Reduction of one-loop tensor 5-point integrals

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Abstract:
A new method for the reduction of one-loop tensor 5-point integrals to related 4-point integrals is proposed. In contrast to the usual Passarino–Veltman reduction and other methods used in the literature, this reduction avoids the occurrence of inverse Gram determinants, which potentially cause severe numerical instabilities in practical calculations. Explicit results for the 5-point tensor coefficients are presented up to rank 4. The expressions for the reduction of the relevant 1-, 2-, 3-, and 4-point tensor coefficients to scalar integrals are also included; apart from these standard integrals no other integrals are needed.

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1 Introduction

High-energy collider experiments reached a high level of accuracy in the last decades. For instance, the LEP, SLC, and Tevatron experiments tested the Standard Model of electroweak and strong interactions as quantum field theories, in the sense that quantum corrections, i.e., higher-order perturbative radiative corrections, had to be taken into account to successfully compare predictions with data. The most precise results typically were obtained from investigations of $1 \rightarrow 2$ particle decays and $2 \rightarrow 2$ scattering reactions. At present and future colliders, such as the Tevatron, the LHC, and an $e^+ e^-$ linear collider, scattering reactions with more final-state particles will gain increasing interest in various contexts, such as in the production of jets, heavy quark flavours, electroweak gauge bosons, and Higgs bosons.

In this paper we focus on $2 \rightarrow 3$ particle reactions and present one of the basic ingredients in the evaluation of radiative corrections at the one-loop level, the calculation of 5-point tensor integrals. Following the well-known procedure of Passarino and Veltman [1], one-loop tensor integrals can be recursively reduced to scalar integrals by solving sets of linear equations for the tensor coefficients in the most general Lorentz-covariant ansatz. Specifically, for each tensor rank this procedure involves a factor of the inverse Gram matrix which is built by the momenta that span the tensor. Since these momenta become linearly dependent at the boundary of phase space, the inverse Gram matrix becomes singular at the boundary. In practice, this leads to numerical instabilities near the phase-space boundary, where the zero in the Gram determinant appearing in the denominator of a tensor coefficient is compensated by a delicate numerical cancellation between scalar integrals in the numerator. The situation is not very problematic in $2 \rightarrow 2$ reactions, where the boundary is reached in forward and backward scattering, i.e., in isolated points in a single phase-space variable (scattering angle $\theta = 0, \pi$). In the three-particle phase space the situation is more involved, and the numerical problems caused by vanishing Gram determinants of four momenta turn out to be serious. Fortunately the appearance of these Gram determinants can be avoided completely if the four-dimensionality of space time is exploited. In the following we describe such a procedure.

Other methods for calculating tensor 5-point integrals via reduction to simpler integrals have been described in Ref. [2]; however, in these approaches either Gram determinants were not avoided completely, or the set of standard integrals had to be extended. Various treatments of scalar 5-point integrals have been presented in Refs. [2–6]. Methods for the numerical integration of scalar 5- and 6-point integrals [7] and of general one-loop integrals with up to six external legs [8] with special treatments of singularities have been proposed recently.

It is known since a long time that the scalar 5-point integral can be reduced to scalar 4-point integrals by using the linear dependence of the integration momentum on the four external momenta in four space–time dimensions [3,4]. In Ref. [5] a generalization of this reduction formula to tensor 5-point integrals has been proposed, but some extra terms were missing in the derivation. We supplement the derivation of Ref. [5] by these missing terms and work out explicit formulas for the tensor coefficients up to rank 4. The relevant formulas for the reduction of the coefficients of the 1-, 2-, 3-, and 4-point functions to standard scalar integrals are also presented. Apart from these standard functions no
integrals are needed. In summary, the results of this paper comprise all one-loop tensor integrals that occur in $1 \to 2$, $2 \to 2$, and $2 \to 3$ particle reactions with up to four external gauge bosons (and an arbitrary number of spin-$\frac{1}{2}$ fermions and scalars) in renormalizable gauge theories. Apart from the six-point functions, the presented set covers also all one-loop tensor integrals appearing in processes with six external particles including up to two external bosons, i.e., in particular, the processes $2f \to 4f$ and $2f \to 2f + 2V$. Only scalar 1-, 2-, 3-, and 4-point integrals have to be calculated in addition; to this end, methods and explicit results can be found in the literature [2,5,9,10].

In its present formulation, the described approach applies to tensor 5-point integrals in four space–time dimensions, i.e., possible infrared (IR) singularities of soft or collinear origin are assumed to be regularized by off-shell or mass regulators.\textsuperscript{1} This procedure is widely used in the calculation of electroweak corrections. However, the present formalism can also be used in dimensional regularization, which is usually adopted in QCD for the treatment of IR singularities. The translation of mass regulators into dimensional regularization is generally described in Ref. [11] for complete QCD and SUSY-QCD amplitudes. A possible strategy for the corresponding treatment of individual integrals can be found in Ref. [12].

The method proposed in this paper has already been tested in actual calculations: it has been used in the calculation of the QCD corrections to the process $gg/\bar{q}q \to t\bar{t}H$ [12] and the electroweak corrections to $e^+e^- \to \nu\bar{\nu}H$ [13]. The results are in numerical agreement with an evaluation that is based on the usual Passarino–Veltman reduction, but a drastic improvement in the numerical stability of the 5-point function was observed.

The paper is organized as follows. In Section 2 we derive the general relation between tensor 5-point integrals and the related 4-point functions. The explicit results for the tensor coefficients of the 5-point integrals up to rank 4 are presented in Section 3. In Section 4 we describe some consistency checks and applications of the presented results; details on the generalizations of the method to dimensional regularization can also be found there. Section 5 contains our conclusion. In App. A we describe the derivation of extra terms that arise from UV divergences in intermediate expressions. Appendices B and C comprise useful results for 1-, 2-, 3-, and 4-point functions.

2 Derivation of the reduction formula for tensor 5-point integrals

In the reduction of tensor 5-point integrals to 4-point integrals we closely follow the strategy and notation of Refs. [5,14]. The one-loop 4- and 5-point functions are defined as

\[
D_{(0,\mu,\nu,\rho,\sigma,\tau)}(p_1, p_2, p_3, m_0, m_1, m_2, m_3) = \frac{(2\pi\mu)^{(4-D)}}{i\pi^2} \int d^Dq \frac{\{1, q_\mu, q_\nu, q_\rho, q_\sigma, q_\tau\}}{N_0 N_1 N_2 N_3},
\]

\[
E_{(0,\mu,\nu,\rho,\sigma,\tau)}(p_1, p_2, p_3, p_4, m_0, m_1, m_2, m_3, m_4) = \frac{1}{i\pi^2} \int d^4q \frac{\{1, q_\mu, q_\nu, q_\rho, q_\sigma, q_\tau\}}{N_0 N_1 N_2 N_3 N_4},
\]

\textsuperscript{1}In 5-point functions, ultraviolet (UV) divergences appear only for tensor rank $\geq 6$, which is not considered in this paper.
with the denominator factors

\[ N_i = (q + p_i)^2 - m_i^2 + i\epsilon, \quad i = 0, \ldots, 4, \quad p_0 = 0, \quad (2.2) \]

where \( i\epsilon (\epsilon > 0) \) denotes an infinitesimal imaginary part. Apart from \( D_{\mu\nu\rho\sigma} \) and \( D_{\mu\nu\rho\sigma\tau} \), all above tensor integrals are UV finite and can be evaluated in four dimensions \((D = 4)\).

The reduction of the 5-point function to 4-point functions is based on the fact that in four dimensions the integration momentum \( q \) depends linearly on the four external momenta \( p_i \) [3]. This gives rise to the identity

\[
0 = \begin{vmatrix} 2q^2 & 2qp_1 & \ldots & 2qp_4 \\ 2p_1q & 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ 2p_4q & 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix} = \begin{vmatrix} 2N_0 + Y_{00} & 2qp_1 & \ldots & 2qp_4 \\ N_1 - N_0 + Y_{10} - Y_{00} & 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ N_4 - N_0 + Y_{40} - Y_{00} & 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix} \quad (2.3)
\]

with

\[ Y_{ij} = m_i^2 + m_j^2 - (p_i - p_j)^2, \quad i, j = 0, \ldots, 4. \quad (2.4) \]

Equation (2.3) implies

\[
0 = \frac{1}{i\pi^2} \int q^4 q \frac{q_{\mu_1} \ldots q_{\mu_P} - \Lambda^2}{N_0N_1 \cdots N_4 q^2 - \Lambda^2} \begin{vmatrix} 2N_0 + Y_{00} & 2qp_1 & \ldots & 2qp_4 \\ N_1 - N_0 + Y_{10} - Y_{00} & 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ N_4 - N_0 + Y_{40} - Y_{00} & 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix}, \quad (2.5)
\]

where \( P \) denotes the number of integration momenta in the numerator. UV divergences that occur in intermediate steps by expanding the determinant are regularized temporarily with a cutoff \( \Lambda \to \infty \), in order to be able to exploit the four-dimensionality of space–time. This approach is valid for all considered 5-point integrals, i.e., for \( P \leq 4 \). Expanding the determinant along the first column, we obtain

\[
0 = \begin{vmatrix} 2D^{(A)}_{\mu_1 \ldots \mu_P} (0) + Y_{00}E^{(A)}_{\mu_1 \ldots \mu_P} \\ 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix} + \sum_{i=1}^{4} (-1)^i \{ D^{(A)}_{\alpha\mu_1 \ldots \mu_P} (i) - D^{(A)}_{\alpha\mu_1 \ldots \mu_P} (0) + (Y_{i0} - Y_{00})E^{(A)}_{\alpha\mu_1 \ldots \mu_P} \}
\]

\[
\times \begin{vmatrix} 2p_1^\alpha & \ldots & 2p_4^\alpha \\ 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix}, \quad (2.6)
\]

\[
0 = \begin{vmatrix} 2p_1^\alpha & \ldots & 2p_4^\alpha \\ 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix},
\]

\[ 0 = \begin{vmatrix} 2p_1^\alpha & \ldots & 2p_4^\alpha \\ 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix},
\]

\[ 0 = \begin{vmatrix} 2p_1^\alpha & \ldots & 2p_4^\alpha \\ 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix},
\]

\[ 0 = \begin{vmatrix} 2p_1^\alpha & \ldots & 2p_4^\alpha \\ 2p_1p_1 & \ldots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \ldots & 2p_4p_4 \\ \end{vmatrix},
\]
where \( D^{(\Lambda)}_{\mu_1 \ldots \mu_P}(i) \) denotes the 4-point function that is obtained from the 5-point function \( E^{(\Lambda)}_{\mu_1 \ldots \mu_P} \) by omitting the \( i \)th propagator \( N_i^{-1} \). The superscript \((\Lambda)\) indicates the regularization as introduced in (2.5). Since all appearing 5-point functions are UV finite (rank \( \leq 5 \)), we can directly perform the limit \( \Lambda \to \infty \) there and omit the superscript \((\Lambda)\) for all \( E \) functions. The same can be done for the UV-finite 4-point functions. The UV-divergent 4-point functions are for asymptotically large \( \Lambda \) split as follows,

\[
D^{(\Lambda)}_{\mu_1 \ldots \mu_P} \xrightarrow{\Lambda \to \infty} D^{(\text{fin})}_{\mu_1 \ldots \mu_P} + \Delta_{\mu_1 \ldots \mu_P},
\]

where \( D^{(\text{fin})}_{\mu_1 \ldots \mu_P} \) is the UV-finite part of the usual 4-point function \( D_{\mu_1 \ldots \mu_P} \) in \( D \) dimensions defined in (2.1). The UV divergence is subtracted from \( D_{\mu_1 \ldots \mu_P} \) as in the \( \overline{\text{MS}} \) scheme.\(^2\) Whenever the functions \( D_{\mu_1 \ldots \mu_P} \) (or their coefficients) are finite, \( D_{\mu_1 \ldots \mu_P} \) and \( D^{(\text{fin})}_{\mu_1 \ldots \mu_P} \) are identical, and we simply write \( D_{\mu_1 \ldots \mu_P} \). The terms \( \Delta_{\mu_1 \ldots \mu_P} \) contain the dependence on \( \Lambda \), but are also finite for finite \( \Lambda \). Inserting (2.7) into (2.6) and taking \( \Lambda \) asymptotically large, results in

\[
0 = \left[ 2D^{(\text{fin})}_{\mu_1 \ldots \mu_P}(0) + 2\Delta_{\mu_1 \ldots \mu_P}(0) + Y_{00}E_{\mu_1 \ldots \mu_P} \right] \begin{vmatrix} 2p_1 p_1 & \ldots & 2p_1 p_4 \\ \vdots & \ddots & \vdots \\ 2p_4 p_1 & \ldots & 2p_4 p_4 \end{vmatrix} + \sum_{i=1}^{4} (-1)^i \left( D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(i) - \left[ D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(0) + p_{1\alpha}D^{(\text{fin})}_{\mu_1 \ldots \mu_P}(0) \right] \right) + p_{1\alpha}D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(0)
\]

\[
+ \Delta_{\alpha \mu_1 \ldots \mu_P}(i) - \Delta_{\alpha \mu_1 \ldots \mu_P}(0) + (Y_{10} - Y_{00})E_{\alpha \mu_1 \ldots \mu_P} \right] \begin{vmatrix} 2p_0^\alpha & \ldots & 2p_4^\alpha \\ 2p_1 p_1 & \ldots & 2p_1 p_4 \\ \vdots & \ddots & \vdots \\ 2p_4 p_1 & \ldots & 2p_4 p_4 \end{vmatrix} \tag{2.8}
\]

The terms involving \( p_{1\alpha}D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(0) \) have been added for later convenience. The extra terms involving \( \Delta_{\mu_1 \ldots \mu_P} \) are absent for \( P \leq 2 \) where no UV-singular integrals appear; they drop out in the final result for \( P = 3 \), but contribute for \( P = 4 \).

The contributions to 4-point functions that involve a momentum \( p_{j\alpha} \) will be simplified in the following. To this end, we introduce the Lorentz-covariant decompositions

\[
D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(i) = \left[ D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(i) \right]^{(p)} + \left[ D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(i) \right]^{(g)}, \quad i = 0, \ldots, 4,
\]

\[
\left[ D^{(\text{fin})}_{\alpha \mu_1 \ldots \mu_P}(i) \right]^{(p)} = \sum_{j=1}^{4} p_{j\alpha} X_{j,\mu_1 \ldots \mu_P}(i),
\]

\(^2\)The final results for the 5-point functions \( E_{\mu_1 \ldots \mu_P} \) do not depend on the details of this subtraction, because these results are finite. We perform the subtraction in order to avoid confusion in products involving metric tensors. Since the subtraction guarantees that all appearing coefficients are finite, the question of whether we take the metric tensor in 4 or \( D \) dimensions concerns only irrelevant terms of \( O(D - 4) \).
\[ [D_{\alpha_1^\ldots\alpha_P}^{(\text{fin})}(i)]^{(g)} = \sum_{j=1}^{P} g_{\alpha_\mu_j} Y_{j,\mu_1^\ldots\mu_j-1,\mu_{j+1}^\ldots\mu_P}(i), \]

\[ [D_{\alpha_1^\ldots\alpha_P}^{(\text{fin})}(0) + p_{1\alpha} D_{\mu_1^\ldots\mu_P}^{(\text{fin})}(0)]^{(p)} = \sum_{j=2}^{4} (p_j - p_1) \alpha Z_{\mu_1^\ldots\mu_P}. \tag{2.9} \]

The operation “\((g)\)” isolates all tensor structures in which the first Lorentz index appears at a metric tensor; the remaining part of the tensor furnishes the “\((p)\)” contribution. The last decomposition in (2.9) becomes obvious after performing a shift \(q \rightarrow q - p_1\) in the integral. From (2.9) it follows immediately that the terms in (2.8) that involve \([D_{\alpha_1^\ldots\alpha_P}^{(\text{fin})}(i)]^{(p)}\) drop out when multiplied with the determinants, because the resulting determinants vanish after summation over \(i\). Finally, the term \(p_{1\alpha} D_{\mu_1^\ldots\mu_P}^{(\text{fin})}(0)\) contributes only for \(i = 1\), where it can be combined with the first term in (2.8). Rewriting the resulting equation using determinants and reinserting the explicit form of the tensor integrals leads to

\[
\left[ \frac{(2\pi\mu)^{4-D}}{1\pi^2} \int d^D q \frac{q_{\mu_1} \ldots q_{\mu_P}}{N_0 N_1 \ldots N_4} \begin{vmatrix} N_0 + Y_{00} & 2q_{p_1} \ldots & 2q_{p_4} \\ N_{10} - Y_{00} & 2p_1 p_1 \ldots & 2p_1 p_4 \\ \vdots & \vdots & \ddots & \vdots \\ Y_{40} - Y_{00} & 2p_4 p_1 \ldots & 2p_4 p_4 \end{vmatrix} \right]^{(\text{fin})} = V_{\mu_1^\ldots\mu_P} + U_{\mu_1^\ldots\mu_P}. \tag{2.10} \]

Here we introduced

\[
V_{\mu_1^\ldots\mu_P} = -\begin{vmatrix} 0 & 2p_1^1 \ldots & 2p_4^1 \\ D_{\alpha_1^\ldots\alpha_P}(1) & 2p_1 p_1 \ldots & 2p_1 p_4 \\ \vdots & \vdots & \ddots & \vdots \\ D_{\alpha_1^\ldots\alpha_P}(4) & 2p_4 p_1 \ldots & 2p_4 p_4 \end{vmatrix} = \sum_{i,j=1}^{4} (-1)^{i+j} \det(\hat{Z}_{ij}^{(4)}) 2p_j^\alpha D_{\alpha_1^\ldots\alpha_P}(i) \tag{2.11} \]

with

\[
D_{\alpha_1^\ldots\alpha_P}(i) = [D_{\alpha_1^\ldots\alpha_P}(0) - D_{\alpha_1^\ldots\alpha_P}(i)]^{(g)}, \quad i = 1, \ldots, 4, \tag{2.12} \]

which collects all terms involving \([D]^{(g)}\). The extra term \(V\) is absent for the reduction of the scalar 5-point integral since \(D_\alpha = [D_\alpha]^{(p)}\). The three-dimensional matrices \(\hat{Z}_{ij}^{(4)}\) result from the four-dimensional Gram matrix

\[
Z^{(4)} = \begin{pmatrix} 2p_1 p_1 & \ldots & 2p_1 p_4 \\ \vdots & \ddots & \vdots \\ 2p_4 p_1 & \ldots & 2p_4 p_4 \end{pmatrix} \tag{2.13} \]

by discarding the \(i\)th row and \(j\)th column. The term \(U_{\mu_1^\ldots\mu_P}\) resulting from the UV divergences reads

\[
U_{\mu_1^\ldots\mu_P} = -2\Delta_{\mu_1^\ldots\mu_P}(0) \det(Z^{(4)}) + \sum_{i,j=1}^{4} (-1)^{i+j} \det(\hat{Z}_{ij}^{(4)}) 2p_j^\alpha [\Delta_{\alpha_1^\ldots\alpha_P}(i) - \Delta_{\alpha_1^\ldots\alpha_P}(0)]. \tag{2.14} \]

\(^3\)In Ref. [5] the \(V\) and \(U\) terms are missing.
The actual evaluation of $U_{\mu_1...\mu_P}$ is described in App. A and yields

$$U_{\mu_1...\mu_P} = \begin{cases} 0 & \text{for } P \leq 3, \\ -\frac{1}{48}(g_{\mu_1\mu_2}g_{\rho_3\mu_4} + g_{\mu_3\mu_2}g_{\rho_4\mu_2} + g_{\rho_1\mu_4}g_{\rho_2\mu_3}) \det(\hat{Z}^{(4)}) & \text{for } P = 4. \end{cases} \quad (2.15)$$

We did not investigate these terms for $P \geq 5$.

Using

$$2p_i p_j = Y_{ij} - Y_{i0} - Y_{0j} + Y_{00}, \quad 2q p_j = N_j - N_0 + Y_{0j} - Y_{00}, \quad (2.16)$$

we can transform the determinant on the left-hand side of (2.10) by adding the first column to each of the other columns, and then enlarging the determinant by one column and one row, resulting in

$$\begin{vmatrix} 1 & Y_{00} & \cdots & Y_{04} \\ 0 & D^{(\text{fin})}_{\mu_1...\mu_P}(0) + Y_{00} E_{\mu_1...\mu_P} & \cdots & D^{(\text{fin})}_{\mu_1...\mu_P}(4) + Y_{04} E_{\mu_1...\mu_P} \\ 0 & Y_{10} - Y_{00} & \cdots & Y_{14} - Y_{04} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Y_{40} - Y_{00} & \cdots & Y_{44} - Y_{04} \end{vmatrix} = V_{\mu_1...\mu_P} + U_{\mu_1...\mu_P}. \quad (2.17)$$

Equation (2.17) is equivalent to

$$\begin{vmatrix} E_{\mu_1...\mu_P} - D^{(\text{fin})}_{\mu_1...\mu_P}(0) & -D^{(\text{fin})}_{\mu_1...\mu_P}(1) & -D^{(\text{fin})}_{\mu_1...\mu_P}(2) & -D^{(\text{fin})}_{\mu_1...\mu_P}(3) & -D^{(\text{fin})}_{\mu_1...\mu_P}(4) \\ 1 & Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 1 & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 1 & Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ 1 & Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ 1 & Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{vmatrix} = V_{\mu_1...\mu_P} + U_{\mu_1...\mu_P}, \quad (2.18)$$

which expresses the tensor 5-point function $E_{\mu_1...\mu_P}$ in terms of five tensor 4-point functions

$$E_{\mu_1...\mu_P} = \sum_{i=0}^{4} \frac{\det(Y_i)}{\det(Y)} D^{(\text{fin})}_{\mu_1...\mu_P}(i) + \sum_{i,j=1}^{4} (-1)^{i+j} \frac{\det(\hat{Z}^{(4)}_{ij})}{\det(Y)} 2p^\alpha D_{\alpha\mu_1...\mu_P}(i) \quad (2.19)$$

where $Y = (Y_{ij}), i = 0, \ldots, 4$, and $Y_i$ is obtained from the 5-dimensional Cayley matrix $Y$ by replacing all entries in the $i$th column with 1.

With the shorthand notation

$$\eta_i = \frac{\det(Y_i)}{\det(Y)}, \quad i = 0, \ldots, 4, \quad \zeta_{ij} = (-1)^{i+j} \frac{\det(\hat{Z}^{(4)}_{ij})}{\det(Y)} = \zeta_{ji}, \quad i, j = 1, \ldots, 4, \quad (2.20)$$
this reads
\[
E_{\mu_1 \ldots \mu_P} = - \sum_{i=0}^{4} \eta_i D_{\mu_1 \ldots \mu_P}^{(\text{fin})}(i) + 2 \sum_{i,j=1}^{4} \zeta_{ij} p_j^\alpha D_{\alpha \mu_1 \ldots \mu_P}(i) + \frac{1}{\det(Y)} U_{\mu_1 \ldots \mu_P}
\]
(2.21)
with \( U \) given in (2.15) for \( P \leq 4 \).

3 Explicit formulas for 5-point tensor coefficients

Here we further exploit (2.21) to derive explicit formulas for the coefficients of tensor 5-point integrals appearing in convenient decompositions into Lorentz covariants. In order to be able to write down the tensor decompositions in a concise way we introduce the notation
\[
T[\mu_1 \ldots \mu_P] = T[\mu_1 \ldots \mu_P] + T[\mu_2 \ldots \mu_{P-1}] + \ldots + T[\mu_P \mu_1],
\]
i.e., Lorentz indices within square brackets represent a sum over all tensors with cyclic permutations of these indices. For example, we have
\[
g_{[\mu \nu \rho \sigma]} = g_{\mu \nu} g_{\rho \sigma} + g_{\nu \rho} g_{\mu \sigma} + g_{\rho \mu} g_{\nu \sigma}.
\]
(3.2)
This notation can be iterated, e.g.,
\[
T[\mu [\nu \rho]] = T[\mu \nu \rho] + T[\mu \rho \nu] = T[\mu \nu \rho + T[\nu \rho \mu] + T[\rho \mu \nu] + T[\mu \rho \nu] + T[\nu \rho \mu] + T[\rho \mu \nu] + T[\mu \rho \nu] + T[\nu \rho \mu] + T[\rho \mu \nu] + T[\mu \rho \nu].
\]
(3.3)
After cancelling the denominator \( N_0 \) the resulting tensor integrals are not in the standard form but can be expressed in terms of standard integrals by shifting the integration momentum. We choose to perform the shift \( q \to q - p_1 \), so that the following 4-point integrals appear:
\[
\tilde{D}_{\{0,\mu,\nu,\rho,\sigma,\tau\}}(0) = \frac{(2\pi\mu)^{(4-D)}}{i2^D} \int d^D q \frac{1}{\tilde{N}_1 \tilde{N}_2 \tilde{N}_3 \tilde{N}_4} \ln_{\tilde{N}_i} (q + p_i - p_1)^2 - m_i^2 + i\epsilon, \quad i = 1, \ldots, 4.
\]
(3.4)
Note that the scalar integral \( D_0 \), and the tensor coefficients \( D_{00} \) and \( D_{0000} \) are invariant under this shift. Therefore, we can omit the tilde on these functions. In the decomposition of \( D_{\{\mu,\nu,\rho,\sigma,\tau\}}(i) \) with \( i = 1, \ldots, 4 \) shifted indices appear which we denote as
\[
j_i = \begin{cases} j & \text{for } i > j, \\ j - 1 & \text{for } i < j. \end{cases}
\]
(3.5)
We also use the notation \( \tilde{\delta}_{ij} = 1 - \delta_{ij}, \) i.e., \( \sum_i \tilde{\delta}_{ij}(\ldots) = \sum_{i\neq j}(\ldots) \).

(i) Scalar integral

For the scalar 5-point function (2.21) reads
\[
E_0 = - \sum_{i=0}^{4} \eta_i D_0(i),
\]
(3.6)
as was also obtained in Refs. [3–5,14].
(ii) Vector integral

For the vector 5-point function and the relevant 4-point functions we have the following covariant decompositions

\[ E^\mu = \sum_{j=1}^{4} p_j^\mu E_j, \]
\[ D^\mu(i) = \sum_{j=1, j \neq i}^{4} p_j^\mu D_{ji}(i), \quad i = 1, \ldots, 4, \]
\[ D^\mu(0) = \tilde{D}^\mu(0) - p_1^\mu D_0(0) = \sum_{j=2}^{4} (p_j - p_1)^\mu \tilde{D}_{j-1}(0) - p_1^\mu D_0(0), \]
\[ D^{\alpha \mu}(i) = g^{\alpha \mu}[D_{00}(i) - D_{00}(0)]. \quad (3.7) \]

Inserting these into (2.21) for the vector integral, we find for the components

\[ E_j = -\sum_{i=1}^{4} \eta_i D_{ji}(i) \delta_{ji} - \eta_0 D_j(0) + 2 \sum_{i=1}^{4} \zeta_{ji} [D_{00}(i) - D_{00}(0)], \quad j = 1, \ldots, 4, \quad (3.8) \]

with

\[ D_1(0) = -\sum_{j=1}^{3} \tilde{D}_j(0) - D_0(0), \quad D_j(0) = \tilde{D}_{j-1}(0), \quad j = 2, 3, 4. \quad (3.9) \]

(iii) Tensor integral of rank 2

For the case of the second rank tensor we introduce the covariant decompositions

\[ E^{\mu \nu} = \sum_{j,k=1}^{4} p_j^\mu p_k^\nu E_{jk} + g^{\mu \nu} E_{00}, \]
\[ D^{\mu \nu}(i) = \sum_{j,k=1}^{4} p_j^\mu p_k^\nu D_{jk}(i) + g^{\mu \nu} D_{00}(i), \quad i = 1, \ldots, 4, \]
\[ D^{\mu \nu}(0) = \tilde{D}^{\mu \nu}(0) - p_1^{[\mu} \tilde{D}^{\nu]}(0) + p_1^\mu p_1^\nu D_0(0) = \sum_{j,k=2}^{4} (p_j - p_1)^{[\mu} (p_k - p_1)^{\nu] \tilde{D}_{j-1,k-1}(0) + g^{\mu \nu} D_{00}(0) \]
\[ - \sum_{j=2}^{4} p_1^{[\mu} (p_j - p_1)^{\nu]} \tilde{D}_{j-1}(0) + p_1^\mu p_1^\nu D_0(0), \]
\[ D^{\alpha \mu \nu}(i) = [D^{\alpha \mu \nu}(i) - D^{\alpha \mu \nu}(0)]^{(g)} \]
\[ = [D^{\alpha \mu \nu}(i) - \tilde{D}^{\alpha \mu \nu}(0) + \tilde{D}^{\alpha \nu}(0)p_1^\mu]^{(g)} \]
\[ = \sum_{j=1}^{4} g^{\alpha [\mu} p_j^\nu D_{00j}(i) - \sum_{j=2}^{4} g^{\alpha [\mu} (p_j - p_1)^{\nu]} \tilde{D}_{00j-1}(0) + p_1^\mu g^{\nu \alpha} D_{00}(0). \quad (3.10) \]
The $E_{00}$ term in the decomposition of $E^{\mu\nu}$ is redundant in the sense that the corresponding covariant, $g^{\mu\nu}$, can be expressed by the four linearly independent vectors $p_j$ up to terms of $O(D - 4)$,

$$g^{\mu\nu} = \sum_{j,k=1}^{4} 2p_j^\mu p_k^\nu (Z^{(4)})_{jk}^{-1} + O(D - 4). \quad (3.11)$$

This means that the set of covariants in the decomposition of $E^{\mu\nu}$ is overcomplete, and the coefficient $E_{00}$ can be defined by convenience. We use this freedom to avoid $\det(Z^{(4)})$ in the denominators.

With the above decompositions we find from (2.21)

$$E_{00} = -4 \sum_{i=1}^{4} \eta_i D_{00}^{(i)} - \eta_0 D_{00}(0),$$

$$E_{jk} = 2 \left\{ \sum_{i=1}^{4} 4 \zeta_{ji} [D_{00k}^{(i)}(i) - D_{00k}(0)] + (j \leftrightarrow k) \right\} - 4 \eta_i D_{jk}(i) \delta_{ji} - \eta_0 D_{jk}(0), \quad j, k = 1, \ldots, 4, \quad (3.12)$$

with

$$D_{11}(0) = 3 \sum_{j,k=1}^{4} D_{jk}(0) + 2 \sum_{j=1}^{4} D_j(0) + D_0(0),$$

$$D_{j1}(0) = -3 \sum_{k=1}^{4} D_{j-1,k}(0) - D_{j-1}(0),$$

$$D_{jk}(0) = D_{j-1,k-1}(0),$$

$$D_{001}(0) = -3 \sum_{j=1}^{4} D_{00j}(0) - D_{00}(0),$$

$$D_{00j}(0) = D_{00,j-1}(0), \quad j, k = 2, 3, 4. \quad (3.13)$$

(iv) Tensor integral of rank 3

The covariant decompositions of the integrals appearing in the reduction of the 3rd rank tensor read

$$E^{\mu\nu\rho} = \sum_{j,k,l=1}^{4} p_j^\mu p_k^\nu p_l^\rho E_{jkl} + \sum_{j=1}^{4} g^{\mu\nu\rho} p_j^\alpha E_{00j},$$

$$D^{\mu\nu\rho}(i) = \sum_{j,k,l=1}^{4} p_j^\mu p_k^\nu p_l^\rho D_{j,k,l}(i) + \sum_{j=1}^{4} g^{\mu\nu\rho} p_j^\alpha D_{00j}(i), \quad i = 1, \ldots, 4,$$

$$D^{\mu\nu\rho}(0) = \tilde{D}^{\mu\nu\rho}(0) - p_1^\mu \tilde{D}^{\nu\rho}(0) + p_1^\nu \tilde{D}^{\mu\rho}(0) - p_1^\rho \tilde{D}^{\mu\nu}(0) - p_1^\mu p_1^\nu p_1^\rho D_0(0)$$

$$= \sum_{j,k,l=2}^{4} (p_j - p_1)^\mu (p_k - p_1)^\nu (p_l - p_1)^\rho \tilde{D}_{j-1,k-1,l-1}(0)$$
with

\[ D_{111}(0) = - \sum_{j,k,l=1}^3 \tilde{D}_{jkl}(0) - 3 \sum_{j,k=1}^3 \tilde{D}_{jk}(0) - 3 \sum_{j=1}^3 \tilde{D}_j(0) - D_0(0), \]

\[ D_{j11}(0) = \sum_{k,l=1}^3 \tilde{D}_{j-1,kl}(0) + 2 \sum_{k=1}^3 \tilde{D}_{j-1,k}(0) + \tilde{D}_{j-1}(0), \]

\[ D_{jkl}(0) = \tilde{D}_{j-1,k-1,l}(0), \]

\[ D_{0011}(0) = \sum_{j,k=1}^3 \tilde{D}_{00jk}(0) + 2 \sum_{j=1}^3 \tilde{D}_{00j}(0) + D_0(0), \]

\[ D_{00j1}(0) = - \sum_{k=1}^3 \tilde{D}_{00,j-1,k}(0) - \tilde{D}_{00,j-1}(0), \]

\[ D_{00j}(0) = - \tilde{D}_{00,j}(0). \]
(v) Tensor integral of rank 4

For the reduction of the 4th rank tensor we introduce

\[
E^{\mu\nu\rho\sigma} = \sum_{j,k,l,m=1}^{4} p_j^\mu p_k^\nu p_l^\rho p_m^\sigma E_{jklm} + \sum_{j,k=1}^{4} (g^{[j\nu} p_j^{\nu]} p_k^\rho + g^\sigma[p_j^{\nu \nu]} p_k^\rho])E_{00jk} + g^{[\mu \rho} g^{\sigma \sigma} E_{0000},
\]

\[
D^{\mu\nu\rho\sigma, (fin)} (i) = \sum_{j,k,l,m=1}^{4} p_j^\mu p_k^\nu p_l^\rho p_m^\sigma D_{jklm}(i) + \sum_{j,k=1}^{4} (g^{[j\nu} p_j^{\nu]} p_k^\rho + g^\sigma[p_j^{\nu \nu]} p_k^\rho])D_{00jk}(i)
\]

\[
+ g^{[\mu \rho} g^{\sigma \sigma} D^{(fin)}_{0000}(i), \quad i = 1, \ldots, 4,
\]

\[
D^{\mu\nu\rho\sigma, (fin)} (0) = \tilde{D}^{\mu\nu\rho\sigma, (fin)} (0) - p_1^{[\mu} P_1^{\nu]} \tilde{D}^{\rho\sigma} (0) + p_1^\mu p_1^\nu \tilde{D}^{\rho\sigma} (0) - p_1^{[\mu} P_1^{\nu]} \tilde{D}^{\rho\sigma} (0) + p_1^\mu p_1^\nu p_1^\rho D_0(0)
\]

\[
= \sum_{j,k,l,m=2}^{4} (p_j - p_1)^\mu (p_k - p_1)^\nu (p_l - p_1)^\rho (p_m - p_1)^\sigma \tilde{D}_{-1,-1,-1,-1}(0)
\]

\[
+ \sum_{j,k=2}^{4} [g^{[j\nu} (p_j - p_1)^\rho (p_k - p_1)^\sigma + g^\sigma[p_j^{\nu \nu]} (p_k - p_1)^\rho] \tilde{D}_{00,j-1}(0)
\]

\[
+ g^{[\mu \rho} g^{\sigma \sigma} D^{(fin)}_{0000}(0) - p_1^\mu \sum_{j,k,l=2}^{4} (p_j - p_1)^\nu (p_k - p_1)^\rho (p_l - p_1)^\sigma \tilde{D}_{-1,-1,-1,-1}(0)
\]

\[- p_1^\mu \sum_{j=2}^{4} g^{[\nu \rho]} (p_j - p_1)^\sigma \tilde{D}_{00,j-1}(0)
\]

\[
+ \sum_{j,k=2}^{4} [p_1^\mu P_1^{\nu]} (p_j - p_1)^\rho (p_k - p_1)^\sigma + \tilde{D}^{\mu \nu \rho \sigma} (0) - \tilde{D}^{\sigma \mu \nu \rho} (0)] \tilde{D}_{-1,-1,-1,-1}(0)
\]

\[- (p_1^{[\mu} P_1^{\nu]} g^{\rho \sigma} + \tilde{D}^{[\rho \sigma]} (0)) D_0(0)
\]

\[- p_1^{[\mu} P_1^{\nu]} P_1^{\rho} \sum_{j=2}^{4} (p_j - p_1)^\sigma \tilde{D}_{-1,-1,-1,-1}(0) + p_1^{[\mu} P_1^{\nu]} P_1^{\rho} p_1^\sigma D_0(0),
\]

\[
D^{\alpha\mu\nu\rho\sigma} (i) = \left[ D^{\alpha\mu\nu\rho\sigma, (fin)} (i) - D^{\alpha\mu\nu\rho\sigma, (fin)} (0) \right]^{(g)}
\]

\[
= \left[ D^{\alpha\mu\nu\rho\sigma, (fin)} (i) - \tilde{D}^{\alpha\mu\nu\rho\sigma, (fin)} (0) + \tilde{D}^{[\nu \rho \sigma]} (0) p_1^\mu \tilde{D}^{[\rho \sigma]} (0) - \tilde{D}^{[\rho \sigma]} (0) p_1^\mu \tilde{D}^{[\rho \sigma]} (0) \right]^{(g)}
\]

\[
= \sum_{j,k,l=1}^{4} g^{[\alpha [\mu} p_j^{\nu] p_k^\rho p_l^\sigma} D_{00jkli}(i) + \sum_{j=1}^{4} g^{[\alpha [\nu \rho] p_j^\sigma]} D^{(fin)}_{0000}(i)
\]

\[- \sum_{j,k,l=2}^{4} g^{[\alpha [\mu} (p_j - p_1)^\nu (p_k - p_1)^\rho (p_l - p_1)^\sigma] \tilde{D}_{00,j-1,k-1,l-1}(0)
\]

\[- \sum_{j=2}^{4} g^{[\alpha [\nu \rho]} (p_j - p_1)^\sigma \tilde{D}_{0000,j-1}(0)
\]
\[ + \sum_{j, k=2}^{4} p_1^{\mu} (p_j - p_1)^{\nu} (p_k - p_1)^{\rho} g^{\sigma} \bar{D}_{00,j-1,k-1}(0) + p_1^{\mu} g^{[\nu \rho g^{\sigma}]\alpha} \bar{D}_{0000}(0) \]
\[ - \sum_{j=2}^{4} \left[ p_1^{\mu} p_1^{\nu} (p_j - p_1)^{\rho} g^{\sigma} + p_1^{\mu} p_1^{\nu} g^{\rho,\alpha} (p_j - p_1)^{\sigma} \right. \]
\[ + \left. p_1^{\nu} p_1^{\rho} [g^{[\mu,\nu]}] \bar{D}_{00,j-1}(0) + p_1^{\mu} p_1^{\rho} g^{\sigma,\alpha} \bar{D}_{00}(0) \right]. \quad (3.17) \]

As above, the coefficients \( E_{00jk} \) and \( E_{0000} \) are redundant and introduced for convenience. The coefficients of the 4th rank tensor 5-point function are then given by

\[ E_{0000} = -\sum_{i=1}^{4} \eta_i D_{0000}(i) - \eta_0 D_{0000}(0) - \frac{1}{48} \frac{\det(Y)}{\det(Z)} \],
\[ E_{00jk} = -\sum_{i=1}^{4} \eta_i D_{00jk}(i) - \eta_0 D_{00jk}(0) \]
\[ + \left\{ 2 \sum_{i=1}^{4} \zeta_{ji} \left[ D_{000k}(i) \bar{D}_{k}(0) - D_{000k}(0) \right] + (j \leftrightarrow k) \right\}, \]
\[ E_{jk1m} = -\sum_{i=1}^{4} \eta_i D_{jk1m}(i) - \eta_0 D_{jk1m}(0) \]
\[ + \left\{ 2 \sum_{i=1}^{4} \zeta_{ji} \left[ D_{00kl}(i) \bar{D}_{l1m}(0) - D_{00kl}(0) \right] + (j \leftrightarrow k) + (j \leftrightarrow l) + (j \leftrightarrow m) \right\}, \quad j, k, l, m = 1, \ldots, 4, \quad (3.18) \]

with

\[ D_{1111}(0) = \sum_{j,k,l,m=1}^{3} \bar{D}_{jklm}(0) + 4 \sum_{j,k,l=1}^{3} \bar{D}_{jkl}(0) + 6 \sum_{j,k=1}^{3} \bar{D}_{jk}(0) + 4 \sum_{j=1}^{3} \bar{D}_{j}(0) + D_{0}(0), \]
\[ D_{j111}(0) = -\sum_{k,l,m=1}^{3} \bar{D}_{j-1,klm}(0) - 3 \sum_{k,l=1}^{3} \bar{D}_{j-1,kl}(0) - 3 \sum_{k=1}^{3} \bar{D}_{j-1,k}(0) - \bar{D}_{j-1}(0), \]
\[ D_{jk11}(0) = \sum_{l,m=1}^{3} \bar{D}_{j-1,k,l,m}(0) + 2 \sum_{l=1}^{3} \bar{D}_{j-1,k,l}(0) + \bar{D}_{j-1,k-1}(0), \]
\[ D_{jkl1}(0) = -\sum_{m=1}^{3} \bar{D}_{j-1,k,l,m}(0) - \bar{D}_{j-1,k-1,l-1}(0), \]
\[ D_{jk1m}(0) = \bar{D}_{j-1,k-1,l-1,m-1}(0), \]
\[ D_{0000}^{(\text{fin})}(0) = -\sum_{j=1}^{3} \bar{D}_{0000j}^{(\text{fin})}(0) - D_{0000}^{(\text{fin})}(0), \]
\[ D_{0000}^{(\text{fin})}(0) = \bar{D}_{0000j}^{(\text{fin})}(0), \]
\[ D_{0011}(0) = -\sum_{j,k,l=1}^{3} \bar{D}_{00jkl}(0) - 3 \sum_{j,k=1}^{3} \bar{D}_{00jk}(0) - 3 \sum_{j=1}^{3} \bar{D}_{00j}(0) - D_{00}(0), \]
\[ D_{00j11}(0) = \sum_{k,l=1}^{3} \tilde{D}_{00,j-1,kl}(0) + 2 \sum_{k=1}^{3} \tilde{D}_{00,j-1,k}(0) + \tilde{D}_{00,j-1}(0), \]

\[ D_{00jk1}(0) = -\sum_{l=1}^{3} \tilde{D}_{00,j-1,k-1,l}(0) - \tilde{D}_{00,j-1,k-1}(0), \]

\[ D_{00jkl}(0) = \tilde{D}_{00,j-1,k-1,l-1}(0), \quad j, k, l, m = 2, 3, 4. \]  

(3.19)

4 Consistency checks, applications, and generalizations

In order to check the explicit results for the 5-point tensor coefficients we have additionally calculated them by applying the usual Passarino–Veltman algorithm [1], which is conceptually completely different from the method of this paper. In the actual comparison we expressed the redundant terms \( E_{00g^{\mu\nu}}, \) etc., in the Lorentz decomposition of \( E_{\mu\nu}, \) etc., in terms of the coefficients \( E_{jk}, \) etc., by exploiting the relation (3.11) for the metric tensor. The numerical comparison of the coefficients showed agreement between the two methods for non-exceptional phase-space points.

Moreover, we have investigated the performance of the presented method in practice. To this end, we have implemented the new method into the calculation of the one-loop QCD corrections to the process \( gg/q\bar{q} \to t\bar{t}H \) [12] and of the one-loop electroweak corrections to \( e^+e^- \to \nu_1\bar{\nu}_1H \) [13], which both were originally evaluated using Passarino–Veltman reduction. We have observed a drastic improvement in the numerical stability of the 5-point function. While near the phase-space boundary the Passarino–Veltman approach could only be rescued by an extrapolation from the inner phase-space region (see Ref. [12] for details), this CPU-time-consuming procedure is practically unnecessary for the approach described above. The results obtained with the two methods are in mutual agreement within the integration errors, but owing to the more extensive use of the extrapolation the calculation employing the Passarino–Veltman methods takes much more time.

Finally, we consider the possibility of generalizing the method of this paper to dimensional regularization. The generalization is trivial in all cases where only abelian soft singularities are involved. Typical examples are pure QED or electroweak processes with an exactly massless photon and no massless charged particles, or QCD processes that do not involve external gluons or massless quarks. In such cases the soft singularity arises from diagrams with photon or gluon exchange between two external (on-shell) massive lines and shows up as a single pole in \( \epsilon = (4 - D)/2. \) Alternatively, if for \( D = 4 \) an infinitesimal photon or gluon mass \( \lambda \) is chosen as IR regulator, the singularity leads to \( \ln \lambda \) terms. The correspondence between these regularizations is well known (see e.g. Ref. [11]):

\[ \ln(\lambda^2) \leftrightarrow \frac{(4\pi\mu^2)^\Gamma(1+\epsilon)}{\epsilon} + \mathcal{O}(\epsilon). \]  

(4.1)

If external massless charged particles are involved but no external massless gluons, i.e., if the IR singularities are due to photons or gluons coupled to massless fermions or sfermions the following approach can be used. The whole calculation can be carried out with mass regulators obeying the hierarchy \( \lambda \ll m \ll |Q|, \) where \( \lambda \) is again an infinitesimal photon or gluon mass, \( m \) is a small fermion or sfermion mass, and \( Q \) denotes
a typical mass scale of the process. Then these results can be translated into dimensional regularization as described in Ref. [11] for the complete QCD and SUSY-QCD amplitudes.

If non-abelian soft singularities or overlapping soft/collinear singularities are involved the situation is more complicated. A convenient possibility to make use of the 4-dimensional approach of this paper is, for instance, described in Ref. [12]. There, a method is presented for translating $D$-dimensional into 4-dimensional integrals by constructing regularization-scheme-independent finite integrals upon subtracting well-defined, simple auxiliary integrals with the same singularity structure. These auxiliary integrals are entirely built from 3-point functions. In summary, this means that $D$-dimensional 5-point integrals are first converted into 4-dimensional 5-point integrals (regularized with masses or off-shell momenta) and 3-point integrals. The 4-dimensional 5-point integrals are then decomposed into 4-point integrals with the method described in this paper.

5 Conclusion

A method for reducing one-loop tensor 5-point integrals to related standard 4-point integrals is proposed that entirely avoids inverse Gram matrices, which are potential sources of numerical instabilities in practice. The presented explicit results for tensor coefficients of 1-, 2-, 3-, 4-, and 5-point functions comprise all expressions needed to deal with $1 \rightarrow 2$, $2 \rightarrow 2$, and $2 \rightarrow 3$ particle reactions with up to four external gauge bosons of renormalizable gauge theories and all expressions, apart from 6-point functions, for $2 \rightarrow 4$ particle reactions with up to two external bosons. The relevant 5-point functions are UV finite, but may contain IR (soft or collinear) singularities. In the explicitly given results a four-dimensional regularization scheme is assumed, but possible ways of translating them into dimensional regularization are described.

Appendix

A Calculation of extra terms resulting from UV-divergent 4-point functions

For the reduction of $E_{\mu \nu \rho}$ and $E_{\mu \nu \rho \sigma}$ we need to evaluate the UV-divergent 4-point functions $D^{(\Lambda)}_{\alpha \mu \nu \rho}$ and $D^{(\Lambda)}_{\alpha \mu \nu \rho \sigma}$ introduced in Section 2. Recall that these integrals are defined in four space–time dimensions. Going to $D$ dimensions and using the large-mass expansion [15] for the regularization parameter $\Lambda$, we obtain for these integrals

$$
D^{(\Lambda)}_{\alpha \mu \nu \rho} = \frac{(2\pi \mu)^{4-D}}{i\pi^2} \left[ \int d^D q \frac{q_\alpha q_\mu q_\nu q_\rho}{N_0 N_1 N_2 N_3 q^2 - \Lambda^2} \right]_{D=4}^{\Lambda \to \infty} + O(\Lambda^{-2} \ln \Lambda) + O(D - 4) \quad (A.1)
$$

and

$$
D^{(\Lambda)}_{\alpha \mu \nu \rho \sigma} = \frac{(2\pi \mu)^{4-D}}{i\pi^2} \left[ \int d^D q \frac{q_\alpha q_\mu q_\nu q_\rho q_\sigma}{N_0 N_1 N_2 N_3 q^2 - \Lambda^2} \right]_{D=4}^{\Lambda \to \infty} + O(\Lambda^{-2} \ln \Lambda) + O(D - 4),
$$

(A.2)
where $D_{\alpha\mu\nu\rho}$ and $D_{\alpha\mu\nu\rho\sigma}$ are the usual 4-point functions in dimensional regularization as defined in (2.1), and we included a non-vanishing momentum $p_0$ in $N_0$ for later convenience. The extra vacuum integrals can be evaluated by standard methods; the results are

$$
\begin{align*}
\frac{(2\pi\mu)^{1-D}}{i\pi^2} \int q^D q_{\alpha} q_{\mu} q_{\nu} q_{\rho} \frac{-\Lambda^2}{(2^D)^4} q^2 - \Lambda^2 = -\frac{1}{24} g_{\alpha[\mu g_{\nu\rho]} \left[ \Delta + \frac{11}{6} + \ln \frac{\mu^2}{\Lambda^2} \right] + O(D - 4),}
\frac{(2\pi\mu)^{1-D}}{i\pi^2} \int q^D q_{\alpha} q_{\mu} q_{\nu} q_{\rho} q_{\sigma} \frac{-\Lambda^2}{(2^D)^4} q^2 - \Lambda^2 \left[ 1 - \sum_{l=0}^{3} \frac{2q p_l}{q^2} \right] = \frac{1}{96} \sum_{l=0}^{3} (p_l)_{[\alpha g_{\mu[\nu g_{\rho\sigma]}]} \left[ \Delta + \frac{25}{12} + \ln \frac{\mu^2}{\Lambda^2} \right] + O(D - 4), (A.3)
\end{align*}
$$

where

$$
\Delta = \frac{2}{4 - D} - \gamma_E + \ln 4\pi (A.4)
$$

represents the UV singularities in dimensional regularization, $\gamma_E$ is Euler’s constant, and $\mu$ is the dimensionful parameter of dimensional regularization.

Since the UV singularities of $D^{(A)}_{\mu_1...\nu P}$ are regularized by $\Lambda$, the poles in $D - 4$ in the vacuum integrals cancel the poles in $D_{\alpha\mu\nu\rho}$ and $D_{\alpha\mu\nu\rho\sigma}$, and we can write

$$
D^{(\Lambda)}_{\mu_1...\nu P} \sim D^{(fin)}_{\mu_1...\nu P} + \Delta_{\mu_1...\nu P} (A.5)
$$

with

$$
\Delta_{\mu\nu\rho\sigma} = -\frac{1}{24} g_{\mu[\nu g_{\rho\sigma]}} \left[ \frac{11}{6} + \ln \frac{\mu^2}{\Lambda^2} \right] + O(D - 4),
\Delta_{\alpha\mu\nu\rho\sigma} = \frac{1}{96} \sum_{l=0}^{3} (p_l)_{[\alpha g_{\mu[\nu g_{\rho\sigma]}]} \left[ \frac{25}{12} + \ln \frac{\mu^2}{\Lambda^2} \right] + O(D - 4) (A.6)
$$

and

$$
D^{(fin)}_{\mu\nu\rho\sigma} = D_{\mu\nu\rho\sigma} - \frac{1}{24} g_{\mu[\nu g_{\rho\sigma]}} \Delta + O(D - 4),
D^{(fin)}_{\alpha\mu\nu\rho\sigma} = D_{\alpha\mu\nu\rho\sigma} + \frac{1}{96} \sum_{l=0}^{3} (p_l)_{[\alpha g_{\mu[\nu g_{\rho\sigma]}]} \Delta + O(D - 4). (A.7)
$$

The quantities $D^{(\Lambda)}_{\mu_1...\nu P}$, $D^{(fin)}_{\mu_1...\nu P}$, and $\Delta_{\mu_1...\nu P}$ are finite in $D$ dimensions, so that we can perform the limit $D \to 4$ there.

In the reduction of $E_{\mu\nu\rho}$, UV-divergent contributions appear only in the difference $[\Delta_{\alpha\mu\nu\rho}(i) - \Delta_{\alpha\mu\nu\rho}(0)]$. Since $\Delta_{\alpha\mu\nu\rho}$ is momentum independent, the difference, and thus $U_{\mu\nu\rho}$, vanishes.

For $E_{\mu\nu\rho\sigma}$, we have contributions from $\Delta_{\mu\nu\rho\sigma}(0)$ and $\Delta_{\alpha\mu\nu\rho\sigma}(i) - \Delta_{\alpha\mu\nu\rho\sigma}(0)$. These terms are given by

$$
\Delta_{\mu\nu\rho\sigma}(0) = -\frac{1}{24} g_{\mu[\nu g_{\rho\sigma]} \left[ \frac{11}{6} + \ln \frac{\mu^2}{\Lambda^2} \right] + O(D - 4),
$$

15
\[ \Delta_{\alpha\mu\nu\rho\sigma}(i) - \Delta_{\alpha\mu\nu\rho\sigma}(0) = \frac{1}{96} \left( \sum_{l=1}^{4} p_l \right)_{[\alpha g_{\mu|\nu} g_{\rho|\sigma}]} \left[ \frac{25}{12} + \ln \frac{\mu^2}{\Lambda^2} \right] \\
- \frac{1}{96} \left( \sum_{l=1}^{4} p_l \right)_{[\alpha g_{\mu|\nu} g_{\rho|\sigma}]} \left[ \frac{25}{12} + \ln \frac{\mu^2}{\Lambda^2} \right] + \mathcal{O}(D - 4) \\
= -\frac{1}{96} p_i_{[\alpha g_{\mu|\nu} g_{\rho|\sigma}]} \left[ \frac{25}{12} + \ln \frac{\mu^2}{\Lambda^2} \right] + \mathcal{O}(D - 4). \quad \text{(A.8)} \]

Inserting (A.8) into (2.14) and using
\[ \sum_{i,j=1}^{4} (-1)^{i+j} \det(\tilde{Z}^{(4)}_{ij}) 2p_{ij}^\alpha p_{i\alpha} = \sum_{i,j=1}^{4} \det(Z^{(4)}) (Z^{(4)})^{-1} 2p_{ij}^\alpha p_{i\alpha} = 4 \det(Z^{(4)}), \quad \text{(A.9)} \]
which employs the relation between \( \det(\tilde{Z}^{(4)}_{ij}) \) and the inverse Gram matrix \((Z^{(4)})^{-1}\), we find
\[ U_{\mu\nu\rho\sigma} = -\frac{1}{48} g_{\mu\nu} g_{\rho\sigma} \det(Z^{(4)}) + \mathcal{O}(D - 4). \quad \text{(A.10)} \]

\section*{B Reduction of tensor 1-, 2-, 3-, and 4-point functions}

One-loop tensor \( N \)-point integrals have the general form
\[ T^{N}_{\mu_1...\mu_F}(p_1, ..., p_{N-1}, m_0, ..., m_{N-1}) = \frac{(2\pi\mu)^{1-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_F}}{N_0 N_1 \cdots N_{N-1}} \quad \text{(B.1)} \]
with \( N_i, i = 0, \ldots, N - 1 \), defined in (2.2). Following the notation of Ref. [9], i.e., \( T^1 \to A, T^2 \to B, T^3 \to C, T^4 \to D \), and using the conventions of Ref. [5], we decompose the genuine tensor integrals into Lorentz-covariant structures:

\[ A^\mu = 0, \quad A^{\mu \nu} = g^{\mu \nu} A_{00}, \]
\[ B^\mu = p_1^\mu B_1, \quad B^{\mu \nu} = g^{\mu \nu} B_{00} + p_1^\mu p_1^\nu B_{11}, \]
\[ B^{\mu \nu \rho} = g^{[\mu \nu} p_1^{\rho]} B_{001} + p_1^{[\mu} p_1^{\nu]} p_1^{\rho]} B_{111}, \]
\[ C^\mu = \sum_{j=1}^{2} p_j^\mu C_j, \quad C^{\mu \nu} = g^{\mu \nu} C_{00} + \sum_{j,k=1}^{2} p_j^\mu p_k^\nu C_{jk}, \]
\[ C^{\mu \nu \rho} = \sum_{j=1}^{2} g^{[\mu \nu \rho]} p_j^{[\rho]} C_{00j} + \sum_{j,k,l=1}^{2} p_j^{[\mu \nu} p_k^{\rho]} C_{jkl}, \]
\[ C^{\mu \nu \rho \sigma} = g^{[\mu \nu \rho \sigma}] C_{0000} + \sum_{j,k,l,m=1}^{2} (g^{[\mu \nu} p_j^{\rho]} p_k^{\sigma]} + g^{[\rho \sigma} p_j^{\mu]} p_k^{\nu]}) C_{00jk} + \sum_{j,k,l,m=1}^{2} p_j^{[\mu \nu \rho} p_k^{\sigma]} C_{jklm}, \]
\[ D^\mu = \sum_{j=1}^{3} p_j^\mu D_j, \quad D^{\mu \nu} = g^{\mu \nu} D_{00} + \sum_{j,k=1}^{3} p_j^\mu p_k^\nu D_{jk}, \]
\[ D^{\mu \nu \rho} = \sum_{j=1}^{3} g^{[\mu \nu \rho]} D_{00j} + \sum_{j,k,l=1}^{3} p_j^{[\mu \nu} p_k^{\rho]} D_{jkl}, \]
formulas. This allows to check the correctness and the numerical stability of the results. Note that several tensor coefficients can be obtained from different reduction defined in (B.2). These results can also be read off from the generic results in Section 4.2 by comparing coefficients.

Because of the symmetry of the tensor $T_{\mu_1...\mu_p}$, all coefficients $B_1,...,D_{jklmn}$ are symmetric under permutation of all indices. They are reduced to scalar integrals recursively by inversion of systems of linear equations. The inhomogeneity of these equations consists of coefficients of lower rank [1]. The equations of this system are obtained by contracting one integration momentum $q_{\mu_1}$ with the $(N - 1)$ external momenta $p^\mu_{i}$ and for $N \geq 2$ also by contraction with the metric $g^\mu_{\mu_2...\mu_p}$. Using

$$2p_i q = N_i - N_0 - f_i \quad \text{with} \quad f_i = p_i^2 - m_i^2 + m_0^2,$$

in (B.1), the first two terms on the right-hand side of (B.3) each cancel exactly one propagator denominator of $q^\mu T_{\mu_1...\mu_p}^N$, the third term is proportional to $T_{\mu_2...\mu_p}^N$. Likewise the contraction with $g^\mu g_{\mu_2...\mu_p}$ yields a factor $q^2$ in the numerator of $g^\mu_{\mu_2...\mu_p} T_{\mu_1...\mu_p}^N$, which can be written as $q^2 = N_0 + m_0^2$. The $N_0$ term cancels the first propagator, the second term leads to the tensor $T_{\mu_2...\mu_p}^N$. With the abbreviation $T_{\mu_1...\mu_p}^{N-1}(i)$, denoting $T_{\mu_1...\mu_p}^N$ with the $i$th denominator omitted, this yields

$$2p_i^{\mu_1} T_{\mu_1...\mu_p}^N = T_{\mu_2...\mu_p}^{N-1}(i) - T_{\mu_2...\mu_p}^{N-1}(0) - f_i T_{\mu_2...\mu_p}^N,$$

$$g^{\mu_1...\mu_p} T_{\mu_2...\mu_p}^N = T_{\mu_3...\mu_p}^{N-1}(0) + m_0^2 T_{\mu_3...\mu_p}^N.$$

Note that for $T_{\mu_1...\mu_p}^{N-1}(0)$ a shift of the integration momentum $q^\mu \rightarrow q^\mu - p^\mu_{i}$ has to be done in order to achieve the standard form (B.1). The tensor integrals with shifted momenta $T_{\mu_1...\mu_p}^{N-1}(0)$ are defined as in (3.4). Expressing the tensors in (B.4) by (B.2), the desired recurrence relations can be read off by comparing coefficients.

In the following we summarize the results for the reduction of all tensor coefficients defined in (B.2). These results can also be read off from the generic results in Section 4.2 of Ref. [5]. Note that several tensor coefficients can be obtained from different reduction formulas. This allows to check the correctness and the numerical stability of the results.

The tensor coefficient of the relevant 1-point function reads

$$A_{00} = \frac{1}{4} m_0^2 A_0 + \frac{1}{8} m_0^4.$$

For the tensor coefficients of 2-point functions we find

$$B_1 = \frac{1}{2p_1^2} \left[ A_0(1) - A_0(0) - f_1 B_0 \right],$$

$$B_{00} = \frac{1}{6} \left[ A_0(0) + f_1 B_1 + 2m_0^2 B_0 + m_0^2 + m_1^2 - \frac{1}{3} p_1^2 \right],$$

$$B_{11} = \frac{1}{3p_1^2} \left[ A_0(0) - 2f_1 B_1 - m_0^2 B_0 - \frac{1}{2} \left( m_0^2 + m_1^2 - \frac{1}{3} p_1^2 \right) \right].$$
\[ B_{001} = \frac{1}{2p_{1}^{2}} [A_{00}(1) - A_{00}(0) - f_{1}B_{00}] \]
\[ = \frac{1}{8} \left[ 2m_{0}^{2}B_{1} - A_{0}(0) + f_{1}B_{11} - \frac{1}{6}(2m_{0}^{2} + 4m_{1}^{2} - p_{1}^{2}) \right], \]
\[ B_{111} = -\frac{1}{4p_{1}^{2}} \left[ A_{0}(0) + 3f_{1}B_{11} + 2m_{0}^{2}B_{1} - \frac{1}{6} \left( 2m_{0}^{2} + 4m_{1}^{2} - p_{1}^{2} \right) \right]. \] (B.8)

For the 3-point functions we obtain for the vector case
\[ C_{i} = \sum_{n=1}^{2} (Z(2))^{-1}_{m} R_{n}^{i}, \quad i = 1, 2, \quad Z(2) = \begin{pmatrix} 2p_{1}p_{1} & 2p_{1}p_{2} \\ 2p_{2}p_{1} & 2p_{2}p_{2} \end{pmatrix}, \] (B.9)

with
\[ R_{n}^{i} = B_{0}(n) - B_{0}(0) - f_{n}C_{0}, \quad n = 1, 2, \] (B.10)

for the second-rank tensor case
\[ C_{00} = \frac{1}{2}m_{0}^{2}C_{0} + \frac{1}{4}B_{0}(0) + \frac{1}{4} \sum_{m=1}^{2} f_{m}C_{m} + \frac{1}{4}, \]
\[ C_{ij} = \sum_{n=1}^{2} (Z(2))^{-1}_{m} [R_{n}^{2} - 2C_{00}\delta_{nj}], \quad i, j = 1, 2, \] (B.11)

with \([i_{n} \text{ is defined in (3.5)})\]
\[ R_{n}^{2} = B_{1}(n)\delta_{ni} - B_{i}(0) - f_{n}C_{i}, \quad n, i = 1, 2, \] (B.12)

and
\[ B_{1}(0) = -\tilde{B}_{1}(0) - B_{0}(0), \quad B_{2}(0) = \tilde{B}_{1}(0), \] (B.13)

for the third-rank tensor case
\[ C_{000} = \frac{1}{3}m_{0}^{2}C_{0} + \frac{1}{6}B_{0}(0) + \frac{1}{6} \sum_{j=1}^{2} f_{j}C_{ij} - \frac{1}{18}, \]
\[ C_{00i} = \sum_{n=1}^{2} (Z(2))^{-1}_{m} R_{n00}^{i}, \]
\[ C_{ijk} = \sum_{n=1}^{2} (Z(2))^{-1}_{m} [R_{n}^{3} - 2C_{00}\delta_{nk} - 2C_{00k}\delta_{nj}], \quad i, j, k = 1, 2, \] (B.14)

with
\[ R_{n00}^{3} = B_{00}(n) - B_{00}(0) - f_{n}C_{00}, \]
\[ R_{nij}^{3} = B_{11}(n)\delta_{ni}\delta_{nj} - B_{ij}(0) - f_{n}C_{ij}, \quad n, i, j = 1, 2, \] (B.15)
and
\[ B_{11}(0) = \tilde{B}_{11}(0) + 2\tilde{B}_1(0) + B_0(0), \]
\[ B_{12}(0) = -\tilde{B}_{11}(0) - \tilde{B}_1(0), \]
\[ B_{22}(0) = \tilde{B}_{11}(0), \]
(B.16)
and for the fourth-rank tensor case
\[ C_{0000} = \frac{1}{4}m_0^2C_{00} + \frac{1}{8}B_{00}(0) + \frac{1}{8}\sum_{i=1}^{m_0} f_iC_{00i} + \frac{1}{48}(m_0^2 + m_1^2 + m_2^2) - \frac{1}{192}[p_1^2 + p_2^2 + (p_1 - p_2)^2], \]
\[ C_{00ij} = \frac{1}{4}m_0^2C_{ij} + \frac{1}{8}B_{ij}(0) + \frac{1}{8}\sum_{k=1}^{m_0} f_kC_{ijk} + \frac{1}{48}(1 + \delta_{ij}), \]
(B.17)
with
\[ R_{n00i}^4 = B_{001}(n)\delta_{ni} - B_{00i}(0) - f_nC_{00i}, \]
\[ R_{nijk}^4 = B_{111}(n)\delta_{ni}\delta_{nj}\delta_{nk} - B_{ijk}(0) - f_nC_{ijk}, \]
(B.18)
and
\[ B_{001}(0) = -\tilde{B}_{001}(0) - B_{00}(0), \]
\[ B_{002}(0) = \tilde{B}_{001}(0), \]
\[ B_{111}(0) = -\tilde{B}_{111}(0) - 3\tilde{B}_1(0) - 3\tilde{B}_1(0) - B_0(0), \]
\[ B_{112}(0) = \tilde{B}_{111}(0) + 2\tilde{B}_1(0) + \tilde{B}_1(0), \]
\[ B_{122}(0) = -\tilde{B}_{111}(0) - \tilde{B}_{11}(0), \]
\[ B_{222}(0) = \tilde{B}_{111}(0). \]
(B.19)
The tensor coefficients of 4-point functions are given by
\[ D_i = \sum_{n=1}^{3} (Z^{(3)})^{-1}_{iin}S_n^1, \quad i = 1, 2, 3, \]
\[ Z^{(3)} = \begin{pmatrix} 2p_1p_1 & 2p_1p_2 & 2p_1p_3 \\ 2p_2p_1 & 2p_2p_2 & 2p_2p_3 \\ 2p_3p_1 & 2p_3p_2 & 2p_3p_3 \end{pmatrix}, \]
(B.20)
with
\[ S_n^1 = C_0(n) - C_0(0) - f_nD_0, \quad n = 1, 2, 3, \]
(B.21)
for the vector case, by
\[ D_{00} = m_0^2D_0 + \frac{1}{2}C_0(0) + \frac{1}{2}\sum_{m=1}^{3} f_mD_m, \]
\[ D_{ij} = \sum_{n=1}^{3} (Z^{(3)})^{-1}_{ijn}S_{nj}^2 - 2D_{00}\delta_{nj}, \quad i, j = 1, 2, 3, \]
(B.22)
with
\[ S_{ni}^2 = C_{in}(n)\delta_{ni} - C_i(0) - f_n D_i, \quad n, i = 1, 2, 3, \] (B.23)
and
\[ C_1(0) = -\sum_{i=1}^{2} \bar{C}_i(0) - C_0(0), \quad C_i(0) = \bar{C}_{i-1}(0), \quad i = 2, 3, \] (B.24)

for the second-rank tensor case, by
\[
D_{00i} = \frac{1}{2} m_0^2 D_i + \frac{1}{4} C_i(0) + \frac{1}{4} \sum_{j=1}^{3} f_j D_{ij},
\]
\[
D_{00i} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} S_{n00}^3,
\]
\[
D_{ijk} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} \left[ S_{njk}^3 - 2D_{00j} \delta_{nk} - 2D_{00k} \delta_{nj} \right], \quad i, j, k = 1, 2, 3, \] (B.25)

with
\[
S_{n00}^3 = C_{00}(n) - C_{00}(0) - f_n D_{00},
\]
\[
S_{nij}^3 = C_{in,jn}(n)\delta_{ni}\delta_{nj} - C_{ij}(0) - f_n D_{ij}, \quad n, i, j = 1, 2, 3, \] (B.26)

and
\[
C_{11}(0) = \sum_{i,j=1}^{2} \bar{C}_{ij}(0) + 2 \sum_{i=1}^{2} \bar{C}_i(0) + C_0(0),
\]
\[
C_{1i}(0) = -\sum_{j=1}^{2} \bar{C}_{i-1,j}(0) - \bar{C}_{i-1}(0),
\]
\[
C_{ij}(0) = \bar{C}_{i-1,j-1}(0), \quad i, j = 2, 3, \] (B.27)

for the third-rank tensor case, by
\[
D_{0000} = \frac{1}{2} m_0^2 D_{000} + \frac{1}{6} C_{00}(0) + \frac{1}{6} \sum_{i=1}^{3} f_i D_{00i} + \frac{1}{60},
\]
\[
D_{00ij} = \frac{1}{2} m_0^2 D_{ij} + \frac{1}{6} C_{ij}(0) + \frac{1}{6} \sum_{k=1}^{3} f_k D_{ijk},
\]
\[
D_{00ij} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} \left[ S_{njk}^4 - 2D_{00j} \delta_{nk} - 2D_{00k} \delta_{nj} \right],
\]
\[
D_{ijkl} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} \left[ S_{nijkl}^4 - 2D_{00j} \delta_{nl} - 2D_{00k} \delta_{nj} - 2D_{00l} \delta_{nk} \right], \quad i, j, k, l = 1, 2, 3, \] (B.28)
with

\[ S^4_{n00i} = C_{00i_n}(n) \delta_{ni} - C_{00i}(0) - f_n D_{n00i}, \]

\[ S^4_{nijkl} = C_{i_n j_k n, i(n)} \delta_{ni} \delta_{nj} \delta_{nk} - C_{ijkl}(0) - f_n D_{ijkl}, \quad n, i, j, k = 1, 2, 3, \]  \hspace{1cm} (B.29)

and

\[ C_{001}(0) = - \sum_{i=1}^{2} \tilde{C}_{00i}(0) - C_{00}(0), \quad C_{00i}(0) = \bar{C}_{00,i-1}(0), \]

\[ C_{111}(0) = - \sum_{i,j,k=1}^{2} \tilde{C}_{ijk}(0) - 3 \sum_{i,j=1}^{2} \tilde{C}_{ij}(0) - 3 \sum_{i=1}^{2} \tilde{C}_{i}(0) - C_{0}(0), \]

\[ C_{11i}(0) = \sum_{j,k=1}^{2} \tilde{C}_{i-1,jk}(0) + 2 \sum_{j=1}^{2} \tilde{C}_{i-1,j}(0) + \tilde{C}_{i-1}(0), \]

\[ C_{1ij}(0) = - \sum_{k=1}^{2} \tilde{C}_{i-1,j-1,k}(0) - \tilde{C}_{i-1,j-1}(0), \]

\[ C_{ij}(0) = \tilde{C}_{i-1,j-1,k-1}(0), \quad i, j, k = 2, 3, \]  \hspace{1cm} (B.30)

for the fourth-rank tensor case, and by

\[ D_{00000} = \frac{1}{4} m^2 D_{00i} + \frac{1}{8} C_{00i}(0) + \frac{3}{8} \sum_{j=1}^{3} f_j D_{00ij} - \frac{1}{192}, \]

\[ D_{00ijk} = \frac{1}{4} m^3 D_{ijk} + \frac{1}{8} C_{ijk}(0) + \frac{3}{8} \sum_{m=1}^{3} f_m D_{ijkm}, \]

\[ D_{00i} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} \frac{1}{8} S_{n0000}, \]

\[ D_{00jk} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} [S_{n00jk} - 2 D_{00000} \delta_{nk} - 2 D_{0000i} \delta_{nj}], \]

\[ D_{ijklm} = \sum_{n=1}^{3} (Z^{(3)})_{in}^{-1} [S_{nijkl} - 2 D_{00ijkl} \delta_{nm} - 2 D_{00klm} \delta_{nj} - 2 D_{00imj} \delta_{nk} - 2 D_{00mjk} \delta_{nl}], \]

\[ i, j, k, l, m = 1, 2, 3, \]  \hspace{1cm} (B.31)

with

\[ S^5_{n0000} = C_{0000}(n) - C_{0000}(0) - f_n D_{00000}, \]

\[ S^5_{n00ij} = C_{00i n,j}(n) \delta_{ni} \delta_{nj} - C_{00i j}(0) - f_n D_{00ij}, \]

\[ S^5_{nijkl} = C_{i n,j k n, l}(n) \delta_{ni} \delta_{nj} \delta_{nk} \delta_{nl} - C_{ijkl}(0) - f_n D_{ijkl}, \quad n, i, j, k, l = 1, 2, 3, \]  \hspace{1cm} (B.32)

and

\[ C_{0011}(0) = \sum_{i,j=1}^{2} \tilde{C}_{00i j}(0) + 2 \sum_{i=1}^{2} \tilde{C}_{00i}(0) + C_{00}(0). \]
for the fifth-rank tensor case.

\section*{C \ UV-divergent parts of tensor integrals}

For practical calculations it is useful to know the UV-divergent parts of the tensor integrals explicitly. We directly give the products of $D - 4$ with all divergent one-loop tensor coefficient integrals appearing in renormalizable theories up to terms of the order $O(D - 4)$

\begin{align*}
(D - 4) A_0(m_0) &= -2m_0^2, \\
(D - 4) A_{00}(m_0) &= -\frac{1}{2}m_0^4, \\
(D - 4) B_0(p_1, m_0, m_1) &= -2, \\
(D - 4) B_1(p_1, m_0, m_1) &= 1, \\
(D - 4) B_{00}(p_1, m_0, m_1) &= \frac{1}{6}(p_1^2 - 3m_0^2 - 3m_1^2), \\
(D - 4) B_{11}(p_1, m_0, m_1) &= -\frac{2}{3}, \\
(D - 4) B_{001}(p_1, m_0, m_1) &= -\frac{1}{12}(p_1^2 - 2m_0^2 - 4m_1^2), \\
(D - 4) B_{111}(p_1, m_0, m_1) &= \frac{1}{2}, \\
(D - 4) C_{00}(p_1, p_2, m_0, m_1, m_2) &= -\frac{1}{2}, \\
(D - 4) C_{001}(p_1, p_2, m_0, m_1, m_2) &= \frac{1}{6}, \\
(D - 4) C_{000}(p_1, p_2, m_0, m_1, m_2) &= \frac{1}{18}[(p_1^2 - p_2)^2 + p_1^2 + p_2^2] \\
&- \frac{1}{12}(m_0^2 + m_1^2 + m_2^2), \\
(D - 4) C_{00ij}(p_1, p_2, m_0, m_1, m_2) &= -\frac{1}{24}(1 + \delta_{ij}), \\
(D - 4) D_{0000}(p_1, p_2, p_3, m_0, m_1, m_2, m_3) &= -\frac{1}{12},
\end{align*}
\[(D - 4) D_{0000i}(p_1, p_2, p_3, m_0, m_1, m_2, m_3) = \frac{1}{4\pi}. \quad (C.1)\]

All other scalar coefficients defined in (B.2) are UV finite.

References

[1] G. Passarino and M. J. Veltman, Nucl. Phys. B 160 (1979) 151.

[2] A. I. Davydychev, Phys. Lett. B 263 (1991) 107; Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B 302 (1993) 299 [Erratum-ibid. B 318 (1993) 649] [hep-ph/9212308] and Nucl. Phys. B 412 (1994) 751 [hep-ph/9306240]; J. M. Campbell, E. W. Glover and D. J. Miller, Nucl. Phys. B 498 (1997) 397 [hep-ph/9612413]; J. Fleischer, F. Jegerlehner and O. V. Tarasov, Nucl. Phys. B 566 (2000) 423 [hep-ph/9907327]; T. Binoth, J. P. Guillet and G. Heinrich, Nucl. Phys. B 572 (2000) 361 [hep-ph/9911342].

[3] D. B. Melrose, Nuovo Cimento XL A (1965) 181.

[4] W. L. van Neerven and J. A. Vermaseren, Phys. Lett. B 137 (1984) 241.

[5] A. Denner, Fortsch. Phys. 41 (1993) 307.

[6] A. T. Suzuki, E. S. Santos and A. G. Schmidt, hep-ph/0210083; F. Tramontano, hep-ph/0211390.

[7] T. Binoth, G. Heinrich and N. Kauer, hep-ph/0210023.

[8] A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, hep-ph/0209219.

[9] G. ’t Hooft and M. J. Veltman, Nucl. Phys. B 153 (1979) 365.

[10] W. Beenakker and A. Denner, Nucl. Phys. B 338 (1990) 349; A. Denner, U. Nierste and R. Scharf, Nucl. Phys. B 367 (1991) 637.

[11] S. Catani, S. Dittmaier and Z. Trócsányi, Phys. Lett. B 500 (2001) 149 [hep-ph/0011222].

[12] W. Beenakker, S. Dittmaier, M. Krämer, B. Plümper, M. Spira and P. M. Zerwas, Phys. Rev. Lett. 87 (2001) 201805 [hep-ph/0107081] and DESY 02-177, hep-ph/0211352.

[13] A. Denner, S. Dittmaier, M. Roth and M. Weber, in preparation.

[14] A. Denner, S. Dittmaier and M. Roth, Nucl. Phys. B 519 (1998) 39 [hep-ph/9710521].

[15] V. A. Smirnov, Phys. Lett. B 394 (1997) 205 [hep-th/9608151] and references therein.