Electromagnetic fields in matter revisited

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Abstract
The force density on matter and the kinetic energy-momentum tensor of the electromagnetic field in matter are obtained starting from Maxwell equations and Lorentz force at microscopic level and averaging over a small region of space-time. The macroscopic force density is taken to depend linearly on the average fields and their first derivatives and is shown to be determined by two phenomenological fields which are subsequently identified with the free current density and the polarization density tensor. It is shown that as expected, the average current density is the sum of the free current density and a dipolar contribution and that the average field fulfill the Maxwell equations. The macroscopic energy-momentum tensor of field is shown to be equal to the standard empty-space energy-momentum tensor built with the macroscopic fields plus a dipolar correction. The density of momentum of the field is confirmed to be given by Minkowski’s expression. The energy-momentum tensor of macroscopic matter is equal to the average of the microscopic energy-momentum tensor of matter plus the difference between the average tensor of microscopic fields and the macroscopic tensor of fields.
1 Introduction

The determination of which is the correct expression for the momentum of photons with energy $E$ in a material media has been, for many years and unsettle issue actively researched \[1, 2, 3, 4\]. The dispute is between the values $nE/c$ and $E/nc$ with $n$ the refraction index, associated respectively with the names of Minkowski and Abraham each of whom proposed at the beginning of the last century an expression for the energy-momentum tensor \[5, 6\]. Although most of the experimental results support Minkowski expression some theoretical arguments related with the movement of the center of mass of matter-field systems prove to have a strong convincing power in the mind of many people \[7, 2, 4\]. These arguments are closely related to the fact that Minkowski tensor is non-symmetric whereas Abraham’s object is symmetric and are consequence of supposing that the center of mass (energy) of a matter-field system should always be inertial.

The theoretical arguments supporting Abraham expression turn out to be wrong \[8, 9\]. Recently, we observed that the usual forms of the center of mass motion theorem may be violated if the energy-momentum of the system is non-symmetric \[8\] and that this violation occurs in some electromagnetic systems. Motivated by this observation we were able \[9\] to present the correct energy-momentum tensor for the electromagnetic field in a material media which is consistent with the microscopic expression of the Lorentz force. It results to be non-symmetric. The obtained expression differs of Minkowski proposal in the diagonal terms but defines the same momentum density. In this form the controversy on the electromagnetic momentum in material media is resolved in favor of Minkowski’s expression. In these works, we also stressed that Abraham’s object is not a Lorentz tensor, a fact that was already pointed out in the literature \[10\] but has not received the attention that it deserves.

In the second mentioned work, we first obtain the Lorentz invariant expression of the force density assuming that the force on a non-polarized but magnetized element of material is equal to the Lorentz force on a magnetic dipole with the same magnetic moment. Then the energy-momentum tensor is obtained by consistency with Maxwell equation and imposing that Newton’s third law between field and matter holds. In this approach we suppose that charges can be separated into free charges and bound charges. The total bound charge of any piece of material is always zero, so that the bound charge at the surface is opposite to the bound charge in the bulk. This leads
to the definition of a polarization field $\mathbf{P}$ such that the densities of dipolar (bound) charges are $\sigma_d = \mathbf{P} \cdot \hat{n}$ in the surface and $\rho_d = -\nabla \cdot \mathbf{P}$ in the bulk. The current density of bound charges must then be $\mathbf{j}_d = \dot{\mathbf{P}} + c \nabla \times \mathbf{M}$.

To understand better the nature of the energy-momentum tensor obtained it is convenient to look for alternative deductions. The traditional treatment of the phenomenological fields, $\mathbf{P}$ and $\mathbf{M}$ which appear in the macroscopic Maxwell equations is to obtain those fields by making averages of microscopic electric and magnetic dipoles. However, in this approach the covariant treatment presents some problems in how to define consistently the dipole moments from microscopic charges and currents, particularly when the dipoles are moving or changing in time [11, 12]. Instead of this in the present paper we present a third approach which is independent of any microscopic model of the material and does not require to use the force on a magnetic dipole. We suppose that the macroscopic quantities may be obtained by taking the average of microscopic fields and forces over small regions of space-time and that the macroscopic force density may be expanded linearly in the macroscopic fields and their derivatives. In the dipolar approximation only the first derivatives are relevant. We define the free charge current density as the four-vector that couples to the average field and the polarization density tensor as the parameters that couple to the first derivatives in the expansion. Then we obtain self-consistently the expressions for the free and bound (dipolar) charge and current densities and for the energy-momentum tensor. Finally we show that the polarization tensor that was defined as the coupling parameters of the force should indeed be interpreted as the polarization-magnetization tensor.

The inconsistencies between Maxwell equations and the theoretical arguments supporting Abraham’s point of view led also to a long lasting controversy about which is the true force that is exerted on the dipole densities and how the total energy-momentum tensor has to be split in the part assigned to the macroscopic electromagnetic field and the part assigned to matter [2]. Some authors even consider that such splitting may be done arbitrarily [2] or that both definitions of the photon momentum may be used meaningfully depending on the context [13]. In our opinion this discussions should be cut off once one realizes that the macroscopic force density is determined by the microscopic Lorentz force and that the use of Newton’s third law between matter and field unequivocally determines the splitting of the energy-momentum tensor.
2 Space-time average

At a microscopic scale, we assume that for the electromagnetic field \( F^{\mu\nu} \) and the current density \( j^\mu \), Maxwell equations and Bianchi’s identity hold

\[
\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu ,
\]

\[
\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 ,
\]

and that the force density is given by Lorentz expression

\[
f^\mu = \frac{1}{c} F^{\mu\alpha} j^\alpha .
\]

We use the metric \( \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) and Gauss units. Correspondingly the energy-momentum tensor for the microscopic electromagnetic field is the standard tensor given by [14],

\[
T^{\mu\nu}_S = -\frac{1}{16\pi} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + \frac{1}{4\pi} F^{\mu\alpha} F^{\nu\alpha} .
\]

which satisfy identically \( \partial_\nu T^{\mu\nu}_S = -f^\mu \) as a consequence of (1), (2) and (3). The force density can alternatively be obtained from the energy-momentum tensor of matter by \( f^\mu = \partial_\nu T^{\mu\nu}_M \) if it is available.

The observable macroscopic quantities correspond to averages of the microscopic quantities over small regions of space-time in the form,

\[
\langle A(x) \rangle = \int A(x') W(x - x') \, dx' ,
\]

where \( W(x) \) is a smooth function that is essentially constant inside a region of size \( R \) and that vanishes outside

\[
W(x) \geq 0 , \quad |x^\mu| > R \implies W(x) = 0 , \quad \int W(x) \, dx = 1 .
\]

Inside the region \( W(x) \approx \langle W \rangle \).

At the scale \( R \) the microscopic fluctuations are washed out, and therefore all the products of averages and fluctuations are negligible. That is, if \( \delta A = A - \langle A \rangle \) then \( \langle \langle B \rangle \delta A \rangle \approx 0 \) and also \( \langle \langle B \rangle \rangle \approx \langle B \rangle \).
With these conditions it follows that

\[ \partial_\nu \langle A \rangle = \int A(x') \partial_\nu W(x - x') \, dx' = - \int A(x') \partial'_\nu W(x - x') \, dx' \]

\[ = - \int \partial'_\nu [A(x') W(x - x')] \, dx' + \int \partial'_\nu A(x') W(x - x') \, dx' \]

\[ = \langle \partial_\nu A \rangle . \]  \hspace{1cm} (7)

We denote the macroscopic fields with a bar. In particular, \( \bar{F}^{\mu\nu} = \langle F^{\mu\nu} \rangle \).

Using (7) one immediately obtains that Maxwell equations (1) are valid for macroscopic fields and the averaged currents

\[ \partial_\nu \bar{F}^{\mu\nu} = \frac{4\pi}{c} \langle j^\mu \rangle \]  \hspace{1cm} (8)

and that Bianchi’s identity is valid for \( \bar{F}^{\mu\nu} \) and also for \( \delta F^{\mu\nu} = F^{\mu\nu} - \bar{F}^{\mu\nu} \).

3 Macroscopic force

As mentioned above, reported apparent inconsistencies in the description of the electromagnetic field in material media led some authors to propose alternative forms of the force which the electromagnetic field exerts on a material media. These where pioneered by Einstein and Laub [15] in 1907 although it seems that some years later Einstein retracted of the suggestion [16]. These proposals have problems either with Lorentz invariance or with being quadratic in the magnetization. The related hypothesis of the existence of a hidden momentum [17] has also problems with the mechanical interpretation of its behavior. In this letter we avoid any controversy about this issue by making the sole hypothesis that the microscopic force is given by the Lorentz expression. From this we obtain a definite expression of the force density in the macroscopic description which depends on a phenomenological tensor \( D^{\mu\nu} \) whose components will be later on identified with the magnetization and polarization vectors. Since we are supposing that momentum is conserved, the matter and field portions of the energy-momentum tensor may be the characterized by supposing that Newton’s third law holds between field and matter.

In order to obtain the average current density and the macroscopic energy-momentum tensors of fields and matter we compute first the average of the
microscopic forces. Since the force (3) is quadratic in the microscopic quantities the result in this case is not straightforward. We assume that the macroscopic force density $\bar{f}^\mu$ on matter shares with the microscopic force density the following four properties that survive the averaging process. 1) It is a local functional of the average field. 2) It is linear in the field. 3) The direction of the force should be determined by the field. 4) It transforms as a four vector. Therefore the force density can be expanded as a sum of terms in which the derivatives of the field tensor are contracted with matter-dependent tensors and the free index belongs to the field derivative tensors. In the dipolar approximation, we neglect the coupling with second and higher order derivatives which corresponds to quadrupolar and higher order couplings. Taking into account Bianchi’s identity, it can be proved that the most general expression for the macroscopic force density that fulfills the four stated above conditions takes the form

$$\bar{f}^\mu = \frac{1}{c} \bar{F}^\mu_\alpha j^\alpha_f + \frac{1}{2} \partial^\mu \bar{F}^{\alpha\beta} D_{\alpha\beta}.$$  \hspace{1cm} (9)$$

It is written in terms of two phenomenological quantities related to matter, a four-vector $j^\alpha_f$ and an antisymmetric four-tensor $D_{\alpha\beta}$ to be identified later in a consistent way respectively with the current density of free charges and the dipolar tensor density. Note that then the temporal component of $D_{\alpha\beta}$ will be identified with the polarization vector $P$, $P_k = D_{0k}$, and the spatial components will be in correspondence with the magnetization vector $M$ by $D_{ij} = \epsilon_{ijk}M_k$.

With the use of some simple algebra and Bianchi’s identity the force expression takes the alternative form

$$\bar{f}^\mu = \frac{1}{c} \bar{F}^\mu_\alpha (j^\alpha_f + c \partial_\beta D^{\alpha\beta}) + \partial_\beta (\bar{F}^\mu_\alpha D^{\beta\alpha}).$$  \hspace{1cm} (10)$$

In the microscopic formulation the total energy momentum tensor $T_{\mu\nu}$ splits naturally in a contribution of the field given by (4) and a contribution $T_{\mu\nu}^M$ of the matter. The average of the total microscopic energy-momentum tensor is the macroscopic energy momentum tensor, but how it splits in a macroscopic matter term $T_{\mu\nu}^M$ and a macroscopic field term $T_{\mu\nu}^F$ should be determined by Newton’s third law. So we write,

$$\bar{T}^\mu_{\alpha\beta} + \bar{T}^\mu_{M\nu} = \langle T_{S}^\mu_{\alpha\beta} + T_{M}^\mu_{\nu} \rangle.$$  \hspace{1cm} (11)$$

and impose,

$$\partial_\beta \bar{T}^{\mu\beta}_{M\nu} = \bar{f}^\mu = -\partial_\beta \bar{T}^{\mu\beta}_{F\nu}.$$  \hspace{1cm} (12)$$
In particular the average of the microscopic energy-momentum tensor of electromagnetic fields is not the macroscopic energy-momentum of the field. It may be expressed as,
\[
\langle T_{\mu\nu}^S \rangle = \bar{T}_{\mu\nu}^S + \langle \delta T_{\mu\nu}^S \rangle ,
\] (13)
where \( \bar{T}_{\mu\nu}^S \) is the standard tensor built with the macroscopic fields and \( \delta T_{\mu\nu}^S \) is the standard tensor built with the fluctuation fields \( \delta F_{\mu\nu} \). For the actual macroscopic energy-momentum tensor of fields inside matter we write,
\[
\bar{T}_{\mu\nu}^F = \bar{T}_{\mu\nu}^S + \Delta T_{\mu\nu}^F ,
\] (14)
where \( \Delta T_{\mu\nu}^F \) is a possible polarization-dependent correction that will be determined self-consistently. The terms linear in the fluctuation were neglected as explained above. To obtain the macroscopic energy-momentum tensor of matter, the average fluctuation tensor of the field should be included and the dipolar correction must be subtracted,
\[
\bar{T}_{\mu\nu}^M = \langle T_{\mu\nu}^M \rangle + \langle \delta T_{\mu\nu}^S \rangle - \Delta T_{\mu\nu}^F .
\] (15)
Note that the fluctuation fields \( \delta F \) should be considered part of the macroscopic matter. This is the reason why \( \bar{f}_\mu \neq \langle f_\mu \rangle \).

Using (12), the macroscopic force density is
\[
\bar{f}_\mu = \partial_\beta \bar{T}_{\mu\beta}^M = \langle f_\mu \rangle + \partial_\beta [\langle \delta T_{\mu\beta}^S \rangle - \Delta T_{\mu\beta}^F] .
\] (16)
Using (11) and (3) the average microscopic force density can be expressed as
\[
\langle f_\mu \rangle = \frac{1}{4\pi} \langle F_{\alpha\beta} \partial_\gamma F^{\alpha\beta} \rangle .
\] (17)
Making the substitution \( F_{\alpha\beta} = \bar{F}_{\alpha\beta} + \delta F_{\alpha\beta} \) one gets
\[
\langle f_\mu \rangle = \frac{1}{4\pi} \bar{F}_{\alpha\beta} \partial_\gamma \bar{F}^{\alpha\beta} + \frac{1}{4\pi} \langle \delta F_{\alpha\beta} \partial_\gamma \delta F^{\alpha\beta} \rangle .
\] (18)
Using (8) in the first term of this last equation and Bianchi’s identity in the second, the average microscopic force density is then
\[
\langle f_\mu \rangle = \frac{1}{c} \bar{F}_{\alpha} \partial_\gamma \bar{F}^{\alpha} + \frac{1}{c} \partial_\gamma \langle j_\alpha \rangle - \langle \partial_\gamma \delta T_{\mu\beta}^S \rangle .
\] (19)
Substituting this result in (16) and equating with (11) one finally gets
\[
\frac{1}{c} \bar{F}_{\alpha} (j_\alpha^\alpha + c \partial_\beta D^{\beta\alpha} - \langle j_\alpha^\alpha \rangle) = -\partial_\gamma (\bar{F}_{\alpha} D^{\beta\alpha} + \Delta T_{\mu\beta}^F) .
\] (20)
This last equation is satisfied identically by setting

\[ \langle j^\alpha \rangle = j^\alpha_f + c \partial_\beta D^{\alpha \beta} \]  

and

\[ \Delta T^\mu_\nu = - F^\mu_\alpha D^{\nu \alpha} . \]  

These solutions are unique. In fact, since (20) must be satisfied for any field tensor, equation (21) follows. A tensor \( X^{\mu \nu} \) whose divergence vanish identically \( (\partial_\nu X^{\mu \nu} \equiv 0) \) could be added to \( \Delta T^\mu_\nu \), but since \( T^\mu_\nu \) can be at most quadratic in the fields, and should vanish when the fields vanish such a tensor also vanishes.

Defining the dipolar current density

\[ j^\alpha_d = c \partial_\beta D^{\alpha \beta} . \]  

Maxwell equations for the macroscopic fields are written in the familiar form,

\[ \partial_\nu \bar{F}^{\mu \nu} = \frac{4 \pi}{c} (j^\mu_f + j^\mu_d) . \]  

Noting that in terms of \( P \) and \( M \) defined above, the also familiar expressions \( \rho_d = - \nabla \cdot P \) and \( j_d = \bar{P} + c \nabla \times M \) are obtained, the identification of \( j^\alpha_f \) as the current of free charges and \( D^{\nu \alpha} \) with the dipolar density is almost completed.

The dipolar or bound charge is conserved identically, \( \partial_\alpha j^\alpha_d = c \partial_\alpha \partial_\beta D^{\alpha \beta} = 0 \). Since the average charge is conserved, equation (21) implies that the free charge is also conserved, \( \partial_\alpha j^\alpha_f = 0 \).

It is worth noting that this whole scheme leaves out processes, like ionization or capture of carriers, in which there is exchange between free charges and bound charges. In those cases (9) does not hold.

Now we proceed to discuss the energy-momentum tensor. It is convenient to define the tensor

\[ H^{\alpha \beta} = \bar{F}^{\alpha \beta} - 4 \pi D^{\alpha \beta} , \]  

which is related to the electric displacement \( \mathbf{D} = \mathbf{E} + 4 \pi \mathbf{P} \) and the magnetizing field \( \mathbf{H} = \mathbf{B} - 4 \pi \mathbf{M} \). \( H^{0k} = D_k \) and \( H^{ij} = \epsilon_{ijk} H_k \). With this tensor Maxwell equations (11) become

\[ \partial_\nu H^{\mu \nu} = \frac{4 \pi}{c} j^\mu_f \]  

(26)
and the macroscopic energy-momentum tensor of fields is

\[ T^\mu\nu_F = T^\mu\nu_S + \Delta T^\mu\nu_F = -\frac{1}{16\pi} \eta^{\mu\nu} \tilde{F}_{\alpha\beta} \tilde{F}^{\alpha\beta} + \frac{1}{4\pi} \tilde{F}_\mu^\alpha H^{\mu\alpha}. \] (27)

This is exactly the result we obtained in Ref. [9].

4 Densities of dipole moment

To completely recover the usual picture let us show that \( P \) and \( M \) are the densities of electric dipole moment and of magnetic dipole moment respectively without relying on the model of microscopic dipoles.

First we find the charges at the surface of a piece of material. Near the surface \( P \), go smoothly to zero in a distance of the order of \( R \). The total dipolar charge is obtained by integrating \( \rho_d = -\nabla \cdot P \) over the volume of the material. Because of Gauss’ theorem such integral is zero since \( P = 0 \) at the surface. Then the total dipolar charge is always zero. It is a bound charge that cannot leave the material. At a macroscopic scale (much bigger than \( R \)) there is a discontinuity at the surface. The dipolar charge at the surface is always opposite to the charge in the bulk. If \( \sigma_d \) is the surface density of polarization, then

\[ 0 = \oint \sigma_d dS - \int \nabla \cdot P dV = \oint [\sigma_d - P \cdot \hat{n}] dS. \] (28)

This can happen for any surface if

\[ \sigma_d = P \cdot \hat{n}, \] (29)

which of course is the usual expression given by the model of microscopic dipoles. The electrical dipole moment of a piece of material is computed as always giving

\[ d = \oint x\sigma_d dS + \int x\rho_d dV = \int P dV \] (30)

This completes the interpretation of \( P \) as the density of dipole moment.

The magnetic current density in the bulk is \( j_M = c\nabla \times M \). In addition there is also a surface current density \( \Sigma_M \) that can be obtained in a way similar to that used for \( \sigma_d \). Let us consider a closed curve that is outside the
piece of material and is the border of a surface that cuts the material. The total magnetic current that crosses the surface is the flux of \( j_M \). Because of Stokes’ theorem such flux is the circulation of \( cM \) in the curve which is zero. Therefore for any surface \( S \) that cuts the material the total magnetic current that crosses the surface is zero. At a macroscopic scale the bulk current is opposite to the surface current. Let \( C \) be the intersection of \( S \) with the surface of the piece of material, \( \mathbf{t} \) the unitary vector tangent to the curve \( C \) and \( \mathbf{n} \) the unitary vector orthogonal to the surface of the piece. The unitary vector which is orthogonal to \( C \) and tangent to the surface of the piece is \( \mathbf{n} \times \mathbf{t} \). Then

\[
0 = \oint_C \Sigma_M \cdot \mathbf{n} \times \mathbf{t} \, dl + \int_S c\nabla \times M \cdot dS
= \oint_C [\Sigma_M \cdot \mathbf{n} \times \mathbf{t} + cM \cdot \mathbf{t}] \, dl
= \oint_C [\Sigma_M \times \mathbf{t} + cM] \cdot \mathbf{t} \, dl .
\]

(31)

The expression that fulfills this equation for any \( \mathbf{t} \) is

\[
\Sigma_M = cM \times \mathbf{n} .
\]

(32)

The usual computation of the magnetic dipole moment of the magnetic currents,

\[
m = \frac{1}{2c} \oint x \times \Sigma_M \, dS + \frac{1}{2c} \int x \times j_M \, dV = \int M \, dV .
\]

(33)

completes the interpretation of \( M \) as the density of magnetic dipole moment.

5 Conclusion

Recently we have presented a deduction of the energy momentum-tensor of the electromagnetic field in matter \cite{9} that should put an end to the long dated Abraham-Minkowski controversy. Our result confirms Minkowski’s expressions \( g_{\text{Min}} = \mathbf{D} \times \mathbf{B}/4\pi c \) for the momentum density of the field in matter and \( p = nE/c \) for the momentum of a photon of energy \( E \), \( n \) being the refraction index. Moreover our result also predicts that an electromagnetic wave incident on a dielectric block will pull the block instead of pushing it.
This invalidates Balazs’s famous argument in favor of Abraham momentum and has the simple physical explanations that dielectric tends to move in the direction of higher field. Due to the many arguments that have been discussed along the years on this issue and also to get a better understanding of the nature of result obtained we found convenient to look for an alternative deduction of the energy-momentum tensor that we report in this paper.

By taking averages over small regions of space-time we have obtained the laws of macroscopic electromagnetism, including the expressions for the force and the energy-momentum tensor that previously were in dispute for a long time [2]. Our derivation is independent of any particular model of microscopic matter. We show that supposing the validity of the Lorentz force at the microscopic level, there is a unique result which is compatible with the validity of Maxwell equations and Bianchi’s identity, both at the microscopic and the macroscopic levels, and for which the macroscopic force is linear in the macroscopic field and its derivatives.

In our approach the polarization tensor is introduced as a phenomenological tensor which couples with the field derivatives in the expansion of the force expression. Then the dipolar current density and the energy-momentum tensor are determined by consistency. Finally it is shown that the polarization tensor can be indeed interpreted as the density of dipolar moments.

The force on the dipoles is not the same as the force on the dipolar charges and currents. To see why this is true, consider a piece of material subdivided into infinitesimal elements. For each element the dipolar moments are determined by the surface charges and currents, and so it is the force. The total force on the piece is the sum of the forces on each element, that is, it can be calculated by integrating the force density (9). In contrast the contributions to the field due to the surface charges and currents of adjacent elements cancel out; what remains are the contributions of the bulk and of the external surface of the piece of material. The present treatment shows that it is inconsistent to assume, as it has been proposed for example in [18], that the force on a polarized material is the Lorentz force on the total current. If the force is of the Lorentz kind then it must be that \( \mathbf{P} = \mathbf{M} = 0 \).

Our results for the force density and the energy-momentum tensor agree exactly with those reported by us in Ref. [9], with the methodological advantage that in the present derivation we do not make use of the expression for the force density on the matter dipoles.
Our expression for the energy density

\[ u = T^0_0 = \frac{1}{8\pi} (E^2 + B^2) + \mathbf{P} \cdot \mathbf{E} \]  

(34)
is different from the energy density of fields without dipolar coupling. The difference \( \mathbf{P} \cdot \mathbf{E} \) is the opposite of the electrostatic energy density of the polarization. This energy is therefore subtracted from the energy of the field and considered part of the energy of matter. Note that there is no similar magnetic term, since no potential magnetic energy exist. Equation (34) is also different from Poynting’s expression, \( u_P = (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})/8\pi \), but our energy current density is the Poynting vector \( \mathbf{S} = c\mathbf{E} \times \mathbf{H}/4\pi \). The energy conservation equation in our formulation

\[ \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}_f + \mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} \]  

(35)
differs also from Poynting conservation equation which is given by

\[ \frac{\partial u_P}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}_f \]  

(36)

Our results are valid for any kind of material in any kind of condition (except when there is exchange between free and bound charges): ferromagnets, saturated paramagnets, electrets, matter moving or at rest, solids, fluids, absorbing and dispersive media, etc. Poynting’s equation is obtained from our result in the particular case of linear polarizabilities. In such a case the dipolar contributions to the power density on the electromagnetic field can be integrated, yielding the opposite of the energy density increase of matter when it is polarized. Such an energy increase should be added to the pure field energy density to obtain Poynting’s energy density. Therefore, Poynting’s energy density corresponds to a mixed entity composed of “fields plus polarizations”. On the other hand Minkowski’s is the only relativistic tensor for which \( T^0_0 = u_P \) and \( cT^{0k} = S^k \) in any frame of reference. For an electromagnetic wave propagating in a medium it makes sense to include the polarization energy of matter as part of the wave energy. So Minkowski’s tensor properly represents the energy-momentum tensor of the EM wave in a non-dispersive medium. It can be useful for calculating the theoretical force on the wave, but it cannot be used for determining the force on matter, since the wave energy has a component which belongs to the matter. For doing that one has to use our tensor. Nevertheless, note that in general, only
the force on matter can be measured. Magnetostriction and electrostriction are also good examples of the advantages of our approach. They are not described by Minkowski’s tensor [1], but may be calculated with our tensor.

Poynting’s equation does not hold when the polarizabilities are not linear. For example it is well known that this is what occurs in the case of optical dispersive media [11, 19]. Our result should be a good starting point for a fresh approach to study such cases.

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