A Novel Reconstruction Algorithm based on Fractional Fourier Transform for Unlimited Sampling

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Abstract

The recovery of bandlimited signals with high dynamic range is a hot issue in sampling research. The unlimited sampling theory expands the recordable range of traditional analog-to-digital converters (ADCs) arbitrarily, and the signal is folded back into a low dynamic range measurement, avoiding the saturation problem. We study the unlimited sampling problem of high dynamic non-bandlimited signals in the Fourier domain (FD) based on the fractional Fourier transform (FRFT). First, the modular nonlinear folding is performed in the fractional Fourier domain (FRFD) for modular arithmetic. Then, the fractional spectrum is estimated for any folding time by means of annihilation filtering. Finally, a novel unlimited sampling algorithm in the FRFD is obtained. The results show that the non-bandlimited signal can be reconstructed in the FD based on the FRFT, and it is not affected by the ADC threshold.

Keywords: Fourier transform, Fractional Fourier transform, unlimited sampling, self-reset ADC, non-linear reconstruction

1. Introduction

In signal processing, sampling \cite{1,2,3} is the primary task faced in the process of digitizing the signal. Since Shannon’s sampling theorem \cite{4} was proposed, sampling theory has been developed for more than 70 years, and its theoretical results \cite{5,6,7,8} are so rich that it has become one of the research hotspots in the field of signal processing. From a practical standpoint, point-wise samples of the function

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are obtained using the analog-to-digital converter (or the ADC), but the ADC [9] has a limited dynamic range \([-\lambda, \lambda]\). Whenever the signal exceeds a certain preset threshold \(\lambda\), the ADC will saturate, and the aliased signal will be clipped due to clipping (Fig. 1(a)) [10, 11]. Since most signals in practical applications are not limited by broadband, the dynamic range is very wide, so self-reset ADC (S-ADC) (Fig. 1(b)) was proposed [12–14]. Each time the input signal reaches the upper (lower) saturation limit, these S-ADCs will be reset to the corresponding other thresholds, it allows the S-ADC to reset rather than saturate, resulting in analog sampling. When the signal reaches the upper (lower) threshold point, it will fold backward (front) by an integer multiple of \(2\lambda\). This phenomenon is equivalent to modulo arithmetic on the input signal, which is very helpful for processing high dynamic range signals.

In view of the S-ADC’s ability to process high dynamic range signals, Bandari et al. [15] recently made the first pioneering contribution. He proposed the unlimited sampling theorem and developed the first provable refactor the guaranteed algorithm. The results of [15] have led to a lot of follow-up work, and the theoretical research of unlimited sampling has gradually enriched [16–21]. In this modular sampling framework, many scholars have studied the sampling and reconstruction of bandlimited functions and smooth functions under different backgrounds. [22] It is shown that the bandlimited function is uniquely characterized by modular samples under certain conditions. [23] The recovery of quantization modulus samples is studied by using edge information. [24] gave the mode sam-
pling theory of S-ADC sparse signal. [25] The unlimited sampling method based on wavelet is suitable for general smooth signals, not limited to bandlimited signals. [26] is mainly applicable to bandlimited signals in the Fourier domain (FD) on unlimited sampling method.

Most of the above unlimited sampling frameworks are based on bandlimited signal, but there are few articles on non-bandlimited signals. For various applications of non-bandlimited signal model, the original results are not directly applicable. According to [27], as a generalized form of the Fourier transform (FT), the fractional Fourier transform (FRFT) can expand the signal range applicable to traditional sampling theory, because the non-bandlimited signal in FD may be bandlimited in the fractional Fourier domain (FRFD). Therefore, the application of traditional sampling theory to non-bandlimited signals cannot achieve optimal results, so it is very necessary to study the unlimited sampling theory of non-bandlimited signals.

Based on the above reasons, we put forward the problem on unlimited sampling of high dynamic non-bandlimited signals in the FD based on the FRFT, we innovate the content of [26], and propose a new FRFD reconstruction theory of high dynamic non-bandlimited signals in the FD based on unlimited sampling framework. First, the modular nonlinear folding is performed in the FRFD for modular arithmetic. Second, the fractional spectrum is estimated for any folding time by means of annihilation filtering. Finally, a novel unlimited sampling algorithm in the FRFD is obtained. This paper is organized as follows. In Section 2, we describe briefly the FRFT, unlimited sampling theorem in the FD and the periodic bandlimited signal model in the FRFD. The FRFD reconstruction theorem on unlimited sampling is proposed in Section 3, we conclude this paper in Section 4.

2. Preliminaries

The Subsection 2.1 will first introduce the FRFT including definition and time shift property in FRFD. Then, the unlimited sampling method of bandlimited signals in the FD will be presented in Subsection 2.2. The periodic bandlimited signal model based on FRFT will be presented in Subsection 2.3.

2.1. Fractional Fourier transform

The FRFT of a signal $x(t)$ with angle $\alpha$ is defined as [28]

$$X_\alpha(u) = \mathcal{F}_\alpha [x(t)] (u) = \int_{-\infty}^{+\infty} x(t) K_\alpha(u,t) dt,$$  (1)
where $F_\alpha$ is the FRFT operator, $u$ stands for fractional frequency, $K_\alpha(u, t)$ denotes the kernel function of the FRFT

$$K_\alpha(u, t) = \begin{cases} A_\alpha e^{j(\cot \frac{\alpha}{2} t^2 - \csc \alpha ut + \cot \frac{\alpha}{2} u^2)}, & \alpha \neq k\pi \\ \delta(t - u), & \alpha = 2k\pi \\ \delta(t + u), & \alpha = (2k - 1)\pi \end{cases}$$

(2)

where $A_\alpha \triangleq \sqrt{\frac{1 - j\cot \alpha}{2\pi}}$. The rotation angle of FRFT is expressed as $\alpha = \frac{p\pi}{2}$ and $p$ is the order of FRFT. The domain $0 < \alpha < \frac{\pi}{2}$ are called fractional Fourier domains in [29], and this definition is also adopted in this paper.

The FRFT can be understood as the rotation of the time-frequency plane. The essence of the FRFT of a signal is to decompose the signal with the chirp signal $K_\alpha(u, t)$ as the basis function. According to the FRFT of the signal $x(t)$, it can be determined whether it is bandlimited in the fractional domain.

The FRFT has linear transform additivity, namely

$$F_{\alpha + \beta} [x(t)] (u) = F_\alpha [x(t)] (u) \cdot F_\beta [x(t)] (u) = X_\alpha(u) \cdot X_\beta(u).$$

(3)

It can be seen that the inverse transform of the FRFT relative to the $\alpha$ angle is the FRFT with the parameter $-\alpha$ angle, we have

$$x(t) = F_{-\alpha} \{X_\alpha(u)\} = \int_{-\infty}^{+\infty} X_\alpha(u)K_{-\alpha}(u, t)du.$$ 

(4)

When $\alpha = -\frac{\pi}{2}$, the FRFT degenerates to the traditional inverse FT. When $\alpha = \frac{\pi}{2}$, the FRFT degenerates to traditional FT, $X_{\frac{\pi}{2}}(u) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ut}dt$. When $\alpha = 0$, the FRFT degenerates to an identity transformation, $X_0(u) = x(t)$. When $\alpha = \pi$, the FRFT degenerates to the inversion of the signal with respect to the time axis, $X_{\pi}(u) = x(-t)$.

A signal $x(t)$ is called $\Omega_\alpha$ bandlimited signal in the FRFT sense [27], which means

$$X_\alpha(u) = 0, \quad |u| > \Omega_\alpha,$$

(5)

where $\Omega_\alpha$ is called the bandwidth of signal $x(t)$ in the FRFD. It has been shown that if the signal is bandlimited in the $\alpha$th FRFD, it can’t be bandlimited in the
FRFT with another angle $\beta$, where $\beta \neq \pm \alpha + n\pi$ for any integer $n$ [27].

The discrete time representation $x(nT_s), n \in \mathbb{Z}$ of the signal $x(t)$ can be obtained by uniformly sampling at intervals of $T_s$. The discrete-time FRFT of the $\alpha$ angle of the discrete-time signal $x(t)$ is defined as follows

$$X_{\alpha,s} = \mathcal{F}_\alpha [x(nT_s)](u) \triangleq \sum_{-\infty}^{+\infty} x(nT_s)K_\alpha(u, nT_s)dt,$$

where $T_s$ is the sampling period.

2.2. Unlimited sampling theorem in Fourier domain

Bandari et al. first proposed the central modular operation mapping definition in [15], which means

$$\mathcal{M}_\lambda: g \mapsto \lambda \left( \left\lfloor \frac{g}{2\lambda} \right\rfloor + \frac{1}{2} \right) - \frac{1}{2}, \quad \left\lfloor g \right\rfloor \overset{\text{def}}{=} g - \lfloor g \rfloor,$$

where $\left\lfloor g \right\rfloor$ defines the fractional part of $g$ and $\lambda > 0$ is the ADC threshold. Note that (7) is a nonlinear modulus mapping, which converts a smooth function into a discontinuous function. It is equivalent to a centered modulo operation since $\mathcal{M}_\lambda(g) \equiv g \mod 2\lambda$. By implementing the mapping (7), it is clear that out of range amplitudes are folded back into the dynamic range $[-\lambda, \lambda]$.

Let’s review some important propositions and conclusions in [16, 26].

Proposition 1 (Modular decomposition property) [16] Let $g \in B_\Omega$ (Space of $\Omega$-bandlimited functions) and $\mathcal{M}_\lambda(\cdot)$ be defined in (7) where $\lambda$ is a fixed, positive constant. Then, the bandlimited function $g(t)$ admits a decomposition

$$g(t) = z(t) + \varepsilon_g(t),$$

where $z(t) = \mathcal{M}_\lambda(g(t))$ and $\varepsilon_g(t)$ is a simple function

$$\varepsilon_g(t) = 2\lambda \sum_{m \in \mathbb{Z}} e[m] \mathbf{1}_{D_m}(t), \quad e[m] \in \mathbb{Z},$$

where $\bigcup_{m \in \mathbb{Z}} D_m = \mathbb{R}$ is a partition of the real line into intervals $D_m$.

In fact, the process of solving discontinuities is very critical. Proposition 1 just proves this problem. Each bandlimited function, whether continuous or discrete, can be decomposed into the sum of the modular function and the stepwise residual of the simple function. Observe that the output signal $z(t)$ is actually the difference between $g(t)$ and a piecewise constant signal $\varepsilon_g(t)$.
Conclusions 1 (Fourier domain reconstruction) [26] Let $g \in B_\Omega$ be a $\tau$-periodic function. Suppose that we are given $Q$ modulo samples of where $y[k] = \mathcal{H}_\lambda(g(kT))$ folded at most $M$ times. Then a sufficient condition for recovery of $g(t)$ from $y[k]$ (up to a constant) is that, $T \leq \frac{\tau}{Q}$ and $Q \geq 2 \left( \frac{\Omega \sigma}{2\pi} + M + 1 \right)$.

As a generalized form of FT, FRFT expands the dimensions of traditional spectrum analysis with FT as the core. Different from the traditional FT which uses complex exponential signal as the basis function, the FRFT uses chirp signal as the basis function. This connotation determines that a non-bandlimited signal in the FD may be bandlimited in the FRFD. Therefore, the unlimited sampling study of non-bandlimited signals in the FD can be transformed into the theoretical study of bandlimited signals in the FRFD. The next step is to study the unlimited sampling theory of bandlimited signals in the FRFD.

2.3. Signal model

Based on the periodic signal model in the FD, this paper proposes a periodic signal model in the FRFD. We consider $\sigma$-periodic signal and $\Omega_\alpha$ bandlimited signal $x(t)$, that is

$$x(t) = x(t + \sigma). \quad (10)$$

It is well known that periodic signals can be extended to fractional Fourier series (FRFS) in FRFD [30]. Therefore, $x(t)$ can be written as [31]

$$x(t) = \sum_{|w| \leq R} \hat{X}_\alpha(w) \Phi_{-\alpha}(w, t), \quad (11)$$

where $\hat{X}_\alpha(w)$ is FRFS coefficient and

$$\Phi_{\alpha}(w, t) = \sqrt{\frac{\sin \alpha - j \cos \alpha}{\sigma}} e^{j \left( \frac{w^2}{\sigma^2} t^2 - \csc \alpha u_0 t + \frac{w^2}{\sigma^2} u_0^2 \right)}, \quad (12)$$

constitutes the basis for FRFS expansion for a $\sigma$-periodic signal $x(t)$, where $R = \left\lfloor \frac{\Omega_\alpha}{u_0} \right\rfloor$, $u_0 = \frac{2\pi \sin \alpha}{\sigma}$.

The sampling signal $x(t)$ obtains $Q$ modular samples at the sampling rate $T$ in the interval $t \in \left[ -\frac{\sigma}{2}, \frac{\sigma}{2} \right]$. The FRFS coefficients of the signal $x(t)$ are given by
in the FRFD

\[ \hat{X}_\alpha(w) = \begin{cases} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x(t) \Phi_\alpha(w, t) dt, & w \in \mathbb{E}_{R,Q}, \\ 0, & w \in \mathbb{I}_Q \setminus \mathbb{E}_{R,Q}, |w| > \Omega_\alpha \end{cases} \quad (13) \]

where the set \( \mathbb{I}_Q = \{0, 1, \ldots, Q - 1\} \) denote the set of \( Q \) contiguous integers, and \( \mathbb{E}_{R,Q} \) is given by

\[ \mathbb{E}_{R,Q} = [0, R] \cup [Q - R, Q - 1], |\mathbb{E}_{R,Q}| = 2R + 1. \quad (14) \]

The well-known Fourier series (FS) is just a special case of FRFS for \( \alpha = \frac{\pi}{2} \), please see [26]. In order to solve for \( \hat{X}_\alpha(w) \) in (13), we must require \( Q \geq 2R + 1 \).

Because of \( QT = \sigma \), so \( T \leq \frac{\sigma}{Q} \leq \frac{\sigma}{2R + 1} \).

The hypothesis of periodic signals in our paper only provides a practical method for recovering signals from folding measurements below. However, when the signal is aperiodic, the theoretical reconstruction guarantees that the aperiodic signal can also be expanded by discrete-time FRFT, but additional requirements are required for sampling samples, and this article will not expand in detail.

3. Reconstruction approach in fractional Fourier domain

In this Section, we will study the unlimited sampling theory of bandlimited signals in the FRFD. Firstly, the nonlinear modulus mapping is shown in Subsection 3.1. Then we compute the folding instants by an annihilation filter in Subsection 3.2. Finally, the fractional Fourier domain reconstruction method is given in Subsection 3.3. Here we make the following symbolic regulations: the sets of real, integer, and complex-valued numbers are denoted by \( \mathbb{R}, \mathbb{Z} \) and \( \mathbb{C} \), respectively.

3.1. Nonlinear modulus mapping

This paper uses the definition and properties of generalized modular non-linear mapping in formula (7) [15, 16, 26] in Subsection 2.2, this phenomenon is equivalent to modulo arithmetic on the input signal. And according to proposition 1 [16] in Subsection 2.2 gives the following form

\[ x(t) = \mathcal{M}_\lambda (x(t)) + v_x(t), \quad (15) \]
\[ v_x(t) = \sum_{m \in \mathbb{Z}} c[m] \mathbf{1}_{[t_m, t_{m+1}]}(t), \]  

(16)

where \( c[m] \in \mathbb{R} \), \( \mathbf{1}_{[t_a, t_b]} \) is the indicator function on \([t_a, t_b] \), \( t_m \in \left[ -\frac{T}{2}, \frac{T}{2} \right] \) denotes the folding instants with \( t_a < t_b \).

Obviously, the output signal \( M_x(x(t)) \) is actually the difference between \( g(t) \) and a residual signal \( v_x(t) \). The \([16]\) requires that the correlation coefficient of the residual function \( v_x(t) \) is an integer multiple of \( 2\lambda \), while \([26]\) don’t need to make assumptions about its correlation coefficient. There is no need to make other assumptions about \( v_x(t) \) in this article.

Without loss of the generality, this article makes the following definitions, \( f[k] = x(kT) \), \( h[k] = M_x x(kT) \), \( v[k] = v_x(kT) \), we have

\[ f[k] = h[k] + v[k]. \]  

(17)

Next, we will study the unlimited sampling theory of bandlimited signals in FRFD in detail. According to \((15)\) and \((17)\), if \( v[k] \) is known, we can recover \( f[k] \) from \( h[k] \). This is the focus of this article. In this paper we develop a method which allows for inferring \( v[k] \) from \( h[k] \).

Similar to the phase unwrapping theory, we can obtain the following fact from the Itoh’s condition \([32]\). when the max-norm of the first-order finite difference of the samples is bounded by \( \lambda \) or \( |f[k+1] - f[k]| \leq \lambda \), the first-order finite difference operator on the modular sequence can be reversed operation to restore. This is a definite result. Therefore, let \( \Delta^N f = \Delta^{N-1} (\Delta f) \) denote the \( N \)th difference operator with \( \Delta f = f(k+1) - f(k) \), we have

\[ \overline{f}[k] = \Delta f[k] = f[k+1] - f[k], \]  

(18)

\[ \overline{h}[k] = \Delta h[k] = h[k+1] - h[k], \]  

(19)

\[ \overline{v}[k] = \Delta v_x(kT). \]  

(20)

From \((17)\), we obtain

\[ \overline{f}[k] = \overline{h}[k] + \overline{v}[k] \]

\[ = \overline{h}[k] + \sum_{m \in M} c[m] \delta(kT - t_m), \quad k \in \mathbb{Q}, \]

(21)
where $\delta$ denote the Dirac distribution, $t_m$ are unknown fold instant, the size of the set $M$ depends on the dynamic range of the signal relative to the threshold $\lambda$.

In this paper, the (21) is written as the FRFD

$$
\mathcal{H}_\alpha[n] = \begin{cases} 
\overline{F}_\alpha[n] - \overline{V}_\alpha[n], & n \in \mathbb{R},Q-1 \\
-\overline{V}_\alpha[n], & n \in \mathbb{I}_{Q-1}\setminus\mathbb{R},Q-1 
\end{cases}
$$

(22)

where $\overline{F}_\alpha, \overline{H}_\alpha, \overline{V}_\alpha$ are the FRFT of $f, h, v$ respectively. At the same time, the discrete FRFT form of $h[k]$ is given

$$
\mathcal{H}_\alpha[n] = \sum_{k \in \mathbb{I}_{Q-1}} \overline{h}[k] \Phi_\alpha(n,k)
$$

$$
= \sum_{k \in \mathbb{I}_{Q-1}} \sqrt{\sin\alpha - j\cos\alpha\sigma} \overline{h}[k] e^{j\left(\frac{\cot\alpha}{2}k^2 - \csc\alpha \overline{\mu_0}kn + \frac{\cot\alpha}{2}\overline{\mu_0}n^2\right)},
$$

(23)

where $\overline{\mu}_0 = \frac{2\pi \sin\alpha}{Q-1}$.

When $\alpha = \frac{\pi}{2}$, this transform is the discrete FT, see [26] for details.

From the above discussion, we can see that the discrete FRFS is divided into two parts in the FRFD, as shown below

i) $n \in \mathbb{R},Q-1$ (Fractional bandlimited part)

ii) $n \in \mathbb{I}_{Q-1}\setminus\mathbb{R},Q-1$ (Fractional domain unlimited sampling part)

If we want to recover $f[k]$ in this paper, we must solve $v[k]$, then (17) is transformed into solving (21), and the key to solving (22) is to find the value of the unknown folding instant $\{c[m], t_m\}_{m \in \mathbb{Z}}$. Need to emphasize here, $\overline{V}_\alpha[n]$ is Dirac function in FRFD, so we can infer that

$$
\overline{V}_\alpha[n] = \sum_{k \in \mathbb{I}_{Q-1}} \sum_{m \in M} c[m] \delta(kT - t_m) \Phi_\alpha(n,kT)
$$

$$
= \sum_{m \in M} c[m] B_\alpha e^{j\left(\frac{\cot\alpha}{2}t_m^2 - \csc\alpha \overline{\mu_0}t_m + \frac{\cot\alpha}{2}\overline{\mu_0}n^2\right)},
$$

(24)

where $B_\alpha = \sqrt{\frac{\sin\alpha - j\cos\alpha}{\sigma}}, \quad M = |M|$.

When $\alpha = \frac{\pi}{2}$, this transform is the discrete FT [26]. The estimation of the unknown parameters in (24) is called the spectral estimation problem [33, 34].
3.2. Computing the folding instants

From the above discussion, if we want to recover $v[k]$, we must find the value of the unknown folding instant $\{c[m], t_m\}_{m \in \mathbb{Z}}$. Thus, the problem (24) is reduced to computing $\{c[m], t_m\}_{m \in \mathbb{Z}}$.

The (24) is the spectral estimation problem. The commonly used spectrum estimation methods are annihilation filter (AF) [33, 35], ESPRIT [36], MUSIC [37], etc. Among them, AF is the most commonly used method in many theoretical analysis and practical applications. In principle, the signal reconstruction process is to use the obtained set of moments or fractional Fourier coefficients of the input signal to solve a spectrum problem to achieve an accurate estimation of the unknown parameter $\{c[m], t_m\}_{m \in \mathbb{Z}}$. Therefore, this paper uses an annihilation filter to solve the problem of spectrum estimation.

For convenience, (24) is written as follows

$$V_\alpha[n] = B_\alpha e^{\frac{j \cot \alpha}{2} \nu_0 n^2} \left( \sum_{m \in M} c[m] e^{\frac{j \cot \alpha}{2} \nu_0 t_m^2} \cdot e^{-j \csc \alpha u_0 T_2 t_m} \cdot \chi_m \cdot e^{-j \csc \alpha u_0 T_2 t_m} \cdot \Im(n) \right), \quad (25)$$

where

$$\Im(n) = \sum_{m \in M} \chi_m \varsigma_m^n. \quad (26)$$

The (26) is a classic spectrum estimation problem, which can be handled by an annihilation filter. When $\alpha = \pi/2$, see [26] for details.

AF is used to estimate the folding moment of the signal, and use the least square method to obtain the amplitude information of the information, so as to realize the reconstruction. It is known from the [33] that we can accurately estimate the unknown parameters $\chi_m$ and $\varsigma_m$ from $2K$ continuous non-zero measured values $\Im(n)$. The following is divided into two parts to solve separately.

i) Construct the filter $\{\Gamma[\vartheta]\}_{\vartheta=0,1,\cdots,M}$ so that its zero point is the parameter $s_m = \left\{ e^{-j \csc \alpha u_0 T_2 t_m} \right\}_{\vartheta=0}^{\vartheta=M-1}$, then the $z$ transform of the filter can be expressed as

$$\Gamma[z] = \prod_{m=0}^{M-1} (1 - s_m z^{-1}) = \sum_{\vartheta=0}^{M} \Gamma[\vartheta] z^{-\vartheta}. \quad (27)$$
It can be seen that the root of the polynomial is the parameter $\varsigma_m$. Therefore, this paper convolutes it directly, so it has

\[
(\Gamma \ast \Im) [n] = \sum_{\vartheta=0}^{M} \Gamma[\vartheta] \Im[n - \vartheta]
\]

\[
= \sum_{\vartheta=0}^{M} \sum_{m=0}^{M-1} c[m] e^{j \frac{\cos \alpha \theta T^2}{2} m} \cdot \Gamma[\vartheta] \cdot e^{-j \frac{\csc \alpha(n-\vartheta)\varsigma_m}{T} t_m}
\]

\[
= \sum_{m=0}^{M-1} c[m] e^{j \frac{\cos \alpha \theta T^2}{2} m} \sum_{\vartheta=0}^{M} \Gamma[\vartheta] e^{j \frac{\csc \alpha \vartheta \varsigma_m}{T} t_m} e^{-j \frac{\csc \alpha n\varsigma_m}{T} t_m} = 0.
\]  

(28)

We write (28) in the form of matrix-vector to obtain

\[
\begin{bmatrix}
\Im[M-1] & \Im[M-2] & \cdots & \Im[0] \\
\Im[M] & \Im[M-1] & \cdots & \Im[1] \\
\vdots & \vdots & \vdots & \vdots \\
\Im[N-1] & \Im[N-2] & \cdots & \Im[N-M]
\end{bmatrix}
\begin{bmatrix}
\Gamma[1] \\
\Gamma[2] \\
\vdots \\
\Gamma[M]
\end{bmatrix}
= -
\begin{bmatrix}
\Im[M] \\
\Im[M+1] \\
\vdots \\
\Im[N]
\end{bmatrix}
\]

(29)

where $\Im = \begin{bmatrix} \Im[0], \Im[1], \cdots, \Im[M] \end{bmatrix}^T$, $\Im[M] = 1$. The unique solution $a$ can be obtained $\Gamma[\vartheta], \vartheta = 1, 2, \cdots, M$, and finally the instantaneous folding time $\{t_m\}_{m \in \mathbb{Z}}$ can be obtained.

ii) In order to estimate the amplitude parameter $\chi_m$, extract $M$ continuous values from the known coefficient $\Im[n]$, that is to say, $m = 0, 1, \cdots, M$, and write the $\Im(n) = \sum_{m=0}^{M-1} \chi_m s_m^n$ in the form of a matrix-vector

\[
U \chi = \Im,
\]

(30)
$$\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
e^{j\frac{\sin \alpha}{2T}t_0^2} \\
e^{j\frac{\sin \alpha}{2T}t_1^2} \\
\vdots \\
e^{j\frac{\sin \alpha}{2T}t_{M-1}^2} \\
\end{bmatrix}
\begin{bmatrix}
c[0] \\
c[1] \\
\vdots \\
c[m-1] \\
\end{bmatrix}
= 
\begin{bmatrix}
\Im[0] \\
\Im[1] \\
\vdots \\
\Im[M-1] \\
\end{bmatrix}$$

(31)

where $U$ is vandermonde matrix, it is a matrix whose columns are geometric series. For any integer $a \neq b$ and $a, b = 0, 1, \cdots, M - 1$ satisfy $U_a \neq U_b$, $U$ is non-singular, at this time, (30) has a unique solution. It needs to be emphasized here that we generally use the least squares method to obtain an estimate of the amplitude information.

Through the above two steps, the instantaneous folding time $\{c[m], t_m\}_{m \in \mathbb{Z}}$ can be obtained. In this way, the key points of this article are solved, we can estimate $v[k]$ and reconstruct $f[k]$ from $h[k]$.

### 3.3. Unlimited sampling theorem in the fractional Fourier domain

Through the above research content, we have found the value of the unknown folding instant $\{c[m], t_m\}_{m \in \mathbb{Z}}$. Now $v[k]$ is known, we can infer $v[k]$ from $h[k]$ and recover $f[k]$ from $h[k]$. Finally we get the sampling density criterion, and the following conclusions can be drawn

$$|\mathbb{I}_{Q-1} \setminus \mathbb{E}_{R,Q-1}| = K - 2R - 2 \geq 2M,$$

(32)

where $M$ is known. Due $QT = \sigma$, $R = \left[\frac{\Omega_\alpha}{u_0}\right]$, $u_0 = \frac{2\pi \sin \alpha}{\sigma}$, we have

$$T = T_{FRFT} \leq \frac{\sigma}{2(R + M + 1)} = \frac{\sigma}{2(\lceil\Omega_\alpha \sigma / 2\pi \sin \alpha\rceil + M + 1)}.$$  

(33)

After performing $M$ folds, (33) can guarantee the restoration and reconstruction of folding moment $\{c[m], t_m\}_{m=0}^{M-1}$. The reconstruction theorem is given below.
Algorithm 1: Unlimited sampling in the fractional Fourier domain

**Input:** $h[k], \sigma, R, T, M = |M|.$

**Output:** $f[k].$

**Method:**

1. Compute the first-order difference $\overline{h}[k].$
2. Compute DFRFT: $\mathbf{H}_\alpha[n].$
3. Compute fold estimation in the FRFD.
   Estimate $\{c[m], t_m\}_{m=0}^{M-1}$ with annihilation filtering method.
4. Estimation $v[k].$
5. Reconstruct $f[k]$: $f[k] = h[k] + v[k].$

**Theorem 1** (Unlimited sampling theorem in the FRFD) Assume a $\sigma$-periodic signal $x(t)$ bandlimited to $(-\Omega_\alpha, \Omega_\alpha)$ in $\alpha$th FRFD and $h[k] = M_\lambda(x(kT))$ folded at most $M$ times. Then a sufficient condition for recovery of $x(t)$ from $h[k]$ is that $T \leq \frac{\sigma}{Q}$ and $Q \geq 2 \left( \frac{\Omega_\alpha \sigma}{2\pi \sin \alpha} + M + 1 \right)$, where $M$ is known.

Proof: From the unlimited sampling part of the fractional domain in (22) and (24), we can get the

$$
\begin{bmatrix}
\mathbb{H}[M-1] & \mathbb{H}[M-2] & \cdots & \mathbb{H}[0] \\
\mathbb{H}[M] & \mathbb{H}[M-1] & \cdots & \mathbb{H}[1] \\
\vdots & \vdots & \vdots & \vdots \\
\mathbb{H}[N-1] & \mathbb{H}[N-2] & \cdots & \mathbb{H}[N-M]
\end{bmatrix}
\begin{bmatrix}
\Gamma[1] \\
\Gamma[2] \\
\vdots \\
\Gamma[M]
\end{bmatrix}
= -
\begin{bmatrix}
\mathbb{H}[M] \\
\mathbb{H}[M+1] \\
\vdots \\
\mathbb{H}[N]
\end{bmatrix},
$$

that is, $\Gamma[\vartheta]$. From

$$(\Gamma \ast \mathbb{H})[n] = 0.$$

We can get the root of $\zeta_m$, bring (24) into (22), use the least square method to estimate $c[m]$, and get $v_k$ in (17). We develop a method which allows for inferring
\( v[k] \) from \( h[k] \) of Algorithm 1. Finally, the modulo sampling and residuals are restored. When \( \alpha = \frac{\pi}{2} \), see [26]. Proof completed.

The unlimited sampling theorem proves that the non-bandlimited signal in the FD based on the FRFT can be recovered from analog sampling as long as it meets (33) whose amplitude exceeds the ADC threshold by orders of magnitude. Special emphasis the signal is not affected by ADC threshold.

The proposed technique to reconstruct \( x(t) \) from \( \mathcal{H}_\lambda(x(t)) \) is referred as unlimited sampling theorem in the fractional Fourier domain and is presented in Algorithm 1. Here, we qualitatively summarize the rationale of Algorithm 1. When the sampling criterion in (33) is satisfied, our recovery method can be used to "unfold" the non-ideal modulus samples of non-bandlimited signals in the Fourier domain based on the fractional Fourier transform. First, compute the first-order difference \( \overline{h}[k] \) in (19). Then through (21) the relationship between \( \overline{f}[k], \overline{h}[k] \) and \( \overline{v}[k] \) is established in the fractional Fourier domain, the parameters \( \{c[m], r_m\} \) in \( \overline{v}[k] \) are obtained. Finally, the inverse operator is used to solve in (17) to complete the reconstruction \( f[k] \).

4. Conclusion

In this article, we study the reconstruction algorithm of bandlimited signals in the fractional Fourier domain based on the unlimited sampling framework of modulo measurement. Our main work is to perform modular operations in the fractional Fourier domain with the folding introduced by modular nonlinearity, and then to deal with the problem of fractional spectrum estimation. It turns out that our reconstruction theory has nothing to do with modulus threshold and can handle arbitrary folding time.

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