Phase diagram and dynamics of $^{164}$Dy dipolar Bose-Einstein condensate in the presence of a rotating anisotropic magnetic field

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We investigate the quench dynamics of quasi-one and two dimensional dipolar Bose-Einstein condensates (dBEC) of $^{164}$Dy atoms under the influence of a rotating external magnetic field. We account for quantum fluctuations, critical to formation of exotic quantum droplet and supersolid phases in the extended Gross-Pitaevskii formalism, which includes the so-called Lee-Huang-Yang (LHY) correction. An analytical variational ansatz allows us to obtain the phase diagrams of the superfluid and droplet phases. A transition from the superfluid to the supersolid phase and to single or array of dipolar droplets with particle number is captured for weaker contact interactions. The dipolar strength is tuned by rotating the magnetic field with subsequent effects on phase boundaries. Following interaction quenches across the aforementioned phases, we monitor the dynamical formation of supersolid clusters or droplet lattices. We include losses to three-body recombination over the parameter crossover regime, where the three-body recombination rate coefficient scales with the fourth power of the scattering length or the dipole length. The three-body recombination leads to the evaporation of such self-bound states, while the anisotropic magnetic field aids to increase the lifetimes of the droplets.

I. INTRODUCTION

Quantum gases 1 2 of atomic species of high spin quantum numbers, such as dysprosium 3 4 or erbium 5 atoms are ideal candidates for probing quantum fluctuations 6. Particularly, the interplay between long-range anisotropic dipole-dipole (DDI) and contact interactions gives rise to a variety of novel many-body phenomena, including anisotropic superfluidity 7 9, appearance of roton excitations 10 15, formation of self-bound quantum droplets 16 23 and supersolid states 24 25. The latter exhibits both global phase coherence and periodic density modulations 26 29, due to breaking of translation invariance, and is associated with the two low-frequency compressive modes 30 of the dipolar Bose-Einstein condensate. This new phase has been widely explored in a series of experiments 11 12 13 31 37, and theoretically in a number of cold atom settings, ranging from Rydberg systems 38 39, lattice trapped atomic mixtures 40 43, and with condensates with spin-orbit coupling 44 45 or coupled to a light field 46. Vorticity patterns of rotating supersolid dipolar gases have been also identified 17 50.

The existence of distinct phases in dBECs 51 is inherently related to the presence of roton excitations 32 53. Specifically, a roton minimum appears in the spectrum due to attractive DDI. This minimum softens for strong DDI such that the excitation energy tends zero and the dipolar gas is set for collapse. Quantum fluctuations stabilize the dipolar gas 6 56 balancing against the attractive DDI 57. To first order, they are commonly described by the LHY interaction energy 58 59. The inclusion of the LHY corrections 60 61 leads to an extended Gross-Pitaevskii equation (eGPE) 18 20 22 62. The treatment can accommodate supersolids as well as single or multiple droplet patterns 17 18 20, whose arrangement depend crucially on the transverse direction 61 63.

In majority of theoretical treatments, the polarizing magnetic field remains fixed, while the contact interactions is tuned 11 22 33. Another possibility, largely unexplored, is to apply an anisotropic (rotating) magnetic field with frequency Ω, to control the DDI 64 65. When Ω is smaller (greater) than the Larmor (trap) frequency, the dipoles follow the external field. In particular, it is possible to use the time-averaged DDI, depending on the angle between the dipole and the field axis, to tune the dipole strength and sign and thus study the impact of rotation on the ground-state phase diagram and on the droplet self-evaporation process.

Manipulating the DDI by an external rotating magnetic field 66 was recently exploited experimentally 67. Another relatively less pursued direction is to monitor the dynamical generation of the self-bound states, traversing the relevant phase boundaries, a scenario that is frequently employed in the experiment 11 23. It would be interesting to explore differences between the quenched states in the long-time dynamics as compared to the respective ground-state configurations. Also, an understanding of the metastable states at intermediate time scales due to specific instabilities and the behavior of the global phase coherence 31 is still far from complete. Motivated by the above intensive experimental and the-
toretical activity, see for instance the reviews [68, 69], we investigate the ground-state phase diagram and quench dynamics of a dBEC under the influence of a rotating magnetic field in both the quasi one-dimensional (quasi-1D) and the quasi two-dimensional (quasi-2D) regimes.

We extract the ground-state phase diagram of the dBEC as a function of the contact interaction and atom number. The emergent phases include the superfluid (SF), the supersolid (SS), and multiple (DL\(_M\)) and single (DL\(_S\)) droplet states. It is shown that SS states are characterized by spatially overlapping density humps while droplets form crystalline patterns arranged either as elongated arrays in quasi-1D or lattices with polygonal characteristics [70] in quasi-2D. These phases have already been shown to occur in fixed magnetic fields [11, 31]. Herein, we determine the explicit boundaries between the different phases, such as the DL\(_M\) (also known as insulating droplet region [33]) and DL\(_S\) by varying the field angle. Interestingly, we find that a tilted magnetic field favors the SF phase and in particular for angles larger than the magic angle, solely the SF and the DL\(_S\) persist, a behavior that is more prominent in the quasi-2D case. Naturally, the anisotropic field alters the configuration of the dBEC, enforcing for instance, an elongated droplet along the z-axis and broader 2D distributions across the x-y plane or, e.g., square and honeycomb lattice structures for angles smaller than the magic angle.

Performing interaction quenches of a SF state across the relevant phase boundaries results in the dynamical nucleation of elongated arrays in quasi-1D as also observed in [33, 57] and lattices in quasi-2D of SS and droplets due to the growth of the roton instability [11]. The latter manifests as ring excitations or elliptic halos in the early times before developing into clusters that then saturate. Phase coherence is not maintained in the course of the evolution and it is fully lost in the droplet regime. Quenches from the SF to the DL\(_S\) phase produce DL\(_M\) lattices.

We demonstrate that the number of droplets contained in a lattice is larger for reduced post-quench contact interactions or tilted fields with an angle smaller than the magic angle. Also, the amount of dynamically nucleated droplets in the long-time quench dynamics is larger as compared to the respective ground-state postquench configuration. Another central feature of our findings is the exploration of the self-evaporation of the above-discussed structures by including three-body recombination processes into our analysis. This mechanism prevails for bound states and raises a nontrivial obstacle in connection with the realization of droplets and especially SS phases [22, 33]. Specifically, we showcase that the anisotropic magnetic field, lying below the magic angle, is a tool to increase the lifetime of self-bound states. These regions were inaccessible in previous studies [16, 17] due to the assumption of an aligned magnetic field along the z-direction.

This work is structured as follows. Section II describes the anisotropic dipolar potential and introduces the eGPE framework. In section III, we extract the ground-state phase diagram of the dBEC in both the quasi-1D (Sec. III A) and the quasi-2D (Sec. III B) regimes. The dynamical generation of self-bound SS and droplet states following interaction quenches is discussed in Sec. IV In Sec. V we monitor the self-evaporation of the quenched states by accounting for three-body recombination processes. A summary of our findings together with future perspectives are provided in Sec. VI. Appendix A is devoted to the construction of the variational approach for confirming the existence of the ground-state phases, while Appendix B delineates the ingredients of our numerical simulations. In Appendix C we briefly analyze the collective excitation processes of the quasi-2D dBEC.

II. BEYOND MEAN-FIELD TREATMENT OF THE DIPOLAR CONDENSATE

Below, we describe the explicit form and properties of the considered DDI potential as well as provide the intrinsic system parameters which closely follow recent experimental settings [11, 22, 33]. Afterwards, we introduce the eGPE framework that we shall use in order to track the phase diagram and subsequently monitor the quench dynamics of the dipolar condensate.

A. Dipolar potential and rotating magnetic field

We consider a harmonically trapped dBEC in three-dimensions (3D) whose atoms possess a magnetic dipole moment \(\mu_m\). The atomic dipoles are polarized, by a rotating uniform magnetic field (in the x-y plane) of strength \(B\), along \(e(t) = \cos \phi e_x + \sin \phi (\cos(\Omega t)e_x + \sin(\Omega t)e_y)\) with \(e_x, e_y, e_z\) being the unit vectors in the \(x, y, z\) spatial directions respectively. The field rotation frequency \(\Omega\) is chosen to be \(\omega_i \ll \Omega \ll \omega_L = \mu B/\hbar\), where \(\omega_i, i = x, y, z\) are the trap frequencies, such that to ensure that the dipoles follow the external field. The tilt angle with respect to the z-axis is \(\phi\), see Fig. 1. This way, the DDI [64] is given by the potential \(U_{dd}(r,t) = \mu_0 \mu^2 \frac{1-3(\cos(\phi) \cdot \hat{r})^2}{4\pi r^2}\), where \(\mu_0\) is the permeability of the vacuum. For \(\Omega = 0\) and \(e_z \cdot \hat{r} = 1\), a head-to-tail arrangement of the dipoles occurs leading to an attractive DDI, i.e. \(U_{dd} < 0\). Where \(e_z \cdot \hat{r} = 0\), the dipoles are located side-by-side and interact repulsively.

The corresponding time-averaged DDI, over a full rotation cycle of the polarizing magnetic field, is

\[
\langle U_{dd}(r) \rangle = \frac{\Omega}{2\pi} \int_{0}^{\pi/2} U_{dd}(r,t)dt = \frac{\mu_0 \mu^2}{4\pi} \left[ \frac{1-3(e_z \cdot \hat{r})^2}{r^2} \right] \left( \frac{3 \cos^2 \phi - 1}{2} \right). \tag{1}
\]

Notice that the last factor in Eq. (1) decreases from 1 to \(-1/2\) when \(0 < \phi < \pi/2\), and vanishes if \(\phi\)
equals the magic angle \[ \phi_m = \cos^{-1} \frac{1}{\sqrt{3}} \approx 54.7^\circ \]
where the time-averaged DDI is attractive even though particles reside side-by-side (also known as anti-dipolar regime\[22, 72\]). Therefore, the dBEC subjected to this rotating magnetic field experiences the time-averaged DDI potential whose strength and sign can be tuned by varying the tilt angle \( \phi \). Such a rotating long-range potential has already been implemented\[4\]. It is worth to mention that the DDI can be tuned this way independently of the tuning of the zero-range interaction which the time-averaged DDI is attractive even though the gas density as \( n \) increases \[n \rightarrow n(r, t) = |\psi(r, t)|^2 \] and the system features supersolid properties characterized by spatially modulated periodic density undulations.

Finally, we remark that the presence of rotation should facilitate the tunability of the DDI, especially in the experiment, since otherwise only a tilted magnetic field (in the \( x-z \) plane) is characterized by two different angles\[1\]. Such a study would be pursued in the future.

### B. Extended Gross-Pitaevskii framework

In the ultracold regime the gas is characterized by a single macroscopic wave function, \( \psi(r, t) = (\hat{\psi}(r, t)) \), whose temporal evolution is described by a suitable eGPE \[18, 22, 61, 72\]. The latter incorporates quantum fluctuations in terms of the first order beyond mean-field LHY correction term and in particular reads

\[
i\hbar \frac{\partial \psi(r, t)}{\partial t} = \left[ -\frac{k^2}{2m} \nabla^2 + V(r) + g|\psi(r, t)|^2 + \gamma(\epsilon_{dd})|\psi(r, t)|^3 + \int dr'U_{dd}(r-r')|\psi(r', t)|^2 \right] \psi(r, t). \tag{2}
\]

Here, the 3D harmonic trap \( V(r) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2 \) and \( m \) is the atom mass. Apart from the long-range time-averaged DDI in Eq. (1), the atoms collide via short-range contact potential, characterized by the effective strength \( g = 4\pi\hbar^2 a_s/m \), with \( a_s \) being the 3D s-wave scattering length. The penultimate term in Eq. (2) denotes the LHY contribution which is crucial for the realization of many-body self-bound states such as the DLs or DLM dipolar droplets, as well as the SS phase. We remark that in 3D, quantum fluctuations scale with the gas density as \( \sim n^{3/2} \). In the harmonic trap, the LHY correction can be incorporated in the eGPE, with the local density approximation, i.e., \( n \rightarrow n(r, t) = |\psi(r, t)|^2 \), and with \( \gamma(\epsilon_{dd}) = \frac{\pi^2 g}{\sqrt{3}} \left( 1 + \frac{3}{8} a_s^2 \right) \) \[59, 72\]. Importantly, the dimensionless parameter \( \epsilon_{dd} = a_{dd}/a_s \) with \( a_{dd} = \mu_B\hbar^2 m/(12\pi\hbar^2) \) being the dipolar length, quantifies the relative strength of the DDI as compared to the contact interaction.

Below, we shall reveal the emergent ground-state phases of the dBEC stemming from the interplay between the dipolar and contact interactions, employing the parameters of the recent experiments, but now accounting for a rotating magnetic field\[11, 12, 33\]. Subsequently, the dynamical deformation of the identified dipolar configurations is monitored upon considering quenches of the s-wave scattering length across the aforementioned phases. A particular emphasis is placed on the role of dimensionality ranging from (i) an elongated quasi-1D trap with frequencies \( (\omega_x, \omega_y, \omega_z) = 2\pi \times (227, 37, 135)\text{Hz} \) \[33\] to (ii) a circularly symmetric quasi-2D trap characterized by \( (\omega_x, \omega_y, \omega_z) = 2\pi \times (45, 45, 133)\text{Hz} \) \[61\], see also Fig. 1(a), (b). Our results can be replicated using a dBEC of \( ^{164}\text{Dy} \) atoms having magnetic moment \( \mu_m = 9.93\mu_B \), where \( \mu_B \) is the Bohr magneton. The dipolar length is \( a_{dd} = 131a_B \), where \( a_B \) is the Bohr radius.

Harmonically trapped 3D dBECs subjected to a static magnetic field along the strongly confined direction, exhibit a superfluid behavior for \( \epsilon_{dd} < 1.4 \) \[38\]. However, for \( \epsilon_{dd} > 1.4 \) and in the mean-field description the system collapses. In sharp contrast, quantum fluctuations (LHY correction which acts repulsively) prevent this collapse and the system features supersolid properties characterized by spatially modulated periodic density preserving coherence\[31\]. Interestingly, upon further increasing \( \epsilon_{dd} \), the dBEC can be maneuvered into the so-called droplet region\[31\] whose coherence is lost.

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1. The DDI for a fixed magnetic field (\( \Omega = 0 \)) reads in the \( x-z \) plane reads \( U_{dd} = -\frac{3}{(m/\hbar^2)}[\cos(\phi)|\hat{\psi}(r)|^2 + \sin(\phi)|\hat{\psi}(r)|^2]^2 \).


III. GROUND-STATE PHASE DIAGRAM OF ROTATIONALLY TUNED DBECS

In the following, we will investigate the ground-state phases of the dBEC arising for different atom numbers and s-wave scattering lengths, while the strength of the DDI is fixed to the one of $^{164}$Dy, i.e. $a_{dd} = 131 a_B$. As such, it is possible to implicitly appreciate the impact of the competition between the dipolar and the contact interaction strength as captured by $\epsilon_{dd} = a_{dd}/a_s$. The discussion is focused on both quasi-1D and quasi-2D regimes aiming to unravel the role of the orientation of the dipoles, as dictated by the titled time-averaged magnetic field [Fig. 1(a), (b)], on the emergent structural configurations. For completeness, in Appendix A we further benchmark the properties of the static phases found within the eGPE approach utilizing a variational ansatz. A similar approach has also been recently leveraged within the dBEC context e.g. in Ref. [73, 74].

To track the spatial distribution of the participating dipoles we invoke integrated density profiles along the x-y plane $n(x, y) = \int dz \ n(x, y, z, t)$ for the quasi-2D geometry and the elongated y- direction $n(y, z) = \int dx \ n(x, y, z)$ in quasi-1D. These are experimentally detectable e.g. via in-situ imaging [6, 75] and herein are normalized to the particle number. It will be shown that the density distributions together with the condensate phase defined by $\Phi = \tan^{-1} [\text{Im}(\psi(r))/\text{Re}(\psi(r))]$, where $\text{Re}(\psi(r))$ [$\text{Im}(\psi(r))$] is the real [imaginary] part of the wave function allow for the identification of the distinct dBEC phases. Indeed, within the SF and DL$_S$ phase, the dBEC possesses a smooth non-modulated density profile and a spatially uniform $\Phi$. Notice that throughout this work the terms density and phase modulations do not refer to variations stemming from the external harmonic trap but rather to spatially periodic undulations caused by interaction effects. However, the SS structure is characterized by a modulated density and a uniform $\Phi$, while the DL$_M$ phase exhibits both a modulated spatial configuration and a non-uniform phase $\Phi$.

Another observable that can be utilized to infer the existence of the different phases is the chemical potential of the system related to the total energy $E$ (see Eq. (4)) via $\mu = \partial E/\partial n$. It is given by

$$
\mu = \int \left[ \frac{\hbar^2}{2m} |\nabla\psi(r)|^2 + V(r)|\psi(r)|^2 + g|\psi(r)|^4 + 
\gamma(\epsilon_{dd})|\psi(r)|^5 + \int dr' U_{dd}(r-r') |\psi(r')|^2 |\psi(r')|^2 \right] dr.
$$

(3)

Naturally, a SF state has $\mu > 0$, whereas the single (DL$_S$) and multiple (DL$_M$) isolated droplet regions occur for $\mu < 0$ since they refer to bound states. In addition we will show that the SS phase appears in the vicinity of $\mu \to 0$.

A. Phases of the quasi-1D dBEC

To realize a quasi-1D dBEC we consider an elongated harmonic trap along the y-axis [Fig. 1(a)] and employ the chemical potential [Eq. (3)] as a “control” parameter for distinguishing the various phases. We should emphasize that the identification of each dBEC phase is also corroborated by relying on all the above-discussed observables (see the discussion below) as well as the momentum distribution of the density (not shown). The corresponding ground-state phase diagram with respect to the atom number, $N$, and the s-wave scattering length, $a_s$, is presented in Fig. 2(a1)-(a4) for different orientations, determined by the angle $\phi$, of the rotating magnetic field. All ground states are obtained by propagating Eq. (2) in imaginary time with a split-step Crank-Nicolson approach, see details in Appendix B.

For $\phi = 0^\circ$ the external field forces the dipoles to be oriented along the z-axis. Inspecting the behavior of the
chemical potential [Fig. 2(a1)] it becomes apparent that the phase arising at large magnitudes of the s-wave scattering length (e.g. \(a_s > 90a_B\) for \(N = 60000\)) has a positive chemical potential energy. As such, a typical SF state emerges characterized by a smooth density profile, see for instance \(n(x, y)\) having a Thomas-Fermi (TF) profile in the case of \(N = 10^5\) and \(a_s = 98a_B\) depicted in Fig. 2(a1). However, for scattering lengths below a certain critical value, indicated by the dashed white line in Fig. 2(a1), the system transitions to a negative \(\mu\) region. As we will argue by tracking the associated density configurations, the latter regime is possible to accommodate distinct phases of matter, namely a SS, a DLs or a DLm.

Indeed, with decreasing \(a_s\) for large atom number \((N > 2 \times 10^4)\) and in the vicinity of \(\mu \rightarrow 0\) (see the bounded area by the solid blue line in Fig. 2(a1)), a periodic density modulated pattern develops along the weakly confined \(y\)-direction. A representative example is shown in Fig. 2(a2) for \(N = 10^5\) and \(a_s = 86a_B\). Importantly, the neighboring density humps exhibit a non-vanishing spatial overlap among each other, thus establishing a global phase coherence across the dBEC [24, 33, 35]. This state is referred to as the SS phase [33, 36, 39]. A further reduction of the scattering length results in a dramatic suppression of the density overlap among the individual density humps for large atom numbers, see in particular Fig. 2(a3), while \(\mu\) acquires large negative values [Fig. 2(a1)]. Here, the system enters the DLm phase (see the bounded area by the blue dashed and solid lines in Fig. 2(a1)) which contains an array of isolated droplets. With smaller atom numbers, the dBEC transits to the single-droplet phase denoted by DLs. It is also evident that for low atom numbers \((N < 10^4)\) where finite size effects are expected to play a crucial role [76], the system deforms from the SF to the DLs state and vice versa with tuning \(a_s\). Notice that the DLs and SF phases, despite being characterized by a zero global phase coherence [see also Fig. 1(c)] differ in their spatial localization and importantly due to the fact the former is self-bound \((\mu < 0)\).

The formation of the above-discussed phases originates from the competition between the distinct energy contributions. In particular the total energy of the dBEC reads

\[
E = E_K + E_V + E_{CI} + E_{DDI} + E_{LHY},
\]

where the energy contributions \(E_{CI} = (1/2) \int dr \psi^\ast(r) \psi(r), E_{DDI} = (1/2) \int dr dr' U_{dd}(r-r')|\psi(r')|^2|\psi(r)|^2, E_{LHY} = \frac{1}{2} \int dr \gamma(\epsilon_{dd})|\psi(r)|^4,\) refer to the contact, the dipolar, and the beyond mean-field LHY interaction energies respectively. These energy terms dictate the generation of the different phases. Also, \(E_K = \frac{\hbar^2}{2} \int dr \nabla^2 |\psi(r)|^2/(2m)\) denotes the kinetic energy and \(E_V = \int dr \nabla^2 |\psi(r)|^2\) is the external potential energy. The interplay of \(E_{CI}, E_{DDI}\) and \(E_{LHY}\) as a function of \(a_s\) for \(N = 10^5\) and \(\phi = 0^\circ\) is depicted in Fig. 3(a). Obviously, \(E_{CI} > 0\) and \(E_{DDI} < 0\) irrespectively of \(a_s\), whilst \(E_{LHY} > 0\) outside the SF phase i.e. \(a_s < 92a_B\), otherwise \(E_{LHY} \rightarrow 0\). Therefore, as expected, \(E_{CI}\) is dominant in the SF state \((a_s > 92a_B)\) in Fig. 3(a), (b) and all other contributions are considerably weaker. In sharp contrast, the terms \(E_{DDI}, E_{LHY}\) determine the emergent phase structures for \(a_s < 92a_B\). Particularly, a reduction of \(a_s\) leads to a rapid increase of \(|E_{DDI}|\) towards negative values, whilst the repulsive \(E_{CI}\) and \(E_{LHY}\) feature an arguably much smaller increase. Notably, the attractive nature of \(|E_{DDI}|\) can not be solely compensated by the repulsive effect of \(E_{CI}\), thus driving the dBEC to collapse. However, this collapse tendency is actually prevented from the enhanced combined repulsive contribution of \(E_{LHY}\) and \(E_{CI}\) occurring for decreasing \(a_s\). This competition leads to the formation of stable supersolid and droplet states in the dBEC in the corresponding contact interaction intervals [6].
Concluding, the DL$_S$ phase can be achieved for every $\phi \neq \phi_m$ and provided that $a_s$ is sufficiently small. Moreover, in the case of $\phi < \phi_m$ the DL$_S$ breaks into a droplet array for larger atom number, a behavior that is absent if $\phi > \phi_m$. This is possible because for $\phi > \phi_m$ the DDI is attractive enforcing the dipoles to reside close with one another in the $x$-$y$ plane thus acting against the formation of density modulations across the $y$-axis [Fig. 2(b$_3$)-(b$_4$)]. In this case, both the total energy and the dipolar interaction energy are minimized when $e_z \cdot \hat{r} = 0$, implying that the atoms self-organize perpendicular to the $z$-axis. Consequently, 2D thin non-modulated density profiles take place even for $\mu < 0$. Strikingly, the size of the single droplet depends crucially on $\phi$. To elucidate this behavior, we display the density profiles, $n(x, y)$, in Fig. 2(b$_1$)-(b$_2$), for $N = 6 \times 10^4$. As it can be seen, the droplet distribution features an enhanced localization as the magnetic field tends to be aligned in the $z$-direction, see e.g. $n(x, y)$ for $\phi = 0^\circ$ and $\phi = 30^\circ$ in Fig. 2(b$_1$)-(b$_2$), but becomes elongated along the $z$-axis in the $y$-$z$ plane due to the magnetic orientation (not shown). This evinces the inherent quasi-1D nature of the DL$_S$. In contrast, when $\phi > \phi_m$ the magnetic field orientation is towards the $x$-$y$ plane and the spatial configuration of the droplet delocalizes along the $y$-axis [Fig. 2(b$_3$)-(b$_4$)]. As a result, the atoms drift towards the $x$-$y$ plane, resulting in an anisotropic 2D density distribution that is stretched along the weakly confined $y$-direction.

In a similar vein, the droplet arrays and the SS states are impacted with tuning $\phi$. Specifically, the number of participating droplets gets larger for increasing $\phi < \phi_m$, ultimately tending to a SS state. On the other hand, the background density of a SS becomes gradually denser for
larger $\phi$, thus destroying its SS nature and finally establishing a SF state. Notice that the transition boundary from a SF to a SS can also be determined from the so-called contrast, $C = (n_{\text{max}} - n_{\text{min}})/(n_{\text{max}} + n_{\text{min}})$, where $n_{\text{max}}$ and $n_{\text{min}}$ are the neighboring density maxima and minima, respectively. Namely, a SF state occurs for $C = 0$, while $C \neq 0$ corresponds to a density modulated state.

**B. Phases of the quasi-2D dBEC**

We now focus on a circularly symmetric quasi-2D trap geometry which can be realized by applying a tight confinement in the transversal $z$-direction. As previously, the magnetic field rotates at an angle $\phi$ relative to the $z$-axis [Fig. 1(b)], while $a_s$ is tuned controlling $\epsilon_{dd}$ and allowing us to enter different phases. The ground-state phase diagrams depicting $\mu$ in the $a_s$-$N$ plane for different orientation angles $\phi$ are provided in Fig. 5(a1)-(a4). We can easily deduce that the emergent phase portrait including also the participating SF, SS, DL$_M$ and DL$_S$ states as well as the shift of their boundaries for increasing $\phi$ are similar to the quasi-1D scenario, see also Fig. 2. For instance, if $\mu > 0$ a smooth 2D density distribution typical of the SF state occurs, whereas for $\mu < 0$ [Figs. 1(h) and 1(i)] and in the interval of $N > 13000$ and $a_s < 92 a_B$, periodic density arrays delineating the SS [DL$_M$] phase appear (see also below). Considering smaller particle numbers, i.e. $N < 12000$, a phase transition from the SF ($\mu > 0$) to DL$_S$ ($\mu < 0$) takes place for decreasing $a_s$. A noticeable difference from the quasi-1D case is the more pronounced shrinking behavior of the DL$_M$ phase as $\phi \to \phi_m$, compare Fig. 2(a2) and Fig. 5(a2). This means that for $\phi > \phi_m$ the isolated droplets are prone to coalesce into a single droplet configuration. However, as in the quasi-1D case, for $\phi > \phi_m$, only the DL$_S$ (with $\mu < 0$) and the SF (having $\mu > 0$) can be formed.

The overall similarity in terms of the phases appearing in both the quasi-2D and quasi-1D regimes can be also verified by inspecting the interplay among the involved energy contributions for varying $a_s$, see e.g. Fig. 4(b) for $N = 10^5$ and $\phi = 0^\circ$. Indeed, as in the quasi-1D case a reduction of $a_s$ leads to an increase of $E_{\text{CI}}$ and $E_{\text{LHY}}$, whereas $E_{\text{DDI}}$ becomes prominently negative when $a_s < 90 a_B$. In this latter regime SS, DL$_M$ and DL$_S$ phases form due to the presence of quantum fluctuations preventing collapse, while if $a_s > 90 a_B$ the contribution of the contact interaction dominates and a SF state occurs. Naturally due to the quasi-2D confinement the aforementioned phases exhibit a different spatial distribution as compared to their quasi-1D counterparts.

To expose the patterns characterizing each phase, we calculate a number of integrated density profiles $n(x,y)$ and $n(y,z)$ in Fig. 5(a1)-(a6) for $N = 6 \times 10^4$, $\phi = 0^\circ$ and distinct values of $a_s$. Focusing on $a_s = 98 a_B$, belonging to the SF region, we observe that a smooth 2D profile appears along the $x$-$y$ plane [Fig. 5(a1)] and an elongated $n(y,z)$ profile in the $y$-direction caused by the tight confinement in the $z$-axis. The finite width in the $z$-direction stems from the presence of the magnetic field along this axis. This behavior would be, of course, absent for a short-range interacting condensate. On the contrary, the density configuration of the SS phase, e.g., occurring at $a_s = 86 a_B$ [Fig. 5(a2)] and Fig. 5(a3)] is arguably more complex. Particularly, four inner localized density humps in $n(x,y)$ arranged in a square pattern are linked by a low density region and they are being

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2 The transition from the SS to the SF state can be equally seen in both the density and the momentum distribution of the dBEC. Indeed, periodic density undulations vanish as the SF is entered. Also, the momentum distribution of a SF is characterized by a single peak structure while for a SS multiple additional peaks appear.

3 A corresponding SF state is essentially insensitive to variations of $\phi$ and only its width increases (decreases) for larger (smaller) $\phi$ as long as $\phi < \phi_m$ ($\phi > \phi_m$).
FIG. 6. Ground-state density profiles (a₁)-(a₃) n(y,z) and (a₁)-(a₆) n(x,y) representing (a₁), (a₂), (a₃) a SS and (a₄), (a₅), (a₆) a droplet cluster. A SS state is characterized by overlapping density humps, while a droplet cluster has a crystal arrangement. The harmonically trapped 2D dBEC with \( \omega_x, \omega_y, \omega_z = 2\pi \times (45, 45, 133) \text{Hz} \) has \( N = 10^5 \) particles and it is subjected to a magnetic field along the \( z \)-direction, i.e. \( \phi = 0^\circ \). The colorbar denotes the density in units of \( m_\omega \ell^{-1} \). \( \ell = 1.37 \mu m^{-2} \).

FIG. 7. Impact of the field orientation \( \phi \) on the spatial distribution of the single droplet state. Densities (a₁)-(a₃) \( n(y,z) \) and (a₁)-(a₆) \( n(x,y) \) of a dBEC with \( N = 6 \times 10^5 \) in a quasi-2D trap with \( \omega_x, \omega_y, \omega_z = 2\pi \times (45, 45, 133) \text{Hz} \). The single droplet becomes narrower (wider) for larger \( \phi \) in the interval \( \phi < \phi_m \) (\( \phi > \phi_m \)) and becomes elongated along the \( z \)-\( (y) \)-direction due to the polarization of the magnetic field. The colorbar denotes the density in units of \( m_\omega \ell^{-1} \). \( \ell = 1.37 \mu m^{-2} \).

scattering length, e.g. to \( a_s = 70a_B \), results in the production of a square crystal structure along the \( x\text{-}y \) plane [Fig. 6(a₅)]. Correspondingly, stripe patterns are evident in \( n(y,z) \), see Fig. 6(a₆).

As in the quasi-1D case, also here the spatial structure of the DL₅ state is altered in terms of \( \phi \). This behavior is visualized in Fig. 7(a₁)-(a₆) displaying \( n(x,y) \) and \( n(y,z) \) for various \( \phi \) and different \( a_s \) residing well inside the DL₅ region. Recall that the emergence of DL₅ is shifted to smaller \( a_s \) intervals as \( \phi \rightarrow \phi_m \) [Fig. 5] due to the accompanied weakening of the effective dipole interaction [Eq. (1)]. The smaller contact interaction is responsible for the decreasing width of \( n(x,y) \) when \( \phi \rightarrow \phi_m \) [Fig. 7(a₁), (a₅)]. On the other hand, the magnetic field for \( \phi > \phi_m \) favors a spreading of the atomic dipoles in the \( x\text{-}y \) plane and thus also the single droplet becomes wider [Fig. 7(a₃)]. Interestingly, we observe that \( n(y,z) \) changes from being highly elongated along \( z \) [Fig. 7(a₄), (a₆)] if \( \phi < \phi_m \), to being so across \( y \) [Fig. 7(a₁), (a₅)] for \( \phi > \phi_m \). The distinct energy terms as a function of \( \phi \) in quasi-2D for fixed \( a_s = 70a_B \) are shown in Fig. 4(c). All energy components (\( E_{CI}, E_{DDI}, E_{LIHY} \)) are positive in the interval \( 22^\circ < \phi < \phi_m \) and thus a SF is formed. However, in the case of \( \phi = 22^\circ \), \( E_{DDI} \) is negative and gradually drops as \( \phi \) lowers. A similar behavior of the energy constituents is observed for \( \phi > \phi_m \) where the single droplet state appears. Therefore in these two regions, namely \( \phi < 22^\circ \) and \( \phi > \phi_m \) droplets are formed.

In a similar vein, the tilt angle has a substantial impact also within the DL₅ and SS phases. Below, we will focus on the deformations of a DL₅ state for varying field orientation which essentially tunes the relative strength, \( \epsilon_{dd} = a_{dd}/a_s \), of the repulsive contact interaction and attractive DDI. In order to explore all possible different arrangements of the emergent droplet clusters we consider here \( N = 2.5 \times 10^5 \) and a fixed \( a_s = 75a_B \), see Figs. 8(a₁)-(a₃). Recall that a decreasing \( \epsilon_{dd} \) (with \( \epsilon_{dd} > 1.4 \)) favors the breaking of each individual droplet into multiple segments, see also Fig. 5(a₁). The parameter \( \epsilon_{dd} \) becomes smaller by either increasing \( a_s \) or decreasing the tilt angle \( \phi \). Its effect on the DL₅ state is presented in Figs. 8(a₁)-(a₃) where we observe that the number of droplets within the cluster increases from six to thirteen for \( \phi = 0^\circ \) and \( \phi = 15^\circ \) respectively due to the larger \( \epsilon_{dd} \). Note that the set \( N = 2.5 \times 10^5, a_s = 75a_B \) lies within the DL₅ regime for both \( \phi = 0^\circ \) and \( 15^\circ \). However, as \( \phi \) increases further to \( \phi = 20^\circ \), the cluster is drastically altered with the number of humps becoming wider while overlapping with each other through a weak background density [Fig. 8(a₃)]. This signals the generation of a SS state distributed in a honeycomb configuration, and the structural deformation of the DL₅ towards a SS by means of \( \phi \). This is attributed to the effective weakening of the DDI for larger \( \phi < \phi_m \).

Such a distribution has been independently identified via tuning \( a_s \) in the presence of a fixed magnetic field along the \( z \)-direction [70]. It is worthwhile to mention that structural deformations of droplet lattices can be surrounded by an outer density ring, see Fig. 6(a₅). This relatively small spatial overlap establishes a global phase coherence across the dBEC [Fig. 1(c)]. The individual density humps are also imprinted in the transversal \( z \)-direction since \( n(y,z) \) shows fringes at their location. A similar peculiar SS pattern has also been recently found in [63, 70] independently. Further reducing the \( s \)-wave
in a lattice. The external quasi-2D geometry is characterized by \( \phi \) fixed \( \phi \). Along these lines, by adjusting the trap ratio \( \omega_x, \omega_y, \omega_z \) such that the SS or the droplet phase is dynamically prepared, the dBEC can be prepared in a SF state where \( a_s = 120a_B \) to \( a_s = 90a_B \) which corresponds to a SS in terms of the ground-state [Sec. IV B]. In both cases, the dBEC is subjected to a magnetic field along the \( z \)-direction, i.e., \( \phi = 0^\circ \). It should, however, be noted that the pattern formation to be described below does not substantially change for \( \phi \neq 0^\circ \). The number of nucleated droplets in a cluster or density peaks in a SS in the long-time dynamics drops as the post quench \( a_s \) is increased, see Table II.

### A. Dynamical nucleation of 2D SS and droplet lattices

Representative instantaneous density profiles \( n(x, y, z = 0; t) \) of the quasi-2D dBEC are presented in Fig. 9(a)–(a6) after a quench from a SF state with \( a_s = 120a_B \) to \( a_s = 90a_B \) which corresponds to a SS in terms of the ground-state [Fig. 5]. As expected, initially \( n(x, y; t = 0) \) has a smooth 2D TF distribution [Fig. 9(a1)]. However, since the quench is performed across a phase boundary it seeds a relevant instability (being here of roton nature [14, 80]) due to roton softening in the postquench phase leading to pattern formation. Indeed, we observe the appearance of ring-shaped density structures which become more pronounced as time-evolves, see Figs. 9(a2)–(a6). For an analysis of the roton instability and its shape in a 3D harmonically trapped dBEC see Ref. [15]. It is in fact the progressive growth of the roton mode, characterized by a non-zero angular momentum, which is responsible in the early time dynamics for the development of these ring structures accompanied by density depleted regions. Later on, for \( t > 20 \) ms, following the interference of the ring densities (stemming from the radial roton) and the growth of azimuthal undulations (originating from the angular roton) the dBEC distribution splits into four overlapping density peaks arranged in a square configuration and surrounding the central density hump which appears to be isolated from the others, see Fig. 9(a6). At these timescales, quantum fluctuations take over and the roton instability growth slows down as has been argued in [33, 37].

The SS pattern, created by the aforementioned spatially overlapping density humps, is characterized by a distorted phase profile. This can be directly seen in Fig. 10(a1)–(a2) taken at \( t = 150 \) ms where the SS has been formed, see in particular the spatial region marked by the pink circle within which the SS resides. Notice that the enhanced distortions at the condensate edges (\( |x| \leq 8 \mu m, |y| \leq 8 \mu m \)) stem from the highly non-smooth low density in conjunction with the high frequency breathing of the entire cloud. Actually, there are

\[ a_s = 75a_B, N = 2.5 \times 10^6 \]

\[ a_s = 62a_B \]

\[ a_s = 72a_B \]

\[ a_s = 84a_B \]

\[ \phi = 0^\circ, N = 2.5 \times 10^6 \]

\[ \phi = 15^\circ, N = 2.5 \times 10^6 \]

\[ \phi = 20^\circ, N = 2.5 \times 10^6 \]

\[ \phi = 20^\circ, N = 2.5 \times 10^6 \]

 Independently achieved by maneuvering the trap ratio among the longitudinal and transversal directions, thus lying in the crossover from 2D to 1D, as it has been demonstrated [78]. Along these lines, by adjusting \( a_s \) it is possible to create a variety of intriguing droplet patterns for fixed \( \phi \) such as square distributions at \( a_s = 62a_B \) [Fig. 9(b1)], pentagon at \( a_s = 72a_B \) [Fig. 9(b2)] or triangle-like lattices at \( a_s = 84a_B \) [Fig. 9(b3)]. Examining the energetics of such configurations, in analogy to what has been done, e.g., for multi-vortex configurations (see, e.g., [79] for an example) would be a particularly intriguing direction for future study.

### IV. QUENCH DYNAMICS

Having addressed the phase diagram of the dBEC, we next investigate its non-equilibrium dynamics. We initially prepare the dBEC in a SF state where \( a_s = 120a_B \). Then, an interaction quench to weaker values of \( a_s \) is considered such that the SS or the droplet phase is dynamically entered. The aim is to inspect whether it is possible to monitor the spontaneous nucleation and properties of these beyond mean-field structures in both the circularly symmetric quasi-2D scenario [Sec. IV A] and the axially elongated quasi-1D geometry [Sec. IV B].

4 The roton modes in our quasi-2D harmonically trapped setup are characterized by the quantum number \( m \). In this sense, \( m = 0 \) refers to the radial roton manifesting as a ring structure and \( m \neq 0 \) are the angular rotors having \( m \) number of azimuthal nodes [15].
two breathing mode frequencies arising in the dynamics, see also Appendix C and Fig. 15(a3), (a4). These are related to the dynamical generation of the SS state as observed in Ref. [31]. The above behavior of the phase profile should be contrasted with the normal SF state. The latter exhibits a uniform phase as demonstrated in Fig. 10(a1) at \( t = 250 \text{ ms} \) after a quench to \( a_s = 110a_B \) at which the initial SF configuration maintains its character. Outside the condensate boundaries denoted by the pink circle in Fig. 10(a1), \( \Phi(x,y,z = 0) \) experiences a slightly increasing tendency caused by the low density region and the accompanying background breathing mode.

Next, we tune to a postquench scattering length \( a_s = 70a_B \), when one DL\( _S \) phase forms. The dBEC is again dynamically distorted showing a two-ring structure and a central density hump due to the involvement of the roton mode. The latter grows at a faster rate when compared to the \( a_s = 90a_B \) quench, see Fig. 9(ba2) and (b2). This larger growth of the roton when entering deeper in the droplet regime is expected from the underlying excitation spectrum [14]. The instability leads then to the disintegration of the inner ring and the central density peak into multiple droplets [Fig. 9(b3)]. The outer ring remains almost intact and a low density circular structure appears at larger radii emerging from the edges of the cloud [Fig. 9(b4)]. This metastable configuration subsequently breaks into a droplet lattice as it can be seen in Fig. 9(b5)-(b6). It should be emphasized that while the postquench ground-state represents a DL\( _S \) we encounter here the spontaneous nucleation of DL\( _M \). These droplet clusters feature, at the early times of their creation, a global breathing motion and thus the distance between individual droplets changes [Fig. 9(b5)-(b6)]. However, in the long-time dynamics (roughly \( t > 70 \text{ ms} \)) the breathing amplitude reduces and the cluster remains practically stationary, see also Appendix C at least up to 500 ms that we have checked signalling that the system approaches a prethermalized state.

The number of isolated droplets contained in the cluster in the long-time dynamics increases for smaller postquench scattering lengths, namely deeper in the droplet regime, see in particular Table II in the case of \( \phi = 0^\circ \). In contrast to this response the number of isolated droplets participating in a cluster realized for a fixed post-quench scattering length is smaller upon increasing the tilt angle. This trend is visualized in the case examples of Table II and it is attributed to the reduced magnitude of the associated DDI. We also note in passing that independently of \( a_s \) the emergent droplet lattice does not form any polygon pattern, in agreement with [18]. The existence of the droplet clusters can also be distinguished by inspecting their phase profiles which are highly non-uniform [Fig. 10(a3)] at \( t = 200 \text{ ms} \) even when compared with the SS phase [Fig. 10(a2)].

An observable that provides further verification for the dynamical creation of the above-discussed beyond mean-field states is the so-called global phase coherence [31, 33, 35], see also Fig. 1(c), which is defined as follows

\[
\beta_c(t) = \int d\mathbf{r} [\Phi(\mathbf{r},t) - \bar{\Phi}(t)]^2.
\]

Here, \( \bar{\Phi}(t) \) is the spatially averaged phase. Accordingly, following adiabatic pulses a SF or SS state has perfect global phase coherence (\( \beta_c(t) = 0 \)), while self-bound

\[5\] The position of the roton minimum \( k_{\text{rot}} \) should satisfy \( k_{\text{rot}} l_z \geq 1 \), where \( l_z = \sqrt{\hbar/ma_z} \) is the harmonic oscillator length scale along the tightly confined direction. In our case, \( k_{\text{rot}} = 3.39 \mu m^{-1} \) with \( l_z = 0.68 \mu m \). Similar findings have been reported e.g. in the experiment of Ref. [14] where \( k_{\text{rot}} = 2.5 \mu m^{-1} \) and \( l_z = 0.625 \mu m \).

\[6\] It is worth mentioning that the number of dynamically nucleated droplets in the long-time evolution is, in general, larger from the one of the respective ground-state configuration. For instance, in the case of \( a_s = 75a_B \) the droplet lattice contains twenty four individual droplets in contrast to four formed in the ground-state for \( N = 10^5 \).
quantum droplets feature $\beta_c(t) \neq 0$. This is expected since as we previously argued the phase of the gas becomes highly distorted for droplet states in contrast of being almost uniform for SS and SF phases. The dynamics of $\beta_c(t)$ is provided in Fig. 10(b1) for various postquench s-wave scattering lengths $a_s$. The prequench (initial) SF state is perfectly coherent, i.e. $\beta_c(t = 0) = 0$. In all cases, the increase of $\beta_c(t)$ at short timescales is inherently related to the quench protocol and becomes more enhanced for larger quench amplitudes where the respective import of energy into the system is naturally larger.

For quenches within the SF phase, e.g., $a_s = 100 a_B$, the global phase coherence fluctuates within $0.13 < \beta_c(t) < 0.2$ after $t > 0.7$ ms. It is not completely suppressed due to the breathing motion of the gas originating from the quench, see also Appendix C and Fig. 15. For the same reason, $\beta_c(t > 0)$ is finite also for quenches towards the SS regime ($a_s = 880 a_B$) while the relatively intensified oscillations compared to the SF stem from the arguably larger amplitude of the underlying breathing mode. The coherence loss in the SS phase can be mitigated by adiabatically reducing the s-wave scattering length. In support of this argument, we utilize a linear ramp $a_s(t) = (a_s^0 - a_s^I) t / \tau$, with $a_s^0$ ($a_s^I$) being the final (initial) scattering lengths respectively, while $\tau$ is the ramp time. An almost adiabatic reduction of the contact interaction here is achieved for $\tau = 300$ ms and then $\beta_c(t) \to 0$ for the SF and SS as shown in Fig. 10(b1).

Turning to quenches in the droplet regime, for instance in the case of $a_s = 700 a_B$, we observe that $\beta_c(t)$ features an increase at short timescales and then fluctuates around $\pi/2$. This saturation tendency of $\beta_c(t)$ deep in the evolution holds equally when performing a linear decrease of $a_s(t)$, see Fig. 10(b1). Note that $\beta_c(t)$ can be at most $\pi/2$ as has also been demonstrated in Ref. [31]. This response implies that the initial phase coherence of the SF phase is rapidly lost and the individual droplets become highly incoherent.

**B. Generation of droplet and SS arrays in the quasi-1D regime**

Subsequently, we study the quench-induced dynamics of the axially elongated dBEC. Characteristic density snapshots across the $x$-$y$ plane when performing a quench within the SS phase, e.g., $a_s = 90 a_B$, are presented in Fig. 11(a1)-(a6). The 2D TF distribution [Fig. 11(a1)] of the initial SF state (with $a_s = 120 a_B$) experiences prominent spatial deformations due to the ensuing roton instability [14] being activated when crossing the SF to SS phase boundary [see also Fig. 2]. Specifically, the reduced postquench value enforces a contraction [Fig. 11(a2), (a3)] and expansion [Fig. 11(a4)-(a6)] of the entire cloud along both $x$ and $y$ directions. This collective motion prevails at short timescales [Fig. 11(a2)] but afterwards the impact of the instability becomes noticeable causing spatial modulations in the density profile along the $y$-axis [Fig. 11(a3)-(a6)]. For instance, six density peaks are detected at $t = 13.5$ ms [Fig. 11(a4)] which break into several ones at $t = 32$ ms [Fig. 11(a6)]. These arrays of overlapping density lumps developing in $n(x, y)$ reveal the dynamical formation of the SS state. It is also worth mentioning that the periodic spatial compression and expansion of the dBEC is characterized by two distinct frequencies; one ascribed to the individual peaks

Note that a non-adiabatic ramp of the scattering length, e.g., in our case realized for $\tau < 350$ ms, always results in a finite global phase coherence.
and another one linked with the background density. We remark that the participation of the two distinct breathing is a characteristic of the emergence of the supersolid phase, a result which has been evinced independently in Ref. [31]. Notably, the SS array becomes stationary only after $t = 100\text{ms}$ (not shown).

Utilizing a quench to $a_s = 70a_B$ leads to a dramatically different response of the BEC as shown in Fig. 11 (b1)-(b6). Already at the early stages of the evolution we observe that the original smooth density configuration [Fig. 11 (b1)] transforms into an elliptic halo profile [Fig. 11 (b2)]. The width of the latter progressively shrinks across the transverse $x$-direction [Fig. 11 (b3)] until the entire cloud becomes highly elongated and then breaks into an array of droplets [Fig. 11 (b4)]. The reason behind the formation of the elliptic halo states is the arising modulational instability due to admixture of different roton modes discussed for instance in Refs. [14, 15, 81] and triggered herein by the quench within the SS phase. As a by-product, the dBEC fragments into multiple highly localized peaks organized in a crystal pattern.

Notice that in contrast to the SS case of post quench interaction $a_s = 90\ a_B$, these density humps are entirely isolated and comprise the self-bound droplets. The droplet array becomes stationary in the long time dynamics e.g. $t > 100\text{ms}$, while at earlier times the inter-droplet distance changes, see e.g. Fig. 11 (b1)-(b6). This phenomenon can be traced back to the collective breathing motion of the cloud due to the interaction quench. Interestingly, unlike the SS phase, here a single-frequency breathing occurs, a property that is attributed to the crystal nature of the droplet phase. These properties of the breathing mode of a SS and a droplet have also been experimentally observed for an elongated dBEC in Ref. [30]. We note that as in the quasi-2D scenario, the stationary array has a larger (lesser) number of droplets for reducing (increasing) post quench $a_s$ ($\phi$) as shown in Table I for fixed $\phi = 0^\circ$ [$a_s = 60a_B$].

| Trap   | $\phi$ | 5$^\circ$ | 10$^\circ$ | 15$^\circ$ | 20$^\circ$ |
|--------|--------|-----------|-----------|-----------|-----------|
| Quasi-2D | 27     | 26        | 25        | 24        | 17        |
| Quasi-1D | 16     | 15        | 14        | 10        | 8         |

TABLE II. Number of isolated droplets in the quasi-2D and the quasi-1D regimes for a specific post quench contact interaction $a_s = 60\ a_B$ and considering different field orientations $\phi$. In both cases the amount of individual droplets in the respective lattice decreases as $\phi$ increases due to the effective weakening of the DDI. Initially, the dBEC resides in a SF state characterized by $a_s = 120a_B$.

The dynamical generation of the different phases can again be probed, as in the quasi-2D scenario, through monitoring the global phase coherence, $\beta_c(t)$. The latter is illustrated in Fig. 10 (b2) for post quench scattering lengths belonging to the SF, SS and droplet phases respectively. As expected for the SF dynamics $\beta_c(t) \neq 0$ taking small values solely due to the collective breathing of the background. A similar behavior of $\beta_c(t)$ occurs also for the SS case, however here it acquires a larger magnitude compared to the SF because the quench strength is larger leading to a relatively enhanced breathing amplitude. Concluding, in the droplet regime $\beta_c(t)$, besides increasing at the initial timescales ($0 < t < 20\text{ms}$) as a result of the expansion of the cloud, it then fluctuates around its maximally attained value $\pi/2$ throughout the time-evolution. This behavior signals the maximal loss of coherence in the emergent droplet array. Moreover, as demonstrated in the quasi-2D regime coherence is also lost during the SS nucleation due to the quench but it can be maintained through an adiabatic ramp of the scattering length (not shown).
FIG. 12. Density snapshots $n(x, y)$ upon considering interaction quenches from $a_s = 120 a_B$ to (a1)-(a3) $a_s = 90 a_B$, (b1)-(b2) $a_s = 70 a_B$ for $\phi = 0^\circ$, and (c1)-(c3) $a_s = 70 a_B$ for $\phi = 90^\circ$. The corresponding dipolar length for $\phi = 0^\circ$ ($\phi = 90^\circ$) is $D = 98.25 ~a_B$ ($D = -98.25 ~a_B$). The BEC consists of $N = 6 \times 10^4$ particles and it is confined in a quasi-2D trap with $(\omega_x, \omega_y, \omega_z) = 2\pi \times (45, 45, 133)$ Hz.

V. SELF-EVAPORATION OF SS AND DROPLETS IN DIFFERENT INTERACTION REGIMES

Previously we have demonstrated the dynamical nucleation of SS and droplet clusters (DL) from an initial SF state after a quench of the $s$-wave scattering length. In fact, experimentally it has been reported [22, 32] that these self-bound structures, due to the highly localized densities, suffer from three-body losses. In the following, we explain the impact of the underlying three-body loss rate in the formation of SS and droplet quasi-2D configurations. Contrary to previous studies, the employed rotating magnetic field [Eq. (1)] allows us to expose the effect of the underlying losses in different interaction regimes compared to the dipole length. The corresponding eGPE [5] has the form of Eq. (2) with the additional imaginary contribution $-i(\hbar K_3/2)\psi(r, t) \partial_t \psi(r, t)$, where $K_3$ denotes the three-body recombination rate [6]. A detailed discussion regarding the competition between the three-body recombination and beyond mean-field processes is provided in Ref. [6].

The important point here is that the scaling of the three-body recombination rate can be determined in terms of the parameter $D = 3a_d(3 \cos^2 \phi - 1)/4$ where $a_d = 131 \ a_B$ herein. It was shown [62] that $K_3 \sim D^4$ for $a_s \ll D$ and $K_3 \sim C a_s^2 (a_s^2 + \beta D^2)$ in the case of $a_s \gg D$. In the above expressions, the constants $C = 33\gamma/3\pi^2\hbar^4/m$ and $\beta \approx 0.44$. It should be emphasized that for $\phi = 0^\circ$, both the SS and droplet phases occur within $a_s \ll D$. This is exactly the situation that has been considered thus far in the literature for interpreting experimental data, see for instance [11, 16]. For our setup this scenario is characterized by a recombination rate $K_3 = 7.1 \times 10^{-4} m^3 / s$. However, since our dBEC is subjected to a tilted magnetic field, it is also possible to enter the interaction regime $a_s > D$, where the three-body recombination rate explicitly depends on the $s$-wave scattering length. Accordingly, below, we will discern among two important scenarios. Namely, utilizing a post quench $a_s$ lying in the droplet regime for $\phi = 0^\circ$, we adjust $\phi$ towards $\phi_m$ and a SF state takes place, since the dipolar interaction is not strong enough to create droplets. Then, $K_3$ becomes $a_s$-dependent. On the other hand, when $\phi \ll \phi_m$, and $a_s < D$, the loss-rate scales explicitly with $D$ and hence $\phi$. This tunability of $K_3$ provides an additional knob to prolong the lifetime of the dynamically accessed self-bound states.

To directly visualize the impact of three-body recombination processes, we present in Fig. 12 instantaneous density profiles during the evolution of the quasi-2D dBEC for different field angles $\phi$ and post quench scattering lengths. Focusing on the $\phi = 0^\circ$ scenario quenches from the SF to either $a_s = 90 a_B$ [Fig. 12(a1)-(a3)] or $a_s = 70 a_B$ [Fig. 12(b1)-(b3)] are considered. Recall that in the absence of losses ($K_3 = 0$) these post quench values lead to the spontaneous formation of SS for $a_s = 90 a_B$ [Fig. 12(a1)-(a3)] and droplet lattices for $a_s = 70 a_B$ [Fig. 12(b1)-(b3)]. However, three-body losses, meaning $K_3 \neq 0$, affect dramatically the emergent density patterns. Particularly, for $a_s = 90 a_B$, the ring like density pattern appears due to the roton instability [Fig. 12(a2)], and we observe the formation of a SS structure already at the early stages of the dynamics [Fig. 12(b1)-(b3)]. These cluster of density humps are not stable structures due to $K_3 \neq 0$ associated with non-negligible atom losses as shown in Fig. 13(a). Eventually after $t > 45$ ms, they coalesce to narrow density peaks around the trap center, and subsequently disappear in the long-time dynamics. Turning to $a_s = 70 a_B$ it becomes apparent that atom losses are much more severe than for $a_s = 90 a_B$, see Fig. 12(b1)-(b3) and Fig. 13(a). As a result the generated droplets [Fig. 12(b1)-(b3)] decay rapidly [Fig. 13(a)] and their degradation results eventually in a highly excited background of the dBEC (not shown). The response of the dBEC quenching to the droplet phase but in the anti-dipolar regime $\phi = 90^\circ$ differs substantially, see Fig. 12(c1)-(c3). It can be easily seen that the dBEC background progressively shrinks due to the presence of $K_3$ [Fig. 12(c2)]. Later on, in conjunction with the attractive nature of the DDI for $\phi = 90^\circ$, a strongly localized distribution appears which is subsequently destroyed. From these findings we can conclude that three-body losses compete with the LHY beyond mean-field...
as explained above, a finite $\phi$ allows to enter the $a_s > D$ region where the loss rate is interaction dependent. As a paradigmatic case, herein, we consider $\phi = 50.07^\circ$ where the loss coefficients are $K_3 = 2.47 \times 10^{-40} \text{m}^6/\text{s}$ and $K_3 = 6.67 \times 10^{-40} \text{m}^6/\text{s}$ for $a_s = 70 a_B$ and $a_s = 90 a_B$ respectively. It is worth mentioning that for $\phi = 50.07^\circ$ the dipolar interaction is not strong enough, thus leading to SF states for $a_s = 70 a_B$ and $a_s = 90 a_B$. Recall that in this regime, where $a_s > D$ for $a_s = 70 a_B$ and $a_s = 90 a_B$, $K_3$ is interaction dependent possessing a greater value for larger $a_s$ and thus leading to an amplified lossy process. However, it is demonstrated that the droplet region, realized at $a_s = 70 a_B$ for both $\phi = 0^\circ$ and $\phi = 90^\circ$ always suffers enhanced losses than the SS ($a_s = 90 a_B$, $\phi = 0^\circ$) and the SF state ($a_s = 90 a_B$, $\phi = 90^\circ$), irrespective of the conditions $a_s < D$ (for $\phi = 90^\circ$) and $a_s > D$ (for $\phi = 90^\circ$). As such the density of the bound state plays here the dominant role for the loss process. Remarkably, we find that the droplet lifetime can be prolonged as long as $\phi < \phi_m$ and $a_s > D$ by increasing $\phi$ [Fig 13(b)]. This suggests that the dynamical persistence of the self-bound patterns can be facilitated with the aid of a tilted magnetic field. As a characteristic example we consider a quench towards $a_s = 60 a_B$, where the loss coefficients are $K_3 = 7.1 \times 10^{-43} \text{m}^6/\text{s}$, $K_3 = 4.66 \times 10^{-43} \text{m}^6/\text{s}$ and $K_3 = 3.28 \times 10^{-43} \text{m}^6/\text{s}$ for $\phi = 0^\circ$, $\phi = 15^\circ$ and $\phi = 20^\circ$ respectively. We deduce that the lifetime of the droplet structures realized at $a_s = 60 a_B$ is prolonged by increasing $\phi$ lying in the interval $\phi < \phi_m$. The above constitute central results of our study and evidently assist the longevity of self-bound states.

VI. CONCLUSIONS & OUTLOOK

We have examined the ground-state phase diagram and the non-equilibrium dynamics of a harmonically trapped dBEC. Importantly, the latter experiences an anisotropic rotating magnetic field around the z-axis. Our considerations are based on the 3D eGPE, including quantum fluctuations to leading order, and cover both the quasi-1D and the quasi-2D regimes. It is found that depending on the value of the s-wave scattering length, the ground-state of the dBEC features SS, SF and single- or multi-droplet states. These phases are identified in the density of the dipolar gas, the phase profiles and the global phase coherence. The properties and structure of the SF and DL$_S$ are further confirmed by constructing a variational model based on a Gaussian ansatz (see Appendix A). The dynamics is triggered by a quench of the short-range contact interaction across the aforementioned phases. This allows us to monitor the progressive generation of different SS and droplet lattices whose coherence is lost. With tuning of the three-body recombination process, we are able to inspect the effect of atom losses across different regimes.

Focus on the ground-state properties of the dBEC,
we reveal that for magnetic fields with tilt angles $\phi$ smaller than the magic one, $\phi_m$, four different phases emerge in terms of the contact interaction and atom number. These include the SF typically occurring for $a_{m} / a_s < 1.4$ and having $\mu > 0$, the SS residing in the vicinity of $\mu = 0$, as well as the DL$_S$ and the DL$_M$ characterized by $\mu < 0$. A SF state exhibits a smooth density distribution in sharp contrast to SS and droplets where substantially modulated patterns emerge. These structures, exhibiting density humps, are spatially overlapping for a SS and tend to crystalline structures deep in the droplet regime. The crystal arrangements correspond to elongated periodic arrays in quasi-1D and droplet clusters in quasi-2D forming canonical polygons. Notably the number of individual droplets contained in a cluster increases by either reducing the contact interaction strength or for larger atom number as well as for a finite magnetic field angle. Transitions among the above-described phases are achieved by appropriately tuning the $s$-wave scattering length or the atom number.

Other parameters can be tuned too, however, in the present work we opted to vary these, by way of example, in connection also with experimentally available “knobs”. Interestingly, as $\phi \rightarrow \phi_m$ the SF state becomes the dominant configuration, a result which is more pronounced in the quasi-2D case. However, for $\phi > \phi_m$, solely the DL$_S$ and SF phases can form. Additionally, the field angle allows for the manipulation of the shape of the dBEC, for instance creating honeycomb and square droplet lattices. Similarly, the single droplet phase becomes elongated along the $z$-axis and therefore quasi-1D in nature for $\phi < \phi_m$ while it features a broad 2D circular distribution in the $x$-$y$ plane when $\phi > \phi_m$.

Next, we track the dynamical generation of SS and droplet structures upon considering quenches of the $s$-wave scattering length from an initial SF state. Elongated arrays of SS and droplets are identified in quasi-1D, whilst SS clusters and droplet lattices form in quasi-2D. These states are nucleated due to the roton instability e.g. manifesting as ring-shaped excitations (in quasi-2D) or elliptic halos (in quasi-1D) at early evolution times and are accompanied by a collective breathing motion of the dBEC background caused by the quench. Soon after their formation, these structures deform into specific arrays or clusters showing a saturated shape in the long-time dynamics. The number of droplets in a lattice is larger for smaller post quench contact interactions or a finite angle such that $\phi < \phi_m$. Coherence is maximally lost for the dynamically formed droplet states but it is nonzero also for the SS. Interestingly, we observe that following quenches to the DL$_S$ phase the system relaxes to a lattice, i.e., the DL$_M$ phase. Moreover, the number of individual droplets participating in a lattice arrangement is larger as compared to the one of the ground-state configuration.

We have analyzed the impact on the dynamical generation of the self-bound states by taking into account three-body recombination processes. It is found that the loss rates are enhanced towards the droplet regime for fields oriented along $\phi = 0$, leading to their self-evaporation, while almost the SS is completely absent. Our results show that employing a tilted magnetic field where the loss coefficient depends on the scattering length, it is possible to prolong the lifetime of droplets as long as $\phi < \phi_m$. Otherwise, the droplet region suffers faster lossy mechanisms than the other states irrespectively of whether the loss coefficient depends or not on the interaction.

Our results pave the way for several future research directions. Motivated by observations [4], we have restricted our study to a particular driving regime, that is $\omega \ll \Omega \ll \omega_L$. However, it would be intriguing to explore the impact of a weak $\Omega \ll \omega$ rotating field, thus extending the present findings to the case where the dipoles cannot instantaneously follow the external magnetic field. Importantly, a quantitative understanding of the pairwise interaction of the droplets in this system and of the pattern formation on the basis of their effective “interacting particle system” SS, would be particularly interesting and relevant in this context. Furthermore, comparison of the results herein with effective lower-dimensional equations describing quasi-1D or quasi-2D dBECs could be of interest as well; see, e.g., [4] for a 1D example. Another straightforward option is to investigate topological pattern formation in the above-discussed ground-state phases utilizing a rotating frame of reference. Moreover, it would be interesting to reveal the respective phase diagram of the dBEC in the presence of nonlinear excitations such as vortex complexes in quasi-2D or solitary waves in quasi-1D. Furthermore, studying the impact of finite temperature effects [85, 86] in the dynamical nucleation of SS and droplet lattices is certainly an intriguing perspective. Here, the dependence of the LHY term on the temperature should be carefully considered. The quench dynamics of a mixture of dipolar condensates across the distinct phases, e.g., discussed in Refs. [57, 58] is a more computationally demanding effort, yet one worthy of consideration.

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FIG. 14. Energy of the quasi-2D dBEC as predicted within the variational approach [Eq. (A7)] in terms of the widths \( \sigma_x \) and \( \sigma_z \). Different panels refer to field orientations (a1) \( \phi = 0^\circ \), (a2) \( \phi = 30^\circ \), (a3) \( \phi = 60^\circ \) and (a4) \( \phi = 90^\circ \). The white crosses denote the location of the respective minimum energy associated with the equilibrium set \((\sigma_x, \sigma_z)\). The sign of the energy subsequently characterizes the equilibrium state as SF or self-bound. The equilibrium state energy between the variational method and the eGPE results are in good agreement exhibiting an average error not more than 10\%. (b1), (b2) The response of (left-axis) \( \sigma_i \) with \( i \equiv \{x, y, z\} \) and (right-axis) the ratio \( \sigma_y/\sigma_i \) for varying \( \phi \) in the (b1) quasi-2D and (b2) quasi-1D regime. In both cases, the \( s \)-wave scattering length is fixed to \( a_s = 75a_B \) and the atom number is \( N = 6 \times 10^8 \). The results obtained through the variational principle (dashed lines) and the eGPE method (solid lines) are in agreement.

Appendix A: Variational treatment

Let us showcase that the SF and DLs ground-state characteristics of the dBEC phase diagram can be obtained within a variational approach instead of numerically solving the time-independent 3D eGPE. Particularly, our model is based on the assumption that the dBEC wave function acquires the Gaussian ansatz [73]:

\[
\psi(x, y, z) = \sqrt{\frac{N}{\pi^{3/2}\sigma_x \sigma_y \sigma_z}} \prod_{\eta=x,y,z} \exp\left(-\frac{\eta^2}{2\sigma_\eta^2}\right), \tag{A1}
\]

where the variational parameters are the widths \( \sigma_\eta \) in the \( \eta = x, y, z \) direction and \( \beta_\eta \) which determines the phase curvature. It is apparent from the functional form of the ansatz that it can not capture a droplet lattice or a SS structure. The Lagrangian \( L(\mathbf{r}) = \int d^3r L(\mathbf{r}) \), with \( L(\mathbf{r}) \) being the Lagrangian density

\[
L = \frac{\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi^* \frac{\partial^2 \psi}{\partial t^2} \right) + \frac{\hbar^2}{2m} |\nabla \psi(\mathbf{r}, t)|^2 + V(\mathbf{r})|\psi(\mathbf{r}, t)|^2
\]

\[
+ \frac{gN^2}{4\sqrt{2}\pi^{3/2}} \frac{1}{\prod_\eta \sigma_\eta^2} + \frac{N^2\hbar^2}{\sqrt{2\pi m}} \frac{a_{dd}}{\prod_\eta \sigma_\eta^2} \left(\frac{3\cos^2 \phi - 1}{2}\right) \times f(k_x, k_y) + \frac{4\sqrt{2}\pi^{3/2}}{25\sqrt{5}\pi^{9/4}} \frac{1}{\prod_\eta \sigma_\eta^{3/2}}, \tag{A2}
\]

Inserting the ansatz of Eq. (A1) into Eq. (A2) and integrating over the spatial coordinates we obtain

\[
L = \sum_{\eta=x,y,z} \left[ \frac{N\hbar^2}{2} \frac{\partial \eta^2}{\partial \sigma_\eta^2} + \frac{N\hbar^2}{2m} \left( \frac{1}{2\sigma_\eta^2} + 2\beta_\eta \sigma_\eta^2 \right) + \frac{Nm}{4} \omega_\eta^2 \sigma_\eta^2 \right]
\]

\[
+ \frac{gN^2}{4\sqrt{2}\pi^{3/2}} \frac{1}{\prod_\eta \sigma_\eta^2} + \frac{N^2\hbar^2}{\sqrt{2\pi m}} \frac{a_{dd}}{\prod_\eta \sigma_\eta^2} \left(\frac{3\cos^2 \phi - 1}{2}\right) \times f(k_x, k_y) + \frac{4\sqrt{2}\pi^{3/2}}{25\sqrt{5}\pi^{9/4}} \frac{1}{\prod_\eta \sigma_\eta^{3/2}}, \tag{A3}
\]

where the parameter \( k_i = \sigma_z/\sigma_i \) (i.e. \( x, y \)) and the function

\[
f(k_x, k_y) = \frac{1}{4\pi} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \left[ \frac{3\cos^2 \theta}{(k_x^2 \cos^2 \phi + k_y^2 \sin^2 \phi) \sin^2 \theta + \cos^2 \theta - 1} \right]. \tag{A4}
\]

Next, by utilizing the Euler-Lagrange equations of motion for \( \sigma_\eta \) and \( \beta_\eta \) we arrive at the coupled set of equations

\[
\beta_\eta = \frac{m}{2\hbar \sigma_\eta} \frac{d\sigma_\eta}{dt}
\]

\[
Nm \frac{d^2 \sigma_\eta}{dt^2} = -\frac{\partial}{\partial \sigma_\eta} U(\sigma_\eta). \tag{A5}
\]

These six Euler-Lagrange equations (A5) constitute exact solutions of the time-independent eGPE. Moreover, in Eq. (A5), the effective potential energy \( U(\sigma_\eta) \) is given
by

\[ U(\sigma_n) = \sum_{\eta} \left[ \frac{Nk^2}{m} \frac{1}{2\sigma_n^2} + \frac{Nm}{2} \omega_n^2 \sigma_n^2 \right] + \frac{gN^2}{2\sqrt{2}\pi^{3/2}} \frac{1}{\prod_{n} \sigma_n} \]

\[ + \frac{2}{\pi} \frac{N^2\rho^2}{m} \frac{\alpha_{dd}}{\prod_{n} \sigma_n} \left( \frac{3\cos^2\phi - 1}{2} \right) f(k_x, k_y) \]

\[ + \frac{8\sqrt{2}\gamma_{QF} N^{5/2}}{25\sqrt{\gamma_{QF}}^{9/4}} \prod_{n} \sigma_n^{3/2}. \]  

(A6)

Apparently, the second set of Eq. (A5) is reminiscent of the classical equations of motion of a particle with coordinates \( \sigma_n \) subjected to the external potential \( U \). As such the total energy of the dBEC reads

\[ E = \frac{N}{2} m \left( \frac{1}{2} \sum_{\eta} [\sigma_n^2] \right) + U(\sigma_n). \]  

(A7)

Therefore, the ground-state energy of the dBEC is simply \( E^{(0)} = U(\sigma_n^0) \), where \( \sigma_n^0 \) denote the equilibrium widths. These are determined through minimization of the energy or equivalently the effective potential \( U(\sigma_n) \).

The resulting energy \( E \) as obtained from Eq. (A7) with respect to \( \sigma_n = \sigma_x, \sigma_y, \sigma_z \) and for various angles \( \phi \) of the magnetic field is provided in Fig. 14(a1)-(a4). To illustrate the equivalence of the eGPE results discussed in the main text to the variational treatment we employ the quasi-2D dBEC with \( a_s = 70 a_B \) and \( a_{dd} = 131 a_B \) containing \( N = 6 \times 10^4 \) atoms in a harmonic trap with \( (\omega_x, \omega_y, \omega_z) = 2\pi \times (45, 45, 133) \text{Hz} \). The equilibrium widths \( (\sigma_x, \sigma_y, \sigma_z) \) of the dBEC are then easily identified by determining the minimum \( E^{(0)} \) of the energy \( E \), see the white crosses in Figs. 14(a1)-(a4). Interestingly, the energy minima [Figs. 14(a1)-(a4)] enable us to appreciate the phase of the dBEC that each angle \( \phi \) favors. Indeed, we find that for \( \phi = 0^\circ \) the minimum energy is negative which is a property associated with the development of a self-bound macro droplet.

In contrast, in the case of either \( \phi = 30^\circ \) [Fig. 14(a2)] or \( \phi = 60^\circ \) [Fig. 14(a3)] \( E^{(0)} \) is positive, thus being representative of the SF phase. Recall that the same behavior has been concluded within the eGPE in the quasi-2D geometry [Fig. 3(a2)]. Turning to \( \phi = 90^\circ \) again the equilibrium state has negative energy [Fig. 14(a4)], a behavior that is related to the single droplet state discussed in Fig. 7(a4)-(a8).

As a further proof-of-principle of our benchmark we present in Figs. 14 (b1) the equilibrium widths \( \sigma_x = \sigma_y = \sigma_x \) and \( \sigma_z \) as predicted in both the variational and the eGPE methods for the quasi-2D geometry in terms of \( \phi \). It becomes evident that the equilibrium widths show almost the same behavior in both methods. A discernible difference is that the variational calculation overestimates (underestimates) the value of \( \sigma_z \) for \( \sigma_y \) until \( \phi \approx 20^\circ \) but underestimates (overestimates) \( \sigma_z \) for \( 20^\circ < \phi < 80^\circ \). Otherwise, they agree. Similar conclusions can be drawn for the quasi-1D dBEC, see Fig. 14(b2).

Appendix B: Further details on the numerical implementation

For the convenience of our numerical simulations we cast the eGPE of Eq. (2) into a dimensionless form. This is achieved by rescaling the length, and time in terms of the harmonic oscillator length \( \hbar = \hbar/m\omega_z \), and the trap frequency \( \omega_r \) respectively, while the transformed wave function obeys \( \Psi(r', t) = N^{1/2}/|\psi(r', t)| \) (for the desired \( N \)). This preserves the normalization of the wave function, while convergence is reached as long as relative deviations of the wave function (at every grid point) and energy between consecutive time-steps are smaller than \( 10^{-6} \) and \( 10^{-8} \) respectively. This solution is then used as an initial state for the quench dynamics where the eGPE is propagated in real time. Since the dipolar potential (Eq. (1)) is divergent at short distances it is calculated in momentum space, see also Ref. [23] for the analytical expression of the Fourier transformation of the dipolar potential. Afterwards, we perform the inverse Fourier transform for obtaining the real space contributions using the convolution theorem. Our simulations are carried out in a 3D box characterized by a grid \((n_x \times n_y \times n_z)\) corresponding to \((256 \times 256 \times 128)\) and \((300 \times 600 \times 300)\) for the quasi-2D and the quasi-1D trap respectively. The employed spatial discretization (grid spacing) refers to \( \Delta x = \Delta y = \Delta z = 0.1 \lambda_{osc} \) while the time-step of the numerical integration is \( \delta t = 10^{-5}/\omega_z \).

Appendix C: Collective excitations of the quenched dBEC

As already mentioned in the main text, the dBEC undergoes a collective breathing motion originated from the interaction quench. A common experimentally relevant measure for estimating the amplitude and frequency of the underlying breathing is the center-of-mass variance along the different spatial directions [71, 70]. It measures a quantity of the instantaneous width of the dBEC and it is defined by

\[ \sigma_\lambda = \sqrt{\int dx dy dz |\psi|^2 \lambda^2}, \]  

(C1)

where \( \lambda = \{x, y, z\} \). Below, we shall analyze the dynamics in the quasi-2D regime and therefore \( \sigma_x = \sigma_y \neq \sigma_z \) because \( \omega_x = \omega_y \neq \omega_z \). However, we should note that
such a breathing dynamics takes place equally also in the quasi-1D case (not shown).

The temporal evolution of the condensate widths in the \( \lambda \)-th direction utilizing a quench from a SF state with \( a_s = 120a_B \) to the SF, SS and DS phases is illustrated in Fig. 15(a1)-(a6). For a final SF state, realized here in the cases of \( a_s = 110a_B \) and \( a_s = 100a_B \), \( \sigma_x(t) \) exhibits an almost constant amplitude oscillatory behavior describing the in-plane compression and expansion dynamics of the cloud Fig. 15(a1)]. As expected, the oscillation (breath- ing) amplitude is reduced for smaller quench amplitudes. For the respective frequency we find that \( \sigma_x(t) \) oscillates in-phase with \( \omega_{SF}^{\lambda(y)} \approx 67\text{Hz} \) at the early stages of the evolution for both \( a_s = 110a_B \) and \( a_s = 100a_B \). Later on, \( \sigma_x(t) \) possesses a smaller frequency \( \omega_{SF}^{\lambda(y)} \approx 64\text{Hz} \) for \( a_s = 100a_B \). Unlike \( \sigma_x(t) \), in the transversal direction \( \sigma_y(t) \) exhibits multifrequency oscillations of time varying amplitude [Fig. 15(a2)]. Generally, the breathing motion does not decay in the SS regime.

Turning to a SS postquench state we observe that \( \sigma_x(t) \) experiences a peculiar beating pattern characterized by two dominant frequencies [Fig. 15(a3)]. They correspond to \( \omega_{SS,1} \approx 73.147\text{Hz} \) and \( \omega_{SS,2} \approx 60.8\text{Hz} \) for \( a_s = 90a_B \), while \( \omega_{SS,1} = 75.961\text{Hz} \) and \( \omega_{SS,2} \approx 17.5\text{Hz} \) for \( a_s = 85a_B \). The mode of higher frequency is related with the deformation of the SS lattice, and the lower one to the collective motion of the background superfluid, see also Fig. 9(a1)-(a6). Evidently, upon reducing \( a_s \), the higher (lower) frequency mode increases (decreases). The involvement of these low frequency compressional (breathing) modes is inherently related to the manifestation of the supersolid state, see also Ref. [31]. This response is anticipated since for a smaller \( a_s \) the background density is more dilute, and the crystal structure becomes more prominent. As such, the compressional mode associated with the crystal hardens whilst the lower one vanishes [30]. Concluding, \( \sigma_x(t) \) initially features an increase while fluctuating and after the formation of the SS lattice it shows a saturation tendency [Fig. 15(a4)].

For the droplet region, a completely different response takes place [Fig. 15(a5)-(a6)]. Particularly, \( \sigma_x(t) \) initially increases and after the formation of the droplet cluster around \( t = 4\text{ms} \) [see also Fig. 9(b1)], it shows an oscillatory trend of decaying amplitude [Fig. 15(a3)], signaling the collective expansion and contraction of the lattice. Afterwards, in the long-time dynamics, \( \sigma_x(t) \) saturates capturing the stationary configuration of the cluster. A similar response can be seen in \( \sigma_y(t) \) [Fig. 15(a1)]; however, unlike \( \sigma_x(t) \), the growth of \( \sigma_y(t) \) is larger for increasing quench \( a_s \), implying a larger amount of transversal excitations.

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