In-situ tunable nonlinearity and competing signal paths in coupled superconducting resonators

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We have fabricated and studied a system of two tunable and coupled nonlinear superconducting resonators. The nonlinearity is introduced by galvanically coupled dc-SQUIDs. We simulate the system response by means of a circuit model, which includes an additional signal path introduced by the electromagnetic environment. Furthermore, we present two methods allowing us to experimentally determine the nonlinearity. First, we fit the measured frequency and flux dependence of the transmission data to simulations based on the equivalent circuit model. Second, we fit the measured power dependence of the transmission data to a model that is predicted by the nonlinear equation of motion describing the system. Our results show that we are able to tune the nonlinearity of the resonators by almost two orders of magnitude via an external coil and two on-chip antennas. The studied system represents the basic building block for larger systems, allowing for quantum simulations of bosonic many-body systems with a larger number of lattice sites.

I. INTRODUCTION

The field of analog quantum simulation [1] has opened up the possibility to experimentally simulate quantum phenomena without the need for universal quantum computing [2, 3]. Especially in the field of quantum many-body physics, where calculations with classical computational approaches are inefficient, analog quantum simulations may lead to a better understanding of the underlying models [4–6]. One of the very promising platforms for such simulations is circuit quantum electrodynamics (QED), where the quantum behaviour of superconducting circuits is used to emulate the quantum system under test [7–12]. Circuit QED allows for a large degree of design flexibility [13–15], experimental control [15–17], in-situ tunability of essential parameters [18, 19], and scalability [8].

A particular example of a quantum many-body system showing rich physical effects to be investigated with quantum simulations is the Bose-Hubbard model (BHM) [20]. It describes the behavior of interacting bosons on a lattice. It has been shown that this model can be simulated in a bottom-up approach by a network of coupled superconducting resonators, each equipped with a direct current superconducting interference device (dc-SQUID) at its current antinode [21–25]. The coupling between SQUID and resonator creates polaritons, which are quasi-particles formed by a superposition of the photonic excitation of the resonator and the matter-like excitation of the SQUID. These polaritons can then be used for simulating the bosonic interaction of the BHM. The implementation of the BHM with superconducting circuits represents an open quantum system, where the particle number at each site can be controlled by the interplay between externally applied microwave drives and local dissipation channels. The natural access to this driven-dissipative regime of the BHM distinguishes superconducting-circuit implementations from cold-atom implementations [5, 7] and is expected to exhibit exciting novel phases of light. A prototypical example in this context is the theoretical prediction of polariton crystallization [26]. Experimentally, a photon ordering phase transition [12] and a dissipation-driven transition between localization and delocalization [27] have been shown. Furthermore, studies on phase transitions in large scale circuits without individual control of each lattice site [10] exist and a single phase, i.e. the Mott insulator phase of photons, could be stabilized in a lossy system [28]. A key prerequisite for the application of artificial circuits in quantum simulation experiments is the ability to accurately design, determine and control the relevant circuit parameters. Hence, a detailed understanding and modeling of quantum circuits consisting of coupled nonlinear superconducting resonator as well as their interaction with the environment [29] is of large importance.

In this work, we address this topic by investigating a system of two coupled resonators with a weak, but tunable nonlinearity. So far, this regime has mostly been investigated in the quite different context of parametrically driven circuits [30–32]. We discuss how to set up a circuit model for the characterization of such a system.

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in the presence of a spurious environment whose microscopic origin does not need to be exactly known. We show that this environment can be modeled with a spurious parallel signal path giving rise to Fano-like resonances. In this way, we gain access to the full parameter space of the coupled systems in a controlled way. Consequently, we can investigate a key property of our system: the nonlinearity of the resonators. Specifically, we employ two different characterization techniques including a direct measurement. We show that the nonlinearity of our resonators can be tuned in situ from values much smaller to values larger than the resonator-resonator coupling. In this way, we provide a technique for a controlled access to promising parameter regimes for future quantum simulations.

II. SAMPLE & EXPERIMENTAL SETUP

In our experiments, we use a sample consisting of two weakly-coupled superconducting resonators fabricated on a 525 µm thick silicon chip using aluminum technology. The whole metal layer including the Josephson junctions is fabricated using double-angle shadow evaporation and lift-off. The overall thickness of the aluminum layer is 140 nm. A photo and micrographs of the sample are shown in Fig. 1. The main part of the superconducting circuit is formed by a series connection of two capacitively coupled coplanar waveguide resonators, which are each intersected by a dc-SQUID. The area of the SQUID loop is 

\[ A_{\text{SQUID}} = (10.5 \times 24.5) \mu m^2. \]

By design, the two Josephson junctions differ in size in order to flatten the flux dependence of the SQUID critical current. In this way, the sensitivity of the system to magnetic flux is decreased. For our junctions, we measure an asymmetry parameter

\[ d = \frac{(I_{c1} - I_{c2})}{(I_{c1} + I_{c2})} \approx 0.13 \]

for both SQUIDs (see Sec. III A). Here, \( I_{c1} \) and \( I_{c2} \) are the critical currents of the two Josephson junctions in the dc-SQUID. We can tune the two resonators in a frequency range between 5.67 GHz and 7.14 GHz by an applied magnetic flux.

We mount the sample to the base plate of a dilution refrigerator with a base temperature of 27 mK and apply a probe signal with a vector network analyzer (VNA) (see Fig. 2). The output signal is amplified with a cryogenic and a room temperature high-frequency amplifier and then detected by the VNA.

We can tune the critical current of our dc-SQUIDS either by means of an external coil or via T-shaped on-chip antennas [see Fig. 1(e)]. The external coil simultaneously tunes both SQUIDS, while the antennas are designed to individually access only one of them. However, we observe non-negligible crosstalk of each antenna to the other SQUID, which we have to account for in our experiments.

III. NONLINEARITY FROM THE CIRCUIT MODEL

In the scope of a quantum simulation experiment, it is vital to know the full parameter set of the underlying circuit in order to precisely predict its behavior. For a system of nonlinear resonators, the nonlinearity is of key interest. We therefore implement two ways to experimentally determine the nonlinearity of our system. In the following, we present a circuit model accurately reproducing our data (see Sec. III A) and allowing us to calculate the nonlinearity (see Sec. III B). In Sec. IV B, we compare these results to a direct measurement of the
nonlinearity based on the power-dependent response of the resonators.

A. Circuit model with competing signal path

In the experiment, we measure the transmission through our two-resonator system as a function of the current $I_{\text{coil}}$ through the coil and of the frequency $\omega d/2\pi$ of the applied microwave drive. The result is shown in Fig. 3(a). As expected, we observe periodic modulations of the two resonance frequencies of the coupled resonators. The maximum resonance frequencies of the two resonators differ by approximately 50 MHz due to inaccuracies in the junction fabrication. For our sample,
this amounts to a difference between the critical currents of the SQUIDs of roughly 15%. A spread in the critical currents of up to 20% is not unusual for our junction process [33]. Resonator 1 couples less strongly to the external magnetic field than resonator 2. This effect cannot be attributed to fabrication inaccuracies because of the relatively large size of the SQUID loop. Instead, we suspect strong local fields caused by asymmetric current flow across the superconducting ground plane.

In order to extract the circuits parameters, we first simulate the response of the sample with a simple circuit model taking into account only the resonators and coupling capacitors (see Fig. 4). Details of the simulation can be found in App. A. The simple circuit model predicts an increased transmission in regions where the resonators are close to resonance and a strongly suppressed transmission elsewhere [see Fig. 3(b)]. Comparing the experimental data to the results of the simulation, we find good agreement when the two resonators are close to resonance with each other, but significant differences otherwise. For example, in the experiment, there is a clear transmission signal of resonator 1 even if resonator 2 is far detuned. The model, on the other hand, predicts a strong damping of the resonance of resonator 2 in this regime. These observations become even more apparent when we look at the transmission signal for certain fixed coil currents. As shown by Fig. 5(b), the simple model can reproduce the measured resonances qualitatively well, despite the fact that the measurement shows a larger background signal. In contrast, Fig. 5(d) shows that, when the two resonators are far detuned, the transmission through the system is predicted to be strongly damped over the whole frequency range. Even on resonance the measured peak is approximately 15 dB higher than predicted.

We can account for these deviations by introducing a generic environment in form of a parasitic signal path (see Fig. 4). This parasitic path consists of a series connection of resistive, inductive, and capacitive elements. They are coupled capacitively to the input and output lines and also inductively to the resonators. As shown in Fig. 3, the model that includes this parasitic path (see App. A for detailed calculations) reproduces the experimental data very well over the whole frequency range. The reason for this significant improvement is the fact that the parasitic path opens up additional transmission channels for the system. First of all, the signal can be directly coupled into the parasitic path via the input line and then be transmitted to the output line leading to an increased constant background even if both resonators are far detuned from the input frequency. Secondly, if the signal is in resonance with resonator 1, it can enter the resonator and then couple inductively to the parasitic path. This can be seen by the increased signal at the resonance frequency of resonator 1 even where resonator 2 is detuned.

Turning back to the frequency-dependent transmission at fixed coil current values, we can clearly distinguish between regions, where the path through the resonator system dominates, and regions, where the parasitic path plays a crucial role. In Fig. 5(a), the two resonators have similar resonance frequencies and therefore transmit most of the signal to the output port. Hence, the parasitic path does not contribute to the shape of the resonances. Away from the resonances, the broadband nature of the parasitic path allows for an increased transmission background as it is observed in the measurements. When the two resonators are far detuned, the presence of the parasitic path also changes the qualitative shape of the resonance, making it Fano-like. While the peaks are symmetric in the simple model at all times [see Fig. 5(d)], the parasitic path model matches the peak-dip feature of the measurement [Fig. 5(e)]. In summary, based on the parasitic path model, we obtain a realistic set of parameters for each resonator (see Tab. I and Tab. II). Additionally, we calculate the capacitance per unit length $C_0 = 18 \text{nF m}^{-1}$ and the inductance per unit length $L_0 = 44 \text{pH m}^{-1}$ of the resonators from their bare resonance frequency. For definitions and further explanations regarding these parameters, please refer to App. A.

B. Calculation of the nonlinearity from the circuit model

Using the parameters extracted from the parasitic-path model discussed in the previous section, we can es-
Table I. Resonator parameters extracted from the circuit model. For each resonator, we show the total critical current $I_c = I_{c1} + I_{c2}$ of the SQUID, the zero current offset $\delta \Phi$ of the flux through the SQUID loop, the flux change $\Delta \Phi$ per applied coil current $\Delta I_{coil}$, the SQUID asymmetry parameter $d$, and the inductive coupling constant $k_L$ to the parasitic path.

![Table I](image)

Table II. Parameters of the parasitic path (see Fig. 4). Here, we show the capacitance $C_p$, resistance $R_p$, and inductance $L_p$ of the parasitic path.

![Table II](image)

Figure 6. Dimensionless envelope $u$ of the first spatial voltage mode at $\Phi_{ext} = 0$ and $\Phi_{ext} = \Phi_0/2$ of a resonator of length $l$ with a SQUID at position $x = 0$. The difference of the spatial mode across the SQUID, $\Delta u$, is a direct measure of the nonlinearity of the resonator.

Figure 7. The nonlinearity $U$ as a function of the current flowing through the external coil. Solid lines represent calculations based on parameters extracted from transmission data and parasitic-path model for resonator 1 (blue) and resonator 2 (yellow). The red dot is the result of a direct measurement for resonator 2 described in Sec. IV B.

by changing the magnetic bias fields. $J$ is extracted from the level splitting at the frequency degeneracy point of the two resonators.

IV. NONLINEARITY FROM A DIRECT POWER-DEPENDENT MEASUREMENT

In addition to the values extracted from the circuit model in the previous section, we present a direct measurement of the nonlinearity of resonator 2. Specifically, we exploit the response of the resonance frequency as a function of the input power. The relevant parameter to determine $U$ is the actual power circulating inside the resonator. Therefore, we first have to extract the external coupling strength between resonator and transmission line to convert the applied power to the field strength inside the resonator.

A. Quality factor

In order to extract the external quality factor of our resonators, we use an input-output formalism [34] and fit the result to the measured reflection signal of the two-resonator chain. In the limit where resonator 1 is far detuned, we find a dependency of the scattering parameter $S_{22}$ on the quality factors of resonator 2.
\[
S_{22} \approx 1 - \frac{2Q_{1,2}/Q_{\text{ext},2}}{1 - 2Q_{1,2} (\omega_{r,2} - \omega_d)}.
\]  

(2)

Here, we have used the loaded quality factor \(Q_{1,2}\), the external quality factor \(Q_{\text{ext},2}\) and the resonance frequency \(\omega_{r,2}\), each of resonator 2. The parameter \(\omega_d\) denotes the angular frequency of the driving field. On resonance of the second resonator, \(\omega_{r,2} - \omega_d = 0\), Eq. (2) further simplifies to

\[
S_{22} \approx 1 - \frac{2Q_{1,2}}{Q_{\text{ext},2}}.
\]  

(3)

Further information on the derivation of these equations can be found in App. C. We first fit the predicted phase dependence, \(\theta = \theta_0 + 2 \arctan(2Q_{1,2}(1 - \omega/\omega_{r,2}))\), to the measured scattering parameter data to extract the loaded quality factor. Then, we fit Eq. (2) to the magnitude and use Eq. (3) to determine the external quality factor (see Fig. 8). For resonator 2, we obtain an external quality factor of \(Q_{\text{ext},2} = 1.35 \cdot 10^5\).

B. Nonlinearity from power-dependent resonance amplitude

In order to get a relation between the directly measurable output voltage of our system and the nonlinearity, we start with the equation of motion for a single resonator. It can be written in terms of the flux \(\Psi = \int V(x,t)dt\), where \(V(x,t)\) is the internal voltage of the resonator,

\[
\frac{\Psi}{L} + C \frac{\dot{\Psi}}{R} + \beta \Psi^3 = F_0 e^{i\omega t}.
\]  

(4)

Here,

\[
\beta = -\frac{1}{24} \left( \frac{2\pi}{\Phi_0} \right)^2 \frac{\Delta u^4}{L_1}
\]  

(5)

is the prefactor of the nonlinear term due to the tunable Josephson junction formed by the SQUID. The parameter \(\beta\) depends on the SQUID inductance \(L_1\) and the drop \(\Delta u\) in the spatial voltage mode across the SQUID (see also Fig. 6). Obviously, the prefactor \(\beta\) is a direct measure for the nonlinearity of the system.

For the Duffing-like equation of motion, we can show that the maximum amplitude \(a\) of the mode is inversely proportional to the nonlinearity for small deviations from the unperturbed resonance frequency \(\omega_0\) [35]

\[
|\Psi|^2 = a^2 = \frac{8}{3} \frac{\omega_0 C}{\beta} \left( \omega - \omega_0 \right).
\]  

(6)

For the experimental output voltage we derive (App. B)

\[
V_{\text{out}}^2 = \frac{8}{3} \frac{\omega_0 C}{\beta} \left( \omega_0 - \omega \right) \frac{Z_G \omega}{2Q_{\text{ext}} L} G,
\]  

(7)

with \(Z_G = 50\ \Omega\) being the characteristic impedance of the circuit and \(G\) the gain of the amplification chain.

In order to obtain information on \(\beta\), we perform power-dependent measurements of the transmission through the
Figure 10. Squared output voltage at the resonance frequency \( \omega_r \) as a function of this frequency for resonator 2 (blue and red dots) and the respective fit (dashed yellow line) of Eq. (7) to data points of low input power (blue dots). If not shown, the error bars are smaller than the symbol size. We estimate the uncertainty of each frequency point to be \( \pm 2 \) MHz.

We have investigated a superconducting circuit consisting of two tunable and coupled nonlinear resonators. The nonlinearity is induced by a dc-SQUID galvanically coupled to each resonator. The system can be fully controlled by means of an external coil and two on-chip antennas, allowing us to tune the nonlinearity by roughly two orders of magnitude. The nonlinearity of resonator 1 (2) can be tuned between a minimum of 0 dBm and a maximum of 10.48 MHz (8.46 MHz). We have shown that we are able to model the response of our two-resonator system with an equivalent circuit including an additional signal path. In this way, we can reliably simulate the experimentally obtained transmission data. We have confirmed the nonlinearity extracted from the circuit model by means of direct, power-dependent transmission measurements. As a result of the demonstrated control of the nonlinearity and the understanding of the environment, the studied system is a promising candidate for quantum simulations of a driven-dissipative Bose-Hubbard physics.

V. CONCLUSION

We have investigated a superconducting circuit consisting of two tunable and coupled nonlinear resonators. The nonlinearity is induced by a dc-SQUID galvanically coupled to each resonator. The system can be fully controlled by means of an external coil and two on-chip antennas, allowing us to tune the nonlinearity by roughly two orders of magnitude. The nonlinearity of resonator 1 (2) can be tuned between a minimum of 0.12 MHz (0.08 MHz) and a maximum of 10.48 MHz (8.46 MHz). We have shown that we are able to model the response of our two-resonator system with an equivalent circuit including an additional signal path. In this way, we can reliably simulate the experimentally obtained transmission data. We have confirmed the nonlinearity extracted from the circuit model by means of direct, power-dependent transmission measurements. As a result of the demonstrated control of the nonlinearity and the understanding of the environment, the studied system is a promising candidate for quantum simulations of a driven-dissipative Bose-Hubbard physics.

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Appendix A: Derivation of the circuit model with a competing path

Here, we present the full circuit model for our two resonator system including a competing path. The standard model without this additional path can be obtained if we neglect all parts containing the parasitic path. For the driving voltage \( V_d \) and output voltage \( V_{\text{out}} \), we find

\[
\begin{align*}
V_d &= Z_0(I_1 + I_p) + V_x \\
V_{\text{out}} &= Z_0(I_3 + I_p),
\end{align*}
\]

with \( I_1 \) (\( I_3 \)) being the current flowing into (out of) the two resonator system. \( V_x \) is the voltage drop across the impedance \( Z_0 \) of the input cable.

We can describe the currents \( I_i \) flowing in our system using the voltages \( V_i \) at the in- and output capacitors (capacity \( C_{i0} \)), the coupling capacitor (capacity \( C_c \)) and the drive frequency \( \omega_d \). The index \( i \) denotes the capacitor starting from driving side and \( j \) the resonator number.

\[
\begin{align*}
I_1 &= (V_x - V_1)i\omega_d C_{i0} \\
I_2 &= (-V_1 - V_2)i\omega_d C_c \\
I_3 &= (-V_2 - V_{\text{out}})i\omega_d C_{i0}.
\end{align*}
\]

As we are only looking at the first voltage modes, the signs of \( V_1 \) and \( V_2 \) change at their second appearance.

In each resonator, the inflowing and outflowing currents have to be the same due to current conservation:

\[
\begin{align*}
I_1 &= (i\omega_d C_1 + \frac{1}{R_1} + \frac{1}{i\omega_d L_1})V_1 - k_{L,1} \sqrt{\frac{L_1}{L_2}} I_p - I_2, \\
I_2 &= (i\omega_d C_2 + \frac{1}{R_2} + \frac{1}{i\omega_d L_2})V_2 - k_{L,2} \sqrt{\frac{L_2}{L_1}} I_p - I_3.
\end{align*}
\]

Here, \( C_j \), \( L_j \) and \( R_j \) are the capacitance, inductance and resistance of resonator \( j = 1, 2 \) and \( k_{L,j} \) is its inductive coupling strength to the parasitic path. \( L_p \), \( C_p \), \( R_p \) and \( I_p \) are the inductance, capacitance and resistance of the parasitic path and the current flowing through it. Additionally we consider the voltage drops inside the parasitic path:

\[
\begin{align*}
&\frac{1}{i\omega_d C_p} I_p - i\omega_d k_{L,1} \sqrt{L_1 L_p} V_1 - \frac{1}{i\omega_d L_1} + i\omega_d k_{L,2} \sqrt{L_2 L_p} V_2 - \frac{1}{i\omega_d L_2} I_p + R_p I_p + \frac{1}{i\omega_d C_p} I_p = V_x - V_{\text{out}}.
\end{align*}
\]

To include the tunability of the circuit introduced by the dc-SQUIDs, we calculate \( C_i \) and \( L_i \) as functions of the external flux \( \Phi_{\text{ext}} \) penetrating the SQUID loops:

\[
C_i = C_0 \int_{-L/2}^{+L/2} u(x, k, \Phi_{\text{ext}})^2 \mathrm{d}x + C_i \Delta u(k, \Phi_{\text{ext}})^2
\]

\[
L_i = \left( \frac{1}{L_0} \int_{-L/2}^{+L/2} \left( \delta_x u(x, k, \Phi_{\text{ext}})^2 \right) \mathrm{d}x + \frac{1}{L_i \Delta u(k, \Phi_{\text{ext}})^2} \right)^{-1}.
\]

With a standard ansatz for the dimensionless envelopes of the spatial mode functions \( u_i \) of the resonator on the left and right side of the SQUID

\[
\begin{align*}
u_i,\ell &\equiv A_i \cos(k(x + L/2)) \\
u_i,\ell &\equiv A_i \cos(k(x - L/2)),
\end{align*}
\]

we can calculate \( u(k) \) and the jump of the mode function \( \Delta u(k) \) at the position of the SQUID after extracting the wave vector \( k \) from

\[
k_{\text{odd}} \tan \left( \frac{k_{\text{odd}} L}{2} \right) = -\frac{Z_0}{\nu} C_i \left( \omega_p^2 - \left( k_{\text{odd}} \nu \right)^2 \right).
\]

As the plasma frequency \( \omega_p = 1/(C_3 L_3) \) depends on the Josephson inductance \( L_3 \), which, in turn, depends on \( \Phi_{\text{ext}} \), the full circuit model is flux dependent. \( L_3 \) can
be described by the supercurrent \( I_s \) flowing through each SQUID loop

\[
I_s = I_c \left| \cos \left( \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \right| \sqrt{1 + d^2 \tan \left( \frac{\Phi_{\text{ext}}}{\Phi_0} \right)} \quad (A14)
\]

Here, \( L_J = \frac{\Phi_0}{2 \pi I_c} \) is the Josephson inductance and \( d = (I_{c1} - I_{c2}) / (I_{c1} + I_{c2}) \) the asymmetry parameter of the dc-SQUID. \( I_{c1} \) is the critical current of junction \( i \) and \( I_c \) the total critical current.

**Appendix B: Maximum of the frequency response function**

In order to use the duffing equation of motion and its frequency response to model our experimental data, we have to transform the equations into experimentally accessible parameters. Therefore we use \( \Psi = \int V dt = V / (i \omega) \) to transform

\[
|\Psi|^2 = a^2 = \frac{8 \omega_0 C}{3 \beta} (\omega - \omega_0) \quad (B1)
\]

into

\[
\frac{V^2}{\omega^2} = \frac{8 \omega_0 C}{3 \beta} (\omega - \omega_0). \quad (B2)
\]

Here, \( V \) is the voltage of the internal mode, which cannot be directly measured. We can use the external quality factor in order to relate the internal and external voltages [36]

\[
Q_{\text{ext}} = \frac{\omega W_m + W_e}{\omega_{\text{loss}}} = \omega \left| V_{\text{int}} \right|^2 / (2 L \omega^2) / |V_{\text{ext}}|^2 / Z_0. \quad (B3)
\]

We assume that, on resonance, the energy \( W_e \) stored in the capacitance is equal to the energy \( W_m \) stored in the inductance and that the power of an electrical signal in our waveguide can be written as \( P = V^2 / Z_0 \). We therefore obtain

\[
\frac{V_{\text{int}}^2}{\omega^2} = \frac{2 V_{\text{out}}^2 Q_{\text{ext}} L}{Z_0 \omega}. \quad (B4)
\]

For the experimentally accessible voltage \( V_{\text{out}} \), we obtain a similar dependence as for the internal flux field, but modified with an additional scaling factor \((Z_0 \omega G) / (2 Q_{\text{ext}} L)\)

\[
\frac{V_{\text{out}}^2}{\omega^2} = \frac{8 \omega_0 C}{3 \beta} (\omega_0 - \omega) \frac{Z_0 \omega}{2 Q_{\text{ext}} L} G. \quad (B5)
\]

Here, we take the power gain \( G \) in our experiment into account as we do not directly measure the output voltage of the resonators but a voltage after amplification (see Fig. 2).

**Appendix C: Calculation of the external quality factor**

In App. B, it is shown that, in order to extract the nonlinearity from a power dependent measurement, we need to know the external quality factor of the resonators. Here, we derive equations, that allow us to extract the external quality factor from a reflection measurement of our two resonator system.

First, we consider a system without intrinsic damping, of which the Hamiltonian reads [37–39]

\[
H_{\text{sys}} = \hbar \omega_0 a^\dagger a + \hbar \omega_b b^\dagger b + \hbar g (a^\dagger b + ab^\dagger) \quad (C1)
\]

\[
H = H_{\text{sys}} + \hbar \int_{-\infty}^{+\infty} d\omega \left\{ \omega t^\dagger (\omega) l (\omega) + i \kappa_l (\omega) [l^\dagger (\omega) a - l (\omega) a^\dagger] \right\}
\]

\[
+ \hbar \int_{-\infty}^{+\infty} d\omega \left\{ \omega t^\dagger (\omega) r (\omega) + i \kappa_r (\omega) [r^\dagger (\omega) b - r (\omega) b^\dagger] \right\} \quad (C2)
\]

Here, by convention, we define the specific type of coupling between the intra-resonator fields, \( a \) for resonator 1 and \( b \) for resonator 2, and the bath, \( l (\omega) \) and \( r (\omega) \), respectively, for the simplicity of derivation. Then, one can derive the following Heisenberg equations of motion for the field operators

\[
\dot{a} = - \frac{i}{\hbar} [a, H_{\text{sys}}] - \int_{-\infty}^{+\infty} d\omega \kappa_l (\omega) l (\omega), \quad (C4)
\]

\[
\dot{r} (\omega) = - \hbar \omega r (\omega) + \kappa_r (\omega) b, \quad (C5)
\]

\[
\dot{b} = - \frac{i}{\hbar} [b, H_{\text{sys}}] - \int_{-\infty}^{+\infty} d\omega \kappa_r (\omega) r (\omega). \quad (C6)
\]

We note that the above equations can be split into two groups, namely Eq. (C3)/(C4) and Eq. (C5)/(C6), each
of which is identical to the input-output formalism of a single system. Following the same procedure as in Ref. 37, we define the input fields
\begin{equation}
I_{\text{in}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \omega e^{-i\omega t} I(\omega), \quad (C7)
\end{equation}
\begin{equation}
R_{\text{in}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \omega e^{-i\omega t} R(\omega). \quad (C8)
\end{equation}

The equations of the intra-resonator fields thus read
\begin{equation}
a = -i\omega a - igb - \frac{\gamma_\ell}{2} a - \sqrt{\gamma_\ell} I_{\text{in}}(t), \quad (C9)
\end{equation}
\begin{equation}
b = -i\omega b - iga - \frac{\gamma_r}{2} b - \sqrt{\gamma_r} R_{\text{in}}(t). \quad (C10)
\end{equation}

Here, we have used the first Markov approximation \( \gamma_\ell = 2\pi\kappa_\ell^2(\omega) \), \( \gamma_r = 2\pi\kappa_r^2(\omega) \) [37]. We also added the internal loss rates \( \gamma_a \) \( \gamma_b \) of resonator 2. The two output fields are
\begin{equation}
I_{\text{out}} = I_{\text{in}} + \sqrt{\gamma_\ell} a(t), \quad (C11)
\end{equation}
\begin{equation}
R_{\text{out}} = R_{\text{in}} + \sqrt{\gamma_r} b(t). \quad (C12)
\end{equation}

Then, we move to the frame rotating with respect to the reference frequency \( \omega_d \) and define \( \Delta_a = \omega_a - \omega_d \), \( \Delta_b = \omega_b - \omega_d \), where \( \omega_d \) is the frequency of the driving field. For steady state solutions, we find
\begin{equation}
S_{11} = \frac{I_{\text{out}}}{I_{\text{in}}} = 1 - \frac{\sqrt{\gamma_\ell}}{(i\Delta_a + \frac{\gamma_\ell^2}{2}) (i\Delta_b + \frac{\gamma_r^2}{2}) + g^2}, \quad (C13)
\end{equation}
\begin{equation}
S_{21} = \frac{R_{\text{out}}}{I_{\text{in}}} = \frac{ig\sqrt{\gamma_\ell}}{(i\Delta_a + \frac{\gamma_\ell^2}{2}) (i\Delta_b + \frac{\gamma_r^2}{2}) + g^2}, \quad (C14)
\end{equation}
\begin{equation}
S_{12} = \frac{I_{\text{out}}}{R_{\text{in}}} = \frac{ig\sqrt{\gamma_r}}{(i\Delta_a + \frac{\gamma_\ell^2}{2}) (i\Delta_b + \frac{\gamma_r^2}{2}) + g^2}, \quad (C15)
\end{equation}
\begin{equation}
S_{22} = \frac{R_{\text{out}}}{R_{\text{in}}} = 1 - \frac{\sqrt{\gamma_r}}{(i\Delta_a + \frac{\gamma_\ell^2}{2}) (i\Delta_b + \frac{\gamma_r^2}{2}) + g^2}. \quad (C16)
\end{equation}

Here, we have used the imaginary unit \( i \), which is related to the imaginary unit in electrical engineering by \( i = -j \). We find that all the damping coefficients, \( \gamma_a \), \( \gamma_b \), \( \gamma_\ell \), and \( \gamma_r \) can be obtained by detuning the two resonators, and measuring the internal and coupling quality factors from the reflection responses \( S_{11} \) and \( S_{22} \), respectively.

We find that the scattering parameter \( S_{22} \) only depends on the external quality factor \( Q_{\text{ext},2} = \omega_b/\gamma_b \) and the loaded quality factor \( Q_{\ell,2} \) of resonator 2, in the limit where resonator 1 is far detuned \( \Delta_a \to \infty \):
\begin{equation}
S_{22} \approx 1 - \frac{2Q_{\ell,2}Q_{\text{ext},2}}{1 - 2iQ_{\ell,2}\Delta_b}. \quad (C17)
\end{equation}

On resonance of the second resonator, \( \Delta_b = 0 \), we then get
\begin{equation}
S_{22} \approx 1 - \frac{2Q_{\ell,2}}{Q_{\text{ext},2}}. \quad (C18)
\end{equation}

Table III. Parameters extracted from the quality factor fitting procedure described in App. C near the maximum resonance frequency of resonator 2. We assume the external quality factor to be frequency-independent.

| parameter   | value                  |
|-------------|------------------------|
| \( Q_{\ell,2} \) | \( 7.4 \cdot 10^5 \)    |
| \( Q_{\text{int},2} \) | \( 7.8 \cdot 10^4 \)    |
| \( Q_{\text{ext},2} \) | \( 1.35 \cdot 10^5 \)    |
| \( \gamma_b \) | \( 2\pi \times 0.91 \text{ MHz} \) |
| \( \gamma_r \) | \( 2\pi \times 0.05 \text{ MHz} \) |

Appendix D: Gain calibration measurements

As we cannot directly measure the gain of our amplification chain while the cryostat is cold, we perform Planck spectroscopy [41, 42] as a photon number calibration measurement. To this end, we use a heatable
attenuator, that emits black body radiation towards the input of the sample. With the resonators far detuned, the signal is reflected to our amplification chain, and we can measure the resulting power with a digitizer card at room temperature. Before digitizing the signal, we convert the signal to an intermediate frequency of 11 MHz with an analog mixer setup. After digitizing, the data is digitally downconverted to DC. As a result, we get the in-phase (I) and quadrature (Q) components of the measured signal as a DC value. We can then write the power of the signal as a function of the temperature of the attenuator

$$P_{\text{signal}} = \frac{I^2 + Q^2}{R} = \frac{\kappa G_{\text{cal}}}{R} \left[ \frac{1}{2} \coth \left( \frac{h f_0}{2 k_B T_{\text{att}}} \right) + n_{\text{noise}} \right],$$

with the Boltzmann constant $k_B$, $\kappa = 2 R \cdot BW \cdot h f_0$, $BW$ the bandwidth of the measurement and $G_{\text{cal}}$ the total gain of the amplification chain. We perform a temperature sweep of the heatable attenuator and measure the resulting output quadrature components, which we can fit to the formula for the black body radiation. The result of this fit is the product $\kappa G_{\text{cal}}$, which relates the number of photons at the sample to the measured voltages and the photon number $n_{\text{noise}}$ of the noise of the amplification chain. In our setup, this photon number is dominated by the noise number from the cryogenic amplifier in the chain. In Fig. 12, we show a temperature sweep from 50 mK to 800 mK. From the fit, we get $\kappa G_{\text{cal}} = 8.1 (\text{mV})^2/\text{photon}$ and a noise number $n_{\text{noise}} = 142$. The high noise number can be explained by aging effects in our HEMT amplifiers. Using the measurement bandwidth of $BW = 2 \text{ MHz}$, we can calculate the gain of the chain and find $G_{\text{cal}} = 109.8 \text{ dB}$. As we perform this calibration measurement with a slightly different setup (including an additional downconversion box, containing an intermediate frequency amplifier with significant gain), we cannot directly use this gain measurement for the interpretation in the experiments performed with a VNA.

In order to estimate the gain of the VNA setup, we measure the gain of the additional room temperature components used in the determination of $G_{\text{cal}}$. We measure the gain of the downconversion box, $G_{\text{box}} = 47.3 \text{ dB}$, and the gain of an additional rf room temperature amplifier $G_{\text{rf-amp}} = 24.6 \text{ dB}$. If we subtract these two values from the determined total gain $G_{\text{cal}}$, we get a value of the gain of the VNA setup $G = 38 \pm 4 \text{ dB}$, which we use in the main text of this paper. The uncertainties stem mainly from the frequency dependency of the gain as the measurement of the gain and of the nonlinearity have been performed at different frequencies (3 dB) and the different cables used in the two measurements, which cannot be reliably accounted for in our estimation (1 dB). The uncertainty in the gain is the main contribution to the uncertainty of the nonlinearity.