One-step replica-symmetry-breaking phase below the de Almeida–Thouless line in low-dimensional spin glasses

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The de Almeida-Thouless (AT) line is the phase boundary in the temperature–magnetic field plane of an Ising spin glass at which a continuous (i.e. second-order) transition from a paramagnet to a replica-symmetry-breaking (RSB) phase occurs, according to mean-field theory. Here, using field-theoretic perturbative renormalization group methods on the Bray-Roberts reduced Landau-Ginzburg-type theory for a short-range Ising spin glass in space of dimension $d$, we show that at nonzero magnetic field the nature of the corresponding transition is modified as follows: a) for $d < 6$ small and positive, with increasing field on the AT line first, the ordered phase just below the transition becomes the so-called one-step RSB, instead of the full RSB that occurs in mean-field theory; the transition on the AT line remains continuous with a diverging correlation length. Then at a higher field, a tricritical point separates the latter transition from a quasi-first-order one, that is one at which the correlation length does not diverge, and there is a jump in part of the order parameter, but no latent heat. The location of the tricritical point tends to zero as $d \to 6$; b) for $d \leq 6$, we argue that the quasi-first-order transition persists down to arbitrarily small nonzero fields, with a transition to full RSB still expected at lower temperature. Whenever the quasi-first-order transition occurs, it is at a higher temperature than the AT transition would be for the same field, preempting it as the temperature is lowered. These results may explain the reported absence of a diverging correlation length in the presence of a magnetic field in low-dimensional spin glasses in simulations and high-temperature series expansions.

I. INTRODUCTION

A. Background and motivation

A transition in a classical Ising spin glass (SG) in a magnetic field within a mean-field treatment was found by de Almeida and Thouless (AT) \cite{1}, who showed that the mean-field solution found by Sherrington and Kirkpatrick (SK) \cite{2} is unstable at sufficiently low temperature $T$ in any magnetic field $h$, and so the instability or transition occurs on a line (now known as the AT line) $T_{AT}(h)$ in the $T$–$h$ plane; the AT line passes through the critical temperature $T_c$ at $h = 0$. In a short-range Ising SG (i.e. the Edwards-Anderson [EA] model \cite{3}), at such a transition the correlation length and the SG susceptibility both diverge, typical of a continuous (or second order) phase transition. The AT instability indicated that in the SK model, or within mean field theory, the symmetry under permutations of the replicas (introduced by EA) must be broken in the phase below the AT line in nonzero as well as in zero magnetic field. The replica-symmetry breaking (RSB) ordering in the low-temperature phase was determined by Parisi \cite{4}, and has been proved to give the correct thermodynamic properties of the SK model \cite{5,6}. Among many reviews, the most relevant to this paper are the books Refs. \cite{7,8}.

In the short-range models, a controversy has remained about whether the RSB picture is a correct description of the ordered phase in each dimension $d$ of space, at least for those $d$ in which a transition at nonzero temperature occurs at $h = 0$. The leading alternative is the scaling droplet picture \cite{9,11}, in which in particular there is no transition at nonzero $h$. Thus the question of the existence and nature of a transition in a magnetic field is important for our understanding of the SG ordered phase, especially in realistic dimensions, say $d = 3$. A number of simulations (in both the nearest-neighbor $d$-dimensional and one-dimensional power law models; see for example Refs. \cite{12,13,14}), and also high-temperature series expansions \cite{15}, found no divergence of the correlation length in a magnetic field in low dimensions ($d < 6$, and for corresponding power-laws in one dimension).

The standard method of studying the effect of fluctuations around mean-field theory in short-range models is to use a statistical field theory with an action obtained from Landau-Ginzburg theory. Perhaps surprisingly, the analysis of the AT line (by which we will always mean at $h > 0$; note that the sign of $h$ is immaterial for Ising spins) in the short-range case within such a treatment encounters difficulties in low dimensions ($d \leq 6$). In an important early paper, Bray and Roberts (BR) \cite{10} formulated a “reduced” action for the fluctuating modes (called “replicas”) that remain massless on the AT line. They found that the perturbative renormalization group (RG) flows for the two coupling constants of this theory experience runaway flows to strong coupling for $d \leq 6$, so that no RG fixed point that could describe the behavior of the AT line for $d \leq 6$ could be found within perturbation theory. They suggested that this might mean that either (i) the transition becomes first order (with no divergence of the correlation length), or (ii) the transition is first-order even in mean-field theory, or (iii) there is no transition in a nonzero magnetic field for $d \leq 6$. The latter possibility has been used as an argument in favor...
of the scaling-droplet theory (see Ref. [17] and references therein, and also the response in Ref. [18]). In a later work, Moore and one of the authors (Ref. [19]; to be referred to as MR) showed that the BR flows also imply that there should be a multicritical point on the AT line as \( d \to 6^+ \).

In general phase-transition theory, the possibility that a transition is first-order can rarely be ruled out entirely, and has frequently been stated to be a possible solution to the problem raised by BR. However, it was pointed out long ago that, within a Landau theory of a SG formulated in terms of replicas, a conventional first-order transition with positive latent heat is not possible; it cannot originate from solving such a theory [20]. The reason is that in replica theory (including Parisi’s RSB scheme), the free energy functional in the limit of \( n = 0 \) replicas must be maximized with respect to Parisi’s \( q(x) \) function, not minimized, and so a crossing of the free energies of extrema of this functional would produce a latent heat that is either zero or negative; the latter is forbidden by conventional thermodynamics.

However, there is a way to obtain a quasi-first-order transition (i.e. first order with zero latent heat) from a RSB Landau theory. It was identified by Gross, Kanter, and Sompolinsky (GKS) [20] when they were studying SGs of spins with either Potts or uniaxial quadrupolar symmetry in zero magnetic field. Assuming isotropic SG order (an assumption that need not concern us here), the SG order parameter becomes a matrix \( Q_{\alpha\beta} \) (\( \alpha, \beta = 1, \ldots, n \)) that is symmetric, with \( Q_{\alpha\alpha} = 0 \) for all \( \alpha \), as for the Ising case. Their Landau theory, that is the free energy expanded in powers of \( Q_{\alpha\beta} \), is

\[
F = \frac{1}{4} \bar{r} \sum_{\alpha\beta} Q_{\alpha\beta}^2 - \frac{1}{6} w_1 \sum_{\alpha\beta\gamma} Q_{\alpha\beta} Q_{\beta\gamma} Q_{\gamma\alpha} - \frac{1}{6} w_2 \sum_{\alpha\beta} Q_{\alpha\beta}^3 - \frac{1}{8} y \sum_{\alpha\beta} Q_{\alpha\beta}^4, \tag{1.1}
\]

the terms up to the cubic order are the most general form allowed by the symmetry of the Potts and quadrupolar models. (The quartic term with coefficient \( y \) is not the most general form; we come to that later.) Ising SGs in zero magnetic field are usually described by the same Landau theory, except that there \( w_2 = 0 \) as a consequence of inversion symmetry in spin space, and \( y > 0 \). GKS found that for \( \bar{r} \) not too large and negative (i) for \( y \leq 0 \) and \( 0 < w_2/w_1 < 1 \), there is a continuous transition at \( \bar{r} = \bar{r}_c = 0 \), but for \( \bar{r} < \bar{r}_c \) the Parisi function \( q(x) \) is a step function (of \( x \in [0,1] \)) instead of the continuous function familiar for the Ising spin glass in mean field theory, and (ii) for \( y < 0 \) and \( w_2/w_1 > 1 \), the transition is discontinuous: \( q(x) \) is again a step function, but \( q(1) \) has a jump at \( \bar{r}_c \), and \( \bar{r}_c \) is now positive; there is no latent heat. In case (ii), the eigenvalues of the Hessian are strictly positive as \( \bar{r} \to \bar{r}_c \) on both sides of the transition, implying that the SG susceptibility and, in a finite-dimensional version, the correlation length do not diverge at \( \bar{r}_c \). The step function form of \( q(x) \) describes what is known as one-step RSB (or 1-RSB), and the quasi-first-order transition in case (ii) has the form of the transition in the random energy model [21][22], though there the extensive part of the entropy is zero in the low-temperature region, which is not the case here.

The non-derivative part of the BR reduced action has the same form as eq. (1.1) through terms of cubic order, except that it involves, in place of \( Q_{\alpha\beta} \), the field \( \tilde{Q}_{\alpha\beta} \) which satisfies the additional conditions \( \sum_{\beta} \tilde{Q}_{\alpha\beta} = 0 \), that define the replicon subspace. (The terms through cubic order give the most general cubic action in the replicon sector.) \( w_2 \) can be nonzero, due to the breaking of inversion symmetry by the magnetic field. Moreover, the BR RG flows for \( w_1, w_2 \) imply that for \( w_2 \neq 0 \) the ratio \( \rho = w_2/w_1 \) tends to a value \( \rho^* = 14.379 \) on the AT line for \( d \leq 6 \). As this is larger than unity, the GKS results could come into play. But then \( y < 0 \) is also necessary. The initial value of \( y \) in BR is positive, but the RG flows might take the parameters into a region where GKS can be applied. Previous works do not seem to have considered the quartic terms that could be included in the BR action. Presumably this was because quartic and higher-order terms are irrelevant in the RG sense near \( d = 6 \) dimensions. However, Fisher and Sompolinsky (FS) [23] explained that, because the quartic terms cannot be dropped at and below a SG transition, they are “dangerously irrelevant”, and moreover they are important for the scaling behavior when \( d < 8 \), because of the form of their RG flows, even though they are irrelevant. For example, the form of the AT line at small \( h \) depends on \( y \), and so is modified for \( 6 < d < 8 \), to interpolate from the mean-field results for \( d > 8 \) to the scaling forms for \( d \leq 6 \) (see also Ref. [24]). The GKS results indicate that an extreme form of dangerous irrelevance could occur, because a negative value of \( y \) causes qualitative changes in the phase transition behavior, not just quantitative changes such as in exponents.

These considerations motivate us to consider a Landau-Ginzburg field theory that extends the BR reduced theory by including quartic terms. The fields in the theory are the same ones, \( \tilde{Q}_{\alpha\beta}(x) \), and the action is now

\[
F[(\tilde{Q}_{\alpha\beta})] = \int d^d x \left[ \frac{1}{4} \sum (\nabla \tilde{Q}_{\alpha\beta})^2 + \frac{1}{4} \bar{r} \sum \tilde{Q}_{\alpha\beta}^2 - \frac{1}{6} w_1 \sum \tilde{Q}_{\alpha\beta} \tilde{Q}_{\beta\gamma} \tilde{Q}_{\gamma\alpha} - \frac{1}{6} w_2 \sum \tilde{Q}_{\alpha\beta}^3 - \frac{1}{8} y \sum \tilde{Q}_{\alpha\beta}^4 \right] \tag{1.2}
\]

(Here the summations are taken freely over all of the distinct indices displayed in each term.) Here we included all possible terms of quartic order, as each is generated by the RG; note that \( y \) has replaced the previous \( y \). We aim to show that in low dimensions the RG flows take the
couplings into a regime where the nature of the transition changes qualitatively.

B. Outline and results

The free energy in eq. (1.2), evaluated with $Q_{\alpha\beta}(x)$ independent of $x$, gives a Landau theory similar to that of GKS, except for the replicon constraint that is now in force, and for the terms with coefficients $y_2, \ldots, y_5$. Because of these differences from the GKS case, our first task is to solve this theory in the various regimes for $y_i$ ($i = 1, \ldots, 5$) and for $w_1, w_2$; this is carried out in Sec. III below. Because of the replicon constraint, the Parisi ansatz for RSB applied to $Q_{\alpha\beta}$ leads to a function $\tilde{q}(x)$ in place of $q(x)$, which obeys $\int_0^1 dx \tilde{q}(x) = 0$ in place of $q(x) \geq 0$. This change makes little difference in practice, and the results are very similar to those of GKS summarized above. The task of including the quartic terms is aided by a paper by Goldbart and Elderfield [25], who considered the full set of quartic terms in the context of GKS. Similar to what they found, the relevant criteria for the extremum to be 1-RSB (in place of $\tilde{y} \leq 0$ or $\tilde{y} < 0$) for $\tilde{r} < \tilde{r}_c$ are that $y_1 - y_3 x + y_5 x^2 \leq 0$ when evaluated at $x = \rho$ for $\rho < 1$, and $y_1 - y_3 + y_5 \rho^2 > 0$ when $\rho \geq 1$. It will be convenient to write these criteria simply as $y \leq 0$ or $\tilde{y} < 0$ respectively, where $y = y_1 - y_3 x_\rho + y_5 x_\rho^2$, evaluated at $x_\rho = \min(\rho, 1)$. For $\rho > 1$, we also find another transition at $\tilde{r} = \tilde{r}_d$ similar to Kirkpatrick and Thirumalai [26], with $\tilde{r}_d > \tilde{r}_c$. (They argued that this is connected with a dynamical transition.)

In Sec. III we then evaluate the RG flows for our theory at one-loop order in perturbation theory, reproducing the results of BR, and extending them to include the important part of the flow equations for $y_i$.

In Sec. IV we consider the consequences of the flows; the results are summarized in the phase diagrams in Fig. 1. We find that for $d > 8$, the transition in weak nonzero magnetic fields takes the same form as in mean-field theory: it is continuous and $q(x)$ below the transition is a continuous function. This occurs because all the couplings included flow to zero, without reaching either $\rho > 1$ or $\tilde{y} < 0$. For $d \leq 8$, the effective values of the couplings $y_i$ become asymptotic at long length scales to expressions quartic in $w_1, w_2$, similar to the discussion in FS. For $6 < d \leq 8$ and at sufficiently small $h$, the $w_i$ couplings flow towards zero, $\rho$ remains small, and $\tilde{y}$ is again positive. The behavior is again as in mean-field theory, except for modifications similar to those of FS.

For $d$ just above 6, we build on the analysis of Moore and the second author [19]. When the magnetic field is not so weak, $\rho$ is driven to larger values, and there is a Lifshitz-type point $\tilde{r}$ beyond which the phase below the transition is 1-RSB, while the transition remains continuous. At higher field, $\rho$ becomes larger than 1, and the transition to 1-RSB becomes quasi-first-order. This implies that there is a tricritical point $T$ on the AT line at $T_{\tilde{r}, h_{\tilde{r}}}$ (with $h_{\tilde{r}} > 0$). The tricritical point $T$ preempts the multicritical point $M$ in MR, as it occurs at slightly lower field in the limit. Likewise, the quasi-first-order transition preempts the AT line for $h > h_{\tilde{r}}$. $h_{\tilde{r}}$ tends to zero rapidly as $d \rightarrow 6^+$. The results mentioned are valid within perturbation theory up to $h$ larger than $h_{\tilde{r}}$ but of similar order.

For $d \leq 6$, unfortunately we cannot quantitatively analyze the flows in perturbation theory with the present methods. We argue by continuity that the quasi-first-order behavior found for $h > h_{\tilde{r}}$ persists to $d \leq 6$ for all $h > 0$. This implies that in these low dimensions, the AT line is preempted by a quasi-first-order transition, with no divergence of the correlation length or SG susceptibility at the transition. This is consistent with a number of simulations, and with high temperature series. Of course, while our arguments are expected to hold at least for $d$ not too far below 6, we cannot rule out further changes in behavior at lower $d$.

In Sec. V we consider some implications of the results and scenario for the metastate $[27, 29]$ in $d < 8$ dimensions. We argue that one or more further transitions must occur as temperature is lowered further below the transition, as shown in Fig. 1. In addition, we further discuss the second solution with 1-RSB that persists to higher temperatures when the quasi-first-order transition occurs; it is similar to a phase that was discussed before for Potts and $p$-spin interaction spin glasses, and connected with a dynamical transition [26, 30]. We speculate that a dynamical transition may occur in the present situation as well. Thus we connect the transition in a magnetic field in Ising spin glasses in low dimensions with the random-energy-model-like discontinuous transition behavior (with a dynamical transition at higher temperature) that is now believed to be somewhat generic, and which includes the random first-order transition (RFOT) theory of structural glasses [31] as well.
We want to point out that our results also apply in cases other than Ising SGs with 2-spin interactions and a uniform magnetic field. For Ising SGs in a random magnetic field of mean zero and standard deviation \( h \), the same (extended) BR action can be derived. In addition, for a SG in which the spins are \( m \)-component unit vectors, inclusion of a mean zero, isotropically-distributed vector-valued random field of standard deviation \( h \) also produces an AT line within mean-field theory [33]. The same BR action applies there, so that similar results are predicted for low-dimensional XY and Heisenberg SGs in a random magnetic field. Similarly, we would find the same for a Potts spin glass in a random magnetic field, though it is possible that there the transition is quasi-first-order even in mean-field theory (depending on \( p \)).

Another family of models are those with \( p \)-spin interaction among Ising spins [21]. A particular 3-spin interaction model has been mapped to the BR theory [33], which is reasonable in view of the lack of inversion symmetry in the \( p \)-spin models for \( p \) odd (however our conclusions differ from that work); see Ref. [35] for another approach.

We also mention that Goldschmidt [36] predicted a fluctuation-driven first-order transition in \( p > 2 \) Potts SGs in zero magnetic field for \( d < 6 \). He predicted a first-order transition simply because the RG flows run to \( p > 1 \); his work predates GKS and does not seem to be a complete analysis.

II. LANDAU THEORY OF EXTENDED BR ACTION

We begin by extremizing the action [12] with respect to \( x \)-independent \( \bar{Q}_{\alpha\beta} \), using the Parisi ansatz [4]; in other words, we consider Landau theory. We drop a factor of volume, and include a Lagrange multiplier \( \lambda \) for the constraints \( \sum_{\beta} Q_{\alpha\beta} = 0 \) for each \( \alpha \) (it turns out that the same value \( \lambda \) is found for each constraint, so we do not include a separate multiplier for each), and divide by \( n \) [3]. Then we need to extremize

\[
F[\{\bar{Q}_{\alpha\beta}\}, \lambda] = \lim_{n \to 0} \frac{1}{n} \left[ -\frac{1}{2} \lambda \sum_{\alpha} \bar{Q}_{\alpha\beta} - \frac{1}{2} \sum_{\alpha\beta} \bar{Q}^2_{\alpha\beta} - \frac{1}{6} w_1 \sum_{\alpha} \bar{Q}_{\alpha\beta} \bar{Q}_{\beta\gamma} \bar{Q}_{\gamma\alpha} - \frac{1}{6} w_2 \sum_{\alpha\beta} \bar{Q}^2_{\alpha\beta} - \frac{1}{8} y_1 \sum_{\alpha\beta} \bar{Q}^2_{\alpha\beta} \bar{Q}^2_{\beta\gamma} - \frac{1}{8} y_2 \sum_{\alpha\beta} \bar{Q}^2_{\alpha\beta} \bar{Q}^2_{\gamma\alpha} - \frac{1}{8} y_3 \sum_{\alpha\beta\gamma} \bar{Q}_{\alpha\beta} \bar{Q}_{\beta\gamma} \bar{Q}_{\gamma\alpha} - \frac{1}{8} y_4 \sum_{\alpha\beta\gamma} \bar{Q}^2_{\alpha\beta} \bar{Q}^2_{\beta\gamma} - \frac{1}{8} y_5 \sum_{\alpha\beta\gamma\delta} \bar{Q}_{\alpha\beta} \bar{Q}_{\beta\gamma} \bar{Q}_{\gamma\delta} \bar{Q}_{\delta\alpha} \right]
\]

(2.1)

with respect to \( \lambda \) and \( \bar{Q}_{\alpha\beta} \), where \( \bar{Q}_{\alpha\alpha} = 0 \) and \( \bar{Q}_{\alpha\beta} = \bar{Q}_{\beta\alpha} \). We set \( \tau = -\bar{r}/2 \) in this section to simplify writing; \( \tau \propto T_{AT}(h) - T \) is positive for \( T < T_{AT}(h) \). The value of \( F \) at the extremum gives the physical free energy density (into which inverse temperature has been absorbed in both Landau-Ginzburg and Landau theory).

The Parisi ansatz involves dividing the symmetric matrix \( Q_{\alpha\beta} \) into square blocks of equal size, and setting all matrix elements in the off-diagonal blocks to one value, and those in the diagonal blocks to another value, except for the entries on the diagonal which are zero. This is then repeated in the same way in each diagonal block (replacing the values originally placed there), and iterated so that it does move \( k \) times in total (giving \( k \)-step RSB, also called \( k \)-RSB). Formally, this means that we choose numbers \( 1 = m_0 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} = n \), where \( m_{i+1}/m_i \) is an integer for \( i = 0, \ldots, k \). Then \( Q_{\alpha\alpha} = 0 \) (all \( \alpha \) ), and

\[
Q_{\alpha\beta} = q_i \text{ for } \left[ \frac{\alpha}{m_i} \right] \neq \left[ \frac{\beta}{m_i} \right] : \left[ \frac{\alpha}{m_{i+1}} \right] = \left[ \frac{\beta}{m_{i+1}} \right]
\]

(2.2)

for \( \alpha \neq \beta \) and \( i = 0, \ldots, k \). (Here \( [a] \) is the ceiling function, the least integer greater than or equal to \( a \).)

When \( n \to 0 \), the \( m_i \)‘s become numbers \( m_i = x_i \) between \( x_{k+1} = 0 \) and \( x_0 = 1 \) and obey the reverse inequalities.

Defining a function of these numbers by \( q(x_i) = q_i \), in the limit \( k \to \infty \) (and assuming the \( x_i \) fill the interval \([0,1]\)) we obtain a function \( q(x) \). Alternatively, for \( k \) finite, we can define \( q(x) \) for all \( x \) to be piecewise constant with steps at \( x = x_i \), with \( q(x) = q_i \) for \( x \in [x_{i+1}, x_i) \). We note that, as \( q(x) \) turns out to be monotonically increasing, we can use it to define a probability measure on overlaps \( q \) (normalized per site) of pure states [37]; using the function \( x \to q(x) \), the measure assigned to an interval \([q_1, q_2]\) is the Lebesgue measure of \( x \) in the inverse image (which is again an interval) of \([q_1, q_2]\). [Equivalently, \((dq/dx)^{-1}\) (with \( \delta \)-functions at \( q \) for which \( q(x) \) has zero derivative) can be viewed as the probability density of overlaps \( q \) of pure states [37].]

In the Parisi ansatz as described, one imposes \( q(x) \geq 0 \). In our case, we assumed (in deriving the BR action [10]) that there is a replica symmetric part \( Q \), and set \( Q_{\alpha\beta} = Q + \bar{Q}_{\alpha\beta} \) for \( \alpha \neq \beta \), where \( \sum_{\beta} \bar{Q}_{\alpha\beta} = 0 \) (the replica constraint). In terms of \( q(x) \), we can associate a function \( \bar{q}(x) \) with \( \bar{Q}_{\alpha\beta} \) exactly as described above for \( Q_{\alpha\beta} \), and then

\[
\bar{q}(x) = q(x) - Q = q(x) - \int_0^1 dx q(x),
\]

(2.3)

as the replica constraint implies that

\[
\int_0^1 dx \bar{q}(x) = 0.
\]

(2.4)

We will assume \( Q \) is nonzero, and that |\( \bar{q}(x) \)| is smaller than \( Q \) at all \( x \). Then we can ignore the condition \( q(x) \geq 0 \), but we include the condition \( \int_0^1 dx \bar{q}(x) = 0 \) instead.
Now evaluating the functional $\mathcal{F}$, we find \[\text{[4, 25]}\]

$$\mathcal{F} = \frac{1}{2} \int_0^1 dx \left[ \lambda \tilde{q}(x) + \tau \tilde{q}(x)^2 + \frac{1}{2} w_2 \tilde{q}(x)^3 + \frac{1}{4} y_1 \tilde{q}(x)^4 \right]$$

$$- \frac{1}{2} w_1 \tilde{q}(x) \left\{ 2 \langle \tilde{q} \rangle \tilde{q}(x) + \int_0^x dx' \left( \tilde{q}(x) - \tilde{q}(x') \right)^2 \right\}$$

$$+ \frac{1}{2} y_2 \left\{ \tilde{q}(x)^4 - 2 \langle \tilde{q} \rangle \tilde{q}(x)^2 \left\{ \int_0^x dx' \left( \tilde{q}(x) - \tilde{q}(x') \right)^2 \right\} \right\}$$

$$- \frac{1}{2} y_3 \left\{ 2 \langle \tilde{q} \rangle ^3 \langle \tilde{q} \rangle + \tilde{q}(x)^2 \int_0^x dx' \left( \tilde{q}(x) - \tilde{q}(x') \right)^2 \right\}$$

$$- \frac{1}{2} y_5 \left\{ \langle \tilde{q} \rangle ^2 - \left\{ 4 \langle \tilde{q} \rangle ^2 \langle \tilde{q} \rangle ^2 \right\} \right\}$$

$$+ 4 \langle \tilde{q} \rangle \langle \tilde{q} \rangle \int_0^x dx' \left( \tilde{q}(x) - \tilde{q}(x') \right)^2$$

$$+ \int_0^x dx' \int_0^x dx'' \left( \tilde{q}(x) - \tilde{q}(x') \right)^2 \left( \tilde{q}(x) - \tilde{q}(x'') \right)^2 \right\} \right\} \right\} \right\}$$

\[\text{(2.5)}\]

Here $\langle f \rangle = \int_0^1 dx f(x)$ for a function $f(x)$ on $x \in [0, 1]$. The term with coefficient $y_3$ disappears on inserting the Parisi ansatz, as $\left( \sum_{\alpha} \tilde{Q}^2_{\alpha} \right)^2$ is of order $n^2$. We note that this term represents randomness in the mass-squared term $\tilde{r}$, and may not be negligible in general, even though it drops out of Landau theory.

**A. Piecewise-differentiable $\tilde{q}(x)$**

We can find variational equations for an extremum, assuming that we can apply ordinary functional differentiation. The derivation is tedious but straightforward, and resembles Refs. \[\text{[4, 25]}\]. Varying $\lambda$ produces the constraint \[\text{[24]}\]; this may be used freely but only after varying $\mathcal{F}$ with respect to $\tilde{q}(x)$. Given the variational equation, which is somewhat complicated because of the quartic terms in $\mathcal{F}$, we can obtain simpler equations by differentiating with respect to $x$ and then dividing by the derivative $\tilde{q}'(x)$, assuming the latter is nonzero \[\text{[4, 25]}\]. Applying this operation twice, we then find that if $\tilde{q}'(x)$ is nonzero at some value of $x$, and if $\tilde{q}(x) \to 0$ as $\tau$ approaches its critical value $\tau_c = 0$ from above (i.e. for a second-order transition), then we must have

$$w_2 - w_1 x = \mathcal{O}(\tau). \quad \text{(2.6)}$$

As $\tau \to 0$, this can be satisfied only at $x = w_2/w_1 \leq 1 \quad \text{[25]}$. Hence such a continuous transition, into a phase with $\tilde{q}(x)$ non-constant and differentiable at $x = w_2/w_1$ just below the transition, can occur only if $0 \leq \rho \leq 1$ (where $\rho = w_2/w_1$). If we take an additional derivative with respect to $x$ before letting $\tau \to 0$, then we find similarly that as $\tau \to 0$, we must have either $\tilde{q}'(x) = 0$ or $\tilde{q}'(x) \to 0$ as $\tau \to 0$.

As $\tilde{q}(x)$ must be monotonically increasing in $x$ (for example, to give the interpretation as a probability, mentioned above), and because both $w_1$ and $w_2$ are positive throughout the paper, we conclude that, defining $\tilde{y} = y_1 - y_3 \rho + y_5 \rho^2$ for $\rho \leq 1$, a nonzero finite slope of $\tilde{q}(x)$ is possible just below the continuous transition only if $0 \leq \rho \leq 1$ and $\tilde{y} > 0$. Otherwise, a piece-wise constant solution is the only non-trivial possibility for a continuous transition (a constant solution would have to be $\tilde{q}(x) = 0$, which means no RSB, and is unstable for $\tau > 0$).

Next, we investigate step-function possibilities when $\tilde{y} \leq 0$. We only consider $\rho > 0$. For $\rho < 1$, we expect a single step to occur, at $x_1 = \rho$ as $\tau \to \tau^+$. In place of the non-zero slope part of $\tilde{q}(x)$, a single step means a one-step or $(k = 1)$-1-RSB solution.

**B. 1-RSB: continuous transition at $\rho < 1$**

The preferred way to consider a step-function (or 1-RSB) solution is to substitute the step-function or 1-RSB matrix into the general Parisi functional $\mathcal{F}$. The function has the assumed form

$$\tilde{q}(x) = \begin{cases} q_0, & x < x_1, \\ q_1, & x \geq x_1, \end{cases} \quad \text{(2.8)}$$

where $x_1 \leq 1$. Then the three parameters $q_0$, $q_1$, and $x_1$ (as well as $\lambda$) can be varied to extremize $\mathcal{F}$. We note that if we used the variational derivative expressions of the previous section, and then substituted the step function, the equation that will be obtained below by varying $x_1$ is not immediately obtained, unless some additional procedure is used. This is why direct evaluation of $\mathcal{F}$ for the step function is the simplest procedure.

$\mathcal{F}$ can be evaluated for the step function to give

$$\mathcal{F} = \frac{1}{2} \lambda \langle \tilde{q} \rangle + \frac{1}{2} \tau \langle \tilde{q}^2 \rangle + \frac{1}{8} w_2 \langle \tilde{q}^2 \rangle + \frac{1}{8} y_1 \langle \tilde{q}^3 \rangle$$

$$- \frac{1}{8} w_1 \left[ 2 \langle \tilde{q} \rangle \langle \tilde{q}^2 \rangle + x_1 (1 - x_1) q_1 (q_1 - q_0)^2 \right]$$

$$+ \frac{1}{8} y_2 \left[ 2 \langle \tilde{q}^3 \rangle - 2 \langle \tilde{q}^2 \rangle^2 - x_1 (1 - x_1) q_1^2 (q_1 - q_0)^2 \right]$$

$$- \frac{1}{8} y_3 \left[ 2 \langle \tilde{q}^3 \rangle + x_1 (1 - x_1) q_1^2 (q_1 - q_0)^2 \right]$$

$$- \frac{1}{8} y_5 \left[ 2 \langle \tilde{q}^3 \rangle - 4 \langle \tilde{q} \rangle^2 \langle \tilde{q}^2 \rangle - 4 \langle \tilde{q} \rangle x_1 (1 - x_1) q_1 (q_1 - q_0)^2 \right.$$\]

$$- \left. x_1^2 (1 - x_1) (q_1 - q_0)^4 \right]. \quad \text{(2.9)}$$

Other than $\langle \tilde{q} \rangle = 0$, the equations obtained by varying the parameters in $\mathcal{F}$ are somewhat involved. We will explicitly solve them in some limiting cases.

First we will suppose that the transition is continuous, and that the quartic terms can be dropped to leading order in $\tau$ (we will see that $\tau_c = 0$). (Note that this is consistent, because $\tilde{y} = 0$ can only lead to a step-function
solution.) In addition to \( \langle \tilde{q} \rangle = 0 \), varying \( q_0, q_1, \) and \( x_1 \)
lead (using \( x_1, 1 - x_1 \neq 0 \)) to
\[
0 = \lambda + 2\tau q_0 + w_2 q_0^2, \quad (2.10)
\]
\[
0 = \lambda + 2\tau q_1 + w_2 q_1^2 - w_1 x_1 (q_1 - q_0)^2, \quad (2.11)
\]
\[
0 = \lambda (q_0 - q_1) + \tau (q_0^2 - q_1^2) + \frac{1}{2} w_2 (q_0^3 - q_1^3) - \frac{1}{3} w_1 \left[ 2(q_0 - q_1)(q_0^2 + (1 - 2x_1)q_1(q_1 - q_0)^2) \right].
\]
Solving, we find to leading order in \( \tau > 0 \) that
\[
x_1 = w_2/w_1 = \rho, \quad (2.13)
\]
\[
q_0 = -\frac{\tau}{w_2}, \quad (2.14)
\]
\[
q_1 = \frac{\tau}{w_1 - w_2}, \quad (2.15)
\]
\[
\lambda = \frac{\tau^2}{w_2}. \quad (2.16)
\]
These results make sense provided \( 0 < \rho < 1 \). They are very similar to corresponding ones of GKS [20]. Higher-order terms can be found as power series in \( \tau \).

We also mention here that there is a continuous transition for \( \rho < 1 \) and \( \tau > 0 \) from full RSB to 1-RSB when \( \tilde{y} \) changes sign. We will not discuss this in further detail. The phase boundaries \( \tau = 0 \) (and any \( \tilde{y} \)) and \( \tilde{y} = 0 \) (for \( \tau > 0 \)) for \( \rho < 1 \) meet to produce a point analogous to a Lifshitz point (though here the transition for \( \tau > 0 \) is continuous, unlike an ordinary Lifshitz point); we refer to this as a Lifshitz-type point.

C. 1-RSB: discontinuous transition at \( \rho > 1 \) and tricritical case \( \rho = 1 \)

In this regime, quartic terms cannot be neglected in general. For \( \rho > 1 \), following GKS [20], we look for a solution with \( x_1 \to 1, q_1 \) tending to a positive constant \( q_{1c}, q_0 = \mathcal{O}(1 - x_1) \) as \( \tau \to \tau_c \) (i.e. a “quasi-first-order” transition, if \( \tau_c \neq 0 \)). Keeping the leading terms in the variational equations, we find \( \lambda \sim -2\tau c q_0 = \mathcal{O}(1 - x_1) \), and the system of quadratic equations
\[
\tilde{y} q_{1c}^2 + (w_2 - w_1) q_{1c} + 2\tau c = 0, \quad (2.17)
\]
\[
\frac{1}{3} \tilde{y} q_{1c}^2 + \frac{1}{3} (w_2 - w_1) q_{1c} + \tau c = 0, \quad (2.18)
\]
which has the unique nonzero solution
\[
\tau_c = \frac{1}{9} \frac{(w_2 - w_1)^2}{\tilde{y}}, \quad (2.19)
\]
\[
q_{1c} = -\frac{6\tau_c}{w_2 - w_1} = -\frac{2}{3} \frac{(w_2 - w_1)}{\tilde{y}}. \quad (2.20)
\]
Here and below, for \( \rho > 1 \) we define \( \tilde{y} = y_1 - y_3 + y_5 \). Then \( \tau_c \) is negative, \( q_{1c} \) is positive (and both are finite) for \( w_2 > w_1, \tilde{y} < 0 \). Again, the results are very similar to GKS [20]. The results are valid within Landau theory provided \( w_2 - w_1 \) is sufficiently small so that \( \tau_c \) and \( q_{1c} \) are small. We remark that, in spite of the jump in \( \tilde{q}(x) \) at \( x = 1 \) that occurs at \( \tau = \tau_c \) (but see the following Sec. 1TD), the latent heat at the transition is zero [20].

For the borderline or tricritical case \( \rho = 1 \), the transition is continuous, but a separate analysis similar to the present section is required. We find that
\[
q_0 = -\frac{2\tau c}{w_1}, \quad (2.21)
\]
\[
1 - x_1 \propto \frac{(-\tilde{y} \tau)^{1/2}}{w_1}, \quad (2.22)
\]
\[
q_1 \propto \left( -\frac{\tau}{y} \right)^{1/2}, \quad (2.23)
\]
\[
\lambda = o(\tau^2) \quad (2.24)
\]
as \( \tau \to 0^+ \) for \( \tilde{y} < 0 \).

D. Additional step solution at \( \rho > 1 \)

A further one-step solution can be found, following Ref. [20]. As \( x_1 \) obeys \( x_1 \leq 1 \), we can look for solutions with \( x_1 = 1 \), ignoring the requirement that \( \partial F/\partial x_1 = 0 \); we still divide the equation \( \partial F/\partial q_1 = 0 \) by \( 1 - x_1 \) before solving with \( x_1 = 1 \). Then \( q_0 = \lambda = 0 \), and a single quadratic is obtained:
\[
\tilde{y} q_1^2 + (w_2 - w_1) q_1 + 2\tau c = 0, \quad (2.25)
\]
with solution
\[
q_1 = -\frac{(w_2 - w_1) - \sqrt{(w_2 - w_1)^2 - 8\tilde{y} \tau c}}{2\tilde{y}}. \quad (2.26)
\]
For this to be real and positive when \( \tilde{y} < 0 \), we require
\[
\tau > \tau_d = \frac{1}{8} \frac{(w_2 - w_1)^2}{\tilde{y}}. \quad (2.27)
\]
At the transition at \( \tau = \tau_d < 0 \), \( q_1 \) is nonzero. Thus if \( q_1 = 0 \) at \( \tau < \tau_d \) (i.e. high \( T \)), then \( q_1 \) becomes nonzero with both a jump and a square-root singularity at \( \tau_d \).
\( \tau_d = \tau_c \) at the tricritical point \( \tau_c = 0, \rho = 1 \). Again, these results are valid within Landau theory provided \( w_2 - w_1 \) is sufficiently small, and are similar to those in Ref. [20]. Thermodynamically, this solution is indistinguishable from the paramagnetic or high-temperature one, \( \tilde{q}(x) = 0 \), as both give \( F = 0 \), because \( \tilde{q}(x) \) differs from zero only on a set of measure zero. Within Landau theory, we have no way to determine which solution is physical other than by maximizing \( F \). Hence it is not clear which of the solutions (the paramagnetic one and the present one) is correct in the region \( \tau_c > \tau > \tau_d \).

We remark that \( 0 > \tau_c > \tau_d \), and that at \( \tau = \tau_c \), the values of \( q_1 \) in the solution here and that in Sec. [11C] are the same. At larger \( \tau \), the earlier discontinuous 1-RSB solution has larger \( F \), so is the physical one. Hence if we accept the \( x_1 = 1 \) solution when \( \tau_d < \tau < \tau_c \), there is no jump of \( q_1 \) at \( \tau_c \), but it is the point such that \( x_1 \) moves away from 1 at larger \( \tau \) (still with no latent heat). We discuss further the meaning of the solution found in this
section in Sec. IV below. For $\rho = 1$, $\tau_c = \tau_d = 0$; this case was discussed at the end of Sec. III C.

To avoid confusion, we will refer to the solution found here for $\tau_d < \tau < \tau_c$, which has $x_1 = 1$ throughout, as the (dynamically-) frozen phase, reserving the term 1-RSB for the region $\tau > \tau_c$ (with any value of $\rho$) in which $x_1 < 1$.

E. $\tilde{\gamma} \geq 0$ and $\rho > 1$, and tetracritical points

The results so far indicate that most features of the Landau-theory phase diagram can be parametrized using only the three variables $\tau$, $\rho$, and $\tilde{\gamma}$. There is also the regime $\rho > 1$ and $\tilde{\gamma} \geq 0$ that we have not discussed. We will not investigate this in detail, but only say that, by elimination of other possibilities, and if solutions exist within Landau theory at all, then a discontinuous (quasi-first-order) transition should be expected, and $\tilde{\gamma}(x)$ will be discontinuous (with break-point $x_1 \rightarrow 1$ as $\tau \rightarrow \tau_\infty$), but not piecewise constant, thus placing it outside the RSB forms we considered here. We might imagine that there would be a transition as $\tilde{\gamma}$ changes sign when $\tau > \tau_c$, and another Lifshitz-type point at $\tau = \tau_c$, this time involving two quasi-first-order boundaries, however as $\tau_c \rightarrow -\infty$ in this limit, Landau theory breaks down before this boundary is reached.

In addition, we mention that the point $\tau = 0$, $\rho = 1$, $\tilde{\gamma} = 0$ is a tetracritical point, from which all the other phases and transitions mentioned already emerge on changing one or more of these parameters. We will not describe it in detail. Note that we did not consider the region $\rho \leq 0$ here at all. There is another tetracritical point at $\tau = 0$, $\rho = 0$, and $\tilde{\gamma} = 0$.

III. CALCULATION OF RG FLOW EQUATIONS

Next we carry out an RG calculation on the extended BR theory at one-loop order in perturbation theory. The method is a standard one [35]: a wavevector cutoff of 1 is assumed, and Fourier components of fields with wavevectors in a shell just below the cutoff are successively integrated out, followed at each step by rescaling to restore both the cutoff and the coefficient of $(\nabla \bar{Q})^2$ to 1. In the calculations reported here, we expand the fluctuations of $Q_{\alpha \beta}$ around $\bar{Q}_{\alpha \beta} = 0$. This should be valid at least in the high-temperature region $\bar{r} > 0$ and at the critical point (AT line) $\bar{r} = 0$. The calculations are standard, except for the role of the replicon constraint [16]. The propagator, or zeroth-order two-point correlation function, $\langle \tilde{Q}_{\alpha \beta}(x)\tilde{Q}_{\alpha \beta}(0) \rangle$, for $\bar{Q}_{\alpha \beta}$ has Fourier transform

$$S_{\alpha \beta, \gamma \delta} = \frac{1}{\vec{k}^2 + \bar{r}}.$$

Here $S$ is a projection operator. In the space of $n \times n$ real matrices with elements $Q_{\alpha \beta}$ that are symmetric and have zeroes on the diagonal, which we equip with norm-square $\sum_{\alpha < \beta} Q_{\alpha \beta}^2$. $S$ is the projection onto the subspace $\sum_{\alpha} Q_{\alpha \beta} = 0$. $S$ can be expressed as

$$S_{\alpha \beta, \gamma \delta} = 1 + 1 \left[ \delta_{\alpha \gamma} + \delta_{\alpha \delta} + \delta_{\beta \gamma} + \delta_{\beta \delta} \right] - \left[ \delta_{\alpha \beta} + \delta_{\gamma \delta} \right] - \left[ \delta_{\alpha \beta} + \delta_{\alpha \delta} + \delta_{\alpha \gamma} + \delta_{\beta \delta} \right] + \left[ \delta_{\alpha \beta} \delta_{\gamma \delta} + \delta_{\alpha \gamma} \delta_{\beta \delta} \right],$$

where generalized Kronecker symbols of rank $k$ are defined as $\delta_{\alpha_1 \ldots \alpha_k} = 1$ if all of $\alpha_i$ ($i = 1, \ldots, k$) are equal, and zero otherwise; the limit $n \rightarrow 0$ has already been taken in the coefficients in this expression.

Then we obtain the one-loop RG flow equations for the effective couplings $\tilde{r}(l)$, $w_i(l)$ ($i = 1, 2$), and $w_i(l)$ ($i = 1, \ldots, 5$) at length scale $\epsilon$ (where $l = 0$ corresponds to the initial cutoff scale; we often refer to $l$ as a scale, though it is in fact the logarithm of the length scale), after setting $n = 0$:

$$\frac{d\tilde{r}}{dl} = \left[ 2 - \tilde{\eta} \right] \tilde{r} - \frac{3\tilde{\eta}}{(1 + \tilde{r})^2} + \ldots,$$

$$\frac{dw_i}{dl} = \frac{1}{2} \left[ \tilde{r} - 3\tilde{\eta} \right] w_i + K_d \left[ w_i^4 - 36w_i^2 - 48w_i^2 + 18w_i^2 + w_i^2 \right] + \ldots,$$

$$\frac{dw_2}{dl} = \frac{1}{2} \left[ \tilde{r} - 3\tilde{\eta} \right] w_2 + K_d \left[ w_2^4 - 60w_1^2 + 36w_1^2 \right] + \ldots,$$

$$\frac{dy_1}{dl} = \left[ 4 - d - 2\tilde{\eta} \right] y_1 + 96K_d w_2^2 \left( w_i - w_i \right) \ldots$$

$$\frac{dy_2}{dl} = \left[ 4 - d - 2\tilde{\eta} \right] y_2 + 16K_d \left[ w_i^4 - 15w_i^2 + w_i^2 \right]$$

$$\frac{dy_3}{dl} = \left[ 4 - d - 2\tilde{\eta} \right] y_3 + 8K_d \left[ w_i^4 - 28w_i^2 \right]$$

$$\frac{dy_4}{dl} = \left[ 4 - d - 2\tilde{\eta} \right] y_4 + K_d \left[ w_i^4 - 32w_i^2 \right]$$

$$\frac{dy_5}{dl} = \left[ 4 - d - 2\tilde{\eta} \right] y_5 + K_d \left[ w_i^4 - 80w_i^2 \right]$$

where $\tilde{\eta} = K_d \left( 4w_i^2 - 16w_i^2 + 11w_i^2 \right) (d - 4)/d$. Here the geometric factor $K_d = 2/\Gamma(d/2)(4\pi)^{d/2}$ arises from integration over the surface of a sphere in $d$ dimensions in the Fourier integrals. We neglected $\tilde{r}$ in denominators arising from one-loop integrals after the first equation, eq. (3.3), and in $\tilde{\eta}$. The $\ldots$ in the flow equations represent possible further one-loop terms, which are of the form $w_i(l)$ ($i = 1, \ldots, 5$) for $\tilde{r}$, $w_i y_j$ ($i = 1, 2$, $j = 1, \ldots, 5$) for the cubic couplings, and of either of the forms $w_i w_j y_k$ ($i, j, k = 1, \ldots, 5$) for the quartic couplings. It will be easily seen from the following that these terms are higher order in the weak-coupling regime of interest in this paper, and can be dropped (as can the $\tilde{\eta}$ term in the flow equations).
for $y_i$ also). We note that the terms kept in the flow equations for $y_i$ are of the “box diagram” form shown in FS. The equations (3.3) agree exactly with those of BR [19] (see also Ref. [39]), though they put $d = 6$ in $\eta$; they (nor, to our knowledge, any later authors) did not consider the quartic couplings $y_i$.

IV. ANALYSIS OF FLOW EQUATIONS

In the RG flow equations (3.3), all coefficients have been left in their general form, without assuming that, say, $d$ is close to 6. This enables us now to analyze the flows in various dimensions. We do this first for the extended BR theory for $d > 6$. After that, we apply the results to the AT line, using the crossover from the unreduced theory to the extended BR theory, beginning with high $d$, and ending with $d < 6$.

A. Weak-coupling flows in extended BR theory

First, we focus on the extended BR Landau-Ginzburg theory. For $d > 6$, the Gaussian fixed point, at which coefficients of all terms beyond the Gaussian ones $(\nabla Q_{\alpha\beta})^2$ and $f(Q_{\alpha\beta})^2$ are zero, is stable, that is all those coefficients flow to zero in perturbation theory (at least for sufficiently weak initial values and when only finitely many are included). Here we use only the leading behavior in each coefficient as it flows toward zero. First, however, we show that the flows for $y_i$ take them to the region $\tilde{y} < 0$ if $d \leq 8$.

1. Flow of $y_i$

The extended BR Landau-Ginzburg theory is in principle valid anywhere near a transition of AT type (among others), provided fluctuations in $Q_{\alpha\beta}$ are small; hence it may be useful even at large magnetic fields. Perturbation theory and perturbative RG calculations in $w_i$, $y_i$ and so on are valid if these couplings are, in an appropriate sense, small. For $d > 6$, there is a nonzero basin of attraction of the Gaussian (zero coupling) fixed point of the RG. Then the theory will be useful at large length scales if the initial values lie inside this basin of attraction. (If $d - 6$ is of order one or greater, then this basin contains all values of the couplings for which perturbation theory would be expected to be valid.) Here we assume the initial values are well inside the basin. For now we take the initial values to be at the scale $l = 0$; later this will be replaced with a scale $l_0 > 0$.

In the regime of linearized flow equations for $w_i$, the solutions for $w_i$ are

$$w_i(l) \approx w_i(0)e^{-\frac{1}{2}l^d},$$

(4.1)

(where $\varepsilon = d - 6$) so the flow lines are radial in the $w_1$-$w_2$ plane: $\rho(l) = \rho(0)$. (These forms are also valid asymptotically whenever $w_i$ lies well inside the basin of attraction of the origin, that is for sufficiently small $w_i$ when $d > 6$.) For now we will make use only of this form.

For $y_i$, it will be useful to look first at the simpler flow equations for the unreduced theory [23],

$$\frac{dy}{dl} = [4 - d]y + Aw^4,$$  

(4.2)

with our approximations, where $w$ has a flow like that of $w_i$. This has solution

$$y(l) = y(0)e^{(4-d)l} + Ae^{(4-d)l}\int_0^l w(l')^4e^{(d-4)l'} dl'.$$  

(4.3)

As $w(l')^4 \approx w(0)^4e^{-2d l'}$, we see that for $d > 8$ the integral converges as $l \to \infty$, and gives only a correction to the initial value $y(0)$; the correction is small if $w_i(0)$ are small. Thus $y(l)$ is governed by its initial value $y(0)$, up to a smaller correction. If we do the same for $y_i$, then for each $i w^4$ is replaced by an expression quartic in the $w_i$s, which has a flow similar to that of $w^4$. It can be written as $w_i^4$ times a quartic polynomial in $\rho = w_2/w_1$, and we have seen that in the present approximation $d\rho/dl \approx 0$. Thus the $y_i$ are proportional to their initial values up to small corrections, and they are simply rescaled by the same $l$-dependent factor as they flow toward zero radially in $y_i$-space. This is in accordance with simple perturbation theory, in which the box diagram for $y_i$ converges in the infrared for $d > 8$, in agreement with Refs. [23] [24].

For $d \leq 8$, the integral does not converge at $l = \infty$, and instead is dominated by its upper limit, to give

$$y(l) \sim Aw(0)^4/(8-d)$$  

(4.4)

as $l \to \infty$ [23] in the case $d < 8$; this decays more slowly than the initial value term. Again, there is similar behavior in our case: the $y_i$ acquire behavior given by the one-loop terms in their RG flow equations (divided by $8-d$). These asymptotics are $y_i \sim y^*_i$ as $l \to \infty$, where

$$y^*_i = 96Kdw_i^4\rho^2(\rho - 1)^2/(8-d),$$  

(4.5)

$$y^*_2 = 16K_d w_1^4(-7\rho + 15\rho^2 - 11\rho^3 + 3\rho^4)/(8-d),$$  

(4.6)

$$y^*_3 = 8K_d w_1^4(14\rho - 28\rho^2 + 13\rho^3 + \rho^4)/(8-d),$$  

(4.7)

$$y^*_4 = K_d w_1^4(11 - 32\rho + 48\rho^2 - 32\rho^3$$

$$+ 8\rho^4)/(8-d),$$  

(4.8)

$$y^*_5 = K_d w_1^4(38 - 80\rho + 40\rho^2 + \rho^4)/(8-d);$$  

(4.9)

these are written in terms of $w_i$ and $\rho = w_2/w_1$, and still depend on $l$ in general. The $y_i(0)$ correction is suppressed by a factor $O(e^{(d-8)l})$. In the regime considered here of radial flows of $w_i$, the $y^*_i(l)$ describe flow lines that are radial in $y_i$-space, as $\rho$ is effectively constant. As the $y^*_i$s depend only on two parameters, the collection of their flow lines form a two-dimensional space; intersected with a sphere of radius $K_d w_1^4$ (i.e. setting $w_1^4$ to a constant), we obtain a curve parametrized by $\rho$. For general initial
conditions \((|y_i(0)| < 1)\) and given \(w_j\), all flow lines of \(y_i\) asymptotically (at large \(l\)) approach the origin along one of these lines. This differs from the case \(d > 8\), in which the flow lines of \(y_i\) do not converge onto a single line, but approach the origin in \(y_i\)-space from all directions. For \(d = 8\), the \(w_i^2\) terms in \(y_i(l)\) are larger by \(l\) than the \(y_i(0)\) terms, due to the linear divergence of the integral. This means that the approach to \(y_i^*(l)\) is logarithmically slow in length scale, but still occurs.

We can now begin to apply the results of Sec. [11] In all cases we need to evaluate

\[
\bar{y} = y_1 - y_3 x_\rho + y_5 x_\rho^2
\]  

(4.10)

where \(x_\rho = \min(\rho, 1)\). For \(d \leq 8\) with \(w_i\) small, the \(y_i\) tend to their asymptotic behavior \(y_i^*(l)\), while \(\rho\) does not change during the flow. When we evaluate \(\bar{y}(l)\) using these \(y_i(l) = y_i^*(l)\), we obtain \(w_i^2\) times a polynomial in \(\rho\), for either \(\rho \leq 1\) or \(\rho \geq 1\). For \(\rho \leq 1\), we have

\[
\bar{y} = K_d w_i^4 \rho^2 (11 - 48 \rho + 32 \rho^2 - 8 \rho^3 + \rho^4)/(8 - d),
\]

(4.11)

and for \(\rho \geq 1\),

\[
\bar{y} = K_d w_i^4 (38 - 192 \rho + 360 \rho^2 - 296 \rho^3 + 89 \rho^4)/(8 - d).
\]

(4.12)

Then \(\bar{y}(l)\) turns out to be positive for small \(\rho > 0\), but negative for 0.8418 < \(\rho < 1.2694\). This is a key point of the analysis. The fact that \(\bar{y} < 0\) in this range of \(\rho\) values means from the results of Sec. [11] that \(\bar{q}(x)\) is discontinuous for \(\tilde{r}\) below \(\tilde{r}_c\), giving 1-RSB; for 0.84 < \(\rho < 1\) the transition remains continuous. This is one of the central results of the paper, and will demonstrate a qualitative change from the behavior in the SK model in dimensions near and below \(d = 6\). For \(\rho < 0.8418\), \(\bar{y}(l) > 0\), and the transition is to full RSB. At \(\rho = 0.8418\), \(\bar{y} = 0\), producing a Lifshitz-type point, with a transition line from full RSB to 1-RSB extending from it into the region \(\tilde{r} < 0\). As \(\bar{y} < 0\) at \(\rho = 1\), there will be a tricritical point there, at which the transition becomes discontinuous. At \(\rho = 1.2694\), again \(\bar{y} = 0\), producing a second Lifshitz-type point, and a transition line in the region \(\tilde{r} < \tilde{r}_c\).

For \(d > 8\), the \(y_i\) flow essentially radially to zero, and the initial values of \(y_i\) (as well as those of \(w_i\)) determine the behavior.

2. Critical phenomena

Next we analyze for \(d > 6\) the Gaussian fixed point with arbitrary perturbations whose asymptotic flows were discussed in Sec. [IV A 1]. As \(\tilde{r}\) is the only relevant parameter at this fixed point when \(d > 6\), it grows with \(l\), and we can stop the flows at \(l = l^*\) such that \(|\tilde{r}(l^*)| = 1\), and solve the Landau-Ginzburg theory with the parameters at \(l = l^*\) to obtain the phase transition behavior. (The closer the initial \(\tilde{r}(0)\) is to zero, the larger the \(l\) that will be required.) This applies for \(\tilde{r} < 0\) as well as \(\tilde{r} > 0\), and the former case will be of particular interest. For \(\tilde{r} < 0\), strictly in the RG one should use propagators in the expansion of the action about its extremum which may be at nonzero \(Q_{\alpha \beta}\), which should be small. In the regime where the Landau-Ginzburg theory is valid (in terms of the action at scale \(l\)), this will always be true sufficiently close to the transition if it is continuous (in fact, when \(\rho < 1\), and one would hope also if it is weakly first order, meaning that the extremum value of \(Q_{\alpha \beta}\) is small as \(\tilde{r} \to \tilde{r}_c\)). In these cases, the only leading order effect of nonzero \(Q_{\alpha \beta}\) will be to make the mass-squared term in the propagator positive; it can be neglected in the flows themselves. In the cases of the phases of interest in this paper, the 1-RSB form holds for \(\tilde{r} < \tilde{r}_c\), and we have checked that the mass-squares are positive and of order (at most) \(|\tilde{r}|\) as \(\tilde{r} \to \tilde{r}_c\), in agreement with GSK [20]; see also Refs. [11] [42]. Then the only change is that we must stop the flows at \(-\tilde{r} = 1\) (within a numerical factor), and the analysis is essentially the same as for \(\tilde{r} > 0\). Then, once \(\tilde{r}(l^*) = 1\), we can apply the Landau theory analysis of Sec. [11] using the effective values of parameters at scale \(l\). We will follow this standard approach.

For these reasons, we now consider the solution of Landau theory as in Sec. [11] but using the effective (\(l\)-dependent) values of the parameters for \(d \leq 8\). (For \(d > 8\), the critical behavior agrees with Landau theory.) Thus in terms of the effective phase diagram of Landau theory which involved three parameters (\(\tilde{r}, \rho\), and \(\bar{y}\)), the system flows onto a two dimensional surface within that space, as \(\bar{y}\) becomes a function of \(\rho\). We recall that \(\tilde{r} = -2\tau\), so here \(\tilde{r}(l) = -2\tau(l)\). We add to the exposition of Sec. [11] by pointing out that if \(q\) stands for any of \(Q_{\alpha \beta}, \bar{q}(x), \rho_0, \) or \(\rho_1\), then at zeroth order in perturbation theory we have

\[
q(l) = e^{2(d-2)l} q(0).
\]

(4.13)

We will mainly assume that \(\bar{y} < 0\).

For \(0 < \rho < 0.8418\), the transition is to full RSB, and the scaling for \(6 < d \leq 8\) is very similar to that of FS for the zero field case, so there is little that we can add here. For 0.8418 < \(\rho < 1\), on inserting \(\tau(l^*) = 1\) and \(w_i(l^*)\) into the expressions in Sec. [11] \(\rho\) we obtain for \(\tilde{r} < 0\) that \(q_0(l^*), q_1(l^*) \sim 1/w(l^*) \sim e^{2\tau l^*}\) (we neglect numerical coefficients and initial values in this section, and \(w\) stands for \(w_1, w_2\), or if \(\rho \neq 1, |w_1 - w_2|\). In terms of \(q\) at \(l = 0\), we find \(q_0(0) \sim e^{-2\tau l^*} = \tau(0)\), which we can characterize with the exponent \(\beta = 1\). We note that \(\rho_0\) do not appear in the expressions for \(q_0, q_1, \) or \(x_1\) at leading order in \(\tilde{r}\), so the \(y_i\) are irrelevant and not dangerous at this continuous transition (i.e. at the Gaussian fixed point for \(\rho < 1\)), provided \(\bar{y} \leq 0\), though they are dangerous in the RSB region as in FS, similar to Refs. [41] [42]. \(w_i\) are still dangerously irrelevant for \(d > 6\), and responsible for the breakdown of the hyperscaling relation implied by \(\beta = 1\).)

For the tricritical case \(\rho = 1\), we can carry out a similar analysis. \(q_0(l^*)\) is of the same order as before, but \(q_1(l^*) \sim e^{2\tau l^*}\) is larger at large \(l^*\). This leads to
Finally, for the case of the discontinuous (quasi-first-order) transition when $1 < \rho < 1.2694$, we can sit at $\tau_c$ or $\tau_d$, which are both negative and scale the same way, so the results are the same for both: we consider $\tau_c$. In order to set $\tau_c(l^*) = (w_2 - w_1)^2/y = -1$, we must have $w_2 - w_1 \sim e^{-c l^*}$, so $\rho - 1 \sim e^{-\frac{1}{2} c l^*}$. This is similar to the role of $1 - x_1$ in the tricritical case. Then $q_{1c}(l^*) \sim e^{-c l^*}$, so $q_{1c}(0) \sim \tau_c(0)(10^{-d}/4)$ for $d < 8$.

Further information about scaling properties can be obtained as well, analogously to Refs. [11, 32]. Note that there may remain modes in the ordered phase that are still massless at $l = l^*$ (i.e. the mass-squares are much smaller than the leading ones, and tend to zero at $\tilde{\tau}_c = 0$ and at $\tilde{\tau}_d$). It would be possible, and necessary, to restrict to a theory of these modes alone (analogous to what BR did in constructing their action), and then carry out the RG at $l > l^*$ on this smaller set of modes. We will not consider that further here.

We have some concerns about the preceding analysis of the scaling behavior at the tricritical point and the discontinuous transition. In addition to $y_i$, we can similarly treat terms of higher order $k > 4$, for example $z_k \sum_{\alpha \beta} \tilde{Q}_{\alpha \beta}^k$; for each of these there are one-loop diagrams that contain a polygon with $k$ sides, and for $d < 2k$ the resulting flows approach $z_k(l) \sim \sum_{\alpha \beta} b_{\alpha \beta} w_{\alpha}^{k-4}(l) w_{\beta}^{2}(l) \sim e^{-\frac{1}{2} k c l} z_k(0)$. We will not need the detailed form of these asymptotics as we did for $y_i$. If we add one of these terms to the action as a perturbation, and look at its effect on $q(l)$, we find a negligible effect in the case of the continuous transition. But for the tricritical point and the discontinuous transition (as $\rho \to 1^+$), we estimate a change in $q_{1c}(l^*)$ or $q_{1c}(l^*)$ of relative size $e^{-\frac{1}{2} k^2 c l^*}$, which increases with $l^*$ for $6 < d < 8$. This means that these higher-order terms should not be neglected, and it would be better to include all of them by summing them up to obtain the one-loop fluctuation correction to the free energy. We suspect that the result will be modifications of the scaling for $6 < d < 8$ in these cases, but we will not pursue this here.

B. Fate of the AT line

Now we come to the central points: the application of the preceding results to the AT line. We work downwards in dimension $d$, from $d > 6$ to $d < 6$.

First, we describe the effects of the nonlinear terms in the flows for $w_i$. The flows for $d > 6$ were shown explicitly in Ref. [17] (see Fig. 2). There is a weak coupling region inside which flows go to the origin, separated by a lobe-shaped separatrix from the region where flows go to infinity (within the leading one-loop RG). There are two pairs of fixed points on the separatrix (the two members of each pair are related by a symmetry, $w_i \to -w_i$, $i = 1, 2$). One pair of fixed points are at $U_1, -U_1$ defined by $w_1 = \pm \sqrt{\varepsilon/2(24K_d)}$, $w_2 = 0$, and both are unstable under a perturbation that remains on the separatrix. The other pair $Z, -Z$ are on the fixed line $\rho = \rho^* = 13.379$, and both are attractive on the separatrix. When the nonlinearities are important within perturbation theory, that is when $d \leq 6 + O(\varepsilon)$ and $w_i$ is order $\varepsilon^{1/2}$ or greater, the value of $\rho$ changes during the flow (unless $\rho = 0$, which is also a fixed line). In the strong-coupling region, or for all nonzero couplings if $d < 6$, $\rho$ flows towards the fixed value, $\rho^*$. Moreover, we should reconsider the RG flows of $y_i$ in light of the flows of $w_i$. We can no longer carry out the integration of the flow equations for $y_i$ directly. But, as each $y_i$ has dimension $4 - d$ to zeroth order, while $w_i$ has dimension $(6 - d)/2 = -\varepsilon/2$ (and the latter is still of the correct order in the vicinity of the separatrix), the integral in the solution for $y_i(l)$ contains a kernel that decays rapidly (as $l^*$ decreases) compared with the inverse rate of change of $w_i(l)$. This means that for large $l$ and with $\varepsilon$ small, again $y_i \sim y_i^*$ as $l \to \infty$. Hence we know the expressions for $y_i^*$ in terms of $w_1$ and $\rho$, and the resulting $y(l)$ is negative for $0.8418 < \rho < 1.2694$, but we must still consider the flows of $w_i$ (or of $w_1$ and $\rho$).

1. $d > 6$ at sufficiently weak magnetic field

In order to describe the AT line, we will need some results about the unreduced Landau-Ginzburg theory as in Ref. [10], and the crossover to the BR reduced theory. First, for the unreduced theory, in which the field is $Q_{\alpha \beta}$ without the restriction to the replicon subspace, the couplings are $r, w$, and some quartic couplings $y, x, u$. The $w$ term has the form of the $w_1$ term in BR, while the $y$ term has the form of the $y_1$ term (and is usually the only quartic coupling kept [3]). For $d > 6$ and zero magnetic field, all couplings (of higher than the quadratic order) flow towards zero at leading order in perturbation theory (including the nonlinear terms for $dw/dl$). When
$h$ is nonzero, at some scale $l = l_0$, $|r(l_0)| = 1$ becomes of order 1, and the nonreplicon modes become massive. That is when the crossover to the BR reduced theory occurs. The initial values of the couplings in the BR action, at the initial scale that is now $l_0$, are then determined as follows [16], with all parameters taking their effective values at $l = l_0$. In non-zero magnetic field there is a replica symmetric order parameter $Q_{\alpha\beta} = Q$ for $\alpha \neq \beta$ [2], and BR showed that the result of expanding the action in terms of $Q_{\alpha\beta} = Q_{\alpha\beta} - Q$ (for $\alpha \neq \beta$) and then imposing the replica constraint on $Q_{\alpha\beta}$ is to produce the values
\[ \tilde{r} = r + 2wQ, \]
\[ w_1 = w - 3uQ, \]
\[ w_2 = 3yQ, \]
and further $y_1$, $y_2$, and $y_0$ will be the same as their counterparts [called $y$, $-2x$, and $u$, respectively, in BR; this $x$ has nothing to do with Parisi’s $x$ in $q(x)$] in the unreduced action up to terms of higher order in $Q$ (that involve terms higher than quartic order in the unreduced Landau-Ginzburg theory); $Q = -r/(2w) = [h^2/(2y)]^{1/3}$ on the AT line [16]. In this region, the initial values of the couplings in the BR action, $\tilde{r}$, $w_1$, and $w_2$, is not continuous. The flows of $\tilde{r}$, $w_1$, and $w_2$ are discussed by MR [19]. From those results, and in the limit as $\varepsilon \to 0$ with $\epsilon l_0$ fixed, we find also from MR that the value of the ratio initially is $\rho(l_0) = O(\varepsilon)$. The important region once the BR flows are close to the unstable fixed point $U$; the initial values lie on a line of vanishing slope as $\varepsilon \to 0$.

For $6 < d \leq 8$ at sufficiently weak magnetic field, as we have seen the flows take $y_1$ to $y_1^*(l)$, and $\rho$ is invariant during the flow. As $\rho$ is initially small and positive, it remains so, and we are in the regime in which $\tilde{y} > 0$. Even though the flow could have driven an initial positive $\tilde{y}$ to negative values, for the initial values in question this does not happen. Hence at sufficiently weak magnetic field and for $d > 6$, the transition remains a continuous one to full RSB, as for $d > 8$. The transition line is still the AT line, $T_c(h) = T_{AT}(h)$, with the FS scaling properties at weak magnetic field, as mentioned just now.

2. Tricritical and Lifshitz-type point on AT line for $d \to 6^+$

For dimensions close to 6, we consider here $\varepsilon > 0$ and small. The flows of $w_1$, $w_2$, in this region were considered by BR and others [16 17 19]. When these couplings are of order $\varepsilon^{1/2}$ or greater, the flows are no longer given by the exponential decay that we saw for larger $d$ (or at weak fields), but instead the flow lines exhibit curvature. The crossover and initial values at $l_0$ were discussed by MR [19]. From those results, and in the limit as $\varepsilon \to 0$ with $\epsilon l_0$ fixed, we find also from MR that the value of the ratio initially is $\rho(l_0) = O(\varepsilon)$. The important region once the BR flows apply is close to the unstable fixed point $U$; the initial values lie on a line of vanishing slope as $\varepsilon \to 0$.

The RG flows take $\rho > 0$ to larger values. For flows inside the separatrix, the flows initially increase $\rho$ and possibly $w_1$, $w_2$, but eventually fall back towards the origin along radial lines with constant slope $\lim_{l \to \infty} \rho(l)$; these were discussed earlier. As $w_1(l_0)$ increases, the value of $\rho(l \to \infty)$ increases and eventually exceeds 1. Once $\rho$ is larger than 0.8418, $\tilde{y}(l \to \infty)$ is negative. This implies that the phase just below the transition becomes 1-RSB, while the transition is still at $T_c(l \to \infty) = 0$ (the AT line), and is continuous.

There is a unique flow line that leaves $U$ and approaches the origin with slope $\rho = 0.8418$ asymptotically as $l \to \infty$. Because $\tilde{y}(l) < 0$ for $\rho(l)$ close to 1, initial values that intersect this flow line define the Lifshitz-type point $\mathcal{E}$ on the AT line in the $T$–$h$ plane. The form of the flows (in particular, the eigenvalues of the linearized equations at $U$) implies that, going back along the flows (i.e. as $l \to -\infty$), this flow line approaches $U$ asymptotically tangent to the separatrix. As $\rho(l_0) \propto \varepsilon$ is small, we see that as $\varepsilon \to 0$, the difference between $w_1/\varepsilon^{1/2}$ on this flow and on the separatrix is a higher-order correction. Consequently, to leading order, we find the exact same asymptotic behavior of the location of this point as $\varepsilon \to 0$ as for the multicritical point $M$ in MR [19] ($M$ arises from the flow into $Z$). There is a similar flow line that approaches $\rho = 1$ as $l \to -\infty$. It defines the tricritical point $T$ at $(T_r, h_T)$ in the $T$–$h$ plane. It too has the same asymptotic behavior as $M$ in MR. Likewise, there is a third flow line that leaves $U$ and approaches the origin with asymptotic slope 1.2694. But because this region undergoes a discontinuous transition
at \( \tilde{r}(l^*) = 1 \), while \( \tilde{r}_c \) in Landau theory at this transition diverges as \( \tilde{y} \to 0 \), the asymptotic region is not reached, and no flow starting at \( h > h_T \) can reach \( \rho(l^*) = 1.2694 \). For both points \( \mathcal{L} \) and \( \mathcal{T} \), we then have the same leading asymptotic location as for \( \tilde{r} \), that is

\[
\begin{align*}
 r(0)_{\mathcal{T}} &= \varepsilon^{-5/3} \left( e^{-\varepsilon_{\text{loT}}} \right)^{2/\varepsilon} \left[ 2K_d w_0^2 (1 - e^{-\varepsilon_{\text{loT}}}) \right]^{5/3}, \quad (4.17) \\
 h(0)_{\mathcal{T}} &= \frac{4\pi^4 (\delta_0 w_0 e^{-\varepsilon_{\text{loT}}})}{8 \left[ 2K_d w_0^2 (1 - e^{-\varepsilon_{\text{loT}}}) \right]^4}, \quad (4.18)
\end{align*}
\]

(where we restored the factors \( K_d \) for consistency; \( K_d \) can be evaluated at \( d = 6 \) here), as \( \varepsilon \to 0 \) with \( \varepsilon_{\text{loT}} \) held constant at \( \varepsilon_{\text{loT}} = \ln 13 = 2.565 \ldots \) \[19\]. Like \( M \), the tricritical point \( \mathcal{T} \) and the Lifshitz-type point \( \mathcal{L} \) tend to the zero magnetic field critical point \( C \) as \( d \to 6^+ \), and do so exponentially fast in \( \varepsilon \).

Although the leading asymptotic positions of \( \mathcal{T} \) and of \( M \) are the same, nonetheless \( \mathcal{T} \) preempts \( M \) as \( h \) is increased on the AT line (i.e. occurs at smaller \( h \) and \( T_c - T \)), because the value of \( w_1 \) at \( l_0 \) is smaller for the flow to \( \mathcal{T} \). (We write \( h(0) \) as simply \( h \) from here on.) At \( h \) slightly larger than \( h_T \), the discontinuous transition sets in, and occurs at \( \tilde{r} > 0 \), that is \( T_c(l) \) is above the AT line \( T_{\text{AT}}(h) \): it preempts the AT transition in this region. In addition, the dynamical transition at \( T_d(h) \) (if it exists) occurs for \( h > h_T \), and \( T_d(h) > T_c(h) > T_{\text{AT}}(h) \) for \( h > h_T \).

In the vicinity of \( \mathcal{T} \) in the \( T-h \) plane, the ordered phase below (but not too far below) \( T_c(h) \) is again of the 1-RSB form. For the critical behavior at and near the tricritical point \( \mathcal{T} \), at \( \varepsilon \) fixed and small, but \( T \) or \( h \) approaching \( \mathcal{T} \), we have the critical properties, described above. For \( h \) just above \( h_T \), the asymptotic \( \rho(l) - 1 \) is small, so the scaling for the jump in \( q_1 \) applies. We note that the critical exponents at \( M \) were non-trivial \[19\], while simple behavior is found for \( \mathcal{T} \), because it is controlled by the Gaussian fixed point. The latter seems more natural than having nontrivial exponents above the upper critical dimension, here \( d = 6 \). For larger \( h \), the initial values at \( l_0 \) would hit, then pass outside of, the separatrix. As the value of \( q_1 \) when the flow stops becomes of order one, we pass out of the domain of validity of Landau theory itself, and should not trust the results this far. In particular, we see no role for the fixed point \( Z \) that controlled the point \( M \) in MR.

The fate of \( \mathcal{T} \) and \( \mathcal{L} \) as \( d \) increases to \( d \leq 6 \) of order one or larger is not known. At some dimension \( d > 6 \) one or both may reach the zero-temperature line and disappear. It is also possible that one or both of them persists at high field, even above \( d = 8 \).

3. Quasi-first-order transition preempts AT line for \( d \leq 6 \)

We have shown that a discontinuous (quasi-first-order) transition occurs for \( d \) slightly above \( 6 \) and at magnetic field \( h > h_T \). By continuity in \( h \), we then expect this to persist at larger \( h \), though we cannot rule out some transition to other behavior at sufficiently large \( h \). (Beyond \( \mathcal{L}' \), the transition is to a phase whose behavior we have not determined here.) Likewise, because of the continuity of RG flows as a function of \( d \) as well as of other parameters (away from fixed points), we expect the discontinuous transition to persist below \( d = 6 \), here for all (sufficiently small) \( h > 0 \), though other behavior might set in for \( 6 - d \) sufficiently large. Again, the discontinuous transition (and the dynamical transition, if it exists) preempts the AT line, and does so now for all \( h > 0 \).

We now consider the quantitative analysis of the transition at small \( h \) for \( d \leq 6 \). We only consider \( d \) not far below \( 6 \). Then the flows of \( y_i \) for \( d = 6 + \varepsilon \) still apply, and \( y_i \sim y_i^* \) for \( l \to \infty \).

For \( d = 6 \), the RG flows for \( w_1 \) exhibit a scaling property: they are invariant if we rescale all of \( w_1 \), \( w_2 \), and \( l^{-1/2} \) by the same factor. Consequently, the flow lines, which can be obtained by solving the equation for \( dw_2/dw_1 \) that results from the flow equations, are mapped to one another by such a rescaling. For \( \rho = w_2/w_1 \) small, we find

\[
\frac{dw_2}{dw_1} = \frac{11}{6} \frac{w_2}{w_1},
\]

to leading order in \( \rho \), with solutions \( w_2 = B w_1^{11/6} \) for \( B \) independent of \( w_1 \), \( w_2 \); these exhibit the scaling property (\( B \) changes under rescaling). Note that here the behavior corresponds to the region outside the separatrix in the flows for \( d = 6 + \varepsilon \), so it is not clear that we can even get close to the flows that control the tricritical point for \( d > 6 \).

The crossover to the BR theory occurs when \( |r(l_0)| = 1 \). The flows for \( w(l) \) in the unreduced theory for \( d = 6 \) are logarithmic, \( w(l) \propto l^{-1/2} \) at weak coupling. For \( \rho \), again \( r(l_0) \sim w(l_0)^2 \), so here \( r(l_0) \sim l_0^{-1} \). To determine the value of \( B \) for the relevant flow, we use \( \rho = B w_1^{5/6} \) at small \( \rho \), giving \( B \sim l_0^{-7/12} \). This is small at weak field, however for flow lines that start near the origin in the \( w_1-w_2 \) plane, the smaller the value of \( B \), the larger the coupling \( w_1 \) at which the flow reaches \( \rho = 1 \). We conclude that for weak field (large \( l_0 \)), the flows pass out of the domain of validity of perturbation theory before reaching \( \rho = 1 \). Hence our approach breaks down, and we cannot evaluate (at least, not with the present method) the dependence of, for example, the size of the jump in \( q_1 \) in terms of \( h(0) \), even though we expect it to be small in the limit \( h \to 0 \). For \( d < 6 \), this effect only becomes worse. It is possible that general methods for fluctuation-driven first-order transitions \[43, 45\] could be useful here.

C. Full RSB at lower \( T \)

For \( d \) of order \( 6 + \varepsilon \) or below, we find a transition from 1-RSB to full RSB as temperature is lowered further below the line \( T_c(h) \), which occurs because \( \tilde{y} \) changes sign.
during the RG flow. If we work near $T_c$, where Landau-Ginzburg theory is valid, then at weak field $\tilde{y}$ (or rather its analog in the unreduced theory) is positive. But we have seen that when the crossover to the BR theory occurs, initially $\rho < 1$, and when $\rho(l)$ is large enough, the RG flows drive $\tilde{y}$ to negative values; this occurs rapidly (over a range of $l$ of order 1). For $d = 6 + \varepsilon$, this occurs only at sufficiently large $h$, and the resulting transition line intersects the AT line in the Lifshitz-type point $L$, the location of which was already discussed. For $d \leq 6$, it occurs right after the crossover. This will occur on a line essentially given by the crossover at which $|\tau(h_0)| = 1$: this line is similar to, but below, the AT line or its replacement $T_c(h)$.

In the literature on Potts and $p$-spin mean-field SG models [20,26], there is a transition from 1-RSB to full RSB that occurs at lower temperature. In these theories, typically the parameters $w_i, y_j$ that occur in Landau theory are treated as constants, while $r, h$ are varied as parameters. Hence the transition here seems to occur in a different way than in those theories.

V. DISCUSSION: THE DYNAMICAL TRANSITION AND THE METASTATE

In the SG models that, in mean-field theory, exhibit a dynamical transition at a temperature $T = T_d$ higher than the thermodynamic transition temperature $T_c$ [20,50], what occurs is a breakdown of the ergodicity of the dynamics at and below $T_d$ in an infinite-size system. We note that, at the microscopic level, a theory using Langevin dynamics requires the use of soft spins. In our short-range models, a time-dependent Landau-Ginzburg theory could be used instead; such a theory would include the static (equilibrium) results of this paper as special cases. We expect that a similar dynamical transition would be found at $T_d$, but we will not attempt this calculation in the present paper. Instead, we turn to the implications for an equilibrium description. A loss of ergodicity in dynamics implies that the configuration space can be broken into a number of ergodic components; by definition, if the (infinite) system were started in a configuration in one of these components, it would never find its way into a distinct component after any finite time. In a short-range system, it is natural to associate each of these ergodic components with a pure state, that is, an extremal Gibbs state of the infinite system (see e.g. Refs. [27,29] and references therein). In the case of the dynamical transition, it appears [26,31] that the entropy of these ergodic components or pure states (calculated from the measure giving the decomposition of the Gibbs state into pure states), is extensive, that is, the entropy (when defined somehow for a finite “window” or subregion of the infinite system; the details of this need not concern us) per unit volume is positive.

Consequently, if one takes the frozen phase of Landau theory seriously, the theory presented here for low-dimensional SGs predicts that in the regime $T_c < T < T_d$, the Gibbs state of the infinite system decomposes into pure states with the associated “configuration” entropy being extensive. For $T < T_c$, the 1-RSB phase (thus, for $T$ not too low), it is believed that the pure states are characterized by random free energies that are independent and exponentially distributed, such that the system is condensed into a countable number of pure states [7]; hence the entropy of the distribution given by the set of corresponding normalized Boltzmann weights is of order one only. Such behavior for $T$ above and below $T_d$ (but below $T_d$ and above some lower transition temperature) closely resembles the random-energy model [21,22], except that energies of the latter are replaced here by free energies, and each individual spin configuration is replaced by a pure state.

There may be legitimate concerns about whether such a picture is truly possible in a short-range, finite-dimensional model. We want to argue that this is no more of an issue for the dynamically-frozen phase than for the 1-RSB phase, and (with one reservation) no more than for RSB in general. (We will leave aside dynamics; there the issue is that, in a short-range system, a diverging relaxation time on approaching some $T_d$ should be accompanied by a diverging length scale, but this does not seem to occur at the dynamical transition of Ref. [20] or here on approaching $T_d$ from above.) The way that a many-(pure)-state picture could break down in a short-range model, when it exists in an infinite range model (or mean field description), is that it may be possible to make a droplet, whose interior is essentially in one pure state, say $a$, as an excitation of another pure state, say $b$. If the increase in free energy on exciting the droplet does not diverge when the size of the droplet goes to infinity, then such droplets will appear thermally on all scales, and the distinction between the pure states $a$ and $b$ will be lost. If instead the droplet free energy does diverge (at least, if it diverges fast enough), then there will only be some density of finite size droplets, and this does not destroy pure state $b$. In a dynamical picture, to get from pure state or component $a$ to $b$ requires thermal activation of a similar droplet; if the droplet free energy diverges with its size then the probability of going from $a$ to $b$ in a finite time will be zero, and such a divergence is a necessary condition for the existence of many ergodic components or pure states. This is the same issue in SG theory that has remained an unresolved controversy for many years, with proponents of RSB and of the scaling-droplet theories on the two sides (the answer to the question may depend on the dimension of space, but we leave this implicit).

The reservation in the present case concerns the extensive entropy of the pure states in the dynamically-frozen phase. It can be shown rigorously that any Gibbs state, and in particular all the pure states, must have the same free energy density; by extending the argument, they have the same density of total entropy also. (This can also be seen heuristically. A difference in free energy density between two Gibbs states implies that a droplet
of the phase with lower free-energy density can be created in the higher one; the probability of such a droplet goes to one as it is made arbitrarily large.) If we define the finite-volume entropy of the pure state decomposition to be that of the Gibbs state minus the average of those of the pure states, then it follows that the entropy of the pure state decomposition cannot be extensive \cite{17}. It should be noted that this does not imply that the decomposition into pure states is trivial; it only means that the related entropy is subextensive. It could be very close to extensive, for example scaling as $\propto W^{d}/\ln W$ in linear size $W$. (The coefficient in such a form would tend to zero as $T \rightarrow T_{c}^{+}$.)

Next we turn to the metastate of the SG. The metastate, introduced in SG theory by Newman and Stein \cite{27} (NS), can characterize the dependence of the Gibbs state in a finite region (or “window”) on the finite size of the system, or alternatively on the disorder far away [the latter giving the Aizenman-Wehr \cite{48} (AW) metastate; the two constructions are known to be equivalent at least in some cases]. Formally, a metastate is a probability distribution on infinite-size Gibbs states, for given disorder. An earlier paper \cite{29} introduced a method for studying the AW metastate using RSB. Here we will apply this to the case of 1-RSB, and allow the possibility of a magnetic field.

Using the AW metastate, we can define the average of the Gibbs state with respect to the metastate for given disorder. For correlation functions, this corresponds to averaging the correlation function over the metastate, for given disorder; we say that this gives a correlator in the “metastate-averaged state” (MAS). To obtain this from finite size, we use several copies of a finite-dimensional system of size $L$ in a box with free boundaries; in an outer region at distance $R$ from the origin ($R < L$) these have independent samples of the disorder (the bonds, and the magnetic fields, if these are random), while in the inner region at distance $< R$ they are identical. For the case of a square of a correlator in the MAS, we can use this construction; taking the average over the disorder in the outer region corresponds to using the MAS. Then we square and average over the disorder in the inner region, and take $L, R \rightarrow \infty$ \cite{29}.

In the present case, with a magnetic field, the MAS correlation function of interest can be represented by first considering (in finite size) the average

$$C_{ij} = \left[\left\langle [\langle s_{i}s_{j}\rangle_{>} - \langle s_{i}\rangle_{>}\langle s_{j}\rangle_{>}]^{2}\right\rangle_{<}\right]. \tag{5.1}$$

Here $\cdot \cdot \cdot$ stands for a disorder average, while $\langle \cdot \cdot \cdot \rangle$ is a thermal average, initially in finite size. The square brackets with subscripts $>\,$ and $<\,$ denote averages over only the disorder in the outer and inner regions, respectively. On taking the limits $L \rightarrow \infty$, then $R \rightarrow \infty$, this gives the correlation function of interest.

This correlator resembles that in the definition of the SG susceptibility, namely

$$\chi_{ij} = \left[\langle s_{i}s_{j}\rangle - \langle s_{i}\rangle\langle s_{j}\rangle\right]^{2}, \tag{5.2}$$

which however differs in that all disorder is averaged over at the end. The SG correlator $\chi_{ij}$ tends to zero at large distance $|x_{i} - x_{j}|$ (where for any $i$, site $i$ is located at $x_{i}$ on the lattice) whenever the thermal average $\langle \cdot \cdot \cdot \rangle$ is taken in a pure state, such as at high temperature. When the Gibbs state is not pure, $\chi_{ij}$ usually tends to a constant, which using RSB can be expressed in terms of integrals of $q(x)$; hence the SG susceptibility, which is the sum of $\chi_{ij}$ with respect to $j$ (say) running over all space, diverges like the volume in any SG ordered phase, for example below the AT line according to the standard full RSB picture. In the present case, in the frozen region $T_{c} < T < T_{d}$, $\chi_{ij}$ tends to zero at large distance, because in integrals the resulting $q(x)$ is indistinguishable from the constant ($= Q$) that occurs in the high-temperature region, and which cancels as for a pure state.

The idea of the correlator $C_{ij}$ is that, if there is no dependence of the two-point spin correlation on the distant disorder, then it becomes the same as the SG correlator $\chi_{ij}$, and so goes to a constant in the SG phase. (This will occur if the metastate is trivial, that is if it is supported on a single Gibbs state.) But if there is such dependence that remains in the limit $R \rightarrow \infty$ then the metastate is non-trivial. The MAS is itself a Gibbs state, and so can be decomposed into pure states, and for a non-trivial metastate it is expected that there are many pure states in that decomposition, and consequently $C_{ij}$ will decay as $|x_{i} - x_{j}| \rightarrow \infty$; RSB theory predicts \cite{29} that in fact it will tend to zero in this limit.

In Ref. \cite{29}, it was argued that the use of distinct disorder in two samples in the outer region breaks the symmetry among the replicas; the replicas are divided into groups that in effect arise from different copies in the outer region. Then a correlation function of spins well in the interior of the inner region as $R \rightarrow \infty$ can be calculated by taking spins with replica indices in different groups. In the replica formalism, using $Q_{\alpha\beta}(x_{i}) \sim s_{i}^{\alpha}s_{i}^{\beta}$, $C_{ij}$ becomes the correlation function

$$C_{ij} = \langle\langle Q_{\alpha\beta}(x_{i})Q_{\alpha\beta}(x_{j})\rangle\rangle - 2\langle\langle Q_{\alpha\beta}(x_{i})\bar{Q}_{\alpha\gamma}(x_{j})\rangle\rangle + \langle\langle \bar{Q}_{\alpha\beta}(x_{i})\bar{Q}_{\gamma\delta}(x_{j})\rangle\rangle, \tag{5.3}$$

where $\langle\langle \cdot \cdot \cdot \rangle\rangle$ stands for an average in the Landau-Ginzburg theory, and each of $\alpha, \ldots, \delta$ belongs to a distinct group. Here we subtracted off the replica symmetric part $Q$ (which in the BR theory is non-fluctuating) to arrive at the fields $\bar{Q}_{\alpha\beta}$; the $Q$ terms cancel. It was argued in Ref. \cite{29} that the components in distinct groups correspond to the outermost blocks in the RSB form (before $n \rightarrow 0$). Thus the expectation $\langle\langle Q_{\alpha\beta}\rangle\rangle$ is equal to $q_{0}$, which is zero above $T_{c}$, and negative below. These expectations again cancel, so we can replace $\bar{Q}_{\alpha\beta}$ by $\delta\bar{Q}_{\alpha\beta} = \bar{Q}_{\alpha\beta} - \langle\langle Q_{\alpha\beta}\rangle\rangle$, which are the fluctuations around the ordered phase.

In the cases studied here, which involve at most 1-RSB, we know that at the Gaussian level, all modes have positive mass-squared, except at some of the transitions. Here we discuss only the non-critical properties. It now
follows, without detailed calculation, that the correlator $C_{ij}$ decays exponentially to zero as $|x_i - x_j| \to \infty$ in all such cases. Note that in the SG phase, this is distinct from the SG correlator $\chi_{ij}$, which goes to a constant. It means that there are many pure states in the MAS; the metastate is highly non-trivial.

In Ref. [29], it was argued that in a SG phase, we would have

$$C_{ij} \sim \frac{1}{|x_i - x_j|^{d-\zeta}},$$

(5.4)

where $\zeta$ is universal and gives information about the metastate; $\zeta \leq d$. For a trivial metastate, $\zeta = d$. (See also Refs. [51]-[53].) Exponential decay should be considered equivalent to $\zeta$ being at its other bound $\zeta = 0$, as in both cases the integral of $C_{ij}$ over $x_i - x_j$ just converges (up to logarithms). It was further argued that $\zeta$ could be interpreted as a fractal dimension for the clusters that flip on passing from one pure state to another. The consequence was that, if we examine the MAS only in a window of size $W$, the logarithm of the number of pure states in the MAS that can be distinguished within the window scales as $W^{d-\zeta}$. For $\zeta = 0$, this means an extensive entropy of pure states. This runs into the same issue as discussed for the frozen phase: the entropy of the pure-state decomposition of the MAS should be non-extensive, but perhaps only just so, in which case we would still have $\zeta = 0$. The argument in Ref. [29] may be too crude to detect this distinction.

In Ref. [29], it was also argued that in short-range models there is a lower bound $\zeta \geq 1$. This bound, which strengthens the result that the entropy of pure states must be sub-extensive, was obtained from the obvious fact that at zero temperature, there cannot be more than $2^{O(W^{d-1})}$ ground states in a window of size $W$, together with the belief that the full RSB phase can be continued to zero temperature, with the exponent $\zeta$ unchanged, so that the same bound arises for full RSB. (It is not clear if, at $T > 0$, a similar bound governs the number of pure states, rather than of ground states, though there are indications that this may be so in a some sense [49].) The present argument that $\zeta = 0$ does not satisfy the bound. Therefore we argue that the 1-RSB phase cannot survive unchanged down to zero temperature. Instead, there must be another transition, presumably to full RSB. Such transitions are also expected for other, seemingly very different, reasons; see Sec. IV C.

In the frozen phase $T_c < T < T_D$, $C_{ij}$ decays exponentially, and so does $\chi_{ij}$. This strongly suggests that the metastate is trivial in this region, even though the Gibbs state contains an entropy of pure states that may be almost extensive. (This form has been discussed as a possible scenario [27]; [29].) This could be tested by checking whether $C_{ij} = \chi_{ij}$, as expected for a trivial metastate. It is tempting to think that on passing through $T_c$, the latter pure states become those involved in the non-trivial metastate of the 1-RSB phase, just as in the REM picture of the transition. We should caution that the idea that the pure states are the “same” should not be taken too literally, due to the expected “temperature chaos” [11; 50], the sensitivity at large length scales of the Gibbs (and pure) states to small changes in temperature.

We also mention here a simpler criterion for non-triviality of the metastate, given in Ref. [29]. If

$$\langle |s_i|^2 \rangle - \langle \langle |s_i|^2 \rangle \rangle > 0$$

(5.5)

(in the limit), then the metastate is non-trivial (the converse may also hold, but that is not completely clear). This means that the conditional variance of $s_i$ due to the distant disorder (for given disorder in the inner region), averaged over disorder in the inner region, is nonzero (in the limit). In terms of RSB, it reduces to

$$\int_0^1 q(x) dx - q(0) > 0,$$

(5.6)

and in terms of $\bar{q}(x)$ becomes $\bar{q}(0) < 0$; thus for the one-step cases in this paper, this is $q_0 < 0$. This immediately gives all the non-triviality results above, however it does not yield the additional information provided by $\zeta$. It is interesting to note that the left-hand side is of order $\tau - \tau_e$ (to leading order) in all cases considered in Landau theory here; thus this order parameter for a non-trivial metastate has exponent $\beta = 1$. In the case of the discontinuous transition, there are no critical fluctuations, so the same should hold even in dimensions $d < 6$ at that transition.

VI. CONCLUSION

In this work, we considered the BR reduced Landau-Ginzburg theory for a SG, extended by the addition of quartic terms. This is the most general one for a RSB transition in the absence of any symmetry. Here it arose because we considered the AT line for an Ising SG in a magnetic field, or for other SGs in random isotropic magnetic fields. This theory is “reduced” in that it contains only the so-called replicon modes, which are the ones involved in RSB [4]. Other fluctuating modes do not generically become massless at the same point as the replicons, in the absence of symmetry; hence they can be neglected or integrated out to leave the reduced theory. Consequently we believe that this theory is of widespread applicability. The case of a SG with no symmetry is a most basic case; SGs with additional symmetry, which usually (except for the Ising case) have additional indices (other than replica indices $\alpha, \beta$) on the $Q_{\alpha\beta}$ field on which the symmetry operations act, are more specialized.

Based on a renormalization-group (RG) analysis of this theory, we showed that for the AT line at $d$ close to 6, fluctuations cause the effective value of the combination $\bar{g}$ of the quartic couplings to become negative, signaling a continuous transition to a one-step RSB (1-RSB) phase
at weak (but not extremely weak) magnetic field. At slightly higher magnetic field, there is a tricritical point \( T \), and the transition becomes a quasi-first-order transition to 1-RSB beyond that point. We expect this latter transition, in particular its quasi-first-order character, to persist for any non-zero field when \( d \leq 6 \), giving a possible resolution of the long-standing problem first posed by BR.

We want to comment here on the possible implications for other techniques for studying the AT line, such as Monte Carlo simulation. The most common method of searching for a transition in a magnetic field has been to look for a divergence of the SG susceptibility, or of the related correlation length. Unfortunately, these methods can only detect a second-order transition, and negative results at \( d < 6 \) (and in the corresponding regime in power-law models) have been interpreted as meaning that there may be no transition. The problem should be studied with methods that can detect a (quasi-)first-order transition. To do so, we suggest use of some of the diagnostics mentioned near the end of Sec. V. These were (i) the divergence (as the volume) of the SG susceptibility; (ii) the MAS correlation function \( C_{ij} \) and its exponent \( \zeta \) (\( \zeta < d \) means a non-trivial metastate, which does not occur in the high-temperature phase), or (iii) perhaps most simply, the single-spin MAS average \( \langle \sum j \rangle \), which again signals that the metastate is non-trivial.

We also comment that Ref. [12] found some evidence of a dynamical transition in low dimensions and near where the AT line would be; this might be consistent with another aspect of our findings (see Ref. [26] and Sec. V above). Results for a range of dimensions were obtained from high-temperature series [15] that studied the SG susceptibility; evidence of an AT line was found for \( d \geq 6 \), but for \( d = 5 \) most Padé approximants showed no divergence of \( \sum j \chi_{ij} \). Clearly these results agree with ours to some degree. The fact that there appeared to be a transition for \( d = 6 \) could be a consequence of it being the borderline case; the quasi-first-order behavior might be very weak there, and the SG susceptibility, though finite, could be large (at the fields studied).

In this work we used the RG approach in a simple way. To consider the possible quasi-first-order transition for \( d \leq 6 \), more powerful RG methods are required. We hope to return to this elsewhere.

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[1] J.R.L. de Almeida and D.J. Thouless, J. Phys. A: Math. Gen. 11, 983 (1978).
[2] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1792 (1975).
[3] S.F. Edwards and P.W. Anderson, J. Phys. F 5, 965 (1975).
[4] G. Parisi, Phys. Rev. Lett. 43, 1754 (1979); J. Phys. A: Math. Gen. 13, L115 (1980); ibid., 13, 1101 (1980); ibid., 13, 1887 (1980).
[5] F. Guerra, Commun. Math. Phys. 233, 1 (2003).
[6] M. Talagrand, C.R. Acad. Sci. Paris, Ser. I 337, 111 (2003); ibid. 337, 625 (2003); Ann. Math. 163, 221 (2006).
[7] M. Mezard, G. Parisi, and M.A. Virasoro, Spin Glass Theory and Beyond, Lecture Notes in Physics, Vol. 9 (World Scientific, Singapore, 1987).
[8] C. De Dominicis and I. Giardina, Random Fields and Spin Glasses (Cambridge University, Cambridge, 2006).
[9] A.J. Bray and M.A. Moore, J. Phys. C 17, L463 (1984); Phys. Rev. B 31, 631 (1985); Heidelberg Colloquium on Glassy Dynamics and Optimization, Lecture Notes in Physics, Vol. 275, eds. J.L. van Hemmen and I. Morgenstern (Springer-Verlag, Berlin, 1986), p. 121.
[10] W.L. McMillan, J. Phys. C 17, 3179 (1984); Phys. Rev. B 30, 476 (1984).
[11] D.S. Fisher and D.A. Huse, Phys. Rev. Lett. 56, 1601 (1986); Phys. Rev. B 38, 386 (1988).
[12] M. Baity-Jesi et al., Phys. Rev. E 89, 032140 (2014).
[13] M. Baity-Jesi et al., J. Stat. Mech. P05014 (2014).
[14] D. Larson, H.G. Katzgraber, M.A. Moore, and A.P. Young, Phys. Rev. B 87, 024414 (2013).
[15] R.R.P. Singh and A.P. Young, Phys. Rev. E 96, 012127 (2017).
[16] A.J. Bray and S.A. Roberts, J. Phys. C 13, 5405 (1980).
[17] M.A. Moore and A.J. Bray, Phys. Rev. B 83, 224408 (2011).
[18] G. Parisi and T. Temesvári, Nucl. Phys. B 858, 293 (2012).
[19] M.A. Moore and N. Read, Phys. Rev. Lett. 120, 130602 (2018).
[20] D.J. Gross, I. Kanter, and H. Sompolinsky, Phys. Rev. Lett. 55, 304 (1985).
[21] B. Derrida, Phys. Rev. B 24, 2613 (1981).
[22] D.J. Gross and M. Mezard, Nucl. Phys. B 240, 431 (1984).
[23] D.S. Fisher and H. Sompolinsky, Phys. Rev. Lett. 54, 1063 (1985).
[24] J.E. Green, M.A. Moore, and A.J. Bray, J. Phys. C 16, L815 (1983).
[25] P. Goldbart and D. Elderfield, J. Phys. C 18, L229 (1985).
[26] T.R. Kirkpatrick and D. Thirumalai, Phys. Rev. Lett. 58, 2091 (1987); Phys. Rev. B 36, 5388 (1987); ibid. 37, 5342 (1988); ibid. 38, 4881 (1988).
[27] C.M. Newman and D.L. Stein, Phys. Rev. Lett. 76, 4821 (1996); Phys. Rev. E 55, 5194 (1997).
[28] C.M. Newman and D.L. Stein, J. Phys.: Condens. Matter 15, R1319 (2003).
[29] N. Read, Phys. Rev. E 90, 032142 (2014).
[30] T.R. Kirkpatrick and P.G. Wolynes, Phys. Rev. B 36, 8552 (1987).
[31] T.R. Kirkpatrick, D. Thirumalai, and P.G. Wolynes, Phys. Rev. A 40, 1045 (1989).
[32] L. Berthier, G. Biroli, P. Charbonneau, E.I. Corwin, S. Franz, and F. Zamponi, J. Chem. Phys. 151, 010901 (2019).
[33] A. Sharma and A.P. Young, Phys. Rev. E 81, 061115 (2010).
[34] M.A. Moore and B. Drossel, Phys. Rev. Lett. 89, 217202 (2002).
[35] M. Campellone, G. Parisi, and P. Ranieri, Phys. Rev. B 59, 1036 (1999).
[36] Y.Y. Goldschmidt, Phys. Rev. B 31, 4369 (1985).
[37] G. Parisi, Phys. Rev. Lett. 50, 1946 (1983).
[38] S.-K. Ma, Modern Theory of Critical Phenomena, (Perseus Publishing, New York, 2000).
[39] I.R. Pimentel, T. Temesvári, and C. De Dominicis, Phys. Rev. B 65, 224420 (2002).
[40] A.B. Harris, T.C. Lubensky, and J.-H. Chen, Phys. Rev. Lett. 36, 415 (1976).
[41] G. Cwilich and T.R. Kirkpatrick, J. Phys. A: Math. Gen. 22, 4971 (1989).
[42] G. Cwilich, J. Phys. A: Math. Gen. 23, 5029 (1990).
[43] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
[44] J. Rudnick, Phys. Rev. B 18, 1406 (1978).
[45] H.H. Jacobson and D.J. Amit, Ann. Phys. (NY) 133, 57 (1981).
[46] E. Gardner, Nucl. Phys. B 257, 747 (1985).
[47] J.L. van Hemmen, Heidelberg Colloquium on Spin Glasses, Lecture Notes in Physics, Vol. 192, eds. J.L. van Hemmen and I. Morgenstern (Springer-Verlag, New York, 1983), p. 203.
[48] M. Aizenman and J. Wehr, Commun. Math. Phys. 130, 489 (1990).
[49] C.M. Newman and D.L. Stein, private communication.
[50] A.J. Bray and M.A. Moore, Phys. Rev. Lett. 58, 57 (1987).
[51] O. White and D.S. Fisher, Phys. Rev. Lett. 96, 137204 (2006).
[52] M. Wittman and A.P. Young, J. Stat. Mech. (2016) 013301.
[53] A. Billoire, L.A. Fernandez, A. Maiorano, E. Marinari, V. Martin-Mayor, J. Moreno-Gordo, G. Parisi, F. Ricci-Tersenghi, and J.J. Ruiz-Lorenzo, Phys. Rev. Lett. 119, 037203 (2017).