ON TAMENESS AND GROWTH CONDITIONS

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Abstract. We study discrete subsets of \( \mathbb{C}^d \), relating “tameness” with growth conditions.

1. Results

A discrete subset \( D \) in \( \mathbb{C}^n \) ( \( n \geq 2 \)) is called “tame” if there exists a holomorphic automorphism \( \phi \) of \( \mathbb{C}^n \) such that \( \phi(D) = \mathbb{Z} \times \{0\}^{n-1} \) (see [3]). If there exists a linear projection \( \pi \) of \( \mathbb{C}^n \) onto some \( \mathbb{C}^k \) ( \( 0 < k < n \)) for which the image \( \pi(D) \) is discrete, then \( D \) is tame ([3]). If \( D \) is a discrete subgroup (e.g. a lattice) of the additive group \( (\mathbb{C}^n,+)) \), then \( D \) must be tame ([1], lemma 4.4 in combination with corollary 2.6). On the other hand there do exist discrete subsets which are not tame (see [3], theorem 3.9).

Here we will investigate how “tameness” is related to growth conditions for \( D \).

Slow growth implies tameness as we well see. On the other hand, rapid growth can not imply non-tameness, since every discrete subset of \( \mathbb{C}^{n-1} \) is tame regarded as subset of \( \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C} \).

The key method is to show that sufficiently slow growth implies that a generic linear projection will have discrete image for \( D \).

The main result is:

Theorem 1. Let \( n \) be a natural number and let \( v_k \) be a sequence of elements in \( V = \mathbb{C}^n \).

Assume that

\[ \sum_k \frac{1}{||v_k||^{2n-2}} < \infty \]

Then \( D = \{v_k : k \in \mathbb{N}\} \) is tame, i.e., there exists a biholomorphic map \( \phi : \mathbb{C}^n \rightarrow \mathbb{C}^n \) such that

\[ \phi(D) = \mathbb{Z} \times \{0\}^{n-1}. \]

This growth condition is fulfilled for discrete subgroups of rank at most \( 2n - 3 \), implying the following well-known fact:
Corollary 1. Let $\Gamma$ be a discrete subgroup of $\mathbb{Z}$-rank at most $2n - 3$ of the additive group $(\mathbb{C}^n, +)$.

Then $\Gamma$ is a tame discrete subset of $\mathbb{C}^n$.

While this is well-known (even with no condition on the $\mathbb{Z}$-rank of $\Gamma$), our approach yields the additional information that these discrete subsets remain tame after a small deformation:

Corollary 2. Let $\Gamma$ be a discrete subgroup of $\mathbb{Z}$-rank at most $2n - 3$ of the additive group $(\mathbb{C}^n, +)$, $0 < \lambda < 1$ and $K > 0$. Let $D$ be a subset of $\mathbb{C}^n$ for which there exists a bijective map $\zeta : \Gamma \to D$ with

$$||\zeta(v) - v|| \leq \lambda ||v|| + K$$

for all $v \in \Gamma$.

Then $D$ is a tame discrete subset of $\mathbb{C}^n$.

This confirms the idea that tame sets should be stable under deformation. Similarly one would hope that the category of non-tame sets is also stable under deformation. Here, however, one has to be careful not to be too optimistic, because in fact the following is true:

Proposition. For every non-tame discrete subset $D \subset \mathbb{C}^n$ ($n > 1$) there is a tame discrete subset $D'$ and a bijection $\alpha : D \to D'$ such that

$$||\alpha(v) - v|| \leq \frac{1}{\sqrt{2}} ||v|| \quad \forall v \in D$$

and

$$||w - \alpha^{-1}(w)|| \leq ||w|| \quad \forall w \in D'.$$

In particular, if $D$ is a tame discrete subset and $\zeta : D \to \mathbb{C}^n$ is a bijective map with $||\zeta(v) - v|| \leq ||v||$ for all $v \in D$, it is possible that $\zeta(D)$ is not tame.

Still, one might hope for a positive answer to the following question:

Question. Let $n \in \mathbb{N}$, $n \geq 2$, let $1 > \lambda > 0$, $K > 0$, let $D$ be a tame discrete subset of $\mathbb{C}^n$ and let $\zeta : D \to \mathbb{C}^n$ be a map such that

$$||\zeta(v) - v|| \leq \lambda ||v|| + K$$

for all $v \in D$. Does this imply that $\zeta(D)$ is a tame discrete subset of $\mathbb{C}^n$?

Technically, the following is the key point for the proof of our main result (theorem):"
Assume that
\[ \sum_k \frac{1}{||v_k||^{2d}} < \infty \]

Then there exists a complex linear map \( \pi : V \to \mathbb{C}^d \) such that the set of all \( \pi(v_k) \) is discrete in \( \mathbb{C}^d \).

In a similar way one can prove such a result for real vector spaces:

**Theorem 3.** Let \( n > d > 0 \). Let \( V \) be a real vector space of dimension \( n \) and let \( v_k \) be a sequence of elements in \( V \).

Assume that
\[ \sum_k \frac{1}{||v_k||^d} < \infty \]

Then there exists a real linear map \( \pi : V \to \mathbb{R}^d \) such that the set of all \( \pi(v_k) \) is discrete in \( \mathbb{R}^d \).

For the proof of the existence of a linear projection \( \pi \) with \( \pi(D) \) discrete we proceed by regarding randomly chosen linear projections and verifying that the image of \( D \) under a random projection has discrete image with probability 1 if the above stated series converges.

2. Proofs

First we deduce an auxiliary lemma.

**Lemma 1.** Let \( k, m > 0 \), \( n = k + m \) and let \( S \) denote the unit sphere in \( \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^m \). Furthermore let
\[ M_\epsilon = \{(v, w) \in \mathbb{R}^k \times \mathbb{R}^m : ||v|| \leq \epsilon, (v, w) \in S \} \]

Then there are constants \( \delta > 0, C_1 > C_2 > 0 \) such that for all \( \epsilon < \delta \) we have
\[ C_1 \epsilon^k \geq \lambda(M_\epsilon) \geq C_2 \epsilon^k \]
where \( \lambda \) denotes the rotationally invariant probability measure on \( S \).

**Proof.** For each \( \epsilon \in ]0, 1[ \) there is a bijection
\[ \phi_\epsilon : B \times S' \to M_\epsilon \]
where
\[ B = \{v \in \mathbb{R}^k : ||v|| \leq 1\}, \quad S' = \{w \in \mathbb{R}^m : ||w|| = 1\} \]
and
\[ \phi_\epsilon(v, w) = \left( \epsilon v; \sqrt{1 - ||\epsilon v||^2} w \right). \]

The functional determinant for \( \phi_\epsilon \) equals
\[ \epsilon^k \left( \sqrt{1 - ||\epsilon v||^2} \right)^m. \]
It follows that
\[ e^k \left( \sqrt{1 - \epsilon^2} \right)^m \text{volume}(S' \times B) \leq \text{volume}(M_\epsilon) \leq e^k \text{volume}(S' \times B), \]
which in turn implies
\[ \lim_{\epsilon \to 0} \epsilon^{-k} \frac{\text{volume}(M_\epsilon)}{\text{volume}(S' \times B)} = 1. \]
Hence the assertion. \[ \square \]

**Lemma 2.** Let $\Gamma$ be a discrete subgroup of $\mathbb{Z}$-rank $d$ in $V = \mathbb{R}^n$.

Then
\[ \sum_{\gamma \in \Gamma} ||\gamma||^{-d-\epsilon} < \infty \]
for all $\epsilon > 0$.

**Proof.** Since all norms on a finite-dimensional vector space are equivalent, there is no loss in generality if we assume that the norm is the maximum norm and $\Gamma = \mathbb{Z}^d \times \{0\}^{n-d}$. Then the assertion is an easy consequence of the fact that $\sum_{n \in \mathbb{N}} n^{-s} < \infty$ if and only if $s > 1$. \[ \square \]

Now we proceed with the proof of theorem 2.

**Proof.** We fix a surjective linear map $L : V \to W = \mathbb{C}^d$. Let $K$ denote $U(n)$ (the group of unitary complex linear transformations of $V$). For each $g \in K$ we define a linear map $\pi_g : V \to W$ as follows:
\[ \pi_g : v \mapsto L(g \cdot v). \]

For $k \in \mathbb{N}$ and $r \in \mathbb{R}^+$ define
\[ S_{k,r} = \{ g \in K : ||\pi_g(v_k)|| \leq r \}, \]
\[ M_{N,r} = \{ g \in K : \# \{ k \in \mathbb{N} : g \in S_{k,r} \} \geq N \} \]
and
\[ M_r = \cap_{N} M_{N,r}. \]

Now for each $g \in K$ the set $\{ \pi_g(v_k) : k \in \mathbb{N} \}$ is discrete unless there is a number $r > 0$ such that infinitely many distinct image points are contained in a ball of radius $r$. By the definition of the sets $M_r$ it follows that $\{ \pi_g(v_k) : k \in \mathbb{N} \}$ is discrete unless $g \in M = \cup M_r$.

Let us now assume that there is no linear map $L' : V \to W$ with $L'(D)$ discrete. Then $K = M$. In particular $\mu(M) > 0$, where $\mu$ denotes the Haar measure on the compact topological group $K$. Since the sets $M_r$ are increasing in $r$, we have
\[ M = \cup_{r \in \mathbb{R}^+} M_r = \cup_{r \in \mathbb{N}} M_r. \]
and may thus deduce that $\mu(M_r) > 0$ for some number $r$. Fix such a
number $r > 0$ and define $c = \mu(M_r) > 0$. Then $\mu(M_{N,r}) \geq c$ for all $N$,
since $M_r = \cap M_{N,r}$. However, for fixed $N$ and $r$ we have

$$N \mu(M_{N,r}) \leq \sum_k \mu(S_{k,r}).$$

Hence

$$\sum_{k \in \mathbb{N}} \mu(S_{k,r}) \geq N \mu(M_{N,r}) \geq Nc$$

for all $N \in \mathbb{N}$. Since $c > 0$, it follows that $\sum_k \mu(S_{k,r}) = +\infty$.

Let us now embedd $\mathbb{C}^d$ into $\mathbb{C}^n$ as the orthogonal complement of
ker $L$. In this way we may assume that $L$ is simply the map which
projects a vector onto its first $d$ coordinates, i.e.,

$$L(w_1, \ldots, w_n) = (w_1, \ldots, w_d; 0, \ldots, 0).$$

Now $g \in S_{k,r}$ is equivalent to the condition that $g(v_k)$ is a real multiple
of an element in $M_\epsilon$ where $M_\epsilon$ is defined as in lemma 1 with $\epsilon = r/||v_k||$. Using lemma 1 we may deduce that $\sum_k \mu(S_{k,r})$ converges if and only if $\sum_k ||v_k||^{-2d}$ converges. □

**Proof of theorem 1.** The growth condition allows us to employ theo-
rem 2 in order to deduce that there is a linear projection onto a space
of complex dimension $d - 1$ which maps $D$ onto a discrete image. By
the results of Rosay and Rudin it follows that $D$ is tame. □

**Proof of the proposition.** We fix a decomposition $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and
write $D$ as the union of all $(a_k, b_k) \in \mathbb{C} \times \mathbb{C}^{n-1} (k \in \mathbb{N})$. We define

$$\alpha(a_k, b_k) = \begin{cases} 
(a_k, 0) & \text{if } ||a_k|| > ||b_k|| \\
(0, b_k) & \text{if } ||a_k|| \leq ||b_k||
\end{cases}$$

Then $D' = \alpha(D)$ is tame because each of the projections to one of the
two factors $\mathbb{C}$ and $\mathbb{C}^{n-1}$ maps $D'$ onto a discrete subset.

The other assertions follow from the triangle inequality. □

The proof of thm. 3 works in the same way as the proof of thm. 2,
simply using the group of all orthogonal transformations instead of the
group of unitary transformations.

**References**

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