A simple derivation of the Schrödinger uncertainty relation

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Abstract
We show how the Schrödinger uncertainty relation for a pair of observables can be deduced using the Cauchy–Schwarz inequality plus successive applications of the commutation relation involving the two observables. Our derivation differs from the original one in the sense that we do not need the expansion of the product of these two observables in a sum of symmetrical and anti-symmetrical operators.

Keywords: quantum mechanics, formalism, uncertainty relations

1. Introduction

In 1930 Erwin Schrödinger presented [1, 2] a lower bound for the product of the dispersion of two non-commuting observables. This lower bound, from now on called the Schrödinger uncertainty relation (SUR), is more general than the usual Heisenberg uncertainty relation (HUR) taught in all quantum mechanics courses. In fact, HUR can be derived from SUR while SUR does not follow from HUR.

In this article we present an alternative derivation of Schrödinger’s relation. Different from the original one, we do not make use of the expansion of the product of two observables in a sum of symmetrical and anti-symmetrical operators.

2. Schrödinger’s derivation

Let $X$ and $P$ be our two non-commuting observables such that $[X, P] = XP - PX = 0$. Now we define the following two states

$$|\psi\rangle = X|\chi\rangle$$

(1)
and
\[ |\phi\rangle = P|\chi\rangle. \quad (2) \]

For the moment we assume \( \langle X \rangle = \langle \chi |X| \chi \rangle = 0 \) and \( \langle P \rangle = \langle \chi |P| \chi \rangle = 0 \). These quantities are, respectively, the mean values of \( X \) and \( P \) for a system described by the normalized state \( |\chi\rangle \). Applying the Cauchy–Schwarz inequality for the states (1) and (2),
\[ \langle \psi |\psi\rangle \langle \phi |\phi\rangle \geq |\langle \psi |\phi\rangle|^2 = \langle \psi |\phi\rangle \langle \phi |\psi\rangle, \]
we get
\[ \langle X^2 \rangle \langle P^2 \rangle \geq \langle XP \rangle \langle PX \rangle. \quad (4) \]

Remembering that the dispersions of \( X \) and \( P \) are \( \Delta X = (\langle X^2 \rangle - \langle X \rangle^2)^{1/2} \) and \( \Delta P = (\langle P^2 \rangle - \langle P \rangle^2)^{1/2} \), equation (4) becomes
\[ (\Delta X)^2 (\Delta P)^2 \geq \langle XP \rangle \langle PX \rangle. \quad (5) \]

Now we make use of Schrödinger’s ingenuity and write \( XP \) as a sum of symmetrical (S) and anti-symmetrical (A) operators:
\[ XP = \frac{XP + PX}{2} + \frac{XP - PX}{2} = S + A, \quad (6) \]
\[ PX = \frac{XP + PX}{2} - \frac{XP - PX}{2} = S - A. \quad (7) \]

Inserting equations (6) and (7) in equation (5) we obtain
\[ (\Delta X)^2 (\Delta P)^2 \geq \langle S + A \rangle \langle S - A \rangle \]
\[ \geq \langle S \rangle^2 - \langle A \rangle^2 = \frac{\langle [X, P] \rangle^2}{4} - \frac{\langle [X, P] \rangle^2}{4}, \quad (8) \]

where \( [X, P] = XP + PX \) is the anti-commutator of \( X \) and \( P \). Moreover, since \( X \) and \( P \) are Hermitian operators (observables) we know that \( [X, P] = iC \), where \( C \) is Hermitian and \( i = \sqrt{-1} \). Therefore, equation (8) can be written as
\[ (\Delta X)^2 (\Delta P)^2 \geq \frac{\langle [X, P] \rangle^2}{4} + \frac{|\langle [X, P] \rangle |^2}{4}. \quad (9) \]

Finally, if we had \( \langle X \rangle \neq 0 \) and \( \langle P \rangle \neq 0 \) the proof can be carried out by making the following substitutions:
\[ X \longrightarrow X - \langle X \rangle, \quad (10) \]
\[ P \longrightarrow P - \langle P \rangle. \quad (11) \]

Repeating the previous procedure we arrive at the general form of SUR:
\[ (\Delta X)^2 (\Delta P)^2 \geq \left( \frac{\langle [X, P] \rangle}{2} - \langle X \rangle \langle P \rangle \right)^2 + \frac{|\langle [X, P] \rangle |^2}{4}. \quad (12) \]

Note that HUR, namely, \( (\Delta X)^2 (\Delta P)^2 \geq |\langle [X, P] \rangle |^2 / 4 \), follows from equation (12) if we drop the first term at the right-hand side.
3. An alternative derivation

We now present a new way of deriving equations (9) and (12) without employing equations (6) and (7), while maintaining the same simplicity of the previous derivation. The key idea behind the following deduction lies on the convenient use of the commutator of $X$ and $P$.

Again, we begin with the simplest situation, i.e., $\langle X \rangle = \langle P \rangle = 0$ and, as before, our starting point is equation (5). Remembering the definition of the commutator of $X$ and $P$ we can write $XP$ and $PX$ as

\begin{align}
XP &= PX + [X, P], \quad (13) \\
PX &= XP - [X, P]. \quad (14)
\end{align}

Using equation (13) we can write equation (5) as

\[ (\Delta X)^2 (\Delta P)^2 \geq \langle PX \rangle^2 + \langle [X, P] \rangle \langle PX \rangle. \quad (15) \]

Now using equation (14) in equation (5) we get

\[ (\Delta X)^2 (\Delta P)^2 \geq \langle XP \rangle^2 - \langle [X, P] \rangle \langle PX \rangle. \quad (16) \]

Finally, using simultaneously equations (13) and (14) in equation (5) we obtain

\begin{align}
(\Delta X)^2 (\Delta P)^2 &\geq \langle XP \rangle \langle PX \rangle + \langle [X, P] \rangle \langle XP \rangle \\
&\quad - \langle [X, P] \rangle \langle PX \rangle - \langle [X, P] \rangle^2. \quad (17)
\end{align}

To finish the proof we add equations (5), (15)–(17), which gives

\[ (\Delta X)^2 (\Delta P)^2 \geq \frac{(\langle XP \rangle + \langle PX \rangle)^2}{4} - \frac{\langle [X, P] \rangle^2}{4}. \quad (18) \]

This equation can be written as

\[ (\Delta X)^2 (\Delta P)^2 \geq \frac{\langle [X, P] \rangle^2}{4} + \frac{\langle [X, P] \rangle^2}{4}. \quad (19) \]

if we remember that $X$ and $P$ are hermitian operators. Finally, if $\langle X \rangle \neq 0$ and $\langle P \rangle \neq 0$ the previous proof also works and we get SUR given by equation (12) if we repeat the previous procedure using equations (10) and (11).

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References

[1] Schrödinger E 1930 Zum Heisenbergschen Unschärfeprinzip Proc. Prussian Acad. Sci. 19 296–303 (in German)
[2] Angelow A and Batoni M C 1999 About Heisenberg uncertainty relation (by E Schrodinger) Bulg. J. Phys. 26 193–203 (arXiv:quant-ph/9903100; this article is an English translation of [1])