Griffiths variational multisymplectic formulation for Lovelock gravity

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Abstract
This work is mainly devoted to constructing a multisymplectic description of Lovelock’s gravity, which is an extension of General Relativity. We establish the Griffiths variational problem for the Lovelock Lagrangian, obtaining the geometric form of the corresponding field equations. We give the unified Lagrangian–Hamiltonian formulation of this model and we study the correspondence between the unified formulations for the Einstein–Hilbert and the Einstein–Palatini models of gravity.

Keywords Field theory · Lagrangian and Hamiltonian formalisms · Jet bundles · Multisymplectic manifolds · Griffiths variational problem · Lovelock gravity · Hilbert–Einstein and Einstein–Palatini actions · Einstein equations

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1 Introduction

The development of the geometric description of classical field theories using multisymplectic [7,20,27,29,44,47] or polysymplectic and k-(co)symplectic manifolds [18,31,32] has rekindled the interest in doing a totally covariant description of many theories in physics and, in particular, General Relativity and other derived from it. Many general aspects as well as specific problems and characteristics of the theory have been studied in this way (see, for instance, [9,12,24,29,33,34,36,37,45,46]).

In particular, the multisymplectic techniques have been applied to describe the most standard models of General Relativity: the Einstein–Hilbert [25] and the Einstein–Palatini (or metric-affine) models (see, for instance, [5,6,26,49]). In some of these applications, a unified formalism which joins the Lagrangian and Hamiltonian formalisms into a single one has been used. This unified Lagrangian–Hamiltonian formalism, introduced for the first time in the pioneering work of Skinner and Rusk [48], is especially useful in mechanics and field theories [3,15,21,43] when the Lagrangian that describes the system is singular. For this reason, such formalism finds immediate application in the study of both, the Einstein–Hilbert and the Einstein–Palatini models of gravity. In the first case, and following the formulation for second order field theories developed in [43], the symmetrized jet-multimomentum bundle is used as framework, which turns out to be a premultisymplectic bundle and therefore admits the use of the premultisymplectic constraint algorithm [16,17] for the study of the field equations. In the second case, an indirect path for the construction of unified formalism is invoked: first, the field theory corresponding to the
Einstein–Palatini model is formulated in [5] as a Griffiths variational problem [30]. Subsequently, and inspired by the work of Gotay [28], a unified formalism is constructed as a Lepage-equivalent problem related to the latter [6]. Although it is known that the Einstein–Palatini and the Hilbert–Einstein Lagrangians essentially lead to the same field equations [13] (the Einstein equations), the way in which the unified formulations correspond to each other is unknown.

In the last decades, new models that extend General Relativity have emerged in theoretical physics [4,10,22]. In particular, Lovelock gravity is a generalization of General Relativity (in vacuum) introduced by Lovelock [39,40] (see also [19] for a previous work on the canonical analysis of Lanczos-Lovelock gravity). His idea was to characterize all the symmetric tensors of order 2, without divergence, that can be constructed from the metric tensor and its derivatives up to second order. In dimension 4, it turns out that the only tensors that verify these properties are the metric and the Einstein tensor. In addition Lovelock proved that this tensor encodes the Euler–Lagrange equations of a Lagrangian density that is a polynomial in the (pseudo) Riemannian curvature. An interesting characterization for the Lovelock Lagrangian is provided in [14]: it is the only Lagrangian that is a polynomial in the (pseudo) Riemannian curvature and is also stable under a procedure called consistent Levi-Civita truncation. Similar considerations can be found in [41,42], where the idea consists in considering the Lagrangian as a function independent of the metric and the curvature, and to find relations between the partial derivatives of the Lagrangian with respect to these variables, induced by the geometry of the problem. As the other aforementioned models in General Relativity, the Lovelock Lagrangian is singular and then the multisymplectic formulation and, in particular, the unified Lagrangian–Hamiltonian formalism are especially suitable for its study.

The objectives of this paper are to state and prove the most general and precise results on the following aspects: to study the correspondence between the unified formulations for the Einstein–Hilbert and the Einstein–Palatini models of gravity, to define the Lovelock Lagrangian in the context of multisymplectic geometry, to characterize geometrically its properties, to establish the Griffiths variational problem for this Lagrangian and to develop the corresponding Lagrangian–Hamiltonian unified formalism.

The organization of the paper is as follows: First, in Sect. 2, we set the basic definitions, notation and canonical structures of the frame bundle, which is widely used in the work. In Sect. 3, the Lovelock Lagrangian is presented and the corresponding variational problem is stated and analyzed. Section 4 is devoted to introduce the infinitesimal symmetries of the system described by the Lovelock Lagrangian and to obtain the field equations that derive from the Griffiths variational problem for this system. Finally, in Sect. 5, the premultisymplectic description of the Lovelock system is carried out using the unified Lagrangian–Hamiltonian formalism. After the conclusions of Sect. 6, an “Appendix” is included where different notations are set and several geometric constructions and definitions used throughout the work are collected.

All manifolds are finite-dimensional, real, paracompact, connected and $C^\infty$. All maps are $C^\infty$. Sum over crossed repeated indices is understood.
2 The frame bundle and its canonical forms

2.1 Basic definitions and notation

Consider a space-time manifold $M$ of dimension $m$. The corresponding bundle of frames (see for example [35]) $\tau : LM \to M$ is the set\(^1\)

$$LM := \bigcup_{x \in M} \text{Iso}(\mathbb{R}^m, T_x M),$$

where $\text{Iso}(V, W)$ is the set of (linear) isomorphisms between vector spaces $V$ and $W$.

It is well-known that the general linear group $G := \text{GL}(m)$ acts naturally on $\mathbb{R}^m$ by automorphisms. This action in turn induces a $G$-right action on $LM$, according to the formula

$$u \cdot A := u \circ A,$$

for every $u \in LM$, $A \in G$, endowing $LM$ with a $G$-principal bundle structure.

Let us fix a matrix (any signature can be used in these considerations; the one chosen here follows closely the signature found in General Relativity)

$$\eta := \text{diag}(-1, 1, \cdots, 1);$$

it can be considered either a map $\eta : \mathbb{R}^m \to (\mathbb{R}^m)^*$ or a bilinear form $\eta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}.$

Associated with the bundle $\tau$, we have the fiber bundle of 1-jets $J^1 \tau$ of sections of $\tau$. Given a section $s \in \Gamma(\tau)$, the 1-jet of $s$ at $x$, denoted $j^1_x s$, is the class of local sections being contact equivalent up to first order at $x$. This space has natural bundle projections $\tau_{10} : J^1 \tau \to LM$ and $\tau_1 : J^1 \tau \to M$.

For every element $A \in G$, right translation $R_A : LM \to LM$ is a bundle isomorphism over the identity and so it can be lifted to a bundle isomorphism of $J^1 \tau$ by taking the 1-jet $j^1 R_A : J^1 \tau \to J^1 \tau$. Accordingly, this defines a right action of $G$ on $J^1 \tau$ and it can be checked that the quotient $C(LM) := J^1 \tau / G$ is a smooth manifold, making $q : J^1 \tau \to C(LM)$ into a $G$-principal bundle fitting the diagram

$$\begin{array}{ccc}
C(LM) & \xrightarrow{\tau_1} & LM \\
\downarrow p & \ & \downarrow \tau \\
M & \xrightarrow{\tau_{10}} & LM
\end{array}$$

The bundle $C(LM)$ is called the bundle of connections of $LM$ (see [8] for an account of the geometry of this bundle). It can be proved that the bundles $J^1 \tau \to C(LM)$ and

\(^1\) Alternatively, we can think of each of the fibers $LM_x$ as the set of ordered bases of the tangent space $T_x M$. 
The bundle of connections. We have just seen the correspondence $\rho \in J^1(\tau) \mapsto \rho_{G(\tau)} := \tau_{10}(\rho)$ and $A \in G$, then $\rho \cdot A = ([\rho]_G, u \cdot A)$.

If $\rho \in \tau^{-1}_i(u)$, then $\rho$ can also be thought of as a linear map $\rho : T_{(\tau)} \rightarrow T_u LM$ such that $T_u \tau \circ \rho = Id_{T_{(\tau)} M}$. The interpretation goes as follows: given a local section $s \in \Gamma(\tau)$ and a tangent vector $X \in T_s M$, then $j_x^1 s(X) = T_x s(X)$. Accordingly, each element $[\rho]_G \in C(LM)$ can be interpreted as a linear map $[\rho]_G : \Gamma(M) \rightarrow (TLM)/G_x$, such that $T\tau \circ [\rho]_G = Id_{T_x M}$, where the action of $G$ in $TLM$ is naturally induced by the action of $G$ on $LM$, and $T\tau : (TLM)/G \rightarrow LM$ is given by $[T\tau](X) = T\tau(X)$.

Coordinates in $LM$ will be denoted using greek indices $(x^\mu)$ and the related fiber coordinates in $LM$ will be denoted $(x^\mu, \xi^\nu)$, where $u(e_k) = \xi^\nu \partial / \partial x^\nu$ and $\{e_1, \ldots, e_m\}$ is the canonical basis in $\mathbb{R}^m$. Accordingly, fiber coordinates in $J^1(\tau)$ will be denoted $(x^\mu, e^\nu_k, e^\nu_{k\sigma})$. Using these coordinates, it can be seen that $(x^\mu, \Gamma^\mu_{\nu\sigma} := -\xi^\nu \xi^\mu)$ define fiber coordinates in $C(LM)$.

### 2.2 The universal principal connection

Now, we define a principal connection on the bundle $q : J^1(\tau) \rightarrow C(LM)$ fulfilling a universal property. First, observe that every element $[\rho]_G \in C(LM)$ can be viewed as a pointwise connection, i.e. every $[\rho]_G$ defines a unique family of projections $\Gamma_u : T_u LM \rightarrow V_u \tau$ for every $u \in \tau^{-1}(\tau(\rho))$. Indeed, if $\rho$ is any representative of the class $[\rho]_G$, set $u = \tau_{10}(\rho)$ and define

$$\Gamma_u := Id_{T_u LM} - \rho \circ T_u \tau.$$ 

It is immediate to see that $\Gamma_u(X) \in V_u \tau$, for every $X \in T_u LM$. For any other element $u' \in \tau^{-1}(\tau(u))$, we use right translation as follows

$$\Gamma_{u'} := T_u R_g \circ \Gamma_u \circ T_{u'} R_{g^{-1}},$$

where $u' = u \cdot g$. It is readily seen that this construction is independent of the choice of the representative $\rho$.

**Remark 1** When treating principal connections, we will use the symbol $\Gamma$ to refer to the family of vertical projections. Furthermore, each principal connection carries a Lie algebra-valued differential form called connection form, which we denote by the symbol $\omega$. In the sequel, we refer to principal connections either through the projections $\Gamma$ or through its connection form $\omega$.

**Remark 2** Taking into account the previous observation, it is clear that the set of principal connections $\Gamma$ on $LM$ is in one-to-one correspondence with the sections of the bundle of connections. We have just seen the correspondence $[\rho]_G \mapsto \Gamma$. The inverse correspondence is given by $\Gamma \mapsto \sigma_\Gamma(x) = [hor^u(x)]_G$, where $hor^u(x) : T_x M \rightarrow T_u LM$ is the horizontal lift related to $u \in \tau^{-1}(x)$ and $\Gamma$. 
Now denote by $\mathfrak{g}$ the Lie algebra of $G$ and define $\omega \in \Omega^1(J^1\tau, \mathfrak{g})$ as

$$\omega|_\rho (Y) := [\rho]_G \left( T_\rho \tau_{10}(Y) \right),$$

where we are using the identification $LM \times \mathfrak{g} \rightarrow V\tau$. The fact that this Lie algebra-valued differential form is indeed the connection form of a principal connection can be easily checked.

To introduce the universal property associated with $\omega$, let us observe that, if $\sigma_\Gamma$ is a principal connection on $LM$ and $\sigma_\Gamma$ is the related section of the bundle of connections, then we can define a section $\tilde{\sigma}_\Gamma \in \Gamma \tau_{10}$ by using the identification $J^1\tau \simeq C(LM) \times_M LM$ as (see the diagram above)

$$\tilde{\sigma}_\Gamma(u) = (\sigma_\Gamma(\tau(u)), u).$$

Then, if $\omega_\Gamma$ is the connection form associated with $\Gamma$, it turns out that $\omega_\Gamma = \tilde{\sigma}_\Gamma^*(\omega)$. In other words, any connection form of a principal connection can be recovered as a pullback of $\omega$ by the section $\tilde{\sigma}_\Gamma$. In this sense we say that $\omega$ is a universal connection. Accordingly, the universal curvature is given by (see Appendix A.4)

$$\Omega := d\omega + [\omega \wedge \omega]$$

and it can be seen that the curvature form associated with $\Gamma$ is $\Omega_\Gamma = \tilde{\sigma}_\Gamma^*(\Omega)$.

If $\{E^i_j\}$ denotes the canonical basis of $\mathfrak{g}$ and $\omega = \omega^i_j E^j_i$, then the coordinate expression of the forms $\omega^i_j$ using fiber coordinates is $\omega^i_j = e^i_\mu (d\epsilon^\mu_j - \epsilon^\mu_j d\epsilon^\sigma_\mu)$ and $\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$.

### 2.3 The canonical form $\theta$

In $LM$ we can define a canonical $\mathbb{R}^m$-valued 1-form $\tilde{\theta}$ as follows. If $X \in T_u LM$, then

$$\tilde{\theta}(X) = u^{-1} \left( T_u \tau(X) \right).$$

This allows us to define a similar form in $J^1\tau$, denoted $\theta$, as the pullback $\tau_{10}^* \tilde{\theta}$.

In terms of the canonical basis of $\mathbb{R}^m$, if we write $\theta = \theta^k e_k$, it can be seen that the coordinate expression of the forms $\theta^k$ is given by $\theta^k = e^\mu_\mu d\epsilon^\mu_j$, where $e^\mu_\mu$ is such that $e^k_\mu e^\mu_j = \delta^k_j$ and $e^i_\mu e^\mu_j = \delta^i_j$. 

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The form $\theta$ turns out to be a tensorial 1-form of type $\text{Ad}$ (for details you can check [35]). We can use the local expressions for $\theta^j$ and $\omega^i_j$ in a trivializing open set $U \subset J^1 \tau$ to prove that these forms are linearly independent. 

The exterior covariant derivative of $\theta$ with respect to $\omega$ gives rise to another differential form fulfilling a new universal property, called the universal torsion form $T$, i.e. (see Appendix A.4)

$$ T = d\theta + \omega^i \wedge \theta. $$

The universal property in this case arise as follows: if $\Gamma$ is a principal connection on $LM$, then its related torsion form $T_\Gamma$ is recovered as the pullback

$$ T_\Gamma = \tilde{\sigma}^* \Gamma(T). $$

As we did before, we can express $T$ in terms of the canonical basis of $\mathbb{R}^m$ by writing

$$ T^k = T^k d + \omega^k_i \wedge \theta^i. $$

A local expression for $T$ in a trivializing open set $U \subset J^1 \tau$ can be obtained using those for $\omega$ and $\theta$. In fact

$$ T^k = d \left( e^k_\mu dx^\mu \right) + e^k_\mu d(e^\mu_i - e^\mu_i \theta^i) \wedge e^i dx^\nu \\
= de^k_\mu \wedge dx^\mu + e^i e^k_\mu d e^\mu_i \wedge dx^\nu - e^k_\mu e^\mu_i e^i \theta^\sigma \wedge dx^\sigma \\
= de^k_\mu \wedge dx^\mu - e^i e^k_\mu d e^\mu_i \wedge dx^\nu + \frac{1}{2} e^k_\mu (e^\mu_i e^i_\sigma - e^\mu_i e^i_\sigma) dx^\sigma \wedge dx^\nu \\
= \frac{1}{2} e^k_\mu e^i_\nu e^i_\sigma - e^\mu_i e^i_\sigma e^i_\nu) dx^\sigma \wedge dx^\nu.
$$

This last expression shows that on the set $e^\mu_i e^i_\sigma - e^\mu_i e^i_\sigma = 0$ of each trivializing neighbourhood the torsion form $T$ vanishes identically. It turns out that all of these sets can be smoothly glued together and define a submanifold $T_0 \subset J^1 \tau$, as the next proposition shows

**Proposition 1** There exists a submanifold $\iota_0 : T_0 \hookrightarrow J^1 \tau$ such that $\iota_0^* T \equiv 0$. 

**Proof** As we anticipated, the manifold $T_0$ is given locally by the conditions

$$ e^\mu_i e^i_\sigma - e^\mu_i e^i_\sigma = 0, \quad (2.1) $$

for every $\mu$, $\nu$, $\sigma$. To see that this is independent of the choice of coordinates, consider another trivializing neighbourhood (having nonempty intersection with the first) with fibered coordinates $(\bar{x}^\mu, \bar{e}^\mu_k, \bar{e}^\mu_k_\sigma)$. Change of coordinates between these two sets is given by

$$ \bar{e}^\mu_k = \frac{\partial \bar{x}^\sigma}{\partial x^\theta} e^\theta_k. $$
\[ \bar{e}^\mu_{kv} = \left( \frac{\partial \bar{x}^\mu}{\partial x^\tau} e^\tau_k + \frac{\partial^2 \bar{x}^\mu}{\partial x^\tau \partial x^\rho} e^\tau_k \right) \frac{\partial x^\rho}{\partial \bar{x}^v} \]

so

\[ \bar{e}^k_{\sigma} e_{kv} - e^k_{\sigma} \bar{e}^\mu_{k\sigma} = \frac{\partial x^\theta}{\partial \bar{x}^\sigma} e^\theta_\sigma \left( \frac{\partial \bar{x}^\mu}{\partial x^\tau} e^\tau_k + \frac{\partial^2 \bar{x}^\mu}{\partial x^\tau \partial x^\rho} e^\tau_k \right) \frac{\partial x^\rho}{\partial \bar{x}^v} \]

\[ - \frac{\partial x^\theta}{\partial x^v} \left( \frac{\partial \bar{x}^\mu}{\partial x^\tau} e^\tau_k + \frac{\partial^2 \bar{x}^\mu}{\partial x^\tau \partial x^\rho} e^\tau_k \right) \frac{\partial x^\rho}{\partial \bar{x}^\sigma} \]

\[ = \frac{\partial x^\theta}{\partial \bar{x}^\sigma} \frac{\partial x^\rho}{\partial x^v} \left( e^k_{\sigma} e^\tau_k - e^k_{\rho} e^\tau_k \right), \]

which implies that the vanishing of the expression (2.1) is independent of the particular trivializing set. \(\square\)

**Remark 3** It is possible to prove that the action of \(G\) preserves the manifold \(T_0\), i.e. \(T_0 \cdot A \subset T_0\) for every \(A \in G\). This allows us to define a \(G\)-action on \(T_0\), making \(T_0 \rightarrow T_0/G\) into a principal \(G\)-bundle. Moreover, denoting \(C_0(LM) := T_0/G\) and using the identification \(J^1 \tau \simeq C(LM) \times LM\), we get the identification \(T_0 \simeq C_0(LM) \times LM\). If we pullback the universal connection \(\omega\) through \(t_0\), we get a new universal property concerning torsionless connections, instead of arbitrary connections. We will use this fact to write the Griffiths variational principle for Lovelock gravity.

### 2.3.1 The Sparling forms \(\theta_{i_1...i_p}\)

For each \(p \leq m\), define

\[ \theta_{i_1...i_p} = \frac{1}{(m-p)!} \varepsilon_{i_1...i_p} i_{p+1}...i_m \theta^{i_p+1} \wedge ... \wedge \theta^{i_m}. \]

It is readily seen that \(\theta_{i_1...i_p}\) is completely antisymmetric in its indices. Additionally we have:

**Lemma 1** Define \(\sigma_0 := \theta^1 \wedge ... \wedge \theta^m\). Then

\[ \theta_{i_1...i_p} = X_{i_1} \cdots X_{i_p} \sigma_0, \]

for any vector fields \(X_{i_k} \in \mathfrak{X}(J^1 \tau)\) projecting to \(u(e_{i_k})\), i.e. \(T \tau_1 (X_{i_k} (j^1 \tau u)) = u(e_{i_k})\).

**Proof** Let us proceed by induction on \(p\). First, observe that

\[ X_{i} \theta^j = (u^{-1} \circ u)^j_i = \delta^j_i. \]  \(\tag{2.2}\)

Then

\[ X_{i} \sigma_0 = X_{i} \left( \theta^1 \wedge ... \wedge \theta^m \right) = \frac{1}{m!} \left( X_{i} \left( \theta^{i_1} \wedge ... \wedge \theta^{i_m} \right) \right) \]

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and renaming the indices

\[ X_i \cdot \sigma_0 = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{k+1} \varepsilon_{i_1 \ldots i_k j_1 \ldots j_m} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \wedge \ldots \wedge \theta^{i_m}, \]

which proves the case \( p = 1 \). Inductively

\[ X_{i_{p+1}} \cdot \theta_{i_1 \ldots i_p} = X_{i_{p+1}} \cdot \left( \frac{1}{(m-p)!} \varepsilon_{i_1 \ldots i_p j_{p+1} \ldots j_m} \theta^{j_{p+1}} \wedge \cdots \wedge \theta^{j_m} \right) \]

\[ = \frac{m-p}{(m-p)!} \sum_{k=1}^{m-p} (-1)^{k+1} \varepsilon_{i_1 \ldots i_p j_{p+1} \ldots j_m \hat{\delta}^{j_{p+k}} j_{p+1} \wedge \cdots \wedge \theta^{j_{p+k}} \wedge \ldots \wedge \theta^{j_m}} \]

\[ = \frac{m-p}{(m-p)!} \sum_{k=1}^{m-p} (-1)^{k+1} \varepsilon_{i_1 \ldots i_p j_{p+1} \ldots j_{p+k-1} j_{p+k+1} \ldots j_m \theta^{j_{p+k}} \wedge \ldots \wedge \theta^{j_m}}, \]

again renaming indices

\[ X_{i_{p+1}} \cdot \theta_{i_1 \ldots i_p} = \frac{m-p}{(m-p)!} \sum_{k=1}^{m-p} (-1)^{2k} \varepsilon_{i_1 \ldots i_p j_{p+2} \ldots j_m} \theta^{j_{p+2}} \wedge \cdots \wedge \theta^{j_m} \]

\[ = \frac{m-p}{(m-p)!} \varepsilon_{i_1 \ldots i_p j_{p+2} \ldots j_m} \theta^{j_{p+2}} \wedge \cdots \wedge \theta^{j_m} = \theta_{i_1 \ldots i_{p+1}}. \]

We use the Sparling forms to write down local expressions for the Lovelock Lagrangian and the equations of motion. To facilitate the related computations it is necessary to know some properties of these forms, so we collect some of them in the next proposition. In the proof (and in the rest of the paper) we use the properties of the Levi-Civita symbol and the generalized Kronecker delta listed in Appendix A.1.

**Proposition 2** The following properties hold (the hat on an index indicates that this index has been suppressed):

\[ \hat{\delta}^{i_1 \ldots i_k} = \varepsilon_{i_1 \ldots i_k j_1 \ldots j_m} \theta^{j_1} \wedge \cdots \wedge \theta^{j_m}, \]

\[ \hat{\delta}^{i_1 \ldots i_k j_1 \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_1 \ldots j_m j_{p+1} \ldots j_m} \theta^{j_{p+1}} \wedge \cdots \wedge \theta^{j_m}, \]

\[ \hat{\delta}^{i_1 \ldots i_k j_{p+1} \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_{p+1} \ldots j_m \theta^{j_{p+1}} \wedge \cdots \wedge \theta^{j_m}} \]

\[ \hat{\delta}^{i_1 \ldots i_k j_{p+1} \ldots j_{p+k} \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_{p+1} \ldots j_{p+k} \ldots j_m \theta^{j_{p+k}} \wedge \cdots \wedge \theta^{j_m}} \]

\[ \hat{\delta}^{i_1 \ldots i_k j_{p+1} \ldots j_{p+k} \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_{p+1} \ldots j_{p+k} \ldots j_m \theta^{j_{p+k}} \wedge \cdots \wedge \theta^{j_m}} \]

\[ \hat{\delta}^{i_1 \ldots i_k j_{p+1} \ldots j_{p+k+1} \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_{p+1} \ldots j_{p+k+1} \ldots j_m \theta^{j_{p+k+1}} \wedge \cdots \wedge \theta^{j_m}} \]

\[ \hat{\delta}^{i_1 \ldots i_k j_{p+1} \ldots j_{p+k+1} \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_{p+1} \ldots j_{p+k+1} \ldots j_m \theta^{j_{p+k+1}} \wedge \cdots \wedge \theta^{j_m}} \]

\[ \hat{\delta}^{i_1 \ldots i_k j_{p+1} \ldots j_{p+k+1} \ldots j_m} = \varepsilon_{i_1 \ldots i_k j_{p+1} \ldots j_{p+k+1} \ldots j_m \theta^{j_{p+k+1}} \wedge \cdots \wedge \theta^{j_m}} \]
1. Let us compute for every \( r \leq s, \theta^i_{1} \wedge \cdots \wedge \theta^i_r \wedge \theta^j_{1, \ldots, j_s} = \frac{(-1)^{r(s-r)}}{(s-r)!} \theta^i_{1, \ldots, j_s} \theta^j_{r+1, \ldots, j_s}. \)

2. \( \theta^k \wedge \theta^i_{1, \ldots, i_p} = \sum_{r=1}^{p} (-1)^{p+r} \delta^k_{i_r} \theta^i_{1, \ldots, i_r} \wedge \theta^i_{r, \ldots, i_p}. \)

3. \( d\theta^i_{1, \ldots, i_p} = T^i \wedge \theta^i_{1, \ldots, i_p} + \sum_{r=1}^{p} (-1)^{p+r} o^i_r \wedge \theta^i_{1, \ldots, i_r} \wedge \theta^i_{r, \ldots, i_p} - o^i_r \wedge \theta^i_{1, \ldots, i_p}. \)

**Proof**

1. Let us compute

\[
\theta^i_{1} \wedge \cdots \wedge \theta^i_r \wedge \theta^j_{1, \ldots, j_s} = \frac{1}{(m-s)!} \theta^i_{1} \wedge \cdots \wedge \theta^i_r \theta^j_{1, \ldots, j_{s+1}} \theta^j_{1, \ldots, j_m} \wedge \cdots \wedge \theta^j_{1, \ldots, j_m}
\]

\[
= \frac{\varepsilon_{j_1, j_s, j_{s+1}, \ldots, j_m} \delta^i_{1, \ldots, i_r} \delta^j_{1, \ldots, j_{s+1}} \cdots \delta^j_{1, \ldots, j_m} \theta^a_1 \wedge \cdots \wedge \theta^a_{m-s+r}}{(m-s)!(m-s+r)!}
\]

\[
= \frac{\varepsilon_{i_{r+1, i_s a_1, \ldots, a_{m-s+r}} \theta^a_1 \wedge \cdots \wedge \theta^a_{m-s+r}}}{(m-s)!(s-r)!}
\]

\[
= \frac{1}{(s-r)!} \delta^i_{j_1, j_s} \theta^i_{1, \ldots, i_r} = \frac{(-1)^{r(s-r)}}{(s-r)!} \delta^i_{j_1, j_s} \theta^i_{1, \ldots, i_r}.
\]

2. From the first point of the proposition and taking \( r = 1 \) and \( s = p, \)

\[
\theta^k \wedge \theta^i_{1, \ldots, i_p} = \frac{(-1)^{p-1}}{(p-1)!} \delta^{k_{i_2, \ldots, i_p}}_{i_{1, \ldots, i_p}} \theta^i_{1, \ldots, i_p} = \frac{(-1)^{p-1}}{(p-1)!} \sum_{r=1}^{p} (-1)^{p+r} \delta^k_{i_r} \delta^{i_{2, \ldots, i_p}}_{j_1, \ldots, j_r} \theta^i_{1, \ldots, i_p}
\]

\[
= \sum_{r=1}^{p} \frac{(-1)^{p+r}}{(p-1)!} \delta^k_{i_r} \delta^{i_{2, \ldots, i_p}}_{j_1, \ldots, j_r} \theta^i_{1, \ldots, i_p} = \sum_{r=1}^{p} \frac{(-1)^{p+r}}{(p-1)!} \delta^k_{i_r} \theta^i_{1, \ldots, i_r} \wedge \theta^i_{j_1, \ldots, j_p}
\]

\[
= \sum_{r=1}^{p} \frac{(-1)^{p+r}}{p-1} \delta^k_{i_r} \theta^i_{1, \ldots, i_r} \wedge \theta^i_{j_1, \ldots, j_p}.
\]

We may also prove this by induction on \( p. \) The case \( p = 1 \) follows from Eq. (2.2). Assuming that the formula holds for \( p - 1, \) then

\[
\theta^k \wedge \theta^i_{1, \ldots, i_p} = \theta^k \wedge (X_p \delta \theta^i_{1, \ldots, i_{p-1}})
\]

\[
= (X_p \theta^k) \wedge \theta^i_{1, \ldots, i_{p-1}} - X_{i_p \wedge} (\theta^k \wedge \theta^i_{1, \ldots, i_{p-1}})
\]

\[
= \delta^k_{i_p} \theta^i_{1, \ldots, i_{p-1}} - X_{i_p \wedge} \left( \sum_{r=1}^{p-1} (-1)^{p-1+r} \delta^k_{i_r} \theta^i_{1, \ldots, i_r \wedge \cdots \wedge \ldots \wedge i_{p-1}} \right)
\]

\[
= (-1)^{p+r} \delta^k_{i_p} \theta^i_{1, \ldots, i_{p-1}} + \sum_{r=1}^{p-1} \frac{(-1)^{p+r}}{p-1} \delta^k_{i_r} \theta^i_{1, \ldots, i_r \wedge \cdots \wedge \ldots \wedge \ldots \wedge i_{p-1}}
\]

\[
= \sum_{r=1}^{p} \frac{(-1)^{p+r}}{p-1} \delta^k_{i_r} \theta^i_{1, \ldots, i_r \wedge \cdots \wedge \ldots \wedge \ldots \wedge i_{p-1}}.
\]
3. Let us compute now the differential of $\theta^i_1 \ldots i_p$

$$d\theta^i_1 \ldots i_p = \frac{1}{(m-p)!} \varepsilon^i_1 \ldots i_{p+1} \ldots i_m d\left(\theta^{i+p+1} \wedge \ldots \wedge \theta^{i+m}\right)$$

$$= \frac{1}{(m-p)!} \varepsilon^i_1 \ldots i_{p+1} \ldots i_m \sum_{k=1}^{m-p} (-1)^{k+1} \theta^{i+p+1} \wedge \ldots \wedge d\theta^{i+p+k} \wedge \ldots \wedge \theta^{i+m}$$

$$= \sum_{k=1}^{m-p} \frac{(-1)^{2k}}{(m-p)!} \varepsilon^i_1 \ldots i_{p+k+1} \ldots i_m d\theta^{i+p+k} \wedge \theta^{i+p+1} \wedge \ldots \wedge \theta^{i+p+k} \wedge \ldots \wedge \theta^{i+m}.$$ 

Renaming the indices and using the first part of the proposition

$$d\theta^i_1 \ldots i_p = \left(T^l - \omega^l_k \wedge \theta^k\right) \wedge \theta^i_1 \ldots i_p$$

$$= T^l \wedge \theta^i_1 \ldots i_p - \omega^l_k \wedge \theta^i_1 \ldots i_p$$

$$= T^l \wedge \theta^i_1 \ldots i_p + \sum_{r=1}^p (-1)^{p+r} \omega^l_r \wedge \theta^i_1 \ldots i_r \wedge \omega^l_r \wedge \theta^i_1 \ldots i_r.$$ 

\[\square\]

3 Variational problem for Lovelock gravity

Griffiths variational problems [30] are posed on triples $(\Lambda \xrightarrow{\pi} M, \lambda, I)$, where $\Lambda \xrightarrow{\pi} M$ is a fiber bundle over the space-time $M$, $\lambda$ is an $m$-form that is $\pi$-horizontal (referred to as the Lagrangian form) and an exterior differential system $I \subset \Omega^*(\Lambda)$ [2] characterizing the admissible sections of the problem.

**Definition 1** The variational problem associated with a variational triple $(\Lambda \xrightarrow{\pi} M, \lambda, I)$ consists in finding the sections $\sigma : M \to \Lambda$ which are integrals for $I$ and are extremals for the functional

$$S[\sigma] := \int_M \sigma^* \lambda.$$ 

Remember that $\sigma$ is integral for $I$ if and only if $\sigma^* \alpha = 0$ for every $\alpha \in I$. In particular, this implies that the variations of $S$ must be performed in such a way that the transformed sections remain integrals of $I$. Hence, we define

**Definition 2** Let $I \subset \Omega^*(\Lambda)$ be an exterior differential system. A local vector field $X \in \mathfrak{X}(\Lambda)$ is an infinitesimal symmetry of $I$ if and only if

$$\mathcal{L}_X I \subset I.$$ 

The set infinitesimal symmetries of $I$ is denoted $\text{Symm}(I)$. 

\[\heartsuit\ Springer\]
With this definition, it can be proved that the solutions to the variational problem associated with the variational triple \((\Lambda \xrightarrow{\pi} M, \lambda, \mathcal{I})\) are those sections \(\sigma\) integral for \(\mathcal{I}\) such that
\[
\sigma^*(X \mu d\lambda) = 0 \quad \text{for every } X \in \mathfrak{X}^{\pi}(\Lambda) \cap \text{Symm}(\mathcal{I})
\]
which are, in turn, the field equations for this problem; here \(\mathfrak{X}^{\pi}(\Lambda)\) indicates the set of vector fields on \(\Lambda\) which are vertical with respect to the projection \(\pi\).

**Remark 4** It could be possible for an exterior differential system to have no infinitesimal symmetries; nevertheless, it will be proved in Sect. 4.1 that the exterior differential system we will use in the variational problem for Lovelock gravity (see Eq. (5.2) below) possesses non trivial infinitesimal symmetries.

**Remark 5** Here we are assuming that \(M\) is a manifold without boundary. Also, in order for \(S\) to be well-defined, the form \(\sigma^* \lambda\) must be compactly supported. In the sequel, we will assume that all the integrals we work with exist.

### 3.1 The Lovelock Lagrangian

Now we are ready to define a Griffiths variational problem for Lovelock gravity [40]. To do that we have to define the corresponding triple introduced in the previous section. The bundle chosen is \(\tau_1 : T_0 \to M\), where \(M\) is the \(m\)-dimensional smooth manifold representing space-time. Here we are writing \(\tau_1\) instead of \(\tau_1|_{T_0}\) only to simplify the notation (we will do the same with the pullbacks of \(\omega\) and \(\theta\) through \(\iota_0\)).

As the exterior differential system restricting the admissible sections we take (see Appendix A.3)
\[
\mathcal{I}_L := \langle \omega_p \rangle_{\text{diff}}.
\]
The subscript \(\text{diff}\) indicates the smallest exterior differential system containing the form \(\omega_p\).

Using the canonical basis of \(\mathbb{R}^m\), we can alternatively describe \(\mathcal{I}_L\) as the exterior differential system generated as follows
\[
\mathcal{I}_L = \left\{ \eta^{ik} \omega_k^i + \eta^{jk} \omega_k^j \right\}_{\text{diff}}. \tag{3.1}
\]
It is also useful to define the forms \(\omega^{ij} := \eta^{ik} \omega_k^j\), in terms of which \(\mathcal{I}_L = \langle \omega^{ij} + \omega^{ji} \rangle_{\text{diff}}\). Then we look for sections \(\sigma \in \Gamma(\tau_1)\) fulfilling the condition
\[
\sigma^* \omega_p = 0.
\]
It is customary to refer to this condition as the metricity condition.
Remark 6 Using the identification $T_0 \simeq C_0(LM) \times_M LM$, every (local) section $\sigma \in \Gamma(\tau_1)$ over $U \subset M$ that is integral for $\mathcal{L}$ can be thought of as a couple of sections $\sigma_1 := q \circ \sigma$ and $\sigma_2 := \tau_{10} \circ \sigma$. As we saw in the previous section, if $\Gamma$ is the principal connection induced by $\sigma_1$ on $\tau$, then $\omega_{\Gamma} = \tilde{\sigma}_1^* \omega$. Hence, the metricity condition implies that $\omega_{\Gamma}$ is a torsionless (pseudo) metric connection.

Following the constructions of Appendix A.4 about vector-valued differential forms, for every $k \leq n$, we can define a $k$-form with values in $\Lambda^k \mathbb{R}^m$ given by

$$\theta^{(k)} := \theta \wedge \cdots \wedge \theta,$$

Hence

$$\theta^{(k)}(X_1, \ldots, X_k) := \theta(X_1) \wedge \cdots \wedge \theta(X_k).$$

Using the canonical basis of $\mathbb{R}^m$ we can write

$$\theta^{(k)} = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Now we can take the Hodge star operator in the second factor (see Appendix A.2), namely

$$\star(\theta^{(k)}) = \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \otimes \frac{1}{(m-k)!} \eta^{i_1 j_1} \cdots \eta^{i_k j_k} e^{j_1 \cdots j_{m-k} e_{j_k+1} \wedge \cdots \wedge e_{j_m}}$$

$$= \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \otimes \frac{1}{(m-k)!} \eta^{i_1 j_1} \cdots \eta^{i_k j_k} e^{j_1 \cdots j_{m-k} \delta^{j_{k+1}}_{j_1} \cdots \delta^{j_m}_{j_k} e_{i_{k+1}} \wedge \cdots \wedge e_{i_m}}$$

$$= \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \otimes \frac{1}{(m-k)!} \eta^{i_1 j_1} \cdots \eta^{i_k j_k} \eta^{r_{k+1} j_{k+1}} \cdots \eta^{r_m j_m} e^{j_1 \cdots j_m} \epsilon_{j_1 \cdots j_{k+1} \cdots j_m}$$

$$\eta^{r_{k+1} j_{k+1}} \cdots \eta^{r_m j_m} e_{i_{k+1}} \wedge \cdots \wedge e_{i_m},$$

where we have used the properties of the Levi-Civita symbol (see Appendix A.1). Now, renaming indices and using the definition of the forms $\theta_{i_1 \cdots i_p}$, we find

$$\star(\theta^{(k)}) = \det(\eta) \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \theta_{i_1 \cdots i_p} \otimes e_{j_1} \wedge \cdots \wedge e_{j_p},$$

(3.2)

with $p = m - k$. Notice that $\theta^{(k)}$ is a $k$-form with values in $\Lambda^k \mathbb{R}^m$, while $\star(\theta^{(k)})$ is a $k$-form with values in $\Lambda^{m-k} \mathbb{R}^m$. Now let $r < \left\lfloor \frac{m}{2} \right\rfloor$ be an integer, where $\lfloor \cdot \rfloor$ denotes the integral part.

Definition 3 Let $V$ be an $m$-dimensional real vector space. We define

$$A_r : \Lambda^{2r} V \to (\Lambda^r V) \otimes (\Lambda^r V)^* \simeq (\text{End}(\Lambda^r V))^*$$
as the unique linear map whose action on elementary $2r$-vectors is given by
\[
\left(v_{j_1} \wedge \cdots \wedge v_{j_r}\right) \wedge \left(v_{j_{r+1}} \wedge \cdots \wedge v_{j_{2r}}\right) \mapsto \frac{1}{(2r)!} \sum_{\sigma \in S_{2r}} \text{sgn}(\sigma) \left(v_{j_{\sigma(1)}} \wedge \cdots \wedge v_{j_{\sigma(r)}}\right) \otimes \hat{\eta}^{\flat} \left(v_{j_{\sigma(r+1)}} \wedge \cdots \wedge v_{j_{\sigma(2r)}}\right),
\]
where $\hat{\eta}$ is the extension of $\eta$ to $\Lambda^r V$ defined on Appendix A.2. It is readily seen that it is in fact well-defined and linear.

Using the linear map $A_r$, we can construct an $(m - 2r)$-form with values in $(\text{End}(\Lambda^r \mathbb{R}^m))^*$ as
\[
\Xi_r = A_r \left(\star \left(\omega^{m-2r}\right)\right).
\]
We can think of $\Xi_r$ as taking values in $(\Lambda^r \mathfrak{g})^*$ rather than $(\text{End}(\Lambda^r \mathbb{R}^m))^*$ because the latter can be viewed as a subspace of the former. That is
\[
(\Lambda^r \mathfrak{g})^* \supset (\text{End}(\Lambda^r \mathbb{R}^m))^*,
\]
which is a consequence of the inclusion $\Lambda^r \mathfrak{g} \subset \text{End}(\Lambda^r \mathbb{R}^m)$ given by the monomorphism $\Gamma : \Lambda^r (\text{End}(\mathbb{R}^m)) \rightarrow \text{End}(\Lambda^r \mathbb{R}^m)$ defined as
\[
A_1 \wedge \cdots \wedge A_r \mapsto \left[ v_1 \wedge \cdots \wedge v_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} A_1 \left(v_{\sigma(1)}\right) \wedge \cdots \wedge A_r \left(v_{\sigma(r)}\right) \right].
\]

We can use these considerations to introduce the Lovelock Lagrangian:

**Definition 4** The Lovelock Lagrangian is the $\tau_1$-horizontal $m$-form
\[
\lambda_L := \sum_{r < [m/2]} a_r \left(\Xi_r \wedge \Omega^r\right),
\]
where $a_r$ are constants and $\Omega^r = \Omega \wedge \cdots \wedge \Omega$.

**Remark 7** (Regularity of the Lovelock Lagrangian) From the viewpoint of classical second order field theory on the bundle of metrics, the Lovelock Lagrangian is a singular Lagrangian: It follows from the fact that the equations of motion are given by the Einstein tensor, which are of second order, although for a regular Lagrangian in a second order theory, they should have been of fourth order.

Nevertheless, it should be stressed that the variational problem posed by $\lambda_L$ and the constraints $T_L$ is not a classical one; in this regard, the extremals of such problem are not necessarily holonomic as sections of the jet bundle $J^1 \tau$. In particular, the notion of regularity of a Lagrangian has not a clear generalization to this case; in fact, it would depend on which feature of this concept we want to highlight:
1. For example, the regularity of a Lagrangian can be seen as a sufficient condition for the existence of solutions for the equations of motion (as it allows us to apply Cauchy–Kovalevskaya theorem). From this viewpoint, the fact that the exterior differential system (5.2) representing these equations of motion admits solutions becomes a necessary condition for the regularity of the variational problem determined by the data \((\lambda_L, I_L)\).

2. For classical variational problems, it could be noted that the regularity of a Lagrangian is also tied to the fact that the associated Legendre transformation is a diffeomorphism. It suggests another way to generalize regularity for a variational problem of the type discussed here. The idea is that, from the unified formalism perspective, Legendre transformation becomes part of the equations of motion, and it can be obtained as a consequence of the direct sum structure of the multimomentum bundle (that in our case is determined by Lemma 5 below). Using the equations of motion (5.2), it results that the map generalizing Legendre transform in this sense is identically zero.

In any case, this generalization would require further research, which is expected to be carried out elsewhere.

### 3.1.1 Expressions in terms of the canonical basis of \(\mathbb{R}^m\)

If we denote by \(\{e^1, \ldots, e^m\}\) the dual basis of the canonical basis in \(\mathbb{R}^m\), we can write

\[
A_r(e_j^1 \wedge \cdots \wedge e_j^{2r}) = \frac{1}{(2r)!} \sum_{\sigma \in S_{2r}} \text{sgn}(\sigma) \eta_{j_{\sigma(1)}}^1 \cdots \eta_{j_{\sigma(r+1)}}^{2r} e_{j_{\sigma(r+1)}} \wedge \cdots \wedge e_{j_{\sigma(2r)}} \otimes e^1 \wedge \cdots \wedge e^r.
\]

Also, using (3.2),

\[
\Xi_r = A_r \left( \star \left( \theta^{(m-2r)} \right) \right)
= \det(\eta) \eta_{j_{\sigma(1)}}^1 \cdots \eta_{j_{\sigma(2r)}}^{2r} \theta_i^{1 \cdots i_{2r}} \otimes A_r \left( e_{j_1} \wedge \cdots \wedge e_{j_{2r}} \right)
= \frac{\det(\eta)}{(2r)!} \sum_{\sigma \in S_{2r}} \text{sgn}(\sigma) \eta_{j_{\sigma(1)}}^1 \cdots \eta_{j_{\sigma(r+1)}}^{2r} \eta_{j_{\sigma(r+1)}}^1 \cdots \eta_{j_{\sigma(r+2)}}^{2r} \theta_{i_1 \cdots i_{2r}} \otimes e_{j_{\sigma(1)}} \wedge \cdots
\]

\[
\cdots \wedge e_{j_{\sigma(2r)}} \otimes e^1 \wedge \cdots \wedge e^r,
\]

but since \(\theta_{i_1 \cdots i_{2r}} = \text{sgn}(\sigma) \theta_{i_{\sigma(1)}} \cdots i_{\sigma(2r)}\),

\[
\Xi_r = \frac{\det(\eta)}{(2r)!} \sum_{\sigma \in S_{2r}} \eta_{j_{\sigma(1)}}^1 \cdots \eta_{j_{\sigma(2r)}}^{2r} \eta_{j_{\sigma(1)}}^1 \cdots \eta_{j_{\sigma(r+2)}}^{2r} \theta_{i_1 \cdots i_{2r}} \otimes e_{j_{\sigma(1)}} \wedge \cdots
\]

\[
\cdots \wedge e_{j_{\sigma(2r)}} \otimes e^1 \wedge \cdots \wedge e^r,
\]
and, as all are dummy indices, we finally get

$$\Xi_r = \det(\eta) \eta^{i_1 j_1 + \ldots + i_r j_r} \theta_{i_1 \ldots i_r} \otimes e_{j_1 \ldots j_r} \wedge \ldots \wedge e_{i_1 \ldots i_r}.$$  

Furthermore, if $$\Omega^{ab} = \Omega^{ab}_q \otimes e^b \otimes e_a,$$ we have

$$\Omega' = \Omega^{a_1 b_1} \wedge \ldots \wedge \Omega^{a_r b_r} \otimes (e^{b_1} \otimes e_{a_1}) \wedge \ldots \wedge (e^{b_r} \otimes e_{a_r}) \in \Omega^{2r} (J^1 \tau) \otimes \Lambda^r g,$$

so that, in view of the inclusion (3.3), we obtain

$$\lambda_L = \sum_{r < [m/2]} a_r \theta_{i_1 \ldots i_r l_1 \ldots l_r} \wedge \Omega^{i_1 l_1} \wedge \ldots \wedge \Omega^{i_r l_r},$$

where $$\Omega^{ab} = \eta^{b q} \Omega^a_q$$ and all the possible multiplicative constants have been absorbed in the constants $$a_r.$$ From now on, we will work with each homogeneous component

$$\lambda^{(r)} = \theta_{i_1 \ldots i_r l_1 \ldots l_r} \wedge \Omega^{i_1 l_1} \wedge \ldots \wedge \Omega^{i_r l_r},$$

which will be indicated with the generic symbol $$\lambda_L.$$

**Remark 8** To simplify the computations, it will be convenient to introduce the following multi-index notation. We use capital letters $$I, J$$ to denote multi-indices $$I = (i_1, \ldots, i_p), J = (j_1, \ldots, j_p).$$ An apostrophe denotes a multi-index formed by removing the first index of a given multi-index, i.e. $$I' = (i_2, \ldots, i_p)$$ if $$I = (i_1, \ldots, i_p).$$ In this case, we use concatenation of indices and multi-indices and write $$I = i_1 I'.$$ Also, we will write $$\Omega^{IJ} = \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_r j_r}$$ and $$\theta_{i_1 \ldots i_r l_1 \ldots l_r} = \theta_{I J}.$$ Thus, the Lovelock Lagrangian can be written

$$\lambda^{(r)} = \theta_{I J} \wedge \Omega^{IJ}.$$  

### 3.1.2 Relation with the metric-affine Lagrangian

To relate $$\lambda_L$$ with the metric-affine formalism, remember that every principal connection $$\Gamma$$ gives rise to a linear connection in $$TM$$ (as an associated vector bundle with fiber $$\mathbb{R}^m$$ and canonical action of $$G$$). Let $$\omega_G$$ be the corresponding connection form (obtained as the pullback of the universal connection $$\omega$$ by a suitable section) and $$\Omega_G$$ its related curvature.

Then, if $$\Omega^{ab}_G = \Omega^{ab}_{\mu \nu} dx^\mu \wedge dx^\nu,$$ we have

$$\Omega^{ab}_{\mu \nu} = R^{a b}_{\mu \nu} e^a_\sigma e^b_\tau,$$  

(3.4)
where \( R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\nu} \) are the components of the curvature tensor with respect to the linear connection induced by \( \Gamma \), i.e.

\[
R \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right) = R_{\rho\mu\nu} \frac{\partial}{\partial x^\rho},
\]

and \( g_{\mu\nu} := e^a_\mu \eta_{ab} e^b_\nu \) is the corresponding metric.

Thus, we can compute locally the pullback of \( \lambda_L \) by a section as follows

\[
\lambda_L = \theta_1 ... \theta_s \wedge \Omega^{i_1 l_1} \wedge \ldots \wedge \Omega^{i_l l_l}
\]

\[
= \varepsilon_{i_1 ... i_l s_1 ... s_k} \theta^{s_1} \wedge \ldots \wedge \theta^{s_k} \wedge \Omega^{i_1 l_1} \wedge \ldots \wedge \Omega^{i_l l_l}
\]

\[
= \varepsilon_{i_1 ... i_l s_1 ... s_k} e^{s_1}_{\rho_1} \ldots e^{s_k}_{\rho_k} \Omega^{i_1 l_1}_{\mu_1 \nu_1} \ldots \Omega^{i_l l_l}_{\mu_l \nu_l} d\mu^1 \wedge \ldots \wedge d\mu^l \wedge d\nu^1 \wedge \ldots \wedge d\nu^l
\]

\[
= \varepsilon_{i_1 ... i_l s_1 ... s_k} e^{s_1}_{\rho_1} \ldots e^{s_k}_{\rho_k} \left( \Omega^{i_1 l_1}_{\mu_1 \nu_1} \ldots \Omega^{i_l l_l}_{\mu_l \nu_l} \right)_{\mu_1 \nu_1} \ldots \Omega^{\nu_l}_{\mu_l \nu_l} d\mu^1 \wedge \ldots \wedge d\mu^l \wedge d\nu^1 \wedge \ldots \wedge d\nu^l
\]

\[
= \varepsilon_{i_1 ... i_l s_1 ... s_k} \left( e^{i_1}_{e_{i_1}} \ldots e^{i_l}_{e_{i_l}} \right) \left( \Omega^{i_1 l_1}_{\mu_1 \nu_1} \ldots \Omega^{i_l l_l}_{\mu_l \nu_l} \right)_{\mu_1 \nu_1} \ldots \Omega^{\nu_l}_{\mu_l \nu_l} d\mu^1 \wedge \ldots \wedge d\mu^l \wedge d\nu^1 \wedge \ldots \wedge d\nu^l
\]

where we have used Eq. (3.4) and the identity

\[
\varepsilon_{i_1 ... i_l s_1 ... s_k} e^{i_1}_{a_1} \ldots e^{i_l}_{a_l} e^{l_1}_{b_1} \ldots e^{l_l}_{b_l} e^{s_1}_{\rho_1} \ldots e^{s_k}_{\rho_k} = \varepsilon_{a_1 ... a_l b_1 ... b_l \rho_1 ... \rho_k} \det(e).
\]

Then, as \( \det(e) = \sqrt{-g} \), we recover the usual Lovelock Lagrangian, i.e.

\[
\lambda_L = \alpha \sqrt{-g} \varepsilon_{a_1 ... a_l b_1 ... b_l \rho_1 ... \rho_k} R^{a_1 b_1}_{a_1 b_1} \ldots R^{a_l b_l}_{a_l b_l} d\mu^1 \wedge \ldots \wedge d\mu^l \wedge d\nu^1 \wedge \ldots \wedge d\nu^l.
\]

### 4 Field equations

As we have said in the previous section, to compute the field equations associated with the Lovelock problem we need to characterize the infinitesimal symmetries of the exterior differential system \( \mathcal{I}_L \). We devote the following section to this task.

#### 4.1 Infinitesimal symmetries of \( \mathcal{I}_L \)

To give a characterization of the infinitesimal symmetries of \( \mathcal{I}_L \), it is useful to introduce a convenient basis of vector fields for the vertical bundle \( V \tau_1 \). Using the identification
$J^1\tau \simeq C(LM) \times_M LM$ we can construct this basis using $q$-vertical and $\tau_{10}$-vertical vector fields.

First, consider the infinitesimal generators associated with the action of $G$ on $J^1\tau$. If $E_j^i$ denote a vector of the canonical basis of $\mathfrak{g}$, we denote these vector fields by $(E_j^i)_{J^1\tau}$. It can be seen that $T_{\tau_{10}}(E_j^i)_{J^1\tau} = (E_j^i)_{LM}$ and hence they are $\tau_1$-vertical vector fields. From the principal bundle structure of $q : J^1\tau \to C(LM)$ it is immediate to see that they are also $q$-vertical.

Furthermore, as $J^1\tau \to LM$ is an affine bundle, we can construct vertical lifts of $\tau$-vertical vector fields. Given a differential form $\alpha \in \Omega^1(M)$ and a $\tau$-vertical vector field $X$, the vertical lift $(\alpha, X)^V$ is defined as the vector field whose flow is given by

$$\theta(t, j^1_x s) = j^1_x s + t\alpha_x \otimes X(s(x)),$$

where the sign “+” must be understood as the affine action of $\tau^*\pi_M \otimes_L M V \tau$ on $J^1\tau$.

In other words,

$$(\alpha, X)^V (j^1_x s) = \frac{d}{dt} \bigg|_{t=0} \left( j^1_x s + t\alpha_x \otimes X(s(x)) \right),$$

We can adapt this definition replacing the differential forms $\alpha$ by differential forms along $J^1\tau$, i.e. $\alpha \in \Gamma(\tau^*\pi_M)$,

$$(\alpha, X)^V (j^1_x s) = \frac{d}{dt} \bigg|_{t=0} \left( j^1_x s + t \alpha|_{j^1_x s} \otimes X(s(x)) \right),$$

In particular, we can use the forms $\theta^r$ and the infinitesimal generators $(E^s_i)_{LM}$, which we denote $(\theta^r, (E^s_i)_{LM})^V$. It is clear that these vector fields are $\tau_{10}$-vertical, and in consequence they are also $\tau_1$-vertical.

It can be proved that the set of vector fields $\{(E^s_i)_{J^1\tau}, (\theta^r, (E^s_i)_{LM})^V\}$ form a basis of the vertical bundle $V\tau_1$ (see [6]).

**Remark 9** It is useful to write down local expressions for the vector fields introduced above. In a trivializing open set, it can be seen that

$$\left( E^k_l \right)_{J^1\tau} (x^\mu, e^\mu_i, e^{\mu}_{i\sigma}) = e^\mu_i \frac{\partial}{\partial e^\mu_k} + e^{\mu}_{i\sigma} \frac{\partial}{\partial e^\mu_{k\sigma}},$$

and

$$\left( \theta^r, (E^s_i)_{LM} \right)^V (x^\mu, e^\mu_i, e^{\mu}_{i\sigma}) = e^r \epsilon^\mu_i \epsilon^{\mu}_{i\sigma} e^\nu \frac{\partial}{\partial e^\nu_{i\sigma}} .$$

---

\footnote{Here $\pi_M : T^*M \to M$ is the cotangent projection.}
Using these expressions we can check that the vector fields \( \{(E^x_i)_{J^1\tau}, (\theta^r, (E^x_i)_{LM})^V\} \), with
\[
(\theta^r, (E^x_i)_{LM})^V = (\theta^r, (E^x_i)_{LM})^V + (\theta^s, (E^r_i)_{LM})^V,
\]
form a basis of the bundle \( V(\tau_1|_{T_0}) \).

**Remark 10** If we fix a principal connection (that may be chosen torsionless) on \( \tau \), it is possible to complete this basis to a full basis of \( T J^1\tau \) by considering the prolongations of the standard horizontal vector fields on \( LM \) (see [35]).

**Proposition 3** The following contractions hold
\[
(\theta^r, (E^x_i)_{LM})^V \overset{\omega}{\wedge} \Omega^k = \delta^k_i \delta^x_j \theta^r,
(\theta^r, (E^x_i)_{LM})^V \overset{\omega}{\wedge} \omega^k_i = 0,
(\theta^r, (E^x_i)_{LM})^V \overset{\omega}{\wedge} \theta^k = 0.
\]

Before moving on, remember that given a vector field \( U \in \mathfrak{X}(LM) \) the first prolongation of \( U \) is the unique vector field \( j^1U \in \mathfrak{X}(J^1\tau) \) that is projectable to \( U \) and is an infinitesimal symmetry of the contact exterior differential system. The next lemma shows that prolongations of \( G \)-invariant vector fields are infinitesimal symmetries of the universal connection [8]:

**Lemma 2** Let \( U \in \mathfrak{X}^V \tau (LM) \) be a vertical \( G \)-invariant vector field on \( LM \). Then
\[
\mathcal{L}_{j^1U} \omega = 0.
\]

**Proof** We know that \( U \) is \( G \)-invariant if and only if its flow \( \Psi^U_1 : LM \to LM \) is an automorphism of \( LM \). Furthermore, for every automorphism \( F : LM \to LM \), we have that
\[
\left( j^1F \right)^* \omega = \omega,
\]
and the lemma follows from this fact. \( \square \)

**Remark 11** Given any element \( u \in LM \), there exists a neighborhood \( V \) containing \( u \) and a set of \( G \)-invariant vector fields \( \{U^i\} \) generating \( \mathfrak{X}^V \tau (V) \) as a \( C^\infty(V) \)-module.

Now we show how to construct infinitesimal symmetries starting from \( G \)-invariant vertical vector fields

**Lemma 3** Let \( \{f^j_i\} \) be a family of arbitrary functions on \( \tau (V) \) and let \( \{U^j_i\} \) be a basis of \( G \)-invariant local vector fields generating \( \mathfrak{X}^V \tau (V) \). Then there exists a (non unique) family of functions \( \{F^k_{ij}\} \) on \( \tau_1^{-1} (V) \subset T_0 \) such that
\[
Z := f^j_i j^1U^j_i + F^k_{ij} (\theta^k, (E^x_i)_{LM})^V.
\]
is an infinitesimal symmetry of $I_L$ tangent to $T_0$.

**Proof** Let $\{ U^i_j \}$ be the basis of $G$-invariant local vector fields generating $X^{V^\tau}(V)$. Since the set of infinitesimal generators $\{ (E^i_j)_LM \}$ form another basis, there exist smooth functions $M^{il}_{jk}, N^{il}_{jk} \in C^\infty(V)$ such that

$$U^i_j = M^{il}_{jk}(E^k_i)_LM \text{ and } M^{ip}_{jq}N^{ql}_{pk} = \delta^i_p \delta^j_q.$$ 

Now, from the formula

$$j^1(fW) = fj^1W + (Df, W)^V, \quad f \in C^\infty(LM), \quad W \in X^{V^\tau}(LM),$$ 

we obtain

$$j^1U^i_j = M^{il}_{jk}(E^k_i)_j V + \left( D_r M^{il}_{jk} \right)(\theta^r, (E^k_i)_LM)_V,$$ 

(4.1)

where $D_r M^{il}_{jk} = \frac{\partial M^{il}_{jk}}{\partial x^\mu}e^\mu_r$. In consequence, in order for $Z$ to be tangent to $T_0$, we must take

$$F^i_{kj} = -f^i_s D_k M^{sl}_{sj} + G^i_{kj}$$ 

(4.2)

with the functions $G^i_{kj}$ fulfilling $G^i_{kj} = G^i_{jk}$.

To compute the Lie derivative, let us write

$$\mathcal{L}_Z\omega^p_{\mu} = \mathcal{L}_{j^1U^i_j}(\omega^p_{\mu}) + \mathcal{L}_{F^i_{kj}(\theta^k, (E^k_i)_LM)}\omega^p_{\mu},$$ 

and compute separately.

First,

$$\mathcal{L}_{f^i_j}(\omega^p_{\mu}) = f^i_j \mathcal{L}_j(\omega^p_{\mu}) + \left( j^1U^i_j(\omega^p_{\mu}) \right)d f^i_j = \mu^{ip}_{jq} D_k f^i_j \theta^k,$$

where $\mu^{ip}_{jq} := j^1U^i_j(\omega^p_{\mu})$ and $D_k f^i_j = \frac{\partial f^i_j}{\partial x^\mu}e^\mu_k$.

On the other hand

$$\mathcal{L}_{F^i_{kj}(\theta^k, (E^k_i)_LM)}\omega^p_{\mu} = F^i_{kj}(\theta^k, (E^k_i)_LM)_V \omega^p_{\mu} = F^i_{kj}(\theta^k, (E^k_i)_LM)_V \omega^p_{\mu},$$

where $(\Omega^p)_q = \frac{1}{2} (\Omega^p_q + \eta_{qa}\Omega^a_p \eta^{bp})$. Using Proposition 3

$$\left( \theta^k, (E^k_i)_LM \right)_V \left( \Omega^p_q + \eta_{qa}\Omega^a_p \eta^{bp} \right) = \left( \delta^p_i \delta^j_q + \eta_{qa}\delta^a_{q} \delta^j_b \eta^{bp} \right) \theta^k = \left( \delta^p_i \delta^j_q + \eta_{qi} \eta^{ip} \right) \theta^k.$$

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from which we deduce [recall (4.2)]

\[ F^i_{kj} \left( \theta^k, \left( E^j_{LM} \right)_V \right) \wedge \Omega^p_{pq} = - \frac{1}{2} f_i^s \left( D_k M^{ip}_{sj} (\eta^{jp} \delta^p_i + \delta^q_i \eta^{jp}) \right) \theta^k + \frac{1}{2} G^i_{kj} \left( \eta^{jq} \delta^p_i + \delta^q_i \eta^{jp} \right) \theta^k \]

\[ = - \frac{1}{2} f_i^s \left( \eta^{jq} D_k M^{ip}_{sj} + D_k M^{iq}_{sj} \eta^{jp} \right) \theta^k + \frac{1}{2} \left( \eta^{jq} G^p_{kj} + G^q_{kj} \eta^{jp} \right) \theta^k. \]

Thus, it is sufficient to take functions \( G^i_{kj} \) fulfilling the equation

\[ \eta^{jq} G^p_{kj} + G^q_{kj} \eta^{jp} = f_i^s \left( \eta^{jq} D_k M^{ip}_{sj} + D_k M^{iq}_{sj} \eta^{jp} \right) - 2 \mu^i_{pq} D_k f_i^j. \tag{4.3} \]

This assures us that \( L_Z (\omega_p)^{pq} = 0. \)

In order to look for a solution to (4.3), consider the decomposition of the set of \( (0,3) \)-tensors of a vector space \( V \) (introduced in [38, Lemma 4.3]), i.e.

\[ T^0_3(V) = \Lambda^3 V \oplus S^3 V \oplus (S_{12} V \cap \ker(\text{Sym})) \oplus (S_{23} V \cap \ker(\text{Sym})), \]

where \( A \in S_{12} V \) if, and only if, \( A(u, v, w) = A(v, u, w) \), and \( B \in S_{23} V \) if, and only if, \( B(u, v, w) = B(u, w, v) \), for every \( u, v, w \in V \) (here \( \text{Sym} \) denotes the symmetrization projector). Such decomposition is given by \( A = \Omega_A + S_A + R_A + T_A \), where \( \Omega_A = \text{Alt}(A) \in \Lambda^3 V \), \( S_A = \text{Sym}(A) \in S^3 V \) and

\[ R_A(u, v, w) = \frac{1}{3} (A(u, v, w) + A(v, u, w) - A(v, w, u) - A(u, w, v)) \]
\[ \in S_{12} V \cap \ker(\text{Sym}), \]

\[ T_A(u, v, w) = \frac{1}{3} (A(u, v, w) + A(u, w, v) - A(v, u, w) - A(w, u, v)) \]
\[ \in S_{23} V \cap \ker(\text{Sym}), \]

or using a basis for \( V \)

\[ (R_A)_{ijk} = \frac{1}{3} (A_{ijk} + A_{jik} - A_{ikj} - A_{kij}), \]

\[ (T_A)_{ijk} = \frac{1}{3} (A_{ijk} + A_{ikj} - A_{jik} - A_{kij}). \]

Now, we want to solve the equation

\[ A_{ijk} = B_{ijk}, \tag{4.4} \]

with \( A_{ijk} = G^q_{ij} \eta_{kq} + G^q_{ik} \eta_{jq} \) and

\[ B_{ijk} = f_i^s \left( \eta_{kp} D_i M^{ip}_{sj} + D_i M^{iq}_{sj} \eta_{jq} \right) - 2 \eta_{kp} \mu^p_{rq} \eta_{qj} D_i f_i^r, \]
for symmetric tensors $G^k_{ij} = G^k_{ji}$. We will use the above mentioned decomposition. It is readily seen that $R_A = R_B = \Omega_A = \Omega_B = 0$. Furthermore

\[(T_A)_{ijk} = \frac{1}{3} \left( G^q_{ij} \eta_{kq} + G^q_{ik} \eta_{jq} + G^q_{iq} \eta_{jk} - G^q_{ji} \eta_{kq} - G^q_{jk} \eta_{iq} - G^q_{kj} \eta_{iq} \right) = \frac{1}{3} \left( A_{ijk} - 2G^q_{jk} \eta_{iq} \right) \]

and

\[(S_A)_{ijk} = \frac{1}{3} \left( G^q_{ij} \eta_{kq} + G^q_{ik} \eta_{jq} + G^q_{iq} \eta_{jk} + G^q_{kj} \eta_{iq} + G^q_{ji} \eta_{kq} \right) = \frac{2}{3} \left( A_{ijk} + G^q_{jk} \eta_{iq} \right), \]

and for $B$ we get

\[(T_B)_{ijk} = \frac{1}{3} \left( B_{ijk} + B_{ikj} - B_{jik} - B_{kij} \right) = \frac{1}{3} \left( 2B_{ijk} - B_{jik} - B_{kij} \right) \]

and

\[(S_B)_{ijk} = \frac{1}{3} \left( B_{ijk} + B_{kij} + B_{jki} \right). \]

Thus, equating each term, we find

\[A_{ijk} - 2G^q_{jk} \eta_{iq} = 2B_{ijk} - B_{jik} - B_{kij}, \]

from $T_A = T_B$, and

\[2A_{ijk} + 2G^q_{jk} \eta_{iq} = B_{ijk} + B_{kij} + B_{jki}, \]

from $S_A = S_B$. But using the initial equation (4.4), we get

\[G^q_{jk} = \frac{1}{2} \eta^{iq} \left( B_{ijk} + B_{kij} - B_{jik} \right) \]

and

\[G^q_{jk} = \frac{1}{2} \eta^{iq} \left( B_{kij} + B_{jki} - B_{ijk} \right). \]

In conclusion, Eq. (4.4) has solutions, one of them being given by the last equation. □

**Remark 12** A consequence of the previous Lemma is that, given a trivializing open set $V$ of $LM$, there exists an infinitesimal symmetry $Z$ of $I_L$ such that $\tau_1 \left( \text{supp} \ Z \right) \subset \tau \left( V \right)$. It allows us to obtain local conditions for the extremals of the Griffiths variational problem.
4.2 Field equations for Lovelock gravity from its Griffiths variational problem

Let us compute the differential of the Lovelock Lagrangian:

\[ d\lambda_L = (d\theta_{i_1 \ldots i_r j_1 \ldots j_r}) \wedge \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_r j_r} \]

+ \((-1)^m \sum_{a=1}^{r} \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{i_1 j_1} \wedge \ldots \wedge d\Omega^{i_a j_a} \wedge \ldots \wedge \Omega^{i_r j_r}.\]

The first term was computed in Proposition 2, now let us work out the second term. If \( \sigma \in S_r \) is the permutation that transpose 1 and \( a \), then

\[ \theta_{i_{\sigma(1)} \ldots i_{\sigma(r)} j_{\sigma(1)} \ldots j_{\sigma(r)}} = \theta_{i_1 \ldots i_r j_1 \ldots j_r}, \]

and we can reorder every summand as follows

\[ \sum_{a=1}^{r} \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{i_1 j_1} \wedge \ldots \wedge d\Omega^{i_a j_a} \wedge \ldots \wedge \Omega^{i_r j_r} \]

\[ = \sum_{a=1}^{r} \theta_{i_{\sigma(1)} \ldots i_{\sigma(r)} j_{\sigma(1)} \ldots j_{\sigma(r)}} \wedge \Omega^{i_{\sigma(1)} j_{\sigma(1)}} \wedge \ldots \wedge d\Omega^{i_{\sigma(a)} j_{\sigma(a)}} \wedge \ldots \wedge \Omega^{i_{\sigma(r)} j_{\sigma(r)}} \]

\[ = \sum_{a=1}^{r} \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{i_a j_a} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge d\Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_r j_r} \]

\[ = \sum_{a=1}^{r} \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge d\Omega^{i_1 j_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r} \]

\[ = r \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge d\Omega^{i_1 j_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r}. \]

Hence

\[ d\lambda_L = (d\theta_{i_1 \ldots i_r j_1 \ldots j_r}) \wedge \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_r j_r} \]

\[ + \((-1)^m r \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge d\Omega^{i_1 j_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r} \]

\[ = \left[ T^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \sum_{s=1}^{r} (-1)^s \omega^l_{i_s} \wedge \theta_{i_1 \ldots i_s \hat{j}_s \ldots i_r j_1 \ldots j_r} \right] \wedge \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_r j_r} \]

\[ + \sum_{s=1}^{r} (-1)^{r+s} \omega^l_{j_s} \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_s \hat{j}_s \ldots i_r j_1 \ldots j_r} \]

\[ + r \left[ T^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{i_1 j_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r} \right] \]

\[ = \left[ T^l \wedge \sum_{s=1}^{r} (-1)^s \omega^l_{i_s} \wedge \theta_{i_1 \ldots \hat{i_s} \ldots i_r j_1 \ldots j_r} \right] \wedge \Omega^{i_1 j_1} \wedge \ldots \wedge \Omega^{i_r j_r}. \]
\[ + \sum_{s=1}^{r} (-1)^{r+s} \omega_{js}^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r l} - \omega_{j}^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r l} \right) \wedge \Omega_{j_1 j_1}^{l_1} \\
+ r \left( \Omega_{q}^{l_1} \wedge \omega_{q}^{j_1} - \omega_{q}^{j_1} \wedge \Omega_{q}^{g_1 j_1} \right) \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \bigg] \wedge \Omega_{i_2 j_2}^{l_2} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r}.
\]

Let us analyze the two sums in the brackets. As above, suppose that \( \sigma \) is the permutation that transpose 1 and \( s \); then

\[ \sum_{s=1}^{r} (-1)^{s} \omega_{i_s}^l \wedge \theta_{i_1 \ldots i_s j_1 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{i_1 l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = \sum_{s=1}^{r} (-1)^{s} \omega_{i_{\sigma(s)}}^l \wedge \theta_{i_{\sigma(1)} \ldots i_{\sigma(s)} j_1 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{i_{\sigma(1)} l_{\sigma(1)}} \wedge \ldots \wedge \Omega_{j_r j_r}^{i_{\sigma(r)} l_{\sigma(r)}} \]

\[ = \sum_{s=1}^{r} (-1)^{s+1} \omega_{i_1}^l \wedge \theta_{i_1 \ldots i_{s+1} j_1 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{i_1 l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = - \sum_{s=1}^{r} \omega_{i_1}^l \wedge \theta_{i_{s+1} \ldots i_r j_1 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{i_1 l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = - r \omega_{i_1}^l \wedge \theta_{i_{1} \ldots i_{r} j_1 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{i_1 l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r}.
\]

Furthermore, using a similar argument,

\[ \sum_{s=1}^{r} (-1)^{r+s} \omega_{i_s}^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = \sum_{s=1}^{r} (-1)^{r+s} \omega_{j_1}^l \wedge \theta_{i_1 \ldots i_s j_2 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = \sum_{s=1}^{r} (-1)^{r+s+1} \omega_{j_1}^l \wedge \theta_{i_1 \ldots i_s j_2 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = r \omega_{j_1}^l \wedge \theta_{i_1 \ldots i_r j_2 \ldots j_r l} \wedge \Omega_{j_1 j_1}^{l_1} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \]

\[ = r \eta_{pq} \omega_{j_1}^l \wedge \theta_{i_1 \ldots i_r j_2 \ldots j_r l} \wedge \Omega_{q}^{l_1} \wedge \Omega_{i_r j_r}^{l_2} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r}.
\]

Now let us simplify the terms \( r \left( \Omega_{q}^{i_1 j_1} \wedge \omega_{q}^{j_1} \wedge \Omega_{q}^{g_1 j_1} \right) \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega_{i_2 j_2}^{l_2} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r} \). First, renaming the dummy indices,

\[ - r \omega_{i_1}^l \wedge \Omega_{j_1 j_1}^{i_1 l_1} \wedge \theta_{i_{1} \ldots i_{r} j_1 \ldots j_r l} \wedge \Omega_{i_2 j_2}^{l_2} \wedge \ldots \wedge \Omega_{i_r j_r}^{l_r}.
\]
which cancels out the first sum. Second
\[
\begin{align*}
    r \omega_{i_1}^j &\wedge \theta_{i_2 \ldots i_r j_1 \ldots j_r l} \wedge \Omega^{i_1 j_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r}, \\
    &= r \Omega_q^{i_1} \wedge \omega_{q l}^{q l} \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r}. 
\end{align*}
\]
and consequently
\[
\begin{align*}
    d\lambda_L &= \left( \eta^{j_1 q} T^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r l} - \eta^{j_1 q} \omega_{l}^j \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \\
    &+ r \left( \eta^{j_1 p} \omega_{p}^q + \eta^{q p} \omega_{p}^q \right) \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \right) \wedge \Omega_q^{i_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r}. 
\end{align*}
\]
These computations amounts to the Lagrangian form on $J^1 \tau$, so we have to take its pullback to $T_0$, i.e.
\[
\begin{align*}
    \iota_0^* d\lambda_L &= \left( \eta^{j_1 q} T^l \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r l} - \eta^{j_1 q} \omega_{l}^j \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \\
    &+ r \left( \eta^{j_1 p} \omega_{p}^q + \eta^{q p} \omega_{p}^q \right) \wedge \theta_{i_1 \ldots i_r j_1 \ldots j_r} \right) \wedge \Omega_q^{i_1} \wedge \Omega^{i_2 j_2} \wedge \ldots \wedge \Omega^{i_r j_r}. 
\end{align*}
\]
Nevertheless, we will omit the pullback $\iota_0$ to simplify notation.

Now we are ready to find the field equations associated with the Griffiths problem $(J^1 \tau, \lambda_L, I_L)$. First we state a lemma we will use later on.

**Lemma 4** If $\Omega$ takes values in $\mathfrak{k}$, then on $T_0$
\[
\Omega_q^{i_1} \wedge \theta_{q i_2 \ldots i_r J} \wedge \Omega^{I J} = -\Omega_{q j_1}^{j_1} \wedge \theta_{i_1 \ldots i_r j_2 \ldots j_r} \wedge \Omega^{I J}.
\]

**Proof** Using the structure equation, we have
\[
d T^q + \omega_{l}^{q l} \wedge T^l = \Omega_q^{q l} \wedge \theta^l,
\]
and since the torsion form annihilates on $T_0$, we have
\[
\Omega_q^{q l} \wedge \theta^l = 0.
\]
Then multiplying both sides by $\theta_{q i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{I J}$,
\[
\Omega_q^{q l} \wedge \theta^l \wedge \theta_{q i_1 \ldots i_r j_1 \ldots j_r} \wedge \Omega^{I J} = 0,
\]
Thus, since \( \Omega^q \) is a two-form. On the other hand, we use the skew-symmetry of the Sparling form and the fact that the curvature vector fields generating \( X^\mathbb{V}_\tau(V) \) on some trivializing open set \( V \subset LM \) and let \( \{ M^{kl}_{ij} \} \)

\[
\Omega^q \land \theta_{qI} \land \Omega^I J
\]

\[
= (-1)^{2r+1} \Omega^q \land \left[ -\delta^q_1 \theta_{qI} + \sum_{a=1}^r \sum_{a=1}^r (-1)^{a+1} \left( \delta^q_i \theta_{qI \ldots i_{a-1}i_a} + (-1)^r \delta^1_j \theta_{qI \ldots j_{a-1}j_a} \right) \right] \land \Omega^I J
\]

\[
= \Omega^q \land \left[ \delta^q_1 \theta_{qI} + \sum_{a=1}^r \sum_{a=1}^r (-1)^a \left( \delta^q_i \theta_{qI \ldots i_{a-1}i_a} + (-1)^r \delta^1_j \theta_{qI \ldots j_{a-1}j_a} \right) \right] \land \Omega^I J
\]

\[
= \left[ \Omega^q_1 \land \theta_{qI} + \sum_{a=1}^r \sum_{a=1}^r (-1)^a \left( \Omega^q_{i_a} \land \theta_{qI \ldots i_{a-1}i_a} + (-1)^r \Omega^q_j \land \theta_{qI \ldots j_{a-1}j_a} \right) \right] \land \Omega^I J.
\]

Let us study both terms in the sum. If \( \sigma \in S_r \) is the permutation transposing 1 and \( a \),

\[
\sum_{a=1}^r (-1)^a \Omega^q_{i_a} \land \theta_{qI \ldots i_{a-1}i_a} \land \Omega^I J = \sum_{a=1}^r (-1)^a \Omega^q_{i_{\sigma(a)}} \land \theta_{qI_{\sigma(1)} \ldots i_{\sigma(a)} \ldots i_{\sigma(r)}} \land \Omega^I \sigma J_{\sigma}
\]

\[
= \sum_{a=1}^r (-1)^{a+1} \Omega^q_{i_a} \land \theta_{qI_{i_{a+1}} \ldots i_{a-1}i_a-1} \land \Omega^I J = \sum_{a=1}^r (-1)^{a+1} (-1)^{a-2} \Omega^q_{i_1} \land \theta_{qI_{i_2} \ldots i_a} \land \Omega^I J
\]

\[
= \sum_{a=1}^r -\Omega^q_{i_1} \land \theta_{qI_{i_2} \ldots i_a} \land \Omega^I J = -r \Omega^q_{i_1} \land \theta_{qI_{i_2} \ldots i_a} \land \Omega^I J
\]

where we use the skew-symmetry of the Sparling form and the fact that the curvature is a two-form. On the other hand,

\[
\sum_{a=1}^r (-1)^{a+r} \Omega^q_{j_{a+1}} \land \theta_{qI_{i_{a+1}} \ldots i_{a-1}i_a-1} \land \Omega^I J = \sum_{a=1}^r (-1)^{a+r} \Omega^q_{i_{\sigma(a)}} \land \theta_{qI_{i_{\sigma(1)}} \ldots j_{\sigma(a)} \ldots j_{\sigma(r)}} \land \Omega^I \sigma J_{\sigma}
\]

\[
= \sum_{a=1}^r (-1)^{a+r} \Omega^q_{j_1} \land \theta_{qI_{j_{a+1}} \ldots j_{a-1}j_a-1} \land \Omega^I J
\]

\[
= \sum_{a=1}^r (-1)^{a+r+1} (-1)^{a-2} \Omega^q_{j_1} \land \theta_{qI_{j_{a+1}} \ldots j_{a-1}j_a} \land \Omega^I J
\]

\[
= (-1)^{r+1} \Omega^q_{j_1} \land \theta_{qI_{j_{2} \ldots j_r}} \land \Omega^I J.
\]

Thus, since \( \Omega^q_1 = 0 \), we get

\[
\Omega^q_{i_1} \land \theta_{qI_{i_2} \ldots i_a} \land \Omega^I J = (-1)^{r+1} \Omega^q_{j_1} \land \theta_{qI_{j_{2} \ldots j_r}} \land \Omega^I J,
\]

or equivalently

\[
\Omega^q_{i_1} \land \theta_{qI_{i_2} \ldots i_a} \land \Omega^I J = -\Omega^q_{j_1} \land \theta_{qI_{j_{2} \ldots j_r}} \land \Omega^I J.
\]

\[\square\]

As we did in the proof of Lemma 3, consider a basis \( \{ U^i_j \} \) of \( G \)-invariant local vector fields generating \( X^\mathbb{V}_\tau(V) \) on some trivializing open set \( V \subset LM \) and let \( \{ M^{kl}_{ij} \} \).
be smooth functions on $V$ such that

$$U^i_j = M^{ii}_{jk} \left( E^k_i \right)_{LM}.$$

Let us compute the contraction $j^1 U^i_{,\tau} \, d\lambda_L$ by using (4.1). First

$$M^{ii}_{rk} \left( E^k_i \right)_{j^1 \tau} \, d\lambda_L$$

$$= j^1 U^i_{,\tau} \left[ r \eta^{jiq} \left( \omega_p \right)^q_j \wedge \theta_{1j} - \frac{1}{2} \eta^{jip} \left( \omega_p \right)^{ij}_i \wedge \theta_{1j} \right] \wedge \Omega^i_q \wedge \Omega^{i_2j_2} \wedge \ldots \wedge \Omega^{i_rj_r}$$

$$= 2 M^{ii}_{rk} \left[ r \eta^{jiq} \frac{1}{2} \left( \delta_i^q \delta_p^q + \eta_{pq} \delta_i^q \eta_{pq} \right) \wedge \theta_{1j} - \frac{1}{2} \eta^{jiq} \delta_i^q \theta_{1j} \right] \wedge \Omega^i_q \wedge \Omega^{i_2j_2} \wedge \ldots \wedge \Omega^{i_rj_r}$$

$$= 2 M^{ii}_{rk} \left[ r \frac{1}{2} \left( \eta^{jiq} \delta_i^q + \delta_i^q \eta_{pq} \right) \wedge \theta_{1j} - \frac{1}{2} \eta^{jiq} \delta_i^q \theta_{1j} \right] \wedge \Omega^i_q \wedge \Omega^{i_2j_2} \wedge \ldots \wedge \Omega^{i_rj_r}$$

$$= 2 M^{ii}_{rk} \left[ \left( \Omega^i_q \eta^{jiq} + \delta_i^q \eta_{pq} \right) \wedge \theta_{1j} \right] \wedge \Omega^{i_2j_2} \wedge \ldots \wedge \Omega^{i_rj_r}.$$

Let us study the terms in brackets. First

$$\left( \Omega^i_q \eta^{jiq} + \delta_i^q \eta_{pq} \Omega^i_q \right) \wedge \theta_{1j} \wedge \Omega^{i_2j_2} = \eta^{jiq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2},$$

and using Lemma 4 and renaming indices,

$$\eta^{jiq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2} + \delta_i^q \eta_{pq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2}$$

$$= \eta^{jiq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2} + \delta_i^q \eta_{pq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2}$$

$$= -\eta^{jiq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2} + \delta_i^q \eta_{pq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2}$$

$$= \eta^{jiq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2} + \delta_i^q \eta_{pq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2}$$

$$= 2 \eta^{jiq} \Omega^i_q \wedge \theta_{1j} \wedge \Omega^{i_2j_2}.$$

Thus

$$M^{ii}_{rk} \left( E^k_i \right)_{j^1 \tau} \, d\lambda_L = 2 M^{ii}_{rk} \left[ r \eta^{jiq} \Omega^i_q \wedge \theta_{1j} - \frac{1}{2} \eta^{jiq} \delta_i^q \theta_{1j} \wedge \Omega^i_q \right] \wedge \Omega^{i_2j_2}$$

$$= 2 r M^{ii}_{rk} \left[ \eta^{jiq} \theta_{1j} - \frac{1}{2r} \eta^{jiq} \delta_i^q \theta_{1j} \right] \wedge \Omega^i_q \wedge \Omega^{i_2j_2}.$$
Now let us compute the other contraction:

\[ D_j M^{i_j}_{rk} \left( \theta^j, \left( E^k_i \right)_{LM} \right)^V d\lambda_L \]

\[ = D_j M^{i_j}_{rk} \left( \theta^j, \left( E^k_i \right)_{LM} \right)^V \eta^q \left[ r \eta^{ijp} (\omega_p)_p^q \wedge \theta_{1j} - \frac{1}{2} \eta^{ijq} (\omega_p)_i^l \wedge \theta_{1j} \right] \wedge \Omega^i_q \wedge \Omega^{j^*} \]

\[ = (-1)^{m+1} D_j M^{i_j}_{rk} \left[ r \eta^{ijp} (\omega_p)_p^q \wedge \theta_{1j} - \frac{1}{2} \eta^{ijq} (\omega_p)_i^l \wedge \theta_{1j} \right] \wedge \tilde{\Omega}^{i^*_j, i^*} \]

where \( \tilde{\Omega}^{i^*_j, i^*} = (\theta^j, \left( E^k_i \right)_{LM})^V \), So, gathering both terms together:

\[ j^1 U^j_r \cdots d\lambda_L = 2r M^{i_j}_{rk} \left[ \eta^{pq} \theta_{1j2...r} - \frac{1}{2} \eta^{iq} \delta^p_j \wedge \theta_{1j} \right] \wedge \Omega^{i_q} \wedge \Omega^{j^*} \]

\[ + (-1)^{m+1} D_j M^{i_j}_{rk} \left[ r \eta^{ijp} (\omega_p)_p^q \wedge \theta_{1j} - \frac{1}{2} \eta^{ijq} (\omega_p)_i^l \wedge \theta_{1j} \right] \wedge \tilde{\Omega}^{i^*_j, i^*} \]

Thus if \( \Sigma : M \rightarrow T_0 \) is a section such that \( \Sigma^* \omega_p = 0 \), we can conclude that

\[ \Sigma^* \left( j^1 U^j_r \cdots d\lambda_L \right) = 2r \left( M^{i_j}_{rk} \circ \Sigma \right) \Sigma^* \left[ \left( \eta^{pq} \theta_{1j2...r} - \frac{1}{2} \eta^{iq} \delta^p_j \wedge \theta_{1j} \right) \wedge \Omega^{i_q} \wedge \Omega^{j^*} \right] \]

from Lemma 3 and Eq. (4.5), we obtain the following result:

**Theorem 1** Let \( \Sigma : M \rightarrow T_0 \) be an extremal for the variational problem associated with the Griffiths triple \( (T_0, \lambda_L, \mathcal{I}_L) \). Then

\[ \Sigma^* \left[ \left( \eta^{pq} \theta_{1j2...r} - \frac{1}{2} \eta^{iq} \delta^p_j \wedge \theta_{1j} \right) \wedge \Omega^{i_q} \wedge \Omega^{j^*} \right] = 0. \]

**Proof** We consider infinitesimal symmetries \( Z \in \mathcal{X}^{\tau_1} (T_0) \) of \( \mathcal{I}_L \) as in Lemma 3, namely

\[ Z = f^l_j j^1 U^j_l + F^k_{kl} \left( \theta^k, \left( E^l_i \right)_{LM} \right)^V, \]

where \( (f^l_j) \) is a family of arbitrary functions on (an open set of) \( M \) such that

\[ \text{supp} \ f^l_j \subset \tau (V) \]

Then by performing the variation induced by \( Z \), we have the formula

\[ \int_{\tau (V)} f^l_r \Sigma^* \left[ M^{i^*_j}_{sk} \left( \eta^{pq} \theta_{1j2...r} - \frac{1}{2} \eta^{iq} \delta^p_j \wedge \theta_{1j} \right) \wedge \Omega^{i_q} \wedge \Omega^{j^*} \right] = 0. \]
and the result follows from the fact that the functions $f_k^i$ are arbitrary and the matrix $(M_{ij}^k)$ is invertible.

\[ \square \]

**Remark 13** It is useful to compare this with the Einstein case. In [6] it is seen that the Einstein equations in vacuum are

\[
\theta_{il} \wedge \Omega_k^l + \theta_{kl} \wedge \Omega_i^l - \eta_{ik} \left( \eta^{pq} \theta_{ql} \wedge \Omega_p^l \right) = 0,
\]

together with the constraints $T = 0 = \omega_p$; the previous theorem, on the other hand, gives us the set of equations

\[
\left( \eta^{kp} \theta_{il} - \frac{1}{2} \delta^k_i \eta^{rp} \theta_{ql} \right) \wedge \Omega_p^l = 0
\]

under the same constraints. Nevertheless, it can be proved (see Corollary 21 in [6]) that under the constraints $T = 0 = \omega_p$, it is true that

\[
\omega_{ik} \wedge \Omega_p^k - \omega_{pk} \wedge \Omega_i^k = 0
\]

as consequence of a Bianchi identity. Therefore, these sets of equations are equivalent.

Theorem 1 gives us a set of necessary conditions for a section $\Sigma : M \to T_0$ to be extremal of the Griffiths variational problem associated with the triple $(T_0, \lambda_L, \tilde{I}_L)$. Our next task is to set the sufficiency of these conditions.

**Proposition 4** Let $\Sigma : M \to T_0$ be a section such that $\Sigma^* \omega_p = 0$ and

\[
\Sigma^* \left[ \left( \eta^{kp} \theta_{lij...j} - \frac{1}{2} \delta^k_i \eta^{rp} \delta^k_l \theta_{ij} \right) \wedge \Omega_q^l \wedge \Omega_{ij}^{ij} \right] = 0.
\]

Then

\[
\Sigma^* (Z, d\lambda_L) = 0,
\]

for every $Z \in \mathfrak{X}^{V_{T_0}} (T_0)$.

**Proof** It is a consequence of the fact that every $Z \in \mathfrak{X}^{V_{T_0}} (T_0)$ can be written in terms of the vector fields

\[
\left\{ j^l U^j_i, (\theta^k, (E_i^j)_{LM})^V \right\}.
\]

\[ \square \]
Thus, in particular, for every $\tau_1$-vertical infinitesimal symmetry $Z$ of the exterior differential system $\mathcal{I}_L$ and any section $\Sigma : M \to J^1 \tau$ fulfilling the hypotheses in the previous proposition, we have that

$$\Sigma^*(Z \lrcorner d\lambda_L) = 0;$$

therefore, $\Sigma$ is an extremal for the Griffiths variational problem $(T_0, \lambda_L, \mathcal{I}_L)$, as required.

5 Unified formalism

5.1 Tautological form on a bundle of forms

The next definitions are quoted from [6]. Let $\pi : P \to N$ be a principal fiber bundle with structure group $H$ and assume that $q$ and $p$ are surjective submersions fitting the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\pi} & N \\
\psi \downarrow & & \downarrow \chi \\
M & & \\
\end{array}$$

Let $V$ be a finite dimensional real vector space $V$ and define the bundle $\tilde{\tau}_{n,q}^k : \bigwedge_{n,q}^k T^* P \otimes V \to P$ of $V$-valued $k$-forms that annihilates when contracted with $n q$-vertical vectors. This bundle has a canonical $V$-valued $k$-form $\Theta_{n,q}^k$ defined through the formula

$$\Theta_{n,q}^k \bigg|_{\alpha} (Z_1, \ldots, Z_k) := \alpha \left(T_{\alpha} \tilde{\tau}_{n,q}^k (Z_1), \ldots, T_{\alpha} \tilde{\tau}_{n,q}^k (Z_k) \right).$$

Given a $H$-representation $(V, \rho)$, it is readily seen that $\bigwedge_{n,q}^k T^* P \otimes V$ is a $H$-space with action given by

$$\Phi_{q}^k (\alpha)(X_1, \ldots, X_k) := \rho(\alpha) \cdot \left(\alpha(T_{u \cdot h} R_{h^{-1}} X_1, \ldots, T_{u \cdot h} R_{h^{-1}} X_k) \right),$$

where $R$ is the right action in $P$ and $h \in H$. It can be proved that the tautological form $\Theta_{n,q}^k$ is then a $H$-equivariant map.

We point out two instances that will be used in the next section. If $H = G = GL(m)$, $P = J^1 \tau$, $N = C(LM)$, $\psi = \tau_1$, $\chi = p$ and $\pi = q$ (that is, the left triangle in the diagram of Sect. 2.1), we have

1. Set $k = m - 2$ and $n = r + 1$, and consider $V_1 = (\mathbb{R}^m)^*$ and $\rho_1$ the natural representation of $G$ on this vector space. Then, we denote the space $E_1 := \bigwedge_{2, \tau_1}^{m-2} J^1 \tau \otimes (\mathbb{R}^m)^*$ and the projection

$$p_1 : E_1 \to J^1 \tau .$$
2. Set $k = m - 1$ and $n = r$, and consider $V_2 = (\mathbb{R}^m)^* \odot (\mathbb{R}^m)^*$ and $\rho_2$ the natural representation of $G$ on this vector space. (The symbol $\odot$ denotes the symmetrized tensor product). Then, we denote the space $E_2 := \bigwedge_{1, \tau_1}^{m-1} J^1 \tau \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^*$ and the projection

$$p_2 : E_2 \rightarrow J^1 \tau.$$ 

To simplify notation we denote by $\Theta_1$ and $\Theta_2$ the corresponding tautological forms on these bundles and, when using the component forms with respect to the canonical bases, we simply write $\Theta_1 = \Theta_1 e^i$ and $\Theta_2 = \Theta_{ij} e^i \otimes e^j$, indicating that a single index corresponds to $\Theta_1$ and two indices to $\Theta_2$.

### 5.2 The multimomentum bundle

The unified formalism for a Griffiths variational problem is built from the idea of a Lepage equivalent problem. Briefly, the construction goes as follows (this idea is inspired in the work of [28] and was proposed in [11]): assume that the differential ideal is locally generated by a subbundle $I \subset \Lambda^* T^* J^1 \tau$ (this means that there is an open cover $\{U_\lambda\}$ such that every $\alpha \in \mathcal{I}$ can be generated by sections of $I|_{U_\lambda}$ when pulled back to $U_\lambda$). Consider an integer $k$ such that $\lambda_L(u) \in \Lambda_k^m(T_u^* J^1 \tau)$ and $I^m_u := I \cap \Lambda_k^m(T_u^* J^1 \tau) \subset \Lambda_k^m(T_u^* J^1 \tau)$, where $\Lambda_k^m(T^* J^1 \tau)$ is the bundle of $m$-forms that annihilate when contracted with $k \tau_1$-vertical vectors. Then define the multimomentum bundle $W_\lambda$ by the equation

$$W_\lambda|_u = \lambda_L(u) + I^m_u,$$

which is an affine subbundle of $\Lambda_k^m(T^* J^1 \tau)$. In the case of Lovelock gravity for a polynomial Lagrangian of degree $r$ in the curvature, it suffices to take $k = r + 1$. (Notice that the form $\theta_{i_1,j_1} \cdots \theta_{i_r,j_r}$ is $\tau_1$-horizontal whereas the form $\Omega_{i_1,j_1} \wedge \cdots \wedge \Omega_{i_r,j_r}$ is only $q$-horizontal, which implies that more than $r \tau_1$-vertical vectors are needed to annihilate $\lambda_L$). In this case we can write any $\rho \in W_\lambda|_u$ as

$$\rho = \lambda_L|_{j^1_\lambda} + \gamma l \wedge T^l|_{j^1_\lambda} + \beta_{ij} \wedge \omega_p^{ij}|_{j^1_\lambda},$$

where $\beta_{ij} \in \Lambda_{r-1}^m(T_{j^1_\lambda}^* J^1 \tau)$ is symmetric in $ij$ and $\gamma_l \in \Lambda_{r+1}^{m-2}(T_{j^1_\lambda}^* J^1 \tau)$.

Observe that an element $\rho$ in $W_\lambda$ is completely determined by an element $j^1_\lambda \in J^1 \tau$ and the forms $\gamma_l$ and $\beta_{ij}$ projecting onto $j^1_\lambda$. Hence we have the following identification:

**Lemma 5** The map

$$\Gamma : \rho \mapsto (\gamma_l e^i, \beta_{ij} e^i \otimes e^j) \simeq (j^1_\lambda e^i, \gamma_l e^i, \beta_{ij} e^i \otimes e^j),$$

where $j^1_\lambda := p_1(\gamma_l e^i) = p_2(\beta_{ij} e^i \otimes e^j)$, induces an isomorphism of the bundles $\tau_\lambda : W_\lambda \rightarrow J^1 \tau$ and $\rho_0 : \tilde{W} \rightarrow J^1 \tau$, with $\tilde{W}_\lambda := E_1 \times j^1_\lambda E_2 \simeq J^1 \tau \times j^1_\lambda E_1 \times j^1_\lambda E_2$. 

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such that

\[ W_\lambda \xrightarrow{\gamma} \hat{W}_\lambda \]

\[ \tau_\lambda \xrightarrow{pr_0\circ pr_1=pr_2 \circ pr_2} J^1 \tau \]

To build the canonical form on \( \hat{W}_\lambda \) we need the tautological forms \( \Theta_1 \) and \( \Theta_2 \), as well as the forms \( \theta, \omega, T \) and \( \Omega \). We use the three projections \( pr_0, pr_1 \), and \( pr_2 \) to pull these forms back to \( \hat{W}_\lambda \), but we do not change the symbols so as to keep notation as simple as possible, e.g. if \( \rho \in \hat{W}_\lambda \), then \( \omega|_\rho \) and \( \Theta_2|_\rho \) must be understood as \( pr_0^*(\omega)|_\rho = \omega|_{u=pr_0(\rho)}(T_\rho pr_0(\cdot), \ldots, T_\rho pr_0(\cdot)) \) and \( pr_2^*(\Theta_2)|_\rho = \Theta_2|_{\beta=pr_2(\rho)}(T_\rho pr_2(\cdot), \ldots, T_\rho pr_2(\cdot)) \), respectively.

Let us now denote by \( \Theta_\lambda \) the pullback of the tautological form \( \Theta \in \Omega^n(\Lambda^m(J^1 \tau)) \) to \( \hat{W}_\lambda \). Then, at any \( \rho = \lambda L|_{j_l^s} + \gamma T^l|_{j_l^s} + \beta_i j \wedge \omega_{ij}|_{j_l^s} \),

\[ \Theta_\lambda|_\rho = \lambda L|_\rho \wedge \Theta|_\rho \wedge T^l|_\rho + \Theta_{ij}|_\rho \wedge \omega_{ij} \]

or, omitting the symbol \( \rho \) and recalling the expression for \( \lambda L \),

\[ \Theta_\lambda = \theta_{IJ} \wedge \Omega^{IJ} + \Theta_I \wedge T^l + \Theta_{ij} \wedge \omega_{ij} \]

In consequence, the differential is given by

\[ \Omega_\lambda = d\Theta_\lambda = d\lambda L + T^l \wedge d\Theta_I + dT^l \wedge \Theta_I + d\Theta_{ij} \wedge \omega_{ij} + (1)m^{-1}\Theta_{ij} \wedge d\omega_{ij} \]

\[ = d\lambda L + T^l \wedge d\Theta_I + \left( \Omega_k \wedge \theta^k - \omega^k \wedge T^k \right) \wedge \Theta_I + d\Theta_{ij} \wedge \omega_{ij} \]

\[ + (1)m^{-1}\Theta_{ij} \wedge \left( -\omega^k \wedge \omega^{ij} + \Omega^ij \right) , \]

and recalling the expression of \( d\lambda L \),

\[ d\lambda L = \left[ \eta^{ij} T^l \wedge \theta_{IJ} - \eta^{ij} \omega_l^l \wedge \theta_{IJ} + r \left( \eta^{ij} \omega^l_p + \eta^{pq} \omega^{li}_{pq} \right) \wedge \theta_{IJ} \right] \wedge \Omega^ij \wedge \Omega^IJ \]

\[ = \left[ \eta^{ij} T^l \wedge \theta_{stlJ^l} \wedge \Omega^IJ - \eta^{ij} \omega_l^l \wedge \theta_{stlJ^l} \wedge \Omega^IJ \right] + r \left( \eta^{ij} \omega^l_p + \eta^{pq} \omega^{li}_{pq} \right) \wedge \theta_{stlJ^l} \wedge \Omega^IJ \]

\[ + \left( -1 \right)^n \eta^{ij} \left[ (1)m d\Theta_{ij} \wedge \Theta_{pj} \wedge (\omega_{\gamma})_i^p - \eta^{ik} \eta_{pj} \Theta_{ij} \wedge (\omega_{\gamma})_k^p \right] \wedge (\omega_p)_l^j . \]
5.3 Field equations

To find the field equations we have to find the contraction of $\Theta_k$, with vertical vectors. We divide this task considering vertical vectors of the form $X = Y + Z$, where $X$ is $pr_0$-projectable and $\tau_1$-vertical, $Y$ is $p_1$-vertical, and $Z$ is $p_2$-vertical. Let us now give a useful description for the vectors $Y$ and $Z$.

Going back to the notation of Sect. 5.1, consider a cross section $\xi \in \Gamma(\tilde{\tau}^k_{n,q})$. Then, since this is a vector bundle, we have the associated vertical lift, which is a $\tilde{\tau}^k_{n,q}$-vertical vector field $\delta\xi \in \mathcal{X}(\Lambda^k_{n,q}T^*P \otimes V)$ given by

$$\delta\xi(\alpha_u) = \frac{d}{dt} \bigg|_{t=0} (\alpha_u + t\xi(u)).$$

In other words, this is the vector field whose flow is given by $\varphi^\xi_t(\alpha_u) = \alpha_u + t\xi(u)$, for every $\alpha_u \in E := \Lambda^k_{n,q}T_u^*P \otimes V$. It is clear that $\delta\xi$ is $\tilde{\tau}^k_{n,q}$-vertical, therefore it annihilates the tautological form $\Theta^k_{n,q}$ (because this is a horizontal form by definition).

Now, let us see the contraction of this type of vectors with the differential $d\Theta^k_{n,q}$.

**Lemma 6** The following identity holds

$$\delta\xi \cdot d\Theta^k_{n,q} = (\tilde{\tau}^k_{n,q})^*(\delta\xi).$$

**Proof** Let us compute

$$\delta\xi \cdot d\Theta^k_{n,q} = \mathcal{L}_{\delta\xi} \Theta^k_{n,q} - d\delta\xi \cdot \Theta^k_{n,q} = \mathcal{L}_{\delta\xi} \Theta^k_{n,q}$$

$$= \lim_{t \to 0} \frac{1}{t} \left[ \Theta^k_{n,q} \bigg|_{\varphi^\xi_t(\alpha_u)} (T_{\alpha_u} \varphi^\xi_t(\cdot), \ldots, T_{\alpha_u} \varphi^\xi_t(\cdot)) - \Theta^k_{n,q} \bigg|_{\alpha_u} (\cdot, \ldots) \right].$$

Now, if $X_1, \ldots, X_k \in T_{\alpha_u}E$,

$$\left. \Theta^k_{n,q} \right|_{\varphi^\xi_t(\alpha_u)} (T_{\alpha_u} \varphi^\xi_t(1), \ldots, T_{\alpha_u} \varphi^\xi_t(X_m))$$

$$= \varphi^\xi_t(\alpha_u) \left( T_{\varphi^\xi_t(\alpha_u)} \tilde{\tau}^k_{n,q} \circ T_{\alpha_u} \varphi^\xi_t(1), \ldots, T_{\varphi^\xi_t(\alpha_u)} \tilde{\tau}^k_{n,q} \circ T_{\alpha_u} \varphi^\xi_t(X_m) \right)$$

$$= (\alpha_u + t\xi(u)) \left( T_{\alpha_u} (\tilde{\tau}^k_{n,q} \circ \varphi^\xi_t)(1), \ldots, T_{\alpha_u} (\tilde{\tau}^k_{n,q} \circ \varphi^\xi_t)(X_m) \right).$$

Then

$$\left. \mathcal{L}_{\delta\xi} \Theta^k_{n,q} \right|_{\alpha_u} (X_1, \ldots, X_m) = \lim_{t \to 0} \frac{1}{t} \left[ (\alpha_u + t\xi(u)) \left( T_{\alpha_u} \tilde{\tau}^k_{n,q}(1), \ldots, T_{\alpha_u} \tilde{\tau}^k_{n,q}(X_m) \right) \right.$$

$$\left. -\alpha_u \left( T_{\alpha_u} \tilde{\tau}^k_{n,q}(1), \ldots, T_{\alpha_u} \tilde{\tau}^k_{n,q}(X_m) \right) \right].$$

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Let us start with the $p_2$-vertical vector fields. Let $Z = \delta \beta$ for some section $\beta \in \Gamma(p_2)$. Observe that the unique non-vanishing contraction for this kind of vectors is with the form $d \Theta_{ij}$, so

$$
\delta \beta \llcorner \Omega_{\lambda} = \eta^{ij} \left( \delta \beta \llcorner d \Theta_{ij} \wedge (\omega_{\rho})_l^i \right) = \eta^{ij} pr_0^* (\beta_{ij} \wedge (\omega_{\rho})_l^i) = \eta^{ij} pr_0^* \left( \beta_{ij} \wedge (\omega_{\rho})_l^i \right).
$$

Now, if $\sigma \in \Gamma(\tau_{\lambda})$ is a cross section, in order to be an extremal for our variational problem it must fulfill

$$
(pr_0 \circ \sigma)^* \left( \eta^{ij} \beta_{ij} \wedge (\omega_{\rho})_l^i \right) = 0, \quad (5.1)
$$

for every section $\beta$. Therefore, it must fulfill the equation

$$
(pr_0 \circ \sigma)^* (\omega_{\rho})_l^i = 0,
$$

which is none other than the metricity condition.

**Remark 14** This can be seen locally as follows. Since Eq. (5.1) must hold for every $\beta_{ij}$, in particular it must hold for horizontal $\beta_{ij}$. Then, writing $\sigma^* \left[ (\omega_{\rho})_l^i \eta^{ij} \right] = f^{ij}_{\lambda} dx^\lambda$ and $\beta_{ij} = p^\mu_{ij} d^{m-1} x_\mu$, for some functions $p^\mu_{ij} \in C^\infty (J^1 \tau)$ and $f^{ij}_{\lambda} \in C^\infty (M)$ (both symmetric in the indices $ij$), we have

$$
\sigma^* \left( \beta_{ij} \wedge (\omega_{\rho})_l^i \eta^{ij} \right) = (-1)^{m-1} (p^\mu_{ij} \circ \sigma) f^{ij}_{\lambda} dx^\lambda \wedge d^{m-1} x_\mu
$$

$$
= (-1)^{m-1} (p^\lambda_{ij} \circ \sigma) f^{ij}_{\lambda} = 0,
$$

which implies

$$
(p^\lambda_{ij} \circ \sigma) f^{ij}_{\lambda} = 0.
$$

then, varying $p^\mu_{ij}$ we conclude $f^{ij}_{\lambda} = 0$ which means $\sigma^* (\omega_{\rho})_l^i \eta^{ij} = 0$ and then $\sigma^* (\omega_{\rho})_l^i = 0$.

Now consider $Y = \delta \gamma$ for some section $\gamma \in \Gamma(p_1)$. Similarly to the previous case, the unique non-vanishing contraction for this kind of vectors is that with the form $d \Theta_l$, so

$$
\delta \gamma \llcorner \Omega_{\lambda} = T_l \wedge (\delta \gamma \llcorner d \Theta_l) = T_l \wedge pr_0^* (\gamma_l) = pr_0^* (T_l \wedge \gamma_l).
$$
If $\sigma \in \Gamma(\tau_\lambda)$ is a cross section, then in order to be an extremal for the variational problem, it must fulfill

$$(pr_0 \circ \sigma)^* \left( T^l \wedge \gamma_l \right) = 0,$$

for every section $\gamma$. Therefore, it must fulfill the equation

$$(pr_0 \circ \sigma)^* T^l = 0,$$

which is in turn the zero torsion condition.

Moving on, we separate several cases for the vector fields $X$. Every vector field can be written as a sum of a $\tau_{10}$-vertical vector field plus a $\tau_{10}$-projectable $\tau$-vertical vector field. The first kind is generated by the vector fields $(\theta^a, (E^b_c)_{LM})^V$, while the second is generated by the set of infinitesimal generators of the action. Consider now a vector field $X = g^c_{ab}(\theta^a, (E^b_c)_{LM})^V$, for some functions $g^c_{ab} \in C^\infty(J^1\tau)$. We only consider those contractions that do not vanish when taking the pullback by a section fulfilling the previous conditions we have found so far. Then

$$g^c_{ab} \left( \theta^a, (E^b_c)_{LM} \right)^V \wedge \Omega_\lambda = g^c_{ab} \left( \theta^q \wedge \Theta + (-1)^{m-1} \eta^t q \Theta_{st} \right) \delta^s_c \delta^b_q \wedge \theta^a = g^c_{aq} \left( \theta^q \wedge \Theta_c + (-1)^{m-1} \eta^t q \Theta_{ct} \right) \wedge \theta^a.$$

Therefore, taking the pullback by a solution to the previous equations and varying the functions $g^c_{ab}$, we conclude that it must also fulfill

$$\sigma^* \left( \Theta_c \wedge \theta^q - \eta^t q \Theta_{ct} \right) = 0,$$

or

$$\sigma^* \left( \eta^t q \Theta_c \wedge \theta^q - \Theta_{ct} \right) = 0,$$

and given the symmetry of $\Theta_{ct}$, we have

$$\sigma^* \left( \eta^t q \Theta_c \wedge \theta^q \right) = \sigma^* \left( \Theta_{ct} \right) = 0,$$

which in turn gives us

$$\sigma^* \left( \Theta_c \right) = 0.$$

Now assume that $X = A_{J^1\tau}$. If $A \in \mathfrak{k}$, then it is easy to see that

$$A_{J^1\tau} \wedge \Omega_\lambda = 0,$$

so we have no new equations.
Finally, consider now \( A \in \mathfrak{p} \). In this case

\[
A_{J^1 \tau^k \Omega} = \left[ 2r \eta^{qp} A^i_p \theta_{stl^1} J^r - \eta^{il} A^l_i \theta_{stl^1} J^r \right] \wedge \Omega^s_q \wedge \Omega^{l^1 l^2},
\]

and simplifying this expression using the symmetry properties we arrive to the last equation

\[
\left[ \theta_{stl^1} J^r \wedge \Omega^s_q + \theta_{stl^1} J^r \wedge \Omega^s_i - \frac{1}{r} \eta_{iq} \left( \eta^{kl} \theta_{stl^1} J^r \wedge \Omega^s_l \right) \right] \wedge \Omega^{l^1 l^2} = 0,
\]

which can be written as

\[
\left( \Omega^{sa} \eta^{tb} + \Omega^{sb} \eta^{ta} - \frac{1}{r} \eta^{ab} \Omega^{st} \right) \wedge \theta_{stl^1} J^r \wedge \Omega^{l^1 l^2} = 0.
\]

This is equivalent to the Euler–Lagrange equations associated with the Lovelock Lagrangian of order \( r \) [39] as the following lemma shows:

**Lemma 7** The vanishing of the pullback of the m-form

\[
\Psi^{ab} := \left( \Omega^{sa} \eta^{tb} + \Omega^{sb} \eta^{ta} - \frac{1}{r} \eta^{ab} \Omega^{st} \right) \wedge \theta_{stl^1} J^r \wedge \Omega^{l^1 l^2}
\]

by a local section \( \sigma \in \Gamma \tau_1 \) is equivalent to the Euler–Lagrange equations associated with the Lovelock Lagrangian.

**Proof** Take fibered coordinates and remember the relation between the coefficients of the curvature form and the curvature tensor [see (3.4)]. Then

\[
e^a_m \Psi^{ab} e^b_v = \left( e^a_m \Omega^{sa} \eta^{tb} e^v_b + e^a_m \Omega^{sb} \eta^{ta} e^v_b - \frac{1}{r} e^a_m \eta^{ab} e^v_b \Omega^{st} \right) \wedge \theta_{stl^1} J^r \wedge \Omega^{l^1 l^2}
\]

\[
= \left( e^a_m \Omega^{sa} \eta^{tb} e^v_b + \Omega^{sb} e^a_m \eta^{ta} e^v_b - \frac{1}{r} g^{\mu \nu} \Omega^{st} \right) \wedge \theta_{stl^1} J^r \wedge \Omega^{l^1 l^2}
\]

\[
= \left( e^a_m \delta^\mu_\rho e^\rho_\eta e^\nu_\sigma e^a_m \rho_\alpha \delta_{\gamma \delta} + e^a_m \delta^\mu_\rho e^\rho_\eta e^\nu_\sigma e^a_m \rho_\alpha \delta_{\gamma \delta} - \frac{1}{r} g^{\mu \nu} e^a_m \right) \wedge \theta_{stl^1} J^r \wedge \Omega^{l^1 l^2}
\]

and renaming some indices

\[
e^a_m \Psi^{ab} e^b_v = \left( e^a_m \delta^\mu_\rho e^\rho_\eta e^\nu_\sigma e^a_m \rho_\alpha \delta_{\gamma \delta} + e^a_m \delta^\mu_\rho e^\rho_\eta e^\nu_\sigma e^a_m \rho_\alpha \delta_{\gamma \delta} - \frac{1}{r} g^{\mu \nu} e^a_m \right) \wedge \theta_{stl^1} J^r \wedge \Omega^{l^1 l^2}
\]
Observe that, renaming indices and using the symmetries of the curvature tensor,

\[ \sum_{k=1}^{r} \frac{1}{2} \delta_{\lambda_1 \ldots \lambda_r}^1 \delta_{\alpha_1 \ldots \alpha_r}^1 \left( g^{\mu \nu} \delta_{\rho}^1 + g^{\nu \rho} \delta_{\mu}^1 \right) R_{\alpha_1 \beta_1} \ldots R_{\alpha_r \beta_r} \, \text{d}^{2n+1} \chi. \]

Hence, we get

\[ e^\mu_a \Psi^{ab} e^\nu_b = \frac{\det(e)}{2r} \left[ \sum_{k=1}^{r} \left( \delta_{\mu \nu}^{\lambda_1 \ldots \lambda_r} \delta_{\alpha_1 \ldots \alpha_r}^{\lambda_1 \ldots \lambda_r} + \delta_{\mu \nu}^{\lambda_1 \ldots \lambda_r} \delta_{\alpha_1 \ldots \alpha_r}^{\lambda_1 \ldots \lambda_r} g^{\rho \nu} \right) R_{\alpha_1 \beta_1} \ldots R_{\alpha_r \beta_r} \, \text{d}^{2n+1} \chi \right]. \]

Therefore, the vanishing of \( \Psi^{ab} \) is equivalent to the equations

\[ \left( \delta_{\mu \nu}^{\lambda_1 \ldots \lambda_r} \delta_{\alpha_1 \ldots \alpha_r}^{\lambda_1 \ldots \lambda_r} g^{\rho \nu} + \delta_{\mu \nu}^{\lambda_1 \ldots \lambda_r} \delta_{\alpha_1 \ldots \alpha_r}^{\lambda_1 \ldots \lambda_r} g^{\rho \mu} \right) R_{\alpha_1 \beta_1} \ldots R_{\alpha_r \beta_r} = 0, \]

which are indeed the Euler–Lagrange equations described by Lovelock in [39]. □
The tensor

$$A^{\mu \nu} = \left( \delta^{\mu \alpha_1 \beta_1 \ldots \alpha_r \beta_r}_{\rho \lambda_1 \theta_1 \ldots \lambda_r \theta_r} g^{\rho \nu} + \delta^{\nu \alpha_1 \beta_1 \ldots \alpha_r \beta_r}_{\rho \lambda_1 \theta_1 \ldots \lambda_r \theta_r} g^{\rho \mu} \right) R^\lambda_{\alpha_1 \beta_1} \cdots R^r_{\alpha_r \beta_r}$$

is the unique tensor (up to a constant) fulfilling the following properties:

- It depends only on the metric tensor $g_{\alpha \beta}$ and its first two derivatives,
- it is symmetric in $\mu \nu$, and
- it has vanishing covariant derivative with respect to the Levi-Civita connection.

This was introduced by Lovelock [39,40] as a generalization of the Einstein tensor for dimensions higher than 4. In this sense, the form $\Psi$ is a global expression for such a tensor.

Gathering all the equations found so far, we conclude that the solutions to the field equations are those cross sections that are integral for the following exterior differential system

$$\mathcal{J} := \left\{ \Theta_I, \Theta_{ij}, T^i, (\omega_p)_j^i, (\Omega_p)_j^i, \Omega_k^i \wedge \theta^i, \Psi^{ij} \right\}. \quad (5.2)$$

Notice that the forms $\Omega_k^i \wedge \theta^i$ and $(\Omega_p)_j^i$ come as a consequence of the first Bianchi identity and the structure equation for the connection $\omega$, respectively.

**Remark 16** It is interesting to note that the forms $\Psi^{ij}$ can be written in terms of the $(m-1)$-forms $\theta_{IJJ} \wedge \Omega^{IJ}$. Indeed

$$(\eta^{ij} \theta^j) \wedge \theta_{IJJ} \wedge \Omega^{IJ} = (\eta^{ij} \theta^j) \wedge \Omega^{st} \wedge \theta_{IJJ} \wedge \Omega^{IJ} = \eta^{ij} \wedge \Omega^{st} \wedge (\theta^j \theta_{IJJ}) \wedge \Omega^{IJ}$$

$$= -\Omega^{st} \wedge \eta^{ij} \left[ -\delta^j_i \theta_{IJJ} \wedge \Omega^{IJ} = \delta^j_i \theta_{IJJ} \wedge \Omega^{IJ} - \delta^j_i \theta_{IJJ} \wedge \Omega^{IJ} \right]$$

$$= -\sum_{a=1}^{r-1} \left( \delta^j_i \theta_{IJJ} \wedge \Omega^{IJ} = \delta^j_i \theta_{IJJ} \wedge \Omega^{IJ} - \delta^j_i \theta_{IJJ} \wedge \Omega^{IJ} \right) \wedge \Omega^{IJ}.$$
and renaming some indices we get

\[ (\eta^{il} \theta^j ) \wedge \theta_{IJ} \wedge \Omega^{IJ} = \eta^{ij} \theta_{IJ} \wedge \Omega^{IJ} - 2r \eta^{it} \theta_{sI} \wedge \Omega^{sj} \wedge \Omega^{IJ}. \]

Therefore

\[- \frac{1}{2r} (\eta^{il} \theta^j + \eta^{jl} \theta^i ) \wedge \theta_{IJ} \wedge \Omega^{IJ} = \left( \eta^{ij} \Omega^{sj} + \eta^{ji} \Omega^{si} - \frac{1}{r} \eta^{ij} \Omega^{st} \right) \wedge \theta_{sI} \wedge \Omega^{IJ}. \]

from where it follows

\[ \Psi^{ij} = - \frac{1}{2r} (\eta^{il} \theta^j + \eta^{jl} \theta^i ) \wedge \theta_{IJ} \wedge \Omega^{IJ}. \]

**Remark 17** In particular, in [5,6], two exterior differential systems where introduced to describe Palatini gravity, the first as a Griffiths variational problem takes the forms \( \theta_{IJ} \wedge \Omega^{IJ} \) among its generators, and the second one by means of the unified formalism associated with the first one, where the forms \( \Psi^{ij} \) are used instead.

### 6 Conclusions and outlook

We have defined the Lovelock Lagrangian in the context of the multisymplectic framework for classical field theories, and we have used this geometric formulation to characterize the properties of this Lagrangian, to establish its Griffiths variational problem and derive the corresponding field equations, and to study the infinitesimal symmetries of the system. As this Lagrangian is singular, this is a (pre-multisymplectic) field theory with constraints and then we have developed the Lagrangian–Hamiltonian unified formalism which is very suitable for its analysis.

Furthermore, if a variational problem has constraints, one can consider applying the constraints before or after performing the variations. In general, these two procedures lead to different sets of equations [1]. When both sets of equations are equal, we say that the variational problem has a **consistent truncation** (by the constraints). In [14] the authors claim that the Lovelock Lagrangians can be characterized by the consistency of the Levi-Civita truncation; that is, replacing the arbitrary connection by the Levi-Civita connection associated to the metric. We hope that the formalism presented in this paper could be very appropriate to analyze this property of gravitational theories, and to study the concept of consistent truncation for variational principles in a geometrical way. This would be a topic for further research.

Finally, the methods and results obtained in this paper are suitable to describe other gravity theories, such as the \( f(R) \) and \( f(T) \) [4,22] models or the BF-gravity [10]. Thus, the multisymplectic formulation of these theories and, eventually, the development of their Lagrangian–Hamiltonian unified formalism are lines of further research.

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A Geometric elements

A.1 Levi-Civita symbol and generalized Kronecker delta

We denote the Levi-Civita symbol in $k$ indices by $\varepsilon_{i_1 \ldots i_k}$ and $\varepsilon^{i_1 \ldots i_k}$

$$\varepsilon_{i_1 \ldots i_k} = \begin{cases} 1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (1, \ldots, k), \\ -1 & \text{if } (i_1, \ldots, i_k) \text{ is an odd permutation of } (1, \ldots, k), \\ 0 & \text{in other case} \end{cases}$$

On the other hand, the generalized Kronecker delta \cite{23} in $k$ indices $\delta_{j_1 \ldots j_k}^{i_1 \ldots i_k}$ is given by

$$\delta_{j_1 \ldots j_k}^{i_1 \ldots i_k} = \begin{cases} 1 & \text{if } (i_1, \ldots, i_k) \text{ is an even permutation of } (j_1, \ldots, j_k), \\ -1 & \text{if } (i_1, \ldots, i_k) \text{ is an odd permutation of } (j_1, \ldots, j_k), \\ 0 & \text{in other case} \end{cases}$$

and can also be expressed as

$$\delta_{j_1 \ldots j_k}^{i_1 \ldots i_k} = \begin{vmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_1}^{i_k} \\ \vdots & \ddots & \vdots \\ \delta_{j_k}^{i_1} & \cdots & \delta_{j_k}^{i_k} \end{vmatrix}.$$ 

Both tensors are completely antisymmetric in their indices and they are seen to fulfill the following properties

1. $\varepsilon_{i_1 \ldots i_k i_{k+1} \ldots i_m} \varepsilon^{i_1 \ldots i_k j_{k+1} \ldots j_m} = k! \delta_{j_{k+1} \ldots j_m}^{i_{k+1} \ldots i_m}$.
2. For any tensor $a^{\nu_1 \ldots \nu_p}$,

$$\frac{1}{p!} \delta^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p} a^{\nu_1 \ldots \nu_p} = a^{[\nu_1 \ldots \nu_p]}.$$

3. If $A$ is a matrix with entries $a_{j_1}^{i}$,

$$\varepsilon_{i_1 \ldots i_m} a_{j_1}^{i_1} \ldots a_{j_m}^{i_m} = \det(A) \varepsilon_{j_1 \ldots j_m}.$$ 

Similar properties hold interchanging lower and upper indices.
A.2 Hodge star operator

Let $V$ be an $m$-dimensional real vector space and $\eta$ a non-degenerate bilinear symmetric form on $V$. For each $k \leq m$, we can define another non-degenerate bilinear and symmetric form $\hat{\eta}$ on $\Lambda^k V$ extending $\eta$ as the unique bilinear form such that on elementary $k$-vectors $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$,

$$\hat{\eta}(\alpha, \beta) := \det[\eta(\alpha_i, \beta_j)].$$

Given that $\Lambda^m V$ is one-dimensional, it follows that there are exactly two $m$-vectors $v$ fulfilling $\hat{\eta}(v, v) = 1$. Let $\omega$ be a preferred unit $m$-vector (observe that fixing such a vector amounts to choosing an orientation for $V$). Then, the star Hodge operator $\star : \Lambda^k V \to \Lambda^{n-k} V$ related to $\eta$ is defined by requiring

$$\alpha \wedge (\star \beta) = \hat{\eta}(\alpha, \beta) \omega.$$ 

It is customary to refer to $\star \beta$ as the Hodge dual of $\beta$.

Given an ordered basis $\{e_1, \ldots, e_m\}$ of $V$ such that $\omega = e_1 \wedge \cdots \wedge e_m$, it is clear that

$$\star(e_i_1 \wedge \cdots \wedge e_{ik}) = e_{ik+1} \wedge \cdots \wedge e_{im}$$

if and only if, $(i_1, \ldots, i_m)$ is an even permutation of $(1, \ldots, m)$.

Lemma 8 If $\beta = \beta^{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{ik}$ $(k \leq m)$. Then

$$\star \beta = \frac{1}{(m-k)!} \beta^{i_1 \cdots i_k} e_{i_1} \eta_{i_1 j_1} \cdots \eta_{i_k j_k} e^{j_1 \cdots j_m} e_{j_{k+1}} \wedge \cdots \wedge e_{j_m}.$$ 

Proof Let $\alpha = \alpha^{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then

$$\alpha \wedge (\star \beta) = \alpha^{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge \left[ \frac{1}{(m-k)!} \beta^{i_1 \cdots i_k} e_{i_1} \eta_{i_1 j_1} \cdots \eta_{i_k j_k} e^{j_1 \cdots j_m} e_{j_{k+1}} \wedge \cdots \wedge e_{j_m} \right]$$

$$= \frac{1}{(m-k)!} \alpha^{i_1 \cdots i_k} e_{i_1} \beta^{i_1 \cdots i_k} \eta_{i_1 j_1} \cdots \eta_{i_k j_k} e^{j_1 \cdots j_m} e_{j_{k+1}} \wedge \cdots \wedge e_{j_m}$$

$$= \frac{1}{(m-k)!} \alpha^{i_1 \cdots i_k} e_{i_1} \beta^{i_1 \cdots i_k} \eta_{i_1 j_1} \cdots \eta_{i_k j_k} e^{j_1 \cdots j_m} e_{j_{k+1}} \wedge \cdots \wedge e_{j_m}$$

$$= \alpha^{i_1 \cdots i_k} e_{i_1} \beta^{i_1 \cdots i_k} \eta_{i_1 j_1} \cdots \eta_{i_k j_k} e^{j_1 \cdots j_k} e_1 \wedge \cdots \wedge e_{j_m}$$

$$= \hat{\eta}(\alpha, \beta) e_1 \wedge \cdots \wedge e_m.$$ 

$\square$
A.3 Cartan decomposition of $\mathfrak{gl}(m)$

Let us use the matrix $\eta$ in order to decompose $\mathfrak{gl}(m)$; in order to get it, consider the involution

$$\theta : \mathfrak{gl}(m, \mathbb{C}) \to \mathfrak{gl}(m, \mathbb{C}) : A \mapsto -\eta A^\dagger \eta;$$

the eigenspaces of $\theta$, associated to the eigenvalues $\pm 1$, induce the decomposition

$$\mathfrak{gl}(m, \mathbb{C}) = u(m-s,s) \oplus \mathfrak{s}(m-s,s)$$

where $s$ is the signature of $\eta$. It should be noted that

$$[u(m-s,s), u(m-s,s)] \subset u(m-s,s),$$

$$[\mathfrak{s}(m-s,s), \mathfrak{s}(m-s,s)] \subset u(m-s,s),$$

and that $\mathfrak{s}(m-s,s)$ is an invariant subspace under the adjoint action of $u(m-s,s)$. This decomposition descends to $\mathfrak{gl}(m) \subset \mathfrak{gl}(m, \mathbb{C})$, namely

$$\mathfrak{gl}(m) = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{k} := u(m-s,s) \cap \mathfrak{gl}(m), \quad \mathfrak{p} := \mathfrak{s}(m-s,s) \cap \mathfrak{gl}(m).$$

Denoting $s := \theta|_{\mathfrak{gl}(m)}$, we have that $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the eigenspace corresponding to the eigenvalue $+1$ (resp. $-1$) for $s$. The projectors in every of these eigenspaces become

$$\pi_{\mathfrak{k}}(A) := \frac{1}{2} \left( A - \eta A^T \eta \right), \quad \pi_{\mathfrak{p}}(A) := \frac{1}{2} \left( A + \eta A^T \eta \right).$$

Given $N$ a manifold and $\gamma \in \Omega^p(N, \mathfrak{gl}(m))$, we define

$$\gamma_k := \pi_{\mathfrak{k}} \circ \gamma, \quad \gamma_p := \pi_{\mathfrak{p}} \circ \gamma.$$

If $\gamma = \gamma^i_j E^i_j$ is the expression of $\gamma$ in terms of the canonical basis of $\mathfrak{gl}(m)$, then we have

$$(\gamma_k)^i_j = \frac{1}{2} \left( \gamma^i_j - \eta_{jp} \gamma^p_q \eta^{qi} \right) \quad \text{and} \quad (\gamma_p)^i_j = \frac{1}{2} \left( \gamma^i_j + \eta_{jp} \gamma^p_q \eta^{qi} \right).$$

These previous considerations are useful when dealing with $\mathfrak{gl}(m)$-valued forms.
A.4 Vector-valued and Lie-algebra-valued differential forms

Let $U$, $V$ and $W$ be finite dimensional real vector spaces and $M$ a smooth manifold and let $B : U \times V \to W$ be a bilinear map. If $\alpha \in \Omega^k(M, U)$ and $\beta \in \Omega^l(M, V)$ are differential forms with values in $U$ and $V$, respectively; we can define a new differential form with values in $W$, $B(\alpha \wedge \beta) \in \Omega^{k+l}(M, W)$, as

$$B(\alpha \wedge \beta)(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) B(\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})).$$

We are interested in a series of particular instances of this definition:

**Pairing:** Consider $U = V^*$, $W = \mathbb{R}$ and $B = \langle \cdot, \cdot \rangle$ the natural pairing between $V^*$ and $V$. In this particular case we denote

$$B(\alpha \wedge \beta) = \langle \alpha \wedge \beta \rangle.$$

**Linear representation:** Consider $U = \text{End}(V)$, $W = V$, and denote $B$ the natural action of $\text{End}(V)$ on $V$. In this particular case we denote

$$B(\alpha \wedge \beta) = \alpha \wedge \beta.$$

**Lie bracket:** Consider $U = V = W = g$, where $g$ is a Lie algebra and $B = [\cdot, \cdot]$ is the related Lie bracket. In this particular case we denote

$$B(\alpha \wedge \beta) = [\alpha \wedge \beta].$$

**Wedge product:** Consider $U = V$, $W = \Lambda^2 V$ and let $B = \cdot \wedge \cdot$ be the usual wedge product. In this particular case we denote

$$B(\alpha \wedge \beta) = \alpha \wedge \beta.$$ 

**Constant linear map:** Consider a zero form $\alpha$ assigning to each $x \in M$ the same linear map $A \in \text{Lin}(V, W)$, in this particular case we denote

$$B(\alpha \wedge \beta) = A(\beta).$$

Similar definitions can be given if $V$ is a vector bundle over $M$ and $\alpha$ and $\beta$ are vector bundle valued differential forms.

References

1. Bergvelt, M.J., De Kerf, E.A.: The Hamiltonian structure of Yang–Mills theories and instantons I. Physica A Stat. Mech. Appl. 139(1), 101–124 (1986). https://doi.org/10.1016/0378-4371(86)90007-5
2. Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., Griffiths, P.A.: Exterior Differential Systems. Springer, Berlin (1991).

3. Cantrijn, F., Vankerschaver, J.: The Skinner-Rusk approach for vakonomic and nonholonomic field theories. In: Differential Geometric Methods in Mechanics and Field Theory, pp. 1–14 (2007). http://hdl.handle.net/1854/LU-375178

4. Capozziello, S., De Laurentis, M.: Extended theories of gravity. Phys. Rep. **509**(4–5), 167–321 (2011). https://doi.org/10.1016/j.physrep.2011.09.003

5. Capriotti, S.: Differential geometry, Palatini gravity and reduction. J. Math. Phys. **55**(1), 012902 (2014). https://doi.org/10.1063/1.4862855

6. Capriotti, S.: Unified formalism for Palatini gravity. Int. J. Geom. Methods Mod. Phys. **15**(3), 1850044 (2018). https://doi.org/10.1142/S0219887818500445

7. Cariñena, J.F., Crampin, M., Ibort, L.A.: On the multisymplectic formalism for first order field theories. Diff. Geom. Appl. **1**, 345–374 (1991). https://doi.org/10.1016/0926-2245(91)90013-Y

8. Castrillón-López, M., Muñoz-Masqué, J.: The geometry of the bundle of connections. Math. Z. **236**(4), 797–811 (2001). https://doi.org/10.1007/PL00004852

9. Castrillón-M., Muñoz-Masqué, J., Rosado, M.E.: First-order equivalent to Einstein–Hilbert Lagrangian. J. Math. Phys. **55**(8), 082501 (2014). https://doi.org/10.1063/1.4890555

10. Celada, M., González, D., Montesinos, M.: BF gravity. Class. Quantum Grav. **33**, 213001 (2016). https://doi.org/10.1088/0264-9381/33/21/213001

11. Cendra, H., Capriotti, S.: Cartan algorithm and Dirac constraints for Griffiths variational problems. arXiv:1309.4080 [math-ph] (2013)

12. Cremaschini, C., Tessarotto, M.: Manifest covariant Hamiltonian theory of General Relativity. App. Phys. Res. **8**(2), 60–81 (2016). https://doi.org/10.5539/apr.v8n2p60

13. Dadhich, N., Pons, J.M.: On the equivalence of the Einstein–Hilbert and the Einstein–Palatini formulations of General Relativity for an arbitrary connection. Gen. Relativ. Gravit. **44**(9), 2337–2352 (2012). https://doi.org/10.1007/s10714-012-1393-9

14. Dadhich, N., Pons, J.M.: Consistent Levi-Civita truncation uniquely characterizes the Lovelock Lagrangians. Phys. Lett. B **705**(1–2), 139–142 (2011). https://doi.org/10.1016/j.physletb.2011.09.108

15. de León, M., Marrero, J.C., Martín de Diego, D.: A new geometric setting for classical field theories. Banach Center Pub. **59**, 189–209 (2003). https://doi.org/10.4064/bc59-0-10

16. de León, M., Marín-Solano, J., Marrero, J.C.: A geometrical approach to classical field theories: a constraint algorithm for singular theories. In: Tamassi, L., Szenthe, J. (eds.) New Developments in Differential Geometry (Debrecen, 1994). Mathematics and Its Applications, vol. 350, pp. 291–312. Springer, Berlin (1996)

17. de León, M., Marín-Solano, J., Marrero, J.C., Muñoz-Lecanda, M.C., Román-Roy, N.: Premultisymplectic constraint algorithm for field theories. Int. J. Geom. Meth. Mod. Phys. **2**(5), 839–871 (2005). https://doi.org/10.1142/S0219887805000880

18. de León, M., Salgado, M., Vilarinho, S.: Methods of differential geometry in classical field theories: k-symplectic and k-cosymplectic approaches. World Sci. (2016). https://doi.org/10.1142/9693

19. Deser, S., Franklin, J.: Canonical analysis and stability of Lanczos-Lovelock gravity. Class. Quantum Gravit. **29**, 072001 (2012). https://doi.org/10.1088/0264-9381/29/7/072001

20. Echeverría-Enríquez, A., Muñoz-Lecanda, M.C., Román-Roy, N.: Geometry of Lagrangian first-order classical field theories. Fortschr. Phys. **44**, 235–280 (1996). https://doi.org/10.1002/prop.2190440304

21. Echeverría-Enríquez, A., López, C., Marín-Solano, J., Muñoz-Lecanda, M.C., Román-Roy, N.: Lagrangian–Hamiltonian unified formalism for field theory. J. Math. Phys. **45**(1), 360–380 (2004). https://doi.org/10.1063/1.1628384

22. Ferraro, R.: f(R) and f(T) theories of modified gravity. AIP Conf. Proc. **1471**, 103–110 (2012). https://doi.org/10.1063/1.4756821

23. Franke, T.: The Geometry of Physics: An Introduction. Cambridge University Press, Cambridge (2001)

24. Gaset, J., Román-Roy, N.: Order reduction, projectability and constraints of second-order field theories and higher-order mechanics. Rep. Math. Phys. **78**(3), 327–337 (2016). https://doi.org/10.1016/j.9490047

25. Gaset, J., Román-Roy, N.: Multisymplectic unified formalism for Einstein–Hilbert gravity. J. Math. Phys. **59**(3), 032502 (2018). https://doi.org/10.1063/1.4998526

26. Gaset, J., Román-Roy, N.: New multisymplectic approach to the Metric-Affine (Einstein–Palatini) action for gravity. J. Geom. Mech. **11**(3), 361–396 (2019). https://doi.org/10.3934/jgm.2019019
27. Giachetta, G., Mangiarotti, L., Sardanashvily, G.: New Lagrangian and Hamiltonian Methods in Field Theory. World Scientific Publishing Co., Inc., River Edge (1997)
28. Gotay, M.J.: An exterior differential system approach to the Cartan form, symplectic geometry and mathematical physics. Actes du Colloque de Géométrie Symplectique et Physique Mathématique en l’honneur de Jean-Marie Souriau (Aix-en-Provence, France, 1990), pp. 160–188 (1991)
29. Gotay, M.J., Isenberg, J., Marsden, J.E., Montgomery, R.: Momentum maps and classical relativistic fields. I. Covariant theory. arXiv:physics/9801019 [math-ph] (2004)
30. Griffiths, P.: Exterior Differential Systems and the Calculus of Variations. Progress in Mathematics. Birkhauser, Basel (1982)
31. Günther, C.: The polysymplectic Hamiltonian formalism in field theory and calculus of variations. I. The local case. J. Differ. Geom. 25(1), 23–53 (1987). https://doi.org/10.4310/jdg/1214440723
32. Kanatchikov, I.V.: Canonical structure of classical field theory in the polymomentum phase space. Rep. Math. Phys. 41, 49–90 (1998). https://doi.org/10.1016/S0034-4877(98)80182-1
33. Kanatchikov, I.V.: De Donder–Weyl Hamiltonian formulation and precanonical quantization of Vielbein gravity. J. Phys. Conf. Ser. 442, 012041 (2013). https://doi.org/10.1088/1742-6596/442/1/012041
34. Kanatchikov, I.V.: On precanonical quantization of gravity. Nonlinear Phenom. Complex Syst. (NPCS) 17, 372–376 (2014)
35. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry. Wiley, New York (1963)
36. Krupka, D.: Introduction to Global Variational Geometry. Atlantis Studies in Variational Geometry. Atlantis Press, Paris (2015). https://doi.org/10.2991/978-94-6239-073-7
37. Krupka, D., Stepankova, O.: On the Hamilton form in second order calculus of variations. In: Proceedings of International Meeting on Geometry and Physics, pp. 85–101. Florence 1982, Pitagora, Bologna (1983)
38. Lewis, A.D.: Notes on energy shaping. In: Proceedings of 43rd IEEE Conference on Decision and Control (CDC), Nassau, 2004, vol. 5, pp. 4818–4823. https://doi.org/10.1109/CDC.2004.1429552
39. Lovelock, D.: Divergence-free tensorial concomitants. Aeq. Math. 4(1–2), 127–138 (1970). https://doi.org/10.1007/BF01817753
40. Lovelock, D.: The Einstein tensor and its generalizations. J. Math. Phys. 12(3), 498–501 (1971). https://doi.org/10.1063/1.1665613
41. Padmanabhan, T.: Some aspects of field equations in generalized theories of gravity. Phys. Rev. D 84(12), 124041 (2011). https://doi.org/10.1103/PhysRevD.84.124041
42. Padmanabhan, T., Kothawala, D.: Lanczos-Lovelock models of gravity. Phys. Rep. 531(3), 115–171 (2013). https://doi.org/10.1016/j.physrep.2013.05.007
43. Prieto-Martínez, P.D., Román-Roy, N.: A new multisymplectic unified formalism for second order classical field theories. J. Geom. Mech. 7(2), 203–253 (2015). https://doi.org/10.3934/jgm.2015.7.203
44. Román-Roy, N.: Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories. Symm. Integ. Geom. Methods Appl. (SIGMA) 5, 100 (2009). https://doi.org/10.3842/SIGMA.2009.100
45. Rosado, M.E., Muñoz-Masqué, J.: Second-order Lagrangians admitting a first-order Hamiltonian formalism. J. Ann. Mat. 197(2), 357–397 (2018). https://doi.org/10.1007/s10231-017-0683-y
46. Rovelli, C.: A note on the foundation of relativistic mechanics. II: Covariant Hamiltonian General Relativity. In: García-Compean, H., Mielnik, B., Montesinos, M., Przanowski, M. (eds.) Topics in Mathematical Physics. General Relativity and Cosmology, vol. 397. World Scientific, Singapore (2006)
47. Saunders, D.J.: The Geometry of Jet Bundles. London Mathematical Society. Lecture Notes Series, vol. 142. Cambridge University Press, Cambridge (1989)
48. Skinner, R., Rusk, R.: Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$. J. Math. Phys. 24(11), 2589–2594 (1983). https://doi.org/10.1063/1.525654
49. Vey, D.: Multisymplectic formulation of Vielbein gravity. De Donder–Weyl formulation, Hamiltonian $(n−1)$-forms. Class. Quantum Gravity 32(9), 095005 (2015). https://doi.org/10.1088/0264-9381/32/9/095005

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