On Attainable Set and Controllability for Abstract Control Problem with Unbounded Input Operator

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Abstract

For linear evolution control system described by \( \dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \) (A generates a strongly continuous semigroup \( \{S(t)\}_{t\geq 0} \) on a Banach space \( X \); \( B \) is a linear unbounded operator), the attainable set \( K(t) \) set is studied. Conditions of the independence of \( t \) for its closure \( \overline{K(t)} \) are established. Controllability conditions for some classes of evolution systems are obtained.

Keywords. Attainable sets, controllability, abstract evolution equations, linear hereditary systems.

AMS MOS subject classification. 35R30, 35J20, 65M30.

1 Introduction.

We consider a system described by linear abstract differential equation of evolution type

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= x_0
\end{align*}
\)

where \( X, U \) be Banach spaces, \( x(t) \in X \) is the current state, \( x_0 \in X \) is the initial state, \( u(t) \in U, u(\cdot) \in L_2([0,t_1],U) \) is the control, \( A \) is a linear
operator generating a strongly continuous semigroup \( \{S(t)\}_{t \geq 0} \) of operators in the class \( C_0 \); \( B : U \to X \) is a linear possibly unbounded operator.

Let \( x(t, x_0, u(\cdot)) \) be the weak solution of equation (1-1)-(1-2), corresponding to the control \( u(\cdot) \).

**Definition 1** A state \( x \in X \) is said to be attainable in the time \( t \) from the origin, if there exists an admissible control \( u(\tau), 0 \leq \tau \leq t \), such that \( x(t, 0, u(\cdot)) = x \).

**Definition 2** The set \( K(t) \) of all states \( x \in X \) attainable in the time \( t \) from the origin is said to be attainable set for equation (1).

Thus the attainable set \( K(t) \) for equation (1) is defined as

\[
K(t) = \{ x \in X : \exists u(\cdot) \in L_2([0,t], U), x = x(t, x_0, u(\cdot)) \}.
\]

(1-3)

It is our main purpose here to study the properties of the set \( K(t) \), and especially, to establish conditions for its independence of \( t \) for sufficiently large \( t \).

These conditions has been established in author’s paper [8, ] for the equation (1-1) with bounded operator \( B \), so this work can be considered as the continuation of [8, ] . The reason to expand the results of [8, ] to equation (1-1) with unbounded operator \( B \) is the existence of the large classes of infinite dimensional control systems adequately described by equation (1-1) with unbounded operator \( B \). Some of these classes are:

- partial differential equations with boundary control;
- functional differential equations with delays both in state and in control variables.

The importance of equation (1-1) with unbounded operator \( B \) (both from theoretical and from practical point of view) has been recognized by many authors. We will use the functional analytic approach developed by Salamon [6].
2 Preliminaries.

Denote by $\sigma$ the spectrum of the operator $A$. Let $\mu \notin \sigma$. We will consider the spaces $W$ and $V$ defined as follows [1], [3], [9]:

- $W$ is the domain $D(A)$ of the operator $A$ with the norm $\|x\|_\mu = \|(\mu I - A)x\|$;
- $V$ is the closure of $X$ with respect to the norm $\|x\|_{-\mu} = \|((\mu I - A)^{-1}x\|_\mu$.

Obviously $W \subset X \subset V$.

It is known that for each $\mu_1, \mu_2$, $\mu_1 \neq \mu_2$ the norm $\|\cdot\|_{\mu_1}$ is equivalent to the norm $\|\cdot\|_{\mu_2}$, the norm $\|\cdot\|_{-\mu_1}$ is equivalent to the norm $\|\cdot\|_{-\mu_2}$, and all $\|\cdot\|_{\mu}$ are equivalent to the appropriate graph norm on $D(A)$, so the Banach spaces $W$ and $V$ do not depend on $\mu$ [9].

We assume

1. the unbounded operator $B$ is bounded as operator from $U$ to $V$;
2. the operator $\Phi(t) : L_2([0, t], U) \to X$ defined by the formula $\Phi(t)u(\cdot) = \int_0^t S(t - \tau)Bu(\tau)\,d\tau$ is bounded for each $t \geq 0$.

If $x \in X$ and $f \in X^*$, we will write $(x, f)$ instead of $f(x)$. The upper superscript $^T$ denotes transposition.

As usual $\mathbb{R}$ is the set of real and $\mathbb{C}$ the set of complex numbers.

For any set $K \subset X$ we denote by $\overline{K}$ the closure of $K$ with respect to the uniform topology of $X$ and by $K^\perp$ the set $\{y \in X^* : (x, y) = 0 \ \forall x \in K\}$.

We assume the operator $A$ to have the following properties:

(I) The domain $D(A^*)$ is dense in $X^*$.

(II) The operator $A$ has a purely point spectrum $\sigma$ which is either finite or has no finite limit points and each $\lambda \in \sigma$ is of a finite multiplicity.

It is known [3], [9], etc.

1. for each $t \geq 0$ the operator $S(t)$ has a continuous extension $\mathcal{S}(t)$ on the space $V$ and the family of operators $\mathcal{S}(t) : V \to V$ is the semigroup in the class $C_0$ with respect to the norm of $V$ and the corresponding infinitesimal generator $\mathcal{A}$ of the semigroup $\mathcal{S}(t)$ is the closed dense extension of the operator $A$ on the space $V$ with domain $D(A) = X$;

\footnote{It is easy to show, that $\Phi(t)u(\cdot) = x(t, 0, u(\cdot))$.}
2. the set of the generalized eigenvectors of operators $A$, $A^*$ and $A$, $A^*$ are the same.

(III) There exists a time moment $T \geq 0$ such that for all $v \in V$ and $t > T$ the function $x(t) = S(t)v$ is expanded in a series of generalized eigenvectors of the operator $A$, converging with respect to the norm of $V$ for a certain grouping of terms uniformly with respect to $t$ on an arbitrary interval $[T_1, T_2]$ ($T_1 > T$).

3 Main results.

Our main task consists of establishing conditions for independence of $K(t)$ at least for sufficiently large $t$.

**Definition 3** A sequence $\{x_i\}_{i \in \mathbb{N}}$ of functions from $L_{2}^{loc}[0, +\infty)$ is called minimal on $[0, \nu]$ ($\nu > 0$) if there is a sequence $\{y_j\}_{j \in \mathbb{N}}$ of functions from $L_2[0, \nu]$ such that

$$\int_0^\nu (x_i(t)y_j(t)) \, dt = \delta_{ij} \quad (i, j \in \mathbb{N})$$

where $\delta_{ij}$ is the Kronecker symbol. The sequence $\{y_j\}_{j \in \mathbb{N}}$ is called a sequence biorthogonal to the sequence $\{x_j\}_{j \in \mathbb{N}}$ on $[0, \nu]$.

Let the numbers $\lambda_j \in \sigma$ ($j \in \mathbb{N}$) be enumerated in the order of non-decreasing absolute values, let $\alpha_j$ be the multiplicity of $\lambda_j \in \sigma$, and let

$$\varphi_{jkl} \quad \text{and} \quad \psi_{jkl}, \quad j \in \mathbb{N}; \quad k = 1, \ldots, m_j; \quad l = 1, 2, \ldots, \beta_{jk}; \quad \sum_{k=1}^{m_j} \beta_{jk} = \alpha_j$$

be the generalized eigenvectors of the operators $A$ and $A^*$, respectively, such that

$$(\varphi_{jp}\delta_{p-t+1}, \psi_{ksq}) = \delta_{jk}\delta_{ps}\delta_{lq} \quad (3-4)$$

$j, k \in \mathbb{N}; \quad p = 1, \ldots, m_j; \quad l = 1, \ldots, \beta_{jp}; \quad s = 1, \ldots, m_k; \quad q = 1, \ldots, \beta_{ks}$.
Theorem 4 If \( t_1 \leq t_2 \), then \( K(t_1) \subseteq K(t_2) \). If the properties (I)–(III) hold and the sequence the sequence \( \{f_{jk}\}_{jk} \) of functions

\[
f_{jk}(t) = (-t)^k \exp(-\lambda_j t), \quad j \in \mathbb{N}; \quad k = 1, \ldots, \alpha_j; \quad t \in [0, +\infty)
\]  

(3-5)
is minimal on \([0, \nu]\), then

\[
K(t_1) = K(t_2) \quad \text{if} \quad t_1, t_2 > T + \nu.
\]

Proof. A weak solution \( x(t) \) of equation (1-1) with the initial condition

(1-2) is defined by the following representation formula\([3], [6], [9]\)

\[
x(t, x_0, u(\cdot)) = S(t) x_0 + \int_0^t S(t - \tau) B u(\tau) d\tau.
\]  

(3-6)
Hence the attainable set \( K(t) \) is defined by the formula

\[
K_t = \left\{ x \in X : \exists u(\cdot) \in L_2([0, t], U), \quad x = \int_0^t S(t - \tau) B u(\tau) d\tau \right\},
\]  

(3-7)
so one can prove the inclusion \( K(t_1) \subseteq K(t_2) \) as well as in \([8]\).

Let \( P_j \) be a projector on the generalized eigenspace of \( A \) at \( \lambda_j \in \sigma \) \( j \in \mathbb{N} \), and let

\[
\Lambda_j = \begin{pmatrix}
\lambda_j & 1 & \ldots & 0 \\
0 & \lambda_j & \ldots & 0 \\
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & \lambda_j
\end{pmatrix}
\]

be the Jordan \((\beta_j \times \beta_j)\)-matrix. We have

\[
(S(t)P_j x, g) = (\Phi_j, g) \exp(\Lambda_j t) (x, \Psi_j)^T, \forall x \in X.,
\]  

(3-8)
where

\[
\Phi_j = \{ \varphi_{jk1}, \varphi_{jk2}, \ldots, \varphi_{jk\beta_j} \} \quad k = 1, \ldots, m_j,
\]

\[
\Psi_j = \{ \psi_{jk1}, \psi_{jk2}, \ldots, \psi_{jk\beta_j} \} \quad k = 1, \ldots, m_j,
\]

\[
(\Phi_j, g) = \{ (\varphi_{jk1}, g), \ldots, (\varphi_{jk\beta_j}, g) \} \quad k = 1, \ldots, m_j,
\]

\[
(x, \Psi_j) = \{ (x, \psi_{jk1}), \ldots, (x, \psi_{jk\beta_j}) \} \quad k = 1, \ldots, m.
\]

\(^2\)Some details of the proof from [Shklyar] are applicable for the case of unbounded operator \( B \) and can be omitted, however we repeat them here for the sake of reader’s convenience.
Now we will prove inclusion $K(t_2) \subseteq K(t_1)$ for all $T \leq t_1 < t_2$. Let $g \in K(t_1)^\perp$. By (3.7)
\[
\left( \int_0^{t_2} S(t_1 - \tau)Bu(\tau) d\tau, g \right) \equiv 0, \forall u \in L_2([0,t_1], U). \tag{3-9}
\]

If the linear operator $B$ is bounded, then
\[
\left( \int_0^{t_1} S(t_1 - \tau)Bu(\tau) d\tau, g \right) = \int_0^{t_1} (S(t_1 - \tau)Bu(\tau), g) d\tau. \tag{3-10}
\]

If the linear operator is unbounded, then we cannot use (3-10) to continue the proof as in [Shklyar], because there exists $u(\cdot) \in L_2([0,t_1], U)$ and $\tau \in [0, t_1]$, such that $S(t - \tau)Bu(\tau) \notin X$, and besides $S(t - \tau)Bu(\tau) \in V$, but we cannot assure $g \in V^\star$.

Let $u_1(t) = \begin{cases} 0, & t_1 - T < t \leq t_1; \\ u(t), & 0 \leq t \leq t_1 - T; \end{cases}$
where $u(\cdot) \in L_2([0, t_1 - T], U)$. If follows from (3-10) that
\[
\left( \int_0^{t_1-T} S(t_1 - \tau)Bu(\tau) d\tau, g \right) \equiv 0, \forall u(\cdot) \in L_2([0, t_1 - T], U). \tag{3-11}
\]

Now let
$u_2(t) = \begin{cases} u(t), & t_1 - T < t \leq t_1; \\ 0, & 0 \leq t \leq t_1 - T; \end{cases}$
where $u(\cdot) \in L_2([t_1 - T, t_1], U)$. Again, it follows from (3-10) that
\[
\left( \int_{t_1-T}^{t_1} S(t_1 - \tau)Bu(\tau) d\tau, g \right) \equiv 0, \forall u(\cdot) \in L_2([t_1 - T, t_1], U). \tag{3-12}
\]

The sets of the generalized eigenvectors of operators $A, A^\ast$ and $A, A^\ast$ are the same, so in accordance with property (III) we have
\[
S(t)v = \sum_{j=1}^{\infty} \Phi_j \exp(\Lambda_j t) (v, \Psi_j)^\top, \forall t > T, \forall v \in V, \tag{3-13}
\]
\[6\]
where $\sum_{j=1}^{\infty}$ is considered with respect to the norm of $V$.

Consider the sequence $S_n(t)v$ of partial sums of series (3-13)

$$S_n(t)v = \sum_{j=1}^{n} \Phi_j \exp(\Lambda_j t) (v, \Psi_j^T).$$  \hspace{1cm} (3-14)

Obviously $S_n(t)v \in X, \forall v \in V$.

One can show that

$$\langle \Phi_j, g \rangle \exp(\Lambda_j t) (v, \Psi_j^T) = \sum_{k=0}^{\beta_j} \exp(\lambda_j t) \langle \Phi_j, g \rangle t^k E_k \langle v, \Psi_j^T \rangle,$$ \hspace{1cm} (3-15)

where $\beta_j \times \beta_j$-matrix $E_j$ is defined by

$$E_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Denote by $g_n(v,t)$ the linear functional (with respect to $v$

$$g_n(v,t) = (S_n(t)v, g) = \sum_{j=1}^{n} \langle \Phi_j, g \rangle \exp(\Lambda_j t) (v, \Psi_j^T).$$  \hspace{1cm} (3-16)

It follows from above considerations that for any $j = 1, 2, ...$ the linear operator acting from $V$ to $\mathbb{R}^{\beta_j}$ defined by $\langle v, \Psi_j^T \rangle$ is bounded. Hence for any natural $n$ the functional $g_n(v,t)$ is a linear bounded functional, and it follows from (3-11) and (3-13), that for $v = Bu(t_1 - \tau)$, where $u(\tau) = 0$ for $\tau \in (t_1 - T, t_1]$

$$\lim_{n \to \infty} \int_{0}^{t_1-T} g_n(Bu(t_1 - \tau), t_1 - \tau) d\tau = 0.$$ \hspace{1cm} (3-17)

Let $w \in U$ and $\gamma_{kl}(t), k = 1, 2, ..., l = 1, 2, ..., \beta_k$ be the sequence of functions biorthogonal to the sequence (3-6) on $[0, t_1 - T]$. Substituting (3-16) to (3-17) and using $u(t) = w_{\gamma_{kl}}(t)$ we obtain after computations

$$\langle \Phi_j, g \rangle E_j^k B^* \Psi_j^T = 0, j = 1, 2, ..., k = 1, ..., \beta_j,$$ \hspace{1cm} (3-18)
On account of (3-18), (3-16) and (3-15) we obtain
\[
\lim_{n \to \infty} \int_0^{t_2 - T} g_n (B u (t_2 - \tau), t_2 - \tau) d\tau = 0,
\]
so
\[
\left( \int_0^{t_2 - T} S(t_2 - \tau) B u (\tau) d\tau, g \right) \equiv 0, \quad \forall u (\cdot) \in L_2 ([0, t_2 - T], U). \tag{3-19}
\]

Joining (3-19) and (3-12), we obtain
\[
\left( \int_0^{t_2} S(t_2 - \tau) B u (\tau) d\tau, g \right) \equiv 0, \quad \forall u \in L_2 ([0, t_1], U). \tag{3-20}
\]

The latter identity imply the inclusion \( g \in K(t_2) \). Thus, \( K(t_1) \subseteq K(t_2) \). Hence \( \overline{K(t_2)} \subseteq K(t_1) \). Since \( K(t_1) \subseteq K(t_2) \) for all \( t_1 \) with \( t_1 < t_2 \), we obtain \( K(t_1) = K(t_2) \) for all \( t_1 \) and \( t_2 \) with \( T + \nu < t_1 < t_2 \). This proves the theorem.

### 3.1 Controllability conditions.

Theorem 4 can be applied for various control problems.

In this section we will show how the proof of Theorem 4 provides a possibility to obtain an approximate null-controllability criterion for the abstract control problem with unbounded input operator. We will consider this kind of controllability only, but other kinds of controllability can be investigated also.

#### 3.1.1 Approximate null-controllability conditions for equation (12).

Denote
\[
\text{Range} \\{ \lambda I - A, R_\mu B \} = \{ z \in X : \exists x \in X, \exists u \in U, \ z = (\lambda I - A)x + R_\mu Bu \}
\]
Theorem 5 Let \( \mu \notin \sigma \). If the properties (I)-(III) hold and the sequence the sequence (3-5) is minimal on \([0,\nu]\), then for equation (1-1) to be approximately null-controllable on \([0,\ t_1]\) it is necessary and, for \( t_1 > T + \nu \), sufficient that

\[
\text{Range}\{\lambda I - A, R_\mu B\} = X, \ \forall \lambda \in \sigma
\]

(3-21)

Proof. Sufficiency. We obtained above that \( g \in K(t_1) \) provided \( t_1 > T \) implies the identity (3-18). One can easy see that the condition (3-21) is equivalent to the condition

\[
B^* R_\mu^* \psi_{j\beta_j} \neq 0, \ j \in \mathbb{N}
\]

(3-22)

Here \( \psi_{j\beta_j} \) is the eigenvector of the adjoint operator \( A^* \) corresponding to the eigenvalue \( \lambda_j \in \sigma \). Since the eigenvectors of the operators \( A^* \) and \( A^* \) are the same, we have \( \psi_{j\beta_j} \in V^* \), so \( B^* \psi_{j\beta_j} \) is well-defined and it follows from (3-22) that

\[
B^* R_\mu^* \psi_{j\beta_j} = (\mu - \lambda) B^* \psi_{j\beta_j} \neq 0, \ j \in \mathbb{N}, \ \mu \notin \sigma, \ \lambda \in \sigma.
\]

(3-23)

Hence (3-23) yields

\[
B^* \psi_{j\beta_j} \neq 0, \ j \in \mathbb{N}.
\]

(3-24)

Solving the linear algebraic system (3-18) provided that (3-24) holds, we obtain

\[
(\Phi_j, g) = 0 \quad (j \in \mathbb{N}).
\]

(3-25)

This and property (III) imply \( S^*(t_1)g = 0 \), therefore, \( g \in \text{Range} S(t_1) \). We have \( K(t_1) \subseteq \text{Range} S(t_1) \), hence \( \text{Range} S(t_1) \subseteq K(t_1) \). The latter relation is equivalent to the approximate null-controllability of equation (1) on \([0, t_1]\). This proves the sufficiency of (3-21).

Necessity. If condition (3-21) does not hold, then there exists \( \lambda \in \sigma \) and \( g \in X^*, \ g \neq 0 \) such that

\[
(\lambda x - Ax, g) = 0, \ \forall x \in D(A),
\]

(3-26)

\[
(R_\mu Bu, g) = 0, \ \forall u \in U.
\]

(3-27)

It follows from (3-26) that the vector \( g \) is the eigenvector of the operator \( A^* \) corresponding to eigenvalue \( \lambda \). Since the eigenvectors of the operators \( A^* \) and
are the same we have $g \in V^*$, the scalar product $(Bu, g)$ is well-defined. and one can write (3-27) in the form

$$(R_{\mu}Bu, g) = (Bu, R_{\mu}^*g) = (Bu, (\mu - \lambda) g) = (\mu - \lambda) (Bu, g). \quad (3-28)$$

It follows from (3-26)–(3-28) that $(S(t)Bu, g) \equiv 0$ for all $t \in [0, +\infty)$ and $u \in U$, but $S^*(t)g \neq 0$ for all $t \in [0, +\infty)$. Hence, $g \in K(t_1)\perp$, but $g \notin \text{Range } S(t_1)\perp$. This proves the necessity of (3-21).

If the operator $A$ is not self adjoint, then it is not trivial problem to calculate the adjoint operator $A^*$. If the operator $A^*$ is calculated then one can use instead condition (3-21) one of the conditions:

1. for any $\lambda \in \sigma$ and any $\mu \notin \sigma$ the system of equations with respect to $g \in X^*$

$$\lambda g - A^*g = 0, \quad B^* R_{\mu}^*g = 0 \quad (3-29)$$

has only trivial solution;

2. for any $\lambda \in \sigma$ the system of equations with respect to $g \in V^*$

$$\lambda g - A^*g = 0, \quad B^* g = 0 \quad (3-30)$$

has only trivial solution;

By (3-26)-(3-30) one can easy obtain rank conditions for approximate null-controllability of equation (1-1).

Remark 1 Since the generalized eigenvectors of the operators $A^*$ and $A^*$ are the same we have $B^*\Psi_j$ to be well-defined, $j = 1, 2, \ldots$.

**Theorem 6** Let $I$ be $\beta_j \times \beta_j$ unit matrix. If the properties (I)–(III) hold and the sequence the sequence (3-5) is minimal on $[0, \nu]$, then for equation (1-1) to be approximately null-controllable on $[0, t_1]$ it is necessary and, for $t_1 > T + \nu$, sufficient that

$$\text{rank } \{\lambda_j I_j - \Lambda_j, B^*\Psi_j\} = \beta_j, \ j = 1, 2, \ldots. \quad (3-32)$$
Proof. It follows from (3-30) that there exists a vector $\eta \in \mathbb{R}^{\beta_j}$ such that $g = \Psi_j \eta$. Since $A^*\Psi_j = \Psi_j \Lambda_j^*$, $j = 1, 2, \ldots$, we obtain from (3-30), that for any $j = 1, 2, \ldots$

$$\Psi_j \left( \lambda_j I_j - \Lambda_j^* \right) \eta = 0,$$

(3-33)

$$B^* \Psi_j \eta = 0.$$  

(3-34)

The condition

$$\text{rank} \Psi_j = \beta_j, \; j = 1, 2, \ldots$$

yields the equivalence between (3-33)-(3-34) and

$$\left( \lambda_j I_j - \Lambda_j^* \right) \eta = 0,$$

(3-35)

$$B^* \Psi_j \eta = 0.$$  

(3-36)

Thus (3-30) is equivalent to (3-33)-(3-34), where $g = \Psi_j \eta$, so (3-33)-(3-36) holds if and only if $\eta = 0$. This shows the validity of (3-32).

3.1.2 Approximate null-controllability conditions for Abstract Boundary Control Problem.

Let $X, U, Y$ be Banach spaces. Consider the abstract boundary control problem

$$\dot{x}(t) = Lx(t),$$

(3-37)

$$Gx(t) = Bu(t),$$

(3-38)

$$x(0) = x_0,$$

(3-39)

where $L : X \rightarrow X$ is a linear unbounded operator with dense domain $Z = D(L)$, $B : U \rightarrow Y$ is a linear bounded injective operator, $G : Z \rightarrow Y$ is a linear bounded operator satisfying the following conditions:

- $G$ is onto, Ker$G$ is dense in $X$;

- there exists a $\mu \in \mathbb{R}$ such that $\mu I - L$ is onto and Ker$(\mu I - L) \cap$ Ker$G = \emptyset$. 

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The problem (3-37)-(3-39) is assumed to be well-posed. This problem is the abstract model for classical control problem described by linear partial differential equations of parabolic type when the control acts through the boundary and if the measurement can only be realized at a few points of the corresponding spatial domain. The conditions (3-38) can be considered as abstract boundary conditions.

Now we transform the abstract boundary control problem (3-37)-(3-39) to the problem (1-1)-(1-2). Consider the space \( W = \text{Ker} G \). We have \( W \subset Z \subset X \) with continuous dense injection. The operator \( A : W \rightarrow X \) is defined by

\[
Ax = Lx \text{ for } x \in W. \quad (3-40)
\]

For any \( y \in Y \) define

\[
\hat{B}y = Lx - Ax, \quad x \in G^{-1} (y) = \{ z \in Z : Gx = y \}. \quad (3-41)
\]

The operator \( \hat{B} : Y \rightarrow V \) defined by (3-41) is a bounded operator, but it is unbounded as an operator \( \hat{B} : Y \rightarrow X \). Given \( u \in U \) denote \( \hat{B}u = \hat{B}Bu \). The operator \( \hat{B} : U \rightarrow V \) is bounded, but the corresponding operator \( \hat{B} : U \rightarrow X \) is unbounded. It follows from (3-41) that

\[
Lx = Ax + \hat{B}u, \quad (3-42)
\]

\[
Gx = Bu. \quad (3-43)
\]

Since the abstract boundary control problem under consideration is uniformly well-posed the operator \( A \) generates the strongly continuous semigroup of bounded operators in the class \( C_0 \). Hence given abstract boundary control problem is equivalent to the control problem

\[
\dot{x}(t) = Ax(t) + \hat{B}u(t), \quad (3-44)
\]

\[
x(0) = x_0. \quad (3-45)
\]

So it follows from above considerations and Theorem that

**Theorem 7** Let \( \mu \notin \sigma \). If the properties (I)-(III) hold and the sequence \( (3-5) \) is minimal on \([0, \nu]\), then for equation (3-37)-(3-38) to be approximately null-controllable on \([0, t_1]\) it is necessary and, for \( t_1 > T + \nu \), sufficient that

\[
\text{Range}\{\lambda I - A, R_\mu \hat{B}B\} = X, \quad \forall \lambda \in \sigma. \quad (3-46)
\]
Together with equation (3-37)-(3-39) consider the abstract elliptic equation

\[ Lx = \mu x \quad (3-47) \]
\[ Gx = y \quad (3-48) \]

Since the problem (3-37)-(3-39) is uniformly well-posed then for any \( y \in Y \) there exists the solution \( x_\mu = D_\mu y \) of the equation (3-47)-(3-48), where \( D_\mu : Y \to X \) is a linear bounded operator. It follows from (3-47)-(3-48) and (3-42) that

\[ Ax + \hat{B}y = \mu x. \]

Hence

\[ x = R_\mu \hat{B}y \]

so

\[ D_\mu = R_\mu \hat{B}. \quad (3-49) \]

Using (3-49) in (3-46) we obtain that the following theorem is valid:

**Theorem 8** Let \( \mu \notin \sigma \). If the properties (I)-(III) hold and the sequence (3-5) is minimal on \([0, \nu]\), then for equation (3-37)-(3-38) to be approximately null-controllable on \([0, t_1]\) it is necessary and, for \( t_1 > T + \nu \), sufficient that

\[ \text{Range} \{ \lambda I - A, D_\mu B \} = X, \forall \lambda \in \sigma. \quad (3-50) \]

There are a lot of methods to calculate the operator \( D_\mu \) for a given concrete boundary control problem [Butkovskii].

**4 Examples.**

In this section we will apply the results obtained in previous sections to a general class of linear neutral functional differential equations, which don’t fit into the framework of [8].
4.1 General neutral functional differential equations.

Consider the linear neutral functional differential equation

\[ \frac{d}{dt} \begin{pmatrix} x(t) - \int_{-h}^{0} dA_0(\tau)x(t + \tau) - \int_{-h}^{0} dB_0(\tau)u(t + \tau) \end{pmatrix} = \int_{-h}^{0} dA(\tau)x(t + \tau) + \int_{-h}^{0} dB(\tau)u(t + \tau), \]

with initial conditions

\[ x(0) - \int_{-h}^{0} dA_0(\tau)\varphi_1(\tau) - \int_{-h}^{0} dB_0(\tau)\varphi_2(\tau) = \varphi^0, x(\tau) = \varphi_1(\tau), u(\tau) = \varphi_2(\tau), \]

where \( x(t) \in \mathbb{R}^n, \varphi_1(\cdot) \in L_2([-h, 0], \mathbb{R}^n), \varphi_2(\cdot) \in L_2([-h, 0], \mathbb{R}^n), u(t) \in \mathbb{R}^r; A_0(\cdot), B_0(\cdot), A(\cdot), B(\cdot) \) are \((n \times n)\) and \((n \times r)\)-matrix-functions of bounded variation, respectively, and the matrix-function \( A_0(\tau) \) satisfies the condition

\[ A_0(0) = \lim_{\tau \to 0^+} A_0(\tau). \]

The state space of the equation under consideration is

\[ X = \mathbb{R}^n \times L_2([-h, 0], \mathbb{R}^n) \times L_2([-h, 0], \mathbb{R}^r). \]

Let \( x_t \) be the function defined by

\[ x_t(\tau) = x(t + \tau), u_t(\cdot) = u(t + \tau), -h \leq \tau \leq 0. \]

Following \cite{6}, we will describe equation (4-51)-(4-53) by well-posed abstract boundary control problem

\[ \frac{d}{dt} \bar{x}(t) = L\bar{x}(t), \]
\[ G\bar{x}(t) = u(t), \]

where \( x(t) \in \mathbb{R}^n, \varphi_1(\cdot) \in L_2([-h, 0], \mathbb{R}^n), \varphi_2(\cdot) \in L_2([-h, 0], \mathbb{R}^n), u(t) \in \mathbb{R}^r; A_0(\cdot), B_0(\cdot), A(\cdot), B(\cdot) \) are \((n \times n)\) and \((n \times r)\)-matrix-functions of bounded variation, respectively, and the matrix-function \( A_0(\tau) \) satisfies the condition

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Following \cite{6}, we will describe equation (4-51)-(4-53) by well-posed abstract boundary control problem

\[ \frac{d}{dt} \bar{x}(t) = L\bar{x}(t), \]
\[ G\bar{x}(t) = u(t), \]

\footnote{The condition (4-54) provides the existence and uniqueness for solutions of (4-51)-(4-53) \cite{Heil}.}

\footnote{The state spaces of \cite{Salamon} is the Hilbert space} \[ \mathbb{R}^n \times L_2([-h, 0], \mathbb{R}^n) \times L_2([-h, 0], \mathbb{R}^r). \]
where

\[ Z = \{ \bar{x} \in X : \varphi^0 \in \mathbb{R}^n, \varphi_1 \in W^{1,2}([-h, 0], \mathbb{R}^n), \varphi_2 \in W^{1,2}([-h, 0], \mathbb{R}^r) \} \]

\[
\begin{align*}
\varphi^0 &= \varphi_1(0) - \int_{-h}^0 dA_0(\tau) \varphi_1(\tau) - \int_{-h}^0 dB_0(\tau) \varphi_2(\tau), \\
\bar{x}(t) &= \left( x(t) - \int_{-h}^0 dA_0(\tau) x(t + \tau) - \int_{-h}^0 dB_0(\tau) u(t + \tau), x_t, u_t \right), \\
\bar{x}(0) &= \bar{x} = \left( \varphi^0 - \int_{-h}^0 dA_0(\tau) \varphi_1(\tau) - \int_{-h}^0 dB_0(\tau) \varphi_2(\tau), \varphi_1(\cdot), \varphi_2(\cdot) \right); \\
L\bar{x} &= \left( \int_{-h}^0 dA(\tau) \varphi_1(\tau) - \int_{-h}^0 dB(\tau) \varphi_2(\tau), \dot{\varphi}_1(\cdot), \dot{\varphi}_2(\cdot) \right), \\
G\bar{x} &= \dot{\varphi}_2(0)
\end{align*}
\]

(4-57)

for which the corresponding operator \( A \) satisfies the conditions (I) - (II) [10], and condition (III) holds for a wide class of (4-51) (see [14: p. 101]). For example, condition (III) holds for

\[ A(\tau) = \sum_{j=0}^{m} A_j \chi_{[-h_{j+1}, -h_j]}(\tau) \quad (0 = h_0 < h_1 < \ldots < h_m = h, -h \leq \tau \leq 0) \]

\[ A_0(\tau) = \sum_{j=0}^{m} A_0 \chi_{[-h_{j+1}, -h_j]}(\tau) \quad (0 = h_0 < h_1 < \ldots < h_m = h, -h \leq \tau \leq 0) \]

where \( A_j \) are \((n \times n)\)-matrices and \( \chi_{[-h_{j+1}, -h_j]}(\tau) \) is the characteristic function of the interval \([-h_{j+1}, -h_j]\).

Let

\[ \Delta(z) = \det \left\{ zI - \int_{-h}^0 dA_0(\tau) z \exp z\tau - \int_{-h}^0 dA(\tau) \exp z\tau \right\}. \]

Denote by \( \omega \) the exponential of the function \( \Delta(z) \) [4], i.e.

\[ \omega = \lim_{|z| \to \infty} \frac{1}{|z|} \log |\det \Delta(z)|. \]
Lemma 9 The sequence (6) is minimal on \([0, \nu]\) for any \(\nu > \omega\).

Proof. The assertion of the lemma has been proven in [Shklyar] for hereditary case and without delays in control \((A_0(\tau)\) and \(B_0(\tau)\) are constant on the segment \([-h, 0] \); \(B(\tau) = \begin{cases} B, & \text{if } \tau = 0, \\ 0, & \text{if } -h < \tau \leq 0. \end{cases}\).

One can use the same proof in the general case. There is only one difference:

\[ \Delta(z) = \sum_{j=0}^{n} r_j(z)z^j, \]

where \(r_j(z)\) is represented as a finite sum of products of numbers

\[ \int_{-h}^{0} da_{jk}(\tau) \exp(-z\tau) \text{ and } \int_{-h}^{0} da_{jk}^0(\tau) \exp(-z\tau) \quad (j = 1, \ldots, n) \]

with \(a_{jk}(\tau)\) and \(a_{jk}^0(\tau)\) being the elements of the matrix \(A(\tau)\) and \(A_0(\tau)\) correspondingly. Further we finish the proof as well as in [8].

4.2 Partial hyperbolic systems.

Consider the wave equation on \([0, t_1] \times [0, \pi]\):

\[ \frac{\partial^2}{\partial t^2} w(t, \theta) = \frac{\partial^2}{\partial \theta^2} w(t, \theta), \quad (4-59) \]

\[ w(t, 0) = u(t), w(t, \pi) = u(t), \]

\[ w(0, \theta) = \varphi_0(\theta), \frac{\partial}{\partial t} w(0, \theta) = \varphi_1(\theta). \]

We assume the weak solution \(w(t, \cdot) \in AC[0, \pi]\), where \(AC[0, \pi]\) is the space of absolutely continuous functions defined on \([0, \pi]\); \(\frac{\partial}{\partial \theta} w(t, \cdot), \varphi_0(\cdot)\) and \(\varphi_1(\cdot) \in L_2[0, \pi]; u(\cdot) \in L_2[0, t_1]\). Define \(v(t, \cdot) = \frac{\partial}{\partial t} w(t, \cdot)\)

\[ x = \left\{ \begin{array}{l} w(t, \cdot) \\ v(t, \cdot) \end{array} \right\} \]

\[ X = W_0^1[0, \pi] \times L_2[0, \pi], \]
where

$$W^1_0 [0, \pi] = \left\{ x \in AC [0, \pi] : x (0) = x (\pi) = 0, \frac{d}{d\theta} x (\cdot) \in L_2 [0, \pi] \right\}. \quad (4-60)$$

We have

$$\dot{x} (t) = L x (t),$$
$$G x (t) = b u (t),$$

where the operators $A, G$ and $b$ are defined by

$$L = \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial \theta^2} & 0 \end{pmatrix},$$

$$G x = \begin{pmatrix} w (0) \\ w (\pi) \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with domain

$$D (A) = \left( W^1_0 [0, \pi] \cap W^2 [0, \pi] \right) \times W^1_0 [0, \pi], D (G) = C \left( [0, \pi], \mathbb{R}^2 \right),$$

where

$$W^2 [0, \pi] = \left\{ x \in AC [0, \pi] : \frac{d}{d\theta} x (\cdot) \in AC [0, \pi], \frac{d^2}{d\theta^2} x (\cdot) \in L_2 [0, \pi] \right\}. \quad (4-62)$$

The above operator $A$ generates a contraction group $S (t)$ on $X$.

The spectrum $\sigma$ of the operator $A$ is defined by

$$\sigma = \{ \lambda \in \mathbb{C} : \lambda = \pm ki, k = 0, 1, 2, ... \}$$

We have the sequence of functions

$$e^{kt}, k = 0, \pm 1, \pm 2, ...$$

be minimal on $[0, 2\pi]$, and all the properties (I)-(III) for any $T > 0$, hence the next theorem follows from Theorem 4.

**Theorem 10** *The closure of the attainable set for equation (4-59) doesn't depend on $t$ for any $t > 2\pi$.*
5 Concluding remarks.

Properties of the attainable set $K(t)$ for equation (1.1) with unbounded input operators are considered. Results of [8] have been generalized for such classes of abstract evolution equations. Based on Theorem 4 the null-controllability criterion for equation (1.1) is obtained. Application to general functional differential equation of neutral type and hyperbolic systems have been considered. As well as in [8], property (III) and the minimality of the functions (3-6) provide the required independence of $t$ for $K(t)$.

By duality principle one can obtain observability conditions for abstract evolution equation with unbounded output operators.

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