GEOMETRY OF QUANTUM PRINCIPAL BUNDLES II

Extended Version

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Abstract. A general noncommutative-geometric theory of principal bundles is presented. Quantum groups play the role of structure groups. General quantum spaces play the role of base manifolds. A differential calculus on quantum principal bundles is studied. In particular, algebras of horizontal and verticalized differential forms on the bundle are introduced and investigated. The formalism of connections is developed. Operators of horizontal projection, covariant derivative and curvature are constructed and analyzed. A quantum generalization of classical Weil’s theory of characteristic classes is sketched. Quantum analogs of infinitesimal gauge transformations are studied. Illustrative examples and constructions are presented.

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1. Introduction

In this study we continue the presentation of the theory of quantum principal bundles.

The theory developed in the previous paper [2] was “semiclassical”: structure groups were considered as quantum objects, however base spaces were classical smooth manifolds. Algebraic formalism developed in the previous paper will be now generalized and incorporated into a completely quantum framework, following general philosophy of non-commutative differential geometry [1]. Base manifolds, structure groups and corresponding principal bundles will be considered as quantum objects.

The paper is organized as follows.

Exposition of the theory begins in Section 3, with a general definition of quantum principal bundles. This definition will translate into a noncommutative-geometric context classical idea that a principal bundle is a space on which the structure group acts freely on the right, such that the base manifold is diffeomorphic to the corresponding orbit space.

After the main definition we pass to questions related to differential calculus on quantum principal bundles. As first, we introduce and analyze a differential *-algebra consisting of “verticalized” differential forms on the bundle. This algebra will be introduced independently of a specification of a complete differential calculus on the bundle.

The calculus on the bundle is based on a graded-differential *-algebra (representing differential forms) possessing two important properties. As first, we require that this differential algebra is generated by “functions” on the bundle. This condition ensures uniqueness of various entities naturally appearing in the study of differential calculus. Secondly, we postulate that the group action on “the functions” on the bundle is extendible to an appropriate differential algebra homomorphism (imitating the corresponding “pull back” of differential forms).

Quantum counterparts of various important entities associated to differential calculus in the classical theory will be introduced in a constructive manner, starting from the algebra of differential forms on the bundle (and from a given differential calculus on the structure quantum group). In particular, a graded *-algebra representing horizontal forms will be introduced and analyzed. Also, graded-differential *-algebras representing differential forms on the base manifold and “verticalized” differential forms on the bundle will be described.

It is important to point out a conceptual difference between this approach to differential calculus and the approach presented in the previous paper, where a differential calculus on the bundle was constructed starting from the standard calculus on the base manifold, and an appropriate calculus on the structure quantum group. The main property of the calculus was a variant of local triviality, in the sense that all local trivializations of the bundle locally trivialize the calculus, too. This property implies certain restrictions on a possible differential calculus on the structure quantum group, as discussed in details in [2]. On the other hand, in this paper we start from a fixed calculus on the group (based on the universal differential envelope of a given first-order differential structure) and the calculus on the base manifold is determined by the calculus on the bundle. However, the calculus on the bundle is not uniquely determined by mentioned initial two conditions.
In Section 4 the formalism of connections will be presented. All corresponding basic “global” constructions and results of the previous paper will be translated into the general quantum context. In particular, operators of horizontal projection, covariant derivative and curvature will be constructed and investigated. Further, two particularly interesting classes of connections will be introduced and analyzed. The first class consists of connections possessing certain multiplicativity property. This is a trivial generalization of multiplicative connections of the previous paper. The second class consists of connections that are counterparts of classical connections introduced in the previous paper. Here, these connections will be called regular. Intuitively speaking, regular connections are “maximally compatible” with the internal geometrical structure of the bundle.

In Section 5 a generalization of classical Weil’s theory of characteristic classes will be presented.

Finally, in Section 6 some examples, remarks and additional constructions are included. In particular, we shall present a general re-construction of differential calculus on the bundle, starting from a given algebra of horizontal forms, and two operators imitating the covariant derivative and the curvature of a regular connection. Further, quantum analogs of infinitesimal gauge transformations will be studied, from two different viewpoints.

We shall also briefly discuss interrelations with a theory of quantum principal bundles presented in [3].

Concerning concrete examples of quantum principal bundles, we shall consider trivial bundles and principal bundles based on quantum homogeneous spaces. The main structural elements of differential calculus and the formalism of connections will be illustrated on these examples.

The paper ends with two appendices. The first appendix is devoted to the analysis of the calculus on the bundle in the case when the higher-order calculus on the structure quantum group is described by the corresponding bicovariant exterior algebra [6]. In particular, it will be shown that if the first-order calculus on the group is compatible with all “transition functions” (in the context of the previous paper) then the higher-order calculus based on the exterior algebra possesses this property too. In fact this is equivalent to a possibility of constructing the calculus on the bundle such that all local trivializations of the bundle locally trivialize the calculus. Further, we shall prove that bicovariant exterior algebras describe, in a certain sense, the minimal higher-order calculus on the group such that the corresponding calculus on the bundle possesses the mentioned trivializability property (universal envelopes always describe the maximal higher-order calculus). We shall also analyze similar questions in the context of general theory.

In the second appendix the structure of the *-algebra representing “functions” on the bundle is analyzed, in the light of the decomposition of the right action of the structure quantum group into multiple irreducible components.

2. Preparatory Material

Before passing to quantum principal bundles we shall fix the notation, and introduce in the game relevant quantum group entities. We shall use the symbol $\hat{\otimes}$ for a graded tensor product of graded (differential *-) algebras.
Here, as in the previous paper, we shall deal with compact matrix quantum groups [5] only (however the compactness assumption is not essential for a large part of the formalism). Let \( G \) be such a group. The algebra of “polynomial functions” on \( G \) will be denoted by \( \mathcal{A} \). The group structure on \( G \) is determined by the comultiplication \( \phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \), the counit \( \epsilon: \mathcal{A} \rightarrow \mathbb{C} \) and the antipode \( \kappa: \mathcal{A} \rightarrow \mathcal{A} \).

The result of an \((n-1)\)-fold comultiplication of an element \( a \in \mathcal{A} \) will be symbolically written as \( a^{(1)} \otimes \cdots \otimes a^{(n)} \). The adjoint action of \( G \) on itself will be denoted by \( \text{ad}: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \). Explicitly, this map is given by

\[
\text{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}. \tag{2.1}
\]

Let \( \Gamma \) be a first-order differential calculus [6] over \( G \) and let \( \Gamma^\wedge = \sum_{k \geq 0} \Gamma^\wedge_k \) be the universal differential envelope ([2]–Appendix B) of \( \Gamma \). Here, the space \( \Gamma^\wedge_k \) consists of \( k \)-th order elements.

For each \( k \geq 0 \) let \( p_k: \Gamma^\wedge \rightarrow \Gamma^\wedge_k \) be the corresponding projection (we shall use the same symbols for projection operators associated to an arbitrary graded algebra built over \( \Gamma \)). Further, let

\[
\Gamma^\otimes = \sum_{k \geq 0} \Gamma^\otimes_k
\]

be the tensor bundle algebra over \( \Gamma \). Here,

\[
\Gamma^\otimes_k = \Gamma \otimes \mathcal{A} \cdots \mathcal{A} \otimes \Gamma
\]

is the tensor product over \( \mathcal{A} \) of \( k \)-copies of \( \Gamma \). The algebra \( \Gamma^\wedge \) can be obtained from \( \Gamma^\otimes \) by factorizing through the ideal \( S^\wedge \) generated by elements \( Q \in \Gamma^\otimes_2 \) of the form

\[
Q = \sum_i da_i \otimes_A db_i
\]

where \( a_i, b_i \in \mathcal{A} \) satisfy

\[
\sum_i a_i db_i = 0.
\]

Let us assume that \( \Gamma \) is left-covariant. Let \( \ell_\Gamma: \Gamma \rightarrow \mathcal{A} \otimes \Gamma \) be the corresponding left action of \( G \) on \( \Gamma \). We shall denote by \( \Gamma^\text{inv} \) the space of left-invariant elements of \( \Gamma \). In other words

\[
\Gamma^\text{inv} = \{ \vartheta \in \Gamma: \ell_\Gamma(\vartheta) = 1 \otimes \vartheta \}.
\]

Further, \( \mathcal{R} \subseteq \ker(\epsilon) \) will be the right \( \mathcal{A} \)-ideal which canonically, in the sense of [6], corresponds to \( \Gamma \). The map \( \pi: \mathcal{A} \rightarrow \Gamma^\text{inv} \) given by

\[
\pi(a) = \kappa(a^{(1)})da^{(2)} \tag{2.2}
\]

is surjective and \( \ker(\pi) = \mathbb{C} \oplus \mathcal{R} \). Because of this there exists a natural isomorphism

\[
\Gamma^\text{inv} \leftrightarrow \ker(\epsilon)/\mathcal{R}.
\]

The above isomorphism induces a right \( \mathcal{A} \)-module structure on \( \Gamma^\text{inv} \), which will be denoted by \( \circ \). Explicitly,

\[
\pi(a) \circ b = \pi(ab - \epsilon(a)b) \tag{2.3}
\]
for each $a, b \in \mathcal{A}$. The maps $\phi$ and $\ell_\tau$ admit common extensions to homomorphisms
\( \ell_\tau^\wedge : \Gamma^\wedge \to \mathcal{A} \otimes \Gamma^\wedge \) and $\ell_\tau : \Gamma^\otimes \to \mathcal{A} \otimes \Gamma^\otimes$ (left actions of $G$ on $\Gamma^\wedge$ and $\Gamma^\otimes$).

The tensor product of $k$-copies of $\Gamma_{\text{inv}}$ will be denoted by $\Gamma_{\text{inv}}^{\otimes k}$. The tensor algebra over $\Gamma_{\text{inv}}$ will be denoted by $\Gamma_{\text{inv}}^\otimes$. It is naturally isomorphic to the space of left-invariant elements of $\Gamma^\otimes$.

The subalgebra of left-invariant elements of $\Gamma^\wedge$ will be denoted by $\Gamma_{\text{inv}}^\wedge$. This algebra is naturally graded. We shall denote by $\Gamma_{\text{inv}}^{\wedge k}$ the space of left-invariant $k$-th order elements. Let $\pi_{\text{inv}} : \Gamma^\wedge \to \Gamma_{\text{inv}}^\wedge$ be the canonical projection map [6] onto left-invariant elements. In the framework of the canonical identification $\Gamma^\wedge \leftrightarrow \mathcal{A} \otimes \Gamma_{\text{inv}}^\wedge$ we have $\pi_{\text{inv}} \leftrightarrow \epsilon \otimes \text{id}$.

The following natural isomorphism holds
\[ \Gamma_{\text{inv}}^\wedge = \Gamma_{\text{inv}}^\otimes / S_{\text{inv}}^\wedge. \]
Here $S_{\text{inv}}^\wedge$ is the ideal in $\Gamma_{\text{inv}}^\otimes$, generated by elements $q \in \Gamma_{\text{inv}}^\otimes$ of the form
\[ q = \pi(a^{(1)}) \otimes \pi(a^{(2)}), \]
where $a \in \mathbb{R}$. This space is in fact the left-invariant part of the ideal $S^\wedge$.

The right $\mathcal{A}$-module structure $\circ$ can be uniquely extended from $\Gamma_{\text{inv}}$ to $\Gamma_{\text{inv}}^\wedge$, such that
\begin{align}
(2.4) & \quad 1 \circ a = \epsilon(a)1 \\
(2.5) & \quad (\vartheta \eta) \circ a = (\vartheta \circ a^{(1)}) (\eta \circ a^{(2)})
\end{align}
for each $\vartheta, \eta \in \Gamma_{\text{inv}}^\wedge$ and $a \in \mathcal{A}$. Explicitly, $\circ$ is given by
\[ \vartheta \circ a = \kappa(a^{(1)}) \vartheta a^{(2)}. \]

The algebra $\Gamma_{\text{inv}}^\wedge \subseteq \Gamma^\wedge$ is $d$-invariant. The following identities hold
\begin{align}
(2.7) & \quad d(\vartheta \circ a) = d(\vartheta) \circ a - \pi(a^{(1)}) (\vartheta \circ a^{(2)}) + (-1)^{\vartheta} \vartheta \circ a^{(1)} \pi(a^{(2)}) \\
(2.8) & \quad d \pi(a) = -\pi(a^{(1)}) \vartheta a^{(2)}. 
\end{align}

If $\Gamma$ is *-covariant then the *-involution $\ast : \Gamma \to \Gamma$ is naturally extendible to $\Gamma^\wedge$ and $\Gamma^\otimes$ (such that $(\vartheta \eta)^* = (-1)^{\vartheta \eta} \vartheta^* \eta^*$ for each $\vartheta, \eta \in \Gamma^\wedge$ ). The maps $\ell_\tau^\otimes \Gamma^\wedge$ are hermitian, in a natural manner. Algebras $\Gamma_{\text{inv}}^\wedge \subseteq \Gamma^\wedge, \Gamma^\otimes$ are *-invariant. We have
\[ \vartheta \circ a = \vartheta^* \circ \kappa(a)^*. \]
for each $a \in \mathcal{A}$ and $\vartheta \in \Gamma_{\text{inv}}^\wedge$.

Explicitly, the *-involution on $\Gamma_{\text{inv}}^\wedge$ is determined by
\[ \pi(a)^* = -\pi(\kappa(a)^*). \]

Let us now assume that the calculus $\Gamma$ is bicovariant, and let $\varphi_\Gamma : \Gamma \to \Gamma \otimes \mathcal{A}$ be the right action of $G$ on $\Gamma$. Maps $\phi$ and $\varphi_\Gamma$ admit common extensions to homomorphisms $\phi_\Gamma, \varphi_\Gamma^\wedge : \Gamma^\wedge, \Gamma^\otimes \to \Gamma \otimes \mathcal{A}$ (right actions of $G$ on corresponding algebras). Let $\varpi : \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \mathcal{A}$ be the adjoint action of $G$ on $\Gamma_{\text{inv}}$. The space $\Gamma_{\text{inv}}$ is right-invariant, that is $\varphi_\Gamma(\Gamma_{\text{inv}}) \subseteq \Gamma_{\text{inv}} \otimes \mathcal{A}$. We have $\varpi = \varphi_\Gamma | \Gamma_{\text{inv}}$. Explicitly,
\[ \varpi \pi = (\pi \otimes \text{id}) \text{ad}. \]
We shall denote by $\varpi^\otimes$, $\varpi^{\otimes k}$, $\varpi^\wedge$ and $\varpi^{\wedge k}$ the adjoint actions of $G$ on the corresponding spaces (coinciding with the corresponding restrictions of $\varpi_1^\otimes$ and $\varpi_1^\wedge$).

The coproduct map $\phi: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ admits the unique extension to the homomorphism $\widehat{\phi}: \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$ of graded-differential algebras. We have

$$\widehat{\phi}(\vartheta) = \ell_F(\vartheta) + \varphi_1(\vartheta),$$

for each $\vartheta \in \Gamma$. Further, we have $\widehat{\phi}(\Gamma^\wedge_{inv}) \subseteq \Gamma^\wedge_{inv} \otimes \Gamma^\wedge$. Let $\widehat{\varpi}: \Gamma^\wedge_{inv} \to \Gamma^\wedge_{inv} \otimes \Gamma^\wedge$ be the corresponding restriction. Explicitly,

$$\widehat{\varpi}(\vartheta) = 1 \otimes \vartheta + \varpi(\vartheta),$$

for each $\vartheta \in \Gamma_{inv}$.

If $\Gamma$ is a bicovariant $\ast$-calculus then the maps $\widehat{\phi}$, $\widehat{\varpi}$ and all the introduced adjoint and right actions are hermitian, in a natural manner.

3. **Quantum Principal Bundles & The Corresponding Differential Calculus**

The aim of this section is twofold. As first, we shall define quantum principal bundles, and briefly describe a geometrical background for this definition. Then, an appropriate differential calculus over quantum principal bundles will be introduced and analyzed. In particular, besides the main algebra consisting of “differential forms” on the bundle, we shall introduce and analyze algebras of “verticalized” and “horizontal” differential forms. Finally, an algebra representing differential forms on the base manifold will be defined.

Let us consider a quantum space $M$, formally represented by a (unital) $\ast$-algebra $V$. At the geometrical level, the elements of $V$ play the role of appropriate “functions” on this space.

**Definition 3.1.** A quantum principal $G$-bundle over $M$ is a triplet of the form $P = (B, i, F)$, where $B$ is a (unital) $\ast$-algebra while $i: V \to B$ and $F: B \to B \otimes A$ are unital $\ast$-homomorphisms such that

**(qpb1)** The map $i: V \to B$ is injective and

$$b \in i(V) \iff F(b) = b \otimes 1$$

for each $b \in B$.

**(qpb2)** The following identities hold

$$\text{id} = (\text{id} \otimes \epsilon)F$$

$$\text{id} \otimes \phi)F = (F \otimes \text{id})F.$$
The elements of $\mathcal{B}$ are interpretable as appropriate “functions” on the quantum space $P$. The map $F$ plays the role of the dualized right action of $G$ on $P$. Condition (qp$\beta$) justifies this interpretation. The map $i: \mathcal{V} \to \mathcal{B}$ plays the role of the dualized “projection” of $P$ on the base manifold $M$. Condition (qp$I$) says that $M$ is identifiable with the corresponding “orbit space” for the right action. Accordingly, the elements of $\mathcal{V}$ will be identified with their images in $i(\mathcal{V})$.

Finally, condition (qp$\beta$) is an effective quantum counterpart of the classical requirement that the action of $G$ on $P$ is free. It is easy to see that this condition can be equivalently formulated as

\[(qp\beta) \text{ For each } a \in \mathcal{A} \text{ there exist elements } b_k, q_k \in \mathcal{B} \text{ such that}\]

\[
1 \otimes a = \sum_k q_k F(b_k). \tag{3.3}
\]

We now pass to questions related to differential calculus on quantum principal bundles. Let $P = (\mathcal{B}, i, F)$ be a quantum principal $G$-bundle over $M$. Let us fix a bicovariant first-order $*$-calculus $\Gamma$ over $G$ and let us consider a graded vector space $\text{ver}(P) = \mathcal{B} \otimes \Gamma^\wedge_{inv}$ (the grading is induced from $\Gamma^\wedge_{inv}$).

**Lemma 3.1.** (i) The formulas

\[
(q \otimes \eta)(b \otimes \vartheta) = \sum_k qb_k \otimes (\eta \circ c_k) \vartheta \tag{3.4}
\]

\[
(b \otimes \vartheta)^* = \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*) \tag{3.5}
\]

\[
d_v (b \otimes \vartheta) = b \otimes d\vartheta + \sum_k b \otimes \pi(c_k) \vartheta \tag{3.6}
\]

where $F(b) = \sum b_k \otimes c_k$, determine the structure of a graded-differential $*$-algebra on $\text{ver}(P)$.

(ii) As a differential algebra, $\text{ver}(P)$ is generated by $\mathcal{B} = \text{ver}^0(P)$.

**Proof.** Let us first check the associativity of the introduced product. Applying (3.4) and (2.5) and elementary properties of $F$ we obtain

\[
\left[ (f \otimes \zeta)(b \otimes \vartheta) \right] (q \otimes \eta) = \sum_k (fb_k \otimes (\zeta \circ c_k) \vartheta) (q \otimes \eta)
\]

\[
= \sum_{kl} f_{bk} q_l \otimes \left[ ((\zeta \circ c_k) \vartheta) \circ d_l \right] \eta
\]

\[
= \sum_{kl} f_{bk} q_l \otimes \left( \zeta \circ (c_k d_l^{(1)}) \right) (\vartheta \circ d_l^{(2)}) \eta
\]

\[
= (f \otimes \zeta) \sum_l b q_l \otimes (\vartheta \circ d_l) \eta
\]

\[
= (f \otimes \zeta) \left[ (b \otimes \vartheta) (q \otimes \eta) \right],
\]

where $F(q) = \sum q_l \otimes d_l$. 


Evidently, \( \mathfrak{vec}(P) \) is a unital algebra, with the unity \( 1 \otimes 1 \). Now we prove that (3.5) determines a \(*\)-algebra structure on \( \mathfrak{vec}(P) \). We have
\[
[(b \otimes \vartheta)^*] = \sum_k (b_k^* \otimes (\vartheta^* \circ c_k^{(2)})^* \circ c_k^{(1)})
\]
\[
= \sum_k b_k \otimes (\vartheta \circ \kappa^{-1}(c_k^{(2)})) \circ c_k^{(1)} = \sum_k b_k \otimes \vartheta \circ (\kappa^{-1}(c_k^{(2)})c_k^{(1)}) = b \otimes \vartheta.
\]
Thus, \( * \) is involutive. Further,
\[
[(q \otimes \eta)(b \otimes \vartheta)]^* = \sum_k (qb_k \otimes (\eta \circ c_k)\vartheta)^*
\]
\[
= \sum_{kl} (q_l b_k)^* \otimes ((\eta \circ c_k^{(2)})\vartheta)^* \circ (d_l c_k^{(1)})^*
\]
\[
= (-1)^{\partial \vartheta \partial \eta} \sum_{kl} b_k^* q_l^* \otimes [(\vartheta^* \circ c_k^{(1)}d_l^{(1)*})[\eta^* \circ (\kappa(c_k^{(3)})c_k^{(2)}d_l^{(2)*})]]
\]
\[
= (-1)^{\partial \vartheta \partial \eta} \sum_{kl} b_k^* q_l^* \otimes (\vartheta^* \circ c_k^{(1)}d_l^{(1)*})(\eta^* \circ d_l^{(2)*})
\]
\[
= (-1)^{\partial \vartheta \partial \eta} \sum_{kl} [b_k^* \otimes (\vartheta^* \circ c_k^{(1)})][q_l^* \otimes (\eta^* \circ d_l^{(2)*})]
\]
\[
= (-1)^{\partial \vartheta \partial \eta}(b \otimes \vartheta)^*(q \otimes \eta)^*.
\]

Let us check that (3.6) defines a hermitian differential on the \(*\)-algebra \( \mathfrak{vec}(P) \). We compute
\[
d_v[(q \otimes \eta)(b \otimes \vartheta)] = \sum_k d_v(q b_k \otimes (\eta \circ c_k)\vartheta) = \sum_k q b_k \otimes d((\eta \circ c_k)\vartheta)
\]
\[
+ \sum_{kl} q_l b_k \otimes \pi(d_l c_k^{(1)})(\eta \circ c_k^{(2)})\vartheta
\]
\[
= \sum_k q b_k \otimes (d(\eta) \circ c_k)\vartheta - \sum_k q b_k \otimes \pi(c_k^{(1)})(\eta \circ c_k^{(2)})\vartheta
\]
\[
+ (-1)^{\partial \eta} \sum_k q b_k \otimes \left( (\eta \circ c_k^{(1)})\pi(c_k^{(2)})\vartheta + (\eta \circ c_k) d\vartheta \right)
\]
\[
+ \sum_{kl} q_l b_k \otimes \pi(d_l \circ c_k^{(1)})(\eta \circ c_k^{(2)})\vartheta
\]
\[
+ \sum_k q b_k \otimes \pi(c_k^{(1)})(\eta \circ c_k^{(2)})\vartheta
\]
\[
= (q \otimes d\eta)(b \otimes \vartheta) + (-1)^{\partial \eta}(q \otimes \eta) \sum_k b_k \otimes \pi(c_k)\vartheta
\]
\[
+ (-1)^{\partial \eta}(q \otimes \eta)(b \otimes d\vartheta) + \sum_l (q_l \otimes \pi(d_l \eta))(b \otimes \vartheta)
\]
\[
= [d_v(q \otimes \eta)](b \otimes \vartheta) + (-1)^{\partial \eta}(q \otimes \eta)d_v(b \otimes \vartheta).
\]
Here, we have used (2.3), (2.5), (2.7) and the main properties of \( F \).
Further
\[ d^*_v(b \otimes \vartheta) = d_v \left( b \otimes d\vartheta + \sum_k b_k \otimes \pi(c_k)\vartheta \right) = \sum_k b_k \otimes \pi(c_k)d\vartheta + \sum_k b_k \otimes (\pi(c_k^{(1)})\pi(c_k^{(2)})\vartheta) + \sum_k b_k \otimes d(\pi(c_k)\vartheta) = 0, \]
according to (2.8). Finally,
\[ d_v[(b \otimes \vartheta)^*] = \sum_k b_k^* \otimes d(\vartheta^* \circ c_k^*) + \sum_k b_k^* \otimes \pi(c_k^{(1)*})(\vartheta^* \circ c_k^{(2)*}) \]
\[ = \sum_k b_k^* \otimes (d\vartheta^*) \circ c_k^* - \sum_k b_k^* \otimes \pi(c_k^{(1)*})(\vartheta^* \circ c_k^{(2)*}) \]
\[ + (-1)^{\partial \vartheta} \sum_k b_k^* \otimes (\vartheta^* \circ c_k^{(1)*})\pi(c_k^{(2)*}) + \sum_k b_k^* \otimes \pi(c_k^{(1)*})(\vartheta^* \circ c_k^{(2)*}) \]
\[ = (b \otimes d\vartheta)^* + (-1)^{\partial \vartheta} \sum_k b_k^* \otimes (\vartheta^* \circ c_k^{(1)*})\pi(c_k^{(2)*}) \]
\[ = (b \otimes d\vartheta)^* + (-1)^{\partial \vartheta} \sum_k b_k^* \otimes (\vartheta^* \circ c_k^{(1)*})\pi(c_k^{(2)*}) \]
\[ = (b \otimes d\vartheta)^* + \sum_k b_k^* \otimes (\pi(c_k^{(2)}\vartheta)^* \circ c_k^{(1)*} \] 
\[ = (b \otimes d\vartheta)^* + \sum_k b_k^* \otimes (\pi(c_k^{(2)}\vartheta)^* \circ c_k^{(1)*} = [d_v(b \otimes \vartheta)]^*. \]

Hence, \( d_v \) is a hermitian differential. To prove (ii) it is sufficient to check that elements of the form \( qd_{\nu}(b) \) linearly generate \( \text{vert}^1(P) \). However, this directly follows from property \( (qp4) \) in the definition of quantum principal bundles.

In the following it will be assumed that \( \text{vert}(P) \) is endowed with the constructed graded-differential \(*\)-algebra structure. The elements of \( \text{vert}(P) \) are interpretable as verticalized differential forms on the bundle. In classical geometry, these entities are obtained by restricting the domain of differential forms (on the bundle) to the Lie algebra of vertical vector fields.

**Lemma 3.2.** There exists the unique homomorphism
\[ \hat{F}_v : \text{vert}(P) \to \text{vert}(P) \otimes \Gamma^\wedge \]
of (graded) differential algebras extending the map \( F \). We have
\[ (\hat{F}_v \otimes \text{id})\hat{F}_v = (\text{id} \otimes \hat{\vartheta})\hat{F}_v. \] (3.7)

The map \( \hat{F}_v \) is hermitian, in the sense that
\[ \hat{F}_{v^*} = (\ast \otimes \ast) \hat{F}_v. \] (3.8)

**Proof.** According to (ii) of the previous lemma, the map \( \hat{F}_v \) is unique, if exists. Let us define a linear map \( \hat{F}_v : \text{vert}(P) \to \text{vert}(P) \otimes \Gamma^\wedge \) by
\[ \hat{F}_v(b \otimes \vartheta) = \sum_{kl} b_{kl} \otimes \vartheta_l \otimes c_kw_l \]
where $F(b) = \sum b_k \otimes c_k$ and $\vartheta(\vartheta) = \sum \vartheta_i \otimes w_i$. It is easy to see that such defined map $\tilde{F}_v$ is a differential algebra homomorphism.

Identities (3.7) and (3.8) directly follow form the fact that $\text{ver}(P)$ is generated by $\mathcal{B}$, as well as from property (3.2) and the hermicity of $d_v$ respectively.

Let us consider a $\ast$-homomorphism $F_v : \text{ver}(P) \to \text{ver}(P) \otimes \mathcal{A}$ given by

$$F_v = (\text{id} \otimes p_0)\tilde{F}_v.$$ 

This map extends the action $F$. It is interpretable as the (dualized) right action of $G$ on verticalized forms. The following identities hold:

\begin{align*}
(3.9) & \quad (F_v \otimes \text{id})F_v = (\text{id} \otimes \varphi)F_v \\
(3.10) & \quad (\text{id} \otimes \epsilon)F_v = \text{id} \\
(3.11) & \quad (d_v \otimes \text{id})F_v = F_v d_v.
\end{align*}

The first two identities justify the interpretation of $F_v$ as an action of $G$. The last identity says that the differential $d_v$ is right-covariant.

So far about verticalized differential forms. We shall assume that a complete differential calculus over the bundle $P$ is based on a graded-differential $\ast$-algebra $\Omega(P)$ possessing the following properties

\begin{enumerate}
\item[(diff1)] As a differential algebra, $\Omega(P)$ is generated by $\mathcal{B} = \Omega_0(P)$.
\item[(diff2)] The map $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is extendible to a homomorphism
\end{enumerate}

$$\tilde{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge$$

of graded-differential algebras.

Let us fix a graded-differential $\ast$-algebra $\Omega(P)$ such that the above properties hold. The elements of $\Omega(P)$ will play the role of differential forms on $P$. It is easy to see that the map $\tilde{F}$ is uniquely determined.

**Lemma 3.3.** We have

\begin{align*}
(3.12) & \quad (\tilde{F} \otimes \text{id})\tilde{F} = (\text{id} \otimes \varphi)\tilde{F} \\
(3.13) & \quad \tilde{F}^\ast = (\ast \otimes \ast)\tilde{F}.
\end{align*}

**Proof.** Both identities directly follow from similar properties of $F$, and from properties (diff$1/2$). \qed

The formula

$$F^\wedge = (\text{id} \otimes p_0)\tilde{F}$$

defines a $\ast$-homomorphism $F^\wedge : \Omega(P) \to \Omega(P) \otimes \mathcal{A}$ extending the action $F$. The following identities hold:

\begin{align*}
(3.15) & \quad (F^\wedge \otimes \text{id})F^\wedge = (\text{id} \otimes \varphi)F^\wedge \\
(3.16) & \quad (\text{id} \otimes \epsilon)F^\wedge = \text{id} \\
(3.17) & \quad (d \otimes \text{id})F^\wedge = F^\wedge d.
\end{align*}
The map $F^\wedge$ is interpretable as the (dualized) right action of $G$ on differential forms. As a homomorphism between algebras, $F^\wedge$ is completely determined by (3.17), and by the fact that it extends $F$.

Now, a very important algebra representing horizontal forms will be introduced in the game. Intuitively speaking, horizontal forms can be characterized as those elements of $\Omega(P)$ possessing “trivial” differential properties along vertical fibers.

**Definition 3.2.** The elements of the graded $*$-subalgebra

$$\mathfrak{hor}(P) = \tilde{F}^{-1}(\Omega(P) \otimes A)$$

of $\Omega(P)$ are called horizontal forms.

Evidently, $B = \mathfrak{hor}^0(P)$.

**Lemma 3.4.** The algebra $\mathfrak{hor}(P)$ is $F^\wedge$-invariant. In other words

$$F^\wedge(\mathfrak{hor}(P)) \subseteq \mathfrak{hor}(P) \otimes A.$$  \hspace{1cm} (3.18)

**Proof.** If $\varphi \in \mathfrak{hor}(P)$ then $(\tilde{F} \otimes \text{id})\tilde{F}(\varphi) = (\text{id} \otimes \hat{\phi})\tilde{F}(\varphi) = (\text{id} \otimes \phi)F^\wedge(\varphi)$ belongs to $\Omega(P) \otimes A \otimes A$. Hence, $\tilde{F}(\varphi) = F^\wedge(\varphi) \in \mathfrak{hor}(P) \otimes A$. \hfill \square

The following technical lemma will be helpful in various considerations.

**Lemma 3.5.** Let us consider a homogeneous element $w \in \Omega^n(P)$. Let $0 \leq k \leq n$ be an integer such that $(\text{id} \otimes \pi_k)\tilde{F}(w) = 0$ for each $l > k$. Then there exist horizontal forms $\varphi_1, \ldots, \varphi_m \in \mathfrak{hor}^{n-k}(P)$ and elements $\vartheta_1, \ldots, \vartheta_m \in \Gamma_{\text{inv}}^k$ such that

$$\sum_{ij} \xi_{ij} \otimes a_{ij} \vartheta_i = (id \otimes \pi_k)\tilde{F}(w)$$

where $\vartheta_i \in \Gamma_{\text{inv}}^k$ are some some linearly independent elements, $\xi_{ij} \in \Omega^{n-k}(P)$ and $a_{ij} \in A$. Applying (2.13), (3.12) and (3.20), and the definition of $k$ we find

$$\sum_{ij} \tilde{F}(\xi_{ij}) \otimes a_{ij} \vartheta_i = (\tilde{F} \otimes \pi_k)\tilde{F}(w) = (id^2 \otimes p_k) \sum_{ij} \xi_{ij} \otimes \tilde{\phi}(a_{ij} \vartheta_i)$$

$$= \sum_{ij} \xi_{ij} \otimes a_{ij}^{(1)} \otimes a_{ij}^{(2)} \vartheta_i.$$

Acting by $id^2 \otimes \pi_{\text{inv}}$ on both sides of the above equality we obtain

$$\sum_i \tilde{F}(\varphi_i) \otimes \vartheta_i = \sum_{ij} \xi_{ij} \otimes a_{ij} \otimes \vartheta_i,$$  \hspace{1cm} (3.21)
where $\varphi_i = \sum_j \xi_{ij} e(a_{ij})$. In other words

$$\hat{F}(\varphi_i) = \sum_j \xi_{ij} \otimes a_{ij}, \quad (3.22)$$

and in particular $\varphi_i \in \mathfrak{hor}^{n-k}(P)$. Finally, combining (3.20) and (3.22) we conclude that (3.19) holds.

We are going to construct a quantum analog for the “verticalizing” homomorphism (in classical geometry, induced by restricting the domain of differential forms on vertical vector fields on the bundle).

**Proposition 3.6.** There exists the unique (graded) differential algebra homomorphism $\pi_v: \Omega(P) \to \mathfrak{ver}(P)$ reducing to the identity map on $\mathcal{B}$. The map $\pi_v$ is surjective and hermitian. Moreover,

$$\hat{F}_v \pi_v = (\pi_v \otimes \text{id}) \hat{F}, \quad (3.23)$$

$$F_v \pi_v = (\pi_v \otimes \text{id}) F^\wedge. \quad (3.24)$$

**Proof.** Let us define a linear (grade-preserving) map $\pi_v: \Omega(P) \to \mathfrak{ver}(P)$ by requiring

$$\pi_v(w) = (\text{id} \otimes \pi_{\text{inv}} p_{k+l}) \hat{F}(w)$$

for each $w \in \Omega^k(P)$. Obviously, $\pi_v$ is reduced to the identity on $\mathcal{B}$.

Let us prove that $\pi_v$ is a differential algebra homomorphism. For given forms $w \in \Omega^k(P)$ and $u \in \Omega^l(P)$ let us choose elements $b_i, q_j \in \mathcal{B}$ and $\vartheta_i, \eta_j \in \Gamma_{\text{inv}}^{k,l}$ such that

$$(\text{id} \otimes p_k) \hat{F}(w) = \sum_i F(b_i) \vartheta_i \quad (\text{id} \otimes p_l) \hat{F}(u) = \sum_j F(q_j) \eta_j,$$

in accordance with the previous lemma.

We have then

$$\pi_v(w) = \sum_i b_i \otimes \vartheta_i \quad \pi_v(u) = \sum_j q_j \otimes \eta_j.$$

A direct computation now gives

$$\pi_v(wu) = (\text{id} \otimes \pi_{\text{inv}} p_{k+l}) \hat{F}(wu)$$

$$\begin{align*}
&= \sum_{ij} (\text{id} \otimes \pi_{\text{inv}})[(F(b_i) \vartheta_i)(F(q_j) \eta_j)] \\
&= \sum_{ij} (\text{id} \otimes \pi_{\text{inv}}) \sum_{rs} [b_{ir} q_{js} \otimes c_{is} d_{js}^{(1)}(\vartheta_i \circ d_{js}^{(2)}) \eta_j] \\
&= \sum_{ijs} b_{ijs} (\vartheta_i \circ d_{js}) \eta_j = \left[ \sum_i b_i \otimes \vartheta_i \right] \left[ \sum_j q_j \otimes \eta_j \right] \\
&= \pi_v(w) \pi_v(u),
\end{align*}$$

where $\varphi_i = \sum_j \xi_{ij} e(a_{ij})$. In other words

$$\hat{F}(\varphi_i) = \sum_j \xi_{ij} \otimes a_{ij}, \quad (3.22)$$

and in particular $\varphi_i \in \mathfrak{hor}^{n-k}(P)$. Finally, combining (3.20) and (3.22) we conclude that (3.19) holds. ☐
where $F(b) = \sum_{i} b_{ir} \otimes c_{ir}$ and $F(q) = \sum_{s} g_{js} \otimes d_{js}$. Further,

$$
\pi_v d(w) = (id \otimes \pi_{inv} p_{k+1}) d\hat{F}(w) = (id \otimes \pi_{inv} d) \left[ \sum_i F(b_i) \theta_i \right]
$$

$$
= (id \otimes \pi_{inv}) \left[ \sum_i b_{ir} \otimes c_{ir} d\theta_i + b_{ir} \otimes d(c_{ir}) \theta_i \right]
$$

$$
= \sum_i b_{ir} \otimes d\theta_i + \sum_i b_{ir} \otimes \pi(c_{ir}) \theta_i
$$

$$
= d_v \left[ \sum_i b_{ir} \otimes \theta_i \right] = d_v \pi_v(w).
$$

Consequently, $\pi_v$ is a homomorphism of differential algebras. The map $\pi_v$ is hermitian, because differentials on $\Omega(P)$ and $\text{ver}(P)$ are hermitian, and the differential algebra $\Omega(P)$ is generated by $B$. To prove (3.23) it is sufficient to observe that its both sides are differential algebra homomorphisms coinciding with $F$ on $B$. Finally, (3.24) follows from (3.23), and definitions of $F^\wedge$ and $F_v$. 

Let us now consider a sequence

$$
0 \to \mathfrak{hor}^1(P) \to \Omega^1(P) \xrightarrow{\pi_v} \text{ver}^1(P) \to 0
$$

of natural homomorphisms of $*$-$\mathcal{B}$-bimodules.

**Lemma 3.7.** The above sequence is exact.

**Proof.** Clearly, $\mathfrak{hor}^1(P) \subseteq \ker(\pi_v) \cap \Omega^1(P)$ and $\pi_v(\Omega^1(P)) = \text{ver}^1(P)$. For each $w \in \Omega^1(P)$ we have

$$(id \otimes p_1) \hat{F}(w) = \sum_i F(b_i) \theta_i,$$

for some $b_i \in B$ and $\theta_i \in \Gamma_{inv}$, according to Lemma 3.5. This implies

$$\pi_v(w) = \sum_i b_i \otimes d\theta_i.$$

Consequently if $w \in \ker(\pi_v)$ then $(id \otimes p_1) \hat{F}(w) = 0$, and hence $w \in \mathfrak{hor}^1(P)$. 

If a differential calculus on the bundle $P$ is given, then it is possible to construct a natural differential calculus on the base space $M$. This calculus is based on a graded-differential $*$-subalgebra $\Omega(M) \subseteq \Omega(P)$ consisting of right-invariant horizontal forms. Equivalently,

$$\Omega(M) = \left\{ w \in \Omega(P) : \hat{F}(w) = w \otimes 1 \right\}. $$

In a special case when the group $G$ is “connected” in the sense that only scalar elements of $\mathcal{A}$ are annihilated by the differential map, the algebra $\Omega(M)$ can be described as

$$\Omega(M) = d^{-1}(\mathfrak{hor}(P)) \cap \mathfrak{hor}(P). $$

The differential algebra $\Omega(M)$ is generally not generated by its 0-order subalgebra $\Omega^0(M) = \mathcal{V}$, in contrast to $\Omega(P)$. However, property (diff1) is not essential for
developing a large part of the formalism. Furthermore, it turns out that the algebra \( \mathfrak{hor}(P) \) is generally not generated by \( \mathcal{B} \) and \( \mathfrak{hor}^1(P) \).

4. The Formalism of Connections

In this section a general theory of connections on quantum principal bundles will be presented. As first, quantum analogs of pseudotensorial forms will be defined. Let \( V \) be a vector space, and let \( v: V \to V \otimes A \) be a representation of \( G \) in this space. Let \( \psi(v, P) \) be the space of all linear maps \( \zeta: V \to \Omega(P) \) such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\zeta} & \Omega(P) \\
v \downarrow & & \downarrow F^\wedge \\
V \otimes A & \xrightarrow{\zeta \otimes \text{id}} & \Omega(P) \otimes A
\end{array}
\]

(4.1)

is commutative (intertwiners between \( v \) and \( F^\wedge \)).

The space \( \psi(v, P) \) is naturally graded

\[
\psi(v, P) = \sum_{k \geq 0} \psi^k(v, P),
\]

(4.2)

where \( \psi^k(v, P) \) consists of maps with values in \( \Omega^k(P) \). The elements of \( \psi^k(v, P) \) can be interpreted as pseudotensorial \( k \)-forms on \( P \) with values in the dual space \( V^* \). Further, \( \psi(v, P) \) is closed with respect to compositions with \( d: \Omega(P) \to \Omega(P) \). It is also a module, in natural manner, over the subalgebra consisting of right-invariant forms. Let

\[
\tau(v, P) = \sum_{k \geq 0} \tau^k(v, P)
\]

(4.3)

be the graded subspace of \( \psi(v, P) \) consisting of pseudotensorial forms having the values in \( \mathfrak{hor}(P) \). The elements of \( \tau(v, P) \) are interpretable as tensorial forms on \( P \) with values in \( V^* \). The space \( \tau(v, P) \) is a module over \( \Omega(M) \).

If the space \( \Gamma_{inv} \) is infinite-dimensional and if \( \Omega(P) \) possesses infinitely non-zero components then, generally, sums figuring in (4.2) and (4.3) will be “larger” then standard direct sums of spaces (and should be interpreted in the appropriate way).

Forms \( \varphi \in \tau(v, P) \) can be also defined by the following equality

\[
\hat{F}\varphi(\vartheta) = (\varphi \otimes \text{id})v(\vartheta).
\]

(4.4)

If the space \( V \) is endowed with an antilinear involution \( *: V \to V \) such that \( v^* = (\ast \otimes \ast)v \) then the formula

\[
\varphi^*(\vartheta) = \varphi(\vartheta^*)^\ast
\]

(4.5)

determines natural \( * \)-involutions on \( \psi(v, P) \) and \( \tau(v, P) \).

For the purposes of this paper, the most important is the case \( V = \Gamma_{inv} \), and \( v = \varpi \). In this case we shall write \( \psi(P) = \psi(\varpi, P) \) and \( \tau(P) = \tau(\varpi, P) \).

We pass to the definition of connection forms.
Definition 4.1. A connection on $P$ is a hermitian map $\omega \in \psi^1(P)$ satisfying
\begin{equation}
\pi_v(\omega(\vartheta)) = 1 \otimes \vartheta
\end{equation}
for each $\vartheta \in \Gamma_{\text{inv}}$.

The above condition corresponds to the classical requirement that connection
forms (understood as $\text{lie}(G)$-valued pseudotensorial ad-type 1-forms) map fundamental vector fields into their generators.

Theorem 4.1. Every quantum principal bundle $P$ admits at least one connection.

Proof. Let us consider the space $W = \pi_v^{-1}(\Gamma_{\text{inv}}) \cap \Omega^1(P)$. By definition this space
is invariant under $*$ and $F^\wedge$, and $\pi_v(W) = \Gamma_{\text{inv}}$. We can write
\begin{equation}
W = \bigoplus_{\alpha \in \mathcal{T}} W^\alpha
\end{equation}
where $W^\alpha$ are corresponding multiple irreducible subspaces (the notation is ex-
plained in Appendix B). Similarly
\begin{equation}
\Gamma_{\text{inv}} = \bigoplus_{\alpha \in \mathcal{T}} \Gamma^\alpha_{\text{inv}}.
\end{equation}
The following decompositions hold
\begin{equation}
W^\alpha \leftrightarrow \text{Mor}(u^\alpha, F^\wedge) \otimes \mathbb{C}^n \quad \Gamma^\alpha_{\text{inv}} \leftrightarrow \text{Mor}(u^\alpha, \varpi) \otimes \mathbb{C}^n
\end{equation}
where $n$ is the dimension of $\alpha$. Further, $\pi_v(W^\alpha) = \Gamma^\alpha_{\text{inv}}$ for each $\alpha \in \mathcal{T}$.

In terms of the above identifications the restriction map $\pi_v : W^\alpha \rightarrow \Gamma^\alpha_{\text{inv}}$ is given by
\begin{equation}
\pi_v(\{\mu \otimes x\}) = \pi_v \mu \otimes x.
\end{equation}
This map is surjective. Let $\tau^\alpha : \text{Mor}(u^\alpha, \varpi) \rightarrow \text{Mor}(u^\alpha, F^\wedge)$ be a left inverse of $\pi_v|W^\alpha$.

Let $\tau : \Gamma_{\text{inv}} \rightarrow W$ be a map defined by
\begin{equation}
\tau = \bigoplus_{\alpha \in \mathcal{T}} \tau^\alpha
\end{equation}
where $\tau^\alpha : \Gamma^\alpha_{\text{inv}} \rightarrow W^\alpha$ are given by $\tau^\alpha = \tau^\alpha \otimes \text{id}$. By construction, $\tau$ intertwines
$\varpi$ and $F^\wedge$ and satisfies $\pi_v(\tau(\vartheta)) = \vartheta$ for each $\vartheta \in \Gamma_{\text{inv}}$. Without a lack of generality
we can assume that $\tau$ is hermitian (if not, we can consider another intertwiner
$*\tau^* : \Gamma_{\text{inv}} \rightarrow W$ and redefine $\tau \mapsto (\tau + *\tau^*)/2$). Finally, composing $\tau$ and the
inclusion map $W \hookrightarrow \Omega(P)$ we obtain a connection on $P$. \qed

Let $\text{con}(P)$ be the set of all connections on $P$. This is a real affine subspace of $\psi^1(P)$. The corresponding vector space consists of hermitian tensorial 1-forms.

Connections can be described in a different, more concise, but equivalent manner,
using an algebraic condition which is a symbiosis of the verticalization condition
(4.6) and the pseudotensoriality property.
Lemma 4.2. A first-order linear map \( \omega : \Gamma_{\text{inv}} \to \Omega(P) \) is a connection on \( P \) iff
\[
\omega(\vartheta^*) = \omega(\vartheta)^*
\]
\[
\hat{F}_\omega(\vartheta) = (\omega \otimes \text{id})\vartheta + 1 \otimes \vartheta,
\]
for each \( \vartheta \in \Gamma_{\text{inv}} \).

Proof. It is clear that above listed properties imply that \( \omega \) is a connection on \( P \). Conversely, let us consider an arbitrary \( \omega \in \text{con}(P) \). Then the pseudotensoriality property and Lemma 3.5 imply that for each \( \vartheta \in \Gamma_{\text{inv}} \),
\[
\hat{F}_\omega(\vartheta) = (\omega \otimes \text{id})\vartheta + 1 \otimes \vartheta,
\]
for some \( b_k \in \mathcal{B} \) and \( \vartheta_k \in \Gamma_{\text{inv}} \). Properties (3.1) and (4.6), and the definition of \( \pi_v \) imply \( 1 \otimes \vartheta = \sum b_k \otimes \vartheta_k \). Hence (4.8) holds. \( \square \)

Every connection \( \omega \) canonically gives rise to a splitting of the sequence (3.25), understood as a sequence of left \( \mathcal{B} \)-modules. Indeed, the map \( \mu_\omega : \text{ver}^1(P) \to \Omega^1(P) \) defined by
\[
\mu_\omega(b \otimes \vartheta) = b\omega(\vartheta)
\]
splits the mentioned sequence. Moreover, the map \( \mu_\omega \) intertwines the corresponding right actions.

For each \( \omega \in \text{con}(P) \) let \( \omega^\oplus : \Gamma_{\text{inv}}^\oplus \to \Omega(P) \) be the corresponding unital multiplicative extension.

Two particularly interesting classes of connections naturally appear in deeper considerations. These classes consist of connections possessing some additional properties that will be called multiplicativity and regularity.

Definition 4.2. A connection \( \omega \) is called multiplicative iff
\[
\omega(\vartheta^{(1)})(a_{\vartheta}^{(1)})\omega(\vartheta^{(2)}) = 0
\]
for each \( a \in \mathcal{R} \). Equivalently, \( \omega \) is multiplicative iff \( \omega^\oplus | S_{\text{inv}}^\wedge = 0 \).

If \( \omega \) is multiplicative then there exists the unique unital multiplicative extension \( \omega^\wedge : \Gamma_{\text{inv}}^\wedge \to \Omega(P) \). This map can be obtained by by factorizing \( \omega^\oplus \) through \( S_{\text{inv}}^\wedge \) (both \( \omega^\wedge \) and \( \omega^\oplus \) are *-preserving).

Condition (4.9) gives a quadratic constraint in the space \( \text{con}(P) \). It is worth noticing that in the classical theory all connections are multiplicative.

Definition 4.3. A connection \( \omega \) is called regular iff
\[
\omega(\vartheta)\varphi = (-1)^{\partial \varphi} \sum_k \varphi_k \omega(\vartheta \circ c_k)
\]
holds for each \( \varphi \in \text{hor}(P) \) and \( \vartheta \in \Gamma_{\text{inv}} \). Here, \( F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k \). Equivalently,
\[
\varphi \omega(\vartheta) = (-1)^{\partial \varphi} \sum_k \omega(\vartheta \circ k^{-1}(c_k)) \varphi_k.
\]
In particular, regular connections graded-commute with forms from \( \Omega(M) \).
Regular connections, if exist, form an affine subspace \( \mathfrak{r}(P) \subseteq \mathfrak{con}(P) \). The corresponding vector space consists of hermitian forms \( \zeta \in \tau^1(P) \) satisfying

\[
\varphi \zeta(\vartheta) = (-1)^{\partial \varphi} \sum_k \zeta(\vartheta \circ c_k^{-1}) \varphi_k,
\]

or equivalently

\[
\zeta(\vartheta) \varphi = (-1)^{\partial \varphi} \sum_k \varphi_k \zeta(\vartheta \circ c_k)
\]

for each \( \varphi \in \mathfrak{hor}(P) \) and \( \vartheta \in \Gamma_{\text{inv}} \).

It is important to mention that if \( \omega \) is regular then the corresponding splitting \( \mu_\omega \) is a splitting of \( \ast \)-bimodules.

If the theory is confined to bundles over classical manifolds and if \( \Gamma \) is the minimal admissible calculus then regular connections are precisely classical connections (in the terminology of the previous paper). In particular, in the case of classical principal bundles all connections are regular.

Let us consider an expression

\[
\ell^\omega(\vartheta, \varphi) = \omega(\vartheta) \varphi - (-1)^{\partial \varphi} \sum_k \varphi_k \omega(\vartheta \circ c_k)
\]

where \( \varphi \in \mathfrak{hor}(P) \) and \( \vartheta \in \Gamma_{\text{inv}} \). The map \( \ell^\omega \) measures the lack of regularity of \( \omega \).

**Lemma 4.3.** We have

\[
\hat{F} \ell^\omega(\vartheta, \varphi) = \sum_{jk} \ell^\omega(\vartheta_j, \varphi_k) \otimes d_j c_k,
\]

where \( \omega(\vartheta) = \sum_j \vartheta_j \otimes d_j \). In particular, \( \ell^\omega \) is a \( \mathfrak{hor}(P) \)-valued map.

**Proof.** A direct computation gives

\[
\hat{F} \ell^\omega(\vartheta, \varphi) = \sum_{jk} \omega(\vartheta_j) \varphi_k \otimes d_j c_k + (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \vartheta c_k
\]

\[
- (-1)^{\partial \varphi} \sum_k \varphi_k \omega(\vartheta_j \circ c_k^{(3)}) \otimes c_k^{(1)} \kappa(c_k^{(2)}) d_j c_k^{(4)}
\]

\[
- (-1)^{\partial \varphi} \sum_k \varphi_k \otimes c_k^{(1)}(\vartheta \circ c_k^{(2)})
\]

\[
= \sum_{jk} \ell^\omega(\vartheta_j, \varphi_k) \otimes d_j c_k. \quad \square
\]

Let \( \sigma: \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \rightarrow \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \) be the (left-invariant part of the) canonical braid operator [6]. This map is explicitly [2] given by

\[
\sigma(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k)
\]

where \( \vartheta(\vartheta) = \sum_k \vartheta_k \otimes c_k \). Let \( m_\Omega \) be the multiplication map in \( \Omega(P) \).

If \( \omega \in \mathfrak{r}(P) \) then

\[
m_\Omega \left\{ \omega \otimes \varphi \right\} = (-1)^{\partial \varphi} m_\Omega \left\{ \varphi \otimes \omega \right\} \sigma,
\]

_____

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[235x698] Regular connections, if exist, form an affine subspace \( \mathfrak{r}(P) \subseteq \mathfrak{con}(P) \). The corresponding vector space consists of hermitian forms \( \zeta \in \tau^1(P) \) satisfying

\[
\varphi \zeta(\vartheta) = (-1)^{\partial \varphi} \sum_k \zeta(\vartheta \circ c_k^{-1}) \varphi_k,
\]

or equivalently

\[
\zeta(\vartheta) \varphi = (-1)^{\partial \varphi} \sum_k \varphi_k \zeta(\vartheta \circ c_k)
\]

for each \( \varphi \in \mathfrak{hor}(P) \) and \( \vartheta \in \Gamma_{\text{inv}} \).

It is important to mention that if \( \omega \) is regular then the corresponding splitting \( \mu_\omega \) is a splitting of \( \ast \)-bimodules.

If the theory is confined to bundles over classical manifolds and if \( \Gamma \) is the minimal admissible calculus then regular connections are precisely classical connections (in the terminology of the previous paper). In particular, in the case of classical principal bundles all connections are regular.

Let us consider an expression

\[
\ell^\omega(\vartheta, \varphi) = \omega(\vartheta) \varphi - (-1)^{\partial \varphi} \sum_k \varphi_k \omega(\vartheta \circ c_k)
\]

where \( \varphi \in \mathfrak{hor}(P) \) and \( \vartheta \in \Gamma_{\text{inv}} \). The map \( \ell^\omega \) measures the lack of regularity of \( \omega \).

**Lemma 4.3.** We have

\[
\hat{F} \ell^\omega(\vartheta, \varphi) = \sum_{jk} \ell^\omega(\vartheta_j, \varphi_k) \otimes d_j c_k,
\]

where \( \omega(\vartheta) = \sum_j \vartheta_j \otimes d_j \). In particular, \( \ell^\omega \) is a \( \mathfrak{hor}(P) \)-valued map.

**Proof.** A direct computation gives

\[
\hat{F} \ell^\omega(\vartheta, \varphi) = \sum_{jk} \omega(\vartheta_j) \varphi_k \otimes d_j c_k + (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \vartheta c_k
\]

\[
- (-1)^{\partial \varphi} \sum_k \varphi_k \omega(\vartheta_j \circ c_k^{(3)}) \otimes c_k^{(1)} \kappa(c_k^{(2)}) d_j c_k^{(4)}
\]

\[
- (-1)^{\partial \varphi} \sum_k \varphi_k \otimes c_k^{(1)}(\vartheta \circ c_k^{(2)})
\]

\[
= \sum_{jk} \ell^\omega(\vartheta_j, \varphi_k) \otimes d_j c_k. \quad \square
\]

Let \( \sigma: \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \rightarrow \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \) be the (left-invariant part of the) canonical braid operator [6]. This map is explicitly [2] given by

\[
\sigma(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k)
\]

where \( \vartheta(\vartheta) = \sum_k \vartheta_k \otimes c_k \). Let \( m_\Omega \) be the multiplication map in \( \Omega(P) \).

If \( \omega \in \mathfrak{r}(P) \) then

\[
m_\Omega \left\{ \omega \otimes \varphi \right\} = (-1)^{\partial \varphi} m_\Omega \left\{ \varphi \otimes \omega \right\} \sigma,
\]
for each $\varphi \in \tau(P)$. The above equality directly follows from (4.16), the tensoriality of $\varphi$, and the definition of regular connections.

Let us fix a linear map $\delta: \Gamma_{inv} \to \Gamma_{inv} \otimes \Gamma_{inv}$ such that
\[
\varpi \otimes \delta = (\delta \otimes \text{id}) \varpi
\]
and such that if
\[
\delta(\vartheta) = \sum_k \vartheta^1_k \otimes \vartheta^2_k
\]
then
\[
d(\vartheta) = \sum_k \vartheta^1_k \vartheta^2_k - \delta(\vartheta^*) = \sum_k \vartheta^2_k \otimes \vartheta^1_k
\]
for each $\vartheta \in \Gamma_{inv}$. Such maps will be called embedded differentials. For each $\vartheta \in \Gamma_{inv}$ there exists $a \in \ker(\epsilon)$ such that
\[
\delta(\vartheta) = -\pi(a^{(1)}) \otimes \pi(a^{(2)})
\]
(4.18)
\[
\pi(a) = \vartheta.
\]

For given linear maps $\varphi, \eta: \Gamma_{inv} \to \Omega(P)$ let us define (as in [2]) new linear maps $\langle \varphi, \eta \rangle, [\varphi, \eta]: \Gamma_{inv} \to \Omega(P)$ by
\[
\langle \varphi, \eta \rangle = m_{\Omega}(\varphi \otimes \eta) \delta
\]
(4.19)
\[
[\varphi, \eta] = m_{\Omega}(\varphi \otimes \eta)c^\top,
\]
(4.20)
where $c^\top: \Gamma_{inv} \to \Gamma_{inv} \otimes \Gamma_{inv}$ is the “transposed commutator” map [6], explicitly given by
\[
c^\top = (\text{id} \otimes \pi)\varpi.
\]
(4.21)

The same brackets can be used for linear maps defined on $\Gamma_{inv}$, with values in an arbitrary algebra.

It is easy to see that if $\varphi \in \psi^i(P)$ and $\eta \in \psi^j(P)$ then $\langle \varphi, \eta \rangle, [\varphi, \eta] \in \psi^{i+j}(P)$ (the same holds for $\tau(P)$). Further,
\[
\langle \varphi, \eta \rangle^* = -(-1)^{ij}(\eta^*, \varphi^*),
\]
(4.22)
as directly follows from (4.5) and the hermicity of $\delta$.

For an arbitrary $\omega \in \text{con}(P)$ let us consider a map $R_\omega: \Gamma_{inv} \to \Omega(P)$ given by
\[
R_\omega = d\omega - \langle \omega, \omega \rangle.
\]
(4.23)
Clearly, this is a pseudotensorial 2-form. Moreover,

**Lemma 4.4.** We have
\[
\hat{\delta}R_\omega(\vartheta) = (R_\omega \otimes \text{id})\varpi(\vartheta)
\]
(4.24)
for each $\vartheta \in \Gamma_{inv}$.
Definition 4.4. The constructed map \( R_\omega \) is called the curvature of \( \omega \).

The curvature \( R_\omega \) implicitly depends on the choice of \( \delta \). This dependence disappears if \( \omega \) is multiplicative.

We are going to introduce the operator of covariant derivative. This operator will be first defined on a restricted domain consisting of horizontal forms. Later on, after constructing the horizontal projection operator, we shall extend the domain of the covariant derivative to the whole algebra \( \Omega(P) \).

For each \( \omega \in \text{con}(P) \) and \( \varphi \in \hat{\text{hor}}(P) \) let us define a new form

\[
D_\omega(\varphi) = d\varphi - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k),
\]

where \( F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k \).

Lemma 4.5. We have

\[
\hat{F} D_\omega(\varphi) = \sum_k D_\omega(\varphi_k) \otimes c_k
\]

for each \( \varphi \in \hat{\text{hor}}(P) \). In particular

\[
D_\omega \hat{\text{hor}}(P) \subseteq \hat{\text{hor}}(P).
\]

Proof. We compute

\[
\begin{align*}
\hat{F} D_\omega(\varphi) &= \hat{F} d\varphi - (-1)^{\partial \varphi} \sum_k \hat{F}(\varphi_k) \hat{F} \omega \pi(c_k) \\
&= d \left( \sum_k \varphi_k \otimes c_k \right) - (-1)^{\partial \varphi} \sum_k \varphi_k \otimes c_k^{(1)} \pi(c_k^{(2)}) \\
&\quad - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k^{(3)}) \otimes c_k^{(1)} \kappa(c_k^{(2)}) c_k^{(4)} \\
&= \sum_k d\varphi_k \otimes c_k - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k^{(1)}) \otimes c_k^{(2)} \\
&= \sum_k D_\omega(\varphi_k) \otimes c_k.
\end{align*}
\]

Hence, \( \hat{\text{hor}}(P) \) is \( D_\omega \)-invariant.\( \square \)
Definition 4.5. The constructed map $D_\omega : \text{hor}(P) \to \text{hor}(P)$ is called the covariant derivative associated to $\omega$.

Proposition 4.6. (i) The diagram

$$
\begin{array}{ccc}
\text{hor}(P) & \xrightarrow{F^\wedge} & \text{hor}(P) \otimes A \\
D_\omega & & D_\omega \otimes \text{id} \\
\text{hor}(P) & \xrightarrow{F^\wedge} & \text{hor}(P) \otimes A \\
\end{array}
$$

is commutative.

(ii) We have

$$
D^2_\omega (\varphi) = - \sum_k \varphi_k \left[ d\omega \pi(c_k) + \omega \pi(c_k^{(1)}) \omega \pi(c_k^{(2)}) \right]
$$

for each $\varphi \in \text{hor}(P)$. In particular, if $\omega$ is multiplicative then

$$
D^2_\omega (\varphi) = - \sum_k \varphi_k \omega R_\omega \pi(c_k).
$$

(iii) If $\omega$ is regular then

$$
D_\omega (\varphi \psi) = D_\omega (\varphi) \psi + (-1)^{\partial \varphi} \varphi D_\omega (\psi)
$$

$$
D_\omega (\varphi^*) = D_\omega (\varphi)^* ,
$$

for each $\varphi, \psi \in \text{hor}(P)$.

(iv) We have

$$
(d - D_\omega)(\Omega(M)) = \{0\}
$$

for each $\omega \in \text{con}(P)$.

Proof. Diagram (4.27) follows from identity (4.26). Property (iv) follows from definitions of $\Omega(M)$ and $D_\omega$. For each $\varphi \in \text{hor}(P)$ we have

$$
D^2_\omega (\varphi) = D_\omega \left( d\varphi - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k) \right)
$$

$$
= - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k) - \sum_k \varphi_k d\omega \pi(c_k)
$$

$$
- (-1)^{\partial \varphi} \sum_k \left( -d\varphi_k \omega \pi(c_k) + (-1)^{\partial \varphi} \varphi_k \omega \pi(c_k^{(1)}) \omega \pi(c_k^{(2)}) \omega \pi(c_k^{(3)}) \omega \pi(c_k^{(4)}) \right)
$$

$$
= - \sum_k \varphi_k \left( d\omega \pi(c_k) + \omega \pi(c_k^{(1)}) \omega \pi(c_k^{(2)}) \right).
$$

If $\omega$ is multiplicative then (4.23) and the definition of $\delta$ imply that (4.29) holds.
Finally, let us assume that $\omega$ is regular. Then

$$D_\omega(\varphi \psi) = (d\varphi)\psi + (-1)^{\partial \varphi} \varphi d\psi$$

$$- (-1)^{\partial \varphi + \partial \psi} \sum_{kl} \left( \varphi_k \psi_l \omega(\pi(c_k) \circ d_l) + \varphi_k \psi_l \epsilon(c_k) \omega \pi(d_l) \right)$$

$$= \left( d\varphi - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k) \right) \psi + (-1)^{\partial \varphi} \varphi \left( d\psi - (-1)^{\partial \psi} \sum_l \psi_l \omega \pi(d_l) \right)$$

$$= D_\omega(\varphi)\psi + (-1)^{\partial \varphi} \varphi D_\omega(\psi),$$

where $\sum_l \psi_l \otimes d_l = F^\wedge(\psi)$. Further,

$$D_\omega(\varphi)^* = d(\varphi^*) + \sum_k \omega \pi(\kappa(c_k)^*) \varphi_k^*$$

$$= d(\varphi^*) + (-1)^{\partial \varphi} \sum_k \varphi_k^* \omega \left[ \pi(\kappa(c_k^{(2)})^*) \circ c_k^{(1)*} \right]$$

$$= d(\varphi^*) - (-1)^{\partial \varphi} \sum_k \varphi_k^* \omega \pi(c_k^*) = D_\omega(\varphi^*).$$

Hence, properties $(iii)$ hold.

Actually, a connection $\omega$ is regular if and only if $D_\omega$ satisfies the graded Leibniz rule. Because of (4.27), the space $\tau(P)$ is closed under taking compositions with $D_\omega$. This fact enables us to define the covariant derivative (which will be denoted by the same symbol) as an operator acting in the space $\tau(P)$.

**Proposition 4.7.** We have

$$D_\omega \varphi = d\varphi - (-1)^{\partial \varphi}[\varphi, \omega]$$

for each $\omega \in \text{con}(P)$ and $\varphi \in \tau(P)$.

**Proof.** This directly follows from the tensoriality of $\varphi$ and from definitions of $D_\omega$ and brackets $[,]$. 

For a given $\omega \in \text{con}(P)$ let $q_\omega : \psi(P) \to \psi(P)$ be a linear map defined by

$$q_\omega(\varphi) = \langle \omega, \varphi \rangle - (-1)^{\partial \varphi} \langle \varphi, \omega \rangle - (-1)^{\partial \varphi}[\varphi, \omega].$$

**Lemma 4.8.** (i) We have

$$\tilde{F} q_\omega(\vartheta) = (q_\omega \otimes \text{id}) F^\wedge(\vartheta)$$

for each $\varphi \in \tau(P)$ and $\vartheta \in \Gamma_{\text{inv}}$. In particular $q_\omega \tau(P) \subseteq \tau(P)$.

(ii) If $\omega \in \tau(P)$ then

$$q_\omega (\tau(P)) = 0.$$
Proof. We compute
\[
\bar{F}q_\omega(\varphi)(\vartheta) = -\bar{F}[\omega \pi(a^{(1)}) \varphi \pi(a^{(2)})] + (\omega, \varphi)(a^{(1)}) \omega \pi(a^{(2)})
\]
\[
- (-1)^{\partial \varphi} \sum_k \bar{F}(\vartheta_k) \omega \pi(c_k)
\]
\[
= -\omega \pi(a^{(2)}) \varphi \pi(a^{(3)}) \otimes \kappa(a^{(1)}) a^{(4)}
\]
\[
- (-1)^{\partial \varphi} \varphi \pi(a^{(3)}) \otimes \pi(a^{(1)}) \kappa(a^{(2)}) a^{(4)}
\]
\[
+ (-1)^{\partial \varphi} \varphi \pi(a^{(2)}) \omega \pi(a^{(3)}) \otimes \kappa(a^{(1)}) a^{(4)}
\]
\[
+ (-1)^{\partial \varphi} \varphi \pi(a^{(2)}) \kappa(a^{(1)}) a^{(3)} \pi(a^{(4)})
\]
\[
- (-1)^{\partial \varphi} \sum_k \left( \varphi(\vartheta_k) \omega \pi(c_k^{(3)}) \otimes c_k^{(1)} \kappa(c_k^{(2)}) c_k^{(4)} + \varphi(\vartheta_k) \otimes c_k^{(1)} \pi(c_k^{(2)}) \right)
\]
\[
= \sum_k \left( \langle \omega, \varphi \rangle(\vartheta_k) \otimes c_k - (-1)^{\partial \varphi} \varphi(\vartheta_k) \otimes c_k \right)
\]
\[
- (-1)^{\partial \varphi} \sum_k \langle \varphi, \omega \rangle(\vartheta_k) \otimes c_k
\]
\[
- (-1)^{\partial \varphi} \sum_k \varphi(\vartheta_k) \otimes dc_k
\]
\[
- (-1)^{\partial \varphi} \varphi \pi(a^{(4)}) \otimes \kappa(a^{(1)}) (da^{(2)}) \kappa(a^{(3)}) a^{(5)}
\]
\[
+ (-1)^{\partial \varphi} \varphi \pi(a^{(2)}) \otimes \kappa(a^{(1)}) da^{(3)} = \sum_k q_\omega(\varphi)(\vartheta_k) \otimes c_k.
\]

Here, \( a \in \ker(\epsilon) \) satisfies (4.18), and \( \sum_k \vartheta_k \otimes c_k = \varpi(\vartheta) \).

Let us assume that \( \omega \) is regular. Applying (2.3), (2.11) and (4.10), and the definition of brackets \( \langle , \rangle \) and \( [ , ] \) we obtain
\[
\langle \omega, \varphi \rangle(\vartheta) = -\omega \pi(a^{(1)}) \varphi \pi(a^{(2)})
\]
\[
= -(-1)^{\partial \varphi} \varphi \pi(a^{(3)}) \omega \pi(a^{(1)}) \circ (\kappa(a^{(2)}) a^{(4)})
\]
\[
= -(-1)^{\partial \varphi} \varphi \pi(a^{(3)}) \omega \pi(a^{(1)}) \circ (\kappa(a^{(2)}) a^{(4)})
\]
\[
= -(-1)^{\partial \varphi} \varphi(\varpi(a^{(1)})) \omega \pi(a^{(2)}) + (-1)^{\partial \varphi} \varphi \pi(a^{(2)}) \omega \pi(\kappa(a^{(1)}) a^{(3)})
\]
\[
= (-1)^{\partial \varphi} \langle \varphi, \omega \rangle(\vartheta) + (-1)^{\partial \varphi} [\varphi, \omega](\vartheta).
\]

Hence, (4.36) holds. \( \square \)

Actually, the following equality holds
\[
q_\omega(\varphi)(\vartheta) = \sum_k \ell^\omega(\vartheta_k^1, \varphi(\vartheta_k^2))
\]
for each \( \vartheta \in \Gamma_{inv} \) and \( \varphi \in \tau(P) \). The next proposition gives a quantum counterpart for the classical Bianchi identity.

**Proposition 4.9.** We have
\[
(D_\omega - q_\omega)(R_\omega) = \langle \omega, \omega \rangle - \langle \omega, \omega \rangle
\]
for each \( \omega \in \con(P) \).
Proof. A direct computation gives
\[(D_\omega - q_\omega)(R_\omega) = dR_\omega - \langle \omega, R_\omega \rangle + \langle R_\omega, \omega \rangle = -d\langle \omega, \omega \rangle
- \langle \omega, d\omega - \langle \omega, \omega \rangle \rangle + \langle d\omega - \langle \omega, \omega \rangle, \omega \rangle
= \langle \omega, \langle \omega, \omega \rangle \rangle - \langle \langle \omega, \omega \rangle, \omega \rangle. \]

If the connection \( \omega \) is multiplicative then the right-hand side of equality \( (4.37) \) vanishes. On the other hand, if \( \omega \) is regular then the second summand on the left-hand side vanishes.

It is important to point out that regular connections are not necessarily multiplicative. However, there exists a common obstruction to multiplicativity for all regular connections, so that if one regular connection is multiplicative, then every regular connection possesses the same property. This obstruction will be now analyzed in more details.

In general, the lack of multiplicativity of a connection \( \omega \) is measured by the map \( r_\omega : \mathcal{R} \to \Omega(P) \) given by \( r_\omega(a) = \omega \pi(a^{(1)}) \omega \pi(a^{(2)}) \).

Lemma 4.10. (i) The following identities hold
\[
\begin{align*}
r_\omega(\kappa(a)^*) &= -r_\omega(a) \\
\pi_* r_\omega(a) &= 0 \\
\hat{F} r_\omega(a) &= F^\kappa r_\omega(a) = (r_\omega \otimes \text{id}) \text{ad}(a).
\end{align*}
\]
In particular \( r_\omega(a) \in \mathfrak{hor}^2(P) \) for each \( a \in \mathcal{R} \).

(ii) Let us assume that \( P \) admits regular connections. The map \( \omega \mapsto r_\omega \) is constant on equivalence classes from the space \( \text{con}(P)/\tau(P) \). If \( \omega \in \tau(P) \) then
\[
r_\omega(a) \varphi = \sum_k \varphi_k r_\omega(ac_k),
\]
for each \( a \in \mathcal{R} \) and \( \varphi \in \mathfrak{hor}(P) \). Furthermore,
\[
tr_\omega(a) = \langle \omega, \omega \rangle \pi(a^{(1)}) \omega \pi(a^{(2)}) - \omega \pi(a^{(1)})\langle \omega, \omega \rangle \pi(a^{(2)}).
\]

Proof. A direct computation gives
\[
r_\omega(a)^* = -\omega \pi(a^{(2)})^* \omega \pi(a^{(1)})^* = -\omega \pi(\kappa(a^{(2)})^*) \omega \pi(\kappa(a^{(1)})^*) = -r_\omega(\kappa(a)^*)
\]
and similarly
\[
\hat{F} r_\omega(a) = \left[ (\omega \otimes \text{id}) \omega \pi(a^{(1)}) + 1 \otimes \pi(a^{(1)}) \right] \left[ (\omega \otimes \text{id}) \omega \pi(a^{(2)}) + 1 \otimes \pi(a^{(2)}) \right]
= \omega \pi(a^{(2)}) \omega \pi(a^{(3)}) \otimes \kappa(a^{(1)}) a^{(4)} + \omega \pi(a^{(2)}) \otimes \kappa(a^{(1)}) a^{(3)} \pi(a^{(4)})
- \omega \pi(a^{(3)}) \otimes \kappa(a^{(1)}) a^{(2)} = \hat{F} r_\omega(a) = (r_\omega \otimes \text{id}) \text{ad}(a).
\]
Let us assume that $\omega$ is regular. Then

$$r_{\omega+\varphi}(a) = \omega\pi(a^{(1)})\omega_{\pi}(a^{(2)}) - \varphi\pi(a^{(3)})\omega\big[\pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)})\big]$$

$$+ \varphi\pi(a^{(1)})\varphi\pi(a^{(2)}) + \varphi\pi(a^{(1)})\omega\pi(a^{(2)})$$

$$= \omega\pi(a^{(1)})\omega_{\pi}(a^{(2)}) + \varphi\pi(a^{(1)})\varphi\pi(a^{(2)}) + \varphi\pi(a^{(2)})\omega\pi\big(\kappa(a^{(1)})a^{(3)}\big)$$

$$= \omega\pi(a^{(1)})\omega_{\pi}(a^{(2)}) + \varphi\pi(a^{(1)})\varphi\pi(a^{(2)})$$

for each $\varphi = \varphi^* \in \tau^1(P)$.

Now if $\zeta = \zeta^* \in \tau^1(P)$ satisfies (4.13) then $\varphi\pi(a^{(1)})\zeta\pi(a^{(2)}) + \zeta\pi(a^{(1)})\varphi\pi(a^{(2)}) = 0$, and in particular $\zeta\pi(a^{(1)})\zeta\pi(a^{(2)}) = 0$ for each $a \in \mathcal{R}$. Hence,

$$r_{\omega+\varphi+\zeta} = r_{\omega+\varphi}.$$

Further, applying (2.3), (4.23) and the tensoriality of the curvature we obtain

$$dr_{\omega}(a) = R_{\omega}\pi(a^{(1)})\omega_{\pi}(a^{(2)}) + \langle \omega,\omega \rangle\pi(a^{(1)})\omega\pi(a^{(2)})$$

$$- \omega\pi(a^{(1)})R_{\omega}\pi(a^{(2)}) - \omega\pi(a^{(1)})\langle \omega,\omega \rangle\pi(a^{(2)})$$

$$= R_{\omega}\pi(a^{(1)})\omega_{\pi}(a^{(2)}) + \langle \omega,\omega \rangle\pi(a^{(1)})\omega\pi(a^{(2)})$$

$$- R_{\omega}\pi(a^{(3)})\omega\big[\pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)})\big] - \omega\pi(a^{(1)})\langle \omega,\omega \rangle\pi(a^{(2)})$$

$$= \langle \omega,\omega \rangle\pi(a^{(1)})\omega\pi(a^{(2)}) - \omega\pi(a^{(1)})\langle \omega,\omega \rangle\pi(a^{(2)})$$

$$+ R_{\omega}\pi(a^{(2)})\omega\pi\big(\kappa(a^{(1)})a^{(3)}\big)$$

$$= \langle \omega,\omega \rangle\pi(a^{(1)})\omega\pi(a^{(2)}) - \omega\pi(a^{(1)})\langle \omega,\omega \rangle\pi(a^{(2)}).$$

Finally,

$$r_{\omega}(a) = \sum_k \varphi_k\omega\big[\pi(a^{(1)}) \circ c_k^{(1)}\big]\omega\big[\pi(a^{(2)}) \circ c_k^{(2)}\big]$$

$$= \sum_k \varphi_k\omega\pi\big((a^{(1)} - \epsilon(a^{(1)})1)c_k^{(1)}\big)\omega\pi\big((a^{(2)} - \epsilon(a^{(2)})1)c_k^{(2)}\big)$$

$$\sum_k \varphi_k\omega\pi(a^{(1)}c_k^{(1)})\omega\pi(a^{(2)}c_k^{(2)}) - \sum_k \varphi_k\omega\pi(ac_k^{(1)})\omega\pi(c_k^{(2)})$$

$$- \sum_k \varphi_k\omega\pi(c_k^{(1)})\omega\pi(ac_k^{(2)}) + \epsilon(a) \sum_k \varphi_k\omega\pi(c_k^{(1)})\omega\pi(c_k^{(2)})$$

$$= \sum_k \varphi_k\omega\pi(a^{(1)}c_k^{(1)})\omega\pi(a^{(2)}c_k^{(2)}) + \epsilon(a) \sum_k \varphi_kr_{\omega}(ac_k)$$

for each $\varphi \in \mathfrak{hor}(P)$. Here, we have used (4.10), and the fact that $\mathcal{R} \subseteq \ker(\epsilon)$ is a right ideal. □

Let us assume that $P$ admits regular connections, and let $\mathcal{S}(P)$ be the left ideal in $\Omega(P)$ generated by the space $r_{\omega}(\mathcal{R})$, for some $\omega \in \mathfrak{r}(P)$. 
Lemma 4.11. (i) The following properties hold:

\begin{align*}
&\mathfrak{T}(P)^* = \mathfrak{T}(P) \\
&\hat{\mathfrak{F}}\mathfrak{T}(P) \subseteq \mathfrak{T}(P) \otimes \Gamma^\wedge \\
&\pi_v \mathfrak{T}(P) = \{0\} \\
&d\mathfrak{T}(P) \subseteq \mathfrak{T}(P).
\end{align*}

(ii) The space \(\mathfrak{T}(P)\) is a two-sided ideal in \(\Omega(P)\).

Proof. Properties (4.44)–(4.45) directly follow from identities (4.39)–(4.40). Concerning (4.46), it follows from (4.42), and the following observations. As first, \(\delta \pi(a) + \pi(a^{(1)}) \otimes \pi(a^{(2)})\) belongs to \(S^\wedge_{inv}\) for each \(a \in A\). Secondly \(\omega \otimes (S^\wedge_{inv}) = r_\omega(\mathcal{R})\). Thirdly, because of (4.11) and (4.40),

\[ r_\omega(a) \omega(\vartheta) \in \mathfrak{T}(P) \]

for each \(a \in \mathcal{R}\) and \(\vartheta \in \mathfrak{hor}(P)\). However, this follows from (4.41) in a straightforward way. Finally, (4.43) follows from (ii) and (4.38). \(\square\)

The ideal \(\mathfrak{T}(P)\) measures the lack of multiplicativity of regular connections. Let \(\omega\) be a regular connection on \(P\), and let us assume that \(\mathfrak{T}(P) = \{0\}\). Then \(\omega\) is multiplicative and the map \(\omega^\wedge : \Gamma^\wedge_{inv} \to \Omega(P)\) possesses the following commutation properties with horizontal forms

\[ \omega^\wedge(\vartheta) \varphi = (-1)^{\partial_\varphi \partial_\vartheta} \sum_k \varphi_k \omega^\wedge(\vartheta \circ c_k) \]

\[ \varphi \omega^\wedge(\vartheta) = (-1)^{\partial_\varphi \partial_\vartheta} \sum_k \omega^\wedge(\vartheta \circ \kappa^{-1}(c_k)) \varphi_k, \]

as easily follows from (4.10) and (4.11).

The formulas

\[ (\psi \otimes \eta)(\varphi \otimes \vartheta) = (-1)^{\partial_\psi \partial_\eta} \sum_k \psi_k \varphi \otimes (\eta \circ c_k) \vartheta \]

\[ (\varphi \otimes \vartheta)^* = \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*), \]

determine the structure of a \(*\)-algebra in the space \(\mathfrak{v}(P) = \mathfrak{hor}(P) \otimes \Gamma^\wedge_{inv}\). The proof is essentially the same as the proof of the \(*\)-algebra properties for \(\mathfrak{ver}(P)\) (actually \(\mathfrak{ver}(P)\) is a \(*\)-subalgebra of \(\mathfrak{v}(P)\)). Here as usual, \(F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k\). The elements of \(\mathfrak{v}(P)\) are interpretable as “vertically-horizontally” decomposed differential forms on \(P\). In the following considerations the space \(\mathfrak{v}(P)\) will be endowed with this (graded) \(*\)-algebra structure.

We shall now construct, starting from an arbitrary connection \(\omega\), an important isomorphism between the spaces \(\Omega(P)\) and \(\mathfrak{v}(P)\), extending the splitting \(\mu_\omega\). This
isomorphism will be the base for the construction and the analysis of horizontal projection operators. Also, some interesting questions related to the structure of horizontal forms will be considered.

Let us fix a splitting of the form

$$\Gamma^\otimes = \Gamma^\wedge \oplus S^\wedge$$

in which $\Gamma^\wedge$ is realized as a complement to the space $S^\wedge$, with the help of a *-preserving section $\iota: \Gamma^\wedge \to \Gamma^\otimes$ (of the factor-projection map) intertwining the adjoint actions. Further, let us assume that the embedded differential $\delta$ is given by

$$\delta(\vartheta) = \iota d(\vartheta).$$

Finally, let us consider a linear map $m_\omega: \mathfrak{vh}(P) \to \Omega(P)$ given by

$$(4.51) \quad m_\omega(\varphi \otimes \vartheta) = \varphi \omega^\wedge(\vartheta),$$

where

$$(4.52) \quad \omega^\wedge = \omega \otimes \iota.$$
We are going to prove, inductively, that the spaces $\Omega_k$ according to Lemma 3.5 where $\psi$ are independent elements. On the other hand, $\omega$ are linearly independent. Consequently, the statement is true for $k = 0$. Let us assume that there exists an element $w \in \Omega_{k+1}(P) \setminus \Omega_k(P)$. Then we have, according to Lemma 3.5

$$ \tilde{F}(w) = \psi + \sum_i F^\wedge(\varphi_i) \vartheta_i,$$

where $\psi \in \Omega(P) \otimes \Gamma_k$ and $\varphi_i \in \mathfrak{hor}(P) \setminus \{0\}$ while $\vartheta_i \in (\Gamma_{inv})_{1+k}$ are linearly independent elements. On the other hand,

$$ \tilde{F} m_\omega \left( \sum_i \varphi_i \otimes \vartheta_i \right) = \psi' + \sum_i F^\wedge(\varphi_i) \vartheta_i,$$

where $\psi' \in \Omega(P) \otimes \Gamma_k$. Hence, $w - m_\omega \left( \sum_i \varphi_i \otimes \vartheta_i \right)$ belongs to $\Omega_k(P)$ and therefor $w \in m_\omega(\mathfrak{hor}(P))$. We conclude that $m_\omega$ is surjective. Thus, (i) holds.

Intertwining property (4.53) directly follows from the definition of $m_\omega$ and from the right-covariance of $\omega^\wedge$.

Finally, let us assume that $\mathcal{Z}(P) = \{0\}$ and let $\omega \in \mathfrak{r}(P)$. Then $\omega$ is multiplicative and $\omega^\wedge$ is a *-homomorphism. Using (4.47) and (4.49)–(4.50) we obtain

$$ m_\omega \left[ (\psi \otimes \eta)(\varphi \otimes \vartheta) \right] = \sum_k (-1)^{\partial \varphi \partial \eta} \psi \varphi_k \omega^\wedge \left[ (\eta \circ c_k) \vartheta \right] = \sum_k (-1)^{\partial \varphi \partial \eta} \psi \varphi_k \omega^\wedge (\eta \circ c_k) \omega^\wedge (\vartheta) = \psi \omega^\wedge (\eta) \omega^\wedge (\vartheta) = m_\omega(\psi \otimes \eta) m_\omega(\varphi \otimes \vartheta),$$

and similarly

$$ m_\omega \left[ (\varphi \otimes \vartheta)^* \right] = \sum_k \varphi_k^* \omega^\wedge (\vartheta^* \circ c_k^*) = (-1)^{\partial \varphi \partial \vartheta} \omega^\wedge (\vartheta^*) \varphi^* = (-1)^{\partial \varphi \partial \vartheta} \omega^\wedge (\vartheta)^* \varphi^* = [\varphi \omega^\wedge (\vartheta)]^* = [m_\omega(\varphi \otimes \vartheta)]^*.$$

In other words, $m_\omega$ is a *-algebra isomorphism. □
The spaces $\Omega_k(P)$ introduced in the above proof form a filtration of $\Omega(P)$, compatible with the graded-differential *-algebra structure. In particular

$$\Omega_k(P) = \sum_{j \geq 0} \Omega_j^k(P).$$

For each $k$ and $\omega \in \con(P)$ the space $\Omega_k(P)$ is linearly spanned by elements of the form $\varphi \omega^\wedge(\vartheta)$, where $\varphi \in \hor(P)$ and $\vartheta \in \Gamma^i_m(P)$, with $i \leq k$. Let $\bar{\mathcal{U}}(P)$ be the graded-differential *-algebra associated to the filtered algebra $\Omega(P)$. The map $\Pi: \mathfrak{vh}(P) \to \bar{\mathcal{U}}(P)$ given by

$$\Pi(\varphi \otimes \vartheta) = [\varphi \omega^\wedge(\vartheta) + \Omega_{k-1}(P)],$$

where $\vartheta \in \Gamma^k_m$, is bijective (and independent of the choice of $\omega$ and $\iota$). Moreover, $\Pi$ is a *-isomorphism, as follows from Lemma 4.3 and the horizontality of $r_\omega$. The introduced filtration is compatible with the map $\hat{F}$, in the sense that $\hat{F}(\Omega_k(P)) \subseteq \Omega_k(P) \otimes \Gamma^k_m$.

In other words, $\hat{F}$ is factorizable through the filtration. The diagrams

$$(4.54) \quad \begin{array}{ccc}
\mathfrak{vh}(P) & \xrightarrow{\Pi} & \bar{\mathcal{U}}(P) \\
\downarrow d_{\mathfrak{vh}} & & \downarrow d \\
\mathfrak{vh}(P) & \xrightarrow{\Pi} & \bar{\mathcal{U}}(P) \\
\downarrow \hat{F}_{\mathfrak{vh}} & & \downarrow \hat{F} \\
\mathfrak{vh}(P) \otimes \Gamma^k_m & \xrightarrow{\Pi \otimes \text{id}} & \bar{\mathcal{U}}(P) \otimes \Gamma^k_m
\end{array}$$

describe the corresponding factorized maps in terms of $\mathfrak{vh}(P)$. The differential $d_{\mathfrak{vh}}: \mathfrak{vh}(P) \to \mathfrak{vh}(P)$ is given by

$$d_{\mathfrak{vh}}(\varphi \otimes \vartheta) = (-1)^{\partial\varphi} \sum_k \varphi_k \otimes \pi(c_k) \vartheta + (-1)^{\partial\varphi} \varphi \otimes d(\vartheta).$$

Similarly, $\hat{F}_{\mathfrak{vh}}: \mathfrak{vh}(P) \to \mathfrak{vh}(P) \otimes \Gamma^k_m$ is given by

$$\hat{F}_{\mathfrak{vh}}(\varphi \otimes \vartheta) = F^k(\varphi) \hat{\omega}(\vartheta).$$

It is worth noticing that $d_{\mathfrak{vh}}|_{\mathfrak{ver}(P)} = d_v$ and $\hat{F}_{\mathfrak{vh}}|_{\mathfrak{ver}(P)} = \hat{F}_v$.

The next lemma gives a more detailed description of higher-order horizontal forms.

**Lemma 4.13.** If the bundle admits regular connections then the algebra

$$\mathfrak{hor}^+(P) = \sum_{k \geq 1} \mathfrak{hor}^k(P)$$

is generated by spaces $\mathfrak{hor}^1(P)$ and $r_\omega(R)$ (where $\omega \in \mathfrak{v}(P)$). In particular, if $\Sigma(P) = \{0\}$ then higher-order horizontal forms are algebraically expressible through the first-order ones.
Proof. The algebra

\[ \Omega^+(P) = \sum_{k \geq 1} \Omega^k(P) \]

is generated by the space \( \Omega^1(P) \) of 1-forms. This fact, together with the *-bimodule splitting \( \Omega^1(P) = \mathfrak{hor}^1(P) \oplus \mathfrak{vec}^1(P) \) determined by an arbitrary \( \omega \in \mathfrak{v}(P) \), can be used to prove that each element \( w \in \Omega^n(P) \) is expressible in the form

\[(4.55) \quad w = \sum_{k=0}^n \left[ \sum_i \varphi_{ik} \omega^\wedge(\theta_{ik}) + \sum_j \psi_{jk} \omega^\wedge(\eta_{jk}) \right],\]

where \( \theta_{ik} \) and \( \eta_{jk} \) are linearly independent elements in the spaces \( \Gamma^{\wedge k}_{mv} \) and \( S^{\wedge k}_{mv} \) respectively, while \( \varphi_{ik}, \psi_{jk} \) are horizontal \( (n-k) \)-forms, expressible as sums of products of \( n-k \) factors from \( \mathfrak{hor}^1(P) \). Now using the facts that \( S^{\wedge 2}_{mv} \) is generated by \( S^{\wedge 2}_{n+2} \) and that \( \omega^\wedge(Q) \) is horizontal for each \( Q \in S^{\wedge 2}_{n+2} \) (because of \( \omega^\wedge(Q) \in r_\omega(\mathcal{R}) \)) we can prove, inductively applying (4.10) and using the definition of \( r_\omega \) and identity (4.40), that the elements \( \omega^\wedge(\eta_{jk}) \) are expressible as sums of products of the form \( r_\omega(a_1) \cdots r_\omega(a_m) \omega^\wedge(\theta_{kl}) \) with \( l + 2m = k \), and a possibly extended set of \( \theta_{ki} \). Inserting this in (4.55) we conclude that

\[(4.56) \quad w = \sum_{k=0}^n \left[ \sum_i \tilde{\varphi}_{ik} \omega^\wedge(\theta_{ik}) \right],\]

where \( \tilde{\varphi}_{ik} \) are horizontal \( (n-k) \)-forms expressible as sums of products of elements from \( \mathfrak{hor}^1(P) \) and \( \text{im}(r_\omega) \). If \( w \in \mathfrak{hor}^1(P) \) then \( \tilde{\varphi}_{ik} = 0 \) for each \( 1 \leq k \leq n \), according to Theorem 4.12 (i). Hence, higher-order horizontal forms are algebraically expressible via the first-order ones, and the 2-forms from the space \( r_\omega(\mathcal{R}) \). \( \square \)

In the general case \( (P) \) is an arbitrary bundle and \( \omega \) is an arbitrary connection) the algebra \( \mathfrak{hor}^1(P) \) is generated by spaces \( \mathfrak{hor}^1(P), r_\omega(\mathcal{R}) \) and horizontal forms obtained by iteratively acting by \( \ell^\omega \) on the elements from \( \mathfrak{hor}^1(P) \) and \( r_\omega(\mathcal{R}) \). It is interesting to observe that in the general case

\[ r_\omega(a) \varphi = \sum_k \varphi_k r_\omega(ac_k) + \ell^\omega(\pi(a^{(1)}), \ell^\omega(\pi(a^{(2)}), \varphi)). \]

If the ideal \( \mathcal{I}(P) \) is non-trivial (if every \( \omega \in \mathfrak{v}(P) \) is not multiplicative) then we can “renormalize” the calculus, passing to the factoralgebra \( \Omega^*(P) = \Omega(P)/\mathcal{I}(P) \) and projecting the whole formalism from \( \Omega(P) \) on \( \Omega^*(P) \). It is worth noticing that such a factorization preserves the first-order calculus. In the framework of this projected calculus regular connections become multiplicative. More precisely, let \( \mathfrak{h}^*(P) \subseteq \Omega^*(P) \) be the corresponding algebra of horizontal forms, and let \( \Pi: \Omega(P) \to \Omega^*(P) \) be the projection map.

**Lemma 4.14.** We have

\[(4.57) \quad \mathfrak{h}^*(P) = \Pi(\mathfrak{hor}(P)). \]

In particular, if \( \omega \) is a regular connection relative to \( \Omega(P) \) then \( \Pi \omega \) is regular in terms of \( \Omega^*(P) \).
Proof. It is evident that $\Pi(\mathfrak{hor}(P)) \subseteq \mathfrak{h}^*(P)$. Let $m^*_\omega$ be the factorization map corresponding to the calculus $\Omega^*(P)$ and to the connection $\Pi\omega$. We have then

$$m^*_\omega[(\Pi(\mathfrak{hor}(P)) \otimes \text{id}] = \Pi m^*_\omega.$$

In particular, (4.57) holds. \(\square\)

With the help of the identification $m^*_\omega$ the corresponding \textit{horizontal projection operator} $h_\omega : \Omega(P) \to \mathfrak{hor}(P)$ can be defined as follows

$$(4.58) \quad h_\omega = (\text{id} \otimes p_0)m^{-1}_\omega.$$

Clearly, $h_\omega$ projects $\Omega(P)$ onto $\mathfrak{hor}(P)$.

The domain of the previously introduced covariant derivative $D_\omega$ will be now extended from $\mathfrak{hor}(P)$ to the whole algebra $\Omega(P)$. Let a map $D_\omega : \Omega(P) \to \mathfrak{hor}(P)$ be defined as follows

$$(4.59) \quad D_\omega = h_\omega d.$$

This is a straightforward generalization of the corresponding classical definition.

The main properties of $h_\omega$ and $D_\omega$ are collected in the following theorem.

\textbf{Theorem 4.15.} (i) The diagrams

\begin{equation}
\begin{array}{ccc}
\Omega(P) & \xrightarrow{F^\wedge} & \Omega(P) \otimes A \\
\downarrow h_\omega & & \downarrow h_\omega \otimes \text{id} \\
\mathfrak{hor}(P) & \xrightarrow{F^\wedge} & \mathfrak{hor}(P) \otimes A
\end{array}
\end{equation}

are commutative.

(ii) The map $D_\omega$ extends the previously defined covariant derivative.

(iii) If $\omega$ is regular and if $\mathfrak{F}(P) = \{0\}$ then $h_\omega$ is a $\ast$-homomorphism and

$$(4.61) \quad D_\omega(wu) = D_\omega(w)h_\omega(u) + (-1)^{\partial w}h_\omega(w)D_\omega(u),$$

$$(4.62) \quad D_\omega(w^*) = D_\omega(w)^*,$$

for each $w, u \in \Omega(P)$.

\textbf{Proof.} The statement (i) follows from the construction of $h_\omega$ and $D_\omega$, and properties (3.17) and (4.53). The statement (iii) is a consequence of property (iii) in Theorem 4.12, and elementary properties of $d : \Omega(P) \to \Omega(P)$. Finally, in the first definition (4.25) of the covariant derivative, the differential of a horizontal form $\varphi$ is written as

$$d\varphi = m^*_\omega\left[D_\omega(\varphi) \otimes 1 + (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \pi(c_k)\right],$$

which implies that the new definition includes the previous one. \(\square\)

According to (4.60), compositions of pseudotensorial forms with $D_\omega$ and $h_\omega$ are tensorial. In particular, it is possible to define, via these compositions, the covariant derivative and the horizontal projection as maps $D_\omega, h_\omega : \psi(P) \to \tau(P)$.

The following proposition gives a more geometrical description of the curvature map, establishing a close analogy with classical geometry.
Proposition 4.16. We have

\[ R_\omega = D_\omega \omega \]

for each \( \omega \in \con(P) \).

Proof. From definitions of \( m_\omega \) and \( R_\omega \), we find

\[ R_\omega(\vartheta) \otimes 1 = m_\omega^{-1}d\omega(\vartheta) - 1 \otimes \delta(\vartheta), \]

for each \( \vartheta \in \Gamma_{inv} \). Hence, \( D_\omega \omega = h_\omega d\omega = R_\omega \).

Let us assume that \( \omega \in \frak{r}(P) \) and \( \frak{T}(P) = \{0\} \). The above proposition implies

\[ R_\omega(\vartheta) = \sum_k \varphi_k R_\omega(\vartheta \circ c_k) \]

(4.64)

for each \( \varphi \in \frak{hor}(P) \) and \( \vartheta \in \Gamma_{inv} \). Evidently, the above commutation relations are mutually equivalent. To obtain the first it is sufficient to act by \( D_\omega \) on (4.10), and apply (4.61) and (4.63).

5. Characteristic Classes

In this section a quantum generalization of classical Weil’s theory of characteristic classes will be presented. Conceptually, we follow the exposition of [8]. As in the classical case, the main result will be a construction of an invariant homomorphism defined on an algebra playing the role of “invariant polynomials” over the “Lie algebra” of \( G \), with values in the algebra of cohomology classes of \( M \). We shall assume that the bundle \( P \) admits regular connections, and that \( \frak{T}(P) = \{0\} \).

For each \( k \geq 0 \) let \( \mathcal{I}^k \subseteq \Gamma_{inv}^k \) be the subspace of \( \varpi^k \)-invariant elements, and let \( \mathcal{I} \) be the direct sum of these spaces. Clearly, \( \mathcal{I} \) is a unital \(*\)-subalgebra of \( \Gamma_{inv}^\infty \).

Let us consider a connection \( \omega \). There exists the unique unital homomorphism \( R^\omega: \Gamma_{inv} \to \Omega(P) \) extending the curvature \( R_\omega \). The map \( R^\omega \) is \(*\)-preserving, horizontally valued, intertwines \( \varpi^\infty \) and \( F^\wedge \), and multiplies degrees by 2. Here, we are interested for the values of the restriction map \( R^\omega \mid \mathcal{I} \).

Proposition 5.1. If \( \vartheta \in \mathcal{I}^k \) then the form \( R^\omega(\vartheta) \) belongs to \( \Omega^{2k}(M) \). Moreover, if \( \omega \in \frak{r}(P) \) then \( R^\omega(\vartheta) \) is closed.

Proof. Equation (4.24) implies

\[ \hat{F}R^\omega(\vartheta) = \sum_k R^\omega(\vartheta_k) \otimes c_k, \]

for each \( \vartheta \in \Gamma_{inv}^\infty \), where \( \sum_k \vartheta_k \otimes c_k = \varpi^\infty(\vartheta) \). Now, the first statement follows from the assumption \( \vartheta \in \mathcal{I}^k \), and from the definition of \( \Omega(M) \). The second statement follows from (4.30), (iv)–Proposition 4.6, and from Bianchi identity \( D_\omega R_\omega = 0 \), which holds for regular multiplicative connections. \( \square \)
Now we prove that the cohomological class of \( R^\otimes_\omega(\vartheta) \) in \( \Omega(M) \) is independent of \( \omega \in \mathfrak{r}(P) \). Let \( \tau \) be another regular connection, and let
\[
\omega_t = \omega + t\varphi
\]
where \( \varphi = \tau - \omega \) and \( t \in [0, 1] \), be the segment in the space \( \mathfrak{r}(P) \) determined by \( \omega \) and \( \tau \).

**Lemma 5.2.** We have
\[
\frac{d}{dt} R_{\omega_t} = D_{\omega_t}(\varphi).
\]

**Proof.** Applying Lemma 4.8 (ii), Proposition 4.7 and (4.23) we obtain
\[
\frac{d}{dt} R_{\omega_t} = \frac{d}{dt}[d\omega + td\varphi - \langle \omega + t\varphi, \omega + t\varphi \rangle]
= d\varphi - \langle \varphi, \omega \rangle - \langle \omega, \varphi \rangle - 2t\langle \varphi, \varphi \rangle
= d\varphi - (\omega + t\varphi, \varphi) - \langle \varphi, \omega + t\varphi \rangle = D_{\omega_t}(\varphi).
\]

Let us consider an element \( \vartheta \in \Gamma^{\otimes k}_{\text{inv}} \) and let \( \vartheta = \sum_i c_i \vartheta_1^i \otimes \cdots \otimes \vartheta_k^i \), where \( c_i \) are complex numbers and \( \vartheta_j^i \in \Gamma_{\text{inv}} \). Applying (5.2) and property (4.30), the definition of \( R^\otimes_\omega \) and the Bianchi identity we obtain
\[
\frac{d}{dt} R_{\omega_t}(\vartheta) = \sum_i c_i [D_{\omega_t}(\varphi)(\vartheta_1^i) \cdots R_{\omega_t}(\vartheta_k^i) + \cdots + R_{\omega_t}(\vartheta_1^i) \cdots D_{\omega_t}(\varphi)(\vartheta_k^i)]
= \sum_i c_i D_{\omega_t} [\varphi(\vartheta_1^i) \cdots R_{\omega_t}(\vartheta_k^i) + \cdots + R_{\omega_t}(\vartheta_1^i) \cdots \varphi(\vartheta_k^i)].
\]

Using the tensoriality property of \( \varphi \) and \( R_{\omega} \), we see that if \( \vartheta \in \mathcal{I}^k \) then the form
\[
\psi_t(\vartheta) = \sum_i c_i [\varphi(\vartheta_1^i) \cdots R_{\omega_t}(\vartheta_k^i) + \cdots + R_{\omega_t}(\vartheta_1^i) \cdots \varphi(\vartheta_k^i)]
\]
belongs to \( \Omega^{2k}(M) \). Hence
\[
\frac{d}{dt} R^\otimes_{\omega_t}(\vartheta) = d\psi_t(\vartheta),
\]
according to (4.32). Integrating the above equality from 0 to 1 we obtain
\[
R_\tau(\vartheta) = R_\omega(\vartheta) + d\left( \int_0^1 \psi_t(\vartheta) \, dt \right).
\]

Let \( H(M) \) be the graded *-algebra of cohomology classes associated to \( \Omega(M) \). We have proved the following theorem.

**Theorem 5.3.** (i) The cohomological class of \( R^\otimes_\omega(\vartheta) \) in \( \Omega(M) \) is independent of the choice of a regular connection \( \omega \), for each \( \vartheta \in \mathcal{I} \).

(ii) The map \( W : \mathcal{I} \to H(M) \) given by
\[
W(\vartheta) = [R^\otimes_\omega(\vartheta)]
\]
is a unital *-homomorphism.
The homomorphism \( W \) plays the role of the Weil homomorphism in classical differential geometry [8].

In fact, in classical geometry the domain of the Weil homomorphism is restricted to the algebra of symmetric invariant elements of the corresponding tensor algebra (invariant polynomials). However, besides simplifying the domain of \( W \) such a restriction gives nothing new: the image of the Weil homomorphism will be the same.

A similar situation holds in the noncommutative case. Let \( S \) be the graded \(*\)-algebra obtained from \( \Gamma \otimes_{inv} \) by factorizing through the ideal \( J \) generated by the space \( \text{im}(I - \sigma) \subseteq \Gamma^{\otimes 2}_{inv} \).

The algebra \( S \) plays the role of polynomials over the “Lie algebra” of \( G \). The adjoint action \( \varpi \) naturally induces the action \( \varpi_S : S \rightarrow S \otimes A \). Let \( I_S \subseteq S \) be the subalgebra consisting of elements invariant under \( \varpi_S \). Clearly,

\[
I_S = I/(I \cap J).
\]

**Lemma 5.4.** If \( \omega \) is regular then

\[
R^{\otimes}_\omega \sigma(\vartheta) = R^{\otimes}_\omega (\vartheta)
\]

for each \( \vartheta \in \Gamma^{\otimes 2}_{inv} \).

**Proof.** Applying (4.64) we find

\[
m_\Omega \left\{ R_\omega \otimes \varphi \right\} = m_\Omega \left\{ \varphi \otimes R_\omega \right\} \sigma
\]

for each \( \varphi \in \tau(P) \). In particular, (5.6) holds. \( \square \)

The above statement implies that both maps \( W \) and \( R^{\otimes}_\omega \) are factorizable through the ideal \( J \) (so that \( I_S \) is the natural domain for the Weil homomorphism).

Let us briefly consider some types of quantum principal bundles, particularly interesting from the point of view of the theory of characteristic classes.

**Bundles over Classical Smooth Manifolds**

Let us assume that \( M \) is a classical (compact) smooth manifold. According to [2] (locally trivial) quantum principal bundles \( P \) over \( M \) are classified by classical \( G_{cl} \)-bundles \( P_{cl} \) (over the same manifold), where \( G_{cl} \) is the classical part of \( G \). Let us assume that \( \Gamma \) is the minimal admissible (bicovariant \(*\)-) calculus over \( G \) (in the sense of [2]). Let \( \Omega(P) \) be the graded-differential \(*\)-algebra canonically associated to \( P \) and \( \Gamma \). Then characteristic classes of \( P \) are naturally interpretable as (classical) characteristic classes of \( P_{cl} \) (however, the converse is generally not true).

**Quantum Line Bundles**

If \( G = U(1) \) and if the calculus on \( G \) is 1-dimensional then there is essentially the only one characteristic class given by the curvature form. It corresponds to the Euler class in the classical theory. The connection actually defines a “global angular form” on the bundle.

The \( \circ \)-structure is characterized by \( \zeta \circ z = \lambda \zeta \), where \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( z : G \rightarrow \mathbb{C} \) is the canonical generator of \( \mathcal{A} \). The higher-order components are trivial, \( S^{0}_{inv} \) is generated by \( \zeta \otimes \zeta \) (if \( \lambda \neq -1 \)) and \( \mathcal{R} \) is generated by \( z^{-1} + z/\lambda - (1 + 1/\lambda) \)'s. Hence
a connection $\omega$ is multiplicative iff $\omega(\zeta)^2 = 0$. The regularity condition can be written as
\[
\omega(\zeta)\varphi = (-1)^\partial\lambda^k\varphi(\zeta),
\]
if $F^\wedge(\varphi) = \varphi \otimes \zeta^k$.

**Special Differential Structures**

Let us assume that the calculus on $G$ is such that elements $\pi(u_{ij})$ linearly generate $\Gamma_{inv}$. Here $u_{ij}$ are matrix elements of the fundamental representation $u: \mathbb{C}^n \to \mathbb{C}^n \otimes A$ of $G$. Let us consider an arbitrary element $\Delta \in \mathcal{I}_S$. Then it is possible to construct a series of characteristic classes, associated to coefficients of the polynomial $p(\lambda, \Delta) = \xi(\Delta)$, where $\xi: \mathcal{S} \to \mathcal{S}$ are automorphisms specified by
\[
\xi(\pi(u_{ij})) = \lambda\delta_{ij}1 - \pi(u_{ij}),
\]
if the above formula is compatible with the ideal $\mathcal{J}$ (in any case $\xi$ are acting in the tensor algebra $\Gamma_{inv}^\otimes$). These automorphisms intertwine the adjoint action $\varpi_S$ and hence preserve the algebra $\mathcal{I}_S$. In particular, if $\Delta$ is a quantum determinant (appropriately defined) then coefficients of $p(\lambda, \Delta)$ define counterparts of classical Chern classes.

Finally, if the structure group $G$ is classical and if the calculus on $G$ is classical then the construction of characteristic classes becomes the same as in the classical case. In particular, regular connections graded-commute with horizontal differential forms. However this gives a relatively strong constraint for differential calculus on the bundle.

6. **Examples, Remarks & Some Additional Constructions**

6.1. **Infinitesimal Gauge Transformations A**

The $*-$-module $\mathcal{E} = \tau^0(P)$ of tensorial 0-forms is definable independently of the choice of a differential calculus on the bundle $P$. The elements of this space are quantum counterparts of infinitesimal gauge transformations (vertical equivariant vector fields on the bundle). The space $\mathcal{E}$ will be here analyzed from this point of view.

Explicitly, $\mathcal{E}$ is consisting of linear maps $\zeta: \Gamma_{inv} \to \mathcal{B}$ such that the diagram
\[
\begin{array}{ccc}
\Gamma_{inv} & \xrightarrow{\zeta} & \mathcal{B} \\
\varpi \downarrow & & \downarrow F \\
\Gamma_{inv} \otimes A & \xrightarrow{\zeta \otimes \text{id}} & \mathcal{B} \otimes A
\end{array}
\]
(6.1)
is commutative.

Let us observe that $\mathcal{E}$ is closed under operations $\langle,\rangle$ and $[\cdot,\cdot]$. In classical geometry, we have $[\cdot,\cdot] = -2\langle\cdot,\cdot\rangle$, and $[\cdot,\cdot]: \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ coincides with the standard commutator of vector fields (up to the sign).

We are going to construct quantum analogs of contraction operators associated to vector fields representing infinitesimal gauge transformations.
For each $\zeta \in \mathcal{E}$ let us consider a map $\iota_\zeta : \Omega(P) \to \Omega(P)$ defined by
\begin{equation}
\iota_\zeta(w) = -(-1)^{\partial w} \sum_k u_k \zeta(\eta_k)
\end{equation}
where $\sum_k u_k \otimes \eta_k = (\text{id} \otimes \pi_{\text{inv} p_1}) \hat{F}(w)$.

**Lemma 6.1.** The diagram
\begin{equation}
\begin{array}{ccc}
\Omega(P) & \xrightarrow{F^\wedge} & \Omega(P) \otimes A \\
\iota_\zeta \downarrow & & \downarrow \iota_\zeta \otimes \text{id} \\
\Omega(P) & \xrightarrow{F^\wedge} & \Omega(P) \otimes A
\end{array}
\end{equation}
is commutative.

**Proof.** A direct computation gives
\begin{align*}
F^\wedge \iota_\zeta (w) &= -(-1)^{\partial w} \sum_i (\text{id} \otimes p_0) \hat{F}(w_i) e(a_i) F \zeta p_1(\vartheta_i) \\
&= -(-1)^{\partial w} \sum_i (w_i \otimes a_i)(\zeta \otimes \text{id}) \varphi p_1(\vartheta_i) \\
&= -(-1)^{\partial w} (m_{\Omega} \otimes \text{id})(\zeta \otimes \pi_{\text{inv} p_1} \otimes p_0) \left( \sum_i w_i \otimes \varphi(a_i \vartheta_i) \right) \\
&= -(-1)^{\partial w} (m_{\Omega} \otimes \text{id})(\zeta \otimes \pi_{\text{inv} p_1} \otimes p_0)(\hat{F} \otimes \text{id}) \hat{F}(w) \\
&= (\iota_\zeta \otimes \text{id}) F^\wedge (w),
\end{align*}
where $\hat{F}(w) = \sum_i w_i \otimes a_i \vartheta_i$, with $w_i \in \Omega(P)$, $a_i \in A$ and $\vartheta_i \in \Gamma_{\text{inv}}$.

The definition of $\iota_\zeta$ implies
\begin{equation}
\iota_\zeta(\varphi w) = (-1)^{\partial \varphi} \varphi \iota_\zeta(w),
\end{equation}
for each $\varphi \in \mathfrak{hor}(P)$ and $w \in \Omega(P)$. In particular,
\begin{equation}
\iota_\zeta(\mathfrak{hor}(P)) = \{0\},
\end{equation}
for each $\zeta \in \mathcal{E}$.

Let us consider linear maps $\ell_\zeta : \Omega(P) \to \Omega(P)$ given by
\begin{equation}
\ell_\zeta = d\iota_\zeta + \iota_\zeta d.
\end{equation}
These maps play the role of the corresponding Lie derivatives.

**Lemma 6.2.** (i) The diagram
\begin{equation}
\begin{array}{ccc}
\Omega(P) & \xrightarrow{F^\wedge} & \Omega(P) \otimes A \\
\ell_\zeta \downarrow & & \downarrow \ell_\zeta \otimes \text{id} \\
\Omega(P) & \xrightarrow{F^\wedge} & \Omega(P) \otimes A
\end{array}
\end{equation}
is commutative.

(ii) The following equality holds

\[ \ell_\zeta(\varphi) = \sum_k \varphi_k \zeta \pi(c_k), \]

where \( \varphi \in \text{hor}(P) \) and \( \sum_k \varphi_k \otimes c_k = F^\wedge(\varphi) \). In particular,

\[ \ell_\zeta(\text{hor}(P)) \subseteq \text{hor}(P). \]

Proof. Diagram (6.7) follows from (6.3), (6.6) and (3.17). Identity (6.8) directly follows from definitions of \( \iota_\zeta \) and \( \ell_\zeta \):

\[ \ell_\zeta \varphi = \iota_\zeta d\varphi = (-1)^{\partial\varphi} m_\Omega(\text{id} \otimes \zeta \pi_{inv}) \left[ \sum_k d\varphi_k \otimes c_k + (-1)^{\partial\varphi} \varphi_k \otimes dc_k \right] = m_\Omega(\text{id} \otimes \zeta \pi_{inv}) \left[ \sum_k \varphi_k \otimes dc_k \right] = \sum_k \varphi_k \zeta \pi(c_k). \]

Covariance properties (6.3) and (6.7) imply that \( \psi(P) \) is invariant under compositions with \( \iota_\zeta \) and \( \ell_\zeta \). In particular, \( \ell_\zeta \tau(P) \subseteq \tau(P) \) for each \( \zeta \in \mathcal{E} \).

Lemma 6.3. We have

\[ \ell_\zeta \varphi = [\varphi, \zeta], \]

for each \( \zeta \in \mathcal{E} \) and \( \varphi \in \tau(P) \).

Proof. It follows from (6.8), definitions of \( c^\top \) and \([,] \), and the tensoriality of \( \varphi \).

Let us compute actions of \( \iota_\zeta \) and \( \ell_\zeta \) on connection forms.

Lemma 6.4. We have

\[ \iota_\zeta \omega = \zeta \]
\[ \ell_\zeta \omega = d\zeta + [\omega, \zeta], \]

for each \( \zeta \in \mathcal{E} \) and \( \omega \in \text{con}(P) \).

Proof. Both identities directly follow from property (4.8) and from definitions of \( \iota_\zeta \) and \( \ell_\zeta \).

In the framework of the theory presented in the previous paper, a very important class of infinitesimal gauge transformations naturally appear. These transformations can be described as infinitesimal generators (in the standard sense) of the group of vertical automorphisms of the bundle \( P \). They form a subspace \( \mathcal{G} \subseteq \mathcal{E} \).

A more detailed geometrical analysis shows that the elements from \( \mathcal{G} \) are naturally identifiable with standard infinitesimal gauge transformations of the classical part \( P_{cl} \) of the bundle \( P \) (vertical automorphisms of \( P \) are in a natural bijection with standard gauge transformations of \( P_{cl} \)). Moreover, the space \( \mathcal{G} \) is closed under brackets \([,]\) and, in terms of the mentioned identification \([- [,]\) becomes the standard Lie bracket of vector fields. The elements from \( \mathcal{G} \) naturally act as derivations on \( \Omega(P) \). However, the action of the derivation generated by an element \( \zeta \in \mathcal{G} \)
generally differs from the action of the corresponding Lie derivative $\ell_\zeta$ introduced in this subsection. This is visible from (6.12), which can be rewritten in the form

$$\ell_\zeta \omega = D_\omega \zeta + [\omega, \zeta] + [\zeta, \omega].$$

The last two summands generally give a nontrivial non-horizontal contribution, even in the case $\zeta \in \mathcal{G}$. Only in classical geometry we have $[\omega, \zeta] + [\zeta, \omega] = 0$ (more generally $[\varphi, \eta] = -(-1)^{\partial \varphi \partial \eta}[\eta, \varphi]$ for each $\varphi, \eta \in \psi(P)$), due to the antisymmetricity of $c^\top$, and the graded-commutativity of $\Omega(P)$.

6.2. Infinitesimal Gauge Transformations B

Motivated by the above remarks, a slightly different approach to defining quantum analogs of the Lie derivative and the contraction operator will be now presented.

The main property of this approach is that in the special case of bundles over smooth manifolds the Lie derivative of an arbitrary element $\zeta \in \mathcal{G}$ coincides with the derivation generated by $\zeta$.

We shall also introduce a general quantum counterpart of the space $\mathcal{G}$, and briefly analyze its properties.

**Lemma 6.5.** (i) For each $\zeta \in \mathcal{E}$ there exists the unique $\varsigma^*_\zeta : \mathfrak{h}(P) \to \mathfrak{h}(P)$ such that

\begin{align}
(6.13) & \quad \varsigma^*_\zeta(\mathfrak{hor}(P)) = \{0\} \\
(6.14) & \quad \varsigma^*_\zeta(w\vartheta) = \varsigma^*_\zeta(w)\vartheta + (-1)^{\partial w} w\varsigma^*_\zeta(\vartheta),
\end{align}

for each $w \in \mathfrak{h}(P)$ and $\vartheta \in \Gamma_{\text{inv}}$.

(ii) Similarly, for each $\zeta \in \mathcal{E}$ there exists the unique $\iota^*_\zeta : \mathfrak{h}(P) \to \mathfrak{h}(P)$ such that

\begin{align}
(6.15) & \quad \iota^*_\zeta(\partial \eta) = -\partial \iota^*_\zeta(\eta) + \sum_k \eta_k \varsigma^*_\zeta(\eta \circ c_k) \\
(6.16) & \quad \iota^*_\zeta(\varphi \eta) = (-1)^{\partial \varphi} \varphi \iota^*_\zeta(\eta),
\end{align}

for each $\varphi \in \mathfrak{hor}(P)$, $\eta \in \Gamma^\wedge_{\text{inv}}$ and $\vartheta \in \Gamma_{\text{inv}}$, where $\sum_k \eta_k \otimes c_k = \varpi^\wedge(\eta)$. In particular,

\begin{align}
(6.17) & \quad \iota^*_\zeta(\mathfrak{hor}(P)) = \{0\}.
\end{align}

(iii) The following identities hold

\begin{align}
(6.18) & \quad F_{\theta} \iota^*_\zeta = (\iota^*_\zeta \otimes \text{id}) F_{\theta} \\
(6.19) & \quad F_{\theta} \varsigma^*_\zeta = (\varsigma^*_\zeta \otimes \text{id}) F_{\theta} \\
(6.20) & \quad \varsigma^*_\zeta(w\vartheta) = \varsigma^*_\zeta(w)\vartheta + (-1)^{\partial w} w\varsigma^*_\zeta(\vartheta),
\end{align}

where $w \in \mathfrak{h}(P)$ and $\vartheta \in \Gamma^\wedge_{\text{inv}}$. 
Proof. We shall prove (i) and properties (6.19) and (6.20). The statements about the map \( \iota^*_\xi \) follow in a similar way. It is clear that conditions (6.13) and (6.14) uniquely fix the values of \( \zeta^*_\xi \), if it exists. Also, (6.20) directly follows from (6.14). In order to establish the existence of \( \zeta^*_\xi \), it is sufficient to check that (6.14) is not in a contradiction with the quadratic constraint generating the ideal \( S^\omega_{inv} \).

For each \( a \in \mathcal{R} \) we have

\[
\left\{ \pi(a^{(1)})\pi(a^{(2)}) \right\} \rightarrow \left( \zeta\pi(a^{(1)})\pi(a^{(2)}) - \pi(a^{(1)})\zeta\pi(a^{(2)}) \right) = (\zeta\pi(a^{(1)}))\pi(a^{(2)}) - (\zeta\pi(a^{(3)}))\pi(a^{(1)}) \circ \{ \kappa(a^{(2)})a^{(4)} \} = (\zeta\pi(a^{(1)}))\pi(a^{(2)}) - (\zeta\pi(a^{(3)}))\pi \left[ (a^{(1)} - \epsilon(a^{(1)})1)\kappa(a^{(2)})a^{(4)} \right] = \zeta\pi(a^{(2)})\pi(\kappa(a^{(1)})a^{(3)}) = 0,
\]

and hence \( \zeta^*_\xi \) exists.

Finally, let us check (6.19). This equality is satisfied trivially on elements from \( \mathfrak{hor}(P) \). The definition of \( \zeta \) implies that it is satisfied on elements from \( \Gamma_{inv} \). Finally, inductively applying (6.14) we conclude that (6.19) holds on the whole algebra \( \mathfrak{vh}(P) \). □

Let us assume that the bundle \( P \) admits regular connections, as well as that \( \mathfrak{T}(P) = \{ 0 \} \).

Lemma 6.6. We have

(6.21)
\[
\iota^*_\zeta = m^w_\omega \iota^*_\zeta m^{-1}_\omega
\]

for each \( \zeta \in \mathcal{E} \) and \( \omega \in \mathfrak{t}(P) \).

Proof. Let us fix a connection \( \omega \in \mathfrak{t}(P) \). For each \( \zeta \in \mathcal{E} \) let \( \iota^*_\zeta : \mathfrak{vh}(P) \rightarrow \mathfrak{vh}(P) \) be a map defined by

\[
\iota^*_\zeta = m^{-1}_\omega \iota^*_\zeta m_\omega.
\]

If \( \varphi \in \mathfrak{hor}(P) \) and \( \eta \in \Gamma_{inv}^\wedge \), then

\[
\iota^*_\zeta(\varphi \eta) = m^{-1}_\omega \iota^*_\zeta (\varphi \omega^\wedge(\eta)) = (-1)^{\partial \varphi} m^{-1}_\omega(\varphi \iota^*_\zeta \omega^\wedge(\eta)) = (-1)^{\partial \varphi} \varphi \iota^*_\zeta \eta
\]

according to (6.4). Further, if \( \vartheta \in \Gamma_{inv} \) then

\[
\iota^*_\zeta(\vartheta \eta) = m^{-1}_\omega \iota^*_\zeta (\vartheta(\omega^\wedge(\eta))) = (-1)^{\partial \vartheta} m^{-1}_\omega(\vartheta m^{-1}_\omega(\vartheta \omega^\wedge(\eta))) = m^{-1}_\omega(\vartheta m^{-1}_\omega \sum_k \omega^\wedge(\eta_k) \otimes \vartheta \circ c_k)
\]

\[
+ (-1)^{\partial \vartheta} m^{-1}_\omega \left\{ \vartheta m^{-1}_\omega \left[ (\vartheta \omega^\wedge(\eta) m^{-1}_\omega(\vartheta(\omega^\wedge(\eta))) \right] \right\}
\]

\[
= \sum_k \eta_k \zeta(\vartheta \circ c_k) - m^{-1}_\omega(\vartheta m^{-1}_\omega(\omega(\vartheta) \iota^*_\zeta \omega^\wedge(\eta)))
\]

\[
= \sum_k \eta_k \zeta(\vartheta \circ c_k) - \partial \iota^*_\zeta(\eta).
\]
Here, $\sum_k \eta_k \otimes c_k = \varpi^\wedge(\eta)$ and we have used elementary properties of entities figuring in the game. Applying (ii) Lemma 6.5 we find that $\iota'_\zeta = \iota^*\zeta$. Hence (6.21) holds.

For each $\omega \in \mathfrak{r}(P)$ and $\zeta \in \mathcal{E}$ let $\varsigma_{\zeta,\omega} : \Omega(P) \to \Omega(P)$ be a map introduced via the diagram

\[
\begin{array}{ccc}
\mathfrak{vh}(P) & \xrightarrow{\varsigma^*_\zeta} & \mathfrak{vh}(P) \\
m_\omega \downarrow & & \downarrow m_\omega \\
\Omega(P) & \xrightarrow{\varsigma_{\zeta,\omega}} & \Omega(P)
\end{array}
\] (6.22)

and let $\ell_{\zeta,\omega} : \Omega(P) \to \Omega(P)$ be a map given by

\[
\ell_{\zeta,\omega} = d\varsigma_{\zeta,\omega} + \varsigma_{\zeta,\omega} d.
\]

(6.23)

It is evident that $\ell_{\zeta,\omega}$ and $\varsigma_{\zeta,\omega}$ are right-covariant maps, in the sense that

\[
F^\wedge \ell_{\zeta,\omega} = (\ell_{\zeta,\omega} \otimes \text{id})F^\wedge
\]

(6.24)

\[
F^\wedge \varsigma_{\zeta,\omega} = (\varsigma_{\zeta,\omega} \otimes \text{id})F^\wedge.
\]

(6.25)

The maps $\ell_{\zeta,\omega}$ and $\varsigma_{\zeta,\omega}$, are also interpretable as quantum counterparts of the Lie derivative and the contraction operator respectively. In contrast to the classical case, these maps are generally connection-dependent. However,

**Lemma 6.7.** If $w \in \Omega_1(P)$ then

\[
\varsigma_{\zeta,\omega}(w) = \iota_{\zeta}(w),
\]

(6.26)

for each $\zeta \in \mathcal{E}$ and $\omega \in \mathfrak{r}(P)$. In particular, operators $\ell_{\zeta}$ and $\ell_{\zeta,\omega}$ possess the same restrictions on $\mathfrak{hor}(P)$.

**Proof.** It follows from the fact that $\iota^*_{\zeta}$ and $\varsigma^*_\zeta$ coincide on the spaces $\mathfrak{hor}(P)$ and $\mathfrak{hor}(P) \otimes \Gamma_{\text{inv}}$. \(\square\)

Covariance properties (6.24)–(6.25) enable us to define actions of $\ell_{\zeta,\omega}$ and $\varsigma_{\zeta,\omega}$ in the space $\psi(P)$.

The $\omega$-dependence of constructed operators becomes explicitly visible if we consider the action of $\ell_{\zeta,\omega}$ on connection forms.

**Lemma 6.8.** We have

\[
\ell_{\zeta,\omega}\tau = D_\tau \zeta + [\tau - \omega, \zeta] + [\zeta, \tau - \omega]
\]

(6.27)

for each $\zeta \in \mathcal{E}$, $\omega \in \mathfrak{r}(P)$ and $\tau \in \text{con}(P)$. In particular $\ell_{\zeta,\omega}\tau$ is always tensorial.
Proof. Using Lemmas 6.3 and 6.7, and properties (6.11) and (6.23) we obtain
\[
\ell_{\zeta,\omega} = d\zeta + [\tau - \omega, \zeta] + \zeta_{\omega,\zeta} d\omega.
\]
On the other hand, (4.23) together with the regularity of \(\omega\), tensoriality of \(\zeta\) and the definition of \(\zeta_{\omega,\zeta}\) gives
\[
\zeta_{\omega,\zeta} d\omega(\theta) = \zeta_{\omega,\zeta}(\langle \omega, \omega \rangle + R_{\omega})(\theta) = -\zeta_{\omega,\zeta}\pi(a^{(1)})\omega + \omega\pi(a^{(1)})\zeta_{\omega,\zeta}\pi(a^{(2)})
\]
\[
= -\zeta_{\omega,\zeta}\pi(a^{(1)})\omega + \zeta_{\omega,\zeta}\pi(a^{(3)})\omega \left[ \pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)}) \right]
\]
\[
= -\zeta_{\omega,\zeta}\pi(a^{(2)})\omega(\kappa(a^{(1)})a^{(3)}) = -[\zeta, \omega](\theta),
\]
where \(a \in A\) satisfies (4.18). Consequently,
\[
\ell_{\zeta,\omega} = D_{\omega,\zeta} + [\tau - \omega, \zeta] = D_{\tau,\zeta} + [\tau - \omega, \zeta] + [\zeta, \tau - \omega].
\]
Finally, let \(G \subseteq \mathcal{E}\) be the space of elements \(\zeta\) satisfying \(\zeta(\varphi) = \sum \varphi_k \zeta(\varnothing \circ c_k)\) for each \(\varphi \in \text{hor}(P)\) and \(\varnothing \in \Gamma_{\text{inv}}\). Let us assume that \(G\) is nontrivial.

Proposition 6.9. (i) The space \(G\) is closed under the action of brackets \([., .]\). We have
\[
\zeta_{\pi(a^{(1)})\pi(a^{(2)})} - [\zeta(\varphi(a^{(1)})\varphi(a^{(2)}))] = [\zeta, [\xi, a]]
\]
for each \(\zeta, \xi \in G\) and \(a \in A\). In particular, brackets \([., .]\) determine a Lie algebra structure on \(G\).

(ii) Operators \(\zeta_{\omega,\zeta}\) and \(\ell_{\zeta,\omega}\) are \(\omega\)-independent, if \(\zeta \in G\).

(iii) The following identities hold
\[
\ell_{\zeta}^*(wu) = \ell_{\zeta}^*(w)u + w\ell_{\zeta}^*(u)
\]
\[
\zeta_{\omega,\zeta}^*(wu) = \zeta_{\omega,\zeta}^*(w)u + (-1)^{d_{\omega,\zeta}} w\zeta_{\omega,\zeta}^*(u)
\]
\[
\zeta_{\omega,\zeta}^* = 0
\]
\[
\ell_{\zeta,\omega}^* - \ell_{\zeta,\omega}^* = -\ell_{[\zeta,\omega]}^*
\]
\[
\ell_{\zeta,\omega}^* = -\ell_{\zeta,\omega}^* = -\ell_{[\zeta,\omega]}^*.
\]
Here \(\zeta, \xi \in G\) and \(w, u \in \Omega(P)\) while \(\zeta_{\omega,\zeta} = \zeta_{\omega,\zeta}^*\) and \(\ell_{\zeta,\omega}^* = \ell_{\zeta,\omega}\).

Proof. We compute
\[
\xi\pi(a^{(1)})\xi\pi(a^{(2)}) = \xi\pi(a^{(3)})\xi\left[ \pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)}) \right]
\]
\[
= \xi\pi(a^{(1)})\xi\pi(a^{(2)}) - [\zeta, \xi]\pi(a).
\]
Let us check that $G$ is closed under the brackets $[,]$. Using properties (6.7)–(6.8) and (6.28)–(6.29) we find

\[
(\langle \zeta, \xi \rangle \pi(a))\varphi = \sum_k \varphi_k \zeta(\pi(a^{(1)}) \circ c_k^{(1)})(\pi(a^{(2)}) \circ c_k^{(2)})
\]

\[
- \sum_k \varphi_k \xi(\pi(a^{(1)}) \circ c_k^{(1)})(\pi(a^{(2)}) \circ c_k^{(2)})
\]

\[
= \sum_k \left( \varphi_k \zeta(\pi(a^{(1)})c_k^{(1)})(\pi(a^{(2)})) - \varphi_k \xi(\pi(a^{(1)}))\zeta(\pi(a^{(2)})) \right)
\]

\[
- \sum_k \ell_\zeta(\varphi_k)\xi(\pi(ac_k)) + \sum_k \xi(\pi(ac_k))\varphi_k \xi(\pi(c_k))
\]

\[
= \sum_k \varphi_k [\zeta, \xi] \pi(ac_k) = \sum_k \varphi_k [\zeta, \xi] (\pi(a) \circ c_k),
\]

where $a \in \ker(\epsilon)$ and $\varphi \in \mathfrak{hor}(P)$.

Now we shall prove identities (6.30)–(6.34). Let us observe that (6.30) directly follows from (6.31) and (6.23). On the other hand the fact that $\zeta_{\xi,\omega}$ is an antiderivation together with Lemma 6.7 shows that $\zeta_{\xi,\omega}$ (and therefore $\ell_\zeta_{\xi,\omega}$) is $\omega$-independent. Evidently, (6.31) is equivalent to the fact that $\zeta^*_{\xi}$ is an antiderivation on $\mathfrak{h}(P)$.

Having in mind identity (6.20) and property (6.13) it is sufficient to check that

\[
\zeta^*_{\xi}(\vartheta \varphi) = \zeta^*_{\xi}(\vartheta) \varphi,
\]

for each $\vartheta \in \Gamma^{\wedge}$, and $\varphi \in \mathfrak{hor}(P)$. However, this easily follows from property (6.28) and the definition of $\zeta^*_{\xi}$.

For each $\zeta, \xi \in G$ the anticommutator of $\zeta^*_{\xi}$ and $\zeta^*_{\xi}$ is an antiderivation on $\mathfrak{h}(P)$. This anticommutator vanishes on $\mathfrak{hor}(P)$ and $\Gamma^{\wedge}_{\text{inv}}$. Therefore it vanishes identically.

Having in mind that $\ell^*_{\zeta}$ are derivations on $\Omega(P)$ commuting with the differential, and that derivations form a Lie algebra, it is sufficient to check that (6.33) holds on elements $b \in B$. We have

\[
(\ell^*_{\zeta} \ell^*_{\xi} - \ell^*_{\xi} \ell^*_{\zeta})(b) = \ell^*_{\xi} \sum_k b_k \xi(\pi(a_k)) - \ell^*_{\zeta} \sum_k b_k \xi(\pi(a_k))
\]

\[
= \sum_k \left[ b_k \xi(\pi(a_k^{(1)}))\zeta(\pi(a_k^{(2)})) - b_k \xi(\pi(a_k^{(1)}))\zeta(\pi(a_k^{(2)})) \right]
\]

\[
= \sum_k b_k [\xi, \zeta] (\pi(a_k)) = \ell^*_{[\xi, \zeta]}(b),
\]

where $F(b) = \sum_k b_k \otimes a_k$. Similarly, it is sufficient to check that (6.34) holds on
elements of the form $\omega(\vartheta)$. We have
\[
\left(\ell^*_\zeta \xi - \zeta \ell^*_\xi\right) \omega = \ell^*_\zeta \xi - \zeta D_\omega \xi = [\xi, \zeta] = -\zeta [\zeta, \xi] \omega,
\]
which completes the proof.

6.3. Some Interrelations

A similar approach to general quantum principal bundles is presented in [3]. Let us briefly consider interrelations between the [3] and the theory developed here. At the level of spaces both formulations coincide (modulo the *-structure). The main difference appears at the level of differential calculus on the bundle. As first, it is assumed in [3] that the higher-order differential calculus uniquely follows from the first-order one, being based on (an appropriate) universal envelope of the first-order differential structure. Secondly, the total “pull back” of the right action does not figure in [3] and horizontal forms are defined in a different way.

More precisely, let $P = (B, i, F)$ be a quantum principal $G$-bundle over $M$ and let $\Omega(P)$ be an arbitrary graded-differential *-algebra satisfying properties (diff1/2). Let $\Omega^1_{\text{hor}} \subseteq \mathfrak{hor}(P)$ be a (*-) subalgebra generated by $B$ and $d(i(V))$. This algebra is a counterpart of horizontal forms introduced in [3]. One of the main conditions postulated in [3] (in the definition of differential calculus) is that the sequence
\[
0 \to \Omega^1_{\text{hor}} \hookrightarrow \Omega^1(P) \xrightarrow{\pi} \text{ver}^1(P) \to 0
\]
is exact. According to Lemma 3.7 this is equivalent to
\[
(6.35) \quad \Omega^1_{\text{hor}} = \mathfrak{hor}^1(P).
\]
This condition can be understood as a (relatively strong) condition for the bundle, if applied to the universal case, for example. Namely, a trivial differential calculus on the bundle can be always constructed by taking the universal differential envelope of $B$ (conditions (diff1/2) hold). However (6.35) does not generally hold (although it holds in various interesting special cases). In particular, not all quantum homogeneous spaces (endowed with universal differential calculus) can be included in the theory presented in [3].

It is worth noticing that if (6.35) holds and if the bundle admits regular and multiplicative connections then $\Omega^1_{\text{hor}} = \mathfrak{hor}(P)$. Also, two horizontal algebras coincide if $\Omega(M)$ is generated (as a differential algebra) by $i(V)$. This follows from the fact that each $\varphi \in \mathfrak{hor}(P)$ can be written in the form
\[
\varphi = \sum_i w_i b_i,
\]
where $b_i \in B$ and $w_i \in \Omega(M)$ (as explained in Appendix B).

The definition of connection forms given in [3] is (modulo the *-structure and differences between differential calculi) equivalent to the definition proposed in this work. On the other hand the embedded differential map $\delta$ does not figure in the formalism of connections. In particular, operators of covariant derivative and curvature described in [3] are generally different from $D_\omega$ and $R_\omega$ constructed here.
6.4. Trivial Bundles

For given quantum space $M$ and compact matrix quantum group $G$ let us define a $*$-algebra $B$ and maps $F: B \to B \otimes A$ and $i: V \to B$ by

$$B = V \otimes A$$

$$i(\cdot) = (\cdot) \otimes 1$$

$\text{id} \otimes \phi = F$.

The triplet $P = (B, i, F)$ is a trivial quantum principal bundle over $M$. Geometrically $P = M \times G$.

The algebra of verticalized forms is isomorphic to the tensor product

$$\text{ver}(P) = V \otimes \Gamma^\wedge,$$

with $d_v \leftrightarrow \text{id} \otimes d$.

Let $\Omega(M)$ be an arbitrary graded-differential $*$-algebra generated by $V = \Omega^0(M)$, representing a differential calculus on $M$. Then it is natural to define the algebra $\Omega(P)$ (representing a differential calculus on the bundle $P$) as the graded tensor product

$$\Omega(P) = \Omega(M) \hat{\otimes} \Gamma^\wedge.$$

We have then

$$\hat{F} = \text{id} \otimes \hat{\phi}.$$

Horizontal forms constitute a $*$-subalgebra

$$\text{hor}(P) = \Omega(M) \otimes A.$$

We shall now analyze in more details the structure of tensorial forms, connection forms, and operators of covariant derivative and curvature, in the special case of trivial bundles.

Let $\mathcal{X}$ be the graded $*\Omega(M)$ module of linear maps $L: \Gamma_{\text{inv}} \to \Omega(M)$.

Lemma 6.10. For each $\varphi \in \tau(P)$ there exists the unique $L^\varphi \in \mathcal{X}$ such that

\begin{equation}
\varphi(\vartheta) = (L^\varphi \otimes \text{id}) \varpi(\vartheta)
\end{equation}

for each $\vartheta \in \Gamma_{\text{inv}}$. The above formula establishes an isomorphism between graded $*\Omega(M)$-modules.

Proof. For a given $L \in \mathcal{X}$ let $\varphi_L: \Gamma_{\text{inv}} \to \Omega(P)$ be a map determined by equality

$$\varphi_L(\vartheta) = (L \otimes \text{id}) \varpi(\vartheta).$$

Evidently, the image of $\varphi_L$ is contained in $\text{hor}(P)$ and

$$F^\wedge \varphi_L = (L \otimes \phi) \varpi = [(L \otimes \text{id}) \varpi \otimes \text{id}] \varpi = (\varphi_L \otimes \text{id}) \varpi.$$

Hence $\varphi_L \in \tau(P)$. It is clear that the map $L \mapsto \varphi_L$ is a monomorphism of $*\Omega(M)$-modules (because $L = (\text{id} \otimes \epsilon) \varphi_L$).

Let us consider an arbitrary tensorial form $\varphi$. Acting by $\text{id} \otimes \epsilon \otimes \text{id}$ on the tensoriality identity for $\varphi$ we find that (6.36) holds, with $L^\varphi = (\text{id} \otimes \epsilon) \varphi$. □

The following lemma gives a similar description of connection forms.
Lemma 6.11. (i) The formula
\[ \omega(\vartheta) = (A^\omega \otimes \text{id})\varpi(\vartheta) + 1 \otimes \vartheta \]
establishes a bijective affine correspondence between connections on \( P \) and hermitian elements of \( \mathcal{X}^1 \).

(ii) A connection \( \omega \) is regular iff
\[ A^\omega(\vartheta)\zeta = (-1)^{\partial_\zeta} \vartheta A^\omega(\vartheta) \]
and
\[ A^\omega(\vartheta \circ a) = \epsilon(a) A^\omega(\vartheta) \]
for each \( a \in A \), \( \vartheta \in \Gamma_{\text{inv}} \) and \( \zeta \in \Omega(M) \).

Proof. The formula \( \omega_0(\vartheta) = 1 \otimes \vartheta \) determines a canonical “flat” connection on \( P \). The statement (i) follows from the previous lemma and the fact that hermitian elements of \( \tau^1(P) \) form the vector space associated to \( \text{con}(P) \).

Let us assume that \( \omega \in \mathfrak{r}(P) \). In other words
\[ \omega(\vartheta)(\zeta \otimes a) = (-1)^{\partial_\zeta} (\zeta \otimes a^{(1)}) \vartheta(\vartheta \circ a^{(2)}) \]
for each \( \zeta \in \Omega(M) \), \( a \in A \) and \( \vartheta \in \Gamma_{\text{inv}} \). This is equivalent to
\[ \sum_k A^\omega(\partial_k)\zeta \otimes c_k a = (-1)^{\partial_\zeta} \sum_k \zeta A^\omega(\partial_k \circ a^{(1)}) \otimes c_k a^{(2)} , \]
where \( \sum c_k \delta_k \otimes c_k = \varpi(\vartheta) \). Acting by \( \text{id} \otimes \epsilon \) on both sides on the above equality we obtain
\[ A^\omega(\vartheta)\epsilon(a) = (-1)^{\partial_\zeta} \vartheta A^\omega(\vartheta \circ a) , \]
which is equivalent to conditions listed in (ii). Conversely (6.41) imply (6.40), evidently. \( \square \)

The bijection \( \omega \leftrightarrow A^\omega \) generalizes the classical correspondence between connections and their gauge potentials. In the previous paper, a similar correspondence was established, at the local level. This was possible because of the local triviality of considered bundles. However, in the general quantum context it is not possible to speak about local domains on the base space, and hence it is not possible to speak about locally trivial bundles.

Lemma 6.12. We have
\[ R^\omega(\vartheta) = (F^\omega \otimes \text{id})\varpi(\vartheta) \]
where \( F^\omega \in \mathcal{X}^2 \) is a hermitian element given by
\[ F^\omega = dA^\omega - \langle A^\omega, A^\omega \rangle . \]

Further, if \( \varphi \in \mathfrak{hor}(P) \) then
\[ D^\omega(\varphi) \leftrightarrow q^{\omega,\varphi} \]
where
\[ q^{\omega,\varphi} = dL^\varphi - (-1)^{\partial_\varphi} [L^\varphi, A^\omega] . \]
Proof. Inserting (6.36) and (6.37) in (4.33) we obtain

\[
(D_\omega \varphi)(\vartheta) = \sum_k dL^\varphi(\vartheta_k) \otimes c_k + (-1)^{\partial \varphi} \sum_k L^\varphi(\vartheta_k) \otimes dc_k
- (-1)^{\partial \varphi} \sum_k \left[ L^\varphi(\vartheta_k) \otimes c_k^{(1)} \pi(c_k^{(2)}) + L^\varphi(\vartheta_k) A^\omega \pi(c_k^{(1)}) \otimes c_k^{(2)} \right]
\]

\[
= \left[ (dL^\varphi - (-1)^{\partial \varphi}[L^\varphi, A^\omega]) \otimes \text{id} \right] \varpi(\vartheta).
\]

Similarly, inserting (6.37) in (4.23) we obtain

\[
R_\omega(\vartheta) = \sum_k dA^\omega(\vartheta_k) \otimes c_k - \sum_k A^\omega(\vartheta_k) \otimes dc_k + 1 \otimes d\vartheta
+ 1 \otimes \pi(a^{(1)}) \pi(a^{(2)}) + A^\omega \pi(a^{(2)}) \otimes \kappa(a^{(1)}) a^{(3)} \pi(a^{(4)})
- A^\omega \pi(a^{(3)}) \otimes \pi(a^{(1)}) \kappa(a^{(2)}) a^{(4)}
+ A^\omega \pi(a^{(2)}) A^\omega \pi(a^{(3)}) \otimes \kappa(a^{(1)}) a^{(4)} = \left[ (dA^\omega - \langle A^\omega, A^\omega \rangle) \otimes \text{id} \right] \varpi(\vartheta).
\]

Here, \( a \in \ker(\epsilon) \) is such that (4.18) holds. \( \square \)

Infinitesimal gauge transformations are in a natural bijection with linear maps \( \gamma : \Gamma_{\text{inv}} \rightarrow \mathcal{V} \). The elements of \( G \) correspond to functions \( \gamma \) satisfying

\[
\gamma(\vartheta \circ a) = \epsilon(a) \gamma(\vartheta)
\]

\[
w \gamma(\vartheta) = \gamma(\vartheta) w
\]

for each \( \vartheta \in \Gamma_{\text{inv}}, a \in \mathcal{A} \) and \( w \in \Omega(M) \).

Let us assume that \( \Gamma \) is the minimal admissible (bicovariant \(*\)-) calculus over \( G \) (in the sense of [2]). The following natural identifications hold

\[
G \leftrightarrow Z^0(M) \otimes \text{lie}(G_{\text{cl}})
\]

\[
\mathfrak{v}(P) \leftrightarrow Z^1(M) \otimes \text{lie}(G_{\text{cl}}),
\]

where \( G_{\text{cl}} \) is the classical part of \( G \) and \( Z(M) \) is the (graded) centre of \( \Omega(M) \).

6.5. Quantum Homogeneous Spaces

Let \( H \) be a compact matrix quantum group and let \( G \) be a (compact) subgroup of \( H \). At the formal level, this presumes a specification of a \(*\)-epimorphism (the corresponding “restriction map”) \( j : \mathcal{B} \rightarrow \mathcal{A} \) such that

\[
(j \otimes j) \phi' = \phi j
\]

\[
\epsilon j = \epsilon'
\]

\[
\kappa j = j \kappa'.
\]

Here \( \mathcal{B} \) is the functional Hopf \(*\)-algebra for \( H \). In what follows entities endowed with the prime will refer to \( H \).

The \(*\)-homomorphism \( F : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A} \) given by

\[
F = (\text{id} \otimes j) \phi'
\]
is interpretable as the right action of $G$ on $H$. Let $M$ be the corresponding “orbit space”. At the formal level, $M$ is represented by the fixed-point $^*$-subalgebra $\mathcal{V} \subseteq \mathcal{B}$.

Let $i : \mathcal{V} \to \mathcal{B}$ be the inclusion map.

**Lemma 6.13.** The triplet $P = (\mathcal{B}, i, F)$ is a quantum principal $G$-bundle over $M$.

**Proof.** It is evident that conditions (qp(bool1)/2) of Definition 3.1 are satisfied. We have

$$1 \otimes j(b) = \kappa(b^{(1)})F(b^{(2)}),$$

for each $b \in \mathcal{B}$. Hence (qp(bool4)) holds.

Because of the inclusion $\phi'(\mathcal{V}) \subseteq \mathcal{B} \otimes \mathcal{V}$ there exists a natural left action of $H$ on $M$, defined by $\phi'i : \mathcal{V} \to \mathcal{B} \otimes \mathcal{V}$. This action is “transitive” in the sense that only scalar elements of $\mathcal{V}$ are invariant. In this sense $M$ is understandable as a quantum homogeneous $H$-space.

Now a construction of a differential calculus on $H$ will be presented, which explicitly takes care about the “fibered” geometrical framework.

Let $\Psi$ be a left-covariant first-order $^*$-calculus over $H$ and let $\mathcal{R}' \subseteq \ker(\epsilon')$ be the corresponding right $\mathcal{B}$-ideal. Let us assume that

\begin{align}
\label{eq:6.46}
j(\mathcal{R}') & \subseteq \mathcal{R} \\
\label{eq:6.47}(\text{id} \otimes j)\text{ad}'(\mathcal{R}') & \subseteq \mathcal{R}' \otimes \mathcal{A}
\end{align}

where $\mathcal{R} \subseteq \ker(\epsilon)$ is the right $\mathcal{A}$-ideal which determines the calculus $\Gamma$ over $G$. Condition (6.46) ensures the existence of the projection map $\rho : \Psi_{\text{inv}} \to \Gamma_{\text{inv}}$, which is determined by the formula

$$\rho \pi' = \pi j.$$

The meaning of condition (6.47) is that the calculus $\Psi$ is right-covariant, relative to $G$. Consequently, there exists the corresponding adjoint action $\chi : \Psi_{\text{inv}} \to \Psi_{\text{inv}} \otimes \mathcal{A}$. This map is explicitly given by

$$\chi \pi' = (\pi' \otimes j)\text{ad}'.
$$

Maps $\rho$ and $\chi$ are hermitian and

\begin{align}
\rho(\vartheta \circ a) & = \rho(\vartheta) \circ j(a) \\
\varpi \rho & = (\rho \otimes \text{id})\chi.
\end{align}

In particular, the space $\mathcal{L} = \ker(\rho)$ is a $^*$- and $\chi$-invariant submodule of $\Psi_{\text{inv}}$.

Let us now assume that the full calculus on the bundle $P$ (⇔ the fibered $H$) is described by a graded-differential $^*$-algebra $\Omega(P)$ built over $\Psi$ which is such that the map $F$ (and therefore $\chi$) can be extended to a differential algebra homomorphism $\bar{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge$ (that is, property (diff2) holds). Let us consider a $^*$-invariant and $\chi$-invariant complement $\mathcal{L}^\perp$ of $\mathcal{L}$ in $\Psi_{\text{inv}}$.

**Lemma 6.14.** A linear map $\omega : \Gamma_{\text{inv}} \to \Omega(P)$ given by

$$\omega(\vartheta) = (\rho|\mathcal{L}^\perp)^{-1}(\vartheta)$$

is a connection on $P$. 

Proof. By construction, it follows that $\omega$ is a hermitian pseudotensorial 1-form. Condition (4.6) directly follows from the observation that

$$\pi_v(\vartheta) = 1 \otimes \rho(\vartheta)$$

for each $\vartheta \in \Psi_{inv}$.

Let us assume that the subspace $L^\perp$ is also a submodule of $\Psi_{inv}$. In other words

$$\rho(\vartheta \circ b) = \rho(\vartheta) \circ b,$$

where $\rho : \Psi_{inv} \to L$ is the projection map, corresponding to the splitting

$$\Psi_{inv} = L \oplus L^\perp.$$

In what follows we shall identify the spaces $L^\perp$ and $\Gamma_{inv}$, via the map $\rho$. The right $B$-module structure on $L^\perp$ can be naturally “projected” to the right $A$-module structure on this space, so that $\rho \leftrightarrow L^\perp$ becomes a right $A$-module isomorphism, because of

$$L^\perp \circ \ker(j) = \{0\}.$$

Further, let us assume that $\Omega(P)$ is left-covariant and let $l(P) \subseteq \Omega(P)$ be a $*$-subalgebra consisting of left-invariant elements. Finally, let us assume that $\omega$ constructed in the above lemma is a regular and multiplicative connection. This assumption implies certain specific algebraic relations between elements of $l(P)$.

It is clear that elements $\eta \in L$ are horizontal. Hence the following relations hold

$$\vartheta \eta + \sum_k \eta_k(\vartheta \circ a_k) = 0$$

where $\vartheta \in L^\perp$ and $\sum_k \eta_k \otimes a_k = \chi(\eta)$.

The action of the covariant derivative on elements from $B$ and $L$ is described by

**Lemma 6.15.** The following identities hold

$$\begin{align*}
\kappa(b) &= \kappa'(b^{(1)}) D_\omega(b^{(2)}) \\
D_\omega(b) &= b^{(1)} \kappa(b^{(2)}) \\
D_\omega \kappa(b) &= -R_\omega \pi j(b) - \kappa(b^{(1)}) \kappa(b^{(2)}),
\end{align*}$$

where $\kappa = \rho \perp \pi'$. 

Proof. Evidently, (6.55) and (6.56) are mutually equivalent. Equation (6.56) directly follows from (2.2), (4.25) and from the definition of $\omega$. Acting by $D_\omega$ on both sides of (6.55) and applying (4.29)–(4.30) we obtain

$$\begin{align*}
D_\omega \kappa(b) &= D_\omega \kappa'(b^{(1)}) D_\omega(b^{(2)}) + \kappa'(b^{(1)}) D^2_\omega(b^{(2)}) \\
&= \{ D_\omega \kappa'(b^{(1)}) \} b^{(2)} \kappa(b^{(3)}) - \kappa'(b^{(1)}) b^{(2)} R_\omega \pi j(b^{(3)}) \\
&= -\kappa(b^{(1)}) \kappa(b^{(2)}) - R_\omega \pi j(b).
\end{align*}$$
In fact, formula (6.57) defines $D_\omega$ and $R_\omega$. If $\pi'(b) \in \mathcal{L}$ then

$$D_\omega(\pi'(b)) = -\varkappa(b^{(1)})\varkappa(b^{(2)})$$

and similarly

$$R_\omega(\pi'(b)) = -\varkappa(b^{(1)})\varkappa(b^{(2)}),$$

if $\varkappa(b) = 0$. The above two formulas are in fact equivalent to (6.57). Both give the same consistency condition for $l(P)$. If $b \in R'$ then

$$\varkappa(b^{(1)})\varkappa(b^{(2)}) = 0.$$  \hfill (6.58)

The above constraint generates a further constraint in the third-order level, because it must be compatible with $D_\omega$ satisfying the graded Leibniz rule. Explicitly,

$$R_\omega\pi j(b^{(1)})\varkappa(b^{(2)}) - \varkappa(b^{(1)})R_\omega\pi j(b^{(2)}) = 0$$

for each $b \in R'$. More generally, it follows that

$$R_\omega(\vartheta)\eta = \sum_k \eta_k R_\omega(\vartheta \circ c_k)$$

where $\vartheta \in \Gamma_{inv}$ and $\eta \in \mathcal{L}$, while $\sum_k \eta_k \otimes c_k = \chi(\eta)$.

It is worth noticing that $\rho$ is extendible to a homomorphism $\rho^\wedge : l(P) \rightarrow \Gamma_{inv}^\wedge$ of graded-differential algebras. In terms of the canonical identification $\Omega(P) \leftrightarrow B \otimes l(P)$ of spaces, the verticalization homomorphism is given by $\pi_v \leftrightarrow \text{id} \otimes \rho^\wedge$.

Motivated by the derived expressions and constraints we shall now construct “the universal” higher-order calculus on the bundle, admitting regular and multiplicative connections of the described geometrical nature. The starting point will be a left-covariant $\ast$-calculus $\Psi$ over $H$, endowed with a splitting of the form $\Psi_{inv} \cong \mathcal{L} \oplus \Gamma_{inv}$. We shall assume that this splitting possesses all above introduced properties. Let $\mathcal{K}_1$ be the ideal in the tensor algebra $\mathcal{L}^\otimes$ generated by elements of the form

$$w = \varkappa(b^{(1)}) \otimes \varkappa(b^{(2)})$$

where $b \in R'$. The formulas

$$D(\pi'(b)) = -\varkappa(b^{(1)})\varkappa(b^{(2)})$$

$$R(\pi'(q)) = -\varkappa(q^{(1)})\varkappa(q^{(2)})$$

consistently define linear maps $D : \mathcal{L} \rightarrow \mathcal{L}^\otimes/\mathcal{K}_1$ and $R : \Gamma_{inv} \rightarrow \mathcal{L}^\otimes/\mathcal{K}_1$. Here, $\pi j(b) = 0$ and $\varkappa(q) = 0$. We have

$$D\varkappa(b) = -R\pi j(b) - \varkappa(b^{(1)})\varkappa(b^{(2)}).$$

Let $\mathcal{K}_2$ be the ideal in $\mathcal{L}^\otimes/\mathcal{K}_1$ generated by relations of the form

$$R(\vartheta)\eta = \sum_k \eta_k R(\vartheta \circ c_k)$$

where $\chi(\eta) = \sum_k \eta_k \otimes c_k$.

The map $D$ can be uniquely extended to a first-order derivation $D : \mathcal{L}^* \rightarrow \mathcal{L}^*$, where $\mathcal{L}^* = \left[\mathcal{L}^\otimes/\mathcal{K}_1\right]/\mathcal{K}_2$. Indeed, it is sufficient to check that the graded Leibniz
rule for $D$ is not in a contradiction with relations generating $K_1$ and $K_2$. This follows from the above derived equations.

Both ideals $K_1$ and $K_2$ are right and $\ast$-invariant, in a natural manner. In other words $L^*$ is a $\ast$-algebra, endowed with the right action $\chi: L^* \to L^* \otimes A$. Let us assume that $R$ is factorized through the ideal $K_2$. By construction, $D$ and $R$ are hermitian and right-covariant maps. In particular, it follows that $DR = 0$.

The $\circ$-structure on $L^\otimes$ can be naturally “projected” to $L^\ast$, through ideals $K_1$ and $K_2$. The following identities hold

\begin{align}
D(\vartheta \circ b) &= D(\vartheta) \circ b - \kappa(b^{(1)})(\vartheta \circ b^{(2)}) + (-1)^{\vartheta b}(\vartheta \circ b^{(1)})\kappa(b^{(2)}) \\
R(\vartheta \circ j(b)) &= R(\vartheta) \circ b.
\end{align}

Finally, let us consider a graded $\ast$-algebra defined as

$$hor_P = B \otimes L^\ast$$

at the level of graded vector spaces, while the product and the $\ast$-structure are given by

\begin{align}
(q \otimes \eta)(b \otimes \vartheta) &= qb^{(1)} \otimes (\eta \circ b^{(2)})\vartheta \\
\ast(b \otimes \vartheta) &= b^{(1)} \ast \otimes (\vartheta^\ast \circ b^{(2)} \ast).
\end{align}

Evidently, $B$ and $L^\ast$ are $\ast$-subalgebras of $hor_P$. The formulas

\begin{align}
D(b \otimes \vartheta) &= b^{(1)} \otimes \kappa(b^{(2)})\vartheta + b \otimes D(\vartheta) \\
F^\ast(b \otimes \vartheta) &= F(b)\chi(\vartheta)
\end{align}

define extensions $D: hor_P \to hor_P$ and $F^\ast: hor_P \to hor_P \otimes A$ of the previously introduced maps. By construction, $F^\ast$ defines the action of $G$ by “automorphisms” of $hor_P$. Further,

\begin{lemma}
The map $D$ is a hermitian right-covariant first-order antiderivation on $hor_P$. The following identities hold

\begin{align}
D^2(\varphi) &= -\sum_k \varphi_k R\pi(c_k) \\
R(\vartheta)\varphi &= \sum_k \varphi_k R(\vartheta \circ c_k),
\end{align}

where $F^\ast(\varphi) = \sum_k \varphi_k \otimes c_k$.

\textbf{Proof.} We compute

\begin{align}
D(\vartheta b) &= D(b^{(1)})\vartheta \circ b^{(2)} + b^{(1)}D(\vartheta \circ b^{(2)}) \\
&= b^{(1)}\kappa(b^{(2)})\vartheta \circ b^{(3)} + b^{(1)}D(\vartheta) \circ b^{(2)} \\
&\quad - b^{(1)}\kappa(b^{(2)})\vartheta \circ b^{(3)} + (-1)^{\vartheta b}(b^{(1)})(\vartheta \circ b^{(2)})\kappa(b^{(3)}) \\
&= D(\vartheta)b + (-1)^{\vartheta b}D(b).
\end{align}
This implies that $D$ is a (first-order) antiderivation on $\mathfrak{b}o\mathfrak{r}_P$. Furthermore, it is sufficient to check that the relation between $D^2$ and $R$ holds on elements from $\mathcal{L}$ and $\mathcal{B}$. We have
\[
D^2(b) = D(b^{(1)}\varphi(b^{(2)})) = b^{(1)}\varphi(b^{(2)})\varphi(b^{(3)}) - b^{(1)}R\pi_j(b^{(2)}) - b^{(1)}\varphi(b^{(2)})\varphi(b^{(3)}) = -b^{(1)}R\pi_j(b^{(2)})
\]
and similarly
\[
D^2\varphi(b) = -D(R\pi_j(b) + \varphi(b^{(1)})\varphi(b^{(2)})) = R\pi_j(b^{(1)})\varphi(b^{(2)}) - \varphi(b^{(1)})R\pi_j(b^{(2)}) = \varphi(b^{(3)})R[\pi_j(b^{(1)}) \circ j(\kappa(b^{(2)})b^{(4)})] - \varphi(b^{(1)})R\pi_j(b^{(2)}) = -\varphi(b^{(2)})R\pi_j(\kappa(b^{(1)})b^{(3)}).
\]
Finally, we have to check the commutation relation between $R$ and $\mathcal{B}$. Direct transformations give
\[
R\pi_j(q)b = -\varphi(q^{(1)})\varphi(q^{(2)})b = -b^{(1)}(\varphi(q^{(1)}) \circ b^{(2)})(\varphi(q^{(2)}) \circ b^{(3)}) = -b^{(1)}\varphi(q^{(1)})\varphi(q^{(2)})\varphi(b^{(3)}) = -b^{(1)}R\pi_j(qb^{(2)}),
\]
where $\varphi(q) = 0$. This completes the proof.

Now the construction of the full differential calculus on the bundle $P$ can be completed applying ideas of Subsection 6. The initial splitting is naturally understandable as a regular and multiplicative connection $\omega$ on $P$. Maps $D$ and $R$ are interpretable as the corresponding operators of covariant derivative and curvature. The full algebra $\Omega(P)$ is left-covariant (over $H$). The associated first-order calculus coincides with $\Psi$. The corresponding differential $^*$-subalgebra $\mathfrak{l}(P)$ of left-invariant elements can be independently described as follows. At the level of (graded) vector spaces
\[
\mathfrak{l}(P) = \mathcal{L}^* \otimes \Gamma^\wedge_{\text{inv}}.
\]
The differential $^*$-algebra structure is specified by
\[
(\eta \otimes \xi)(\vartheta \otimes \zeta) = (-1)^{|\xi||\vartheta|} \sum_k \eta \vartheta_k \otimes (\xi \circ c_k)\zeta
\]
\[
(\vartheta \otimes \zeta)^* = \sum_k \vartheta_k^* \otimes (\zeta^* \circ c_k^*)
\]
\[
d^\wedge(\vartheta \otimes \zeta) = D(\vartheta) \otimes \zeta + (-1)^{|\vartheta|} \sum_k \vartheta_k \otimes \pi(c_k)\zeta + (-1)^{|\vartheta|} \vartheta d^\wedge(\zeta).
\]
Here $d^\wedge : \mathfrak{l}(P) \rightarrow \mathfrak{l}(P)$ is the corresponding differential and $\sum_k \vartheta_k \otimes c_k = \chi(\vartheta)$. Finally,
\[
d^\wedge(\zeta) = d(\zeta) + R(\zeta)
\]
for $\zeta \in \Gamma^\wedge_{\text{inv}}$. The map $\rho^\wedge$ is given by $\rho^\wedge : \epsilon_{\mathcal{L}} \otimes \text{id}$, where $\epsilon_{\mathcal{L}}$ is a character on $\mathcal{L}^*$ specified by $\epsilon_{\mathcal{L}}(\mathcal{L}) = \{0\}$.

As a concrete example, let us consider the quantum Hopf fibering [7]. This bundle is described by the quantum group $H = SU(2)$, and its subgroup $G = H_{cl} = U(1)$. The base manifold is the quantum 2-sphere [7].
By definition [4], $\mathcal{B}$ is the *-algebra generated by elements $\alpha$ and $\gamma$ and the following relations

$$
\alpha\alpha^* + \mu^2\gamma\gamma^* = 1 \quad \alpha^*\alpha + \gamma^*\gamma = 1
$$
$$
\alpha\gamma = \mu\gamma\alpha \quad \alpha\gamma^* = \mu\gamma^*\alpha \quad \gamma\gamma^* = \gamma^*\gamma.
$$

The fundamental representation is given by

$$
u = (\nu^\dagger)^{-1} = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}
$$

where $\mu \in (-1, 1) \setminus \{0\}$.

Let us first assume that the first-order differential structure $\Psi$ over $H$ coincides with the 3D-calculus [4], based on a right $\mathcal{B}$-ideal $\mathcal{R}' = \text{gen}\{\gamma^2, \gamma\gamma^*, \gamma^*\gamma, \alpha\gamma - \gamma, \alpha\gamma^* - \gamma^*, \mu^2\alpha + \alpha^* - (1 + \mu^2)1\}$.

The space $\Psi_{inv}$ is 3-dimensional and spanned by elements

$$
\eta = \pi'(\alpha - \alpha^*) \quad \eta_+ = \pi'(\gamma) \quad \eta_- = \pi'(\gamma^*).
$$

The corresponding right $\mathcal{B}$-module structure $\circ$ is specified by

$$
\mu^2\eta \circ \alpha = \eta \quad \eta \circ \alpha^* = \mu^2\eta
$$
$$
\mu\eta_\pm \circ \alpha = \eta_\pm \quad \eta_\pm \circ \alpha^* = \mu\eta_\pm
$$

with $\Psi_{inv} \circ \gamma = \Psi_{inv} \circ \gamma^* = \{0\}$.

The *-algebra $\mathcal{A}$ of polynomial functions on $G$ is generated by the canonical unitary element $z = j(\alpha)$ (and we have $j(\gamma) = j(\gamma^*) = 0$). Let us assume that $\Gamma$ is a left-covariant calculus over $G$ based on the right $\mathcal{A}$-ideal $\mathcal{R} = j(\mathcal{R}')$. The space $\Gamma_{inv}$ is 1-dimensional, and spanned by $\zeta = \rho(\eta) = \pi(z - z^*)$. We have $\rho(\eta_+) = \rho(\eta_-) = 0$ and $\Gamma^{\wedge k} = \{0\}$ for $k \geq 2$. However, the calculus based on $\Gamma^{\wedge}$ differs from the classical differential calculus, because the right $\mathcal{A}$-module structure on $\Gamma_{inv}$ is given by

$$
\mu^2\zeta \circ z = \zeta \quad \zeta \circ z^* = \mu^2\zeta.
$$

The space $\mathcal{L}$ is spanned by $\eta_+$ and $\eta_-$. Let us define a splitting $\Psi_{inv} \cong \mathcal{L} \oplus \Gamma_{inv}$ (⇔ the space $\mathcal{L}^\wedge$) by identifying $\eta$ and $\zeta$. The corresponding relations determining the algebra $\mathcal{L}^\wedge$ are

$$
\eta_+^2 = \eta_-^2 = 0
$$
$$
\eta_+\eta_- = -\mu^2\eta_-\eta_+
$$

while the third-order relations are trivialized. Additional relations determining the algebra $(\mathcal{L})$ are

$$
\eta_+ = -\frac{1}{\mu^2}\eta_+\eta \quad \eta_-^2 = 0 \quad \eta_+ = -\mu^4\eta_-\eta.
$$

It is worth noticing that $\Omega(P) = \Psi^\wedge$ and hence the higher-order calculus on the bundle coincides with the calculus constructed in [4]. Modulo differences between general formulations, the canonical regular connection $\omega$ (associated to the fixed
splitting of $\Psi_{\text{inv}}$) coincides with a connection constructed in [3]. The curvature of $\omega$ is given by

\begin{equation}
R_\omega(\xi) = d\eta = \mu(1 + \mu^2)\eta_-\eta_+.
\end{equation}

Next, let us consider the case of the $4D^+$-calculus over $S^\mu U(2)$. This calculus $\Psi$ is bicovariant and $^*$-covariant. By definition [6] the corresponding inv splitting of $\Psi_{\text{inv}}$ is given by

\begin{equation}
1 = \left\{ a(\mu^2\alpha + \alpha^* - (1 + \mu^2)1) \right\} \quad 3 = \left\{ a\gamma, a(\alpha - \alpha^*), a\gamma^* \right\}
\end{equation}

\begin{equation}
5 = \left\{ \gamma^2, \gamma(\alpha - \alpha^*), \mu^2\alpha^2 - (1 + \mu^2)(\alpha\alpha^* - \gamma\gamma^*) + \alpha^2, \gamma^*(\alpha - \alpha^*), \gamma^*2 \right\}
\end{equation}

where $a = \mu^2\alpha + \alpha^* - (\mu^3 + 1/\mu)1$. The space $\Psi_{\text{inv}}$ is 4-dimensional. A natural basis is given by elements

\begin{equation}
\tau = \pi'(\mu^2\alpha + \alpha^*)
\end{equation}

\begin{equation}
\eta_+ = \pi'(\gamma) \quad \eta = \pi'(\alpha - \alpha^*) \quad \eta_- = \pi'(\gamma^*).
\end{equation}

Elements $\eta_+$, $\eta_-$ and $\eta$ form a triplet (relative to the adjoint action). The element $\tau$ is $\pi'$-invariant. We have ([2]–Section 6)

\begin{align*}
\eta_+ \circ \gamma^* &= \eta_- \circ \gamma = \frac{(1 + \mu)(1 - \mu^2)}{\mu(1 + \mu^2)(1 - \mu^3)}\tau - \frac{1 - \mu^2}{\mu(1 + \mu^2)}\eta \\
\eta_+ \circ \gamma &= \eta_+ \circ \gamma^* = \eta_- \circ \gamma^* = \eta_- \circ \alpha^* - \frac{1 - \mu^2}{\mu}\eta_-
\end{align*}

\begin{align*}
-\eta \circ \alpha^* &= \left(\frac{(1 + \mu)(1 - \mu^2)}{\mu(1 + \mu^2)(1 - \mu^3)^2} - \frac{2\mu}{1 + \mu^2}\right)\tau - \frac{2\mu}{1 + \mu^2}\eta \\
\eta \circ \alpha &= \frac{\mu(1 + \mu)(1 - \mu^2)}{(1 + \mu^2)(1 - \mu^3)^2}\tau + \frac{2\mu}{1 + \mu^2}\eta
\end{align*}

\begin{align*}
\tau \circ \gamma &= \frac{(1 - \mu)(1 - \mu^3)}{\mu}\eta_+ \quad \tau \circ \alpha^* &= \frac{1 + \mu^4}{\mu(1 + \mu^2)}\tau - \frac{\mu(1 - \mu)(1 - \mu^3)}{1 + \mu^2}\eta \\
\tau \circ \gamma^* &= \frac{(1 - \mu)(1 - \mu^3)}{\mu}\eta_- \quad \tau \circ \alpha &= \frac{1 + \mu^4}{\mu(1 + \mu^2)}\tau + \frac{(1 - \mu)(1 - \mu^3)}{\mu(1 + \mu^2)}\eta.
\end{align*}

It turns out that the ideal $\mathcal{R} = j(\mathcal{R}')$ is generated by the element $z + \mu z^* - (1 + \mu)1$. The space $\mathcal{L}$ is spanned by elements $\eta_+$, $\eta_-$ and $\xi = \tau + ((1 - \mu^3)/(1 + \mu))\eta$. It is worth noticing that $\xi$ is (the unique) common characteristic vector for all operators of the form $\circ b$ (where $b \in \mathcal{B}$). Explicitly,

\begin{align*}
\xi \circ \alpha &= \frac{1}{\mu}\xi \quad \xi \circ \alpha^* = \mu\xi \quad \xi \circ \gamma = \xi \circ \gamma^* = 0.
\end{align*}

However, the space $\mathcal{L}$ does not possess a $\circ$-invariant complement. A natural choice of the complement $\mathcal{L}^\perp$ is to consider the subspace spanned by $\eta$ (following
a weak analogy with the previous example). The calculus on $G = U(1)$ is non-
standard. The higher-order calculus is trivial. The corresponding right $A$-module
structure is given by

$$\zeta \circ z = \mu \zeta \quad \zeta \circ z^* = \frac{1}{\mu} \zeta$$

where $\zeta = \rho(\eta)$. Let us assume that the higher-order calculus on the bundle is based
on the bicovariant exterior algebra $\Psi^\vee$ [6]. Let $\omega$ be the connection corresponding
to the above splitting. This connection is not multiplicative.

The most general form of a connection (coming from a complement $\mathcal{L}^\perp$) is

\begin{equation}
\omega(\zeta) = \eta + t \xi
\end{equation}

where $t \in \mathbb{R}$. All these connections are non-regular. A connection $\omega$ is multiplicative
iff $(\eta + t \xi)^2 = 0$, which is equivalent to $t = -(1 + \mu)/(1 - \mu^3)$, in other words the
corresponding complement is spanned by $\tau$. However it is worth noticing that if
the higher-order calculus is described by $\Psi^\wedge$ then $\omega$ is not multiplicative, because
$\tau^2 \neq 0$ in this case.

The curvature is given by

\begin{equation}
R_\omega = d(\eta + t \xi) = \left(\frac{\mu t}{1 - \mu^2} + \frac{\mu}{(1 - \mu)(1 - \mu^3)}\right)(\tau \eta + \eta \tau).
\end{equation}

We see that for the above mentioned special value of $t$ the curvature vanishes.

Finally, let us assume that the calculus on $G$ is \textit{classical} (based on standard
differential forms). Let us assume that $\Psi$ is a bicovariant *-calculus. This implies
$(X \otimes \text{id})\text{ad}'(\mathcal{R}') = \{0\}$, where $X$ is the canonical generator of $\text{lie}(G)$. In other
words, $\Psi$ is \textit{admissible} (in the sense of the previous paper). Let us assume that $\Psi$ is
the \textit{minimal} admissible (bicovariant *) calculus. The space $\Psi_{inv}$ can be naturally
identified with the algebra $\mathcal{D}$ consisting of all elements $b \in \mathcal{B}$ invariant under the
left action of $G$ on $H$ (polynomial functions on a quantum 2-sphere). In terms of
this identification

$$\varpi' \leftrightarrow (\phi'|\mathcal{D}): \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{B} \quad (\ ) \circ b \leftrightarrow \kappa(b^{(1)})(b^{(2)}).$$

Let $\tau$ be the element corresponding to $1 \in \mathcal{D}$ and let us assume that $\mathcal{L}^\perp$ is spanned
by $\tau$. Explicitly,

\begin{equation}
\tau = \frac{2}{\mu^2 - 1} \pi'(\mu^2 \alpha + \alpha^*).
\end{equation}

The element $\tau$ is right-invariant ($\mathcal{L}^\perp$ consists precisely of right-invariant elements
of $\Psi_{inv}$ and each integer-valued irreducible multiplet appears without degeneracy
in the decomposition of $\varpi'$ into irreducible components), and

\begin{equation}
\tau \circ b = e'(b)\tau.
\end{equation}

In particular

\begin{equation}
\sigma(\tau \otimes \vartheta) = \vartheta \otimes \tau \\
\sigma(\vartheta \otimes \tau) = \tau \otimes \vartheta.
\end{equation}
Let us assume that the higher-order calculus on the bundle is described by the exterior algebra $\Psi^\vee$. We have
\[
\tau^2 = 0 \\
\tau d\tau = 0
\]
Identities (6.67) and (6.68) imply
\[
\tau w = (-1)^{\partial w} w \tau
\]
for each $w \in \Psi^\vee$. The connection $\omega$ corresponding to $\mathcal{L}^\perp$ is regular and multiplicative. Moreover, $\omega$ is flat in the sense that $R_\omega = 0$.

6.6. A Constructive Approach to Differential Calculus

Every regular connection $\omega$ induces the isomorphism $m_\omega : \mathfrak{vh}(P) \leftrightarrow \Omega(P)$ (if $\mathcal{X}(P) = \{0\}$). Moreover, if two algebras are identified with the help of $m_\omega$ then it is possible to express the differential structure on $\Omega(P)$ in terms of the algebra structure on $\mathfrak{vh}(P)$, and the following maps:
\[
D_\omega : \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P) \\
R_\omega : \Gamma_{\text{inv}} \rightarrow \mathfrak{hor}(P) \\
d : \Gamma_{\text{inv}}^\wedge \rightarrow \Gamma_{\text{inv}}^\wedge \\
F^\wedge : \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P) \otimes \mathcal{A}.
\]
Explicitly,
\[
d^\wedge (\varphi \otimes \vartheta) = D_\omega (\varphi) \otimes \vartheta + (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \pi(c_k) \vartheta + (-1)^{\partial \varphi} \varphi d^\wedge (\vartheta),
\]
where $d^\wedge$ is the corresponding differential on $\mathfrak{vh}(P)$ and $d^\wedge |\Gamma_{\text{inv}}^\wedge$ is fixed by
\[
d^\wedge (\vartheta) = d(\vartheta) + R_\omega (\vartheta),
\]
for $\vartheta \in \Gamma_{\text{inv}}^\wedge$, and extended on $\Gamma_{\text{inv}}^\wedge$ by the graded Leibniz rule.

It is worth noticing that the curvature $R_\omega$ is completely determined by the covariant derivative, as easily follows from (4.29) and property (qpb4) from Section 3. Indeed, we have
\[
R_\omega \pi(a) = - \sum_k q_k D^2_\omega (b_k)
\]
where $q_k, b_k \in \mathcal{B}$ are such that (3.3) holds.

In this subsection an “opposite” construction will be presented, which builds the algebra $\Omega(P)$ and a regular multiplicative connection $\omega$ starting from a $*$-algebra playing the role of horizontal forms, and three operators imitating the covariant derivative, curvature and the right action.

Let $P = (\mathcal{B}, i, F)$ be a quantum principal $G$-bundle over $M$. Let
\[
\mathfrak{hor}_P = \bigoplus_{k \geq 0} \mathfrak{hor}_P^k
\]
be a graded *-algebra such that $\mathfrak{hor}^0_P = \mathcal{B}$. Further, let us assume that a grade-preserving *-homomorphism $F^* : \mathfrak{hor}_P \to \mathfrak{hor}_P \otimes A$ extending the map $F$ is given such that

\begin{align}
(F^* \otimes \text{id})F^* &= (\text{id} \otimes \phi)F^* \\
(\text{id} \otimes \epsilon)F^* &= \text{id}.
\end{align}

(6.71) \hspace{1cm} (6.72)

Let us assume that a linear first-order map $D : \mathfrak{hor}_P \to \mathfrak{hor}_P$ is given such that the following properties hold

\begin{align}
D(\varphi \psi) &= D(\varphi)\psi + (-1)^{\partial \varphi} D(\psi) \\
D^* &= * D \\
F^* D &= (D \otimes \text{id}) F^*.
\end{align}

(6.73) \hspace{1cm} (6.74) \hspace{1cm} (6.75)

Finally, let us assume that there exists a linear map $R : \Gamma_{inv} \to \mathfrak{hor}_P$ such that

\begin{align}
D^2(\varphi) &= - \sum_k \varphi_k R\pi(c_k),
\end{align}

(6.76)

for each $\varphi \in \mathfrak{hor}_P$, where $F^*(\varphi) = \sum_k \varphi_k \otimes c_k$.

Evidently, $D$ plays the role of the covariant derivative. The map $R$ is determined uniquely by the above condition. It plays the role of the curvature map. Explicitly

\begin{align}
R\pi(a) &= - \sum_k q_k D^2(b_k)
\end{align}

(6.77)

where $q_k, b_k \in \mathcal{B}$ are such that (3.3) holds.

We pass to a “reconstruction” of the calculus $\Omega(P)$. Let us first analyze the map $R$ in more details.

**Proposition 6.17.** The following identities hold

\begin{align}
F^* R &= (R \otimes \text{id}) \varpi \\
DR &= 0 \\
R^* &= * R \\
R(\vartheta) \varphi &= \sum_k \varphi_k R(\vartheta \circ c_k).
\end{align}

(6.78) \hspace{1cm} (6.79) \hspace{1cm} (6.80) \hspace{1cm} (6.81)

*Proof.* Acting by $D \otimes \text{id}$ on equality (3.3) and using (6.73) we obtain

\[ 0 = \sum_k D(q_k)F(b_k) + \sum_k q_k(D \otimes \text{id})F(b_k). \]

This, together with (6.75) and (6.76), implies

\[ 0 = \sum_k D(q_k)D^2(b_k) + \sum_k q_k D^3(b_k) = D\left(\sum_k q_k D^2(b_k)\right) = -DR\pi(a). \]

Hence, (6.79) holds. Further, (3.3) implies

\[ 1 \otimes 1 \otimes a = \sum_{kj} F(q_k)F(b_{kj}) \otimes a_{kj} \]
and hence
\[
1 \otimes \kappa(a^{(1)}) \otimes a^{(2)} = \sum_{ijk} q_{ki} b_{kj} \otimes z_{ki} \otimes a_{kj},
\]
where \(F(b_k) = \sum_j b_{kj} \otimes a_{kj}\) and \(F(q_k) = \sum_i q_{ki} \otimes z_{ki}\). From (6.82) we find
\[
(6.83) \\
1 \otimes \kappa(a) = \sum_k F(q_k) b_k
\]
\[
(6.84) \\
e(a) 1 = \sum_k q_k b_k
\]
\[
(6.85) \\
1 \otimes a^{(2)} \otimes \kappa(a^{(1)}) a^{(3)} = \sum_{ijk} q_{ki} F(b_{kj}) \otimes z_{ki} a_{kj}.
\]

Identity (6.84) also directly follows from (3.3). A direct computation now gives
\[
R[\pi(a)^*] = -R\pi(\kappa(a)^*) = \sum_k b_k^* D^2(q_k^*) = -\sum_k D^2(b_k^*) q_k^* = [R\pi(a)]^*,
\]
and similarly
\[
(R \otimes \text{id}) \varpi \pi(a) = R\pi(a^{(2)}) \otimes \kappa(a^{(1)}) a^{(2)} = -\sum_{ijk} q_{ki} D^2(b_{kj}) \otimes z_{ki} a_{kj}
\]
\[
= -F^* \left( \sum_k q_k D^2(b_k) \right) = F^* R \pi(a).
\]

This proves (6.78) and (6.80). Finally, let us prove (6.81). We have
\[
\sum_{nkj} q_{kj} b_{kj} \varphi_n \otimes a_{kj} c_n = \sum_n \varphi_n \otimes ac_n,
\]
for each \(\varphi \in \text{hor}_p\). Hence
\[
\sum_{nkj} q_{kj} b_{kj} \varphi_n \otimes \pi(a_{kj} c_n) = \sum_n \varphi_n \otimes \pi(a) \circ c_n,
\]
if \(a \in \ker(\epsilon)\). The above equality, together with (6.76) implies
\[
\sum_{nkj} q_{kj} b_{kj} \varphi_n R\pi(a_{kj} c_n) = -\sum_k q_k D^2(b_k \varphi) = \sum_n \varphi_n R[\pi(a) \circ c_n].
\]

On the other hand, (6.77) and (6.84) imply
\[
[R\pi(a)] \varphi = -\sum_k q_k D^2(b_k \varphi) = -\sum_k q_k D^2(b_k \varphi).
\]

Hence, property (6.81) holds. \(\square\)
Let us consider the graded space \( \Omega(P) = \mathfrak{hor}_P \otimes \Gamma^\wedge \), endowed with the following \(*\)-algebra structure

\[
(\psi \otimes \eta)(\varphi \otimes \vartheta) = (-1)^{\partial \varphi \partial \eta} \sum_k \psi \varphi_k \otimes (\eta \circ c_k) \vartheta
\]

(6.86)

\[
(\varphi \otimes \vartheta)^* = \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*),
\]

(6.87)

where \( F^*(\varphi) = \sum_k \varphi_k \otimes c_k \). Algebras \( \mathfrak{hor}_P \) and \( \Gamma^\wedge \) are understandable as \(*\)-sub-algebras of \( \Omega(P) \), in a natural manner.

**Lemma 6.18.** There exists the unique antiderivation \( d^\wedge : \Gamma^\wedge \to \Omega(P) \) satisfying

\[
d^\wedge(\vartheta) = R(\vartheta) + d(\vartheta)
\]

(6.88)

for each \( \vartheta \in \Gamma^\wedge \).

**Proof.** The graded Leibniz rule implies that the values of \( d^\wedge \) on higher-order forms are completely determined by the restriction \( d^\wedge |\Gamma^\wedge \) (and we have \( d^\wedge 1 = 0 \)). Hence, \( d^\wedge \) is unique, if exists.

Let us prove that \( d^\wedge \) can be consistently constructed by extending, with the help of the graded Leibniz rule, a linear map (acting on \( \Gamma^\wedge \)) given by (6.88). The extension exists if \( d^\wedge \) does not appear at the level of second-order constraints defining the algebra \( \Gamma^\wedge \). Simple transformations give

\[
\left\{ \pi(a^{(1)})\pi(a^{(2)}) \right\} \to R\pi(a^{(1)})\pi(a^{(2)})
\]

\[
- \pi(a^{(1)})\pi(a^{(2)})\pi(a^{(3)})
\]

\[
+ \pi(a^{(1)})\pi(a^{(2)})\pi(a^{(3)}) - \pi(a^{(1)})R\pi(a^{(2)})
\]

\[
= R\pi(a^{(1)})\pi(a^{(2)})
\]

\[
- R\pi(a^{(3)})[\pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)})]
\]

\[
= R\pi(a^{(2)})\pi(\kappa(a^{(1)})a^{(3)}) = 0,
\]

for each \( a \in R \). Hence, \( d^\wedge \) exists. \( \Box \)

The formula

\[
d^\wedge(\varphi \otimes \vartheta) = D(\varphi) \otimes \vartheta + (-1)^{\partial \varphi} \varphi d^\wedge(\vartheta) + (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \pi(c_k) \vartheta
\]

(6.89)

defines a linear first-order map \( d^\wedge : \Omega(P) \to \Omega(P) \) extending \( d^\wedge \) introduced in the previous lemma.

**Proposition 6.19.** The following identities hold

\[
d^\wedge * = *d^\wedge
\]

(6.90)

\[
(d^\wedge)^2 = 0
\]

(6.91)

\[
d^\wedge(wu) = d^\wedge(w)u + (-1)^{\partial w} wd^\wedge(u).
\]

(6.92)
Proof. A direct calculation gives for \( w \in \Gamma_{\text{inv}} \) and \( u = \phi \otimes \vartheta \),

\[
d^\wedge(wu) = (-1)^{\partial w + \partial u} \sum_{k} w_k \phi_k \otimes (d \circ c_k) \vartheta + (-1)^{\partial w + \partial u} w \varphi d^\wedge(\vartheta) + D(w\varphi) \otimes \vartheta
\]

\[
= -w[D(\varphi) \otimes \vartheta] + (-1)^{\partial w} w \varphi d^\wedge(\vartheta) + d^\wedge(w) u
\]

\[
- (-1)^{\partial w} \sum_{k} w_k \varphi_k \otimes (c_k) \vartheta = d^\wedge(w) u - wd^\wedge(u).
\]

A similar computation shows that (6.92) holds for \( w \in \mathfrak{hor}_P \),

\[
d^\wedge(wu) = (-1)^{\partial w + \partial u} \sum_{l} w_l \varphi_l \otimes (d_l c_k) \vartheta + (-1)^{\partial w + \partial u} w \varphi d^\wedge(\vartheta) + D(w\varphi) \otimes \vartheta
\]

\[
= D(w) \phi \otimes \vartheta + (-1)^{\partial w} w D(\phi) \otimes \vartheta + (-1)^{\partial w + \partial u} w \varphi d^\wedge(\vartheta)
\]

\[
+ (-1)^{\partial w + \partial u} \sum_{k} w_k \varphi_k \otimes (c_k) \vartheta + (-1)^{\partial w + \partial u} \sum_{k} w_l \varphi_k \otimes (\pi(d_l) \circ c_k) \vartheta
\]

\[
= d^\wedge(w) u + (-1)^{\partial w} wd^\wedge(u),
\]

where \( \sum_{l} w_l \otimes d_l = F^*(w) \). It follows that (6.92) holds for arbitrary \( w, u \in \Omega(P) \).

Let us prove that the square of \( d^\wedge \) vanishes. We have

\[
(d^\wedge)^2 \pi(a) = d^\wedge(-\pi(a^{(1)})\pi(a^{(2)}) + R \pi(a)) = DR \pi(a)
\]

\[
+ \pi(a^{(1)})\pi(a^{(2)})\pi(a^{(3)}) - \pi(a^{(1)})\pi(a^{(2)})\pi(a^{(3)})
\]

\[
- [R \pi(a^{(1)})] \pi(a^{(2)}) + \pi(a^{(1)})R \pi(a^{(2)})
\]

\[
+ R \pi(a^{(2)})\pi(\kappa(a^{(1)})a^{(3)})
\]

\[
= R \pi(a^{(3)})[\pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)})] - (R \pi(a^{(1)})\pi(a^{(2)})
\]

\[
+ R \pi(a^{(2)})\pi(\kappa(a^{(1)})a^{(3)}) = 0
\]

for each \( a \in \mathcal{A} \). Further,

\[
(d^\wedge)^2 \varphi = d^\wedge \left( D \varphi + (-1)^{\partial \varphi} \sum_{k} \varphi_k \pi(c_k) \right) = D^2 \varphi - (-1)^{\partial \varphi} \sum_{k} D(\varphi_k)\pi(c_k)
\]

\[
+ (-1)^{\partial \varphi} \sum_{k} D(\varphi_k)\pi(c_k) + \sum_{k} \left\{ \varphi_k \pi(c_k^{(1)})\pi(c_k^{(2)}) + \varphi_k R \pi(c_k) + \varphi_k d \pi(c_k) \right\} = 0,
\]

for each \( \varphi \in \mathfrak{hor}_P \). Having in mind that spaces \( \mathfrak{hor}_P \) and \( \Gamma_{\text{inv}} \) generate \( \Omega(P) \), and using the fact that the square of an antiderivation is a derivation we conclude that \((d^\wedge)^2 = 0\).
Finally,
\[
d^\wedge(\varphi^*) = D(\varphi^*) + (-1)^{\partial\varphi} \sum_k \varphi_k \pi(c_k^*)
\]
\[
= (D\varphi)^* + (-1)^{\partial\varphi} \sum_k \varphi_k^* \pi(c_k^{(2)})^* \circ c_k^{(1)*}
\]
\[
= (D\varphi + (-1)^{\partial\varphi} \sum_k \varphi_k \pi(c_k))^* = d^\wedge(\varphi)^*
\]
for each \(\varphi \in \mathfrak{hor}_P\). It follows that \(d^\wedge\) is a hermitian map. \(\square\)

We are going to prove that \(\Omega(P)\) satisfies condition \((\text{diff}2)\). The formula
\[
(6.93) \quad \hat{F}(\varphi \otimes \vartheta) = F^*(\varphi) \hat{\vartheta}(\vartheta)
\]
determines a linear map \(\hat{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge\) extending \(F^*\) and \(\hat{\vartheta}\).

**Proposition 6.20.** The map \(\hat{F}\) is a homomorphism of differential \(\ast\)-algebras.

**Proof.** For each \(\varphi \in \mathfrak{hor}_P\) and \(\vartheta \in \Gamma^{\text{inv}}\) we have
\[
\hat{F}(\varphi \wedge \vartheta) = \sum_k \hat{F}(\varphi_k \wedge c_k) = \sum_k \varphi_k (\vartheta \circ c_k^{(2)})
\]
\[
+ (-1)^{\partial\varphi} \sum_{kl} (\varphi_k \wedge c_k^{(1)})(\vartheta_l \circ c_k^{(3)} \otimes c_k^{(2)} a_k c_k^{(4)})
\]
\[
= (1 \otimes \vartheta + \vartheta \wedge (\vartheta)) \sum_k \varphi_k \wedge c_k = \hat{F}(\vartheta) \hat{F}(\varphi).
\]
Here, \(\sum_k \varphi_k \wedge c_k = F^*(\varphi)\) and \(\vartheta \wedge (\vartheta) = \sum_l \vartheta_l \otimes a_l\). Using the facts that \(F^*\) and \(\hat{\vartheta}\) are multiplicative, and that \(\mathfrak{hor}_P\) and \(\Gamma^{\text{inv}}\) generate \(\Omega(P)\), we conclude that \(\hat{F}\) is multiplicative, too.

Let us prove that \(\hat{F}\) intertwines differentials. We have
\[
\hat{F}d^\wedge(\varphi) = \hat{F}\left(D(\varphi) + (-1)^{\partial\varphi} \sum_k \varphi_k \pi(c_k^*)\right)
\]
\[
= \sum_k D\varphi_k \otimes c_k + (-1)^{\partial\varphi} \sum_k \varphi_k \otimes c_k^{(1)} \pi(c_k^{(2)})
\]
\[
+ (-1)^{\partial\varphi} \sum_k \varphi_k \pi(c_k^{(3)})^{(1)} \otimes c_k^{(2)} \pi(c_k^{(4)})
\]
\[
= \sum_k D\varphi_k \otimes c_k + (-1)^{\partial\varphi} \sum_k \varphi_k \otimes \omega c_k
\]
\[
+ (-1)^{\partial\varphi} \sum_k \varphi_k \otimes d c_k
\]
\[
= \sum_k d^\wedge(\varphi_k) \otimes c_k + (-1)^{\partial\varphi} \sum_k \varphi_k \otimes d c_k
\]
\[
= (d^\wedge \otimes \text{id} + (-1)^{\partial\varphi} \text{id} \otimes d) \hat{F}(\varphi).
\]
Further, if $\vartheta \in \Gamma^{\text{inv}}$ then
\[
\hat{F}d^\vartheta(\vartheta) = F^*R(\vartheta) + \hat{\omega}(d\vartheta) \\
= (R \otimes \text{id})\omega(\vartheta) + 1 \otimes d\vartheta + (d \otimes \text{id})\omega(\vartheta) - (\text{id} \otimes d)\omega(\vartheta) \\
= (d^\vartheta \otimes \text{id})\omega(\vartheta) - (\text{id} \otimes d)\omega(\vartheta) + 1 \otimes d\vartheta \\
= [d^\vartheta \otimes \text{id} + (-1)^{\vartheta*}\text{id} \otimes d][1 \otimes \vartheta + \omega(\vartheta)] \\
= [d^\vartheta \otimes \text{id} + (-1)^{\vartheta*}\text{id} \otimes d]\hat{\omega}(\vartheta).
\]
Hence, $\hat{F}$ preserves differential structures. Finally, $\hat{F}$ preserves differential structures.

Thus, $\hat{F}$ is a hermitian map.

Clearly, $\mathcal{B} = \Omega^0(P)$. The algebra of horizontal forms corresponding to $\Omega(P)$ coincides with the initial one. In other words, we can write $\mathfrak{hor}_P = \mathfrak{hor}(P)$. Further, $F^*$ coincides with (the restriction of) the corresponding right action $F^\vartheta$.

Let us consider a map $\omega: \Gamma^{\text{inv}} \rightarrow \Omega(P)$ given by $\omega(\vartheta) = 1 \otimes \vartheta$.

**Proposition 6.21.** (i) The map $\omega$ is a regular multiplicative connection on $P$. In particular, $\Omega(P) = \{0\}$.

(ii) We have $R = R_\omega$ and $D = D_\omega$.

**Proof.** It is evident that $\omega$ is a hermitian map. According to the definition of $\hat{F}$, we have
\[
\hat{F}\omega(\vartheta) = \hat{\omega}(\vartheta) = \omega(\vartheta) + 1 \otimes \vartheta = (\omega \otimes \text{id})\omega(\vartheta) + 1 \otimes \vartheta.
\]
Hence, $\omega$ is a connection on $P$. Multiplicativity of $\omega$ directly follows from the fact that $\Gamma^{\text{inv}}$ is a subalgebra of $\Omega(P)$. In particular, $\omega^\vartheta(\vartheta) = 1 \otimes \vartheta$ for each $\vartheta \in \Gamma^{\text{inv}}$. Regularity follows from the definition of the product in $\Omega(P)$. Finally, (ii) follows from definitions of $R_\omega$, $D_\omega$ and $d^\vartheta$. 

The corresponding factorization map $m_\omega$ reduces to the identity. It is worth noticing that the construction of the algebra $\mathfrak{ver}(P)$ of verticalized forms can be understood as a trivial special case of the construction presented in this subsection. Indeed, if we define $\mathfrak{hor}_P = \mathcal{B}$ (with $\mathfrak{hor}_P^k = \{0\}$ for $k \geq 1$), $D = 0$ (and hence $R = 0$) and $F^* = F$ then $\Omega(P) = \mathfrak{ver}(P)$ and $\hat{F} = \hat{F}^v$. The algebra $(\mathfrak{ver}(P),d_{\mathfrak{nh}})$ can be viewed in a similar way.
Appendix A. On Bicovariant Exterior Algebras

In the presented theory we have assumed that the higher-order calculus on the
structure quantum group is described by the corresponding universal envelope.
This assumption is conceptually the most natural. However all the formalism can
be repeated (straightforwardly, or with natural modifications) if the higher-order
calculus on $G$ is described by an appropriate non-universal differential structure.

Here, it will be assumed that the higher-order calculus on $G$ is based on the
corresponding bicovariant exterior algebra [6]. The appendix is devoted to the
analysis of some aspects of these structures, interesting from the point of view of
differential calculus on quantum principal bundles.

Let $\Gamma$ be a bicovariant first-order differential calculus over $G$ and let us consider
the canonical flip-over automorphism $\sigma : \Gamma \otimes \Lambda \Gamma \to \Gamma \otimes \Lambda \Gamma$ (its “left-invariant”
restriction is given by (4.16)).

The corresponding exterior algebra [6] $\Gamma^\vee$ can be constructed by factorising $\Gamma^\otimes$
through the ideal

$$S^\vee = \ker(A).$$

Here $A = \sum_n A_n$ is the “total antisymmetrizer” map, with $A_n : \Gamma^\otimes n \to \Gamma^\otimes n$
given by

$$A_n = \sum_{\pi \in S_n} (-1)^\pi \sigma_\pi$$

where $\sigma_\pi$ is the operator obtained by replacing transpositions $i \leftrightarrow i + 1$
figuring in a minimal decomposition of $\pi$, by the corresponding $\sigma$-twists (this definition is
consistent, due to the braid equation for $\sigma$). By definition, $A$ acts as the identity transformation on $\Lambda$ and $\Gamma$. The following decomposition holds

$$(A.1) \quad A_{k+l} = (A_k \otimes A_l)A_{kl}$$

where

$$A_{kl} = \sum_{\pi \in S_{kl}} (-1)^\pi \sigma_{\pi^{-1}}$$

and $S_{kl} \subseteq S_{k+l}$ is the subset consisting of permutations preserving the order of the
first $k$ and the last $l$ factors.

The differential map $d : \Lambda \to \Gamma$ can be naturally extended to the differential on
the whole $\Gamma^\vee$. By universality, there exists the unique graded-differential homomorphism $\tilde{\phi} : \Gamma^\wedge \to \Gamma^\vee$ reducing to identity maps on $\Lambda$ and $\Gamma$. If $\Gamma$ is $*$-covariant
then the $*$-involutions on $\Gamma$ and $\Lambda$ can be extended to the $*$-structure on $\Gamma^\vee$ (so
that $\tilde{\phi}$ is a hermitian map).

Proposition A.1. (i) There exists the unique differential algebra homomorphism
$\phi^\vee : \Gamma^\vee \to \Gamma^\vee \otimes \Gamma^\vee$ extending the map $\phi$.

(ii) There exists the unique graded-antimultiplicative extension $\kappa^\vee : \Gamma^\vee \to \Gamma^\vee$ of
the antipode $\kappa$, satisfying

$$(A.2) \quad \kappa^\vee d = d \kappa^\vee.$$
(iii) The following identities hold

\[(A.3) \quad (\phi^\vee \otimes \text{id})\phi^\vee = (\text{id} \otimes \phi^\vee)\phi^\vee\]

\[(A.4) \quad m^\vee(\kappa^\vee \otimes \text{id})\phi^\vee = m^\vee(\text{id} \otimes \kappa^\vee)\phi^\vee = 1\epsilon^\vee\]

\[(A.5) \quad (\epsilon^\vee \otimes \text{id})\phi^\vee = (\text{id} \otimes \epsilon^\vee)\phi^\vee = \text{id}\]

where \(m^\vee\) is the product map in \(\Gamma^\vee\) and \(\epsilon^\vee\) is a linear functional specified by

\[\epsilon^\vee(\vartheta) = \epsilon p_0(\vartheta)\].

(iv) If \(\Gamma\) is \(*\)-covariant then

\[(A.6) \quad (\ast \kappa^\vee)^2(\vartheta) = \vartheta\]

\[(A.7) \quad \phi^\vee\ast = (\ast \otimes \ast)\phi^\vee\].

Proof. Uniqueness of maps \(\phi^\vee\) and \(\kappa^\vee\) is a consequence of the fact that \(\Gamma^\vee\) is generated, as a differential algebra, by \(\mathcal{A}\). Let \(\sharp^\vee : \Gamma \rightarrow \Gamma\) be the canonical extension of the antipode map [2]–Appendix B. There exists the unique graded-antimultiplicative extension \(\kappa^{\otimes} : \Gamma^{\otimes} \rightarrow \Gamma^{\otimes}\) of \(\kappa\) and \(\sharp^\vee\). We have

\[(A.8) \quad \sigma_{\kappa^{\otimes}}(\vartheta) = \kappa^{\otimes} \sigma(\vartheta)\]

for each \(\vartheta \in \Gamma \otimes \mathcal{A} \Gamma\). This directly follows from the fact that \(\sharp^\vee\) maps left-invariant to right-invariant elements, and conversely. Hence,

\[(A.9) \quad \kappa^{\otimes}_n \sigma_\pi = \sigma_{j\pi j}\kappa^{\otimes}_n\]

for each \(\pi \in S_n\), where \(j\) is the “total inverse” permutation. This implies that operators \(\kappa^{\otimes}_n\) commute with \(A_n\). Therefore \(\kappa^{\otimes}\) can be factorized through the ideal \(S^\vee\). In such a way we obtain a graded-antimultiplicative map \(\kappa^\vee : \Gamma^\vee \rightarrow \Gamma^\vee\) satisfying (A.2).

Let us now consider a \(\mathcal{A} \otimes \mathcal{A}\)-module \(\Psi\) given by

\[\Psi = (\mathcal{A} \otimes \Gamma) \oplus (\Gamma \otimes \mathcal{A})\]

It is easy to see that \(\Psi\) is a bicovariant bimodule over the group \(G \times G\). In particular, the corresponding right and left actions of \(G \times G\) on \(\Psi\) are given by

\[\varphi_\Psi((a \otimes \vartheta) \oplus (\eta \otimes b)) = (\phi(a)\varphi_T(\vartheta)) \oplus (\varphi_T(\eta)\phi(b))\]

\[\ell_\Psi((a \otimes \vartheta) \oplus (\eta \otimes b)) = (\phi(a)\ell_T(\vartheta)) \oplus (\ell_T(\eta)\phi(b)),\]

where on the right-hand side the tensor multiplication is assumed. The following natural isomorphism holds

\[\Psi_{\text{inv}} \cong \Gamma_{\text{inv}} \oplus \Gamma_{\text{inv}}\]

Further, \(\Psi\) is a first-order calculus over \(G \times G\), in a natural manner. The corresponding differential map \(D : \mathcal{A} \otimes \mathcal{A} \rightarrow \Psi\) is given by \(D = d \otimes \text{id} + \text{id} \otimes d\). In terms of the above identification the corresponding right \(\mathcal{A} \otimes \mathcal{A}\)-module structure on \(\Psi_{\text{inv}}\) is given by

\[(A.10) \quad (\vartheta \oplus \eta) \circ (a \otimes b) = \epsilon(a)(\vartheta \circ b) \oplus (\eta \circ a)\epsilon(b)\]
and the action of the corresponding flip-over operator \( \Sigma: \Psi^\otimes_2 \to \Psi^\otimes_2 \) is determined by the block-matrix

\[
\Sigma = \begin{pmatrix}
\sigma & 0 & 0 & 0 \\
0 & 0 & \tau & 0 \\
0 & \tau & 0 & 0 \\
0 & 0 & 0 & \sigma
\end{pmatrix}
\]

(A.11)

where \( \tau: \Gamma^\otimes_2 \to \Gamma^\otimes_2 \) is the standard transposition. This implies

\[
\Psi^\sim \cong \Gamma^\sim \otimes \Gamma^\sim.
\]

Let \( \sharp: \Gamma \to \Psi \) be a map given by

\[
\sharp = \ell + \wp.
\]

The following identities hold

\[
D\phi = \sharp d
\]

(A.12)

\[
\sharp (da) = \sharp (\vartheta) \phi(a)
\]

(A.13)

\[
\sharp (a \vartheta) = \phi(a) \sharp (\vartheta).
\]

(A.14)

Equalities (A.13)–(A.14) imply that \( \phi \) and \( \sharp \) can be consistently extended to a homomorphism \( \phi^\otimes: \Gamma^\otimes \to \Psi^\otimes \). The following inclusion holds

\[
\phi^\otimes (\ker(A)) \subseteq \ker(A^\Sigma),
\]

where \( A^\Sigma \) is the antisymmetrizer corresponding to \( \Sigma \). Hence \( \phi^\otimes \) can be projected through ideals \( \ker(A) \) and \( \ker(A^\Sigma) \). In such a way we obtain the homomorphism \( \phi^\sim: \Gamma^\sim \to \Gamma^\sim \otimes \Gamma^\sim \). Equality (A.12) implies that \( \phi^\sim \) intertwines the corresponding differentials.

Properties (A.3)–(A.5) as well as (A.6)–(A.7) simply follow from analogous properties for \( \phi \) and \( \kappa \). It is worth noticing that \( \kappa^\sim \) and \( \phi^\sim \) can be obtained by projecting \( \kappa^\wedge \) and \( \phi^\wedge \) from \( \Gamma^\wedge \) to \( \Gamma^\sim \).

Let us now turn to the conceptual framework of the previous paper, and assume that \( M \) is a classical compact smooth manifold, and \( P \) a principal \( G \)-bundle over \( M \). Further, let us assume that \( \Gamma \) is the minimal admissible left-covariant calculus over \( G \) (this calculus is bicovariant and \( * \)-covariant, too). Let \( \tau = (\pi_U)_{U \in \mathcal{U}} \) be an arbitrary trivialization system for \( P \), and let \( \mathcal{C}_\tau \) be the corresponding \( G \)-cocycle, consisting of “transition functions” \( \psi_{UV}: S(U \cap V) \otimes A \to S(U \cap V) \otimes A \). The restrictions \( \varphi_{UV} = \psi_{UV} | \mathcal{A} \) are \( * \)-homomorphisms explicitly given by

\[
\varphi_{UV} = (g_{UV} \otimes \text{id}) \phi
\]

where \( g_{UV}: U \cap V \to G_{cl} \) represent the classical \( G_{cl} \)-cocycle corresponding to \( \mathcal{C}_\tau \) (understood here as \( * \)-homomorphisms \( g_{UV}: \mathcal{A} \to S(U \cap V) \)).

There exists the unique map \( \sharp_{UV}: \Gamma \to \left[ \Omega^1(U \cap V) \otimes \mathcal{A} \right] \otimes \left[ S(U \cap V) \otimes \Gamma \right] \) satisfying

\[
\sharp_{UV}(a \xi) = \varphi_{UV}(a) \sharp_{UV}(\xi)
\]

\[
\sharp_{UV}(da) = [d \otimes \text{id} + \text{id} \otimes d] \varphi_{UV}(a),
\]
for each $a \in A$ and $\xi \in \Gamma$. This implies also

$$\sharp_{UV}(\xi a) = \sharp_{UV}(\xi) \varphi_{UV}(a)$$

and hence there exists (the unique) homomorphism $\varphi_{UV}^\oplus: \Gamma^\oplus \to \Omega(U \cap V) \hat{\otimes} \Gamma^\oplus$ extending both $\varphi_{UV}$ and $\sharp_{UV}$.

All antisymetrizing operators are left and right covariant in a natural manner. In particular they are reduced in the corresponding spaces of left-invariant elements. In what follows we shall denote by the same symbols their restrictions in $\Gamma_{\text{inv}}$ (if there is no ambiguity from the context).

**Proposition A.2.** We have

(A.15) $\varphi_{UV}^\oplus[S^\vee] \subseteq \Omega(U \cap V) \hat{\otimes} S^\vee$

for each $(U,V) \in N^2(\mathcal{U})$.

**Proof.** The ideal $S^\vee$ is bicovariant. In particular, it has the form

$$S^\vee \cong A \otimes S^\vee_{\text{inv}}$$

where $S^\vee_{\text{inv}}$ is the left-invariant part of $S^\vee$. The following equality holds

(A.16) $\sharp_{UV}(\vartheta) = 1 \otimes \vartheta + \sum_k \partial^{UV}(\vartheta_k) \otimes c_k$

for each $\vartheta \in \Gamma_{\text{inv}}$. Here, $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$ and $\partial^{UV}: \Gamma_{\text{inv}} \to \Omega(U \cap V)$ is the map specified by

(A.17) $\partial^{UV} \pi(a) = g_{UV}(a^{(1)})d(g_{UV}(a^{(2)}))$.

Let us observe that

(A.18) $\partial^{UV}(\eta)\partial^{UV}(\zeta) = -\sum_k \partial^{UV}(\zeta_k)\partial^{UV}(\eta_k)$

for each $\eta, \zeta \in \Gamma_{\text{inv}}$, where $\sum_k \zeta_k \otimes \eta_k = \sigma(\eta \otimes \zeta)$. Indeed this follows from (4.16), and from the identity

(A.19) $\partial^{UV}(\vartheta \circ a) = \epsilon(a)\partial^{UV}(\vartheta)$.

Now for an arbitrary $\vartheta \in \Gamma_{\text{inv}}^\oplus$ let us consider the elements $\vartheta_k = (\id \otimes p_k)\varphi_{UV}^\oplus(\vartheta)$ where $0 \leq k \leq n$. It follows from (4.16) and (A.16) that these elements have the form

(A.20) $\vartheta_k = (\nabla^{UV}_l \otimes \id^k)A_{lk}(\vartheta)$

with $l = n - k$. Here we have identified $\Gamma^\oplus$ and $A \otimes \Gamma_{\text{inv}}^\oplus$, and $\nabla^{UV}_l$ are components of the unital multiplicative extension of the map $\nabla^{UV} = (\partial^{UV} \otimes \id)\varpi$. According to (A.18) the above expression can be rewritten as

$$\vartheta_k = \frac{1}{l!}(\nabla^{UV}_l A_l \otimes \id^k)A_{lk}(\vartheta).$$
In particular
\[(\text{id} \otimes A_k)(\vartheta_k) = \frac{1}{\Lambda} (\nabla^U \otimes \text{id}^k) A_n(\vartheta),\]
according to (A.1). Hence, (A.15) holds. \(\square\)

The map \(\varphi_{UV}^\otimes\) can be factorized through the ideal \(S^\vee\). In such a way we obtain a homomorphism \(\varphi_{UV}^\vee : \Gamma^\vee \to \Omega(U \cap V) \otimes \Gamma^\vee\) of graded-differential *-algebras. Now the formula
\[(A.21) \quad \psi_{UV}^\vee (\alpha \otimes \vartheta) = \alpha \varphi_{UV}^\vee (\vartheta)\]
defines a graded-differential *-automorphism of \(\Omega(U \cap V) \otimes \Gamma^\vee\) ( extending both \(\psi_{UV}\) and \(\varphi_{UV}^\vee\)). These maps are \(\Omega(U \cup V)\)-linear and satisfy the following cocycle conditions
\[(A.22) \quad \psi_{UV}^\vee \psi_{VW}^\vee (\varphi) = \psi_{UW}^\vee (\varphi)\]
for each \((U, V, W) \in N^3(U)\) and \(\varphi \in \Omega(U \cap V \cap W) \otimes \Gamma^\vee\).

Applying the above results, and using similar constructions as in the previous paper, it is possible to construct a graded-differential *-algebra \(\Lambda(P)\), representing the corresponding “differential forms” on the bundle. This algebra is locally trivial, in the sense that any local trivialization \((U, \pi_U)\) of the bundle can be “extended” to a local representation of the form \(\Omega(U) \otimes \Gamma^\vee \leftrightarrow \Lambda(P|_U)\).

By construction, the right action \(F\) can be (uniquely) extended to the homomorphism \(F^\vee : \Lambda(P) \to \Lambda(P) \otimes \Gamma^\vee\) of graded-differential *-algebras. Moreover, all entities naturally appearing in the differential calculus on \(P\) constructed from the universal envelope \(\Gamma^\wedge\) have counterparts in the calculus based on \(\Lambda(P)\), and all algebraic relations are preserved (because the formalism can be obtained by “projecting” the first calculus on \(\Lambda(P)\)).

In a certain sense, \(\Gamma^\vee\) is the minimal bicovariant graded algebra (built over \(\Gamma\)) compatible with all possible “transition functions” for the bundle \(P\). Namely, let us assume that \(\mathcal{N} \subseteq \Gamma^\otimes\) is a bicovariant graded-ideal satisfying \(\mathcal{N}^k = \{0\}\) for \(k \in \{0, 1\}\) and
\[(A.23) \quad \varphi_{UV}^\otimes (\mathcal{N}) \subseteq \Omega(U \cap V) \otimes \mathcal{N}\]
for each trivialization system \(\tau = (\pi_U)_{U \in \mathcal{U}}\) and each \((U, V) \in N^2(U)\). This ensures the possibility to construct the corresponding global algebra for the bundle \(P\), which locally will be of the form \(\Omega(U) \otimes \left[\Gamma^\otimes / \mathcal{N}\right]\).

**Lemma A.3.** Under the above assumptions we have
\[(A.24) \quad \mathcal{N} \subseteq S^\vee.\]

**Proof.** We shall prove inductively that
\[(A.25) \quad \mathcal{N}_{inv}^k \subseteq S_{inv}^k,\]
for each \(k \geq 2\). Let us assume that \(\sum \vartheta_i \otimes \eta_i = \psi \in \mathcal{N}_{inv}^2\). Applying (A.23) and (A.16) we obtain
\[0 = \sum_{ikl} \partial^{UV} (\vartheta_{ik}) \partial^{UV} (\eta_{il}) \otimes c_{ik} d_{il} + \sum_{ik} \partial^{UV} (\vartheta_{ik}) \otimes c_{ik} \eta_i - \sum_{il} \partial^{UV} (\eta_{il}) \otimes \vartheta_i d_{il}\]
where \( \varpi(\vartheta_i) = \sum_k \vartheta_{ik} \otimes c_{ik} \) and \( \varpi(\eta_i) = \sum_l \eta_{il} \otimes d_{il} \). In particular

\[
0 = \sum_k \vartheta^{UV}(\vartheta_{ik}) \otimes c_{ik} \eta_i - \sum_l \vartheta^{UV}(\eta_{il}) \otimes \vartheta_i d_{il}.
\]

In other words, modulo the identification \( \Gamma \leftrightarrow A \otimes \Gamma^\text{inv} \), we have

\[
(\nabla^{UV} \otimes \text{id})(I - \sigma)(\psi) = 0.
\]

Having in mind that the family of maps \( \nabla^{UV} \) distinguishes elements of \( \Gamma^\text{inv} \) (a consequence of the minimality of \( \Gamma \)) we conclude that \( \psi = \sigma(\psi) \). In other words

\[
N_{k+1}^\text{inv} \subseteq \ker(I - \sigma) = S_{k+1}^\text{inv}.
\]

Let us assume that \( (A.25) \) holds for some \( k \geq 2 \). Then

\[
0 = (\text{id} \otimes A_k p_k) \hat{F} \omega \otimes \varpi((A.26)) = (\nabla^{UV} \otimes A_k) A_{1k}(\psi)
\]

for each \( \psi \in N_{k+1}^\text{inv} \). Because of the arbitrariness of \( \tau \) it follows that

\[
A_{k+1}(\psi) = (\text{id} \otimes A_k) A_{1k}(\psi) = 0,
\]

in other words \( N_{k+1}^\text{inv} \subseteq (S_{k+1}^\text{inv}) \).

\[
\square
\]

We pass to the study of the problem of passing from \( \Gamma^\wedge \) to \( \Gamma^\vee \), in the framework of the general theory.

Let \( P = (B, i, F) \) be a quantum principal \( G \)-bundle over a quantum space \( M \).

**Lemma A.4.** Modulo the natural identification \( \Gamma^\wedge \leftrightarrow A \otimes \Gamma^\wedge_{\text{inv}} \) we have

\[
(A.26) \quad (\text{id} \otimes p_k) \hat{F} \omega \otimes \varpi = (F \wedge \omega \otimes [\ ]_k^\wedge) A_{1k}
\]

where \( n = k + l \), and \( \omega \) is an arbitrary connection on \( P \).

**Proof.** Essentially the same reasoning as in the proof of Proposition A.2, applying identity (4.8) instead of (A.16).

The above equation implies

\[
(A.27) \quad \hat{F} \omega \otimes (S_{\text{inv}}^\vee) \subseteq \omega \otimes (S_{\text{inv}}^\vee) \otimes \Gamma^\wedge + \Omega(P) \otimes [S^\vee]^\wedge;
\]

for each \( \omega \in \text{con}(P) \). Let us assume that the calculus (described by \( \Omega(P) \)) admits regular connections, and that \( \mathcal{T}(P) = \{0\} \) (multiplicativity of regular connections).

Let \( \Upsilon(P) \subseteq \Omega(P) \) be the space linearly generated by elements of the form \( \varphi \omega \otimes (\vartheta) \) where \( \vartheta \in S_{\text{inv}}^\vee \) and \( \varphi \in \text{hor}(P) \), while \( \omega \in \mathcal{V}(P) \).

**Lemma A.5.** (i) The space \( \Upsilon(P) \) is a (two-sided) graded-differential *-ideal in \( \Omega(P) \), independent of the choice of a regular connection \( \omega \).

(ii) We have

\[
(A.28) \quad \hat{F}(\Upsilon(P)) \subseteq \Upsilon(P) \otimes \Gamma^\wedge + \Omega(P) \otimes [S^\vee]^\wedge.
\]
Proof. The space \( m_\omega^{-1}(\Upsilon(P)) = \sHom(P) \otimes [S^\omega_{inv}]^\wedge \) is a graded two-sided \(^*\)-ideal in \( \sV(P) \). Hence the space \( \Upsilon(P) \) is the graded \(^*\)-ideal in \( \Omega(P) \). Inclusion (A.28) directly follows from (A.27).

Let us prove that \( d(\Upsilon(P)) \subseteq \Upsilon(P) \). It is sufficient to check that \( d\omega^\otimes(\vartheta) \in \Upsilon(P) \) for each \( \vartheta \in S^\omega_{inv} \). The following equality holds

\[
(A.29) \quad d\omega^\otimes(\vartheta) = \omega^\otimes \delta^*(\vartheta) + m_\Omega \left( R_\omega \otimes \omega^\otimes \right) A_{1k-1}(\vartheta)
\]

for each \( k \geq 2 \) and \( \vartheta \in \Gamma^\otimes_{inv} \). Here, \( \delta^*: \Gamma^\otimes_{inv} \to \Gamma^\otimes_{inv} \) is the unique (hermitian) antiderivation extending a given embedded differential map \( \delta \). In particular, if \( \vartheta \in S^\otimes_{inv} \) then both summands on the right-hand side of the above equality belong to \( \Upsilon(P) \), because of \( \delta^*(S^\otimes_{inv}) \subseteq (S^\otimes_{inv})^{k+1} \) and \( A_{1k-1}(S^\otimes_{inv}) \subseteq \Gamma_{inv} \otimes (S^\otimes_{inv})^{k-1} \).

Finally, let \( \zeta \) be an arbitrary (hermitian) tensorial 1-form satisfying (4.13). We have then

\[
(\omega + \zeta)^\otimes(\vartheta) = \sum_{k+l=n} \frac{1}{k!l!} m_\Omega(\zeta^\otimes k A_k \otimes \omega^\otimes l) A_{kl}(\vartheta)
\]

for each \( \vartheta \in \Gamma^\otimes_{inv} \). This shows that \( \Upsilon(P) \) is \( \omega \)-independent. \( \square \)

Hence, it is possible to pass jointly to the factoralgebra \( \Lambda(P) = \Omega(P)/S^\otimes \) (as a representative of the calculus on the bundle), and to the exterior bicovariant algebra \( \Gamma^\otimes \) (representing the calculus on \( \mathcal{C} \)). This factorization does not change the first-order differential structure. For this reason the spaces of connection forms associated to both calculi on \( P \) are the same. Further, the spaces \( \sHom(P) \) and \( \Omega_1(P) \) are preserved. This implies that regular connections relative to \( \Omega(P) \) and \( \Lambda(P) \) are the same. By construction, regular connections are multiplicative, relative to \( \Lambda(P) \), too.

Lemma A.3 establishing the “minimality” of the exterior algebra has a general quantum counterpart.

Let us assume that the calculus on the bundle is such that tensorial 1-forms \( \zeta \) satisfying (4.13) distinguish elements of \( \Gamma^\otimes_{inv} \). In the case of bundles over classical compact manifolds the algebra \( \Omega(P) \) built from the minimal admissible calculus \( \Gamma \) possesses this property.

Let us consider a bicovariant graded-ideal \( \mathcal{N} \subseteq \Gamma^\otimes \) satisfying \( \mathcal{N}^k = \{0\} \) for \( k \in \{0,1\} \). The space \( \mathcal{N}_\omega = m_\omega(\sHom(P) \otimes [\mathcal{N}_{inv}]^\wedge) \) is a two-sided ideal in \( \Omega(P) \). Let us assume that \( \mathcal{N}_\omega \) is independent of \( \omega \in \tau(P) \).

**Lemma A.6.** Under the above assumptions \( \mathcal{N} \subseteq S^\otimes \). \( \square \)

The construction of “global” differential calculus on the bundle given in Subsection 6 can be applied to the exterior algebra case, too.

Starting from algebras \( \sHom \), and \( \Gamma^\otimes_{inv} \), it is possible to construct a graded \(^*\)-algebra structure on \( \Lambda(P) = \sHom \otimes \Gamma^\otimes_{inv} \). Using maps \( R, D \), and \( d: \Gamma^\otimes_{inv} \to \Gamma^\otimes_{inv} \) it is possible to construct a natural differential \( d^\otimes \) on \( \Lambda(P) \). All constructions are the same as in the universal envelope case. The only nontrivial point is to prove the analog of Lemma 6.18.

Let \( R^*: \Gamma^\otimes_{inv} \to \Lambda(P) \) be the unique antiderivation extending \( R \). The following equality holds

\[
(A.30) \quad R^*(\eta) = (R \otimes [\cdot]^\wedge n) A_{1n}(\eta)
\]
for each $\eta \in (\Gamma_{\text{inv}} \otimes \inv)^{n+1}$. This implies that $R^*$ can be factorized through $S^\vee_{\text{inv}}$. In such a way we obtain the map $R^\vee: \Gamma_{\text{inv}} \rightarrow \Lambda(P)$. Finally, the map $d^\vee: \Gamma_{\text{inv}} \rightarrow \Lambda(P)$ is defined by

$$d^\vee(\emptyset) = d(\emptyset) + R^\vee(\emptyset).$$

**APPENDIX B. MULTIPLE IRREDUCIBLE SUBMODULES**

In this appendix the structure of the $\ast$-algebra $\mathcal{B}$ of functions on a quantum principal bundle $\mathcal{P} = (\mathcal{B}, i, F)$ will be analyzed from the viewpoint of the representation theory [5] of the structure quantum group.

Let $\mathcal{T}$ be the set of (equivalence classes of) irreducible unitary representations of $G$.

The representation $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ of $G$ in $\mathcal{B}$ is highly reducible. For each $\alpha \in \mathcal{T}$ let $\mathcal{B}^\alpha \subseteq \mathcal{B}$ be the multiple irreducible subspace corresponding to $\alpha$. Evidently, $\mathcal{B}^\alpha$ is a bimodule over $\mathcal{V}$. We have

$$\mathcal{B} = \sum_{\alpha \in \mathcal{T}} \mathcal{B}^\alpha.$$

For an arbitrary $\alpha \in \mathcal{T}$, let $u^\alpha: \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathcal{A}$ be a representative of this class (where $n$ is the dimension of $\alpha$). Let $(e_1, \ldots, e_n)$ be the absolute basis in $\mathbb{C}^n$. We have

$$u^\alpha(e_j) = \sum_{j=1}^{n} e_j \otimes u^\alpha_{ji},$$

where $u^\alpha_{ij} \in \mathcal{A}$ are the corresponding matrix elements.

Let us consider the space $\text{Mor}(u^\alpha, F)$ consisting of intertwiners (morphisms) $\varphi: \mathbb{C}^n \rightarrow \mathcal{B}$ between representations $u^\alpha$ and $F$. The space $\text{Mor}(u^\alpha, F)$ is a $\mathcal{V}$-bimodule, in a natural manner. The maps $\varphi$ take values from $\mathcal{B}^\alpha$.

Bimodules $\text{Mor}(u^\alpha, F) \otimes \mathbb{C}^n$ and $\mathcal{B}^\alpha$ are naturally isomorphic, via the correspondence

$$\varphi \otimes x \leftrightarrow \varphi(x).$$

This identification intertwines $F|_{\mathcal{B}^\alpha}$ and $\text{id} \otimes u^\alpha$. Irreducible $G$-multiplets in $\mathcal{B}^\alpha$ are of the form $\{\varphi(e_1), \ldots, \varphi(e_n)\}$, for some intertwiner $\varphi$.

Let us fix $i, j \in \{1, \ldots, n\}$. There exist intertwiners $\mu_1, \ldots, \mu_d \in \text{Mor}(u^\alpha, F)$ and numbers $c_{klpq} \in \mathbb{C}$ (where $k, l \in \{1, \ldots, d\}$ and $p, q \in \{1, \ldots, n\}$) such that

$$\sum_{klpq} c_{klpq} b^\alpha_{kp} F(b^\alpha_{lq}) = 1 \otimes u^\alpha_{ij},$$

where $b^\alpha_{kp} = \mu_k(e_p)$. Equivalently

$$\sum_{klpq} c_{klpq} F(b^\alpha_{kp}) b^\alpha_{lq} = 1 \otimes u^\alpha_{ij}.$$
The above equalities imply that the summation over indexes satisfying \((p,q) \neq (i,j)\) can be dropped. In other words,

\[
\sum_{kl} c_{kl} b_{ki}^\alpha F(b_{kj}^\alpha) = 1 \otimes u_{ij}^\alpha
\]

(B.4)

\[
\sum_{kl} c_{kl} F(b_{ki}^\alpha)^* b_{ij}^\alpha = 1 \otimes u_{ji}^{\alpha*}
\]

(B.5)

where \(c_{kl} = c_{klij}\). From the fact that \(\{b_{ki}^\alpha, \ldots, b_{kn}^\alpha\}\) form a \(G\)-multiplet it follows that the above equalities hold for arbitrary \(i,j \in \{1, \ldots, n\}\).

Equalities (B.4)–(B.5) imply that the numbers \(c_{kl}\) can be always chosen such that the matrix \((c_{kl})\) is hermitian.

Further, without a lack of generality we can assume that the matrix \((c_{kl})\) is nonsingular. In what follows, it will be assumed that the matrix \((c_{kl})\) is positive.

Diagonalizing this matrix, and redefining in the appropriate way intertwiners \(\mu_k\) we obtain

\[
\sum_{k} b_{ki}^\alpha F(b_{kj}^\alpha) = 1 \otimes u_{ij}^\alpha
\]

(B.6)

\[
\sum_{k} F(b_{ki}^\alpha)^* b_{kj}^\alpha = 1 \otimes u_{ji}^{\alpha*}
\]

(B.7)

Equivalently, the following identities hold

\[
\sum_{k} b_{ki}^\alpha b_{kj}^\alpha = \delta_{ij}.
\]

(B.8)

In particular, all irreducible representations appear in the decomposition of \(F\) into irreducible components (all \(\mathcal{V}\)-bimodules \(B_{\alpha}\) are non-trivial).

Let us consider, for each \(f \in \mathcal{V}\), the elements

\[
\varrho_{kl}(f) = \sum_{i} b_{ki}^\alpha f b_{li}^{\alpha*}.
\]

(B.9)

These elements are \(F\)-invariant and the following identities hold

\[
\varrho_{kl}(f)^* = \varrho_{lk}(f^*)
\]

(B.10)

\[
\sum_{n} \varrho_{kn}(f) \varrho_{nl}(g) = \varrho_{kl}(fg).
\]

(B.11)

Indeed, we have \(F(b_{ki}^\alpha) = \sum_{j} b_{kj}^\alpha \otimes u_{ji}^\alpha\), and a direct computation gives

\[
F(\varrho_{kl}(f)) = \sum_{i} F(b_{ki}^\alpha f b_{li}^{\alpha*}) = \sum_{imn} b_{km}^\alpha f b_{ln}^{\alpha*} \otimes u_{mi}^\alpha u_{ni}^{\alpha*}
\]

\[
= \sum_{mn} b_{km}^\alpha f b_{ln}^{\alpha*} \otimes \delta_{mn} = \varrho_{kl}(f) \otimes 1,
\]

and similarly

\[
\sum_{n} \varrho_{kn}(f) \varrho_{nl}(g) = \sum_{nj} b_{ki}^\alpha f b_{nj}^{\alpha*} b_{lj}^{\alpha*} = \sum_{ij} b_{ki}^\alpha f g b_{lj}^{\alpha*} \delta_{ij} = \sum_{i} b_{ki}^\alpha f g b_{li}^{\alpha*} = \varrho_{kl}(fg).
\]
In other words, the maps $φ_{kl} : V → V$ realize a $*$-homomorphism $φ : V → M_d(V)$, where $M_d(V)$ is the $*$-algebra of $d × d$-matrices over $V$. In particular, $q_* = φ(1)$ is a projector in $M_d(V)$.

Let us consider the free left $V$-module $V^d$, with the absolute basis $(ε_1, \ldots, ε_d)$. The elements of the algebra $M_d(V)$ are understandable as endomorphisms (acting on the right) of $V^d$, in a natural manner. Explicitly, this realization is given by

(B.12) \[(ε_k)A = \sum_l A_{kl}ε_l.\]

Let $E ⊆ V^d$ be the left $V$-submodule determined by the projector $q_*$. Evidently, $E$ is $φ$-invariant and $φ$, together with the left multiplication, determine a (unital) $V$-bimodule structure on $E$.

Let $♯_* : V^d → \text{Mor}(u^α, F)$ be the left $V$-module homomorphism given by

(B.13) \[♯_* (ε_k) = µ_k.\]

The following identity holds

(B.14) \[♯_* (ψφ(f)) = ♯_* (ψ)f,\]

in particular $♯_* (ψq_*) = ♯_* (ψ)$.

This implies that the restriction $(♯_* | E) : E → \text{Mor}(u^α, F)$ is a homomorphism of unital $V$-bimodules.

Now we shall prove that $♯_* | E$ is bijective. Let us assume that $ψ ∈ \ker(♯_*).$ This implies

\[\sum_{kl} ψ_k b^α_{ki}b^α_{li}^* = 0,\]

for each $l \in \{1, \ldots, d\}$ where $ψ = \sum_k ψ_k ε_k$. In other words, $ψq_* = 0$, which means that $♯_* | E$ is injective. We have

\[µ = \sum_k q_k µ_k,\]

for each $µ ∈ \text{Mor}(u^α, F)$, where $q_k ∈ V$ are elements given by

\[q_k = \sum_i µ(e_i)b^α_{ki}.\]

In other words, elements $µ_k$ span the left $V$-module $\text{Mor}(u^α, F)$. This implies that $♯_*$ is surjective.

Hence, $\text{Mor}(u^α, F)$ are finite and projective, as left $V$-modules. This implies that left $V$-modules $B^α$ are finite and projective.

Relations (B.8) can be rewritten in the form

(B.15) \[B^\dagger B = I_n,\]

where $B$ is a $d × n$ matrix with coefficients $b^α_{ki}$, and $I_n$ is the unit matrix in $M_n(B)$.

Let us assume that the following additional relations hold

(B.16) \[(ZBC^{-1})^\dagger B^* = I_n,\]
where $C \in M_n(C)$ is the canonical intertwiner [5] between $u^\alpha$ and its second contragradient $u^\alpha_{cc}$, and $Z \in M_d(C)$ is a strictly positive matrix.

Relations of this type naturally appear in a $C^*$-algebraic version of the theory of quantum principal bundles. The matrix $Z$ is connected with modular properties of an appropriate invariant integral on the bundle.

Let us consider a map $\lambda: V \to M_d(V)$ given by

$$\lambda_{kl}(f) = \sum_i b^\alpha_{ki} f[ZBC^{-1}]_li.$$  

We have then

$$\lambda(f)^\dagger = Z^* \lambda(f^*)(Z^*)^{-1}$$  

$$\lambda(f)\lambda(g) = \lambda(fg).$$  

Further, the elements of $M_d(V)$ are naturally identifiable with endomorphisms of the right $V$-module $V$ (acting on the left). Let $\mathcal{F} \subseteq V^d$ be a right $\mathcal{V}$-submodule determined by a projection $q^\alpha = \lambda(1)$. The map $\lambda$ induces a unital left $\mathcal{V}$-module structure on $\mathcal{F}$, so that $\mathcal{F}$ becomes a $\mathcal{V}$-bimodule. Let $\sharp^\alpha: \mathcal{V}^d \to \text{Mor}(u^\alpha, F)$ be a right $\mathcal{V}$-module homomorphism given by $\sharp^\alpha(\varepsilon_k) = \nu_k$, where $\nu_k = \sum_i Z_{ki}\mu_i$. Then the restriction $(\sharp^\alpha|\mathcal{F}): \mathcal{F} \to \text{Mor}(u^\alpha, F)$ is a bimodule isomorphism.

A similar consideration can be applied to all covariant algebras figuring in the game. In particular

$$\mathfrak{hor}(P) = \bigoplus_{\alpha \in \mathcal{T}} \mathcal{H}^\alpha$$

where $\mathcal{H}^\alpha$ are corresponding $\alpha$-multiple irreducible subspaces (relative to the decompositions of $F^\wedge$). These spaces are $\Omega(M)$-bimodules. The following natural decompositions hold

$$\mathcal{H}^\alpha = \text{Mor}(u^\alpha, F^\wedge) \otimes \mathbb{C}^n.$$  

Every $\varphi \in \mathfrak{hor}(P)$ can be written in the form

$$\varphi = \sum_k w_k b_k$$  

where $w_k \in \Omega(M)$ and $b_k \in B$. Indeed, it is sufficient check the statement for elements of some irreducible multiplet.

Let us assume that $\{\varphi_1, \ldots, \varphi_n\} \in \mathcal{H}^\alpha$ is an irreducible $\alpha$-multiplet. We have then

$$\varphi_i = \sum_{jk} \varphi_{jk} b^\alpha_{kj} b^\alpha_{ki}$$

for each $i \in \{1, \ldots, n\}$. On the other hand the elements $\sum_{jk} \varphi_{jk} b^\alpha_{kj}$ belong to $\Omega(M)$.

In particular, if $\mathfrak{hor}^+(P)$ is generated by $\mathfrak{hor}(P)$ and if every first-order horizontal form $\varphi$ can be written as $\varphi = \sum_k b_k d(g_k)$, where $b_k \in B$ and $g_k \in \mathcal{V}$, then the spaces $\mathcal{H}^\alpha$ are linearly spanned by elements of the form $h^\alpha = b^\alpha d(f_1) \ldots d(f_n)$, where $f_k \in \mathcal{V}$ and $b^\alpha \in B^\alpha$. 
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