UNIVERSAL CURVATURE IDENTITIES III

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Abstract. We examine universal curvature identities for pseudo-Riemannian manifolds with boundary. We determine the Euler-Lagrange equations associated to the Chern-Gauss-Bonnet formula and show that they are given solely in terms of curvature and the second fundamental form and do not involve covariant derivatives thus generalizing a conjecture of Berger to this context.

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1. Introduction

The study of curvature is central to modern differential geometry and mathematical physics. Properties of the curvature operator have been examined by many authors – see, for example, the discussion in [1, 12]. Eta Einstein geometry has been investigated [10, 24]. Curvature plays an important role in spectral geometry – see, for example, [2]. The Lorentzian and higher signature settings are of special importance [7, 23].

The Gauss Bonnet theorem has many physical applications as are the associated Euler-Lagrange Equations (see, for example, [5, 15, 25]). This paper deals with universal curvature identities arising from the Euler-Lagrange equations for the Chern-Gauss-Bonnet theorem for manifolds with boundary. In Section 1.1 and in Section 1.2, we discuss the Gauss-Bonnet theorem for closed pseudo-Riemannian manifolds, present the associated Euler-Lagrange equations, and discuss some of the historical context of the problem that we shall be considering. In Section 1.3, we recall the Gauss-Bonnet theorem for manifolds with boundary; we shall always assume the restriction of the pseudo-Riemannian metric to the boundary to be non-degenerate. In Section 1.4, we state Theorem 1.4 – this is the first result of the paper. It gives the associated Euler-Lagrange equations for the Gauss-Bonnet integrand for manifolds with boundary.

The remainder of the paper is devoted to the proof of Theorem 1.4. Section 2 treats basic invariance theory; the question of universal curvature identities is central. In Theorem 2.3, we shall summarize previous results concerning universal curvature identities in the scalar case (both in the interior and on the boundary) and in the symmetric 2-tensor case in the interior. Theorem 2.4 is the second main result of this paper. In it, we extend the results of Theorem 2.3 to discuss universal curvature identities for symmetric 2-tensors defined by the geometry of the embedding $\partial M \subset M$. There is a technical fact we shall need in the proof of Theorem 2.4 that we postpone until Section 4 to avoid breaking the flow of the discussion. In Section 3, we use Theorem 2.4 to complete the proof of Theorem 1.4.

Section 4 provides the technical results which are central to the discussion. Rather than using H. Weyl’s theory of invariants [26] to treat the universal curvature identities which arise in Theorem 1.4, we have chosen to adopt the approach of [13] which was originally used to give a heat equation proof of the Gauss-Bonnet theorem. This seemed easier rather than having to introduce an additional complicated formalism to use results of [24].
1.1. The Gauss-Bonnet Theorem for closed manifolds. Let \((M, g)\) be a compact pseudo-Riemannian manifold of signature \((p, q)\) and dimension \(m = p + q\) with smooth boundary \(\partial M\); if the signature is indefinite, we assume the restriction of the metric to the boundary to be non-degenerate. Let \(dx\) be the Riemannian element of volume. Let \(\vec{e} := \{e_1, ..., e_m\}\) be a local orthonormal frame for the tangent bundle of \(M\). Set
\[
\varepsilon_{i_1...i_n} = \varepsilon(g, e_i)_{i_1...i_n} := g(e^{i_1} \wedge ... \wedge e^{i_n}, e^{j_1} \wedge ... \wedge e^{j_n}) = \det (g(e^i, e^j)) = \pm 1.
\]
(1.a)

Clearly this vanishes if the indices are not distinct and thus, in particular, is zero if \(n > m\). We adopt the Einstein convention and sum over repeated indices. Let \(R_{ijkl} = R(g, e^i)_{ijkl}\) denote components of the curvature tensor of the Levi-Civita connection \(\nabla\) relative to the local orthonormal frame \(\vec{e}\). If \(n = 2\bar{n}\) is even, define the Euler form or the Pfaffian \(E^{(p,q)}_{m,n} = E^{(p,q)}_{m,n}(g)\) by setting:
\[
E^{(p,q)}_{m,n} := \frac{R_{i_1j_2k_3l_4...i_n} - R_{i_1j_2k_3l_4...j_n}}{(8\pi)^{\bar{n}}\bar{n}!}e^{i_1...i_n}.
\]
(1.b)

This is independent of the choice of the local orthonormal frame field \(\vec{e}\). We set \(E^{(p,q)}_{m,n} = 0\) if \(n\) is odd. Let \(\chi(M)\) be the Euler-Poincaré characteristic. The generalized Gauss-Bonnet theorem \([11, 8, 9]\) states:

**Theorem 1.1.** Let \((M, g)\) be a closed pseudo-Riemannian manifold of signature \((p, q)\) and dimension \(m = p + q\). Then
\[
\chi(M) = \int_M E^{(p,q)}_{m,n}(g)dx_g.
\]

We note that \(\chi(M) = 0\) in this setting if \(m\) is odd and it was for that reason we set \(E^{(p,q)}_{m,n} = 0\) if \(n\) was odd. If \(p\) or \(q\) is odd, then \(\chi(M) = 0\) even though \(E^{(p,q)}_{m,n}(g)\) need not vanish locally in this setting.

1.2. The Euler-Lagrange Equations for manifolds without boundary. We examine these formulas in dimensions \(m > n\). Let \(h\) be a symmetric 2-cotensor. We consider the variation of the metric \(g_t := g + th\); this is a non-degenerate metric of signature \((p, q)\) for small \(t\). We integrate by parts to express:
\[
\partial_t \left\{ \int_M E^{(p,q)}_{m,n}(g_t)dx_g \right\} \bigg|_{t=0} = \int_M h_{ij}Q^{(p,q),2}_{m,n,ij}(g)dx_g.
\]

Here \(Q^{(p,q),2}_{m,n,ij} = Q^{(p,q),2}_{m,n,ij}(g)\) is a canonically defined symmetric 2-tensor field. Since \(E^{(p,q)}_{m,n} = 0\) for \(m < n\) or for \(n\) odd, we set \(Q^{(p,q),2}_{m,n,ij} = 0\) in these cases. Furthermore, Theorem \([11]\) shows \(Q^{(p,q),2}_{m,n,ij} = 0\) if \(m = n\). Thus only the case \(m > n\) and \(n\) even is relevant.

In the Riemannian setting, Berger \([3]\) conjectured that \(Q^{(p,q),2}_{m,n,ij}\) could be expressed only in terms of curvature; the higher covariant derivatives did not enter. This was subsequently verified by Kuz’mina \([19]\) and Labbi \([20, 21, 22]\). Set \(\varepsilon^{(p,q),2}_{m,n} = 0\) if \(n\) is odd. If \(n = 2\bar{n}\) is even, define \(\varepsilon^{(p,q),2}_{m,n,ij} = \varepsilon^{(p,q),2}_{m,n,ij}(g)\) by setting:
\[
\varepsilon^{(p,q),2}_{m,n,ij} := \frac{R_{i_1j_2k_3l_4...i_n} - R_{i_1j_2k_3l_4...j_n}}{(8\pi)^{\bar{n}}\bar{n}!}e^{i_1...i_n}.
\]
(1.c)

This symmetric 2-tensor valued function is independent of the choice of \(\vec{e}\). One then has \([16, 17]\):
Theorem 1.2. If $M$ is a closed pseudo-Riemannian manifold of signature $(p, q)$ and dimension $m = p + q > n$, then:

$$\partial_t \left\{ \int_M E_{m,n}^{(p,q)}(g_t) \, dx_g \right\} \bigg|_{t=0} = \frac{1}{2} \int_M h_{ij} E_{m,n,i}^{(p,q),2}(g) \, dx_g.$$  

1.3. The Gauss-Bonnet Theorem for manifolds with boundary. There are boundary correction terms which appear if $\partial M$ is non-empty. We assume the restriction of the pseudo-Riemannian metric $g$ to the boundary is non-degenerate. Normalize the choice of $\tilde{e}$ near $\partial M$ so that $e_1$ is the inward pointing unit geodesic vector field. Let indices $(a,b)$ range from 2 through $m$ and index the induced orthonormal frame for the tangent bundle of the boundary. Let $\nabla$ denote the Levi-Civita connection. Let $L_{ab} := L(g, \tilde{e})_{ab}$ give the components of the second fundamental form:

$$L_{ab} := g(\nabla_a e_b, e_1).$$

The parity of $n$ plays no role. For $0 \leq 2\nu \leq n - 1$, set:

$$F_{m,n-1,\nu}^{(p,q),\partial M} := \frac{R_{a_1 a_2 b_2 \ldots a_{2\nu - 1} b_2 b_{2\nu - 1} L_{a_{2\nu - 1} b_{2\nu - 1} b_{2\nu + 1} \ldots L_{a_n b_n - 1}} b_{n - 1} \ldots b_1 b_n}}{(8\pi)^{\nu} \nu! \text{Vol}(S^{n-1-2\nu})(n - 1 - 2\nu)!},$$

$$F_{m,n-1}^{(p,q),\partial M} := \sum_{\nu} F_{m,n-1,\nu}^{(p,q),\partial M}.$$  \hfill (1.d)

The scalar functions $\{F_{m,n-1}^{(p,q),\partial M}, F_{m,n-1,\nu}^{(p,q),\partial M}\}$ are independent of the choice of the local orthonormal frame $\tilde{e}$. Theorem 1.4 generalizes to this setting to yield:

Theorem 1.3. Let $(M, g)$ be a compact pseudo-Riemannian manifold of signature $(p, q)$ and of dimension $m = p + q$. Assume the restriction of the metric to the boundary is non-degenerate. Then:

$$\chi(M) = \int_M E_{m,n}^{(p,q)}(g) \, dx_g + \int_{\partial M} F_{m,n-1}^{(p,q),\partial M}(g) \, dy_g.$$  

1.4. The Euler-Lagrange Equations for manifolds with boundary. Assume the restriction of the pseudo-Riemannian metric to the boundary is non-degenerate. For $0 \leq 2\nu \leq n - 1$, define:

$$F_{m,n-1,\nu,ab}^{(p,q),2,\partial M} := \frac{R_{a_1 a_2 b_2 \ldots a_{2\nu - 1} b_2 b_{2\nu - 1} L_{a_{2\nu - 1} b_{2\nu - 1} b_{2\nu + 1} \ldots L_{a_n b_n - 1}} b_{n - 1} \ldots b_1 b_n}}{(8\pi)^{\nu} \nu! \text{Vol}(S^{n-1-2\nu})(n - 1 - 2\nu)!},$$

$$F_{m,n-1,ab}^{(p,q),2,\partial M} := \sum_{\nu} F_{m,n-1,\nu,ab}^{(p,q),2,\partial M}.$$  \hfill (1.e)

These symmetric 2-tensor valued functions on the boundary are independent of the choice of the local orthonormal frame $\tilde{e}$. The first main new result of this paper is to generalize Theorem 1.4 to the case of manifolds with boundary; the remainder of this paper is devoted to the proof of the following result:

Theorem 1.4. Let $M$ be a compact pseudo-Riemannian manifold with boundary of dimension $m \geq n + 1$. Assume the restriction of the metric to the boundary is non-degenerate. Then:

$$\partial_t \left\{ \int_M E_{m,n}^{(p,q)}(g_t) \, dx_g + \int_{\partial M} F_{m,n-1}^{(p,q),\partial M}(g_t) \, dy_g \right\} \bigg|_{t=0} = \frac{1}{2} \int_M h_{ij} E_{m,n,i}^{(p,q),2}(g) \, dx_g + \frac{1}{2} \int_{\partial M} h_{ab} F_{m,n-1,ab}^{(p,q),2,\partial M}(g) \, dy_g.$$
2. INVARIANCE THEORY

Section 2 is devoted to the discussion of invariance theory. We begin in Section 2.1 with a discussion of local formulas; this is central to the matter at hand. In Section 2.2, we introduce the spaces of invariants with which we shall be working. We shall be dealing with formal expressions in the covariant derivatives of the curvature tensor (and also the tangential covariant derivatives of the second fundamental form when considering the boundary geometry) which are invariant under the action of the orthogonal group (and thus independent of the choice of local orthonormal frame $\mathbf{e}$) and which are either scalar valued or symmetric 2-tensor valued. They are universally defined by contractions of indices. In Section 2.3, we give various examples of scalar and symmetric 2-tensor valued invariants both in the interior and on the boundary. We also begin the discussion of universal curvature identities. In Section 2.4, we introduce the restriction map. An expression which is non-zero in dimension $m$ may vanish when restricted to dimension $m - 1$ and thus becomes a universal curvature identity in dimension $m - 1$. We give both a geometric definition and then subsequently an algebraic definition of the restriction map. We analyze in Theorem 2.3 the kernel of the restriction map in certain contexts. Section 2.5 treats symmetric 2-tensor valued invariants on the boundary and contains, in Theorem 2.4, the second main result of this paper which is of independent interest.

2.1. Local formulas. It is worth saying a bit about the general framework in which we shall be working. We first consider invariants defined on the interior of a pseudo-Riemannian manifold $(M, g)$. Let $X = (x_1, ..., x_m)$ be a system of local coordinates on $M$. Let $\alpha$ be a multi-index. We introduce formal variables $g_{ij} := g(\partial x_i, \partial x_j)$ and $g_{ij/\alpha} := d^\alpha x g_{ij}$ for the derivatives of the metric. We consider expressions $P = P(g_{ij}, g_{ij/\alpha})$ which are polynomial in the variables $\{g_{ij/\alpha} \}_{|\alpha| > 0}$ with coefficients which depend smoothly on the $\{g_{ij}\}$ variables. Given a system of local coordinates $X$ and a point $P$, we can evaluate $P(g, X)$; we say $P$ is invariant if this evaluation is independent of the particular coordinate system chosen and depends only on the point of the manifold and on the metric $g$. We introduce special notation for the first and second derivatives of the metric setting:

$$g_{ij/k} := \partial_{x_k} g_{ij} \quad \text{and} \quad g_{ij/kl} := \partial_{x_k} \partial_{x_l} g_{ij}.$$  

If we take geodesic polar coordinates centered at a point $x_0$ of $M$, then we have

$$g_{ij}(g, X)(x_0) = \pm \delta_{ij} \quad \text{and} \quad g_{ij/k}(g, X)(x_0) = 0.$$  

Furthermore, all the higher derivatives of the metric at $x_0$ can be evaluated in terms of the components of the covariant derivatives of the curvature tensor at $x_0$. Thus we can equally well regard $P = P(g_{ij}, R_{ijkl}, R_{ijkl,n}, ...)$.

2.2. Spaces of invariants. We say that the curvature $R_{ijkl}$ is of degree 2 since it is linear in the 2-jets of the metric and quadratic in the 1-jets of the metric. The second fundamental form is of degree 1. Each covariant derivative increases the degree by 1. We define the spaces with which we shall be working as follows:
Lemma 2.1. The metric takes the form of an orthonormal basis, let ξ

Remark 2.1. In the indefinite setting, we must take more care with raising and lowering indices and there is a slight additional bit of technical fuss. If the metric is positive definite, then:

Assertions (1)-(3) of Lemma 2.1 by:

By H. Weyl’s first theorem of invariants, all invariants arise from contraction of indices. For example, we have (see, for example, [14]):

Lemma 2.1. If the metric is positive definite, then:

(1) \( \mathcal{I}^{(0,m)}_{m,0} = \text{Span} \{ 1 \} \).
(2) \( \mathcal{I}^{(0,m)}_{m,2} = \text{Span} \{ R_{ijji} \} \).
(3) \( \mathcal{I}^{(0,m)}_{m,4} = \text{Span} \{ R_{ijji,kkkl}, R_{ijji,kkll}, R_{ijji,kllk}, R_{ijji,lklk} \} \).
(4) \( \mathcal{I}^{(p,q),2}_{m,n} = \text{Span} \{ R_{ijji,kkll}, R_{ijji,kllk}, R_{ijji,lklk} \} \).

By H. Weyl’s first theorem of invariants, all invariants arise from contraction of indices. For example, we have (see, for example, [14]):

Remark 2.1. In the indefinite setting, we must take more care with raising and lowering indices and there is a slight additional bit of technical fuss. If \( \{ e_i \} \) is an orthonormal basis, let \( \xi_i := g(e_i, e_i) = \pm 1 \). We must then, for example, replace Assertions (1)-(3) of Lemma 2.1 by:

\[
\mathcal{I}^{(p,q)}_{m,0} = \text{Span} \{ 1 \}, \quad \mathcal{I}^{(p,q)}_{m,2} = \text{Span} \{ \sum_{ij} \xi_i \xi_j R_{ijji} \}, \\
\mathcal{I}^{(p,q)}_{m,4} = \text{Span} \{ \sum_{ijkl} \xi_i \xi_j \xi_k \xi_l R_{ijkl}, \sum_{ijkl} \xi_i \xi_j \xi_k \xi_l R_{ijkl} \}.
\]

If \( \eta_1 \) and \( \eta_2 \) are covectors, define the symmetric product \( \eta_1 \circ \eta_2 \) by setting:

\[
\eta_1 \circ \eta_2 = \frac{1}{2} (\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1).
\]

The metric takes the form \( g = e^i \circ e^i \); the map \( P \rightarrow P \cdot g \) embeds \( \mathcal{I}^{(p,q)}_{m,n} \) into \( \mathcal{I}^{(p,q),2}_{m,n} \).

Lemma 2.1 generalizes to be [16]:

Lemma 2.2. If the metric is positive definite, then:

(1) \( \mathcal{I}^{(0,m),2}_{m,n} = \text{Span} \{ e^k \circ e^k \} \).
(2) \( \mathcal{I}^{(0,m),2}_{m,2} = \text{Span} \{ R_{ijji} e^k \circ e^k, R_{ijji} e^j \circ e^k \} \).
(3) \( \mathcal{I}^{(0,m),2}_{m,4} = \text{Span} \{ R_{ijji,kkll} e^i \circ e^i, R_{ijji,kkll} e^j \circ e^k, R_{ijji,kllk} e^j \circ e^k, R_{ijji,kllk} e^j \circ e^k \} \).

Near the boundary, we normalized the local orthonormal frame so that \( e_1 \) is the inward unit geodesic normal vector field. Let indices \([a, b]\) range from 2 through \( m \) and index the induced orthonormal frame \( \{e_2, ..., e_m\} \) for the tangent bundle of the boundary. Let \( \cdot' \) denote multiple covariant differentiation with respect to the Levi-Civita connection of the boundary. We have \( [9] \):

**Lemma 2.3.** If the metric is positive definite, then:

1. \( T_{m,0}^{(0,m),\partial M} = \text{Span}\{1\} \).
2. \( T_{m,1}^{(0,m),\partial M} = \text{Span}\{L_{aa}\} \).
3. \( T_{m,2}^{(0,m),\partial M} = \text{Span}\{R_{abab}, R_{a11a}, L_{ab}L_{bb}, L_{ab}L_{ab}\} \).
4. \( T_{m,3}^{(0,m),\partial M} = \text{Span}\{R_{a11a1}, L_{aa}L_{bb}L_{cc}, L_{ab}L_{ab}L_{cc}, L_{ab}L_{bc}L_{ac}\} \).

It is not difficult to establish the result that follows out our treatment; we omit the proof in the interests of brevity since we shall not need it in discussion:

**Lemma 2.4.** If the metric is positive definite, then:

1. \( T_{m,0}^{(0,m),2,\partial M} = \text{Span}\{e^1 \circ e^1, e^a \circ e^b\} \).
2. \( T_{m,1}^{(0,m),2,\partial M} = \text{Span}\{L_{aa}e^1 \circ e^1, L_{aa}e^b \circ e^b, L_{ab}e^a \circ e^b\} \).
3. \( T_{m,2}^{(0,m),2,\partial M} = \text{Span}\{R_{a11a}e^1 \circ e^1, L_{ab}L_{bb}e^1 \circ e^1, L_{ab}L_{ab}e^1 \circ e^1, R_{ab}e^b \circ e^b, R_{a11a}e^b \circ e^b, L_{aa}L_{bb}e^c \circ e^c, L_{ab}L_{ab}e^c \circ e^c, R_{ab}e^a \circ e^c, R_{a11a}e^a \circ e^c, L_{ab}L_{ac}e^b \circ e^c\} \).
4. \( T_{m,3}^{(0,m),3,\partial M} = \text{Span}\{R_{a11a1}e^1 \circ e^1, R_{a11a1}e^b \circ e^b, R_{a11a1}e^c \circ e^c, R_{ab}e^b \circ e^c, R_{a11a1}e^b \circ e^c, L_{aa}L_{bb}e^c \circ e^c, L_{ab}L_{ab}e^c \circ e^c, R_{ab}e^b \circ e^c, R_{a11a1}e^b \circ e^c, L_{ab}L_{ac}e^c \circ e^c, R_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, R_{ab}e^b \circ e^c, R_{a11a1}e^b \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}e^c \circ e^c, L_{ab}L_{bc}L_{cc}e^d \circ e^d, L_{ab}L_{bc}L_{cc}e^d \circ e^d, L_{ab}L_{bc}L_{cc}e^d \circ e^d, L_{ab}L_{bc}L_{cc}e^d \circ e^d\} \).

2.3. **Universal curvature identities.** The invariants of Lemma 2.1 are not linearly independent in low dimensions. To simplify matters, we work in the Riemannian context; similar results hold if we introduce the tensor \( \xi \). We have the following relations in dimensions 1, 3, and 5:

**Lemma 2.5.** In the Riemannian setting:

1. If \( m = 1 \), then \( 0 = R_{ijij} \).
2. If \( m = 3 \), then \( 0 = R_{ijiil}R_{abab} - 4R_{aija}R_{bijh} + R_{ijkt}R_{ijkl} \).
3. If \( m = 5 \), then \( 0 = R_{ijiil}R_{abab} - 12R_{aija}R_{bijh} + 3R_{a11a}R_{ijkl} + 24R_{aija}R_{bklb}R_{ijlh} + 16R_{aija}R_{bijk}R_{ckl} - 24R_{aija}R_{jklb}R_{nik} + 2R_{ijkt}R_{kltan}R_{amij} - 8R_{kaij}R_{nikt}R_{jl} \).
Let \( n \) be even. If \( \sigma \) is a permutation and if \( R \) is the curvature tensor, let

\[
P_{n,\sigma}(R) := g^{ij} s_{\sigma}^1 \cdots g^{i_2n-1} s_{\sigma}^{i_2} R_{i_1 i_2} s_{\sigma}^{i_2} \cdots s_{\sigma}^{i_{2n-2}} R_{i_{2n-1} i_{2n}} s_{\sigma}^{i_{2n-1}} \,.
\]

If \( \mathcal{C} = \{ C_\sigma \} \) is a collection of constants, let:

\[
P_{n,\mathcal{C}} := \sum_{\sigma \in \text{Perm}(\mathcal{U})} C_\sigma P_{n,\sigma}.
\]

Let \((p, q)\) be arbitrary. H. Weyl’s first theorem of invariants, when applied to the curvature tensor, yields that if \( P \in \mathcal{T}_{m,n}^{(p,q)} \) is a scalar invariant of the curvature tensor of degree \( \delta \) which does not involve the covariant derivatives of the curvature tensor, then \( P = P_{n,\mathcal{C}} \) for some \( \mathcal{C} \); if \( P \neq 0 \), necessarily \( \delta \) is even. Similar expressions appear if the covariant derivatives of \( R \) are involved; the notation becomes more complicated and as we shall not in any event need such expressions, we omit details in the interests of brevity.

We say that \( P_{n,\mathcal{C}} \) is a universal curvature identity in signature \((p, q)\) if \( P_{n,\mathcal{C}} \) vanishes for all pseudo-Riemannian metrics of signature \((p, q)\). The following is a useful observation [17]; it shows only the dimension is relevant.

**Theorem 2.2.** If \( P_{n,\mathcal{C}} \) is a universal scalar curvature identity in the curvature tensor in signature \((p, q)\), then \( P_{n,\mathcal{C}} \) is a universal scalar curvature in any other signature \((\tilde{p}, \tilde{q})\) if \( p + q = \tilde{p} + \tilde{q} \).

**Remark 2.2.** A similar assertion holds for elements of \( \mathcal{I}_{m,n}^{(p,q),2} \). If we consider polynomials in \( \{ L, R \} \), a similar assertion holds for elements of \( \mathcal{I}_{m,n}^{(p,q),\partial M} \) and \( \mathcal{I}_{m,n}^{(p,q),2,\partial M} \). In the discussion of [17], we first passed to the algebraic setting and then used analytic continuation. The reason to avoid dealing with the covariant derivatives of the curvature tensor was the relation

\[
R_{ijkl;uv} - R_{ijkl;vu} = R_{uvw} R_{ijkl} - R_{uv} R_{ijkl} + R_{uvw} R_{ijkl} + R_{uvw} R_{ijkl}.
\]

This is quadratic in the curvature. Thus the space of possible tensors is not a linear space. As we shall not need Theorem 2.2, we shall omit the proof and present it simply for the sake of completeness.

There are trivial identities that arise from the curvature symmetries

\[
R_{ijkl} = -R_{ijkl} = R_{klij} \quad \text{and} \quad R_{ijkl} + R_{jkl} + R_{kij} = 0. \tag{2.a}
\]

Thus, for example, \( R_{ij1} + R_{ij1} = 0 \) is a universal curvature identity; this expression defines the zero local formula. The identities of Lemma 2.5 do not arise in this fashion; they are dimension specific. These local formulas are zero in dimensions \( \{1, 3, 5\} \) but are non-zero in dimensions \( \{2, 4, 6\} \), respectively.

### 2.4. The Restriction Map

We introduce some additional notation to describe universal curvature identities which are dimension specific. Fix a signature \((p, q)\).

Let \( P \in \mathcal{T}_{m,n}^{(p,q)} \). Set \( s_-(p, q) = (p - 1, q) \) and \( s_+(p, q) = (p, q - 1) \). If \( p = 0 \), then set \( r_-(P) = 0 \) and if \( q = 0 \), then set \( r_+(P) = 0 \) since there are no manifolds of this signature. Otherwise, let \((N_{m-1}^\pm, g_N)\) be an \(m-1\) dimensional pseudo-Riemannian manifold of signature \( s_\pm(p, q) \). Define \( r_\pm(P) \in \mathcal{T}_{m-1,n}^{s_\pm(p,q)} \) by setting:

\[
r_\pm(P)(N_{m-1}^\pm, g_N)(x) = P(N_{m-1}^\pm \times S^1, g_N \pm d\theta^2)(x, \theta).
\]

The particular angle \( \theta \in S^1 \) is irrelevant as \( S^1 \) is a homogeneous space. This yields an invariant in dimension \( m - 1 \) of the appropriate signature and defines maps:

\[
r_- : \mathcal{T}_{m,n}^{(p,q)} \to \mathcal{T}_{m-1,n}^{(p-1,q)} \quad \text{and} \quad r_+ : \mathcal{T}_{m,n}^{(p,q)} \to \mathcal{T}_{m-1,n}^{(p,q-1)}.
\]
We use a similar construction to define
\[ r_- : T^{(p,q),2}_{m,n} \rightarrow T^{(p-1,q),2}_{m-1,n}, \quad r_+ : T^{(p,q),2}_{m,n} \rightarrow T^{(p-1,q-1),2}_{m-1,n}, \]
\[ r_- : T^{(p,q),2,\partial M}_{m,n} \rightarrow T^{(p-1,q),2,\partial M}_{m-1,n}, \quad r_+ : T^{(p,q),2,\partial M}_{m,n} \rightarrow T^{(p-1,q-1),2,\partial M}_{m-1,n}. \]

We emphasize that if \( p = 0 \), then \( r_- = 0 \) and if \( q = 0 \), then \( r_+ = 0 \) since there are no manifolds of the indicated signature.

Elements of \( T^{(p,q)}_{m,n} \) are defined by summations that range from 1 to \( m \); the corresponding elements of \( T^{(p,q)}_{m-1,n} \) are defined by restricting the range of the summation. For example:

\[ \tau_m = \sum_{i,j=1}^{m} R_{i j i j} \quad \text{and} \quad \tau_{m-1} = \sum_{i,j=1}^{m-1} R_{i j i j}. \]

Thus \( r(\tau_m) = \tau_{m-1} \) since the form of the invariant is the same. The scalar curvature is universal in this sense - it is invariant under restriction. Furthermore, it is now clear that the restriction of a universal curvature identity in dimension \( m \) generates a corresponding universal curvature identity in dimension \( m - 1 \). One has \([13,15,17]\):

**Theorem 2.3.** Adopt the notation established above.

1. If \( n < m \), then \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q)}_{m,n} = \{ 0 \} \).
2. If \( n \) is odd, then \( T^{(p,q)}_{m,m-1} = \{ 0 \} \). If \( n \) is even, then the invariant \( \{ E^{(p,q)}_{m,m} \} \) of Equation \( \{ L \} \) is a basis for \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q)}_{m,m} \).
3. If \( n < m - 1 \), then \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q),2}_{m,n} = \{ 0 \} \).
4. If \( n \) is odd, then \( T^{(p,q),2}_{m,m-1} = \{ 0 \} \). If \( n \) is even, then the invariant \( \{ E^{(p,q),2}_{m,m-1} \} \) of Equation \( \{ L \} \) is a basis for \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q),2}_{m,m-1} \).
5. If \( n < m - 1 \), then \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q),2,\partial M}_{m,n} = \{ 0 \} \).
6. The invariants \( \{ F^{(p,q),2,\partial M}_{m,m-1}\} \) of Equation \( \{ L \} \) for \( 0 \leq 2\nu \leq m - 1 \) are a basis for \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q),2,\partial M}_{m,m-1} \).

2.5. **Symmetric 2-tensor valued invariants on the boundary.** The following result provides a characterization of the invariants \( \{ F^{(p,q),2,\partial M}_{m,n} \} \) which appear in Theorem 1.3. It is the appropriate extension of Theorem 2.3 to this setting and is the second main result of this paper:

**Theorem 2.4.** Adopt the notation established above.

1. If \( n = 0 \), then \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q),2,\partial M}_{m,n} = \{ 0 \} \).
2. The invariants \( \{ F^{(p,q),2,\partial M}_{m,m-2}\} \) of Equation \( \{ L \} \) for \( 0 \leq 2\nu \leq m - 2 \) are a basis for \( \ker \{ r_+ \} \cap \ker \{ r_- \} \cap T^{(p,q),2,\partial M}_{m,m-2} \).

**Proof.** Let \( 0 \neq P \in T^{(p,q),2,\partial M}_{m,n} \), i.e. \( P \) defines a non-zero local formula of degree \( n \) on the boundary in signature \( (p,q) \). Express \( P \) as a polynomial in the derivatives of the metric. Fix a point \( Y \in \partial M \). Recall that \( e_1 \) is the inward pointing geodesic normal vector field. Choose geodesic polar coordinates \( (x_2, \ldots, x_m) \) on the boundary centered at \( Y \) and then use the exponential map

\[ \mathcal{F} \rightarrow \exp(x_1 \epsilon_1(x_2, \ldots, x_m)) \]

to define coordinates \( (x_1, \ldots, x_m) \) near \( Y \). Let \( e_i := \partial_{x_i}(Y) \); we choose the coordinates on the boundary so that these form an orthonormal basis for \( T_Y M \) for \( 1 \leq i \leq m \) and the \( e_i \) for \( 2 \leq i \leq m \) form an orthonormal basis for \( T_Y (\partial M) \). We
normalize the orthonormal basis so that the indices \( \{e_2, \ldots, e_q\} \) are timelike and the indices \( \{e_{q+1}, \ldots, e_m\} \) are spacelike; here \( (\tilde{p}, \tilde{q}) = (p-1, q) \) if the normal vector \( e_1 \) is timelike and \( (\tilde{p}, \tilde{q}) = (p, q-1) \) if the normal vector \( e_1 \) is spacelike. We then have the relations:

\[
g_{1a} \equiv 0, \quad g_{ab}(Y) = \pm \delta_{ab}, \quad g_{ab/c}(Y) = 0, \quad g_{ab/1}(Y) = L_{a1}.
\]

Let \( P = P_{ij}e^i \circ e^j \in \mathbb{R}^{(p,q),2,\mathcal{O}_M} \). Express \( P_{ij} \) as a polynomial in terms of the derivatives of the metric. Let \( A \) be a monomial of \( P_{ij} \). After taking into account the normalizations given above, we may express

\[
A = g_{a_1b_1/1} \cdots g_{a_kb_k/n_1} g_{i_1j_1/\alpha_1} \cdots g_{i_\ell j_\ell/\alpha_\ell} e^i \circ e^j \quad \text{for}
\]

\[
n = k + |\alpha_1| + \ldots + |\alpha_\ell| \quad \text{and} \quad |\alpha_i| \geq 2 \quad \text{for} \quad 1 \leq i \leq \ell.
\]

If we assume that \( r_+(P) = 0 \) and \( r_-(P) = 0 \), then every index \( a \) for \( 2 \leq a \leq m \) must appear in \( A \). In particular \( \deg_a(A) \geq 2 \) for \( 2 \leq a \leq m \). We count indices:

\[
2(m - 1) \leq \sum_{a=2}^{m} \deg_a\{g_{a_1b_1/1} \cdots g_{a_kb_k/n_1} g_{i_1j_1/\alpha_1} \cdots g_{i_\ell j_\ell/\alpha_\ell} e^i \circ e^j\}
\]

\[
= 2 + 2k + \ell \sum_{a=2}^{m} \deg_a(g_{i_\nu j_\nu/\alpha_\nu}) \leq 2 + 2k + \sum_{a=1}^{m} \deg_1(g_{i_\nu j_\nu/\alpha_\nu})
\]

\[
= 2 + 2k + 2\ell + |\alpha_1| + \ldots + |\alpha_\ell| \leq 2 + 2k + 2(|\alpha_1| + \ldots + |\alpha_\ell|) = 2 + 2n.
\]

Consequently \( m - 1 \leq n + 1 \). This is, of course, not possible if \( n < m - 2 \) which proves Assertion (1).

In the limiting case that \( n = m - 2 \), all the inequalities given above must have been equalities. This implies that \( \deg_1(A) = 0 \) and that \( |\alpha_i| = 2 \) for \( 1 \leq i \leq \ell \). Consequently,

\[
P = P_{ab}(g_{c_1c_2/1}, g_{c_3c_4/\tilde{c}_3\tilde{c}_4}) e^a \circ e^b.
\]

At the point in question, we can express the variables \( g_{c_1c_2/\tilde{c}_3\tilde{c}_4} \) in terms of the curvature \( R_{\mathcal{O}_M} \). Since \( R_{abcd} = R_{\mathcal{O}_M} \) modulo quadratic terms in the second fundamental form, \( P = P_{ab}(L_{c_1c_2}, R_{\mathcal{O}_M} d_{\tilde{d}_3} d_{\tilde{d}_4}) \).

We now return to Equation (2.4). We consider symmetric 2-tensor valued monomials of the form:

\[
A = L_{a_1b_1} \cdots L_{a_kb_k} g_{c_1d_1/\epsilon_1} \cdots g_{c_\ell d_\ell/\epsilon_\ell} e^u \circ e^v
\]

where \( k + 2\ell = m - 2 \). We say that \( A \) is \textit{admissible} if each index \( 2 \leq a \leq m \) appears exactly twice in \( A \). Let \( A \) be the space of admissible monomials. Let \( c(A, P) \) be the coefficient of \( A \) in \( P \). We may then express

\[
P = \sum_{A \in A} c(A, P) \cdot A.
\]

Clearly \( P \) is invariant under the action of the subgroup of the orthogonal group fixing the normal vector \( e_1 \); we let \( \mathcal{J}_{\tilde{p}, \tilde{q}} \) be the subspace of all such polynomials. Let \( \tilde{m} : = m - 1 \) be the dimension of the boundary. We will show presently in Lemma [1.2] that

\[
\dim\{\mathcal{J}_{\tilde{p}, \tilde{q}}\} = \begin{cases} 1 + \frac{\tilde{m} - 1}{2} & \text{if } \tilde{m} \text{ is odd} \\ 1 + \frac{\tilde{m}}{2} & \text{if } \tilde{m} \text{ is even} \end{cases}
\]

(2.c)

On the other hand the invariants \( I_{m,m-2,\nu}^{(p,q),2,\mathcal{O}_M} \) give rise to admissible polynomials and there are exactly this many of them. To complete the proof, we must establish linear independence. This may be done as follows. We have \( n = m - 2 \). Take \((M, g) = (N^{m-1} \times S^1, g_N \pm d\theta^2)\). It is then immediate that

\[
g_M\{\mathcal{J}_{m,m-2,\nu}^{(p,q),2,\mathcal{O}_M} (g_M)\}, \pm d\theta^2) = I_{m-1,m-2,\nu}^{(p,q),\mathcal{O}_M} (g_N).
\]

(2.d)
Since the invariants \( \{ F_{m-1,m-2,\nu} \} \) are linearly independent local formulas, the desired result follows.

\[ \square \]

3. The proof of Theorem 1.4

3.1. Euler Lagrange equations for a manifold with boundary. We may integrate by parts to compute:

\[
\partial_t \left\{ \int_M F_{m,n}^{(p,q)}(g_t)dx + \int_{\partial M} F_{m,n-1}^{(p,q),\partial M}(g_t)dy \right\}
\]

\[
= \int_M h_{ij} Q_{m,n,i,j}^{(p,q),2}dx + \sum_{k=0}^{n-1} \int_{\partial M} (\nabla^k h_{ij}) Q_{m,n-1-k,i,j}^{(p,q),2,\partial M}dy .
\]

We examined \( Q_{m,n,i,j}^{(p,q),2} \) previously in Section 1.2. Suppose first that \( n = m - 2 \). If we consider a product manifold of the form \((M, g_M) = (N \times S^1, g_N \pm d\theta^2)\) and take the perturbation \( h = h_N + 0 \), then the Gauss-Bonnet theorem shows that the Euler-Lagrange equations are trivial. Consequently

\[ r(Q^{(p,q),2}_{m,n,i,j}) = 0 \quad \text{and} \quad r(Q^{(p,q),2,\partial M}_{m,m-2-k,i,j}) = 0 . \]

Consequently, by Theorem 2.4, \( Q^{(p,q),2,\partial M}_{m,m-2-k,i,j} = 0 \) for \( k > 0 \) while there are universal constants \( d_{m,\nu} \) so that

\[ Q^{(p,q),2,\partial M}_{m,m-2} = \sum_\nu d_{m,\nu} F^{(p,q),2,\partial M}_{m,m-2,\nu} . \]

The precise normalizing constants can then be evaluated and shown to be \( \frac{1}{2} \) by applying Equation (2.d) to the example

\[ (M, g, h) := (N \times S^1, g_N \pm d\theta^2, \pm d\theta^2) . \]

The point being that one uses the Gauss-Bonnet theorem and notes that the variation of the volume element \( \partial_t dx_{g_t} = \frac{1}{2} dx_{g} \). This completes the proof of Theorem 1.4 if \( n = m - 2 \).

Next suppose \( n = m - 3 \). We consider

\[ Q^{(p,q),2,\partial M}_{m,m-3} - \frac{1}{2} \sum_\nu F^{(p,q),2,\partial M}_{m,m-3,\nu} . \]

We use the case \( n = m - 2 \) already established to see this vanishes under \( r_{\pm} \) and, hence, by Theorem 2.4 this invariant vanishes. This completes the proof if \( n = m - 3 \). The general case now follows by induction. This completes the proof of Theorem 1.4.

\[ \square \]

4. The algebraic context

Section 4 is devoted to establishing the estimate of Equation (2.c) which was used in the proof of Theorem 2.4. We work in a purely algebraic context. In Section 4.1, we introduce the basic algebraic formalism. In Section 4.2, we define the relevant structure groups. In Section 4.3, we prove the Exchange Lemma – this is a lemma related to orthogonal invariance. In Section 4.4, we complete our discussion by deriving the estimate of Equation (2.b).
4.1. Notational conventions. Let \((V, \tilde{g})\) be an inner product space of signature \((\tilde{p}, \tilde{q})\) and dimension \(\tilde{m} := \tilde{p} + \tilde{q}\); to relate the discussion in this section to the results needed in Section 2, we need only set \(\tilde{m} = m - 1\) and take \(V = T_Y(\partial M)\) and \(\tilde{g} = \tilde{g}|_V\). We change notation slightly and let all indices range from 1 to \(\tilde{m}\) rather than from 2 to \(m = \tilde{m} + 1\). Let \(\{e_u\}\) be an orthonormal basis for \(V\). Let \(\tilde{L}_{ab}\) and \(\tilde{g}_{ab/cd}\) be formal variables where we impose the symmetries:

\[
\tilde{L}_{ab} = \tilde{L}_{ba}, \quad \tilde{g}_{ab/cd} = \tilde{g}_{ba/cd} = \tilde{g}_{ab/dc}.
\]  (4.a)

Let

\[
\tilde{L} := \tilde{L}_{ab}e^a \circ e^b \in S^2(V^*),
\]

\[
\tilde{D}^2 \tilde{g} := \tilde{g}_{ab/cd}(e^a \circ e^c) \otimes (e^b \circ e^d) \in S^2(V^*) \otimes S^2(V^*).
\]

If \(T\) is a linear transformation of \(V\), then the natural action of \(T\) on \(S^2(V^*)\) defines the action of \(T\) on these variables. More precisely, if \(Te_u = \tilde{T}_u e_v\), then

\[
(T \tilde{L})_{ab} = \tilde{T}_{a}^{\delta} \tilde{T}_{b}^{\gamma} \tilde{L}_{\delta \gamma} \quad \text{and} \quad (T \tilde{D}^2 \tilde{g})_{ab/cd} = \tilde{T}_{a}^{\delta} \tilde{T}_{b}^{\gamma} \tilde{T}_{c}^{\ell} \tilde{T}_{d}^{\mu} \tilde{g}_{\delta \gamma / \ell \mu}.
\]

In other words, we simply expand \(\tilde{L}\) and \(\tilde{D}^2 \tilde{g}\) multi-linearly. This is, of course, exactly the usual action of the general linear group on the second fundamental form and on the 2-jets of the metric.

Consider a symmetric 2-tensor valued monomial of the form:

\[
A = \tilde{L}_{aa_1b_1} \cdots \tilde{L}_{aa_kb_k} \tilde{g}_{c_1d_1}/e_{c_1f_1} \cdots \tilde{g}_{c_\ell d_\ell}/e_{c_\ell f_\ell} e^u \circ e^v.
\]

We say an index \(a\) touches itself in \(A\) if \(A\) is divisible by one of the variables \(\tilde{L}_{aa}\), \(\tilde{g}_{aa/\star \star}\), \(\tilde{g}_{\star \star /aa}\), or \(e^a \circ e^a\). Let \(\delta\) be the Kronecker symbol. We let

\[
\deg_{\mathcal{G}}(A) := \sum_{\mu=1}^{k} \{\delta_{a_\mu, w} + \delta_{b_\mu, w}\} + \sum_{\nu=1}^{\ell} \{\delta_{c_\nu, w} + \delta_{d_\nu, w} + \delta_{e_\nu, w} + \delta_{f_\nu, w}\}
+ \delta_{a, w} + \delta_{b, w},
\]

\[
\ord_{L}(A) := k, \quad \ord_{\mathcal{G}}(A) = 2\ell \quad \ord(A) := k + 2\ell,
\]

\(\deg_{\mathcal{G}}(A)\) is the number of times that the index \(w\) appears in \(A\). Motivated by the discussion of the proof of Theorem 2.4 we say that \(A\) is admissible if

\[
\deg_{\mathcal{G}}(A) = 2 \quad \text{for} \quad 1 \leq a \leq \tilde{m}.
\]

Let \(A\) be the set of admissible monomials. If \(A \in A\), then \(1 + k + 2\ell = \tilde{m}\) so:

\[
\ord(A) = \ord_{L}(A) + \ord_{\mathcal{G}}(A) = k + 2\ell = \tilde{m} - 1.
\]

Let \(C := \{c(A, P)\}_{A \in A}\) be a collection of constants. We form the associated admissible polynomial:

\[
P = P_C := \sum_{A \in A} c(A, P) \cdot A.
\]

We say that \(A\) is a monomial of \(P\) if \(c(A, P) \neq 0\). We may expand an admissible polynomial \(P\) in the form:

\[
P = \sum_{k=\tilde{m}-1}^{\tilde{m}} P_k \quad \text{where} \quad P_k = P_{k, C} := \sum_{A \in A, \ord(A) = k} c(A, P) \cdot A. \quad (4.b)
\]

The symmetries of Equation (4.a) mean that we can regard \(P_k \in \otimes^{k+2\ell+1} S^2(V^*)\).
4.2. Structure groups. Let $GL$ be the general linear group of $V$ and let $O$ be the associated orthogonal group:

$$O := \{ T \in GL : g(Tx, Ty) = g(x, y) \text{ for all } x, y \in V \}.$$ 

For $1 \leq k \leq \tilde{m}$, let

$$O_k := \{ T \in O : Te_b = e_b \text{ for all } b < k \}.$$ 

If $\{a, b\}$ are distinct indices, let $V_{a,b} := \text{Span}\{e_a, e_b\}$. If $V_{a,b}$ has signature $(2,0)$ or has signature $(0,2)$, we consider the rotations $T_{a,b}(\theta) \in O$: defined by setting:

$$T_{a,b}(\theta)e_c := \begin{cases} 
\cos \theta e_a + \sin \theta e_b & \text{if } c = a \\
-\sin \theta e_a + \cos \theta e_b & \text{if } c = b \\
e_c & \text{if } c \neq a \text{ and } c \neq b 
\end{cases}.$$ 

Similarly, if $V_{a,b}$ has signature $(1,1)$, we consider the hyperbolic boosts $T_{a,b}(\theta) \in O$ defined by setting:

$$T_{a,b}(\theta)e_c := \begin{cases} 
\cosh \theta e_a + \sinh \theta e_b & \text{if } c = a \\
\sinh \theta e_a + \cosh \theta e_b & \text{if } c = b \\
e_c & \text{if } c \neq a \text{ and } c \neq b 
\end{cases}.$$ 

The transformations $\{T_{a,b}(\theta)\}$ for $\theta \in \mathbb{R}$ and $1 \leq a < b \leq \tilde{m}$ generate the connected component of the identity of $O$.

Each index appears exactly twice in any admissible monomial. If $V_{a,b}$ has signature $(1,1)$, then we replace the two ‘$a$’ indices by ‘$\cosh \theta_a + \sinh \theta_b$’, we replace the two ‘$b$’ indices by ‘$\sinh \theta_a + \cosh \theta_b$’, and we expand multi-linearly to determine $T_{a,b}(\theta)A$: if $V_{a,b}$ has signature $(2,0)$ or $(0,2)$, then there are sign changes. We replace the two ‘$a$’ indices by $\cos \theta_a + \sin \theta_b$ and we replace the two ‘$b$’ indices by $-\sin \theta_a + \cos \theta_b$ before expanding multi-linearly. Thus each admissible monomial gives rise to 16 different monomials which must be combined and simplified in computing the action of $T_{a,b}(\theta)$ on an admissible polynomial. We do not sum over repeated indices in what follows in the remainder of Section 4.2. If $V_{a,b}$ has signature $(1,1)$, we have:

$$T_{a,b}(\theta)\tilde{g}_{aa/ab} = \cosh^4 \theta \tilde{g}_{aa/ab} + \cosh^3 \theta \sinh \theta(2\tilde{g}_{ab/ab} + 2\tilde{g}_{aa/ab})$$
$$+ \cosh^2 \theta \sin^2 \theta(\tilde{g}_{aa/ab} + \tilde{g}_{bb/ab} + 4\tilde{g}_{ab/ab})$$
$$+ \cosh \theta \sin \theta(2\tilde{g}_{ab/ab} + 2\tilde{g}_{bb/ab} + 4\tilde{g}_{ab/ab}) + \sinh^4 \theta \tilde{g}_{bb/ab}.$$ 

The 4 terms (counted with multiplicity) with a coefficient of $\cosh^3 \theta \sin \theta$ arise from changing a single index $a \rightarrow b$ or $b \rightarrow a$; if $V_{a,b}$ has signature $(2,0)$ or $(0,2)$, then there are appropriate changes of sign:

$$T_{a,b}(\theta)\tilde{g}_{aa/ab} = \cosh^4 \theta \tilde{g}_{aa/ab} + \cosh^3 \theta \sin \theta(2\tilde{g}_{ab/ab} - 2\tilde{g}_{aa/ab})$$
$$+ \cosh^2 \theta \sin^2 \theta(\tilde{g}_{aa/ab} + \tilde{g}_{bb/ab} - 4\tilde{g}_{ab/ab})$$
$$+ \cosh \theta \sin \theta(2\tilde{g}_{ab/ab} - 2\tilde{g}_{bb/ab}) + \sinh^4 \theta \tilde{g}_{bb/ab}.$$ 

Suppose that $\tilde{m} = 2$ so indices range from 1 to 2. Let

$$P = \det(L) = \tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{12}.$$ 

Assume $V_{1,2}$ has signature $(1,1)$. Then:

$$T_{12}(\theta)\tilde{L}_{11} = \cosh^2 \theta \tilde{L}_{11} + \sinh^2 \theta \tilde{L}_{22} + 2 \cosh \theta \sinh \theta \tilde{L}_{12},$$
$$T_{12}(\theta)\tilde{L}_{12} = (\cosh^2 \theta + \sinh^2 \theta)\tilde{L}_{12} + \cosh \theta \sinh \theta(\tilde{L}_{11} + \tilde{L}_{22}),$$
$$T_{12}(\theta)\tilde{L}_{22} = \sinh^2 \theta \tilde{L}_{11} + \cosh^2 \theta \tilde{L}_{22} + 2 \cosh \theta \sinh \theta \tilde{L}_{12}.$$
We compute:
\[
T_{12}\tilde{L}_{11}\tilde{L}_{22} = \cosh^4 \theta \tilde{L}_{11}\tilde{L}_{22} + \cosh^3 \theta \sinh \theta (2\tilde{L}_{12}\tilde{L}_{22} + 2\tilde{L}_{11}\tilde{L}_{12})
\]
\[
+ \cosh^2 \theta \sinh^2 \theta (\tilde{L}_{11}\tilde{L}_{11} + \tilde{L}_{22}\tilde{L}_{22} + 4\tilde{L}_{12}\tilde{L}_{12})
\]
\[
+ \cosh \theta \sinh^3 \theta (2\tilde{L}_{12}\tilde{L}_{22} + 2\tilde{L}_{11}\tilde{L}_{12}) + \sinh^4 \theta \tilde{L}_{11}\tilde{L}_{22},
\]
\[
T_{12}\tilde{L}_{12}\tilde{L}_{12} = \cosh^4 \theta \tilde{L}_{12}\tilde{L}_{12} + \cosh^3 \theta \sinh \theta (2\tilde{L}_{12}\tilde{L}_{22} + 2\tilde{L}_{11}\tilde{L}_{12})
\]
\[
+ \cosh^2 \theta \sinh^2 \theta (2\tilde{L}_{12}\tilde{L}_{12} + \tilde{L}_{11}\tilde{L}_{11} + \tilde{L}_{22}\tilde{L}_{22} + 2\tilde{L}_{12}\tilde{L}_{12})
\]
\[
+ \cosh \theta \sinh^3 \theta (2\tilde{L}_{11}\tilde{L}_{12} + 2\tilde{L}_{22}\tilde{L}_{12}) + \sinh^4 \theta \tilde{L}_{12}\tilde{L}_{12}
\]
\[
T_{12}(\tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{12}) = \cosh^4 \theta (\tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{12})
\]
\[
- 2 \cosh^2 \theta \sinh^2 \theta (\tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{12}) + \sinh^4 \theta (\tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{12})
\]
\[
= \tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{12}.
\]

This shows, not surprisingly, that \(\det(L)\) is invariant under the action of the orthogonal group in dimension 2.

4.3. **The exchange lemma.** Suppose given a monomial \(C\) with \(\text{deg}_{\theta}(C) = 3\), \(\text{deg}_a(C) = 1\), and \(\text{deg}_c(C) = 2\) for \(c \neq a, b\). We suppose that the index \(a\) touches itself in \(C\). There are then exactly 2 admissible monomials \(A\) and \(B\) which transform to \(C\) by changing a single index \(a \rightarrow b\); there are no admissible monomials which transform to \(C\) by changing a single index \(b \rightarrow a\) since such a monomial would have the index \(a\) appearing 4 times. There are several possibilities:

1. If \(C = L_{aa}A_1\), then \(A = L_{ab}A_1\) and \(B = L_{aa}B_1\) where to define \(B_1\), the index \(a\) appearing in \(A_1\) is exchanged for the index \(b\).
2. If \(C = g_{aa/cd}A_1\) (resp. \(g_{cd/aa}A_1\)), then \(A = g_{ab/cd}A_1\) (resp. \(g_{cd/ab}A_1\)) and \(B = g_{aa/cd}B_1\) (resp. \(g_{cd/aa}B_1\)) where to define \(B_1\), the index \(a\) appearing in \(cdA_1\) is exchanged for the index \(b\).
3. If \(C = A_1e^a \circ e^b\), then \(A = A_1e^b \circ e^a\) and \(B = B_1e^a \circ e^a\) where to define \(B_1\), the index \(a\) appearing in \(A_1\) is exchanged for the index \(b\).

The following technical Lemma appeared first in the discussion of [13] of the heat equation proof of the Gauss-Bonnet formula when Theorem 2.3 was first proved; we modify the proof given there to make it applicable to the present context.

**Lemma 4.1.** Let \(P = P_C\) be an admissible polynomial and let \(\{a, b\}\) be distinct indices. Assume that \(T_{a,b}(\theta)P = P\) for all \(\theta\). Let \(\{C, A, B\}\) be as above. Then \(A\) is a monomial of \(P\) if and only if \(B\) is a monomial of \(P\).

**Proof.** Suppose first that \(V_{a,b}\) has signature (1, 1). We expand \(T_{a,b}(\theta)A\) by replacing each index \(a\) by \(\cosh \theta a + \sinh \theta b\) and each index \(b\) by \(\sinh \theta a + \cosh \theta b\) and expanding multi-linearly. To obtain \(C\), we must change an odd number of indices. We consider the coefficient of \(\cosh^3 \theta \sinh \theta\) in \(c(C, T_{a,b}(\theta)A)\); this arises from changing exactly one index and leaving the other three indices the same. Since \(\text{deg}_a(C) = 1\) and \(\text{deg}_b(A) = 2\), the index that is being changed is \(b \rightarrow a\). This may, of course, be done either in one way (in which case we set \(\sigma_1 = 1\)) or two ways (in which case we set \(\sigma_1 = 2\)). To illustrate this, we consider the case (1) above. For example, if we have that \(A = L_{aa}L_{bc}L_{ab}A_3\) for \(\{a, b, c, d\}\) distinct indices, then \(\sigma_1 = 1\) while if \(A = L_{aa}L_{bc}L_{ab}A_2\) for \(\{a, b, c\}\) distinct indices, then \(\sigma_1 = 2\). The arguments for the other possible cases (2) and (3) are essentially similar.

The argument given above shows that
\[
c(C, T_{a,b}(\theta)A) = \sigma_1 \cosh^3 \theta \sinh \theta + * \cosh \theta \sinh^3 \theta\]
for \(\sigma_1 \in \{1, 2\}\)
and where \(*\) is a coefficient which is not of interest. Similarly
\[
c(C, T_{a,b}(\theta)B) = \sigma_2 \cosh^3 \theta \sinh \theta + * \cosh \theta \sinh^3 \theta\]
for \(\sigma_2 \in \{1, 2\}\).
Suppose that $X$ is an admissible monomial so that $X$ transforms to $C$ by changing exactly one index $a \to b$ or $b \to a$. Since $\deg_a(X) = 2$ and $\deg_b(C) = 3$, $X$ can not transform to $C$ by changing the index $a$ to $b$ and thus transforms to $C$ by changing the index $b$ to $a$; conversely $X$ is obtained from $C$ by changing exactly one index $a$ to $b$ and leaving the other indices alone. Consequently $X = A$ or $X = B$ since these are the only monomials obtained in this way. Since the number of indices which must be changed is odd, the powers of cosh and sinh are odd and we have:

$$0 = c(C, T_{a,b}(\theta) P) = (\sigma_1 c(A, P) + \sigma_2 c(B, P)) \cosh\theta \sinh\theta + \star \cosh\theta \sinh^3\theta.$$  

We eliminate the coefficient of $\cosh\theta \sinh^3\theta$ by examining:

$$0 = \lim_{\theta \to 0} \frac{c(C, T_{a,b}(\theta) P)}{\sinh\theta} = \sigma_1 c(A, P) + \sigma_2 c(B, P).$$

Since $\sigma_i \in \{1, 2\}$, $c(A, P)$ is non-zero if and only if $c(B, P)$ is non-zero which proves the Lemma if $V_{a,b}$ has signature $(1, 1)$. If $V_{a,b}$ has signature $(0, 2)$ or $(2, 0)$, then we have similarly that:

$$0 = -(\sigma_1 c(A, P) + \sigma_2 c(B, P)) \cosh^3\theta \sin\theta + \star \cosh\theta \sin^3\theta$$

and the argument again is similar. \[\square\]

### 4.4 The estimate of Equation (2.2).

Let $\mathcal{J}$ be the space of admissible polynomials which are invariant under the action of the orthogonal group $O$. For $0 \leq k \leq \bar{m} - 1$ and $k \equiv \bar{m} - 1 \pmod{2}$, set

$$Q_k := L_{a_1} b_k \cdots L_{a_k} b_k \bar{g}_{a_k+2b_k+1/a_k+2b_k+2} \cdots \bar{g}_{a_{\bar{m}}-2b_{\bar{m}}-2/a_{\bar{m}}-1b_{\bar{m}}-1} e^{a_m} \circ e^{b_m} e^{a_1 \cdots a_{\bar{m}}}.$$  

**Lemma 4.2.** The polynomials $Q_k$ for $k \equiv \bar{m} - 1 \pmod{2}$ are a basis for $\mathcal{J}$. Thus

$$\dim(\mathcal{J}) = \begin{cases} 1 + \frac{\bar{m} - 1}{2} & \text{if } \bar{m} \text{ is odd} \\ 1 + \frac{\bar{m}}{2} & \text{if } \bar{m} \text{ is even} \end{cases}.$$  

**Proof.** Let $P \in \mathcal{J}$. Adopt the notation of Equation (4.1) to express $P = \sum_k P_k$. Then the $P_k$ are each invariant under the action of $O$ separately. Let $\mathcal{J}_k$ be the span of such polynomials; $\mathcal{J} = \oplus_k \mathcal{J}_k$. We will complete the proof by showing $\dim(\mathcal{J}_k) \leq 1$. Let $0 \neq P_k \in \mathcal{J}_k$. Let

$$A = L_{a_1} b_k \cdots L_{a_k} b_k \bar{g}_{a_k+2b_k+1/a_k+2b_k+2} \cdots \bar{g}_{a_{\bar{m}}-2b_{\bar{m}}-2/a_{\bar{m}}-1b_{\bar{m}}-1} e^{a_m} \circ e^{b_m}$$

be a monomial of $P$. We can apply the exchange Lemma to assume $a_1 = b_1$. If $a_1 = 1$, fine. Otherwise, we can apply the exchange Lemma to assume $b_1 = 1$ and then apply the exchange Lemma again to construct a monomial $A_1$ of $P$ so that $a_1 = b_1 = 1$. Decompose

$$P_k = \tilde{L}_{11} P_{k,1} + \tilde{P}_{k,1}$$

where $\tilde{P}_{k,1} := P_k - \tilde{L}_{11} P_{k,1}$ and where

$$P_{k,1} = \frac{1}{\tilde{L}_{11}} \sum_{B \in A, \tilde{L}_{11} \text{ divides } B} c(B, P) \cdot B.$$  

Then $0 \neq P_{k,1}$ and $P_{k,1}$ is invariant under the action $O_2$ since we have fixed the index ‘1’. Thus, in particular, $P_{k,1}$ is invariant under the action of $T_{a,b}(\theta)$ for $2 \leq a < b \leq \bar{m}$. We can then apply the exchange Lemma to show that $\tilde{L}_{22}$ divides some monomial of $P_{k,1}$. We continue in this fashion to construct an admissible polynomial $0 \neq P_{k,k}$ which is invariant under the action $O_{k+1}$ and so that $P_{k,k}$ is divisible by $\tilde{L}_{11} \cdots \tilde{L}_{kk}$. We express

$$P_{k,k} = \tilde{L}_{11} \cdots \tilde{L}_{kk} \bar{g}_{a_{k+1}b_{k+1/a_{k+2}b_{k+2}}} \cdots \bar{g}_{a_{\bar{m}}-2b_{\bar{m}}-2/a_{\bar{m}}-1b_{\bar{m}}-1} e^{a_m} \circ e^{b_m}.$$
Of course, if $k = 0$, then $P_{k,k} = P$ where as if $k = \tilde{m} - 1$, then we shall not proceed further.

If $k < \tilde{m} - 1$, we apply the exchange Lemma twice to choose a monomial so $a_{k+1} = b_{k+1} = k + 1$ and continue in this fashion finally to show that

$$\tilde{L}_{11} \ldots \tilde{L}_{kk} \tilde{g}_{k+1,k+1/k+2,k+2/\ldots} \tilde{g}_{\tilde{m} - 2,\tilde{m} - 2/\tilde{m} - 1,\tilde{m} - 1} e^{a_{\tilde{m}}} \circ e^{b_{\tilde{m}}}$$

is a monomial of $P$. Since the index $\tilde{m}$ can only appear in $\{a_{\tilde{m}}, b_{\tilde{m}}\}$ we have $a_{\tilde{m}} = b_{\tilde{m}} = \tilde{m}$. We conclude therefore that

$$A_k := \tilde{L}_{11} \ldots \tilde{L}_{kk} \tilde{g}_{k+1,k+1/k+2,k+2/\ldots} \tilde{g}_{\tilde{m} - 2,\tilde{m} - 2/\tilde{m} - 1,\tilde{m} - 1} e^{a_{\tilde{m}}} \circ e^{b_{\tilde{m}}}$$

is a monomial of $P_k$. We summarize. If $0 \neq P_k \in \mathcal{J}_k$, then $c(A_k, P_k) \neq 0$. This shows that $\dim(\mathcal{J}_k) \leq 1$.

Remark 4.1. H. Weyl’s second theorem of invariants states that all relations amongst orthogonal linear invariants arising from contractions of indices can be constructed using the tensor $\varepsilon$ described in Equation (1.3). When passing from the case of general orthogonal linear invariants to the case of universal curvature invariants, one must also include the curvature symmetries of Equation (2.2). This plays a crucial role in the proof of Theorem 2.3 given in [16, 17]. However, rather than using H. Weyl’s theorems directly (which would necessitate extending the discussion to include covariant derivatives of the second fundamental form), we have chosen to give a proof of Theorem 1.4 based on the analysis of [13], which was first developed to give a heat equation proof of the Gauss-Bonnet theorem.

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