COMPACT HYPERBOLIC COXETER FIVE-DIMENSIONAL POLYTOPES WITH NINE FACETS

JIMING MA AND FANGTING ZHENG

Abstract. In this paper, we obtain a complete classification of compact hyperbolic Coxeter five-dimensional polytopes with nine facets.

1. Introduction

A Coxeter polytope in the spherical, hyperbolic or Euclidean space is a polytope whose dihedral angles are all integer submultiples of $\pi$. Let $X^d$ be $\mathbb{E}^d$, $S^d$, or $H^d$. If $\Gamma \subset \text{Isom}(X^d)$ is a finitely generated discrete reflection group, then its fundamental domain is a Coxeter polytope in $X^d$. On the other hand, if $\Gamma = \Gamma(P)$ is generated by reflections in the bounding hyperplanes of a Coxeter polytope $P \subset X^d$, then $\Gamma$ is a discrete group of isometries of $X^d$ and $P$ is its fundamental domain.

There is an extensive body of literature in this field. In early work, [Cox34] has proved that any spherical Coxeter polytope is a simplex and any Euclidean Coxeter polytope is either a simplex or a direct product of simplices. See, for example, [Cox34, Bou68] for full lists of spherical and Euclidean Coxeter polytopes.

However, for hyperbolic Coxeter polytopes, the classification remains an active research topic. It was proved by Vinberg [Vin85] that no compact hyperbolic Coxeter polytope exists in dimensions $d \geq 30$, and non-compact hyperbolic Coxeter polytope of finite volume does not exist in dimensions $d \geq 996$ [Pro87]. These bounds, however, may not be sharp. Examples of compact polytopes are known up to dimension 8 [Bug84, Bug92]; non-compact polytopes of finite volume are known up to dimension 21 [Vin72, VK78, Bor98]. As for the classification, complete results are only available in dimensions less than or equal to three. Poincare finished the classification of 2-dimensional hyperbolic polytopes in [P1882]. That result was important to the work of Klein and Poincare on discrete groups of isometries of the hyperbolic plane. In 1970, Andreev proved an analogue for the 3-dimensional hyperbolic convex finite volume polytopes [And70(1), And70(2)]. This theorem played a fundamental role in Thurston’s work on the geometrization of 3-dimensional Haken manifolds.

In higher dimensions, although a complete classification is not available, interesting examples have been displayed in [Mak65, Mak66, Vin67, Mak68, Vin69, Rus89, ImH90, All06]. In addition, enumerations are reported for the cases in which the differences between the
numbers of facets $m$ and the dimensions $d$ of polytopes are fixed to some small number. When $m - d = 1$, Lannér classified all compact hyperbolic Coxeter simplices $\text{Lan50}$. The enumeration of non-compact hyperbolic simplices with finite volume has been reported by several authors, see e.g. $\text{Bow68}$, $\text{Vin67}$, $\text{Kos67}$. For $m - d = 2$, Kaplinskaja described all hyperbolic Coxeter simplicial prisms $\text{Kap74}$. Esselmann $\text{Ess96}$ later enumerated other possibilities in this family, which are named $\text{Esselmann polytopes}$. Tumarkin $\text{Tum04}$ classified all non-compact hyperbolic Coxeter $d$-dimensional polytopes with $n + 2$ facets. In the case of $m - d = 3$, Esselman proved in 1994 that compact hyperbolic Coxeter $d$-polytopes with $d + 3$ facets only exist when $d \leq 8$ $\text{Ess94}$. By expanding the techniques derived by Esselmann in $\text{Ess94}$ and $\text{Ess96}$, Tumarkin completed the classification of compact hyperbolic Coxeter $d$-polytopes with $d + 3$ facets $\text{Tum07}$. In the non-compact case, Tumarkin proved in $\text{Tum04}$ such that polytopes do not exist in dimensions greater than or equal to 17. Furthermore, Roberts provided a list for the family with exactly one non-simple vertex $\text{Rob15}$. In the case of $m - d = 4$, Flikson-Tumarkin showed in $\text{FT08}$ that compact hyperbolic Coxeter $d$-polytope with $d + 4$ facets does not exist when $d$ is greater than or equal to 8. This bound is sharp because of the example constructed by Bugeanko $\text{Bug84}$. In addition, Flikson-Tumarkin showed that Bugeanko’s example is the only 7-dimensional polytope with 11 facets. However, complete classifications for $d = 4, 5, 6$ are not presented. Recently, Burcroff $\text{Bur22}$ and Ma-Zheng $\text{MZ22}$ provided a complete list of 348 compact Coxeter 4-polytopes with 8 polytopes independently.

Besides, some scholars have also considered polytopes with small numbers of disjoint pairs $\text{FT08}$, $\text{FT09}$, $\text{FT14}$ or of certain combinatoric types, such as $d$-pyramid $\text{Tum04}$ and $d$-cube $\text{Jac17}$, $\text{JT18}$. An updating overview of the current knowledge for hyperbolic Coxeter polytopes is available on Anna Felikson’s webpage $\text{F}$.

In this paper, the following is proved:

**Theorem 1.1.** There are exactly 51 compact hyperbolic Coxeter 5-polytopes with 9 facets.

We remark that Burcroff has also carried out 50 such polytopes independently $\text{Bur22}$. We have communicated with Burcroff our results of 51 polytopes when her preprint appeared. Burcroff replied that she lost the case where the weights are all less than or equal to five. We now all agree that 51 is the correct number. The correspondence between the notions of our and Burcroff’s polytopes is presented in Section $\text{Section 7}$.

The paper $\text{JT18}$ is the main inspiration for our recent work on enumerating hyperbolic Coxeter polytopes. In comparison with $\text{JT18}$, we use a more universal “block-pasting” algorithm, which is first introduced in $\text{MZ18}$, rather than the “tracing back” algorithm. More geometric obstructions are adopted and programmed to considerably reduce the computational load. Our algorithm efficiently enumerates hyperbolic Coxeter polytopes over arbitrary combinatoric type rather than merely the $n$-cube.

Last but not the least, our main motivation in studying the hyperbolic Coxeter polytopes is for the construction of high-dimensional hyperbolic manifolds. However, this is not the theme here. Readers can refer to, for example, $\text{KM13}$, for interesting hyperbolic manifolds built via special hyperbolic Coxeter polytopes.
This paper is organized as follows. We provide in Section 2 some preliminaries about hyperbolic (compact Coxeter) polytopes. Then, in Section 3 we recall the 2-phases procedure and some terminologies introduced by Jacquemet and Tschantz [Jac17, JT18] for numerating hyperbolic Coxeter $n$-cubes. The 322 combinatorial types of simple 5-polytopes with 9 facets are reported in Section 4. The enumerations of all the “SEILper”-potential matrices are in Section 5. The sectional restrictions are also introduced there to further restrict the number of SEILper matrices. Next, signature obstructions are applied for Gram matrices of actual hyperbolic Coxeter polytopes in Section 6. Validations and the complete lists of the resulting Coxeter diagrams and hyperbolic lengths of Theorem 1.1 are shown in Section 7.

Acknowledgment

We would like to thank Amanda Burcroff for communicating with us about her result after we posted our preprint [MZ22]. The computations is pretty delicate and complex, and the list now is much more convincing due to the mutual check. We are also grateful to Nikolay Bogachev for his interest and discussion about the results. The computations throughout this paper are performed on a cluster of server of PARATERA, engrid12, line priv_para (CPU:Intel(R) Xeon(R) Gold 5218 16 Core v5@2.3GHz).

2. PRELIMINARY

In this section, we recall some essential facts about compact Coxeter hyperbolic polytopes, including Gram matrices, Coxeter diagrams, characterization theorems, etc. Readers can refer to, for example, [Vin93] for more details.

2.1. Hyperbolic space, hyperplane and convex polytope. We first describe a hyperboloid model of the $d$-dimensional hyperbolic space $\mathbb{H}^d$. Let $\mathbb{E}^{d,1}$ be a $d + 1$-dimensional Euclidean vector space equipped with a Lorentzian scalar product $\langle \cdot, \cdot \rangle$ of signature $(d, 1)$. We denote by $C_+$ and $C_-$ the connected components of the open cone

$$C = \{ x = (x_1, \ldots, x_d, x_{d+1}) \in \mathbb{E}^{d,1} : \langle x, x \rangle < 0 \}$$

with $x_{d+1} > 0$ and $x_{d+1} < 0$, respectively. Let $R_+$ be the group of positive numbers acting on $\mathbb{E}^{d,1}$ by homothety. The hyperbolic space $\mathbb{H}^d$ can be identified with the quotient set $C_+/R_+$, which is a subset of $PS^d = (\mathbb{E}^{d,1}\setminus\{0\})/R_+$. The natural projection is denoted by

$$\pi : (\mathbb{E}^{d,1}\setminus\{0\}) \to PS^d.$$ 

We denote $\overline{\mathbb{H}}^d$ as the completion of $\mathbb{H}^d$ in $PS^d$. The points of the boundary $\partial \overline{\mathbb{H}}^d = \overline{\mathbb{H}}^d \setminus \mathbb{H}^d$ are called the ideal points. The affine subspaces of $\overline{\mathbb{H}}^d$ of dimension $d - 1$ are hyperplanes. In particular, every hyperplane of $\overline{\mathbb{H}}^d$ can be represented as

$$H_e = \{ \pi(x) : x \in C_+, \langle x, e \rangle = 0 \},$$

where $e$ is a vector with $\langle e, e \rangle = 1$. The half-spaces separated by $H_e$ are denoted by $H_e^+$ and $H_e^-$, where

$$H_e^+ = \{ \pi(x) : x \in C_+, \langle x, e \rangle \leq 0 \}.$$ 

The mutual disposition of hyperplanes $H_e$ and $H_f$ can be described in terms of the corresponding two vectors $e$ and $f$ as follows:
• The hyperplanes $H_e$ and $H_f$ intersect if $|\langle e, f \rangle| < 1$. The value of the dihedral angle of $H_e^* \cap H_f^*$, denoted by $\angle H_e H_f$, can be obtained via the formula
  \[
  \cos \angle H_e H_f = -\langle e, f \rangle;
  \]
• The hyperplanes $H_e$ and $H_f$ are ultraparallel if $|\langle e, f \rangle| = 1$;
• The hyperplanes $H_e$ and $H_f$ diverge if $|\langle e, f \rangle| > 1$. The distance $\rho(H_e, H_f)$ between $H_e$ and $H_f$, when $H_e^+ \subset H_f^*$ and $H_f^+ \subset H_e^*$, is determined by
  \[
  \cosh \rho(H_e, H_f) = -\langle e, f \rangle.
  \]

We say a hyperplane $H_e$ supports a closed bounded convex set $S$ if $H_e \cap S \neq 0$ and $S$ lies in one of the two closed half-spaces bounded by $H_e$. If a hyperplane $H_e$ supports $S$, then $H_e \cap S$ is called a face of $S$.

**Definition 2.1.** A $d$-dimensional convex hyperbolic polytope is a subset of the form

\[
(2.2) \quad P = \bigcap_{i \in I} H_i^* \in \mathbb{H}^d,
\]

where $H_i^*$ is the negative half-space bounded by the hyperplane $H_i$ in $\mathbb{H}^d$ and the line “—” above the intersection means taking the completion in $\mathbb{H}^d$, under the following assumptions:

• $P$ contains a non-empty open subset of $\mathbb{H}^d$ and is of finite volume;
• Every bounded subset $S$ of $P$ intersects only finitely many $H_i$.

A convex polytope of the form (2.2) is called acute-angled if for distinct $i, j$, either $\angle H_i H_j \leq \frac{\pi}{2}$ or $H_i^+ \cap H_j^+ = \emptyset$. It is obvious that Coxeter polytopes are acute-angled. We denote $e_i$ as the corresponding unit vector of $H_i$, namely $e_i$ is orthogonal to $H_i$ and point away from $P$. The polytope $P$ has the following form in the hyperboloid model.

\[
P = \pi(K) \cap \mathbb{H}^d,
\]

where $K = K(P)$ is the convex polyhedral cone in $\mathbb{E}^{d,1}$ given by

\[
K = \{x \in \mathbb{E}^{d,1} : \langle x, e_i \rangle \leq 0 \text{ for all } i\}.
\]

In the sequel, a $d$-dimensional convex polytope $P$ is called a $d$-polytope. A $j$-dimensional face is named a $j$-face of $P$. In particular, a $(d-1)$-face is called a facet of $P$. We assume that each of the hyperplane $H_i$ intersects with $P$ on its facet. In other words, the hyperplane $H_i$ is uniquely determined by $P$ and is called a bounding hyperplane of the polytope $P$. A hyperbolic polytope $P$ is called compact if all of its 0-faces, i.e., vertices, are in $\mathbb{H}^d$. It is called of finite volume if some vertices of $P$ lie in $\partial \mathbb{H}^d$.

2.2. Gram matrices, Perron-Frobenius Theorem, and Coxeter diagrams. Most of the content in this subsection is well-known by peers in this field. We present them for the convenience of the readers. In particular, Theorem 2.3 and 2.4 are extremely important throughout this paper.

For a hyperbolic Coxeter $d$-polytope $P = \bigcap_{i \in I} H_i^*$, we define the Gram matrix of polytope $P$ to be the Gram matrix $(\langle e_i, e_j \rangle)$ of the system of vectors $\{e_i \in \mathbb{E}^{d,1} | i \in I\}$ that determine
The Gram matrix of $P$ is the $m \times m$ symmetric matrix $G(P) = (g_{ij})_{1 \leq i,j \leq m}$ defined as follows:

\[
g_{ij} = \begin{cases} 
1 & \text{if } j = i, \\
-\cos \frac{\pi}{k_{ij}} & \text{if } H_i \text{ and } H_j \text{ intersect at a dihedral angle } \frac{\pi}{k_{ij}}, \\
-\cosh \rho_{ij} & \text{if } H_i \text{ and } H_j \text{ diverge at a distance } \rho_{ij}, \\
-1 & \text{if } H_i \text{ and } H_j \text{ are ultraparallel.}
\end{cases}
\]

Other than the Gram matrix, a Coxeter polytope $P$ can also be described by its Coxeter graph $\Gamma = \Gamma(P)$. Every node $i$ in $\Gamma$ represents the bounding hyperplane $H_i$ of $P$. Two nodes $i_1$ and $i_2$ are joined by an edge with weight $2 \leq k_{ij} \leq \infty$ if $H_i$ and $H_j$ intersect in $\mathbb{H}^n$ with angle $\frac{\pi}{k_{ij}}$. If the hyperplanes $H_i$ and $H_j$ have a common perpendicular of length $\rho_{ij} > 0$ in $\mathbb{H}^n$, the nodes $i_1$ and $i_2$ are joined by a dotted edge, sometimes labelled $\cosh \rho_{ij}$. In the following, an edge of weight 2 is omitted, and an edge of weight 3 is written without its weight. The rank of $\Gamma$ is defined as the number of its nodes. In the compact case, $k_{ij}$ is not $\infty$, and we have $2 \leq k_{ij} < \infty$.

A square matrix $M$ is said to be the direct sum of the matrices $M_1, M_2, \cdots, M_n$ if by some permutation of the rows and of columns, it can be brought to the form

\[
\begin{pmatrix}
M_1 & 0 \\
& M_2 \\
& & \ddots \\
& & & M_n
\end{pmatrix}
\]

A matrix $M$ that cannot be represented as a direct sum of two matrices is said to be indecomposable\(^1\). Every matrix can be represented uniquely as a direct sum of indecomposable matrices, which are called (indecomposable) components. We say a polytope is indecomposable if its Gram matrix $G(P)$ is indecomposable.

In 1907, Perron found a remarkable property of the the eigenvalues and eigenvectors of matrices with positive entries. Frobenius later generalized it by investigating the spectral properties of indecomposable non-negative matrices.

**Theorem 2.2** (Perron-Frobenius, [Gan59]). An indecomposable matrix $A = (a_{ij})$ with non-positive entries always has a single positive eigenvalue $r$ of $A$. The corresponding eigenvector has positive coordinates. The moduli of all of the other eigenvalues do not exceed $r$.

It is obvious that Gram matrices $G(P)$ of an indecomposable Coxeter polytope is an indecomposable symmetric matrix with non-positive entries off the diagonal. Since the diagonal elements of $G(P)$ are all 1s, $G(P)$ is either positive definite, semi-positive definite or indefinite. According to the Perron-Frobenius theorem, the defect of a connected semi-positive definite matrix $G(P)$ does not exceed 1, and any proper submatrix of it is positive definite. For a Coxeter $n$-polytope $P$, its Coxeter diagram $\Gamma(P)$ is said to be elliptic if $G(P)$ is positive definite; $\Gamma(P)$ is called parabolic if any indecomposable component of $G(P)$ is degenerate and every subdiagram is elliptic. The elliptic and connected parabolic diagrams are exactly

\(^1\)It is also referred to as “irreducible” in some references.
| Connected elliptic diagrams | Connected parabolic diagrams |
|-----------------------------|-------------------------------|
| $A_n$ ($n \geq 1$)         | $\tilde{A}_1$                |
| $\ldots$                   | $\tilde{n}$                  |
| $A_n$ ($n \geq 2$)         | $\ldots$                    |
| $B_n = C_n$ ($n \geq 2$)   | $\tilde{B}_n$ ($n \geq 3$) |
| $D_n$ ($n \geq 4$)         | $\tilde{D}_n$ ($n \geq 4$) |
| $G_2^{(m)}$                | $\tilde{G}_2$               |
| $F_4$                      | $\tilde{F}_4$               |
| $E_6$                      | $\tilde{E}_6$               |
| $E_7$                      | $\tilde{E}_7$               |
| $E_8$                      | $\tilde{E}_8$               |
| $H_3$                      | $\ldots$                   |
| $H_4$                      | $\ldots$                   |

**Figure 1.** Connected elliptic (left) and connected parabolic (right) Coxeter diagrams.

the Coxeter diagrams of spherical and Euclidean Coxeter simplices, respectively. They are classified by Coxeter [Cox34] as shown in Figure [1].

A connected diagram $\Gamma$ is a Lannér diagram if $\Gamma$ is neither elliptic nor parabolic; any proper subdiagram of $\Gamma$ is elliptic. Those diagrams are irreducible Coxeter diagrams of compact hyperbolic Coxeter simplices. All such diagrams, reported by Lannér [Lan50], are listed in Figure [2].

Although the full list of hyperbolic Coxeter polytopes remains incomplete, some powerful algebraic restrictions are known [Vin85(1)]:

**Theorem 2.3.** ([Vin85(2), Th. 2.1]. Let $G = (g_{ij})$ be an indecomposable symmetric matrix of signature $(d,1)$, where $g_{ii} = 1$ and $g_{ij} \leq 0$ if $i \neq j$. Then there exists a unique (up to isometry of $\mathbb{H}^d$) convex hyperbolic polytope $P \subset \mathbb{H}^d$, whose Gram matrix coincides with $G$.  

6
**Theorem 2.4.** ([Vin85][2], Th. 3.1, Th. 3.2) Let $P = \bigcap_{i \in I} H_i^- \in \mathbb{H}^d$ be a compact acute-angled polytope and $G = G(P)$ be the Gram matrix. Denote $G_J$ the principal submatrix of $G$ formed from the rows and columns whose indices belong to $J \subset I$. Then,

1. The intersection $\bigcap_{j \in J} H_j^-, J \subset I$, is a face $F$ of $P$ if and only if the matrix $G_J$ is positive definite.
2. For any $J \subset I$ the matrix, $G_J$ is not parabolic.

A convex polytope is said to be simple if each of its faces of codimension $k$ is contained in exactly $k$ facets. By Theorem 2.4, we have the following corollary:

**Corollary 2.5.** Every compact acute-angled polytope is simple.

## 3. Potential hyperbolic Coxeter matrices

In order to classify all of the compact hyperbolic Coxeter 5-polytopes with 9 facets, we firstly enumerate all Coxeter matrices of simple 5-polytope with 9 facets that satisfy spherical conditions around all of the vertices. These are named potential hyperbolic Coxeter matrices in [JT18]. Almost all of the terminology and theorems in this section are proposed by Jacquemet and Tschantz. We recall them here for reference, and readers can refer to [JT18] for more details.
3.1. **Coxeter matrices.** The *Coxeter matrix* of a hyperbolic Coxeter polytope $P$ is a symmetric matrix $M = (m_{ij})_{1 \leq i, j \leq N}$ with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$
m_{ij} = \begin{cases} 
1, & \text{if } j = i, \\
k_{ij}, & \text{if } H_i \text{ and } H_j \text{ intersect in } \mathbb{H}^n \text{ with angle } \frac{\pi}{k_{ij}}, \\
\infty, & \text{otherwise.}
\end{cases}
$$

Note that, compared with Gram matrix, the Coxeter matrix does not involve the specific information of the distances of the disjoint pairs.

**Remark 3.1.** In the subsequent discussions, we refer to the Coxeter matrix $M$ of a graph $\Gamma$ as the Coxeter matrix $M$ of the Coxeter polyhedron $P$ such that $\Gamma = \Gamma(P)$.

3.2. **Partial matrices.**

**Definition 3.2.** Let $\Omega = \{n \in \mathbb{Z} | n \geq 2\} \cup \{\infty\}$ and let $\star$ be a symbol representing an undetermined real value. A *partial matrix* of size $m \geq 1$ is a symmetric $m \times m$ matrix $M$ whose diagonal entries are 1, and whose non-diagonal entries belong to $\Omega \cup \{\star\}$.

**Definition 3.3.** Let $M$ be an arbitrary $m \times m$ matrix, and $s = (s_1, s_2, \ldots, s_k)$, $1 \leq s_1 < s_2 < \cdots < s_k \leq m$. Let $M^s$ be the $k \times k$ submatrix of $M$ with $(i, j)$-entry $m_{s_i, s_j}$.

**Definition 3.4.** We say that a partial matrix $M = (p_{ij})_{1 \leq i, j \leq m}$ is a *potential matrix* for a given polytope $P$ if

- There are no entries with the value $\star$;
- There are entries $\infty$ in positions of $M$ that correspond to disjoint pair;
- For every sequence $s$ of indices of facets meeting at a vertex $v$ of $P$, the matrix, obtained by replacing value $n$ with $\cos \frac{\pi}{n}$ of submatrix $M^s$, is elliptic.

For brevity, we use a *potential vector*

$$C = (p_{12}, p_{13}, \cdots, p_{1m}, p_{23}, p_{24}, \cdots, p_{2m}, \cdots, p_{ij}, \cdots, p_{m-1,m}), \quad p_{ij} \neq \infty$$

to denote the potential matrix, where $1 \leq i < j \leq m$ and non-infinity entries are placed by the subscripts lexicographically. The potential matrix and potential vector $C$ can be constructed one from each other easily. In general, an arbitrary Coxeter matrix corresponds to a Coxeter vector following the same manner. We mainly use the language of vectors to explain the methodology and report the enumeration results. It is worthy to remark that, for a given Coxeter diagram, the corresponding (potential / Gram) matrix and vector are not unique in the sense that they are determined under a given labelling system of the facets and may vary when the system changed. In Section 5, we apply a permutation group to the nodes of the diagram and remove the duplicates to obtain all of the distinct desired vectors.

For each rank $r \geq 2$, there are infinitely many finite Coxeter groups, because of the infinite 1-parameter family of all dihedral groups, whose graphs consist of two nodes joined by an edge of weight $k \geq 2$. However, a simple but useful truncation can be utilized:

**Proposition 3.5.** There are finitely many finite Coxeter groups of rank $r$ with Coxeter matrix entries at most seven.
It thus suffices to enumerate potential matrices with entries at most seven, and the other candidates can be obtained from substituting integers greater than seven with the value seven. In other words, we now have more variables, that are restricted to be integers larger than or equal to seven, besides length unknowns. In the following, we always use the terms “Coxeter matrix” or “potential matrix” to mean the one with integer entries less than or equal to seven unless otherwise mentioned.

In [JT18], the problem of finding certain hyperbolic Coxeter polytopes is solved in two phases. In the first step, potential matrices for a particular hyperbolic Coxeter polytope are found; the “Euclidean-square obstruction” is used to reduce the number. Secondly, relevant algebraic conditions are solved for the admissible distances between non-adjacent facets. In our setting, additional universal necessary conditions, except for the vertex spherical restriction and Euclidean square obstruction, are adopted and programmed to reduce the number of the potential matrices.

4. Simple 5-polytopes with 9 facets

K. Fukuda, H. Miyata and S. Moriyama completed the enumeration of small realizable oriented matroids and, as a corollary, provided a list of 322 5-dimensional simplicial polytopes with 9 vertices in [FMM13], denoted by $P_1$, $P_2$, $P_3$, $P_4$, $P_{322}$, which is dual to the simple 5-dimensional polytopes with 9 facets. Each line in data file provided in [FMM13] is for one combinatorial type. The data on line 322 is as follows:

\[
[9,8,7,6,5] \ [9,8,7,6,4] \ [9,8,7,5,4] \ [9,7,6,5,4] \ [8,7,6,5,3] \ [8,7,5,4,3] \ [8,6,5,4,3] \ [7,6,5,3,2] \ [7,6,4,3,2] \ [7,5,4,3,2] \ [6,5,4,3,1] \ [6,5,4,2,1] \ [6,5,3,2,1] \ [6,5,3,1,2] \ [5,4,3,2,1]
\]

where the number 1, 2, \cdots, 9 denote the nine facets and each square bracket corresponds to one vertex that is incident to the enclosed five facets. For example, there are 18 vertices of the above polytope $P_{322}$.

From the original information, we can search out the following data for each polytope:

(1) The permutation subgroup $g_i$ of $S_9$ that is isomorphic to the symmetry group of $P_k$;
(2) The set $d_k$ of pairs of the disjoint facets;
(3) The set $l_4 / l_5$ of sets of four / five facets of which the intersection is of the combinatorial type of a tetrahedron / 4-simplex.
(4) The set $l_5\text{ basis} / l_4\text{ basis}$ of sets of five / four facets of which bound a 4-simplex / 3-simplex facet. The label of the bounded 4-somplex / 3-simplex facet is recorded as well. Note that $l_5\text{ basis} / l_4\text{ basis}$ is a subset of $l_5 / l_4$. The set $l_j\text{ basis}$ can be non-empty only for a $j$-dimensional polytope.
(5) The set $i_2$ of sets of facets of which the intersection is of the combinatorial type of a 2-cube.
(6) The set $s_3 / s_4 / s_5$ of sets of three / four / five facets of which the intersection is not a face / an edge / a vertex of $P_k$, and no disjoint pairs are included.
(7) The set $e_3 / e_4 / e_5$ of sets of three / four / five facets of which no disjoint pairs are included.
(8) The set $se_6 / se_7$ of sets of six / seven facets of which no disjoint pairs are included.
Table 1. Combinatorics of $P_{322}$.

For example, for the polytope $P_{322}$, the above sets are as shown in Table 1.

It is worthy to mention that the set $l_5.basis$, if not empty, can help to reduce the computation since the list of simplicial 5-prisms is available. For example, suppose $\{2, 4, 5, 6, 7\}$ (referring to facets $F_2, F_4, F_5, F_6, F_7$) bound a facet 8 (means $F_8$) of 4-simplex. Then we can assume that $F_8$ is orthogonal to $F_2, F_4, F_5, F_6, F_7$ (i.e., $m_{28} = m_{48} = m_{58} = m_{68} = m_{78} = 2$). The vectors obtained this way can be treated as bases, named basis vectors, and
all of the other potential vectors that may lead to a Gram vector can be realized by gluing the simplicial 5-prisms, as shown in Figure 3 at their orthogonal ends.

Moreover, among all of the 322 polytopes, we only need to study those with number of hyperparallel pairs larger than or equal to two due to the following theorems:

**Theorem 4.1** ([FT08](#), part of Theorem A). *Every compact hyperbolic Coxeter d-polytope, \( d > 4 \), has a pair of disjoint facets.*

**Theorem 4.2** ([FT09](#), Main Theorem A). *A compact hyperbolic Coxeter d-polytope with exactly one pair of non-intersecting facets has at most \( d + 3 \) facets.*

There are 109 polytopes with two or more hyperparallel facets. We group them by the number \( d_k \) of disjoint pairs as illustrated in Table 2.

| \( d_k \) | labels of polytopes                   |
|--------|--------------------------------------|
| 6      | 312 319 322                           |
| 5      | 254 302 311 315 318 320 321           |
| 4      | 249 255 280 291 292 293 295 300 301 303 310 313 314 316 317 |
| 3      | 2 10 175 240 241 242 246 247 248 252 256 264 268 271 272 273 279 282 284 286 287 289 290 294 296 |
|        | 297 298 299 304 305 306 307 308 309   |
| 2      | 4 9 18 61 68 110 173 174 176 179 180 182 209 210 212 214 218 216 225 234 239 243 244 245 |
|        | 250 251 253 257 258 259 260 261 262 263 265 266 267 269 270 274 275 276 277 278 281 283 285 288 |

**Table 2.** Five groups with respect to different numbers of disjoint pairs.

5. **Block-pasting algorithms for enumerating all of the candidate matrices over a certain combinatorial type.**

We now use the *block-pasting algorithm* to determine all of the potential matrices for the 109 compact combinatorial types reported in Section 4. Recall that the entries have only finite options, i.e., \( k_{ij} \in \{1, 2, 3, \ldots, 7\} \cup \{\infty\} \). Compared to the backtracking search algorithm raised in [JT18](#), “block-pasting” algorithm is more efficient and universal. Generally speaking, the backtracking search algorithm uses the method of “a series circuit”, where the potential matrices are produced one by one. Whereas, the block-pasting algorithm adopts
the idea of “a parallel circuit”, where different parts of a potential matrix are generated simultaneously and then pasted together.

For each vertex \( v_i \) of a 5-dimensional hyperbolic Coxeter polytope \( P_k \), we define the \( i \)-chunk, denoted as \( k_i \), to be the ordered set of the five facets intersecting at the vertex \( v_i \) with increasing subscripts. For example, for the polytope \( P_{322} \) discussed above, there are 18 chunks as it has 18 vertices. We may also use \( k_i \) to denote the ordered set of subscripts, i.e., \( k_i \) is referred to as a set of integers of length five.

Since the compact hyperbolic 5-dimensional polytopes are simple, each chunk possesses \( \binom{5}{2} = 10 \) dihedral angles, namely the angles between every two adjacent facets. For every chunk \( k_i \), we define an \( i \)-label set \( e_i \) to be the ordered set \( \{10a + b \mid \{a, b\} \in E_i \} \), where \( E_i \), named the \( i \)-index set, is the ordered set of pairs of facet labels. These are formed by choosing every two members from the chunk \( k_i \) where the labels increase lexicographically.

For example, suppose the five facets intersecting at the first vertex are \( F_1, F_2, F_3, F_5, \) and \( F_6 \). Then, we have

\[
k_1 = \{F_1, F_2, F_3, F_5, F_6 \}(\text{or } \{1, 2, 3, 5, 6 \}),
\]
\[
E_1 = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{5, 6\}\}
\]
\[
e_1 = \{12, 13, 15, 16, 23, 25, 26, 35, 36, 56\}.
\]

Next, we list all of the Coxeter vectors of the elliptic Coxeter diagrams of rank 5. Note that we have made the convention of considering only the diagrams with integer entries less than or equal to seven. The qualified Coxeter diagrams are shown in Figure 4.

![Figure 4. Spherical Coxeter diagram of rank 5 with labels less than or equal to seven.](image)

We apply the permutation group on five letters \( S_5 \) to the labels of the nodes of the Coxeter diagrams in Figure 4. This produces all of the possible vectors when varying the order of the five facets. For example, there are 60 vectors for the single diagram \( D_5 \) as shown in Figure 5. There are 1946 distinct such vectors of rank 5 elliptic Coxeter diagrams. The set
of all of these vectors is called the pre-block; it is denoted by \( S \). The set \( S \) can be regarded as a 1946 \( \times \) 10 matrix in the obvious way. In the following, we do not distinguish these two viewpoints and may refer to \( S \) as either a set or a matrix.

![Diagram of D5]

\[
\text{chunk:} \{F_1, F_2, F_3, F_4, F_5\} \\
\text{(or } \{1, 2, 3, 4, 5\}\text{)}
\]

\[
\text{block:} \{3, 2, 2, 2, 3, 2, 3, 2, 3, 2\}
\]

\[
\text{after permuting the labels:} \\
\{2, 2, 2, 3, 2, 3, 2, 3, 2, 3\} \\
\{2, 2, 2, 3, 2, 3, 2, 3, 3, 2\} \\
\{2, 2, 2, 3, 2, 3, 2, 3, 2, 3\} \\
\{2, 2, 2, 3, 2, 3, 2, 3, 2, 3\} \\
\{3, 3, 3, 2, 2, 2, 2, 2, 2, 2\} \\
\{3, 3, 3, 2, 2, 2, 2, 2, 2, 2\}
\]

\( \text{60 in total} \)

Figure 5. Building the pre-block \( S \).

We then generate every dataframe \( B_i \), named the \( i \)-block, of size 1946 \( \times \) 10, corresponding to every chunk \( k_i \), for a given polytope \( P_k \), where \( 1 \leq i \leq |V_k| \) and \( |V_k| \) is the number of vertices of \( P_k \). Firstly we evaluate \( B_i \) by \( S \) and take the ordered set \( e_i \) defined above as the column names of \( B_i \). For example, for \( e_1 = \{12, 13, 15, 16, 23, 25, 26, 35, 36, 36\} \), the columns of \( B_i \) are referred to as (12)-, (13)-, (15)-, (16)-, (23)-, (25)-, (26)-, (35)-, (36)-, (56)-columns.

Denote \( L \) to be a vector of length 36 as follows:

\[
L = \{12, 13, ..., 19, 23, 24, ..., 29, 34, 35, ..., 39, 45, 46, ..., 49, 56, 57, ..., 59, 67, 68, 69, 78, 79, 89\}.
\]

Then all of the numbers in the label set of \( d_k \) (the set of disjoint pair of facets of \( P_k \)) are excluded from \( L \) to obtain a new vector. For brevity, the new vector is also denoted by \( L \). For example, the numbers excluded for the polytope \( P_{322} \) are 17, 18, 28, 19, 29, and 39 as illustrated in Table 1. The length of \( L \) is denoted by \( l \).

Next, we extend every 1946 \( \times \) 10 dataframe to a 1946 \( \times \) \( l \) one, with column names \( L \), by simply putting each \((ij)\)-column to the position of corresponding labelled column, and filling in the value zero in the other positions. We continue to use the same notation \( B_i \) for the extended dataframe. In the rest of the paper, we always mean the extended dataframe when using the notation \( B_i \).

After preparing all of the the blocks \( B_i \) for a given polytope \( P_k \), we proceed to paste them up. More precisely, when pasting \( B_1 \) and \( B_2 \), a row from \( B_1 \) is matched up with a row of \( B_2 \) where every two entries specified by the same index \( i \), where \( i \in e_1 \cap e_2 \), have the same values. The index set \( e_1 \cap e_2 \) is called a linking key for the pasting. The resulting new row is actually the sum of these two rows in the non-key positions; the values are retained in the key positions. The dataframe of the new data is denoted by \( B_1 \cup^s B_2 \).

We use the following example to explain this process. Suppose

\[
B_1 = \{x_1, x_2\} = \{(1, 2, 4, 4, 2, 6, 0, 0, \cdots, 0), (1, 2, 4, 5, 2, 6, 0, 0, \cdots, 0)\},
\]

\[
B_2 = \{y_1, y_2, y_3\} = \{(1, 2, 4, 4, 0, 0, 1, 7, 0, 0, \cdots, 0), (1, 2, 4, 4, 0, 0, 6, 5, 0, 0, \cdots, 0),
(1, 2, 3, 4, 0, 0, 1, 7, 0, 0, \cdots, 0)\}.
\]

In this example, \( x_1 \) has the same values with \( y_1 \) and \( y_2 \) on the (12)-, (13)-, (14)- and (15)- positions. In other words, the linking key here is \( \{12, 13, 14, 15\} \). Thus, \( y_1 \) and \( y_2 \) can paste to \( x_1 \), forming the Coxeter vectors

\[
(1, 2, 4, 4, 2, 6, 1, 7, 0, \cdots, 0) \text{ and } (1, 2, 4, 4, 2, 6, 6, 5, 0, \cdots, 0),
\] respectively.
In contrast, $x_2$ cannot be pasted to any element of $B_2$ as there are no vectors with entry 5 on the (15)-position. Therefore,

$$B_1 \cup^* B_2 = \{(1, 2, 4, 2, 6, 0, \cdots, 0), (1, 2, 4, 2, 6, 5, 0, \cdots, 0)\}.$$

We then move on to paste the sets $B_1 \cup^* B_2$ and $B_3$. We follow the same procedure with an updated index set. Namely the linking key, is now $e_1 \cup e_2 \cap e_3$. We conduct this procedure until we finish pasting the final set $B_{|V_k|}$. The set of linking keys used in this procedure is

$$\{e_1 \cap e_2, e_1 \cup e_2 \cap e_3, \cdots, e_1 \cap e_2 \cup \cdots \cup e_{i-1} \cap e_i, \cdots, e_1 \cup e_2 \cup \cdots \cup e_{|V|-1} \cap e_{|V|}\}.$$

After pasting the final block $B_{|V_k|}$, we obtain all of the potential vectors of the given polytope. This approach has been Python-programmed on a PARATERA server cluster.

When carrying out this approach, all of the potential vectors of $P_{322}$ are enumerated successfully. However it encounters a memory error in some cases. For example, for the polytope $P_{10}$, the computation incurs an error when pasting $B_{16}$ as illustrated in Table 3.

| i-chunk (1 ≤ i ≤ 24) | data size after pasting $b_i$ | time consumed (s) |
|-----------------------|-------------------------------|-------------------|
| $h_1$                 | 12345                         | 1946              | 0 |
| $h_2$                 | 12346                         | 32780             | 0.05 |
| $h_3$                 | 12356                         | 46851             | 0.05 |
| $h_4$                 | 12457                         | 909145            | 0.7 |
| $h_5$                 | 12467                         | 1295489           | 1.06 |
| $h_6$                 | 12567                         | 774519            | 0.69 |
| $h_7$                 | 13457                         | 1619039           | 1.32 |
| $h_8$                 | 13467                         | 1137673           | 1.06 |
| $h_9$                 | 13567                         | 1052454           | 0.79 |
| $h_{10}$              | 23458                         | 20832842          | 16.68 |
| $h_{11}$              | 23468                         | 34761662          | 24.93 |
| $h_{12}$              | 23568                         | 20789435          | 17.63 |
| $h_{13}$              | 24578                         | 40260708          | 28.12 |
| $h_{14}$              | 24678                         | 29788033          | 21.62 |
| $h_{15}$              | 25678                         | 27886613          | 16.95 |
| ...                   | ...                           | Memory error      | ... |

Table 3. Pasting of $P_{10}$.

Therefore, a refined algorithm is needed to continue this research. The philosophy of the refinement is to introduce more necessary conditions other than the vertex spherical and Euclidean square restrictions, to reduce the number of vectors during the block-pasting. Firstly, we collect data sets $L_4$, $L_5$, $L_5\text{basis}$, $S_3$, $S_4$, $S_5$, $S_6$, $S_7$, $E_3$, $E_4$, $E_5$, $E_6$, $E_7$, $I_2$, as claimed in Table 4, by the following two steps:

1. Prepare Coxeter diagrams of rank $r$, as assigned in Table 4 and write down the Coxeter vectors under an arbitrary system of node labeling.
2. Apply the permutation group $S_r$ to the labels of the nodes and produce the desired data set consisting of all of the distinct Coxeter vectors under all of the different labelling systems.
Note that the set $\mathcal{S}_5$ is exactly the pre-block $\mathcal{S}$ we construct before. Readers can refer to the process of producing $\mathcal{S}$ for the details of building these data sets.

| Types of Coxeter diagrams                      | # Coxeter diagrams | # distinct Coxeter Vectors after permutation on nodes | data sets |
|-----------------------------------------------|--------------------|--------------------------------------------------------|-----------|
| Coxeter diagrams of compact hyperbolic 4-simplex | 5                  | 420                                                    | $\mathcal{L}_4$ |
| Coxeter diagrams of compact hyperbolic 3-simplex | 9                  | 108                                                    | $\mathcal{L}_3$ |
| rank 3 elliptic Coxeter diagrams              | 9                  | 31                                                     | $\mathcal{S}_3$ |
| rank 4 elliptic Coxeter diagrams              | 29                 | 242                                                    | $\mathcal{S}_4$ |
| rank 5 elliptic Coxeter diagrams              | 47                 | 1946                                                   | $\mathcal{S}_5$ |
| rank 6 elliptic Coxeter diagrams              | 117                | 20206                                                  | $\mathcal{S}_6$ |
| rank 7 elliptic Coxeter diagrams              | 196                | 227676                                                 | $\mathcal{S}_7$ |
| rank 3 connected parabolic Coxeter diagrams   | 3                  | 10                                                     | $\mathcal{E}_3$ |
| rank 4 connected parabolic Coxeter diagrams   | 3                  | 27                                                     | $\mathcal{E}_4$ |
| rank 5 connected parabolic Coxeter diagrams   | 5                  | 257                                                    | $\mathcal{E}_5$ |
| rank 6 connected parabolic Coxeter diagrams   | 4                  | 870                                                    | $\mathcal{E}_6$ |
| rank 7 connected parabolic Coxeter diagrams   | 5                  | 6870                                                   | $\mathcal{E}_7$ |
| Coxeter diagrams of Euclidean 2-cube          | 4                  | 3                                                      | $\mathcal{I}_2$ |

Table 4. Data sets used to reduce the computational load.

Next, we modify the block-pasting algorithm by using additional metric restrictions. More precisely, remarks 5.1–5.4, which are practically reformulated from Theorem 2.4, must be satisfied.

Remark 5.1. ("$l_4/l_5$-condition") The Coxeter vector of the six/ten dihedral angles formed by the four/five facets with the labels indicated by the data in $l_4/l_5$ is IN $\mathcal{L}_4/\mathcal{L}_5$.

Remark 5.2. ("$s_3/s_4/s_5/s_6/s_7$-condition") The Coxeter vector of the three/six/ten/fifteen/twenty-one dihedral angles formed by the three/four/five/six/seven facets with the labels indicated by the data in $s_3/s_4/s_5/s_6/s_7$ is NOT IN $\mathcal{S}_3/\mathcal{S}_4/\mathcal{S}_5/\mathcal{S}_6/\mathcal{S}_7$.

Remark 5.3. ("$e_3/e_4/e_5/e_6/e_7$-condition") The Coxeter vector of the three/six/ten/fifteen/twenty-one dihedral angles formed by the three/four/five/six/seven facets with the labels indicated by the data in $e_3/e_4/e_5/e_6/e_7$ is NOT IN $\mathcal{E}_3/\mathcal{E}_4/\mathcal{E}_5/\mathcal{E}_6/\mathcal{E}_7$.

Remark 5.4. ("$i_2$-condition") The Coxeter vector formed by the four facets with the labels indicated by the data in $i_2$ is NOT IN $\mathcal{I}_2$.

The "IN" and "NOT IN" tests are called the "saving" and the "killing" conditions, respectively. The "saving" condition is much more efficient than the "killing" condition because the "what kinds of vectors are qualified" is much more restrictive than the "what kinds of vectors are not qualified". The more delicate "saving conditions" are introduced in next section. Moreover, we remark that the $l_3$-condition and the sets $l_3$ and $\mathcal{L}_3$, which can be defined analogously as the $l_4/l_5$ setting, are not introduced. This is because the effect of using both $s_3$- and $e_3$- conditions is equivalent to adopting the $l_3$-condition.

We now program these conditions and insert them into the appropriate layers of the pasting to reduce the computational load. Here the "appropriate layer" means the layer
where the dihedral angles are non-zero for the first time. For example, for \( \{3, 6, 7\} \in e_3 \), we find the \( j \)-th block pasting where the data in columns (36, 37, 67) of the dataframe \( B_1 \cup^* B_2 \cdots \cup^* B_j \) become non-zero. Therefore, the \( e_3 \)-condition for \( \{3, 6, 7\} \) is inserted immediately after the \( j \)-th block pasting. The symmetries of the polytopes are factored out when the pastes are finished. The matrices (or vectors) after all these conditions (metric restrictions and symmetry equivalence) are called “SEILper”-potential matrices (or vectors) of certain combinatorial types. All of the numbers of the results are reported in Tables 5–9. In those tables, the notation \( \text{grp}_i(j,k) \) means there are \( j \) polytopes with \( i \) pairs of disjoint facets, and \( k \) of them admit SEILper potential vectors, i.e., the number of SEILper potential vectors that are non-zero. The letters “s, m, h, and d” on the lines named “time” represent “seconds, minutes, hours, and days”, respectively. The numbers in red indicate that the corresponding polytopes have a non-empty set \( l_5 \)-basis; therefore, the results obtained are basis SEILper potential vectors. This calculation is called the basis approach. We confirm these cases without using the \( l_5 \)-condition, called the direct approach, in the validation part presented in Section 7.

| grp6 (3/3) | 312 319 322 |
|------------|------------|
| #SEILper   | 1 5 3      |
| time       | 37.755 s 26.48 s 23.495 s |

Table 5. Result of group 6.

| grp5 (7/7) | 254 302 311 315 318 320 321 |
|------------|----------------------------|
| #SEILper   | 9467 5 288 49 2 6 3          |
| time       | 2.925 h 45.49 s 3.405 m 42.06 s 33.81 s 53.115 s 8.685 m |

Table 6. Result of group 5.

| gr4 (15/9) | 249 255 280 291 292 293 295 300 301 |
|------------|-------------------------------------|
| #SEILper   | 3910 282 148 37 10 0 0 383 5        |
| time       | 5.49 m 5.6 m 1.46 h 2.63 m 8.91 m 4.2 m 5.11 m 9.755 h 2.675 m |
|            | 303 310 313 314 316 317             |
|            | 0 0 68 0 8 0                        |
|            | 5.23 m 5.64 m 8.215 m 1.805 m 0.855 m 8.5 m |

Table 7. Result of group 4.

So far, we have restricted the number of candidates and perform the computations in a reasonable amount of time. However, when passing to the second phase of calculating the signature, there are cases, such as the 9467 SEILper potential matrices of the polytope \( P_{254} \), that require an unsatisfying amount of time. In the next section, the restriction of “intersection types” is introduced to facilitate the computation of signature in 30s per polytope.

6. Signature Constraints of Hyperbolic Coxeter \( n \)-Polytopes

After preparing all of the SEILper matrices, we proceed to calculate the signatures of the potential Coxeter matrices to determine if they lead to the Gram matrix \( G \) of an actual hyperbolic Coxeter polytope.
Firstly, we modify every SEILper matrix $M$ as follows:

1. Replace $\infty$s by unknowns of $x_i$;
2. Replace 2, 3, 4, 5, and 6 by 0, $-\frac{1}{2}$, $-\frac{l}{2}$, $-\frac{m}{2}$, $-\frac{n}{2}$, where $l^2 - 1 = 2$, $l > 0$, $m^2 - m - 1 = 0$, $m > 0$, $n^2 - 3 = 0$, $n > 0$;
3. Replace 7s by unknowns of $y_i$.

By Theorem 2.3, the resulting Gram matrix must have signature $(5, 1)$. This implies that the determinant of every $6 \times 6$ minor of each modified $9 \times 9$ SEILper matrix is zero. Therefore, we have the following system of equations and inequality on $x_i$, $l$, $m$, $n$, and $y_i$ to
further restrict and lead to the Gram matrices of the desired polytopes:

\[
\begin{align*}
2 \det(M_i) &= 0, \text{for any of the } 5/2 \times 5/2, \\
1.8 < y_i < 2 &\text{ for all } y_i \\
x_i < -1 &\text{ for all } x_i \\
l^2 - 1 = 2, \ m^2 - m - 1 = 0, \ m > 0, \ n^2 - 3 = 0, \ n > 0
\end{align*}
\]

The above conditions are initially stated by Jacquemet and Tschantz in [JT18]. Due to practical constraints in Mathematica, we denote \(2 \cos(\pi/4)\), \(2 \cos(\pi/5)\), \(2 \cos(\pi/6)\) by \(l_2\), \(m_2\), \(n_2\), rather than \(l, m, n\) and set \(2 \det(M_i) = 0\) rather than \(\det(M_i) = 0\). Delicate reasons for doing so can be found in [JT18]. Moreover, we first find the Gröbner bases of the polynomials involved, i.e., \(2 \det(M_i), l^2 - 1, m^2 - m - 1, n^2 - 3\), before solving the system. This quickly passes over the cases that have no solution. However, when dealing with some combinatorial types, like \(P_{254}, P_{249}, P_2, P_{10}\), etc., the computation cannot be accomplished in reasonable amount of time. In some cases, a single matrix can require more than two hours to compute, which is costly and impedes the validation process. Hence, we introduce the “intersection restrictions” to further reduce the number of SEILper matrices.

Let \(P^3, P^4,\) and \(D^4\) denote the combinatorial types of simplicial 3-prism, simplicial 4-prism, and the product of two 2-simplices. We prepare some data for these three combinatorial type as shown in Table 10.

| Combinatorial type | \(P^3\) | \(P^4\) | \(D^4\) |
|--------------------|--------|--------|--------|
| # facets           | 5      | 6      | 6      |
| datum              | s_3 s_4 | s_3 s_4 s_5 | s_3 s_4 s_5 |
| \(d_*\): set of disjoint pair of facets; | e_3 e_4 | e_3 e_4 e_5 | e_3 e_4 e_5 |
| \(g_*\): symmetry group | i_2 d_{p^3} g_{p^3} | i_2 l_4 l_4 basis d_{p^4} g_{p^4} | i_2 l_4 l_4 basis d_{p^4} g_{p^4} |

Table 10. Combinatorics of \(P^3, P^4,\) and \(D^4\) that are used to obtain SEILper matrices.

We use \(S_3, S_4,\) and \(S_4\) as the pre-blocks of \(P^3, P^4\) and \(D^4\), and apply remarks 5.1–5.4 to the data presented in Table 10 to obtain the SEILper potential matrices. The calculation results are arranged on the first line of Table 11.

Next, we generate the sets \(P_3, P_4,\) and \(D_4\) as defined below:

1. The set \(P_3\) consists of all of the SEILper potential matrices of the simplicial 3-prism \(P^3\) with the signature \((3, 1)\) or \((4, 1)\). Equivalently, we are finding the Coxeter groups corresponding to \(P^3\) that admit the Fuchsian or quasi-Fuchsian representations into \(PO_{4,1}(\mathbb{R})\).
2. The set \(P_4\) consists of all of the SEILper potential matrices of the simplicial 4-prism \(P^4\) with the signature \((4, 1)\) or \((5, 1)\).
3. The set \(D_4\) consists of all of the SEILper potential matrices of the product of two 2-simplices \(D^4\) with the signature \((4, 1)\) or \((5, 1)\).

We now explain the signature calculation method for \(D^4\), and it is analogous for \(P^3\) and \(P^4\). From the combinatorial type \(D^4\), the signature of the potential matrix has at least
four positive eigenvalues. Therefore the signature is one of \((4, 0), (4, 1), (5, 0), (4, 2), (5, 1),\) and \((6, 0)\). We modify the \text{SEILper matrices} as described in the beginning of this section and suppose the eigenvalue of the given matrix to be \(r_1, r_2, \ldots, r_6\). Then, the determinant is a function of \(a_1, x_i, y_i, l, m,\) and \(n\); it can be written as:

\[
\phi(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_6 x^6,
\]

where \(c_i, 1 \leq i \leq 6\), is a function on \(a_1, x_i, y_i, l, m,\) and \(n\). Next, the conditions in (6.1) are extended as follows, which leads to different signatures:

- if \(c_0 < 0\), \(\text{sig}=(5,1)\);
- if \(c_0 = 0\) and \(c_1 < 0\), \(\text{sig}=(4,1)\)

We apply the permutation groups \(S_5, S_6\), and \(S_6\) to the labels of the nodes in the Coxeter diagrams corresponding to \(P_3, P_4,\) and \(D_4\). This produces the desired data set consisting of all of the distinct Coxeter vectors as presented in Table 11.

| polytope  | \(P^3\) | \(P^4\) | \(D^4\) |
|-----------|---------|---------|---------|
| \# \text{SEILper matrix} | 192     | 26      | 1200    |
| \(3, 1\)  | 148     | 17      | 7       |
| \(4, 1\)  | 148     | 17      | 10      |
| total     | 148     | 17      | 10      |
| \# \text{distinct Coxeter vector after permuting nodes} | 1122    | 540     | 864     |

**Table 11.** Fushian and quasi-Fuchisian representations corresponding to \(P^3, P^4,\) and \(D^4\).

In the following, we continue to use notations \(P_3, P_4\), and \(D_4\) to denote the corresponding set of vectors. Firstly, we check all of the 109 polytopes for the intersection types of \(P^3, P^4,\) and \(D^4\) in their combinatorics. For example, regarding \(P^3\), we are finding the five facets of \(P_3\) such that the lattice formed by their intersection coincides with the lattice formed by the intersection of the five 2-dimensional faces of the 3-prism \(P^3\). The polytopes having the intersection types of \(P^3, P^4,\) and \(D^4\) in their combinatorics are listed in Table 12. Next, we use the data sets \(P^3, P^4,\) and \(D^4\) to filter the set of the \text{SEILper matrices}. Note that the three conditions for restriction types are “saving” conditions, which foreshow their efficiency. The aforementioned “saving” conditions, \(l_4\)- and \(l_5\)-conditions, can be viewed as restrictions for the intersection types of 3-simplex and 4-simplex, respectively. The results after this procedure are arranged in column 6 of Table 12.

After deducing and modifying all of these limited \text{SEILper potential matrices}, we calculate (6.1) for each candidate. The following logic is adopted:

1. Prepare conditions (6.1) for a given potential vector
2. Calculate the Gröbner bases of the polynomials involved and modify the condition (6.1) correspondingly.
3. Solve the equations and inequality in (2) simultaneously and restrict the running time to a maximum of 30s.
4. If the calculation does not finish in 30s, the program aborts and turns to solve the equations and inequality in (1) directly.
Table 12. The “intersection type (#)” means there are #-many combinatorial types of $P^3$, $P^4$, or $D^4$, possessed by the corresponding polytope. For example, “$P^4(5)$” in the second line means there are 5 sets of six facets of $P_{254}$ such that the lattice formed by them is a 4-prism $P^4$. The “#7” means the maximal of the numbers of 7 of potential vectors in the corresponding set of SEILper potential vectors. It is obvious that the numbers of SEILper vectors decrease dramatically after this step. In addition, the values of #7 decrease as well.

Although we set a threshold of 30 s, for almost all of the vectors, the calculations can be done within 1 s. Moreover, we use the logic above (1)-(4) rather than continuing to wait for the solutions of the equations and inequality in (2). This is because the function GroebnerBasis in Mathematica either works quite quickly or consumes an unaffordable amount of time. Alternatives are therefore needed. The last step is to check whether the signature is indeed $(5, 1)$ and whether $\frac{\pi}{\arccos\left(y_i/2\right)}$ is an integer.

After conducting all of these procedures in Mathematica, we find that only six of all simple 5-polytopes with 9 facets admit compact hyperbolic structure. We also glue the 4-prism to the one with orthogonal 4-prism ends. The results are reported in Table 13. The polytopes with labels in red are the “basis” polytopes, and the ones on last line can be obtained by gluing prism ends to those of the second line from the bottom. The Coxeter diagrams and length information are shown in the end (pages 23–26).
Table 13. Results of the compact hyperbolic Coxeter 5-polytopes with 9 facets. The value of polytope labeled by 284 on the last line is “N”, which means the $l_5$ basis set of $P_{284}$ is empty and we are not be able to glue it with any prism end.

7. Validation and Results

1. “Basis approach” vs. “direct approach”.

We calculate SEILper potential matrices without using $l_5$ basis-conditions for those polytopes having 4-simplex facets. The restriction sets regarding the intersection types of $P^3$, $P^4$, and $D^4$ are then applied. After checking for the signature obstructions, we finally obtain the Gram matrices corresponding to all of the possible compact hyperbolic polytopes. The results are reported in Table 14. They are the same as the previous work generated via “basis approach”.

| $d_5$ | 6 | 5 | 4 | 3 |
|-------|---|---|---|---|
| polytope | 312 | 319 | 322 | 302 | 313 | 284 |
| # (selected) SEILper potential | 1 | 3 | 5 | 5 | 68 | 60 |
| # gram of basis vectors | 1 | 3 | 5 | 1 | 1 | 1 |
| # gram after suitably gluing 4-prisms to the orthogonal ends | 1 | 22 | 18 | 6 | 3 | N |

Table 14. Results obtained by direct approach.

2. A. Burcroff presents 50 polytopes in [Bur22]. A careful check reveals that we have covered all of these polytopes and obtained one more. The correspondence of notions of the polytopes, which admit a hyperbolic structure, between our list and Burcroff’s are presented in Table 15.
Table 15. Notion correspondence between our list and the list in [Bur22].

| MZ    | 322 | 319 | 312 | 302 | 313 | 284 |
|-------|-----|-----|-----|-----|-----|-----|
| A. Burcroft | $H_2$ | $H_1$ | $H_3$ | $H_4$ | $H_5$ |

[Bur22]: [Bur22]
1. Coxeter diagrams for $P_{322}$.

Figure 6. The 18 compact hyperbolic Coxeter 5-polytopes with 9 facets over polytope $P_{322}$. 

23
2. Coxeter diagrams for $P_{319}$ (part 1).

Figure 7. 16 of the 22 compact hyperbolic Coxeter 5-polytopes with 9 facets over polytope $P_{319}$. 

3. Coxeter diagrams for $P_{319}$ (part 2), $P_{312}$, $P_{302}$, $P_{313}$.

Figure 8. 6 of 22, 1, 6, 3 compact hyperbolic Coxeter 5-polytopes with 9 facets over polytopes $P_{319}$, $P_{312}$, $P_{302}$, $P_{313}$, respectively.
4. Coxeter diagram for $P_{284}$.

![Coxeter Diagram](image)

**Figure 9.** The only compact hyperbolic Coxeter 5-polytopes with 9 facets over polytope $P_{284}$

|   |   |   |   |
|---|---|---|---|
| a | b | c | d |
| $P_{122}$.1 | $\frac{1}{2}\sqrt{1(6 + \sqrt{5})}$ | $\frac{1}{2}\sqrt{31 + 15\sqrt{5}}$ | $\frac{1}{2}\sqrt{23 + 8\sqrt{5}} + \sqrt{5(63 + 26\sqrt{5})}$ | $\frac{1}{2}\sqrt{946 + 423\sqrt{5} + \sqrt{1749835 + 782550\sqrt{5}}}$ |
| $P_{122}$.4 | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 + \sqrt{5} + 2\sqrt{8 + 3\sqrt{5}})$ | $\frac{1}{2}(1 + \sqrt{5} + 2\sqrt{8 + 3\sqrt{5}})$ | $\frac{1}{2}\sqrt{9 + 4\sqrt{5} + \sqrt{132 + 59\sqrt{5}}}$ |
| $P_{122}$.13 | $\sqrt{\frac{26}{3} + \frac{25\sqrt{3}}{3}}$ | $\frac{1}{2}\sqrt{6 + \sqrt{3}}$ | $\sqrt{23 + 8\sqrt{5}}$ | $\frac{1}{2}(15 + 7\sqrt{5})$ |
| $P_{19}$.1 | $\frac{1}{2}(3 + \sqrt{5})$ | $\frac{1}{2}\sqrt{6 + \sqrt{3}}$ | $\frac{1}{2}(9 + 1\sqrt{5} + \sqrt{132 + 59\sqrt{5}})$ | $\frac{1}{2}\sqrt{19 + 9\sqrt{5}}$ |
| $P_{19}$.7 | $\sqrt{\frac{2}{3} (23 + 9\sqrt{5})}$ | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(9 + 1\sqrt{5} + \sqrt{132 + 59\sqrt{5}})$ | $\frac{1}{2}\sqrt{47 + 17\sqrt{5} + 2\sqrt{570 + 259\sqrt{5}}}$ |
| $P_{19}$.10 | $\frac{1}{2}(3 + \sqrt{5})$ | $\frac{1}{2}(3 + \sqrt{5})$ | $\frac{1}{2}(3 + \sqrt{5})$ | $\frac{1}{2}(3 + \sqrt{5})$ |
| $P_{19}$.16 | $\frac{1}{2}(19 + 9\sqrt{5})$ | $\frac{1}{2}(19 + 9\sqrt{5})$ | $\frac{1}{2}(19 + 9\sqrt{5})$ | $\frac{1}{2}(19 + 9\sqrt{5})$ |
| $P_{19}$.17 | $\frac{1}{2}(119 + 55\sqrt{5})$ | $\frac{1}{2}(119 + 55\sqrt{5})$ | $\frac{1}{2}(119 + 55\sqrt{5})$ | $\frac{1}{2}(119 + 55\sqrt{5})$ |
| $P_{12}$.1 | $\frac{1}{2}\sqrt{\frac{2}{3} + \sqrt{5}}$ | $\frac{1}{2}\sqrt{15 + \sqrt{5}}$ | $\frac{1}{2}\sqrt{15 + \sqrt{5}}$ | $\frac{1}{2}\sqrt{15 + \sqrt{5}}$ |
| $P_{102}$.1 | $\frac{1}{2}\sqrt{\frac{2}{3} + \sqrt{5}}$ | $\frac{1}{2}\sqrt{30 + 13\sqrt{5}}$ | $\frac{1}{2}\sqrt{30 + 13\sqrt{5}}$ | $\frac{1}{2}(31 + 15\sqrt{5})$ |
| $P_{113}$.1 | $\frac{1}{2}\sqrt{\frac{2}{3} + \sqrt{5}}$ | $\frac{1}{2}\sqrt{5 + \sqrt{5}}$ | $\frac{1}{2}\sqrt{5 + \sqrt{5}}$ | $\frac{1}{2}(3 + \sqrt{5})$ |
| $P_{284}$ | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 + \sqrt{5})$ |

**Table 16.** The cosh $\rho_{ij}$, denoted by letter alphabetically, of the result polytopes.
References

[All06] D. Allcock. Infinitely many hyperbolic Coxeter groups through dimension 19. Geometry & Topology 10 (2006), 737–758.

[And70(1)] E. M. Andreev. Convex polyhedra in Lobachevskii spaces (Russian). Math. USSR Sbornik 10 (1970), 413–440.

[And70(2)] E. M. Andreev. Convex polyhedra of finite volume in Lobachevskii space (Russian). Math. USSR Sbornik 12 (1970), 255–259.

[Bor98] R. Borchers. Coxeter groups, Lorentzian lattices, and K3 surfaces. IMRN 19 (1998), 1011–1031.

[Bou68] N. Bourbaki. Groupes et alg`ebres de Lie. Ch. IV–VI, Hermann, Paris, 1968.

[Bur22] A. Burcroff. Near classification of compact hyperbolic Coxeter d-polytopes with $d + 4$ facets and related dimension bounds. arXiv:2201.03437.

[Bug84] V. O. Bugaenko. Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring $\mathbb{Z}[\frac{\sqrt{5}+1}{2}]$. Moscow Univ. Math. Bull., 39 (1984), 6–14.

[Bug92] V. O. Bugaenko. Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices. Adv. Sov. Math., 8 (1992), 33–55.

[Cox34] H. S. M. Coxeter. Uber kompakte hyperbolische Coxeter-Polytope mit wenigen Facetten. Universit at Bielefeld, SFB 343, (1994) no. 94–087.

[Ess94] F. Esselmann. Über kompakte hyperbolische Coxeter-Polytope mit wenigen Facetten. Universit at Bielefeld, SFB 343, (1994) no. 94–087.

[Ess96] F. Esselmann. The classification of compact hyperbolic Coxeter $d$-polytopes with $d + 2$ facets. Comment. Math. Helvetici 71 (1996), 229–242.

[F] A. Flikson http://www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html

[FT08(1)] A. Felikson, P. Tumarkin. On hyperbolic Coxeter polytopes with mutually intersecting facets. J. Combin. Theory A 115 (2008), 121–146.

[FT08(2)] A. Felikson, P. Tumarkin. On compact hyperbolic Coxeter $d$-polytopes with $d + 4$ facets. Trans. Moscow Math. Soc. 69 (2008), 105-151.

[FT09] A. Felikson, P. Tumarkin. Coxeter polytopes with a unique pair of non-intersecting facets. J. Combin. Theory A 116 (2009), 875–902.

[FT14] A. Felikson, P. Tumarkin. Essential hyperbolic Coxeter polytopes. Israel Journal of Mathematics 199 (2014) no.1, 113–161.

[FMM13] K. Fukuda, H. Miyata, S. Moriyama. Complete enumeration of small realizable oriented matroids. Discrete & Computational Geometry 49 (2013) no.2, 359–381.

[Gan59] F. R. Gantmacher. The theory of matrix. Chelsea Publishing Company (1959), New York.

[ImH90] H.-C. Im Hof. Napier cycles and hyperbolic Coxeter groups. Bull. Soc. Math. de Belg. S’erie A, XLII (1990), 523–545.

[Jac17] M. Jacquemet. On hyperbolic Coxeter $n$-cubes. Europ. J. Combin. 59 (2017), 192–203.

[JT18] M. Jacquemet and S. Tschantz. All hyperbolic Coxeter $n$-cubes. J. Combin. Theory A, 158 (2018), 387–406.

[Kap74] I. M. Kaplinskaya. Discrete groups generated by reflections in the faces of simplicial prisms in Lobachevskian spaces. Math. Notes 15 (1974), 88–91.

[KM13] A. Kolpakov and B. Martelli. Hyperbolic four-manifolds with one cusp. Geom & Funct. Anal., 23, 2013, 1903–1933.

[Kos67] J. L. Koszul. Lectures on hyperbolic Coxeter group. University of Notre Dame, 1967.

[Lan50] F. Lanner. On complexes with transitive groups of automorphisms. Comm. Sem. Math. Univ. Lund, 11 (1950), 1–71.

[MZ18] J. Ma and F. Zheng. Orientable hyperbolic 4-manifolds over the 120-cell. Math. Comp. 90 (2021), no. 331, 2463–2501.

[MZ22] J. Ma and F. Zheng. Compact hyperbolic Coxeter 4-polytopes with 8 facets. arXiv:2201.00154.

[Mak65] V. S. Makarov. On one class of partitions of Lobachevskian space. Dokl. Akad. Nauk SSSR, 161 no.2, (1965), 277–278. English transl.: Sov. Math. Dokl. 6, 400–401 (1965), Zbl.135,209.
[Mak66] V. S. Makarov. On one class of discrete groups of Lobachevskian space having an infinite fundamental region of finite measure. Dokl. Akad. Nauk SSSR, 167 no.2, (1966), 30–33. English transl.: Sov. Math. Dokl. 7, 328–331 (1966), Zbl.146,165.

[Mak68] V. S. Makarov. On Fedorov groups of four- and five-dimensional Lobachevsky spaces. Issled. po obsch. algeb. no. 1, Kishinev St. Univ., 1970, 120–129 (Russian).

[P1882] H. Poincaré. Théoréme des groupes fuchsiens (French). Acta Math. 1, 1882, no. 1, 1–76.

[Pro87] M. N. Prokhorov. The absence of discrete reflection groups with non-compact fundamental polyhedron of finite volume in Lobachevsky space of large dimension. Math. USSR Izv., 28 (1987), 401–411.

[Rob15] M. Roberts. A classification of non-compact Coxeter polytopes with $n + 3$ facets and one non-simple vertex. arXiv:1511.08451.

[Rus89] O. P. Rusmanov. Examples of non-arithmetic crystallographic Coxeter groups in $n$-dimensional Lobachevsky space for $6 \leq n \leq 10$. Problems in Group Theory and Homology Algebra, Yaroslavl, 1989, 138–142 (Russian).

[Tum03] P. Tumarkin. Non-compact hyperbolic Coxeter $n$-polytopes with $n+3$ facets, short version (3 pages). Russian Math. Surveys, 58 (2003), 805–806.

[Tum04(1)] P. Tumarkin. Hyperbolic Coxeter $n$-polytopes with $n+2$ facets. Math. Notes 75 (2004), 848–854.

[Tum04(2)] P. Tumarkin. Hyperbolic Coxeter $n$-polytopes with $n+3$ facets. Trans. Moscow Math. Soc. (2004), 235–250.

[Tum07] P. Tumarkin. Compact hyperbolic Coxeter $n$-polytopes with $n + 3$ facets. The electronic journal of combinatorics 14 (2007), #R69.

[Vin67] E. B. Vinberg. Discrete groups generated by reflections in Lobachevskii spaces. Mat. USSR Sb. 1 (1967), 429–444.

[Vin69] E. B. Vinberg. Some examples of crystallographic groups in Lobachevskian spaces. 78(120), no. 4,(1969), 633–639.

[Vin72] E. B. Vinberg. On groups of unit elements of certain quadratic forms. Math. USSR Sb., 16 (1972), 17–35.

[Vin85(1)] E. B. Vinberg. The absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension. Trans. Moscow Math. Soc., 47 (1985), 75–112.

[Vin85(2)] E. B. Vinberg, Hyperbolic reflection groups. Russian Math. Surveys 40 (1985), 31–73.

[Vin93] E. B. Vinberg, editor. Geometry II - Spaces of Constant Curvature. Springer, 1993.

[VK78] E. B. Vinberg and I. M. Kaplinskaya. On the groups $O_{18,1}(Z)$ and $O_{19,1}(Z)$. Soviet Math. Dokl. 19 (1978), 194–197.