CELLULAR STRUCTURE FOR THE HERZOG–TAKAYAMA RESOLUTION

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ABSTRACT. The Herzog–Takayama resolution is the minimal free resolution for the so called ideals with a regular linear quotient. This class contains all matroidal and stable ideals. The resolutions of matroidal and stable ideals are known to be cellular. In this note we show that the Herzog–Takayama resolution is also cellular.

1. Introduction

In this section we describe our problem. The exact definitions will appear in the next sections. Also, we refer to the books by Hatcher [8], Peeva [14] and Milller and Sturmfels [12], for undefined terminology.

The idea of determining free resolutions of monomial modules over a polynomial ring using labelled chain complexes of topological objects was introduced by Bayer, Peeva and Sturmfels [2] by considering the labeled chain complex of simplicial complexes. Every monomial module admits such a resolution, namely the Taylor resolution, although it is most of the time far from being minimal. This method was generalized in [3], where the authors constructed a regular cell complex (the Hull complex) for any monomial modules which from its labeled chain complex one gets a resolution of the module that is much shorter than the Taylor resolution in general. Batzies and Welker in [1] developed an algebraic analogue of Forman’s discrete Morse theory to minimize the resolutions.

Unfortunately, even if we consider the more general case of CW complexes, as the authors did in [1], it is not the case that one can describe the minimal free resolution of a monomial module using these methods. A family of non-examples is given in [17]. However, for a monomial ideal with a linear quotient it is shown in [1, Proposition 4.3,] that there exists a CW-complex which supports the minimal free resolution of the ideal. The minimal free resolution of a class of ideals with a linear quotient, the so called ideals with a regular linear quotient (see Section 2, for definition), is determined by Herzog and Takayama in
This class contains all matroidal ideals, stable ideals and square-free stable ideals. In [13] the authors have given a polytopal complex that supports the minimal free resolution of matroidal ideals. In [11], Mermin constructed a regular cell complex that supports the Eliahou-Kervaire resolution for stable ideals. On the other hand, since the square free part of the Eliahou-Kervaire resolution resolves square-free stable ideals minimally, such a construction for square-free stable ideals exists. So it is natural to ask whether the Herzog-Takayama resolution is cellular or not? The aim of this paper is to give a positive answer to the question above by explicitly constructing a regular cell complex that supports the Herzog-Takayama resolution for ideals with a regular linear quotient.

2. Monomial Ideals with a Linear Quotient and Cellular Resolution

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ be polynomial ring in $n$ variables. For a monomial ideal $I$ in $S$, we denote by $M(I)$ the set of all monomials in $I$. We also denote by $G(I)$ the unique minimal set of generators of $I$. We say that $I$ has a linear quotient, if $G(I)$ admits an admissible order, that is a linear ordering $u_1, \ldots, u_m$ of monomials in $G(I)$ such that the ideal $\langle u_1, \ldots, u_{j-1} \rangle : u_j$ is generated by a subset $q(u_j)$ of variables for all $2 \leq j \leq m$. To any admissible order of $I$ one can associate a unique decomposition function, that is, a function $g : M(I) \rightarrow G(I)$ that maps a monomial $v$ to $u_j$, if $j$ is the smallest index for which $v \in I_j$, where $I_j := \langle u_1, \ldots, u_j \rangle$ (see [9] for more on decomposition functions).

It is not difficult to check that an ordering $u_1, \ldots, u_m$ of $G(I)$ is an admissible order if and only if it satisfies the following shelling type condition which considered by Batzies and Welker [1, Page 157].

For all $j$ and $i < j$ there exists $k < j$ such that

\[ \text{lcm}(u_k, u_j) = x_t \cdot u_j \text{ for some } x_t \text{ that divides } u_i. \]  

It is known [10] Lemma 2.1] that for a monomial ideal with a linear quotient there always exists a degree increasing admissible order. So, throughout this note all admissible orders are considered to be increasing.

The following result was proved in [1] Proposition 4.3] by using the algebraic discrete Morse theory, here we give an alternative proof.

**Proposition 2.1.** For an ideal $I$ with a linear quotient, there exists a CW complex $X_I$ that supports the minimal free resolution of $S/I$. 
Proof. Let \( u_1, \ldots, u_m \) be a degree increasing admissible order of \( G(I) \). Assume that we have inductively constructed a CW complex \( X_{j-1} \) that supports the minimal free resolution of \( S/I_{j-1} \). Also, let \( \Delta \) be the simplex associated to the Koszul resolution of \( S/\langle q(u_j) \rangle \). Lifting the left non-zero map of the short exact sequence

\[
0 \rightarrow S/\langle q(u_j) \rangle \rightarrow S/I_{j-1} \rightarrow S/I_j \rightarrow 0,
\]

to the minimal free resolutions of \( S/\langle q(u_j) \rangle \) and \( S/I_{j-1} \), then induces a cellular map of \( \Delta \) into \( X_{j-1} \). Therefore the resolution of \( S/I_j \) obtained by the homological mapping cone has a cellular structure that is the topological mapping cone of the cellular map of \( \Delta \) into \( X_{j-1} \). The fact that this resolution is minimal follows from [9, Lemma 1.5] \( \square \)

Remark 2.2. Reiner and Welker [15, Section 5] gave an example of an ideal with a linear quotient for which the differential matrices in the minimal free resolution cannot be written using only \( \pm 1 \) coefficients. This shows that the cell complex \( X_I \) in Proposition 2.1 is not regular in general.

3. Monomial Ideals with a Regular Linear Quotient and Associated Regular Cell Complexes

The difficulty of constructing the cell complex \( X_I \) in the practice is then how to define the cellular maps in each step, which is indeed as difficult as defining the comparison maps \( S/\langle q(u_j) \rangle \rightarrow S/I_{j-1} \). This has been already observed by Herzog and Takayama in [9], where the authors defined a subfamily of ideals with a linear quotient that have decomposition function with a similar behaviour as of the stable ideals'. A decomposition function \( g \) is said to be regular, if \( q(g(yu_j)) \subseteq q(u_j) \), for any \( j \) and any \( y \in q(u_j) \). We say that \( I \) has a regular linear quotient, if it admits an admissible order with a regular decomposition function. The following equation for the decomposition function of ideals having regular linear quotient is proved in [9], we will frequently use it

\[
(2) \quad g(yg(zu)) = g(zg(yu)) \quad \forall u \in G(I) \quad \forall z, y \in q(u).
\]

It can be easily checked that the set of ideals with a regular linear quotient contains all stable and square-free stable ideals. On the other hand it is known [9, Theorem 1.10] that the reverse lexicographic order on minimal generators of a matroidal ideal is an admissible order with a regular decomposition function. The minimal free resolution of ideals with a regular linear quotient was given in [9, Theorem 1.12]. We shall call their construction the Herzog and Takayama resolution.
Construction 3.1. Let $I$ be a monomial ideal with a regular linear quotient with respect to the admissible order $u_1, \ldots, u_m$ of $G(I)$. Also let $g$ be its decomposition function. We will construct a regular labelled cell complex $X_I$ inductively, as follows

(i) Let $X_1$ be the 0-simplex with the labelled vertex $\{u_1\}$,
(ii) assume that the regular labeled cell complex $X_{j-1}$ with vertices $u_1, \ldots, u_{j-1}$ is constructed,
For what follows, let $u = u_j$ be a point outside $X_{j-1}$ and for a subset $\sigma$ of $q(u)$, define $g(\sigma; u) := g(u \cdot \prod_{y \in \sigma} y)$. Also, denote by $X(u)$ the subcomplex of $X_{j-1}$, induced by $\{g(\sigma; u)\}$ for all subsets $\sigma$ of $q(u)$.
(iii) glue an $l$-ball $B(u)$ along its boundary to $X(u) \cup \{u\} * \partial X(u)$, where $l = |q(u)|$.
Having defined the new maximal cell $B(u)$, what is remained to have the complete description of $X_j$ is to give a cell decomposition for $\{u\} * \partial X(u)$. For this purpose let us for any proper subset $\sigma$ of $q(u)$, denote by $X(\sigma, u)$ the subcomplex of $X_{j-1}$ induced by the vertices $\{g(\tau; u)\}$ for all subsets $\tau$ of $\sigma$.
(iv) define $X_j$ to be $X_{j-1} \cup \{B(u)\} \cup_{\sigma \subseteq q(u)} \{\{u\} * X(\sigma, u)\}$.

Note that in our construction above, we need $X(u)$ to be a ball. The proof of this, as well as more technical details on it, is the topic of the next section.

4. A Simplicial Subdivision

In this section, combinatorial properties of the cell complex $X_I$ constructed in 3.1 are studied via a simplicial subdivision of it. We follow the notation in Construction 3.1. Let us start by giving the simplicial subdivision of $X_I$.

Construction 4.1. We construct a simplicial complex $\Lambda_I$ inductively as follows:

(i) Let $\Lambda_1$ be the 0-simplex $\{u_1\}$,
(ii) Assume that $\Lambda_{j-1}$ is constructed,
(iii) Take a cone with apex $\{u\}$ over the subcomplex $\Lambda(u)$ of $\Lambda_{j-1}$, induced by $\{g(\sigma; u)\}$, for all subsets $\sigma$ of $q(u)$, to obtain $\Lambda_j = \Lambda_{j-1} \cup \{\{u\} * \Lambda(u)\}$.

The following is immediate.

Lemma 4.2. $\Lambda_j$ (respectively $\Lambda(u)$) is a simplicial subdivision of $X_j$ (respectively $X(u)$).
A closure operator on a finite set $E$ is a function $c : 2^E \to 2^E$, such that for all $\sigma, \tau \subseteq E$,

1. ($CO_1$) $\sigma \subseteq c(\sigma)$,
2. ($CO_2$) $\sigma \subseteq \tau$ implies $c(\sigma) \subseteq c(\tau)$,
3. ($CO_3$) $c(c(\sigma)) = c(\sigma)$.

A closure operator $c$ is said to be an anti-exchange closure if, in addition, for all $a \neq b$ in $E$, it satisfies the following anti-exchange axiom:

(AE) If $a, b \notin c(\sigma)$ and $a \in c(\{b\} \cup \sigma)$, then $b \notin c(\{a\} \cup \sigma)$.

A finite set $E$ together with an anti-exchange closure $c$ on it is called a convex geometry. See Björner and Ziegler [6], for a comprehensive introduction to a more general topic, greedoids.

Now let us define a closure operator on $\langle q(u), c \rangle$: For a subset $\sigma$ of $q(u)$ we let $c(\sigma)$ to be the largest subset $\tau$ of $q(u)$ such that $g(\sigma; u) = g(\tau; u)$. The fact that this operator is well defined follows from the simple fact that for any $\delta_1$ and $\delta_2$ if $g(\delta_1; u) = g(\delta_2; u)$, then $g(\delta_1; u) = g(\delta_1 \cup \delta_2; u)$.

**Proposition 4.3.** The pair $\langle q(u), c \rangle$ is a convex geometry.

**Proof.** The axioms ($CO_1$) and ($CO_2$) clearly hold. Assume that $\tau$ is the union $\delta \cup \sigma$. From the equation $g(\delta \cup \sigma; u) = g(\delta; g(\sigma; u))$, it follows that $g(\tau; u) = g(\delta \cup c(\sigma); u)$ and in particular $c(\sigma) \subseteq c(\tau)$.

Now, we shall verify the anti-exchange axiom. Let $\sigma$ be any subset of $u$ and denote by $v$ the monomial minimal generator $g(\sigma; u)$. The condition that $a \notin c(\sigma)$ is equivalent to $a \in q(v)$. Now for a variable $b$ in $q(v)$ different from $a$, let $a \in c(\{b\} \cup \sigma)$, or equivalently $g(abv) = g(bv)$. The fact that $g(abv)$ is a minimal monomial generator different from $v$ that divides $av$ implies that $g(abv) \neq g(bv) = g(abv)$ and therefore $b \notin c(\{a\} \cup \sigma)$.

The set of all closed sets of a convex geometry, i.e. those subsets that are fixed under the closure operator, forms a lattice when the partial ordering is inclusion. This lattice is known to be meet-distributive, see e.g. [7] Theorem 3.3].

**Proposition 4.4.** $\Lambda(u)$ is a shellable $(l - 1)$-dimensional ball.

**Proof.** Let $L$ be the lattice of closed sets of $\langle q(u), c \rangle$ and $\hat{L} = L \setminus \{\emptyset\}$. The decomposition function $g(\cdot; u)$ can be seen as a bijection between the elements of $\hat{L}$ and vertices of $\Lambda(u)$, that maps every closed set $\sigma$ to $g(\sigma; u)$. Furthermore, via this map the faces of $\Lambda(u)$ are precisely chains in $\hat{L}$ and hence $\Lambda(u)$ is the order complex of $\hat{L}$. Now, since the order complex of the proper part of any meet-distributive lattice is a shellable ball (see, e.g., [4], $\Lambda(u)$ is a shellable ball. The argument about dimension follows from [6] Proposition 8.7.2.(iv)].
A simplicial complex $K$ is said to be a PL–ball if a subdivision of $K$ is a subdivision of a simplex. A regular cell complex $\Gamma$ is a PL–ball if its barycentric subdivision is a PL-ball. Many properties that we expect balls to have fail when they are not PL, for instance the complex obtained by gluing two balls with the same dimension along a common codimension one ball lying on their boundaries is not necessarily a ball, if they are not PL. We refer to [5, Section 4.7.(d)] and the references therein for a detailed discussion on the subject.

We mentioned that for our construction to be well-defined we need $X(u)$ to be a ball, but actually the inductive construction requires more, that is

**Corollary 4.5.** $X(u)$ is an $(l-1)$-dimensional PL–ball.

**Proof.** It follows from Lemma 4.2, Proposition 4.4 and [4, Corollary 2.2].

□

5. Main Result

In this section we prove our main result. We begin by proving two auxiliary lemmas.

**Lemma 5.1.** $X_I$ is contractible.

**Proof.** Let $f : X(u) \rightarrow X_{j-1}$ be the inclusion map. Then $X_I$ is homotopy equivalent to the mapping cone $C_f$ of $f$. Now since $X(u)$ and $X_{j-1}$ are contractible, by [16, Page 27] $f$ is a homotopy equivalence and therefore $C_f$ is contractible (see, e.g., [8, Section 4.2] for more details).

□

Let $I \subset S$ be a monomial ideal and $\mu$ a monomial in $S$. Then we denote by $I_{\leq \mu}$ the ideal generated by those monomials in $G(I)$ that divide $\mu$.

**Lemma 5.2.** Let $I \subset S$ be a monomial ideal with linear quotient and let $\mu$ be a monomial in $S$. Then $I_{\leq \mu}$ has linear quotient. Moreover if $I$ is regular, then so is $I_{\leq \mu}$.

**Proof.** Let $u_1, \ldots, u_m$ be an admissible order of minimal generators of $I$ and assume that $u_{i_1} \ldots, u_{i_t}$ generates $I_{\leq \mu}$, where $i_1 < \ldots < i_t$. We show that $u_{i_1} \ldots, u_{i_t}$ is an admissible order for $I_{\leq \mu}$. For $s < t$ there exists $l < i_t$ such that $\text{lcm}(u_{i_t}, u_{i_s}) = u_{i_t} x_r$ for some $x_r$ which divides $u_{i_s}$. In particular $u_l$ divides $\mu$ and hence $u_l \in I_{\leq \mu}$.

For the second part denote by $g'$ the decomposition function of $I_{\leq \mu}$. Let $y$ be a variable such that $g'(yu_{i_j}) \neq u_{i_j}$ and $z$ another variable such that $g'(zu_{i_j}) \neq g'(yu_{i_j})$. To show that $g'$ is regular, then it is
enough to show that $g'(zu_{ij}) \neq u_{ij}$. Assume not. Then for any $l < i_j$ that $u_l$ divides $zu_{ij}$, $u_l$ does not divide $\mu$ and in particular $\deg_z(u_{ij}) = \deg_z(\mu)$, where $\deg_z$ stands for the degree of $z$ in the monomial. On the other hand, $v = g'(zu(yu_{ij}))$ divides $zyu_{ij}$. So, $v$ divides $yu_{ij}$, since $\deg_z(v) \leq \deg_z(\mu) = \deg_z(u_{ij})$. A contradiction, since $v = g'(zu(yu_{ij}))$ appears earlier than $g'(yu_{ij})$ in the admissible order.

□

Now we are in the position to prove our main result.

**Theorem 5.3.** If $I$ is has regular linear quotient, then the labeled regular cell complex $X_I$ supports the minimal free resolution of $I$.

**Proof.** First observe that for any monomial $\mu$ the subcomplex $X_{\leq \mu}$ of $X_I$, consists of all cells with a label that divides $\mu$, is the same as the complex $X_{I_{\leq \mu}}$ and hence is contractible by lemmas 5.1 and 5.2. Now we shall show that any two cells with a non trivial containment relation have different labels. Let $c$ be a cell of $X_I$. Then the vertices of $c$ are $v$ together with $\{g(\sigma; v)\}_{\sigma \subseteq \tau}$, for some $v$ and some $\tau \subseteq q(v)$ and in particular the label of $c$ is $v \prod_{z \in \tau} z$. For a maximal cell $c'$ of $c$, we have only two possibilities: either $c'$ contains $v$ or not. In the former case the other vertices of $c'$ are $\{g(\sigma; v)\}_{\sigma \subseteq \tau'}$, where $\tau' = \tau \setminus \{y\}$ for some variable $y$. So, the label of $c'$ is different from the label of $c$. In the latter case, the vertices of $c'$ are $g(yv)$ together with $\{g(\sigma; g(yv))\}_{\sigma \subseteq \tau'}$, for some $y \in q(v)$, where $\tau' = \tau \setminus \{y\}$. The label of $c'$ is then $g(yv) \prod_{z \in \tau'} z$ which is different from the label of $c$, since $\deg(g(yv)) \leq \deg(v)$ and $|\tau'| < |\tau|$.

□

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