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To cite this version:
Rémi Carles, Satoshi Masaki. Semiclassical Analysis for Hartree equation. Asymptotic Analysis, IOS Press, 2008, 58 (4), pp.211-227. 10.3233/ASY-2008-0882. hal-00195618

HAL Id: hal-00195618
https://hal.archives-ouvertes.fr/hal-00195618
Submitted on 11 Dec 2007

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SEMiclassical analysis for Hartree equation

RÉMI CARLES and SATOSHI MASAKI

Abstract. We justify WKB analysis for Hartree equation in space dimension at least three, in a régime which is supercritical as far as semiclassical analysis is concerned. The main technical remark is that the nonlinear Hartree term can be considered as a semilinear perturbation. This is in contrast with the case of the nonlinear Schrödinger equation with a local nonlinearity, where quasilinear analysis is needed to treat the nonlinearity.

1. Introduction

We consider the semiclassical limit $\varepsilon \to 0$ for the Hartree equation

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = \lambda(|x|^{-\gamma} \ast |u^\varepsilon|^2)u^\varepsilon, \gamma > 0, \lambda \in \mathbb{R}, x \in \mathbb{R}^n,$$

in space dimension $n \geq 3$. We consider initial data of WKB type,

$$u^\varepsilon(0, x) = a^\varepsilon_0(x)e^{i\phi_0(x)/\varepsilon},$$

where $a^\varepsilon_0$ typically has an asymptotic expansion as $\varepsilon \to 0$,

$$a^\varepsilon_0 \sim a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots, \quad a_j \text{ independent of } \varepsilon \in ]0, 1].$$

The approach that we follow is closely related to the pioneering works of P. Gérard [12], and E. Grenier [15], for the nonlinear Schrödinger equation with local nonlinearity:

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = f(|u^\varepsilon|^2)u^\varepsilon,$$

where the function $f$ is smooth, and real-valued. In [13], the assumption $f' > 0$ is necessary for the arguments of the proof. More recently, this assumption was relaxed in [3], allowing to consider the case $f(y) = +y^\sigma$, $\sigma \in \mathbb{N}$. Moreover, it is noticed in [3] that to carry out a WKB analysis in Sobolev spaces for (1.3), the assumption $f' \geq 0$ is essentially necessary. Typically, in the case $f' < 0$, working with analytic data is necessary, and sufficient as shown in [12, 21]. The reason is that the local nonlinearity is analyzed through quasilinear arguments in [14, 5], and $f'$ determines the velocity of a wave equation: if $f' > 0$, then the wave equation is hyperbolic, and $f' < 0$, the underlying operator becomes elliptic.

The above discussion is altered for the Hartree type nonlinearity. Typically, no assumption is made on the sign of $\lambda$ here. As noticed in [1] in the special case of the Schrödinger–Poisson system, the nonlocal nonlinearity in (1.1) can be handled by semilinear arguments. However, a quasilinear analysis is needed to handle the convective coupling.
There are at least two motivations to study this question, besides the general picture of justifying approximations motivated by physics. As remarked in [3] in the case of a local nonlinearity, WKB analysis and a geometrical transform can help understand the behavior of a wave function near a focal point, in a supercritical régime. In [20], other informations were obtained thanks to a different approach, in the case of a Hartree type nonlinearity.

The approach of [20] and the results of the present paper will certainly be helpful to improve the understanding of the focusing phenomenon in semiclassical analysis. Another application of the WKB analysis for (1.1) concerns the Cauchy problem for the Hartree equation, that is (1.1) with semiclassical analysis. In [20], other informations were obtained thanks to a different approach, in the case of a Hartree type nonlinearity.

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\[ \phi \]

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\[ \phi + \phi_{\text{quad}} \]

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where 

\[ \phi_{\text{quad}} \]

is a polynomial of degree at most two.

For \( s > n/2 \), we denote by \( X^s(\mathbb{R}^n) \) the Zhidkov space

\[ X^s(\mathbb{R}^n) = \{ u \in L^\infty(\mathbb{R}^n)\mid \nabla u \in H^{s-1}(\mathbb{R}^n) \}. \]

This space was introduced in [22] (see also [23]) in the case \( n = 1 \), and its study was generalized to the multidimensional case in [14]. We denote

\[ \| u \|_{X^s} := \| u \|_{L^\infty} + \| \nabla u \|_{H^{-1}}. \]

We write \( H^s = H^s(\mathbb{R}^n) \) and \( X^s = X^s(\mathbb{R}^n) \).

**Theorem 1.4.** Let Assumption 1.1 be satisfied. There exists \( T > 0 \) independent of \( \varepsilon \) and \( s > 1+n/2 \), and a unique solution \( u^\varepsilon \in C([0, T]; H^s) \) to the equation (1.1)–(1.2). Moreover, it can be written in the form 

\[ u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}, \]

where \( a^\varepsilon \) is complex-valued, \( \phi^\varepsilon \) is real-valued, with

\[ a^\varepsilon \in C([0, T]; H^s), \quad \phi^\varepsilon \in C \left( [0, T]; L^\infty \cap L^{nq_0/\gamma} \right), \]

and

\[ \nabla \phi^\varepsilon \in C \left( [0, T]; X^{s+1} \cap L^{p_0} \right). \]

Moreover, if \( \phi_0 \in L^{p_0} \) for some \( p_0 \in [n/\gamma, nq_0/(n-q_0)] \) then \( \phi^\varepsilon \in C([0, T]; L^{p_0}) \).
Note that obtaining a local existence time $T$ which is independent of $\varepsilon \in [0, 1]$ is already a non-trivial information, at least for a focusing nonlinearity $\lambda < 0$. Using classical results on the Cauchy problem for Hartree equation \((\ref{eq:hartree})\), and a scaling argument, would yield an existence time that goes to zero with $\varepsilon$. Taking $q_0 = 2$, we immediately obtain the following corollary:

**Corollary 1.5.** Let Assumption 1.6 be satisfied. Let $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon / \varepsilon}$ be the solution given in Theorem 1.4. If $\gamma \in [n/2 - 1, n - 2]$ and $\nabla \phi_0 \in H^{s+1}$, then

$$\phi^\varepsilon \in C \left([0, T]; X^{s+2} \cap L^{\frac{2n}{s-2}}\right).$$

With this local existence result, we can justify a WKB expansion, provided that the initial data have a suitable expansion as $\varepsilon \to 0$.

**Assumption 1.6.** Let $N$ be a positive integer. We suppose Assumption 1.6 with some $s > n/2 + 2N + 1$. Moreover, the initial amplitude $a_0^\varepsilon$ writes

$$a_0^\varepsilon = a_0 + \sum_{j=1}^{N} \varepsilon^j a_j + \varepsilon^N r_N^\varepsilon,$$

where $a_j \in H^s$ ($0 \leq j \leq N$) and $\|r_N^\varepsilon\|_{H^s} \to 0$ as $\varepsilon \to 0$.

**Theorem 1.7.** Let Assumption 1.6 be satisfied. Let $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon / \varepsilon}$ be the unique solution given in Theorem 1.4. Then, there exist $(b_j, \varphi_j)_{0 \leq j \leq N}$, with

$$b_j \in C([0, T]; H^{s-2j}),$$

$$\varphi_j \in C \left([0, T]; L^\infty \cap L^{\frac{nq_0}{n-q}}\right) \quad \text{and} \quad \nabla \varphi_j \in C \left([0, T]; X^{s-2j+1} \cap L^{q_0}\right),$$

such that:

$$a^\varepsilon = b_0 + \sum_{j=1}^{N} \varepsilon^j b_j + o (\varepsilon^N) \quad \text{in} \quad C([0, T]; H^{s-2N}),$$

$$\phi^\varepsilon = \varphi_0 + \sum_{j=1}^{N-1} \varepsilon^j \varphi_j + o (\varepsilon^N) \quad \text{in} \quad C([0, T]; L^\infty \cap L^{\frac{nq_0}{n-q}}),$$

$$\nabla \phi^\varepsilon = \nabla \varphi_0 + \sum_{j=1}^{N} \varepsilon^j \nabla \varphi_j + o (\varepsilon^N) \quad \text{in} \quad C \left([0, T]; X^{s-2N+1} \cap L^{q_0}\right).$$

Moreover, for $j \geq 1$, $\varphi_j \in L^p$ for all $p > n/\gamma$, and $\nabla \varphi_j \in L^q$ for all $q > n/(\gamma + 1)$.

**Corollary 1.8.** Let Assumption 1.6 be satisfied. The solution $u^\varepsilon$ given in Theorem 1.7 has the following asymptotic expansion, as $\varepsilon \to 0$:

$$u^\varepsilon = e^{i\phi_0 / \varepsilon} \left(\beta_0 + \varepsilon \beta_1 + \ldots + \varepsilon^{N-1} \beta_{N-1} + \varepsilon^{N-1} \rho^\varepsilon\right), \quad ||\rho^\varepsilon||_{H^{s-2N+2}} \xrightarrow{\varepsilon \to 0} 0,$$

where $\varphi_0$ is given by Theorem 1.7 and $\beta_j \in C([0, T]; H^{s-2j})$ is a smooth function of $(b_k, \varphi_{k+1})_{0 \leq k \leq j}$. For instance,

$$\beta_0 = b_0 e^{i\varphi_1}; \quad \beta_1 = b_1 e^{i\varphi_1} + i\varphi_2 b_0 e^{i\varphi_1}.$$
Corollary 1.9. Let \( n \geq 5, \lambda \in \mathbb{R}, \max(n/2 - 2, 2) < \gamma \leq n - 2 \) and \( 0 < s < s_c = \gamma/2 - 1 \). There exists a sequence of initial data
\[
(\psi_0^h)_{0 < h \leq 1}, \quad \|\psi_0^h\|_{H^s} \to 0,
\]
a sequence of times \( t^h \to 0 \), such that the solution to
\[
i\partial_t \psi^h + \frac{1}{2} \Delta \psi^h = \lambda \left( |x|^{-\gamma} * |\psi^h|^2 \right) \psi^h; \quad \psi^h|_{t=0} = \psi_0^h
\]
satisfies
\[
\|\psi^h(t^h)\|_{H^s} \to +\infty, \quad \forall k > \frac{s}{1 + s_c - s} = \frac{s}{\gamma/2 - s}.
\]

Note that unlike in the case of Schrödinger equations with local nonlinearity, considering a large space dimension is necessary to observe this phenomenon: in low space dimensions, Hartree equations are locally well-posed in Sobolev spaces of positive regularity (see e.g. [8, 14]).

Using Sobolev embedding, one could infer a loss of regularity at the level of the energy space (consider \( s > 1 \) and \( s_{c-s} = 1 \), hence \( \gamma = 4s > 1 \)), in the spirit of [19] (see also [2, 21] for Schrödinger equations), provided that the space dimension is \( n \geq 7 \).

The rest of this paper is organized as follows. In the next paragraph, we present the general strategy adopted in this paper. In §3, we collect some technical estimates. Theorem 1.4 is proved in §4, and Theorem 1.7 is proved in §5, as well as Corollary 1.8. Finally, Corollary 1.9 is inferred in §6.

2. General strategy

To prove Theorem 1.4 and Theorem 1.7, we follow the same strategy as in [15]. Seek a solution \( u^\varepsilon \) to (1.1)–(1.2) represented as
\[
u^\varepsilon(t, x) = a^\varepsilon(t, x) e^{i\phi^\varepsilon(t, x)/\varepsilon},
\]
with a complex-valued space-time function \( a^\varepsilon \) and a real-valued space-time function \( \phi^\varepsilon \). Note that \( a^\varepsilon \) is expected to be complex-valued, even if its initial value \( a_0^\varepsilon \) is real-valued. We remark that the phase function \( \phi^\varepsilon \) also depends on the parameter \( \varepsilon \). Substituting the form (2.1) into (1.1), we obtain
\[
i\varepsilon \left( \partial_t a^\varepsilon + a^\varepsilon i \frac{\partial \phi^\varepsilon}{\varepsilon} \right) + \frac{\varepsilon^2}{2} \left( \Delta a^\varepsilon + 2 \left( \nabla a^\varepsilon \cdot i \frac{\nabla \phi^\varepsilon}{\varepsilon} \right) - a^\varepsilon \frac{|\nabla \phi^\varepsilon|^2}{\varepsilon^2} + a^\varepsilon i \frac{\Delta \phi^\varepsilon}{\varepsilon} \right) = \lambda \left( |x|^{-\gamma} * |a^\varepsilon|^2 \right) a^\varepsilon.
\]

To obtain a solution of the above equation (hence, of (1.1)), we choose to consider the following system:
\[
\begin{cases}
\partial_t a^\varepsilon + \nabla a^\varepsilon \cdot \nabla \phi^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i\varepsilon \frac{\Delta a^\varepsilon}{2}, \\
\partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + \lambda \left( |x|^{-\gamma} * |a^\varepsilon|^2 \right) = 0.
\end{cases}
\]

This choice is essentially the same as the one introduced by E. Grenier [15]. We consider this with the initial data
\[
(2.3) \quad a^\varepsilon|_{t=0} = a_0^\varepsilon, \quad \phi^\varepsilon|_{t=0} = \phi_0.
\]
From now on, we work only on \(|\mathbb{R}^n|\). We first prove that it admits a unique solution with suitable regularity (see Theorem 1.4), hence providing a solution to (1.1)–(1.2). The asymptotic expansion (Theorem 1.7) then follows by the similar arguments.

To conclude this paragraph, we remark that uniqueness for (1.1)–(1.2) in the class \(C([0, T]; H^s)\), \(s > n/2 + 1\), is a straightforward consequence of Lemma 3.1 and Sobolev inequality, Assumption 1.1 implies that \(\nabla \phi \in L^\infty([0, T]; \mathbb{R}^n)\), \(\phi \in C_0^\infty([0, T]; \mathbb{R})\), \(\gamma = 1\), and uniqueness follows from standard energy estimates in 

\[ \langle \nabla |x|^{-\gamma} * |u|^2 \rangle_{L^\infty(\mathbb{R}^n)} \leq \left( \frac{2}{\gamma} \right) \langle \nabla |x|^{-\gamma} \rangle_{L^1(\mathbb{R}^n)} \langle |u|^2 \rangle_{L^\infty(\mathbb{R}^n)} + \langle (1 - \gamma) |x|^{-\gamma} \rangle_{L^\infty(\mathbb{R}^n)} \langle |u|^2 \rangle_{L^2(\mathbb{R}^n)}. \]

Since we work with an \(H^s\) regularity, \(s > n/2 + 1\), the above right hand side is bounded, and uniqueness follows from standard energy estimates in \(L^2\).

3. Preliminary estimates

We first recall a consequence of the Hardy-Littlewood-Sobolev inequality, which can be found in [10] Th. 4.5.9 or [13] Lemma 7:

**Lemma 3.1.** If \(\varphi \in \mathcal{D}'(\mathbb{R}^n)\) is such that \(\nabla \varphi \in L^p(\mathbb{R}^n)\) for some \(p \in [0, n]\), then there exists a constant \(\gamma\) such that \(\varphi - \gamma \in L^p(\mathbb{R}^n)\), with \(1/p = 1/q + 1/n\).

**Remark 3.2.** The limiting case \(\gamma = n - 2\) corresponds to the Schrödinger–Poisson system considered in [14], with suitable conditions at infinity to integrate the Poisson equation.

**Remark 3.3.** By Lemma 3.1 and Sobolev inequality, Assumption 1.4 implies \(\phi_0 \in L^{n_0/2} \cap L^\infty\), and \(\nabla \phi_0 \in L^{n_0} \cap X^{s+1}\). Note that \(2n/(n - 2) < n\) if \(n > 5\). Therefore, in this case, we can always find \(q_0\) in \([n/(\gamma + 1), n]\) such that \(\nabla \phi_0 \in L^{q_0}\).

The next two lemmas can be found in [17]:

**Lemma 3.4** (Commutator estimate). Let \(s \geq 0\) and \(1 < p < \infty\). Set \(\Lambda = (1 - \Delta)^{1/2}\). Then, it holds that

\[ \| \Lambda^s(f g) - f \Lambda^s g \|_{L^p} \leq c(\| \nabla f \|_{L^\infty} \| \Lambda^{s-1} g \|_{L^p} + \| \Lambda^s f \|_{L^p} \| g \|_{L^\infty}). \]

**Lemma 3.5.** Let \(s > 0\) and \(1 < p < \infty\). There exists \(C > 0\) such that

\[ \| \Lambda^s(f g) \|_{L^p} \leq C(\| \Lambda^s f \|_{L^p} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| \Lambda^s g \|_{L^p}), \quad \forall f, g \in W^{s,p} \cap L^\infty. \]

The following lemma is crucial for our analysis:

**Lemma 3.6.** Let \(n \geq 3\), \(k \geq 0\), and \(s_1, s_2 \in \mathbb{R}\). Let \(\gamma > 0\) satisfying \(n/2 - k < \gamma \leq n - k - s_1 + s_2\). Then, there exists \(C_\gamma\) such that

\[ \left\| \nabla^k(|x|^{-\gamma} * f) \right\|_{L_{H^1}} \leq C_\gamma(\| f \|_{H^{s_2}} + \| f \|_{L^1}), \quad \forall f \in L^1 \cap H^{s_2}. \]
Proof. Since $\mathcal{F}|x|^{-\gamma} = C|\xi|^{-n+\gamma}$, it holds that
\[
\left\| |\nabla|^k (|x|^\gamma * f) \right\|_{L^2} = C \left\| \xi^k |\xi|^{-n+\gamma+k} \mathcal{F} f \right\|_{L^2}.
\]
The high frequency part ($|\xi| > 1$) is bounded by $C \|f\|_{L^2}$ if $-n + \gamma + k + s_1 - s_2 \leq 0$. On the other hand, the low frequency part ($|\xi| \leq 1$) is bounded by
\[
C \|\mathcal{F} f\|_{L^\infty} \int_{|\xi| \leq 1} |\xi|^{2(n-\gamma+k)} d\xi \leq C \|f\|_{L^1}
\]
if $2(-n + \gamma + k) > -n$, that is, if $\gamma > n/2 - k$. \qed

4. Existence result: proof of Theorem 4.1

Operating $\nabla$ to the equation for $\phi^\varepsilon$ in (2.2) and putting $v^\varepsilon := \nabla \phi^\varepsilon$, we obtain the following system:
\[
(4.1) \quad \begin{cases}
\partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = \frac{i}{2} \Delta a^\varepsilon, & a^\varepsilon|_{t=0} = a^\varepsilon_0, \\
\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + \lambda \nabla (|x|^{-\gamma} * |a^\varepsilon|^2) = 0, & v^\varepsilon|_{t=0} = \nabla \phi_0.
\end{cases}
\]

We first construct the solution $(a^\varepsilon, v^\varepsilon)$ to the system (4.1).

Proposition 4.1. Let Assumption 1.1 be satisfied. There exists $T > 0$ independent of $\varepsilon$ and $s$, such that for all $\varepsilon \in [0, 1]$, (4.1) has a unique solution
\[
(a^\varepsilon, v^\varepsilon) \in C \left([0, T]; H^s \times \left(X^{s+1} \cap L^{\frac{2n}{n-2}}\right)\right).
\]
Moreover, the norm of $(a^\varepsilon, v^\varepsilon)$ is bounded uniformly for $\varepsilon \in [0, 1]$.

4.1. Regularized system. We shall prove the existence of the solution to the system (4.1) by taking the limit of the solutions to the corresponding regularized system. We take $\varphi \in C_0^{\infty}(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \varphi(x) dx = 1$ and $\varphi \geq 0$ and set
\[
(4.2) \quad J_\delta f = \varphi_\delta * f
\]
where $\varphi_\delta = \delta^{-n} \varphi(x/\delta)$. We first treat the following regularized system:
\[
(4.3) \quad \begin{cases}
\partial_t a_\delta^\varepsilon + J_\delta (v_\delta^\varepsilon \cdot \nabla J_\delta a_\delta^\varepsilon) + \frac{1}{2} a_\delta^\varepsilon \nabla \cdot J_\delta v_\delta^\varepsilon = \frac{i}{2} \Delta J_\delta a_\delta^\varepsilon, & a_\delta^\varepsilon|_{t=0} = a_\delta^\varepsilon_0, \\
\partial_t v_\delta^\varepsilon + J_\delta (v_\delta^\varepsilon \cdot \nabla J_\delta v_\delta^\varepsilon) + \lambda \nabla J_\delta (|x|^{-\gamma} * |a_\delta^\varepsilon|^2) = 0, & v_\delta^\varepsilon|_{t=0} = \nabla \phi_0.
\end{cases}
\]

The point is that the regularized equations (4.3) have been chosen so that the Cauchy problem can be solved as in the standard framework of Sobolev and Zhidkov spaces:

Lemma 4.2. Let Assumption 1.1 be satisfied. For all $\varepsilon \in [0, 1]$ and $\delta \in [0, 1]$, there exists $T_\varepsilon^\delta > 0$ such that the Cauchy problem (4.3) has a unique solution $(a_\delta^\varepsilon, v_\delta^\varepsilon) \in C^1([0, T_\varepsilon^\delta], H^{s+1} \times X^{s+2} \cap L^{\frac{2n}{n-2}})$.

Proof. The proof is based on the usual theorem for ordinary differential equations. We use the following estimates
\[
\|J_\delta (v_\delta^\varepsilon \cdot \nabla J_\delta a_\delta^\varepsilon)\|_{H^{s+1}} \leq C \|v_\delta^\varepsilon\|_{H^{s+1}} \|\nabla J_\delta a_\delta^\varepsilon\|_{H^{s+1}} \\
\leq C \delta^{-1} \|v_\delta^\varepsilon\|_{H^{s+2}} \|a_\delta^\varepsilon\|_{H^{s+1}},
\]

and
\[ \| a_\delta^s \nabla \cdot J_\delta v_\delta^s \|_{H^{s+1}} \leq C \delta^{-1} \| a_\delta^s \|_{H^{s+1}} \| v_\delta^s \|_{X^{s+2}}, \]
\[ \| \Delta J_\delta^2 a_\delta^s \|_{H^{s+1}} \leq C \delta^{-2} \| a_\delta^s \|_{H^{s+1}}, \]
\[ \| J_\delta (v_\delta^s \cdot \nabla J_\delta v_\delta^s) \|_{X^{s+2}} \leq C \delta^{-1} \| v_\delta^s \|_{X^{s+2}}^2, \]
\[ \| \nabla J_\delta (|x|^{-7} * |a_\delta^s|^2) \|_{X^{s+2}} \leq C \| \Delta(|x|^{-7} * |a_\delta^s|^2) \|_{H^{s+1}} \]
\[ \leq C \| a_\delta^s \|_{H^{s+1}}^2. \]

We have applied Lemma 3.6 with \( k = 2 \) and \( s_1 = s_2 = s + 1 \). We note that the space \( X^{s+2} \cap L^\infty \) with norm \( \| \|_{X^{s+2}} \) is complete. \( \square \)

4.2. Uniform bound. We shall establish an upper bound for the \( H^s \) norm and \( X^{s+1} \) norm of \( a_\delta^s \) and \( v_\delta^s \) for \( s > n/2 + 1 \), respectively. We first estimate the \( H^s \) norm of \( a_\delta^s \). We use the following convention for the scalar product in \( L^2 \):
\[ \langle \varphi, \psi \rangle := \int_{\mathbb{R}^n} \varphi(x) \overline{\psi(x)} \, dx. \]

Set \( \Lambda = (I - \Delta)^{1/2} \). We shall estimate
\[ \frac{d}{dt} \| a_\delta^s \|_{H^s}^2 = 2 \text{Re} \langle \partial_t \Lambda^s a_\delta^s, \Lambda^s a_\delta^s \rangle. \]

Since \( [\Lambda^s, \nabla] = 0 \) and \( [\Lambda^s, J_\delta] = 0 \), by commuting \( \Lambda^s \) with the equation for \( a_\delta^s \), we find:
\[ \partial_t \Lambda^s a_\delta^s + J_\delta \Lambda^s (v_\delta^s \cdot \nabla J_\delta a_\delta^s) + \frac{1}{2} \Lambda^s (a_\delta^s \nabla J_\delta \cdot v_\delta^s) - i \frac{\varepsilon}{2} J_\delta \Delta J_\delta \Lambda^s a_\delta^s = 0. \]

The coupling of the second term and \( \Lambda^s a_\delta^s \) is written as
\[ \langle \Lambda^s (v_\delta^s \cdot \nabla J_\delta a_\delta^s), J_\delta \Lambda^s a_\delta^s \rangle = \langle v_\delta^s \cdot \nabla J_\delta \Lambda^s a_\delta^s, J_\delta \Lambda^s a_\delta^s \rangle \]
\[ + \langle [\Lambda^s, v_\delta^s] \cdot \nabla J_\delta a_\delta^s, J_\delta \Lambda^s a_\delta^s \rangle, \]
where we have used the fact that \( \langle J_\delta f, g \rangle = \langle f, J_\delta g \rangle \) for any \( f \) and \( g \). We see from the integration by parts that
\[ \| \text{Re} \langle v_\delta^s \cdot \nabla J_\delta \Lambda^s a_\delta^s, J_\delta \Lambda^s a_\delta^s \rangle \| \leq \frac{1}{2} \| \nabla v_\delta^s \|_{L^\infty} \| J_\delta \Lambda^s a_\delta^s \|_{L^2}^2. \]

Moreover, the commutator estimate shows that
\[ \| \text{Re} \langle [\Lambda^s, v_\delta^s] \cdot \nabla J_\delta a_\delta^s, J_\delta \Lambda^s a_\delta^s \rangle \| \]
\[ \leq C(\| \nabla v_\delta^s \|_{H^{s-1}} \| J_\delta \nabla a_\delta^s \|_{L^\infty} + \| \nabla v_\delta^s \|_{L^\infty} \| J_\delta \nabla a_\delta^s \|_{H^{s-1}}) \| J_\delta \Lambda^s a_\delta^s \|_{L^2}. \]

We estimate the third term of (4.4) by the Kato–Ponce inequality as
\[ \| \text{Re} \langle \Lambda^s (a_\delta^s \nabla J_\delta \cdot v_\delta^s), \Lambda^s a_\delta^s \rangle \| \]
\[ \leq C(\| a_\delta^s \|_{L^\infty} \| J_\delta \nabla v_\delta^s \|_{H^s} + \| a_\delta^s \|_{H^s} \| J_\delta \nabla v_\delta^s \|_{L^\infty}) \| \Lambda^s a_\delta^s \|_{L^2} \]
and the last term vanishes since
\[ \text{Re} \langle -i \Delta J_\delta \Lambda^s a_\delta^s, J_\delta \Lambda^s a_\delta^s \rangle = \text{Re} i \| \nabla J_\delta \Lambda^s a_\delta^s \|_{L^2}^2 = 0. \]

Therefore, summarizing (4.4)–(4.8), we end up with
\[ \frac{d}{dt} \| a_\delta^s \|_{H^s}^2 \leq C(\| a_\delta^s \|_{W^{1,\infty}} + \| \nabla v_\delta^s \|_{L^\infty}) (\| a_\delta^s \|_{H^s} + \| v_\delta^s \|_{X^{s+1}}) \| a_\delta^s \|_{H^s}, \]
For the estimate of the Hartree nonlinearity, we use Lemma 3.6 with (4.11).

Integration by parts shows

\[ \sum_{(4.10)} \text{–} (4.13), \]

we deduce that

\[ \text{and the commutator estimate also shows} \]

\[ (4.14) \]

We consider the coupling of this equation and \( Q \). From the equation for \( v_3^\epsilon \), we have

\[ \partial_t Q v_3^\epsilon + J_\delta Q (v_3^\epsilon \cdot \nabla J_\delta v_3^\epsilon) + Q \nabla \Lambda^a J_\delta \left( |x|^{-\gamma} * |a_3^\epsilon|^2 \right) = 0 \]

We consider the coupling of this equation and \( Q v_3^\epsilon \). The second term can be written as

\[ \langle Q (v_3^\epsilon \cdot \nabla J_\delta v_3^\epsilon), J_\delta Q v_3^\epsilon \rangle = \langle v_3^\epsilon \cdot \nabla J_\delta Q v_3^\epsilon, J_\delta Q v_3^\epsilon \rangle \]

and so that

\[ \text{As the previous case, integration by parts shows} \]

\[ (4.11) \]

and the commutator estimate also shows

\[ (4.12) \]

For the estimate of the Hartree nonlinearity, we use Lemma 3.6 with \( k = 2 \) and \( s_1 = s_2 = s \), to obtain

\[ \left\| \lambda J_\delta \Lambda^a \nabla^2 (|x|^{-\gamma} * |a_3^\epsilon|^2) \right\|_{L^2} \leq C(\|a_3^\epsilon\|_{H^s} + \|a_3^\epsilon\|_{H^s}^2) \]

Summarizing (4.10)–(4.13), we deduce that

\[ \frac{d}{dt} \| \nabla v_3^\epsilon \|_{H^s}^2 \leq C(\|a_3^\epsilon\|_{L^2} + \|a_3^\epsilon\|_{L^\infty} + \|\nabla v_3^\epsilon\|_{L^\infty})(\|a_3^\epsilon\|_{H^s} + \|\nabla v_3^\epsilon\|_{H^s}) \]

and so that

\[ (4.14) \]

Using Lemma 3.1, we see that the above estimate yields an \( L^{2n/(n-2)} \) estimate for \( v_3^\epsilon \). Interpolating with a suitable \( H^k \) norm shows that the \( L^\infty \) norm of \( v_3^\epsilon \) is estimated as above. Alternatively, integrating the second equation of (4.1) with respect to time, Sobolev embedding directly yields a similar estimate for \( v_3^\epsilon \) in \( L^\infty(\mathbb{R}^n) \).

Now, putting \( M^2_\delta(t) := \|a_3^\epsilon(t)\|_{H^s}^2 + \|\nabla v_3^\epsilon(t)\|_{H^s}^2 + \|v_3^\epsilon(t)\|_{L^\infty}^2 \), we conclude from (4.14) and (4.12) that

\[ (4.15) \]

We obtain the following Lemma.

**Lemma 4.3.** Let Assumption 4.1 be satisfied with \( s > n/2 + 1 \). There exists \( T \) independent of \( \delta \) and \( \epsilon \) such that the solution \( (a_3^\epsilon, v_3^\epsilon) \) is bounded in \( C([0, T]; H^s \times X^{s+1}) \) uniformly in \( \delta \in [0, 1] \).
Proof. We only estimate the above $M_δ(τ)$. Note that $v^δ_0$ vanishes at spatial infinity. It implies that $\|v^δ_0\|_{L^∞}$ is bounded by $\|∇v^δ_0\|_{H^s}$, with some constant, since $n \geq 3$. and $s > n/2 + 1$. Sobolev embedding and (4.15) yield

\begin{equation}
M_δ(τ) \leq C + C \int_0^τ (M_δ(τ))^{3/2} \, dτ.
\end{equation}

Therefore, there exists $T^*_δ > 0$ depending only on $M_δ(0)$ such that $M_δ(t)$ is bounded by constant times $M_δ(0)$ uniformly in $t \in [0, T^*_δ]$. Since $M_δ(0)$ is bounded independent of $δ$ and $ε$ by assumption, $T^*_δ$ can be taken independent of $δ$ and $ε$, as well as the upper bound of $M_δ(t)$.

\end{proof}

4.3. Existence of the solution to the nonlinear hyperbolic system.

Next we prove the existence of the solution to (4.1). From Lemma 4.3, we see the continuities of $a^δ$ and $v^δ$.

\begin{equation}
\begin{align*}
t & \to \ δ' \to 0. \\
\text{Moreover, we have } (a^ε, v^ε) & \in C_w([0, T]; H^s) \times X^{s+1} \cap L^\infty, \\
\text{we shall show that } (a^ε, v^ε) & \text{satisfies } (4.1) \text{ in } D^′([0, T] \times \mathbb{R}^n). \text{ We fix some } t. \text{ We choose some } s' \text{ so that } s > s' > n/2 + 1. \text{ Then, the above convergences imply } (a^δ, \nabla v^δ) \to (a^ε, \nabla v^ε) \text{ strongly in } C([0, T]; H^s_{\text{loc}} \times H^s_{\text{loc}}) \text{ as } δ' \to 0. \\
\text{Then, we deduce that } a^δ, \nabla a^δ, v^δ, \text{ and } \nabla v^δ \text{ converge uniformly in any } H^s_{\text{loc}} \text{ with some constant. Then, we can pass to the limit in all the terms in } (4.1), \text{ except possibly the Hartree term. Since } (f * g) h = f * (g * h) \text{ and } (f * g, h) = (\tilde{f}, \tilde{g} * h) \text{ with } \tilde{g}(x) = \tilde{g}(-x), \text{ the Hartree term can be rewritten as }
\end{align*}
\end{equation}

\begin{equation}
\langle λ \nabla J_δ(|x|^{-γ} * |a^δ|^2, \varphi) \rangle = -λ \langle J_δ |a^δ|^2, |x|^{-γ} * ∇\varphi \rangle.
\end{equation}

The function $|x|^{-γ} * ∇\varphi$ is not compactly supported, but an $ε/3$-argument shows that the right hand side tends to $-λ \langle |a^ε|^2, |x|^{-γ} * ∇\varphi \rangle$. Thus, we obtain the solution $(a^ε, v^ε) \in C_w([0, T]; H^s \times X^{s+1} \cap L^\infty)$. We now claim that this solution is strongly continuous in time. To prove this, we only have to show that the solution is norm continuous, that is, the function $M^ε(t) := \|a^ε(t)\|^2_{H^s} + \|∇v^ε(t)\|^2_{H^s}$ is continuous in time. In the same way as (4.15), we have

\begin{equation}
\frac{d}{dt} M^ε \leq C (\|a^ε\|_{L^2} + \|a^ε\|_{W^{1,∞}} + \|∇v^ε\|_{L^∞}) M^ε.
\end{equation}

Since the right hand side is bounded, $M^ε$ is upper semi-continuous. Weak continuities of $a^ε$ and $v^ε$ imply the lower semi-continuity of $M^ε$. Hence, $M^ε$ is continuous.

Lemma 4.4. Let Assumption 4.1 be satisfied. Suppose $s > n/2 + 1$. Let $T$ be given in Lemma 4.3. For all $ε \in [0, 1]$, there exists $(a^ε, v^ε) \in C([0, T]; H^s \times X^{s+1} \cap L^\infty)$ which solves (4.1) in $D^′$.
4.4. Uniqueness. We next prove the uniqueness of the solution \((a^c, v^c)\) by showing that if \((a^1_1, v^1_1)\) and \((a^2_2, v^2_2)\) are solutions to (4.11), in the class \(C([0, T]; H^s \times X^{s+1} \cap L^{2n/(n-2)})\) for some \(s > n/2 + 1\), then the distance \((a^1_1 - a^2_2, v^1_1 - v^2_2)\) is equal to zero in \(L^\infty([0, T]; L^2 \times H^1)\) sense. Denote \((d^a_1, d^v_1) := (a^1_1 - a^2_2, v^1_1 - v^2_2)\). Then, from (4.11), the system for \((d^a_1, d^v_1)\) is rewritten as

\begin{align}
\partial_t d^a_1 + d^c_v \cdot \nabla a^1_1 + v^2_1 \cdot \nabla d^a_1 + \frac{1}{2} d^a_1 \cdot \nabla v^2_1 + \frac{1}{2} a^2_2 \cdot \nabla d^c_v = -d^a_1, \\
\partial_t d^v_1 + d^c_v \cdot \nabla v^2_1 + v^2_1 \cdot \nabla d^v_1 + \lambda \nabla (|x|^{\gamma} * (d^c_v a^2_1 + a^2_2 d^c_v)) = 0.
\end{align}

Now estimate the \(L^2\) norm of \(d^a_1\). From the equation in (4.18), it holds that

\[
\frac{d}{dt} \|d^a_1\|^2_{L^2} = 2 \operatorname{Re} \langle \partial_t d^a_1, d^a_1 \rangle 
\]

\[
\leq C \left| \operatorname{Re} \langle d^c_v \cdot v^1_1, d^a_1 \rangle \right| + C \left| \operatorname{Re} \langle v^2_1 \cdot \nabla d^a_1, d^a_1 \rangle \right| + C \left| \operatorname{Re} \langle a^2_2 \cdot d^c_v, d^a_1 \rangle \right| \]

\[
+ C \left| \operatorname{Re} \langle a^2_2 \cdot v^2_1, d^a_1 \rangle \right| + | \operatorname{Re} \langle i \Delta d^a_1, d^a_1 \rangle |.
\]

Now, Hölder’s inequality and integration by parts show that

\[
| \operatorname{Re} \langle a^2_2 \cdot \nabla d^a_1, d^a_1 \rangle | \leq \|a^2_2 \cdot \nabla d^a_1\|_{L^\infty} \|\nabla d^a_1\|_{L^2} \|d^a_1\|_{L^2},
\]

\[
| \operatorname{Re} \langle v^2_1 \cdot \nabla d^a_1, d^a_1 \rangle | + | \operatorname{Re} \langle d^a_1 \cdot \nabla v^2_1, d^a_1 \rangle | \leq (\|\nabla v^2_1\|_{L^\infty} + \|\nabla v^2_1\|_{L^\infty}) \|d^a_1\|^2_{L^2},
\]

\[
\operatorname{Re} \langle i \Delta d^a_1, d^a_1 \rangle = 0.
\]

Another use of Hölder’s and Sobolev inequalities shows

\[
| \langle d^c_v \cdot \nabla a^1_1, d^a_1 \rangle | \leq \|d^c_v\|_{L^n} \|\nabla a^1_1\|_{L^2} \|d^a_1\|_{L^2}
\]

Thus, we end up with the estimate

\[
\frac{d}{dt} \|d^a_1\|^2_{L^2} \leq C \left( \|d^c_v\|^2_{L^2} + \|\nabla d^c_v\|^2_{L^2} \right),
\]

where the constant \(C\) depends on \(\|a^1_1\|_{H^{n/2}}, \|a^2_2\|_{H^{n/2}}, \|a^1_1\|_{L^\infty}, \|a^2_2\|_{L^\infty}\), and \(\|\nabla v^2_k\|_{L^2} (k = 1, 2)\).

Similarly, for all \(1 \leq i, j \leq n\), we have the estimates for \(\partial_i d^c_v,j\):

\[
| \langle (\partial_i d^c_v,j) \cdot \nabla v^1_{i,j}, \partial_i d^c_v,j \rangle \rangle | \leq C \|\nabla v^1_{i,j}\|_{L^\infty} \|\partial_i d^c_v,j\|^2_{L^2},
\]

\[
| \langle (d^c_v \cdot \partial_i v^2_{i,j}, \partial_i d^c_v,j) \rangle \rangle | \leq C \|\nabla v^1_{i,j}\|_{L^n} \|\nabla d^c_v,j\|_{L^2} \|\partial_i d^c_v,j\|_{L^2},
\]

\[
| \langle (\partial_i v^2_1 \cdot \nabla d^c_v,j, \partial_i d^c_v,j) \rangle \rangle | \leq C \|\partial_i v^2_1\|_{L^\infty} \|\nabla d^c_v,j\|^2_{L^2},
\]

\[
| \langle \partial_i \partial_i (|x|^{-\gamma} * (d^c_v a^2_1)), \partial_i d^c_v,j \rangle \rangle | \leq C (\|a^1_1\|_{L^\infty} + \|a^1_1\|_{L^2}) \|d^c_v,j\|_{L^2} \|\partial_i d^c_v,j\|_{L^2},
\]

\[
| \langle \partial_i \partial_i (|x|^{-\gamma} * (a^2_2 d^c_v)), \partial_i d^c_v,j \rangle \rangle | \leq C (\|a^2_2\|_{L^\infty} + \|a^2_2\|_{L^2}) \|d^c_v,j\|_{L^2} \|\partial_i d^c_v,j\|_{L^2},
\]

where \(v^1_{i,j}\) and \(d^c_v,j\) denote the \(j\)-th components of \(v^1_1\) and \(d^c_v\), respectively. Summing up over \(i\) and \(j\), we obtain

\[
\frac{d}{dt} \|d^c_v\|^2_{L^2} \leq C \left( \|d^a_1\|^2_{L^2} + \|\nabla d^a_1\|^2_{L^2} \right).
\]

Denote \(D(t) := \|d^a_1(t)\|^2_{L^2} + \|\nabla d^a_1(t)\|^2_{L^2}\): since \(D(0) = 0\), Gronwall lemma shows that \(D(t) = 0\) for all \(t \in [0, T]\).
**Lemma 4.5.** The solution \((a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})\) to the system (1.1) given in Lemma 4.4 is unique.

4.5. Completion of the proof of Proposition 4.1. Now we complete the proof of existence result.

**Proof of Proposition 4.1.** We have already shown that the system (4.1) has a unique solution \((a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{\frac{2n}{n-2}})\).

We show that the existence time \(T\) is independent of \(s\), thanks to tame estimates. In the above proof, the existence time \(T\) depends on \(s\). However, once we show the existence of the solution in \([0, T_{s_0}]\) for some \(s_0 > n/2 + 1\), then, for any \(s_1 > n/2 + 1\), we deduce from (4.17) that

\[
M_{s_1}^\varepsilon(t) \leq M_{s_1}^\varepsilon(0) \exp \left( C t \sup_{0 \leq \tau \leq t} (\|a^\varepsilon(\tau)\|_{L^2} + \|a^\varepsilon(\tau)\|_{W^{1,\infty}} + \|\nabla v^\varepsilon(\tau)\|_{L^\infty}) \right)
\]

and so that \(M_{s_1}^\varepsilon(t) < \infty\) holds for \(t \in [0, T_{s_0}]\). It means that the solution \((a^\varepsilon, v^\varepsilon)\) extends to time \(T_{s_0}\) as a \(H^{s_1} \times X^{s_1+1} \cap L^{\frac{2n}{n-2}}\)-valued function, that is, \(T_{s_0} \geq T_{s_0}\). The same argument also shows \(T_{s_0} \geq T_{s_1}\). Therefore, \(T\) does not depend on \(s\).

4.6. Construction of \(\phi^\varepsilon\). We finally construct \(\phi^\varepsilon\) from \(v^\varepsilon\) defined in Proposition 4.1. Since \(v^\varepsilon\) is known, in view of (4.22), it is natural to define \(\phi^\varepsilon\) as

\[
\phi^\varepsilon(t, x) = \phi_0(x) - \int_0^t \left( \frac{1}{2} |v^\varepsilon(\tau, x)|^2 + \lambda \left( |x|^{-\gamma} * |a^\varepsilon|^2 \right)(\tau, x) \right) d\tau.
\]

By Assumption 1.1 and Lemma 3.1, \(\phi_0 \in L^\infty \cap L^{\frac{2nq_0}{n-q_0}}\). Proposition 1.1 shows that

\[
|v^\varepsilon|^2 \in C \left( [0, T]; X^{s+1} \cap L^{\frac{2nq_0}{n-q_0}} \right).
\]

Lemma 3.6 with \(k = 2\) and \(s_1 = s_2 = s\) shows that

\[
\nabla^2 \left( |x|^{-\gamma} * |a^\varepsilon|^2 \right) \in C \left( [0, T]; H^s \right).
\]

By the Hardy–Littlewood–Sobolev inequality, for all \(n/(\gamma + 1) < q < \infty\), it holds that

\[
\|\nabla (|x|^{-\gamma} * |a^\varepsilon|^2)\|_{L^q} \leq C \|(|x|^{-\gamma-1} * |a^\varepsilon|^2)\|_{L^q} \leq C \|a^\varepsilon\|^2_{H^s},
\]

and \(\nabla (|x|^{-\gamma} * |a^\varepsilon|^2) \in C \left( [0, T]; H^{s+1} \right)\). Moreover, the Sobolev inequality shows \(|x|^{-\gamma} * |a^\varepsilon|^2 \in L^\infty\). Therefore, \(\phi^\varepsilon\) has the regularity announced in Theorem 1.3. To conclude, we simply notice the identity

\[
\partial_t (\nabla \phi^\varepsilon - v^\varepsilon) = \nabla \partial_t \phi^\varepsilon - \partial_t v^\varepsilon = 0,
\]

so that \(v^\varepsilon = \nabla \phi^\varepsilon\), and (4.22) yields the second equation in (4.22). This completes the proof of Theorem 1.3.
5. Asymptotic expansion

Proof of Theorem 4.7. First order. Suppose that Assumption 1.6 is satisfied with $N \geq 1$. We already know that the equation (4.11) has a unique solution $(a^\varepsilon, v^\varepsilon) \in C([0, T]; H^s \times X^{s+1} \cap L^{\infty})$ for all $\varepsilon \in [0, 1]$. Denote $(b_0, w_0) := (a^\varepsilon, v^\varepsilon)_{|\varepsilon=0}$. We define

$$b^\varepsilon = \frac{a^\varepsilon - b_0}{\varepsilon}, \quad w^\varepsilon = \frac{v^\varepsilon - w_0}{\varepsilon}.$$  \hspace{1cm} (5.1)

Substituting $a^\varepsilon = b_0 + \varepsilon b^\varepsilon$ and $v^\varepsilon = w_0 + \varepsilon w^\varepsilon$ into the system (4.11), we obtain the system for $(b^\varepsilon, w^\varepsilon)$:

$$\begin{cases}
\partial_t b^\varepsilon + w^\varepsilon \cdot \nabla b_0 + w_0 \cdot \nabla b^\varepsilon + \frac{1}{2} b_0 \nabla \cdot w^\varepsilon + \frac{1}{2} b^\varepsilon \nabla \cdot w_0 \\
+ \varepsilon w^\varepsilon \cdot \nabla b^\varepsilon + \frac{\varepsilon}{2} b^\varepsilon \nabla \cdot w^\varepsilon - i \frac{\varepsilon}{2} \Delta b_0 = i \frac{\varepsilon}{2} \Delta b^\varepsilon,
\end{cases}$$ \hspace{1cm} (5.2)

$$\begin{cases}
\partial_t w^\varepsilon + w^\varepsilon \cdot \nabla w_0 + w_0 \cdot \nabla w^\varepsilon + \lambda \nabla (|x|^{-\gamma} \ast 2 \Re(b_0 b^\varepsilon)) \\
+ \varepsilon w^\varepsilon \cdot \nabla w^\varepsilon + \lambda \varepsilon (|x|^{-\gamma} \ast |b^\varepsilon|^2) = 0,
\end{cases}$$  \hspace{1cm} (5.3)

where we have used the fact that $(b_0, w_0)$ is the solution to the system (4.11) with $\varepsilon = 0$ and the assumption that the initial data of $a^\varepsilon$ is written as $a^\varepsilon_0 = a_0 + \varepsilon a_1 + \sum_{j=2}^N \varepsilon^j a_j + \varepsilon^N r_N^\varepsilon$. Since we know that $b^\varepsilon \in C([0, T]; H^s)$ and $w^\varepsilon \in C([0, T]: X^{s+1} \cap L^{\infty})$ for $\varepsilon > 0$, we just need to prove a priori estimates which are independent of $\varepsilon$. Mimicking the energy estimates (4.9) and (4.10), we obtain

$$\begin{align*}
\frac{d}{dt}(\|b^\varepsilon\|^2_{H^s \times 2} + \|\nabla w^\varepsilon\|^2_{H^{s-2}}) \\
\leq C + C(1 + \varepsilon(\|b^\varepsilon\|_{L^2} + \|b^\varepsilon\|_{W^{1, \infty}} + \|w^\varepsilon\|_{L^\infty}))(\|b^\varepsilon\|^2_{H^{s-2}} + \|\nabla w^\varepsilon\|^2_{H^{s-2}}),
\end{align*}$$  \hspace{1cm} (5.4)

where the constant $C$ depends on $\|b_0\|_{H^s}$ and $\|\nabla w_0\|_{H^s}$. Indeed, the quadratic terms of the system can be handled by the same way since they are exactly the same as those in the system (4.11) up to the constant $\varepsilon$. We estimate linear terms essentially by the same way. Note that the integration by parts does not work well, and so that we need the $H^{s-1}$-boundedness of $b_0$ and $\nabla w_0$. The term $i \frac{\varepsilon}{2} \Delta b_0$ is also new. By the presence of this term, $b_0$ is required to be bounded in $H^s$.

Mimicking the proof of Proposition 4.1, we can show the existence of a unique solution $(b^\varepsilon, w^\varepsilon) \in C([0, T], H^{s-2} \times X^{s-1} \cap L^{\infty})$ for all $\varepsilon \in [0, 1]$. Since $b^\varepsilon(0)$ is uniformly bounded in $H^{s-2}$ by assumption, we see that the $H^{s-2} \times X^{s-1}$-bound of $(b^\varepsilon, w^\varepsilon)$ is independent of $\varepsilon$. It proves $\|a^\varepsilon - b_0\|_{H^{s-2}} + \|\nabla v^\varepsilon - \nabla w_0\|_{H^{s-2}} = O(\varepsilon)$. Moreover, the existence time $T$ is also independent of $\varepsilon$. Then, we see from (5.1) that, for $\varepsilon > 0$, the existence time for $(b^\varepsilon, w^\varepsilon)$ must be equal to that for $(a^\varepsilon, v^\varepsilon)$. Hence, we conclude that the existence time for $(b^\varepsilon, w^\varepsilon)$ with $\varepsilon = 0$ is also the same.
Thus, putting \((b_1, w_1) = (b^\varepsilon, w^\varepsilon)|_{\varepsilon = 0}\), we obtain the solution to the system

\begin{align}
\partial_t b_1 + w_1 \cdot \nabla b_0 + w_0 \cdot \nabla b_1 + \frac{1}{2} b_1 \nabla \cdot w_0 - \frac{i}{2} \Delta b_0 &= 0, \\
\partial_t w_1 + w_1 \cdot \nabla w_0 + w_0 \cdot \nabla w_1 + \lambda \nabla(|x|^{-\gamma} * 2 \text{Re}(b_0 \overline{b_1})) &= 0,
\end{align}

\begin{align}
b_{1|t=0} &= a_1, \quad w_{1|t=0} = 0.
\end{align}

Since \(w_{1|t=0} \equiv 0 \in L^r\) for all \(r\) and \(b_1 \in H^{s-2}\), we see that \((b_1, w_1) \in C([0, T]; H^{s-2-1} \times X^{s-1} \cap L^q)\) for all \(q \in [n/(\gamma + 1), \infty]\). By the similar way as the construction of \(\phi^\varepsilon\), we can construct \(\phi_1\) so that \(\phi_1 \in L^p\) for all \(p \in [n/\gamma, \infty]\) and \(\nabla \phi_1 = v_1\).

**Higher order.** Let \(N \geq 2\) and Assumption 1.6 be satisfied. Take \(m \in [2, N]\) and assume that, for \(1 \leq k \leq m - 1\), the system

\begin{align}
\partial_t b_k + \sum_{i+j=k} w_i \cdot \nabla b_j + \frac{1}{2} \sum_{i+j=k} b_i \nabla \cdot w_j - \frac{i}{2} \Delta b_{k-1} &= 0, \\
\partial_t w_k + \sum_{i+j=k} w_i \cdot \nabla w_j + \lambda \sum_{i+j=k} \nabla(|x|^{-\gamma} * 2 \text{Re}(b_i \overline{b_j})) &= 0,
\end{align}

\begin{align}
b_{k|t=0} = a_k, \quad w_{k|t=0} = 0,
\end{align}

has a unique solution \((b_k, w_k) \in C([0, T]; H^{s-2k} \times X^{s-2k+1} \cap L^q)\), where \(q \in [n/(\gamma + 1), \infty]\). Denote

\begin{align}
b^\varepsilon &= \frac{a^\varepsilon - \sum_{j=0}^{m-1} \varepsilon^j b_j}{\varepsilon^m}, \quad w^\varepsilon = \frac{v^\varepsilon - \sum_{j=0}^{m-1} \varepsilon^j w_j}{\varepsilon^m}.
\end{align}

Then, \((b^\varepsilon, w^\varepsilon)\) satisfies the following system:

\begin{align}
\partial_t b^\varepsilon + \sum_{\ell=0}^{m-1} \varepsilon^\ell \left( w^\varepsilon \cdot \nabla b^\ell + w^\ell \cdot \nabla \nabla b^\varepsilon + \frac{1}{2} b^\varepsilon \nabla \cdot w^\ell + \frac{1}{2} b^\ell \nabla \cdot w^\varepsilon \right)
+ \sum_{\ell=0}^{m-1} \varepsilon^\ell \sum_{i,j<m, i+j=m+l} \left( w_i \cdot \nabla b_j + \frac{1}{2} b_i \nabla \cdot w_j \right)
+ \varepsilon^m w^\varepsilon \cdot \nabla b^\varepsilon + \frac{\varepsilon^m}{2} b^\varepsilon \nabla \cdot w^\varepsilon - \frac{i}{2} \Delta b_{m-1} = \frac{i}{2} \Delta b^\varepsilon,
\end{align}

\begin{align}
\partial_t w^\varepsilon + \sum_{\ell=0}^{m-1} \varepsilon^\ell \left( w^\varepsilon \cdot \nabla w^\ell + w^\ell \cdot \nabla w^\varepsilon \right) + \lambda \nabla(|x|^{-\gamma} * 2 \text{Re}(b_0 \overline{b^\varepsilon}))
+ \sum_{\ell=0}^{m-1} \varepsilon^\ell \sum_{i,j<m, i+j=m+l} \left( w_i \cdot \nabla w_j + \lambda \nabla(|x|^{-\gamma} * 2 \text{Re}(b_i \overline{b_j})) \right)
+ \varepsilon^m w^\varepsilon \cdot \nabla w^\varepsilon + \lambda \varepsilon^m \nabla(|x|^{-\gamma} * |b^\varepsilon|^2) = 0,
\end{align}

\begin{align}
b^\varepsilon_{|t=0} = a_m + \sum_{j=1}^{N-m} \varepsilon^j a_{j+m} + \varepsilon^{N-m} r^\varepsilon_N, \quad w^\varepsilon_{|t=0} = 0.
\end{align}

By induction on \(m\), the system 5.5-10 has a unique solution

\((b^\varepsilon, w^\varepsilon) \in C \left([0, T]; H^{s-2m} \times X^{s-2m+1}\right) \quad (s - 2m + 1 > n/2)\).
As the first order, we see that the upper bound of \((b^\varepsilon, w^\varepsilon)\) and the existence time \(T\) is independent of \(\varepsilon\). Furthermore, \(T\) is the same one as for \((a^\varepsilon, v^\varepsilon)\). Note that we need the \(H^{s-2m+2}\) boundedness of \(\Delta b_{m-1}\) to solve the system \((5.3) - (5.10)\). Denote \((b_m, w_m) := (b^\varepsilon, w^\varepsilon)|_{\varepsilon=0}\). Then, \((b_m, w_m)\) satisfies \((5.7) - (5.8)\) with \(k = m\).

Since \(w_m|_{t=0} \equiv 0 \in L^r\) for all \(r\) and \(b_m \in H^{s-2m}\) with \(s - 2m > n/2 + 2(N - m) + 1 \geq n/2 + 1\), we see that \(w_m \in C([0, T]; X^{s-2m+1} \cap L^q)\) for all \(q \in [n/(\gamma + 1), \infty]\), and that there exists \(\varphi_m\) which satisfies \(\varphi_m \in C([0, T]; L^p)\) for all \(p \in [n/\gamma, \infty]\) and \(\nabla \varphi_m = w_m\).

**Proof of Corollary 1.8.** Corollary 1.8 is a straightforward consequence of Theorem 1.7 by considering the asymptotic expansion of

\[
e^{i\varphi_0/\varepsilon + iv^3} (b_0 + \varepsilon b_1 + \ldots + \varepsilon^N b_N) e^{i\varepsilon^2 + i\varepsilon^2 v^3 + \ldots + i\varepsilon^{N-1} v^N}\]

in powers of \(\varepsilon\). The first exponential is not modified, but one considers the asymptotic expansion of the last two terms in this product.

Note the shift in precision, between Theorem 1.7 and Corollary 1.8: the initial order of precision \(o(\varepsilon^N)\) becomes \(o(\varepsilon^{N-1})\) (in \(L^2 \cap L^\infty\)). This is because the phase \(\phi^\varepsilon\) is divided by \(\varepsilon\) to go back to \(u^\varepsilon\). This phenomenon has several consequences, see e.g. [7] for instabilities. □

**6. Proof of Corollary 1.9.**

To see that Corollary 1.9 is a consequence of Corollary 1.8, we resume the same approach as in [7]. Let \(a_0 \in \mathcal{S}(\mathbb{R}^n)\) be non-trivial (independent of \(h\)), and consider

\[
\psi_0^h(x) = h^{s-n/2} a_0 \left(\frac{x}{h}\right).
\]

Let \(\varepsilon = h^{s-\sigma - 1} = h^{\gamma/2 - 1 - s}: h\) and \(\varepsilon\) go to zero simultaneously by assumption. Consider the change of unknown function

\[
\psi^h(t, x) = h^{s-n/2} u^\varepsilon \left(\frac{t}{\varepsilon h^2}, \frac{x}{h}\right).
\]

Then the Cauchy problem for \(\psi^h\) is equivalent to:

\[
iv\partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = \lambda(|x|^{-\gamma} * |u^\varepsilon|^2) u^\varepsilon \quad ; \quad u^\varepsilon|_{t=0} = a_0.
\]

This is \((1.1) - (1.2)\) with \(\phi_0 = 0\), and \(a_0^\varepsilon = a_0\) independent of \(\varepsilon\). By construction, the phase \(\varphi_0\) provided in Theorem 1.7 is such that

\[
\varphi_0|_{t=0} = 0 \quad ; \quad \partial_t \varphi_0|_{t=0} = -\lambda |x|^{-\gamma} * |a_0|^2,
\]

where we have used the equation determining \(\varphi_0\), that is, \((2.2)\) with \(\varepsilon = 0\). Therefore, there exists \(\tau > 0\) independent of \(\varepsilon\) such that \(\varphi_0|_{t=t_0}^{t_0} = \) is non-trivial on the support of \(a_0\): at time \(t = \tau\), \(u^\varepsilon\) is exactly \(\varepsilon\)-oscillatory, from Corollary 1.8. Back to \(\psi^h\), this yields Corollary 1.9 up to replacing \(a_0\) with \(|\log h|^{-1} a_0\) (this makes no trouble, since the competition in terms of \(h\) is logarithmic decay vs. algebraic decay).

**Acknowledgments.** The authors express their deep gratitude to Professor Yoshio Tsutsumi for his helpful advice.
References

1. T. Alazard and R. Carles, Semi-classical limit of Schrödinger–Poisson equations in space dimension $n \geq 3$, J. Differential Equations 233 (2007), no. 1, 241–275.
2. T. Alazard and R. Carles, Loss of regularity for super-critical nonlinear Schrödinger equations, preprint, archived as math.AP/0701857.
3. T. Alazard and R. Carles, Supercritical geometric optics for nonlinear Schrödinger equations, preprint, archived as arXiv:0704.2488.
4. N. Burq, P. Gérard, and N. Tzvetkov, An instability property of the nonlinear Schrödinger equation on $S^d$, Math. Res. Lett. 9 (2002), no. 2-3, 323–335.
5. N. Burq, P. Gérard, and N. Tzvetkov, Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations, Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 2, 255–301.
6. R. Carles, Cascade of phase shifts for nonlinear Schrödinger equations, J. Hyperbolic Differ. Equ. 4 (2007), no. 2, 207–231.
7. R. Carles, Geometric optics and instability for semi-classical Schrödinger equations, Arch. Ration. Mech. Anal. 183 (2007), no. 3, 525–553.
8. T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.
9. M. Christ, J. Colliander, and T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. 125 (2003), no. 6, 1235–1293.
10. M. Christ, J. Colliander, and T. Tao, Ill-posedness for nonlinear Schrödinger and wave equations, archived as arXiv:math.AP/0311048.
11. C. Gallo, Schrödinger group on Zhidkov spaces, Adv. Differential Equations 9 (2004), no. 5-6, 509–538.
12. P. Gérard, Remarques sur l’analyse semi-classique de l’équation de Schrödinger non linéaire, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.
13. P. Gérard, The Cauchy problem for the Gross-Pitaevskii equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 5, 765–779.
14. J. Ginibre and G. Velo, Sur une équation de Schrödinger non linéaire avec interaction non locale, Nonlinear partial differential equations and their applications, Collège de France Seminar (H. Brézis and J.-L. Lions, eds.), vol. 2, Research Notes in Math., no. 60, Pitman, 1982, pp. 155–199.
15. E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, Proc. Amer. Math. Soc. 126 (1998), no. 2, 523–530.
16. L. Hörmander, The analysis of linear partial differential operators. I, second ed., Springer Study Edition, Springer-Verlag, Berlin, 1990, Distribution theory and Fourier analysis.
17. D. Lannes, Sharp estimates for pseudo-differential operators with symbols of limited smoothness and commutators, J. Funct. Anal. 232 (2006), no. 2, 495–539.
18. G. Lebeau, Non-linear optic and supercritical wave equation, Bull. Soc. Roy. Sci. Liège 70 (2001), no. 4-6, 267–306 (2002), Hommage à Pascal Laubin.
19. G. Lebeau, Perte de régularité pour les équations d’ondes sur-critiques, Bull. Soc. Math. France 133 (2005), 145–157.
20. S. Masaki, Semi-classical analysis for Hartree equations in some supercritical cases, Ann. Henri Poincaré 8 (2007), no. 6, 1037–1069.
21. L. Thomann, Instabilities for supercritical Schrödinger equations in analytic manifolds, preprint, archived as arXiv:0707.1785.
22. P. E. Zhidkov, The Cauchy problem for a nonlinear Schrödinger equation, JINR Commun., P5-87-373, Dubna (1987), (in Russian).
23. P. E. Zhidkov, Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001.
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