Material Facts Obscured in Hansen’s Modern Gauss-Markov Theorem

H D Vinod *

May 4, 2022

Abstract

We show that the abstract and conclusion of Hansen’s Econometrica paper, Hansen (2022), entitled a modern Gauss-Markov theorem (MGMT), obscures a material fact, which in turn can confuse students. The MGMT places ordinary least squares (OLS) back on a high pedestal by bringing in the Cramer-Rao efficiency bound. We explain why linearity and unbiasedness are linked, making most nonlinear estimators biased. Hence, MGMT extends the reach of the century-old GMT by a near-empty set. It misleads students because it misdirects attention back to the unbiased OLS from beneficial shrinkage and other tools, which reduce the mean squared error (MSE) by injecting bias.

1 Introduction

The century-old Gauss-Markov theorem (GMT) showed that the ordinary least squares (OLS) estimator (sample mean) is the (minimum variance) best (efficient) linear unbiased estimator (BLUE). Stein (1955) shocked statisticians when he proved the inadmissibility of OLS, and threw the first salvo toward the break-up of the long-dominating GMT. Numerous subsequent papers established that the so-called Stein-rule (nonlinear shrinkage) estimator is guaranteed to reduce the mean squared error (MSE) of OLS. Efron and Morris (1973) explain the practical significance of Stein-rule in

*Email vinod@fordham.edu.
a Scientific American magazine article (May 1977) using baseball batting averages. The key point of the Stein-rule is that despite the GMT BLUE result, shrinkage estimation beats OLS.

Ridge regression is another strand of literature rejecting OLS. It is also a biased nonlinear shrinkage estimator. Vinod (2020) explains the “big idea” behind ridge regression, and concludes with a quote from Hastie (2020): “What started out as a simple fix for wayward linear regression models has evolved into a large collection of tools for data modeling. It would be hard to imagine the life of a data scientist without them.”

With considerable ingenuity, Hansen (2022) claims to generalize GMT, calling it modern. He drops the modifier “L=linear” in BLUE. He claims to improve upon other efforts by Kariya, Kurata, Berk, and Hwang to generalize GMT. Others are too restrictive on the class of allowed nonlinearity or error distributions. The abstract and conclusion sections of Hansen (2022) give the false impression that OLS remains best even in a class of nonlinear estimators. However, the eleven pages of Econometrica devoted to MGMT do not include any Efron-Morris type example of a nonlinear unbiased estimator.

Hansen fails to reveal that the adjectives linear and unbiased (L&U) are like conjoint twins in the present context, since all nonlinear estimators used in practice are biased in finite samples. Hence, MGMT dropping the modifier L while keeping its twin U (unbiased) is primarily an empty change of little practical consequence. The gap between BLUE and BUE is hairline-thin. I understand that group theory in pure mathematics has translation-invariant groups. These ‘groups’ can yield esoteric quadratic functions to design an unbiased nonlinear estimator. The exception proves the rule that the class of nonlinear unbiased estimators is not simply “small,” as Hansen admits, but near empty.

2 Linearity & Unbiasedness Conjoint twins

Now we explain why it is hard to separate the conjoint twins (L&U). We begin with the linear regression model with \( p \) regressors. In standard notation, it is \( y = X\beta + \epsilon \). The OLS estimator is \( b = (X'X)^{-1}X'y \). The linearity of the OLS model \( y = X\beta + \epsilon \) allows us to write

\[
b = (X'X)^{-1}X'y = (X'X)^{-1}X'X\beta + \epsilon \tag{1}
\]
where $b$ and $\beta$ are $p \times 1$ vectors. Now compute the expectation of $b$ using the assumption that $E\epsilon = 0$. We have $Eb = \beta$, where $(X'X)^{-1}(X'X) = I$.

Also, OLS residuals $\hat{\epsilon}_t$ add up to zero in finite samples, $\sum \hat{\epsilon}_t = 0$.

It is important to note that one can compute the expectation of each regression coefficient separately, only because of the linearity of the OLS model. The separation difficulty creates the conjoint twins of linearity and unbiasedness (L&U) in the context of individual coefficient estimates in finite samples.

A corresponding nonlinear model is $y_t = f(X\beta)_t + u_t$, where the nonlinear regression function $f(X\beta)_t$ depends on a vector of coefficients $\beta$. Unlike linear models, components of $\beta$ can include complicated functions, including various powers and trigonometric transformations. Unbiasedness requires proof that each estimated nonlinear coefficient based on finite samples be equal to its true population value, $E(\hat{\beta}) = \beta$.

Since there is no commonly accepted method for estimating the nonlinear parameters $\hat{\beta}$, which can be sensitive to starting values, one cannot simply write the expression for the expectation of a single component $\beta_j$ which does not also depend on other components $\beta_k, k \neq j$. Clean expressions for only one nonlinear coefficient at a time may not exist. It is thus established that most individual nonlinear estimators are biased, $E(\hat{\beta}) \neq \beta$. Hence, our claim of conjoint twins (L&U) is supported.

Consider a textbook example from the R code snippet #R1.2.1 in Vinod (2008). Let $y$ denote output, $K$ denotes capital input, and $L$ denotes labor input. A nonlinear Cobb-Douglas production function is

$$y = AK^\alpha L^\beta + \epsilon. \quad (2)$$

The exact implication of Hansen’s modern GMT to the nonlinear least squares (NLS) estimator of (2) is unclear. The textbook snippet uses metals data to estimate NLS coefficients ($A, \alpha, \beta$). If one plugs in the NLS estimates to define the fitted value of output $\hat{y}$, the residuals, $\hat{\epsilon} = y - \hat{y}$, do not add up to zero, or $\Sigma \hat{\epsilon} \neq 0$. Then, least squares estimates ($\hat{A}, \hat{\alpha}, \hat{\beta}$) must be biased. Hence, Hansen’s modern GMT does not apply to the estimation of Cobb-Douglas nonlinear functions.
3 Final Remarks

Hansen’s conclusion states, “The Gauss-Markov Theorem is a core efficiency result but restricts attention to linear estimators—and this is an inherently uninteresting restriction.” We agree. However, Hansen fails to disclose that his MGMT does not at all cover most of the interesting nonlinear estimators used in Econometrics. Hansen’s MGMT has not reestablished the efficiency of OLS, except for an uninteresting, near-empty set of nonlinear unbiased estimators.

The abstract and conclusion of a research paper are akin to advertisements. A Federal Trade Commission (FTC) employee, Fair (2014), formally defines what is expected from advertisers. They must disclose all “material” facts about a product, so that they satisfy a mnemonic of four P’s: (i) Prominent display, (ii) understandable Presentation, (iii) a visible Placement, and (iv) a Proximity to the claim being modified. This note shows that Hansen (2022) fails to satisfy these norms.

Hansen is careful to include a caveat in an obscure part of his paper stating that “the class of nonlinear unbiased estimators is small.” The caveat fails to clarify that the relevant set is near empty, not just small. More important, the caveat is “material” to the main point of MGMT. In the interest of ethical disclosure norms, the caveat should have been more conspicuously presented, proximate to the claims in the abstract and conclusion of Hansen’s paper. Otherwise, readers may be tempted to conclude that OLS beats many nonlinear estimators, a patently false impression.

Instead of citing the more relevant Stein (1955), Hansen cites the insight by Stein (1956). This should not give a false impression that Hansen has co-opted the Stein-rule. This note hopes to explain the presence of conjoint twins (L&U) in regression model estimation from finite samples. It is hard to separate linearity from unbiasedness. Hence Hansen’s MGMT extends the set of estimators where OLS is more efficient than nonlinear estimators by an uninteresting, nearly empty set. A generalization that extends the applicability of GMT by one esoteric unbiased nonlinear estimator is not a worthy generalization.
References

Efron, B. and Morris, C. (1973), “Stein’s Estimation Rule and its Competitors: An Empirical Bayes Approach,” *Journal of the American Statistical Association*, 68, 117–130.

Fair, L. (2014), *Full Disclosure*, Washington, DC: The Federal Trade Commission.

Hansen, B. (2022), “A Modern Gauss-Markov Theorem,” *Econometrica*, 90(?), 1–18.

Hastie, T. (2020), “Ridge Regularization: An Essential Concept in Data Science,” *Technometrics*, 0, 1–8.

Stein, C. (1955), “Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution,” in *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, CA, pp. 197–206.

Vinod, H. (2020), “What’s the big idea? Ridge regression and regularization,” *Significance*, 17, 41.

Vinod, H. D. (2008), *Hands-on Intermediate Econometrics Using R: Templates for Extending Dozens of Practical Examples*, Hackensack, NJ: World Scientific, ISBN 10-981-281-885-5.