Near-optimal Reinforcement Learning in Factored MDPs: Oracle-Efficient Algorithms for the Non-episodic Setting

Ziping Xu \(^1\) Ambuj Tewari \(^1\)

**Abstract**

We study reinforcement learning in factored Markov decision processes (FMDPs) in the non-episodic setting. We focus on regret analyses providing both upper and lower bounds. We propose two near-optimal and oracle-efficient algorithms for FMDPs. Assuming oracle access to an FMDP planner, they enjoy a Bayesian and a frequentist regret bound respectively, both of which reduce to the near-optimal bound \(\tilde{O}(DS\sqrt{AT})\) for standard non-factored MDPs. Our lower bound depends on the span of the bias vector rather than the diameter \(D\) and we show via a simple Cartesian product construction that FMDPs with a bounded span can have an arbitrarily large diameter, which suggests that bounds with a dependence on diameter can be extremely loose. We, therefore, propose another algorithm that only depends on span but relies on a computationally stronger oracle. Our algorithms outperform the previous near-optimal algorithms on computer network administrator simulations.

1. Introduction

Designing exploration strategies that are both computationally and statistically efficient is an important problem in Reinforcement Learning (RL). A common performance measure to theoretically evaluate exploration strategies is regret, the difference between the highest achievable cumulative reward and an agent’s actual rewards. There is a rich line of work that focuses on regret analysis in tabular MDPs where state and action space are finite and small (Jaksch et al., 2010; Osband et al., 2013; Dann & Brunskill, 2015; Kearns & Singh, 2002). In tabular setting, the lower bound on regret is \(\sqrt{DSAT}\), with \(D, S, A, T\) as the diameter, state size, action size and total steps, respectively. This lower bound has been achieved by Tossou et al. (2019) in tabular setting.

A current challenge in RL is dealing with large state and action spaces where even polynomial dependence of regret on state and action space size is unacceptable. One idea to meet this challenge is to consider MDP with structure that allows us to represent them with a smaller number of parameters. One such example is a factored MDP (FMDP) (Boutilier et al., 2000) whose transitions can be represented by a compact Dynamic Bayesian network (DBN) (Ghahramani, 1997).

There is no FMDP planner that is both computationally efficient and accurate. Guestrin et al. (2003) proposed approximate algorithms with pre-specified basis functions. They use approximate linear programming and approximate policy iteration with max-norm projections, where the max-norm errors of the approximated value functions can be upper bounded by the projection error. For the even harder online learning setting, we study oracle-efficient algorithms, which can learn an unknown FMDP efficiently by assuming an efficient planning oracle. In this, we follow prior work in online learning where the goal is to design efficient online algorithms that only make a polynomial number of calls to an oracle that solves the associated offline problem. For example, oracle-based efficient algorithms have been proposed for the contextual bandit problem (Syrgkanis et al., 2016; Luo et al., 2017).

The only available near-optimal regret analysis of FMDPs is by Osband & Van Roy (2014). They consider the episodic setting and part of our motivation is to extend their work to the more challenging non-episodic setting. They proposed two algorithms, PSRL (Posterior Sampling RL) and UCRL-factored. The two algorithms enjoy near-optimal Bayesian and frequentist regret bounds, respectively. However, their UCRL-factored algorithm relies on solving a Bounded FMDP (Givan et al., 2000) with no computationally efficient solution for the factored case yet. Other computationally efficient algorithms either have some high order terms in their analysis (Strehl, 2007) or depend on some strong connectivity assumptions, e.g. mixing time (Kearns & Koller, 1999).

This paper makes three main contributions which are sum-
We solve non-episodic MDP by splitting the $T$ whenever the number of visits of some state-action pair is doubled. The doubling trick leads to a random splitting and epochs and change policies at the start of each epoch. $K$ is the number of epochs and $A$ is the number of actions. We provide a posterior sampling algorithm (PSRL) with a near-optimal Bayesian regret bound for non-episodic MDPs. The PSRL has better empirical performance than DORL and the previous sample-efficient algorithms. We define the regret of a reinforcement learning algorithm that our setting only polynomial number of times. The upper bound, when specialized to the standard non-factored MDP setting, matches that of UCRL2 (Jaksch et al., 2010).

2. Preliminaries

2.1. Non-episodic MDP

We consider the non-episodic and undiscounted Markov decision process (MDP), represented by $M = \{S, A, P, R\}$, with the finite state space $S$, the finite action space $A$, the transition probability $P \in P_{S \times A \times S}$ and reward distribution $R : P_{S \times A \times [0,1]}$. Here $\Delta(X)$ denotes a distribution over the space $X$ and $\mathbb{P}_{X_1 \to X_2}$ is the class of all the mappings from space $X_1$ to $\Delta(X_2)$. Let $S := \{S\}$ and $A := \{A\}$.

An MDP $M$ and an algorithm $\mathcal{L}$ operating on $M$ with an initial state $s_1 \in S$ constitute a stochastic process described by the states $s_t$ visited at time step $t$, the actions $a_t$ chosen by $\mathcal{L}$ at step $t$, the rewards $r_t \sim R(s_t, a_t)$ and the next state $s_{t+1} \sim P(s_t, a_t)$ obtained for $t = 1, \ldots, T$. Let $H_T = \{s_1, a_1, r_1, \ldots, s_{t-1}, a_{t-1}, r_{t-1}\}$ be the trajectory up to time $t$.

To learn an non-episodic and undiscounted MDP with sub-linear regret, we need some connectivity constraint. There are several subclasses of MDPs corresponding to different types of connectivity constraints (e.g., see the discussion in Bartlett & Tewari (2009)). This paper focuses on the class of communicating MDPs, i.e., the diameter of the MDP, which is defined below, is upper bounded by some $D < \infty$.

**Definition 2.1 (Diameter).** Consider the stochastic process defined by a stationary policy $\pi : S \to A$ operating on an MDP $M$ with initial state $s$. Let $T(s' | M, \pi, s)$ be the random variable for the first time step in which state $s'$ is reached in this process. Then the diameter of $M$ is defined as

$$D(M) := \max_{s \neq s' \in S} \min_{\pi : S \to A} \mathbb{E}[T(s' | M, \pi, s)].$$

A stationary policy $\pi$ on an MDP $M$ is a mapping $S \mapsto A$. An average reward (also called gain) of a policy $\pi$ on $M$ with an initial distribution $s_1$ is defined as

$$\lambda(M, \pi, s_1) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{T} \left[ \sum_{t=1}^{T} r(s_t, \pi(s_t)) \right],$$

where the expectation is over trajectories $H_T$ and the limitation may be a random value. We restrict the choice of policies within the set of all policies that give fixed average rewards, $\Pi$. It can be shown that for a communicating MDP the optimal policies with the highest average reward are in the set and neither of optimal policy and optimal reward depends on the initial state. Let $\pi^*(M) = \arg \max_{\pi \in \Pi} \lambda(M, \pi, s_1)$ denote the optimal policy for MDP $M$ starting from $s$ and $\lambda^*(M)$ denote the optimal average reward or optimal gain of the optimal policy.

We define the regret of a reinforcement learning algorithm $\mathcal{L}$ operating on MDP $M$ up to time $T$ as

$$\text{Regret}(T, \mathcal{L}, M) := \sum_{t=1}^{T} \left( \lambda^*(M) - r_t \right).$$

To illustrate the usefulness of span as a difficulty parameter for FMDPs, we show that simple fixed epochs can also have a near-optimal gain with bounded span within a confidence guarantee, PSRL has better empirical performance than PSRL by Ouyang et al. (2017) when specialized to standard non-episodic MDPs.

2.2. Non-factored MDPs

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**Optimal equation for undiscounted MDP.** We let \( R(M, \pi) \) denote the \( S \)-dimensional vector with each element representing \( E_{r \sim R(s, \pi(s))}[r] \) and \( P(M, \pi) \) denote the \( S \times S \) matrix with each row as \( P(s, \pi(s)) \). For any communicating MDP \( M \) using the optimal policy \( \pi^* \), there exists a vector \( h \in \mathbb{R}^S \), such that the optimal gain satisfies the following equation (Puterman, 2014):

\[
1 + h = R(M, \pi^*) + P(M, \pi^*)h. \tag{1}
\]

We let \( h(M) \) be the vector satisfying the equation. Let \( sp(h) := \max_{s_1, s_2} h(s_1) - h(s_2) \).

**2.2. Factored MDP**

Factored MDP is modeled with a DBN (Dynamic Bayesian Network) (Dean & Kanazawa, 1989), where transition dynamic and rewards are factored and each factor only depends on a finite scope of state and action space. We use the definition in Osband & Van Roy (2014). We call \( \mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \).

**Definition 2.2** (Scope operation for factored sets). For any subset of indices \( Z \subseteq \{1, 2, \ldots, n\} \), let us define the scope set \( \mathcal{X}[Z] := \bigotimes_{i \in Z} \mathcal{X}_i \). Further, for any \( x \in \mathcal{X} \) define the scope variable \( x[Z] \in \mathcal{X}[Z] \) to be the value of the variables \( x_i \in \mathcal{X}_i \) with indices \( i \in Z \). For singleton sets \( \{i\} \), we write \( x[\{i\}] \) in the natural way.

**Definition 2.3** (Factored reward distribution). A reward distribution \( R \) is factored over \( \mathcal{X} \) with scopes \( Z_1^R, \ldots, Z_n^R \) if and only if, for all \( x \in \mathcal{X} \), there exists distributions \( \{R_i \in \mathcal{P}[\mathcal{X}[Z_i^R], [0, 1]) \}_{i=1}^m \) such that any \( r \sim R(x) \) can be decomposed as \( \sum_{i=1}^m r_i \), with each \( r_i \sim R_i(x[Z_i^R]) \) individually observable. Throughout the paper, we also let \( R(x) \) denote reward function of the distribution \( R(x) \), which is the expectation \( E_{r \sim R(x)}[r] \).

**Definition 2.4** (Factored transition probability). A transition function \( P \) is factored over \( \mathcal{X} \times \mathcal{A} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \) and \( \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_m \) with scopes \( Z_1^P, \ldots, Z_m^P \) if and only if, for all \( x \in \mathcal{X}, s \in \mathcal{S} \) there exists some \( \{P_i \in \mathcal{P}[\mathcal{X}[Z_i^P], \mathcal{S}_i] \}_{i=1}^m \) such that,

\[
P(s|x) = \prod_{i=1}^m P_i(s[i] | x[Z_i]).
\]

For simplicity, let \( P(s) \) also denote the vector for the probability of each next state from current pair \( x \). We define \( P_i(x) \) in the same way.

**Assumptions on FMDP.** To ensure a finite number of parameters, we assume that \( |\mathcal{X}[Z_i^R]| \leq L \) for \( i \in [n] \), \( |\mathcal{X}[Z_i^P]| \leq L \) for \( i \in [m] \) and \( |\mathcal{S}_i| \leq W \) for all \( i \in [m] \) for some finite \( L \) and \( W \). Furthermore, we assume that \( r \sim R \) is in \([0, 1]\) with probability 1.

**Empirical estimates.** We first define number of visits for each factored set. Let \( N_{R_i}^k(x) := \sum_{t=1}^{k-1} 1 \{ x \sim R_i^k(x) \} \) be the number of visits to \( x \in \mathcal{X}[Z_i^R] \) until \( t \), \( N_{P_i}(x) \) be the number of visits to \( x \in \mathcal{X}[Z_i^P] \) until \( t \) and \( N_{P_i}^k(s, x) \) be the number of visits to \( x \in \mathcal{X}[Z_i^P], s \in \mathcal{S}_i \) until \( t \). The empirical estimate for \( R_i(x) \) is \( \hat{R}_i^k(x) = \frac{\sum_{t=1}^{k-1} \mathbb{1} \{ x \sim R_i^k(x) \}}{N_{P_i}^k(s, x)} \) for \( i \in [m] \). Estimate for transition probability is \( \hat{P}_i^k(s \mid x) = \frac{N_{P_i}(s, x)}{\max(1, N_{P_i}(s, x))} \) for \( i \in [m] \). We let \( \hat{N}_{R_i}^k, \hat{R}_i^k \) and \( \hat{P}_i^k \) to be \( N_{R_i}^k, R_i^k, \) and \( P_i^k \) with \( tk \) be the first step of episode \( k \).

**3. Algorithm**

We use PSRL (Posterior Sampling RL) and a modified version of UCRL-factored, called DORL (Discrete Optimism RL). The term “discrete” here means taking possible transition probabilities from a discrete set (instead of a continuous one used by UCRL2). The main difference between UCRL and DORL version is that DORL only relies on a planner for FMDP, while UCRL needs to solve a bounded-parameter FMDP (Givan et al., 2000). Both PSRL and DORL use a fixed policy within an episode. For PSRL (Algorithm 2), we apply optimal policy for an MDP sampled from posterior distribution of the true MDP. For DORL (Algorithm 1), instead of optimizing over a bounded MDP, we construct a new extended MDP, which is also factored with the number of parameters polynomial in that of the true MDP. Then we find the optimal policy for the new factored MDP and map it to the policy space of the true MDP. Instead of using dynamic episodes, we show that a simple fixed episode scheme can also give us near-optimal regret bounds.

**3.1. Extended FMDP**

Previous near-optimal algorithms on regular MDP depend on constructing an extended MDP with high probability of being optimism, i.e., the optimal gain of the extended MDP is higher than that of the true MDP. There are two constructions for non-factored MDPs. Jaksch et al. (2010) constructs the extended MDP with a continuous action space to allow choosing any transition probability in a confidence set, whose width decrease with an order \( O(1/\sqrt{N}) \), where \( N \) is the number of visits. This construction generates a bounded-parameter MDP. Agrawal & Jia (2017) instead samples transition probability only from the extreme points of the confidence set and combined them by adding extra discrete actions.

Solving the bounded-parameter MDP by the first construction, which requires storing and ordering the \( S \)-dimensional bias vector, is not feasible for FMDP. There is no direct adaptation that mitigates this computation issue. We show that the second construction, by removing the sampling
part, can be solved with a much lower complexity in FMDP setting.

We formally describe the construction. For simplicity, we ignore the notations for $k$ in this session. First define the error bounds as an input. For every $x \in \mathcal{X}[Z_i^t]$, $s \in \mathcal{S}$, we have an error bound $W_{P_i}(s \mid x)$ for transition probability $P_i(s \mid x)$. For every $x \in \mathcal{X}[Z_i^R]$, we have an error bound $W_{R_i}(x)$ for $R_i(x)$. At the start of episode $k$ the construction takes the inputs of $M_k$ and the error bounds, and outputs the extended MDP $M_k$.

**Extreme transition dynamic.** We first define the extreme transition probability mentioned ahead in factored setting. Let $P_i(x)^{s+}$ be the transition probability that encourages visiting $s \in S_i$, be
\[
P_i(x)^{s+} = P_i(x) - W_{P_i}(\cdot \mid x) + \mathbb{I}_s \sum_j W_{P_i}(j \mid x),
\]
where $\mathbb{I}_j$ is the vector with all zeros except for one on the $j$-th element. By this definition, $P_i(x)^{s+}$ is a new transition probability that puts all the uncertainty onto the direction $s$. Our construction assigns an action for each of the extreme transition dynamic.

**Construction of extended FMDP.** Our new factored MDP $M_k = \{S, \tilde{\mathcal{A}}, P, R\}$, where $\tilde{\mathcal{A}} = \mathcal{A} \times \tilde{\mathcal{S}}$ and the new scopes $\{Z_i^R\}_{i=1}^m$ and $\{Z_i^P\}_{i=1}^m$ are the same as those for the original MDP.

Let $\tilde{\mathcal{X}} = \mathcal{X} \times \mathcal{S}$. The new transition probability is factored over $\tilde{\mathcal{X}} = \bigotimes_{i \in [m]} (\mathcal{X}[Z_i^P] \times S_i)$ and $\tilde{\mathcal{S}} = \bigotimes_{i \in [m]} S_i$ with the factored transition probability to be $\tilde{P}_i(x, s[i]) := P_i(x)^{s[i]+}$, for any $x \in \mathcal{X}[Z_i^P], s \in \mathcal{S}$.

The new reward function is factored over $\tilde{\mathcal{X}} = \bigotimes_{i \in [l]} (\mathcal{X}[Z_i^P] \times S_i)$, with reward functions to be $\tilde{R}_i(x, s[i]) := R_i(x) + W_{R_i}(x)$, for any $x \in \mathcal{X}[Z_i^P], s \in \mathcal{S}$.

**Claim 3.1.** The factored set $\tilde{\mathcal{X}} = \mathcal{X} \times \tilde{\mathcal{A}}$ of the extended MDP $M_k$ satisfies each $\mathcal{X}[Z_i^P] \leq LW$ for any $i \in [m]$ and each $|\tilde{\mathcal{X}}[Z_i^P]| \leq LW$ for any $i \in [l]$.

By Claim 3.1, any planner that efficiently solves the original MDP, can also solve the extended MDP. We find the best policy $\tilde{\pi}_k$ for $M_k$ using the planner. To run a policy $\pi_k$ on original action space, we choose $\pi_k$ such that $(s, \pi_k(s)) = f(s, \tilde{\pi}_k(s))$ for every $s \in S$, where $f : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ maps any new state-action pair to the pair it is extended from, i.e. $f(x^*) = x$ for any $x^* \in \tilde{\mathcal{X}}$.

**Algorithm 1** DORL (Discrete Optimism RL)

**Input:** $\mathcal{S}, \mathcal{A}$, accuracy $\rho$, $T$, encoding $\mathcal{G}$ and $L$, upper bound on the size of each factor set.

$k \leftarrow 1$; $t \leftarrow 1$; $t_k \leftarrow 1$; $T_k \leftarrow 1$; $\mathcal{H} = \{\}$.

repeat

Construct the extended MDP $M_k$ using error bounds:
\[
W^k_{P_i}(s \mid x) = \min\{\sqrt{18P_i(s|x)\log(c_{i,k})} + 18\log(c_{i,k}) \max\{N_{P_i}^k(x), 1\}, \tilde{P}_i^k(s|x)}.
\]

for $c_{i,k} = 6mS_i|\mathcal{X}[Z_i^P]|t_k/\rho$ and
\[
W^k_{R_i}(x) = \sqrt{12\log(6L|\mathcal{X}[Z_i^P]|t_k/\rho)} \max\{N_{R_i}^k(x), 1\}.
\]

Compute $\tilde{\pi}_k = \pi(M_k)$ and find corresponding $\pi_k$ in original action space.

for $t = t_k$ to $t_k + T_k - 1$ do

Apply action $a_t = \tilde{\pi}_k(s_t)$

Observe new state $s_{t+1}$.

Observe new rewards $r_{t+1} = (r_{t+1,1}, \ldots r_{t+1,l})$.

$\mathcal{H} = \mathcal{H} \cup \{(s_t, a_t, r_{t+1,1}, s_{t+1})\}$.

end for

$k \leftarrow k + 1$.

$T_k \leftarrow \lfloor k/L \rfloor$; $t_k \leftarrow t + 1$.

until $t_k > T$

4. Results

We achieve the near-optimal Bayesian regret bound by PSRL and frequentist regret bound by DORL, respectively.

**Theorem 4.1** (Regret of PSRL). Let $M$ be the factored MDP with graph structure $\mathcal{G} = (\{S_i\}_{i=1}^m; \{\mathcal{X}_i\}_{i=1}^m; \{Z_i^R\}_{i=1}^m; \{Z_i^P\}_{i=1}^m), \text{ all } |\mathcal{X}_i[Z_i^R]|$ and $|\mathcal{X}_i[Z_i^P]| \leq L, |S_i| \leq W$ and diameter upper bounded by $D$. Then if $\phi$ is the true prior distribution over the set of MDPs with diameter $\leq D$, then we bound Bayesian regret of PSRL:
\[
\mathbb{E}[R_T] = \tilde{O}(D(l + m\sqrt{W})\sqrt{TL}).
\]

**Theorem 4.2** (Regret of DORL). Let $M$ be the factored MDP with graph structure $\mathcal{G} = (\{S_i\}_{i=1}^m; \{\mathcal{X}_i\}_{i=1}^m; \{Z_i^R\}_{i=1}^m; \{Z_i^P\}_{i=1}^m), \text{ all } |\mathcal{X}_i[Z_i^R]|$ and $|\mathcal{X}_i[Z_i^P]| \leq L, |S_i| \leq W$ and diameter upper bounded by $D$. Then, with high probability, regret of DORL is upper bounded by:
\[
R_T = \tilde{O}(D(l + m\sqrt{W})\sqrt{TL}).
\]

The two bounds match the frequentist regret bound in Jaksch.
Theorem 4.3. For any algorithm, the optimal policy of the true MDP, \(\pi(M)\), and the Bayesian bound in Theorem 4.2 also holds.\\\\Remark. Replacing the episode length in Algorithm 1 and 2 with any \(\{T_k\}_{k=1}^{K}\) that satisfies \(K = O(\sqrt{LT})\) and \(T_k = O(\sqrt{T/L})\) for all \(k \in [K]\), the frequential bound in Theorem 4.2 still holds. Furthermore, if \(\{T_k\}_{k=1}^{K}\) is fixed the Bayesian bound in Theorem 4.2 also holds.

Our lower bound restricts the scope of transition probability, i.e. the scope contains itself, which we believe is a natural assumption.

Theorem 4.3. For any algorithm, any graph structure \(G\) and \(n\) with \(|S_i| = \lambda_i\), \(|A_i| = \lambda_i\), \(|Z_i^R| = \lambda_i\), \(|Z_i^P| = \lambda_i\) with \(|S_i|\leq L_i\), \(|X[Z_i^R]|\leq W_i\), \(|X[Z_i^P]|\leq W_i\) and \(i \in [n]\), there exist an FMDP with the span of bias vector \(sp(h^*)\), such that for any initial state \(s \in S\), the expected regret of the algorithm after \(T\) step is \(O(\sqrt{sp(h^*)WT})\).

For a tighter regret bound depending on span, we use a factored REGAL.C, which replaces all the diameter \(D\) with an upper bound \(H\) on span. The discussion of factored REGAL.C is in Appendix F.

5. Analysis

A standard regret analysis consists of proving the optimism, bounding the deviations and bounding the probability of failing the confidence set. Our analysis follows the standard procedure while adapting them to a FMDP setting. The novelty is on the proof of the general episode-assigning criterion and the lower bound.

Some notations. For simplicity, we let \(\pi^*\) denote the optimal policy of the true MDP, \(\pi(M)\). Let \(t_k\) be the starting time of episode \(k\) and \(K\) be the total number of episodes. Since \(\hat{R}^k(x, s)\) for any \((x, s) \in \mathcal{X}\) does not depends on \(s\), we also let \(\hat{R}^k(x, s)\) denote \(\hat{R}^k(x, s)\) for any \(s\). Let \(\lambda^*\) and \(\lambda_k\) denote the optimal average reward for \(M\) and \(M_k\).

Regret decomposition. We follow the standard regret analysis framework by Jaksch et al. (2010). We first decompose the total regret into three parts in each episode:

\[
R_T = \sum_{t=1}^{T} (\lambda^* - \lambda_t) = \sum_{k=1}^{K} \sum_{t=k}^{k+1} (\lambda^* - \lambda_k) + \sum_{k=1}^{K} \sum_{t=k}^{k+1} (\lambda_k - R(s_t, a_t)) + \sum_{k=1}^{K} \sum_{t=k}^{k+1} (R(s_t, a_t) - \lambda_t). \tag{6}
\]

Using Hoeffding’s inequality, the regret caused by (6) can be upper bounded by \(\sqrt{2T \log \left(\frac{8}{\rho} \right)}\), with probability at least \(\frac{1}{2}\).

Confidence set. Let \(M_k\) be the confidence set of FMDPs at the start of episode \(k\) with the same factorization, such that for and each \(i \in [l]\),

\[
|\mathcal{R}_i(x) - \hat{\mathcal{R}}_i^k(x)| \leq W_{\mathcal{R}_i}^k(x), \forall x \in X[Z_i^R],
\]

where \(W_{\mathcal{R}_i}^k(x) := \sqrt{12 \log(6 \lambda_i |X[Z_i^R]| |t_k/\rho|) / \max\{N_{\mathcal{R}_i}^k(x), 1\}}\) as defined in (3); and for each \(j \in [m]\),

\[
|\mathcal{P}_j(s|x) - \hat{\mathcal{P}}_j^k(s|x)| \leq W_{\mathcal{P}_j}^k(s|x), \forall x \in X[Z_j^P], s \in S_j,
\]

where \(W_{\mathcal{P}_j}^k(s|x)\) is defined in (2). It can be shown that

\[
|\mathcal{P}_j(x) - \hat{\mathcal{P}}_j^k(x)| \leq 2 \sqrt{18 |S_i| \log(6 S_i m |X[Z_j^P]| |t_k/\rho|) / \max\{N_{\mathcal{P}_j}^k(x), 1\}}
\]

where \(W_{\mathcal{P}_j}^k(x) := 2 \sqrt{18 |S_i| \log(6 S_i m |X[Z_j^P]| |t_k/\rho|) / \max\{N_{\mathcal{P}_j}^k(x), 1\}}\).

In the following analysis, we all assume that true MDP \(M\) for both PSRL and DORL are in \(M_k\) and \(M_k\) by PSRL are in \(M_k\) for all \(k \in [K]\). In the end, we will bound the regret caused by the failure of confidence set.

5.1. Regret caused by difference in optimal gain

We further bounded the regret caused by (4). For PSRL, since we use fixed episodes, we show that the expectation of (4) equals to zero.
Lemma 5.1 (Lemma 1 in Osband et al. (2013)). If \( \phi \) is the distribution of \( M \), then, for any \( \sigma(H_{t_k}) \) – measurable function \( g \),
\[
\mathbb{E}[g(M) \mid H_{t_k}] = \mathbb{E}[g(M_k) \mid H_{t_k}].
\]

We let \( g = \lambda(M, \pi(M)). \) As \( g \) is a \( \sigma(H_{t_k}) \) – measurable function. Since \( t_k \) is fixed value for each \( k \), we have \( \mathbb{E}[\sum_{k=1}^{K} T_k (g(M) - g(M_k))] = 0. \)

For DORL, we need to prove optimism, i.e. \( \lambda(M_k, \tilde{\pi}_k) \geq \lambda^* \) with high probability. Given \( M \in \mathcal{M}_k \), we show that there exists a policy for \( M_k \) with an average reward \( \geq \lambda^* \).

Lemma 5.2. For any policy \( \pi \) for \( M \) and any vector \( h \in \mathbb{R}^2 \), let \( \hat{\pi} \) be the policy for \( M_k \) satisfying \( \hat{\pi}(s) = (\pi(s), s^*) \), where \( s^* = \arg \max_s h(s) \). Then, given \( M \in \mathcal{M}_k \), \( P(M, \hat{\pi}) - P(M, \pi) \)\( h \geq 0. \)

Corollary 5.3. Let \( \hat{\pi}^* \) be the policy that satisfies \( \hat{\pi}^*(s) = (\pi^*(s), s^*) \), where \( s^* = \max_s h(M) \). Then \( \lambda(M_k, \hat{\pi}^*, s_1) \geq \lambda^* \) for any starting state \( s_1 \).

Proof of Lemma 5.2 and Corollary 5.3 are shown in Appendix A. Thereon, \( \lambda(M_k, \hat{\pi}_k) \geq \lambda(M_k, \tilde{\pi}_k, s_1) \geq \lambda^* \). The total regret of (4) \( \leq 0. \)

5.2. Regret caused by deviation

We further bound regret caused by (5), which can be decomposed into the deviation between our brief \( M_k \) and the true MDP. We first show that the diameter of \( M_k \) can be upper bounded by \( D \).

Bounded diameter. We need diameter of extended MDP to be upper bounded to give a sublinear regret. For PSRL, since prior distribution has no mass on MDP with diameter greater than \( D \), the diameter of MDP from posterior is upper bounded by \( D \) almost surely. For DORL, we have the following lemma, the proof of which is given in Appendix B.

Lemma 5.4. When \( M \) is in the confidence set \( \mathcal{M}_k \), the diameter of the extended MDP \( D(M_k) \) \( \leq D \).

Deviation bound. Let \( \nu_k(s, a) \) be the number of visits on \( s, a \) in episode \( k \) and \( \nu_k \) be the row vector of \( \nu_k(\cdot, \pi_k(\cdot)) \). Let \( \Delta_k = \sum_{s,a} \nu_k(s, a)(\lambda(M_k, \pi_k) - R(s, a)) \). Using optimal equation,
\[
\Delta_k = \sum_{s,a} \nu_k(s, a) \left[ \lambda(M_k, \pi_k) - \tilde{R}_k(s, a) \right] + \sum_{s,a} \nu_k(s, a) \left[ \tilde{R}_k(s, a) - R(s, a) \right] = \nu_k(\tilde{p}_k - I) h_k + \nu_k(\tilde{R}_k - R^k)
\]
\[
= \nu_k(P^k - I) h_k + \nu_k(\hat{p}_k^k - P^k) h_k + \nu_k(\tilde{R}_k^k - R^k),
\]
where \( \tilde{p}_k := P(M_k, \tilde{\pi}_k), P^k := P(M, \pi_k), h_k := h^*(M_k), \) and \( \tilde{R}_k := R(M_k, \tilde{\pi}_k), R^k := R(M, \pi_k). \)

Using Azuma-Hoeffding inequality and the same analysis in (Jaksch et al., 2010), we bound (1) with probability at least \( 1 - \frac{\rho}{2} \).
\[
\sum_k (1) = \sum_k \nu_k(\tilde{p}_k^k - I) h_k \leq D \sqrt{\frac{5}{2} \log \left( \frac{8}{\rho} \right)} + KD.
\]

To bound (2) and (3), we analysis the deviation in transition and reward function between \( M \) and \( M_k \). For DORL, the deviation in transition probability is upper bounded by
\[
\max_{s'} |\tilde{p}_k^k(s, s') - \hat{p}_k^k(s, s')| \leq \min\{2 \sum_{s \in S_i} W_{r_i}^k(s \mid x), 1\}
\]
\[
\leq \min\{2 \bar{W}_{r_i}^k(x), 1\} \leq 2 \bar{W}_{r_i}^k(x),
\]

The deviation in reward function \( |\tilde{R}_i^k - \hat{R}_i^k(x)| \leq W_{r_i}^k(x) \).

For PSRL, since \( M_k \in \mathcal{M}_k \), \( |\hat{p}_k^k - \hat{p}_k^k(x)| \leq W_{r_i}^k(x) \) and \( |\tilde{R}_i^k - \hat{R}_i^k(x)| \leq W_{r_i}^k(x) \).

Decomposing the bound for each scope provided by \( M \in \mathcal{M}_k \) and \( M_k \) for PSRL \( \in \mathcal{M}_k \), it holds for both PSRL and DORL that:
\[
\sum_k (2) \leq 3 \sum_k D \sum_{i=1}^{m} \sum_{x' \in X[Z_i]} \nu_k(x) W_{r_i}^k(x), \quad (8)
\]
\[
\sum_k (3) \leq 2 \sum_k \sum_{i=1}^{m} \sum_{x \in X[Z_i]^k} \nu_k(x) W_{r_i}^k(x); \quad (9)
\]

where with some abuse of notations, define \( \nu_k(x) = \sum_{x' \in X[Z_i]} \nu_k(x') \) for \( x \in X[Z_i] \). The second inequality is from the fact that \( |\hat{p}_k^k(\cdot | x) - P^k(\cdot | x)|_1 \leq \sum_{i=1}^{m} |\hat{p}_k^k(\cdot | x[Z_i]^k) - P^k(\cdot | x[Z_i]^k)|_1 \) (Osband & Van Roy, 2014).

5.3. Balance episode length and episode number

We give a general criterion for bounding (7), (8) and (9).
Lemma 5.5. For any fixed episodes \( \{T_k\}_{k=1}^K \), if there exists an upper bound \( T \), such that \( T_k \leq T \) for all \( k \in [K] \), we have the bound
\[
\sum_{x \in \mathcal{X}[Z]} \sum_k v_k(x) / \sqrt{\max\{1, N_k(x)\}} \leq LT + \sqrt{LT},
\]
where \( Z \) is any scope with \( |\mathcal{X}[Z]| \leq L \), and \( v_k(x) \) and \( N_k(x) \) are the number of visits to \( x \) in and before episode \( k \). Furthermore, total regret of (7), (8) and (9) can be bounded by \( \tilde{O}(\sqrt{WDM} + 1)(LT + \sqrt{LT}) + KD) \).

Lemma 5.5 implies that bounding the deviation regret is to balance total number of episodes and the length of the longest episode. The proof, as shown in Appendix C, relies on defining the last episode \( k_0 \), such that \( N_{k_0}(x) \leq \nu_{k_0}(x) \).

Instead of using the doubling trick that was used in (Jaksch et al., 2010). We use an arithmetic progression: \( T_k = [k/L] \) for \( k \geq 1 \). As in our algorithm, \( T \geq \sum_{k=1}^{K-1} T_k \geq \sum_{k=1}^{K-1} [k/L] = k/L \cdot \frac{K^2}{2} \), we have \( K \leq \sqrt{3LT} \) and \( T_k \leq KT \leq K/L \leq \frac{\sqrt{3T}}{L} \) for all \( k \in [K] \). Thus, by Lemma 5.5, putting (6), (7), (9), (8) together, the total regret for \( M \in \mathcal{M}_k \) is upper bounded by
\[
\tilde{O}(\sqrt{WDM} + 1)\sqrt{LT}),
\]
with a probability at least \( 1 - \frac{p}{T} \).

For the failure of confidence set, we prove the following Lemma in Appendix D.

Lemma 5.6. For all \( k \in [K] \), with probability greater than \( 1 - \frac{2p}{T} \), \( M \in \mathcal{M}_k \) holds.

Combined with (10), with probability at least \( 1 - \frac{2p}{T} \) the regret bound in Theorem 4.2 holds.

For PSRL, \( M_k \) and \( M \) has the same posterior distribution. The expectation of the regret caused by \( M \in \mathcal{M}_k \) and \( M_k \notin \mathcal{M}_k \) are the same. Choosing sufficiently small \( \rho \leq \sqrt{1/T} \), Theorem 4.1 follows.

6. Lower Bound

The usefulness of the span of the optimal bias vector as a difficulty parameter for standard MDP has been discussed in the literature (Bartlett & Tewari, 2009). Here we further argue that the span is a better notion of difficulty for FMDPs since it scales better when we generate rather simple FMDPs that decompose into independently evolving MDPs. For such FMDPs, span grows in a controlled way whereas the diameter can blow up. We also provide a proof sketch for the lower bound of \( \sqrt{sp(h)}WT \) stated earlier in Theorem 4.3. Since our upper bounds are stated in terms of the diameter, we also observe that the REGAL.C algorithm of Bartlett & Tewari (2009) can be extended to FMDPs and shown to enjoy a regret bound that depends on the span, not the diameter. However, doing this does not lead to an oracle efficient solution which is the main focus of this paper.

Large diameter case. We consider a simple FMDP with infinite diameter but still solvable. Let \( S_1 = S_2 = \{0, 1, 2, 3\} \), \( A_1 = A_2 = \{1, 2\} \). State space \( S = S_1 \times S_2 \) and action space \( A = A_1 \times A_2 \) are factored with indices \( \{1, 2\} \). The transition probability is factored over \( S = \bigotimes_i S_i \) and \( S \times A = \bigotimes_i (S_i \times A_i) \) with scope \( Z^R_i = \{i\} \) for \( i = 1, 2 \). The reward function is factored over \( S \times A = \bigotimes_i (S_i \times A_i) \) with scope \( Z^R_i = \{i\} \) for \( i = 1, 2 \). By this definition, the FMDP can be viewed as two independent MDP, \( M_1 \) and \( M_2 \) that are set to be communicating with bounded diameter. Each factored transition probability is chosen such that from any state and action pair, the next state will either move forward or move backward with probability one (state 0 is connected with state 3 as a circle).

This FMDP can be easily solved by solving \( M_1 \) and \( M_2 \) independently. However, since \( s[1] + s[2] \) always keeps the same parity, \( (0, 0) \) cannot be transmitted to \( (0, 1) \). Thus, the FMDP has an infinite diameter. The span of optimal policy, on the other hand, is upper bounded by \( D(M_1) + D(M_2) \), which is tight in this case. To ensure an communicating MDP, we can simply add an extra action for each MDP with small probability \( \delta > 0 \) to stay unchanged for each factored state. In this way, the diameter can be arbitrarily large.

Lower bound with only dependency on span. Let formally state the lower bound. Our lower bound casts some restrictions on the scope of transition probability, i.e. the scope contains itself, which we believe is a natural assumption. We provide a proof sketch for Theorem 4.3 here.

Proof sketch. Let \( l = |\cup_i Z^R_i| \). As \( i \in Z^R_i \), a special case is the FMDP with graph structure \( G = \left( \{S_i\}_{i=1}^n, \{S_i \times A_i\}_{i=1}^n, \{\{i\}\}_{i=1}^l, \{\emptyset\}_{l+1}^n \right) \), which can be decomposed into \( n \) independent MDPs as in the previous example. Among the \( n \) MDPs, the last \( n-l \) MDPs are trivial. By simply setting the rest \( l \) MDPs to be the construction used by (Jaksch et al., 2010), which we refer to as "JAO MDP", the regret for each MDP with the span \( sp(h) \), is \( \Omega(\sqrt{sp(h)WT}) \) for \( i \in [l] \). The total regret is \( \Omega(l\sqrt{sp(h)WT}) \).

Lemma 6.1. Let \( M^+ \) be the Cartesian product of \( n \) independent MDPs \( \{M_i\}_{i=1}^n \), each with a span of bias vector \( sp(h_i) \). The optimal policy for \( M^+ \) has a span \( \sum_i sp(h_i) \).

Using Lemma 6.1 (proved in E), \( sp(h^*) = l sp(h) \) and the total expected regret is \( \Omega(\sqrt{l sp(h)WT}) \). Normalizing the reward function to be in \([0, 1]\), the expected regret of the FMDP is \( \Omega(\sqrt{sp(h^*)WT}) \), which completes the proof.
Closer to the lower bound. Even if regret bounds of implementable algorithms depends on the diameter, which was shown to be not optimal, we can simply modify REGAL.C (Bartlett & Tewari, 2009) to replace the diameter with the span. The proof of which is shown in Appendix F.

7. Simulation

While both of the algorithms we proposed are oracle-efficient, DORL has higher computational complexity and we found that it performs worse on regular MDP compared with PSRL and UCRL2 (see Appendix G). For comparisons, we choose the previous sample efficient and implementable algorithms, factored $E^3$ and factored Rmax (f-Rmax). F-Rmax was shown to have better empirical performance (Guessrin et al., 2002). Thus, we only compare between PSRL and f-Rmax. For PSRL, at the start of each episode, we simply sample each factored transition probability and reward functions from a Dirichlet distribution and a Gaussian distribution, i.e. $P^f_i(x) \sim \text{Dirichlet}(N^f_i, x)/c$ and $R^f_i(x) \sim N(\tilde{R}^f_i(x), c/N^f_i(x)$, where $c$ is optimized as 0.75. The total number of samplings for PSRL in each round is upper bounded by the number of parameters of the FMDP. For f-Rmax, $m$, the number of visits needed to be known are chosen from 100, 300, 500, 700 and the best choice is selected for each experiment.

For the approximate planner used by our algorithm, we implemented approximate linear programming (Guessrin et al., 2003) with the basis $h_i(s) = s_i$ for $i \in [m]$. However, for regret evaluation, we use an accurate planner to find the true optimal average reward.

We compare two algorithms, PSRL and f-Rmax on computer network administrator domain with a circle and a three-leg structure (Guessrin et al., 2001; Schuurmans & Patrasu, 2002). To avoid the extreme case in our lower bound, both the MDPs are set to have limited diameters.

Figure 1 shows the regret of the two algorithms on circle structure with a size 7, 11 and on three-leg structure with a size 7, 10, respectively. Each experiment is run five times, with which mean and standard deviation are computed. As it is the case for regular MDP, PSRL performs best among all the three methods we tested. Optimal parameter for PSRL is stable in the way that $c$ around 0.75 is the optimal parameter for all the experiments. As the true optimal reward is used for regret evaluating, we observe that PSRL could always find the true optimal policy despite the use of an approximate planner.

8. Discussion

In this paper, we provide two oracle-efficient algorithms PSRL and DORL for non-episodic FMDP, with a Bayesian and frequentist regret bound of $D(l + m\sqrt{W})\sqrt{LT}$, respectively. PSRL outperforms previous near-optimal algorithm f-Rmax on computer network administration domain. The regret is still small despite using an approximate planner. We prove an lower bound of $\sqrt{sp(h^*)WT}$ for non-episodic MDP. Our large diameter example shows that diameter $D$ can be arbitrary larger than the span $sp(h^*)$. It can be shown that by directly replacing the $M_k$ with the $M_k^*$ in REGAL.C (Bartlett & Tewari, 2009), the diameter $D$ could be improved to $sp(h^*)$.

REGAL.C relies on a harder computational oracle that is not efficiently solvable yet. Fruit et al. (2018) achieved the same regret bound using an implementable Bias-Span-Constrained value iteration on non-factored MDP. It remains unknown whether a upper bound depending on span of optimal bias vector can be derived using an oracle-efficient algorithm in the FMDP setting.

On the lower bound of non-episodic FMDP, it remains an open problem whether the dependency of the span could be increased to the diameter if we restrict to oracle-efficient algorithms and whether an arbitrary graph structure gives the same lower bound.

Our algorithms require the full knowledge of the graph structure of a FMDP, which can be impractical. The structural learning scenario has been studied by Strehl et al. (2007); Chakraborty & Stone (2011); Hallak et al. (2015). Their algorithm either relies on an admissible structure learner or does not have regret or sample complexity guarantee. It remains an open problem of whether an efficient algorithm with theoretical guarantees exists for FMDP with unknown graph structure.

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Figure 1. Regrets of PSRL and f-Rmax on circle MDP with a size 7, 11 and on three-leg MDP with a size 7, 10. For PSRL, \( c = 0.75 \). For f-Rmax, \( m = 300 \) or 500.

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A. Optimism (Proof of Lemma 5.2 and Corollary 5.3)

**Lemma A.1.** For any policy \( \pi \) for \( M \) and any vector \( \mathbf{h} \in \mathbb{R}^S \), let \( \hat{\pi} \) be the policy for \( M_k \) satisfying \( \hat{\pi}(s) = (\pi(s), s^*) \), where \( s^* = \arg \max_s \mathbf{h}(s) \). Then, given \( M \in \mathcal{M}_k \), \( (P(M_k, \hat{\pi}) - P(M, \pi)) \mathbf{h} \geq 0 \).

**Proof.** We fix some \( s \in S \) and let \( x = (s, \pi(s)) \in X \). Recall that for any \( s_i \in S_i \), \( \Delta^k_i(s_i|x) = \min \left\{ \frac{18 \hat{P}^k_i(s_i|x) \log(c_{i,k})}{\max \left\{ N^k_{P_i}(x), 1 \right\}}, \frac{18 \log(c_{i,k})}{\max \left\{ N^k_{P_i}(x), 1 \right\}} \right\} \).

and define \( P^+(\cdot|x) = \hat{P}^k_i(\cdot|x) - \Delta^k_i(\cdot|x) \). Slightly abusing the notations, let \( \hat{P} = P(M_k, \hat{\pi})_s \), \( P = P(M, \pi)_s \). Define two \( S \)-dimensional vectors \( \hat{P} \) and \( P^- \) with \( \hat{P}(\bar{s}) = \prod_{i} P_i(\bar{s}|Z_i^P) |x \) and \( P^-(\bar{s}) = \Pi_i P_i^-(\bar{s}|Z_i^P) |x \) for \( \bar{s} \in S \).

As \( M \in \mathcal{M}_k \), \( P^- \leq P \). Define \( \alpha := \hat{P} - P \leq \hat{P} - P^- =: \Delta \). Without loss of generality, we let \( \max_i \mathbf{h}(s) = D \).

\[
\sum_i \hat{P}(i) \mathbf{h}(i) = \sum_i P(i) - \hat{P}(i) \mathbf{h}(i) + D \left( 1 - \sum_j P(j) \right)
\]

\[
= \sum_i P(i) \mathbf{h}(i) + D \sum_j \Delta(j)
\]

\[
= \sum_i \left( \hat{P}(i) - \Delta(i) \right) \mathbf{h}(i) + D \Delta(i)
\]

\[
= \sum_i P(i) \mathbf{h}(i) + (D - \hat{P}(i)) \Delta(i)
\]

\[
\geq \sum_i \hat{P}(i) \mathbf{h}(i) + (D - \hat{P}(i)) \alpha(i)
\]

\[
= \sum_i \left( P(i) - \alpha(i) \right) \mathbf{h}(i) + D \alpha(i)
\]

\[
= \sum_i P(i) \mathbf{h}(i) + D \sum_i \alpha(i) = \sum_i P(i) \mathbf{h}(i)
\]

**Corollary A.2.** Let \( \hat{\pi}^* \) be the policy that satisfies \( \hat{\pi}^*(s) = (\pi^*(s), s^*) \), where \( s^* = \max_s \mathbf{h}(M) \). Then \( \lambda(M_k, \hat{\pi}^*, s_1) \geq \lambda^* \) for any starting state \( s_1 \).

**Proof.** Let \( \mathbf{d}(s_1) := d(M_k, \hat{\pi}^*, s_1) \in \mathbb{R}^{1 \times S} \) be the row vector of stationary distribution starting from some \( s_1 \in S \). By optimal equation,

\[
\lambda(M_k, \hat{\pi}^*, s_1) - \lambda^* = \mathbf{d}(s_1) \mathbf{R}(M_k, \hat{\pi}^*) - \lambda^* \mathbf{d}(s_1) \mathbf{1} \\
= \mathbf{d}(s_1) \mathbf{R}(M_k, \hat{\pi}^*) - \lambda \mathbf{d}(s_1) \mathbf{1} \\
= \mathbf{d}(s_1) \mathbf{R}(M_k, \hat{\pi}^*) - \mathbf{R}(M_k, \hat{\pi}^*) \mathbf{d}(s_1) \mathbf{1} \\
= \mathbf{d}(s_1) \left( \mathbf{L}(M_k, \hat{\pi}^*) - \mathbf{R}(M_k, \hat{\pi}^*) \right) \mathbf{1} \\
\geq \mathbf{d}(s_1) \left( \mathbf{L}(M_k, \hat{\pi}^*) - \mathbf{R}(M_k, \hat{\pi}^*) \right) \mathbf{1} \\
+ \mathbf{d}(s_1) \left( \mathbf{P}(M_k, \hat{\pi}^*) - \mathbf{P}(M_k, \hat{\pi}^*) \right) \mathbf{1} \\
\geq 0,
\]

where the last inequality is by Lemma 5.2 and Corollary 5.3 follows.

B. Proof of Lemma 5.4

**Lemma B.1.** Given \( M \) in the confidence set \( \mathcal{M}_k \), the diameter of the extended MDP \( D(M_k) \leq D \).
Proof. Fix a \( s_1 \neq s_2 \), there exist a policy \( \pi \) for \( M \) such that the expected time to reach \( s_2 \) from \( s_1 \) is at most \( D \), without loss of generality we assume \( s_2 \) is the last state. Let \( E \) be the \((S - 1) \times 1\) vector with each element to be the expected time to reach \( s_2 \) except for itself. We find \( \tilde{\pi} \) for \( M_k \) such that the expected time to reach \( s_2 \) from \( s_1 \) can be bounded by \( D \). We choose the \( \tilde{\pi} \) that satisfies \( \tilde{\pi}(s) = (\pi(s), s_2) \).

Let \( Q \) be the transition matrix under \( \tilde{\pi} \) for \( M_k \). Let \( Q^- \) be the matrix removing \( s_2 \)-th row and column and \( P^- \) defined in the same way for \( M \). We immediately have \( P^{-1}E \geq Q^{-1}E \), given \( M \in M_k \). Let \( \tilde{E} \) be the expected time to reach \( s_2 \) from every other states except for itself under \( \tilde{\pi} \) for \( M_k \).

We have \( \tilde{E} = 1 + Q^- \tilde{E} \). The equation for \( E \) gives us \( E = 1 + P^- E \geq 1 + Q^- E \). Therefore,

\[
\tilde{E} = (1 - Q^-)^{-1} 1 \leq E,
\]

and \( \tilde{E}_{s_1} \leq E_{s_1} \leq D \). Thus, \( D(M_k) \leq D \).

C. Deviation bound (Proof of Lemma 5.5)

Lemma C.1. For any fixed episodes \( \{T_k\}_{k=1}^K \), if there exists an upper bound \( \tilde{T} \), such that \( T_k \leq \tilde{T} \) for all \( k \in [K] \), we have the bound

\[
\sum_{x \in \mathcal{X}[Z]} \sum_k \nu_k(x) / \sqrt{\max\{1, N_k(x)\}} \leq \tilde{T} + \sqrt{LT},
\]

where \( Z \) is any scope, and \( \nu_k(x) \) and \( N_k(x) \) are the number of visits to \( x \) in and before episode \( k \). Furthermore, total regret of (7), (8) and (9) can be bounded by \((\sqrt{WDm} + 1)(\tilde{T} + \sqrt{LT}) + KD\).

Proof. We bound the random variable \( \sum_{k=1}^K \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}} \) for every \( x \in \mathcal{X}[Z] \), where \( \nu_k(x) = \sum_{t=t_k}^{t_{k+1}-1} \mathbb{I}(x_t = x) \) and \( N_k(x) = \sum_{t=1}^{k-1} \nu_k(x) \).

Let \( k_0(x) \) be the largest \( k \) such that \( N_k(x) \leq \nu_k(x) \). Thus \( \forall k \geq k_0(x), N_k(x) > \nu_k(x) \), which gives \( N_t(x) := N_k(x) + \sum_{t=t_k}^{t} \mathbb{I}(x_t = x) < 2N_k(x) \) for \( t_k \leq t < t_{k+1} \).

Conditioning on \( k_0(x) \), we have

\[
\sum_{k=1}^K \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}} \\
\leq N_{k_0(x)}(x) + \nu_{k_0(x)}(x) + \sum_{k > k_0(x)} \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}} \\
\leq 2\nu_{k_0(x)}(x) + \sum_{k > k_0(x)} \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}} \\
\leq 2\tilde{T} + \sum_{k > k_0(x)} \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}}.
\]

where the first inequality uses \( \max\{N_k(x), 1\} \geq 1 \) for \( k = 1, \ldots, k_0(x) \), the second inequality is by the fact that \( N_{k_0(x)}(x) \leq \nu_{k_0(x)}(x) \) and the third one is by \( \nu_{k_0(x)} \leq T_{k_0(x)} \leq T_K \).
And letting $k_1(x) = k_0(x) + 1$ and $N(x) := N_k(x) + \nu_k(x)$, we have

$$\sum_{k > k_0(x)}^T \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}} \leq \sum_{t = k_1(x)}^T 2 \frac{1(x_t = x)}{\sqrt{\max\{N_t(x), 1\}}} \leq \sum_{t = k_1(x)}^T 2 \frac{1(x_t = x)}{\sqrt{\max\{N_t(x) - N_{k_1}(x), 1\}}} \leq 2 \int_1^{N(x) - N_{k_1}(x)} \frac{1}{\sqrt{x}} dx \leq (2 + \sqrt{2}) \sqrt{N(x)}.$$

Given any $k_0(x)$, we can bound the term with a fixed value $2T + (2 + \sqrt{2}) \sqrt{N(x)}$. Thus, the random variable $\sum_{k=1}^K \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}}$ is upper bounded by $2T + (2 + \sqrt{2}) \sqrt{N(x)}$ almost surely. Finally, $\sum_x \sum_{k=1}^K \frac{\nu_k(x)}{\sqrt{\max\{N_k(x), 1\}}} \leq LT + (2 + \sqrt{2}) \sqrt{LT}$. The regret by (8) is

$$\sum_k 3D \sum_{i \in [m]} \sum_{x \in \mathcal{X}[Z_i^R]} \nu_k(x) \hat{W}_{P_i}^k(x) = O(\sqrt{WDm(2T + \sqrt{LT}) + KD}).$$

The regret by (9) is

$$\sum_k 2 \sum_{i \in [k]} \sum_{x \in \mathcal{X}[Z_i^R]} \nu_k(x) \hat{W}_{R_i}^k(x) = o(\sqrt{2} \sqrt{MDm(2T + \sqrt{LT}) + KD}).$$

The last statement is completed by directly summing (7), (8) and (9).

**D. Regret caused by failing confidence bound**

**Lemma D.1.** *For all $k \in [K]$, with probability greater than $1 - \frac{3\rho}{N}$, $M \in \mathcal{M}_k$ holds.*

**Proof.** We first deal with the probabilities, with which in each round a reward function of the true MDP $M$ is not in the confidence set. Using Hoeffding’s inequality, we have for any $t, i$ and $x \in \mathcal{X}[Z_i^R]$,

$$\mathbb{P}\left\{ |\hat{R}_i^t(x) - R_i(x)| \geq \sqrt{\frac{12 \log (6l |\mathcal{X}[Z_i^R]| t/\rho)}{\max\{1, N_{R_i}(x)\}}} \right\} \leq \frac{\rho}{3l |\mathcal{X}[Z_i^R]| t^5}, \text{ with a summation } \leq \frac{3}{12} \rho.$$

Thus, with probability at least $1 - \frac{3\rho}{T^2}$, the true reward function is in the confidence set for every $t \leq T$.

For the transition probability, we use a different concentration inequality.

**Lemma D.2** (Multiplicative Chernoff Bound (Kleinberg et al., 2008) Lemma 4.9). *Consider $n$, i.i.d random variables $X_1, \ldots, X_n$ on $[0, 1]$. Let $\mu$ be their mean and let $X$ be their average. Then with probability $1 - \rho$,

$$|X - \mu| \leq \sqrt{\frac{3 \log (2/\rho) X}{n}} + \frac{3 \log (2/\rho)}{n}.$$
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Using Lemma D.2, for each $x, i, k$, it holds that with probability $1 - \rho/(6m |\mathcal{X} | Z^*_i | t_k^0)$,

$$|\hat{P}_i(\cdot|x) - P_i(\cdot|x)|_1 \leq \sqrt{\frac{18S_i \log(c_i,k)}{\max\{N_{P_i}^k(x), 1\}}} + \frac{18 \log(c_i,k)}{\max\{N_{P_i}^k(x), 1\}}.$$

Then with a probability $1 - \frac{3\rho}{24}$, it holds for all $x, i, k$. Therefore, with a probability $1 - \frac{3\rho}{8}$, the true MDP is in the confidence set for each $k$.

E. Span of Cartesian product of MDPs

**Lemma E.1.** Let $M^+$ be the Cartesian product of $n$ independent MDPs $\{M_i\}_{i=1}^n$, each with a span of bias vector $sp(h_i)$. The optimal policy for $M^+$ has a span $sp(h^+) = \sum_i sp(h_i)$.

**Proof.** Let $\lambda^*_i$ for $i \in [n]$ be the optimal gain of each MDP. Optimal gain of $M^+$ is directly $\lambda^* = \sum_{i \in [n]} \lambda^*_i$. Recall that the definition of bias vector is also

$$h_i(s) = \mathbb{E}[\sum_{t=1}^{\infty} (r^i_t - \lambda^*_i) \mid s_1 = s], \quad \forall s \in \mathcal{S}_i,$$

where $r^i_t$ is the reward of the $i$-th MDP at time $t$ and $s_1^i := s_1[i]$.

The lemma is directly by

$$h^+(s) = \mathbb{E}[\sum_{t=1}^{\infty} (r_t - \lambda^*) \mid s_1 = s]$$

$$= \mathbb{E}[\sum_{t=1}^{\infty} (\sum_{i \in [n]} (r^i_t - \lambda^*_i)) \mid s_1 = s]$$

$$= \sum_{i \in [n]} \mathbb{E}[\sum_{t=1}^{\infty} (r^i_t - \lambda^*_i) \mid s_1^i = s[i]]$$

$$= \sum_{i \in [n]} h_i(s[i]).$$

We immediately have $sp(h^+) = \sum_i sp(h_i)$.

F. Algorithms with dependency on span

In this section, we show that a frequentist regret bound that depends on span rather than diameter can be achieved with a factored REGAL.C (Bartlett & Tewari, 2009) as shown in Algorithm 3. For factored REGAL.C, we define the same confidence set $\mathcal{M}_k$ as what we use in the previous analysis. Given $M \in \mathcal{M}_k$ and $sp(h^+(M)) \leq H$, we have $\lambda^*(M_k) \geq \lambda^*(M)$, which upper bounds the regret by (4) with 0. For the regret by (5) and (6), since $sp(h^+(M_k)) \leq H$ and it is easy to see that all the $D$ in the analysis for deviation can be replaced by $H$. Thus, we have the same final regret bound with all $D$ replaced by $H$. 
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Algorithm 3 Factored REGAL.C

Input: $S, A$, Prior $\phi$, $T$, encoding $G$ and $H$.

$k \leftarrow 1; t \leftarrow 1; t_k \leftarrow 1; T_k = 1; \mathcal{H} \leftarrow \{\}$

repeat

Choose $M_k \in \mathcal{M}_k$ by solving the following optimization over $M \in \mathcal{M}_k$,

$$\max \lambda^*(M) \text{ subject to } sp(h^*(M)) \leq H.$$ 

Compute $\tilde{\pi}_k = \pi(M_k)$.

for $t = t_k$ to $t_k + T_k - 1$

Apply action $a_t = \pi_k(s_t)$

Observe new state $s_{t+1}$

Observe new rewards $r_{t+1} = (r_{t+1,1}, \ldots, r_{t+1,l})$

$\mathcal{H} = \mathcal{H} \cup \{(s_t, a_t, r_{t+1,1}, s_{t+1})\}$

$t \leftarrow t + 1$

end for

$k \leftarrow k + 1$.

$T_k \leftarrow \lceil k/L \rceil; t_k \leftarrow t + 1.$

until $t_k > T$

G. DORL on regular MDPs

We compared DORL with UCRL2 and PSRL on the standard example RiverSwim (Filippi et al., 2010). PSRL outperforms DORL and UCRL2 significantly.

![Figure 2. Regret of DORL and UCRL on RiverSwim problem with a size of 6.](image)