ON STABILITY OF N-TIMES INTEGRATED SEMIGROUPS WITH NONQUASIANALYTIC GROWTH

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ABSTRACT. We discuss the behaviour at infinity of n-times integrated semigroups with nonquasianalytic growth and invertible generator. The results obtained extend in this setting a theorem of O. El Mennaoui on stability of bounded once integrated semigroups, and (partially) a theorem of Q. P. Vţu on stability of $C_0$-semigroups.

1. Introduction

Let $A$ be a closed operator on a Banach space $X$ with domain $D(A)$. The solution $u: [0, \infty) \to X$ of the Cauchy equation

$$u'(t) = Au(t), \quad t \geq 0; \quad u(0) = x \in D(A)$$

is given by $u(t) = T_0(t)x$ where $T_0(t) := e^{tA}, \ t \geq 0$, is the $C_0$-semigroup generated by $A$, provided $A$ satisfies the Hille-Yosida condition; see [2, Section 3.1]. There still are other important cases where $A$ does not satisfy that condition but it is the generator of an exponentially bounded $n$-times integrated semigroup in the following sense:

There exist a family $(T_n(t))_{t \geq 0}$ of bounded operators on $X$ and $C, w \geq 0$ such that $\|T_n(t)\| \leq Ce^{wt}, \ t \geq 0$, and

$$(\lambda - A)^{-1}x = \lambda^n \int_0^{\infty} e^{-\lambda t} T_n(t)x \ dt, \quad \Re \lambda > w, \ x \in X.$$ 

Every $C_0$-semigroup is a 0-times integrated semigroup; for more information on integrated semigroups and examples see [2 Sections 3.2 and 8.3], [3] and references therein.

Put $u(t) := (d/dt)^nT_n(t)x$, so that

$$T_n(t)x = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} u(s)x \ ds, \quad x \in X.$$ 

Then the function $u(t)$ is the unique solution to the equation (1) and the limit $\lim_{t \to \infty} T_n(t)$ -or alternatively its ergodic version $\lim_{t \to \infty} t^{-n}T_n(t)$-

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reflects the asymptotic behaviour of the solution $u$ at infinity. In this respect, when $n = 0$, a uniformly bounded $C_0$-semigroup $(T_0(t))_{t \geq 0}$ is said to be stable if

$$\lim_{t \to \infty} T_0(t)x = 0, \quad x \in X.$$ 

The stability of uniformly bounded $C_0$-semigroups on Banach spaces, under certain spectral assumptions on their infinitesimal generators, is proven in [1] and [8], with different proofs and independently one from each other. We refer to this stability result as the Arendt-Batty-Lyubich-V̆u theorem. It states that $(T_0(t))_{t \geq 0}$ is stable whenever

$$\sigma(A) \cap i\mathbb{R} \text{ is countable and } \sigma_P(A^*) \cap i\mathbb{R} = \emptyset.$$ 

Here $\sigma(A)$ is the spectrum of the generator $A$ and $\sigma_P(A^*)$ is the point spectrum of the adjoint operator $A^*$ of $A$. For the asymptotic behaviour and stability of operator semigroups we refer the reader to [4] and [9].

It seems interesting to have a result like the Arendt-Batty-Lyubich-V̆u theorem for $n$-times integrated semigroups. In this setting, a notion of stability has been defined for once integrated semigroups as follows. Suppose that $(T_0(t))_{t \geq 0}$ is a $C_0$-semigroup and let $(T_1(t))_{t \geq 0}$ denote the (trivial, say) once integrated semigroup induced by $(T_0(t))_{t \geq 0}$, which is defined by

$$T_1(t)x := \int_0^t T_0(s)x \, ds, \quad x \in X.$$ 

By a well known property of $C_0$-semigroups,

$$T_0(t)x - x = AT_1(t)x = T_1(t)Ax, \quad \text{for } x \in D(A).$$

Assume in addition that the $C_0$-semigroup $(T_0(t))_{t \geq 0}$ is stable. Then, provided $A$ is invertible, one gets that there exists the limit

$$\lim_{t \to \infty} T_1(t)x = -A^{-1}x, \quad x \in \overline{D(A)}.$$ 

Motivated by this observation, a (nontrivial) general once integrated semigroup $T_1(t)$ is called stable in [5, p. 363] when $\lim_{t \to \infty} T_1(t)x$ exists for every $x \in \overline{D(A)}$. Moreover, it is also shown in [5, Prop. 5.1] that if a once integrated semigroup $T_1(t)$ is stable in the sense of that definition then $A$ must be invertible, which is to say that 0 belongs to the resolvent set $\rho(A)$ of $A$. The following result is [5, Theorem 5.6]. It gives a version of the Arendt-Batty-Lyubich-V̆u theorem for once integrated semigroups.

**Theorem 1.1.** Let $A$ be the generator of a once integrated semigroup $(T_1(t))_{t \geq 0}$ such that $\sup_{t > 0} \|T_1(t)\| < +\infty$. Assume in addition that $\sigma(A) \cap i\mathbb{R}$ is countable, $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$ and $0 \in \rho(A)$. Then $(T_1(t))_{t \geq 0}$ is stable.
The boundedness condition assumed on \((T_1(t))_{t \geq 0}\) in Theorem 1.1 looks somehow restrictive: For a uniformly bounded \(C_0\)-semigroup \((T_0(t))_{t \geq 0}\) and its \(n\)-times integrated semigroup

\[
T_n(t)x = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1}T_0(s)x \, ds, \quad t > 0, \, x \in X,
\]

the derived boundedness condition on \((T_n(t))_{t \geq 0}\) which is to be expected from the integral expression is \(\sup_{t > 0} t^{-n} \|T_n(t)\| < +\infty\), so that for \(n = 1\) is \(\sup_{t > 0} t^{-1} \|T_1(t)\| < +\infty\) instead of boundedness.

The purpose of this note is to extend Theorem 1.1 to \(n\)-times integrated semigroups for every natural \(n\) and a fairly wide boundedness condition involving nonquasianalytic weights. We say that a positive weight \(\omega\) is countable, 

As in [11, Section 1] we assume that \(\lim \inf_{t \to \infty} \omega(t)^{-1} \omega(s + t) \geq 1\) for all \(s > 0\). Then one can define the function \(\tilde{\omega}\) on \([0, \infty)\) given by

\[
\tilde{\omega}(s) := \lim \sup_{t \to \infty} \frac{\omega(t + s)}{\omega(t)}, \quad s \geq 0, \quad \text{and} \quad \tilde{\omega}(s) := 1, \quad s < 0,
\]

is a weight function. Clearly, \(\tilde{\omega}(t) \leq \omega(t)\) for every \(t \geq 0\).

Our main result is the following. In the statement, and throughout the paper, the symbol “\(\sim\)” in \(a(t) \sim b(t)\) as \(t \to \infty\) means that \(\lim_{t \to \infty} b(t)^{-1} a(t) = c > 0\) as \(t \to \infty\).

**Theorem 1.2.** Let \(A\) be the generator of a \(n\)-times integrated semigroup \((T_n(t))_{t \geq 0}\) such that \(\sigma(A) \cap i \mathbb{R}\) is countable, \(\sigma_p(A^*) \cap i \mathbb{R} = \emptyset\) and \(0 \in \rho(A)\). Assume that

\[
\sup_{t \geq 1} \omega(t)^{-1} \|T_n(t)\| < +\infty,
\]

for some nonquasianalytic weight \(\omega\) on \([0, \infty)\) for which \(\tilde{\omega}(t) = O(t^k)\), as \(t \to \infty\), for some \(k \geq 0\).

We have:

(i) If \(\omega(t)^{-1} = o(t^{-n+1})\) as \(t \to \infty\), then

\[
\lim_{t \to \infty} \omega(t)^{-1}T_n(t)x = 0, \quad x \in D(A^n).
\]

(ii) If \(\omega(t) \sim t^{n-1}\) as \(t \to \infty\), then

\[
\lim_{t \to \infty} t^{-n+1}T_n(t)x = -\frac{1}{(n-1)!} A^{-1}x, \quad x \in D(A^n).
\]

**Remark 1.1.** For \(n = 1\), Theorem 1.2 (ii) is [5, Theorem 5.6]. So any \(n\)-times integrated semigroup \((T_n(t))_{t \geq 0}\) satisfying the equality of Theorem 1.2 (ii) might well be called stable. Alternatively, the ergodic
type equality \( \lim_{t \to \infty} \omega(t)^{-1} T_n(t)x = 0 \), for \( \omega(t) \sim t^n \) at infinity, defines a property on \( (T_n(t))_{t \geq 0} \) which corresponds to stability of \( C_0 \)-semigroups when \( n = 0 \). Then one could say that a \( n \)-times integrated semigroup satisfying Theorem 1.2 (i) for \( \omega(t) \sim t^n \) as \( t \to \infty \) is stable of order \( n \), and stable under \( \omega \) in general.

2. Proof of Theorem 1.2

In order to establish Theorem 1.2 one needs to extend [11, Theorem 7] and [5, Theorem 5.6]. Firstly, and more precisely, Theorem 2.1 below is an improvement of [11, Theorem 7], which is in turn an extension of the Arendt-Batty-Lyubich-Vu theorem. In fact [11, Theorem 7] is recovered in the particular case that the operator \( R \) is the identity operator in Theorem 2.1. The case \( \beta(t) \equiv 1 \) in Theorem 2.1 appears in [1, Remark 3.3].

For a Banach space \( (Y, \| \cdot \|) \), let \( B(Y) \) denote the Banach algebra of bounded operators on \( Y \).

\textbf{Theorem 2.1.} Let \( (U(t))_{t \geq 0} \subset B(Y) \) be a \( C_0 \)-semigroup of positive exponential type with generator \( L \). Let \( \beta \) be a nonquasianalytic weight on \( [0, \infty) \) such that \( t \beta(t) = O(t^k) \) as \( t \to \infty \), for some \( k \geq 0 \). Assume that there exists \( R \in B(Y) \) such that \( U(t)R = RU(t) \) for all \( t \geq 0 \) and \( \|U(t)R\| \leq \beta(t) \) for \( t \geq 0 \).

If \( \sigma(L) \cap i\mathbb{R} \) is countable and \( \sigma_P(L^*) \cap i\mathbb{R} = \emptyset \) then

\[ \lim_{t \to \infty} \frac{1}{\beta(t)} U(t)Ry = 0, \quad y \in Y. \]

\textit{Proof.} The overall argument goes along similar lines as in [11, Theorem 7], lemmata included. Next, we outline that argument for convenience of prospective readers and give details, when necessary, to extend the corresponding assertions to our setting.

Put

\[ q(y) := \limsup_{t \to \infty} \beta(t)^{-1} \|U(t)Ry\|, \quad y \in Y. \]

Then \( q \) is a seminorm on \( Y \) such that \( q(y) \leq \|y\| \) for all \( y \in Y \). Moreover, \( q(U(s)y) \leq \tilde{\beta}(s)q(y) \) for every \( s \geq 0 \), \( y \in Y \), and so \( N := \{ y \in Y : q(y) = 0 \} \) is a \( U(t) \)-invariant closed subspace of \( Y \). Hence one can define a norm \( \tilde{q} \) on \( Y/N \) given by

\[ \tilde{q}(\pi(y)) := q(y), \quad y \in Y, \]

and an operator \( \hat{U}(t) \) on \( Y/N \) given by

\[ \hat{U}(t)(\pi(y)) := \pi(U(t)y), \quad y \in Y, t \geq 0, \]

where \( \pi \) is the projection \( Y \to Y/N \).

It is straightforward to show that \( (\hat{U}(t))_{t \geq 0} \) is a strongly continuous semigroup in the norm \( \tilde{q} \) on \( Y/N \). Let \( (Z, \| \cdot \|_Z) \) be the \( \tilde{q} \)-completion
of $Y/N$, and let $V(t)$ be the continuous extension on $Z$ of $\hat{U}(t)$ for all $t > 0$. Then:

(a) $\|\pi(y)\|_Z = \limsup_{t \to \infty} \frac{1}{\beta(t)} \|U(t) Ry\|$ for $y \in Y$. This is obvious.

(b) $\|V(t)\|_{Z \to Z} \leq \tilde{\beta}(t)$, $t \geq 0$, and from this fact one readily obtains that $(V(t))_{t \geq 0}$ is a $C_0$-semigroup in $B(Z)$. The above bound follows by continuity and density from the estimate

$$\hat{q}(\hat{U}(t)\pi(y)) = \hat{q}(\pi(U(t)y)) = q(U(t)y)$$

$$\leq \tilde{\beta}(t)q(y) \leq \tilde{\beta}(t)\hat{q}(\pi(y)), \quad y \in Y, t \geq 0.$$

(c) $\|V(t)z\|_Z \geq \|z\|_Z$ for all $z \in Z$: For $y \in Y$ and $t \geq 0$,

$$\hat{q}(\hat{U}(t)\pi(y)) = \limsup_{t \to \infty} \frac{\beta(t + s)\|U(t + s)Ry\|_Y}{\beta(t + s)} \geq \hat{q}(\pi(y)).$$

Then we apply continuity and density.

(d) $V(t) \circ \pi = \pi \circ U(t)$ ($t \geq 0$) and then one easily obtains that

$$\pi(D(L)) \subseteq D(H)$$

and $H \circ \pi = \pi \circ L$ on $D(L)$, where $H$ is the infinitesimal generator of $(V(t))_{t \geq 0}$.

(e) $\sigma(H) \subseteq \sigma(L)$: By hypothesis, $(\hat{U}(t))_{t \geq 0}$ is of exponential type $\delta > 0$ whence, as is well known, for $y \in Y$ and $\lambda \in \mathbb{C}$, $\Re \lambda > \delta$,

$$R(\lambda, L)y := -(\lambda - L)^{-1}y = -\int_0^\infty e^{-\lambda t}U(t)y \, dt.$$

Similarly, since $\|V(t)\|_{Z \to Z} \leq \tilde{\beta}(t)$ for all $t \geq 0$, the semigroup $(V(t))_{t \geq 0}$ is of exponential type 0, and therefore we have for $z \in Z$ and $\lambda \in \mathbb{C}$, $\Re \lambda > 0$,

$$R(\lambda, H)z := -(\lambda - H)^{-1}z = -\int_0^\infty e^{-\lambda t}V(t)z \, dt.$$

On the other hand, $R$ commutes with $U(t)$, $t \geq 0$, by assumption and so $R$ commutes with $R(\lambda, L)$ for $\Re \lambda > \delta$. Then

$$q(R(\lambda, L)y) \leq \|R(\lambda, L)\|q(y)$$

for all $y \in Y$, which implies that $N$ is $R(\lambda, L)$-invariant. Hence one can define the bounded operator $\hat{R}(\lambda, L)$ on $Z$ given by $\hat{R}(\lambda, L)(\pi(y)) := \pi(R(\lambda, L)y)$, $y \in Y$. Thus,

$$\hat{R}(\lambda, L)\pi(y) = \pi(R(\lambda, L)y) = -\int_0^\infty e^{-\lambda t}U(t)y \, dt$$

$$= -\int_0^\infty e^{-\lambda t}V(t)\pi(y) \, dt = R(\lambda, H)\pi(y)$$

where (d) has been applied in the last but one equality. Hence $\hat{R}(\lambda, L) = R(\lambda, H)$, for $\Re \lambda > \delta$.

Now, for $\Re \lambda > \delta$ and any $\mu \in \rho(L)$, by using the resolvent identity

$$R(\lambda, L) - R(\mu, L) = (\lambda - \mu)R(\lambda, L)R(\mu, L)$$
on $Y$ and its corresponding identity for $\hat{R}(\lambda, L)$ and $\hat{R}(\mu, L)$ on $Z$, one readily finds that there exists $R(\mu, H)$ with

$$ R(\mu, H) = \hat{R}(\mu, L), $$

see more details in [11] p. 234. Thus $\mu \in \rho(H)$. Hence $\rho(L) \subseteq \rho(H)$ as we claimed.

(f) $\sigma_P(H^*) \subseteq \sigma_P(L^*)$. This is straightforward to see, using restrictions of functionals; see [11] p. 234.

Suppose, if possible, that $Z \neq \{0\}$. By (e) above, we have that $\sigma(H) \cap i\mathbb{R}$ is countable and then $i\mathbb{R} \setminus \sigma(H) \neq \emptyset$. So, by (c) above and [11] Lemma 2, the $C_0$-semigroup $(V(t))_{t \geq 0}$ can be extended to a $C_0$-group $(V(t))_{t \in \mathbb{R}}$ such that $\|V(-t)\|_{Z \rightarrow Z} \leq 1 (t > 0)$ and $\|V(t)\|_{Z \rightarrow Z} = O(t^k)$, as $t \to +\infty$. Also, $\sigma(H)$ is nonempty by (b) above and [11] Lemma 5.

Then reasoning as in [11] Theorem 7 one gets $\sigma_P(H^*) \cap i\mathbb{R} \neq \emptyset$ whence $\sigma_P(L^*) \cap i\mathbb{R} \neq \emptyset$ by (f) above. This is a contradiction and so we have proved that $Z = \{0\}$. By (a) above, we get the statement. $\Box$

The following theorem is the quoted extension of [5, Theorem 5.6].

**Theorem 2.2.** Let $\omega$ be a nonquasianalytic weight such that $\tilde{\omega}$ is of polynomial growth at infinity. Let $(X, \| \cdot \|)$ be a Banach space and $(T_n(t))_{t \geq 0}$ be a $n$-times integrated semigroup in $\mathcal{B}(X)$ with generator $(A, D(A))$ such that $\|T_n(t)\| \leq \omega(t), \ t \geq 0$. Let assume that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.

For every $\mu > 0$ we have:

(i) If $\omega(t)^{-1} = o(t^{-(n-1)})$, as $t \to \infty$, then

$$ \lim_{t \to \infty} \omega(t)^{-1}T_n(t)A^n(\mu - A)^{-2n-1}x = 0, \ x \in X. $$

(ii) If $\omega(t) \sim t^{n-1}$, as $t \to \infty$, then

$$ \lim_{t \to \infty} \frac{1}{t^{n-1}}T_n(t)A^n(\mu - A)^{-2n-1}x = -\frac{A^{n-1}(\mu - A)^{-n}x}{(n-1)!}, \ x \in X. $$

**Proof.** Take $\mu > \delta > 0$. For $x \in X$ define

$$ \|x\|_Y := \sup_{t \geq 0} \|e^{-\delta t}(T_n(t)A^n(\mu - A)^{-n}x + \sum_{j=0}^{n-1} \frac{t^j}{j!}A^j(\mu - A)^{-n}x)\|. $$

Note that $A(\mu - A)^{-1} = -I + \mu(\mu - A)^{-1}$ is a bounded operator on $X$ and $T_n(0) = 0$, so $\| \cdot \|_Y$ is a norm on $X$ and there exists a constant $M_\delta > 0$ such that

(2) $$ \|(\mu - A)^{-n}x\| \leq \|x\|_Y \leq M_\delta \|x\|, \ x \in X. $$

Let $Y$ be the Banach space obtained as the completion of $X$ in the norm $\| \cdot \|_Y$. By the Extrapolation Theorem [3, Theorem 0.2], there exists a closed operator $B$ on $Y$ which generates a $C_0$-semigroup $(S(t))_{t \geq 0} \subset \mathcal{B}(Y)$ of positive exponential type such that $D(B^n) \hookrightarrow X \hookrightarrow Y$, $A = B_X$ where the operator $B_X$ is given by the conditions
\[ D(B_X) := \{ x \in D(B) \cap X : Bx \in X \}, \quad B_X(x) := B(x) \ (x \in X). \]

Moreover, \( \sigma_p(B^*) \subseteq \sigma_p(A^*) \), and also \( \rho(A) = \rho(B) \) with

\[ (\lambda - A)^{-1}x = (\lambda - B)^{-1}x, \quad \lambda \in \rho(A) = \rho(B), x \in X; \]

see [3, Remark 3.1].

Let \((S_n(t))_{t \geq 0}\) be the \(n\)-times integrated semigroup generated by \(B\) on \(Y\), given by

\[ S_n(t)y := \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} S(s)y \, ds, \quad y \in Y. \]

Then \(S_n(t)x = T_n(t)x\) for all \(x \in X\) and \(t \geq 0\). To see this, note that \((T_n(t))_{t \geq 0}\) and \((S_n(t))_{t \geq 0}\) are of exponential type so one can rewrite (3) above in terms of the Laplace transforms of \((T_n(t))_{t \geq 0}\) and \((S_n(t))_{t \geq 0}\) respectively, for \(\Re \lambda\) large enough. Then it suffices to apply the uniqueness of the Laplace transform.

From the above identification between \(T_n(t)\) and \(S_n(t)\), it readily follows that

\[ \|S_n(u)x\|_Y \leq \|T_n(u)\| \|x\|_Y \leq \omega(u)\|x\|_Y, \quad u \geq 0, x \in X, \]

which is to say, by density, that \(\|S_n(u)\| \leq \omega(u)\), for all \(u \geq 0\).

Now, by reiteration of the well known equality

\[ S(t)y - y = \int_0^t BS(s)y \, ds \quad (t \geq 0, \ y \in D(B)), \]

we have

\[ S(t)y = S_n(t)B^ny + \sum_{j=0}^{n-1} \frac{t^j}{j!} B^jy, \quad y \in D(B^n). \]

Hence, for every \(y \in Y\),

\[ S(t)(\mu - B)^{-n}y = S_n(t) \left( \frac{B}{\mu - B} \right)^n y + \sum_{j=0}^{n-1} \frac{t^j}{j!} \left( \frac{B}{\mu - B} \right)^j (\mu - B)^{-(n-j)} y \]

and therefore there exists a constant \(C_\mu > 0\) such that

\[ \|S(t)(\mu - B)^{-n}\|_{Y \to Y} \leq C_\mu \omega(t), \quad t \geq 0. \]

Then, by applying Theorem 1.2 with \(U(t) = S(t), \ B = L \) and \(R = (\mu - A)^{-n}\), we obtain

\[ \lim_{t \to \infty} \frac{1}{\omega(t)} \|S(t)(\mu - B)^{-n}y\|_Y = 0, \quad y \in Y, \]
whence, by (2), (3) and (4),
\[
0 = \lim_{t \to \infty} \frac{1}{\omega(t)} \|T_n(t)A^n(\mu - A)^{-n}x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j(\mu - A)^{-n}x\|_Y
\geq \limsup_{t \to \infty} \frac{1}{\omega(t)} \|T_n(t)A^n(\mu - A)^{-2n}x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j(\mu - A)^{-2n}x\|_X,
\]
for every \(x \in X\).

Thus we get
\[
\lim_{t \to \infty} \frac{1}{\omega(t)} T_n(t)A^n(\mu - A)^{-2n}x = -\lim_{t \to \infty} \frac{1}{\omega(t)} \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j(\mu - A)^{-2n}x
\]
in \(X\), and the statement follows readily. \(\square\)

**Proof of Theorem 2.2.** In the setting of Theorem 2.2, let assume in addition that \(0 \in \rho(A)\). Since the resolvent function of \(A\) is holomorphic-so continuous- in the open subset \(\rho(A) \subseteq \mathbb{C}\) we have that
\[
\lim_{\mu \to 0^+} A^n(\mu - A)^{-n} = \lim_{\mu \to 0^+} (-I + \mu(\mu - A)^{-1})^n = (-1)^n I.
\]

Now, to prove (i) and (ii) of the theorem it suffices to notice that
\[
\sup_{t > 0} \omega(t)^{-1} \|T_n(t)\| < \infty \text{ in both cases.}
\]

\(\square\)

### 3. Final Comments and Remarks

It looks desirable to find out the behavior of a \(n\)-times integrated semigroup at infinity when its generator \(A\) is not assumed to be invertible. According to the remark prior to Theorem 1.1 the existence of \(\lim_{n \to \infty} T_n(x)\) (for \(n = 1\)) entails invertibility of \(A\). Thus the type of convergence at infinity of \(T_n(t)\), if there is some, that one can expect if \(A\) is not invertible must be weaker than the existence of limit.

In [7], under the assumptions
\[
\sup_{t > 0} t^{-n} \|T_n(t)\| < \infty \text{ and } \lim_{t \to 0^+} n! t^{-n} T_n(t)x = x \quad (x \in X),
\]

it has been proved that
\[
\lim_{t \to \infty} t^{-n} T_n(t)\pi_n(f) = 0, \quad f \in \mathcal{S}_n,
\]
in the operator norm, where \(\mathcal{S}_n\) is the subspace of functions of \(\mathcal{T}_+(n)(t^n)\) which are of spectral synthesis in \(\mathcal{T}(n)(|t|^n)\) with respect to the subset \(\sigma(A) \cap \mathbb{R}\), and \(\pi_n(f) = (-1)^n \int_0^\infty f^{(n)}(t)T_n(t) \, dt\). Here, \(\mathcal{T}(n)(|t|^n)\) is the convolution Banach algebra obtained as the completion of the Schwarz class in the norm \(f \mapsto \int_0^\infty |f^{(n)}(t)||t|^n \, dt\), and \(\mathcal{T}_+(n)(t^n)\) is the restriction of \(\mathcal{T}(n)(|t|^n)\) on \((0, \infty)\). This result is an extension of the Esterle-Strouse-Vu-Zouakia theorem, which corresponds to the case \(n = 0\); see [6] and [10]. In [6] it is shown that, under the assumptions that \(\sigma(A) \cap i\mathbb{R}\) is countable and \(\sigma_P(A^*) \cap i\mathbb{R} = \emptyset\), the subspace \(\pi_0(\mathcal{S}_0)X\)
is dense in $X$ so one gets another way –for it is different from the original one– to establish the Arendt-Batty-Lyubich-Vũ theorem. The proof of that density is attained by methods of harmonic analysis.

We wonder if in the case when $A$ is not invertible the argument considered in [6] to deduce the Arendt-Batty-Lyubich-Vũ theorem works for $n$-times integrated semigroups; that is, if $\pi_n(\mathcal{S}_n)X$ is dense in $X$ (under the conditions $\sigma(A) \cap i\mathbb{R}$ countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$). This would give us the ergodic type property
\[
\lim_{t \to \infty} t^{-n}T_n(t)x = 0, \quad x \in X.
\]
Notice that (5) is a consequence of Theorem 1.2 (i) when $A$ is invertible; on the other hand, the ergodicity of a $n$-times integrated semigroup $(T_n(t))_{t \geq 0}$ such that $\sup_{t \geq 1} t^{-n}\|T_n(t)\| < \infty$ is characterized in [5] in terms of Abel-ergodicity or/and ergodic decompositions of the Banach space $X$. Such an approach will be considered in a forthcoming paper.

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