A dynamical approach to the cosmological constant

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1. Introduction

Recent observations of large scale structure, such as the Hubble diagram of type 1a supernovae (SNe) and the angular power spectrum of the cosmic microwave background (CMB), etc., indicate that the universe is flat and in a stage of accelerated expansion [1,2]. One immediate conclusion is that the universe is dominated by a form of matter with negative pressure which is widely referred to as dark energy today. Nowadays, nearly all observational data are in agreement with the simplest possibility of the dark energy being a cosmological constant Λ, characterized by a pressure equal in magnitude and opposite in sign to the energy: pΛ = −ρΛ. Its energy density and pressure are strictly constant throughout space and time. However, there are still several issues unsolved with the introduction of this theory. The first issue is known as the coincidence problem. Evidence from observations suggests that the matter density ρm and the dark energy density ρΛ have similar values today. As to the cosmological constant Λ, the energy density ρΛ associated with the cosmological constant remains constant while the energy density in matter ρm decreases as ρm ∝ a−3 during the expansion of the universe. Accordingly, it has been only a very small period of time when these two densities are within an order of magnitude of each other. It is difficult to explain why it is comparable with the critical density today. This becomes the ‘coincidence problem’. For this coincidence to happen, it appears that the ratio must be set to a specific, infinitesimal value in the very small period of time when these two densities are within an order of magnitude of each other. It is difficult to explain why it is comparable with the critical density today. This becomes the ‘coincidence problem’. For this coincidence to happen, it appears that the ratio must be set to a specific, infinitesimal value in the very early universe. The second issue is the infamous cosmological constant problem. The energy density of the cosmological constant Λ is unchanged during the expansion of the universe, which suggests that it can be identified as a vacuum energy density. However, the vacuum energy, according to relative theory, should be a value that is 120 orders of magnitude greater than the energy density required for the cosmological constant. The absence of a fundamental mechanism which can set the cosmological constant to zero or to a very small value is the cosmological constant problem. One candidate to explain these problems is the quintessence, which is a slowly rolling scalar field ϕ and may have a potential [3]. For the quintessence field, the equation of state is typically a function of redshift ωϕ(z) = pϕ/ρϕ, whose value differs from −1. It seems that the contribution of this scalar field leads the universe to reach the critical energy and to accelerate its expansion [1]. Time varying models of dark energy, to a certain extent, can ameliorate the fine tuning problem (faced by Λ), but do not resolve the puzzle of cosmic coincidence.

If the explanation for the accelerating Universe ultimately fits within Einstein’s framework, it will be a stunning new triumph for General Relativity [4]. However, in most cases the occurrence of a self-interaction potential for the scalar field makes it difficult to solve the field equations analytically, although some techniques for deriving solutions have been developed. In this Letter, we studied the dynamical behavior of a simple cosmological model in presence of both nonrelativistic matter and scalar field by applying the method developed in our previous paper [5]. The exact analytic cosmological solution of the Einstein’s equations is derived by assuming a particular relation between the time derivative of the scalar field and that of the Hubble function. The evolution of the expansion scale factor and its potential are then determined with this assumption. A dynamical solution to the cosmological constant problem is introduced. In this method, Einstein’s cosmological constant Λ appears as an integration constant. The pressure of the...
dark energy behaves as Einstein’s cosmological constant, but the dark energy density decreases as the universe expands. Unlike the cosmological constant \( \Lambda \) which is ascribed to vacuum energy, the dark energy density and the energy density of the ordinary matter decrease at same rates during the expansion of the universe. Therefore, the model is free of the coincidence problem. Comparing such model with the current cosmological observations shows that its predictions are consistent with all astrophysical observations.

2. The cosmological dynamics of scalar-field dark energy model

We will consider a scalar field with a potential energy density \( V(\phi) \) evolving in a spatially-flat Friedmann–Robertson–Walker (FRW) universe containing the energy density of the ordinary matter \( \rho_m \). We use the system of units in which \( 8 \pi G = c = 1 \).

The evolution equations for the flat FRW cosmology read

\[
3H^2 = \rho_\phi + \rho_m, \tag{1}
\]

\[
2\dot{H} + 3H^2 = -p_\phi, \tag{2}
\]

where overdot denotes time derivative and \( H = \dot{a}/a \) is the Hubble parameter. In Eqs. (1) and (2), \( \rho_\phi \) and \( p_\phi \) represent the energy density and pressure of the scalar field respectively:

\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \tag{3}
\]

\[
p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi). \tag{4}
\]

The evolution equation for scalar field is

\[
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{5}
\]

where prime denotes the derivative with respect to \( \phi \). In the case of dust, the energy density of the ordinary matter \( \rho_m \) (including both baryon matter and cold dark matter) is related with the scale factor through

\[
\rho_m = \rho_m0 \left( \frac{a_0}{a} \right)^3, \tag{6}
\]

where the lower index “0” indicating present day values of the corresponding quantity. It is worth stressing that this model cannot be used from the very beginning of the universe, but only since decoupling of radiation and dust.

In general, the Hubble parameter \( H \) will not be exactly constant, but will vary as the field \( \phi \) evolves along the potential \( V(\phi) \). A more efficient approach to the more general case is to express the Hubble parameter directly as a function of the scalar field \( \phi \), \( H = H(\phi) \) [6]. From Eqs. (2) and (4) we obtain

\[
\dot{\phi} = -2H\pm \Phi(\phi), \tag{7}
\]

where

\[
\Phi(\phi) \equiv \sqrt{4H^2 + 2V - 6H^2}. \tag{8}
\]

Eq. (8) can be inverted and rewritten as

\[
V(\phi) = 3H^2 - 2H^2 + \frac{1}{2} \dot{\phi}^2. \tag{9}
\]

Let us combine Eq. (1) with Eq. (2) to obtain

\[
2\dot{H} + \rho_m + \phi^2 = 0. \tag{10}
\]

From Eqs. (1), (5)–(7) and (9)–(10), we have

\[
2(H' + \Phi)\Phi' + (2H^2 - 3H)\Phi = 0. \tag{11}
\]

Now this differential equation convolves two time-dependent functions: \( H(\phi) \) and \( \Phi(\phi) \) (and their derivatives). In general, it is impossible to solve both of them. To solve the field equations analytically, we assume the following relationship

\[
\Phi(\phi) = \pm (2 - \alpha)H', \tag{12}
\]

instead of assuming a particular potential for the scalar field or a particular form of the scale factor [7], where \( \alpha \) is a positive constant. It is obviously that a simple solution of Eq. (11) is found to be

\[
2\alpha H'' - 3H = 0, \tag{13}
\]

Eq. (13) can be easily integrated to give

\[
H(\phi) = Ce^{\sqrt{3/(2\alpha)\phi}} + Be^{-\sqrt{3/(2\alpha)\phi}}, \tag{14}
\]

where \( C \) and \( B \) are positive integration constants. Substituting Eq. (12) into Eq. (7), we get

\[
\dot{\phi} = -\alpha H'. \tag{15}
\]

For the quintessence field, great effort has been made to determine the appropriate scalar potential that could explain current cosmological observations. From Eqs. (9), (12) and (14), the form of the potential takes the form as

\[
V(\phi) = \frac{3\alpha}{4}(Ce^{\sqrt{3/(2\alpha)\phi}} - Be^{-\sqrt{3/(2\alpha)\phi}})^2 + \Lambda, \tag{16}
\]

where \( \Lambda = 12CB \). Such potential is a common functional form for the self-interaction potential [8]. It is to be found in higher dimension theories, supergravity and superstring models [9].

A reliable model should explain why the present amount of the dark energy is so small compared with the fundamental scale (fine-tuning problem) and why it is comparable with the critical density today (coincidence problem). From Eqs. (3), (9), (12) and (15) we have

\[
\rho_\phi = 3H^2 + \alpha(\alpha - 2)H^2. \tag{17}
\]

The energy density of the ordinary matter is given by

\[
\rho_m = \alpha(2 - \alpha)H^2. \tag{18}
\]

From Eqs. (14) and (18), we get

\[
3H^2 = \frac{2}{2-\alpha} \rho_m + \Lambda. \tag{19}
\]

In the case of dust, the energy density of the ordinary matter \( \rho_m \) varies with the expansion of the universe as \( \rho_m = \rho_m0(1 + z)^3 \), where the redshift \( z \) is defined by \( 1 + z = a_0/a \). In terms of redshift, the expression of the energy density of the scalar field is given by

\[
\rho_\phi = \frac{\alpha\rho_m0}{2-\alpha}(1 + z)^3 + \Lambda. \tag{20}
\]

From Eq. (20) it is clear that the dark energy density decreases monotonously as \( a^{-3} \) as universe is expanding. Eq. (20) also implies that the dark energy density and the energy density of the ordinary matter decrease at same rates during the expansion of the universe. Thus, the model is free of the coincidence problem.

As is well known, there is no convincing fundamental physics idea for why the dark energy dominance happened only recently. From Eqs. (4), (9), (12), and (14), (15) the expression of the pressure of the scalar field is given by

\[
p_\phi = -\Lambda = \text{const.} \tag{21}
\]

Eqs. (20) and (21) show that the positive energy density of the ordinary matter, together with the positive energy density of the scalar field, tend to decrease. But the negative pressure of the scalar field holds constant as the universe is expanding. These properties allow the scalar field to eventually dominate the energy of the universe, giving rise to a late-time epoch of accelerated expansion. In this model the dark energy behaves like a matter with small negative pressure at early times, but drives acceleration of the universe at the current epoch.
3. A dynamical approach to the cosmological constant

In Ref. [10] they find a very good agreement between supernovae observations and the results from baryon acoustic oscillation peak (BAO) for $\Omega_{m0} \approx 0.276 \pm 0.023$, where $\Omega_{m0} = \rho_{m0}/3H_0^2$ refers to densities at the present day ($z = 0$) in units of the critical density. The estimated values of $\Omega_{m0}$ are completely model-independent and are only based on observational data. Since at present, the accepted value of the Hubble constant $H_0$ is $(72 \pm 8) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [11]. To be more quantitative in all the calculations we select $\Omega_{m0} = 0.27$, the present value of the Hubble parameter $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ with $h = 0.70$ and $\alpha = 0.2$.

While the cosmological constant $\Lambda$ is so incredibly tiny that particle physicists have traditionally assumed that it must be mathematically zero, observational cosmologists steadily refined their measurements and changed their upper bound to an approximate equality $\Lambda \sim (10^{-5} \text{ eV})^4$ [12]. It is difficult to find a mechanism which gives the value of the cosmological constant, as the supernovae observations suggest. From Eq. (19), the constant $\Lambda$ is given by

$$\Lambda = 3H_0^2 \left(1 - \frac{2\Omega_{m0}}{2 - \alpha}\right) = (2.3 \times 10^{-3} \text{ eV})^4.$$  \hfill (22)

It behaves as the cosmological constant in Einstein’s equations. Eq. (22) means that in our model Einstein’s cosmological constant $\Lambda$ appears as an integration constant.

It is well known that a very small current value of $\rho_{\phi}$ is in agreement with observations. In terms of Eq. (20) we have

$$\rho_{\phi} = \frac{1}{9} \rho_{m0}(1 + z)^3 + \Lambda.$$  \hfill (23)

From Eq. (23) we obtain $\rho_{\phi}$ had a large value at an early time ($z \to \infty$). At the present epoch, the redshift $z$ is zero, we have $\rho_{\phi} \approx \Lambda$. It is shown that the energy density of the scalar field can be very small now and tends to the value $\Lambda$.

4. Comparing with astrophysical observations

In the absence of compelling theoretical guidance, there is a simple way to parameterize dark energy, by its equation of state $\omega_{\phi}$. As is well known, the simplest model of the quintessence is to assume a constant $\omega_{\phi}$. However, in our model the dark energy $\omega_{\phi}$ parameter evolves as a function of the redshift according to

$$\omega_{\phi} = \frac{\rho_{\phi}}{\rho_m} = -\frac{2 - \alpha - 2\Omega_{m0}}{\alpha\Omega_{m0}(1 + z)^3 + 2 - \alpha - 2\Omega_{m0}}.$$  \hfill (24)

In a flat universe, the combination of WMAP and the Supernova Legacy Survey (SNLS) data yield a significant constraint on the equation of state of the dark energy, $\omega = -0.97 \pm 0.09$ [13]. From Eq. (24) we find out the present value of $\omega_{\phi}$ is $\omega_{\phi} = -0.96$. Moreover, the behavior of the evolving dark energy can determine the fate of the universe. From Eq. (24) we obtain that $\omega_{\phi}$ approaches arbitrarily close to $-1$ as the scalar field continues to evolve. So, our universe will experience an eternal acceleration.

Using Eq. (2) the deceleration parameter $q = -\ddot{a}/aH^2$ is determined by

$$q = \frac{1}{2} \left( \frac{\Lambda}{H^2} - 1 \right).$$  \hfill (25)

from which one can get the deceleration parameter today is $q_0 = -0.55$. The universe switched from the deceleration phase to the acceleration phase when $q = 0$. From Eq. (25) we obtain that the transition from a decelerating towards an accelerating universe is at the redshift of $z_r = 0.67$. The acceleration of the universe is therefore a very recent phenomenon.

In terms of Eq. (20) the radio of the dark energy density to the matter density is

$$\frac{\rho_{\phi}}{\rho_m} = \frac{\alpha}{2 - \alpha} + \left[ \frac{1}{\Omega_{m0}} - \frac{2 - \alpha}{\alpha} \right] \frac{1}{1 + z}.$$  \hfill (26)

It is shown that the dark energy density evolves rapidly and is dynamically unimportant even at moderate redshift (e.g., $\rho_{\phi}/\rho_m = 0.2$ at $z = 2$). Thus, the early structure ($z > 1$) is unaffected, but structure will stop growing sooner [14]. It is obvious that the negative constant pressure of the scalar field can drive the accelerated expansion, but also weakly enough so that the structure formation scenario does not get spoiled in recent past.

The current central question in cosmology research is what theoretical models are consistent with the currently detected form of $H(z)$. The simplest model consistent with the currently detected form of $H(z)$ is the flat cosmological constant model. In this model, it predicts an expansion history of the universe which is described by a Hubble parameter $H(z)$ as a function of the redshift $z$ given by

$$H(z) = H_0 \left[ \Omega_{m0}(1 + z)^3 + \Omega_{\Lambda} \right]^{1/2},$$  \hfill (27)

where $\Omega_{\Lambda}$ is a constant density due to the cosmological constant $\Lambda$ and $H_0$ is the present value of the Hubble parameter. Its disadvantage is lack of theoretical motivation and fine tuning. From Eq. (19) the expression of the Hubble parameter is given by

$$H(z) = H_0 \left[ \frac{10}{9} \Omega_{m0}(1 + z)^3 + \left(1 - \frac{10}{9} \Omega_{m0}\right) \right]^{1/2}. \hfill (28)$$

It is similar as Eq. (27). So our model is a dynamical approach to the cosmological constant and is going to be in agreement with the observed value of any quantity depending only on the expansion history.

The fundamental test of the background dynamics of a cosmological model is the SNe magnitude-redshift, based on the luminosity distance [15]. For a flat universe the luminosity distance $d_L$ as a function of the redshift is given by

$$d_L = (1 + z) \int_0^z \frac{dz'}{H(z')}.$$  \hfill (29)

The redshift distance $r(z_{\text{dec}})$ is

$$r(z_{\text{dec}}) = \int_0^{z_{\text{dec}}} dz \frac{H(z)}{H(z)}, \hfill (30)$$

where $z_{\text{dec}} = 1100$. The currently measured value is $r(z_{\text{dec}}) = 13.7 \pm 0.5 \text{ Gpc}$ [16]. Using Eqs. (28) and (30) we can derive $r(z_{\text{dec}}) = 13.69 \text{ Gpc}$.

For the CMB data, the shift parameter $S$ can be used to constrain the dark energy models and is given by

$$S = \sqrt{\Omega_{m0} H_0} \frac{d_L(z_r)}{1 + z_r}.$$  \hfill (31)

Wang and Mukherjee [17] have used the WMAP 3-year data [13] to derive $S = 1.70 \pm 0.03$ with $z_r = 1090$. Using Eqs. (29) and (31) we can obtain $S = 1.66$.

Recently Eisenstein et al. [18] successfully found the size of baryonic oscillation peak employing a large spectroscopic sample of luminous red galaxy from the SDSS and obtained a parameter $A$, which is independent of dark energy models. A parameter $A$ is given by [15]

$$A = \sqrt{\Omega_{m0} \left[ \frac{H_0^2 d_L^2(z_1)}{H_1 z_1^2 (1 + z_1)^2} \right]^{1/3}}, \hfill (32)$$
and Eisenstein et al. [18] found that $A = 0.469 \pm 0.017$ where $z_1 = 0.35$. Adopting parameter $A$ we can obtain the constraint on dark energy models from the SDSS. From Eqs. (28), (29) and (32) we find $A = 0.462$.

The age of universe is another observable parameter that can be used to constrain the parameters of dark energy models. Richer et al. [19] and Hansen et al. [20] proposed an age of $12.7 \pm 0.7$ Gyr, by applying the white dwarf cooling sequence method. The inflationary and radiation-dominated eras is so tiny compared with the matter-dominated period that we could assume that $z \to \infty$ when $t = 0$. The age of the universe at redshift $z$ is given by

$$t_z = \int_{z}^{\infty} \frac{dz'}{(1+z')H(z')}.$$  

(33)

At the present epoch, the redshift $z$ is zero. Using Eq. (28), the age of universe at the present time is $t_0 = 13.5$ Gyr. It is fully consistent with the recent estimation of $13.7 \pm 0.2$ Gyr reported by WMAP team [16].

The SNe results show that although the universe is undergoing accelerated expansion now, this has not always been the case. Up until about 5 billion years, the universe was matter dominated and was decelerating. Using $z_T = 0.67$ we find that the age of the universe when it starts to accelerate is $t_T = 7.9$ Gyr, i.e., the Universe had undergone the transition from deceleration phase to acceleration phase 5.6 Gyrs ago. These values are reasonable and in good agreement with the recent estimation of WMAP team [16].

As commented in the introduction, the $\Lambda$-term model is in agreement with all the available observational data. Our model with $\Omega_{m0} = 0.27$ and $\Omega_{\phi0} = 0.73$ is practically equivalent to a $\Lambda$-term model with $\Omega_{m0} = 0.3$ and $\Omega_{\Lambda} = 0.7$ in accounting for present day observations. It is well known that the parameter $\Omega_{m0}$ is very important, not only because of its physical meaning, but also because its value can, in principle, be measured independently from the SNe observations [21]. So distinguishing our model from the $\Lambda$-term model by means of observational data of the parameter $\Omega_{m0}$ is possible.

5. Conclusions

The dynamical behavior of a simple cosmological model is discussed in presence of both nonrelativistic matter and scalar field. The exact solution of the Einstein’s equations is found by assuming a particular relation between the time derivative of the scalar field and that of the Hubble function. The evolution of the expansion scale factor and its potential are then determined. In this Letter, we have discussed a dynamical solution to the cosmological constant problem. The analysis has shown that Einstein’s cosmological constant $\Lambda$ appears as an integration constant. The pressure of the dark energy behaves as Einstein’s cosmological constant, but the dark energy density decreases during the expansion of the universe. Unlike Einstein’s cosmological constant $\Lambda$, which is ascribed to vacuum energy, the dark energy density and the energy density of the ordinary matter decrease at same rates as the universe expands. Thus, the model is free of the coincidence problem.

This model allows a very stringent comparison with experimental data on supernovae. In our model, there are only two free independent parameters, namely the present Hubble parameter $H_0$ and the matter density $\Omega_{m0}$. In terms of $H_0$ and $\Omega_{m0}$, the present age of the universe is obtained as $t_0 = 13.5$ Gyr. In particular, the present values of observational interest are derived as $\Omega = (2.3 \times 10^{-3} \text{ eV}^4$ and $q_0 = -0.55$. Moreover, the universe began accelerating only recently, at redshifts about 0.67. Comparing such model with the current cosmological observations shows that its predictions are consistent with all astrophysical observations.

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