Tensor methods for strongly convex strongly concave saddle point problems and strongly monotone variational inequalities

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Abstract In this paper we propose two $p$-th order tensor methods for $\mu$-strongly-convex-strongly-concave saddle point problems. The first method is based on the assumption of $L_p$-smoothness of the gradient of the objective and it achieves a convergence rate of $O((L_p R_p^p / \mu)^{2p+1} \log(\mu R^2 / \varepsilon))$, where $R$ is an estimate of the initial distance to the solution. Under additional assumptions of $L_1$, $L_2$ and $L_p$-smoothness of the gradient of the objective we connect the first method with a locally superlinear converging algorithm and develop a new method with the complexity of $O((L_p R_p^p / \mu)^{2p+1} + \log \log(L_3^3 / 2\mu^2 \varepsilon))$. Since we treat saddle-point problems as a particular case of variational inequalities, we also propose two methods for strongly monotone variational inequalities with the same complexity as the described above.

Keywords Variational inequality · Saddle point problem · High-order smoothness · Tensor methods

1 Introduction

In this work we focus on the classic min-max saddle point problem:

$$\min_{x \in X} \max_{y \in Y} g(x, y),$$

where $g : X \times Y \to \mathbb{R}$ is a convex over $X$ and concave over $Y$, and the sets $X, Y$ are convex. This is a particular case of a more general problem, called

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monotone variational inequality (MVI). In MVI we have a monotone operator 
\( F : Z \rightarrow \mathbb{R}^n \) over a convex set \( Z \subset \mathbb{R}^n \) and we need to find 
\[
Z^* \in Z : \forall z \in Z, (F(z), z^* - z) \leq 0.
\] (2)

If we set \( Z = X \times Y \) and 
\( F(z) = (\nabla_x g(x, y), -\nabla_y g(x, y)) \), then MVI is equivalent 
to the min-max saddle point problem (1).

In this paper we consider unconstrained min-max saddle point 
problem (1) with \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^m \). Additionally, we assume 
\( g(x, y) \) is \( \mu \)-strongly 
convex in \( x \in \mathbb{R}^n \) and \( \mu \)-strongly concave in \( y \in \mathbb{R}^m \).

There is a number of papers on numerical methods for saddle point 
problem (1) in convex-concave setting \([12,17,19,26,27]\). One of the most popular among 
first-order methods for this setting is the Mirror-Prox algorithm \([17]\), 
which treats saddle-point problems via solving the corresponding MVI. According 
to \([18]\), this method achieves optimal complexity of 
\( O(1/\varepsilon) \) iterations for first-
order methods applied to smooth convex-concave saddle point problems in 
large dimensions.

Additional assumption of strong convexity and strong concavity leads to 
better results. The algorithms from \([5,14,22,23,26]\) achieve iteration 
complexity of \( O(L/\mu \log(1/\varepsilon)) \). In \([13]\) the authors proposed an algorithm with 
complexity \( O(L/\sqrt{\mu_x \mu_y} \log^3(1/\varepsilon)) \), which matches up to a logarithmic factor 
the lower bound, obtained in \([28]\). It worths to mention that \( \log^3(1/\varepsilon) \) factor 
can be improved, namely, it is possible to achieve iteration complexity of 
\( O(L/\sqrt{\mu_x \mu_y} \log(1/\varepsilon)) \) (see \([5]\)).

The methods listed above use first-order oracles, and it is known from 
opimization that tensor methods, which use higher-order derivatives, 
have faster convergence rate, yet for the price of more expensive iteration. The idea 
of using derivatives of high order in optimization is not new (see \([9]\)). The most 
common type of high-order methods use second-order oracles, for example 
Newton method \([20,23]\) and its modifications such as the cubic regularized 
Newton method \([21]\). Recently the idea of exploiting oracles beyond the second 
order started to attract increased attention, especially in convex optimization 
\([1,3,4,6,7]\).

However, much less is known on high-order methods for saddle point prob-
lems and MVIs. In \([16]\) the authors propose a second-order method based on 
their Hybrid Proximal Extragradient framework \([15]\). The resulting complexity 
is \( O(1/\varepsilon^{3/2}) \). A recent work \([2]\) shows how to modify Mirror-Prox method using 
oracles beyond second order and improves complexity to reach duality gap \( \varepsilon \) to 
\( O(1/\varepsilon^{3/2}) \) for convex-concave problems with \( p \)-th order Lipshitz derivatives. 
The paper \([10]\) proposes a cubic regularized Newton method for solving sad-
dle point problem, which has global linear and local superlinear convergence 
rate if \( \nabla g(x, y) \) and \( \nabla^2 g(x, y) \) are Lipschitz-continuous and \( g(x, y) \) is strongly 
convex in \( x \) and strongly concave in \( y \).

In our work we make a next step and propose a Tensor method for 
strongly monotone variational inequalities and, as a corollary, a Tensor method for sad-
dle point problems with strongly-convex-strongly-concave objective. Standing
on the ideas from [2] and [10], our work can be split into two parts. Firstly, we apply restart technique [25] to the HighOrderMirrorProx Algorithm 1 from [2], which is possible thanks to strong convexity and strong concavity of the objective. Such a modification improves the algorithm complexity to 
\[O((L_p R_p / \mu)^{2p+1} \log(\mu R^2/\varepsilon)),\]
where \(R\) is an upper bound for the initial distance to the solution \(\| (x_1, y_1) - (x^*, y^*) \|_2\), and \(L_p\) is the Lipschitz constant of the \(p\)-th derivative. Secondly, using an estimate of the area of local super-linear convergence, when the algorithm reaches this area, we switch to the Cubic-Regularized Newton Algorithm 3 from [10] to obtain local super-linear convergence of our algorithm. The total complexity of the final Algorithm 4 becomes
\[O((L_p R_p / \mu)^{2p+1} + \log \log(L_3/2\mu^2\varepsilon)).\]
We want to emphasize, that the obtained \(\log \log(1/\varepsilon)\) dependency on \(\varepsilon\) cannot be improved even in convex optimization [11].

Our paper is organized as follows. First of all, in Section 2 we provide necessary notations, assumptions (Section 2.1) and lemmas (Section 2.2). Then, we present the new algorithm and obtain its convergence rate in Section 3. In Section 3.1 we talk only about restarted algorithm from [2] and get its complexity. Then, in Section 3.2 we describe how to connect it to Algorithm 3 from [10] in its quadratic convergence area and get the final Algorithm 4 convergence rate. Finally, in Section 4 we discuss our results and present some possible directions for future work.

## 2 Preliminaries

We use \(z \in \mathbb{R}^n \times \mathbb{R}^m\) to denote the pair \((x, y), \nabla^p g(z)[h_1, ..., h_p], p \geq 1\) to denote directional derivative of \(g\) at \(z\) along directions \(h_i \in \mathbb{R}^n \times \mathbb{R}^m, i = 1, ..., p\). The norm of the \(p\)-th order derivative is defined as
\[\|\nabla^p g(z)\|_2 = \max_{h_1, ..., h_p \in \mathbb{R}^n \times \mathbb{R}^m} \{|\nabla^p g(z)[h_1, ..., h_p]| : \|h_i\|_2 \leq 1, i = 1, ..., p\}\]
or equivalently
\[\|\nabla^p g(z)\|_2 = \max_{h \in \mathbb{R}^n \times \mathbb{R}^m} \{|\nabla^p g(z)[h]^p| : \|h\|_2 \leq 1\}.
\]
Here we denote \(\nabla^p g(z)[h, ..., h]\) as \(\nabla^p g(z)[h]^p\). Also here and below \(\| \cdot \|_2\) is a Euclidean norm for vectors.

### 2.1 Assumptions

As mentioned earlier, we consider unconstrained saddle point problem

\[
\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} g(x, y)
\]

with strongly-convex-strongly-concave and \(p\)-times differentiable objective \(g\).
Assumption 1 \( g(x, y) \) is \( \mu \)-strongly convex in \( x \) and \( \mu \)-strongly concave in \( y \).

Recall that the definition of strong convexity and strong concavity is as follows.

Definition 1 \( g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) is called \( \mu \)-strongly convex and \( \mu \)-strongly concave if
\[
\forall x_1, x_2 \in \mathbb{R}^n, \ y \in \mathbb{R}^m \ (\nabla_x g(x_1, y) - \nabla_x g(x_2, y), x_1 - x_2) \geq \mu \|x_1 - x_2\|^2_2 , \tag{3}
\]
\[
\forall y_1, y_2 \in \mathbb{R}^m, \ x \in \mathbb{R}^n \ (-\nabla_y g(x, y_1) + \nabla_y g(x, y_2), y_1 - y_2) \geq \mu \|y_1 - y_2\|^2_2 . \tag{4}
\]

The problem (1) is usually solved in terms of the duality gap
\[
\Phi_{X \times Y}(x, y) := \max_{y' \in Y} g(x, y') - \min_{x' \in X} g(x', y) . \tag{5}
\]
Since in our case \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^m \), we drop the notations of these sets from index of the duality gap and denote duality gap just as \( \Phi(x, y) \).

Before showing the connection between problem (1) and MVI (2) we need the definition of strong monotonicity.

Definition 2 \( F : Z \rightarrow \mathbb{R}^n \) is strongly monotone if
\[
\langle F(z_1) - F(z_2), z_1 - z_2 \rangle \geq \mu \|z_1 - z_2\|^2_2 . \tag{6}
\]

Denote \( z = \begin{pmatrix} x \\ y \end{pmatrix} \), and operator \( F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m \):
\[
F(z) = F(x, y) := \begin{pmatrix} \nabla_x g(x, y) \\ -\nabla_y g(x, y) \end{pmatrix} . \tag{7}
\]

According to these definitions, the min-max problem (1) can be tackled via solving the MVI problem (2) with the specific operator \( F \) given in (7). In our work we use the following assumptions.

Assumption 2 \( F(z) \) satisfies first order Lipschitz condition (\( L_1 \)-smooth):
\[
\|F(z_1) - F(z_2)\|_2 \leq L_1 \|z_1 - z_2\|_2 . \tag{8}
\]

Assumption 3 \( F(z) \) satisfies second order Lipschitz condition (\( L_2 \)-smooth):
\[
\|\nabla F(z_1) - \nabla F(z_2)\|_2 \leq L_2 \|z_1 - z_2\|_2 . \tag{9}
\]

Assumption 4 \( F(z) \) satisfies \( p \)-th order Lipschitz condition (\( L_p \)-smooth):
\[
\|\nabla^{p-1} F(z_1) - \nabla^{p-1} F(z_2)\|_2 \leq L_p \|z_1 - z_2\|_2 . \tag{10}
\]
2.2 Auxiliary lemmas, notations and propositions

To be consistent with the works \cite{2} and \cite{10} we need to introduce some additional lemmas and notations. One of the key components of Algorithm 1 is the $p$-th order Taylor approximation of $F$ at $z$:

$$T_{p,z}(\hat{z}) := \sum_{i=1}^{p} \frac{1}{i!} \nabla^i F(z)[\hat{z} - z]^i.$$  

Since our goal is an approximate solution to MVI, we define its $\varepsilon$-approximate solution as

$$z^* \in \mathcal{Z} : \forall z \in \mathcal{Z} \langle F(z), z^* - z \rangle \leq \varepsilon.$$  \hspace{1cm} (11)

At the same time the bounds of Algorithm 1 is of the form

$$\forall z \in \mathcal{Z}, \frac{1}{T} \sum_{t=1}^{T} \gamma_t \langle F(z_t), z_t - z \rangle \leq \varepsilon,$$  \hspace{1cm} (12)

where the points $z_t$ are produced by algorithm, $\gamma_t$ are some positive weights and $T = \sum_{t=1}^{T} \gamma_t$. The following lemma establishes the relation between (11) and (12).

\textbf{Lemma 1 (Lemma 2.7 from \cite{2})} Let $F : \mathcal{Z} \to \mathbb{R}^n$, be monotone, $z_t \in \mathcal{Z}, t = 1, ..., T$, and let $\gamma_t > 0$. Let $\bar{z}_t = \frac{1}{T} \sum_{t=1}^{T} \gamma_t z_t$. Assume (12) holds. Then $\bar{z}_t$ is an $\varepsilon$-approximate solution to MVI problem (2), which is sometimes called "weak MVI", is closely connected to strong MVI problem, where we need to find

$$z^* \in \mathcal{Z} : \forall z \in \mathcal{Z}, \langle F(z^*), z^* - z \rangle \leq 0.$$  \hspace{1cm} (13)

If $F$ is continuous and monotone, the problems (2) and (13) are equivalent.

By $D : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^n$ we denote Bregman divergence induced by a function $d : \mathcal{Z} \to \mathbb{R}$, which is continuously-differentiable and 1-strongly convex. The definition of Bregman divergence is

$$D(z_1, z_2) := d(z_1) - d(z_2) - \langle \nabla d(z_2), z_1 - z_2 \rangle.$$  

In our paper we use half of squared Euclidean distance as Bregman divergence

$$D(z_1, z_2) = \frac{1}{2} \|z_1 - z_2\|^2.$$  \hspace{1cm} (14)

A unique solution $z^*$ to a saddle point problem \cite{11} exists, and $F(z^*) = 0$, owing to strong convexity and strong concavity of $g(x,y)$. Thus, we can use the following merit function during analysis of Algorithm 3 complexity.

$$m(z) := \frac{1}{2} \|F(z)\|^2 = \frac{1}{2} \|\nabla_x g(x,y)\|^2 + \|\nabla_y g(x,y)\|^2.$$  \hspace{1cm} (15)

This merit function is introduced in \cite{10}.

Finally, we use the relation between the merit function $m(z)$ and the duality gap under assumptions \cite{11} and \cite{2}.
Algorithm 1: HighOrderMirrorProx [Algorithm 1 in [2]]

1: Input $z_1 \in \mathbb{Z}, p \geq 1, 0 < \varepsilon < 1, T > 0$.
2: for $t = 1$ to $T$ do
3: Determine $\gamma_t, \hat{z}_t$ such that:
\begin{align*}
\hat{z}_t &= \arg \min_{z \in \mathbb{Z}} \{ \gamma_t(T_p, z_t(z)), z - z_t \} + D(z, z_t), \\
\frac{p!}{32L_p \|z_t - z_t\|_2^{p-1}} &\leq \gamma_t \leq \frac{p!}{16L_p \|z_t - z_t\|_2^{p-1}}, \\
z_{t+1} &= \arg \min_{z \in \mathbb{Z}} (\gamma_t F(\hat{z}_t), z - \hat{z}_t) + D(z, z_t).
\end{align*}
4: Define $I_T \overset{\text{def}}{=} \sum_{t=1}^T \gamma_t$
5: return $\bar{z}_T \overset{\text{def}}{=} \frac{1}{I_T} \sum_{t=1}^T \gamma_t \hat{z}_t$.

Proposition 1 (Proposition 2.5 from [10]) Let assumptions [1] and [2] hold. For problem (1) and any point $z = (x, y)$ the duality gap (5) and the merit function (15) satisfy the following inequalities
\begin{equation}
\frac{\mu}{L_1} m(z) \leq \Phi(x, y) \leq \frac{L_1}{\mu^2} m(z).
\end{equation}

3 Main results

Firstly, in this section we propose the algorithm for finding $\varepsilon$-approximate solution to problem (2), where $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ is $L_p$-smooth and $\mu$-strongly monotone operator (assumptions [4] and [1]), which allows to achieve iteration complexity of $O \left( \left( \frac{L_p R^p}{\mu} \right) \log \frac{\mu R^2}{\varepsilon} \right)$, where $R \geq \|z_1 - z^*\|_2$. This algorithm is a restarted modification of Algorithm 1.

Secondly, we develop the algorithm for tackling the same problem, with $F$ being $L_1, L_2$ and $L_p$-smooth and $\mu$-strongly monotone operator (all assumptions [1], [2], [3], [4]). It involves the idea of exploiting previous algorithm until it reaches the area of quadratic convergence of Algorithm 3 and then switching to it. This algorithm allows to achieve iteration complexity of $O \left( \left( \frac{L_p R^p}{\mu} \right)^{\frac{p-1}{p}} \log \frac{L_1 L_2 R}{\mu^2} + \log \frac{\frac{L_1^2}{\mu^2}}{\frac{\mu^2}{\varepsilon}} \right)$.

3.1 Restarted HighOrderMirrorProx

First of all, we need lemma about convergence rate of the Algorithm 1.

Lemma 2 (Lemma 4.1 from [2]) Suppose $F : \mathbb{Z} \rightarrow \mathbb{R}^n$ is $p$th-order $L_p$-smooth and let $I_T = \sum_{t=1}^T \gamma_t$. Then, the iterates $\{\hat{z}_t\}_{t \in [T]}$ generated by Algorithm 1, satisfy for all $z \in \mathbb{Z} \subset \mathbb{R}^n$, 

Algorithm 2 Restarted HighOrderMirrorProx

1: Input $z_1 \in \mathbb{Z}, p \geq 1, 0 < \varepsilon < 1, R$ such that $R \geq \|z_1 - z^*\|_2$.
2: $k = 1$
3: $\hat{z} = z_1$
4: for $i \in [n]$, where $n = \left\lceil \frac{1}{2} \log \frac{\mu R^2}{\varepsilon} \right\rceil$
do
5: Set $T = \left\lfloor \frac{R^2}{2} \left( \frac{64}{L_p} \frac{\mu R^2}{\varepsilon} \right)^{\frac{1}{p+1}} \right\rfloor$
6: Run Algorithm 1 from $\hat{z}$ for $T$ iterations
7: $\bar{z} = \check{z}_T$
8: return $\check{z}$

\[
\frac{1}{T} \sum_{t=1}^{T} (\gamma_t F(\hat{z}_t), \hat{z}_t - z) \leq \frac{16L_p}{p!} \left( \frac{D(z, z_1)}{T} \right)^{\frac{p+1}{p}}. \tag{17}
\]

From Lemma 1 we know, that $\hat{z}_t = \frac{1}{1} \sum_{t=1}^{T} \gamma_t z_t$, where $z_t$ and $\gamma_t$ are from Algorithm 1 is an $\varepsilon$-solution to regular MVI \(\text{(11)}\), if the right hand side of \(\text{(17)}\) is smaller than $\varepsilon$. Hence, it is also a solution to a convex-concave saddle point problem. The natural way to improve the method for convex (concave) problem in tighter strongly convex (strongly concave) setting is to use restarts \[25\]. As a result we obtain Algorithm 2.

**Theorem 1** Suppose $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, that is defined in \(\text{(7)}\), is $L_p$-smooth and $\mu$-strongly monotone (Assumptions \[7\] and \[4\] hold). Denote $R$ such that $R \geq \|z_1 - z^*\|_2$. Then Algorithm 2 complexity is

\[
O \left( \frac{L_p R^p}{\mu} \log \frac{\mu R^2}{\varepsilon} \right). \tag{18}
\]

**Proof** From \(\text{(13)}\) and \(\text{(12)}\) we get the following:

\[
\sum_{t=1}^{T} \gamma_t (F(\hat{z}_t) - F(z^*); \hat{z}_t - z^*) \leq \frac{16L_p}{p!} \left( \frac{\|\hat{z}_t - z^*\|_2^2}{2T} \right)^{\frac{p+1}{p}}. \tag{19}
\]

From this and the fact that $F(x)$ is $\mu$-strongly monotone we have

\[
\mu \|\bar{z}_T - z^*\|_2 \leq \frac{\mu}{T} \sum_{t=1}^{T} \gamma_t \|\hat{z}_t - z^*\|_2^2 \leq \frac{1}{T} \sum_{t=1}^{T} \gamma_t (F(\hat{z}_t) - F(z^*); \hat{z}_t - z^*) \tag{20}
\]

\[
\leq \frac{16L_p}{p!} \left( \frac{\|\hat{z}_t - z^*\|_2^2}{2T} \right)^{\frac{p+1}{p}},
\]

where \(\text{(*)}\) follows from convexity of $\|z\|_2^2$. 

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Algorithm 3 CRN-SPP [Algorithm 1 in [10]]

1: \textbf{Input} $z_0, \varepsilon, \bar{\gamma} > 0, \rho, \alpha \in (0, 1)$, $g$ satisfies Assumptions 1 and 2.
2: \textbf{while} $m(z_k) > \varepsilon$ \textbf{do}
3: \hspace{1em} $\gamma_k = \bar{\gamma}$
4: \hspace{1em} \textbf{while} True \textbf{do}
5: \hspace{2em} Solve the subproblem $(\tilde{x}_{k+1}, \tilde{y}_{k+1}) = \arg\min_x \max_y g_k(x, y; \gamma_k)$
6: \hspace{2em} if $\gamma_k(\|\tilde{x}_{k+1} - x_k\| + \|\tilde{y}_{k+1} - y_k\|) > \mu$ then
7: \hspace{3em} $\gamma_k = \rho \gamma_k$
8: \hspace{2em} else \textbf{break}
9: \hspace{1em} $d_k = (\tilde{x}_{k+1} - x_k; \tilde{y}_{k+1} - y_k)$
10: \hspace{1em} if $m(z_k + ad_k) < m(z_k + d_k)$ then
11: \hspace{2em} $z_{k+1} = z_k + d_k$, $k = k + 1$
12: \hspace{1em} \textbf{else if} $m(z_k + ad_k) \geq m(z_k + d_k)$ then
13: \hspace{2em} $z_{k+1} = z_k + d_k$
14: \hspace{1em} \textbf{return} $z_k$

Now we restart the method every time the distance to solution decreases twice. Let $T$ be such that $\|\bar{z}_T - z^*\|_2 \leq \frac{\|z_1 - z^*\|_2}{2}$ and denote $R$ such that $R \geq \|z_1 - z^*\|_2$. Then the number of iterations before next restart is

$$\mu \|\bar{z}_T - z^*\|_2^2 \leq \frac{\mu R^2}{4} \leq \frac{16L_p}{p^\mu} \left(\frac{R^2}{2T}\right)^{\frac{1}{p+1}} \Leftrightarrow T \leq \left[\frac{R^2}{2} \left(\frac{64L_p}{p^\mu R}\right)^{\frac{1}{p+1}}\right].$$

Next we need to obtain the number of restarts, required to achieve the desired accuracy

$$\varepsilon = \frac{1}{T_{\text{max}}} \sum_{t=1}^{T_{\text{max}}} \gamma_t (F(\tilde{z}_t) - F(z^*)) \geq \mu \|\bar{z}_{T_{\text{max}}} - z^*\|_2^2 = \mu \left(\frac{\|z_1 - z^*\|_2^2}{2^n}\right) \Leftrightarrow n \geq \frac{1}{2} \log \frac{\mu R^2}{\varepsilon} \Leftrightarrow n = \left\lceil \frac{1}{2} \log \frac{\mu R^2}{\varepsilon}\right\rceil.$$

Finally, the total number of iterations is

$$N = T n = \left\lceil\frac{R^2}{2} \left(\frac{64L_p}{p^\mu R}\right)^{\frac{1}{p+1}}\right\rceil \cdot \left\lceil\frac{1}{2} \log \frac{\mu R^2}{\varepsilon}\right\rceil = O\left(\frac{L_p R^p}{\mu}\right)^{\frac{1}{p+1}} \log \frac{\mu R^2}{\varepsilon}.$$

This completes the proof. \hfill \qed

3.2 Local quadratic convergence

As a next step we introduce Algorithm 3 and theorem, that proves its local quadratic convergence.
Algorithm 4 Repeated HighOrderMirrorProx with local quadratic convergence

1: Input $z_1 \in X, p \geq 1, 0 < \varepsilon < 1, R = \|z_1 - z^*\|_2$.
2: $k = 1$
3: $\tilde{z} = z_1$
4: for $i \in [n]$, where $n = \left\lceil \log \frac{L_1L_2R}{\mu^2} \right\rceil$ do
5: Set $T = \left\lfloor \frac{\mu^2}{2} \left( \frac{64L_p}{\mu^2} \right)^p \right\rfloor$
6: Run Algorithm 1 from $\tilde{z}$ for $T$ iterations
7: $\bar{z}_T$
8: Run Algorithm 3 from $\tilde{z}$ while $m(z_k) > \tilde{\varepsilon} = \frac{\mu^2}{L_1L_2}$
9: return $z_k$

Theorem 2 (Theorem 3.6 from [10]) Suppose $F : X \to \mathbb{R}^n$ is $L_1$- and $L_2$-smooth $\mu$-strongly monotone operator (assumptions 1, 2 and 3 hold). Let $\{z_k\}$ be generated by Algorithm 3 with $\tilde{\gamma} = \frac{L_1L_2}{\mu^2}$, then

$$\exists K > 0 : \forall k \geq K \|z_{k+1} - z^*\|_2 \leq \frac{L_1L_2}{\mu^2} \|z_k - z^*\|_2.$$ (21)

The area of quadratic convergence is

$$\|z_k - z^*\|_2 \leq \frac{\mu^2}{L_1L_2}. \quad (22)$$

Our idea is to use Algorithm 3 until it reaches this area and then switch to Algorithm 4, which in the end defines Algorithm 4. From Proposition 1, our Theorem 1 and Theorem 2, we obtain the complexity of Algorithm 4.

Theorem 3 Suppose $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, that is defined in (7), is $L_1$-, $L_2$- and $L_p$-smooth and $\mu$-strongly monotone operator (all assumptions 1, 2, 3, 4 hold). Denote $R$ such that $R \geq \|z_1 - z^*\|_2$. Then the complexity of Algorithm 3 is

$$O \left( \left( \frac{L_pR}{\mu} \right)^{\frac{1}{p+1}} \log \frac{L_1L_2R}{\mu^2} + \log \frac{\log \frac{L_1L_2R}{\mu^2}}{\log \frac{L_1L_2R}{\mu^2}} \right). \quad (23)$$

Proof First of all, we need to find the number of iterations of Algorithm 2 to reach the area of local quadratic convergence of Algorithm 3

$$\|\tilde{z}_n - z^*\|_2 \leq \frac{\mu^2}{L_1L_2} \Leftrightarrow \frac{\|\tilde{z}_1 - z^*\|_2}{2^n} \leq \frac{\mu^2}{L_1L_2} \Leftrightarrow n = \left\lceil \log \frac{L_1L_2R}{\mu^2} \right\rceil.$$

Next we switch to Algorithm 3 and we need to obtain its number of iterations until convergence. Denote by $\varepsilon'$ the accuracy of solution in terms of the merit function (15), Owing to $L_1$-smoothness of $F(z)$ and the fact that $F(z^*) = 0$ we can get that

$$\varepsilon' = m(z_k) = \frac{1}{2} \|F(z_k)\|_2^2 = \frac{1}{2} \|F(z_k) - F(z^*)\|_2^2 \leq \frac{L_1^2}{2} \|z_k - z^*\|_2^2. \quad (24)$$
Now we establish a connection between the solution in terms of merit function $m(z)$ and the duality gap $\Phi(x, y)$. From (24) and (16) we get the following:

$$
\varepsilon = \Phi(x, y) = \max_{y' \in \mathbb{R}^n} f(x, y') - \min_{x' \in \mathbb{R}^n} f(x', y) \leq \frac{L_1}{\mu^2} m(z_k) = \frac{L_1}{\mu^2} \varepsilon' \Leftrightarrow \frac{\mu^2 \varepsilon}{L_1} \leq \varepsilon'.
$$

(25)

Then, from (21), (22), (24) and (25) we can obtain the needed number of iterations $k$

$$
\frac{\mu^2 \varepsilon}{L_1} \leq \frac{L_2}{2} \|z_k - z^\star\|_2^2 \leq \frac{L_2}{\mu^2} \left( \frac{L_1 L_2}{\mu^2} \|z_{k-2} - z^\star\|_2^2 \right)^2 \leq \cdots
$$

$$
\leq \frac{L_2}{2} \left( \frac{L_1 L_2}{\mu^2} \right)^{2^k - 2} \|z_1 - z^\star\|_2^2 \leq \frac{L_2}{2} \left( \frac{L_1 L_2}{\mu^2} \right)^{2^{k-1} - 2} \left( \frac{\mu^2}{L_1 L_2} \right)^{2^k}
$$

$$
\Leftrightarrow \frac{2 \mu^2 \varepsilon}{L_1^2} \leq \left( \frac{\mu^2}{L_1 L_2} \right)^{2^k - 1 + 2} \Leftrightarrow \log \frac{2 \mu^2 \varepsilon}{L_1} \leq (2^{k-1} + 2) \log \frac{\mu^2}{L_1 L_2}
$$

Since $\log(\mu^2/L_1 L_2) < 0$,

$$
\log \frac{2 \mu^2 \varepsilon}{L_1} \leq 2^{k-1} \log \frac{\mu^2}{L_1 L_2} \Rightarrow k = \left\lceil \log \frac{\log \frac{L_1^3}{\mu \varepsilon}}{\log \frac{L_1 L_2}{\mu^2}} \right\rceil + 1.
$$

Finally, the total number of iterations of Algorithm 4 is

$$
N = T n + k = \left\lceil \frac{R^2}{2} \left( \frac{64 L_p}{p \mu R} \right)^{\frac{1}{p+1}} \right\rceil \cdot \left\lceil \log \frac{L_1 L_2 R}{\mu^2} \right\rceil + \left\lceil \log \frac{\log \frac{L_1^3}{\mu \varepsilon}}{\log \frac{L_1 L_2}{\mu^2}} \right\rceil + 1
$$

$$
= O \left( \frac{L_p R^p}{\mu} \right)^{\frac{1}{p+1}} \log \frac{L_1 L_2 R}{\mu^2} + \log \frac{\log \frac{L_1^3}{\mu \varepsilon}}{\log \frac{L_1 L_2}{\mu^2}} \right)

\square

4 Discussion

In this work we propose a $p$-th order tensor method for strongly-convex-strongly-concave saddle point problem with local quadratic convergence. Our method is based on the ideas, developed in the works [2] and [10]. In [2] the authors use $p$-th order oracle to construct an algorithm for MVI with monotone operator. As a corollary, this algorithm allows to solve saddle point problems with convex-concave objective. Because of strong convexity and strong concavity of our problem, we can apply a restart technique to the method from [2] and get better algorithm complexity. To further improve local convergence rate we switch to the algorithm from [10] in the area of its
quadratic convergence. This way we get rid of the multiplicative logarithmic factor and get additive log log factor in the final complexity estimate. In spite of all the improvements, we should remind about many additional assumptions about the problem, which reduces number of real problems, that can suit to it.

In future we plan some numerical experiments and, as another future direction, it may be interesting to use the proposed method inside the framework from [4] and investigate its convergence rate in terms of norm of the gradient. Additional possible directions of further research are the more general Hélder conditions instead of Lipschitz conditions and uniformly convex case.

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