GOOD REDUCTION OF UNITARY GROUPS OF QUATERNIONIC SKEW-HERMITIAN FORMS

SRIMATHY SRINIVASAN*

ABSTRACT. Given a field $K$ equipped with a set of discrete valuations $V$, we develop a general theory to relate ramification of skew-hermitian forms over a quaternion $K$-algebra $Q$ to ramification of quadratic forms over the function field $K(Q)$ obtained via Morita equivalence. Using this we show that if $(K,V)$ satisfies certain conditions, then the number of $K$-isomorphism classes of special unitary groups of quaternionic skew-hermitian forms that have good reduction at all valuations in $V$ is finite and bounded by a value that depends on size of a quotient of the Picard group of $V$ and the size of the kernel and cokernel of residue maps in Galois cohomology of $K$ with finite coefficients. As a corollary we prove a conjecture of Chernousov, Rapinchuk, Rapinchuk for groups of this type.

1. INTRODUCTION

Given a discrete valuation $v$ on a field $K$, let $O_v, K_v$ and $K^{(v)}$ denote respectively the valuation ring, the completion of $K$ and the residue field. Let $G$ be an absolutely almost simple linear algebraic group defined over $K$. In a recent paper, Chernousov, Rapinchuk, Rapinchuk (CRR19) asked the following question:

"Can one equip $K$ with a set of discrete valuations $V$ such that, the set of $K$-isomorphism classes of $K$-forms of $G$ having good reduction at all $v \in V$ is finite?"

Here $G$ is said to have good reduction at $v$ if there exists a reductive group scheme $\tilde{G}$ over $O_v$ with generic fiber $\tilde{G} \otimes_{O_v} K_v$ isomorphic to $G \otimes_K K_v$. An affirmative answer to the question has important implications such as the properness of local-global map in Galois cohomology, finiteness of the genus of an algebraic group (see [CRR13], [CRR19] for the definition of genus and its relation to good reduction) and in eigenvalue rigidity problems. A detailed explanation of why this question is important and interesting can be found in [CRR16a] and [CRR19]. Here we just highlight some of the main points and what is known for the context of this paper.

It is well known that the answer to the question is affirmative for algebraic groups over a number field where $V$ can be chosen to be the set containing almost all non-archimedean places([Gro96], [Con15], [JL15]). The paper [CRR19] discusses the following cases.

(i) $K$ is a two dimensional global field i.e, function field of a smooth geometrically integral curve over a number field or function field of a smooth geometrically integral surface over a finite field and $V$ is a divisorial set of valuations (i.e valuations arising from prime divisors of an integral model of $K$).

(ii) $K$ is a function field of a smooth geometrically integral curve $C$ over a field $k$ of characteristic $\neq 2$ that satisfies condition $(F'_{m})$ and $V$ is the set of valuations corresponding to closed points of $C$.

Regarding (ii), see [5.1] for the definition, properties and examples of fields of type $(F'_{m})$.

For the cases $(K, V)$ as above, we list below groups of each type such that the number of $K$-isomorphism classes with the given description having good reduction at all valuations in $V$ is known to be finite. All the underlying forms in the list below are non-degenerate. The details can be found in [CRR13], [CRR16a], [CRR16b] and [CRR19].

1. Type A:

*This material is based upon work supported by the National Science Foundation under Grant No. DMS - 1638352.
(a) **Inner forms:** $SL_m(D)$, $D$ a division algebra of index $d$ and $md - 1$ prime to $\text{char } K$. (In fact it is true for all $K$ that are finitely generated and for a divisorial set of places $V$).

(b) **Outer forms:** $SU_n(L/K, h)$ where $L/K$ is a quadratic extension and $h$ is an hermitian form.

(c) **Other outer forms:** Open

(2) **Type B:**

(a) $Spin_{2n+1}(q)$, $n \geq 2$,

(3) **Type C:**

(a) $SU_n(h, Q)$, $n \geq 2$, where $Q$ is a quaternion algebra over $K$ with symplectic involution and $h$ is a hermitian form over $Q$.

(b) Other forms: Open

(4) **Type D:**

(a) Spinor groups: $Spin_{2n}(q)$, $n \geq 3$,

(b) Other forms: Open (See below)

(5) **All simple groups of type $G_2$.**

(6) **All other exceptional groups:** Open

We will briefly outline the steps involved in the proof of the above statements. First for $(K, V)$ as above one proves finiteness of certain unramified cohomologies which yields I(a) and (5). For other types, the underlying (hermitian) forms in the description of groups is reduced to quadratic forms via Jacobson’s theorem ([Jac40]). Finally, the proof is derived by associating unramified quadratic forms to unramified cohomologies via the Milnor isomorphism.

Based on the above evidence, Chernousov, Rapinchuk, Rapinchuk conjectured the following.

**Conjecture 1.1.** (Conjecture 7.3 in [CRR19]) : Let $K = k(C)$ be the function field of a smooth affine geometrically integral curve over a field $k$ and let $V_0$ be the set of discrete valuations associated with the closed points of $C$. Furthermore, let $G$ be an absolutely almost simple simply connected algebraic $K$-group and let $m$ be the order of the automorphism group of its root system. Assume that $\text{char } k$ is prime to $m$ and that $k$ satisfies $(F_m')$. Then the set of $K$-isomorphism classes of $K$-forms of $G$ that have good reduction at all $v \in V_0$ is finite.

The main goal of this paper is to develop a general theory that relates ramification of skew-hermitian forms over a quaternion algebra $Q$ to the ramification of the corresponding quadratic form over the function field of the conic associated to $Q$ obtained via Morita equivalence. This general theory allows us to throw some light on the case 4(b) in the list above. We now give a brief sketch of this philosophy. Consider special unitary groups $SU_n(h, Q)$ of non-degenerate $n$-dimensional skew-hermitian forms $h$ over a quaternion $K$- algebra $Q$ with the canonical (symplectic) involution. Let $K(Q)$ denote the function field of the Severi Brauer variety associated to $Q$. For a skew-hermitian form $h$ over $Q$, let $q_{hK(Q)}$ be the quadratic form over $K(Q)$ obtained via Morita theory (See [6]). Given a valuation $v$ on $K$, we describe a method to extend it to a valuation $\tilde{v}$ on $K(Q)$ in such a way that if $h$ is unramified (see [17] for the notion of ramification of skew-hermitian forms over quaternions) with respect to the valuation $v$ then the associated quadratic form $d_{hK(Q)}$ is unramified with respect to $\tilde{v}$. This is then used to prove the required finiteness statements and as a consequence we obtain a proof of Conjecture 1.1 for these groups.

2. Notations

All the fields in this paper have characteristic $\neq 2$. For a discrete valuation $v$ on a field $F$, let $\mathcal{O}_v$ (or $\mathcal{O}_F$) denote the valuation ring in $F$ when the underlying field $F$ (resp. the underlying valuation) is clear. Let $F_v$ denote the completion of $F$ with respect to $v$ and let $F_v(\mu_2)$ denote its residue field. For a quaternion algebra $Q$ over $F$, we denote its class in $2Br(F) \simeq H^2(F, \mu_2)$ by $[Q]$. The involution on $Q$ is the canonical (symplectic) involution. Let $Q_v := Q \otimes_FF_v$. For a ring $R$, let $R^\sigma$ denote its group of
units. All the forms considered in this paper are finite dimensional and non-degenerate. The Witt ring of $F$ is denoted by $W(F)$ and $\mathcal{T}(F)$ denotes its fundamental ideal.

3. Main Results

We recall the following notations and facts from [CRR19]. Let $V$ be a set of discrete valuations on a field $K$ that satisfies the following conditions.

(A) For any $a \in K^*$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite.
(B) $\text{char } K(v) \neq 2 \forall v \in V$.

As noted in [CRR19], condition (A) is satisfied for a divisorial set of valuations $V$ on a finitely generated $K$. Let $\text{Div}(V)$ be the free abelian group on the set $V$ called divisors. Because of condition (A), for any $a \in K^*$ one has the notion of subgroup if principle divisor, $P(V)$ generated by

$$(a) = \sum_{v \in V} v(a) v$$

The Picard group of $V$ denoted $\text{Pic}(V)$ is given by

$$\text{Pic}(V) = \text{Div}(V)/P(V)$$

Due to conditions (A) and (B), for $l$ a power of 2, we have residue maps in Galois cohomology (See §5.2 for a discussion on this)

$$(3.1) \quad r_l^i : H^i(K, \mu_l^{\otimes i-1}) \to \bigoplus_{v \in V} H^{i-1}(K(v), \mu_l^{\otimes i-2})$$

where $\mu_l$ denotes $l$-th roots of unity. For $l = 2$, this is just

$$(3.2) \quad r_2^i : H^i(K, \mu_2) \to \bigoplus_{v \in V} H^{i-1}(K(v), \mu_2)$$

Let $H^i(K, \mu_l^{\otimes i-1})_V$ and $H^i(K, \mu_l^{\otimes i-1})_V$ denote respectively the $\ker r_l^i$ and $\text{coker } r_l^i$. The main goal of the paper is to prove the following.

**Theorem 3.1.** Let $V$ be a set of discrete valuations on $K$ satisfying conditions (A) and (B). Suppose the following holds:

1. the quotient $\text{Pic}(V)/2\text{Pic}(V)$ is finite and
2. the cohomology groups $H^2(K, \mu_2)_V$, $H^i(K, \mu_2)_V$ and $H^i(K, \mu_l^{\otimes i-1})_V$ are finite for all $i = 1, 2, \cdots l := \lceil \log_2 2n \rceil + 1$.

Then the number of $K$-isomorphism classes of special unitary groups of $n$-dimensional skew-hermitian forms $SU_n(h)$, where $h$ is skew-hermitian over some quaternion (not necessarily division) algebra with the canonical involution, having good reduction at all $v \in V$ is finite and bounded above by

$$(3.3) \quad |H^2(K, \mu_2)_V| \cdot |\text{Pic}(V)/2\text{Pic}(V)| \cdot \prod_{i=1}^l |H^i(K, \mu_2)_V| \cdot |H^i(K, \mu_l^{\otimes i-1})_V|$$

A situation where the hypothesis of Theorem 3.1 holds is the following.

**Proposition 3.2.** Let $K = k(C)$ be the function field of a geometrically integral curve over a field $k$ of characteristic $\neq 2$ that satisfies (F) (see §5.1) and let $V$ be the set of discrete valuations on $K$ corresponding to closed points of $C$. Then $(K, V)$ satisfies the hypothesis of Theorem 3.1.

**Proof.** Conditions (A) and (B) are easily seen to be satisfied. The finiteness of $\text{Pic}(V)/2\text{Pic}(V)$ is shown in the proof of Theorem 1.4 in [CRR19]. We will now show finiteness of $H^2(K, \mu_2)_V$, $H^i(K, \mu_2)_V$ and
5.1. Fields of type \((F')\). For any \(l\) coprime to \(\text{char } K'^{(v)}\), the Bloch-Ogus spectral sequence and Kato complexes yield a long exact sequence in cohomology (see §4 in [Rap19])

\[ \cdots \to H^i_{\text{ét}}(C, \mu_{l^i-1}) \to H^i(K, \mu_{l^i-1}) \to \bigoplus_{v \in V} H^{i-1}(K'^{(v)}, \mu_{l^i-2}) \to H^{i+1}_{\text{ét}}(C, \mu_{l^i-1}) \to \cdots \]

By Lemma 5.1 and Corollary 3.2 in [Rap19], \(H^i_{\text{ét}}(C, \mu_{l^j})\) is finite for all \(i \geq 0\) and all \(j\). This proves the claim.

The Conjecture of Chernousov, Rapinchuk, Rapinchuk for these groups easily follows.

**Corollary 3.3.** (Conjecture 7.3 in [CRR19] for the universal coverings of special unitary groups of quaternionic skew-hermitian forms) Let \(K = k(C)\) be the function field of a geometrically integral curve over a field \(k\) of characteristic \(\neq 2\) that satisfies \((F'_0)\) and let \(V\) be the set of discrete valuations on \(K\) corresponding to closed points of \(C\). Then the number of \(K\)-isomorphism classes of (the universal coverings of) special unitary groups \(SU_m(h)\) of non-degenerate \(m\)-dimensional skew-hermitian forms \(h\) over some quaternion \(K\)-algebra with the canonical involution, having good reduction at all places in \(V\) is finite.

**Proof.** This easily follows from Proposition 3.2 and Theorem 3.1.

4. Outline

The paper is organized as follows. In §5 we briefly recall the necessary results from the literature that will lay foundation for the rest of the paper. In §6 we use Morita theory to reduce skew-hermitian forms over quaternions to quadratic forms and give an explicit example of the correspondence which will be used later. In §7 we define the notion of unramified skew-hermitian forms and relate it to good reduction of unitary groups of skew-hermitian forms. Next in §8 we describe a method to extend valuations from a complete discrete valued field to the function field of Severi-Brauer variety associated to a quaternion algebra over the field and discuss the properties of this extension. Finally, in §9 we prove the main theorem of the paper by relating the ramification of skew-hermitian forms to ramification of associated quadratic forms.

5. Preliminaries

5.1. Fields of type \((F'_m)\). The notion of fields of type \((F'_m)\) is introduced in [Rap19]. Recall from [Rap19] that for \(m\) prime to \(\text{char } k\), a field \(k\) is said to be of type \((F'_m)\) if for every finite separable extension \(L/k\), the quotient \(L^*/(L^*)^m\) is finite (here \(L^*\) is the multiplicative group of units). This notion generalizes Serre’s condition \((F)\) ([Ser02]) and is useful for many applications. It is shown that over fields of type \((F'_m)\), certain Galois cohomology groups are finite (see Theorem 1.1 in [Rap19]), which is useful for computations of unramified cohomologies (Proposition 4.2 in [Rap19]). Examples of such fields include finite fields, local fields and higher dimensional local fields such as \(\mathbb{Q}_p((t_1)) \cdots ((t_n))\) (note that the last two are not finitely generated). See also Example 2.9 in [Rap19]. We now make the following observation (I thank P. Deligne to remark about this).

**Lemma 5.1.** Let \(k\) be a field and let \(m\) be prime to its characteristic. Then \(k\) is of type \((F'_m)\) if and only if it is of type \((F'_p)\) for every prime \(p\) dividing \(m\).

**Proof.** By definition, it is clear that if \(k\) is of type \((F'_m)\), it is of type \((F'_p)\) for every prime \(p\) dividing \(m\). We will prove the other direction by induction on the number of primes dividing \(m\). Let \(r\) be the number of primes dividing \(m\). Consider the case \(r = 1\). Assume that \(k\) is of type \((F'_p)\). Let \(L\) be a finite separable
extension of \( k \). For every \( j \geq 1 \) we have exact sequences of groups

\[
0 \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{p^j}} \to \frac{(L^*)}{(L^*)^{p^j}} \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{p^j}} \to 0
\]

\( x \mapsto x \)

\[
0 \to \frac{\mu_{p^{j-1}}(L^*)}{\mu_{p^j}(L^*) \cap (L^*)^{p^j}} \to \frac{(L^*)}{(L^*)^{p^j}} \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{p^j}} \to 0
\]

\( x \mapsto x^{p^j} \)

where \( \mu_n(L^*) \) denotes \( n \)-th roots of unity in \( L^* \). By hypothesis, these sequences imply that \( \frac{(L^*)}{(L^*)^{p^j}} \) is finite if \( \frac{(L^*)^{p^{j-1}}}{(L^*)^{p^j}} \) is finite. Thus by induction on \( j \) we conclude that \( k \) is of type \( (F'_{p^j}) \). Therefore the the statement of the lemma is true for \( r = 1 \). Assume that \( m \) is arbitrary and \( k \) is of type \( (F'_{p^j}) \) for every prime \( p \) dividing \( m \). Let \( m = p^j n \) where \( n \) is coprime to \( p \) and \( j \geq 1 \). We have the following exact sequences

\[
0 \to \frac{(L^*)^{p^j}}{(L^*)^{m}} \to \frac{(L^*)}{(L^*)^{m}} \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{m}} \to 0
\]

\( x \mapsto x \)

\[
0 \to \frac{\mu_{p^j}(L^*)}{\mu_{p^j}(L^*) \cap (L^*)^{n}} \to \frac{(L^*)}{(L^*)^{n}} \to \frac{(L^*)^{p^{j-1}}}{(L^*)^{n}} \to 0
\]

\( x \mapsto x^{p^j} \)

By induction hypothesis on \( r \), \( \frac{(L^*)}{(L^*)^{m}} \) is finite and by the case \( r = 1 \), \( \frac{(L^*)}{(L^*)^{p^j}} \) is finite. Therefore we conclude that \( \frac{(L^*)}{(L^*)^{m}} \) is finite. This proves that \( k \) is of type \( (F'_{m}) \).

\[ \blacksquare \]

5.2. **Residue maps.** Let \( K \) be a field with discrete valuation \( v \) and let \( l \) be prime to \( char \ K^{(v)} \). Recall from Chapter II in [GMS03] that for every integer \( j \) we have residue maps in Galois cohomology

\[
r_{i,v}^j : H^i(K, \mu_i^{\otimes j}) \to H^{i-1}(K^{(v)}, \mu_i^{\otimes j-1}), \ i \geq 1
\]

where \( \mu_l \) is the group of \( l \)-th roots of unity in the separable closure of \( K^{(v)} \) and \( \mu_i^{\otimes j} \) is the \( j \)-th Tate twist of \( \mu_l \) as described in [GMS03] (Chapter II, §7.8). An element of \( H^i(K, \mu_i^{\otimes j}) \) is said to be unramified at \( v \) if it is in the kernel of the above residue map. Now assume that \( char \ K^{(v)} \neq 2 \). Note that when \( l = 2 \), we simply have

\[
r_{2,v}^1 : H^1(K, \mu_2) \to H^0(K^{(v)}, \mu_2)
\]

Let \( \pi \) be a uniformizer of \( K_v \). Let \( W(K) \) denote the Witt ring of \( K \). Recall from [Lam05], Chapter VI, §1 that we have residue homomorphisms

\[
\partial_{1,v} : W(K) \xrightarrow{Res_{K_v/K}} W(K_v) \to W(K^{(v)})
\]

The residue homomorphisms can be described as follows. Let \( q \) be a quadratic form over \( K \) and let \( Res_{K_v/K}(q) = \langle u_1, u_2, \cdots u_m, \pi u_{m+1} \cdots \pi u_n \rangle \), \( u_i \in \mathcal{O}_{K_v} \). Then

\[
\partial_{1,v}(q) = \langle \overline{u}_1, \cdots, \overline{u}_m \rangle
\]

\[
\partial_{2,v}(q) = \langle \overline{u}_{m+1}, \cdots, \overline{u}_n \rangle
\]
Here $\pi_i$ denotes the image of $u_i$ in $K^{(v)}$. We say that $q$ is unramified at $v$ if it is in the kernel of $\partial_{2,v}$. Let $W_0(K_v)$ be the subring of $W(K_v)$ generated by classes $<u>$, $u \in O_{K_v}^*$. Then we have a split exact sequence (see §5 in [Mil70])

$$0 \rightarrow W_0(K_v) \rightarrow W(K_v) \xrightarrow{\partial_{2,v}} W(K^{(v)}) \rightarrow 0$$

Recall now that due to Voevodsky’s proof of the Milnor conjecture ([OVV07] , [Voe03]), for any field $F$ with characteristic $\neq 2$ we have natural isomorphisms

$$(5.2) \quad e_n : \mathcal{I}(F)^n/\mathcal{I}(F)^{n+1} \rightarrow H^n(F, \mu_2)$$

where $\mathcal{I}(F)$ denotes the fundamental ideal in $W(F)$. Moreover, the isomorphisms $e_n$ commute with the respective residue homomorphisms (see Satz 4.11 in [AP71]), that is for $[q] \in W(K)$, we have

$$e_{n-1}(\partial_{2,v}(q)) = r_{2,v}(e_n(q))$$

6. REDUCTION TO QUADRATIC FORMS VIA MORITA EQUIVALENCE

From now on, let $Q$ denote a (not necessarily division) quaternion algebra with center $K$.

6.1. General theory. The general theory of Morita equivalence for Hermitian modules can be found in Knus’ book [Knu91] (See Chapter 1, §9). In particular, by Morita theory, a non-degenerate skew-hermitian form of rank $n$ over $Q$ gives rise to a non-degenerate quadratic form of rank $2n$ over $K$ whenever $Q$ is split. In this case, let us denote the quadratic form associated to the skew-hermitian from $h$ by $q_h$. By the properties of Morita equivalence, $h$ is determined by $q_h$, and moreover, two such skew-hermitian forms are isometric if and only if the associated quadratic forms are isometric. So whenever $Q$ is split the skew-hermitian forms over $Q$ can be completely studied by studying the associated quadratic forms over $K$. For an explicit description of Morita equivalence in this case see [Sch85], p. 361-362.

Let $h$ be a skew-hermitian form over a non-split $Q$. A generic way to split $Q$ is by extending the base field to the function field of the associated Severi-Brauer variety. Let $K(Q)$ denote the function field of the Severi-Brauer variety associated to $Q$. Now $Q_{K(Q)}$ is isomorphic to the matrix algebra $M_2(K(Q))$ with involution given by

$$M \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

Since $Q_{K(Q)}$ is split, the skew-hermitian form $h_{K(Q)} := h \otimes_K K(Q)$ can be reduced to a quadratic form $q_{h_{K(Q)}}$ via Morita equivalence. This reduction has nice properties due to the following result from [PSS01].

Proposition 6.1. (Proposition 3.3 in [PSS01]) Let $W^{-1}(Q)$ denote the Witt group of skew-hermitian forms over $Q$. With the notations as above, the canonical homomorphism

$$W^{-1}(Q) \rightarrow W^{-1}(Q \otimes_K K(Q))$$

is injective.

This means that $h$ is hyperbolic if and only if $h_{K(Q)}$ is hyperbolic if and only if, by Morita equivalence, $q_h$, is hyperbolic. This philosophy of understanding the skew-hermitian form $h$ over $Q$ by studying the quadratic form $q_h$ is cleverly employed in Berhuy’s paper [Ber07] to find cohomological invariants. We will be using this philosophy to study ramification properties of these forms.
6.2. An explicit example.

Example 6.2. As before let $Q = (d, t)$ be a quaternionic (not necessarily division) algebra over $K$ with basis $<1, i, j, ij | i^2 = d, j^2 = t, ij = -ji>$. Then the Severi-Brauer variety of $Q$ has function field $K(Q)$ given by the fraction field of $K[x, y]/(dx^2 + ty^2 - 1)$. An explicit splitting of $Q$ over $K(Q)$ is given by the following.

\[
Q \otimes_K K(Q) \timesrightarrow{\exists} M_2(K(Q))
\]

\[
i \mapsto \begin{bmatrix} dx & -y \\ -dty & -dx \end{bmatrix}, j \mapsto \begin{bmatrix} ty & x \\ dtx & -ty \end{bmatrix}, ij \mapsto \begin{bmatrix} 0 & 1 \\ -dt & 0 \end{bmatrix}
\]

Let $h$ be a non-degenerate skew-hermitian form over $Q$ of rank $n$. Then it is well-known that $h$ has a diagonal matrix representation over $Q$. By abuse of notation, let us denote the matrix also by $h$. Since $h$ is skew-hermitian, the diagonal entries are pure quaternions. Let

\[
h \simeq \bigoplus_{i=1}^{n} a_i i + b_i j + c_i k
\]

Then,

\[
h_{K(Q)} \simeq \bigoplus_{i=1}^{n} a_i \begin{bmatrix} dx & -y \\ -dty & -dx \end{bmatrix} + b_i \begin{bmatrix} ty & x \\ dtx & -ty \end{bmatrix} + c_i \begin{bmatrix} 0 & 1 \\ -dt & 0 \end{bmatrix}
\]

Now we use the explicit description of Morita equivalence from [Sch85, p. 361-362] to conclude that the quadratic form associated to $h_{K(Q)}$ has matrix given by (again by abuse of notation)

\[
q_{h_{K(Q)}} \simeq \bigoplus_{i=1}^{n} a_i \begin{bmatrix} -dty & -dx \\ -dx & y \end{bmatrix} + b_i \begin{bmatrix} dtx & -ty \\ -ty & -x \end{bmatrix} + c_i \begin{bmatrix} -dt & 0 \\ 0 & -1 \end{bmatrix}
\]

Let $N_i = Nrd_Q(a_i i + b_i j + c_i k) \in K$ denote the reduced norm of the quaternion $(a_i i + b_i j + c_i k)$ in $Q$. Then diagonalizing the above matrix yields

\[
q_{h_{K(Q)}} \simeq \bigoplus_{i=1}^{n} \begin{bmatrix} (a_i y - b_i x - c_i) & 0 \\ 0 & -(a_i y - b_i x - c_i)N_i \end{bmatrix}
\]

We will be using this matrix representation of $q_{h_{K(Q)}}$ later.

Remark 6.3. From the above description of Morita equivalence, it is clear that for $\lambda \in K^*$,

\[
q_{(\lambda h)_{K(Q)}} = \lambda q_{h_{K(Q)}}
\]

7. Unramified Quaternionic Skew-Hermitian Forms

Recall from §5.2 that when $\text{char } K^{(v)} \neq 2$, we have residue map (see [Sal99, §10, Ser02 Chapter II, Appendix or GMS03, Chapter II])

\[
2Br(K) \simeq H^2(K, \mu_2) \xrightarrow{r_v} H^1(K^{(v)}, \mu_2)
\]

The kernel of this map denoted by $2Br(K)^v$ consists of classes of quaternions unramified at $v$. If $Q$ is in the kernel of $r_v$, we say that $Q$ is unramified at $v$. By Theorem 10.3 in [Sal99], $Q$ is unramified at $v$ if and only if $[Q]$ is obtained from an Azumaya algebra over $O_K$. Let $Q_v := Q \otimes_K K_v$. Then $Q_v$ is either split i.e, a matrix algebra or is a quaternion division algebra over $K_v$. Suppose $Q_v$ is not split. Since $K_v$
is Henselian, one can extend the valuation $v$ on $K_v$ to a (necessarily unique) valuation on $Q_v$ (Corollary 2.2 in [Wad02]), which by abuse of notation is also denoted by $v$. The extended valuation on $Q_v$ is given by (equation (2.7) in [Wad02])

\[
(7.2) \quad v(a) = \frac{1}{2} v(Nrd(a)) \quad \forall a \in Q_v^*
\]

where $Nrd$ denotes the reduced norm on $Q_v$. Let

\[
A_v = \{ a \in Q_v^* | v(a) \geq 0 \} \cup 0
\]

be the valuation ring of $Q_v$. Note that its group of units is given by

\[
A_v^* = \{ a \in Q_v^* | v(a) = 0 \}
\]

If $Q_v$ is unramified at $v$, then $A_v$ is an Azumaya algebra over $O_{K_v}$ and $Q_v \cong A_v \otimes_{O_{K_v}} K_v$. Since $[A_v : O_{K_v}] = [Q_v : K_v] = 4$, $A_v$ is quaternionic and has representation given by $A_v = (d_v, t_v)$ for some $d_v, t_v \in O_{K_v}^*$. (See Theorem 3.2 in [Wad02] and Example 2.4 (ii) and Proposition 2.5 in [JW90]).

Recall now the following exact sequence (see Prop 7.7 in [GMS03], §3 in [Wad02] and Theorem 2, §3 Chapter XII in [Ser79])

\[
(7.3) \quad 0 \to H^2(K^{(v)}, \mu_2) \to H^2(K_v, \mu_2) \to H^1(K^{(v)}, \mu_2) \to 0
\]

where $r$ is the residue map and $s$ is the canonical map resulting from $G_{K_v} \to G_{K^{(v)}}$ (Here $G_F$ denotes the absolute Galois group of a field $F$). This yields an isomorphism (see equation (3.7) in [Wad02])

\[
s^{-1} : \ker(r) := 2Br(K_v)v \xrightarrow{\sim} 2Br(K^{(v)})
\]

\[
Q_v \mapsto Q^{(v)}
\]

where $Q^{(v)}$ is the residue quaternion algebra given by $(\overline{d}_v, \overline{t}_v)$ (Here $\overline{d}_v, \overline{t}_v \in K^{(v)}$ are the residues of $d_v, t_v$ modulo the maximal ideal in $O_{K^{(v)}}$).

Now let $h$ be a non-degenerate skew-hermitian form over $Q$. Let $h_v := h \otimes_K K_v$ be the form over $Q_v$. We say that $h$ is unramified at $v$ if $Q$ is unramified at $v$ and if either $Q_v$ is split and the quadratic form $q_{h_v}$ is unramified at $v$ or if $Q_v$ is not split and $h_v$ is obtained from a non-degenerate skew-hermitian form $\tilde{h}$ over the Azumaya algebra $A_v$ i.e., $h_v = \tilde{h} \otimes_{A_v} K_v$. Since skew-hermitian forms over $Q$ can be diagonalized, we easily get the following

**Proposition 7.1.** Let $h$ be a non-degenerate skew-hermitian form over $Q$. Then $h$ is unramified at $v$ if and only if $Q_v$ is unramified and

(i) $q_{h_v}$ is unramified at $v$ if $Q_v$ is split (see §5.2) or

(ii) $h_v$ has a diagonal matrix representation with diagonal entries taking values in $A_v^*$ if $Q_v$ is not split.

As mentioned in §4 recall from [CRR19] (Notations and conventions section) that an algebraic group $G$ has good reduction at $v$ if there exists a reductive group scheme $\mathcal{G}$ over the valuation ring $O_{K_v}$ whose generic fiber $\mathcal{G} \otimes_{O_{K_v}} K_v$ is isomorphic to $G \otimes_K K_v$. Given a skew-hermitian form $h$ over $Q$, it is easy to see that algebraic group $SU(h, Q)$ has good reduction at $v$ if and only if the form $\alpha_v h_v$ is unramified at $v$ for some $\alpha_v \in K_v^*$.

8. Extension of valuation from $K_v$ to $K_v(Q_v)$

In this section assume that $Q$ is a quaternion division algebra over $K$ unramified at $v$. We extend the valuation $v$ from $K_v$ to $K_v(Q_v)$. There are two cases:
Good Reduction of Unitary Groups of Quaternionic Skew-Hermitian Forms

Proposition 8.1. The valuation in \( \tilde{v} \) on \( K_v(Q_v) \) defined above has the following properties.

(i) The residue field of the valuation \( \tilde{v} \), denoted by \( K_v(Q_v)_{(\tilde{v})} \) is isomorphic to \( K^{(v)}(Q^{(v)}) \), the function field of the residue division algebra \( Q^{(v)} \) over \( K^{(v)} \) (see [7]).

(ii) The ramification index \( e_{\tilde{v}/v} = 1 \).

(iii) The valuation \( \tilde{v} \) is the unique one extending \( v' \). In particular, for \( \alpha \in K_v(Q_v) \)

\[ \tilde{v}(\alpha) = \frac{1}{2}v'(\text{Norm}_{K_v(Q_v)/K_v(x)}(\alpha)) \]

(iv) For \( a_i \in \mathcal{O}_{K_v}, \tilde{v}(a_1 y + a_2 x + a_3) = 0 \) if and only if \( \min\{v(a_i)\} = 0 \).

Proof. All of the above claims are clear when \( Q_v \) splits. So assume that \( Q_v \) is not split. To prove (i), note that \( \tilde{v}(y^2) = v'(\frac{1}{t_v} - \frac{d_v}{t_v}x^2) = 0 \) since \( d_v, t_v \) are units in \( \mathcal{O}_{K_v} \). Hence \( \tilde{v}(y) = 0 \) and \( y \notin \mathcal{O}_{\tilde{v}} \). Let \( \overline{f}, \overline{g}, \overline{d_v}, \overline{t_v}, \overline{v} \) denote the corresponding residues at \( \tilde{v} \). Then we have an embedding,

\[ F := K^{(v)}(\overline{v})/((\overline{d_v} \overline{x}^2 + \overline{t_v} \overline{y}^2 - 1) \hookrightarrow K_v(Q_v)_{(\tilde{v})} \]

Now \( (\overline{d_v} \overline{x}^2 + \overline{t_v} \overline{y}^2 - 1) \) is the conic corresponding to the residue quaternion algebra \( Q^{(v)} \). So \( F \cong K^{(v)}(Q^{(v)}) \). Moreover \( Q^{(v)} \) is not split because \( \overline{Q} \) is unramified at \( \overline{v} \) and due to injectivity of \( s \) in the exact sequence of \( (7.3) \) in [7]. Therefore the conic \( (\overline{d_v} \overline{x}^2 + \overline{t_v} \overline{y}^2 - 1) \) is not hyperbolic over \( K^{(v)}(\overline{v}) \) and hence \( [K^{(v)}(Q^{(v)}) : K^{(v)}(\overline{v})] = 2 \). But since \( [K_v(Q_v)_{(\tilde{v})} : K^{(v)}(\overline{v})] \leq [K_v(Q_v) : K_v(x)] = 2 \), we conclude that \( K_v(Q_v)_{(\tilde{v})} \) is isomorphic to \( K^{(v)}(Q^{(v)}) \).

We now prove (ii) and (iii). Let \( g \) be the number of distinct valuations on \( K_v(Q_v) \) extending \( v' \). From the above argument we see that \( f_{\tilde{v}/v'} = [K_v(Q_v)_{(\tilde{v})} : K^{(v)}(\overline{v})] = 2 \). This implies \( e_{\tilde{v}/v'} = 1 \) and \( g = 1 \) from the inequality

\[ [K_v(Q_v) : K_v(x)] = e_{\tilde{v}/v'} f_{\tilde{v}/v'} g \]

Also as mentioned before \( e_{\tilde{v}/v'} = 1 \) (from Example 2.3.3 in [FJ08]). This proves that \( e_{\tilde{v}/v} = 1 \). From the fact the Galois group \( G(K_v(Q_v)/K_v(x)) \) acts transitively on the extensions on the valuation \( v' \) on \( K_v(Q_v) \) (Exercise 8, Chapter 2 in [FJ08]) and \( g = 1 \), we get (iii).
Note that by (iii) we have
\[
\tilde{v}(a_1y + a_2x + a_3) = \frac{1}{2} v'\left(\text{Norm}_{K_v(Q_v)}(a_1y + a_2x + a_3)\right) \\
= \frac{1}{2} v'\left(-a_1^2\frac{1}{t_v} - \frac{d_v}{t_v} x^2 + a_2^2 x^2 + a_3^2 + 2a_2a_3 x\right) \\
= \frac{1}{2} v'\left((a_2^2 + \frac{d_v}{t_v} a_3^2) x^2 + 2a_2a_3 x + (a_3^2 - \frac{1}{t_v} a_1^2)\right)
\]  
From this (iv) easily follows. \qed

By abuse of notation, let \( \tilde{v} \) also denote the valuation on \( K(Q) \) obtained by restriction via the embedding \( K(Q) \hookrightarrow K(Q) \otimes_K K_v \simeq K_v(Q_v) \). We now relate ramification of skew-hermitian forms over quaternions to quadratic forms.

**Theorem 8.2.** Let \( h \) be a skew-hermitian form over a quaternionic division algebra \( Q \) with center \( K \). Let \( v \) be a discrete valuation on \( K \) such that \( \text{char } K^{(v)} \neq 2 \). If \( h \) is unramified at \( v \), then the corresponding quadratic form \( q_{h,K(Q)} \) is unramified \( \tilde{v} \) where \( \tilde{v} \) is the extension of \( v \) as above.

**Proof.** Since \( h \) is unramified at \( v \), \( Q \) is necessarily unramified at \( v \). We have two cases

(i) \( Q_v \) is split. Since \( K(Q)_v \) contains \( K_v(Q_v) \) as a subfield, it suffices to show that \( q_{h,K_v(Q_v)} \) is unramified at \( \tilde{v} \). But by functoriality of Morita equivalence, we have \( q_{h,K_v(Q_v)} = q_{h_v} \otimes_{K_v} K_v(Q_v) \). Together with Proposition 7.1(i), we get that \( q_{h,K(Q)} \) is unramified \( \tilde{v} \).

(ii) \( Q_v \) is not split. With notation as in §7, \( Q_v = A_v \otimes_{\mathcal{O}_{K_v}} K_v \), where \( A_v \), the valuation ring of \( Q_v \), is a quaternionic Azumaya algebra over \( Q_{K_v} \) given by \( A_v = \langle 1, i, j, k \mid i^2 = d_v, j^2 = t_v, ij = -ji, d_v, t_v \in A_v^* \rangle \). By Proposition 7.1(ii), \( h_v \) has a matrix representation that is diagonal with diagonal entries taking values in \( A_v^* \). Consider one such representation

\[
h_v \simeq \bigoplus_{l=1}^n a_l i + b_l j + c_l i j
\]

where \( a_l i + b_l j + c_l i j \in A_v^* \). Let \( N_l = N_{rdQ}(a_l i + b_l j + c_l i j) \) be the reduced norm. Then by (7.2),

\[
0 = v(a_l i + b_l j + c_l i j) = \frac{1}{2} v(N_l) = \frac{1}{2} v(-a_l^2 d_v - b_l^2 t_v + c_l^2 d_v t_v)
\]

This implies that for each \( l \) that \( \min\{v(a_l), (b_l), (c_l)\} = 0 \). Now recall that by Example 6.2 in §6.2

\[
q_{h,K_v(Q_v)} \simeq \bigoplus_{l=1}^n \left[ \begin{array}{cc} a_l y - b_l x - c_l & 0 \\ 0 & -(a_l y - b_l x - c_l) N_l \end{array} \right]
\]

We are now done by Proposition 8.1(iv). \qed

9. **Proof of Theorem 3.1**

We begin with an easy lemma. We stick to the notations as before.

**Lemma 9.1.** Let \( Q \) be unramified at \( v \). Then we have

\[
r_2^1([Q] \cup H^{i-2}(K, \mu_2)) \subseteq [Q^{(v)}] \cup H^{i-3}(K^{(v)}, \mu_2)
\]

where \( r_2^1 \) is the residue map given by (3.2) in §3.
Proof. Since $Q$ is unramified at $v$, $r^i_2([Q]) = 0$. Therefore by the exact sequence (see Proposition 7.7 in [GMS03])

$$0 \to H^i(K^{(v)}, \mu_2) \xrightarrow{s^i_2} H^i(K_v, \mu_2) \xrightarrow{r^i_2} H^{i-1}(K^{(v)}, \mu_2) \to 0$$

we have that $[Q_v]$ is uniquely identified as the image of the residue quaternion algebra $[Q^{(v)}]$ over $K^{(v)}$ under $s^i_2$. The result now follows from Chapter II §7, Exercise 7.12 in [GMS03]. □

Now let us recall some of the facts discussed in §1.2.2 of Berhuy’s paper [Ber07]. To simplify notations, let $F(Q)$ be the function field of the Severi-Brauer variety associated to a quaternion algebra $Q$ over an arbitrary field $F$. Now consider the valuations $W$ on $F(Q)$ arising from closed points on the conic defined by $Q$. Then for every $l \geq 1$, the kernel of the corresponding residue map

$$r^i_l : H^i(F(Q), \mu_{l}^{\otimes i-1}) \to \bigoplus_{w \in W} H^{i-1}(F(Q)^{(w)}, \mu_{l}^{\otimes i-2})$$

is the unramified cohomology group with respect to the valuations in $W$ denoted by $H_{nr}^i(F(Q), \mu_{l}^{\otimes i-1})$. We now let

$$\mathbb{Q}/\mathbb{Z}(i - 1) = \lim_{\to} \mu_{l}^{\otimes i-1}$$

where the limit is taken over all the integers prime to the characteristic of $F$. Then $H^i(F(Q), \mathbb{Q}/\mathbb{Z}(i - 1))$ is the direct limit of the groups $H^i(F(Q), \mu_{l}^{\otimes i-1})$ with respect to the maps

$$H^i(F(Q), \mu_{n}^{\otimes i-1}) \to H^i(F(Q), \mu_{l}^{\otimes i-1}), m|n$$

The corresponding residue maps are compatible to each other yielding the unramified cohomology $H_{nr}^i(F(Q), \mathbb{Q}/\mathbb{Z}(i - 1))$. Moreover for any field $E$, $H^{i-1}(E, \mu_{2m}^{\otimes i-1})$ is identified with the $2^m$ torsion subgroup of $H^i(E, \mathbb{Q}/\mathbb{Z}(i - 1))$ and hence the canonical map of change of coefficients

$$(9.1) H^{i-1}(E, \mu_{2m}^{\otimes i-1}) \hookrightarrow H^{i-1}(E, \mu_{2^m}^{\otimes i-1}) m|n$$

is injective. We now recall the following results from [Ber07]

**Proposition 9.2.** (Proposition 7 and Proposition 9 in [Ber07]) Let $h$ be a skew-hermitian form over a field $F$. Then

(i) $[\gamma_{h,F(Q)}] \in H_{nr}^i(F(Q), \mu_2)$

(ii) For $i \geq 1$, the restriction map yields an isomorphism

$$\text{Res}_{F(Q)/F} : H^i(F, \mathbb{Q}/\mathbb{Z}(i - 1))/([Q] \cup H^{i-2}(F, \mu_2)) \simeq H_{nr}^i(F(Q), \mathbb{Q}/\mathbb{Z}(i - 1))$$

where $[Q] \cup H^{i-2}(F, \mu_2)$ is viewed as a subgroup of $H^i(F, \mathbb{Q}/\mathbb{Z}(i - 1))$ (if $i = 1$, $[Q] \cup H^{i-2}(F, \mu_2) = 0$). In particular, the inverse image denoted by $S$, of $2$-torsion subgroup of the unramified cohomology group $H_{nr}^i(F(Q), \mu_2)$ under $\text{Res}_{F(Q)/F}$ is a subgroup of $H^i(F, \mu_{2m}^{\otimes i-1})/([Q] \cup H^{i-2}(F, \mu_2))$.

Thus we have an isomorphism $j$ obtained by restricting $\text{Res}_{F(Q)/F}$ to $S$,

$$j : S \xrightarrow{\cong} H_{nr}^i(F(Q), \mu_2)$$

**Notation:** Let $\tilde{V}$ denote the collection of valuations $\tilde{v}$ on $K(Q)$ obtained by extending the valuations $v \in V$ on $K$ as described in §8. To simplify notations from now on let $L = K(Q)$.

We now prove that the hypothesis of Theorem 3.1 implies the following finiteness theorem.

**Theorem 9.3.** Let $(K, V)$ satisfy the hypothesis of Theorem 3.1. Then for each $i \geq 1$, the kernel of the residue map

$$R^i_2 : H_{nr}^i(L, \mu_2) \to \bigoplus_{\tilde{v} \in \tilde{V}} H^{i-1}(L^{(\tilde{v})}, \mu_2)$$
denoted by $H^n_{nr}(L, \mu_2)_{\tilde{\mathcal{V}}}$, is finite and is bounded by

$$|H^n_{nr}(L, \mu_2)_{\tilde{\mathcal{V}}}| \leq |H^{i}(K, \mu_2)^{\mathcal{V}}| \cdot |H^{i}(K, \mu_{4}^{\otimes i-1})^{\mathcal{V}}|$$

**Proof.** For each $v \in V$, the following diagram with arrows representing the natural restriction maps commutes by the functoriality.

\[
\begin{array}{c}
H^i(K, \mu_{4}^{\otimes i-1}) \quad \xrightarrow{\delta} \quad H^i(K_v, \mu_{4}^{\otimes i-1}) \\
|\quad \downarrow \quad | \\
H^i(L, \mu_{4}^{\otimes i-1}) \quad \xrightarrow{\delta} \quad H^i(L_v, \mu_{4}^{\otimes i-1}) \\
\end{array}
\]

Moreover by Chapter II, §8, Proposition 8.2 in [GMS03], the functoriality of restriction yields

\[
\begin{array}{c}
H^i(K_v, \mu_{4}^{\otimes i-1}) \quad \xrightarrow{\delta} \quad H^{i-1}(K^{(v)}, \mu_{4}^{\otimes i-2}) \\
|\quad \downarrow \quad | \\
H^i(L_v, \mu_{4}^{\otimes i-1}) \quad \xrightarrow{\delta} \quad H^{i-1}(L^{(v)}, \mu_{4}^{\otimes i-2}) \\
\end{array}
\]

where the horizontal arrows represent the residue maps. Combining the above commutative diagrams for each $v \in V$, together with Proposition 9.2, Lemma 9.1 and (9.1), we get that the following diagram commutes where $i$ is injective and $j$ is an isomorphism.

\[
\begin{array}{c}
S \xrightarrow{i} H^i(K, \mu_{4}^{\otimes i-1}) \xrightarrow{r^i_4} \bigoplus_{v \in V} \left( (Q^{(v)} \cup H^{i-2}(K, \mu_{2}))^{\mathcal{V}} \right) \\
\downarrow \quad \downarrow \quad \downarrow \\
H^i_{nr}(L, \mu_{2}) \xrightarrow{P^i_2} \bigoplus_{v \in V} H^{i-1}(L^{(v)}, \mu_{2}) \\
\end{array}
\]

\[
e_{\tilde{v}/v} = 1
\]

where $e_{\tilde{v}/v} = 1$. 


Now $L_{\tilde{v}}$ is also the completion of $K_v(Q_v)$ at $\tilde{v}$. So by Proposition 8.1(ii), for the extension $L_{\tilde{v}}/K_v$, we have ramification index $e_{\tilde{v}/v} = 1$ and the residue field $L^{(\tilde{v})} \simeq K^{(v)}(Q^{(v)})$. Hence the right vertical map is injective by Proposition 9.2.

By the hypothesis on $(K, V)$, it is easy to see that $\text{Ker } r_4^i$ is finite and

$$|H^i_{nr}(L, \mu_2)_{\tilde{v}}| = |\text{Ker } R_2^i| \leq |\text{Ker } r_4^i| = |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|$$

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1**: Since the 2-torsion subgroup of unramified division algebras denoted by $2Br(K)_V$ is isomorphic to $H^2(K, \mu_2)_V$ which is finite by hypothesis, there are only finitely many quaternion algebras unramified at all $v \in V$. So it suffices to show that for a fixed unramified $Q$ over $K$, the number of $K$-isomorphism classes of special unitary groups $SU_n(h, Q)$ of $n$-dimensional skew-hermitian forms $h$ over $Q$ that have good reduction at all $v \in V$ is upper bounded by

$$|\text{Pic}(V)/2\text{Pic}(V)| \cdot \prod_{i=1}^l |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|$$

We have two cases.

(i) $Q$ is a split quaternion. In this case $SU(h, Q) \simeq SO_{2n}(q_h, K)$. Then by the proof of Theorem 2.1 in [CRR19], we conclude that the number of $K$-isomorphism classes of $SO_{2n}(q_h, K)$ that have good reduction at all $v \in V$ is finite and bounded above by

$$|\text{Pic}(V)/2\text{Pic}(V)| \cdot \prod_{i=1}^l |H^i(K, \mu_2)_V| \leq |\text{Pic}(V)/2\text{Pic}(V)| \cdot \prod_{i=1}^l |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^{\otimes i-1})_V|$$

The above inequality is due to (9.1).

(ii) $Q$ is a quaternionic division algebra unramified at all $v \in V$. The idea of the proof in this case is to go back and forth between $h$ and $q_{h,F(Q)}$ and using arguments similar to the one in [CRR19].

**Notation:** In order to avoid notational complexity, we will be simplifying some notations as follows.

- For a skew-hermitian form $h$ over a quaternion algebra $Q$ with center a field $F$, we will make a slight abuse of notation and write $q_h$ instead of $q_{h,F(Q)}$ for the quadratic form corresponding to $h_{F(Q)}$ obtained via Morita theory.
- For a field $F$ and a class $[q] \in \mathcal{I}(F)^m$, we denote by $<q> \in H^m(F, \mu_2)$, its image under the natural map

$$\mathcal{I}(F)^m \to \mathcal{I}(F)^m/\mathcal{I}(F)^{m+1} \simeq H^m(F, \mu_2)$$

where the first map is the natural projection and the second one is the Milnor isomorphism as mentioned in (5.2).

As before let $L := K(Q)$. Let $\{h_i\}_{i \in I}$ denote a family of $n$-dimensional non-degenerate skew-hermitian forms over $Q$ such that

- for each $i \in I, G_i = SU_n(h_i, Q)$ has good reduction at all $v \in V$ and
- for $i, j \in I, i \neq j$, the forms $h_i$ and $h_j$ are not similar i.e., $h_i \napprox \lambda h_j, \lambda \in K^*$. We claim that

$$|I| \leq \prod_{i=0}^l d_i$$
where \( d_0 = \text{Pic}(V)/2\text{Pic}(V) \) and for \( 1 \leq i \leq l = [\log_2 2n] + 1 \),
\[
d_i = |H^i(K, \mu_2)^V| \cdot |H^i(K, \mu_4^\otimes i - 1)^V|
\]

Note that the above conditions imply that for each \( i \in I \) and any \( v \in V \), there exists \( \lambda_v^{(i)} \in K_v^* \) such that the form \( \lambda_v^{(i)} h_i \) over \( Q_v \) is unramified at \( v \). Also because of condition (A) on \( K \), we can assume that \( \lambda_v^{(i)} = 1 \) for almost all \( v \in V \). Recall by Lemma 2.2 in [CRR19] that there is a natural isomorphism
\[
\text{Pic}(V)/2\text{Pic}(V) \simeq \mathbb{I}(K, V)/\mathbb{I}(K, V)^2 \mathbb{I}_0(K, V) K^*
\]
which is the group of idèles and
\[
\mathbb{I}_0(K, V) = \prod_{v \in V} \mathcal{O}_v^*
\]
is the subgroup of integral idèles.

So \( \lambda^{(i)} := (\lambda_v^{(i)})_{v \in V} \in \mathbb{I}(K, V) \). Since \( d_0 \) is finite by hypothesis, using (9.2), we conclude that there exists a subset \( J_0 \subseteq I \) of size \( \geq I/d_0 \) (if \( I \) is infinite so is \( J_0 \)) such that all \( \lambda_v^{(i)}, i \in J_0 \) have the same image in \( \mathbb{I}(K, V)/\mathbb{I}(K, V)^2 \mathbb{I}_0(K, V) K^* \). Fix \( j_0 \in J_0 \). For any \( j \in J_0 \), we can write
\[
\lambda^{(j)} = \lambda^{(j_0)}(\alpha^{(j)})^2 \beta^{(j)} \delta^{(j)}
\]
with \( \alpha^{(j)} \in \mathbb{I}(K, V), \beta^{(j)} \in \mathbb{I}_0(K, V) \) and \( \delta^{(j)} \in K^* \). Then set
\[
H_j = \delta^{(j)} h_j
\]
\[
\Lambda_j = (\delta^{(j)})^{-1} \lambda^{(j)} = \lambda^{(j_0)}(\alpha^{(j)})^2 \beta^{(j)}
\]
It is easy to see that for \( j \neq j', H_j \not\subseteq H_{j'} \) and hence \( q_{H_j} \not\subseteq q_{H_j'} \) as quadratic forms over \( L \) (see [6.1]). Moreover, \( \Lambda_v^{(j)} H_j = \lambda_v^{(j)} h_j \) and hence \( \Lambda_v^{(j)} H_j \) is unramified \( v \). Therefore by Theorem 8.2
\[
q_{\Lambda_v^{(j)} H_j} \in W_0(L_{\tilde{v}}), \ j \in J_0
\]
(See [5.2] for the definition of \( W_0(L_{\tilde{v}}) \)). Also note that
\[
q(j, \tilde{v}) := \Lambda_v^{(j_0)}(q_{H_j} \perp -q_{H_{j_0}})
\]
\[
= \Lambda_v^{(j_0)} \cdot (\Lambda_v^{(j)})^{-1} \cdot \Lambda_v^{(j)} q_{H_j} \perp -\Lambda_v^{(j_0)} q_{H_{j_0}}
\]
\[
= (\alpha^{(j)})^{-2} (\beta^{(j)})^{-1} q_{\Lambda_v^{(j)} H_j} \perp -q_{\Lambda_v^{(j_0)} H_{j_0}} \quad \text{(by Remark 6.3)}
\]
\[
\cong (\beta^{(j)})^{-1} q_{\Lambda_v^{(j)} H_j} \perp -q_{\Lambda_v^{(j_0)} H_{j_0}}
\]
As \( (\beta^{(j)})^{-1} \in \mathcal{O}_v^* \), we see that
\[
[q(j, \tilde{v})] = \Lambda_v^{(j_0)}([q_{H_j} \perp -[q_{H_{j_0}}]]) \in W_0(L_{\tilde{v}}) \cap \mathcal{I}(L_v)
\]
Now by Lemma 3.3 in [CRR19], we get \( < q_{H_j} > = < q_{H_{j_0}} > \in H^1(L_{\tilde{v}}, \mu_2) \) is unramified at \( v \). Since \( v \) was arbitrary we conclude that
\[
< q_{H_j} > = < q_{H_{j_0}} > \in H^1(L_{\mu_2})_{\tilde{v}}
\]
Then we can find a subset $J_1 \subseteq J_0$ of size $\geq |J_0|/d_1 \geq |I|/d_0d_1$ such that for $j \in J_1$, the classes $[qH_j] - [qH_{j_0}] \in \mathcal{I}(L)$ all have the same image in $H^1(L, \mu_2)$. Fix $j_1 \in J_1$. Then for any $j \in J_1$, we have

$$< qH_j > - < qH_{j_1} > = ( < qH_j > - < qH_{j_0} > ) - ( < qH_{j_1} > - < qH_{j_0} > ) = 0 \in H^1(L, \mu_2)$$

Hence $[qH_j] - [qH_{j_1}] \in \mathcal{I}(L)^2$. Moreover

$$\Lambda^{(j)} = \Lambda^{(j_1)}(\overline{\alpha}^{(j)})^2(\overline{\beta}^{(j)}), \quad \overline{\alpha}^{(j)} \in \Pi(K, V), \overline{\beta}^{(j)} \in \Pi_0(K, V)$$

Then as before we conclude that for every $\tilde{v} \in \tilde{V}$

$$\Lambda_v^{(j_1)}([qH_j] \perp [qH_{j_1}]) \in W_0(L_{\tilde{v}}) \cap \mathcal{I}(L_{\tilde{v}})^2$$

Again by using Lemma 3.3 in [CRRT19], we get

$$< qH_j > - < qH_{j_1} > \in H^2(L, \mu_2)_{\tilde{v}}$$

Then there exists a subset $J_2 \subseteq J_1$ of size $\geq |I|/d_1d_1d_2$ such that for each $j \in J_2$, the classes $[qH_j] - [qH_{j_1}]$ have the same image in $H^2(L, \mu_2)$. Fixing $j_2 \in J_2$, we have

$$[qH_j] - [qH_{j_2}] \in \mathcal{I}(L)^3 \quad j \in J_2$$

Proceeding inductively we get a nested chain of subsets

$$I \supseteq J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_l$$

such that for any $m = 1, 2, \cdots, l$,

- $|J_m| \geq |I|/d_0d_1 \cdots d_m$ and
- for $j \in J_m$, we have

$$[qH_j] - [qH_{j_0}] \in \mathcal{I}(L)^{m+1} \quad j \in J_m$$

But by a theorem of Arason and Pfister ([AP71], also see [Lam05], Chapter X, Hauptsatz 5.1), the dimension of any positive dimensional anisotropic form in $\mathcal{I}(K)^{l+1}$ is $\geq 2^{l+1} > 2^{\log_2 n} = 4n$. Thus $[qH_j] - [qH_{j_0}] \in \mathcal{I}(L)^{l+1}$ implies that $qH_j \equiv qH_{j_0}, \forall j \in J_l$. But as seen before the forms $qH_j$ are pairwise inequivalent. Hence we conclude that $|J_l| = 1$ and

$$|I| \leq |Pic(V)/2Pic(V)| \cdot \prod_{i=1}^l |H^i(K, \mu_2)V| \cdot |H^i(K, \mu_4^{\otimes i})_V|$$

ACKNOWLEDGEMENTS

I thank Andrei Rapinchuk for the many useful discussions I had with him while he was at the Institute for Advanced Study, which inspired the research presented in this paper. I would also like to thank Daniel Krashen for the fruitful conversations on this topic and for his feedback on this work. Finally I am very grateful to Pierre Deligne for suggesting some corrections and making insightful remarks on the results which not only enhanced the quality of this manuscript but also helped me understand math better.

REFERENCES

[AP71] Jón Kristinn Arason and Albrecht Pfister. Beweis des Krullschen Durchschnittsatzes für den Wittring. Invent. Math., 12:173–176, 1971.

[Ber07] Grégory Berhuy. Cohomological invariants of quaternionic skew-Hermitian forms. Arch. Math. (Basel), 88(5):434–447, 2007.

[Con15] Brian Conrad. Non-split reductive groups over $\mathbb{Z}$. In Autours des schémas en groupes. Vol. II, volume 46 of Panor. Synthèses, pages 193–253. Soc. Math. France, Paris, 2015.

[CRRT13] Vladimir I. Chernousov, Andrei S. Rapinchuk, and Igor A. Rapinchuk. The genus of a division algebra and the unramified Brauer group. Bull. Math. Sci., 3(2):211–240, 2013.
[CRR16a] Vladimir I. Chernousov, Andrei S. Rapinchuk, and Igor A. Rapinchuk. On some finiteness properties of algebraic groups over finitely generated fields. *C. R. Math. Acad. Sci. Paris*, 354(9):869–873, 2016.

[CRR16b] Vladimir I. Chernousov, Andrei S. Rapinchuk, and Igor A. Rapinchuk. On the size of the genus of a division algebra. *Tr. Mat. Inst. Steklova*, 292(Algebra, Geometriya i Teoriya Chisel):69–99, 2016. Reprinted in *Proc. Steklov Inst. Math.* 292 (2016), no. 1, 63–93.

[CRR19] Vladimir I. Chernousov, Andrei S. Rapinchuk, and Igor A. Rapinchuk. Spinor groups with good reduction. *Compos. Math.*, 155(3):484–527, 2019.

[FJ08] Michael D. Fried and Moshe Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, third edition, 2008. Revised by Jarden.

[GMS03] Skip Garibaldi, Alexander Merkurjev, and Jean-Pierre Serre. *Cohomological invariants in Galois cohomology*, volume 28 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[Gro96] Benedict H. Gross. Groups over \( \mathbb{Z} \). *Invent. Math.*, 124(1-3):263–279, 1996.

[Jac40] N. Jacobson. A note on hermitian forms. *Bull. Amer. Math. Soc.*, 46:264–268, 1940.

[JL15] A. Javanpeykar and D. Loughran. Good reduction of algebraic groups and flag varieties. *Arch. Math. (Basel)*, 104(2):133–143, 2015.

[JW90] Bill Jacob and Adrian Wadsworth. Division algebras over Henselian fields. *J. Algebra*, 128(1):126–179, 1990.

[Knu91] Max-Albert Knus. *Quadranatic and Hermitian forms over rings*, volume 294 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni.

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Mil70] John Milnor. Algebraic \( K \)-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970.

[OVV07] D. Orlov, A. Vishik, and V. Voevodsky. An exact sequence for \( K^M_*/2 \) with applications to quadratic forms. *Ann. of Math. (2)*, 165(1):1–13, 2007.

[PSS01] R. Parimala, R. Sridharan, and V. Suresh. Hermitian analogue of a theorem of Springer. *J. Algebra*, 243(2):780–789, 2001.

[Rap19] Igor A. Rapinchuk. A generalization of Serre’s condition (F) with applications to the finiteness of unramified cohomology. *Math. Z.*, 291(1-2):199–213, 2019.

[Sal99] David J. Saltman. *Lectures on division algebras*, volume 94 of *CBMS Regional Conference Series in Mathematics*. Published by American Mathematical Society, Providence, RI; on behalf of Conference Board of the Mathematical Sciences, Washington, DC, 1999.

[Sch85] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[Voe03] Vladimir Voevodsky. Motivic cohomology with \( \mathbb{Z}/2 \)-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (98):59–104, 2003.

[Wad02] A. R. Wadsworth. Valuation theory on finite dimensional division algebras. In *Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999)*, volume 32 of *Fields Inst. Commun.*, pages 385–449. Amer. Math. Soc., Providence, RI, 2002.

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON NJ, USA - 08540

E-mail address: srimathy@ias.edu