Sudden Singularities in Brans-Dicke Cosmology
and its Generalisations

John D. Barrow
DAMTP, Centre for Mathematical Sciences
University of Cambridge, Wilberforce Rd.,
Cambridge CB3 0WA, United Kingdom

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Abstract

We show that cosmological sudden singularities that respect the energy conditions can occur at finite times in Brans-Dicke and more general scalar-tensor theories of gravity. We construct these explicitly in the Friedmann universes. Higher-order versions of these singularities are also possible, including those that arise with when scalar fields have a self-interaction potential of power-law form.

1 Introduction

Following the realisation that finite-time ‘sudden’ singularities can arise in general relativistic cosmologies where the scale factor, its first time derivative and the fluid density can remain finite whilst its second derivative and fluid pressure diverge although the strong energy condition remains unviolated [1, 2], there has been extensive study of this possibility and its close relatives. Generalisations were found in ref [3] and examples appeared in anisotropic cosmologies [4] and higher-order gravity theories with $f(R)$ lagrangians [3]. These are weak singularities in the senses of Tipler [5] and Krolak [6] and their conformal diagrams have been constructed in ref. [7]. Geodesics are left unscathed by sudden singularities, [8] and the general behaviour (in the function-counting sense) of the Einstein and geodesic equations in their neighbourhood was found in refs. [9, 10, 11]. This behaviour appears robust in the presence of quantum particle production [12]. The first examples were existence proofs that required unmotivated pressure-density relations, but more recently generalised singularities of this sort have been found by Barrow and Graham [13] to appear in simple flat Friedmann universes with scalar fields having power-law self-interaction potential $V(\phi) = V_0 \phi^n$, $0 < n < 1$. They always develop a finite-time singularity, where the Hubble rate and its first derivative are finite, but its second derivative
diverges. For non-integer values of \( n > 1 \), there is a class of models with even weaker higher-derivative singularities: infinities occur first at a finite time in the \((k + 2)^{th}\) time derivative of the Hubble expansion rate, where \( k < n < k + 1 \) and \( k \) is a positive integer [13]. These models inflate but inflation ends in a singular fashion.

In this paper we investigate whether finite-time singularities can occur in Brans-Dicke theory [14], which generalises general relativity by admitting the possibility of spacetime variation in the Newtonian gravitational ‘constant’, \( G \), via its promotion to the status of a scalar field that can vary in space and time. We show that sudden singularities can also appear in these theories and we display the effects that they have on the time-evolution of \( G(t) \). We show that our conclusions hold also in more general scalar-tensor theories in which the Brans-Dicke coupling parameter, \( \omega \), is no longer a constant and where a self-interaction potential is also introduced for the scalar field. Other investigations of the effect finite-time singularities on varying constants have been made [15] in the context of the BSBM theory for varying fine structure ‘constant’ [16], but not with varying \( G \), although the two could easily be combined, as in ref. [17].

2 Brans-Dicke Sudden Singularities

We assume the spacetime metric to be of isotropic and homogeneous form, with expansion scale factor \( a(t) \), where \( t \) is the comoving proper time coordinate, \((r, \theta, \phi)\) are polar coordinates, the curvature parameter, \( k \), takes values 0 or \( \pm 1 \) depending on the curvature of the space sections of constant time, and the speed of light is set to unity, so

\[
ds^2 = dt^2 - a^2(t)\left\{ \frac{dr^2}{1 - kr^2} + r^2(\sin^2\theta + d\phi^2) \right\}.
\] (1)

In standard notation, the field equations of Brans-Dicke theory with energy-momentum tensor \( T_{ab} \) and Brans-Dicke scalar field \( \phi(t) \) and scalar field coupling constant \( \omega \) are [14]:

\[
G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi_{,c}) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi),
\] (2)

\[
\Box\phi = \frac{8\pi}{3 + 2\omega}T_a^a,
\] (3)

\[
T_{a,b} = 0.
\] (4)

The Einstein-Brans-Dicke field equations for a Friedmann universe with metric (1), containing fluid with pressure \( p \) and density \( \rho \) are:
\[
\begin{align*}
3 \ddot{a} &= -\frac{8\pi}{(3 + 2\omega)\phi} [(2 + \omega)p + 3(1 + \omega)] - \omega \frac{\ddot{\phi}}{\phi^2} - \frac{\dot{\phi}}{\phi}, \\
\ddot{\phi} + 3 \frac{\dot{a}}{a} \phi &= \frac{8\pi}{(3 + 2\omega)} (\rho - 3p), \\
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) &= 0, \\
\frac{\dot{a}^2}{a^2} &= \frac{8\pi \rho - k}{3\phi} - \frac{\dot{\phi}}{\phi a} + \frac{\omega \dot{\phi}^2}{6 \phi^2}, \\
G &= \left(\frac{2\omega + 4}{2\omega + 3}\right) \phi^{-1}.
\end{align*}
\]

We are interested first to see if there can occur a sudden singularity at a finite time where \(\phi, \dot{\phi}, a, \ddot{\phi}, \rho\) are all finite, but where \(\ddot{\phi}, \rho\) can be infinite. Henceforth, we drop the curvature term \((k/a^2)\) and assume the flat geometry with \(k = 0\) since the curvature turns out to play no essential role in the discussion. In principle, singularities in second time-derivatives of the Brans-Dicke scalar field, \(\ddot{\phi}\), could occur at a different time to any in \(\ddot{a}\) and \(p\), but it is easy to show (as we will see below) that all finite-time singularities of this type have to occur at the same time in a Friedmann universe, which we label \(t_s\). At such a finite-time sudden singularity, we see that all terms in (8) are finite as \(t \to t_s\) from below and the dominant divergent terms in the remaining equations give the asymptotic system:

\[
\begin{align*}
3 \ddot{a} &\to -\frac{24\pi p(1 + \omega)}{(3 + 2\omega)\phi} \frac{\ddot{\phi}}{\phi}, \\
\ddot{\phi} &\to -\frac{24\pi p}{(3 + 2\omega)}, \\
\dot{\rho} &\to -3 \frac{\dot{a}}{a} \rho.
\end{align*}
\]

From (10) and (11), eliminating \(p\), we have a consistency relation:

\[
3 \ddot{a} a \to -\frac{24\pi p(1 + \omega)}{(3 + 2\omega)\phi} \ddot{\phi} - \frac{\dot{\phi}}{\phi} = \frac{\ddot{\phi}}{2\phi} (3 + 2\omega) - \frac{3\ddot{\phi}}{2\phi},
\]

and so, as \(t \to t_s\), we have

\[
\frac{\ddot{a}}{a} \to \frac{\omega \ddot{\phi}}{3\phi}.
\]

This require singularities in second derivatives of \(a(t)\) and \(\phi(t)\) to be simultaneous. We pick the following forms for the \(a(t)\) and \(\phi(t)\) evolution:

\[
\phi = \phi_s \left(\frac{t}{t_s}\right)^n - C \left(1 - \frac{t}{t_s}\right)^n,
\]

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with $0 < r < 1 < n < 2$, and

\[ a(t) = \left( \frac{t}{t_s} \right)^q (a_s - 1) + 1 - \left( 1 - \frac{t}{t_s} \right)^\lambda, \]  

(16)

with $0 < q < 1 < \lambda < 2$. Hence, as $t \to t_s$ we have,

\[ a \to a_s + q(1 - a_s)(1 - \frac{t}{t_s}) \to a_s, \]  

(17)

\[ \phi \to \phi_s(1 - r[1 - \frac{t}{t_s}]) \to \phi_s, \]  

(18)

\[ \frac{\ddot{\phi}}{\phi} \to -\frac{Cn(n-1)}{t_s^2}(1 - \frac{t}{t_s})^{n-2} \to \infty, \]  

(19)

\[ \frac{\ddot{a}}{a} \to -\frac{\lambda(\lambda-1)}{a_s t_s^2}(1 - \frac{t}{t_s})^{\lambda-2} \to \infty. \]  

(20)

The consistency condition (14) is satisfied if we take:

\[ n = \lambda, \]  

(21)

\[ C = \frac{3}{\omega a_s}. \]  

(22)

Therefore, the final form of the solution with the required simultaneous sudden singularity in second derivatives of $a(t)$ and $\phi(t)$ is:

\[ \phi = \phi_s \left( \frac{t}{t_s} \right)^r - \frac{3}{\omega a_s} \left( 1 - \frac{t}{t_s} \right)^n, \]  

(23)

\[ a(t) = \left( \frac{t}{t_s} \right)^q (a_s - 1) + 1 - \left( 1 - \frac{t}{t_s} \right)^\lambda, \]  

(24)

with $1 < n < 2$. We note that $\frac{\ddot{a}}{a} < 0$ and so the strong energy condition is still satisfied. We note also that if we wish to shift the sudden singularities up to appear in derivatives of $a$ and $\phi$ that are higher than second then this can be arranged choosing the range for $n$ suitably, with $r - 1 < n < r$ in order to have a generalised sudden singularity [3] with an infinity in the $r^{th}$ time derivatives of $a(t)$ and $\phi(t)$. Other varieties of sudden singularity (see [23] for a classification of types of finite-time singularity involving infinities in different combinations of cosmological variables in theories other than Brans-Dicke) can also be engineered by suitable choice of these parameters and their ranges.

If at early times, $t \to 0$, the solution behaves like the special Brans-Dicke exact solutions of eqs. (5)-(8) with ‘Machian’ initial condition [14] $\phi(0) = 0$, then

\[ a \propto \left( \frac{t}{t_s} \right)^q, \]  

(25)

\[ \phi \propto \left( \frac{t}{t_s} \right)^r. \]  

(26)
The relation between $q$ and $r$ that exists for the exact power-law Brans-Dicke solutions of (8) in the presence of a perfect-fluid source with equation of state [18]

$$p = (\gamma - 1)\rho$$

is

$$3\gamma q + r = 2$$

This ensures that $\rho/\phi \propto t^{-2}$, and all terms in the Friedmann-like equation (8) fall as $t^{-2}$ when $k = 0$ since $\rho \propto a^{-3\gamma}$ from (7).

However, these ‘Machian’ solutions are not the general solutions of eqs. (5)-(8). In the general solution of eqs. (5)-(8), [20, 22], then $\phi(0) \neq 0$ and the solution is dominated by the scalar field dynamics, rather than by the matter term, $\rho/\phi$, as $t \to 0$: it is ‘non-Machian’ in this sense. In that case, at early times we have $a \propto t^{(1-\beta)/(3-\beta)}$ and $\phi \propto t^{2\beta/3}$ with $\beta \equiv \left(\frac{3}{2\omega+1}\right)^{1/2}$, ensuring approach to the vacuum solution of O’Hanlon and Tupper [21] as $t \to 0$. For large $\omega$, on approach of the theory to general relativity, we have $\beta \to 0$, and hence,

$$a \propto t^{1/3}; \quad \phi \propto t^{2\beta/3},$$

and so we have $q = 1/3$ and $r = 2\beta/3$ for the possible early time behaviour in general if the vacuum stresses dominate, as we would expect.

As $t \to t_s$, we have the asymptotic forms

$$a(t) \to a_s + q(1-a_s)(1 - \frac{t}{t_s}),$$

$$\phi(t) \to \phi_s \left[1 - r\left(1 - \frac{t}{t_s}\right)\right],$$

$$G(t) \to \phi^{-1} \to \frac{G_s}{1 - r\left(1 - \frac{t}{t_s}\right)} \to G_s \left[1 + r\left(1 - \frac{t}{t_s}\right)\right],$$

$$\frac{\dot{\phi}}{\phi} = -\frac{\dot{G}}{G} \to \frac{r\left[1 - (r - 1)(1 - \frac{t}{t_s})\right]}{t_s\left[1 - r\left(1 - \frac{t}{t_s}\right)\right]} \to \frac{r}{t_s}.$$  

Present-day observations bound $r$ with $G/G \sim r/t_0 < 10^{-12}yr^{-1}$ when $t_0 << t_s$. The usual power-law fall-off in $G$ tails off to a constant value, $G_s$, which is smaller than the present value by a factor $t_0/t_s$.

### 3 More General Scalar-tensor Theories

It is straightforward to see the consequences of generalising from Brans-Dicke theory, where the coupling parameter, $\omega$, is constant, to a scalar-tensor gravity
theory where $\omega = \omega(\phi)$, as described in refs. \[25, 22, 24\]. The essential field equations \[??\] are generalised in this case to become \[25\]:

\[\begin{align*}
3 \ddot{a}/a &= -\frac{8\pi}{(3 + 2\omega)\phi} [(2 + \omega)\rho + 3p(1 + \omega)] - \omega \frac{\dot{\phi}^2}{\phi^2} - \frac{\dot{\phi}}{\phi} - \frac{\omega' \dot{\phi}^2}{2(3 + 2\omega)\phi} \\
\dot{\phi} + 3 \frac{\dot{a}}{a} \phi &= \frac{8\pi}{(3 + 2\omega)(\rho - 3p)} - \frac{\dot{\phi}^2}{(3 + 2\omega)} \phi'.
\end{align*}\]

(33) (34)

We can see that the appearance of the new non-zero $\omega'(\phi)$ terms does not affect the finite-time singularities created by the divergences of the $\dot{\phi}$ and $\ddot{a}$ terms above because the $\phi$ and $\dot{\phi}$ terms that multiply them tend to constants as $t \to t_s$ at the sudden singularity. Hence, we expect the behaviour at sudden singularities in general scalar-tensor theories to be as was described above for Brans-Dicke theory. Infinities can occur in $\omega(\phi)$ at some finite time (even the present day) but are usually harmless. The $\omega(\phi) \to \infty$ limit is part of the general relativity limit. The other requirement in this limit is that $\omega'/\omega^3 \to 0$ as $\omega \to \infty$. So, for example, if $\omega(\phi) \propto (\phi - \phi_0)^n$, \[24\], then we require $n < 0$ for $\omega \to \infty$ as $\phi \to \phi_0$ and $n \leq -1/2$ to ensure $\omega'/\omega^3 \to 0$. We have not included the potential term \[19\] in the general scalar-tensor theory and expect new features will enter with its presence when its form is suitably chosen. The effects will mirror those of power-law scalar field potentials in general relativity found in ref \[13\] and lead to infinities in higher than second powers of the scale factor.

When a self-interaction potential, $V(\phi)$ is also present in these theories, it adds terms of the form

\[\frac{1}{2\omega + 3} (\phi V'(\phi) - 2V(\phi)),\]

to the right-hand side of eq. \[34\]. Therefore, we expect the new higher-order singularities found by Barrow and Graham \[13\] in general relativity with power-law scalar field potentials to occur also for scalar-tensor cosmologies with $V(\phi) \propto \phi^n$, $n \neq 2$.

### 4 Discussion

We have extended the study of finite-time singularities of 'sudden' type from general relativity and associated $f(R)$ gravity theories to scalar-tensor theories which incorporate varying $G$. We have constructed the form of these singularities in Brans-Dicke gravity theory and find that more general scalar-tensor theories introduce no new features.

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1The requirement is $\left| \frac{\omega'}{(3 + 2\omega)^2(4 + 2\omega)} \right| \to 0$ when $\omega \to \infty$. In that limit this reduces to $\omega'/\omega^3 \to 0$. 
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