RIGIDITY OF SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE IN SPACE FORMS*

HONG-WEI XU AND JUAN-RU GU

Abstract

Let $M$ be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in the simply connected space form $F^{n+p}(c)$ with $c + H^2 > 0$, where $H$ is the mean curvature of $M$. We prove that if the Ricci curvature of $M$ satisfies $\text{Ric}_M \geq (n-2)(c + H^2)$, then $M$ is either a totally umbilic sphere, the Clifford hypersurface $S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ in $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$, or $\mathbb{C}P^2(\frac{1}{3}(c + H^2))$ in $S^7(\frac{1}{\sqrt{c+H^2}})$. In particular, if $\text{Ric}_M > (n-2)(c + H^2)$, then $M$ is a totally umbilic sphere.

1 Introduction

The investigation of rigidity of submanifolds with parallel mean curvature attracts a lot of attention of differential geometers. After the pioneering work on compact minimal submanifolds in a sphere due to Simons [19], Lawson [10] and Chern-do Carmo-Kobayashi [3] obtained a classification of $n$-dimensional oriented compact minimal submanifolds in $S^{n+p}$ whose squared norm of the second fundamental form satisfies $S \leq C(n, p, H)$. It was partially extended to submanifolds with parallel mean curvature in a sphere by Okumura [15, 16], Yau [26] and others. In 1990, Xu [22] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

**Theorem A.** Let $M$ be an $n$-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere $S^{n+p}$. Denote by $S$ and $H$ the squared norm of the second fundamental form and the mean curvature of $M$. If $S \leq C(n, p, H)$, then $M$ is either a totally umbilic sphere, a Clifford hypersurface in $S^{n+1}(r)$, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C(n, p, H)$ is defined by

$$C(n, p, H) = \begin{cases} 
\alpha(n, H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\
\frac{n}{2-p}, & \text{for } p \geq 2 \text{ and } H = 0, \\
\min \left\{ \alpha(n, H), \frac{n-pH^2}{2-p} + nH^2 \right\}, & \text{for } p \geq 3 \text{ and } H \neq 0,
\end{cases}$$

*2010 Mathematics Subject Classification. 53C24; 53C40; 53C42.
Keywords: Submanifolds, rigidity theorem, Ricci curvature, mean curvature, second fundamental form.
Research supported by the NSFC, Grant No. 11071211; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China.
where
\[
\alpha(n, H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.
\]

In 1991, A. M. Li and J. M. Li [12] improved Simons’ pinching constant for \(n\)-dimensional compact minimal submanifolds in \(S^{n+p}\) to \(\max\{\frac{n}{2}, \frac{n}{2p} + \frac{2}{3}\}\}. Using Li-Li’s matrix inequality [12], Xu [23] improved the pinching constant \(C(n, p, H)\) in Theorem A to
\[
C'(n, p, H) = \begin{cases} 
\alpha(n, H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\
\min\{\alpha(n, H), \frac{n}{2}(2n + 5nH^2)\}, & \text{otherwise}.
\end{cases}
\]

The rigidity theorem for compact minimal submanifolds with pinched sectional curvature in a sphere was initiated by Yau [26], then by Itoh [9], and finally by Gu and Xu [5]. It was extended to compact submanifolds with parallel mean curvature in space forms by Shen, Han and the authors [5, 17, 24].

In 1979, Ejiri [4] obtained the following rigidity theorem for \(n\geq 4\)-dimensional oriented compact simply connected minimal submanifolds with pinched Ricci curvatures in a sphere.

**Theorem B.** Let \(M\) be an \((n\geq 4)\)-dimensional oriented compact simply connected minimal submanifold in an \((n+p)\)-dimensional unit sphere \(S^{n+p}\). If the Ricci curvature of \(M\) satisfies \(\text{Ric}_M \geq n - 2\), then \(M\) is either the totally geodesic submanifold \(S^n\), the Clifford torus \(S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})\) in \(S^{n+1}\) with \(n = 2m\), or \(\mathbb{C}P^2(4/3)\) in \(S^7\). Here \(\mathbb{C}P^2(4/3)\) denotes the 2-dimensional complex projective space minimally immersed into \(S^7\) with constant holomorphic sectional curvature \(\frac{4}{3}\).

The pinching constant above is the best possible in even dimensional cases. It’s better than the pinching constants of Simons [19] and Li-Li [12] in the sense of the average of Ricci curvatures. The following problem seems very attractive, which has been open for many years.

**Problem A.** Is it possible to generalize Ejiri’s rigidity theorem for minimal submanifolds to the cases of submanifolds with parallel mean curvature in a sphere?

In 1987, Sun [20] gave a partial answer to the problem above and showed that if \(M\) is an \((n\geq 4)\)-dimensional compact oriented submanifold with parallel mean curvature in \(S^{n+p}\) and its Ricci curvature is not less than \(\frac{n(n-2)}{n+1}(1 + H^2)\), then \(M\) is a totally umbilic sphere. Afterward, Shen [18] and Li [13] extended Ejiri’s rigidity theorem to the case of \(3\)-dimensional compact minimal submanifolds in a sphere.

Let \(F^{n+p}(c)\) be an \((n+p)\)-dimensional simply connected space form with constant curvature \(c\). Recently, Xu and Tian [25] proved a vanishing theorem for the fundamental group of a submanifold, which says that if \(M\) is an \((n\geq 3)\)-dimensional compact submanifold in the space form \(F^{n+p}(c)\) with \(c \geq 0\), and if the Ricci curvature of \(M\) satisfies \(\text{Ric}_M \geq \frac{n-1}{n}c + \frac{n^2H^2}{8}\), then \(M\) is simply connected. This implies that the assumption that \(M\) is simply connected in the Ejiri rigidity theorem can be taken off. Therefore, we have the following refined version of the Ejiri rigidity theorem.
**Theorem C.** Let $M$ be an $n(\geq 3)$-dimensional oriented compact minimal submanifold in an $(n+p)$-dimensional unit sphere $S^{n+p}$. If the Ricci curvature of $M$ satisfies $\text{Ric}_M \geq n-2$, then $M$ is either the totally geodesic submanifold $S^n$, the Clifford torus $S^m(\frac{1}{\sqrt{\frac{1}{2}c + H^2}}) \times S^m(\frac{1}{\sqrt{\frac{1}{2}c + H^2}})$ in $S^{n+1}$ with $n = 2m$, or $\mathbb{C}P^2(4/3)$ in $S^7$. Here $\mathbb{C}P^2(4/3)$ denotes the 2-dimensional complex projective space minimally immersed into $S^7$ with constant holomorphic sectional curvature $\frac{4}{3}$. 

Motivated by Theorem C, we would like to propose the following problem.

**Problem B.** Is it possible to generalize the refined version of the Ejiri rigidity theorem for minimal submanifolds to the cases of submanifolds with parallel mean curvature in space forms?

The purpose of the present paper is to give affirmative answers to Problems A and B. More precisely, we will prove the following rigidity theorem for submanifolds with parallel mean curvature in space forms.

**Main Theorem.** Let $M$ be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in the space form $F^{n+p}(c)$ with $c + H^2 > 0$. If

$$\text{Ric}_M \geq (n-2)(c + H^2),$$

then $M$ is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{\frac{1}{2}c + H^2}})$, a Clifford hypersurface $S^m(\frac{1}{\sqrt{\frac{1}{2}c + H^2}}) \times S^m(\frac{1}{\sqrt{2(c + H^2)}})$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{\frac{1}{2}c + H^2}})$ with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3}(c + H^2))$ in $S^7(\frac{1}{\sqrt{\frac{1}{2}c + H^2}})$. Here $\mathbb{C}P^2(\frac{4}{3}(c + H^2))$ denotes the 2-dimensional complex projective space minimally immersed into $S^7(\frac{1}{\sqrt{\frac{1}{2}c + H^2}})$ with constant holomorphic sectional curvature $\frac{4}{3}(c + H^2)$.

### 2 Notation and lemmas

Throughout this paper, let $M^n$ be an $n$-dimensional compact Riemannian manifold isometrically immersed in an $(n + p)$-dimensional complete and simply connected space form $F^{n+p}(c)$ with constant curvature $c$. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n + p; \quad 1 \leq i, j, k, \ldots \leq n; \quad n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p.$$

Choose a local field of orthonormal frames $\{e_A\}$ in $F^{n+p}(c)$ such that, restricted to $M$, the $e_i$'s are tangent to $M$. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection
1-forms of $F^{n+p}(c)$ respectively. Restricting these forms to $M$, we have
\[
\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha,
\]
\[
h = \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha, i} h_{ii}^\alpha e_\alpha,
\]
\[
R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \sum_{\alpha} (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),
\]
\[
R_{\alpha \beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta),
\]
where $h$, $\xi$, $R_{ijkl}$ and $R_{\alpha \beta kl}$ are the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of $M$. Denote by $Ric(u)$ the Ricci curvature of $M$ in direction of $u \in UM$. From the Gauss equation, we have
\[
Ric(e_i) = (n - 1)c + \sum_{\alpha} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2].
\]
For a matrix $A = (a_{ij})$, we denote by $N(A)$ the square of the norm of $A$, i.e.,
\[
N(A) = tr(AA^T) = \sum a_{ij}^2.
\]
We define
\[
S = |h|^2, \quad H = |\xi|, \quad H_\alpha = (h_{ij}^\alpha)_{n \times n}.
\]
Then the scalar curvature $R$ of $M$ is given by
\[
R = n(n-1)c + n^2H^2 - S.
\]
Let $M$ be a submanifold with parallel mean curvature vector $\xi$. Choose $e_{n+1}$ such that it is parallel to $\xi$, and
\[
trH_{n+1} = nH, \quad trH_\alpha = 0, \quad \text{for } \alpha \neq n+1.
\]
Set
\[
S_H = trH^2_{n+1}, \quad S_I = \sum_{\alpha \neq n+1} trH^2_\alpha.
\]
The following lemma will be used in the proof of our results.

**Lemma 1 (26).** If $M^n$ is a submanifold with parallel mean curvature in $F^{n+p}(c)$, then either $H \equiv 0$, or $H$ is non-zero constant and $H_{n+1}H_\alpha = H_\alpha H_{n+1}$ for all $\alpha$.

We denote the first and the second covariant derivatives of $h_{ij}^\alpha$ by $h_{ijk}^\alpha$ and $h_{ijkl}^\alpha$ respectively. The Laplacian $\Delta h_{ij}^\alpha$ of $h_{ij}^\alpha$ is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ij}^{\alpha kk}$. Following [20], we have
\[
\Delta h_{ij}^{n+1} = \sum_{k, m} (h_{mk}^{n+1} R_{mijk} + h_{im}^{n+1} R_{mkj}).
\]
3 Proof of Main Theorem

To verify Main Theorem, we need to prove the following theorem.

**Theorem 1.** Let $M$ be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature ($H \neq 0$) in the space form $F^{n+p}(c)$. If

$$Ric_M \geq (n-2)(c+H^2),$$

where $c+H^2 > 0$, then $M$ is pseudo-umbilical.

**Proof.** By the Gauss equation (1) and (6), we have

$$\frac{1}{2} \Delta S_H = \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1}$$

$$= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \sum_{i,j,k,m} h_{ij}^{n+1} h_{km}^{n+1} \left[ (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij})c + \sum_{\alpha} (h_{mj}^{\alpha} h_{ik}^{\alpha} - h_{mk}^{\alpha} h_{ij}^{\alpha}) \right]$$

$$+ \sum_{i,j,k,m} h_{ij}^{n+1} h_{km}^{n+1} \left[ (\delta_{mj} \delta_{kk} - \delta_{mk} \delta_{jk})c + \sum_{\alpha} (h_{mj}^{\alpha} h_{kk}^{\alpha} - h_{mk}^{\alpha} h_{jk}^{\alpha}) \right]$$

$$= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + nc \sum_{i,j} (h_{ij}^{n+1})^2 - \left[ \sum_{i,j} (h_{ij}^{n+1})^2 \right]^2 - n^2 cH^2$$

$$+ nH \sum_{i,j,k} h_{ij}^{n+1} h_{jk}^{n+1} h_{ki}^{n+1} - \sum_{\alpha \neq n+1} \left[ \sum_{i,j} (h_{ij}^{n+1} - H \delta_{ij}) h_{ij}^{\alpha} \right]^2. \quad (7)$$

Let $\{e_i\}$ be a frame diagonalizing the matrix $H_{n+1}$ such that $h_{ij}^{n+1} = \lambda_i^{n+1} \delta_{ij}$, for all $i, j$. Set

$$f_k = \sum_i (\lambda_i^{n+1})^k,$$

$$\mu_i^{n+1} = H - \lambda_i^{n+1}, \quad i = 1, 2, ..., n,$$

$$B_k = \sum_i (\mu_i^{n+1})^k.$$

Then

$$B_1 = 0, \quad B_2 = S_H - nH^2,$$

$$B_3 = 3HS_H - 2nH^3 - f_k.$$

This together with (7) implies that

$$\frac{1}{2} \Delta S_H = \sum_{i,j,k} (h_{ijk}^{n+1})^2 + nc S_H - S_H^2 - n^2 cH^2 + nH f_3 - \sum_{\alpha \neq n+1} \left( \sum_i \mu_i^{n+1} h_{ii}^{\alpha} \right)^2$$

$$= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + nc S_H - S_H^2 - n^2 cH^2$$

$$+ nH (3HS_H - 2nH^3 - B_3) - \sum_{\alpha \neq n+1} \left( \sum_i \mu_i^{n+1} h_{ii}^{\alpha} \right)^2$$

$$= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + B_2 [nc + 2nH^2 - S_H] - nHB_3 - \sum_{\alpha \neq n+1} \left( \sum_i \mu_i^{n+1} h_{ii}^{\alpha} \right)^2. \quad (8)$$
Let $d$ be the infimum of the Ricci curvature of $M$. Then we have
\[
Ric(e_i) = (n - 1)c + nH\lambda_i^{n+1} - (\lambda_i^{n+1})^2 - \sum_{\alpha \neq n+1,j} (h_{ij}^\alpha)^2 \geq d. \tag{9}
\]

This implies that
\[
S - nH^2 \leq n[(n - 1)(c + H^2) - d], \tag{10}
\]
and
\[
(n - 2)H(\lambda_i^{n+1} - H) - (\lambda_i^{n+1} - H)^2 + (n - 1)(c + H^2) - \sum_{\alpha \neq n+1,j} (h_{ij}^\alpha)^2 - d \geq 0. \tag{11}
\]

It follows from (11) that
\[
H(\lambda_i^{n+1} - H) \geq \frac{(\lambda_i^{n+1} - H)^2}{n - 2} + \sum_{\alpha \neq n+1,j} (h_{ij}^\alpha)^2 + \frac{d}{n - 2} - \frac{n - 1}{n - 2}(c + H^2).
\]

So,
\[
-nHB_3 \geq \frac{n}{n - 2} \sum_i (\mu_i^{n+1})^4 + \frac{n}{n - 2} \sum_{\alpha \neq n+1} \sum_{i,j} (h_{ij}^\alpha)^2 (\mu_i^{n+1})^2 + \frac{n}{n - 2} [d - (n - 1)(c + H^2)] |B_2|. \tag{12}
\]

From (8) and (12), we get
\[
\frac{1}{2} \Delta S_H \geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + B_2 \left\{ nc + 2nH^2 - S_H + \frac{n}{n - 2} [d - (n - 1)(c + H^2)] \right\}
\]
\[
+ \frac{n}{n - 2} \sum_i (\mu_i^{n+1})^4 + \sum_{\alpha \neq n+1} \left[ \frac{n}{n - 2} \sum_i (h_{ii}^\alpha)^2 (\mu_i^{n+1})^2 - \left( \sum_i h_{ii}^{n+1} \mu_i^\alpha \right)^2 \right]
\]
\[
\geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + B_2 \left\{ nc + 2nH^2 - S_H + \frac{n}{n - 2} [d - (n - 1)(c + H^2)] \right\}
\]
\[
+ \frac{B_2^2}{n - 2} - \frac{n - 3}{n - 2} \sum_{\alpha \neq n+1} \left( \sum_i h_{ii}^{n+1} \mu_i^\alpha \right)^2
\]
\[
\geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + B_2 \left\{ nc + nH^2 - \frac{n - 3}{n - 2}(S - nH^2) \right\}
\]
\[
+ \frac{n}{n - 2} [d - (n - 1)(c + H^2)]. \tag{13}
\]

This together with (10) implies that
\[
\frac{1}{2} \Delta S_H \geq \sum_{i,j,k} (h_{ijk}^{n+1})^2 + \frac{n}{n - 2} B_2 \left\{ (n - 2)(c + H^2) \right\}
\]
\[
-(n - 3)[(n - 1)(c + H^2) - d] + [d - (n - 1)(c + H^2)]
\]
\[
= \sum_{i,j,k} (h_{ijk}^{n+1})^2 + nB_2 [d - (n - 2)(c + H^2)]. \tag{14}
\]
By the assumption, we have \( d \geq (n - 2)(c + H^2) \). This together with (14) and the maximum principal implies that \( S_H \) is a constant, and
\[
(S_H - nH^2)[d - (n - 2)(c + H^2)] = 0. \tag{15}
\]
Suppose that \( S_H \neq nH^2 \). Then \( d = (n - 2)(c + H^2) \). We consider the following two cases:

(i) If \( n = 3 \), then the inequalities in (13) and (14) become equalities. Thus, we have
\[
h_{ij}^\alpha = 0, \text{ for } \alpha \neq n + 1, \ i \neq j,
\]
\[
|\mu_i^{n+1}| = |\mu_j^{n+1}|, \ \mu_i^{n+1} = h_{ii}^\alpha, \text{ for } \alpha \neq n + 1, \ 1 \leq i, j \leq n. \tag{16}
\]
This implies \( \mu_i^{n+1} = 0, \ i = 1, 2, \cdots, n \). It follows from Gauss equation that \( c + H^2 = 0 \). This contradicts with assumption.

(ii) If \( n \geq 4 \), then the inequalities in (13) and (14) become equalities and we have
\[
\text{Ric}_M \equiv (n - 2)(c + H^2),
\]
\[
h_{ij}^\alpha = 0, \text{ for } \alpha \neq n + 1, \ i \neq j,
\]
\[
|\mu_i^{n+1}| = |\mu_j^{n+1}|, \ \mu_i^{n+1} = h_{ii}^\alpha, \text{ for } \alpha \neq n + 1, \ 1 \leq i, j \leq n. \tag{17}
\]
It follows from Gauss equation that \( \mu_i^{n+1} = 0 \) and \( c + H^2 = 0 \). This contradicts with assumption.

Therefore, \( S_H = nH^2 \), i.e., \( M \) is a pseudo-umbilical submanifold. This completes the proof of Theorem 1.

The following result due to Yau \[26\] will be used in the proof of our main theorem.

**Theorem 2.** Let \( N^{n+p} \) be a conformally flat manifold. Let \( N_1 \) be a subbundle of the normal bundle of \( M^n \) with fiber dimension \( k \). Suppose \( M \) is umbilical with respect to \( N_1 \) and \( N_1 \) is parallel in the normal bundle. Then \( M \) lies in an \((n+p-k)\)-dimensional umbilical submanifold \( N' \) of \( N \) such that the fiber of \( N_1 \) is everywhere perpendicular to \( N' \).

**Proof of Main Theorem.** When \( H = 0 \), the assertion follows from Theorems C.

When \( p = 1 \) and \( H \neq 0 \), we get the conclusion from Theorem 1.

Now we assume that \( p \geq 2 \) and \( H \neq 0 \). It follows from the assumption and Theorem 1 that \( M \) is pseudo-umbilical. It is seen from Theorem 2 that \( M \) lies in an \((n+p-1)\)-dimensional totally umbilical submanifold \( F^{n+p-1}(\tilde{c}) \) of \( F^{n+p}(c) \), i.e., the isometric immersion from \( M \) into \( F^{n+p}(c) \) is given by
\[
i \circ \varphi : M \to F^{n+p-1}(\tilde{c}) \to F^{n+p}(c),
\]
where \( \varphi : M^n \to F^{n+p-1}(\tilde{c}) \) is a isometric immersion with mean curvature vector \( \xi_1 \), and \( i : F^{n+p-1}(\tilde{c}) \to F^{n+p}(c) \) is the totally umbilic submanifold with mean curvature vector \( \xi_2 \). Denote by \( h_2 \) the second fundamental form of isometric immersion \( i \). Set
\[
H_1 = |\xi_1|, \ H_2 = |\xi_2|. \tag{18}
\]
We know that $\xi = \xi_1 + \eta$, where $\eta = \frac{1}{n} \sum_i h_2(e_i, e_i)$ and $\{e_i\}$ is a local orthonormal frame field in $M$. Since $\xi_1 \perp \xi$, and $\eta \parallel \xi$, we obtain $\xi_1 = 0$, and $\eta = \xi$. Noting that $F^{n+p-1}(\tilde{c})$ is a totally umbilic submanifold in $F^{n+p}(c)$, we have $|\eta| = H_2$. Thus,

$$H^2 = H_1^2 + |\eta|^2 = H_2^2. \quad (19)$$

This together with the Gauss equation implies that

$$\tilde{c} = c + H^2. \quad (20)$$

Hence, $M$ is an oriented compact minimal submanifold in $S^{n+p-1}(\frac{1}{\sqrt{c+H^2}})$. It follows from Theorem C that $M$ is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, a Clifford hypersurface $S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$, or $CP^2(\frac{4}{3}(c + H^2))$ in $S^n(\frac{1}{\sqrt{c+H^2}})$. This completes the proof of Main Theorem.

As a consequence of Main Theorem, we get the following

**Corollary 1.** Let $M^n$ be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in the space form $F^{n+p}(c)$ with $c + H^2 > 0$. If

$$Ric_M > (n-2)(c + H^2),$$

then $M$ is the totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$.

Motivated our main theorem, we would like to propose the following differentiable rigidity theorem for submanifolds in space forms.

**Conjecture A.** Let $M$ be an $n(\geq 3)$-dimensional compact oriented submanifold in the space form $F^{n+p}(c)$ with $c + H^2 > 0$. If

$$Ric_M \geq (n-2)(c + H^2),$$

then $M$ is diffeomorphic to either the standard $n$-sphere $S^n$, the Clifford hypersurface $S^m(\frac{1}{\sqrt{2}}) \times S^m(\frac{1}{\sqrt{2}})$ in $S^{n+1}$ with $n = 2m$, or $CP^2$. In particular, if $Ric_M > (n-2)(c + H^2)$, then $M$ is diffeomorphic to $S^n$.

To get an affirmative answer to Conjecture A, we hope to prove the following conjecture on the Ricci flow.

**Conjecture B.** Let $(M, g_0)$ be an $n(\geq 4)$-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$ with $c + H^2 > 0$. If the Ricci curvature of $M$ satisfies

$$Ric_M > (n-2)(c + H^2),$$

then the normalized Ricci flow with initial metric $g_0$

$$\frac{\partial}{\partial t} g(t) = -2Ric_g(t) + \frac{2}{n} \text{tr}_g(t) g(t),$$

satisfies the above conditions.
exists for all time and converges to a constant curvature metric as \( t \to \infty \). Moreover, \( M \) is diffeomorphic to a spherical space form. In particular, if \( M \) is simply connected, then \( M \) is diffeomorphic to \( S^n \).

Making use of the convergence results of Hamilton [6] and Brendle [2] for Ricci flow and the nonexistence theorem of stable currents due to Lawson-Simons [11] and Xin [21], Xu and Tian [25] gave the partial affirmative answers to Conjectures A and B.

Recently, Andrews and Baker [1], Liu, Xu, Ye and Zhao [14] obtained the convergence theorems for the mean curvature flow of higher codimension in Euclidean spaces. Motivated by our main theorem and the conjectures in [5, 14], we would like to propose the following conjecture on the mean curvature flow in higher codimensions.

**Conjecture C.** Let \( F_0 : M \to F^{n+p}(c) \) be an \( n \)-dimensional compact submanifold in an \( (n+p) \)-dimensional space form \( F^{n+p}(c) \) with \( c + H^2 > 0 \). If the Ricci curvature of \( M \) satisfies

\[
Ric_M > (n-2)(c + H^2),
\]

then the mean curvature flow

\[
\begin{aligned}
\frac{\partial}{\partial t} F(x, t) &= n\xi(x, t), \ x \in M, \ t \geq 0, \\
F(\cdot, 0) &= F_0(\cdot),
\end{aligned}
\]

has a unique smooth solution \( F : M \times [0, T) \to F^{n+p}(c) \) on a finite maximal time interval, and \( F_t(\cdot) \) converges uniformly to a round point \( q \in F^{n+p}(c) \) as \( t \to T \). In particular, \( M \) is diffeomorphic to \( S^n \).

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