Rectilinear and $O$-convex hull with minimum area*

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Abstract

Let $P$ be a set of $n$ points in the plane and $O$ be a set of $k$ lines passing through the origin. We show: (1) How to compute the $O$-hull of $P$ in $\Theta(n \log n)$ time and $O(n)$ space, (2) how to compute and maintain the rotated hull $OH_\theta(P)$ for $\theta \in [0, 2\pi)$ in $O(kn \log n)$ time and $O(kn)$ space, and (3) how to compute in $\Theta(n \log n)$ time and $O(n)$ space a value of $\theta$ for which the rectilinear convex hull, $RH_\theta(P)$, has minimum area, thus improving the previously best $O(n^2)$ algorithm presented by Bae et al. in 2009.

1 Introduction

Restricted-orientation convexity is a generalization of traditional convexity that stems from the interest in restricted-orientation geometry, where the geometric objects under study comply with restrictions related to a fixed set of orientations. Restricted-orientation geometry started with the work of Güting [15] in the early eighties, as a generalization of the study of orthogonal polygons (whose edges are parallel to the coordinate axes). In particular, the interest in the rectilinear convex hull of sets of geometric objects arises from the study of orthogonal convexity [19, 20, 23], a non-traditional notion of convexity that restricts convex sets to those (known as ortho-convex) whose intersection with any line parallel to a coordinate axis is either empty or connected. For geometric object sets in the plane, the rectilinear convex hull has been extensively studied since its formalization in the early eighties, and it has found applications in several research fields including illumination [1], polyhedron reconstruction [11], geometric search [25], and VLSI circuit layout design [26]. Researchers have also studied relations between rectilinear convex hulls of colored point sets [3], and developed generalizations of orthogonal convexity [13, 14, 18] along with related computational results [4, 5, 6].

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We consider here the generalization of Fink and Wood [13] of ortho-convexity where convexity is defined by having intersection either empty or connected with any line parallel to those in a given set $O$ of lines through the origin. The corresponding $O$-convex hull, or $O$-hull for short, is then an orientation-dependent enclosing shape of a finite set of points in the plane. As every convex set is $O$-convex but not vice versa, the $O$-hull of a point set is always contained in the standard convex hull of the same point set (and therefore, in traditional enclosing shapes such as disks, ellipses or rectangles), so it is relevant in applications where the enclosing shape is required to have minimum area. Applications with such requirements can be found in pattern recognition from digital images: rotation dependent and minimum area enclosing shapes are commonly used in form shape analysis [12, 22, 27], and keep being studied because of their applications to feature classification [17, 24].

1.1 Definitions and preliminaries

Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points in the plane in general position. Let $\mathcal{C}H(P)$ denote the convex hull of $P$ and let $V = \{p_1, \ldots, p_h\}$ be the set of vertices of the boundary of the convex hull, $\partial(\mathcal{C}H(P))$, as we meet them when traversing $\partial(\mathcal{C}H(P))$ in counterclockwise order starting at an arbitrary vertex $p_1$. Further, let $E = \{e_1, \ldots, e_h\}$ be the set of edges of $\partial(\mathcal{C}H(P))$, where $e_i = p_ip_{i+1}$ and the indices are taken modulo $h$.

An open quadrant of the plane is the intersection of two open half planes whose supporting lines are parallel to the $x$- and $y$-axes. Further, an open quadrant is said to be $P$-free if it contains no elements of $P$. Let $W$ denote the set of all $P$-free open quadrants, then the orthogonal convex hull of $P$, also called rectilinear convex hull, is the set (see Figure 1 for an example):

$$\mathcal{R}H(P) = \mathbb{R}^2 \setminus \bigcup_{W \in W} W,$$

Figure 1: The orthogonal convex hull of a set of points.

Orthogonal convexity can be generalized by considering a finite set $O$ of $k$ different lines passing through the origin and saying that a set is $O$-convex if its intersection with any line parallel to an element of $O$ is either connected or empty.

The present work deals with the $O$-convex hull of a set $P$ of $n$ points. Let us label the lines in $O$ as $\ell_1, \ldots, \ell_k$ so that $i < j$ implies that the slope of $\ell_i$ is smaller than the slope of $\ell_j$. The origin splits each $\ell_i$ into two rays $r_i$ and $r_{i+k}$, generating a set of $2k$ rays. Hereinafter, indices are taken modulo $2k$ and sometimes $r_{i+k}$ will denoted as $-r_i$, i.e., the opposite ray of $r_i$. Given two indexes $i$ and $j$, we define the wedge $W_{i,j}$ to be the region spanned as we rotate $r_i$ in the counterclockwise direction until it reaches $r_j$. A translation of a $W_{i,j}$ wedge will be called a $W_{i,j}$ wedge, and one of these will be said to be $P$-free if it does not contain any point of $P$ in its interior, i.e., in the open $W_{i,j}$ wedge. Of particular interest to us is the set of open $W_{i+i+k}$ wedges, see Figure 2 for an example.
We denote by $W^i$ the union of all the $P$-free open $W^{i,k}_{i+1}$ wedges. Thus, by analogy with the orthogonal case, the $O$-hull of $P$ is (see Figure 3 for an example):

$$O\mathcal{H}(P) = \mathbb{R}^2 \setminus \bigcup_{i=1}^{2k} W^i.$$ 

In this paper we deal with rotations of the $O$-hull: Let $O_\theta$ be the set of lines obtained by rotating the elements of $O$ by an angle $\theta$. Clearly, the $O_\theta$-hull of $P$, called the rotated hull and denoted as $O\mathcal{H}_\theta(P)$, is different from $O\mathcal{H}(P)$: As we rotate $O$ by an angle of $\theta$, the wedges $W_{i,j}$ will also rotate and the sets $W^i$ will also change accordingly. We will denote the resulting set as $W^i_\theta$, so that the rotated hull $O\mathcal{H}_\theta(P)$ is now defined as (see Figure 4 for an example):

$$O\mathcal{H}_\theta(P) = \mathbb{R}^2 \setminus \bigcup_{i=1}^{2k} W^i_\theta.$$ 

Throughout this paper we will consider only sets $O$ such that the angle $\alpha_i$, defined by two consecutive rays $r_i$ and $r_{i+1}$ ($\alpha_{2k}$ defined by rays $r_{2k}$ and $r_1$) is at most $\frac{\pi}{2}$. We also denote by $sector(r_i, r_{i+1})$ to the sector formed by the rays $r_i$ and $r_{i+1}$. Thus, we assume that $\alpha_i \leq \frac{\pi}{2}$, implying that no $W^{i,k}_{i+1}$ wedge will have an aperture angle smaller than $\frac{\pi}{2}$. In other words, we do not allow the wedges to be smaller than a quadrant.
Figure 4: Changes on the rotated $O_\theta$-hull $O\mathcal{H}_\theta(P)$ while varying $\theta$.

1.2 Our contribution

In this paper we present the following results:

1. An algorithm to compute the $O$-hull, $O\mathcal{H}(P)$, of a set of $n$ points $P$ in $\Theta(n \log n)$ time and $O(n)$ space, regardless of the number $k$ of elements in $O$.

2. An algorithm to compute and maintain the rotated $O$-hull, $O\mathcal{H}_\theta(P)$, of a set of $n$ points $P$ for $\theta \in [0, 2\pi)$ in $O(kn \log n)$ time and $O(kn)$ space. Further, this allows to compute, within the same complexities, the orientation $\theta$ for which the boundary of $O\mathcal{H}_\theta(P)$ has minimum number of steps, of staircases, or of connected components.

3. For the particular case in which $O$ is composed by two orthogonal lines, our algorithm computes the value of $\theta \in [0, 2\pi)$ for which the rectilinear convex hull of $P$, $R\mathcal{H}_\theta(P)$, has minimum area, also returning $R\mathcal{H}_\theta(P)$ in $\Theta(n \log n)$ time. This improves the $O(n^2)$ time algorithm presented by Bae et al. [8].

2 Computing $O\mathcal{H}(P)$

In this section we give a $\Theta(n \log n)$ time and $O(n)$ space algorithm to compute the $O$-hull, $O\mathcal{H}(P)$, of a set of $n$ points $P$.

2.1 Computing the vertices

For each $r_i$, compute first the directed line parallel to $r_i$ which supports $CH(P)$ leaving $P$ on its right side. Suppose, without loss of generality, that each of these lines intersects $CH(P)$ at a single point, labelled $p_s$, $i = 1, \ldots, 2k$. Notice that it is not necessarily true that $p_s$ is different from $p_{s+1}$. Thus, $p_{s1}, p_{s2}, \ldots, p_{sk}$ are vertices of the boundary of the $O$-hull, $\partial(O\mathcal{H}(P))$, as we meet them in counterclockwise order. Note that these $p_s$ might not give all the vertices of $\partial(O\mathcal{H}(P))$, see Figure 5.

Because of the definition $O\mathcal{H}(P) = \mathbb{R}^2 \setminus \bigcup_{i=1}^{2k} W_i$, we need to compute $\partial W_i$ and this requires knowing when a wedge in $W_i$ can intersect the interior of $CH(P)$. It is easy to see that there are wedges in $W_i$ that intersect the interior of $CH(P)$ if, and only if, $p_s \neq p_{s+1}$, and that any wedge in $W_i$ intersecting the interior of $CH(P)$ necessarily does so by intersecting an edge of $\partial(CH(P))$ whose endpoints $p_j, p_{j+1}$ fulfill $s_i \leq j, j + 1 < s_{i+1}$. Let us denote by $[s_i, s_{i+1}]$ the
closed interval of those indices of vertices on $\partial(CH(P))$ between $s_i$ and $s_{i+1}$, called the **stabbing interval** of $W^i$. Figure 5 shows an example, where the first part of the following observation can also be checked.

![Figure 5: The O-hull of P, OH(P), in Figure 8 showing which edges of $\partial(CH(P))$, if any, are intersected by each $W^i$. Note that $W^i$ and $W^{i+1}$, lined instead of solid, do not intersect the interior of $CH(P)$. For $p_1$ being the uppermost point and moving counterclockwise, the stabbing intervals are $[1, 4]$, $[4, 6]$, $[6, 7]$, and $[7, 1]$.](image)

**Observation 1.** If $s$ belongs to the stabbing interval $[s_i, s_{i+1}]$ of a wedge in $W^i$, then the orientation of the edge $e_s$ of $\partial(CH(P))$ belongs to sector$(r_i, r_{i+1})$ in $O$. Also note that, if $O$ contains the supporting lines of the $h$ edges in $\partial(CH(P))$, then the stabbing interval of each of all the $W^i$ is a point and therefore $O\mathbb{H}(P) = CH(P)$.

It is easy to see that we can calculate the elements $p_{s_1}, \ldots, p_{s_{2k}}$ on $CH(P)$ in $O(n \log n)$ time. This gives us now the endpoints of the stabbing interval $[s_i, s_{i+1}]$. (Actually, only those intervals not being a single index will be needed, so that the others can be discarded.) Next, we calculate the alternating polygonal chain which joins them, which we refer to as staircase.

### 2.2 Computing the staircases between vertices

The staircase joining $p_{s_i}$ to $p_{s_{i+1}}$ is determined by wedges in $W^i$, with aperture angle $\Theta_i = \pi - \alpha_i \geq \frac{\pi}{2}$. See again Figure 5. Actually, such a staircase is contained in the boundary $\partial W^i$ where, in counterclockwise direction around $O\mathbb{H}(P)$, right turns arise at apexes of $O$-wedges in $W^i$, called extremal, and left turns arise at points of $P$ which are the supporting points of those extremal wedges. See Figure 6.

Before presenting how to compute the staircase, let us note that $O\mathbb{H}(P)$ can be disconnected: We will say that a pair of extremal wedges are opposite to each other if one of them is defined by rays $r_{i+1}$ and $r_{i+k}$, and the other by rays $r_{i+k+1}$ and $r_i$, indices taken modulo $2k$. As can be seen in Figure 6, right, a non-empty intersection between two opposite $O$-wedges results in $O\mathbb{H}(P)$ being disconnected. In such case we say that the intersecting wedges overlap, and refer to their intersection as their overlapping region.
Lemma 1. Only opposite O-wedges can overlap. If two of them overlap, then the O-hull is disconnected.

Proof. In any pair of non-opposite O-wedges, one of them contains a ray parallel to a bounding ray of the other. As every wedge in the boundary of OH(P) is supported by at least two points in P, two non-opposite wedges that intersect would inevitably make one of them non-P-free, contradicting the definition of O-hull. See again Figures 2 and 3.

We now proceed with the computation of the staircase, starting by the computation of the supporting points. In order to do so, we will make use of an algorithm by Avis et al. [7]. Given an angle \( \Theta \geq \frac{\pi}{2} \) this algorithm finds, in \( O(n \log n) \) time and \( O(n) \) space, the maximal \( \Theta \)-escaping wedges, those having as apex a point \( p \in P \), having aperture angle at least \( \Theta \), and being P-free, i.e., not containing any point of \( P \) in its interior. (In other words, maximal \( \Theta \)-escaping wedges are the maximal wedges allowing an angle \( \Theta \) to escape from \( p \) without hitting other points of \( P \).) Points \( p \) being apexes of a maximal \( \Theta \)-escaping wedge are called \( \Theta \)-maxima. The algorithm provides, indeed, for each such wedge its two defining rays. Note that if \( \Theta \geq \frac{\pi}{2} \), then any element of \( P \) is the apex of at most three \( \Theta \)-escaping wedges.

We apply the algorithm of Avis et al. [7] to our point set \( P \) for the angle \( \Theta = \min\{\Theta_i : i = 1, \ldots, 2k\} \geq \frac{\pi}{2} \). This gives at most three maximal \( \Theta \)-escaping intervals for every \( p \in P \), hence a linear number in total. We store these intervals in a circular table, see Figure 7, in which we also include the stabbing wedges \( W_{i+1}^{i+k} \).

Doing so, when a stabbing wedge \( W_{i+1}^{i+k} \) fits into the escaping interval of a point \( p \), we know that \( p \) is not only a \( \Theta \)-maxima, but actually a \( \Theta_i \)-maxima, which is indeed equivalent to be a supporting point in \( \partial W^i \). (In Figure 7 right, the gray stabbing interval from the wedge \( W_2^4 \) does not fit into the black escaping interval, because in the left picture the wedge \( W_2^4 \) cannot escape from \( p \).)

Thus, in \( O(n \log n) \) time and \( O(n) \) space we can sort the endpoints of the two types of intervals and sweep circularly the table, stopping at the defining rays of the wedges \( W_{i+1}^{i+k} \), to check if the corresponding \( p \) supports a staircase \( \partial W^i \). This gives the set \( V(P) \) of vertices of \( OH(P) \). It just remains to obtain the boundary of \( OH(P) \), for which standard techniques (see [21]) can be used in order to compute the staircases \( \partial W^i \) from their supporting points and to join them in \( O(n \log n) \) time and \( O(n) \) space. Hence, we have computed \( OH(P) \) in \( O(n \log n) \) time and \( O(n) \) space. This time complexity is optimal, since given \( OH(P) \) we can compute in
Figure 7: Left: Escaping intervals for a point \( p \) and \( \Theta = \frac{\pi}{2} \) (which is the case in the example of Figure 2). Right: Circular table where the solid circles correspond to the points \( p_1, \ldots, p_n \) from the inside to the outside. On them, the \( \Theta \)-escaping intervals, where the one for \( p \) depicted in the left is highlighted. In gray, the stabbing interval corresponding to the wedge \( W_2 \) from Figure 2 (the other stabbing intervals are omitted for the sake of clarity). Finally, the innermost circle reflects, as small marks, the vertex events in \([0, 2\pi)\) corresponding to the endpoints of the escaping intervals.

linear time \( \mathcal{CH}(\mathcal{OH}(P)) = \mathcal{CH}(P) \), and it is known that computing the convex hull of a set of points in the plane has an \( \Omega(n \log n) \) time lower bound (see [21]). Thus, we get the following result:

**Theorem 1.** For a set \( \mathcal{O} \) of \( k \) lines such that \( \Theta \geq \frac{\pi}{2} \), \( \mathcal{OH}(P) \) can be computed in \( \Theta(n \log n) \) time and \( O(n) \) space, and these complexities are independent of the number \( k \).

Furthermore, the algorithm from Avis et al. [7] is \( \Theta \)-dependent, in the sense that the complexities above stand for \( \Theta \geq \frac{\pi}{2} \), but it also works for \( \Theta < \frac{\pi}{2} \) in \( O(\frac{n}{\Theta} \log n) \) time and \( O(n) \) space. Thus, we can construct a circular table as above for \( \Theta = \min\{\Theta_i : i = 1, \ldots, 2k\} \), storing at most \( \frac{2n}{\Theta} \) circular intervals for each \( p \), hence using \( O(\frac{2n}{\Theta}) \) space in total. Thus, the previous result is extended to a general set \( \mathcal{O} \) as follows:

**Theorem 2.** For any set \( \mathcal{O} \) of \( k \) lines, \( \mathcal{OH}(P) \) can be computed in \( O(\frac{n}{\Theta} \log n) \) time and \( O(\frac{n}{\Theta}) \) space, and these complexities are independent of the number \( k \).

3 Computing and maintaining \( \mathcal{OH}_\Theta(P) \)

Recall that, as we rotate \( \mathcal{O} \) by an angle of \( \theta \) getting \( \mathcal{O}_\theta \), the wedges \( W_{i,j} \) also rotate. Thus, the sets \( W^\theta_i \) change accordingly, giving rise to the sets \( W^\theta_\theta \). The rotated hull \( \mathcal{OH}_\theta \)-hull of \( P \) is then (recall Figure 4):

\[
\mathcal{OH}_\theta(P) = \mathbb{R}^2 \setminus \bigcup_{i=1}^{2k} W^\theta_i.
\]

Let \( \partial W^\theta_\theta \) denote the boundary of \( W^\theta_\theta \). As in Subsection 2.2, \( \partial W^\theta_\theta \) is an alternating polygonal chain, or staircase, with interior angle \( \Theta_i = \pi - \alpha_i \geq \frac{\pi}{2} \) where, in counterclockwise direction around \( \mathcal{OH}_\theta(P) \), right turns arise at apexes of \( \mathcal{OH}_\theta \)-wedges in \( W^\theta_\theta \), called extremal, and left turns arise at points of \( P \) which are the supporting points of those extremal wedges. Recall Figure 5.

The following lemma is a straightforward generalization of Lemma 1 and will be needed in Section 4.
Lemma 2. Only opposite $O_\theta$-wedges can overlap. If two of them overlap, then the $O_\theta$-hull is disconnected.

We next show how to maintain $\partial OH_\theta(P)$ for $\theta \in [0,2\pi)$. We will denote by $T_\theta(P)$ the set of overlapping regions in $\partial OH_\theta(P)$, and by $V_\theta(P)$ the set of vertices of $\partial OH_\theta(P)$ in circular order while sequentially traversing $\partial W_\theta^i$, $i = 1, \ldots, 2k$, in the counterclockwise direction.

3.1 The boundary of $\partial OH_\theta(P)$

Applying a rotation of angle $\theta$ to the set $O$ changes the rotated hull of $P$, $\partial OH_\theta(P)$. In particular, the supporting vertices of the staircases $\partial W_\theta^i$ might change. We now aim to update those staircases, in $O(\log n)$ time per insertion or deletion of a point. In order to do so, we need to maintain the (at most) $2k$ staircases into (at most) $2k$ different balanced trees, one for each staircase. Notice that some of the staircases may appear and/or disappear during the rotation. The total insertion or deletion operation can be done in $O(kn\log(kn)) = O(kn\log n)$ time.

By using the circular table in Figure 7 we can rotate the (gray) stabbing wedges $W_{i+1}^{i+k}$, stopping at events arising when a defining ray of a stabbing wedge hits a vertex event in the innermost circle, i.e., entering or leaving an escaping interval (black). This provides the information about whether the stabbing wedges fit or not into the escaping intervals and this, the innermost circle, i.e., entering or leaving an escaping interval (black). This provides the stopping at events arising when a defining ray of a stabbing wedge hits a vertex event in $V_\theta(P)$ of vertices of $\partial OH_\theta(P)$ (i.e., the points on the staircases). Since the number of escaping intervals for a point is at most three and during the rotation these can arise in any of the $2k$ wedges corresponding to rotated $W_{i+1}^{i+k}$, there are $O(kn)$ events. Thus, we get the following result:

Theorem 3. For any set $O$ of $k$ lines, maintaining the boundary of $\partial OH_\theta(P)$ during a complete rotation for $\theta \in [0,2\pi)$ can be done in $O(kn\log n)$ time and $O(kn)$ space.

Notice that when $k$ is constant our result gives time and space complexities $\Theta(n \log n)$ and $O(n)$, respectively. This includes the case of the rectilinear convex hull, $RCH_\theta(P)$.

Corollary 1. For any set $O$ of $k$ lines, computing the orientation $\theta$ such that the boundary of $\partial OH_\theta(P)$ has minimum number of steps, or minimum number of staircases, or it is connected, or it has the minimum (or maximum) number of connected components, can be done in $O(kn\log n)$ time and $O(kn)$ space.

3.2 The area of $\partial OH_\theta(P)$

For a fixed value of $\theta$, we can compute the area of $\partial OH_\theta(P)$ using the fact that

$$\text{area}(\partial OH_\theta(P)) = \text{area}(P(\theta)) - \text{area}(P(\theta) \setminus \partial OH_\theta(P)), \quad (1)$$

where $P(\theta)$ denotes the filled polygon having the points in $V_\theta(P)$ as vertices and an edge connecting two vertices if they are consecutive elements in $V_\theta(P)$, see Figure 8. We will compute the area of $P(\theta) \setminus \partial OH_\theta(P)$ by decomposing it into a linear number of two types of regions: the triangles defined by every pair of consecutive elements in $V_\theta(P)$, and the overlapping regions in $T_\theta(P)$ (more details will be given in Section 5). See again Figure 8.

While $\theta$ increases from $0$ to $2\pi$, the set $V_\theta(P)$ changes at the values of angles $\theta$ where a point of $P$ becomes (resp. is no longer) a vertex of $\partial OH_\theta(P)$. We call these angles insertion (resp. deletion) events. Analogously, the set $T_\theta(P)$ changes at overlap (resp. release) events; that is, the values of $\theta$ where a pair of opposite extremal $O_\theta$-wedges start (resp. stop) overlapping.

Our approach is based on the efficient computation of the vertex event sequence and the overlap event sequence generated by all the points in $P$. (Notice that there exist point configurations where overlap events do not coincide with vertex events.) Clearly, the vertex event
Figure 8: Computing the area of $\mathcal{O}H_\theta(P)$. The dash-dotted lines indicate the border of $P(\theta)$. Left: The region $P(\theta) \setminus \mathcal{O}H_\theta(P)$. Right: Highlighted, a triangular region defined by two consecutive elements in $V_\theta(P)$, and an overlapping region.

sequence generated by all the points in $P$ can be easily obtained from the discussion and techniques of the Subsection [3.1] above. However, the overlap event sequence generated by all the points in $P$ needs a deeper insight, to which we will devote the next section.

4 The overlap events sequence

As we observed above (see Lemma 2) only opposite staircases $\partial W^i_\theta$ and $\partial W^{i+k}_\theta$ can intersect (recall Figure 6). Notice that, for any $\theta$, only one pair $\{i, i + k\}$ of opposite staircases can intersect, and that in this case $\mathcal{O}H_\theta(P)$ becomes disconnected. The corresponding intersection $W^i_\theta \cap W^{i+k}_\theta$ can be composed of several overlapping regions. We will show next how to maintain $W^i_\theta \cap W^{i+k}_\theta$ as $\theta$ increases.

Thus, the main goal of this section is to obtain the sequence of overlapping events, from which we will compute the overlapping regions appearing during a complete rotation. In the next two subsections, we will show that the number of overlapping events is linear. Then, we will illustrate an algorithm for computing them in an optimal way.

For the sake of clarity, and since the generalization to other cases is straightforward, in the rest of this paper we will stick to the orthogonal case, i.e., to $\mathcal{O}$ being composed by a horizontal and a vertical line. Thus, the given wedges $W^2_2$, $W^3_3$, $W^4_4$, and $W^1_1$ will be actually the second, third, fourth, and first quadrants and the extremal $\mathcal{O}_\theta$-wedges will be extremal $\theta$-quadrants.

4.1 The chain of arcs

Let the arc chain of $P$, denoted by $A(P)$, be the curve composed by the points $a$ in the plane which are apexes of a $P$-free extremal $\theta$-quadrant $w_a$ for some $\theta \in [0, 2\pi)$. Notice that, if $a \notin P$, then $w_a$ is supported by two points of $P$. The sub-chain associated to an edge $e_i$ of $\partial(CH(P))$ is then the curve $A_{e_i}$ composed by those points $a$ such that $w_a$ intersects $e_i$. See Figure 9 left. This sub-chain $A_{e_i}$ is monotone with respect to $e_i$, since it is composed by arcs of circles, which have to be monotone in order for the $w_a$ to intersect $e_i$, and two consecutive monotone arcs whose extremal $\theta$-quadrant intersect $e_i$ can only form a monotone curve. Finally, since a sub-chain may have vertices not belonging to $P$, we call link to the part of a sub-chain which lies between two points of $P$. See Figure 9 right.

Note that, if a pair of opposite wedges generates an overlapping region, then their apexes lie on intersecting links (see Figure 9 left). Hence, in order to prove that the set $\mathcal{T}_\theta(P)$ of
overlapping regions can be maintained in linear time and space, we will first prove that there is a linear number of intersections between links.

4.2 The number of intersections between links is in $O(n)$

Let us outline the flow of ideas in this subsection. We will construct a weighted graph whose vertices are the sub-chain disks having an edge of $\partial(CH(P))$ as diameter. The edges of the weighted graph will join those sub-chain disks whose corresponding sub-chains intersect. The number of intersections will be, precisely, the weight of the edge. Then, the total number of intersections equals the sum of weights, which we are proving to be linear.

Each point $p \in P$ can be in at most four sub-chain disks because $p$ can be the apex of at most four $P$-free wedges $w_a$ of size $\frac{\pi}{2}$. Thus, each point $p \in P$ can be in the intersection of at most $\binom{4}{2} = 6$ pairs of sub-chain disks, therefore contributing to the weight of at most 6 edges of the weighted graph. We will prove that the weight of every edge in the graph is linear on the number of points from $P$ contained in the corresponding sub-chain disks (Theorem 4). Therefore, the sum of weights in the graph will be linear on the total number of points in $P$, as wanted.

We first need a series of three lemmas:

**Lemma 3.** For any three points $a, b, c$ appearing from left to right on a link, the angle $\angle abc$ lies in $[\frac{\pi}{2}, \pi)$. In particular, every link with endpoints $p, q \in P$ is contained in the disk of diameter $pq$, called its link disk.

**Proof.** Let $p, q$ be the endpoints of the link, and hence consecutive points of $P$ along the chain. Therefore, $b \not\in P$ and thus $w_b$ is an extremal $\theta$-quadrant. That $\angle abc \geq \frac{\pi}{2}$ follows from $a, c$ not being in the interior of $w_b$ (otherwise this would not be $P$-free, either because some of $a, c$ is in $P$ or because one of the points of $P$ supporting the extremal $\theta$-quadrants with apexes $a, c$ is in the interior of $w_b$). That $\angle abc < \pi$ follows from the orthogonal projections of $p$ and $q$ over

Figure 9: Left: The arc-chain of $P$, highlighting the sub-chain associated to $e_i$. Right: Highlighted, a link of that sub-chain.
the corresponding edge of $\partial(CH(P))$ being inside the intersection of that edge with $w_b$. See Figure 10.

![Figure 10: Illustration of Lemma 3](image)

In the following lemma we identify the diameter of a link with that of its link disk.

**Lemma 4.** Consider the link disks in the two sub-chains associated to a pair of edges of $\partial(CH(P))$. The link disk $D$ of smallest diameter can be intersected by at most five links from the other sub-chain $A_e$.

**Proof.** Let $R$ be the strip bounded by the lines that orthogonally project $D$ over the edge $e$ associated to the sub-chain $A_e$. Because of the monotonicity, only the part of the sub-chain being inside $R$ can intersect $D$ (see Figure 11, left).

![Figure 11: Left: Only the part of the sub-chain being inside $R$ can intersect $D$. Middle: There are no peaks at points of $P$ inside $R$. Right: At most 5 links intersect $D$.](image)

If no arc in the sub-chain $A_e$ has endpoints inside $R$, then at most one link can intersect $D$. Otherwise, we will see that the sub-chain $A_e$ has no peaks at points of $P$ inside $R$: If there were a peak $p \in R$, let $q, r$ be its neighbors, being $r$ the one closer to the edge $e$. The segment obtained intersecting the parallel to $e$ through $q$ with the strip $R$ determines a disk which does not contain the peak $p$, since the length of $pq$ equals the diameter of a link-disk and, hence, has to be greater than the diameter of $D$, which equals the width of $R$. See Figure 11 middle. Then $\angle qpr < \frac{\pi}{2}$, a contradiction since $w_b$ has to be $P$-free.

Since $A_e$ has no peaks at points of $P$ inside $R$, it can have at most one valley inside $R$ and, therefore, at most five links from $A_e$ can intersect $D$ since this is inside $R$. See Figure 11 right. 


Lemma 5. There are $O(n)$ pairs of intersecting links in the two sub-chains associated to a pair of edges of $\partial(CH(P))$.

**Proof.** Let $\mathcal{L}$ be the list of all those links, ordered by increasing diameter. From Lemma 4, the first link in $\mathcal{L}$ is intersected by at most five of the remaining links in $\mathcal{L}$. By removing this link from $\mathcal{L}$, we get that the next link in the list is also intersected by at most constant number of links. As there is a linear number of extremal arcs and each arc belongs to a single link, there is also a linear number of elements in $\mathcal{L}$. Therefore, by recursively removing the link with smallest diameter from $\mathcal{L}$, the total number of intersecting pairs adds up to $O(n)$.

We are now ready to prove the main result of this section, Theorem 4, which implies that the weight of every edge in the weighted graph defined above is linear on the number of points from $P$ contained in the corresponding sub-chain disks.

**Theorem 4.** There are $O(n)$ intersection points between the links in the two sub-chains associated to a pair of edges of $\partial(CH(P))$.

**Proof.** Because of the monotonicity, we know that two links within the same sub-chain can intersect only at one of their endpoints. By Lemma 5, we just have to prove that links from two different sub-chains intersect at most twice.

Suppose that there exist at least three intersection points $a, b, c$ between two links from sub-chains associated to $e_i$ and $e_s$. Without loss of generality, assume that $a, b, c$ appear from left to right on the link associated to $e_i$. Note that then they also appear from left to right on the link associated to $e_s$, since otherwise at least one of the points cannot belong to this link, as the three of them would form an angle either smaller than $\frac{\pi}{2}$ (Figure 12(a)) or greater than $\pi$ (Figure 12(b)), in contradiction with Lemma 3.

Let $e_l$ and $e_m$ be respectively, the edges of $\partial(CH(P))$ intersected by the rays from $b$ passing through $a$ and $c$. While traversing the edge set of $\partial(CH(P))$ in any direction, $e_s$ lies between either $e_l$ and $e_i$, or $e_i$ and $e_m$ (see Figure 13(a)). Consider $e_s$ to be in the first case (the argument for the second case is symmetric) and denote with $\ell$ the line perpendicular to $e_i$ passing through $a$. Since $w_a$ is a maximal wedge bounded by rays intersecting $e_i$, as in the proof of Lemma 3, $w_a$ does not contain any other point from the link associated to $e_i$ (see Figure 13(b)). Note that $c$ and $p_{s+1}$ are in opposite sides of $\ell$ and are not contained in $w_a$ and thus, $\angle p_{s+1}ac \geq \frac{\pi}{2}$ and $\angle acp_{s+1} \leq \frac{\pi}{2}$. Since $a,c,p_{s+1}$ appear from left to right on the link associated to $e_s$, we get from Lemma 3 that $c$ cannot belong to $A_s$. \qed
4.3 Computing the overlap event sequence

Next, we outline the algorithm to compute the overlap event sequence.

**Event-sequence algorithm**

1. Compute the arc chain of $P$.
   Each extremal arc should be specified in terms of the points supporting the corresponding extremal $\theta$-quadrant and also in terms of the angular interval defined by these points called the *tracing interval*. The elements in $A(P)$ should be grouped by links. To compute the arc chain of $P$, we extend the traversal algorithm outlined in Section 2:
   
   (a) At each insertion event, at most two extremal arcs are generated, and at most one extremal arc is interrupted. Pointers should be set up from the interrupted extremal arcs to the ones just generated. If an extreme of a new arc is a point in $P$, a new link should be initialized with the respective extremal arc.
   
   (b) At each deletion event, at most one extremal arc is generated, and at most two extremal arcs are interrupted. One of the interrupted extremal arcs will be always ending at a point in $P$, so a link is completed. As before, pointers from the interrupted to the newly created extremal arcs should be set up.

2. Color extremal arcs.
   Traverse $A(P)$ in such way that the vertices of $\partial(CH(P))$ are visited in counterclockwise circular order, while assigning colors to each extremal arc: red if its subchain corresponds to an edge in the upper chain $\partial(CH(P))$, and blue otherwise (see Figure 14). Note that regarding the value of $\theta$, a pair of extremal $\theta$-quadrants intersecting an edge in the upper chain (resp. lower chain) of $\partial(CH(P))$ are not opposite to each other. If there is an intersection between monochromatic links then, they do not admit overlapping extremal $\theta$-quadrants.

3. Identify bichromatic intersecting links.
   Note that the largest possible extremal arc is a semicircle and therefore, any extremal arc can be partitioned in at most three segments to get a set of curves monotone with respect to an arbitrary direction. The arcs in $A(P)$ can thus be transformed into a set $A'(P)$ of curves monotone with respect to the same direction. The Bentley and Ottmann
plane sweep algorithm \cite{9} can then be applied on $A'(P)$ to compute the intersection points between extremal arcs. We discriminate from these points those belonging to bichromatic pairs of arcs. Pointers to the links containing the involved extremal arcs should be set up, so we can obtain the set of all bichromatic pairs of intersecting links in $A(P)$.

4. Compute the overlap event sequence.

Consider two extremal $\theta$-quadrants denoted as $Q_{\theta}(p,q)$ and $Q_{\theta}(r,s)$, and a pair of maximal arcs $\overline{ab} \in C(p,q)$ and $\overline{cd} \in C(r,s)$ with their corresponding tracing intervals $(\alpha_a, \alpha_b)$ and $(\alpha_c, \alpha_d)$. See Figure \ref{fig:15}. We say that $\overline{ab}$ and $\overline{cd}$ admit overlapping $\theta$-quadrants, if $Q_{\varphi}(p,q)$ and $Q_{\psi}(r,s)$ overlap for some $\varphi \in (\alpha_a, \alpha_b)$ and $\psi \in (\alpha_c, \alpha_d)$.

Assume that $\overline{ab}$ and $\overline{cd}$ admit overlapping $\theta$-quadrants and, without loss of generality, suppose that $p$ precedes $q$ in $V_{\theta}(P)$ for all $\theta \in (\alpha_a, \alpha_b)$, and that $r$ precedes $s$ for all $\theta \in (\alpha_c, \alpha_d)$. It is not hard to see that, since the extremal $\theta$-quadrants $Q_{\theta}(p,q)$ and $Q_{\theta}(r,s)$ are opposite to each other, $(\alpha_a, \alpha_b) \cap (\alpha_c + \pi, \alpha_d + \pi)$ is not empty and, during this interval, the ray of $Q_{\theta}(p,q)$ passing through $p$ (resp. $q$) is parallel to the ray of $Q_{\theta}(r,s)$ passing through $r$ (resp. $s$). Note that $q$ and $s$ lie on different sides of the line $\ell_{p,r}$ passing through $p$ and $r$, as otherwise $Q_{\theta}(p,q) \cap Q_{\theta}(r,s)$ could not be $P$-free. For the same reason, the points $p, r$ lie on opposite sides of $\ell_{q,s}$ and, therefore, the line segments $\overline{pr}$ and $\overline{qs}$ intersect with each other. It is easy to see that this intersection is contained in the overlapping region generated by $Q_{\theta}(p,q)$ and $Q_{\theta}(r,s)$ and, thus, we have that $\overline{pr} \cap \overline{qs} \subset D(p,q) \cap D(r,s)$. Note that the angular interval of maximum size where $Q_{\theta}(p,q)$ and $Q_{\theta}(r,s)$ may overlap, called the maximum overlapping interval, is bounded by the orientations where $X_{\theta}$ is parallel to $\overline{pr}$ and $Y_{\theta}$ is parallel to $\overline{qs}$.

![Figure 14: The colored arc chain of $P$.](image)

![Figure 15: The arcs $\overline{ab}$ and $\overline{cd}$ (highlighted) admit overlapping $\theta$-quadrants. Figures (a) and (b) show respectively, the overlap and the release event of the corresponding overlapping region.](image)
Observation 2. The arcs $\tilde{a}b$ and $\tilde{c}d$ admit overlapping $\theta$-quadrants if, and only if, $Q_\theta(p,q)$ and $Q_\theta(r,s)$ define a maximum overlapping interval $(\theta_1, \theta_2)$, and

$$(\theta_1, \theta_2) \cap (\alpha_a, \alpha_b) \cap (\alpha_c + \pi, \alpha_d + \pi) \neq \emptyset.$$ 

Let $(a_1\tilde{a}_2, a_2a_3, \ldots, a_k\tilde{a}_{k+1})$ be the set of extremal arcs for all $\theta \in [0, 2\pi)$, where $k = O(n)$, labeled while traversing $A(P)$ in such way that the vertices of $\partial(CH(P))$ are visited in counterclockwise circular order. We denote with $\ell_{u,v}$ the subsequence $\langle (a_u, a_{u+1}), \ldots, (a_v, a_{v+1}) \rangle$ of consecutive arcs in $A(P)$ forming a link. Note that the extremal intervals of the arcs in $\ell_{u,v}$ define the sequence $\langle \alpha_{a_u}, \ldots, \alpha_{a_{v+1}} \rangle$ of increasing angles.

Based on the Observation 2 above, we can compute the overlapping regions generated by the extremal arcs belonging to a pair $\ell_{u,v}$ and $\ell_{s,t}$ of intersecting links as an extension of the well-known linear-time merge procedure that operates on the lists $\langle \alpha_{a_u}, \ldots, \alpha_{a_{v+1}} \rangle$ and $\langle \alpha_{a_s} + \pi, \ldots, \alpha_{a_{t+1}} + \pi \rangle$, and their corresponding arc sequences: the intersection between a pair of non-consecutive maximal intervals in the merged list is empty. These pairs can be ignored as they do not comply with Observation 2 and therefore, at most a linear number of pairs of extremal arcs in $\ell_{u,v}$ and $\ell_{s,t}$ admit overlapping $\theta$-quadrants.

Let $\ell_{u,v}$ and $\ell_{s,t}$ be two intersecting links containing respectively, $n_{u,v} = u - v + 1$ and $n_{s,t} = t - s + 1$ extremal arcs. At most $O(n_{u,v} + n_{s,t})$ pairs of extremal arcs admit overlapping $\theta$-quadrants. The overlapping regions generated by the admitted extremal $\theta$-quadrants can be computed using $O(n_{u,v} + n_{s,t})$ time and space.

By Theorem 3 we know that Item 1 takes $O(n \log n)$ time and $O(n)$ space, as a constant number of additional operations are performed at each event while traversing the vertex event sequence using the algorithm outlined in Section 2. Item 2 takes $O(n)$ time and space, as the number of extremal arcs in $A(P)$ is linear in the number of elements in $P$. To compute $A(P)$ we require a linear run on $A(P)$ and, by Theorem 4, the Bentley and Ottmann plane sweep processes $A(P)$ in $O(n \log n)$ time and $O(n)$ space. Additional linear time is needed to discriminate from the resulting intersection points those belonging to bichromatic intersecting links, Item 3 requires a total of $O(n \log n)$ time and $O(n)$ space. Finally, from Lemma 5 and Theorem 4 and the facts that there is a linear number of extremal arcs and each arc belongs to a single link, Item 4 requires $O(n \log n)$ time and $O(n)$ space.

Theorem 5. The overlap event sequence can be computed in $O(n \log n)$ time and $O(n)$ space.

4.4 Sweeping the overlap event sequence

We now store the overlap event sequence as points on a circle $[0, 2\pi)$, over which we represent the wedges $W_{i+1}$ in a similar way as we did in the innermost circle in Figure 15 right (where we stored the vertex events instead).

We also store $T_\theta(P)$ in a hash table using tuples with the points supporting the overlapping $\theta$-quadrants as keys. The tuples will contain the supporting points in the same order as they are found while traversing $V_\theta(P)$. For an example, the overlapping region in Figure 15(a) would be stored in $T_\theta(P)$ using as key the tuple $(p, q, r, s)$.

As in Subsection 3.1, we now rotate the wedges $W_{i+1}$ simultaneously around the center and we stop when one of their defining rays passes over an overlap event in the innermost circle, to update $T_\theta(P)$ accordingly. It is easy to see that, at any fixed value of $\theta$ there are $O(n)$ overlapping regions in $O(H_\theta(P))$. Clearly, these regions can be computed in linear time from $V_\theta(P)$. As there are $O(n)$ overlap events, we obtain the following result.
Theorem 6. Using the overlap event sequence, the set $T_\theta(P)$ can be maintained while $\theta$ increases its value in $[0, 2\pi)$ using $O(n)$ time and $O(n)$ space.

Observation 3. The results above can also be adapted to $O$ being composed by $k$ lines without a sector greater than $\frac{\pi}{2}$. Then, the overlap event sequence can be computed in $O(kn \log n)$ time and $O(kn)$ space, and the set $T_\theta(P)$ can be maintained while $\theta$ increases in $[0, 2\pi)$ using $O(kn)$ time and $O(kn)$ space.

5 Minimum area

In this section we adapt the results from Bae et al. [8] to compute the value of $\theta$ that minimizes the area of $O_H(\theta)(P)$ in $\Theta(n \log n)$ time and $O(n)$ space. For simplicity, we assume $O$ to be composed by an horizontal and a vertical line.

Let $(\alpha, \beta)$ be an angular interval in $[0, 2\pi)$ containing no events. Extending Equation (1) we express the area of $O_H(\theta)(P)$ for any $\theta \in (\alpha, \beta)$ as

$$\text{area}(O_H(\theta)(P)) = \text{area}(P(\theta)) - \sum_j \text{area}(\triangle_j(\theta)) + \sum_k \text{area}(\square_k(\theta)).$$

(2)

Remember that the term $P(\theta)$ denotes the polygon having the points in $V_\theta(P)$ as vertices, with an edge connecting two vertices if they are consecutive elements in $V_\theta(P)$. The term $\triangle_j(\theta)$ denotes the triangular region bounded by the line through two consecutive vertices $v_j, v_{j+1} \in V_\theta(P)$, the line through $v_j$ parallel to $X_\theta$, and the line through $v_{j+1}$ parallel to $Y_\theta$. Finally, the term $\square_k(\theta)$ denotes the $k$-th overlapping region in $T_\theta(P)$. See Figure 16.

Figure 16: Computing the area of $O_H(\theta)(P)$. The polygon $P(\theta)$ is bounded by the dotted line. A triangle $\triangle_j(\theta)$ and an overlapping region $\square_k(\theta)$ are filled with blue.

We now show that at any particular value of $\theta$ we can evaluate Equation (2) in linear time and, as $\theta$ increases from 0 to $2\pi$, a constant number of terms need to be updated at each event, regardless of its type.

The polygon. At any fixed value of $\theta$ the area of $P(\theta)$ can be computed from $V_\theta(P)$ in $O(n)$ time. The term $\text{area}(P(\theta))$ changes only at vertex events. These events can be handled in constant time: at an insertion (resp. deletion) event, the area of a triangle needs to be subtracted (resp. added) to the previous value of $\text{area}(P(\theta))$. See Figure 17.

The triangular regions. According to Bae et al. [8] the area of $\triangle_j(\theta)$ can be expressed as

$$\text{area}(\triangle_j(\theta)) = b_j^2 \cdot \cos(c_j + (\theta - \alpha)) \cdot \sin(c_j + (\theta - \alpha)),$$

(3)
where \( b_j^2 \) and \( c_j \) are constant values depending on the coordinates of the vertices supporting the quadrant bounding \( \triangle_j(\theta) \). Contracting Equation (3) we have that

\[
\text{area}(\triangle_j(\theta)) = \frac{1}{2} b_j^2 \cdot \sin(2c_j + (\theta - \alpha)) \\
= \frac{1}{2} b_j^2 \cdot [\sin(2c_j) \cdot \cos(2(\theta - \alpha)) + \cos(2c_j) \cdot \sin(2(\theta - \alpha))] \\
= B_j \cdot \cos(2(\theta - \alpha)) + C_j \cdot \sin(2(\theta - \alpha)),
\]

(4)

where \( B_j = \frac{1}{2} b_j^2 \cdot \sin(2c_j) \) and \( C_j = \frac{1}{2} b_j^2 \cdot \cos(2c_j) \). Since Equation 4 can be computed in constant time and there are \( O(n) \) triangles (because so is the number of elements in \( V_\theta(P) \), recall Subsection 3.1), at any fixed value of \( \theta \) the term \( \sum_j \text{area}(\triangle_j(\theta)) \) can be computed in \( O(n) \) time. At an insertion event the term for one triangle is removed from \( \sum_j \text{area}(\triangle_j(\theta)) \) and, as a vertex supports at most two extremal \( \theta \)-quadrants, the terms of at most two triangles are added. The converse occurs for deletion events. The term \( \sum_j \text{area}(\triangle_j(\theta)) \) is not affected by overlap events. See Figure 18.

The overlapping regions. According to Bae et al. [8] the area of the \( k \)-th overlapping region can be expressed as

\[
\text{area}(\square_k(\theta)) = B_k + C_k \cos(2(\theta - \alpha)) + D_k \sin(2(\theta - \alpha)),
\]

(5)

where \( B_k, C_k, \) and \( D_k \) are constants depending on the coordinates of the vertices supporting the overlapping \( \theta \)-quadrants that generate \( \square_k(\theta) \). Equation 5 can be computed in constant
time and there are $O(n)$ overlapping regions in $T_\theta(P)$, so at any fixed value of $\theta$ the term $\sum_k \text{area}(\square_k(\theta))$ can be computed in $O(n)$ time. Overlap events require the term of a single overlapping region to be added to or deleted from $\sum_k \text{area}(\square_k(\theta))$. As a vertex supports at most two extremal $\theta$-quadrants, at a vertex event the terms of a constant number of overlapping regions are added or deleted.

Before describing the minimum area algorithm, we need the next three important properties of $\text{area}(\mathcal{OH}_\theta(P))$. First of all, from Lemma 4 in Bae et al. [8] the value of $\theta$ for which $\text{area}(\mathcal{OH}_\theta(P))$ is minimum in $(\alpha, \beta)$ could not be $\alpha$ nor $\beta$. Second, Equation (2) has $O(n)$ terms for any $\theta \in (\alpha, \beta)$ and thus, it can be reduced to

$$\text{area}(\mathcal{OH}_\theta(P)) = C + D \cos(2(\theta - \alpha)) + E \sin(2(\theta - \alpha))$$

(6)

in $O(n)$ time. The terms $C$, $D$ and $E$ denote constants resulting from adding up the constant values in $\text{area}(P(\theta))$, and Equations (4) and (5). Finally, as Equation (6) has a constant number of inflection points in $[0, 2\pi)$, a constant number of operations suffice to obtain the value of $\theta$ that minimizes $\text{area}(\mathcal{OH}_\theta(P))$ in $(\alpha, \beta)$.

The search algorithm. We outline next the minimum area algorithm.

1. Compute the events sequence.

   Compute the vertex event sequence, as described in Subsection 3.1, and the overlap event sequence, as described in Subsection 4.3. Merge both sequences into a single circular sequence of angles $\{\theta_1, \ldots, \theta_m, \theta_1\}, m \in O(n)$, which we can represent in a circular table as in the innermost circle of Figure 7, right. Clearly, while $\theta$ increases in $[0, 2\pi)$ the relevant features of $\mathcal{OH}_\theta(P)$ remain unchanged during each interval $(\theta_i, \theta_{i+1})$, and each angle $\theta_i$ is an insertion, deletion, overlap, or release event.

2. Initialize the angular sweep.

   Place the four wedges $W^3_2$, $W^4_3$, $W^1_4$, and $W^2_1$ over the circular table, as we did in Subsection 3.1 and Subsection 4.3. Without loss of generality, assume that the first (counter-clockwise) defining ray of the wedge $W^2_1$ intersects the angular interval $(\theta_m, \theta_1)$. Compute the sets $V_{\theta_1}$ and $T_{\theta_1}$ for the current $\theta_1$ as in Subsection 3.1 and express $\text{area}(\mathcal{OH}_\theta(P))$ for $\theta \in [\theta_1, \theta_2)$ using Equation (6). Compute the constant values in this equation considering the restriction $\theta \in [\theta_1, \theta_2)$. Optimize the resulting equation to compute the angle $\theta_{\min}$ of minimum area.

3. Perform the angular sweep.

   Rotate simultaneously the four wedges $W^3_2$, $W^4_3$, $W^1_4$, and $W^2_1$ as we did in Subsection 4.3. During the sweeping process, update $V_\theta(P)$ and $T_\theta(P)$ as explained in Subsections 3.1 and 4.4. Additionally, at each event:

   (a) Update Equation (6) by adding or subtracting terms as explained before.

   (b) Optimize the updated version of Equation (6) to obtain the local angle of minimum area, and replace $\theta_{\min}$ if $\text{area}(\mathcal{OH}_\theta(P))$ is improved.

From Theorems 3 and 5, computing the vertex and the overlap event sequences takes $\Theta(n \log n)$ time and $O(n)$ space. As both sequences have $O(n)$ events, we require linear time to merge them into the sequence of events and thus, item (i) consumes a total of $\Theta(n \log n)$ time and $O(n)$ space. At item (2), $V_\theta(P)$ can be computed in $\Theta(n \log n)$ time and $O(n)$ space.
(see \[16\]), and \(T_\theta(P)\) can be easily computed from \(V_\theta(P)\) in linear time. An additional linear time is required to obtain Equation (6), while \(\theta_{\text{min}}\) can be computed in constant time. This gives a total of \(\Theta(n \log n)\) time and \(O(n)\) space. Finally, by Theorems 3 and 6 respectively, maintaining \(V_\theta(P)\) and \(T_\theta(P)\) requires \(\Theta(n \log n)\) time and linear space for each. Items 3(a) and 3(b) are repeated \(O(n)\) times (one per event in the sequence) and, as we described before, each repetition takes constant time. Therefore, to perform Item 3(a) a total of \(\Theta(n \log n)\) time and \(O(n)\) space is required. Notice that, after the sweeping process is finished, in additional \(O(n \log n)\) time and \(O(n)\) space we can compute both \(OH_\theta(P)\) and area\((OH_\theta(P))\) for the angle \(\theta_{\text{min}}\) giving the minimum area. From this analysis we obtain our main result.

**Theorem 7.** Let \(\mathcal{O}\) composed by the horizontal and vertical lines. Computing the \(\mathcal{O}\)-convex hull \(OH_\theta(P)\) (i.e., the unoriented rectilinear convex hull of \(P\)) of minimum area over all \(\theta \in [0, 2\pi)\) requires \(\Theta(n \log n)\) time and \(O(n)\) space.

**Observation 4.** The results above can also be adapted to the case of a set \(\mathcal{O}\) composed by \(k\) lines without a sector greater than \(\frac{\pi}{2}\). Then, computing the \(\mathcal{O}\)-convex hull of \(P\) of minimum area over all \(\theta \in [0, 2\pi)\) requires \(\Theta(kn \log n)\) time and \(O(kn)\) space.

### 6 Concluding remarks

We show how to compute the \(\mathcal{O}\)-hull of \(P\) in \(\Theta(n \log n)\) time and \(O(n)\) space and, then, how to compute and maintain the rotated hull \(OH_\theta(P)\) while \(\theta\) increases in \([0, 2\pi)\) in \(O(kn \log n)\) time and \(O(kn)\) space, provided that no two consecutive lines in \(\mathcal{O}\) span an angle greater than \(\frac{\pi}{2}\). Moreover, we also solve the problem of computing an orientation of the plane for which the rectilinear convex hull of \(P\) has minimum area in \(\Theta(n \log n)\) time and \(O(n)\) space, thus improving the \(O(n^2)\) time algorithm presented by Bae et al. \[8\]. Notice that, if there is more than one optimal orientation, a trivial modification allows us to report all the orientations with no additional complexity.

Without much effort, our algorithm can be extended to optimize (both minimize or maximize) other properties of \(OH_\theta(P)\). Examples of such properties are the perimeter, the number of connected components, or the number of elements of \(P\) in the interior or over the boundary of \(OH_\theta(P)\). Interesting extensions of the “property optimization” family of problems are to consider dynamic or moving point sets. If we consider extensions to three dimensional point sets, the sole problem of maintaining the rectilinear convex hull over all orientations of the coordinate system seems interesting and non-trivial.

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