The exceptional set in a generalized Goldbach’s problem

Dongho Byeon* and Keunyoung Jeong

Abstract

In this paper, we compute the size of the exceptional set in a generalized Goldbach problem and show that for a given polynomial \( f(x) \in \mathbb{Z}[x] \) with a positive leading coefficient, positive integers \( A, B, g, i, j \) with \( 0 < i, j < g \), there are infinitely many positive integers \( n \) which satisfy \( 2f(n) = Ap_1 + Bp_2 \) for primes \( p_1 \equiv i, p_2 \equiv j \) (mod \( g \)) under a mild condition.

1 Introduction

Brüdern, Kawada and Wooley [BKW] computed the size of the exceptional set of a polynomial-type generalization of Goldbach problem.

Theorem 1. [BKW, Theorem 1] Let \( f(x) \in \mathbb{Z}[x] \) be a polynomial which has a positive leading coefficient with degree \( k \) and \( \mathcal{E}_k(N, f) \) be the number of positive integers \( n \) with \( 1 \leq n \leq N \) for which the equation \( 2f(n) = p_1 + p_2 \) has no solution in primes \( p_1, p_2 \). Then there is an absolute constant \( c > 0 \) such that

\[
\mathcal{E}_k(N, f) \ll f^{-\frac{c}{k}} N^{1-\frac{c}{k}}.
\]

This theorem implies that there are infinitely many positive integers \( n \) which satisfy \( 2f(n) = p_1 + p_2 \) for primes \( p_1, p_2 \). Similarly, one can ask if for given positive integers

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A, B, g, i, j with 0 < i, j < g, there are infinitely many positive integers n which satisfy $2f(n) = A p_1 + B p_2$ for primes $p_1 \equiv i$, $p_2 \equiv j \pmod{g}$.

To answer this question, we will prove the following theorem.

**Theorem 2.** Let $f(x) \in \mathbb{Z}[x]$ be a polynomial which has a positive leading coefficient with degree $k$. Let $A, B$ be positive odd integers and $g, i, j$ positive integers with $0 < i, j < g < N^{24k}$ for a sufficiently small positive real number $\delta$ to be chosen later and $(i, g) = (j, g) = 1$. Suppose that there is at least one integer $m$ such that

$$2f(m) \equiv Ai + Bj \pmod{g}.$$ 

Let $\Gamma = \{A, B, g, i, j\}$ and let $E_{k, \Gamma}(N, f)$ be the number of positive integers $n \in [1, N]$ with $2f(n) \equiv Ai + Bj \pmod{g}$ for which the equation $2f(n) = A p_1 + B p_2$ has no solution in primes $p_1 \equiv i, p_2 \equiv j \pmod{g}$. Then there is an absolute constant $c > 0$ such that

$$E_{k, \Gamma}(N, f) \ll k, \Gamma N^{1 - c/k}.$$ 

This immediately implies the positive answer of the above question.

**Corollary 3.** Let $f(x) \in \mathbb{Z}(x)$ be a polynomial which has a positive leading coefficient.

Let $A, B$ be positive odd integers and $g, i, j$ positive integers with $0 < i, j < g$ and $(i, g) = (j, g) = 1$. If there is at least one integer $m$ such that

$$2f(m) \equiv Ai + Bj \pmod{g},$$

then there are infinitely many positive integers $n$ which satisfy

$$2f(n) = A p_1 + B p_2$$

for primes $p_1 \equiv i$, $p_2 \equiv j \pmod{g}$.

Let $N$ be a large positive integer, $\delta$ a sufficiently small positive real number to be chosen later, $X := 2f(N)$, $P := X^{6\delta}$, $Q := X/P$ and $\kappa := 2^{-\frac{1}{4}}$. Let $A, B$ be positive odd integers and $g, i, j$ positive integers with $0 < i, j < g < P^4$ and $(i, g) = (j, g) = 1$. Let $\Gamma = \{A, B, g, i, j\}$. We define the exponential sum $S_i(\alpha)$ by

$$S_i(\alpha) := \sum_{\substack{P < p \leq X \\ p \equiv i \pmod{g}}} (\log p)e(a p),$$
where \( e(ap) := e^{2\pi ap} \) and the summation is over primes \( p \) with \( P < p \leq X \) and \( p \equiv i \pmod{g} \). When \( T \subseteq [0,1] \), we write

\[
\Gamma(n; T) := \int_T S_i(Aa)S_j(Ba)e(-an)d\alpha
\]

and \( \Gamma(n) := \Gamma(n;[0,1]) \). Then \( \Gamma(2f(n)) \) counts the number of solutions of the equation \( 2f(n) = Ap_1 + Bp_2 \) in primes \( p_1 \equiv i, p_2 \equiv j \pmod{g} \) with weight \( \log p_1 \log p_2 \).

Let \( \mathfrak{M} \subset [0,1] \) be the major arc defined by

\[
\mathfrak{M} = \bigcup_{0 \leq a \leq q \leq P, (a,q) = 1} \mathfrak{M}(q,a),
\]

where

\[
\mathfrak{M}(q,a) = \{ \alpha \in [0,1] : \left| \alpha - \frac{a}{q} \right| \leq \frac{P}{qX} \},
\]

and \( m \subset [0,1] \) be the minor arc defined by

\[
m = [0,1] \setminus \mathfrak{M}.
\]

In Section 2 we compute \( \Gamma(2f(n);m) \), in Section 3 we compute \( \Gamma(2f(n);\mathfrak{M}) \), and in Section 4 combining these, we prove Theorem 2. Basically we follow [BKW] and [MV].

Finally we mention that some special forms of Corollary 3 are applied to the arithmetic of elliptic curves. See [BJ] and [BJK]. One of the aims of this paper is to give a full proof of a full generalization of the special forms for future applications.

## 2 Minor Arc

In [BKW, Lemma 1], the authors proved that there exists a positive real number \( a = a(\delta) \) depending on \( \delta \) such that

\[
\sum_{\kappa N \leq n \leq N} |r(2f(n);m)| \ll XN^{1-\delta},
\]

where \( r(2f(n);m) = \int_m S(\alpha)^2 e(-\alpha \cdot 2f(n))d\alpha \) and \( S(\alpha) = \sum_{p \leq X} \log p e(ap) \).

In this section, we show that the same result holds for \( \Gamma(2f(n);m) \). To do this, we need the following lemma which concerns the residue class condition; \( i, j \pmod{g} \) and the coefficient condition; \( A, B \). For the proof of the lemma, we follow the proof
of [IK, Theorem 13.6]. A new ingredient in our proof is the orthogonality relations of Dirichlet characters.

**Lemma 4.** Suppose that there exist integers \( a \) and \( q \) such that \( (a, q) = 1 \) and \( \left| a - \frac{a}{q} \right| \leq \frac{1}{q} \). Then for \( x \geq 2 \) we have

\[
\sum_{p \leq x \atop p \equiv i \pmod{g}} (\log p) e(aAp) \ll (x^\frac{2}{5} + xq^{-\frac{1}{2}} + x^\frac{1}{2} q^\frac{1}{2} )(\log x)^3,
\]

where the summation is over primes \( p \) with \( p \leq x \) and \( p \equiv i \pmod{g} \).

**Proof.** Let \( \chi \) be a Dirichlet character modulo \( g \). The orthogonality relations of Dirichlet characters imply that

\[
\sum_{p \leq x \atop p \equiv i \pmod{g}} (\log p) e(aAp) = \sum_{p \leq x} \frac{1}{\varphi(g)} \sum_{\chi} \chi(i)(\log p) e(aAp)
\]

\[
\ll \sum_{\chi} \left| \sum_{p \leq x} \chi(p)(\log p) e(aAp) \right|.
\]

Thus it is enough to show that

\[
\sum_{p \leq x} \chi(p)(\log p) e(aAp) \ll (x^\frac{2}{5} + xq^{-\frac{1}{2}} + x^\frac{1}{2} q^\frac{1}{2} )(\log x)^3.
\]

Let \( \Lambda(n) \) be the von Mangoldt function which defined as follows:

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k, \\
0 & \text{otherwise.}
\end{cases}
\]

By the fact \( \sum_{p \leq x} \log p \ll x \),

\[
\left| \sum_{p^k \leq x} \chi(p^k)(\log p) e(aAp^k) \right| \leq \sum_{p^k \leq x} \log p = \sum_{p \leq x^\frac{1}{k}} \log p \ll x^\frac{1}{k}.
\]

Hence

\[
\sum_{p \leq x} \chi(p)(\log p) e(aAp) = \sum_{n \leq x} \chi(n) \Lambda(n)e(aAn) + O(x^\frac{1}{k}).
\]

Thus it is enough to show that

\[
\sum_{n \leq x} \chi(n) \Lambda(n)e(aAn) = \sum_{n \leq Ax \atop \Lambda|n} \chi\left(\frac{n}{\Lambda}\right) \Lambda\left(\frac{n}{\Lambda}\right)e(an) \ll (x^\frac{4}{5} + xq^{-\frac{1}{2}} + x^\frac{1}{2} q^\frac{1}{2} )(\log x)^3.
\]

From the Vaughan’s identity, we have that for \( y, z \geq 1 \) and \( n \) such that \( A|n \) and \( \frac{n}{A} > z \),

\[
\Lambda\left(\frac{n}{A}\right) = \sum_{b | \frac{n}{A}} \mu(b) \log \frac{n}{Ab} - \sum_{b | \frac{n}{A}} \mu(b) \Lambda(c) + \sum_{b | \frac{n}{A}} \mu(b) \Lambda(c).
\]
Then
\[
\sum_{\substack{n \leq Ax \atop A/n}} \chi(n) \Lambda(n) e(an) = \sum_{lm \leq Ax} \sum_{m \leq M} \chi(l/m) \mu(m) (\log \frac{l}{A}) e(alm) - \sum_{lmn \leq Ax} \sum_{m \leq M, n \leq N} \chi(l/mn) \mu(m) \Lambda(n) e(almn) + O(N).
\]

We need
\[
\sum_{m \leq M} \left| \sum_{mn \leq Ax} \chi(n/m) e(amn) \right| \ll (M + xq^{-1} + q) \log 2q x.
\] (1)

It is derived as follow:
\[
\sum_{m \leq M} \left| \sum_{mn \leq Ax} \chi(n/m) e(amn) \right| = \sum_{m \leq M} \left| \sum_{i=0}^{g-1} \chi(i) \sum_{mn \leq x} e(amn') \right|
\leq g \sum_{m \leq M} \min\left(\frac{x}{mA}, 2\|mA\|\right) \leq g \sum_{m \leq AM} \min\left(\frac{x}{m}, \frac{1}{2\|m\|}\right)
\ll (AM + xq^{-1} + q) \cdot \log 2q Ax \ll (M + xq^{-1} + q) \cdot \log 2q x,
\]

where \(\|a\| = \min_{u \in \mathbb{Z}} |a - u|\). It is known [IK, Theorem 13.6] that
\[
\sum_{n \leq x} \Lambda(n) e(an) \ll (x^\frac{2}{3} + xq^{-\frac{1}{2}} + x^\frac{1}{2} q^{\frac{1}{2}})(\log x)^3.
\]

If we use (1) instead of (13.46) in the proof of [IK, Theorem 13.6] and take \(M = N = x^\frac{2}{3}\), then by the same argument in the proof of [IK, Theorem 13.6], we get
\[
\sum_{lm \leq Ax} \sum_{m \leq M} \chi(l/m) \mu(m) (\log \frac{l}{A}) e(alm) \ll (x^\frac{2}{3} + xq^{-1} + q) \log qx \cdot \log x,
\]
\[
\sum_{lmn \leq Ax} \sum_{m \leq M, n \leq N} \chi(l/mn) \mu(m) \Lambda(n) e(almn) \ll (x^\frac{4}{3} + xq^{-1} + q) \log qx \cdot \log x,
\]
\[
\sum_{lmn \leq Ax} \sum_{m \leq M, n \leq N} \chi(l/mn) \mu(m) \Lambda(n) e(almn) \ll (x^\frac{4}{3} + xq^{-\frac{1}{2}} + x^\frac{1}{2} q^{\frac{1}{2}})(\log x)^3,
\]
which prove the lemma.

Now we can prove the following analogue of [BKW, Lemma 1]. For the proof, we follow the proof of [BKW, Lemma 1]. A new ingredient in our proof is the bound of \(S_f(Aa)\) in Lemma [4].
Proposition 5. There is a positive real number \( a = a(\delta) \) such that

\[
\sum_{\kappa N < n \leq N} |r_\Gamma(2f(n); m)| \ll XN^{1-\frac{\delta}{2}}.
\]

Proof. By the Hölder inequality, we have

\[
\sum_{\kappa N < n \leq N} |r_\Gamma(2f(n); m)| \leq \sup_{a \in m} |S_i(Aa)S_j(Ba)|^{\frac{1}{t}} \left( \int_0^1 |S_i(Aa)S_j(Ba)|da \right)^{\frac{1}{t-1}} \left( \int_0^1 |K(-a)|^t da \right)^{\frac{1}{t}}.
\]

where

\[
K(a) = \sum_{\kappa N < n \leq N} \eta(2f(n))e(2f(n)a) \quad \text{and} \quad \eta(u) = \begin{cases} 1 & \text{if } r_\Gamma(u, m) \geq 0, \\ -1 & \text{otherwise}. \end{cases}
\]

By Lemma 4, we get for \( a \in m \),

\[
S_i(Aa) = \sum_{P < p \leq X \pmod{g}} (log p)e(aAp) \ll (X^\delta + Xq^{-\frac{1}{2}} + X^\frac{1}{2}q^\frac{1}{2})(log X)^{3} \ll X^{1-3\delta}(log X)^3.
\]

This implies that

\[
\sup_{a \in m} |S_i(Aa)S_j(Ba)| \ll [X^{1-3\delta}(log X)^3]^2.
\]

From the proof of [BKW, Lemma 1], we know that

\[
\int_0^1 |S_i(Aa)S_j(Ba)| da \ll XlogX \quad \text{and} \quad \int_0^1 |K(-a)|^t da \ll N^{t-k(1-\delta)}.
\]

By combining these bounds, we have

\[
\sum_{\kappa N < n \leq N} |r_\Gamma(2f(n); m)| \ll (XlogX)^{1-\frac{1}{t}N^{1-\frac{(1-\delta)k}{2}}(X^{1-3\delta}(log X)^3)^{\frac{1}{t}}}
\]

\[
\ll NX^{1-\frac{\delta}{2}}(log X)^2,
\]

which proves the lemma. \( \square \)

3 Major Arc

Let \( Y \) be a real number with \( 1 \leq Y \leq X^{\frac{\delta}{2}}. \) In [BKW, Lemma 2], the authors proved that for all \( n \) satisfying \( \kappa N < n \leq N \), with the possible exception of \( O(N^{1+\epsilon}Y^{-1}) \) values of \( n \)

\[
r(2f(n); \mathfrak{m}) \gg XY^{-\frac{1}{2}}(log X)^{-1},
\]
where \( r(2f(n); \mathfrak{M}) = \int_{\mathfrak{M}} S(a)^2 e(-a \cdot 2f(n)) \, da \) and \( S(a) = \sum_{p < p \leq X} (\log p) e(ap) \).

In this section, we show that the same result holds for \( r_\Gamma(2f(n); \mathfrak{M}) \). To do this, we need some lemmas which concern the residue class condition; \( i, j \mod g \) and the coefficient condition; \( A, B \).

First we state the basic properties of exceptional characters, which are established by Davenport [Dav].

**Lemma 6.** There is a constant \( c_1 > 0 \) such that \( L(\sigma, \chi) \neq 0 \) whenever

\[
\sigma \geq 1 - \frac{c_1}{\log P},
\]

for all primitive Dirichlet characters \( \chi \) of modulus \( q \leq P \), with the possible exception of at most one primitive character \( \tilde{\chi} \mod \tilde{r} \). If it exists, \( \tilde{\chi} \) is quadratic, and the unique exceptional real zero \( \tilde{\beta} \) of \( L(s, \tilde{\chi}) \) satisfies

\[
\frac{c_2}{r^2 \log^2 r} \leq 1 - \tilde{\beta} \leq \frac{c_1}{\log P},
\]

for a constant \( c_2 > 0 \).

The following lemma is a modification of [Gal, Theorem 7]. For the proof of the lemma, we follow the proof of [Gal, Theorem 7]. A new ingredient in our proof is the orthogonality relations of Dirichlet characters.

**Lemma 7.** Suppose that \( \frac{x}{P} \leq h \leq x \) and \( \exp(\log^{\frac{1}{2}} x) \leq P \leq x^b \). If there is no exceptional character, we have

\[
\sum_{q \leq P} \sum_{\chi \neq 1}^{\ast} \sum_{p \equiv i \mod g}^{x+h} \chi(p) \log p \ll h \exp(-c_3 \frac{\log x}{\log P})
\]

for a constant \( c_3 \), where \( \sum^\ast \) denotes that the sum is taken over all primitive Dirichlet characters of modulus \( q \) and if there is the exceptional character, the right hand side may be replaced by \( h(1 - \tilde{\beta}) \log P \exp(-c_3 \frac{\log x}{\log P}) \). Here the term with \( q = 1 \) is read as follows: if there is no exceptional character, it is

\[
\sum_{p \equiv i \mod g}^{x+h} \log p - \sum_{n \equiv i \mod g}^{x < n \leq x + h} 1
\]

and if there is the exceptional character, it is

\[
\sum_{p \equiv i \mod g}^{x+h} \tilde{\chi}(p) \log p + \sum_{n \equiv i \mod g}^{x < n \leq x + h} n^{\tilde{\beta} - 1}.
\]
Proof. Let
\[
\psi(x) := \sum_{n \leq x} \Lambda(n), \quad \psi(x, \chi) := \sum_{n \leq x} \chi(n) \Lambda(n), \quad \text{and} \quad \psi(x, \chi; i, g) := \sum_{n \leq x} \chi(n) \Lambda(n) \mod g.
\]
Using the orthogonality relations of Dirichlet characters, we have
\[
\psi(x, \chi; i, g) = \frac{1}{\varphi(g)} \sum_{n \leq x} (\sum_{\chi} \overline{\chi}(i) \chi(n)) \chi(n) \Lambda(n) = \frac{1}{\varphi(g)} \sum_{\chi} \overline{\chi}(i) \psi(x, \chi \cdot \chi'),
\]
where \(\chi'\) varies in the set of Dirichlet characters of modulus \(g\) and \(\chi \cdot \chi'(n) := \chi(n) \chi'(n)\).

For \(q \leq T \leq x^\frac{1}{2}\)
\[
\psi(x, \chi) = \delta_\chi x - \sum_{\rho} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right),
\]
where \(\delta_\chi = 1\) or 0 according to whether \(\chi = \chi_0\) or not, and the sum on the right is over the zeros \(\rho\) of \(L(s, \chi)\) in \(0 \leq \text{Re}(\rho) \leq 1, |\text{Im}(\rho)| \leq T\). By (2)
\[
\psi(x, \chi; i, g) = \frac{1}{\varphi(g)} \sum_{\chi} \overline{\chi}(i) (\delta_\chi x - \sum_{\rho} \frac{x^\rho}{\rho}) + O\left(\frac{x \log^2 x}{T}\right),
\]
where the second sum is over the zeros \(\rho\) of \(L(s, \chi \cdot \chi')\) in \(0 \leq \text{Re}(\rho) \leq 1, |\text{Im}(\rho)| \leq T\).

Since
\[
\psi(x + h, \chi; i, g) - \psi(x, \chi; i, g) = \sum_{p \equiv i \pmod{g}} \chi(p) \log p + O(x^{\frac{1}{2}}),
\]
by (3)
\[
\sum_{p \equiv i \pmod{g}} \chi(p) \log p = \frac{1}{\varphi(g)} \left(\sum_{\chi} \overline{\chi}(i) (\delta_\chi x - h - \sum_{\rho} \frac{(x + h)^\rho - x^\rho}{\rho})\right) + O\left(\frac{x \log^2 x}{T}\right).
\]
Thus
\[
\sum_{q \leq P} \sum_{\chi}^* \sum_{p \equiv i \pmod{g}}^{x+h} \chi(p) \log p \ll \sum_{q \leq P} \sum_{\chi}^* \sum_{\chi}^* h \sum_{\rho} \frac{x^{\rho-1} + P^2}{T},
\]
where the fourth sum of the right hand side is over the zeros \(\rho = \beta + \gamma i\) of \(L(s, \chi \cdot \chi')\) in \(0 \leq \text{Re}(\rho) \leq 1, |\text{Im}(\rho)| \leq T\). Let \(N_x(\alpha, T)\) be the number of zeros \(\rho\) of \(L(s, \chi)\) in the rectangle
\[
\{\rho \in \mathbb{C} : \alpha \leq \text{Re}(\rho) \leq 1, |\text{Im}(\rho)| \leq T\}.
\]
Then the quadruple sum on the right hand side of (4) is
\[
- \int_0^1 x^{\alpha-1} \frac{\partial}{\partial \alpha} (\sum_{q \leq P} \sum_{\chi}^* \sum_{\chi}^* N_{\chi}(\alpha, T)) d\alpha
= \int_0^1 x^{\alpha-1} \log x \sum_{q \leq P} \sum_{\chi}^* \sum_{\chi}^* N_{\chi}(\alpha, T) d\alpha + \frac{1}{x} \sum_{q \leq P} \sum_{\chi}^* \sum_{\chi}^* N_{\chi}(0, T).
\]
Consider the decomposition of a character $\chi$ of modulus $q = \prod_p p^{a_p}$

$$\chi = \prod_p \chi_{p^{a_p}},$$

where $\chi_{p^{a_p}}$ is a character of modulus $p^{a_p}$. Assume $g$ is a prime and $g|q$. Then the conductor of $\chi' \cdot \chi$ is $q$ except the case that $u_g(q) = 1$ and the $p$-parts of decompositions of $\chi$ and $\chi'$ are inverse each other, in this case the conductor of $\chi' \cdot \chi$ is $\frac{q}{g}$. Therefore,

$$\sum_{\chi}^* \sum_{\chi'}^* N_{\chi' \chi}(a,T) = (\varphi(g) - 1)(\sum_{\chi} N_{\chi,1}(a,T) + \sum_{\chi} N_{\chi,2}(a,T)),$$

where the first sum on the right hand side varies in the set of non-primitive characters of modulus $q$ which are induced by a primitive character of modulus $\frac{q}{g}$ and the second one on the right hand side varies in the set of primitive characters of modulus $q$. Let $\chi$ be a Dirichlet character of modulus $q$ induced by primitive character $\chi^*$ of modulus $q^*$. Then,

$$L(s, \chi) \prod_{p|q^*} \frac{1}{1 - \chi^*(p)p^{-s}} = L(s, \chi^*).$$

Each factor $\frac{1}{1 - \chi^*(p)p^{-s}}$ has a pole at

$$s = \frac{2\pi i}{\log p} (l + \frac{m_p}{\varphi(q^*)}),$$

where $m_p$ be the smallest positive integer satisfying $\chi^*(p) = e^{\frac{m_p}{\varphi(q^*)}}$ and $l$ be an integer such that $\frac{2\pi i}{\log p} (l + \frac{m_p}{\varphi(q^*)}) < T$. Therefore

$$N_{\chi}(a,T) = N_{\chi^*}(a,T) + c_{a,q,q^*,T},$$

where $c_{a,q,q^*,T} = 0$ if $a > 0$ and $c_{0,q,q^*,T} = \sum_{p|q} (\frac{\varphi(p)}{p}) + 1$. Hence, for a prime $g$

$$\sum_{\frac{q}{g} \leq P}^{*} \sum_{\chi} N_{\chi'}(a,T) = (\varphi(g) - 1)(\sum_{q \leq \frac{P}{g}} \sum_{\chi} c_{a,q,q^*,T} + \sum_{q \leq \frac{P}{g}} \sum_{\chi} N_{\chi}(a,T))$$

$$< 2\varphi(g) \sum_{q \leq \frac{P}{g}} \sum_{\chi} N_{\chi}(a,T) + \varphi(g) \sum_{q \leq \frac{P}{g}} \sum_{\chi} c_{a,q,q^*,T}.$$
where $d(m)$ be the number of divisors of $m$. Therefore if $a > 0$,

$$
\sum_{q \leq P} \sum_{\chi}^{*} N_{T} \chi'(a, T) = \sum_{q \leq P} \sum_{\chi}^{*} N_{T} \chi'(a, T) + \sum_{q \leq gP} \sum_{\chi}^{*} N_{T} \chi'(a, T) \\
\leq \sum_{q \leq gP} \sum_{\chi}^{*} N_{T} \chi(a, T) + d(g)^2 \varphi(g) \sum_{q \leq gP} \sum_{\chi}^{*} N_{T} \chi(a, T) \\
\ll \sum_{q \leq gP} \sum_{\chi}^{*} N_{T} \chi(a, T),
$$

and if $a = 0$ since $c_{0, q, \frac{P}{\pi}} \leq \frac{T \log m}{\pi} + d(m)$,

$$
\sum_{q \leq P} \sum_{\chi}^{*} N_{T} \chi'(0, T) \ll \sum_{q \leq gP} \sum_{\chi}^{*} N_{T}(0, T) + \sum_{q \leq gP} \sum_{\chi}^{*} \left( \varphi(g) \sum_{q \leq gP} \sum_{\chi}^{*} N_{T}(0, T) \right) + \sum_{q \leq gP} \sum_{\chi}^{*} N_{T}(0, T) + \frac{P^2 T}{x}.
$$

Thus (5) is

$$
\ll \int_{0}^{1} x^{a-1} \log x \sum_{q \leq gP} \sum_{\chi}^{*} N_{T} \chi(a, T) d\alpha + \frac{1}{x} \sum_{q \leq gP} \sum_{\chi}^{*} N_{T}(0, T) + \frac{1}{x} \sum_{q \leq gP} \sum_{\chi}^{*} N_{T}(0, T) + \frac{gP^2 T}{x}
$$

$$
\ll x^{-\theta(T)} + \frac{P^2 T}{x},
$$

where

$$
\theta(T) = \begin{cases} 
\frac{1}{\log T} & \text{if there is no exceptional character,} \\
\frac{c_2}{\log T} \log \frac{e \cdot c_1}{(1 - \beta) \log T} & \text{otherwise.}
\end{cases}
$$

For (6), we used [Gal, Theorem 6];

$$
\sum_{q \leq T} \sum_{\chi}^{*} N_{T} \chi(a, T) \ll T^{c(1 - a)}
$$

and assumed $T^c \leq x^{\frac{1}{2}}$ and $T > gP$. If we choose $T = P^5$ and $b = \frac{1}{10c}$, then $\frac{P^2 T}{x} \ll x^{-\frac{1}{2}}$ and the lemma follows.

From Lemma 7 and the argument below [MV, Lemma 4.3], we have the following modification of [MV, Lemma 4.3].

**Lemma 8.** If the exceptional character does not occur, there are positive absolute constants $c_4, c_5$ which satisfy

$$
\sum_{q \leq P} \sum_{\chi}^{*} \max \left( h + \frac{N}{P} \right)^{-1} \left| \sum_{x \leq \chi(p) \log p} \chi(p) \right| \ll \exp(-c_4 \frac{\log N}{\log P})
$$

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for \( \exp(\log^{\frac{1}{2}} N) \leq P \leq N^{c_3} \) and if the exceptional character occurs, the right hand side may be replaced by \((1 - \tilde{\beta}) \log P \exp(-c_6 \frac{\log N}{\log P})\). Here the term with \( q = 1 \) is read as follows: if there is no exceptional character, it is

\[
\sum_{x-h \leq p \leq x} \log p - \sum_{x - h < n \leq x \atop n \equiv i \pmod{g}} 1
\]

and if there is the exceptional character, it is

\[
\sum_{x-h \leq p \leq x} \tilde{\chi}(p) \log p + \sum_{x - h < n \leq x \atop n \equiv i \pmod{g}} n^{\tilde{\beta} - 1}.
\]

For a Dirichlet character \( \chi \) modulo \( q \), define

\[
S_i(\chi, \eta) := \sum_{P < p \leq X \atop p \equiv i \pmod{g}} (\log p) \chi(p) e(p\eta),
\]

and

\[
T_i(\eta) := \sum_{n \equiv i \pmod{g} \atop n \leq X} e(n\eta), \quad \tilde{T}_i(\eta) := - \sum_{P < n \leq X \atop n \equiv i \pmod{g}} n^{\tilde{\beta} - 1} e(n\eta),
\]

where the last one is defined only if there is an exceptional character.

Let \( \chi_0 \) be the principal character modulo \( q \). Define

\[
W_i(\chi, \eta) := \begin{cases} 
S_i(\chi, \eta) - T_i(\eta) & \text{if } \chi = \chi_0, \\
S_i(\chi, \eta) - \tilde{T}_i(\eta) & \text{if } \chi = \tilde{\chi}\chi_0, \\
S_i(\chi, \eta) & \text{otherwise.}
\end{cases}
\]

Suppose that a Dirichlet character \( \chi \pmod{q} \) is induced by a primitive character \( \chi^* \pmod{r} \). Put

\[
W_i^A(\chi) = \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} |W_i(\chi, A\eta)|^2 \, d\eta \right)^{\frac{1}{2}} \quad \text{and} \quad W_i^A = \sum_{q \leq P} \sum_{\chi} W_i^A(\chi).
\]

We note that \( W_i^A(\chi) = W_i^A(\chi^*) \). Then we have the following lemma which is a modification of [MV] (7.1).

**Lemma 9.** If the exceptional character does not occur, there is an absolute constant \( c_6 \) which satisfies

\[
W_i^A \ll X^{\frac{1}{2}} \exp(-c_6 \frac{\log X}{\log P}),
\]

and if the exceptional character occurs, the right hand side may be replaced by \( X^{\frac{1}{2}}(1 - \tilde{\beta}) \log P \exp(-c_6 \frac{\log X}{\log P}) \).
Proof. First assume that $\chi$ be a primitive character which is not equal to $\chi_0$ nor $\tilde{\chi}\chi_0$.

Then

$$W_i^A(\chi)^2 = \int_{\frac{1}{q^2}}^{\frac{1}{q}} \left| \sum_{\substack{P < p \leq X \atop p \equiv i \mod g}} \chi(p) \log p \cdot e(Ap\eta) \right|^2 d\eta.$$  

Applying [MV, Lemma 4.2] to the real numbers

$$u_n := \begin{cases} 
\chi(p) \log p & \text{if } n = Ap, P < p \leq X, p \equiv i \mod g, \\
0 & \text{otherwise,}
\end{cases}$$

we get

$$W_i^A(\chi)^2 \ll \int_{0}^{2AX} \left| \frac{1}{qQ} \sum_{\substack{x - \frac{qQ}{2} \leq Ap \leq x \atop p \equiv i \mod g}} \chi(p) \log p \right|^2 dx.$$  

Thus

$$W_i^A(\chi) \ll \left(2AX\right)^{\frac{1}{2}} \max_{x \leq 2AX} \left| \frac{1}{qQ} \sum_{x - \frac{qQ}{2} \leq Ap \leq x \atop p \equiv i \mod g} \chi(p) \log p \right|$$

$$\ll X^{\frac{1}{2}} \max_{x \leq 2AX} \left( \frac{Q}{2} \right)^{-1} \left| \sum_{x - \frac{qQ}{2} \leq Ap \leq x \atop p \equiv i \mod g} \chi(p) \log p \right|$$

$$\leq X^{\frac{1}{2}} \max_{x \leq 2AX} \max_{h \leq X} (Q + h)^{-1} \left| \sum_{x - h \leq Ap \leq x \atop p \equiv i \mod g} \chi(p) \log p \right| \quad (\because \frac{qQ}{2} \leq X)$$

$$\leq X^{\frac{1}{2}} \max_{x \leq 2AX} \max_{h \leq 2X} (2Q + h)^{-1} \left| \sum_{x - h \leq Ap \leq x \atop p \equiv i \mod g} \chi(p) \log p \right|.$$  

When $\chi = \chi_0$ or $\tilde{\chi}\chi_0$, $W_i(\chi, \eta)$ is exactly the term of the case of $q = 1$ in Lemma \[8\].

Hence

$$W_i^A = \sum_{q \leq P} \sum_{\chi}^* W_i^A(\chi)$$

$$\ll X^{\frac{1}{2}} \sum_{q \leq P} \sum_{\chi}^* \max_{x \leq 2X} \max_{h \leq 2X} \left( \frac{2X}{P} \right)^{-1} \left| \sum_{x - h \leq Ap \leq x \atop p \equiv i \mod g} \chi(p) \log p \right|.$$  

So the lemma is proved by Lemma \[8\].

Now we can prove the following analogue of [BKW, Lemma 2]. For the proof, we follow the proof of [BKW, Lemma 2]. New ingredients in our proof are the estimations of the bounds of new terms which do not appear in the proof of [BKW, Lemma 2], using Lemma 9 if needed.
Proposition 10. Suppose that \( Y \) is a real number with \( 1 \leq Y \leq X^{\frac{1}{2}} \). If there is at least one integer \( m \) such that

\[
2f(m) = Ai + Bj \pmod{g},
\]

then

\[
r_T(2f(n);\mathfrak{M}) \gg XY^{-\frac{1}{2}}(\log X)^{-1}
\]

for all \( n \in (\kappa N, N] \) with \( 2f(n) \equiv Ai + Bj \pmod{g} \) except at most \( O(N^{1+\epsilon}Y^{-1}) \) numbers.

Proof. Let \( n \in (\kappa N, N] \) be an integer with \( 2f(n) \equiv Ai + Bj \pmod{g} \). The proof is divided into three steps. In the first step, we will prove that if there is no exceptional character, then \( r_T(2f(n);\mathfrak{M}) \gg X \). Next we will show that even if there is the exceptional character, the same lower bound holds when \( (2f(n), \tilde{r}) = 1 \). Finally we will show that the number of integers \( n \) for which

\[
(2f(n), \tilde{r}) > 1 \text{ and } r_T(2f(n);\mathfrak{M}) \ll XY^{-\frac{1}{2}}(\log X)^{-1}
\]

is at most \( O(N^{1+\epsilon}Y^{-1}) \).

First we assume that there is no exceptional character. For \( \alpha \in \mathfrak{M}(q, a) \) we write \( \alpha = \frac{a}{q} + \eta \) for \( (a, q) = 1, \ 0 \leq \eta < 1 \) and \( q < P \). Let \( \chi \) be a Dirichlet character of modulus \( q \) and \( \tau(\chi) = \sum_{n=1}^{q} \chi(n)e(\frac{n}{q}) \) the Gaussian sum. Since \( e(\frac{a}{q}) = \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a)\tau(\chi) \), we have

\[
S_i(Aa) = \frac{\mu(q)}{\varphi(q)} T_i(A\eta) + \frac{1}{\varphi(q)} \sum_{\chi} \chi(Aa)\tau(\chi)W_i(\chi, A\eta).
\]

Thus

\[
r_T(2f(n);\mathfrak{M}) = \sum_{q \leq P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2f(n)) \left[ \int_{\frac{1}{q}}^{\frac{1}{2}} T_i(A\eta)T_j(B\eta)e(-2f(n)\eta) d\eta \right] (7)
\]

\[
+ \sum_{q \leq P} \frac{\mu(q)^2}{\varphi(q)^2} \sum_{\chi} \chi'(B)c_{\chi'}(-2f(n))\tau(\bar{\chi}') \left[ \int_{\frac{1}{q}}^{\frac{1}{2}} T_i(A\eta)W_j(\chi', B\eta)e(-2f(n)\eta) d\eta \right] (8)
\]

\[
+ \sum_{q \leq P} \frac{\mu(q)^2}{\varphi(q)^2} \sum_{\chi} \chi(A)c_{\chi}(-2f(n))\tau(\bar{\chi}) \left[ \int_{\frac{1}{q}}^{\frac{1}{2}} T_j(B\eta)W_i(\chi, A\eta)e(-2f(n)\eta) d\eta \right] (9)
\]

\[
+ \sum_{q \leq P} \left[ \frac{1}{\varphi(q)^2} \sum_{\chi, \chi'} \chi(A)\chi'(B)c_{\chi, \chi'}(-2f(n))\tau(\bar{\chi})\tau(\bar{\chi}') \right. 
\]

\[
\left. \times \left[ \int_{\frac{1}{q}}^{\frac{1}{2}} W_i(\chi, A\eta)W_j(\chi', B\eta)e(-2f(n)\eta) d\eta \right] \right] (10)
\]

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Thus condition \( q \gg 0 \) and by some elementary computations, we have

\[ \sum_{h=1}^{q} \chi(h) e \left( \frac{hm}{q} \right) \quad \text{and} \quad \sum_{h=1}^{q} \chi(h) e \left( \frac{hm}{q} \right). \]

Using \([MV, \text{Lemma 5.5}]\) and the same argument in \([MV, \text{Section 6}]\), we have

\[ \left( \sum_{h=1}^{q} \chi(h) e \left( \frac{hm}{q} \right) \right) W^A X_i^j \quad \text{and} \quad \left( \sum_{h=1}^{q} \chi(h) e \left( \frac{hm}{q} \right) \right) W^B X_j^i. \]

Let us compute the bound of (7). Since \( T_i(A\eta) \ll \frac{1}{\|gA\eta\|} \), by assuming the harmless condition \( qQ > 2gA \) and \( A \geq B \), we have

\[ \int_{1/\| g \|}^{1/\| gA \|} T_i(A\eta)T_j(B\eta)e(-2f(n)\eta)d\eta \ll \int_{1/\| g \|}^{1/\| gA \|} \int_{1/\| gB \|}^{1/\| gA \|} d\eta \ll \int_{1/\| g \|}^{1/\| gA \|} \frac{1}{g^2AB\eta^2}d\eta \ll qQ. \]

By some elementary computations, we have

\[ \int_{1/\| g \|}^{1/\| gA \|} T_i(A\eta)T_j(B\eta)e(-2f(n)\eta)d\eta - \sum_{1 \leq t \leq gA} \left( \sum_{P \leq k, l \leq X} \frac{1}{Ak + Bl - 2f(n)} [e(Ak + Bl - 2f(n)) - e(Ak + Bl - 2f(n))] \right) \]

\[ = \sum_{1 \leq t \leq gA} \left( \sum_{P \leq k, l \leq X} \frac{1}{Ak + Bl - 2f(n)} - \sum_{P \leq k, l \leq X} 1 \right) \]

\[ = O(\log X), \]

and

\[ \int_{1/\| g \|}^{1/\| gA \|} T_i(A\eta)T_j(B\eta)e(-2f(n)\eta)d\eta = \int_{1/\| g \|}^{1/\| gA \|} e((Ak + Bl - 2f(n))\eta)d\eta \]

\[ = \sum_{1 \leq t \leq gA} \frac{1}{g} = \frac{2f(n)}{g^2AB} + O(1). \]

Together with these three estimations, we have

\[ \int_{1/\| g \|}^{1/\| gA \|} T_i(A\eta)T_j(B\eta)e(-2f(n)\eta)d\eta = \frac{2f(n)}{g^2AB} + O(qQ). \]

Thus

\[ (7) = \sum_{q \neq p} \frac{\mu(q)^2}{q^2} c_q(-2f(n)) \left( \frac{2f(n)}{g^2AB} + O(qQ) \right). \]
Note that the last term appears when

\[ r_T(2f(n); \mathfrak{N}) = \mathcal{S}(2f(n)) \frac{2f(n)}{g^2 AB} + O(X^{1+\delta} P^{-1}) + O\left( \frac{2f(n)}{\varphi(2f(n))} (W_i X^{1/2} + W_j X^{1/2} + W_i W_j) \right), \]

where

\[ \mathcal{S}(n) = \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-n) = \prod_{p|n} (1 - \frac{1}{(p-1)^2}) \prod_{p|n} (1 + \frac{1}{p-1}). \]

In this equation, the first error term \( X^{1+\delta} P^{-1} \) is negligible. Also from Lemma 9, the second error term is less than \( \frac{6f(n)}{\varphi(2f(n))} X e^{-\frac{\alpha}{\log n}} \). If we choose a sufficiently small positive real number \( \delta \), then \( r_T(2f(n); \mathfrak{N}) \geq (1 - c_7) \mathcal{S}(2f(n)) f(n) \). These imply that \( r_T(2f(n); \mathfrak{N}) \gg X \), which is the conclusion of the first step.

Next, assume that there is the exceptional character. In this case, we have

\[ S_1(A\alpha) = \frac{\mu(q)}{\varphi(q)} T_i(A\eta) + \frac{1}{\varphi(q)} \sum_{\chi} \chi(A) \tau(\xi) W_i(\chi, A\eta) + \frac{\chi(A) \tau(\xi) \chi(0)}{\varphi(q)} T_i(A\eta). \]

Note that the last term appears when \( q \) divides \( \tilde{r} \), the modulus of the exceptional character. This makes additional terms in \( r_T(2f(n); \mathfrak{N}) \) which are

\[
\begin{align*}
\sum_{q \equiv P \mod{\tilde{r}}} & \frac{\tau(\xi) \chi(0)^2}{\varphi(q)} \tilde{r}(AB) c_q(-2f(n)) \int_{-1/\tilde{q}}^{1/\tilde{q}} \tilde{T}_i(A\eta) \tilde{T}_j(B\eta) e(-2f(n)\eta) d\eta \\
+ & \sum_{q \equiv P \mod{\tilde{r}}} \frac{\mu(q) \tau(\xi) \chi(0)}{\varphi(q)^2} \tilde{r}(B) c_{\tilde{r}}(\tilde{r}(0) - 2f(n)) \int_{-1/\tilde{q}}^{1/\tilde{q}} T_i(A\eta) \tilde{T}_j(B\eta) e(-2f(n)\eta) d\eta \\
+ & \sum_{q \equiv P \mod{\tilde{r}}} \frac{\mu(q) \tau(\xi) \chi(0)}{\varphi(q)^2} \tilde{r}(A) c_{\tilde{r}}(\tilde{r}(0) - 2f(n)) \int_{-1/\tilde{q}}^{1/\tilde{q}} T_j(B\eta) \tilde{T}_i(A\eta) e(-2f(n)\eta) d\eta \\
+ & \sum_{q \equiv P \mod{\tilde{r}}} \frac{\tilde{r}(B) \tau(\xi) \chi(0)}{\varphi(q)^2} \sum_{\chi} \chi(\tilde{r}(-2f(n)) \tau(\xi) \chi(A)) \int_{-1/\tilde{q}}^{1/\tilde{q}} W_i(\chi, A\eta) \tilde{T}_j(B\eta) e(-2f(n)\eta) d\eta \\
+ & \sum_{q \equiv P \mod{\tilde{r}}} \frac{\tilde{r}(A) \tau(\xi) \chi(0)}{\varphi(q)^2} \sum_{\chi} \chi(\tilde{r}(-2f(n)) \tau(\xi) \chi(B)) \int_{-1/\tilde{q}}^{1/\tilde{q}} W_j(\chi, B\eta) \tilde{T}_i(A\eta) e(-2f(n)\eta) d\eta. \tag{11}
\end{align*}
\]

By the same argument in the previous step, the last two terms of (11) are bounded by

\[
2X^{1/2} \sum_{q \equiv P \mod{\tilde{r}}} \frac{1}{\varphi(q)^2} \sum_{\chi} c_{\tilde{r}}(-2f(n)) \tau(\xi) \chi(0) |W_k^C\chi| \ll \frac{2f(n)}{\varphi(2f(n))} X^{1/2} W_k^C,
\]
where \( C = A \) or \( B \) and \( k = i \) or \( j \). And the first three terms in (11) turn out to be

\[
\begin{align*}
\sum_{q \equiv P \atop \pi q} \frac{\tau(\tilde{\chi}_0)^2}{q(q^2)} \tilde{\chi}(ABa^2)c_q(-2f(n))(\tilde{T}_{ij}^{AB}(2f(n)) + O(qQ)) \\
+ \sum_{q \equiv P \atop \pi q} \frac{\mu(q)r(\tilde{\chi}_0)}{q(q^2)} \tilde{\chi}(B)c_{\tilde{\chi}_0}(-2f(n))(\tilde{J}_{ji}^{BA}(2f(n)) + O(qQ)) \\
+ \sum_{q \equiv P \atop \pi q} \frac{\mu(q)r(\tilde{\chi}_0)}{q(q^2)} \tilde{\chi}(A)c_{\tilde{\chi}_0}(-2f(n))(\tilde{J}_{ij}^{AB}(2f(n)) + O(qQ)),
\end{align*}
\]

where

\[
\tilde{T}_{ij}^{AB}(n) := \int_0^1 \tilde{T}(An)\tilde{T}(B\eta)e(-n\eta)d\eta \\
\tilde{J}_{ij}^{AB}(n) := \int_0^1 \tilde{T}(An)\tilde{T}(B\eta)e(-n\eta)d\eta.
\]

Let \( \tilde{S}(n) := \sum_{q \equiv P \atop \pi q} \frac{\tau(\tilde{\chi}_0)^2}{q(q^2)}c_q(-n) \). Then we can prove

\[
\begin{align*}
r_{1}(2f(n); \mathfrak{M}) & = \tilde{S}(2f(n))\frac{2f(n)}{g^2AB} + \tilde{\chi}(ABa^2)\tilde{S}(2f(n))\tilde{T}_{ij}^{AB}(2f(n)) \\
& + O(\tilde{\chi}(2f(n))^2\tilde{\chi}(2f(n))X_{\tilde{\chi}^{2}}) + O(X_{1+\delta}P^{-1}(2f(n), \tilde{\phi})) \\
& + O(\frac{2f(n)}{\tilde{\chi}(2f(n))}(X_{\tilde{\chi}}^{2}(W_{i}^{A} + W_{j}^{B}) + W_{i}^{A}W_{j}^{B})).
\end{align*}
\]

(12)

by just following [MV, p.364]. Our assumption \((2f(n), \tilde{\phi}) = 1\) means that the fourth term of (12) is less than \( X^{1-5\delta} \). Using the same method in [MV, section 8], we have

\[
\tilde{S}(n) \ll o(1) \quad \text{and} \quad \tilde{T}_{ij}^{AB}(n) = \sum_{P < k < n - P \atop k \equiv i, \frac{n - Ak}{B} \in \mathbb{Z}} (k\frac{n - Ak}{B})^{\beta - 1} \leq \sum_{P < k < n - P \atop \frac{n - Ak}{B} \in \mathbb{Z}} (k\frac{n - Ak}{B})^{\beta - 1} \leq n^\beta.
\]

These facts and Lemma 9 imply that

\[
r_{1}(2f(n); \mathfrak{M}) \gg X \quad \text{if} \quad (2f(n), \tilde{\phi}) = 1,
\]

which is the conclusion of the second step.

Finally we assume \((2f(n), \tilde{\phi}) > 1\), so we have \( \tilde{\chi}(2f(n)) = 0 \). Then by Lemma 9 and (12) there is a constant \( c_{7} \) satisfying

\[
|r_{1}(2f(n); \mathfrak{M}) - \tilde{S}(2f(n))\frac{2f(n)}{g^2AB} - \tilde{S}(2f(n))\tilde{T}_{ij}^{AB}(-2f(n))| \leq c_{7}(T_{1} + T_{2}),
\]

(13)

where

\[
T_{1} = X^{1+\delta}P^{-1}(2f(n), \tilde{\phi}) \quad \text{and} \quad T_{2} = \frac{2f(n)}{\tilde{\phi}(2f(n))}(1 - \tilde{\beta})Xe^{-\tilde{\phi}} \log P.
\]

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All the arguments in [BKW, p.122–123] can be used for $r_\Gamma(2f(n);m)$ since we know that $\tilde{I}_{ij}^{AB}(2f(n)) \leq n^{\tilde{\alpha}}$. As a consequence, we conclude that if $1 < (2f(n), \tilde{r}) < Y$, then

$$r_\Gamma(2f(n);m) \gg XY^{-1}(\log X)^{-1}$$

and there are at most $O(N^{1+\epsilon}Y^{-1})$–exceptions which are $n$ with $(2f(n), \tilde{r}) > Y$. This completes the proof of the proposition. \hfill \Box

4 Proof of Theorem 2

We note that if there is at least one integer $m$ such that $2f(m) \equiv Ai + Bj \pmod{g}$, the set of $n \in (\kappa N, N]$ with $2f(n) \equiv Ai + Bj \pmod{g}$ has a positive density in the set of $n \in (\kappa N, N]$. Then the proof of Theorem 2 is exactly same as the proof of [BKW, Theorem 1] since Proposition 5 and Proposition 10 give the same results for $r_\Gamma(2f(n);m)$ and $r_\Gamma(2f(n);m)$ as [BKW, Lemma 1] and [BKW, Lemma 2].

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Department of Mathematics, Seoul National University, Seoul, Korea
E-mail: dhbyeon@snu.ac.kr

Department of Mathematics, Seoul National University, Seoul, Korea
E-mail: kyjeongg@gmail.com