Multibubble cavitation inception

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The inception of cavitation in multibubble cases is studied numerically and theoretically to show that it is different from that in single-bubble cases in several aspects. Using a multibubble model based on the Rayleigh-Plesset equation with an acoustic interaction term, we confirmed that the recently reported suppression of cavitation inception due to the interaction of non-identical bubbles can take place not only in liquid mercury but also in water, and we found that a relatively large bubble can significantly decrease the cavitation threshold pressure of a nearby small bubble. By examining in detail the transition region where the dynamics of the suppressed bubble changes drastically as the inter-bubble distance changes, we determined that the explosive expansion of a bubble under negative pressure can be interrupted and turn into collapse even though the far-field liquid pressure well exceeds the bubble’s threshold pressure. Numerical results suggest that the interruption of expansion occurs when the bubble radius is exceeded by the instantaneous unstable equilibrium radius of the bubble determined using the total pressure acting on the bubble. When we extended the discussion to systems of larger numbers of bubbles, we found that a larger number of bubbles have a stronger suppression effect. The present findings would be useful in understanding the complex behavior of cavitation bubbles in practical applications where in general many cavitation nuclei exist and may interact with each other.

I. INTRODUCTION

Cavitation in liquids is a common phenomenon that has been observed and used in a wide range of fields including mechanical and chemical engineering and nuclear and medical applications.\cite{1,2,3,4,5,6,7,8,9,10,11} When cavitation occurs in a liquid under negative pressure, many gas or vapor bubbles emerge and explosively expand. The cavitation bubbles then collapse violently after the negative pressure is released, emitting high-speed liquid jets and/or shock waves. Because of its practical importance and rich physics, cavitation has long been studied after the pioneering work by Rayleigh.\cite{12} However, cavitation is not yet fully understood, and there remain many unclear aspects of even its fundamental characteristics, such as the inception processes, threshold pressures at which cavitation occurs, and lifetime of emerged bubbles.\cite{3,7,13,14,15}

One of the key factors in cavitation bubble dynamics is bubble-bubble interaction. It has long been known that, when bubbles interact with each other, their dynamics are sometimes significantly different from that of isolated bubbles. In Ref.\cite{16}, for example, Smith and Mesler studied experimentally the dynamics of a spark-generated vapor bubble interacting with a gas bubble in an effort to search for a method to reduce cavitation damage. They demonstrated that energy transfer between the bubbles alters their dynamics, and that positive pressure pulses emitted by the collapsing vapor bubble are reduced when the gas bubble stays near the vapor bubble. In Ref.\cite{17} Mørch studied the collapse of cavitation bubble clusters using an energy-balance equation and found that the collapse intensity depends on the volume fraction of bubbles. He also studied experimentally the temperature dependence of cavitation bubble dynamics and showed that the number of cavitation bubbles increases but their size decreases as the liquid temperature increases. Using a detailed theoretical model, Chahine and Liu\cite{21,22} studied the growth of a vapor bubble cluster in a superheated liquid, and found that the expansion rate of bubbles and the temperature drop at the bubble surface decrease as the number of interacting bubbles increases. In a numerical study of shock-wave propagation in a bubble cloud, Wang and Brennen\cite{23} showed that the expansion rate of bubbles at the cloud center is significantly smaller that that on the cloud surface. In a study of multibubble sonoluminescence, Mettin et al.\cite{20} showed numerically that in a strong acoustic field, the sign of the secondary Bjerknes force (an interaction force acting between pulsating bubbles) depends in a complicated manner on the driving pressure, ambient radii of bubbles, and inter-bubble distance.

In this paper we examined the inception processes of cavitation in water in multibubble cases or, in other words, multi-nuclei cases. The present study was motivated by a recent numerical study of cavitation in liquid mercury and a technique to suppress it. Ida et al.\cite{24,25} showed numerically that bubble-bubble interaction through pressure waves can have a significant impact on the inception of cavitation. Using a nonlinear multibubble model based on the Keller-Miksis equation, it was found that in certain situations, the explosive expansion of a bubble (i.e., cavitation inception) is completely suppressed by the positive pressure waves that larger neighboring bubbles emit when they are growing. This theoretical prediction was confirmed by an experiment, the results of which showed that microbubble injection into liquid mercury suppresses cavitation inception and significantly reduces cavitation damage.\cite{24} This finding implies that the inception processes of cavitation in multibubble cases can be significantly different from that...
We present in this paper a detailed discussion of the effect of bubble-bubble interaction on cavitation inception in water. The theoretical model used in this study is a nonlinear multibubble model in which Rayleigh-Plesset equations for single bubbles are coupled through an interaction term that represents the bubble-emitted pressure waves. From a number of relevant nonlinear models, we select the classical Rayleigh-Plesset equations, which are for gas bubbles in an incompressible liquid, coupled through an acoustic interaction term not taking into account time delay effects due to the finite sound speed of water. The time history of the far-field liquid pressure is assumed to be a constant negative pressure following a sinusoidal decompression. When the period of the decompression process is set to zero, this pressure-time history is reduced to a step change that has been frequently considered in cavitation study. Based on the model and setting, we first consider the interaction of two non-identical bubbles under negative pressure. The numerical results suggest that the suppression of cavitation inception reported in Ref. 21 can occur in water as well if the bubbles’ ambient radii and inter-bubble distance are properly set. Then, we discuss several details of the results. Particular attention is focused on the effective cavitation pressure (a dynamic Blake threshold) of the suppressed bubble and the transition region in parameter space where the bubble’s behavior drastically changes as, for example, the inter-bubble distance changes. We also discuss the dynamics of larger numbers of identical bubbles to show that they have a stronger suppression effect. In this discussion, we derive an exact theoretical formula that relates the surface velocity of interacting bubbles to their instantaneous radii. The theoretical formula is a direct extension of the single-bubble formula derived previously and has high accuracy and wide applicability. Using the formula we discuss how bubble-bubble interaction changes the expansion rate of bubbles and the negative pressure amplitude in water. The obtained conclusions are consistent with previous numerical and experimental observations. In the present investigation we found that the inception process of cavitation in multibubble cases can be much more complex than in single-bubble cases and that a variety of patterns of inception are possible.

The rest of this paper is organized as follows: In Sec. II a brief review of cavitation bubble dynamics is presented, and in Sec. III the model equations used in the present study are introduced. Section IV presents numerical and theoretical results and discussion of these results, and Sec. V presents our concluding remarks, including comments about applications and future works.
pressure, determined from \(dp_L/dR = 0\), is

\[
p_C = p_v - \frac{32\sigma^3}{27\left(\frac{P_0}{R_0} + \frac{2\sigma}{R_0} - p_v\right)R_0^3},
\]

and the corresponding critical radius is

\[
R_C = \sqrt{\frac{3}{2\sigma} \left(\frac{P_0}{R_0} + \frac{2\sigma}{R_0} - p_v\right)R_0^3},
\]

When \(p_L\) slowly decreases and exceeds \(p_C\), the bubble begins to explosively expand. Then, if \(p_L\) holds constant at \(p_L < p_C\), the bubble will be in a steady growth state, where the expansion rate, \(dR/dt\) (\(t\) being time), is nearly constant. The asymptotic expansion rate of such a bubble follows a simple formula:\1,2

\[
dR/dt \approx \sqrt{\frac{2(p_v - p_L)}{3\rho}},
\]

where \(\rho\) is the liquid density and, in this case, \(p_L\) is the liquid pressure in the far field where the bubble-emitted pressure is negligible.

For \(p_C < p_L < p_v\), Eq. (1) has two roots which correspond to equilibrium radii.\2,3,9,20-23,33 The smaller root represents the stable equilibrium radius, around which the bubble can undergo damped oscillation. On the other hand, the larger root represents the unstable equilibrium radius, where a small deviation in bubble radius results in the breakdown of equilibrium: The bubble will shrink quickly towards the stable equilibrium radius if the deviation is negative but will expand without bound if the deviation is positive. As shown in Sec. [IV], the unstable equilibrium radius plays an important role in our study.

In dynamic cases where the liquid pressure varies rapidly and the transient motion of bubbles plays a role, however, the above scenario is only a rough description. Detailed theoretical and numerical investigations of single-bubble dynamics have revealed several dynamic effects on cavitation inception. Researchers have found, for example, that even if \(\min[p_L(t)] < p_C\), the instantaneous bubble radius can in certain conditions be greater than the unstable equilibrium radius in the transient period, resulting in the inception of cavitation.\9,20,21,31,33,34,35 This observation says that the effective threshold pressure and the effective critical radius in dynamic cases are different from those in the quasistatic case. Researchers also found that in a dynamic case the liquid viscosity, which obviously does not appear in the quasistatic formula, can alter the threshold pressure and the critical radius (see Ref. 33 and references therein). Since in the present study we consider the effect of the pressure waves emitted by growing bubbles, our problem is naturally dynamic.

As mentioned above, a number of useful insights into single-bubble cavitation have been published. In reality, however, not only one but many cavitation nuclei exist and may interact with each other if they are sufficiently close to each other. Single-bubble study is thus not sufficient to fully understand cavitation dynamics in practical situations. As shown in Sec. [IV] bubble-bubble interaction can (sometimes drastically) alter the inception processes of cavitation in several ways.

### III. MODEL EQUATIONS

In the present study we assume the cavitation nuclei to be spherical gas microbubbles. The theoretical model used to describe their evolution is the coupled Rayleigh-Plesset equations, which read

\[
R_{i} \dddot{R}_{i} + \frac{3}{2} \dot{R}_{i}^{2} = \frac{1}{\rho} \frac{\partial p_{s,i}}{\partial R_{i}} - \sum_{j=1,j \neq i}^{N} \frac{1}{D_{ij}} \frac{d(R_{i}^{2} \dot{R}_{j})}{dt},
\]

\[
p_{s,i} = p_{b,i} - \frac{2\sigma}{R_{i}} - \frac{4\mu \dot{R}_{i}}{R_{i}} - p_{L}(t),
\]

where \(R_{i} = R_{i}(t)\) is the time-dependent radius of bubble \(i\), \(D_{ij}\) is the distance between the centers of bubbles \(i\) and \(j\), \(p_{L}(t)\) is the liquid pressure in the far field, \(N\) is the number of bubbles, and the overdots denote the time derivative \(d/dt\). The surrounding liquid is assumed to be water of density \(\rho = 1000\, \text{kg/m}^3\), viscosity \(\mu = 1.002 \times 10^{-3}\, \text{Pa s}\), and surface tension \(\sigma = 0.0728\, \text{N/m}\). The compressibility of water is neglected, since we are not interested in the details of the collapse of bubbles. The bubble content is assumed to be an ideal gas, and the internal pressure of bubble \(i\) \((p_{b,i})\) is thus given by

\[
p_{b,i} = \left(\frac{P_0}{R_0} + \frac{2\sigma}{R_0} \right) \left(\frac{R_0}{R_i}\right)^{3\kappa_i},
\]

where \(R_0\) and \(\kappa_i\) are the ambient radius and polytropic exponent, respectively, of bubble \(i\). The vapor pressure and mass exchange across the bubble surface are neglected. We did not consider the adiabatic behavior of violently collapsing bubbles, since we are interested only in the expansion phase. Thus, \(\kappa_i\) is set to unity in most cases. Since translation of bubbles and high-order terms depending on the translational velocity\2,6,9,12 are neglected in this model, we set the inter-bubble distances \(D_{ij}\) to be much greater than the ambient radii \([\text{e.g., as } D_{ij} \geq 10(R_0 + R_0)]\). Using essentially the same model, Bremond et al.\29 recently showed that the coupled Rayleigh-Plesset equation can accurately describe the first rapid expansion and collapse, even when the instantaneous bubble radii reach 75% of half of \(D_{ij}\). Based on this finding, we allow the maximum bubble radii to be comparable to their observation.

This nonlinear system of equations describes the radial motion of \(N\) spherical bubbles coupled through the
bubble-emitted pressure waves. In this system, the bubbles’ radial motion is driven not only by the change in $p_L(t)$ but also by the pressures from the neighboring bubbles described by the last term of Eq. (6); that is, the total driving pressure on bubble $i$ ($p_{T_i}$) is

$$p_{T_i} = p_L(t) + \rho \sum_{j=1,j\neq i}^N \frac{1}{D_{ij}} \frac{d(R_j^2 \dot{R}_j)}{dt}.$$ \hspace{1cm} (9)

The last term of Eq. (6) was derived by a well-known formula for the pressure wave emitted by a pulsating sphere\textsuperscript{2,20,21,40}

$$p_j = \rho \frac{d(R_j^2 \dot{R}_j)}{dt} + O \left( \frac{1}{r_j^4} \right),$$ \hspace{1cm} (10)

where $r_j$ is the distance measured from the center of bubble $j$.

The coupling of bubbles through pressure waves, a kind of bubble-bubble interaction, is known to lead to a variety of phenomena that single bubbles can never exhibit, such as attraction and repulsion of pulsating bubbles\textsuperscript{20,25,36,38,39,41,42,43,44,45} acoustic localization in bubbly liquids\textsuperscript{46,47} avoided crossings of resonance frequencies\textsuperscript{48} large phase delays\textsuperscript{45} and filamentary structure formation in a strong sound field\textsuperscript{49,50} In the following section we discuss the effect of bubble-bubble interaction on cavitation processes.

IV. NUMERICAL AND THEORETICAL INVESTIGATIONS

A. Pressure-time history and initial conditions

The time history of the far-field liquid pressure is assumed as follows:

$$p_L(t) = \begin{cases} P_0 & \text{for } t < -T, \\ P_0 + W(P_{ng} - P_0) & \text{for } -T \leq t \leq 0, \\ P_{ng} & \text{otherwise,} \end{cases}$$ \hspace{1cm} (11)

with

$$W = \frac{1 - \cos \left( \frac{\pi}{T}(t + T) \right)}{2},$$ \hspace{1cm} (12)

where $P_{ng}$ is a constant negative value and $T$ is the period of the decompression process from $P_0$ to $P_{ng}$. This function represents a constant negative pressure following a sinusoidal decompression and is continuous up to the first time derivative (see Fig. 2). In the following discussions, $P_{ng}$ and $T$ are used as control parameters. When one sets $T \to 0$, Eq. (11) is reduced to a step change like that considered in Refs. 2,21,30,33,35,51.

The initial conditions assumed in the present study are

$$R_i(t = -T) = R_{i0},$$ \hspace{1cm} (13)

$$\dot{R}_i(t = -T) = 0,$$ \hspace{1cm} (14)

that is, the bubbles are initially at equilibrium.

B. Competitive growth of non-identical bubbles

Let us consider the dynamics of two non-identical bubbles (bubbles 1 and 2) under negative pressure. An example of a single-bubble case (i.e., for $D_{12} \to \infty$) is shown in
The threshold pressure of bubble 1 in the present case is different from the value given by the quasistatic theory. In the rest of this subsection and in the next subsection, we discuss details of this phenomenon.

The numerical result just described suggests that the effective threshold pressure of bubble 1 in the case of $D_{12} = 10(R_{10} + R_{20})$ is much lower than that predicted by the quasistatic theory and a more intense negative pressure is thus needed to cavitate bubble 1. Figure 4 shows the radius-time curves of bubble 1 for different $D_{12}$ values. The numbers denote $D_{12}/(R_{10} + R_{20})$. The curves of $R_2$ for different $D_{12}$ values are indistinguishable. For $D_{12} = 10(R_{10} + R_{20})$, bubble 1 cannot grow significantly although $p_L$ is well below the threshold pressure of the bubble.

![Figure 4](image1.png)

**FIG. 4:** Radius-time curves of bubble 1 for different $P_{ng}$ values selected around the dynamics threshold pressure. The numbers shown in the panel denote $P_{ng}/P_0$.

![Figure 5](image2.png)

**FIG. 5:** (Color online) Radius-time curves for $P_{ng} = -0.25P_0$ and four different $D_{12}$ values. The numbers denote $D_{12}/(R_{10} + R_{20})$. The curves of $R_2$ for different $D_{12}$ values are indistinguishable. For $D_{12} = 10(R_{10} + R_{20})$, bubble 1 cannot grow significantly although $p_L$ is well below the threshold pressure of the bubble.

Since $P_{ng}$ well exceeds the threshold pressures, both bubbles undergo explosive expansion. The response of bubble 2 in the decompression process is in this case faster than that of bubble 1, because bubble 2 has a much higher threshold pressure than that of bubble 1. Hence, the explosive expansion of bubble 2 begins earlier than that of bubble 1. After the transient motion has decayed, the expansion rates of both bubbles converge to an almost constant value determined by Eq. (5). These observations are consistent with the well-known behavior of single cavitation bubbles. As mentioned in Sec. IV, the threshold pressure in dynamic cases is in general different from the value given by the quasistatic theory. The threshold pressure of bubble 1 in the present case ($T = 10\,\mu s$) is slightly higher than the theoretical prediction. Figure 4 shows the radius-time curves of bubble 1 for different $P_{ng}$. The bubble cannot grow significantly for $P_{ng} \leq -0.175P_0$, but it expands without bound for $P_{ng} \geq -0.177P_0$. From this observation, the dynamic threshold pressure is found to be about $-0.176P_0$, which is only 1.7% higher than the theoretical prediction. The dynamic threshold comes closer to the quasistatic prediction as $T$ increases.

For finite $D_{12}$, the dynamics of the bubbles can have a different pattern. In Fig. 5 we show the results for $P_{ng} = -0.25P_0$ with four different values of $D_{12}$. From the figure, one finds that the expansion rate of bubble 1 is decreased as $D_{12}$ decreases, and the explosive expansion of this bubble is finally suppressed for $D_{12} = 10(R_{10} + R_{20})$, although the negative pressure considered here clearly exceeds its threshold pressure: the expansion ratio of bubble 1, $\max[R_1(t)]/R_{10}$, for $D_{12} = 10(R_{10} + R_{20})$ is only about 2.57. No considerable change occurs in the dynamics of bubble 2 in the shown period, because bubble 1 is too small to cause it. This result proves that the suppression phenomenon reported in Refs. 21,22 is not inherent in liquid mercury but can also occur in water, whose material properties (e.g., density, surface tension) are greatly different from those of mercury. In the rest of this subsection and in the next subsection, we discuss details of this phenomenon.

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proximate formula for the effective threshold pressure of bubble 1 (as shown in Sec. IV D, a larger number of bubbles cause a greater pressure rise). Equating this to the need for a more intense negative (far-field) pressure to cavitate. The positive pressure from bubble 2 is estimated by the following simple formula:

\[
p_2(t) = -\frac{4R_2(t)}{3D_{12}} p_{ng},
\]

(17)

which is given using Eqs. (10) and (5) under the assumption of \( \dot{R}_2 \approx 0 \). Since \( p_{ng} \) is negative, \( p_2(t) \) is positive. The total pressure acting on bubble 1 is thus

\[
P_{ng} + p_2 = \left(1 - \frac{4R_2(t)}{3D_{12}}\right) p_{ng},
\]

(18)

which is clearly higher than the far-field liquid pressure \( P_{ng} \) (as shown in Sec. IV D a larger number of bubbles cause a greater pressure rise). Equating this to the threshold pressure of bubble 1 (\( p_{c1} \)), we have an approximate formula for the effective threshold pressure of bubble 1:

\[
P_{ng} = \frac{p_{c1}}{\frac{4R_2}{3D_{12}}}. \tag{19}
\]

Since the denominator of the right-hand side is smaller than unity, this gives a pressure lower than \( p_{c1} \). This equation suggests that because the problem is essentially dynamic, the effective threshold pressure of a bubble cannot be determined uniquely but depends on the instantaneous radius of the neighboring bubble. Rearranging Eq. (19), one obtains a formula for the instantaneous radius as follows:

\[
R_2 = \frac{3}{4} \left(1 - \frac{p_{c1}}{p_{ng}}\right) D_{12}, \tag{20}
\]

For \( p_{c1} = -0.179P_0 \) and \( D_{12} = 10(R_{10} + R_{20}) \) with \( p_{ng} = -0.25P_0 \), this formula gives \( R_2 = 0.213D_{12} = 46.9\mu m \). This criterion appears to be fulfilled in the example of Fig. 6, the radius of bubble 2 becomes greater than this criterion before bubble 1 begins to expand explosively.

The observed suppression of explosive expansion, or competitive growth of bubbles, occurs also for other couples of bubbles. In Fig. 7 we show the expansion ratios of bubbles, max\( [R_j(t)]/R_{j0} \), for \( R_{10} = 2 \mu m \), \( p_{ng} = -0.25P_0 \), and \( t \leq 20 \mu s \) as functions of \( R_{20} \). Here the inter-bubble distance was fixed as \( D_{12} = 220 \mu m \), which corresponds to \( 10(R_{10} + R_{20}) \) in the previous example. From this figure one finds that the explosive expansion of bubble 1 is completely suppressed when \( R_{20} \geq 17 \mu m \). This figure also suggests that when \( R_{20} < R_{10} \) the expansion of bubble 2 can be suppressed by bubble 1, and that for \( R_{20} > R_{10} \) the expansion ratio of bubble 2 decreases monotonically as \( R_{20} \) increases. The latter is due to the fact that the expansion rate of bubble 2 has an almost constant value determined by Eq. (5) although \( R_{20} \) changes, and hence a larger \( R_{20} \) gives a smaller expansion ratio. In Fig. 8 we show phase diagrams of max\( [R_j(t)]/R_{j0} \) in a wider parameter range. Here, both \( R_{20} \) and \( D_{12} \) are used as parameters, and three different values of \( R_{10} \) [(a) 1.6, (b) 2, and (c) 3 \( \mu m \)] are assumed. The white regions seen in the right side of the panels denote cases where the explosive expansion of bubble 1 is completely suppressed. As can be clearly seen, the white region becomes narrower as \( R_{10} \) increases. This is because \( p_{c1} \) increases and consequently the pressure from bubble 2 needed for the suppression of bubble 1...
becomes greater as $R_{10}$ increases. From the same figure, one can also see that larger $R_{20}$ and smaller $D_{12}$ values are more effective for cavitation suppression. This is consistent with the result predicted by Eq. (17) or (19), and also with the previous numerical finding. The above results prove that the suppression of explosive expansion can occur if the bubbles’ ambient radii and inter-bubble distance are appropriately set.

The results shown in Fig. 5 imply that a transition of bubble dynamics takes place in a parameter range between $D_{12} = 10(R_{10} + R_{20})$ and $15(R_{10} + R_{20})$. Here we clarify what occurs in the transition region. In Fig. 9 we show the radius-time curves of bubble 1 for $R_{10} = 2 \mu m$, $R_{20} = 20 \mu m$, $P_{ng} = -0.25P_0$, and five different $D_{12}$ values selected from the above-mentioned parameter range. The other parameters were set as in the above examples. As shown previously, decreasing $D_{12}$ results in the decrease of the expansion rate of bubble 1. In the parameter range considered here, however, one more interacting change can be found: The expansion of bubble 1 is interrupted at a moment. In the results for $D_{12} = 12(R_{10} + R_{20}) \sim 12.1(R_{10} + R_{20})$, the bubble collapses although $p_L$ (the lower panel) holds constant at a negative value. Such a behavior is not allowed for isolated bubbles, which can only expand without bound. This observation suggests that bubble-bubble interaction sometimes causes a significant change in the lifetime of cavitating bubbles.

Let us consider the mechanism underlying this obser-
viation. As known from Eq. (17), the total pressure acting on bubble 1 increases as time goes on [i.e., as \( R_2(t) \) becomes greater]. Indeed, the total pressure at bubble 1’s position determined numerically by

\[
p_{T1} = p_L + \frac{\rho}{D_{12}} \frac{d(R_2^2 R_2)}{dt}
\]

increases for \( t > 0 \) (see Fig. 10). From this, one may presume that the interruption of bubble expansion occurs when the total pressure rises above the quasistatic threshold pressure of bubble 1. This conjecture is, however, incorrect or insufficient. The total pressure is clearly higher than the threshold pressure in most periods except for a short duration around \( t = 0 \), and their crossing points are far from the time of interruption (see Fig. 10). The notion of threshold pressure is therefore useless for the present purpose.

We suggest using the unstable equilibrium radius to understand the interrupted expansion. As mentioned in Sec. 11 a bubble has an unstable equilibrium radius under the condition of \( p_{C1} < p_L < p_C \) and begins unbounded expansion when the bubble’s instantaneous radius becomes greater than the unstable equilibrium radius, even though \( p_L > p_C \). We reveal here that the unstable equilibrium radius plays an essential role in the occurrence of interrupted expansion. Figure 11 shows the radius-time curves of bubble 1 for three different \( D_{12} \) values and the corresponding unstable equilibrium radius (called hereafter \( R_{U1} \)). Here \( R_{U1} \) was determined using Eq. (11) by replacing \( p_L \) with \( p_{T1} \) (21). Since \( p_{T1} \) is time dependent, \( R_{U1} \) varies in time. For \( D_{12} = 12.5(R_{10} + R_{20}) \), \( R_1 \) well exceeds \( R_{U1} \) for \( 0 \mu s < t < 37 \mu s \), and hence bubble 1 can expand rapidly as a single bubble does. For \( D_{12} = 12.05(R_{10} + R_{20}) \), \( R_1 \) is slightly larger than \( R_{U1} \) for \( 0 \mu s < t < 22 \mu s \) and bubble 1 expands mildly during this period. However, \( R_1 \) is exceeded by \( R_{U1} \) at about \( t = 22 \mu s \), and then bubble 1 stops expanding and begins collapsing. For \( D_{12} = 12(R_{10} + R_{20}) \), \( R_1 \) is smaller than or almost equal to \( R_{U1} \) in most periods, and hence bubble 1 cannot expand considerably and behaves as a stable bubble under positive absolute pressure. As can be seen in Fig. 11(b), the crossing point of \( R_1 \) with \( R_{U1} \) correlates well with the timing for bubble 1 to stop accelerating. This observation proves that the instantaneous unstable equilibrium radius can be used as a probe for the interruption of bubble expansion.

Until now we have only considered cases of \( T = 10 \mu s \). This time length is greater than the characteristic time of bubble 1, \( T_{b1} = 0.296 \mu s \) for \( R_{10} = 2 \mu m \), given by

\[
T_{b1} = \frac{\pi}{\omega_1}
\]
with $\omega_1$ being the isothermal eigenfrequency of bubble 1:

$$\omega_1 = \sqrt{-\frac{1}{\rho R_{10}^2} \left( 3 P_0 + \frac{4\sigma}{R_{10}} \right)}.$$  \hspace{1cm} (23)

Here we briefly examine the effect of smaller $T$. For smaller $T$, $p_L(t)$ exceeds the threshold pressures of the bubbles earlier than for larger $T$ and the bubbles’ dynamics and interaction effect would thus be altered. Also, as $T$ approaches $T_{b1}$, the response of bubble 1 to the decompression process should change. Figures 12 and 13 show results for $T = 2 \mu s$ and $0.3 \mu s$ ($\approx T_{b1}$), respectively, with $P_{ng} = -0.25P_0$. In the case of $T = 2 \mu s$, one finds that the inter-bubble distance of $D_{12} = 12(R_{10} + R_{20})$, which was sufficient to cause interrupted expansion when $T = 10 \mu s$, is now not sufficient and a smaller $D_{12}$ is required. This is because the rapid expansion of bubble 1 begins in an earlier stage than in the previous example, where the radius and expansion rate of bubble 2 are not so large. This result reveals that the dynamics of interacting bubbles depends on the time history of the motion. The same tendency can be found in the case of $T = 0.3 \mu s$, where an even smaller $D_{12}$ is needed to interrupt the expansion of bubble 1. However, these results are qualitatively the same as that for $T = 10 \mu s$: Bubble 1 can grow considerably only when $R_1 > R_{ue1}$.

D. The case of larger numbers of bubbles

So far we have considered double-bubble cases. In realistic cavitation, however, a large number of bubbles emerge at the same time and should interact with each other in a complicated manner. As a first step toward
understanding the inception processes in many-bubble cases, in this subsection we discuss how a larger number of gas bubbles change the negative pressure amplitude in water. As shown below, a larger number of bubbles decrease the negative pressure amplitude more significantly and thus have a stronger suppression effect.

In this discussion, we categorize the bubbles considered into two groups. The bubbles in the first group are assumed to be very small, and their rapid expansion is completely suppressed by other bubbles. The bubbles in the second group are relatively large and can expand rapidly under an assumed negative pressure. This assumption allows us to neglect the impact from the first group on the second group, because the amplitudes of the pressure waves from the bubbles in the first group are negligibly small. We furthermore assume, as done in the local homogeneity assumption, that the Rayleigh-Plesset equation for single bubbles can be integrated analytically if the external pressure is constant and the liquid viscosity is neglected.

The theoretical result is verified numerically for few-bubble systems using a test problem similar to that considered by Chahine and Liu. We derive the theoretical formula for $\dot{R}$ of the bubbles in the second group and use it to examine the influence of bubble-bubble interaction on the expansion rate and negative pressure amplitude. Since the expansion rate depends on the total negative pressure on the bubble, one can see from the theoretical formula how the bubbles change the negative pressure value. The theoretical result is verified numerically for few-bubble systems using a test problem similar to that considered by Chahine and Liu.

We derive the theoretical formula for $\dot{R}$ by integrating the coupled Rayleigh-Plesset equation. It is known that the Rayleigh-Plesset equation for single bubbles can be integrated analytically if the external pressure is constant and the liquid viscosity is neglected. The derived formulas, which describe the relation between $\dot{R}(t)$ and $R(t)$, have provided several important insights into bubble dynamics, such as the asymptotic expansion rate under constant negative pressure and the minimum size of a bubble at collapse. Here we extend the formulas to multibubble systems. A theoretical formula for multibubble systems was given in Ref. [1] under the assumptions of $\dot{R} \approx 0$ and $R_i \gg R_{i0}$. The present formula is derived without these assumptions and thus has higher accuracy and wider applicability. We first consider a system of two identical bubbles and then extend to systems of larger numbers of identical bubbles arranged symmetrically.

When $N = 2$ and $R_1(t) = R_2(t)$, Eq. (3) is reduced to a single differential equation,

$$R_1 \ddot{R}_1 + \frac{3}{2} \dot{R}_1^2 = \frac{1}{\rho} \dot{s}_{1} - \frac{1}{D_{12}} \frac{d}{dt} \left( R_1^2 \dot{R}_1 \right),$$

which can be rewritten as

$$\frac{1}{2} \left( 2R_1 \ddot{R}_1 + 3 \dot{R}_1^2 \right) + \frac{R_1}{2D_{12}} \left( 2R_1 \dot{R}_1 + 4 \dot{R}_1^2 \right) = \frac{1}{\rho} \dot{s}_{1}.$$  

We attempt here to transform the left-hand side of this equation to a time derivative. Using the identity of

$$\frac{1}{R_1^{m-1}} \frac{d}{dt} \left( R_1^m \dot{R}_1 \right) = 2R_1 \ddot{R}_1 + m \dot{R}_1^2,$$

where $m$ is an integer, the left-hand side of Eq. (25) is rewritten as

$$\frac{1}{2} \left( 2R_1 \ddot{R}_1 + 3 \dot{R}_1^2 \right) + \frac{R_1}{2D_{12}} \left( 2R_1 \dot{R}_1 + 4 \dot{R}_1^2 \right) = \frac{1}{\rho} \dot{s}_{1}.$$  

From Eqs. (24) and (25) we have

$$\frac{d}{dt} \left( R_1^2 \dot{R}_1 + \frac{R_1^2 \dot{R}_1^2}{D_{12}} \right) = \frac{2R_1^2 \dot{R}_1}{\rho} \dot{s}_{1}.$$  

Assuming $dp_L/dt = 0$ and $\mu = 0$, the right-hand side of this equation can also be rewritten into a time derivative,

$$\frac{d}{dt} \left[ \frac{-2p_L}{3\rho} R_1^3 + \frac{1}{\rho} \left( P_0 + \frac{2\sigma}{R_{10}} \right) R_1^3 F - \frac{2\sigma}{R_1^2} \right],$$

where

$$F = \begin{cases} \frac{2}{3(1 - \kappa_1)} \left( \frac{R_{10}}{R_1} \right)^{3\kappa_1} & \text{for } \kappa_1 \neq 1, \\ \frac{2}{3} \left( \frac{R_{10}}{R_1} \right)^{3} \log R_1 & \text{for } \kappa_1 = 1. \end{cases}$$

Substituting Eq. (29) into Eq. (28) and integrating, we finally have

$$\left( 1 + \frac{R_1}{D_{12}} \right) \ddot{R}_1^2 = -\frac{2p_L}{3\rho} + \frac{1}{\rho} \left( P_0 + \frac{2\sigma}{R_{10}} \right) \frac{\alpha}{\rho R_1^2}$$

Here, $\alpha$ is the constant of integration which is determined by an initial condition. For $R_1(t = 0) = R_{10}$ and $\dot{R}_1(t = 0) = 0$, for instance,

$$\alpha = \frac{2p_L R_{10}^3}{3\rho} - \frac{2}{3(1 - \kappa)} \left( P_0 + \frac{2\sigma}{R_{10}} \right) R_{10}^3 \frac{2\sigma R_{10}^2}{\rho}$$

for $\kappa \neq 1$, or

$$\alpha = \frac{2p_L R_{10}^3}{3\rho} - \frac{2}{\rho} \left( P_0 + \frac{2\sigma}{R_{10}} \right) R_{10}^3 \log R_{10} + \frac{2\sigma R_{10}^2}{\rho}$$

for $\kappa = 1$. We should note that the vapor pressure $p_v$ is neglected in this theory but it can easily be taken into account by only replacing $P_0$ and $p_L$ with $P_0 - p_v$ and $p_L - p_v$, respectively.

To confirm the accuracy of the presented formula, we performed a numerical test. In Fig. 14 we show numerical and theoretical results for $R_{10} = 20 \mu m$, $D_{12} = 20R_{10}$, $\kappa_1 = 1$, $P_{sg} = -0.25P_0$, $\mu = 0$ Pa s, and $T = 0 \mu s$. The numerical results were obtained by directly solving the coupled Rayleigh-Plesset equation (24), and the theoretical results were obtained using Eq. (31) with $R_1$ given numerically and $\alpha$ at $t = 0 \mu s$. These results are indistinguishable, proving the accuracy of Eq. (31). Figure 14 also shows the result for $D_{12} \to \infty$, from which one finds that bubble-bubble interaction decreases the expansion rate of the bubbles. In Fig. 15 we show the numerical
Expansion rates [m/s]  

FIG. 14: (Color online) Expansion rates of coupled bubbles for $R_{10} = 20 \mu m$, $D_{12} = 20R_{10}$, and $\mu = 0$ Pa s. The lower panel shows $P_L/P_0$ assumed in this example, normalized by $P_0$ ($P_{in} = -0.25P_0$, $T = 0 \mu s$). The numerical (the solid curve) and theoretical (the dots) results of the expansion rate are indistinguishable. The dash-dotted curve shown in the upper panel is the expansion rate of a single bubble with the same ambient radius.

Expansion rates [m/s]  

FIG. 15: (Color online) Same as Fig. 14 but for $\mu = 1.002 \times 10^{-3}$ Pa s. The lower panel shows the instantaneous Reynolds number given by Eq. (32).

result of a viscous case with $\mu = 1.002 \times 10^{-3}$ Pa s. The result is in close agreement with the theory. This is because the viscosity has, in the present case, only a minor effect in most periods; see Fig. (b) which reveals that the instantaneous Reynolds number at the bubble surface, defined here by

$$ Re = \frac{\text{inertial term}}{\text{viscous term}} = \frac{3}{2} \frac{\dot{R}^2_i}{4\mu R_i} = \frac{3}{8} \frac{\rho R_i \dot{R}_i}{\mu}, $$

is large except for in the early stage.

In Eq. (31), the effect of bubble-bubble interaction is concentrated in the term of

$$ \left(1 + \frac{R_1}{D_{12}}\right). $$

For $D_{12} \to \infty$, this converges to unity and the formula is reduced to that for single bubbles. Obviously a smaller $D_{12}$ leads to a smaller expansion rate. This indicates that bubble-bubble interaction decreases the expansion rate as observed in the above numerical test. This conclusion is consistent with earlier numerical observations. The decrease in expansion rate obviously indicates the reduction of the negative pressure amplitude at the bubbles’ position. As discussed in Sec. IV B this reduction is caused by the positive pressure waves emitted by the bubbles themselves, that is, bubbles expanding under negative pressure emit positive pressure waves that reduce the negative pressure amplitude in the liquid.

Next, we briefly consider cases of larger numbers of bubbles using the theoretical formula. Here, following Ref. 18, we discuss systems of identical bubbles arranged symmetrically. This configuration allows us to use a model equation similar to Eq. (24). Examples considered are three bubbles arranged as a regular triangle (Case 3), four bubbles arranged as a regular tetragon (Case 4a), four bubbles arranged as a regular tetrahedron (Case 4b), and eight bubbles arranged as a regular hexahedron (Case 8). We assume that the center-to-center distances between nearest-neighbor bubbles (i.e., the length of the sides of the regular arrangements) in all cases are the same, $D_{12}$. These assumptions reduce Eq. (6) to a single differential equation,

$$ R_1 \ddot{R}_1 + \frac{3}{2} \frac{\dot{R}^2_1}{R_1} = \frac{1}{\rho} - 1 \frac{d(R^2_1 \dot{R}_1)}{dt}, $$

where $E_\alpha$ is the effective inter-bubble distance given as follows:

$$ E_\alpha = \begin{cases} E_3 = \frac{D_{12}}{2} & \text{for Case 3}, \\ E_{4a} = \frac{2}{3} + \frac{2}{3} & \text{for Case 4a}, \\ E_{4b} = \frac{D_{12}}{4} & \text{for Case 4b}, \\ E_8 = \frac{3}{4} & \text{for Case 8}. \end{cases} $$

Obviou
Fig. 16: (Color online) Bubble radii and total driving pressures on bubble 1 for different bubble populations. The parameters are the same as those used in Fig. 14. The numbers shown in the panels indicate the case numbers (“1” and “2” are for the single- and double-bubble cases, respectively). As indicated by the arrows, the radius decreases and the total driving pressure increases as the bubble population increases. Since Eq. (36) has the same form as the double-bubble formula (24), the exact formula (31) can be used for the present cases by replacing \( D_{12} \) with \( E_\alpha \). From Eq. (37), one easily finds

\[
D_{12} > E_3 > E_{4a} > E_{4b} > E_8,
\]

that is, a larger number of bubbles or a denser population leads to a stronger interaction effect and a smaller expansion rate (note that Case 4b, whose mean inter-bubble distance is \( D_{12} \), is denser than Case 4a). This is confirmed by the numerical result shown in Fig. 16 (a) given by directly solving Eq. (36), which reveals the same tendency as expected. The present theoretical result is consistent with the numerical finding by Chahine and Liu.

The above theoretical result can also be interpreted as indicating that the presence of a larger number of bubbles causes a greater reduction in the negative pressure amplitude. In a system of a larger number of bubbles, the total amplitude of the bubble-emitted positive pressures should be greater and hence the negative pressure in the liquid, the driving force on the bubbles, is more reduced compared to the case of two bubbles [see Fig. 16 (b)]. This naturally results in a greater reduction of expansion rate, as seen above. From this argument one can conclude that a larger number of bubbles have a stronger suppression effect on neighboring (smaller) bubbles than that of single bubbles.

Although we have only considered few-bubble systems, the theoretical formula for \( R \) (31) would be applicable to a larger cluster of bubbles. In recent years, there has been an effort to use a very simple model to study the dynamics of large bubble clusters. In those works, the degree of freedom of a bubble cluster is significantly reduced by a local homogeneity assumption and a model equation very similar to Eq. (24) is derived. The present theoretical formula provides a solution of the simple model equation.

V. CONCLUDING REMARKS

We have studied the inception processes of cavitation in multibubble cases, where multiple cavitation nuclei interact with each other, and have shown that bubble-bubble interaction changes the inception processes in a variety of ways. Performing numerical simulations of the dynamics of non-identical bubbles under negative pressure, we have demonstrated that the suppression of the explosive expansion of small bubbles by bubbles expanding earlier, recently reported for liquid mercury, is possible in water as well. To more deeply understand the numerical observation, we have discussed the effective threshold pressure of interacting bubbles. We found that even a bubble can significantly decrease (by about 50%) the effective threshold pressure of a smaller neighboring bubble. This change of threshold value is much more significant than that caused by the dynamic effect due to rapid change in the far-field liquid pressure, which is only about 1.7% in our case.

From a detailed analysis of the transition region where the dynamics of the suppressed bubble drastically changes as the inter-bubble distance changes, we have revealed that the explosive expansion of a bubble under negative pressure can be interrupted and turn into collapse even though the far-field liquid pressure remains well below the bubble’s (quasistatic) threshold pressure. Using the notion of unstable equilibrium radius, we have found that the interruption of bubble expansion takes place when the instantaneous bubble radius is exceeded by the instantaneous unstable equilibrium radius determined using the total pressure acting on the bubble. Both the suppression and interruption of bubble expansion are caused by the pressure wave that a neighboring bubble emits when it grows.

By analytically integrating the acoustically coupled Rayleigh-Plesset equations under the assumption of con-
stant far-field liquid pressure, we have derived an exact formula for the surface velocity of interacting identical bubbles. The formula shows clearly that bubble-bubble interaction decreases the expansion rate of bubbles and that the decrease is more prominent for systems of a larger number of bubbles. This theoretical result is consistent with earlier numerical findings.\textsuperscript{18} This change in the expansion rate implies that the presence of a larger number of bubbles reduces the negative pressure amplitude in water to a greater degree than in the case with two bubbles, and hence a larger number of bubbles have a stronger suppression effect. The presented exact formula would be applicable to a large bubble cluster by incorporating the local homogeneity assumption.\textsuperscript{22}

The present findings could be a key to understanding the complex behavior of cavitation bubbles in practical situations where a large number of cavitation nuclei exist and interact with each other. From the present results, one can for example speculate as follows: Even if a liquid involves a large number of cavitation nuclei and one imposes a strong negative pressure that can cavitate all of the individual nuclei, only part of them will grow and be observed because bubbles growing earlier suppress the expansion of the remaining neighboring nuclei. If this scenario is true, prediction of the actual population of cavitation bubbles in practical situations will require that, among other things, one know not only the initial distribution of cavitation nuclei but also their dynamic behavior in which bubble-bubble interaction plays a significant role. Also, the present findings may be relevant to the study of the superstability of nanobubbles,\textsuperscript{53} where the interaction between nanobubbles on a hydrophobic surface and cavitation microbubbles is seen. In that study, Borkent \textit{et al.} found experimentally that surface nanobubbles do not cavitate even for a sufficiently strong negative pressure, while cavitation bubbles originating from microscopic cracks are rapidly expanding.

More detailed analysis of multibubble cavitation dynamics by bifurcation and perturbation theories or other theories that have been used for single-bubble study\textsuperscript{30,31,32,33,34} would be an interesting subject to pursue. Also, detailed parametric study of the effective threshold pressure in multibubble cases could provide useful insights into the cavitation threshold pressure in practical applications. Statistical analysis of the actual population of explosively expanding bubbles in cases where non-identical nuclei interact would also be a meaningful area of study. Combining the presented pictures of cavitation inception with a scenario of nuclei merging, like that proposed by Marschall \textit{et al.}\textsuperscript{13} may provide a more realistic picture of multibubble cavitation inception.

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