Corrected Version of ‘A Unifying Variational Perspective on Some Fundamental Information Theoretic Inequalities’

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Abstract

This paper proposes a unifying variational approach for proving some fundamental information theoretic inequalities. Fundamental information theory results such as maximization of differential entropy, minimization of Fisher information (Cramér-Rao inequality), worst additive noise lemma, and extremal entropy inequality (EEI) are interpreted as functional problems and proved within the framework of calculus of variations. Several applications and possible extensions of the proposed results are briefly mentioned.

Index Terms

Maximizing Entropy, Minimizing Fisher Information, Worst Additive Noise, Extremal Entropy Inequality, Calculus of Variations

I. INTRODUCTION

In the information theory realm, it is well-known that given the second-order moment (or variance), a Gaussian density function maximizes the differential entropy. Similarly, given the second-order moment, the Gaussian density function minimizes the Fisher information, a result which is referred to as the Cramér-Rao inequality in the signal processing literature. Surprisingly, the proofs proposed in the literature for these fundamental results are relatively quite diverse, and no unifying feature exists. Since differential entropy or Fisher information is a functional with respect to a probability density function, the most natural way to establish these results is by approaching them from the perspective of functional analysis. This paper presents a unifying variational framework to address these results as well as numerous other fundamental information theoretic results. A challenging information theoretic inequality, referred to as the extremal entropy inequality (EEI) [2], can be dealt with successfully in the proposed framework of functionals. Furthermore, the proposed variational calculus perspective is useful in establishing other novel results, applications and extensions of the existing information theoretic inequalities.

The main theme of this paper is to illustrate how some tools from calculus of variations can be used successfully to prove some of the fundamental information theoretic inequalities, which have been widely used in information theory and other fields, and to establish some applications. The proposed variational approach provides alternative proofs for some of the fundamental information theoretic inequalities and enables finding novel extensions of the existing results. This statement is strengthened by the fact that the proposed variational framework is quite general and powerful, and it allows easy integration of various linear and inequality constraints into the functional that is to be optimized. Therefore, we believe that a large number of applications could benefit of these tools. The proposed variational approach offers also a potential guideline for finding the optimal solution for many open problems.

Variational calculus techniques have been used with great success in solving important problems in image processing and computer vision [4] such as image reconstruction (denoising, deblurring), inverse problems, and image segmentation. Recently, variational techniques were also advocated for optimization of multiuser communication.
systems [4], for deriving analytical wireless channel models using the maximum entropy principle when only limited information about the environment is available [5], and for designing optimal training sequences for radar and sonar applications [6]-[7]. Maximum entropy principle found also applications in spectral estimation (e.g., Burg’s maximum entropy spectral density estimator [11]) and Bayesian statistics [8].

The major results of this paper are enumerated as follows. First, using calculus of variations, the maximizing differential entropy and minimizing Fisher information theorems are proved under the classical (standard) assumptions found in the literature as well as under a different set of assumptions. It is shown that a Gaussian density function maximizes the differential entropy but it minimizes the Fisher information, given the second-order moment. It is also shown that a half normal density function maximizes the differential entropy over the set of non-negative random variables, given the second-order moment. Furthermore, it is shown that a half normal density function minimizes the Fisher information over the set of non-negative random variables, provided that the regularity condition [9] is ignored and the second-order moment is given. It is also shown that a chi density function minimizes the Fisher information over the set of non-negative random variables, under the assumption that the regularity condition holds and the second-order moment is given.

Second, a novel proof of the worst additive noise lemma [9] is provided in the proposed functional framework. Previous proofs of the worst additive noise lemma were based on Jensen’s inequality or data processing inequality [9], [10]. Unlike the previous proofs, our approach is purely based on calculus of variations techniques, and the vector version of the lemma is treated.

Third, EEI is studied from the perspective of a functional problem. The main advantage of the proposed new proof is that neither the channel enhancement technique and the entropy power inequality (EPI), adopted in [2], nor the equality condition of data processing inequality and the technique based on the moment generating functions, used in [11], are required. Using a technique based on calculus of variations, an alternative proof of EEI is provided. Finally, several applications and extensions of the proposed results are discussed.

The rest of this paper is organized as follows. Some variational calculus preliminary results and their corollaries are first reviewed in Section III. Maximizing differential entropy theorem and minimizing Fisher information theorem (Cramér-Rao inequality) are proved in Section III. In Section IV, the worst additive noise lemma is introduced and proved based on variational arguments. EEI is proved in Section V. In Section VI some additional applications of the proposed variational techniques are briefly mentioned. Finally, Section VII concludes this paper.

II. SOME PRELIMINARY CALCULUS OF VARIATIONS RESULTS

In this section, we will review some of the fundamental results from variational calculus, and establish the concepts, notations and results that will be used constantly throughout the rest of the paper. These results are standard and therefore will be described briefly without further details. Additional details can be found in calculus of variations books such as [12]-[14].

**Definition 1:** A functional $U[f]$ might be defined as

$$U[f] = \int_a^b K(x, f(x), f'(x))dx,$$

which is defined on the set of continuous functions $(f(x))$ with continuous first-order derivatives $(f'(x) = df(x)/dx)$ on the interval $[a, b]$. The function $f(x)$ is assumed to satisfy the boundary conditions $f(a) = A$ and $f(b) = B$. The functional $K(\cdot, \cdot, \cdot)$ is also assumed to have continuous first-order and second-order (partial) derivatives with respect to (wrt) all of its arguments.

**Definition 2:** The increment of a functional $U[f]$ is defined as

$$\Delta U[t] = U[f + t] - U[f],$$

where the function $t(x)$, that satisfies the boundary conditions $t(a) = t(b) = 0$, represents the admissible increment of $f(x)$, and it is assumed independent of the function $f(x)$ and twice differentiable.

**Definition 3:** Suppose that given $f(x)$, the increment in (2) is expressed as

$$\Delta U[t] = \varphi[t] + \epsilon \|t\|,$$

\[1\] The regularity condition is defined in Theorems 6 and 7.
where $\varphi [t]$ is a linear functional, $\epsilon$ goes to zero as $\|t\|$ approaches zero, and $\| \cdot \|$ denotes a norm defined in the case of a function $f(x)$ as:

$$\|f\| = \sum_{i=0}^{n} \max_{a \leq x \leq b} \left| f^{(i)}(x) \right|,$$

(4)

where $f^{(i)}(x) = d^i f(x)/dx^i$ are assumed to exist and be continuous for $i = 0, \ldots, n$ on the interval $[a, b]$, and the summation upper index $n$ might vary depending on the normed linear space considered (e.g., if the normed linear space consists of all continuous functions $f(x)$, which have continuous first-order derivative on the interval $[a, b]$, $\|f\| = \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} |f'(x)|$, and in this case $n = 1$; see e.g., [12] for further details). Under the above assumptions, the functional $U [f]$ is said to be differentiable, and the major part of the increment $\varphi [t]$ is called the (first-order) variation of the functional $U [f]$ and it is expressed as $\delta U [f]$.

Based on Definitions 1, 2, 3 and Taylor’s theorem (see e.g., [12]-[14] for additional justifications), the first-order and the second-order variations of a functional $U [f]$ can be expressed as

$$\delta U [f] = \int \left[ K'_{f'} (x, f, f') t(x) + K'_{f'} (x, f, f') t'(x) \right] dx$$

(5)

$$\delta^2 U [f] = \frac{1}{2} \int \left[ K''_{f'} t(x)^2 + 2K''_{f'} t(x) t'(x) + K''_{f'} t'(x)^2 \right] dx$$

(6)

where $K'_{f'}$ and $K''_{f'}$ stand for the first-order partial derivatives wrt $f$ and $f'$, respectively, $K''_{f'}$ denotes the second-order partial derivative wrt $f$ and $f'$, $K''_{f'}$ represents the second-order partial derivative wrt $f$, and $K'_{f'}$ is the second-order partial derivative wrt $f'$. Throughout the paper to simplify the exposition, the arguments of functionals or functions are omitted unless the arguments are ambiguous or confusing. Also, the range of integration in various integrals will not be explicitly marked unless the range is ambiguous.

**Theorem 1 ([12]):** A necessary condition for the functional $U[f]$ in (1) to have an extremum (or local optimum) for a given function $f = f^*$ is that its first variation vanishes at $f = f^*$:

$$\delta U [f^*] = 0,$$

(7)

for all admissible increments. This implies

$$K'_{f^*} - \frac{d}{dx} K''_{f^*} = 0,$$

(8)

a result which is known as Euler’s equation. When the functional in (1) includes multiple functions (e.g., $f_1, \ldots, f_m$) and multiple integrals wrt $x_1, \ldots, x_n$, i.e.,

$$\int \cdots \int K \left( x_1, \ldots, x_n, f_1, \ldots, f_m, f'_1, \ldots, f'_m \right) dx_1 \cdots dx_n,$$

then Euler’s equation in (8) takes the form of the system of equations:

$$K'_{f_i} - \sum_{j=1}^{n} \frac{d}{dx_j} K'_{f_i} = 0, \quad i = 1, \ldots, m.$$

(9)

In particular, when the functional does not depend on the first-order derivative of the functions $f_1, \ldots, f_m$, the equations in (9) reduce to

$$K'_{f_i} = 0, \quad i = 1, \ldots, m.$$

(10)

**Proof:** Details of the proof of this theorem can be found, e.g., in [12]-[14].

**Theorem 2 ([12]):** A necessary condition for the functional $U[f]$ in (1) to have a minimum for a given $f = f^*$ is that the second variation of functional $U[f]$ be nonnegative:

$$\delta^2 U [f^*] \geq 0,$$

(11)
for all admissible increments. This implies
\[ K''_{f^* f^*} \geq 0. \] (12)
In particular, when the functional in (1) does not depend on the first-order derivative of the function \( f \), (12) simplifies to
\[ K''_{f^* f^*} \geq 0. \] (13)
When the functional in (1) includes multiple functions (e.g., \( f_1, \ldots, f_m \)) and multiple integrals wrt \( x_1, \ldots, x_n \), i.e.,
\[
\int \cdots \int K(x_1, \ldots, x_n, f_1, \ldots, f_m) \, dx_1 \cdots dx_n,
\]
then the condition in (13) is expressed in terms of the positive semi-definiteness of the matrix:
\[
\begin{bmatrix}
    K''_{f_1 f_1} & \cdots & K''_{f_1 f_m} \\
    \vdots & \ddots & \vdots \\
    K''_{f_m f_1} & \cdots & K''_{f_m f_m}
\end{bmatrix} \succeq 0.
\]
(14)

Proof: The inequality in (13) is easily derived from the inequality in (12) since \( K''_{f^*_x f^*_x} \) and \( K''_{f^*_x f^*_x} \) are vanishing in (6) when the functional in (1) does not depend on the first-order derivative of the function \( f_x \). The remaining details of the proof can be tracked in (12).

Theorem 3 ([12]): Given the functional
\[ U[f_1, f_2] = \int_a^b K(x, f_1, f_2, f'_1, f'_2) \, dx, \] (15)
assume that the admissible functions satisfy the following boundary conditions:
\[
\begin{align*}
f_1(a) &= A_1, \quad f_1(b) = B_1, \quad f_2(a) = A_2, \quad f_2(b) = B_2, \\
k(x, f_1, f_2) &= 0, \\
L[f_1, f_2] &= \int_a^b \bar{L}(x, f_1, f_2, f'_1, f'_2) \, dx = l,
\end{align*}
\] (16) (17)
where \( a, b, A_1, B_1, A_2, B_2 \), and \( l \) are constants, \( k(x, f_1, f_2) \) is a functional wrt \( f_1 \) and \( f_2 \), and \( U[f_1, f_2] \) is assumed to have an extremum for \( f_1 = f^*_1 \) and \( f_2 = f^*_2 \).

If \( f^*_1 \) and \( f^*_2 \) are not extremals of \( L[f_1, f_2] \), or \( k'_1 \) and \( k'_2 \) do not vanish simultaneously at any point in (16), there exist a constant \( \lambda \) and a function \( \lambda(x) \) such that \( f^*_1 \) and \( f^*_2 \) are extremals of the functional
\[
\int_a^b (K(x, f_1, f_2, f'_1, f'_2) + \lambda \bar{L}(x, f_1, f_2, f'_1, f'_2) + \lambda(x)k(x, f_1, f_2)) \, dx.
\] (18)

Based on Theorem 3, the following corollary is derived.

Corollary 1: Given the functional
\[ U[f_x, f_y] = \int_a^b \int_a^b K(x, y, f_x, f_y) \, dx \, dy, \] (19)
assume that the admissible functions satisfy the following boundary conditions:
\[
\begin{align*}
f_x(a) &= A_x, \quad f_x(b) = B_x, \quad f_y(a) = A_y, \quad f_y(b) = B_y, \\
k(y, f_x, f_y) &= g(y, f_y) - \int_a^b \bar{k}(x, y, f_x) \, dx = 0, \\
L[f_x, f_y] &= \int_a^b \int_a^b \bar{L}(x, y, f_x, f_y) \, dx \, dy = l,
\end{align*}
\] (20)
where \( a, b, A_x, B_x, A_y, \) and \( B_y \) stand for some constants, \( f_x \) is a function of \( x \), \( f_y \) is a function of \( y \), \( g(y, f_y) \) is a function of \( f_y \), and \( \bar{k}(x, y, f_x) \) is a function of \( f_x \). The functional \( U[f_x, f_y] \) is assumed to have an extremum at \( f_x = f^*_x \) and \( f_y = f^*_y \).
Unless \( f_{x^*} \) and \( f_{y^*} \) are extremals of \( L[f_x, f_y] \), or \( k'_{f_{x^*}} \) and \( k'_{f_{y^*}} \) simultaneously vanish at any point of \( k(y, f_x, f_y) \), there exist a constant \( \lambda \) and a function \( \lambda(y) \) such that \( f_x = f_{x^*} \) and \( f_y = f_{y^*} \) is an extremal of the functional

\[
\int_a^b \left\{ \int_a^b \left[ (K(x, y, f_x, f_y) + \lambda \tilde{L}(x, y, f_x, f_y) - \lambda(y) \tilde{k}(x, y, f_x))dx \right] + \lambda(y)g(y, f_y) \right\}dy. \tag{21}
\]

**Proof:** This corollary is a simple extension of Theorem 3 for multiple integrals. Therefore, the detailed proof is omitted. Based on Theorems [1, 2] and Corollary [1] we can derive the following corollary, which will be repeatedly used throughout this paper.

**Corollary 2:** Based on the functional defined in (21), the following necessary conditions are derived for the optimal solutions \( f_{x^*} \) and \( f_{y^*} \):

\[
\int K_{f_{x^*}}(x, y, f_{x^*}, f_{y^*}) + \lambda \tilde{L}_{f_{x^*}}(x, y, f_{x^*}, f_{y^*}) - \lambda(y) \tilde{k}_{f_{x^*}}(x, y, f_{x^*})dy = 0, \tag{22}
\]

\[
K_{f_{y^*}}(x, y, f_{x^*}, f_{y^*}) + \lambda \tilde{L}_{f_{y^*}}(x, y, f_{x^*}, f_{y^*})dx + \lambda(y)g_{f_{y^*}}(y, f_{y^*}) = 0, \tag{23}
\]

and the matrix

\[
\begin{bmatrix}
    G''_{f_{x^*} f_{x^*}} & G''_{f_{x^*} f_{y^*}} \\
    G''_{f_{y^*} f_{x^*}} & G''_{f_{y^*} f_{y^*}}
\end{bmatrix},
\]

is positive semi-definite. The functional \( G \) is defined as

\[
G(x, y, f_{x^*}, f_{y^*}) = K(x, y, f_{x^*}, f_{y^*}) + \lambda \tilde{L}(x, y, f_{x^*}, f_{y^*}) - \lambda(y) \tilde{k}(x, y, f_{x^*}) + \lambda(y)g(y, f_{y^*})q(x),
\]

and \( q(x) \) is a (arbitrary but fixed) function which satisfies \( \int_a^b q(x)dx = 1 \), and it is introduced to homogenize the functional in (21).

**Proof:** The equations in (22) and (23) are derived from the first-order variation condition in Theorem 1. The equations in (22) and (23) are Euler’s equations for multiple integrals. The positive semi-definiteness of the matrix in (24) is derived from the second-order variation condition in Theorem 2. This condition is the same as the one in (14). The details of the proof are omitted here.

### III. MAX ENTROPY AND MIN FISHER INFORMATION

This simple but significant result–given the second-order moment (or variance) of a random vector, a Gaussian random vector maximizes the differential entropy–is well-known. In this section, a completely rigorous and general derivation of the distribution achieving the maximum entropy will be first provided. This proof sets up the variational framework for establishing a second important result in this section, namely the Cramér-Rao bound, which states that for a given mean and correlation matrix, a normally distributed random vector minimizes the Fisher information matrix.

**Theorem 4 ([1], [10]):** Given a mean vector \( \mu_x \) and a correlation matrix \( \Omega_x \), a Gaussian random vector \( X \) with the correlation matrix \( \Omega_x \) (and the mean vector \( \mu_x \)) maximizes the differential entropy, i.e.,

\[
h(X) \leq h(X_0), \tag{25}
\]

where \( h(\cdot) \) denotes differential entropy, \( X \) is an arbitrary (but fixed) random vector with the correlation matrix \( \Omega_x \).

**Proof:** We first construct a functional, which represents the inequality in (25) and required constraints, as follows:

\[
\min_{f_x} \int f_x(x) \log f_x(x)dx, \tag{26}
\]

\[
\text{s. t.} \quad \int f_x(x)dx = 1, \tag{27}
\]

\[
\int xx'' f_x(x)dx = \Omega_x. \tag{28}
\]

Using Theorem 3, the functional in (26) is expressed as

\[
\min_{f_x} U[f_x], \tag{29}
\]
where \( U[f_x] = \int K(x, f_x)dx = \int f_x(x) \left( \log f_x(x) + \alpha + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_i x_j \right) dx \), \( \alpha \) is the Lagrange multiplier associated with the constraint (27), and \( \lambda_{ij} \) stand for the Lagrange multipliers corresponding to the constraints (28).

Based on Theorem [1] by checking the first-order variation condition, we can find the optimal solution \( f_{x^*}(x) \) as follows:

\[
K'_{fx} \bigg|_{f_x = f_{x^*}} = 1 + \log f_{x^*}(x) + \alpha + x^T \Lambda x = 0 \tag{30}
\]

with the matrix \( \Lambda = [\lambda_{ij}], \ i, j = 1, \ldots, n. \) Considering the constraints in (27) and (28), from (30) it turns out that

\[
f_{x^*}(x) = \exp \left\{ -x^T \Lambda x - \alpha - 1 \right\} = (2\pi)^{-\frac{n}{2}} |\Omega_x|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} x^T \Omega_x^{-1} x \right\}, \tag{31}
\]

where

\[
\alpha = -1 + \frac{1}{2} \log (2\pi)^n |\Omega_x|, \quad \Lambda = \frac{1}{2} \Omega_x^{-1}. \tag{32}
\]

Two remarks are now in order. First, the correlation matrix \( \Omega_x \) is assumed to be invertible. When the correlation matrix is non-invertible, similar to the method shown in [2], we can equivalently re-write the functional problem in (26) as

\[
\min_{f_x(x)} -h(X) \iff \min_{f_x(x)} -h(X), \tag{33}
\]

where \( X = Q_{0} \tilde{X}, \) and in the spectral factorization \( \Omega_x = Q_{0} \Lambda_x Q_{0}^T, \) \( Q_{0} \) is an orthogonal matrix, \( \Lambda_x = \text{diag}(\Lambda_1, \ldots, \Lambda_m, 0, \ldots, 0) \) and \( \text{diag} \) denotes a diagonal matrix.

Let \( \tilde{X} = [\tilde{X}_a^T, \tilde{X}_b^T]^T, \) where the dimensions of \( \tilde{X}_a \) and \( \tilde{X}_b \) are \( m \) and \( n - m, \) respectively. Since the correlation matrix (or covariance matrix) of \( X, \Omega_x, \) is equal to the diagonal matrix \( \Lambda_x, \) \( \tilde{X}_a \) and \( \tilde{X}_b \) are statistically independent. Therefore, \( h(\tilde{X}) = h(\tilde{X}_a, \tilde{X}_b) = h(\tilde{X}_a) + h(\tilde{X}_b). \) Since the correlation of \( \tilde{X}_b \) is a zero matrix, we can consider \( \tilde{X}_b \) as a deterministic vector, and we only need to consider \( h(\tilde{X}_a). \) Therefore, the equation in (33) and constraints in (27)-(28) are equivalently re-written as

\[
\min_{f_{\tilde{X}_a}(x)} -h(\tilde{X}_a), \quad \text{s.t.} \quad \int f_{\tilde{X}_a}(x)dx = 1, \quad \int xx^T f_{\tilde{X}_a}(x)dx = \Lambda_{\tilde{X}_a}, \tag{34}
\]

where \( \Lambda_{\tilde{X}_a} = \text{diag}(\Lambda_1, \ldots, \Lambda_m) > 0 \) is a positive-definite matrix. Therefore, without loss of generality, we assume that the correlation matrix \( \Omega_x \) is invertible. Second, if an additional constraint, related to the mean vector of \( X, \) \( \mu_x, \) is enforced, the optimal solution is a multi-variate Gaussian density function with mean \( \mu_x, \) instead of the multi-variate Gaussian density function with zero mean (31).

Based on Theorem [2] since

\[
K''_{fx} \bigg|_{f_x = f_{x^*}} = \frac{1}{f_{x^*}(x)} > 0, \tag{35}
\]

the second-order variation \( \delta^2 U[f_{x^*}] \) is positive, and the optimal solution \( f_{x^*} \) is a minimal solution for the variational problem in (26).

Therefore, the negative of differential entropy \( -h(X) \) is miniimized (or equivalently \( h(X) \) is maximized) when \( X \) is a multi-variate Gaussian random vector with zero mean and covariance matrix \( \Sigma_x. \) Even though Theorems [1] [2] are necessary conditions for the minimum, in this case, a multi-variate Gaussian density function is the actual solution since there is only one solution, namely the multi-variate Gaussian density function, in the feasible set.
An alternative justification of global optimality of multi-variate Gaussian pdf can be achieved by exploiting the convexity of $K(x, f_x)$ wrt $f_x$.

Remark 1: The proposed proof is different from the ones mentioned in [1], [10] in the sense that the proposed proof relies only on variational calculus techniques, and it turns out that the constraints related to the first-order moment or non-singularity of correlation matrix are not necessary.

The maximum entropy result can be extended in various ways. A simple variation consists in considering only non-negative random variables. Then it turns out that a Gaussian random variable is not anymore the solution which maximizes the differential entropy. The following theorem can be easily established and states that a half-normal random variable maximizes the differential entropy over the set of non-negative random variables.

**Theorem 5:** Within the class of non-negative random variables with given second-order moment $m^2_x$, a half-normal random variable $X_{HN}$ maximizes the differential entropy, i.e.,

$$h(X) \leq h(X_{HN}),$$  \hspace{1cm} (34)

where $X$ is an arbitrary (but fixed) non-negative random variable with the second-order moment $m^2_x$, and $h(\cdot)$ denotes differential entropy.

**Proof:** The proof is omitted since it can be established following similar steps to the proof of Theorem 4.

Adopting a similar variational approach to the one in Theorem 4, we can also determine the probability density function that minimizes the Fisher information matrix as shown by the following theorem.

**Theorem 6 (Cramér-Rao Inequality (a vector version)):** Given a mean vector $\mu_x$ and a correlation matrix $\Omega_x$, the Gaussian density function with the mean vector $\mu_x$ and the correlation matrix $\Omega_x$ minimizes the Fisher information matrix, i.e.,

$$\mathbb{J}(X) \succeq \mathbb{J}(X_G),$$ \hspace{1cm} (35)

where $X$ and $X_G$ stand for an arbitrary (but fixed) random vector and Gaussian random vector, respectively, with given mean $\mu_x$ and correlation matrix $\Omega_x$, and $\mathbb{J}(\cdot)$ denotes the Fisher information matrix:

$$\mathbb{J}(X) = \begin{bmatrix}
  s_{11} & \cdots & s_{1n} \\
  \vdots & \ddots & \vdots \\
  s_{n1} & \cdots & s_{nn}
\end{bmatrix},$$ \hspace{1cm} (36)

$$s_{ij} = \int \left( \frac{d}{dx_i} f_x(x) \right) \left( \frac{d}{dx_j} f_x(x) \right) f_x(x) dx.$$

The following regularity condition is assumed to be satisfied:

$$\int \nabla f_x(x) dx = 0,$$

where $\nabla$ denotes the gradient, and the covariance matrix is supposed to be non-singular.

**Proof:** We first represent the inequality in (35) as a functional with the required constraints as follows:

$$\min_{f_x} \int \xi^T \nabla f_x(x) \nabla f_x(x) \xi \frac{1}{f_x(x)} dx,$$ \hspace{1cm} (37)

s. t. \hspace{0.5cm} $\int f_x(x) dx = 1,$

$$\int x f_x(x) dx = \mu_x,$$

$$\int xx^T f_x(x) dx = \Omega_x,$$ \hspace{1cm} (38)

where $\xi$ is an arbitrary but fixed non-zero vector, and it is defined as $\xi = [\xi_1, \ldots, \xi_n]^T$.

Using Theorem 3, the functional problem in (37) is expressed as

$$\min_{f_x} U[f_x],$$ \hspace{1cm} (39)
where $U[f_x] = \int K(x, f_x, \nabla f_x)dx$. $K(x, f_x, \nabla f_x) = (\xi^T \nabla f_x(x) \nabla f_x(x)^T \xi / f_x(x)) + \alpha f_x(x) + f_x(x) \sum_{i=1}^n \zeta_i x_i + f_x(x) \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} x_i x_j$, and $\alpha$, $\zeta_i$, and $\lambda_{ij}$ are the Lagrange multipliers corresponding to the three constraints in (38).

Based on Theorem 1 by confirming the first-order variation condition, i.e., $\delta U[f_{x^*}] = 0$, we can find the optimal solution $f_{x^*}(x)$ as follows:

$$K'_{f_x} - \sum_{i=1}^n \frac{\partial}{\partial x_i} K'_{f_x_i} \bigg|_{f_x = f_{x^*}} = 0,$$

(40)

where

$$K'_{f_x} = -\frac{\xi^T \nabla f_x(x) \nabla f_x(x)^T \xi}{f_x(x)^2} + \alpha + \zeta^T x + x^T \Lambda x,$$

$$\frac{\partial}{\partial x_i} K'_{f_x_i} = \frac{\partial}{\partial x_i} \left( \frac{2 n \sum_{j=1}^n \frac{\partial}{\partial x_j} f_x(x) \zeta_i \zeta_j}{f_x(x)} \right)$$

$$= \frac{2 n \sum_{j=1}^n \frac{\partial}{\partial x_j} f_x(x) \zeta_i \zeta_j}{f_x(x) - 2 \frac{\sum_{j=1}^n \frac{\partial}{\partial x_j} f_x(x) \zeta_i \zeta_j \frac{\partial}{\partial x_i} f_x(x)}{f_x(x)^2}}.$$  

(41)

Therefore, the left-hand side of the equation in (40) is expressed as

$$K'_{f_x} - \sum_{i=1}^n \frac{\partial}{\partial x_i} K'_{f_x_i} = \frac{\sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i} f_x(x) \frac{\partial}{\partial x_j} f_x(x) \zeta_i \zeta_j}{f_x(x)^2} - \frac{2 \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_j} f_x(x) \zeta_i \zeta_j}{f_x(x)} + \alpha + \sum_{i=1}^n \zeta_i x_i + \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} x_i x_j$$

$$= 0.$$  

(42)

Unlike Theorem 4, we cannot directly calculate $f_{x^*}(x)$ from (40). Fortunately, the first two parts in equation (42) are expressed as quadratic forms when $f_{x^*}(x)$ is a multi-variate Gaussian density function, and therefore, the multi-variate Gaussian density function satisfies the equality in (42). When $f_{x^*}(x)$ is a multi-variate Gaussian density function:

$$f_{x^*}(x) = (2\pi)^{-n/2} |\Sigma_x|^{-\frac{1}{2}} e^{-\frac{1}{2} (x - \mu_x)^T \Sigma_x^{-1} (x - \mu_x)}$$

with $\Sigma_x = \Omega_x - \mu_x \mu_x^T$ and

$$\Sigma_x^{-1} = \begin{bmatrix}
\sigma_{x_{11}}^2 & \cdots & \sigma_{x_{1m}}^2 \\
\vdots & \ddots & \vdots \\
\sigma_{x_{m1}}^2 & \cdots & \sigma_{x_{mm}}^2
\end{bmatrix},$$

(43)

its partial derivatives can be expressed as follows:

$$\frac{\partial}{\partial x_i} f_{x^*}(x) = \frac{-1}{2} \left( \sum_{l=1}^n \sigma_{x_{il}}^2 (x_l - \mu_{x_l}) + \sum_{m=1}^n \sigma_{x_{im}}^2 (x_m - \mu_{x_m}) \right) f_{x^*}(x),$$

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f_{x^*}(x) = -\frac{1}{2} \left( \sigma_{x_{ij}}^2 + \sigma_{x_{ji}}^2 \right) f_{x^*}(x) + \frac{1}{4} \left( \sum_{l=1}^n \sigma_{x_{il}}^2 (x_l - \mu_{x_l}) + \sum_{m=1}^n \sigma_{x_{im}}^2 (x_m - \mu_{x_m}) \right)$$

$$\cdot \left( \sum_{l=1}^n \sigma_{x_{jl}}^2 (x_l - \mu_{x_l}) + \sum_{m=1}^n \sigma_{x_{jm}}^2 (x_m - \mu_{x_m}) \right) f_{x^*}(x)$$

(44)
By substituting (44) into (42), it turns out that
\[
K_{f_X}^\prime - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} K_{f_X}^\prime = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \xi_j \left( \sum_{l=1}^{n} \left( \sigma_{x_i}^2 + \sigma_{x_j}^2 \right) (x_l - \mu_{x_i}) \right) \left( \sum_{m=1}^{n} \left( \sigma_{x_m}^2 + \sigma_{x_j}^2 \right) (x_m - \mu_{x_m}) \right) + \frac{n}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sigma_{x_i}^2 + \sigma_{x_j}^2 \right) \xi_i \xi_j + \alpha + \sum_{i=1}^{n} \xi_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} x_i x_j
\]
\[
= \frac{n}{2} \sum_{l=1}^{n} \sum_{m=1}^{n} \omega_{lm} \left( x_l - \mu_{x_l} \right) \left( x_m - \mu_{x_m} \right) + \frac{n}{2} \sum_{l=1}^{n} \sum_{m=1}^{n} \left( \sigma_{x_l}^2 + \sigma_{x_m}^2 \right) \xi_l \xi_m + \alpha + \sum_{l=1}^{n} \xi_l x_l + \sum_{l=1}^{n} \sum_{m=1}^{n} \lambda_{lm} x_l x_m
\]
\[
= x^T \Omega x + x^T \Lambda x + \zeta^T \xi - 2 \mu_x^T \Omega x + \mu_x^T \Omega \mu_x + \xi^T \Psi \xi + \alpha
\]
\[
= 0,
\]
where

\[
\Sigma_{x_{lm}} = \begin{bmatrix}
\Sigma_{x_{11}} & \cdots & \Sigma_{x_{1n}} \\
\vdots & \ddots & \vdots \\
\Sigma_{x_{n1}} & \cdots & \Sigma_{x_{nn}}
\end{bmatrix}, \quad \Lambda = \begin{bmatrix}
\lambda_{11} & \cdots & \lambda_{1n} \\
\vdots & \ddots & \vdots \\
\lambda_{n1} & \cdots & \lambda_{nn}
\end{bmatrix},
\]
\[
\Psi = \begin{bmatrix}
\psi_{11} & \cdots & \psi_{1n} \\
\vdots & \ddots & \vdots \\
\psi_{n1} & \cdots & \psi_{nn}
\end{bmatrix}, \quad \Omega = \begin{bmatrix}
\omega_{11} & \cdots & \omega_{1n} \\
\vdots & \ddots & \vdots \\
\omega_{n1} & \cdots & \omega_{nn}
\end{bmatrix}
\]
\[
\Sigma_{x_{ij}} = \frac{1}{4} \left( \sigma_{x_i}^2 + \sigma_{x_j}^2 \right) \left( \sigma_{x_{jm}}^2 + \sigma_{x_{mj}}^2 \right)
\]
\[
= \sigma_{x_i}^2 \sigma_{x_j}^2 \quad i, j = 1, \ldots, n, \quad l, m = 1, \ldots, n
\]
\[
\psi_{ij} = 2 \sigma_{x_i} \sigma_{x_j}, \quad i, j = 1, \ldots, n
\]
\[
\omega_{lm} = \xi^T \Sigma_{x_{lm}} \xi, \quad l, m = 1, \ldots, n.
\]

Therefore, the Lagrange multipliers \( \alpha \) and \( \lambda_{ij} \) must be selected as
\[
\alpha = -\mu_x^T \Omega \mu_x - \xi^T \Psi \xi,
\]
\[
\zeta = 2 \Omega \mu_x,
\]
\[
\Lambda = -\Omega.
\]

Since the second-order variation is positive:
\[
K_{f_X}'' \bigg|_{f_X = f_{X^*}} = 2 \frac{\xi^T \xi}{f_{X^*}(x)} \geq 0,
\]
based on Theorem 2, the Gaussian distribution \( f_{X^*}(x) \) is necessarily optimal for the variational problem in (37). Even though Theorems 1 and 2 are necessary conditions for the minimum, in this case, the multi-variate Gaussian density function is sufficiently the global minimum solution since this is a convex optimization problem (the objective function is strictly convex and its constraint set is convex).

Using similar variational arguments, one can show that a half-normal and a chi density function minimize the Fisher information over the set of non-negative random variables as shown by the following two theorems.

*Theorem 7:* Within the class of non-negative continuous random variables with fixed second-order moment \( m_{x}^2 \), the Fisher information is minimized by a half-normal random variable \( X_{HN} \):
\[
J(X) \geq J(X_{HN}),
\]
where \( X \) is an arbitrary (but fixed) non-negative random variable with the second-order moment \( m_x^2 \), and \( J(\cdot) \) denotes the Fisher information.

*Remark 2:* Theorem 7 does not assume the following regularity condition:
\[
\int_{0}^{\infty} \frac{d}{dx} f(x) dx = 0.
\]
for the Fisher information.

The following result establishes the counterpart of Theorem 7 for the class of non-negative random variables with fixed second order moment and whose distribution satisfies the regularity condition in (50).

**Theorem 8 ([15]):** Within the class of non-negative continuous random variables $X$ with fixed second-order moment and whose distributions satisfy the regularity condition in (50), the Fisher information is minimized by a chi-distributed random variable $X_c$:

$$J(X) \succeq J(X_c),$$

where $J(\cdot)$ stands for the Fisher information.

**Proof:** Unlike the proof in [15], by considering the first-order and the second-order moments instead of variance, we construct a variational problem and address the problem using the first-order and second-order necessary conditions, as well as the convexity property of the problem. The details of the proof are omitted because of the similar steps to those encountered in the proof of Theorem 6.

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**IV. WORST ADDITIVE NOISE LEMMA**

Worst additive noise lemma was introduced and exploited in several references [9], [10], [18], and it has been widely used in numerous other applications. One of the main applications of the worst additive noise lemma pertains to the capacity calculation of a wireless communication channel subject to different constraints such as Gaussian MIMO broadcasting, Gaussian MIMO wire-tap, etc. In this section, the worst additive noise lemma for random vectors will be proved solely based on variational arguments.

**Theorem 9:** Assume $X$ is an arbitrary but fixed random vector and $X_c$ is a Gaussian random vector, whose correlation matrix is identical to that of $X$, denoted as $\Omega_x$. Given a Gaussian random vector $W_c$, assumed independent of both $X$ and $X_c$ and with the correlation matrix $\Omega_w$, then the following relation holds:

$$I(X + W_c; W) \geq I(X_c + W_c; W).$$

**Proof:** Our novel proof is entirely anchored in the variational calculus framework. A summary of our proof runs as follows. First, we construct a variational problem, which represents the inequality in (52) and required constraints in a functional form. Second, using the first-order variation condition, we find necessary optimal solutions, which satisfy Euler’s equation. Third, using the second-order variation condition, we show that the optimal solutions are necessarily local minima. Finally, we justify that the local minimum is also global.

By setting $Y = X + W_c$, where $X$ and $W_c$ are independent of each other, in (52), the mutual information $I(X + W_c; W)$ can be expressed as

$$I(X + W_c; W) = h(Y) - h(Y|W_c) = h(Y) - h(X).$$

Then, we consider the functional:

$$\min_{f_x} \quad \int \int f_x(x) f_{y|x}(y|x) \log \left( \int f_x(x) f_{y|x}(y|x) dx \right) dy + \int f_x(x) f_{y|x}(y|x) \log f_x(x) dx \int dy$$

s. t. $\int f_x(x) dx = 1,$

$$\int x f_x(x) dx = \mu_x,$$

$$\int x x^T f_x(x) dx = \Omega_x.$$  

The density function $f_y(y)$ and conditional density function $f_{y|x}(y|x)$ are expressed as

$$f_y(y) = \int f_x(x) f_{y|x}(y|x) dx,$$

$$f_{y|x}(y|x) = f_w(y-x),$$
Based on the first-order variation condition, we can find the optimal solution $(f^*_y(y))$ for $\int f_x(x)f_{y,x}(y|x)dx$ and $f_w(y-x)$ for $f_{y|x}(y|x)$, respectively, and appropriately changing the constrains in (54), the variational problem in (53) is expressed as

$$\min_{f_x,f_y} \int f_x(x)f_w(y-x) \left[ -\log f_y(y) + \log f_x(x) \right] dxdy$$

s. t. $\int f_x(x)f_w(y-x)dxdy = 1$,

$$\int xf_x(x)f_w(y-x)dxdy = \mu_x,$$

$$\int x^xf_x(x)f_w(y-x)dxdy = \Omega_x,$$

$$f_y(y)dy = 1,$$

$$\int yf_y(y)dy = \mu_y,$$

$$\int y^yf_y(y)dy = \Omega_y,$$

$$f_y(y) = \int f_x(x)f_w(y-x)dx.$$  

The functional problem in (57) can be re-cast into the following equivalent form:

$$\min_{f_x,f_y} \int \left( \int K(x,y,f_x,f_y)dx \right) + \tilde{K}(y,f_y)dy,$$

where $x^T = [x_1, \ldots, x_n]$, $y^T = [y_1, \ldots, y_n]$, and $\alpha_0, \zeta, \gamma_{ij}, \alpha_1, \eta_i, \theta_{ij}$, and $\lambda(y)$ stand for the Lagrange multipliers corresponding to the constraints (58), (59), (60), (61), (62), (63), and (64), respectively.

Define now the functional $U$ as

$$U[f_x,f_y] = \int \left( \int K(x,y,f_x,f_y)dx \right) + \tilde{K}(y,f_y)dy,$$

where

$$K(x,y,f_x,f_y) = f_x(x)f_w(y-x)\left[ -\log f_y(y) + \log f_x(x) + \alpha_0 + \sum_{i=1}^n \zeta_i x_i + \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} x_i x_j - \lambda(y) \right],$$

$$\tilde{K}(y,f_y) = f_y(y)[\alpha_1 + \sum_{i=1}^n \eta_i y_i + \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} y_i y_j + \lambda(y)].$$

Based on the first-order variation condition, we can find the optimal solution $f_{x^*}$ and $f_{y^*}$ as follows:

$$\int K'_{f_x}|_{f_x=f_{x^*},f_y=f_{y^*}} dy = \int f_w(y-x)(-\log f_{y^*}(y) + \log f_{x^*}(x) + \alpha_0 + \zeta x^T + x^T \Gamma x + 1 - \lambda(y))dy = 0,$$

$$\int K'_{f_y} dx + \tilde{K}'_{f_y}|_{f_x=f_{x^*},f_y=f_{y^*}} = - \int \frac{f_{x^*}(x)f_w(y-x)dx}{f_{y^*}(y)} + \alpha_1 + \eta^T y + y^T \Theta y + \lambda(y) = 0,$$

where

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix}, \ \Theta = \begin{bmatrix} \theta_{11} & \cdots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \cdots & \theta_{nn} \end{bmatrix}$$

respectively.
\[ \zeta = [\zeta_1, \ldots, \zeta_n]^T \text{ and } \eta = [\eta_1, \ldots, \eta_n]^T. \]

The following relationships satisfy the necessary conditions (67) and (68):

\[
\begin{align*}
0 &= -\log f_{v^*}(y) - \lambda(y), \\
0 &= \log f_{x^*}(x) + \alpha_0 + \zeta^T x + x^T \Gamma x + 1, \\
\lambda(y) &= 1 - \alpha_1 - \eta^T y - y^T \Theta y,
\end{align*}
\]

and hence:

\[
\begin{align*}
f_{x^*}(x) &= \exp(-\alpha_0 - \zeta^T x - x^T \Gamma x - 1), \\
f_{v^*}(y) &= \exp(-1 + \alpha_1 + \eta^T y + y^T \Theta y).
\end{align*}
\]

Considering the constraints in (58)-(64), \( f_{x^*}(x) \) and \( f_{v^*}(y) \) in (71) are expressed as

\[
\begin{align*}
f_{x^*}(x) &= (2\pi)^{-\frac{n}{2}} |\Sigma_x|^{-\frac{1}{2}} e^{-(x-\mu_x)^T \Sigma_x^{-1} (x-\mu_x)} \\
&= \exp(-\alpha_0 - \zeta^T x - x^T \Gamma x - 1), \\
f_{v^*}(y) &= (2\pi)^{-\frac{n}{2}} |\Sigma_v|^{-\frac{1}{2}} e^{-(y-\mu_y)^T \Sigma_v^{-1} (y-\mu_y)} \\
&= \exp(-1 + \alpha_1 + \eta^T y + y^T \Theta y),
\end{align*}
\]

where \( \Sigma_x = \Omega_x - \mu_x \mu_x^T, \Sigma_v = \Sigma_x + \Sigma_w, \) and \( \Sigma_w \) is the covariance matrix of \( W_c. \) Based on the equations in (72), it turns out that

\[
\begin{align*}
\alpha_0 &= -1 + \frac{1}{2} \log (2\pi)^n |\Sigma_x| - \frac{1}{2} \mu_x^T \Sigma_x^{-1} \mu_x, \\
\alpha_1 &= 1 - \frac{1}{2} \log (2\pi)^n |\Sigma_v| - \frac{1}{2} \mu_y^T \Sigma_v^{-1} \mu_y, \\
\Gamma &= \frac{1}{2} \Sigma_x^{-1}, \\
\zeta &= -\Sigma_x^{-1} \mu_x, \\
\Theta &= \frac{1}{2} \Sigma_v^{-1}, \\
\eta &= -\Sigma_v^{-1} \mu_y.
\end{align*}
\]

Therefore, \( f_{x^*} \) and \( f_{v^*} \) are multi-variate Gaussian density functions (without loss of generality, and we can assume that the covariance matrix \( \Sigma_x \) is invertible due to the arguments mentioned in Appendix A).

Now, by confirming the second-order variation condition, we will show that the optimal solutions \( f_{x^*} \) and \( f_{v^*} \) are necessarily local minima. Using Theorem 2 we will show that the following matrix is positive semi-definite:

\[
\begin{bmatrix} K''_{f_{x^*} f_{x^*}} & K''_{f_{x^*} f_{v^*}} \\ K''_{f_{v^*} f_{x^*}} & K''_{f_{v^*} f_{v^*}} \end{bmatrix} \bigg|_{f_{x^*} = f_{x^*}, f_{v^*} = f_{v^*}} \succeq 0.
\]

(74)

Since the elements of the matrix in (74) are defined as

\[
\begin{align*}
K''_{f_{x^*} f_{x^*}} |_{f_{x^*} = f_{x^*}, f_{v^*} = f_{v^*}} &= \frac{f_w(y - x)}{f_{x^*}(x)}, \\
K''_{f_{x^*} f_{v^*}} |_{f_{x^*} = f_{x^*}, f_{v^*} = f_{v^*}} &= \frac{f_{x^*}(x)f_w(y - x)}{f_{v^*}(y)^2}, \\
K''_{f_{v^*} f_{x^*}} |_{f_{x^*} = f_{x^*}, f_{v^*} = f_{v^*}} &= \frac{f_w(y - x)}{f_{v^*}(y)}, \\
K''_{f_{v^*} f_{v^*}} |_{f_{x^*} = f_{x^*}, f_{v^*} = f_{v^*}} &= \frac{f_w(y - x)}{f_{v^*}(y)},
\end{align*}
\]

(75)

the matrix is a positive semi-definite matrix, and therefore \( \delta^2 U \succeq 0. \) Because of the convexity of functional \( K(x, y, f_{x^*}, f_{v^*}) \) wrt variables \( f_{x^*} \) and \( f_{v^*} \), the optimal solutions \( f_{x^*} \) and \( f_{v^*} \) actually globally minimize the variational functional in (57). Even though these optimal solutions are necessarily optimal, there exists only one solution, which
is the multi-variate Gaussian density function and it satisfies Euler’s equation in (67) and (68). Therefore, \( f_x \) and \( f_y \) are also sufficient in this case.

An alternative more detailed proof of the fact that \( f_x \) and \( f_y \) represent global optimal solutions is to show that 
\[
U[f_x, f_y] \geq U[f_{x*}, f_{y*}],
\]
where \( f_x, f_y \) denote any arbitrary functions satisfying the boundary conditions and the constraints. First, the following functionals are defined:
\[
F(x, y, f_x, f_y) = f_x(x) f_y(y - x) [- \log f_y(y) + \log f_x(x)] ,
\]
\[
F_0(x, y, f_x) = f_x(x) f_y(y - x) ,
\]
\[
F_1(x, y, f_x) = x f_x(x) f_y(y - x) ,
\]
\[
F_2(x, y, f_x) = x_2 f_x(x) f_y(y - x) ,
\]
and thus \( K(x, y, f_x, f_y) \) can be expressed as
\[
K(x, y, f_x, f_y) = F(x, y, f_x, f_y) + \alpha_0 F_0(x, y, f_x)
\]
\[+ \sum_{i=1}^{n} \int F_1(x, y, f_x) + \sum_{i=1}^{n} \int \int F_2(x, y, f_x) - \lambda(y) f_x(x) f_y(y - x) .
\]
Since the Hessian matrix of \( K(x, y, f_x, f_y) \) wrt \( f_x \) and \( f_y \) is given by
\[
\begin{bmatrix}
  f_w(y - x)/f_x(x) & -f_w(y - x)/f_y(y)
\end{bmatrix}
\]
which is positive semi-definite, \( K(x, y, f_x, f_y) \) is convex wrt \( f_x \) and \( f_y \), and the following inequality holds
\[
K(x, y, f_x, f_y) - K(x, y, f_{x*}, f_{y*}) \geq [(f_x - f_{x*})K'_{f_x} + (f_y - f_{y*})K'_{f_y}] \bigg|_{f_x = f_{x*}, f_y = f_{y*}} ,
\]
due to the fact that the convex function lies above its tangents. Therefore, it follows that
\[
U[f_x, f_y] - U[f_{x*}, f_{y*}] = \int \int F(x, y, f_x, f_y) - F(x, y, f_{x*}, f_{y*}) dxdy
\]
\[+ \int \int \int \int F_1(x, y, f_x) + \sum_{i=1}^{n} \int \int \int F_2(x, y, f_x) - \lambda(y) f_x(x) f_y(y - x) dxdy
\]
\[+ \sum_{i=1}^{n} \int \int F_1(x, y, f_x) + \sum_{i=1}^{n} \int \int \int F_2(x, y, f_x) - \lambda(y) f_x(x) f_y(y - x) dxdy
\]
\[+ \lambda(y) \bigg[ f_y(y) - f_{y*}(y) \bigg] dx - \lambda(y) \bigg[ f_{y*}(y) - f_y(y) \bigg] dx
\]
\[= \int \int K(x, y, f_x, f_y) - K(x, y, f_{x*}, f_{y*}) dxdy + \int \int (f_y - f_{y*}) (\alpha_1 + y^T \Theta y + \lambda(y)) dxdy
\]
\[= \int \int [f_x - f_{x*}] K'_f dx + \int \int [f_y - f_{y*}] K'_f dy
\]
\[= 0 ,
\]
Based on (76), the righthand side of (77) can be lower bounded as follows:
\[
U[f_x, f_y] - U[f_{x*}, f_{y*}] \geq \int \int [f_x - f_{x*}] K'_f dx + \int [f_y - f_{y*}] (\alpha_1 + y^T \Theta y + \lambda(y)) dy
\]
\[= \int [f_x - f_{x*}] K'_f dx + \int [f_y - f_{y*}] (\alpha_1 + y^T \Theta y + \lambda(y)) dy
\]
\[= 0 ,
\]
where (a) follows from the fact that
\[
\tilde{K}_{f_v}' \bigg|_{f_x=f_{x^*}, f_Y=f_{Y^*}} = \alpha_1 + \eta^T y + y^T \Theta y + \lambda(y),
\]
and (b) is due to (67) and (68). This proves the sufficiency of the Gaussian distributions, and therefore, \(f_{x^*}\) and \(f_{y^*}\) minimize the variational problem.

**Remark 3:** The constraints related to the vector means in (59) and (62) are unnecessary. Without these constraints, the optimal solutions are still multi-variate Gaussian density functions but the vector means are equal to zero.

---

V. EXTREMAL ENTROPY INEQUALITY

Extremal entropy inequality, proposed by Liu and Viswanath [2], was motivated by multi-terminal information theoretic problems such as the vector Gaussian broadcast channel and the distributed source coding with a single quadratic distortion constraint. EEI is an entropy power inequality which includes a covariance constraint. Because of the covariance constraint, the extremal entropy inequality could not be proved directly by using the classical Entropy Power Inequality (EPI). Therefore, new techniques ([16], [11]) were adopted in the proofs reported in [2], [11]. In this section, the extremal entropy inequality will be proved using a variational approach.

**Theorem 10:** Assume that \(\mu \geq 1\) is an arbitrary but fixed constant and \(\Sigma\) is a positive semi-definite matrix. A Gaussian random vector \(W\) with positive definite covariance matrix \(\Sigma_w\) is assumed to be independent of an arbitrary random vector \(X\) whose covariance matrix \(\Sigma_x\) satisfies \(\Sigma_x \preceq \Sigma\). Then, there exists a Gaussian random vector \(X^*_\alpha\) with covariance matrix \(\Sigma_{x^*}\) which satisfies the following inequality:
\[
h(X) - \mu h(X + W) \leq h(X^*_\alpha) - \mu h(X^*_\alpha + W),
\]
(79)
where \(\Sigma_{x^*} \preceq \Sigma\).

**Proof:** By setting \(Y = X + W\), we first consider the following variational problem (without loss of generality, we assume that \(X\), \(W\), and \(Y\) have zero mean):

\[
\min_{f_x, f_Y} \int \int f_x(x) f_w(y - x) (-\mu \log f_Y(y) + \log f_x(x) + \mu (\mu - 1) \log f_w(y - x)) dxdy
\]
(80)

s.t. \(\int \int f_x(x) f_w(y - x) dxdy = 1\),
\[
\int \int xx^T f_x(x) f_w(y - x) dxdy = \int \int yy^T f_x(x) f_w(y - x) dxdy + \int \int (y - x)(y - x)^T f_x(x) f_w(y - x) dxdy + \int \int xx^T f_x(x) f_w(y - x) dxdy \preceq \Sigma,
\]
\[
\int \int yy^T f_x(x) f_w(y - x) dxdy = \Sigma_{y^*},
\]
\[- \int \int f_\alpha(x) f_w(y - x) \log f_x(x) dxdy \geq p_x,
\]
\[
f_Y(y) = \int f_x(x) f_w(y - x) dx,
\]
(81)
where \(p_x\) is a constant, and \(\Sigma_{y^*}\) stands for the covariance matrix of the optimal solution \(Y\). In addition, the term \(\mu(\mu - 1) \int \int f_x(x) f_w(y - x) \log f_w(y - x) dxdy = \mu(\mu - 1) h(W)\) is added to the objective functional (80), and being a constant, it does not affect the optimization problem. Without loss of generality, the matrix \(\Sigma\) is assumed to be a positive definite matrix due to the same arguments mentioned in [2].
The optimization problem (80) can be better addressed by re-casting it as follows:

\[
\min_{f_x, f_y} \int f_x(x) f_w(y - x) [-\mu \log f_y(y) + \log f_x(x) + \mu (\mu - 1) \log f_w(y - x)] \, dxdy
\]

\(\text{s.t.} \quad \int f_x(x) f_w(y - x) dxdy = 1, \tag{82}\)

\[
\int \left( y_i y_j - x_i x_j - (y - x)_i (y - x)_j \right) f_x(x) f_w(y - x) dxdy = 0, \tag{83}\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int x_i x_j \xi_i \xi_j f_x(x) f_w(y - x) dxdy \right) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^2 \xi_i \xi_j, \tag{84}\]

\[
\int y_i y_j f_x(x) f_w(y - x) dxdy = \sigma_{ij}^2, \tag{85}\]

\[
-\int f_x(x) f_w(y - x) \log f_x(x) dxdy \geq p_x, \tag{86}\]

\[
f_y(y) = \int f_x(x) f_w(y - x) dx, \tag{87}\]

where the arbitrary deterministic non-zero vector \(\xi\) is defined as \([\xi_1, \ldots, \xi_n]^T\), \(\sigma_{ij}^2\) denotes the \(i^{th}\) row and \(j^{th}\) column entry of \(\Sigma_{y^*}\), \(i = 1, \ldots, n\), and \(j = 1, \ldots, n\).

Using Lagrange multipliers, as shown in Corollary 1, the functional problem in (82) and the constraints in (83)-(88) can be expressed in terms of the Lagrangian:

\[
\min_{f_x, f_y} \int \left( \int K(x, y, f_x, f_y) \, dx \right) + \tilde{K}(y, f_y) \, dy, \tag{89}\]

where

\[
K(x, y, f_x, f_y) = f_x(x) f_w(y - x) [-\mu \log f_y(y) + \log f_x(x) + \mu (\mu - 1) \log f_w(y - x) + \alpha_0
\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{n} (\gamma_{ij} y_i y_j - \gamma_{ij} x_i x_j - \gamma_{ij} (y - x)_i (y - x)_j + \theta x_i x_j \xi_i \xi_j + \phi_{ij} y_i y_j) - \alpha_1 \log f_x(x) - \lambda(y)], \tag{90}\]

\[
\tilde{K}(y, f_y) = \lambda(y) f_y(y).
\]

The Lagrange multipliers \(\alpha_0, \gamma_{ij}, \theta, \phi_{ij}, \alpha_1,\) and \(\lambda(y)\) correspond to the constraints in (83), (84), (85), (86), (87), and (88), respectively.

To find the optimal solutions, based on Corollary 2, the first-order variation condition is checked as follows:

\[
\int K'_x f_x + \tilde{K}'_y f_y \bigg|_{f_x = f_x^*, f_y = f_y^*} \, dx = \int f_w(y - x) [-\mu \log f_y(y) + (1 - \alpha_1) \log f_x(x) + \mu (\mu - 1) \log f_w(y - x) + \alpha_0
\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{n} (\gamma_{ij} y_i y_j - \gamma_{ij} x_i x_j - \gamma_{ij} (y - x)_i (y - x)_j + \theta x_i x_j \xi_i \xi_j + \phi_{ij} y_i y_j) - \lambda(y) + 1 - \alpha_1] \, dy
\]

\[= 0. \tag{91}\]

\[
\int K'_y f_y + \tilde{K}'_y f_y \bigg|_{f_x = f_x^*, f_y = f_y^*} = -\mu \int f_x(x) f_w(y - x) \, dx f_y(y) + \lambda(y) = 0. \tag{92}\]
The following expressions satisfy the equalities in (91) and (92):
\[ \lambda(y) = \mu, \]
\[ f_y(y) = (2\pi)^{-\frac{n}{2}} \left| -\frac{\mu}{2} (\Gamma + \Phi)^{-1} \right|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} y^T \left( -\frac{\mu}{2} (\Gamma + \Phi)^{-1} \right)^{-1} y \right\} (2\pi)^{\frac{n}{2}} \left| -\frac{\mu}{2} (\Gamma + \Phi)^{-1} \right|^{\frac{1}{2}} \exp \left\{ \frac{c_y}{\mu} \right\}, \]
\[ f_y(x) = (2\pi)^{-\frac{n}{2}} \left| -\frac{\mu}{2} (\Gamma - \Xi)^{-1} \right|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} x^T \left( -\frac{1}{2} (\Gamma - \Xi)^{-1} \right)^{-1} x \right\}, \]
\[ f_x(x) = (2\pi)^{-\frac{n}{2}} \left| -\frac{\mu}{2} (\Gamma - \Xi)^{-1} \right|^{-\frac{1}{2}} \exp \left\{ -\alpha_0 + \mu - 1 + \alpha_1 + c_w + c_y \right\}, \]
where
\[ \Phi = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix}, \]
\[ \Xi = \begin{bmatrix} \xi_1 \xi_1 & \cdots & \xi_1 \xi_n \\ \vdots & \ddots & \vdots \\ \xi_n \xi_1 & \cdots & \xi_n \xi_n \end{bmatrix}, \]
\[ x = [x_1, \cdots, x_n]^T, \]
\[ y = [y_1, \cdots, y_n]^T. \]

Now considering the constraints in (83)-(88), the equations in (93) are further processed as follows:
\[ f_y(y) = (2\pi)^{-\frac{n}{2}} \left| \Sigma_y \right|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} y^T \Sigma_y^{-1} y \right\}, \]
\[ f_y(x) = (2\pi)^{-\frac{n}{2}} \left| \Sigma_x \right|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} x^T \Sigma_x^{-1} x \right\}. \]
where
\[
\alpha_0 = \mu - (1 - \alpha_1) + \frac{\mu(\mu - 1)}{2} \log (2\pi)^n |\Sigma_w| \\
- \frac{\mu}{2} \log (2\pi)^n |\Sigma_v| + \frac{1 - \alpha_1}{2} \log (2\pi)^n |\Sigma_x|,
\]
\[
\Gamma = - \frac{\mu(\mu - 1)}{2} \Sigma_w^{-1},
\]
\[
\Phi = - \Gamma - \frac{\mu}{2} \Sigma_v^{-1},
\]
\[
\Sigma_{x^*} = - \frac{1 - \alpha_1}{2} \Gamma^{-1},
\]
\[
|\Sigma_{x^*}| = \left(\frac{1}{2\pi e} \exp \left\{ \frac{2}{n} \right\} \right)^n.
\]

The inequality in (96) is due to the second-order variation condition, which will be presented later in this proof. The inequality (95) is based on the theory of KKT conditions since the multiplier associated with the inequality constraint is nonnegative. Moreover, the complementary slackness condition in the KKT conditions leads to the following relationship:
\[
\theta \left[ \int \int \left( \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \xi_i \xi_j \right) f_{x^*}(x) f_w(y-x) dx dy - \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^2 \xi_i \xi_j \right] = 0.
\]

Based on Theorem 2, to make the second variation nonnegative, the positive semi-definiteness of the following matrix is required:
\[
\begin{bmatrix}
K''_{fx^*fx^*} & K''_{fy^*fx^*} \\
K''_{fy^*fx^*} & K''_{fy^*fy^*}
\end{bmatrix},
\]
which further reduces to the following condition:
\[
\begin{bmatrix}
h_x & h_y \\
h_x & h_y
\end{bmatrix} \begin{bmatrix}
K''_{fx^*fx^*} & K''_{fy^*fx^*} \\
K''_{fy^*fx^*} & K''_{fy^*fy^*}
\end{bmatrix} \begin{bmatrix}
h_x \\
h_y
\end{bmatrix}
= K''_{fx^*fx^*} h_x^2 + K''_{fy^*fx^*} h_x^2 + (K''_{fx^*fy^*} + K''_{fy^*fx^*}) h_y h_x \\
\geq 0,
\]

where \( h_x \) and \( h_y \) are arbitrary admissible functions. Since \( K''_{fx^*fx^*}, K''_{fy^*fx^*}, K''_{fy^*fy^*}, \) and \( K''_{fy^*fx^*} \) are defined as
\[
K''_{fx^*fx^*} = \frac{(1 - \alpha_1) f_w(y-x)}{f_{x^*}(x)},
\]
\[
K''_{fx^*fy^*} = - \frac{\mu f_w(y-x)}{f_{y^*}(y)},
\]
\[
K''_{fy^*fx^*} = - \frac{\mu f_w(y-x)}{f_{y^*}(y)},
\]
\[
K''_{fy^*fy^*} = \frac{\mu f_{x^*}(x) f_w(y-x)}{f_{y^*}(y)^2},
\]

(100)
which holds true if \(1 - \alpha_1 \geq \mu\) (i.e., \(\alpha_1 \leq 1 - \mu \leq 0\)). Condition \(\alpha_1 \leq 0\) is also imposed by the KKT complementary slackness condition corresponding to the constraint \((87)\). Therefore, the optimal solutions \(f_{x^*}\) and \(f_{y^*}\) minimize the functional problem in \((82)\), and the proof is completed because of convexity of the functional \(K(x, y, f_x, f_y)\) wrt variables \(f_x\) and \(f_y\).

A more detailed alternative justification of the fact the Gaussian distributions \(f_{x^*}\) and \(f_{y^*}\) are global minima is next presented. We will prove the sufficiency of the Gaussian-distributions by showing \(U[f_x, f_y] = 0\), \(f_{x^*}\), \(f_{y^*}\) denote any arbitrary functions satisfying the boundary conditions and the constraints. First, the following functionals are defined:

\[
F(x, y, f_x, f_y) = f_x(x) f_w(y - x) (-\mu \log f_x(y) + \log f_x(x) + \mu (\mu - 1) \log f_w(y - x)),
\]

\[
F_0(x, y, f_x) = f_x(x) f_w(y - x),
\]

\[
F_1(x, y, f_x) = (y_i y_j - x_i x_j - (y - x)_i (y - x)_j) f_x(x) f_w(y - x),
\]

\[
F_2(x, y, f_x) = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \xi_i \xi_j \right) f_x(x) f_w(y - x),
\]

\[
F_3(x, y, f_x) = y_i y_j f_x(x) f_w(y - x),
\]

\[
F_4(x, y, f_x) = -f_x(x) f_w(y - x) \log f_x(x),
\]

and thus

\[
K(x, y, f_x, f_y) = F(x, y, f_x, f_y) + \alpha_0 F_0(x, y, f_x) + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} F_1(x, y, f_x) + \theta F_2(x, y, f_x)
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij} F_3(x, y, f_x) + \alpha_1 F_4(x, y, f_x) - \lambda(y) f_x(x) f_w(y - x).
\]

It can be verified that the Hessian matrix of \(K(x, y, f_x, f_y)\) wrt \(f_x\) and \(f_y\) is given by

\[
\begin{bmatrix}
(1 - \alpha_1) f_w(y - x) / f_x(x) & -\mu f_w(y - x) / f_y(y) \\
-\mu f_w(y - x) / f_x(x) & \mu f_x(x) f_w(y - x) / f_y(y)^2
\end{bmatrix},
\]

which is positive semi-definite due to \((96)\). The convexity property of \(K(x, y, f_x, f_y)\) yields that

\[
K(x, y, f_{x^*}, f_y) - K(x, y, f_x, f_{y^*}) \geq \left[(f_{x^*} - f_x) K'_{f_x} + (f_y - f_{y^*}) K'_{f_y}\right]_{f_x = f_{x^*}, f_y = f_{y^*}},
\]

\[(102)\]
and it follows that

\[ U[f_x, f_y] - U[f_{x^*}, f_{y^*}] = \int \int F(x, y, f_x, f_y) - F(x, y, f_{x^*}, f_{y^*}) \, dxdy \]

\[ \geq \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \left[ \int \int F_1(x, y, f_x) - F_1(x, y, f_{x^*}) \, dxdy \right] + \theta \left[ \int \int F_2(x, y, f_x) - F_2(x, y, f_{x^*}) \, dxdy \right] 
\]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij} \left[ \int \int F_3(x, y, f_x) - F_3(x, y, f_{x^*}) \, dxdy \right] + \alpha_1 \left[ \int \int F_4(x, y, f_x) - F_4(x, y, f_{x^*}) \, dxdy \right] 
\]

\[ + \lambda(y) \left[ f_v(y) - \int f_x(x)f_w(y-x) \, dx \right] \, dy - \lambda(y) \left[ f_{v^*}(y) - \int f_{x^*}(x)f_w(y-x) \, dx \right] \, dy \]

\[ = \int \int K(x, y, f_x, f_{x^*}) - K(x, y, f_{x^*}, f_{x^*}) \, dxdy + \lambda(y) (f_v(y) - f_{v^*}(y)) \, dy \]

\[ \geq \int \int \left[ (f_x - f_{x^*})K_{f_x} + (f_v - f_{v^*})K_{f_{v^*}} \right] \mid f_x = f_{x^*}, f_v = f_{v^*} \, dxdy \]

\[ \geq \int (f_x - f_{x^*}) \left[ K_{f_x} \mid f_x = f_{x^*}, f_v = f_{v^*} \right] \, dx + \int (f_v - f_{v^*}) \left[ K_{f_{v^*}} \mid f_x = f_{x^*}, f_v = f_{v^*} \right] \, dx + \lambda(y) \, dy \]

\[ \geq 0, \]

where the inequality (a) follows from the complementary slackness condition in the KKT conditions (97). Indeed, since \( f_x \) only represents an arbitrary feasible solution and \( \theta \geq 0 \), it follows that

\[ \theta \left[ \int \int F_2(x, y, f_{x^*}) \, dxdy - \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^2 \xi_i \xi_j \right] = 0, \]

and

\[ \theta \left[ \int \int F_2(x, y, f_x) \, dxdy - \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^2 \xi_i \xi_j \right] \leq 0, \]

and therefore, \( \theta \left[ \int \int F_2(x, y, f_x) - F_2(x, y, f_{x^*}) \, dxdy \right] \leq 0 \). Similarly, the complementary slackness condition associated with (87) leads to \( \alpha_1 \left[ \int \int F_4(x, y, f_x) - F_4(x, y, f_{x^*}) \, dxdy \right] \leq 0 \). In addition, (b) is due to (102), and (c) follows from (91) and (92). This proves the sufficiency of Gaussian distributions.

Remark 4: The proposed proof only exploits calculus of variations tools. Unlike the previous proofs, this proof does not adopt neither the channel enhancement technique and EPI as in [2] nor the EPI and data processing inequality as in [11].

Theorem 11: Assume that \( \mu \geq 1 \) is an arbitrary but fixed constant and \( \Sigma \) is a positive semi-definite matrix. Independent Gaussian random vectors \( W_{\xi} \) with covariance matrix \( \Sigma_w \) and \( V_{\xi} \) with covariance matrix \( \Sigma_v \) are assumed to be independent of an arbitrary random vector \( X \) with covariance matrix \( \Sigma_x \preceq \Sigma \). Both covariance matrices \( \Sigma_w \) and \( \Sigma_v \) are assumed to be positive definite. Then, there exists a Gaussian random vector \( X_{\xi} \) with covariance matrix \( \Sigma_{X_{\xi}} \) which satisfies the following inequality:

\[ h(X + W_{\xi}) - \mu h(X + V_{\xi}) \leq h(X_{\xi} + W_{\xi}) - \mu h(X_{\xi} + V_{\xi}), \]

where \( \Sigma_{X_{\xi}} \preceq \Sigma \).

Proof: See Appendix B

Remark 5: The proposed proof does not borrow any techniques from [2]. Even though the proposed proof adopts the equality condition for the data processing inequality, a result which was also exploited in [11], the proposed proof is different from the one in [11] due to the following features. First, the proposed proof uses the equality condition of the data processing inequality only once while the proof in [11] uses it twice. The proof in [2]...
exploited the channel enhancement technique twice, which is equivalent to using the equality condition in the data processing inequality. Second, the proposed proof does not use the moment generating function technique unlike the proof proposed in [11]; instead the current proof directly exploits a property of the conditional mutual information pertaining to a Markov chain.

VI. APPLICATIONS

Because of the easiness to incorporate a broad class of constraints, the proposed variational framework finds applicability in a large number of applications. Herein section, we will briefly illustrate some potential applications in this regard and state several open research problems which might be also addressed within the considered functional framework.

A. Gaussian Wire-tap Channel

The secrecy capacity of Gaussian wire-tap channel has been studied by many researchers [20], [33]. We will approach the Gaussian wire-tap problem from the estimation viewpoint, rather than considering the secrecy capacity from an information theoretic perspective.

The following scalar Gaussian wire-tap channel is considered:

\[
\begin{align*}
Y_1 &= aX + W_G, \\
Y_2 &= aX + W_G + Z_G,
\end{align*}
\]  

(106)

where \( X \) is an arbitrary but fixed random variable with zero mean and unit variance, \( a \) is a constant, and \( W_G \) and \( Z_G \) are Gaussian random variables with variances \( \sigma^2_W \) and \( \sigma^2_Z \), respectively. The random variables \( W_G \) and \( Z_G \) are independent of each other, and they have zero mean. In the channel model (106), \( Y_1 \) and \( Y_2 \) are considered as a legitimate receiver and as an eavesdropper, respectively. The goal of this problem is the following. Assume that both receivers use minimum mean square error (MMSE) estimators. Given the value of the mean square error (MSE), which allows to correctly decode the legitimate receiver, what is the optimal distribution which maximizes the difference between the MSE in the legitimate receiver and the MSE in the eavesdropper?

The above mentioned problem adopts both practical and reasonable assumptions due to the following reasons. First, the MMSE estimator is an optimal estimator in the sense that it minimizes the MSE. Therefore, it is reasonable to use such an optimal estimator. Second, to prevent from eavesdropping, finding the signal distribution that maximizes the difference between the MSEs corresponding to the legitimate receiver and the eavesdropper, respectively, represents a legitimate design objective. To find the optimal distribution, the following functional problem is constructed:

\[
\begin{align*}
\max_{f_X(x)} \ & \text{Var}(X|Y_2) - \text{Var}(X|Y_1), \\
\text{s.t.} \ & \text{Var}(X|Y_1) = R,
\end{align*}
\]

(107)

where \( \text{Var}(X|Y) = \mathbb{E} \left[ (X - \mathbb{E}[X|Y])^2 \right] \), \( \mathbb{E}[] \) denotes the expectation operator, and \( R \) is a constant.

The optimization problem in (107) is expressed as

\[
\begin{align*}
\max_{f_X(x)} \ & \text{Var}(\mathbb{E}[X|Y_1]|Y_2), \\
\text{s.t.} \ & \mathbb{E} \left[ \mathbb{E}[X|Y_1]^2 \right] = 1 - R.
\end{align*}
\]

(108)

The equation in (108) is due to the total law of variance and the Markov chain \( X \rightarrow Y_1 \rightarrow Y_2 \). Since \( \mathbb{E}[X^2] = 1 \), the equation (109) follows from the constraint in (107).

The objective function in (108) is further expressed as

\[
\text{Var}(\mathbb{E}[X|Y_1]|Y_2) = \mathbb{E} \left[ \mathbb{E}[X|Y_1]^2 \right] - \mathbb{E} \left[ \mathbb{E}[X|Y_2]^2 \right]
\]

(110)
and using the equations (109), (110), the optimization problem in (108) is re-formulated in terms of the following variational problem:

\[
\min_{f_{Y^2}, g} \int \frac{1}{f_{Y^2}(y)} g(y)^2 dy, \tag{111}
\]

\[
\int g^2 f_{Y^2}(y) dy = m_{Y^2}^2, \tag{112}
\]

\[
g(y) = \int x f_{Y^2|x}(y|x) f_X(x) dx, \tag{113}
\]

where \( f_X(x) \) and \( f_{Y^2}(y) \) are the probability density functions of \( X \) and \( Y^2 \), respectively, and \( m_{Y^2}^2 \) stands for the second-order moment of \( Y^2 \).

Since the first term in (110) is given and

\[
E\left[ E[X|Y^2]\right] = \int f_{Y^2}(y) \left( \int x f_{Y^2|x}(y|x) f_X(x) dx \right)^2 dy,
\]

the objective function in (111) is derived from the equation (108). Also, the additional constraint in (112) is required to solve this variational problem.

Considering the Lagrange multipliers \( \lambda_1 \) and \( \lambda(y) \) to account for the constraints in (112) and (112), respectively, the following variational problem is constructed:

\[
\int K(y, f_{Y^2}, g) dy,
\]

where

\[
K(y, f_{Y^2}, g) = \frac{g(y)^2}{f_{Y^2}(y)} + \lambda_1 y^2 f_{Y^2}(y) + \lambda(y) \left( g(y) - \int x f_{Y^2|x}(y|x) f_X(x) dx \right). \tag{114}
\]

In accordance with Theorem [1] we can determine \( g^* \) and \( f_{Y^2}^* \) to enforce the first-order variation to be zero:

\[
K_{f_{Y^2}} = -\frac{g^*(y)^2}{f_{Y^2}^*(y)^2} + \lambda_1 y^2 = 0, \tag{115}
\]

\[
K_{g^*} = 2\frac{g^*(y)}{f_{Y^2}^*(y)} + \lambda(y) = 0,
\]

Taking into account (115), it follows further that

\[
E[X^*|Y^2^*] = \frac{g^*(y)}{f_{Y^2}^*(y)} = \sqrt{\lambda_1} y. \tag{116}
\]

Since the equation in (116) is a linear function of \( y \) and \( Y^2^* \) is a Gaussian random variable, therefore, \( X^* \) is also a Gaussian random variable. Based on Theorem [2] it can be verified that the second-order variation is nonnegative. Moreover, due to the convexity of \( K(y, f_{Y^2}, g) \) wrt \( f_{Y^2} \) and \( g \), we can confirm that the solution is optimal, and the proof is completed.

**B. Additional Applications**

The importance of the variational framework in establishing some fundamental information theoretic inequalities was already illustrated herein paper. At their turn, these information theoretic inequalities played a fundamental role in establishing other important results and applications. For example, the minimum Fisher information theorem (Cramér-Rao inequality) and maximum entropy theorem were used for developing min-max robust estimation techniques [25], results which were recently further extended to the more general framework of noise with arbitrary distribution (and correlation) in [27] and used to explain why the MIMO channel estimation scheme proposed in [26] exhibits a min-max robustness property. Along the same line of potential applications, the extensions of the maximum entropy and minimum Fisher information results to positive random variables, as stated in Theorems [5, 7, 8] play a fundamental role in developing robust clock synchronization algorithms for wireless sensor networks and other wireless networks that rely on message exchanges to acquire the timing information. A large
class of clock synchronization protocols (see e.g., TPSN, Internet, PBS [28]) rely on the two-way message exchange mechanism and for which the timing synchronization approach reduces to estimating a linear regression model for which the distribution of additive noise has positive support but it is otherwise arbitrary [28]. Designing robust timing synchronization algorithms for such protocols is difficult, because of the variability of delay distributions caused by the variable network traffic. However, this problem can now be resolved at the light of the results brought by Theorems 5, 7 and 8. By optimizing the design of timing messages for the scenario of a chi or log-normal distributed delay, then min-max robust time synchronization algorithms could be developed.

The extremal entropy inequality was used in the vector Gaussian broadcast channel [2], the distributed source coding with a single quadratic distortion constraint problem [2], the Gaussian wire-tap channel [11], and many other problems. Even though these applications were traditionally addressed using the information theoretic inequalities, one can directly approach these applications by means of the proposed variational calculus techniques. One of the benefits of such a variational approach is the fact that it can cope with many types of constraints as opposed to the EEI which is still quite rigid in its formulation. As Prof. Max Costa suggested the authors of this paper in a private communication, in the context of Z Gaussian interference channels, such a variational approach might be helpful to develop novel entropy-power-like inequalities, where the limiting variables are Gaussian and independent but not anymore identically distributed, and to assess the capacity of the Z-Gaussian interference channel.

Additional important extensions of maximum entropy theorem, minimum Fisher information theorem, additive worst noise lemma, and extremal entropy inequality might be envisioned within the proposed variational framework by imposing various restrictions on the range of values assumed by random variables/vectors (e.g., random variables whose support is limited to a finite length interval or finite set of values) or on their second or higher-order moments and correlations. For example, the problem of finding the worst additive noise under a covariance constraint [9] as well as establishing multivariate extensions of Costa’s entropy power inequality [30] along the lines mentioned by Liu et al. [21] and Palomar [31], [32] might be also addressed within the proposed variational framework. However, all these challenges together with finding a variational proof of EPI remain open research problems for future study.

VII. CONCLUSIONS

In this paper, we derived several fundamental information theoretic inequalities using a functional analysis framework. The main benefit for employing calculus of variations for proving information theoretic inequalities is the fact that the global optimal solution is obtained from the necessary conditions for optimality. A brief summary of this paper contributions is the following. First, the entropy maximizing theorem and Fisher information minimizing theorem were derived under different assumptions. Second, the worst additive noise lemma was proved from the perspective of a functional problem. Third, the extremal entropy inequality was derived using calculus of variations techniques. Finally, applications and possible extensions that could be addressed within the proposed variational framework were briefly presented. Many open research problems were also formulated.

APPENDIX A

NON-INVERTIBLE CORRELATION (OR COVARIANCE) MATRIX

Let $\Omega_X = Q_\Omega \Lambda_\Omega Q_\Omega^T$ and $\bar{X} = Q_\Omega^T X = [\bar{X}_a^T, \bar{X}_b^T]$, where $\Lambda_\Omega = diag(\Lambda_1, \ldots, \Lambda_m, 0, \ldots, 0)$, $\Omega_X$ is a singular matrix, $Q_\Omega$ is an orthogonal matrix, and $diag(\cdot)$ denotes a diagonal matrix. The correlation matrix of $X_b$ is the zero matrix, and therefore, it is considered as a deterministic vector. Without loss of generality, we can assume $\bar{X}_b = 0$. The following matrices are also considered:

$$Q_\Omega^T \Omega_X Q_\Omega = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix},$$

$$D = \begin{bmatrix} I & -B^T C^{-1} \\ 0 & I \end{bmatrix}.$$  (117)
where the dimensions of $A$, $B$, and $C$ are $m \times m$, $(n - m) \times m$, and $(n - m) \times (n - m)$, respectively. Then,

$$DQ^T_{\eta_1}X = \begin{bmatrix} I & -B^T C^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{X}_a \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{X}_a \\ 0 \end{bmatrix},$$

$$DQ^T_{\eta_2}W_{\bar{\alpha}} = \begin{bmatrix} \hat{W}_{a_\bar{\alpha}} \\ \hat{W}_{c_\bar{\alpha}} \end{bmatrix},$$

$$E[ DQ^T_{\eta_1}W_{\bar{\alpha}} W_{\bar{\alpha}}^T Q_{\bar{\alpha}} D_{\bar{\alpha}}] = \begin{bmatrix} A - B^T C^{-1}B & 0 \\ 0 & C \end{bmatrix}. \tag{118}$$

Due to (118), the random vectors $\hat{W}_{a_\bar{\alpha}}$ and $\hat{W}_{c_\bar{\alpha}}$ are statistically independent of each other.

The left-hand side of the equation in (105) can be re-expressed as

$$h(X + W_{\bar{\alpha}}) - h(X) = h(DQ^T_{\eta_1}X + DQ^T_{\eta_2}W_{\bar{\alpha}}) - h(DQ^T_{\eta_1}X)$$

$$= h(\hat{X}_a + W_{a_\bar{\alpha}}, \hat{X}_b + W_{c_\bar{\alpha}}) - h(\hat{X}_a, \hat{X}_b)$$

$$= h(\hat{X}_a + W_{a_\bar{\alpha}}) - h(\hat{X}_a) + h(\hat{X}_b + W_{c_\bar{\alpha}}) - h(\hat{X}_b). \tag{119}$$

In (119), $\hat{X}_b$ is considered as a deterministic variable, $\hat{W}_{a_\bar{\alpha}}$ is given, the term $(a)$ can be ignored in the optimization, and the correlation matrix of $\hat{X}_a$ is non-singular. Therefore, we can always assume the correlation matrix to be invertible.

**APPENDIX B**

**Proof of Theorem III**

Proof: First, choose a Gaussian random vector $\hat{W}_{\bar{\alpha}}$ whose covariance matrix $\Sigma_{\bar{\alpha}}$ satisfies $\Sigma_{\bar{\alpha}} \preceq \Sigma_{\bar{\omega}}$ and $\Sigma_{\bar{\alpha}} \preceq \Sigma_{\bar{\bar{\alpha}}}$. Since the Gaussian random vectors $V_{\bar{\omega}}$ and $W_{\bar{\alpha}}$ can be represented as the summation of two independent random vectors $\hat{W}_{\bar{\alpha}}$ and $\hat{V}_{\bar{\alpha}}$, and the summation of two independent random vectors $\hat{W}_{\bar{\omega}}$ and $\hat{W}_{\bar{\alpha}}$, respectively, the left-hand side of the equation in (105) is written as follows:

$$\mu h(X + V_{\bar{\alpha}}) - h(X + W_{\bar{\alpha}})$$

$$\geq \mu h(X + V_{\bar{\alpha}}) - h(X + \hat{W}_{\bar{\alpha}}) - h(W_{\bar{\alpha}}) + h(\hat{W}_{\bar{\alpha}})$$

$$= \mu h(X + \hat{W}_{\bar{\alpha}} + \hat{V}_{\bar{\alpha}}) - h(X + \hat{W}_{\bar{\alpha}}) - h(W_{\bar{\alpha}} + \hat{W}_{\bar{\alpha}}) + h(\hat{W}_{\bar{\alpha}}). \tag{120}$$

Since the expression will be minimized over $f_{\bar{x}}, f_{\bar{y}}$, the last two terms in (120) are ignored, and by substituting $Y$ and $X$ for $X + W_{\bar{\alpha}} + V_{\bar{\alpha}}$ and $X + W_{\bar{\alpha}}$, respectively, the inequality in (105) is equivalently expressed as the following variational problem:

$$\min_{f_{\bar{x}}, f_{\bar{y}}} \mu h(Y) - h(X) - \mu (\mu - 1) h(\hat{V}_{\bar{\alpha}})$$

s. t. $\int \int f_{\bar{x}}(x)f_{\bar{y}}(y-x)dx dy - 1 = 0,$

$$\int \int f_{\bar{x}}(x)f_{\bar{y}}(y-x)xx^T dx dy - \Sigma_{\bar{x}} \preceq 0,$$

$$\int \int f_{\bar{x}}(x)f_{\bar{y}}(y-x)yy^T dx dy - \Sigma_{\bar{y}} = 0,$$

$$\int \int f_{\bar{x}}(x)f_{\bar{y}}(y-x)(yy^T - xx^T - (y - x)(y - x)^T) dx dy = 0,$$

$$- \int \int f_{\bar{x}}(x)f_{\bar{y}}(y-x) \log f_{\bar{x}}(x) dx dy \geq p_x$$

$$f_{\bar{y}}(y) = \int f_{\bar{x}}(x)f_{\bar{y}}(y-x) dx,$$

where $\hat{X} = X + \hat{W}_{\bar{\alpha}}, Y = \hat{X} + \hat{V}_{\bar{\alpha}}, W_{\bar{\alpha}} = \hat{W}_{\bar{\alpha}} + \hat{W}_{\bar{\alpha}}, V_{\bar{\alpha}} = \hat{W}_{\bar{\alpha}} + \hat{V}_{\bar{\alpha}}, \Sigma_{\bar{x}} = \Sigma + \Sigma_{\bar{\omega}}, \Sigma_{\bar{y}} = \Sigma + \Sigma_{\bar{\bar{\alpha}}}, \Sigma_{\bar{y}} = \Sigma_{\bar{\bar{\alpha}}}, \Sigma_{\bar{y}},$ and $\Sigma_{\bar{y}}$ is the covariance matrix of the optimal solution $X^*$. 


The variational problem in (121) is exactly the same as the one in (82). Therefore, using the same method as in the proof of Theorem 10, we obtain the following inequality (see the details in the proof of Theorem 10):

\[ \mu h(X + \tilde{W}_a + \hat{V}_a) - h(X + \tilde{W}_a) - h(\tilde{W}_a) + h(\hat{V}_a) \geq \mu h(X_a^* + \tilde{W}_a + \hat{V}_a) - h(X_a^* + \tilde{W}_a) - h(\tilde{W}_a) + h(\hat{V}_a). \] (122)

By appropriately choosing \( X_a^* \) and \( \tilde{W}_a \), the right-hand side of the equation in (122) is expressed as

\[ \mu h(X_a^* + \tilde{W}_a + \hat{V}_a) - h(X_a^* + \tilde{W}_a) - h(\tilde{W}_a) + h(\hat{V}_a) = \mu h(X_a^* + \tilde{W}_a + \hat{V}_a) - h(X_a^* + \tilde{W}_a). \] (123)

The equality in (123) is due to the equality condition of the data processing inequality in [11]. For the completeness of the proof, we introduce a technique, which is slightly different from the one in [11].

To satisfy the equality in the equation (123), the equality condition in the following lemma must be satisfied.

**Lemma 1 (Data Processing Inequality in [11]):** When three random vectors \( Y_1, Y_2, \) and \( Y_3 \) represent a Markov chain \( Y_1 \rightarrow Y_2 \rightarrow Y_3 \), the following inequality is satisfied:

\[ I(Y_1; Y_3) \leq I(Y_1; Y_2). \] (124)

The equality holds if and only if \( I(Y_1; Y_2 | Y_3) = 0 \).

In Lemma 1, \( Y_1, Y_2, \) and \( Y_3 \) are defined as \( X_a^*, X_a^* + \tilde{W}_a, \) and \( X_a^* + \tilde{W}_a + \hat{V}_a \), respectively. Therefore, the equality condition, \( I(Y_1; Y_2 | Y_3) = 0 \) is expressed as

\[
I(Y_1; Y_2 | Y_3) = h(Y_1 | Y_3) - h(Y_1 | Y_2, Y_3) \\
= \frac{1}{2} \log (2\pi e)^n |\Sigma_{y_1 | y_3}| - \frac{1}{2} \log (2\pi e)^n |\Sigma_{y_1 | y_2}| \\
= \frac{1}{2} \log (2\pi e)^n |\Sigma_{y_1} - \Sigma_{y_1} \Sigma_{y_3}^{-1} \Sigma_{y_1}| - \frac{1}{2} \log (2\pi e)^n |\Sigma_{y_1} - \Sigma_{y_1} \Sigma_{y_2}^{-1} \Sigma_{y_1}| \\
= \frac{1}{2} \log (2\pi e)^n |\Sigma_{X^*} - \Sigma_{X^*} (\Sigma_{X^*} + \Sigma_{w} + \Sigma_{w})^{-1} \Sigma_{X^*}| - \frac{1}{2} \log (2\pi e)^n |\Sigma_{X^*} - \Sigma_{X^*} (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*}| \\
= \frac{1}{2} \log (2\pi e)^n |\Sigma_{X^*}| |I - (\Sigma_{X^*} + \Sigma_{w} + \Sigma_{w})^{-1} \Sigma_{X^*}| - \frac{1}{2} \log (2\pi e)^n |I - (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*}| \\
= \frac{1}{2} \log (2\pi e)^n |I - (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*}| - \frac{1}{2} \log (2\pi e)^n |I - (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*}| \\
= 0. \] (125)

If \( (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*} = (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*} \), the equality in (125) is satisfied, the equality condition in Lemma 1 holds, and therefore, the equality in (123) is proved. The validity of \( (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*} = (\Sigma_{X^*} + \Sigma_{w})^{-1} \Sigma_{X^*} \) is proved by Lemma 8 in [11].

Therefore, \( I(Y_1; Y_2 | Y_3) = 0 \), and from the equations in (120), (122), and (123), we obtain the following extremal entropy inequality:

\[ \mu h(X + V_a) - h(X + W_a) \geq \mu h(X + \tilde{W}_a + \hat{V}_a) - h(X + \tilde{W}_a) - h(\tilde{W}_a) + h(\hat{V}_a) \]

and the proof is completed.

\[ \blacksquare \]
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