VECTORS AND GENERALIZED DHAMEN-MICCHELLI SPACES

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Abstract. This is the first of a series of papers on partition functions and the index theory of transversally elliptic operators. In this paper we only discuss algebraic and combinatorial issues related to partition functions. The applications to index theory will appear in [4].

Here we introduce a space of functions on a lattice which generalizes the space of quasi-polynomials satisfying the difference equations associated to cocircuits of a sequence of vectors \( X \). This space \( \mathcal{F}(X) \) contains the partition function \( P_X \). We prove a "localization formula" for any \( f \) in \( \mathcal{F}(X) \). In particular, this implies that the partition function \( P_X \) is a quasi-polynomial on the sets \( c - B(X) \) where \( c \) is a big cell and \( B(X) \) is the zonotope generated by the vectors in \( X \).

1. Introduction

Recall some notions. We take a lattice \( \Gamma \) in a vector space \( V \) and \( X := [a_1, \ldots, a_m] \) a list of non zero elements of \( \Gamma \), spanning \( V \) as vector space. If \( X \) generates a pointed cone \( C(X) \), the partition function \( P_X(\gamma) \) counts the number of ways in which a vector \( \gamma \in \Gamma \) can be written as \( \sum_{i=1}^{m} k_i a_i \) with \( k_i \) non negative integers.

A quasi-polynomial is a function on \( \Gamma \) which coincides with a polynomial on each coset of some sublattice of finite index in \( \Gamma \). A theorem [6,10], generalizing the theory of the Ehrhart polynomials [5], shows that \( P_X(\gamma) \) is a quasi-polynomial on certain regions \( c - B(X) \), where \( B(X) := \{ \sum_{i=1}^{m} t_i a_i, \ 0 \leq t_i \leq 1 \} \) is the zonotope generated by \( X \) while \( c \) denotes a big cell, that is a connected component of the complement in \( V \) of the singular vectors which are formed by the union of all cones \( C(Y) \) for all the sublists \( Y \) of \( X \) which do not span \( V \). The complement of \( C(X) \) is a big cell. The other cells are inside \( C(X) \) and are convex.

The quasi-polynomials describing the partition function belong to a remarkable finite dimensional space introduced and described by Dahmen-Micchelli [6] and which in this paper will be denoted by \( DM(X) \). This is the space of solutions of a system of difference equations. In order to describe it, let us call a subspace \( r \) of \( V \) rational if \( r \) is the span of a sublist of \( X \). We need to recall that a cocircuit \( Y \) in \( X \) is a sublist of \( X \) such that \( X \setminus Y \) does not span \( V \) and \( Y \) is minimal with this property. Thus

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\( Y \) is of the form \( Y = X \setminus H \) where \( H \) is a rational hyperplane. Given \( a \in \Gamma \), the difference operator \( \nabla_a \) is the operator on functions defined by 
\[
\nabla_a(f)(b) := f(b) - f(b-a).
\]
For a list \( Y \) of vectors, we set 
\[
\nabla_Y := \prod_{a \in Y} \nabla_a.
\]
We then define:
\[
DM(X) := \{ f | \nabla_Y f = 0, \text{ for every cocircuit } Y \text{ in } X \}.
\]
It is easy to see that \( DM(X) \) is finite dimensional and consists of quasi-polynomial functions (cf. [3]).

In this article, we introduce 
\[
\mathcal{F}(X) := \{ f | \nabla_{X \setminus \mathfrak{r}} f \text{ is supported on } \mathfrak{r} \text{ for every rational subspace } \mathfrak{r} \}.
\]
Clearly \( \mathcal{P}_X \) as well as \( DM(X) \) are contained in \( \mathcal{F}(X) \). The space \( \mathcal{F}(X) \) is of interest even when \( X \) does not span a pointed cone and occurs in studying indices of transversally elliptic operators on a vector space.

The main result of this article is a “localization formula” for an element \( f \) in \( \mathcal{F}(X) \). In particular, given a chamber \( \mathfrak{c} \), our localization formula allows us to write explicitly the partition function \( \mathcal{P}_X \) as a sum of a quasi-polynomial function \( \mathcal{P}_X^\mathfrak{c} \in DM(X) \) and of other functions \( f_\mathfrak{r} \in \mathcal{F}(X) \) supported outside \( \mathfrak{c} - B(X) \). This allows us to give a short proof of the quasi-polynomiality of \( \mathcal{P}_X \) on the regions \( \mathfrak{c} - B(X) \). Furthermore this decomposition implies Paradan’s wall crossing formulae [8] for the quasi-polynomials \( \mathcal{P}_X^\mathfrak{c} \). Our approach is strongly inspired by Paradan’s localization formula in Hamiltonian geometry, but our methods here are elementary. We wish to thank Michel Duflo and Paul-Emile Paradan for several suggestions and corrections.

2. Special functions

2.1. Basic notations. Let \( \Gamma \) be a lattice and \( E = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \). Consider the space \( \mathcal{C}_E[\Gamma] \) of \( E \) valued functions on \( \Gamma \). When \( E = \mathbb{Z} \), we shall simply write \( \mathcal{C}[\Gamma] \). We display such a function \( f(\gamma) \) also as a formal series
\[
\Theta(f) := \sum_{\gamma \in \Gamma} f(\gamma) e^{\gamma}.
\]
Of course, under suitable convergence conditions, the series \( \sum_{\gamma \in \Gamma} f(\gamma) e^{\gamma} \) is a function on the torus \( G \) whose character group is \( \Gamma \), and it is the Laplace–Fourier transform of \( f \). In fact the functions that we shall study are Fourier coefficients of some important generalized functions on \( G \). This fact and the several implications for the index theory of transversally elliptic operators will be the subject of a subsequent paper [4].

The space \( \mathcal{C}_E[\Gamma] \) is in an obvious way a module over the group algebra \( E[\Gamma] \), multiplication by \( e^\lambda \) on the series \( \Theta(f) \) corresponding to the translation operator \( \tau_\lambda \) defined by
\[
(\tau_\lambda f)(\gamma) := f(\gamma - \lambda)
\]
on the function \( f \). Thus multiplication by \( 1 - e^\lambda \) corresponds to the difference operator \( \nabla_\lambda \).
We denote by $\delta_0$ the function on $\Gamma$ identically equal to 0 on $\Gamma$, except for $\delta_0(0) = 1$. Remark that the product of two formal series $\Theta(f_1)\Theta(f_2)$, whenever it is defined, corresponds to convolution $f_1 * f_2$ of the functions $f_1$ and $f_2$. The function $\delta_0$ is the unit element.

Remark 2.2. Notice that, for a difference operator $\nabla_a$ acting on a convolution, we have:

$$\nabla_a(f_1 * f_2) = \nabla_a(f_1) * f_2 = f_1 * \nabla_a(f_2).$$

Let now $X := [a_1, \ldots, a_m]$ be a list of non zero elements of $\Gamma$ and let $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space generated by $\Gamma$. We assume that $X$ generates the vector space $V$, but we do not necessarily assume that $X$ generates a pointed cone in $V$.

If $X$ generates a pointed cone, then we can define

$$\Theta_X = \prod_{a \in X} \sum_{k=0}^{\infty} e^{ka}.$$

We write

$$\Theta_X = \sum_{\gamma \in \Gamma} p_X(\gamma)e^\gamma$$

where $p_X \in \mathcal{C}[\Gamma]$ is the partition function. “Morally” the series $\Theta_X$ is equal to $\prod_{a \in X} \frac{1}{1-e^a}$, but $\frac{1}{1-e^a}$ has to be understood as the geometric series expansion $\sum_{k=0}^{\infty} e^{ka}$.

Remark 2.3. We easily see that the partition function satisfies the difference equation $\nabla_X p_X = \delta_0$. Clearly this equation has infinitely many solutions. The fact that $p_X$ is uniquely determined by the recursion expressed by this equation comes from the further property of this solution of having support in the cone $C(X)$. We shall see other functions of the same type appearing in this paper.

Definition 2.4. i) A subspace of $V$ generated by a subset of the elements of $X$ will be called rational (relative to $X$).

ii) Given a rational subspace $\mathcal{L}$, we denote by $\mathcal{C}[\Gamma, \mathcal{L}]$ the set of elements in $\mathcal{C}[\Gamma]$ which have support in the lattice $\Gamma \cap \mathcal{L}$.

iii) Given a rational subspace $\mathcal{L}$, we set $\nabla_X \setminus \mathcal{L} := \prod_{a \in X \setminus \mathcal{L}} \nabla_a$.

With these notations, the space $DM_E(X)$ defined by Dahmen–Micchelli is formed by the set of functions $f \in \mathcal{C}_E[\Gamma]$ satisfying the system of difference equations $\nabla_X \setminus \mathcal{L} f = 0$ as $\mathcal{L}$ varies among all proper rational subspaces relative to $X$. It is easy to see that $DM_E(X)$ consists of quasi–polynomials.

It follows from their theory (see also [3]) that, for each $E$, the space $DM_E(X)$ is a free $E$–module of dimension $\delta(X)$, the volume of the zonotope $B(X)$. In particular $DM_E(X) = E \otimes_{\mathbb{Z}} DM_{\mathbb{Z}}(X)$ for all $E$. Therefore from now on we shall work directly over $\mathbb{Z}$ and drop the subscript $E$. 
The smallest sub–lattice of $\Gamma$ for which each function of $DM(X)$ is a polynomial on its cosets is the intersection of all the sublattices of $\Gamma$ generated by all the bases of $V$ that one can extract from $X$ (the least common multiple).

Given a rational subspace $r$, we will identify the space $C[\Gamma \cap r]$ with the subspace $C[\Gamma, r]$ of $C[\Gamma]$ by extending the functions with $0$ outside $r$.

2.5. The special functions $P^F_X$. Given a rational subspace $r$, $X \setminus r$ defines a hyperplane arrangement in the space $\mathbb{R} U$ orthogonal to $r$. Take an open face $F_L$ in $\mathbb{R} U$ with respect to this hyperplane arrangement. We shall call such a face a regular face for $X \setminus r$. If $F_L$ is a regular face, then $-F_L$ is also a regular face. By definition a vector $u \in \mathbb{R} U$ and such that $\langle u, a \rangle \neq 0$ for all $a \in X \setminus r$ lies in a unique such regular face $F_L$ and $u$ will be called a regular vector for $X \setminus r$.

Given a regular face $F_L$ for $X \setminus r$, we divide $X \setminus r$ into two parts $A, B$, according to whether they take positive or negative values on our face. We denote the cone $C(A, -B)$, generated by the list $[A, -B]$, by $C(F_L, X \setminus r)$.

If we take $u \in F_L$, $C(F_L, X \setminus r)$ is contained in the closed half space of vectors $v$ where $u$ is non negative.

We are going to consider the series $\Theta^F_X$ which is characterized by the following two properties:

**Lemma 2.6.** There exists a unique element

$$\Theta^F_X = \sum_{\gamma} P^F_X(\gamma)e^\gamma$$

such that

i) $\prod_{a \in X \setminus r}(1 - e^a)\Theta^F_X = 1$, equivalently $\nabla X \setminus r P^F_X = \delta_0$.

ii) $P^F_X$ is supported in $-\sum_{b \in B} b + C(F_L, X \setminus r)$.

**Proof.** Set:

$$\Theta^F_X = (-1)^{|B|}e^{-\sum_{b \in B} b} \prod_{a \in A} \left(\sum_{k=0}^{\infty} e^{ka}\right) \prod_{b \in B} \left(\sum_{k=0}^{\infty} e^{-kb}\right).$$

It is easily seen that this element satisfies the two properties and is unique.

In particular, if $r = U$, we have $F_U = \{0\}$ and $P^0_X = \delta_0$.

Morally, $\Theta^F_X = \prod_{a \in X \setminus r} \frac{1}{1-e^a} = \prod_{a \in A} \frac{1}{1-e^a} \prod_{b \in B} \frac{1}{1-e^b}$. We indeed need to reverse the sign of some of the vectors in $X \setminus r$ in order that the convolution product of the corresponding geometric series makes sense.

Although a function $f \in C[\Gamma, r]$ may have infinite support, we easily see that the convolution $P^F_X \ast f$ is well defined. In fact we claim that, given any $\gamma \in \Gamma$, we can write $\gamma = \lambda + \mu$ with $\mu \in r \cap \Gamma$, and $\lambda \in (-\sum_{b \in B} b +$
Choose two rational spaces $\mathbb{F}_x, \mathbb{F}_y$ and an regular face $F_x$ for $X \setminus r$ in $\mathbb{F}^\perp$. The image of $F_x$ modulo $\mathbb{F}^\perp$ is a regular face for $(X \cap r) \setminus r$. To simplify notations, we still denote this face by $F_x$. We have:

**Proposition 2.7.**

1) $\nabla_{(X \setminus r) \cap \mathbb{F}} P_{X \setminus \mathbb{F}} F_x = P_{(X \setminus \mathbb{F}) \cap \mathbb{F}} F_x$.

2) For $g \in C[\mathbb{F} \cap \mathbb{F}]$:

\[
\nabla_{(X \setminus r) \cap \mathbb{F}} (P_{X \setminus \mathbb{F}} F_x \ast g) = P_{(X \setminus \mathbb{F}) \cap \mathbb{F}} (\nabla_{(X \setminus r) \cap \mathbb{F}} (g \ast \Theta (F_x))) - \frac{\Theta (F_x)}{\Theta (F_x)} \nabla_{(X \setminus r) \cap \mathbb{F}} (g \ast \Theta (F_x)).
\]

Proof. 1) From Equation (1), we see that the series associated to the function $\nabla_{(X \setminus r) \cap \mathbb{F}} P_{X \setminus \mathbb{F}} F_x$ equals

\[
\Theta (F_x) = (-1)^{|B \cap r|} e^{-\sum_{b \in B \cap r} b} \prod_{a \in A \cap \mathbb{F}} \sum_{k=0}^{\infty} e^{ka} \prod_{b \in B \cap \mathbb{F}} \sum_{k=0}^{\infty} e^{-kb}.
\]

2) Let $g \in C[\mathbb{F} \cap \mathbb{F}]$. Take any rational subspace $r$, we have that $\nabla_{X \setminus r} = \nabla_{(X \setminus r) \cap \mathbb{F}} \nabla_{(X \setminus r) \setminus \mathbb{F}}$, thus

\[
\nabla_{(X \setminus r) \cap \mathbb{F}} (P_{X \setminus \mathbb{F}} F_x \ast g) = (\nabla_{(X \setminus r) \cap \mathbb{F}} (P_{X \setminus \mathbb{F}} F_x) \ast (\nabla_{(X \setminus r) \setminus \mathbb{F}} (g \ast \Theta (F_x)))) - \frac{\Theta (F_x)}{\Theta (F_x)} \nabla_{(X \setminus r) \cap \mathbb{F}} (g \ast \Theta (F_x)).
\]

As $\nabla_{(X \setminus r) \cap \mathbb{F}} (P_{X \setminus \mathbb{F}} F_x) = P_{(X \setminus \mathbb{F}) \cap \mathbb{F}} F_x$ from part 1), we obtain Formula (2), which is the mother of all other formulæ of this article.

In particular, for $r = t$, Formula (2) implies the following.

**Proposition 2.8.** If $f \in C[\mathbb{F} \cap \mathbb{F}]$, we have $f = \nabla_{X \setminus \mathbb{F}} (P_{X \setminus \mathbb{F}} F_x \ast f)$.

### 3. A remarkable space

#### 3.1. The space $\mathcal{F}(X)$

We let $S_X$ denote the set of all rational subspaces relative to $X$.

**Definition 3.2.** We define the space of interest for this article by:

\[
\mathcal{F}(X) := \{ f \in C[\mathbb{F}] \mid \nabla_{X \setminus \mathbb{F}} f \in C[\mathbb{F}], \text{ for all } r \in S_X \}.
\]

One of the equations (corresponding to $r = \{0\}$) that must satisfy $\Theta (f)$ when $f \in \mathcal{F}(X)$ is the relation $\prod_{a \in X} (1 - e^c) \Theta (f) = c$, where $c$ is a constant. This equation was the motivation for introducing the space $\mathcal{F}(X)$. Indeed the first important fact on this space is the following:

**Lemma 3.3.**

1) If $F$ is a regular face for $X$, then $P_{X \setminus \mathbb{F}} F_x$ lies in $\mathcal{F}(X)$.

2) The space $DM(X)$ is contained in $\mathcal{F}(X)$. 


Proof. i) Indeed, $\nabla_{X \setminus \underline{r}} P^F_X = P^F_{X \cap \underline{r}} \in \mathcal{C}[\Gamma, r]$. 

ii) Is clear from the definitions.

In particular, if $X$ generates a pointed cone, then the partition function $P_X$ lies in $\mathcal{F}(X)$.

Example 3.4. Let us give a simple example. Let $\Gamma = \mathbb{Z} \omega$ and $X = [2\omega, -\omega]$. Then it is easy to see that $\mathcal{F}(X)$ is a free $\mathbb{Z}$ module of dimension 4, with corresponding basis

$$
\theta_1 = \sum_{n \in \mathbb{Z}} e^{n \omega}, \quad \theta_2 = \sum_{n \in \mathbb{Z}} n e^{n \omega},
\theta_3 = \sum_{n \in \mathbb{Z}} \left(\frac{n}{2} + \frac{1 - (-1)^n}{4}\right) e^{n \omega}, \quad \theta_4 = \sum_{n \geq 0} \left(\frac{n}{2} + \frac{1 - (-1)^n}{4}\right) e^{n \omega}.
$$

Here $\theta_1, \theta_2, \theta_3$ are a $\mathbb{Z}$ basis of $\mathcal{D}(X)$.

In fact, there is a much more precise statement of which Lemma 3.3 is a very special case and which will be the object of Theorem 3.8.

3.5. Some properties of $\mathcal{F}(X)$. Let $r$ be a rational subspace and $F_{\underline{r}}$ be a regular face for $X \setminus \underline{r}$.

Proposition 3.6. i) The map $g \mapsto P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast g$ gives an injection from $\mathcal{F}(X \cap \underline{r})$ to $\mathcal{F}(X)$. Moreover

(4) $\nabla_{X \setminus \underline{r}} (P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast g) = g, \forall g \in \mathcal{F}(X \cap \underline{r})$.

ii) $\nabla_{X \setminus \underline{r}}$ maps $\mathcal{F}(X)$ surjectively to $\mathcal{F}(X \cap \underline{r})$.

iii) If $g \in \mathcal{D}(X \cap \underline{r})$, then $\nabla_{X \setminus \underline{r}} (P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast g) = 0$ for any rational subspace $\underline{l}$ such that $\underline{l} \cap \underline{r} \neq \underline{r}$.

Proof. i) If $g \in \mathcal{F}(X \cap \underline{r})$, then $\nabla_{(X \cap \underline{r}) \setminus (\underline{r} \cap \underline{r})} g \in \mathcal{C}[\Gamma \cap \underline{r} \cap \underline{r}]$, hence Formula (2) in Proposition 2.7 shows that $\nabla_{X \setminus \underline{r}} (P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast g) \in \mathcal{C}[\Gamma, \underline{l}]$, so that $P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast g \in \mathcal{F}(X)$ as desired. Formula (4) follows from the fact that $\nabla_{X \setminus \underline{r}} P^{F_{\underline{r}}}_{X \setminus \underline{r}} = \delta_0$.

ii) If $f \in \mathcal{F}(X)$, we have $\nabla_{X \setminus \underline{r}} f \in \mathcal{F}(X \cap \underline{r})$. In fact take a rational subspace $\underline{l}$ of $\underline{r}$, we have that $\nabla_{(X \cap \underline{r}) \setminus (\underline{r} \cap \underline{r})} f = \nabla_{X \setminus \underline{r}} f \in \mathcal{C}[\Gamma \cap \underline{r} \cap \underline{r}]$. The fact that $\nabla_{X \setminus \underline{r}}$ is surjective is a consequence of Formula (4).

iii) Similarly, if $g \in \mathcal{D}(X \cap \underline{r})$, Formula (2) in Proposition 2.7 implies the third assertion of our proposition.

Proposition 3.6 allows us to associate, to a rational space $\underline{r}$ and a regular face $F_{\underline{r}}$ for $X \setminus \underline{r}$, the operator

$$
\Pi^{F_{\underline{r}}}_{X \setminus \underline{r}} : f \mapsto P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast (\nabla_{X \setminus \underline{r}} f)
$$
on $\mathcal{F}(X)$. From Formula (4), it follows that the operator $\Pi^{F_{\underline{r}}}_{X \setminus \underline{r}}$, on $\mathcal{F}(X)$, is a projector with image $P^{F_{\underline{r}}}_{X \setminus \underline{r}} \ast \mathcal{F}(X \cap \underline{r})$. 

3.7. The main theorem. Choose, for every rational space \( r \), a regular face \( F_\Sigma \) for \( X \setminus r \). The following theorem is the main theorem of this section.

**Theorem 3.8.** With the previous choices, we have:

\[
\mathcal{F}(X) = \bigoplus_{\Sigma \in S_X} P_{X,\Sigma}^F \ast DM(X \cap \Sigma).
\]

**Proof.** Denote by \( S_X^{(i)} \) the subset of subspaces \( \Sigma \in S_X \) of dimension \( i \). Consider \( \nabla_{X \setminus \Sigma} \) as an operator on \( \mathcal{F}(X) \) with values in \( C[\Gamma] \). Define the spaces

\[
\mathcal{F}(X)_i := \cap_{\Sigma \in S_X^{(i-1)}} \ker(\nabla_{X \setminus \Sigma}).
\]

Notice that by definition \( \mathcal{F}(X)_{\{0\}} = \mathcal{F}(X) \), that \( \mathcal{F}(X)_{dim V} \) is the space \( DM(X) \) and that \( \mathcal{F}(X)_{i+1} \subseteq \mathcal{F}(X)_i \).

**Lemma 3.9.** Let \( \Sigma \in S_X^{(i)} \).

i) The image of \( \nabla_{X \setminus \Sigma} \) restricted to \( \mathcal{F}(X)_i \) is contained in the space \( DM(X \cap \Sigma) \).

ii) If \( f \) is in \( DM(X \cap \Sigma) \), then \( P_{X,\Sigma}^F \ast f \in \mathcal{F}(X)_i \).

**Proof.** i) First we know, by the definition of \( \mathcal{F}(X) \), that \( \nabla_{X \setminus \Sigma} \mathcal{F}(X)_i \) is contained in the space \( C[\Gamma, \Sigma] \). Let \( \Sigma_t \) be a rational hyperplane of \( \Sigma \), so that \( \Sigma_t \) is of dimension \( i - 1 \). By construction, we have that for every \( f \in \mathcal{F}(X)_i \)

\[
0 = \prod_{a \in X \setminus \Sigma} \nabla_a f = \prod_{a \in (X \cap \Sigma) \setminus \Sigma} \nabla_a \nabla_{X \setminus \Sigma} f.
\]

This means that \( \nabla_{X \setminus \Sigma} f \) satisfies the difference equations given by the cocircuits of \( X \cap \Sigma \), that is, it lies in \( DM(X \cap \Sigma) \).

ii) Follows from the third item of Proposition 3.6.

Consider the map \( \mu_i : \mathcal{F}(X)_i \to \bigoplus_{\Sigma \in S_X^{(i)}} DM(X \cap \Sigma) \) given by

\[
\mu_i f := \bigoplus_{\Sigma \in S_X^{(i)}} \nabla_{X \setminus \Sigma} f
\]

and the map \( P_i : \bigoplus_{\Sigma \in S_X^{(i)}} DM(X \cap \Sigma) \to \mathcal{F}(X)_i \) given by

\[
P_i(\bigoplus_{\Sigma} g_\Sigma) := \sum P_{X,\Sigma}^F \ast g_\Sigma.
\]

**Theorem 3.10.** The sequence

\[
0 \to \mathcal{F}(X)_{i+1} \to \mathcal{F}(X)_i \xrightarrow{\mu_i} \bigoplus_{\Sigma \in S_X^{(i)}} DM(X \cap \Sigma) \to 0
\]

is exact. Furthermore, the map \( P_i \) provides a splitting of this exact sequence: \( \mu_i P_i = \text{Id} \).

**Proof.** By definition, \( \mathcal{F}(X)_{i+1} \) is the kernel of \( \mu_i \), thus we only need to show that \( \mu_i P_i = \text{Id} \). Given \( \Sigma \in S_X^{(i)} \) and \( g \in DM(X \cap \Sigma) \), by Formula [4] we have \( \nabla_{X \setminus \Sigma}(P_{X,\Sigma}^F \ast g) = g \). If instead we take \( \Sigma_t \neq \Sigma \) another subspace of \( S_X^{(i)} \), \( \Sigma \cap \Sigma_t \) is a proper subspace of \( \Sigma_t \). Item iii) of Proposition 3.6 says that for
g \in DM(X \cap r), \nabla_{X \setminus r}(F^{F_{\overline{X} \setminus \overline{r}}}_{X \setminus \overline{r}} \ast g) = 0. Thus, given a family \( g_\overline{r} \in DM(X \cap r) \), the function \( f = \sum_{i \in S_X} F_{X \setminus \overline{r}} \ast g_i \) is such that \( \nabla_{X \setminus r} f = g_\overline{r} \) for all \( \overline{r} \in S_X^{(i)} \). This proves our claim that \( \mu_i \mathbf{P}_i = \text{Id} \).

Putting together these facts, Theorem 3.8 follows. □

A collection \( \mathbf{F} = (F_{\overline{r}}) \) of faces \( F_{\overline{r}} \subset r^\perp \) regular for \( X \setminus r \), indexed by the rational subspaces \( r \in S_X \) will be called a \( X \)-regular collection.

Given a \( X \)-regular collection \( \mathbf{F} \), we can write an element \( f \in \mathcal{F}(X) \) as

\[
f = \sum_{\overline{r} \in S_X} f_{\overline{r}}, \quad \text{(Theorem 3.8)}
\]

with \( f_{\overline{r}} \in \mathcal{P}^{F_{\overline{r}}}_{X \setminus \overline{r}} \ast DM(X \cap r) \). This expression for \( f \) will be called the \( \mathbf{F} \) decomposition of \( f \). In this decomposition, we always have \( F_V = \{0\} \), \( \mathcal{P}^{F_V}_{X} = \delta_0 \) and the component \( f_V \) is in \( DM(X) \).

The space \( \mathcal{P}^{F_{\overline{r}}}_{X \setminus \overline{r}} \ast DM(X \cap r) \) will be referred to as the \( F_{\overline{r}} \)-component of \( \mathcal{F}(X) \).

From Lemma 3.10 it follows that the operator \( \text{Id} - \mathbf{P}_i \mu_i \) projects \( \mathcal{F}(X)_i \) to \( \mathcal{F}(X)_{i+1} \) with kernel \( \bigoplus_{\overline{r} \in S_X^{(i)}} \mathcal{P}^{F_{\overline{r}}}_{X \setminus \overline{r}} \ast DM(X \cap r) \) (this operator depends of \( \mathbf{F} \)). Thus the ordered product

\[
\Pi_i^{\mathbf{F}} := (\text{Id} - \mathbf{P}_{i-1} \mu_{i-1})(\text{Id} - \mathbf{P}_{i-2} \mu_{i-2}) \cdots (\text{Id} - \mathbf{P}_0 \mu_0)
\]

projects \( \mathcal{F}(X) \) to \( \mathcal{F}(X)_i \); therefore, we have

**Proposition 3.11.** Let \( \mathbf{F} \) be a \( X \)-regular collection and \( r \) a rational subspace of dimension \( i \). The operator

\[
P^{\mathbf{F}}_r = \Pi_i^{\mathbf{F}} \ast (\text{Id} - \mathbf{P}_{i-1} \mu_{i-1})(\text{Id} - \mathbf{P}_{i-2} \mu_{i-2}) \cdots (\text{Id} - \mathbf{P}_0 \mu_0) = \Pi_i^{\mathbf{F}} \ast \Pi_i^{\mathbf{F}}
\]

is the projector of \( \mathcal{F}(X) \) to the \( F_r \)-component \( \mathcal{P}^{F_{\overline{r}}}_{X \setminus \overline{r}} \ast DM(X \cap r) \) of \( \mathcal{F}(X) \).

In particular, if \( \dim(V) = s \), the operator

\[
P_V := (\text{Id} - \mathbf{P}_{s-1} \mu_{s-1})(\text{Id} - \mathbf{P}_{s-2} \mu_{s-2}) \cdots (\text{Id} - \mathbf{P}_0 \mu_0)
\]

is the projector \( \mathcal{F}(X) \to DM(X) \) associated to the direct sum decomposition:

\[
\mathcal{F}(X) = DM(X) \oplus \left( \bigoplus_{\overline{r} \in S_X \setminus \overline{r}} \mathcal{P}^{F_{\overline{r}}}_{X \setminus \overline{r}} \ast DM(X \cap r) \right).
\]

Let \( \mathbf{F} = (F_{\overline{r}}) \) be a \( X \)-regular collection. If \( \overline{r} \) is a rational subspace and, for each \( \overline{r} \in S_X \setminus \overline{r} \), we take the image of \( F_{\overline{r}} \) modulo \( \overline{r} \) we get a \( X \cap \overline{r} \)-regular collection. We still denote it by \( \mathbf{F} \) in the next proposition. The proof of this proposition is skipped, as it is very similar to preceding proofs.
Proposition 3.12. Let $t$ be a rational subspace. Let $f \in \mathcal{F}(X)$ and $f = \sum_{x \in S(X)} f_x$ be the $\mathcal{F}$ decomposition of $f$ and $\nabla_{X \setminus t} f = \sum_{x \in S_X \cap t} g_x$ be the $\mathcal{F}$ decomposition of $\nabla_{X \setminus t} f$, then

- $\nabla_{X \setminus t} f_x = 0$ if $x \notin S_X \cap t$,
- $\nabla_{X \setminus t} f_x = g_x$ if $x \in S_X \cap t$.

Remark 3.13. It follows from the previous theorems and the properties of $DM(X)$ that, for every $E$, we could define a space $\mathcal{F}_E(X)$ of $E$ valued functions as in Definition 3.2 and we have $\mathcal{F}_E(X) = E \otimes \mathbb{Z} \mathcal{F}(X)$.

3.14. Localization theorem.

Definition 3.15. By the word tope, we mean a connected component of the complement in $V$ of the union of the hyperplanes generated by subsets of $X$.

We introduce also $B(X) := \{ \sum_{i=1}^m t_i a_i, \ 0 \leq t_i \leq 1 \}$ the zonotope generated by $X$. $B(X)$ is a compact convex polytope which appears in several ways in the theory and plays a fundamental role.

We will show that, for every element $f \in \mathcal{F}(X)$, the function $f(\gamma)$ coincides with a quasi–polynomial on the sets $(\tau - B(X)) \cap \Gamma$ as $\tau$ varies over all topes (we simply say $f$ is a quasi–polynomial on $\tau - B(X)$).

Definition 3.16. Let $\tau$ be a tope and $r$ be a proper rational subspace. We say that a regular face $F_r$ for $X \setminus r$ is non–positive on $\tau$ if there exists $u_r \in F_r$ and $x_0 \in \tau$ such that $\langle u, x_0 \rangle < 0$.

Let $\mathcal{F} = \{ F_r \}$ be a $X$–regular collection. We shall say that $\{ \mathcal{F} \}$ is non–positive on $\tau$ if each $F_r$ is non–positive on $\tau$.

Let $f \in \mathcal{F}(X)$ and let $f = \sum f_x$ be the $\mathcal{F}$ decomposition of $f$.

Remark 3.17. This choice of $\mathcal{F}$ has the effect of pushing the supports of the elements $f_x$ (for $r \neq V$) away from $\tau$. See Figures 1, 2 which describe the $\mathcal{F}$ decomposition of the partition function $\mathcal{P}_X$ for $X := [a,b,c]$ with $a := \omega_1, b := \omega_2, c := \omega_1 + \omega_2$ in the lattice $\Gamma := \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2$. Thus the content of Theorem 3.18 is very similar to Paradan’s localization theorem [9].

Our previous claim then follows from the explicit construction below.

Theorem 3.18 (Localization theorem). Let $\tau$ be a tope. Let $\mathcal{F} = \{ F_r \}$ be a $X$–regular collection non–positive on $\tau$.

The component $f_V$ of the $\mathcal{F}$ decomposition $f = \sum_{x \in S_X} f_x$ is a quasi–polynomial function in $DM(X)$ such that $f = f_V$ on $\tau - B(X)$. 
Figure 1. The partition function of $X := [a, b, c]$

Proof. Let $\underline{r}$ be a proper rational space and $f_{\underline{r}} = \mathcal{P}_{X \cap \underline{r}}^{F_{\underline{r}}} * k_{\underline{r}}$ where $k_{\underline{r}} \in DM(X \cap \underline{r})$. In the notation of Lemma 2.6, the support of $f_{\underline{r}}$ is contained in the polyhedron $\underline{r} - \sum_{b \in B} b + C(F_{\underline{r}}, X \setminus \underline{r}) \subset \underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})$. This last polyhedron is convex and, by construction, it has a boundary limited by hyperplanes which are rational with respect to $X$. Thus, either $\tau \subset \underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})$ or $\tau \cap (\underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})) = \emptyset$. Take $u_{\underline{r}} \in F_{\underline{r}}$ and $x_0 \in \tau$ so that $u_{\underline{r}}(x_0) < 0$. As $u_{\underline{r}} \geq 0$ on $\underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})$ it follows that $\tau$ is not a subset of $\underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})$, so that $\tau \cap (\underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})) = \emptyset$.

In fact, we claim that $\tau - B(X)$ does not intersect the support $\underline{r} - \sum_{b \in B} b + C(F_{\underline{r}}, X \setminus \underline{r})$ of $f_{\underline{r}}$. Indeed, otherwise, we would have an equation $v - \sum_{x \in X} t_x x = s + \sum_{a \in A} k_a a + \sum_{b \in B} h_b (-b)$ with $v \in \tau$, $0 \leq t_x \leq 1$, $k_a \geq 0$, $h_b \geq 1$, $s \in \underline{r}$. This would imply that $v \in \underline{r} + C(F_{\underline{r}}, X \setminus \underline{r})$, a contradiction. Thus $f$ coincides with the quasi–polynomial $f_V$ on $\tau - B(X)$. \qed

One should remark that a quasi–polynomial is completely determined by the values that it takes on $\tau - B(X)$, thus $f_V$ is independent on the construction.

Definition 3.19. We shall denote by $f^\tau$ the quasi–polynomial coinciding with $f$ on $\tau - B(X)$.

Let us remark that the open subsets $\tau - B(X)$ cover $V$, when $\tau$ runs over the topes of $V$ (with possible overlapping). Thus the element $f \in \mathcal{F}(X)$ is entirely determined by the quasi–polynomials $f^\tau$.

Example 3.20. In Figure 3, for each tope $\tau$, the set of integral points in $\tau - B(X)$ is contained in one of the affine closed cones limited by thick
Figure 2. $F$ decomposition of the partition function of $X := [a, b, c]$ for $F$ non-positive on $\tau$ lines. We are showing in color the convex envelop of the integral points in $\tau - B(X)$ and not the larger open set $\tau - B(X)$. 
For collections arising from scalar products, one can give an explicit formula for the decomposition of an element $f \in \mathcal{F}(X)$.

Let us choose a scalar product on $V$, and identify $V$ and $U$ with respect to this scalar product. Given a point $\beta$ in $V$ and a rational subspace $\tau$ in $S_X$, we write

$$\beta = p_\tau \beta + p_\tau^\perp \beta$$

with $p_\tau \beta \in \tau$ and $p_\tau^\perp \beta$ in $\tau^\perp$.

**Definition 3.21.** We say that $\beta \in V$ is generic with respect to $\tau$ if $p_\tau \beta$ is in a tope $\tau(p_\tau \beta)$ for the sequence $X \cap \tau$, and $p_\tau^\perp \beta \in \tau^\perp$ is regular for $X \setminus \tau$.

We clearly have

**Proposition 3.22.** The set of $\beta$ which are not generic with respect to $\tau$ is a union of finitely many hyperplanes.

By Theorem 3.18, if $f \in \mathcal{F}(X)$, the element $\nabla_{X \setminus \tau} f$ is in $\mathcal{F}(X \cap \tau)$ and coincides with a quasi–polynomial $(\nabla_{X \setminus \tau} f)^\tau \in DM(X \cap \tau)$ on each tope $\tau$ for the system $X \cap \tau$.

**Theorem 3.23.** Let $\beta \in V$ be generic with respect to all the rational subspaces $\tau$. Let $F_\tau^\beta$ be the unique regular face for $X \setminus \tau$ containing $p_\tau^\perp \beta$.

Then

$$f = \sum_{\tau \in S_X} \mathcal{P}_{X \setminus \tau}^{-F_\tau^\beta} \ast (\nabla_{X \setminus \tau} f)^\tau(p_\tau \beta).$$
Proof. By the hypotheses made on $\beta$ the collection $F = \{F_{c}\}$ is a $X$–regular collection. Set $f = \sum_{c \in S_{X}} P_{X \backslash H}^{F_{c}} \ast q_{L}$ with $q_{L} \in DM(X \cap \ell)$. We apply Proposition 3.12. It follows that $q_{L}$ is the component in $DM(X \cap \ell)$ in the decomposition of $\nabla_{X \backslash H} f \in F(X \cap \ell)$, with respect of the $X \cap \ell$–regular collection induced by $F$. Set $u_{\ell} = -p_{\ell} \beta$. Remark that, for $\ell \subset X$, we have $\langle u_{\ell}, p_{\ell} \beta \rangle = -\|u_{\ell}\|^{2}$, so that each $u_{\ell}$ is negative on $p_{\ell} \beta$. Thus the formula follows from Theorem 3.18. \hfill $\square$

3.24. Wall crossing formula. We first develop a general formula describing how the functions $f^{\tau}$ change when crossing a wall. Then we apply this to the partition function $P_{X}$ and deduce that it is a quasi–polynomial on $c - B(X)$, where $c$ is a big cell.

Let $H$ be a rational hyperplane, and let $u \in U$ be an equation of the hyperplane. Then the two open faces in $H^\perp$ are the half-lines $F_{H} = R_{>0}u$ and $-F_{H}$.

Lemma 3.25. If $q \in DM(X \cap H)$, then $w := (P_{X \setminus H}^{F_{H}} - P_{X \setminus H}^{-F_{H}}) \ast q$ is an element of $DM(X)$.

Remark 3.26. In [1], a one-dimensional residue formula is given for $w$ allowing us to compute it.

Proof. If $\ell \in S_{X}$ is different from $H$, $\nabla_{X \setminus H}(P_{X \setminus H}^{F_{H}} \ast q) = \nabla_{X \setminus H}(P_{X \setminus H}^{-F_{H}} \ast q) = 0$, as follows from Proposition 3.6 (iii). If $\ell = H$, then $\nabla_{X \setminus H}(P_{X \setminus H}^{F_{H}} \ast q - P_{X \setminus H}^{-F_{H}} \ast q) = q - q = 0$. \hfill $\square$

Assume that $\tau_{1}$, $\tau_{2}$ are two adjacent topes, namely $\tau_{1} \cap \tau_{2}$ spans a hyperplane $H$. The hyperplane $H$ is a rational subspace. Let $\tau_{12}$ be the unique tope for $X \cap H$ such that $\tau_{1} \cap \tau_{2} \subset \tau_{12}$ (see Figure 4).

Example 3.27. Let $C$ be the cone generated by the vectors $a := \omega_{3} + \omega_{1}$, $b := \omega_{3} + \omega_{2}$, $c := \omega_{3} - \omega_{1}$, $d := \omega_{3} - \omega_{2}$ in a 3-dimensional space $V := \mathbb{R}\omega_{1} \oplus \mathbb{R}\omega_{2} \oplus \mathbb{R}\omega_{3}$. Figure 4 represents the section of $C$ cut by the affine hyperplane containing $a, b, c, d$. We consider $X := [a, b, c, d]$.

We show in section, on the left of the picture, two topes $\tau_{1}, \tau_{2}$ adjacent along the hyperplane $H$ generated by $b, d$ and, on the right, the tope $\tau_{12}$. The list $X \cap H$ is $[b, d]$. The closure of the tope $\tau_{12}$ is "twice bigger" than $\tau_{1} \cap \tau_{2}$.

Let $f \in F(X)$. The function $\nabla_{X \setminus H} f$ is an element of $F(H \cap X)$, thus, by Theorem 3.18, there exists a quasi–polynomial $(\nabla_{X \setminus H} f)^{\tau_{12}}$ on $H$ such that $\nabla_{X \setminus H} f$ agrees with $(\nabla_{X \setminus H} f)^{\tau_{12}}$ on $\tau_{12}$.

Theorem 3.28. Let $\tau_{1}, \tau_{2}, H, \tau_{12}$ be as before and $f \in F(X)$. Let $F_{H}$ be the half line in $H^\perp$ positive on $\tau_{1}$. Then

$$f^{\tau_{1}} - f^{\tau_{2}} = (P_{X \setminus H}^{F_{H}} - P_{X \setminus H}^{-F_{H}}) \ast (\nabla_{X \setminus H} f)^{\tau_{12}}.$$
Proof. Let \( x_0 \) be a point in the relative interior of \( \tau_1 \cap \tau_2 \) in \( H \). Then \( x_0 \) does not belong to any \( X \)-rational hyperplane different from \( H \) (see Figure 4). Therefore we can choose a regular vector \( u_\tau \) for \( X \setminus H \) for every rational subspace \( \tau \) different from \( H \). Consider the sequences \( u^1 = (u^1_\tau) \) where \( u^1_\tau = u_\tau \) for \( \tau \neq H \) and \( u^1_H = -u \) and \( u^2 = (u^2_\tau) \) where \( u^2_\tau = u_\tau \) for \( \tau \neq H \) and \( u^2_H = u \). Correspondingly we get two \( X \)-regular collections \( F^1 \) and \( F^2 \).

For \( i = 1, 2 \) let \( f = f^i_V + f^i_H + \sum_{\tau \in H \setminus V} f^i_\tau \) be the \( F^1 \) decomposition of \( f \).

We write \( f^i_H = \mathcal{P}^{-F^i_H}_{X \setminus H} q^{(i)} \) with \( q^{(i)} \in DM(X \cap H) \). Now the sequence \( u^1 \) takes a negative value at the point \( x_1 \) of \( \tau_1 \), thus by Theorem 3.11 the component \( f^1_V \) is equal to \( f^{\tau_1} \). By Proposition 3.12,

\[
\nabla_{X \setminus H} f = q^{(1)} + \sum_{\tau \in H \setminus \tau \# H} \nabla_{X \setminus H} f^1_\tau
\]

is the \( F^1 \) decomposition of \( \nabla_{X \setminus H} f \) so that, again by Theorem 3.11, \( q^{(1)} = (\nabla_{X \setminus H} f)^{\tau_1} \). We thus have \( f^1_V = f^{\tau_1} \) and \( f^1_H = \mathcal{P}^{-F^1_H}_{X \setminus H} (\nabla_{X \setminus H} f)^{\tau_1} \).

Similarly \( u^2 \) takes a negative value at the point \( x_2 \) of \( \tau_2 \), so \( f^2_V = f^{\tau_2} \) and \( f^2_H = \mathcal{P}^{F^2_H}_{X \setminus H} (\nabla_{X \setminus H} f)^{\tau_2} \).

Now from Proposition 3.11 when \( \dim(\tau) = i \),

\[
\begin{align*}
\Pi^1_X f &= \Pi^{F^1_X} X f, \\
\Pi^2_X f &= \Pi^{F^2_X} X f,
\end{align*}
\]

and, for any \( i < \dim V \), the operators \( \Pi^1_X f \) and \( \Pi^2_X f \) are equal. Thus \( f^1_\tau = f^2_\tau \) for \( \tau \neq V, H \). So we obtain \( f^1_V + f^1_H = f^2_V + f^2_H \), and our formula.

\[\square\]

Consider now the case where \( X \) spans a pointed cone. Let us interpret Formula (6) in the case in which \( f = \mathcal{P}_X \). We know that for a given tope \( \tau \), \( \mathcal{P}_X \) agrees with a quasi–polynomial \( \mathcal{P}^{X \setminus H}_X \) on \( \tau - B(X) \). Recall that \( \nabla_{X \setminus H}(\mathcal{P}_X) = \mathcal{P}_{X \cap H} \) as we have seen in Lemma 3.3. It follows that given...
two adjacent topes \( \tau_1, \tau_2 \) as above, \((\nabla_{X \setminus H} f)^{\tau_1} f^{\tau_2}\) equals \((P_{X \cap H})^{\tau_1} f^{\tau_2}\) (extended by zero outside \( H \)). So we deduce the identity

\[
(7) \quad P_{X}^{\tau_1} - P_{X}^{\tau_2} = (P_{X \setminus H}^{F_H} - P_{X \setminus H}^{-F_H}) \ast P_{X \cap H}^{\tau_1} f^{\tau_2}.
\]

This is Paradan’s formula ([8], Theorem 5.2).

**Example 3.29.** Assume \( X = [a, b, c] \) as in Remark 3.17. We write \( v \in V \) as \( v = v_1 \omega_1 + v_2 \omega_2 \). Let \( \tau_1 = \{ v_1 > v_2 > 0 \} \), \( \tau_2 = \{ 0 < v_1 < v_2 \} \). Then one easily (see Figure 1) sees that

\[
P_{X}^{\tau_1} = (n_2 + 1), \quad P_{X}^{\tau_2} = (n_1 + 1), \quad P_{X \cap H}^{\tau_1} = 1.
\]

Equality (7) is equivalent to the following identity of series which is easily checked:

\[
\sum_{n_1, n_2} (n_2 - n_1) x_1^{n_1} x_2^{n_2} = (- \sum_{n_1 \geq 0, n_2 < 0} x_1^{n_1} x_2^{n_2} + \sum_{n_1 < 0, n_2 \geq 0} x_1^{n_1} x_2^{n_2})(\sum_k x_1^k x_2^k).
\]

Recall that a big cell is a connected component of the complement in \( V \) of the singular vectors, which are formed by the union of all cones \( C(Y) \) for all the sublists \( Y \) of \( X \) which do not span \( V \). A big cell is usually larger than a tope. See Figure 5 which shows a section of a cone in dimension 3 generated by 3 independent vectors \( a, b, c \). Here \( X = [a, b, c, a + b + c] \). On the drawing, the vertices \( a, b, c, d \) represents the intersection of the section with the half lines \( \mathbb{R}^+ a, \mathbb{R}^+ b, \mathbb{R}^+ c, \mathbb{R}^+ d \).

![Figure 5. Topes and cells inside \( C(X) \) for \( X := [a, b, c, d := a + b + c] \).](image)

Let us now consider a big cell \( c \). We need

**Lemma 3.30.** Given a big cell \( c \), let \( \tau_1, \ldots, \tau_k \) be all the topes contained in \( c \). Then:

\[
c - B(X) = \bigcup_{i=1}^k (\tau_i - B(X)).
\]

**Proof.** Notice that \( \bigcup_{i=1}^k \tau_i \) is dense in \( c \). Given \( v \in c - B(X) \), \( v + B(X) \) has non empty interior and thus its non empty intersection with the open set \( c \) has non empty interior. It follows that \( v + B(X) \) meets \( \bigcup_{i=1}^k \tau_i \) proving our claim. \(\square\)

Now in order to prove the statement for big cells, we need to see what happens when we cross a wall between two adjacent topes.
Theorem 3.31. On $\mathfrak{c} - B(X)$, the partition function $P_X$ agrees with a quasi–polynomial $f^\mathfrak{c} \in DM(X)$.

Proof. By Lemma 3.30 it suffices to show that given two adjacent topes $\tau_1$, $\tau_2$ in $\mathfrak{c}$, $P_{X}^{\tau_1} = P_{X}^{\tau_2}$.

But now notice that the positive cone spanned by $X \cap H$, support of $P_{X \cap H}$, is formed of singular vectors and therefore it is disjoint from $\mathfrak{c}$ by definition of big cells. Therefore $P_{X \cap H}$ vanishes on $\tau_{12}$. Thus $P_{X \cap H}^{\tau_{12}} = 0$ and our claim follows from Formula (7).

□

This theorem was proven \[6\] by Dahmen-Micchelli for topes, and by Szenes-Vergne \[10\] for cells. In many cases, the sets $\mathfrak{c} - B(X)$ are the maximal domains of quasi–polynomiality for $P_X$.

Remark 3.32. If $\mathfrak{c}$ is a big cell contained in the cone $C(X)$, the open set $\mathfrak{c} - B(X)$ contains $\mathfrak{r}$ so that the quasi–polynomial $f^\mathfrak{c}$ coincides with $P_X$ on $\mathfrak{r}$.

This is usually not so for $f \in \mathcal{F}(X)$ and a tope $\tau$: the function $f$ does not usually coincide with $f^\mathfrak{r}$ on $\mathfrak{r}$. Figure 6 describes the function $f := -P_{X}^{\tau_2}$ in $\mathcal{F}(X)$ with $X := [a, b, c]$ as in Remark 3.17 and $u \in F_\mathfrak{r}$ strictly negative on $a, b$. We see for example that $f$ is equal to 0 on the set $n_1 = 0, n_2 \leq 0$ which is in the closure of the tope $\tau_3 := \{v_2 < v_1 < 0\}$ while the quasi–polynomial $f^{\tau_3} = (n_1 + 1)$ takes the value $-1$ there.

We finally give a formula for $P_X$ due to Paradan.

Let us choose a scalar product on $V$. We use the notations of Theorem 3.23. As $\nabla_{X \setminus \mathfrak{r}} P_X = P_{X \cap \mathfrak{r}}$, we obtain as a corollary of Theorem 3.23...
Theorem 3.33. (Paradan). Let $\beta \in V$ be generic with respect to all the rational subspaces $r$. Then, we have

$$\mathcal{P}_X = \sum_{r \in S_X} \mathcal{P}_{X \setminus r} \ast (\mathcal{P}_{X \cap r})^r(\mathcal{P}_X \setminus r).$$

Remark 3.34. The set of $\beta \in V$ generic with respect to all the rational subspaces $r$ decomposes into finitely many open polyhedral cones. The decomposition depends only on the cone in which $\beta$ lies.

Finally if $X$ spans a pointed cone and $\beta$ has negative scalar product with $X$, the decomposition reduces to $\mathcal{P}_X = \mathcal{P}_X$, the component for $r = \{0\}$.

4. A second remarkable space

4.1. A decomposition formula. In this section, we want to present the analogue for distributions, the proofs are essentially the same or simpler than in the previous case, so we skip them. We shall freely use the notations of the previous sections.

Let $V$ be a finite dimensional vector space, consider the space $\mathcal{D}(V)$ of distributions on $V$. We denote by $\delta_0$ the delta distribution on $V$.

$\mathcal{D}(V)$ is in an obvious way a module over the algebra of distributions with compact support, under convolution. Let now $X := [a_1, \ldots, a_m]$ be a list of non–zero elements of $V$.

Definition 4.2. i) Given a rational subspace $r$, we denote by $\mathcal{D}(V, r)$ the set of elements in $\mathcal{D}(V)$ which vanish on all test functions vanishing on $r$.

ii) Given a vector $a \neq 0$, we denote by $\partial_a$ the directional derivative associated to $a$. For a list $Y$ of non–zero vectors, we denote by $\partial_Y := \prod_{a \in Y} \partial_a$.

The restriction map $\mathcal{C}_c^\infty(V) \to \mathcal{C}_c^\infty(r)$ on test functions induces, by duality, an identification between the space of distributions on $r$ and the space $\mathcal{D}(V, r)$.

If $X$ spans $V$, we consider the space defined by Dahmen–Micchelli, which we denote $D(X)$, formed by the distributions $f \in \mathcal{D}(V)$ satisfying the system of differential equations $\partial_Y f = 0$, as $Y$ varies among all cocircuits of $X$. It is easy to see that an element of $D(X)$ is a polynomial density $P(x)dx$ on $V$.

Assume that $X$ spans a pointed cone. Recall that the multivariate spline $T_X$ is the tempered distribution defined, on test functions $f$, by:

$$\langle T_X | f \rangle = \int_0^\infty \cdots \int_0^\infty f \left( \sum_{i=1}^m t_i a_i \right) dt_1 \cdots dt_m.$$
If $W$ is the span of $X$ and if we choose a Lebesgue measure $dx$ on $W$, we may interpret $T_X$ as a function on $W$ supported in the cone $C(X)$ by writing $\langle T_X | f \rangle = \int_W f(x)T_X(x)dx$.

If $Y$ is a sublist of $X$, one has that $\partial_{X \setminus Y} T_X = T_Y$.

We next define the vector space of interest for this section:

$$G(X) := \{ f \in D(V) | \partial_{X \cap \tau} f \in D(V, \tau), \text{ for all } \tau \in S_X \}. \tag{9}$$

**Lemma 4.3.**

i) If $X$ generates a pointed cone, the multivariate spline $T_X$ lies in $G(X)$.

ii) The space $D(X)$ is contained in $G(X)$.

**Proof.**  

i) $\partial_{X \setminus \tau} T_X = T_X \cap \tau \in D(V, \tau)$. 

ii) is clear from the definition. \qed

As for the partition functions, this lemma is a very special case of Theorem 4.5 which follows.

Given a rational subspace $\tau$, take a regular face $F_{\tau}$ for $X \setminus \tau$. Divide as before the set $X \setminus \tau$ into two parts $A, B$, of positive and negative vectors on $F_{\tau}$.

We want to define an element $T_{X \setminus \tau} \in D(V)$ which is characterized by the following two properties:

**Lemma 4.4.** There exists a unique element $T_{X \setminus \tau}^{F_{\tau}}$ characterized by the properties $\partial_{X \setminus \tau} T_{X \setminus \tau}^{F_{\tau}} = \delta_0$ and $T_{X \setminus \tau}^{F_{\tau}}$ is supported in $C(A, -B)$.

**Proof.** Set:

$$T_{X \setminus \tau}^{F_{\tau}} = (-1)^{|B|} T_{[A, -B]}.$$

It is easily seen that this element satisfies the two properties. The uniqueness is also clear. \qed

Identify the space of Dahmen–Micchelli $D(X \cap \tau)$ with a subspace of $D(V, \tau)$. Although a distribution $f \in D(V, \tau)$ may have non compact support, we easily see that the convolution product $T_{X \setminus \tau}^{F_{\tau}} * f$ is well defined. In fact, given any $\gamma \in V$, we can write $\gamma = \lambda + \mu$ with $\mu \in \tau$, and $\lambda \in C(A, -B)$ only in a bounded polytope.

The analog of the “mother formula” (2) of Proposition 2.7 is the following formula.

For $g \in D(V, \tau)$:

$$\partial_{X \setminus \tau}(T_{X \setminus \tau}^{F_{\tau}} * g) = T_{(X \setminus \tau) \cap \tau}^{F_{\tau}} * (\partial_{(X \cap \tau) \setminus (\tau \cap \gamma)} g). \tag{10}$$

Following the same scheme of proof as for Theorem 3.8 the following theorem follows:
Theorem 4.5. Choose for every rational space $r$, a regular face $F_r$ for $X \setminus r$. Then:

$$G(X) = \oplus_{r \in S_X} T^F_{X \setminus r} \ast D(X \cap r).$$

We associate, to a rational space $r$ and a regular face $F_r$ for $X \setminus r$, the operator on $G(X)$ defined by

$$\pi^r_{F_r} : f \mapsto T^F_{X \setminus r} \ast (\partial_{X \setminus r} f).$$

This is well defined. Indeed $\partial_{X \setminus r} f$ is supported on $r$ so that the convolution is well defined. We see that $\pi^r_{F_r}$ maps $G(X)$ to $G(X)$ and that it is a projector.

Given a $X$–regular collection $F$, we can write an element $f \in G(X)$ as

$$f = \sum_{r \in S_X} f_r$$

with $f_r \in T^F_{X \setminus r} \ast D(X \cap r)$. This expression for $f$ will be called the $F$ decomposition of $f$. In this decomposition, the component $f_V$ is in $D(X)$.

The space $T^F_{X \setminus r} \ast D(X \cap r)$ will be referred to as the $F_r$-component of $G(X)$.

Let $F$ be a $X$–regular collection. One can write in the same way as in Proposition 3.11 the explicit projectors to the various components.

4.6. Polynomials. Let $f \in G(X)$ and let $\tau$ be a tope.

Proposition 4.7 (Localization theorem). Let $F$ be a $X$-regular collection non-positive on $\tau$. Let $f = \sum f_r$ be the $F$-decomposition of $f$. Then the component $f_V$ of this decomposition is a polynomial density in $D(X)$ such that $f = f_V$ on $\tau$.

Proof. Write $f = \sum_{r \in S_X} f_r$ with $f_r = T^F_{X \setminus r} \ast k_r$ where $k_r \in D(X \cap r)$. The distribution $f_r = T^F_{X \setminus r} \ast k_r$ is supported on $r + C(F_r, X \setminus r)$. As in Theorem 3.18 we know that $\tau \cap (r + C(F_r, X \setminus r)) = \emptyset$.  

Remark 4.8. Thus the distribution $f$ is a locally polynomial density on $V$. In particular this is a tempered distribution. Here the distribution $f$ coincides with the polynomial density $f_V$ only on $\tau$ and not on the bigger open set $\tau - B(X)$. This extension property is replaced the regularity property that $f$ is of class $C^{r-1}$, where $r$ is the minimum of the cardinality of the cocircuits of $X$ (see [3]).

We shall denote by $f^\ast$ the polynomial density in $D(X)$ coinciding with $f$ on the tope $\tau$.

Let $H$ be a rational hyperplane, let $u \in U$ be an equation of the hyperplane and $F_H$ the half-line containing $u$. 
Lemma 4.9. If \( q \in D(X \cap H) \), then \( w := (T_{X \setminus H}^{F_H} - T_{X \setminus H}^{-F_H}) * q \) is an element of \( D(X) \).

Let us use the notations \( \tau_1, \tau_2, H, \tau_{12} \) as in Theorem 3.28. Let \( f \in G(X) \). The distribution \( \partial_{X \setminus H} f \) is an element of \( G(H \cap X) \), thus by Proposition 4.7 there exists a polynomial density \( (\partial_{X \setminus H} f)^{\tau_{12}} \in D(X \cap H) \) on \( H \) such that \( \partial_{X \setminus H} f \) agrees with \( (\partial_{X \setminus H} f)^{\tau_{12}} \) on \( \tau_{12} \).

Theorem 4.10. Let \( F_H \) be the half line in \( H^+ \) positive on \( \tau_1 \). Then
\[
(11) \quad f^{\tau_1} - f^{\tau_2} = (T_{X \setminus H}^{F_H} - T_{X \setminus H}^{-F_H}) * (\partial_{X \setminus H} f)^{\tau_{12}}.
\]

When \( X \) spans a pointed cone, we interpret Formula (11) for \( f = T_X \).

On a given tope \( \tau \), \( T_X \) agrees with a polynomial density \( T_X \).

Since \( \partial_{X \setminus H}(T_X) = T_{X \cap H} \) we deduce, the identity
\[
(12) \quad T_X^{\tau_1} - T_X^{\tau_2} = (T_{X \setminus H}^{F_H} - T_{X \setminus H}^{-F_H}) \ast T_X^{\tau_{12}}.
\]

Now the statement for big cells:

Theorem 4.11. On \( c \), the multivariate spline \( T_X \) agrees with a polynomial density in \( D(X) \).

Remark 4.12. Once Theorems 3.31 and 4.11 have been proven, it is easy to deduce from them that the generalized Khovanskii–Pukhlikov formula relating volumes and number of points holds [2]. Indeed, one can prove it easily sufficiently far from the walls (cf. [3]).

Finally, using a scalar product, we give Paradan’s formula for \( T_X \).

Theorem 4.13. (Paradan). Let \( \beta \in V \) be generic with respect to all the rational subspaces \( r \). Then, we have
\[
T_X = \sum_{r \in S_X} T_{X \setminus r}^{F_r^\beta} \ast (T_{X \setminus r})^{\tau(p_r^\beta)}.
\]

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