Abstract. We report here a computation giving the complete list of facets for the cut polytopes over several very symmetric graphs with 15 – 30 edges, including $K_8, K_{3,3,3}, K_{1,4,4}, K_{5,5},$ some other $K_{l,m}, K_{l,l,m},$ Prism$_7, A$Prism$_6,$ Möbius ladder $M_{12},$ Icosahedron, Heawood and Petersen graphs.

For $K_8$, it shows that the huge lists of facets of the cut polytope $\text{CUTP}_8$ and cut cone $\text{CUT}_8$, given in [9] is complete. We also confirm the conjecture that any facet of $\text{CUTP}_8$ is adjacent to a triangle facet.

The lists of facets for $K_{1,l,m}$ with $(l, m) = (4, 4), (3, 5), (3, 4)$ solve a problem (see, for example, [27]) in quantum information theory.

1. Introduction

Polyhedra, generated by cuts and by finite metrics are central objects of Discrete Mathematics; see, say, [16]. In particular, they are tightly connected with the well-known NP-hard optimization problems such as the max-cut problem and the unconstrained quadratic 0, 1 programming problem. To find their (mostly unknown) facets is the main approach to these problems.

Given a graph $G = (V, E)$, for a vertex subset $S \subseteq V = \{1, \ldots, n\}$, the cut semimetric $\delta_S(G)$ is a vector (actually, a symmetric $\{0, 1\}$-matrix) defined as

$$\delta_S(x, y) = \begin{cases} 1 & \text{if } xy \in E \text{ and } |S \cap \{x, y\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So, $\delta_S$ can be seen also as the adjacency matrix of a cut (into $S$ and $\overline{S}$) subgraph of $G$. Clearly, $\delta_{\{1,\ldots,n\}-S} = \delta_S$. A cut polytope $\text{CUTP}(G)$ and a cut cone $\text{CUT}(G)$ are defined as the convex hull of all such semimetrics and the positive span of all non-zero ones among them, respectively.

The dimension of $\text{CUTP}(G)$ and $\text{CUT}(G)$ is equal to the size $|E|$ of $G$. The most interesting and complicated case is $\text{CUTP}(K_n)$ and $\text{CUT}(K_n)$, denoted simply $\text{CUTP}_n, \text{CUT}_n$ and called the cut polytope and the cut cone.

In Table 2 we list information on the cut polytopes of several graphs. For all polytopes computed here, the number of vertices of $\text{CUTP}(G)$ was $2^n - 1$, where $n$ is the order $|V|$ of $G$. The data file of the groups and orbits of facets of considered polytopes is available from [17].

The hypermetric cone $\text{HYP}_n$ is the set of functions $d : \{1, \ldots, n\}^2 \rightarrow \mathbb{R}$ (actually, symmetric matrices over $\mathbb{R}_{\geq 0}$ having only zeroes on the diagonal), such that

$$H(b, d) = \sum_{1 \leq i < j \leq n} b_i b_j d(i, j) \leq 0 \text{ for all } b \in \mathbb{Z}^n, \sum_i b_i = 1.$$ 

The metric cone $\text{MET}_n$ is the set of all semimetrics on $n$ points, i.e., those of above functions, which satisfy all triangle inequalities, i.e., above inequality with $b$ being a
permutation of $(1, 1, -1, 0, \ldots, 0)$. If $b$ is a permutation of $(1, 1, 1, -1, -1, 0, \ldots, 0)$ or of $(1, 1, 1, -1, -1, 1, 0, \ldots, 0)$, the inequality is called pentagonal or 7-gonal, respectively. The bounding of $\text{MET}_n$ by $\binom{n}{2}$ perimeter inequalities $d_{ij} + d_{ik} + d_{jk} \leq 2$ produces the metric polytope $\text{METP}_n$.

We have the evident inclusions $\text{CUT}_n \subseteq \text{HYP}_n \subseteq \text{MET}_n$ and $\text{CUTP}_n \subseteq \text{METP}_n$ with $\text{CUT}_n = \text{MET}_n$ and $\text{CUTP}_n = \text{METP}_n$ only for $3 \leq n \leq 4$.

Also, $\text{CUT}_n = \text{HYP}_n$ only for $3 \leq n \leq 6$; the proper cones $\text{HYP}_7$ and $\text{HYP}_8$ are described in [14] and forthcoming [15], respectively.

In fact, $\text{CUT}_n$ is the set of all $n$-vertex semimetrics, which embed isometrically into some metric space $l_1$, and rational-valued elements of $\text{CUT}_n$ correspond exactly to the $n$-vertex semimetrics, which embed isometrically, up to a scale $\lambda \in \mathbb{N}$, into the path metric of some $N$-cube $K^N_2$. It shows importance of this cone in Analysis and Combinatorics.

Given a polyhedral cone or polytope $P$, let $e_P$ denote the number of extreme rays of cone $P$ or vertices of polytope $P$, while $o_P$ denote the number of corresponding orbits under the symmetry group of $P$. Let $f_P$ denote the number of facets of $P$ and $o_P$ denote the number of corresponding orbits. Table 1 gives $e_P(o_P)$ and $f_P(o_P)$ for $P = \text{CUTP}_n, \text{CUT}_n, \text{MET}_n, \text{METP}_n$ with $3 \leq n \leq 8$.

The enumeration of orbits of facets of $\text{CUT}_n$ and $\text{CUTP}_n$ for $n \leq 7$ was done in [26], [23] for $n = 5, 6$ and 7, respectively. The enumeration of orbits of extreme rays of $\text{MET}_7$ was done in [24]. The orbits of vertices of $\text{METP}_n$ were enumerated in [11] for $n = 7$ and in [12] for $n = 8$. For $n \leq 6$ such enumeration is easy.

The symmetry group of a graph $G = (V, E)$ induces symmetry of $\text{CUTP}(G)$. For any $U \subset \{1, \ldots, n\}$, the map $\delta_S \mapsto \delta_{U \Delta S}$ also defines a symmetry of $\text{CUTP}(G)$. Together those form the restricted symmetry group $\text{ARes}(\text{CUTP}(G))$ of order $2^{|V|} |\text{Aut}(G)|$. The full symmetry group $\text{Aut}(\text{CUTP}(G))$ may be larger. For $n \neq 4$, $\text{ARes}(K_n) = \text{Aut}(\text{CUTP}(K_n))$ but $\text{Aut}(\text{CUTP}(K_4)) = \text{Aut}(K_{4,4})$ ([13]). So, $|\text{Aut}(\text{CUTP}(K_n))|$ is $2^{n-1}n!$ for $n \neq 4$ and $6 \times 2^4 4!$ for $n = 4$. The symmetry group of the cones $\text{CUT}_n$ and $\text{MET}_n$ is simply the symmetric group $\text{Sym}(n)$. Clearly, it holds:

1. If $G$ is $K_{l,m}$ (and so, $|\text{Aut}(K_{l,m})| = l!m!$ for $2 \leq l < m$ and twice it for $l = m$), then $\text{Aut}(\text{CUTP}(G)) = \text{ARes}(\text{CUTP}(G)) = 2^{l+m-1} |\text{Aut}(K_{l,m})|$ for $2 < l \leq m$, while $\text{Aut}(\text{CUTP}(G))$ is $m!2^m-1$ times larger if $2 < l < m$ and $m$ times larger if $m = l = 2$.

2. $|\text{Aut}(\text{CUTP}(K_{l,m}))| = 2|\text{Aut}(\text{CUTP}(K_{l,m}))|$. 

3. If $G$ is $\text{Prism}_m$ ($m \neq 4$), $\text{APrism}_m$ ($m \neq 3$) or Möbius ladder $m$, then $|\text{Aut}(\text{CUTP}(G))| = |\text{ARes}(\text{CUTP}(G))| = 2^{2m-1} |\text{Aut}(G)| = m2^{2m+1}$.

2. Computational methods

The second author has developed over the years an effective computer program ([13]) for enumerating facets of polytopes which are symmetric. The technique used is adjacency decomposition method, which was originally introduced in [2] and applied to the Transporting Salesman polytope, the Linear Ordering polytope and the cut polytope.

The algorithm is detailed in Algorithm 1 and surveyed in [8]. The initial facet of the polytope $P$ is obtained via linear programming. The tests of equivalence are done via the GAP functionality of permutation group and their implementation of partition backtrack. The problematic aspect is computing the facets adjacent to a
TABLE 1. The number of facets and vertices (or extreme rays) in the cut and metric polytopes (or cones) for $n \leq 8$.

| $P$            | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ |
|----------------|---------|---------|---------|---------|---------|---------|
| CUTF$e$        | 4(1)    | 8(1)    | 16(1)   | 32(1)   | 64(1)   | 128(1)  |
| CUTF$e,f$      | 3(1)    | 7(2)    | 15(2)   | 31(3)   | 63(3)   | 127(4)  |
| MET$e$         | 3(1)    | 12(1)   | 40(2)   | 210(4)  | 38,780(36) | 49,604,520(2,169) |
| MET$e,f$       | 3(1)    | 7(2)    | 25(3)   | 296(7)  | 55,226(46) | 119,269,588(3,918) |
| METP$e$        | 4(1)    | 8(1)    | 32(2)   | 554(3)  | 275,840(13) | 1,550,825,600(533) |
| METP$e,f$      | 4(1)    | 16(1)   | 40(1)   | 80(1)   | 140(1)  | 224(1)  |

TABLE 2. The number of facets of the cut polytopes CUTF($G$) of some graphs. $|\text{Aut(CUTF($G$))}| > 2^{|V|-1} |\text{Aut($G$)}|$ is indicated by $^*$.

| $G = (V, E)$ | $|V|$ | $|E|$ | $|\text{Aut(CUTF($G$))}|$ | Number of facets (orbits) |
|--------------|------|------|----------------------------|---------------------------|
| $K_8$        | 8    | 28   | 5,160,960                  | 217,093,472(147)          |
| $K_{3,3,3}$  | 9    | 27   | 331,776                    | 624,406,788(2015)         |
| $K_{1,4,4}$  | 9    | 24   | 294,912                    | 36,391,264(175)           |
| $K_{1,3,5}$  | 9    | 23   | 184,320                    | 71,340(7)                 |
| $K_{1,3,4}$  | 8    | 19   | 18,432                     | 12,480(6)                 |
| $K_{1,3,3}$  | 7    | 15   | 4,608                      | 684(3)                    |
| $K_{1,2,m}, 2 \leq m \leq 6$; $m+3$ & $3m+2$ | $2^m \cdot |\text{Aut} (K_{2,m})|$ | $8m + 8 \binom{m}{2}(2)$ |
| $K_{5,5}$    | 10   | 25   | 14,745,600                 | 16,482,678,610(1,282)     |
| $K_{4,7}$    | 11   | 28   | 123,863,040                | 271,596,584(15)           |
| $K_{4,6}$    | 10   | 20   | 8,847,360                  | 23,179,008(12)            |
| $K_{4,5}$    | 9    | 20   | 737,280                    | 983,560(8)                |
| $K_{4,4}$    | 8    | 16   | 147,456                    | 27,968(4)                 |
| $K_{3,m}, 3 \leq m \leq 6$; $m+3$ & $3m$ | $2^{m+1} \cdot |\text{Aut} (K_{3,m})|$ | $6m + 24 \binom{m}{2}(2)$ |
| $K_{2,m}, 2 \leq m \leq 8$; $m+2$ & $2m$ | $2^{m+1} \cdot |\text{Aut} (K_{2,m})|$ | $4m^2(1)$                 |
| Icosahedron  | 12   | 30   | 245,760                    | 1,552(4)                  |
| Cube         | 8    | 12   | 6,144                      | 200(3)                    |
| Cuboctahedron| 12   | 24   | 98,304                     | 1,360(5)                  |
| Tr. Tetrahedron| 12  | 18   | 4,152                      | 540(4)                    |
| APRism$_6$   | 12   | 24   | 49,152                     | 2,032(5)                  |
| Prism$_7$    | 14   | 21   | 229,376                    | 7,394(6)                  |
| Möbius ladder $M_{12}$ | 12   | 18   | 491,152                    | 26,452(6)                 |
| Heawood graph| 14   | 21   | 2,752,512                  | 5,361,194(9)              |
| Petersen graph| 10  | 15   | 61,440                     | 3,614(4)                  |

facet. This is itself a dual description problem for the polytope defined by the facet $F$. 

An interesting problem is the check when $\mathcal{R}$ is complete. Of course, if all the orbits are treated, i.e. if $\mathcal{R} = \mathcal{D}$, then the computation is complete. However, sometimes we can conclude before that:

**Theorem 1.** Let $G(P)$ be the skeleton graph of a $m$-dimensional polytope.

(i) $G(P)$ is at least $m$-connected.

(ii) If we remove all the edges contained in a given face $F$, then the remaining graph is still connected.

Hence, if the total set of facets, equivalent to a facet in $\mathcal{R} \setminus \mathcal{D}$, has size at most $m - 1$ or contains a common vertex, then we can conclude that $\mathcal{R}$ is complete. The requirements for applying any of those two criterions are rather severe, but they are easy to check and, when applicable, the benefits are large. The geometry underlying Theorem 1 is illustrated in Figure 1.

**Algorithm 1:** The adjacency decomposition method

The adjacency decomposition method works well when the polytope is symmetric but it still relies on computing a dual description. This is easy when the incidence of the facet is low but become more and more problematic for facets of large incidence. However, quite often high incidence facets also have large symmetry groups. Therefore, a natural extension of the technique is to apply the method

**Figure 1.** Illustration of connectivity results of Theorem 1.
recursively and so, obtain the *recursive adjacency decomposition method*, which is again surveyed in [8].

In order to work correctly, the method requires several ingredients. One is the ability to compute easily automorphism group of the polytope, see [7] for details. Another is good heuristics for deciding when to apply the method recursively or not. Yet another is a storing system for keeping dual description that may be reused, and this again depends on some heuristics. The method has been applied successfully on numerous problems [19, 20, 21] and here.

The framework, that we have defined above, can be applied, in order to sample facets of a polytope, say, $P$. A workable idea is to use linear programming and then get some facets of $P$. But doing so, we overwhelmingly get facets of high incidence, while we may be interested in obtaining facets of low incidence. One can adapt the adjacency decomposition method to do such a sampling. Let us call two facets *equivalent* if their incidence is the same. By doing so, we remove the combinatorial explosion, which is the main difficulty of such dual-description problem. At the end, we get a number of orbits of facets of different incidence, which give an idea of the complexity of the polytope.

3. The facets of CUTP$_8$

In [9], the adjacency decomposition method was introduced and was applied to the Transporting Salesman polytope, the Linear Ordering polytope and the cut polytope. For the cut polytope CUTP$_8$, the authors found 147 orbits, consisting of 217,093,472 facets, but this list was potentially incomplete, since they were not able to the treat the triangle, pentagonal and 7-gonal inequalities (defined in Section 3) at that time. Therefore, they only prove that the number of orbits is at least 147. The enumeration is used in [4, 3] for work in quantum mechanics.

Sometimes ([16, 1]) this is incorrectly understood and it is reported that the number of orbits is exactly 147 with [9] as a reference.

Here we show that Christof-Reinelt’s list is complete. We had to treat the 3 remaining orbits of facets since we could not apply Theorem [1]. However, we could apply the theorem in deeper levels of the recursive adjacency method and this made the enumeration faster.

**Conjecture 1.** ([16, 10]) Any facet of CUTP$_n$ is adjacent to a triangle inequality facet.

The conjecture was checked for $n \leq 7$; here we confirm it for $n = 8$.

Looking for a counterexample to this conjecture, we applied our sampling framework to CUTP$_n$ for $n = 10, 11$ and 12. We got initial facets of low incidence and then we complemented this with random walks in the set of all facets. This allowed us to find many simplicial facets (more than 10,000 for each) of these CUTP$_n$ but all of them were adjacent to at least one triangle inequality facet.

4. Correlation polytopes of $K_{n,m}$

In quantum physics and quantum information theory, Bell inequalities, involving joint probabilities of two probabilistic events, are exactly inequalities valid for the *correlation polytope* CORP$(G)$ (called also *Boolean quadric polytope* $BQP(G)$) of a graph, say, $G$. In particular, CORP$(K_{n,m})$ is seen in quantum theory as the set of possible results of a series of Bell experiments with a non-entangled (separable)
quantum state shared by two distant parties, where one party (Alice) has \( n \) choices of possible two-valued measurements and the other party (Bob) has \( m \) choices. Here, a valid inequality of \( \text{CORP}(K_{n,m}) \) is called a Bell inequality and if facet inducing, a tight Bell inequality. This polytope is equivalent (linearly isomorphic via the covariance map) to the cut polytope of \( K_{1,n,m} \). [10] Section 5.2]. Similarly, \( \text{CUTP}(K_{1,n,m,l}) \) represents three-party Bell inequalities.

The symmetry group of \( \text{CORP}(K_{n,n}) \) and \( \text{CUTP}(K_{1,n,m}) \) is of order \( 2^{1+n+m}n!m! \).

We computed all facets of \( \text{CORP}(K_{n,m}) \) with \((n, m) = (4, 4), (3, 5), (3, 4) \) and confirmed known enumerations for \((n, m) = (2, 2), (3, 3) \). In fact, the page [27] collects the progress on finding Bell inequalities. The case of \( \text{CORP}(K_{n,n}) \) is called there \((n, n, 2)\)-setting. The cases \( n = 2, 3 \) were settled in [22] and [25], respectively. For \( n = 4 \), only partial lists of facets were known; our 175 orbits of 36, 391, 264 facets of \( \text{CORP}(K_{4,4}) \) finalize this case.

In contrast to the Bell’s inequalities, which probe entanglement between spatially-separated systems, the Leggett-Garg inequalities test the correlations of a single system measured at different times. The polytope, defined by those inequalities for \( n \) observables, is, actually (2), the cut polytope \( \text{CUTP}_n \).

\[ \text{Remark 1. Given a graph } G = (V, E), \text{ an edge } (v_1 v_2) \in E, \text{ an integer } s \geq 3 \text{ and a } s \text{-cycle } (v_1, \ldots, v_s) \text{ of } G, \text{ we call } x(v_1, v_2) \geq 0 \text{ and } \sum_{i=1}^{s-1} x(v_i, v_{i+1}) - x(v_1, v_s) \geq 0 \text{ edge face and } s \text{-cycle face, respectively. Clearly, these inequalities are face-defining, since they are valid on any cut of } G. \text{ There are } 2|E| \text{ edge faces and their incidence is } 2^{|V| - 2}. \text{ Each } s \text{-cycle of } G \text{ gives } 2^{s-1} \text{ } s \text{-cycle faces, which are of incidence } s2^{|V| - s}. \text{ Depending on the cycle structure and the decomposition into orbits, this gives many orbits of putative facets of } \text{CUTP}(G). \]

5. Acknowledgments

Second author gratefully acknowledges support from the Alexander von Humboldt foundation. Second author also thanks Edward Dahl from DWave for suggesting the problem of computing the dual description of \( \text{CORP}(K_{4,4}) \).

References

[1] B. Assarf, E. Gawrilow, K. Herr, M. Joswig, A. Paffenholz, and T. Rehn, polymake in linear and integer programming, preprint at arxiv:arXiv:1408.4653, August 2014.
[2] D. Avis, P. Hayden, and M. M. Wilde, Leggett-Garg inequalities and the geometry of the cut polytope, Phys. Rev. A (3) 82 (2010), no. 3, 030102, 4, URL: http://dx.doi.org/10.1103/PhysRevA.82.030102.
[3] D. Avis, H. Imai, and T. Ito, On the relationship between convex bodies related to correlation experiments with dichotomic observables, J. Phys. A 39 (2006), no. 36, 11283–11299, URL: http://dx.doi.org/10.1088/0305-4470/39/36/010.
[4] D. Avis, H. Imai, T. Ito, and Y. Sasaki, Two-party Bell inequalities derived from combinatorics via triangular elimination, J. Phys. A 38 (2005), no. 50, 10971–10987, URL: http://dx.doi.org/10.1088/0305-4470/38/50/007.
[5] D. Avis and Mutti, All the facets of the six-point Hamming cone, European J. Combin. 10 (1989), no. 4, 309–312, URL: http://dx.doi.org/10.1016/S0195-6698(89)80002-2.
[6] M. L. Balinski, On the graph structure of convex polyhedra in n-space, Pacific J. Math. 11 (1961), 431–434.
[7] D. Bremner, M. Dutour Sikirić, D. V. Pasechnik, T. Rehn, and A. Schürmann, Computing symmetry groups of polyhedra, LMS J. Comput. Math. 17 (2014), no. 1, 565–581.
[8] D. Bremner, M. Dutour Sikirić, and A. Schürmann, *Polyhedral representation conversion up to symmetries*, Polyhedral computation, CRM Proc. Lecture Notes, vol. 48, Amer. Math. Soc., Providence, RI, 2009, pp. 45–71.

[9] T. Christof and G. Reinelt, *Decomposition and parallelization techniques for enumerating the facets of combinatorial polytopes*, Internat. J. Comput. Geom. Appl. 11 (2001), no. 4, 433–440, URL: [http://dx.doi.org/10.1142/S0218195901000560](http://dx.doi.org/10.1142/S0218195901000560).

[10] A. Deza and M. Deza, *On the skeleton of the dual cut polytope*, Discrete and computational geometry, Lecture Notes in Comput. Sci., vol. 2866, Springer, Berlin, 2003, pp. 118–128, URL: [http://dx.doi.org/10.1007/978-3-540-44400-8_12](http://dx.doi.org/10.1007/978-3-540-44400-8-12).

[11] A. Deza and M. Deza, *The contact polytope of the Leech lattice*, Discrete Comput. Geom. 44 (2010), no. 4, 904–911, URL: [http://dx.doi.org/10.1007/s00454-010-9266-z](http://dx.doi.org/10.1007/s00454-010-9266-z).

[12] M. Dutour Sikirić, *Cut polytopes*, URL: [http://mathieudutour.altervista.org/CutPolytopes/](http://mathieudutour.altervista.org/CutPolytopes/).

[13] M. Dutour Sikirić, *Polyhedral*, URL: [http://mathieudutour.altervista.org/Polyhedral/](http://mathieudutour.altervista.org/Polyhedral/).

[14] V. P. Grishukhin, *All faces of the cut cone \( C_n \) for \( n = 7 \) are known*, European J. Combin. 11 (1990), no. 2, 115–117, URL: [http://dx.doi.org/10.1016/S0195-6698(13)80064-9](http://dx.doi.org/10.1016/S0195-6698(13)80064-9).

[15] V. P. Grishukhin, *Computing extreme rays of the metric cone for seven points*, European J. Combin. 13 (1992), no. 3, 153–165, URL: [http://dx.doi.org/10.1016/0195-6698(92)90021-Q](http://dx.doi.org/10.1016/0195-6698(92)90021-Q).

[16] M. E. Tylkin (=M. Deza), *On Hamming geometry of unitary cubes*, Soviet Physics. Dokl. 5 (1960), 940–943.
