Big-Bang Nucleosynthesis and WIMP dark matter in modified gravity

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In the present work the primordial Big-Bang Nucleosynthesis (BBN) and weakly interacting massive particle (WIMP) dark matter are discussed in a certain class of modified gravitational theories, namely \(f(R) \sim R^n\) gravity. The new gravitational model is characterized by a single parameter \(n\). First we determine the conditions under which the theoretical predictions for the \(^4\)He abundance are in agreement with the observations. More precisely, during BBN the physics is known and all the parameters are known. The only free parameter to be constrained is the power \(n\) related to the new gravitational model. After that, for cold dark matter we use the value of \(n\) determined from the BBN considerations and determine how the mass of the dark matter particle is related to the annihilation cross section in order for the cold dark matter constraint to be satisfied.
I. INTRODUCTION

There is accumulated evidence both from astrophysics and cosmology that about 1/4 of the energy budget of the universe consists of so called dark matter, namely a component which is non-relativistic and does not feel the electromagnetic nor the strong interaction. For a review on dark matter see e.g. [1]. Although the list of possible dark matter candidates is long (for a nice list see e.g. [2]), it is fair to say that the most popular dark matter particle is the LSP in supersymmetric models with R-parity conservation [3]. The superpartners that have the right properties for playing the role of cold dark matter in the universe are the axino, the gravitino and the lightest neutralino. By far the most discussed case in the literature is the case of the neutralino (see the classical review [4]), probably because of the prospects of possible detection. On the other hand, primordial Big-Bang nucleosynthesis (BBN) is one of the cornerstones of modern cosmology. In the old days, BBN together with Hubble’s law and CMB supported and strengthened the Hot Big-Bang idea. Nowadays, BBN can be used to test and constrain possible new physics beyond the standard model. The new physics may be either due to exotic particles predicted by particle physics model or due to a new expansion law for the universe predicted by a new gravitational model. For a recent review on BBN see e.g. [5]. In the present work we shall be interested in a class of new gravitational models of the form $f(R) \sim R^n$, where the power $n$ is the only parameter that characterizes this class of models. Although in the literature the authors usually discuss this kind of modified gravitational models in the late times universe (see e.g. [6, 7]), here we wish to discuss this class of models in the early universe. In [8] the authors were interested in the baryon asymmetry in the framework of gravitational baryogenesis proposed a few years ago [9].

In this Letter we wish to study this class of gravity models in two respects, namely primordial Big-Bang nucleosynthesis (BBN) and WIMP dark matter. Our investigation will allow us to first derive the allowed range for the power $n$, and to see how different this class of models can be compared to general relativity. Then for these values of $n$ we determine how the WIMP mass has to be related to its annihilation cross section so that the cold dark matter constraint is satisfied. As a matter of fact, already in [8], the authors have mentioned that obtaining the right baryon asymmetry in agreement with BBN requires a value of $n$ close to unity. Their discussion was based on the argument that the temperature relevant for BBN should be within the range $0.1 – 100$ MeV. Here, however, we perform a more accurate investigation by actually computing the cosmological helium abundance employing the semi-analytical method introduced in [10]. We remark that one can use numerical codes [11] for a proper treatment of BBN and accurate computation of the light nuclei abundances. However, the final density of $^4He$ is very weakly sensitive to the whole nuclear network [5]. Therefore, in the present investigation we shall employ the semi-analytical treatment
of [10], computing the Helium abundance to a very good approximation avoiding sophisticated computer softwares. See also [12] for a recent example of a published work in which the same semi-analytical method was used to constrain the higher dimensional Planck mass in a brane model. Our results show that BBN requires the models considered in the present work to be only slightly different from the usual Einstein’s general relativity, whereas dark matter consideration alone does not seem to constrain this class of new gravity theories due to the degeneracy in parameter space of the underlying particle physics models. However, from the BBN consideration we give a precise range for the allowed values of the power $n$ and confirm the result of [13] using a different approach based on physics of the early universe. Finally, we remark at this point that according to our findings, certain scenarios that require a value of $n$ considerably different than one cannot work. Furthermore, the models that satisfy our constraints do not lead to the late cosmic acceleration.

Our work is organized as follows. The article consists of five sections, of which this introduction is the first. The modified gravitational model is described in the next section. The analysis based on BBN is discussed in section 3, while the investigation based on WIMP dark matter is presented in section 4. Finally we conclude in the last section.

II. THE MODIFIED GRAVITATIONAL MODEL

Here we shall present the model of $f(R)$ gravity that will be discussed in this paper, and we shall summarize the basic formulas following [8]. The model is described by the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m[g_{\mu\nu}, \phi_m],$$

(1)

where $G$ is Newton’s constant, $\kappa^2 = 8\pi G$, and $S_m$ is the action of the matter field, $\phi_m$. Varying this action with respect to the metric we obtain the field equations for gravity, which generalize the usual Einstein’s equations,

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' = \kappa^2 T_{\mu\nu},$$

(2)

where $T_{\mu\nu}$ is the energy-momentum tensor for the matter, and a prime denotes differentiation with respect to $R$. For the gravity part we consider the spatially flat Robertson-Walker (RW) line element

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2),$$

(3)

while for the matter part we consider a cosmological fluid characterized by a time-dependent energy density $\rho(t)$ and pressure $p(t)$

$$T^\mu_\nu = \text{diag}(\rho, -p, -p, -p).$$

(4)
The 0 − 0 component of (2) gives
\[-3 \ddot{a} f' - \frac{1}{2} f + 2 \dot{a} f'' \dot{R} = \kappa^2 \rho,\]  
while the \(i - i\) components give
\[\left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right) f' + \frac{1}{2} f - 2 \frac{\dot{a}}{a} f'' \dot{R} - f''' \ddot{R}^2 - 3 f'' \dot{R} = \kappa^2 p.\]  
Here a dot denotes differentiation with respect to the cosmic time \(t\). In addition to the above cosmological equations, we have as usual the energy conservation law
\[\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0.\]  
We restrict ourselves to the models of the form
\[f(R) = \left( \frac{R}{A} \right)^n,\]  
where \(A\) is a constant, \(A \sim M_p^{2-2/n}\), with \(M_p = 1.22 \times 10^{19}\) GeV the Planck mass. The power \(n\) is the unique parameter of this class of modified gravitational models, and \(n = 1\) corresponds to the usual Einstein’s theory. Since we are interested in the physics of the early universe, we consider the radiation dominated era in which \(p = \rho / 3\), and \(\rho \sim a^{-4}\). Searching for a power law solution for the scale factor, \(a(t) \sim t^\alpha\), the cosmological equations determine the unknown power \(\alpha\) in terms of \(n\) as follows:
\[\alpha = \frac{n}{2}.\]  
Notice that when \(n = 1\) we recover the known result \(a(t) \sim t^{1/2}\) for the radiation era in the usual Einstein’s general relativity. Then using the expression for the energy density
\[\rho = \frac{\pi^2}{30} g_\ast T^4,\]  
one obtains the relation between time and temperature
\[T = \left( \frac{15}{4\pi^3 g_\ast} \right)^{1/4} g_\alpha^{1/4} M_p^{1/2} \frac{1}{\alpha^2 A^{\alpha/2}},\]  
where \(g_\ast\) counts the relativistic degrees of freedom for energy density, and
\[g_\alpha \equiv 6^{2\alpha} a^{2\alpha} - 10 \alpha^2 + 8 \alpha - 1 \frac{2(1-2\alpha)}{(1-2\alpha)^{1-2\alpha}}.\]  
Since the quantity \(g_\alpha\) must be positive, the allowed range of \(\alpha\) is \(0.155 \lesssim \alpha \lesssim 1/2\), and \(\alpha = 1/2\) corresponds to Einstein’s theory. Finally the Hubble parameter is given by
\[H(T) = \frac{\alpha A^{1/2}}{g_\alpha M_p^{1/2}} \left( \frac{4\pi^3 g_\ast}{15} \right)^{1/4} T^{\frac{1}{2}},\]  
which generalizes the usual formula \(H(T) \sim T^2\), valid in the standard cosmology of Einstein’s general relativity.
III. BIG-BANG NUCLEOSYNTHESIS

In this section we briefly review the sequence of basic events leading to the synthesis of primordial Helium during the early stages of the expansion of the universe in the standard cosmology based on Einstein’s general relativity, following [10]. Then we shall present the corresponding discussion and our results for the modified gravity case. Since all the parameters are fixed and the power $n$ is the only free parameter of the model, our discussion will allows us to determine the allowed range for $n$.

When the rates of the weak interactions keeping baryons in chemical equilibrium with leptons become comparable to the Hubble parameter, the neutron fraction $X = n_n / (n_n + n_p)$ is frozen at some value $X(T ≃ 0)$ to be determined below, where $n_n$ and $n_p$ are neutron and proton number density respectively. Once the temperature has fallen below about $1/25$ of the Deuterium binding energy, the Deuterium bottleneck opens up, and nearly all of the original neutrons present at the decoupling time are captured in $^4He$. Taking into account the neutron decay, the final helium mass fraction is given by

$$Y_4 \approx 2\exp(-t_c/\tau)X(T \approx 0)$$

where $\tau = 885.7 \pm 0.8$ sec [14] is neutron’s lifetime, and $t_c \sim 3$ min is the capture time at which neutrons are captured into Deuterium.

Now we discuss how to compute $X(T \approx 0)$ and $t_c$. To this end, we employ a semi-analytical method (see e.g. [10], [15]) which is sufficiently accurate and very useful, since the physics is very transparent and the dependence of the abundances on input parameters can be clearly worked out without sophisticated computer softwares. Indeed, the abundance of $^4He$ is very weakly sensitive to the whole nuclear network [5], and therefore a sufficiently accurate result can be obtained without using computer codes. Our semi-analytical approach relies on the work of [10], and we shall not present here in detail all relevant formulas, as they are quite involved. To compute $X(T \approx 0)$ we need to integrate the following rate equation

$$\frac{dX(t)}{dt} = \lambda_{pn}(t)(1 - X(t)) - \lambda_{np}X(t).$$

Here we denote by $\lambda_{pn}$ the rate for the weak processes to convert protons into neutrons and by $\lambda_{np}$ the rate for the reverse processes that convert neutrons into protons. These rates are time dependent because of their temperature dependence. The rate $\lambda_{np}$ is the sum of the rates of three processes

$$\lambda_{np} = \lambda(v + n \rightarrow p + e^-) + \lambda(e^+ + n \rightarrow p + \bar{\nu}) + \lambda(n \rightarrow p + \bar{\nu} + e^-),$$

each of which is computed using standard field-theoretic techniques. After a few simplifications the rate $\lambda_{np}$ is computed as follows [10].

$$\lambda_{np}(y) = \left(\frac{255}{\tau y^5}\right)(12 + 6y + y^2),$$

where $y = T/T_0$ is the temperature of the universe normalized to the Deuterium binding energy.
where \( y = \Delta m / T \) with \( \Delta m = m_n - m_p = 1.29 \) MeV being the neutron-proton mass difference. The detailed balance relation gives

\[
\lambda_{pn}(y) = e^{-y} \lambda_{np}(y).
\] (18)

We now rewrite (15) in terms of \( y \) instead of time as

\[
\frac{dX(y)}{dy} = \frac{dt}{dy} \left( \lambda_{pn}(y)(1 - X(y)) - \lambda_{np}(y)X(y) \right),
\] (19)

where \( dt/dy \) can be computed using the definition of \( y \), \( y = \Delta m / T \), and the fact that \( \dot{T}/T = -H \). With the initial condition \( X(y = 0) = 1/2 \), the rate equation for \( X(y) \) can be integrated numerically, and from the graphical solution one can compute \( X(T \simeq 0) \), which we denote by \( \bar{X} \).

Finally, let us add a few words regarding the capture time. The bottleneck opens up when the main reaction converting Deuterium into heavier elements

\[
D + D \rightarrow T + p
\] (20)

become efficient. First we introduce

\[
z = \frac{\epsilon_D}{T},
\] (21)

where \( \epsilon_D = m_p + m_n - m_D = 2.23 \) MeV is the Deuterium binding energy. The another quantity that is important in estimating capture time is the Deuterium abundance, \( X_D \equiv n_D / (n_n + n_p) \), where \( n_D \) is the Deuterium number density. From the Saha equation \( X_D \) is given by

\[
X_D = 2.8 \times 10^{-14} \eta_{10} T_{MeV}^{3/2} \exp(z) X_p X,
\] (22)

where \( X_p = n_p / (n_n + n_p) \) is the proton fraction. In the above formula \( T_{MeV} \) is the temperature in MeV units, and we have parametrized the baryon-to-photon ratio by

\[
\eta_{10} \equiv 10^{10} \times \frac{n_b}{n_\gamma},
\] (23)

where we use the observational value \( n_b/n_\gamma = 6.1 \times 10^{-10} \) from WMAP [18]. The condition that determines the temperature (or time) at which the Deuterium bottleneck opens up reads as follows(for more details we refer the reader to [10]).

\[
2X_D R_{DD} \simeq 1,
\] (24)

where

\[
R_{DD} = \frac{dt}{dz} < \sigma v > n_b = 2.9 \times 10^7 z^{-4/3} \exp(-1.44z^{1/3})
\] (25)
and \(<\sigma v>\) is the thermal average of the relevant cross section times relative velocity, which is a function of \(z\). Then one can solve (24) with respect to \(z\) to get \(t_c\), and then from (14) one can finally obtain \(Y_4\).

We now consider the constraints on \(f(R)\) gravity coming from BBN by computing the Helium mass fraction at the conclusion of the BBN. To this end, we will follow [10] for standard cosmology as described above, and the modifications will be done by adopting the relation (11) instead of the standard one.

In the modified gravity model, the basic physics governing the details of primordial nucleosynthesis remains the same, and the only thing that is different now is the new time-temperature relation (11), from which one obtains

\[
t = \left(\frac{15}{4\pi^3 10.75}\right)^{1/4\alpha} \frac{M_\nu^{1/2\alpha}}{A^{1/2}} \left(\frac{y}{\Delta m}\right)^{1/\alpha} \frac{1}{8^{1/4\alpha}}
\]

(26)

\[
t = \left(\frac{15}{4\pi^3 3.37}\right)^{1/4\alpha} \frac{M_p^{1/2\alpha}}{A^{1/2}} \left(\frac{z}{\epsilon_D}\right)^{1/\alpha} \frac{1}{8^{1/4\alpha}}
\]

(27)

In (26)-(27) we have taken into account the appropriate value for \(g_*\) at the relevant temperature. Finally, the \(^4\text{He}\) mass fraction is still given by (14), but now both freeze-out abundance \(x(T \simeq 0) = \bar{X}\) and capture time \(t_c\) are modified due to the new time-temperature relation.

| \(\delta = 1 - n\) | \(\bar{X}\) | \(t_c\) (sec) | \(Y_4\) |
|------------------|--------|-------------|--------|
| 0                | 0.1529 | 176.76      | 0.2504 |
| \(10^{-5}\)      | 0.1528 | 176.94      | 0.2503 |
| \(10^{-4.5}\)    | 0.1526 | 177.34      | 0.2499 |
| \(10^{-4}\)      | 0.1521 | 178.59      | 0.2486 |
| \(10^{-3.9}\)    | 0.1519 | 179.06      | 0.2482 |
| \(10^{-3.8}\)    | 0.1516 | 179.66      | 0.2476 |
| \(10^{-3.7}\)    | 0.1513 | 180.42      | 0.2468 |
| \(10^{-3.6}\)    | 0.1509 | 181.37      | 0.2459 |
| \(10^{-3.5}\)    | 0.1504 | 182.58      | 0.2448 |
| \(10^{-3.4}\)    | 0.1497 | 184.12      | 0.2433 |
| \(10^{-3.3}\)    | 0.1489 | 186.07      | 0.2414 |
| \(10^{-3.2}\)    | 0.1479 | 188.54      | 0.2391 |
| \(10^{-3.1}\)    | 0.1466 | 191.71      | 0.2362 |
| \(10^{-3}\)      | 0.1450 | 195.77      | 0.2326 |
| \(10^{-2.9}\)    | 0.1430 | 201.01      | 0.2280 |

Table I: Helium abundance, capture time, and freeze-out neutron mass fraction for several values of \(\delta = 1 - n\).

First we integrate (19) with the initial condition \(X(y = 0) = 1/2\) to obtain \(\bar{X}\) (it is enough to evaluate \(\bar{X}\) at \(y = 15\) because it freezes out). Then we use the condition (24) to compute \(t_c\). Note that now the function \(t(T)\)
is the one predicted by the new gravitational model. Finally we compute $Y_4$ from (14) for several values of $\alpha$ (or $n$). Our results are shown in Table I and Fig. 1. The results show that $Y_4$ is very sensitive to $1-n$. The freeze-out abundance decreases, while the capture time increases, as $1-n$ increases (or $\alpha$ decreases). These both effects bring about decrease in $Y_4$ as $1-n$ increases. The reason is as follows. The decrease of the freeze-out abundance is due to the decrease of the freeze-out temperature, which is calculated by equating the Hubble parameter (13) with the weak interaction rate $\Gamma \equiv \lambda_{np}(T) + \lambda_{pn}(T) \sim T^5$ [16]. The increase of the capture time is mainly due to the temperature-time relation: For fixed temperature, the corresponding time increases as $\alpha$ decreases.

We have interpolated the computed helium abundances as a function of $n$ (Fig. 1) and obtained the lower bound of $n$ in comparison with the observational data. The allowed range for $n$ is

$$1 - n \lesssim 0.00016 \quad \text{or equivalently} \quad \alpha \gtrsim 0.49992 \quad (28)$$

according to the observational constraint [17]

$$Y_4 = 0.2516 \pm 0.0040. \quad (29)$$

We see that according to our results the power $n$, compared to the value found in [8], should be pushed even closer to unity. This implies that the predicted baryon asymmetry within the framework of gravitational baryogenesis using this class of models becomes now too low. Furthermore, several existing scenarios that require a value of $n$ considerably different than unity cannot work.

IV. WIMP DARK MATTER

Recent cosmological observations [18] have established the allowed range of the normalized density of cold dark matter in the universe

$$0.075 \lesssim \Omega_{cdm} h^2 \lesssim 0.126. \quad (30)$$

In the present section we assume that the role of cold dark matter in the universe is played by weakly-interacting massive particles (WIMPs) proposed by physics beyond the standard model. For concreteness one can think of the lightest neutralino. The discussion to follow is a model-independent one, in which we have considered a generic WIMP assuming that its mass is $(100-500)$ GeV, and that its typical cross section of the relevant processes in which it participates is not very different than $\sigma \sim \alpha_{em}^2 / M_{ew}^2$, where $\alpha_{em} = 1/137$ is the electromagnetic fine structure constant, and $M_{ew} \sim 100$ GeV is the electroweak scale. The relic density of the dark matter particle depends on its mass $m$, its annihilation cross section $\sigma_0$, and finally on the power
characterizing the gravitational model. Since BBN has already determined the allowed range for $n$, we fix it to a given value and therefore the cold dark matter constraint $\Omega h^2 \simeq 0.1$ gives a certain relation between the WIMP mass and the annihilation cross section.

The evolution of the number density $n$ of the dark matter particle in an expanding universe is determined by solving the Boltzmann equation \[3, 16\]

$$\dot{n} + 3Hn = -\langle \sigma v \rangle (n^2 - n_{EQ}^2),$$

(31)

where $H$ is the Hubble parameter, $n_{EQ}$ is the number density at equilibrium, $v$ is the relative velocity, and $\sigma$ is the total annihilation cross section. The thermal average of the total annihilation cross section times the relative velocity $\langle \sigma v \rangle$ is given by

$$\langle \sigma v \rangle = \frac{1}{n_{EQ}^2} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} f(E_1)f(E_2)\sigma v,$$

(32)

where $f(E)$ is the fermion distribution function, $f(E) = 1/(1 + \exp(E/T))$. Finally the number density at equilibrium is given by

$$n_{EQ} = \int \frac{d^3 p}{(2\pi)^3} f(E).$$

(33)

Now it is convenient to introduce new variables, namely dimensionless quantities

$$x = \frac{m}{T},$$

(34)

$$Y = \frac{n}{s},$$

(35)

where $T$ is the temperature and $s$ is the entropy density

$$s = h_s \frac{2\pi^2}{45} T^3$$

(36)

with $h_s$ being the number of relativistic degrees of freedom for entropy density. Assuming entropy conservation, the Boltzmann equation can be written down equivalently as follows

$$\frac{dY}{dx} = -s \frac{sxH}{xH} \langle \sigma v \rangle (Y^2 - Y_{EQ}^2).$$

(37)

The yield at equilibrium $Y_{EQ}$ for non-relativistic (cold, $x \gg 3$) relics is given by the approximate expression

$$Y_{EQ} \simeq g \frac{45}{2\pi^4} \left( \frac{\pi}{8} \right)^{1/2} x^{3/2} \exp(-x) \frac{1}{h_s},$$

(38)

where $g = 2$ is the spin polarizations of the dark matter particle. In standard cosmology during the radiation dominated era, the Hubble parameter as a function of the temperature is given by $H(T) = 1.67 g_{*}^{1/2} T^2/M_p$. Parameterizing $\langle \sigma v \rangle$ as

$$\langle \sigma v \rangle = \sigma_0 x^{-l}$$

(39)
the Boltzmann equation takes the final compact form
\[ \frac{dY}{dx} = -\lambda x^{-1/2} (Y^2 - Y_{EQ}^2), \quad (40) \]

where \( \lambda \) is a constant given by
\[ \lambda = \left( \frac{x < \sigma v >}{H(m)} \right)_{x=1} = 0.264\left(\frac{h_s/\sqrt{g_s}}{g_{1/2}}\right)M_p m\sigma_0. \quad (41) \]

We can obtain an approximate analytical solution of Boltzmann equation by the following arguments. Initially, for large temperatures the annihilation rate is larger than the expansion rate of the universe and the WIMP abundance follows the equilibrium abundance. At some point \( x_f \) the annihilation rate becomes comparable to the expansion rate and the dark matter particle decouples from the thermal bath. For \( x \gg x_f \) we can neglect the \( Y_{EQ} \) term in the Boltzmann equation. Then the equation can be easily integrated and the solution \( Y_\infty \equiv Y(x = \infty) \) is given by
\[ Y_\infty = \frac{l + 1}{\lambda} x_f^{-1}, \quad (42) \]

where the freeze-out temperature \( x_f \) is given \[16\] by
\[ x_f = \ln\left[0.038(l + 1)\left(\frac{g_s/\sqrt{g_s}}{g_{1/2}}\right)M_p m\sigma_0 \right] \]
\[ - \left(l + \frac{1}{2}\right) \ln\{\ln\left[0.038(l + 1)\left(\frac{g_s/\sqrt{g_s}}{g_{1/2}}\right)M_p m\sigma_0 \right]\}. \quad (43) \]

After having integrated the Boltzmann equation for \( Y(x) \), then the relic abundance for the dark matter particle is given by
\[ \Omega_{cdm} h^2 = \frac{m Y_\infty s(T_0) h^2}{\rho_{cr}}, \quad (44) \]

where \( T_0 \) is the today’s temperature. Here we make use of the following values:
\[ T_0 = 2.73K = 2.35 \times 10^{-13} \text{ GeV} \quad (45) \]
\[ h_s(T_0) = 3.91 \quad (46) \]
\[ \rho_{cr}/h^2 = 8.1 \times 10^{-47} \text{ GeV}^4 \quad (47) \]

So far we discussed the case of the standard cosmology. Now let us take into account the modification of gravity. Taking into account \( a \sim t^\alpha \) and time-temperature relation \[11\], one can express the Hubble parameter as
\[ H = H_\alpha(m) x^{-1/\alpha}, \quad (48) \]
where

\[ H_\alpha(m) = \frac{\alpha A_1^2}{8\alpha M_p^2} \left( \frac{4\pi^3 g_*}{15} \right)^{1/3} m^{1/3}. \]  
(49)

For \( \alpha = 1/2 \), this reduces to the usual \( H(m) \) parameter for the standard cosmology \[16\]

\[ H_{\alpha=1/2}(m) = 1.67 g_*^{1/2} m^2/M_p = H(m). \]  
(50)

For \( x \gtrsim 3 \) the temperature dependence of the annihilation cross section is parameterized as

\[
<\sigma v> \equiv \sigma_0 x^{-l},
\]  
(51)

where \( l = 0 \) corresponds to s-wave annihilation, \( l = 1 \) to p-wave annihilation, etc. Then the Boltzmann equation for the abundance of dark matter becomes

\[
\frac{dY}{dx} = -\tilde{\lambda} x^{-l}(Y^2 - Y_{EQ}^2),
\]  
(52)

where

\[
\tilde{\lambda} = \left( x <\sigma v > s \right)_{H_\alpha(m)} = \frac{H(m)}{H_\alpha(m)} \lambda = 0.264(h_* g_*^{1/2}) M_p m \tilde{\sigma}_0.
\]  
(53)

Here we introduced new parameter \( \tilde{l} \) and \( \tilde{\sigma}_0 \) to clearly show the effect of the modification of gravity on Boltzmann equation:

\[
\tilde{l} = l + \left( 2 - \frac{1}{\alpha} \right),
\]  
(54)

\[
\tilde{\sigma}_0 = \frac{H(m)}{H_\alpha(m)} \sigma_0.
\]  
(55)

It follows that \( \tilde{l} = l \) and \( \tilde{\sigma}_0 = \sigma_0 \) for \( \alpha = 1/2 \). It is clear that the modification coming from \( f(R) \) is implemented entirely by the correction on these two parameters. By comparison to (40)-(41), one can easily see that the Boltzmann equation (52) together with (53) exactly corresponds to one of the standard cosmology with the averaged product of annihilation cross section and velocity

\[
<\sigma v> = \tilde{\sigma}_0 x^{-\tilde{l}}.
\]  
(56)

Since the Boltzmann equation has exactly the form as in standard cosmology, one would get the same results as in the standard case, but with the replacement, \( \sigma_0 \rightarrow \tilde{\sigma}_0 \) and \( l \rightarrow \tilde{l} \).

The most important quantity in estimating the relic density is \( x_f \), which is the time when \( Y \) ceases to track \( Y_{EQ} \), or equivalently, when \( Y - Y_{EQ} \) becomes of order \( Y_{EQ} \). This quantity is computed by (43) as

\[
x_f = \ln[0.038(\tilde{l} + 1)(g/g_*^{1/2}) M_p m \tilde{\sigma}_0] \\
- \left( \tilde{l} + \frac{1}{2} \right) \ln\{\ln[0.038(\tilde{l} + 1)(g/g_*^{1/2}) M_p m \tilde{\sigma}_0]\}.
\]  
(57)
Then the present yield \( Y_\infty \) and relic density \( \Omega_{cdm}h^2 \) are given by

\[
Y_\infty = \frac{3.79(\tilde{l} + 1)x_f^{\tilde{l}+1}}{(h_*/g_*/^{1/2})M_p m \tilde{\sigma}_0},
\]

(58)

\[
\Omega_{cdm}h^2 = 1.07 \times 10^y \frac{(\tilde{l} + 1)x_f^{\tilde{l}+1} \text{GeV}^{-1}}{(h_*/g_*/^{1/2})M_p \tilde{\sigma}_0}.
\]

(59)

Since the cosmic temperature during the period of interest is \( T \simeq (\text{a few})\text{GeV} \), we set \( h_* = g_* \simeq 100 \). Eq. (57) - (59) are one of the main results of this section. It is remarkable that they are expressed as an analytical function of \( \alpha \) through \( \tilde{l}, \tilde{\sigma}_0 \) and \( x_f \). Thus once \( l, \sigma_0 \) and \( m \) together with \( \alpha \) are given, the relic density is directly obtained from (59). We will use s-wave approximation \( (l = 0) \), so \( \tilde{l} = 2 - 1/\alpha \). In this case \( \tilde{l} \) is negative because \( \alpha < 1/2 \), but since we know from the consideration in the previous section that BBN allows tiny deviation of \( n \) from 1 in (8), we assume that \( \tilde{l} + 1 > 0 \) in (57)-(59).

We are finally in a position to present our numerical results in figures, showing the relation between the annihilation cross section and WIMP mass. The dark matter abundance is a function of three parameters, namely \( n,m,\sigma_0 \). If the power \( n \) is fixed according to the BBN results, and we impose the cold dark matter constraint \( 0.075 < \Omega_{cdm}h^2 < 0.126 \), it is possible to obtain a certain relation between the WIMP mass \( m \) and its annihilation cross section \( \sigma_0 \). Our results can be shown in the figures 2 and 3 below. In particular, in Fig. 2 we show the annihilation cross section as a function of the WIMP mass for fixed values of \( n \) (corresponding either to general relativity or to the new gravitational model for the range determined from BBN), and for the upper limit \( \Omega_{cdm}h^2 = 0.126 \). In Fig. 3 we show \( \sigma_0 \) as a function of \( m \) for fixed values of \( n \) and for the lower limit \( \Omega_{cdm}h^2 = 0.075 \). From these figures one can see what the lower bound (Fig. 2), and upper bound (Fig. 3) of the annihilation cross section should be for a given WIMP mass.

V. CONCLUSIONS

In the present work we have studied primordial Big-Bang nucleosynthesis and WIMP dark matter in a class of modified gravitational theories. This class of gravitational models predict a novel expansion law for the early universe. For BBN we have employed a semi-analytical computation in which the basic physics is quite transparent. For WIMP dark matter we have given a model independent discussion applying the usual treatment found in standard textbooks or reviews. Concerning BBN, by comparing the theoretical predictions to the available observational data we were able to put bounds on the unique parameter appearing in this class of modified gravitational theories. We have found that the models considered in the present work are allowed to be only slightly different from the usual Einstein’s general relativity. In the dark matter
section we have obtained an analytical expression for the cold dark matter abundance as a function of $n$. After that we fixed the power $n$ according to the BBN results, and we have shown in figures how the annihilation cross section and the WIMP mass are related in order that the cold dark matter constraint is satisfied. The predicted baryon asymmetry within the framework of gravitational baryogenesis using this class of models is too low for our BBN range of $n$, and therefore this mechanism for baryon asymmetry does not seem to be consistent with BBN constraints. Finally we remark in passing that the models that satisfy our bounds do not lead to the late cosmic acceleration.

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[1] C. Munoz, Int. J. Mod. Phys. A **19** (2004) 3093 [arXiv:hep-ph/0309346].
[2] M. Taoso, G. Bertone and A. Masiero, JCAP **0803** (2008) 022 [arXiv:0711.4996 [astro-ph]].
[3] J. L. Feng, *In the Proceedings of 31st SLAC Summer Institute on Particle Physics: Cosmic Connection to Particle Physics (SSI 2003), Menlo Park, California, 28 Jul - 8 Aug 2003*, pp L11 [arXiv:hep-ph/0405215].
[4] G. Jungman, M. Kamionkowski and K. Griest, Phys. Rept. **267** (1996) 195 [arXiv:hep-ph/9506380].
[5] F. Iocco, G. Mangano, G. Miele, O. Pisanti and P. D. Serpico, [arXiv:0809.0631 [astro-ph]].
[6] S. Capozziello, V. F. Cardone, S. Carloni and A. Troisi, Phys. Lett. A **326** (2004) 292 [arXiv:gr-qc/0404114];
C. F. Martins and P. Salucci, Mon. Not. Roy. Astron. Soc. **381** (2007) 1103 [arXiv:astro-ph/0703243].
[7] S. Capozziello, V. F. Cardone, S. Carloni and A. Troisi, Int. J. Mod. Phys. D **12** (2003) 1969 [arXiv:astro-ph/0307018];
S. Capozziello, V. F. Cardone and A. Troisi, JCAP **0608** (2006) 001 [arXiv:astro-ph/0602349].
[8] G. Lambiase and G. Scarpetta, Phys. Rev. D **74** (2006) 087504 [arXiv:astro-ph/0610367].
[9] H. Davoudiasl, R. Kitano, G. D. Kribs, H. Murayama and P. J. Steinhardt, Phys. Rev. Lett. **93** (2004) 201301 [arXiv:hep-ph/0403019].
[10] J. Bernstein, L. S. Brown and G. Feinberg, Rev. Mod. Phys. 61, 25 (1989).
[11] R. V. Wagoner, W. A. Fowler and F. Hoyle, Astrophys. J. **148** (1967) 3; L. Kawano, “Let’s Go: Early Universe. Guide To Primordial Nucleosynthesis Programming,” FERMILAB-PUB-88-034-A;
L. Kawano, “Let’s go: Early universe. 2. Primordial nucleosynthesis: The Computer way,” FERMILAB-PUB-92-004-A;
O. Pisanti, A. Cirillo, S. Esposito, F. Iocco, G. Mangano, G. Miele and P. D. Serpico, Comput. Phys. Commun.
178 (2008) 956 [arXiv:0705.0290 [astro-ph]].

[12] J. C. Fabris and J. A. O. Marinho, Grav. Cosmol. 9(2003)270 [arXiv:astro-ph/0306051].

[13] A. F. Zakharov, A. A. Nucita, F. De Paolis and G. Ingrosso, Phys. Rev. D 74 (2006) 107101 [arXiv:astro-ph/0611051].

[14] PARTICLE DATA GROUP collaboration, W.M. Yao et al., Review of particle physics, J. Physics. G 33(2006) 1.

[15] V. Mukhanov, Int. J. Theor. Phys. 43 (2004) 669 [arXiv:astro-ph/0303073].

[16] E. W. . Kolb and M. S. . Turner, “THE EARLY UNIVERSE. REPRINTS,” REDWOOD CITY, USA: ADDISON-WESLEY (1988) 719 P. (FRONTIERS IN PHYSICS, 70)

[17] Y. I. Izotov, T. X. Thuan and G. Stasinska, Astrophys. J. 662, 15 (2007) [arXiv:astro-ph/0702072].

[18] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 170 (2007) 377 [arXiv:astro-ph/0603449].
Figure 1: Theoretical helium 4 abundance versus $\delta = 1 - n$. The strip shows the allowed observational range.

Figure 2: Annihilation cross section versus WIMP mass for CDM abundance $\Omega_{cdm}h^2 = 0.126$. Shown are $n = 1$ (solid), $n = 1 - 10^{-4}$ (dashed), and $n = 1 - 2 \times 10^{-4}$ (dotted).
Figure 3: Same as figure 3, but for CDM abundance $\Omega_{cdm}h^2 = 0.075$. 