Elliptic Partial Differential Equation Involving Singularity

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Abstract

The aim of this paper is to prove existence of solution for a partial differential equation involving a singularity with a general nonnegative, Radon measure $\mu$ as its nonhomogenous term which is given as

\[-\Delta u = f(x)h(u) + \mu \text{ in } \Omega \]
\[u = 0 \text{ on } \partial \Omega \]
\[u > 0 \text{ on } \Omega, \]

where $\Omega$ is a bounded domain of $\mathbb{R}^N$. $f$ is a nonnegative function over $\Omega$.

**keywords:** elliptic PDE; Sobolev space; Schauder fixed point theorem.

**AMS classification:** 35J35, 35J60.

1. Introduction

Problems involving singularity has of late become a hugely popular interest of research amongst the Mathematical community. A good amount of research has been done to prove the existence of a solution to the problem

\[-\Delta u = f(x)h(u) \text{ in } \Omega \]
\[u = 0 \text{ on } \partial \Omega. \quad (1.1)\]

A few noteworthy results on such problems can be found in [1], [3, 4], [10], [11], [12], [13] and the references therein. An existence result due to Lazer and Mc Kenna [1], pertaining to the case of $h(s) = \frac{1}{s^\gamma}$ with $f$ being sufficiently regular, has a unique solution obtained by the application of the sub and the super solution method. The authors of this article in [1] have proved that the problem in (1.2) has a solution iff

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They have also shown that for \( \gamma > 1 \), solutions to the problem in (1.2) with infinite energy exists. A weaker condition on the function \( f \) can be considered by picking \( f \) from \( L^p(\Omega) \), for \( p \geq 1 \), or from the space of Radon measures. In a study due to Boccardo and Orsina \([5]\), they have proved the existence and uniqueness of solution to the problem

\[
-\Delta u = \frac{f}{u^{\gamma}} \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \). They considered \( f \geq 0 \) in \( \Omega \) and \( \gamma > 0 \). The result depend on the \( L^p \) space in which \( f \) is chosen from. The value of \( \gamma \) also decide the space in which the function belongs to - if \( \gamma < 1 \) then \( u \in W^{1,1}_0(\Omega) \), if \( \gamma = 1 \) then \( u \in H^1_0(\Omega) \), if \( \gamma > 1 \) then \( u \in H^1_{\text{loc}}(\Omega) \) where the zero Dirichlet boundary condition is assumed in a weaker sense, \( u^{2+1\gamma} \in H^1_0(\Omega) \), than the usual sense of trace. It is worth mentioning the work due to Giachetti et al. \([3, 4]\) and the references therein. When \( f \) is a measure, the problem may not possess a solution in general and in this case the question of nonexistence is of great importance as seen in \([5]\). In \([6]\) the authors have considered a nonlinear elliptic boundary value problem with a general singular lower order term. The authors here have shown the existence of a distributional solution. A slight improvement of the result in \([1]\) can be found in \([2]\). A series of noteworthy contributions to the semilinear problem with a singularity has been made by Canino et al \([7, 8, 9, 11, 12, 10]\). In \([7]\) a minimax method is used to address the ‘jumping problem’ for a singular semilinear elliptic equation. A symmetry of solutions have been shown in \([8]\) for some semilinear equations with singular nonlinearities. In \([9]\) the authors have considered quasilinear elliptic equations involving the p-Laplacian and singular nonlinearities. They have deduced a few comparison principles and have proved some uniqueness results. The readers may also refer to \([10, 11, 12]\) and the references therein. In this paper we will prove the existence of nonnegative weak solution to the following pde.

\[
-\Delta u = f(x)h(u) + \mu \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega, \\
u > 0 \quad \text{on } \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) for \( N \geq 2 \), \( f > 0 \) and \( \mu \) is a nonnegative Radon measure.

\[\text{1.1 Notations}\]

In this subsection we have explained the notations which will be used throughout this article. We denote a Sobolev space as \( W^{k,p}(\Omega) \) \([22]\), where \( \Omega \) is an open set of \( \mathbb{R}^N \),
consists of all locally summable functions \( u : \Omega \rightarrow \mathbb{R} \) such that for each multiindex \( \alpha \) with \( |\alpha| \leq k \), \( D^\alpha u \) exists in the weak sense and belongs to \( L^p(\Omega) \). If \( u \in W^{k,p}(\Omega) \), we define its norm as

\[
||u||_{W^{k,p}(\Omega)} = \begin{cases} 
\left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{\frac{1}{p}} & (1 \leq p < \infty), \\
\sum_{|\alpha| \leq k} ||D^\alpha u||_{L^\infty(\Omega)} & (p = \infty). 
\end{cases}
\]

We denote by \( W^{k,p}_0(\Omega) \) the closure of \( C_\infty^c(\Omega) \) in \( W^{k,p}(\Omega) \) and \( W^{k,p}_{loc}(\Omega) \) to be the local Sobolev space such that for any \( u \in W^{k,p}_{loc}(\Omega) \) and any compact \( K \subset \Omega \), \( u \in W^{k,p}(K) \).

The Hölder Space [22] is \( C^{k,\beta}(\overline{\Omega}) \) with \( 0 < \beta \leq 1 \) consists of all functions \( u \in C^k(\overline{\Omega}) \) such that the norm \( \sum |\alpha| \leq k \sup |D^\alpha u| + \sup_{x \neq y} \left\{ \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\beta} \right\} \) is finite. We will use the truncation function for fixed \( k > 0 \),

\[
T_k(s) = \max\{-k, \min\{k, s\}\},
\]

and

\[
G_k(s) = (|s| - k)^+ \text{sign}(s),
\]

with \( s \in \mathbb{R} \). Observe that \( T_k(s) + G_k(s) = s \) for any \( s \in \mathbb{R} \) and \( k > 0 \).

We will also use the notation

\[
\int_\Omega f(x) dx := \int f.
\]

We will denote the space of all finite Radon measures on \( \Omega \) as \( \mathcal{M}(\Omega) \). If \( \mu \in \mathcal{M}(\Omega) \), then we define the norm as

\[
||\mu||_{\mathcal{M}(\Omega)} = \int_\Omega d|\mu|.
\]

We will use the Marcinkiewicz space \( M^q(\Omega) \) [18] (or the weak \( L^q(\Omega) \) space) defined for every \( 0 < q < \infty \), as the space of all measurable functions \( f : \Omega \rightarrow \mathbb{R} \) such that the corresponding distribution functions satisfy an estimate of the form

\[
m(\{x \in \Omega : |f(x)| > t\}) \leq \frac{C}{t^q} \quad t > 0, C < \infty.
\]

For bounded \( \Omega \) we have \( M^q \subset M^\overline{q} \) if \( q \geq \overline{q} \), for some fixed positive \( \overline{q} \). We recall here the following useful continuous embeddings

\[
L^q(\Omega) \hookrightarrow M^q(\Omega) \hookrightarrow L^q(\Omega), \quad (1.4)
\]

for every \( 1 < q < \infty \) and \( 0 < \epsilon < q - 1 \). We organize the paper as follows. In Section 2 we state and prove the main results which pertains to the cases \( \gamma \leq 1 \) and \( \gamma > 1 \). In Section 3 we make a few remarks for the case \( \gamma < 1 \). In Section 4 we discuss the problem with a few relaxation on the assumptions made on \( f \).
2. Assumptions, Definitions and the main results

Let us consider the following boundary value problem.

\[
-\Delta u = h(u)f + \mu \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial\Omega,
\]
\[
u > 0 \quad \text{in } \Omega,
\]

where \(\Omega\) is an open bounded subset of \(\mathbb{R}^N\), \(N > 2\), \(\mu\) is a nonnegative, bounded Radon measure on \(\Omega\), \(f \ge 0 \in L^m(\Omega)\) for \(m > 1\), which could be a measure. We make sure that both \(f\) and \(\mu\) are nonzero.

The function \(h : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is a nonlinear, non-increasing function which we suppose to be continuous such that

\[
\lim_{s \to 0^+} h(s) \in (0, \infty], \quad \text{and} \quad \lim_{s \to \infty} h(s) = h(\infty) < \infty.
\]

We assume the growth condition near zero as

\[
\exists C_1, K > 0 \quad \text{such that} \quad h(s) \le \frac{C_1}{s^\gamma} \quad \text{if} \quad s < K,
\]

with \(\gamma > 0\). We will later observe that the behavior of \(h\) at infinity influences the regularity of the solution \(\nu\). Hence, we need to assume the following.

\[
\exists C_2, \overline{K} > 0 \quad \text{such that} \quad h(s) \le \frac{C_2}{s^\theta} \quad \text{if} \quad s > \overline{K},
\]

for \(\theta > 0\). We now give two important definitions which is essential to our study of the problem in (2.1).

**Definition 2.1.** Let \((\mu_n)\) be the sequence of measurable functions in \(\mathcal{M}(\Omega)\). We say \((\mu_n)\) converges to \(\mu \in \mathcal{M}(\Omega)\) in the sense of measure i.e. \(\mu_n \rightharpoonup \mu \in \mathcal{M}(\Omega)\), if

\[
\int_\Omega f d\mu_n \to \int_\Omega f d\mu, \quad \forall f \in C_0(\Omega).
\]

**Definition 2.2.** If \(\gamma < 1\), then a weak solution to the problem in (2.1) is a function in \(W^{1,1}_0(\Omega)\) such that

\[
\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega fh(u)\varphi + \int_\Omega \varphi d\mu, \quad \forall \varphi \in C^1_c(\Omega)
\]

and

\[
\forall K \subset \subset \Omega, \exists C_K \quad \text{such that} \quad u \ge C_K > 0
\]

If \(\gamma \ge 1\), then a weak solution to the problem is a function \(u \in W^{1,1}_{loc}(\Omega)\) satisfying (2.5) and (2.6) such that \(T_k^{\frac{2+\gamma}{2}} u \in W^{1,2}_0(\Omega)\) for each fixed \(k > 0\).
In both the cases, i.e. $\gamma \leq 1$ and $\gamma > 1$, we will show the existence of solutions for problem (1.3) in the subsection 2.1 and 2.2. In order to prove this, we begin by considering a sequence of the following problems.

$$
-\Delta u_n = h_n \left( u_n + \frac{1}{n} \right) f_n + \mu_n \text{ in } \Omega,
$$

$$
u_n = 0 \text{ on } \partial \Omega,
$$

(2.7)

where $(\mu_n)$ is a sequence of smooth nonnegative functions bounded in $L^1(\Omega)$ and converging weakly to $\mu$ in the sense of Definition 2.1, $h_n = T_n(h)$ and $f_n = T_n(f)$ are the truncations at level $n$. The weak formulation of (2.7) is

$$
\int_{\Omega} \nabla u_n \nabla \varphi = \int_{\Omega} h_n \left( u_n + \frac{1}{n} \right) f_n \varphi + \int_{\Omega} \mu_n \varphi, \quad \forall \varphi \in C^1_c(\Omega).
$$

(2.8)

We now prove the existence of a solution to the problem (2.7) in the following lemma.

**Lemma 2.3.** Problem (2.7) admits a nonnegative weak solution $u_n \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$.

**Proof.** We will apply the Schauder’s fixed point argument used in [21] to prove the lemma. For a fixed $n \in \mathbb{N}$ let us define a map,

$$
G : L^2(\Omega) \rightarrow L^2(\Omega),
$$

such that, for any $v \in L^2(\Omega)$ gives the weak solution $w$ to the following problem

$$
-\Delta w = h_n \left( |v| + \frac{1}{n} \right) f_n + \mu_n \text{ in } \Omega,
$$

$$
w = 0 \text{ on } \partial \Omega.
$$

(2.9)

The existence of a unique $w \in W^{1,2}_0(\Omega)$ corresponding to a $v \in L^2(\Omega)$ is guaranteed by the Lax-Milgram theorem. Thus we can choose $w$ as a test function in the weak formulation of (2.9) with the test function space $W^{1,2}_0(\Omega)$. We begin by using the Poincaré inequality to get

$$
\lambda_1^2 \int_\Omega |w|^2 \leq \int_\Omega |\nabla w|^2
$$

$$
= \int_\Omega h_n \left( |v| + \frac{1}{n} \right) f_n w + \int_\Omega w \mu_n \text{ by the weak formulation of (2.9)}
$$

$$
\leq C_1 \int_{(|v| + \frac{1}{n} \leq \lambda)} |f_n w| + \max \frac{h(s)}{K} \int_{\lambda \leq |v| + \frac{1}{n} \leq K} f_n w
$$

$$
+ C_2 \int_{(|v| + \frac{1}{n} \geq K)} |f_n w| + C(n) \int_\Omega |w|
$$
\[
\leq C_1 n^{1+\gamma} \int_{|v| + \frac{1}{n} \leq K} |w| + n \max_{K \subset \subset \Omega} h(s) \int_{(K \subset \subset \Omega)} |w| + C_2 n^{1+\theta} \int_{|v| + \frac{1}{n} \geq K} |w| \\
+ C(n) \int_{\Omega} |w| \\
\leq C(n, \gamma) \int_{\Omega} |w| \\
\leq C'.C(n, \gamma) ||w||_2 \text{ by using the Hölder's inequality.} \tag{2.10}
\]

This shows that
\[
||w||_{L^2(\Omega)} \leq C'.C(n, \gamma), \tag{2.11}
\]
where, \(C'\) and \(C(n, \gamma)\) are independent of \(v\). We will next prove that the map \(G\) is continuous over \(L^2(\Omega)\). Consider a sequence \((v_k)\) that converges to \(v\) in \(L^2(\Omega)\)-norm. Then by the dominated convergence theorem we get
\[
\| (h_n (v_k + \frac{1}{n}) f_n + \mu_n) - (h_n (v + \frac{1}{n}) f_n + \mu_n) \|_{L^2(\Omega)} \to 0.
\]
Hence, by the uniqueness of the weak solution, we can say that \(w_k = G(v_k)\) converges to \(w = G(v)\) in \(L^2(\Omega)\). Thus \(G\) is continuous over \(L^2(\Omega)\).

What finally needs to be checked is whether \(G(L^2(\Omega))\) is relatively compact in \(L^2(\Omega)\) or not?. We proved in \(2.11\) that
\[
\int_{\Omega} |\nabla w|^2 = \int_{\Omega} |\nabla G(v)|^2 \leq C'.C(n, \gamma),
\]
for any \(v \in L^2(\Omega)\), so that, \(G(L^2(\Omega))\) is relatively compact in \(L^2(\Omega)\) by Rellich-Kondrachov theorem. Therefore, we proved that \(G(L^2(\Omega))\) is relatively compact in \(L^2(\Omega)\). Now, on applying the Schauder fixed point theorem we obtain that \(G\) has a fixed point \(u_n \in L^2(\Omega)\) that is a weak solution to \((2.7)\) in \(W_0^{1,2}(\Omega)\). Since, \((h_n (u_n + \frac{1}{n}) f_n + \mu_n) \geq 0\) then by the maximum principle \(u_n \geq 0\). Furthermore, for a fixed \(n\), since the right-hand side of \((2.7)\) is in \(L^\infty(\Omega)\) we have \(u_n\) belongs to \(L^\infty(\Omega)\) by Théorème 4.2, page 215 in \[19\] and this concludes the proof.

The next step is to prove that \((u_n)\) is uniformly bounded from below on compact subsets of \(\Omega\).

**Lemma 2.4.** The sequence \((u_n)\) is such that for every \(K \subset \subset \Omega\) there exists \(C_K\) (independent of \(n\)) such that \(u_n(x) \geq C_K > 0, \text{ a.e. in } K\), and for every \(n \in \mathbb{N}\).

**Proof.** Let us consider the problem
\[
-\Delta v_n = h_n \left( v_n + \frac{1}{n} \right) f_n \text{ in } \Omega, \\
v_n = 0 \text{ on } \partial \Omega. \tag{2.12}
\]
We first show the existence of a weak solution \( v_n \) to the problem in (2.12) such that \( \forall K \subset \subset \Omega, \exists C_K \) such that \( v_n \geq C_K > 0 \), for almost every \( x \) in \( K \) and \( C_K \) is independent of \( n \). The existence of a weak solution to (2.12) follows the same proof as in Lemma 2.3. Since \( 0 \leq f_n \leq f_{n+1} \), \( h \) is non-increasing and hence \( h_n \) is non-increasing, we have

\[
-\Delta v_n = f_n h_n \left( v_n + \frac{1}{n} \right) \\
\leq f_{n+1} h_n \left( v_n + \frac{1}{n+1} \right). \tag{2.13}
\]

We also know that \( u_{n+1} \) is a weak solution to

\[
-\Delta v_{n+1} = f_{n+1} h_{n+1} \left( v_{n+1} + \frac{1}{n+1} \right) \text{ in } \Omega, \\
v_{n+1} = 0 \text{ on } \partial \Omega. \tag{2.14}
\]

The difference between the weak formulations of the problems in (2.13), (2.14) with the choice of a test function as \( (v_n - v_{n+1})^+ \) we obtain

\[
\int_{\Omega} \nabla (v_n - v_{n+1}) \cdot \nabla (v_n - v_{n+1})^+ = \int_{\Omega} |\nabla (v_n - v_{n+1})^+|^2 \\
\leq \int_{\Omega} f_{n+1} \left[ h_n \left( v_n + \frac{1}{n+1} \right) \\
- h_{n+1} \left( v_{n+1} + \frac{1}{n+1} \right) \right] (v_n - v_{n+1})^+ \\
= \int_{\Omega} f_{n+1} \left[ \left\{ h_n \left( v_n + \frac{1}{n+1} \right) \\
- h_{n+1} \left( v_{n+1} + \frac{1}{n+1} \right) \right\} \chi_{[v_n \leq v_{n+1}]} (v_n - v_{n+1})^+ \\
+ \left\{ h_n \left( v_n + \frac{1}{n+1} \right) \\
- h_{n+1} \left( v_{n+1} + \frac{1}{n+1} \right) \right\} \chi_{[v_n > v_{n+1}]} (v_n - v_{n+1})^+ \right] \\
\leq 0. \tag{2.15}
\]

Therefore, \( (v_n - v_{n+1})^+ = 0 \) almost everywhere in \( \Omega \), thus implying that \( v_n \leq v_{n+1} \).

We use the Théorème 4.2, in page 215 [19] to obtain

\[
\|v_1\|_\infty \leq K_1 \|f_1 h_1 (v_1 + 1)\|_\infty + K_2 \|v_1\|_2 \\
\leq K_1 + K_2 = C \text{ say.} \tag{2.16}
\]
Thus we have

\[-\Delta v_1 = f_1 h_1(v_1 + 1) \geq f_1 h_1(||v_1||_\infty + 1) \geq f_1 h_1(C + 1) > 0.\]  

(2.17)

Since \(f_1 h_1(C+1)\) is identically not equal to zero, hence by the strong maximum principle over \(-\Delta\) we have \(v_1 > 0\). Since we have considered a relatively compact subset \(K\) of \(\Omega\), there exists a constant \(C_K\) such that \(v_1 \geq C_K > 0\).

Coming back to the proof of the lemma, we first take the difference between the weak formulations of (2.7) and (2.12) respectively with the choice of test function being \((u_n - v_n)^-\). It is easy to show that \(u_n \geq v_n\) almost everywhere in \(\Omega\), for if not, i.e., if \(u_n < v_n\) in \(\Omega\). Then we have

\[-\int_\Omega |\nabla(u_n - v_n)^-|^2 = \int_\Omega \nabla(u_n - v_n) \cdot \nabla(u_n - v_n)^-\]

\[= \int_\Omega \left(h_n \left(u_n + \frac{1}{n}\right) - h_n \left(v_n + \frac{1}{n}\right)\right) f_n \cdot (u_n - v_n)^-\]

\[+ \int_\Omega \mu_n \cdot (u_n - v_n)^-\]

\[\geq 0.\]

This implies \(u_n \geq v_n\) almost everywhere in \(\Omega\) and hence in \(K\). We have also showed that for every \(K \subset \subset \Omega\) there exists \(C_K\) such that \(v_n \geq C_K > 0\). Hence \(\forall K \subset \subset \Omega \exists C_K\) such that \(u_n \geq C_K > 0\), a.e. in \(K\).

We are now in a position to prove the existence of a solutions to the problem (2.1). In order to do this we divide the problem into the following two cases.

2.1 The case of \(\gamma < 1\)

In this subsection, we consider the problem in (2.7) for the case of \(\gamma < 1\).

**Lemma 2.5.** Let \(u_n\) be a solution of (2.7), where \(h\) satisfy (2.3) and (2.4), with \(\gamma < 1\) and \(\theta \geq 1\). Then \((u_n)\) is bounded in \(W^{1,q}_0(\Omega)\) for every \(q < \frac{N}{N-1}\).

**Proof.** We will first prove that \((\nabla u_n)\) is bounded in \(M^{N,N}(\Omega)\). For this, we take \(\varphi = T_k(u_n)\) as a test function in the weak formulation of (2.7) and get

\[\int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega h_n \left(u_n + \frac{1}{n}\right) T_k(u_n) f_n + \int_\Omega T_k(u_n) \mu_n.\]  

(2.18)
Now, \( \frac{T_k(u_n)}{(u_n + \frac{1}{n})^\gamma} \leq \frac{u_n^\gamma}{(u_n + \frac{1}{n})^\gamma u_n^\gamma} \leq u_n^{1-\gamma} \).

Using (2.3) and (2.4) in the right hand side of (2.18) we have,

\[
\int_\Omega h_n \left( u_n + \frac{1}{n} \right) f_n T_k(u_n) \leq C_1 \int_{(u_n + \frac{1}{n} \leq K)} \frac{f_n T_k(u_n)}{(u_n + \frac{1}{n})^\gamma} + \max h(s) \int_{(K \leq (u_n + \frac{1}{n}) \leq K)} f_n T_k(u_n) \\
+ C_2 \int_{(u_n + \frac{1}{n} \geq K)} \frac{f_n T_k(u_n)}{(u_n + \frac{1}{n})^\theta} \\
\leq C_1 K^{1-\gamma} \int_{(u_n + \frac{1}{n} \leq K)} f + k \max h(s) \int_{(K \leq (u_n + \frac{1}{n}) \leq K)} f \\
+ \frac{C_2 k}{K^\theta} \int_{(u_n + \frac{1}{n} \geq K)} f \\
\leq Ck
\]

and

\[
\int_\Omega T_k(u_n) \mu_n \leq k ||\mu_n||_{L^1(\Omega)} \leq Ck.
\]

Combining the previous results we obtain,

\[
\int_\Omega |\nabla T_k(u_n)|^2 \leq Ck. \tag{2.19}
\]

Consider

\[
\{|\nabla u_n| \geq t \} = \{|\nabla u_n| \geq t, u_n < k \} \cup \{|\nabla u_n| \geq t, u_n \geq k \} \\
\subset \{|\nabla u_n| \geq t, u_n < k \} \cup \{u_n \geq k \} \subset \Omega.
\]

Then using the subadditivity property of Lesbegue measure \( m \) we have,

\[
m(|\nabla u_n| \geq t) \leq m(|\nabla u_n| \geq t, u_n < k) + m(|u_n \geq k|). \tag{2.20}
\]

Therefore, from the Sobolev inequality

\[
\frac{1}{\lambda_1^2} \left( \int_\Omega |T_k(u_n)|^{2*} \right)^{\frac{2}{2*}} \leq \int_\Omega |\nabla T_k(u_n)|^2 \leq Ck,
\]

\( \lambda_1 \) is the first eigen value of the Laplacian operator. Now, on restricting the integral on the left hand side on \( I_1 = \{x \in \Omega : u_n \geq k \} \), on which \( T_k(u_n) = k \), we then obtain

\[
k^2 m(|u_n \geq k|) \leq Ck,
\]

which implies

\[
m(|u_n \geq k|) \leq \frac{C}{k^{\frac{N}{N-2}}} \quad \forall k \geq 1.
\]

Hence, \( (u_n) \) is bounded in \( M^{\frac{N}{N-2}}(\Omega) \). Proceeding similarly for \( I_2 = \{|\nabla u_n| \geq t, u_n < k \} \), we get
\[
m(\{|\nabla u_n| \geq t, u_n < k\}) \leq \frac{1}{t^2} \int_{\Omega} |\nabla T_k(u_n)|^2 \leq \frac{C_k}{t^2}, \forall k > 1.
\]

Now (2.20) becomes
\[
m(\{|\nabla u_n| \geq t\}) \leq m(\{|\nabla u_n| \geq t, u_n < k\}) + m(\{u_n \geq k\}) \leq \frac{Ck}{t^2} + \frac{C}{k^{\frac{N}{N-2}}}, \forall k > 1.
\]

We then choose \(k = t^{\frac{N-2}{N-1}}\) and hence we get
\[
m(\{|\nabla u_n| \geq t\}) \leq \frac{C}{t^{\frac{N}{N-1}}}, \forall t \geq 1,
\]

We have proved that \((\nabla u_n)\) is bounded in \(M^{\frac{N}{N-1}}(\Omega)\). This implies by property (1.4) that \((u_n)\) is bounded in \(W^{1,q}_0(\Omega)\) for every \(q < \frac{N}{N-1}\).

\[\square\]

**Theorem 2.6.** There exists a weak solution \(u\) of (2.1) in \(W^{1,q}_0(\Omega)\) for every \(q < \frac{N}{N-1}\).

**Proof.** With the consideration of Lemma 2.5, it implies that there exists \(u\) such that the sequence \((u_n)\) converges weakly to \(u\) in \(W^{1,q}_0(\Omega)\) for every \(q < \frac{N}{N-1}\). This implies that for \(\varphi \in C^1_c(\Omega)\)
\[
\lim_{n \to \infty} \int_{\Omega} \nabla u_n \cdot \nabla \varphi = \int_{\Omega} \nabla u \cdot \nabla \varphi.
\]

In addition to this, by compact embeddings we can assume that \(u_n\) converges to \(u\) both strongly in \(L^1(\Omega)\) and up to a subsequence almost everywhere in \(\Omega\). Thus, taking \(\varphi \in C^1_c(\Omega)\), we have,
\[
0 \leq |h_n \left( u_n + \frac{1}{n} \right) f_n \varphi| \leq \begin{cases} 
C_1 \|\varphi\|_{L^\infty(\Omega)} f, & \text{if } u_n + \frac{1}{n} \leq K \\
M \|\varphi\|_{L^\infty(\Omega)} f, & \text{if } K \leq u_n + \frac{1}{n} \leq K_f \\
C_2 \|\varphi\|_{L^\infty(\Omega)} f, & \text{if } u_n + \frac{1}{n} \geq K_f
\end{cases}
\]

where, \(M > 0\) and \(K\) is the set \(\{x \in \Omega : \varphi(x) \neq 0\}\). This is sufficient to apply the dominated convergence theorem to obtain
\[
\lim_{n \to \infty} \int_{\Omega} h_n \left( u_n + \frac{1}{n} \right) f_n \varphi = \int_{\Omega} h(u) f \varphi.
\]

Hence, we can pass the limit \(n \to \infty\) in the last term of (2.8) involving \(\mu_n\). This concludes the proof of the result as it is easy to pass to the limit in (2.8). Therefore, we obtain a weak solution of (2.1) in \(W^{1,q}_0(\Omega)\) for every \(q < \frac{N}{N-1}\). \[\square\]
2.2 The case of $\gamma \geq 1$

As this is a strongly singular case, so we can hold some local estimates on $u_n$ in the Sobolev space. We shall give global estimates on $T_k^{\frac{\gamma + 1}{2}}(u_n)$ in $W_0^{1,2}(\Omega)$ with the aim of giving sense, at least in a weak sense, to the boundary values of $u$.

Lemma 2.7. Let $u_n$ be a solution of (2.7) with $\gamma \geq 1$. Then $T_k^{\frac{\gamma + 1}{2}}(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ for every fixed $k > 0$.

Proof. Consider $\varphi = T_k^\gamma(u_n)$ as a test function in (2.7). We have

$$\gamma \int_\Omega \nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma - 1}(u_n) = \int_\Omega h_n \left( u_n + \frac{1}{n} \right) f_n T_k^\gamma(u_n) + \int_\Omega T_k^\gamma(u_n) \mu_n. \quad (2.21)$$

Since $\gamma \geq 1$ and by the definition of $T_k u_n$, we can estimate the term on the left hand side of (2.21) as

$$\gamma \int_\Omega \nabla u_n \cdot \nabla T_k(u_n) T_k^{\gamma - 1}(u_n) \geq \gamma \int_\Omega |\nabla T_k^{\frac{\gamma + 1}{2}}(u_n)|^2.$$

Recalling that $\frac{T_k^\gamma(u_n)}{(u_n + \frac{1}{n})^\gamma} \leq \frac{u_n^\gamma}{(u_n + \frac{1}{n})^\gamma} \leq 1$, the term on the right hand side of (2.21) can be estimated as

$$\int_\Omega h_n \left( u_n + \frac{1}{n} \right) f_n T_k^\gamma(u_n) + \int_\Omega T_k^\gamma(u_n) \mu_n \leq C_1 \int_{(u_n + \frac{1}{n})^{\frac{1}{\gamma}} \leq K} \frac{f_n T_k^\gamma(u_n)}{(u_n + \frac{1}{n})^\gamma} + C_2 \int_{(u_n + \frac{1}{n})^{\frac{1}{\gamma}} \geq K} \frac{f_n T_k^\gamma(u_n)}{(u_n + \frac{1}{n})^\gamma}
+ \max_{[K, K]} \int_{(u_n + \frac{1}{n})^{\frac{1}{\gamma}} \leq K} f_n T_k^\gamma(u_n) + k^\gamma \int_\Omega \mu_n
\leq C_1 \int_{(u_n + \frac{1}{n})^{\frac{1}{\gamma}} \leq K} f + C_2 k^\gamma \int_{(u_n + \frac{1}{n})^{\frac{1}{\gamma}} \geq K} f
+ k^\gamma \max_{[K, K]} \int_{(u_n + \frac{1}{n})^{\frac{1}{\gamma}} \leq K} f + k^\gamma \int_\Omega \mu_n
\leq C(k, \gamma) k^\gamma.$$

On combining the previous inequalities we get

$$\int_\Omega |\nabla T_k^{\frac{\gamma + 1}{2}}(u_n)|^2 \leq C k^\gamma \quad (2.22)$$

then, $\left( T_k^{\frac{\gamma + 1}{2}}(u_n) \right)$ is bounded in $W_0^{1,2}(\Omega)$ for every fixed $k > 0$. \hfill $\square$

Now, so as to pass to the limit $n \to \infty$ in the weak formulation (2.8), we require to prove some local estimates on $u_n$. We first prove the following.
Lemma 2.8. Let \( u_n \) be a solution of (2.7) with \( \gamma \geq 1 \). Then \( (u_n) \) is bounded in \( W_{\text{loc}}^1(\Omega) \) for every \( q < \frac{N}{N-1} \).

Proof. We follow [18] to prove this theorem in two steps.

Step 1. We claim that \( (G_1(u_n)) \) is bounded in \( W_{0}^{1,q}(\Omega) \) for every \( q < \frac{N}{N-1} \).

It is apparent that \( G_1(u_n) = 0 \) when \( 0 \leq u_n \leq 1 \), \( G_1(u_n) = u_n - 1 \), otherwise, i.e., when \( u_n > 1 \). So \( \nabla G_1(u_n) = \nabla u_n \) for \( u_n > 1 \).

Now, we need to show that \( (\nabla G_1(u_n)) \) is bounded in \( M_{\infty}^N(\Omega) \), where \( M_{\infty}^N(\Omega) \) is the Marcinkiewicz space. Then we have

\[
\{|\nabla u_n| > t, u_n > 1\} = \{|\nabla u_n| > t, 1 < u_n \leq k + 1\} \cup \{|\nabla u_n| > t, u_n > k + 1\}
\]

\[
\subset \{|\nabla u_n| > t, 1 < u_n \leq k + 1\} \cup \{u_n > k + 1\} \subset \Omega.
\]

Hence, by the subadditivity of Lebesgue measure \( m \), we have

\[
m(\{|\nabla u_n| > t, u_n > 1\}) \leq m(\{|\nabla u_n| > t, 1 < u_n \leq k + 1\}) + m(\{u_n > k + 1\}). \tag{2.23}
\]

In order to estimate (2.23) we take \( \varphi = T_k(G_1(u_n)) \), for \( k > 1 \), as a test function in (2.7).

We observe that \( \nabla T_k(G_1(u_n)) = \nabla u_n \) only when \( 1 < u_n \leq k + 1 \), otherwise is zero, and \( T_k(G_1(u_n)) = 0 \) on \( \{u_n \leq 1\} \). Hence we have

\[
\int_{\Omega} |\nabla T_k(G_1(u_n))|^2 = \int_{\Omega} h_n \left( u_n + \frac{1}{n} \right) f_n T_k(G_1(u_n)) + \int_{\Omega} T_k(G_1(u_n)) \mu_n
\]

\[
\leq C_1 \int_{(u_n + \frac{1}{n} \leq K)} \left( u_n + \frac{1}{n} \right)^\gamma \mathfrak{f}(\frac{k}{K}) + \max h(s) \int_{(k \leq (u_n + \frac{1}{n} \leq K))} f_n T_k(G_1(u_n))
\]

\[
+ C_2 \int_{(u_n + \frac{1}{n} \geq K)} \mathfrak{f}(\frac{k}{K}) + k \int \mu_n
\]

\[
\leq C_1 k \int_{(u_n + \frac{1}{n} \leq K)} \left( 1 + \frac{1}{n} \right)^\gamma + \max h(s) \int_{(k \leq (u_n + \frac{1}{n} \leq K))} f_n
\]

\[
+ C_2 k \int_{(u_n + \frac{1}{n} \geq K)} f_n + k \int \mu_n
\]

\[
\leq C k,
\]

and by restricting the above integral on \( I_1 = \{1 < u_n \leq k + 1\} \) we get,

\[
\int_{\{1 < u_n \leq k + 1\}} |\nabla T_k(G_1(u_n))|^2 = \int_{\{1 < u_n \leq k + 1\}} |\nabla u_n|^2
\]

\[
\geq \int_{\{\nabla u_n > t, 1 < u_n \leq k + 1\}} |\nabla u_n|^2
\]

\[
\geq t^2 m(\{|\nabla u_n| > t, 1 < u_n \leq k + 1\})
\]
so that,
\[ m(\{|\nabla u_n| > t, 1 < u_n \leq k + 1\}) \leq \frac{Ck}{t^2} \quad \forall k \geq 1. \]

According to (2.22) in the proof of Lemma 2.7 one can see that
\[ \int_{\Omega} |\nabla T_k^{\gamma+1}(u_n)|^2 \leq Ck^\gamma \quad \forall k > 1, \]
Therefore, from the Sobolev inequality
\[ \frac{1}{\lambda_1^2} \left( \int_{\Omega} |T_k^{\gamma+1}(u_n)|^{2^*} \right)^\frac{2}{2^*} \leq \int_{\Omega} |\nabla T_k^{\gamma+1}(u_n)|^2 \leq Ck^\gamma, \]
where, \( \lambda_1 \) is the first eigen value of the laplacian operator. Now, if we restrict the integral on the left hand side on \( I_2 = \{x : u_n(x) > k + 1\} \), on which \( T_k(u_n) = k \), we then obtain
\[ k^{\gamma+1}m(\{u_n > k + 1\})^{\frac{2}{2^*}} \leq Ck^\gamma, \]
so that
\[ m(\{u_n > k + 1\}) \leq \frac{C}{k^{\frac{N}{N-1}}} \quad \forall k \geq 1. \]

So, \((u_n)\) is bounded in \( M^{\frac{N}{N-2}}(\Omega) \), i.e., \((G_1(u_n))\) is also bounded in \( M^{\frac{N}{N-2}}(\Omega) \). Now (2.23) becomes
\[ m(\{|\nabla u_n| > t, u_n > 1\}) \leq m(\{|\nabla u_n| > t, 1 < u_n \leq k + 1\}) + m(\{u_n > k + 1\}) \leq \frac{Ck}{t^2} + \frac{C}{k^{\frac{N}{N-2}}}, \forall k > 1. \]
We then choose \( k = t^{\frac{N}{N-2}} \) and we get
\[ m(\{|\nabla u_n| > t, u_n > 1\}) \leq \frac{C}{t^{\frac{N}{N-1}}} \quad \forall t \geq 1, \]

We just proved that \((\nabla u_n) = (\nabla G_1(u_n))\) is bounded in \( M^{\frac{N}{N-2}}(\Omega) \). Thus by the property in (1.4) that \((G_1(u_n))\) is bounded in \( W^{1,q}_0(\Omega) \) for every \( q < \frac{N}{N-1} \).

Step 2. We claim that \( T_1(u_n) \) is bounded in \( W^{1,q}_{lo}(\Omega) \) for every \( q < \frac{N}{N-1} \).

We have to examine the behaviour, for small values, of \( u_n \) for each \( n \). We want to show that for every \( K \subset \subset \Omega \)
\[ \int_K |\nabla T_1(u_n)|^2 \leq C. \quad (2.24) \]
We have already proved that \( u_n \geq C_K \) on \( K \subset \subset \Omega \) in Lemma 2.4. We will use \( \varphi = T_1^\gamma(u_n) \) as a test function in (2.8) to get
\[ \int_{\Omega} \nabla u_n \cdot \nabla T_1(u_n) T_1^{\gamma-1}(u_n) = \int_{\Omega} h_n \left( u_n + \frac{1}{n} \right) f_n T_1^\gamma(u_n) + \int_{\Omega} T_1^\gamma(u_n) \mu_n \leq C. \quad (2.25) \]
Now observe that
\[ \int_{\Omega} \nabla u_n \cdot \nabla T_1(u_n)T_1^{\gamma-1}(u_n) \geq \int_K |\nabla T_1(u_n)|^2 T_1^{\gamma-1}(u_n) \geq C_K^{\gamma-1} \int_K |\nabla T_1(u_n)|^2. \] (2.26)

Combining (2.25) and (2.26) we get (2.24). Since \( u_n = T_1(u_n) + G_1(u_n) \), hence, \( (u_n) \) is bounded in \( W^{1,q}_{\text{loc}}(\Omega) \) for every \( q < \frac{N}{N-1} \).

Now, we can finally state and prove the existence result.

**Theorem 2.9.** Let \( \gamma \geq 1 \). Then there exists a weak solution \( u \) of (2.1) in \( W^{1,q}_{\text{loc}}(\Omega) \) for every \( q < \frac{N}{N-1} \).

**Proof.** The proof of this theorem is a straightforward application of the results in Theorem 2.6, Lemma 2.7 and Lemma 2.8.

### 3. Further discussion of the case \( \gamma < 1 \).

In this section we will consider \( \Omega \) to be a bounded open subset of \( \mathbb{R}^N (N \geq 2) \), with boundary \( \partial \Omega \) of class \( C^{2,\beta} \) for some \( 0 < \beta < 1 \). We consider the following semilinear elliptic problem

\[
-\Delta u = h(u)f + \mu \text{ in } \Omega,
\]
\[
u = 0 \text{ on } \partial \Omega,
\] (3.1)

where \( 0 < \gamma < 1 \), \( f \in C^\beta(\bar{\Omega}) \) such that \( f > 0 \) in \( \bar{\Omega} \) and \( \mu \) is a nonnegative bounded Radon measure on \( \Omega \).

**Definition 3.1.** A very weak solution to problem (3.1) is a function \( u \in L^1(\Omega) \) such that \( u > 0 \) a.e. in \( \Omega \), \( fh(u) \in L^1(\Omega) \), and

\[
-\int_{\Omega} u \Delta \varphi = \int_{\Omega} h(u)f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\Omega).
\] (3.2)

We will show the existence of a non-negative very weak solution to the problem (3.1).

**Definition 3.2.** A function \( \bar{u} \) is a subsolution for (3.1) if \( \bar{u} \in L^1(\Omega) \), \( \bar{u} > 0 \) in \( \Omega \), \( fh(\bar{u}) \in L^1(\Omega) \) and

\[
-\int_{\Omega} \bar{u} \Delta \varphi \leq \int_{\Omega} h(\bar{u})f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\Omega), \ \varphi \geq 0.
\] (3.3)

Equivalently, \( \bar{u} \) is said to be a supersolution for the problem (3.1) if \( \bar{u} \in L^1(\Omega) \), \( \bar{u} > 0 \) in \( \Omega \), \( fh(\bar{u}) \in L^1(\Omega) \) and

\[
-\int_{\Omega} \bar{u} \Delta \varphi \geq \int_{\Omega} h(\bar{u})f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\Omega), \ \varphi \geq 0.
\] (3.4)
**Theorem 3.3.** Let \( u \) is a subsolution and \( \bar{u} \) is a supersolution to the problem (3.1) with \( u \leq \bar{u} \) in \( \Omega \), then there exists a solution \( u \) to (3.1) according to the Definition 3.1 such that \( u \leq u \leq \bar{u} \).

**Proof.** We prove the theorem by following an argument due to Ponce [17]. We define \( \bar{g} : \Omega \times \mathbb{R} \to \mathbb{R} \) as

\[
\bar{g}(x,t) = \begin{cases} 
  f(x)h(u(x)) & \text{if } t < u(x), \\
  f(x)h(t) & \text{if } u(x) \leq t \leq \bar{u}(x), \\
  f(x)h(\bar{u}(x)) & \text{if } t > \bar{u}(x).
\end{cases}
\]

Moreover, \( u > 0 \) and hence \( \bar{g} \) is well defined a.e. in \( \Omega \). For each fixed \( v \in L^1(\Omega) \) we have that \( \bar{g}(x,v(x)) \in L^1(\Omega) \). We divide the proof of the theorem into two steps.

**Step 1.** We claim that if \( u \) satisfies

\[
-\Delta u = \bar{g}(x,u) + \mu \text{ in } \Omega,
\]

\[
u = 0 \text{ on } \partial \Omega,
\]

then \( u \leq u \leq \bar{u} \). Thus \( \bar{g}(.,u) = fh(u) \in L^1(\Omega) \), and \( u \) is a solution to (3.1).

The very weak formulation of (3.5) is given by

\[
-\int_\Omega u \Delta \varphi = \int_\Omega \bar{g}(x,u) \varphi + \int_\Omega \varphi d\mu, \ \forall \varphi \in C^2_0(\Omega).
\]

We only show that \( u \leq \bar{u} \) in \( \Omega \). The proof of the other side of the inequality, \( u \leq u \), follows similarly.

We will show that \( u \) is a solution to (3.5), and \( \bar{u} \) is a supersolution to (3.1). Subtracting equation (3.6) from (3.4) we have, for every \( \varphi \in C^2_0(\Omega) \) such that \( \varphi \geq 0 \),

\[
-\int_\Omega (u - \bar{u}) \Delta \varphi \leq \int_\Omega (\bar{g}(x,u) - fh(\bar{u})) \varphi = \int_\Omega \chi_{\{u \leq \bar{u}\}} (\bar{g}(x,u) - fh(\bar{u})) \varphi.
\]

Now applying Kato type inequality (4.2) from the Appendix we get,

\[
\int_\Omega (u - \bar{u})^+ \leq \int_\Omega \chi_{\{u \leq \bar{u}\}} (\bar{g}(x,u) - fh(\bar{u})) (\text{sign}_+(u - \bar{u})) \varphi = 0,
\]

which further implies that

\[
\int_\Omega (u - \bar{u})^+ \leq 0.
\]

Thus \( u \leq \bar{u} \) a.e. in \( \Omega \), and the proof of the claim is complete.

**Step 2.** We now show that a solution to problem (3.5) does exist. Let us define

\[ G : L^1(\Omega) \to L^1(\Omega), \]
This map assigns to every $v \in L^1(\Omega)$ the solution $u$ to the following linear problem

$$
-\Delta u = \bar{g}(x, v) + \mu \text{ in } \Omega,
$$

$$
u = 0 \text{ on } \partial \Omega.
$$

(3.7)

The problem in (3.7) admits a unique solution for a given Radon measure due to [19]. We need to show that this map is continuous in $L^1(\Omega)$. Let us choose a sequence $(v_n)$ converging to some function $v$ in $L^1(\Omega)$, then by the definition of $\bar{g}$ and $h$ being a non-increasing, continuous function we get

$$
|\bar{g}(x, v_n(x))| \leq f h(u).
$$

Hence, using the dominated convergence theorem, we conclude that

$$
||\bar{g}(x, v_n) - \bar{g}(x, v)||_{L^1(\Omega)} \to 0.
$$

By [20] the linear problem (3.7) has a unique very weak solution corresponding to this $v$. Thus

$$
\lim_{n \to \infty} -\int_{\Omega} u_n \Delta \phi = \lim_{n \to \infty} \int_{\Omega} fh(v_n) \phi + \int_{\Omega} \phi d\mu = \int_{\Omega} fh(v) \phi + \int_{\Omega} \phi d\mu = -\int_{\Omega} u \Delta \phi.
$$

Hence $u = G(v)$. It can be seen from Théorème 9.1 in [19] that $||u_n - u||_{1} \leq ||u_n - u||_{W_0^{1,q}(\Omega)} \leq ||\bar{g}(x, v_n) + \mu - (\bar{g}(x, v) + \mu)||_{\mathcal{M}(\Omega)} = ||\bar{g}(x, v_n) - \bar{g}(x, v)||_{1} \to 0$ as $n \to \infty$. Hence $||u_n - u||_{L^1(\Omega)} = ||G(v_n) - G(v)||_{L^1(\Omega)} \to 0$ and therefore, we proved that $G$ is continuous.

We are still left to prove that the set $G(L^1(\Omega))$ is bounded and relatively compact in $L^1(\Omega)$. For every $v \in L^1(\Omega)$ we have

$$
||\bar{g}(x, v) + \mu||_{\mathcal{M}(\Omega)} \leq ||\bar{g}(x, v)||_{\mathcal{M}(\Omega)} + ||\mu||_{\mathcal{M}(\Omega)} \leq ||fh(u)||_{L^1(\Omega)} + ||\mu||_{\mathcal{M}(\Omega)}.
$$

Again, by Théorème 9.1 in [19], we see that $G(v)$ is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ and therefore, by Rellich-Kondrachov theorem we get $G(L^1(\Omega))$ is bounded and hence relatively compact in $L^1(\Omega)$.

Now we can apply Schauder fixed point theorem to see that $G$ has a fixed point $u \in L^1(\Omega)$. According to the result from step 1, we conclude that $u$ is a solution to (3.1) such that $\underline{u} \leq u \leq \bar{u}$. \qed
**Theorem 3.4.** There exists a solution to the problem (3.1) in the sense of Definition 3.1.

**Proof.** We want to find both a subsolution and a supersolution to problem (3.1) in the sense of Definition 3.2. Then we will use the result in Theorem 3.3 to prove the existence of a solution to the problem (3.1) in the sense of Definition 3.1. We first find a subsolution. Let us consider the problem

\[-\Delta v = h(v) f \text{ in } \Omega,\]
\[v = 0 \text{ on } \partial \Omega.\]  

(3.8)

The existence of a very weak solution in \(L^1(\Omega)\) to the problem in (3.8) can be proved as in the argument in Theorem 3.3 using the Schauder fixed point theorem. Consider the eigen function \(\phi_1 > 0\) of \(-\Delta\) corresponding to the smallest eigen value \(\lambda_1\) with \(\phi_1|_{\partial \Omega} = 0\) [22]. Observe that

\[-\Delta \phi_1 - h(\phi_1) f < 0\]

\[= -\Delta v - h(v) f\]

due to the facts (i) that \(\phi_1 > 0\), the non-increasing nature of \(h\) and (ii) \(v\) being a solution to (3.8). Hence we have \(v > 0\) in \(\Omega\). Since \(\mu\) is a nonnegative Radon measure we get the following inequality,

\[-\int_{\Omega} v \Delta \varphi \leq \int_{\Omega} h(v) f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\bar{\Omega}), \varphi \geq 0,\]

and hence \(v\) is a subsolution to the problem (3.1). Now we look for a supersolution of the problem in (3.1). Let \(w\) be the solution of

\[-\Delta w = \mu \text{ in } \Omega,\]
\[w = 0 \text{ on } \partial \Omega.\]  

(3.9)

Since \(\mu \geq 0\), by the maximum principle on Laplacian we have \(w \geq 0\). Let us denote \(z = w + v\), where \(v\) is a solution to (3.8). Then we get

\[-\int_{\Omega} z \Delta \varphi = -\int_{\Omega} w \Delta \varphi - \int_{\Omega} v \Delta \varphi = \int_{\Omega} h(v) f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\bar{\Omega}).\]

We know that \(w\) is nonnegative, then we have \(0 < h(z) \leq h(v)\). Thus, we have

\[\int_{\Omega} h(z) f \varphi + \int_{\Omega} \varphi d\mu \leq \int_{\Omega} h(v) f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\bar{\Omega}), \varphi \geq 0,\]

i.e., \(z\) is a positive function in \(L^1(\Omega)\) such that \(h(z) \leq h(v) \in L^1(\Omega)\) and

\[-\int_{\Omega} z \Delta \varphi \geq \int_{\Omega} h(z) f \varphi + \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^2_0(\bar{\Omega}), \varphi \geq 0.\]
Therefore, $z$ is a supersolution to (3.1). We can now apply Theorem 3.3 to get the conclusion that there exists a solution $u$ to problem (3.1) in the sense of Definition 3.1.

3.1 Relaxation on assumptions on $f$

We proved the Theorem 3.4 by assuming a strong regularity on $f$ i.e. $f$ belongs to $C^\beta(\bar{\Omega})$ for some $0 < \beta < 1$. In this section we do some relaxation on our assumption on $f$ in order to prove the existence of solution.

For a fix $\delta > 0$, let us define $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$, and let $f$ be an almost everywhere positive function in $L^1(\Omega) \cap L^\infty(\Omega_\delta)$.

**Theorem 3.5.** Let $f \in L^1(\Omega) \cap L^\infty(\Omega_\delta)$ such that $f > 0$ a.e. in $\Omega$ for some fixed $\delta > 0$. Then there exists a solution to the problem (3.1) in the sense of Definition 3.1.

**Proof.** We consider the following sequence of problems

$$
-\Delta v_n = h \left( v_n + \frac{1}{n} \right) f_n \text{ in } \Omega,
$$

$$
v_n = 0 \text{ on } \partial\Omega,
$$

where, $f_n = T_n(f)$. In Lemma 2.4 we proved that the nondecreasing sequence $(v_n)$ converges to a solution of problem (3.8) and for each fixed $n$, the function $v_n$ belongs to $L^\infty(\Omega)$. So we observe that the function $h(v_n + 1)f_n$ also belongs to $L^\infty(\Omega)$. Now we can apply Lemma 3.2 in [16] so as to obtain

$$
\frac{v_1(x)}{d(x)} \geq C \int_\Omega d(y)f_1(y)h \left( \| v_1 \|_{L^\infty(\Omega)} + 1 \right) dy \geq C > 0
$$

for every $x$ in $\Omega$. where $d(x) = d(x, \partial\Omega)$ is the distance function of $x$ from $\partial\Omega$. Thus, we have

$$v(x) \geq v_1(x) \geq Cd(x), \text{ a.e. on } \Omega.$$

Therefore, as $f \in L^\infty(\Omega_\delta)$, we have $h(v)f \in L^1(\Omega)$ due to the facts (i) $h(v)f \leq h(Cd(x))f$ and (ii) $h(Cd(x)f$ is integrable for every $\gamma < 1$. Hence the subsolution is bounded from below and this allows us to proceed as in the proof of Theorem 3.4.

Thus we conclude that there exists a solution to the problem in (3.1). \qed
4. Appendix

We prove the Kato type inequality for the problem

\[-\Delta u = h(u)f + \mu \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial \Omega,\]
\[u > 0 \text{ in } \Omega,\]  

(4.1)

where \(f > 0\) and \(u \in L^1(\Omega)\) is a very weak solution with \(u > 0\) a.e. in \(\Omega\) and \(fh(u) \in L^1(\Omega)\).

Let \(u_1\) and \(u_2\) are two very weak solutions to the problem (4.1) with measure sources \(\mu_1\) and \(\mu_2\), respectively. Hence, \(u_1, u_2 \in L^1(\Omega)\) and \(h(u_1)f, h(u_2)f \in L^1(\Omega)\). Then for every \(\phi \in C_0^2(\bar{\Omega})\), the very weak formulations corresponding to the problem (4.1) are

\[-\int_{\Omega} u_1 \Delta \phi = \int_{\Omega} h(u_1)f \phi + \int_{\Omega} \phi d\mu_1,\]

and

\[-\int_{\Omega} u_2 \Delta \phi = \int_{\Omega} h(u_2)f \phi + \int_{\Omega} \phi d\mu_2.\]

Taking the difference between two formulations we get

\[-\int_{\Omega} (u_1 - u_2) \Delta \phi = \int_{\Omega} f(h(u_1) - h(u_2))\phi + \int_{\Omega} \phi (d\mu_1 - d\mu_2),\]

and this a very weak formulation of the problem

\[-\Delta(u_1 - u_2) = f(h(u_1) - h(u_2)) + (\mu_1 - \mu_2) \text{ in } \Omega,\]
\[u_1 - u_2 = 0 \text{ on } \partial \Omega,\]

Now refering to the Proposition 1.5.4 (Kato’s inequality) of [15], we can observe that

\[-\int_{\Omega} (u_1 - u_2)^+ \Delta \phi \leq \int_{\Omega} f(h(u_1) - h(u_2))(\text{sign}_+(u_1 - u_2))\phi + \int_{\Omega} \phi (d\mu_1 - d\mu_2),\]

where, \(\text{sign}_+(u_1 - u_2) = \chi_{\{x \in \Omega: u_1(x) \geq u_2(x)\}}\). Let us consider a standard choice \(\phi_0\) such that \(-\Delta \phi_0 = 1\) in \(\Omega\) and \(\phi_0 = 0\) on \(\partial \Omega\). Now the above inequality becomes

\[\int_{\Omega} (u_1 - u_2)^+ \leq \int_{\Omega} f(h(u_1) - h(u_2))(\text{sign}_+(u_1 - u_2))\phi_0 + \int_{\Omega} \phi_0 (d\mu_1 - d\mu_2) \quad (4.2)\]

Equation (4.2) is our required Kato type inequality .

Now as \(f > 0\) and \(\phi_0 \geq 0\) we get

\[\int_{\Omega} (u_1 - u_2)^+ + \int_{\Omega} f(h(u_2) - h(u_1))\phi_0 \leq \int_{\Omega} \phi_0 (d\mu_1 - d\mu_2).\]

So if \(\mu_1\) becomes equal to \(\mu_2\), then

\[\int_{\Omega} f(h(u_2) - h(u_1))\phi_0 \leq 0.\]

As \(h(u_2) - h(u_1) \geq 0\) for \(u_1 \geq u_2\). So we reach at the conclusion that \(h(u_1) = h(u_2)\).
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