On Generalized Super-Coherent States

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Abstract

A set of operators, the so-called $k$-fermion operators, that interpolate between boson and fermion operators are introduced through the consideration of an algebra arising from two non-commuting quon algebras. The deformation parameters $q$ and $1/q$ for these quon algebras are roots of unity with $q = \exp(2\pi i/k)$ and $k \in \mathbb{N}\{0, 1\}$. The case $k = 2$ corresponds to fermions and the limiting case $k \to \infty$ to bosons. Generalized coherent states (connected to $k$-fermionic states) and super-coherent states (involving a $k$-fermionic sector and a purely bosonic sector) are investigated.

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1 Introduction

The interest of $q$-deformations for statistical physics is still very high in the community of physicists and mathematicians. In recent years, many works have been devoted to statistics of $q$-bosons, $q$-fermions and quons (see, for instance, Ref. [1] and references therein). This paper is devoted to $k$-fermions which are objects interpolating between fermions (corresponding to $k = 2$) and bosons (corresponding to $k \to \infty$).

The material in the present paper is organized as follows. We first discuss (in Section 2) the $k$-fermionic algebra $\Sigma_q$, where $q := \exp(2\pi i/k)$ with $k \in \mathbb{N} \setminus \{0, 1\}$, in terms of generalized Grassmann variables. Then, we introduce (in Section 3) generalized coherent states. Finally, the notion of fractional super-coherent states is introduced (in Section 4) from a certain limit of the well-known deformed coherent states.

2 The $k$-fermions

2.1 The $k$-fermionic algebra $\Sigma_q$

We first introduce the $k$-fermionic algebra $\Sigma_q$. The algebra $\Sigma_q$ is generated by five operators $a_+, a_-, a_+^\dagger, a_-^\dagger$ and $N$. We assume that $N$ is an Hermitean operator, that $a_+^\dagger$ (respectively, $a_-^\dagger$) is the adjoint of $a_+$ (respectively, $a_-$) and that these operators satisfy

\[
\begin{align*}
    a_-a_+ - qa_+a_- &= 1 \iff a_+^\dagger a_- - \bar{q}a_-^\dagger a_+ = 1 \quad (1a) \\
    Na_+ - a_+N &= +a_+ \iff Na_+^\dagger - a_+^\dagger N = -a_+^\dagger \quad (1b) \\
    Na_- - a_-N &= -a_- \iff Na_-^\dagger - a_-^\dagger N = +a_-^\dagger \quad (1c) \\
    (a_+)^k = (a_-)^k &= 0 \iff (a_+^\dagger)^k = (a_-^\dagger)^k = 0 \quad (1d) \\
    a_-a_+^\dagger = q^\ast a_+^\dagger a_- \iff a_+a_-^\dagger = q^\ast a_-^\dagger a_+ \quad (1e)
\end{align*}
\]

where the complex number

\[ q := \exp\left(\frac{2\pi i}{k}\right) \quad \text{with} \quad k \in \mathbb{N} \setminus \{0, 1\} \]

is a root of unity. (In Eq. (1), $\bar{q}$ stands for the complex conjugate of $q$.) The algebra $\Sigma_q$ clearly involves two non-commuting quon algebras $A_q$ (spanned by $a_+, a_-$ and $N$) and $A_{\bar{q}}$ (spanned by $a_+^\dagger, a_-^\dagger$ and $N$).

In view of the defining relations (1), the operators $a_+, a_-, a_+^\dagger, a_-^\dagger$ and $N$ act on a Fock space $\mathcal{F} := \{|n\rangle : n = 0, 1, \cdots, k-1\}$ with card $\mathcal{F} = k$. Furthermore, we chose a representation of $\Sigma_q$ in the following way. The action of $N$ is standard in the sense that

\[ N|n\rangle = n|n\rangle \]
while the action of the remaining operators is given by
\[ a_-|n\rangle = ([n]_q)^{1/2} |n - 1\rangle \quad \text{with} \quad a_-|0\rangle = 0 \]
\[ a_+^+|n\rangle = ([n]_q)^{1/2} |n - 1\rangle \quad \text{with} \quad a_+^+|0\rangle = 0 \]
and
\[ a_+|n\rangle = ([n+1]_q)^{1/2} |n + 1\rangle \quad \text{with} \quad a_+|k - 1\rangle = 0 \]
\[ a_-^+|n\rangle = ([n+1]_q)^{1/2} |n + 1\rangle \quad \text{with} \quad a_-^+|k - 1\rangle = 0 \]
where the symbol \([\ ]_q\) is defined by
\[ [X]_q := \frac{1 - q^X}{1 - q} \]
for any operator or number \(X\). Thus, the operators \(a_-\) and \(a_+^+\) behave like annihila-
tion operators, the operators \(a_+\) and \(a_-^+\) like creation operators and the operator \(N\) like a number oper-
ator.

The state vector \(|n\rangle\) can be written as
\[ |n\rangle = \frac{(a_+)^n}{([n]_q!)^{1/2}} |0\rangle \quad \text{or} \quad |n\rangle = \frac{(a_-^+)^n}{([n]_q!)^{1/2}} |0\rangle \quad \text{for} \quad n = 0, 1, \cdots, k - 1 \]
where, as usual, the \(p\)-deformed factorial \([n]_p\) is defined by (with \(p = q\) and \(\bar{q}\))
\[ [n]_p! := [1]_p [2]_p \cdots [n]_p \quad \text{for} \quad n \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad [0]_p! := 1 \]

In the specific case \(k = 2\), the algebra \(\Sigma_{-1}\) corresponds to ordinary fermion oper-
ators with \(a_+ = a_-\) and \(a_-^+ = a_+\) for which we have \((a_-)^2 = (a_+)^2 = 0\), a relation that reflects the Pauli exclusion principle. In the limiting case \(k \to \infty\), the algebra \(\Sigma_{+1}\) corresponds to ordinary boson operators with \(a_+^+ = a_-\) and \(a_-^+ = a_+\). For \(k\) arbitrary, the algebra \(\Sigma_q\) corresponds to \(k\)-fermion operators \(a_-\) and \(a_+\) (with their adjoint \(a_-^+\) and \(a_+^+\), respectively) that interpolate between fermion and boson oper-
ators; the space \(\mathcal{F}\) is of dimension \(k\) for the \(k\)-fermionic algebra \(\Sigma_q\) (i.e., two-
dimensional for the fermionic algebra \(\Sigma_{-1}\) and infinite-dimensional for the bosonic algebra \(\Sigma_{+1}\)).

2.2 Grassmannian realization of \(\Sigma_q\)

We give here some preliminaries useful for obtaining a Grassmannian realization of the algebra \(\Sigma_q\). Equation (1d) suggests that we use generalized Grassmann variables (see Refs. [2-5]) \(z\) and \(\bar{z}\) such that
\[ z^k = \bar{z}^k = 0 \quad \text{(2)} \]
(The particular case \( k = 2 \) corresponds to ordinary Grassmann variables.) We then introduce the \( \partial_z \)- and \( \partial_{\bar{z}} \)-derivatives via
\[
\partial_z f(z) := \frac{f(qz) - f(z)}{(q - 1)z}, \quad \partial_{\bar{z}} g(\bar{z}) := \frac{g(q\bar{z}) - g(\bar{z})}{(q - 1)\bar{z}}
\]
where \( f : z \mapsto f(z) \) and \( g : \bar{z} \mapsto g(\bar{z}) \) are arbitrary functions. The linear operators \( \partial_z \) and \( \partial_{\bar{z}} \) satisfy
\[
\partial_z z^n = [n]_q z^{n-1}, \quad \partial_{\bar{z}} \bar{z}^n = [n]_{\bar{q}} \bar{z}^{n-1}
\]
for \( n = 0, 1, \ldots, k - 1 \). Therefore, when \( f(z) \) and \( g(\bar{z}) \) can be developed as
\[
f(z) = \sum_{n=0}^{k-1} a_n z^n, \quad g(\bar{z}) = \sum_{n=0}^{k-1} b_n \bar{z}^n
\]
where the coefficients \( a_n \) and \( b_n \) in the expansions are complex numbers, we check that
\[
(\partial_z)^k f(z) = (\partial_{\bar{z}})^k g(\bar{z}) = 0
\]
Consequently, we shall assume that the conditions
\[
(\partial_z)^k = (\partial_{\bar{z}})^k = 0
\]
hold in addition to Eq. (2).

From Eqs. (2) and (4), the correspondences
\[
a_- \rightarrow \partial_z, \quad a_+ \rightarrow z, \quad a_+^+ \rightarrow \partial_{\bar{z}}, \quad a_-^+ \rightarrow \bar{z}
\]
clearly provide us with a realization of Eqs. (1a) and (1d). Note that Eq. (1e) leads to
\[
\partial_z \partial_{\bar{z}} = \frac{q^2}{4} \partial_{\bar{z}} \partial_z, \quad \bar{z}z = q^2 \bar{z}z
\]
in the realization based on Eq. (5).

3 Generalized coherent states

There exists several methods for introducing coherent states. We can use the action of a displacement operator on a reference state [6] or the construction of an eigenstate for an annihilation operator [7,8] or the minimisation of uncertainty relations [9]. In the case of the ordinary harmonic oscillator, the three methods lead to the same result (when the reference state is the vacuum state). Here, the situation is a little bit more intricate (as far as the equivalence of the three methods is concerned) and we chose to define the generalized coherent states or \( k \)-fermionic coherent states \( |z\rangle \) and \( |\bar{z}\rangle \) as follows
\[
|z\rangle := \sum_{n=0}^{k-1} \frac{z^n}{([n]_q!)^{1\over 2}} |n\rangle, \quad |\bar{z}\rangle := \sum_{n=0}^{k-1} \frac{\bar{z}^n}{([n]_{\bar{q}}!)^{1\over 2}} \langle n|
\]
where \( z \) and \( \bar{z} \) are generalized Grassmann variables that satisfy Eq. (2). It can be easily checked that the state vectors \(|z\rangle\) and \(|\bar{z}\rangle\) are eigenvectors of the operators \(a_-\) and \(a_+^\dagger\), respectively. More precisely, we have

\[
a_-|z\rangle = z|z\rangle, \quad a_+^\dagger|\bar{z}\rangle = \bar{z}|\bar{z}\rangle
\]

The case \( k = 2 \) corresponds to fermionic coherent states while the limiting case \( k \to \infty \) to bosonic coherent states.

We define

\[
|z\rangle := \sum_{n=0}^{k-1} \langle n | \bar{z}^n \frac{\bar{z}^n}{([n]_q!)^{\frac{1}{2}}}, \quad |\bar{z}\rangle := \sum_{n=0}^{k-1} \langle n | \bar{z}^n \frac{z^n}{([n]_q!)^{\frac{1}{2}}}
\]

Then, the ‘scalar products’ \((z'|z)\) and \((\bar{z}'|\bar{z})\) follow from the ordinary scalar product \(\langle n'|n \rangle = \delta(n',n)\). For instance, we get

\[
(z'|z) = \sum_{n=0}^{k-1} \bar{z}^n z^n \frac{\bar{z}^n}{([n]_q!)^{\frac{1}{2}}}
\]

In view of the relationship

\[
[n]_q! = q^{-\frac{1}{2}} n^{n-1} [n]_q!
\]

and of the property

\[
\bar{z}^n z^n = q^{-\frac{1}{2}} n^{n-1} (\bar{z} z)^n
\]

we obtain the following result

\[
(z|z) = \sum_{n=0}^{k-1} \frac{(\bar{z} z)^n}{[n]_q!}
\]

Similarly, we have

\[
(\bar{z}|\bar{z}) = \sum_{n=0}^{k-1} \frac{(z \bar{z})^n}{[n]_q!}
\]

By defining the \(q\)-deformed exponential \(e_q\) by

\[
e_q : x \mapsto e_q(x) := \sum_{n=0}^{k-1} \frac{x^n}{[n]_q!}
\]

we can rewrite Eqs. (6) and (7) as

\[
(z|z) = e_q(\bar{z} z), \quad (\bar{z}|\bar{z}) = e_q(z \bar{z})
\]

(Observe that the summation in the exponential \(e_q\) is finite, for \( k \) finite, rather than infinite as is usually the case in \(q\)-deformed exponentials.)

We guess that the \(k\)-fermionic coherent states \(|z\rangle\) and \(|\bar{z}\rangle\) form overcomplete sets with respect to some integration process accompanying the derivation process.
inherent to Eq. (3). Following Majid and Rodríguez-Plaza [5], we consider the integration process defined by

\[ \int dz z^p = \int d\bar{z} \bar{z}^p := 0 \quad \text{for} \quad p = 0, 1, \ldots, k - 2 \]  

(8a)

and

\[ \int dz z^{k-1} = \int d\bar{z} \bar{z}^{k-1} := 1 \]  

(8b)

Clearly, the integrals in (8) generalize the Berezin integrals corresponding to \( k = 2 \).

In the case where \( k \) is arbitrary, we can derive the overcompleteness property

\[ \int dz |z\rangle \mu(z, \bar{z}) (z) d\bar{z} = \int d\bar{z} |\bar{z}\rangle \mu(\bar{z}, z) (\bar{z}) dz = 1 \]

where the function \( \mu \) defined through

\[ \mu(z, \bar{z}) := \sum_{n=0}^{k-1} \left( \frac{[n_q]![n_q]!}{k!} \right)^\frac{1}{2} z^{k-1-n} \bar{z}^{k-1-n} \]

may be regarded as a measure.

## 4 Fractional super-coherent states

We now switch to \( Q \)-deformed coherent states of the type

\[ |Z\rangle := \sum_{n=0}^{\infty} \frac{Z^n}{([n]_Q)!} |n\rangle \]  

(9)

associated to a quon algebra \( A_Q \) where \( Q \in \mathbb{C} \setminus S^1 \). The latter states are simple deformations of the bosonic coherent states (cf. Ref. [10]). The coherent state \( |Z\rangle \) may be considered to be an eigenstate, with the eigenvalue \( Z \in \mathbb{C} \), of an annihilation operator \( b_- \) in a representation such that the operator \( b_- \) and the associated creation operator \( b_+ \) satisfy

\[ b_- |n\rangle = \left( \frac{[n]_Q}{[n]_Q} \right)^\frac{1}{2} |n - 1\rangle \quad \text{with} \quad b_- |0\rangle = 0 \]

\[ b_+ |n\rangle = \left( \frac{[n + 1]_Q}{[n + 1]_Q} \right)^\frac{1}{2} |n + 1\rangle \]

with \( n \in \mathbb{N} \).

For \( Q \rightarrow q \), we have \( [k]_Q! \rightarrow 0 \). Therefore, the term \( Z^k/([k]_Q!)^\frac{1}{2} \) in Eq. (9) makes sense for \( Q \rightarrow q \) only if \( Z \rightarrow z \), where \( z \) is a generalized Grassmann variable with \( z^k = 0 \). This type of reasoning has been invoked for the first time in Ref. [11]. (In [11], the authors show that there is an isomorphism between the braided line and the one-dimensional super-space.)
It is the aim of this section to determine the limit

$$|\xi| := \lim_{Q \to q} \lim_{Z \to z} |Z|$$

when $Q$ goes to the root of unity $q = \exp(2\pi i/k)$ and $Z$ to a Grassmann variable $z$. The starting point is to rewrite Eq. (9) as

$$|Z| = \sum_{r=s=0}^{k-1} \frac{Z^{r+s}}{([r+s]_Q!)^{1/2}} |rk+s\rangle$$

Then, by making use of the formulas

$$\frac{[k]_Q}{[rk]_Q} \to \frac{1}{r} \quad \text{for} \quad Q \to q \quad \text{with} \quad r \neq 0$$

and

$$\frac{[s]_Q}{[rk+s]_Q} \to 1 \quad \text{for} \quad Q \to q \quad \text{with} \quad s = 0, 1, \cdots, k-1$$

do we find that

$$\lim_{Q \to q} \lim_{Z \to z} \frac{Z^{r+s}}{([r+s]_Q!)^{1/2}} \frac{\alpha^r}{([s]_Q!)^{1/2} (r!)^{1/2}}$$

works for $s = 0, 1, \cdots, k-1$ and $r \in \mathbb{N}$. The complex variable $\alpha$ in Eq. (10) is defined by

$$\alpha := \lim_{Q \to q} \lim_{Z \to z} \frac{Z^k}{([k]_Q!)^{1/2}}$$

Therefore, we obtain

$$|\xi| = \sum_{r=0}^{k-1} \sum_{s=0}^{k-1} \frac{z^s}{([s]_Q!)^{1/2} (r!)^{1/2}} \alpha^r |rk+s\rangle$$

Finally, by employing the symbolic notation

$$|rk+s\rangle \equiv |r\rangle \otimes |s\rangle$$

we arrive at the formal expression

$$|\xi| = \sum_{r=0}^{k-1} \alpha^r |r\rangle \otimes \sum_{s=0}^{k-1} \frac{z^s}{([s]_Q!)^{1/2}} |s\rangle$$

We thus end up with the product of a bosonic coherent state by a $k$-fermionic coherent state. This product shall be called a fractional super-coherent state. In the particular case $k = 2$, it reduces to the product of a bosonic coherent state by a fermionic coherent state, i.e., to the super-coherent state associated to a superoscillator [12]. In the framework of field theory, Eq. (11) means that in the limit $Q \to q$, every field $\psi$ with values $\psi(Z)$ is transformed into a fractional super-field $\Psi$ with value $\Psi(z, \alpha)$, $z$ being a generalized Grassmann variable and $\alpha$ a bosonic variable.
5 Concluding remarks

As a main result, the $k$-fermions introduced in the present paper can be ranged between fermions (for $k = 2$) and bosons (for $k \to \infty$). This result is further emphasized by calculating the coherence factor $g^{(m)}$ for an assembly of $k$-fermions: We find that $g^{(m)} = 0$ for $m > k - 1$ so that, in a many-particle scheme, a given state of fractional spin $S = \frac{1}{k}$ cannot be occupied by more than $k - 1$ identical $k$-fermions. The $k$-fermions thus satisfy a generalized Pauli exclusion principle.

We close this paper by mentioning two open questions. First, does the $W_\infty$ algebra described by Fairlie, Fletcher and Zachos [13] play an important role in the symmetries inherent to $k$-fermions (see also Ref. [14])? Second, what is the connection between $k$-fermions and fractional super-symmetry for anyons [15,16], especially the anyons constructed from unitary representations of the group diffeomorphisms of the plane [16]? These matters should be the object of future works.

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