Electrons-Holes on Noncommutative Plane and Hall Effect

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Abstract

By considering $N_e$-electrons and $N_h$-holes together in uniform external magnetic and electric fields, we end up with a total Hall conductivity $\sigma_{H}^{\text{tot}}$, which is depending to the difference between $N_e$ and $N_h$ and becomes null when $N_e = N_h$. Dealing with the same system but requiring that the coordinates of plane are noncommuting, we obtain a new Hall conductivity $\sigma_{H}^{(\text{tot},\text{nc})}$. In the limit $N_e = N_h$, we find that $\sigma_{H}^{(\text{tot},\text{nc})}$ is only noncommutativity parameters $\theta_i$-dependent, which means that theoretically it is possible to have Hall effect without $B$. Moreover, at the critical points $\theta_e = l^2$ and $\theta_h = -l^2$, we find that $\sigma_{H}^{(\text{tot},\text{nc})}$ becomes two times the usual Hall conductivity for an noninteracting mixing system.

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1 Introduction

Two-dimensional (2D) systems of particles under strong magnetic fields exhibiting some beautiful and exciting phenomena like fractional quantum Hall effect (FQHE) \[1, 2\]. As example of system where FQHE can be appeared, one can mention \(GaAs/AlGaAs\) heterostructure \[3\] and \(GaAs-Al_{0.3}Ga_{0.7}\) \[4\]. As consequence, one may can think to consider a mixing system of electrons and holes moving together in the plane. As shown in \[4\], such kind of system exists actually in nature and it is just a doped semiconductor either with electrons or holes.

On the other hand, experimentally a combined system of 2D electron gas (2DEG) and 2D hole gas (2DHG) can also be formed by \(InAs/AlSb/GaSb\) \[5\]. In fact, when \(GaSb\) and \(InAs\) are brought together, negative charge is transferred from \(GaSb\) into \(InAs\), creating holes and electrons, respectively, at the interface of those layers and \(AlSb\) is used as a barrier. Theoretically, many studies have been carried on a system formed by separate 2DEG and 2DHG \[3, 4, 6, 7\]. Because it offers the possibility of forming a gas of excitons aligned along the two parallel planes \[4\].

Recently with Dayi, we proposed \[10\] an approach based on noncommutative geometry tools \[11\] to describe FQHE of a system of electrons. In fact, the corresponding filling factor is found to be

\[
\nu_{DJ} = \frac{\pi}{2} \rho(l^2 - \theta)
\]  

and it is identified to the observed fractional values \(f = 1/3, 2/3, 1/5, \cdots\). Also this approach allowed us to make a link with composite fermion theory \[12, 13\] of FQHE by setting an effective magnetic field

\[
B_{DJ} = \frac{B}{1 - \theta l^2} 
\]  

similar to that felt by composite fermions.

Motivated by the above results, we would like to consider an noninteracting system of electrons and holes living on the plane and in the presence of uniform magnetic field and electric field. Calculating the total Hall conductivity \(\sigma^\text{tot}_H\), we find that this latter vanishes when the number of electrons \(N_e\) coincides with the number of holes \(N_h\). However, another results can be found when we consider the present system on noncommutative plane. Indeed, after deriving the corresponding total Hall conductivity \(\sigma^{(\text{tot, nc})}_H\), we show that in the limit \(N_e = N_h\), this quantity is noncommutativity parameters \(\theta_l\)-dependent and \(B\)-independent. On the contrary, it does not vanish as noticed in the standard case. Moreover, we present other discussions and we show
that (1) can be recovered from the present study. Taking into account of the critical points
\( \theta_e = l^2 \) and \( \theta_h = -l^2 \), we find that \( \sigma_{\text{H}}^{(\text{tot}, \text{nc})} \) becomes \( B \)-dependent.

In section 2, we give the energy spectrum and the eigenstates of a Hamiltonian describing noninteracting electron-hole living on plane and in the presence of magnetic and electric fields. By using the standard definition we determine the corresponding total current operator leading to a total Hall conductivity. In section 3, we deal with the same problem but in noncommutative plane and we calculate its total Hall conductivity. We discuss and comment the obtained results in section 4 and we consider the critical points of the noncommutative analysis in section 5.

## 2 Hall conductivity on plane

We would like to study a mixing system, which may can be the interface between two layers of 2DEG and 2DHG or a doped semiconductor. In doing so, let us start by considering a system of an electron and a hole living together on the plane \( (x, y) \) and in the presence of an uniform external \( \vec{B} \) and \( \vec{E} \) fields. Without interaction, this system is described by the Hamiltonian

\[
H^{\text{tot}} = H_e + H_h
\]

as sum of two independent Hamiltonian’s corresponding to an electron \( H_e \) and a hole \( H_h \), respectively

\[
H_e = \frac{1}{2m_e}(\vec{p}_e + e\vec{A}_e)^2 + eEx_e
\]

\[
H_h = \frac{1}{2m_h}(\vec{p}_h - e\vec{A}_h)^2 - eEx_h
\]

where \( (\vec{p}_i, \vec{x}_i) \) are electron and hole phase spaces, \( i = e, h \) denotes electron and hole. In the symmetric gauges

\[
\vec{A}_e = \frac{B}{2}(y_e, x_e)
\]

\[
\vec{A}_h = \frac{B}{2}(y_h, x_h)
\]

(3) takes the form

\[
H^{\text{tot}} = \frac{1}{2m_e} \left[ \left( p_{x_e} - \frac{eB}{2e} y_e \right)^2 + \left( p_{y_e} + \frac{eB}{2e} x_e \right)^2 \right] + eEx_e
\]

\[
+ \frac{1}{2m_h} \left[ \left( p_{x_h} + \frac{eB}{2e} y_h \right)^2 + \left( p_{y_h} - \frac{eB}{2e} x_h \right)^2 \right] - eEx_h.
\]

From the last equation, one may notice that the sign of charge of the carrier \( (e, h) \) is important and will play a crucial role in the next. This point will be clear when we will derive the Hall current and then the Hall conductivity.
The Hamiltonian $H_{\text{tot}}$ can be diagonalised simply by considering the following operators corresponding to electron \[10\]

\[
\begin{align*}
    b^\dagger_e &= -2ip_{ze} + \frac{eB}{2c} z_e + \lambda_e \\
    b_e &= 2ip_{ze} + \frac{eB}{2c} z_e + \lambda_e \\
    d_e &= 2ip_{ze} - \frac{eB}{2c} z_e \\
    d^\dagger_e &= -2ip_{ze} - \frac{eB}{2c} z_e
\end{align*}
\]  

and another set related to hole

\[
\begin{align*}
    b^\dagger_h &= -2ip_{zh} - \frac{eB}{2c} z_h + \lambda_h \\
    b_h &= 2ip_{zh} + \frac{eB}{2c} z_h + \lambda_h \\
    d_h &= 2ip_{zh} + \frac{eB}{2c} z_h \\
    d^\dagger_h &= -2ip_{zh} + \frac{eB}{2c} z_h.
\end{align*}
\]  

The $\lambda_i$’s are fixed to be

\[
\begin{align*}
    \lambda_e &= \frac{m_e e F}{B} \\
    \lambda_h &= \frac{m_h e F}{B}.
\end{align*}
\]  

These sets satisfy the commutation relations

\[
\begin{align*}
    [b_i, b_i^\dagger] &= \pm 2m_i \hbar \omega_i \\
    [d_i^\dagger, d_i] &= \pm 2m_i \hbar \omega_i
\end{align*}
\]  

and other commutators vanish, where plus refers to electron and minus to hole. The cyclotron frequencies are given by

\[
\omega_i = \frac{eB}{m_i c}.
\]

Actually, $H_{\text{tot}}$ can be expressed in terms of the above operators as follows

\[
H_{\text{tot}} = \frac{1}{4m_e} (b^\dagger_e b_e + b_e b^\dagger_e) - \frac{\lambda_e}{2m_e} (d^\dagger_e + d_e) - \frac{\lambda_e^2}{2m_e} \\
+ \frac{1}{4m_h} (b^\dagger_h b_h + b_h b^\dagger_h) - \frac{\lambda_h}{2m_h} (d^\dagger_h + d_h) - \frac{\lambda_h^2}{2m_h}.
\]  

From the eigenvalue equation

\[
H_{\text{tot}} \vert \Psi_{\text{tot}} \rangle = E_{\text{tot}} \vert \Psi_{\text{tot}} \rangle
\]

we can obtain the energy spectrum and eigenstates

\[
\Psi_{\text{tot}}^{n_e, n_h, \alpha_e, \alpha_h} = \Phi_{n_e} \otimes \Phi_{n_h} \otimes \phi_{\alpha_e} \otimes \phi_{\alpha_h} \equiv \vert n_e, n_h, \alpha_e, \alpha_h \rangle
\]

\[
E_{\text{tot}}^{n_e, n_h, \alpha_e, \alpha_h} = \frac{\hbar \omega_e}{2} (2n_e + 1) + \frac{\hbar \omega_h}{2} (2n_h + 1) - \frac{\hbar \lambda_e}{m_e} \alpha_e - \frac{\lambda_e^2}{2m_e} - \frac{\hbar \lambda_h}{m_h} \alpha_h - \frac{\lambda_h^2}{2m_h}
\]

\[3\]
where
\[ \Phi_{n_i} = \frac{1}{\sqrt{(2m_i\hbar \omega_i)^{n_i} n_i!}} (b_i^\dagger)^{n_i} |0> \]
\[ \phi_{\alpha_i} = e^{i(\alpha_i y_i + \frac{m_i \hbar \omega_i}{2} x_i y_i)} . \]

\( n_e, n_h = 0, 1, 2 \ldots \) and \( \alpha_e, \alpha_h \in \mathbb{R} \). \( \otimes \) denotes the direct product.

At this stage, we would like to determine the corresponding total Hall conductivity \( \sigma_{H}^{\text{tot}} \) in order to have some informations about the behaviour of the mixing system. To derive this physical quantity, one can use directly the definition of the related total current operator \( \vec{J}_{\text{tot}} \), such as
\[ \vec{J}_{\text{tot}} = \vec{J}_e + \vec{J}_h \]
where \( \vec{J}_e \) and \( \vec{J}_h \) are defined to be
\[ \vec{J}_e = -\frac{e \rho_e}{m_e} (\vec{p}_e + \frac{e}{c} \vec{A}_e) \]
\[ \vec{J}_h = e \frac{\rho_h}{m_h} (\vec{p}_h - \frac{e}{c} \vec{A}_h) \]
and \( \rho_e \) and \( \rho_h \) are, respectively, electron and hole densities. Let us emphasis here that the sign of charge of different particles is taken account. Moreover, the expectation value of \( \vec{J}_{\text{tot}} \) can be calculated with respect to the eigenstates \( |n_e, n_h, \alpha_e, \alpha_h> \). Therefore, we obtain
\[ < J_{x}^{\text{tot}} > = 0 \]
\[ < J_{y}^{\text{tot}} > = \frac{ec}{B} \left( \rho_e - \rho_h \right) E . \]
The second equation determines the so-called Hall conductivity and then leads us to have a total \( \sigma_{H}^{\text{tot}} \) as
\[ \sigma_{H}^{\text{tot}} = \frac{ec}{B} \left( \rho_e - \rho_h \right) \]
which is actually sum of two contributions coming from electrons and holes, respectively
\[ \sigma_{H}^e = \frac{ec \rho_e}{B} \]
\[ \sigma_{H}^h = -\frac{ec \rho_h}{B} . \]
At this level, let us notice two remarks. First, one may can see that once we have \( \rho_h = \rho_e \), \( \sigma_{H}^{\text{tot}} \) becomes null. Then in this case the mixing system is behaving like an insulator. Second, if one drops \( \sigma_{H}^{\text{tot}} \) as function of the magnetic field \( B \), we end up with straight line. However, the experiment observation claimed that at strong \( B \), we have plateaus. Consequently, when the straight line meets the plateaus, it is equivalent to have
\[ \frac{ec}{B} \left( \rho_e - \rho_h \right) = \nu^{\text{tot}} \frac{e^2}{h} \]
where \( \nu^{\text{tot}} \) is the total filling factor, which is characterising quantum Hall effect. This equation leads us to have
\[
\nu^{\text{tot}} = \frac{N_e - N_h}{N_{\Phi_B}} \tag{22}
\]
it is nothing but the definition of \( \nu^{\text{tot}} \), i.e. the ratio between number of particles ensuring the conductivity and number of quantum flux \( N_{\Phi_B} = \frac{\Phi_B}{\Phi_0} \), where \( \Phi_0 = \hbar c/e \) is unit of flux.

\section{Hall conductivity on noncommutative plane}

The results derived in the last section can be generalised in terms of noncommutative geometry \cite{11} and lead us to have more informations, especially for total Hall conductivity. To clarify these points, let us start by demanding that the coordinates of the plane are noncommuting, which means that the spacial commutator is now broken such that
\[
[x^i, x^j] = i\theta^{ij} \tag{23}
\]
where \( \theta^{ij} = \epsilon^{ij}\theta \) is the noncommutativity parameter and \( \epsilon^{12} = -\epsilon^{21} = 1 \). Basically, we are forced in this case to replace \( fg(x) = f(x)g(x) \) by the relation
\[
f(x) \star g(x) = \exp\left[\frac{i}{2} \theta^{ij} \partial_x^i \partial_y^j\right] f(x)g(y)|_{x=y} \tag{24}
\]
where \( f \) and \( g \) are two arbitrary functions, supposed to be infinitely differentiable. As consequence, now we are going to deal with quantum mechanics by considering the following algebra
\[
[x^i, x^j] = i\theta^{ij} \\
[p^i, x^j] = -i\delta^{ij} \tag{25} \\
[p^i, p^j] = 0
\]

At this level, one may can use the above receipt to deform the electron’s and hole’s phase space independently \cite{14}. Thus, instead of using (23), we consider two different relations, namely one with respect to electrons
\[
[x_e, y_e] = i\theta_e \tag{26}
\]
and another one related to holes
\[
[x_h, y_h] = i\theta_h \tag{27}
\]
where $\theta_i$’s are real parameter, and the usual canonical quantization similar to the two last equations given in (25).

Actually, we can write down the noncommutative version of the Hamiltonian (3). In doing on, let us notice $H_{\text{tot}}$ acts on an arbitrary function $\Psi(\vec{r}, t)$ as

$$H_{\text{tot}} \ast \Psi(\vec{r}, t) = H^{(\text{tot, nc})}(\vec{r}, t)$$

which implies that $H^{(\text{tot, nc})}$ is

$$H^{(\text{tot, nc})} = \frac{1}{2me} \left[ \left( \gamma_e p_{x_e} - \frac{eB}{2c} y_e \right)^2 + \left( \gamma_e p_{y_e} + \frac{eB}{2c} x_e \right)^2 \right] + eE(x_e - \frac{\theta_e}{2\hbar} y_e)$$

$$+ \frac{1}{2mh} \left[ \left( \gamma_h p_{x_h} + \frac{eB}{2c} y_h \right)^2 + \left( \gamma_h p_{y_h} - \frac{eB}{2c} x_h \right)^2 \right] - eE(x_h - \frac{\theta_h}{2\hbar} y_h)$$

(29)

where the $\gamma_i$’s are defined to be

$$\gamma_e = 1 - \theta_e l^{-2}$$

$$\gamma_h = 1 + \theta_h l^{-2}$$

(30)

and $l = 2l_0$, $l_0 = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length. Note in passing that (29) is also a sum of two noncommutative parts of $H_e$ and $H_h$:

$$H^{(\text{tot, nc})} = H^{\text{nc}}_e + H^{\text{nc}}_h.$$  

(31)

Now, one can process as before to diagonalise the noncommutative Hamiltonian $H^{(\text{tot, nc})}$. For that, let us define the following operators for electron

$$\tilde{b}^\dagger_e = -2i\gamma_e p_{z_e} + \frac{eB}{2c} z_e + \lambda_e$$

$$\tilde{b}_e = 2i\gamma_e p_{z_e} + \frac{eB}{2c} z_e + \lambda_e$$

(32)

and

$$\tilde{d}_e = 2i\gamma_e p_{z_e} - \frac{eB}{2c} z_e$$

$$\tilde{d}^\dagger_e = -2i\gamma_e p_{z_e} - \frac{eB}{2c} z_e.$$  

(33)

Also in similar way one can define another set of operators for hole

$$\tilde{b}^\dagger_h = -2i\gamma_h p_{z_h} - \frac{eB}{2c} z_h + \lambda_h$$

$$\tilde{b}_h = 2i\gamma_h p_{z_h} - \frac{eB}{2c} z_h + \lambda_h$$

(34)

and

$$\tilde{d}_h = 2i\gamma_h p_{z_h} + \frac{eB}{2c} z_h$$

$$\tilde{d}^\dagger_h = -2i\gamma_h p_{z_h} + \frac{eB}{2c} z_h.$$  

(35)
The sets of operators \((\tilde{b}_i, \tilde{b}_i^\dagger)\) and \((\tilde{d}_i, \tilde{d}_i^\dagger)\) commute with each other. Moreover, they verify the commutation relations

\[
[\tilde{b}_i, \tilde{b}_i^\dagger] = \pm 2m_i \hbar \tilde{\omega}_i
\]

\[
[\tilde{d}_i, \tilde{d}_i^\dagger] = \pm 2m_i \hbar \tilde{\omega}_i
\]

(36)

where the \(\tilde{\omega}_i\)'s and the \(\lambda_{\pm i}\)'s are given by

\[
\tilde{\omega}_e = \gamma_e \omega_e
\]

\[
\tilde{\omega}_h = \gamma_h \omega_h
\]

\[
\lambda_{\pm e} = \lambda_e \pm \frac{em_e E\theta_e}{4\gamma_e \hbar}
\]

\[
\lambda_{\pm h} = \lambda_h \pm \frac{em_h E\theta_h}{4\gamma_h \hbar}
\]

(37)

To make these equations holding and for further analysis, we assume that the conditions \(\theta_e \neq l^2\) and \(\theta_h \neq -l^2\) are satisfied. We will come back to this assumption in the last section and discuss its consequence. In terms of the above creation and annihilation operators, the Hamiltonian \(H^{(\text{tot, nc})}\) takes the form

\[
H^{(\text{tot, nc})} = \frac{1}{4m_e} (\tilde{b}_e^\dagger \tilde{b}_e + \tilde{b}_e \tilde{b}_e^\dagger) - \frac{\lambda_{e\pm}}{2m_e} (\tilde{d}_e^\dagger + \tilde{d}_e) - \frac{\lambda_{e\pm}^2}{2m_e^2}
\]

\[
+ \frac{1}{4m_h} (\tilde{b}_h^\dagger \tilde{b}_h + \tilde{b}_h \tilde{b}_h^\dagger) - \frac{\lambda_{h\pm}}{2m_h} (\tilde{d}_h^\dagger + \tilde{d}_h) - \frac{\lambda_{h\pm}^2}{2m_h^2}
\]

(38)

As before, we can solve the eigenvalue equation

\[
H^{(\text{tot, nc})} |\Psi^{(\text{tot, nc})}\rangle = E^{(\text{tot, nc})} |\Psi^{(\text{tot, nc})}\rangle
\]

(39)

to get the eigenstates:

\[
|\Psi^{(\text{tot, nc})}\rangle = |\Psi^{(\text{nc})}_{(n_e, \alpha_e, \theta_e)}\rangle \otimes |\Psi^{(\text{nc})}_{(n_h, \alpha_h, \theta_h)}\rangle \equiv |n_e, n_h, \alpha_e, \alpha_h, \theta_e, \theta_h\rangle
\]

(40)

where

\[
|\Psi^{(\text{nc})}_{(n_i, \alpha_i, \theta_i)}\rangle = \frac{1}{\sqrt{(2m_i \hbar \tilde{\omega}_i)^{n_i}}} e^{i(\alpha_i y_i + \frac{m_i \theta_i}{2\hbar} x_i y_i)} (\tilde{b}_i^\dagger)^{n_i} |0\rangle
\]

(41)

and the corresponding eigenvalues:

\[
E^{(\text{tot, nc})}_{(n_e, n_h, \alpha_e, \alpha_h, \theta_e, \theta_h)} = E^{(\text{nc})}_{(n_e, \alpha_e, \theta_e)} + E^{(\text{nc})}_{(n_h, \alpha_h, \theta_h)}
\]

(42)

where

\[
E^{(\text{nc})}_{(n_e, \alpha_e, \theta_e)} = \frac{\hbar^2}{2} (2n_e + 1) - \frac{\hbar \gamma_e \lambda_{e\pm}}{m_e} \alpha_e - \frac{m_e}{2} \lambda_{e\pm}^2
\]

\[
E^{(\text{nc})}_{(n_h, \alpha_h, \theta_h)} = \frac{\hbar^2}{2} (2n_h + 1) - \frac{\hbar \gamma_h \lambda_{h\pm}}{m_h} \alpha_h - \frac{m_h}{2} \lambda_{h\pm}^2
\]

(43)

with \(n_i = 0, 1, 2...\) and \(\alpha_i \in \mathbb{R}\).
Here one may can ask about the corresponding conductivity to see the difference with the commutative case. Actually this quantity resulting from the Hamiltonian $H^{\text{tot,nc}}$ is determined by defining the total current operator $\vec{J}^{\text{tot,nc}}$ on noncommutative plane as

$$\vec{J}^{\text{tot,nc}} = -\frac{e}{m_e} \rho_e \gamma_e (\gamma_e \vec{p}_e + \frac{e}{c} \vec{A}_e + \vec{a}_e) + \frac{e}{m_h} \rho_h \gamma_h (\gamma_h \vec{p}_h - \frac{e}{c} \vec{A}_h + \vec{a}_h)$$

(44)

where the $\vec{a}_i$ vectors are

$$\vec{a}_e = (0, -\frac{m_e e E\theta_e}{2\hbar \gamma_e})$$

$$\vec{a}_h = (0, \frac{m_h e E\theta_h}{2\hbar \gamma_h}).$$

(45)

Its expectation value are calculated with respect to the eigenstates $|n_e, n_h, \alpha_e, \alpha_h, \theta_e, \theta_h>$ and it found to be

$$\langle J_x^{\text{tot,nc}} \rangle = 0$$

$$\langle J_y^{\text{tot,nc}} \rangle = \frac{e c}{B} (\rho_e \gamma_e - \rho_h \gamma_h) E.$$ 

(46)

Therefore, the total Hall conductivity on noncommutative plane of electrons and holes, denoted by $\sigma^{\text{tot,nc}}_H$, is

$$\sigma^{\text{tot,nc}}_H = \frac{e c}{B} (\rho_e \gamma_e - \rho_h \gamma_h).$$

(47)

To close this section, let us notice that the commutative analysis is recovered if the noncommutativity parameters $\theta_i$ are switched off.

4 Discussions

In this section, we would like to comment the obtained result (47) and then discuss its consequence by considering some special cases either related to different densities or noncommutativity parameters. Indeed, let us distinguish some relevant cases:

First: (47) can be written as follows

$$\sigma^{\text{tot,nc}}_H = \sigma^{\text{tot}}_H + \frac{\pi}{2} \left( \theta_h \rho_h - \theta_e \rho_e \right) \frac{e^2}{\hbar}.$$ 

(48)

Then, the second term can be interpreted as a quantum correction to standard Hall effect, which is noncommutativity parameters dependent. It is obvious to see that

$$\sigma^{\text{tot,nc}}_H |_{\theta_e=\theta_h=0} = \sigma^{\text{tot}}_H$$

(49)

once we have $\theta_e = \theta_h = 0$. 

8
Second: Let us make contact with quantum Hall effect by defining the corresponding total effective filling factor

\[ \nu_{\text{eff}}^{\text{tot}} = \frac{\pi}{2} l^2 \left( \rho_e \gamma_e - \rho_h \gamma_h \right) \]  

which implies that

\[ \sigma_H^{(\text{tot,nc})} = \nu_{\text{eff}}^{\text{tot}} e^2 \frac{\hbar}{\hbar}. \]  

It is similar to the ordinary relation for Hall conductivity \( \sigma_H^{\text{tot}} = \nu^{\text{tot}} e^2 \hbar \). Then, one may can interpret (47) as a result of QHE by setting an effective density and magnetic field, such that their ratio is

\[ \frac{\rho_{\text{eff}}}{B_{\text{eff}}} = \frac{\rho_e \gamma_e}{B} \left( \rho_h \gamma_h \gamma_h \right) \]  

and leading to

\[ \sigma_H^{(\text{tot,nc})} = \frac{ec \rho_{\text{eff}}}{B_{\text{eff}}} = \nu_{\text{eff}}^{\text{tot}} e^2 \frac{\hbar}{\hbar}. \]

Third: One can see easily that (47) vanishes when the relation is satisfied

\[ \frac{\gamma_e}{\gamma_h} = \frac{\rho_h}{\rho_e} \]  

which is independent to the relationship between different densities. It is on the contrary with the standard case where the total Hall conductivity is found to be zero whether \( N_e = N_h \).

Fourth: If we assume that

\[ \rho_e \gamma_e = \beta \rho_h \gamma_h \]  

for any \( \beta \), then (47) gives us

\[ \sigma_H^{(\text{tot,nc})} = (\beta - 1) \frac{ec \rho_{\text{eff}}}{B_{\text{eff}}} \rho_h \gamma_h. \]  

It is \((\beta - 1)\) times of what we obtained [10] for one system and coincides with our former results when \( \beta = 2 \).

Fifth: If we demand that the numbers of electrons and holes are the same, namely \( \rho_h = \rho_e = \rho \), then (47) becomes

\[ \sigma_H^{(\text{tot,nc})} |_{\rho_h=\rho_e=\rho} = -\frac{\pi \rho}{2} \left( \theta_e + \theta_h \right) e^2 \frac{\hbar}{\hbar}. \]  

and therefore

\[ \nu_{\text{eff}}^{\text{tot}} |_{\rho_h=\rho_e=\rho} = -\frac{\pi \rho}{2} \left( \theta_e + \theta_h \right). \]
As consequence, if \( \theta_e = \theta_h = \theta \), we find

\[
\sigma_H^{(\text{tot,nc})}_{|\rho_e=\rho_h=\rho} = -\pi \rho \theta \frac{e^2}{h}
\]  

and obviously (57) becomes zero when \( \theta_e = -\theta_h \). These results are completely different from standard case where \( \sigma_H^{\text{tot}} \) is found to be null at the present limit (\( \rho_e = \rho_h = \rho \)). These equations tell us about the possibility to have theoretically Hall effect without magnetic field just by controlling the noncommutativity parameter \( \theta \) for fixed densities. As consequence, one may can ask about the experiment realisation of such kind of result, otherwise is it possible to prove (57,59) experimentally?

5 Critical points

Another result can be obtained by taking into account of critical points. In fact, following the above analysis we have to consider two different critical values of \( \theta_i \), namely \( \theta_e = l^2 \) and \( \theta_h = -l^2 \). Then, let us see what is the consequence of these particular values of \( \theta_i \) on the above results of the noncommutative study.

(i) \( \theta_e = l^2 \): The relation (26) reads as

\[
[x_i^e, x_j^e] = il^2 \delta^{ij}.
\]  

In this case the whole Hamiltonian becomes

\[
H^{(\text{tot,nc})}_{|\theta_e=l^2} = \frac{m \omega_e^2}{4} \left( x_e^2 + y_e^2 \right) + \frac{2Ec}{B} \left( \frac{eB}{2c} x_e - p_ye \right) + H^{\text{nc}}_h.
\]  

Using the same techniques as before, we find that (47) now reads as

\[
\sigma_H^{(\text{tot,nc})}_{|\theta_e=l^2} = \frac{2ec \rho_e}{B} - \frac{ec \rho_h}{B} \gamma_h,
\]  

which is \( \theta_h \)-dependent. Here also one may can report the above discussion as concerning non-commutativity parameter.

(ii) \( \theta_h = -l^2 \): A similar result as before can be found just by switching the indices’s. Indeed, now the deformed hole space is characterised by

\[
[x_i^h, x_j^h] = -il^2 \delta^{ij}
\]  

\[\text{(63)}\]
instead of (27). Then, the corresponding Hall conductivity is

\[
\sigma_{H}^{(\text{tot,nc})}|_{\theta_{h}=-l^{2}} = \frac{e\rho_{e}}{B} \gamma_{e} - \frac{2e\rho_{h}}{B}
\]  

(64)

which is now \(\theta_{e}\)-dependent.

(iii) \(\theta_{e} = l^{2} = -\theta_{h}\): (28) reduces to the following one

\[
H^{(\text{tot,nc})}|_{\theta_{e}=l^{2}=-\theta_{h}} = \frac{m\omega_{e}^{2}}{4} \left( x_{e}^{2} + y_{e}^{2} \right) + \frac{2E_{e}}{B} \left( \frac{eB}{2e} x_{e} - p_{y_{e}} \right) \\
+ \frac{m\omega_{h}^{2}}{4} \left( x_{h}^{2} + y_{h}^{2} \right) - \frac{2E_{h}}{B} \left( \frac{eB}{2e} x_{h} - p_{y_{h}} \right).
\]

(65)

Therefore, we end up with

\[
\sigma_{H}^{(\text{tot,nc})}|_{\theta_{e}=l^{2}=-\theta_{h}} = \frac{2e\rho}{B} \left( \rho_{e} - \rho_{h} \right).
\]

(66)

Note that the obtained relation is actually two times the standard result \(\sigma_{H}^{\text{tot}}\) and becomes zero once we have \(N_{e} = N_{h}\).

6 Conclusion

By considering a system of the electrons and holes living together on noncommutative plane and in the presence of magnetic and electric fields, we ended up with the corresponding total Hall conductivity. This latter has some particularities in sense that it is noncommutativity parameters dependent and showed some other behaviour than the usual one. In fact, on the contrary to standard case, it does not vanish whether we have \(N_{e} = N_{h}\) and then became magnetic field independent. Therefore, this result suggested the possibility to have Hall effect without magnetic field. Moreover, by taking into account of different critical points, we got the usual Hall conductivity for a mixing system times a factor two.

One may can ask about the possibility to deal with an interacting system of electrons and holes in the framework of noncommutative geometry. This problem and related questions will be considered in the feature.

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References

[1] D.C. Tsui, H.L. Störmer and A.C. Gossard, *Phys. Rev. Lett.* **48** (1982) 1555.

[2] R.E. Prange and S.M. Girvin (editors), “The Quantum Hall Effect”, (New York, Springer 1990).

[3] G.S. Boebinger, A.M. Chang, H.L. Stormer and D.C. Tsui, *Phys. Rev. B* **32** (1985) 4268.

[4] A.M. Chang, B. Berghund, D.C. Tsui, H.L. Stormer and J.C.M. Hwang, *Phys. Rev. Lett.* **53** (1984) 997.

[5] Y. Lin, E.E. Mendez and A.G. Abanov, *Tunneling Characteristics of an Electron-Hole Trilayer under an In-plane Magnetic Field*, cond-mat/0207070.

[6] S.M. Girvin and A.H. MacDonald, “Multicomponent Quantum Hall Systems: The Sum of Their Parts and More”, (John Willey & Sons, New York, 1997), Chap. 5, p. 161.

[7] A.H. MacDonald, *Physica B* **298** (2001) 129.

[8] E. Demler, C. Nayak and S.D. Sarma, *Phys. Rev. Lett.* **86** (2001) 1853.

[9] A. Mitra and S.M. Girvin, *Phys. Rev. B* **64** (2001) 041309.

[10] Ö.F. Dayi and A. Jellal, *Hall Effect in Noncommutative Coordinates*, hep-th/0111267, to appear in *J. Math. Phys.* (2002).

[11] A. Connes, *Noncommutative Geometry* (Academic Press, London 1994).

[12] J.V. Jain, *Phys. Rev. Lett.* **63** (1989) 199; *Phys. Rev. B* **41** (1990) 7653, *Adv. Phys.* **41** (1992) 105.

[13] O. Heinonen (editor), “Composite Fermions: A Unified View of Quantum Hall Regime”, (World Scientific, 1998).

[14] Ö.F. Dayi, Private Communication.