Division theorems for the rational cohomology of some discriminant complements

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Abstract

The main purpose of this paper is to show that the mixed Hodge polynomial of the “space of equations” for smooth complete intersections of given multidegree in $\mathbb{C}P^n$ is divisible by the mixed Hodge polynomial of the group $\text{GL}_{n+1}^+(\mathbb{C})$, the quotient being the mixed Hodge polynomial of the corresponding quotient space. As a by-product of the method used in the proof, we obtain expressions divisible by the order the automorphism group of any smooth projective hypersurface of given dimension and degree.

1 Introduction and main results

Let $n$ and $k$ be integers satisfying $1 \leq k \leq n+1$. Set $\mathbf{d} = (d_1, \ldots, d_k)$ to be a collection of integers such that $2 \leq d_1 \leq \cdots \leq d_k$. Denote by $\Pi_{\mathbf{d}, n}$ the $\mathbb{C}$-vector space of all $k$-tuples $(f_1, \ldots, f_k)$, where $f_i, i=1, \ldots, k$, is a homogeneous polynomial in $n+1$ variables of degree $d_i$ with coefficients in $\mathbb{C}$. For every $(f_1, \ldots, f_k) \in \Pi_{\mathbf{d}, n}$ denote by $\text{Sing}(f_1, \ldots, f_k)$ the projectivisation of the set of all $x \in \mathbb{C}^{n+1} \setminus \{0\}$ such that
- $f_i(x) = 0, i = 1, \ldots, k$,
- the gradients of $f_i, i = 1, \ldots, k$ at $x$ are linearly dependent.

Set $\Sigma_{\mathbf{d}, n}$ to be the subset of $\Pi_{\mathbf{d}, n}$ consisting of all $(f_1, \ldots, f_k)$ such that $\text{Sing}(f_1, \ldots, f_k) \neq \emptyset$. If $(f_1, \ldots, f_k) \in \Pi_{\mathbf{d}, n} \setminus \Sigma_{\mathbf{d}, n}$, then the subvariety $X$ of $\mathbb{C}P^n$ defined by $f_1 = \cdots = f_k = 0$ is smooth, and $f_1, \ldots, f_k$ generate the homogeneous ideal of $X$. For this reason the space $\Pi_{\mathbf{d}, n} \setminus \Sigma_{\mathbf{d}, n}$ can be viewed as the space of equations for some smooth complete intersections of multidegree $\mathbf{d}$ in $\mathbb{C}P^n$. The case $k = n+1$ does not quite agree with this interpretation (it would correspond to “empty complete intersections”), but we include it nonetheless, since we shall need it later as a starting point for some computations.

The group $\text{GL}_{n+1}^+(\mathbb{C})$ acts on $\Pi_{\mathbf{d}, n}$ in an obvious way:

$$\text{GL}_{n+1}^+(\mathbb{C}) \times \Pi_{\mathbf{d}, n} \ni (A, (f_1, \ldots, f_k)) \mapsto (f_1 \circ A, \ldots, f_k \circ A);$$

(this action preserves $\Sigma_{\mathbf{d}, n}$ (and hence, $\Pi_{\mathbf{d}, n} \setminus \Sigma_{\mathbf{d}, n}$).

The main purpose of the paper is to prove the following theorem.

**Theorem 1.** Suppose $\mathbf{d} \neq (2)$. Then the geometric quotient of $\Pi_{\mathbf{d}, n} \setminus \Sigma_{\mathbf{d}, n}$ by $\text{GL}_{n+1}^+(\mathbb{C})$ exists, and the Leray spectral sequence of the corresponding quotient map degenerates over $\mathbb{Q}$ (or modulo a sufficiently large prime) at the second term.
This theorem generalises a recent result of J. Steenbrink and C. Peters for the case \( k = 1 \) \( \square \). Our general strategy will be the same as in \( \square \); however, the details will be different and more elementary (or so we hope).

For a complex algebraic variety \( V \), we define the mixed Hodge polynomial of \( V \) to be

\[
P_{mHdg}(V) = \sum_{n,p,q} t^n u^p v^q \dim_C(\Gr^p_F \Gr^{W}_{p+q} H^n(V, \mathbb{C})).
\]

By setting in this expression \( v = u \), respectively, \( u = v = 1 \), we get the Poincaré-Serre polynomial, respectively, the Poincaré polynomial, of \( V \). The mixed Hodge polynomial of \( V \) with compact supports (which we denote by \( P_{mHdg,c} \)) is obtained by replacing \( H^n \) by \( H^n_c \) in the definition of \( P_{mHdg} \); by specialising \( P_{mHdg,c} \) at \( t = -1 \) we get the Serre characteristic of \( V \).

An easy corollary of theorem \( \square \) is

**Corollary 1.** We have

\[
P_{mHdg}(\Pi_{d,n} \setminus \Sigma_{d,n}) = P_{mHdg}((\Pi_{d,n} \setminus \Sigma_{d,n})/\text{GL}_{n+1}(\mathbb{C})),
\]

\[
P_{mHdg,c}(\Pi_{d,n} \setminus \Sigma_{d,n}) = P_{mHdg,c}((\Pi_{d,n} \setminus \Sigma_{d,n})/\text{GL}_{n+1}(\mathbb{C})).
\]

By the Leray-Hirsch principle, in order to prove theorem \( \square \) it suffices to construct global cohomology classes on \( \Pi_{d,n} \setminus \Sigma_{d,n} \) (over \( \mathbb{Q} \) or modulo a prime \( p, p \gg 0 \)) such that their pullbacks under any orbit map generate the cohomology of the group \( \text{GL}_{n+1}(\mathbb{C}) \) (as a topological space). We realise such classes as linking numbers with some natural subvarieties of \( \Sigma_{d,n} \).

It turns out that in our situation working with integer coefficients is just a little bit more difficult than with the rationals. However, taking this little extra effort pays off, since it enables one to determine explicitly which multiple of the generator of the highest cohomology group of \( \text{GL}_{n+1}(\mathbb{C}) \) comes from the cohomology of \( \Pi_{d,n} \setminus \Sigma_{d,n} \) via an orbit map. This (together with some simple computations) implies the following results:

**Theorem 2.** Let \( d \) be an integer \( > 2 \). Then the order of the subgroup of \( \text{GL}_{n+1}(\mathbb{C}) \) consisting of the transformations that fix \( f \in \Pi_{d,n} \setminus \Sigma_{d,n} \) divides

\[
\prod_{i=0}^{n}((-1)^{n-i} + (d-1)^{n-i+1})(d-1)^i.
\]

Actually, we prove in section \( \square \) an analogous statement for arbitrary \( d \) (theorem \( \square ' \)), but the resulting formula is a bit messy (and was therefore banned from the introduction).

**Theorem 3.** The order of the subgroup \( \text{PGL}_{n+1}(\mathbb{C}), n \geq 1 \), consisting of the transformations that preserve a smooth hypersurface of degree \( d \geq 2 \) divides

\[
\frac{1}{n+1} \prod_{i=0}^{n-1} \frac{1}{C_{n+1}^i}((-1)^{n-i} + (d-1)^{n-i+1})\text{LCM}(C_{n+1}^i(d-1)^i, (n+1)(d-1)^n).
\]

(Here \( \text{LCM} \) stands for the least common multiple.)

By the Lefschetz principle, the statements of theorems \( \square \) and \( \square ' \) are in fact true over any algebraically closed field of characteristic 0.
In the case \( n = 1 \), theorem 3 is equivalent to saying that the order of the group of linear fractional transformations that preserve a (given) subset of \( d > 2 \) points of \( \mathbb{C}P^1 \) divides \( d(d-1)(d-2) \); this should be easy to prove directly. Notice that if \( d, d-1 \) and \( d-2 \) are pairwise coprime, one can not expect to have a strictly stronger result.

For curves in \( \mathbb{C}P^2 \), surfaces in \( \mathbb{C}P^3 \) and threefolds in \( \mathbb{C}P^4 \) the expression (2) amounts to

\[
d^2(d^2 - 3d + 3)(d-2)
\]

and

\[
\frac{1}{3} d^3(d-1)^8(d^3 - 4d^2 + 6d - 4)(d^2 - 3d + 3)(d-2)\text{LCM}(3, 2(d-1))
\]

respectively.

In the case of plane curves, a result similar to formula (3) was obtained in [1] by P. Aluffi and C. Faber by studying the degrees of the \( \text{PGL}_3(\mathbb{C}) \)-orbits of smooth curves. Namely, in [1] the authors produce, given a smooth plane curve \( C \), a certain expression depending on the degree and the nature of flexes of \( C \) and divisible by the order of the stabiliser \( \subset \text{PGL}_3(\mathbb{C}) \) of \( C \).

If \( n \geq 3, d \geq 3 \) and \( (d, n) \neq (4, 3) \), then any automorphism of a smooth hypersurface of degree \( d \) in \( \mathbb{C}P^n \) is known [10, theorem 2] to be the restriction of a projective transformation, so in these cases theorem 3 implies that the order of the full automorphism group divides (2).

The expression (2) is majorated by \( d^{2n(n+1)(n+1)^{-1}} \); since (2) is divisible by the order of the projective automorphism group of any smooth hypersurface of degree \( d \) in \( \mathbb{C}P^n \), it can hardly be expected to be a sharp bound. Indeed, smaller bounds are known; the best one known to the author is

\[
J(n+1)d^n
\]

given by A. Howard and A. J. Sommese [2] (here \( J \) is the Jordan function, i.e., \( J(m) \) is the minimal integer such that any finite subgroup of \( \text{GL}_m(\mathbb{C}) \) contains a normal Abelian subgroup of index \( \leq J(m) \); B. Weisfeiler proved [15] that \( J(m) \leq (m+1)!m^a \ln m + b \) for some \( a, b \in \mathbb{R} \)). However, theorem 3 gives additional information on the orders of automorphism groups; in a sense, asymptotically as \( d \to \infty \), it provides much more restrictions than [2], since the number of divisor of \( x \in \mathbb{Z} \) grows more slowly than any power of \( x \) as \( x \to \infty \) (see, e.g., [7, theorem 317]). More on theorem 3 can be found in section 6.

Upper bounds for automorphism groups of arbitrary varieties of general type are given in [13]. See also G. Xiao’s papers [19, 20] where it is shown that the automorphism group of a surface \( S \) of general type contains at most \( 1764(K_S)^2 \) elements (this bound amounts to \( 1764(d(d-4)^2 \) for surfaces of degree \( d > 4 \) in \( \mathbb{C}P^3 \)).

The idea of the proof of theorem 1 presented here came from the following remark. The first columns of the Vassiliev spectral sequences that compute the Borel-Moore homology of the determinant varieties [15] (i.e., the spaces of degenerate \( (n+1) \times (n+1) \)-matrices) and of the discriminant varieties \( \Sigma_{2,n} \) (see [14, 11, 13]) coincide up to a dimension shift. This paper may be viewed as an attempt to understand the relationship between the corresponding cohomology classes.
The (rest of the) paper is organised in the following way. In section 2 we introduce some notation and formulate and/or prove several preliminary results. Then we study in sections 3, 4 and 5 the way homology classes of the GL_{n+1}(C)-orbits in \Pi_{d,n} \setminus \Sigma_{d,n} are linked with certain subvarieties of \Sigma_{d,n}; the results of these sections are then used in section 6 to prove theorems 1-3. In section 6 we list the values of (2) for small n and d and discuss several particular cases of theorem 3. We also give there an analogue of theorem 3 for the groups of deck transformations of ramified coverings \mathbb{C}P^n \to \mathbb{C}P^n and an application of theorem 3 to real algebraic geometry and discuss some open questions. In the end we give an index of some non-self-explanatory notation used throughout the paper.

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2 Notation and preliminaries

In the sequel the coefficients of all (co)homology groups are assumed to be integer, unless stated otherwise. Whenever different sign choices are possible (e.g., in the definition of the \sim-product), we shall use the classical conventions (e.g., as in [5]).

Any topological space considered in 2.1 and 2.2 is assumed to have the form “a finite CW-complex minus a point”. Notice that any complex algebraic variety satisfies this condition.

All algebraic varieties that we shall consider will be defined over \mathbb{C}, unless stated otherwise. As usual, we denote by I(V) the ideal of an affine variety V.

We shall denote by \text{tot}(\xi) the total space of a vector bundle \xi.

The first Chern class of the cotautological bundle on \mathbb{C}P^n will be called the canonical generator of \text{H}^*(\mathbb{C}P^n).

In the sequel we consider elements of \mathbb{C}^m as column vectors, unless stated otherwise.

When talking about the cohomology of an algebraic variety or a Lie group we shall always mean the cohomology of the corresponding topological space.

2.1 Borel-Moore homology and linking numbers

Let X be a topological space. We denote by \tilde{X} the one-point compactification of X. The Borel-Moore homology groups of X will be denoted by \text{H}_*(X). These groups can be viewed either as the homology groups of the complex of locally finite singular chains or as the homology groups of \tilde{X} modulo the infinity.

We shall use the symbol \text{H}^*_\text{tot} to denote the integer cohomology groups modulo torsion.

Let M be a smooth oriented manifold of dimension p, let X \subset M a closed subspace, and let c \in \text{ker}(\text{H}_{p-q}(X) \to \text{H}_{p-q}(M)). Suppose \text{H}_{q-1}(M) = 0. Then the group \text{H}^{q-1}(M) \cong \text{H}_{p-q+1}(M) is finite, and c defines a unique element of \text{H}_{p-q+1}(M \setminus X) / \text{torsion} \cong \text{H}^{q-1}_\text{tot}(M \setminus X). This element will be called the linking number with c in M and denoted by \text{lk}_{c,X,M}.

Here is another equivalent definition: suppose that c is represented by a smooth singular chain \tilde{c}, and consider the function \tilde{c} \in \text{H}_{q-1}(M \setminus X) = \text{ker}(\text{H}_{q-1}(M \setminus X) \to \text{H}_{q-1}(M)), represent it by a smooth singular chain
z, find a smooth singular chain w in M that is bounded by z and transversal to c, and calculate the intersection index #(w, ̄c). This function also defines a unique element of $H^q_{\text{tor}}(M \setminus X)$, which coincides with $(-1)^q \text{lk}_c, M$.

If Y is a closed subspace of M containing X, then an easy check shows that $\text{lk}_{c, X, M}$ restricted to $M \setminus Y$ is $\text{lk}_{c', Y, M}$, where $c'$ is the image of c in $\bar{H}_{p-q}(Y)$:

\[
\begin{array}{cccc}
H^{q-1}(M \setminus Y) & \overset{\text{Poincaré}}{\longrightarrow} & H_{p-q+1}(M \setminus Y) & \overset{\cong}{\longrightarrow} & \bar{H}_{p-q+1}(M, Y) & \longrightarrow & \bar{H}_{p-q}(Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q-1}(M \setminus X) & \overset{\text{Poincaré}}{\longrightarrow} & H_{p-q+1}(M \setminus X) & \overset{\cong}{\longrightarrow} & \bar{H}_{p-q+1}(M, X) & \longrightarrow & \bar{H}_{p-q}(X)
\end{array}
\] (5)

In particular, if $V_1 \subset V_2 \subset \mathbb{C}^m$ are irreducible affine subvarieties, then the restriction of $\text{lk}_{[V_1], V_2, \mathbb{C}^m}$ to $\mathbb{C}^m \setminus V_2$ is Alexander dual to the image of $[V_1]$ in $\bar{H}_*(V_2)$.

In the sequel, if $X \subset Y \subset M$, we shall often write $\text{lk}_{c, X, M}$ instead of $\text{lk}_{c, X, M}|_{M \setminus Y}$. We shall also drop X from the notation whenever it is clear which space $X \subset M$ we are considering.

Suppose that $X_1$ and $X_2$ are topological spaces, $f : X_1 \to X_2$ is a locally trivial fibration with fibre F, which we assume to be a smooth orientable manifold. Suppose that f is homologically trivial, and introduce an orientation on F. There exists a preimage map $f_+^* : \bar{H}_*(X_2) \to \bar{H}_{\dim F}(X_1)$ defined as follows: we associate to an element $a \in \bar{H}_s(X_2)$ the image in $\bar{H}_s(X_1)$ of what has remained in $\bar{E}^\infty$ of the class $a \otimes [F] \in E^2$ (where $(E^\ast)$ is the Leray spectral sequence converging to the Borel-Moore homology of $X_1$).

If $X_2$ is a smooth manifold, and a is represented by a smooth singular chain $\tilde{a}$, then $f_+^*(a)$ can be represented by the (appropriately triangulated) preimage of the support of $\tilde{a}$ (hence the term “preimage map”).

The basic properties of linking numbers that we shall need can be summarised as follows.

**Proposition 1.**

1. Suppose that $M_1$ and $M_2$ are smooth algebraic varieties, $f : M_1 \to M_2$ is a locally trivial holomorphic fibration with fibre F, and let $N_2$ be a (closed) subvariety of $M_2$. Set $N_1 = f^{-1}(N_2)$. Assume that $f$ is homologically trivial. We have then $f^*(\text{lk}_{c, N_2, M_2}) = \text{lk}_{f_+^*(c), N_1, M_1}$ for any $c \in \bar{H}_*(N_2)$ whose image in $\bar{H}_*(M_2)$ is zero.

2. Suppose that $V \subset \mathbb{C}^m$ is an irreducible affine subvariety, and let $E \subset \mathbb{C}^m$ be an affine plane such that $\text{codim}_{\mathbb{C}^m} V = \text{codim}_{E} \cap E$. Let $C_1, \ldots, C_l$ be the components of $V \setminus E$ of maximal dimension. Denote by $\alpha_i$ the intersection multiplicity of $V$ and $C$ along $C_i, i = 1, \ldots, l$. Then in $H^*(E \setminus V)$ the following holds:

$$\text{lk}_{[V], \mathbb{C}^m}|_{E \setminus V} = \sum_{i=1}^l \alpha_i \text{lk}_{[C_i], E}.$$ 

3. Suppose that $V \subset \mathbb{C}^m$ is an irreducible affine variety, set $l = \text{codim}_{\mathbb{C}^m} V$, and let $F : \mathbb{C}^l \to \mathbb{C}^m$ be a polynomial mapping whose restriction to the unit ball $U \subset \mathbb{C}^l$ is an embedding and such that $F^{-1}(V) \cap U = \{0\}$. Then the pullback under $F|_U$ of $\text{lk}_{[V], \mathbb{C}^m}$ is the canonical generator of $H^{2l-1}(U \setminus \{0\})$ times intersection multiplicity $\mu$ of $F(U)$ and $V$ at $F(0)$. We have

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{0, \mathbb{C}^l}/F^*(I(V))_{m_0},$$
where \( m_0 \) is the ideal formed by the polynomials in \( \mathcal{O}_{\mathbb{C}^1} \) that vanish at the origin. If moreover \( F^{-1}(V) = \{0\} \), then

\[
\mu = \dim_{\mathbb{C}}(\mathcal{O}_{0,\mathbb{C}^1}/F^*(I(V))_{m_0}) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^1}/F^*(I(V))).
\]

\[\clubsuit\]

**Corollary 2.** If \( f : E_1 \to E_2 \) is a surjective linear map of complex vector spaces, and \( V \subset E_2 \) is an irreducible affine subvariety, then

\[
f^*(\text{lk}_{[V],E_2}) = \text{lk}_{[f^{-1}(V)],E_1}.
\]

\[\clubsuit\]

### 2.2 Some facts about vector bundles

All propositions in this subsection are standard exercises, but proofs are given nonetheless for the sake of completeness.

#### 2.2.1 The image of the zero section in the Borel-Moore homology

**Proposition 2.** Let \( \eta \) be a real oriented vector bundle of rank \( l \) over a real smooth oriented manifold \( X \) of dimension \( d \), and let \( Y \subset X \) be a oriented submanifold Poincaré dual to the Euler class \( e(\xi) \) of \( \xi \). Set \( E = \text{tot}(\eta), E' = \text{tot}(\eta|_Y) \). The image of \([X]\) in \( \tilde{H}_*(\mathbb{C}) \) under the zero section embedding is equal to \([E']\).

(We orient \( E \) and \( E' \) by the usual rule “first the fibre, then the base”.)

**Proof.** Equip \( \eta \) with a Riemannian metric, and set \( E_0 \) and \( E_0' \) to be the union of all elements of \( E \), respectively, of \( E' \), of length \( \geq 1 \). Denote by \( u \in H^l(E, E_0) \) the Thom class of \( \eta \), and set \( e' \) to be the restriction of \( u \) to \( E \). By definition, \( e(\eta) \) is the image of \( e' \) under the isomorphism \( H^l(E) \cong H^l(X) \).

Let \( \tilde{B} \) be some contractible compact neighbourhood of \( \infty \) in \( \tilde{X} \), and set \( B = \tilde{B}\setminus\{\infty\}, Z = \text{tot}(\eta|_B) \). Due to the functoriality of the \( -\times \)-product, the following diagram is commutative.

\[
\begin{array}{c}
\tilde{H}_{d-l}(Y) \xrightarrow{\cong} \tilde{H}_{d-l}(X) \xrightarrow{\cong} H_{d-l}(E, Z) \\
\downarrow \cong \downarrow \cong \downarrow \cong \\
\tilde{H}_d(X) \xrightarrow{\cong} H_d(E, B) \xrightarrow{\cong} H_d(E, Z) \\
\downarrow \cong \downarrow \cong \downarrow \cong \\
\tilde{H}_d(E') \xrightarrow{\cong} \tilde{H}_d(E) \xrightarrow{\cong} H_d(E, E_0 \cup Z) \\
\end{array}
\]

The manifold \( Y \) is chosen so that the images of \([X]\) and \([Y]\) in \( H_{d-l}(E, Z) \) coincide. The Thom isomorphism \( \bullet \sim u : H_d(E, E_0 \cup Z) \to H_{d-l}(E, Z) \) takes the image of \([E']\) to the image of \([Y]\), which proves the proposition.\[\clubsuit\]
2.2.2 Degree of some varieties swept by linear subspaces

Proposition 3. Let \( \eta \) be an holomorphic vector subbundle of rank \( l \) of the trivial bundle \( \mathbb{C}^N \times X \) over an irreducible projective variety \( X \) of dimension \( d \). Set \( V \subset \mathbb{C}^N \) be the union of all fibres of \( \eta \), and denote by \( A \) and \( v \) the matrix

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & -c_d(\eta) \\
1 & 0 & \cdots & 0 & -c_{d-1}(\eta) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & -c_2(\eta) \\
0 & \cdots & 0 & 1 & -c_1(\eta)
\end{pmatrix}
\]

and the vector \((0, \ldots, 0, 1)^T\) respectively. Let \( w \in H^{2d}(X) \) be the last coordinate of \( A^dv \). Then \( w([X]) \geq 0 \); if \( w([X]) > 0 \), then \( V \) has the expected dimension \( d + l \), and \( \deg V = w([X]) \).

Proof. Set \( Y \) to be the total space of the projectivisation of \( \eta \). We shall consider \( Y \) as a subset of \( \mathbb{C}P^{N-1} \times X \). Let \( a \) be the canonical generator of \( H^*(\mathbb{C}P^{N-1}) \). The proposition would follow if we manage to show that \( a^{l+d-1} \otimes 1 = (a^{l-1} \otimes 1)p^*(w) \), where \( p: Y \to X \) is the bundle projection.

Identify \( H^*(X) \) with its image under \( p^* \); the ring \( H^*(Y) \) is generated over \( H^*(X) \) by \( b = a \otimes 1 \) with the relation

\[
b^l + \sum_{i=l-d}^{l-1} b^i c_{l-i}(\eta) = 0.
\]

Using this relation we obtain

\[
b^{l+j-1} = \sum_{i=l-d}^{l-1} u_{i,j} b^i,
\]

where \( u_{i,j} \in H^{2(l+j-i-1)}(X) \) are such that \((u_{l-1,j}, \ldots, u_{l-1,0})^T = A^dv \). In particular, \( b^{l+d-1} = b^{l-1}w \), since \( u_{l-1,d} = 0 \) for dimension reasons. The proposition is proven.

2.2.3 Chern classes of some bundles

Let \( d \) be a positive integer, and set \( \eta \) to be the vector bundle over \( \mathbb{C}P^n \) with total space

\[
\{(f, x) \in \Pi_{(d),n} \times \mathbb{C}P^n \mid \text{Sing } f \ni x\}
\]

Proposition 4. The total Chern class of \( \eta \) is equal to \((1 + (d-1)a)^{-n-1}\), where \( a \) is the canonical generator of \( H^*(\mathbb{C}P^n) \).

2.3 Tautological principal bundles

We denote by \( G_m(\mathbb{C}^N) \) the Grassmann manifold consisting of all \( m \)-dimensional complex vector subspaces of \( \mathbb{C}^N \). Let \( F_m(\mathbb{C}^N) \) be the total space of the corresponding tautological principal bundle, i.e.,

\[
F_m(\mathbb{C}^N) = \{(E, (v_1, \ldots, v_m)) \in G_m(\mathbb{C}^N) \times (\mathbb{C}^N)^m \mid v_1, \ldots, v_m \text{ span } E\}.
\]
Let \( m_1, \ldots, m_l \) and \( N_1, \ldots, N_l \) be sequences of positive integers, and let \( S \) be an irreducible subvariety of \( \prod_{i=1}^l G_{m_i}(\mathbb{C}^{N_i}) \). Denote by \( p \) the projection

\[
\prod_{i=1}^l F_{m_i}(\mathbb{C}^{N_i}) \to \prod_{i=1}^l G_{m_i}(\mathbb{C}^{N_i}).
\]

Set \( G = \prod_{i=1}^l GL_{m_i}(\mathbb{C}) \).

**Proposition 5.** For any \( c \in H^{>0}(\prod_{i=1}^l G_{m_i}(\mathbb{C}^{N_i})) \), we have \( p_1^+([S] \sim c|_S) = 0 \) (where \( p_1^+ \) is the preimage map defined in [27]).

In the sequel we shall need only a particular case of this proposition. However, we give a general version, since it could be useful in further applications (and the proof takes two lines anyway).

**Proof.** It is easy to check that \( p_1^+([S] \sim c|_S) = p_1^+([S]) \sim p^*(c) \); this class vanishes, since \( p^*(c) = 0 \).

### 2.4 The cohomology of \( GL_m(\mathbb{C}) \)

Denote by \( e_m \) the canonical generator of \( H^{2m-1}(\mathbb{C}^m \setminus \{0\}) \). We shall call the map \( GL_i(\mathbb{C}) \to GL_j(\mathbb{C}), i < j \), given by

\[
A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}
\]

the canonical inclusion.

Let \( c_1^m, \ldots, c_{m}^m, c_i^m \in H^{2i-1}(GL_m(\mathbb{C})) \) be the system of multiplicative generators such that

- \( c_{m}^m \) is the pullback of \( e_m \) under the map \( A \mapsto \) the last column of \( A \),
- the pullback of \( c_i^m \) under the canonical inclusion \( GL_{m-1}(\mathbb{C}) \subset GL_m(\mathbb{C}) \) is \( c_i^{m-1} \) for \( m \geq 2, 1 \leq i < m \).

We shall write \( o(c_i^m) \) to denote a linear combination of monomials in \( c_j^m, j < i \).

**Proposition 6.** 1. The cohomology map induced by \( GL_m(\mathbb{C}) \ni A \mapsto A^T \in GL_m(\mathbb{C}) \) takes \( c_i^m \) to \((-1)^{i+1}c_i^m \).

2. The cohomology map induced by \( GL_m(\mathbb{C}) \ni A \mapsto A^{-1} \in GL_m(\mathbb{C}) \) takes \( c_i^m \) to \(-c_i^m \).

**Proof:** an easy induction on \( m \). The induction step is performed in either case as follows. We know the image of \( c_i^m, i < m \), since the restriction \( H^*(GL_m(\mathbb{C})) \to H^*(GL_{m-1}(\mathbb{C})) \) is injective in dimensions \( < 2m - 1 \). Write the image of \( c_i^m \) as \( ace_i^m + o(c_i^m) , a \in \mathbb{Z} \). Since \( c_i^m \) generates (over \( \mathbb{Z} \)) the degree \( 2m - 1 \) part of \( \ker(H^*(GL_m(\mathbb{C})) \to GL_{m-1}(\mathbb{C})) \), we see that \( o(c_i^m) = 0 \). The coefficient \( a \) is fixed by looking at the action on the highest cohomology group; this action is the identity, if the corresponding map restricted to the standard \( U_m \subset GL_m(\mathbb{C}) \) is orientation-preserving, and minus identity otherwise.
2.5 Spaces of matrices and polynomials

2.5.1 Subvarieties of spaces of matrices

We denote by Mat\(_{i,j}(\mathbb{C})\) the space of all complex matrices with \(i\) rows and \(j\) columns.

Recall that \(k \leq n + 1\). Set

\[
W_{k,n} = \{ A \in \text{Mat}_{n+1,k}(\mathbb{C}) \mid \text{rk} A < k \},
\]

\[
W_{k,n} = \{ (A, x) \mid \text{rk} A < k, x^T A = 0 \}.
\]

The codimension of \(W_{k,n}\) in \(\text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}\) is \(n + 1\).

Set

\[
X_{k,n} = \{ A \in \text{Mat}_{n+1,k}(\mathbb{C}) \mid \text{rk} A < k, \text{the last row of } A \text{ is zero} \}
\]

and (for \(k > 1\))

\[
Y_{k,n} = \{ A \in \text{Mat}_{n+1,k}(\mathbb{C}) \mid \text{the last } k-1 \text{ columns of } A \text{ form a matrix from } X_{k-1,n} \}.
\]

Finally, set \(X_{k,n} = \{(A, x) \in W_{k,n} \mid A \in X_{k,n}\}\) and \(Y_{k,n} = \{(A, x) \in W_{k,n} \mid A \in Y_{k,n}\}\).

**Proposition 7.** Suppose that \(n + 1 > k > 1\). Then the intersection of \(W_{k,n}\) and the vector subspace \(E\) of \(\text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}\) formed by all \((A, x), A = (a_{i,j})_{1 \leq i \leq n}^{0 \leq j \leq k}\) such that

\[
a_{n,2} = \cdots = a_{n,k} = 0
\]

is \(X_{k,n} \cup Y_{k,n}\); the intersection multiplicity along each one of these components is 1.

**Proof.** Set \(E\) to be the vector subspace of \(\text{Mat}_{n+1,k}(\mathbb{C})\) defined by (7). The variety \(W_{k,n}\) (respectively, \(X_{k,n}\) and \(Y_{k,n}\)) contains an open dense subset that is (the total space of) a vector bundle over the subset of \(W_{k,n}\) (respectively, of \(X_{k,n}\) and \(Y_{k,n}\)) formed by matrices of rank \(k-1\). Hence, to prove the proposition, it is sufficient to show that the intersection multiplicity of \(W_{k,n}\) and \(E\) along both \(X_{k,n}\) and \(Y_{k,n}\) is 1.

Let \(T_1\) be an affine mapping of the unit ball \(U \subset \mathbb{C}^{n-k+2}\) to \(E\) such that \(T_1(U)\) intersects \(X_{k,n}\) transversally at one smooth point; let us assume that this point is \(T_1(0)\) and that the last \(k-1\) columns of \(T_1(0)\) are linearly independent.

It is well known (see, e.g., [3, theorem 2.10]) that the ideal of \(W_{k,n}\) in \(\text{Mat}_{n+1,k}(\mathbb{C})\) is generated by all \(k \times k\)-minors; analogously, the ideal of \(X_{k,n}\) in \(E\) is generated by the polynomial \(A \mapsto a_{n,1}\) (where \(A = (a_{i,j})_{1 \leq i \leq n}^{0 \leq j \leq k}\)) and the \(k \times k\)-minors involving the first \(n\) rows. An immediate check shows that the localisations at 0 of the pullbacks of these ideals under \(T_1\) coincide, and hence, the intersection multiplicity of \(W_{k,n}\) and \(E\) along \(X_{k,n}\) is 1.

The case of \(Y_{k,n}\) can be considered in an analogous way. Namely, let \(T_2 : U \to E\) be an affine mapping such that \(T_2(U)\) intersects \(Y_{k,n}\) transversally at one smooth point, which is \(T_2(0)\). Assume that the left bottom item of \(T_2(0)\) is nonzero. The ideal of \(Y_{k,n}\) in \(E\) is generated by all \((k-1) \times (k-1)\)-minors involving the first \(n\) rows and the last \(k-1\) columns, and we proceed as above to conclude that the intersection multiplicity of \(W_{k,n}\) and \(E\) along \(Y_{k,n}\) is also 1.

An alternative, albeit longer proof of proposition 4 can be given as follows. One could start by showing that the ideal of \(W_{k,n}\) in \(\text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}\) is generated by the polynomials

\[
(A, x) \mapsto a k \times k\text{-minor of } A
\]
2.5.2 Some subvarieties of $\Sigma_{d,n}$

For any $X \subseteq \mathbb{C}P^n$ denote by $V_{d,n,X}$ the subset of $\Pi_{d,n}$ consisting of all $(f_1, \ldots, f_k)$ such that $\text{Sing}(f_1, \ldots, f_k) \cap X \neq \emptyset$. Notice that $\Sigma_{d,n} = V_{d,n,\mathbb{C}P^n}$.

**Proposition 8.** For any $i = 0, \ldots, n$, and any projective subspace $L$ of dimension $i$, $V_{d,n,L}$ is an irreducible affine subvariety of $\Pi_{d,n}$ of codimension $n-i+1$.

In the sequel, whenever it is clear (or irrelevant), which projective subspace $L \cong \mathbb{C}P^i$ we are considering, we shall write $V_{d,n,\mathbb{C}P^i}$ instead of $V_{d,n,L}$.

Set $a_{d,n}^i = \text{lk}_{V_{d,n,\mathbb{C}P^n-i+1}, \Pi_{d,n}}$.

If $f$ is a homogeneous polynomial in $n+1$ variables, $x \in \mathbb{C}^{n+1}$, we set $df|_x$ to be the vector

$$
\left( \frac{\partial f}{\partial x_0}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)^T;
$$

notice that if $A \in \text{GL}_{n+1}(\mathbb{C})$, and $g = f \circ A$, then

$$
dg|_x = A^T df|_{Ax}. \quad (8)
$$

Take $(f_1^0, \ldots, f_k^0) \in \Pi_{d,n} \setminus \Sigma_{d,n}$; denote by $b_{d,n}^i$ the pullback of $a_{d,n}^i$ under the corresponding orbit map $\text{GL}_{n+1}(\mathbb{C}) \to \Pi_{d,n} \setminus \Sigma_{d,n}$. We obviously have

$$
b_{d,n}^i = m_{d,n}^i c_{i+1} + o(c_{i+1}),
$$

where $m_{d,n}^i$ are integers (notice that these integers do not depend on the choice of $(f_1^0, \ldots, f_k^0) \in \Pi_{d,n} \setminus \Sigma_{d,n}$). One of our main tasks in the sequel will be obtaining explicit expressions for $m_{d,n}^i$.

2.5.3 Miscellany

For any $i = 1, \ldots, k$ define the suspension map $S_{d,n}^i : \Pi_{d,n} \to \Pi_{d,n+1}$ by the formula

$$(f_1, \ldots, f_k) \mapsto (f_1, \ldots, f_i + x_{i+1}^d, \ldots, f_k) \quad (9)$$

(here $f_1, \ldots, f_k$ are polynomials in $x_0, \ldots, x_n$).

Recall also the following identity (sometimes called the Euler formula):

$$
f(x_0, \ldots, x_n) = \frac{1}{d} \sum_{j=0}^n x_j \frac{\partial f}{\partial x_j}(x_0, \ldots, x_n), \quad (10)
$$

where $f$ is a homogeneous complex polynomial of degree $d$. 





3 The action of $\text{GL}_{n+1}(\mathbb{C})$ on $\Pi_{\underline{d},n} \setminus \Sigma_{\underline{d},n}$

This section and the following one are devoted to the calculation of $m^{\underline{d}}_{n+1}$ (and hence, the details get somewhat technical at times).

3.1 The calculation of $m^{\underline{d}}_{n+1}$

Take $(f^0_1, \ldots, f^0_k) \notin \Sigma_{\underline{d},n}$. Let $F_{\underline{d},n}$ be the mapping $\mathbb{C}^{n+1} \rightarrow \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}$ defined by

$$x \mapsto (df^0_1|x, \ldots, df^0_k|x, x).$$

(11)

Due to (10), $F_{\underline{d},n}(x) \in W_{k,n}$, iff $x = 0$. Define $N(\underline{d},n)$ by the formula

$$F_{\underline{d},n}^+(k|W_{k,n}|, \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}) = N(\underline{d},n)e_{n+1}.$$

In the sequel we give (proposition 12 and lemma 3) explicit formulae for $N(\underline{d},n)$; however we present here one basic property of these numbers, since, on the one hand, it follows directly from the definition, and on the other hand, it is all we shall need for the proof of theorem 11.

Proposition 9. We have $N(\underline{d},n) > 1$, unless $\underline{d} = (2)$.

Proof. If $k > 1$, the degree of $W_{k,n}$ is $> 1$, and zero belongs to the singular locus of $W_{k,n}$. Hence, the intersection multiplicity of $W_{k,n}$ and the image of $F_{\underline{d},n}$ at $0$ is $> 1$.

If $k = 1$, the sequence $\underline{d}$ contains just one element, $\underline{d} = (d)$. The degree of $W_{1,n}$ is $1$, and $N((d),n)$ is equal to the degree of the mapping $\mathbb{C}^{n+1} \ni x \mapsto df^0|_x$, $f^0 \in \Pi(\underline{d},n) \setminus \Sigma(\underline{d},n)$, which is $(d - 1)^{n+1} > 1$, unless $\underline{d} = (2)$. \hfill \-box

Lemma 1. We have $b^{\underline{d}}_{n+1} = (N(\underline{d},n) + (-1)^{n+k+1})e_{n+1}$, and hence, $m^{\underline{d}}_{n+1} = N(\underline{d},n) + (-1)^{n+k+1}$.

Proof. Take $(f^0_1, \ldots, f^0_k) \notin \Sigma_{\underline{d},n}$, and set $x_0 = (0, \ldots, 0, 1)^T$.

Recall that the class $a^{\underline{d},n+1}$ is the linking number with the variety

$$\{ (f_1, \ldots, f_k) | f_1(x_0) = \cdots = f_k(x_0) = 0, df_1|_{x_0}, \ldots, df_k|_{x_0} \text{are linearly dependent} \}.$$

Due to (10), this variety is the preimage of $W_{k,n}$ under

$$(f_1, \ldots, f_k) \mapsto (df_1|_{x_0}, \ldots, df_k|_{x_0}, x_0).$$

This mapping $\Pi_{\underline{d},n} \rightarrow \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}$ can be included into the following commutative diagram

$$\begin{array}{ccc}
\text{GL}_{n+1}(\mathbb{C}) & \rightarrow & \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1} \\
\downarrow & & \\
\Pi_{\underline{d},n} & \rightarrow & \\
\end{array}$$

where the vertical arrow is the orbit map, and the diagonal one is the map

$$A \mapsto (A^T df^0_1|_{Ax_0}, \ldots, A^T df^0_k|_{Ax_0}, x_0).$$

(12)

11
Denote by $\alpha$ the linking number

\[ \text{lk}([W_{k,n}]_{\cap ([\text{Mat}_{n+1,k}(\mathbb{C}) \times \{x_0\}], \text{Mat}_{n+1,k}(\mathbb{C}) \times \{x_0\}}. \]

Applying corollary 2, we see that $b_{n+1}$ is the pullback of $\alpha$ under 12.

Set $E = \text{Mat}_{n+1,k}(\mathbb{C}) \times (\mathbb{C}^{n+1} \setminus \{0\})$, and set $V = E \cap W_{k,n}$. Denote by $\beta$ the restriction of $\text{lk}[W_{k,n}]_{\cap \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{n+1}}$ to $E \setminus V$. Since $V$ fibers over $\mathbb{C}^{n+1} \setminus \{0\}$, the intersection of $V$ and $\text{Mat}_{n+1,k}(\mathbb{C}) \times \{x\}$ is transversal for any $x \in \mathbb{C}^{n+1} \setminus \{0\}$, which implies (due to the second assertion of proposition 1) that the restriction of $\beta$ to

\[ (\text{Mat}_{n+1,k}(\mathbb{C}) \times \{x_0\}) \setminus W_{k,n} = (\text{Mat}_{n+1,k}(\mathbb{C}) \times \{x_0\}) \setminus V \]

is $\alpha$.

The group $\text{GL}_{n+1}(\mathbb{C})$ acts on $E$ by the formula

\[ A \cdot (B, x) = ((A^T)^{-1}B, Ax). \]

This action preserves $V$; indeed, if $x^T B = 0$, then $(Ax)^T (A^T)^{-1}B = 0$. Let us compute the pullback of $\beta$ under the action map

\[ \text{GL}_{n+1}(\mathbb{C}) \times (E \setminus V) \to E \setminus V. \]

The nontrivial cohomology groups of $E \setminus V$ start in dimension $2n + 1$, which implies that the pullback of $\beta$ has the form $\gamma \otimes 1 + 1 \otimes \beta$, where $\gamma \in H^{2n+1}(\text{GL}_{n+1}(\mathbb{C}))$.

**Lemma 2.** We have $\gamma = (-1)^{n+k}c_{n+1}^{n+1}$.

We shall prove this lemma a little later.

The mapping (12) can be represented as the composition

\[ \text{GL}_{n+1}(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C}) \times (\mathbb{C}^{n+1} \setminus \{0\}) \xrightarrow{F \times F} \text{GL}_{n+1}(\mathbb{C}) \times (E \setminus V) \to E \setminus V \]

(13)

(where the first map is $A \mapsto (A^{-1}$, the last column of $A$), and the third one is the action map). Applying lemma 2 and the second assertion of proposition 1 we obtain that the pullback of $\beta$ under 12 is $-\gamma + N(d_n) c_{n+1}^{n+1} = (N(d_n) + (-1)^{n+k+1})c_{n+1}^{n+1}$, which implies lemma 2

3.1.1 **Proof of lemma 2**

We keep the notation of the proof of lemma 1.

Define the (linear) polynomial $g_i$, $i = 1, \ldots, k$, by $g_i = x_{n-i+1}$, and set

\[ F_{\text{lin}}(x) = (dg_1|_x, \ldots, dg_k|_x, x), \]

\[ s(A) = (A^T d_0|_{Ax_0}, \ldots, A^T d_k|_{Ax_0}, x_0) = (\text{the last } k \text{ columns of } A^T, x_0) \]

(recall that $x_0 = (0, \ldots, 0, 1)^T$). The mapping $s$ can be represented as the composition

\[ \text{GL}_{n+1}(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C}) \times (\mathbb{C}^{n+1} \setminus \{0\}) \to \text{GL}_{n+1}(\mathbb{C}) \times (E \setminus V) \to E \setminus V, \]

(13)
where the first and the last arrows are the same as in (13), and the middle one is $Id \times F_{\text{in}}$. The image of $F_{\text{in}}$ does not meet $W_{k,n}$, hence, pullback of $\beta$ under (12) is $-\gamma$. We shall now calculate this pullback directly, which will complete the proof of lemma 2.

Recall that above we have defined

$$W_{k,n} = \{ A \in \text{Mat}_{n+1,k}(\mathbb{C}) \mid \text{rk} \, A < k \},$$

$$X_{k,n} = \{ A \in \text{Mat}_{n+1,k}(\mathbb{C}) \mid \text{rk} \, A < k, \text{the last row of } A \text{ is zero} \}.$$ Notice that $X_{k,n} \times \{ x_0 \} = W_{k,n} \cap (\text{Mat}_{n+1,k}(\mathbb{C}) \times \{ x_0 \})$. Let $\xi_0, \xi_1$ and $\xi_2$ be the vector bundles on $\mathbb{C}^{P^{k-1}}$ such that

$$\text{tot}(\xi_0) = \{(y_1, \ldots, y_k), (z_1 : \ldots : z_k) \in \mathbb{C}^k \times \mathbb{C}^{k-1} \mid \sum y_iz_i = 0 \},$$

$$\text{tot}(\xi_1) = \{(A, z) \in \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{k-1} \mid Az = 0 \},$$

$$\text{tot}(\xi_2) = \{(A, z) \in \text{Mat}_{n+1,k}(\mathbb{C}) \times \mathbb{C}^{k-1} \mid Az = 0, \text{the last row of } A \text{ is zero} \}.$$ Denote by $a$ the image of $e_{n+1} \in H^2n+1(\mathbb{C}^{n+1} \setminus \{ 0 \})$ under the map

$$(\text{Mat}_{n+1,k}(\mathbb{C}) \setminus W_{k,n}) \ni A \mapsto \text{the last column of } A,$$

and let $b$ be the maximal power of the the canonical generator of $H^*(\mathbb{C}^{P^{k-1}})$. Set $b'$ to be the image of $b$ under the isomorphism $H^*(\mathbb{C}^{P^{k-1}}) \rightarrow H^*(\text{tot}(\xi_2))$.

**Proposition 10.** We have $c_{k-1}(\xi_0) = c(\xi_0) = (-1)^{k+1}b$.

**Proof.** The direct sum of $\xi_0$ and the cotautological bundle is isomorphic (as a topological vector bundle) to the trivial rank $k$ bundle on $\mathbb{C}^{P^{k-1}}$.

**Proposition 11.** The restriction of $\text{lk}_{[X_{k,n}], \text{Mat}_{n+1,k}(\mathbb{C})}$ to $\text{Mat}_{n+1,k}(\mathbb{C}) \setminus W_{k,n}$ is $(-1)^{k+1}a$.

**Proof of proposition 11.** Let us first calculate the image of $[X_{k,n}]$ in $\tilde{H}_*(W_{k,n})$. Set $\xi$ to be the pullback of $\xi_0$ to $\xi_2$. Notice that $\text{tot} \, \xi$ can be naturally identified with $\text{tot} \, \xi_1$.

Let $F$ be the fibre of $\xi_2$ over $(0 : \cdots : 0 : 1)$; clearly, $F$ is Poincaré dual to $b'$ in $\text{tot}(\xi_2)$, hence, by propositions 2 and 10 the image of $[\text{tot}(\xi_2)]$ in $\tilde{H}_*(\text{tot}(\xi_1))$ is $(-1)^{k+1}[\text{tot}(\xi')] = (-1)^{k+1}[\text{the fibre of } \xi_1 \text{ over } (0 : \cdots : 0 : 1)]$. The diagram

$$\begin{array}{ccc}
\text{tot}(\xi_2) & \longrightarrow & X_{k,n} \\
\text{tot}(\xi_1) & \longleftarrow & \text{tot}(\xi) & \longleftarrow & W_{k,n}
\end{array}$$

implies then that the image of $[X_{k,n}]$ in $\tilde{H}_*(W_{k,n})$ is $(-1)^{k+1}(\text{the image of } [U])$, where $U$ is is the union of all matrices $\in \text{Mat}_{n+1,k}(\mathbb{C})$ with zero last column.

The restrictions of $\text{lk}_{[X_{k,n}], \text{Mat}_{n+1,k}(\mathbb{C})}$ and $(-1)^{k+1}\text{lk}_{[U], \text{Mat}_{n+1,k}(\mathbb{C})}$ to $\text{Mat}_{n+1,k}(\mathbb{C}) \setminus W_{k,n}$ coincide (cf. diagram 15). Applying corollary 2 we obtain that the restriction of $\text{lk}_{[U], \text{Mat}_{n+1,k}(\mathbb{C})}$ to $\text{Mat}_{n+1,k}(\mathbb{C}) \setminus W_{k,n}$ is $a$, and proposition 11 follows.

Proposition 11 and the first assertion of proposition 6 imply that the pullback of $\beta$ under (12) is $(-1)^n+1 c_{n+1}^{n+1} = -\gamma$. Lemma 2 is now proven.
3.2 The calculation of the remaining $m_{d}^{d,n}$.

If $k > 1$, set $d^{\prime} = (d_{2}, \ldots, d_{k})$. Suppose we have proven that $b_{i}^{d_{i}n-1} = m_{i}^{d_{i}n-1}c_{i}^{n}$ for $k < n+1, i \leq n$ and $b_{k}^{d_{k}n-1} = m_{k}^{d_{k}n-1}c_{k}^{n}$. We shall express here $m_{d}^{d,n}$ in terms of $m_{i}^{d_{i}n-1}$ and $m_{k}^{d_{k}n-1}$.

Consider first the “generic” case $k < n+1$. Take an element $(f_{1}, \ldots, f_{k}) \in \mathbb{P}_{d,n-1}$, and take a point $(0 : \cdots : 0 : x_{i-1} : \cdots : x_{n-1}) \in \mathbb{C}P^{n-1}$. There exists $x_{n}$ such that $\text{Sing}(f_{1} + x_{n}^{d_{1}}, \ldots, f_{k}) \ni (0 : \cdots : 0 : x_{i-1} : \cdots : x_{n-1} : x_{n})$, iff

$$(0 : \cdots : 0 : x_{i-1} : \cdots : x_{n-1}) \in \text{Sing}(f_{1}, \ldots, f_{k}) \cup \text{Sing}(f_{2}, \ldots, f_{k}).$$

This implies that the preimage of $V_{d,n,\mathbb{C}P^{n+1}}$ under the suspension map $S_{d}^{d_{i}n-1}$ is the union of $V_{d,n-1,\mathbb{C}P^{n+1}}$ and

$$\{(f_{1}, \ldots, f_{k}) \mid (f_{2}, \ldots, f_{k}) \in V_{d,n-1,\mathbb{C}P^{n+1}}\}$$

(if $k = 1$, the second set is empty). Denote by $\gamma_{i}^{d_{i}n}$ and $\gamma_{i}^{d_{k}n}$ the corresponding multiplicities (set $\gamma_{d}^{d_{i}n} = 0$, if $k = 1$). Identify $\text{GL}_{n}(\mathbb{C})$ with the image of the canonical inclusion $\text{GL}_{n}(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C})$; under this identification the map $S_{d}^{d_{i}n}$ becomes $\text{GL}_{n}(\mathbb{C})$-equivariant, and we obtain (using the second assertion of proposition 1) that the restriction of $b_{i}^{d_{i}n}, i = 1, \ldots, n$, to $\text{GL}_{n}(\mathbb{C})$ is $(\gamma_{i}^{d_{i}n} m_{i}^{d_{i}n-1} + \delta_{i}^{d_{i}n} m_{i}^{d_{i}n-1})c_{i}^{n}$.

Suppose now $k = n+1 > 1$ (the case $k = 1, n = 0$ corresponding to empty hypersurfaces in $\mathbb{C}P^{d}$ has in fact already been considered in the previous subsection). Set $f_{1} = x_{i}^{d_{i}}$. We replace the suspension map $S_{d}^{d_{i}n-1}$ by the map

$$(f_{2}, \ldots, f_{k}) \mapsto (f_{1}, f_{2}, \ldots, f_{k}). \quad (14)$$

A point $(0 : \cdots : 0 : x_{i-1} : \cdots : x_{n-1}) \in \text{Sing}(f_{2}, \ldots, f_{k})$ (i.e., the polynomials $f_{2}, \ldots, f_{k}$ have a common zero at $(0, \ldots, 0, x_{i-1}, \ldots, x_{n-1}) \in \mathbb{C}^{n}$), iff there exists $x_{n}$ such that $f_{1}, f_{2}, \ldots, f_{k}$ have a common zero at $(0, \ldots, 0, x_{i-1}, \ldots, x_{n-1}, x_{n}) \in \mathbb{C}^{n+1}$ (in fact, such $x_{n}$ is necessarily zero). This implies that the intersection of the image of (13) and $V_{d,n,\mathbb{C}P^{n+1}}$ is

$$\{(f_{1}, \ldots, f_{k}) \mid (f_{2}, \ldots, f_{k}) \in V_{d,n-1,\mathbb{C}P^{n+1}}\}.$$

Denote by $\gamma_{d}^{d_{i}n}$ the corresponding multiplicity. If we identify as above $\text{GL}_{n}(\mathbb{C})$ with the image of the canonical inclusion $\text{GL}_{n}(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C})$, the map (14) becomes $\text{GL}_{n}(\mathbb{C})$-equivariant, hence $b_{i}^{d_{i}n}, i = 1, \ldots, n$, restricted to $\text{GL}_{n}(\mathbb{C})$ is $\gamma_{i}^{d_{i}n} m_{i}^{d_{i}n-1}c_{i}^{n}$.

We summarise the results of this subsection in the following lemma (where $\delta_{i}^{d_{i}n}$ is set to be zero in the case $k = n+1$).

**Lemma 3.** Suppose that $n > 0$. For any $i = 1, \ldots, n$, there exist integers $\gamma_{i}^{d_{i}n}, \delta_{i}^{d_{i}n} \geq 0$ at least one on which is positive and such that $b_{i}^{d_{i}n} = (\gamma_{i}^{d_{i}n} m_{i}^{d_{i}n-1} + \delta_{i}^{d_{i}n} m_{i}^{d_{i}n-1})c_{i}^{n+1}$, and hence, $m_{d}^{d,n} = \gamma_{i}^{d_{i}n} m_{i}^{d_{i}n-1} + \delta_{i}^{d_{i}n} m_{i}^{d_{i}n-1}$.

4 Fixing the coefficients

In the previous section we have introduced a lot of various coefficients. Here we convert them into numbers.
4.1 The calculation of $N(d, n)$

Let us first consider the “extreme” cases $k = 1$ and $k = n + 1$.

Proposition 12. 1. For any $d \geq 2$ we have $N((d), n) = (d - 1)^{n+1}$.

2. If $k = n + 1$, then $N(d, n) = \prod_{i=1}^{d} d_{i} - 1$.

(Notice that in the most extreme case $k = n + 1 = 1$ both formulae coincide.)

Proof. The first one of these formulae has already been obtained in the proof of proposition 4. Indeed, set $x_{0} = (0 : \cdots : 0 : 1)$; clearly, $x_{0} \in \text{Sing}(f_{1}, \ldots, f_{n+1})$, iff $f_{1}(x_{0}) = \cdots = f_{n+1}(x_{0}) = 0$. Hence, $V_{d,n}(x_{0})$ is the preimage of $0 \in \mathbb{C}^{n+1}$ under the mapping

$$(f_{1}, \ldots, f_{n+1}) \mapsto (f_{1}(x_{0}), \ldots, f_{n+1}(x_{0}))^{T}.$$ 

Hence, the class $b_{n+1}^{d}$ is equal to $c_{n+1}^{d}$ times the degree $\lambda$ of the mapping

$$x \mapsto (f_{1}^{0}(x), \ldots, f_{n+1}^{0}(x))^{T},$$

where $(f_{0}^{0}, \ldots, f_{0}^{0}) \in \Pi_{d,n} \setminus \Sigma_{d,n}$ (cf. the proof of lemma 11). By setting $f_{i}^{0} = x_{i-1}^{d_{i}}$, we see that $\lambda = \prod_{i=1}^{d} d_{i}$. The required formula for $N(d, n)$ follows now from lemma 14.

Proposition 13. Suppose that $1 < k < n + 1$. Then we have $N(d, n) = (d_{1} - 1)N(d_{1}, n - 1) + d_{1}N(d_{1}, n - 1)$.

Proof. Take $(g_{1}, \ldots, g_{k}) \in \Pi_{d,n-1} \setminus \Sigma_{d,n}$ such that $(g_{2}, \ldots, g_{k}) \in \Pi_{d,n-1} \setminus \Sigma_{d,n-1}$, and set in (11) $f_{i}^{0} = g_{i} + x_{i}^{d_{i}}$, $f_{i}^{0} = g_{i}$, $i = 2, \ldots, k$. The image of $F_{d,n}(x)$ will be then contained in the vector subspace of $E \subset \text{Mat}_{n+1,k} \times \mathbb{C}^{n+1}$ defined by (7). Applying proposition 4 and the second assertion of proposition 11 we have

$$F_{d,k}^{*}(\text{lk}[\text{Sing}, \text{Mat}_{n+1,k} \times \mathbb{C}^{n+1}]) = F_{d,k}^{*}(\text{lk}[\text{Mat}_{n,k}], E + \text{lk}[\text{Sing}, E]).$$

Let us calculate $F_{d,k}^{*}(\text{lk}[\text{Sing}, E]) \in H^{2n+1}(\mathbb{C}^{n+1} \setminus \{0\})$. Consider the isomorphism $R : E \to (\text{Mat}_{n,k} \times \mathbb{C}^{n}) \times \mathbb{C}^{2}$ defined as follows: take an element $(A, x) \in E$, where

$$A = (a_{i,j})_{0 \leq i \leq n, 1 \leq j \leq k}, x = (x_{0}, \ldots, x_{n})^{T},$$

to $((A', x'), a_{n,1}, x_{n})$, where

$$A' = (a_{i,j})_{0 \leq i \leq n-1, 1 \leq j \leq k}, x' = (x_{0}, \ldots, x_{n-1})^{T}$$

(see figure 11).

Under this identification, the variety $X_{k,n}$ is taken to $W_{k,n-1} \times \{(z_{1}, z_{2}) \mid z_{1} = 0\}$, and the mapping $F_{d,n}$ is written as

$$F_{d,n} \times (\text{the map } (x_{0}, \ldots, x_{n})^{T} \mapsto (d_{1}x_{n}^{d_{1}-1}, x_{n})),
$$

where $F_{d,n-1}$ is obtained by setting in $f_{i}^{0} = g_{i}, i = 1, \ldots, k$, in (11). This proves that

$$F_{d,k}^{*}(\text{lk}[\text{Sing}, E]) = (d_{1} - 1)N(d_{1}, n - 1)e_{n+1}.$$
In order to complete the proof of the proposition, it is sufficient to show that

$$F^*_{d,k}(|Y_{k,n}|, E) = d_1 N(d', n - 1) e_{n+1}. \quad (15)$$

Indeed, consider the isomorphism $T : E \to C^n \times (\text{Mat}_{n,k-1}(C) \times C^n) \times C^2$ defined as follows: we take $(A, x) \in E$, where

$$A = (a_{i,j})_{0 \leq i \leq n, 1 \leq j \leq k} \text{ and } x = (x_0, \ldots, x_n)^T,$$

to $(v, (A', x'), a_{n,1}, x_n)$, where

$$A' = (a_{i,j})_{0 \leq i \leq n-1, 2 \leq j \leq k} \text{ and } x' = (x_0, \ldots, x_{n-1})^T, v = (a_{1,1}, \ldots, a_{n,1})^T$$

(see figure 2).

Under this identification, $Y_{d,n}$ is transformed into

$$\{(v, (A', x'), (z_1, z_2)) \mid (A', x') \in W_{k-1,n-1}, v^T \cdot x' + z_1 z_2 = 0\}, \quad (16)$$

and the mapping $F_{d,n}$ is becomes

$$\left( \text{the map } C^{n+1} \ni \begin{pmatrix} x \\ 0 \end{pmatrix} \mapsto d_1 | x \in C^n \right) \times F_{d',n-1} \times (\text{the map } (x_0, \ldots, x_n)^T \mapsto (d_1 x_{d_1-1}, x_n)), $$

where $F_{d',n-1}$ is obtained by setting $f^0_{i} = g_{i+1}, i = 1, \ldots, k - 1$ in \ref{13}. The variety \ref{15} projects onto $C^n \times W_{k-1,n-1}$, the preimage of a point being a curve $\subset C^2$ of the form

$$\{(z_1, z_2) \mid z_1 z_2 = a\}. \quad \text{Hence, the intersection multiplicity of the image of } F_{d,n} \text{ and } Y_{k,n} \text{ is equal to } d_1 \text{ times the intersection multiplicity of the image of } F_{d',n-1} \text{ and } W_{k-1,n-1}, \text{ which proves } \ref{16}. \quad \Box$$

Let us present the results of this subsection in a (more or less) compact way.
Lemma 4. We have for \( k > 1 \)

\[
N(d, n) = (d_1 - 1)^{n-k+2} + \sum_{i=2}^{k} \frac{d_i \cdots d_{i-1}(d_i - 1)}{(n-k+1)!} \frac{d^{n-k+1}}{dt^{n-k+1}} \bigg|_{t=0} \frac{1}{(1 - (d_1 - 1)t) \cdots (1 - (d_i - 1)t)}.
\]

This expression does not look symmetric in \( d_1, \ldots, d_k \) but in fact it is.

**Proof.** We can arrange the \( N(d, n) \)'s into a table the following one (represented here for \( n = 5, k = 3 \)):

|   | \((d_3 - 1)^4\) | \((d_2, d_3), 4\) | \((d_1, d_2, d_3), 5\) |
|---|----------------|-------------------|-------------------|
| 2 | 1              | \((d_2, d_3), 3\) | \((d_1, d_2, d_3), 4\) |
| 1 | 0              | \((d_2, d_3), 2\) | \((d_1, d_2, d_3), 3\) |
| 0 | \(-1\)         | \(-1\)            | \(-1\)            |
| -2| 2              | 1                 | 1                 |

(Here the \( x \)-coordinate is \( k \), the \( y \)-coordinate is \( n - k \), and the bottom line of 1’s is added for formal reasons.)

A path connecting two boxes of this table will be called a **staircase**, if it goes only downwards or to the left. A **segment** is a staircase that joins two neighbouring boxes. Let us associate the **weight** \( d_{k-a+1} \), respectively, \( d_{k-a+1} - 1 \), to the segment joining the boxes \( (a, b) \) and \( (a - 1, b) \), respectively, the boxes \( (a, b) \) and \( (a, b - 1) \); the weight of a staircase is set to be the product of the weights of all of its segments. Due to propositions 12 and 13, \( N(d, n) \) is equal to the sum of the weights of all staircases that descend from the box containing \( N(d, n) \) to the bottom line of 1’s and whose last segment is vertical \( \blacklozenge \).

4.2 The multiplicities \( \gamma_i^{d, n} \) and \( \delta_i^{d, n} \)

**Lemma 5.** If \( k < n + 1 \), then \( \delta_i^{d, n} = d_i - 1 \).

**Proof.** We can obtain \( \delta_i^{d, n} \) as follows. Let \( U \subset \Pi_{d, n-1} \) be a small \( i \)-dimensional ball that intersects \( V_{d, n-1, \mathbb{C}^{P^{n-1}}} \) transversally at a smooth point \( \{(f_1^0, \ldots, f_k^0)\} \) (the subspace \( \mathbb{C}^{P^{n-1}} \) in the definition of \( V_{d, n-1, \mathbb{C}^{P^{n-1}}} \) is taken to be

\[
\{0 : \cdots : 0 : x_{i-1} : \cdots : x_{n-1}\} \in \mathbb{C}^{P^{n-1}},
\]

as in the proof of lemma 3).

Assume that \( \text{Sing}(f_1^0, \ldots, f_k^0) \cap \mathbb{C}^{P^{n-1}} = \{(0 : \cdots : 0 : 1)\} \). Moreover, \( V_{d, n-1, \mathbb{C}^{P^{n-1}}} \text{ and } \Pi_{d, n-1, \mathbb{C}^{P^{n-1}}} \times V_{d, n-1, \mathbb{C}^{P^{n-1}}} \) are distinct irreducible subvarieties of \( \Pi_{d, n-1} \) of the same dimension, hence, we can also assume \( U \cap (\Pi_{d, n-1, \mathbb{C}^{P^{n-1}}} \times V_{d, n-1, \mathbb{C}^{P^{n-1}}}) = \emptyset \).

Clearly, \( \delta_i^{d, n} \) is equal to the intersection multiplicity \( \mu \) of \( U' = S_{d, n-1}^{P^{n-1}}(U) \) and \( V_{d, n, \mathbb{C}^{P^{n-1}}} \) at \( S_{d, n-1}^{P^{n-1}}(f_1^0, \ldots, f_k^0) = (f_1 + x_n, f_2, \ldots, f_k) \) (once again, we take the subspace

\[
\mathbb{C}^{P^{n-1}} = \{(0 : \cdots : 0 : x_{i-1} : \cdots : x_n) \in \mathbb{C}^{P^n}\}
\]

used in the proof of lemma 3 as the subspace \( \mathbb{C}^{P^{n-1}} \) in the definition of \( V_{d, n, \mathbb{C}^{P^{n-1}}} \). Set

\[
U_1 = \{(0 : \cdots : 0 : z_1 : \cdots : z_{n-1} : 1 : z_{n-1+1})\}.
\]
For any \( x = (0 : \cdots : 0 : z_1 : \cdots : z_{n-i} : 1 : z_{n-i+1}) \in U_1 \) set \( \tilde{x} \) to be the lifting 
\((0, \ldots, 0, z_1, \ldots, z_{n-i}, 1, z_{n-i+1})^T \) of \( x \). Finally, set \( V_{d, n, CP^{n-i+1}} \) to be the natural smooth resolution \( V_{d, n, CP^{n-i+1}} \), i.e., 
\[
\tilde{V}_{d, n, CP^{n-i+1}} = \{(f_1, \ldots, f_k, x) \in \Pi_{d, n} \times CP^{n-i+1} \mid x \in \text{Sing}(f_1, \ldots, f_k)\}.
\]

Obviously, \( \mu \) is equal to the intersection multiplicity of \( U' \times U_1 \) and \( \tilde{V}_{d, n, CP^{n-i+1}} \) at 
\[
((f_1 + d_1 z_1^2, f_2, \ldots, f_k), (0 : \cdots : 0 : 1 : 0)),
\]
which is equal to the the intersection multiplicity of \( W_{d, n} \) and the image of the mapping 
\( F: U \times U_1 \to \text{Mat}_{n+1, k}(\mathbb{C}) \times \mathbb{C}^{n+1} \) given by 
\[
((f_1, \ldots, f_k), x) \mapsto (df_1|_{\tilde{x}}, \ldots, df_k|_{\tilde{x}}).
\]

It can be readily seen that \( F \) is a local embedding at \((f_1^0, \ldots, f_k^0), (0 : \cdots : 0 : 1 : 0)\). The image of \( F \) is contained in the vector subspace \( E \subset \text{Mat}_{n+1, k}(\mathbb{C}) \times \mathbb{C}^{n+1} \) defined by \( 7 \), and it does not intersect \( Y_{d, n} \), since we have assumed \((f_2, \ldots, f_k) \notin V_{d, n-1, CP^{n-i+1}} \) for any \((f_2, \ldots, f_k) \in U \). Hence, due to proposition \( 4 \) the intersection multiplicity of \( F(U \times U_1) \) and \( W_{d, n} \) is equal to the intersection multiplicity of \( F(U \times U_1) \) and \( X_{d, n} \).

Recall that in the proof of proposition \( 13 \) we have introduced the isomorphism \( R : E \to (\text{Mat}_{n, k} \times \mathbb{C}^n) \times \mathbb{C}^2 \); the composition \( R \circ F \) takes 
\[
((f_1, \ldots, f_k), (0 : \cdots : 0 : z_1 : \cdots : z_{n-i} : 1 : z_{n-i+1}))
\]
to 
\[
(F'((f_1, \ldots, f_k), y), d_1 z_1^{d_1-1}, z_{n-i+1}),
\]
where \( y = (0, \ldots, 0, z_1, \ldots, z_{n-i}, 1)^T \), and \( F' \) is the mapping \( U \times \mathbb{C}^n \to \text{Mat}_{n, k}(\mathbb{C}) \times \mathbb{C}^n \) given by 
\[
((f_1, \ldots, f_k), y) \mapsto (df_1|_y, \ldots, df_k|_y, y).
\]

Since \( U \cap V_{d, n-1, CP^{n-i+1}} \), the image of \( F' \) intersects \( W_{k, n-1} \) transversally, which implies the lemma.\\(\spadesuit\)

**Lemma 6.** If \( k > 1 \), we have \( \eta_{d, n} = d_1 \).

**Proof.** We repeat with minor modifications the proof of the previous lemma. Let \( U \subset \Pi_{d, n-1} \) be a small disc transversal to \( V_{d, n-1, CP^{n-i+1}} \) at a smooth point \((f_2^0, \ldots, f_k^0)\), and let \( U_1 \) be the neighbourhood of \((0 : \cdots : 0 : 1 : 0) \) in \( CP^{n-i+1} \subset CP^{n} \) introduced in the proof of lemma \( 5 \). Recall that \( \eta_{d, n} \) was defined in a slightly different way in the cases \( k = n + 1 \) and \( k < n + 1 \).

Consider first the case \( k < n + 1 \). Choose \( f_1^0 \) so that \( \{f_1^0\} \times U \cap V_{d, n-1, CP^{n-i+1}} = \emptyset \) (where \( CP^{n-i} \) is given by \( 7 \)). Assume that \( \text{Sing}(f_2, \ldots, f_k) \cap CP^{n-i} = \{0 : \cdots : 0 : 1\} \). Clearly, \( \eta_{d, n} \) is equal to the intersection multiplicity of \( U' = S_{d, n}^1(U) \) and \( V_{d, n, CP^{n-i+1}} \) (the projective subspace \( CP^{n-i+1} \) being given by \( 13 \)). We proceed then as in the proof of lemma \( 5 \) except that this time we use the isomorphism \( T \) (and not \( R \)) introduced in the proof of proposition \( 13 \) instead of \( R \) (cf. ibid., the computation of \( F_{d, k}^1(lk[Y_{k, n}], E) \)).
If \( k = n + 1 \), then \( \gamma^d_i^{\alpha} \) is equal to the intersection multiplicity of \( \{0\} \times \mathbb{C}^{n+1} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \) and the image of the mapping \( U \times U_1 \rightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \) that takes \(((f_2, \ldots, f_k), x), x = (0 : \cdots : 0 : z_1 : \cdots : z_{n-i} : 1 : z_{n-i+1}) \) to

\[
((f_2(\bar{x}), \ldots, f_k(\bar{x}), z_{d_i}^{d-1})^T, \bar{x}).
\]

It can be easily seen that \( \nu = d_1 \), which completes the proof of the lemma.

### 4.3 Explicit formulae for \( m_i^{d,n} \)

Let us summarise the results of sections \( \S3 \) and \( \S4 \)

**Lemma 7.** The pullback \( b_i^{d,n} \in H^{2i-1}(GL_{n+1}(\mathbb{C})) \) of any class \( a_i^{d,n} \in H^{2i-1}(\Pi_{d,n}) \) \( \sum_{d,n}, i = 1, \ldots, n + 1 \), under an orbit map is equal to \( m_i^{d,n} \) with

\[
m_i^{d,n} = N(d_i, n) + (-1)^{n-k+1}, \text{ if } i \geq n - k + 2, \text{ and } \]

\[
m_i^{d,n} = N(d_i, n) + (-1)^{i+1}N(d_i, n - i), \text{ if } i \leq n - k + 1. \tag{19}
\]

Moreover, \( m_i^{d,n} > 0 \), unless \( d = (2) \).

**Proof.** Due to lemmas \( \S3 \) and proposition \( \S4 \) we have \( b_i^{d,n} = m_i^{d,n} \mathbf{c}_n^{n+1} \), \( i = 1, \ldots, n + 1 \), and \( m_i^{d,n} > 0 \) for \( d \neq (2) \), so it remains only to prove (19). If \( k = n + 1 \) or \( k = 1 \) (and \( i \) is arbitrary), these formulae follow immediately from lemma \( \S4 \) proposition \( \S2 \) and lemmas \( \S6 \) and \( \S3 \) If \( d \) and \( n \) are arbitrary, and \( i = n + 1 \), then (19) is given by lemma \( \S4 \).

Let us now fix an \( i \). Assume that \( k \geq i \) (otherwise replace \( d \) by a sufficiently long sequence of the form \((2, \ldots, 2, d_1, \ldots, d_k))\). Consider the following three arrays of numbers:

\[
\begin{align*}
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i + l & (d_k - 1)^{i+l+2} & N((d_k-1, d_k, i + l + 2) & \cdots & N(d_i, i + l + k) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i - 2 & (d_k - 1)^i & N((d_k-1, d_k, i) & \cdots & N(d_i, i + k - 2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & (d_k - 1)^2 & N((d_k-1, d_k, 2) & \cdots & N(d_i, k) \\
-1 & d_k - 1 & d_k-1d_k-1 & \cdots & d_k \cdots d_k - 1 \\
1 & & & & & & & \\
2 & & & & & & & \\
\cdots & & & & & & & \\
& & & & & & & \\
1 & & & & & & & \\
& & & & & & & \\
2 & & & & & & & \\
\cdots & & & & & & & \\
& & & & & & & \\
1 & & & & & & & \\
2 & & & & & & & \\
\cdots & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i + l & (-1)^{i+1}(d_k - 1)^{i+1} & (-1)^{i+1}(N(d_k-1, d_k, 2 + l) & \cdots & (-1)^{i+1}N(d_i, l) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i & (-1)^{i+1}(d_k - 1)^2 & (-1)^{i+1}N((d_k-1, d_k, 2) & \cdots & (-1)^{i+1}N(d_i, k) \\
i - 1 & (-1)^{i+1}(d_k - 1) & (-1)^{i+1}(d_k-1d_k-1) & \cdots & (-1)^{i+1}(d_k \cdots d_k - 1) \\
i - 2 & (-1)^{i+1} & (-1)^{i+1} & \cdots & (-1)^{i+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 1 & 1 & \cdots & 1 \\
1 & & & & & & \\
2 & & & & & & \\
\cdots & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\end{align*}
\]

\( (21) \)
Let us denote the items of the arrays (20), (21) and (22) with coordinates \((a, b)\) by \(x_1(a, b)\), \(x_2(a, b)\) and \(x_3(a, b)\) respectively. We have already seen that \(x_2(a, b) = x_1(a, b) + x_2(a, b)\), when \(b = -1\), or \(a = 1\) or \(a + b = i - 1\). Due to proposition 13 and lemma 3, the items of the arrays satisfy the recursive relation \(x_j(a, b) = (d_{k-a+1}) + x_j(a, b-1) + d_{k-a+1}x_j(a-1, b)\), \(j = 1, 2, 3\). Hence, any \(x_3(a, b)\) is equal to \(x_1(a, b) + x_2(a, b)\).

This proves the formulae (19) for \(\mathbf{m}_k\) such that \(k = \text{length of } \mathbf{d} \geq i\), and in fact, for any \(\mathbf{m}_k\), where \(\mathbf{c}\) is a sequence of the form \((d_{k-j}, \ldots, d_k)\), and \(l \geq \max(i - 1, \text{length} (\mathbf{c}) - 1) = \max(i-1, j)\). Hence, the assumption \(k \geq i\) that we made does not restrict the generality. The lemma is proven. ♣

5 Proofs of the theorems

5.1 Proof of theorem 1

Proposition 14. Suppose \(\mathbf{d} \neq (2)\). Then the stabiliser \(G \subset \text{GL}_{n+1}(\mathbb{C})\) of an element of \((f_1, \ldots, f_k) \in \Pi_{d,n} \setminus \Sigma_{d,n}\) is finite.

Proof. Denote by \(H\) the connected component of the identity of \(G\). To prove the proposition, it suffices to show that \(H\) is trivial. If \(k = n+1\), this follows from the fact that any element of \(H\) acts identically on the preimage of a generic point in \(\mathbb{C}^{n+1}\) under the ramified covering \(\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}\) defined by \(x \mapsto (f_1(x), \ldots, f_{n+1}(x))^T\).

Suppose now that \(k < n+1\). Notice that the elements of \(H\) diagonalise simultaneously in some basis of \(\mathbb{C}^{n+1}\). Indeed, if \((\mathbf{d}, n) \neq ((3), 2), ((2), 3)\), this follows easily from the absence of nonzero holomorphic vector fields on a smooth complete intersection of multidegree \(\mathbf{d}\) in \(\mathbb{C}P^n\) (see, e.g., [17], proposition 2.111); the remaining two cases (which correspond to elliptic curves in \(\mathbb{C}P^2\) or on a quadric in \(\mathbb{C}P^3\) can be treated directly.

Hence, \(H\) is in fact a complex torus. If \(H \neq \{\text{Id}\}\), the rational cohomology mapping induced by \(\text{GL}_{n+1}(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C})/H\) (and hence, the rational cohomology mapping induced by \(\text{GL}_{n+1}(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C})/\text{G}\) is not surjective, which contradicts lemma 17. ♣

It should not be very difficult to show directly that the stabilisers of the elements of \(\Pi_{d,n} \setminus \Sigma_{d,n}\) do not contain unipotent transformations, which would enable one to prove proposition 14 without using the nonexistence of holomorphic vector fields on smooth complete intersections.

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1 I am grateful to J. Steenbrink for this reference.
The first part of theorem 1 (the existence of the geometric quotient) follows from proposition 14 and from the fact that $\Pi_{d,n} \setminus \Sigma_{d,n}$ is a hypersurface complement (and hence, an affine variety).

The second part of theorem 1 follows now from the Leray-Hirsch principle and lemma 7.

5.2 Proofs of theorems 2 and 3

Both proofs are based on the following trivial observation.

Proposition 15. Suppose that $\rho : GL_l(\mathbb{C}) \to GL_N(\mathbb{C})$ is a representation that takes scalar matrices to scalar matrices. Let $X \subset \mathbb{C}^N$ be a connected open subset invariant both under $\rho(GL_l(\mathbb{C}))$ and $\mathbb{C}^\ast$. Suppose that all stabilisers of both the action of $GL_l(\mathbb{C})$ on $X$ and the action of $PGL_l(\mathbb{C})$ on $X/\mathbb{C}^\ast$ are finite.

1. Let $a_1, \ldots, a_l$ be cohomology classes of $X$ such that the pullback of any $a_i$ under an orbit map is $m_i c_i$ with $m_i \neq 0$. Then the order of the stabiliser of any $x \in X$ divides $\prod_{i=1}^l m_i$.
2. If moreover there exist nonzero $u_2, \ldots, u_l$ such that any $u_i a_i$ descends into $X/\mathbb{C}^\ast$, then $l$ divides $\prod_{i=2}^l u_i m_i$, and the stabiliser of any $\bar{x} \in X/\mathbb{C}^\ast$ divides $\prod_{i=2}^l u_i m_i$.

The first assertion of this proposition together with lemma 7 imply the following theorem

Theorem 2'. Let $G \subset GL_{n+1}(\mathbb{C})$ be the stabiliser of an element $(f_1, \ldots, f_k) \in \Pi_{d,n} \setminus \Sigma_{d,n}$. If $k = n + 1$, then the order of $G$ divides

$$\prod_{i=1}^{n+1} (N(d, n) + 1) = (d_1 \cdots d_k)^{n+1};$$

if $k < n + 1$, then the order of $G$ divides

$$\left(\prod_{i=1}^{n-k+1} (N(d, n) + (-1)^{i+1} N(d, n - i))\right) \left(\prod_{i=n-k+2}^{n+1} (N(d, n) + (-1)^{n-k+1})\right).$$

Recall that explicit expressions for $N(d, n)$ are given by proposition 12 and lemma 4. If $d \geq 3$, then, due proposition 12, we have $N((d), i) = (d-1)^{i+1}$. By substituting this in theorem 2', we obtain immediately the assertion of theorem 2 (we make the change of variable $i \to n + 1 - i$ in the product to make it look nicer).

Passing from the vector case to the projective one requires a little more work. It is only here that we make use of the general definition of linking numbers given in 2.1.

Let us introduce some additional notation. Let $m \leq n$ be an integer, and set

$$x_{m,n}(d) = \left. \frac{1}{m!} \frac{d^m}{dt^m} \right|_{t=0} \frac{1}{(1 + (d - 1)t)^{n+1}}.$$
Let $y_{m,n}(d)$ be the right bottom item of the matrix $A^m$, where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -x_{m,n}(d) \\ 1 & 0 & \cdots & 0 & -x_{m-1,n}(d) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 & -x_{2,n}(d) \\ 0 & \cdots & 0 & 1 & -x_{1,n}(d) \end{pmatrix}$$

Notice that, due to propositions 4 and 3, we have $y_{m,n}(d) = \deg V_{d,n,C \mathbb{P}^m}$.

**Proposition 16.** We have $\deg V_{d,n,C \mathbb{P}^m} = y_{m,n}(d) = C_{n+1}^m(d-1)^m$.

**Proof.** Assume that $m \geq 2$ (otherwise, the assertion is obvious). Write $A^i$ as $(a_{i,j})$. The proposition would follow, if we prove the following claim for any $i = 1, \ldots, m$:

**Claim:** We have $a_{i,m-m+i} = (d-1)^i C_{n+1}^l$ for $l = 0, \ldots, i$, and $a_{i,j} = 0$, if $j < m - i$.

We proceed by induction on $i$. The case $i = 1$ is clear. Suppose that the claim is proven for some $i \leq m - 1$. By writing $A^{i+1} = A^i A = A^i (N + A')$, where $N$ is nilpotent, and all columns of $A'$ are zero, except for the last one, which consists of $-x_{m,n}, \ldots, -x_{1,n}$, one obtains

$$a_{i,m-j}^{i+1} = a_{i,m-j+1}^i = 0 \text{ for } j < m - (i + 1),$$

$$a_{i,m-j}^{i+1} = a_{i,m-j+1}^i = (d-1)^i C_{n+1}^l, l = 0, \ldots, i,$$

$$a_{i,m}^{i+1} = - \sum_{j=m-i}^{m} a_{i,j} x_{m-j+1,n} = - \sum_{l=0}^{i} (d-1)^l C_{n+1}^l x_{i-l+1,n}.$$

Since $i + 1 \leq m$ and

$$(1 + (d-1)t)^{n+1} (1 + \sum_{j=1}^{m} x_{j,n} t^j) = 1 + o(t^m), t \to 0,$$

we conclude that

$$\sum_{l=0}^{i} (d-1)^l C_{n+1}^l x_{i-l+1,n} + (d-1)^{i+1} C_{n+1}^{i+1} = 0,$$

which completes the proof of the claim.

Now set $\Pi'_{\{d\},n}$ to be the projectivisation of $\Pi_{\{d\},n}$. Let $i$ be an integer such that $2 \leq i \leq n + 1$. Set $\Sigma_{\{d\},n}$ respectively, $V'_{\{d\},n,C \mathbb{P}^{n-i+1}}$ to be the image of $\Sigma_{\{d\},n}$, respectively, of $V_{\{d\},n,C \mathbb{P}^{n-i+1}}$, in $\Pi'_{\{d\},n}$.

Let $a \in H^2(\Pi'_{\{d\},n})$ be the canonical generator, and set $b = [\Sigma_{\{d\},n}] \sim a^{i-1} |_{\Sigma_{\{d\},n}}$. If $p = \deg V_{\{d\},n,C \mathbb{P}^{n-i+1}} = \deg V'_{\{d\},n,C \mathbb{P}^{n-i+1}}$, $q = \deg \Sigma_{\{d\},n} = \deg \Sigma'_{\{d\},n}$, then the image of the class

$$c = \frac{1}{p} \text{LCM}(p,q) [V'_{\{d\},n,C \mathbb{P}^{n-i+1}}] - \frac{1}{q} \text{LCM}(p,q) b \in H_*(\Sigma'_{\{d\},n})$$

in $H_*(\Pi'_{\{d\},n})$ is zero, hence the linking number with this class can be viewed as a well-defined element of $H^* (\Pi'_{\{d\},n} \setminus \Sigma'_{\{d\},n})$.  

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Let \( pr : \Pi(d), n \setminus \{0\} \to \Pi(d), n \) be the natural projection. Due to the first assertion of proposition \( \Box \), the pullback of \( \text{lk}_{\nu}, \Sigma(d), n, \Pi(d), n \) to \( \Pi(d), n \setminus \{0\} \) is the linking number with

\[
\frac{1}{p} \text{LCM}(p, q)|V(d), n, C \Pi^{n-i+1} \setminus \{0\}| - \frac{1}{q} \text{LCM}(p, q)pr^+_i(b)
\]

in \( \Pi(d), n \setminus \{0\} \) (here \( pr^+_i \) is the preimage map defined in \( \Box \)). By proposition \( \Box \) (in fact, this is the only time we use proposition \( \Box \)), \( pr^+_i(b) = 0 \), and hence, the class

\[
\frac{\text{LCM}(p, q)}{p} \text{lk}_{\nu(V(d), n), \Pi(d), n} = \frac{\text{LCM}(C_{n+i+1}^{n-i+1}(d-1)^{n-1}, (n+1)(d-1)^n)}{C_{n+i+1}^{n-i+1}(d-1)^{n-i+1}} \text{lk}_{\nu(d), n}
\]

is the pullback of an element of \( H^*(\Pi(d), n \setminus \Sigma(d), n) \). The proof of theorem \( \Box \) is now completed using the second assertion of proposition \( \Box \) (as above, we make the change of variable \( i \to n+1-i \) in the product). \( \Box \)

6 Discussion

6.1 Some particular cases of theorem \( \Box \)

Here we list the values of \( \Box \) for \( n = 2, 3, 4 \) and \( 3 \leq d \leq 10 \).

| \( d \) | \( n \) | 2 | 3 | 4 |
|---|---|---|---|---|
| 3 | \( 432 = 2^4 \cdot 3^3 \) | \( 414720 = 2^{10} \cdot 3^4 \cdot 5 \) | \( 218972160 = 2^{14} \cdot 3^7 \cdot 5 \cdot 11 \) |
| 4 | \( 18144 = 2^5 \cdot 3^4 \cdot 7 \) | \( 2^{10} \cdot 3^8 \cdot 5 \cdot 7 \) | \( 2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 61 \) |
| 5 | \( 2^8 \cdot 3^3 \cdot 5 \cdot 13 \) | \( 2^{19} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17 \) | \( 2^{20} \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 17 \cdot 41 \) |
| 6 | \( 2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \) | \( 2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \) | \( 2^9 \cdot 3^5 \cdot 5^6 \cdot 7 \cdot 13 \cdot 521 \) |
| 7 | \( 2^4 \cdot 3^1 \cdot 5^7 \cdot 2^3 \cdot 31 \) | \( 2^{10} \cdot 3^8 \cdot 5^2 \cdot 2^7 \cdot 31 \cdot 37 \) | \( 2^{14} \cdot 3^6 \cdot 5^2 \cdot 2^7 \cdot 11 \cdot 31 \cdot 37 \cdot 101 \) |
| 8 | \( 2^7 \cdot 3^3 \cdot 5^4 \cdot 7 \cdot 43 \) | \( 2^{13} \cdot 3^8 \cdot 5^2 \cdot 2^7 \cdot 79 \cdot 43 \) | \( 2^{15} \cdot 3^2 \cdot 5^2 \cdot 2^7 \cdot 11 \cdot 43 \cdot 191 \) |
| 9 | \( 2^2 \cdot 3^5 \cdot 5 \cdot 7 \cdot 19 \) | \( 2^8 \cdot 3^7 \cdot 5 \cdot 2^3 \cdot 7^2 \cdot 13 \cdot 19 \) | \( 2^{26} \cdot 3^9 \cdot 5 \cdot 2^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 331 \) |
| 10 | \( 2^5 \cdot 3^8 \cdot 5^2 \cdot 7^3 \) | \( 2^{11} \cdot 3^7 \cdot 5^3 \cdot 41 \cdot 73 \) | \( 2^{11} \cdot 3^2 \cdot 5^3 \cdot 41 \cdot 73 \cdot 1181 \) |

An explicit description of all automorphism groups of smooth projective hypersurfaces of given degree \( > 2 \) is known in very few cases; in fact, to the author’s knowledge, there are three such cases: plane cubics and quartics and cubic surfaces.

A smooth cubic curve in \( \mathbb{C}P^2 \) can have 18, 36 or 54 projective automorphisms, depending on the value of the \( j \)-invariant. The least common multiple of these numbers is \( 2^2 \cdot 3^3 \), which is 4 times smaller than the corresponding item of table \( \Box \).

For \( n = 2, d = 4 \) the value of \( \Box \) is 18144, which is 9 times the LCM of the orders of the projective automorphism groups of smooth plane quartics (the list of these groups is given e.g. in \( \Box \) section 6.5.2\(^2\)).

\(^2\)The reference was communicated to me by O. Tommasi.
The list of automorphism groups of smooth cubics in $\mathbb{C}P^3$ is given in [8] by T. Hosoh who corrected an earlier classification by B. Segre [12]; the least common multiple of the orders of those groups is $3240 = 2^3 \cdot 3^4 \cdot 5$; compare this with the $n = 3, d = 3$ item of table 1.

In fact, the expression (2) seems (at least for small $d$ and $n$) not to contain “parasitic” primes, i.e., primes that do not actually occur as orders of automorphisms of smooth degree $d$ hypersurfaces in $\mathbb{C}P^n$.

6.2 Odds and ends

Here we consider some questions that do not fit into other sections, but are related to the topics discussed of the paper.

6.2.1 Deck transformations

The proof of the following statement is completely analogous to the proof of theorem 3.

**Theorem 4.** Let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a ramified covering of degree $dn$. Then the order of the group formed by the automorphisms $g : \mathbb{C}P^n \to \mathbb{C}P^n$ such that $f \circ g = f$ divides

$$d^{n^2-1} \prod_{i=2}^{n+1} \frac{1}{C_i^{n+1}} \text{LCM}(C_i^{n+1}, (n+1)d^{i-1}).$$

This result should be easy to generalise to the case of ramified coverings from $\mathbb{C}P^n$ to arbitrary weighted projective spaces.

6.2.2 Actions of the orthogonal groups

Here we present two corollaries of theorem 1 following from the well-known fact that for odd $l$ the inclusion $\text{SO}_l(\mathbb{R}) \subset \text{GL}_l(\mathbb{C})$ induces an epimorphism of the rational cohomology groups (see, e.g., [2]).

**Corollary 3.** Let $q$ be a nonzero quadratic polynomial defining a (possibly singular) quadratic hypersurface $\subset \mathbb{C}P^n$. Set $m = \text{dim ker } q$. Suppose that $n + m$ is even and that $m \leq 1$. The assertion of the division theorem remains true, if we replace $\text{GL}_{n+1}(\mathbb{C})$ by $\text{Aut}^{0}(q)$ and $\Pi_{d,n} \setminus \Sigma_{d,n}$ by the space of all $(f_1, \ldots, f_k)$ of multidegree $d$ such that $\text{Sing}(q, f_1, \ldots, f_k) = \emptyset$.

(Here $\text{Aut}^{0}(q)$ is the connected component of the identity of $\text{Aut}(q)$; notice that $\text{Aut}^{0}(q)$ contracts to $\text{SO}_{n+1-m}(\mathbb{R}) \times U_m$, hence the requirement for $n + m$ to be even.)

The condition that $n + \text{dim ker } q$ should be even can probably be removed. Obvious as it is, this corollary indicates that one should be able to prove an analogue of theorem 3 for smooth hypersurfaces of quadrics. This will be the subject of a further work.

**Corollary 4.** Denote by $\Pi_{d,n}(\mathbb{R}) \subset \Pi_{d,n}$ the set of fixed points of the (standard) complex conjugation, and suppose that $n$ is even. The assertion of theorem 4 remains true, if we replace $\Pi_{d,n} \setminus \Sigma_{d,n}$ by $\Pi_{d,n}(\mathbb{R}) \setminus \Sigma_{d,n}$ and $\text{GL}_{n+1}(\mathbb{C})$ by $\text{GL}_{n+1}(\mathbb{R})$.

It is not known to the author if this statement holds for odd $n$. 
6.2.3 Possible generalisations and some open questions

Notice that besides $\text{GL}_{n+1}(C)$ there are other groups acting naturally on $\Pi_{d,n} \setminus \Sigma_{d,n}$. For instance, set $G_{d,n}$ to be the group generated by all transformations

$$(f_1, \ldots, f_i, \ldots, f_k) \mapsto (f_1, \ldots, a f_i + g f_j, \ldots, f_k)$$

(where $i \neq j$ are indices such that $d_i \geq d_j$, $a \in C^*$, and $g$ is a homogeneous polynomial of degree $d_i - d_j$).

The geometric quotient of $\Pi_{d,n} \setminus \Sigma_{d,n}$ by $G_{d,n}$ obviously exists; its elements parametrise smooth complete intersections themselves, rather than their equations (since we quotient out all possible ways to pick up a minimal system of generators for the homogeneous ideal of the variety given by $f_1(x) = \cdots = f_k(x) = 0, (f_1, \ldots, f_k) \in \Pi_{d,n} \setminus \Sigma_{d,n}$).

Finally, notice that the subgroup of $\text{GL}_{n+1}(C) \times G_{d,n}$ acting identically on $\Pi_{d,n} \setminus \Sigma_{d,n}$ is isomorphic to $C^*$. Denote by $G_{d,n}$ the corresponding quotient group. The answers to the following questions are unknown to the author, if $k > 1$.

- Do we have a division theorem for the action of $G_{d,n}$ on $\Pi_{d,n} \setminus \Sigma_{d,n}$?

- Suppose that $d_1 = \cdots = d_k$. Do we have then a division theorem for the action of $G_{d,n}$ on $\Pi_{d,n} \setminus \Sigma_{d,n}$? Notice that one can not expect such a result to hold for the sequences $\underline{d}$ that contain at least two different items, since the cohomology of $G_{d,n}$ will then have too many generators in dimension 1 (and $\Sigma_{d,n}$ is an irreducible hypersurface).

- Notice that the group $\text{PGL}_{n+1}(C)$ acts on the quotient $(\Pi_{d,n} \setminus \Sigma_{d,n})/G_{d,n}$; do we have a division theorem for that action?

Index of notation

| $k, \underline{i}$ | 4 |
| $n, \underline{1}$ | 1 |
| $d, \underline{0}$ | 11 |
| $\Pi_{d,n}, \underline{1}$ | 11 |
| $\Sigma_{d,n}, \underline{1}$ | 11 |
| $\text{Sing}(f_1, \ldots, f_k), \underline{1}$ | 11 |
| $\text{lk}_{c,X,M}, \text{lk}_{c,M}, \underline{1}$ | 11 |
| $\text{tot}(\xi), \underline{1}$ | 11 |
| $e_m, \underline{8}$ | 8 |
| $c^m_{i}, \underline{8}$ | 8 |
| $o(e^m_{i}), \underline{8}$ | 8 |
| $\text{Mat}_{i,j}(C), \underline{1}$ | 13 |
| $W_{k,n}, \underline{9}$ | 9 |
| $X_{k,n}, \underline{9}$ | 9 |
| $Y_{k,n}, \underline{9}$ | 9 |
| $X_{k,n}, \underline{9}$ | 9 |
| $Y_{k,n}, \underline{9}$ | 9 |
| $a_{\underline{d},n}, \underline{10}$ | 10 |
| $B_{\underline{d},n}, \underline{10}$ | 10 |
| $m_{\underline{d},n}, \underline{10}$ | 10 |
| $S_{\underline{d},n}, \underline{10}$ | 10 |
| $F_{\underline{d},n}, \underline{11}$ | 11 |
| $N(d,n), \underline{13}$ | 13 |
| $d', \underline{14}$ | 14 |
| $\gamma_{\underline{d},n}, \underline{14}$ | 14 |
| $\delta_{\underline{d},n}, \underline{14}$ | 14 |

References

[1] P. Aluffi, C. Faber, “Linear orbits of smooth plane curves”, J. Algebraic Geom., 2, 1993, no. 1, 155–184. [arXiv:math.AG/9206001]
[2] A. Borel, “Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts”, Ann. of Math. (2) 57, 1953, 115–207.

[3] W. Bruns, U. Vetter, “Determinantal rings”, Lecture Notes in Mathematics, 1327, Springer-Verlag, Berlin, 1988.

[4] I. V. Dolgachev, “Topics in classical algebraic geometry, Part I”, available\footnote{As of October the 24th, 2005.} at http://www.math.lsa.umich.edu/~idolga/topics1.pdf

[5] A. T. Fomenko, D. B. Fuchs, “A course in homotopic topology”, Nauka, Moscow, 1989 (in Russian).

[6] A. G. Gorinov, “Real cohomology groups of the space of nonsingular curves of degree 5 in $\mathbb{CP}^2$”, Ann. Fac. Sci. Toulouse Math., 14, no. 3, 2005, 395–434, arXiv:math.AT/0105108

[7] G. H. Hardy, E. M. Wright, “An introduction to the theory of numbers”, fourth edition, the Clarendon Press, Oxford University Press, London, 1975.

[8] T. Hosoh, “Automorphism groups of cubic surfaces”, J. Algebra 192, 1997, no. 2, 651–677.

[9] A. Howard, A. J. Sommese, “On the orders of the automorphism groups of certain projective manifolds”, Manifolds and Lie groups (Notre Dame, Ind., 1980), 145–158, Progr. Math., 14, Birkhäuser, Boston, Mass., 1981.

[10] H. Matsumura, P. Monsky, “On the automorphisms of hypersurfaces”, J. Math. Kyoto Univ., 3, 1963/1964, 347–361.

[11] C.A.M. Peters, J.H.M. Steenbrink, “Degeneration of the Leray spectral sequence for certain geometric quotients”, Mosc. Math. J. 3, 2003, no. 3, 1085–1095, 1201, arXiv:math.AG/0112093

[12] B. Segre, “The non-singular cubic surfaces,” Oxford Univ. Press, London, 1942.

[13] E. Szabó, “Bounding automorphism groups”, Math. Ann. 304, 1996, no. 4, 801–811.

[14] O. Tommasi, “Rational cohomology of the moduli space of genus 4 curves”, Compos. Math. 141 (2000), no. 2, 359–384, arXiv:math.AG/0312055

[15] V. A. Vassiliev, “Geometric realization of the homology of classical Lie groups, and complexes that are $S$-dual to flag manifolds” (in Russian). Algebra i Analiz 3, 1991, no. 4, 113–120, translation in St. Petersburg Math. J. 3, 1992, no. 4, 809–815.

[16] V. A. Vassiliev, “How to calculate homology groups of spaces of nonsingular algebraic projective hypersurfaces in $\mathbb{CP}^n$”, Proc. Steklov Math. Inst., 1999, 225, 121-140.

[17] J. Wahl, “Derivations, automorphisms and deformations of quasihomogeneous singularities”, Singularities, Part 2 (Arcata, Calif., 1981), 613–624, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.
[18] B. Weisfeiler, “Post-classification version of Jordan’s theorem on finite linear groups” Proc. Nat. Acad. Sci. U.S.A. 81, 1984, no. 16, Phys. Sci., 5278–5279.

[19] G. Xiao, “Bound of automorphisms of surfaces of general type. I”, Ann. of Math. (2) 139, 1994, no. 1, 51–77.

[20] G. Xiao, “Bound of automorphisms of surfaces of general type. II”, J. Algebraic Geom. 4, 1995, no. 4, 701–793.

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