Behavior and dynamics of the set of absolute nilpotent and idempotent elements of chain of evolution algebras depending on the time

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BEHAVIOR AND DYNAMICS OF THE SET OF ABSOLUTE NILPOTENT AND IDEMPOTENT ELEMENTS OF CHAIN OF EVOLUTION ALGEBRAS DEPENDING ON THE TIME

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Abstract

In this paper we construct some families of three-dimensional evolution algebras which satisfies Chapman-Kolmogorov equation. For all of these chains we study the behavior of the baric property, the behavior of the set of absolute nilpotent elements and dynamics of the set of idempotent elements depending on the time.

Keywords: Evolution algebras, Chapman-Kolmogorov equation, baric algebra, property transition, idempotent, nilpotent.

Mathematics Subject Classification (2010): 13J30, 13M05.

1 Introduction

Historically, mathematical methods have long been successfully used in population genetics. Research from mathematics to population genetics is based on Mendel’s laws (Gregor Johann Mendel, 1822-1884), where he used symbols that, from an algebraic point of view, suggest the expression of his genetic laws. Between 1856 and 1863, Mendel studied pea hybridization and gave a fundamental concept of classical genetics, a gene called "constant character", to explain the observed inheritance statistics. In 1866, Gregor Mendel published the results of many years of experiments on the selection of pea plants [12]. He showed that both parents must transmit discrete physical factors that convey information about their characteristics to their offspring at conception. Mendel was the first to use symbols that, from an algebraic point of view, express their genetic laws. Thus, mathematicians and geneticists once used non-associative algebras to study Mendelian genetics, and later some other authors called it "Mendelian algebras".

A notion of evolution algebra is introduced by J.P. Tian in [20]. This evolution algebra is defined as follows. Let \((E, \cdot)\) be an algebra over a field \(K\). If it admits a countable basis \(e_1, e_2, \ldots\), such that \(e_i \cdot e_j = 0\), if \(i \neq j\) and \(e_i \cdot e_i = \sum_k a_{ik}e_k\), for any \(i\), then this algebra is called an evolution algebra. The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebra; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc.

The Kolmogorov-Chapman equation gives the fundamental relationship between the probability transitions (kernels). Namely, it is known that (see e.g. [19]) if
each element of a family of matrices satisfying the Kolmogorov-Chapman equation is stochastic, then it generates a Markov process.

There are many random processes which cannot be described by Markov processes of square stochastic matrices (see for example [4, 5, 9, 13]).

To have non-Markov process one can consider a solution of the Kolmogorov-Chapman equation which is not stochastic for some time as in [2, 15, 16], where a chain of evolution algebras (CEA) is introduced and investigated. Later, this notion of CEA was generalized in [10], where a concept of flow of arbitrary finite-dimensional algebras (i.e. their matrices of structural constants are cubicmatrices) is introduced. In [3] considered Markov processes of cubic stochastic (in a fixed sense) matrices which are also called quadratic stochastic process (QSPs).

In the book [20], the foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed. In [18] the algebraic structures of function spaces defined by graphs and state spaces equipped with Gibbs measures by associating evolution algebras are studied. Results of [18] also allow a natural introduction of thermodynamics in studying of several systems of biology, physics and mathematics by theory of evolution algebras. There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. In [7] classified a family of three-dimensional evolution algebras, and in [8] and [6] considered a notion of approximation of an algebra by evolution algebras. In [2] a notion of a chain of evolution algebras is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The chain is defined by the sequence of matrices of the structural constants (of the evolution algebras considered in [20]) which satisfies the Chapman-Kolmogorov equation. In [17] studied chains generated by two-dimensional evolution algebras.

In this paper we continue investigation of chain of evolution algebras, in more detail we study chains generated by three-dimensional evolution algebras.

2 Preliminaries

Following [2] we consider a family \( \{E_{s,t} : s, t \in \mathbb{R}, 0 \leq s \leq t \} \) of \( n \)-dimensional evolution algebras over the field \( \mathbb{R} \), with basis \( e_1, e_2, \ldots, e_n \), and the multiplication table

\[
e_i e_i = \sum_{j=1}^{n} a_{i,j}^{[s,t]} e_j, \quad i = 1, \ldots, n; \quad e_i e_j = 0, \quad i \neq j.
\]

Here parameters \( s, t \) are considered as time.

Denote by \( \mathcal{M}_{[s,t]} = (a_{i,j}^{[s,t]})_{i,j=1,\ldots,n} \) the matrix of structural constants.

**Definition 1.** A family \( \{E_{s,t} : s, t \in \mathbb{R}, 0 \leq s \leq t \} \) of \( n \)-dimensional evolution algebras over the field \( \mathbb{R} \), is called a chain of evolution algebras CEA if the matrix \( \mathcal{M}_{[s,t]} \) of structural constants satisfies the Chapman-Kolmogorov equation

\[
\mathcal{M}_{[s,t]} = \mathcal{M}_{[s,\tau]} \mathcal{M}_{[\tau,t]}, \quad \text{for any } s < \tau < t.
\]
A character for an algebra $A$ is a nonzero multiplicative linear form on $A$, that is, a nonzero algebra homomorphism from $A$ to $\mathbb{R}$ [11]. Not every algebra admits a character. For example, an algebra with the zero multiplication has no character.

**Definition 2.** A pair $(A, \sigma)$ consisting of an algebra $A$ and a character $\sigma$ on $A$ is called a baric algebra. The homomorphism $\sigma$ is called the weight (or baric) function of $A$ and $\sigma(x)$ the weight (baric value) of $x$.

In [11] for the evolution algebra of a free population it is proven that there is a character $\sigma(x) = \sum_i x_i$, therefore that algebra is baric. But the evolution algebra $E$ introduced in [20] is not baric, in general. The following theorem gives a criterion for an evolution algebra $E$ to be baric.

**Theorem 1.** An $n$-dimensional evolution algebra $E$, over the field $\mathbb{R}$, is baric if and only if there is a column $(a_{1i_0}, \ldots, a_{ni_0})^T$ of its structural constants matrix $M = (a_{ij})_{i,j=1, \ldots, n}$, such that $a_{i_0i_0} \neq 0$ and $a_{i_i_0} = 0$, for all $i \neq i_0$. Moreover, the corresponding weight function is $\sigma(x) = a_{i_0i_0}x_{i_0}$.

In [2] several concrete examples of chains of evolution algebras are given and time-dynamics are studied. We continue the research of CEAs, in more detail, we study the CEAs generated by three-dimensional evolution algebras.

To construct a chain of three-dimensional evolution algebras one has to solve equation (1) for the $3 \times 3$ matrix $M_{s,t}$. This equation gives the following system of functional equations (with nine unknown functions):

\begin{align}
& a_{11}^{s,t} = a_{11}^{s,t} a_{11}^{s,t} + a_{12}^{s,t} a_{21}^{s,t} + a_{13}^{s,t} a_{31}^{s,t}, \\
& a_{12}^{s,t} = a_{11}^{s,t} a_{12}^{s,t} + a_{12}^{s,t} a_{22}^{s,t} + a_{13}^{s,t} a_{32}^{s,t}, \\
& a_{13}^{s,t} = a_{11}^{s,t} a_{13}^{s,t} + a_{12}^{s,t} a_{23}^{s,t} + a_{13}^{s,t} a_{33}^{s,t}, \\
& a_{21}^{s,t} = a_{21}^{s,t} a_{11}^{s,t} + a_{22}^{s,t} a_{21}^{s,t} + a_{23}^{s,t} a_{31}^{s,t}, \\
& a_{22}^{s,t} = a_{21}^{s,t} a_{12}^{s,t} + a_{22}^{s,t} a_{22}^{s,t} + a_{23}^{s,t} a_{32}^{s,t}, \\
& a_{23}^{s,t} = a_{21}^{s,t} a_{13}^{s,t} + a_{22}^{s,t} a_{23}^{s,t} + a_{23}^{s,t} a_{33}^{s,t}, \\
& a_{31}^{s,t} = a_{31}^{s,t} a_{11}^{s,t} + a_{32}^{s,t} a_{21}^{s,t} + a_{33}^{s,t} a_{31}^{s,t}, \\
& a_{32}^{s,t} = a_{31}^{s,t} a_{12}^{s,t} + a_{32}^{s,t} a_{22}^{s,t} + a_{33}^{s,t} a_{32}^{s,t}, \\
& a_{33}^{s,t} = a_{31}^{s,t} a_{13}^{s,t} + a_{32}^{s,t} a_{23}^{s,t} + a_{33}^{s,t} a_{33}^{s,t}.
\end{align}

But analysis of the system (2) is difficult. In [15] it was solved when matrix $M_{s,t}$ has upper-triangular view. We will consider several cases where the system is solvable. For this solutions corresponds some evolution algebras. And we will check which of these algebras are isomorphic to 3-dimensional classified algebras.
3 Construction of chains of three-dimensional evolution algebras

In this section we find solves of system of equations (2) in particular cases.

3.1 Three-dimensional CEAs corresponding to matrices with same columns

Let
\[ a_{11}^{s,t} = a_{12}^{s,t} = a_{13}^{s,t} = \alpha(s,t), \quad a_{21}^{s,t} = a_{22}^{s,t} = a_{23}^{s,t} = \beta(s,t), \quad a_{31}^{s,t} = a_{32}^{s,t} = a_{33}^{s,t} = \gamma(s,t). \]

Then equation (2) reduced to
\[
\begin{align*}
\alpha(s,t) &= \alpha(s,\tau) (\alpha(\tau,t) + \beta(\tau,t) + \gamma(\tau,t)), \\
\beta(s,t) &= \beta(s,\tau) (\alpha(\tau,t) + \beta(\tau,t) + \gamma(\tau,t)), \\
\gamma(s,t) &= \gamma(s,\tau) (\alpha(\tau,t) + \beta(\tau,t) + \gamma(\tau,t)).
\end{align*}
\]

(3)

Denote
\[
\begin{align*}
\delta(s,t) &= \alpha(s,t) + \beta(s,t) + \gamma(s,t), \\
\zeta(s,t) &= \alpha(s,\tau) - \beta(\tau,t) + \gamma(s,t), \\
\eta(s,t) &= \alpha(s,\tau) - \beta(\tau,t) - \gamma(s,t).
\end{align*}
\]

(4)

Then the last system of functional equations can be written as
\[
\delta(s,t) = \delta(s,\tau)\delta(\tau,t), \quad \zeta(s,t) = \zeta(s,\tau)\delta(\tau,t), \quad \eta(s,t) = \eta(s,\tau)\delta(\tau,t), \quad s \leq \tau \leq t.
\]

The first equation is Cantor’s second equation which has a very rich family of solutions:

a) \( \delta(s,t) \equiv 0; \)

b) \( \delta(s,t) = \frac{h(t)}{h(\tau)} \), where \( h \) is an arbitrary function with \( h(s) \neq 0; \)

c) \( \delta(s,t) = \begin{cases} 
1, & \text{if } s \leq t < a, \\
0, & \text{if } t \geq a, \end{cases} \) where \( a > 0. \)

Using these solutions from the second and third equations we find \( \zeta \) and \( \eta \):

a') \( \zeta(s,t) = \eta(s,t) = 0; \)

b') \( \zeta(s,t) = g(s)h(t), \quad \zeta(s,t) = f(s)h(t) \) where \( g \) and \( f \) are an arbitrary functions;

c') \( \zeta(s,t) = \begin{cases} 
\varphi(s), & \text{if } s \leq t < a, \\
0, & \text{if } t \geq a, \end{cases} \quad \eta(s,t) = \begin{cases} 
\psi(s), & \text{if } s \leq t < a, \\
0, & \text{if } t \geq a, \end{cases} \) where \( a > 0 \) and \( \varphi(s), \psi(s) \) are arbitrary functions.

Substituting these solutions into (4) and finding \( \alpha(s,t), \beta(s,t) \) and \( \gamma(s,t) \) we get the following matrices
\[
\mathcal{M}_{0}^{[s,t]} = \begin{pmatrix} 
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix};
\]
Lemma 1. For some $s, \tau, t \in S$, let $M[s, \tau, t] = P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}$.

Case 1. $s, \tau, t \in S$. Let $M[s, \tau, t] = P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}$.

Case 2. $s, \tau, t \notin S$. Let $M[s, \tau, t] = P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}$.

Case 3. $s, t \in S, \tau \notin S$. Let $M[s, \tau, t] = P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}$.

Case 4. $s, \tau, t \in S$. Let $M[s, \tau, t] = P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}$.

Case 5. $s, t \notin S, \tau \in S$. Let $M[s, \tau, t] = P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}M[s, \tau, t]P_{123}$.
Let \( M^{[s,t]} \) be defined as

\[
M^{[s,t]} = \begin{cases}
P_{321}M^{[s,t]} & \text{if } s \in \tilde{S}, t \notin \tilde{S}, \\
P_{321}M^{[s,t]}P_{321} & \text{if } s \in \tilde{S}, t \in \tilde{S}, \\
M^{[s,t]}P_{321} & \text{if } s \notin \tilde{S}, t \in \tilde{S}, \\
M^{[s,t]} & \text{if } s \notin \tilde{S}, t \notin \tilde{S}.
\end{cases}
\]

Then \( M^{[s,\tau]}M^{[\tau,t]} = M^{[s,t]} \) if and only if \( \tilde{M}^{[s,\tau]}\tilde{M}^{[\tau,t]} = \tilde{M}^{[s,t]} \). Where \( P_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \).

**Conclusion.** By applying Lemma 2 and Lemma 3 if necessary it is not restrictive to assume that \( M^{[s,\tau]}M^{[\tau,t]} = M^{[s,t]} \) with \((1,0,0)M^{[s,t]} \neq (0,0,0)\) for any \((s,t)\).

Note that if \( P \) is nonsingular and \( \tilde{M} = PMP \) then \( M = p^{-1}MP^{-1} \). Consequently, if you solve one of the systems then you solve the other.

Let now we solve the equation (1) by assuming that \((1,0,0)M^{[s,t]} \neq (0,0,0)\) for all \((s,t)\) in the case that \( \text{rank} M^{[s,t]} = 1 \) for all \((s,t)\).

**Remark 1.** If \( \text{rank} M^{[s,0,t]} = 1 \) then \( \text{rank} M^{[s,t]} = 1 \) for all \((s,t)\).

Because, \( \text{rank} M^{[s,0,t]}M^{[0,t]} \leq 1 \) and if \( \text{rank} M^{[s,0,t]}M^{[0,t]} = 0 \) then \( \text{rank} M^{[s,t]} = 0 \) so then \( M^{[s,0,t]} = 0 \) for all \((s,t)\). Therefore \( \text{rank} M^{[s,0,t]} = 1 \) \( \forall t \) and hence \( \text{rank} M^{[s,s]}M^{[s,0,t]} = 1 \) so that \( \text{rank} M^{[s,t]} = 1 \) for all \((s,t)\). If \( \text{rank} M^{[s,t]} = 1 \) for all \((s,t)\) then

\[
M^{[s,t]} = \begin{pmatrix}
\alpha(s,t) & \beta(s,t) & \gamma(s,t) \\
k_1(s,t)\alpha(s,t) & k_1(s,t)\beta(s,t) & k_1(s,t)\gamma(s,t) \\
k_2(s,t)\alpha(s,t) & k_2(s,t)\beta(s,t) & k_2(s,t)\gamma(s,t)
\end{pmatrix}.
\]

We denote \( \tilde{\nu}(s,t) = (\alpha(s,t), \beta(s,t), \gamma(s,t)) \) and \( \tilde{k}(s,t) = (1, k_1(s,t), k_2(s,t)) \).
Then for this case Kolmogorov-Chapman equation has the following view

\[
\begin{align*}
< \mathbf{v}(s, \tau), \alpha(\tau, t) \mathbf{k}(\tau, t) > &= \alpha(s, t), \\
< \mathbf{v}(s, \tau), \beta(\tau, t) \mathbf{k}(\tau, t) > &= \beta(s, t), \\
< \mathbf{v}(s, \tau), \gamma(\tau, t) \mathbf{k}(\tau, t) > &= \gamma(s, t), \\
< k_1(s, \tau) \mathbf{v}(s, \tau), \alpha(\tau, t) \mathbf{k}(\tau, t) > &= k_1(s, t)\alpha(s, t), \\
< k_1(s, \tau) \mathbf{v}(s, \tau), \beta(\tau, t) \mathbf{k}(\tau, t) > &= k_1(s, t)\beta(s, t), \\
< k_1(s, \tau) \mathbf{v}(s, \tau), \gamma(\tau, t) \mathbf{k}(\tau, t) > &= k_1(s, t)\gamma(s, t), \\
< k_2(s, \tau) \mathbf{v}(s, \tau), \alpha(\tau, t) \mathbf{k}(\tau, t) > &= k_2(s, t)\alpha(s, t), \\
< k_2(s, \tau) \mathbf{v}(s, \tau), \beta(\tau, t) \mathbf{k}(\tau, t) > &= k_2(s, t)\beta(s, t), \\
< k_2(s, \tau) \mathbf{v}(s, \tau), \gamma(\tau, t) \mathbf{k}(\tau, t) > &= k_2(s, t)\gamma(s, t).
\end{align*}
\]

or

\[
\begin{align*}
\alpha(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= \alpha(s, t), \\
\beta(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= \beta(s, t), \\
\gamma(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= \gamma(s, t), \\
k_1(s, \tau)\alpha(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= k_1(s, t)\alpha(s, t), \\
k_1(s, \tau)\beta(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= k_1(s, t)\beta(s, t), \\
k_1(s, \tau)\gamma(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= k_1(s, t)\gamma(s, t), \\
k_2(s, \tau)\alpha(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= k_2(s, t)\alpha(s, t), \\
k_2(s, \tau)\beta(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= k_2(s, t)\beta(s, t), \\
k_2(s, \tau)\gamma(\tau, t) < \mathbf{v}(s, \tau), \mathbf{k}(\tau, t) > &= k_2(s, t)\gamma(s, t).
\end{align*}
\]

where \(< -, - >\) is scalar product of two vectors.

If we put first three equations to other equations on (5) then we give

\[
\begin{align*}
k_1(s, \tau)\alpha(s, t) &= k_1(s, t)\alpha(s, t), \\
k_1(s, \tau)\beta(s, t) &= k_1(s, t)\beta(s, t), \\
k_1(s, \tau)\gamma(s, t) &= k_1(s, t)\gamma(s, t), \\
k_2(s, \tau)\alpha(s, t) &= k_2(s, t)\alpha(s, t), \\
k_2(s, \tau)\beta(s, t) &= k_2(s, t)\beta(s, t), \\
k_2(s, \tau)\gamma(s, t) &= k_2(s, t)\gamma(s, t).
\end{align*}
\]

Since \(\mathbf{v}(s, t) \neq 0\), therefor from the first three equations of the system we get 

\(k_1(s, \tau) = k_1(s, t)\), this means that \(k_1(s, t)\) does not depend on \(t\). Thus we can take it
as an arbitrary function $\psi(s) : k_1(s, t) = \psi(s)$. Similarly, from the last three equations we get $k_2(s, t) = \varphi(s)$ for an arbitrary $\varphi(s)$.

If we put these values to the first three equations of (5), then we get $\frac{\alpha(s, t)}{\alpha(t, t)} = \frac{\beta(s, t)}{\beta(t, t)} = \frac{\gamma(s, t)}{\gamma(t, t)} = < \vec{v}(s, \tau), (1, \psi(\tau), \varphi(\tau)) >$. Because of right hand does not depend on $t$, it means $\frac{\alpha(s, t)}{\alpha(t, t)}$, $\frac{\beta(s, t)}{\beta(t, t)}$, and $\frac{\gamma(s, t)}{\gamma(t, t)}$ are must not depend on $t$. Therefore we can assume that $\alpha(s, t) = f_1(s)g_1(t)$, $\beta(s, t) = f_2(s)g_2(t)$ and $\gamma(s, t) = f_3(s)g_3(t)$, where $f_i$ and $g_i$, $i = 1, 2, 3$ are arbitrary functions.

Since $\frac{\alpha(s, t)}{\alpha(t, t)} = \frac{\beta(s, t)}{\beta(t, t)}$, we get $\frac{f_1(s)}{f_1(t)} = \frac{f_2(s)}{f_2(t)}$ that is $\frac{f_2(t)}{f_1(t)}$ is constant function and we denote it by $C_1$. Similarly, from $\frac{\alpha(s, t)}{\alpha(t, t)} = \frac{\gamma(s, t)}{\gamma(t, t)}$ we get that $\frac{f_3(t)}{f_1(t)}$ is also constant function and we denote it by $C_2$.

Therefore, $\alpha(s, t) = f_1(s)g_1(t)$, $\beta(s, t) = C_1f_1(s)g_2(t)$ and $\gamma(s, t) = C_2f_1(s)g_3(t)$. Without loss of generality we can denote $C_1g_2(t)$ by $g_2(t)$ and $C_2g_3(t)$ by $g_3(t)$ respectively. Thus

$$\alpha(s, t) = f_1(s)g_1(t), \quad \beta(s, t) = f_1(s)g_2(t), \quad \gamma(s, t) = f_1(s)g_3(t).$$

From $\frac{\alpha(s, t)}{\alpha(t, t)} = < \vec{v}(s, \tau), (1, \psi(\tau), \varphi(\tau)) >$ we get

$$\frac{f_1(s)}{f_1(t)} = f_1(s)g_1(\tau) + \psi(\tau)f_1(s)g_2(\tau) + \varphi(\tau)f_1(s)g_3(\tau)$$

and if we put $\tau = s$ we get

$$f_1(s) = \frac{1}{g_1(s) + \psi(s)g_2(s) + \varphi(s)g_3(s)}.$$

Therefore we have a family of solutions in a view

$$\mathcal{M}_3^{[s, t]} = f_1(s)
\begin{pmatrix}
g_1(t) & g_2(t) & g_3(t) \\
p(s)g_1(t) & p(s)g_2(t) & p(s)g_3(t) \\
p(s)g_1(t) & p(s)g_2(t) & p(s)g_3(t)
\end{pmatrix}.$$
3.3 Three-dimensional CEAs in some other cases

Case 1. \(a_{11}^{[s,\tau]} = a_{12}^{[s,\tau]} = a_{21}^{[s,\tau]} = 0\). In this case system of equations (2) has the following view:

\[
0 = a_{13}^{[s,\tau]} a_{31}^{[\tau,\tau]},
0 = a_{13}^{[s,\tau]} a_{32}^{[\tau,\tau]},
\]

\[
a_{13}^{[s,t]} = a_{13}^{[s,\tau]} a_{33}^{[\tau,\tau]},
0 = a_{23}^{[s,\tau]} a_{31}^{[\tau,\tau]},
\]

\[
a_{22}^{[s,\tau]} = a_{22}^{[s,\tau]} a_{22}^{[\tau,\tau]} + a_{23}^{[s,\tau]} a_{32}^{[\tau,\tau]},
\]

\[
a_{23}^{[s,\tau]} = a_{22}^{[s,\tau]} a_{23}^{[\tau,\tau]} + a_{23}^{[s,\tau]} a_{33}^{[\tau,\tau]},
\]

\[
a_{31}^{[s,\tau]} = a_{33}^{[\tau,\tau]} a_{31}^{[\tau,\tau]},
\]

\[
a_{32}^{[s,\tau]} = a_{32}^{[s,\tau]} a_{22}^{[\tau,\tau]} + a_{33}^{[\tau,\tau]} a_{32}^{[\tau,\tau]},
\]

\[
a_{33}^{[s,\tau]} = a_{31}^{[s,\tau]} a_{33}^{[\tau,\tau]} + a_{32}^{[s,\tau]} a_{23}^{[\tau,\tau]} + a_{33}^{[\tau,\tau]} a_{33}^{[\tau,\tau]}.
\]

(6)

Case 1.1. \(a_{13}^{[s,\tau]} = a_{23}^{[s,\tau]} = 0\). Consequently \(a_{13}^{[s,\tau]} = 0\). Then system of equation (6) has the following view:

\[
a_{22}^{[s,\tau]} = a_{22}^{[s,\tau]} a_{22}^{[\tau,\tau]},
\]

\[
a_{23}^{[s,\tau]} = a_{22}^{[s,\tau]} a_{23}^{[\tau,\tau]},
\]

\[
a_{31}^{[s,\tau]} = a_{33}^{[\tau,\tau]} a_{31}^{[\tau,\tau]},
\]

\[
a_{32}^{[s,\tau]} = a_{32}^{[s,\tau]} a_{22}^{[\tau,\tau]} + a_{33}^{[\tau,\tau]} a_{32}^{[\tau,\tau]},
\]

\[
a_{33}^{[s,\tau]} = a_{32}^{[s,\tau]} a_{23}^{[\tau,\tau]} + a_{33}^{[\tau,\tau]} a_{33}^{[\tau,\tau]}.
\]

(7)

The first equation of (7) has the following solutions:

a) \(a_{22}^{[s,\tau]} \equiv 0\);

b) \(a_{22}^{[s,\tau]} = \frac{h(t)}{h(s)}\), where \(h(s) \neq 0\);

c) \(a_{22}^{[s,\tau]} = \begin{cases} 1, & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a, \end{cases}\) where \(a > 0\).

Case 1.1. \(a_{22}^{[s,\tau]} \equiv 0\). Consequently, \(a_{23}^{[s,\tau]} = 0\). Then we the following system of equations:

\[
a_{31}^{[s,\tau]} = a_{33}^{[\tau,\tau]} a_{31}^{[\tau,\tau]},
\]

\[
a_{32}^{[s,\tau]} = a_{33}^{[\tau,\tau]} a_{32}^{[\tau,\tau]},
\]

\[
a_{33}^{[s,\tau]} = a_{33}^{[\tau,\tau]} a_{33}^{[\tau,\tau]}.
\]

(8)

The last equation of (8) has the following solutions:
a') \( a_{s,t}^{[s,t]} \equiv 0; \)

b') \( a_{s,t}^{[s,t]} = \frac{g(t)}{g(s)}, \) where \( g \) is an arbitrary function with \( g(s) \neq 0; \)

c') \( a_{s,t}^{[s,t]} = \begin{cases} 
1, & \text{if } s \leq t < a, \\
0, & \text{if } t \geq a, 
\end{cases} \) where \( a > 0. \)

**Case 1.1.a.a'.** \( a_{s,t}^{[s,t]} \equiv 0. \) Consequently \( a_{31}^{[s,t]} = a_{32}^{[s,t]} = 0. \) Thus in this case we doesn’t give new algebra.

**Case 1.1.a.b'.** \( a_{s,t}^{[s,t]} = \varphi(t)g(s) \). By put this value to first and second equations of (8) we give that \( a_{31}^{[s,t]} = \psi(t)g(s) \) and \( a_{32}^{[s,t]} = f(t)g(s). \) Thus we give the following new CEA:

\[
\mathcal{M}_{5}^{[s,t]} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\varphi(t) & \psi(t) & 1 \\
g(s) & g(s) & g(s)
\end{pmatrix},
\]

**Case 1.1.a.c'.** \( a_{s,t}^{[s,t]} = \begin{cases} 
1, & \text{if } s \leq t < a, \\
0, & \text{if } t \geq a, 
\end{cases} \) where \( a > 0. \) Then by putting this solution to first and second equations of (8) we give that \( a_{31}^{[s,t]} = \varphi(t), \) if \( s \leq t < a, \) and \( a_{32}^{[s,t]} = \psi(t), \) if \( s \leq t < a. \) Thus we give \( \mathcal{M}_{6}^{[s,t]} \) and the following new CEA:

\[
\mathcal{M}_{6}^{[s,t]} = \begin{cases} 
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\varphi(t) & \psi(t) & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, & \text{if } s \leq t < a, \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, & \text{if } t \geq a.
\end{cases}
\]

Cases 1.1.b, 1.1.c and 1.2, 1.3 are more difficult. So we don’t include to this paper these cases.

### 4 Baric property transition

In [2] a notion of property transition for CEAs defined.

**Definition 3.** Assume a CEA, \( E^{[s,t]}, \) has a property, say \( P, \) at pair of times \((s_0, t_0)\); one says that the CEA has \( P \) property transition if there is a pair \((s, t) \neq (s_0, t_0)\) at which the CEA has no the property \( P. \)
Denote
\[ T = \{(s, t) : 0 \leq s \leq t\}; \]
\[ T_P = \{(s, t) \in T : E^{[s,t]} \text{ has property } P\}; \]
\[ T_P^0 = T \setminus T_P = \{(s, t) \in T : E^{[s,t]} \text{ has no property } P\}. \]

The sets have the following meaning:
- \( T_P \) — the duration of the property \( P \);
- \( T_P^0 \) — the lost duration of the property \( P \);

The partition \( \{T_P, T_P^0\} \) of the set \( T \) is called \( P \) property diagram.

For example, if \( P \) = commutativity then since any evolution algebra is commutative, we conclude that any CEA has not commutativity property transition.

Since an evolution algebra is not a baric algebra, in general, using Theorem 1 we can give baric property diagram. Let us do this for the above given chains \( E_i^{[s,t]} \), \( i = 0, \ldots, 6 \).

**Theorem 2.**

(i) (There is no non-baric property transition) The algebras \( E_i^{[s,t]} \), \( i = 0, 3 \) are not baric for any time \( (s, t) \in T \);

(ii) (There is baric property transition) The CEAs \( E_i^{[s,t]} \), \( i = 1, 2, 4, 5 \) have baric property transition with baric property duration sets as the following
\[ T_b^{(1)} = \{(s, t) \in T : g(s) = f(s) = \pm \frac{1}{\kappa(s)}\} \cup \{(s, t) \in T : g(s) = -f(s) = \frac{1}{\kappa(s)}\}; \]
\[ T_b^{(2)} = \{(s, t) \in T : \varphi(s) = \psi(s) = \pm 1\} \cup \{(s, t) \in T : \varphi(s) = 1, \ \psi(s) = -1\}; \]
\[ T_b^{(4)} = \{(s, t) \in T : g(t) \neq 0\}; \quad T_b^{(5)} = \{(s, t) \in T : s \leq t < a\}. \]

**Proof.** By Theorem 1 a three-dimensional evolution algebra \( E^{[s,t]} \) is baric if and only if \( a_{11}^{[s,t]} \neq 0, a_{21}^{[s,t]} = a_{31}^{[s,t]} = 0 \) or \( a_{22}^{[s,t]} \neq 0, a_{12}^{[s,t]} = a_{32}^{[s,t]} = 0 \) or \( a_{33}^{[s,t]} \neq 0, a_{13}^{[s,t]} = a_{23}^{[s,t]} = 0 \).

By detailed checking of these conditions over algebras \( E_i^{[s,t]} \), \( i = 0, \ldots, 5 \) we give proof of the Theorem 2. \( \square \)

## 5 Absolute nilpotent and idempotent elements transition

In this section we answer to problem of existence of "uniqueness of absolute nilpotent element" property transition.

**Definition 4.** The element \( x \) of an algebra \( A \) is called an absolute nilpotent if \( x^2 = 0 \).

Let \( E \) be an \( n \)-dimensional evolution algebra over the field \( \mathbb{R} \) with matrix of structural constants \( \mathcal{M} = (a_{ij}) \), then for arbitrary \( x = \sum_i x_i e_i \) we have
\[ x^2 = \sum_j \left( \sum_i a_{ij} x_i^2 \right) e_j. \]
The algebra $I$ solve this problem for the CEA’s time-dynamics of the idempotent elements for algebras

Theorem 4. \(V\) evolution algebra are especially important, because they are the fixed points (i.e. elements of an algebra $E$ as $E_i^{[s,t]}$ denote

Then we have the following theorem which contain answer to problem of existence of "uniqueness of absolute nilpotent element" property transition.

Theorem 3. (1) There CEA’s $E_i^{[s,t]}$, \(i = 0,4,5\) have infinitely many of absolute nilpotent elements for any time $(s,t) \in T$;

(2) The CEA’s $E_i^{[s,t]}$, \(i = 1,2\) have "uniqueness of absolute nilpotent element" property transition with the property duration sets as the following

\[ T_{nil}^{(1)} = \{(s,t) \in T : h(t) \neq 0, f(s) < g(s) < \frac{1}{h(s)}, -f(s) < \frac{1}{h(s)}\}; \]

\[ T_{nil}^{(2)} = \{(s,t) \in T : -1 < \psi(s) < \varphi(s) < 1, t < a\}; \]

\[ T_{nil}^{(3)} = \{(s,t) \in T : g_1(t), g_2(t), g_3(t) - the\ signs\ are\ difference\}. \]

Proof. The proof consists the simple analysis of the solutions of the system (9) for each $E_i^{[s,t]}$, \(i = 0, \ldots, 5\) \(\square\)

An element $x$ of an algebra $A$ is called idempotent if $x^2 = x$. Such points of an evolution algebra are especially important, because they are the fixed points (i.e. $V(x) = x$) of the evolution operator $V$. We denote by $Id(E)$ the set of idempotent elements of an algebra $E$. Since $x = \sum_i x_i e_i$ then the equation $x^2 = x$ can be written as

\[ x_j = \sum_{i=1}^n a_{ij} x_i^2, \quad j = 1, \ldots, n. \]  

The general analysis of the solutions of the system (10) is very difficult. We shall solve this problem for the CEA’s $E_i^{[s,t]}$, \(i = 0, \ldots, 5\). The following theorem gives the time-dynamics of the idempotent elements for algebras $E_i^{[s,t]}$, \(i = 0, \ldots, 5\).

Theorem 4.

(1) The algebra $E_0^{[s,t]}$ have unique idempotent $(0,0)$.

(2) $Id(E_1^{[s,t]}) = \left\{ \begin{array}{ll} (0,0,0), & \text{if } (s,t) \in \{(s,t) \in T : h(t) = 0\} \\ (0,0,0), & \text{if } (s,t) \in \{(s,t) \in T : h(t) \neq 0\} \end{array} \right.$

(3) $Id(E_2^{[s,t]}) = \left\{ \begin{array}{ll} (0,0,0), & \text{if } (s,t) \in \{(s,t) \in T : t \geq a\} \\ (0,0,0),(1,1,1), & \text{if } (s,t) \in \{(s,t) \in T : s \leq t < a\} \end{array} \right.$
we get \( x \) for each solution \((s, t) \in \mathcal{T} : F(s, t) = 0 \) 
where \( F(s, t) = f_1(s)(g_1(t) + \psi(s)(g_2(t))^2 + \varphi(s)(g_3(t))^2). \)

\[ (4) \quad \mathcal{I}(E_3^{[s,t]}) = \begin{cases} 
(0, 0, 0), & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : F(s, t) = 0 \} \\
(0, 0, 0), & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : g(t) = 0 \} \\
\end{cases}, \]

\[ (5) \quad \mathcal{I}(E_4^{[s,t]}) = \begin{cases} 
(0, 0, 0), & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : g(t) = 0 \} \\
\end{cases}, \]

\[ (6) \quad \mathcal{I}(E_5^{[s,t]}) = \begin{cases} 
(0, 0, 0), & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : t \geq a \} \\
\end{cases}. \]

**Proof.** The proof contains detailed analysis of solutions of the system (10) for each \( E_i^{[s,t]} \). We shall give here proof of the assertion (4) which is more substantial.

In this case the system (10) has the form

\[
\begin{align*}
  x_1 &= f_1(s)g_1(t)x_1^2 + \psi(s)x_2^2 + \varphi(s)x_3^2, \\
  x_2 &= f_1(s)g_2(t)x_1^2 + \psi(s)x_2^2 + \varphi(s)x_3^2, \\
  x_3 &= f_1(s)g_3(t)x_1^2 + \psi(s)x_2^2 + \varphi(s)x_3^2. 
\end{align*}
\]

It is clear that the system (10) for \( E_3^{[s,t]} \) has a solution \((0, 0, 0)\).

Now we assume that \( g_1(t) \neq 0 \), then \( f_1(s)(x_1^2 + \psi(s)x_2^2 + \varphi(s)x_3^2) = x_1 \)
and therefore \( x_2 = x_1 \frac{g_2(t)}{g_1(t)}, x_3 = x_1 \frac{g_3(t)}{g_1(t)} \). If we put these values to first equation of
(11) we get \( x_1 = \frac{g_1(t)}{F(s,t)}, \) where \( F(s, t) = f_1(s)(g_1(t) + \psi(s)(g_2(t))^2 + \varphi(s)(g_3(t))^2) \).

Consequently, \( x_2 = \frac{g_2(t)}{F(s,t)} \) and \( x_3 = \frac{g_3(t)}{F(s,t)} \). Therefore the system (10) for \( E_3^{[s,t]} \) has a
solution \(( \frac{g_1(t)}{F(s,t)}, \frac{g_2(t)}{F(s,t)}, \frac{g_3(t)}{F(s,t)} \).

Proofs of the assertions \((i), i = 1, 2, 4, 5, 6 \) are similar analysis of the system (10) for each \( E_i^{[s,t]} \). \( \square \)

6 Conclusion

In conclusion we note that we constructed chains of three-dimensional evolution algebras in the present paper and studied behavior of the property to be baric for each chains constructed in the Section 3. We showed that some of the chains are never baric. For other chains which have (baric property transition) we defined a baric property controller function. For each 5 chains of evolution algebras in Section 5 we studied the behavior and dynamics of the set of absolute nilpotent and idempotent elements depending on the time respectively.

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