ON ALGEBRAIC DIVISION RINGS OVER HENSELIAN FIELDS OF FINITE ABSOLUTE BRAUER $p$-DIMENSIONS AND RESIDUALLY ARITHMETIC TYPE

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Abstract. Let $(K, v)$ be a Henselian field with a residue field $\bar{K}$ and a value group $v(K)$, and let $\mathbb{P}$ be the set of prime numbers. This paper finds conditions on $K$, $v(K)$ and $\bar{K}$ under which every algebraic associative central division $K$-algebra $R$ contains a central $K$-subalgebra $\bar{R}$ decomposable into a tensor product of central $K$-subalgebras $R_p$, $p \in \mathbb{P}$, of finite $p$-primary dimensions $[R_p : K]$, such that each finite-dimensional $K$-subalgebra $\Delta$ of $R$ is isomorphic to a $K$-subalgebra $\Delta$ of $\bar{R}$.

1. Introduction

All algebras considered in this paper are assumed to be associative with a unit. Let $E$ be a field, $E_{\text{sep}}$ its separable closure, $F_E(E)$ the set of finite extensions of $E$ in $E_{\text{sep}}$, $\mathbb{P}$ the set of prime numbers, and for each $p \in \mathbb{P}$, let $E(p)$ be the maximal $p$-extension of $E$ in $E_{\text{sep}}$, i.e. the composite of all finite Galois extensions of $E$ in $E_{\text{sep}}$ whose Galois groups are $p$-groups. It is known, by the Wedderburn-Artin structure theorem (cf. [22], Theorem 2.1.6), that an Artinian $E$-algebra $\mathbb{A}$ is simple if and only if it is isomorphic to the full matrix ring $M_n(\mathbb{D})$ of order $n$ over a division $E$-algebra $\mathbb{D}$. This holds, $n$ is uniquely determined by $\mathbb{A}$, and so is $\mathbb{D}$, up-to isomorphism; $\mathbb{D}$ is called an underlying division $E$-algebra of $\mathbb{A}$. The $E$-algebras $\mathbb{A}$ and $\mathbb{D}$ share a common centre $Z(\mathbb{A})$; we say that $\mathbb{A}$ is a central $E$-algebra if $Z(\mathbb{A}) = E$.

Denote by $\text{Br}(E)$ the Brauer group of $E$, by $s(E)$ the class of finite-dimensional central simple algebras over $E$, and by $d(E)$ the subclass of division algebras $D \in s(E)$. For each $A \in s(E)$, let $\text{deg}(A)$, $\text{ind}(A)$ and $\exp(A)$ be the degree, the Schur index and the exponent of $A$, respectively.

It is well-known (cf. [33], Sect. 14.4) that $\exp(A)$ divides $\text{ind}(A)$ and shares with it the same set of prime divisors; also, $\text{ind}(A) \mid \text{deg}(A)$, and $\deg(A) = \text{ind}(A)$ if and only if $A \in d(E)$. Note that $\text{ind}(B_1 \otimes_E B_2) = \text{ind}(B_1)\text{ind}(B_2)$ whenever $B_1, B_2 \in s(E)$ and $\text{g.c.d.}\{\text{ind}(B_1), \text{ind}(B_2)\} = 1$; equivalently, $B_1' \otimes_E B_2' \in d(E)$, if $B_1' \in d(E)$, $j = 1, 2$, and $\text{g.c.d.}\{\text{deg}(B_1'), \text{deg}(B_2')\} = 1$ (see [33], Sect. 13.4). Since $\text{Br}(E)$ is an abelian torsion group, and $\text{ind}(A)$, $\exp(A)$ are invariants both of $A$ and its equivalence class $[A] \in \text{Br}(E)$, these results indicate that the study of the restrictions on the pairs $\text{ind}(A)$, $\exp(A)$, $A \in s(E)$, reduces to the special case of $p$-primary pairs, for an arbitrary fixed $p \in \mathbb{P}$. The Brauer $p$-dimensions $\text{Br}_p(E)$, $p \in \mathbb{P}$, contain essential

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information on these restrictions. We say that $\text{Brd}_p(E) = n < \infty$, for a given $p \in \mathbb{P}$, if $n$ is the least integer $\geq 0$, for which $\text{ind}(A_p) \mid \exp(A_p)^n$ whenever $A_p \in s(E)$ and $[A_p]$ lies in the $p$-component $\text{Br}(E)_p$ of $\text{Br}(E)$; if no such $n$ exists, we put $\text{Brd}_p(E) = \infty$. For instance, $\text{Brd}_p(E) \leq 1$, for all $p \in \mathbb{P}$, if and only if $E$ is a stable field, i.e. $\deg(D) = \exp(D)$, for each $D \in d(E)$: $\text{Brd}_p(E) = 0$, for some $p' \in \mathbb{P}$, if and only if $\text{Br}(E)_{p'}$ is trivial.

The absolute Brauer $p$-dimension of $E$ is defined to be the supremum $\text{abrd}_p(E)$ of $\text{Brd}_p(R) : R \in \text{Fe}(E)$. It is a well-known consequence of Albert-Hochschild’s theorem (cf. [31], Ch. II, 2.2) that $\text{abrd}_p(E) = 0$, $p \in \mathbb{P}$, if and only if $E$ is a field of dimension $\leq 1$, i.e. $\text{Br}(R) = \{0\}$, for every finite extension $R/E$. When $E$ is a perfect field, we have $\dim(E) \leq 1$, if and only if the absolute Galois group $G_E = \hat{G}(E_{\text{sep}}/E)$ is a projective profinite group, in the sense of [34]. Also, class field theory shows that $\text{Brd}_p(E) = \text{abrd}_p(E) = 1$, $p \in \mathbb{P}$, if $E$ is a global or local field.

It follows from well-known general properties of the basic types of algebraic extensions (cf. [27], Ch. V, Sects. 4 and 6) that if $E$ is a field of characteristic $q$, then $\text{Brd}_p(E') \leq \text{abrd}_p(E)$, for any algebraic extension $E'/E$, and any $p \in \mathbb{P}$ not equal to $q$ (see also [8], (1.2), and [33], Sect. 13.4). When $q > 0$, we restrict our considerations to the special case where $E$ is a virtually perfect field, i.e. $E$ is a finite extension of its subfield $E^q = \{\lambda^q : \lambda \in E\}$. Algebraic extensions of $E$ preserve the property of being virtually perfect; this is implied by the following result (cf. [27], Ch. V, Sect. 6):

\[(1.1) \ [E' : E^q] = [E : E^q], \text{ for every finite extension } E'/E.\]

Statement (1.1) enables one to deduce from Albert’s theorem (cf. [1], Ch. VII, Theorem 28) and the former conclusion of [10], Lemma 4.1, that if $[E : E^q] = q^\kappa$, where $\kappa \in \mathbb{N}$, then $\text{Brd}_q(E') \leq \kappa$. In order to present the main results of our research in a unified way, we say that $E$ is virtually perfect also in the case of $q = 0$. It is easily verified (cf. [7], Proposition 2.4) that every virtually perfect field $K$ of finite absolute Brauer $p$-dimensions, for all $p \in \mathbb{P}$, is an FC-field, in the sense of [5] and [7]. As shown in [5] (see also [2]), this sheds light on the structure of locally finite-dimensional (abbr., LFD) central division $K$-algebras, as follows:

**Proposition 1.1.** Let $K$ be a virtually perfect field with $\text{abrd}_p(K) < \infty$, for each $p \in \mathbb{P}$, and suppose that $R$ is a central division LFD-algebra over $K$, i.e. finitely-generated $K$-subalgebras of $R$ are finite-dimensional. Then $R$ possesses a $K$-subalgebra $\bar{R}$ with the following properties:

(a) $\bar{R}$ is $K$-isomorphic to the tensor product $\otimes_{p \in \mathbb{P}} R_p$, where $\otimes = \otimes_K$ and $R_p \in d(K)$ is a $K$-subalgebra of $R$ of $p$-primary degree $p^{k(p)}$, for each $p$;

(b) Every $K$-subalgebra $\Delta$ of $R$ of at most countable dimension is embeddable in $\bar{R}$ as a $K$-subalgebra; hence, for each $p \in \mathbb{P}$, $k(p)$ is the greatest integer for which there exists $r_p \in R$ of degree $[K(r_p) : K]$ divisible by $p^{k(p)}$;

(c) $\bar{R}$ is isomorphic to $R$ if the dimension $[R : K]$ is at most countable.

By the main result of [5], the conclusion of Proposition 1.1 remains valid whenever $K$ is an FC-field; in particular, this holds if $\text{char}(K) = q$, $\text{abrd}_p(K) < \infty$, $p \in \mathbb{P} \setminus \{q\}$, and in case $q > 0$, there exists $\mu \in \mathbb{N}$, such that $\text{Brd}_q(K') \leq \mu$, for every finite extension $K'/K$. As already noted, the latter
condition is satisfied if \( q > 0 \) and \([K : K^q] = q^n\). It is worth mentioning, however, that the existence of an upper bound \( \mu \) as above is sometimes possible in case \([K : K^q] = \infty\). More precisely, for each \( q \in \mathbb{P} \), there are fields \( E_n, n \in \mathbb{N} \), with the following properties, for each \( n \) (see Proposition 3.9):

\[
(1.2) \quad \text{char}(E_n) = q, \quad [E_n : E_n^q] = \infty, \quad \text{Br}_p(E_n) = \text{abrd}_p(E_n) = [n/2], \quad \text{for all} \quad p \in \mathbb{P} \setminus \{q\}, \quad \text{and} \quad \text{Br}_p(E_n') = n - 1, \quad \text{for every finite extension} \quad E_n'/E_n.
\]

In particular, FC-fields of characteristic \( q > 0 \) need not be virtually perfect. Therefore, it should be pointed out that if \( F/E \) is a finitely-generated extension of transcendence degree \( \nu > 0 \), where \( E \) is a field of characteristic \( q > 0 \), then \( \text{Br}_q(F) < \infty \) if and only if \([E : E^q] < \infty\) [10], Theorem 2.2; when \([E : E^q] = q^\mu < \infty\), we have \([F : F^q] = q^{\nu + \mu}\), which means that \( \text{abrd}_q(F) \leq \nu + u \). This attracts interest in the following open problem:

**Problem 1.2.** Let \( E \) be a field with \( \text{abrd}_p(E) < \infty \), for some \( p \in \mathbb{P} \) different from \( \text{char}(E) \). Find whether \( \text{abrd}_p(F) < \infty \), for any finitely-generated transcendental field extension \( F/E \).

Global fields and local fields are virtually perfect (cf. [17], Example 4.1.3) of absolute Brauer \( p \)-dimensions one, for all \( p \in \mathbb{P} \), so they satisfy the conditions of Proposition 1.1. In view of a more recent result of Matzri [28], Proposition 1.1 also applies to any field \( K \) of finite Tsen rank, that is, to any field \( K \) of type \( C_m \), in the sense of Lang, for some integer \( m \geq 0 \). By type \( C_m \), we mean that every nonzero homogeneous polynomial \( f \) of degree \( d \) and with coefficients from \( K \) has a nontrivial zero over \( K \), provided that \( f \) depends on \( n > d^m \) algebraically independent variables over \( K \). For example, algebraically closed fields are of type \( C_0 \); finite fields are of type \( C_1 \), by the Chevalley-Warning theorem (cf. [20], Theorem 6.2.6). Complete discrete valued fields with algebraically closed residue fields are also of type \( C_1 \) (Lang’s theorem, see [31], Ch. II, 3.3), and so are pseudo algebraically closed (abbr., PAC) fields of characteristic zero (cf. [19], Remark 21.3.7); in characteristic \( q > 0 \), perfect PAC fields are of type \( C_2 \) [19], Theorem 21.3.6.

The present research is essentially a continuation of [7]. Since the class of fields of finite Tsen ranks consists of virtually perfect fields and it is closed under the formation of both field extensions of finite transcendency degree (by the Lang-Nagata-Tsen theorem [31]) and formal Laurent power series fields in one variable [21], the above-noted result of [28] significantly extends the scope of applicability of the main result of [5]. It allows to view Proposition 1.1 as a step in the development of a theory of noncommutative algebraic division rings, which is able to reduce the study of algebraic central division algebras over finitely-generated extensions \( E \) of fields \( E_0 \) distinguished by their arithmetic, algebraic, diophantine, topological, or other properties to the study of \( E \)-algebras \( D \in d(E) \) (results illustrating this can be found, e.g., in [7], Sect. 5). In view of Problem 1.2, it is presently unknown whether one may take as \( E_0 \) any virtually perfect field with \( \text{abrd}_p(E_0) < \infty \), \( p \in \mathbb{P} \).

The motivation for our research also comes from the following conjectural generalization of Proposition 1.1 and [7], Theorem 4.1, in the spirit of the primary tensor product decomposition theorem for finite-dimensional central division algebras over arbitrary fields (see, e.g., [33], Sect. 14.4):
Conjecture 1.3. Assume that $K$ is a field satisfying the conditions of Proposition 1.1, and let $R$ be an algebraic central division $K$-algebra. Then $R$ possesses a $K$-subalgebra $\tilde{R}$ with the following properties:

(a) $\tilde{R}$ is $K$-isomorphic to the tensor product $\otimes_{p \in \mathbb{P}} R_p$, where $\otimes = \otimes_K$ and $R_p \in d(K)$ is a $K$-subalgebra of $R$ of $p$-primary degree $p^{k(p)}$, for each $p \in \mathbb{P}$;

(b) Every $K$-subalgebra $\Delta$ of $R$, which is LFD of at most countable dimension, is embeddable in $\tilde{R}$ as a $K$-subalgebra; in particular, $K$ equals the centralizer $C_R(\tilde{R}) = \{ c \in R : \tilde{c} r = \tilde{r} c, \tilde{r} \in \tilde{R} \}$, and for each $p \in \mathbb{P}$, $k(p)$ is the maximal integer for which there is $\rho_p \in R$ such that $p^{k(p)}$ divides $[K(\rho_p) : K]$.

Clearly, one may consider the main aspects of Conjecture 1.3 restricting to the case of $[R : K] = \infty$. For reasons clarified in the sequel, in this paper we assume further that $R$ belongs to the class of $K$-algebras of linearly bounded degree, in the sense of Amitsur [2]. This class is defined as follows:

Definition 1. An algebraic algebra $\Psi$ over a field $F$ is said to be an algebra of linearly (or locally) bounded degree (briefly, an LBD-algebra), if the following condition holds, for any finite-dimensional $F$-subspace $V$ of $\Psi$: there exists $n(V) \in \mathbb{N}$, such that $[F(v) : F] \leq n(V)$, for each $v \in V$.

It is presently unknown whether every algebraic associative division algebra $\Psi$ over a field $F$ is LFD. This problem has been posed by Kurosh as a division ring-theoretic analog of the Burnside problem for torsion groups [20]. Clearly, if the stated problem is solved affirmatively, then Conjecture 1.3 will turn out to be a restatement of Proposition 1.1. The solution will be the same if and only if the following two questions have positive answers:

Questions 1.4. Let $F$ be a field.

(a) Find whether algebraic division $F$-algebras are LBD-algebras over $F$.

(b) Find whether division LBD-algebras over $F$ are LFD.

Although Questions 1.4 (a) and (b) are closely related to the Kurosh problem, each of them makes interest in its own right. For example, the main results of [7] indicate that an affirmative answer to Question 1.4 (a) would prove Conjecture 1.3 in the special case where $K$ is a virtually perfect field which is of arithmetic type, in the following sense:

Definition 2. A field $K$ is said to be of arithmetic type, if $\text{abrd}_p(K)$ is finite and $\text{abrd}_p(K(p)) = 0$, for each $p \in \mathbb{P}$.

It is a well-known consequence of class field theory that every algebraic extension of a global field $K_0$ is a field of arithmetic type. For the present research, however, it is more significant that the answer to Question 1.4 (a) is positive when $F$ is a noncountable field. More generally, by Amitsur’s theorem [2], algebraic associative algebras over such $F$ are LBD-algebras. Furthermore, it follows from Amitsur’s theorem that if $A$ is an arbitrary LBD-algebra over any field $E$, then the tensor product $A \otimes_E E'$ is an LBD-algebra over any extension $E'$ of $E$ (see [2]). These results are repeatedly used in the present paper for proving Conjecture 1.3 under the hypothesis that $K$ lies in some classes of Henselian fields of finite absolute Brauer $p$-dimensions and with virtually perfect residue fields of arithmetic type.
2. Statement of the main result

Assume that $K$ is a virtually perfect field with abrd$_p(K) < \infty$, for all $p \in \mathbb{P}$, and let $R$ be an algebraic central division $K$-algebra. Evidently, if $R$ possesses a $K$-subalgebra $\tilde{R}$ with the properties described by Conjecture 1.3, then there is a sequence $k(p)$, $p \in \mathbb{P}$, of integers $\geq 0$, such that $p^{k(p)+1}$ does not divide $[K(r): K]$, for any $r \in R$, $p \in \mathbb{P}$. The existence of such a sequence is guaranteed if $R$ is an LBD-algebra over $K$ (cf. [7], Lemma 3.9). When $k(p) = k(p)_R$ is the minimal integer satisfying the stated condition, it is called a $p$-power of $R/K$. In this setting, the notion of a $p$-splitting field of $R/K$ is defined as follows:

**Definition 3.** Let $K'$ be a finite extension of $K$, $R'$ the underlying (central) division $K'$-algebra of $R \otimes_K K'$, and $\gamma(p)$ the integer singled out by the Wedderburn-Artin $K'$-isomorphism $R \otimes_K K' \cong M_{\gamma(p)}(R')$. We say that $K'$ is a $p$-splitting field of $R/K$ if $p^{k(p)}$ divides $\gamma(p)$.

Note that the class of $p$-splitting fields of a central division LBD-algebra $R$ over a virtually perfect field $K$ with abrd$_p(K) < \infty$, $p' \in \mathbb{P}$, is closed under the formation of finite extensions. Indeed, it is well-known (cf. [22], Lemma 4.1.1) that $R \otimes_K K'$ is a central simple $K'$-algebra, for any field extension $K'/K$. This algebra is a left (and right) vector space over $R$ of dimension equal to $[K': K]$, which implies it is Artinian whenever $[K': K]$ is finite. Since $R \otimes_K K_2$ and $(R \otimes_K K_1) \otimes_K K_2$ are isomorphic $K_2$-algebras, for any tower of field extensions $K \subseteq K_1 \subseteq K_2$ (cf. [23], Sect. 9.4, Corollary (a)), these observations enable one to deduce our assertion about $p$-splitting fields of $R/K$ from the Wedderburn-Artin theorem (and well-known properties of tensor products of matrix algebras, see [23], Sect. 9.3, Corollary b). Further results on $k(p)$ and the $p$-power of the underlying division $K'$-algebra of $R \otimes_K K'$, obtained in the case where $K'/K$ is a finite extension, are presented at the beginning of Section 4 (see Lemma 4.1). They have been proved in [7], Sect. 3, under the extra hypothesis that $\dim(K_{sol}) \leq 1$, where $K_{sol}$ is the compositum of finite Galois extensions of $K$ in $K_{sep}$ with solvable Galois groups. These results partially generalize well-known facts about finite-dimensional central division algebras over arbitrary fields, leaving open the question of whether the validity of the derived information depends on the formulated hypothesis (see Remark 4.2).

The results of Section 4, combined with Amitsur’s theorem referred to in Section 1, form the basis of the proof of the main results of the present research. Our proof also relies on the theory of Henselian fields and their finite-dimensional division algebras (cf. [23]). Taking into consideration the generality of Amitsur’s theorem, we recall that the class $\mathcal{HNF}$ of Henselian noncountable fields contains every maximally complete field, i.e. any nontrivially valued field $(K, v)$ which does not admit a valued proper extension with the same value group and residue field. For instance, $\mathcal{HNF}$ contains the generalized formal power series field $K_0((\Gamma))$ over a field $K_0$, where $\Gamma$ is a nontrivial ordered abelian group, and $v$ is the standard valuation of $K_0((\Gamma))$ trivial on $K_0$ (see [17], Example 4.2.1 and Theorem 18.4.1). Moreover, for each $m \in \mathbb{N}$, $\mathcal{HNF}$ contains every complete $m$-discretely valued...
field with respect to its standard \(\mathbb{Z}^m\)-valued valuation, where \(\mathbb{Z}^m\) is viewed as an ordered abelian group by the inverse-lexicographic ordering.

By a complete 1-discretely valued field, we mean a complete discrete valued field, and when \(m \geq 2\), by definition, a complete \(m\)-discretely valued field with an \(m\)-th residue field \(K_0\) means a field \(K_m\) which is complete with respect to a discrete valuation \(v_m\), such that the residue field \(\tilde{K}_m := K_{m-1}\) of \((K_m, v_m)\) is itself a complete \((m-1)\)-discretely valued field with an \((m-1)\)-th residue field \(K_0\). If \(m \geq 2\) and \(\omega_{m-1}\) is the standard \(\mathbb{Z}^{m-1}\)-valued valuation of \(K_{m-1}\), then the composite valuation \(\omega_m = \omega_{m-1} * v_m\) is the standard \(\mathbb{Z}^m\)-valued valuation of \(K_m\). It is known that \(\omega_m\) is Henselian (cf. [13], Proposition A.15) and \(K_0\) equals the residue field of \((K_m, \omega_m)\). This applies to the important special case where \(K_0\) is a finite field, i.e. \(K_m\) is an \(m\)-dimensional local field, in the sense of Kato and Parshin.

The purpose of this paper is to prove Conjecture [13] for two types of Henselian fields. Our first main result can be stated as follows:

**Theorem 2.1.** Let \(K = K_m\) be a complete \(m\)-discretely valued field with a virtually perfect \(m\)-th residue field \(K_0\), for some integer \(m > 0\), and let \(R\) be an algebraic central division \(K\)-algebra. Suppose that \(\text{char}(K_0) = q\) and \(K_0\) is of arithmetic type. Then \(R\) possesses a \(K\)-subalgebra \(\hat{R}\) with the properties claimed by Conjecture [13].

When \(\text{char}(K_m) = \text{char}(K_0)\), \((K_m, \omega_m)\) is isomorphic to the iterated formal Laurent power series field \(\hat{K}_m := \hat{K}_0((X_1)) \ldots ((X_m))\) in \(m\) variables, considered with its standard \(\mathbb{Z}^m\)-valued valuation, say \(\hat{w}_m\), acting trivially on \(K_0\); in particular, this is the case where \(\text{char}(K_0) = 0\). It is known that \((K_m, \hat{w}_m)\) is maximally complete (cf. [17], Sect. 18.4). As \(K_0\) is a virtually perfect field, whence, so is \(K_m\), this enables one to prove the assertion of Theorem 2.1 by applying our second main result to \((K_m, \hat{w}_m)\):

**Theorem 2.2.** Let \((K, v)\) be a Henselian field with \(\hat{K}\) of arithmetic type. Assume also that \(\text{char}(K) = \text{char}(\hat{K}) = q\), \(K\) is virtually perfect and \(\text{abrd}_p(K)\) is finite, for each \(p \in \mathbb{P} \setminus \{q\}\). Then every central division LBD-algebra \(R\) over \(K\) has a central \(K\)-subalgebra \(\hat{R}\) admissible by Conjecture [13].

The assertions of Theorems 2.1 and 2.2 are known in case \(R \in \text{d}(K)\). When \([R : K] = \infty\), they can be deduced from the following lemma, by the method of proving Theorem 4.1 of [7] (see Remark 4.7 and Section 8).

**Lemma 2.3.** Assume that \(K\) is a field and \(R\) is a central division \(K\)-algebra, which satisfy the conditions of some of Theorems 2.1 and 2.2. Then, for each \(p \in \mathbb{P}\), \(K\) has a finite extension \(E_p\) in \(K(p)\) that is a \(p\)-splitting field of \(R/K\); equivalently, \(p\) does not divide \([E_p(p) : E_p]\), for any element \(p\) of the underlying division \(E_p\)-algebra \(R_p\) of \(R \otimes_K E_p\).

The fulfillment of the conditions of Lemma 2.3 ensures that \(\text{dim}(K_{\text{sol}}) \leq 1\) (see Lemmas 5.3 and 5.3 below). This plays an essential role in the proof of Theorems 2.1 and 2.2 which also relies on the following known result:
Given an HDV-field $(\mathcal{K}, v)$, the scalar extension map $\text{Br}(\mathcal{K}) \to \text{Br}(\mathcal{K}_v)$, where $\mathcal{K}_v$ is a completion of $\mathcal{K}$ with respect to the topology of $v$, is an injective homomorphism which preserves Schur indices and exponents (cf. [13], Theorem 1); hence, $\text{Br}_{p'}(\mathcal{K}) \leq \text{Br}_{p'}(\mathcal{K}_v)$, for every $p' \in \mathbb{P}$.

The earliest draft of this paper is contained in the manuscript [6]. Here we extend the scope of results of [6], obtained before the theory of division algebras over Henselian fields in [23], and the progress in absolute Brauer $p$-dimensions made in [28], [32] and other papers allowed us to consider the topic of the present research in the desired generality (including, for example, $m$-dimensional local fields which are not of arithmetic type, in the sense of Definition 2, for any $m \geq 2$, see Lemmas 3.3, 6.4 and Remark 6.5).

The basic notation, terminology and conventions kept in this paper are standard and virtually the same as in [22], [27], [33] and [10]. Throughout, Brauer groups, value groups and ordered abelian groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. For any algebra $A$, we consider only subalgebras containing its unit. Given a field $E$, $E^*$ denotes its multiplicative group, $E^{**} = \{a^n : a \in E^*\}$, for each $n \in \mathbb{N}$, and for any $p \in \mathbb{P}$, $p\text{Br}(E)$ stands for the maximal subgroup $\{ p \in \text{Br}(E) : pb_p = 0 \}$ of $\text{Br}(E)$ of period dividing $p$. We denote by $I(E'/E)$ the set of intermediate fields of any field extension $E'/E$, and by $\text{Br}(E'/E)$ the relative Brauer group of $E'/E$ (the kernel of the scalar extension map $\text{Br}(E) \to \text{Br}(E')$). In case $\text{char}(E) = q > 0$, we write $[a, b]_E$ for the $q$-symbol $E$-algebra generated by elements $\xi$ and $\eta$, such that $\eta^q = (\xi + 1)\eta$, $\xi^q - \xi = a \in E$ and $\eta^q = b \in E^*$.

Here is an overview of the rest of this paper. Section 3 includes preliminaries on Henselian fields used in the sequel. It also shows that a Henselian field $(\mathcal{K}, v)$ satisfies the condition $\text{abrd}_p(\mathcal{K}) < \infty$, for some $p \in \mathbb{P}$ not equal to $\text{char}(\mathcal{K})$, if and only if $\text{abrd}_p(\mathcal{K}) < \infty$ and the subgroup $\nu(v)$ of the value group $v(\mathcal{K})$ is of finite index. When $(\mathcal{K}, v)$ is maximally complete with $\text{char}(\mathcal{K}) = q > 0$, we prove in addition that $\text{abrd}_q(\mathcal{K}) < \infty$ if and only if $\mathcal{K}$ is virtually perfect. These results fully characterize generalized formal power series fields (and, more generally, maximally complete equicharacteristic fields) of finite absolute Brauer $p$-dimensions, for all $p \in \mathbb{P}$, and so prove their admissibility by Proposition 1.1 (see Corollary 3.6 and Lemma 3.5 (b)(ii)). Section 4 contains lemmas which provide the main Galois-theoretic and ring-theoretic ingredients of the proofs of Lemma 2.3 and our main results. Most of them have been extracted from [17], wherefore, they are stated here without proof. As noted above, we also show in Section 4 how to deduce Theorems 2.1 and 2.2 from Lemma 2.3 by the method of proving 4.1. In Section 5 we prove that Henselian fields $(\mathcal{K}, v)$ with $\text{char}(\mathcal{K}) = q > 0$ satisfy $\text{abrd}_q(\mathcal{K}(q)) = 0$, and so do HDV-fields of residual characteristic $q$. Section 6 collects valuation-theoretic ingredients of the proof of Lemma 2.3; these include a tame version of the noted lemma, stated as Lemma 6.3. Sections 7 and 8 are devoted to the proof of Lemma 2.3. For this purpose, we adapt to Henselian fields the method of proving [17], Lemma 8.3, relying on Lemmas 6.3 and 6.6 and on results of Sections 4.
and 5. In the setting of Theorem 2.1 when $m \geq 2$ and $\text{char}(K_m) = 0 < q$, we apply Lemma 3.3 at a crucial point of our proof.

3. Preliminaries on Henselian fields and their finite-dimensional division algebras

Let $K$ be a field with a nontrivial valuation $v$, $O_v(K) = \{a \in K : v(a) \geq 0\}$ the valuation ring of $(K, v)$, $M_v(K) = \{\mu \in K : v(\mu) > 0\}$ the maximal ideal of $O_v(K)$, $O_v(K)^* = \{u \in K : v(u) = 0\}$ the multiplicative group of $O_v(K)$, $v(K)$ and $\hat{K} = O_v(K)/M_v(K)$ the value group and the residue field of $(K, v)$, respectively; put $\nabla_0(K) = \{\alpha \in K : \alpha - 1 \in M_v(K)\}$. We say that $v$ is Henselian if it extends uniquely, up-to equivalence, to a valuation $v_L$ on each algebraic extension $L/K$. This holds, for example, if $K = K_v$ and $v(K)$ is a valuation of $\Omega$ extending $v$. When this holds, the Galois groups $G$ and Corollary A.25): Henselizations of valued fields are their immediate extensions (see, e.g., [17], Proposition 15.3.7, or [35], Corollary A.28). In order that $v$ be Henselian, it is necessary and sufficient that any of the following two equivalent conditions is fulfilled (cf. [17], Theorem 18.1.2, or [35], Theorem A.14):

(3.1) (a) Given a polynomial $f(X) \in O_v(K)[X]$ and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where $f'$ is the formal derivative of $f$, there is a zero $c \in O_v(K)$ of $f$ satisfying the equality $v(c - a) = v(f(a)/f'(a))$;

(b) For each normal extension $\Omega/K$, $v'(\tau(\mu)) = v'(\mu)$ whenever $\mu \in \Omega$, $v'$ is a valuation of $\Omega$ extending $v$, and $\tau$ is a $K$-automorphism of $\Omega$.

When $v$ is Henselian, so is $v_L$, for any algebraic field extension $L/K$. In this case, we put $O_v(L) = O_v(L), M_v(L) = M_v(L), v(L) = v_L(L)$, and denote by $\hat{L}$ the residue field of $(L, v_L)$. Clearly, $\hat{L}/\hat{K}$ is an algebraic extension and $v(L)$ is an ordered subgroup of $v(L)$; the index $e(L/K)$ of $v(K)$ in $v(L)$ is called a ramification index of $L/K$. By Ostrowski’s theorem (see [17], Sects. 17.1 and 17.2) if $[L : K]$ is finite, then $\hat{L}: \hat{K}[e(L/K)]$ divides $[L : K]$, and the integer $[L : K][\hat{L}: \hat{K}]^{-1}e(L/K)^{-1}$ is not divisible by any $p \in \mathbb{P}$ different from $\text{char}(\hat{K})$. The extension $L/K$ is defectless, i.e. $[L : K] = [\hat{L}: \hat{K}]e(L/K)$, in the following three cases:

(3.2) (a) If $\text{char}(\hat{K}) \nmid [L : K]$ (apply Ostrowski’s theorem);

(b) If $(K, v)$ is HDV and $L/K$ is separable (see [35], Theorem A.12);

(c) When $(K, v)$ is maximally complete (cf. [37], Theorem 31.22).

Assume that $(K, v)$ is a Henselian field and $R/K$ is a finite extension. We say that $R/K$ is inertial, if $[R : K] = [\hat{R}: \hat{K}]$ and $\hat{R}/\hat{K}$ is a separable extension; $R/K$ is said to be totally ramified, if $e(R/K) = [R : K]$. Inertial extensions of $K$ have the following useful properties (see [35], Theorem A.23 and Corollary A.25):

Lemma 3.1. Let $(K, v)$ be a Henselian field. Then:

(a) An inertial extension $R'/K$ is Galois if and only if $\hat{R}'/\hat{K}$ is Galois. When this holds, the Galois groups $G(R'/K)$ and $G(\hat{R}'/\hat{K})$ are isomorphic.
(b) The compositum \( K_{\text{ur}} \) of inertial extensions of \( K \) in \( K_{\text{sep}} \) is a Galois extension of \( K \) with \( G(K_{\text{ur}}/K) \cong G_{\hat{K}} \).

(c) Finite extensions of \( K \) in \( K_{\text{ur}} \) are inertial, and the natural mapping of \( I(K_{\text{ur}}/K) \) into \( I(\hat{K}_{\text{sep}}/\hat{K}) \) is bijective.

(d) For each \( K_1 \subseteq \text{Fe}(K) \), the intersection \( K_0 = K_1 \cap K_{\text{ur}} \) equals the maximal inertial extension of \( K \) in \( K_1 \); in addition, \( \hat{K}_0 = \hat{K}_1 \).

When \((K, v)\) is Henselian, the finite extension \( R/K \) is called tamely ramified, if \( \text{char}(\hat{K}) \nmid e(R/K) \). The following lemma gives an account of some basic properties of tamely ramified extensions of \( K \) in \( K_{\text{sep}} \) (see (3.1) (b) and [35], Theorems A.9 (i),(ii) and A.24):

**Lemma 3.2.** Let \((K, v)\) be a Henselian field with \( \text{char}(\hat{K}) = q \), \( K_{\text{tr}} \) the compositum of tamely ramified extensions of \( K \) in \( K_{\text{sep}} \), \( \mathbb{P}' = \mathbb{P} \setminus \{q\} \), and let \( \hat{e}_p \) be a primitive \( p \)-th root of unity in \( \hat{K}_{\text{sep}} \), for each \( p \in \mathbb{P}' \). Then \( K_{\text{tr}}/K \) is a Galois extension with \( G(K_{\text{tr}}/K_{\text{ur}}) \) abelian, and the following holds:

(a) All finite extensions of \( K \) in \( K_{\text{tr}} \) are tamely ramified.

(b) There is \( T(K) \subseteq I(K_{\text{tr}}/K) \) with \( T(K) \cap K_{\text{ur}} = K \) and \( T(K).K_{\text{ur}} = K_{\text{tr}} \); hence, finite extensions of \( K \) in \( T(K) \) are tamely and totally ramified.

(c) The field \( T(K) \) singled out in (b) is isomorphic as a \( K \)-algebra to \( \otimes_{p \in \mathbb{P}' \setminus \{q\}} T_p(K) \), where \( \otimes = \otimes_K \), and for each \( p \), \( T_p(K) \in I(T(K)/K) \) and every finite extension of \( K \) in \( T_p(K) \) is of \( p \)-primary degree; in particular, \( T(K) \) equals the compositum of the fields \( T_p(K), p \in \mathbb{P}' \).

(d) With notation being as in (e), \( T_p(K) \neq K, \) for some \( p \in \mathbb{P}' \), if and only if \( v(K) \neq pv(K) \); when this holds, \( T_p(K) \in I(K(p)/K) \) if and only if \( \hat{e}_p \in \hat{K} \) (equivalently, if and only if \( K \) contains a primitive \( p \)-th root of 1).

The Henselian property of \((K, v)\) guarantees that \( v \) extends to a unique, up-to equivalence, valuation \( v_D \) on each \( D \in d(K) \) (cf. [35], Sect. 1.2.2).

Put \( v(D) = v_D(D) \) and denote by \( \hat{D} \) the residue division ring of \((D, v_D)\). It is known that \( \hat{D} \) is a division \( \hat{K} \)-algebra, \( v(D) \) is an ordered abelian group and \( v(K) \) is an ordered subgroup of \( v(D) \) of finite index \( e(D/K) \) (called the ramification index of \( D/K \)). Note further that \([\hat{D}: \hat{K}] < \infty \), and by Ostrowski-Draxl’s theorem (cf. [16] and [35], Propositions 4.20 and 4.21), \([\hat{D}: \hat{K}]e(D/K) \mid [D: K] \) and \([D: K][\hat{D}: \hat{K}]^{-1}e(D/K)^{-1} \) has no prime divisor \( p \neq \text{char}(\hat{K}) \). The division \( K \)-algebra \( D \) is called inertial, if \([D: K] = [\hat{D}: \hat{K}] \); it is called totally ramified, if \([D: K] = e(D/K) \). We say that \( D/K \) is defectless, if \([D: K] = [\hat{D}: \hat{K}] \); this holds in the following two cases:

(3.3) (a) If \( \text{char}(\hat{K}) \nmid [D: K] \) (apply the Ostrowski-Draxl theorem);

(b) If \((K, v)\) is an HDV-field (see [36], Proposition 2.2).

The division \( K \)-algebra \( D \) is called nicely semi-ramified (abbr., NSR), in the sense of [24], if \( e(D/K) = [\hat{D}: \hat{K}] = \deg(D) \) and \( \hat{D}/\hat{K} \) is a separable field extension. As shown in [24], when this holds, \( \hat{D}/\hat{K} \) is a Galois extension, \( G(\hat{D}/\hat{K}) \) is isomorphic to the quotient group \( v(D)/v(K) \), and \( D \) decomposes into a tensor product of cyclic NSR-algebras over \( K \) (see also [35], Propositions 8.40 and 8.41). The result referred to allows to prove our next lemma, stated as follows:
Lemma 3.3. Let \((K, v)\) be a Henselian field, such that \(\text{abrd}_p(\hat{K}(p)) = 0\), for some \(p \in \mathbb{P}\) not equal to \(\text{char}(\hat{K})\). Then every \(\Delta_p \in d(K)\) of \(p\)-primary degree has a splitting field that is a finite extension of \(K\) in \(K(p)\).

Lemma 3.3 shows that if \(R\) is a central division LBD-algebra over a field \(K\) satisfying the conditions of some of the main results of the present paper, and if there is a \(K\)-subalgebra \(R\) of \(R\) with the properties claimed by Conjecture 1.3, then for each \(p \in \mathbb{P}\) with at most one exception, \(K\) has a finite extension \(E_p\) in \(K(p)\) that is a \(p\)-splitting field of \(R/K\). This leads to the idea of proving Theorems 2.1 and 2.2 on the basis of Lemma 2.3 (for further support of the idea and a step to its implementation, see Lemma 5.3).

Proof of Lemma 3.3. The assertion is obvious if \(\Delta_p\) is an NSR-algebra over \(K\), or more generally, if \(\Delta_p\) is Brauer equivalent to the tensor product of cyclic division \(K\)-algebras of \(p\)-primary degrees. When the \(K\)-algebra \(\Delta_p\) is inertial, we have \(\hat{\Delta}_p \in d(\hat{K})\) (cf. [23], Theorem 2.8), so our conclusion follows from the fact that \(\text{abrd}_p(\hat{K}(p)) = 0\) and \(\hat{K}(p) = \hat{K}(p)\), which ensures that \([\Delta_p] \in \text{Br}(K_v \cap K(p)/K)\). Since, by [23], Lemmas 5.14 and 6.2, \([\Delta_p] = [I_p \otimes_K N_p \otimes_K T_p]\), for some inertial \(K\)-algebra \(I_p\), an NSR-algebra \(N_p/K\), and a tensor product \(T_p\) of totally ramified cyclic division \(K\)-algebras, such that \([I_p], [N_p]\) and \([T_p] \in \text{Br}(K(p))\), these observations prove Lemma 3.3.

The following two lemmas give a valuation-theoretic characterization of those Henselian virtually perfect fields \((K, v)\) with \(\text{char}(K) = \text{char}(\hat{K})\), which satisfy the condition \(\text{abrd}_p(K) < \infty\), for some \(p \in \mathbb{P}\).

Lemma 3.4. Let \((K, v)\) be a Henselian field. Then \(\text{abrd}_p(K) < \infty\), for a given \(p \in \mathbb{P}\) different from \(\text{char}(\hat{K})\), if and only if \(\text{abrd}_p(\hat{K}) < \infty\) and the quotient group \(v(K)/qv(K)\) is finite.

Proof. We have \(\text{abrd}_p(\hat{K}) \leq \text{abrd}_p(K)\) (by [23], Theorem 2.8, and [35], Theorem A.23), so our assertion can be deduced from [12], Proposition 6.1, Theorem 5.9 and Remark 6.2 (or [12], (3.3) and Theorem 2.3).

Lemma 3.4 and our next lemma show that a maximally complete field \((K, v)\) with \(\text{char}(K) = \text{char}(\hat{K})\) satisfies \(\text{abrd}_p(K) < \infty\), \(p \in \mathbb{P}\), if and only if \(\hat{K}\) is virtually perfect and for each \(p \in \mathbb{P}\), \(\text{abrd}_p(\hat{K}) < \infty\) and \(v(K)/qv(K)\) is finite. When this holds, \(K\) is virtually perfect as well (see [12], Lemma 3.2).

Lemma 3.5. Let \((K, v)\) be a Henselian field with \(\text{char}(\hat{K}) = q > 0\). Then:
(a) \([\hat{K} : \hat{K}^q]\) and \(v(K)/qv(K)\) are finite, provided that \(\text{Brd}_q(K) < \infty\);
(b) The inequality \(\text{abrd}_q(K) < \infty\) holds, in case \(\hat{K}\) is virtually perfect and some of the following two conditions is satisfied:
(i) \(v\) is discrete;
(ii) \(\text{char}(K) = q\) and \(K\) is virtually perfect; in particular, this occurs if \(\text{char}(K) = q\), \(v(K)/qv(K)\) is finite and \((K, v)\) is maximally complete.
Proof. Statement (a) is implied by [11], Proposition 3.4, so one may assume that \( \hat{K} : K^q = q^\mu \) and \( v(K)/qv(K) \) has order \( q^\tau \), for some integers \( \mu \geq 0 \), \( \tau \geq 0 \). We prove statement (b) of the lemma. Suppose first that \( v \) is discrete. Then \( \text{Brd}_p(K) \leq \text{Brd}_q(K_v) \), by (2.1), so it is sufficient to prove that \( \text{abrd}_p(K) < \infty \), provided that \( K = K_v \). If \( \text{char}(K) = 0 \), this is contained in [22], Theorem 2, and when \( \text{char}(K) = q \), the finitude of \( \text{abrd}_p(K) \) is obtained as a special case of Lemma 3.5 (b) (ii) and the fact that \( (K, v) \) is maximally complete (cf. [27], Ch. XII, page 488). It remains for us to prove Lemma 3.5 (b) (ii). Our former assertion follows from [1], Ch. VII, Theorem 28, statement (1.1) and [10], Lemma 4.1 (which ensure that \( K_v \) is maximally complete (cf. [27], Ch. XII, page 488). It remains for us to prove Lemma 3.5 (b) (ii). Our former assertion follows from [1], Ch. VII, Theorem 28, statement (1.1) and [10], Lemma 4.1 (which ensure that \( \text{abrd}_p(K) \leq \log_q|K : K^q| \)). Observe finally that if \( (K, v) \) is maximally complete with \( \text{char}(K) = q \), then \( K : K^q = q^{\mu + \tau} \). This can be deduced from (3.2) (c), since \( (K^q, v_q) \) is maximally complete, \( v_q(K^q) = qv(K) \) and \( K^q \) is the residue field of \( (K^q, v_q) \), where \( v_q \) is the valuation of \( K^q \) induced by \( v \). More precisely, it follows from (3.2) (c) and the noted properties of \( (K^q, v_q) \) that the degrees of finite extensions of \( K^q \) in \( K \) are at most equal to \( q^{\mu + \tau} \), which yields \( K : K^q \leq q^{\mu + \tau} \) and so allows to conclude that \( K : K^q = q^{\mu + \tau} \) (whence, \( \text{abrd}_p(K) \leq \mu + \tau \)). Thus Lemma 3.5 (b) (ii) is proved.

Corollary 3.6. Let \( K_0 \) be a field and \( \Gamma \) a nontrivial ordered abelian group. Then the formal power series field \( K = K_0((\Gamma)) \) satisfies the inequalities \( \text{abrd}_p(K) < \infty \), \( p \in \mathbb{P} \), if and only if \( K_0 \) is virtually perfect, \( \text{abrd}_p(K_0) < \infty \), \( p \in \mathbb{P} \setminus \{\text{char}(K_0)\} \), and the quotient groups \( \Gamma/p\Gamma \) are finite, for all \( p \in \mathbb{P} \).

Proof. Let \( v_\Gamma \) be the standard valuation of \( K \) inducing on \( K_0 \) the trivial valuation. Then \( (K, v_\Gamma) \) is maximally complete (cf. [17], Theorem 18.4.1) with \( v(K) = \Gamma \) and \( \hat{K} = K_0 \) (see [17], Sect. 2.8 and Example 4.2.1), so Corollary 3.6 can be deduced from Lemmas 3.5 and 3.5. □

Remark 3.7. Given a field \( K_0 \) and an ordered abelian group \( \Gamma \neq \{0\} \), the standard realizability of the field \( K_1 = K_0((\Gamma)) \) as a maximally complete field, used in the proof of Corollary 3.6 allows us to determine the sequence \( (b, a) = \text{Brd}_p(K_1) \), \( \text{abrd}_p(K_1) : p \in \mathbb{P} \), in the following two cases: (i) \( K_0 \) is a global or local field (see [12], Proposition 5.1, and [11], Corollary 3.6 and Sect. 4, respectively); (ii) \( K_0 \) is perfect and \( \text{dim}(K_0) \leq 1 \) (see [11], Proposition 3.5, and [12], Propositions 5.3, 5.4). In both cases, \( (b, a) \) depends only on \( K_0 \) and \( \Gamma \). Moreover, if \( (K, v) \) is Henselian with \( \hat{K} = K_0 \) and \( v(K) = \Gamma \), then: (a) \( (b, a) = \text{Brd}_p(K), \text{abrd}_p(K) : p \in \mathbb{P} \), provided that \( (K, v) \) is maximally complete and \( \text{char}(K) = \text{char}(K_0) \); (b) \( \text{Brd}_p(K) = \text{Brd}_p(K_1) \) and \( \text{abrd}_p(K) = \text{abrd}_p(K_1) \), for each \( p \neq \text{char}(K_0) \). When \( K_0 \) is finite and \( p \neq \text{char}(K_0) \), \( \text{Brd}_p(K) \) has also been computed in [4], Sect. 7.

Next we show that \( \text{abrd}_p(K_m) < \infty \), \( p \in \mathbb{P} \), if \( K_m \) is a complete \( m \)-discretely valued field with \( m \)-th residue field admissible by Theorem 2.12.

Lemma 3.8. Let \( K_m \) be a complete \( m \)-discretely valued field with an \( m \)-th residue field \( K_0 \). Then \( \text{abrd}_p(K_m) < \infty \), for all \( p \in \mathbb{P} \), if and only if \( K_0 \) is virtually perfect with \( \text{abrd}_p(K_0) < \infty \), for every \( p \in \mathbb{P} \setminus \{\text{char}(K_0)\} \).
Proposition 3.9. Let $F_0$ be an algebraically closed field of nonzero characteristic $q$ and $F_n$: $n \in \mathbb{N}$, be a tower of extensions of $F_0$ defined inductively as follows: when $n > 0$, $F_n = F_{n-1}((T_n))$ is the formal Laurent power series field in a variable $T_n$ over $F_{n-1}$. Then the following holds, for each $n \in \mathbb{N}$:

(a) $F_n$ possesses a subfield $\Lambda_n$ that is a purely transcendental extension of infinite transcendency degree over the rational function field $F_{n-1}((T_n))$.

(b) The maximal separable (algebraic) extension $E_0$ of $\Lambda_n$ in $F_n$ satisfies the equalities $[E_n:E_0^n] = \infty$, $\text{brd}_p(E_n) = \text{brd}_p(E_n) = [n/2]$, for all $p \in \mathbb{P} \setminus \{q\}$, and $\text{brd}_q(E'_n) = n - 1$, for every finite extension $E'_n/E_n$.

Proof. The assertion of Proposition 3.9 (a) is known (cf., e.g., [3]), and it implies $[E_n:E_0^n] = \infty$. Let $w_n$ be the natural discrete valuation of $F_n$, trivial on $F_{n-1}$, and $v_n$ be the valuation of $E_n$ induced by $w_n$. Then $(F_n, w_n)$ is complete and $E_n$ is dense in $F_n$, which yields $v_n(E_n) = w_n(F_n)$ and $F_{n-1}$ is the residue field of $(E_n, v_n)$ and $(F_n, w_n)$; hence, $v_n$ is discrete. Similarly, if $n \geq 2$, then the natural $\mathbb{Z}^n$-valued valuation $\theta_n'$ of $F_n$ (trivial on $F_0$) is Henselian and induces on $E_0$ a valuation $\theta_n$. Also, $v_n$ is Henselian (cf. [17], Corollary 18.3.3), and $\theta_n$ extends the natural $\mathbb{Z}^{n-1}$-valued valuation $\theta_n'$ of $F_{n-1}$. As $\theta_n'$ is Henselian and $F_{n-1}((T_n)) \subset E_n$, this ensures that so is $\theta_n$ (see [35], Proposition A.15), $F_0$ is the residue field of $(E_n, \theta_n)$, and $\theta_n(E_n) = \mathbb{Z}^n$. At the same time, it follows from (2.1) and the Henselian property of $v_n$ that $\text{brd}_p(E_n) \leq \text{brd}_p(F_n)$, for each $p$. In addition, $(F_n, \theta_n')$ is maximally complete with a residue field $F_0$ (cf. [17], Theorem 18.4.1), whence, by [11], Proposition 3.5, $\text{brd}_q(F_n) = \text{brd}_q(F_n) = n - 1$. Since $\theta_n(E_n') \cong \mathbb{Z}^n \cong \theta_n'(F_n')$ and $F_0$ is the residue field of $(E_n', \theta_n, E_0)$ and $(F_n, \theta_n', E_n')$ whenever $E_n'/E_n$ and $F_n'/F_n$ are finite extensions, one obtains from [12], Proposition 5.3 (b), and [10], Lemma 4.2, that $\text{brd}_q(E_n') \geq n - 1$ and $\text{brd}_p(E_n') = \text{brd}_p(F_n') = [n/2]$, for each $p \neq q$. Note finally that $v_n(E_n') \cong \mathbb{Z} \cong w_n(F_n')$, and the completion of $(E_n', v_n, E_n')$ is a finite extension of $F_n$ (cf. [27], Ch. XII, Proposition 3.1), so it follows from (2.1) and the preceding observations that $\text{brd}_q(E_n') = n - 1$, for all $n$. \hfill \Box

4. Lemmas on $p$-powers and finite-dimensional central subalgebras of division LBD-algebras

Let $R$ be a central division LBD-algebra over a virtually perfect field $K$ with $\text{brd}_p(K) < \infty$, $p \in \mathbb{P}$. The existence of finite $p$-powers $k(p)$ of $R/K$, $p \in \mathbb{P}$, imposes essential restrictions on a number of algebraic properties.
of $R$, especially, on those extensions of $K$ which are embeddable in $R$ as $K$-subalgebras. For example, it turns out that if $K(p) \neq K$, for some $p > 2$, then $K(p)/K$ is an infinite extension (the additive group $\mathbb{Z}_p$ of $p$-adic integers, endowed with its natural topology, is a homomorphic image of $\mathcal{G}(K(p)/K)$, see \cite{38}), whence, $K(p)$ is not isomorphic to a $K$-subalgebra of $R$. In this Section we present results on $p$-powers and $p$-splitting fields, obtained in the case of $\dim(K_{sol}) \leq 1$. These results form the basis for the proofs of Theorems \ref{2.1} \ref{2.2} and Lemma \ref{2.3}. The first one is an immediate consequence of \cite{7}, Lemmas 3.12 and 3.13, and it also implies Lemma 4.1 (b) and (c).

As a matter of fact, Lemma \ref{4.1} (a) is identical in content with \cite{7}, Lemmas 3.12 and 3.13, and it also implies Lemma \ref{4.1} (b) and (c).

Lemma 4.1. Assume that $R$ is a central division LBD-algebra over a virtually perfect field $K$ with $\dim(K_{sol}) \leq 1$ and $\text{abrd}_p(K) < \infty$, $p \in \mathbb{P}$. Let $K'/K$ be a finite extension, $R'$ the underlying (central) division $K'$-algebra of the LBD-algebra $R \otimes_K K'$, $\gamma$ the integer for which $R \otimes_K K'$ and the full matrix ring $M_\gamma(R')$ are isomorphic as $K'$-algebras, and for each $p \in \mathbb{P}$, let $k(p)$ and $k(p)'$ be the $p$-powers of $R/K$ and $R'/K'$, respectively. Then:

(a) The greatest integer $\mu(p) \geq 0$ for which $p^{\mu(p)} \mid \gamma$ is equal to $k(p) - k(p)'$; hence, $k(p) \geq k(p)'$ and $\gamma \mid p^{1+k(p)}$, for any $p \in \mathbb{P}$;

(b) The equality $k(p) = k(p)'$ holds if and only if $p \nmid \gamma$; specifically, if $k(p) = 0$, then $k(p)' = 0$ and $p \nmid \gamma$.

(c) $K'$ is a $p$-splitting field of $R/K$ if and only if $k(p)' = 0$, that is, $p \nmid [K'(r') : K']$, for any $r' \in R'$.

Remark 4.2. The proofs of \cite{7}, Lemmas 3.12 and 3.13, rely essentially on the condition that $\dim(K_{sol}) \leq 1$, more precisely, on its restatement that $\text{abrd}_p(K_p) = 0$, for each $p \in \mathbb{P}$, where $K_p$ is the fixed field of a Hall pro-$\mathbb{P}$-subgroup $H_p$ of $\mathcal{G}(K_{sol}/K)$. It is not known whether the assertions of Lemma \ref{4.1} remain valid if this condition is dropped; also, the question of whether $\dim(E_{sol}) \leq 1$, for every field $E$ (posed in \cite{25}) is open. Here we note that the conclusion of Lemma \ref{4.1} holds if the assumption that $\dim(K_{sol}) \leq 1$ is replaced by the one that $R \otimes_K K'$ is a division $K'$-algebra. Then it follows from \cite{7}, Proposition 3.3, that $k(p) = k(p)'$, for every $p \in \mathbb{P}$.

Lemma 4.3. Assuming that $K$ and $R$ satisfy the conditions of Lemma \ref{4.1}, let $D \in d(K)$ be a $K$-subalgebra of $R$, and let $k(p)$ and $k(p)'$, $p \in \mathbb{P}$, be the $p$-powers of $R/K$ and $C_R(D)/K$, respectively. Then:

(a) For each $p \in \mathbb{P}$, $k(p) - k(p)'$ equals the power of $p$ in the primary decomposition of $\deg(D)$; in particular, $k(p) \geq k(p)'$;

(b) $k(p) = k(p)'$ if and only if $p \nmid \deg(D)$; in this case, a finite extension $K'$ of $K$ is a $p$-splitting field of $R/K$ if and only if so is $K'$ for $C_R(D)/K$;

(c) If $k(p)' = 0$, for some $p \in \mathbb{P}$, then a finite extension $K'$ of $K$ is a $p$-splitting field of $R/K$ if and only if $p \nmid \text{ind}(D \otimes_K K')$.

Proof. It is known (cf. \cite{33}, Sect. 13.1, Corollary b) that if $K_1$ is a maximal subfield of $D$, then $K_1/K$ is a field extension, $[K_1 : K] = \deg(D) := d$...
and $D \otimes_K K_1 \cong M_d(K_1)$ as $K_1$-subalgebras. Also, by the Double Centralizer Theorem (see [22], Theorems 4.3.2 and 4.4.2), $R = D \otimes_K C_R(D)$ and $C_R(D) \otimes_K K_1$ is a central division $K_1$-algebra equal to $C_R(K_1)$. In view of [7], Propositions 3.1 and 3.3, this ensures that $k(p)^e$ equals the $p$-power of $(C_R(D) \otimes_K K_1)/K_1$, for each $p \in \mathbb{P}$. Applying now Lemma 4.1 one proves Lemma 4.3 (a). Lemma 4.3 (b)-(c) follows from Lemmas 4.1 and 4.3 (a), combined with [7], Lemma 3.5, and [33], Sect. 9.3, Corollary b. □

The following lemma (for a proof, see [7], Lemma 7.4) can be viewed as a generalization of the uniqueness part of the primary tensor product decomposition theorem for algebras $D \in d(K)$ over an arbitrary field $K$.

**Lemma 4.4.** Let $\Pi$ be a finite subset of $\mathbb{P}$, and let $S_1, S_2$ be central division LBD-algebras over a field $K$ with $\text{abrd}_p(K) < \infty$, for all $p \in \mathbb{P}$. Assume that $k(p)_{S_1} = k(p)_{S_2} = 0$, $p \in \Pi$, the $K$-algebras $R_1 \otimes_K S_1$ and $R_2 \otimes_K S_2$ are $K$-isomorphic, where $R_i \in s(K)$, $i = 1, 2$, and $\deg(R_1)\deg(R_2)$ is not divisible by any $p \in \mathbb{P} \setminus \Pi$. Then $R_1 \cong R_2$ as $K$-algebras.

For a proof of our next lemma, we refer the reader to [7], Lemma 8.3, which has been proved under the assumption that $R$ is a central division LBD-algebra over a field $K$ of arithmetic type. Therefore, we note that the proof in [7] remains valid if the assumption on $K$ is replaced by the one that $\text{abrd}_p(K) < \infty$, $p \in \mathbb{P}$, $\dim(K_{sol}) \leq 1$, $K$ is virtually perfect, and there exist $p$-splitting fields $E_p : p \in \mathbb{P}$, of $R/K$ with $E_p \subseteq K(p)$, for each $p$.

**Lemma 4.5.** Let $K$ be a field with $\dim(K_{sol}) \leq 1$, $R$ a central division LBD-algebra over $K$, and $k(p) : p \in \mathbb{P}$, the sequence of $p$-powers of $R/K$. Assume that, for each $p \in \mathbb{P}$, $E_p$ is a finite extension of $K$ in $K(p)$, which is a $p$-splitting field of $R/K$. Then:

(a) The full matrix ring $M_{\gamma(p)}(R)$, where $\gamma(p) = \left[ E_p : K \right] p^{-k(p)}$, is an Artinian central simple LBD-algebra over $K$, which possesses a subalgebra $\Delta_p \in s(K)$, such that $\deg(\Delta_p) = \left[ E_p : K \right]$ and $E_p$ is isomorphic to a $K$-subalgebra of $\Delta_p$. Moreover, if $\left[ E_p : K \right] = p^{k(p)}$, i.e. $E_p$ is embeddable in $R$ as a $K$-subalgebra, then $\Delta_p$ is a $K$-subalgebra of $R$.

(b) The centralizer of $\Delta_p$ in $M_{\gamma(p)}(R)$ is a central division $K$-algebra of $p$-power zero.

The following lemma generalizes [7], Lemma 8.5, to the case where $K$ and $R$ satisfy the conditions of Lemma 4.5. For this reason, we take into account that the proof of the lemma referred to, given in [7], remains valid under the noted weaker conditions. Our next lemma can also be viewed as a generalization of the well-known fact that, for any field $E$, $D_1 \otimes_E D_2 \in d(E)$ whenever $D_i \in d(E)$, $i = 1, 2$, and $\gcd\{\deg(D_1), \deg(D_2)\} = 1$ (see [33], Sect. 13.4). Using this lemma and the uniqueness part of the Wedderburn-Artin theorem, one obtains that, in the setting of Lemma 4.5, the underlying central division $K$-algebra of $\Delta_p$ is embeddable in $R$ as a $K$-subalgebra.

**Lemma 4.6.** Let $K$ be a field, $R$ a central division LBD-algebra over $K$, and $E_p$, $p \in \mathbb{P}$, be extensions of $K$ satisfying the conditions of Lemma 4.5. Also,
let $D \in d(K)$ be a division $K$-algebra such that $\gcd\{(\deg(D)), [K(\alpha) : K]\} = 1$, for each $\alpha \in R$. Then $D \otimes_K R$ is a central division LBD-algebra over $K$.

**Remark 4.7.** Assume that $K$ is a field and $R$ is a central division $K$-algebra satisfying the conditions of some of Theorems 2.1 and $E_p$: $p \in \mathbb{P}$, are $p$-splitting fields of $R/K$ with the properties required by Lemma 2.3. Then it follows from Lemmas 4.3 and 4.6 that, for each $p \in \mathbb{P}$, there exists a unique, up-to $K$-isomorphism, $K$-subalgebra $R_p \in d(K)$ of $R$ of degree $\deg(R_p) = p^{k(p)}$, where $k(p)$ is the $p$-power of $R/K$. Moreover, Lemma 4.3 implies $R_p$, $p \in \mathbb{P}$, can be chosen so that $R_{p'} \subseteq C_R(R_{p''})$ whenever $p', p'' \in \mathbb{P}$ and $p' \neq p''$. Therefore, there exist $K$-subalgebras $T_n$, $n \in \mathbb{N}$, of $R$, such that $T_n \cong \otimes_{j=1}^n R_{p_j}$ and $T_n \subseteq T_{n+1}$, for each $n$; here $\otimes = \otimes_K$ and $\mathbb{P}$ is presented as a sequence $p_n$: $n \in \mathbb{N}$. Hence, the union $\hat{R} = \bigcup_{n=1}^{\infty} T_n := \otimes_{n=1}^{\infty} R_{p_n}$ is a central $K$-subalgebra of $R$. Note further that $R = T_n \otimes_K C_R(T_n)$, for every $n \in \mathbb{N}$, which enables one to deduce from Lemmas 4.1, 4.4, 4.6, and 7, Lemma 3.5, that a finite-dimensional $K$-subalgebra $T$ of $R$ is embeddable in $T_n$ as a $K$-subalgebra in case $p_n' \mid [T : K]$, for any $n' > n$. One also sees that $K = \cap_{n=1}^{\infty} C_R(T_n) = C_R(R)$, and by [7], Lemma 9.3, every LFD-subalgebra of $R$ (over $K$) of countable dimension is embeddable in $\hat{R}$.

The following two lemmas are used at crucial points of our proof of Lemma 2.3. The former one has not been formally stated in [7]. However, special cases of it have been used in the proof of [7], Lemma 8.3.

**Lemma 4.8.** Let $K$ and $R$ satisfy the conditions of Lemma 2.1 and let $K_1$, $K_2$ be finite extensions of $K$ in an algebraic closure of $K_{sep}$. Denote by $R_1$ and $R_2$ the underlying division algebras of $R \otimes_K K_1$ and $R \otimes_K K_2$, respectively, and suppose that there exist $D_i \in d(K_i)$, $i = 1, 2$, such that $D_i$ is a $K_i$-subalgebra of $R_i$ and $\deg(D_i) = p^{k(p)}$, for a given $p \in \mathbb{P}$ and each index $i$, where $k(p)$ is the $p$-power of $R/K$. Then:

(a) The underlying division $K_1K_2$-algebras of $R_1 \otimes_K K_1K_2$, $R_2 \otimes_K K_1K_2$ and $R \otimes_K K_1K_2$ are isomorphic;

(b) $p$ does not divide $[K_i(c_i) : K_i]$, for any $c_i \in C_{R_i}(D_i)$, and $i = 1, 2$.

(c) If $p \nmid [K_1K_2 : K]$, then $D_1 \otimes_{K_1} K_1K_2$ and $D_2 \otimes_{K_2} K_1K_2$ are isomorphic central division $K_1K_2$-algebras; for example, this holds in case $p \nmid [K_1 : K]$, $i = 1, 2$, and $\gcd\{[K_1 : K_0], [K_2 : K_0]\} = 1$, where $K_0 = K_1 \cap K_2$.

**Proof.** Note that $R \otimes_K K_1K_2$ and $(R \otimes_K K_1) \otimes_K K_1K_2$, $i = 1, 2$, are isomorphic $K_1K_2$-algebras. These algebras are central simple and Artinian, which enables one to deduce Lemma 4.8 (a) from Wedderburn-Artin’s theorem and [3], Sect. 9.3, Corollary b. In addition, it follows from Lemma 2.1 the assumptions on $D_1$ and $D_2$, and the Double Centralizer Theorem that $k(p)$ equals the $p$-powers of $R_i/K_i$, and $C_{R_i}(D_i)$ is a central division $K_i$-subalgebra of $R_i$, for each $i$. Hence, by Lemma 4.3 (a), $C_{R_i}(D_i)/K_i$ and $C_{R_i}(D_i)/K_2$ are of $p$-power zero, which proves Lemma 4.8 (b).

We turn to the proof of Lemma 4.8 (c). Assume that $p \nmid [K_1K_2 : K]$ and denote by $R'$ the underlying division $K_1K_2$-algebra of $R \otimes_K K_1K_2$. Then, by Lemma 4.1 $k(p)$ equals the $p$-power of $R'/K_1K_2$. Applying [7], Lemma 3.5
(or results of [33], Sect. 13.4), one also obtains that $D_i \otimes_{K_i} K_1 K_2 \in d(K_1 K_2)$ and $D_i \otimes_{K_i} K_1 K_2$ are embeddable in $R'$ as $K_1 K_2$-subalgebras, for $i = 1, 2$. Let $D_1'$ and $D_2'$ be $K_1 K_2$-subalgebras of $R'$ isomorphic to $D_1 \otimes_{K_1} K_1 K_2$ and $D_2 \otimes_{K_2} K_1 K_2$, respectively. As above, then it follows that, for each index $i$, $R'$ coincides with $D_i' \otimes_{K_1 K_2} C_{R'}(D_i')$, $C_{R'}(D_i')$ is a central division algebra over $K_1 K_2$, and $C_{R'}(D_i')/K_1 K_2$ is of zero $p$-power; thus $p \parallel [K_1 K_2(c') : K_1 K_2]$, for any $c' \in C_{R'}(D_i') \cup C_{R'}(D_2')$. Therefore, by Lemma 4.9, $D_i' \cong D_2'$, whence, $D_1 \otimes_{K_1} K_1 K_2 \cong D_2 \otimes_{K_2} K_1 K_2$ as $K_1 K_2$-algebras. The latter part of our assertion is obvious, so Lemma 4.8 is proved.

Lemma 4.9. Let $D$ be a finite-dimensional simple algebra over a field $K$. Suppose that the centre $B$ of $D$ is a compositum of extensions $B_1$ and $B_2$ of $K$ of relatively prime degrees, and the following conditions are fulfilled:

(a) $[D : B] = n^2$ and $D$ possesses a maximal subfield $E$ such that $[E : B] = n$ and $E = B\hat{E}$, for some separable extension $\hat{E}/K$ of degree $n$;
(b) $p > n$, for every $p \in \mathbb{P}$ dividing $[B : K]$;
(c) $D \cong D_i \otimes_{B_i} B$ as a $B$-algebra, for some $D_i \in s(B_i)$, $i = 1, 2$.

Then there exist $\hat{D} \in s(K)$ with $[\hat{D} : K] = n^2$, and isomorphisms of $B_i$-algebras $\hat{D} \otimes_{K} B_i = D_i$, $i = 1, 2, 3$, where $B_3 = B$ and $D_3 = D$.

Lemma 4.9 has been proved as [7], Lemma 8.2 (see also [5]). It has been used for proving [7], Lemma 8.3, and the main result of [5]. In the present paper, the application of Lemma 4.9 in the proof of Lemma 7.2 gives us the possibility to deduce Lemma 2.3 by the method of proving [7], Lemma 8.3.

5. Henselian fields $(K, v)$ with $\text{char}(\hat{K}) = q > 0$ and $\text{abrd}_q(K(q)) \leq 1$

The question of whether $\text{abrd}_q(\Phi(q)) = 0$, for every field $\Phi$ of characteristic $q > 0$ seems to be open. This Section gives a criterion for a Henselian field $(K, v)$ with $\text{char}(\hat{K}) = q$ and $\hat{K}$ of arithmetic type to satisfy the equality $\text{abrd}_q(K(q)) = 0$. To prove this criterion we need the following two lemmas.

Lemma 5.1. Let $(K, v)$ be a Henselian field with $\text{char}(\hat{K}) = q > 0$ and $\hat{K} \neq \hat{K}^q$, and in case $\text{char}(K) = 0$, suppose that $v$ is discrete and $v(q) \in qv(K)$. Let also $\hat{K}/\hat{K}$ be an inseparable extension of degree $q$. Then there is $\Lambda \in I(K(q)/K)$, such that $[\Lambda : K] = q$ and $\Lambda$ is $\hat{K}$-isomorphic to $\hat{\Lambda}$.

Proof. By the assumption on $\hat{K}/\hat{K}$, there exists $\hat{a} \in \hat{K} \setminus \hat{K}^q$, such that $\hat{\Lambda} = \hat{K}(\sqrt[q]{\hat{a}})$. Hence, by the Artin-Schreier theorem, one may take as $\Lambda$ the extension of $K$ in $K_{\text{sep}}$ obtained by adjunction of a root of the polynomial $X^q - X - a\pi^{-q}$, for any fixed $\pi \in K$ with $v(q) > 0$. When $\text{char}(K) = 0$, our assertion is contained in [13], Lemma 5.4, so Lemma 5.4 is proved.

Lemma 5.2. Let $(K, v)$ be a Henselian field, $L/K$ an inertial extension of degree $m$, $N(L/K)$ the norm group of $L/K$, and $\alpha$ an element of the multiplicative group $\mathfrak{N}_0(K)$. Then $\alpha \in N(L/K)$.
Proof. This is a special case of [18], Proposition 2.

Next we show that a Henselian field \((K, v)\) with \(\text{char}(\hat{K}) = q > 0\) satisfies \(\text{abrd}_q(K(q)) = 0\), provided that \(\text{char}(K) = q\) or the valuation \(v\) is discrete. As noted in the Introduction, the inequality \(\text{abrd}_q(K(q)) = 0\) ensures that \(\text{Br}(K(q)) = \{0\}\), for every finite extension \(K(q')/K(q)\).

Lemma 5.3. Let \((K, v)\) be a Henselian field with \(\text{char}(\hat{K}) = q > 0\), and in case \(\text{char}(K) = 0\), let \(v\) be discrete. Then \(v(K(q)) = qv(K(q))\), the residue field \(\hat{K}(q) = \hat{K}(q)\) of \((K(q), v_{K(q)})\) is perfect, and \(\text{abrd}_q(K(q)) = 0\).

Since the proof of Lemma relies on the presentability of cyclic \(K\)-algebras of degree \(q\) as \(q\)-symbol algebras over \(K\), we recall some basic facts related to such algebras over any field \(E\) with \(\text{char}(E) = q > 0\). Firstly, for each pair \(a \in E, b \in E^r, [a, b]_E \in s(E)\) and \(\text{deg}([a, b]_E) = q\) (cf. [20], Corollary 2.5.5). Secondly, if \([a, b]_E \in d(E)\), then the polynomial \(f_a(X) = X^q - X - a \in E[X]\) is irreducible over \(E\). This follows from the Artin-Schreier theorem (see [27], Ch. VI, Sect. 6), which also shows that if \(f_a(X)\) is irreducible over \(E\), then \(E(\xi)/E\) is a cyclic field extension, \([E(\xi) : E] = q\), and \(([a, b]_E\) is isomorphic to the cyclic \(E\)-algebra \((E(\xi)/E, \sigma, b)\), where \(\sigma\) is the \(E\)-automorphism of \(E(\xi)\) mapping \(\xi\) into \(\xi + 1\); hence, by [33], Sect. 15.1, Proposition b, \([a, b]_E \in d(E)\) if and only if \(b \notin N(E(\xi)/E)\).

Proof of Lemma. It is clear from Galois theory, the definition of \(K(q)\) and the closeness of the class of pro-\(q\)-groups under the formation of profinite group extensions that \(\hat{K}(q) = K(q)\), for every \(\hat{K} \in I(K(q)/K)\); in particular, \(K(q)(q) = K(q)\), which means that \(K(q)\) does not admit cyclic extensions of degree \(q\). As \((K(q), v_{K(q)})\) is Henselian, this allows to deduce from [10], Lemma 4.2, and [13], Lemma 2.3, that \(v(K(q)) = qv(K(q))\). We show that the field \(\hat{K}(q) = \hat{K}(q)\) is perfect. It follows from Lemma 5.1 and [13], Lemma 2.3, that in case \(\text{char}(K) = 0\) (and \(v\) is discrete), one may assume without loss of generality that \(v(q) \in qv(K)\). Denote by \(\Sigma\) the set of those fields \(\tilde{U} \in I(K(q)/K)\), for which \(v(\tilde{U}) = v(K), \tilde{U} \neq \hat{K}\) and \(\tilde{U}/\hat{K}\) is a purely inseparable extension. In view of Lemma 5.1, our extra hypothesis ensures that \(\Sigma \neq \emptyset\). Also, \(\Sigma\) is a partially ordered set with respect to set-theoretic inclusion, so it follows from Zorn’s lemma that it contains a maximal element, say \(U'\). Using again Lemma 5.1 one obtains that \(U'\) is a perfect field. Since \((K(q), v_{K(q)})/(U', v_{U'})\) is a valued extension, whence, \(\hat{K}(q)/\hat{U}'\) is an algebraic extension, this proves that \(\hat{K}(q)\) is perfect as well.

It remains to be seen that \(\text{abrd}_q(K(q)) = 0\). Suppose first that \(\text{char}(K) = q\), fix an algebraic closure \(\overline{K}\) of \(K_{\text{sep}}\), and put \(\bar{v} = v_{\overline{K}}\). It is known [1], Ch. VII, Theorem 22, that if \(K\) is perfect, then \(\text{Br}(K') = \{0\}\), for every finite extension \(K'/K\). We assume further that \(K\) is imperfect and \(K_{\text{ins}}\) is the perfect closure of \(K\) in \(\overline{K}\). It is easily verified that \(K_{\text{ins}}\) equals the union \(\bigcup_{\nu=1}^{\nu} K^q_{-\nu}\) of the fields \(K^q_{-\nu} = \{\beta \in K: \beta^q \in K\}, \nu \in \mathbb{N}\), and \([K^q_{-\nu} : K] \geq q^\nu\), for each index \(\nu\). To prove the equality \(\text{abrd}_q(K(q)) = 0\) it suffices to show that \(\text{Br}(L'/q) = \{0\}\), for an arbitrary \(L' \in \text{Fe}(K(q))\). Clearly,
We prove (5.1) by showing that, for any fixed \( K \)-primary degree, there is a finite extension \( K_1 \) of \( K(q) \) (depending on \( D \)), such that \( [D] \in \text{Br}(LK_1/L) \), i.e. the compositum \( LK_1 \) is a splitting field of \( D \). Our proof relies on the fact that \( K_{\text{ins}} \) is perfect. This ensures that \( \text{Br}(K_{\text{ins}})_q = \{0\} \) whenever \( K_{\text{ins}} \in I(K/K_{\text{ins}}) \), which implies \( \text{Br}(L_k)_q = \text{Br}(L_kK_{\text{ins}}/L_k) \), for every \( L_k \in \text{Fe}(L) \). Thus it turns out that \( [D] \in \text{Br}(L, J'/L') \), for some finite extension \( J' \) of \( K \) in \( K_{\text{ins}} \). In particular, \( J' \) lies in the set, say \( D \), of finite extensions \( I' \) of \( K \) in \( K_{\text{ins}} \), for which \( K \) has a finite extension \( \Delta' \) of \( K(q) \), such that \( [D \otimes_L \Delta'] \in \text{Br}(L\Delta'/L\Delta') \). Choose \( J \in D \) to be of minimal degree over \( K \). We prove that \( J = K \), by assuming the opposite. For this purpose, we use the following fact:

\[
(5.2) \text{For each } \beta \in K_{\text{ins}}^* \text{ and any nonzero element } \pi \in M_v(K(q)), \exists \beta' \in K(q)^*, \text{ such that } v(\beta' - \beta) > v(\pi).
\]

To prove (5.2) it is clearly sufficient to consider only the special case of \( v(\beta) \geq 0 \). Note also that if \( \beta \in K \), then one may put \( \beta' = \beta(1 + \pi^2) \), so we assume further that \( \beta \notin K \). A standard inductive argument leads to the conclusion that, one may assume, for our proof, that \( [K(\beta): K] = q^n \) and the assertion of (5.2) holds for any pair \( \beta_1 \in K_{\text{ins}}^*, \pi_1 \in M_v(K(q)) \setminus \{0\} \) satisfying \( [K(\beta_1): K] < q^n \). Since \( [K(\beta^n): K] = q^{n-1} \), our extra hypothesis ensures the existence of an element \( \tilde{\beta} \in K(q) \) with \( v(\tilde{\beta} - \beta^n) > qv(\pi) \). Applying Artin-Schreier’s theorem to the polynomial \( X^q - X - \tilde{\beta}^n \pi^{-q} \), one proves that the polynomial \( X^q - \pi^q(q-1)X - \tilde{\beta} \in K(q)[X] \) has a root \( \beta' \in K(q) \). In view of the inequality \( v(\tilde{\beta}) \geq 0 \), this implies consecutively that \( v(\beta') \geq 0 \) and \( v(\beta'^n - \tilde{\beta}) \geq q^2(q-1)v(\pi) \). As \( v(\beta' - \beta^n) > qv(\pi) \), it is now easy to see that \( v(\beta'^n - \tilde{\beta}) > qv(\pi) \), whence, \( v(\beta' - \beta) > v(\pi) \), as claimed by (5.2).

We continue with the proof of (5.1). The assumption that \( J \neq K \) shows that there exists \( I \in I(J/K) \) with \( [I: K] = [J: K]/q; \) this means that \( I \notin D \). Take an element \( b \in I \) so that \( J = I(\sqrt{b}) \) and \( v(b) \geq 0 \), and put \( \Lambda = \Lambda_J \otimes I, \Lambda' = \Lambda \Lambda \). As \( \tilde{K}(q) \) is a perfect field (i.e. \( \tilde{K}(q) = \tilde{K} \) and \( v(K(q)) = qv(K\tilde{q}) \)), one may assume, for our proof, that \( \Lambda_J \) is chosen so that \( b = b_J^2 \tilde{b} \), for some \( b_J \in O_v(\Lambda_J) \) and \( \tilde{b} \in \nabla(1_{\Lambda_J}) \).

Let now \( \Delta \) be the underlying division \( \Lambda' \)-algebra of \( D \otimes_L \Lambda' \). Then follows from [33], Sect. 13.4, Corollary, and the choice of \( J \) that \( \Delta \neq \Lambda' \) and \( [\Delta] \in \text{Br}(\Lambda'J/\Lambda') \). This implies \( \Delta \cong [a, b]_{\Lambda'} \) as \( \Lambda' \)-algebras, for some \( a \in \Lambda'^* \) (see, for instance, the end of the proof of [22], Theorem 3.2.1). It is therefore clear that the polynomial \( h_a(X) = X^q - X - a \in \Lambda'[X] \) has no root in \( \Lambda' \), so it follows from the Artin-Schreier theorem (see [27], Ch. VI, Sect. 6) that \( h_a \) is irreducible over \( \Lambda' \), and the field \( W_a = \Lambda'(\xi_a) \) is a degree \( q \) cyclic extension of \( K \).
\(\Lambda',\) where \(\xi_\alpha \in \mathbb{K}\) and \(h_\alpha(\xi_\alpha) = 0\). One also sees that \(W_\alpha\) is embeddable in \(\Delta\) as a \(\Lambda'\)-subalgebra, and \(\Delta\) is isomorphic to the cyclic \(\Lambda'\)-algebra \((W_\alpha/\Lambda', \sigma, b)\), for a suitably chosen generator \(\sigma\) of \(\mathcal{G}(W_\alpha/\Lambda')\). Because of the above-noted presentation \(b = b'\hat{b}\), this indicates that \(\Delta \cong (W_\alpha/\Lambda', \sigma, \hat{b})\). Note further that the extension \(W_\alpha/\Lambda'\) is not inertial. Assuming the opposite, one obtains from Lemma 5.2 that \(\hat{b} \in N(W_\alpha/K)\) which means that \(|\Delta| = 0\) (cf. [33], Sect. 15.1, Proposition b). Since \(\Delta \in d(\Lambda')\) and \(\Delta \neq \Lambda'\), this is a contradiction, proving our assertion. In view of Ostrowski’s theorem and the equality \([W_\alpha : \Lambda'] = q\), the considered assertion can be restated by saying that \(W_\alpha/\Lambda'\) is a purely inseparable extension of degree \(q\) unless \(\hat{W}_\alpha = \hat{\Lambda}'\).

Next we observe, using (3.1) (b), that \(\eta = (\xi_\alpha + 1)\xi_\alpha^{-1}\) is a primitive element of \(W_\alpha/\Lambda'\) and \(\eta \in O_v(W_\alpha)^*\); also, we denote by \(f_\eta(X)\) the minimal polynomial of \(\eta \in \Lambda'\), and \(D(f_\eta)\) the discriminant of \(f_\eta\). It is easily verified that \(f_\eta(X) \in O_v(W_\alpha)[X]\), \(f_\eta(0) = (-1)^q\), \(D(f_\eta) \neq 0\), and \(\nu(D(f_\eta)) = q\nu(f'_\eta(\eta)) > 0\) (the inequality is strict, since \([W_\alpha : K] = q\) and \(W_\alpha/\Lambda'\) is not inertial). Moreover, it follows from Ostrowski’s theorem that there exists \(\pi_0 \in O_v(K)\) of value \(\nu(\pi_0) = [K(D(f_\eta)): K]\nu(D(f_\eta))\). Note also that \(b\nu(b^{-1}) \in K^*\) (whence, \(q^{-1}\nu(b) \in v(K)\)), put \(\lambda' = \pi_0 b\nu(b^{-1})\), and let \(b'\) be the \(\eta\)-th root of \(b\) lying in \(K_{\text{ins}}\). Applying (5.2) to \(b'\) and \(\lambda'\) (which is allowed because of the inequalities \(\nu(\lambda') \geq \nu(\pi_0) > 0\)), one obtains that there is \(\lambda \in K(q)^*\) with \(\nu(\lambda' - b) > q\nu(\lambda')\). Consider now the fields \(\Lambda_J(\lambda), \Lambda(\lambda)\) and \(\Lambda'(\lambda)\) instead of \(\Lambda_J, \Lambda,\) and \(\Lambda',\) respectively. Clearly, \(\Lambda_J(\lambda)\) is a finite extension of \(K\) in \(\nu(\lambda)\), \(\Lambda(\lambda) = \Lambda_J(\lambda)\), and \(\Lambda'(\lambda) = L\Lambda(\lambda)\), so our choice of \(J\) indicates that one may assume, for the proof of (5.1), that \(\lambda \in \Lambda_J\).

We can now rule out the possibility that \(J \neq K\), by showing that \([a, b]_{\Lambda'} \notin d(\Lambda')\) (in contradiction with the choice of \(J\) which requires that \(I \notin \mathcal{D}\)). Indeed, the norm \(N_{\Lambda'}(\lambda \eta)\) is equal to \(\lambda^q\), and it follows from the equality \(\nu(\lambda' - b) > q\nu(\lambda')\) that \(v(\lambda' - b) > v(\pi_0)\). Thus it turns out that
\[
\nu(\lambda^q b^{-1} - 1) > q\nu(\lambda') - v(b) > v(\pi_0) \geq \nu(D(f_\eta)) = q\nu(f'_\eta(\eta)).
\]

Therefore, applying (3.1) to the polynomial \(f_\eta(X) + (-1)^q(\lambda^q b^{-1} - 1)\) and the element \(\eta\), one obtains that \(\lambda^q b^{-1} \) and \(b\) are contained in \(N(W_\alpha/\Lambda')\), which means that \([a, b]_{\Lambda'} \notin d(\Lambda')\), as claimed. Hence, \(J = K\), and by the definition of \(\mathcal{D}\), there exists a finite extension \(\Lambda_K\) of \(K\) in \(\nu(\lambda)\), such that \([\mathcal{D} \otimes_L \Lambda K] \in \text{Br}(L\Lambda_K/L\Lambda_K) = \{0\}\). In other words, \([\mathcal{D}] \in \text{Br}(L\Lambda_K/L)\), so (5.1) and the equality abrd\(_q\)(\(K(q)\)) = 0 are proved in case \(\text{char}(K) = q\).

Our objective now is to prove Lemma 5.3 in the special case where \(v\) is discrete. Clearly, one may assume, for our proof, that \(\text{char}(K) = 0\). Note that there exist fields \(\Psi_\nu \in I(K(q)/K), \nu \in \mathbb{N}\), such that \(\Psi_\nu/K\) is a totally ramified Galois extension with \([\Psi_\nu : K] = q^\nu\) and \(\mathcal{G}(\Psi_\nu/K)\) abelian of period \(q\), for each index \(\nu\), and \(\Psi_{\nu'} \cap \Psi_{\nu''} = K\) whenever \(\nu' \neq \nu'', \nu' \in \mathbb{N}\) and \(\nu' \neq \nu''\). This follows from [33], Lemma 2.3 (and Galois theory, which ensures that each finite separable extension has finitely many intermediate fields). Considering, if necessary, \(\Psi_1\) instead of \(K\), one obtains further that it is sufficient to prove Lemma 5.3 under the extra hypothesis that \(v(\lambda) \in q\nu(K)\).

In addition, the proof of the \(q\)-divisibility of \(\nu(K(q))\) shows that, for the proof of Lemma 5.3 one may consider only the special case where \(\hat{K}\) is perfect.
Let now $\Phi$ be a finite extension of $K$ in $K_{\text{sep}}$, and $\Omega \in d(\Phi)$ a division algebra, such that $[\Omega] \in \text{Br}(\Phi)_q$ and $[\Omega] \neq 0$. We complete the proof of Lemma 5.3 by showing that $[\Omega] \in \text{Br}(\Psi, \Phi \widetilde{\Phi})$, for every sufficiently large $v \in \mathbb{N}$. As $v$ is discrete and Henselian with $K$ perfect, the prolongation of $v$ on $\Phi$ (denoted also by $v$) and its residue field $\Phi$ preserve the same properties, so it follows from the assumptions on $\Omega$ that it is a cyclic NSR-algebra over $\Phi$, in the sense of [23]. In other words, there exists an inertial cyclic extension $Y$ of $\Phi$ in $K_{\text{sep}}$ of degree $[Y : \Phi] = \deg(\Omega)$, as well as an element $\pi \in \Phi^*$ and a generator $y$ of $\mathcal{G}(Y/\Phi)$, such that $v(\pi) \not\in qv(\Phi)$ and $\Omega$ is isomorphic to the cyclic $\Phi$-algebra $(Y/\Phi, y, \pi)$. It follows from Galois theory and our assumptions on the fields $\Psi, ~ \nu$, that $\Psi, \nu \cap Y = K$, for all $\nu$, with, possibly, finitely many exceptions. Fix $\nu$ so that $\Psi, \nu \cap Y = K$ and $\deg(\Omega).q^\mu \leq q^\nu$, where $\mu$ is the greatest integer for which $q^\mu \mid [\Phi : K]$. Put $\Omega_\nu = \Omega \otimes_\Phi \Psi, \nu$ and denote by $v_\nu$ the valuation of $\Psi, \nu \Phi$ extending $v$. It is easily obtained from Galois theory and the choice of $\nu$ (cf. [27], Ch. VI, Theorem 1.12) that $\Psi, \nu Y/\Psi, \nu \Phi$ is a cyclic extension, $[\Psi, \nu Y : \Psi, \nu \Phi] = [Y : \Phi] = \deg(\Omega)$, $y$ extends uniquely to a $\Psi, \nu \Phi$-automorphism $y_\nu$ of $\Psi, \nu Y$, $y_\nu$ generates $\mathcal{G}(\Psi, \nu Y/\Psi, \nu \Phi)$, and $\Omega_\nu$ is isomorphic to the cyclic $\Psi, \nu \Phi$-algebra $(\Psi, \nu Y/\Psi, \nu Y, y_\nu, \pi)$. Also, the assumptions on $\Psi, \nu$ show that $v_\nu(\pi) \in q^{\nu-\mu}v_\nu(\Psi, \nu \Phi)$. Therefore, by the theory of cyclic algebras (cf. [33], Sect. 15.1), and the divisibility $\deg(\Omega) \mid q^{\nu-\mu}$, $\Omega_\nu$ is $\Psi, \nu \Phi$-isomorphic to $(\Psi, \nu Y/\Psi, \nu \Phi, y_\nu, \nu_\nu, \lambda_\nu)$, for some $\lambda_\nu \in O_{\nu_\nu}(\Psi, \nu \Phi)^*$. Since $K$ is perfect (that is, $K = K^{\nu, q}$, for each $\nu \in \mathbb{N}$), a similar argument shows that $\lambda_\nu$ can be chosen to be an element of $\mathcal{V}_0(\Psi, \nu \Phi)$. Taking also into account that $\Psi, \nu Y/\Psi, \nu \Phi$ is inertial, one obtains from Lemma 5.2 that $\lambda_\nu \in N(\Psi, \nu Y/\Psi, \nu \Phi)$. Hence, by the cyclicity of $\Psi, \nu Y/\Psi, \nu \Phi$, $[\Omega_\nu] = 0$, i.e. $[\Omega] \in \text{Br}(\Psi, \nu Y/\Psi, \nu \Phi)$. As $\Psi, \nu \in I(K(q)/K)$ and $\Omega \in d(\Phi)$ represents an arbitrary nonzero element of $\text{Br}(\Phi)_q$, now it becomes clear that $\text{Br}(\Phi)_q = \text{Br}(K(q)/\Phi)$, for each $\Phi \in \text{Fe}(K)$, so Lemma 5.3 is proved.

At the end of this Section, we prove two lemmas which show that $\dim(K_{\text{sol}}) \leq 1$ whenever $K$ is a field satisfying the conditions of Lemma 2.3.

**Lemma 5.4.** Let $(K, v)$ be a Henselian field with $\text{char}(\widehat{K}) = q$ and $\dim(K_{\text{sol}}) \leq 1$, and in case $\text{char}(K) = 0 < q$, let $v$ be discrete. Then $\dim(K_{\text{sol}}) \leq 1$.

**Proof.** Put $\mathbb{P}' = \mathbb{P} \setminus \{q\}$, and for each $p \in \mathbb{P}'$, fix a primitive $p$-th root of unity $\varepsilon_p \in K_{\text{sep}}$ and a field $T_p(K) \in I(K_{ur}/K)$ in accordance with Lemma 5.2 (b) and (c). Note first that the compositum $T(K)$ of fields $T_p(K)$, $p \in \mathbb{P}'$, is a subfield of $K_{\text{sol}}$. Indeed, $T_p(K) \in I(K(\varepsilon_p)(p)/K)$, for each $p \in \mathbb{P}'$, so our assertion follows from Galois theory, the cyclicity of the extension $K(\varepsilon_p)/K$, and the fact that finite solvable groups form a closed class under taking subgroups, quotient groups and group extensions. Secondly, Lemma 3.1 implies the field $K_{ur} \cap K_{\text{sol}} := U$ satisfies $\widehat{U} = \widehat{K}_{\text{sol}}$. Observing also that $v(T(K)) = pv(T(K))$, $p \in \mathbb{P}'$, and $\dim(K_{\text{sol}}) \leq 1$, and using (3.2) (a), (3.3) (a) and [23], Theorem 2.8, one obtains consecutively that $v(K') = pv(K')$, $\text{Br}(K') \cong \text{Br}(K')_p \cong \text{Br}(K')_p$ and $\text{Br}_{d}(K') = \text{Br}_{d}(K') = 0$, for each $p \in \mathbb{P}'$ and every finite extension $K'$ of $K_{\text{sol}}$. When $q = 0$, this proves Lemma 5.3 and in case $q > 0$, our proof is completed by applying Lemma 5.3. \qed
Lemma 5.5. Let $K_m$ be a complete $m$-discretely valued field with \( \dim(K_{0, \text{sol}}) \leq 1 \), $K_0$ being the $m$-th residue field of $K_m$. Then $\dim(K_{m, \text{sol}}) \leq 1$.

Proof. In view of Lemma 5.4, one may consider only the case where $m \geq 2$. Denote by $K_{m-j}$ the $j$-th residue field of $K_m$, for $j = 1, \ldots, m$. Suppose first that $\text{char}(K_m) = \text{char}(K_0)$. Using repeatedly Lemma 5.4, one obtains that $\dim(K_{m, \text{sol}}) \leq 1$, which allows to assume, for the rest of our proof, that $\text{char}(K_m) = 0$ and $\text{char}(K_0) = q > 0$. Let $\mu$ be the maximal integer for which $\text{char}(K_{m-\mu}) = 0$. Then $0 \leq \mu < m$, $\text{char}(K_{m-\mu-1}) = q$, and in case $\mu < m-1$, $K_{m-\mu-1}$ is a complete $m-1$-discrete valued field with last residue field $K_0$; also, $K_{m-\mu}$ is a complete discrete valued field with a residue field $K_{m-\mu-1}$. Therefore, Lemma 5.4 yields $\dim(K_{m-\mu', \text{sol}}) \leq 1$, for $\mu' = \mu, \mu + 1$. Note finally that if $\mu > 0$, then $K_m$ is a complete $\mu$-discretely valued field with $\mu$-th residue field $K_{m-\mu}$, and by Lemma 5.4 (used repeatedly), $\dim(K_{m-m', \text{sol}}) \leq 1$, $m' = 0, \ldots, \mu - 1$, as required. 

6. Tame version of Lemma 2.3 for admissible Henselian fields

Let $(K, v)$ be a Henselian field with $\hat{K}$ of arithmetic type and characterisite $q$, put $\mathbb{P}_q = \mathbb{P} \setminus \{q\}$, and suppose that $\text{abrd}_p(K) < \infty$, $p \in \mathbb{P}$, and $R$ is a central division LBD-algebra over $K$. Our main objective in this Section is to prove a modified version of Lemma 2.3, where the fields $E_p$, $p \in \mathbb{P}$, are replaced by tamely ramified extensions $V_p$, $p \in \mathbb{P}_q$, of $K$ in $K_{\text{sep}}$, chosen so as to satisfy the following conditions, for each $p \in \mathbb{P}_q$: $V_p$ is a $p$-splitting field of $R/K$, $[V_p : K]$ is a $p$-primary number, and $V_p \cap K_{\text{ur}} \subseteq K(p)$. The desired modification is stated as Lemma 6.6 and is also called a tame version of Lemma 2.3. Our first step towards this goal can be formulated as follows:

Lemma 6.1. Let $(K, v)$ be a Henselian field and let $T/K$ be a tamely totally ramified extension of $p$-primary degree $[T : K] > 1$, for some $p \in \mathbb{P}$. Then there exists a degree $p$ extension $T_1$ of $K$ in $T$. Moreover, $T_1/K$ is a Galois extension if and only if $K$ contains a primitive $p$-th root of unity.

Proof. Our assumptions show that $v(T)/v(K)$ is an abelian $p$-group of order equal to $[T : K]$, whence, there is $\theta \in T$ with $v(\theta) \neq v(K)$ and $pv(\theta) \in v(K)$. Therefore, it follows that $K$ contains elements $\theta_0$ and $a$, such that $v(\theta_0) = pv(\theta) = v(\theta^p)$, $v(a) = 0$ and $v(\theta^p - \theta_0a) > 0$. This implies the existence of an element $\theta' \in T$ satisfying $v(\theta') > 0$ and $\theta^p = \theta_0a(1 + \theta')$. Note further that, by the assumption on $T/K$, $p \neq \text{char}(T)$ and $T = \hat{K}$; hence, by (3.1) (a), applied to the binomial $X^p - (1 + \theta')$, $1 + \theta' \in T^{sp}$. More precisely, $1 + \theta' = (1 + \theta_1)^p$, for some $\theta_1 \in T$ of value $v(\theta_1) > 0$. Observing now that $v(\theta_0a) \notin pv(K)$ and $(\theta(1 + \theta_1)^{-1})^p = \theta_0a$, one obtains that the field $T_1 = K(\theta(1 + \theta_1)^{-1})$ is a degree $p$ extension of $K$ in $T$. Suppose finally that $\varepsilon$ is a primitive $p$-th root of unity lying in $T_{\text{sep}}$. It is clear from the noted properties of $T_1$ that $T_1(\varepsilon)$ is the Galois closure of $T_1$ in $T_{\text{sep}}$ over $K$. Since $[K(\varepsilon) : K] | p - 1$ (see [27], Ch. VI, Sect. 3), this ensures that $T_1/K$ is a Galois extension if and only if $\varepsilon \in K$, so Lemma 6.1 is proved. 

□
The fields $V_p(K)$, $p \in \mathbb{P} \setminus \{\text{char}(\hat{K})\}$, singled out by the next lemma play the same role in our tame version of Lemma 2.3 as the role of the maximal $p$-extensions $K(p)$, $p \in \mathbb{P}$, in the original version of Lemma 2.3.

**Lemma 6.2.** Let $(K,v)$ be a Henselian field with $\text{abrd}_p(\hat{K}(p)) = 0$, for some $p \in \mathbb{P}$ different from $\text{char}(\hat{K})$. Fix $T_p(K) \in I(T(K)/K)$ in accordance with Lemma 5.2 (c), and put $K_0(p) = K(p) \cap K_{ur}$. Then $\text{abrd}_p(V_p(K)) = 0$, where $V_p(K) = K_0(p)T_p(K)$.

**Proof.** It follows from Lemma 3.2 (b) and (c) that $v(T') = pv(T')$, for any $T' \in I(K_{\text{sep}}/T_p(K))$; therefore, if $D' \in d(T')$ is of $p$-primary degree $\geq p$, then it is neither totally ramified nor NSR over $T'$. As $p \neq \text{char}(\hat{K})$, this implies in conjunction with Decomposition Lemmas 5.14 and 6.2 of [23], that $D'/T'$ is inertial. Thus it turns out that $\text{Br}(\hat{T}_p)$ must be nontrivial. Suppose now that $T' \in I(K_{\text{sep}}/V_p(K))$. Then $\hat{T}'/\hat{K}(p)$ is a separable field extension, so the condition that $\text{abrd}_p(\hat{K}(p)) = 0$ requires that $\text{Br}(\hat{T}'_p) = \{0\}$, i.e. $\text{Br}_p(T') = 0$. Since the field $T'$ is an arbitrary element of $I(K_{\text{sep}}/V_p(K))$, this proves Lemma 6.2. $\square$

The following lemma presents the main properties of finite extensions of $K$ in $V_p(K)$, which are used for proving Lemma 2.3.

**Lemma 6.3.** In the setting of Lemma 6.2, let $V$ be an extension of $K$ in $V_p(K)$ of degree $p^j > 1$. Then there exist fields $\Sigma_0, \ldots, \Sigma_\ell \in I(V/K)$, such that $[\Sigma_j : K] = p^j$, $j = 0, \ldots, \ell$, and $\Sigma_{j-1} \subset \Sigma_j$ for every index $j > 0$.

**Proof.** By Lemma 3.1 (d), the field $K$ has an inertial extension $V_0$ in $V$ with $\hat{V}_0 = \hat{V}$. Moreover, it follows from (3.2) (a) and the inequality $p \neq \text{char}(\hat{K})$ that $V/V_0$ is totally ramified. Considering the extensions $V_0/K$ and $V/V_0$, one concludes that it is sufficient to prove Lemma 6.3 in the special case where $V_0 = V$ or $V_0 = K$. If $V_0 = V$, then our assertion follows from Lemma 5.1 (c), Galois theory and the subnormality of proper subgroups of finite $p$-groups (cf. [27], Ch. I, Sect. 6). When $V_0 = K$, by Lemma 6.2, there is a degree $p$ extension $V_1$ of $K$ in $V$. As $V/V_1$ is totally ramified, this allows to complete the proof of Lemma 6.3 by a standard inductive argument. $\square$

Lemma 5.3 and our next lemma characterize the fields of arithmetic type among all fields admissible by some of Theorems 2.1 and 2.2. These lemmas show that an $m$-dimensional local field is of arithmetic type if and only if $m = 1$. They also prove that if $(K,v)$ is a Henselian field with $\text{char}(K) = \text{char}(\hat{K})$, then $K$ is a field of arithmetic type, provided that it is virtually perfect, $\hat{K}$ is of arithmetic type, $v(K)/pv(K)$ are finite groups, for all $p \in \mathbb{P}$, and $\hat{K}$ contains a primitive $p$-th root of unity, for each $p \in \mathbb{P} \setminus \{\text{char}(\hat{K})\}$.

**Lemma 6.4.** Assume that $(K,v)$ and $p$ satisfy the conditions of Lemma 6.3, $\hat{K}_p = \hat{K}_{\text{sep}}$ is a primitive $p$-th root of unity, and $\tau(p)$ is the dimension of the group $v(K)/pv(K)$, viewed as a vector space over the field $\mathbb{Z}/p\mathbb{Z}$. Then $\text{abrd}_p(K(p)) = 0$ unless $\hat{e} \notin \hat{K}$ and $\tau(p) + \text{cd}_p(G_{K(p)}) \geq 2$. 

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Proof. It follows from (3.2) (a) and Lemma 3.1 that if \( v(K) = pv(K) \), then \( K(p) \subseteq K_{ur} \), whence, \( K(p) = K_0(p) = V_p \), and by Lemma 6.2 \( \text{abrd}_p(K(p)) = 0 \). This agrees with the assertion of Lemma 6.3 in case \( v(K) = pv(K) \), since \( p \neq \text{char}(\hat{K}) \) and, by Galois cohomology, we have \( \text{abrd}_p(\hat{K}(p)) = 0 \) if and only if \( \text{cd}_p(G_{\hat{K}(p)}) \leq 1 \) (see [20], Theorem 6.1.8, or [34], Ch. II, 3.1). Therefore, we assume in the rest of the proof that \( v(K) \neq pv(K) \). Fix a primitive \( p \)-th root of unity \( \varepsilon \in K_{sep} \), and as in Lemma 6.3, consider a finite extension \( V \) of \( K \) in \( V_p \). It is easily verified that \( \varepsilon \in K \) if and only if \( \hat{\varepsilon} \in \hat{K} \), and this holds if and only if \( \varepsilon \in V_p(K) \). Suppose that \( [V: K] = p^f > 1 \) and take fields \( \Sigma_j \subseteq I(V/K) \), \( j = 0, 1, \ldots, \ell \), as required by Lemma 6.3. Observing that \( K(p) = K_1(p) \), for any \( K_1 \in I(K(p)/K) \), and using Galois theory and the normality of maximal subgroups of nontrivial finite \( p \)-groups, one obtains that \( V \subseteq I(K(p)/K) \) if and only if \( \Sigma_j/\Sigma_{j-1} \) is a Galois extension, for every \( j \geq 1 \). In view of Lemma 6.1, this occurs if and only if \( \varepsilon \in K \) or \( V \subseteq I(K_0(p)/K) \). It is now easy to see that \( K(p) = V_p(K) \) if \( \varepsilon \in K \), and \( K(p) = K_0(p) \), otherwise. Hence, by Lemma 6.2 \( \text{abrd}_p(K(p)) = 0 \) in case \( \varepsilon \in K \), as claimed by Lemma 6.1.

Assume finally that \( v(K) \neq pv(K) \) and \( \varepsilon \notin K \) (in this case, \( p > 2 \)). It is easy to see that if \( \text{cd}_p(G_{\hat{K}(p)}) = 1 \), then there is a finite extension \( Y \) of \( K_0(p) \) in \( K_{ur} \), such that \( \hat{Y}(p) \neq Y \). Therefore, there exists a degree \( p \) cyclic extension \( Y' \) of \( Y \) in \( K_{ur} = Y.K_{ur} \), which ensures the existence of a nicely semi-ramified \( Y \)-algebra \( \Lambda \in d(Y) \), in the sense of [23], of degree \( p \): this yields \( \text{abrd}_p(K_0(p)) \geq \text{Brd}_p(Y') \geq 1 \). The inequality \( \text{abrd}_p(K_0(p)) \geq 1 \) also holds if \( \tau(p) \geq 2 \), i.e. \( v(K)/pv(K) \) is noncyclic. Indeed, then \( \text{Brd}_p(K_0(p)\varepsilon) \geq 1 \); this follows from the fact that \( v(K_0(p)\varepsilon) = v(K) \), which implies the symbol \( K_0(p)\varepsilon \)-algebra \( A_\varepsilon(a_1, a_2; K_0(p)\varepsilon) \) (defined, e.g., in [23]) is a division one whenever \( a_1 \) and \( a_2 \) are elements of \( K^* \) chosen so that the cosets \( v(a_i) + pv(K) \), \( i = 1, 2 \), generate a subgroup of \( v(K)/pv(K) \) of order \( p^2 \).

In order to complete the proof of Lemma 6.3 it remains to be seen that \( \text{abrd}_p(K_0(p)) = 0 \) in case \( \text{cd}_p(G_{\hat{K}(p)}) = 0 \) and \( \tau(p) = 1 \). Since \( p \neq \text{char}(\hat{K}) \), this is the same as to prove that \( \text{cd}_p(G_{\hat{K}_0(p)}) \leq 1 \). As \( K_0(p) = K_{ur} \cap K(p) \), we have \( v(K_0(p)) = v(K) \) and \( K_0(p) = \hat{K}(p) \), so it follows from [9], Lemma 1.2, that \( \text{cd}_p(G_{\hat{K}_0(p)}) \leq \text{cd}_p(G_{\hat{K}(p)}) + \tau(p) = 1 \), as claimed.

Remark 6.5. Summing-up Lemmas 3.4, 3.5 and 6.4, one obtains a complete valuation-theoretic characterization of the fields of arithmetic type among the maximally complete fields \( (K, v) \) with \( \text{abrd}_p(K) < \infty \), for every \( p \in \mathbb{P} \).

As demonstrated in the proof of Corollary 6.6 this fully describes the class \( C_0 \) of those fields of arithmetic type, which lie in the class \( C \) of generalized formal power series fields of finite absolute Brauer \( p \)-dimensions. Note that \( C \) is considerably larger than \( C_0 \). For example, if \( K_0 \) is a finite field and \( \Gamma \) is an ordered abelian group with finite quotients \( \Gamma/p\Gamma \), for all \( p \in \mathbb{P} \), then \( K_0(\Gamma) \in C \setminus C_0 \) in case \( \Gamma/p\Gamma \) are noncyclic, for infinitely many \( p \).

The conclusion of Lemma 6.4 remains valid if \( K \) is an arbitrary field, \( p \in \mathbb{P} \), and \( V \) is a finite extension of \( K \) in \( K(p) \) of degree \( p^f > 1 \); then the extensions \( \Sigma_j/\Sigma_{j-1} \), \( j = 1, \ldots, \ell \), are Galois of degree \( p \) (see [27], Ch. I, ...
Sect. 6). Considering the proof of Lemma 6.4, one also sees that, in the setting of Lemma 6.3, \( V_p(K) \subseteq K(p) \) if and only \( \hat{K} \) contains a primitive \( p \)-th root of unity or \( v(\hat{K}) = pv(K) \). These observations and Lemma 6.4 allow to view the following result as a tame version of Lemma 2.3.

**Lemma 6.6.** Assume that \( K, q \) and \( R \) satisfy the conditions of Theorem 2.1 or Theorem 2.2. Put \( \mathcal{P}' = \mathcal{P} \setminus \{q\} \), and for each \( p \in \mathcal{P}' \), denote by \( k(p) \) the \( p \)-power of \( R/K \), and by \( V_p(K) \) the extension of \( K \) in \( K_{sep} \) singled out by Lemma 6.2. Then there exist finite extensions \( V_p \) of \( K \), \( p \in \mathcal{P}' \), with the following properties, for each \( p \):

(c) \( V_p \) is a \( p \)-splitting field of \( R/K \), i.e. \( p \) does not divide \( [V_p(\delta_p) : V_p] \), for any element \( \delta_p \) of the underlying central division \( V_p \)-algebra \( \Delta_p \) of \( R \otimes_K V_p \);

(c) \( V_p \in I(V_p(K)/K) \), so \( [V_p : K] = p^{k(p)} \), for some integer \( k(p) \geq p \), and the maximal inertial extension \( U_p \) of \( K \) in \( V_p \) is a subfield of \( K(p) \).

**Proof.** It is clearly sufficient to show that \( R \otimes_K V_p(K) \cong M_{p^{k(p)}}(R') \), for some central division \( V_p(K) \)-algebra \( R' \). Our proof relies on the inclusion \( V_p(K) \subseteq K_{sol} \). In view of the \( V_p(K) \)-isomorphism \( R \otimes_K V_p(K) \cong (R \otimes_K Y_p) \otimes_{Y_p} V_p(K) \), for each \( Y_p \in I(V_p(K)/K) \), this enables one to obtain from Lemma 6.1 that \( R \otimes_K V_p(K) \) is \( V_p(K) \)-isomorphic to \( M_{s(p)}(R_p) \), for some central division \( V_p(K) \)-algebra \( R_p \) and some \( s(p) \in \mathbb{N} \) dividing \( p^{k(p)} \). In order to complete the proof of Lemma 6.6 we show that \( p^{k(p)} \mid s(p) \). Since, by Lemma 6.2, abrds \( V_p(K) = 0 \), it can be deduced from [7], Lemma 3.6, that for any finite extension \( Y' \) of \( V_p(K) \), \( R_p \otimes_{V_p(K)} Y' \) is isomorphic as an \( Y' \)-algebra to \( M_{y'}(R') \), for some \( y' \in \mathbb{N} \) not divisible by \( p \), and some central division \( LBD \)-algebra \( R' \) over \( Y' \). Note further that there is an \( Y' \)-isomorphism \( R \otimes_K Y' \cong (R \otimes_K Y) \otimes_Y Y' \), for any \( Y \in I(Y'/K) \). This, applied to the case where \( Y = V_p(K) \), and together with the Wedderburn-Artin theorem and [33], Sect. 9.3, Corollary b, leads to the conclusion that \( R \otimes_K Y' \cong M_{s(p),y'}(R') \) as \( Y' \)-algebras. Considering again an arbitrary \( Y \in I(Y'/K) \), one obtains similarly that if \( R_Y \) is the underlying division \( Y \)-algebra of \( R \otimes_K Y \), then there exists an \( Y \)-isomorphism \( R \otimes_K Y \cong M_{y}(R_Y) \), for some \( y \in \mathbb{N} \) dividing \( s(p),y' \). Suppose now that \( Y' = V_p(K) \), for some finite extension \( Y \) of \( K \) in an algebraic closure of \( V_p(K) \), such that \( p^{k(p)} \mid [Y : K] \) and \( Y \) embeds in \( R \) as a \( K \)-subalgebra. Then, by the previous observation, \( p^{k(p)} \mid s(p),y' \); since \( p \nmid y' \), this implies \( p^{k(p)} \mid s(p) \) and so completes the proof of Lemma 6.6. □

Lemmas 6.2, 6.3, 6.6 and the results of Sections 4 and 5 give us the possibility to deduce Lemma 2.3 by the method of proving [7], Lemma 8.3. This is done in the following two Sections in two steps.

### 7. A special case of Lemma 2.3

Let \( K \) be a field and \( R \) a central division \( LBD \)-algebra over \( K \) satisfying the conditions of Theorem 2.1 or Theorem 2.2, and put \( q = \text{char}(K_0) \) in the former case, \( q = \text{char}(K) \) in the latter one. This Section gives a proof of Lemma 2.3 in the case where \( q \) does not divide \( [K(r) : K] \), for any \( r \in R \). In order to achieve this goal we need the following two lemmas:
Lemma 7.1. Let \((K, v)\) be a field with \(\dim(K_{ad}) \leq 1\) and \(\text{abrd}_\ell(K) < \infty\), for all \(\ell \in \mathbb{P}\). Fix \(p \in \mathbb{P}\) and a field \(M \in I(M'/K)\), for some finite Galois extension \(M'\) of \(K\) in \(K_{\text{sep}}\) with \(G(M'/K)\) nilpotent and \([M': K]\) not divisible by \(p\). Assume that \(R\) is a central division LBD-algebra over \(K\), \(R_M\) is the underlying division \(M\)-algebra of \(R \otimes_K M\), and there is an \(M\)-subalgebra \(\Delta_M\) of \(R_M\), such that all the following (equivalent) conditions hold:

(a) \(M\) is a \(p\)-splitting field of \(R/K\), for every \(p' \in \mathbb{P}\) dividing \([M : K]\);

(b) \(\Delta_M \in d(M)\) and \(\deg(\Delta_M) = p^k(p)\), where \(k(p)\) is the \(p\)-power of \(R/K\);

(c) \(\Delta_M \not\sim M\) is a maximal subgroups of nilpotent finite groups (established by the Burnside-Wielandt theorem, see [24], Theorem 17.1.4) and from the existence of an \(R\)-isomorphism \((R \otimes_K M_0) \otimes_{M_0} M\) are isomorphic, one concludes that conditions (c) and (cc) of Lemma 7.1 are fulfilled again. Therefore, a standard inductive argument shows that it suffices to prove Lemma 7.1 under the extra hypothesis that there exists a subalgebra \(\Delta_0 \in d(M_0)\) of \(R_0\), such that \(\Delta_0 \otimes_{M_0} M \cong \Delta_M\) as \(M\)-algebras. Let \(\varphi\) be a \(K\)-automorphism of \(M_0\) of order \(p\), and let \(\hat{\varphi}\) be the unique \(K\)-isomorphism of \(R \otimes_K M_0\) extending \(\varphi\) and acting on \(R\) as the identity. Then it follows from the Skolem-Noether theorem (cf. [22], Theorem 4.3.1) and from the existence of an \(M_0\)-isomorphism \(R \otimes_K M_0 \cong M_0 \otimes_{M_0} R_0\) (where \(p^*_0 = p\) or \(p^* = 1\) depending on whether or not \(M_0\) is embeddable in \(R\) as a \(K\)-subalgebra) that \(R_0\) has a \(K\)-automorphism \(\hat{\varphi}\) extending \(\varphi\). Note also that \(p \nmid [M_0(z_0): M_0]\), for any \(z_0 \in C_{R_0}(\Delta_0)\). This is implied by Lemma 7.1, condition (cc) of Lemma 7.1 and the fact that \(C_{R_M}(\Delta_M)\) is the underlying division \(M\)-algebra of \(C_{R_0}(\Delta_0) \otimes_{M_0} M\). Hence, by Lemma 7.1, \(\Delta_0\) is \(M_0\)-isomorphic to its image \(\Delta'_0\) under \(\hat{\varphi}\), so it follows from the Skolem-Noether theorem that \(\varphi\) extends to a \(K\)-automorphism of \(\Delta_0\). Since \(p \nmid [M_0 : K]\) and \(\deg(\Delta_0) = \deg(\Delta) = p^k(p)\), this enables one to deduce from Teichmüller’s theorem (cf. [13], Sect. 9, Theorem 4) and [7], Lemma 3.5, that there exists an \(M_0\)-isomorphism \(\Delta_0 \cong \Delta \otimes_K M_0\), for some central \(K\)-subalgebra \(\Delta\) of \(R\).

Lemma 7.2. Let \((K, v)\) be a Henselian field with \(\text{abrd}_\ell(K) < \infty\), \(\ell \in \mathbb{P}\), and \(\hat{K}\) of arithmetic type, and let \(R\) be a central division LBD-algebra over \(K\). Fix a primitive \(p\)-th root of unity \(\varepsilon \in K_{\text{sep}}\), for some \(p \in \mathbb{P}\), \(p \neq \text{char}(\hat{K})\), and suppose that \(\dim(K_{ad}) \leq 1\) and \(R\) satisfies the following conditions:

(i) \(p^2\) and \(\text{char}(\hat{K})\) do not divide the degree \([K(\delta): K]\), for any \(\delta \in R\);

(ii) There is a \(K\)-subalgebra \(\Theta\) of \(R\), which is a totally ramified extension of \(K\) of degree \([\Theta : K] = p\).

Then there exists a central \(K\)-subalgebra \(\Delta\) of \(R\), such that \(\deg(\Delta) = p\) and \(\Delta\) possesses a \(K\)-subalgebra isomorphic to \(\Theta\). Moreover, if \(\varepsilon \notin K\), then \(\Delta\) contains as a \(K\)-subalgebra an inertial cyclic extension of \(K\) of degree \(p\).
Proof. As in the proof of Lemma 6.1 one obtains from the assumption on \( \Theta/K \) that \( \Theta = K(\xi) \), where \( \xi \) is a \( p \)-th root of an element \( \theta \in K^* \) of value \( v(\theta) \not\in pv(K) \). Suppose first that \( \varepsilon \in K \). Then \( \Theta/K \) is a cyclic extension, so it follows from the Skolem-Noether theorem that there exists \( \eta \in R^* \), such that \( \eta \xi \eta^{-1} = \varepsilon \xi \). As a first step towards our proof, we show that \( \eta \) can be chosen so as to satisfy the following:

(7.1) The field extension \( K(\eta^p)/K \) is inertial.

Put \( \eta^p = \rho, B = K(\rho) \), and \( r = [B : \mathcal{B}] \), where \( \mathcal{B} \) is the maximal inertial extension of \( K \) in \( B \). It is easily verified that \( \xi \rho = \rho \xi \). Since \( \xi \eta \neq \xi \xi \) and \( \varepsilon \in K \), this means that \( \eta \notin B \) and \( [K(\eta) : K] = p \). Observing that \( [K(\eta) : K] = [K(\eta) : B][B : K] \), and by assumption, \( p^2 \mid [K(\eta) : K] \), one also obtains that \( p \mid [B : K] \). Therefore, \( p \mid r \), whence, the pairs \( \xi, \eta \) and \( \xi, \eta^r \) generate the same \( K \)-subalgebra of \( R \). Similarly, condition (i) of Lemma 7.2 shows that \( \text{char}(\hat{K}) \mid r \), which leads to the conclusion that the set of those \( b \in B \), for which \( v(b) = r v(B) \) equals the inner group product \( B^* \mathcal{N}_v(B) \).

Since, by the Henselian property of \( (B, v_B) \), \( \mathcal{N}_v(B) \subset B^{*r} \), this observation indicates that there exists a pair \( \rho_0 \in B^*, \rho_1 \in B^*, \) such that \( \rho^r = \rho_0 \rho_1^r \).

Putting \( \eta_1 = \eta \rho_1^{-1} r^r \), for a fixed \( r^r \in \mathbb{N} \) satisfying \( r^r \equiv 1 \pmod{p} \), one obtains that \( \eta_1 \xi \eta_1^{-1} = \varepsilon \xi \) and \( \eta_1^p = \rho_0^r \in B \), which proves (7.1).

Our objective now is to prove the existence of a \( K \)-subalgebra \( \Delta \) of \( R \) with the properties required by Lemma 7.2. Let \( \mathbb{P}' = \mathbb{P} \setminus \{ \text{char}(\hat{K}), p \} \), and for each \( p' \in \mathbb{P}' \), take an extension \( V_{p'} = K \) in \( K_{p'} \) in accordance with Lemma 6.6 and put \( \mathcal{U}_{p'} = V_{p'} \cap K_{ur} \). Consider a sequence \( \Pi_n, n \in \mathbb{N} \), of pairwise distinct finite subsets of \( \mathbb{P}' \), such that \( \cup_{n=1}^\infty \Pi_n = \mathbb{P}' \) and \( \Pi_n \subset \Pi_{n+1} \), for each index \( n \). Denote by \( W_n \) the compositum of the fields \( V_{p_n}, p_n \in \Pi_n \), and by \( R_n \) the underlying division \( W_n \)-algebra of \( R \otimes_K W_n \), for any \( n \). We show that \( W_n, R_n \) and the compositum \( \Theta_n = \Theta W_n \) satisfy conditions (i) and (ii) of Lemma 7.2. It is easily verified that \( [W_n : K] = \prod_{p_n \in \Pi_n} [V_{p_n} : K] \); in view of (3.2) (a), this ensures that \( \Theta_n/W_n \) is a totally ramified extension of degree \( p \). Using the fact that \( R \otimes_K \Theta W_n \) is isomorphic to the \( W_n \)-algebras \( (R \otimes_K \Theta) \otimes_{\Theta} \Theta W_n \) and \( (R \otimes_K W_n) \otimes_{\Theta} W_n \) (cf. \[ \{ \} \], Sect. 9.4, Corollary a), one obtains from \[ \{ \} \], Lemma 3.5, and the uniqueness part of the Wedderburn-Artin theorem, that \( \Theta_n \) embeds in \( R_n \) as a \( W_n \)-subalgebra. Note also that \( W_n \) and \( R_n \) satisfy condition (i) of Lemma 7.2 since \( p \) and \( \text{char}(\hat{K}) \) do not divide \( [W_n : K] \), this follows from Lemma 6.1 and \[ \{ \} \], Lemma 3.5.

The next step towards our proof of the lemma can be stated as follows:

(7.2) When \( n \) is sufficiently large, \( R_n \) has a \( W_n \)-subalgebra \( \Delta_n \in d(W_n) \), such that \( \text{deg}(\Delta_n) = p \) and \( \Theta_n \) embeds in \( \Delta_n \) as a \( W_n \)-subalgebra.

Our proof of (7.2) relies on Lemma 4.1 and the choice of the fields \( W_\nu, \nu \in \mathbb{N} \), which indicate that for any \( \nu \) and each \( \delta_\nu \in R_\nu \), \( [W_\nu(\delta_\nu) : W_\nu] \) is not divisible by any \( p_\nu \in \Pi_\nu \). Arguing as in the proof of Lemma 4.3, given in \[ \{ \} \], one obtains from (7.1) the existence of a finite dimensional \( W_\nu \)-subalgebra \( \Lambda_\nu \) of \( R_\nu \) satisfying the following:

(7.3) (i) The centre \( B_\nu \) of \( \Lambda_\nu \) is an inertial extension of \( W_\nu \) of degree not divisible by \( \text{char}(\hat{K}) \), \( p \) and any \( p_\nu \in \Pi_\nu \); moreover, by (3.2) (a) and
Lemma \([8.1]\) (d), \(B_\nu = B_\nu W_\nu\) and \([B_\nu : W_\nu] = [B_\nu : W_\nu]\), where \(B_\nu\) and \(W_\nu\) are the maximal inertial extensions of \(K\) in \(B_\nu\) and \(W_\nu\), respectively.

(ii) \(\Lambda_\nu\) has degree \(p\) as an algebra in \(d(B_\nu), C_{R_\nu}(\Lambda_\nu)\) is a central division \(B_\nu\)-algebra, and \(C_{R_\nu}(\Lambda_\nu)/B_\nu\) is of \(p\)-power zero (see Lemmas \([4.1]\) and \([4.3]\)).

It is easily verified that the field \(W_\nu\) defined in (7.3) (i) equals the compositum of the fields \(U_{\nu'}, \nu' \in \Pi_\nu\), for each \(\nu \in \mathbb{N}\); in view of Lemma \([6.3]\) (cc), this means that the Galois closures in \(K_{\text{sep}}\) over \(K\) of \(W_\nu, \nu \in \mathbb{N}\), are finite Galois extensions of \(K\) with nilpotent Galois groups (of orders not divisible by any \(p' \in \mathbb{P} \setminus \Pi_\nu\), for any \(\nu\)). Considering a \(W_\nu\)-isomorphic copy \(B'_\nu\) of \(B_\nu\) in \(W_{\text{sep}}\), taking into account that proper subgroups of nilpotent finite groups are subnormal (by the Burnside-Wielandt theorem), and using Galois theory and (7.3) (i), one obtains that:

(7.4) For any pair of indices \(\nu, \nu'\) with \(\nu < \nu', \ [B'_\nu W_{\nu'}/W_\nu]\) is a field extension of degree dividing \([B_\nu : W_\nu]\).

It is clear from (7.3) (i) and the assumptions on \(\Pi_\nu, \nu \in \mathbb{N}\), that there exists an index \(\nu_0\), such that all prime divisors of \([B_\nu : W_\nu]\) are greater than \(p\), for each \(\nu > \nu_0\). Similarly, for any \(\nu\), one can find \(\xi(\nu) \in \mathbb{N}\) satisfying the condition \(\text{gcd}\{[B_{\nu'} : W_\nu], [B_\nu : W_\nu]\} = 1\) whenever \(\nu' \in \mathbb{N}\) and \(\nu' > \nu + \xi(\nu)\).

Thus it follows that, for each pair \(\nu, n \in \mathbb{N}\) with \(\nu_0 < \nu < \xi(\nu) < n - \nu\), we have \(\text{gcd}\{[B_{\nu'} : W_\nu], [B_n : W_n]\} = 1\), and also, \([B_{\nu'} : W_\nu][B_n : W_n]\) is not divisible by any prime number less than or equal to \(p\).

We show that \(R_n\) possesses a central \(W_n\)-subalgebra \(\Delta_n\) with the properties claimed by (7.2). Take a generator \(\varphi\) of \(G(\Theta/K)\), and for each \(\xi \in \mathbb{N}\), let \(\varphi_\xi\) be the unique \(W_\xi\)-automorphism of \(\Theta_\xi\) extending \(\varphi\). Fix an embedding \(\psi_\xi\) of \(B_\xi\) in \(K_{\text{sep}}\) as a \(W_\xi\)-algebra, and denote by \(B'_\xi\) the image of \(B_\xi\) under \(\psi_\xi\). Clearly, \(\psi_\xi\) gives rise to a canonical bijection of \(s(B_\xi)\) upon \(s(B'_\xi)\), which in turn induces a homomorphism \(\psi_\xi' : \text{Br}(B_\xi) \to \text{Br}(B'_\xi)\). Denote by \(\Sigma_\xi\) and \(\Sigma'_\xi\) the underlying division algebras of \(R_\xi \otimes_{W_\xi} B_\xi\) and \(R_\xi \otimes_{W_\xi} B'_\xi\) respectively, and let \(\bar{\psi}_\xi\) be a \(W_\xi\)-isomorphic copy of \(B_\xi\) in the full matrix \(W_\xi\)-algebra \(M_{\bar{b}_\xi}(W_\xi)\), where \(\bar{b}_\xi = [B_\xi : W_\xi]\). Using the fact that \(M_{\bar{b}_\xi}(R_\xi) \cong M_{\bar{b}_\xi}(W_\xi) \otimes_{W_\xi} R_\xi\) over \(W_\xi\), and applying the Skolem-Noether theorem to \(B_\xi\) and \(B'_\xi\), one obtains that \(R_\xi \otimes_{W_\xi} B_\xi\) and \(M_{\bar{b}_\xi}(C_{R_\xi}(B_\xi))\) are isomorphic as \(B_\xi\)-algebras. Hence, by the Wedderburn-Artin theorem, so are \(\Sigma_\xi\) and \(C_{R_\xi}(B_\xi)\). These observations allow to identify the \(B'_\xi\)-algebras \(R_\xi \otimes_{W_\xi} B'_\xi\) and \(M_{\bar{b}_\xi}(\Sigma'_\xi)\) and to prove the following fact:

(7.5) There exists a \(W_\xi\)-isomorphism \(\hat{\psi}_\xi : M_{b_\xi}(C_{R_\xi}(B_\xi)) \to (R_\xi \otimes_{W_\xi} B'_\xi)\), which extends \(\psi_\xi\) and maps \(C_{R_\xi}(B_\xi)\) upon \(\Sigma'_\xi\). The image \(\Lambda'_\xi\) of \(\Lambda_\xi\) under \(\hat{\psi}_\xi\) is a central \(B'_\xi\)-subalgebra of \(\Sigma'_\xi\) of degree \(p\), which is a representative of the equivalence class \(\psi_\xi(\Lambda_\xi)\) in \(\text{Br}(B'_\xi)\).

Now fix a pair \(\nu, n\) so that \(n_0 < \nu < \xi(\nu) < n - \nu\). Retaining notation as in (7.5), we turn to the proof of the following assertion:

(7.6) The tensor products \(\Lambda'_\nu \otimes_{B'_\nu} (B'_\nu B'_n), \ (\Lambda'_\nu \otimes_{B'_\nu} B'_\nu W_n) \otimes_{B'_\nu W_n} (B'_\nu B'_n)\) and \(\Lambda'_\nu \otimes_{B'_\nu} (B'_\nu B'_n)\) are isomorphic central division \(B'_\nu B'_n\)-algebras.

The statement that \(\Lambda'_\nu \otimes_{B'_\nu} (B'_\nu B'_n) \cong \Lambda'_\nu \otimes_{B'_\nu} B'_\nu W_n \otimes_{B'_\nu W_n} (B'_\nu B'_n)\) as
Hence, the former assertion of Lemma 7.2 can be deduced from Lemma 7.1 if there exists an algebra of the central simple algebra of $\nu,n$.

Denote by $\Sigma$ is a direct product of finite groups, indexed by $\Pi$. Let now $\nu,n \in \nu,n$, assume that $\nu,n$ is chosen so that $\bar{\phi}(\Theta)$ is central division $B'_n$-$n$-algebras which are embeddable in $\Sigma_{\nu,n}$ as $B'_n$-$n$-algebras. At the same time, it follows from Lemma 7.1 and the observation on $[B'_n]:K$ and $p$ that $\Sigma_{\nu,n}/(B'_n)$ is of $p$-power one. Now the proof of (7.6) is completed by Lemma 4.8. Since, by (7.4) and the choice of the indices $\nu,n$, we have $\gcd([B'_n^0]:W_n, [B'_n^0]:W_n^0) = 1$, statement (7.6) and Lemma 4.9 imply the following:

(7.7) There exists $\Delta_n \in d(W_n)$, such that $\Delta_n \otimes_{W_n} B'_n \cong \Delta_n$ and $\Delta_n \otimes_{W_n} (B'_n^0 W_n) \cong \Delta_n' \otimes_{B'_n} (B'_n^0 W_n)$ (over $B'_n$ and $B'_n^0 W_n$, respectively).

It is clear from (7.7) and the $W_n$-isomorphism $B_n \cong B'_n$ that the $B_n$-algebras $\Delta_n \otimes_{W_n} B_n$ and $\Delta_n$ are isomorphic. This proves (7.2). Applying (7.1), one obtains that $\Delta_{n,0} \otimes_{W_n} W_n = \Delta_n$ as $W_n$-algebras, for some $\Delta_{n,0} \in d(W_n)$ (here $W_n = K_{ur} \cap W_n$). Note also that if $W_n^0$ is the Galois closure of $W_n$ in $K_{sep}$ over $K$, then $G(W_n^0/K)$ is nilpotent and $p \nmid |W_n^0:K|$. This follows from Galois theory and the definition of fields $W_n, n \in \mathbb{N}$, which imply $G(W_n^0/K)$ is a direct product of finite $p_n$-groups, indexed by $\Pi_n$, for each $n$. Now the former assertion of Lemma 7.2 can be deduced from Lemma 7.1 if $\varepsilon \in K$.

Let now $\varepsilon \notin K$, $[K(\varepsilon):K] = m$, and $R_\varepsilon$ be the underlying division algebra of the central simple $K(\varepsilon)$-algebra $R \otimes_K K(\varepsilon)$. Then $K(\varepsilon)/K$ is a cyclic field extension and $m \nmid p-1$, which implies $\Theta(\varepsilon)/K(\varepsilon)$ is a totally ramified Kummer extension of degree $p$. Observing also that $R_\varepsilon$ is a central LBD-algebra over $K(\varepsilon)$, one obtains that $\Theta(\varepsilon)$ embeds in $R_\varepsilon$ as a $K(\varepsilon)$-subalgebra. At the same time, it follows from Lemma 1.1 that the $p$-power $k(p)_\varepsilon$ of $R_\varepsilon/K(\varepsilon)$ is less than $2$, i.e. $p^2 \nmid [K(\varepsilon, \delta^\prime):K(\varepsilon)]$, for any $\delta^\prime \in R_\varepsilon$.

Hence, $k(p)_\varepsilon = 1$, and by the already considered special case of our lemma, $R_\varepsilon$ possesses a central $K(\varepsilon)$-subalgebra $\Delta_\varepsilon$, such that $\deg(\Delta_\varepsilon) = p$ and there exists a $K(\varepsilon)$-subalgebra of $\Delta_\varepsilon$ isomorphic to $\Theta(\varepsilon)$. Let now $\varphi$ be a generator of $G(K(\varepsilon)/K)$. Then $\varphi$ extends to an automorphism $\varphi$ of $R_\varepsilon$ (as a $K$-algebra), so Lemma 4.3 ensures that $\Delta_\varepsilon$ is $K(\varepsilon)$-isomorphic to its image under $\varphi$. Together with the Skolem-Noether theorem, this shows that $\varphi$ can be chosen so that $\varphi(\Delta_\varepsilon) = \Delta_\varepsilon$. Now it follows from Teichmüller’s theorem that there is a $K(\varepsilon)$-isomorphism $\Delta_\varepsilon \cong \Delta_\varepsilon \otimes_K K(\varepsilon)$, for some $\Delta \in s(K)$ with $\deg(\Delta) = p$. As $\gcd(m,p) = 1$, one obtains that $\Delta \in d(K)$. Finally, it can be deduced from [7], Lemma 3.5, that $\Delta$ is isomorphic to a $K$-subalgebra of $R$, which in turn has a $K$-subalgebra isomorphic to $\Theta$. Hence, by Albert’s criterion (see [33], Sect. 15.3), $\Delta$ is a cyclic $K$-algebra. Observe finally that cyclic degree $p$ extensions of $K$ are inertial. Since $p \neq \text{char}(K)$ and $\varepsilon \notin K$, this is implied by (3.2) (a) and Lemma 6.1 so Lemma 7.2 is proved. □
The main lemma of the present Section can be stated as follows:

**Lemma 7.3.** Let \((K, v)\) be a Henselian field with abrd\(_p(K) < \infty\), \(p \in \mathbb{P}\), \(\dim(K_{sol}) \leq 1\), and \(\widehat{K}\) of arithmetic type, and let \(R\) be a central division LBD-algebra over \(K\), such that \(\text{char}(\widehat{K}) \mid [K(\delta) : K]\), for any \(\delta \in R\). Then, for any \(p \in \mathbb{P}\) not equal to \(\text{char}(\widehat{K})\), there exists a \(p\)-splitting field \(E_p\) of \(R/K\), that is included in \(K(p)\).

**Proof.** Fix an arbitrary \(p \in \mathbb{P} \setminus \{\text{char}(\widehat{K})\}\), as well as a primitive \(p\)-th root of unity \(\varepsilon = \varepsilon_p\) in \(K_{sep}\), take \(T_p(K)\) as in Lemma 5.1 (c), and put \(V_p(K) = K_0(p).T_p(K)\), where \(K_0(p) = K(p) \cap K_{ur}\). For each \(z \in \mathbb{P}\), denote by \(k(z)\) the \(z\)-power of \(R/K\), and let \(\ell\) be the minimal integer \(\ell(p) \geq 0\), for which there exists an extension \(V_p\) of \(K\) in \(V_p(K)\) satisfying conditions (c) and (cc) of Lemma 6.6. As shown in the proof of Lemma 6.4, \(K(p) = V_p(K)\) if \(\varepsilon \in K\) or \(v(K) = pv(K)\), and \(K(p) = K_0(p)\), otherwise. In the former case, \(V_p\) clearly has the properties claimed by Lemma 7.3, so we suppose, for the rest of our proof, that \(\varepsilon \notin K\), \(v(K) \neq pv(K)\) and \(V_p/K\) is chosen so that \([V_p : K] = p^\ell\) and the ramification index \(e(V_p/K)\) is minimal. Let \(E_p\) be the maximal inertial extension of \(K\) in \(V_p\). Then it follows from the inequality \(p \neq \text{char}(\widehat{K})\) that \(\widehat{E}_p = \widehat{V}_p\) (cf. [33], Proposition A.17); using also (3.1) (a), one sees that \(V_p/E_p\) is totally ramified and \([V_p : E_p] = e(V_p/K)\). Note further that \(E_p \subseteq K_0(p)\), by Lemma 6.6 so it suffices for the proof of Lemma 7.3 to show that \(V_p = E_p\) (i.e. \(e(V_p/K) = 1\)). Assuming the opposite and using Lemma 6.4 with its proof, one obtains that there is an extension \(\Sigma\) of \(E_p\) in \(V_p\), such that \([\Sigma : K] = p^{\ell−1}\).

The main step towards the proof of Lemma 7.3 is to show that \(p\), the underlying division \(\Sigma\)-algebra \(R_\Sigma\) of \(R \otimes_K \Sigma\), and the field extension \(V_p/\Sigma\) satisfy the conditions of Lemma 7.2. Our argument relies on the assumption that \(\dim(K_{sol}) \leq 1\). In view of Lemma 4.1 it guarantees that, for each \(z \in \mathbb{P} \setminus \{p\}\), \(k(z)\) is the \(z\)-power of \(R_\Sigma/\Sigma\). Thus it turns out that \(\text{char}(\widehat{K}) \mid [\Sigma(\rho') : \Sigma]\), for any \(\rho' \in R_\Sigma\). At the same time, it follows from the Wedderburn-Artin theorem and the choice of \(V_p\) and \(\Sigma\) that there exist isomorphisms \(R \otimes_K \Sigma \cong M_\gamma(R_\Sigma)\) and \(R \otimes_K V_p \cong M_{\gamma'}(V_p)\) (as algebras over \(\Sigma\) and \(V_p\), respectively), where \(\gamma' = p^{k(p)}\), \(\gamma \mid p^{k(p)−1}\) and \(R_{V_p}\) is the underlying division \(V_p\)-algebra of \(R \otimes_K V_p\). Note further that the \(\Sigma\)-algebras \(M_\gamma(R_\Sigma)\) and \(M_{\gamma'}(\Sigma) \otimes_\Sigma R_\Sigma\) are isomorphic, which enables one to deduce from the existence of a \(V_p\)-isomorphism \(R \otimes_K V_p \cong (R \otimes_K \Sigma) \otimes_\Sigma V_p\) (cf. [33], Sect. 9.4, Corollary a) that \(M_{\gamma'}(R_{V_p}) \cong M_\gamma(V_p) \otimes_{V_p}(R_\Sigma \otimes_\Sigma V_p)\) as \(V_p\)-algebras; hence, by Wedderburn-Artin’s theorem and the inequality \(\gamma < \gamma'\), \(R_\Sigma \otimes_\Sigma V_p\) is not a division algebra. This, combined with 7, Lemma 3.5, and the equality \([V_p : \Sigma] = p\), proves that \(R_\Sigma \otimes_\Sigma V_p \cong M_{\gamma}(R_{V_p})\), for some central division \(V_p\)-algebra \(R_{V_p}\) (which means that \(V_p\) is embeddable in \(R_\Sigma\) as a \(\Sigma\)-subalgebra). It is now easy to see that

\[
M_{\gamma}(R_{V_p}) \cong M_{\gamma}(V_p) \otimes (M_p(V_p) \otimes V_p) R'_{V_p} \cong (M_{\gamma}(V_p) \otimes V_p M_p(V_p)) \otimes_{V_p} R'_{V_p} \\
\cong M_{\gamma}(V_p) \otimes_{V_p} R'_{V_p} \cong M_{\gamma}(R_{V_p}).
\]

Using Wedderburn-Artin’s theorem, one obtains that \(\gamma = \gamma'/p = p^{k(p)−1}\).
and \( R_v \cong R_{v_p} \) over \( V_p \). Therefore, by Lemma 4.1, \( p^2 \nmid [\Sigma(\rho') : \Sigma] \), for any \( \rho' \in R_\Sigma \), which completes the proof of the fact that \( p, R_\Sigma \) and \( V_p/R_\Sigma \) satisfy the conditions of Lemma 7.2. Furthermore, it follows that a finite extension \( E \) of \( R_\Sigma/\Sigma \) if and only if it is a such a field for \( R/K \).

We are now in a position to complete the proof of Lemma 7.3 in the case where \( \varepsilon \notin K \) and \( v(K) \neq pv(K) \). By Lemma 7.2, \( R_\Sigma \) possesses a central \( \Sigma \)-subalgebra \( \Delta \), such that \( \deg(\Delta) = p \) and \( V_p \) embeds in \( \Delta \) as a \( \Sigma \)-subalgebra; hence, by \cite{22}, Theorem 4.4.2, \( R_\Sigma = \Delta \otimes_\Sigma C(\Delta) \), where \( C(\Delta) \) is the centralizer of \( \Delta \) in \( R_\Sigma \). In addition, \( C(\Delta) \) is a central division \( \Sigma \)-algebra, and since \( p^2 \nmid [\Sigma(\rho') : \Sigma] \), for any \( \rho' \in R_\Sigma \), it follows from Lemma 4.3 that \( p \nmid [\Sigma(c) : \Sigma] \), for any \( c \in C(\Delta) \). Note also that \( \varepsilon \notin \Sigma \), since \( \gcd([K(\varepsilon) : K], [\Sigma : K]) = 1 \) (whence, \( K(\varepsilon) \cap \Sigma = K \)). Therefore, Lemma 7.2 requires the existence of a degree \( p \) cyclic extension \( \Sigma' \) of \( \Sigma \) in \( K_{\text{sep}} \), which is inertial over \( \Sigma \), by Lemma 6.1 and embeds in \( \Delta \) as a \( \Sigma \)-subalgebra. This implies \( \Sigma' \) is a \( p \)-splitting field of \( R_\Sigma/\Sigma \) and \( R/K \) (see Lemma 13.4 and Corollary 13.4), \( [\Sigma' : K] = p^k \), \( e(\Sigma'/K) = e(V_p/K)/p \), and \( \Sigma'/\Sigma \) is a cyclic extension of degree \( p \). Taking finally into consideration that \( \Sigma \subseteq I(\hat{K}(p)/K) \), and using Lemma 5.1 and \( 35 \), Proposition A.17, one obtains consecutively that \( \Sigma' \subseteq I(\hat{K}(p)/K) \) and \( E_p \) has a degree \( p \) extension \( E'_p \) in \( \Sigma' \cap K_0(p) \). It is now easy to see that \( \Sigma' = E'/\Sigma \) and \( \Sigma' \subseteq I(V_p(K)/K) \). The obtained properties of \( \Sigma' \) show that it satisfies conditions (c) and (cc) of Lemma 6.6 As \( e(\Sigma'/K) < e(V_p/K) \), this contradicts our choice of \( V_p \) and thereby yields \( e(V_p/K) = 1 \), i.e. \( V_p = E_p \), so Lemma 7.3 is proved. \( \square \)

8. Proof of Lemma 2.3 and the main results

We begin this Section with a lemma which shows how to deduce Lemma 2.3 in general from its validity in the case where \( q > 0 = k(q) \) (\( q \) is defined at the beginning of Section 7, and \( k(q) \) is the \( q \)-power of \( R/K \)).

Lemma 8.1. Let \((K,v)\) be a Henselian field with \( \hat{K} \) of arithmetic type, \( \text{char} (\hat{K}) = q \), \( \dim(K_{\text{sol}}) \leq 1 \) and \( \text{abrd}_{p}(K) < \infty \), \( p \in \mathbb{P} \). Put \( \mathbb{P}' = \mathbb{P} \setminus \{ q \} \), take a central division LBD-algebra \( R \) over \( K \), and in case \( q > 0 \), assume that \( K \) has an extension \( E_q \) in \( K(q) \) that is a \( q \)-splitting field of \( R/K \). Then, for each \( p \in \mathbb{P}' \), there is a \( p \)-splitting field \( E_p \) of \( R/K \), lying in \( I(\hat{K}(p)/K) \).

Proof. Our assertion is contained in Lemma 7.3 if \( q = 0 \), so we assume that \( q > 0 \). Let \( R_{E_q} \) be the underlying division \( E_q \)-algebra of \( R \otimes_K E_q \), and for each \( p \in \mathbb{P} \), let \( k(p)' \) be the \( p \)-power of \( R_{E_q} \). Lemma 4.1 (c) and the assumption on \( E_q \) ensure that \( k(q)' = 0 \), and \( k(p)' \) equals the \( p \)-power of \( R/K \) whenever \( p \in \mathbb{P}' \). Therefore, by Lemma 7.3 for each \( p \in \mathbb{P}' \), there is an extension \( E'_p \) of \( E_q \) in \( E_q(p) \), which is a \( p \)-splitting field of \( R_{E_q} \). This enables one to deduce from Lemmas 4.5 and 4.6 that there exist \( E_q \)-algebras \( \Delta_p \in d(E_q) \), \( p \in \mathbb{P}' \), embeddable in \( R_{E_q} \), and such that \( \deg(\Delta_p') = p^{k(p)} \), for every \( p \in \mathbb{P}' \). Hence, by Lemma 7.1 \( R \) possesses central \( K \)-subalgebras \( \Delta_p \in d(K) \), \( p \in \mathbb{P}' \), with \( \Delta_p \otimes_K E_q \cong \Delta_p' \) as \( E_q \)-algebras, for each index \( p \). In view of Lemmas 8.3 and 4.3 (c), this proves Lemma 8.1. \( \square \)
We are now prepared to complete the proof of Lemma 2.3 in general, and thereby to prove Theorems 2.1 and 2.2. If \((K, v)\) and \(\hat{K}\) satisfy the conditions of Theorem 2.2, then the conclusion of Lemma 2.3 follows from Lemmas 5.3 and 8.1. As noted in Remark 4.7, this leads to a proof of Theorem 2.2.

Remark 8.2. Let \((K, v)\) be an HDV-field with \(\hat{K}\) of arithmetic type and virtually perfect, and let \(R\) be a central division LBD-algebra over \(K\). Then it follows from Lemmas 5.3 and 8.1 that, for each \(p \in \mathbb{P}\), there exists a finite extension \(E_p\) of \(K\) in \(K(p)\), which is a \(p\)-splitting field of \(R/K\). Therefore, as in Remark 1.7, one concludes that \(R\) has a central \(K\)-subalgebra \(\hat{R}\) subject to the restrictions of Conjecture 1.3. This proves Theorem 2.1 in case \(m = 1\).

In the rest of the proof of Lemma 2.3 we assume that \(m \geq 2\) and \(K = K_m\) is a complete \(m\)-discretely valued field whose \(m\)-th residue field \(K_0\) is virtually perfect of characteristic \(q\) and arithmetic type. Denote by \(K_{m-m'}\) the \(m'\)-th residue field of \(K_m\), for \(m' = 1, \ldots, m\). Arguing as in the proof of Theorem 2.2, one obtains that the conclusions of Theorem 2.1 and Lemma 2.3 hold if \(q = 0\) as well as in the case where \(\text{char}(K_{m-1}) = q > 0\). Therefore, it remains for us to prove Theorem 2.1 under the hypothesis that \(q > 0\) and \(\text{char}(K_{m-1}) = 0\) (so \(\text{char}(K_m) = 0\)). Denote by \(\mu\) the maximal index for which \(\text{char}(K_{m-\mu}) = 0\), fix a primitive \(q\)-th root of unity \(\varepsilon \in K_{\text{sep}}\), put \(K' = K(\varepsilon)\), and denote by \(R'\) the underlying division \(K'\)-algebra of \(R \otimes_K K'\).

It is clear from Lemma 6.4 that \(R'\) satisfies the condition of Lemma 8.1, whence, for each \(p \in \mathbb{P}\), there exists a finite extension \(E'_p\) of \(K'\) in \(K'(p)\), which is a \(p\)-splitting field of \(R'/K'\). Similarly to the proof of Lemma 8.1, this allows to show that, for each \(p \in \mathbb{P}\), \(R'\) possesses a \(K'\)-subalgebra \(\Delta'_q \subseteq d(K')\) of degree \(p^k[p]\), where \(k(p)\) is the \(p\)-power of \(R'/K'\). Observing now that \(K'/K\) is a cyclic field extension with \([K': K] = q - 1\), and applying Lemma 7.1 to the \(K'\)-subalgebra \(\Delta'_q \subset R'\), one concludes that there is a \(K\)-subalgebra \(\Delta_q \subseteq d(K)\) of \(R\), such that \(\Delta_q \otimes_K K' \cong \Delta'_q\) as a \(K'\)-subalgebra. In addition, it follows from Lemma 1.1 and the divisibility \([K': K] = q - 1\) that \(k(q)\) equals \(q\)-power \(k(q)\) of \(R/K\). Observe now that \(\text{abrd}_q(K_{m-\mu})(q) \leq 1\) and \(\text{abrd}_q(K_{m-\mu}) < \infty\). As \(\text{char}(K_{m-\mu-1}) = q\), the former inequality is implied by Lemma 5.3 and the existence of a Henselian discrete valuation \(w\) of \(K_{m-\mu}\) with a residue field \(K_{m-\mu-1}\). The latter one is proved by applying 3.2, Corollary 2.5, to \((K_{m-\mu}, w)\), which is allowed if \(K_{m-\mu-1}\) is virtually perfect.

Note also that \(K = K_m\) has a Henselian valuation \(\omega_{\mu}\) with a residue field \(K_{m-\mu}\) and \(\omega_{\mu}(K) \cong \mathbb{Z}^\mu\). In view of Lemma 6.4, this means that \(\text{abrd}_q(K_{m}) < \infty\). Observe finally that the standard \(\mathbb{Z}^m\)-valued valuation \(v\) of \(K\) (with \(\hat{K} = K_0\)) equals the composite valuation \(\tilde{v} \circ \omega_{\mu}\), where \(\tilde{v}\) is the standard \(\mathbb{Z}^m\)-valued valuation of \(K_{m-\mu}\). Applying Lemma 8.3 to \(\Delta_q/K\), and Lemma 1.3 to \(R\) and \(C_R(\Delta_q)\), one concludes that \((K, v)\) and \(R/K\) satisfy the conditions of Lemma 8.1 with respect to \(q\). Therefore, for each \(p \in \mathbb{P}\), \(K\) has a finite extension \(E_p\) in \(K(p)\), which is a \(p\)-splitting field of \(R/K\), as claimed by Lemma 2.3. In view of Remark 1.7, this enables one to complete the proof of Theorem 2.2.

Note finally that, in the setting of Conjecture 1.3 it is unknown whether there exists a sequence \(E_p\), \(p \in \mathbb{P}\), of \(p\)-splitting fields of \(R/K\), such that
$E_p \subseteq K(p)$, for each $p$. In view of Proposition 1.1 and [29], Conjecture 1 (see also the end of [33], Ch. 15), and since Questions 1.4 (a) and (b) are open, the answer is affirmative in all presently known cases. When $R$ is an LFD-algebra and $K$ contains a primitive $p$-th root of unity, for every $p \in \mathbb{P}$, $p \neq \text{char}(K)$, such an answer follows from Proposition 1.1, combined with [1], Ch. VII, Theorem 28, and the Merkur’ev-Suslin theorem [30], (16.1) (see also [20], Theorem 9.1.4 and Ch. 8, respectively). This supports the idea to make further progress in the study on Conjecture 1.3 by extending the scope of Lemma 2.3 to more general fields $K$ with $\text{ad}_{K}(K) \lessgtr \infty$, $p \in \mathbb{P}$. To conclude with, it would surely be of interest to understand whether a proof of Conjecture 1.3 for a field $E$ admissible by Proposition 1.1 could lead to an answer to Question 1.4 (b), for central division LBD-algebras over $E$.

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