Surprising Examples of Manifolds in Toric Topology!

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Abstract

We investigate small covers and quasitoric over the duals of neighborly simplicial polytopes with small number of vertices in dimensions 4, 5, 6 and 7. In the most of the considered cases we obtain the complete classification of small covers. The lifting conjecture in all cases is verified to be true. The problem of cohomological rigidity for small covers is also studied and we have found a whole new series of weakly cohomologically rigid simple polytopes. New examples of manifolds provide the first known examples of quasitoric manifolds in higher dimensions whose orbit polytopes have chromatic numbers $\chi(P^n) \geq 3n - 5$.

Introduction

Quasitoric manifolds and their real analogues appeared in a seminal paper [14] of Davis and Januszkiewicz as a topological generalizations of non-singular projective toric varieties and real toric varieties. The manifolds have a locally standard $(S^d)^n$ action where $d = 0$ in the case of small covers and $d = 1$ in the case of quasitoric manifolds, such that the orbit space of the action is identified with a simple polytope as a manifold with corners. The simplest examples are manifolds over the $n$-dimensional simplex $\Delta^n$, $\mathbb{C}P^n$ for quasitoric manifolds and $\mathbb{R}P^n$ for small covers and they are unique up to homeomorphism.

In the last decades, toric topology experienced an impressive progress. The most significant results are summarized in recent remarkable monograph [8] by Buchstaber and Panov. However one of the most interesting problems in toric topology such as classification of simple polytopes that can appear as the orbit spaces of some quasitoric manifolds and classification of quasitoric manifolds and small covers over given simple polytope are still open. In dimension 2 and 3 every simple polytope is the orbit space for quasitoric manifolds, but in dimensions larger than 3 our knowledge is still limited on some particular classes of polytopes and examples. Recent progress in the problem of classification is done by Hasui in [22] who studied toric topology of cyclic polytopes. The class of cyclic polytopes is subclass of the class of neighborly polytopes which is for many reasons very important class of polytopes, and appears in combinatorics, enumerative geometry, probability, etc.

Hasui’s progress motivated us to attempt to say something about toric topology of neighborly polytopes. Our work is based on recent results of Moritz Firchung about enumeration of neighborly simplicial polytopes in dimensions 4, 5, 6 and 7, [16] and [17]. Thus, we were able to completely classify small covers over neighborly simple 4-polytopes with up to 12 facets (it is trivial to see that there is no small cover over neighborly simple 4-polytopes with over 15 facets), neighborly simple 5-polytopes with up to 9 facets and neighborly simple 6-polytopes with up to 10 facets. We
verified that the Lifting conjecture for small covers is true for all of them and partially answered on some rigidity questions. Our computer search successfully found neighborly simple 5-polytopes with up to 10 facets and neighborly simple 7-polytopes with up to 11 facets appearing as the orbit space of some small cover.

These new examples contradict the intuition that one can get on the first glance from Hasui’s work that small covers and quasitoric manifolds over neighborly polytopes are rarely structures. It is pointed out that using our examples we can obtain a simple polytope $P^n$ in any dimension $n$ for which $\chi(P^n) \geq 3n - 5$. Thus, these new examples of manifolds and combinatorial structures of these polytopes certainly deserve further study in mathematics since they give new light on our current knowledge in toric topology.

In Section 1 we review basic facts about simple polytopes, neighborly polytopes and small covers and quasitoric manifolds. Section 2 is devoted to the classification problem in toric topology and review of the current knowledge, while in Section 3 we explain the algorithm we used in our computer search for the characteristic matrices over neighborly polytopes. Sections 4, 5, 6 and 7 are devoted to studying simple neighborly polytopes in dimensions 4, 5, 6 and 7 and related problems from toric topology. In Section 8 we illustrate some interesting examples from the previous sections.

1 Basic Constructions

In this section we define quasitoric manifolds and small covers and describe some of their properties.

1.1 Simplicial and Simple Polytopes

Let us start with recalling basic concepts and constructions in the theory of polytopes. For more advanced topics and further reading, we refer reader to the classical monographs [21] and [31].

A point set $K \subseteq \mathbb{R}^n$ is convex if for any two points $x, y \in K$, we have that the straight line segment $[x, y] = \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$ lies entirely in $K$ (see Figure 1). Clearly, the intersection of two convex sets is again a convex set and $\mathbb{R}^n$ is itself convex. The ‘smallest’ convex set containing a given set $K$ is called the convex hull of $K$ and is equal to the intersection of all convex sets that contain $K$:

$$\text{conv}(K) := \bigcap \{L \subseteq \mathbb{R}^n \mid K \subseteq L, L \text{ is convex}\}.$$

**Definition 1.1.** A convex polytope is the convex hull of a finite set of points in some $\mathbb{R}^n$.

Using the definition of convexity, we can show by induction on $k$ that for any finite set of points $\{x_1, \ldots, x_k\} \subseteq K$, the convex hull of $K$ contains the set

$$\left\{ \lambda_1 x_1 + \cdots + \lambda_k x_k \mid \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

This means that each point $x$ of a convex polytope $P$ has a presentation

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1,$$

where $x_1, \ldots, x_k$ are the points whose convex hull is the polytope $P$. 
Example 1.1. The standard $n$-simplex $\Delta^n$ is convex hull of $n + 1$ points $\{O, e_1, \ldots, e_n\}$ where $O$ is origin and $e_1, \ldots, e_n$ the standard base in $\mathbb{R}^n$.

The dimension of a polytope is the dimension of its affine hull. Every linear form $l = l_a : \mathbb{R}^n \to \mathbb{R}$ has the form $x \mapsto ax$, where $a \in (\mathbb{R}^n)^*$ and $ax$ is the scalar obtained as the matrix product assuming that the point $x$ is represented with a column vector in $\mathbb{R}^n$ and $a$ is represented by a row vector in $(\mathbb{R}^n)^*$. We say that a linear inequality $mx \leq r$ is valid for a convex polytope $P \subseteq \mathbb{R}^n$ if it is satisfied for all points $x \in P$. A face of $P$ is any set of the form

$$F = P \cap \{x \in \mathbb{R}^n \mid mx = r\}.$$ 

The dimension of a face is the dimension of its affine hull.

From the obvious inequalities $0x \leq 0$ and $0x \leq 1$, we deduce that the polytope $P$ itself and $\emptyset$ are faces of $P$. All other faces of $P$ are called proper faces. The faces of dimension 0, 1, $\dim(P) - 2$, and $\dim(P) - 1$ are called vertices, edges, ridges and facets, respectively. The faces of a polytope $P$ are polytopes of smaller dimension and every intersection of finite number of faces is a face of $P$. [31] Proposition 2.3. The faces of a convex polytope $P$ form a partially ordered structure with respect to inclusion.

Definition 1.2. The face lattice of a convex polytope $P$ is the poset $L := L(P)$ of all faces of $P$, partially ordered by inclusion.

We say that two polytopes $P_1$ and $P_2$ are combinatorially equivalent if their face lattices $L(P_1)$ and $L(P_2)$ are isomorphic. A combinatorial polytope is a class of combinatorially equivalent polytopes.

Definition 1.3. A polytope $P$ is called simplicial if all its proper faces are simplices.

For any convex polytope $P \subseteq \mathbb{R}^n$ we define its polar set $P^* \subseteq (\mathbb{R}^n)^*$ by

$$P^* := \{c \in (\mathbb{R}^n)^* \mid cx \leq 1 \text{ for all } x \in P\}.$$ 

It is well known fact from convex geometry that the polar set $P^*$ is a convex set in $(\mathbb{R}^n)^*$ that contains $0$ in its interior. Moreover, if $O \in P$ then $P^*$ is convex polytope and $(P^*)^* = P$. We refer to the combinatorial polytope $P^*$ as the dual of the combinatorial polytope $P$. The face lattice $L(P^*)$ is the opposite of the face lattice $L(P)$ of $P$. 

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Definition 1.4. A polytope $P$ is called simple if its dual polytope $P^*$ is simplicial.

Now we observe that polytope $P^n$ is simple if there are exactly $n$ facets meeting at each vertex of $P^n$ and each face of simple polytope is again a simple polytope. Any combinatorially polar polytope of a simple polytope is simplicial.

The notion of $f$-vector is a fundamental concept in the combinatorial theory of polytopes. It has been extensively studied for last four centuries.

Definition 1.5. Let $P$ be a simplicial $n$-polytope. The $f$-vector is the integer vector

$$f(P) = (f_{-1}, f_0, f_1, \ldots, f_{n-1}),$$

where $f_{-1} = 1$ and $f_i = f_i(P)$ denotes the number of $i$-faces of $P$, for all $i = 1, \ldots, n - 1$.

With the $f$-vector it is naturally associated the notion of the $f$-polynomial. The $f$-polynomial of a simplicial polytope $P$ is

$$f(t) = t^n + f_0 t^{n-1} + \cdots + f_{n-1}.$$

Another important notion in combinatorics of polytopes is the $h$-vector. We will introduce it by defining the $h$-polynomial first. The $h$-polynomial is the polynomial

$$h(t) = f(t - 1),$$

and the coefficients $h_0, \ldots, h_n$ of the $h$-polynomial $h(t) = h_0 t^n + \cdots + h_{n-1} t + h_n$ define the $h$-vector by

$$h(P) = (h_0, h_1, \ldots, h_n).$$

The $f$-vector and the $h$-vector are combinatorial invariants of $P$ and depend only on the face lattice. They carry the same information about the polytope and mutually determine each other by means of linear relations coming from the equation (1)

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{n-i}{n-k} f_{i-1}, f_{n-k-1} = \sum_{j=k}^n \binom{j}{k} h_{n-j}, k = 0, \ldots, n.$$

It is a natural question to describe which integer vectors may appear as the $h$-vectors of simple polytopes. The answer to the question is provided by the renowned $g$-theorem, that was conjectured by McMullen in [26]. The necessity part of the theorem was proved by Stanley in [30] while the sufficiency part was proved by Billera and Lee in [5].

Theorem 1.1. An integer vector $(f_{-1}, f_0, f_1, \ldots, f_{n-1})$ is the $f$-vector of a simple $n$-polytope if and only if the corresponding sequence $(h_0, h_1, \ldots, h_n)$ determined by (1) satisfies the following conditions:

1. $h_i = h_{n-i}$, $i = 0, \ldots, n$ (the Dehn-Sommerville equations);
2. $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$, $i = 0$;
3. $h_0 = 1$, $h_{i+1} - h_i \leq (h_i - h_{i-1})^{(i)}$, $i = 0, \ldots, \lfloor n/2 \rfloor - 1$. 


Recall that for any two integers \(a\) and \(i\), \(a^{(i)}\) is defined as

\[
a^{(i)} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j + 1}{j + 1},
\]

where \(a_i > a_{i-1} > \cdots > a_j \geq j \geq 1\) are the unique integers such that

\[a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}.
\]

The latter representation of \(a\) is known as the binomial \(i\)-expansion of \(a\).

### 1.2 Neighborly polytopes

An \(n\)-polytope \(P\) is said to be \(k\)-neighborly if any subset of \(k\) or less vertices is the vertex set of a face of \(P\). It is straightforward to check that for \(k > \left\lceil \frac{n}{2} \right\rceil\), the simplex \(\Delta^n\) is the only \(k\)-neighborly polytope. Thus, polytopes that are \(\left\lfloor \frac{n}{2} \right\rfloor\)-neighborly are of particular interests and are called neighborly polytopes.

Note that for a neighborly \(n\)-polytope \(P^n(m)\) with \(m\) vertices it holds

\[
f_i(P^n(m)) = \binom{m}{i + 1} \text{ for } i = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

By the Dehn-Sommerville equations (1.1) we straightforwardly deduce the following claim.

**Lemma 1.1.** The \(h\)-vector of a neighborly \(n\)-polytope \(P^n(m)\) with \(m\) vertices is given by

\[
h_i(P^n(m)) = h_{n-i}(P^n(m)) = \binom{m-n+i-1}{i} \text{ for } i = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.
\]

Lemma 1.1 can be reformulated in terms of the \(f\)-vector as:

**Corollary 1.1.** The \(f\)-vector of a neighborly \(n\)-polytope \(P^n(m)\) with \(m\) vertices is given by

\[
f_i(P^n(m)) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{j}{n-1-i} \binom{m-n+j-1}{j} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k}{i+1-k} \binom{m-n+k-1}{k},
\]

for \(i = -1, \ldots, n - 1\), where we assume \(\binom{k}{j} = 0\) for \(k < j\).

The neighborly polytopes are very important objects in combinatorics because they are solutions of various extremal properties. They satisfy the upper bound predicted by Motzkin for maximal number of \(i\)-faces of an \(n\)-polytope with \(m\) vertices. This statement is known as the Upper Bound Theorem and it is first proved by McMullen in [25].

**Theorem 1.2** (The Upper Bound Theorem). From all simplicial \(n\)-polytopes \(Q\) with \(m\) vertices, any simplicial neighborly \(n\)-polytope \(P^n(m)\) with \(m\) vertices has the maximal number of \(i\)-faces, \(2 \leq i \leq n - 1\). That is

\[f_i(Q) \leq f_i(P^n(m)) \text{ for } i = 0, \ldots, n - 1.
\]

The equality in the above formula holds if and only if \(Q\) is a simplicial neighborly \(n\)-polytope with \(m\) vertices.
The Upper Bound Theorem implies that for a simplicial \( n \)-polytope \( Q \) with \( m \) vertices the following inequalities for \( h \)-vector are true

\[
h_i(Q) \leq \binom{m - n + i - 1}{i}, \quad i = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

A classical example of a neighborly \( n \)-polytope with \( m \) vertices is the cyclic polytope \( C^n(m) \). Recall that the moment curve \( \gamma \) in \( \mathbb{R}^n \) is defined by \( \gamma: \mathbb{R} \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = (t, t^2, \ldots, t^n) \in \mathbb{R}^n \). The cyclic polytope \( C^n(m) \) is the convex hull

\[
C^n(m) := \text{conv} \{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_m)\},
\]

for \( m \) distinct points \( \gamma(t_i) \) with \( t_1 < t_2 < \cdots < t_m \) on the moment curve. The combinatorial class of \( C^n(m) \) does not depend on the specific choices of the parameters \( t_i \) due to Gale’s evenness condition, see \[31\] Theorem 0.7.]

**Theorem 1.3** (Gale’s evenness condition). Let \( m > d \geq 2 \) and \( C^n(m) \) be the cyclic polytope with vertices \( \gamma(t_i) \) with \( t_1 < t_2 < \cdots < t_m \) on the moment curve. An \( n \)-subset \( S \subseteq \{1, 2, \ldots, m\} \) forms a facet of \( C^n(m) \) if and only if the following ‘evenness condition’ is satisfied:

If \( i < j \) are not in \( S \) then the number of \( k \in S \) such that \( k < k < j \) is even.

Cyclic polytopes are simplicial polytopes and it can be proved that even-dimensional neighborly polytopes are necessarily simplicial, but this is not true in general. For example, any 3-dimensional polytope is neighborly by definition.

If the number of vertices \( m \) of a neighborly \( n \)-polytope is not grater than \( n + 3 \) then combinatorially the polytope is isomorphic to a cyclic polytope. However, there are many neighborly polytopes which are not cyclic. Barnette in \[11\] constructed an infinite family of duals of neighborly \( n \)-polytopes by using an operation called ‘facet splitting’ and Shemer in \[29\] introduced a sewing construction that allows to add a vertex to a neighborly polytope in such a way as to obtain a new neighborly polytope. Both constructions show that for a fixed \( n \) the number of combinatorially different neighborly polytopes grows superexponentially with the number of vertices \( m \). The number of combinatorial types of neighborly polytopes in dimensions 4, 5, 6 and 7 with ‘small’ number of vertices is extensively studied in the last decades. For more informations see \[16\].

Duals of simplicial neighborly \( n \)-polytopes are simple polytopes with property that each \( \left\lfloor \frac{n}{2} \right\rfloor \) facets have nonempty intersections. We shall call such polytopes also neighborly and in the rest of the paper under term neighborly we assume simple neighborly polytope.

### 1.3 Quasitoric Manifolds and Small Covers

Quasitoric manifolds and small covers are extensively studied in toric topology in the last twenty years. A detailed exposition on them can be found in Buchstaber and Panov’s monographs \[7\] and \[8\]. Here we briefly review the main definition and results about them.

Let

\[
G_d = \begin{cases} 
S^0, & \text{if } d = 1 \\
S^1, & \text{if } d = 2 \\
\mathbb{R}_d, & \text{if } d = 1 \\
\mathbb{Z}_d, & \text{if } d = 2.
\end{cases}
\]

and \( \mathbb{K}_d = \begin{cases} 
\mathbb{R}, & \text{if } d = 1 \\
\mathbb{C}, & \text{if } d = 2.
\end{cases} \)

where \( S^0 = \{-1, 1\} \) and \( S^1 = \{z \mid |z| = 1\} \) are multiplicative subgroups of real and complex numbers, respectively. The standard action of group \( G^0_d \) on \( \mathbb{K}^0_d \) is given as

\[
G^0_d \times \mathbb{K}^0_d \to \mathbb{K}^0_d : (t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) \mapsto (t_1 x_1, \ldots, t_n x_n).
\]

A \( G^0_d \)-manifold is a differentiable manifold with a smooth action of \( G^0_d \).
Definition 1.6. A map \( f : X \to Y \) between two \( G \)-spaces \( X \) and \( Y \) is called weakly equivariant if for any \( x \in X \) and \( g \in G \) holds
\[
 f(g \cdot x) = \psi(g) \cdot f(x),
\]
where \( \psi : G \to G \) is some automorphism of group \( G \).

Let \( M^{dn} \) be a \( d \times n \)-dimensional \( G_d^n \)-manifold. A standard chart on \( M^{dn} \) is an ordered pair \((U, f)\), where \( U \) is a \( G_d^n \)-stable open subset of \( M^{dn} \) and \( f \) is a weakly equivariant diffeomorphism from \( U \) onto some \( G_d^n \)-stable open subset of \( \mathbb{R}_d^n \). A standard atlas is an atlas which consists of standard charts. A \( G_d^n \) action on a \( G_d^n \)-manifold \( M^{dn} \) is called locally standard if manifold \( M^{dn} \) has a standard atlas. The orbit space for a locally standard action is naturally regarded as a manifold with corners.

Definition 1.7. A \( G_d^n \)-manifold \( \pi_d : M^{dn} \to P^n \) \((d = 1, 2)\) is a smooth closed \( (d \times n) \)-dimensional \( G_d^n \)-manifold admitting a locally standard \( G_d^n \)-action such that its orbit space is a simple convex \( n \)-polytope \( P^n \) regarded as a manifold with corners. If \( d = 1 \) such a \( G_d^n \)-manifold is called a small cover and if \( d = 2 \) a quasitoric manifold.

It is a standard fact that we do not have to distinguish combinatorially equivalent simple polytopes in the above definition. Moreover, it is straightforward to check the following proposition.

Proposition 1.1. Let \( M_1^{dn} \) and \( M_2^{dn} \) be \( G_d^n \)-manifolds over simple polytopes \( P \) and \( P' \) such that there is a weakly equivariant homeomorphism \( f : M_1^{dn} \to M_2^{dn} \). Then \( f \) descends to a homeomorphism from \( P \) to \( P' \) as manifolds with corners.

Let \( P^n \) be a simple polytope with \( m \) facets \( F_1, \ldots, F_m \). By Definition 1.7 it follows that every point in \( \pi^{-1}(\text{rel.int}(F_i)) \) has the same isotropy group which is an one-dimensional subgroup of \( G_d^n \). We denote it by \( G_d(F_i) \). Each \( G_d^n \)-manifold \( \pi_d : M^{dn} \to P^n \) determines a characteristic map \( l_d \) on \( P^n \)
\[
l_d : \{F_1, \ldots, F_m\} \to \mathbb{R}^n_d
\]
defined by mapping each facet of \( P^n \) to nonzero elements of \( \mathbb{R}^n_d \) such that
\[
l_d(F_i) = \lambda_i = (\lambda_{1,i}, \ldots, \lambda_{n,i})^t \in \mathbb{R}^n_d, \text{ where } \lambda_i \text{ is a primitive vector such that }
\]
\[
 G_d(F_i) = \left\{ (t^{\lambda_{1,i}}, \ldots, t^{\lambda_{n,i}}) | t \in \mathbb{K}_d, |t| = 1 \right\}.
\]
From the characteristic map we obtain an integer \((n \times m)\)-matrix \( \Lambda_{\mathbb{R}_d}(M^{dn}) := (\lambda_{i,j}) \) which is called the characteristic matrix of \( M^{dn} \). For \( d = 2 \) each \( \lambda_i \) is determined up to a sign. Since the \( G_d^n \)-action on \( M^{dn} \) is locally standard, the characteristic matrix \( \Lambda_{\mathbb{R}_d}(M^{dn}) \) satisfies the non-singular condition for \( P^n \), i.e. if \( n \) facets \( F_{i_1}, \ldots, F_{i_n} \) of \( P^n \) meet at vertex, then
\[
 | \det \Lambda_{\mathbb{R}_d}(i_1, \ldots, i_n, M^{dn}) | = 1,
\]
where \( \Lambda_{\mathbb{R}_d}(i_1, \ldots, i_n, M^{dn}) := (\lambda_{i_1}, \ldots, \lambda_{i_n}) \). Any integer \((n \times m)\)-matrix satisfying the non-singular condition for \( P^n \) is also called the characteristic matrix on \( P^n \).

The construction of a small cover and a quasitoric manifold from the characteristic pair \((P^n, \Lambda_{\mathbb{R}_d})\) where \( \Lambda_{\mathbb{R}_d} \) is a characteristic matrix is described in [6] and [7] Construction 5.12. For each point \( x \in P^n \), we denote the minimal face containing \( x \) in its relative interior by \( F(q) \). The characteristic map \( l_d \) corresponding to \( \Lambda_{\mathbb{R}_d} \) is a map from the set of the faces of \( P^n \) to the set of subtori of \( G_d^n \) defined by
\[
l_d(F_{i_1} \cap \ldots \cap F_{i_k}) := \text{im} \left( l_d^{i_1, \ldots, i_k} : G_d^k \to G_d^n \right),
\]
where \( l_d^{i_1, \ldots, i_k} \) is the characteristic map corresponding to \( \Lambda_{\mathbb{R}_d}(i_1, \ldots, i_n, M^{dn}) \).
where \( l_d^{(i_1,...,i_k)} \) is the map induced from the linear map determined by \( A_{\mathbb{K}_d}^{(i_1,...,i_k)} \). A \( G_d^n \)-manifold \( M^{dn}(A_{\mathbb{K}_d}) \) over simple polytope \( P^n \) is obtained by setting
\[
M^{dn}(A_{\mathbb{K}_d}) := (G_d^n \times P^n) / \sim_{l_d},
\]
where \( \sim_{l_d} \) is an equivalence relation defined by \( (t_1, p) \sim_{l_d} (t_2, q) \) if and only if \( p = q \) and \( t_1 t_2^{-1} \in l_d(F(q)) \). The free action of \( G_d^n \) on \( G_d^n \times P^n \) obviously descends to an action on \( (G_d^n \times P^n) / \sim_{l_d} \) with quotient \( P^n \). Simple polytope \( P^n \) is covered by the open sets \( U_v \) obtained by deleting all faces not containing vertex \( v \) of \( P^n \). Clearly, \( U_v \) is diffeomorphic to \( \mathbb{R}^n_+ \), so the space \( (G_d^n \times P^n) / \sim_{l_d} \) is covered by open sets \( (G_d^n \times U_v) / \sim_{l_d} \) homeomorphic to \( \mathbb{K}_d^n \). We easily see that the transition maps are diffeomorphic, so \( G_d^n \)-action on \( (G_d^n \times P^n) / \sim_{l_d} \) is locally standard and \( M^{dn}(A_{\mathbb{K}_d}) \) is a \( G_d^n \)-manifold \( \pi_d : M^{dn} \to P^n \) over simple convex \( n \)-polytope \( P^n \).

### 1.4 Lifting problem

Every quasitoric manifold \( M^{2n} \) admits an involution called conjugation such that its fixed point set is homeomorphic to a small cover \( M^n \) over the same polytope \( P^n \), [14, Corollary 1.9]. We may assume that \( M^{2n} \) is given by a characteristic pair \( (P^n, \Lambda) \) where \( \Lambda \) is an integer matrix satisfying the non-singular condition for \( P^n \). This means that \( M^{2n}(\Lambda) := (T^n \times P^n) / \sim_{l_d} \), where \( \sim_{l_d} \) is an equivalence relation defined by \( (t_1, p) \sim_{l_d} (t_2, q) \) if and only if \( p = q \) and \( t_1 t_2^{-1} \in l_d(F(q)) \). However, an obvious involution \( (t, p) \mapsto (t^{-1}, p) \) on \( T^n \times P^n \) with the fixed point set \( \mathbb{Z}_2^n \times P^n \) descends to an involution \( \tau \) on \( M^{2n}(\Lambda) \) with the fixed point set \( M^n := (\mathbb{Z}_2^n \times P^n) / \sim_{l_d} \), i.e. a small cover over \( P^n \). In this case, the real characteristic matrix \( \Lambda_{\mathbb{Z}_2}(M^n) \) is exactly the modulo 2 reduction of the characteristic matrix \( \Lambda_{\mathbb{Z}_2}(M^{2n}) \). The following problem posted by Zhi Lü, known as the lifting problem asks if the converse is true.

**Problem 1.** Let \( P \) be a simple polytope and let \( M^n \) be a small cover over \( P \). Is it true that there is a quasitoric manifold \( M^{2n} \) such that \( M^n \) is the fixed point set of the conjugation on \( M^{2n} \)?

The problem can be reformulated in the following way: For any real characteristic matrix \( \Lambda_{\mathbb{Z}_2}(M^n) = \Lambda \) where \( M^n \) is a small cover over simple polytope \( P^n \), is it true that there is a characteristic matrix \( \tilde{\Lambda} \) such that \( (P^n, \tilde{\Lambda}) \) is the characteristic pair of a quasitoric manifold? In other terms, does the diagram
\[
\begin{array}{ccc}
\tilde{\Lambda} & \xrightarrow{(\text{mod } 2)} & \mathbb{Z}_2^n \\
F_i \Lambda & \xrightarrow{(\text{mod } 2)} & \mathbb{Z}_2^n \\
\end{array}
\]
commute for each facet \( F_i \) of \( P^n \)?

Since the determinant of any three times three matrix with entries 0 and 1 is between −2 and 2, the non-singular condition for \( \Lambda_{\mathbb{Z}_2}(M^n) \) is satisfied with the same matrix but viewed as the characteristic manifold of a quasitoric manifold. Therefore, the lifting conjecture is true for all simple polytopes in dimension 2 and 3.

The answer to the lifting problem is also affirmative for all small covers over dual cyclic polytopes, see [22]. The hypothesis is also true for the products of simplices by [10].

### 1.5 Cohomology of Quasitoric Manifolds and Small Covers

In their seminal paper Davis and Januszkiewicz [14] calculated the cohomology ring and the characteristic classes of quasitoric manifolds and small covers. They are closely related with combinatorics of underlying simple polytope \( P^n \).
Theorem 1.4. Let $P^n$ be a simple polytope and $(h_0, h_1, \ldots, h_d)$ be its $h$-vector. Suppose that there exists a $G^n_d$-manifold $M^{dn}$ with locally standard $G^n_d$-action with $P^n$ as the orbit space of this action. Let $b_{di}(M^{dn})$ be the $i$-th Betti number of $M^{dn}$. Then $b_{di}(M^{dn}) = \dim_{\mathbb{R}_d} H_i(M^{dn}; \mathbb{R}_d) = h_i$. The homology of $M^{dn}$ vanishes in odd dimensions and is free in even dimensions in the case of quasitoric manifolds ($d = 2$).

We will define two ideals naturally assigned to $P^n$ and the characteristic matrix $\Lambda_d$. Let $F_1, \ldots, F_m$ be the facets of $P^n$. Let $\mathbb{R}_d[v_1, \ldots, v_m]$ be the polynomial algebra over $\mathbb{R}_d$ on $m$ generators with grading $\deg(v_i) = d$. The Stanley-Reisner ideal $\mathcal{I}_P$ is the ideal generated by all square-free monomials $v_{i1}, v_{i2}, \ldots, v_{in}$ such that $F_{i1} \cap \cdots \cap F_{in} = \emptyset$. Let $\Lambda_d = (\lambda_{ij})$ be a characteristic $n \times m$ matrix over $P^n$. We define linear forms 

$$\theta_i := \sum_{j=1}^m \lambda_{ij} v_j$$

and define $\mathcal{J}$ to be the ideal in $\mathbb{R}_d[v_1, \ldots, v_m]$ generated by $\theta_i$ for all $i = 1, \ldots, n$. Let $M^{dn}$ be a $G^n_d$ manifold corresponding to the characteristic pair $(P^n, \Lambda_d)$ and $\pi : M^{dn} \rightarrow P^n$ be the orbit map. From Definition 1.7 we obtain that each $\pi^{-1}(F_i)$ is a closed submanifold of dimension $d(n - 1)$ which is itself a $G^{n-1}_d$-manifold over $F_i$. Let $v_i \in H^d(M^{dn}; \mathbb{R}_d)$ denote its Poincaré dual. The ordinary cohomology of small covers and quasitoric manifolds has the following ring structure (see [13]).

Theorem 1.5. Let $M^{dn}$ be a $G^n_d$-manifold corresponding to the characteristic pair $(P^n, \Lambda_d)$. Then the cohomology ring of $M^{dn}$ is given by

$$H^*(M^{dn}) \cong \mathbb{R}_d[v_1, \ldots, v_m]/(\mathcal{I}_P + \mathcal{J}).$$

(4)

The total Stiefel-Whitney class can be described by the following Davis-Januszkiewicz formula:

$$w(M^{dn}) = \prod_{i=1}^m (1 + v_i) \in H^*(M^{dn}; \mathbb{Z}_2),$$

(5)

(where $v_i$ is the $\mathbb{Z}_2$-reduction of the corresponding class over $\mathbb{Z}$ coefficients in the case $d = 2$).

1.6 Cohomological rigidity

A simple polytope $P$ is called cohomologically rigid if its combinatorial structure is determined by the cohomology ring of a $G^n_d$-manifold over $P$. In general, an arbitrary simple polytope does not have this property, but some important polytopes such as simplices or cubes are known to be cohomologically rigid, see [23]. Another classes of cohomologically rigid polytopes are studied in [12]. We shall refer to such $P$ simply as rigid throughout the paper.

Definition 1.8. A simple polytope $P$ is cohomologically rigid if there exists a $G^n_d$-manifold $M^{dn}$ over $P$, and whenever there exists a $G^n_d$-manifold $N^{dn}$ over another polytope $Q$ with a graded ring isomorphism $H^*(M^{dn}; \mathbb{R}_d) \cong H^*(N^{dn}; \mathbb{R}_d)$ there is a combinatorial equivalence $P \sim Q$.

We already explained that the cohomology ring of a $G^n_d$-manifold over $P$ reveals a lot of combinatorics of $P$. If a simple polytope $P$ supports a $G^n_d$-manifold and there is no other simple polytope with the same $f$-vector as $P$, then $P$ is automatically rigid. Thus, polygons are always rigid. For more results on rigidity question, we address the reader to [12].

Another important question in toric topology related to rigidity is the following one:
Problem 2. Suppose \( M^{dn} \) and \( N^{dn} \) are \( G^n_d \)-manifolds such that \( H^*(M^{dn}; \mathbb{R}_d) \cong H^*(N^{dn}; \mathbb{R}_d) \) as graded rings. Is it true that \( M^{dn} \) and \( N^{dn} \) are homeomorphic?

Problem 2 is studied in several papers of Suyong Choi, Mikiya Masuda, Taras Panov, Dong Youp Suh and others [12], [23], [10], ... A nice exposition on the topic is given in a survey article [11]. Slightly weaker version of Problem 2 is also extensively studied in the last ten years.

Problem 3. Suppose \( M^{dn} \) and \( N^{dn} \) are \( G^n_d \)-manifolds over simple polytope \( P^n \) such that \( H^*(M^{dn}; \mathbb{R}_d) \cong H^*(N^{dn}; \mathbb{R}_d) \) as graded rings. Is it true that \( M^{dn} \) and \( N^{dn} \) are homeomorphic?

If the answer to Problem 3 for a simple polytope \( P^n \) is affirmative, then we say that \( P \) is weakly cohomologically rigid. It is known that the dodecahedron, the product of simplices, \( k \)-gonal prisms are all weakly cohomologically rigid, see [11]. Hasui studied in [22] cohomological rigidity and weakly cohomological rigidity of the cyclic polytopes.

2 Classification problem

Classification problem of \( G^n_d \)-manifolds over a given combinatorial simple polytope \( P^n \) is intractable. Moreover, it is not clear whether a combinatorial simple polytope is the orbit space of some \( G^n_d \)-manifold. From the previous discussion we know that this problem is equivalent to the existence of a characteristic map over \( P \).

Problem 4. Find a combinatorial description of the class of polytopes \( P^n \) admitting a characteristic map.

We know that the class admitting a characteristic map contains some important combinatorial simple polytopes such as the simplex, the cube, the permutahedron, polygons, 3-dimensional polyhedrons etc. They all belong to the class of simple polytopes with ‘small’ chromatic number.

Definition 2.1. The coloring into \( k \) colors of a simple polytope \( P^n \) with \( m \) facets \( F_1, \ldots, F_m \) is a map
\[
c : \{F_1, \ldots, F_m\} \to [k]
\]
such that for every \( i \) and \( j, i \neq j \) and \( F_i \cap F_j \) is a codimension-two face of \( P_n \) holds \( c(F_i) \neq c(F_j) \). The least \( k \) for which there exist a coloring of the simple polytope \( P^n \) is called the chromatic number \( \chi(P^n) \).

Obviously, \( \chi(P^n) \geq n \) for any simple polytope \( P^n \). The chromatic number of a 2-dimensional simple polytope is clearly equal to 2 or 3, depending on the parity of the number of its facets. By the famous Four Color Theorem we deduce that the chromatic number of a 3-dimensional polytope is 3 or 4. But, for \( n \geq 4 \) in general it does not hold that \( \chi(P^n) \leq n + 1 \).

Clearly, the class of simple polytopes \( P^n \) whose chromatic number is equal to \( n \) or \( n + 1 \), allows the characteristic map.

Example 2.1. The coloring with \( n \) colors gives rise to a canonical characteristic function \( \lambda \) where \( \lambda(F_i) = e_{c(i)} \), while in the case of colorings with \( n + 1 \) colors for all the facets \( F_i \) such that \( c(F_i) = n + 1 \) we assign \( \lambda(F_i) = -e_1 - \ldots - e_n \), where \( e_1, \ldots, e_n \) are the standard base vectors of \( \mathbb{R}_d^n \). The quasitoric manifold arising from this construction is referred to as the canonical quasitoric manifold of the pair \( (P^n, c) \). That means there exist small covers and quasitoric manifolds over such polytopes.
Theorem 2.1. For any simple polytope $P^n$, the following inequality holds:
\[ \chi(P^n) \leq 2^n - 1. \]  

However, there are also examples of polytopes which do not admit a characteristic maps, see [13] and [22]. Their examples are neighborly simple polytopes with large number of facets.

Example 2.2. Let $P^n$ be a 2-neighborly simple polytope with $m \geq 2^n$. The chromatic number of $P^n$ is equal to the number of its facets $m$. But then the existence of a $G^n_d$ manifold over $P^n$ would contradict the inequality (6).

There are two main classification problems of $G^n_d$-manifolds over a given simple polytope: up to a weakly equivariant diffeomorphism (the equivariant classification) and up to a diffeomorphism (the topological classification). The notion of characteristic matrix plays the central role in the classification of $G^n_d$-manifolds. We assume that the facets of $P^n$ are ordered in such a way that the first $n$ of them share a common vertex. The following technical lemma is useful for the classification problem.

Let $M^{dn}$ be a $G^n_d$-manifold over $P^n$ with characteristic map $l_d$.

Lemma 2.1. There exist a weakly equivariant diffeomorphism $f : M^{dn} \to M^{dn}$ induced by an automorphism $\psi$ of $G^n_d$ such that the characteristic matrix induced by $f$ has the form $(I_n \times n|\ast)$ where $\ast$ denotes some $n \times (m - n)$ matrix.

For a given polytope $P^n$, let us denote the set of all weakly equivariant homeomorphism classes of $G^n_d$-manifolds over $P$ by $\mathbb{R}_d M_P$ and by $\mathbb{R}_d M_P^{homeo}$ the set of all homeomorphism classes of $G^n_d$-manifolds over $P$. Define $\mathbb{R}_d M_P$ the set of all $\mathbb{R}_d$ characteristic matrices over $P$. The map $\Lambda_{\mathbb{R}_d} \to M^{dn}(\Lambda_{\mathbb{R}_d})$ is a surjection of $\mathbb{R}_d M_P$ onto $\mathbb{R}_d M_P$, [14] Proposition 1.8].

Let us denote by $\text{Aut}(P)$ the group of all automorphisms of the face poset of $P$, that are bijections from the set of the facets of $P$ to itself which preserve the structure of all faces of $P$. Group $GL(n, \mathbb{R}_d)$ acts on $\mathbb{R}_d M_P$ by left multiplication. In the case $d = 2$, the group $\mathbb{Z}_2^m$ acts on $\mathbb{R}_d M_P$ by multiplication with $-1$ in each column. Also, the group $\text{Aut}(P)$ acts on $\mathbb{R}_d M_P$ by permuting columns. Let

\[ \mathbb{R}_d \mathcal{X}_P = \begin{cases} 
GL(n, \mathbb{Z}_2) \backslash \mathbb{Z}_2 M_P, & \text{if } d = 1 \\
GL(n, \mathbb{Z}) \backslash \mathbb{Z}_2 M_P / \mathbb{Z}_2^m, & \text{if } d = 2
\end{cases} \]

The action of $\text{Aut}(P)$ on $\mathbb{R}_d M_P$ descends to the action of $\text{Aut}(P)$ on $\mathbb{R}_d \mathcal{X}_P$, see [22] Proposition 2.12]. Let us denote with $[\Lambda_{\mathbb{R}_d}]$ the orbit of $\Lambda_{\mathbb{R}_d}$ in $\mathbb{R}_d \mathcal{X}_P \backslash \text{Aut}(P)$.

Classical result in the classification of $G^n_d$-manifolds up to weakly equivariant homeomorphism is the following theorem

Theorem 2.1. For any simple polytope $P^n$, the map $[\Lambda_{\mathbb{R}_d}] \mapsto M^{dn}(\Lambda_{\mathbb{R}_d})$ is a bijection between $\mathbb{R}_d \mathcal{X}_P \backslash \text{Aut}(P)$ and $\mathbb{R}_d M_P$.

For a rigorous proof, we address reader to [22].

The following results are known about classification of quasitoric manifolds and small covers over a given simple polytope $P^n$. 

11
Proposition 2.1.  
• Any small cover over \( \Delta^n \) is weakly equivariant (and topologically) diffeomorphic to \( \mathbb{R}P^n \).

• Any quasitoric manifold over \( \Delta^n \) is weakly equivariant (and topologically) diffeomorphic to \( \mathbb{C}P^n \).

Classification of small covers and quasitoric manifolds over polygons with \( m \geq 4 \) sides is obtained by Orlik and Raymond [28].

Theorem 2.2.

• A small cover over a convex polygon is homeomorphic to the connected sums of \( S^1 \times S^1 \) and \( \mathbb{R}P^2 \).

• A quasitoric manifold over a convex polygon is homeomorphic to the connected sums of \( S^2 \times S^2 \), \( \mathbb{C}P^2 \) and \( \mathbb{C}P^2 \).

Topological classification of quasitoric manifolds over the product of two simplices \( \Delta^n \times \Delta^m \) is obtained in [13]. General case of the product of \( m \) simplices is studied in several papers [10], [23] and [15] and its relation to generalized Bott manifolds is explained.

Garrison and Scott found 25 small covers up to homeomorphism over dodecahedron [20] using computer search.

Sho Hasui studied the question for the case of dual cyclic polytopes \( C^n(m)^* \). He obtained the following theorem.

Theorem 2.3.

• If \( n \geq 4 \) and \( m \geq n+4 \), or \( n \geq 6 \) and \( m \geq n+3 \), there exist no \( G^n_d \) manifolds over \( C^n(m)^* \).

• There exists 1 small cover and 4 different quasitoric manifolds over \( C^4(7) \).

• There exists 1 small cover and 46 different quasitoric manifolds over \( C^5(8) \).

Our research is motivated by recent progress in understanding of combinatorics of neighborly polytopes in low dimensions [16]. In contrast with the results of Hasui which states that \( G^n_d \) structures are in general rare over the duals of cyclic polytopes, this is not true for the dual of neighborly polytopes in general, at least in dimensions 4, 5 and 6.

3 Algorithm for \( G^n_d \)-structures over simple polytopes

Significant progress in understanding the properties of polytopes in higher dimensions is achieved using computers. Classification problem for \( G^n_d \)-manifolds over given simple polytope \( P \) for \( d = 1 \) is reduced to checking whether the non-singularity condition is fulfilled for each vertex of \( P \) for every \( m \times n \) matrix with entries 0 and 1. The possible set of solutions can be reduced using the results of Section 2, but for general polytopes \( P \) that is not of significantly helpful since the number of matrices we need to check is still too big for computers. However, this approach in toric topology contributed to significant progress in our understanding of \( G^n_d \)-manifolds, see [20] and [22].

Following these ideas, we already modified known algorithms to be more convenient. At first, the non-singularity condition depends on the face poset of \( P \) and the program is set to find the facets of a polytope which is given by the coordinates of vertices of its dual polytope. After that, since by Lemma 2.1 we assume that the first \( n \) rows of the characteristic matrix form the identity
Algorithm 3.1 Pseudocode describing the search method

**input:**
- \( n \): number of columns
- \( m \): number of rows
- \( P_d \): vertex coordinates of dual polytope

**output:**
- \( B \): list of binary matrices satisfying nonsingularity condition

```plaintext
1: function MatrixSearch\((n, m, \text{posets})\)
2: \hspace{1cm} P ← \text{ConvexHull}(P_d) \quad \triangleright \text{calculate convex hull of coordinates}
3: \hspace{1cm} \lambda_1 ← (1, 0, 0, \ldots, 0, 0, 0) \quad \triangleright \text{the first row of the identity matrix } I_n
4: \hspace{1cm} \lambda_2 ← (0, 1, 0, \ldots, 0, 0, 0)
5: \hspace{1cm} : 
6: \hspace{1cm} \lambda_n ← (0, 0, 0, \ldots, 0, 0, 1)
7: \hspace{1cm} B ← [] \quad \triangleright \text{initialize } B \text{ to an empty list}
8: \hspace{1cm} \text{for all possible binary states of } \lambda_{m-n+1} \text{ do}
9: \hspace{1cm} \quad \text{for all possible binary states of } \lambda_{m-n+2} \text{ do}
10: \hspace{1cm} \quad : 
11: \hspace{1cm} \quad \text{for all possible binary states of } \lambda_m \text{ do}
12: \hspace{1cm} \quad \quad nonsingular ← TRUE
13: \hspace{1cm} \quad \quad \text{for all } p \text{ in } P \text{ do}
14: \hspace{1cm} \quad \quad \quad \text{if } \text{Determinant}(\lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_n}) = 0 \text{ then}
15: \hspace{1cm} \quad \quad \quad \quad nonsingular ← FALSE
16: \hspace{1cm} \quad \quad \quad \text{break}
17: \hspace{1cm} \quad \quad \text{if } nonsingular \text{ then}
18: \hspace{1cm} \quad \quad \quad \text{INSERT}(B, \lambda)
19: \hspace{1cm} \text{return } B
```
matrix $I^n$, we tested all possible combinations for the remaining $m - n$ rows and checked the non-singularity condition for each vertex of $P$. The pseudocode for this algorithm is given in Algorithm 3.

Algorithm takes coordinates of the dual polytope given at [18] and calculates the convex hull of its vertices (line 2). The resulting convex hull is stored as an array of points $P$, where each point is represented as an array of indices $P_i = (p_1, p_2, \ldots, p_n)$, meaning that $i$-th point of polytope is $P_i = \bigcap_{k=1}^n F_{p_k}$. For more detailed explanation, see section 4.1. After calculating the convex hull, binary matrix $\lambda$ is created such that its first $n$ rows define an identity matrix $I_n$, and the other rows are initially set to zeroes. For each possible binary state of $\lambda_{n+1}$ through $\lambda_m$ we calculate the singularity condition (lines 13-16), and if the current state of $\lambda$ satisfies the condition, it is added to the list of solutions $B$ (line 18). The complete code for this procedure can be found at [32].

The program is successfully used for checking the existence of small covers over the duals of simplicial neighborly polytopes in dimensions 4, 5, 6 and 7 and ‘small number’ of vertices. The starting point for our search were the results of Moritz Firsching obtained in his thesis [17] where he enumerated the combinatorial types of simplicial polytopes in low dimensions and ‘small’ number of vertices. From [17, Table 1.1] we see that this number grows fast as the number of vertices increases. The complete classification of neighborly polytopes is done in few cases, see [1], [2], [3], [6], [19] and [27].

We discussed the significance of neighborly polytopes in Subsection 1.2, which together with the recent results of Hasui [22] motivated us to look for small covers over the polytopes studied in [16] and [17]. Contrary to our conjecture based on current examples and especially on complete classification over dual cyclic polytopes, we obtained explicit examples which show that $G^n_d$-structures can exist on dual neighborly polytopes even if there is no small cover over dual cyclic polytope with the same number of vertices. Some of the obtained small covers seemed as good candidates for a counterexample to the lifting conjecture, but the lifting hypothesis is verified to be true for all considered polytopes. In the next sections we present those interesting results.

## 4 Neighborly 4-polytopes

The simplex is the only neighborly polytope with 5 vertices and the cyclic polytope $C^4(6)$ is the unique neighborly polytope on 6 vertices, so small covers over these polytopes are $\mathbb{R}P^4$, and $\mathbb{R}P^2 \times \mathbb{R}P^2$ and a small cover which is the total space of the projective bundle of sum of three line bundles where two of them are trivial and the other is Hopf line bundle, respectively see [24]. Similarly, $\mathbb{C}P^4$ is the unique quasitoric manifold over $\Delta^4$ and a family of some Bott manifolds are quasitoric manifolds over $C^4(6)$, [10]. Also, $C^4(7)$ is the only neighborly 4-polytope with 7 vertices and by Hasui’s result, Theorem 2.3 there is only one small cover and 4 different quasitoric manifolds over the polytope.

### 4.1 Neighborly 4-polytopes with 8 facets

There are 3 combinatorially distinct neighborly 4-polytopes with 8 vertices. One of them is $C^4(8)$ and it is already known that it is not the orbit space of a small cover. But, the other 2 polytopes allow the characteristic maps. Let us denote by $P^4_0(8)$ and $P^4_1(8)$ the duals of neighborly 4-polytopes with 8 vertices:
Note that the vertices of $P_0^4(8)$ are

\[
P_0^4(8) = \left\{ P_{0} \cap F_{1} \cap F_{2} \cap F_{3}, P_{0} \cap F_{1} \cap F_{2} \cap F_{7}, P_{0} \cap F_{1} \cap F_{3} \cap F_{4}, P_{0} \cap F_{1} \cap F_{4} \cap F_{5}, P_{0} \cap F_{2} \cap F_{3} \cap F_{4}, P_{0} \cap F_{2} \cap F_{4} \cap F_{5}, P_{0} \cap F_{2} \cap F_{5} \cap F_{6}, F_{0} \cap F_{1} \cap F_{2} \cap F_{7}, F_{0} \cap F_{1} \cap F_{3} \cap F_{4}, F_{0} \cap F_{1} \cap F_{4} \cap F_{5}, F_{0} \cap F_{2} \cap F_{3} \cap F_{4}, F_{0} \cap F_{2} \cap F_{4} \cap F_{5}, F_{0} \cap F_{2} \cap F_{5} \cap F_{6}, F_{0} \cap F_{3} \cap F_{4} \cap F_{5}, F_{0} \cap F_{3} \cap F_{5} \cap F_{6}, F_{0} \cap F_{4} \cap F_{5} \cap F_{6}, F_{0} \cap F_{4} \cap F_{5} \cap F_{7}, F_{0} \cap F_{4} \cap F_{6} \cap F_{7}, F_{0} \cap F_{5} \cap F_{6} \cap F_{7}, F_{0} \cap F_{5} \cap F_{6} \cap F_{7}, F_{1} \cap F_{2} \cap F_{3} \cap F_{4}, F_{1} \cap F_{2} \cap F_{3} \cap F_{7}, F_{1} \cap F_{2} \cap F_{4} \cap F_{5}, F_{1} \cap F_{2} \cap F_{6} \cap F_{7}, F_{1} \cap F_{3} \cap F_{4} \cap F_{5}, F_{1} \cap F_{3} \cap F_{5} \cap F_{6}, F_{1} \cap F_{4} \cap F_{5} \cap F_{6}, F_{1} \cap F_{4} \cap F_{5} \cap F_{7}, F_{1} \cap F_{5} \cap F_{6} \cap F_{7}, F_{2} \cap F_{3} \cap F_{4} \cap F_{5}, F_{2} \cap F_{3} \cap F_{5} \cap F_{6}, F_{2} \cap F_{4} \cap F_{5} \cap F_{6}, F_{2} \cap F_{4} \cap F_{5} \cap F_{7}, F_{2} \cap F_{5} \cap F_{6} \cap F_{7}, F_{3} \cap F_{4} \cap F_{5} \cap F_{6}, F_{3} \cap F_{4} \cap F_{5} \cap F_{7}, F_{3} \cap F_{5} \cap F_{6} \cap F_{7} \right\}.
\]
and the vertices of \( P_1^4(8) \) are

\[
\begin{align*}
F_0 \cap F_1 \cap F_2 \cap F_3, & \quad F_0 \cap F_1 \cap F_2 \cap F_4, & \quad F_0 \cap F_1 \cap F_3 \cap F_7, & \quad F_0 \cap F_1 \cap F_4 \cap F_5, \\
F_0 \cap F_1 \cap F_5 \cap F_6, & \quad F_0 \cap F_1 \cap F_6 \cap F_7, & \quad F_0 \cap F_2 \cap F_3 \cap F_4, & \quad F_0 \cap F_3 \cap F_4 \cap F_5, \\
F_0 \cap F_3 \cap F_5 \cap F_6, & \quad F_0 \cap F_3 \cap F_6 \cap F_7, & \quad F_1 \cap F_2 \cap F_3 \cap F_7, & \quad F_1 \cap F_2 \cap F_4 \cap F_5, \\
F_1 \cap F_2 \cap F_5 \cap F_7, & \quad F_1 \cap F_5 \cap F_6 \cap F_7, & \quad F_2 \cap F_3 \cap F_4 \cap F_6, & \quad F_2 \cap F_3 \cap F_6 \cap F_7, \\
F_2 \cap F_4 \cap F_5 \cap F_7, & \quad F_2 \cap F_4 \cap F_6 \cap F_7, & \quad F_3 \cap F_4 \cap F_5 \cap F_6, & \quad F_4 \cap F_5 \cap F_6 \cap F_7. 
\end{align*}
\]

The following propositions are obtained by computer search, but it is straightforward to prove them using the method from [22]. The complete output of computer search can be found at [33].

**Proposition 4.1.** \( \mathbb{R}X_{P_0^4(8)} \) has exactly 7 elements, and they are represented by the matrices

\[
\begin{align*}
a_1 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \\
a_2 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \\
a_3 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \\
a_4 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \\
a_5 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \\
a_6 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \\
a_7 \left[ P_0^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.
\end{align*}
\]

**Proposition 4.2.** \( \mathbb{R}X_{P_1^4(8)} \) has exactly 3 elements and they are represented by the matrices

\[
\begin{align*}
a_1 \left[ P_1^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \\
a_2 \left[ P_1^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \\
a_3 \left[ P_1^4(8) \right] & = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.
\end{align*}
\]

An immediate observation is that the real characteristic matrices from Propositions 4.1 and 4.2 considered with \( \mathbb{Z} \)-coefficients are the characteristic matrices of quasitoric manifolds over \( P_0^4(8) \) and \( P_1^4(8) \).
**Corollary 4.1.** Neighborly 4-polytopes $P^4_0(8)$ and $P^4_1(8)$ are the orbit spaces of some quasitoric manifolds.

**Corollary 4.2.** The lifting conjecture holds for all duals of neighborly 4-polytopes with 8 vertices.

Now we classify the small covers over $P^4_0(8)$ and $P^4_1(8)$.

**Theorem 4.1.** There are exactly 3 different small covers up to weakly equivariant homeomorphism $M^4(a_1 [P^4_0(8)])$, $M^4(a_2 [P^4_0(8)])$ and $M^4(a_7 [P^4_0(8)])$ over $P^4_0(8)$ and exactly 3 different small covers up to weakly equivariant homeomorphism $M^4(a_1 [P^4_1(8)])$, $M^4(a_2 [P^4_1(8)])$ and $M^4(a_3 [P^4_1(8)])$ over $P^4_1(8)$.

**Proof:** From the face poset of $P^4_0$ we identify 2-faces of $P^4_0(8)$. $F_0 ∩ F_1$, $F_9 ∩ F_2$, $F_2 ∩ F_5$, $F_2 ∩ F_7$, $F_3 ∩ F_4$ and $F_5 ∩ F_6$ are hexagons, $F_1 ∩ F_3$, $F_1 ∩ F_6$, $F_3 ∩ F_7$ and $F_6 ∩ F_7$ are pentagons, $F_0 ∩ F_4$, $F_0 ∩ F_6$, $F_1 ∩ F_4$, $F_1 ∩ F_7$, $F_2 ∩ F_3$, $F_2 ∩ F_4$, $F_2 ∩ F_5$, $F_3 ∩ F_6$ and $F_5 ∩ F_7$ are quadrilaterals and $F_0 ∩ F_3$, $F_0 ∩ F_7$, $F_1 ∩ F_5$, $F_2 ∩ F_6$, $F_3 ∩ F_5$, $F_4 ∩ F_5$, $F_4 ∩ F_6$ and $F_4 ∩ F_7$ are triangles. Thus we observe that if Aut($P^4_0(8)$) is nontrivial, the elements of this group act on the facets fixing the sets $\{F_0, F_2, F_4, F_5\}$ and $\{F_1, F_3, F_6, F_7\}$. By considering all eventual images of $F_0$ by the action of Aut($P^4_0(8)$), we straightforwardly check that only the 4 following permutations

$$
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 7 & 0 & 6 & 5 & 4 & 3 & 1 \\
2 & 7 & 0 & 6 & 5 & 4 & 3 & 1
\end{bmatrix}
$$

belong to Aut($P^4_0(8)$). Thus, Aut($P^4_0(8)$) is generated by $\tau = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 4 & 1 & 0 & 2 & 7 & 3 \end{bmatrix}$.

By direct calculation we examine the action of Aut($P^4_0(8)$) on $\mathbb{R}X_{P^4_0(8)}$,

$$
\begin{align*}
\tau(a_1 [P^4_0(8)]) &= a_5 [P^4_0(8)], \quad \tau(a_2 [P^4_0(8)]) = a_6 [P^4_0(8)], \quad \tau(a_3 [P^4_0(8)]) = a_4 [P^4_0(8)], \quad \tau(a_4 [P^4_0(8)]) = a_2 [P^4_0(8)], \quad \tau(a_5 [P^4_0(8)]) = a_1 [P^4_0(8)], \quad \tau(a_6 [P^4_0(8)]) = a_3 [P^4_0(8)]
\end{align*}
$$

and the claim for $P^4_0(8)$ follows from Theorem 2.1.

Similarly, we obtain that Aut($P^4_0(8)$) is trivial and the claim is therefore proved.

\[ \square \]

### 4.2 Neighborly 4-polytopes with 9 facets

Now we proceed to the duals of neighborly 4-polytopes on 9 vertices. According to [16] and [17] there are 23 different such polytopes. Instead of listing all coordinates of their vertices, we fix the following notation

$$
P^m_i(m)
$$

where $m$ is the dimension of the polytope, $m$ is the number of its vertices and $i$ is the index of the polytope as in [18]. The complete output of computer search together with their face posets can be found at [33].

Using explicit computer search we obtain the following propositions.
Proposition 4.3. The polytopes $P_{40}^4(9)$, $P_{41}^4(9)$, $P_{42}^4(9)$, $P_{43}^4(9)$, $P_{44}^4(9)$, $P_{45}^4(9)$, $P_{46}^4(9)$, $P_{47}^4(9)$, $P_{48}^4(9)$, $P_{49}^4(9)$, $P_{10}^4(9)$, $P_{11}^4(9)$, $P_{12}^4(9)$, $P_{13}^4(9)$, $P_{15}^4(9)$, $P_{16}^4(9)$, $P_{18}^4(9)$, $P_{19}^4(9)$ and $P_{20}^4(9)$ do not admit real characteristic maps and thus are not the orbit spaces of a small cover or a quasitoric manifold.

Proposition 4.4. $\mathbb{R}X_{P_{14}^4}^4(9)$ has exactly 1 element, and it is represented by the matrix

$$a_1 \begin{bmatrix} P_{14}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition 4.5. $\mathbb{R}X_{P_{16}^4}^4(9)$ has exactly 1 element and it is represented by the matrix

$$a_1 \begin{bmatrix} P_{16}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Proposition 4.6. $\mathbb{R}X_{P_{21}^4}^4(9)$ has exactly 15 elements and they are represented by the matrices

$$a_1 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, a_2 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$a_3 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, a_4 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$a_5 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, a_6 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$a_7 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, a_8 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$a_9 \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, a_{10} \begin{bmatrix} P_{21}^4(9) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
Proposition 4.7. \( \mathbb{R} \mathcal{X}_{P_{21}^4(9)} \) has exactly 1 element and it is represented by the matrix

\[
a_{15} \left[ P_{21}^4(9) \right] = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Propositions 4.4, 4.5 and 4.7 directly imply the following results:

**Corollary 4.3.** There is only one small cover \( M^4(a_1 \left[ P_{14}^4(9) \right]) \) over \( P_{14}^4(9) \).

**Corollary 4.4.** There is only one small cover \( M^4(a_1 \left[ P_{16}^4(9) \right]) \) over \( P_{16}^4(9) \).

**Corollary 4.5.** There is only one small cover \( M^4(a_1 \left[ P_{22}^4(9) \right]) \) over \( P_{22}^4(9) \).

**Corollary 4.6.** The polytopes \( P_{14}^4(9), P_{16}^4(9) \) and \( P_{22}^4(9) \) are weakly cohomologically rigid.

We complete full classification of small covers over the duals of neighborly polytopes with 9 vertices.

**Theorem 4.2.** There are exactly 4 small covers up to weakly equivariant diffeomorphism \( M^4(a_1 \left[ P_{21}^4(9) \right]), M^4(a_2 \left[ P_{21}^4(9) \right]), M^4(a_5 \left[ P_{21}^4(9) \right]) \) and \( M^4(a_{10} \left[ P_{21}^4(9) \right]) \) over \( P_{21}^4(9) \).

\[ \begin{align*}
a_{11} \left[ P_{21}^4(9) \right] = & \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}, \\
& \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
& \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}.
\end{align*} \]

**Proof:** The proof is similar to the proof of Theorem 4.1. The key steps are to observe that \( \text{Aut}(P_{21}^4(9)) \) is generated by two generators

\[
\tau = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 6 & 5 & 4 & 8 & 7 & 0 & 2 & 3
\end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 4 & 5 & 2 & 3 & 6 & 8 & 7
\end{pmatrix},
\]

and that the action of \( \text{Aut}(P_{21}^4(9)) \) on \( \mathbb{Z}_2 \mathcal{X}_{P_{21}^4(9)} \) is given by the following diagram:
An intriguing question not solved by Theorem 4.2 is topological classification of small covers over $P_{21}^4(9)$.

**Question 4.1.** Is $P_{21}^4(9)$ a weakly $\mathbb{Z}_2$-cohomologically rigid polytope?

Again, the real characteristic matrices from Propositions 4.4, 4.5, 4.6 and 4.7 considered with $\mathbb{Z}$-coefficients are the characteristic matrices of quasitoric manifolds over $P_{14}^4(9), P_{16}^4(9), P_{21}^4(9)$ and $P_{22}^4(9)$.

**Corollary 4.7.** Neighborly 4-polytopes $P_{14}^4(9), P_{16}^4(9), P_{21}^4(9)$ and $P_{22}^4(9)$ are the orbit spaces of some quasitoric manifolds.

**Corollary 4.8.** The lifting conjecture holds for all duals of neighborly 4-polytopes with 9 vertices.

### 4.3 Neighborly 4-polytopes with 10 facets

There are 431 different simply neighborly polytopes with 10 facets, listed on [18]. By computer search we obtain

**Theorem 4.3.** The polytopes $P_{50}^4(10), P_{57}^4(10), P_{57}(58), P_{74}^4(10), P_{75}^4(10), P_{104}^4(10), P_{147}^4(10), P_{152}^4(10), P_{192}^4(10), P_{221}^4(10), P_{223}^4(10), P_{233}^4(10), P_{270}^4(10), P_{273}^4(10), P_{278}^4(10), P_{28}^4(10), P_{290}^4(10), P_{304}^4(10), P_{305}^4(10), P_{319}^4(10), P_{325}^4(10), P_{340}^4(10), P_{345}^4(10), P_{349}^4(10), P_{350}^4(10), P_{356}^4(10), P_{360}^4(10), P_{374}^4(10), P_{381}^4(10), P_{384}^4(10), P_{395}^4(10), P_{397}^4(10), P_{399}^4(10), P_{401}^4(10), P_{404}^4(10), P_{405}^4(10), P_{415}^4(10), P_{426}^4(10), P_{429}^4(10)$ and $P_{430}^4(10)$ allow a characteristic map while the other simply neighborly polytopes with 10 facets are not the orbit spaces of small covers.

Now we proceed to the classification of small covers over simply neighborly polytopes with 10 facets. For complete list of matrices, see [33].

**Proposition 4.8.** $\mathbb{R}X_{P_{50}^4(10)}$ has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{50}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{vmatrix}.
\]

**Proposition 4.9.** $\mathbb{R}X_{P_{57}^4(10)}$ has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{57}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{vmatrix}.
\]
Proposition 4.10. $\mathcal{X}_{P_{58}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{58}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \ \end{vmatrix}.$$

Proposition 4.11. $\mathcal{X}_{P_{74}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{74}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \ \end{vmatrix}.$$

Proposition 4.12. $\mathcal{X}_{P_{75}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{75}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \ \end{vmatrix}.$$

Proposition 4.13. $\mathcal{X}_{P_{104}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{104}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \ \end{vmatrix}.$$

Proposition 4.14. $\mathcal{X}_{P_{147}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{147}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \ \end{vmatrix}.$$

Proposition 4.15. $\mathcal{X}_{P_{152}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{152}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \ \end{vmatrix}.$$

Proposition 4.16. $\mathcal{X}_{P_{171}^4(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{171}^4(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \ \end{vmatrix}.$$

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Proposition 4.17. $\mathcal{X}_{P_{192}^{4}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{192}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{vmatrix}.$$

Proposition 4.18. $\mathcal{X}_{P_{221}^{4}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{221}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{vmatrix}.$$

Proposition 4.19. $\mathcal{X}_{P_{223}^{4}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{223}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{vmatrix}.$$

Proposition 4.20. $\mathcal{X}_{P_{233}^{4}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{233}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{vmatrix}.$$

Proposition 4.21. $\mathcal{X}_{P_{270}^{4}(10)}$ has exactly 2 elements and they are represented by the matrices

$$a_1 \left[ P_{270}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{vmatrix} \text{ and } a_2 \left[ P_{270}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{vmatrix}.$$

Proposition 4.22. $\mathcal{X}_{P_{273}^{4}(10)}$ has exactly 2 elements and they are represented by the matrices

$$a_1 \left[ P_{273}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{vmatrix} \text{ and } a_2 \left[ P_{273}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{vmatrix}.$$

Proposition 4.23. $\mathcal{X}_{P_{278}^{4}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P_{278}^{4}(10) \right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{vmatrix}.$$

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Proposition 4.24. $\mathcal{R}X_{P^4_{288}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P^4_{288}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$ 

Proposition 4.25. $\mathcal{R}X_{P^4_{290}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P^4_{290}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$ 

Proposition 4.26. $\mathcal{R}X_{P^4_{304}(10)}$ has exactly 2 elements and they are represented by the matrices

$$a_1 \left[ P^4_{304}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad a_2 \left[ P^4_{304}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$ 

Proposition 4.27. $\mathcal{R}X_{P^4_{305}(10)}$ has exactly 2 elements and they are represented by the matrices

$$a_1 \left[ P^4_{305}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a_2 \left[ P^4_{305}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$ 

Proposition 4.28. $\mathcal{R}X_{P^4_{319}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P^4_{319}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$ 

Proposition 4.29. $\mathcal{R}X_{P^4_{325}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P^4_{325}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$ 

Proposition 4.30. $\mathcal{R}X_{P^4_{340}(10)}$ has exactly 1 element and it is represented by the matrix

$$a_1 \left[ P^4_{340}(10) \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$
Proposition 4.31. \( \mathcal{E}_{P_{345}(10)} \) has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{345}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.32. \( \mathcal{E}_{P_{349}(10)} \) has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{349}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.33. \( \mathcal{E}_{P_{350}(10)} \) has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{350}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.34. \( \mathcal{E}_{P_{356}(10)} \) has exactly 2 elements and they are represented by the matrices

\[
a_1 \left[ P_{356}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\quad \text{and} \quad
a_2 \left[ P_{356}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.35. \( \mathcal{E}_{P_{360}(10)} \) has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{360}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.36. \( \mathcal{E}_{P_{374}(10)} \) has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{374}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.37. \( \mathcal{E}_{P_{381}(10)} \) has exactly 1 element and it is represented by the matrix

\[
a_1 \left[ P_{381}(10) \right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}.
\]
Proposition 4.38. \( \mathbb{R} \mathcal{X}_{P_{384}^4(10)} \) has exactly 1 element and it is represented by the matrix
\[
a_1 [P_{384}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.39. \( \mathbb{R} \mathcal{X}_{P_{395}^4(10)} \) has exactly 1 element and it is represented by the matrix
\[
a_1 [P_{395}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.40. \( \mathbb{R} \mathcal{X}_{P_{397}^4(10)} \) has exactly 2 elements and they are represented by the matrices
\[
a_1 [P_{397}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{vmatrix}, \quad a_2 [P_{397}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.41. \( \mathbb{R} \mathcal{X}_{P_{399}^4(10)} \) has exactly 2 elements and they are represented by the matrices
\[
a_1 [P_{399}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1
\end{vmatrix}, \quad a_2 [P_{399}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.42. \( \mathbb{R} \mathcal{X}_{P_{401}^4(10)} \) has exactly 1 element and it is represented by the matrix
\[
a_1 [P_{401}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.43. \( \mathbb{R} \mathcal{X}_{P_{404}^4(10)} \) has exactly 2 elements and they are represented by the matrices
\[
a_1 [P_{404}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{vmatrix}, \quad a_2 [P_{404}^4(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{vmatrix}.
\]
Proposition 4.44. $\mathcal{X}_{P_{405}^{(10)}}$ has exactly 1 element and it is represented by the matrix

$$a_1 [P_{405}^4(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.$$

Proposition 4.45. $\mathcal{X}_{P_{415}^{(10)}}$ has exactly 2 elements and they are represented by the matrices

$$a_1 [P_{415}^4(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{bmatrix} \text{ and } a_2 [P_{415}^4(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}.$$

Proposition 4.46. $\mathcal{X}_{P_{426}^{(10)}}$ has exactly 1 element and it is represented by the matrix

$$a_1 [P_{426}^4(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.$$

Proposition 4.47. $\mathcal{X}_{P_{429}^{(10)}}$ has exactly 1 element and it is represented by the matrix

$$a_1 [P_{429}^4(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.$$

Proposition 4.48. $\mathcal{X}_{P_{430}^{(10)}}$ has exactly 1 element and it is represented by the matrix

$$a_1 [P_{430}^4(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}.$$

We have to finish the classification of small covers over neighborly 4-polytopes with 10 facets. In order to do that, we have to determine the symmetry groups of the polytopes $P_{270}^4(10)$, $P_{273}^4(10)$, $P_{304}^4(10)$, $P_{305}^4(10)$, $P_{356}^4(10)$, $P_{397}^4(10)$, $P_{404}^4(10)$, $P_{415}^4(10)$, $P_{426}^4(10)$, $P_{429}^4(10)$ and $P_{430}^4(10)$. By direct examination from their posets we found that all of them are trivial, except for the symmetry group of $P_{397}^4(10)$ which is $\mathbb{Z}_2$. Using the method in the proof of Theorem 4.1 we deduce:

Theorem 4.4. There are exactly 3 different small covers $M^4(a_1 [P_{397}^4(10)])$, $M^4(a_2 [P_{397}^4(10)])$ and $M^4(a_3 [P_{397}^4(10)])$ over $P_{397}^4(10)$.

For other neighborly 4-polytopes with 10 facets small covers are classified by the characteristic matrices from the above propositions. This could be summarized in the following theorem

Theorem 4.5. Each of the polytopes $P_{30}^4(10)$, $P_{37}^4(10)$, $P_{57}^4(58)$, $P_{74}^4(10)$, $P_{75}^4(10)$, $P_{104}^4(10)$, $P_{147}^4(10)$, $P_{152}^4(10)$, $P_{171}^4(10)$, $P_{192}^4(10)$, $P_{221}^4(10)$, $P_{223}^4(10)$, $P_{233}^4(10)$, $P_{278}^4(10)$, $P_{288}^4(10)$, $P_{290}^4(10)$, $P_{319}^4(10)$, $P_{325}^4(10)$, $P_{340}^4(10)$, $P_{345}^4(10)$, $P_{349}^4(10)$, $P_{350}^4(10)$, $P_{366}^4(10)$, $P_{374}^4(10)$, $P_{381}^4(10)$, $P_{384}^4(10)$, $P_{395}^4(10)$, $P_{401}^4(10)$, $P_{405}^4(10)$, $P_{426}^4(10)$, $P_{429}^4(10)$ and $P_{430}^4(10)$ are the orbit spaces of exactly 1 small cover.
Each of the polytopes $P^4_{270}(10)$, $P^4_{273}(10)$, $P^4_{304}(10)$, $P^4_{305}(10)$, $P^4_{356}(10)$, $P^4_{399}(10)$, $P^4_{404}(10)$, and $P^4_{415}(10)$ are the orbit spaces of exactly 2 small covers.

Directly from Theorems 4.5 and 4.3 we deduce majority of simple neighborly 4-polytopes with 10 facets are weakly cohomologically $\mathbb{Z}_2$ rigid. In the remaining cases the question is open.

**Question 4.2.** Are the polytopes $P^4_{270}(10)$, $P^4_{273}(10)$, $P^4_{304}(10)$, $P^4_{305}(10)$, $P^4_{356}(10)$, $P^4_{399}(10)$, $P^4_{404}(10)$, $P^4_{415}(10)$ weakly cohomologically $\mathbb{Z}_2$ rigid?

Now we are going to verify the lifting conjecture for the polytopes above.

**Proposition 4.49.** Small cover $M^4(a_1[P^4_{50}(10)])$ from Proposition 4.8 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{50}(10)$ given by the characteristic matrix

$$\tilde{a}_1[P^4_{50}(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{vmatrix}.$$

**Proposition 4.50.** Small cover $M^4(a_1[P^4_{104}(10)])$ from Proposition 4.13 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{104}(10)$ given by the characteristic matrix

$$\tilde{a}_1[P^4_{104}(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.$$

**Proposition 4.51.** Small cover $M^4(a_1[P^4_{152}(10)])$ from Proposition 4.15 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{152}(10)$ given by the characteristic matrix

$$\tilde{a}_1[P^4_{152}(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{vmatrix}.$$

**Proposition 4.52.** Small cover $M^4(a_1[P^4_{233}(10)])$ from Proposition 4.20 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{233}(10)$ given by the characteristic matrix

$$\tilde{a}_1[P^4_{233}(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0
\end{vmatrix}.$$

**Proposition 4.53.** Small cover $M^4(a_1[P^4_{340}(10)])$ from Proposition 4.30 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{340}(10)$ given by the characteristic matrix

$$\tilde{a}_1[P^4_{340}(10)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{vmatrix}.$$

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Proposition 4.54. Small cover $M^4(a_1 [P^4_{397}(10)])$ from Proposition 4.40 and Theorem 4.4 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{397}(10)$ given by the characteristic matrix

$$
\tilde{a}_1 [P^4_{397}(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix},
$$

while $M(\tilde{a}_2 [P^4_{397}(10)])$ and $M(\tilde{a}_3 [P^4_{397}(10)])$ are the fixed points of conjugation subgroup of $T^4$ for quasitoric manifolds coming from their respective $\mathbb{Z}_2$-characteristic matrices assumed that coefficients are in $\mathbb{Z}$.

Proposition 4.55. Small cover $M^4(a_1 [P^4_{404}(10)])$ from Proposition 4.43 is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifold over $P^4_{404}(10)$ given by the characteristic matrix

$$
\tilde{a}_1 [P^4_{404}(10)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix},
$$

while $M(a_2 [P^4_{404}(10)])$ is the fixed point set of conjugation subgroup of $T^4$ for quasitoric manifolds coming from their respective $\mathbb{Z}_2$-characteristic matrices assumed that coefficients are in $\mathbb{Z}$.

The real characteristic matrices for $P^4_{57}(10)$, $P^4_{55}(10)$, $P^4_{74}(10)$, $P^4_{75}(10)$, $P^4_{147}(10)$, $P^4_{171}(10)$, $P^4_{192}(10)$, $P^4_{221}(10)$, $P^4_{223}(10)$, $P^4_{270}(10)$, $P^4_{273}(10)$, $P^4_{278}(10)$, $P^4_{288}(10)$, $P^4_{290}(10)$, $P^4_{304}(10)$, $P^4_{305}(10)$, $P^4_{319}(10)$, $P^4_{325}(10)$, $P^4_{345}(10)$, $P^4_{349}(10)$, $P^4_{350}(10)$, $P^4_{356}(10)$, $P^4_{360}(10)$, $P^4_{374}(10)$, $P^4_{381}(10)$, $P^4_{384}(10)$, $P^4_{395}(10)$, $P^4_{399}(10)$, $P^4_{401}(10)$, $P^4_{405}(10)$, $P^4_{426}(10)$, $P^4_{429}(10)$, $P^4_{430}(10)$ and $P^4_{439}(10)$ from Propositions 4.9, 4.10, 4.11, 4.12, 4.14, 4.16, 4.18, 4.19, 4.21, 4.22, 4.23, 4.24, 4.25, 4.26, 4.27, 4.28, 4.29, 4.31, 4.32, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39, 4.42, 4.44, 4.45, 4.46, 4.47 and 4.48 seen as the characteristic matrices with coefficients in $\mathbb{Z}$ verify the lifting conjecture for these polytopes.

Corollary 4.9. Neighborly 4-polytopes $P^4_{50}(10)$, $P^4_{57}(10)$, $P^4_{58}(10)$, $P^4_{74}(10)$, $P^4_{75}(10)$, $P^4_{104}(10)$, $P^4_{147}(10)$, $P^4_{152}(10)$, $P^4_{171}(10)$, $P^4_{192}(10)$, $P^4_{221}(10)$, $P^4_{223}(10)$, $P^4_{233}(10)$, $P^4_{270}(10)$, $P^4_{273}(10)$, $P^4_{278}(10)$, $P^4_{288}(10)$, $P^4_{290}(10)$, $P^4_{304}(10)$, $P^4_{305}(10)$, $P^4_{319}(10)$, $P^4_{325}(10)$, $P^4_{345}(10)$, $P^4_{349}(10)$, $P^4_{350}(10)$, $P^4_{356}(10)$, $P^4_{360}(10)$, $P^4_{374}(10)$, $P^4_{381}(10)$, $P^4_{384}(10)$, $P^4_{395}(10)$, $P^4_{399}(10)$, $P^4_{401}(10)$, $P^4_{405}(10)$, $P^4_{426}(10)$, $P^4_{429}(10)$, $P^4_{430}(10)$ and $P^4_{439}(10)$ are the orbit spaces of some quasitoric manifolds.

Corollary 4.10. The lifting conjecture holds for all duals of neighborly 4-polytopes with 10 vertices.

4.4 Neighborly 4-polytopes with 11 facets

There are 13935 neighborly 4-polytopes with 11 facets. By extensive computer search we determined the face posets of each of them and found all real characteristic matrices over them and it turned out that only 31 of them are the orbit spaces of a small cover. Using the same approach as in the previous cases we classified all small covers over those polytopes. As in the previous cases we follow the notation of neighborly polytopes from [18].
Proposition 4.56. $\mathcal{R}X_{P_{14}(11)}^4$ has exactly 2 elements and they are represented by the matrices
\[
a_1[P_{14}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]
and
\[
a_2[P_{14}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}.
\]

Proposition 4.57. $\mathcal{R}X_{P_{231}(11)}^4$ has exactly 1 element and it is represented by the matrix
\[
a_1[P_{231}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}.
\]

Proposition 4.58. $\mathcal{R}X_{P_{328}(11)}^4$ has exactly 1 element and it is represented by the matrix
\[
a_1[P_{328}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}.
\]

Proposition 4.59. $\mathcal{R}X_{P_{396}(11)}^4$ has exactly 1 element and it is represented by the matrix
\[
a_1[P_{396}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

Proposition 4.60. $\mathcal{R}X_{P_{491}(11)}^4$ has exactly 1 element and it is represented by the matrix
\[
a_1[P_{491}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}.
\]

Proposition 4.61. $\mathcal{R}X_{P_{623}(11)}^4$ has exactly 1 element and it is represented by the matrix
\[
a_1[P_{623}^4(11)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}.
\]

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Proposition 4.62. \( \mathcal{X}_{P_{1044}^4(11)} \) has exactly 2 elements and they are represented by the matrices
\[
a_1[P_{1044}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]
and
\[
a_2[P_{1044}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]

Proposition 4.63. \( \mathcal{X}_{P_{1369}^4(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{1369}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

Proposition 4.64. \( \mathcal{X}_{P_{1478}^4(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{1478}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

Proposition 4.65. \( \mathcal{X}_{P_{1896}^4(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{1896}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.66. \( \mathcal{X}_{P_{3681}^4(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{3681}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.67. \( \mathcal{X}_{P_{3687}^4(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{3687}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}.
\]

Proposition 4.68. \( \mathcal{X}_{P_{3752}^4(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{3752}^4(11))] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}.
Proposition 4.69. \( R \times_{3760} P \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{3760}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.70. \( R \times_{4897} P \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{4897}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.71. \( R \times_{5013} P \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{5013}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.72. \( R \times_{5431} P \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{5431}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.73. \( R \times_{7266} P \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{7266}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.74. \( R \times_{7304} P \) has exactly 2 elements and they are represented by the matrices
\[
a_1[P_{7304}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{vmatrix} \quad \text{and}
\]
\[
a_2[P_{7304}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.75. \( R \times_{7375} P \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{7375}(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{vmatrix}.
\]
Proposition 4.76. \( R \mathcal{X}_{P_{7503}(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{7503}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.77. \( R \mathcal{X}_{P_{7771}(11)} \) has exactly 2 elements and they are represented by the matrices
\[
a_1[P_{7771}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{vmatrix}

and
\[
a_2[P_{7771}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.78. \( R \mathcal{X}_{P_{8955}(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{8955}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.79. \( R \mathcal{X}_{P_{9121}(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{9121}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.80. \( R \mathcal{X}_{P_{10072}(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{10072}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.81. \( R \mathcal{X}_{P_{10378}(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{10378}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.82. \( R \mathcal{X}_{P_{12021}(11)} \) has exactly 1 element and it is represented by the matrix
\[
a_1[P_{12021}(11))] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{vmatrix}.
\]
Proposition 4.83. \( \mathcal{X}_{P_{12710}}^4(11) \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{12710}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.84. \( \mathcal{X}_{P_{13226}}^4(11) \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{13226}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{vmatrix}.
\]

Proposition 4.85. \( \mathcal{X}_{P_{13351}}^4(11) \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{13351}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.86. \( \mathcal{X}_{P_{13494}}^4(11) \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{13494}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.
\]

Symmetry groups of the polytopes \( P_{14}^4(11), P_{1044}^4(11), P_{7304}^4(11) \) and \( P_{7771}^4(11) \) are trivial so we deduce that the real characteristic matrices over simply neighborly 4-polytopes with 11 facets listed above determine non-diffeomorphic small covers.

Theorem 4.6.

- \( P_{231}^4(11), P_{328}^4(11), P_{396}^4(11), P_{491}^4(11), P_{623}^4(11), P_{1369}^4(11), P_{1478}^4(11), P_{1896}^4(11), P_{3081}^4(11), P_{3087}^4(11), P_{3752}^4(11), P_{3760}^4(11), P_{4897}^4(11), P_{5013}^4(11), P_{5431}^4(11), P_{7266}^4(11), P_{7375}^4(11), P_{7503}^4(11), P_{8955}^4(11), P_{9121}^4(11), P_{10072}^4(11), P_{10378}^4(11), P_{12921}^4(11), P_{12710}^4(11), P_{13226}^4(11), P_{13351}^4(11) \) and \( P_{13494}^4(11) \) are the orbit spaces for 1 small cover.

- \( P_{14}^4(11), P_{1044}^4(11), P_{7304}^4(11) \) and \( P_{7771}^4(11) \) are the orbit spaces for 2 small covers.

- All other simply neighborly 4-polytopes with 11 facets are not the orbit space of a small cover.

Theorem 4.6 implies that all simply neighborly 4-polytopes with 11 facets, except maybe four of them, are cohomologically \( \mathbb{Z}_2 \) rigid.

Question 4.3. Are \( P_{14}^4(11), P_{1044}^4(11), P_{7304}^4(11) \) and \( P_{7771}^4(11) \) cohomologically \( \mathbb{Z}_2 \) rigid?

Finally we verify the Lifting conjecture for small covers over neighborly 4-polytopes with 11 facets. The real characteristic matrices for small covers seen with \( \mathbb{Z} \) coefficients are the characteristic matrices of quasitoric manifolds, except for \( a_1[P_{328}^4(11)], a_1[P_{396}^4(11)], a_1[P_{491}^4(11)], a_1[P_{7771}^4(11)], \) and \( a_1[P_{8955}^4(11)] \).
Proposition 4.87. Small cover \( M^4(a_1 [P_{328}^4(11)]) \) from Proposition 4.58 is the fixed point set of conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{328}^4(11) \) given by the characteristic matrix
\[
\tilde{a}_1 [P_{328}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{vmatrix}.
\]

Proposition 4.88. Small cover \( M^4(a_1 [P_{396}^4(11)]) \) from Proposition 4.59 is the fixed point set of conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{396}^4(11) \) given by the characteristic matrix
\[
\tilde{a}_1 [P_{396}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{vmatrix}.
\]

Proposition 4.89. Small cover \( M^4(a_1 [P_{491}^4(11)]) \) from Proposition 4.60 is the fixed point set of conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{491}^4(11) \) given by the characteristic matrix
\[
\tilde{a}_1 [P_{491}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{vmatrix}.
\]

Proposition 4.90. Small cover \( M^4(a_1 [P_{7771}^4(11)]) \) from Proposition 4.77 is the fixed point set of conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{7771}^4(11) \) given by the characteristic matrix
\[
\tilde{a}_1 [P_{7771}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{vmatrix}.
\]

while \( M(a_2 [P_{7771}^4(11)]) \) is the fixed point of conjugation subgroup of \( T^4 \) for quasitoric manifolds coming from their respective \( \mathbb{Z}_2 \)-characteristic matrices assumed that coefficients are in \( \mathbb{Z} \).

Proposition 4.91. Small cover \( M^4(a_1 [P_{8955}^4(11)]) \) from Proposition 4.78 is the fixed point set of conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{8955}^4(11) \) given by the characteristic matrix
\[
\tilde{a}_1 [P_{8955}^4(11)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{vmatrix}.
\]

Corollary 4.11. The Lifting conjecture holds for small covers over neighborly 4-polytopes with 11 facets.

4.5 Neighborly 4-polytopes with 12 facets

Intuition based on known examples of the orbit spaces of small covers make us feel reserved towards possibility of obtaining examples of polytopes with high chromatic numbers (close to the
upper bound predicted by inequality \(\Box\)). Thus, our complete classification of small covers over neighborly 4-polytopes with 12 facets gave 22 ‘exotic’ examples of small covers. There are 556062 combinatorially different simple neighborly 4-polytopes with 12 facets, but only following polytopes:

- \(P_4^{40558}(12)\), \(P_4^{27589}(12)\), \(P_4^{33229}(12)\), \(P_4^{85576}(12)\), \(P_4^{115259}(12)\), \(P_4^{126807}(12)\), \(P_4^{178178}(12)\), \(P_4^{187125}(12)\), \(P_4^{20848}(12)\), \(P_4^{27589}(12)\), \(P_4^{33229}(12)\), \(P_4^{85576}(12)\), \(P_4^{115259}(12)\), \(P_4^{126807}(12)\), \(P_4^{178178}(12)\), \(P_4^{187125}(12)\), \(P_4^{210848}(12)\), \(P_4^{238110}(12)\), \(P_4^{260526}(12)\), \(P_4^{328152}(12)\), \(P_4^{328999}(12)\), \(P_4^{347872}(12)\), \(P_4^{377800}(12)\), \(P_4^{415765}(12)\), \(P_4^{446898}(12)\), \(P_4^{449639}(12)\), \(P_4^{458015}(12)\), \(P_4^{460700}(12)\) and \(P_4^{496733}(12)\) allow unique (up to \(GL(4, \mathbb{Z}_2)\) action) real characteristic map.

**Theorem 4.7.** All small covers over simply neighborly 4-polytopes with 12 facets are the following 22 small covers given by their real characteristic matrices and their respective orbit polytopes (in the notation from [18]):

- \(a_1[P_4^{24058}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix},

- \(a_1[P_4^{27589}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix},

- \(a_1[P_4^{33229}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix},

- \(a_1[P_4^{85576}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix},

- \(a_1[P_4^{115259}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix},

- \(a_1[P_4^{126807}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix},

- \(a_1[P_4^{178178}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},

- \(a_1[P_4^{187125}(12)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.

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\[
\begin{align*}
\mathbf{a}_1[P_{210848}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}, \\
\mathbf{a}_1[P_{238110}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}, \\
\mathbf{a}_1[P_{260526}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}, \\
\mathbf{a}_1[P_{286350}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}, \\
\mathbf{a}_1[P_{323818}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}, \\
\mathbf{a}_1[P_{323999}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}, \\
\mathbf{a}_1[P_{347872}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}, \\
\mathbf{a}_1[P_{377800}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}, \\
\mathbf{a}_1[P_{415765}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}, \\
\mathbf{a}_1[P_{446898}^4(12)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]
Theorem 4.7 is obtained by substantive computer search.

**Corollary 4.12.** Neighborly 4-polytopes with 12 facets are all weakly cohomologically \( \mathbb{Z}_2 \) rigid.

Finally we verify the Lifting conjecture for small covers over neighborly 4-polytopes with 12 facets. The matrices from Theorem 4.7 are the characteristic matrices seen with \( \mathbb{Z} \) coefficients, except \( a_1[P_{449639}(12)] \) and \( a_1[P_{458015}(12)] \).

**Proposition 4.92.** Small cover \( M^4(a_1[P_{323818}(12)]) \) from Theorem 4.7 is the fixed point set of conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{323818}(12) \) given by the characteristic matrix

\[
\tilde{a}_1 [P_{323818}(12)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{vmatrix}.
\]

**Proposition 4.93.** Small cover \( M^4(a_1[P_{347872}(12)]) \) from Theorem 4.7 is the fixed point of set conjugation subgroup of \( T^4 \) for quasitoric manifold over \( P_{347872}(12) \) given by the characteristic matrix

\[
\tilde{a}_1 [P_{347872}(12)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{vmatrix}.
\]

**Corollary 4.13.** The Lifting conjecture holds for small covers over neighborly 4-polytopes with 12 facets.
5 Neighborly 5-polytopes

Classification of small covers over duals of neighborly 5-polytopes on 6, 7 and 8 vertices is already known because these polytopes are $\Delta^5$, $\Delta^2 \times \Delta^3$ and dual of $C^5(8)$. There are 126 combinatorially different simple neighborly 5-polytopes with 9 facets. By computer search we found that known because these polytopes are

Proposition 5.1. $\mathbb{R} \mathcal{X}_{P_4^5(9)}$ has exactly 1 element and it is represented by the matrix

$$a_1[P_4^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}.$$

Theorem 5.1. There is only one small cover $M^5(a_1[P_4^5(9)])$ over the polytope $P_4^5(9)$.

Proof: It is an immediate consequence of Proposition 5.1.

Proposition 5.2. $\mathbb{R} \mathcal{X}_{P_5^5(9)}$ has exactly 3 elements and they are represented by the matrices

$$a_1[P_5^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}, \quad a_2[P_5^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix},$$

and

$$a_3[P_5^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}.$$

Theorem 5.2. There are exactly three covers $M^5(a_1[P_5^5(9)])$, $M^5(a_2[P_5^5(9)])$ and $M^5(a_3[P_5^5(9)])$ over the polytope $P_5^5(9)$.

Proof: The symmetry group of $P_5^5(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.2.

Proposition 5.3. $\mathbb{R} \mathcal{X}_{P_6^5(9)}$ has exactly 1 element and it is represented by the matrix

$$a_1[P_6^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.$$

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Theorem 5.3. There is only one small cover $M^5(a_1[6^5(9)])$ over the polytope $P^5_6(9)$.

Proof: It is an immediate consequence of Proposition 5.3. □

Proposition 5.4. $\mathcal{X}_{P^5_7(9)}$ has exactly 3 elements and they are represented by the matrices

\[
\begin{align*}
a_1[7^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \
\end{pmatrix},
a_2[7^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \
\end{pmatrix},
a_3[7^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \
\end{pmatrix}.
\end{align*}
\]

Theorem 5.4. There are exactly three small covers $M^5(a_1[7^5(9)])$, $M^5(a_2[7^5(9)])$ and $M^5(a_3[7^5(9)])$ over the polytope $P^5_7(9)$.

Proof: The symmetry group of $P^5_7(9)$ is trivial by direct examination of its poset, so the theorem is an immediate consequence of Proposition 5.4. □

Proposition 5.5. $\mathcal{X}_{P^5_8(9)}$ has exactly 7 elements and they are represented by the matrices

\[
\begin{align*}
a_1[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{pmatrix},
a_2[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix},
a_3[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{pmatrix},
a_4[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\end{pmatrix},
a_5[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{pmatrix},
a_6[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{pmatrix},
a_7[8^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{pmatrix}.
\end{align*}
\]
Theorem 5.5. There are exactly 7 small covers $M^5(a_1[P_8^5(9)])$, $M^5(a_2[P_8^5(9)])$, $M^5(a_3[P_8^5(9)])$, $M^5(a_4[P_8^5(9)])$, $M^5(a_5[P_8^5(9)])$, $M^5(a_6[P_8^5(9)])$ and $M^5(a_7[P_8^5(9)])$ over the polytope $P_8^5(9)$.

Proof: Direct examination of the face poset of $P_8^5(9)$ shows that its symmetry group is trivial so the claim follows from Proposition 5.5.

Proposition 5.6. $\mathbb{R} \cdot \chi_{P_{10}^5(9)}$ has exactly four elements and they are represented by the matrices

\[
a_1[P_{10}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad a_2[P_{10}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad a_3[P_{10}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad \text{and} \quad a_4[P_{10}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}
\]

Theorem 5.6. There are exactly four small covers $M^5(a_1[P_{10}^5(9)])$, $M^5(a_2[P_{10}^5(9)])$, $M^5(a_3[P_{10}^5(9)])$ and $M^5(a_4[P_{10}^5(9)])$ over the polytope $P_{10}^5(9)$.

Proof: The symmetry group of $P_{10}^5(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.6.

Proposition 5.7. $\mathbb{R} \cdot \chi_{P_{11}^5(9)}$ has exactly 6 elements and they are represented by the matrices

\[
a_1[P_{11}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad a_2[P_{11}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad a_3[P_{11}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad a_4[P_{11}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}, \quad a_5[P_{11}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}, \quad \text{and} \quad a_6[P_{11}^5(9)] = \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Theorem 5.7. There are exactly 6 small covers $M^5(a_1[P_{11}^5(9)])$, $M^5(a_2[P_{11}^5(9)])$, $M^5(a_3[P_{11}^5(9)])$, $M^5(a_4[P_{11}^5(9)])$, $M^5(a_5[P_{11}^5(9)])$ and $M^5(a_6[P_{11}^5(9)])$ over the polytope $P_{11}^5(9)$.
Proof: The symmetry group of \( P_{12}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.7. □

Proposition 5.8. \( \mathcal{X}_{P_{12}^5} \) has exactly four elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{12}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}, \\
a_2[P_{12}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}, \\
a_3[P_{12}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \quad \text{and} \\
a_4[P_{12}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}.
\end{align*}
\]

Theorem 5.8. There are exactly four small covers \( M^5(a_1[P_{12}^5(9)]) \), \( M^5(a_2[P_{12}^5(9)]) \), \( M^5(a_3[P_{12}^5(9)]) \) and \( M^5(a_4[P_{12}^5(9)]) \) over the polytope \( P_{12}^5(9) \).

Proof: The symmetry group of \( P_{12}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.6. □

Proposition 5.9. \( \mathcal{X}_{P_{13}^5} \) has exactly 10 elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \\
a_2[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}, \\
a_3[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \quad \text{and} \\
a_4[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \\
a_5[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \quad \text{and} \\
a_6[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \\
a_7[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \quad \text{and} \\
a_8[P_{13}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}.
\end{align*}
\]
Theorem 5.9. There are exactly 10 small covers $M^5(a_1[P_{13}^5(9)])$, $M^5(a_2[P_{13}^5(9)])$, $M^5(a_3[P_{13}^5(9)])$, $M^5(a_4[P_{13}^5(9)])$, $M^5(a_5[P_{13}^5(9)])$, $M^5(a_6[P_{13}^5(9)])$, $M^5(a_7[P_{13}^5(9)])$, $M^5(a_8[P_{13}^5(9)])$, $M^5(a_9[P_{13}^5(9)])$ and $M^5(a_{10}[P_{13}^5(9)])$ over the polytope $P_{13}^5(9)$.

Proof: The symmetry group of $P_{13}^5(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.9.

Proposition 5.10. $\mathcal{X}_{P_{15}^5(9)}$ has exactly 7 elements and they are represented by the matrices

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
.
and \( a_3[P_{19}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix} \).

**Theorem 5.11.** There are exactly three small covers \( M^5(a_1[P_{19}^5(9)]) \), \( M^5(a_2[P_{19}^5(9)]) \) and \( M^5(a_3[P_{19}^5(9)]) \) over the polytope \( P_{19}^5(9) \).

Proof: The symmetry group of \( P_{19}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.11.

**Proposition 5.12.** \( R X_{P_{22}^5(9)} \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{22}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix} .
\]

**Theorem 5.12.** There is only one small cover \( M^5(a_1[P_{22}^5(9)]) \) over the polytope \( P_{22}^5(9) \).

Proof: It is an immediate corollary of Proposition 5.12.

**Proposition 5.13.** \( R X_{P_{24}^5(9)} \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{24}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix} .
\]

**Theorem 5.13.** There is only one small cover \( M^5(a_1[P_{24}^5(9)]) \) over the polytope \( P_{24}^5(9) \).

Proof: It is an immediate consequence of Proposition 5.13.

**Proposition 5.14.** \( R X_{P_{25}^5(9)} \) has exactly two elements and they are represented by the matrices

\[
a_1[P_{25}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix} \quad \text{and} \quad a_2[P_{25}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix} .
\]

**Theorem 5.14.** There are exactly two small covers \( M^5(a_1[P_{25}^5(9)]) \) and \( M^5(a_2[P_{25}^5(9)]) \) over the polytope \( P_{25}^5(9) \).

Proof: The symmetry group of \( P_{25}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.14.
Proposition 5.15. \( R_{P_{26}^5(9)} \) has exactly 1 element and it is represented by the matrix

\[
a_1[P_{26}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}.
\]

Theorem 5.15. There is only one small cover \( M^5(a_1[P_{26}^5(9)]) \) over the polytope \( P_{26}^5(9) \).

Proof: It is an immediate consequence of Proposition 5.15. \(\square\)

Proposition 5.16. \( R_{P_{28}^5(9)} \) has exactly 6 elements and they are represented by the matrices

\[
a_1[P_{28}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \quad a_2[P_{28}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
\]

\[
a_3[P_{28}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \quad a_4[P_{28}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
\]

\[
a_5[P_{28}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \quad a_6[P_{28}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}.
\]

Theorem 5.16. There are exactly 6 small covers \( M^5(a_1[P_{28}^5(9)]), M^5(a_2[P_{28}^5(9)]), M^5(a_3[P_{28}^5(9)]), M^5(a_4[P_{28}^5(9)]), M^5(a_5[P_{28}^5(9)]) \) and \( M^5(a_6[P_{28}^5(9)]) \) over the polytope \( P_{28}^5(9) \).

Proof: The symmetry group of \( P_{28}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.16. \(\square\)

Proposition 5.17. \( R_{P_{29}^5(9)} \) has exactly three elements and they are represented by the matrices

\[
a_1[P_{29}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}, \quad a_2[P_{29}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix},
\]

\[
and \quad a_3[P_{29}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}.
\]
Theorem 5.17. There are exactly three small covers \( M^5(a_1[P_{29}(9)]) \), \( M^5(a_2[P_{29}(9)]) \) and \( M^5(a_3[P_{29}(9)]) \) over the polytope \( P_{29}^5(9) \).

Proof: The symmetry group of \( P_{29}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.17. \( \square \)

Proposition 5.18. \( \mathbb{R}X_{P_{31}}^5(9) \) has exactly two elements and they are represented by the matrices

\[
a_1[P_{31}^5(9)] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}
\text{ and } a_2[P_{31}^5(9)] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}.
\]

Theorem 5.18. There are exactly two small covers \( M^5(a_1[P_{31}^5(9)]) \) and \( M^5(a_2[P_{31}^5(9)]) \) over the polytope \( P_{31}^5(9) \).

Proof: The symmetry group of \( P_{31}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.18. \( \square \)

Proposition 5.19. \( \mathbb{R}X_{P_{32}}^5(9) \) has exactly two elements and they are represented by the matrices

\[
a_1[P_{32}^5(9)] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\text{ and } a_2[P_{32}^5(9)] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}.
\]

Theorem 5.19. There are exactly two small covers \( M^5(a_1[P_{32}^5(9)]) \) and \( M^5(a_2[P_{32}^5(9)]) \) over the polytope \( P_{32}^5(9) \).

Proof: The symmetry group of \( P_{32}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.19. \( \square \)

Proposition 5.20. \( \mathbb{R}X_{P_{34}}^5(9) \) has exactly two elements and it is represented by the matrices

\[
a_1[P_{34}^5(9)] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}
\text{ and } a_2[P_{34}^5(9)] = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}.
\]

Theorem 5.20. There are exactly two small covers \( M^5(a_1[P_{34}^5(9)]) \) and \( M^5(a_2[P_{34}^5(9)]) \) over the polytope \( P_{34}^5(9) \).

Proof: From the face poset of \( P_{34}^5(9) \) the symmetry group of \( P_{34}^5(9) \) is \( \mathbb{Z}_2 \). However, it acts trivially on \( \mathbb{R}X_{P_{34}^5(9)} \) so the theorem follows from Proposition 5.20. \( \square \)
Proposition 5.21. \( \mathcal{RX}_{P_{35}^5(9)} \) has exactly three elements and they are represented by the matrices

\[
a_1[P_{35}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix},
a_2[P_{35}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix},
a_3[P_{35}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}, \quad \text{and} \quad a_4[P_{35}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]

Theorem 5.21. There are exactly two small covers \( M^5(a_1[P_{35}^5(9)]) \) and \( M^5(a_3[P_{35}^5(9)]) \) over the polytope \( P_{35}^5(9) \).

Proof: From the face poset of \( P_{35}^5(9) \) the symmetry group of \( P_{35}^5(9) \) is \( \mathbb{Z}_2 = \langle \sigma | \sigma^2 = 1 \rangle \) and it acts on \( \mathcal{RX}_{P_{35}^5(9)} \) by \( \sigma(a_1[P_{35}^5(9)]) = a_2[P_{35}^5(9)], \sigma(a_2[P_{35}^5(9)]) = a_1[P_{35}^5(9)] \) and \( \sigma(a_3[P_{35}^5(9)]) = a_4[P_{35}^5(9)] \).

□

Proposition 5.22. \( \mathcal{RX}_{P_{36}^5(9)} \) has exactly one element and it is represented by the matrix

\[
a_1[P_{36}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}.
\]

Theorem 5.22. There is only one small cover \( M^5(a_1[P_{36}^5(9)]) \) over the polytope \( P_{36}^5(9) \).

Proof: It is an immediate consequence of Proposition 5.22.

□

Proposition 5.23. \( \mathcal{RX}_{P_{39}^5(9)} \) has exactly four elements and they are represented by the matrices

\[
a_1[P_{39}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
a_2[P_{39}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
a_3[P_{39}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}, \quad \text{and} \quad a_4[P_{39}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Theorem 5.23. There are exactly four small covers \( M^5(a_1[P_{39}^5(9)]) \), \( M^5(a_2[P_{39}^5(9)]) \), \( M^5(a_3[P_{39}^5(9)]) \) and \( M^5(a_4[P_{39}^5(9)]) \) over the polytope \( P_{39}^5(9) \).
Proof: The symmetry group of $P_{39}^5(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.23. \hfill \square

**Proposition 5.24.** $\mathbb{R}X_{P_{30}^5(9)}$ has exactly 1 element and it is represented by the matrix

$$a_1[P_{30}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{vmatrix}.$$ 

**Theorem 5.24.** There is only one small cover $M^5(a_1[P_{30}^5(9)])$ over the polytope $P_{30}^5(9)$.

Proof: It is an immediate consequence of Proposition 5.24. \hfill \square

**Proposition 5.25.** $\mathbb{R}X_{P_{41}^5(9)}$ has exactly two elements and they are represented by the matrices

$$a_1[P_{41}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{vmatrix} \quad \text{and} \quad a_2[P_{41}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{vmatrix}.$$ 

**Theorem 5.25.** There are exactly two small covers $M^5(a_1[P_{41}^5(9)])$ and $M^5(a_2[P_{41}^5(9)])$ over the polytope $P_{41}^5(9)$.

Proof: From the face poset of $P_{41}^5(9)$ the symmetry group of $P_{41}^5(9)$ is $\mathbb{Z}_2$. However, it acts on $\mathbb{R}X_{P_{41}^5(9)}$ so the theorem follows from Proposition 5.25. \hfill \square

**Proposition 5.26.** $\mathbb{R}X_{P_{43}^5(9)}$ has exactly three elements and they are represented by the matrices

$$a_1[P_{43}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{vmatrix} \quad , \quad a_2[P_{43}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{vmatrix} \quad \text{and} \quad a_3[P_{43}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{vmatrix}.$$ 

**Theorem 5.26.** There are exactly three small covers $M^5(a_1[P_{43}^5(9)])$, $M^5(a_2[P_{43}^5(9)])$ and $M^5(a_3[P_{43}^5(9)])$ over the polytope $P_{43}^5(9)$.

Proof: The symmetry group of $P_{43}^5(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.26. \hfill \square
Proposition 5.27. \( \mathbb{R}X_{P_{45}^5(9)} \) has exactly five elements and they are represented by the matrices

\[
\begin{align*}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
\end{align*}
\]

Theorem 5.27. There are exactly five small covers \( M^5(a_1[P_{45}^5(9)]) \), \( M^5(a_2[P_{45}^5(9)]) \), \( M^5(a_3[P_{45}^5(9)]) \), \( M^5(a_4[P_{45}^5(9)]) \) and \( M^5(a_5[P_{45}^5(9)]) \) over the polytope \( P_{45}^5(9) \).

Proof: The symmetry group of \( P_{45}^5(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.27. \( \square \)

Proposition 5.28. \( \mathbb{R}X_{P_{47}^5(9)} \) has exactly 33 elements represented by the matrices

\[
\begin{align*}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}.
\end{align*}
\]
$$a_9[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
$$

$$a_{10}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
$$

$$a_{11}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{12}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
$$

$$a_{13}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{14}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{15}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{16}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{17}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{18}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{19}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{20}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
$$

$$a_{21}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{22}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
$$

$$a_{23}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

$$a_{24}[P^5_{47}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}.$$
Theorem 5.28. There are exactly 7 small covers \( M^5(a_1[p^5_{47}(9)]) \), \( M^5(a_2[p^5_{47}(9)]) \), \( M^5(a_3[p^5_{47}(9)]) \), \( M^5(a_5[p^5_{47}(9)]) \), \( M^5(a_9[p^5_{47}(9)]) \), \( M^5(a_{10}[p^5_{47}(9)]) \), \( M^5(a_{12}[p^5_{47}(9)]) \) and \( M^5(a_{33}[p^5_{47}(9)]) \) over the polytope \( P^5_{47}(9) \).

Proof: We determine the symmetry group of \( P^5_{47}(9) \). Let us denote by \( F_0, \ldots, F_8 \) the facets of \( P^5_{47}(9) \) in such a way that the \( i \)-th column in a real characteristic matrix corresponds to the facet \( F_{i-1} \). From the face poset we easily obtain the number of vertices of 3-polytope \( F_i \cap F_j \). An immediate observation is that every element \( \theta \in \text{Aut}(P^5_{47}(9)) \) sends \( F_i \cap F_j \) to \( \theta(F_i) \cap \theta(F_j) \). By careful examination of the face poset structure of \( P^5_{47}(9) \) we find that \( \text{Aut}(P^5_{47}(9)) \) is isomorphic to the permutation group \( S_3 \) and it is generated by two generators

\[
\tau = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 0 & 7 & 6 & 1 & 2 & 8 & 5 & 3
\end{pmatrix}
\quad \text{and} \quad
\sigma = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
7 & 5 & 4 & 3 & 2 & 1 & 8 & 0 & 6
\end{pmatrix}.
\]
and that the action of \(\text{Aut}(P^5_{47}(9))\) on \(\mathcal{X}_{P^5_{47}(9)}\) is given by the following diagram

![Diagram](image-url)

and the claim directly follows. \(\square\)

**Proposition 5.29.** \(\mathcal{X}_{P^5_{49}(9)}\) has exactly two elements and they are represented by the matrices

\[
\begin{align*}
a_1[P^5_{49}(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\text{and } a_2[P^5_{49}(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

**Theorem 5.29.** There are exactly two small covers \(M^5(a_1[P^5_{49}(9)])\) and \(M^5(a_2[P^5_{49}(9)])\) over the polytope \(P^5_{49}(9)\).

**Proof:** From the face poset of \(P^5_{49}(9)\) the symmetry group of \(P^5_{49}(9)\) is \(Z_2\). However, it acts trivially on \(\mathcal{X}_{P^5_{49}(9)}\) so the theorem follows from Proposition 5.29. \(\square\)
Proposition 5.30. \( \mathbb{R} \mathcal{X}_{P_{50}^5(9)} \) has exactly two elements and they are represented by the matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Theorem 5.30. There is exactly one small cover \( M^5(a_1[P_{50}^5(9)]) \) over the polytope \( P_{50}^5(9) \).

Proof: From the face poset of \( P_{50}^5(9) \) the symmetry group of \( P_{50}^5(9) \) is \( \mathbb{Z}_2 \) and its generator is represented by the permutation \( \sigma = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8) \). It acts on \( \mathbb{R} \mathcal{X}_{P_{50}^5(9)} \) by \( \sigma(a_1[P_{50}^5(9)]) = a_2[P_{50}^5(9)] \) and \( \sigma(a_2[P_{50}^5(9)]) = a_1[P_{50}^5(9)] \) so the theorem follows from Proposition 5.30.

Proposition 5.31. \( \mathbb{R} \mathcal{X}_{P_{51}^5(9)} \) has exactly 1 element and it is represented by the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Theorem 5.31. There is only one small cover \( M^5(a_1[P_{51}^5(9)]) \) over the polytope \( P_{51}^5(9) \).

Proof: It is an immediate consequence of Proposition 5.31.

Proposition 5.32. \( \mathbb{R} \mathcal{X}_{P_{52}^5(9)} \) has exactly two elements and they are represented by the matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Theorem 5.32. There is exactly one small cover \( M^5(a_1[P_{52}^5(9)]) \) over the polytope \( P_{52}^5(9) \).

Proof: From the face poset of \( P_{52}^5(9) \) the symmetry group of \( P_{52}^5(9) \) is \( \mathbb{Z}_2 \) and its generator is represented by the permutation \( \sigma = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8) \). It acts on \( \mathbb{R} \mathcal{X}_{P_{52}^5(9)} \) by \( \sigma(a_1[P_{52}^5(9)]) = a_2[P_{52}^5(9)] \) and \( \sigma(a_2[P_{52}^5(9)]) = a_1[P_{52}^5(9)] \) so the theorem follows from Proposition 5.32.

Proposition 5.33. \( \mathbb{R} \mathcal{X}_{P_{54}^5(9)} \) has exactly 6 elements and they are represented by the matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Theorem 5.33. There are exactly 6 small covers $M_5^{P_5^{54}}(9)$, $M_5^{P_5^{54}}(9)$, $M_5^{P_5^{54}}(9)$, $M_5^{P_5^{54}}(9)$, $M_5^{P_5^{54}}(9)$ and $M_5^{P_5^{54}}(9)$ over the polytope $P_5^{54}(9)$.

Proof: The symmetry group of $P_5^{54}(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.33. □

Proposition 5.34. $\mathcal{X}_{P_5^{55}}(9)$ has exactly three elements and it is represented by the matrices

$$a_1^{P_5^{55}}(9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad a_2^{P_5^{55}}(9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$\quad a_3^{P_5^{55}}(9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$

Theorem 5.34. There are exactly three small covers $M_5^{P_5^{55}}(9)$, $M_5^{P_5^{55}}(9)$ and $M_5^{P_5^{55}}(9)$ over the polytope $P_5^{55}(9)$.

Proof: The symmetry group of $P_5^{55}(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.34. □

Proposition 5.35. $\mathcal{X}_{P_5^{56}}(9)$ has exactly three elements and they are represented by the matrices

$$a_1^{P_5^{56}}(9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad a_2^{P_5^{56}}(9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$\quad a_3^{P_5^{56}}(9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$
**Theorem 5.35.** There are exactly three small covers $M^5(a_1[P_{56}^5(9)])$, $M^5(a_2[P_{56}^5(9)])$ and $M^5(a_3[P_{56}^5(9)])$ over the polytope $P_{56}^5(9)$.

*Proof:* The symmetry group of $P_{56}^5(9)$ is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.35. □

**Proposition 5.36.** $\mathcal{X}_{P_{57}^5(9)}$ has exactly 19 elements represented by the matrices

\[
\begin{align*}
a_1[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_2[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_3[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_4[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_5[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix},
a_6[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_7[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix},
a_8[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{pmatrix},
a_9[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_{10}[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_{11}[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_{12}[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix},
a_{13}[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{pmatrix},
a_{14}[P_{57}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{pmatrix},
\end{align*}
\]
so the theorem follows from Proposition 5.32. □

Proposition 5.37.

Theorem 5.36. There are exactly 12 small covers $M^5(a_1[P^5_{57}(9)])$, $M^5(a_2[P^5_{57}(9)])$, $M^5(a_3[P^5_{57}(9)])$, $M^5(a_4[P^5_{57}(9)])$, $M^5(a_5[P^5_{57}(9)])$, $M^5(a_6[P^5_{57}(9)])$, $M^5(a_7[P^5_{57}(9)])$, $M^5(a_8[P^5_{57}(9)])$, $M^5(a_9[P^5_{57}(9)])$, $M^5(a_{10}[P^5_{57}(9)])$, $M^5(a_{11}[P^5_{57}(9)])$, $M^5(a_{12}[P^5_{57}(9)])$, $M^5(a_{13}[P^5_{57}(9)])$, $M^5(a_{14}[P^5_{57}(9)])$, $M^5(a_{15}[P^5_{57}(9)])$ and $M^5(a_{16}[P^5_{57}(9)])$ over the polytope $P^5_{57}(9)$.

Proof: From the face poset of $P^5_{57}(9)$ the symmetry group of $P^5_{57}(9)$ is $\mathbb{Z}_2$ and its generator is represented by the permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 2 & 3 & 4 & 8 & 2 & 7 & 5 \end{pmatrix}$. Its action on $\mathbb{X}P^5_{52}(9)$ is described by the following diagram

$$a_1[P^5_{57}(9)] \xrightarrow{\sigma} a_2[P^5_{57}(9)] \xrightarrow{\sigma} a_3[P^5_{57}(9)] \xrightarrow{\sigma} a_4[P^5_{57}(9)] \xrightarrow{\sigma} a_5[P^5_{57}(9)] \xrightarrow{\sigma} a_6[P^5_{57}(9)] \xrightarrow{\sigma} a_7[P^5_{57}(9)]$$

$$a_8[P^5_{57}(9)] \xrightarrow{\sigma} a_9[P^5_{57}(9)] \xrightarrow{\sigma} a_{10}[P^5_{57}(9)] \xrightarrow{\sigma} a_{11}[P^5_{57}(9)] \xrightarrow{\sigma} a_{12}[P^5_{57}(9)]$$

$$a_{13}[P^5_{57}(9)] \xrightarrow{\sigma} a_{14}[P^5_{57}(9)] \xrightarrow{\sigma} a_{15}[P^5_{57}(9)] \xrightarrow{\sigma} a_{16}[P^5_{57}(9)]$$

so the theorem follows from Proposition 5.32.

Proposition 5.37. $\mathbb{X}P^5_{52}(9)$ has exactly three elements and they are represented by the matrices

$$a_1[P^5_{58}(9)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, a_2[P^5_{58}(9)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, a_3[P^5_{58}(9)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$
Theorem 5.37. There are exactly two small covers $M^5(a_1[P_{58}^5(9)])$ and $M^5(a_2[P_{58}^5(9)])$ over the polytope $P_{58}^5(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P_{58}^5(9)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and its generators are represented by the permutations $\sigma = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right)$ and $\tau = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 8 & 2 & 3 & 4 & 0 & 7 & 6 \end{array} \right)$. $\sigma$ fixes elements of $\mathbb{R}X_{P_{58}^5(9)}$ and $\tau$ acts by $\tau(a_1[P_{58}^5(9)]) = a_1[P_{58}^5(9)]$, $\tau(a_2[P_{58}^5(9)]) = a_3[P_{58}^5(9)]$ and $\tau(a_3[P_{58}^5(9)]) = a_2[P_{58}^5(9)]$ and the claim follows from Proposition 5.37. □

Proposition 5.38. $\mathbb{R}X_{P_{59}^5(9)}$ has exactly two elements and they are represented by the matrices

$$a_1[P_{59}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad a_2[P_{59}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Theorem 5.38. There is exactly one small cover $M^5(a_1[P_{59}^5(9)])$ over the polytope $P_{59}^5(9)$.

Proof: From the face poset of $P_{59}^5(9)$ the symmetry group of $P_{59}^5(9)$ is $\mathbb{Z}_2$ and its generator is represented by the permutation $\sigma = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 0 & 6 & 8 & 4 & 2 \end{array} \right)$. It acts on $\mathbb{R}X_{P_{59}^5(9)}$ by $\sigma(a_1[P_{59}^5(9)]) = a_2[P_{59}^5(9)]$ and $\sigma(a_2[P_{59}^5(9)]) = a_1[P_{59}^5(9)]$ so the theorem follows from Proposition 5.38. □

Proposition 5.39. $\mathbb{R}X_{P_{60}^5(9)}$ has exactly 1 element and it is represented by the matrix

$$a_1[P_{60}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Theorem 5.39. There is only one small cover $M^5(a_1[P_{60}^5(9)])$ over the polytope $P_{60}^5(9)$.

Proof: It is an immediate consequence of Proposition 5.39. □

Proposition 5.40. $\mathbb{R}X_{P_{62}^5(9)}$ has exactly one element and it is represented by the matrix

$$a_1[P_{62}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Theorem 5.40. There is only one small cover $M^5(a_1[P_{62}^5(9)])$ over the polytope $P_{62}^5(9)$.

Proof: It is an immediate consequence of Proposition 5.40. □
Proposition 5.41. $\mathbb{R}X_{P_{64}^5(9)}$ has exactly one element and it is represented by the matrix

$$a_1[P_{64}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{vmatrix}.$$

Theorem 5.41. There is only one small cover $M^5(a_1[P_{64}^5(9)])$ over the polytope $P_{64}^5(9)$.

Proof: It is an immediate consequence of Proposition 5.41.

Proposition 5.42. $\mathbb{R}X_{P_{65}^5(9)}$ has exactly three elements and they are represented by the matrices

$$a_1[P_{65}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{vmatrix}, \quad a_2[P_{65}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{vmatrix}.$$

and $a_3[P_{65}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{vmatrix}$.

Theorem 5.42. There are exactly two small covers $M^5(a_1[P_{65}^5(9)])$ and $M^5(a_2[P_{65}^5(9)])$ over the polytope $P_{65}^5(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P_{65}^5(9)$ is $\mathbb{Z}_2$ and its generator is represented by the permutation $\sigma = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$.

The claim follows from Proposition 5.42.

Proposition 5.43. $\mathbb{R}X_{P_{66}^5(9)}$ has exactly one element and it is represented by the matrix

$$a_1[P_{66}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{vmatrix}.$$

Theorem 5.43. There is only one small cover $M^5(a_1[P_{66}^5(9)])$ over the polytope $P_{66}^5(9)$.

Proof: It is an immediate consequence of Proposition 5.43.

Proposition 5.44. $\mathbb{R}X_{P_{67}^5(9)}$ has exactly three elements and they are represented by the matrices

$$a_1[P_{67}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{vmatrix}, \quad a_2[P_{67}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{vmatrix}.$$
Theorem 5.44. There are exactly three small covers \( M^5(a_1[P^5_{67}(9)]) \), \( M^5(a_2[P^5_{67}(9)]) \) and \( M^5(a_3[P^5_{67}(9)]) \) over the polytope \( P^5_{67}(9) \).

Proof: The symmetry group of \( P^5_{67}(9) \) is trivial by direct checking from its poset, so the theorem is an immediate consequence of Proposition 5.44.

Proposition 5.45. \( \mathbb{R}X^5_{P^5_{68}(9)} \) has exactly five elements and they are represented by the matrices

\[
\begin{align*}
a_1[P^5_{68}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_2[P^5_{68}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}, \\
a_3[P^5_{68}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}, \\
a_4[P^5_{68}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_5[P^5_{68}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
\end{align*}
\]

Theorem 5.45. There are exactly three small covers \( M^5(a_1[P^5_{68}(9)]) \), \( M^5(a_2[P^5_{68}(9)]) \) and \( M^5(a_3[P^5_{68}(9)]) \) over the polytope \( P^5_{68}(9) \).

Proof: As in the previous proofs we find that the symmetry group of \( P^5_{68}(9) \) is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and its generators are represented by the permutations \( \sigma = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8) \) and \( \tau = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8) \). The action of \( \text{Aut}(P^5_{68}(9)) \) on \( \mathbb{R}X^5_{P^5_{68}(9)} \) is depicted on the following diagram

\[
a_1[P^5_{68}(9)] \xrightarrow{\sigma} a_4[P^5_{68}(9)], \quad a_2[P^5_{68}(9)] \xrightarrow{\tau} a_5[P^5_{68}(9)], \quad a_3[P^5_{68}(9)] \xrightarrow{\sigma} a_5[P^5_{68}(9)]
\]

and the claim directly follows by Proposition 5.45.

Proposition 5.46. \( \mathbb{R}X^5_{P^5_{69}(9)} \) has exactly three elements and they are represented by the matrices

\[
\begin{align*}
a_1[P^5_{69}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}, \\
a_2[P^5_{69}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
\end{align*}
\]
Theorem 5.46. There are exactly three small covers $M^5(a_1[P_{69}^5(9)])$, $M^5(a_2[P_{69}^5(9)])$ and $M^5(a_3[P_{69}^5(9)])$ over the polytope $P_{69}^5(9)$.

Proof: The symmetry group $\text{Aut}(P_{69}^5(9))$ is $\mathbb{Z}_2$ and its generator is represented by the permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 3 & 0 & 5 & 6 & 7 & 8 \end{pmatrix}$, but it acts trivially on $\mathbb{R}X_{P_{69}^5(9)}$, so the theorem is an immediate consequence of Proposition 5.46. \qed

Proposition 5.47. $\mathbb{R}X_{P_{70}^5(9)}$ has exactly seven elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix},
a_2[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix},
a_3[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix},
a_4[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix},
a_5[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix},
a_6[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix},
a_7[P_{70}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{vmatrix}.
\end{align*}
\]

Theorem 5.47. There are exactly 7 small covers $M^5(a_1[P_{70}^5(9)])$, $M^5(a_2[P_{70}^5(9)])$, $M^5(a_3[P_{70}^5(9)])$, $M^5(a_4[P_{70}^5(9)])$, $M^5(a_5[P_{70}^5(9)])$, $M^5(a_6[P_{70}^5(9)])$ and $M^5(a_7[P_{70}^5(9)])$ over the polytope $P_{70}^5(9)$.

Proof: The symmetry group $\text{Aut}(P_{70}^5(9))$ is $\mathbb{Z}_2$ and its generator is represented by the permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{pmatrix}$, but it acts trivially on $\mathbb{R}X_{P_{70}^5(9)}$, so the theorem is an immediate consequence of Proposition 5.47. \qed
Proposition 5.48. \( \mathcal{X}_{P_{71}^5(9)} \) has exactly one element and it is represented by the matrix

\[
a_1[P_{71}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

Theorem 5.48. There is only one small cover \( M^5(a_1[P_{71}^5(9)]) \) over the polytope \( P_{71}^5(9) \).

Proof: It is an immediate consequence of Proposition 5.48. \( \square \)

Proposition 5.49. \( \mathcal{X}_{P_{72}^5(9)} \) has exactly four elements and they are represented by the matrices

\[
a_1[P_{72}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{bmatrix}, \quad a_2[P_{72}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{bmatrix},
\]

\[
a_3[P_{72}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix} \quad \text{and} \quad a_4[P_{72}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

Theorem 5.49. There are exactly two small covers \( M^5(a_1[P_{72}^5(9)]) \) and \( M^5(a_2[P_{72}^5(9)]) \) over the polytope \( P_{72}^5(9) \).

Proof: As in the previous proofs we find that the symmetry group of \( P_{72}^5(9) \) is \( \mathbb{Z}_3 \) its generator is represented by the permutation \( \sigma = (0 1 2 3 4 5 6 7 8) \). The action of \( \text{Aut}(P_{72}^5(9)) \) on \( \mathcal{X}_{P_{72}^5(9)} \) is depicted on the following diagram

\[
\begin{array}{cccccccc}
a_1[P_{72}^5(9)] & a_2[P_{72}^5(9)] & \sigma & a_3[P_{72}^5(9)] & \sigma & a_4[P_{72}^5(9)] \\
\sigma & \sigma & \sigma & \sigma & \sigma & \sigma \\
\end{array}
\]

and the claim directly follows by Proposition 5.49. \( \square \)

Proposition 5.50. \( \mathcal{X}_{P_{73}^5(9)} \) has exactly seven elements and it is represented by the matrices

\[
a_1[P_{73}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}, \quad a_2[P_{73}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
\]
Theorem 5.50. There are exactly 7 small covers $M^5(a_3[P_{73}^5(9)])$, $M^5(a_4[P_{73}^5(9)])$, $M^5(a_5[P_{73}^5(9)])$, $M^5(a_6[P_{73}^5(9)])$ and $M^5(a_7[P_{73}^5(9)])$ over the polytope $P_{73}^5(9)$.

Proof: From the face poset of $P_{73}^5(9)$ the symmetry group Aut($P_{73}^5(9)$) is trivial. □

Proposition 5.51. $\mathcal{X}_{P_{74}^5(9)}$ has exactly six elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{74}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\quad a_2[P_{74}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\end{align*}
\]

and

\[
\begin{align*}
a_3[P_{74}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\quad a_4[P_{74}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\quad a_5[P_{74}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\quad a_6[P_{74}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}.
\end{align*}
\]

Theorem 5.51. There are exactly 6 small covers $M^5(a_1[P_{74}^5(9)])$, $M^5(a_2[P_{74}^5(9)])$, $M^5(a_3[P_{74}^5(9)])$, $M^5(a_4[P_{74}^5(9)])$, $M^5(a_5[P_{74}^5(9)])$ and $M^5(a_6[P_{74}^5(9)])$ over the polytope $P_{74}^5(9)$.

Proof: From the face poset of $P_{74}^5(9)$ the symmetry group Aut($P_{74}^5(9)$) is trivial. □

Proposition 5.52. $\mathcal{X}_{P_{76}^5(9)}$ has exactly two elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{76}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix},
\quad a_2[P_{76}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}.
\end{align*}
\]
Theorem 5.52. There are exactly two small covers $M^5(a_1[P_{76}^5(9)])$ and $M^5(a_2[P_{76}^5(9)])$ over the polytope $P_{76}^5(9)$.

Proof: From the face poset of $P_{76}^5(9)$ the symmetry group $\text{Aut}(P_{76}^5(9))$ is trivial. \hfill $\square$

Proposition 5.53. $\mathbb{R}X_{P_{79}^5(9)}$ has exactly two elements and they are represented by the matrices

$$
a_1[P_{79}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{vmatrix} \quad \text{and} \quad
a_2[P_{79}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{vmatrix}.
$$

Theorem 5.53. There are exactly two small covers $M^5(a_1[P_{79}^5(9)])$ and $M^5(a_2[P_{79}^5(9)])$ over the polytope $P_{79}^5(9)$.

Proof: From the face poset of $P_{79}^5(9)$ the symmetry group $\text{Aut}(P_{79}^5(9))$ is trivial. \hfill $\square$

Proposition 5.54. $\mathbb{R}X_{P_{81}^5(9)}$ has exactly four elements and they are represented by the matrices

$$
a_1[P_{81}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{vmatrix} , \quad
a_2[P_{81}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{vmatrix} , \quad
a_3[P_{81}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{vmatrix} , \quad
a_4[P_{81}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{vmatrix}.
$$

Theorem 5.54. There are exactly two small covers $M^5(a_1[P_{81}^5(9)])$ and $M^5(a_3[P_{81}^5(9)])$ over the polytope $P_{81}^5(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P_{81}^5(9)$ is $\mathbb{Z}_2$ whose generator is represented by the permutation $\sigma = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8)$.

The action of $\text{Aut}(P_{81}^5(9))$ on $\mathbb{R}X_{P_{81}^5(9)}$ is depicted on the following diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\sigma \\
\rightarrow \\
\alpha \\
\rightarrow \end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\sigma \\
\rightarrow \\
\alpha \\
\rightarrow \end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\sigma \\
\rightarrow \\
\alpha \\
\rightarrow \end{array}
\end{array}
\end{array}
$$

and the claim directly follows by Proposition 5.54. \hfill $\square$

Proposition 5.55. $\mathbb{R}X_{P_{83}^5(9)}$ has exactly three elements and they are represented by the matrices

$$
a_1[P_{83}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{vmatrix} , \quad
a_2[P_{83}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{vmatrix}.
$$
and $a_3[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.

**Theorem 5.55.** There are exactly three small covers $M^5(a_1[P_{83}^5(9)])$, $M^5(a_2[P_{83}^5(9)])$ and $M^5(a_3[P_{83}^5(9)])$ over the polytope $P_{83}^5(9)$.

**Proof:** The symmetry group $\text{Aut}(P_{83}^5(9))$ is trivial. $\square$

**Proposition 5.56.** $\mathcal{X}_{P_{83}^5(9)}$ has exactly six elements and they are represented by the matrices

\[
a_1[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad a_2[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},

\]

\[
a_3[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad a_4[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},

\]

\[
a_5[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad a_6[P_{83}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.

\]

**Theorem 5.56.** There are exactly 6 small covers $M^5(a_1[P_{88}^5(9)])$, $M^5(a_2[P_{88}^5(9)])$, $M^5(a_3[P_{88}^5(9)])$, $M^5(a_4[P_{88}^5(9)])$, $M^5(a_5[P_{88}^5(9)])$ and $M^5(a_6[P_{88}^5(9)])$ over the polytope $P_{88}^5(9)$.

**Proof:** From the face poset of $P_{88}^5(9)$ the symmetry group $\text{Aut}(P_{88}^5(9))$ is trivial. $\square$

**Proposition 5.57.** $\mathcal{X}_{P_{88}^5(9)}$ has exactly one element and it is represented by the matrix

\[
a_1[P_{88}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.

\]

**Theorem 5.57.** There is only one small cover $M^5(a_1[P_{88}^5(9)])$ over the polytope $P_{88}^5(9)$.

**Proof:** It is an immediate consequence of Proposition 5.57. $\square$
Proposition 5.58. \( \mathcal{X}_{P_{89}(9)} \) has exactly 22 elements and they are represented by the matrices

\[
\begin{align*}
\alpha_1[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ \end{bmatrix}, & \alpha_2[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_3[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ \end{bmatrix}, & \alpha_4[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_5[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \end{bmatrix}, & \alpha_6[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_7[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \end{bmatrix}, & \alpha_8[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_9[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ \end{bmatrix}, & \alpha_{10}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_{11}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, & \alpha_{12}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_{13}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, & \alpha_{14}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, \\
\alpha_{15}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}, & \alpha_{16}[P_{89}(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \end{bmatrix}.
\end{align*}
\]
Theorem 5.58. There are exactly 10 small covers $M^5(a_1[P^5_{89}(9)])$, $M^5(a_2[P^5_{89}(9)])$, $M^5(a_3[P^5_{89}(9)])$, $M^5(a_4[P^5_{89}(9)])$, $M^5(a_5[P^5_{89}(9)])$, $M^5(a_6[P^5_{89}(9)])$, $M^5(a_7[P^5_{89}(9)])$, $M^5(a_8[P^5_{89}(9)])$, $M^5(a_9[P^5_{89}(9)])$ and $M^5(a_{10}[P^5_{89}(9)])$ over the polytope $P^5_{89}(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P^5_{89}(9)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and its generators are represented by the permutations $\sigma = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 8 & 3 & 4 & 5 & 1 & 0 \\ 2 & & & & & & & \end{array} \right)$ and $\tau = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 5 & 4 & 3 & 6 & 7 \end{array} \right)$. The action of $\text{Aut}(P^5_{89}(9))$ on $\mathbb{R}^\mathcal{X}_{P^5_{89}(9)}$ is depicted on the following diagram:

![Diagram of permutations]

and the claim directly follows by Proposition 5.58. \qed
Proposition 5.59. $\mathcal{X}_{P_{94}^5(9)}$ has exactly 10 elements and they are represented by the matrices

\[
\begin{align*}
 a_1[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, & \quad a_2[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, \\
 a_3[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, & \quad a_4[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, \\
 a_5[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, & \quad a_6[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, \\
 a_7[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, & \quad a_8[P_{94}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}.
\end{align*}
\]

Theorem 5.59. There are exactly 10 small covers $M^5(a_1[P_{94}^5(9)]), M^5(a_2[P_{94}^5(9)]), M^5(a_3[P_{94}^5(9)]), M^5(a_4[P_{94}^5(9)]), M^5(a_5[P_{94}^5(9)]), M^5(a_6[P_{94}^5(9)]), M^5(a_7[P_{94}^5(9)]), M^5(a_8[P_{94}^5(9)]), M^5(a_9[P_{94}^5(9)])$ and $M^5(a_{10}[P_{94}^5(9)])$ over the polytope $P_{94}^5(9)$.

Proof: From the face poset of $P_{94}^5(9)$ the symmetry group $\text{Aut}(P_{94}^5(9))$ is trivial. \qed

Proposition 5.60. $\mathcal{X}_{P_{97}^5(9)}$ has exactly 8 elements and they are represented by the matrices

\[
\begin{align*}
 a_1[P_{97}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}, & \quad a_2[P_{97}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}, \\
 a_3[P_{97}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}, & \quad a_4[P_{97}^5(9)] &= \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{bmatrix},
\end{align*}
\]
Theorem 5.60. There are exactly 10 small covers $M^5(a_1[P_{97}^5(9)])$, $M^5(a_2[P_{97}^5(9)])$, $M^5(a_3[P_{97}^5(9)])$, $M^5(a_4[P_{97}^5(9)])$, $M^5(a_5[P_{97}^5(9)])$, $M^5(a_6[P_{97}^5(9)])$, $M^5(a_7[P_{97}^5(9)])$ and $M^5(a_8[P_{97}^5(9)])$ over the polytope $P_{97}^5(9)$.

Proof: From the face poset of $P_{97}^5(9)$ the symmetry group $\text{Aut}(P_{97}^5(9))$ is $\mathbb{Z}_2$ whose generator is represented by permutation $\sigma = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 1 & 3 & 4 & 5 & 6 & 7 \end{array} \right)$, but its action on $\mathbb{R}_2X_{P_{97}^5(9)}$ is trivial. \hfill $\square$

Proposition 5.61. $\mathbb{R}X_{P_{98}^5(9)}$ has exactly 8 elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \\
a_2[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \\
a_3[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \\
a_4[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \\
a_5[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \\
a_6[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \\
a_7[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \\
a_8[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \\
a_9[P_{98}^5(9)] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \\
\end{align*}
\]

Theorem 5.61. There are exactly 10 small covers $M^5(a_1[P_{98}^5(9)])$, $M^5(a_2[P_{98}^5(9)])$, $M^5(a_3[P_{98}^5(9)])$, $M^5(a_4[P_{98}^5(9)])$, $M^5(a_5[P_{98}^5(9)])$, $M^5(a_6[P_{98}^5(9)])$, $M^5(a_7[P_{98}^5(9)])$ and $M^5(a_8[P_{98}^5(9)])$ over the polytope $P_{98}^5(9)$. 

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Proof: From the face poset of $P_{98}^5(9)$ the symmetry group $\text{Aut}(P_{98}^5(9))$ is $\mathbb{Z}_2$ whose generator is represented by permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 2 & 3 & 4 & 1 & 6 & 7 & 8 \end{pmatrix}$, but its action on $\mathbb{R}\mathcal{X}_{P_{98}^5(9)}$ is trivial. □

Proposition 5.62. $\mathbb{R}\mathcal{X}_{P_{100}^5(9)}$ has exactly 7 elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}, & a_2[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \\
a_3[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, & a_4[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \\
a_5[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, & a_6[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \\
a_7[P_{100}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}
\end{align*}
\]

and $a_7[P_{100}^5(9)]$ is represented by the matrix

\[
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.
\]

Theorem 5.62. There are exactly five small covers $M^5(a_1[P_{100}^5(9)])$, $M^5(a_2[P_{100}^5(9)])$, $M^5(a_3[P_{100}^5(9)])$, $M^5(a_4[P_{100}^5(9)])$ and $M^5(a_8[P_{100}^5(9)])$ over the polytope $P_{100}^5(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P_{100}^5(9)$ is $\mathbb{Z}_2$ whose generator is represented by the permutation $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 2 & 3 & 4 & 1 & 6 & 7 & 8 \end{pmatrix}$. The action of $\text{Aut}(P_{100}^5(9))$ on $\mathbb{R}\mathcal{X}_{P_{100}^5(9)}$ is depicted on the following diagram

\[
\begin{align*}
a_1[P_{100}^5(9)] \xrightarrow{\sigma} a_5[P_{100}^5(9)] & \quad a_2[P_{100}^5(9)] \xrightarrow{\sigma} a_6[P_{100}^5(9)] & \quad a_3[P_{100}^5(9)] \xrightarrow{\sigma} a_7[P_{100}^5(9)] \\
a_4[P_{100}^5(9)] & \quad a_8[P_{100}^5(9)]
\end{align*}
\]

and the claim directly follows by Proposition 5.62. □

Proposition 5.63. $\mathbb{R}\mathcal{X}_{P_{101}^5(9)}$ has exactly one element and it is represented by the matrix

\[
a_1[P_{101}^5(9)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.
\]
Theorem 5.63. There is only one small cover \(M^5(a_1[P_{101}^5(9)])\) over the polytope \(P_{101}^5(9)\).

Proof: It is an immediate consequence of Proposition 5.63. □

Proposition 5.64. \(\mathcal{X}_{P_{102}^5(9)}\) has exactly two elements and they are represented by the matrices

\[
a_1[P_{102}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
a_2[P_{102}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

Theorem 5.64. There are exactly two small covers \(M^5(a_1[P_{102}^5(9)])\) and \(M^5(a_2[P_{102}^5(9)])\) over the polytope \(P_{102}^5(9)\).

Proof: From the face poset of \(P_{102}^5(9)\) the symmetry group \(\text{Aut}(P_{102}^5(9))\) is trivial. □

Proposition 5.65. \(\mathcal{X}_{P_{104}^5(9)}\) has exactly one element and it is represented by the matrix

\[
a_1[P_{104}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]

Theorem 5.65. There is only one small cover \(M^5(a_1[P_{104}^5(9)])\) over the polytope \(P_{104}^5(9)\).

Proof: It is an immediate consequence of Proposition 5.65. □

Proposition 5.66. \(\mathcal{X}_{P_{105}^5(9)}\) has exactly one element and it is represented by the matrix

\[
a_1[P_{105}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

Theorem 5.66. There is only one small cover \(M^5(a_1[P_{105}^5(9)])\) over the polytope \(P_{105}^5(9)\).

Proof: It is an immediate consequence of Proposition 5.66. □

Proposition 5.67. \(\mathcal{X}_{P_{107}^5(9)}\) has exactly three elements and they are represented by the matrices

\[
a_1[P_{107}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix},
a_2[P_{107}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix},
\quad
a_3[P_{107}^5(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]
Theorem 5.67. There are exactly three small covers $M^5(a_1[P^5_{107}(9)])$, $M^5(a_2[P^5_{107}(9)])$ and $M^5(a_3[P^5_{107}(9)])$ over the polytope $P^5_{107}(9)$.

Proof: The symmetry group $\text{Aut}(P^5_{107}(9))$ is trivial. □

Proposition 5.68. $\mathcal{X}_{P^5_{109}(9)}$ has exactly one element and it is represented by the matrix

$$a_1[P^5_{109}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Theorem 5.68. There is only one small cover $M^5(a_1[P^5_{109}(9)])$ over the polytope $P^5_{109}(9)$.

Proof: It is an immediate consequence of Proposition 5.68. □

Proposition 5.69. $\mathcal{X}_{P^5_{111}(9)}$ has exactly one element and it is represented by the matrix

$$a_1[P^5_{111}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Theorem 5.69. There is only one small cover $M^5(a_1[P^5_{111}(9)])$ over the polytope $P^5_{111}(9)$.

Proof: It is an immediate consequence of Proposition 5.69. □

Proposition 5.70. $\mathcal{X}_{P^5_{112}(9)}$ has exactly 36 elements and they are represented by the matrices

$$a_1[P^5_{112}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad a_2[P^5_{112}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$a_3[P^5_{112}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad a_4[P^5_{112}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$a_5[P^5_{112}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad a_6[P^5_{112}(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$
\[
\begin{align*}
a_7[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_8[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_9[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{10}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_{11}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{12}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_{13}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{14}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_{15}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{16}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_{17}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{18}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_{19}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{20}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, \\
a_{21}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}, & a_{22}[P_{112}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}.
\end{align*}
\]
Theorem 5.70. There are exactly 18 small covers $P^5_{112}(9)$, $M^5(a_1[P^5_{112}(9)])$, $M^5(a_2[P^5_{112}(9)])$, $M^5(a_3[P^5_{112}(9)])$, $M^5(a_4[P^5_{112}(9)])$, $M^5(a_5[P^5_{112}(9)])$, $M^5(a_6[P^5_{112}(9)])$, $M^5(a_7[P^5_{112}(9)])$, $M^5(a_8[5^5_{112}(9)])$, $M^5(a_9[5^5_{112}(9)])$, $M^5(a_{10}[5^5_{112}(9)])$, $M^5(a_{11}[5^5_{112}(9)])$, $M^5(a_{12}[5^5_{112}(9)])$, $M^5(a_{13}[5^5_{112}(9)])$, $M^5(a_{14}[5^5_{112}(9)])$, $M^5(a_{15}[5^5_{112}(9)])$, $M^5(a_{16}[5^5_{112}(9)])$, $M^5(a_{17}[5^5_{112}(9)])$, $M^5(a_{18}[5^5_{112}(9)])$, $M^5(a_{19}[5^5_{112}(9)])$, $M^5(a_{20}[5^5_{112}(9)])$, $M^5(a_{21}[5^5_{112}(9)])$ and $M^5(a_{31}[5^5_{112}(9)])$ over the polytope $P^5_{112}(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P^5_{112}(9)$ is $Z_2 \oplus Z_2$ and its generators are represented by the permutations $\tau = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 3 & 7 & 1 & 4 & 5 & 6 & 2 \end{array} \right)$ and

\[
\tau = \left( \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 3 & 7 & 1 & 4 & 5 & 6 & 2 \end{array} \right)
\]
\[ \sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 5 & 4 & 6 & 7 & 8 \end{pmatrix}. \] The action of \( \text{Aut}(P_{112}^5(9)) \) on \( \mathfrak{X}_{P_{112}^5(9)} \) is depicted on the following diagram

\[
\begin{array}{cccc}
\sigma \circ a_1[P_{112}^5(9)] & \sigma \circ a_2[P_{112}^5(9)] & \sigma \circ a_3[P_{112}^5(9)] & \sigma \circ a_4[P_{112}^5(9)] \\
\sigma \circ a_5[P_{112}^5(9)] & \sigma \circ a_6[P_{112}^5(9)] & \sigma \circ a_7[P_{112}^5(9)] & \sigma \circ a_8[P_{112}^5(9)] \\
\sigma \circ a_9[P_{112}^5(9)] & \sigma \circ a_{10}[P_{112}^5(9)] & \sigma \circ a_{11}[P_{112}^5(9)] & \sigma \circ a_{12}[P_{112}^5(9)] \\
\sigma \circ a_{13}[P_{112}^5(9)] & \sigma \circ a_{14}[P_{112}^5(9)] & \sigma \circ a_{15}[P_{112}^5(9)] & \sigma \circ a_{16}[P_{112}^5(9)] \\
\sigma \circ a_{17}[P_{112}^5(9)] & \sigma \circ a_{18}[P_{112}^5(9)] & \sigma \circ a_{19}[P_{112}^5(9)] & \sigma \circ a_{20}[P_{112}^5(9)] \\
\sigma \circ a_{21}[P_{112}^5(9)] & \sigma \circ a_{22}[P_{112}^5(9)] & \sigma \circ a_{23}[P_{112}^5(9)] & \sigma \circ a_{24}[P_{112}^5(9)] \\
\sigma \circ a_{25}[P_{112}^5(9)] & \sigma \circ a_{26}[P_{112}^5(9)] & \sigma \circ a_{27}[P_{112}^5(9)] & \sigma \circ a_{28}[P_{112}^5(9)] \\
\sigma \circ a_{29}[P_{112}^5(9)] & \sigma \circ a_{30}[P_{112}^5(9)] & \sigma \circ a_{31}[P_{112}^5(9)] & \sigma \circ a_{32}[P_{112}^5(9)] \\
\sigma \circ a_{33}[P_{112}^5(9)] & \sigma \circ a_{34}[P_{112}^5(9)] & \sigma \circ a_{35}[P_{112}^5(9)] & \sigma \circ a_{36}[P_{112}^5(9)]
\end{array}
\]

and the claim directly follows by Proposition 5.70.

**Proposition 5.71.** \( \mathfrak{X}_{P_{113}^5(9)} \) has exactly 16 elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{113}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, & a_2[P_{113}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \\
a_3[P_{113}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, & a_4[P_{113}^5(9)] &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}
\end{align*}
\]
Theorem 5.71. There are exactly 7 small covers $M^5(a_1[P_{113}^S(9)])$, $M^5(a_2[P_{113}^S(9)])$, $M^5(a_3[P_{113}^S(9)])$, $M^5(a_4[P_{113}^S(9)])$, $M^5(a_8[P_{113}^S(9)])$, $M^5(a_{12}[P_{113}^S(9)])$ and $M^5(a_{13}[P_{113}^S(9)])$ over the polytope $P_{113}^S(9)$.

Proof: As in the previous proofs we find that the symmetry group of $P_{113}^S(9)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and its generators are represented by the permutations $\tau = \left( \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 6 & 7 & 4 & 0 & 2 & 3 & 8 \end{array} \right)$ and $\sigma = \left( \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 8 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{array} \right)$. The action of $\text{Aut}(P_{113}^S(9))$ on $\mathcal{X}_{P_{113}^S(9)}$ is depicted on the following diagram

\[
\begin{array}{cccccccccccc}
a_{15}[P_{113}^S(9)] & \xrightarrow{\tau} & a_{7}[P_{113}^S(9)] & \xrightarrow{\sigma} & a_{2}[P_{113}^S(9)] & \xrightarrow{\tau} & a_{6}[P_{113}^S(9)] & \xrightarrow{\sigma} & a_{3}[P_{113}^S(9)] & \xrightarrow{\tau} & a_{15}[P_{113}^S(9)] \\
a_{15}[P_{113}^S(9)] & \xrightarrow{\sigma} & a_{7}[P_{113}^S(9)] & \xrightarrow{\tau} & a_{2}[P_{113}^S(9)] & \xrightarrow{\sigma} & a_{6}[P_{113}^S(9)] & \xrightarrow{\tau} & a_{3}[P_{113}^S(9)] & \xrightarrow{\sigma} & a_{15}[P_{113}^S(9)] \
\end{array}
\]
and the claim directly follows by Proposition 5.71.

Proposition 5.72. \( \mathcal{X}_{P_{114}^5(9)} \) has exactly 9 elements and they are represented by the matrices

\[
\begin{align*}
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_1[P_{114}^5(9)]) = 10000011100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_2[P_{114}^5(9)]) = 01000001101, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_3[P_{114}^5(9)]) = 001000111100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_4[P_{114}^5(9)]) = 01000011110, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_5[P_{114}^5(9)]) = 001000111100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_6[P_{114}^5(9)]) = 000100110110, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_7[P_{114}^5(9)]) = 00001011100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_8[P_{114}^5(9)]) = 000001111100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_9[P_{114}^5(9)]) = 000001111100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{10}[P_{114}^5(9)]) = 100000111111, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{11}[P_{114}^5(9)]) = 010000111100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{12}[P_{114}^5(9)]) = 000100110110, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{13}[P_{114}^5(9)]) = 00001011100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{14}[P_{114}^5(9)]) = 10000011111, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{15}[P_{114}^5(9)]) = 010000111100, \\
\sigma|P_{114}^5(9)| & \mapsto \sigma(a_{16}[P_{114}^5(9)]) = 00001011100.
\end{align*}
\]

and\( a_9|P_{114}^5(9)| \mapsto a_9[P_{114}^5(9)] = 100000111111. \]

Theorem 5.72. There are exactly 8 small covers \( M^5(a_1[P_{114}^5(9)]) \), \( M^5(a_2[P_{114}^5(9)]) \), \( M^5(a_3[P_{114}^5(9)]) \), \( M^5(a_4[P_{114}^5(9)]) \), \( M^5(a_5[P_{114}^5(9)]) \), \( M^5(a_6[P_{114}^5(9)]) \), \( M^5(a_7[P_{114}^5(9)]) \) and \( M^5(a_8[P_{114}^5(9)]) \) over the polytope \( P_{114}^5(9) \).

Proof: As in the previous proofs we find that the symmetry group of \( P_{114}^5(9) \) is \( \mathbb{Z}_2 \) and its generator is represented by the permutation \( \sigma = \left(\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 6 & 3 & 4 & 5 & 2 & 7 & 8 \end{array}\right) \). The action of \( \text{Aut}(P_{114}^5(9)) \) on \( \mathbb{R}\mathcal{X}_{P_{114}^5(9)} \) is \( \sigma(a_2[P_{114}^5(9)]) = a_9[P_{114}^5(9)] \), \( \sigma(a_9[P_{114}^5(9)]) = a_2[P_{114}^5(9)] \) and other elements are fixed.

\( \Box \)
Proposition 5.73. \( \mathcal{X}_{P^{5}_{115}(9)} \) has exactly three elements and they are represented by the matrices

\[
\begin{align*}
\alpha_1[P^{5}_{115}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{bmatrix},
\alpha_2[P^{5}_{115}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{bmatrix},
\end{align*}
\]

and \( \alpha_3[P^{5}_{115}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{bmatrix}. \]

Theorem 5.73. There are exactly three small covers \( M^5(\alpha_1[P^{5}_{115}(9)]) \), \( M^5(\alpha_2[P^{5}_{115}(9)]) \) and \( M^5(\alpha_3[P^{5}_{115}(9)]) \) over the polytope \( P^{5}_{115}(9) \).

Proof: The symmetry group \( \text{Aut}(P^{5}_{115}(9)) \) is \( \mathbb{Z}_2 \) and its generator is represented by the permutation \( \sigma = \left( \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 1 & 2 & 3 & 4 & 5 & 0 & 7 & 8
\end{array} \right) \), but its action on \( \mathcal{X}_{P^{5}_{115}(9)} \) is trivial. \( \Box \)

Proposition 5.74. \( \mathcal{X}_{P^{5}_{116}(9)} \) has exactly six elements and they are represented by the matrices

\[
\begin{align*}
\alpha_1[P^{5}_{116}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
\alpha_2[P^{5}_{116}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
\alpha_3[P^{5}_{116}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
\alpha_4[P^{5}_{116}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
\alpha_5[P^{5}_{116}(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \text{and} \alpha_6[P^{5}_{116}(9)] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}.
\end{align*}
\]

Theorem 5.74. There are exactly five small covers \( M^5(\alpha_1[P^{5}_{116}(9)]) \), \( M^5(\alpha_2[P^{5}_{116}(9)]) \), \( M^5(\alpha_3[P^{5}_{116}(9)]) \), \( M^5(\alpha_4[P^{5}_{116}(9)]) \) and \( M^5(\alpha_5[P^{5}_{116}(9)]) \) over the polytope \( P^{5}_{116}(9) \).

Proof: As in the previous proofs we find that the symmetry group of \( P^{5}_{116}(9) \) is \( \mathbb{Z}_2 \) and its generator is represented by the permutations \( \sigma = \left( \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & 7 & 3 & 4 & 5 & 6 & 2 & 8
\end{array} \right) \). The action of \( \text{Aut}(P^{5}_{116}(9)) \) on \( \mathcal{X}_{P^{5}_{116}(9)} \) is \( \sigma(\alpha_4[P^{5}_{116}(9)]) = \alpha_6[P^{5}_{116}(9)] \), \( \sigma(\alpha_6[P^{5}_{116}(9)]) = \alpha_4[P^{5}_{116}(9)] \) and other elements are fixed. \( \Box \)
Proposition 5.75. \( \mathbb{R}X_{P_{117}^5(9)} \) has exactly 16 elements and they are represented by the matrices

\[
\begin{align*}
\mathbf{a}_1[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, & \mathbf{a}_2[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \\
\mathbf{a}_3[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, & \mathbf{a}_4[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \\
\mathbf{a}_5[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, & \mathbf{a}_6[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \\
\mathbf{a}_7[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{bmatrix}, & \mathbf{a}_8[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \\
\mathbf{a}_9[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, & \mathbf{a}_{10}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \\
\mathbf{a}_{11}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, & \mathbf{a}_{12}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \\
\mathbf{a}_{13}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, & \mathbf{a}_{14}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \\
\mathbf{a}_{15}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \quad \text{and} \quad \mathbf{a}_{16}[P_{117}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\end{bmatrix}.
\end{align*}
\]
**Theorem 5.75.** There are exactly 7 small covers $M^5(a_1[P_{117}(9)])$, $M^5(a_2[P_{117}(9)])$, $M^5(a_3[P_{117}(9)])$, $M^5(a_4[P_{117}(9)])$, $M^5(a_5[P_{117}(9)])$, $M^5(a_6[P_{117}(9)])$ and $M^5(a_7[P_{117}(9)])$ over the polytope $P_{117}^5(9)$.

**Proof:** As in the previous proofs we find that the symmetry group of $P_{117}^5(9)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and its generators are represented by the permutations $\tau = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 2 & 4 & 0 & 8 & 7 & 6 \end{array}\right)$ and $\sigma = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 7 & 2 & 3 & 4 & 5 & 6 & 1 & 8 \end{array}\right)$. The action of $\text{Aut}(P_{117}^5(9))$ on $\mathcal{X}_{P_{117}^5(9)}$ is depicted on the following diagram

and the claim directly follows by Proposition 5.75.

**Proposition 5.76.** $\mathcal{X}_{P_{118}^5(9)}$ has exactly five elements and they are represented by the matrices

$$a_1[P_{118}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad a_2[P_{118}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$a_3[P_{118}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad a_4[P_{118}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

and $a_5[P_{118}^5(9)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$.

**Theorem 5.76.** There are exactly 5 small covers $M^5(a_1[P_{118}(9)])$, $M^5(a_2[P_{118}(9)])$, $M^5(a_3[P_{118}(9)])$, $M^5(a_4[P_{118}(9)])$ and $M^5(a_5[P_{118}(9)])$ over the polytope $P_{118}^5(9)$.

**Proof:** The symmetry group $\text{Aut}(P_{118}^5(9))$ is trivial.

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Proposition 5.77. \(\mathbb{R}X_{P_{119}^5(9)}\) has exactly three elements and they are represented by the matrices

\[
\begin{align*}
\mathbf{a}_1[P_{119}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix},
\mathbf{a}_2[P_{119}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix},
\mathbf{a}_3[P_{119}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\end{align*}
\]

Theorem 5.77. There are exactly two small covers \(M^5(\mathbf{a}_1[P_{119}^5(9)])\) and \(M^5(\mathbf{a}_2[P_{119}^5(9)])\) over the polytope \(P_{119}^5(9)\).

Proof: The symmetry group \(\text{Aut}(P_{119}^5(9))\) is \(\mathbb{Z}_2\) and its generator is represented by the permutation \(\sigma = \left(\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
7 & 8 & 2 & 3 & 3 & 5 & 6 & 0 & 1
\end{array}\right)\). It acts on \(\mathbb{R}X_{P_{119}^5(9)}\) by \(\sigma(\mathbf{a}_1[P_{119}^5(9)]) = \mathbf{a}_1[P_{119}^5(9)], \sigma(\mathbf{a}_2[P_{119}^5(9)]) = \mathbf{a}_3[P_{119}^5(9)]\) and \(\sigma(\mathbf{a}_3[P_{119}^5(9)]) = \mathbf{a}_2[P_{119}^5(9)]\).

Proposition 5.78. \(\mathbb{R}X_{P_{120}^5(9)}\) has exactly three elements and they are represented by the matrices

\[
\begin{align*}
\mathbf{a}_1[P_{120}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{bmatrix},
\mathbf{a}_2[P_{120}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix},
\mathbf{a}_3[P_{120}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\end{align*}
\]

Theorem 5.78. There are exactly three small covers \(M^5(\mathbf{a}_1[P_{120}^5(9)])\), \(M^5(\mathbf{a}_2[P_{120}^5(9)])\) and \(M^5(\mathbf{a}_3[P_{120}^5(9)])\) over the polytope \(P_{120}^5(9)\).

Proof: The symmetry group \(\text{Aut}(P_{120}^5(9))\) is trivial, so the claim is corollary of Proposition 5.78.

Proposition 5.79. \(\mathbb{R}X_{P_{122}^5(9)}\) has exactly three elements and they are represented by the matrices

\[
\begin{align*}
\mathbf{a}_1[P_{122}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix},
\mathbf{a}_2[P_{122}^5(9)] &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}.
\end{align*}
\]
\[
\begin{align*}
\text{and } a_3[P_{122}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix}.
\end{align*}
\]

**Theorem 5.79.** There is exactly one small cover \(M^5(a_1[P_{122}^5(9)])\) over the polytope \(P_{122}^5(9)\).

**Proof:** As in the previous proofs we find that the symmetry group of \(P_{122}^5(9)\) is \(\mathbb{Z}_3\) its generator is represented by the permutation \(\sigma = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 4 & 5 & 0 & 6 & 8 & 1 & 5 & 8
\end{pmatrix}\). The action of \(\text{Aut}(P_{122}^5(9))\) on \(\mathcal{X}_{P_{122}^5(9)}\) is depicted on the following diagram

\[
\begin{array}{c}
a_1[P_{122}^5(9)] \\
\downarrow \sigma \downarrow \sigma \downarrow \sigma \\
a_2[P_{122}^5(9)] \\
\downarrow \sigma \downarrow \sigma \downarrow \sigma \\
a_3[P_{122}^5(9)]
\end{array}
\]

and the claim directly follows by Proposition 5.79. \(\square\)

**Proposition 5.80.** \(\mathcal{X}_{P_{123}^5(9)}\) has exactly nine elements and they are represented by the matrices

\[
\begin{align*}
a_1[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{vmatrix},
\quad a_2[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{vmatrix},
\end{align*}
\]

\[
\begin{align*}
a_3[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix},
\quad a_4[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix},
\end{align*}
\]

\[
\begin{align*}
a_5[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix},
\quad a_6[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix},
\end{align*}
\]

\[
\begin{align*}
a_7[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{vmatrix},
\quad a_8[P_{123}^5(9)] &= \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix},
\end{align*}
\]

\[
\text{and } a_9[P_{123}^5(9)] = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{vmatrix},
\]

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Theorem 5.80. There are exactly 8 small covers \( M^5(a_1[p_{123}(9)]) \), \( M^5(a_2[p_{123}(9)]) \), \( M^5(a_3[p_{123}(9)]) \), \( M^5(a_4[p_{123}(9)]) \), \( M^5(a_5[p_{123}(9)]) \), \( M^5(a_6[p_{123}(9)]) \), \( M^5(a_7[p_{123}(9)]) \) and \( M^5(a_8[p_{123}(9)]) \) over the polytope \( p_{123}(9) \).

Proof: As in the previous proofs we find that the symmetry group of \( p_{123}(9) \) is \( Z_2 \) and its generator is represented by the permutations \( \sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{pmatrix} \). \( \text{Aut}(p_{123}(9)) \) acts on \( \mathbb{R}X_{p_{123}(9)} \) is \( \sigma(a_8[p_{123}(9)]) = a_9[p_{123}(9)], \sigma(a_9[p_{123}(9)]) = a_8[p_{123}(9)] \) and other elements are fixed.

\[ \square \]

Proposition 5.81. \( \mathbb{R}X_{p_{124}(9)} \) has exactly two elements and they are represented by the matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

Theorem 5.81. There is exactly one small cover \( M^5(a_1[p_{124}(9)]) \) over the polytope \( p_{124}(9) \).

Proof: From the face poset of \( p_{124}(9) \) the symmetry group \( \text{Aut}(p_{124}(9)) \) is \( Z_2 \oplus Z_2 \) whose generators are represented by permutations \( \tau = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 0 & 2 & 6 & 4 & 5 & 3 & 5 & 8 \end{pmatrix} \) and

\[
\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 8 & 3 & 4 & 5 & 6 & 7 & 2 \end{pmatrix} \]. \( \sigma \) is fixing all elements in \( \mathbb{R}X_{p_{124}(9)} \), but \( \tau(a_1[p_{124}(9)]) = a_2[p_{124}(9)] \) and \( \tau(a_2[p_{124}(9)]) = a_1[p_{124}(9)] \)

\[ \square \]

Proposition 5.82. \( \mathbb{R}X_{p_{125}(9)} \) has exactly 10 elements and they are represented by the matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

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Theorem 5.82. There are exactly 10 small covers \( M^S(a_1[P_{125}(9)]), M^S(a_2[P_{125}(9)]), M^S(a_3[P_{125}(9)]), M^S(a_4[P_{125}(9)]), M^S(a_5[P_{125}(9)]), M^S(a_6[P_{125}(9)]), M^S(a_7[P_{125}(9)]), M^S(a_8[P_{125}(9)]), M^S(a_9[P_{125}(9)]) \) and \( M^S(a_{10}[P_{125}(9)]) \) over the polytope \( P_{125}(9) \).

Proof: As in the previous proofs we find that the symmetry group of \( P_{125}(9) \) is \( \mathbb{Z}_2 \) and its generator is represented by the permutation \( \sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 6 & 3 & 4 & 5 & 2 & 7 & 8 \end{pmatrix} \). However, the action of \( \text{Aut}(P_{125}(9)) \) on \( \mathbb{R} X_{P_{125}(9)} \) is trivial. We summarize the results about the classification of small covers in the following theorem.

Theorem 5.83. \( P_{6}^5(9), P_{1}^5(9), P_{2}^5(9), P_{3}^5(9), P_{9}^5(9), P_{16}^5(9), P_{17}^5(9), P_{18}^5(9), P_{20}^5(9), P_{21}^5(9), P_{27}^5(9), P_{30}^5(9), P_{31}^5(9), P_{37}^5(9), P_{38}^5(9), P_{42}^5(9), P_{44}^5(9), P_{46}^5(9), P_{48}^5(9), P_{53}^5(9), P_{61}^5(9), P_{63}^5(9), P_{75}^5(9), P_{77}^5(9), P_{78}^5(9), P_{80}^5(9), P_{82}^5(9), P_{84}^5(9), P_{86}^5(9), P_{87}^5(9), P_{90}^5(9), P_{91}^5(9), P_{92}^5(9), P_{93}^5(9), P_{95}^5(9), P_{96}^5(9), P_{99}^5(9), P_{103}^5(9), P_{106}^5(9), P_{108}^5(9), P_{110}^5(9) \) and \( P_{121}^5(9) \) are not the orbit spaces of a small cover.

\( P_{4}^5(9), P_{6}^5(9), P_{22}^5(9), P_{24}^5(9), P_{29}^5(9), P_{36}^5(9), P_{40}^5(9), P_{50}^5(9), P_{51}^5(9), P_{52}^5(9), P_{59}^5(9), P_{60}^5(9), P_{62}^5(9), P_{64}^5(9), P_{66}^5(9), P_{71}^5(9), P_{88}^5(9), P_{101}^5(9), P_{104}^5(9), P_{105}^5(9), P_{109}^5(9), P_{111}^5(9), P_{122}^5(9) \) and \( P_{124}^5(9) \) are the orbit spaces for 1 small cover.

\( P_{25}^5(9), P_{32}^5(9), P_{34}^5(9), P_{35}^5(9), P_{41}^5(9), P_{49}^5(9), P_{58}^5(9), P_{65}^5(9), P_{72}^5(9), P_{76}^5(9), P_{79}^5(9), P_{81}^5(9), P_{102}^5(9) \) and \( P_{119}^5(9) \) are the orbit spaces for 2 small covers.

\( P_{5}^5(9), P_{7}^5(9), P_{19}^5(9), P_{29}^5(9), P_{31}^5(9), P_{43}^5(9), P_{55}^5(9), P_{56}^5(9), P_{67}^5(9), P_{68}^5(9), P_{69}^5(9), P_{73}^5(9), P_{83}^5(9), P_{107}^5(9), P_{115}^5(9) \) and \( P_{120}^5(9) \) are the orbit spaces for 3 small covers.

\( P_{10}^5(9), P_{12}^5(9) \) and \( P_{35}^5(9) \) are the orbit spaces for 4 small covers.

\( P_{45}^5(9), P_{109}^5(9), P_{116}^5(9) \) and \( P_{118}^5(9) \) are the orbit spaces for 5 small covers.

\( P_{1}^5(9), P_{28}^5(9), P_{34}^5(9), P_{54}^5(9) \) and \( P_{85}^5(9) \) are the orbit spaces for 6 small covers.

\( P_{8}^5(9), P_{15}^5(9), P_{47}^5(9), P_{50}^5(9), P_{113}^5(9) \) and \( P_{117}^5(9) \) are the orbit spaces for 7 small covers.

\( P_{97}^5(9), P_{114}^5(9) \) and \( P_{123}^5(9) \) are the orbit spaces for 8 small covers.

\( P_{13}^5(9), P_{89}^5(9), P_{94}^5(9), P_{98}^5(9) \) and \( P_{125}^5(9) \) are the orbit spaces for 10 small covers.

\( P_{57}^5(9) \) is the orbit space for 12 small covers.

\( P_{112}^5(9) \) is the orbit space for 18 small covers.

In the following theorem we verify the lifting conjecture for neighborly simple 5-polytopes with 9 facets.

Theorem 5.84. The lifting conjecture holds for all neighborly simple 5-polytopes with 9 facets.

\[
a_9[P_{125}(9)] = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \end{vmatrix} \quad \text{and} \quad a_{10}[P_{125}(9)] = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \end{vmatrix} .
\]
Proof: The conjecture obviously holds for polytopes not admitting a real characteristic map. The same matrices representing the small covers over neighborly simple 5-polytopes with 9 facets viewed with $\mathbb{Z}$-coefficients are the characteristic matrices of quasitoric manifolds, except for $M^5(a_1[P^5_{26}(9)])$, $M^5(a_3[P^5_{54}(9)])$, $M^5(a_1[P^5_{64}(9)])$, $M^5(a_1[P^5_{97}(9)])$, $M^5(a_1[P^5_{101}(9)])$, $M^5(a_1[P^5_{118}(9)])$ and $M^5(a_3[P^5_{118}(9)])$.

However, it turns out they are respectively the fixed points of conjugation subgroup of $T^6$ for quasitoric manifolds $M^5(\tilde{a}_1[P^5_{125}(9)])$, $M^5(\tilde{a}_3[P^5_{54}(9)])$, $M^5(\tilde{a}_1[P^5_{64}(9)])$, $M^5(\tilde{a}_1[P^5_{97}(9)])$, $M^5(\tilde{a}_1[P^5_{101}(9)])$, $M^5(\tilde{a}_1[P^5_{118}(9)])$ and $M^5(\tilde{a}_3[P^5_{118}(9)])$ given with their characteristic matrix respectively

\[
\begin{align*}
\tilde{a}_1[P^5_{26}(9)] &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{vmatrix}, \\
\tilde{a}_3[P^5_{54}(9)] &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{vmatrix}\end{align*}
\]

\[
\begin{align*}
\tilde{a}_1[P^5_{64}(9)] &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{vmatrix}, \\
a_1[P^5_{97}(9)] &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{vmatrix}\end{align*}
\]

\[
\begin{align*}
\tilde{a}_1[P^5_{101}(9)] &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{vmatrix}, \\
a_1[P^5_{118}(9)] &= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{vmatrix}\end{align*}
\]

and \(\tilde{a}_3[P^5_{118}(9)]\) is given in a similar manner.

\[\square\]

5.1 Neighborly 5-polytopes with 10 facets

Our computer search found that 36015 out 159374 of simple neighborly 5-polytopes with 10 facets admit a real characteristic map (see [33]). Complete classification of small covers using the methods above is of course possible, but due to large numbers difficult to be done by hand. The results are available at [33].

6 Neighborly 6-polytopes

The cases of duals of neighborly 6-polytopes with 7, 8 and 9 vertices are already solved. By [22, Corollary 6.5] there are no small covers over the dual of $C^6(9)$ which is the only neighborly 6-polytope with 9 facets. There are 37 combinatorially different neighborly 6-polytopes with 10 facets, [16] and [17]. By computer search and following the notation of [18] we find that all of them except $P^6_6(10)$ do not allow any real characteristic map.
Proposition 6.1. $z_2 X_{P_6^6(10)}$ has exactly 1 element and it is represented by the matrix

$$
\lambda = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.
$$

Theorem 6.1. There is exactly 1 small cover $M^6(\lambda)$ over $P_6^6(10)$ and no small covers over other 36 neighborly 6-polytopes with 10 facets.

It is straightforward to see that $\Lambda$ from Proposition 6.1 can serve as the characteristic matrix with $\mathbb{Z}$ coefficients, and so:

Corollary 6.1. The lifting conjecture is true for all neighborly 6-polytopes with 10 facets.

Corollary 6.2. All simply neighborly 6-polytopes with 10 facets are weakly cohomologically $\mathbb{Z}_2$ rigid.

7 Neighborly 7-polytopes

Small covers and quasitoric manifolds over simple neighborly 7-polytopes with 8, 9 and 10 are already studied. We recall that by [22] there is no characteristic man over $(C^7(9))^*$. Our computer search found that 108 out of 35993 distinct simple neighborly 7-polytopes with 11 facets admit some characteristic map. The characteristic matrices and the polytopes are available at [33].

8 Applications to higher dimensions

Finding a combinatorial condition for a simple polytope to admit a characteristic map is one of the most attractable open problem in toric topology. Small covers and quasitoric manifolds are interesting class of manifolds and although we understand the whole classes of those manifolds such as generalized Bott manifolds, in general our knowledge on the orbit spaces of these manifolds is still incomplete. In this section we illustrate few new examples in higher dimensions showing that small covers and quasitoric manifolds can exist even over polytopes having ‘high’ chromatic numbers compared to the dimension of polytope.

Buchstaber and Ray’s [9, Proposition 4.7] says that if $M^{dn}$ and $N^{dn}$ are $G^n_d$-manifolds over simple polytopes $P^m$ and $Q^n$ then $M^{dn} \times N^{dn}$ is a $d(m + n)$-dimensional $G^n_d$-manifold over the polytope $P^m \times Q^n$.

It is straightforward to prove that $\chi(P^m \times Q^n) = \chi(P^m) + \chi(Q^n)$ for simple polytopes $P^m$ and $Q^n$.

Using these two facts we have that the polytope $(P_{24058}^4(12))^k = P_{24058}^4(12) \times \ldots P_{24058}^4(12)$ has the chromatic number equal to $12k$ and is the orbit space of $G^n_d$-manifold $M^{12dk} = (M^{dk}(a_1[P_{24058}^4(12)]))^k$. Thus, for $n = 4k$, there is a polytope with chromatic number $3n$ admitting a characteristic map.

If $n = 4k + 3$ one can take the polytope $(P_{24058}^4(12))^k \times \Delta^2$ admitting a characteristic map and having chromatic number equal to $12k + 4 = 3n - 5$. If $n = 4k + 2$ one can take the polytope $(P_{24058}^4(12))^k \times \Delta^2$ which is also the orbit space of a quasitoric manifold and a small cover whose
chromatic number is equal to $12k + 3 = 3n - 3$. If $n = 4k + 1$ one can take the polytope $(P_{24058}^{12})^{k-1} \times \Delta^1$ which is also the orbit space of a quasitoric manifold and a small cover whose chromatic number is equal to $12k + 1 = 3n - 2$.

**Corollary 8.1.** For every $n \in \mathbb{N}$, $n \geq 2$ there is a simple polytope $P^n$ with $\chi(P^n) \geq 3n - 5$ which is the orbit space of a quasitoric manifold.

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