1 Introduction

For a faithful presentation of the problem and result discussed in [1] we quote from this paper:

"The fourth-order polynomial defined by
\[ H(x) := \nu/2(1/2x^2 - \lambda)^2, \]
where \( x \in \mathbb{R}, \nu, \lambda \) are positive constants (1) is the well-known Landau’s second-order free energy, each of its local minimizers represents a possible phase state of the material, while each local maximizer characterizes the critical conditions that lead to the phase transitions.

The purpose of this paper is to find the extrema of the following nonconvex total potential energy functional in 1D,
\[ I[u] := \left( \int_a^b H \left( \frac{du}{dx} \right) - fu \right) dx. \] (2)
The function \( f \in C[a, b] \) satisfies the normalized balance condition
\[ \int_a^b f(x)dx = 0, \] (3)
and there exists a unique zero root for \( f \) in \([a, b]\). (4)
Moreover, its \( L^1 \)-norm is sufficiently small such that..."
\[ \|f\|_{L^1(a,b)} < 2\lambda \nu \sqrt{2\lambda / (3\sqrt{3})}. \] (5)

The above assumption is reasonable since large \( \|f\|_{L^1(a,b)} \) may possibly lead to instant fracture, which is represented by nonsmooth solutions. The deformation \( u \) is subject to the following two constraints,

\[ u \in C^1[a,b], \] (6)

\[ \frac{du}{dx}(a) = \frac{du}{dx}(b) = 0. \] (7)

Before introducing the main result, we denote

\[ F(x) := -\int_a^x f(\rho) d\rho, \quad x \in [a,b]. \]

Next, we define a polynomial of third order as follows,

\[ E(y) := 2y^2(\lambda + y/\nu), \quad y \in [-\nu\lambda, +\infty). \]

Furthermore, for any \( A \in [0,8\lambda^3\nu^2/27) \),

\[ E_3^{-1}(A) \leq E_2^{-1}(A) \leq E_1^{-1}(A) \]

stand for the three real-valued roots for the equation \( E(y) = A \).

At the moment, we would like to introduce the main theorem.

**Theorem 1.1.** For any function \( f \in C[a,b] \) satisfying (3)–(5), one can find the local extrema for the nonconvex functional (2).

- For any \( x \in [a,b], \overline{u}_1 \) defined below is a local minimizer for the nonconvex functional (2),

\[ \overline{u}_1(x) = \int_a^x F(\rho)/E_1^{-1}(F^2(\rho)) d\rho + C_1, \quad \forall C_1 \in \mathbb{R}. \] (9)

- For any \( x \in [a,b], \overline{u}_2 \) defined below is a local minimizer for the nonconvex functional (2),

\[ \overline{u}_2(x) = \int_a^x F(\rho)/E_2^{-1}(F^2(\rho)) d\rho + C_2, \quad \forall C_2 \in \mathbb{R}. \] (10)

- For any \( x \in [a,b], \overline{u}_3 \) defined below is a local maximizer for the nonconvex functional (2),

\[ \overline{u}_3(x) = \int_a^x F(\rho)/E_3^{-1}(F^2(\rho)) d\rho + C_3, \quad \forall C_3 \in \mathbb{R}. \] (11)

As mentioned in [1], in getting the above result the authors use “the canonical duality method”.

Let us observe from the beginning that nothing is said about the norm (and the corresponding topology) on \( C^1[a,b] \) when speaking about local extrema (minimizers or maximizers).

In the following we discuss a slightly more general problem and compare our conclusions with those of Theorem 1.1 in [1]. We don’t analyze the method by which the conclusions in Theorem 1.1 of [1] are obtained even if this is worth being done. Similar problems are considered by Gao and Ogden in [2] and [3] which are discussed by Voisei and Zălinescu in [5] and [6], respectively.

More precisely consider \( \theta \in C[a,b] \) such that \( \theta(x) > 0 \) for \( x \in [a,b] \), the polynomial \( H \) defined by \( H(y) := \frac{1}{2}(\frac{1}{4}y^2 - \lambda)^2 \) with \( \lambda > 0 \), and the function

\[ J := J_f : C^1[a,b] \rightarrow \mathbb{R}, \quad J_f(u) := \int_a^b \theta \cdot (H \circ u' - fu), \]

where, \( \int_a^b \) denotes the Riemann integral \( \int_a^b h(x) dx \) of the function \( h : [a,b] \rightarrow \mathbb{R} \) (when it exists). Of course, taking \( \theta \) the constant function \( \nu (> 0) \) and replacing \( f \) by \( \nu^{-1}f \) we get the functional \( I \) considered in [1].
Let us set
\[ X := C_0[a, b] := \{ v \in C[a, b] \mid v(a) = v(b) = 0 \}, \]
\[ Y := C_{1,0}[a, b] := \{ u \in C^1[a, b] \mid u' := du/dx \in C_0[a, b] \}. \]

Of course \( X \) is a linear subspace of \( C[a, b] \); it is even a closed subspace (and so a Banach space) if \( C[a, b] \) is endowed with the supremum norm \( \| \cdot \|_\infty \). Clearly, other norms could be considered on \( X \).

Observe that the function \( F \) defined in [1] (and quoted above) is in \( C^1[a, b] \cap X \) with \( F' := dF/dx = -f \). Moreover, condition (5) implies that \( \| F \|_\infty < 2\lambda \sqrt{2\lambda}/(3\sqrt{3}) = (2\lambda/3)^{3/2} \) because
\[ |F(x)| = \left| \int_a^x f(\xi) d\xi \right| \leq \int_a^x |f(\xi)| d\xi \leq \int_a^b |f(\xi)| d\xi = \| f \|_{L^1(a,b)}. \]

Furthermore, condition (4) implies that \( F(x) > 0 \) for \( x \in (a, b) \), or \( F(x) < 0 \) for \( x \in (a, b) \).

For \( u \in Y \) and \( v := u' \) we have that
\[ \int_a^b uf = -\int_a^b uF' = -u(x)F(x) \big|_a^b + \int_a^b u' F = \int_a^b vF. \tag{1} \]

Using this fact, for \( u \) satisfying the constraints (6) and (7), and \( v := u' \), one has
\[ J(u) = \int_a^b \theta (H \circ v - Fv) =: K(v). \]

## 2 Study of local extrema of the function \( K \)

As mentioned above, in the sequel \( H : \mathbb{R} \to \mathbb{R} \) is defined by \( H(y) := \frac{1}{2} \left( \frac{1}{2} y^2 - \lambda \right) \) with \( \lambda > 0 \), \( \theta \in C[a, b] \) is such that \( \mu := \min_{x \in [a,b]} \theta(x) > 0 \); moreover \( F \in C^1[a, b] \cap X \) is such that \( F(x) \neq 0 \) for \( x \in (a, b) \) and \( \| F \|_\infty < (2\lambda/3)^{3/2} \).

Our first purpose is to find the local extrema of
\[ K := K_F : X \to \mathbb{R}, \quad K_F(v) := \int_a^b \theta \cdot (H \circ v - Fv) \tag{2} \]
on \( X = C_0[a, b] \) endowed with the norm \( \| \cdot \|_p \) where \( p \in [1, \infty] \).

First we study the Fréchet and Gâteaux differentiability of \( K \).

**Lemma 1** Let \( g \in C[a, b] \setminus \{ 0 \} \), \( s \in \mathbb{N}^* \setminus \{ 1 \} \) and \( p \in [1, \infty] \). Then, with \( h \in X \),
\[ \lim_{\| h \|_p \to 0} \frac{1}{\| h \|_p} \int_a^b gh^s = 0 \iff p \geq s. \]

**Proof.** Set \( \gamma := \| g \|_\infty (> 0) \). For \( s < p < \infty \) and \( h \in X \) we have that
\[ \left| \int_a^b gh^s \right| \leq \gamma \int_a^b |h|^s \cdot 1 \leq \gamma \left( \int_a^b (|h|^s)^{s/p} \right)^{s/p} \left( \int_a^b 1^{p/(p-s)} \right)^{(p-s)/p}, \]

and so
\[ \left| \int_a^b gh^s \right| \leq \gamma (b-a)^{(p-s)/p} \| h \|_p^s \quad \forall h \in X. \tag{3} \]
The above inequality is true also, as easily seen, for \( p = s \) and \( p = \infty \) (setting \( (p - s)/p = 1 \) in the former case); from it we get \( \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \int_a^b gh^s = 0 \) because \( s > 1 \).

Assume now that \( p < s \). Since \( g \in C[a, b] \setminus \{0\} \), there exist \( \delta > 0 \), and \( a', b' \in [a, b] \) with \( a' < b' \) such that \( g(x) \geq \delta \) for \( x \in [a', b'] \) or \( g(x) \leq -\delta \) for \( x \in [a', b'] \). Doing a translation, we suppose that \( a' = 0 \). For \( n \in \mathbb{N}^* \) with \( n \geq n_0 \) (\( \geq 2/b' \)) consider

\[
h_n(x) := \begin{cases} 
\alpha_n x & \text{if } x \in [0, 1/n], \\
\alpha_n (2/n - x) & \text{if } x \in (1/n, 2/n), \\
0 & \text{if } x \in [a, 0) \cup [2/n, b],
\end{cases}
\]

with \( \alpha_n := n^{1+\gamma/p} > 0 \), where \( \frac{2}{b' - 1} \leq \gamma < 1 \). Clearly, \( h_n \in X = C_0[a, b] \). In this situation

\[
\left| \int_a^b g h_n^s \right| = \int_0^{2/n} |g| h_n^s \geq 2\delta \int_0^{1/n} (\alpha_n x)^s dx = 2\delta \alpha_n^s \frac{1}{s + 1 n^{s+1}} = \frac{2\delta}{s + 1} n^{\frac{\gamma - p}{p}},
\]

while a similar argument gives

\[
\|h_n\|_p = \left( \frac{2\alpha_n^p}{p + 1 n^{p+1}} \right)^{1/p} = \left( \frac{2}{p + 1} \right)^{1/p} n^{\frac{\gamma - 1}{p}} \to 0.
\]

On the other hand,

\[
\frac{1}{\|h_n\|_p} \left| \int_a^b g h_n^s \right| \geq \frac{2\delta}{s + 1} \left( \frac{p + 1}{2} \right)^{1/p} n^{\frac{\gamma - (p - 1)}{p}} \to \infty,
\]

which proves our assertion. The proof is complete.

**Proposition 2** Let \( X = C_0[a, b] \) be endowed with the norm \( \|\cdot\|_p \), where \( p \in [1, \infty] \). Then \( K \) is Gâteaux differentiable; moreover, for \( v \in X \), \( K \) is Fréchet differentiable at \( v \) if and only if \( p \geq 4 \).

Proof. Let \( v \in X \) and \( g_2 := \frac{1}{2} \theta (\frac{3}{4} v^2 - \lambda) \), \( g_3 := \frac{1}{2} \theta v \) and \( g_4 := \frac{1}{4} \theta v^2 \); of course, \( g_2, g_3, g_4 \in C[a, b] \).

Set also \( \beta := \max\{\|g_2\|_\infty, \|g_3\|_\infty\} \).

Observe that for all \( v, h \in X \) we have that

\[
K(v + h) = K(v) + \int_a^b \theta \left[ v \left( \frac{3}{4} v^2 - \lambda \right) - F \right] h + \int_a^b \frac{1}{2} \theta \left( \frac{3}{4} v^2 - \lambda \right) h^3 + \int_a^b \frac{1}{2} \theta vh^3 + \int_a^b \frac{1}{8} \theta h^4.
\]

For \( v \in X \) consider

\[
T_v : X \to \mathbb{R}, \quad T_v(h) := \int_a^b \theta \left[ v \left( \frac{3}{4} v^2 - \lambda \right) - F \right] h \quad (h \in X).
\]

Clearly, \( T_v \) is a linear operator; \( T_v \) is also continuous for every \( p \in [1, \infty] \). Indeed, setting \( \gamma_v := \|\theta \left[ v \left( \frac{3}{4} v^2 - \lambda \right) - F \right] \|_\infty \in \mathbb{R}_+ \) we have that

\[
|T_v(h)| \leq \gamma_v \int_a^b |h| \leq \gamma_v \|h\|_p \cdot \|1\|_{p'} = \gamma_v (b - a)^{1/p'} \|h\|_p \quad \forall h \in X
\]

for \( p, p' \in [1, \infty] \) with \( p' \) the conjugate of \( p \), that is \( p' := p/(p - 1) \) for \( p \in (1, \infty) \), \( p' := \infty \) for \( p = 1 \) and \( p' := 1 \) for \( p = \infty \). Hence \( T_v \) is continuous.
Let \( p \in [1, \infty] \) and \( v \in X \) be fixed. Using (5) we have that

\[
\frac{|K(v + h) - K(v) - T_v(h)|}{\|h\|_p} \leq \frac{1}{\|h\|_p} \left( \left| \int_a^b g_2 h^2 \right| + \left| \int_a^b g_3 h^3 \right| + \left| \int_a^b g_4 h^4 \right| \right)
\]

for \( h \neq 0 \). Using Lemma 1 for \( p \geq 4 \), we obtain that \( \lim_{\|h\|_p \to 0} \frac{K(v + h) - K(v) - T_v(h)}{\|h\|_p} = 0 \). Hence \( K \) is Fréchet differentiable at \( v \).

Assume now that \( p < 4 \). Using again (5) we have that

\[
K(v + h) - K(v) - T_v(h) \geq \frac{\mu}{8} \int_a^b h^4 - \beta \int_a^b |h|^3 - \beta \int_a^b h^2 \quad \forall h \in X.
\]

Take \( a = a’ = 0 < b’ = b \) (possible after a translation), \( \alpha_n := n^{1+\gamma/p} \) with \( \frac{3}{8} < \gamma < 1 \) and \( h := h_n \) defined by (4). Using the computations from the proof of Lemma 1, we get

\[
\int_a^b |h_n|^s = 2 \int_0^{1/n} (\alpha_n x)^{s} dx = \frac{2}{s+1} n^{\frac{s}{s-1} - p}, \quad \|h_n\|_p = \left( \frac{2}{p+1} \right)^{1/p} \frac{1}{n^{(1-\gamma)/p}} \to 0,
\]

whence

\[
\frac{K(v + h_n) - K(v) - T_v(h_n)}{\|h_n\|_p} \geq \left( \frac{\mu+1}{2} \right)^{1/p} \frac{1}{n^{1-\gamma/p}} \left( \frac{\mu}{8} \cdot \frac{2}{s+1} n^{\frac{s}{s-1} - p} - \frac{2}{3} \beta_3 n^{\frac{s}{s-1} - p} - \frac{2}{3} \beta_2 n^{\frac{s}{s-1} - p} \right)
\]

\[
= \left( \frac{\mu+1}{2} \right)^{1/p} n^{\frac{s}{s-1} - p} \left( \frac{2}{s+1} - \frac{1}{2} \beta_3 n^{\frac{\gamma}{\gamma-1}} - \frac{2}{3} \beta_2 n^{\frac{2\gamma}{\gamma-1}} \right) \to \infty.
\]

This shows that \( K \) is not Fréchet differentiable at \( v \).

Because \( K : (X, \|\cdot\|_\infty) \to \mathbb{R} \) is Fréchet differentiable at \( v \in X \), it follows that

\[
\lim_{t \to 0} \frac{K(v + t h) - K(v)}{t} = T_v(h) \in \mathbb{R} \quad \forall h \in X.
\]

Because \( T_v : (X, \|\cdot\|_p) \to \mathbb{R} \) is linear and continuous, it follows that \( K \) is Gâteaux differentiable at \( v \) for every \( p \in [1, \infty] \) with \( \nabla K(v) = T_v \). \( \square \)

We consider now the problem of finding the stationary points of \( K \), that is those points \( v \in X \) with \( T_v = 0 \).

**Proposition 3** The functional \( K \) has only one stationary point \( \bar{v} \). More precisely, for each \( x \in [a, b] \), \( \bar{v}(x) \) is the unique solution from \((-\sqrt{2\lambda/3}, \sqrt{2\lambda/3})\) of the equation \( z(\frac{1}{2}z^2 - \lambda) = F(x) \).

Proof. Assume that \( v \in X \) is stationary; hence \( T_v h = \int_a^b V h = 0 \) for every \( h \in X \), where \( V := \theta v(\frac{1}{2}v^2 - \lambda) - F \) (\( \in X \subset C[a,b] \)). We claim that \( V = 0 \). In the contrary case, since \( V \) is continuous, there exists \( x_0 \in (a, b) \) with \( V(x_0) \neq 0 \). Suppose that \( V(x_0) > 0 \). By the continuity of \( V \) there exist \( a’, b’ \in \mathbb{R} \) such that \( a < a’ < x_0 < b’ < b \) and \( V(x) > 0 \) for every \( x \in [a’, b] \). Take

\[
\bar{h} : [a, b] \to \mathbb{R}, \quad \bar{h}(x) := \begin{cases} \frac{x-a’}{b’-a} & \text{if } x \in (a’, \frac{1}{2}(a’ + b’)) \\ \frac{b’-x}{b’-a} & \text{if } x \in (\frac{1}{2}(a’ + b’), b’), \\ 0 & \text{if } x \in [a, a’] \cup (b’, b].
\end{cases}
\]
Moreover, the mappings $\| \cdot \|_\infty$ is a local maximizer for $K$ with respect to $\| \cdot \|_\infty$, and $\overline{\nu}$ is not a local extremum point of $K$ with respect to $\| \cdot \|_p$ for $p \in [1,4)$.

Proof. Let us consider first the case $p = \infty$. From (5) we get

$$K(\overline{\nu} + h) - K(\overline{\nu}) = \int_a^b \theta \left[ \frac{1}{2} (\frac{3}{2} \overline{\nu}^2 - \lambda) + \frac{1}{2} \overline{\nu} h + \frac{1}{2} h^2 \right] h^2 \forall h \in X. \quad (12)$$

Since $F \in C[a,b]$, there exists some $x_0 \in [a,b]$ such that $\| F \|_\infty = |F(x_0)| < (2\lambda/3)^{3/2}$, and so $|\overline{\nu}(x)| < |\overline{\nu}(x_0)| =: \gamma < \sqrt{2\lambda/3}$ for $x \in [a,b]$. It follows that $\frac{1}{2} (\frac{3}{2} \overline{\nu}^2 - \lambda) \leq \frac{1}{2} (\frac{3}{2} \gamma^2 - \lambda) =: -\eta < \frac{1}{2} (\frac{3}{2} \gamma^2 - \lambda) = 0$. Hence

$$\frac{1}{2} (\frac{3}{2} \overline{\nu}^2 - \lambda) + \frac{1}{2} \overline{\nu} h + \frac{1}{2} h^2 \leq -\eta + \frac{1}{2} \gamma \| h \|_\infty + \frac{1}{8} \| h \|_\infty^2 < 0 \forall h \in X, \quad \| h \|_\infty < \varepsilon, \quad (13)$$

where $\varepsilon := 2(\sqrt{\gamma^2 + 2\eta} - \gamma)$. It follows that $\overline{\nu}$ is a (strict) local maximizer of $K$.

Assume now that $p \in [1,4)$. Of course, there exists a sequence $(h_n)_{n \geq 1} \subset X \setminus \{0\}$ such that $\| h_n \|_\infty \to 0$. Taking into account (13), we have that $K(\overline{\nu} + h_n) < K(\overline{\nu})$ for large $n$. Since $\| h_n \|_p \to 0$, $\overline{\nu}$ is not a local minimizer of $K$ with respect to $\| \cdot \|_p$. In the proof of Proposition 2 we found a sequence $(h_n)_{n \geq 1} \subset X \setminus \{0\}$ such that $\| h_n \|_p \to 0$ and $\| h_n \|_p^{-1} (K(\overline{\nu} + h_n) - K(\overline{\nu}) - T_{\overline{\nu}} h_n) \to \infty$. Since $T_{\overline{\nu}} = 0$, we obtain that $K(\overline{\nu} + h_n) - K(\overline{\nu}) > 0$.
for large $n$, proving that $\pi$ is not a local maximizer of $K$. Hence $\pi$ is not a local extremum point of $K$. □

We don’t know if $\pi$ is a local maximizer of $K$ for $p \in [4, \infty)$; having in view (13), surely, $\pi$ is not a local minimizer of $K$.

Proposition 4 shows the importance of the norm (and more generally, of the topology) on a space when speaking about local extrema.

Let us establish now the relations between the local extrema of $J$ with the constraints (6) and (7) in [1], that is, local extrema of $J$ restricted to $C_{1,0}[a,b]$, and the local extrema of $K$ in the case in which $C^1[a,b]$ is endowed with the (usual) norm defined by

$$
\|u\| := \|u\|_\infty + \|u'\|_\infty \quad (u \in C^1[a,b]),
$$

and $C_0[a,b]$ is endowed with the norm $\|\cdot\|_\infty$.

**Proposition 5** Consider the norm $\|\cdot\|$ (defined in (14)) on $C^1[a,b]$ and the norm $\|\cdot\|_\infty$ on $C_0[a,b]$. If $\pi$ is a local minimizer (maximizer) of $J$ on $C_{1,0}[a,b]$, then $\pi'$ is a local minimizer (maximizer) of $K$. Conversely, if $\pi$ is a local minimizer (maximizer) of $K$, then $\pi \in C^1[a,b]$ defined by $\pi(x) := u_0 + \int_a^x \pi'(\xi)d\xi$ for $x \in [a,b]$ and a fixed $u_0 \in \mathbb{R}$ is a local minimizer (maximizer) of $J$ on $C_{1,0}[a,b]$.

Proof. Assume that $\pi$ is a local minimizer of $J$ on $C_{1,0}[a,b]$; hence $\pi \in C_{1,0}[a,b]$. It follows that there exists $r > 0$ such that $J(\pi') \leq J(u)$ for every $u \in C_{1,0}[a,b]$ with $\|u - \pi\| < r$. Set $\pi' := \pi'$ and take $v \in X = C_0[a,b]$ with $\|v - \pi\|_\infty < r' := r/(1 + b - a)$. Define $u : [a,b] \to \mathbb{R}$ by $u(x) := \pi(a) + \int_a^x v(\xi)d\xi$ for $x \in [a,b]$. Then $u \in C_{1,0}[a,b]$ and $u' = v$. Since $\pi(x) = \pi(a) + \int_a^x \pi'(\xi)d\xi$, we get

$$
\|u - \pi\| = \|u - \pi\|_\infty + \|u' - \pi'\|_\infty \leq (b - a) \|v - \pi\|_\infty + \|v - \pi\|_\infty < r'(1 + b - a) = r.
$$

Hence $K(\pi') = J(\pi') \leq J(u) = K(v)$. This shows that $\pi$ is a local minimizer for $K$.

Conversely, assume that $\pi$ is a local minimizer for $K$. Then there exists $r > 0$ such that $K(\pi') \leq K(v)$ for $v \in C_0[a,b]$ with $\|v - \pi\| < r$, and take $u_0 \in \mathbb{R}$ and $\pi : [a,b] \to \mathbb{R}$ defined by $\pi(x) := u_0 + \int_a^x \pi'(\xi)d\xi$ for $x \in [a,b]$. Then $\pi \in C_{1,0}[a,b]$. Consider $u \in C_{1,0}[a,b]$ with $\|u - \pi\| < r$, that is

$$
\|u - \pi\|_\infty + \|u' - \pi'\|_\infty = \|u - \pi\|_\infty + \|u' - \pi'\|_\infty < r; \quad \text{then } \|u' - \pi'\|_\infty < r.
$$

Since $u' \in C_0[a,b]$, it follows that $J(u) = K(u') \geq K(\pi') = J(\pi)$, and so $\pi$ is a local minimizer of $J$ on $C_{1,0}[a,b]$. The case of local maximizers for $J$ and $K$ is treated similarly. □

Putting together Propositions 3, 4 and 5 we get the next result.

**Theorem 6** Consider the norm $\|\cdot\|$ (defined in (14)) on $C^1[a,b]$ and the norm $\|\cdot\|_\infty$ on $C_0[a,b]$. Let $\pi \in C_{1,0}[a,b]$ and set $\pi := \pi$. Then the following assertions are equivalent:

(i) $\pi$ is a local maximum point of $J$ restricted to $C_{1,0}[a,b]$.

(ii) $\pi$ is a local extremum point of $J$ restricted to $C_{1,0}[a,b]$.

(iii) $\pi$ is a stationary point of $K$.

(iv) $\pi$ is a local extremum point of $K$.

(v) $\pi$ is a local maximum point of $K$.

(vi) $\pi = z_2 \circ F$, where $z_2(A)$ is the unique solution of the equation $z \left(\frac{1}{2}z^2 - \lambda\right) = A$ in the interval $(-\sqrt{2\lambda}/3, \sqrt{2\lambda}/3)$ for $A \in (-2\lambda/3)^{3/2}, (2\lambda/3)^{3/2})$.

(vii) there exists $u_0 \in \mathbb{R}$ such that $\pi(x) = u_0 + \int_a^x z_2(F(\rho))d\rho$ for every $x \in [a,b]$.
3 Discussion of Theorem 1.1 from Gao and Lu’s paper [1]

First of all, we think that in the formulation of [1, Th. 1.1], “local extrema for the nonconvex functional (2)” must be replaced by “local extrema for the nonconvex functional (2) with the constraints (6) and (7)”, “local minimizer for the nonconvex functional (2)” must be replaced by “local minimizer for the nonconvex functional (2) with the constraints (6) and (7)” (2 times), and “local maximizer for the nonconvex functional (2)” must be replaced by “local maximizer for the nonconvex functional (2) with the constraints (6) and (7)” (2 times). Below, we interpret [1, Th. 1.1] with these modifications.

As pointed in Introduction, no norms are considered on the spaces mentioned in [1]. For this reason in Theorem 5 we considered the usual norms on $C^1[a, b]$ and $C_0[a, b]$; these norms are used in this discussion. Moreover, let $\theta(x) := 1$ for $x \in [a, b]$ in Theorem 6 and $\nu = 1$ in [1, Th. 1.1]. In the conditions of [1, Th. 1.1] $F(x) > 0$ for $x \in (a, b)$ or $F(x) < 0$ for $x \in (a, b)$. For the present discussion we take the case $F > 0$ on $(a, b)$.

Assume that the mappings

$$\rho \mapsto F(\rho)/E_j^{-1}(F^2(\rho)) =: v_j(\rho)$$  \hspace{1cm} (15)

[where “$E_3^{-1}(A) \leq E_2^{-1}(A) \leq E_1^{-1}(A)$” stand for the three real-valued roots for the equation $E(y) = A^2$ with $E(y) = 2y^2(y + \lambda)$ and $A \in [0, 8\lambda^3/27]$ are well defined for $\rho \in \{a, b\}$ [there are no problems for $\rho \in (a, b)$].

If [1, Th. 1.1] is true, then $v_1, v_2, v_3 \in C_0[a, b]$; moreover, $v_1$ and $v_2$ are local minimizers of $K$, and $v_3$ is a local maximizer of $K$. This is of course false taking into account Theorem 6 because $K$ has not local minimizers.

Because $z_2 \circ F$ is the unique local maximizer of $K$, we must have that $v_3 = z_2 \circ F$. Let us see if this is true. Because $z_i(A)$ are solutions of the equation $G(z) = A$ and $E_j^{-1}(A)$ are solutions of the equation $E(y) = A$, we must study the relationships among these numbers.

First, the behavior of $E$ is given in the next table.

| $y$ | $E'(y)$ | $E(y)$ |
|-----|---------|--------|
| $-\infty$ | $+$ | $-\infty$ |
| $-\lambda$ | $+$ | $0$ |
| $-\frac{2\lambda}{3}$ | $0$ | $-\frac{2\lambda}{3}$ |
| $0$ | $+$ | $0$ |
| $\frac{\lambda^2}{2}$ | $-\infty$ | $\frac{\lambda^2}{2}$ |
| $\infty$ | $	riangleright$ | $	riangleright$ |

Secondly, for $y, z, A \in \mathbb{C} \setminus \{0\}$ such that $yz = A$ we have that

$$G(z) = A \iff A = y \left(\frac{1}{2} \frac{A^2}{y^2} - \lambda\right) = A \iff 2y^2(y + \lambda) = A^2 \iff E(y) = A^2.$$  \hspace{1cm} (16)

Analyzing the behavior of $G$ and $E$ (recall that $\kappa = \sqrt{2\lambda/3}$), and the relation $yz = A$ for $A \neq 0$ (mentioned above), the correspondence among the solutions of the equations $G(z) = A$ and $E(y) = A^2$ for $A \in (0, (2\lambda/3)^{3/2})$ is:

$$z_1(A) = A/E_2^{-1}(A^2), \quad z_2(A) = A/E_3^{-1}(A^2), \quad z_3(A) = A/E_1^{-1}(A^2)$$  \hspace{1cm} (17)

for all $A \in (0, (2\lambda/3)^{3/2})$. This shows that only the third assertion of [1, Th. 1.1] is true (of course, considering the norm defined in [13] on $C^1[a, b]$).
4 Discussion of Theorem 1.1 from Lu and Gao’s paper [4]

A similar problem to that in [1], discussed above, is considered in [4]. In the abstract of this paper one finds:

“In comparison with the 1D case discussed by D. Gao and R. Ogden, there exists huge difference in higher dimensions, which will be explained in the theorem”.

More precisely, in [4] it is said:

“In this paper, we consider the fourth-order polynomial defined by

\[ H(\bar{\gamma}) := \nu/2 \left(1/2|\bar{\gamma}|^2 - \lambda\right)^2, \bar{\gamma} \in \mathbb{R}^n, \nu, \lambda > 0 \text{ are constants, } |\bar{\gamma}|^2 = \bar{\gamma} \cdot \bar{\gamma}. \]

... The purpose of this paper is to find the extrema of the following nonconvex total potential energy functional in higher dimensions,

\[ I[u] := \int_{\Omega} (H(|\nabla u|) - fu) \, dx, \]

where \( \Omega = \text{Int} \{ \mathbb{B}(O, R_1) \setminus \mathbb{B}(O, R_2) \}, R_1 > R_2 > 0, \mathbb{B}(O, R_1) \text{ and } \mathbb{B}(O, R_2) \text{ denote two open balls with center } O \text{ and radii } R_1 \text{ and } R_2 \text{ in the Euclidean space } \mathbb{R}^n, \text{ respectively. } “\text{Int}” \text{ denotes the interior points. In addition, let } \Sigma_1 := \{ x : |x| = R_1 \}, \text{ and } \Sigma_2 := \{ x : |x| = R_2 \}, \text{ then the boundary } \partial \Omega = \Sigma_1 \cup \Sigma_2. \]

The radially symmetric function \( f \in C(\overline{\Omega}) \) satisfies the normalized balance condition

\[ (2) \int_{\Omega} f(|x|) \, dx = 0, \]

and

\[ (3) f(|x|) = 0 \text{ if and only if } |x| = R_3 \in (R_2, R_1). \]

Moreover, its \( L^1 \)-norm is sufficiently small such that

\[ (4) \|f\|_{L^1(\Omega)} < 4\nu R_2^{n-1}\sqrt{2\lambda\pi^n}/(3\sqrt{3}\Gamma(n/2)), \]

where \( \Gamma \) stands for the Gamma function. This assumption is reasonable since large \( \|f\|_{L^1(\Omega)} \) may possibly lead to instant fracture. The deformation \( u \) is subject to the following three constraints,

\[ (5) u \text{ is radially symmetric on } \overline{\Omega}, \]

\[ (6) u \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega}), \]

\[ (7) \nabla u \cdot \bar{n} = 0 \text{ on both } \Sigma_1 \text{ and } \Sigma_2, \]

where \( \bar{n} \) denotes the unit outward normal on \( \partial \Omega \).

By variational calculus, one derives a correspondingly nonlinear Euler–Lagrange equation for the primal nonconvex functional, namely,

\[ (8) \text{div} (\nabla H(|\nabla u|)) + f = 0 \text{ in } \Omega, \]

equipped with the Neumann boundary condition (7). Clearly, (8) is a highly nonlinear partial differential equation which is difficult to solve by the direct approach or numerical method [2, 15]. However, by the canonical duality method, one is able to demonstrate the existence of solutions for this type of equations.

... Before introducing the main result, we denote

\[ F(r) := -1/r^n \int_{R_2}^r f(\rho)\rho^{n-1} \, d\rho, \quad r \in [R_2, R_1]. \]

Next, we define a polynomial of third order as follows,

\[ E(y) := 2y^2(\lambda + y/\nu), \quad y \in [-\nu\lambda, +\infty). \]

Furthermore, for any \( A \in [0, 8\lambda^3\nu^2/27], \)

\[ E_3^{-1}(A) \leq E_2^{-1}(A) \leq E_1^{-1}(A) \]

stand for the three real-valued roots for the equation \( E(y) = A. \)

At the moment, we would like to introduce the theorem of multiple extrema for the nonconvex functional (2).
Theorem 1.1. For any radially symmetric function $f \in C(\bar{\Omega})$ satisfying (2)–(4), we have three solutions for the nonlinear Euler–Lagrange equation (8) equipped with the Neumann boundary condition, namely

- For any $r \in [R_2, R_1]$, $\overline{\psi}_1$ defined below is a local minimizer for the nonconvex functional (2),

$$\overline{\psi}_1(|r|) = \overline{\psi}_1(r) := \int_{R_2}^r F(\rho)/E_1^{-1}(F^2(\rho)\rho^2)\,d\rho + C_1, \quad \forall C_1 \in \mathbb{R}.$$  

- For any $r \in [R_2, R_1]$, $\overline{\psi}_2$ defined below is a local minimizer for the nonconvex functional (2) in 1D. While for the higher dimensions $n \geq 2$, $\overline{\psi}_2$ is not necessarily a local minimizer for (2) in comparison with the 1D case.

$$\overline{\psi}_2(|r|) = \overline{\psi}_2(r) := \int_{R_2}^r F(\rho)/E_2^{-1}(F^2(\rho)\rho^2)\,d\rho + C_2, \quad \forall C_2 \in \mathbb{R}.$$  

- For any $r \in [R_2, R_1]$, $\overline{\psi}_3$ defined below is a local maximizer for the nonconvex functional (2),

$$\overline{\psi}_3(|r|) = \overline{\psi}_3(r) := \int_{R_2}^r F(\rho)/E_3^{-1}(F^2(\rho)\rho^2)\,d\rho + C_3, \quad \forall C_3 \in \mathbb{R}.$$  

In the final analysis, we apply the canonical duality theory to prove Theorem 1.1.”

First, observe that one must have (1) instead of (2) just before the statement of [4, Th. 1.1], as well as in its statement, excepting for (2)–(4). Secondly, (even from the quoted texts) one must observe that the wording in [1] and [4] is almost the same; the mathematical part is very, very similar, too.

To avoid any confusion, in the sequel the Euclidian norm on $\mathbb{R}^n$ will be denoted by $|\cdot|_n$ instead of $|\cdot|$.

Remark that it is said $f \in C(\bar{\Omega})$, which implies $f$ is applied to elements $x \in \bar{\Omega}$, while a line below one considers $f(|x|)$ (that is $f(|x|_n)$ with our notation); because the (Euclidean) norm $|x|_n$ of $x \in \bar{\Omega}$ belongs to $[R_2, R_1]$, writing $f(|x|)$ shows that $f : [R_2, R_1] \to \mathbb{R}$. Of course, these create ambiguities. Probably the authors wished to say that a function $g : \bar{\Omega} \to \mathbb{R}$ is radially symmetric if there exists $\psi : [R_2, R_1] \to \mathbb{R}$ such that $g(x) = \psi(|x|_n)$ for every $x \in \bar{\Omega}$, that is $g = \psi \circ |\cdot|_n$ on $\Omega$; observe that $\psi$ is continuous if and only if $\psi \circ |\cdot|_n$ is continuous. Because also the functions $u$ in the definition of $I$ are asked to be radially symmetric on $\Omega$ (see [4] (5)), it is useful to observe that for a Riemann integrable function $\psi : [R_2, R_1] \to \mathbb{R}$, using the usual spherical change of variables, we have that

$$\int_\Omega (\psi \circ |\cdot|_n)(x)\,dx = \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \int_{R_2}^{R_1} r^{n-1}\psi(r)\,dr = \gamma_n \int_{R_2}^{R_1} \theta \psi, \quad \gamma_n := \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad \theta : [R_2, R_1] \to \mathbb{R}, \quad \theta(r) := r^{n-1}.$$  

So, in the sequel we consider that $f : [R_2, R_1] \to \mathbb{R}$ is continuous. Condition [4] (2) becomes $\int_{R_2}^{R_1} \theta f = 0$ [for the definition of $\theta$ see (19)], condition [4] (3) is equivalent to the existence of a unique $R_3 \in (R_2, R_1)$ such that $f(R_3) = 0$ (that is $(\theta f)(R_3) = 0$), while condition [4] (4) is equivalent to $\|\theta f\|_{L^1[R_2, R_1]} < \nu R_2^{n-1}(2\lambda/3)^{3/2}$.

Moreover, condition [4] (5) is equivalent to the existence of $v : [R_2, R_1] \to \mathbb{R}$ such that $u = v \circ |\cdot|_n$, while the condition $u \in C(\bar{\Omega})$ is equivalent to $v \in C[R_2, R_1]$. Which is the meaning of $\nabla u(x)$ in condition [4] (7) for $u \in W^{1,\infty}(\Omega)$ and $x \in \Sigma_1$ (or $x \in \Sigma_2$)? For example, let us consider $v : [1, 3] \to \mathbb{R}$ defined by $v(t) := (t - 1)^2 \sin\frac{1}{t-1}$ for
$t \in (1, 2)$. Is $u := v \circ |.|_n$ in $W^{1,\infty}(\Omega)$ for $R_2 := 1$ and $R_1 := 2$? If YES, which is $\nabla u(x)$ for $x \in \mathbb{R}^n$ with $|x|_n = 1$?

Let us assume that $v \in C^1(R_2 - \varepsilon, R_1 + \varepsilon)$ for some $\varepsilon \in (0, R_2)$ and take $u := v \circ |.|_n$. Then clearly $u \in C^1(\Delta)$, where $\Delta := \{x \in \mathbb{R}^n \mid |x|_n \in (R_2 - \varepsilon, R_1 + \varepsilon)\}$, and

$$\nabla u(x) = v'(|x|_n) |x|_n^{-1} x, \quad |\nabla u(x)|_n = |v'(|x|_n)|$$

(20)

for all $x \in \Delta$. Without any doubt, $u|_{\Omega} \in W^{1,\infty}(\Omega)$; moreover, $\nabla u(x)\bar{n} = v'(|x|_n) |x|_n^{-1} x \cdot (|x|_n^{-1} x) = v'(R_1)$ for every $x \in \Sigma_1$ and $\nabla u(x)\bar{n} = -v'(R_2)$ for $x \in \Sigma_2$. Hence such a $u|_{\Omega}$ satisfies condition \[4\ (7)] if and only if $v'(R_1) = v'(R_2) = 0$.

Having in view the remark above, we discuss the result in \[4\ Th. 1.1\] for $W^{1,\infty}(\Omega)$ replaced by $C^1(\overline{\Omega})$, more precisely the result in \[4\] concerning the local extrema of $I$ defined in \[4\ (1)\] (quoted above) on the space

$$U := \{u := v \circ |.|_n \mid v \in C^1[R_2, R_1], \quad v'(R_1) = v'(R_2) = 0\}$$

$$= \{v \circ |.|_n \mid v \in C_{1,0}[R_2, R_1]\} \subset C^1(\overline{\Omega})$$

when $C^1(\overline{\Omega})$ (and $U$) is endowed with the norm

$$\|u\| := \|u\|_\infty + \|\nabla u\|_\infty;$$

(21)

moreover, in the sequel, $V := C_0[R_2, R_1]$ is endowed with the norm $\|\|_\infty$.

Unlike \[4\], let us set

$$F(r) := -\frac{1}{r^{n-1}} \int_{R_2}^r f(\rho)\rho^{n-1}d\rho = -\frac{1}{r^{n-1}} \int_{R_2}^r \theta f, \quad r \in [R_2, R_1],$$

(22)

where $\theta$ is defined in \[19\].

**Remark 7** Notice that our $F(r)$ is $r$ times the one introduced in \[4\].

From (22) and the hypotheses on $f$, we have that $F(R_1) = F(R_2) = 0$ and $(\theta f)' = -\theta f$ on $[R_2, R_1]$. Since

$$(\theta f)'(r) = 0 \iff (\theta f)(r) = 0 \iff f(r) = 0 \iff r = R_3$$

and $(\theta f)(R_1) = (\theta f)(R_2) = 0$, it follows that $\theta F > 0$ or $\theta F < 0$ on $(R_2, R_1)$, that is $F > 0$ or $F < 0$ on $(R_2, R_1)$. Moreover, from the definition of $F$ we get

$$R_2^{n-1} |F(r)| \leq |r^{n-1} F(r)| = \int_{R_2}^r |\theta f| = \|\theta f\|_{L^1[R_2, R_1]} < R_2^{n-1}(2\lambda/3)^{3/2}$$

for every $r \in [R_2, R_1]$, whence $|F(r)| < (2\lambda/3)^{3/2}$ for $r \in [R_2, R_1]$.

Let $u \in U$, that is $u := v \circ |.|_n$ with $v \in C_{1,0}[R_2, R_1]$, and set $v := v'$ $(\in C_0[R_2, R_1])$. We have that

$$\int_{R_2}^{R_1} \theta f v = -\int_{R_2}^{R_1} (\theta f) v = -(|\theta f| v)_{R_2}^{R_1} + \int_{R_2}^{R_1} \theta f v' = \int_{R_2}^{R_1} \theta F v.$$
Using (18) and (20) we get
\[ I[u] = \int_{\Omega} [H(|\nabla u(x)|) - f(|x|)u(x)] \, dx = \int_{\Omega} [H(\|v'(|x|)\|) - f(|x|)v(|x|)] \, dx \]
\[ = \gamma_n \int_{R_1}^{R_2} \theta(H \circ |v| - Fv) = \gamma_n \int_{R_2}^{R_1} \theta(H \circ v - Fv), \]
that is
\[ I[u] = \gamma_n K(v), \]
where \( K \) is defined in (2) and \([a, b] := [R_2, R_1]\). Therefore, Theorem 6 applies also in this situation. Applying it we get that \( I \) defined in [4, (1)] has no local minimizers and \( \overline{u} \in C^1(\overline{\Omega}) \) is a local maximizer of \( I|U \) if and only if there exists \( u_0 \in \mathbb{R} \) such that \( \overline{u}(x) = u_0 + \int_{|x|}^{R_2} z_2(F(\rho)) \, d\rho \) for every \( x \in \overline{\Omega} \), where \( z_2(A) \) is the unique solution of the equation
\[ z \left( \frac{1}{2} z^2 - \lambda \right) = A \text{ in the interval } (-\sqrt{2\lambda/3}, \sqrt{2\lambda/3}) \text{ for } A \in (-2\lambda/3)^{3/2}, (2\lambda/3)^{3/2}). \]

For the present discussion we take the case in which \( F > 0 \) on \((a, b)\). In this case observe that
\[ z_2(A) = A/E_2^{-1}(A^2) \text{ for } A \in (0, (2\lambda/3)^{3/2}). \]
This proves that the first and second assertions of [4, Th. 1.1] are \textit{false}; in particular, \( \overline{u}_2 \) is not a local minimizer of \( I|U \) (exactly as in the 1D case).
Moreover, from the discussion above, we can conclude that also the assertion “In comparison with the 1D case discussed by D. Gao and R. Ogden, there exists huge difference in higher dimensions” from the abstract of [4] is false.

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