LARGE M ASYMPTOTICS FOR MINIMAL PARTITIONS OF THE DIRICHLET EIGENVALUE

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ABSTRACT. In this paper, we study large $m$ asymptotics of the $l^1$ minimal $m$-partition problem for Dirichlet eigenvalue. For any smooth domain $\Omega \subset \mathbb{R}^n$ such that $|\Omega| = 1$, we prove that the limit $\lim_{m \to \infty} l^1_m(\Omega) = c_0$ exists, and the constant $c_0$ is independent of the shape of $\Omega$. Here $l^1_m(\Omega)$ denotes the minimal value of the normalized sum of the first Laplacian eigenvalues for any $m$-partition of $\Omega$.

1. INTRODUCTION

Let $\Omega$ be a bounded, smooth domain in $\mathbb{R}^n$, and $m > 1$ be a positive integer. We consider the following so-called $l^1$-minimal partition problem:

**Problem P.** Find a partition of $\Omega$ into $m$, mutually disjoint subsets $\Omega_j$, $j = 1, 2, \ldots, m$, such that $\Omega = \bigcup_{j=1}^m \Omega_j$, and it minimizes the $l^1$ energy functional $\sum_{j=1}^m \lambda_1(\Omega_j)$ among all admissible partitions. Here $\lambda_1(A)$ denotes the first eigenvalue of Laplacian $\Delta$ on $A$ with the zero Dirichlet boundary condition on $\partial A$.

The existence of the minimal partition and regularity of free interfaces have been studied by many authors, see [7, 9, 16, 5, 4, 14, 10, 11] and survey articles [1 2, 2, 13]. In [10], Cafferelli and the second author proved the equivalence between Problem P and the following problem:

**Problem P*.** Let

$$\Sigma^m = \{ y \in \mathbb{R}^m, \sum_{k \neq l} y_k^2 y_l^2 = 0 \}$$

Find $u \in H^1_0(\Omega, \Sigma^m)$ such that

$$\int_{\Omega} u_j^2 dx = 1 \text{ for any } j = 1, \ldots, m,$$

and that $u$ minimize $\int_{\Omega} |\nabla u|^2 dx$ among all such maps in $H^1_0(\Omega, \Sigma^m)$.

Problem (P*) obviously admits a minimizer $u = (u_1, u_2, \ldots, u_m)$. It is proved in [10] that $u$ is locally Lipschitz continuous in $\Omega$ (and Lipschitz continuous up to the boundary when $\partial \Omega$ is smooth), and $\Omega_j = \{ x \in \Omega : u_j(x) > 0 \}$ ($j = 1, \ldots, m$) are open subsets of $\Omega$ whose boundaries $\partial \Omega_j$ are smooth away from a relatively closed subset $S \subset \Omega$, of Hausdorff dimension at most $n-2$. Moreover $\{ \Omega_j \}_{j=1}^m$ gives a partition of $\Omega$ that minimizes $\sum_{j=1}^m \lambda_1(\Omega_j)$. It is shown later by O. Alper that the set $S$ is rectifiable and of bounded $(n-2)$ dimensional Hausdorff measure, [1].

In this note we are interested in the asymptotic behavior of the minimal partition as $m \to \infty$. Our main theorem is the following:

**Theorem 1.1.** Let $\Omega$ be a bounded, smooth domain in $\mathbb{R}^n$ with $|\Omega| = 1$. Then

$$\lim_{m \to \infty} l^1_m(\Omega) = c_0 \text{ for some positive constant } c_0 \text{ independent of } \Omega.$$
Here

\[ l^1_m(\Omega) = \frac{\sum_{j=1}^{m} \lambda_1(\Omega_j)}{m^{1+\frac{2}{n}}}, \]

\[ \Omega = \bigcup_{j=1}^{m} \Omega_j \] is a \(l^1\)-minimal \(m\)-partition.

Remark 1.1. For \(\Omega \subset \mathbb{R}^2\), by hexagonal tiling construction and Faber-Krahn inequality, one can easily get the following lower bound and upper bound for the constant \(c_0\):

\[ \lambda_1(D) \leq c_0 \leq \lambda_1(H), \]

where \(D\) is the 2-D unit-area disk and \(H\) is the unit-area regular hexagon.

It should be noted, in the above theorem, the smoothness of \(\Omega\) does not play any role here, and the smoothness assumption is just for convenience. The problem of large \(m\) asymptotics was considered first in \([10]\) and they prove that \(\sum_{j=1}^{m} \lambda_1(\Omega_j) \sim m \lambda_m(\Omega)\), where \(\lambda_m(\Omega)\) is the \(m\)-th Dirichlet eigenvalue of \(\Omega\). They also made a conjecture that the limit \(\lim_{m \to \infty} l^1_m(\Omega)\) exists and for the case \(\Omega \subset \mathbb{R}^2\), the minimal partitions for large \(m\) will be close to a regular Hexagon packing pattern and the constant \(c_0\) equals to \(\lambda_1(H)\). Theorem 1.1 here verifies the first part of the conjecture, while the second part (regular Hexagon pattern) remains open though one can very well expect it in a stochastic sense. In recent years some attempts have been made to a related issue. For examples, Bourgain \([3]\) and Steinerberger \([15]\) have improved the lower bound in (1.2) by showing that \(l^1_m(\Omega) > \lambda_1(D) + \varepsilon_0\) for some sufficiently small constant \(\varepsilon_0\). Their tools are a quantitative Faber-Krahn inequality and some packing properties of disks in \(\mathbb{R}^2\). In \([8]\), Bucur, Fragalà, Velichkov and Verzini study this so-called “honeycomb conjecture”, and they give a proof under the assumption that every \(\Omega_j (j = 1, \ldots, m)\) is convex and regular hexagon minimizes \(\lambda_1\) among all convex hexagons with the same area, which is itself an interesting open problem.

In Section 2 we will prove Theorem 1.1. The proof will be concentrated on the case \(n = 2\). For \(n \geq 3\) one can apply the same arguments with only some obvious modifications. We first prove the limit exists for the unit cube, and then we prove the statement for general domain \(\Omega\) by approximating it using smaller dyadic cubes of the same size.

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2. Proof of Theorem 1.1

We first prove the following lemma:

Lemma 2.1. Let \(Q\) be a unit cube in \(\mathbb{R}^2\). For any \(m > 0\) and \(k \geq 1\), it holds that

\[ l^1_m(Q) \geq l^1_{mk^2}(Q). \]

Proof. Let \(s = l^1_m(Q)\). By existence of the \(l^1\)-minimal \(m\)-partition, there is a \(m\)-partition \(\{\Omega_j\}_{j=1}^{m}\) of \(Q\) such that \(\sum_{j=1}^{m} \lambda_1(\Omega_j) = m^2 s\). Now we divided \(Q\) into \(k^2\) identical cubes \(\{Q_i\}_{i=1}^{k^2}\) with edge length \(\frac{1}{k}\). In each \(Q_i\), we put a translated and scaled copy of the same \(m\)-partition as \(\{\Omega_j\}_{j=1}^{m}\), which is denoted by \(\{\Omega_j^i\}_{j=1}^{m}\). As a result we get a \(mk^2\)-partition of \(Q\), and we have

\[ l^1_{mk^2}(Q) \leq \left( \sum_{1 \leq i \leq k^2} ( \sum_{1 \leq j \leq m} \lambda_1(\Omega_j^i)) \right) / (mk^2)^2 = s. \]

Here we have used the degree \(-2\)-homogeneity of \(\lambda_1\) with respect to scalings.

\[ \square \]
Proof of Theorem 1.1

Step 1. Consider the unit cube $Q$, we show that there exists $c_0$ such that

$$\lim_{m \to \infty} l^1_m(Q) = c_0.$$ 

Define

$$a(Q) = \liminf_{m \to \infty} l^1_m(Q).$$

For any $\varepsilon > 0$, there exists an integer $m_\varepsilon$ such that $l^1_{m_\varepsilon}(Q) \leq a(Q) + \frac{\varepsilon}{2}$. For any $m \geq m_\varepsilon$, there exists $k \in \mathbb{N}$ such that $k^2 m_\varepsilon \leq m \leq (k+1)^2 m_\varepsilon$. By Lemma 2.1, we have

$$l^1_{(k+1)^2 m_\varepsilon}(Q) \leq l^1_{m_\varepsilon}(Q).$$

Let $\{\Omega_j\}^{(k+1)^2 m_\varepsilon}_{j=1}$ be the minimal $(k+1)^2 m_\varepsilon$-partition of $Q$. By grouping together some of the subdomains $\Omega_j$, we can obtain a new $m$-partition of $Q$, denoted by $\{\Omega'_j\}^m_{j=1}$, then we deduce that

$$l^1_m(Q) \leq \frac{\sum_{j=1}^m \lambda_1(\Omega'_j)}{m^2} \leq \frac{(k+1)^2 m_\varepsilon}{m^2} l^1_{m_\varepsilon}(Q) \leq \frac{(k+1)^4}{k^2} (a(Q) + \varepsilon).$$

Let $k_\varepsilon$ be sufficiently large such that $(\frac{k+1}{k^2})^4 (a(Q) + \frac{\varepsilon}{2}) \leq a(Q) + \varepsilon$. Then for any $m \geq k_\varepsilon^2 m_\varepsilon$, we have

$$l^1_m(Q) \leq a(Q) + \varepsilon,$$

which implies that $\limsup_{m \to \infty} l^1_m(Q) \leq a(Q)$. One can deduce from the above proof that $\lim_{m \to \infty} l^1_m(Q) = \lim_{m \to \infty} l^1_{m+o(m)}(Q)$.

Step 2. For any bounded, smooth domain $\Omega \subset \mathbb{R}^2$ such that $|\Omega| = 1$, we prove

$$\limsup_{m \to \infty} l^1_m(\Omega) \leq \lim_{m \to \infty} l^1_m(Q) = a(Q).$$

For any $\varepsilon > 0$, there is $k \in \mathbb{N}$, such that

$$(2.4) \quad \bigcup_{j=1}^k Q_j \subset \Omega \subset \left( \bigcup_{j=1}^k Q_j \right) \cup \left( \bigcup_{i=1}^l Q_{k+i} \right).$$

Here $\{Q_j\}_{j=1}^k$ and $\{Q_{k+i}\}_{i=1}^l$ are smaller dyadic cubes of the same size and satisfies

$$\sum_{i=1}^l |Q_{k+i}| \leq \frac{\varepsilon}{4}.$$

If $k > 1$, we let $m = (k-1)n + t$, where $m, n, t \in \mathbb{N}$ and $t < (k-1)$. Then we have

$$l^1_m(\Omega) \leq l^1_m(\bigcup_{j=1}^k Q_j) \leq \left( \frac{(k-1)n^2 l^1_m(Q)}{|Q_j|} + \frac{t^2 l^1_m(Q)}{|Q_j|} \right) / m^2.$$

Here the second inequality comes from the construction of the partition that divide each of $Q_j$ ($j = 1, \ldots, (k-1)$) into $n$ subdomains and divide the last cube $Q_k$ into $t$ subdomains. Let $n$ be sufficiently large or equivalently $m$ sufficiently large, we can guarantee that the value of the last line is less than $a(Q)(1 + \varepsilon)$, which leads to that $\limsup_{m \to \infty} l^1_m(\Omega) \leq a(Q)$. 

Step 3. We are left to prove \( \liminf_{m \to \infty} l_m^1(\Omega) \geq a(Q) \). Given \( \varepsilon > 0 \), by (2.4), \( \Omega \) can be approximated by smaller dyadic cubes. Then we have

\[
l_m^1(\Omega) \geq l_m^1\left((\cup_{j=1}^k Q_j) \cup (\cup_{i=1}^l Q_{k+i})\right)
\]

It suffices to show that given \( m \) large enough,

\[
l_m^1\left((\cup_{j=1}^k Q_j) \cup (\cup_{i=1}^l Q_{k+i})\right) \geq (1 - \varepsilon)a(Q).
\]

Actually, (2.5) is implied by the following Lemma 2.2.

**Lemma 2.2.** Let \( \Omega \) be a domain in \( \mathbb{R}^2 \) with \( |\Omega| = 1 \). \( \Gamma \) is a straight line that separates \( \Omega \) into two sub-domains \( D_1, D_2 \), with area \( \alpha, 1 - \alpha \) respectively. Assume there exists a constant \( c \) such that

\[
\lim_{m \to \infty} l_m^1\left(\frac{1}{\sqrt{\alpha}} D_1\right) = \lim_{m \to \infty} l_m^1\left(\frac{1}{\sqrt{1 - \alpha}} D_2\right) = c
\]

Then

\[
\lim_{m \to \infty} l_m^1(\Omega) = c.
\]

Let’s assume this lemma and proceed with our proof. Note that \( (\cup_{j=1}^k Q_j) \cup (\cup_{i=1}^l Q_{k+i}) \) is the union of \( k + l \) small cubes, whose areas added up to \( (1 + \delta) \) for some \( \delta < \frac{1}{4} \). By proper scalings and by repetitive applications of Lemma 2.2, we can then get that

\[
\lim_{m \to \infty} l_m^1\left((\cup_{j=1}^k Q_j) \cup (\cup_{i=1}^l Q_{k+i})\right) = \lim_{m \to \infty} l_m^1(Q) \geq (1 - \varepsilon)a(Q),
\]

which yields the conclusion (2.5). The proof of the theorem is then completed.

**Proof of Lemma 2.2.** Without loss of generality, one assumes \( \Gamma = \{x = 0\} \) and \( D_1 = \{z = (x, y) \in \Omega: x < 0\}, D_2 = \{z = (x, y) \in \Omega: x > 0\} \). Note that by the same arguments as in the Step 2, we have

\[
\limsup_{m \to \infty} l_m^1(\Omega) \leq c.
\]

It suffices to prove for any \( \varepsilon > 0 \), there exists \( m_{\varepsilon} \) such that if \( m \geq m_{\varepsilon} \), then

\[
l_m^1(\Omega) \geq c(1 - \varepsilon).
\]

In the rest of proof we will always fix \( \varepsilon > 0 \) and we always assume \( m \) is large enough (depending on \( \varepsilon \) that it will be specified later). We need to study Problem (P*), which is the equivalent formulation of the minimal partition Problem P. Let \( u = (u_1, ..., u_m) \in H^1_0(\Omega, \Sigma^m) \) be a minimizer of Problem (P*), then \( \{\text{supp}(u_j)\}_{j=1}^m \) gives a minimal \( m \)-partition of \( \Omega \). Denote

\[
\Omega_j = \text{supp}(u_j)
\]

Take a fixed small number \( \delta \) (also depends on \( \varepsilon \) only, will be determined later). We define the following regions:

\[
S_\delta = \{z = (x, y) \in \Omega: \text{dist}(z, \Gamma) < \frac{\delta}{2}\} = \{z = (x, y) \in \Omega, |x| < \frac{\delta}{2}\}
\]

\[
D'_1 = D_1 \setminus S_\delta, \quad D'_2 = D_2 \setminus S_\delta
\]

Then we classify the subdomains in the partition \( \{\Omega_j\}_{j=1}^m \) according to their intersections with \( S_\delta, D'_1, D'_2 \).

\[
A_\delta = \{\Omega_k : \Omega_k \cap D'_2 = \emptyset\},
\]

\[
B_\delta = \{\Omega_k : \Omega_k \cap D'_1 = \emptyset\},
\]

\[
C_\delta = \{\Omega_k : \Omega_k \cap D'_1 \neq \emptyset, \Omega_k \cap D'_2 \neq \emptyset\}.
\]
We are mostly interested in subdomains in $C_\delta$. Take $\Omega_j \in C_\delta$. Define the sub-region of $S_\delta$:

$$S_{(r,r+\frac{\delta}{2})} = \{ z \in S_\delta : r < x < r + \frac{\delta}{2}, \quad r \in [-\frac{\delta}{2},0] \}.$$ 

Noted that for each $r$, $S_{(r,r+\frac{\delta}{2})}$ is a region with half width of $S_\delta$. Obviously, there exists $r_j \in [-\frac{\delta}{2},0]$ such that

$$\int_{\Omega_j \cap S_{(r_j,r_j+\frac{\delta}{2})}} |u_j|^2 \leq \frac{1}{2} \int_{\Omega_j} u_j^2.$$ 

Let $\xi_j$ be a smooth cut-off function such that

$$\xi_j(z) \equiv 1 \text{ if } z \not\in S_{(r_j,r_j+\frac{\delta}{2})}; \quad \xi_j(z) \equiv 0 \text{ on } x = r_j + \frac{\delta}{4}; \quad |\nabla \xi_j| \leq \frac{8}{\delta}.$$ 

**Claim:** If \( \frac{\int_{\Omega_j} |\nabla (\xi_j u_j)|^2}{\int_{\Omega_j} |\xi_j u_j|^2} \geq (1 + \frac{\epsilon}{5}) \lambda_1(\Omega_j) \), then there exists a constant $C_1$ which depends on $\epsilon$, such that $\lambda_1(\Omega_j) \leq C_1(\epsilon)$.

**Proof of the Claim:** we calculate directly:

$$\frac{\int_{\Omega_j} |\nabla (\xi_j u_j)|^2}{\int_{\Omega_j} |\xi_j u_j|^2} \geq (1 + \frac{\epsilon}{5}) \lambda_1(\Omega_j)$$

$$\Rightarrow \int_{\Omega_j} |\nabla \xi_j|^2 u_j^2 + 2 \xi_j u_j \nabla \xi_j \cdot \nabla u_j + \xi_j^2 |\nabla u_j|^2 \geq (1 + \frac{\epsilon}{5}) \lambda_1(\Omega_j) \int_{\Omega_j} (u_j \xi_j)^2$$

By integration by parts, we have

$$\int_{\Omega_j} |\nabla u_j|^2 \xi_j^2 = \int_{\Omega_j} \lambda_1(\Omega_j) (u_j \xi_j)^2 - \int_{\Omega_j} 2 \xi_j u_j \nabla \xi_j \cdot \nabla u_j.$$ 

Thus (2.9) implies

$$\int_{\Omega_j} |\nabla \xi_j|^2 u_j^2 \geq \frac{\epsilon}{5} \lambda_1(\Omega_j) \int_{\Omega_j} (u_j \xi_j)^2$$

By assumption on $\xi_j$, we conclude that

$$\lambda_1(\Omega_j) \leq \frac{640}{\epsilon \delta^2} =: C_1(\epsilon).$$

Let $D_\delta$ be the subset of $C_\delta$ that is consisted of all these sub-domains that satisfies (2.8). According to the above claim, for any $\Omega_j \in D_\delta$, $\lambda_1(\Omega_j) \leq C_1(\epsilon)$, and by the well-known Faber-Krahn inequality, there exists a constant $C_2(\epsilon)$ such that $|\Omega_j| \geq C_2(\epsilon)$. Then we can control the number of sub-domains in $D_\delta$ by a constant only depends on $\epsilon$, but is independent of $m$, i.e. $\#D_\delta \leq C_3(\epsilon)$.

Based on $u$, $A_\delta$, $B_\delta$, $C_\delta$, $D_\delta$, we can then define modified vector-valued functions $v,w$ such that $\text{supp}(v) \subset D'_1 \cup S_\delta$ and $\text{supp}(w) \subset D'_2 \cup S_\delta$. We follow the following schemes:

(i) If $\Omega_j \in A_\delta$, then $v_j = u_j$;
(ii) If $\Omega_j \in B_\delta$, then $w_j = u_j$;
(iii) If $\Omega_j \in C_\delta \backslash D_\delta$, then we have by definition

$$\frac{\int_{\Omega_j} |\nabla (\xi_j u_j)|^2}{\int_{\Omega_j} |\xi_j u_j|^2} \leq (1 + \frac{\epsilon}{5}) \lambda(\Omega_j)$$

Note that $\xi = 0$ on $\{ x = r_j + \frac{\delta}{4} \}$, the line $\{ x = r_j + \frac{\delta}{4} \}$ divides $\Omega_j$ into two sub-domains $\Omega_j^1, \Omega_j^2$, where

$$\Omega_j^1 \subset D'_1 \cup S_\delta, \quad \Omega_j^2 \subset D'_2 \cup S_\delta.$$
Moreover, we have $u_j \xi_j|_{\Omega_1} \in H_0^1(\Omega_1^1)$ and $u_j \xi_j|_{\Omega_2} \in H_0^1(\Omega_2^2)$. We denote

$$\tau_1 := \frac{\int_{\Omega_1^1} |\nabla (\xi_j u_j)|^2}{\int_{\Omega_1^1} |\xi_j u_j|^2}, \quad \tau_2 := \frac{\int_{\Omega_2^2} |\nabla (\xi_j u_j)|^2}{\int_{\Omega_2^2} |\xi_j u_j|^2}.$$  

Clearly (2.10) implies that

$$\min\{\tau_1, \tau_2\} \leq (1 + \frac{\delta}{5}) \lambda(\Omega_j).$$

If $\tau_1 \leq \tau_2$, then we let

$$v_j = \frac{\xi_j u_j}{\sqrt{\int_{\Omega_1^1} |\xi_j u_j|^2}} \text{ on } \Omega_1^1, \quad v_j = 0 \text{ elsewhere.}$$

Otherwise, let

$$w_j = \frac{\xi_j u_j}{\sqrt{\int_{\Omega_2^2} |\xi_j u_j|^2}} \text{ on } \Omega_2^2, \quad w_j = 0 \text{ elsewhere.}$$

We also denote

$$E_\delta = \{\Omega_j^1 : \Omega_j \in C_\delta \setminus D_\delta, \tau_1 \leq \tau_2\}$$

$$F_\delta = \{\Omega_j^2 : \Omega_j \in C_\delta \setminus D_\delta, \tau_1 > \tau_2\}.$$

(iv) Finally, we rearrange the vector of functions $v, w$ such that

$$\text{supp } v_j \neq \emptyset, \quad \text{supp } v_j \in A_\delta \cup E_\delta, \text{ for all } j = 1, ..., m_1.$$  

$$\text{supp } w_j \neq \emptyset, \quad \text{supp } w_j \in B_\delta \cup F_\delta, \text{ for all } j = 1, ..., m_2.$$  

Here $m_1 = \# A_\delta + \# E_\delta, m_2 = \# B_\delta + \# F_\delta$.  

Now we are ready to prove (2.6). One calculates

$$\frac{1}{m^2} \sum_{j=1}^m \int_{\Omega_j} |\nabla u_j|^2 \geq \frac{1}{m^2} \left( \sum_{\Omega_j \in A_\delta} \int_{\Omega_j} |\nabla u_j|^2 + \sum_{\Omega_j \in B_\delta} \int_{\Omega_j} |\nabla u_j|^2 + \sum_{\Omega_j \in C_\delta \setminus D_\delta} \int_{\Omega_j} |\nabla u_j|^2 \right)$$

$$\geq \frac{1}{m^2} \left( \sum_{\Omega_j \in A_\delta} \int_{\Omega_j} |\nabla v_j|^2 + \sum_{\Omega_j \in B_\delta} \int_{\Omega_j} |\nabla v_j|^2 + \sum_{\Omega_j \in E_\delta} \int_{\Omega_j} |\nabla v_j|^2 + \frac{\sum_{\Omega_j \in E_\delta} \int_{\Omega_j} |\nabla u_j|^2}{1 + \varepsilon/5} + \frac{\sum_{\Omega_j \in F_\delta} \int_{\Omega_j} |\nabla u_j|^2}{1 + \varepsilon/5} \right)$$

$$\geq \frac{1}{m^2(1 + \varepsilon/5)} \left( \sum_{\Omega_j \in A_\delta} \int_{\Omega_j} |\nabla v_j|^2 + \sum_{\Omega_j \in E_\delta} \int_{\Omega_j} |\nabla v_j|^2 + \sum_{\Omega_j \in F_\delta} \int_{\Omega_j} |\nabla w_j|^2 + \frac{\sum_{\Omega_j \in E_\delta} \int_{\Omega_j} |\nabla v_j|^2}{1 + \varepsilon/5} + \frac{\sum_{\Omega_j \in F_\delta} \int_{\Omega_j} |\nabla w_j|^2}{1 + \varepsilon/5} \right)$$

(2.11)

Define

$$\tilde{D}_1 = \bigcup_{\Omega_j \in A_\delta \cup E_\delta} \Omega_j, \quad \tilde{D}_2 = \bigcup_{\Omega_j \in B_\delta \cup F_\delta} \Omega_j.$$  

By the construction above we have

$$\tilde{D}_i \subset D_i' \cup S_\delta \subset (1 + \frac{\varepsilon}{10}) D_i, \quad \text{for } i = 1, 2.$$
where $\delta < \delta(\varepsilon)$ is small enough. Hence we obtain that

$$\lim_{m \to \infty} l_{m1}^1(\tilde{D}_1) \geq (1 - \frac{\varepsilon}{5}) \lim_{m \to \infty} l_{m1}^1(D_1) = \frac{1 - \varepsilon/5}{\alpha}c$$

$$\lim_{m \to \infty} l_{m1}^1(\tilde{D}_2) \geq (1 - \frac{\varepsilon}{5}) \lim_{m \to \infty} l_{m1}^1(D_2) = \frac{1 - \varepsilon/5}{1 - \alpha}c$$

Note that by our construction, $v \in H_0^1(\tilde{D}_1, \Sigma^{m_1})$ and $w \in H_0^1(\tilde{D}_2, \Sigma^{m_2})$, $m_1 + m_2 = m - C_3(\varepsilon)$. We take $m$ sufficiently large such that

$$\left(\frac{m - C_3(\varepsilon)}{m}\right)^2 \geq 1 - \frac{\varepsilon}{5}, \quad l_{m1}^1(\tilde{D}_1) \geq \frac{1 - \varepsilon/4}{\alpha}c, \quad l_{m2}^1(\tilde{D}_2) \geq \frac{1 - \varepsilon/4}{1 - \alpha}c.$$

Here we have assumed that $m_1, m_2$ also go to infinity when $m$ goes to infinity. If the latter is not true, then it is even easier to conclude (2.6), and we shall omit the details to the readers. By combining (2.11), (2.12), (2.13) and (2.14) we can deduce that

$$\frac{1}{m^2} \sum_{j=1}^{m} \int_{\Omega_j} |\nabla u_j|^2 \geq \frac{1}{m^2(1 + \varepsilon/5)} \left( m_1^2 l_{m1}^1(\tilde{D}_1) + m_2^2 l_{m2}^1(\tilde{D}_2) \right) \geq \frac{c(1 - \varepsilon/4)}{(1 + \varepsilon/5)m^2} \left( \frac{m_1^2}{\alpha} + \frac{m_2^2}{1 - \alpha} \right) \geq \frac{1 - \varepsilon/4}{1 + \varepsilon/5} \left( \frac{m - C_3(\varepsilon)}{m} \right)^2 c \geq \frac{(1 - \varepsilon/4)(1 - \varepsilon/5)}{1 + \varepsilon/5}c \geq (1 - \varepsilon)c.$$

It completes the proof. \hfill \Box

**References**

[1] O. Alper, On the singular set of free interface in an optimal partition problem, *Comm. Pure Appl. Math.* 73.4 (2020), 855–915.

[2] V. Bonnaillie-Noël and B. Helffer, Nodal and spectral minimal partitions - The state of the art in 2016 -, Shape optimization and spectral theory (2017), 353–397. Warsaw, Poland: De Gruyter Open.

[3] J. Bourgain, On Pleijel’s Nodal Domain Theorem, *Int. Math. Res. Not.* 2015.6 (2015), 1601–1612.

[4] D. Bucur, Minimization of the k-th eigenvalue of the Dirichlet Laplacian, *Arch. Ration. Mech. Anal.* 206.3 (2012), 1073–1083.

[5] D. Bucur, G. Buttazzo, and A. Henrot, Existence results for some optimal partition problems, *Adv. Math. Sci. Appl.* Tokyo 8.2 (1998), 571–579.

[6] D. Bucur, I. Fragalà, On the honeycomb conjecture for Robin Laplacian eigenvalues, *Commun. Contemp. Math.* 21.02 (2019), 1850007.

[7] D. Bucur and J. P. Zolesio, N-dimensional shape optimization under capacity constraints, *J. Diff. Eqs.* 123 (1995), 504–522.

[8] D. Bucur, I. Fragalà, B. Velichkov and G. Verzini, On the honeycomb conjecture for a class of minimal convex partitions, *Trans. Amer. Math. Soc.* 370.10 (2018), 7149–7179.

[9] G. Buttazzo, and G. Dal Maso, Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions, *Appl. Math. Optim.* 23 (1991), 17–49.

[10] L. Caffarelli and F.H. Lin, An optimal partition problem for eigenvalues, *J. Sci. Comput.*, 31 (2007), 5–18.

[11] L. Caffarelli and F.H. Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, *J. Amer. Math. Soc.* 21 (2008), 847–862.

[12] B. Helffer, On spectral minimal partitions: A survey, *Milan J. Math.*, 78 (2010), 575–590.

[13] F.H. Lin, Extremum problems of Laplacian eigenvalues and generalized Polya conjecture, *Chinese Annals of Mathematics, Series B* 38.2 (2017), 497–512.
[14] D. Mazzoleni and A. Pratelli, Existence of Minimizers for spectral problems. *J. Math. Pures Appl.* **100.3** (2013), 433–453.

[15] S. Steinerberger, A geometric uncertainty principle with an application to Pleijel’s estimate, *Annales Henri Poincaré** 15(12) (2014), 2299–2319 Springer Basel.

[16] V. Sverak, On optimal shape design. *J. Math. Pures Appl.* **72** (1993), 537–551.

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