A SPECIAL CASE OF sl\((n)\)-FUSION COEFFICIENTS

GEANINA TUDOSE

ABSTRACT. We give a combinatorial description of \(sl(n)\)-fusion coefficients in the case where one partition has at most two columns. As a result we establish some properties for this case including solving the conjecture that fusion coefficients are increasing with respect to the level \(k\).

1. Introduction

Fusion coefficients first appeared in the literature as the structure constants of the Verlinde (fusion) algebra associated to an affine Kac-Moody algebra \(\hat{\mathfrak{g}}\) in the Wess-Zumino-Witten model of conformal field theory. Since then, many equivalent interpretations have been found in other contexts such as quantum groups and Hecke algebras at root of unity [8], quantum cohomology of the grassmannian [4], spaces of generalized theta functions, spaces of intertwiners in vertex operator algebras [19], knot invariants for 3-manifolds [16] and others.

If \(\mathfrak{g}\) is a semi-simple finite dimensional Lie algebra and \(L(\lambda)\) is the integrable representation of \(\mathfrak{g}\) with highest weight \(\lambda\), the tensor product coefficients \(N_{\lambda\mu}^\nu\) are defined by the relation \(L(\lambda) \otimes L(\mu) = \bigoplus N_{\lambda\mu}^\nu L(\nu)\). For \(\mathfrak{g} = sl(n)\) the integrable representations are indexed by partitions and the tensor product coefficients are the well-known Littlewood-Richardson coefficients \(c_{\lambda\mu}^\nu\). Given a positive level \(k\), the fusion coefficients \(N^{(k)}_{\lambda\mu}^\nu\) are defined by

\[
L(\lambda) \otimes_k L(\mu) = \bigoplus N^{(k)}_{\lambda\mu}^\nu L(\nu)
\]

where the fusion product \(\otimes_k\) is the reduction of the tensor product via the representation at level \(k\) of the algebra \(\hat{\mathfrak{g}}\). A more detailed approach to fusion coefficients arising in conformal field theory is given in [17, 18]. For our purposes we will give in Section 3 an equivalent definition for the case \(\mathfrak{g} = sl(n)\).

By some representation theoretic arguments it is known that these coefficients are non-negative but a general combinatorial description is still lacking even for type \(A\). Only some particular cases are known: the cases \(n = 2\) and \(n = 3\) [1, 2, 10] where the combinatorial objects used are the Berenstein-Zelevinski triangles, and more recently the case where all partitions in the product are rectangles [14, 15] in which affine crystal theory for perfect crystals was used. In addition, a \(q\)-analogue of fusion coefficients has also been introduced [5].

To date, the most effective algorithm for computing fusion coefficients for any type is the Kac-Walton algorithm [9, 17]. In this algorithm, the fusion coefficients are expressed in

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terms of the tensor product coefficients

\[ N_{\lambda \mu}^{(k) \nu} = \sum_{w \in \hat{W}_k \atop w.\nu \in P^+} \det(w) N_{\lambda \mu}^{w.\nu} \]

where \( \hat{W}_k \) is the affine Weyl group, \( w.\nu = w(\nu + \rho) - \rho \), \( P^+ \) is the set of dominant weights and \( \rho \) is the sum of fundamental weights. In this notation the affine Weyl groups are isomorphic, and only the action of the reflection \( s_0 \) on the weight lattice of the algebra \( \hat{g} \) is different with respect to the level \( k \) i.e.

\[ s_0(\hat{\lambda}) = \hat{\lambda} - (k - (\lambda, \theta))\hat{\alpha}_0 \]

where \( \theta \) is the highest root of \( \hat{g} \), \( \{\hat{\alpha}_i, \ i = 0, 1, \ldots n-1\} \) are the simple roots, and \( (\cdot | \cdot) \) is the symmetric bilinear form on the Cartan subalgebra of \( \hat{g} \).

In this paper we use the interpretation given by Goodman and Wenzl [8] to give a combinatorial description for \( sl(n) \) fusion coefficients where the partition \( \mu \) has two columns.

Our main result is Theorem 12 where we show that the fusion coefficients count paths in the Young’s lattice with some extra conditions. An equivalent interpretation in terms of Littlewood-Richardson tableaux is given in Remark 13. The tool for finding this description is the pairing technique for proving the classical Littlewood-Richardson rule by means of a sign-reversing involution. Therefore we include in Section 2 a proof of the classical rule so that in Section 3 we can construct the involution for fusion coefficients. In Section 4 we establish some interesting properties of these coefficients in our specific case. Some of these confirm known properties such as positivity and the inequality \( N_{\lambda \mu}^{(k) \nu} \leq N_{\lambda \mu}^{\nu} \), but most importantly we confirm the increasing property as function of \( k \) conjectured in [18] i.e. \( N_{\lambda \mu}^{(k) \nu} \leq N_{\lambda \mu}^{(k+1) \nu} \). We conclude our paper with Section 5 where we propose another avenue for approaching the problem.

2. Proof of the Littlewood-Richardson rule

The proof of the LR-rule is based on the Jacobi-Trudi determinantal identities and uses a sign reversing involution which yields a combinatorial characterization of the LR-coefficients in terms of paths in the Young’s lattice. The involution is an adaptation of the involution constructed by Remmel and Shimozono [13].

The LR-coefficients are the structure constants \( c_{\lambda \mu}^\nu \) for the ring of symmetric polynomials with respect to the basis of Schur functions:

\[ s_\lambda s_\mu = \sum_\nu c_{\lambda \mu}^\nu s_\nu. \]

We intend to give a characterization of these coefficients of the form

\[ c_{\lambda \mu}^\nu = \# \left\{ \text{paths in the Young’s lattice from } \lambda \text{ to } \nu \right\} \]

where \( \# \) denotes the number of paths. There are many ways of getting to this result depending on which determinantal formula we use. We shall choose the one expressing the Schur functions in terms of the elementary symmetric polynomials \( e_k \). The reason for this choice is accounted for in the proof of the rule for fusion coefficients.
In order to prove the LR-rule we first need some definitions. Most of those not given here and results concerning symmetric functions that we use can be found in [12]. For a partition \( \lambda \) we consider its diagram to be the set of points \((i, j) \in \mathbb{Z}^2\) such that \(1 \leq i \leq \lambda_j\), where \(1 \leq j \leq \text{length}(\lambda)\).

We denote a path \( P \) in the Young’s lattice from \( \lambda \) to \( \nu \) by a chain of partitions

\[
P : \lambda^{(0)} = \lambda \subseteq \lambda^{(1)} \subseteq \cdots \lambda^{(n)} = \nu
\]

where each partition \( \lambda^{(k)} \) differs from the previous one \( \lambda^{(k-1)} \) by exactly one box. We also denote by \(|P| = n\) the length of the path \( P \).

Sometimes we need paths from \( \lambda \) to \( \nu \) made from successive paths i.e. \( P = P_1 * P_2 * \cdots * P_m \), where each \( P_i \) is a path from \( \lambda^{(i)} \) to \( \lambda^{(j)} \) with \( i \leq j \), and \( * \) denotes the concatenation of the paths.

![Figure 1.](image1.png)

Figure 1. \( P : \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \lambda^{(3)} \) and we can also write, say

\[
P = P_1 * P_2, \text{ where } P_1 = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \text{ and } P_2 = \lambda^{(2)} \subseteq \lambda^{(3)}.
\]

Next we introduce a labeling of each box in a partition in order to define a 1-1 correspondence between the paths and the sequence of labels such that the boxes on the diagonals \( x - y = i \) are indexed by \( i \).

![Figure 2.](image2.png)

Figure 2. Labeling of \( \lambda = (5, 4, 3, 2, 1) \).

Using this labeling we identify the path \( P = \lambda \subseteq \lambda^{(1)} \subseteq \cdots \lambda^{(n)} \) with the sequence of boxes added in each step. From here we shall write the labels of these boxes as

\[
l(P) = (l_1, l_2, \ldots, l_n),
\]

where \( l_i \) is the label of \( \lambda^{(i)}/\lambda^{(i-1)} \), for \( i = 1, \ldots, n \). We say that \( P \) is a decreasing path if \( l(P) \) is decreasing. If \( \alpha \) is a sequence of integers \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) with \( \alpha_1 + \alpha_2 + \cdots + \alpha_k = |P| \) we say that \( P \) has ascents included in positions \( \alpha \) if

\[
(l_{\alpha_0+\alpha_1+\cdots+\alpha_i+1}, l_{\alpha_0+\alpha_1+\cdots+\alpha_i+2}, \ldots, l_{\alpha_0+\alpha_1+\cdots+\alpha_i+1})
\]

is a decreasing sequence for every \( i \in \{0, 1, 2, \ldots, k-1\} \), where \( l_j \) is the \( j^{th} \) component of \( l(P) \) and \( \alpha_0 = 0 \).
We make the convention that if \( \alpha \) contains negative integers the set of paths with ascents included in positions \( \alpha \) is the empty set. We also note that a path \( P \) can have ascents included in different \( \alpha \)'s.

**Example 1:** \( P \) in Figure 1 has \( l(P) = (3,1,2) \) and \( P \) has ascents in \( \alpha = (2,1) \) and also ascents included in positions \( (1,1,1) \).

For each general path \( P \) and a sequence \( \alpha = (\alpha_1, \ldots, \alpha_k) \) such that \( P \) has ascents included in positions \( \alpha \), we cut the path \( P \) into \( k \) consecutive paths each of length \( \alpha_i \), \( i = 1 \ldots k \); then we associate a tableau \( T_P \) whose columns are made from the sequence of labels of \( P \) written top-to-bottom. Sometimes when \( \alpha \) is understood we will make no distinction between \( P \) and \( T_P \). We say that a path \( P \) fits a partition \( \mu \), if \( T_P \in CS(\mu) \), where \( CS(\mu) \) represents the Young tableaux of shape \( \mu \), strictly increasing in columns and weakly increasing in rows.

**Theorem 1.** (Littlewood-Richardson rule)

\[
c^\nu_{\lambda\mu} = \# \{ \text{paths } P \text{ from } \lambda \text{ to } \nu \text{ that fit } \mu \}.
\]

**Proof.**

Let \( \mu' \) denote the conjugate partition of \( \mu \).

Using the Jacobi-Trudi identity to express \( s_\mu \) in terms of the elementary symmetric functions given in Equation (1) we get

\[
s_\lambda s_\mu = s_\lambda \det(e_{\mu'_i+j})_{1 \leq i,j \leq n} = \sum_{\nu} c^\nu_{\lambda\mu}s_\nu \quad \text{where } n \geq l(\mu')
\]

and when we expand the determinant we have

\[
s_\lambda \sum_{\sigma \in S_n} (-1)^\sigma e_{\sigma,\mu'} = \sum_{\nu} c^\nu_{\lambda\mu}s_\nu
\]

where \( \sigma,\mu' = \sigma(\rho + \mu') - \rho \) and \( \rho = (n-1, n-2, \ldots, 1, 0) \). On the other hand multiplying a Schur function with an elementary symmetric function we get

\[
s_\lambda e_k = \sum_{\nu/\lambda=k-\text{column strip}} s_\nu.
\]

We can also view this equality in terms of paths in the Young’s lattice as

\[
s_\lambda e_k = \sum_{\nu} a^\nu_{\lambda(k)}s_\nu
\]

where \( a^\nu_{\lambda(k)} = \# \{ \text{decreasing paths from } \lambda \text{ to } \nu \text{ of length } k \} \). It is not difficult to see that, indeed

\[
a^\nu_{\lambda(k)} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is a } k \text{ column strip} \\ 0 & \text{otherwise.} \end{cases}
\]

Using rule (3) repeatedly, the left-hand side of Equation (3) becomes

\[
\sum_{\sigma \in S_n} (-1)^\sigma \sum_{\nu} s_\nu e_{\sigma,\mu'} = \sum_{\sigma \in S_n} \sum_{\nu} (-1)^\sigma a^\nu_{\lambda(\sigma,\mu')}s_\nu
\]

where

\[
a^\nu_{\lambda(\sigma,\mu')} = \# \{ \text{paths from } \lambda \text{ to } \nu \text{ with ascents in positions } \sigma,\mu' \}.
\]
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Hence

\begin{equation}
\lambda \mu \nu \chi \sigma = \sum_{\sigma \in S_n} (-1)^{\sigma \chi} a_{\lambda \mu \nu} = \sum_{(\sigma, P) \in \Omega} (-1)^{\sigma \chi}
\end{equation}

where $\Omega$ is the set of pairs $(\sigma, P)$, $\sigma \in S_n$ and $P$ is a path from $\lambda$ to $\nu$ with ascents in positions $\sigma, \mu'$. Since $P$ has ascents included in positions $\sigma, \mu'$, we can write $P = P_1 \cdots P_n$ in which $P_i$ is a decreasing path of length $|P_i| = (\sigma, \mu')_i$. 

The next step is to construct a sign reversing involution on the set $\Omega$. The involution uses the crystal operators defined by a pairing and was constructed in [13] which also contains further details.

Suppose that a path $P$ is made from two successive paths $P = P_1 \ast P_2$ of lengths $p$ and $q$. The word of $P$ denoted by $w$ is the sequence of all labels in $P$ sorted in increasing order. A label can appear at most twice, i.e. once in each column and if this happens we consider the first occurrence corresponding to the first column in $T_P$ and the second occurrence corresponding to the second column.

We construct $\hat{w}$ in the following way

- Replace every letter in $w$ which is a label in $l(P_1)$ by a left parenthesis
- Replace every letter in $w$ which is a label in $l(P_2)$ by a right parenthesis.

Example 2: The path $P$ with $T_P = \begin{array}{cccc}
3 & 2 & 4 & \\
0 & 1 & & \\
-1 & -1 & & \\
-2 & -3 & & 
\end{array}$ has the word $w = 321101234$, where $\bar{n} = -n$.

The parentheses structure is

$\hat{w} = (())()()$. 

We say that a letter is paired if it corresponds to a parenthesis that is matched under the usual rule of parenthesization. Otherwise we call it unpaired. We say that a word $w$ has type $(l, r)$ if there are $l$ unpaired left parenthesis and $r$ unpaired right parenthesis.

Next we define two operators on words which will be partial functions, the raising operator $e$ and the lowering operator $f$ where

$e$ changes the rightmost unpaired right parenthesis into a left one.

$f$ changes the leftmost unpaired left parenthesis into a right one.

It is clear that for $e$ or $f$ to be applied we need $r > 0$ (resp. $l > 0$). We shall also write $e(P)$ or $f(P)$ and understand that $e$ or $f$ is applied to the word of $P$ with $e(P) = P'_1 \ast P'_2$, where $|P'_1| = |P_1| + 1$, $|P'_2| = |P_2| - 1$ and also $f(P) = P''_1 \ast P''_2$ with $|P''_1| = |P_1| - 1$, $|P''_2| = |P_2| + 1$.

Example 3: For $P$ in Example 2, $e(w)$, $f(w)$ have the parentheses structure

$\hat{e}(w) = ((()))()$ and $\hat{f}(w) = ()()()$.
and the results of these operators on the tableau of $P$ are

$$T_e(P) = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 4 \\ -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad T_f(P) = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 2 \\ -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}. $$

The next result helps us to establish that $e$ and $f$ define an involution.

**Proposition 2** (Proposition 3 of [13]). If $\eta$ is any of the operators $e$, $f$, then the unpaired subwords of $\eta(w)$ and of $w$ occupy the same positions (assuming $\eta$ is defined) and if $w$ has at least $m$ unpaired left parentheses then $f^m e^m(w) = w$. A similar property holds for the unpaired right parentheses.

Therefore we can consider that $e^{-1} = f$ and $f^{-1} = e$ where they are defined. The following useful result is a reformulation of Proposition 5 of [13].

**Proposition 3.** A path $P$ fits $\mu$ (i.e. $T_P \in CS(\mu)$) if and only if there are no unpaired right parentheses for every two columns $(P_i, P_{i+1})$ in $\mu$, where $i = 1, \ldots, \mu_1 - 1$.

**Remark 4.** If $P$ does not fit a partition then there exists two consecutive columns $P_i$ and $P_{i+1}$ for which we have

- at least $|P_{i+1}| - |P_i| - 1$ unpaired right parentheses, if $|P_{i+1}| - |P_i| - 1 > 0$
- at least $-(|P_{i+1}| - |P_i| - 1)$ unpaired left parentheses, if $|P_{i+1}| - |P_i| - 1 < 0$.

It is an easy consequence of the expansion of the determinant in (3) that $|P_{i+1}| - |P_i| - 1 \neq 0$.

We use Proposition 3 to construct the involution $\Psi$ on the right-hand side of (3) as follows.

1. If $T_P \in CS(\mu)$ then $\sigma = id$ and define $\Psi(id, P) = (id, P)$.
2. If $T_P \notin CS(\mu)$ then let $(r, r+1)$ be the pair of consecutive columns where a violation of the column-strict tableau property occurs while reading $T_P$ from right to left, bottom to top, row-wise. We call this position canonical. Define

$$\Psi(\sigma, P) := ((r, r+1), \sigma, P_1 \ast \cdots \ast P_{r-1} * e^{|P_{r+1}| - |P_r| - 1}(P_r * P_{r+1}) \cdots * P_m).$$

We must show that $\Psi$ is a well-defined involution. We first check that $\Psi(\sigma, P) \in \Omega$. This is trivial when $T_P \in CS(\mu)$, so we shall assume that $T_P \notin CS(\mu)$. From Proposition 3 and the Remark 4, it is clear that we can define the operator $e^{|P_{r+1}| - |P_r| - 1}(P_r * P_{r+1})$. If $\Psi(\sigma, P) = (\sigma', P')$, then $P' = P_1 \ast \cdots \ast P_r \ast P'_{r+1} \cdots \ast P_m$ is indeed a path with ascents included in positions $\sigma', \mu'$, since both $P'_r$ and $P'_{r+1}$ are decreasing paths and $|P'_r| = (\sigma', \mu')_i$, for any $i$. Thus $\Psi$ is well-defined.

Next we shall show that $\Psi$ is an involution. This is again obvious for $P$ a partition that fits $\mu$. Let $P$ be a path such that $T_P \notin CS(\mu)$ and let $(\sigma', P') = \Psi(\sigma, P)$. To see that $\Psi(\sigma', P') = (\sigma, P)$ it is necessary to show that the violation of the column-strict tableau property occurs in the same place for both $T_P$ and $T_{P'}$. This violation can be either a non-increasing pair on a row $k$ and columns $r$ and $r + 1$, or the associated tableau is not a shape.
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If $T_P$ satisfies first situation, then all the $i^{th}$ columns, $i \neq r, r + 1$ in $T_{\Psi(P)}$ remain unchanged. For the columns $r$ and $r + 1$

```
   k-th row
```

everything under the $k^{th}$ row is also unchanged since all these labels are paired parentheses in the word of $P_r \ast P_{r+1}$. For this row the only change that can occur is of the type

```
   \|
   \|
```

which is a violation of the shape property. Thus the canonical position for $T_{\Psi(P)}$ is the same as for $T_P$.

If $T_P$ satisfies second situation, i.e. a violation of the shape property occurs on the $k^{th}$ row and the columns $r$ and $r + 1$, we have the reverse of the above situation and as a result the same canonical position for both the tableau and the image.

Therefore $\Psi$ is a well-defined involution, is sign-reversing and by definition, its only fixed points are $(id, P)$, where $P \in CS(\mu)$. This proves the characterization given in Theorem 1.

**Remark 5.** The characterization of LR-coefficients given here is equivalent to the characterization where $c^{\nu/\lambda}_{\mu}$ counts the number of row-strict tableaux of shape $\nu/\lambda$, content $\mu'$ whose word is lattice (read column-wise). We say that a word is lattice if every initial subword has (the number of $i$'s) ≥ (the number of $(i + 1)$'s), for every $i$. To see the equivalence we note the 1-1 correspondence obtained by labeling all boxes from $P_i$ with $i$, for any $i$.

3. **THE LR-RULE FOR FUSION COEFFICIENTS WHERE ONE PARTITION HAS AT MOST TWO COLUMNS**

The fusion coefficients we consider are the structure constants for the fusion algebra of WZW conformal field theories associated to $\widehat{sl(n)}$ at level $k$. This algebra $\mathcal{F}^{(n,k)}$ is isomorphic to the algebra of symmetric polynomials $\mathbb{Q}(x_1, \ldots, x_n)^{S_n}/\mathcal{I}^{(n,k)}$ where $\mathcal{I}^{(n,k)}$ is the ideal of $\mathbb{Q}(x_1, \ldots, x_n)^{S_n}$ generated by the Schur functions $s_\lambda$ for which $\lambda_1 - \lambda_n = k + 1$, and $s_{(1^n)} - 1$.

The interpretation of the fusion algebra and many results that we will use here rely on the paper of Goodman and Wenzl [8].

Before we proceed we require some more notation and definitions most of which can also be found in [4] or [3]. In fact the interpretation of the fusion algebra we use is taken from [3] as are many results which we will manipulate.

We say that a partition $\lambda$ is $(n, k)$-restricted, if $l(\lambda) \leq n$ and $0 < \lambda_1 - \lambda_n \leq k$. We denote the set of $(n, k)$-restricted partitions by $\Pi^{(n,k)}$. If $\lambda$ is such that $l(\lambda) \leq n$ and $\lambda_1 - \lambda_n = k + 1$ we call it a border diagram and if $\lambda$ is such that $l(\lambda) \leq n$ and $\lambda_1 - \lambda_n = k$ we call it an edge diagram.

We say that a row-strict tableau $T$ is $(n, k)$-restricted if the shape of $T$ is a $(n, k)$-restricted partition and the row-strict property is preserved when we align the $n^{th}$ row and the first row on the right of $k$ boxes. We denote by $RS\Pi^{(n,k)}(\lambda, \mu)$ the set of row-strict $(n, k)$-restricted
tableaux of shape $\lambda$ and content $\mu$. Similarly, we define the column-strict $(n, k)$-restricted tableaux and denote their set by $\text{CS}\Pi^{(n,k)}(\lambda, \mu)$.

Example: A row-strict $(4, 4)$-restricted tableau

\[
\begin{array}{cccc}
1 & 7 & 8 & \\
1 & 3 & 4 & \\
1 & 2 & 3 & \\
1 & 2 & 3 & 4 & 5 & 7 \\
1 & 7 & 8 & \\
\end{array}
\]

The fusion algebra $\mathcal{F}^{(n,k)}$ has a linear basis indexed by the set $\bar{\Pi}^{(n,k)} = \{ \lambda, l(\lambda) \leq n-1, \lambda_1 \leq k \}$. We can define the quotient map in the following way

\[
\Pi^{(n,k)} \longrightarrow \bar{\Pi}^{(n,k)}
\]

$\lambda \rightarrow \bar{\lambda} = (\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)$.

The product of two Schur functions indexed by $\bar{\Pi}^{(n,k)}$ can be recovered from the product of Schur functions indexed by $\Pi^{(n,k)}$. Therefore we can instead work with the basis $\{s_\lambda\}_{\lambda \in \Pi^{(n,k)}}$.

The structure constants of the fusion algebra are defined by

\[
s_\lambda s_\mu = \sum_\nu N^{(k)}_{\lambda \mu} s_\nu \quad \text{where } s_\lambda, s_\mu, s_\nu \in \Pi^{(n,k)}.
\]

By their equivalent interpretation to the Hecke algebras at root of unity [8] it is known that these coefficients are nonnegative. Here we are able to give a combinatorial characterization for them in the case $\mu_1 \leq 2$ and in addition prove some properties one of which was conjectured in [18]. Using the notations from Lie algebras this means that the weight $\mu$ has the form $\mu = \Lambda_i + \Lambda_j$, where $1 \leq i, j \leq n-1$ and $\Lambda_i$ are the fundamental weights of $sl(n)$. In order to proceed we need the following result from [8].

**Proposition 6** (Corollary 3.3 of [8]). If $\mu \in \Pi^{(n,k)}$, then

\[
s_\mu = \det(e_{\mu'_i - i + j})_{1 \leq i,j \leq m}
\]

where $m \geq l(\mu')$ and $e_r = 0$ for $r > n$ or $r < 0$.

Multiplying a Schur function by an elementary symmetric function within the fusion algebra (Proposition 2.6 of [8]) we get

\[
s_\lambda e_r = \sum_{\nu/\lambda = r \text{-column strip}} s_\nu.
\]

If in Equation (6) we have $\mu_1 = 1$ and hence $s_\mu = e_r$, then the above expression gives the fusion coefficients to be

\[
N^{(k)}_{\lambda \mu} = \begin{cases} 
1 & \text{if } \nu/\lambda \text{ is a } r\text{-column strip and } \nu \in \Pi^{(n,k)} \\
0 & \text{otherwise}.
\end{cases}
\]

For $\mu_1 > 1$, using Proposition 6 on the left-hand side of Equation (6) we obtain

\[
s_\lambda s_\mu = s_\lambda \det(e_{\mu'_i - i + j})_{1 \leq i,j \leq m}.
\]
By expanding the determinant and using Equation (7) we get
\begin{equation}
\sum_{\sigma \in S_m} (-1)^\sigma \sum_{\nu \in \Pi(n,k)} s_{\nu} e_{\sigma,\nu} = \sum_{\sigma \in S_m} \sum_{\nu \in \Pi(n,k)} (-1)^\sigma a^{(k)\nu}_{\lambda(\sigma,\mu')},
\end{equation}
where $a^{(k)\nu}_{\lambda(\sigma,\mu')}$ is the number of paths in the Young’s lattice from $\lambda$ to $\nu$ with ascents included in positions $(\sigma,\mu')$ and for which the partitions corresponding to these positions are $(n,k)$-restricted.

We denote by $P_{(n,k)}^{(\sigma,\mu')}$ the set of all such paths.

If we equate the coefficient of $s_{\nu}$ in both Equations (6) and (9) we get
\begin{equation}
N_{\lambda\mu}^{\nu(k)} = \sum_{(\sigma, P) \in \Omega_k} (-1)^\sigma
\end{equation}
where $\Omega_k$ is the set of pairs $(\sigma, P)$, $\sigma \in S_m$ and $P \in P_{(n,k)}^{(\sigma,\mu')}$. Our aim is to construct an involution $\Phi$ on the set $\Omega_k$ that cancels the negative terms on the right-hand side of Equation (10) and that will yield a combinatorial description for the coefficients $N_{\lambda\mu}^{\nu(k)}$.

In this paper we consider $\mu = m = 2$.

Remark 7. We exclude here the case $l(\mu) = \mu_1 = n$. In this case $s_{\mu} = e_n e_{\mu_2}$, so $N_{\lambda\mu}^{\nu(k)} = a^{(k)\nu}_{\lambda(\sigma,\mu')} = \text{card} P_{(n,k)}^{(\sigma,\mu_2)}$.

We may assume in what follows that $l(\mu) < n$.

Let $\lambda$ and $\nu$ be two $(n,k)$-restricted partitions and $P$ a decreasing path from $\lambda$ to $\nu$ with labels $l(P) = (l_1, \ldots, l_t)$, where $l_1 > l_2 > \ldots > l_t$, so that $\nu/\lambda$ is a column strip. We say that $P$ has a $\perp$-label if $P$ has a label corresponding to the first row of the diagram $\lambda$ and $P$ has a $\top$-label if there is a label corresponding to the $n$th row of the diagram $\lambda$, where $n$ is given by the definition of $(n,k)$-restricted partition. We will denote these labels simply by $\perp$ and $\top$.

**Example 1**: Suppose $\nu/\lambda = \begin{array}{ccccccc} -5 & -4 & -3 & -2 & -1 & 1 & 2 \end{array}$, where $n = 6$ and $l(P) = (3, 2, 1, 1, 4, 5)$, so 3 represents the $\perp$-label and 5 represents the $\top$-label.

We shall write these labels in the tableau of $P$ as
\begin{equation}
\begin{array}{ccccccc} 3 & 2 & 1 & 1 & 4 & 5 \end{array} \begin{array}{c} \downarrow \perp \downarrow \top \end{array}
\end{equation}

Since $\perp$ is the largest label and $\top$ is the smallest, in figures where we do not specify the filling, we omit the symbols and we just use grey boxes to indicate their presence. The four
possible situations are

\[\begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\end{array}\]

(12)

The involution that we construct primarily uses the crystal operators introduced in the previous section, however when this is not possible we define a new modified operator.

Let \( P \) be a path \( P = P_1 \ast P_2 \) from \( \lambda \) to \( \nu \) with the intermediate diagram \( \eta \) such that all \( \lambda, \eta, \nu \) are \((n,k)\)-restricted partitions, that is

\[\lambda \xrightarrow{P_1} \eta \xrightarrow{P_2} \nu.\]

(13)

Consider the following sets:

- \( \mathcal{A} = \{\text{paths } P = P_1 \ast P_2 \in \mathcal{P}_{\mu'}(21), \text{ such that } |P_1| < |P_2|\}\)
- \( \mathcal{B} = \{\text{paths } P = P_1 \ast P_2 \in \mathcal{P}_{\mu}, \text{ such that } |P_1| \geq |P_2|\}\).

It is not difficult to see that the set \( \Omega_k \) of Equation (10) is in fact

\[\Omega_k = \{(21), P, P \in \mathcal{A}\} \cup \{(id), P, P \in \mathcal{B}\}\]

and by an abuse of notation we will write \( \Omega_k = \mathcal{A} \cup \mathcal{B} \). The involution \( \Phi \) that we will construct will have the property that \( \Phi(\mathcal{A}) \subseteq \mathcal{B} \) and \( \Phi(\mathcal{B}) \subseteq \mathcal{A} \). We will start be defining \( \Phi \) on the set \( \mathcal{A} \).

If \( P \in \mathcal{A} \) is a path as in (13) the image \( \Phi(P) = P'_1 \ast P'_2 \) will be

\[\lambda \xrightarrow{P'_1} \eta' \xrightarrow{P'_2} \nu.\]

Since we want \( \Phi(P) \in \mathcal{B} \) we must ensure that \( \eta' \) is a restricted partition. We denote by \( \Psi \) the involution for the classical LR-rule constructed previously. We consider the following two cases.

**Case 1.** Suppose \( \nu \) is not an edge diagram i.e. \( \nu_1 - \nu_n < k \).

In this case let \( \Phi(P) = \Psi(P) \). To show that \( \Phi \) is well-defined recall that the rightmost \( |P_2| - |P_1| - 1 \) unpaired right parentheses from the word of \( P \) must change into left parentheses and hence this number of labels from the column \( P_2 \) move into the first column. We note that if the largest label of \( P_2 \) corresponds to an unpaired parenthesis, then this is the first to move. We must therefore check that if this unpaired label is \( \perp \) we still obtain a partition \( \eta' \in \Pi_{(n,k)} \). When the \( \Psi \)-operator is applied to \( P \), the image of the intermediate partition denoted by \( \eta' \) satisfies \( \eta'_1 - \eta'_n = \eta_1 - \eta_n + \{-1, 0, 1\} \). Now since \( \nu_1 - \nu_n < k \) we only need to see what happens when \( \nu_1 - \nu_n = k - 1 \) and \( \eta_1 - \eta_n = k \). In other words we have \( \top \in P_2 \), and \( \perp \notin P_2 \) i.e.

\[TP_2 = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\end{array}\]
In this case the length of the first row of the diagram $\eta'$ will be equal to the length of the first row of the diagram $\eta$, so $\eta' \in \Pi^{(n,k)}$, too. Regardless of the presence of $\perp$ or $\top$ in $P_1$, the image

$$T_{\Phi(P)} = \begin{array}{c|c|c|c|c|c|c} \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline \end{array}$$

is not a column strict tableau because it belongs to the image of the operator $\Psi$.

**Case 2.** The partition $\nu$ is an edge diagram i.e. $\nu_1 - \nu_n = k$.

From (12) it follows that there are 16 cases to be studied depending on whether $\perp$ or $\top$ appears in $P_1$ or $P_2$.

**A.** $P_2$ contains both $\perp$ and $\top$:

$$T_{P_2} = \begin{array}{c|c|c|c|c|c|c} \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline \end{array}$$

**A-I.** $P_1$ has also contains $\perp$ and $\top$:

$$T_P = T_{P_1 \ast P_2} = \begin{array}{c|c|c|c|c|c|c} \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline \end{array}$$

so $\nu_1 - \nu_n = k$ and $\eta_1 - \eta_n = k$. In this case define $\Phi(P) = \Psi(P)$. Again $\Phi$ is well-defined because the $\perp$-labels in $P_1$ and $P_2$ will actually be consecutive letters in the word of $P = P_1 \ast P_2$, so they will be paired with each other. As a result the $\perp$-label of the second column will not move into the first column. This means that the first row of $\eta'$ has the same length as the first row of $\eta$ and therefore $\eta' \in \Pi^{(n,k)}$. The image has the form

$$T_{\Phi(P)} = \begin{array}{c|c|c|c|c|c|c} \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline \end{array}$$

which is not a column strict tableau.

**A-II.** $P_1$ only contains $\perp$:

$$T_P = \begin{array}{c|c|c|c|c|c|c} \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline \end{array}$$
so \( \nu_1 - \nu_n = k, \eta_1 - \eta_n = k \) and \( \lambda_1 - \lambda_n = k - 1 \). Using the same argument as before \( \Phi(P) = \Psi(P) \) is well-defined. Since \( \bot \) in \( P_2 \) will not move it follows that the image

\[
T_{\Phi(P)} = 
\]

is not a column strict tableau.

**A-III.** The case when \( P_1 \) only contains \( \top \):

\[
T_P = 
\]

is not possible as we have \( \nu_1 - \nu_n = k, \eta_1 - \eta_n = k \) and \( \lambda_1 - \lambda_n = k + 1 \), so \( \lambda \notin \Pi(n,k) \).

**A-IV.** \( P_1 \) does not contain either \( \bot \) or \( \top \):

(14)

\[
T_P = 
\]

This is a case when, by applying the operator \( \Psi \), it is possible that \( \bot \) from the second column will move into the first column, and hence the possibility that \( \eta' \notin \Pi(n,k) \). The operator that we therefore need to construct here will be a modification of the operator \( \Psi \).

There are two subcases to consider depending on whether \( \bot \) is a paired parenthesis in the word of \( P \) or not.

**a).** If \( \bot \) is paired then \( \Phi(P) = \Psi(P) \) is well defined since the pairing of this \( \bot \)-label means that it remains in the second column i.e. the intermediate image partition \( \eta' \) will have \( \eta'_1 = \eta_1 \), and hence \( \eta' \in \Pi(n,k) \).

Example 2: If \( \nu/\lambda = \begin{array}{ccc}
-5 \\
-4 \\
-3 \\
-2 \\
0 \\
1
\end{array} \), \( n = 6, k = 2 \), and the tableau \( T_P = \begin{array}{ccc}
0 \\
-4 \\
-1 \\
-5
\end{array} \), then \( w = 543102 \), and its parentheses structure is \( \hat{w} = ))((()) \). Thus \( \hat{\Phi(w)} = ))((()) \) and

\[
T_{\Phi(P)} = 
\]


The image has the form

\[ T_{\Phi(P)} = \]

and \( T_{\Phi(P)} \) is not a column strict tableau.

b). The \( \perp \)-label is not a paired parenthesis in the word of \( P \). This is a case when \( \Psi \) cannot be applied since \( \perp \) would move into the first column, which means that \( \eta'_1 = \eta_1 + 1 \) and \( \eta'_n = \eta_n \), so \( \eta'_1 - \eta'_n = k + 1 \). For this case we will define a new operator.

We denote by \( D_1 \subseteq A \) the subset of paths satisfying

i) \( \nu \) is an edge diagram \( (\nu_1 - \nu_n = k) \)

ii) the \( (\perp, \top) \)-structure as described by (14)

iii) \( \perp \) is not a paired parenthesis.

We write the word of \( P \) in the same manner as before and assign parentheses. The first letter in this word is actually \( \top \) from \( P_2 \) and the last letter is \( \perp \) from \( P_2 \), since these numbers are the smallest and the largest of all labels, respectively i.e.

\[ w = \top \ldots \ldots \perp . \]

Suppose that in this word we have \( w = \ldots \ldots a_1 \ldots a_2 \ldots a_s \), where \( a_s = \perp \), and the sequence \( a_1 a_2 \ldots a_s \) represents the labels corresponding to the last column of \( \nu/\lambda \).

**Example 3:**

\[ \nu/\lambda = \]

\[ \]

where \( \square \) represents boxes in \( P_1 \) and \( \bigcirc \) represents boxes in \( P_2 \).

Let \( i_0 = \min \{ i \mid a_i \text{ is unpaired letter} \} \). We note that in this case all left parentheses will be paired, since the first and last letter in the word \( w \) are right unpaired parentheses. Thus \( w \) has the following parentheses structure.

\[ \hat{w} = ) ) ( ) ( ) ( ) ( ) ( ) . \]

\[ w = a_1 a_2 a_{i_0} . \]

In the above description of the word \( w \), we highlighted the parenthesis associated to label \( a_{i_0} \). Since all left parentheses (in \( P_1 \)) are paired, the number of unpaired (right) parentheses is \( |P_2| - |P_1| \).

We define the operator \( \phi_1 : D_1 \rightarrow B \) on \( w \) by specifying the changes with respect to the parentheses structure so \( \phi_1(w) = \) changes all right unpaired parentheses into left parentheses except the label \( a_{i_0} \):

\[ \phi_1(w) = (((())) \ldots ())) . \]
Example 4:

For \( \nu / \lambda = \begin{array}{c}
\hline
-3 \\
1 \\
2 \\
3 \\
\hline
\end{array} \), \( n = 4, k = 3 \) and \( T_P = \begin{array}{c}
\hline
3 \\
2 \\
1 \\
0 -3 \\
\hline
\end{array} \) we have \( w = \bar{3} 0 1 2 3 \) and the parentheses structure is \( \hat{w} = (()) \). Thus \( \hat{\phi}_1(w) = (()) \) and \( T_{\phi_1(P)} = \begin{array}{c}
\hline
3 \\
2 \\
-3 \\
1 \\
\hline
\end{array} \).

In the case when \( a_{i_0} = a_s = \perp \) the situation is slightly different.

Example 5:

For \( \nu / \lambda = \begin{array}{c}
\hline
-2 \\
0 \\
2 \\
\hline
\end{array} \), \( n = 3, k = 2 \) and \( T_P = \begin{array}{c}
\hline
2 \\
0 \\
-2 \\
\hline
\end{array} \) we have \( w = \bar{2} 0 2 \) and the parentheses structure is \( \hat{w} = )() \). Thus \( \hat{\phi}_1(w) = (()) \) and \( T_{\phi_1(P)} = \begin{array}{c}
\hline
0 \\
-2 \\
2 \\
\hline
\end{array} \).

As we have seen in Example 4 and 5 the image has the form

\[
(16) \quad T_{\phi_1(P)} = \begin{array}{c}
\hline
\text{OR} \\
\text{OR} \\
\hline
\end{array}.
\]

Proposition 8. The operator \( \phi_1 \) is well defined.

Proof.

There are two things that we need to check. Given that \( \phi_1(P) = P_1' \ast P_2' \) with intermediate diagram \( \eta' \), so \( \lambda \xrightarrow{P_1'} \eta' \xrightarrow{P_2'} \nu \), we have to see that

- \( \eta' \) is a partition
- the skew shapes \( \eta'/\lambda \) and \( \nu/\eta' \) are column-strips.

We show first that \( \eta' \) is a partition. Assume that \( \eta' \) is not, that is there exists \( l \) such that \( \eta'_l < \eta'_{l+1} \). Since the operator \( \phi_1 \) removes labels from a column strip we must have

\[
\eta'_{l+1} = \eta'_l + 1
\]

For simplicity let us denote the labels in the first column of \( P \) and \( \phi_1(P) \) by 1 and the ones in the second column by 2. Generically \( \phi_1 \) transforms some “2 → 1”.

We obtain the above situation only if

\[
\begin{array}{c}
\hline
\ \ \ \ 2 \\
\ \ \ \ 2 \\
\ hline
\end{array} \xrightarrow{\phi_1} \begin{array}{c}
\hline
\ \ \ \ 1 \\
\ \ \ a \\
\ \ \ 2 \\
\ hline
\end{array}
\]

This means that the label \( a \) is not a paired parenthesis in \( w \) and the label \( b \), which is a right parenthesis, is paired or is the label \( a_{i_0} \). Let us consider these two situations.
–If $b(\sim 2)$ is paired, then its pair, a label $b'(\sim 1)$, must be to its left in $w$. Two situations may occur

1). $b' \leq a$. We have

$$\hat{w} = \ldots (\ldots ) \ldots$$
$$w = \ldots b' \ a \ b \ldots .$$

which shows that the label $a$ would be paired. This is not possible since $a$ is assumed to be unpaired.

2). $b' > b$. Since $b'$ must be on $b$’s left we have $b' = b$.

$$\hat{w} = \ldots ) ( \ldots$$
$$w = \ldots a \ b' \ b \ldots .$$

In the shape $\nu/\lambda$ this corresponds to

$$\begin{array}{c}
2 \\
\nu/ \ 2 \\
\nu' \\
\ 2 \\
\ 1 \\
\ 1 \\
\ 1 \\
\ 1
\end{array}$$

In the figure we also indicated the pairing. The pairing that we illustrated above is a consequence of the fact that labels on the same diagonal are in fact equal so they are consecutive letters in the word. In this case we note that there must exist a label from the first path $(\sim 1)$ that is above $b'$. But since this label and $a$ are on the same diagonal (equal) they will pair, a contradiction.

–Therefore assume $b$ is $a_{i_0}$ (the special label). In this case the labels $a$ and $b$ are part of the last column of $\nu$, and $a = a_{i_0} - 1$, $b = a_{i_0}$. Since $a_{i_0}$ was defined to be the smallest label in the last column to be unpaired, the smaller labels in this column $a_i < a_{i_0}$ are paired parentheses. Thus, in particular $a$ would be paired and we again obtain a contradiction. Hence $\eta'$ is indeed a partition.

We now prove that $\eta'/\lambda$ and $\nu/\eta'$ are column strips.

Since $\phi_1$ moves labels from the second path into the first path we have

$$\nu/\eta' \subset \nu/\eta$$

and because $\nu/\eta$ was a column strip, $\nu/\eta'$ is a column strip as well.

Next we show that $\eta'/\lambda$ is a column strip. Assume it is not. This occurs when in two consecutive columns and the same row in $T_P$ we have first a label $a (\sim 1)$ followed by $b (\sim 2)$ changed by $\phi_1$ into two 1’s i.e.

$$a \begin{array}{c}
1 \\
2 \\
b
\end{array} \xrightarrow{\phi_1} a \begin{array}{c}
1 \\
1 \\
1 \\
b
\end{array} .$$

Let us study the situation in $T_P$. We note that in $\hat{w}$ the label $a$ is paired (since it is a left parenthesis and all of them are paired) but the label $b$ is not a paired parenthesis and it is not $a_{i_0}$. The situation above has the following features.

–There is no other label 1 (in the first path) underneath the label $a$. If there were any, say

$$c \begin{array}{c}
1 \\
2 \\
b
\end{array} .$$
the label \( c(\sim 1) \) just below \( a \) would pair with \( b \).

- There must be other labels from the second path, (2’s) above the label \( b \). If there were none, then \( a \) would be paired with \( b \) in \( \hat{w} \) since \( b \) is the first right parenthesis on the right of \( a \) i.e.

\[
\hat{w} = \ldots ( \ldots \\
w = a \ b \ldots 
\]

We observe that the number of 1’s in the first column above the label \( a \) exceeds or is equal to the number of 2’s in the second column above \( b \) i.e.

\[
\begin{array}{c}
\begin{array}{c}
2 \\
v_1' 2 \\
v_2' 2 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
a \\
v_1 2 \\
v_2 2 \\
\end{array}
\end{array}
\]

Now since the labels on diagonals are the first to pair, the label \( b \) will pair with the label 1 in the first column situated on the same row as the last 2 in the second column. This again contradicts our requirement that \( b \) is not paired. Thus \( \eta'/\lambda \) is a column strip.

This concludes the proof of Proposition 8. \( \square \)

Our next task is to find a complete characterization of the image inside \( \mathcal{B} \) and to define \( \Phi \) in this case. Consider \( w \) to be the word of \( T_P \), for \( P \in \mathcal{B} \) and let \( w' = a_1a_2 \ldots a_s \) be the subword of \( w \) made with the labels in the last column of the partition \( \nu \) so

\[
\hat{w} = ( ( ( ( ( ( ( ( ( ( w' = ) ) ) ) ) ) ) ) ) ) ) \\
w' = a_1a_2\ldots a_s
\]

This subword \( w' \) might contain labels from both \( P_1 \) and \( P_2 \). Since this is a word of a column we must first have the boxes from \( P_1 \) on top of which are the boxes from \( P_2 \).

**Example 6:** Consider

\[
\nu = \begin{array}{ccc}
\{ & \{ \\
\{ & \{ \\
\{ & \{ \\
\end{array}
\end{array}
\}
\}
\}
\}
\]

in \( P_2 \)

\[
\begin{array}{c}
\begin{array}{c}
\{ & \\
\} & \\
\} & \\
\} & \\
\end{array}
\end{array}
\]

in \( P_1 \)

Thus the parentheses structure is

\[
\hat{w}' = ( ) \ldots ) ( ( ( .... ( \\
w = a_1 \ldots a_{i0}
\]

i.e. the right are followed by the left parentheses. We denote by \( a_{i0} \) the rightmost right parenthesis of \( w' \). We also identify the label \( (a_1 - 1) \) (if it exists in \( w \)) which will play a role in the next definition. This is the label situated on the penultimate column and on the same row as the last box in the last column, i.e. the label \( a_1 \).

**Example 7:**
Definition. The subset $D_2 \subseteq B$ of paths $P = P_1 \ast P_2$ is defined by the paths satisfying

1. $T_P$ is a column strict tableaux
2. the $(\perp, \top)$-label structure of $T_P$ is described by (16)
3. the last column contains labels from $P_2$ and the label $a_{i_0}$ is not paired with $(a_1 - 1)$ (if the latter exists in $w$)
4. the smallest label $\top$ in $w$, is either an unpaired left parenthesis or paired with the label $a_{i_0}$.

The operator $\phi_2 : D_2 \rightarrow A$ applied to $w$ is defined via

$\hat{\phi_2(w)}$ which changes all unpaired left parentheses into right ones including the left parenthesis $b_{i_0}$ paired with $a_{i_0}$ i.e.

$$
\begin{align*}
    w &= \ldots b_{i_0} \ldots \ldots a_{i_0} \ldots \\
    \hat{w} &= ( ( ) ( ( ) ( ) ) ) ( ( ) ) \\
    \hat{\phi_2(w)} &= ( ) ( ) ( ( ) ( ) ) ) .
\end{align*}
$$

Remark 9. The following properties of the image $T_{\phi_2(P)}$ are easy consequences of the above definition.

i) The number of parenthesis to be changed is now $|P_1| - |P_2| - 1$. If $|P_1| = p$, $|P_2| = q$ then $T_{\phi_2(P)} = P''_1 \ast P''_2$ with the intermediate diagram $\eta'' : \lambda \xrightarrow{P''_1} \eta'' \xrightarrow{P''_2} \nu$ and $|P''_1| = p - (p - q + 1) = q - 1$, $|P_2| = q + p - q + 1 = p + 1$.

ii) The first letter $\top$ in $w$, is moved by the operator $\phi_2$ in $P''_2$.

iii) The biggest letter $\perp$ in $w$, is either an unpaired label or is the label $a_{i_0}$. In both cases this label is in $P''_2$ and is unpaired in $\phi_2(w)$.

These characteristics prove that $\text{Im}(\phi_2) \subseteq \text{Dom}(\phi_1) = D_1$.

Proposition 10. The operator $\phi_2$ is well defined.

Proof. As in Proposition 8 we have to check that the intermediate diagram $\eta''$ is a partition and that both skew diagrams $\eta''/\lambda$ and $\nu/\eta''$ are column strips. The proof that $\eta''$ is a partition is similar to the one in Proposition 8 and we leave it to the reader.

We shall prove that $\eta''/\lambda$ and $\nu/\eta''$ are column-strips. Since $\phi_2$ moves labels from the first path into the second path we have that $\eta''/\lambda \subset \eta/\lambda$ so $\eta''/\lambda$ is a column strip.
We now show that \( \nu/\eta'' \) is a column strip. Assume it is not. This occurs when in two consecutive columns and the same row in \( T_P \) we first have a label \( a(\sim 1) \) followed by \( b(\sim 2) \) changed by \( \phi_2 \) into two 2’s i.e.

\[
a [1\ 2\ b] \xrightarrow{\phi_2} a [2\ 2\ b].
\]

This means the label \( a \) is either an unpaired left parenthesis or is the label \( b_{i_0} \).

- Assume that \( a \) is an unpaired left parenthesis. We first note that there are no other 2’s above the label \( b \) (if there were any, the first label 2 above \( b \) would be on the same diagonal with \( a \), so it would pair with it). Another useful observation is that the number of 2’s in the second column must exceed or be equal to the number of 1’s in the first column below the label \( a \) i.e.

\[
\begin{array}{c}
a \\
\downarrow \varepsilon \\
\downarrow \varepsilon \\
2
\end{array}
\]

In this case the label \( a \) pairs with the label 2 in the second column situated on the same row as the first 1 in the first column.

- Assume that \( a \) is the label \( b_{i_0} \), i.e. the label paired with \( a_{i_0} \). As before we claim that there are no labels 2 in the second column above \( b \). If there were any, the first one above \( b \) would pair with \( a \), so this label must be \( a_{i_0} \). This is not possible since there are no labels 2 below \( a_{i_0} \), by the definition of \( a_{i_0} \). As above we also have that the number of 2’s in the second column must exceed or be equal to the number of 1’s in the first column below the label \( a \). We note that \( a = b_{i_0} \) pairs with the label \( a_{i_0} (\sim 2) \) in the second column situated on the same row as the first label 1 in the first column. This shows that the value of the label \( a = b_{i_0} \) is \( a_1 - 1 \), where \( a_1 \) is the last label in the last column of \( \nu \), which is also the label \( b \).

This situation contradicts condition (3) in the definition of \( D_2 \).

This concludes the proof of Proposition 10.

The following result shows that the operators \( \phi_1 \) and \( \phi_2 \) are inverse to each other.

**Proposition 11.**

a). \( \text{Im}(\phi_1) \subset D_2 \) and \( \text{Im}(\phi_2) \subset D_1 \).

b). \( \phi_1 \circ \phi_2 = id_{D_2} \) and \( \phi_2 \circ \phi_1 = id_{D_1} \).

**Proof.**

a). In Remark 9 we showed that \( \text{Im}(\phi_2) \subset D_1 \). We next show that \( \text{Im}(\phi_1) \subset D_2 \).

1. In \( \phi_1(w) \) all right parentheses will be paired (including \( a_{i_0} \) so \( T_{\phi_1(p)} \) is a column strict tableau.

2. We also establish in description [16] the \((\top, \bot)\)-label structure.

3. The label \( a_{i_0} \) in \( \phi_1(w) \) cannot pair with the label \( (a_1 - 1) \) (see Figure 3). If this happens, then in \( w \) the label \( (a_1 - 1) \) was a right parenthesis i.e in the second path. However in \( w \) the label \( a_1 \), which is situated on the same row, is also in the second path. This cannot be possible since the second path must represent a column-strip.
4. The \( \top \)-label, which is the first letter in \( w \) (or \( \phi_1(w) \)) is unpaired or it pairs with \( a_{i_0} \) if there were no other unpaired right parentheses between \( \top \) and \( a_{i_0} \).

Hence \( \text{Im}(\phi_1) \subset D_2 \).

b). The relation \( \phi_2 \circ \phi_1 = \text{id}_{D_1} \) is obvious by the definition of the operators \( \phi_1 \) and \( \phi_2 \).

We illustrate this by the following example. Let \( T_P \in D_1 \) and \( w \) be its word with the parentheses structure

\[
\hat{w} =))))(((((()(())())))
\]

where \( a_{i_0} \) is highlighted. Applying \( \phi_1 \) we get.

\[
\hat{\phi_1(w)} = (((((())((((())()())
\]

Since we showed that \( T_{\phi_1(P)} \in D_2 \) we can apply the operator \( \phi_2 \) to \( \phi_1(w) \) to get

\[
\phi_2(\hat{\phi_1(w)}) = )))((((((())())))
\]

Therefore we have \( \phi_2(\phi_1(w)) = w \). Similarly we have that \( \phi_1 \circ \phi_2 = \text{id}_{D_2} \).

\( \square \)

We define all column strict tableaux of shape \( \mu \) that do not belong to the set \( D_2 \) to be \( k \)-fusion and we denote their set by \( \text{CSF}_k(\mu) = \text{CS}(\mu) \setminus D_2 \).

We finish the case that we studied (A-IV,b) by letting \( \Phi = \phi_1 \).

**B.** \( P_2 \) contains the \( \perp \)-label but not the \( \top \)-label:

\[
T_{P_2} = \begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\]

**B-I.** \( P_1 \) contains both labels:

\[
T_P = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

In this case \( \eta_1 - \eta_n = k - 1 \) and \( \lambda_1 - \lambda_n = k - 1 \) so define \( \Phi(P) = \Psi(P) \) which is again well defined by a similar argument to the one in **A-I.** Therefore the tableau of the image:

\[
T_{\Phi(P)} = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

is not a column strict tableau.

**B-II.** \( P_1 \) contains the \( \perp \)-label but not the \( \top \)-label:

\[
T_P = \begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}
\]
Here it is clear that $\Phi(P) = \Psi(P)$ is well-defined and the image

$$T_{\Phi(P)} = \text{\[image\]}$$

is again not a column strict tableau.

**B-III.** $P_1$ contains the $\top$-label but not the $\bot$-label:

$$T_P = \text{\[image\]}.$$

In this case $\eta_1 - \eta_n = k - 1$ and $\lambda_1 - \lambda_n = k$. Define $\Phi(P) = \Psi(P)$. Here it is possible that the $\bot$-label will move into the first column if it is not a paired parenthesis, which means that the length of the first row of $\eta'$ will increase by one, so $\eta'_1 - \eta'_n = (k - 1) + 1 = k$, and $\eta'$ is still a $(n, k)$-restricted partition. In this case the image can be

$$T_{\Phi(P)} = \text{\[image\]}.$$

In both cases the image is not a column strict tableau.

**B-IV.** $P_1$ does not contain either labels:

$$T_P = \text{\[image\]}.$$

This is a case similar to the previous one, so $\Phi$ will be defined in the same way. The image can be

$$T_{\Phi(P)} = \text{\[image\]}.$$

but, again, in both cases the image is not a column strict tableau.

For the remaining cases C where $P_2$ has the $\top$-label but not the $\bot$-label and D when $P_2$ does not have either labels, since the $\bot$-label is not present there is no danger in increasing the first row of the diagram $\eta'$, so in all these cases we define $\Phi(P) = \Psi(P)$. The structure of the $(\bot, \top)$-label of the images will look the same as for $P$ and the associated tableaux are not column strict.

**Observation:** As we have seen in A–I, many of these 8 remaining cases will not be possible. To conclude we have found only one case where we introduce a new operator.

To finish this analysis we must also consider the situation $P = P_1 \ast P_2$ for which $|P_1| \geq |P_2|$. Since we have already seen the structure of the image of $\Phi$, we have the following.
i) If there is a violation of the $CS$-property for $T_P$ and $P \notin D_1$ then $\Phi(P) = \Psi(P)$.

ii) If $P \in D_1$, then $\Phi(P) = \phi_1(P)$.

iii) If $T_P \in CS(\mu) \setminus CSF_k(\mu)$, i.e. $P \in D_2$ then $\Phi(P) = \phi_2(P)$.

iv) In any other case, i.e. $T_P \in CSF_k(\mu)$, we have $\Phi(P) = P$.

In fact we also proved that $\Phi$ is an involution on the set of paths made from two decreasing paths whose intermediate partitions are $(n, k)$-restricted.

The fusion coefficients, which are the number of fixed points of the involution $\Phi$, count the number of $k$-fusion tableaux. Therefore we have:

**Theorem 12.** For $\mu$ a two-column partition and any level $k$ we have

\[
N^{(k)}_{\lambda\mu} = \# \{\text{paths } P \text{ from } \lambda \text{ to } \nu \text{ in } \mathcal{P}^{(n,k)}_{(\mu')} \text{ that fit } \mu \text{ and } T_P \in CSF_k(\mu')\}.
\]

**Remark 13.** By replacing every label from $P_1$ by 1 and the labels from $P_2$ by 2 in the partition $\nu$ and reinterpreting the conditions in the definition of $D_2$ we get the following characterization for fusion coefficients in the case $\mu$ has two columns.

The coefficient $N^{(k)}_{\lambda\mu}$ counts the number tableaux in $RS\Pi^{(n,k)}(\nu/\lambda, \mu)$ whose word (read column-wise) is lattice, except the tableaux for which

- $\nu_1 - \nu_n = k$,
- the first row contains exactly one of 1 or 2 and the last row contains exactly a 1,
- the last column contains 2's,
- the number of 1's in the penultimate column under the height of the last column is strictly less than the number of 2's in the last column (see Figure 4 under the thick line),
- the number of 1's in the reading word is always strictly bigger than the number of 2's except (perhaps) when the last 2 is counted.

4. Applications

We shall now give some consequences of the last theorem.

**Corollary 14** (part of Prop(2.2) of [8]). For any level $k$, if $\mu$ is a one or two-column partition, we have

a). $N^{(k)}_{\lambda\mu} \leq c_{\lambda\mu}'$, where the latter are the classical Littlewood-Richardson coefficients.

b). If all the paths in $\Omega_k$ are only passing through $(n, k)$-restricted partitions (e.g. $\lambda_1 - \lambda_n \leq k - 1$), then $N^{(k)}_{\lambda\mu} = c_{\lambda\mu}'$.

**Proof.**

a). This is an obvious consequence of Theorem 12.
b). With this condition, the case where \( T_P \not\in \text{CSF}_k(\mu) \) cannot occur, so \( \Phi = \Psi \).

The next result proves the conjecture (2.4) in [18], in our special case.

**Theorem 15.** If \( \mu \) is a one or two-column partition, then we have

\[
N_{\lambda\mu}^{(k)\nu} \leq N_{\lambda\mu}^{(k+1)\nu}.
\]

**Proof.**

From the way we constructed the involution \( \Phi \) we know that

\[
N_{\lambda\mu}^{(k)\nu} = \#\{ P \in \mathcal{P}_{(n,k)}^{(\mu)} \text{, from } \lambda \text{ to } \nu \text{ such that } \Phi(P) = P \}.
\]

In order to prove the inequality it suffices to see that if \( \Phi_{(k)}(P) = P \) then \( \Phi_{(k+1)}(P) = P \) as well. Note we index the operators by the levels that we consider.

Suppose this is not the case. Since \( T_P \in \text{CSF}_k(\mu) \subset \text{CS}(\mu) \), it is possible that \( P \) is not \((k + 1)\)-fusion. In this case the partition \( \nu \) must be an edge diagram for level \((k + 1)\) i.e. \( \nu_1 - \nu_n = k + 1 \). This cannot happen since \( \nu \) is also a \((n, k)\)-restricted partition i.e. \( \nu_1 - \nu_n \leq k \).

We yield another application by using the rank-level duality. Recall [8] that we can define a bijection between \((n, k)\)-restricted partitions and \((k, n)\)-restricted partitions as follows.

For \( \lambda \in \Pi_{(n,k)} \), cut the rectangle \( \lambda_1 \times n \) into rectangles of sides \( k \times n \). Conjugate each rectangle separately and then glue the resulting partitions back together.

**Example:**

![Example Diagram]

It is clear that the resulting partition \( \tilde{\lambda} \) constructed in this way is a \((k, n)\)-restricted partition. Goodman and Wenzl [8] showed that the fusion coefficients are invariant under this bijection i.e. \( N_{\lambda\mu}^{(k)\nu} = N_{\tilde{\lambda}\tilde{\mu}}^{(n)\tilde{\nu}} \) and as a result we have the following theorem.

**Theorem 16.** For any level \( k \) if \( n \geq 3 \), and \( \mu \) is a partition with one or two rows, then the fusion coefficients \( N_{\lambda\mu}^{(k)\nu} = N_{\tilde{\lambda}\tilde{\mu}}^{(n)\tilde{\nu}} \) are given in Theorem 12.

**Proof.** If \( n \geq 3 \) then \( \mu_1 \leq k \). Therefore \( \tilde{\mu} = \mu' \), where \( \mu' \) is the conjugate of \( \mu \). It is now clear that we are in the setting of Theorem 12 and as a result we can determine the coefficients \( N_{\lambda\mu}^{(k)\nu} \).

\[
N_{\lambda\mu}^{(k)\nu} = \begin{cases} c_{\lambda\mu}^{\nu} & \text{if } k \geq \lambda_1 - \lambda_2 + \mu_1 - \mu_2 + \nu_1 - \nu_2 \nonumber \\ 0 & \text{otherwise} \end{cases}.
\]

**Remark 17.** For \( n = 2 \) the fusion coefficients are given by the Gepner-Witten formula [7]
5. Conclusions

The goal of this paper was to find an appropriate involution, in the same manner as for the Littlewood-Richardson rule, which would give a much desired combinatorial description for the fusion coefficients. As for the LR-rule we started by defining the involution in the case where one partition has at most two columns. We were able to prove that except in one case, the involution remained the same. In this special case we argued that we must construct a different operator, somehow similar with the classical one, and we were successful in doing so. The obstruction in defining the involution in the general case is the fact that we could not find a canonical position in the partition where the operator for the 2-column case is to be applied. A reason for this is that it seems there is no specific area in the 2-column part that remained unchanged by the involution.

Another question one can ask about fusion coefficients is does there exist an equivalent Robinson-Schensted correspondence? This question seems legitimate since we can establish a result similar to the following equality \[3\].

**Proposition 18.** If \( \lambda \subseteq \nu \) are two partitions then

\[
\# \{ \text{ paths from } \lambda \text{ to } \nu \} = \sum_{\mu \vdash |\nu/\lambda|} c_{\lambda\mu}^\nu f^\lambda
\]

where \( f^\lambda \) denotes the number of standard tableaux of shape \( \lambda \).

We extend some of the definitions given previously. We say that a path is \((n, k)\)-restricted if it only passes through \((n, k)\)-restricted partitions. Let \( T_\lambda \) be a standard tableau of shape \( \lambda \). We can identify the standard tableau with a path from \( \emptyset \) to \( \lambda \). The \( i^{th} \) partition in the path is obtained obtained from the previous one by adding the box indexed \( i \) in the tableau \( T_\lambda \).

**Example:** The tableau \( T_\lambda = \begin{array}{ccc} 3 \\ 2 & 4 \\ 1 & 3 & 8 \end{array} \) represents the path

\[
\begin{array}{cccccccc}
\emptyset & \leftrightarrow & \begin{array}{c} 3 \end{array} & \leftrightarrow & \begin{array}{cc} 2 & 4 \end{array} & \leftrightarrow & \begin{array}{ccc} 1 & 3 & 8 \end{array} & \leftrightarrow \\
\end{array}
\]

We say that a standard tableau is \((n, k)\)-restricted if the associated path is \((n, k)\)-restricted. We note that this definition is consistent with the definition of a column-strict restricted tableau. We denote by \( f_k^\lambda \) the number of \((n, k)\)-restricted standard tableaux of shape \( \lambda \). A future task is to find some sort of expression for these numbers. We have the following result.

**Theorem 19.** For \( \lambda \subseteq \nu \), partitions in \( \Pi^{(n,k)} \) we have

\[
\# \{ \text{restricted paths from } \lambda \text{ to } \nu \} = \sum_{\mu \vdash |\nu/\lambda|, \mu \in \Pi^{(n,k)}} N_{\lambda\mu}^{(k)} f_k^\mu.
\]

**Proof.**
First we prove by induction that
\[(e_1)^m = \sum_{\mu \vdash m} s_\mu f^\mu_k.\] (21)

We stress here that all equalities take place in the fusion algebra \(F^{(n,k)}\) and all partitions involved are \((n,k)\)-restricted.
If \(m = 1\) the right hand side of Equation (21) is \(s_{(1)}f^{(1)}_k = e_1 \cdot 1.\)

Assume that the equality is true for \(m\) and we shall prove it for \(m + 1\). First observe
\[(e_1)^{m+1} = \left(\sum_{\mu \vdash m} s_\mu f^\mu_k\right) \cdot e_1 = \sum_{\mu \vdash m} (s_\mu e_1) f^\mu_k = \sum_{\mu \vdash m} \left(\sum_{|\nu/\lambda|=1} s_\nu f^\nu_k\right) = \sum_{\mu \vdash m+1} s_\nu \left(\sum_{|\nu/\mu|=1} f^\nu_k\right).\]

To show that \(\sum_{|\nu/\lambda|=1} f^\nu_k = f^\nu_k\) we note that each \((n,k)\)-restricted standard tableau of shape \(\mu\) determines a unique \((n,k)\)-restricted standard tableau of shape \(\nu\) by adding the corresponding box with the entry \((m + 1)\). This process is reversible since the box filled with \((m + 1)\), which is the largest number of the standard tableau, is an exterior corner of the shape \(\nu\).
Therefore we get that \((e_1)^{m+1} = \sum s_\nu f^\nu_k.\)

Now we proceed to prove Equation (20). If we multiply \(s_\lambda\) successively with \(e_1\) in the fusion algebra we get
\[s_\lambda(e_1)^m = \sum_{|\nu/\lambda|=m} \#\{\text{restricted paths from } \lambda \text{ to } \nu\} s_\nu.\] (22)

Using (21), the left-hand side of this equality becomes
\[s_\lambda(e_1)^m = s_\lambda \sum_{\mu \vdash m} s_\mu f^\mu_k = \sum_{\mu \vdash m} s_\lambda s_\mu f^\mu_k = \sum_{\mu \vdash m} (\sum_{|\nu/\lambda|=m} N^{(k)\nu}_{(k)\mu} s_\nu f^\nu_k) = \sum_{\nu} \left(\sum_{|\nu/\mu|=1} N^{(k)\nu}_{(k)\mu} f^\nu_k\right)s_\nu.\]

Equating the coefficient of \(s_\nu\) in the right-hand side of Equation (22) and the line above we get
\[\#\{\text{restricted paths from } \lambda \text{ to } \nu\} = \sum_{\nu} \left(\sum_{\mu \vdash |\nu/\lambda|, \mu \in \Pi^{(n,k)}} N^{(k)\nu}_{(k)\mu} f^\nu_k\right).\]

In view of the last equation one could hope to define fusion-Knuth relations among the words of the restricted paths. This might happen since in the classical case, the Knuth relations and Equation (19) determine the Littlewood-Richardson coefficients as the number of equivalence classes. In a similar way, fusion coefficients would count equivalence classes under fusion-Knuth relations. This, however, remains to be the subject of further investigation.

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Department of Mathematics and Statistics, York University, North York, Ont., M3J 1P3, CANADA
E-mail address: gtudose@mathstat.yorku.ca