A lattice fermion without doubling

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Abstract

A free fermion without doubler is formulated on 1+D dimensional discrete Minkowski space-time. The action is not hermitian but causes no harm. In 1+3 dimensional massless case the equation describes a single species of Dirac particle in the continuous space-time limit. In 1+1 dimensional massless case the equation is the same as the automaton equation by 't Hooft and describes a chiral fermion. The time evolution operator is unitary and the norm is conserved. For interacting fermions with gauge fields the evolution operator is not unitary. If it is considered as an approximation for the theory on continuous space-time, the path integral formalism can be applied, where the fermion without doubling is used. Consequences of loosening the unitarity condition on the time evolution operator is discussed.

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I. INTRODUCTION

As is well known, we necessarily meet the difficulty of so-called fermion doubling when we formulate a fermion on lattice space-time. Nielsen-Ninomiya proved under certain assumptions that a chiral fermion cannot be formulated on Kogut-Susskind lattice (=continuous time and discrete space) [1]. At present it is widely believed that fermion doubling is inevitable on discrete space-time in general. In order to get rid of this unwanted partner of the fermion Wilson introduced an additional term in Lagrangian to make the mass of the partner very large [2]. Hence the partner has no effect at low energy. Another attempt is one by Susskind [3]. He showed these two fermions could be interpreted as isodoublets (=fermions of two different flavours).

In the present paper we want to show that a fermion can be formulated without doubling on Minkowski lattice space-time (Sec.2). In order to avoid the no-go theorem we do not assume the hermiticity of action which was one of the assumptions of Nielsen-Ninomiya. The non-hermitian action may lead to some problems, but in our case it seems not so serious. For example, the action and the hermitian conjugate action yield two consistent field equations. This means both the real and imaginary parts of the action take stationary values at the same time, if the equation is satisfied. However, it is not yet clear whether the non-hermiticity of the action produces further difficulties.

The equation we obtained on 1+D dimensional discrete Minkowski space-time has no fermion doubling solution. In 1+3 dimensions it tends to Dirac equation in the continuous limit. The time evolution operator is unitary and therefore, the norm is conserved. In 1+1 dimensions a chiral fermion can be described in massless case which is the same as the automaton model of ’t Hooft [4]. However, in 1+3 dimensions a completely chiral fermion cannot be described as in the case of Susskind. This situation is not improved even if the unitarity of time evolution operator is not assumed (Sec.4).

In the above mentioned we treated only a free fermion field. In the case of interacting fields, however, the situation is completely different. It is difficult to assume the unitarity
of the time evolution operator. If we stand on the viewpoint that the discrete space-time is an approximation of continuous case, we may disregard this point, because the unitarity of time evolution operator is recovered in the continuous space-time limit. In the lattice gauge theory which is formulated on Euclidean space-time, the unitarity of time evolution operator is not considered seriously. Using the method of path integral formalism we can calculate any quantities in principle without the effect of fermion doubling, though the non-hermiticity might bring some practical difficulties (Sec.3).

II. NON-INTERACTING FERMIONS

In this section we develop a lattice theory of non-interacting fermions without doublers in $1 + D$ dimensional discrete space-time. We take a hyper-rectangular lattice with lattice constants $\tau$ and $\sigma$ for the time and space direction, respectively. Lattice points are represented by a set of $1 + D$ integers $(t, x^1, \ldots, x^D) \equiv (t, \mathbf{x}) \equiv x$. At each site is attached a spinor variable $\Psi(x)$, whose number of components is not specified for the moment.

A. Equations of motion

We assume that equations of motion for $\Psi$ take the form:

$$\Psi(t + 1, \mathbf{x}) = U \Psi(t, \mathbf{x}),$$

where $U$ is the time evolution operator acting on the $\mathbf{x}$ and spinor space. Postulating some properties on $U$, we will determine $U$ and the minimal dimension of the spinor space for $\Psi$.

The time evolution operator $U$ is assumed to be linear in the unit shift operators in the $\mathbf{x}$-space so that only the nearest neighbor sites are coupled in the action, which will be given in the next subsection. Thus we have

$$U = \sum_{i=1}^{D} \left( A_i S_i + B_i S_i^\dagger \right) + C,$$

where the unit shift operators $S_i$’s are defined by
and $A_i$, $B_i$ and $C$ are matrices with respect to spinor indices.

We furthermore assume that the time evolution operator is unitary:

$$UU^\dagger = 1.$$ (2.4)

This condition is necessary for the theory to be well-defined quantum theory on discrete time. However, if the theory is considered to be an approximation to the continuum theory, we could loosen the condition in such a way that the unitarity should be recovered only in the continuum limit. In this section we assume the unitarity, and consequences of loosening the condition will be discussed in Sec. 4.

We require that each component of $\Psi$ satisfies the discrete version of the Klein-Gordon equation:

$$\frac{U - 2 + U^{-1}}{\tau^2} - \sum_{i=1}^{D} \frac{S_i - 2 + S_i^\dagger}{\sigma^2} + M^2 = 0,$$ (2.5)

where $M$ is the hermitian mass matrix. The dispersion relation implied by this equation is given by

$$\frac{4}{\tau^2} \sin^2 \frac{k^0}{2} - \frac{4}{\sigma^2} \sum_{i=1}^{D} \sin^2 \frac{k_i}{2} - M^2 = 0,$$ (2.6)

where $k^0$ and $k$ are introduced through the Fourier transform:

$$\Psi(t, x) = \int_{-\pi}^{\pi} dk^0 dk e^{-i(k^0 t - k x)} \Psi_{k^0 k}.$$ (2.7)

Evidently there is no doubling problem with Eq. (2.6), which has also been pointed out by Yamamoto \[5\]. Therefore, it is this equation (2.5) which assures that no doublers appear in our formalism. In the following we assume $M = 0$ and at the end of this subsection we will discuss the case of $M \neq 0$.

Substituting $U$ given by Eq. (2.2) in the Klein-Gordon equation (2.3), where $U^{-1}$ is replaced by $U^\dagger$, we find the relations:
\[ A_i + B_i^\dagger = r, \]
\[ A_i^\dagger + B_i = r, \]
\[ C + C^\dagger - 2 = -2D, \quad (2.8) \]

where \( r \) is defined as \( (\tau/\sigma)^2 \).

On the other hand the unitarity condition \( (2.4) \) requires that
\[ A_i B_j^\dagger + A_j B_i^\dagger = 0, \]
\[ A_i A_j^\dagger + B_j B_i^\dagger = 0, \quad \text{(for } i \neq j \text{)} \]
\[ A_i C^\dagger + C B_i^\dagger = 0, \]
\[ \sum_{i=1}^{D} (A_i A_i^\dagger + B_i B_i^\dagger) + CC^\dagger = 1. \quad (2.9) \]

Combining those relations \( (2.8) \) and \( (2.9) \), we find that \( U \) takes the following form:
\[ U = \sum_{i=1}^{D} \left\{ \frac{r}{2} (1 + a_i) S_i + \frac{r}{2} (1 - a_i^\dagger) S_i^\dagger \right\} + (1 - Dr) + i C_I, \quad (2.10) \]

where the operators in the spinor space \( a_i \)'s and \( C_I \) (the imaginary part of \( C \)) are yet to be determined by the following algebra:
\[ \{a_i, a_j\} = 2, \]
\[ \{a_i, C_I\} = -2i (1 - Dr), \]
\[ \{a_i, a_j^\dagger\} = -2, \quad \text{(for } i \neq j \text{)} \]
\[ \sum_{i=1}^{D} \{a_i, a_i^\dagger\} = D \left( \frac{8}{r} - 4D - 2 \right) \frac{4}{r^2} C_I^2. \quad (2.11) \]

Here a set of curly brackets \( \{ , \} \) is an anticommutator.

Using this algebra, we find that \( C_I \) is not linearly independent of \( a_i \) but expressed as
\[ C_I = i \frac{r}{2} \sum_{i=1}^{D} (a_i - a_i^\dagger), \quad (2.12) \]

which can be readily verified by showing the square of the difference between both sides of the equation vanishes. The relation \( (2.12) \) has an interesting consequence on the lattice constants \( \tau \) and \( \sigma \). Calculating \( C_I^2 \) by Eq. \( (2.12) \), one finds
\[ C_f^2 = D r (1 - D r), \] (2.13)

which should be positive definite since \( C_I \) is hermitian. Thus we have

\[ r \equiv \left( \frac{\tau}{\sigma} \right)^2 \leq \frac{1}{D}. \] (2.14)

Yamamoto has obtained this inequality as well as the more general one mentioned below involving the finite mass \([5]\). The argument is that the \( k^0 \) should be real in the dispersion relation (2.6), which is consistent with our requirement of the unitarity of \( U \).

It is convenient to decompose \( a_i \) into the real and imaginary part as \( a_i = X_i + i Y_i \), where \( X_i \) and \( Y_i \) are hermitian. By Eqs. (2.12) and (2.13) the algebra for \( X_i \) and \( Y_i \) is reduced to be

\[
\{X_i, X_j\} = 2 \frac{1}{r} \delta_{ij},
\]

\[
\{X_i, Y_j\} = 0,
\]

\[
\{Y_i, Y_j\} = 2 \left( \frac{1}{r} \delta_{ij} - 1 \right). \tag{2.15}
\]

Although the anticommutation relation between \( X_i \)'s is diagonal with respect to the indices \( i \) and \( j \), the one between \( Y_i \)'s is not. The matrix \( 2(\delta_{ij}/r - 1) \), however, can be easily diagonalized. The eigenvalues are \( 2(1/r - D) \) for the eigenvector \((1,1,\ldots,1)\) and \( 2/r \) for any vectors orthogonal to \((1,1,\ldots,1)\). Thus the latter eigenvalue is \((D-1)\)-fold degenerate.

Therefore, when \( r < 1/D \), we have \( 2D \) non-zero hermitian operators whose anticommutation relation is diagonal. Namely \( X_i \) and \( Y_i \), after the diagonalization, form the Clifford algebra up to trivial normalization factors:

\[
\Gamma_i^\dagger = \Gamma_i, \quad i = 1, \ldots, N
\]

\[
\{\Gamma_i, \Gamma_j\} = 2 \delta_{ij} \tag{2.16}
\]

with \( N \) being \( 2D \) in this case. So the dimension of the irreducible representation for \( X \) and \( Y \) is \( 2^D \) and \( \Psi(x) \) should have \( 2^D \) components at least.

In the case of \( r = 1/D \), where \( r \) takes the maximal value allowed by Eq. (2.14), one of the eigenvalues \( 2(1/r - D) \) vanishes and the linear relation \( \sum Y_i = 0 \) holds. In this case
we have the Clifford algebra with $N = 2D - 1$, whose irreducible representation has the dimension of $2^{D-1}$. Consequently $\Psi(x)$ has $2^{D-1}$ components.

Assuming $r = 1/D$, so that $\Psi(x)$ has the smaller number of components $2^{D-1}$, let us investigate the explicit form of the time evolution operator $U$ for the cases $D = 1$ and $D = 3$.

1+1 dimensional case

$\Psi(t, x^1)$ is one-component field and $U$ is given by

$$U = \frac{1}{2}(1 + X_1)S_1 + \frac{1}{2}(1 - X_1)S_1^\dagger,$$  \hspace{1cm} (2.17)

where $X_1^2 = 1$. When $X_1 = 1$, we have $U = S_1$ and the equation of motion is simply $\Psi(t + 1, x^1) = \Psi(t, x^1 + 1)$, which clearly simulates without doublers the chiral left moving fermion $(\partial_0 - \partial_1)\Psi = 0$ in the 1+1 continuum theory. It is interesting that the correspondence is established by replacing the time derivative $\partial_0 \Psi$ by the forward difference quotient but the spatial derivative $\partial_1 \Psi$ by the backward one. This equation of motion has also been given by ’t Hooft et al. in the cellular automaton approach [4]. In the same way, the case of $X_1 = -1$ describes the right moving chiral fermion without doublers.

1+3 dimensional case

The ratio of lattice constants $\tau/\sigma = \sqrt{r}$ is given by $1/\sqrt{3}$ and $\Psi(t, \mathbf{x})$ is 4-component field. The time evolution operator $U$ then takes the form:

$$U = 1 + \sum_{i=1}^{D} \tau \Gamma_i \frac{S_i - S_i^\dagger}{2\sigma} + \sum_{i=1}^{D} \frac{1}{6}(1 + iY_i)(S_i - 2 + S_i^\dagger),$$  \hspace{1cm} (2.18)

where $Y_i$’s are given by

$$Y_1 = \sqrt{2}\Gamma_4,$$

$$Y_2 = -\sqrt{\frac{1}{2}}\Gamma_4 + \sqrt{\frac{3}{2}}\Gamma_5,$$

$$Y_3 = -Y_1 - Y_2.$$  \hspace{1cm} (2.19)

More explicitly those $\Gamma_i$’s $(i = 1, \ldots, 5)$ are expressed by the Dirac matrices:
\[ \Gamma_i = \gamma^0 \gamma^i \quad (i = 1, 2, 3), \quad \Gamma_4 = \gamma^0, \quad \Gamma_5 = \gamma^5. \quad (2.20) \]

The second term in the time evolution operator (2.18) is reduced to \( \tau \gamma^0 \gamma^i \partial_i \) in the continuum limit. The last term in \( U \) goes to zero as \( \tau^2 \) in the naïve continuum limit, and it reminds us of the Wilson term except for the spinor operators \( Y_i \) in front. It is this spinor dependence that eliminates doublers completely, whereas the Wilson term only gives doublers a large mass.

Now we briefly mention only the results for the case of \( M \neq 0 \), which is more involved. Instead of the algebra (2.11), we have

\[
\begin{align*}
\{a_i, a_j\} &= 2, \\
\{a_i, C_I\} &= -2i C_R, \\
\{a_i, a_i^\dagger\} &= -2, \quad \text{(for } i \neq j) \\
\sum_{i=1}^{D} \{a_i, a_i^\dagger\} &= (1 - C^\dagger C) \frac{4}{r^2} - 2D, \quad (2.21)
\end{align*}
\]

and the real part of \( C \) is given by \( C_R = (1 - Dr) - \tau^2 M^2 / 2 \). An interesting consequence is that \( M^2 \) commutes with \( a_i \) and \( C_I \), and therefore, \( M^2 \) is just a number in the irreducible representation. We can furthermore show that \( r \leq (1 - \tau^2 M^2 / 4) / D \) and even if the equality holds, \( \Psi(x) \) has \( 2^D \) components at least.

**B. Action**

We have shown that one can construct equations of motion for \( \Psi(x) \) without doublers by the time evolution operator \( U \) of the form (2.2), which is local and unitary. Now we construct the action on the lattice which leads to the equations of motion.

We immediately see that the desired equations of motion derive from the variation with respect to \( \Psi^\dagger \) of the action:

\[
S = i \sum_{t, x} \Psi^\dagger(t + k, x) (\Psi(t + 1, x) - U \Psi(t, x)), \quad (2.22)
\]
where \( k \) is an arbitrary integer. The above action is evidently local but not hermitian. It should be noted, however, that the variation of the action with respect to \( \Psi \) also gives the equivalent equations of motion, which is a consequence of the unitarity of \( U \).

As we have seen in the last subsection, the single chiral fermion can safely reside in 1+1 dimensional discrete space-time. This does not contradict the Nielsen-Ninomiya theorem, since our action is local but not hermitian.

We will now show that the quantization through the path integral with the action (2.22) is equivalent to the equations of motion (2.1) with the canonical equal-time commutation relations \( \{ \hat{\Psi}(t, x), \hat{\Psi}^\dagger(t, x') \} = \delta_{xx'} \). Here and in what follows, \( \hat{\Psi} \) and \( \hat{\Psi}^\dagger \) should be understood to be canonically quantized fields whereas \( \Psi \) and \( \Psi^\dagger \) are Grassmannian variables.

First we introduce the time evolution operator \( \hat{U} \) in the fermion Fock space, which should not be confused with \( U \), as follows:

\[
\hat{U} = \exp \left( -i \sum_x \hat{\Psi}^\dagger(t, x) H \hat{\Psi}(t, x) \right),
\]

where the hermitian operator \( H \) in the \( x \) and spinor space is defined through \( U = \exp(-iH) \). The \( \hat{U} \) is evidently unitary and reproduces the equations of motion:

\[
\hat{\Psi}(t + 1, x) = \hat{U}^{-1} \hat{\Psi}(t, x) \hat{U} = U \hat{\Psi}(t, x).
\]

We also find that \( \hat{U} \) is independent of \( t \).

Now we calculate the matrix element of \( \hat{U} \) between fermion coherent states \(| \Psi(\cdot) \rangle \) and \(| \Psi'(\cdot) \rangle \). Here the coherent states are defined as

\[
| \Psi(\cdot) \rangle \equiv \exp \left( - \sum_x \Psi(x) \hat{\Psi}^\dagger(0, x) \right) | 0 \rangle, \quad \hat{\Psi}(0, x) | 0 \rangle = 0.
\]

The matrix element of \( \hat{U} \) can be readily calculated as

\[
< \Psi'(\cdot) | \hat{U} | \Psi(\cdot) > = \exp \left( \sum_x \Psi^\dagger(x) U \Psi(x) \right).
\]

Noting that the completeness relation of the coherent states is given by

\[
1 = \int \prod_x [d\Psi^\dagger(x) d\Psi(x)] \exp \left( - \sum_x \Psi^\dagger(x) \Psi(x) \right) | \Psi(\cdot) > < \Psi(\cdot) |,
\]
for the transition amplitude from the initial state $|\Psi(t_0, \cdot)\rangle$ to the final state $|\Psi(t_n, \cdot)\rangle$ we have

$$< \Psi(t_n, \cdot) | \hat{U}^n | \Psi(t_0, \cdot) > = \exp \left( \sum_{x} \Psi^\dagger(t_n, x) \Psi(t_n, x) \right) \int D\Psi D\bar{\Psi} \exp \left( - \sum_{t=t_0, x} \Psi^\dagger(t + 1, x) (\Psi(t + 1, x) - U \Psi(t, x)) \right),$$

(2.28)

which establishes the equivalence of the two ways of quantization after redefining $\Psi^\dagger(t + k, x)$ as $\Psi^\dagger(t + 1, x)$ in the action (2.22).

Finally we will give the explicit form of the propagator, which is the inverse of the kernel in the action (2.22). Defining the unit shift operator in the time direction as $S_0 \Psi(t, x) = \Psi(t + 1, x)$, we have

$$< T \Psi(x) \Psi^\dagger(x') > = \left( \frac{1}{1 - S_0^\dagger U} \right)_{x,x'}
= \left( \frac{S_0 - U^\dagger}{(S_0 - 2 + S_0^\dagger) - (U - 2 + U^\dagger)} \right)_{x,x'}.$$

(2.29)

Evidently the propagator has no extra poles, since in the momentum space its denominator takes the form of the left hand side of the dispersion relation (2.6), due to the Klein-Gordon equation (2.3).

**III. INTERACTION WITH GAUGE FIELDS**

We have seen in the previous section that the lattice fermion without doubling can be formulated for the free case in 1+D dimensions. In this section we consider the interaction of the fermion with gauge fields.

The interaction is introduced by replacing the shift operators by covariant ones:

$$S_0 \rightarrow S_0(x) \equiv V(x, x + \hat{0})S_0,$$

(3.1)

$$S_i \rightarrow S_i(x) \equiv V(x, x + i\hat{i})S_i,$$

(3.2)
where $\hat{0}$ and $\hat{i}$ are unit vectors along the time and $i$-th space direction, respectively, and $V(x, y)$ is a link variable connecting sites $x$ and $y$. After these replacements the action is written as

$$S = i \sum_x \Psi^\dagger(x) \{ 1 - S_0^\dagger(x) U(x) \} \Psi(x) + S_{\text{gauge}},$$

(3.3)

where $U(x)$ is defined by replacing $S_i$ with $S_i(x)$ in Eq. (2.2) and $S_{\text{gauge}}$ is the action for gauge fields. It is easily seen that the action is invariant under the lattice gauge transformation with a gauge group element $g(x)$:

$$\Psi(x) \rightarrow g(x) \Psi(x),$$

(3.4)

$$V(x, y) \rightarrow g(x) V(x, y) g(y)^{-1}.$$

(3.5)

It is also checked that the new action has the correct continuum limit.

Using this new action and path integral formalism, we can calculate the vacuum expectation value of an arbitrary operator $O$ by the formula:

$$< O > = \int \mathcal{D} \Psi^\dagger \mathcal{D} \Psi \mathcal{D} V \ O \exp(iS) \ / \ \int \mathcal{D} \Psi^\dagger \mathcal{D} \Psi \mathcal{D} V \exp(iS).$$

(3.6)

In the usual case the gauge interaction is introduced in an analogous way using the symmetric discretization of time. But the unitarity of the time evolution of fermion fields is not clear for the symmetric discretization, since the field at time $t + 1$ depends on fields at $t$ and $t - 1$. Usually this is not considered seriously because the unitarity is recovered in the continuum limit. However, in our case we have the explicit time evolution operator for a finite lattice spacing, and therefore, we can go a little further in the discussion of unitarity.

We assume the Klein-Gordon condition and the unitarity condition also in this case. The Klein-Gordon condition gives the same relations Eq. (2.8) for $A_i$, $B_j$ and $C$ as in the free case. From the unitarity requirement we have the following relations:

$$A_i B_j^\dagger V(x, x + \hat{i}) V(x + \hat{i}, x + \hat{i} + \hat{j})$$

$$+ A_j B_i^\dagger V(x, x + \hat{j}) V(x + \hat{j}, x + \hat{i} + \hat{j}) = 0,$$

(3.7)
\[ A_i A_j^\dagger V(x, x + \hat{i})V(x + \hat{i}, x + \hat{i} - \hat{j}) \]
\[ + B_j B_i^\dagger V(x, x - \hat{j})V(x - \hat{j}, x + \hat{i} - \hat{j}) = 0, \quad (i \neq j) \quad (3.8) \]

\[ A_i C_i^\dagger + CB_i^\dagger = 0, \]

\[ \sum_{i=1}^{D} (A_i A_i^\dagger + B_i B_i^\dagger) + CC^\dagger = 1, \quad (3.10) \]

which correspond to Eq. (2.9) in the free case.

The last two relations are the same as before, but the first two contain link variables and depend on sites. The site dependence of the conditions means the unitarity of the time evolution operator \( U(x) \) cannot be satisfied by simple algebraic relations between \( A \) and \( B \) in general.

**1+1 dimensional case**

In 1+1 dimensions the relations reduce to the previous ones owing to the unidimensionality of space. The unitarity of the time evolution operator is kept by the same choice of \( A \) and \( B \) as in the free case. So the path integral formulation is well established keeping the unitarity at a finite lattice spacing.

**1+D dimensional case with \( D \geq 2 \)**

We cannot determine \( A \) and \( B \) keeping the unitarity of the time evolution operator in this case. If we adopt the relations (2.9) for \( A \) and \( B \) in the free case, we obtain from the two relations (3.7) and (3.8)

\[ (A_i A_j^\dagger - B_j B_i^\dagger) \{ V(x, x + \hat{i})V(x + \hat{i}, x + \hat{i} - \hat{j}) \]
\[ - V(x, x - \hat{j})V(x - \hat{j}, x + \hat{i} - \hat{j}) \} = 0, \quad (i \neq j) \quad (3.11) \]
\[(A_i B_j^\dagger - A_j B_i^\dagger)\{V(x, x + \hat i)V(x + \hat i, x + \hat i + \hat j)
- V(x, x + \hat j)V(x + \hat j, x + \hat i + \hat j)\} = 0. \quad (3.12)\]

We can see the origin of the unitarity violation comes from the difference between clockwise and anti-clockwise parallel transports around a plaquette. This factor is higher order in the lattice spacing $\sigma$ and expected to be harmless in the continuum limit.

**IV. DISCUSSION ON UNITARITY**

In Sec.2 we imposed the unitarity condition (2.4) and the Klein-Gordon condition (2.5) on the time evolution operator $U$ of the spinor field

\[U = \sum_{i=1}^{D} (A_i S_i + B_i S_i^\dagger) + C. \quad (4.1)\]

We could not find the spinor field on discrete space-time with the same dimension as one on continuous space-time.

In this section we try to find the spinor field with lower dimension than the one in Sec.2 without imposing the unitarity condition, since the Klein-Gordon condition ensures that the spinor field has no doubling.

As in Sec.2 we assume the Klein-Gordon equation:

\[U^{-1} - 2 + U = r \sum_{i=1}^{D} (S_i - 2 + S_i^\dagger) - \tau^2 M^2, \quad (4.2)\]

where we supposed the existence of inverse matrix for the time evolution operator. From the above equation we can see that the inverse matrix $U^{-1}$ is of the same form as $U$:

\[U^{-1} = \sum_{i=1}^{D} (\bar A_i S_i + \bar B_i S_i^\dagger) + C. \quad (4.3)\]

Generally the term proportional to $(S_i^{(i)})^n \ (n > 1)$ may be included in $U^{-1}$, but these terms are not allowed, because in Eq. (4.2) there is no term which cancels out those terms.

From Eq. (4.2) we obtain the following conditions:
\[ A_i + \bar{A}_i = r, \]
\[ B_i + \bar{B}_i = r, \]  \hspace{1cm} (4.4)
\[ C + \bar{C} = K, \]

where \( K \) is
\[ K = 2(1 - rD) - \tau^2 M^2, \]  \hspace{1cm} (4.5)

and from \( UU^{-1} = 1 \) we obtain
\[ A_i \bar{A}_j + A_j \bar{A}_i = 0, \quad \text{(for all } i, j) \]
\[ B_i \bar{B}_j + B_j \bar{B}_i = 0, \quad \text{(for all } i, j) \]
\[ A_i \bar{C} + C \bar{A}_i = 0, \quad \text{(for all } i) \]  \hspace{1cm} (4.6)
\[ B_i \bar{C} + C \bar{B}_i = 0, \quad \text{(for all } i) \]
\[ A_i B_j + B_j \bar{A}_i = 0, \quad (i \neq j) \]
\[ \sum_{i=1}^{D} (A_i \bar{B}_i + B_i \bar{A}_i) + C \bar{C} = 1. \]

We find that the matrix \((C - K/2)^2\) commutes with \( A_i, B_i \) and \( C \), and we can regard this matrix as the unit matrix up to a constant factor \( DQ \):
\[ (C - \frac{K}{2})^2 = DQ \mathbb{I}. \]  \hspace{1cm} (4.7)

Using Eqs. (4.4) ∼(4.7) and assuming the isotropy of spatial lattice directions we can get the anticommutation relations between \( A_i, B_i \) and \( C \)
\[ \{M_a, M_b\} = F_a \delta_{a,b} \mathbb{I}, \]  \hspace{1cm} (4.8)

where
\[
F_a = \begin{cases} 
-2 \left( \frac{K^2}{4D} - \frac{1}{D} - Q \right), & (1 \leq a \leq D) \\
2 \left( \frac{K^2}{4D} - \frac{1}{D} - Q \right), & (D + 1 \leq a \leq 2D - 1) \\
\lambda_+, & (a = 2D) \\
\lambda_-, & (a = 2D + 1) 
\end{cases}
\]  \hspace{1cm} (4.9)
and $\lambda_- \pm$ are eigenvalues of the $2 \times 2$ matrix $\mathcal{M}$, which is defined by anticommutation relation between $C - K/2$ and $\sum_{i=1}^{D} (A_i + B_i)/\sqrt{D}$

$$
\mathcal{M} = \begin{pmatrix}
\{\sum_{i=1}^{D} \frac{(A_i + B_i)}{\sqrt{D}}, \sum_{i=1}^{D} \frac{(A_i + B_i)}{\sqrt{D}}\} & \{\sum_{i=1}^{D} \frac{(A_i + B_i)}{\sqrt{D}}, C - \frac{K}{2}\} \\
\{C - \frac{K}{2}, \sum_{i=1}^{D} \frac{(A_i + B_i)}{\sqrt{D}}\} & \{C - \frac{K}{2}, C - \frac{K}{2}\} \\
\end{pmatrix}
= \begin{pmatrix}
2(Dr^2 - Q + \frac{K^2 - 4}{4D}) \sqrt{DKr} \\
\sqrt{DKr} & 2DQ \\
\end{pmatrix}.
$$

(4.10)

Clearly $\lambda_- \pm$ are real since the matrix $\mathcal{M}$ is hermitian. The matrices $M_a$'s are defined by

$$
M_a = \begin{cases}
A_a - B_a, & (1 \leq a \leq D) \\
\sum_{i=1}^{D} \xi_{a-D}^i (A_i + B_i), & (D + 1 \leq a \leq 2D - 1) \\
\eta_+^{(1)} \sum_{i=1}^{D} \xi_{0}^i (A_i + B_i) - \sqrt{Dr} + \eta_+^{(2)} (C - \frac{K}{2}), & (a = 2D) \\
\eta_-^{(1)} \sum_{i=1}^{D} \xi_{0}^i (A_i + B_i) - \sqrt{Dr} + \eta_-^{(2)} (C - \frac{K}{2}), & (a = 2D + 1)
\end{cases}
$$

(4.11)

where $\eta_\pm^{(p)}$ are the eigenvectors of $\mathcal{M}$ with eigenvalues $\lambda_\pm$:

$$
\mathcal{M} \begin{pmatrix}
\eta_+^{(1)} \\
\eta_+^{(2)} \\
\eta_-^{(1)} \\
\eta_-^{(2)}
\end{pmatrix} = \begin{pmatrix}
\lambda_+ \\
\lambda_+
\end{pmatrix}.
$$

(4.12)

and the $\xi_i$'s, ($\alpha = 0, 1, ..., D - 1$) are orthogonal vectors:

$$
\sum_{i=1}^{D} \xi_{\alpha}^i \xi_{\beta}^i = \delta_{\alpha, \beta}.
$$

(4.13)

We do not give the definite form of these vectors except for

$$
\xi_0^i = \frac{1}{\sqrt{D}} \begin{pmatrix}
1 \\
1 \\
1 \\
\cdot \\
\cdot \\
1
\end{pmatrix},
$$

(4.14)
since we do not need those forms in the later discussion.

From these equations we obtain

$$\{ \hat{M}^\dagger_a, \hat{M}^\dagger_b \} = F_a \delta_{ab} \mathbb{1}. \quad (4.15)$$

There is no condition which prescribes the anticommutation relations between $M_a$ and $M_a^\dagger$. For simplicity we assume the following anticommutation relation:

$$\{ M_a, M_b^\dagger \} = D_{ab} \mathbb{1}. \quad (4.16)$$

Noticing that $M_a$ tends to a linear combination of $\gamma$ matrices in continuous space-time limit, we can easily see that the case with $Q = (K^2/4 - 1)/D$ is undesired. In this case all matrices $M_a$ vanish in the continuous space-time limit except for the two matrices $M_{2D}$ and $M_{2D+1}$, although we need $D + 1$ matrices in order that the equation of spinor field goes to Dirac equation in the limit $\sigma, \tau \to 0$.

We consider the case with $F_a \neq 0 \ (a = 1, ..., 2D + 1)$. It is convenient to use

$$\tilde{M}_a = \frac{M_a}{\sqrt{F_a}}$$

$$= \tilde{X}_a + i \tilde{Y}_a. \quad (4.17)$$

The anticommutation relations between these matrices become

$$\left\{ \begin{pmatrix} \tilde{X}_a \\ \tilde{Y}_a \end{pmatrix}, \begin{pmatrix} \tilde{X}_b \\ \tilde{Y}_b \end{pmatrix} \right\} = \begin{pmatrix} \delta_{ab} + G_{ab} & F_{ab} \\ -F_{ab} & G_{ab} \end{pmatrix}, \quad (4.18)$$

where

$$G_{ab} = -\frac{1}{2} (\delta_{ab} - \tilde{D}_{ab} - \tilde{D}_{ab}^* ) \mathbb{1},$$

$$F_{ab} = \frac{1}{2i} (\tilde{D}_{ab} - \tilde{D}_{ba}^* ) \mathbb{1}, \quad (4.19)$$

and $\tilde{D}_{ab}$ is

$$\tilde{D}_a = \frac{D_{ab}}{\sqrt{F_a F_b}}. \quad (4.20)$$
As the eigenvalues of matrix $\mathcal{G}_{ab}$ are non-negative, the matrix $\delta_{ab} + \mathcal{G}_{ab}$ is invertible. Using the inverse matrix of $\delta_{ab} + \mathcal{G}_{ab}$, we define

$$\tilde{Y}'_a = \sum_{b=1}^{\mathcal{F}(\mathcal{I} + \mathcal{G})^{-1}}_{ab} X_b + Y_a,$$

(4.21)

and then we have

$$\left\{ \begin{pmatrix} \tilde{X}_a \\ \tilde{Y}'_a \end{pmatrix}, \left( \begin{pmatrix} \tilde{X}_b \\ \tilde{Y}'_b \end{pmatrix} \right) \right\} = \begin{pmatrix} \delta_{ab} + \mathcal{G}_{ab} & 0 \\ 0 & \mathcal{D}_{ab} + (\mathcal{F}(\mathcal{I} + \mathcal{G})^{-1})_{ab} \end{pmatrix}. $$

(4.22)

As the upper-left submatrix of the right hand side of the above equation is invertible, we cannot make the number of independent matrices $X_a$ and $Y_a$ smaller than $2D+1$. Thus the dimension of these matrices is larger than $2^D$ (for example, the lower limit of the dimension is 8 for $D = 3$).

If $\lambda_+$ and $\lambda_-$ are zero, the dimension of matrices is equal to or larger than $2^{D-1}$. If either $\lambda_+$ or $\lambda_-$ is zero, the dimension of matrices is equal to or larger than $2^D$. For example, the limit of the dimension is 4 or 8 for $D = 3$. The massless spinor field in Sec.2 corresponds to the former case.

In conclusion we found under the assumption (4.16) that the dimension of spinor field without the unitarity condition is not smaller than the dimension with the condition.

V. SUMMARY

We have formulated a free fermion without doubling on 1+D dimensional Minkowski lattice space-time. We required there the unitarity of time evolution operator and Klein-Gordon equation on lattice space-time. This means the norm is conserved and the fermion has no doubler. We showed that the minimal number of components of massless field $\Psi$ is $2^{D-1}$. In 1+1 dimensional case the equation is the same as that of the cellular automaton by ’t Hooft et al. In 1+3 dimensional massless case the time evolution operator was expressed in an explicit form using usual $\gamma$ matrices, which tends to Dirac operator in the continuous space-time limit.
The action is not hermitian. We proved the equivalence of canonical quantization and that through the path integral. We have given the explicit form of the fermion propagator, which has no extra poles of doublers.

In the case where the fermion interacts with gauge fields the action was also written in a gauge invariant form. The vacuum expectation value of an arbitrary operator is calculated by the path integral formulation in principle. The time evolution operator is not unitary in this case, and the reason was considered briefly.

We have tried to find the spinor field with lower dimension than $2^{D-1}$ without imposing the unitarity condition. However, we found the spinor field with the smallest dimension is $2^{D-1}$ under a certain condition.
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