Analysis of the quadratic criterion of identification quality when using RLC models of electrotechnical devices and systems

A N Kirichenko¹, A V Bubnov¹, V V Barskov¹ and A V Schekochikhin²
¹Omsk State Technical University, 11 Mira ave., Omsk, 644050, Russia
²Nizhnevartovsk State University, Nizhnevartovsk, Russia

Abstract. The article presents the results of mathematical analysis of quadratic criterion of identification quality for RLC models, which are used in determination of parameters and characteristics of electrotechnical devices and systems. It is shown, that for RLC models of voltages resonance and of currents resonance in the presence of at least two harmonic components of the current in a given frequency band, the problem of minimizing the identification quality criterion has only thing a solution. For each of these models, the conditions for convergence to the optimal solution are determined.

1. Introduction

The solution of problems of identifying the parameters and characteristics of electric power systems, the separate electrotechnical devices (ED) in many cases is based on the use of identification models in the form of passive, linear RLC two-terminal networks. [1-3]. The use of such models makes it possible to increase the reliability of determining the parameters and characteristics of electrotechnical systems and their elements under conditions of distortion the measurements by interferences of different nature based on a more complete account of a priori information [4,5]. Also, the use of RLC models is relevant for modeling oscillatory processes in mechanical systems using the method of electro-mechanical analogies, objects and structures of microelectronics and nanotechnology [6,7].

In work [8] as basic elements for structural synthesis of RLC models of identification it is offered to use models of voltages resonance (VR) and of currents resonance (CR) which schemes are presented in Figure 1. The simplicity and versatility of these models allows to provide an effective solution to a wide range of tasks related to parameter diagnostics and the control ED and systems.

![Figure 1. Models of voltages resonance (VR) and of currents resonance (CR)](image)

2. Statement of the problem
For a given model structure (model VR or CR), the problem of identification of ED parameters can be solved as a problem of minimization of the quadratic criterion of identification quality [8]

\[ F(x) = \frac{1}{T} \int_0^T \left[ u^*(t) - u(t, x) \right]^2 dt \rightarrow \min, \quad (1) \]

where \( u^*(t) \) is the voltage measured at the input terminals of ED; \( u(t, x) \) is the voltage at the input of the identification model; \( x \) is the parameter vector of the model; \( T \) is a registration period of voltage and current signals.

The purpose of this work is to determine the necessary and sufficient conditions for the existence of the solution of the problem (1), as well as the conditions for convergence of algorithms for optimal parametric synthesis of VR and CR models to this solution. To achieve this goal, it is necessary to obtain the gradient \( \nabla F(x) \) and the Hessian matrix \( \nabla^2 F(x) \) of the quadratic identification quality criterion [8] for each model and analyze the corresponding function of the RLC model parameters based on the determination of the characteristics of the Hessian matrix at stationary points.

3. VR model

After substituting the right side of VR model equation

\[ u(t) = \dot{R}i(t) + L \frac{di(t)}{dt} + \frac{I}{C} \int i(t)dt \quad (2) \]

into the formula (1) the criterion of identification quality would be written as follows:

\[ F(x) = \frac{1}{T} \int_0^T \left[ u(t) - (\dot{R}i(t) + L \frac{di(t)}{dt} + \frac{I}{C} \int i(t)dt) \right]^2 dt. \quad (3) \]

If we will denote \( G = \frac{I}{LC} \) and square the integrand, then, with provision for the sine functions’ property of orthogonality, the quality criterion would be transposed into the integrals’ sum

\[ F(x) = \frac{1}{T} \int_0^T \left[ u^2(t) dt - \frac{2R}{T} \int_0^T u(t)i(t)dt - \frac{2L}{T} \int_0^T u(t) \frac{di(t)}{dt} dt - \frac{2G}{T} \int_0^T (u(t)) \int i(t)dt dt + \frac{R^2}{T} \int i^2(t)dt \right] \]

\[ + \frac{L^2}{T} \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt + \frac{2LG}{T} \int_0^T \left( \frac{di(t)}{dt} \right) i(t)dt + \frac{G^2}{T} \int \left( i(t) \right)^2 dt. \quad (4) \]

Analytical expressions for determining the first and second partial derivatives of the quality criterion (4) are given in the Appendix.

First of all, it should be noted that the identification quality criterion (4) is a quadratic function, since the values of the second partial derivatives \( F(x) \) do not depend on the model parameters.

In order to determine the conditions for convergence of problem (1) to the optimal solution, it is advisable to use the Sylvester’s criterion [9]. In accordance with criterion the Hessian matrix \( \nabla^2 F(x) \) is determined positively if all elements in its principal diagonal are positive and all its leading principal minors are positive as well.

Principal diagonal’s elements values of \( \nabla^2 F(x) \) matrix are constant and positive for all values \( x \):
\[
\frac{\partial^2 F(x)}{\partial R^2} = \frac{2}{T} \int_0^T i^2(t) dt > 0; \tag{5}
\]

\[
\frac{\partial^2 F(x)}{\partial L^2} = \frac{2}{T} \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt > 0; \tag{6}
\]

\[
\frac{\partial^2 F(x)}{\partial G^2} = \frac{2}{T} \int_0^T \left( \int i(t) dt \right)^2 dt > 0. \tag{7}
\]

Respectively, the first and the second leading principal minors would be positive as well:

\[
\frac{\partial^2 F(x)}{\partial R^2} = \frac{2}{T} \int_0^T i^2(t) dt > 0; \tag{8}
\]

\[
\frac{\partial^2 F(x)}{\partial R^2} \cdot \frac{\partial^2 F(x)}{\partial L^2} = \frac{4}{T^2} \int_0^T i^2(t) dt \cdot \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt > 0. \tag{9}
\]

The third leading principal minor could be written as

\[
\det \nabla^2 F(x) = \frac{\partial^2 F(x)}{\partial R^2} \cdot \frac{\partial^2 F(x)}{\partial L^2} \cdot \frac{\partial^2 F(x)}{\partial G^2} - \frac{\partial^2 F(x)}{\partial R^2} \cdot \left( \frac{\partial^2 F(x)}{\partial L \partial G} \right)^2 = \frac{8}{T^3} \int_0^T i^2(t) dt \left( \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt \cdot \int_0^T \left( \int i(t) dt \right)^2 dt - \int_0^T \left( \frac{di(t)}{dt} \right) \left( \int i(t) dt \right) dt \right)^2. \tag{10}
\]

Here, in accordance with the Cauchy-Bunyakovsky inequality [10]:

\[
\int_0^T \left( \frac{di(t)}{dt} \right)^2 dt \cdot \int_0^T \left( \int i(t) dt \right)^2 dt \geq \int_0^T \left( \frac{di(t)}{dt} \right) \left( \int i(t) dt \right) dt. \tag{11}
\]

The inequality (11) is being transposed to the absolute equality, that is, the third leading principal minor (\( \nabla^2 F(x) \) matrix’s determinant) turns into zero only if there is one harmonic component of the affecting current in the defined frequency band. In that case, \( \nabla^2 F(x) \) matrix is semi-defined for all \( x \) values, that is, the problem (1) has the infinite amount of solutions. With a few (at least two) harmonic components of the affecting current in the defined frequency band, the difference between the left and the right sides of inequality (11) greater than zero, that is, the third leading principal minor of the Hessian matrix is positive.

For example, if the frequency band contains two current’s harmonic components

\[
i(t) = I_1 \sin(k_1 \omega t + \psi_1) + I_2 \sin(k_2 \omega t + \psi_2), \tag{12}\]

the inequality (11) would transpose into the following form after the fundamental frequency period integration actions:

\[
I_1^2 k_1^2 + I_2^2 k_2^2 + I_1 I_2 \left( \frac{k_1^2 + k_2^2}{k_1 k_2} \right) \geq (I_1^2 k_1 + I_2^2 k_2)^2. \tag{13}
\]
Since \( k_1 \neq k_2 \), the difference between the left and the right sides of inequality (13) is always greater than zero:

\[
I_1^2 T_2 \left( k_1^2 - k_2^2 \right)^2 \overline{k_i k_j} > 0.
\]  

(14)

So, with a few harmonic components of the affecting current in the defined frequency band, the Hessian matrix \( \nabla^2 F(x) \) is constant and positively determined for all \( x \) values. Consequently, the identification quality criterion (4) is a quadratic, convex function of the VR model’s parameters and the problem of optimal parametric synthesis has only thing a solution, correspondent to the minimum of the function.

4. CR model

After substituting the right side of CR model equation

\[
u(t) = Ri(t) - RC \frac{du(t)}{dt} + L \frac{di(t)}{dt} - LC \frac{d^2 u(t)}{dt^2}.
\]

(15)

into the formula (1) the criterion of identification quality would be written as follows:

\[
F(x) = \frac{1}{T} \int_0^T \left[ u(t) - (Ri(t) - RC \frac{du(t)}{dt} + L \frac{di(t)}{dt} - LC \frac{d^2 u(t)}{dt^2}) \right]^2 dt.
\]

(16)

If we will square the integrand, then, with provision for the property of orthogonality, the quality criterion would be transposed into the integrals’ sum

\[
F(x) = \frac{1}{T} \int_0^T u^2(t) dt - \frac{2R}{T} \int_0^T u(t)i(t) dt - \frac{2L}{T} \int_0^T i^2(t) dt + \frac{R^2}{T} \int_0^T u^2(t) dt + \frac{2LC}{T} \int_0^T \frac{d^2 u(t)}{dt^2} dt
\]

\[
+ \left( \frac{2L^2 C^2}{T} \right) \int_0^T \left( \frac{d^2 u(t)}{dt^2} \right)^2 dt - \left( \frac{2L^2 C^2}{T} \right) \int_0^T \frac{d^2 u(t)}{dt^2} dt \int_0^T \frac{d^2 u(t)}{dt^2} dt + \frac{R^2 C^2}{T} \int_0^T \left( \frac{du(t)}{dt} \right)^2 dt
\]

\[
- \frac{2R^2 C^2}{T} \int_0^T \frac{du(t)}{dt} dt.
\]

(17)

The same as in the previous case, in order to determine the conditions for convergence to the optimal solution of problem (1) we will use the Sylvester’s criterion.

Principal diagonal’s elements values of \( \nabla^2 F(x) \) matrix are constant and positive for all \( x \) values.

In fact

\[
\frac{\partial^2 F(x)}{\partial R^2} = \frac{2}{T} \int_0^T (i(t) - C \frac{du(t)}{dt})^2 dt > 0;
\]

(18)

\[
\frac{\partial^2 F(x)}{\partial L^2} = \frac{2}{T} \int_0^T \left( \frac{di(t)}{dt} - C \frac{d^2 u(t)}{dt^2} \right)^2 dt > 0;
\]

(19)
\[
\frac{\partial^2 F(x)}{\partial C^2} = \frac{2L^2}{T} \int_0^T \left( \frac{d^2 u(t)}{dt^2} \right)^2 dt + \frac{2R^2}{T} \int_0^T \left( \frac{du(t)}{dt} \right)^2 dt > 0. \tag{20}
\]

Respectively, the first and the second leading principal minors would be positive as well
\[
\frac{\partial^2 F(x)}{\partial R^2} = \frac{2T}{T^2} \int_0^T \left( i(t) - C \frac{du(t)}{dt} \right)^2 dt > 0; \tag{21}
\]
\[
\frac{\partial^2 F(x)}{\partial R^2} \cdot \frac{\partial^2 F(x)}{\partial L^2} = \frac{4T}{T^2} \int_0^T \left( i(t) - C \frac{du(t)}{dt} \right)^2 dt \int_0^T \left( \frac{di(t)}{dt} - C \frac{d^2 u(t)}{dt^2} \right)^2 dt > 0. \tag{22}
\]

The third leading principal minor could be determined from the formula
\[
\text{det } \nabla^2 F(x) = \frac{\partial^2 F(x)}{\partial R^2} \cdot \frac{\partial^2 F(x)}{\partial L^2} - \frac{\partial^2 F(x)}{\partial R \partial C} \cdot \frac{\partial^2 F(x)}{\partial L \partial C} - \frac{\partial^2 F(x)}{\partial R \partial C} \cdot \frac{\partial^2 F(x)}{\partial L \partial C} \cdot \frac{\partial^2 F(x)}{\partial R \partial C}
\]
\[
= \frac{8}{T^3} \left[ \int_0^T \left( i(t) - C \frac{du(t)}{dt} \right)^2 dt \int_0^T \left( \frac{di(t)}{dt} - C \frac{d^2 u(t)}{dt^2} \right)^2 dt \int_0^T \left( \frac{du(t)}{dt} \right)^2 dt - R \int_0^T \left( i(t) \frac{du(t)}{dt} \right)^2 dt \right. \\
\left. - \int_0^T \left( i(t) - C \frac{du(t)}{dt} \right)^2 dt \int_0^T \frac{du(t)}{dt} \left( \frac{du(t)}{dt} - C \frac{d^2 u(t)}{dt^2} \right)^2 dt - 2RC \int_0^T \frac{du(t)}{dt} \left( \frac{du(t)}{dt} - C \frac{d^2 u(t)}{dt^2} \right)^2 dt \int_0^T \frac{du(t)}{dt} \right] \tag{23}
\]

As a result of multiple calculations it was established that with the defined harmonic influence on the CR model input, the third leading principal minor (23) could be positive or negative, depending on R, L and C parameters’ values. For example, if voltage on the CR model’s input has three harmonic components
\[
u(t) = \sin \omega t + 0.8 \sin(2\omega t + \frac{\pi}{8}) + 0.5 \sin(3\omega t + \frac{\pi}{10}), \tag{24}\]

then
\[
\text{det } \nabla^2 F(x) = \begin{cases} 
-30.42 & \text{for } x = (100; 0.1; 1.10^{-3})^T; \\
2.38 & \text{for } x = (25; 0.08; 0.16 \cdot 10^{-5})^T. 
\end{cases} \tag{25}
\]

Thus, with different x values the Hessian matrix \( \nabla^2 F(x) \) is determined positive, if \( \text{det } \nabla^2 F(x) > 0 \), or its sign could be undefined, if \( \text{det } \nabla^2 F(x) < 0 \) [9]. Subsequently, the identification quality criterion (16) represents a non-convex function of the CR model parameters. However, in connection with the fact that the Hessian matrix could not be determined negatively (the first leading principal minor is positive for all x values), the quality criterion (17) presents a single-extremum function, the stationary points of which correspond either the minimum point or the saddle points. That is, just as with the use of the VR model, the problem of minimizing the quadratic quality criterion of identification for the CR model has a unique solution. However, when searching for this solution, it is necessary to take into account that the condition of convergence does not hold for x values located in the vicinity of the saddle points of the identification quality criterion.
5. Conclusion
In the result of the held analysis it was established that when using the VR model, the quadratic quality criterion of identification is a convex, quadratic function of the model parameters. Therefore, provided that there are at least two harmonic components of the affecting current in a given frequency band, the problem of minimizing the identification quality criterion has a unique solution corresponding to the minimum of this function.

When using the CR model, the quality criterion is a single-extremum, non-convex function of the model parameters. Therefore, the task of minimizing the quality criterion also has a unique solution corresponding to the minimum of this function. However, in this case, it is necessary to provide conditions for convergence to the optimal solution for the values of model parameters located in the neighborhood of the saddle points of the identification quality criterion.

On the basis of the obtained results developed the effective algorithms of the optimal parametric synthesis of the VR and CR models of ED [8].

6. Appendix

6.1. Analytical expressions for determining the values of partial derivatives of the identification quality criterion when using VR model (G = I IC)

6.1.1. First partial derivatives

\[
\frac{\partial F(x)}{\partial R} = 2 \left( \frac{T}{T} \int_0^T u(t) i(t) dt + R \int_0^T i^2(t) dt \right);
\]

\[
\frac{\partial F(x)}{\partial L} = 2 \left( \frac{T}{T} \int_0^T u(t) i(t) dt + L \left( \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt + G \int_0^T \left( \frac{di(t)}{dt} \right) i(t) dt \right) dt \right); \quad (A.1)
\]

\[
\frac{\partial F(x)}{\partial G} = 2 \left( \frac{T}{T} \int_0^T u(t) i(t) dt + G \left( \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt + \int_0^T i(t) dt \right) dt \right). \quad (A.2)
\]

6.1.2. Second partial derivatives

\[
\frac{\partial^2 F(x)}{\partial R^2} = 2 \frac{T}{T} \int_0^T i^2(t) dt; \quad (A.4)
\]

\[
\frac{\partial^2 F(x)}{\partial L^2} = 2 \frac{T}{T} \int_0^T \left( \frac{di(t)}{dt} \right)^2 dt; \quad (A.5)
\]

\[
\frac{\partial^2 F(x)}{\partial G^2} = 2 \frac{T}{T} \int_0^T \left( \int_0^T i(t) dt \right)^2 dt; \quad (A.6)
\]

\[
\frac{\partial F^2(x)}{\partial R \partial L} = \frac{\partial F^2(x)}{\partial L \partial R} = 0; \quad (A.7)
\]

\[
\frac{\partial F^2(x)}{\partial R \partial G} = \frac{\partial F^2(x)}{\partial G \partial R} = 0; \quad (A.8)
\]

\[
\frac{\partial F^2(x)}{\partial L \partial G} = \frac{\partial F^2(x)}{\partial G \partial L} = 2 \frac{T}{T} \int_0^T \left( \frac{di(t)}{dt} \right) i(t) dt. \quad (A.9)
\]
6.2. Analytical expressions for determining the values of partial derivatives of the identification quality criterion when using CR model

6.2.1. First partial derivatives

\[
\begin{align*}
\frac{\partial F(x)}{\partial R} &= \frac{2}{T} \left( -T \int_0^T u(t) i(t) dt + R \int_0^T (i(t) dt)^2 - RC \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt - 2RC \int_0^T \frac{d}{dt} u(t) i(t) dt \right) ; \\
\frac{\partial F(x)}{\partial L} &= \frac{2}{T} \left( -T \int_0^T \frac{d}{dt} u(t) u(t) dt + C \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt + L \int_0^T \left( \frac{d}{dt} i(t) \right)^2 dt + L \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt \right) \\
&\quad - 2LC \int_0^T \frac{d}{dt} u(t) \frac{d}{dt} i(t) dt ; \\
\frac{\partial F(x)}{\partial C} &= \frac{2}{T} \left( L \int_0^T \frac{d}{dt} u(t) u(t) dt + CL \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt + R \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt - R^2 \int_0^T \frac{d}{dt} u(t) i(t) dt \right) \\
&\quad - L^2 \int_0^T \frac{d}{dt} u(t) \frac{d}{dt} i(t) dt .
\end{align*}
\]

(A.10) \hspace{1cm} (A.11) \hspace{1cm} (A.12)

6.2.2. Second partial derivatives

\[
\begin{align*}
\frac{\partial^2 F(x)}{\partial R^2} &= \frac{2}{T} \left( T \int_0^T \frac{d}{dt} (i(t) dt)^2 - 2C \int_0^T \frac{d}{dt} u(t) i(t) dt + C^2 \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt \right) ; \\
\frac{\partial^2 F(x)}{\partial L^2} &= \frac{2}{T} \left( T \int_0^T \left( \frac{d}{dt} i(t) \right)^2 dt - 2C \int_0^T \frac{d}{dt} i(t) \frac{d}{dt} u(t) dt + C^2 \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt \right) ; \\
\frac{\partial^2 F(x)}{\partial C^2} &= \frac{2}{T} \left( L^2 \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt + R^2 \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt \right) ; \\
\frac{\partial^2 F(x)}{\partial R \partial L} &= \frac{\partial^2 F(x)}{\partial L \partial R} = 0 ; \\
\frac{\partial^2 F(x)}{\partial R \partial C} &= \frac{\partial^2 F(x)}{\partial C \partial R} = \frac{2}{T} \left( 2RC \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt - 2R \int_0^T \frac{d}{dt} u(t) i(t) dt \right) . \\
\frac{\partial^2 F(x)}{\partial L \partial C} &= \frac{\partial^2 F(x)}{\partial C \partial L} = \frac{2}{T} \left( T \int_0^T \frac{d}{dt} u(t) i(t) dt + 2LC \int_0^T \left( \frac{d}{dt} u(t) \right)^2 dt - 2L \int_0^T \frac{d}{dt} u(t) \frac{d}{dt} i(t) dt \right) .
\end{align*}
\]

(A.13) \hspace{1cm} (A.14) \hspace{1cm} (A.15) \hspace{1cm} (A.16) \hspace{1cm} (A.17) \hspace{1cm} (A.18)

7. References

[1] Liu S, Li Y, Xiang J and Ji F 2016 *IET Power Electronics* Vol. 9 Iss. 10 pp 2095-2102

[2] Ferrigno L, Laracca M and Pietrosanto A 2008 *IEEE Trans. on Industrial Electronics* Vol. 57 Iss. 11 pp 2513-2521

[3] Herrera W, Aponte G, Pleite J and Gonzalez-Garcia C 2013 *Int. Journal of Electrical Power & Energy Systems* Vol. 53 pp 643-648
[4] Lemoine M 1977 *E.D.F. Bull. de la direction des etudes et rech.* pp. 5-27

[5] Heindl M, Tenbohlen S, Velásquez J, Kraetge A and Wimmer R 2010 *Int. Conf. on Condition Monitoring and Diagnosis (Tokio)* A7-3 pp 201-204

[6] Körner J, Reiche1 C F, Büchner B, Mühl T and Gerlach G 2016 *Journal of Sensors and Sensor Systems* 5, pp 245-259

[7] Gusev V G, Chernikov I G and Mirina T V 2007 *Measurement Techniques* Vol. 50, Iss. 11, pp 1197–1200

[8] Kirichenko A N, Goryunov V N, Schekochikhin A V and Barskov V V 2014 *IEEE Conf. Dynamics of Systems, Mechanisms and Machines (Omsk)*

[9] Vasilyev F P 1980 *Numerical Methods of Extreme Problems Solution* (Moscow: Science) 518 p

[10] Bronstein I N and Semendyaev K A 1986 *Reference Book of Mathematics for Engineers and University Students* (Moscow: Science) 544 p