Function spaces and capacity related to a Sublinear Expectation: application to G-Brownian Motion Paths

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Abstract In this paper we give some basic and important properties of several typical Banach spaces of functions of $G$-Brownian motion paths induced by a sublinear expectation–$G$-expectation. Many results can be also applied to more general situations. A generalized version of Kolmogorov’s criterion for continuous modification of a stochastic process is also obtained. The results can be applied in continuous time dynamic and coherent risk measures in finance in particular for path-dependence risky positions under situations of volatility model uncertainty.

Keywords Capacity · Sublinear expectation · $G$-Expectation · $G$-Brownian motion · Dynamical programming principle

1 Introduction

How to measure the risk of financial losses in a financial market is still a challenging problem. In a seminal paper [1] a basic notion of coherent risk measures...
was introduced: Let $\mathcal{H}$ be a linear space of financial losses, considered as a space of random variables. A coherent risk measure $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is a real valued (monetary value) functional with the properties of constant preserving (called cash invariance), monotonicity, convexity and positive homogeneity. Namely, a coherent risk measure is in fact a sublinear expectation $\mathbb{E}$ defined on $\mathcal{H}$ (see Theorem 9 or Definition 37). It was proved that a sublinear expectation has the following representation (see [1, 10, 18]): There exists a family of linear expectations $\{\mathbb{E}_\theta\}_{\theta \in \Theta}$ such that

$$\mathbb{E}[X] = \max_{\theta \in \Theta} \mathbb{E}_\theta[X], \quad X \in \mathcal{H}.$$ 

The meaning in economics of this representation is that the risk measure $\mathbb{E}$ is in fact the robust super-expectation over the family of uncertainty of linear expectations $\{\mathbb{E}_\theta\}_{\theta \in \Theta}$.

As an example, let us consider a typical situation in a financial market where the price of a stock satisfies the equation

$$dS_t = S_t(\gamma^\theta_t dt + \sigma^\theta_t dW_t),$$

where $W$ is a standard Brownian motion, and $(\gamma^\theta_t, \sigma^\theta_t)_{t \geq 0}$, $\theta \in \Theta$, are unknown processes parameterized by $\theta \in \Theta$. A financial loss $X$ is formulated as a given random variable depending on the path of $S$ or, equivalently, on the path of $B^\theta = \int_0^t (\gamma^\theta_s ds + \sigma^\theta_s dW_s)$. For each fixed $\theta \in \Theta$, let $P_\theta$ be the probability measure on the space of continuous paths $(\Omega, \mathcal{F}) = (C(0, \infty), \mathcal{B}(C(0, \infty)))$ induced by $(B^\theta_t)_{t \geq 0}$, and $\mathbb{E}_\theta$ the corresponding expectation. The risk measure of $X$ under the above uncertainty is formulated as

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} \mathbb{E}_\theta[X].$$

A typical situation is that each probability $P_\theta$ is absolutely continuous with respect to the ‘reference measure’ $P_0$ corresponding to the case when $B^\theta_t \equiv W_t$. In this case the uncertainty can only comes from $\gamma^\theta$ and thus is called drift uncertainty. Chen and Epstein [6] proposed to use $g$-expectation, (small $g$, introduced in [23]) for a robust valuation: $\mathbb{E}_g[X] = \sup_{\theta \in \Theta} \mathbb{E}_\theta[X]$. It was also proved (see [15, 6]) that this corresponds to the case where the uncertain drift has the form $\{\gamma^\theta_s \in K, s \geq 0\}$, for some $K \subset \mathbb{R}$. [20, 19] proposed to use $g$-expectation as a time consistent risk measure. [11] proved that any coherent and time consistent risk measure absolutely continuous with respect to $P_0$ can be approximated by a $g$-expectation.

But in finance there is an important situation called ‘volatility uncertainty’ in which the uncertainty comes from the “volatility coefficient” $\{\sigma^\theta, \theta \in \Theta\}$. A major difficulty here is that the probabilities $\{P_\theta\}_{\theta \in \Theta}$ are mutually singular and thus the corresponding $\mathbb{E}$ cannot be dominated by any $g$-expectation. This type of uncertainty was initially studied by Avellaneda, Levy and Paras [3] and Lyons [22], for the superhedging of European options with payoffs depending only on the terminal value $B^\theta_T$, the discrete-time case has been also studied in [10]. But for the superhedging of a general path-dependence option, the difficulty was dramatically increased. This situation was studied independently by [24] and [14] with very different approaches. Motivated by the problem of coherent
risk measures under the volatility uncertainty, [26] introduced a sublinear expectation on a well-defined space $L^1_G(\Omega)$ under which the increments of the canonical process $(B_t)_{t \geq 0}$ are zero-mean, independent and stationary and can be proved to be ‘$G$-normally distributed’ (see [29]). This type of processes is called ‘$G$-Brownian motion’ and the corresponding sublinear expectation $\mathbb{E}[\cdot]$ is called ‘$G$-expectation’ (capital $G$). Recently, we have discovered a strong link between the framework of [24], [25], [26], [27] and the one introduced in [14].

A well-known and fundamentally important fact in probability theory is that the linear space $L^1_B(\Omega)$ coincides with the $E_{P_0}[|\cdot|]$-norm completion of the space of bounded and continuous functions $C_b(\Omega)$ or bounded and $\mathcal{F}$-measurable functions $B_b(\Omega)$, or even smaller one, the space $L_{ip}(\Omega) \subset C_b(\Omega)$ of bounded and Lipschitz cylinder functions (see Section 3 for its definition).

Similar problems arise in the theory of $G$-Brownian motion: The space $L^1_G(\Omega)$ is defined as the $\mathbb{E}[\cdot]|\cdot|^{-}\text{n}orm completion of $L_{ip}(\Omega)$. Can we prove that each element $X \in L^1_G(\Omega)$ can be identified as an element of $L^1$, the space of all $\mathcal{F}$-measurable random variables $X$ such that $\mathbb{E}[|X|] < \infty$? Furthermore, what is the relation between the $\mathbb{E}[\cdot]|\cdot|^{-}\text{n}orm completions of $B_b(\Omega)$, $C_b(\Omega)$ and $L_{ip}(\Omega)$?

In this paper we give an affirmative answer to the first problem. For the second problem, we will prove that, in fact, the $\mathbb{E}[\cdot]|\cdot|^{-}\text{n}orm completions of $L_{ip}(\Omega)$ and $C_b(\Omega)$ are the same, but they are strict subspace of the $\mathbb{E}[\cdot]|\cdot|^{-}\text{n}orm$-completion of $B_b(\Omega)$.

In this paper a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ is constructed so that the $G$ expectation is the upper expectation of $\mathcal{P}$, i.e.:

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for each } X \in L_{ip}(\Omega).$$

Following [21], we define the corresponding regular Choquet capacity:

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).$$

We then can prove that each element $X \in L^1_G(\Omega)$ has a $c$-quasi continuous version on $\Omega$. Moreover we have $C_b(\Omega) \subset L^1_G(\Omega) \subset L^1$ (see also [17], [14] for a different approach).

This paper is organized as follows: in Section 2, we use a family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ to define a sublinear expectation as the upper expectation of $\mathcal{P}$, as well as the related capacity, especially, we use a weakly compact family of probability measures to define the corresponding regular sublinear expectation and regular capacity. Here $\Omega$ is assumed to be a general complete separable metric space of which $C(0, \infty)$ and $D(0, \infty)$ (the path space of càdlàg processes) are typical examples. Each element of $\mathbb{E}[\cdot]|P|^{1/p}$-completion of $C_b(\Omega)$ is proved to has a quasi-continuous version. Concrete characterizations of completions of different function spaces are given. As a by-product, we obtain a generalized version of Kolmogorov’s criterion for continuous modification of a stochastic process. In Section 3, we let $\Omega = C^d[0, \infty)$ and use a method of stochastic control to prove that $G$-expectation is a upper expectation associated to a weakly compact family $\mathcal{P}$ and then apply the results of Section 2 to the $G$-expectation and the corresponding functional spaces.
2 Integration theory associated to an upper probability

Let $\Omega$ be a complete separable metric space equipped with the distance $d$, $\mathcal{B}(\Omega)$ the Borel $\sigma$-algebra of $\Omega$ and $\mathcal{M}$ the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$.

- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions;
- $B_b(\Omega)$: all bounded functions in $L^0(\Omega)$;
- $C_b(\Omega)$: all continuous functions in $B_b(\Omega)$.

All along this section, we consider a given subset $P \subseteq \mathcal{M}$.

2.1 Capacity associated to $P$

We denote $c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega)$.

One can easily verify the following theorem.

**Theorem 1.** The set function $c(\cdot)$ is a Choquet capacity, i.e. (see [12, 13]),

1. $0 \leq c(A) \leq 1, \quad \forall A \subset \Omega$.
2. If $A \subset B$, then $c(A) \leq c(B)$.
3. If $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{B}(\Omega)$, then $c(\bigcup A_n) \leq \sum c(A_n)$.
4. If $(A_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{B}(\Omega)$: $A_n \uparrow A = \bigcup A_n$, then $c(\bigcup A_n) = \lim_{n \to \infty} c(A_n)$.

Furthermore, we have

**Theorem 2.** For each $A \in \mathcal{B}(\Omega)$, we have

$$c(A) = \sup \{ c(K) : K \text{ compact } K \subset A \}.$$  

**Proof.** It is simply because

$$c(A) = \sup_{P \in \mathcal{P}} \sup_{K \text{ compact } K \subset A} P(K) = \sup_{K \text{ compact } P \in \mathcal{P}} P(K) = \sup_{K \text{ compact } K \subset A} c(K).$$

**Definition 3.** We use the standard capacity-related vocabulary: a set $A$ is polar if $c(A) = 0$ and a property holds “quasi-surely” (q.s.) if it holds outside a polar set.
Remark 4. In other words, $A \in \mathcal{B}(\Omega)$ is polar if and only if $P(A) = 0$ for any $P \in \mathcal{P}$.

We also have in a trivial way a Borel-Cantelli Lemma.

**Lemma 5.** Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of Borel sets such that

$$\sum_{n=1}^{\infty} c(A_n) < \infty.$$  

Then $\limsup_{n \to \infty} A_n$ is polar.

**Proof.** Applying the Borel-Cantelli Lemma under each probability $P \in \mathcal{P}$. $\square$

The following theorem is Prohorov’s theorem.

**Theorem 6.** $\mathcal{P}$ is relatively compact if and only if for each $\varepsilon > 0$, there exists a compact set $K$ such that $c(K^c) < \varepsilon$.

The following two lemmas can be found in [21].

**Lemma 7.** $\mathcal{P}$ is relatively compact if and only if for each sequence of closed sets $F_n \downarrow \emptyset$, we have $c(F_n) \downarrow 0$.

**Proof.** We outline the proof for the convenience of readers.

“$\Rightarrow$” part: It follows from Theorem 6 that for each fixed $\varepsilon > 0$, there exists a compact set $K$ such that $c(K^c) < \varepsilon$. Note that $F_n \cap K \downarrow \emptyset$, then there exists an $N > 0$ such that $F_n \cap K = \emptyset$ for $n \geq N$, which implies $\lim_{n \to \infty} c(F_n) < \varepsilon$. Since $\varepsilon$ can be arbitrarily small, we obtain $c(F_n) \downarrow 0$.

“$\Leftarrow$” part: For each $\varepsilon > 0$, let $(A^k_i)_{i=1}^{\infty}$ be a sequence of open balls of radius $1/k$ covering $\Omega$. Observe that $(\bigcup_{i=1}^{n_k} A^k_i)^c \downarrow \emptyset$, then there exists an $n_k$ such that $c((\bigcup_{i=1}^{n_k} A^k_i)^c) < \varepsilon 2^{-k}$. Set $K = \cap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A^k_i$. It is easy to check that $K$ is compact and $c(K^c) < \varepsilon$. Thus by Theorem 6 $\mathcal{P}$ is relatively compact. $\square$

**Lemma 8.** Let $\mathcal{P}$ be weakly compact. Then for each sequence of closed sets $F_n \downarrow F$, we have $c(F_n) \downarrow c(F)$.

**Proof.** We outline the proof for the convenience of readers. For each fixed $\varepsilon > 0$, by the definition of $c(F_n)$, there exists a $P_n \in \mathcal{P}$ such that $P_n(F_n) \geq c(F_n) - \varepsilon$. Since $\mathcal{P}$ is weakly compact, there exist $P_{n_k}$ and $P \in \mathcal{P}$ such that $P_{n_k}$ converge weakly to $P$. Thus

$$P(F_m) \geq \limsup_{k \to \infty} P_{n_k}(F_m) \geq \limsup_{k \to \infty} P_{n_k}(F_{n_k}) \geq \lim_{n \to \infty} c(F_n) - \varepsilon.$$  

Letting $m \to \infty$, we get $P(F) \geq \lim_{n \to \infty} c(F_n) - \varepsilon$, which yields $c(F_n) \downarrow c(F)$. $\square$

Following [21] (see also [9][18]) the upper expectation of $\mathcal{P}$ is defined as follows: for each $X \in L^0(\Omega)$ such that $E_P[X]$ exists for each $P \in \mathcal{P}$,

$$\mathbb{E}[X] = \mathbb{E}^\mathcal{P}[X] := \sup_{P \in \mathcal{P}} E_P[X].$$  

It is easy to verify
Theorem 9. The upper expectation $\mathbb{E}[\cdot]$ of the family $\mathcal{P}$ is a sublinear expectation on $B_0(\Omega)$ as well as on $C_b(\Omega)$, i.e.,

1. for all $X, Y$ in $B_0(\Omega)$, $X \geq Y \implies \mathbb{E}[X] \geq \mathbb{E}[Y]$.
2. for all $X, Y$ in $B_0(\Omega)$, $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
3. for all $\lambda \geq 0$, $X \in B_0(\Omega)$, $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.
4. for all $c \in \mathbb{R}$, $X \in B_0(\Omega)$, $\mathbb{E}[X + c] = \mathbb{E}[X] + c$.

Moreover, it is also easy to check

Theorem 10. We have

1. Let $\mathbb{E}[X_n]$ and $\mathbb{E}[\sum_{n=1}^{\infty} X_n]$ be finite. Then $\mathbb{E}[\sum_{n=1}^{\infty} X_n] \leq \sum_{n=1}^{\infty} \mathbb{E}[X_n]$.
2. Let $X_n \uparrow X$ and $\mathbb{E}[X_n], \mathbb{E}[X]$ be finite. Then $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$.

Definition 11. The functional $\mathbb{E}[\cdot]$ is said to be regular if for each $\{X_n\}_{n=1}^{\infty}$ in $C_b(\Omega)$ such that $X_n \downarrow 0$ on $\Omega$, we have $\mathbb{E}[X_n] \downarrow 0$.

Similar to Lemma 7 we have:

Theorem 12. $\mathbb{E}[\cdot]$ is regular if and only if $\mathcal{P}$ is relatively compact.

Proof. “$\Rightarrow$” part: For each sequence of closed subsets $F_n \downarrow \emptyset$ such that $F_n, n = 1, 2, \cdots$, are non-empty (otherwise the proof is trivial), there exists $\{g_n\}_{n=1}^{\infty} \subset C_b(\Omega)$ satisfying

$$0 \leq g_n \leq 1, \quad g_n = 1 \text{ on } F_n \text{ and } g_n = 0 \text{ on } \{\omega \in \Omega : d(\omega, F_n) \geq \frac{1}{n}\}.$$ 

We set $f_n = \bigwedge_{i=1}^{n} g_i$, it is clear that $f_n \in C_b(\Omega)$ and $1_{F_n} \leq f_n \downarrow 0$. $\mathbb{E}[\cdot]$ is regular implies $\mathbb{E}[f_n] \downarrow 0$ and thus $c(F_n) \downarrow 0$. It follows from Lemma 7 that $\mathcal{P}$ is relatively compact.

“$\Leftarrow$” part: For each $\{X_n\}_{n=1}^{\infty} \subset C_b(\Omega)$ such that $X_n \downarrow 0$, we have

$$\mathbb{E}[X_n] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X_n] = \sup_{P \in \mathcal{P}} \int_0^\infty P(\{X_n \geq t\})dt \leq \int_0^\infty c(\{X_n \geq t\})dt.$$ 

For each fixed $t > 0$, $\{X_n \geq t\}$ is a closed subset and $\{X_n \geq t\} \downarrow \emptyset$ as $n \uparrow \infty$. By Lemma 7 $c(\{X_n \geq t\}) \downarrow 0$ and thus $\int_0^\infty c(\{X_n \geq t\})dt \downarrow 0$. Consequently $\mathbb{E}[X_n] \downarrow 0$. \hfill $\square$

### 2.2 Functional spaces

We set, for $p > 0$,

- $\mathcal{L}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[|X|^p] < \infty\}$;
- $\mathcal{N}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = 0\}$;
• $\mathcal{N} := \{X \in L^0(\Omega) : X = 0, \text{c-q.s.}\}$.

It is seen that $\mathcal{L}^p$ and $\mathcal{N}^p$ are linear spaces and $\mathcal{N}^p = \mathcal{N}$, for each $p > 0$. We denote $\mathbb{L}^p := \mathcal{L}^p/\mathcal{N}$. As usual, we do not take care about the distinction between classes and their representatives.

**Lemma 13.** Let $X \in \mathbb{L}^p$. Then for each $\alpha > 0$

$$c(\{|X| > \alpha\}) \leq \frac{E[|X|^p]}{\alpha^p}. $$

**Proof.** Just apply Markov inequality under each $P \in \mathcal{P}$.\hfill $\Box$

Similar to the classical results, we get the following proposition and the proof is omitted which is similar to the classical arguments.

**Proposition 14.** We have

1. For each $p \geq 1$, $\mathbb{L}^p$ is a Banach space under the norm $\|X\|_p := (E[|X|^p])^{\frac{1}{p}}$.

2. For each $p < 1$, $\mathbb{L}^p$ is a complete metric space under the distance $d(X,Y) := E[|X - Y|^p]$.

We set $\mathcal{L}^\infty := \{X \in L^0(\Omega) : \exists$ a constant $M$, s.t. $|X| \leq M$, q.s.$\}$; $\mathbb{L}^\infty := \mathcal{L}^\infty/\mathcal{N}$.

**Proposition 15.** Under the norm $\|X\|_\infty := \inf \{M \geq 0 : |X| \leq M, \text{ q.s.}\}$, $\mathbb{L}^\infty$ is a Banach space.

**Proof.** From $\{|X| > \|X\|_\infty\} = \bigcup_{n=1}^{\infty} \{|X| \geq \|X\|_\infty + \frac{1}{n}\}$ we know that $|X| \leq \|X\|_\infty$, q.s., then it is easy to check that $\|\cdot\|_\infty$ is a norm. The proof of the completeness of $\mathbb{L}^\infty$ is similar to the classical result.\hfill $\Box$

With respect to the distance defined on $\mathbb{L}^p$, $p > 0$, we denote by

- $\mathbb{L}^p_b$ the completion of $B_b(\Omega)$.
- $\mathbb{L}^p_c$ the completion of $C_b(\Omega)$.

By Proposition 14 we have

$$\mathbb{L}^p_c \subset \mathbb{L}^p_b \subset \mathbb{L}^p, \quad p > 0.$$ 

The following Proposition is obvious and the proof is left to the reader.
Proposition 16. We have

1. Let \( p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \). Then \( X \in \mathbb{L}^p \) and \( Y \in \mathbb{L}^q \) implies
   \[
   XY \in \mathbb{L}^1 \quad \text{and} \quad \mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} (\mathbb{E}[|Y|^q])^{\frac{1}{q}};
   \]

Moreover \( X \in \mathbb{L}^p_c \) and \( Y \in \mathbb{L}^q_c \) implies \( XY \in \mathbb{L}^1_c \);

2. \( \mathbb{L}^{p_1} \subset \mathbb{L}^{p_2}, \mathbb{L}^{q_1}_b \subset \mathbb{L}^{q_2}_b, \mathbb{L}^{p_1}_c \subset \mathbb{L}^{p_2}_c, 0 < p_2 \leq p_1 \leq \infty \);

3. \( \|X\|_p \uparrow \|X\|_\infty \), for each \( X \in \mathbb{L}^\infty \).

Proposition 17. Let \( p \in (0, \infty) \) and \( (X_n) \) be a sequence in \( \mathbb{L}^p \) which converges to \( X \) in \( \mathbb{L}^p \). Then there exists a subsequence \( (X_{n_k}) \) which converges to \( X \) quasi-surely in the sense that it converges to \( X \) outside a polar set.

Proof. Let us assume \( p \in (0, \infty) \), the case \( p = \infty \) is obvious since the convergence in \( \mathbb{L}^\infty \) implies the convergence in \( \mathbb{L}^p \) for all \( p \).

One can extract a subsequence \( (X_{n_k}) \) such that
\[
\mathbb{E}(|X - X_{n_k}|^p) \leq 1/k^{p+2}, \quad k \in \mathbb{N}.
\]

We set for all \( k \)
\[
A_k = \{|X - X_{n_k}| > 1/k\},
\]
then as a consequence of the Markov property (Lemma 13) and the Borel-Cantelli Lemma 5, \( c \lim_{k \to \infty} A_k = 0 \). As it is clear that on \( (\lim_{k \to \infty} A_k)^c \), \( (X_{n_k}) \) converges to \( X \), the proposition is proved.

We now give a description of \( \mathbb{L}^p_b \).

Proposition 18. For each \( p > 0 \),
\[
\mathbb{L}^p_b = \{X \in \mathbb{L}^p : \lim_{n \to \infty} \mathbb{E}[|X|^p 1_{(|X| > n)}] = 0\}.
\]

Proof. We denote \( J_p = \{X \in \mathbb{L}^p : \lim_{n \to \infty} \mathbb{E}[|X|^p 1_{(|X| > n)}] = 0\} \). For each \( X \in J_p \) let \( X_n = (X \land n) \lor (-n) \in \mathbb{B}_b(\Omega) \). We have
\[
\mathbb{E}[|X - X_n|^p] \leq \mathbb{E}[|X|^p 1_{(|X| > n)}] \to 0, \text{ as } n \to \infty.
\]

Thus \( X \in \mathbb{L}^p_b \).

On the other hand, for each \( X \in \mathbb{L}^p_b \), we can find a sequence \( \{Y_n\}_{n=1}^{\infty} \) in \( \mathbb{B}_b(\Omega) \) such that \( \mathbb{E}[|X - Y_n|^p] \to 0 \). Let \( y_n = \sup_{\omega \in \Omega} |Y_n(\omega)| \) and \( X_n = (X \land y_n) \lor (-y_n) \).

Since \( |X - X_n| \leq |X - Y_n| \), we have \( \mathbb{E}[|X - X_n|^p] \to 0 \). This clearly implies that for any sequence \( (\alpha_n) \) tending to \( \infty \), \( \lim_{n \to \infty} \mathbb{E}[|X - (X \land \alpha_n) \lor (-\alpha_n)|^p] = 0 \).

Now we have, for all \( n \in \mathbb{N} \),
\[
\mathbb{E}[|X|^p 1_{(|X| > n)}] = \mathbb{E}[(|X| - n + n)^p 1_{(|X| > n)}] \leq (1 + 2^{p-1}) \mathbb{E}[(|X| - n)^p 1_{(|X| > n)}] + n^p c(|X| > n).
\]
The first term of the right hand side tends to 0 since
\[
\mathbb{E}[(|X| - n)^p 1_{\{|X| > n\}}] = \mathbb{E}[(|X| - (X \wedge n) \vee (-n))^p] \to 0.
\]
For the second term, since
\[
\frac{n^p}{2^p} 1_{\{|X| > n\}} \leq (|X| - \frac{n}{2^p})^p 1_{\{|X| > n\}} \leq (|X| - \frac{n}{2^p} 1_{\{|X| > \frac{n}{2^p}\}}),
\]
we have
\[
\frac{n^p}{2^p} c(|X| > n) = \frac{n^p}{2^p} \mathbb{E}[1_{\{|X| > n\}}] \leq \mathbb{E}[(|X| - \frac{n}{2^p})^p 1_{\{|X| > \frac{n}{2^p}\}}] \to 0.
\]
Consequently \( X \in J_p \).

**Proposition 19.** Let \( X \in \mathbb{L}^1_p \). Then for each \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that for all \( A \in \mathcal{B}(\Omega) \) with \( c(A) \leq \delta \), we have \( \mathbb{E}||X|_{\mathbb{A}}| \leq \varepsilon \).

**Proof.** For each \( \varepsilon > 0 \), by Proposition [18] there exists an \( N > 0 \) such that \( \mathbb{E}[|X|_{\mathbb{A}}|_{\{|X| > N\}}] \leq \frac{\varepsilon}{2} \). Take \( \delta = \frac{\varepsilon}{2N} \). Then for a subset \( A \in \mathcal{B}(\Omega) \) with \( c(A) \leq \delta \), we have
\[
\mathbb{E}[|X|_{\mathbb{A}}|] \leq \mathbb{E}[|X|_{\mathbb{A}}|_{\{|X| > N\}}] + \mathbb{E}[|X|_{\mathbb{A}}|_{\{|X| \leq N\}}] \\
\leq \mathbb{E}[|X|_{\mathbb{A}}|_{\{|X| > N\}}] + Nc(A) \leq \varepsilon.
\]
\[\square\]

It is important to note that not every element in \( \mathbb{L}^p \) satisfies the condition \( \lim_{n \to \infty} \mathbb{E}[|X|^p 1_{\{|X| > n\}}] = 0 \). We give the following two counterexamples to show that \( \mathbb{L}^1 \) and \( \mathbb{L}^1_0 \) are different spaces even under the case that \( \mathcal{P} \) is weakly compact.

**Example 20.** Let \( \Omega = \mathbb{N} \), \( \mathcal{P} = \{P_n : n \in \mathbb{N}\} \) where \( P_1(\{1\}) = 1 \) and \( P_n(\{1\}) = 1 - \frac{1}{n} \), \( P_n(\{n\}) = \frac{1}{n} \), for \( n = 2, 3, \ldots \). \( \mathcal{P} \) is weakly compact. We consider a function \( X \) on \( \mathbb{N} \) defined by \( X(n) = n, n \in \mathbb{N} \). We have \( \mathbb{E}[|X|] = 2 \) but \( \mathbb{E}[|X|_{\mathbb{A}}|] = 1 \not\to 0 \). In this case, \( X \in \mathbb{L}^1 \) but \( X \not\in \mathbb{L}^1_0 \).

**Example 21.** Let \( \Omega = \mathbb{N} \), \( \mathcal{P} = \{P_n : n \in \mathbb{N}\} \) where \( P_1(\{1\}) = 1 \) and \( P_n(\{1\}) = 1 - \frac{1}{n} \), \( P_n(\{kn\}) = \frac{1}{n} \), \( k = 1, 2, \ldots, n, \) for \( n = 2, 3, \ldots \). \( \mathcal{P} \) is weakly compact. We consider a function \( X \) on \( \mathbb{N} \) defined by \( X(n) = n, n \in \mathbb{N} \). We have \( \mathbb{E}[|X|] = \frac{n^2}{2n} \) and \( n\mathbb{E}[1_{\{|X| \geq n\}}] = \frac{1}{n} \not\to 0 \), but \( \mathbb{E}[|X|_{\mathbb{A}}|_{\{|X| \geq n\}}] = \frac{1}{2} + \frac{1}{2n} \not\to 0 \). In this case, \( X \) is in \( \mathbb{L}^1 \), continuous and \( n\mathbb{E}[1_{\{|X| \geq n\}}] \to 0 \), but it is not in \( \mathbb{L}^1_0 \).

### 2.3 Properties of elements in \( \mathbb{L}^1_p \)

**Definition 22.** A mapping \( X \) on \( \Omega \) with values in a topological space is said to be quasi-continuous (q.c.) if
\[
\forall \varepsilon > 0, \text{ there exists an open set } O \text{ with } c(O) < \varepsilon \text{ such that } X|_O \text{ is continuous.}
\]
**Definition 23.** We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s..

**Proposition 24.** Let $p > 0$. Then each element in $L_p^c$ has a quasi-continuous version.

**Proof.** Let $(X_n)$ be a Cauchy sequence in $C_b(\Omega)$ for the distance on $\mathbb{L}^p$. Let us choose a subsequence $(X_{n_k})_{k \geq 1}$ such that
\[
\mathbb{E}[|X_{n_{k+1}} - X_{n_k}|^p] \leq 2^{-2k}, \quad \forall k \geq 1,
\]
and set for all $k$,
\[
A_k = \bigcup_{i=k}^{\infty}\{|X_{n_{i+1}} - X_{n_i}| > 2^{-i/p}\}.
\]
Thanks to the subadditivity property and the Markov inequality, we have
\[
c(A_k) \leq \sum_{i=k}^{\infty}\mathbb{E}[|X_{n_{i+1}} - X_{n_i}|^p] \leq \sum_{i=k}^{\infty}2^{-i} = 2^{-k+1}.
\]
As a consequence, $\lim_{k \to \infty} c(A_k) = 0$, so the Borel set $A = \bigcap_{k=1}^{\infty} A_k$ is polar.

As each $X_{n_k}$ is continuous, for all $k \geq 1$, $A_k$ is an open set. Moreover, for all $k$, $(X_{n_k})$ converges uniformly on $A_k^c$ so that the limit is continuous on each $A_k^c$.

This yields the result. \(\square\)

The following theorem gives a concrete characterization of the space $\mathbb{L}_c^p$.

**Theorem 25.** For each $p > 0$,
\[
\mathbb{L}_c^p = \{X \in \mathbb{L}^p : X has a quasi-continuous version, \lim_{n \to \infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X| > n\}}] = 0\}.
\]

**Proof.** We denote
\[
J_p = \{X \in \mathbb{L}^p : X has a quasi-continuous version, \lim_{n \to \infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X| > n\}}] = 0\}.
\]

Let $X \in \mathbb{L}_c^p$, we know by Proposition 24 that $X$ has a quasi-continuous version. Since $X \in \mathbb{L}_c^p$, we have by Proposition 18 that $\lim_{n \to \infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X| > n\}}] = 0$. Thus $X \in J_p$.

On the other hand, let $X \in J_p$ be quasi-continuous. Define $Y_n = (X \wedge n) \vee (-n)$ for all $n \in \mathbb{N}$. As $\mathbb{E}[|X|^p \mathbf{1}_{\{|X| > n\}}] \to 0$, we have $\mathbb{E}[|X - Y_n|^p] \to 0$.

Moreover, for all $n \in \mathbb{N}$, as $Y_n$ is quasi-continuous, there exists a closed set $F_n$ such that $c(F_n) < \frac{1}{n^{p+1}}$ and $Y_n$ is continuous on $F_n$. It follows from Tietze’s extension theorem that there exists $Z_n \in C_b(\Omega)$ such that $|Z_n| \leq n$ and $Z_n = Y_n$ on $F_n$.

We then have
\[
\mathbb{E}[|Y_n - Z_n|^p] \leq (2n)^p c(F_n) \leq \frac{(2n)^p}{n^{p+1}}.
\]
So $\mathbb{E}[|X - Z_n|^p] \leq (1 + 2^{p-1})(\mathbb{E}[|X - Y_n|^p] + \mathbb{E}[|Y_n - Z_n|^p]) \to 0$, and $X \in \mathbb{L}_c^p$. \(\square\)
We give the following example to show that $L_c^p$ is different from $L_b^p$ even under the case that $\mathcal{P}$ is weakly compact.

**Example 26.** Let $\Omega = [0,1]$, $\mathcal{P} = \{\delta_x : x \in [0,1]\}$ is weakly compact. It is seen that $L_b^p = C_b(\Omega)$ which is different from $L_c^p$.

We denote $L_c^\infty := \{X \in L^\infty : X$ has a quasi-continuous version\}, we have

**Proposition 27.** $L_c^\infty$ is a closed linear subspace of $L^\infty$.

**Proof.** For each Cauchy sequence $\{X_n\}_{n=1}^\infty$ of $L^\infty$ under $\|\cdot\|_\infty$, we can find a subsequence $\{X_{n_i}\}_{i=1}^\infty$ such that $\|X_{n_{i+1}} - X_{n_i}\|_\infty \leq 2^{-i}$. We may further assume that each $X_n$ is quasi-continuous. Then it is easy to prove that for each $\varepsilon > 0$, there exists an open set $G$ such that $c(G) < \varepsilon$ and $|X_{n_{i+1}} - X_{n_i}| \leq 2^{-i}$ for all $i \geq 1$ on $G^c$, which implies that the limit belongs to $L_c^\infty$. \(\square\)

As an application of Theorem 25, we can easily get the following results.

**Proposition 28.** Assume that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version and that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying $\lim_{t \to \infty} \frac{t^p}{f(t)} = \infty$ and $E[f(|X|)] < \infty$. Then $X \in L_c^p$.

**Proof.** For each $\varepsilon > 0$, there exists an $N > 0$ such that $\frac{t^p}{f(t)} \geq \frac{1}{\varepsilon}$, for all $t \geq N$. Thus

$$E[|X|^p 1_{\{|X| > N\}}] \leq \varepsilon E[f(|X|) 1_{\{|X| > N\}}] \leq \varepsilon E[f(|X|)].$$

Hence $\lim_{N \to \infty} E[|X|^p 1_{\{|X| > N\}}] = 0$. From Theorem 25 we infer $X \in L_c^p$. \(\square\)

**Lemma 29.** Let $\{P_n\}_{n=1}^\infty \subset \mathcal{P}$ converge weakly to $P \in \mathcal{P}$. Then for each $X \in L^\infty_c$, we have $E_{P_n}[X] \to E_P[X]$.

**Proof.** We may assume that $X$ is quasi-continuous, otherwise we can consider its quasi-continuous version which does not change the value $E_Q$ for each $Q \in \mathcal{P}$. For each $\varepsilon > 0$, there exists an $N > 0$ such that $E[|X|^p 1_{\{|X| > N\}}] < \frac{\varepsilon}{2}$. Set $X_N = (X \wedge N) \vee (-N)$. We can find an open subset $G$ such that $c(G) < \frac{\varepsilon}{3N}$ and $X_N$ is continuous on $G^c$. By Tietze’s extension theorem, there exists $Y \in C_b(\Omega)$ such that $|Y| \leq N$ and $Y = X_N$ on $G^c$. Obviously, for each $Q \in \mathcal{P},$

$$|E_Q[X] - E_Q[Y]| \leq E_Q[|X - X_N|] + E_Q[|X_N - Y|] \leq \frac{\varepsilon}{2} + 2N\frac{\varepsilon}{4N} = \varepsilon.$$

It then follows that

$$\limsup_{n \to \infty} E_{P_n}[X] \leq \lim_{n \to \infty} E_{P_n}[Y] + \varepsilon = E_P[Y] + \varepsilon \leq E_P[X] + 2\varepsilon,$$

and similarly $\liminf_{n \to \infty} E_{P_n}[X] \geq E_P[X] - 2\varepsilon$. Since $\varepsilon$ can be arbitrarily small, we then have $E_{P_n}[X] \to E_P[X]$. \(\square\)

**Remark 30.** For continuous $X$, the above lemma is Lemma 3.8.7 in [4].
Theorem 36. Let $X$ be a process indexed by $I$. In the above definition, quasi-modification is also called modification in some papers.

Remark 32. It is important to note that $X$ does not necessarily belong to $L_1^1$.

Proof. For the case $E[X] > -\infty$, if there exists a $\delta > 0$ such that $E[X_n] > E[X] + \delta$, $n = 1, 2, \ldots$, we then can find a $P_n \in \mathcal{P}$ such that $E_{P_n}[X_n] > E[X] + \delta - \frac{1}{n}$, $n = 1, 2, \ldots$. Since $\mathcal{P}$ is weakly compact, we then can find a subsequence $\{P_{n_i}\}_{i=1}^{\infty}$ that converges weakly to some $P \in \mathcal{P}$. From which it follows that

$$E_P[X] = \lim_{j \to \infty} E_{P_n}[X_{n_j}] \geq \limsup_{j \to \infty} E_{P_{n_j}}[X_{n_j}] \geq \limsup_{j \to \infty} \{E[X] + \delta - \frac{1}{n_j}\} = E[X] + \delta, \quad i = 1, 2, \ldots.$$  

Thus $E_P[X] \geq E[X] + \delta$. This contradicts the definition of $E[\cdot]$.

We immediately have the following corollary.

Corollary 33. Let $\mathcal{P}$ be weakly compact and let $\{X_n\}_{n=1}^{\infty}$ be a sequence in $L_1^1$ decreasingly converging to 0 q.s.. Then $E[X_n] \downarrow 0$.

2.4 Kolmogorov’s criterion

Definition 34. Let $I$ be a set of indices, $(X_t)_{t \in I}$ and $(Y_t)_{t \in I}$ be two processes indexed by $I$. We say that $Y$ is a quasi-modification of $X$ if for all $t \in I$, $X_t = Y_t$ q.s..

Remark 35. In the above definition, quasi-modification is also called modification in some papers.

We now give a Kolmogorov criterion for a process indexed by $\mathbb{R}^d$ with $d \in \mathbb{N}$.

Theorem 36. Let $p > 0$ and $(X_t)_{t \in [0,1]^d}$ be a process such that for all $t \in [0,1]^d$, $X_t$ belongs to $L^p$. Assume that there exist positive constants $c$ and $\varepsilon$ such that

$$E[|X_t - X_s|^p] \leq c|t - s|^{d+\varepsilon}.$$ 

Then $X$ admits a modification $\tilde{X}$ such that

$$E\left[\left(\sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\alpha}\right)^p\right] < \infty,$$ 

for every $\alpha \in [0, \varepsilon/p]$. As a consequence, paths of $\tilde{X}$ are quasi-surely Hölder continuous of order $\alpha$ for every $\alpha < \varepsilon/p$ in the sense that there exists a Borel set $N$ of capacity 0 such that for all $w \in N^c$, the map $t \to \tilde{X}(w)$ is Hölder continuous of order $\alpha$ for every $\alpha < \varepsilon/p$. Moreover, if $X_t \in L_1^p$ for each $t$, then we also have $\tilde{X}_t \in L_1^p$.  

12
Proof. Let $D$ be the set of dyadic points in $[0, 1]^d$:

$$D = \left\{ \left( \frac{i_1}{2^n}, \ldots, \frac{i_d}{2^n} \right); \ n \in \mathbb{N}, i_1, \ldots, i_d \in \{0, 1, \ldots, 2^n\} \right\}.$$ 

Let $\alpha \in [0, \varepsilon/p)$. We set

$$M = \sup_{s, t \in D, s \neq t} \frac{|X_t - X_s|}{|t - s|^\alpha}.$$ 

Thanks to the classical Kolmogorov’s criterion (see Revuz-Yor [31]), we know that for any $P \in \mathcal{P}$, $E_P[M]$ is finite and uniformly bounded with respect to $P$ so that

$$E[M] = \sup_{P \in \mathcal{P}} E_P[M] < \infty.$$ 

As a consequence, the map $t \mapsto X_t$ is uniformly continuous on $D$ quasi-surely and so we can define

$$\forall t \in [0, 1]^d, \hat{X}_t = \lim_{s \to t, s \in D} X_s.$$ 

It is now clear that $\hat{X}$ satisfies the enounced properties. \hfill \Box

3 $G$-Brownian motion under $G$-expectations

In this section we consider the following path spaces: $\Omega = C^d_0(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left( \max_{t \in [0, i]} \{|\omega^1_t - \omega^2_t|\} \land 1 \right).$$ 

It is clear that $(\Omega, \rho)$ is a complete separable metric space. We also denote $\Omega_T = \{ \omega, \lambda T : \omega \in \Omega \}$ for each fixed $T \in [0, \infty)$.

Let $\mathcal{H}$ be a vector lattice of real functions defined on $\Omega$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{b, \text{Lip}}(\mathbb{R}^n)$, where $C_{b, \text{Lip}}(\mathbb{R}^n)$ denotes the space of all bounded and Lipschitz functions on $\mathbb{R}^n$.

**Definition 37.** A functional $E : \mathcal{H} \mapsto \mathbb{R}$ is called a sublinear expectation on $\mathcal{H}$ if it satisfies:

1. **Monotonicity:** for all $X, Y$ in $\mathcal{H}$, $X \geq Y \implies E[X] \geq E[Y]$.
2. **Sub-additivity:** for all $X, Y$ in $\mathcal{H}$, $E[X + Y] \leq E[X] + E[Y]$.
3. **Positive homogeneity:** for all $\lambda \geq 0$, $X \in \mathcal{H}$, $E[\lambda X] = \lambda E[X]$.
4. **Constant translatability:** for all $c \in \mathbb{R}$, $X \in \mathcal{H}$, $E[X + c] = E[X] + c$. 

13
A $d$-dimensional random vector $X$ with each component in $\mathcal{H}$ is said to be $G$-normally distributed under the sublinear expectation $\mathbb{E}[\cdot]$ if for each $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, the function $u$ defined by

$$u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}X)], \quad t \geq 0, \quad x \in \mathbb{R}^d$$

satisfies the following $G$-heat equation:

$$\frac{\partial u}{\partial t} - G(D^2 u) = 0, \quad \text{on } (t, x) \in [0, \infty) \times \mathbb{R}^d,$$

$$u(0, x) = \varphi(x),$$

where $D^2 u$ is the Hessian matrix of $u$, i.e., $D^2 u = (\partial^2_{x_i x_j} u)_{i,j=1}^d$ and

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Theta} \text{tr}[\gamma \gamma^T A], \quad A = (A_{ij})_{i,j=1}^d \in \mathbb{S}_d. \quad (1)$$

$\mathbb{S}_d$ denotes the space of $d \times d$ symmetric matrices. $\Theta$ is a given non empty, bounded and closed subset of $\mathbb{R}^{d \times d}$ which is the space of all $d \times d$ matrices.

**Remark 38.** The above $G$-heat equation has a unique viscosity solution. We refer to [3] for the definition, existence, uniqueness and comparison theory of this type of parabolic PDE (see also [29] for our specific situation). If $G$ is non-degenerate, i.e., there exists a $\beta > 0$ such that $G(A) - G(B) \geq \beta \text{tr}[A - B]$ for each $A, B \in \mathbb{S}_d$ with $A \succeq B$, then the above $G$-heat equation has a unique $C^{1,2}$-solution (see e.g. [22]).

We consider the canonical process: $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$. We introduce the space of finite dimensional cylinder random variables: for each fixed $T \geq 0$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) : \forall n \geq 1, t_1, \ldots, t_n \in [0, T], \forall \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})\},$$

It is clear that $L_{ip}(\Omega_T) \subseteq L_{ip}(\Omega_T) \subseteq C_b(\Omega_T)$, for $t \leq T$. We also denote

$$L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n) \subseteq C_b(\Omega).$$

We can construct (see [20,27]) a consistent sublinear expectation called $G-$expectation $\mathbb{E}[\cdot]$ on $L_{ip}(\Omega)$, such that $B_t$ is $G$-normally distributed under $\mathbb{E}[\cdot]$ and for each $s, t \geq 0$ and $t_1, \ldots, t_N \in [0, t]$ we have

$$\mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_N}, B_{t+s} - B_t)] = \mathbb{E}[\psi(B_{t_1}, \ldots, B_{t_N})], \quad (2)$$

where $\psi(x_1, \ldots, x_N) = \mathbb{E}[\varphi(x_1, \ldots, x_N, \sqrt{s}B_t)]$. Under $G-$expectation $\mathbb{E}[\cdot]$, the canonical process $\{B_t : t \geq 0\}$ is called $G-$Brownian motion.

**Remark 39.** Relation [3] implies that the increments of $B$ are independent and stationary distributed with respect to the sublinear expectation $\mathbb{E}[\cdot]$. The condition that $B_t$ is $G$-normally distributed can be also automatically obtained provided that $\mathbb{E}[\|B_t\|^2] \leq Ct^2$ (see [29]).

14
The topological completion of $L_{ip}(\Omega_T)$ (resp. $L_{ip}(\Omega)$) under the Banach norm $\mathbb{E}[\cdot]$ is denoted by $L_{ip}^c(\Omega_T)$ (resp. $L_{ip}^c(\Omega)$). $\mathbb{E}[\cdot]$ can be extended uniquely to a sublinear expectation on $L_{ip}^c(\Omega)$.

In the previous section the sublinear expectation $\mathbb{E}[\cdot]$ is induced as an upper expectation associated to a family $\mathcal{P}$ of probability measures. In this Section $\mathbb{E}[\cdot]$ will always be the $G$-expectation. We will prove that $C_b(\Omega) \subset L_{ip}^c(\Omega)$ and that, in fact, $\mathbb{E}[\cdot]$ is the upper expectation of a weakly compact family $\mathcal{P}$ on $\Omega$, thus all results in Section 2 hold true.

### 3.1 G-Expectation as an upper-Expectation

In this subsection we will construct a family $\mathcal{P}$ of probability measures on $\Omega$, for which the upper expectation coincides with the $G$-expectation $\mathbb{E}[\cdot]$ on $L_{ip}(\Omega)$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t \geq 0} = (W_t^i)_{i=1, t \geq 0}$ a $d$-dimensional Brownian motion in this space. The filtration generated by $\bar{W}$ is denoted by

$$\mathcal{F}_t := \sigma\{W_u, 0 \leq u \leq t\} \vee \mathcal{N}, \quad \mathcal{F}_t^s := \sigma\{W_{s+u} - W_s, 0 \leq u \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the collection of $P$-null subsets. We also denote, for a fixed $s \geq 0$,

$$\mathcal{F}_t^s := \sigma\{W_{s+u} - W_s, 0 \leq u \leq t\} \vee \mathcal{N}, \quad \mathcal{F}_t^s := \{\mathcal{F}_t^s\}_{t \geq 0}.$$

Let $\Theta$ be a given bounded and closed subset in $\mathbb{R}^{d \times d}$. We denote by $\mathcal{A}_{t,T}^{\Theta}$, the collection of all $\Theta$-valued $\mathcal{F}$-adapted process on an interval $[t, T] \subset [0, \infty)$. For each fixed $\theta \in \mathcal{A}_{t,T}^{\Theta}$ we denote

$$B_{t,T}^{i,\theta} := \int_t^T \theta_s dW_s.$$ 

In this section we will prove that, for each $n = 1, 2, \ldots$, $\varphi \in C_b, \text{Lip}(\mathbb{R}^{n \times n})$ and $0 \leq t_1, \ldots, t_n < \infty$, the $G$-expectation defined in [26, 27] can be equivalently defined by

$$\mathbb{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \sup_{\theta \in \mathcal{A}_{t,T}^{\Theta}} E_P[\varphi(B_{t_1}^{i,\theta}, B_{t_2}^{i,\theta}, \ldots, B_{t_n}^{i,\theta})].$$

Given $\varphi \in C_b, \text{Lip}(\mathbb{R}^n \times \mathbb{R}^d)$, $0 \leq t \leq T < \infty$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we define

$$\Lambda_{t,T}[\zeta] = \text{ess sup}_{\theta \in \mathcal{A}_{t,T}^{\Theta}} E_P[\varphi(\zeta, \int_t^T \theta_s dW_s)|\mathcal{F}_t]. \quad (3)$$

**Lemma 40.** For each $\theta^1$ and $\theta^2$ in $\mathcal{A}_{t,T}^{\Theta}$, there exists $\theta \in \mathcal{A}_{t,T}^{\Theta}$ such that

$$E_P[\varphi(\zeta, B_{t,T}^{i,\theta})|\mathcal{F}_t] = E_P[\varphi(\zeta, B_{t,T}^{i,\theta^1})|\mathcal{F}_t] \vee E_P[\varphi(\zeta, B_{t,T}^{i,\theta^2})|\mathcal{F}_t]. \quad (4)$$

Consequently, there exists a sequence $\{\theta^i\}_{i=1}^\infty$ of $\mathcal{A}_{t,T}^{\Theta}$, such that

$$E_P[\varphi(\zeta, B_{t,T}^{i,\theta^i})|\mathcal{F}_t] \nearrow \Lambda_{t,T}[\zeta], \quad P\text{-a.s..} \quad (5)$$
We also have, for each \( s \leq t \),

\[
E_P[\operatorname{ess} \sup_{\theta \in \mathcal{A}_t^{\theta, \omega}} E_P[\varphi(\zeta, \int_t^T \theta_s dW_s)|\mathcal{F}_t]|\mathcal{F}_s] = \operatorname{ess} \sup_{\theta \in \mathcal{A}_t^{\theta, \omega}} E_P[\varphi(\zeta, \int_t^T \theta_s dW_s)|\mathcal{F}_s].
\]

(6)

Proof. We set \( A = \left\{ \omega : E_P[\varphi(\zeta, B_t^{\theta, \omega})|\mathcal{F}_t](\omega) \geq E_P[\varphi(\zeta, B_T^{\theta, \omega})|\mathcal{F}_t](\omega) \right\} \) and take \( \theta_s = I_{[t,T]}(s)(I_A \theta_s^1 + I_A \theta_s^2) \). Since

\[
\varphi(\zeta, B_t^{\theta, \omega}) = I_A \varphi(\zeta, B_t^{\theta, \omega}) + I_A \varphi(\zeta, B_T^{\theta, \omega}),
\]

we derive (41) and then (43). (6) follows from (41) and Yan’s commutation theorem (cf \( \text{[33]} \) in Chinese and Thm. a3 in the Appendix of \( \text{[24]} \)).

Lemma 41. The mapping \( \Lambda_{t,T}[] : L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n) \rightarrow L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}) \) has the following regularity properties: for each \( \zeta, \zeta' \in L^2(\mathcal{F}_t) \):

(i) \( \Lambda_{t,T}[\zeta] \leq C_\varphi \).

(ii) \( |\Lambda_{t,T}[\zeta] - \Lambda_{t,T}[\zeta']| \leq k_\varphi |\zeta - \zeta'| \).

where \( C_\varphi = \sup_{(x,y)} \varphi(x,y) \) and \( k_\varphi \) is the Lipschitz constant of \( \varphi \).

Proof. We only need to prove (ii). We have

\[
\Lambda_{t,T}[\zeta] - \Lambda_{t,T}[\zeta'] \leq \operatorname{ess} \sup_{\mathcal{A}_t^{\theta, \omega}} E_P[\varphi(\zeta, \int_t^T \theta_s dW_s) - \varphi(\zeta', \int_t^T \theta_s dW_s)|\mathcal{F}_t]
\]

\[
\leq k_\varphi |\zeta - \zeta'|
\]

and, symmetrically, \( \Lambda_{t,T}[\zeta'] - \Lambda_{t,T}[\zeta] \leq k_\varphi |\zeta - \zeta'| \). Thus (ii) follows.

Lemma 42. For each \( x \in \mathbb{R}^n \), \( \Lambda_{t,T}[x] \) is a deterministic function. Moreover,

\[
\Lambda_{t,T}[x] = \Lambda_{0,T-t}[x].
\]

(7)

Proof. Since the collection of processes \( (\theta_s)_{s \in [t,T]} \) with

\[
\left\{ \theta_s = \sum_{j=1}^N I_{A_j \theta}^j s : \{A_j\}_{j=1}^N \text{ is an } \mathcal{F}_t \text{-partition of } \Omega, \theta^j \in \mathcal{A}_{t,T}^{\theta} \text{ is } (\mathcal{F}_t) \text{-adapted} \right\}
\]

is dense in \( \mathcal{A}_{t,T}^{\theta} \), we can take a sequence \( \theta^j s = \sum_{j=1}^N I_{A_j \theta}^j s \) of this type of processes such that \( E_P[\varphi(x, B_t^{\theta, \omega})|\mathcal{F}_t] \rightarrow \Lambda_{t,T}[x] \). But

\[
E_P[\varphi(x, B_t^{\theta, \omega})|\mathcal{F}_t] = \sum_{j=1}^N I_{A_j \theta}^j E_P[\varphi(x, B_t^{\theta, \omega})|\mathcal{F}_t] = \sum_{j=1}^N I_{A_j \theta}^j E_P[\varphi(x, B_t^{\theta, \omega})]
\]

\[
\leq \max_{1 \leq j \leq N} E_P[\varphi(x, B_t^{\theta, \omega})] = E_P[\varphi(x, B_t^{\theta, \omega})],
\]

16
where, for each $i$, $j$, is a maximizer of $\{E_P[\varphi(x, B_T^{t,\theta^{ij}})]\}_{j=1}^{N_i}$. This implies that

$$\lim_{i \to \infty} E_P[\varphi(x, B_T^{t,\theta^{ij}})] = \Lambda_{t,T}[x], \quad a.s.$$ 

and thus $\Lambda_{t,T}[x]$ is a deterministic number. In the above proof, we know that

$$ess \sup_{\theta \in \Theta_0} E_P[\varphi(x, B_T^{t,\theta})] = ess \sup_{\theta \in \Theta_0} E_P[\varphi(x, \int_0^{T-t} \theta_s dW_s^1)],$$

where $W_t^i = W_{t+s} - W_t$, $s \geq 0$, and $\Theta_0$ is the collection of $\Theta$-valued and $\mathbb{F}$-adapted processes on $[0, T-t]$. Thus (7) follows.

We will denote $u_{t,T}(x) := \Lambda_{t,T}[x]$, $t \leq T$. By Lemma 43, $u_{t,T}(\cdot)$ is a bounded and Lipschitz function.

**Lemma 43.** For each $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have

$$u_{t,T}(\zeta) = \Lambda_{t,T}[\zeta], \quad a.s.$$ 

**Proof.** By the above regularities of $\Lambda_{t,T}[]$ and $u_{t,T}(\cdot)$ we only need to check the situation where $\zeta$ is a step function, i.e., $\zeta = \sum_{j=1}^{N} I_{A_j} x_j$, where $x_j \in \mathbb{R}^n$ and $\{A_j\}_{j=1}^{N}$ is an $\mathcal{F}_t$-partition of of $\Omega$. For each $x_j$, let $\{\theta^{ij}\}_{i=1}^{\infty}$ of $\Theta_0$ be $(\mathbb{F}_t)$-adapted process such that

$$\lim_{i \to \infty} E_P[\varphi(x_j, B_T^{t,\theta^{ij}})] = \Lambda_{t,T}[x_j] = u_{t,T}(x_j).$$

Setting $\theta^i = \sum_{j=1}^{N} \theta^{ij} I_{A_j}$, we have

$$\Lambda_{t,T}[\zeta] = \sum_{j=1}^{N} I_{A_j} E_P[\varphi(x_j, B_T^{t,\theta^{ij}})] = E_P[\varphi(\sum_{j=1}^{N} I_{A_j} x_j, B_T^{t,\sum_{j=1}^{N} I_{A_j} \theta^{ij}})|\mathcal{F}_t]$$

$$= \sum_{j=1}^{N} I_{A_j} E_P[\varphi(x_j, B_T^{t,\theta^{ij}})|\mathcal{F}_t] \to \sum_{j=1}^{N} I_{A_j} u_{t,T}(x_j) = u_{t,T}(\zeta).$$

On the other hand, for each given $\theta \in \Theta_0$, we have

$$E_P[\varphi(\zeta, B_T^{t,\theta})|\mathcal{F}_t] = E_P[\varphi(\sum_{j=1}^{N} I_{A_j} x_j, B_T^{t,\theta})|\mathcal{F}_t]$$

$$= \sum_{j=1}^{N} I_{A_j} E_P[\varphi(x_j, B_T^{t,\theta})|\mathcal{F}_t]$$

$$\leq \sum_{j=1}^{N} I_{A_j} u_{t,T}(x_j) = u_{t,T}(\zeta).$$

We thus have $ess \sup_{\theta \in \Theta_0} E_P[\varphi(\zeta, B_T^{t,\theta})|\mathcal{F}_t] \leq u_{t,T}(\zeta)$. The proof is complete.

17
The following result generalizes the well-known dynamical programming principle:

**Theorem 44.** For each \( \varphi \in C_{b,\text{Lip}}(\mathbb{R}^n \times \mathbb{R}^d) \), \( 0 \leq s \leq t \leq T \) and \( \zeta \in L^2(\Omega, \mathcal{F}_s, P; \mathbb{R}^n) \) we have

\[
\text{ess sup}_{\theta \in A_{s,T}^\theta} E_P[\varphi(\zeta, B^s_{t+h}, B^t_{T}) | \mathcal{F}_s] = \text{ess sup}_{\theta \in A_{s,T}^\theta} E_P[\psi(x, y, B^s_{t+h}, B^t_{T}) | \mathcal{F}_s],
\]

where \( \psi \in C_{b,\text{Lip}}(\mathbb{R}^n \times \mathbb{R}^d) \) is given by

\[
\psi(x, y) := \text{ess sup}_{\bar{\theta} \in A_{T-T}^{\bar{\theta}}} E_P[\varphi(x, y, B^s_{T}, B^t_{T}) | \mathcal{F}_T] = \sup_{\bar{\theta} \in A_{T-T}^{\bar{\theta}}} E_P[\varphi(x, y, B^s_{T}, B^t_{T})].
\]

**Proof.** It is clear that

\[
\text{ess sup}_{\theta \in A_{s,T}^\theta} E_P[\varphi(\zeta, B^s_{t+h}, B^t_{T}) | \mathcal{F}_s] = \text{ess sup}_{\theta \in A_{s,T}^\theta} \left\{ \text{ess sup}_{\bar{\theta} \in A_{T-T}^{\bar{\theta}}} E_P[\varphi(\zeta, B^s_{t+h}, B^t_{T}) | \mathcal{F}_T] \right\}.
\]

It follows from (6) and Lemma 43 that

\[
\text{ess sup}_{\theta \in A_{s,T}^\theta} E_P[\varphi(\zeta, B^s_{t+h}, B^t_{T}) | \mathcal{F}_s] = E_P[\psi(\zeta, B^s_{t+h}, B^t_{T}) | \mathcal{F}_s],
\]

We thus have (8). \( \square \)

For each given \( \varphi \in C_{b,\text{Lip}}(\mathbb{R}^d) \) and \( (t, x) \in [0, T] \times \mathbb{R}^d \), we set

\[
v(t, x) := \sup_{\theta \in A_{t,T}^\theta} E_P[\varphi(x + B^t_{T})].
\]

Since for each \( h \in [0, T-t] \),

\[
v(t, x) = \sup_{\theta \in A_{t,T}^\theta} E_P[\varphi(x + B^t_{T})] = \sup_{\theta \in A_{t,T}^\theta} E_P[\varphi(x + B^t_{t+h}, B^t_{T})] = \sup_{\theta \in A_{t,T}^\theta} E_P[v(t+h, x + B^t_{t+h})].
\]

This gives us the well-known dynamic programming principle:

**Proposition 45.** We have

\[
v(t, x) = \sup_{\theta \in A_{t,T}^\theta} E_P[v(t+h, x + B^t_{t+h})],
\]

(9)
Lemma 46. $v$ is bounded by $\sup |\varphi|$. It is a Lipschitz function in $x$ and $\frac{1}{2}$-holder function in $t$.

Proof. We only need to prove the regularity in $t$.

$$
\sup_{\theta \in \mathcal{A}^0_{t,t+h}} E_P[v(t+h, x + B_{t+h}^\theta) - v(t+h, x)] = v(t, x) - v(t + h, x).
$$

Since $v$ is a Lipschitz function in $x$, the absolute value of the left hand is bounded by

$$
C \sup_{\theta \in \mathcal{A}^0_{t,t+h}} E_P[|B_{t+h}^\theta|] \leq C_1 h^{1/2}.
$$

The $\frac{1}{2}$-holder of $v$ in $t$ is obtained. 

Theorem 47. $v$ is a viscosity solution of the $G$-heat equation:

$$
\frac{\partial v}{\partial t} + G(D^2 v) = 0, \quad \text{on } (t, x) \in [0, T) \times \mathbb{R}^d,
$$

$$
v(T, x) = \varphi(x),
$$

where the function $G$ is given in (1).

Proof. Let $\psi \in C^{2,3}_b((0,T) \times \mathbb{R}^d)$ be such that $\psi \geq v$ and, for a fixed $(t, x) \in (0,T) \times \mathbb{R}^d$, $\psi(t, x) = v(t, x)$. From the dynamic programming principle (9) it follows that

$$
0 = \sup_{\theta \in \mathcal{A}^0_{t,t+h}} E_P[v(t+h, x + B_{t+h}^\theta) - v(t, x)]
$$

$$
\leq \sup_{\theta \in \mathcal{A}^0_{t,t+h}} E_P[\psi(t+h, x + B_{t+h}^\theta) - \psi(t, x)]
$$

$$
= \sup_{\theta \in \mathcal{A}^0_{t,t+h}} E_P \left[ \int_t^{t+h} \left( \frac{\partial \psi}{\partial s} + \frac{1}{2} \text{tr}[\theta_s \theta_s^T D^2 \psi] \right) (s, x + \int_t^s \theta_r dW_r) ds \right].
$$

Since $(\frac{\partial \psi}{\partial s} + \frac{1}{2} \text{tr}[\theta_s \theta_s^T D^2 \psi])(s, y)$ is uniformly Lipschitz in $(s, y)$, we have for small $h > 0$

$$
E_P \left[ \frac{\partial \psi}{\partial s} + \frac{1}{2} \text{tr}[\theta_s \theta_s^T D^2 \psi] \right] (s, x + \int_t^s \theta_r dW_r) \leq E_P \left[ \frac{\partial \psi}{\partial s} + \frac{1}{2} \text{tr}[\theta_s \theta_s^T D^2 \psi] \right] (t, x) + Ch^{1/2}.
$$

Thus

$$
\sup_{\theta \in \mathcal{A}^0_{t,t+h}} E_P \int_t^{t+h} \left( \frac{\partial \psi}{\partial s} + \frac{1}{2} \text{tr}[\theta_s \theta_s^T D^2 \psi] \right) (t, x) ds + Ch^{3/2} \geq 0.
$$

Thus

$$
\left( \frac{\partial \psi}{\partial s} + \frac{1}{2} \sup_{\gamma \in \Theta} \text{tr}[\gamma \gamma^T D^2 \psi] \right) (t, x) h + Ch^{3/2} \geq 0
$$
and then $\frac{\partial \psi}{\partial t} + G(D^2 \psi))(t, x) \geq 0$. By the definition, $v$ is a viscosity subsolution. Similarly we can prove that it is also a supersolution.

We observe that $u(t, x) := v(T - t, x)$, thus $u$ is a viscosity solution of $\frac{\partial u}{\partial t} - G(D^2 u) = 0$, with Cauchy condition $u(0, x) = \varphi(x)$.

From the uniqueness of the viscosity solution of $G$-heat equation and Theorem 44 we get immediately:

**Proposition 48.**

$$
E[\varphi(B^0_{t_1}, B^0_{t_2}, \ldots , B^0_{t_n})] = \sup_{\theta \in A^{\theta}_{0,T}} E_{P_\theta}[\varphi(B^0_{t_1}, B^0_{t_2}, \ldots , B^0_{t_n})] \\
= \sup_{\theta \in A^{\theta}_{0,T}} E_{P_\theta}[\varphi(B^0_{t_1}, B^0_{t_2}, \ldots , B^0_{t_{n-1}})],
$$

where $P_\theta$ is the law of the process $B^0_{t} = \int_0^t \theta_s dW_s$, $t \geq 0$, for $\theta \in A^{\theta}_{0,\infty}$.

Now we prove that $\{P_\theta, \theta \in A^{\theta}_{0,\infty}\}$ is tight, this is important in the following subsection.

**Proposition 49.** The family of probability measures $\{P_\theta, \theta \in A^{\theta}_{0,\infty}\}$ on $C^d_0(\mathbb{R}^+) \quad$ is tight.

**Proof.** We apply Itô’s formula to $(B^0_{t})_{t \geq s}$:

$$
|B^0_{t}|^4 = \int_s^t 4 |B^0_{r}|^2 B^0_{r} \, dB^0_{r} + 2 \int_s^t \text{tr}[\theta_r \theta_r^T (I_d |B^0_{r}|^2 + 2 B^0_{r} \otimes B^0_{r})] \, dr.
$$

We thus have

$$
E_{P_\theta}[|B_t - B_s|^4] = 2E \int_s^t \text{tr}[\theta_r \theta_r^T (I_d |B^0_{r}|^2 + 2 B^0_{r} \otimes B^0_{r})] \, dr \\
= 2E \int_s^t (|\theta_r|^2 |B^0_{r}|^2 + 2(\theta_r, B^0_{r})^2) \, dr \\
\leq C \int_s^t |B^0_{r}|^2 \, dr \leq Cd \int_s^t (r - s) \, dr \\
= C d \frac{(t - s)^2}{2}.
$$

We then apply the well-known result of moment criterion for tightness of Kolmogorov-Chentsov’s type to conclude that $\{P_\theta, \theta \in A\}$ is tight.

3.2 Capacity related to $G$-expectation

We denote $\mathcal{P}_1 = \{P_\theta : \theta \in A^{\theta}_{0,\infty}\}$ and $\mathcal{P} = \overline{\mathcal{P}_1}$ the closure of $\mathcal{P}_1$ under the topology of weak convergence. By Proposition 19 $\mathcal{P}_1$ is tight and then $\mathcal{P}$ is weakly compact. We set

$$
c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).
$$
For each \( X \in L^0(\Omega) \) such that \( E_P[X] \) exists for each \( P \in \mathcal{P} \), we set

\[
\hat{E}[X] = \mathbb{E}^\mathcal{P}[X] = \sup_{P \in \mathcal{P}} E_P[X].
\]

Now we prove that

\[
L^1_\mathcal{G}(\Omega_T) = \{ X \in L^0(\Omega_T) : X \text{ has a q.c. version, } \lim_{n \to \infty} \hat{E}[|X|1_{\{|X|>n\}}] = 0 \},
\]

\[
L^1_\mathcal{G}(\Omega) = \{ X \in L^0(\Omega) : X \text{ has a q.c. version, } \lim_{n \to \infty} \hat{E}[|X|1_{\{|X|>n\}}] = 0 \},
\]

\[
\mathbb{E}[X] = \hat{E}[X], \quad \forall X \in L^1_\mathcal{G}(\Omega),
\]

where q.c. denotes quasi-continuous for simplicity.

For proving this we need the following lemma.

**Lemma 50.** Let \( K \) be a compact subset of \( \Omega_T \) equipped with the distance \( \rho(\omega^1,\omega^2) = \max_{0 \leq t \leq T} |\omega^1_t - \omega^2_t| \). Then for each \( \Phi \in C_b(\Omega_T) \), there exists a sequence \( \{\Phi_n\}_{n=1}^\infty \subset L_{ip}(\Omega_T) \) with \( \|\Phi_n\|_{\sup} \leq \|\Phi\|_{\sup} \) such that \( \Phi_n \) converges uniformly to \( \Phi \) on \( K \).

**Proof.** This is just the consequence of the Stone-Weierstrass theorem. \( \square \)

**Theorem 51.** We have

\[
L^1_\mathcal{G}(\Omega_T) = \{ X \in L^0(\Omega_T) : X \text{ has a q.c. version, } \lim_{n \to \infty} \hat{E}[|X|1_{\{|X|>n\}}] = 0 \},
\]

\[
\mathbb{E}[X] = \hat{E}[X], \quad \forall X \in L^1_\mathcal{G}(\Omega_T).
\]

**Proof.** It follows from Proposition [48] that

\[
\mathbb{E}[X] = \hat{E}[X], \quad \forall X \in L_{ip}(\Omega_T).
\]

Thus \( L^1_\mathcal{G}(\Omega_T) \) can be seen as the completion of \( L_{ip}(\Omega_T) \) under the norm \( \hat{E}[|\cdot|] \). For any fixed \( \psi \in C_b(\Omega_T) \), since \( \mathcal{P} \) is tight, we have for each \( n \in \mathbb{N} \), there exists a compact set \( K_n \subset \Omega_T \) such that \( c(K_n^c) < \frac{1}{n} \). For this \( K_n \), by Lemma 50 there exists a \( \varphi_n \in L_{ip}(\Omega_T) \) such that

\[
\|\varphi_n\|_{\sup} \leq \|\psi\|_{\sup} \quad \text{and} \quad \sup_{\omega \in K_n} |\varphi_n(\omega) - \psi(\omega)| < \frac{1}{n}.
\]

Thus

\[
\hat{E}[|\varphi_n - \psi|] \leq 2 \|\psi\|_{\sup} c(K_n^c) + \frac{1}{n} c(K_n) < (2 \|\psi\|_{\sup} + 1) \frac{1}{n} \to 0.
\]

It then follows that \( C_b(\Omega_T) \subset L^1_\mathcal{G}(\Omega_T) \), by Theorem 25 we obtain the result. \( \square \)

**Remark 52.** The above results also hold for \( L^1_\mathcal{G}(\Omega) \), the proof is similar.
We also set
\[ \bar{c}(A) := \sup_{P \in \mathcal{P}_1} P(A), \quad A \in \mathcal{B}(\Omega). \]

It is easy to verify the following
1. \( \bar{c}(A) \leq c(A) \) for each \( A \in \mathcal{B}(\Omega) \).
2. \( \bar{c}(O) = c(O) \) for each open set \( O \subset \Omega \).

Thus, a function is \( c \)-quasi-continuous if and only if it is \( \bar{c} \)-quasi-continuous, so we simply write quasi-continuous function. For each \( X \in L^0(\Omega) \) such that \( E_P[X] \) exists for each \( P \in \mathcal{P}_1 \), we set
\[ \bar{E}[X] = E^{\mathcal{P}_1}[X] = \sup_{P \in \mathcal{P}_1} E_P[X]. \]

It is easy to verify the following
1. \( \bar{E}[X] \leq \hat{E}[X] \) for each \( X \) which makes both expectation meaningful.
2. \( \bar{E}[X] = \hat{E}[X] \) for each bounded quasi-continuous function \( X \).

Similar to the proof of Theorem 51, we get the following theorem.

**Theorem 53.** We have
\[
L^1_G(\Omega_T) = \{ X \in L^0(\Omega_T) : X \text{ has a q.c. version}, \lim_{n \to \infty} \bar{E}[|X|1_{\{|X|>n\}}] = 0 \},
\]
\[
L^1_G(\Omega) = \{ X \in L^0(\Omega) : X \text{ has a q.c. version}, \lim_{n \to \infty} \bar{E}[|X|1_{\{|X|>n\}}] = 0 \},
\]
\[
\bar{E}[X] = \hat{E}[X], \quad \forall X \in L^1_G(\Omega).
\]

**Remark 54.** Theorem 51 holds for \( \bar{E}[\cdot] \) under the capacity \( c(\cdot) \). But it does not necessarily hold for \( \hat{E}[\cdot] \) under the capacity \( \bar{c}(\cdot) \).

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