Dynkin Diagrams of $CP^1$ Orbifolds

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We investigate $N = 2$ supersymmetric sigma model orbifolds of the sphere in the large radius limit. These correspond to $N = 2$ superconformal field theories. Using the equations of topological-anti-topological fusion for the topological orbifold, we compute the generalized Dynkin diagrams of these theories - i.e., the soliton spectrum - which was used in the classification of massive superconformal theories. They correspond to the extended Dynkin diagrams associated to finite subgroups of $SO(3)$.

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1. Introduction and Summary

The recent classification of $N = 2$ superconformal field theories with massive deformations [1] provides a major step towards zeroing in on phenomenologically desirable string vacua [2]. Orbifolds are known to be consistent string vacua as well [3]. The superconformal orbifold theories considered here are constructed by taking the large radius (Ricci flat) limit of massive theories - supersymmetric sigma model orbifolds of the sphere (CP$^1$). Topological orbifolds of sigma models were described in [4]. The theories we consider are asymptotically free. As such, they are somewhat less interesting than the Calabi-Yau spaces. However, their classification reveals some structure. We will analyze the topological sectors of these theories to compute the soliton spectrum. This is then used to classify the theory. Specifically, the numbers of solitons connecting two ground states can be organized in a matrix. The classification program puts Diophantine constraints on the allowable integer matrices. These constraints are satisfied, for example, by the Cartan matrices of Lie groups [1]. We can thus ask the question: which matrices are associated to the CP$^1$ orbifolds? We find for the dihedral orbifolds that these matrices correspond to associated affine Lie groups, and expect the same of the exceptional cases (for the cyclic case, see [1][5]).

We first briefly review topological orbifolds on sigma models (section two) and the classification of “massive” $N = 2$ superconformal theories (section three). We then discuss dihedral orbifolds of CP$^1$ (section four) and solve the equations of topological-anti-topological fusion (tt$^*$) [6] to compute the soliton spectrum for orbifolds of CP$^1$ by discrete subgroups of SO(3), and show that the generalized Dynkin diagrams are those of the associated affine Lie algebras (section five). As an example, the $D_5$ orbifold is computed in the appendix.

2. Topological Orbifolds

The topological field theories associated to orbifolds of sigma models were discussed in [4]. In the usual topological sigma model associated to a Kahler manifold, $K$, of complex dimension $d$, the observables are the cohomology classes. The Hodge grading corresponds to chiral fermion numbers $(p, q) \leftrightarrow (f_L, d - f_R)$. In the orbifold model, in which we have an action of a group, $G$, by holomorphic isometries of $K$, we project to group invariant cohomology classes. In addition, we have twisted operators, which create twisted states from the NS vacuum. These observables are described by the cohomology classes of the
manifolds fixed under the group elements. So, for example, the $g$–twisted observables coincide with $H^*(M_g)$, where $M_g = \{m \in K | gm = m\}$. If the group is nonabelian, $g$ is understood to represent a conjugacy class; all $M_g$ are homeomorphic for a given conjugacy class, so any representative suffices (the actual observables are appropriate invariant combinations of operators associated to differential forms for the various $M_g, g \in \{g\}$). The fermions of the theory have tangent space indices, so at fixed points, where there is a non-trivial action of the group on the normal bundle to the manifold, they obey twisted boundary conditions. The fermionic vacuum, in such a case, undergoes a chiral fermion number shift \[7\] \[8\]. If we consider the observables as differential forms on the orbifold, the grading of the twisted forms gets shifted. This is expressed in the compact formula

$$H^{p,q}(K/G) \equiv \bigoplus_{\{g\}} H^{p-F_g,q-F_g}(M_g),$$

where $F_g$ represents the fermion number shift, and is given by the sum of the phases of the eigenvalues of the $g$ action on the normal bundle of $M_g$. $C(g)$ is the centralizer of $g$, i.e. $C(g) = \{h \in G | hg = gh\}$. Here $p$ and $q$ can be fractional, but $p - F_g$ and $q - F_g$ are integers.

The ring of observables - which coincides with the chiral ring of the nontopological theory - can be calculated from the topological three point functions, which are calculated from an appropriate moduli space. In the usual sigma model, this space is just the set of holomorphic maps from the world sheet to target space. In the orbifold case, the notion of holomorphic maps is ill-defined, since the orbifold is not a manifold. Instead, we take equivariant holomorphic maps with respect to an appropriate branched cover of the sphere (see \[9\]).\footnote{Throughout this paper, we take the philosophy of defining the theory through factorization; thus we only consider genus zero correlation functions. This avoids problems of compactifying moduli space for higher genus maps.} That is, we choose a Riemann surface, $\tilde{\Sigma}$, which is a branched cover by $G$ over the world sheet $\Sigma$. The insertion points of operators correspond to the branch points, and the branching elements correspond to the twisted sector of the operator. Then, equivariant maps obey $g\phi(x) = \phi(gx)$ for all group elements $g$. They uniquely define maps from the sphere to the orbifold, which are analytic on nonsingular regions (see below).
The correlation functions are a sum over contributions from the different components of instanton moduli space (holomorphic, equivariant maps).

3. Classification of $N = 2$ Superconformal Theories

We briefly review the classification by Cecotti and Vafa of $N = 2$ superconformal theories with massive deformations [1], and its relation to the $tt^*$ equations [3].

Any $N = 2$ theory yields a topological theory from the $Q$–closed modulo $Q$–exact observables. Likewise, the $CPT$ conjugate fields create an “anti-topological” theory. The states of the topological theory are denoted $|a\rangle$, and correspond to the observables $\phi_a$. The anti-topological states are related by a change of basis, effected by the real structure matrix: $|\bar{a}\rangle = M^{\bar{b}|b}. The quantum field theory defines a metric on the Hilbert space, $\mathcal{H}$, which descends to a metric on the topological theory, since $Q$–exact terms are zero in correlators:

$$g_{a\bar{b}} = \langle \bar{b}|a\rangle.$$ 

There is also a topological metric defined by intersections in an appropriate moduli space. These structures are defined for any $N = 2$ theory, and become geometrical structures on the space of theories. We can coordinatize this space by coupling constants $\{t_i\}$. Choosing an action $S_0$, we write

$$S(t) = S_0 + \left[ \int d^2\theta t_i \phi_i + c.c. \right].$$

At each $t$, we have a chiral ring, isomorphic to the Ramond ground states of the theory. We thus have a vector bundle - the bundle of ground states - with the metric given above (now $t$–dependent). A ground state, characterized by its $U(1)$ charge, is then a section of this bundle; its wave function, then, is $t$–dependent, and we can thus consider the connection defined by

$$(A_i)_{a\bar{b}} = \langle \bar{b}|\partial_i|a\rangle.$$ 

Then $D_i = \partial_i - A_i$. In fact we can consider a family of connections indexed by a “spectral parameter,” $x$ :

$$\nabla_i = D_i - x C_i,$$

$$\nabla^\tau_i = D^\tau_i - x^{-1} C^\tau_i,$$

where $C_i$ represents the action of $\phi_i$; that is, $\phi_i \phi_j = (C_i)_j^k \phi_k$ ($C^\tau_i = g C_i^\tau g^{-1}$). The $tt^*$ equations, conditions on the metric and the $C_i$, are then summarized by the statement that $\nabla$ and $\nabla^\tau$ are flat for all $x$. 

The solutions to the $tt^*$ equations encode the number of solitons which saturate the Bogolmonyi bound and connect the ground states. In the Landau-Ginzburg case, the soliton numbers have a topological description in terms of intersection numbers of vanishing cycles over families of varieties. The vacua correspond to points satisfying $dW = 0$, and solitons connecting them travel along straight lines in the $W$ plane. The inverse image of the values of $W$ near a critical point form spheres in $\mathbb{C}^n$. When these spheres intersect, one can build a soliton path between the vacua. Because of this interpretation, the soliton numbers must behave as the intersection numbers under $t-$dependent perturbations of the superpotential, in particular when the vacua become colinear via the perturbation. In [4], the authors developed the analog of this interpretation for a general $N = 2$ theory.

As we saw above, the $tt^*$ equations can be formulated as flatness conditions on a family of connections. The equations have the built-in requirement that the hermitian metric is independent of the overall phase of the generalized superpotential. By generalized superpotential, we mean the values $w_a$ which can be assigned to the different vacua such that the Bogolmonyi soliton masses (the central terms of the $N = 2$ algebra) are given by the differences of the $w_a$. These are the canonical coordinates. That this independence should hold follows from the freedom to redefine the phases of the fermions. The equations are given in terms of the connections

$$\nabla_i = \partial_i + (g \partial_i g^{-1}) - x C_i,$$

$$\nabla^*_i = \overline{\partial_i} - x^{-1} \overline{C_i},$$

written here in the $A_i = 0$ gauge. We consider the set of equations

$$\nabla_i \Psi(x, w_a) = \nabla^*_i \Psi(x, w_a) = 0.$$  

In order to solve these equations simultaneously, we must require that $\nabla$ and $\nabla^*$ commute, i.e. they are flat; this consistency condition is $tt^*$. In general, there will be $n$ solutions to (3.2), so we take $\Psi$ to be an $n \times n$ matrix whose columns are solutions. The equations are singular at $x = 0, \infty$, which means the columns of $\Psi$ will mix under monodromy $x \rightarrow e^{2\pi i x} : \Psi \rightarrow H \cdot \Psi$. If we consider $\beta \rightarrow 0$ with $x$ small, i.e. the conformal limit, then these equations indicate that the phases eigenvalues of the monodromy around zero are precisely the Ramond charges. Since these charges must be real, the eigenvalues $\lambda_i = e^{2\pi i q_i}$ of the monodromy must satisfy $|\lambda_i| = 1$. Because the equations of $tt^*$ are flatness equations, they describe isomonodromic deformations. That is, the monodromy is a constant. Indeed it is
calculable in the $\beta \to \infty$ limit, where the monodromy $H$ of $\Psi$ is expressable in terms of the soliton numbers $A_{ij}$. The relation is

$$H = S(S^{-1})^t,$$

$$S = 1 - A.$$  \hfill (3.3)

Statements about the charges (e.g. CPT, unique highest/lowest charge vacua) are then conditions on the possible matrices $A$. This is detailed in section six of [1]. To us, the important result is that simply laced Lie groups lead to solutions to the Diophantine equations of classification.

The simply laced Lie groups are related to possible solutions for $A$ as follows. Suppose the matrix $B = S + S^t$ is positive definite. Then $HBH^t = SS_t(S + S^t)S^{-1}S^t = B$, which means that $H$ is in the orthogonal group to the quadratic form, $B$, which tells us that $H$ is simple and $|\lambda_i| = 1$. The simply laced Lie groups correspond to positive definite integral matrices by constructing a Dynkin diagram from the matrix. $B$ defines an inner product on $\mathbb{R}^n$, and if we take $A$ to be upper triangular, with $A_{ij} = -B_{ij}/2, i < j$, then $H = (1 - A)(1 - A)^{-t}$ satisfies the Diophantine constraints. These matrices correspond to the $N = 2 A - D - E$ minimal models. Weyl reflections of the lattice vectors produce different, though equivalent solutions to the Diophantine equations. These reflections correspond to perturbations of the vacua through collinear configurations in the $W$ plane. The affine models correspond to the case where $B = S + S^t$ has a single zero eigenvector, $v$. Then $B$ defines a reduced matrix $\hat{B}$ on the orthogonal complement to $\mathbb{R}v$, which solves the Diophantine equations. Noting that $H^t v = -v$, so $\lambda_v = 1$ and we see that all the eigenvalues $\lambda$ of $H$ have $|\lambda| = 1$.

4. Orbifolds of $\mathbb{C}P^1$.

Discrete subgroups of $SO(3)$ act naturally on $\mathbb{C}P^1$, which is topologically a sphere. The description is simplest in homogeneous coordinates. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ sends the point $(x, y)$ to $(ax + by, cx + dy)$. By the projective identification, the center $Z_2$ of $SU(2)$ acts trivially, so $G \subset SU(2)$ acts by the image under the covering $SU(2) \to SO(3)$. The topological orbifolds of these models were considered in [4]. For genus zero three-point functions, from which the operator ring is derived, we can use the sphere itself to represent a branched cover of the (worldsheet) sphere, with the action of the group given by the
fundamental $SU(2)$ representation, as for the orbifold above. Then $\mathcal{M}_k$, the holomorphic maps of degree $k$, is represented by pairs of degree $k$ homogeneous polynomials:

$$\Phi : (X, Y) \mapsto \left( \sum_{l=0}^{k} \phi_{0l}X^{k-l}Y^l, \sum_{l=0}^{k} \phi_{1l}X^{k-l}Y^l \right).$$

So $\Phi$ is represented by a $2 \times (k+1)$ matrix (defined up to overall multiplication by a scalar), acting on $(X^k, X^{k-1}Y, ..., Y^k)$, and the equivariant maps obey

$$\Phi \cdot \rho_k(g) = \lambda \rho_1(g) \cdot \Phi,$$

where $\rho_1$ is the fundamental $SU(2)$ representation, and $\rho_k$ is the $(k+1)$–dimensional representation on degree $k$ homogeneous polynomials induced by $\rho_1$ ($\rho_k = (\otimes^k \rho_1)_{symm}$). $\lambda$ is an arbitrary, possibly $g$–dependent factor. Finding such maps amounts to finding intertwining maps of projective representations $[10]$.†

Using the formula (2.1), we determine the ring of observables for the $D_N$ to be as follows. If $N = 2k$, there are two operators in the identity sector: 1 and $X$, which descend from the original sigma model on $\mathbb{CP}^1$. Associated to each conjugacy class $\theta_j$ of rotations of the $N$–gon by $\pm2\pi j/N$, $j = 1...k-1$, are two operators: call them $\phi_j, \phi_{N-j}$. The rotation by $\pi$ is a central element of the group and has associated with it a single operator, $\phi_k$. In addition, there are two conjugacy classes of flips (at a vertex or midpoint diagonal of the polygon), which have one operator apiece, $\rho$ and $\tau$. The two fixed points under $\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for example, are $(1,1)$ and $(1,-1)$ and are related by the $\pi$ rotation $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, which centralizes $\rho$. The ring was shown in $[4]$ to be generated by $\rho$ and $\phi \equiv \phi_1$, and is given by

$$\rho^2 \phi^2 = 4\rho$$

$$\rho^2 = 1 + \frac{1}{2}W_{2k}(\phi) + \sum_{l=1}^{k-1} W_{2l}(\phi)$$

$$\phi W_{2k}(\phi) = 2W_{2k-1}(\phi),$$

† Note: $\Phi$ must not be identically zero. Where the polynomials in (4.1) have $r$ common roots, the roots are divided out to get a lower degree map. These maps, technically, are in $\mathcal{M}_{k-r}$, not $\mathcal{M}_k$, and make up the compactification divisor. The compactified $\overline{\mathcal{M}}_k$ is then $\mathbb{CP}^{2(k+1)-1}$. 

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where the functions $W_n(\phi)$ are the Chebyshev polynomials, defined here to be

$$W_n(X = 2\cos(z)) = 2\cos(nz). \quad (4.4)$$

In terms of $\phi$, the operators $\phi_j$ are

$$\phi_j = W_j(\phi). \quad (4.5)$$

We also have $1 \equiv \frac{1}{2} \phi_0 = \frac{1}{2} W_0, \chi = \frac{1}{2} W_N,$ and $\tau = \frac{1}{2} \rho \phi$ (here $\chi = X$ up to normalization by the one-instanton action, i.e. the area of the target sphere: $\chi = \beta^{-\frac{1}{2}} X, \beta = e^{-A}$).

In the odd case, $N = 2k + 1$, there are again $N + 3$ operators. The two untwisted operators remain, as before; in each of the $k$ rotation classes, there are two operators, $\phi_k, \phi_{N-k}$ (there is no central element); and the lone flip conjugacy class has two operators, $\rho$ and $\tau$. The ring is then given by

$$\rho \phi^2 = 4 \rho \quad (4.6a)$$

$$\rho^2 = \frac{1}{2} W_{2k+1}(\phi) + \sum_{l=1}^{k} W_{2l-1}(\phi) \quad (4.6b)$$

$$\phi W_{2k+1}(\phi) = 2 W_{2k}(\phi), \quad (4.6c)$$

where the same expressions for the operators (as we’ve defined them) in terms of $\phi$ and $\rho$ still hold.

In fact, both rings can be given by the same equations. Using the recursion relation obeyed by both the $\phi_j$ and $W_j$:

$$x W_j(x) = W_{j+1}(x) + W_{j-1}(x) \quad j \geq 1,$$

we can write the last equation as $W_{N+1}(\phi) = W_{N-1}(\phi)$. Writing $\phi = 2\cos(z)$ this reads $2\cos[(N + 1)z] = 2\cos[(N - 1)z]$, we can solve this equation as if it were numerical, and get $z = \frac{2\pi}{N}$, so the $x$ solutions are $x_j = 2\cos(\frac{j\pi}{N})$. Most of these roots ($j \neq 2, -2$), and $x = 0$, are roots of the right hand side of (4.3b) and (4.6b), from which we get

$$RHS(4.3b) \propto \phi \prod_{j=1}^{N-1} (\phi - 2\cos(j\pi/N)) \propto \frac{\phi}{\phi^2 - 4} (W_{N+1}(\phi) - W_{N-1}(\phi))$$

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up to an overall constant, which can be determined by L'Hôpital's rule. The result is the same in the odd and even case. We use the last form of the above equation and the recursion relation to write, for \( N \) odd or even,

\[
\rho \phi^2 = 4 \rho \quad (4.7a)
\]

\[
\rho^2 = \frac{1}{2} \frac{W_{N+2}(\phi) - W_{N-2}(\phi)}{\phi^2 - 4} \quad (4.7b)
\]

\[
W_{N+1}(\phi) = W_{N-1}(\phi). \quad (4.7c)
\]

We stress that the right hand side of (4.7b) is a polynomial.

In order to compute the matrix of soliton numbers associated to this theory, it is convenient to work in a particular basis for the chiral ring, the canonical basis \([11]\). Such a basis exists for finite \( \beta \). In this basis the operator algebra reads

\[
A_i \cdot A_j \propto \delta_{ij} A_j, \quad (4.8)
\]

where the constant of proportionality is determined by requiring that the topological metric obeys

\[
\eta_{ij} = \delta_{ij}. \quad (4.9)
\]

That \( \eta \) is diagonal is a simple consequence of (4.8). To find this basis, we use a trick associated to the chiral ring of a Landau Ginzburg model (off criticality). In particular, the point basis is obtained by writing the derivative of the superpotential \( W'(x) = \prod_i (x - r_i) \) and defining \( A_j = \prod_{i \neq j} (x - r_i) / W'(r_j) \). The canonical basis differs only in normalization.

In the above, the \( r_j \) are complex numbers which satisfy the ring relations. In the Landau-Ginzburg case, they represent vacuum expectation values of \( W \). We can perform this trick for our rings as well, though the physical interpretation of the \( r_j \) seems to be lost.

The numerical solutions to (4.7) are as follows. The first equation implies \( \rho = 0 \) or \( \phi^2 = 4 \). If \( \phi^2 = 4 \), so \( \phi = 2 \epsilon \), where \( \epsilon = \pm 1 \), then since \( 2 \epsilon = 2 \cos(\frac{\epsilon - 1}{2} \pi) \), the last equation reads \( 2 \cos(\frac{\epsilon - 1}{2} (N + 1) \pi) = 2 \cos(\frac{\epsilon - 1}{2} (N - 1) \pi) \) and is satisfied. Equation (4.7b) can be evaluated by L'Hôpital's rule to be \( \rho^2 = (-1)^N (\frac{\epsilon - 1}{2}) N \). The point basis, which satisfies \( A_i \cdot A_j = \delta_{ij} A_j \), assigns to each solution a ring element. Let \((\rho_a, \phi_a)\) represent the \( a^{th} \) solution obtained above \((a = 1 \ldots N + 3)\). The corresponding point basis element is then

\[
A_a = \prod_{i,j=1}^{N+3} \frac{(\rho - \rho_i)(\phi - \phi_j)}{(\rho_a - \rho_i)(\phi_a - \phi_j)} \quad (4.10)
\]
This expression can simplify greatly. For example, if there is an overall factor of \( \rho \), we can make the replacement \( \phi^2 \to 4 \), by virtue of (4.3). It will be convenient for us to focus on the \( \phi \) pieces of the basis elements. In fact, for all the points with \( \rho = 0 \), the ring elements only contain factors of \( \phi \). For the other four points, \( \rho^2 = \pm N \), only the “angular,” i.e. \( \phi \)-dependent, pieces are important for soliton number computations, as we shall see below. We thus define an “effective” basis \( \hat{A}_a \), \( a = 0 \ldots N \), labeled by the \( N + 1 \) values of \( \phi : \phi_a = 2 \cos(\frac{2\pi a}{N}) \). They have the simpler expression

\[
\hat{A}_a = \varepsilon_a \prod_{i=0}^{N} \frac{(\phi - \phi_i)}{(\phi_a - \phi_i)}; \tag{4.11}
\]

where \( \varepsilon_a = \frac{1}{2} \) if \( a = 0 \) or \( a = N \), \( \varepsilon_a = 1 \), otherwise.

5. Dynkin Diagrams and Dihedral Orbifolds

In this section we will solve the \( tt^* \) equations and compute the soliton numbers between the vacua. The resulting generalized Dynkin diagram corresponds to the affine Lie group associated to the dihedral group.

We recall a connection between discrete subgroups of \( SU(2) \) and affine Lie groups. Each subgroup of \( SU(2) \) has associated with it a two-dimensional fundamental representation, \( R \). The tensor product of any irreducible representation, \( V_i \), with \( R \) decomposes as

\[
V_i \otimes R \cong \bigoplus_j A_{ij} V_j.
\]

A theorem due to McKay [12] states that the matrix \( A \) is the adjacency matrix of a Dynkin diagram for an affine Lie algebra. For example, consider the fundamental (though reducible) representation of \( Z_N \); the generator \( g \) acts by \( g = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \), where \( \gamma = e^{2\pi i/N} \). The irreducible representations, \( k \) of \( Z_N \) are just \( \gamma^k \), and it is clear that \( k \otimes R \cong (k - 1) \oplus (k + 1) \). Now for a discrete subgroup of \( SO(3) \), we use the fundamental representation of the double cover of the subgroup. So for \( Z_N \) we use \( Z_{2N} \), and the same decomposition rule applies, with \( k \) now ranging from zero to \( 2k - 1 \). Thus we have the correspondence shown in fig. [4]. The correspondence between the discrete dihedral groups of \( SO(3) \) and extended Dynkin diagrams of the \( D \)-series is

\[
D_N \leftrightarrow \hat{D}_{N+2};
\]
the $D_5 \leftrightarrow \hat{D}_7$ case is shown in fig. 4.

Now we wish to compute the soliton matrix for the dihedral orbifolds. By the discussion in section three, we know that we can compute a Dynkin diagram from this object. We will compute it in the canonical basis, described in section four. However, another basis is more convenient for solving the $tt^*$ equations, which we do below.

Implementing the symmetries of the orbifold simplifies the computation of the ground state metric. Namely, the product of all of the twists of the operators must be the identity (considered as incoming states, in genus zero). For nonabelian orbifolds, the product of two conjugacy classes contains a sum of other conjugacy classes: the decomposition of the classes is called the group ring. In order to have a nonzero correlation function, the products of the classes in the group ring must contain the identity. For two point functions, this says that the metric is block diagonal for conjugacy classes (the dual of $\{h\}$ is $\{h^{-1}\}$, which is $\{h\}$, when viewed as an outgoing state). Now no conjugacy class contains more than two operators. In addition to this constraint, we have the reality constraint, i.e. $(CPT)^2 = 1$. This can be used to show that the metric is indeed diagonal, and in each sector has two real positive components, $a, b$, satisfying $ab = 1$. Thus there is one real parameter:

$$g_{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \quad (5.1)$$

We study behavior of $g$ as a function of the instanton action $|\beta| = e^{-A}$. Through a proper change of variables, all of the equations are equivalent to the sinh-Gordon equation. The $tt^*$ equation of interest, derivable from the flatness of (3.1), is

$$\bar{\sigma}_i(g \partial_j g^{-1}) = [C_j, \bar{C}_i].$$

We are interested in the variations with respect to scale: $i = j = \beta$.

The operator corresponding to $\beta$ is $-X/\beta$. Since $X^2 \sim 1$ (and since its product with other fields always contains a single field) we see that $C_\beta$ decomposes into $2 \times 2$ blocks; acting on operators of fermion number $\frac{k}{N}$ and $\frac{N-k}{N}$ it has the block form

$$C_\beta = \frac{1}{\beta} \begin{pmatrix} 0 & \beta^k \\ \beta^{N-k} & 0 \end{pmatrix}$$

(recall that $\beta$ can be assigned a chiral fermion number of two; then this number is conserved in the ring products). Note that for $N = 2k$, the operator $\phi_k$ obeys $X\phi_k = \beta^{\frac{k}{2}}\phi_k$, meaning
that \( C_\beta \) is a \( 1 \times 1 \) matrix in this block, giving that \( \langle \overline{\phi_k} | \phi_k \rangle \) is essentially constant - it is a pure power of \( \beta \), due to our “dimensionless” definition of fields.\(^\dagger\)

Defining
\[
x = 4|\beta|^\frac{1}{2}, \quad u(x) = 2\log \left(a|\beta|^\frac{N-2k}{2N}\right),
\]
we get
\[
u'' + \frac{1}{x}u' = 4 \sinh u.
\]
We must require finiteness of \( g_{ij} \) in the conformal \( (\beta \to 0) \) limit. This tells us that as \( x \to 0 \), \( u \) behaves as
\[
u \to r \log x + s, \quad r = 2 \left( \frac{N-2k}{N} \right).
\]
The \( x \to \infty \) behavior gives us the the matrix of soliton numbers. Namely, the metric should obey
\[
g_{ij} \sim \delta_{ij} - \frac{i}{\pi} \mu_{ij} K_0(m_{ij} \beta).\]
The \( x \to \infty \) asymptotic behavior is known \([13]\). The solution to (5.3) obeying (5.4) contributes
\[
2i \sin \left( \pi \frac{N-2k}{2N} \right) \nu_k, \quad \nu_k = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \ldots (N-1) \\ 4, & k = N \end{cases}
\]
to \( \mu \) in this expansion. The factor \( \nu_k \) is due to our choice of basis (see footnote) and the fact that \( X = \frac{1}{2} \beta^\frac{1}{2} W_N(\phi) \). Note that the operators corresponding to flips all appear with \( \frac{k}{N} = \frac{1}{2} \) (even when \( N \) is odd), meaning they do not contribute to the soliton numbers!

This was our justification for isolating our analysis on the \( \phi \) parts of the canonical basis.

Before calculating the soliton numbers, though, we wish to choose a standard basis - the canonical basis, discussed in section four. To do this, we express the expansions of \( A_l \) in terms of the above basis, which solved the \( tt^* \) equations. We have that \( \phi_k = W_k(\phi) \). So we express products of the form \((4.10)\) as
\[
\tilde{A}_l = \sum_{i=0}^{N} (c_i)_j \phi^j = \sum_{j=0}^{N} (a_l)_j W_j(\phi).
\]

\(^\dagger\) We have removed the appearance of \( \beta \) in the ring by defining “dimensionless” ring elements. For example, in the \( k^{th} \) conjugacy class we have \( \phi_k = \beta^{-k/2N}(\theta_A^k + \theta_B^{N-k}) \), where \( A \) and \( B \) label the fixed points (likewise for \( \phi_{N-k} \)). We choose our sub-basis here so that \( \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). This was used in deriving (5.1).
The following identity facilitates this change of basis. From the recursion relations, one can derive

\[ x^n = \sum_{j=1}^{[\frac{n}{2}]} \binom{n}{j} \tilde{W}_{n-2j}(x) \]  

(half of Pascal’s triangle, in a way), where we have defined

\[ \tilde{W}_n(x) = \begin{cases} 
1, & n = 0, \\
W_n(x), & n > 0.
\end{cases} \]

We still need to make one correction to the normalization, so that \( \eta_{ij} = \delta_{ij} \). This is simple since \( X \) is the only operator with nonzero topological correlation, and \( \phi^n = \tilde{W}_n(\phi) + \ldots \), where \( \ldots \) represent lower degree polynomials (recall \( 2X \sim \tilde{W}_n \)). The result is that the canonical basis elements (just the \( \phi \) parts), which we denote \( \hat{A}_l \) are

\[ \hat{A}_l = N_l \tilde{A}_l = \pm \left[ \frac{1}{2\epsilon_a} \prod_{i=0}^{N} (\phi_a - \phi_i) \right]^{\frac{1}{2}}. \]  

(5.9)

Writing

\[ \hat{A}_l = \sum_{i=0}^{N} (\hat{a}_l)_i \tilde{W}_i(\phi), \]  

(5.10)

and using the diagonal property of the metric and equations (5.5) and (5.6), we arrive at the expression for the soliton adjacency matrix

\[ \hat{A}_{rs} = \sum_{j=0}^{N} (\hat{a}_r)_j (\hat{a}_s)_j 2i\nu_j \cos \left( \pi \frac{j}{N} \right). \]  

(5.11)

The result is that the matrix \( A \), obtained from \( \hat{A} \) by restoring the two ring elements or \( \hat{A}_0 \) and \( \hat{A}_N \), is the adjacency matrix of the extended Dynkin diagram of the corresponding affine Lie group, up to choices of signs for the \( N_l \). To be precise, the matrix \( A \), corresponding to the monodromy in (3.3) is the upper triangular part of the matrix we obtained above. We have checked our results explicitly for the first several values of \( N \), and we obtain the expected form of \( A \), with integer (ones and zeros) coefficients. For higher \( N \), we have evaluated the expressions numerically, and have obtained ones and zeros, though we know of no mathematical proof that this must be so. In the appendix, we check the \( N = 5 \) case explicitly.
Another check we can perform is obtaining $H$ from $A$, i.e. $H = (1 - A)(1 - A)^{-1}$, and computing its characteristic polynomial. We know the Ramond charges of the chiral ring, and these should be the phases of the eigenvalues. Indeed this is the case.

It is natural to guess that orbifolds by the exceptional discrete groups are described by the exceptional affine Lie groups. Indeed, a simple check of the characteristic polynomials of the matrices $H$ yields the correct Ramond charges, though the full quantum ring and $tt^*$ equations have not been computed.

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Appendix.

In this appendix, we show a detailed computation of some of the matrix elements of $A$ for the $D_5$ orbifold of $\mathbb{CP}^1$.

By taking the real part of $\sum_{j=1}^{5} e^{2\pi i j/5} = 0$, one easily calculates

\[
x \equiv \cos(\pi/5) = \frac{1 + \sqrt{5}}{4}, \quad y \equiv \cos(2\pi/5) = \frac{-1 + \sqrt{5}}{4}
\]

Note $xy = 1/4; x^2 + y^2 = 3/4; x^2 - y^2 = \sqrt{5}/4; 2x^2 = 1 + y; 2y^2 = 1 - x$. Then, using (4.10), (5.9), and (5.8) we get

\[
\hat{A}_0 = N_0 c_0 \tilde{A}_0
\]

\[
= N_0 \frac{1}{2} \prod_{l=1}^{5} (\phi - 2\cos(l\pi/5))/(2 - 2\cos(l\pi/5))
\]

\[
= N_0 \frac{1}{2} (\phi - 2x)(\phi - 2y)(\phi + 2y)(\phi + 2x)(\phi + 2)/20
\]

\[
= \pm \frac{\sqrt{20}}{40} (\phi + 2)(\phi^2 - 4x^2)(\phi^2 - 4y^2)
\]

\[
= \pm \frac{\sqrt{5}}{20} [\phi^5 + 2\phi - 3\phi^3 - 6\phi^2 + \phi + 2]
\]

\[
= \pm \frac{\sqrt{5}}{20} [W_5(\phi) + 2W_4(\phi) + 2W_3(\phi) + 2W_2(\phi) + 2W_1(\phi) + 2].
\]
Similarly, making use of the identities above, one finds

\[ \hat{A}_1 = \pm \frac{i\sqrt{5}}{10} [W_5(\phi) + 2xW_4(\phi) + 2yW_3(\phi) - 2yW_2(\phi) - 2xW_1(\phi) - 2]. \]  

(A.2)

Plugging into (5.11) yields

\[ A_{01} = \pm \left( \frac{\sqrt{5}}{20} \right) \left( \frac{\pm i\sqrt{5}}{10} \right) (2i)[1 \cdot 4\cos(5\pi/5) + 4x \cdot 2\cos(4\pi/5) + 4y \cdot 2\cos(3\pi/5) - 4y \cdot 2\cos(2\pi/5) - 4x \cdot 2\cos(\pi/5) - 4\cos(0)] \]

\[ = \pm \left( \frac{1}{5} \right) [\cos(\pi) + 2x\cos(4\pi/5) + 2y\cos(3\pi/5) - 2y\cos(2\pi/5) - 2x\cos(\pi/5) - \cos(0)] \]

\[ = \pm \left( \frac{1}{5} \right) [-1 - 2x^2 - 2y^2 - 2y^2 - 2x^2 - 1] \]

\[ = 1, \]

(A.3)

where we have used the freedom of signs to choose +1.

One finds \( A_{12} = A_{23} = A_{34} = A_{45} = 1 \) as well. Recalling that there are really two ring elements corresponding to both \( \hat{A}_0 \) and \( \hat{A}_5 \) gives us the Dynkin diagram of \( \hat{D}_8 \), shown in fig. 2. Specifically, to solve the Diophantine equations, we take the upper triangular matrix \( A_{i<j} \), remembering that \( \hat{A}_0 \) and \( \hat{A}_N \) each represent two operators. Specifically, for \( D_5 \), we obtain

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]  

(A.4)

The monodromy matrix is

\[ H = (1 - A)(1 - A)^{-t} \]

and its characteristic equation is

\[
det(z - H) = z^8 - z^7 - z^6 + z^5 + z^3 - z^2 - z - 1 \]

\[ = (z - 1)^2 (z + 1)^2 (z^4 - z^3 + z^2 - z + 1) \]

\[ = \Psi_1(z)^2 \Psi_2(z)^2 \Psi_{10}(z), \]

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where $\Psi_n(z)$ is the $n^{th}$ cyclotomic polynomial. We can easily read off the Ramond charges of the superconformal theory to be

$$\{q_i\} = \left\{-\frac{1}{2}, -\frac{3}{10}, -\frac{1}{10}, 0, 0, \frac{1}{10}, \frac{3}{10}, \frac{1}{2}\right\},$$

which are the NS charges shifted by $-\frac{\hat{c}}{2} = -\frac{1}{2}$, as expected.
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Figure Captions

Fig. 1. The double cover of $Z_N$ is $Z_{2N}$, from which we determine the associated affine Lie algebra. We find $Z_N \leftrightarrow \hat{A}_{2N-1}$. The case $N = 4$ is shown.

Fig. 2. The extended Dynkin diagram of the affine Lie algebra $\hat{D}_7$. This corresponds to the dihedral group $D_5$. 