Chiral symmetry breaking and restoration from
holography

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Abstract
We study intersection of $N_c$ color D4 branes with topology $R^{1,3} \times S^1$ with radius $R_\tau$ of $S^1$ with $N_f$ Dp-branes and anti-Dp branes in the strong coupling limit in the probe approximation. The resulting model has $U(N_f) \times U(N_f)$ global symmetry. This is interpreted as the chiral symmetry group in this geometric setting. We see an $n$ dimensional theory for $n$ overlapping directions between color and flavor branes. At zero temperature we do see the breakdown of chiral symmetry, but there arises a puzzle: we do not see any massless NG boson to the break down of this symmetry group for $n = 2, 3$ for a specific p. At finite temperature we do see the restoration of chiral symmetry group and the associated phase transitions are of first order. The asymptotic distance of separation, $L$, between the quarks increases as one moves deep into the bulk at zero temperature case. At finite temperature the distance between the quarks do not vary much but as one comes close to the horizon the distance starts to decrease very fast. The chiral symmetry restoration is described by a curve, which connects $L$ with the temperature, $T$. In general this quantity is very difficult to compute but if we evaluate it numerically then the curve is described by an equation $LT = c$, where $c$ is a constant and is much smaller than one. It means for $L/R_\tau$ above $2\pi c$ there occurs the deconfined phase along with the chiral symmetry restored phase.

We also study another system in which we have added an electric field to the color D4-branes at finite temperature. This model shows up chiral symmetry breaking depending on the value of the electric field and a dimensionless parameter, which is related to the temperature of the background. But in this case there appears a puzzle: it exhibits chiral symmetry restoration in the confining phase.
1 Introduction

The study of “QCD” through ADS/CFT has been an interesting step to understand some of the mysteries of the universe. This QCD that we are after do not looks to be in the same class of real QCD. From the study of strong nuclear interaction there are three important properties that we need to understand better. Those are Confinement, Chiral symmetry breaking and asymptotic freedom and most importantly all these in the limit of small number of color degrees of freedom. However, our study till now has been limited only to the case when we have large number color degrees of freedom i.e. in the class of Maldacena limit or duality. In this latter class of models where we try to view all these interesting properties obeyed by the strong nuclear interaction in terms of geometric language and this geometric language is necessarily a low energy effective language. So it means that it is harder to see asymptotic freedom in this language. More importantly, to reproduce real QCD means one need to go to small color limit. Nevertheless the confinement and chiral symmetry breaking can be studied at low energy effective theories in the large color limit. Even with this shortcomings of large color limit where we have to sacrifice one of the interesting properties of strong nuclear interaction in the geometric sense, we still have the other two interesting properties with us and the hope is that one might be able to understand asymptotic freedom of QCD by making an ultraviolet completion of this theory i.e. by embedding this low energy effective theory described in a geometrical way in a dual theory which has got a nice UV behavior in the sense of seeing the asymptotic freedom. If that is going to be case then why not try to understand in a much better way the properties that one would like to see in the low energy effective theory. With this aim to understand the confinement and chiral symmetry breaking let us proceed and get some understanding of these phenomena and their phase diagram by considering some toy models [1, 16]. Before getting into that we should not forget that we are studying these two phenomena in a theory that is plagued by the presence of Kaluza-Klein modes. In order to get a pure gluonic theory one need a procedure to make these KK modes heavy and can be integrated out, but in practice its becoming very difficult to achieve [5, 6, 7], which is studied in the context of [4]. Hence, we shall assume that they are not there or are integrated out in some region of parameter space. With all these drawbacks in mind we hope to understand some of mass formula, relation among couplings and some other properties that could be useful in trying to understand the real QCD.

The model that we shall be considering are those that are described in a geometrical
way in the presence of fluxes coming from either NS-NS or R-R sector of closed string or both. So, the degrees of freedom are those coming from metric, $g_{MN}$, dilaton, and various p-from field strength. We shall call these as the background geometry which may or may not be showing confinement. This background geometry is obtained by taking the near horizon limit of $N_c$ Dp-branes in the large 't Hooft coupling limit and large $N_c$ limit. For this background geometry the fields that reside on the world volume of the p-brane transforms adjointly under the color group.

To add matter fields which transform (bi-)fundamentally a prescription was proposed by Karch-Katz [2]. This involves adding some branes to this background geometry. The result of this is that the fields that reside on the open string that stretch between these two branes transform under the product gauge group as bifundamentally. The addition of branes is going to change the energy momentum of the system and will back react on the background. In order to avoid it we shall take the number of these later branes to be very small in comparison to the branes that gives us the background geometry. This approximation is called as probe-brane approximation [3, 9–16]. In this setting the chiral symmetry breaking/restoration is understood in two different ways for a specific background geometry. In one case [8] the common directions along which both the color and probe (flavor) branes are not extended correspond to the plane of rotation of some group and it is this global rotation group correspond to the chiral symmetry group. For an example, this group is U(1) if the background geometry is described by the near horizon limit of a stack D4 branes and the flavor brane is a bunch of D6 branes with 4 overlapping directions and 4 non-overlapping directions. In the other case [16] the chiral symmetry group is again a global symmetry group but realized as the gauge symmetry of the flavor branes. In this paper we shall use the later approach to chiral symmetry group to study some phase diagram involving chiral symmetry breaking/restoration and confinement/deconfinement transition. To understand chiral symmetry breaking/restoration, using this geometric language, we need to compute the action of the flavor brane with different embeddings which will give us information about this symmetry.

By taking the background geometry as that of the near horizon limit of D4 branes [1] gives us a confining theory which is geometrically described in terms of metric, dilaton and a 4-form magnetic flux. This smooth background has an interesting property which is not shared by any other known confining background geometry is that the metric is completely diagonal and computation in this geometry becomes very easy. The radial coordinate do not stay in the range from zero to infinity rather from some non-zero value
$u_0$ to $\infty$. This is taken to avoid closed time like curves in the background. The topology of this geometry roughly is $R^{1,3} \times S^1 \times R \times S^4$, where the color D4 branes are extended along $R^{1,3}$ and wrapped around the $S^1$. Now putting a bunch of probe Dp-branes whose worldvolume directions can be extended along any directions except the $S^1$, which we have mentioned very clearly later. The study of this probe brane is achieved by taking a bunch of Dp-$\bar{D}p$ branes and it gives an interesting way to break the chiral symmetry. This symmetry is generated by realizing the gauge symmetry of $N_f$ Dp and $\bar{D}p$ branes, which is $U(N_f) \times U(N_f)$. The break down of chiral symmetry means breaking this product group down to its diagonal subgroup. From the dynamics of the Dp-brane point view, if we had put $N_f$ Dp and anti-Dp branes at two different positions on $S^1$ whose other ends are stuck to the boundary of the background geometry then this corresponds to a situation with $U(N_f) \times U(N_f)$ global symmetry seen on the $N_c$ number of coincident D4-branes. From the study of the action of the flavor branes, which is governed by DBI and CS action, one gets a configuration in which the flavor branes are not anymore attached to $S^1$ i.e. not sitting at $u_0$ but are rather taking a $U$-shaped configuration which is at a $\bar{u}_0$ distance away from the core of the $N_c$ D4-branes. For this kind of configuration it means that the global symmetry group is now broken down to its diagonal subgroup $U(N_f)$. The topology of the flavor brane configurations is that of a $U$-shaped configuration in which the Dp-branes and anti-Dp branes are joined together at some radial coordinate $\bar{u}_0$. This $\bar{u}_0$ can be greater or equal to $u_0$ as the later one corresponds to the minimum value of radial coordinate. Among these $U$-shaped configurations, the one that has highest energy when $\bar{u}_0 = u_0$ and the energy starts to decreases when $\bar{u}_0$ deviates away from $u_0$, at zero temperature. So, there will be breakdown of chiral symmetry at zero temperature due to the choice of our brane embeddings. Hence, we see that the study of chiral symmetry in the confining phase yields the break down of the chiral symmetry group. i.e. the chiral symmetry breaking phase stays in the the confining phase.

If we Wick rotate this background geometry then we generate a black hole solution at a temperature $T$ and it breaks confinement. Now probing again this background by a bunch of coincident $N_f$ Dp-branes and anti-Dp branes, as previously, we see that there is a restoration of chiral symmetry group. In this case there arises two distinct configurations: one whose topology is that of a U-shaped configuration and the other is two straight parallel branes sitting at two different point on a circle. So, we can say depending on the value of $T$ we have a nice phase diagram. In the current scenario we can have two phase transitions, one is associated to confinement-deconfinement transition solely between the
zero temperature background and the black hole background. The second is the chiral
symmetry breaking or restoration as seen through the flavor branes. These different
phases can be combined among themselves as Confining (C) + Chiral symmetry braking
($\chi_{sb}$), deconfinement (DC)+($\chi_{sb}$), DC+chiral symmetry restoration ($\chi_{sr}$) and finally C +
$\chi_{sr}$. In the above D4/Dp/$\bar{D}p$ model there is no C+ $\chi_{sr}$ phase in which the system can
stay, which supports (21) in the sense of large $N_c$ limit.

The confinement-deconfinement transition happens at a temperature $T = 1/(2\pi R_\tau)$,
where $R_\tau$ is the radius of $S^1$ and the chiral symmetry breaking-restoration happens at
T=$c/L$, where L is the asymptotic distance of separation between Dp and anti-Dp branes
and $c$ is numerical factor, which in our case is less than one. From this it is interesting
to note that for $L = 2\pi c R_\tau$ one can’t distinguish the phases, whereas for L staying a
bit above $L = 2\pi c R_\tau$ the DC and $\chi_{sr}$ happen together and for L staying from 0 to a
bit less than $L = 2\pi c R_\tau$ the dominated phase is that of C+$\chi_{sb}$, DC+$\chi_{sb}$. The C+$\chi_{sb}$,
DC+$\chi_{sb}$ phases can be separated by the temperature T of the background. For T less than
$1/(2\pi R_\tau)$ the dominating phase is C+$\chi_{sb}$. When the temperature is above $1/(2\pi R_\tau)$ but
less than T=$c/L$ then the preferred phase is DC+$\chi_{sb}$. Hence there is a clear distinction
between the C+$\chi_{sb}$, DC+$\chi_{sb}$ and DC+$\chi_{sr}$ phases.

Another result that follows from this study is that the asymptotic distance of separa-
tion, L, between the quarks increases as one moves deep into the bulk at zero temperature
case. At finite temperature the distance between the quarks do not vary much but as one
approaches close to the horizon the distance starts to decrease very fast. It is in this
region, which is very deep in the bulk, the distance changes very fast both at zero and
finite temperature.

The scale of confinement depends on the parameters of geometry, for $N_c$ color D4-
branes it depends on $g_s N_c$ and on the minimum value of the radial coordinate $u_0$ by
eq(3.5). The chiral symmetry breaking scale depends on the expectation value of fermion
bilinear. If we take it roughly as the distance of separation between the lowest point of
the U-shaped configuration to the minimum value of the radial coordinate. Then one
can distinguish these two scales. More interestingly one can make the chiral symmetry
braking as high as possible by just moving up this lowest point towards the boundary.
Explicitly, the mass , in units of $\alpha' = 1$, is

$$m = \frac{1}{2\pi} \int_{u_0}^{\bar{u}_0} du \sqrt{-g_{\bar{u}0} g_{uu}} = u_0[0.137 - \frac{1}{6\pi} \text{Beta}[l^3, 2/3, 1/2]] \tag{1.1}$$

where $0 \leq l \leq 1$. From eq (3.3) and using the figures (6), (7) and (8), we can rewrite
the tension of fundamental string as $T_{F1} < \frac{4\pi}{2\pi L_{\text{lt}}}$ for some value of $l$, where $L_{\text{lt}}$ is the asymptotic separation between the flavor branes in the zero temperature phase.

The plan of the paper is: in section 2, we shall give the brane configuration and the limit in which we are going to work. In section 3, the background geometry and in section 4, the chiral symmetry breaking in the zero temperature phase and the associated NG boson and the puzzle. The puzzle is that for a specific kind of brane configuration, the CS action becomes important to study the fluctuation associated to gauge fields and the massless NG boson. But unfortunately we do not see NG boson for some specific cases, which is very puzzling. In section 5, we shall study the chiral symmetry restoration at finite temperature. In section 6, we shall try to understand the chiral symmetry breaking/restoration in a different background and confinement-deconfinement transitions. In particular by introducing another electric charge to the near horizon limit of $N_c$ D4-branes by the charging up procedure used in black hole physics. The phase transition point in the phase diagram is studied with respect to the temperature $T$ and the charge, which is related to the boost parameter. So, the confinement-deconfinement transition is between the electrically charged non-confining black hole background and the confining background and the parameter that describes this transition are: boost parameter and $\frac{\beta}{2\pi R_c}$. In this case we see that when the boost parameter takes a value above 0.064 then the system stays in the confining phase irrespective of the value of $\frac{\beta}{2\pi R_c}$. The chiral symmetry breaking/restoration transition is achieved with both the temperature and the boost parameter. For any finite boost we do see there is a chiral symmetry breaking and restoration. It means we see the appearance of both the confining phase and the chiral symmetry restored phase together, which is very confusing. We conclude in section 8. In section 9, we have studied chiral symmetry breaking/restoration and massless NG boson for $QCD_4$ case in detail and in the last section we have worked out the details of the inclusion of the electric field in the in the D4 background by uplifting the solution to 11 dimensional theory, then boosting the solution and again reducing to generate the new solution.

2 The brane configuration

To study the chiral symmetry breaking, we are considering an intersecting brane configuration in Type IIA theory. The way we shall proceed is by considering a bunch of coincident $N_c$ D4-branes wrapped around the 4th directions with radius $R_c$ and extended
along (0123) directions. After taking the low energy limit we see the background is magnetically charged under a 4-form antisymmetric field. This background shows confining behavior. Let us add matter fields in the (bi)-fundamental representation to this via probe-brane approximation.

The kind of branes that we shall add are of the coincident $N_f$ Dp-brane and $N_f$ anti-Dp branes stuck at two different points along the 4th direction and also are separated along some other spatial directions. The configuration looks as

| Branes(p) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $QCD_n$ |
|-----------|---|---|---|---|---|---|---|---|---|---|--------|
| D4:       | x | x | x | x |   |   |   |   |   |   |        |
| D4, $\bar{D}4$: | x | x | x | x |   | x |   |   |   |   | $QCD_4$ |
| D6, $\bar{D}6$: | x | x | x | x |   | x | x |   |   |   | $QCD_4$ |
| D8, $\bar{D}8$: | x | x | x | x | x | x | x |   |   |   | $QCD_4$ |
| D4, $\bar{D}4$: | x | x |   | x |   |   |   | x |   |   | $QCD_3$ |
| D6, $\bar{D}6$: | x | x | x |   | x | x | x | x |   |   | $QCD_3$ |
| D2, $\bar{D}2$: | x | x |   | x |   |   |   |   | x |   | $QCD_2$ |
| D4, $\bar{D}4$: | x | x | x | x | x | x | x | x | x |   | $QCD_2$ |
| D6, $\bar{D}6$: | x | x | x | x | x | x | x | x | x |   | $QCD_2$ |

(2.2)

The 4th direction is compact, 5th direction is the radial direction and 6, 7, 8 and 9th direction makes an $S^4$. The $SO(1,9)$ symmetry is broken by the color $N_c$ D4-branes to $SO(1,4) \times SO(5)$. The flavor $N_f$ Dp-branes and anti-Dp branes also break $SO(1,9)$ down to its subgroup. Let $n$ be the number of directions common to both color and flavor branes. Then the field theory, before taking the low energy limit, is $n$-dimensional. Which we have written as $QCD_n$. The symmetry that is preserved both by the color and flavor brane in general has a structure that of $SO(1,n-1) \times SO(p-n+1)$, for $n=2,3,4$. There could possibly be another group and its not difficult to figure out its structure. For an example the $D2 - \bar{D}2$ for $QCD_2$ case the $SO(1,9)$ symmetry is broken to $SO(1,1)_{01} \times SO(2)_{23} \times SO(4)_{6789}$. The flavor branes are separated by a distance $L$ and the directions along which they are separated can be figured out from (2.2), for example the $D2 - \bar{D}2$ for $QCD_2$ case the flavor branes are separated along (2346789) directions. There is one more important point to note is that there is no chirality for $QCD_3$ case, in the sense of having Weyl fermions.
This brane intersection in the low energy limit gives various fields that arises from p-p, p-\bar{p}, \bar{p}-\bar{p}, 4-p, 4-\bar{p} strings. As suggested in [19] in the limit of $N_c \to \infty, g_s \to 0$ and fixing $g_s N_c, N_f$ with $N_f << N_c, L >> l_s$, the coupling of p-p, p-\bar{p}, and \bar{p}-\bar{p}, strings goes to zero and becomes non-dynamical sources. Hence, the degrees of freedom in the low energy limit and in the above limiting case is described by the 4-4, 4-p and 4-\bar{p} strings. The gauge symmetry of the flavor brane $U(N_f) \times U(N_f)$ becomes a global symmetry of the n dimensional theory at the intersection of the brane. The fermions that appear from the 4-p intersection transform as $(\bar{N}_f, 1)$ of the global symmetry and that of 4-\bar{p} intersection as $(1, \bar{N}_f)$. Both fermions transform in the fundamental $N_c$ representation of the color group.

The 't Hooft coupling is dimensionful and has got unit length dimension. The theory becomes weakly coupled in the $L >> \lambda$ and strongly coupled in the $\lambda >> L$ limit. Before taking the 4th direction as compact, in the near horizon limit the string coupling goes to infinity in UV. It means this kind of theories are non-normalisable. The UV completed theory [18] is described by a six dimensional (0,2) CFT compactified on a circle of radius $g_s \sqrt{\alpha'}$. In the weak coupling limit its been shown that the single gluon exchange dominates over multiple gluons between the fermions coming from the intersection of color and flavor branes, for details see [19]. We shall study this intersection in the opposite limit i.e. in the strongly coupled limit.

3 The background

In the strong coupling $\lambda >> L$ limit, it means the easiest way to study the system in this regime is by bringing the flavor branes closer. The background geometry that we are considering is that of a near horizon limit of a bunch of $N_c$ coincident D4 branes which acts as the source for the 4-form magnetic field. The topology of the geometry is $R^{1,3} \times S^1 \times S^4 \times R^1$. The geometry along $R^{1,3}$ is flat but the presence of magnetic field, which has got fluxes along $S^4$, back reacts to create warping on the 10-dimensional geometry. Hence it makes conformally flat along $R^{1,3}$. Moreover, the size of $S^1$ vanishes at the lower range of radial coordinate and there is non-trivial dilaton. Writing down the geometry explicitly in string frame

$$ds^2 = \left(\frac{u}{R}\right)^{3/2}(\eta_{\mu\nu}dx^\mu dx^\nu + f(u)d\tau^2) + \left(\frac{R}{u}\right)^{3/2}\left(\frac{du^2}{f(u)} + u^2d\Omega_4^2\right),$$
\[ e^{\phi} = g_s(u/R)^{3/4}, \quad F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \left(\frac{u_0}{u}\right)^3. \] (3.3)

The D4-branes are extended along \( x^\mu, \tau \) directions and smeared over \( S^4 \). \( \epsilon_4, V_4 \) are the unit volume form and volume of \( S^4 \) respectively. From the expression to \( f(u) \) it follows that the radial coordinate stays from \( u_0 \) to \( \infty \). Since the \( \tau \) direction is compact one can compactify the fermions to satisfy anti-periodic boundary condition [1], so as to break supersymmetry, before taking the near horizon limit. In what follows we shall also break supersymmetry explicitly by taking a situation where a brane and anti-brane has been placed on the \( S^1 \). Another important point to note that the periodicity of the \( S^1 \) is not \( 2\pi \) but rather

\[ \tau \sim \tau + \delta \tau, \quad \delta \tau = \frac{4\pi}{3} \left(\frac{R^3}{u_0}\right)^{1/2}. \] (3.4)

Given this background one can convince one self that this is a confining background with the tension of the flux tube evaluated at the minimum value to the radial coordinate is

\[ T_{F_1} = \frac{1}{2\pi \alpha'} \left(\frac{u_0}{R}\right)^{3/2}, \quad R^3 = \pi g_s N_c \left(\alpha'\right)^{3/2}. \] (3.5)

With this background and following the prescription of [13], which is one of the way to understand chiral symmetry breaking and restoration, is to probe this background with a bunch of \( N_f \) coincident Dp-branes localized on some point on \( S^1 \) and putting another bunch of \( N_f \) anti-Dp-branes on some other point of \( S^1 \). Now the global symmetry as seen from the D4-brane point of view is \( U(N_f) \times U(N_f) \), which means the two set of branes do not touch each other and breaking of this global symmetry down to its diagonal subgroup \( U(N_f) \) corresponds to joining the branes with the antibranes. It means the break down of chiral symmetry. Here there arises a question what if we had taken \( N_f \) Dp-branes and \( \tilde{N}_f \) anti-Dp branes with \( N_f \) not necessarily same as \( \tilde{N}_f \)? The possibilities are there could be a configuration with unbroken product global symmetry, i.e. \( U(N_f) \times U(\tilde{N}_f) \to U(n_f) \times U(\tilde{n}_f) \) with \( N_f, \tilde{N}_f > n_f, \tilde{n}_f \), which means the chiral symmetry is unbroken. But, if \( U(N_f) \times U(\tilde{N}_f) \to U(n_f) \) then there is a signature of chiral symmetry breaking. For \( N_f \) is not same as \( \tilde{N}_f \), it seems we can’t study that in this near horizon limit. For our purpose we shall be taking the case where we have got equal number of branes and anti-branes. This is the setting which has been studied recently in [19] in the weak coupling limit and [22, 23] in the finite temperature case for \( p=8 \) case.
4 Strongly coupled action: in the low temperature phase

The dynamics of flavor branes in the near horizon limit of $N_c$ D4 branes are described by the DBI and CS action. For the flavor branes that we are considering, it is not difficult to convince that there won’t be any CS term in the action. To compute the DBI part, we need to compute the induced metric on the worldvolume of the flavor Dp-brane. The induced metric is

$$ds_{Dp}^2 = \left(\frac{u}{R}\right)^{(3/2)}[-dt^2 + dx_{n-1}^2] + \left(\frac{u}{R}\right)^{(3/2)}[f\tau'^2 + (R/u)^3 \frac{1}{f}]du^2 + R^{(3/2)}u^{(1/2)}d\Omega_{p-n}^2, \quad (4.6)$$

where $0 \leq p-n \leq 4$ and we have fixed the p-branes at $\theta_i = \pi/2$ for simplicity, wherever it required, in particular we use the metric of $S^4$ as $d\Omega_4^2 = d\theta_1^2 + sin^2\theta_1 d\theta_2^2 + sin^2\theta_1 sin^2\theta_2 (d\theta_3^2 + sin^2\theta_3 d\phi^2)$. The DBI part of the action now becomes

$$S_{Dp}(QCD_n; LTP) = -\frac{T_p}{g_s} V_{p-n} V_n R^{(p-2n)/4} \int du (\frac{u}{R})^{(p+2n)/4} \sqrt{f(\frac{d\tau}{du})^2 + (R/u)^3 \frac{1}{f}}. \quad (4.7)$$

Since the action do not depend on $\tau(u)$ explicitly, implies the conjugate momentum associated to it is constant. Assuming the boundary condition $u(\tau = 0) \rightarrow \bar{u}_0$ with $u'(\tau = 0) = 0$ gives

$$u' = f(\frac{u}{R})^{(3/2)}[\frac{u^{(p+2n)/2} f}{\bar{u}_0^{(p+2n)/2} f_0} - 1]^{(1/2)}. \quad (4.8)$$

Integrating this one can find the most general solution. The choice of our boundary condition tells us that this configuration is a symmetric U-shaped curve in (U, $\tau$) plane. There is a special U-shaped configuration for which it goes all the way to the bottom of the $\tau$ circle, but by a slight abuse of notation we shall denote it as $||$, for which the momentum vanishes i.e. $\tau' = 0$. This means $u' = \frac{1}{\tau} = \infty$ and it happens when $u_0 \rightarrow u_0$. The function $f_0 = 1 - (\frac{u_0}{\bar{u}_0})^3$. In both the configuration the flavor branes are located at $\tau = \pm \frac{L}{2}$, asymptotically. In this case the chiral symmetry is broken explicitly before making any calculation because we have put flavor branes at two different points on the $\tau$ circle and this circle shrinks to zero size. However, to figure out which of the U-shaped configuration has got the lower energy, i.e. is it the one which stays close to the shrinking $\tau$ circle or the one which stays away from it. These two different configurations are plotted in figure (1).
Computing the value of the action for these two types of configurations and taking their difference one end up with

\[ S(U) - S(||) = -\left[ \int_{0}^{1} dz \frac{z^{\frac{p+2n+10}{12}}}{ \sqrt{1-l^3 z - (1-l^3)z^{\frac{p+2n}{6}}} - \sqrt{1-l^3 z}} \right] - \int_{1}^{\frac{u_{0}}{\bar{u}_{0}}} dz \frac{z^{\frac{p+2n+10}{12}}}{\sqrt{1-l^3 z}} \],

(4.9)

where \( l = \frac{u_{0}}{u_{0}} \). In general it is very difficult to find the difference between the two configurations analytically. So, we shall do it numerically by using Mathematica package. The parameter \( l \) can take maximum value 1, when the turning point \( \bar{u}_{0} \) coincides with \( u_{0} \), the minimum value to \( u \), and it takes minimum value when the turning point approaches the boundary i.e. \( 0 \leq l \leq 1 \). We have not given the expression to \( \left[ \right] \). This factor is positive and comes from the prefactor that multiplies the DBI action, integration over the compact and non-compact directions and doing some change of variables. we find it is given by \( \left[ \right] = \frac{T_{p}}{3g_{s}} V_{p-n} V_{n} R \frac{3(p-2n+2)}{4} u_{0}^{\frac{p+2n-2}{4}} \).

The difference between these two U-shaped configurations, \( S(||) - S(U) \) vs \( l \), is plotted in figure (2) and (3).
The difference between the actions, $S_{Dp}(||) - S_{Dp}(U)$, is plotted against $l = \frac{u}{u_0}$ for $QCD_2$ case and for $D2-\bar{D}2, D4-\bar{D}4$ and $D6-\bar{D}6$ branes in the zero temperature case.

From these plots we see that it is the action of the U-shaped configuration which stays away from the bottom of the $\tau$ circle ,||, dominates over the one which stays close to the bottom of the $\tau$ circle configuration. Stating it in terms of the energy, we find that ||-configuration has got more energy, as in the Minkowski signature the action is the negative of energy. Hence we find that at zero temperature the global symmetry $U(N_f) \times U(N_f)$ is broken to its diagonal subgroup $U(N_f)$, which is described by this U-shaped configuration. In this case the breakdown of the chiral symmetry is guaranteed due to the presence of the shrinking $\tau$-circle. However, in the cases when there is no shrinking of $\tau$-circle, still one can achieve chiral symmetry breaking as in [19]. Which means at zero temperature there is break down of chiral symmetry.

4.1 NG Boson

It is known that the break down of a global symmetry would mean the presence of a massless Nambu-Goldstone boson in the spectrum. The way to see this massless scalar in the break down of the chiral symmetry is to look for the fluctuation to the U(1) gauge potential that live on the worldvolume of flavor brane by [16]. Let us go through the procedure and see whether there appears any NG boson in the spectrum.

The fluctuation in the gauge potential to the DBI action of a Dp brane can be written
Figure 3: The difference between the straight and U-shaped action, $S_{Dp}(\|) - S_{Dp}(U)$, is plotted against $l = \frac{u_0}{u_0}$ for QCD$_3$ case for D4-$\bar{D}4$ and D6-$\bar{D}6$ branes in the zero temperature case.

as

$$S_{DBI} = T_p \int e^{-\phi} \sqrt{-\det[g]} \left( \frac{1}{4} [g]^{ab} F_{bc} [g]^{cd} F_{da} \right) + O(F^3), \quad (4.10)$$

where we have used the notation $[g]$ as the pullback of the metric onto the worldvolume of the p-brane. Exciting the fields only along the S0(1,n-1) and the radial direction sets the DBI action as

$$S_{DBI} = T_p \int e^{-\phi} \sqrt{-\det[g]} \frac{1}{4} \left( [g]^{\mu\nu} F_{\nu\rho} [g]^{\rho\sigma} F_{\sigma\mu} + 2 [g]^{\mu\nu} F_{\nu\mu} [g]^{uu} F_{uu} \right), \quad (4.11)$$

where $\mu, \nu = 0, 1, \ldots, n - 1$. There could in general be the fluctuation to gauge potential in CS action and for our purpose it is easy to see that only for D6-$\bar{D}6$ in the QCD$_2$ and QCD$_3$ case there can be a term in the CS action which supports fluctuations to quadratic order. For a D6-brane the action is

$$S_{CS} = \frac{\mu_6}{2} \int [C_3] \wedge F^2 + O(F^3) = \frac{3}{4} \mu_6 k N_c \int F^2. \quad (4.12)$$

For the QCD$_3$ case this indeed is the form of CS action, whereas for the QCD$_2$ case

$$S_{CS} = \frac{\mu_6}{2} \int [F_4] \wedge A \wedge F + O(F^3) = \mu_6 N_c \pi \int A \wedge F \quad (4.13)$$

this form of the action is useful. k is a constant that appear in the three form antisymmetric tensor. Note the branes have been fixed at $\theta_i = \pi/2$. Let us rewrite the flavor
From equation 4.19 there arises two relations for the mode \( \psi \) where

\[
F_{\mu \nu} = \frac{1}{4} \left( [g]^{\mu \nu} F_{\nu \rho} [g]^{\rho \sigma} F_{\sigma \mu} + 2 [g]^{\mu \nu} F_{\nu \alpha} [g]^{\alpha \mu} F_{\mu \nu} \right) + \frac{b}{2} \int d^3 x du e^{\rho a} (A_u F_{\rho \sigma} + 2 A_u F_{\sigma u}),
\]

where \( b = \mu_6 N_c \pi \) and \( b \) vanishes for other branes. Similarly, for \( QCD_3 \) case the action for flavor D6-brane is

\[
S_{D6}(QCD_3) = T_6 \int e^{-\phi} \sqrt{-det[g]} \left( [g]^{\mu \nu} F_{\nu \rho} [g]^{\rho \sigma} F_{\sigma \mu} + 2 [g]^{\mu \nu} F_{\nu \alpha} [g]^{\alpha \mu} F_{\mu \nu} \right) + a \int d^3 x du e^{\mu \nu \rho} (F_{\mu \rho} F_{\nu u}),
\]

with \( a = \frac{3}{4} \mu_6 k N_c \) and \( a \) vanishes for other branes. So, the fluctuated action of the flavor brane is described by the sum of DBI and CS action. Computing the DBI part of the action for our purpose we find it can be rewritten as

\[
S_{D6}(QCD_n; LTP) = - \left[ \right] \int d^m x du \left[ \frac{K_1(u)}{4} F_{\mu \nu} F^{\mu \nu} + \frac{K_2(u)}{2} \eta^{\mu \nu} F_{\mu \nu} F_{\nu \mu} \right],
\]

where

\[
K_1(u) = \frac{R^3 u^{\frac{p+2n-9}{2}}}{\sqrt{u^{\frac{p+2n}{4}} f - \bar{u}_0^\frac{p+2n}{2} f_0}}, \quad K_2(u) = R^3 u^{\frac{3}{2}} \sqrt{u^{\frac{p+2n}{2}} f - \bar{u}_0^\frac{p+2n}{2} f_0}.
\]

As usual \( \left[ \right] = \frac{T_{6g}}{4} V_{p-n} R^{\frac{p-2n+4}{4}} \) contains some over all positive constants. Expanding the gauge potential as

\[
A_{\mu}(x^\mu, u) = \sum_{n=1} B^{(n)}_{\mu}(x^\mu) \psi_n(u), \quad A_u(x^\mu, u) = \sum_{n=0} \pi^{(n)}(x^\mu) \phi_n(u),
\]

\[
F_{\mu \nu} = \sum_n F^{(n)}_{\mu \nu}(x^\mu) \psi_n(u), \quad F_{\mu u} = \sum_n (\partial_\mu \pi^{(n)}(x) \phi_n(u) - B^{(n)}_{\mu}(x) \partial_u \psi_n(u)),
\]

where \( F^{(n)}_{\mu \nu} = \partial_\mu B^{(n)}_{\nu} - \partial_\nu B^{(n)}_{\mu} \). Let us impose proper eigenvalue equation for \( \psi_n \) and normalization conditions for the mode \( \psi_n \) and \( \phi_n \). The conditions are

\[
-\partial_u (K_2 \partial_u \psi_n) = K_1 \lambda_n \psi_n, \quad \left[ \right] \int du K_1 \psi_n \psi_m = \delta_{nm}
\]

\[
\left[ \right] \int du K_2 \phi_m \phi_n = \delta_{mn}, \quad \left[ \right] \int du K_2 \partial_u \psi_n \partial_u \psi_m = \lambda_n \delta_{nm} = m_n^2 \delta_{nm}.
\]

From equation 4.19 there arises two relations

\[
\phi_n = \sqrt{\frac{K_1}{K_2}} \psi_n, \quad \phi_n = \frac{1}{m_n} \partial_u \psi_n.
\]
In order to evaluate the fluctuated action we need one more information about the zero mode of \( \phi_0 \). From the equation of motion to the DBI and CS action we find that

\[
\phi_0 = c_1/K_2(u),
\]

where \( c_1 \) is a constant. The asymptotic behavior of \( K_1(u) \) and \( K_2(u) \) are

\[
K_1(u) \to 4R^{(3/2)}u^{\frac{p+2n-18}{4}}, \quad K_2(u) \to 4R^{-(3/2)}u^{\frac{p+2n-6}{4}}.
\]

We have defined the normalization condition for \( \phi_n \) in such a way so as to generate the canonical kinetic term for \( \pi^{(n)} \), which is true for the zero mode of \( A_u \) along the radial direction i.e. \( \phi_0 \) obeys

\[
\int du K_2 \phi_0^2 = 1
\]

Now the LHS of equation (4.23) in the asymptotic limit of \( K_2 \) becomes

\[
\sim \int du \frac{1}{u^{\frac{p+2n-6}{4}}}. \quad (4.24)
\]

The range of \( p+2n \) is \( 0 \leq \frac{p+2n-6}{4} \leq \frac{5}{2} \), which means that the above integral is not normalisable in contrast to our assumption earlier. In fact for some flavor Dp-branes it diverges linearly and for some other logarithmically. This happens when \( P+2n=6 \) and \( p+2n=10 \) respectively. So, for \( QCD_2 \) case and for flavor D2 brane it diverges linearly and for flavor D6-branes it diverges logarithmically, whereas for \( QCD_3 \) case, for flavor D4-branes it diverges logarithmically. For other cases the mode \( \phi_0 \) is normalisable.

Does this mean the absence of a normalisable zero mode \( \phi_0 \) for these flavor brane cases do not have any chiral symmetry breaking? But we saw the chiral symmetry breaking in the low temperature phase explicitly. So, then where is the NG boson?

The fluctuated actions for \( QCD_2 \) and \( QCD_3 \) are

\[
S(QCD_3) = \left. \int d^3x \left[ \sum_{n=1} \left( \frac{1}{4} F_{\mu\nu}^{(n)} F^{(n)\mu\nu} + \frac{1}{2} m_n^2 B_{\mu}^{(n)} B^{(n)\mu} + \frac{1}{2} \partial_\mu \pi^{(0)} \partial^\mu \pi^{(0)} \right) \right. \right]
\]

\[
- \sum_{n,m=1} \frac{M_n^2 M_m^2}{4} \int d^3x e^{\mu\rho} F^{(n)}_{\mu\nu} B^{(m)}_{\rho\nu} + a \sum_{n=1} M_n^2 \int d^3x e^{\mu\rho} F^{(n)}_{\mu\nu} \partial_\nu \pi^{(0)}
\]

\[
S(QCD_2) = \left. \int d^3x \left[ \sum_{n=1} \left( \frac{1}{4} F_{\mu\nu}^{(n)} F^{(n)\mu\nu} + \frac{1}{2} m_n^2 B_{\mu}^{(n)} B^{(n)\mu} + \frac{1}{2} \partial_\mu \pi^{(0)} \partial^\mu \pi^{(0)} \right) \right. \right]
\]

\[
- \frac{b}{2} \sum_{n,m=1} \int d^3x e^{\rho\sigma} \left[ M_n^2 (\pi^{(0)} F^{(n)}_{\rho\sigma} + 2 B^{(n)}_{\rho} + \frac{1}{m_n} \partial_\rho \pi^{(n)} \partial_\sigma \pi^{(0)} \right]
\]

\[
M_m^2 (\frac{1}{m_n} \pi^{(m)} F^{(m)}_{\rho\sigma} - 2 B^{(m)}_{\rho} ( B^{(m)}_{\rho} + \frac{1}{m_n} \partial_\rho \pi^{(m)} )) \right],
\]

\[
(4.25)
\]
where
\[ \int du \psi_n \partial_u \psi_m = M_{nm}^2 = -M_{mn}^2, \quad \int du \phi_0 \psi_n = M_n^2. \tag{4.26} \]

From this we do get massless boson for QCD with D6-brane and anti-D6 brane case but the kinetic term for \( \pi^{(0)} \) is not canonically normalized. However we do not get any massless NG boson for QCD case. For the QCD with \( D2 - \bar{D}2 \) flavor branes there is no CS term but the mode \( \phi_0 \) is not normalized. Hence the \( \pi^{(0)} \) that appear in the DBI do not make sense and the same is true for \( D4 - \bar{D}4 \) brane in the QCD case. The \( \pi^{(0)} \) field in these cases can’t be interpreted as the NG bosons.

Let us work in a different gauge choice, \( A_u = 0 \). In this gauge choice, \( A_\mu = A_\mu - \partial_\mu \Lambda \), we can take \( \Lambda = \sum m_n \pi^{(n)} \psi_n \). Then \( A_\mu = \sum_n \left( B_\mu^{(n)} - \frac{1}{m_n} \partial_\mu \pi^{(n)} \right) \psi_n \). Computing the fluctuated action for QCD, for example, we find
\[ S(QCD) = -\int d^2 x \left( \frac{1}{4} F_{\mu \nu}^{(0)} F^{(0) \mu \nu} + \sum_n \left( \frac{1}{4} F_{\mu \nu}^{(n)} F^{(n) \mu \nu} + \frac{1}{2} m_n^2 B_\mu^{(n)} B^{(n) \mu} \right) \right) - b \sum_{n,m} \int d^2 x \epsilon^{\rho \sigma} M_{nm}^2 B_\rho^{(n)} B_\sigma^{(m)}, \tag{4.27} \]

where we have absorbed that extra factor into \( B_\mu^{(n)} \) and the sum in the expansion of \( A_\mu(x,u) \) in terms of modes runs from 0 to \( \infty \). Similarly one can compute for the QCD case in this gauge. The result is
\[ S(QCD) = -\int d^3 x \left( \frac{1}{4} F_{\mu \nu}^{(0)} F^{(0) \mu \nu} + \sum_n \left( \frac{1}{4} F_{\mu \nu}^{(n)} F^{(n) \mu \nu} + \frac{1}{2} m_n^2 B_\mu^{(n)} B^{(n) \mu} \right) \right) - a \sum_{n,m} \int d^3 x \epsilon^{\mu \nu \rho} M_{nm}^2 F_{\mu \nu}^{(n)} B_\rho^{(m)}. \tag{4.28} \]

The conclusion is again as before, we do not see any massless NG boson for QCD case.

5 Chiral symmetry at finite temperature

The probe brane computation of a bunch of coincident Dp-brane in the supergravity limit with the background that of near horizon limit of a bunch coincident \( N_c \) D4-branes at finite temperature is
\[ ds^2 = \left( \frac{u}{R} \right)^{3/2} \left( f(u) dt^2 + \sum_{i=1}^3 \delta_{ij} dx^i dx^j + d\tau^2 \right) + \left( \frac{R}{u} \right)^{3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right), \]
\[ e^\phi = g_s \left( \frac{u}{R} \right)^{3/4}, \quad F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \left( \frac{u_T}{u} \right)^3. \tag{5.29} \]
The periodicity of the Euclidean time direction is
\[ \beta = \frac{4\pi}{3} \left( \frac{R^3}{u_T} \right)^{1/2}. \] (5.30)

The Euclidean action of probe Dp brane in this background is
\[ S_{Dp}(QCD_n) = \frac{T_p}{g_s} V_{p-n} R^{(3p-2n)}/4 \int du \sqrt{f u^{p+2n} \sqrt{\tau^2} + (1/f)(R/u)^3}, \] (5.31)
where the induced metric is
\[ ds_{Dp}^2 = \left( \frac{u}{R} \right)^{(3/2)} [f dt^2 + dx_{n-1}^2] + \left( \frac{u}{R} \right)^{(3/2)} \left[ \tau^2 + (R/u)^3 \frac{1}{f} \right] du^2 + R^{(3/2)} u^{(1/2)} d\Omega_{p-n}^2. \] (5.32)
and we have fixed the flavor brane at \( \theta_i = \pi/2 \), whenever required, for simplicity. Since the action do not depend upon \( \tau(u) \) explicitly, it means the corresponding momentum is conserved. As before, there are two independent configurations which solve the equation of motion. One is when \( \tau' = 0 \) or its inverse \( \frac{du}{d\tau} \) diverges. This corresponds to a straight configuration and in this case we do not break the global symmetry \( U(N_f) \times U(N_f) \). To remind about this symmetry, we have taken Dp-brane and anti-Dp brane and put them at two different points on the compact \( \tau = x^4 \) circle. The second configuration is that of a U-shaped solution. This solution is described by a curve in the \( u, \tau \) plane and is
\[ u' = \sqrt{f(u/R)^{3/2} \left[ u^{p+2n} f - \frac{1}{f_0} \right]}^{1/2}, \] (5.33)
where \( f_0 = 1 - \left( \frac{u_T}{u_0} \right)^3 \). The U-shaped brane configuration joins Dp-brane with anti-Dp brane at \( \tilde{u}_0 \).

In order to determine the lowest energy configuration, we have to find the difference between the actions and determine which configuration is preferred from it. Substituting this velocity in the action for the U-shaped configuration, we find
\[ S(U) = \left[ \right] \int_{u_T}^{\infty} du \frac{u^{p+2n-6} \sqrt{f}}{\sqrt{f - \left( \frac{\tilde{u}_0}{u} \right)^{2n} f_0}}. \] (5.34)

Similarly the action for the straight Dp-brane and anti-Dp-brane is
\[ S(||) = \left[ \right] \int_{u_0}^{\infty} du \frac{u^{p+2n-6}}{4}. \] (5.35)

Note the integration for the straight configuration case can be written as a sum of two pieces one from \( u_0 \) to \( \tilde{u}_0 \) and the other from \( \tilde{u}_0 \) to \( \infty \). Doing some change of variables and
finding the difference between the actions, we find

\[ S(U) - S(||) = \left[ \int_0^1 dz z^{\frac{p+2n+10}{12}} \left( \frac{1 - t^3 z}{1 - t^3 z - (1 - t^3) z^{\frac{p+2n}{6}}} - 1 \right) - \right. \]
\[ \left. \frac{12}{p + 2n - 2} \left( 1 - t^{\frac{p+2n-2}{4}} \right) \right], \]  

(5.36)

where \( 0 \leq t = \frac{u_T}{u_0} \leq 1 \). Now, finding the difference is not easy, so instead we plot the difference using Mathematica package. The difference between the action is plotted separately for different value of \( n \) in figures 4 and 5.

![Graph showing the difference between two possible solutions](image)

Figure 4: The difference between the two possible solutions, \( S_{Dp}(U) - S_{Dp}(||) \), is plotted against \( t = \frac{u_T}{u_0} \) for QCD2 case and for different flavor branes: D2-\( \bar{D}2 \), D4-\( \bar{D}4 \) and D6-\( \bar{D}6 \), in the finite temperature phase.

Let the asymptotic distance between the flavor Dp-branes and anti-Dp branes be \( L_{HTP} \) in the finite temperature case and \( L_{LTP} \) for the zero temperature case. The expressions for them are

\[ L_{HTP} = \frac{2}{3} \sqrt{\frac{R^3}{u_0}} \sqrt{1 - t^3} \int_0^1 dz \frac{z^{\frac{p+2n-10}{12}}}{\sqrt{1 - t^3 z} \left( 1 - t^3 z - (1 - t^3) z^{\frac{p+2n}{6}} \right)} \]  

(5.37)

\[ L_{LTP} = \frac{2}{3} \sqrt{\frac{R^3}{u_0}} \sqrt{1 - l^3} \int_0^1 dz \frac{z^{\frac{p+2n-10}{12}}}{\left( 1 - l^3 z \right) \left( 1 - l^3 z - (1 - l^3) z^{\frac{p+2n}{6}} \right)} \]  

(5.38)

Let us summarize the behavior of chiral symmetry breaking with the dependence on \( n \) and \( p \) and mention the value of \( t \) where this happens, the temperature, the asymptotic
Figure 5: The difference between the actions for the two possible solutions, $S_{Dp}(U) - S_{Dp}(||)$, is plotted against $t = \frac{u_T}{u_0}$ for $QCD_3$ and for flavor brane: $D4-\bar{D}4$ and $D6-\bar{D}6$ in the finite temperature case.

In general for a given type of color brane, in our case the temperature is related to $t = \frac{u_T}{u_0}$ as

$$ T = \frac{3}{4\pi} \sqrt{t} \frac{1}{\sqrt{R^3/u_0}}. \quad (5.40) $$

| $QCD_n$ | Dp-$\bar{D}p$ | $t_c = (\frac{u_T}{u_0})_c$ | $T_c \sqrt{\frac{R^3}{u_0}}$ | $\frac{L_c}{\sqrt{R^3/u_0}}$ | $L_c T_c$ | $\frac{L_c}{R_c}$ |
|--------|-------------|-----------------|-----------------|-----------------|--------|---------|
| 2      | 2           | 0.47951         | 0.165314        | 1.68167         | 0.278  | 1.74675 |
| 2      | 4           | 0.55317         | 0.177558        | 1.3218          | 0.234696 | 1.47464 |
| 2      | 6           | 0.613           | 0.186914        | 1.09676         | 0.205  | 1.28805 |
| 3      | 4           | 0.613           | 0.186914        | 1.09676         | 0.205  | 1.28805 |
| 3      | 6           | 0.6622          | 0.19427         | 0.9439          | 0.183371 | 1.15216 |
| 4      | 4           | 0.6621          | 0.194255        | 0.943903        | 0.183358 | 1.15207 |
| 4      | 6           | 0.7025          | 0.2             | 0.833931        | 0.166786 | 1.04795 |
| 4      | 8           | 0.73573         | 0.204772        | 0.751283        | 0.153842 | 0.966616 |
For $p=2,4,6$ and $n=2$ in the LTP

$\frac{L}{\sqrt{\frac{R^3}{u_0}}}$ is plotted against $l = \frac{u_0}{\bar{u}_0}$ for D2-$\bar{D}2$, D4-$\bar{D}4$ and D6-$\bar{D}6$ flavor branes for the $QCD_2$ case in zero temperature phase.

Denoting $T_c$ as the critical temperature at which there occurs phase transition and imposing the condition that at this temperature there also occurs a confinement-deconfinement phase transition. The confinement-deconfinement gives an inverse relation between the critical temperature $T_c$ and the radius of the compact circle $R_\tau$.

$$T_c = \frac{1}{2\pi R_\tau}. \quad (5.41)$$

From figures (6), (7), and (8) it can observed that the asymptotic separation between the flavor branes increases as the turning point $\bar{u}_0$ approaches the minimum value i.e. $u_0$, which means the end points of the flavor Dp-branes and anti-Dp branes move away from each other as we allow these flavor branes to move deep into the bulk and when these flavor branes stay close to the boundary their separation takes the minimum value. This is true in the zero temperature phase and is true for all $n$ of $QCD_n$. However, in the finite temperature case the situation is just opposite. As long as these flavor Dp-branes and anti-Dp branes stay close to the boundary their asymptotic separation attains maximum value and as these flavor branes move deep into the bulk and comes close to the horizon their asymptotic separation between the end points decreases and takes a minimum value at the horizon.

The absolute value of the difference between the actions $||S_{Dp}(U) - S_{Dp}(||)||$ for flavor branes in the zero temperature case starts from a non-zero value then increases attains a maximum value and then decreases towards zero. It means that deep in the bulk
Figure 7: $\frac{L}{\sqrt{R^3/u_0}}$ is plotted against $l = \frac{u_0}{u_0}$ for flavor branes: D4-$\bar{D}$4 and D6-$\bar{D}$6 for the QCD$_3$ case in zero temperature case.

both the actions take same value at $u_0$ for the zero temperature case, which it should by construction, and at the horizon for the high temperature case. But there is some thing markedly different happens to this difference only at finite temperature phase that is for some special value of the turning point the actions take same value, even before reaching the horizon and the difference between the actions vanish. It is this point where there happens a phase transition. The phase transition is first order.

From the table (5.39), we see that this transition point depends on the number of common directions that these flavor branes share with the color branes along with the dimensionality of the flavor brane. For a given $n$ the transition point increases with the increases of the dimensionality of the flavor brane. From the table we also see that the transition point for a given $n$ with the maximum value of $p$ almost coincides with the minimum value of $p$ for $n+1$. The transition temperature in a given unit increases with the increase of $p$ and $n$. Whereas the asymptotic distance of separation between the flavor Dp-brane and anti-Dp brane decreases with the increase of $p$ for a given $n$. The dimensionless quantity $L_cT_c$ decreases with the increase of $p$, so also the ratio between the asymptotic separation of flavor branes to the radius of the compact $\tau$ circle.
Figure 8: $\frac{L}{\sqrt{R^3/T_0}}$ is plotted against $l = \frac{u_0}{u_0}$ for different flavor branes, D2-$\bar{D}2$, D4-$\bar{D}4$ and D6-$\bar{D}6$ for the $QCD_4$ case at in the zero temperature phase.

6 D4+D0 background

Recently, there has been various attempts to study the phase diagram of various geometric backgrounds by introducing charges through boosting the simple solutions and also been used to generate black hole solutions with different charges. Here we shall adopt the same technique to generate additional charges to the D4 branes. The prescription is to boost the solution of D4-branes along $t, \tau$ directions and then uplift the solution to 11-dimension and reducing the solution along $\tau$ direction to generate the new solution which is electrically charged. The result will be same if we interchange the first two steps. The extra direction that we add while uplifting the solution to 11-dimension is taken to satisfy anti-periodic boundary conditions for fermions after reducing it to 10-dimension.

The system that we are going to study is a non-confining background generated from the background studied in the previous sections by introducing another charge, which would be D0-brane charge, by first boosting the solution and then uplifting it to 11-dimension and reducing it to generate a D4+D0 brane solution. We are interested to study the chiral symmetry restoration in the above mentioned probe brane approximation with temperature and with the D0-brane charge. To simplify our life henceforth, we shall be saying the boost parameter not the D0-brane charge.

The non-confining and near extremal solution is generated from the confining solution by a double Wick rotation, $t \rightarrow -it$, $\tau \rightarrow -i\tau$ of (3.3). Now applying the above mentioned
Figure 9: \( \frac{L}{\sqrt{R^3/u_0}} \) is plotted against \( t = \frac{u_T}{u_0} \) for D2-\( \bar{D} \)2, D4-\( \bar{D} \)4 and D6-\( \bar{D} \)6 in the QCD\(_2\) case and in the finite temperature phase.

The boost parameter\(^1\) \( \alpha_T \) is as usual and can take value from 0 to \( \infty \). It is interesting to note that the periodicity of the Euclidean time circle is

\[ t \sim t + \beta, \quad \beta = \beta_o c h \alpha_T, \quad \beta_o = \frac{4\pi}{3} \left( \frac{R^3}{u_T} \right)^{1/2} \]

(6.43)

and the periodicity associated to the Euclidean time of the near extremal black hole solution generated by double Wick rotation of (3.3) is \( \beta_0 \), which is the zero boost limit of (6.43).

Let us go through this procedure of boosting, uplifting and then reducing to generate an extra electric charge for the confining solution (3.3). The result of all this generates a solution which is non-confining due to the presence of electric charge. The explicit form

\(^1\)Since \( \tau \) is compact the boost should be thought to have taken in the the covering space. To avoid cluttering of the trigonometric function, we are using \( \text{Sinh}x = \text{sh}x, \text{Cosh}x = \text{ch}x \) etc.
For $p=4,6$ and $n=3$ in the HTP

Figure 10: $\frac{L}{\sqrt{R^3/u_0}}$ is plotted against $t = \frac{u_T}{u_0}$ for D4-$\bar{D}4$ and D6-$\bar{D}6$, QCD, and in the finite temperature phase.

of the solution is

$$ds^2 = (B/h)^{1/2}(u)[-Fdt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dy^2] + (Bh)^{1/2}(u)\left[\frac{du^2}{f} + u^2d\Omega_4^2\right],$$

$$e^\phi = (B^3/h)^{1/4}(u), \quad C_1 = \frac{(f(u) - 1)}{B(u)}sh_{\alpha_0}ch_{\alpha_0}dt, \quad F(u) = f/B$$

$$F_4 = \frac{2\pi N_c}{V_4}e_4, \quad f(u) = 1 - \left(\frac{u_0}{u}\right)^3, \quad h = (R/u)^3, \quad B(u) = 1 - (u_0/u)^3(ch_{\alpha_0})^2.$$  \hspace{1cm} (6.44)

The radial coordinate $u$ stays from $u_0$ to $\infty$ and when $u$ take the lowest value $u_0$, $B$ becomes negative and the solution do not make much sense as the dilaton becomes complex and the metric contains an overall phase. So, we shall be only dealing with eq \ref{6.42} for our study of chiral symmetry breaking. Also, while reducing along $\tau$ the periodicity of it is not $2\pi$.

### 6.1 Chiral symmetry restoration

We shall study chiral symmetry restoration by putting a stack of coincident D8-brane and anti-D8 brane on the compact direction $y$. As before there arises two different brane embeddings one which is straight and the other one which is curved. In the former case each set of coincident branes do not do much whereas in the latter case there is a turning point for some value of radial coordinate $u = \bar{u}_0$ and the end result is that there is U-shaped configuration and it is this configuration of probe brane gives the chiral symmetry
For $p=2, 4, 6$ and $n=4$ in the HTP

Figure 11: $\frac{L}{\sqrt{R^3/u_0}}$ is plotted against $t = \frac{u_T}{u_0}$ for flavor branes: D2-$\bar{D}$2, D4-$\bar{D}$4 and D6-$\bar{D}$6 for the $QCD_4$ case and in the finite temperature phase.

breaking. Stating all these in a different way is: the gauge symmetry on the probe D8-branes is interpreted from the D4+D0-brane point of view as a global symmetry which is nothing but the chiral symmetry. For simplicity our calculation will be done by taking a brane and anti-brane.

The probe D8-brane action which we shall use has the as usual DBI action and there won’t be any CS action as we are considering zero U(1) flux on the worldvolume of the brane. It is easy to see that this is a consistent solution. The induced metric on the D8-brane, in the static gauge choice and exciting the lone scalar along the radial direction gives

$$ds^2_{D8} = (M/h)^{1/2}(u)[-Hdt^2+dx_1^2+dx_2^2+dx_3^2]+(Mh)^{1/2}u^2d\Omega_4^2+(M/h)^{1/2}u^{4}(1+\frac{h}{f}u^2)du^2,$$

with $u' = \frac{du}{dy}$. The D8-brane action in the Euclidean signature becomes

$$S_{D8} = T_8 \int dtdx_1dx_2dx_3du^4d\Omega e^{-\phi}\sqrt{-detg_s}$$

$$= T_8V_{3+1}V_4 \int dy(HM^3)^{(1/2)}u^4\sqrt{1+\frac{h}{f}u'^2},$$

where $V_{3+1}, V_4$ are the volumes associated to $R^{1,3}, S^4$ respectively. Note we have assumed $g_s = 1$, for simplicity. The Lagrangian of the D8 brane do not depend explicitly on $y$ coordinate, which means there is a conserved quantity and is

$$\frac{d}{dy}(\frac{\sqrt{HM^3u^4}}{\sqrt{1+\frac{h}{f}u'^2}}) = 0.$$
The boundary condition at \( y = 0 \) for the \( U \)-shaped solution is \( u(0) = \bar{u}_0 \) and the condition on the ‘velocity’ is \( u'(0) = 0 \). With this condition the velocity

\[
u' = \sqrt{\frac{f/h}{f_0} \left( \frac{M}{M_0} \right)^2 \left( \frac{u}{\bar{u}_0} \right)^8 - 1}^{1/2},
\]

where \( f_0 = 1 - \left( \frac{u_T}{\bar{u}_0} \right)^3 \) and \( M_0 = 1 + \left( \frac{u_T}{\bar{u}_0} \right)^3 s h^2 \alpha_T \). The value of \( u \) for which the D8 branes turns around and merge with the anti D8brane is \( \bar{u}_0 \).

For the straight probe brane configuration in which the chiral symmetry is unbroken has got \( f_0 = 0 \), which means \( M_0 \to c h^2 \alpha_T, H_0 \to 0 \).

The energy associated to these configurations can be derived from the value of the action. Rewriting the action for the straight and \( U \)-shaped configurations, with appropriate ranges for the integrals one ends up with

\[
S_{D8}(||) = -\left[ \int_{u_T}^{\infty} u(-1/2) (u^3 + u_T^3 s h^2 \alpha_T) \right]
\]

\[
S_{D8}(U) = -\left[ \int_{\bar{u}_0}^{\infty} \frac{(u^3 + u_T^3 s h^2 \alpha_T)^2 u_T^{-3/2} \sqrt{u^3 - u_T^3}}{(u^3 - u_T^3)(u^3 + u_T^3 s h^2 \alpha_T)^2 - u f_0 M_0^2 u_0^2} \right],
\]

where \( [\ . \ ] = T_3 V_3 + V_4 R^{3/2} \). By doing some change of variables and defining \( t = \frac{u_T}{u_0} \), we can integrate the \( U \)-shaped configuration and express the integral in terms of a variable which lies from 0 to 1, whereas for the straight configuration the \( u \) integral can be written in terms of two pieces one from \( u_T \) to \( \bar{u}_0 \) and the other is from \( \bar{u}_0 \) to \( \infty \). The result of doing all these gives us

\[
\Delta S_{D8} = S_{D8}(U) - S_{D8}(||) = \left[ \int_0^{1/2} -\frac{(2/7)(1 - \alpha_T^2)}{2 s h^2 \alpha_T (t^3 - \alpha_T^2)} - \frac{1}{3} \int_0^1 dz \right]
\]

\[
z^{-13/6} (1 + t^3 s h^2 \alpha_T) \left( \frac{(1 - t^3 z)(1 + t^3 z s h^2 \alpha_T)^2}{(1 - t^3)(1 + t^3 z s h^2 \alpha_T)^2 z^{8/3} - 1} \right)
\]

(6.50)

Let the asymptotic distance between the D8-brane and anti D8-brane be denoted as \( L \), which depends on the boost parameter, and \( t \) as

\[
L = \frac{2}{3} \left( R^3 \sqrt{1 - t^3 (1 + t^3 s h^2 \alpha_T)} \right) \int_0^1 dz \frac{z^{1/2}}{(1 + t^3 z s h^2 \alpha_T)^2 z^{8/3} - (1 - t^3 z)^{8/3}}
\]

(6.51)

The chiral symmetry restoration in this case can be seen from the figure [12] for two different boosts, \( \alpha_T = 0.01, 1 \),

while that of the asymptotic separation, \( L \), between the D8-branes and anti-D8 branes, when plotted is showing the usual behavior that we saw in earlier sections, is that when
Figure 12: The difference between the two allowed configuration, $S(U) - S(||)$, is plotted against $t = \frac{u^T}{u_0}$ for D8-$\bar{D}$8 flavor branes for the QCD$_4$ case and in the finite temperature phase for $\alpha_T = 0.01$, 1.0.

the flavor brane moves very deep into the bulk i.e. into the horizon the L decreases and it increases when the flavor branes stay close to the boundary. This can be seen from the figure 13.

Figure 13: $L/\sqrt{R^3/u_0}$ is plotted against $t = \frac{u^T}{u_0}$ for D8-$\bar{D}$8 flavor branes for the QCD$_4$ case and in the finite temperature phase for $\alpha_T = 0.01$, 1.0.

7 Spacetime transition

In this section we would like to study whether there would be any spacetime transition between different geometries. There is indeed a transition between the confining geometry
and the finite temperature version of it, \(10.71\). However, the transition between either \(3.3\) and \(6.42\) or between \(10.71\) and \(6.42\) do not looks possible due to the presence of extra charge associated to boosting. But, that’s not correct. We shall show by explicit computation that there indeed happens a spacetime transition.

The on-shell action for a set of coincident D4-branes which gives confinement, when evaluated in the Einstein frame with Euclidean time gives

\[
S_{\text{confinement}} = V_4 V_3 \beta 2\pi R_\tau \frac{3}{16 k^2} \left( \frac{2\pi N_c}{V_4} \right)^2 R^{-6} \int_{u_0}^{R_\tau} du u^2, \tag{7.52}
\]

where \(V_4\) comes from integrating the four unit sphere, \(S^4\), \(V_3\) comes from integrating the three flat non-compact directions, \(\beta\) is the periodicity of Euclidean time, \(2\pi R_\tau\) is the periodicity associated to the compact \(\tau\) direction. The rest of the terms in \(7.52\) comes from evaluating the Ricci curvature, the dilaton kinetic term and the 4-form term. The integral is being regulated as there is a UV divergence and is taken from the minimum value of \(u\) which is \(u_0\) to some large value of \(u\), \(R_\tau\).

Now evaluating the on-shell action for the boosted near extremal solution \(6.42\), in the Einstein frame with Euclidean time is

\[
S_{\text{Boosted}} = V_4 V_3 \beta 2\pi R_y \frac{3}{16 k^2} \int_{u_T}^{R_\tau} du \left[ \frac{3u^2 u_T^3 s h_{\alpha_T} c h_{\alpha_T}}{u^3 + u_T^3 s h_{\alpha_T}^2} u^{-2} + 3 \left( \frac{2\pi N_c}{V_4} \right)^2 R^{-6} u^2 \right]. \tag{7.53}
\]

Again the factors like \(V_4\), \(V_3\), \(\beta\) arise from \(S^4\), three flat directions and from the periodicity of Euclidean time direction respectively. The periodicity of the compact \(y\) direction is \(2\pi R_y\). First term in \(7.53\) arises, roughly, from the 2-from field. The range of the radial coordinate is from \(u_T\) to a UV cut-off \(R_\tau\).

Computing the periodicities following Wittens prescription [1] we find that the divergences associated to each action are same and equal. So, by taking the difference between the actions one ends up with

\[
S_{\text{Boosted}} - S_{\text{confinement}} = V_4 V_3 \beta_0 c h_{\alpha_T} 2\pi R_\tau \frac{3}{16 k^2} \frac{3 u_T^3}{8 \pi^3} \left[ (u_0/u_T)^3 - \left( \frac{3}{4} + 8\pi^3 \right) s h_{\alpha_T}^2 \right]. \tag{7.54}
\]

It is easy to see that for for zero boost there is a phase transition for \(u_0 = u_T\), which is the case between the confining and deconfining phase. [22]. It means when \(u_0\) is smaller then \(u_T\), the near extremal black hole without any D0-brane charge is preferred and when \(u_T\) becomes smaller then \(u_0\), the confined or zero temperature phase is preferred.

It is interesting to note that the square bracket in \(7.54\) can be written as

\[
\left( \frac{\beta_0}{2\pi R_\tau} \right)^6 - 1 + \left( \frac{3}{4} + 8\pi^3 \right) s h_{\alpha_T}^2 = \begin{cases} 
-ve & \text{for } \alpha_T < 0.064 \text{ and for some value of } \frac{\beta_0}{2\pi R_\tau} \\
+ve & \text{for } \alpha_T \geq 0.064 \text{ and for any value of } \frac{\beta_0}{2\pi R_\tau}
\end{cases} \tag{7.55}
\]
The conclusion of this spacetime transition is that for the boost above 0.064 and for any value to $\beta_0/2\pi R_\tau$, the preferred phase is the confining phase or the zero temperature phase whereas if the boost is below 0.064 and $\beta_0/2\pi R_\tau$ is less than 0.093 then the preferred phase is that of the near extremal geometry of D4+D0 brane. If the boost is below 0.064 but $\beta_0/2\pi R_\tau$ is above 0.093 then the preferred phase is the zero temperature phase.

There appears a puzzle: for boost $\alpha_T = 1.0$ we see from figure 12, there occurs a chiral symmetry restoration but from eq. (7.55) we see that for a boost above 0.064 the system stays in the confining phase. Then how come we have chiral symmetry restoration in the confining phase?

8 Conclusion

We have studied the Sakai-Sugimoto model of chiral symmetry breaking and restoration in the near horizon limit of $N_c$ color D4 brane using Dp-brane and anti-Dp brane as flavor branes in the holographic setting. The result of phase transition, the asymptotic distance between the quarks $L$, depends on the dimensionality of the flavor branes as well as on the number of directions that color and flavor brane share. If the number of common directions that they share is $n$, then the quantities that is mentioned above depends on $p$ and $n$ in a typical way. It appears that it goes as $p+2n$. So, for $p=6$, $n=2$ and $p=4$, $n=3$, the phase diagrams and the distance $L$ are identical.

The distance $L$, at zero temperature behaves in a different way than at finite temperature. At zero temperature $L$ increases as we move away from the boundary. It takes maximum value deep in the bulk. Whereas at finite temperature $L$ decreases but very slowly as we move away from the boundary and it starts to decrease very fast when we approach the horizon. The transition point, which describes chiral symmetry restoration, increases with the increase of the dimensionality of the flavor brane in a given $n$, so also the critical temperature associated to this transition point.

The chiral symmetry restoration is described by a curve, which relates the asymptotic distance of separation between the quarks with the temperature. In general this quantity is very difficult to compute but if we evaluate it numerical then the curve is described by an equation $L T = c$, where $c$ is a constant and is much smaller than one. It means for $L/R_\tau$ above $2\pi c$ there occurs the deconfined phase along with the chiral symmetry restored phase.

There appears two puzzles in this study of confinement-deconfinement transition and
chiral symmetry breaking and restoration. One is the absence of NG boson for the $QCD_2$ when $p=2,6$ and $QCD_3$ for D4-$\bar{D}4$ case. This is due to the absence of a normalisable zero mode $\phi_0$. There could be a possibility to avoid this for $QCD_3$ case, as in odd dimensions one can’t have Weyl fermions. The other is the simultaneous existence of confinement and chiral symmetry restoration phase for boost above 0.064.

**Note added:** As we were finishing the paper there appeared [24], with which there are some overlaps.

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## 9 Appendix

### 9.1 Chiral symmetry breaking

In this section we shall study in detail the chiral symmetry breaking as discussed in the introduction in the strong coupling limit. The near horizon bulk geometry of $N_c$ D4 branes is to be probed by a bunch of coincident Dp and anti-Dp branes, for $p=4,6,8$. The worldvolume direction of D4 are along $x^\mu, u$, and that of D6 brane are $x^\mu, u$, and wrapped over the $S^2$ of $S^4$. The D8 brane is wrapped along $S^4$ and extended along $x^\mu, u$. The $S^4$ is described by $d\Omega_4^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\Omega_2^2$. D6 branes are kept at $\theta_2 = \theta_1 = \pi/2$. This means that there is no Chern-Simon part of the action to probe Dp-brane. For single probe brane the action is now described by only the DBI part

$$S_{Dp} = -\frac{T_p V_4 V_p^{-4}}{g_s} R^{3(p-8)/4} \int d\tau (p+8)/4 \sqrt{f \left( \frac{d\tau}{du} \right)^2 + \left( \frac{R}{u} \right)^2 f}.$$  \hfill (9.56)

We have excited the only scalar $\tau$, radially. It’s easy to see that the zero momentum configuration is a solution and it corresponds to a U-shaped configuration. It means the branes and anti-branes gets joined at the minimum value to $u$ which is $u_0$. By a slight abuse of notation we shall denote this by $\bar{u}$. There exists another configuration which is that of U-shaped but with non-zero momentum. The boundary condition on $u(\tau)$: as $u(\tau = 0) \rightarrow \bar{u}_0$ and $\frac{du}{d\tau}|_{\tau=0} = 0$, which makes the U-shaped configuration symmetric.
The difference to the actions in the Minkowskian signature is

\[ S_{Dp}(U) - S_{Dp}(||) = -\left[ \int_1^{u_0} dz z^{-(p+18)/12} \left( \frac{1}{\sqrt{1-l^3 z}} - \frac{1}{\sqrt{1-l^3 z}} - \frac{1}{\sqrt{1-l^3 z}} \right) - \int_1^{u_0} \frac{z^{-18/12}}{\sqrt{1-l^3 z}} \right] \]

where the U-shaped configuration with non-zero momentum has been integrated over the radial direction \( u \) from the turning point \( u_0 \) to \( \infty \) and the U-shaped with zero momentum i.e. \(|\|\)-configuration is integrated from \( u_0 \) to \( \infty \). Also, we have the changed the variable and \( l = \frac{u_0}{u} \). The factor \([\ ]\) comes from the prefactor of the Dp brane and some other numerical factor that appear from the change of variables and more importantly these terms are positive.

In general its very difficult to find the difference analytically. So, we do it numerically using Mathematica package. The difference is plotted in the following figure 14.

![Figure 14: \( S_{Dp}(||) - S_{Dp}(U) \) is plotted against \( l = \frac{u_0}{u} \) for \( p=4,6,8 \).](image)

It is interesting to note that in Minkowskian signature \( S_{Dp}(U) - S_{Dp}(||) \sim E(||) - E(U) \), for static configurations. Hence from the plot we conclude that for the embeddings in which the D4, D6 and D8 branes along with their antibranes are stuck to the \( \tau \) direction, prefers to take the U-shaped configuration, which is staying away from the \( u_0 \) line, there by minimizing its energy. In the zero temperature case the chiral symmetry \( U(N_f) \times U(N_f) \) is broken to its diagonal subgroup \( U(N_f) \) with the above choice of embeddings.

### 9.2 Goldstone boson

The breakdown of the global symmetry implies there must appear Nambu-Goldstone boson. Since, we start with the case with zero gauge potential, which is a solution to
equation of motion. Now fluctuate the gauge potential with the assumption that we have only turned on the gauge potential along $A_\mu$ and $A_u$. No gauge potential along the sphere directions. The quadratic fluctuation to gauge potential is governed by

$$ S_{D\mu} = \int d^4 x du \left[ \frac{K_1(u)}{4} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\nu\rho} F_{\mu\sigma} + \frac{K_2(u)}{2} \eta^{\mu\nu} F_{\nu\mu} F_{\mu\nu} \right], \quad (9.58) $$

where

$$ \frac{K_1(u)}{4} = \frac{u^{(p-4)/4} \sqrt{f}}{u'} \sqrt{1 + \left( \frac{R}{u} \right)^3 \frac{u^2}{f^2}} $$

$$ \frac{K_2(u)}{4} = \frac{u^{(p-4)/4} \sqrt{f}}{u'} \frac{1}{\sqrt{1 + \left( \frac{R}{u} \right)^3 \frac{u^2}{f^2}}}. \quad (9.59) $$

The gauge potentials $A_\mu(x^\mu, u)$ and $A_u(x^\mu, u)$ are assumed to be expanded in terms of complete sets of ortho-normal functions as i.e. it is assumed that this expansion solves the equation motion that follows from the quadratic action $[9.58]$

$$ A_\mu(x^\mu, u) = \sum_{n=1} B_\mu^{(n)}(x^\mu) \psi_n(u), \quad A_u(x^\mu, u) = \sum_{n=0} \pi^{(n)}(x^\mu) \phi_n(u) $$

$$ F_{\mu\nu} = \sum_{n} F_{\mu\nu}^{(n)}(x^\mu) \psi_n(u), \quad F_{\mu\nu} = \sum_{n} \left( \partial_\mu \pi^{(n)}(x) \phi_n(u) - B_\mu^{(n)}(x) \partial_\nu \psi_n(u) \right), \quad (9.60) $$

where $F(x)_{\mu\nu} = \partial_\mu B^{(n)}_{\nu} - \mu \leftrightarrow \nu$. Let us also assume the following eigenvalue equation and the orthonormalisation condition

$$ -\partial_u (K_2 \partial_u \psi_n) = K_1 \lambda_n \psi_n, \quad [ ] R^3 \int du K_1 \psi_n \psi_m = \delta_{nm} $$

$$ [ ] R^3 \int du K_2 \phi_m \phi_n = \delta_{mn} \text{for all } n \quad (9.61) $$

The condition on $\phi_n$ gives a canonically normalized kinetic term. The condition hold for $n \geq 1$, using $\lambda_n = m_n^2$

$$ [ ] R^3 \int du K_2 \partial_u \psi_n \partial_u \psi_m = \lambda_n \delta_{nm}, \quad \partial_u \psi_n = \sqrt{\lambda_n} \phi_n = m_n \phi_n, \quad \phi_n = \sqrt{\frac{K_1}{K_2}} \psi_n. \quad (9.62) $$

Substituting the above decomposition and the eigenvalue equation along the orthonormalization condition and some identification from $[9.62]$, one ends up with

$$ S_{D\mu} = -[ ] R^3 \int d^4 x \left( \sum_{n=1} \frac{1}{4} F_{\mu\rho}^{(n)} F_{\nu\rho}^{(n)} \eta^{\mu\nu} + \frac{1}{2} m_n^2 B_\mu^{(n)} B_\nu^{(n)} \eta^{\mu\nu} + \frac{1}{2} \partial_\mu \pi^{(0)} \partial_\nu \pi^{(0)} \eta^{\mu\nu} \right), \quad (9.63) $$

where we have made a gauge transformation associated to $B_\mu^{(n)}$. From the above expression it follows that the massless gauge boson becomes massive by eating the $\pi^{(n)}$ for $n \geq 1$ and one left with the massless $\pi^{(0)}$, which is interpreted as the Nambu-Goldstone boson associated to the chiral symmetry breaking.
9.3 Chiral symmetry restoration

In this section we shall study the strong coupling analysis of the dual field theory to the background studied in the previous section in the context of probe brane approximation but in the high temperature or non-extremal solution of a bunch of coincident D4 branes. 

To begin, let us do a Wick continuation of the solution (3.3) and the final non-extremal solution in Euclidean space is

\[ ds^2 = \left(\frac{u}{R}\right)^{3/2} (f(u) dt^2 + \sum_1^3 \delta_{ij} dx^i dx^j + d\tau^2) + \left(\frac{R}{u}\right)^{3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right), \]

\[ e^\phi = g_s (u/R)^{3/4}, \quad F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \left(\frac{u_T}{u}\right)^3. \] (9.64)

The periodicity of the Euclidean solution is

\[ \beta = \frac{4\pi}{3} \left(\frac{R^3}{u_T}\right)^{(1/2)}. \] (9.65)

We want to probe this background using \( D4 - \bar{D}4, D6 - \bar{D}6, D8 - \bar{D}8 \) branes i.e. a bunch of coincident Dp and anti-Dp branes have been put at two different points along the \( \tau \) circle for \( p=4, 6 \) and 8. The DBI action that govern a part of the dynamics of these branes in this background is

\[ S_{DBI} = \frac{T_p}{g_s} V_4 V_{p-4} R^{3(p-8)/4} \int u^{(p+8)/4} \sqrt{f} \sqrt{\left(\frac{d\tau}{du}\right)^2 + \left(\frac{R}{u}\right)^2} \frac{1}{f}. \] (9.66)

It is interesting to note that we are considering a single Dp -anti Dp brane. The D4 branes are extended along \( t, x^i, u \), D6 branes are extended along \( t, x^i, u \) and wrapped along an \( S^2 \) of \( S^4 \). Keeping things a bit explicitly, the \( d\Omega_4^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\Omega_2^2 \). D6 branes are kept at \( \theta_2 = \theta_1 = \pi/2 \). The term \( V_4 \) that appear in the DBI action comes from integrating suitably \( t, x^i \) directions, \( V_{p-4} \) comes from integrating the appropriate part of \( S^4 \), sphere integral. Also, we have excited only one scalar field \( \tau \) along the radial direction \( u \). There is no CS action for the case we are interested in.

From (9.66) it follows trivially that

\[ \frac{du}{d\tau} = \left[ \frac{u^{(p+8)/2} f}{f_0^{(p+8)/2} f_0} - 1 \right]^{(1/2)} \sqrt{f(u/R)^{(3/2)}}, \] (9.67)

where we have chosen the boundary condition as \( u(\tau = 0) \to u_0 \) and \( u'(\tau = 0) = 0 \). Here prime denotes derivative with respect to \( \tau \) and \( f_0 = 1 - \left(\frac{u_T}{u_0}\right)^3 \). The vanishing of the
derivative of $u$ means there is a turning point and the configuration of the flavor brane looks: it turned around at $\bar{u}_0$ and joins with the anti Dp branes forming a U-shaped configuration. There exists another configuration which is a zero 'momentum' configuration with $u'$ blowing up or the vanishing of $\frac{dr}{du}$. The previous U-shaped configuration from the point of view of $N_c$ D4 branes is that there is break down of $U(N_f) \times U(N_f)$ to its diagonal subgroup, whereas the latter zero momentum configuration implies there is no break down of this global symmetry. The following conclusion can be drawn from these two kinds of configuration. If the U-shaped configuration has got lower energy then there is a break down of chiral symmetry and if the zero momentum configuration has got lower energy then there is no break down of chiral symmetry.

Substituting back the expression of $u'$ into the action and doing change of variables, one ends up with following actions for the U-shaped and zero momentum configuration

$$S(U-\text{shaped}) = \left[ \frac{1}{3} \int_0^1 dz^3 z^{-\frac{p+18}{12}} \left( \frac{1-t^3 z}{1-t^3 z - \left(1-t^3 \right) z^{\frac{p+8}{6}}} \right)^{1/2} \right] ,$$

$$S(|| - \text{shaped}) = \left[ (4/p + 6) \left[ 1 - t^{(p+6)/4} \right] + (1/3) \int_0^1 z^{-\frac{p+18}{12}} \right] . \tag{9.68}$$

Note, we have used the notation $||$-shaped to designate the zero momentum configuration, also the meaning of $\left[ \right]$ is that there are some less important positive term comes from both the change of variable and from the prefactor that multiplies the action in (9.66).

In calculating the integrals the range of integration for the U-shaped configuration is, the radial coordinate stays from $\bar{u}_0$ to $\infty$, whereas for the $||$-shaped the radial coordinate stays from $u_T$ to $\infty$. The parameter $t = \frac{u_T}{\bar{u}_0}$.

Finally taking the difference between the actions we end up with

$$S(U) - S(||) = \left[ \frac{1}{3} \int dzz^3 \frac{1-t^3 z}{1-t^3 z - \left(1-t^3 \right) z^{\frac{p+8}{6}}} \right] - \frac{12}{p+6} (1-t^{\frac{p+6}{4}}) . \tag{9.69}$$

Analytically its very difficult to proceed with the above integral. So to draw any conclusion we shall use Mathematica package to find the difference as a function of $t$. Note that this parameter stays from 0 to 1. The zero value to $t$ means that the configuration is close to the boundary of the bulk geometry and the latter means the configuration is touching the horizon of the bulk geometry.

The difference between the actions for the U-shaped and parallel configuration is plotted in figure (15).

For $p=4, 6, 8$, the critical value to $T$ are $t_c(p = 4) = 0.6221, t_c(p = 6) = 0.7025, t_c(p = 8) = 0.73573$. So, the pattern that emerges is that for increasing the the number of world-
volume direction of the brane makes the critical value of $t_c$ to move towards right i.e. increases, which means the configuration move towards the horizon of the bulk geometry. Naively, we can interpret this as: increasing the value of $p$ means increasing the tension of the brane and more massive it becomes and hence move away from the boundary.

Let the distance between the two Dp and anti-Dp brane be $L$. Then this distance $L$ is related to the turning point $\bar{u}_0$ and the horizon of the solution $u_T$ as

$$L = \frac{2}{3}(\frac{R^3}{\bar{u}_0})^{(1/2)} \sqrt{1 - t^3} \int_0^1 dz \frac{z^{\frac{p-2}{12}}}{\sqrt{1 - t^3 z} \sqrt{1 - t^3 z - (1 - t^3)z^{\frac{p-2}{12}}}} \quad (9.70)$$

Computing the distance of separation at the critical point $T_c$ we see that the distance of separation decreases with the increase of the dimension of the world volume. For our case $L_c(p = 4) = 0.943903, L_c(p = 6) = 0.833931, L_c(p = 8) = 0.751283$ in units of $(R^3/\bar{u}_0)^{(1/2)}$.

The critical inverse temperature at which there occurs the chiral symmetry breaking is $\beta_c(p = 4)5.14867, \beta_c(p = 6) = 5.0, \beta_c(p = 8) = 4.89065$ in units of $(R^3/\bar{u}_0)^{(1/2)}$. It means that the critical temperature at which there occurs chiral symmetry breaking increases with increasing the dimension of the world-volume direction in units of $(\bar{u}_0/R^3)^{(1/2)}$.

The confinement-deconfinement transition occurs at temperature $\beta_{c-dec} = 2\pi R_T = 1/T_{c-dec}$. Comparing both the confinement-deconfinement transition temperature and the chiral symmetry breaking temperature we find that $\frac{L_c(p)}{R_T}$ decreases with increasing $p$, the dimension of the world-volume directions. For $\frac{L_c(p=4)}{R_T} = 1.15207, \frac{L_c(p=6)}{R_T} = 1.04795, \frac{L_c(p=8)}{R_T} = 0.966616$. From this analysis we see that for the transition to happen the Dp brane and anti-Dp brane do not necessarily sit at the diametrically opposite point of the $\tau$ circle.

Figure 15: $S_{Dp}(U) - S_{Dp}(||)$ is plotted against $t = \frac{u}{\bar{u}_0}$ for $p=4,6,8$. 

| $p$ | $L_c$ in units of $(R^3/\bar{u}_0)^{(1/2)}$ |
|-----|-------------------------------------|
| 4   | 0.943903 |
| 6   | 0.833931 |
| 8   | 0.751283 |





The order of the transition from the chiral symmetry breaking phase to the chiral restoration phase is first order, which follows from the figure 15.

### 10 Charging up the solution

The charging up procedure that we follow is carried out by boosting and using U-dualities. We start with a near extremal black hole solution in IIA and add an electric charge by first uplifting it to 11 dimension and then apply boost along the time and the $\tau$ direction. Finally reduce the solution along the $\tau$ direction to generate the solution which is charged under the electric field.

Having spelled out the procedure let us move to find the form of the solution. The starting point is the near extremal solution of a stack of coincident D4 brane

$$ds_{10}^2 = (u/R)^{3/2}(-f(u)dt^2 + \sum_{i,j=1}^{3} \delta_{ij}dx^i dx^j + d\tau^2) + (R/u)^{3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_2^2 \right) ,$$

$$e^\phi = g_s (u/R)^{3/4}, \quad F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \left( \frac{u_T}{u} \right)^3 .$$

Consider the Lorentz boost along $\tau$ direction as

$$\left( t_{\text{new}} \atop \tau_{\text{new}} \right) = \left( \begin{array}{cc} \cosh \alpha_T & \sinh \alpha_T \\ \sinh \alpha_T & \cosh \alpha_T \end{array} \right) \left( t_{\text{old}} \atop \tau_{\text{old}} \right) .$$

Henceforth, we shall drop the subscript from $(t, \tau)$. The metric in the 11 dimensional space is

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dy + A_\mu dx^\mu)^2.$$  (10.73)

Using (10.71) in (10.73), and boosting as mentioned above we find the 11 dimensional solution as

$$ds_{11}^2 = e^{-2\phi/3} (R/u)^{-3/2} \left( -(L + \frac{N^2}{M}) dt^2 + \sum_{i,j=1}^{3} \delta_{ij} dx^i dx^j \right) + M(d\tau + \frac{N}{M} dt)^2 +$$

$$e^{-2\phi/3} (R/u)^{3/2} \left( \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right) + e^{4\phi/3} dy^2 ,$$

$$F_4 = \frac{2\pi N_c}{V_4} \epsilon_4,$$  (10.74)

where

$$M = 1 + \left( \frac{u_T}{u} \right)^3 \sinh^2 \alpha_T , \quad N = \left( \frac{u_T}{u} \right)^3 \sinh \alpha_T \sinh \alpha_T , \quad L = 1 - \left( \frac{u_T}{u} \right)^3 \cosh^2 \alpha_T .$$  (10.75)
Reducing along $\tau$ direction with assuming $g_s = 1$, gives
\[
ds^2 = (M/h)^{1/2}(u)[-H dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dy^2] + (Mh)^{1/2}(u)[du^2 + u^2 d\Omega_4^2],
\]
\[e^\phi = (M^3/h)^{1/4}(u), \quad C_1 = \frac{(1 - f(u))}{M(u)} sh_{\alpha_T} c_{\alpha_T} dt, \quad H(u) = f/M
\]
\[F_4 = \frac{2\pi N_c}{V_4} \epsilon_4, \quad f(u) = 1 - \left(\frac{u_T}{u}\right)^3, \quad h = (R/u)^3, \quad M(u) = 1 + (u_T/u)^3 (sh_{\alpha_T})^2.
\]

(10.76)

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