ON THE BOUNDED NEGATIVITY CONJECTURE AND SINGULAR PLANE CURVES

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Abstract. We show that two questions raised by Brian Harbourne related to the bounded negativity conjecture and singular plane curves have a negative answer in some cases. For rational curves having only ordinary singularities, this question is shown to be related to strong new bounds on the number of singularities of multiplicity greater or equal to 3 such a curve may have. This fact suggests a conjecture on the non-existence of rational curves of degree \( d > 8 \) having only ordinary triple points as singularities. We also give lower bounds for the \( H \)-constant \( H(C) \) in terms of the maximal multiplicity of the singularities of \( C \), or when \( C \) has only singularities of type \( A_s \) with \( 1 \leq s \leq 5 \) and \( D_4 \).

1. Introduction

The following conjecture, which seems to go back at least to F. Enriques, see [2], is still open.

Conjecture 1.1. (Bounded Negativity Conjecture). Let \( X \) be any smooth projective rational surface. Then there is a bound \( B_X \) such that for every reduced curve \( C \subset X \) we have \( C^2 \geq B_X \).

An equivalent conjecture, see [2, Proposition 5.1] is the following.

Conjecture 1.2. (Bounded Negativity Conjecture for irreducible curves). Let \( X \) be any smooth projective rational surface. Then there is a bound \( b_X \) such that for every reduced irreducible curve \( C \subset X \) we have \( C^2 \geq b_X \).

The concept of \( H \)-constants was introduced to explore the Bounded Negativity Conjectures (see for example [1, 16, 20]). Given a reduced singular curve \( C \subset \mathbb{P}^2 \), let \( S = \{ p_1, \ldots, p_r \} \) be the set of singular points of \( C \), let \( m_i \) be the multiplicity of \( p_i \) and let \( d \) be the degree of \( C \). Recall that the singularity \( p_i \) is called an ordinary point if it has \( m_i \) smooth branches, with distinct tangents. Define

\[
H(C) = \frac{d^2 - \sum_{i=1}^{r} m_i^2}{r} = \frac{(C')^2}{r},
\]

where \( C' \) is the proper transform of \( C \) under the blowing up of the points \( p_i \). The following questions occur in [14], see Question 2.21 and Question 2.25.

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Question 1.3. Does there exist an irreducible reduced singular curve $C \subset \mathbb{P}^2$ such that

$$H(C) \leq -2$$

Question 1.4. Does there exist a reduced singular curve $C \subset \mathbb{P}^2$ such that

$$H(C) \leq -4$$

To continue our discussion, it is convenient to introduce the notation

$$\sigma_k(C) := d^2 - \sum_{i=1}^{r} m_i^2 + kr.$$  

With this notation, a negative answer to Question (1.3) is equivalent to the following.

Conjecture 1.5. With the above notation, for any irreducible reduced singular curve $C \subset \mathbb{P}^2$ one has

$$\sigma_2(C) > 0.$$  

And a negative answer to Question (1.4) is equivalent to the following.

Conjecture 1.6. With the above notation, for any reduced singular curve $C \subset \mathbb{P}^2$ one has

$$\sigma_4(C) > 0.$$  

There are examples of reducible curves $C_n$ for which the sequence of $H$-constants $H(C_n)$ decreases asymptotically to $-4$, see [3, 17, 18]. When $C$ is a union of lines, one also has $H(C) > -4$, and the known smallest value of $H(C)$ in this situation is $H(C) = -\frac{225}{67} = -3.358...$, see [1]. In this note we prove some inequalities for the invariant $\sigma_k(C)$. Our first result is the following.

Theorem 1.7. With the above notation, assume that $C$ is irreducible. Then one has

$$\sigma_2(C) \geq 3d - E(\tilde{C}) - \sum_{i=1}^{r} (m_i - 2),$$

where $\tilde{C}$ is the normalization of the curve $C$ and $E(\tilde{C})$ denotes the corresponding Euler characteristic of $\tilde{C}$. Moreover, the equality holds if the curve $C$ has only ordinary singularities. Therefore Conjecture (1.3) holds when

$$3d - E(\tilde{C}) > \sum_{i=1}^{r} (m_i - 2).$$

Corollary 1.8. Conjecture (1.3) holds when the irreducible curve $C$ has only singularities of multiplicity 2, and more generally when

$$(1.3) \quad 3d - 2 > \sum_{i=1}^{r} (m_i - 2).$$

This inequality holds for any irreducible curve whenever $d \leq 20$. 
The fact that Conjecture 1.5 holds for irreducible curves of degree \( d \leq 20 \) was known, for instance it is stated towards the end of section 2 in [1]. See Example 2.1 below for curves not satisfying the inequality (1.3).

**Corollary 1.9.** Conjecture 1.5 holds when all the singularities of the curve \( C \) have multiplicity \( \leq 3 \) and all the singularities of multiplicity 3 are situated on a curve of degree 8. In particular, this holds when the number of singularities of \( C \) of multiplicity 3 is strictly smaller than 45.

Let \( m_1 = \max\{m_i : 1 \leq i \leq r\} \). In terms of \( m_1 \) we have the following easy result.

**Proposition 1.10.** For a reduced plane curve \( C \) one has

\[
\sigma_{2m_1-1}(C) \geq 2d - 1 > 0 \quad \text{and in particular} \quad H(C) > -2m_1 + 1.
\]

For an irreducible plane curve \( C \) one has

\[
\sigma_{m_1}(C) \geq 3d - 2 > 0 \quad \text{and in particular} \quad H(C) > -m_1.
\]

In particular, both Conjectures 1.5 and 1.6 hold when the curve \( C \) has only singularities of multiplicity 2.

Using more technicalities, we can improve marginally the above result in the case of reducible curves.

**Theorem 1.11.** For a reduced plane curve \( C \) having only singularities of multiplicity \( \leq m_1 \), one has \( H(C) > -k \) where

\[
k = \begin{cases} 
\frac{11}{3} = 3.66... & \text{if } m_1 = 3 \\
2m_1 - \frac{4}{3} & \text{if } m_1 \text{ is even}, \\
2m_1 - \frac{10}{3} & \text{if } m_1 \text{ is odd and } m_1 \geq 5.
\end{cases}
\]

In particular, Conjecture 1.6 holds for reduced curves having only singularities of multiplicity \( \leq 3 \).

We have also the following result, an improvement of Theorem 1.11 for reducible curves with very restrictive types of singularities.

**Theorem 1.12.** For a reduced plane curve \( C \), having only singularities of type \( A_s \) for \( 1 \leq s \leq 5 \) and \( D_4 \), one has \( H(C) > -k \) where

\[
k = \frac{45}{13} = 3.46...
\]

We recall that \( A_1 \) denotes a node, \( A_2 \) a simple cusp, \( A_3 \) a tacnode, \( A_4 \) a rampoid cusp, in general the singularity \( A_s \) has a local equation \( u^2 - v^{s+1} = 0 \), and \( D_4 \) an ordinary triple point.

Finally we discuss some consequences of Theorem 1.7 above.

**Corollary 1.13.** With the above notation, assume that all the singularities of the irreducible curve \( C \) are ordinary. Then

\[
\sigma_2(C) = 3d - E(C) - \sum_{i=1}^{r}(2m_i - 3).
\]
Hence, in this situation, Conjecture 1.5 is equivalent to the inequality

$$3d - \sum_{i=1}^{r} (2m_i - 3) > E(C).$$

**Corollary 1.14.** If $C$ is a rational curve having only ordinary singularities, then Conjecture 1.5 is equivalent to the inequality

$$3d - 2 > \sum_{i=1}^{r} (m_i - 2) = n_3 + 2n_4 + 3n_5 + \ldots,$$

where $n_j$ denotes the number of singularities of $C$ of multiplicity $j$.

In spite of the large number of known results on the possible configurations of singularities of an irreducible plane curve, see for instance [12, 13], the inequality in Corollary 1.14 does not seem to be available.

**Remark 1.15.** (i) Corollary 1.14 should be compared to the known fact that there are degree $d$ rational nodal curves, for any integer $d \geq 3$, whose number of nodes is

$$n_2 = \frac{(d - 1)(d - 2)}{2},$$

see for instance [15].

(ii) Using Lemma 3.12 in [1], in the situation of Corollary 1.14 one gets the inequality

$$\sum_{i=1}^{r} m_i = 2n_2 + 3n_3 + 4n_4 + \ldots \leq \frac{r + \sqrt{r^2 + 4r(d - 1)(d - 2)}}{2},$$

which is an equality exactly when all the singular points of $C$ have the same multiplicity.

**Remark 1.16.** Assume that $C$ is a rational plane curve, having only nodes and ordinary triple points as singularities. Then the condition $n_3 < 3d - 2$ in Corollary 1.14 is clearly equivalent with the condition

$$n_2 > \frac{d^2 - 21 + 14}{2}.$$ 

Hence one way to construct counterexamples to Conjecture 1.5 is to construct a rational plane curve, having only nodes and ordinary triple points as singularities, of degree $d > 20$ with as few nodes as possible. We have no idea whether such curves exist, but lower degree examples suggest the following.

**Conjecture 1.17.** For any rational plane curve, having only nodes and ordinary triple points as singularities, of degree $d > 8$, the number $n_2$ of nodes is strictly positive.

This conjecture is obviously true when $d \equiv 0 \pmod{3}$, and it holds for $d \geq 21$ if Conjecture 1.5 holds in this range. In view of the existence of rational nodal curves, as recalled in Remark 1.15 (i), Conjecture 1.17 looks quite surprising. When $d = 4$, the quartic $C : z(x^3 - y^3) + x^4 + y^4 = 0$ is a rational curve with one ordinary triple
point and no other singularity. The existence of a rational octic curve with \( n_3 = 7 \) triple points in general position was communicated to us by Brian Harbourne, hence our condition \( d > 8 \) above in Conjecture 1.17. Note that in degrees \( d > 8 \), \( d \not\equiv 0 \pmod{3} \), the non-existence of plane curves with triple generic points follows from [9], since the expected dimension in this case is negative. In low degrees \( d \leq 7 \), we have the following easy result.

**Proposition 1.18.** For a rational plane curve, having only nodes and ordinary triple points as singularities, of degree \( d \in \{5, 6, 7\} \), the number \( n_2 \) of nodes is strictly positive.

We would like to thank Brian Harbourne for many useful discussions related to this topic.

2. The proof of Theorem 1.7, Corollary 1.8 and Corollary 1.9

With the above notations, let \( \delta_i \) be the \( \delta \)-invariant of the singularity \( p_i \) on the curve \( C \). Then it is known that the genus \( g(\tilde{C}) \) of the normalization \( \tilde{C} \) of the curve \( C \) is given by

\[
g(\tilde{C}) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{r} \delta_i.
\]

Let now \( P_i = \{p_{ij} : j = 1, ..., r'_i\} \) be the set of all infinitely near points of the singularity \( p_i \), with multiplicity \( m_{ij} = \text{mult}(p_{ij}) \geq 2 \). Note that the first infinitely near point of the singularity \( p_i \) is just the point \( p_i \) itself, which we denote by \( p_{i1} \) and hence \( m_{i1} = m_i \geq 2 \), since \( p_i \) is a singular point of \( C \). The set \( P_i \) consists only of the point \( p_i \) if and only if the singularity \( p_i \) is an ordinary point. There is a formula for \( \delta_i \) in terms of the multiplicities \( m_{ij} = \text{mult}(p_{ij}) \geq 2 \), namely

\[
\delta_i = \sum_{j=1}^{r'_i} \left( \frac{m_{ij}}{2} \right).
\]

see the Remark after Theorem 6.5.9 in [21]. Let \( \mu_i \) be the Milnor number of the singularity \( p_i \). Then is known that

\[
\mu_i = \sum_{j=1}^{r'_i} m_{ij}(m_{ij} - 1) - r_i + 1
\]

where \( r_i \) is the number of branches of the singularity \( p_i \), see [21 Theorem 6.5.9]. It follows that \( \mu_i + r_i \geq m_i(m_i - 1) + 1 \) and hence

\[
2\delta_i = \mu_i + r_i - 1 \geq m_i^2 - m_i.
\]
Moreover, the equality holds if the singularity \(p_i\) is an ordinary point. If we use this inequality and the equality (2.1) we get

\[\sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i \leq d^2 - 3d + 2 - 2g(\tilde{C}),\]

with equality if all the singularities of \(C\) are ordinary. Finally this yields

\[\sigma_2(C) = d^2 - \sum_{i=1}^{r} m_i^2 + 2r \geq 3d - E(\tilde{C}) - \sum_{i=1}^{r} m_i + 2r,\]

which is equivalent to the main claim in Theorem 1.7.

Next we prove Corollary 1.8. Since \(E(\tilde{C}) \leq 2\), the first claim is obvious. Formulas (2.1) and (2.2) imply that

\[\frac{(d-1)(d-2)}{2} \geq \sum_{i=1}^{r} \delta_i \geq \sum_{i=1}^{r} \left(\frac{m_i}{2}\right).\]

Since

\[\left(\frac{m}{2}\right) \geq 3(m - 2)\]

for any integer \(m \geq 2\), it follows that

\[\frac{(d-1)(d-2)}{6} \geq \sum_{i=1}^{r} (m_i - 2).\]

To end the proof of Corollary 1.8 we just notice that

\[3d - 2 > \frac{(d-1)(d-2)}{6}\]

for any degree \(d \leq 20\).

**Example 2.1.** There are irreducible curves \(C\) for which the inequality in (1.3) does not hold. Here is such an example. Let \(3 \leq p < q\) be two relatively prime integers and consider the irreducible curve

\[C_{p,q} : (x^p + y^q)^q - (y^q + z^q)^p = 0.\]

Then one has \(d = r = pq\) and \(m_i = p\) for all \(i\)’s. Hence

\[3d - 2 - \sum_{i=1}^{r} (m_i - 2) = 3pq - 2 - pq(p - 2) = pq(5 - p) - 2 < 0\]

if \(p \geq 5\). On the other hand, one has

\[\sigma_2(C) = p^2q^2 - p^3q + 2pq = p^2q(q - p) + 2pq > 0\]

for any \(p\) and \(q\) as above, and hence Conjecture 1.5 holds for the curves in this example. Moreover, one has \(E(\tilde{C}) = E(C)\) since all the singularities of \(C\) are irreducible, and hence

\[E(\tilde{C}) = 2 - (d - 1)(d - 2) + \sum_{i=1}^{r} \mu_i = pq(4 - p - q),\]
where \( \mu_i = (p - 1)(q - 1) \) is the Milnor number of the singularity \( p_i \). This implies that
\[ pq(p + q - 1) = 3d - E(\tilde{C}) > \sum_{i=1}^{r} (m_i - 2) = pq(p - 2), \]

One can see that in this example the difference between the two terms of the above inequality can be quite large.

We prove now Corollary 1.9. In view of Corollary 1.8, we may assume that \( d > 20 \). Let \( C' \) be a curve of degree 8 containing the singularities of \( C \) of multiplicity 3. We apply Bezout Theorem to the pair of curves \( C \) and \( C' \), having no irreducible component in common. Hence the intersection number \( C \cdot C' \) is 8d. On the other hand, all the singularities of \( C \) of multiplicity 3 are in the intersection \( C \cap C' \), and at each of these points, the local intersection multiplicity of \( C \) and \( C' \) is at least 3. It follows that
\[ 8d \geq 3n_3, \]
where \( n_3 \) denotes the number of singular points of multiplicity 3 on \( C \). It follows that
\[ n_3 \leq \frac{8d}{3} < 3d - 2 \]
and the first claim follows from Corollary 1.8. For the second claim, note that the space of homogeneous polynomials in 3 variables of degree 8 has dimension
\[ \binom{10}{2} = 45. \]
Hence there is such a nonzero homogeneous polynomial of degree 8 vanishing at all the triple points of \( C \), since \( n_3 < 45 \).

3. The proof of Proposition 1.10 of Theorem 1.11 and of Theorem 1.12.

To prove the first claim in Proposition 1.10, we recall that the total Milnor number \( \mu(C) \) is given by the sum of all the local Milnor numbers of the singularities of \( C \), namely
\[ \mu(C) = \sum_{i=1}^{r} \mu(C, p_i). \]
When the singularity \( p_i \) is an ordinary \( m_i \)-multiple point, then
\[ \mu(C, p_i) = (m_i - 1)^2. \]
In general, by the semicontinuity of the local Milnor number, we always have
\[ \mu(C, p_i) \geq (m_i - 1)^2. \]
If we set \( k = 2m_1 - 1 \), then one has
\[ \sigma_k(C) = d^2 - \sum_{i=1}^{r} (m_i^2 - k) \geq d^2 - \sum_{i=1}^{r} (m_i - 1)^2 \geq d^2 - \mu(C) \geq d^2 - (d - 1)^2 = 2d - 1. \]
Indeed, one clearly has
\[ m_i^2 - k \leq (m_i - 1)^2 \]
for all \( i = 1, \ldots, r \) and the inequality \( \mu(C) \leq (d - 1)^2 \) is well known, see for instance the embedding of Milnor lattices (4.4.1) on p. 161 in [7], or look at the papers [5, 6].

For the second claim, when \( C \) is irreducible, we follow the same idea, just replacing the Milnor numbers with the \( \delta \)-invariants. We get, for \( k = m_1 \), the following inequalities
\[
\sigma_k(C) = d^2 - \sum_{i=1}^{r} (m_i^2 - k) \geq d^2 - \sum_{i=1}^{r} (m_i - 1)m_i \geq d^2 - 2\sum_{i=1}^{r} \delta_i \geq d^2 - (d-1)(d-2) = 3d-2.
\]

Here we have used the genus formula (2.1), the inequality (2.3) and the obvious inequality
\[ m_i^2 - m_1 \leq m_i(m_i - 1). \]

We prove now Theorem 1.11 following the same idea. For a plane curve singularity \((C, p)\), given in local analytic coordinates by an equation \( g(u, v) = 0 \), we consider the Milnor lattice \( L(\hat{C}, p) \) of the surface singularity given by
\[
(\hat{C}, p) : g(u, v) + w^2 = 0.
\]
We denote by \( \mu_-(\hat{C}, p) \) the rank of a maximal sublattice in \( L(\hat{C}, p) \) which is negative definite. We have the following.

**Lemma 3.1.** If \( m \) is the multiplicity of the singularity \((C, p)\), then
\[
\mu_-(\hat{C}, p) \geq \begin{cases} 
\frac{3m^2 - 6m + 4}{4} & \text{if } m \text{ is even,} \\
\frac{3m^2 - 4m + 1}{4} & \text{if } m \text{ is odd,}
\end{cases}
\]
and the equality holds when \((C, p)\) is an ordinary \( m \)-multiple point.

**Proof.** When \((C, p)\) is an ordinary \( m \)-multiple point, using the fact that the Milnor lattice is a topological invariant and that \( \mu \)-constant deformations do not change the topology, we see that \( \mu_-(\hat{C}, p) \) can be computed using \( g(u, v) = u^m + v^m \). The claim follows then from [19]. If \((C, p)\) is not an ordinary \( m \)-multiple point, we can deform it to an ordinary \( m \)-multiple point \((C', p)\), and this deformation gives an embedding of lattices \( L(\hat{C}', p) \to L(\hat{C}, p) \), which proves our claim in general.

Now we continue the proof of Theorem 1.11. We have
\[
\sigma_k(C) = d^2 - \sum_{i=1}^{r} (m_i^2 - k) \geq d^2 - \sum_{i=1}^{r} \mu_-(\hat{C}, p_i) \geq d^2 - \frac{4}{3} \sum_{i=1}^{r} \mu_-(\hat{C}, p_i) \geq d^2 - \frac{4}{3} \mu_-(\hat{C}_d, 0).
\]
Here \((C_d, 0)\) is an ordinary singularity of multiplicity \( d \) obtained as the cone over the intersection of \( C \) with a generic line \( L \), and the inequality
\[
\sum_{i=1}^{r} \mu_-(\hat{C}, p_i) \leq \mu_-(\hat{C}_d, 0)
\]
comes from an inclusion of lattices
\[ \bigoplus_{i=1}^r L(\hat{C}, p_i) \to L(\hat{C}, d, 0), \]
see for instance [5] [6]. Now one can use Lemma 3.1 to compute \( \mu_-(\hat{C}, d, 0) \) and this implies \( \sigma_k(C) > 0 \).

To prove Theorem 1.12, we recall that the total Milnor number \( \mu(C) \) is the same as the total Tjurina number
\[ \tau(C) = \sum_{i=1}^r \tau(C, p_i) \]
of \( C \), since all the singularities of \( C \) are weighted homogeneous. Moreover, for the local Milnor numbers, one has
\[ \mu(C, p_i) = (m_i - 1)^2 \]
when \( p_i \) is a singularity of type \( A_1 \) or \( D_4 \), and
\[ 1 = (m_i - 1)^2 < \mu(A_s) = s, \]
when \( p_i \) is a singularity of type \( A_s \) with \( 2 \leq s \leq 5 \). We have the following obvious inequalities
\[ (3.1) \quad d^2 - \sum_{i=1}^r m_i^2 + kr = d^2 - \sum_{i=1}^r (m_i^2 - k) \geq d^2 - \frac{18}{13} \sum_{i=1}^r (m_i - 1)^2 > d^2 - \frac{18}{13} \tau(C). \]
Indeed, it is easy to check that
\[ m^2 - k \leq \frac{18}{13} (m - 1)^2 \]
where \( m = 2, 3 \) and \( k = \frac{45}{13} \).

We recall now that for a plane curve \( C : f = 0 \) having only singularities of type \( A_s \) for \( 1 \leq s \leq 5 \) and \( D_4 \), the minimal degree \( \rho = mdr(f) \) of a Jacobian syzygy satisfies
\[ (3.2) \quad \rho \geq \rho_0 = \frac{2(d + \epsilon)}{3} - 2, \]
where \( \epsilon = 0 \) if \( d \equiv 0 \pmod{3} \), \( \epsilon = 2 \) if \( d \equiv 1 \pmod{3} \) and \( \epsilon = 1 \) if \( d \equiv 2 \pmod{3} \), see [8]. Since
\[ \rho_0 \geq \frac{d - 2}{2}, \]
it follows from [10] that one has
\[ (3.3) \quad \tau(C) \leq \tau(d, \rho_0) = (d - 1)(d - \rho_0 - 1) + \rho_0^2 - \left( \frac{2\rho_0 - d + 2}{2} \right). \]
Using the formula for \( \rho_0 \) given in (3.2), we get
\[ (3.4) \quad \tau(d, \rho_0) = \frac{13d^2 - 21d - 4d\epsilon - 8\epsilon^2 + 24\epsilon}{18}. \]
It follows that
\[ d^2 - \frac{18}{13} \tau(C) \geq d^2 - \frac{18}{13} \tau(d, \rho_0) = \frac{21d + 4d\epsilon + 8\epsilon^2 - 24\epsilon}{13} > 0. \]
This proves Theorem 1.12.

4. The proof of Corollary 1.13, Corollary 1.14 and Proposition 1.18

First we consider Corollary 1.13. Let \( n : \tilde{C} \to C \) be the normalization map. Using the additivity properties of the Euler characteristic for complex constructible partitions, see for instance [11], we get
\[ E(C) = E(C_0) + r, \]
where \( C_0 \) denotes the smooth part of \( C \). Similarly, one has
\[ E(\tilde{C}) = E(n^{-1}(C_0)) + E(n^{-1}(C \setminus C_0)) = E(C_0) + \sum_{i=1}^{r} m_i, \]
if we assume in addition that all the singularities of \( C \) are ordinary. Hence in this situation we have
\[ E(\tilde{C}) = E(C) - r + \sum_{i=1}^{r} m_i, \]
which proves the first part of Corollary 1.13 using Theorem 1.7.

To prove the last claim, recall that
\[ \sum_{i=1}^{r} m_i(m_i - 1) + 2g(\tilde{C}) = (d - 1)(d - 2). \]
This implies
\[ \sigma_3(C) = 3d - 2 - (2n_2 + 3n_3) + 3(n_2 + n_3) + 2g(\tilde{C}) = 3d - 2 + n_2 + 2g(\tilde{C}) > 0. \]

Note that Corollary 1.14 is a direct consequence of Theorem 1.7.

Finally we prove Proposition 1.18. Assume that \( C \) is a rational plane curve, having only nodes and ordinary triple points as singularities. Let \( d \) be the degree of \( C \) and recall that
\[ n_2 + 3n_3 = \frac{(d - 1)(d - 2)}{2}. \]
We have the following case-by-case discussion.

(i) When \( d = 5 \), there is no irreducible rational quintic with only 2 triple points. Indeed, the line \( L \) joining these two points would have intersection \( C \cdot L \geq 3 + 3 = 6 \), and hence \( C \) and \( L \) would have a common irreducible component. This is not possible since \( C \) is irreducible.

(iii) When \( d = 6 \), there is an irreducible sextic having \( n_2 = 1 \) and \( n_3 = 3 \), see Example 3.5 and Example 3.11 in [4]. The case \( n_2 = 0 \) is not possible by obvious reason, since \( n_2 + 3n_3 = 10 \).

(iv) When \( d = 7 \), there is no irreducible rational curve \( C \) with only 5 triple points. Indeed, if such a curve exists, then choose a conic \( Q \) passing through these 5 triple
points. We have \( C \cdot Q \geq 3 \times 5 = 15 > 14 = \deg(C) \cdot \deg(Q) \). Hence \( C \) and \( Q \) would have a common irreducible component. This is not possible since \( C \) is irreducible.

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