop diagrams. In two dimensions the propagator is logarithmic, so we might expect the \( L \) in the exponent of (15) to be replaced by \( \ln L \). The resulting expression cannot lead to differences in perturbation expansions, so the effect explained by (15) is almost certainly unique to \( d = 1 \).

Having seen that OPT on a finite lattice gives the correct expansion for the energy density as \( L \to \infty \), in one dimension, we now examine what happens when \( L \) is taken to be infinite at the outset. In this case, as noted previously, the Faddeev-Popov terms in (6) are insufficient for regulating the \( \pi \) propagator; the field must be given a mass, \( \mu \), to be eventually removed. But this spoils the \( O(N) \) symmetry, and so one is not justified to insert gauge-fixing terms: \( \Delta_F \delta(F) \) should no longer ap-
appear in (6). (Actually, for the gauge choice $F = \sum_i \pi_i$, it turns out not to matter whether one keeps the $\Delta\pi \delta(F)$ factor, when $L = \infty$.) It is straightforward to show that the only effect of these changes on the result, (9), are (1) to replace the second term by its $L \to \infty$ limit (recall that $D(x)$ is IR-finite); and (2) to replace the first term, the $\Delta\pi$ contribution, by $-\frac{1}{2}g^2(N - 1)|\pi|^2$ (zero) when $d = 1$ ($d > 1$). This agrees fully with the OPT calculation on an infinite lattice for $d = 2$ which was done by Elitzur [2]. Since the present method was designed to correctly measure the contributions of the long spin waves, yet it agrees with OPT for $d \geq 2$, we conclude that the long spin waves do not cause OPT to give incorrect results when $d \geq 2$. Furthermore, the problem in $d = 1$ is clearly seen to be due to the noncommutativity of the two limits $g \to 0$ and $\mu \to 0$ (where by $g \to 0$ I mean developing the asymptotic expansion), just as was suggested earlier. On the other hand, this analysis shows that these limits do commute for $d \geq 2$. (More precisely, the order $\mu \to 0$, $g \to 0$, $L \to \infty$ commutes with $L \to \infty$, $g \to 0$, $\mu \to 0$ for $d \geq 2$.)

This conclusion derives further support from comparison with the $N = \infty$ limit of the model, where again exact solutions are known [8]. This limit exists if the coupling is rescaled so that $\beta \equiv 1/gN$ is held fixed as $N \to \infty$. Then, for example, (12) would become

$$f_{N \to \infty}^{d=1} = 1 - \frac{1}{2\beta} + \frac{1}{8\beta^2} + O(\beta^{-3})$$

(16)

and using the fact that $D(1) = -\frac{d}{4}$ for $d = 2$, (5) and (9) give us

$$f_{N \to \infty}^{d=2} = \langle S_0 \cdot S_1 \rangle = 1 - \frac{1}{4\beta} + O(\beta^{-3})$$

(17)

There should be no problem with first doing the small $g$ expansion and then taking the $L \to \infty$ limit, since the latter requires $g$ to be infinitesimal. Indeed, the exact solution for $N = \infty$, in one dimension [6] is

$$f_{N \to \infty}^{d=1} = \frac{2\beta}{1 + (1 + 4\beta^2)^{1/2}}$$

(18)

which agrees with (16) when expanded in powers of $1/\beta$. In two dimensions, the exact solution [9] is

$$f_{N \to \infty}^{d=2} = \langle S_0 \cdot S_1 \rangle = \frac{1 - \frac{1}{2\beta} + O(\beta^{-3})}{\beta} \times$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{4}{4} \sum_\mu \cos p_\mu + \frac{1}{8} \sum_\mu \cos p_\mu$$

(19)

where $m$ is a dynamically generated mass, which for large $\beta$ is given by

$$m^2 = 32 \exp(-4\pi\beta) + O(\exp(-8\pi\beta))$$

(20)

For large $\beta$, hence small $m$, the integral is dominated by its $p \equiv 0$ contributions, and we can approximate it by

$$\int_0^\Lambda dp \frac{p}{p^2 + m^2} \approx \frac{1}{4\pi} \ln(\Lambda^2/m^2)$$

$$= \beta - \frac{1}{4\pi} \ln(32/m^2) + O(\exp(-4\pi\beta))$$

(21)

where $\Lambda = O(1)$. The important thing to notice is that when this is substituted into (19), the expansion of $f$ in powers of $1/\beta$ terminates after the $1/\beta^2$ term. This agrees with the OPT result, (17), whose $1/\beta^2$ term is seen to vanish. In contrast, Patrasciu’s result for the energy density in $d = 2$ contains a piece of order $g^2N^2$ which when reexpressed in the large-$N$ limit is a $1/\beta^2$ term. There seems to be no subtlety of ordering limits in this comparison, since (19) was derived for an infinite lattice, and as has already been mentioned, $g \to 0$ and $N \to \infty$ are expected to commute.

Finally, the $\beta$ function calculation of Polyakov [1] merits attention because it is on a completely independent footing from the OPT method, yet it gets the same answer, $\beta(g) = -g^2(N - 2)/2\pi + O(g^4)$. The salient point is that it uses Wilson’s method of integrating out the high-momentum components of the spin field and seeing how the coupling is renormalized in the effective action for the remaining low-momentum components. Thus, $\beta(g)$ is determined by the short wavelength fluctuations, and is insensitive to the long spin wave effects by which Patrasciu explains the difference between his $\beta$ function and the standard result.

In conclusion, I have shown that OPT on a finite lattice is capable of giving the correct asymptotic weak-coupling expansion for $d = 1$, even in the infinite volume limit; therefore the same procedure should work in two dimensions, corroborating the standard result: asymptotic freedom for $N \geq 3$. The fact that the $M \to \infty$ limit of the linear $\sigma$ model (performed here), the known $N = \infty$ solutions, and Wilson’s renormalization group (ref. [1]) lead to the same answer gives one yet more confidence in ordinary perturbation theory, as applied to the $O(N)$ nonlinear $\sigma$ models.

I am indebted to John Preskill for suggesting this problem and for many stimulating discussions, and also to J. Feng and B. Warr for the latter. Lately I learned that H.D. Politzer and G. Siopsis were carrying out related investigations [10]. I thank them for helpful exchanges of ideas.

References

[1] A.M. Polyakov, Phys. Lett. B 59 (1975) 79.
[2] S. Elitzur, Nucl. Phys. B212 (1983) 501.
[3] N.D. Mermin and H. Wagner, Phys. Rev. Lett. 17 (1966) 1133.
[4] A. Patrasciou, Phys. Rev. Lett. 56, 1023 (1986).
[5] Y. Brihaye and P. Rossi, Nucl. Phys. B255 (1984) 226.
[6] H.E. Stanley, Phys. Rev. 179 (1969) 570.
[7] P. Hasenfratz, Phys. Lett. B 141 (1984) 385.
[8] H.E. Stanley, Phys. Rev. 176 (1968) 718.
[9] P. Di Vecchia, R. Musto, F. Nicodemi, R. Pettorino, P. Rossi and P. Salomonson, Phys. Lett. B 127 (1983) 109.
[10] H.D. Politzer and G. Siopsis, unpublished.