As an unstable light pure leptonic system, positronium is a very specific probe atom to test bound state QED. In contrast to ordinary QED for free leptons, the bound state QED theory is not so well understood and bound state approaches deserve highly accurate tests. We present a brief overview of precision studies of positronium paying special attention to uncertainties of theory as well as comparison of theory and experiment. We also consider in detail advantages and disadvantages of positronium tests compared to other QED experiments.

Keywords: Bound state; Quantum electrodynamics; positronium.

1. Introduction

Quantum electrodynamics (QED) is the only quantum field theory which can be successfully applied to a broad range of effects (bound states, scattering, decay) and energies from microwave radiation to high energies in the GeV range and deliver us various accurate predictions for measurable quantities with an uncertainty reaching the ppt level.

QED of photons and leptons is an absolutely correct theory in a sense that its Lagrangian is well defined and in principle there is no problem for performing any calculations.

- However, that is not sufficient since exact calculations are strongly limited by increasing difficulties in the calculation of higher-order effects and we always have to deal with a finite number of terms in the perturbative expansion. A question therefore arises how to estimate terms, which are too complicated to be calculated and sometimes that is in part art.

- However, QED is in some way incomplete since electromagnetic interactions may involve hadrons and strong interactions which cannot be calculated \textit{ab initio}. The weak interactions may also be involved, but usually it is not a problem to find related contributions.

- However, there is a basic theoretical problem while performing comparison to experiment. Theory is not in position to predict any numbers. What theory can

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only do is to express some measurable quantity in terms of others. In particular, to produce any quantitative prediction within bound state QED, we need to be first able to determine with a proper accuracy values of basic fundamental constants (the Rydberg constant $R_\infty$, the fine structure constant $\alpha$ as well as different masses and magnetic moments) and auxiliary parameters (such as the proton charge radius) due to the hadronic sector, which are necessary input data for bound state QED calculations.

• However, there is no way to solve the bound state problem in general. The application of QED to the bound state forms a field called bound state QED, which experiences its own difficulties, additionally to QED problems for free particles.

2. Bound state QED

Bound state QED is quite attractive as a training field to solve the bound state problem in quantum theory. It may be helpful for few-nucleon nuclei and for hadronic particles. In particular, there is a similarity in physics of positronium and quark-antiquark systems (mesons).

The difficulties of precision QED calculations for free particles are mainly due to an increasing number (up to one thousand) of complicated diagrams (the four-loop level). The bound state QED theory deals with much simpler diagrams, however, the charged particles are bound there rather than free, and thus the Coulomb exchange may be not a small effect. A detailed review on QED calculations for light atoms can be found in [1].

The free QED involves only one small parameter $\alpha$, while the bound state QED theory needs at least three and all three expansions are not quite good[2].

• Indeed, we still have to deal with $\alpha$, the power of which indicates the number of QED loops involved. The expansion is asymptotical but that is not important since the bound state calculations mainly need one-loop and two-loop contributions, with three-loop effects being important rather seldom.

• The Coulomb strength $Z\alpha$ appears because of binding effects and the parameters $\alpha$ and $Z\alpha$ behave in a quite different way. There is a number of contributions where we need to sum over an infinite number of Coulomb exchanges (e.g. the Bethe logarithm). Importance of all the exchanges assumes their essentially non-relativistic behaviour and still allows a $Z\alpha$ expansion, which cannot be avoided since calculations exact in $Z\alpha$ are possible for a few contributions only. However, that is not a well behaving expansion, because the limit $Z\alpha \to 0$ is related to an unbound two-body system and thus leads to a non-analytic behaviour of perturbative expressions. The non-analyticity in Coulomb systems is usually accompanied with numerous logarithmic factors. For $Z = 1$ (hydrogen, muonium, positronium) one can find: $\ln(1/Z\alpha) \simeq 5$. The corrections known up to now may include up to cube of this logarithm, $\ln^3(1/Z\alpha) \simeq 120$ (for the Lamb shift), and up to logarithm squared for hyperfine structure and positronium physics[3]. For several reasons the non-logarithmic contributions also involve big coefficients[2].
• A bound state problem supposes that we deal with an atom consisting of an orbiting particle(s) and an attractor (nucleus) and that involves one more parameter, a ratio $m/M$ of masses of the orbiting particle (mainly an electron) and the nucleus which for conventional atoms is $m/M \simeq 0.5 \cdot 10^{-3} A^{-1}$, for muonium is $m/M \simeq 1/207$ and for positronium is $m/M = 1$. The behaviour of the expansion in $m/M$ is not good since the limit $m/M \to 0$ is related to a kind of bound “neutrino”. The non-analytic behaviour shows itself in logarithmic terms and, e.g., for muonium $\ln(M/m) \simeq 5.5$.

• Some more parameters are involved due to nuclear effects, such as, e.g., a ratio of the Bohr radius to the nuclear radius.

Thus, the bound state QED theory involves a rather rich spectrum of problems and it deserves to be tested, particularly in spite of the lack of well established universal prescriptions appropriate for the two-body bound state problem in general. There are two basic problems of the precision bound state QED theory: Lamb shift and hyperfine structure.

• The Lamb shift calculations mainly need an external field approximation, while recoil corrections are less important and only the simplest of them are involved.

• In contrast, calculations of the hyperfine interval are crucially affected by the recoil effects, while some external field effects (e.g., the higher-order two-loop corrections) are relatively less important.

3. Hyperfine structure in bound state QED

Most of interest to positronium properties is due to recoil effects which are crucial since $m/M = 1$. For this reason we consider here in more detail studies of the hyperfine structure in light atoms.

Magnetic effects are relativistic effects and thus, in contrast to the Coulomb interaction responsible for the Lamb shift, the higher momentum transfers and shorter distances are more important. At shorter distances the recoil and nuclear-structure effects are enhanced. The nuclear effects in hydrogen and other light atoms dominate over the bound state QED. Still there are three possible QED tests with the hyperfine structure in which the problem of nuclear effects can be avoided.

• A comparison of the $1s$ and $2s$ hyperfine intervals that offers a specific difference

$$D_{21} = 8\nu_{\text{HFS}}(2s) - \nu_{\text{HFS}}(1s),$$

which is immune to leading effects of the nuclear structure.

• A comparison of conventional and muonic atoms for the same nucleus.

• A study of a pure leptonic atomic system such as muonium and positronium.

Determination of the $2s$ hyperfine interval in muonic hydrogen is in part the goal of an PSI experiment, which is now in progress. The HFS interval in the $1s$ and $2s$ states was successfully studied for several light atoms. The more complicated


measurement is related to the metastable $2s$ state. $D_{21}$ theory is compared to experiment in Fig. 1. The theory of $D_{21}$ for several atoms is summarized in Table 1. The dominant uncertainty of QED theory is due to higher-order one-loop and two-loop corrections in order $\alpha(Z\alpha)^3$ and $\alpha^2(Z\alpha)^2$ and recoil corrections in order $\alpha(Z\alpha)^2(m/M)$ (in units of the $1s$ HFS splitting). The higher-order nuclear effects also substantially contribute to the uncertainty.

![Fig. 1. Determination of the $D_{21}$ difference in hydrogen and helium-3 ion. The references can be found in Table 10.](image)

| Contribution     | Hydrogen [kHz] | Deuterium [kHz] | $^3\text{He}^+$ ion [kHz] |
|------------------|----------------|-----------------|---------------------------|
| $D_{21}$ (QED3)  | 48.937         | 11.305          | -1189.252                 |
| $D_{21}$ (QED4)  | 0.018(3)       | 0.0043(5)       | -1.137(53)                |
| $D_{21}$ (nucl)  | -0.002         | 0.0026(2)       | 0.317(36)                 |
| $D_{21}$ (theo)  | 48.953(3)      | 11.312(5)       | -1190.083(63)             |

The muonium theory of the $1s$ hyperfine interval is summarized in Table 2. The dominant uncertainty of QED theory is due to higher-order recoil corrections in order $\alpha(Z\alpha)^2(m/M)$ and $(Z\alpha)^3(m/M)$ which are also in part responsible for the uncertainty of $D_{21}$ (see above). The other important part of the uncertainty is related to the determination of the leading term (so-called Fermi energy)

$$
\frac{E_F}{\hbar} = 16 \frac{\alpha^2 \mu_\mu}{\mu_B} R_\infty \left( \frac{m_\mu}{m_\mu + m_e} \right)^3
$$

in terms of fundamental constants $\mu_\mu/\mu_B$ and $\alpha$ and inaccuracy in their determination.
Table 2. Theory of the $1s$ hyperfine splitting in muonium. The calculations\textsuperscript{13} have been adjusted to \( \alpha^{-1} = 137.03599876(52) \)\textsuperscript{14} and \( \mu_\mu/\mu_p = 3.18334517(36) \) which was obtained from the analysis of the data on Breit-Rabi levels in muonium\textsuperscript{15} and other less accurate experiments.

| Term       | \( \Delta E \) [kHz] |
|------------|-----------------------|
| \( E_F \)  | 4 459 031.88(50)      |
| \( a_e \)  | 5 170.926(1)          |
| QED2       | - 873.147             |
| QED3       | - 26.410              |
| QED4       | - 0.551(218)          |
| Hadronic   | 0.240(4)              |
| Weak       | - 0.065               |
| Total      | 4 463 302.73(55)      |

4. Why positronium?

As mentioned before, the recoil effects are better seen in positronium. Below we consider the positronium spectrum and a comparison of theory to experiment. We find that the uncertainty in calculating all experimentally studied transitions is related to the same recoil contributions as for the muonium HFS interval.

The recoil effects play a crucial role in a two-body bound problem showing how closely the bound system is to a real two-body system. However, the significance of positronium is not limited by the possibility to verify the theory of recoil corrections.

- Since the corrections of interest are enhanced (\( m/M \) is not a suppressing factor any longer), the fractional accuracy for successful high-precision tests is now relatively low. As a result, in contrast to hydrogen, the interpretation of the measurements of the $1S - 2S$ interval does not crucially involve knowledge of the Rydberg constant with high accuracy. A study of the hyperfine interval does not require a value of the fine structure constant with high accuracy as it is in muonium. Since \( m/M = 1 \), it is not necessary to determine either \( m/M \) or \( \mu_{\text{Nuc}}/\mu_B \) in an additional experiment. In other words, positronium offers several high precision tests of bound state QED without determinations of fundamental constants with high accuracy.

- The hyperfine and recoil effects are enhanced and thus can be seen not only in a direct study of the hyperfine structure but also in the investigation of the gross or fine structure in contrast to hydrogen and muonium. Thus, positronium offers a few transitions which can be studied with high accuracy (see Table\textsuperscript{3}). Adding to that an opportunity of different experiments on positronium annihilation, we find a big variety of properties to be studied.

- It is not even necessary to mention that as a light pure leptonic atomic system, the positronium atom is free of hadronic effects. It is important that positronium
is light because hadronic effects in leptonic systems involve a high momentum transfer. They can be seen in muonium and in the muon anomalous magnetic moment they are responsible for a dominant part of the theoretical uncertainty, while they are strongly suppressed for positronium and the anomalous magnetic moment of electron.

Since positronium is a very specific atom, the notation for positronium is slightly different from other two-body atoms. First, in two-body atoms, even in hydrogen and muonium, it is customary to keep the value of the nuclear charge \( Z \) in order to recognize exchange photons and photons of QED radiative effects and thus to trace the origin of different corrections. In positronium there is an interference between both kinds of photons because of annihilation diagrams and thus it is meaningless to keep \( Z \).

Hydrogen and other two-body atoms are one-electron systems and one can use for them both small (e.g., 1\( s \)) and capital (e.g., 1\( S \)) letters to denote levels. The former are used for an electron, while the latter are for all electrons in an atom and that is indeed the same for single-electron atomic systems. Here, we prefer to use small-case letters for hydrogen and others. Since the nuclear spin effects are not suppressed, positronium has a structure of energy levels (in respect to their spin and angular momentum) rather similar to a two-electron system (such as the neutral helium atom) and one has to use only capital letter for its orbital momentum.

We summarize theoretical predictions for various transitions and decay rates in Table 3. The Table does not contain experimental results but only references to figures where a comparison of theory to experiment is performed. Our results for decay rates and the HFS interval are slightly different from those quoted in the literature because we include into the decay rates contributions related to the decay to non-minimal number of photons (four for parapositronium and five for orthopositronium) which are sometimes omitted and because of our conservative estimation of uncertainty. To be conservative, we estimate the final uncertainty by half of the value of the leading logarithmic terms, if the non-leading term is
unknown or reduces the logarithmic contribution, and by half of the value of the whole logarithmic contribution, if the non-leading term enhances the leading term (cf. [16]).

5. Positronium spectrum

Let us now consider the positronium spectrum in more detail. The hyperfine interval in positronium has been measured with the highest absolute accuracy among positronium transitions. A comparison of theory and experiment is presented in Fig. 2 while theory is summarized in Table 4. The theoretical contributions are classified in units of the Fermi energy

$$E_F = \frac{7}{48} \alpha^4 mc^2,$$

which is defined including the virtual one-photon annihilation. E.g., QED2 is related to corrections of relative order $\alpha^2$. This notation is helpful for a comparison to the theory of decay rates, while the absolute units, also presented in the Table, allow a simple and direct comparison with theory of the $1S - 2S$ interval. Agreement with experiment is not perfect (within 2.5-3 standard deviations) which demonstrates that positronium is not very well understood.

The $1S - 2S$ interval was measured in orthopositronium by a method of three-photon ionization which has a resonance related to the two-photon excitation of the metastable $2S$ state from the ground state. The absolute uncertainty of the $1S - 2S$ interval is compatible with the hyperfine splitting although somewhat lower. The uncertainty of the theoretical predictions for both hyperfine interval and the $1S - 2S$ transition (see Table 5) is determined by a correction of order $\alpha^7 mc^2$. That is related to $\alpha^3 E_F$ in the positronium hyperfine interval (QED3).
Table 4. QED contributions to the 1\(S\) hyperfine interval in positronium. The uncertainty is presented following [16].

| Term | \(\Delta E\) [MHz] |
|------|------------------|
| \(E_F\) \(\alpha^4 mc^2\) | 204386.6 |
| QED1 \(\alpha^5 mc^2\) | -1005.5 |
| QED2 \(\alpha^6 mc^2\) | -11.817 |
| QED3 \(\alpha^7 mc^2\) | -1.2(6) |
| Total | 203391.7(6) |

and thus to QED4 corrections in atoms with heavy nucleus \((m/M \ll 1)\), namely, \(\alpha (Z\alpha)^2(m/M)E_F\) and \((Z\alpha)^3(m/M)E_F\). A comparison of theory to experiment is presented in Fig. 3.

Table 5. Theory of the \(1^3S_1 - 2^3S_1\) interval in positronium.

| Term | \(\Delta E\) [MHz] |
|------|------------------|
| \(\alpha^4 mc^2\) | 1233690735.1 |
| \(\alpha^5 mc^2\) | -82005.6 |
| \(\alpha^6 mc^2\) | -1501.4 |
| \(\alpha^7 mc^2\) | -7.24 |
| Total | 1233607222.2(6) |

Fig. 3. Theory and experiment (a – [25], b – [26]) for determination of the \(1^3S_1 - 2^3S_1\) interval in positronium.
The theoretical uncertainty for the \( n = 2 \) fine structure in positronium (see Fig. 4) is compatible with that for the hyperfine interval and the \( 1S - 2S \) transition, but the experiment on \( 2S - 2P \) transitions cannot provide competitive results (see Fig. 5). However, progress is possible.

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**Fig. 4.** Fine structure in positronium. The scale does not allow to see a width of all levels, but only of the broad \( 2S \) singlet state. The \( 2^3S_1 - 2^1P_1 \) transition is forbidden, however, it may be observed by applying a magnetic field.

**Fig. 5.** Measurements of the fine structure transitions at \( n = 2 \). Experiments were performed at Brandeis University (a – [27]), University of Michigan (b – [28] and d – [30]) and Mainz University (c – [29] and e – [31]).

6. **Positronium annihilation**

Positronium is an unstable atom because of electron and positron annihilation. The \( 1S \) state of positronium decays mainly into two or three photons depending on its spin. Thus, the orthopositronium (\( S = 1 \)) annihilates into an odd number of photons, while the parapositronium (\( S = 0 \)) into an even number. The leading
rates are

\[ \Gamma^{(o)}_{3\gamma} (\text{oPs}) = \frac{2(\pi^2 - 9)}{9\pi} \frac{\alpha^6 m_e^2}{\hbar} , \quad (4) \]

\[ \Gamma^{(o)}_{2\gamma} (\text{pPs}) = \frac{1}{2} \alpha^5 \frac{m_e^2}{\hbar} . \quad (5) \]

The lifetime of orthopositronium was responsible for a long-standing discrepancy of theory and most accurate experiments performed at the University of Michigan\(^{32,33}\), while theory was in fair agreement with less accurate experiments, which disagreed with the Michigan data. The present situation is summarized in Fig. 6. Hopefully, the crisis seems to be resolved since the new Michigan vacuum result\(^{38}\) now agrees with theory and the reevaluation\(^{36}\) of the former gas experiment\(^{32}\) shifted the value towards theory. A further reexamination of systematic effects is in progress and the final uncertainty will probably be higher\(^{40}\).

Meanwhile, the lifetime of parapositronium also measured at the University of Michigan\(^{41}\) is in fair agreement with the theoretical prediction (see Fig. 7). The theoretical predictions for the annihilation decay rates for ortho- and parapositronium are summarized in Table 6.

We do not consider here exotic decay modes related to possible new physics, but rather rare modes possible within QED and the Standard Model. Two rare QED modes are allowed at a detectable level. Their theory and experiments are summarized in Table 7. We also include in the Table a photonless annihilation of orthopositronium into a pair of neutrino-antineutrino. The other modes with a pair of \(\nu \bar{\nu}\) are also possible but their branching ratios are a few orders of magnitude lower.
Table 6. Theory of annihilation decay rate of ortho- and parapositronium (the 1S state). The leading contributions are defined above in Eqs. (4) and (5). The decay rate of ortho/parapositronium into five/four photons is included into QED2 terms.

| Contribution | Decay rate of orthopositronium [µsec⁻¹] | Decay rate of parapositronium [µsec⁻¹] |
|--------------|-----------------------------------------|----------------------------------------|
| $\Gamma(0)$  | 7.21117                                 | 8.63250                                |
| $\alpha \cdot \Gamma(0)$ | -0.17230 | -47.25 |
| $\alpha^2 \cdot \Gamma(0)$ | 0.001111(1) | 4.43(1) |
| $\alpha^3 \cdot \Gamma(0)$ | -0.0000111(2) | -0.08(4) |
| **Total**    | 7.03996(2)                                | 7989.62(4)                             |

Table 7. Rare positronium decay modes. The theoretical uncertainty for $4\gamma$ decay is related to unknown higher-order corrections.

| Rare mode | Leading mode | Branching (theor) | Branching (exp) |
|-----------|--------------|-------------------|-----------------|
| oPs → 5\gamma | oPs → 3\gamma | 1.0 \times 10^{-6}, Ref. [49, 50] | $2.2^{+2.6}_{-1.8} \times 10^{-6}$, Ref. [47] |
| oPs → νν | oPs → 3\gamma | 6.2 \times 10^{-21}, Ref. [49] | < $5.8 \times 10^{-4}$ (90% CL), Ref. [51] |
| pPs → 4\gamma | pPs → 2\gamma | 1.439(2) \times 10^{-6} Ref. [52] | 1.3(30) \times 10^{-6}, Ref. [53] |

We have to note that the branching fraction of the orthopositronium decay into neutrinos is so low, that this decay is unlikely to be soon detected, however, a mode $oPs \rightarrow nothing$ (since the neutrino is not detectable) can still be of interest if the
neutrino has a non-vanishing magnetic moment (see, e.g., Fig. 8). This problem was discussed in part in [57] but we do not like to present here any numbers. In our opinion, results of such analysis can depend on a model introducing neutrino mass and magnetic moment. Still, studies of the pure neutrino modes can provide a limit on the $\tau$-neutrino magnetic moment at a level above (but possibly not much above) the current limits [58]. Since systematic effects are different and the interpretation depends on the model, we think it may be important to have several independent limitations.

![Diagram](image)

Fig. 8. Annihilation of orthopositronium into a pair of $\nu\bar{\nu}$: via $Z$-boson (left) and photon (right). The former diagram is related to the Standard Model, while the latter is present only if $\mu_\nu \neq 0$.

7. Positronium and other QED tests

Tests with positronium are quite different from other tests. Advantages and disadvantages of positronium studies and some other QED experiments are listed in Table 8.

Table 8. Advantages and disadvantages of different QED tests for conventional light atoms, pure leptonic bound systems and free leptons.

| System          | Value | Dominant uncertainties                              |
|-----------------|-------|-----------------------------------------------------|
| Hydrogen        | Lamb shift | Nuclear size, higher-order two-loop effects, $R_\infty$ |
| Hydrogen        | 1s HFS | Huge uncertainty due to nuclear structure           |
| Hydrogen        | $D_{21}$ | Experiment                                          |
| $^3$He$^+$      | $D_{21}$ | Higher-order two-loop, recoil, nuclear effects, experiment |
| $^4$He$^+$      | Lamb shift | Nuclear size, higher-order two-loop effects, experiment |
| Muonium         | 1s HFS | Higher-order recoil effects, $\mu_\mu/\mu_B$, $\alpha$ |
| Positronium     | 1S HFS | Higher-order recoil effects, experiment              |
| Positronium     | 1S − 2S | Higher-order recoil effects, experiment              |
| Positronium     | 2S − 2P | Experiment                                          |
| Orthopositronium | $\Gamma(1s)$ | Experiment                                       |
| Parapositronium | $\Gamma(1s)$ | Experiment                                       |
| Electron        | $g − 2$ | Uncertainty: $\alpha$, cavity QED effects          |
| Muon            | $g − 2$ | Uncertainty: hadronic effects                       |

To compare different QED tests with/without positronium, we also check which contributions are crucial for a comparison of theory to experiments. They are summarized in Table 8.
Table 9. Comparison of the bound state QED theory and experiment: crucial orders of magnitude for the energy levels and decay rates needed for the comparison \(^2\) (in units of \(mc^2\)).

| Atom, value                        | Crucial order(s) |
|------------------------------------|------------------|
| Hydrogen (gross structure)         | \(\alpha(Z\alpha)^7, \alpha^2(Z\alpha)^6\) |
| Hydrogen (fine structure)          | \(\alpha(Z\alpha)^7, \alpha^2(Z\alpha)^6\) |
| Hydrogen (Lamb shift)              | \(\alpha(Z\alpha)^7, \alpha^2(Z\alpha)^6\) |
| \(^3\)He\(^+\) ion (2s HFS)        | \(\alpha(Z\alpha)^7 m/M, \alpha^2(Z\alpha)^6 m/M, \alpha(Z\alpha)^6 (m/M)^2, (Z\alpha)^7 (m/M)^2\) |
| \(^4\)He\(^+\) ion (Lamb shift)   | \(\alpha(Z\alpha)^7, \alpha^2(Z\alpha)^6\) |
| Muonium (1s HFS)                   | \(\alpha(Z\alpha)^7 m/M, \alpha(Z\alpha)^6 (m/M)^2, (Z\alpha)^7 (m/M)^2\) |
| Positronium (1S HFS)               | \(\alpha^7\) |
| Positronium (1S – 2S)              | \(\alpha^7\) |
| Positronium (2S – 2P)              | \(\alpha^7\) |
| Para-positronium (decay rate)      | \(\alpha^7\) |
| Ortho-positronium (decay rate)     | \(\alpha^8\) |
| Para-positronium (4\(\gamma\) branching) | \(\alpha^8\) |
| Ortho-positronium (5\(\gamma\) branching) | \(\alpha^8\) |

8. Higher-order logarithmic corrections and uncertainty of positronium calculations

To consider HFS tests in detail, we summarize the results on a study of the hyperfine structure in light two-body atoms in Table 10. In these tests the QED uncertainty has never been a limiting factor for comparison of theory and experiment. The following uncertainties may also be involved: due to nuclear effects, determination of fundamental constants or measurement. One of them is always bigger than the theoretical uncertainty.

The theory and experiment are in general in good agreement. The uncertainty for the 1s hyperfine interval in muonium and positronium and in part of \(D_{21}\) for helium-3 ion is related to the same recoil corrections in order \(\alpha^3(m/M)E_F\) (see Table 10). The positronium uncertainty in fractional units is much higher than that for experiments with heavy atoms \((m/M \ll 1)\), however, its sensitivity to higher-order recoil effects is at approximately the same level.

The higher-order recoil contributions are known only in the logarithmic approximation and we estimate the uncertainty as half the value of the leading term if the next-to-leading term is unknown or cancels a part of the leading term. However, if the leading and the next-to-the leading terms are of the same sign, we estimate the uncertainty as a half value of the whole logarithmic contribution (cf. \(59\) \& \(13\), \(16\)). Thus, a calculation of the next-to-leading terms does not reduce an uncertainty. A reason for such a conservative estimation is that the leading logarithmic term is mostly a result of a single contribution without any cancellation. For the \(ns\) state it is state-independent. For these reasons it has a kind of natural value and can be used to estimate the non-leading terms. In contrast, the non-leading term has quite accidental value and may be sometimes quite below the natural level. Thus, it cannot be used alone to estimate properly the next term of the logarithmic expansion. However, we cannot simply ignore the next-to-leading term, particularly in the case
when it is not too small. In the present paper we follow a compromise which allows to achieve a conservative estimation of the uncertainty. A calculation of the next of the non-leading terms helps us to check the reliability of this estimation using only the leading term and meantime is a necessary step towards a calculation of the whole contribution beyond the logarithmic approximation which is strongly needed for muonium and positronium HFS.

Table 10. Comparison of experiment and theory for hyperfine structure in hydrogen-like atoms. In the $D_{21}$ case the reference is given only for the $2s$ hyperfine interval.

| Atom          | Exp. $\Delta/\sigma$ | Theory $\sigma/E_F$ |
|---------------|------------------------|-----------------------|
| Hydrogen, $D_{21}$ | 49.13(13) | 48.953(3) | 1.4 | 0.09 |
| Hydrogen, $D_{21}$ | 48.53(23) | -1.8 | 0.16 |
| Hydrogen, $D_{21}$ | 49.13(40) | 0.4 | 0.28 |
| Deuterium, $D_{21}$ | 11.16(16) | 11.3125(5) | -1.0 | 0.49 |
| $^3\text{He}^+$ ion, $D_{21}$ | -1189.979(71) | -1190.083(63) | 1.10 | 0.01 |
| $^3\text{He}^+$, $D_{21}$ | -1190.116(11) | 0.0 | 0.18 |
| Muonium, 1s | 4 463 302.78(5) | 4 463 302.88(55) | -0.18 | 0.11 |
| Positronium, 1S | 203 389 100(740) | 203 391 700(600) | -2.9 | 4.4 |
| Positronium, 1S | 203 397 500(1600) | -2.5 | 8.2 |

9. Soft and hard QED effects

Not only essential orders of QED corrections offer a possibility of comparing the efficiency of different experiments. The bound state QED theory clearly recognizes two kinds of contributions: the soft-photon contribution and the hard-photon contribution. The latter are very similar to free QED, while the former essentially involve binding effects. There are two most important soft-photon contributions.

- Crucial corrections in the external field approximation are due to the higher-order two-loop self-energy. The inaccuracy in its calculation determines an uncertainty of the Lamb shift calculations for hydrogen and hydrogen-like ions, while similar effects involving the magnetic field significantly contribute to the uncertainty of $D_{21}$ in the helium-3 ion.

- Recoil corrections are crucially important for tests with hyperfine structure and they determine the uncertainty of muonium HFS, positronium spectrum (hyperfine interval, $1s - 2s$ transition, fine structure) and a part of the uncertainty of $D_{21}$ for the helium-3 ion.

Effects of the hard-photon exchange are very similar to effects of free QED. We note, however, that the regions of integration in momentum space are different. The most accurate free QED calculations which may be compared to experiment are related to the anomalous magnetic moments of electron and muon. The integration for them is performed over a kind of isotropic region in Euclidean space ($|k_0| \sim |k|$)
and the crucial level is the four-loop approximation. For the bound state problems there are two other specific regions of integration.

- For some problems it is sufficient to apply an external field approximation to hard-photon corrections and thus loops of exchange photons are related to zero energy transfer \((k_0 = 0)\). Even for recoil effects the integration is essentially not covariant in Euclidean space including a contribution from a specific region \(|k_0| \ll |k|\). The highest crucial orders are related to four-loop corrections for the external-field approximation and three-loop corrections for recoil effects. In contrast to the anomalous magnetic moment, that is a calculation for two different particles with its own simplifications and difficulties.

- The other specific situation for integration is related to the positronium annihilation when some photons are real \((k_0 = |k|)\). The orthopositronium studies (main decay mode and branching fraction for five-photon decay) allow to check calculations of four-loop corrections in such a non-isotropic region of integration. The accuracy of the branching fraction determination for the four-photon decay of parapositronium is at the level of a few percent. Because of a relatively large \(\alpha\)-contribution to the \(4\gamma\) annihilation rate, \[ \Gamma(pPs \rightarrow 4\gamma) = \Gamma^{(0)}(pPs \rightarrow 4\gamma) \times \left(1 - 14.5(6) \frac{\alpha}{\pi}\right) \] the experiments are approaching a level, where four-loop diagrams are important.

Thus, we see that the bound state problems supply us with a possibility to check modern algorithms for four-loop calculations and that is competitive to the anomalous magnetic moment of electron. We note, however, that the difficulties which appear at the four-loop level are a large number of diagrams with high-dimension integrations and overlapping UV divergencies. Neither annihilation nor exchange loops are ultraviolet divergent, but the exchange loops involve strong infrared divergence.

One can also compare soft-QED effects of the virtual one-photon annihilation and the real two- and three-photon annihilation. Soft Coulomb corrections to the hard annihilation block can be presented in fractional units. A crucial theoretical uncertainty related to \(\alpha^3\) corrections (QED3 term) is at the level of few ppm, which can be hardly achieved for a measurement of the annihilation rates.

10. Summary

Thus, we summarize our paper with the following statements.

- Positronium spectroscopy offers a reliable test of our understanding of higher-order recoil corrections within bound state QED, which play a significant role in other QED tests with the hyperfine structure. The theoretical uncertainty for positronium is of a pure QED origin and other effects such as the nuclear structure effects or inaccuracy in determination of fundamental constants are not involved.
Studies of recoil effects are of particular interest because their significance answers a question whether we study a really two-body system. In conventional atoms (hydrogen, helium) the role of the nucleus as a particle is reduced. The positronium is a truly two-body system which is the closest to the neutral helium atom. In contrast to two-body atoms, the QED uncertainty in helium is bigger than that of experiment and theory needs a significant improvement. Thus, positronium theory serves as an intermediate step between hydrogen and helium.

- A study of positronium annihilation including rare decay modes allows to test approaches to the calculation of four-loop diagrams but for a specific integration region. Today the four-loop approximation is the highest level for precision calculations of any measurable QED quantities (if indeed no big logarithms are involved such as those in scattering kinematics).
- There are exotic decay modes of positronium beyond QED, but within the Standard Model and in particular a decay of orthopositronium into a pair of neutrino and antineutrino, which is a dominant mode beyond QED. The branching fraction is very low and it cannot be detected presently. Still, a study of this channel can probably give a limit for the magnetic moment of the $\tau$ neutrino which, although somewhat weaker than the current limits, will, however, have completely different systematic effects.

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