From Luttinger liquid to non-Abelian quantum Hall states

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We formulate a theory of non-Abelian fractional quantum Hall states by considering an anisotropic system consisting of coupled, interacting one dimensional wires. We show that Abelian anyons are not sufficient for universal quantum computation. The quantum Hall effect is a promising venue for non Abelian states. There is growing evidence that the pfaffian state introduced by Moore and Read describes the quantum Hall plateau observed at filling \( \nu = 5/2 \). The Moore Read state gives the simplest non-Abelian state, with quasiparticles that exhibit Ising non-Abelian statistics. While the observation and manipulation of Ising anyons is an important goal, Ising anyons are not sufficient for universal quantum computation. The \( Z_3 \) parafermion state introduced by Read and Rezayi is a candidate for the quantum Hall plateau at \( \nu = 12/5 \). The quasiparticles of the Read Rezayi state are related to Fibonacci anyons, which have a more intricate structure that in principle allows universal quantum computation.

There is currently great interest in realizing quantum Hall physics in materials without an external magnetic field or Landau levels. This possibility was inspired by Haldane’s realization that a zero field integer quantum Hall effect can occur in graphene, provided time reversal symmetry is broken. Though such an anomalous quantum Hall effect has not yet been observed, related physics occurs in topological insulators, which have been predicted and observed in both two and three dimensional systems. Recently, there have been suggestions for generalizations of this idea to zero field fractional quantum Hall states, as well as fractional topological insulators. The question naturally arises whether it is possible to engineer zero field non-Abelian quantum Hall states, which exhibit the full Ising, or even Fibonacci anyons.

A difficulty with answering this question is a lack of methods for dealing with strongly interacting systems. Moore and Read, and later Read and Rezayi, built on Laughlin’s idea and constructed trial many body wave functions as correlators of non trivial conformal field theories. This allowed most properties of the quasiparticle and edge states to be deduced, and it had the virtue of allowing the construction of interacting electron Hamiltonians with the desired ground state. However, since this approach relied on the structure of the lowest Landau level, it is not clear how it can be applied to a lattice system. Effective topological quantum field theories and parton constructions provide an elegant framework for classifying quantum Hall states and provide a description of their low energy properties. However, since the original electronic degrees of freedom are replaced by more abstract variables, these theories provide little guidance for what kind of electronic Hamiltonian can lead to a given state.

In this paper we introduce a new method for describing non Abelian quantum Hall states by considering an anisotropic system consisting of an array of coupled one dimensional wires. Study of the anisotropic limit of quantum Hall states dates back to Thouless et al., who used this limit to evaluate the Chern invariant in the integer quantum Hall effect. The Chalker Coddington model for the integer quantum Hall effect also has a simple anisotropic limit, which is closely related to the coupled wire model. A coupled wire construction for Abelian fractional quantum Hall states was introduced in Ref. Here we build on that work and show how the coupled wire construction can be adapted to describe the Moore Read pfaffian state, as well as the more general Read Rezayi sequence of quantum Hall states.

The coupled wire construction has a number of desirable features. First, it allows for the definition of a simple Hamiltonian, expressed in terms of electronic degrees of freedom, that can be transformed, via Abelian bosonization, into a form that for certain special param-
eters can be solved essentially exactly. The quasiparticle spectrum, as well as the edge state structure follow in a straightforward manner. The coupled wire construction thus provides a direct link between a microscopic electron Hamiltonian and the low energy conformal field theory description of the edge states. As such, it provides an intermediate between the wave function approach to quantum Hall states and the effective field theory approach.

The coupled wire construction also provides a simple picture for the quantum entanglement present in quantum Hall states\cite{34,35}. When the electron Hamiltonian is transformed via Abelian bosonization, it becomes identical to a theory of strips of quantum Hall fluid coupled via electron tunneling between their edge states. It thus provides a concrete setting for the more abstract coupled edge state models considered by Gils, et al\cite{36}. The coupled wire model is also similar in spirit to the AKLT model of quantum spin chains\cite{37}, which provides a similarly intuitive and solvable model for understanding fractionalization in one dimension. For the non Abelian quantum Hall states, the coupled wire construction provides a concrete interpretation for the coset construction, which is a powerful (albeit abstract) mathematical tool for describing non Abelian quantum Hall states\cite{38}.

A final virtue of the coupled wire construction is that it can be applied to zero field anisotropic lattice models. The effect of the magnetic field in the coupled wire model is to modify the momentum conservation relations when electrons tunnel between wires. A similar effect could arise due to scattering from a periodic potential. The coupled wire construction may thus provide some guidance for the construction of lattice models for the fractional and non Abelian quantum Hall states. Note that our construction is somewhat different from the proposals for Chern insulators Refs. \cite{17,18,19} because we do not require nearly flat bands with a non zero Chern number.

The outline of the paper is as follows. We will begin in section II with an extended introduction to the coupled wire construction for Abelian quantum Hall states. Much of this material was contained either explicitly or implicitly in Ref. \cite{36}. Here, since we are free from the constraints of a short paper, we will fill in some details that were absent in Ref. \cite{36}. In particular, we will explain the generalization of the coupled wire construction to describe systems of bosons, we will demonstrate the Abelian fractional statistics of quasiparticles described in our approach, and we will explicitly construct second level hierarchical fractional quantum Hall states.

Section III is devoted to the Moore Read state. We will begin with a construction of this state for bosons at filling $\nu = 1$. This leads to a bosonized model that can be solved using via fermionization. We will then show that for a special set of parameters the model has a particularly simple form, which can be interpreted in the framework of the coset construction of conformal field theory. We will conclude section III by showing how to construct the more general Moore Read state at filling $\nu = 1/(1 + q)$ for $q$ even (odd) for bosons (fermions).

In section IV we will generalize our construction to describe the Read Rezayi sequence at level $k$. Again, the formulation is simplest for bosons at filling $\nu = k/2$, where the coupled wire model is closely related to the coset construction of these states. We show that the coupled wire model leads maps to a bosonized representation of the critical point of a $Z_k$ statistical mechanics model, which for $k = 3$ reduces to the 3 state Potts model. This bosonized representation allows us to identify the $Z_k$ parafermion primary fields and fully characterize the edge states of the Read Rezayi states. Finally, as in section II, we conclude by generalizing our results to describe bosonic (fermionic) level $k$ Read Rezayi states at filling $k/(2 + qk)$ for $q$ even (odd).

Some of the technical details are presented in the appendices. Appendix A gives a careful treatment of Klein paradox, while Appendices B and C contain some of the conformal field theory calculations discussed in section IV.

II. ABELIAN QUANTUM HALL STATES

A. Coupled Wire Construction for Fermions

In this section we review the coupled wire construction for fermions introduced in Ref. \cite{36}. We begin by considering an array of identical uncoupled spinless non interacting one dimensional wires, as shown in Fig. 1a, with a single particle electronic dispersion $E(k)$. We assume each wire is filled to the same density, characterized by Fermi momentum $k_F$. The two dimensional electron density is then $n_e = k_F/\pi a$, where $a$ is the separation between wires. A perpendicular magnetic field, represented in the Landau gauge $A = -Byx$, shifts the momentum of each wire. The right and left moving Fermi momenta of the $j$'th wire are then

$$k_{Fj}^{L/R} = \pm k_F^0 \pm bj,$$

(2.1)

where $b = |e|aB/h$. The filling factor $hn_e/eB$ is then

$$\nu = 2k_F/b.$$  

(2.2)

The low energy Hamiltonian, linearized about the Fermi momenta is

$$\mathcal{H}_0 = \sum_j \int dx \psi_j^0 \left( \psi_j^{R/L}(-i\partial_x - k_{Fj}^{R/L})\psi_j^{R/L} - \psi_j^{R/L\dagger}(-i\partial_x - k_{Fj}^{R/L})\psi_j^{R/L\dagger} \right),$$

(2.3)

where $\psi_j^{R/L}$ describe the fermion modes of the $j$'th wire in the vicinity of the Fermi points $k_{Fj}^{R/L}$.

We next bosonize by introducing bosonic fields $\varphi_j(x)$ and $\theta_j(x)$ that satisfy

$$[\partial_x \theta_j(x), \varphi_{j'}(x')] = i\pi \delta_{jj'}\delta_{xx'},$$

(2.4)
where we use the shorthand notation $\delta_{x,x'} = \delta(x - x')$. $\varphi_j(x)$ is a bosonic phase field, while $\theta(x)$ describes density fluctuations. The long wavelength density fluctuations on the $j$’th wire are

$$\rho_j(x) = \sum_a \psi_{j,a}^\dagger (x) \psi_j^a (x) = \partial_x \theta_j(x)/\pi. \quad (2.5)$$

The electron creation and annihilation operators may be written

$$\psi_j^R (x) = \frac{\kappa_j}{\sqrt{2\pi x_c}} e^{i(k_{F,j} x + \varphi_j(x) + \theta_j(x))},$$

$$\psi_j^L (x) = \frac{\kappa_j}{\sqrt{2\pi x_c}} e^{i(k_{F,j} x + \varphi_j(x) - \theta_j(x))}.$$

where $x_c$ is a regularization dependent short distance cut-off and $\kappa_j$ is a Klein factor that assures the anticommutation of the fermion operators on different wires. Eq. 2.4 hides the zero momentum parts of $\theta_j$ and $\phi_j$, which must be accounted for in order to correctly treat the Klein factors. Since this issue tends to obscure the simplicity of our construction, we will not dwell on it in the text of the paper. Appendix A contains a careful discussion of the zero modes and Klein factors, which shows when they can be safely ignored.

In terms of the density and phase variables, the Hamiltonian for non interacting electrons is

$$H_{\text{non int.}}^0 = \frac{v_F}{2\pi} \sum_j \int dx \left[(\partial_x \varphi_j)^2 + (\partial_x \theta_j)^2\right]. \quad (2.7)$$

Interactions between electrons as well as electron tunneling between the wires can be added. In general, there are two classes of terms: forward scattering and inter channel scattering. The forward scattering terms conserve the number of electrons in each channel and can be expressed as the interactions between densities and currents. This leads to a Hamiltonian that is quadratic in the boson variables,

$$H_{\text{SLL}}^0[\theta, \varphi] = \sum_{jk} \int dx \left( \frac{\partial_x \varphi_j}{\partial_x \theta_j} \right) \mathbf{M}_{jk} \left( \frac{\partial_x \varphi_k}{\partial_x \theta_k} \right). \quad (2.8)$$

Here the $2 \times 2$ matrix $\mathbf{M}_{jk} = \delta_{jk} \mathbf{I}v_F/2\pi + \mathbf{U}_{jk}$, where $\mathbf{U}_{jk}$ describes the forward scattering interactions. $H_{\text{SLL}}^0$ describes a gapless anisotropic conductor in a sliding Luttinger liquid phase.

Symmetry allowed inter channel scattering terms must be added to $H_{\text{SLL}}^0$. They can open a gap and lead to interesting phases. The allowed terms are built from products of single electron operators, and have the form

$$O_j^{\{m_p,n_p\}} (x) = \prod_p \psi_{j+p}^R (x) \psi_{j+p}^L (x)^\dagger.$$

where $s_p^{R/L}$ are integers such that $\psi_{j+p}^{R/L} (x) \psi_{j+p}^{R/L\dagger}$ appears $|s_p^{R/L}|$ times for $s_p^{R/L} > 0$ ($< 0$). It is convenient to write $s_p^{R/L}$ in terms of a new set of integers,

$$s_p^R = (n_p + m_p)/2 \quad (2.10)$$

$$s_p^L = (n_p - m_p)/2. \quad (2.11)$$

Then $O_j^{\{m_p,n_p\}} (x)$ takes the form

$$O_j^{\{m_p,n_p\}} (x) = c\kappa_j R \exp \left( \sum_p n_p \varphi_{j+p} + m_p \theta_{j+p} \right). \quad (2.12)$$

where $c$ is a non universal constant. The product of Klein factors is

$$\kappa_j^{\{m_p,n_p\}} = \prod_p \kappa_p^{n_p} \kappa_{j+p}^{m_p}. \quad (2.13)$$

The oscillatory factor describing the net momentum of $O_j^{\{m_p,n_p\}}$ is

$$R^{\{m_p,n_p\}} (x) = \exp \left( \sum_p b_{m_p} n_p + k_{F,p} m_p x \right). \quad (2.14)$$

The operators $O_j^{\{m_p,n_p\}}$ define a term in the Hamiltonian,

$$V^{\{m_p,n_p\}} = \sum_j \int dx \left( \psi_{j+p}^{\{m_p,n_p\}} O_j^{\{m_p,n_p\}} (x) + h.c. \right). \quad (2.15)$$

There are physical constraints on the allowed $\{m_p,n_p\}$. Since $s_p^{R/L}$ must be integers, we require

$$m_p = n_p \mod 2. \quad (2.16)$$

Charge conservation requires that

$$\sum_p n_p = 0. \quad (2.17)$$
Momentum conservation implies
\[ \sum_p (bp_n + k_F m_p) = 0, \tag{2.18} \]
so that the oscillatory term in \(\nu_{\{m_p, n_p\}}\) vanishes.

The Hamiltonian
\[ \mathcal{H} = \mathcal{H}_0^{\text{SLL}} + \sum_{\{m_p, n_p\}} V_{\{m_p, n_p\}} \tag{2.19} \]
can be studied perturbatively using the standard renormalization group analysis. The lowest order RG flow equation for \(\nu_{\{m_p, n_p\}}\) is
\[ dv_{\{m_p, n_p\}}/dt = (2 - \Delta_{\{m_p, n_p\}})\nu_{\{m_p, n_p\}}, \tag{2.20} \]
The scaling dimension \(\Delta_{\{m_p, n_p\}}\) depends on the forward scattering interactions \(M_{jk}\) in \(\mathcal{H}_0^{\text{SLL}}\). When \(\Delta_{\{m_p, n_p\}} > 2\) the operator \(O_{\{m_p, n_p\}}\) is irrelevant and \(\nu_{\{m_p, n_p\}}\) does not destabilize the gapless sliding Luttinger liquid fixed point. When \(\Delta_{\{m_p, n_p\}} < 2\), \(O_{\{m_p, n_p\}}\) is relevant and \(\nu_{\{m_p, n_p\}}\) grows at low energy, destabilizing the sliding Luttinger liquid. In principle \(M_{jk}\) can be parameterized given an underlying model of the electron-electron interactions. However, \(M_{jk}\) may be renormalized by irrelevant and/or momentum non conserving operators, so it may not resemble the bare interactions. Here we follow the approach of Ref. 36 and assume that \(M_{jk}\) has values such that a particular operator (or set of operators) \(O_{\{m_p, n_p\}}\) is relevant. Our object is to characterize the resulting non trivial strong coupling phases. There are special values of \(M_{jk}\) that lead to particularly simple boson Hamiltonians that can be solved exactly. These solvable points provide a powerful way to characterize the resulting strong coupling phases.

As shown in Ref. 36 a number of non trivial 2D phases can be analyzed using this approach, including Abelian fractional quantum Hall states, superconductors and crystals of electrons, quasiparticles or vortices. In particular, Abelian quantum Hall states are described by a single relevant operator \(\{m_p, n_p\}\) satisfying \(\sum_p m_p \neq 0\). From (2.2) and (2.18) this corresponds to a filling factor
\[ \nu = 2 - \frac{\sum_p m_p}{\sum_p m_p}, \tag{2.21} \]
In Section II.C we will review this construction for the Laughlin states and the Abelian hierarchy states. But first, we will show that the coupled wire construction can also be straightforwardly applied to systems of bosons.

### B. Coupled Wire Construction for Bosons

We now consider coupled wires of one dimensional bosons. The low energy excitations of a single wire can be described by “bosonizing the bosons”, to express them in terms of a slowly varying phase \(\varphi(x)\) and a conjugate density variable \(\theta(x)\) satisfying \(\varphi(0) = 0\). The density fluctuations have important contributions near wavevectors \(q_n = 2\pi n \bar{\rho}\) that are multiples of the average 1D density \(\bar{\rho}\).
\[ \rho(x) = \bar{\rho} + \sum_n \rho_n(x) \tag{2.22} \]
As with the fermions, the long wavelength density fluctuation is \(\rho_0(x) = \partial_x \theta(x)/\pi\). The density wave at \(q \sim 2\pi \bar{\rho}n\) is
\[ \rho_n(x) \propto e^{i(2k_F x + 2\theta(x))}. \tag{2.23} \]
Here and in the following we will denote the 1D density \(\bar{\rho}\) in terms of \(\nu = 2k_F^2 \equiv 2\pi \bar{\rho}\). This allows us to proceed analogously with the fermions and use formulas (2.2) and (2.18) for the filling factor.

The Hamiltonian for bosonic wires coupled only by long wavelength interactions has exactly the same form as \(\mathcal{H}_0^{\text{SLL}}\). The only difference is that the non interacting Pauli compressibility term \(\mathcal{H}^{\text{non int.}}_0\) is absent. Tunneling a boson between wire \(j\) and \(j + p\) in the presence of a magnetic field is described by the operator
\[ \Phi_j^{\dagger}(x)\Phi_j(x)e^{i px}, \tag{2.24} \]
where \(\Phi_j^{\dagger}(x) \propto \exp i \varphi_j(x)\) is the boson creation operator. Due to interactions this process can involve scattering from \(2k_F n\) density fluctuations of the bosons. The most general coupling term thus has the form
\[ O_j^{\{m_p, n_p\}} = cR^{\{m_p, n_p\}} \exp i \left( \sum_p n_p \varphi_{j+p} + m_p \theta_{j+p} \right), \tag{2.25} \]
where \(R\) is given in (2.14). This is almost identical to the inter channel scattering terms for fermions. The only differences are the absence of Klein factors and the constraints on the allowed values of \(\{m_p, n_p\}\). Charge and momentum conservation still requires (2.17, 2.18), but unlike for fermions, where \(m_p\) and \(n_p\) obey (2.16), the corresponding constraint for bosons is
\[ m_p = 0 \mod 2. \tag{2.26} \]
The analysis of bosonic states then follows in exactly the same manner as fermionic states, as described in Eqs. (2.19)-(2.21).

### C. Laughlin States \(\nu = 1/m\)

Here we will examine the coupled wire construction for the Laughlin states in some detail. We include the details here because the Laughlin states provide the simplest non trivial application of the coupled wire construction. We begin by introducing the relevant interaction term, and then characterize the bulk quasiparticles and edge states.
The decoupling can be explicitly seen from the commutation algebra,
\[ \left[ \partial_x \phi_j^R(x), \phi_j^L(x') \right] = 2\pi i m \hbar \delta_{pp'} \delta_{jj'} \delta_{xx'}. \] (2.29)

The interaction term is now \( O_{j+1/2} = \exp i (\phi_{j+1/2}^R - \phi_{j+1/2}^L) / (2\pi m) \).

The charge density is \( \rho_j = (\partial_x \phi_j^R - \partial_x \phi_j^L) / (2\pi m) \).

It is also convenient to introduce new density and phase variables defined on the links \( \ell = j + 1/2 \) between wires,
\[ \tilde{\theta}_\ell = (\phi_{j+1}^R - \phi_{j+1}^L) / 2, \]
\[ \tilde{\varphi}_\ell = (\phi_{j+1}^R + \phi_{j+1}^L) / 2. \] (2.30)

These satisfy \( [\partial_x \tilde{\theta}_\ell(x), \tilde{\varphi}_\ell(x')] = [\partial_x \tilde{\varphi}_\ell(x), \tilde{\varphi}_\ell(x')] = 0 \) and
\[ [\partial_x \tilde{\theta}_\ell(x), \tilde{\varphi}_\ell(x')] = i\pi m \delta_{\ell \ell'} \delta_{xx'}. \] (2.31)

The charge density associated with the link \( \ell \) can be written
\[ \tilde{\rho}_\ell = \partial_x \tilde{\theta}_\ell / (m\pi). \] (2.32)

In terms of the new variables, the Hamiltonian becomes,
\[ H = \tilde{H}_\text{SLL}_0[\tilde{\theta}, \tilde{\varphi}] + \sum_{\ell} \int dx \cos 2\tilde{\theta}_\ell, \] (2.33)

where without loss of generality we have assumed \( v \) is real. As shown in Appendix A, there is no Klein factor, provided the zero momentum component of \( \tilde{\theta} \) is correctly defined.

Provided the forward scattering interactions defining \( \tilde{H}_\text{SLL}^0 \) are such that \( v \) is relevant, the system will flow at low energy to a gapped phase in which \( \tilde{\theta} \) is localized in a well of the cosine potential. As argued in Ref. 36, it is always possible to find such interactions. In particular, consider a simple interaction such that \( \tilde{H}_\text{SLL}^0 \) has the decoupled form
\[ \tilde{H}_\text{SLL}^0 = \frac{v_0}{2\pi} \sum_{\ell} \int dx \left( \frac{1}{g} (\partial_x \tilde{\theta}^2 + g (\partial_x \tilde{\varphi})^2) \right). \] (2.34)

The scaling dimension of \( \cos 2\tilde{\theta} \) is \( \Delta = mg \). It follows that for \( g < 2/m \), \( v \) is relevant. It should be emphasized that \( \tilde{H}_\text{SLL}^0 \) can be expressed in terms of the original fermion operators, which includes a specific four fermion forward scattering interaction. For special values of \( g \) this model can be solved exactly. In the limit \( g \to 0 \), the variable \( \tilde{\varphi} \) becomes a stiff classical variable, so that the approximation of replacing \( -\cos 2\tilde{\theta} \) by \( 2\tilde{\theta}^2 \) becomes exact. For larger \( g \), we rely on our understanding that \( g \) is renormalized downward by \( v \), so that at \( \tilde{\theta} \) stiffens at low energy. For \( g = 1/m \) there is another exact solution because it is possible to define new variables such that the Hamiltonian has precisely the form of (2.7). The problem can then be reformalized and expressed in terms of non interacting fermions which have a single particle energy gap. We will not dwell on these exact solutions any further in this paper. We will be content with our understanding that any \( g < 2/m \) leads to a gapped state.

The gapped phase is the Laughlin state. This can be seen by examining the quasiparticle excitations and the edge states.
2. Bulk Quasiparticles

Quasiparticles occur when $\tilde{\theta}_j(x)$ has a kink where it jumps by $\pi$. From (2.22) it can be seen that such a kink is associated with a charge $e/m$. This makes the charge fractionalization in the fractional quantum Hall effect appear similar to the fractionalization that occurs in the one dimensional Su Schrieffer Heeger (SSH) model. However, there is a fundamental difference. The solitons in the SSH model occur at domain walls separating physically distinct states. This prevents solitons from hopping between wires via a local operator. In contrast, the states characterized by $\theta_\ell$ and $\theta_\ell + \pi$ are physically equivalent. They are related by a gauge transformation in which, say, $\varphi_j \rightarrow \varphi_j + 2\pi$, which takes $\tilde{\theta}_{j+1/2}$ to $\tilde{\theta}_{j+1/2} \mp \pi$. This allows quasiparticles to hop via a local operator without the nonlocal string. Though SSH solitons and Laughlin quasiparticles are distinct, they become equivalent on a cylinder with finite circumference. The Tao Thouless “thin torus” limit can be described the extreme case in which the “cylinder” consists of a single wire with electron tunneling “around” the cylinder. In this case, our theory maps precisely to an $m$ state version of the SSH model.

The local operator that hops quasiparticles between links $j = 1/2$ and $j - 1/2$ is simply the backscattering of a bare electron on wire $j$, $\psi_j^L \psi_j^R$ (or equivalently for bosons the $2k_F$ density operator). Using the transformations (2.28) and (2.30) it is straightforward to show that

$$\chi_j(x') = e^{2i\hat{\theta}_j(x')/m} e^{i\tilde{\theta}_j(x')/m} \rho_{j}(x, x') = e^{i(\tilde{\theta}_j(x)-\tilde{\theta}_j(x'))/m} e^{i\tilde{\theta}_j(x')/m}$$  (2.35)

From (2.31) it can be seen that this operator takes $\partial_x \tilde{\theta}_{j+1/2}$ to $\partial_x \tilde{\theta}_{j+1/2} \mp \pi \sigma(x-x')$, transferring a quasiparticle from $j + 1/2$ to $j - 1/2$. The operator that transfers a quasiparticle along wire from $x_j$ to $x_{j+1}$ along wire link $\ell$ is

$$\rho_\ell(x, x') = e^{i(\hat{\theta}_\ell(x)-\hat{\theta}_\ell(x'))/m} e^{i\int_x^{x'} dx' \partial_x \tilde{\theta}_\ell/m}$$  (2.36)

which can also be expressed in terms of the bare electron densities and currents.

A quasiparticle operator may be defined as

$$\Psi^{R/L}_{QP, \ell} (x) = e^{i\hat{\theta}_\ell/\ell + 1/m} e^{i\tilde{\theta}_\ell(x')/m}$$  (2.37)

In the bulk, since $\hat{\theta}$ is gapped, we have $\Psi^{R/L}_{QP, \ell} = \Psi^{\ell}_{QP} e^{2i\hat{\theta}_\ell}$. Of course, since $\Psi^{R/L}_{QP, \ell}$ can not be locally built out of bare electron operators, it is not by itself a physical operator. However, the operator that transfers a quasiparticle from one location to another can be built from a string of local operators like (2.35) and (2.36). This allows the fractional statistics of the quasiparticles to be seen quite simply.

To move a quasiparticle from $x_1$ to $x_2$ on link $\ell$ and then to $x_2$ on $\ell_2$, use the operator

$$\rho_{\ell_2-1/2} \prod_{j=\ell_1+1/2}^{\ell_2-1/2} \chi_j(x_j)$$  (2.38)

Since $\hat{\theta}$ is gapped, this can be written

$$\Psi^{R}_{QP, \ell} (x_2) \Psi^{L}_{QP, \ell_1} (x_1) \prod_{\ell=\ell_1+1}^{\ell_2-1} e^{2i(\hat{\theta}_\ell(x))} e^{2i\hat{\theta}_\ell(x)/m}$$  (2.39)

The string of $\langle \hat{\theta} \rangle$ is responsible for the fractional statistics.

Consider moving a quasiparticle through a closed loop. The operator that takes a quasiparticle around the rectangle formed by $x_1$, $x_2$, $\ell_1$ and $\ell_2$ can be constructed by doing (2.38) twice, which eliminates the quasiparticle operators. This then gives a phase

$$\prod_{\ell=\ell_1+1}^{\ell_2-1} e^{2i(\hat{\theta}_\ell(x))} e^{2i\hat{\theta}_\ell(x)/m} = e^{2\pi i N_{QP}/m}$$  (2.40)

where $N_{QP}$ is the number of quasiparticles enclosed by the rectangle. Here we have used the fact that $\langle \hat{\theta}_\ell(x_1) - \hat{\theta}_\ell(x_2) \rangle / 2\pi$ simply counts the number of quasiparticles on link $\ell$ between $x_1$ and $x_2$.

3. Edge States

For a finite array of wires with open boundary conditions, the edge states are apparent, since there are extra chiral modes left over on the first and last wire. From (2.29), it can be seen that these modes have precisely the chiral Luttinger liquid structure of $\nu = 1/m$ edge states.

$$\mathcal{H}_{edge} = \mu v_p \left( \partial_x \phi^L_j \right)^2$$  (2.41)

with $\left[ \partial_x \phi^L_j (x), \phi^L_i (x') \right] = 2\pi i \delta_{xx'}$. The electron operator on the $j = 1$ edge is

$$\Psi^L_1 = e^{i\hat{\phi}_1^L}$$  (2.42)

It is straightforward to show that this operator has the expected dimension $\Delta = m/2$, characteristic of the chiral Luttinger liquid.

One can view the change of variables (2.28) as a transformation between a sliding Luttinger liquid built out of bare electrons and a sliding Luttinger liquid built out of $\nu = 1/m$ edge states. The correlated tunneling term for the bare electrons becomes the electron tunneling operator coupling the edge states. The array of wires then becomes an array of strips of $\nu = 1/m$ quantum Hall fluid coupled by electron tunneling, as shown in Fig. 2. When the electron tunneling is relevant the strips merge...
FIG. 3. (a) Schematic of tunneling processes in (2.45) that lead to 2nd level Abelian hierarchy states. (b) After the transformation (2.47) the model describes coupled strips of ν = 2n/(m₀ + m₁) quantum Hall state coupled by tunneling electrons between the two channels of edge states.

The quasiparticle operator at the j = 1 edge is
\[
\Psi_{QP,j}^{L} = e^{i\phi_{j}/m}. \tag{2.43}
\]

As discussed above, since \(\Psi_{QP}\) cannot be made out of bare electron operators, it is not by itself a physical operator. However, quasiparticle tunneling from the top to the bottom edge can be built from a string of backscattering operators (2.35). When the gapped bulk degrees of freedom are integrated out, this string of operators becomes
\[
\prod_{j=1}^{N} \chi_{j} \sim \Psi_{QP,1}^{L} \Psi_{QP,N}^{R}. \tag{2.44}
\]

D. Hierarchy States

In this section we show how the coupled wire construction describes hierarchical Abelian fractional quantum Hall states.\(^{52,53}\) We restrict ourselves to second level states, which are characterized by a 2 × 2 K matrix.\(^{12}\) Generalization to higher levels is straightforward.

2nd level hierarchy states arise from an interaction term that involve three coupled wires. A generic term is shown in Fig. 3 and can be described by the operator
\[
\mathcal{O}_{j} = \exp \left[ n (\varphi_{j+1} - \varphi_{j+1}) + 2m_{0}\theta_{j} + m_{1} (\theta_{j+1} + \theta_{j-1}) \right]. \tag{2.45}
\]

Here n and m₀ are any integers, while m₁ (m₁ + n) is an even integer for bosons (fermions). Again, we defer discussion of the Klein factors to Appendix A. From (2.21) this interaction conserves momentum at a filling factor
\[
\nu = \frac{2n}{m_{0} + m_{1}}. \tag{2.46}
\]

This set of states corresponds to the standard Halperin-Halperin hierarchy states at filling \((p_{0} + 1/p_{1})^{-1}\) \((p_{0}\) is even (odd) for bosons (fermions) and \(p_{1}\) is even) for the choice, \(n = p_{1}/2, m_{0} = p_{0}p_{1}/2\) and \(m_{1} = p_{0}p_{1}/2 + 1\).

To analyze this state, we group the wires into pairs \(j = 2k\) and \(j = 2k + 1\). Pair \(k\) is connected to pair \(k + 1\) by two tunneling terms, \(O_{2k}\) and \(O_{2k+1}\). As in (2.28) we define new variables that decouple right and left moving modes on the pairs of wires.

\[
\begin{align*}
\phi_{R_{k,1}}^L &= n \varphi_{2k-1} + m_{1} \theta_{2k-1} + 2m_{0} \theta_{2k} \\

\phi_{R_{k,1}}^R &= n \varphi_{2k-1} - m_{1} \theta_{2k-1} \\

\phi_{R_{k,2}}^R &= n \varphi_{2k} + m_{1} \theta_{2k} \\

\phi_{R_{k,2}}^L &= n \varphi_{2k} - m_{1} \theta_{2k} - 2m_{0} \theta_{2k-1}.
\end{align*}
\]

The new fields obey the commutation algebra
\[
\left[ \partial_{x}\phi_{k,a}^{p}(x), \phi_{k',b}^{p'}(x') \right] = 2\pi i\rho_{pp'}\delta_{kk'}K_{ab}\delta_{xx'}. \tag{2.48}
\]

where the \(K\) matrix is
\[
K_{ab} = n \left( \begin{array}{cc} m_{1} & m_{0} \\ m_{0} & m_{1} \end{array} \right). \tag{2.49}
\]

The charge density is
\[
\rho_{k} = \sum_{a} t_{a} \partial_{x}(\phi_{k,a}^{R} - \phi_{k,a}^{L})/2\pi \tag{2.50}
\]

with
\[
t_{a} = \frac{1}{m_{0} + m_{1}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right). \tag{2.51}
\]

We next define variables on links \(\ell = k + 1/2,\)
\[
\tilde{\theta}_{a,\ell=\ell+1/2} = (\phi_{k,a}^{R} - \phi_{k+1,a}^{L})/2 \tag{2.52}
\]

\[
\tilde{\varphi}_{a,\ell=\ell+1/2} = (\phi_{k,a}^{L} + \phi_{k+1,a}^{R})/2. \tag{2.53}
\]

These satisfy, \(\partial_{x}\tilde{\theta}_{a,\ell},\tilde{\varphi}_{\ell,a}=0\) and
\[
\left[ \partial_{x}\tilde{\theta}_{a,\ell},\tilde{\varphi}_{\ell',b} \right] = i\pi K_{ab}\delta_{\ell\ell'}\delta_{xx'}. \tag{2.54}
\]

In terms of these new variables, the Hamiltonian in the presence of the correlated \(n\)-electron tunneling operators becomes
\[
\mathcal{H} = \mathcal{H}_{\text{SLL}}^{0}[\tilde{\theta}_{a,\ell},\tilde{\varphi}_{\ell,a}] + \sum_{\ell} \int dxv \left( \cos 2\tilde{\theta}_{a,\ell} + \cos 2\tilde{\varphi}_{a,\ell} \right). \tag{2.55}
\]

If \(\nu\) flows to strong coupling, we have a gapped bulk, describing a \(\nu = 2n/(m_{0} + m_{1})\) quantum Hall fluid characterized by the \(K\) matrix (2.49). As in Section II.C.3, this can be interpreted as quantum Hall strips with edge states coupled by the charge \(ne\) tunneling operators
\[
\Psi_{k,a}^{ne,RL} = e^{i\phi_{k,a}^{RL}}. \tag{2.56}
\]
Quasiparticles, given by π kinks in $\tilde{\theta}_{\ell,1}$ or $\tilde{\theta}_{\ell,2}$ are created by

$$
\Psi_{Q_{P,a,k+1/2}}^{R/L} = e^{i \sum_k K_{ak}^b \phi_{k+1/2}^b}. \quad (2.57)
$$

They have charge $e/(m_0 + m_1)$. The bare electron backscattering operator corresponds to quasiparticle tunneling,

$$
\chi_{k,a} = e^{-i2\theta_{2k-2+a}} = \Psi_{Q_{P,a,k+1/2}^L}^{R/L} \Psi_{Q_{P,a,k+1/2}^L}^{R/L}. \quad (2.58)
$$

At the edge of a semi infinite system, where will be two chiral modes left over described by $\phi_{1,a}$. From (2.48) it can be seen that these give precisely the chiral Luttinger liquid edge states characterized by the K matrix (2.49).

III. MOORE READ STATE

We now generalize the coupled wire construction to describe the Moore Read state. Our approach was motivated by the observation by Fradkin, Nayak, Shoutens [22] that the Moore Read state for bosons at filling $\nu = 1$ has a simple interpretation in terms of two coupled copies of bosons at $\nu = 1/2$. Each copy is described by a $SU(2)_1$ Chern Simons theory, and the coupling between them introduces the symmetry breaking $SU(2)_1 \times SU(2)_1 \rightarrow SU(2)_2$.

We therefore first consider the problem of coupled wires of bosons at filling $\nu = 1$, where the bosons on each wire have two flavors, each at $\nu = 1/2$. The allowed boson tunneling and backscattering terms in our construction have a simple representation in the low energy bosonized theory. Moreover, by fermionizing the bosons, the Majorana fermions associated with the Moore Read state emerge naturally.

There is a special set of values for the interactions in which the problem is particularly simple. In this case, the Hilbert space associated with the two right (left) moving chiral modes on each wire decouples into two sectors. One of the sectors is coupled to the corresponding sector of the left (right) moving modes on the same wire, while the other sector is coupled to the corresponding sector of the left (right) moving modes on the neighboring wire. Both of these couplings introduce gaps, but the two sectors are gapped in “opposite directions”. This gives a kind of hybrid between the insulating phase, in which all chiral modes are paired on the same wire, and the quantum Hall states, in which all the chiral modes are coupled on neighboring wires. What gets left behind on the edge is a fraction of the original chiral modes.

This fractionalization of the original chiral modes described mathematically in terms of the coset construction in conformal field theory [22]. The original pair of chiral modes are described by a $SU(2)_1 \times SU(2)_1$ theory with central charge $c = 2$. These modes decompose into three sectors: $SU(2)_1 \times SU(2)_1/ SU(2)_2$, $SU(2)_2/U(1)$ and $U(1)$ with $c = 1/2$, 1/2 and 1, respectively. The independent sectors are then gapped “in different directions”.

We will describe this construction in Section III.B. This, in effect, gives a concrete and somewhat more explicit implementation of the Fradkin, Nayak, Shoutens construction.

After establishing the Moore Read state for $\nu = 1$ bosons, we will go on to generalize our construction to account for fermions, and the $q$-pfaffian state at filling $\nu = 1/(q+1)$, where $q$ is even (odd) for bosons (fermions).

A. Bosons at $\nu = 1$

We begin with a Hamiltonian $H = H_{\text{SLL}} + V$ describing coupled wires of two component bosons, which can be viewed as a double layer system, as in Fig. 4. Each component has a density corresponding to filling $\nu = 1/2$. $H^0$ has the same form as (2.8), except now each wire has two bosons, $\theta_{j,a}$ and $\varphi_{j,a}$, for $a = 1, 2$, which satisfy

$$
[\partial_x \theta_{j,a}(x), \varphi_{j,a'}(x')] = i \pi \delta_{jj'} \delta_{aa'} \delta_{xx'}. \quad (3.1)
$$

The interaction terms $V$ consist of boson tunneling and backscattering operators that are consistent with momentum conservation. We consider three such terms, depicted in Fig. 4

$$
V = \sum_j \int dx \left( \sum_{ab=1}^2 t_{ab} \phi_{j,ab} + u \phi_{j}^u v \phi_{j}^v \right) + h.c. \quad (3.2)
$$

The first term involves coupling between channel a on one wire and channel b on the neighboring wire.

$$
\phi_{j,ab}^{\dagger} = e^{i(\varphi_{j,a} - \varphi_{j+1,b} + 2(\theta_{j,a} + \theta_{j+1,a}))}. \quad (3.3)
$$

This term is similar to (2.27). The coefficient 2 of the $\theta$ terms is fixed by the filling factor. In addition, there are allowed terms that couple the two channels on a single wire. These include a Josephson like coupling between the two channels,

$$
\phi_{j}^u = e^{i(\varphi_{j,1} - \varphi_{j,2})}, \quad (3.4)
$$

as well as an interaction that locks the “$2k_F$” densities of the two channels,

$$
\phi_{j}^v = e^{i(2\theta_{j,1} - 2\theta_{j,2})}. \quad (3.5)
$$

These three terms (as well as combinations of them) are the only allowed interaction terms at $\nu = 1/2$ that include up to first neighbor coupling.

It is now useful to introduce right and left moving chiral fields,

$$
\tilde{\phi}_{j,a}^R = \varphi_{j,a} + 2\theta_{j,a}, \quad \tilde{\phi}_{j,a}^L = \varphi_{j,a} - 2\theta_{j,a}, \quad (3.6)
$$

as well as “charge” and “spin” fields,

$$
\tilde{\phi}_{j,a}^p = (\tilde{\phi}_{j,1} + \tilde{\phi}_{j,2})/2, \quad \tilde{\phi}_{j,a}^p = (\tilde{\phi}_{j,1} - \tilde{\phi}_{j,2})/2. \quad (3.7)
$$
we have

\begin{equation}
\sum_{\sigma} \left( \delta_{J+1,\sigma} + \delta_{J,\sigma} \right) = 2 \sum_{\sigma} \left[ (\delta_{J+1,\sigma} - \delta_{J,\sigma}) \cos 2 \tilde{\phi}_{J,\sigma} \right].
\end{equation}

Note that in passing between the first and second lines of

\((3.9)\) \((3.11)\) getting the factors of \(i\) right requires care in splitting the exponential. This is explained in Appendix

A.3, where the zero momentum components of \(\hat{\phi}^{R/L}_{J,\sigma}\) are properly taken into account.

The latter fields satisfy

\begin{equation}
\left[ \partial_x \tilde{\phi}^{R,\mu}_{j,\mu}(x), \tilde{\phi}^{R,\mu'}_{j,\mu'}(x') \right] = 2\pi i \delta_{\mu \mu'} \delta_{J+1} \delta_{xx'}
\end{equation}

for \(\mu = \rho, \sigma\).

For simplicity, we will first focus on the case in which \(t_{ab} = t\), independent of \(a\) and \(b\), and \(t, u\) and \(v\) are real.

We will comment on the more general case later. In this case, in terms of the new variables, we have

\[
\sum_{ab} it_{ab}O_{j,ab}^{\nu} + h.c. = 8t \cos(\tilde{\phi}_{J,\rho} - \tilde{\phi}_{J+1,\rho}) \cos \tilde{\phi}_{J,\sigma} \cos \tilde{\phi}_{J+1,\sigma}.
\]

and

\[
u O_{j,\nu}^{\nu} + h.c. = 2u \cos(\tilde{\phi}_{J,\rho} + \tilde{\phi}_{J,\rho}) = 2i u (\sin \tilde{\phi}_{J,\rho} \sin \tilde{\phi}_{J,\rho} - \cos \tilde{\phi}_{J,\rho} \cos \tilde{\phi}_{J,\rho})
\]

\[
u O_{j,\nu}^{\nu} + h.c. = 2v \cos(\tilde{\phi}_{J,\rho} - \tilde{\phi}_{J,\rho}) = 2iv (\sin \tilde{\phi}_{J,\rho} \sin \tilde{\phi}_{J,\rho} + \cos \tilde{\phi}_{J,\rho} \cos \tilde{\phi}_{J,\rho}).
\]

Since the interaction term \(V\) is a sum of non commuting terms, analysis of this state is more complicated than it was for the Abelian quantum Hall states. However, a tremendous simplification occurs when the forward scattering interactions in \(\mathcal{H}^0_{\text{Q Hall}}\) are such that \(\tilde{\phi}^{R/L}_{J,\rho}\) and \(\tilde{\phi}^{R/L}_{J,\sigma}\) are decoupled, and the Hamiltonian for \(\tilde{\phi}^{R/L}_{J,\rho}\) has the non interacting form, \(\mathcal{H}^0_{\text{cont. int.}}\) in (2.7). In this case, the operator \(\exp i\tilde{\phi}_{J,\sigma}\) has precisely the form of a bosonized Dirac fermion. This allows us to fermionize, by writing

\[
\psi_{J+1/2,\sigma}^{R/L} = \xi_{J,\sigma}^{R/L} + i\eta_{J,\sigma}^{R/L} = \frac{\kappa_{J,\sigma}}{2\pi} \exp i\tilde{\phi}_{J,\sigma}^{R/L} \psi_{J,\sigma}^{R/L}
\]

where \(\psi_{J,\sigma}^{R/L}\) is a Dirac fermion operator, and \(\xi_{J,\sigma}, \eta_{J,\sigma}\) are Majorana fermion operators. For the charge sector, we define

\[
\begin{align*}
\hat{\theta}_{J+1/2,\rho} &= \frac{\hat{\phi}^{R}_{J,\rho} - \hat{\phi}^{L}_{J+1,\rho}}{2} \\
\hat{\varphi}_{J+1/2,\sigma} &= \frac{\hat{\phi}^{R}_{J,\sigma} + \hat{\phi}^{L}_{J+1,\sigma}}{2}.
\end{align*}
\]

They satisfy

\[
\begin{align*}
\left[ \partial_x \hat{\theta}_{\ell,\rho}(x), \hat{\theta}_{\ell',\rho}(x') \right] &= 0 \text{ and } \\
\left[ \partial_x \hat{\varphi}_{\ell,\sigma}(x), \hat{\varphi}_{\ell',\sigma}(x') \right] &= i\pi \delta_{\ell \ell'} \delta_{xx'}.
\end{align*}
\]

The Hamiltonian may now be written,

\[
\mathcal{H} = \mathcal{H}^0_{\text{Q Hall}}(\hat{\theta}_{\ell,\rho}, \hat{\varphi}_{\ell,\sigma}) + \mathcal{H}^0_{\text{M}1}[\eta_{\ell,\sigma}] + \mathcal{H}^0_{\text{M}2}[\xi_{\ell,\sigma}] + V
\]

where \(\mathcal{H}^0_{\text{Q Hall}}[\hat{\theta}_{\ell,\rho}, \hat{\varphi}_{\ell,\sigma}]\) has the form (2.8). For a special value of the interactions in the charge sector it is also possible to fermionize \(\hat{\theta}_{J,\rho}\) and \(\hat{\varphi}_{J,\sigma}\), though that is not necessary for our purposes. The free fermion Hamiltonian for the Majorana fermion \(\eta_{J,\sigma}^{R/L}\) is

\[
\mathcal{H}^0_{\text{M}1}[\eta_{\ell,\sigma}] = \sum_{\ell} \int dx \left( \eta_{\ell,\sigma} \partial_x \eta_{\ell,\sigma} - \eta_{\ell,\sigma} \partial_x \eta_{L,\sigma} \right).
\]

with a similar expression for \(\mathcal{H}_{\text{M}}[\xi_{\ell,\sigma}]\). The interaction term is

\[
V = \sum_{J} \int dx \left[ \hat{\theta}_{J+1,\rho} \hat{\varphi}_{J+1,\sigma} \left( \xi_{J,\sigma}^{R/L} \xi_{J+1,\sigma}^{R/L} + \left( \tilde{u} - \tilde{v} \right) \xi_{J,\sigma}^{R/L} \xi_{J+1,\sigma}^{R/L} + \left( \tilde{u} + \tilde{v} \right) \eta_{J,\sigma}^{R/L} \eta_{J+1,\sigma}^{R/L} \right) \right].
\]

We now assume \(\mathcal{H}^0_{\text{Q Hall}}[\hat{\theta}_{\ell,\rho}, \hat{\varphi}_{\ell,\sigma}]\) is in a regime such that \(\hat{\theta}\) is relevant, and \(\hat{\theta}_{\ell,\rho}\) is pinned in a self consistent minimum of cos \(\tilde{\phi}_{\ell,\rho}\). \(\mathcal{H}\) then describes independent free fermion problems for \(\xi_{\ell,\sigma}\) and \(\eta_{\ell,\sigma}\). The \(\eta\) sector has a gap with a \(k_y\) independent dispersion \(E = \pm \sqrt{\tilde{u}^2 k_x^2 + \left( \tilde{u} - \tilde{v} \right)^2}\). The \(\xi\) sector has dispersion \(E = \pm \sqrt{\tilde{v}^2 k_x^2 + \left| \tilde{u} \right|^2 \eta_{\ell,\sigma}^{R/L} \eta_{L,\sigma}^{R/L}}\) with a gap that closes at a point \(\tilde{v} = \pm \left| \tilde{v} \right|\) signaling a quantum phase transition. The phase diagram is shown in Fig. [3].

We identify the \(\tilde{v} > \tilde{u} - \tilde{v}\) phase with the Moore Read states [34]. Its physics is most transparent at the special point \(\tilde{u} = \tilde{v}\) where the chiral Majorana modes \(\eta_{J,\sigma}^{R/L}\) and
The elementary quasiparticle is thus associated with a kink in which \( \tilde{\phi} \). The present case is slightly different, though because the transformation \( \tilde{\phi}_{j+1,R/L} \rightarrow \tilde{\phi}_{j,R/L} + 2\pi \) (which connects equivalent states) translates to \( \tilde{\phi}_{j,R/L} \rightarrow \tilde{\phi}_{j,R/L} + \pi \). It then follows that the transformation \( \tilde{\theta}_{j,R/L} \rightarrow \tilde{\theta}_{j,R/L} + \pi/2 \), \( (\tilde{\xi}_{j,R/L},\tilde{\eta}_{j,R/L}) \rightarrow (\tilde{\xi}_{j,R/L},\tilde{\eta}_{j,R/L}) \) connects equivalent states. The elementary quasiparticle is thus associated with a kink in which \( \tilde{\phi}_{j,R/L} \) jumps by \( \pi/2 \), corresponding to a charge \( e/2 \). This introduces a domain wall where the mass term coupling \( \tilde{\xi}_{j,R/L} \) and \( \tilde{\xi}_{j+1,R/L} \) changes sign. This binds a zero energy Majorana bound state, as is expected for the charge \( e/2 \) quasiparticles of the bosonic Moore Read state. As in Section II.C.2, the quasiparticle tunneling operators can be related to the backscattering of bare electrons. We defer the discussion of this to section IV.C.

The \( \ell < |\tilde{u} - \tilde{v}| \) phase corresponds to a strongly paired quantum Hall state of charge \( 2e \) bosons at filling \( \nu = 1/4 \). It is most easily understood in the limit \( \ell < |\tilde{u} - \tilde{v}| \). In this case, the Majorana modes pair up with the pattern in Fig. 5, so that there are no gapless Majorana modes at the edge. In this limit, individual bosons can not tunnel between wires because it excites the gapped \( \tilde{\xi}_{j,R/L} \) modes. However, a pair of bosons can tunnel without disturbing the \( \tilde{\xi}_{j,R/L} \) sector. The charge modes thus pair up leaving a single gapless chiral charge mode at the edge.

We now briefly discuss the case in which we relax our assumption about the equality of the different \( t_{ab} \). In this case, the \( t_{ab} \) terms will have the general structure

\[
\sum_{ab} t_{ab} O_{j,R/L} + h.c. = i \left( \xi_{j,R/L}^{R/L} \eta_{j,R/L}^{R/L} \right) T \left( \tilde{\xi}_{j+1,R/L}^{R/L} \tilde{\eta}_{j+1,R/L}^{R/L} \right),
\]

where the \( 2 \times 2 \) matrix \( T_{nm} = \tilde{t}_{nm} \cos(2\tilde{\theta}_{j,R/L} + \tilde{\beta}_{ab}) \) is characterized by a magnitude \( \tilde{t}_{nm} \) and a phase \( \tilde{\beta}_{ab} \), which depend on \( t_{ab} \). When \( \tilde{\theta}_{j,R/L} \) is stiff, we may again analyze the problem by putting \( \tilde{\theta}_{j,R/L} \) in a self consistent minimum and solving the non interacting fermion problem.

It is clear that due to the existence of a bulk energy gap, the phases discussed above will persist for a finite range in the more general parameter space. However, another possible phase is possible where the neutral Majorana modes pair up in the manner shown in Fig 5. In this case the edge has a gapless charge mode and two gapless Majorana modes, or equivalently two gapless bosonic modes.

To see this, consider another special limit where \( t_{ab} = t\delta_{ab} \), with \( t \) real. It then follows that \( \tilde{t}_{nm} = \delta_{nm} \cos(2\tilde{\theta}_{j,R/L} + \tilde{\beta}_{ab}) \), so that the \( V \) has a term \( t \cos(2\tilde{\theta}_{j,R/L} + \tilde{\beta}_{ab}) \tilde{\xi}_{j,R,L}^{R/L} \tilde{\eta}_{j,R,L}^{R/L} \). The phase diagram for this case is shown in Fig 5b. When \( \tilde{u} = \tilde{v} \), the \( \xi \) sector is gapped by \( \tilde{t} \), while the \( \eta \) sector involves competition between \( \tilde{t} \) and \( \tilde{u} + \tilde{v} \). For \( \tilde{t} < \tilde{u} + \tilde{v} \) we have the pairing in Fig 5, giving the Moore Read state, while for \( \tilde{t} > \tilde{u} + \tilde{v} \), we have the pairing in Fig 5, which has two bosonic edge modes. This is most easily understood when \( t_{ab} = t\delta_{ab} \) and \( \tilde{u} = \tilde{v} = 0 \), which simply corresponds to two decoupled \( \nu = 1/2 \) bosonic quantum Hall states.

### B. Coset Construction

The results of the previous section can be understood within the framework of the coset construction in conformal field theory. This is useful because it helps us make contact with the work of Fradkin, Nayak and Shouten as well as subsequent generalizations. It also
introduces a framework that will allow us to generalize our construction to the Read Rezayi (1D) sequence of quantum Hall states. Here we give a brief introduction to this well developed, but somewhat abstract, mathematical construction that emphasizes its physical meaning in the context of our coupled wire theory.

Our construction began with the SLL fixed point, which has bosonic modes $\phi_{j,a}^R$ on each wire. The two right moving chiral modes on each wire define a conformal field theory with central charge $c = 2$. Through the sequence of transformations, the Hilbert space of these two modes was split into three pieces, described by $\phi_{j,a}^R$, $\xi_{j,a}$ and $\eta_{j,\sigma}$. The bosonic mode corresponds to $c = 1$, while each of the Majorana modes has $c = 1/2$. At the point $u = v > 0$ and $t_{ab} = t$, the decoupling is perfect. The decomposition of the Hilbert space can be summarized by

$$2 = 1/2 + 1/2 + 1.$$  \hspace{1cm} (3.19)

The coupling terms we introduced allow the modes in the different sectors to pair up in different directions, as shown in Fig. [6]. This leads to a bulk gap, but leaves behind edge states in some of the sectors. The Moore Read state thus has $c = 3/2$ at the edge.

To understand this decomposition more generally, it is important to realize that at the special filling factor $\nu = 1/2$ the original chiral modes $\phi_{a}^R$ have an extra symmetry because $\exp i \phi_{a}^R$ has scaling dimension $\Delta = 1$. It follows that the operators

$$J_{a,\pm} = e^{\pm i \phi_{a}^R}/(2\pi x_c),$$  \hspace{1cm} (3.20)

$$J_{a,z}^R = \partial_z \phi_{a}^R/(2\pi).$$  \hspace{1cm} (3.21)

generate an $SU(2)$ symmetry. Each channel is thus described by a $SU(2)_1$, Wess Zumino Witten model. The two channels together have $SU(2)_1 \times SU(2)_1$ symmetry. $SU(2)_1 \times SU(2)_1$ has a diagonal subgroup, $SU(2)_2$, generated by $J_R^R = J_1^R + J_2^R$. In terms of the boson operators we have

$$J_{\pm}^R = e^{\pm i \phi_{a}^R} \cos \phi_{a}^R/(2\pi x_c),$$  \hspace{1cm} (3.22)

$$J_z^R = \partial_z \phi_{a}^R/(2\pi).$$  \hspace{1cm} (3.23)

$SU(2)_2$, in turn, has a subgroup $U(1)$ generated by $J_z^R$.

The coset construction allows a Wess Zumino Witten (WZW) model described by a group $G$ with subgroup $H$ to be divided into two pieces described by $G/H$ and $H$. This means that the Hamiltonian can be written as the sum of two commuting terms, $H_G = H_{G/H} + H_H$, so that the Hilbert space of eigenstates factorizes. In the language of conformal field theory the energy momentum tensor can be written $T_G = T_{G/H} + T_H$, and the components $T_{G/H}$ and $T_H$ have no singularities in their operator product expansion. It follows that the central charge of a coset theory given simply by $c_{G/H} = c_G - c_H$. Applied to the present problem, we have

$$T_{SU(2)_1 \times SU(2)_1} = T_{SU(2)_1 \times SU(2)_1/SU(2)_2} + T_{SU(2)_2/U(1)} + T_{U(1)}.$$  \hspace{1cm} (3.24)

Using the fact that $c_{SU(2)_2} = 3k/(k + 2)$ it is simple to see that (3.24) is equivalent to (3.19).

Consider the hopping term between wires, which for $t_{ab} = t$ can be written

$$i \sum_{ab} O_{j,ab}^t + h.c. = 8t J_{j,\pm}^R J_{j+1,\pm}^L.$$  \hspace{1cm} (3.25)

FIG. 6. Coupling of edge states at the decoupled point described by the dashed line in Fig. [3]. The right and left moving $U(1)$ charge modes and the $SU(2)_1/U(1)$ Majorana fermion ($Z_2$ parafermion) modes are coupled on neighboring wires, while the right and left moving $SU(2)_1 \times SU(2)_1/SU(2)_2$ modes are coupled on the same wire. This pattern leaves $U(1)$ and $SU(2)_2/U(1)$ chiral modes at the edge. This provides a concrete interpretation for the coset construction for the Moore Read state.
C. Generalization to fermions

We now consider coupled wires of fermions. Unfortunately, for a uniform magnetic field there is no simple coupled wire construction for the $\nu = 1/2$ fermionic Moore Read state. The tunneling terms that are allowed by momentum conservation either lead to a strongly paired Abelian quantum Hall state of charge $2e$ bosons, or they involve pairs of non-commuting terms that can not be easily analyzed using the present methods. Evidently, the Moore Read state is not sufficiently “close” to the SLL fixed point for uniform field.

However, we found that if the magnetic field is staggered, so that the flux between neighboring wires alternates between two values, then a construction similar to the preceding section can be developed. One can view this as a generalization of the two channel construction in the preceding section, where instead of having the two layers directly on top of one another, one layer is slid over relative to the other. Equivalently, this can be viewed as a single layer system with a staggered field as in Fig. 4a.

The system in Fig. 4a has a magnetic flux per unit length in units of the flux quantum $b = e\alpha B/\hbar$ that alternates between $b$ and $0$. Our more general construction then corresponds to sliding one layer relative to the other, so that the flux per length in units of the flux quantum alternates between two values $b_1 = b + \delta b$ and $b_2 = b - \delta b$. The average flux $\bar{b}$ is related to the filling factor, $\nu = 2k_F/\bar{b}$. The two channel bosonic problem then corresponds to $\delta b = b$, while the uniform field corresponds to $\delta b = 0$. We will show that when $\delta b = 2k_F$ the allowed tunneling terms have a structure similar to that in (3.3-3.5).

This construction gives the $\nu = 1/2$ fermionic Moore Read state for $b = 4k_F$, as well as the more general “q-Pfaffian” state at $\nu = 1/(1 + q)$ for $b = 2k_F(1 + q)$, where $q$ is even (odd) for bosons (fermions). The state in this series with $q = -1$ is special, and corresponds to a $p + ip$ superconductor in zero net magnetic field. In our construction, the modification of the state by changing the uniform component of the field $\bar{b}$ is reminiscent of modifying the Moore Read wavefunction by including a Jastrow factor that compensates the change in magnetic field.\]

1. $q$-pfaffian state

Consider an array of wires with alternating magnetic flux, shown in Fig 7. We parameterize the two fluxes as $2k_F(2 + q)$ and $2k_Fq$. We group the wires into pairs, indexed by $j$ and $a = 1, 2$. The interaction terms then have a form similar to (3.3-3.5):

$$V = \sum_j \int dx \left( \sum_{ab=1}^{2} t_{ab} O_{j,ab}^t + uO_j^u + vO_j^v \right) + h.c. \quad (3.26)$$

There are four terms coupling pair $j$ to $j + 1$,

$$O_{j,ab}^t = e^{i(\phi_{j,a} - \phi_{j+1,b} + (q + 2)(\theta_{j,a} + \theta_{j+1,b}))} Q_{j,ab} \quad (3.27)$$

with

$$Q_j = \begin{pmatrix} e^{i2q\theta_{j,2} + i2q\theta_{j+1,1}} & e^{i2q\theta_{j+1,1}} \\ 1 & e^{i2q\theta_{j,1}} \end{pmatrix}. \quad (3.28)$$

Two terms operate within a single pair. The first involving tunneling an electron between the two wires

$$O_j^u = e^{i(\phi_{j,1} - \phi_{j+1,2} + q(\theta_{j,1} + \theta_{j+1,2})).} \quad (3.29)$$

The second giving an interaction between the 2$k_F$ densities.

$$O_j^v = e^{i(2\theta_{j,1} - 2\theta_{j+1,2}).} \quad (3.30)$$

From (2.16) and (2.26), it is clear that these interactions are appropriate for bosons (fermions) when $q$ is even (odd).

We can write the first term as

$$O_{j,ab}^t = e^{i\tilde{\phi}_{j,a} - \tilde{\phi}_{j+1,b}} \quad (3.31)$$

with

$$\tilde{\phi}_{j,1} = \varphi_{j,1} + (q + 2)\theta_{j,1} + 2q\theta_{j,2}$$

$$\tilde{\phi}_{j,2} = \varphi_{j,2} + (q + 2)\theta_{j,2} \quad (3.32)$$

We next define the sum and difference variables

$$\tilde{\phi}_{j,P} = (\tilde{\phi}_{j,1} + \tilde{\phi}_{j,2}) / 2,$$

$$\tilde{\phi}_{j,\sigma} = (\tilde{\phi}_{j,1} - \tilde{\phi}_{j,2}) / 2. \quad (3.33)$$
These variables obey the commutation relations,

\[
\begin{align*}
[\partial_x \tilde{\phi}_{j,\rho}^R(x), \tilde{\phi}_{j',\rho}^R(x')] &= 2\pi i(1 + q) p \delta_{jj'} \delta_{xx'} \delta_{xx'} \tag{3.34} \\
[\partial_x \tilde{\phi}_{j,\rho}^L(x), \tilde{\phi}_{j',\rho}^L(x')] &= 2\pi i p \delta_{jj'} \delta_{xx'} \delta_{xx'} \tag{3.35}
\end{align*}
\]

Note that the commutation relation for the \( \sigma \) sector is identical to (3.3). This allows us to proceed in the same manner as section III.A. For the \( \sigma \) sector to be unaltered, it was essential that the staggered field satisfy \( \delta b = 2k_F \).

The charge sector, on the other hand is modified, and resembles that of the Laughlin state with \( \sigma \) identical to (3.8). This allows us to proceed in the same way that of the Moore Read state, however, for the charge mode, the gapless edge mode in the quantum Hall case is replaced by a gapless bulk collective mode.

### IV. READ REZAYI SEQUENCE

In this section, we will generalize the coupled wire construction to describe the Read Rezayi sequence of stated. This sequence includes the Moore Read state for \( k = 2 \), as well as other states, which are described in terms of the \( Z_k \) parafermion conformal field theory. As in the previous section, the analysis is simplest for bosons. Following the analysis of Fradkin, Nayak and Shoutens, we thus consider \( k \) channels of bosons, which are each at filling \( \nu = 1/2 \), so that the total filling factor is \( k/2 \). At the end of this section we will briefly describe the generalization, similar to section III.C, which gives the known Read Rezayi states at \( \nu = k/(2 + kq) \), where \( q \) is even (odd) for bosons (fermions).

#### A. Bosons at \( \nu = k/2 \)

Consider coupled wires of \( k \) channel bosons. The analysis is similar to Section III.A, except now each wire is characterized by \( \theta_{j,a} \) and \( \varphi_{j,a} \) satisfying (3.1) for \( a = 1, \ldots, k \). The Hamiltonian is again \( H_{SLL}(\theta, \phi) + V \) with

\[
V = \sum_j dx \left( \sum_{ab=1}^k t_{ab} O^t_{j,ab} + \sum_{a<b=1}^k u_{ab} O^u_{j,ab} + v_{ab} O^v_{j,ab} \right) + \text{h.c.} \tag{4.1}
\]

The interaction coupling neighboring wires

\[
O^t_{j,ab} = e^{i(\varphi_{j,a} - \varphi_{j+1,b}) + 2(\theta_{j,a} + \theta_{j+1,b})} \tag{4.2}
\]

is the same as before, while the interactions operating within a single wire come in more varieties,

\[
O^u_{j,ab} = e^{i(\varphi_{j,a} - \varphi_{j,b})}, \tag{4.3}
\]

as well as an interaction that locks the “2\( k_F \)” densities of the two channels,

\[
O^v_{j,ab} = e^{i(2(\theta_{j,a} - \theta_{j,b})} \tag{4.4}
\]

As in section III.A we first define chiral boson modes

\[
\tilde{\phi}_{j,a}^R = (\varphi_{j,a} + 2\theta_{j,a})/\sqrt{2}
\]

\[
\tilde{\phi}_{j,a}^L = (\varphi_{j,a} - 2\theta_{j,a})/\sqrt{2}. \tag{4.5}
\]
For later convenience, this definition differs by a factor of \( \sqrt{2} \) from the modes defined in Eq. 3.6. We then introduce a charge mode \( \tilde{\phi}^R_{j,\rho} \) and \( k - 1 \) neutral modes \( \tilde{\phi}^R_{j,\sigma} \) by writing

\[
\tilde{\phi}^R_{j,\mu} = \left( \frac{\tilde{\phi}^R_{j,\rho}}{\tilde{\phi}^R_{j,\sigma}} \right)_\mu,
\]

with

\[
\tilde{\phi}^R_{j,\mu} = \sum_{a=1}^{k} O_{\mu a} \tilde{\phi}^R_{j,a}.
\]

\( O_{\mu a} \) is an orthogonal matrix that has the form

\[
O_{\mu a} = \left( \frac{1}{\sqrt{k}} \bar{d}_a \right)_\mu,
\]

where \( \bar{d}_a \) are a set of \( k \) vectors with \( k - 1 \) components that satisfy

\[
\sum_a \bar{d}_a = 0 \quad (4.9)
\]

\[
\sum_a \bar{d}_a^\alpha \bar{d}_a^\beta = \delta_{\alpha\beta} \quad (4.10)
\]

\[
\bar{d}_a \cdot \bar{d}_b = \delta_{ab} - 1/k. \quad (4.11)
\]

\( \bar{d}_a \) may be viewed as the unit vector in the \( a \) direction projected into the plane perpendicular to \( (1, 1, ..., 1) \). For example, for \( k = 3 \) they form 3 planar vectors oriented at 120°,

\[
\bar{d}_1 = \left( \frac{\sqrt{2}}{\sqrt{6}} \right), \quad \bar{d}_2 = \left( \frac{-1}{\sqrt{6}} \right), \quad \bar{d}_3 = \left( \frac{0}{\sqrt{6}} \right). \quad (4.12)
\]

The transformation then has the explicit form,

\[
\tilde{\phi}^R_{j,\rho} = \frac{1}{\sqrt{k}} \sum_{a=1}^{k} \phi^R_{j,a} \quad (4.13)
\]

\[
\tilde{\phi}^R_{j,\sigma} = \sum_{a=1}^{k} \bar{d}_a \phi^R_{j,a}.
\]

along with

\[
\tilde{\phi}^R_{j,a} = \left( \frac{1}{\sqrt{k}} \phi^R_{j,\rho} + \bar{d}_a \cdot \phi^R_{j,\sigma} \right). \quad (4.14)
\]

For \( k = 2 \) the charge and spin modes \( \tilde{\phi}^R_{j,\mu} \) are identical to the corresponding modes defined in Eq. 3.7 in Section III.A (though \( \tilde{\phi}^R_{j,\sigma} \) differ by \( \sqrt{2} \)). The charge and neutral modes satisfy

\[
\left[ \partial_x \tilde{\phi}^R_{j,\mu}(x), \tilde{\phi}^R_{j',\mu'}(x') \right] = 2\pi i \epsilon_{\mu\mu'} \delta_{jk}\delta_{j'k'} \delta_{xx'} \quad (4.15)
\]

Expressed in these variables, the interaction terms have the form,

\[
\mathcal{O}_{j,ab}^t = e^{i\sqrt{2/k}(\tilde{\phi}^R_{j,\rho} - \tilde{\phi}^L_{j+1,\rho})} e^{i\sqrt{2/(k-1)}(\tilde{\phi}^R_{j,\sigma} - \tilde{\phi}^L_{j+1,\sigma})} \quad (4.16)
\]

\[
\mathcal{O}^u_{j,ab} = e^{i(\tilde{d}_a - \tilde{d}_b) \cdot (\tilde{\phi}^R_{j,\rho} + \tilde{\phi}^L_{j,\rho})/\sqrt{2}} \quad (4.17)
\]

\[
\mathcal{O}^u_{j,ab} = e^{i(\tilde{d}_a - \tilde{d}_b) \cdot (\tilde{\phi}^R_{j,\sigma} - \tilde{\phi}^L_{j,\sigma})/\sqrt{2}}. \quad (4.18)
\]

We will now focus on the special case in which \( t_{ab} = t \) are independent of \( a \) and \( b \). Then,

\[
V = \sum_j \int dx e^{i\sqrt{2/k}(\tilde{\phi}^R_{j,\rho} - \tilde{\phi}^L_{j+1,\rho})} \Psi_R^j \Psi_L^j + \sum_{ab}(u_{ab} + v_{ab}) i \Upsilon^R_{j,ab} \Upsilon^L_{j,ab} + (u_{ab} - v_{ab}) i \Xi^R_{j,ab} \Xi^L_{j,ab}, \quad (4.19)
\]

where

\[
\Psi_R^j = \sum_a \exp \left[ i\sqrt{2} \bar{d}_a \cdot \tilde{\phi}^R_{j,\sigma} \right] \quad (4.20)
\]

and

\[
\Upsilon^R_{j,ab} = \sin \left[ \frac{1}{\sqrt{2}} (\bar{d}_a - \bar{d}_b) \cdot \tilde{\phi}^R_{j,\sigma} \right] \quad (4.21)
\]

\[
\Xi^R_{j,ab} = \cos \left[ \frac{1}{\sqrt{2}} (\bar{d}_a - \bar{d}_b) \cdot \tilde{\phi}^R_{j,\sigma} \right]. \quad (4.22)
\]

In the following we will show that at the special point \( u = v \) a decoupling similar to what occurred in Section III.B occurs. To establish this, we will first use the coset construction to show how the \( k \) chiral modes on each wire decouple into separate sectors. We will then show that \( \Psi^R_{j,ab} \) acts only in one sector, while \( \Upsilon^R_{j,ab} \) acts only in the other. The coupling terms in (4.11) then lead to gaps in which the different sectors are paired in different directions, leaving behind non trivial edge states. \( \Psi^R_{j,ab} \) will be identified as a \( Z_k \) parafermion operator. The coupling term \( \Upsilon^R_{j,ab} \), on the other hand, leads to a theory on an individual wire which can be identified with the critical point of a \( Z_k \) model, which is a particular \( k \) state generalization of the Ising and 3 state Potts model.

### B. Coset Construction and Primary Fields

Each wire is characterized by \( k \) right and left moving chiral modes, which individually are described by a \( SU(2)_1 \) WZW model. As in the previous section, \( [SU(2)_1]^k \) can be decoupled by considering the diagonal sub algebra \( SU(2)_k \). This leads to the following decomposition of the energy momentum tensor

\[
T|SU(2)_k|^k = T|SU(2)_1|^k + T|SU(2)_k|U(1) + T|U(1)|. \quad (4.23)
\]

In terms of the central charge, this is equivalent to

\[
k = \frac{k(k-1)}{k+2} + \frac{2(k-1)}{k+2} + 1 \quad (4.24)
\]

Clearly \( k = 2 \) reduces to (3.19). For \( k = 3 \), we have \( z = 6/5+4/5+1 \). The \( SU(2)_k/U(1) \) sector is precisely the \( Z_k \) parafermion theory introduced by Zamolodchikov and...
The non trivial content of this decoupling is that the three energy momentum tensors have no singular terms in their operator product expansion. In a Hamiltonian formalism, this means that the Hamiltonian \( (4.25) \) splits into three commuting pieces, \( \mathcal{H} = \mathcal{H}_{U(1)} + \mathcal{H}_{SU(2)_k/U(1)} + \mathcal{H}_{[SU(2)_1]^k/SU(2)_k} \). This decoupling has also appeared in a somewhat different context in Ref. [58].

We now show that \( \Psi \) and \( T \) act only in a single sector. This is done by computing the operator product expansion with the \( T \)'s. Details of the calculation are in Appendix C. We find that for \( z \to w \) the singular terms in the operator product expansion are

\[
T_{SU(2)_k/U(1)}(z)\Psi(w) = \frac{1 - 1/k}{(z-w)^2} \Psi(z) + \frac{1}{(z-w)} \partial_z \Psi(z), \tag{4.30}
\]

\[
T_{U(1)}(z)\Psi(w) = T_{[SU(2)_1]^k/SU(2)_k} \Psi(w) = 0. \tag{4.31}
\]

Eqs. \( (4.30) \) and \( (4.31) \), shows that \( \Psi \) is the primary field of the \( SU(2)_k/U(1) \) theory with scaling dimension \( 1 - 1/k \), known as a \( Z_k \) parafermion operator. \( \Psi \) is a generalization of the Majorana fermion, which can be regarded as a \( Z_2 \) parafermion. The fact that there no singular terms in the OPE for the other two sectors means that \( \Psi \) acts only in the \( SU(2)_k/U(1) \) parafermion sector. In a Hamiltonian formulation, we would have \( [\Psi, \mathcal{H}_{U(1)}] = [\Psi, \mathcal{H}_{[SU(2)_1]^k/SU(2)_k}] = 0 \). A mass term \( \Psi R \Psi^L \) has scaling dimension \( 2 - 2/k \), and is relevant. It leads to an energy gap in the parafermion sector.

For \( \Upsilon_{ab} \) we find

\[
T_{[SU(2)_1]^k/SU(2)_k} \Upsilon_{ab}(w) = \frac{1/2}{(z-w)^2} \Upsilon_{ab}(z) + \frac{1}{(z-w)} \partial_z \Upsilon_{ab}(z), \tag{4.32}
\]

\[
T_{U(1)}(z)\Upsilon_{ab}(w) = T_{[SU(2)_1]^k/SU(2)_k} \Upsilon_{ab}(w) = 0. \tag{4.33}
\]

This shows that \( \Upsilon_{ab} \) are primary fields of the \( [SU(2)_1]^k/SU(2)_k \) sector with scaling dimension \( 1/2 \), and do not act in the \( SU(2)_k/U(1) \) or the \( U(1) \) sectors. A mass term \( \Upsilon^R \Upsilon^L \) has dimension \( 1 \) and leads to an energy gap in the \( [SU(2)_1]^k/SU(2)_k \) sector.

We have also computed the OPE’s for \( \Xi_{ab} \), defined in \( (4.22) \). Unlike \( \Psi \) and \( \Upsilon \), though, \( \Xi \) is not primary and acts non trivially in both the \( [SU(2)_1]^k/SU(2)_k \) and the \( SU(2)_k/U(1) \) sectors. Thus, unlike the \( k = 2 \) case, it is not clear how \( \Xi^R \Xi^L \) competes with the other terms. Nonetheless, on the special line \( u_{ab} = v_{ab}, t_{ab} = t \) the \( \Xi_{ab} \) term is absent, and we have the decoupling shown in Fig. 9 in which the \( U(1) \) and \( SU(2)_k/U(1) \) sectors are gapped across wires, while the \( [SU(2)_1]^k/SU(2)_k \) sector is gapped within a wire. This gives the Read Rezayi state, which has a left over gapless edge state with a \( U(1) \) charge mode and a \( SU(2)_k/U(1) Z_k \) parafermion mode.
C. Quasiparticle Operators

To construct quasiparticle operators we follow the logic of Section II.C.2 and consider the $2k_F$ backscattering of bare particles on channel $a$ of wire $j$,

$$\chi_{j,a} = e^{2i\theta_{j,a}}$$
$$= e^{i\sqrt{2/k}(\phi_{j,a} - \phi_{j,a}')} + i\sqrt{2/k} (\phi_{j,a} - \phi_{j,a}') (4.35)$$

We thus define

$$\Psi^{R/L}_{QP,j+1/2,a} = e^{i\sqrt{2/k}\phi_{j+1/2,a}^{R/L}}$$

where

$$\Sigma_{j,a}^{R/L} = \exp \left[i\sqrt{2k}d_a \cdot \sigma_j^{R/L}\right].$$

$\Psi^{R/L}_{QP,j+1/2,a}$ creates a quasiparticle with charge $1/2$ (check!) with a non trivial action in the neutral sector.

Like $\Xi_{ab}$, the operators $\Sigma_a$ are not primary, and acts in both the $[SU(2)^k]/SU(2)_k$ and $SU(2)_k/U(1)$ sectors. Nonetheless, in the next section we will argue that when $[SU(2)^k]/SU(2)_k$ sector is gapped, $\Sigma_a$ acts as a primary field projected into the parafermion sector, which corresponds to the spin operator $\sigma$.

D. Relation to $Z_k$ Statistical Mechanics Model

On a single wire, the mass term $((u_{ab} + v_{ab})Y_{ab}^R Y_{ab}^L$ opens a gap and leaves behind a $SU(2)_k$ gapless edge state with a charge mode and a $Z_k$ parafermion. The $Z_k$ parafermion conformal field theory is known to describe the critical point of a $Z_k$ generalization of the Ising model\textsuperscript{[23]} For $k = 3$ it is the 3 state Potts model. Similarly, for $k > 3$ it is a particular version of a $Z_k$ symmetric $k$ state generalized Potts model. In this section we show that our bosonization representation provides a simple and intuitively appealing way to understand this connection. This allows us to identify the projected operators $\Sigma_a$ with the primary fields $\sigma$ of the $Z_k$ parafermion model.

We begin by rewriting the mass term by introducing new variables for the single wire,

$$\varphi_\sigma = (\phi_{ab}^R + \phi_{ab}^L)/2,$$
$$\theta_\sigma = (\phi_{ab}^R - \phi_{ab}^L)/2.$$

These variables satisfy

$$[\partial_x \theta_\sigma(x), \varphi_\sigma^\beta(x')] = i\pi \delta_{\alpha\beta} \delta_{xx'}. (4.39)$$

The Hamiltonian for the neutral sector of a single wire then has the form

$$\mathcal{H} = \frac{v}{2\pi} \left((\partial_x \theta_\sigma)^2 + (\partial_x \varphi_\sigma)^2\right)$$
$$+ \sum_{ab} u_{ab} \cos\sqrt{2d_{ab}} \cdot \theta_\sigma + v_{ab} \cos\sqrt{2d_{ab}} \cdot \varphi_\sigma (4.40)$$

where we have abbreviated $d_{ab} = d_a - d_b$.

First consider the simplest case $k = 2$, where $\theta_\sigma$ and $\varphi_\sigma$ have a single component, and $d_{12} - d_{21} = \sqrt{2}$. Then we have

$$\mathcal{H} = \frac{v}{2\pi} \left((\partial_x \theta_\sigma)^2 + (\partial_x \varphi_\sigma)^2\right) - u \cos 2\theta_\sigma - v \cos 2\varphi_\sigma$$

Viewed as a transfer matrix for the partition function of an anisotropic statistical mechanics problem, this Hamiltonian gives a well known representation of the 2D Ising model\textsuperscript{[25]}. This can be understood by first considering $u = 0$. This describes the 2D XY model with order parameter $\cos \theta_\sigma, \sin \theta_\sigma$. For $v = 0$, $\theta_\sigma$ is a non compact variable, so there are no vortices. From (4.39) it can be seen that $\exp \pm 2i\varphi_\sigma(x, \tau)$ creates a vortex where $\theta_\sigma$ winds by $2\pi$ around $(x, \tau)$. $v$ is thus the fugacity for vortices, and its presence makes $\theta_\sigma$ an angular variable defined modulo $2\pi$. Integrating out $\theta_\sigma$ gives the sine gordon representation of the XY model. For nonzero $v$, $u - \cos 2\theta_\sigma$ introduces an Ising anisotropy into the XY model. For large $u$ $\theta_\sigma$ is pinned in the minima of this potential at $\theta = n\pi$. Due to the presence of $v$, only two of those minima are distinct. For $u \neq v$, since both $u$ and $v$ are relevant, the system flows at low energy to a strong coupling phase in which either $\theta_\sigma$ or $\varphi_\sigma$ is pinned. The symmetry under $u \leftrightarrow v$ and $\theta_\sigma \leftrightarrow \varphi_\sigma$ is precisely the Kramers Wannier duality of the Ising model. At the self dual point $u = v$, the system at low energy flows to the fixed point of the Ising critical point.

For $k > 2$, a similar interpretation is possible. Now, however, $\theta_\sigma$ lives in $k - 1$ dimensions. $\exp i\sqrt{2d_{ab}} \cdot \varphi_\sigma$ creates vortices around which $\theta \rightarrow \theta_\sigma + \sqrt{2\pi}d_{ab}$. This compactifies $\theta_\sigma$, so that it is defined on a $k - 1$ dimensional torus. $\cos 2d_{ab} \cdot \theta_\sigma$ introduces a periodic potential for $\theta_\sigma$, and pins $\theta$ in its minima.

For $k = 3$ the minima of the periodic potential are shown in Fig. 9. They form a triangular lattice with lattice constant $2\pi/\sqrt{3}$. The compactification of $\theta_\sigma$ is associated with a larger triangular lattice with lattice constant $2\pi$. This identifies points on the original lattice.
that are on the same \( \sqrt{3} \) sublattice. There are 3 distinct minima. We thus have a two dimensional generalization of the XY model (where the order parameter is defined on a torus \( T^2 \)), with a 3 state anisotropy. Since both the vortices and the anisotropy are relevant (the periodic potential has scaling dimension 1), this leads to a 3 state generalization of the Ising model with \( Z_3 \) symmetry, which is uniquely specified by the 3 state Potts model. Again, the critical point appears at the self dual point \( u_{ab} = v_{ab} \).

For \( k = 4 \), the minima of the periodic potential form a three dimensional fcc lattice. The fcc lattice can be viewed as a larger bcc lattice with a 4 site basis. The compactification is associated with the larger bcc lattice. There are thus 4 states, so we have a 4 state generalization of the Ising model with \( Z_4 \) symmetry, which is known as the Ashkin Teller model. This model has a parameter, which for different values gives, for example, the 4 state Potts model and the 4 state clock model. It is not immediately obvious what the value of that parameter should be based on the form of (4.41). However, from the analysis of the previous section, we know that the critical point is described by the \( Z_4 \) parafermion conformal field theory. We thus expect that this model describes the Fateev Zamolodchikov point of the Ashkin Teller model.

For general \( k \) the minima occur on a \( k-1 \) dimensional lattice formed by combinations of \( \sqrt{2\pi} d_{ab} \). There are \( k \) distinct but equivalent minima to this potential, which can be located at \( \vec{\theta}_n = n \sqrt{2\pi} d_{1} \), for \( n = 0,...,k-1 \). The minima at \( n = k \) is equivalent to the one at \( n = 0 \) because from (4.9) \( k d_1 = \sum_{a=1}^{k} d_{aj} \). All other minima of \( \cos \sqrt{2d_{ab}} \cdot \vec{\theta}_n \) can also be reduced to these \( k \) minima with a suitable combination of \( \sqrt{2\pi} d_{ab} \). This model thus describes a \( k \) state system with \( Z_k \) symmetry. \( Z_k \) models have extra parameters for \( k \geq 4 \), but as discussed above, since the critical point is described by the \( Z_k \) parafermion theory we conclude that our model describes the Fateev Zamolodchikov point of the \( Z_k \) model.

Now we consider the operators \( \Sigma_a \) discussed above, which has a simple interpretation. When \( (u_{ab} + v_{ab}) Y_{ab} R_{ab} R_{ab} \) opens a large gap, then we can restrict \( \vec{\theta}_n \) to the minima \( \vec{\theta}_n = n \sqrt{2\pi} d_{1} \). It is then straightforward to see that

\[
\Sigma^R_{j,a} \Sigma^L_{j,a} = e^{i \sqrt{2d_{ab}} \cdot \vec{\theta}_n} = e^{-2\pi i n/k},
\]

(4.42)

independent of \( a \). This is precisely the spin order parameter of the \( Z_k \) model, which gives different values \( e^{-2\pi i n/k} \) for the different ordered states specified by \( n \). We thus conclude that when the \( \left[ SU(2) \right]^k / SU(2)_k \) sector is gapped, the operator \( \Sigma^R_{j,a} \), when projected into the \( SU(2)_k / U(1) \) sector corresponds precisely to the spin field \( \sigma \) of the \( Z_k \) model.

FIG. 10. Schematic diagram for the generalized Read Rezayi state at filling \( \nu = k/(2+ kq) \). Groups of \( k \) wires are coupled to one another by \( O_{j+1/2,ab}^u \), which require a specific staggered magnetic field for momentum conservation. Wires within a group are coupled by \( O_{j,ab}^{v,u} \), which are independent of the field. Representative examples of \( O_{j+1/2,ab}^u \) and \( O_{j,ab}^{v,v} \) are shown.

E. Generalization

As in Section III.C our construction for the Read Rezayi sequence can be generalized by introducing a staggered component to the field. Again, the way to think about it is to start with the bosonic state at \( \nu = k/2 \), which can be viewed as a staggered field, in which the field between neighboring wires is \( b \) for one out of every \( k \) neighbors and 0 for the other \( k - 1 \) neighbors. Keeping this staggered field fixed, we now add a uniform field \( \bar{b} \), and find that for certain values, which correspond to filling factor

\[
\nu = \frac{k}{2 + kq}
\]

(4.43)

there are allowed tunneling processes, which have a structure similar to (4.11-4.14). Expressed in terms of charge/neutral variables, as in (4.13), we find that the neutral sector is independent of \( q \), while the charge sector is modified, as in (3.34).

Rather than repeating the algebra in Section III.C, we simply display the diagram, analogous to Fig 7, in Fig 10.

V. CONCLUSION

In this paper we have introduced a new formulation of non-Abelian fractional quantum Hall states, in which electronic models built from coupled interacting one dimensional wires can be analyzed using Abelian bosonization. The picture that emerges from this analysis is summarized in Figs. 9 and 10. Non-Abelian states can be viewed as systems in which the original one dimensional chiral fermion modes are split into fractionalized sectors, in accordance with the coset construction of conformal field theory. The different coset sectors are then coupled to one another in “opposite directions”. This leads to
be treated separately. Define
\[ \psi_j^p(x) = \frac{\kappa_j^p}{\sqrt{2\pi x_c}} e^{i\delta_j^p(x)}. \]  
(A1)

Here, \( p = R/L = +1/ -1 \) describes the right and left moving chiral modes, and
\[ [\phi_j^p(x), \phi_j^{p'}(x')] = ip\pi \delta_{pp'} \delta_{jj'} s_{xx'}, \]  
(A2)

where \( s_{xx'} = \text{sgn}(x - x') \). The Klein factors are necessary to assure that fermion fields associated with different channels anticommute. They may be represented by the factors
\[ \kappa_i^p = (-1)^{\sum_{j<q} N_j^q}, \]  
(A3)

where the number operator for each chiral channel
\[ N_i^p = p \int dx \partial_x \phi_i^p / 2\pi, \]  
(A4)

satisfies \([N_j^p, \phi_j^{p'}] = i\delta_{jj'} \delta_{pp'} \) and has integer eigenvalues. We have chosen an ordering of the chiral modes with direction \( p = R/L \) on wire \( i \), such that \((i, L)<(i, R)<(i + 1, L)<(i + 1, R)\). Defined in this way, the Klein factors mutually commute \([\kappa_j^p, \kappa_j^{p'}] = 0 \), but the fermion operators anticommute, \([\psi_j^p, \psi_j^{p'}] = 0 \). Other choices for the phase factors in (A3) are also possible.

The density and phase fields defined on each wire may be introduced as
\[ \phi_j^0 = (\phi_j^R + \pi N_j^L)/2 \]  
(A5)
\[ \phi_j^L = (\phi_j^R - \phi_j^L + \pi N_j^L)/2 \]  
(A6)

These satisfy \([\theta_j(x), \theta_{j'}(x')] = [\varphi_j(x), \varphi_{j'}(x')] = 0 \), along with
\[ [\varphi_j(x), \varphi_{j'}(x')] = i\pi \delta_{jj'} \Theta(x - x'). \]  
(A7)

The electron operators are then
\[ \psi_j^p(x) = \frac{\kappa_j^p}{\sqrt{2\pi x_c}} e^{i(\varphi_j + p\theta_j)}. \]  
(A8)

where the Klein factor (with no superscript) used in (2.7), is
\[ \kappa_i = (-1)^{\sum_{j<q} N_j^r + N_j^L} \]  
(A9)

is now independent of \( p \).

Consider the backscattering operator on an individual wire,
\[ \mathcal{O}_j = \psi_j^L \psi_j^R = \frac{1}{2\pi ix_c} e^{2i\theta_j} \]  
(A10)

The Klein factors are absent, and can safely be ignored. In the subsections 1 and 2 we will apply this analysis to the Laughlin states and hierarchy states. The bosonic Moore Read state does not require Klein factors in the original model, however, they require care when refermionizing. This is discussed in subsection 3.
1. Laughlin States

The electron operator responsible for the fermionic Laughlin state \( \nu = 1/m \), for \( m \) odd, is

\[
O_{Lj+1/2}^{\nu/m} = (\psi_{Lj+1}^L)^{m+1} (\psi_{Lj+1}^R)^{m-1} (\psi_{Lj}^L)^{m+1} (\psi_{Lj}^R)^{m+1}
\]

(A11)

We write this as

\[
O_{Lj+1/2}^{\nu/m} = \tilde{\psi}_j^L \tilde{\psi}_j^R
\]

(A12)

where

\[
\tilde{\psi}_j^L = (\psi_{Lj}^L)^{m+1} (\psi_{Lj}^R)^{m+1},
\]

\[
\tilde{\psi}_j^R = (\psi_{Lj}^L)^{m-1} (\psi_{Lj}^R)^{m+1}.
\]

(A13)

We now keep the Klein factors and define \( \tilde{\theta}_{j+1/2} \) and \( \tilde{\varphi}_{j+1/2} \) such that

\[
e^{2i\tilde{\theta}_{j+1/2}} = \tilde{\psi}_j^L \tilde{\psi}_j^R
\]

\[
e^{2i\tilde{\varphi}_{j+1/2}} = \tilde{\psi}_j^L \tilde{\psi}_j^R.
\]

(A14)

Then,

\[
2\tilde{\theta}_{j+1/2} = \tilde{\theta}_j^L - \tilde{\theta}_j^R + \pi \tilde{N}_j^\theta
\]

\[
2\tilde{\varphi}_{j+1/2} = \tilde{\varphi}_j^L + \tilde{\varphi}_j^R + \pi \tilde{N}_j^\varphi.
\]

(A15)

where

\[
\tilde{\theta}_j^R = \frac{1 + m}{2} \phi_j^R + \frac{1 - m}{2} \phi_j^L,
\]

\[
\tilde{\theta}_j^L = \frac{1 - m}{2} \phi_j^R + \frac{1 + m}{2} \phi_j^L.
\]

(A16)

\( \tilde{N}_j^\theta/\varphi \) are sums of \( N_j^{L/R} \) determined by the Klein factors, using \([A1, A3, A13]\). Defined in this way, the commutation relations obeyed by \( \tilde{\psi}_j^{L/R} \) guarantee that \([e^{2iA_1(x)}, e^{2iA_1(x')}] = 0 \) for \( A = \tilde{\theta} \) and \( x \neq x' \). This means that \([A_{\nu}(x), B_{\nu}(x')] = i P^{AB}_{L/R} x/2 \), where \( P^{AB}_{L/R} \) is an integer. However, there is freedom in how \( \tilde{N}_j^\theta/\varphi \) is defined because \([A3]\) is unchanged when \( \tilde{N}_j^\theta/\varphi \) is increased by an even integer (which could depend on \( N_{i}^{L/R} \)). This freedom can be exploited to define \( \tilde{\theta} \) and \( \tilde{\varphi} \) so that they obey a standard commutation relation. While a general method for determining \( \tilde{N}_j^\theta/\varphi \) remains to be developed, we have by trial and error found (non unique) solutions. For

\[
\tilde{N}_j^\theta = \frac{m - 1}{2} N_j^L + \frac{m + 1}{2} N_j^R
\]

\[
\tilde{N}_j^\varphi = \frac{m - 1}{2} N_j^L + m N_j^R + \frac{m - 1}{2} N_{j+1}^L
\]

(A17)

(A18)

the fields \( \tilde{\theta}_j \) and \( \tilde{\varphi}_j \) defined in \([A15, A3, A13] \) and \([A14] \), as well as \([\tilde{\theta}_j(x), \tilde{\varphi}_j(x')] = \Theta(x - x') \).

(A19)

2. Hierarchy States

The procedure for defining the Klein factors for the hierarchy states is similar to that in the preceding section. Here we just sketch the process. We may again write the tunneling operators, defined in \([2.45]\) as

\[
O_{2k} = \tilde{\psi}_{k+1}^L \tilde{\psi}_{k,1}^R
\]

\[
O_{2k+1} = \tilde{\psi}_{k+1}^L \tilde{\psi}_{k,2}^R
\]

(A20)

with

\[
\tilde{\psi}_{k,1} = (\psi_{2k-1}^L)^{m+1} (\psi_{2k-1}^R)^{m-1} (\psi_{2k}^L)^{m+1} (\psi_{2k}^R)^{m+1}
\]

\[
\tilde{\psi}_{k,2} = (\psi_{2k}^L)^{m+1} (\psi_{2k}^R)^{m+1}.
\]

(A21)

We now define

\[
e^{2i\tilde{\theta}_{k+1/2}} = \tilde{\psi}_{k+1}^L \tilde{\psi}_{k+1}^R.
\]

(A22)

Then,

\[
2\tilde{\theta}_{k+1/2} = \tilde{\theta}_{k,1}^L - \tilde{\theta}_{k,1}^R + \pi \tilde{N}_{k,1}^\theta
\]

\[
2\tilde{\theta}_{k+1/2} = \tilde{\theta}_{k,2}^L + \tilde{\theta}_{k,2}^R + \pi \tilde{N}_{k,2}^\varphi.
\]

(A23)

with

\[
\tilde{\theta}_{k,1}^L = \frac{n + m_1}{2} \phi_{2k-1}^L + \frac{n - m_1}{2} \phi_{2k-1}^L + m_0 (\phi_{2k-1}^L - \phi_{2k-1}^L)
\]

\[
\tilde{\theta}_{k,1}^R = \frac{n + m_1}{2} \phi_{2k-1}^L + \frac{n + m_1}{2} \phi_{2k-1}^L + m_0 (\phi_{2k-1}^L - \phi_{2k-1}^L)
\]

\[
\tilde{\theta}_{k,2}^L = \frac{n + m_1}{2} \phi_{2k}^L + \frac{n + m_1}{2} \phi_{2k}^L + m_0 (\phi_{2k-1}^L - \phi_{2k-1}^L).
\]

(A24)

and

\[
\tilde{N}_{k+1/2,1} = \frac{n - m_1}{2} (N_{2k-1}^L + N_{2k-1}^R) - (n + m_0) N_{2k}^L - n (N_{2k}^R + N_{2k+1}^R)
\]

\[
\tilde{N}_{k+1/2,2} = \frac{n - m_1}{2} (N_{2k}^L + N_{2k}^R) - (n + m_0) N_{2k+1}^L - n (N_{2k+1}^R + N_{2k+1}^R)
\]

(A25)

These fields then satisfy

\[
[\tilde{\theta}_{k,a}(x), \tilde{\theta}_{l,b}(x')] = [\tilde{\varphi}_{k,a}(x), \tilde{\varphi}_{l,b}(x')] = 0.
\]

(A26)

In addition

\[
[\tilde{\theta}_{k,a}(x), \tilde{\varphi}_{l,b}(x')] = i \pi \delta_{kl} (K_{ab} \Theta(x - x') + W_{ab}).
\]

(A27)
with $K$ given in (2.49) and $W_{ab}$ are integers. It may be possible to find another choice for $N_{k,a}^{R/L}$ in which $W_{ab} = 0$. However, for the purpose of this paper, this choice suffices.

3. Moore-Read state

Here we provide the details of the zero modes and Klein factors for the $\nu = 1/2$ bosonic Moore Read state discussed in Section III.A. Since it is a model of bosons, there are no Klein factors in the original model. However, it is necessary to keep track of the zero modes in order to correctly fermionize the theory. The analysis in this appendix leads to the appropriate (and slightly counterintuitive) factors of $i$ in Eqs. (3.9), (3.11).

Recall $\Phi^1_{j,a}(x) \sim e^{i\phi_{j,a}(x)}$ is the boson creation operator and $\rho_a \sim e^{i\psi_{j,a}(x)}$ represents the density wave at $q \sim 2\pi \bar{m}$. In order for these fields to commute for $x \neq x'$, we require

$$\theta_{j,a}(x), \varphi'_{j,a}(x') = i\pi \delta_{j,j'} \delta_{a,b} \Theta(x-x') \quad (A28)$$

where $\Theta(x-x')$ is the step function. To define the chiral modes, with the appropriate commutation relations, we must augment (3.6) with the appropriate factors as in (A6). This may be written

$$\theta_{j,a} = (\phi_{j,a}^R - \phi_{j,a}^L + \pi N_{j,a}^L)/2$$
$$\varphi_{j,a} = (\phi_{j,a}^R + \phi_{j,a}^L + \pi N_{j,a}^L)/2. \quad (A29)$$

Here, $N_{j,a}^p = p \int dx \partial_x \phi_{j,a}^p/(2\pi)$. These fields obey commutation relations identical to (A2). The extra $N_{j,a}^L$ term was necessary to make $\phi_{j,a}^R$ and $\phi_{j,a}^L$ commute. Charge and spin modes can then be defined, as in (3.7), and the correction terms involve $N_{j,a}^p = p \int dx \partial_x \phi_{j,a}^p/(2\pi)$, which satisfy

$$[N_{j,a}^p, \tilde{\phi}_{j,b}^p] = i \quad (A30)$$

for $\mu = \rho, \sigma$.

In terms of these variables we then write

$$O^u_{j,a} = \exp[i(\tilde{\phi}_{j,a}^R + \tilde{\phi}_{j,a}^L + \pi N_{j,a}^L)]$$
$$O^v_{j,a} = \exp[i(\tilde{\phi}_{j,a}^R - \tilde{\phi}_{j,a}^L + \pi N_{j,a}^L)] \quad (A31)$$

We now separate the chiral modes in the exponential, keeping track of the commutator, and obtain

$$O^u_{j,a} = i e^{i(\tilde{\phi}_{j,a}^R + \pi N_{j,a}^L)} e^{i\tilde{\phi}_{j,a}^L}$$
$$O^v_{j,a} = i e^{i(\tilde{\phi}_{j,a}^R + \pi N_{j,a}^L)} e^{-i\tilde{\phi}_{j,a}^L}. \quad (A32)$$

It then follows that

$$O^u_{j,a} + h.c. = i[\cos(\tilde{\phi}_{j,a}^R + \pi N_{j,a}^L) \cos \tilde{\phi}_{j,a}^L$$
$$- \sin(\tilde{\phi}_{j,a}^R + \pi N_{j,a}^L) \sin \tilde{\phi}_{j,a}^L]$$
$$O^v_{j,a} + h.c. = i[\cos(\tilde{\phi}_{j,a}^R + \pi N_{j,a}^L) \cos \tilde{\phi}_{j,a}^L$$
$$+ \sin(\tilde{\phi}_{j,a}^R + \pi N_{j,a}^L) \sin \tilde{\phi}_{j,a}^L] \quad (A33)$$

Thus, the Josephson and inter wire coupling terms are given in (3.10), (3.11). A similar analysis gives (3.9). The factor of $N_{j,a}^L$ in (A31) precisely gives the necessary factor to define the Klein factor for the fermions in (3.12), as in Eqs. (A1) and (A3).

A similar analysis can be applied to the Read Rezayi states for general $k$. For example, (4.17) is modified to be

$$O^u_{j,a} = e^{i\tilde{d}_{ab}(\phi_{j,a}^R + \phi_{j,a}^L + \pi N_{j,a}^L)/\sqrt{2}}$$
$$= e^{i\tilde{d}_{ab}(\phi_{j,a}^R + N_{j,a}^L)} e^{i\tilde{d}_{ab} \tilde{\phi}_{j,a}^L}. \quad (A34)$$

where $N_{j,a}^p = p \int dx \partial_x \phi_{j,a}^p/2\pi$ and $\tilde{d}_{ab} = \tilde{d}_a - \tilde{d}_b$.

Appendix B: Decoupling of Energy Momentum Tensor

In this appendix we review how the energy momentum tensor for $[SU(2)_1]^k$ is decomposed into $[SU(2)_1]^k/SU(2)_k$, $SU(2)_k/U(1)$ and $U(1)$.

Since the notations in the conformal field theory literature and the condensed matter literature are somewhat different, we first review the translation between the two for a single non interacting (right moving) fermion mode. In the CFT notation, this is written as

$$\psi(z) = e^{i\phi(z)}; \quad (B1)$$

where $z$ is a complex coordinate and the colons denote normal ordering. The chiral boson field $\phi$ satisfies

$$\langle \phi(z) \phi(w) \rangle = -\log(z-w) \quad (B2)$$

so that $\psi$ satisfies $\langle \psi(z) \psi(w) \rangle = 1/(z-w)$. The energy momentum tensor is

$$T(z) = -\frac{1}{2} \langle \partial_z \phi \partial_z \phi \rangle. \quad (B3)$$

Using $\langle \phi(w) \rangle = 1/(z-w)$ and Wick’s theorem, it can then be shown that the singular terms in the operator product expansion (OPE) of $T(z)$ with $\psi(w)$ are

$$T(z) \psi(w) = \frac{\Delta}{(z-w)^2} \psi(w) + \frac{1}{z-w} \partial_w \psi(w). \quad (B4)$$

with $\Delta = 1/2$. This shows that $\psi$ is a primary field with dimension $\Delta = 1/2$.

In the condensed matter literature the chiral fermion field is often written as

$$\psi(x, \tau) = \frac{1}{\sqrt{2\pi x_c}} e^{i\phi(x, \tau)}, \quad (B5)$$

where the operator is not normal ordered and $x_c$ is a convergence factor in divergent momentum integrals, which plays the role of a short distance cutoff. Since
exp \, i \phi = (x_c / L)^{1/2} \exp \, i \phi \cdot \; \text{the cutoff} \, x_c \text{can be eliminated by writing } (B5) \text{using a normal ordered exponential.} \, \psi \text{satisfies } \langle \psi(x, \tau) \psi(0,0) \rangle = [2\pi(v \tau + ix)]^{-1}. \text{The dynamics is governed by the Hamiltonian}

\begin{equation}
H = \int dx \frac{v_F}{4\pi} : (\partial_\tau \phi)^2 : . \tag{B6}
\end{equation}

To make contact with the CFT notation, we first observe that the normalization of \( \psi \) in (B5) differs by a factor of \( \sqrt{2\pi} \) from (B1). Consider a finite system with periodic boundary conditions of length \( L \), so that \( v \tau + ix \) is defined on a cylinder, and introduce the radial variable

\begin{equation}
z(x, \tau) = e^{2\pi(v \tau + ix)/L}, \tag{B7}
\end{equation}

The fermion field on the cylinder then has the form similar to (B1),

\begin{equation}
\psi(x, \tau) = \sqrt{\frac{2\pi z}{L}} : e^{i\phi(z)} : |z = z(x, \tau). \tag{B8}
\end{equation}

Aside from the \( \sqrt{2\pi} \) difference in the normalization, this is equivalent to (B5) for \( L \to \infty \).

The Hamiltonian (B6) corresponds to the lowest mode of the energy momentum tensor in the radial quantization.

\begin{equation}
H = \frac{2\pi v_F}{L} L_0, \tag{B9}
\end{equation}

with

\begin{equation}
L_0 = \frac{1}{2\pi i} \int dz z T(z) \tag{B10}
\end{equation}

where the integral is on a circle \( |z| = e^{2\pi v_F \tau / L} \). It can readily be seen that (B7), (B9) and (B10) imply that (B3) and (B6) are equivalent.

Now consider the \( k \) fields \( \phi_a \), along with \( \phi_{\mu = \rho = 0} \) considered in Section IV.A, which are related by (4.13). Again, we consider only a single (ie right moving) chiral sector, and omit the superscript \( R \). In the notation defined above, these satisfy

\begin{equation}
\langle \tilde{\phi}_a(z) \tilde{\phi}_b(w) \rangle = -\delta_{ab} \log(z - w)
\end{equation}

\begin{equation}
\langle \tilde{\phi}_\mu(z) \tilde{\phi}_\nu(w) \rangle = -\delta_{\mu\nu} \log(z - w), \tag{B11}
\end{equation}

and the energy momentum tensor is

\begin{equation}
T_{[SU(2)]^k} = -\frac{1}{2} \sum_a : (\partial_\tau \tilde{\phi}_a)^2 : \tag{B12}
\end{equation}

\begin{equation}
= -\frac{1}{2} : \left( (\partial_\tau \tilde{\phi}_a)^2 + (\partial_\tau \tilde{\phi}_a)^2 \right) : \tag{B13}
\end{equation}

For each of the \( k \) channels, the operators

\begin{equation}
J^z_a = i \partial_\tau \tilde{\phi}_a
\end{equation}

\begin{equation}
J^a_a = J^a = \pm i J^a = \sqrt{2} \langle z = w \rangle \tilde{\phi}_a \tag{B14}
\end{equation}

form an \( SU(2)_1 \) current algebra. Using the fact that \( \langle J_a \cdot J_a := -3 : (\partial_\tau \tilde{\phi}_a)^2 : \), we may write the energy momentum tensor in a way that reflects the \( [SU(2)]^k \) symmetry,

\begin{equation}
T_{[SU(2)]^k} = \frac{1}{6} \sum_a : J_a \cdot J_a : . \tag{B15}
\end{equation}

The diagonal subalgebra generated by \( J = \sum_a J_a \) forms a \( SU(2)_k \) current algebra. The corresponding energy momentum tensor will be proportional to \( J \cdot J \) (we now suppress the normal ordering symbols, for brevity). The coefficient can be deduced by using (B14) to compute

\begin{equation}
J \cdot J = -(k + 2)(\partial_\tau \tilde{\phi}_a)^2 - 2(\partial_\tau \tilde{\phi}_a)^2 + 2 \sum_{a \neq b} e^{i\pi(v \tau + ix)}(\partial_\tau \tilde{\phi}_a - \partial_\tau \tilde{\phi}_b) \tag{B16}
\end{equation}

If we require that the \( (\partial_\tau \tilde{\phi}_a)^2 \) terms in (B13) and (B16) coincide, then an expression similar to (B4) shows that \( J^z = i\sqrt{2} \partial_\tau \tilde{\phi}_a \) has the appropriate scaling dimension \( \Delta = 1 \). We then recover the Sugawara energy momentum tensor,

\begin{equation}
T_{SU(2)_k} = \frac{1}{2(k + 2)} : J \cdot J : . \tag{B17}
\end{equation}

It follows that we may express \( T_{[SU(2)]^k} \) in (B13) as \( T_{SU(2)_k} + T_{[SU(2)]^k} / SU(2)_k \), where \( T_{SU(2)_k} \) is given in (B17), and \( T_{[SU(2)]^k} / SU(2)_k \) is the remainder. Moreover, \( T_{SU(2)_k} \) may be further decomposed into \( T_{1(1)} + T_{SU(2)_k} / U(1) \), where the \( U(1) \) term, generated by \( J^z \), is simply is the \( (\partial_\tau \tilde{\phi}_a)^2 \) term in (B16), and \( T_{SU(2)_k} / U(1) \) is the rest. This leads to the final results quoted in (4.27 4.28 4.29).

Appendix C: Operator Product Expansions

In this appendix we provide the details of the calculations for Eqs. (4.30 - 4.32) that show that the operators \( \Psi \) and \( \Phi \) at the end of Section IV.A are primary fields of the \( SU(2)_k / U(1) \) parafermion sector and the \( [SU(2)]^k / SU(2)_k \) sectors respectively. The key is to compute the singular terms in the operator product expansion of the energy momentum tensors for the coset sectors (given in Eqs. 4.27 4.28 4.29 and discussed in the previous Appendix) with these operators.

The necessary terms involve two kinds of products, which it is useful to consider separately. First, using Wick’s theorem and (B2), the OPE of the quadratic terms in \( T \) with an exponential operator takes the form

\begin{equation}
-\frac{1}{2} (\partial_\tau \tilde{\phi}_a(z))^2 e^{i\bar{D}_a \tilde{\phi}_a(w)}
\end{equation}

\begin{equation}
= \left( |\bar{D}_a^2 / (z - w)^2 | + i \bar{D}_a \tilde{\phi}_a(z) / (z - w) \right) e^{i\bar{D}_a \tilde{\phi}_a(w)} \tag{C1}
\end{equation}

For brevity we have suppressed the normal ordering symbols. This shows that the operator \( e^{i\bar{D}_a \tilde{\phi}_a} \) is a primary field of the \( [SU(2)]^k \) theory (described by (B13))
with scaling dimension $\Delta = |\vec{D}|^2/2$. In particular, using (4.11), this shows that $\Delta_{\Psi} = |d_a|^2 = 1 - 1/k$ and $\Delta_{\Xi} = \Delta_{\Xi} = |d_a - d_b|^2/4 = 1/2$.

Using the Baker Hausdorf formula and (B2), we may show that

$$e^{i\vec{D}_1 \cdot \vec{\phi}_a(z) - i\vec{D}_2 \cdot \vec{\phi}_b(w)} = \frac{1}{(z-w)^{\frac{1}{2}}} e^{i(\vec{D}_1 \cdot \vec{\phi}_a(z) + \vec{D}_2 \cdot \vec{\phi}_b(w))} \quad \text{(C2)}$$

The OPE’s for $\Psi$ involve (C1) with $\vec{D} = \sqrt{2\vec{d}_a}$, along with (C2) with $\vec{D}_1 = \sqrt{2(d_a - d_b)}$ and $\vec{D}_2 = \sqrt{2\vec{d}_c}$. In this case, $\vec{D}_1 \cdot \vec{D}_2 = 2(\delta_{ac} - \delta_{bc})$, so there are singular terms in the OPE only for $c = b \neq a$. Using the fact (from Eq. 4.9) that $\sum_{b \neq a} d_a - d_b = k\vec{d}_a$, we then find

$$\sum_{a \neq b} e^{i\sqrt{2}(d_a - d_b) \cdot \vec{\phi}_a(z)} \sum_c e^{i\sqrt{2}\vec{d}_c \cdot \vec{\phi}_c(w)} = \sum_{a \neq b} \left( \frac{1}{(z-w)^2} + \frac{i\sqrt{2}(d_a - d_b) \cdot \partial_w \vec{\phi}_a}{z-w} \right) e^{i\sqrt{2}\vec{d}_a \cdot \vec{\phi}_a}$$

Combining (C1) (with $|\vec{D}|^2/2 = 1 - 1/k$) and (C3), leads immediately to the results (4.30) and (4.31) quoted in Section IV.B. That $T_{U(1)} \Psi = 0$ is obvious because $\Psi$ does not depend on $\vec{\phi}_a$.

Thus, we have established that $\Psi$ is a primary field of the $SU(2)_k/U(1)$ sector. $\Psi$ are a bosonized representation for $Z_k$ parafermions operators. They can be combined to define more general operators,

$$\Psi_\ell = A_{k,l} \sum_{a_1 < \ldots < a_l} e^{i\sqrt{2}\sum_{i=1}^l d_{a_i} \cdot \vec{\phi}_a} \quad \text{(C4)}$$

For an appropriate normalization $A_{k,l} = \sqrt{(l-1)!/k!}$, these can be shown using (C2) to satisfy the OPE’s for $Z_k$ parafermions discovered by Zamolodchikov and Fateev.

The OPE’s for $\Psi$ and $\Xi$ involve (C1) with $\vec{D} = (d_a - d_b)/\sqrt{2}$ and $|\vec{D}|^2 = 1$, along with (C2) with $\vec{D}_1 = \sqrt{2}(d_a - d_b)$ and $\vec{D}_2 = (d_c - d_d)/\sqrt{2}$. It follows that $\vec{D}_1 \cdot \vec{D}_2 = \delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}$. This leads to a $1/(z-w)^2$ singularity for $a = d$ and $b = c$. In addition, there is a $1/(z-w)$ singularity for $a = d = b \neq c$ or $b = c = a \neq d$. After an analysis similar to (C3) it follows that

$$\sum_{a \neq b, c \neq d} e^{i\sqrt{2}(d_a - d_b) \cdot \vec{\phi}_a(z)} - \sum_{a \neq b} e^{i\sqrt{2}(d_a - d_b) \cdot \vec{\phi}_a(w)/\sqrt{2}}$$

Combining (C1) (with $|\vec{D}|^2/2 = 1/2$) then leads to (4.32) and (4.33). Again, the $U(1)$ term vanishes because $\vec{\phi}$ is independent of $\vec{\phi}_a$. Thus, $\Xi$ is a primary field of the $[SU(2)_k]/SU(2)_k$ sector.

For $\Xi$, the last term does not cancel. The OPE's for both $T_{SU(2)_k/U(1)}$ and $T_{[SU(2)_k]}$ are both non zero and do not have the form of a primary field. Presumably, $\Xi$ can be written as a sum of terms that are products of primary fields in the $SU(2)_k/U(1)$ and $[SU(2)_k]/SU(2)_k$ sectors.

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1. C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
2. A. Kitaev, Ann. Phys. 303, 2 (2003).
3. G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
4. N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
5. M. Greiter, X.G. Wen, and F. Wilczek, Nucl. Phys. B 374, 567 (1992).
6. M. Dolev, M. Heiblum, V. Umansky, A. Stern, and D. Mahalu, Nature 452, 829 (2008).
7. P. Radu, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, Science 320, 899 (2008).
8. R.L. Willett, L. N. Pfeiffer, and K. W. West, Proc. Natl. Acad. Sci. 106, 8853 (2009).
9. M.H. Freedman, M. Larsen and Z. Wang, Commun. Math. Phys. 227, 605 (2002).
10. N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
11. J. K. Slingerland and F. A. Bais, Nucl. Phys. B 612, 229 (2001).
12. S. Trebst, M. Troyer, Z. Wang and A.W.W. Ludwig, Prog. Theor. Phys. Suppl. 176, 384 (2008).
13. L. Hormoni, G. Zikos, N. E. Bonesteel and S. H. Simon, Phys. Rev. B 75, 165310 (2007).
14. F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).
15. M.Z. Hasan and C.L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
16. X.L. Qi and S.C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
17. T. Neupert, L. Santos, C. Chamoun, and C. Mudry, Phys. Rev. Lett. 106, 236804 (2011).
18. D. N. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nat Commun 2, 389 (2011).
19. X. L. Qi, Phys. Rev. Lett. 107, 126803 (2011).
20. N. Regnault and B. A. Bernevig, ArXiv: 1105.4867 (2011).
21. M. Levin and A. Stern, Phys. Rev. Lett. 103, 196803 (2009).
J. Maciejko, X.L. Qi, A. Karch and S.C. Zhang, Phys. Rev. Lett. 105, 246809 (2010).

B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B 83, 195139 (2011).

M. Levin, F. J. Burnell, M. Koch-Janusz, A. Stern, arXiv:1108.4954 (2011).

E. Kapit, P. Ginsparg, E. Mueller arXiv:1109.4561 (2011).

M. Levin, F. J. Burnell, M. Koch-Janusz, A. Stern, arXiv:1108.4954 (2011).

B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B 83, 195139 (2011).

R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).

S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. 58, 1252 (1987).

S. C. Zhang, T. H. Hansson and S. A. Kivelson, Phys. Rev. Lett. 62, 82 (1989).

N. Read, Phys. Rev. Lett. 65, 1502 (1990).

A. Lopez and E. Fradkin, Phys. Rev. B 44, 5246 (1991).

X. G. Wen and A. Zee, Phys. Rev. B 46, 2290 (1992).

X. G. Wen, Phys. Rev. B 60, 8827 (1999).

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).

J. T. Chalker and P. D. Coddington, J. Phys. C 21, 2665 (1988).

C.L. Kane, R. Mukhopadhyay, and T.C. Lubensky, Phys. Rev. Lett. 88, 36401 (2002).

S. Dong, E. Fradkin, R. G. Leigh and S. Nowling, J. High Energy Phys. 05, 016 (2008).

H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2008).

X. Li, Qi, H. Katsura and A.W.W. Ludwig, arXiv:1103.5437 (2011).

C. Gils, E. Ardonne, S. Trebst, A. W. W. Ludwig, M. Troyer and Z. Wang, Phys. Rev. Lett. 103, 070401 (2009).

I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. 59, 799 (1987).

E. Fradkin, C. Nayak and K. Schoutens, Nucl. Phys. B 546, 711 (1999).

E. Fradkin, C. Nayak, A. Tsvelik and F. Wilczek, Nucl. Phys. B 516, 704 (1998).

C. S. O’Hern, T. C. Lubensky and J. Toner, Phys. Rev. Lett. 83, 2745 (1999).

V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, Phys. Rev. Lett. 85, 2160 (2000).

A. Vishwanath and D. Carpentier, Phys. Rev. Lett. 86, 676 (2001).

S.L. Sondhi and K. Yang, Phys. Rev. B 63, 054430 (2001).

R. Mukhopadhyay, C. L. Kane, and T.C. Lubensky, Phys. Rev. B 63, 081103(R) (2001).

W.P. Su, J.R. Schrieffer and A.J. Heeger, Phys. Rev. B 22, 2099 (1980).

R. Tao and D. J. Thouless, Phys. Rev. B 28, 1142 (1983).

X. G. Wen, Phys Rev. B 43, 11025 (1991); Phys. Rev. Lett. 64, 2206 (1990).

F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983).

B.I. Halperin, Phys. Rev. Lett. 52, 1583 (1984).

P. Fendley, M. P. A. Fisher, and C. Nayak, Phys. Rev. B 75, 045317 (2007).

P. Goddard, A. Kent and D. Olive, Phys. Lett. 152B, 88 (1985).

For reviews of conformal field theory, see P. Ginsparg, in *Fields, Strings and Critical Phenomena*, (Les Houches, Session XLIX, 1988) ed. by E. Brezin and J. Zinn Justin, 1989; P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Springer-Verlag, New York, 1997.

A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP 62, 215 (1985).

I. Affleck, M. Oshikawa and H. Saleur, Nucl. Phys. B 594, 535 (2001).

J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B 16, 1217 (1977).

P. Bonderson and J. K. Slingerland, Phys. Rev. B 78, 125323 (2008).

E. Ardonne and K. Schoutens, Phys. Rev. Lett. 82, 5096 (1999).

A. Kitaev, Ann. Phys. 321, 2 (2006).

P. Griffin and D. Nemeschansky, Nucl. Phys. B 323, 545 (1989).