On Bohr’s theorem for general Dirichlet series

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Abstract
We present quantitative versions of Bohr’s theorem on general Dirichlet series
\[ D = \sum a_n e^{-\lambda_n s} \]
assuming different assumptions on the frequency \( \lambda = (\lambda_n) \),
including the conditions introduced by Bohr and Landau. Therefore, using the
summation method by typical (first) means invented by M. Riesz, without any
condition on \( \lambda \), we give upper bounds for the norm of the partial sum operator
\[ S_N(D) := \sum_{n=1}^{N} a_n(D) e^{-\lambda_n s} \]
of length \( N \) on the space \( D^\infty(\lambda) \) of all somewhere convergent \( \lambda \)-Dirichlet series, which allow a holomorphic and bounded extension to the open right half plane \([Re > 0]\). As a consequence for some classes of \( \lambda \)’s we obtain a Montel theorem in \( D^\infty(\lambda) \); the space of all \( D \in D^\infty(\lambda) \) which converge on \([Re > 0]\). Moreover, following the ideas of Neder we give a construction of frequencies \( \lambda \) for which \( D^\infty(\lambda) \) fails to be complete.

KEYWORDS
convergence abscissas, Dirichlet series, polynomials, typical means

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1 INTRODUCTION

A general Dirichlet series is a formal sum \( \sum a_n e^{-\lambda_n s} \), where \( (a_n) \) are complex coefficients (called Dirichlet coefficients),
\( s \) a complex variable and \( \lambda := (\lambda_n) \) a strictly increasing nonnegative real sequence tending to \(+\infty\) (called frequency).
To see first examples we choose \( \lambda = (\log n) \) and obtain ordinary Dirichlet series \( \sum a_n n^{-s} \), whereas the choice \( \lambda = (n) = (0, 1, 2, \ldots) \) leads to formal power series \( \sum a_n z^n \) regarding the substitution \( z = e^{-s} \).

Within the last two decades the theory of ordinary Dirichlet series had a sort of renaissance which in particular led to the solution of some long-standing problems (see [11] and [20] for more information). A fundamental object in these investigations is given by the Banach space \( H_\infty \) of all ordinary Dirichlet series \( D := \sum a_n n^{-s} \), which converge and define bounded functions on \([Re > 0]\).

One of the main tools in this theory is the fact that every ordinary Dirichlet series \( D \in H_\infty \) converges uniformly on \([Re > \varepsilon]\) for all \( \varepsilon > 0 \), which is a consequence of what is called Bohr’s theorem and was proven by Bohr in [4].

Bohr’s theorem (qualitative version). Let \( D = \sum a_n n^{-s} \) be a somewhere convergent ordinary Dirichlet series having a holomorphic and bounded extension \( f \) to \([Re > 0]\). Then \( D \) converges uniformly on \([Re > \varepsilon]\) for all \( \varepsilon > 0 \).
Several years ago in [1] this “ordinary” result was improved by a quantitative version.

**Bohr’s theorem** (qualitative version). There is a constant \( C > 0 \) such that for all somewhere convergent \( D = \sum a_n n^{-s} \) allowing a holomorphic and bounded extension \( f \) to \([Re > 0]\) and \( N \in \mathbb{N} \) with \( N \geq 2 \)

\[
\sup_{[Re>0]} \left| \sum_{n=1}^{N} a_n n^{-s} \right| \leq C \log(N) \sup_{[Re>0]} |f(s)|. \tag{1.1}
\]

An important consequence is that Bohr’s theorem implies that \( H_\infty \) is a Banach space (see [11, §1.4]).

The natural domain of Bohr’s theorem for general Dirichlet series is the space \( \mathcal{D}_\infty^\text{ext}(\lambda) \) of all somewhere convergent \( \lambda \)-Dirichlet series \( D = \sum a_n e^{-\lambda_n s} \) allowing a holomorphic and bounded extension \( f \) to \([Re > 0]\) and \( N \in \mathbb{N} \) with \( N \geq 2 \).

\[
\sup_{[Re>0]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq C \log(N) \sup_{[Re>0]} |f(s)|. \tag{1.1}
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The inclusion \( \mathcal{D}_\infty(\lambda) \subseteq \mathcal{D}_\infty^\text{ext}(\lambda) \) in general is strict (see e.g. the frequencies constructed in [19, §1]). A natural norm on \( \mathcal{D}_\infty^\text{ext}(\lambda) \) (and on \( \mathcal{D}_\infty(\lambda) \)) is given by \( \|D\|_\infty := \sup_{[Re>0]} |f(s)| \), where \( f \) is the (unique) extension of \( D \). Note that a priori, \( \|\cdot\|_\infty \) is only a semi norm, that is it could be possible for a particular Dirichlet series \( D = \sum a_n e^{-\lambda_n s} \) with some \( a_n \neq 0 \) to have a bounded holomorphic extension \( f \) with \( \|D\|_\infty = 0 \), or equivalently, it is not clear whether \( \mathcal{D}_\infty^\text{ext}(\lambda) \) can be considered as a subspace of \( H_\infty[Re > 0] \), the Banach space of all holomorphic and bounded functions on \([Re > 0]\). Here it is important to distinguish Dirichlet series from their limit function, and to prove that \( \|\cdot\|_\infty \) in fact is a norm on \( \mathcal{D}_\infty^\text{ext}(\lambda) \) requires to check that all Dirichlet coefficients of \( D \) vanish provided \( \|D\|_\infty = 0 \) (see Corollary 3.9).

We say that a frequency \( \lambda \) satisfies Bohr’s theorem (or Bohr’s theorem holds for \( \lambda \)) if every \( D \in \mathcal{D}_\infty^\text{ext}(\lambda) \) converges uniformly on \([Re > \epsilon]\) for all \( \epsilon > 0 \). It was a prominent question in the beginning of the 20th century for which \( \lambda \)’s Bohr’s theorem holds.

Actually Bohr proves his theorem not only for the case \( \lambda = (\log n) \) but for the class of \( \lambda \)’s satisfying the following condition (we call it Bohr’s condition \((BC)\)):

\[
\exists l = l(\lambda) > 0 \forall \delta > 0 \exists C > 0 \forall n \in \mathbb{N} : \lambda_{n+1} - \lambda_n \geq C e^{-(l+\delta)\lambda_n}; \tag{1.2}
\]

roughly speaking this condition prevents the \( \lambda_n \)’s from getting too close to fast. Then in [4] Bohr shows that if \( \lambda \) satisfies \((BC)\), then Bohr’s theorem holds for \( \lambda \). Note that \( \lambda = (\log n) \) satisfies \((BC)\) with \( l = 1 \).

In [18] Landau gives another sufficient condition (we call it Landau’s condition \((LC)\)), which is weaker than \((BC)\) and extends the class of frequencies for which Bohr’s theorem holds:

\[
\forall \delta > 0 \exists C > 0 \forall n \in \mathbb{N} : \lambda_{n+1} - \lambda_n \geq C e^{-\epsilon \lambda_n}; \tag{1.3}
\]

We like to mention that in [19, §1] Neder considered \( \lambda \)’s satisfying

\[
\exists x > 0 \exists C > 0 \forall n \in \mathbb{N} : \lambda_{n+1} - \lambda_n \geq C e^{-x \lambda_n},
\]

and then proved that this condition is not sufficient for satisfying Bohr’s theorem by constructing, giving some \( x > 0 \), a Dirichlet series \( D \) (belonging to some frequency \( \lambda \)) for which \( \sigma_x(D) = \sigma_0(D) = x \) and \( \sigma_0^\text{ext}(D) \leq 0 \) hold. In particular this shows that the inclusion \( \mathcal{D}_\infty(\lambda) \subseteq \mathcal{D}_\infty^\text{ext}(\lambda) \) is strict for these \( \lambda \)’s.

Like Bohr, Landau under his condition \((LC)\) only proves the qualitative version of Bohr’s theorem. Of course, establishing quantitative versions means to control the norm of the partial sum operator

\[
S_N : \mathcal{D}_\infty^\text{ext}(\lambda) \to \mathcal{D}_\infty(\lambda), \quad D \mapsto \sum_{n=1}^{N} a_n(D) e^{-\lambda_n s}, \quad N \in \mathbb{N}, \tag{1.4}
\]

since by definition

\[
\sup_{[Re>0]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq \|S_N\| \|D\|_\infty.
\]
Then using the summation method of typical means of order $k > 0$ invented by M. Riesz (Proposition 3.4), our main result gives an estimate of $\|S_N\|$ without assuming any condition on $\lambda$ (Theorem 3.2).

**Main result.** For all $0 < k \leq 1$ and $N \in \mathbb{N}$ we have

$$\|S_N\| \leq C \frac{\Gamma(k+1)}{k} \left( \frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^k,$$

where $\Gamma$ is the Gamma function and $C > 0$ a universal constant.

As a consequence assuming Bohr's condition (1.2) on $\lambda$ the choice $k_N := \frac{1}{\lambda_N}$, $N \geq 2$ (since $\lambda_1 = 0$ is possible), leads to

$$\|S_N\| \leq C_1(\lambda)\lambda_N,$$

which reproves (1.1) for $\lambda = (\log n)$. Under Landau's condition (1.3) using (1.5) with $k_N := e^{-\delta n}$, $\delta > 0$, we obtain

$$\|S_N\| \leq C_2(\lambda, \delta)e^{\delta \lambda_N};$$

the quantitative version of Bohr’s theorem under $(LC)$. As a consequence of (1.6) we extend Bayart’s Montel theorem from the ordinary case (see [2, §4.3.3, Lemma 18]) to $D_\infty(\lambda)$ in the case of $\lambda$’s satisfying $(LC)$ (Theorem 4.10).

Another application of the summation method of typical means gives an alternative proof of the fact that $\mathbb{Q}$-linearly independent $\lambda$’s (that is $\sum q_n \lambda_n = 0$ implies $q = 0$ for all finite rational sequences $q = (q_n)$) imply Bohr’s theorem, which was proven by Bohr in [6]. More precisely we show that in this case the space $D_\infty^{\text{xt}}(\lambda)$ equals $\ell_1$ (as Banach spaces) via $\sum a_n e^{-\lambda_n s} \mapsto (a_n)$ (Theorem 4.7).

Moreover, we would like to consider $D_\infty(\lambda)$ as a Banach space. Unfortunately it may fail to be complete. Based on ideas of Neder we give a construction of $\lambda$’s for which $D_\infty(\lambda)$ is not complete (Theorem 5.2). But there are sufficient conditions on $\lambda$, including $(BC)$ and $\mathbb{Q}$-linearly independence, we present in Theorem 5.1.

Before we start let us mention that recently in [8] given a frequency $\eta$ the authors introduced the space $H_\infty(\eta)$ of all series of the form $\sum b_n \eta^{-s}$, which converge and define a bounded function on $[Re > 0]$. Then defining $\lambda := (\log(\eta_n))$ we have $H_\infty(\eta) = D_\infty(\lambda)$ and so both approaches are equivalent in this sense. All results on $H_\infty(\eta)$ in [8] are based on the assumption that $\lambda$ satisfies the condition $(BC)$. In contrast to this article, we here try to avoid assumptions on $\lambda$ as much as possible.

This text is inspired by the work of (in alphabetical order) Besicovitch, Bohr, Hardy, Landau, Neder, Perron, and M. Riesz. In Section 3 we prove our main result and in Section 4 we apply it to obtain quantitative variants of Bohr’s theorem under different assumptions on $\lambda$, including $(BC)$ and $(LC)$. We finish by Section 5, where we face completeness of $D_\infty(\lambda)$. We start recalling some basics on Dirichlet series.

## 2 GENERAL DIRICHLET SERIES

As already mentioned in the introduction a strictly increasing nonnegative real sequence $\lambda := (\lambda_n)$ tending to $+\infty$ we call a frequency. Then general Dirichlet series $D = \sum a_n e^{-\lambda_n s}$ belonging to some $\lambda$ we call $\lambda$-Dirichlet series and we define $D(\lambda)$ to be the space of all (formal) $\lambda$-Dirichlet series. Moreover, the (complex) coefficient $a_n$ is called the $n$th Dirichlet coefficients of $D$. Finite sums $\sum_{n=1}^N a_n e^{-\lambda_n s}$ are called Dirichlet polynomials.

Recall that the natural domains of convergence of Dirichlet series are half spaces (see [13, Theorem 1, p. 3]). The following “abscissas” rule the convergence theory of general Dirichlet series.

$$\sigma_c(D) = \inf \{ \sigma \in \mathbb{R} \mid D \text{ converges on } [Re > \sigma] \},$$

$$\sigma_a(D) = \inf \{ \sigma \in \mathbb{R} \mid D \text{ converges absolutely on } [Re > \sigma] \},$$

$$\sigma_u(D) = \inf \{ \sigma \in \mathbb{R} \mid D \text{ converges uniformly on } [Re > \sigma] \}.$$
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\[ \sigma_b(D) = \inf \{ \sigma \in \mathbb{R} \mid D \text{ converges and defines a bounded function on } \{ Re > \sigma \} \}. \]

Additionally, we define, provided \( \sigma_c(D) < \infty \),

\[ \sigma^{ext}_b(D) = \inf \{ \sigma \in \mathbb{R} \mid \text{the limit function of } D \text{ allows a holomorphic and bounded extension to } \{ Re > \sigma \} \}. \]

By definition \( \sigma_c(D) \leq \sigma_b(D) \leq \sigma_u(D) \leq \sigma_a(D) \). In general all these abscissas differ. For instance an example of Bohr shows that \( \sigma_c(D) = \sigma^{ext}_b(D) = \sigma_b(D) = -\infty \) and \( \sigma_u(D) = +\infty \) is possible (see [7]). Moreover, general Dirichlet series define holomorphic functions on \( \{ Re > \sigma_c(D) \} \), which relies on the fact that they converge uniformly on all compact subsets of \( \{ Re > \sigma_c(D) \} \) (see [13, Theorem 2, p. 3]). Let us recall the spaces of Dirichlet series already defined in the introduction.

**Definition 2.1.** Let \( \lambda \) be a frequency. We define the space \( D_\infty(\lambda) \) as the space of all \( \lambda \)-Dirichlet series \( D \) which converge on \( \{ Re > 0 \} \) and define a bounded function there.

**Definition 2.2.** We define \( D^{ext}_\infty(\lambda) \) to be the space of all somewhere convergent Dirichlet series \( D \in D(\lambda) \), which allow a holomorphic and bounded extension \( f \) to \( \{ Re > 0 \} \).

We endow \( D^{ext}_\infty(\lambda) \) with the semi norm \( ||D||_\infty := \sup_{\{ Re > 0 \}} |f(s)| \), where \( f \) is the (unique) extension of \( D \). Corollary 3.9 proves that \( || \cdot ||_\infty \) in fact is a norm. Bohr’s theorem (say in the ordinary case) motivates to give the following definition.

**Definition 2.3.** We say that a frequency \( \lambda \) satisfies Bohr’s theorem (or Bohr’s theorem holds for \( \lambda \)), whenever every \( D \in D^{ext}_\infty(\lambda) \) converges uniformly on \( \{ Re > \varepsilon \} \) for all \( \varepsilon > 0 \).

Observe that \( \lambda \) satisfies Bohr’s theorem if and only if \( \sigma^{ext}_b = \sigma_u \) for all somewhere convergent \( \lambda \)-Dirichlet series.

Let us consider again \( \lambda = (n) = (0, 1, 2, \ldots) \) to give an easy example. Then (BC) holds and via the substitution \( z = e^{-s} \), which maps the open right half plane \( \{ Re > 0 \} \) to the open punctured unit disc \( D \setminus \{0\} \), the space \( D_\infty((n)) \) coincides with \( H_\infty(D) \); the space of all holomorphic and bounded functions on \( D \). In this case Bohr’s theorem states the fact, that if a power series \( P(z) = \sum c_n z^n \) converging on some neighbourhood of the origin allows an extension \( g \in H_\infty(D) \), then \( P \) actually converges in \( D \) and coincides with \( g \) with uniform convergence on each closed disk contained in \( D \).

**2.1 A Bohr–Cahen formula**

There are useful Bohr–Cahen formulas for the abscissas \( \sigma_c \) and \( \sigma_a \), that are, given \( D = \sum a_n e^{-\lambda_n s} \),

\[ \sigma_c(D) \leq \limsup_N \frac{\log\left( \sum_{n=1}^{N} |a_n| \right)}{\lambda_N} \quad \text{and} \quad \sigma_u(D) \leq \limsup_N \frac{\log\left( \sum_{n=1}^{N} |a_n| \right)}{\lambda_N}, \]

where in each case equality holds if the left hand side is nonnegative. See [13, Thm. 7 and 8, p. 8] for a proof. The formula for \( \sigma_u \) (and its proof) extends from the ordinary case in [11, §1.1, Prop. 1.6] canonically to arbitrary \( \lambda \)’s:

\[ \sigma_u(D) \leq \limsup_N \frac{\log\left( \sup_{t \in \mathbb{R}} \sum_{n=1}^{N} |a_n e^{-\lambda_n i t}| \right)}{\lambda_N}, \tag{2.1} \]

where again the equality holds if the left hand side is nonnegative. In this section we derive (2.1) from the following proposition concerning uniform convergence of sequences of Dirichlet series and take advantage of both in Section 4. Then the particular case of a sequence of partial sums will reprove (2.1).
Therefore given a sequence of (formal) $\lambda$-Dirichlet series $D_j = \sum a^j_n e^{-\lambda_n s}$ we define

$$\Delta = \Delta((D_j)) : = \limsup_{(N,j) \in \mathbb{N}^2} \log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a^j_n e^{-\lambda_n it} \right| \right),$$

where we endow $\mathbb{N}^2$ with the product order, that is $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$.

**Proposition 2.4.** Let $\lambda$ be frequency and let $D_j = \sum a^j_n e^{-\lambda_n s}$ be a sequence of $\lambda$-Dirichlet series, such that the limits $a_n := \lim_{j \to \infty} a^j_n$ exist for all $n$. Then $(D_j)$ converges uniformly on $[\Re > \Delta + \varepsilon]$ to $D := \sum a_n e^{-\lambda_n s}$ for all $\varepsilon > 0$.

Dealing with uniform convergence on half spaces it is enough to check on vertical lines, since for any finite complex sequence $(a_n)$ we for all $x \geq 0$ have

$$\sup_{[\Re > x]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| = \sup_{[\Re = x]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right|,$$

which is a consequence of the modulus maximum principle (see e.g. the Phragmén–Lindelöf Theorem from [3, p. 137] or [11, Lemma 1.7, §1.1] for $\lambda = (\log(n))$, where the proof extends to the general case).

**Proof of Proposition 2.4.** Assume, that $\Delta < \infty$ (otherwise the claim is trivial) and let $\varepsilon > 0$. Then by definition of $\Delta$

$$\exists N_1 \exists j_1 \forall N \geq N_1 \forall j \geq j_1 : \log \left( \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a^j_n e^{-\lambda_n it} \right| \right) \lambda_N < \Delta + \varepsilon,$$

and so for all $N \geq N_1$ and $j \geq j_1$

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a^j_n e^{-\lambda_n it} \right| \leq e^{\lambda_N (\Delta + \varepsilon)}.$$

Now we fix $M, N \geq N_1, j \geq j_1$, write for simplicity $S^j_M(it) := \sum_{k=1}^{N} a^j_k e^{-\lambda_k it}$ and we set $\sigma_0 := \Delta + 2\varepsilon$. Then by Abel summation and (2.2)

$$\left| \sum_{n=N+1}^{M} a^j_n e^{-\lambda_n (\sigma_0 + it)} \right| \leq \left| S^j_M(it) e^{-\lambda_N \sigma_0} \right| + \left| S^j_M(it) e^{-\lambda_M \sigma_0} \right| + \sum_{n=N}^{M-1} \left| S^j_M(it) \right| \left| e^{-\lambda_n \sigma_0} - e^{-\lambda_{n+1} \sigma_0} \right|$$

$$\leq e^{-\lambda_N \varepsilon} + e^{-\lambda_M \varepsilon} + \sum_{n=N}^{M-1} \left| S^j_M(it) \right| \left| e^{-\lambda_n \sigma_0} - e^{-\lambda_{n+1} \sigma_0} \right|.$$

Since

$$\left| e^{-\lambda_n \sigma_0} - e^{-\lambda_{n+1} \sigma_0} \right| = \left| \sigma_0 \int_{\lambda_n}^{\lambda_{n+1}} e^{-\sigma_0 x} dx \right| \leq |\sigma_0| \int_{\lambda_n}^{\lambda_{n+1}} e^{-\sigma_0 x} dx,$$

we obtain

$$\sum_{n=N}^{M-1} \left| S^j_M(it) \right| \left| e^{-\lambda_n \sigma_0} - e^{-\lambda_{n+1} \sigma_0} \right| \leq |\sigma_0| \sum_{n=N}^{M-1} e^{\lambda_n (\Delta + \varepsilon)} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\sigma_0 x} dx$$
\[ \leq |\sigma_0| \sum_{n=N}^{M-1} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\sigma_0 x} e^{x(\Delta + \epsilon)} \, dx \]
\[ = |\sigma_0| \sum_{n=N}^{M-1} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\epsilon x} \, dx \]
\[ = |\sigma_0| \int_{\lambda_N}^{\lambda_M} e^{-\epsilon x} \, dx \leq |\sigma_0| \int_{\lambda_N}^{\infty} e^{-\epsilon x} \, dx = |\sigma_0| \frac{1}{\epsilon} e^{-\lambda_N \epsilon}. \]

So together we for all \( M \geq N \geq N_1 \) and \( j \geq j_1 \) have

\[ \sup_{\text{Re}=\sigma_0} \left| \sum_{n=N}^{M} a_n e^{-\lambda_n s} \right| \leq e^{-\lambda_N \epsilon} \left( 2 + \frac{1}{\epsilon} \right). \]

Now tending \( j \to \infty \) gives

\[ \sup_{\text{Re}=\sigma_0} \left| \sum_{n=N}^{M} a_n e^{-\lambda_n s} \right| \leq e^{-\lambda_N \epsilon} \left( 2 + \frac{1}{\epsilon} \right), \]

which implies that \( D \) converges on \( [\text{Re} > \Delta] \). Moreover for all \( j \geq j_1 \) and \( N \geq N_1 \) we have

\[ \left| \sum_{n=1}^{\infty} (a_n - a_n) e^{-\lambda_n s} \right| \leq \sum_{n=1}^{N} |a_n - a_n| + \sum_{n=N}^{\infty} (a_n - a_n) e^{-\lambda_n s} \]
\[ = \sum_{n=1}^{N} |a_n - a_n| + \lim_{M \to \infty} \lim_{k \to \infty} \left| \sum_{n=N}^{M} (a_n - a_n) e^{-\lambda_n s} \right| \]
\[ \leq \sum_{n=1}^{N} |a_n - a_n| + 2 e^{-\lambda_N \epsilon} \left( 2 + \frac{1}{\epsilon} \right), \]

and so

\[ \limsup_{j \to \infty} \sup_{\text{Re}=\sigma_0} \left| \sum_{n=1}^{\infty} (a_n - a_n) e^{-\lambda_n s} \right| \leq e^{-\lambda_N \epsilon} \left( 2 + \frac{1}{\epsilon} \right). \]

which proves the claim tending \( N \to \infty \). \( \square \)

**Corollary 2.5.** Let \( D = \sum a_n e^{-\lambda_n s} \) be a \( \lambda \)-Dirichlet series. Then

\[ \sigma_u(D) \leq \limsup_{N \to \infty} \frac{\log(\sup_{t \in \mathbb{R}} |\sum_{n=1}^{N} a_n e^{-\lambda_n i t}|)}{\lambda_N}. \]

**Proof.** Defining \( D_j = \sum_{n=1}^{j} a_n e^{-\lambda_n s} = \sum a_n e^{-\lambda_n s} \) we obtain

\[ \Delta(D_j) \leq \limsup_{N \to \infty} \frac{\log(\sup_{t \in \mathbb{R}} |\sum_{n=1}^{N} a_n e^{-\lambda_n i t}|)}{\lambda_N}. \]
Indeed, if \( j, N \in \mathbb{N} \) and \( m := \min(j, N) \), then

\[
\log \left( \frac{\sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_{n} e^{-\lambda_{n} t} \right|}{\lambda_{N}} \right) \leq \log \left( \frac{\sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{m} a_{n} e^{-\lambda_{n} t} \right|}{\lambda_{m}} \right).
\]

\[\square\]

### 3 MAIN RESULT AND APPROXIMATION BY TYPICAL RIESZ MEANS

Recall that by Definition 2.3 a frequency \( \lambda \) satisfies Bohr’s theorem if every \( D \in D_{\infty}^{\text{ext}}(\lambda) \) converges uniformly on \([Re > \varepsilon]\) for all \( \varepsilon > 0 \) or equivalently the equality

\[
\sigma_{b}^{\text{ext}} = \sigma_{u}
\]

holds for all somewhere convergent \( \lambda \)-Dirichlet series. As already mentioned in the introduction it was a prominent question in the beginning of the 20th century for which \( \lambda \)’s the equality (3.1) holds. The following remark shows how control of the norm of the partial sum operator

\[
S_{N} : D_{\infty}^{\text{ext}}(\lambda) \to D_{\infty}(\lambda), \quad D \mapsto \sum_{n=1}^{N} a_{n}(D)e^{-\lambda_{n}s}, \quad N \in \mathbb{N},
\]

is linked with (3.1).

**Remark 3.1.** By Corollary 2.5 the equality (3.1) holds if

\[
\limsup_{N \to \infty} \frac{\log \left( \| S_{N} : D_{\infty}^{\text{ext}}(\lambda) \to D_{\infty}(\lambda) \| \right)}{\lambda_{N}} = 0.
\]

As announced, our main result gives bounds of \( \| S_{N} \| \) without any assumptions on \( \lambda \), which is a sort of uniform version of [13, Thm. 21, p. 36].

**Theorem 3.2.** For all \( 0 < k \leq 1, N \in \mathbb{N} \) and \( D = \sum a_{n} e^{-\lambda_{n}s} \in D_{\infty}(\lambda) \) we have

\[
\sup_{[Re > 0]} \left| \sum_{n=1}^{N} a_{n} e^{-\lambda_{n}s} \right| \leq C \frac{\Gamma(k + 1)}{k} \frac{\lambda_{N+1} - \lambda_{N}}{\lambda_{N+1}} ^{k} \| D \|_{\infty},
\]

where \( C > 0 \) is a universal constant and \( \Gamma \) denotes the Gamma function.

**Remark 3.3.** According to Theorem 3.2 and Corollary 2.5 the equality \( \sigma_{b}^{\text{ext}} = \sigma_{u} \) holds if there is a zero sequence \( (k_{N}) \) such that

\[
\limsup_{N \to \infty} \frac{\log \left( \frac{\lambda_{N}}{k_{N} \lambda_{N+1} - \lambda_{N}} ^{k_{N}} \right)}{\lambda_{N}} = 0.
\]

In Section 4 we revisit the conditions \((BC)\) and \((LC)\) of Bohr and Landau (see (1.2) and (1.3) in the introduction), and show that they are sufficient for (3.1) by choosing suitable sequences \( (k_{N}) \) (Theorems 4.2 and 4.4).

Now let us prepare the proof of Theorem 3.2. We need several ingredients, and start with the following result, which is of independent interest.
Proposition 3.4. Let $D = \sum a_n e^{-\lambda_n s} \in D^x_{\infty}(\lambda)$ with extension $f$. Then for all $k > 0$ the Dirichlet polynomials

$$R^k(D) = \sum_{\lambda_n < x} a_n \left(1 - \frac{\lambda_n}{x}\right)^k e^{-\lambda_n s}$$

converge uniformly to $f$ on $[Re > \varepsilon]$ as $x \to \infty$ for all $\varepsilon > 0$. Moreover,

$$\sup_{x \geq 0} \|R^k_x(D)\|_{\infty} \leq \frac{e}{\pi} \Gamma(k + 1) \left(1 + \frac{1}{k}\right) \|D\|_{\infty}.$$ (3.2)

Proposition 3.4 is indicated after the proof of [13, Thm. 41, p. 53] (without inequality (3.2)). In the language of [13] it states that on every smaller halfplane $[Re > \varepsilon]$ the limit functions of Dirichlet series $D \in D^x_{\infty}(\lambda)$ are uniform limits of their typical (first) means of any order $k > 0$. The proof relies on a formula of Perron (see [13, Thm. 39, p. 50]). We give an alternative proof of this formula (Lemma 3.6) using the Fourier inversion formula and we first deduce (3.2) from it. Then, using a Bohr–Cahen type formula for the abscissa of uniform summability by typical first means (Lemma 3.8), we show that (3.2) implies the first part of Proposition 3.4. Note that $\lambda$ satisfies Bohr’s theorem, if the first part of Proposition 3.4 is valid for $k = 0$.

The second main ingredient for the proof of Theorem 3.2 links partial sums of a Dirichlet series to its typical means.

Lemma 3.5. For any choice of complex numbers $a_1, \ldots, a_N$ and $0 < k \leq 1$ we have

$$\left|\sum_{n=1}^N a_n \right| \leq 3 \left(\frac{1}{\lambda_{N+1} - \lambda_N}\right)^k \sup_{0 \leq x \leq \lambda_{N+1}} \left|\sum_{\lambda_n < x} a_n (x - \lambda_n)^k\right|.$$ (3.3)

Let us first show how Proposition 3.4 and Lemma 3.5 gives Theorem 3.2 before we prove them.

Proof of Theorem 3.2. Let $s \in [Re > 0]$. Then by (3.2) from Proposition 3.4

$$\sup_{0 \leq x \leq \lambda_{N+1}} \left|\sum_{\lambda_n < x} a_n e^{-\lambda_n s} (x - \lambda_n)^k\right| \leq \lambda_{N+1}^k \sup_{0 \leq x \leq \lambda_{N+1}} \left|\sum_{\lambda_n < x} a_n e^{-\lambda_n s} \left(1 - \frac{\lambda_n}{x}\right)^k\right|$$

$$\leq \lambda_{N+1}^k \frac{2e}{\pi} \frac{\Gamma(k + 1)}{k} \|D\|_{\infty}.$$ (3.4)

So together with Lemma 3.5 we obtain

$$\sup_{[Re > 0]} \left|\sum_{n=1}^N a_n e^{-\lambda_n s}\right| \leq \frac{6e}{\pi} \frac{\Gamma(k + 1)}{k} \left(\frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N}\right)^k \|D\|_{\infty}. \quad \square$$ (3.5)

Now let us prepare the proofs of Proposition 3.4 and Lemma 3.5. The former relies on the following formula of Perron from [13, Thm. 39, p. 50].

Lemma 3.6. Let $D = \sum a_n e^{-\lambda_n s} \in D^x_{\infty}(\lambda)$ with extension $f$. Then for all $\varepsilon > 0$ and $k \geq 0$

$$\sum_{\lambda_n < x} a_n \left(1 - \frac{\lambda_n}{x}\right)^k = \frac{\Gamma(k + 1)}{2\pi i} \frac{1}{x^k} \int_{\varepsilon - i \infty}^{\varepsilon + i \infty} f(s) e^{xs} \frac{ds}{s^{1+k}}.$$ (3.6)

for all $x \in \mathbb{R}$ if $k > 0$ and $x \notin \{\lambda_n \mid n \in \mathbb{N}\}$ whenever $k = 0$.

For the sake of completeness we like to mention, that actually, if $k = 0$ and $x = \lambda_n$ for some $n$, then (3.6) also holds true, if the integral on the right hand side is defined by its principle value (see [13, Thm. 13, p. 12]). This case (which is
not needed in the following) is not covered by our alternative proof of Lemma 3.6, where we use the Fourier inversion formula. We need the following observation.

**Lemma 3.7.** Let \( D = \sum a_n e^{-\lambda_n s} \in D(\lambda) \) with \( \sigma_c(D) < \infty \). Then for all \( \sigma > \max(0, \sigma_c(D)) \) and \( k \geq 0 \) the function

\[
x \mapsto e^{-\sigma x} \sum_{\lambda_n < x} a_n (x - \lambda_n)^k
\]

belongs to \( L_1(\mathbb{R}) \).

**Proof.** Fix \( \sigma > \max(0, \sigma_c(D)) \) and write \( A^k(t) := \sum_{\lambda_n < t} a_n (t - \lambda_n)^k \). Let first \( k = 0 \). Then by Abel summation for all \( t > 0 \)

\[
A^0(t) = e^{\sigma t} \sum_{\lambda_n < t} a_n e^{-\sigma \lambda_n} - \sigma \int_0^t e^{\sigma y} \sum_{\lambda_n < y} a_n e^{-\sigma \lambda_n} dy.
\]

(3.4)

Let \( \varepsilon > 0 \). Since \( D \) converges at \( \sigma \) we for all \( 0 < t \leq x \) obtain multiplying by \( e^{-(\sigma + \varepsilon) x} \)

\[
|A^0(t)| e^{-(\sigma + \varepsilon) x} \leq C(\sigma) e^{-\varepsilon x} + \sigma C(\sigma) e^{-\varepsilon x} \int_0^\infty e^{-\sigma u} du = 2C(\sigma) e^{-\varepsilon x}.
\]

(3.5)

In particular \( |A^0(x)| e^{-(\sigma + \varepsilon) x} \leq 2C(\sigma) e^{-\varepsilon x} \) for all \( x \in \mathbb{R} \) and so \( A^0 e^{-(\sigma + \varepsilon)} \in L_1(\mathbb{R}) \). Now let \( k > 0 \). By Abel summation for all \( x \geq 0 \) we have

\[
A^k(x) = \sum_{\lambda_n < x} a_n (x - \lambda_n)^k = k \int_0^x (x - t)^{k-1} A^0(t) dt,
\]

(3.6)

which is taken from [13, Ch. IV, §2, p. 21]. Again by multiplying by \( e^{-(\sigma + \varepsilon) x} \) we obtain by using (3.5)

\[
e^{-(\sigma + \varepsilon) x} |A^k(x)| \leq 2C(\sigma) e^{-\varepsilon x} k \int_0^x (x - t)^{k-1} dt = 2C(\sigma) e^{-\varepsilon x} x^k,
\]

and so \( e^{-(\sigma + \varepsilon)} A^k \in L_1(\mathbb{R}) \).

\( \Box \)

**Proof of Lemma 3.6.** Let first \( \sigma > \max(0, \sigma_c(D)) \). Then for all \( s \in [Re > \sigma] \) (see [13, Thm. 24, p. 39], where \( k > 0 \) is considered, but the case \( k = 0 \) follows in the same way applying Abel summation)

\[
f(s) = \frac{1}{\Gamma(k+1)} \int_0^\infty s^{k+1} e^{-st} \sum_{\lambda_n < t} a_n (t - \lambda_n)^k dt
\]

and so

\[
\frac{\Gamma(k+1)}{2\pi i} \int_{2\sigma - i\infty}^{2\sigma + i\infty} \frac{f(s)e^{xs}}{s^{k+1}} ds = \frac{1}{2\pi i} \int_{2\sigma - i\infty}^{2\sigma + i\infty} \int_0^\infty e^{s(x-t)} \sum_{\lambda_n < t} a_n (t - \lambda_n)^k dt ds.
\]

Now again for simplicity we write \( A^k(t) := \sum_{\lambda_n < t} a_n (t - \lambda_n)^k \), which is a differentiable function on \( \mathbb{R} \), if \( k > 0 \). The function \( A^0 \) is not differentiable at \( \lambda_n \) for all \( n \), but elsewhere else. Moreover, by Lemma 3.7 we know that \( A^k e^{-2\sigma} \in L_1(\mathbb{R}) \) for all \( k \geq 0 \). Now, denoting by \( F_{L_1(\mathbb{R})} \) the Fourier transform on \( L_1(\mathbb{R}) \), the Fourier inversion formula (see [16, §1.2]) gives

\[
\frac{\Gamma(k+1)}{2\pi i} \int_{2\sigma - i\infty}^{2\sigma + i\infty} \frac{f(s)e^{xs}}{s^{k+1}} ds = \frac{1}{2\pi i} \int_{2\sigma - i\infty}^{2\sigma + i\infty} \int_0^\infty e^{s(x-t)} A^k(t) dt ds.
\]
\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{x(2\sigma + iy)} \int_{0}^{\infty} A^k(t) e^{-t(2\sigma + iy)} \, dt \, dy
\]
\[
= e^{2x\sigma} \int_{-\infty}^{\infty} F_{\lambda_1} (A^k e^{-2\sigma \cdot}) (y) \, e^{iyx} \, dy
\]
\[
= e^{2x\sigma} F_{\lambda_1} (F_{\lambda_1} (A^k e^{-2\sigma \cdot})) (-x) = e^{2x\sigma} A^k(x) e^{-2x\sigma} = A^k(x),
\]
where \( x \notin \lambda \) if \( k = 0 \) and \( x \) is arbitrary if \( k > 0 \). This implies (3.3) for \( \varepsilon = 2\sigma \), where \( \sigma > \max(0, \sigma_c(D)) \). For \( \varepsilon > 0 \) arbitrary by Cauchy’s integral theorem (and boundedness of \( f \)) we have
\[
\int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{f(s)e^{xs}}{s^{1+k}} \, ds = \int_{2(\varepsilon+\max(0,\sigma_c(D)))+i\infty}^{2(\varepsilon+\max(0,\sigma_c(D)))-i\infty} \frac{f(s)e^{xs}}{s^{1+k}} \, ds,
\]
which finishes the proof.

□

For the proof of Proposition 3.4 it remains to verify the following Bohr–Cahen type formula for the abscissa \( \sigma_k^u(D) \) of uniform summability by typical first means of order \( k \), which is defined to be the infimum of all \( \sigma \in \mathbb{R} \) such that \( (R^k_s(D)) \) converges uniformly on \( \{Re > \sigma\} \) as \( x \to \infty \).

Lemma 3.8. Let \( 0 < k \leq 1 \) and \( D \in D(\lambda) \). Then
\[
\sigma_k^u(D) \leq \limsup_{x \to \infty} \frac{\log \left( \|R^k_s(D)\|_\infty \right)}{x},
\]
(3.7)
where equality holds whenever \( \sigma_k^u(D) \) is nonnegative.

Proof. Let \( L \) denote the right hand side of (3.7). We first show (3.7) and assume that \( L < \infty \), since otherwise the claim is trivial. Hence, fixing \( \varepsilon > 0 \), there is a constant \( C \) such that for all \( x \)
\[
\left\| R^k_s(D) \right\|_\infty \leq Ce^{(L+\varepsilon)x}.
\]
(3.8)
Let \( u := L + 3\varepsilon \) and we claim that \( (R^k_s(D)) \) converges uniformly on \( \{Re = u\} \), tending \( x \to \infty \). For \( w \in \mathbb{C} \) for notational simplicity we write \( A^k_0(t) = \sum_{\lambda_j < t} a_n e^{-u\lambda_n(t-\lambda_n)^k} \), where \( 0 \leq k \leq 1 \). Following the lines of [13, §VI.3 b), p. 42] by Abel’s summation we for all \( s, w \in \mathbb{C} \) obtain
\[
R^k_s(D)(s+w) = e^{-sx} R^k_s(D)(w) - \frac{1}{x^k} \int_{0}^{x} A^k_0(t) \frac{d}{dt} ((e^{-st} - e^{-sx})(x-t)^k) \, dt.
\]
(3.9)
In particular, with the choice \( s = u \) and \( w = i\tau \), where \( \tau \in \mathbb{R} \), by (3.8) the first term on the right hand side of (3.9) vanishes whenever \( x \to \infty \). Applying integration by parts we obtain
\[
-\frac{1}{x^k} \int_{0}^{x} A^k_0(t) \frac{d}{dt} ((e^{-st} - e^{-sx})(x-t)^k) \, dt = \frac{1}{x^k} \int_{0}^{x} A^k_1(t) \frac{d^2}{dt^2} ((e^{-ut} - e^{-ux})(x-t)^k) \, dt,
\]
(3.10)
where the second derivative appearing is given by
\[
u^2 e^{-ut} (x-t)^k + 2kue^{-ut} (x-t)^{k-1} + k(k-1)(e^{-ut} - e^{-ux})(x-t)^{k-2} = g_1(t) + g_2(t) + g_3(t).
\]
If \( k = 1 \), then \( g_3 \) vanishes and by assumption we have
\[
|A^k_1(t)| = |tR^k_1(D)(i\tau)| \leq C_1 e^{(L+2\varepsilon)|t|}
\]
with some constant $C_1 = C_1(\varepsilon)$. If $0 < k < 1$, then by [13, Lem. 6, p. 27]

$$
\Gamma(k + 1)\Gamma(1 - k)A_{1i}^1(t) = \int_0^t A_{1i}^1(y)(t - y)^{-k} \, dy,
$$

and so for all $t > 0$ with $C_2 = C_2(k) = (\Gamma(k + 1)\Gamma(1 - k)(1 - k))^{-1}$

$$
|A_{1i}^1(t)| \leq C_2 t^{-k} \sup_{0 < y < t} |A_{1i}^1(y)| = C_2 t^{-k} \sup_{0 < y < t} |\mathcal{K}^R_k(i\tau)| \leq C_3 e^{(1 + 2\varepsilon)t},
$$

where $C_3 = C_3(k, \varepsilon)$. With $C_4 := \max(C_1, C_3)$ for $0 < k \leq 1$ we have

$$
\left| \frac{1}{x^k} \int_0^x A_i^1(i\tau)g_2(t) \, dt \right| \leq C_4 \frac{2ku}{x^k} \left( \int_0^\frac{x}{2} e^{-ut}(x - t)^{k-1} \, dt + \int_\frac{x}{2}^x e^{-ut}(x - t)^{k-1} \, dt \right)
$$

$$
\leq C_4 \frac{2ku}{x^k} \left( \frac{x}{2} \right)^{k-1} \int_0^\infty e^{-ut} \, dt + e^{-\frac{x}{2}} \int_\frac{x}{2}^x (x - t)^{k-1} \, dt
$$

$$
= C_4 \frac{2ku}{x^k} \left( \frac{x}{2} \right)^{k-1} \frac{1}{\varepsilon} + e^{-\frac{x}{2}} \frac{1}{k} \left( \frac{x}{2} \right)^k \leq C_5 \left( \frac{1}{x} + e^{-\frac{x}{2}} \right),
$$

which tends to zero uniformly in $\tau$ as $x \to \infty$. Analogously, using $|e^{-ut} - e^{-ux}| \leq u(x - t)e^{-ut}$, we obtain

$$
\left| \frac{1}{x^k} \int_0^x A_i^1(i\tau)g_3(t) \, dt \right| \leq C_4 \frac{k(1 - k)}{x^k} \int_0^x e^{-ut}u(x - t)(x - t)^{k-2} \, dt
$$

$$
= C_4 \frac{k(1 - k)u}{x^k} \int_0^x e^{-ut}(x - t)^{k-1} \, dt,
$$

which also vanishes uniformly in $\tau$ tending $x \to \infty$. It remains to consider the integral with $g_1$. The dominated convergence theorem implies for all $\tau$

$$
\lim_{x \to \infty} \frac{1}{x^k} \int_0^x u^2 e^{-ut}(x - t)^k A_{1i}^1(i\tau) \, dt = \int_0^\infty u^2 e^{-ut} A_{1i}^1(i\tau) \, dt,
$$

and we claim that the convergence is uniform in $\tau$. Indeed, we have

$$
\left| \frac{1}{x^k} \int_0^x u^2 e^{-ut}(x - t)^k A_{1i}^1(i\tau) \, dt - \int_0^\infty u^2 e^{-ut} A_{1i}^1(i\tau) \, dt \right| \leq \left| \int_0^\infty u^2 e^{-ut} A_{1i}^1(i\tau) \left( 1 - \frac{t}{x} \right)^k \, dt \right|
$$

$$
\leq C_4 \int_0^\infty u^2 e^{-ut} \left( 1 - \chi_{(0, x)}(t) \left( 1 - \frac{t}{x} \right)^k \right) \, dt.
$$

Now, since the function $h_x(t) := \left( 1 - \chi_{(0, x)}(t) \left( 1 - \frac{t}{x} \right)^k \right)$ vanishes as $x \to \infty$ and $|h_x(t)| \leq 1$, by the dominated convergence theorem we obtain

$$
\lim_{x \to \infty} \frac{1}{x^k} \int_0^x u^2 e^{-ut}(x - t)^k A_{1i}^1(i\tau) \, dt = \int_0^\infty u^2 e^{-ut} A_{1i}^1(i\tau) \, dt
$$

(3.12)

uniformly in $\tau$. In summary, the sequence $(R^C_k(D))$ on $[Re = u]$ converges uniformly to (3.12) tending $x \to \infty$, which proves (3.7). Now assume that $\sigma^C_k(D) \geq 0$. Let $\varepsilon > 0$ and $u := \sigma^C_k(D) + \varepsilon$. Then $(R^C_k(D))$ converges uniformly on $[Re = u]$, and so $\sup_x \sup_{[Re = u]} |R^C_k(D)(s)| =: C_6 < \infty$. Let first $0 < k < 1$. Then by (3.9) with the choice $s = -u$ and $w = u + i\tau$, following
the calculation we used before (in particular using the adapted variant of (3.11)) we obtain

\[
|R_k^x(D)(i\tau)| \leq C_6 \left( e^{ux} + \frac{1}{x^k} \frac{1}{\Gamma(1+k)\Gamma(1-k)} \int_0^x \left| A_k^y(y) \right| (t-y)^{-k} dy \left| \frac{d^2}{dt^2} ((e^{ut} - e^{ux})(x-t)^k) \right| dt \right)
\]

\[
\leq C_6 \left( e^{ux} + \frac{1}{x^k} \frac{1}{\Gamma(1+k)\Gamma(1-k)} \int_0^x y^k (t-y)^{-k} dy \left| \frac{d^2}{dt^2} ((e^{ut} - e^{ux})(x-t)^k) \right| dt \right).
\]

By substitution with \( v = \frac{y}{t} \) we for every \( \alpha, \beta, t > 0 \) obtain

\[
\frac{1}{t^{\alpha+\beta}} \int_0^t y^x (t-y)^{\beta-1} dy = \int_0^1 v^x (1-v)^{\beta-1} dv = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}.
\]

(3.13)

In particular, if \( 0 < k < 1 \), then the choice \( \alpha = k \) and \( \beta = 1 - k \) gives

\[
\int_0^t y^k (t-y)^{-k} dy = \Gamma(k+1)\Gamma(1-k)t.
\]

(3.14)

Using that, we continue estimating

\[
|R_k^x(D)(i\tau)| \leq C_6 \left( e^{ux} + \frac{1}{x^k} \frac{1}{\Gamma(1+k)\Gamma(1-k)} \int_0^x t \left| \frac{d^2}{dt^2} ((e^{ut} - e^{ux})(x-t)^k) \right| dt \right)
\]

\[
\leq C_6 e^{ux} \left( 1 + \frac{1}{x^{k-1}} \int_0^x u^2 (x-t)^k + (2k'u + k(1-k)u)(x-t)^{k-1} dt \right)
\]

\[
\leq C_7 e^{ux} (1 + x^2 + x).
\]

Hence for all \( \varepsilon > 0 \)

\[
\limsup_{x \to \infty} \frac{\log \left( \|R_k^x(D)\|_\infty \right)}{x} \leq u + \limsup_{x \to \infty} \frac{\log \left( C_7(1 + x^2 + x) \right)}{x} = \sigma_u(D) + \varepsilon,
\]

and so \( L \leq \sigma_u(D) \), if \( 0 < k < 1 \). The remaining case \( k = 1 \) follows in a similar way. Indeed, (3.9) with the choice \( s = -u \) and \( w = u + i\tau \) together with the adapted variant of (3.10) leads to

\[
|R_1^x(D)(i\tau)| \leq C_6 \left( e^{ux} + \frac{1}{x} \int_0^x \left| \frac{d}{dt} (ue^{ut} - xe^{ux})(x-t)^k \right| dt \right) \leq C_8 e^{ux} (1 + x^3),
\]

which finally completes the proof. \( \square \)

Proof of Proposition 3.4. First we prove the stated inequality. Let \( x \geq 0 \) and \( \varepsilon = \frac{1}{x} \). Then applying for fixed \( s_0 \in [Re > 0] \) Lemma 3.6 to the translated Dirichlet series \( D_{s_0}(s) = \sum a_n e^{-\lambda_n x_0} e^{-\lambda_n s} \) with extension \( f_{s_0}(s) := f(s + s_0) \) we obtain

\[
\sup_{s_0 \in [Re > 0]} \left| \sum a_n \left( 1 - \frac{x_0}{x} \right)^{\lambda_n} e^{-\lambda_n s_0} \right| \leq \|D\|_{\infty} \frac{1}{x^k} \frac{\Gamma(k+1)}{2\pi} e^{\frac{\pi}{2} \varepsilon} \int_0^\infty \frac{1}{|\varepsilon + it|^{1+k}} dt
\]

\[
\leq \|D\|_{\infty} \frac{1}{x^k} \frac{\Gamma(k+1)}{\pi} e^{\left( 1 + \frac{1}{k} \right) x^k}
\]

\[
= \|D\|_{\infty} \frac{e}{\pi} \Gamma(k+1) \left( 1 + \frac{1}{k} \right).
\]
Now having \((3.2)\), Lemma 3.8 implies that for \(0 < k \leq 1\) the Dirichlet polynomials \(R^k_x(D)\) converge uniformly on \([\text{Re} > \varepsilon]\) as \(x \to \infty\) for every \(\varepsilon > 0\). Then \(h(s) := \lim_{x \to \infty} R^k_x(D)(s)\) defines a holomorphic function (see e.g. [13, Thm. 27, p. 44]) and, since \(D\) converges at some \(s_0 = \sigma_0 + it_0\), \(h\) coincides with \(f\) on \([\text{Re} > \sigma_0]\) (see also [13, Thm. 16, p. 29]). Hence \(h = f\) and \((R^k_x(D))\) converges uniformly to \(f\) on every smaller halfplane contained in \([\text{Re} > 0]\). So the claim holds for \(0 < k \leq 1\). Let \(k > 1\) and write \(k = l + k'\), where \(0 < k' \leq 1\) and \(l \in \mathbb{N}\). Then by [13, Lem. 6, p. 27], or by direct inspection

\[
R^k_x(D)(s) = \frac{\Gamma(k + 1)}{\Gamma(k') + 1} \frac{1}{\Gamma(l)} \int_0^x \left( \sum_{\lambda_n \leq t} a_n e^{-\lambda_n s}(t - \lambda_n)^k \right)(x - t)^{l-1} dt. \tag{3.15}
\]

Now by (3.13) with \(\alpha = k'\) and \(\beta = l\) we have

\[
\frac{\Gamma(k' + 1) \Gamma(l)}{\Gamma(k + 1)} = \frac{1}{x^k} \int_0^x t^{k'}(x - t)^{l-1} dt. \tag{3.16}
\]

Then writing \(C := \frac{\Gamma(k + 1)}{\Gamma(k' + 1) \Gamma(l)}\) and using (3.16) we obtain for every \(s \in [\text{Re} > 0]\)

\[
R^k_x(D)(s) - f(s) = R^k_x(D)(s) - Cf(s) \frac{1}{x^k} \int_0^x t^{k'}(x - t)^{l-1} dt \\
= \frac{C}{x^k} \int_0^x (x - t)^{l-1} t^{k'} (R^k_{r_t}(D)(s) - f(s)) dt.
\]

So, given \(\varepsilon > 0\) and \(u > 0\), choose \(x_0\) such that \(|R^k_{r_t}(D)(s) - f(s)| \leq \varepsilon\) for all \(t > x_0\) and \(s \in [\text{Re} > u]\). Then for all \(x > x_0\) and \(s \in [\text{Re} > u]\) we have

\[
|R^k_x(D)(s) - f(s)| \leq \sup_{y \geq 0} \|R^y_{r_t}(D) - f\|_\infty \frac{C}{x^k} \int_{x_0}^x (x - t)^{l-1} t^{k'} dt + \varepsilon \frac{C}{x^k} \int_{x_0}^x (x - t)^{l-1} t^{k'} dt \\
\leq 2C\|D\|_\infty \frac{x_0^{k'}}{x^k} \frac{2x^l}{l} + \varepsilon \frac{Cx^{k'} x_l}{x^k l} \\
\leq 4C\left(\|D\|_\infty \left(\frac{x_0}{x}\right)^{k'} + \varepsilon\right),
\]

which finishes the proof tending \(x \to \infty\). \(\square\)

Proposition 3.4 gives a direct link to the theory of almost periodic functions on \(\mathbb{R}\) and proves that the space \((D^*_{\infty}(\lambda), \| \cdot \|_\infty)\) actually is a normed space. Recall that by definition a continuous function \(f : \mathbb{R} \to \mathbb{C}\) is called (uniformly) almost periodic, if to every \(\varepsilon > 0\) there is a number \(l > 0\) such that for all intervals \(I \subset \mathbb{R}\) with \(|I| = l\) there is a translation number \(\tau \in I\) such that \(\sup_{x \in \mathbb{R}} |f(x + \tau) - f(x)| \leq \varepsilon\) (see [3] for more information). Then by a result of Bohr a bounded and continuous function \(f\) is almost periodic if and only if it is the uniform limit of trigonometric polynomials on \(\mathbb{R}\), which are of the form \(p(t) := \sum_{n=1}^N a_n e^{-i\lambda_n t}\), where \(x_n \in \mathbb{R}\) (see e.g. [20, §1.5.2.2, Thm. 1.5.5]). In particular, the Dirichlet polynomials \(R^k_x(D)\) stated in Proposition 3.4 considered as functions on vertical lines \([\text{Re} = \sigma]\) are almost periodic.

**Corollary 3.9.** If \(D = \sum a_n e^{-\lambda_n s} \in D^*_{\infty}(\lambda)\) with extension \(f\), then the function \(f_{\sigma}(t) := f(\sigma + it) : \mathbb{R} \to \mathbb{C}\) is almost periodic and for all \(\sigma > 0\)

\[
a_n = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T f(\sigma + it)e(\sigma + it)\lambda_n dt.
\]

In particular, \(\sup_{n \in \mathbb{N}} |a_n| \leq \|D\|_\infty\) and \(D^*_{\infty}(\lambda)\) is a normed space.
Proof. Since on $\text{Re} = \sigma$ the limit function $f$ is the uniform limit of $R_1^\nu(D)$ tending $x \to \infty$ (Proposition 3.4), $f_\sigma$ is almost periodic. Then by [3, Ch. I, §3.11, p. 21] we have

$$
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it) e^{(\sigma + it)\lambda_n} dt = \lim_{x \to \infty} \frac{1}{2T} \int_{-T}^{T} R_1^\nu(\sigma + it) e^{(\sigma + it)\lambda_n} dt
$$

$$
= \lim_{x \to \infty} a_n \left( 1 - \frac{\lambda_n}{x} \right) = a_n
$$

for all $\sigma > 0$ and so $|a_n| \leq \|D\|_\infty$.

Another property of almost periodic functions is that they allow a unique continuous extension to the Bohr compactification $\overline{\mathbb{R}}$ of $\mathbb{R}$ (see [20, §1.5.2.2, Thm. 1.5.5]). In particular, the monomials $e^{-i\lambda_n}$ extend uniquely to characters on $\overline{\mathbb{R}}$.

To complete our proof of Theorem 3.2, it remains to verify Lemma 3.5.

Proof of Lemma 3.5. We again use (3.6) and obtain for all $N \in \mathbb{N}$

$$
(\lambda_{N+1} - \lambda_N)^k \left| \sum_{n=1}^{N} a_n \right| = \left| \sum_{n=1}^{N} a_n \int_{\lambda_N}^{\lambda_{N+1}} k(\lambda_{N+1} - t)^{k-1} dt \right|
$$

$$
= \left| \int_{\lambda_N}^{\lambda_{N+1}} \left( \sum_{\lambda_n < t} a_n \right) k(\lambda_{N+1} - t)^{k-1} dt \right|
$$

$$
= \left| \int_{0}^{\lambda_{N+1}} \left( \sum_{\lambda_n < t} a_n \right) k(\lambda_{N+1} - t)^{k-1} dt - \int_{0}^{\lambda_N} \left( \sum_{\lambda_n < t} a_n \right) k(\lambda_{N+1} - t)^{k-1} dt \right|
$$

$$
\leq \sum_{n=1}^{N} a_n (\lambda_{N+1} - \lambda_n)^k + \left| \int_{0}^{\lambda_N} \left( \sum_{\lambda_n < t} a_n \right) k(\lambda_{N+1} - t)^{k-1} dt \right|
$$

$$
\leq \sup_{0 \leq x \leq \lambda_{N+1}} \left| \sum_{\lambda_n < x} a_n (x - \lambda_n)^k \right| + \left| \int_{0}^{\lambda_N} \left( \sum_{\lambda_n < t} a_n \right) k(\lambda_{N+1} - t)^{k-1} dt \right|
$$

Now, applying [13, Lemma 7, p. 28] to the real and imaginary parts of the integral we obtain

$$
\left| \int_{0}^{\lambda_N} \left( \sum_{\lambda_n < t} a_n \right) k(\lambda_{N+1} - t)^{k-1} dt \right| \leq 2 \sup_{0 \leq x \leq \lambda_N} \left| \sum_{\lambda_n < x} a_n (x - \lambda_n)^k \right|
$$

and so

$$
\left| \sum_{n=1}^{N} a_n \right| \leq 3 \left( \frac{1}{\lambda_{N+1} - \lambda_N} \right)^k \sup_{0 \leq x \leq \lambda_N} \left| \sum_{\lambda_n < x} a_n (x - \lambda_n)^k \right|.
$$

□

4 ON BOHR’S THEOREM

Now we apply our main result Theorem 3.2 to prove quantitative variants of Bohr’s theorem for certain classes of $\lambda$’s (including $(BC)$ and $(LC)$) by giving bounds for $\|S_N\|$, $N \in \mathbb{N}$. Observe that by Corollary 3.9 we always have the trivial bound $\|S_N\| \leq N$. Hence by Remark 3.1 $\lambda$ satisfies Bohr’s theorem (or equivalently equality (3.1) holds), if

$$
L(\lambda) := \limsup_{N \to \infty} \frac{\log(N)}{\lambda_N} = 0.
$$

(4.1)
For instance \( \lambda = (n) = (0, 1, 2, \ldots) \) fulfils \( L((n)) = 0 \) and we (again) see as a consequence that for power series we cannot distinguish between uniform convergence and boundedness of the limit function up to \( \epsilon \).

We like to mention that the number \( L(\lambda) \) also has a geometric meaning. Bohr shows in [5, §3, Hilfssatz 3, Hilfssatz 2] that

\[
L(\lambda) = \sigma_c \left( \sum \lambda_n e^{-\lambda_n s} \right) = \sup_{D \in \mathcal{D}(\lambda)} \sigma_a(D) - \sigma_c(D),
\]

where the latter is the maximal width of the so-called strip of pointwise and not absolutely convergence.

**Remark 4.1.** We summarize relations of \((BC), (LC)\) and “\(L(\lambda) < \infty\).”

(1) \((BC)\) implies \(L(\lambda) < \infty \) and \((LC)\).
(2) \((LC)\) and \(L(\lambda) < \infty \) does not necessarily imply \((BC)\).
(3) \(L(\lambda) < \infty \) does not necessarily imply nor \((LC)\) and so neither \((BC)\).
(4) \((LC)\) does not necessarily imply nor \(L(\lambda) < \infty \) and so neither \((BC)\).

**Proof.**

(1) The implication \((BC) \Rightarrow (LC)\) is clear and the fact that \(L(\lambda) < \infty\), if \(\lambda \in (BC)\), is done in [5, §3, Hilfssatz 4].
(2) Take \(\lambda\) defined by \(\lambda_{2n} = n + e^{-n^2}\) and \(\lambda_{2n+1} = n\). Then \((LC)\) is satisfied with \(L(\lambda) = 0\), but \(\lambda\) fails for \((BC)\).
(3) Define \(\lambda_{2n} = n + e^{-n^2}\) and \(\lambda_{2n-1} = n\). Then \(L(\lambda) = 0\) and \(\lambda\) does not satisfy \((LC)\).
(4) Consider \(\lambda := \left( \sqrt{\log(n)} \right)\). Then \(L(\lambda) = +\infty\) (and so \((BC)\) fails) but \((LC)\) is satisfied: We claim that \(\lambda_{n+1} - \lambda_n \geq C e^{-2\lambda_n^2}\) for some \(C\), that is \(\sqrt{\log(n+1)} - \sqrt{\log(n)} \geq \frac{C n}{2 \sqrt{\log(n+1)}} \to +\infty\).

Since to every \(\delta > 0\) there is a constant \(C = C(\delta)\) such that \(e^{-2\lambda_n^2} \geq C(\delta) e^{-\delta \lambda_n}\), the claim follows. \(\square\)

### 4.1 Landau’s condition

**Theorem 4.2.** Let \((LC)\) hold for \(\lambda\). Then for all \(\delta > 0\) there is \(C = C(\delta, \lambda)\) such that for all \(D = \sum a_n e^{-\lambda_n s} \in D^{\infty}_\infty(\lambda)\) and all \(N \in \mathbb{N}\) we have

\[
\sup_{|\Re s| > 0} \left| \sum_{n=1}^N a_n e^{-\lambda_n s} \right| \leq C e^{\delta \lambda_N} \|D\|_\infty.
\]

In particular, \(\lambda\) satisfies Bohr’s theorem.

**Proof.** W.l.o.g. we may assume that \(\lambda_{n+1} - \lambda_n \leq 1\) for all \(n\). Indeed, suppose that \(\lambda_{n+1} - \lambda_n > 1\) for some \(n\), and let \(l\) be the smallest natural number such that \(l \geq \lambda_{n+1} - \lambda_n\). Then we define a new frequency \(\lambda^2\) by adding more numbers to \(\lambda\), so that \(\lambda^2\) satisfies \((LC)\) and \(\sup_n \lambda^2_{n+1} - \lambda^2_n \leq 1\). We follow two steps. First, whenever \(l \geq 3\), we add \(\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{n+l-2}\) to \(\lambda\). Since \(1 < |\lambda_{n+1} - (\lambda_n + l - 2)| \leq 2\), this procedure gives a new frequency, say \(\lambda^1\), such that \(\sup_n \lambda^1_{n+1} - \lambda^1_n \leq 2\). Moreover, \(\lambda^1\) has \((LC)\), since \(\lambda\) satisfies \((LC)\) by assumption. Now, we add more numbers to \(\lambda^1\) to obtain the announced frequency \(\lambda^2\). If \(\lambda^1_{n+1} - \lambda^1_n > 1\) for some \(n\), then we add the number \(w := \lambda^1_{n+1} + \lambda^1_n/2\) to \(\lambda^1\). In this way the new frequency \(\lambda^2\) fulfills \(\sup_n \lambda^2_{n+1} - \lambda^2_n \leq 1\) and \((LC)\), which follows from the observation \(\lambda^1_{n+1} - w = w - \lambda^1_n\). Now
let $\delta > 0$ and set $k_N = e^{-\delta \lambda_N}$. Then by Theorem 3.2 and assuming $\lambda_{n+1} - \lambda_n \leq 1$ for all $n$ we obtain

$$||S_N|| \leq C \left( \frac{\Gamma(k_N + 1)}{k_N} \left( \frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^{k_N} \right) \leq C_1 e^{\delta \lambda_N} (1 + \lambda_N)^{e^{\lambda_{N+1}/\lambda_N} - 1} \leq C_2 e^{\delta \lambda_N}.$$

Now Corollary 2.5 gives $\sigma_u(D) \leq \delta$ for all $\delta > 0$ and so $\sigma_u(D) \leq 0$.

**Corollary 4.3.** Let $\lambda$ satisfy (LC). Then to every $\sigma > 0$ there is a constant $C_1 = C_1(\sigma, \lambda)$ such that for all $N \in \mathbb{N}$ and $D = \sum a_n e^{-\lambda_n s} \in D^{ex}_{\infty} (\lambda)$ we have

$$\sup_{[Re > \sigma]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq C_1 ||D||_{\infty}.$$

**Proof.** Let us write $S_N(it) := \sum_{n=1}^{N} a_n e^{-it \lambda_n}$ for simplicity and fix $\sigma > 0$. Then by Abel summation and Theorem 4.2, choosing $\delta = \sigma$ in (LC), we for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$ have

$$\sup_{[Re > \sigma]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n (2\sigma + it)} \right| = \left| S_N(it) e^{-\lambda_N 2\sigma} + \sum_{n=1}^{N-1} S_n(it)(e^{-\lambda_n 2\sigma} - e^{-\lambda_{n+1} 2\sigma}) \right| \leq C ||D||_{\infty} \left( e^{\sigma \lambda_N} e^{-2\lambda_N \sigma} + \frac{1}{2\sigma} \sum_{n=1}^{N-1} e^{\sigma \lambda_n} \int_{\lambda_n}^{\lambda_{n+1}} e^{-2\sigma x} \, dx \right) \leq C ||D||_{\infty} \left( e^{-\lambda_N \sigma} + \frac{1}{2\sigma} \sum_{n=1}^{N-1} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\sigma x} \, dx \right) \leq C ||D||_{\infty} \left( 1 + \frac{1}{2\sigma^2} \int_{0}^{\infty} e^{-\sigma x} \, dx \right) = C ||D||_{\infty} \left( 1 + \frac{1}{2\sigma^2} \right).$$

**4.2 | Bohr’s condition**

We already know from Theorem 4.2 that if (BC) holds for $\lambda$, then $\lambda$ satisfies Bohr’s theorem, since (BC) implies (LC) (Remark 4.1). But the stronger assumption (BC) improves the bound for the norm of $S_N$.

**Theorem 4.4.** Let (BC) hold for $\lambda$. Then there is a constant $C = C(\lambda) > 0$ such that for all $D = \sum a_n e^{-\lambda_n s} \in D^{ex}_{\infty} (\lambda)$ and all $N \in \mathbb{N}$ with $N \geq 2$

$$\sup_{[Re > 0]} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq C \lambda_N ||D||_{\infty}.$$

**Proof of Theorem 4.4.** Again w.l.o.g. we assume that $\lambda_{n+1} - \lambda_n \leq 1$ following the same procedure as in the proof of Theorem 4.2. Then choosing $k_N = \frac{1}{\lambda_N}$, $N \geq 2$ (since $\lambda_1 = 0$ is possible), by Theorem 3.2 we obtain

$$||S_N|| \leq \frac{C_1}{k_N \left( \frac{\lambda_{N+1}}{\lambda_{N+1} - \lambda_N} \right)^{k_N}} \leq C_2 \lambda_N^{k_N} \lambda_N \leq C_3 \lambda_N.$$

$\square$
Remark 4.5. For particular cases the bounds in Theorem 4.4 (and Theorem 4.2) may be bad (which is not surprising since this is an abstract result for all λ’s satisfying (BC) respectively (LC)). For instance in the case λ = (n) = (0, 1, 2, ...), since the projection \( S_N(f) = \sum_{n=0}^{N} c_n(f)z^n : H_\infty(D) \to H_\infty(D) \) corresponds to convolution with the Dirichlet kernel \( D_N(z) = \sum_{n=-N}^{N} z^n \) (after using the identification \( H_\infty(D) = H_\infty(T) \)), we obtain \( \|S_N\| = \|D_N\|_1 \sim \log(N) \). To our knowledge the question about optimality of the bounds in the ordinary case λ = (log n) is still open.

To put it differently the conditions (BC) and (LC) states that the sequence \( \left( \log(\frac{1}{\lambda_{n+1} - \lambda_n}) \right) \) increases at most linearly respectively exponentially, and the quality of the growth gives different bounds for \( \|S_N\| \). We consider now λ’s whose growth is somewhat in between:

\[ \exists \ l, \ d > 0 \ \forall \ \delta > 0 \ \exists \ C > 0 \ \forall \ n \in \mathbb{N} : \lambda_{n+1} - \lambda_n \geq Ce^{-(l+\delta)\lambda^d_n}. \]  

(4.2)

Clearly (4.2) implies (LC) and so Theorem 4.2 holds, but for this class of frequencies Theorem 3.2 gives an improved bound for \( \|S_N\| \). Recall that \( \lambda = (\sqrt{\log(n)}) \) satisfies (4.2) with \( d = 2 \) (see proof of Remark 4.1).

Theorem 4.6. If λ satisfies (4.2) with \( d > 0 \), then there is a constant \( C = C(d, \lambda) \) such that for all \( D = \sum a_n e^{-\lambda_n s} \in D_\infty^{ext}(\lambda) \) and \( N \in \mathbb{N} \) with \( N \geq 2 \)

\[ \sup_{|Re>0|} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| \leq C\lambda^d_N \|D\|_{\infty}. \]

Proof of Theorem 4.4. As before w.l.o.g. we assume that \( \lambda_{n+1} - \lambda_n \leq 1 \). Then choosing \( k_N = \frac{1}{\lambda^d_N} \), \( N \geq 2 \), we obtain (with Theorem 3.2)

\[ \|S_N\| \leq C_1 k_N \lambda^d_N \leq C_2 \lambda^d_N. \]

\[ \square \]

4.3 \ Q-linearly independent frequencies

In [6] Bohr proves that Q-linearly independent λ’s satisfy the equality

\[ \sigma^{ext}_b = \sigma_a \]

for all somewhere convergent λ-Dirichlet series. In this section we give an alternative proof to Bohr’s using Proposition 3.4 and the so-called Kronecker’s theorem, which states that the set \{\( (e^{-\lambda_n t}) \mid t \in \mathbb{R} \) \} is dense in \( T^\infty \), whenever the real sequence \( (\lambda_n) \) is Q-linearly independent. The latter is equivalent to the fact that for every choice of complex coefficient \( a_1, \ldots, a_N \) the equality

\[ \sup_{|Re>0|} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| = \sum_{n=1}^{N} |a_n| \]  

holds. For a proof of the equivalence of Kronecker’s theorem and (4.3) see [17, §VI.9] and for a proof of Kronecker’s theorem see e.g. [9, §3.1, Example 3.7] or again [17, §VI.9].

Theorem 4.7. Let \( D = \sum a_n e^{-\lambda_n s} \in D_\infty^{ext}(\lambda) \) and let λ be Q-linearly independent. Then \( (a_n) \in \ell_1 \) and \( \|(a_n)\|_1 = \|D\|_\infty \). Moreover, isometrically

\[ D_\infty^{ext}(\lambda) = D_\infty(\lambda) = \ell_1 \]
and \( \lambda \) satisfies Bohr's theorem. In particular,

\[
\sup_{N \in \mathbb{N}} \sup_{\Re s > 0} \left| \sum_{n=1}^{N} a_n e^{-\lambda_n s} \right| = \| D \|_{\infty}.
\]

**Proof.** Let \( f \) be the extension of \( D \). Then by Proposition 3.4 for every \( \sigma > 0 \) the polynomials

\[
R_x^{\lambda}(D) = \sum_{\lambda_n < x} a_n \left(1 - \frac{\lambda_n}{x} \right) e^{-\lambda_n s}
\]

converge to \( f \) uniformly on \( \{ \Re s = \sigma \} \) as \( x \to \infty \). Together with (4.3) we for all \( N \in \mathbb{N} \) obtain

\[
\sum_{n=1}^{N} |a_n| = \sup_{\sigma > 0} \lim_{x \to \infty} \sum_{n=1}^{N} \left| a_n \left(1 - \frac{\lambda_n}{x} \right) e^{-\lambda_n \sigma} \right|
\]

\[
\leq \sup_{\sigma > 0} \lim_{x \to \infty} \sum_{\lambda_n < x} |a_n| \left(1 - \frac{\lambda_n}{x} \right) e^{-\lambda_n \sigma}
\]

\[
= \sup_{\sigma > 0} \lim_{x \to \infty} \sup_{t \in \mathbb{R}} \left| \sum_{\lambda_n < x} a_n \left(1 - \frac{\lambda_n}{x} \right) e^{-\lambda_n (\sigma + it)} \right|
\]

\[
= \sup_{\sigma > 0} \sup_{t \in \mathbb{R}} |f(\sigma + it)| = \| D \|_{\infty}.
\]

So \( (a_n) \in \ell_1 \) with \( \|(a_n)\|_1 \leq \| D \|_{\infty} \) and \( \sigma_0(D) \leq 0 \). Hence

\[
\| D \|_{\infty} = \sup_{\sigma > 0} \sup_{\Re s > \sigma} |D(s)| \leq \sup_{\sigma > 0} \sum_{n=1}^{\infty} |a_n| e^{-\sigma \lambda_n} \leq \|(a_n)\|_1.
\]

\( \square \)

Let us summarize the results of Theorem 4.2, Theorem 4.7 and (4.1).

**Remark 4.8.** A frequency \( \lambda \) satisfies Bohr's theorem if one the following conditions holds:

1. \( L(\lambda) = 0 \),
2. \( \lambda \) is \( \mathbb{Q} \)-linearly independent,
3. \( (LC) \).

In the theory of ordinary Dirichlet series, given \( D = \sum a_n n^{-s} \), the so called \( m \)-homogeneous part of \( D \) is the (formal) sum \( \sum a_n n^{-\lambda n} \), where \( a_n \neq 0 \) implies \( n \) only has \( m \) prime factors counting multiplicity. Recall that \( D_\infty((\log n)) \) and the space \( H_\infty(B_{c_0}) \) of all holomorphic and bounded functions on the open unit ball of \( c_0 \) are isometrically isomorphic via Bohr's transform

\[
B : H_\infty(B_{c_0}) \to D_\infty((\log n)), \ f \mapsto \sum a_n n^{-\lambda n},
\]

where \( a_n := c_{\alpha}(f) \) (the \( \alpha \)th Taylor coefficient of \( f \)) whenever \( n = p^\alpha \) in its prime number decomposition. This identification links the space of all \( m \)-homogeneous Dirichlet series \( D^{(m)}_\infty((\log n)) \) to the space of \( m \)-homogenous polynomials (or equivalently bounded \( m \)-linear forms) on \( c_0 \) (see [11, §2, §3]). In particular, \( D^{(1)}_\infty((\log n)) \) equals the space of all 1-linear forms on \( c_0 \). Hence \( D^{(1)}_\infty((\log n)) = c'_0 = \ell_1 \). Since \( (\log p_n) \), where \( p_n \) is the \( n \)th prime number, is \( \mathbb{Q} \)-linearly independent, Theorem 4.7 recovers this result.
Corollary 4.9.

\[ D_\infty^{(1)}((\log n)) = D_\infty((\log p_n)) = \ell_1. \]

4.4 | A Montel theorem

In [2, Lemma 18] Bayart proves that every bounded sequence \((D^N) \subset D_\infty((\log n))\) allows a subsequence \((D^{N_k})\) and some \(D \in D_\infty((\log n))\) such that \((D^{N_k})\) converges uniformly to \(D\) on \([Re > \varepsilon] \) for all \(\varepsilon > 0\); a fact which is called “Montel theorem” and extends to the following classes of \(\lambda\)’s.

Theorem 4.10. Let \(\lambda\) be a frequency statisfying \((LC)\), \(L(\lambda) = 0\) or \(\lambda\) be \(\mathbb{Q}\)-linearly independent. Let \((D_j) \subset D_\infty(\lambda)\) be a bounded sequence with Dirichlet coefficients \((a_j)\). Then there is a subsequence \((D_{j_k})\) such that \((D_{j_k})\) converges uniformly on \([Re > \varepsilon]\) to \(D = \sum a_ne^{^{-\lambda n}s} \in D_\infty(\lambda)\) for all \(\varepsilon > 0\), where \(a_n := \lim_{k \to \infty} a_{j_k}n\).

Proof. By Corollary 3.9 we for all \(n, j \in \mathbb{N}\) have

\[ |||a_{j_n}||| \leq ||D_j||_\infty \leq \sup_j |||D_j|||_\infty =: C_1 < \infty. \]

Hence by diagonal process we find a subsequence \((j_k)\) such that \(\lim_k a_{j_k}n =: a_n\) exists for all \(n \in \mathbb{N}\). We (formally) define \(D := \sum a_ne^{^{-\lambda n}s}\). If \(L(\lambda) = 0\), then \(\sigma_a(D) \leq 0\) and \(\sigma_a(D_j) \leq 0\) for all \(j\), since the Dirichlet coefficients are bounded, and the claim follows easily. In the remaining cases Theorem 4.2 respectively Theorem 4.7 together with Proposition 2.4 applied to the sequence \((D_{j_k})\) gives the claim. Indeed, let \(\varepsilon > 0\) and let us first assume that \(\lambda\) fulfils \((LC)\). Then for all \(k\) and \(N\) by Theorem 4.2

\[ \sup_{[Re > \varepsilon]} \left| \sum_{n=1}^N a_{j_k}ne^{-\lambda n}s \right| \leq C(\varepsilon)e^{\varepsilon\lambda N}|||D_{j_k}|||_\infty \leq C(\varepsilon)C_1e^{\varepsilon\lambda N}. \]

Now Proposition 2.4 implies that \((D_{j_k})\) converges to \(D\) on \([Re > 2\varepsilon]\). If \(\lambda\) is \(\mathbb{Q}\)-linearly independent, the claim follows in the same way replacing Theorem 4.2 by Theorem 4.7.

5 | ABOUT COMPLETENESS

Recall that from Corollary 3.9 we know that \((D_\infty^{ex}(\lambda), ||\cdot||_\infty)\) is a normed space. In this section we face completeness. We first state sufficient conditions on \(\lambda\) for completeness of \(D_\infty(\lambda)\) and \(D_\infty^{ex}(\lambda)\). Then we give a construction of \(\lambda\)’s for which \(D_\infty(\lambda)\) fails to be complete. We like to mention that in [8] it is already proven that \((BC)\) is sufficient for completeness of \(D_\infty(\lambda)\) by introducing the following condition, which is equivalent to \((BC)\):

\[ \exists p \geq 1: \inf_{n \in \mathbb{N}} e^{p\lambda_{n+1}} - e^{p\lambda_n} > 0. \]

(5.1)

Their proof (see [8, §2, Prop. 2.1]) shows that given (5.1) the choice \(l(\lambda) := p\) succeeds in \((BC)\) and given \(l(\lambda)\) from \((BC)\) the choice \(p := l + \delta\) for every \(\delta > 0\) is admissible for (5.1).

Recall that being an isometric subspace of \(H_\infty[Re > 0]\) (the space of all bounded and holomorphic functions on \([Re > 0]\)) the spaces \(D_\infty^{ex}(\lambda)\) and \(D_\infty(\lambda)\) are complete if and only if they are closed in \(H_\infty[Re > 0]\).

Theorem 5.1. If \(L(\lambda) < \infty\), then \(D_\infty^{ex}(\lambda)\) is complete. The space \(D_\infty(\lambda)\) is a Banach space if one of the following conditions hold:

1. \(\lambda\) is \(\mathbb{Q}\)-linearly independent,
2. \(L(\lambda) = 0\).
(3) \( \sigma_b^{ext} = \sigma_c \) and \( L(\lambda) < \infty \). In particular, this holds for \( \lambda \)'s satisfying
\[(3.1) \text{(BC)},\]
\[(3.2) \text{(LC)} \) and \( L(\lambda) < \infty \).

Moreover, all of the stated conditions are not necessary for completeness.

Because of the different nature of the stated sufficient conditions on \( \lambda \), it seems like we are far away from a characterization. In particular, it would be interesting to find a condition on \( \lambda \) sufficient for \( \sigma_b^{ext} = \sigma_c \), which is weaker than (LC).

**Proof of Theorem 5.1.** If \( \lambda \) is \( Q \)-linearly independent, then \( D_{\infty}(\lambda) = \ell_1 \) as Banach spaces by Theorem 4.7. So let us assume \( L(\lambda) < \infty \). Then we claim that \( D_{\infty}(\lambda) \) is a closed subspace of \( H_\infty(\{Re > 0\}) \). Indeed if \( (D^K) \) is a sequence in \( D_{\infty}(\lambda) \) with Dirichlet coefficients \( (a_n^K) \), which converges to some \( f \in H_\infty(\{Re > 0\}) \), then \( |a_n^K| \leq \|D^K\|_\infty \) for all \( n, K \in \mathbb{N} \) (Corollary 3.9).

Hence the limits \( a_n := \lim_K a_n^K \) exist and \( (a_n) \) is bounded. So the Dirichlet series \( D := \sum a_n e^{-\lambda_n s} \) converges absolutely on \( \{Re > L(\lambda)\} \) and we claim that \( D \) and \( f \) coincide on \( \{Re > L(\lambda)\} \). Let \( s \in \{Re > L(\lambda)\} \), \( \text{Re} s = \sigma \) and \( \varepsilon > 0 \). Then there is \( K_0 \) such that \( |a_n - a_n^K| \leq \varepsilon \) for all \( K \geq K_0 \) and all \( n \in \mathbb{N} \). Then for such \( K \) we obtain for large \( N \)

\[
|D(s) - f(s)| \leq |D(s) - \sum_{n=1}^N a_n e^{-\lambda_n s}| + \sum_{n=1}^N |(a_n - a_n^K)e^{-\lambda_n s}| + \sum_{n=1}^N |a_n^K e^{-\lambda_n s} - D^K(s)| + |D^K(s) - f(s)|
\]

\[
\leq \varepsilon + \varepsilon \sum_{n=1}^\infty e^{-\lambda_n \sigma} + \varepsilon + \varepsilon,
\]

which implies \( D(s) = f(s) \) on \( \{Re > L(\lambda)\} \). Hence \( D \in D_{\infty}(\lambda) \) with extension \( f \) and \( D_{\infty}(\lambda) \) is complete, which coincides with \( D_{\infty}(\lambda) \) assuming \( \sigma_b^{ext} = \sigma_c \).

Now we come to the “Moreover” part. Since \( L((\log n)) = 1 \) and \( D_{\infty}((\log n)) \) is complete, the condition “\( L(\lambda) = 0 \)” is not necessary. Moreover, the frequency defined by \( \lambda_{2n} = n + e^{-n^2} \) and \( \lambda_{2n-1} = n \) does not satisfy (BC) but \( L(\lambda) = 0 \). Hence the condition (BC) is not necessary. The frequency defined by \( \lambda_{2n} = n + e^{-n^2} \) and \( \lambda_{2n-1} = n \) does not satisfy (LC) but \( L(\lambda) = 0 \). Finally, by choosing a \( Q \)-linearly independent \( \lambda \) increasing slowly enough we see that the condition “\( L(\lambda) < \infty \)” is not necessary.

On the other hand in the following sense there are infinitely \( \lambda \)'s for which \( D_{\infty}(\lambda) \) fails to be complete.

**Theorem 5.2.** Let \( \lambda \) be a frequency. Then there is a strictly increasing sequence \( (s_n) \) of natural numbers such that the space \( D_{\infty}(\tilde{\eta}) \), where the frequency \( \eta \) is obtained by ordering the set

\[
\left\{ \lambda_n + \frac{j}{s_n} (\lambda_{n+1} - \lambda_n) \mid n \in \mathbb{N}, \ j = 0, \ldots, s_n - 1 \right\}
\]

increasingly, is not complete.

**Proof.** First we explain why it is sufficient to assume \( \lambda_{n+1} - \lambda_n \leq 1 \) for all \( n \). Therefore, suppose \( \sup_{n \in \mathbb{N}} \lambda_{n+1} - \lambda_n > 1 \). Then for each \( n \) we consider the interval \( [\lambda_n, \lambda_{n+1}] \) and add equidistantly new numbers to \( \lambda \) in \( [\lambda_n, \lambda_{n+1}] \), such that the distance of these new numbers (inclusive the edge point \( \lambda_n \) and \( \lambda_{n+1} \)) is less than 1. Since for each interval \( I_n \) we only add finitely many new numbers we obtain in this way a new frequency (with subsequence \( \lambda \)), say \( \tilde{\lambda} \), satisfying \( \lambda_{n+1} - \lambda_n \leq 1 \) for all \( n \). Suppose now that we are able to find a sequence \( (s_n) \) for \( \tilde{\lambda} \) as stated in the theorem and we denote by \( \tilde{\eta} \) the corresponding new frequency. Then we already know that \( D_{\infty}(\tilde{\eta}) \) is not complete. Now we want to add again more numbers to \( \tilde{\eta} \) in such a way that the frequency \( \eta \) obtained is of the form (5.2) for some suitable sequence \( (s_n) \). Then clearly \( D_{\infty}(\eta) \) remains incomplete. Note that each interval \( [\tilde{\lambda}_n, \tilde{\lambda}_{n+1}] \) contains now finitely many intervals of the form \( [\tilde{\lambda}_j, \tilde{\lambda}_{j+1}] \).

By assumption each of these interval is decomposed into equidistant parts by \( s_j \). Since there are only finitely many of them, choose the smallest distance and add (finitely) many numbers to the interval \( [\tilde{\lambda}_n, \tilde{\lambda}_{n+1}] \) such that within \( [\tilde{\lambda}_n, \tilde{\lambda}_{n+1}] \).
all added numbers are equidistant (this is always possible). Denoting by \( s_n \) the number in \([\lambda_n, \lambda_{n+1}]\) we add to \( \lambda \) by this procedure, we obtain the desired frequency \( \eta \) of the form (5.2).

So we may assume that \( \sup_{n \leq m} \lambda_{n+1} - \lambda_n \leq 1 \). Now let \( I_k := \{ \lambda_n \mid k \leq \lambda_n < k + 1 \}, k \in \mathbb{N}_0 \), and \( x > 0 \) arbitrary. If \( |I_k| \neq 0 \), where \( |I_k| \) denotes the number of elements in \( I_k \), we define \( b_k = e^{-xk}|I_k|^{-1} \), and if \( |I_k| = 0 \), then \( b_k = 0 \). Note that \( |I_k| \leq 1 \), since \( \lambda \) tends to infinity. If \( \lambda_n \in I_k \) for some \( k \) we define \( r_{n,k} \) to be the largest natural number smaller than \( e^{2x\lambda_n}|I_k| \). Following the spirit of Neder in [19, §1] we define the new frequency \( \eta \) given by the following set

\[
\bigcup_{k \in \mathbb{N}} \left\{ \lambda_n + \frac{j}{2r_{n,k}}(\lambda_{n+1} - \lambda_n) \mid \lambda_n \in I_k, \; j = 0, ..., 2r_{n,k} - 1 \right\}.
\]

Then we (formally) define the \( \eta \)-Dirichlet series \( D = \sum a_m e^{-\eta_m s} \), by setting \( a_m = b_k \) if \( \eta_m = \lambda_n + \frac{j}{2r_{n,k}}(\lambda_{n+1} - \lambda_n) \) for some \( \lambda_n \in I_k \) and \( j \in \{1, ..., 2r_{n,k} - 1\} \setminus \{r_{n,k}\} \). The remaining \( a_m \)'s equal zero. Now we claim that \( \sigma_c(D) \geq x > 0 \), which shows \( D \notin D_{\infty}(\eta) \). If \( \lambda_n \in I_k \), then by the choice of \( b_k \) and \( r_{n,k} \)

\[
\sum_{\lambda_n \leq m < \lambda_{n+1} - \frac{1}{2}(\lambda_{n+1} - \lambda_n)} a_m e^{-x\eta_m} = b_k \sum_{j=1}^{r_{n,k}-1} \frac{1}{r_{n,k} - j} e^{-x\left(\lambda_n + \frac{j}{2r_{n,k}}(\lambda_{n+1} - \lambda_n)\right)}
\]

\[
\geq b_k e^{-x}e^{-x\lambda_n} \sum_{j=1}^{r_{n,k}-1} \frac{1}{j} \geq b_k e^{-x} e^{-x\lambda_n} \log(r_{n,k})
\]

\[
\geq b_k e^{-x} e^{-x\lambda_n} \frac{1}{4} e^{\lambda_n 2x} |I_k|
\]

\[
= e^{-x} e^{-xk} e^{\lambda_n x} \geq e^{-x} > 0.
\]

Hence \( D \) does not converge in \( x \). Now we claim that \( D_{\infty}(\eta) \) is not complete. Therefore recall (see [12, §4]) that the Fejér polynomials \( F_m(z) := \sum_{j=1}^{2m-1} \frac{1}{m-j}z^j \), \( m \in \mathbb{N} \), where \( \frac{1}{0} := 0 \), satisfy

\[
\sup_{m \in \mathbb{N}} \sup_{|z| < 1} |F_m(z)| =: C < \infty.
\]

Now consider the sums

\[
D^K(s) := \sum_{k=1}^{K} b_k \sum_{\lambda_n \in I_k} e^{-\lambda_n s} F_{r_{n,k}} \left( e^{-x \frac{j}{2r_{n,k}}(\lambda_{n+1} - \lambda_n)} \right)
\]

\[
= \sum_{n=1}^{K} b_k \sum_{j=1}^{2r_{n,k}-1} \frac{1}{r_{n,k} - j} e^{-x \left(\lambda_n + \frac{j}{2r_{n,k}}(\lambda_{n+1} - \lambda_n)\right)} s,
\]

which are partial sums of \( D \) defined before and \( (D^K) \subset D_{\infty}(\eta) \). Moreover, for each \( s \in [\Re > 0] \) and \( K > L \)

\[
|D^K(s) - D^L(s)| \leq C \sum_{k=K}^{L} b_k |I_k| \leq \sum_{k=K}^{\infty} (e^{-x})^k
\]

and so \( (D^K) \) is Cauchy in \( D_{\infty}(\eta) \). Assuming now that \( D_{\infty}(\eta) \) is complete, the limit of \( (D^K) \) in \( D_{\infty}(\eta) \) is \( D \), since \( a_m = \lim_{K} a_m(D^K) \). But we already know that \( D \notin D_{\infty}(\eta) \), which gives a contradiction.

\[ \square \]

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