Large deviations principle for 2D Navier-Stokes equations with space time localised noise

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Abstract

We consider a stochastic 2D Navier-Stokes equation in a bounded domain. The random force is assumed to be non-degenerate and periodic in time, its law has a support localised with respect to both time and space. Slightly strengthening the conditions in the pioneering work about exponential ergodicity by Shirikyan [Shi15], we prove that the stochastic system satisfies Donsker-Varadhan type large deviations principle. Our proof is based on a criterion of [JNPS15] in which we need to verify uniform irreducibility and uniform Feller property for the related Feynman-Kac semigroup.

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1 Introduction

In this paper, we consider a 2D Navier-Stokes system in a bounded domain $D \subseteq \mathbb{R}^2$ with smooth boundary $\partial D$:

\[
\begin{align*}
\partial_t u + \langle u, \nabla u \rangle - \nu \Delta u + \nabla p &= \eta(t, x), \quad \text{div} u = 0, \quad x \in D \\
u u|_{\partial D} &= 0, \\
(1.1) \\
u(0, x) &= u_0(x).
\end{align*}
\]

Here $u = (u_1, u_2)$ and $p$ are the velocity and the pressure of the fluid respectively, $\nu > 0$ is the viscosity, and $\eta$ is the external random force to be specified below.

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Let $I_k$ be the indicator function of the interval $(k - 1, k)$ and assume that $\eta_k$ is a sequence of i.i.d random variables in $L^2([0, 1] \times D)$ which is localised in both time and space; see the hypothesis (H1) below for details. In this paper, $\eta$ is a random force with the form

$$\eta(t, x) = \sum_{k=1}^{\infty} I_k(t)\eta_k(t - k + 1, x), \quad t \geq 0.$$  \hspace{1cm} (1.2)

Let us denote by $n$ the outward normal to the boundary $\partial D$ and introduce the space

$$H = \{ u \in L^2(D, \mathbb{R}^2) : \text{div} u = 0 \text{ in } D, \langle u, n \rangle = 0 \text{ on } \partial D \} \hspace{1cm} (1.3)$$

which will be endowed with the usual $L^2$ norm $\| \cdot \|$.

We fix an open set $Q \subseteq [0, 1] \times D$ and denote by $\{\phi_j\} \subseteq H^1(Q, \mathbb{R}^2)$ an orthonormal basis in $L^2(Q, \mathbb{R}^2)$. Let $\chi \in C_0^\infty(Q)$ be a non-zero function and let $\psi_j = \chi\phi_j$. The following hypotheses (H1) and (H2) are both from [Shi15].

**(H1) Structure of the noise.** The random variables $\eta_k$ have the following form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk}\psi_j(t, x),$$ \hspace{1cm} (1.4)

where $\xi_{jk}$ are i.i.d. scalar random variables such that $|\xi_{jk}| \leq 1$ with probability 1, and $\{b_j\} \subseteq \mathbb{R}$ is a non-negative sequence such that $B := \sum_{j=1}^{\infty} b_j\|\psi_j\|_1 < \infty$,

where $\| \cdot \|_1$ denotes usual Sobolev norm on $H^1(Q, \mathbb{R}^2)$. Moreover, $\xi_{jk}$ has a $C^1$-smooth density $\rho_j$ with respect to the Lebesgue measure on the real line.

Let us denote by $K \subseteq L^2(Q, \mathbb{R}^2)$ the support of the law of $\eta_k$. The hypotheses imposed on $K$ imply that $K$ is a compact subset in $H^1_0(Q, \mathbb{R}^2)$.

**(H2) Approximate controllability.** There exists a $\hat{u} \in H$ such that for any positive constants $R$ and $\varepsilon$ one can find an integer $\ell \geq 1$ with the following property: given $v \in B_H(R) := \{ u \in H, \| u \| \leq R \}$, there are $\theta_1, \ldots, \theta_\ell \in K$ such that

$$\| S_\ell(v, \theta_1, \ldots, \theta_\ell) - \hat{u} \| \leq \varepsilon,$$ \hspace{1cm} (1.5)

where $S_\ell(v, \theta_1, \ldots, \theta_\ell)$ is the vector $u_\ell$ defined by (1.6) with $\eta_k = \theta_k$ and $u_0 = v$.

In order to prove the large deviations principle, we need the following additional condition:
(H3) Nondegenerate noise. $b_j \neq 0$ for all $j$ with $b_j$ being defined in (H1).

By [KS12, Theorem 2.1.18], (1.1) admits a unique strong solution $u(t)$. Since $u(1)$ depends on $u_0$ and $\eta_1$, we use $S(u_0, \eta_1)$ to denote it. With these notations, we have

$$u_k = S(u_{k-1}, \eta_k), \quad k \geq 1. \quad (1.6)$$

As $\eta_k$ are i.i.d random variables in $L^2([0, 1] \times D)$, (1.6) defines a homogeneous family of Markov chains in $H$. We denote it by $(u_k, \mathbb{P}_u)$, $u \in H$ and use $P_k(u, \Gamma)$ to denote the transition function for $(u_k, \mathbb{P}_u)$.

The objective of this paper is to prove the large deviations of the following occupation measures under hypotheses (H1)-(H3)

$$\zeta_k = \frac{1}{k} \sum_{j=0}^{k-1} \delta_{u_j}, \quad k \geq 1.$$ 

See Theorem 2.2 below for details.

There have been many literatures on the ergodicity of stochastic 2D Navier-Stokes system, see [FM95, EMS01, HM06, HM08] for the Gaussian noises case and [KS00a] for the kick noises case, the monograph [KS12] gives a comprehensive review on the researches in these two directions. When the driven noises are Lévy type, we refer the reader to [DX11], [DXZ11] and the references therein.

There are very few papers on the ergodicity of stochastic partial differential equations (SPDEs) driven by physically localized noises. To the best of our knowledge, there exist only four papers in this direction. Besides [Shi15], Shirikyan [Shi17] proved the exponentially mixing for one-dimensional Burgers equation perturbed by a stochastic forcing which is white in time and localized in space. When the noise is localized in physical space and degenerate in Fourier space, Nersesyan [Ner21] proved that the complex Ginzburg-Landau equation is exponential mixing. For the 2D Navier-Stokes system driven by a random force acting through the boundary, Shirikyan [Shi21] established an exponential mixing property.

However, there seem no works concerning the Donsker-Varadhan type large deviations principle (LDP) for SPDEs driven by physically localized noise, the main motivation of this paper is to partly fill in this gap. Continuing the pioneering work in [Shi15] and slightly strengthening the conditions therein, we prove that the system (1.1) satisfies Donsker-Varadhan type LDP.

Our proof is based on a criterion established in [JNPS15] by verifying uniform irreducibility and uniform Feller properties for the related Feynman-Kac semi-group. Note that Donsker-Varadhan type LDPs have been extensively studied since the work [DV]. When Markov processes are strong Feller and irreducible, Wu [Wu01] established the hyper-recurrence criterion which were applied to study several SPDEs, see [Gou07a, Gou07b, WXX21].
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Notation

In this paper, we use the following notation

\( H \) is the Hilbert space defined by (1.3), \( B_H(a,R) \) is the closed ball in \( H \) of radius \( R \) centred at \( a \). When \( a = 0 \), we simply write it as \( B(H) \).

For an open set \( Q \) of a Euclidean space, \( H^m = H^m(Q) \) is the Sobolev space of order \( m \). We endow the space \( H^m \) with the usual Sobolev norms which are denoted by \( \| \cdot \|_m \) or \( \| \cdot \|_{H^m(Q)} \). \( H^m_0 = H^m_0(Q) \) is the closure in \( H^m \) of the space of infinitely smooth functions with compact support.

If \( X \) is a metric space, \( L^\infty(X) \) denotes the space of bounded Borel-measurable functions \( f \colon X \to \mathbb{R} \) endowed with the norm \( \| f \|_{\infty} = \sup_{u \in X} |f(u)| \).

\( B(X) \) is the Borel \( \sigma \)-algebra of \( X \). \( \mathcal{P}(X) \) denotes the set of probability measures on \( X \) endowed with the topology of the weak convergence. For \( \mu \in \mathcal{P}(X) \) and \( f \in L^\infty(X) \), we denote \( \langle f, \mu \rangle = \int_X f(u)\mu(du) \).

We denote by \( E_m \) the vector span of \( \psi_1, \ldots, \psi_m \) endowed with the \( L^2 \) norm and by \( B_R \) the ball in \( H^1(Q) \) of radius \( R \) centred at origin. \( P_m \) stands the orthogonal projection in \( L^2(Q, \mathbb{R}^2) \) onto the \( m \)-dimensional space \( E_m \).

Let \( \{ e_j \}_{j=1}^\infty \) be an orthonormal basis in \( H \) which consists of the eigenfunctions of the Stokes operator \( L = -\nu \Pi \Delta \), where \( \Pi \) stands for the Leray projection in \( L^2(D, \mathbb{R}^2) \) onto the closed space \( H \). Let \( \Pi_N \) be the orthogonal projection in \( H \) on the vector space \( H_N \) spanned by \( e_1, \ldots, e_N \).

2 Main results

Let \( X \) be a polish space and let \( \mathcal{P}(X) \) be the space of probability measures on \( X \) endowed with the topology of weak convergence. A random probability measure on \( X \) is defined as a measurable mapping from a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) to \( \mathcal{P}(X) \). A mapping \( I : \mathcal{P}(X) \to [0, +\infty] \) is called a rate function if it is lower-semicontinuous, and a rate function \( I \) is said to be good if its level set \( \{ \sigma \in \mathcal{P}(X) : I(\sigma) \leq \alpha \} \) is compact for any \( \alpha \geq 0 \). A good rate function \( I \) is nontrivial if its effective domain \( D_I = \{ \sigma \in \mathcal{P}(X) : I(\sigma) < \infty \} \) is not a singleton.
Definition 2.1. Let \( \{\zeta_n, n \geq 1\} \) be a sequence of random probability measures on \( X \). We say that \( \{\zeta_n, n \geq 1\} \) satisfies an LDP with a rate function \( I : \mathcal{P}(X) \to [0, +\infty] \), if the following two bounds hold:

Upper bound. For any closed subset \( F \subset \mathcal{P}(X) \), we have
\[
\limsup_{k \to \infty} \frac{1}{k} \log \sup \mathbb{P}\{\zeta_k \in F\} \leq -\inf_{\sigma \in F} I(\sigma).
\]

Lower bound. For any open subset \( G \subset \mathcal{P}(X) \), we have
\[
\liminf_{k \to \infty} \frac{1}{k} \log \inf \mathbb{P}\{\zeta_k \in G\} \geq -\inf_{\sigma \in G} I(\sigma).
\]

Let \( u_0 \) be an arbitrary random variable in \( H \) which is independent of \( \{\eta_k\} \) and define a family of occupation measures \( \{\zeta_k\} \) by
\[
\zeta_k = \frac{1}{k} \sum_{j=0}^{k-1} \delta_{u_j}, \tag{2.1}
\]
where \( \delta_u \) is the delta measure concentrated at \( u \).

In order to state our main result, let us first give the definition of \( A \) as below. For a closed subset \( B \subset H \), define the sequence of sets
\[
A_0(B) = B, \quad A_k(B) = \{S(u, \eta) : u \in A_{k-1}(B), \eta \in K\}, \quad k \geq 1,
\]
and denote
\[
A(B) = \bigcup_{k \geq 0} A_k(B),
\]
where the closure is in \( H \). We shall call \( A(B) \) the domain of attainability from \( B \). Let \( \hat{u} \in H \) be defined in (H2), we denote
\[
A_k = A_k(\{\hat{u}\}), \quad \forall k \geq 0, \quad A = A(\{\hat{u}\}).
\]

Since \( A \) a close subset of the Hilbert space \( (H, \|\cdot\|) \), \( (A, \|\cdot\|) \) is a closed metric space. For this space, we use \( C(A) \) to denote the space of continuous functions on \( A \), and use \( C^1(A) \) to denote the space of functions on \( A \) that are continuously Fréchet differentiable.

Shirikyan [Shi15] proved that the stochastic system (1.6) possesses a unique stationary distribution \( \mu \), which is exponentially mixing. In Proposition 3.5 below, we further show that prove that the support of measure \( \mu \) is \( A \), i.e.,
\[
supp(\mu) = A.
\]

The following theorem is the main result of this paper, its proof will be given in Section 3.
Theorem 2.2. Let Hypotheses (H1), (H2) and (H3) hold and let \( u_0 \) be an arbitrary random variable in \( H \) whose law is supported by \( A \). Then the family \( \{ \zeta_k, k \geq 1 \} \) of random measures on \( A \) satisfies the LDP with a good rate function \( I \) defined by

\[
I(\sigma) = \sup_{V \in C(A)} \left( \langle V, \sigma \rangle - Q(V) \right), \quad \sigma \in \mathcal{P}(A),
\]

where \( Q(V) \) is defined as

\[
Q(V) = \lim_{k \to \infty} \frac{1}{k} \log \mathbb{E} \exp \left( \sum_{j=1}^{k} V(u_j) \right),
\]

this limits exists for any \( V \in C(A) \)\(^1\) and does not depend on the initial point \( u_0 \).

Remark 2.3. The sequence \( \{ \zeta_k \} \) also satisfies an LDP on \( H \) under the conditions of Theorem 2.2. More precisely, there is a lower-semicontinuous mapping \( I : \mathcal{P}(H) \to [0, +\infty] \) such that

\[
- \inf_{\lambda \in \hat{\Gamma}} I(\lambda) \leq \liminf_{k \to \infty} \frac{1}{k} \log \mathbb{P}(\zeta_k \in \Gamma) \leq \limsup_{k \to \infty} \frac{1}{k} \log \mathbb{P}(\zeta_k \in \Gamma) \leq - \inf_{\lambda \in \overline{\Gamma}} I(\lambda),
\]

where \( \Gamma \subset \mathcal{P}(H) \) is an arbitrary Borel subset and \( \hat{\Gamma} \) and \( \overline{\Gamma} \) denote its interior and closure, respectively.

Remark 2.4. Note that [Shi15] assumes that the random force has the form \( f(t, x) = h(t, x) + \eta(t, x) \) with \( h(t, x) \) being some periodic function in \( H^1(Q, \mathbb{R}^2) \). Because \( h(t, x) \) will not play an essential role in proving large deviations, without loss of generality we assume \( h(t, x) \equiv 0 \) in this paper.

3 Proof of Theorem 2.2

First, let us show the compactness of \( A \) in \( H \) which plays an important role in our proof. By [KS12, Proposition 2.1.21], one has

\[
\| S(u, \eta_k) \| \leq \kappa \| u \| + C_1 \| \eta_k \|_{L^2(Q)},
\]

where \( \kappa < 1, C_1 > 0 \) are some universal constants. Thanks to hypothesis (H1), we can choose \( r > 0 \) big enough so that

\[
\| \eta_k \|_{L^2(Q)} \leq r
\]

with probability 1. Then, with probability 1, one has

\[
\| S(u, \eta_k) \| \leq \kappa \rho + C_1 r, \quad \forall u \in B_H(\rho).
\]

\(^1\)In the beginning of Section 3, we will prove that \( A \) is a compact set. Therefore, if \( V \in C(A) \), then \( V \) is also bounded.
It follows that if $R \geq \frac{C_{1}r}{1 - \kappa}$, then the ball $B_{H}(R)$ is invariant for the Markov chain $(u_{k}, P_{u})$. Let us take $R$ big enough such that

$$R > \frac{C_{1}r}{1 - \kappa}$$

and $\hat{u} \in B_{H}(R)$. Let $X$ be the image of the set $B_{H}(R) \times B_{L^{2}(Q)}(r)$ under the mapping $S$, then, for any $u \in X$ and $\eta_{k} \in B_{L^{2}(Q)}(r)$, $S(u, \eta_{k}) \in X$. By the compact embedding theorem (see [Eva10, Section 5.7] for example) and [KS12, (2.52)], $X$ also is a compact set in $H$. Therefore, we obtain the compactness of $\mathcal{A}$ from the fact that $\mathcal{A} \subseteq X$ is a close set.

We endow the set $\mathcal{A}$ with distance $d(u, v) = \|u - v\|$, then $\mathcal{A}$ is a compact metric space. Theorem 2.2 immediately follows from Proposition 3.1, as long as we verify uniform irreducibility and uniform Feller property which are given in Section 3.1 and Section 3.2 respectively. Hence, we complete the proof of Theorem 2.2.

**Proposition 3.1.** Let $\{u_{k}, k \geq 1\}$ be a homogeneous family of Markov chains on a compact metric space $(\mathcal{A}, d(\cdot, \cdot))$ with transition function $P_{k}(u, \cdot)$. The family $\{\xi_{k}, k \geq 1\}$ of random measures on $\mathcal{A}$ is defined by (2.1). Assume the following two conditions hold:

**Uniform Irreducibility.** For any $r > 0$, there is an integer $m \geq 1$ and a constant $p > 0$ such that

$$P_{m}(x, B(a, r)) \geq p, \quad \forall x, a \in \mathcal{A}, \quad (3.1)$$

where $B(a, r) = \{y \in \mathcal{A} : d(y, a) \leq r\}$.

**Uniform Feller property:** For any $V, f \in C^{1}(\mathcal{A})$, the sequence $\{\|B_{n}^{V}f\|_{\infty}^{-1}B_{n}^{V}f\}$ is uniformly equicontinuous on $\mathcal{A}$. Here, for any $u_{0} \in \mathcal{A}$ and $V, f \in C(\mathcal{A})$, $B_{n}^{V}f(u_{0})$ and $\|B_{n}^{V}f\|_{\infty}$ are defined by

$$B_{n}^{V}f(u_{0}) = E_{u_{0}}\left[\exp\left(\sum_{i=1}^{n}V(u_{i})\right)f(u_{n})\right] \quad \text{and} \quad \|B_{n}^{V}f\|_{\infty} = \sup_{u_{0} \in \mathcal{A}}|B_{n}^{V}f(u_{0})|,$$

respectively. In the above, the subscript $u$ in $E_{u}$ is the initial state of the Markov chain $\{u_{k}\}$. We call $B_{n}^{V}$ the Feynman-Kac semigroup with respect to $V$.

Then the family $\{\xi_{k}, k \geq 1\}$ satisfies the LDP with a good rate function $I$ defined by

$$I(\sigma) = \sup_{V \in C(\mathcal{A})} \left(\langle V, \sigma \rangle - Q(V)\right), \quad \sigma \in \mathcal{P}(\mathcal{A}), \quad (3.2)$$

where $Q(V)$ is defined as

$$Q(V) = \lim_{k \to \infty} \frac{1}{k} \log E \exp\left(\sum_{j=1}^{k}V(u_{j})\right),$$

this limit exists for any $V \in C(\mathcal{A})$ and does not depend on the initial point $u_{0}$.

\footnote{The uniformly equicontinuous of $\{\|B_{n}^{V}f\|_{\infty}^{-1}B_{n}^{V}f\}$ on $\mathcal{A}$ means that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\|B_{n}^{V}f\|_{\infty}^{-1}B_{n}^{V}f(u_{1}) - \|B_{n}^{V}f\|_{\infty}^{-1}B_{n}^{V}f(u_{2})\| \leq \varepsilon$ for any $u_{1}, u_{2} \in \mathcal{A}$ with $d(u_{1}, u_{2}) \leq \delta$.}
The arguments below is essentially from [JNPS15], we only sketch its main ingredients.

By the abstract result established by [Kif90](see also [JNPS15, Theorem A.1]), we only need to prove the following two properties.

**Property 1.** For any $V \in C(A)$, the limit
\[
Q(V) = \lim_{k \to \infty} \frac{1}{k} \log \mathbb{E} \exp \{ \sum_{n=1}^{k} V(u_n) \}
\]
exists and does not depend on the initial condition $u_0$.

**Property 2.** There is a dense vector space $V \subseteq C(A)$ such that there exists unique $\sigma_V \in \mathcal{P}(A)$ satisfying
\[
Q(V) = \langle V, \sigma_V \rangle - I(\sigma_V),
\]
where the $I$ is defined by (3.2).

For any $V \in \mathcal{V} := C^1(A)$, the generalized Markov kernal $P^V(u, \cdot)$ is given by
\[
P^V(u, \Gamma) = \mathbb{E}_u (I_\Gamma(u_1)e^{V(u_1)}), \quad u \in A, \Gamma \in \mathcal{B}(A),
\]
where $I_\Gamma(\cdot)$ is the indicator function of set $\Gamma$. We define $P^V_k(u, \Gamma)$ by the relations $P^V_0(u, \cdot) = \delta_u$, $P^V_1(u, \Gamma) = P^V(u, \Gamma)$ and
\[
P^V_k(u, \cdot) := \int_A P^V_{k-1}(u, dv)P^V(v, \cdot), \quad k \geq 2.
\]
The operators $B^V_k : C(A) \to C(A)$ and $B^V_k : \mathcal{M}_+(A) \to \mathcal{M}_+(A)$ are respectively defined by
\[
B^V_k f(u) = \int_A P^V_k(u, dv)f(v), \quad B^V_k \mu(\Gamma) = \int_A P^V_k(u, \Gamma)\mu(du),
\]
where $f \in C(A)$, $\mu \in \mathcal{M}_+(A)$, $\Gamma \in \mathcal{B}(A)$ and $\mathcal{M}_+(A)$ is the space of non-negative Borel measures on $A$ endowed with the topology of weak convergence. By (3.1) and the following inequality
\[
P^V_k(u, \cdot) \geq e^{-k\|V\|}_\infty P_k(u, \cdot) \quad \text{for any } u \in A,
\]
one finds that for some $C = C(k, V) > 0$,
\[
C^{-1} \leq P^V_k(u, A) \leq C, \quad \text{for all } u \in A.
\]
By this inequality and the uniform Feller property, and using [JNPS15, Theorem 2.1] for $B^V_k$, $B^V_k \mu$ and $P^V_k(u, \cdot)$, there are a number $\lambda_V > 0$, a strictly positive function $h_V \in C(A)$, and a measure $\mu_V \in \mathcal{P}(A)$ satisfying
\[
B^V_k h_V = \lambda_V h_V, \quad B^V_k \mu_V = \lambda_V \mu_V, \quad \langle h_V, \mu_V \rangle = 1,
\]
\[
8
\]
such that for any $f \in C(A)$ and $\nu \in M_+(A)$ we have

$$\lambda_k^{-k}B_k^V f \to \langle f, \mu_V \rangle h_V \text{ in } C(A) \text{ as } k \to +\infty, \quad (3.3)$$

$$\lambda_k^{-k}B_k^{V_*} \nu \to \langle h_V, \nu \rangle \mu_V \text{ in } M_+(A) \text{ as } k \to +\infty. \quad (3.4)$$

Following the argument in [JNPS15, Pages 2134–2135], we can obtain the Property 1 and Property 2 from (3.3) and (3.4). The proof is complete. \qed

### 3.1 Proof of Uniform Irreducibility

The proof of uniform irreducibility is inspired by the method in [KS00b].

**Lemma 3.2.** For any $\rho > 0$ and integer $M \geq 1$ there is $p_0 = p_0(\rho, M) > 0$ such that

$$\mathbb{P}(\|\eta_j - x_j\|_{L^2(Q)} < \rho, 1 \leq j \leq M) \geq p_0$$

uniformly for $x_1, \ldots, x_M \in K$.

**Proof.** Recall that $K$ is a compact subset of $H^1_0(Q, \mathbb{R}^2)$ and thus a compact subset of $L^2(Q)$, there exist $n_0$ elements $y_1, \ldots, y_{n_0} \in K$ such that $B_H(y_1, \rho/2), \ldots, B_H(y_{n_0}, \rho/2)$ cover $K$. Since $\{\eta_j\}$ are i.i.d random variables with support in $K$, we know

$$\mathbb{P}(\|\eta_j - x_j\|_{L^2(Q)} < \rho, 1 \leq j \leq M) = \prod_{j=1}^M \mathbb{P}(\|\eta_j - x_j\|_{L^2(Q)} < \rho)$$

As $K$ is the support of $\eta_1$, we know $\tilde{p}_0 := \min_{1 \leq j \leq n_0} \mathbb{P}(\|\eta_j - y_j\|_{L^2(Q)} < \rho/2) > 0$. It is easy to see that for all $j$ $\mathbb{P}(\|\eta_j - x_j\|_{L^2(Q)} < \rho) \geq \mathbb{P}(\|\eta_j - y_{n(j)}\|_{L^2(Q)} < \rho/2) \geq \tilde{p}_0$ where $y_{n(j)}$ is the point in $\{y_1, \ldots, y_{n_0}\}$ such that $x_j \in B_H(y_{n(j)}, \rho/2)$. Taking $p_0 = \tilde{p}_0^M$, we immediately conclude the proof. \qed

**Lemma 3.3.** For any $r > 0$ there is an integer $k \geq 0$ such that $A$ is contained in the $r$-neighbourhood of $A_k$, i.e., for any $a \in A$ there exists $a_k \in A_k$ such that $a_k \in B_H(a, r)$.

**Proof.** We claim that $A_{j+1} \supset A_j$ for all $j$. To see this, for any $u \in A_j$, we know $u \in A_{j+1}$ since $0 \in K$ and thus $u$ itself is a (constant) solution to the corresponding system (1.1).

Since $A$ is the closure of $\cup_{j=0}^\infty A_j$, for any $r > 0$ we have

$$A \subset \cup_{j=0}^\infty O_j,$$

where $O_j$ is the open $r$-neighbourhood of $A_j$ in $H$. Since $\{A_j\}$ are an increasing sequence, so are $\{O_j\}$. By the compactness of $A$, we complete the proof. \qed
Lemma 3.4. For any \( r > 0 \) there are positive constants \( \varepsilon, \delta \) and an integer \( k > 0 \) such that

\[
P_k(u, B_H(a, r)) \geq \varepsilon, \quad \forall u \in B_H(\hat{u}, \delta) \text{ and } a \in A,
\]

where \( \hat{u} \) is defined in (H2) and \( \{P_k\} \) is the transition probability family of the Markov chain \( \{u_k\} \).

Proof. By Lemma 3.3, for any \( r > 0 \), there is a \( k = k(r) \geq 1 \) such that for any \( a \in A \), the following

\[\|a - a_k\| \leq \frac{r}{2}\]

holds for some \( a_k \in A_k \). Since \( a_k \in A_k \), there are \( \theta_j^0 \in K \) with \( j = 1, \ldots, k \) such that

\[S_k(\hat{u}, \theta_1^0, \ldots, \theta_k^0) = a_k,\]

where \( S_k \) is given in hypothesis (H2). Noticing that \( S \) is a continuous mapping from \( H \times L^2(Q) \) to \( H \), the following holds

\[S_k(u, \theta_1, \ldots, \theta_k) \in B_H(a_k, \frac{r}{2})\]

if

\[\|u - \hat{u}\| < \delta, \quad \|\theta_j - \theta_j^0\|_{L^2(Q)} < \delta, \quad j = 1, \ldots, k,\]

for sufficiently small \( \delta \). For any \( u \in B_H(\hat{u}, \delta) \), with the help of

\[P_k(u, B_H(a, r)) \geq P(S_k(u, \eta_1, \ldots, \eta_k) \in B_H(a_k, \frac{r}{2}))\]

\[\geq P(\|\eta_j - \theta_j^0\|_{L^2(Q)} < \delta, \quad j = 1, \ldots, k)\]

and Lemma 3.2, we complete our proof immediately. \( \square \)

Proposition 3.5. The Markov chain \( \{u_k, k \geq 0\} \) associated with \( (1.6) \) has a unique stationary measure \( \mu \). Moreover, the followings holds:

(i) Uniform Irreducibility on \( A \): for any \( r > 0 \), there is an integer \( m \geq 1 \) and a constant \( p > 0 \) such that

\[P_m(x, B_H(a, r)) \geq p, \quad \forall x, a \in A;\]

(ii) \( A = \text{supp } \mu \).

Proof. (i) By the approximate controllability hypothesis (H2) and the compactness of \( A \), for any \( \delta > 0 \), there exist some \( \varepsilon_1 > 0 \) and some integer \( L_1 > 0 \) such that

\[P_{L_1}(x, B_H(\hat{u}, \delta)) \geq \varepsilon_1, \quad \forall x \in A. \quad (3.5)\]
In view of Lemma 3.4, for any \( r > 0 \), there exist positive constants \( \varepsilon_2, \delta \) and integer \( L_2 > 0 \) such that
\[
P_{L_2}(u, B_H(a,r)) \geq \varepsilon_2, \quad \forall a \in A, \forall u \in B_H(\hat{u}, \delta). \tag{3.6}
\]
By (3.5) and (3.6), we obtain
\[
P_{L_1+L_2}(x, B_H(a,r)) = \int_H P_{L_1}(x, du) P_{L_2}(u, B_H(a,r))
\geq P_{L_1}(x, B_H(\hat{u}, \delta)) \inf_{u \in B_H(\hat{u}, \delta)} P_{L_2}(u, B_H(a,r))
\geq \varepsilon_1 \varepsilon_2 > 0,
\tag{3.7}
\]
which implies (i).

(ii) Since \( A \) is an invariant subset for (1.6), it carries a stationary measure. Since the stationary measure is unique, we must have
\[\text{supp } \mu \subset A.\]
On the other hand, by (3.7), for any \( a \in A \), one sees that
\[
\mu(B_H(a,r)) = \int_A \mu(dx) P_{L_1+L_2}(x, B_H(a,r)) \geq \varepsilon_1 \varepsilon_2 > 0,
\]
which implies \( A \subset \text{supp } \mu \). The proof of (ii) is complete. \(\square\)

### 3.2 Proof of uniform Feller property

Nersesyan et. al [NPX22] proved uniform Feller property by Malliavin calculus for stochastic 2D Navier-Stokes equation driven by highly degenerate noise, which cannot be applied in our setting because the noise of (1.1) is not Gaussian type. In contrast, our approach is via a coupling established by Shirikyan [Shi15], it takes a role similar to the Malliavin calculus in [NPX22].

#### 3.2.1 A coupling by Shirikyan [Shi15]

For any \( \varepsilon > 0 \), define a symmetric function \( d_\varepsilon : H \times H \to \mathbb{R} \) by the relation
\[
d_\varepsilon(u_1, u_2) = \begin{cases} 1, & \text{if } ||u_1 - u_2|| > \varepsilon, \\ 0, & \text{if } ||u_1 - u_2|| \leq \varepsilon \end{cases}
\]
and introduce the following function on the space \( \mathcal{P}(H) \times \mathcal{P}(H) \):
\[
K_\varepsilon(\mu_1, \mu_2) = \sup_{f,g} (\langle f, \mu_1 \rangle - \langle g, \mu_2 \rangle),
\]
where the supremum is taken over all functions \( f, g \in C(H) \) satisfying
\[
f(u_1) - g(u_2) \leq d_\varepsilon(u_1, u_2), \forall u_1, u_2 \in H.
\]
We have the following proposition, whose proof is very similar to the argument in [Shi15], but we give its details for the completeness.
\textbf{Proposition 3.6.} For any \( q \in (0, 1) \), there exist positive constants \( d = d(q) > 0 \) and \( C = C(d, q) \) such that for any two points \( u_0, u'_0 \in A \) satisfying the inequality \( \| u_0 - u'_0 \| \leq d \) the pair \( (P_1(u_0, \cdot), P_1(u'_0, \cdot)) \) admits a coupling \( (V(u_0, u'_0), V'(u_0, u'_0)) \) such that
\[
\mathbb{P}\{ \| V(u_0, u'_0) - V'(u_0, u'_0) \| > q \| u_0 - u'_0 \| \} \leq C \| u_0 - u'_0 \| \tag{3.8}
\]
and the functions \( V, V' : \Omega \times Z \to H \) are measurable, where
\[
Z = \{ (u_0, u'_0) \in A \times A : \| u_0 - u'_0 \| \leq d \}. \tag{3.9}
\]

\textbf{Proof.} We fix \( \hat{R} > 0 \) so large that \( A \subseteq B_H(\hat{R} - 1) \) and \( \| \eta_k \|_{H^1(Q)} \leq \hat{R} - 1 \) with probability 1.

By [Shi15, Theorem 3.1], for \( q \in (0, 1) \), there exist positive constants \( C, d, \) integer \( m \) and a continuous mapping
\[
\Phi : B_{\hat{R}} \times B_H(\hat{R}) \to \mathcal{L}(H, \mathcal{E}_m)
\]
\textsuperscript{3}such that the following properties hold:

**Contraction.** For any \( h \in B_{\hat{R}} \) and \( u_0, u'_0 \in B_H(\hat{R}) \) with \( \| u_0 - u'_0 \| \leq d \), we have
\[
\| S(u_0, h) - S(u'_0, h + \Phi(h, u_0)(u'_0 - u_0)) \| \leq \frac{q \| u_0 - u'_0 \|}{2}. \tag{3.10}
\]

**Regularity.** The mapping \( \Phi \) is infinitely smooth in the Fréchet sense.

**Lipschitz continuity.** The mapping \( \Phi \) is Lipchitz continuous with the constant \( C, \) i.e.,
\[
\| \Phi(h_1, u_1) - \Phi(h_2, u_2) \|_{\mathcal{L}} \leq C(\| h_1 - h_2 \|_{H^1(Q)} + \| u_1 - u_2 \|),
\]
where \( \| \cdot \|_{\mathcal{L}} \) stands for the norm in the space \( \mathcal{L}(H, \mathcal{E}_m) \).

By [Shi15, Proposition 5.3], there exists a coupling \( (V(u_0, u'_0), V'(u_0, u'_0)) \) such that
\[
\mathbb{P}\{ \| V(u_0, u'_0) - V'(u_0, u'_0) \| > \varepsilon \} \leq K_{\varepsilon/2}(P_1(u_0, \cdot), P_1(u'_0, \cdot)) \tag{3.11}
\]
holds with \( \varepsilon = q \| u_0 - u'_0 \| \) and the functions \( V, V' : \Omega \times Z \to H \) are measurable.

It suffices to bound \( K_{\varepsilon/2}(P_1(u_0, \cdot), P_1(u'_0, \cdot)) \). Define the transformation \( \Psi = \Psi_{u_0, u'_0} \) of the space \( H^1(Q) \) by the following relation
\[
\Psi(h) = h + \varrho(\| h \|_{H^1(Q)}) \Phi(h, u_0)(u'_0 - u_0) \tag{3.12}
\]
where \( \varrho \) is a smooth function such that \( \varrho(a) = 1 \) for \( a \leq \hat{R} - 1 \) and \( \varrho(a) = 0 \) for \( a \geq \hat{R} \). We denote the law of \( \eta_k \) and \( \Psi(\eta_k) \) on \( B_{\hat{R}} \) by \( \lambda \) and \( \Psi_\lambda \), respectively. With the help of (3.10) and [Shi15, Proposition 5.2], one has
\[
K_{\varepsilon/2}(P_1(u_0, \cdot), P_1(u'_0, \cdot)) \leq 2 \| \lambda - \Psi_\lambda \|. \tag{3.13}
\]
With the help of the Lipschitz continuity of functions $\Phi$, $\Psi$, by the definition (3.12), we can verify the conditions of [Shi15, Proposition 5.6] with a constant $\kappa$ proportional to $\|u_0 - u'_0\|$. Therefore,

$$\|\lambda - \Psi_\kappa(\lambda)\| \leq C\|u_0 - u'_0\|. \tag{3.14}$$

Combining (3.13), (3.14) and (3.11), we obtain the desired result (3.8). \qed

### 3.2.2 Proof of uniform Feller property

**Theorem 3.7.** Let Hypotheses (H1), (H2) and (H3) hold, then for any $V, f \in C^1(A)$, the family $\{B_n f, n \geq 0\}$ is equicontinuous on $A$. Furthermore, there is a $\gamma > 0$ such that

$$|B_n f(u_0) - B_n f(u'_0)| \leq C\|B_n f\|_\infty (\|f\|_\infty + e^{-\gamma n}\|\nabla f\|_\infty)\|u_0 - u'_0\|, \tag{3.15}$$

holds for any $u_0, u'_0 \in A$ and $n \geq 1$. Here $C = C(\gamma, \|V\|_\infty, \|\nabla V\|_\infty)$. In particular, uniform Feller property holds.

**Proof.** Uniform Feller property follows from (3.15) immediately. Let us prove (3.15) in the following four steps.

**Step 1: construction of coupling processes.** This coupling is borrowed from [Shi15], here we give the details for the completeness. Let $q = q(\|V\|_\infty) \in (0, 1)$ be a constant that will be determined later. By Proposition 3.6, there exist positive constants $d = d(q) > 0$, $C = C(d, q)$ such that for any two points $u_0, u'_0 \in A$ with $\|u_0 - u'_0\| \leq d$, the pair $(\eta_1(u_0, \cdot), P_1(u'_0, \cdot))$ admits a coupling $(V(u_0, u'_0), V'(u_0, u'_0))$ such that (3.8) holds and that the functions $V, V' : \Omega \times Z \to H$ are measurable, here $Z$ is given by (3.9). We now define a coupling operator by the relation

$$R(u_0, u'_0) = \begin{cases} (V(u_0, u'_0), V'(u_0, u'_0)), & \text{for } \|u_0 - u'_0\| \leq d, \\ (S(u_0, \zeta), S(u'_0, \zeta')), & \text{for } \|u_0 - u'_0\| > d, \end{cases} \tag{3.16}$$

where $\zeta$ and $\zeta'$ are independent random variables whose law coincides with that of $\eta_1$. Without loss of generality, we assume that $\zeta, \zeta', V$ and $V'$ are all defined on the same probability space. To stress the dependence on $\omega$, we shall sometimes write $R(u_0, u'_0; \omega)$ instead of $R(u_0, u'_0)$.

Let $(\Omega_k, \mathcal{F}_k, \mathbb{P}_k), k \geq 1$ be countable many copies of the probability space on which $R$ is defined and let $(\Omega, \mathcal{F}, \mathbb{P})$ be the direct product of these spaces. For $u_0, u'_0 \in A$, set

$$(u_k, u'_k) = R(u_{k-1}, u'_{k-1}; \omega_k), \quad k \geq 1. \tag{3.17}$$

For the filtration generated by $\{(u_i, u'_i), 0 \leq i \leq k\}$, we denote it by $\mathcal{F}_k$. We
define the following events which will be used below:

\[A_0 = \{\omega : \|u_1 - u_1'\| > q\|u_0 - u_0'\|\},\]
\[B_i = \{\omega : \|u_i - u_i'\| \leq q\|u_{i-1} - u_{i-1}'\|\},\]
\[A_k = \bigcap_{i=1}^k B_i \cap B_{k+1}^c, \quad \forall 1 \leq k \leq n - 1,\]
\[A_n = \bigcap_{i=1}^n B_i.\]

**Step 2: stratification.** We write

\[B_n^V f(u_0) - B_n^V f(u_0') = \sum_{k=0}^n J_k,\]

where

\[J_k = \mathbb{E}\{I_{A_k}\{\exp\{\sum_{i=1}^n V(u_i)\} f(u_n) - \exp\{\sum_{i=1}^n V(u_i')\} f(u_n')\}\}, \quad 0 \leq k \leq n.\]

In the below, we will use the notation \(\sum_{i=m_1}^{m_2} c_i = 0\) if \(m_1 > m_2\).

**Step 3: estimates of \(J_k\), \(0 \leq k \leq n - 1\).** By triangle inequality,

\[|J_k| \leq |J_{k,1}| + |J_{k,2}|,\]

with \(J_{k,1} = \mathbb{E}\{I_{A_k}\{\exp\{\sum_{i=1}^{k+1} V(u_i)\} f(u_n)\}\}\) and \(J_{k,2} = \mathbb{E}\{I_{A_k}\{\exp\{\sum_{i=1}^n V(u_i')\} f(u_n')\}\}\)

Noticing that \(A\) is an invariant subset for (1.6), we have

\[|J_{k,1}| = \left|\mathbb{E}\{I_{A_k}\{\exp\{\sum_{i=1}^{k+1} V(u_i)\}\} \mathbb{E}\left[\exp\{\sum_{i=k+2}^n V(u_i)\} f(u_n)\mid \mathcal{F}_{k+1}\right]\}\right| \leq \|f\|_\infty e^{\|V\|_\infty (k+1)} \|B_n^V 1\|_\infty \mathbb{P}(A_k).\quad (3.17)\]

By Proposition 3.6, for \(1 \leq k \leq n - 1\),

\[\mathbb{P}(A_k) = \mathbb{P}\left((\bigcap_{i=1}^k B_i) \cap B_{k+1}^c\right) = \mathbb{E}\{I_{\bigcap_{i=1}^k B_i} \mathbb{P}(B_{k+1}^c \mid \mathcal{F}_k)\} \leq C_q \mathbb{E}\{I_{\bigcap_{i=1}^k B_i} \|u_k - u_k'\|\} \leq C_q q^k \|u_0 - u_0'\|,\]

here \(C_q\) is a constant independent of \(k\). Obvious, by Proposition 3.6, the above estimate also holds for \(k = 0\). Combining this inequality with (3.17), we get

\[|J_{k,1}| \leq C_q \|f\|_\infty \|B_n^V 1\|_\infty e^{\|V\|_\infty (k+1)} q^k \|u_0 - u_0'\|.\quad (3.18)\]

It is obvious that the same inequality holds for \(J_{k,2}\). Hence,

\[|J_k| \leq C_q \|f\|_\infty \|B_n^V 1\|_\infty e^{\|V\|_\infty (k+1)} q^k \|u_0 - u_0'\|.\]
Step 4: estimate of $J_n$. Using the fact

$$\prod_{k=1}^{\ell} a_k - \prod_{k=1}^{\ell} b_k = \sum_{k=1}^{\ell} a_1 \cdots a_{k-1} (a_k - b_k) b_{k+1} \cdots b_{\ell}$$

with $\ell = n + 1$, $a_{n+1} = f(u_n)$, $b_{n+1} = f(u'_n)$ and

$$a_k = \exp\{V(u_k)\}, \quad b_k = \exp\{V(u'_k)\}, \quad 1 \leq k \leq n,$

we arrive at

$$J_n = \sum_{k=1}^{n} \mathbb{E}\left\{I_{A_n} \left[ \exp\left\{\sum_{i=1}^{k} V(u_i)\right\} - \exp\{V(u'_k)\} \right]\right\}
$$

with

$$\ell = n + 1, \quad a_{n+1} = f(u_n), \quad b_{n+1} = f(u'_n)$$

and

$$a_k = \exp\{V(u_k)\}, \quad b_k = \exp\{V(u'_k)\}, \quad 1 \leq k \leq n,$$

we obtain

$$J_n = \sum_{k=1}^{n} \mathbb{E}\left\{I_{A_n} \left[ \exp\left\{\sum_{i=1}^{k} V(u_i)\right\} \left( \exp\{V(u'_k)\} - \exp\{V(u_k)\} \right) \right]\right\}
$$

First, consider the terms $J_{n,k}, 1 \leq k \leq n$. Notice that, on the event $A_n$, one has

$$\|u_k - u'_k\| \leq q_k \|u_0 - u'_0\|. \quad (3.19)$$

Therefore,

$$J_{n,k} \leq e^{\|V\|_\infty} \mathbb{E}\left\{ I_{A_n} \left[ \|u_k - u'_k\| \exp\left\{ \sum_{i=k+1}^{n} V(u'_i) \right\} \right]\right\}
$$

$$\leq e^{\|V\|_\infty} \mathbb{E}\left\{ q_k \|u_0 - u'_0\| \mathbb{E}\left\{ \exp\left\{ \sum_{i=k+1}^{n} V(u'_i) \right\} 1_{F_k} \right]\right\}
$$

$$\leq \|f\|_\infty e^{\|V\|_\infty} \|B_{n+1} V\|_\infty q_k \|u_0 - u'_0\|.$$

Next, consider the term $J_{n,n+1}$. By a similar argument, we have

$$J_{n,n+1} \leq e^{\|V\|_\infty} \|\nabla f\|_\infty q^n \|u_0 - u'_0\|.$$

Combining the above estimates of $J_{n,k}, 1 \leq k \leq n + 1$, we obtain

$$J_n \leq \sum_{k=1}^{n} \|f\|_\infty e^{\|V\|_\infty} \|B_{n+1} V\|_\infty q_k \|u_0 - u'_0\|
$$

$$+ e^{\|V\|_\infty} \|\nabla f\|_\infty q^n \|u_0 - u'_0\|. \quad (3.20)$$

By setting $q$ small enough such that $q e^{\|V\|_\infty} \leq e^{-\gamma}$, by (3.20) and (3.18), we complete our proof. □
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