An adaptive multilevel Monte Carlo algorithm for the stochastic drift-diffusion-Poisson system

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Abstract

We present an adaptive multilevel Monte Carlo algorithm for solving the stochastic drift-diffusion-Poisson system with non-zero recombination rate. The a-posteriori error is estimated to enable goal-oriented adaptive mesh refinement for the spatial dimensions, while the a-priori error is estimated to guarantee quadratic convergence of the $L^2$-error. In the adaptive mesh refinement, efficient estimation of the error indicator gives rise to better error control. For the stochastic dimensions, we use the multilevel Monte Carlo method to solve this system of stochastic partial differential equations. Finally, the advantage of the technique developed here compared to uniform mesh refinement is discussed using a realistic numerical example.

Keywords: Stochastic partial differential equation (SPDEs), stochastic drift-diffusion-Poisson system, adaptive mesh refinement, a-priori error estimation, a-posteriori error estimation, multilevel Monte Carlo.

AMS subject classifications: 60H15, 35R60, 65M50, 82D37

1. Introduction

The stochastic drift-diffusion-Poisson (DDP) system is a general model for charge transport in random environments. A leading example is the field-effect transistor (FET), where the stochastic coefficients can describe process variations, noise, and fluctuations in devices as diverse as transistors and sensors. Process variations, noise, and fluctuations are significantly important especially in devices scaled into the deca-nanometer regime, as random effects become more important in smaller devices. Among the many sources of noise, random-dopant fluctuations (RDF) are one of the most important. Random-dopant fluctuations stem from the fact that the doping process in the semiconductors leads to a random number and random position of dopants. Therefore, each impurity atom influences the charge transport and the mobilities. A schematic diagram is shown in Figure 1.

The finite-element method (FEM) is employed here to discretize the spatial dimensions. In the error analysis, only the discretization error, i.e., insufficient mesh density to properly capture the solution, is considered. A-priori error

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estimates yield knowledge about convergence and stability of the solvers and information on the asymptotic behavior of errors for different mesh sizes [1]. A-posteriori error estimates make it possible to control the mesh on the entire computational domain by using adaptive algorithms, i.e., by focusing computational effort on the parts of the domain which contribute most to the total error [2]. In adaptive mesh refinement, a-posteriori error estimators are used to indicate where the error is particularly high, and then more mesh elements are placed in those locations. Here we estimate the local error for a coupled system of equations. The error estimate indicates which elements should be refined or coarsened simultaneously for the Poisson equation and the drift-diffusion equations.

The mentioned stochastic problem is computationally very expensive; in order to obtain an acceptable error, thousands of simulations are necessary. In stochastic PDE, the Monte Carlo (MC) method is one of the most popular and straightforward numerical techniques. However, the main drawback of the MC method is its well-known convergence rate. The multilevel Monte Carlo finite-element (MLMC-FE) method [3, 4] is an efficient numerical alternative. In [5], we introduced an optimal MLMC-FE method to model the random effects in a charge-transport model. The optimality indicates the best choice of mesh sizes and the number of samples to minimize the computational complexity given a total error. In [6], the efficiency of the method for three-dimensional simulations of various nanoscale devices was investigated in detail. Convergence can be improved by using a randomized rank-1 lattice rule [7, 8].

The first analysis of a finite-element method, a one-dimensional one, for solving the (deterministic) DDP system can be found in [9]. An extension of the analysis to the two-dimensional problem was presented in [10]. In [11], fixed points of finite-element discretizations were used to approximate the solutions of the steady-state drift-diffusion system, and the convergence rate in the energy norm was estimated. In [12], the optimal convergence rate and its stability were shown. However, in all these publications, the systems are in thermal equilibrium, i.e., the recombination rate is...
zero. In the present work, the first purpose is an a-priori error estimate of the stationary stochastic DDP system with a nonzero recombination rate.

In [13], an adaptive stochastic Galerkin finite-element method for linear elliptic boundary-value problems was presented. The idea of using an adaptive MLMC method for weak approximations of solutions of stochastic differential equations was explained in [14]. In [15], an adaptive MLMC algorithm was introduced for PDEs with stochastic data.

In [5], we showed the effectiveness of using an MLMC method as an alternative to the MC method in the solving the stochastic DDP system. There the meshes were refined uniformly. The novelty of the present work is computing an accurate local-error estimator based on goal oriented error estimation for the coupled system of equations. Therefore, a smaller number of meshes must be refined and the method is more computationally effective. Randomness in the model problem considered here stems from the random position of dopants in a transistor, which affect the error and hence the refinement process. Since MLMC is a variance-reduction method, the faster decay in the variance of the MLMC method leads to a reduction of the statistical error and therefore the total computational cost. Here the effect of adaptive mesh refinement on the reduction of the variance is also taken into account.

The remainder of this manuscript is as follows. In Section 2, we give the system of model equations with stochastic coefficients and explain its boundary conditions. In Section 3, we present the error estimates in the finite-element space, namely an a-priori error estimate and an a-posteriori error estimate. In Section 4, we introduce the MLMC finite-element method for the DDP system and define an optimization problem to minimize the total computational cost. Then, we present numerical results for a transistor and quantify the random-dopants effect in Section 5. The adaptive MLMC-FE method is used to approximate the expected value of the solution of the system of equations with random coefficients. Also, the method is compared with MLMC-FE method with uniform mesh refinement. Finally, conclusions are drawn in Section 6.

2. The Stochastic Model Problem

The stochastic Poisson equation is used generally for the electrostatic potential

$$- \nabla \cdot (A(x, \omega) \nabla V(x, \omega)) = \rho(x, \omega),$$

on the bounded Lipschitz domain $D \subset \mathbb{R}^d$ ($d \leq 3$). In the equation, $V$ indicates the electrostatic potential, $A$ denotes the dielectric constant (permittivity), and $\rho$ is the charge concentration. In (1), $x \in D$ is the spatial variable, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is an $n$-dimensional random variable defined on the complete probability space $(\Omega, \mathcal{A}, P)$ equipped with $\mathcal{A} \subset 2^\Omega$ as the $\sigma$-algebra of events, $P: \mathcal{A} \rightarrow [0, 1]$ as a probability measure, the sample space $\Omega$. The randomness arises from the random distribution of dopant atoms (uniformly distributed) in source and drain areas (shown in Figure 1).
In a semiconductor, the charge concentration is derived by the free electron and hole densities (i.e., $n$ and $p$) and the doping concentration $C$; the total charge concentration is therefore

$$\rho = q(p - n + C).$$

We change the concentrations $n$ and $p$ to the so-called Slotboom variables $u$ and $v$, which are given by

$$n(x, \omega) = n_i e^{V(x, \omega)/U_T} u(x, \omega), \quad (2a)$$

$$p(x, \omega) = n_i e^{-V(x, \omega)/U_T} v(x, \omega). \quad (2b)$$

Here, $n_i$ is the intrinsic carrier density of the semiconductor (in the numerical examples, a value of $1.5 \times 10^{10}$ cm$^{-3}$ is used for silicon) and $U_T$ indicates the thermal voltage, which is at room temperature is about 26 mV.

A schematic diagram of a sample computational geometry is shown in Figure 1. The domain is partitioned into two subdomains, i.e.,

$$D = D_{Si} \cup D_{ox}.$$  

The first subdomain $D_{Si}$ consists of silicon, i.e., the channel and the source and drain areas, where the drift-diffusion-Poisson system models the charge transport. The gate contact is separated from the channel by an insulating silicon dioxide layer $D_{ox}$. In $D_{ox}$, there is no charge transport and therefore we only have the Poisson equation.

The boundary $\partial D$ of the domain $D$ is separated into $\partial D_D$ and $\partial D_N$, which denote the surfaces where the Dirichlet and Neumann boundary conditions

$$V|_{\partial D_D} = V_D \quad \text{and} \quad n \cdot \nabla V|_{\partial D_N} = 0 \quad (3)$$

hold, where $n$ is the outward pointing unit normal on the boundary $\partial D_N$. Dirichlet boundary conditions $V|_{\partial D_D}$ are applied for the potential at the source, drain, and gate contacts, i.e., $V = V_S$, $V = V_D$, and $V = V_G$.

Neumann boundary conditions are applied on all other boundaries. On the Neumann parts $\partial D_N$ of the boundary the currents and the electric field are assumed to vanish in the normal direction to the surface. This yields the three Neumann boundary conditions

$$\partial V(x) \cdot n = 0 \quad \forall x \in \partial D_N,$$

$$J_n(x) \cdot n = 0 \quad \forall x \in \partial D_N,$$

$$J_p(x) \cdot n = 0 \quad \forall x \in \partial D_N.$$
The stochastic DDP system

\[-\nabla \cdot (A(x, \omega) \nabla V(x, \omega)) = q \left(C(x, \omega) + p(x, \omega) - n(x, \omega)\right), \quad (4a)\]

\[\nabla \cdot J_n(x, \omega) = qR(x, \omega), \quad (4b)\]

\[-\nabla \cdot J_p(x, \omega) = qR(x, \omega), \quad (4c)\]

\[J_n(x, \omega) = q(D_n \nabla n(x, \omega) - \mu_n n(x, \omega) \nabla V(x, \omega)), \quad (4d)\]

\[J_p(x, \omega) = q(-D_p \nabla p(x, \omega) - \mu_p p(x, \omega) \nabla V(x, \omega)) \quad (4e)\]

is employed to model self-consistent charge transport, where \(J_n\) and \(J_p\) are the current densities, \(\mu_n\) and \(\mu_p\) are the mobilities, and \(D_n\) and \(D_p\) are diffusion coefficients, which can be calculated by the Einstein relations \(D_p = U_T \mu_p\) and \(D_n = U_T \mu_n\).

Furthermore, \(R\) is Shockley-Read-Hall recombination rate, which is defined by

\[R_{SRH}(n, p) := \frac{np - n_i^2}{\tau_n (p + n_i) + \tau_p (n + n_i)},\]

where \(\tau_n\) and \(\tau_p\) are the lifetimes of the free carriers (absolutely positive). For the purposes of the present work, any other recombination rate can be used as long as it satisfies modest assumptions [5].

Using the Slotboom variables \(u\) and \(v\) defined in [7], the DDP system [4] takes the form

\[-\nabla \cdot (A(x) \nabla V(x, \omega)) = q \left(C(x, \omega) - n_i \left(e^{U_T V(x, \omega)} - e^{(-U_T V(x, \omega))}\right)\right) \quad (5a)\]

\[U_T n_i \nabla \cdot (\mu_n e^{U_T V(x, \omega)} u(x, \omega)) = R(x, \omega), \quad (5b)\]

\[U_T n_i \nabla \cdot (\mu_p e^{-U_T V(x, \omega)} v(x, \omega)) = R(x, \omega) \quad (5c)\]

with the recombination rate

\[R(x, \omega) = n_i \frac{u(x, \omega)v(x, \omega) - 1}{\tau_p(e^{U_T V(x, \omega)} + 1) + \tau_n(e^{-U_T V(x, \omega)} + 1)}.\]

The Dirichlet boundary conditions for the Slotboom variables are

\[u(x, \omega)|_{\partial D_{h,0}} = u_D(x) \quad \text{and} \quad v(x, \omega)|_{\partial D_{h,0}} = v_D(x). \quad (6)\]

Zero Neumann conditions are used for the Slotboom variables as well. The interface conditions

\[V(0+, y, \omega) - V(0-, y, \omega) = 0 \quad \text{on} \; \Gamma,\]

\[A(0+) \partial_y V(0+, y, \omega) - A(0-) \partial_y V(0-, y, \omega) = 0 \quad \text{on} \; \Gamma\]

can be used to model the presence of a layer of charge carriers at the surface of a FET after homogenization [16]. Here \(\Gamma\) is the interface or surface between \(D_{h1}\) and \(D_{h0}\), and the notation \(0^+\) and \(0^-\) denotes the limits from both sides of the interface \(\Gamma\) located at \(x = 0\). The directions of \(y\) are along the interface. We make these assumptions regarding the existence and uniqueness of the solution [5].
Assumptions 1. 1. The bounded computational domain $D \subset \mathbb{R}^3$ has a $C^2$ Dirichlet boundary $\partial D_D$, the Neumann boundary $\partial D_N$ consists of $C^2$ segments, and the Lebesgue measure of the Dirichlet boundary $\partial D_D$ which is nonzero. The $C^2$ manifold $\Gamma \subset D$ separates the domain $D$ into two nonempty regions $D^+$ and $D^-$; therefore, $\text{meas}(\Gamma \cap \partial D) = 0$ and $\Gamma \cap \partial D \subset \partial D_N$ hold.

2. The coefficient $A(x, \omega)$ is assumed to be a strongly measurable mapping from $\Omega$ into $L^\infty(D)$. It is uniformly elliptic and bounded function of position $x \in D$ and the elementary event $\omega \in \Omega$, i.e., there exist constants $0 < A^- < A^+ < \infty$ such that

$$0 < A^- \leq \text{ess inf}_{x \in D} A(x, \omega) \leq \|A(\cdot, \omega)\|_{L^\infty(D)} \leq A^+ < \infty \quad \forall \omega \in \Omega.$$  

Moreover, $A(x, \omega)|_{D^+ \times \Omega} \in C^1(D^+ \times \Omega, \mathbb{R}^{3 \times 3})$ and $A(x, \omega)|_{D^- \times \Omega} \in C^1(D^- \times \Omega, \mathbb{R}^{3 \times 3})$.

3. The doping concentration $C_{\text{dep}}(x, \omega)$ is bounded above and below with the bounds

$$\underline{C} := \inf_{x \in D} C_{\text{dep}}(x, \omega) \leq C_{\text{dep}}(x, \omega) \leq \sup_{x \in D} C_{\text{dep}}(x, \omega) =: \overline{C} \quad \forall \omega \in \Omega.$$  

4. There is a constant $R \ni K \geq 1$ which satisfies

$$\frac{1}{K} \leq u_D(x), v_D(x) \leq K \quad \forall x \in \partial D_{\text{SI}, D}.$$  

5. The functionals $M_a : L^2(\Omega; H^1(D)) \cap L^\infty(D \times \Omega) \to L^2(\Omega; H^{1/2}(\Gamma)) \cap L^\infty(\Gamma \times \Omega)$ and $M_p : L^2(\Omega; H^1(D)) \cap L^\infty(D \times \Omega) \to L^\infty(\Gamma \times \Omega)$ are continuous.

6. The electron and hole mobilities are uniformly bounded functions of $x \in D$ and $\omega \in \Omega$. Therefore, $\forall x \in D$ as well as $\forall \omega \in \Omega$ we have

$$0 < \mu^-_e \leq \mu_e(x, \omega) \leq \mu^+_e < \infty$$

$$0 < \mu^-_h \leq \mu_h(x, \omega) \leq \mu^+_h < \infty,$$

where $\mu_p(x, \omega), \mu_h(x, \omega) \in C^1(D_{\text{SI}} \times \Omega, \mathbb{R}^{3 \times 3})$.

Moreover, the inclusions $f(x, \omega) \in L^2(\Omega; L^2(D)) \cap L^\infty(D \times \Omega)$, $V_D(x) \in H^{1/2}(\partial D) \cap L^\infty(\Gamma)$, $u_D, v_D(x) \in H^{1/2}(\partial D_S)$, $\sigma(x, \omega) \in L^2(\Omega; H^{1/2}(\Gamma))$, and $\gamma(x, \omega) \in L^2(\Omega; L^2(\Gamma))$ hold.

The mentioned assumptions also guarantee the uniform ellipticity of the Poisson and the continuity equations.
Finally, we can write the following boundary-value problem

\[-\nabla \cdot (A(x)\nabla V(x,\omega)) = q\left(C(x,\omega) - n_i\left(e^{V(x,\omega)/\tau}u(x,\omega) - e^{-V(x,\omega)/\tau}v(x,\omega)\right)\right) \text{ in } D_{S_1}, \tag{7a}\]

\[-\nabla \cdot (A(x)\nabla V(x,\omega)) = 0 \text{ in } D_{S_0}, \tag{7b}\]

\[U_T \nabla \left(\mu_0 e^{-\frac{V(x,\omega)}{\tau}} \nabla u(x,\omega)\right) = \frac{u(x,\omega) v(x,\omega) - 1}{\tau_p\left(e^{\frac{\tau}{\tau_p}} u(x,\omega) + 1\right) + \tau_n\left(e^{\frac{\tau}{\tau_n}} v(x,\omega) + 1\right)} \text{ in } D_{S_1}, \tag{7c}\]

\[U_T \nabla \left(\mu_0 e^{-\frac{V(x,\omega)}{\tau}} \nabla v(x,\omega)\right) = \frac{u(x,\omega) v(x,\omega) - 1}{\tau_p\left(e^{\frac{\tau}{\tau_p}} u(x,\omega) + 1\right) + \tau_n\left(e^{\frac{\tau}{\tau_n}} v(x,\omega) + 1\right)} \text{ in } D_{S_1}, \tag{7d}\]

\[V(0^+, y, \omega) - V(0^-, y, \omega) = 0 \text{ on } \Gamma, \tag{7e}\]

\[A(0+) \partial_2 V(0^+, y, \omega) - A(0-) \partial_2 V(0^-, y, \omega) = 0 \text{ on } \Gamma, \tag{7f}\]

\[V(x, \omega) = V_D(x) \text{ on } \partial D_D, \tag{7g}\]

\[n \cdot \nabla V(x, \omega) = 0 \text{ on } \partial D_N, \tag{7h}\]

\[u(x, \omega) = u_D(x), \quad v(x, \omega) = v_D(x) \text{ on } \partial D_{D,S_1}, \tag{7i}\]

\[n \cdot \nabla u(x, \omega) = 0, \quad n \cdot \nabla v(x, \omega) = 0 \text{ on } \partial D_{N,S_1} \tag{7j}\]

for all \(\omega \in \Omega\).

3. Error Estimates for the Finite-Element Method

The finite-element method can be used to estimate approximations of the solution of (7). Here, we derive both a-priori and a-posteriori error estimates. The main feature of an a-priori error estimate is providing knowledge about the asymptotic behavior of the discretization error. A-posteriori estimates use the approximate solution \((V_h, u_h, v_h)\) to give an estimate for the error of the solutions. In adaptive mesh refinement, a-posteriori error estimators are then used to denote where the error is specifically high. Therefore, mesh refinement is used to control the discretization error.

**Definition 3.1.** Suppose \(D \subseteq \mathbb{R}^d \ (d \in \mathbb{N})\). For \(1 \leq p \leq \infty\), let us define

\[L^p(D) := \{ u \mid u \text{ is measurable on } D \text{ and } \| u \|_{L^p(D)} < \infty \},\]

where \(\| u \|_{L^p(D)} := \left(\int_D |u(X)|^p dX\right)^{\frac{1}{p}}\). Particularly, the space \(L^2(D)\) is a Hilbert space equipped with the inner product

\[(u, v)_D := \int_D u(x)v(x)dx \quad \forall u, v \in L^2(D)\]

and hence the norm

\[\| u \|_D = \left(\int_D |u(x)|^2 dx\right)^{\frac{1}{2}} \quad \forall u \in L^2(D).\]
Definition 3.2. Suppose $D \subseteq \mathbb{R}^d$. We define

$$L^1_{\text{loc}}(D) := \{f \mid f \in L^1(K) \forall \text{ compact } K \subset \text{ interior } D\},$$

and assume $k$ be a non-negative integer and $f \in L^1_{\text{loc}}(D)$. Assume that the weak derivatives $D^k f$ exist for all $|\alpha| \leq k$. We define the Sobolev norm

$$\|f\|_{H^k(D)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^2(D)}^2\right)^{\frac{1}{2}},$$

and hence the Sobolev spaces

$$H^k(D) := \{f \in L^1_{\text{loc}}(D) \mid \|f\|_{H^k(D)} < \infty \}.$$

The Sobolev space $H^k(D)$ is a Hilbert space with respect to the inner product

$$(u, v)_{k,D} := \sum_{0 \leq |\alpha| \leq k} \int_D D^\alpha u(x) D^\alpha v(x) dx,$$

which induces the norm above.

We suppose that the coefficient function $A$ is assumed to be uniformly elliptic, bounded, and self-adjoint with $A|_{D^{+}} \in C^1(\Gamma^{+}, \mathbb{R}^{d \times d})$, $A|_{D^{-}} \in C^1(\Gamma^{-}, \mathbb{R}^{d \times d})$. Because $A$ is self-adjoint, there exists an $A^* \in C^1(\Gamma^{-}, \mathbb{R}^{d \times d})$, $A|_{D^{+}} \in C^1(\Gamma^{+}, \mathbb{R}^{d \times d})$, and $A = A^*$. The weak formulation of (7a)–(7d), i.e., the Poisson equation on $D$ and the drift-diffusion equations for electrons and holes on $D_{Si}$ for all $\omega \in \Omega$ can be written as

$$\left(\hat{A} \nabla V(\omega), \hat{A} \nabla \varphi_1\right)_{D_{Si}} = (qC(\omega), \varphi_1)_{D_{Si}} - \left(qn_e e^{\frac{qV}{k_B T}} u(\omega) - qn_h e^{\frac{qV}{k_B T}} v(\omega), \varphi_1\right)_{D_{Si}} \forall \varphi_1 \in X_1, \quad (8a)$$

$$\left(\hat{A} \nabla V(\omega), \hat{A} \nabla \varphi_1\right)_{D_{Si}} = 0 \forall \varphi_1 \in X_1, \quad (8b)$$

$$- \left(U_T \mu_e e^{\frac{qV}{k_B T}} \nabla u(\omega), \nabla \varphi_2\right)_{D_{Si}} = \left(\frac{u(\omega)V(\omega) - 1}{\tau_p e^{\frac{qV}{k_B T}} u(\omega) + 1 + \tau_n e^{\frac{qV}{k_B T}} v(\omega) + 1}\right) \cdot \varphi_2_{D_{Si}}, \quad \forall \varphi_2 \in X_2, \quad (8c)$$

$$- \left(U_T \mu_p e^{\frac{qV}{k_B T}} \nabla v(x, \omega), \nabla \varphi_3\right)_{D_{Si}} = \left(\frac{u(\omega)V(\omega) - 1}{\tau_p e^{\frac{qV}{k_B T}} u(\omega) + 1 + \tau_n e^{\frac{qV}{k_B T}} v(\omega) + 1}\right) \cdot \varphi_3_{D_{Si}}, \quad \forall \varphi_3 \in X_2, \quad (8d)$$

where we have the Hilbert spaces

$$X_1 := H^1(D \setminus \Gamma) \cap L^\infty(D \setminus \Gamma), \quad (9)$$

$$X_2 := H^1(D_{Si}) \cap L^\infty(D_{Si}). \quad (10)$$

We employ the finite-element spaces $X_{1,h} \subset X_1$ and $X_{2,h} \subset X_2$, which are piecewise polynomials on $D$ and $D_{Si}$ of order $k + 1$ ($k \geq 0$), respectively. We assume that $T_h$ is a partition of $D$ into non-overlapping triangles such that no vertex of one triangle lies in the interior of another triangle (i.e., $D = \bigcup_{K \in T_h} K$) and $h$ is the maximum diameter.
of the partition. Hence, based on the weak formulation (8) and the finite-element space, we define the finite-element
method for (7a–7d). It means to find \((V_h, u_h, v_h) \in (X_{1,h}, X_{2,h}, X_{3,h})\) such that

\[
\left( \hat{\nabla} V_h(\omega), \hat{\nabla} \psi_1 \right)_{D_h} = (q C(\omega), \psi_1)_{D_h} - \left( q n e^{\frac{V_h}{\tau}} u_h(\omega) - q n e^{\frac{V_h}{\tau}} v_h(\omega), \psi_1 \right)_{D_h} \quad \forall \psi_1 \in X_{1,h},
\]

\[
\left( \hat{\nabla} V_h(\omega), \hat{\nabla} \psi_1 \right)_{D_h} = 0 \quad \forall \psi_1 \in X_{1,h}.
\]

Then there exists a constant \(c\) such that the inequality

\[
\| V - \Pi_{1,h} V \|_D \leq c h^\alpha \| V \|_{H^\alpha(D)} \quad \forall V \in H^\alpha(D) \cap X_1, \quad 1 \leq \alpha \leq k + 1,
\]

\[
\| V - \Pi_{1,h} V \|_{H^\alpha(D)} \leq c h^{\alpha-1} \| V \|_{H^{\alpha-1}(D)}, \quad \forall V \in H^{\alpha+1}(D) \cap X_1 \quad 1 \leq \alpha \leq k + 1.
\]

The same projection (i.e., \(\Pi_{2,h} : X_2 \rightarrow X_{2,h}\)) can be defined on \(D_{Si}\). Moreover, Let \(\{m_j\}_{j=1}^m\) be the set of vertices in \(T_h\), we define the interpolation

\[
I_{1,h} : X_1 \rightarrow X_{1,h}
\]

by

\[
I_{1,h} V(m_j) = V(m_j) \quad j = 1, \ldots, m,
\]

where we have the following approximation results for \(I_{1,h}\)

\[
\| V - I_{1,h} V \|_D \leq c h^\alpha \| V \|_{H^\alpha(D)} \quad \forall V \in H^\alpha(D) \cap X_1, \quad 1 \leq \alpha \leq k + 1,
\]

\[
\| V - I_{1,h} V \|_{H^\alpha(D)} \leq c h^{\alpha-1} \| V \|_{H^{\alpha-1}(D)}, \quad \forall V \in H^{\alpha+1}(D) \cap X_1 \quad 1 \leq \alpha \leq k + 1.
\]

Again, we can define the same interpolation on \(D_{Si}\) (i.e., \(I_{2,h} : X_2 \rightarrow X_{2,h}\)) in the same way as \(I_{1,h}\) and the same approximation results can be concluded.

Next, we state and prove an a-priori error estimate.

**Theorem 2** (A-priori error estimate). Let \((V(\omega), u(\omega), v(\omega)) \in \left( H^1(D \setminus \Gamma) \cap L^\infty(D \setminus \Gamma) \right) \times \left( H^1(D_{Si}) \cap L^\infty(D_{Si}) \right)^2\) be the solution of (7) and \((V_h(\omega), u_h(\omega), v_h(\omega)) \in X_{1,h} \times X_{2,h} \times X_{2,h}\) be the solution of (17). Suppose further that \(\alpha \in \{1, \ldots, k\}\). Then there exists a constant \(c \in \mathbb{R}^+\) such that the inequality

\[
\| e_{1,h}(\omega) \|_D^2 + \| e_{2,h}(\omega) \|_{D_h}^2 + \| e_{3,h}(\omega) \|_{D_h}^2 \leq c h^{2\alpha} \left( \| u(\omega) \|_{H^1(D_{Si})}^2 + \| v(\omega) \|_{H^1(D_{Si})}^2 + \| V(\omega) \|_D^2 \right).
\]
holds for all $\omega \in \Omega$, where

$$e_{1,h}(\omega) := V_h(\omega) - \Pi_{1,h} V(\omega),$$
$$e_{2,h}(\omega) := u_h(\omega) - \Pi_{2,h} u(\omega),$$
$$e_{3,h}(\omega) := v_h(\omega) - \Pi_{2,h} v(\omega).$$

**Proof.** To simplify notation, we drop the random variable $\omega$ from the arguments of all functions in the following.

Substituting $\varphi_1 = e_{1,h}$ into (5a) and $\varphi_1 = e_{1,h}$ into (11a) as well as subtracting these two equations, we have the following error equation

$$\left( \tilde{A} \nabla e_{1,h}, \tilde{A} \nabla e_{1,h} \right)_{D_h} = q_n \left( e^{\frac{\tau}{\nu} \Pi_{2,h} u - e^{\frac{\tau}{\nu}} u_h} - e^{\frac{\tau}{\nu} \Pi_{2,h} v - e^{\frac{\tau}{\nu}} v_h}, e_{1,h} \right)_{D_h}$$
$$+ q_n \left( e^{\frac{\tau}{\nu} \Pi_{2,h} v - e^{\frac{\tau}{\nu}} v_h} - e^{\frac{\tau}{\nu} \Pi_{2,h} u - e^{\frac{\tau}{\nu}} u_h}, e_{1,h} \right)_{D_h}$$
$$+ \left( \tilde{A} \nabla (V - \Pi_{1,h} V), \tilde{A} \nabla e_{1,h} \right)_{D_h}. \tag{19}$$

Substituting $\varphi_1 = e_{1,h}$ into (8b) and $\varphi_1 = e_{1,h}$ into (11b) as well as subtracting these two equations results in

$$\left( \tilde{A} \nabla e_{1,h}, \tilde{A} \nabla e_{1,h} \right)_{D_n} = \left( \tilde{A} \nabla (V - \Pi_{1,h} V), \tilde{A} \nabla e_{1,h} \right)_{D_n}. \tag{20}$$

Furthermore, substituting $\varphi_2 = e_{2,h}$ into (8a) and $\varphi_2 = e_{2,h}$ into (11a) and subtracting these two equations leads to

$$U_T \mu \left( e^{\frac{\nu}{\tau} V_h} \nabla u_h - e^{\frac{\nu}{\tau} V} \nabla \Pi_{2,h} u, \nabla e_{2,h} \right)_{D_h} = \left( \frac{1 - u_h}{\tau_p} e^{\frac{\nu}{\tau} V_h + 1} + \tau_n \left( e^{\frac{\nu}{\tau} V_h + 1} \right), e_{2,h} \right)_{D_h}$$
$$+ \left( \frac{1 - \Pi_{2,h} u}{\tau_p} e^{\frac{\nu}{\tau} \Pi_{2,h} u + 1} + \tau_n \left( e^{\frac{\nu}{\tau} \Pi_{2,h} u + 1} \right), e_{2,h} \right)_{D_h}$$
$$+ \left( e^{\frac{\nu}{\tau} \nabla \Pi_{2,h} u - e^{\frac{\nu}{\tau} \nabla u}, e_{2,h}} \right)_{D_n}. \tag{21}$$

Again, substituting $\varphi_3 = e_{3,h}$ into (8b) and $\varphi_3 = e_{3,h}$ into (11b) and subtracting these two equations yields

$$U_T \mu \left( e^{\frac{\nu}{\tau} V_h} \nabla V_h - e^{\frac{\nu}{\tau} V} \nabla \Pi_{2,h} V, \nabla e_{3,h} \right)_{D_h} = \left( \frac{1 - u_h}{\tau_p} e^{\frac{\nu}{\tau} V_h + 1} + \tau_n \left( e^{\frac{\nu}{\tau} V_h + 1} \right), e_{3,h} \right)_{D_h}$$
$$+ \left( \frac{1 - \Pi_{2,h} u}{\tau_p} e^{\frac{\nu}{\tau} \Pi_{2,h} u + 1} + \tau_n \left( e^{\frac{\nu}{\tau} \Pi_{2,h} u + 1} \right), e_{3,h} \right)_{D_h}$$
$$+ \left( e^{\frac{\nu}{\tau} \nabla \Pi_{2,h} V - e^{\frac{\nu}{\tau} \nabla V}, e_{2,h}} \right)_{D_n}. \tag{22}$$

Let us define the following functions

$$L_1(a, b, c) := e^{\frac{a}{\tau}} (b - c),$$
$$L_2(a, b, c, d) := \left( e^{\frac{a}{\tau}} - e^{\frac{b}{\tau}} \right) (c - d),$$
$$L_3(a, b, c) := \left( e^{\frac{a}{\tau}} - e^{\frac{b}{\tau}} \right) c. \tag{23}$$
Then, we can write

\[
e^{-\frac{v}{\tau}} \Pi_{2,h}u - e^{\frac{v}{\tau}} u_h = L_1(V, \Pi_{2,h}u, u_h) + L_2(\Pi_{1,h}V, \Pi_{2,h}u, u_h) + L_3(\Pi_{1,h}V_h, \Pi_{2,h}u).
\]

(24)

\[
e^{-\frac{v}{\tau}} \Pi_{2,h}u - e^{\frac{v}{\tau}} u = e^{-\frac{v}{\tau}} (\Pi_{2,h}u - u) + \left( e^{-\frac{v}{\tau}} - e^{\frac{v}{\tau}} \right) u.
\]

(25)

\[
e^{-\frac{v}{\tau}} \nabla u_h - e^{\frac{v}{\tau}} \nabla \Pi_{2,h}u = L_1(V, \nabla u_h, \nabla \Pi_{2,h}u) + L_2(V, \Pi_{1,h}V, \nabla \Pi_{2,h}u, \nabla u_h)
\]

\[+ L_3(V_h, \Pi_{1,h}V, \nabla \Pi_{2,h}u) + L_4(V_h, \Pi_{1,h}V, \nabla \Pi_{2,h}u).
\]

(26)

We can use the similar procedure to rewrite \(e^{-\frac{v}{\tau}} \Pi_{1,h}V - e^{\frac{v}{\tau}} V_h\), \(e^{-\frac{v}{\tau}} \Pi_{1,h}V - e^{\frac{v}{\tau}} V\), and \(e^{\frac{v}{\tau}} \nabla V - e^{-\frac{v}{\tau}} \nabla \Pi_{1,h}V\).

Next, we define

\[
f(a, b, c) := \frac{-bc + 1}{\tau_p \left( e^{\frac{v}{\tau}} b + 1 \right) + \tau_n \left( e^{\frac{v}{\tau}} c + 1 \right)}.
\]

Now substituting (24)--(26) into (19)--(20), we find

\[
\left\| \tilde{A} \nabla e_{1,h} \right\|_{D_n}^2 + U_T \mu_s \left( L_1(V, \nabla u_h, \nabla \Pi_{2,h}u), \nabla e_{2,h} \right)_{D_n} + U_T \mu_p \left( L_1(V, \nabla V_h, \nabla \Pi_{1,h}V), \nabla e_{3,h} \right)_{D_n} =
\]

\[
qn \left( L_2(\Pi_{1,h}V, V, \Pi_{1,h}u, u_h) + L_2(\Pi_{1,h}V, V, u_h, \Pi_{2,h}u) - L_2(-\Pi_{1,h}V, -V, \Pi_{1,h}V, V_h) - L_2(-\Pi_{1,h}V, V, -\Pi_{1,h}V, V_h) \right) + L_4(\Pi_{1,h}V_h, -V_h, V_h, \Pi_{1,h}V, e_{1,h})_{D_n}
\]

\[
+ \left( \tilde{A} \nabla (V - \Pi_{1,h}V), \tilde{A} \nabla e_{1,h} \right)_{D_n} + \left( e^{-\frac{v}{\tau}} (\Pi_{2,h}u - u), e_{2,h} \right)_{D_n} + \left( e^{-\frac{v}{\tau}} - e^{\frac{v}{\tau}} \right) u, e_{2,h} \right)_{D_n} + \left( e^{\frac{v}{\tau}} (\Pi_{1,h}V - e), e_{3,h} \right)_{D_n}
\]

\[
+ \left( e^{-\frac{v}{\tau}} \nabla \Pi_{2,h}u - e^{\frac{v}{\tau}} \nabla \Pi_{1,h}V, e_{3,h} \right)_{D_n} + \left( e^{\frac{v}{\tau}} \nabla \Pi_{2,h}u - e^{-\frac{v}{\tau}} \nabla \Pi_{1,h}V, e_{3,h} \right)_{D_n}
\]

(28)

Now we define the functional \(M_1 : X_h \times X_h \times X_h \rightarrow \mathbb{R}\) as

\[
M_1(\Pi_{1,h}V, V_h)(e_{3,h}) := (f(\Pi_{1,h}V, \Pi_{2,h}u, \Pi_{2,h}v) - f(V_h, u_h, v_h), e_{3,h})_{D_n}.
\]

Next, we define

\[
\theta_1 := \theta V_h + (1 - \theta) \Pi_{1,h}V = \theta e_{1,h} + e_{1,h} - V_h = (1 + \theta) e_{1,h} - V_h.
\]

where \(0 < \theta < 1\). Using the Taylor expansion for the functional \(M_1\) over \(\Pi_{1,h}V\) leads to

\[
M_1(\Pi_{1,h}V, V_h)(e_{3,h}) = M_1(\Pi_{1,h}V, \Pi_{1,h}V)(e_{3,h}) + \int_0^1 M'_1(\Pi_{1,h}V, \theta_1)(e_{3,h}) \, d\theta.
\]

(29)
Since the function $f$ is differentiable, we take the directional derivative in the projection direction which lead to

$$M'_1(\Pi_{1,b}V, \theta_1)(e_{3,b}) = \lim_{r \to 0} \frac{M_1(\Pi_{1,b}V, \theta_1 + r \Pi_{1,b}V)(e_{3,b}) - M_1(\Pi_{1,b}V, \theta_1)(e_{3,b})}{r}$$

\[
\leq \lim_{r \to 0} \frac{(f(\Pi V_h, \Pi_{2,b}u, \Pi_{2,b}v) - f(\theta_1 + r\Pi_{1,b}V, u_h, v_h), e_{3,b})_{D_1}}{r} - \lim_{r \to 0} \frac{(f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(\theta_1, u_h, v_h), e_{3,b})_{D_1}}{r}
\]

\[
\leq c(\theta_1)(e_{1,b}, e_{3,b})_{D_3}. \tag{30}
\]

Here, we should note that the function $f$ satisfies the Lipschitz condition \[13\] and the constant $c(\theta_1)$ is independent of $h$. Considering \[29\] and \[30\], we conclude that

\[
|M_1(\Pi_{1,b}V, V_h)(e_{3,b})| \leq |(f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(\Pi_{1,b}V, u_h, v_h), e_{3,b})_{D_1}| + \int_0^1 M'_1(\Pi_{1,b}V, \theta_1)(e_{3,b}) \, d\theta.
\]

Therefore, Eq. \[30\] and the triangle inequality give rise to

\[
|M_1(\Pi_{1,b}V, V_h)(e_{3,b})| \leq |(f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(\Pi_{1,b}V, u_h, v_h), e_{3,b})_{D_1}| + |(e_{2,b}, e_{3,b})_{D_3}|. \tag{31}
\]

In the next step, we define the functional

$$M_2(\Pi_{2,b}u, u_h)(e_{3,b}) := (f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(\Pi_{1,b}V, u_h, v_h), e_{3,b})_{D_1}.$$

Similarly, we define the variable

$$\theta_2 := (1 + \theta)e_{1,b} - u_h,$$

where $0 < \theta < 1$. Taylor expansion of $M_2$ over $\Pi_{2,b}u$ shows that

$$M_2(\Pi_{2,b}u, u_h)(e_{3,b}) = M_2(\Pi_{2,b}u, \Pi_{2,b}u)(e_{3,b}) + \int_0^1 M'_2(\Pi_{2,b}u, \theta_2)(e_{3,b}) \, d\theta. \tag{32}$$

Again, since the function $f$ is differentiable, the directional derivative of $M_2$ is given by

$$M'_2(\Pi_{2,b}u, \theta_2)(e_{3,b}) = \lim_{r \to 0} \frac{M_2(\Pi_{2,b}u, \theta_2 + r \Pi_{2,b}u)(e_{3,b}) - M_2(\Pi_{2,b}u, \theta_2)(e_{3,b})}{r}$$

\[
\leq \lim_{r \to 0} \frac{(f(\Pi V_h, \Pi_{2,b}u, \Pi_{2,b}v) - f(V_h, \theta_2 + r\Pi_{2,b}u, v_h), e_{3,b})_{D_1}}{r} - \lim_{r \to 0} \frac{(f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(V_h, \theta_2, v_h), e_{3,b})_{D_1}}{r}
\]

\[
\leq c(\theta_2)(e_{1,b}, e_{3,b})_{D_3}. \tag{33}
\]

where again the constant $c(\theta_2)$ is independent of $h$. Therefore, we have

$$|M_2(\Pi_{2,b}u, u_h)(e_{3,b})| \leq |(f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(\Pi_{1,b}V, u_h, v_h), e_{3,b})_{D_1}| + \int_0^1 M'_2(\Pi_{2,b}u, \theta_2)(e_{3,b}) \, d\theta.$$
and applying the triangle inequality to (32) results in

$$|M_2(\Pi_{2,b}u, u_b)(e_{3,b})| \lesssim |f(\Pi_{1,b}V, \Pi_{2,b}u, \Pi_{2,b}v) - f(\Pi_{1,b}V, u_b, v_b), e_{3,b})_{D_b}| + |e_{1,b}, e_{3,b})_{D_b}|. \tag{34}$$

Now applying Lemma 1 using triangle inequality and the stability properties of $\Pi_{1,b}$ and $\Pi_{2,b}$, we have

$$|(L_1(V, \Pi_{2,b}u, u_b) - L_1(-V, \Pi_{2,b}v, v_b))_{D_b} | \leq |L_1(V, \Pi_{2,b}u, u_b) - L_1(-V, \Pi_{2,b}v, v_b, \Pi_{2,b}v))_{L^2(D_b)}| \leq \left(\|\bar{e} V\|_{L^2(D_b)}\|e_{2,b}\|_{L^2(D_b)} + \|e V\|_{L^2(D_b)}\|e_{3,b}\|_{L^2(D_b)}\right)\|e_{1,b}\|_{L^2(D_b)}, \tag{35}$$

and

$$|(L_2(\Pi_{1,b}V, \Pi_{2,b}u, u_b) + L_2(\Pi_{1,b}V, \Pi_{2,b}u, u_b) - L_2(-\Pi_{1,b}V, -\Pi_{2,b}V, v_b, \Pi_{2,b}v))_{e_{1,b})_{D_b} | \leq \left(\|\bar{e} V - V\|_{L^2(D_b)}\|e_{2,b}\|_{L^2(D_b)} + \|e V\|_{L^2(D_b)}\|e_{3,b}\|_{L^2(D_b)}\right)\|e_{1,b}\|_{L^2(D_b)} \tag{36}$$

and

$$|(L_3(\Pi_{1,b}V, \Pi_{2,b}u, -\Pi_{1,b}V, -\Pi_{2,b}V, v_b, \Pi_{2,b}v))_{e_{1,b})_{D_b} | \leq \left(\|\bar{e} V - V\|_{L^2(D_b)}\|e_{2,b}\|_{L^2(D_b)} + \|e V\|_{L^2(D_b)}\|e_{3,b}\|_{L^2(D_b)}\right)\|e_{1,b}\|_{L^2(D_b)} \tag{37}$$

Moreover, using Lemma 1 and the stability properties of $\Pi_{2,b}u$, we can write

$$\|e_{1,b}\|_{D_b} + \|\nabla e_{2,b}\|_{D_b} + \|\nabla e_{3,b}\|_{D_b} \lesssim \|e_{1,b}\|_{L^2(D_b)} + \|e_{2,b}\|_{L^2(D_b)} + \|e_{3,b}\|_{L^2(D_b)} \lesssim h^{2\alpha - 2}\|e_{1,b}\|_{H^{\alpha}(D_b)} + h^{2\alpha}\|e_{2,b}\|_{H^{\alpha}(D_b)} + h^{2\alpha}\|e_{3,b}\|_{H^{\alpha}(D_b)} \tag{40}$$

Also, we can get the similar results for $\|e_{1,b}\|_{D_b} + \|\nabla e_{2,b}\|_{D_b} + \|\nabla e_{3,b}\|_{D_b}$ as well as $\|e_{1,b}\|_{D_b} + \|\nabla e_{2,b}\|_{D_b}$. Combination of the above arguments gives us

$$\|e_{1,b}\|_{D_b} + \|\nabla e_{2,b}\|_{D_b} + \|\nabla e_{3,b}\|_{D_b} \lesssim \|e_{1,b}\|_{L^2(D_b)} + \|e_{2,b}\|_{L^2(D_b)} + \|e_{3,b}\|_{L^2(D_b)} \lesssim h^{2\alpha - 2}\|e_{1,b}\|_{H^{\alpha}(D_b)} + h^{2\alpha}\|e_{2,b}\|_{H^{\alpha}(D_b)} + h^{2\alpha}\|e_{3,b}\|_{H^{\alpha}(D_b)} \lesssim h^{2\alpha}\|e_{1,b}\|_{H^{\alpha}(D_b)} + h^{2\alpha}\|e_{2,b}\|_{H^{\alpha}(D_b)} + h^{2\alpha}\|e_{3,b}\|_{H^{\alpha}(D_b)} \tag{41}$$

Finally, employing (41) as well as the Poincaré inequality (since $e_{1,b} \equiv 0$ on $\partial D$, $e_{2,b} \equiv 0$ and $e_{3,b} \equiv 0$ on $\partial D_b$), we find

$$\|e_{1,b}\|_{D_b} + \|\nabla e_{2,b}\|_{D_b} + \|\nabla e_{3,b}\|_{D_b} \lesssim h^{2\alpha}\left(\|V\|_{H^{\alpha}(D_b)} + \|u\|_{H^{\alpha}(D_b)} + \|v\|_{H^{\alpha}(D_b)}\right), \tag{42}$$

\[\square\]
In the next step, we use a residual based a-posteriori error estimation technique to estimate the local error $\eta_T$ on each finite element $T \in \mathcal{T}_h$. The error indicator will serve as the foundation for a refinement strategy in order to control and minimize the errors in the Poisson equation (7.8-7.11) and in the continuity equations (7.9-7.10).

**Theorem 3 (A-posteriori error estimate).** For $\omega \in \Omega$ let $(V(\omega), u(\omega), v(\omega)) \in (H^1(D \setminus \Gamma) \cap L^\infty(D \setminus \Gamma) \times H^1(D_{Si}) \cap L^\infty(D_{Si}))^2$ be the solution of (8) and $(V_h(\omega), u_h(\omega), v_h(\omega)) \in X_{1,h} \times X_{2,h} \times X_{3,h}$ be the solution of (11). Suppose that for some $\{c_1, \ldots, c_6\} \subset \mathbb{R}^+$ the following inequality holds where $\mathcal{T}^{Si}_h := \mathcal{T}_h \cap D_{Si}$ and

$$
\begin{align*}
\| (V_h - V)(\omega) \|_{T_{\delta}}^2 + \| (u_h - u)(\omega) \|_{T_{\delta}}^2 + \| (v_h - v)(\omega) \|_{T_{\delta}}^2 & \leq c_1 \sum_{\zeta \in \mathcal{T}^{Si}_h} h_\zeta^2 \| r_1(\omega) \|_{T_{\delta}}^2 + c_2 \sum_{\zeta \in \mathcal{T}^{Si}_h} h_\zeta^2 \| r_2(\omega) \|_{T_{\delta}}^2 \\
& + c_3 \sum_{\zeta \in \mathcal{T}^{Si}_h} h_\zeta^2 \| r_3(\omega) \|_{T_{\delta}}^2 + c_4 \sum_{\gamma \in \partial \mathcal{T}^{Si}_h} \left[ \frac{\partial V_h}{\partial \nu}(\omega) \right]_{\gamma}^2 h_\gamma^2 \\
& + c_5 \sum_{\gamma \in \partial \mathcal{T}^{Si}_h} \left[ U_{T\mu_p} \frac{\partial u_h}{\partial \nu}(\omega) \right]_{\gamma}^2 h_\gamma^2 + c_6 \sum_{\gamma \in \partial \mathcal{T}^{Si}_h} \left[ U_{T\mu_p} \frac{\partial v_h}{\partial \nu}(\omega) \right]_{\gamma}^2 h_\gamma^2
\end{align*}
$$

\text{Proof.} \quad \text{In the following, the dependence of the solutions on the random variable $\omega$ is not indicated in order to simplify notation. We first define}

$$
\begin{align*}
epsilon_{1,h} := V - V_h, \\
epsilon_{2,h} := u - u_h, \\
epsilon_{3,h} := v - v_h.
\end{align*}
$$

Here $\zeta$ denotes the area of an element in $\mathcal{T}_h$ or $\mathcal{T}^{Si}_h$, $\gamma$ denotes the boundary of the element, and the brackets $[\cdot]$ indicate the jump at the element boundary.
Using the test functions $\varphi_1 \in X_1$, $\varphi_2 \in X_2$, and $\varphi_3 \in X_2$, the weak formulation yields

\[
(\hat{A} \nabla_{E,h}, \hat{A} \nabla \varphi_1)_{D_u} = (qC, \varphi_1)_{D_u} - \left( q_n e^{\frac{v_T}{\tau_T} u} - q_n e^{\frac{v_T}{\tau_T} u_{h}}, \varphi_1 \right)_{D_u} + \left( q_n e^{\frac{v_T}{\tau_T} v} - q_n e^{\frac{v_T}{\tau_T} v_{h}}, \varphi_1 \right)_{D_u} - \left( q_n e^{\frac{\gamma_T}{\tau_T} u_{h}} - q_n e^{\frac{\gamma_T}{\tau_T} v_{h}}, \varphi_1 \right)_{D_u} - (\hat{A} \nabla V_h, \hat{A} \nabla \varphi_1)_{D_u},
\]

(44a)

\[
(\hat{A} \nabla_{E,h}, \hat{A} \nabla \varphi_1)_{D_u} = - (\hat{A} \nabla V_h, \hat{A} \nabla \varphi_1)_{D_u},
\]

(44b)

\[
(U_{T} h e^{\frac{v_T}{\tau_T} \nabla u} - U_{T} H e^{\frac{v_T}{\tau_T} \nabla u_{h}}, \nabla \varphi_2)_{D_u} = - \left( U_{T} h e^{\frac{v_T}{\tau_T} \nabla u_{h}}, \nabla \varphi_2 \right)_{D_u} - \left( U_{T} H e^{\frac{v_T}{\tau_T} \nabla u}, \nabla \varphi_2 \right)_{D_u} - \left( \tau_p e^{\frac{v_T}{\tau_T} u} + \tau_n e^{\frac{v_T}{\tau_T} v} + 1 \right)\varphi_2 - \frac{u_n v_{h} - 1}{\tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1}}\varphi_2 - \left( \tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1} \right)\varphi_2 - \left( \tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1} \right)\varphi_2
\]

(44c)

\[
(U_{T} H e^{\frac{v_T}{\tau_T} \nabla v} - U_{T} H e^{\frac{v_T}{\tau_T} \nabla v_{h}}, \nabla \varphi_3)_{D_u} = - \left( U_{T} h e^{\frac{v_T}{\tau_T} \nabla v_{h}}, \nabla \varphi_3 \right)_{D_u} - \left( U_{T} H e^{\frac{v_T}{\tau_T} \nabla v}, \nabla \varphi_3 \right)_{D_u} - \left( \tau_p e^{\frac{v_T}{\tau_T} u} + \tau_n e^{\frac{v_T}{\tau_T} v} + 1 \right)\varphi_3 - \frac{u_n v_{h} - 1}{\tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1}}\varphi_3 - \left( \tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1} \right)\varphi_3 - \left( \tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1} \right)\varphi_3
\]

(44d)

Next we substitute $\psi_1 := I_{1,0} \varphi_1$ into (11b) and (11d), $\psi_2 := I_{2,0} \varphi_2$ into (11c), and $\psi_3 := I_{2,0} \varphi_3$ into (11d), and sum up the equations. Then we subtract them from (44a) – (44d), which leads to

\[
(\hat{A} \nabla_{E,h}, \hat{A} \nabla \varphi_1)_{D_u} + \left( U_{T} h e^{\frac{v_T}{\tau_T} \nabla u} - U_{T} H e^{\frac{v_T}{\tau_T} \nabla u_{h}}, \nabla \varphi_2 \right)_{D_u} - \left( U_{T} H e^{\frac{v_T}{\tau_T} \nabla v} - U_{T} H e^{\frac{v_T}{\tau_T} \nabla v_{h}}, \nabla \varphi_3 \right)_{D_u} + \left( q_n e^{\frac{v_T}{\tau_T} u} - q_n e^{\frac{v_T}{\tau_T} u_{h}}, \varphi_1 \right)_{D_u} + \left( q_n e^{\frac{v_T}{\tau_T} v} - q_n e^{\frac{v_T}{\tau_T} v_{h}}, \varphi_1 \right)_{D_u} - \left( q_n e^{\frac{\gamma_T}{\tau_T} u_{h}} - q_n e^{\frac{\gamma_T}{\tau_T} v_{h}}, \varphi_1 \right)_{D_u} - (\hat{A} \nabla V_h, \hat{A} \nabla (\varphi_1 - I_{1,0} \varphi_1))_{D_u} + \left( U_{T} H e^{\frac{v_T}{\tau_T} \nabla u}_{h}, \nabla (I_{2,0} \varphi_2 - \varphi_2) \right)_{D_u} + \left( U_{T} H e^{\frac{v_T}{\tau_T} \nabla v_{h}}, \nabla (I_{2,0} \varphi_3 - \varphi_3) \right)_{D_u} - \left( \tau_p e^{\frac{v_T}{\tau_T} u} + \tau_n e^{\frac{v_T}{\tau_T} v} + 1 \right)\varphi_2 - \frac{u_n v_{h} - 1}{\tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1}}\varphi_2 - \left( \tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1} \right)\varphi_2 - \left( \tau_p e^{\frac{v_T}{\tau_T} u_{h}} + \tau_n e^{\frac{v_T}{\tau_T} v_{h} + 1} \right)\varphi_2
\]

(45)
on [45] leads to

\[
\left( \hat{A} \nabla \varphi_1, \hat{A} \nabla \varphi_1 \right)_D = \left( \nabla u \cdot \nabla \varphi_1 - U_T \mu e^{-\frac{r}{\mu}} \nabla \varphi_1 - U_T \mu e^{-\frac{r}{\mu}} \nabla \varphi_2 \right)_D - \left( U_T \mu e^{-\frac{r}{\mu}} \nabla \varphi_1 - U_T \mu e^{-\frac{r}{\mu}} \nabla \varphi_2 \right)_D,
\]

\[
+ \left( q e^{-\frac{r}{\mu}} \nabla V - q e^{-\frac{r}{\mu}} \nabla h \cdot \nabla \varphi_1 \right)_D + \left( q e^{-\frac{r}{\mu}} \nabla V - q e^{-\frac{r}{\mu}} \nabla h \cdot \nabla \varphi_1 \right)_D \leq \sum_{\zeta \in \mathcal{T}_h} \| r \|_0 \| \varphi_1 - I_1 \varphi_1 \|_c + \sum_{\zeta \in \mathcal{T}_h} \| r \|_0 \| \varphi_2 - I_2 \varphi_2 \|_c + \sum_{\zeta \in \mathcal{T}_h} \| r \|_0 \| \varphi_2 - I_2 \varphi_2 \|_c,
\]

where

\[
\begin{align*}
r_1 & := -\nabla \cdot (A \nabla V_h) - q \left( e^{-\frac{r}{\mu}} u - e^{-\frac{r}{\mu}} v_h \right), \\
r_2 & := U_T \nabla \cdot (\mu e^{-\frac{r}{\mu}} \nabla u_h) - \frac{u_h v_h - 1}{\tau_p} \left( e^{-\frac{r}{\mu}} u_h + 1 \right), \\
r_3 & := U_T \nabla \cdot (\mu e^{-\frac{r}{\mu}} \nabla v_h) - \frac{u_h v_h - 1}{\tau_p} \left( e^{-\frac{r}{\mu}} u_h + 1 \right).
\end{align*}
\]

It can be easily seen that

\[
\begin{align*}
e^{-\frac{r}{\mu}} u - e^{-\frac{r}{\mu}} u_h = & \left( e^{-\frac{r}{\mu}} - e^{-\frac{r}{\mu}} \right) e_{2,h} + \left( e^{-\frac{r}{\mu}} - e^{-\frac{r}{\mu}} \right) u, \\
e^{-\frac{r}{\mu}} v - e^{-\frac{r}{\mu}} v_h = & \left( e^{-\frac{r}{\mu}} - e^{-\frac{r}{\mu}} \right) e_{3,h} + \left( e^{-\frac{r}{\mu}} - e^{-\frac{r}{\mu}} \right) v,
\end{align*}
\]

Now substituting \( \varphi_i = e_{i,h} \) for \( i \in \{1, 2, 3\} \) into (46) and using (47a)-(47d). Equation (44) and Lemma 4 yield

\[
\begin{align*}
\| \nabla e_{1,h} \|^2 + \| \nabla e_{2,h} \|^2 + \| \nabla e_{3,h} \|^2 & \leq \sum_{\zeta \in \mathcal{T}_h} h_{\zeta}^2 \| r \|_0 + \sum_{\zeta \in \mathcal{T}_h} h_{\zeta}^2 \| r \|_0 + \sum_{\zeta \in \mathcal{T}_h} h_{\zeta}^2 \| r \|_0 + \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| \hat{A} \frac{\partial V_h}{\partial y} \right| \| h_{\gamma} \|^2 \\
+ & \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| U_T \mu \frac{\partial u_h}{\partial y} \right| \| h_{\gamma} \|^2 + \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| U_T \mu \frac{\partial v_h}{\partial y} \right| \| h_{\gamma} \|^2 + \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| \hat{A} \frac{\partial V_h}{\partial y} \right| \| h_{\gamma} \|^2,
\end{align*}
\]

where \( T_{h} := T_h \cap D_{St} \). Applying the Poincaré inequality (since \( e_{1,h} \equiv 0 \) on \( \partial D \), \( e_{2,h} \equiv 0 \) and \( e_{3,h} \equiv 0 \) on \( \partial D_{St} \)) to the above equation, we conclude

\[
\begin{align*}
T_{h}^2 \left( \| e_{1,h} \|^2 + \| e_{2,h} \|^2 + \| e_{3,h} \|^2 \right) & \leq c_1 \sum_{\zeta \in \mathcal{T}_h} h_{\zeta}^2 \| r \|_0 + c_2 \sum_{\zeta \in \mathcal{T}_h} h_{\zeta}^2 \| r \|_0 + c_3 \sum_{\zeta \in \mathcal{T}_h} h_{\zeta}^2 \| r \|_0 + c_4 \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| \hat{A} \frac{\partial V_h}{\partial y} \right| \| h_{\gamma} \|^2 \\
+ & c_5 \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| U_T \mu \frac{\partial u_h}{\partial y} \right| \| h_{\gamma} \|^2 + c_6 \sum_{\gamma \in \mathcal{E}_{\zeta}} \left| U_T \mu \frac{\partial v_h}{\partial y} \right| \| h_{\gamma} \|^2 + \frac{3}{2} \sum_{\gamma \in \mathcal{E}_{\zeta}} \| e_{\gamma} \|_D^2 \\
+ & \| e_{1,h} \|^2 + \| e_{2,h} \|^2 + \| e_{3,h} \|^2.
\end{align*}
\]
where \(u\) are diameters. Therefore a nested family of the tetrahedra in the mesh. Also, the mesh is regularly refined using a geometric sequence of maximum uniform tetrahedra and thus construct a regular mesh solution of the model equation or equations. The standard MC estimator \(E_{\text{MC}}\) for \(\mathbb{E}[u_h]\) is the sample mean

\[
E_{\text{MC}}[u_h] := \bar{u}_h := \frac{1}{M} \sum_{i=1}^{M} u^{(i)}_h,
\]

where \(u^{(i)}_h = u_h(x, \omega^{(i)})\) is the \(i\)-th sample (independent random variable) of the solution \(u\).

In a multilevel Monte Carlo (MLMC) method, we start by partitioning the computational domain \(D\) into quasi-uniform tetrahedra and thus construct a regular mesh \(T_{h_0}\), that is also the coarsest one. Here \(h_0\) indicates the maximum diameter of the tetrahedra in the mesh. Also, the mesh is regularly refined using a geometric sequence of maximum diameters. Therefore a nested family \(\{T_{h_l}\}_{l=0}^{\infty}\) of regular tetrahedra is constructed. We define

\[
h_{\ell} := \max_{K \in T_{h_\ell}} \text{diam}(K),
\]

where

\[
h_{\ell} = r^{-\ell} h_0, \quad r > 1,
\]

holds for the \(\ell\)-th refined mesh according to construction. Obviously \(h_0 > h_1 > \cdots > h_L\) holds, where \(h_0\) indicates the mesh size of the coarsest level 0, and \(r > 1\) is independent of the level \(\ell\).

Instead of calculating the expected value \(\mathbb{E}[u]\) by \(\mathbb{E}[u_h]\) on a constant triangulation \(T_h\), the MLMC method approximates the expected value \(\mathbb{E}[u]\) using several \(\mathbb{E}[u_h]_\ell\), \(\ell \in \{0, 1, \ldots, L\}\), estimated on the nested family \(\{T_{h_\ell}\}_{\ell=0}^{\infty}\). In fact, to overcome the drawback of the MC method, the MLMC estimator avoids prohibitively many expensive evaluations of \(\mathbb{E}[u_h]\) on the finest level \(L\).

The FE approximation of the expected value of \(V_{h_\ell}\) at level \(L\) can be written as

\[
\mathbb{E}[u_{h_L}] = \mathbb{E}[u_{h_0}] + \mathbb{E} \left[ \sum_{\ell=1}^{L} (u_{h_{\ell-1}} - u_{h_{\ell-1}}) \right] = \mathbb{E}[u_{h_0}] + \sum_{\ell=1}^{L} \mathbb{E}[u_{h_{\ell-1}} - u_{h_{\ell-1}}].
\]
The mean square error (MSE) is estimated by:

\[
\text{MSE} \leq M_0^1 \sigma_0^2[ u_{h_0}] + \sum_{\ell=1}^{L} M_\ell^{-1} \sigma_\ell^2[ u_{h_\ell} - u_{h_{\ell-1}}] + \| E[u_{h_0}] - E[u] \|_{L^2(\Omega,D)}^2
\]

as shown for example in [5], where the variance is given by \( \sigma[ u]^2 := \| E[u] - u \|_{L^2(\Omega,D)}^2 \). The first and second terms of (55) are the statistical error, while the last term is the discretization error. In the next section, we study the effect of mesh refinement, i.e., uniform refinement (according to (53)) and adaptive refinement (using the error indicator (50)) on both terms of errors.

In order to estimate the computational errors, we continue with the degrees of freedom. It enables us to draw a fair comparison between adaptive MLMC-FE and uniform MLMC-FE methods. According to the error bound in (18), we define the discretization error as

\[
E_\ell := \left( \| E[e_{1,h_\ell}] \|_{L^2(\Omega,D)}^2 + \| E[e_{2,h_\ell}] \|_{L^2(\Omega,D)}^2 + \| E[e_{3,h_\ell}] \|_{L^2(\Omega,D)}^2 \right)^{1/2}
\]

Furthermore, at level \( \ell \), we assume that

\[
E_\ell^2 \leq C_1 N_\ell^{-2\alpha} := C_1 (N_{p,\ell} + 2N_{D,\ell})^{-2\alpha}
\]

where \( N_{p,\ell} \) is the number of unknowns or the (degrees of freedom) for the Poisson equation and \( N_{D,\ell} \) indicates the number of unknowns for the two continuity equations. The exponent \( \alpha \) is the convergence rate of the error. For the statistical error, the following inequality

\[
\sigma^2[\Delta V_{h_\ell}] + \sigma^2[\Delta u_{h_\ell}] + \sigma^2[\Delta v_{h_\ell}] \leq C_2 N^{-\beta}_\ell
\]

is assumed to show the convergence of the statistical error (at level \( \ell \)). For \( \ell = 0 \), the assumption

\[
\sigma^2[\Delta V_{h_0}] + \sigma^2[\Delta u_{h_0}] + \sigma^2[\Delta v_{h_0}] \leq C_0
\]

is used as well.

Due to the computational challenge of solving a system of SPDEs, an effective computational strategy is crucial. We strive to determine the optimal number \( M_\ell \) of samples which minimize the computational work when \( \text{MSE} \leq \varepsilon^2/2 \). In other words, the optimal number of samples are defined such that the statistical error is less than \( \varepsilon^2/2 \). The optimal value of \( L \) (the lowest possible number) determined in the sense that the discretization error \( (E_L) \) is less than \( \varepsilon/\sqrt{2} \). For this, the following optimization problem is solved

minimize \( M_\ell \) subject to

\[
f(M_\ell) := \sum_{\ell=0}^{L} M_\ell N_\ell,
\]

\[
g(M_\ell) := \frac{C_0}{M_0} + C_2 \sum_{\ell=1}^{L} \frac{N_{\ell}^{-\beta}}{M_\ell} \leq \frac{\varepsilon^2}{2},
\]

where

\[
\frac{C_0}{M_0} + C_2 \sum_{\ell=1}^{L} \frac{N_{\ell}^{-\beta}}{M_\ell} \leq \frac{\varepsilon^2}{2},
\]

is the number of unknowns for the two continuity equations. The exponent \( \alpha \) is the convergence rate of the error. For the statistical error, the following inequality

\[
\sigma^2[\Delta V_{h_\ell}] + \sigma^2[\Delta u_{h_\ell}] + \sigma^2[\Delta v_{h_\ell}] \leq C_2 N^{-\beta}_\ell
\]

is assumed to show the convergence of the statistical error (at level \( \ell \)). For \( \ell = 0 \), the assumption

\[
\sigma^2[\Delta V_{h_0}] + \sigma^2[\Delta u_{h_0}] + \sigma^2[\Delta v_{h_0}] \leq C_0
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\[
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\]

\[
g(M_\ell) := \frac{C_0}{M_0} + C_2 \sum_{\ell=1}^{L} \frac{N_{\ell}^{-\beta}}{M_\ell} \leq \frac{\varepsilon^2}{2},
\]
where the optimization is over all $M_\ell > 0$. Moreover, since the optimal numbers $M_\ell$ of samples at level $\ell$ are in general not integers, they are rounded up and replaced by $\lceil M_\ell \rceil$. The details of the optimal approach were given in Giles’s MLMC paper [7].

5. Numerical Example and Results

In this work, we have chosen a double-gate MOSFET (DG-MOSFET) as a realistic example to implement the multilevel adaptive method developed above and to investigate its behavior. In these semiconductor transistors, the width of the silicon channel is very small and two gate contacts are used in the both sides of the channel to control the channel efficiently [20, 21, 22]. Hence, the current can potentially be twice the current through a single-gate device, since inversion layers can exist at both gates. This device structure suppresses short-channel effects and leads to higher currents as compared to the usual MOSFET structure having only one gate.

The FET device (see Figure 1 for a schematic diagram) consists of two materials, namely silicon ($D_{Si}$) in the channel and source and drain regions and silicon dioxide ($D_{ox}$) as the insulator. The purpose of the insulator is to suppress direct charge flow from the gate into the channel and vice versa. The permittivities of the materials are $A_{Si} = 11.7A_0$ and $A_{ox} = 3.9A_0$, where the vacuum permittivity (dielectric constant) is $8.85 \cdot 10^{-12}$ Fm$^{-1}$. Moreover, the gates have a length of 30 nm and are separated from the silicon channel by a 2 nm thick oxide layer. The channel width is $W = 15$ nm and it is connected to the heavily n-type doped source and drain regions of length $L_{SD} := 10$ nm in each region.

Regarding the boundary conditions of the model equations, we apply Dirichlet boundary conditions at the gates ($V_g = 0.2$ V) and at the source and drain contacts ($V_{SD} = 0.1$ V). The contacts are illustrated in Figure 1. For the rest of the transistor, we apply zero Neumann boundary conditions. In the highly doped source and drain regions, the dopant atoms are randomly distributed and indicated in the figure by blue circles. Therefore, random-dopant effects are included due to the random position of the dopant atoms.

In the common models, the doping concentration is modeled as a macroscopic, deterministic quantity which averages out any microscopic non-uniformities due to the random placement and random number of dopants. For large devices with a large number of dopants, this continuum model is physically reasonable, since the electrostatic potential appears spatially homogeneous and is sufficiently well described by the averaged charge density. However, in nanoscale transistors, the randomly distributed dopant atoms lead to inevitable variations between the billions of transistors in an integrated circuit. In fact, in nanoscale FETs, each dopant has a significant effect on the quantum state.

As mentioned already, the main source of device variation is the random motion of impurity atoms during the fabrication procedure of implantation and annealing. In order to model the stochastic coefficients in the model equations, each dopant is modeled as a Gaussian distribution such that the doping concentration at point $x$ is given by [23]

$$C(x, \omega) := \sum_j \frac{C_j}{(2\pi\sigma^2)^{3/2}} \exp \left( -\frac{(x - x_j(\omega))^2}{2\sigma^2} \right),$$

(60)
where \( x_j \) and \( C_j \) are the position and the charge of the \( j \)-th dopant, respectively. In the source and drain, to determine the position of random dopant, two random points (according to the two-dimensional problem) are used to translate it. For instance, the random variable \( \omega = (\frac{1}{2}, \frac{1}{2}) \) transforms the dopants to the center of a region \( (x_j(\omega)) \). Here, we assume that both regions have same equal number of dopants. Also, \( \sigma := 0.35 \text{ nm} \) corresponds to the extent of the electrostatic influence, and the results are not significantly sensitive to the value of \( \sigma \). Finally, the source and drain regions contain n-type dopants corresponding to a continuous doping concentration of \( 1 \cdot 10^{19} \text{ cm}^{-3} \) (heavily doped) and the doping concentration of the channel is \( 1 \cdot 10^{16} \text{ cm}^{-3} \).

![Figure 2: Initial mesh (left) and uniformly refined mesh (right) for a DG-MOSFET.](image)

In the following, we strive to draw a fair comparison between adaptive and uniform MLMC-FE methods. Uniform and refined meshes corresponding to the device shown in Figure 1 are depicted in Figure 2. In adaptive refinement, we use the marking strategy introduced in [19]. Here, for each element \( T \in T_{S_i} \) and \( T_{S_i}^h \), the local refinement indicator \( \eta_T \) satisfies

\[
\sum_T \eta_T^2 \geq \theta \eta^2,
\]

where the associated error estimator is defined as

\[
\eta := \left( \sum_{T \in T_{S_i}^h} \eta_T^2 \right)^{1/2}.
\]

In other words, we refine the smallest subset of elements whose corresponding error indicators in sum exceed the threshold \( \theta \eta^2 \).

The adaptive algorithm for the boundary-value problem is shown Algorithm 1. In the multilevel setting, the mesh is refined as long as \( \mathcal{E}_h^2 \) is greater than or equal to \( \varepsilon^2/2 \) and the number of samples are obtained according to the optimization problem [59]. Also, the same number and positions of random variables are used on all levels \( \ell \in \{1, \ldots, L\} \). In the numerical example, we set \( \theta_0 := 0.6 \) and the initial mesh \( T_0 \) and its uniform refinement are depicted in Figure 2.

The adaptively refined meshes for \( \ell \in \{1, \ldots, 6\} \) for the coupled system of equations are shown in Figures 3, 4, and 5. As shown, most of the meshes have been refined due to the randomness in the source and drain areas. Similarly, the interface condition between the insulator and the channel \( \Gamma \) gives rise to more refinements, where less
Initialization ($\ell = 0$): 
Initial mesh $T_0$, and $\theta := \theta_0$.

while $E_{2,\ell}^2 > \epsilon^2/2$ do

for $i = 1, \ldots, M_\ell$

Solve the boundary-value problem (7) to find $V_h^{(i)}, v_h^{(i)}$, and $u_h^{(i)}$ according to $M_\ell$.

end

Estimate the expected values $E[V_h], E[v_h], \text{ and } E[u_h]$ of the solutions.

Calculate the error indicator $\eta_\ell$ by (50) for the expected values on all elements.

Determine the triangles to be refined or coarsened using the marking strategy (61).

$T_{\ell+1} := \text{refine}(T_\ell, \eta_\ell)$

$\ell := \ell + 1$.

end

Algorithm 1: The adaptive MLMC-FE strategy for the coupled system of equations (7).

mesh refinement occurred in the channel (green triangles). The corresponding degree of freedom for the Poisson ($N_{PF}$) and drift-diffusion ($N_{DD}$) equations are summarized in Table 1 and Table 2, respectively. We compare the obtained degrees of freedom for adaptive and uniform refinement, where the initial mesh is the same in both cases.

![Figure 3: Adaptive mesh refinement for a DG-MOSFET with random dopants at $\ell = 1$ (left) and $\ell = 2$ (right).](image)

| $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|---|
| uniform | 1 685 | 4 749 | 11 626 | 27 084 | 60 569 | 131 819 | 280 264 |
| adaptive | 1 685 | 2 839 | 5 531 | 10 876 | 22 128 | 45 885 | 98 123 |

Table 1: Degrees of freedom $N_\ell$ for different levels comparing uniform MLMC-FE and adaptive MLMC-FE methods.

In the next step, we compare the discretization error of uniform and adaptive refinement with 100 number of
Figure 4: Adaptive mesh refinement for a DG-MOSFET with random dopants at $\ell = 3$ (left) and $\ell = 4$ (right).

Figure 5: Adaptive mesh refinement of a DG-MOSFET with random dopants at $\ell = 5$ (left) and $\ell = 6$ (right).

Table 2: Degrees of freedom $N_{DD}$ for different levels comparing uniform MLMC-FE and adaptive MLMC-FE methods.

| $\ell$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|--------|----|----|----|----|----|----|----|
| uniform | 755 | 2 141 | 5 293 | 12 372 | 27 743 | 60 509 | 128 833 |
| adaptive | 755 | 898 | 1 640 | 3 166 | 6 614 | 13 553 | 29 305 |

Figure 6: The discretization error of the drift-diffusion-Poisson system as a function of the degrees of freedom.
Figure 7: The statistical error of the drift-diffusion-Poisson system (58) as a function of the degrees of freedom.

| $\varepsilon$ | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ |
|---------------|-------|-------|-------|-------|-------|-------|
| 0.080         | 17    | 5     | –     | –     | –     | –     |
| 0.040         | 90    | 26    | 8     | –     | –     | –     |
| 0.020         | 418   | 121   | 35    | 11    | –     | –     |
| 0.010         | 1 671 | 481   | 137   | 42    | –     | –     |
| 0.007         | 3 759 | 1 080 | 308   | 94    | 30    | –     |
| 0.005         | 7 366 | 2 117 | 604   | 183   | 58    | –     |
| 0.002         | 5 3764| 17 113| 4 163 | 1 012 | 573   | 57    |

Table 3: The optimal number of samples for the uniform MLMC-FE method.

samples. Figure 6 indicates that the adaptive refinement reduces the error and obtains a convergence rate of $\alpha = 1.46$. However, the uniform refinement leads to a smaller convergence rate of $\alpha = 1.11$.

Moreover, we compare the statistical error of both multilevel methods. Figure 7 illustrates the decay of the variance for different degrees of freedom. The results show that similar to the discretization error, the variance in the adaptive approach is reduced faster ($\beta = 2.27$) compared to uniform refinement ($\beta = 1.73$). Again, the efficiency of the adaptive method is shown by the numerical results. Also, $C_0 = 0.041$ is obtained as the variance of level $\ell = 0$.

In order to estimate the optimal computational complexity, we solve the optimization problem (59). An interior point method can be used to solve the global optimization problem [5], where the results are the optimal number of samples. For different tolerances $\varepsilon$, the optimal values are summarized in Table 3 and Table 4 for uniform and adaptive refinements, respectively. In multilevel methods, most of the work is performed on coarse levels. The main reason is
the reduction of variance on the finer grids.

The computational work $\sum_{l=0}^{L} M_l N_l$ for both refinement methods are depicted in Figure 8. We here observe a significantly better efficiency of the adaptive model compared with the uniform approach. As depicted in the figure, the computational cost asymptotically behaves like $O(\varepsilon^{-2})$ for both multilevel techniques, which agrees with [24]. A more interesting computational result achieved regarding the CPU time. As Figure 8 shows since in the adaptive approach (compared to the uniform refinement) fewer degrees of freedom are needed, in different levels, lower computational time is used (to obtain same error tolerance). The difference between two computational times is more pronounced for lower prescribing errors that indicates the adaptive technique efficiency.

| $\varepsilon$ | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|--------------|------|------|------|------|------|------|------|
| 0.080        | 18   | 6    | 3    | –    | –    | –    | –    |
| 0.040        | 78   | 26   | 10   | 4    | –    | –    | –    |
| 0.020        | 334  | 111  | 44   | 14   | 5    | –    | –    |
| 0.010        | 1369 | 472  | 186  | 41   | 20   | –    | –    |
| 0.007        | 2856 | 949  | 372  | 119  | 42   | 14   | –    |
| 0.005        | 5597 | 1859 | 729  | 233  | 81   | 27   | –    |
| 0.002        | 36068| 11979| 4692 | 1502 | 520  | 170  | 54   |

Table 4: The optimal number of samples for the adaptive MLMC-FE method.

Figure 8: At the computational cost (left) and computational cost (right) of uniform MLMC-FE and adaptive MLMC-FE methods for different prescribed tolerances.
6. Conclusions

We have presented an adaptive MLMC-FE method for the numerical solution of the stochastic drift-diffusion-Poisson system. First, we proved an a-priori error estimate for the coupled system of equations with non-zero recombination rate. The error estimate points out how fast the error decreases as the mesh size decreases and can be considered as a useful measure of the efficiency of a given finite-element method. Also, using the stochastic numerical example, we estimated the convergence rate of the discretization error.

Secondly, a practically useful a-posteriori error indicator to bound the discretization error for the coupled system of equations was derived. From a computational point of view, the error estimator is inexpensive to estimate and guarantees the bounds on the error on all points of the geometry. The error indicator was used to design an adaptive refinement strategy to refine the mesh, where all coefficients in the system of equations can be random.

Regarding numerical examples, we implemented this adaptive MLMC-FE method to quantify noise and variations in nanoscale transistors as a real-world example. To this end, we defined a strategy to refine the meshes in the stochastic setting. The new technique was compared to the multilevel method with uniform refinement as a useful benchmark. Better convergence of the discretization error and better decay of variance were observed indicating the efficiency of the new approach. Finally, we employed an optimization problem to minimize the computational complexity. The optimal numbers of samples are obtained as the solution of the global optimization problem. The results indicate that in addition to a better control of error, a noticeable reduction of the computational work/time are achieved by the adaptive method.

7. Acknowledgments

The first and the last authors acknowledge support by FWF (Austrian Science Fund) START project no. Y660 PDE Models for Nanotechnology. The second author also acknowledges support by FWE project no. P28367-N35. The authors appreciate useful comments by Markus Melenk (TU Wien).

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