Chains of mini-boson stars

Shi-Xian Sun, Li Zhao\textsuperscript{1} and Yong-Qiang Wang\textsuperscript{1}

Institute of Theoretical Physics and Research Center of Gravitation, Lanzhou University, 222 Tianshui South Road, Lanzhou 730000, China
Key Laboratory of Quantum Theory and Applications of MoE, Lanzhou University, 222 Tianshui South Road, Lanzhou 730000, China
Lanzhou Center for Theoretical Physics and Key Laboratory of Theoretical Physics of Gansu Province, Lanzhou University, 222 Tianshui South Road, Lanzhou 730000, China

E-mail: 120220908811@lzu.edu.cn, lizhao@lzu.edu.cn, yqwang@lzu.edu.cn

Abstract: In this paper, we re-investigate the static, soliton-like solutions in the model of the Einstein gravity coupled to a free and complex scalar field, which have been known as mini-boson stars. With the numerical methods, we have discovered a new family of solutions in addition to the typical single mini-boson star solution. These solutions can be interpreted as chains of boson stars, consisting of multiple boson stars along the symmetry axis. We demonstrate the configuration of two types of chains, one with an even number of constituents and the other with an odd number of constituents. Furthermore, we also study the effect of the frequency of the complex scalar field on the ADM mass $M$ and the U(1) scalar charge $Q$. It is noteworthy that the existence of chains of boson stars does not require the introduction of a complex scalar field with self-interacting potential.

Keywords: Black Holes, Classical Theories of Gravity

ArXiv ePrint: 2210.09265
1 Introduction

It is well known that the most general stationary solution of the vacuum Einstein equation in a four-dimensional spacetime is described by a so-called Kerr geometry, which is a rotating black hole only with two parameters, the black-hole mass and angular momentum. The above uniqueness of the Kerr black hole is sometimes referred to as the “no-hair” theorem [1]. When considering the model of a free scalar field minimally coupled to Einstein’s gravity, it has been difficult to obtain analytical or numerical black hole solutions for a long time. Until the last few years, Herdeiro and Radu presented a novel family of solutions of Kerr black holes with scalar hair in refs. [2, 3]. Furthermore, when restrictions on the stationary solution of compact object with event horizon are lifted, there exist a family of horizonless solutions, which have become well-known as Boson stars (BSs).

The study of boson stars (BSs) has a long history. In the 1960s, Kaup discovered a family of spherically symmetric BSs in four-dimensional Einstein gravity coupled to a free, complex scalar field with the constant angular frequency of the phase of the field in the complex plane [4, 5]. However, due to little astrophysical interest in directed searches for a boson star, BSs with rotational symmetry were not studied until the 1990s. The first rotating solutions of boson stars in the Einstein-Klein-Gordon theory were found in [6]. BSs with a free scalar field without self-interaction were well known as mini-BSs, which can then be extended to the self-interacting case [7], the excited case of scalar field with nodes [8] and the multistate BSs [9, 10]. There are also studies on the solutions of the self-gravitating solitons in Einstein-Proca or Einstein-Dirac models [11–13]. Another interest of BSs in astronomy is their application to axions [14, 15], cosmic dark matter [16, 17], and black hole mimickers [18–20]. In addition, the collisions of binary BSs have been studied extensively [21–27], which offers a possible way to detect the BSs with the gravitational waves generated by the merger of binary stars.
Recently, by introducing the self-interaction potential with the type of quartic and sextic terms, which was studied in literatures on the Q-balls in the field theory, a novel family of solutions interpreted as chains of boson stars, which is made of some boson stars along the symmetry axis, was obtained in the Einstein-Klein-Gordon theory \cite{28}. Such solutions can be divided into two classes, including even chains and odd chains according to the parity at the two sides of the equatorial plane. The obvious difference is that in the case of even-numbered chains, the curve of the relationship between mass and frequency could form a spiral pattern similar to the case of a single boson star, while odd-numbered star chains can form a kind of loop pattern, which is very different from the case of a single boson star. Moreover, in the second branch of odd-numbered star chains with loop patterns, the system of star chains turn into a radially excited spherically symmetric boson star. Rotating generalizations of chains of boson stars with the sextic potential also been studied in \cite{29}.

It is worth noting that in flat spacetime there does not exist the chain of multisoliton in the free scalar field model. So, it will be interesting to see whether there are solutions of the chains of mini-BSs without the self-interaction potential in the model of free scalar field coupled to gravity. In the present paper, we numerically solve the coupled system of nonlinear partial differential equations of scalar field and Einstein equations, and obtain a family of chains of mini-BSs, which can be divided into two classes, including even chains and odd chains. Moreover, we also study the effect of the frequency of the complex scalar field on the ADM mass and the Noether charge.

This paper is organized as follows. In section 2, we briefly review the model of a free, complex scalar field coupled to Einstein’s gravity. The boundary conditions for the chains of BSs and some relevant physical quantities for the analysis of the solutions are given in section 3. We show the numerical results of two types of BSs chains in section 4. The conclusion and discussion are given in the last section. Throughout this paper, we adopt the conventions: (i) Roman letters a,b,c, \ldots denote spacetime indices running over 0, 1, 2, 3; (ii) the metric signature is (−,+,+,+).

**Note added.** When we are finishing this project, we notice that there appears a paper \cite{30}, which overlaps with our results of chains of BSs with two constituents.

## 2 The model

Let us introduce the model of Einstein gravity coupled to a free, complex massive scalar field in four-dimensional spacetime. The action of this theory reads

\[
S = \int_{\Omega} d^4 x \sqrt{-g} \left( \frac{R}{16\pi G} - g^{ab} \nabla_a \psi^* \nabla_b \psi - \mu^2 \psi^* \psi \right),
\]  

(2.1)

where \( \Omega \) is the spacetime manifold, \( R \) is the Ricci scalar of the spacetime metric \( g_{ab} \), and \( G \) is the Newton’s constant. The mini-BSs are obtained without self-interaction potential, and the term proportional to \( \mu^2 \) is known as a mass term of the complex scalar field \( \psi \). The above action is invariant under a global U(1) transformation \( \psi \to \psi e^{i\theta} \), where \( \theta \) is constant, which implies the conservation of the total particle number. Variation of the action (2.1)
with respect to the complex scalar field \( \psi^* \) and the metric \( g^{ab} \) could lead to the following Klein-Gordon (KG) equation of the scalar field

\[
\Box\psi = \mu^2 \psi ,
\tag{2.2}
\]

and Einstein field equations

\[
R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G (\nabla_a \psi^* \nabla_b \psi + \nabla_b \psi^* \nabla_a \psi - g_{ab} (\nabla_c \psi^* \nabla^c \psi + \nabla_c \psi \nabla^c \psi^* + \mu^2 \psi^* \psi)) ,
\tag{2.3}
\]

where \( \Box = \nabla_a \nabla^a \) represents the d’Alembert operator.

In the absence of the complex scalar field \( \psi \), one of exact solutions of Einstein equations (2.3) is known as Kerr black hole, which is a stationary and axisymmetric compact object with the event horizon. A particular limit of Kerr black hole with the scalar hair, in the absence of an event horizon, i.e., \( r_H = 0 \), is the rotating BSs. In what follows, we will only consider a non-rotating family of chains of mini-BSs with complex scalar field \( \psi \) that has an axisymmetric distribution. In axially symmetric spacetimes, these solutions can be naturally described by the line element ansatz which was derived from Weyl \[31, 32\]

\[
ds^2 = -e^{-2U} dt^2 + e^{2U} \left[ e^{2k} (d\rho^2 + dz^2) + P^2 d\varphi^2 \right],
\tag{2.4}
\]

where \( (t, \rho, \varphi, z) \) are the cylindrical coordinates in 3+1 dimensional spacetime, and \( U, k \) and \( P \), are unknown functions of the spatial non-angular coordinates \( \rho \) and \( z \) only. Indeed, it is difficult to solve these functions using numerical methods in cylindrical coordinates. In order to make use of the numerical method in solving the chains of BSs, it is conventional to convert cylindrical coordinates \( (\rho, z) \) into spherical coordinates \( (r, \theta) \) by the formulas

\[
r = \sqrt{\rho^2 + z^2}, \quad \theta = \arctan \frac{\rho}{z}.
\tag{2.5}
\]

Moreover, by defining the new functions \( f \equiv e^{-2U}, \ l \equiv \frac{P^2}{r^2 \sin^2 \theta} \), and \( m \equiv e^{2k} \), the line element (2.4) is converted into the Weyl-Lewis-Papapetrou form \[33\]

\[
ds^2 = -f dt^2 + \frac{m}{f} \left( dr^2 + r^2 d\theta^2 \right) + \frac{l}{f} r^2 \sin^2 \theta d\varphi^2 ,
\tag{2.6}
\]

where three metric functions \( f, l, \) and \( m \) are functions of the variables \( r \) and \( \theta \) only.

In addition, one could introduce the following ansatz of matter fields:

\[
\psi = \phi(r, \theta)e^{-i\omega t}.
\tag{2.7}
\]

Here, the amplitude \( \phi \) depends on the radial distance \( r \) and polar angle \( \theta \), and the constant \( \omega \) is the angular frequency of the complex scalar field, which means that scalar field possesses a harmonic time dependence. Under the metric assumption (2.6), the Einstein-Klein-Gordon system gives rise to the set of six coupled partial differential equations (PDEs). Three of them form the combinations of Einstein equations: \( E^t_t - E^r_r - E^\theta_\theta - E^\varphi_\varphi = 0 \), \( E^r_r + E^\theta_\theta = 0 \) and \( E^\varphi_\varphi = 0 \), one forms the Klein-Gordon equation, and the others form two more Einstein
equations which need not to be solved. In terms of the line element (2.6), the explicit forms of the KG equation and the combined Einstein equations are given by

\[ \phi_{,rr} + \phi_{,\theta \theta} + \left( \frac{2}{r} + \frac{l_r}{2l} \right) \phi_{,r} \right. + \frac{\cot \theta + \frac{l_{,\theta}}{2l}}{r^2} \phi_{,\theta} + \left. \left( \frac{\omega^2}{f} - \mu^2 \right) \frac{m}{f} \right) \phi = 0, \tag{2.8} \]

\[ f_{,rr} + f_{,\theta \theta} r^2 + 2f_{,r} r \frac{f_{,r}}{r^2} - \frac{1}{f} \left( f_{,r}^2 + \frac{f_{,\theta}^2}{r^2} \right) + \frac{1}{2f} \left( f_{,r} l_{,r} + f_{,\theta} l_{,\theta} \right) \]

\[ + 16\pi G \left( \mu^2 \phi^2 - \frac{2\omega^2 \phi^2}{f} \right) m = 0, \tag{2.9} \]

\[ l_{,rr} + \frac{l_{,\theta \theta}}{r^2} + \frac{3l_{,r}}{r} + 2 \cot \theta l_{,\theta} + \frac{1}{2l} \left( f_{,r}^2 + \frac{f_{,\theta}^2}{r^2} \right) + 32\pi G \left( \mu^2 \phi^2 - \frac{\omega^2 \phi^2}{f} \right) \frac{lm}{f} = 0, \tag{2.10} \]

\[ m_{,rr} + \frac{m_{,\theta \theta}}{r^2} + \frac{m_{,r}}{r} + \frac{m_{,\theta}}{2f^2} \left( f_{,r}^2 + \frac{f_{,\theta}^2}{r^2} \right) - \frac{1}{m} \left( f_{,r}^2 + \frac{f_{,\theta}^2}{r^2} \right) \frac{m^2}{f} \]

\[ + 16\pi G \left[ \frac{f}{m} \left( \frac{\phi^2}{r^2} + \frac{\phi_{,\theta}^2}{r^2} \right) + \mu^2 \phi^2 - \frac{\omega^2 \phi^2}{f} \right] \frac{m^2}{f} = 0, \tag{2.11} \]

where the commas denote differentiation with respect to \( r \) or \( \theta \). These four equations are second-order elliptic PDEs, which can be solved as a boundary value problem when appropriate boundary conditions are imposed.

3 Boundary conditions

Before numerically solving the coupled PDEs, we should obtain the asymptotic behaviors of the four functions \( f, l, m \) and \( \phi \), which is equivalent to give the boundary conditions we need. We will still use the boundary conditions by following the same steps given in refs. \([3, 28]\).

Considering an axial symmetry system, we have polar angle reflection symmetry \( \theta \rightarrow \pi - \theta \) on the equatorial plane, and thus it is convenient to consider the coordinate range \( \theta \in [0, \pi/2] \). In addition, the scalar field can have either even or odd parity with respect to reflections on the equatorial plane. So, we require

\[ \begin{cases} \partial_\theta \phi \big|_{\theta=\pi/2} = 0, & \text{even parity} \\ \phi \big|_{\theta=\pi/2} = 0, & \text{odd parity} \end{cases} \tag{3.1} \]

The geometry remains invariant under the reflection on the equatorial plane, thus the metric functions satisfy the following boundary conditions

\[ \partial_\theta f \big|_{\theta=\pi/2} = \partial_\theta l \big|_{\theta=\pi/2} = \partial_\theta m \big|_{\theta=\pi/2} = 0. \tag{3.2} \]

To preserve the regularity at the origin \( r = 0 \), the boundary conditions for the metric functions is given by

\[ \begin{cases} \partial_r f \big|_{r=0} = \partial_r l \big|_{r=0} = \partial_r m \big|_{r=0} = 0, \end{cases} \tag{3.3} \]

together with

\[ \begin{cases} \partial_r \phi \big|_{r=0} = 0, & \text{even parity} \\ \phi \big|_{r=0} = 0, & \text{odd parity} \end{cases} \tag{3.4} \]
While on the symmetry axis at $\theta = 0, \pi$, the regularity must be imposed Neumann boundary conditions on the functions

$$\partial_\theta f|_{\theta=0,\pi} = \partial_\theta l|_{\theta=0,\pi} = \partial_\theta m|_{\theta=0,\pi} = \partial_\theta \phi|_{\theta=0,\pi} = 0. \quad (3.5)$$

In addition, the asymptotic behaviors near the boundary $r \to \infty$ are

$$f|_{r\to\infty} = m|_{r\to\infty} = l|_{r\to\infty} = 1, \quad \phi|_{r\to\infty} = 0, \quad (3.6)$$

where the Minkowski spacetime background is recovered and the scalar field comes into its vacuum value.

The chains of mini-BSs possess two global “charges”: the ADM mass $M$ and the Noether charge $Q$. The ADM mass $M$ can be obtained in two ways: (i) from the asymptotic behavior of the solutions. Near the boundary $r \to \infty$, the metric function $f$ has the following forms

$$f \to 1 - \frac{2GM}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (3.7)$$

where the parameters $M$ is the ADM mass of the BSs; (ii) from the respective Komar expressions [34],

$$M = \frac{1}{4\pi G} \int_{\Sigma} R_{\mu\nu} n^\mu \xi^\nu dV, \quad (3.8)$$

where $\Sigma$ denotes a spacelike hypersurface (with the volume element $dV$), $n^\mu$ is vector normal to $\Sigma$ with $n^\mu n_\mu = -1$ and $\xi^\nu$ is the everywhere timelike Killing vector field. Besides the ADM mass, the conserved Noether charge $Q$ is given by

$$Q = \int_{\Sigma} j_\mu n^\mu dV, \quad (3.9)$$

where $j_\mu$ is Noether 4-current associated with the global $U(1)$ symmetry of the complex scalar field,

$$j_\mu = -i(\psi \partial_\mu \psi^* - \psi^* \partial_\mu \psi). \quad (3.10)$$

These two physical quantities are related by the “first law of thermodynamics” [35]

$$dM = \omega dQ. \quad (3.11)$$

4 Numerical results

In this section, we will solve the above second-order elliptic PDEs numerically. We introduce a new compactified radial coordinate $x = \frac{r}{r_T}$, which maps the semi-infinite interval $r \in [0, \infty)$ to a finite one $x \in [0, 1]$. Considering the reflection symmetry $\theta \to \pi - \theta$, it is sufficient to consider the range of angular variables $\theta \in [0, \frac{\pi}{2}]$. All numerical calculations are based on the finite element method, performed by using the multi-physical field simulation software COMSOL [36]. The number of grid points ranges between $80 \times 100$ and $100 \times 100$ in the integration region $0 \leq x \leq 1$ and $\theta \in [0, \frac{\pi}{2}]$. Our iterative process is the Newton-Raphson method, and the relative error for the numerical solutions in this work is estimated to be below $10^{-5}$. 

- 5 -
Figure 1. Plots of the scalar field amplitude $\phi$ in coordinates $\rho = r \sin \theta$ and $z = r \cos \theta$, for chains of mini-BSs with one to five constituents. The horizontal direction is the $z$-axis, while the vertical direction is the $\rho$-axis. The bright spots with red and blue colour indicate the positive and negative value for the amplitude $\phi$ of scalar field, respectively. The parameter is set to $\omega = 0.9$.

In the numerical results below, we have fixed the parameters $\mu = 1$ and $G = 1/8\pi$. The only remaining input parameter is the scalar field’s frequency $\omega$. Physical quantities of interest, such as ADM mass and Noether charge, can be calculated from the numerical solutions. The spatial distribution of the scalar field in this model can be either symmetric or antisymmetric under reflections along the equatorial plane. However, the distribution of metric function and energy density is always symmetric. When the spatial distribution of the scalar field is symmetric under reflections along the equatorial plane, it has a parity value of $P = +1$, which corresponds to an even-parity configuration. Conversely, when the spatial distribution of the scalar field is antisymmetric under reflections along the equatorial plane, it has a parity value of $P = -1$ corresponding to an odd-parity configuration.

In figure 1, we plot the distribution of scalar field function on the first branch for both even-parity and odd-parity configurations. The figure shows chains of mini-BSs with one to five constituents, where the bright spots with red and blue colors indicate positive and negative values for the amplitude $\phi$ of scalar field, respectively. The single mini-BS is spherically symmetric, while the higher numbered mini-BSs are uniformly distributed along the $z$-axis. In figure 2, we provide 3-dimensional plots for the metric function $f(z, \rho)$ for one to five troughs located symmetrically along the $z$-axis. We can observe that the metric functions $f(z, \rho)$ are affected by the maximum or minimum values of the scalar fields at
their corresponding positions. Additionally, the number of constituents is determined by the number of extreme values of the scalar field, and the profile of \( f(z, \rho) \) shows as many minima as the number of constituents. However, except at the central region, the minima of \( f(z, \rho) \) do not precisely coincide with the extrema of scalar field. They are also influenced by the scalar field distribution in other regions, leading to a displacement towards the center.

4.1 Even constituents

In this subsection, we will analyze the solutions of BSs with even number of constituents located symmetrically along the symmetry \( z \)-axis. These configurations represent chains of BSs have an odd-parity scalar field. The \( \omega \)-dependences for even chains are shown in figure 3. In all panels, the red and blue lines indicate chains of BSs with two constituents and four constituents. In order to compare with the results of the single BS, we also draw black lines, which represents the \( \omega \)-dependences for the single BS.

It has been observed that on the first branch the mass of chains of BSs with even constituents initially increases with decreasing frequency \( \omega \), reaches a maximum value, and then decreases to a minimum value at the turning point. After the turning point between the first and second branches, the ADM mass decreases with frequency, which corresponds to the second branch of chains of BSs with even constituents. The Noether charge \( Q \) also shows a similar spiral dependence on frequency, initially increasing until it reaches a maximum and then decreasing until a turning point is reached. The second branch starts where the Noether charge decreases with increasing frequency. Three spiral curves originate from the vacuum and revolve into the center of the graph. As the spiral curves approach the central region, the numerical error increases, requiring a finer mesh for accurate computations. Although the \( \omega \)-dependences of ADM mass and Noether charge...
Figure 3. Plots of the dependence of ADM mass $M$, Noether charge $Q$, minimal value of the metric function $f_{\text{min}}$ and maximal value of the scalar field $\phi_{\text{max}}$ on the frequency $\omega$. In all plots, the black, red and blue lines denote one, two or four constitutes of BSs, respectively. In the three upper panel, the solid line represents the relationship between $M$ and $\omega$, while the dashed line represents the relationship between $Q$ and $\omega$.

of two or four constituents exhibit spiraling behavior similar to that of single BSs, there is an overall upward shift in the spiral pattern with an increase in the number of BSs corresponding to the same frequency. From the perspective of binding energy, solutions with $M > Q$ are considered unstable. As shown in the three upper panals of figure 3, we can observe that for the single BS and chains of BSs with two constituents, the solutions at the first branch already exhibit unstable. However, for chains of BSs with four constituents, there are no existences of $M > Q$ in the first branch.

The lower two panels of the figure display the minimal value of the metric function $f_{\text{min}}$ and maximal value of scalar field function $\phi_{\text{max}}$ at the center of BSs. The minimal value of the metric function $f_{\text{min}}$ initially decreases as the frequency decreases, and then after the turning point between the first and second branches, $f_{\text{min}}$ continues to decrease to zero as the frequency increases. On the other hand, the scalar field function $\phi_{\text{max}}$ initially increases with decreasing frequency and then increases with increasing frequency. The higher even number of constituents exhibit a similar pattern as single BS, as shown in figure 3.

4.2 Odd constituents

In this subsection, we turn to the solutions of odd number of constituents with an even-parity scalar field. In figure 4, we present the frequency dependence of chains of BSs with three constituents, which is drawn with black line. We also draw a blue line in each panel, which represents the $\omega$-dependences for a first excited spherically symmetric single BS with one
Figure 4. Plots of the influence of frequency on ADM mass $M$, Noether charge $Q$, minimal value of the metric function $f_{\text{min}}$, the value of the scalar field $\phi(0)$. The inset in the upper panel shows an enlarged image of the area with the red dot. In all plots, the black and blue lines denote chains of BSs with three constitutes and first excited state, respectively.
single radial node. Generally, the scalar field solution of $n$-th excited state possesses $n$ nodes, such that the profile of homogeneous scalar field solution of the first excited state is given by an initial guess that has one node.

The subfigures in figure 4 on upper, middle and lower correspond to the influence of frequency on ADM mass $M$, Noether charge $Q$, minimal value of the metric function $f_{\text{min}}$ and the value of the scalar field $\phi(0)$. Interestingly, the properties of such chains are very different from those of single BS, all of the diagrams for $(\omega, Q)$, $(\omega, M)$, $(\omega, f_{\text{min}})$ and $(\omega, \phi(0))$ of chains of BSs with three constituents have a nontrivial loop structure, which is formed by extending the second branch back to the first branch at the upper limit of frequency value $\omega_{\text{max}}$.

From panels in figure 4, we see that the loop structure of the ADM mass $M$ (upper) and the Noether charge $Q$ (middle) of the chains with three constituents overlaps on their second branch with the first branch of a first excited spherical single BS. The red dot is a point where they intersect, and the corresponding critical frequency is $\omega_{\text{cr}} = 0.863$. The red point is the only spherically symmetric solution on the black line as a special case, precisely corresponding to a solution of the first excited state spherically symmetric single BS. However, for the other intersection points on these panels, it is important to note that they have the same $M$ or $Q$ values at that frequency, but they do not possess the same field configuration. When $\omega$ deviates a little from $\omega_{\text{cr}}$, the images of the scalar field distribution are plotted in the left and right subfigures shown in figure 5. From this figure, we can clearly see the result of this model, when $\omega$ is below $\omega_{\text{cr}}$, the low-frequency distribution of chains of BSs is squeezed in the axial direction; after that, in the case that $\omega = \omega_{\text{cr}}$, the central dominant BSs will be surrounded by one “boson shell”, which means the profile of the chains of BSs is spherically symmetric. When $\omega$ is greater than $\omega_{\text{cr}}$, the chains of BSs tends to be distributed along the equatorial plane, where the system deviates from spherical configuration and a single “boson torus” encloses the central dominant BS.

In the middle right panel we show that the metric function $f_{\text{min}}$ of the chains with three constituents decreases with decreasing frequency, which forms a first branch. Then the metric function $f_{\text{min}}$ increases with increasing frequency which forms the second branch,
returning back to the first branch at its upper limit $\omega_{\text{max}}$. The scalar field $\phi(0)$ of the chains with three constituents increases with decreasing frequency and decreases with increasing frequency, eventually decreasing to zero. The second branch of their images and the first branch of a single excited state BS have only one intersection point.

5 Discussion

In this paper, we have investigated the chains of BSs within a model with a free complex scalar field coupled to gravity, and obtained the bound state of chains of mini-BSs with up to five constituents located symmetrically along the symmetry axis. Our results indicate that the existence of chains of BSs does not require the introduction of a complex scalar field with self-interacting potential.

We have used the finite element method to numerically solve the partial differential equation for chains of mini-BSs. For chains with higher even numbers of constituents, they typically exhibit patterns similar to a single spherical BS. The $(\omega, M)$ and $(\omega, Q)$ curves for chains with an even number of constituents show a spiraling behavior similar to that of a single boson star, but with larger ADM mass and Noether charge as the number of constituents increases. The minimal value of the metric function $f_{\text{min}}$ and maximal value of scalar field function $\phi_{\text{max}}$ exhibit oscillating behavior as the frequency varies. For chains with an odd number of constituents, the $(\omega, M), (\omega, Q), (\omega, \phi(0))$ and $(\omega, f_{\text{min}})$ curves exhibit a nontrivial loop structure formed by extending the second branch back to the first branch at the upper limit of frequency value $\omega_{\text{max}}$, which is different from the case of a single spherical BS. There is only one intersection of these profiles on the second branch with the spiraling curve of the first excited state BS, where the central BS is surrounded by one “Boson shell”.

There are some interesting extensions of this work. Firstly, the generalization of stationary solution of chains of rotating BSs are also analyzed in ref. [29], we would like to extend their study to construct the rotating generalizations of chains of boson stars with the free scalar field. Secondly, it would be interesting to construct the multistated chains of BSs, where coexisting states of the several scalar fields are presented, including the ground and excited states. Finally, the dynamic stability of chains of BSs is still not discussed in this paper, which will be leave to our further study.

Acknowledgments

We sincerely thank associate professor L. Qiao for his support in the software. This work is supported by National Key Research and Development Program of China (Grant No. 2020YFC2201503 and No. 2021YFC2203003) and the National Natural Science Foundation of China (Grants No. 12275110 and No. 12247101). Parts of computations were performed on the shared memory system at institute of computational physics and complex systems in Lanzhou university.
Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] R. Ruffini and J.A. Wheeler, *Introducing the black hole*, Phys. Today **24** (1971) 30 [inSPIRE].

[2] C.A.R. Herdeiro and E. Radu, *Kerr black holes with scalar hair*, Phys. Rev. Lett. **112** (2014) 221101 [arXiv:1403.2757] [inSPIRE].

[3] C. Herdeiro and E. Radu, *Construction and physical properties of Kerr black holes with scalar hair*, Class. Quant. Grav. **32** (2015) 144001 [arXiv:1501.04319] [inSPIRE].

[4] D.J. Kaup, *Klein-Gordon Geon*, Phys. Rev. **172** (1968) 1331 [inSPIRE].

[5] R. Ruffini and S. Bonazzola, *Systems of selfgravitating particles in general relativity and the concept of an equation of state*, Phys. Rev. **187** (1969) 1767 [inSPIRE].

[6] S. Yoshida and Y. Eriguchi, *Rotating boson stars in general relativity*, Phys. Rev. D **56** (1997) 762 [inSPIRE].

[7] F. Kling and A. Rajaraman, *Profiles of boson stars with self-interactions*, Phys. Rev. D **97** (2018) 063012 [arXiv:1710.02057] [inSPIRE].

[8] L.G. Collodel, B. Kleihaus and J. Kunz, *Excited Boson Stars*, Phys. Rev. D **96** (2017) 084066 [arXiv:1708.02057] [inSPIRE].

[9] A. Bernal, J. Barranco, D. Alic and C. Palenzuela, *Multi-state Boson Stars*, Phys. Rev. D **81** (2010) 044031 [arXiv:0908.2435] [inSPIRE].

[10] H.-B. Li et al., *Rotating multistate boson stars*, Phys. Rev. D **101** (2020) 044017 [arXiv:1906.00420] [inSPIRE].

[11] R. Brito, V. Cardoso, C.A.R. Herdeiro and E. Radu, *Proca stars: Gravitating Bose-Einstein condensates of massive spin 1 particles*, Phys. Lett. B **752** (2016) 291 [arXiv:1508.05395] [inSPIRE].

[12] C.A.R. Herdeiro, J.M.S. Oliveira and E. Radu, *A class of solitons in Maxwell-scalar and Einstein-Maxwell-scalar models*, Eur. Phys. J. C **80** (2020) 23 [arXiv:1910.11021] [inSPIRE].

[13] F. Finster, J. Smoller and S.-T. Yau, *Particle-like solutions of the Einstein-Dirac equations*, Phys. Rev. D **59** (1999) 104020 [gr-qc/9801079] [inSPIRE].

[14] D. Guerra, C.F.B. Macedo and P. Pani, *Axion boson stars*, JCAP **09** (2019) 061 [Erratum ibid. **06** (2020) E01] [arXiv:1909.05515] [inSPIRE].

[15] J.F.M. Delgado, C.A.R. Herdeiro and E. Radu, *Rotating Axion Boson Stars*, JCAP **06** (2020) 037 [arXiv:2005.05982] [inSPIRE].

[16] A. Suárez, V.H. Robles and T. Matos, *A Review on the Scalar Field/Bose-Einstein Condensate Dark Matter Model*, Astrophys. Space Sci. Proc. **38** (2014) 107 [arXiv:1302.0903] [inSPIRE].

[17] L. Hui, J.P. Ostriker, S. Tremaine and E. Witten, *Ultralight scalars as cosmological dark matter*, Phys. Rev. D **95** (2017) 043541 [arXiv:1610.08297] [inSPIRE].

[18] V. Cardoso and P. Pani, *Testing the nature of dark compact objects: a status report*, Living Rev. Rel. **22** (2019) 4 [arXiv:1904.05363] [inSPIRE].
[19] K. Glampedakis and G. Pappas, *How well can ultracompact bodies imitate black hole ringdowns?*, Phys. Rev. D 97 (2018) 041502 [arXiv:1710.02136] [insPIRE].

[20] C.A.R. Herdeiro et al., *The imitation game: Proca stars that can mimic the Schwarzschild shadow*, JCAP 04 (2021) 051 [arXiv:2102.01703] [insPIRE].

[21] C. Palenzuela, I. Olabarrieta, L. Lehner and S.L. Liebling, *Head-on collisions of bosons*, Phys. Rev. D 75 (2007) 064005 [gr-qc/0612067] [insPIRE].

[22] C. Palenzuela, L. Lehner and S.L. Liebling, *Orbital Dynamics of Binary Boson Star Systems*, Phys. Rev. D 77 (2008) 044036 [arXiv:0706.2435] [insPIRE].

[23] T. Helfer, E.A. Lim, M.A.G. Garcia and M.A. Amin, *Gravitational Wave Emission from Collisions of Compact Scalar Solitons*, Phys. Rev. D 99 (2019) 044046 [arXiv:1802.06733] [insPIRE].

[24] M. Bezares, C. Palenzuela and C. Bona, *Final fate of compact boson star mergers*, Phys. Rev. D 95 (2017) 124005 [arXiv:1705.01071] [insPIRE].

[25] C. Palenzuela et al., *Gravitational Wave Signatures of Highly Compact Boson Star Binaries*, Phys. Rev. D 96 (2017) 104058 [arXiv:1710.09432] [insPIRE].

[26] N. Sanchis-Gual et al., *Head-on collisions and orbital mergers of Proca stars*, Phys. Rev. D 99 (2019) 024017 [arXiv:1806.07779] [insPIRE].

[27] J. Calderón Bustillo et al., *GW190521 as a Merger of Proca Stars: A Potential New Vector Boson of 8.7 × 10^{−13} eV*, Phys. Rev. Lett. 126 (2021) 081101 [arXiv:2009.05376] [insPIRE].

[28] C.A.R. Herdeiro et al., *Chains of Boson Stars*, Phys. Rev. D 103 (2021) 065009 [arXiv:2101.06442] [insPIRE].

[29] R. Gervalle, *Chains of rotating boson stars*, Phys. Rev. D 105 (2022) 124052 [arXiv:2206.03982] [insPIRE].

[30] P. Cunha, C. Herdeiro, E. Radu and Y. Shnir, *Two boson stars in equilibrium*, Phys. Rev. D 106 (2022) 124039 [arXiv:2210.01833] [insPIRE].

[31] H. Weyl, *The theory of gravitation*, Annalen Phys. 54 (1917) 117 [insPIRE].

[32] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact solutions of Einstein’s field equations*, Cambridge University Press (1980).

[33] B. Kleihaus and J. Kunz, *Static axially symmetric Einstein-Yang-Mills dilaton solutions: Addendum asymptotic solutions*, Phys. Rev. D 61 (2000) 107503 [hep-th/9909160] [insPIRE].

[34] R.M. Wald, *General Relativity*, Chicago University Press (1984) [DOI:10.7208/chicago/9780226870373.001.0001] [insPIRE].

[35] T.D. Lee and Y. Pang, *Nontopological solitons*, Phys. Rept. 221 (1992) 251 [insPIRE].

[36] M. Tabatabaian, *COMSOL for Engineers*, Mercury Learning and Information (2014).