On the pseudo-Hermitian nondiagonalizable Hamiltonians

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Abstract

We consider a class of (possibly nondiagonalizable) pseudo-Hermitian operators with discrete spectrum, showing that in no case (unless they are diagonalizable and have a real spectrum) they are Hermitian with respect to a semidefinite inner product, and that the pseudo-Hermiticity property is equivalent to the existence of an antilinear involutory symmetry. Moreover, we show that a typical degeneracy of the real eigenvalues (which reduces to the well known Kramers degeneracy in the Hermitian case) occurs whenever a fermionic (possibly nondiagonalizable) pseudo-Hermitian Hamiltonian admits an antilinear symmetry like the time-reversal operator $T$. Some consequences and applications are briefly discussed.

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1 Introduction

Non Hermitian Hamiltonians play by now a relevant rôle in physics, in that they appear in several completely different problems \cite{1}. Among them, a remarkable subclass is given by the pseudo-Hermitian operators \cite{2}, i.e., those operators which satisfy

$$\eta H \eta^{-1} = H^\dagger$$

(1)

with \(\eta = \eta^\dagger\) [instead, whenever (1) holds without any constraint on the (linear and invertible) operator \(\eta\), \(H\) is called weakly pseudo-Hermitian \cite{3}]. Of course, Hermiticity is a particular case of pseudo-Hermiticity, corresponding to \(\eta = 1\).

Pseudo-Hermiticity also represents the mathematical background of a recent proposal on a complex extension of Quantum Mechanics \cite{4,5}.

The essential feature of the pseudo-Hermitian operators is the peculiarity of their spectrum, which can be constituted by real as well as complex (but grouped in complex-conjugate pairs) eigenvalues \cite{3,6}. This property, originally stated with reference to diagonalizable operators with discrete spectrum, has been recently extended to a class of nondiagonalizable Hamiltonians \cite{7}. Such Hamiltonians can arise, for instance, for some critical parameter values, whenever a physical system undergoes a perturbation which preserves the pseudo-Hermiticity, but not the diagonalizability, of its Hamiltonian. An example of such a situation is shown in Sec. 5.
The aim of this paper is just to carry on a systematic and deep study on nondiagonalizable pseudo-Hermitian operators.

To this end, we recall (and partly refine) in Sec. 2 the basic results on this subject. Next, we inquire in Sec. 3 into the definiteness or the indefiniteness of the metric induced by $\eta$, concluding that for any pseudo-Hermitian operator $H$ with discrete spectrum, the metric is always indefinite unless $H$ is diagonalizable with real spectrum. This result disproves a recently stated theorem on the subject [7].

Successively, in Sec. 4, we take into account another characteristic feature of the pseudo-Hermiticity property, i.e., its connection with the existence of antilinear symmetries, which has been already enlightened in the case of diagonalizable operators [3, 8], showing that such connection holds also for the nondiagonalizable case. Sec. 5 is devoted to a discussion on the time-reversal invariance of fermionic Hamiltonians, extending a result on the (generalized) Kramers degeneracy that we have already proven for diagonalizable operators [9]. Finally, some concluding remarks and possible applications of the previous results are briefly presented in Sec. 6.

2 The spectrum of nondiagonalizable pseudo-Hermitian operators

According to [12] we consider here only linear operators $H$ acting in a separable Hilbert space $\mathcal{H}$ and having discrete spectrum. Moreover, throughout this...
paper we shall assume that all the eigenvalues $E_n$ of $H$ have finite algebraic multiplicity $g_n$ and that there is a basis of $\mathcal{H}$ in which $H$ is block-diagonal with finite-dimensional diagonal blocks. Then, a complete biorthonormal basis $\mathcal{E} = \{ |\psi_n, a, i \rangle, |\phi_n, a, i \rangle \}$ exists such that the operator $H$ can be written in the following form [7]:

$$H = \sum_n d_n \sum_{a=1}^{p_{n,a}} (E_n \sum_{i=1}^{p_{n,a}} |\psi_n, a, i \rangle \langle \phi_n, a, i| + \sum_{i=1}^{p_{n,a}-1} |\psi_n, a, i \rangle \langle \phi_n, a, i + 1|)$$

(2)

where $d_n$ denotes the geometric multiplicity (i.e., the degree of degeneracy) of $E_n$, $a$ is a degeneracy label and $p_{n,a}$ represents the dimension of the simple Jordan block $J_a(E_n)$ associated with the labels $n$ and $a$ (hence, $\sum_{a=1}^{d_n} p_{n,a} = g_n$). Furthermore, we denote by $k(n,a)$ the total number of identical simple blocks $J_a(E_n)$ occurring in the above decomposition of $H$.

Hence, $|\psi_n, a, 1 \rangle$ (respectively, $|\phi_n, a, p_{n,a} \rangle$) is an eigenvector of $H$ (respectively, $H^\dagger$):

$$H |\psi_n, a, 1 \rangle = E_n |\psi_n, a, 1 \rangle, \quad H^\dagger |\phi_n, a, p_{n,a} \rangle = E_n^* |\phi_n, a, p_{n,a} \rangle,$$

(3)

and the following relations hold:

$$H |\psi_n, a, i \rangle = E_n |\psi_n, a, i \rangle + |\psi_n, a, i-1 \rangle, \quad i \neq 1,$$

(4)

$$H^\dagger |\phi_n, a, i \rangle = E_n^* |\phi_n, a, i \rangle + |\phi_n, a, i+1 \rangle, \quad i \neq p_{n,a}.$$  

(5)
The elements of the biorthonormal basis obey the usual relations:

$$\langle \psi_m, a, i | \phi_n, b, j \rangle = \delta_{mn} \delta_{ab} \delta_{ij}, \quad (6)$$

$$\sum_n d_n \sum_{a=1}^{p_n} p_{n,a} \sum_{i=1}^{\delta_n} |\psi_n, a, i \rangle \langle \phi_n, a, i | = \sum_n d_n \sum_{a=1}^{p_n} p_{n,a} \sum_{i=1}^{\delta_n} |\phi_n, a, i \rangle \langle \psi_n, a, i | = 1. \quad (7)$$

The following theorem has been proven in [7]:

**Theorem 1.** Let $H$ be a linear operator acting in a Hilbert space $\mathcal{H}$. Suppose that the spectrum of $H$ is discrete, that its eigenvalues have finite algebraic multiplicity, and that (2) holds. Then, the following conditions are equivalent:

i) the eigenvalues of $H$ are either real or come in complex-conjugate pairs and the geometric multiplicity and the Jordan dimensions of the complex-conjugate eigenvalues coincide;

ii) $H$ is pseudo-Hermitian.

In order to fix our notation, and for the benefit of the reader, we prefer to provide here a (somewhat different) proof of the implication $i) \Rightarrow ii)$, which allows us to obtain a useful decomposition of $\eta$.

Let us therefore assume that condition $i)$ holds, and use (whenever it is necessary) the subscript "0" to denote real eigenvalues, and the subscripts "±" to denote the complex eigenvalues with positive or negative imaginary part,
respectively. Then, $H$ assumes the following form (see Eq. (2)):

$$
H = \sum_{n_0} \sum_{a=1}^{d_{n_0}} (E_{n_0} \sum_{i=1}^{p_{n_0,a}} |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i| + \sum_{i=1}^{p_{n_0,a}-1} |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i+1|) + \\
\sum_{n_+} \sum_{n_- a=1}^{d_{n_+}} \sum_{a=1}^{p_{n_+, a}} (E_{n_+} |\psi_{n_+}, a, i\rangle \langle \phi_{n_+}, a, i| + E_{n_-} |\psi_{n_-}, a, i\rangle \langle \phi_{n_-}, a, i|) + \\
\sum_{i=1}^{p_{n_+, a}-1} (|\psi_{n_+}, a, i\rangle \langle \phi_{n_+}, a, i+1| + |\psi_{n_-}, a, i\rangle \langle \phi_{n_-}, a, i+1|)).
$$

(8)

Furthermore, given any complete orthonormal basis $\mathcal{F} = \{|u_n, a, i\rangle\}$ in our space (that we denote by the same $n, a, i$ labels used for the elements of $\mathcal{E}$), let us pose

$$
S = \sum_{n} \sum_{a=1}^{d_{n}} \sum_{i=1}^{p_{n,a}} |\phi_{n}, a, i\rangle \langle u_{n}, a, i| ,
$$

(9)

and $\tilde{H} = S^\dagger H S^\dagger$. By a straightforward calculation one obtains

$$
\tilde{H} = \sum_{n_0} \sum_{a=1}^{d_{n_0}} (E_{n_0} \sum_{i=1}^{p_{n_0,a}} |u_{n_0}, a, i\rangle \langle u_{n_0}, a, i| + \sum_{i=1}^{p_{n_0,a}-1} |u_{n_0}, a, i\rangle \langle u_{n_0}, a, i+1|) + \\
\sum_{n_+} \sum_{n_- a=1}^{d_{n_+}} \sum_{a=1}^{p_{n_+, a}} (E_{n_+} |u_{n_+}, a, i\rangle \langle u_{n_+}, a, i| + E_{n_-} |u_{n_-}, a, i\rangle \langle u_{n_-}, a, i|) + \\
\sum_{i=1}^{p_{n_+, a}-1} (|u_{n_+}, a, i\rangle \langle u_{n_+}, a, i+1| + |u_{n_-}, a, i\rangle \langle u_{n_-}, a, i+1|)).
$$

(10)

Then, let us consider the involutory operators $U$ and $V$ defined respectively
as follows:

$$U|u_{n^\pm}, a, i\rangle = |u_{n^\pm}, a, i\rangle, \quad U|u_n, a, i\rangle = |u_n, a, i\rangle,$$  \hspace{1cm} (11)

and

$$V|u_n, a, i\rangle = |u_n, a, p_n, a + 1 - i\rangle.$$ \hspace{1cm} (12)

The explicit forms of $U$ and $V$ are:

$$U = U1 = \sum_{n_0, a, i} |u_{n_0}, a, i\rangle \langle u_{n_0}, a, i| + \sum_{n_+, n^-, a, i} (|u_{n^-}, a, i\rangle \langle u_{n^+}, a, i| + |u_{n^+}, a, i\rangle \langle u_{n^-}, a, i|).$$  \hspace{1cm} (13)

and

$$V = V1 = \sum_{n_0, a, i} |u_{n_0}, a, p_n, a + 1 - i\rangle \langle u_{n_0}, a, i| + \sum_{n_+, n^-, a, i} (|u_{n^+}, a, p_{n^+}, a + 1 - i\rangle \langle u_{n^+}, a, i| + |u_{n^-}, a, p_{n^-}, a + 1 - i\rangle \langle u_{n^-}, a, i|).$$  \hspace{1cm} (14)

Moreover both $U$ and $V$ are clearly Hermitian operators, and (recalling that, by hypothesis, $p_{n^+, a} = p_{n^-, a}$)
\[ UV = UV1 = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} |u_{n_0,a}, p_{n_0,a} + 1 - i \rangle \langle u_{n_0,a}, a, i| + \]
\[ \sum_{n_+} \sum_{n_-, a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} (|u_{n_+,a}, p_{n_+,a} + 1 - i \rangle \langle u_{n_-,a}, a, i| + |u_{n_-,a}, p_{n_-,a} + 1 - i \rangle \langle u_{n_+,a}, a, i|) = VU. \]

Thus, one can easily verify that \( \tilde{H} \) is a pseudo-Hermitian operator:

\[ \tilde{\eta} \tilde{H} \tilde{\eta}^{-1} = \tilde{H}^\dagger \]

where \( \tilde{\eta} = UV \). Hence, finally,

\[ \eta H \eta^{-1} = H^\dagger \]

where

\[ \eta = S \tilde{\eta} S^\dagger = SU VS^\dagger = \sum_{n_0} \sum_{a=1}^{d_{n_0}} \sum_{i=1}^{p_{n_0,a}} |\phi_{n_0,a}, p_{n_0,a} + 1 - i \rangle \langle \phi_{n_0,a}, a, i| + \]
\[ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+}} \sum_{i=1}^{p_{n_+,a}} (|\phi_{n_+,a}, p_{n_+,a} + 1 - i \rangle \langle \phi_{n_-,a}, a, i| + |\phi_{n_-,a}, p_{n_-,a} + 1 - i \rangle \langle \phi_{n_+,a}, a, i|) = \eta^\dagger. \] (15)

In conclusion we see that the spectrum of a pseudo-Hermitian operator is real if and only if \( U \equiv 1 \) (hence, by Eq. (15), \( \eta = SV S^\dagger \)), and that a pseudo-Hermitian operator is diagonalizable if and only if \( V \equiv 1 \) (hence, again by Eq.
(15), \( \eta = S \Sigma S^\dagger \).

**Remark.**

We stress here that in order to prove the implication \( ii) \implies i) \) only the invertibility of \( \eta \) is needed, while the Hermiticity property \( \eta = \eta^\dagger \) does not come into play [7]. Hence, by the same arguments one can prove that even the spectrum of a *weakly pseudo-Hermitian* operator \[ \text{i.e., an operator which satisfies Eq.}(1) \text{ without any constraint on the (linear and invertible) operator } \eta \], satisfies condition \( i) \). On the other hand, the above proof shows that condition \( i) \) implies that an Hermitian operator \( \eta \) exists which fulfils Eq. (1). Thus, if we just consider operators having a discrete spectrum, the (possibly broader) class of weakly pseudo-Hermitian operators actually coincides with the one of pseudo-Hermitian operators. Nevertheless, we recall that the weak pseudo-Hermiticity is a more useful notion, in that, for instance, it simplifies checking Eq. (1).

### 3 Nondiagonalizability and metric indefiniteness

We have seen in the previous section that an Hermitian operator \( \eta \) always exists such that a nondiagonalizable operator \( H \) (whose spectrum obeys condition \( i) \) in Theorem 1) is pseudo-Hermitian; moreover, it is well known that in this case one can define a new inner product \[ \text{[7]} \]

\[
\langle\langle \psi, \phi \rangle \rangle_{\eta} := \langle \psi | \eta | \phi \rangle ,
\]  

(16)
and, correspondingly, a $\eta$-pseudonorm $\langle\langle \psi, \psi \rangle \rangle_{\eta}$. Then, one may of course inquire into the definiteness or the indefiniteness of the metric induced by $\langle\langle , \rangle \rangle_{\eta}$.

Eq. (15) in the previous section clearly shows that the metric associated with such an $\eta$ cannot be a definite (nor a semidefinite) operator; indeed, being $\tilde{\eta}$ involutory and non-identical (unless $H$ is a diagonalizable operator with real spectrum), some of its eigenvalues (but not all) must be negative, hence the same happens (by the Sylvester’s law of inertia [10]) for the eigenvalues of the operator $\eta$. This fact can suggest that in all cases of nondiagonalizable (or else, diagonalizable with complex spectrum) pseudo-Hermitian operators, the metric must be indefinite; however, as Eq. (15) does not provide us the more general form of $\eta$, we must resort to some other argument in order to confirm this conjecture.

Let us then consider the simplest $2 \times 2$ nondiagonalizable operator $A$:

$$A = \begin{pmatrix} E & 1 \\ 0 & E \end{pmatrix} (E \in \mathbb{R})$$

By a straightforward calculation one can verify that $A$ is pseudo-Hermitian and the more general operator $\eta$ which fulfils Eq. (1) is

$$\eta = \begin{pmatrix} 0 & k \\ k & k' \end{pmatrix} (k \neq 0);$$

moreover $\eta = \eta^\dagger$ if and only if $k, k' \in \mathbb{R}$. The eigenvalues of $\eta$ have with certainty opposite signs, and obviously the same happens for the $\eta$-pseudonorm
of the corresponding eigenvectors; hence some state exists with a negative \( \eta \)-pseudonorm, beside other states with a positive \( \eta \)-pseudonorm.

This simple example disproves a recently stated theorem according to which “(a nondiagonalizable operator) \( H \) is pseudo-Hermitian if and only if it is Hermitian with respect to a positive semi-definite inner product”\(^7\).

Actually, the following theorem holds.

**Theorem 2.** Let \( H \) be a \( \eta \)-pseudo-Hermitian operator with discrete spectrum. Then, the operator \( \eta \) is definite if and only if \( H \) is diagonalizable with real spectrum.

**Proof.** Let \( H \) be a pseudo-Hermitian operator. We preliminarily observe that, being in any case \( \eta \) an invertible operator, all its eigenvalues must be different from zero, so that the metric induced by the inner product (16) either is definite or is indefinite. Now, let us suppose that a positive (respectively, negative) definite operator \( \eta \) exists which fulfils condition (1); then, an \( R \) exists such that \( \eta = R^{\dagger}R \) (respectively, \( \eta = -R^{\dagger}R \))\(^10\), and by Eq. (1) we obtain

\[
RHR^{-1} = R^{\dagger-1}H^{\dagger}R^{\dagger} = (RHR^{-1})^{\dagger},
\]

i.e., \( RHR^{-1} \) is Hermitian, hence it is diagonalizable and it has a real spectrum. Since the similarity transformations preserve the properties of the spectrum, the same occurs for \( H \). Conversely, if \( H \) is diagonalizable with real spectrum, then by Eq. (15) in the previous section a positive definite metric \( \eta = SS^{\dagger} \) exists which fulfils condition (1) (since in this case \( U = V \equiv 1 \) ).\( \blacksquare \)
4 Nondiagonalizable operators and antilinear symmetries

A very intriguing feature of the pseudo-Hermiticity property is its connection with the existence of antilinear symmetries. This connection was already acknowledged to hold in the case of diagonalizable operators with discrete spectrum \([3, 8]\): indeed, the pseudo-Hermiticity property is a necessary and sufficient condition for a (diagonalizable) operator \(H\) to admit an antilinear (involutory) symmetry \([3]\). Considering the great physical interest in the study of such symmetries (we recall that the time-reversal symmetry is associated, in complex quantum mechanics, with an antilinear operator), we intend here to inquire the above-mentioned connection in the case of nondiagonalizable pseudo-Hermitian operators. To this end, let us premise a definition.

**Definition.** Given the complete orthonormal basis \(\mathcal{F} = \{ |u_{m, a, i}\rangle \}\) in a Hilbert space, we call conjugation associated with it the involutory antilinear operator

\[
\Theta_{\mathcal{F}} = \sum_{m, a, i} |u_{m, a, i}\rangle K \langle u_{m, a, i}|,
\]

where the operator \(K\) acts transforming each complex number on the right into its complex conjugate.

Analogously, in the case of a complete biorthonormal basis \(\mathcal{E} = \{ |\psi_{n, a, i}\rangle, |\phi_{n, a, i}\rangle \}\),
we call conjugation associated with it the involutory antilinear operator \[ \Theta \]

\[ \Theta = \sum_{n,a,i} |\psi_{n,a,i}\rangle K \langle \phi_{n,a,i}|. \] (18)

Then, the following theorem holds.

**Theorem 3.** Let \( H \) be a linear operator. Suppose that the spectrum of \( H \) is discrete, that its eigenvalues have finite algebraic multiplicity and that (2) holds. Then the following conditions are equivalent:

i) an antilinear invertible operator \( \Omega \) exists such that \( [H, \Omega] = 0 \);

ii) \( H \) is (weakly) pseudo-Hermitian;

iii) an antilinear involutory operator \( \hat{\Omega} \) exists such that \( [H, \hat{\Omega}] = 0 \);

iv) a basis exists in which \( H \) assumes a real form.

**Proof.** i) \( \Rightarrow \) ii). Let \( \Omega \) exist such that \( [H, \Omega] = 0 \). This implies that \( [\tilde{H}, \tilde{\Omega}] = 0 \), where \( \tilde{H} = S^\dagger HS^\dagger - 1 \) and \( \tilde{\Omega} = S^\dagger \Omega S^\dagger - 1 \). Then, the linear operator

\[ \tilde{\eta} = V \Theta \tilde{\Omega} \]

(where \( \tilde{\Omega} \) is the orthonormal basis associated with \( \tilde{H} \) (see Eq. (10)), while \( V \) and \( \Theta \) are defined as in Eqs.(12) and (17), respectively) fulfils the condition stated by Eq. (1), hence \( \tilde{H} \) is (weakly) pseudo-Hermitian; indeed,
\[
V \Theta_3 \tilde{\Omega} \tilde{H} \tilde{\Omega}^{-1} \Theta_3^{-1} V^{-1} = V \Theta_3 \tilde{H} \Theta_3 V = \\
= \sum_{n_0} d_{n_0} \sum_{a=1}^{p_{n_0, a}} (E_{n_0} \sum_{i=1}^{p_{n_0, a}}} |u_{n_0}, a, i\rangle \langle u_{n_0}, a, i| + \sum_{i=1}^{p_{n_0, a}}} |u_{n_0}, a, i + 1\rangle \langle u_{n_0}, a, i|) + \\
\sum_{n_+} \sum_{n_-, a=1}^{p_{n_+, a}} (E_{n_+}^* |u_{n_+, a, i}\rangle \langle u_{n_+, a, i}| + E_{n_-} |u_{n_-}, a, i\rangle \langle u_{n_-}, a, i|) + \\
\sum_{n_+} \sum_{n_-} \sum_{a=1}^{p_{n_+, a}} (E_{n_+}^* |u_{n_+, a, i + 1}\rangle \langle u_{n_+, a, i + 1}| + E_{n_-} |u_{n_-}, a, i + 1\rangle \langle u_{n_-}, a, i + 1|) = \tilde{H}^\dagger,
\]

Finally, posing \( \eta = S \tilde{\eta} S^\dagger = SV \Theta_3 S^\dagger \Omega \) one obtains

\[
\eta H \eta^{-1} = SV \Theta_3 S^\dagger \Omega (SV \Theta_3 S^\dagger \Omega)^{-1} = S \tilde{H}^\dagger S^{-1} = H^\dagger.
\]

(ii) \( \Rightarrow \) (iii). If \( H \) is (weakly) pseudo-Hermitian, the eigenvalues of \( H \) are either real or come in complex-conjugate pairs and the geometric multiplicity and the Jordan dimensions of the complex-conjugate eigenvalues coincide (see the remark below Theorem 1). Then, one can easily see, recalling the definition of the operator \( U \) provided in the proof of Theorem 1 (Eq.(11)) and Eqs. (10) and (17), that

\[
\Theta_3 \tilde{H} \Theta_3 = U \tilde{H} U.
\]

Hence the antilinear operator
\[ \tilde{\Omega} = \Theta_\theta U = U \Theta_\theta = \sum_{n_0} d_{n_0} \sum_{a=1}^{p_{n_0,a}} |u_{n_0}, a, i\rangle K \langle u_{n_0}, a, i| + \]

\[ \sum_{n_+, n_-} \sum_{a=1}^{d_{n_+} p_{n_+,a}} \sum_{i=1}^{d_{n_-} p_{n_-,a}} (|u_{n_+}, a, i\rangle K \langle u_{n_-}, a, i| + |u_{n_-}, a, i\rangle K \langle u_{n_+}, a, i|) \]

commutes with \( \tilde{H} \). Moreover, \( \tilde{\Omega} \) is involutory, as one can immediately verify by using the explicit expression of \( \tilde{\Omega} \) in the previous equation. Then, it follows immediately (recalling Eq. (18) and observing that \( \Theta_\theta S^\dagger = S^\dagger \Theta_\xi \) ) that

\[ \tilde{\Omega} = S^\dagger - 1 \tilde{\Omega} S^\dagger = S^\dagger - 1 U \Theta S^\dagger = S^\dagger - 1 U S^\dagger \Theta_\xi \]

(19)

commutes with \( H \) and is involutory.

\( iii \) \( \Rightarrow \) \( iv \). (See Prop.5 in [3], where an analogous statement has been proven, referring to diagonalizable operators).

If we denote by \( L \) the linear part of \( \tilde{\Omega} \), i.e., \( \tilde{\Omega} = L K \) (where \( K \) is the complex conjugation operator), then \( \tilde{\Omega}^2 = 1 \) implies \( LL^* = 1 \) and this is possible if and only if an \( M \) exists such that \( L = MM^{*-1} \) [1]. Then \( [H, \tilde{\Omega}] = 0 \) implies \( HMM^{*-1} = MM^{*-1}H^* \), hence

\[ M^{-1}HM = (M^{*-1}H^*M^*) = (M^{-1}HM)^*. \]

\( iv \) \( \Rightarrow \) \( i \). Trivially, every operator which assumes a real form in some basis \( \mathfrak{B} \) commutes with the conjugation associated with \( \mathfrak{B} \).\]
Remark. Note that the equivalence $i) \iff iv)$ we proven above clearly restates precisely a similar (seemingly, more general) result in literature, according to which whenever $H$ commutes with an antiunitary symmetry $A$ such that $A^{2k} = 1$ ($k$ odd), it is possible to construct a basis in which the matrix elements of $H$ are real. \[12\]

5 The Kramers degeneracy

On the basis of the above-stated theorem (in particular, by the implication $i) \Rightarrow ii)$ ) one can conclude that any time-reversal invariant (diagonalizable or not) Hamiltonian $H$ must belong to the class of pseudo-Hermitian Hamiltonians. The converse does not hold in general, since not always one can interpret the antilinear symmetry $\Omega$ of $H$ as the time-reversal operator $T$ ; furthermore, it is well known that in case of fermionic systems

\[ T^2 = -1 \]

and the above theorem, whereas it assures the existence of an involutory antilinear symmetry, does not say anything about the existence of a symmetry like $T$.

In order to go more deeply into the matter, we can now state the following

Theorem 4. Let $H$ be a linear operator with a discrete spectrum. Then, the following conditions are equivalent:

The following conditions are equivalent:
i) an antilinear operator $\mathcal{T}$ exists such that $[H, \mathcal{T}] = 0$, with $\mathcal{T}^2 = -1$;

ii) $H$ is pseudo-Hermitian and the Jordan blocks associated with any real eigenvalue occur in pair [i.e., for any couple $E_{n_0}, a$, the number $k(n_0, a)$ is even (see Sec. 2)].

**Proof.** Let us assume that condition i) holds; then, by Theorem 3, $H$ is pseudo-Hermitian, hence its eigenvalues are either real or come in complex-conjugate pairs and the geometric multiplicity and the Jordan dimensions of the complex-conjugate eigenvalues coincide (see Theorem 2).

Let now $|\psi_{n_0}, a, 1\rangle$ be an eigenvector of $H$: then, $\mathcal{T}|\psi_{n_0}, a, 1\rangle$ too is an eigenvector of $H$, corresponding to the same eigenvalue $E_{n_0}$, and linearly independent from $|\psi_{n_0}, a, 1\rangle$. (Indeed, assume that $\mathcal{T}|\psi_{n_0}, a, 1\rangle = \alpha |\psi_{n_0}, a, 1\rangle$ for some $\alpha \in \mathbb{C}$; applying $\mathcal{T}$ one gets $|\psi_{n_0}, a, 1\rangle = -|\alpha|^2 |\psi_{n_0}, a, 1\rangle$, which is impossible.)

If $|\psi_{n_0}, b, 1\rangle$ is another eigenvector of $H$, linearly independent from $|\psi_{n_0}, a, 1\rangle$ and $\mathcal{T}|\psi_{n_0}, a, 1\rangle$, also $\mathcal{T}|\psi_{n_0}, b, 1\rangle$ is linearly independent from all three; otherwise, applying once again $\mathcal{T}$ to the relation

$$
\alpha |\psi_{n_0}, a, 1\rangle + \beta \mathcal{T}|\psi_{n_0}, a, 1\rangle + \gamma |\psi_{n_0}, b, 1\rangle + \delta \mathcal{T}|\psi_{n_0}, b, 1\rangle = 0
$$

we could eliminate, for instance, $\mathcal{T}|\psi_{n_0}, b, 1\rangle$, thus obtaining a linear dependence between $|\psi_{n_0}, a, 1\rangle, \mathcal{T}|\psi_{n_0}, a, 1\rangle$ and $|\psi_{n_0}, b, 1\rangle$, contrary to the previous hypothesis.

We can conclude, iterating this procedure, that the geometric multiplicity
$d_{n_0}$ of $E_{n_0}$ must be necessarily even. Moreover, one can always assume that, for a suitable choice of the basis vectors, $\mathcal{T}\psi_{n_0}, a, 1 \equiv \psi_{n_0}, a', 1$ for some $a'$.

Let us consider now the subset of vectors $\{|\psi_{n_0}, a, i\}, i = 1, ... p_{n_0, a}\}$. They constitute a basis in the subspace associated with the Jordan block $J_a(E_{n_0})$; then by hypothesis one has

$$\sum_{i=1}^{p_{n_0, a}} \alpha_i |\psi_{n_0}, a, i\rangle = 0 \iff \alpha_i = 0 \forall i = 1, ... p_{n_0, a}.$$ 

Applying $\mathcal{T}$ to the previous equation, one obtains

$$\sum_{i=1}^{p_{n_0, a}} \alpha_i^* \mathcal{T}\psi_{n_0}, a, i \rangle = 0 \iff \alpha_i = 0 \forall i = 1, ... p_{n_0, a}.$$ 

hence, the vectors $\{|\mathcal{T}\psi_{n_0}, a, i\rangle \equiv |\psi_{n_0}, a', i\rangle, i = 1, ... p_{n_0, a}\}$ too are linearly independent, and $p_{n_0, a} = \dim J_a(E_{n_0}) \leq p_{n_0, a'} = \dim J_{a'}(E_{n_0})$. On the other hand, applying $\mathcal{T}$ to the basis vectors $\{|\mathcal{T}\psi_{n_0}, a, i\rangle\}$ of the subspace associated with $J_{a'}(n_0)$, one obtains that the dimensions of the two blocks must coincide, hence $J_a(n_0)$ and $J_{a'}(n_0)$ are identical.

(Alternatively, the same result can be obtained by applying $\mathcal{T}$ to both members of Eq. (4)).

Conversely, let condition ii) hold; then $H$ assumes the form
\[ H = \sum_{n_0} \sum_{a=1}^{d_{n_0}/2} E_{n_0} \sum_{i=1}^{p_{n_0,a}} \left( |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i| + |\psi_{n_0}, a + d_{n_0,a}/2, i\rangle \langle \phi_{n_0}, a + d_{n_0,a}/2, i| \right) + \right. \\
\left. \sum_{n_0} \sum_{a=1}^{p_{n_0,a}} \left( |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i+1| + |\psi_{n_0}, a + d_{n_0,a}/2, i\rangle \langle \phi_{n_0}, a + d_{n_0,a}/2, i+1| \right) + \right. \\
\left. \sum_{n_0} \sum_{a=1}^{p_{n_0,a}} \left( |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i| + E^*_{n_0} |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i| \right) + \right. \\
\left. \sum_{n_0} \sum_{a=1}^{p_{n_0,a}} \left( |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i+1| + |\psi_{n_0}, a, i+1\rangle \langle \phi_{n_0}, a, i| \right) \right]. \\
\]

Let us denote by \( \mathcal{T} \) the following antilinear operator:

\[ \mathcal{T} = \sum_{n_0} \sum_{a=1}^{d_{n_0}/2} \sum_{i=1}^{p_{n_0,a}} \left( |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a + d_{n_0,a}/2, i| - |\psi_{n_0}, a + d_{n_0,a}/2, i\rangle \langle \phi_{n_0}, a, i| \right) + \right. \\
\left. \sum_{n_0} \sum_{a=1}^{d_{n_0}/2} \sum_{i=1}^{p_{n_0,a}} \left( |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a, i| - |\psi_{n_0}, a, i\rangle \langle \phi_{n_0}, a + d_{n_0,a}/2, i| \right) \right], \tag{20} \]

where the operator \( K \) acts transforming each complex number on the right into its complex-conjugate. Then, one easily obtains, by inspection, that \([H, \mathcal{T}] = 0\) and \( \mathcal{T}^2 = -1 \). ■

Recalling that the algebraic multiplicity of any \( E_n \) is \( g_n = \sum_{a=1}^{d_a} p_{n,a} \), from Theorem 4 in particular it follows that whenever a pseudo-Hermitian operator \( H \) admits an antilinear symmetry \( \mathcal{T} \) with \( \mathcal{T}^2 = -1 \), both the geometric and the algebraic multiplicity of any real eigenvalue of \( H \) is even.

The above-mentioned theorem generalizes an analogous theorem stated from the authors (and referring to diagonalizable pseudo-Hermitian operators)\(^9\).
which in turn generalizes from various point of view the Kramers theorem on
the degeneracy of any fermionic (Hermitian) Hamiltonian. Hence, by an abuse
of language, we will continue to denote as "Kramers degeneracy" this typical
feature of real eigenvalues of pseudo-Hermitian operators admitting a symmetry
like $\mathcal{T}$.

6 Concluding remarks

Basing on Theorem 4, we can quickly test the $T$-invariance properties of pseudo-
Hermitian Hamiltonians. Indeed, let us consider for instance the operator

$$H_{\text{eff}} = \begin{pmatrix} E & ir \\ is & E \end{pmatrix} (E, r, s \in \mathbb{R})$$

which we already discussed elsewhere [9], and which arises in the modified Mash-
hoon model [13], where one introduces a ($T$-violating) spin-rotation coupling to
explain the muon’s anomalous $g$ factor.

This Hamiltonian (as long as it is diagonalizable) is time-reversal violating
[9]; however, for some choice of parameter values (for instance, $r \neq s = 0$),
$H_{\text{eff}}$ is no longer diagonalizable. Now, on the basis of Theorem 4 we can
conclude that also for such values $H_{\text{eff}}$ cannot admit an antilinear symmetry
$\mathcal{T}$ such that $\mathcal{T}^2 = -1$ (hence, $H_{\text{eff}}$ cannot be $T$-invariant). In fact, being
the geometric multiplicity of its eigenvalue $E$ odd, condition ii) of Theorem
4 does not hold. We recall however that we obtained the same result by a
straightforward calculation [4].

Finally, we note that in a symmetry-adapted basis \{ |\psi_n \rangle , \mathcal{X} |\psi_n \rangle \} the matrix of any pseudo-Hermitian operator \( H \), satisfying condition \( ii \) of Theorem 4, assumes a symplectic form. This property, in the Hermitian case, is often used in order to simplify some electronic-structure calculations occurring for instance in molecular or solid-state physics. [14]

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