The Exact Renormalization Group and Approximations∗

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ABSTRACT

We review the Exact Renormalization Group equations of Wegner and Houghton in an approximation which permits both numerical and analytical studies of nonperturbative renormalization flows. We obtain critical exponents numerically and with the local polynomial approximation (LPA), and discuss the advantages and shortcomings of these methods, and compare our results with the literature. In particular, convergence of the LPA is discussed in some detail. We finally integrate the flows numerically and find a $c$-function which determines these flows to be gradient in this approximation.

1. Introduction

The Exact Renormalization Group (ERG) is an old yet almost unexplored approach to non-perturbative computations in quantum field theory (for a detailed review of the method see Ref. 4). Recently, some authors have pursued the idea of considering the projection of the exact equations on local actions for constant fields.5 Though restrictive, this approximation maintains a good deal of non-perturbative physics – enough to find satisfactory estimates of critical exponents5–7 and can be improved systematically.8,9

To start with, we consider one of the physically equivalent formulation of the ERG as originally studied by Wegner and Houghton.2 In this contribution we consider a scalar theory in $d$ dimensions with a single scalar field $\phi(q)$ (we work in the momentum representation). The ERG equation, describing how the effective action

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\[
\frac{\partial S}{\partial t} = \frac{1}{2t} \int q \left\{ \ln \left( \frac{\partial^2 S}{\partial \phi(q) \partial \phi(-q)} \right) - \frac{\partial S}{\partial \phi(q)} \frac{\partial S}{\partial \phi(-q)} \left( \frac{\partial^2 S}{\partial \phi(q) \partial \phi(-q)} \right)^{-1} \right\} - \int q \phi(q) \frac{\partial S}{\partial \phi(q)} - dS + (1 - \frac{d}{2} - \eta) \int q \phi(q) \frac{\partial S}{\partial \phi(q)} + \text{const.},
\]

where \( \eta \) is the anomalous dimension of \( \phi \), the prime in the first integral above indicates integration only over the infinitesimal shell of momenta \( e^{-t} \Lambda_0 \leq q \leq \Lambda_0 \), and the prime in the derivative indicates that it does not act on the \( \delta \)-functions in \( \partial S/\partial \phi(q) \). This equation is known as the “sharp cut-off” version of the ERG because the integration of modes is reduced to a shell. Here we use the approximation proposed in Ref. 5, constraining the effective action to have no other derivative pieces than the canonical kinetic term, that is, in coordinate space,

\[
S = \int d^d x \left( \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right).
\]

To exploit the ERG Eq. (1) within this approximation, Hasenfratz and Hasenfratz set up the local, constant mode projection via the action of the operator

\[
e^x \frac{\partial}{\partial \phi(0)}
\]

obtaining the following differential equation for the effective potential

\[
\dot{V}(x, t) = \frac{A_d}{2} \ln(1 + V''(x, t)) + d \cdot V(x, t) + (1 - \frac{d}{2} - \eta) x V'(x, t) + \text{const.},
\]

where \( A_d/2 = [(4\pi)^{d/2} \Gamma(d/2)]^{-1} \), the dot is a scale derivative \( \partial/\partial t \), \( x \) is the constant mode \( \phi(0) \), and we again refer the reader to the derivation in Ref. 5. In the approximation we are using, Eq. (2), we actually leave no room for wavefunction renormalization, and this turns out to imply that \( \eta = 0 \) above. For greater ease of calculations, we will actually study the equation for \( f(x, t) = V'(x, t) \), trivially found from the above:

\[
\dot{f}(x, t) = \frac{A_d}{2} \frac{f''(x, t)}{(1 + f'(x, t))} + (1 - \frac{d}{2}) x f'(x, t) + (1 + \frac{d}{2}) f(x, t),
\]

with \( \eta \) already set to 0. The constant \( A_d \) can be absorbed by a rescaling of \( x \) thus disappearing from the equation above, a fact we will make use of later. This is a reflection of universality in Eq. (4), whereby the shape of \( f^* \) will depend on \( A_d \) but the critical exponents will not.

The numerical treatment of this equation can be carried out in two fashions. On one hand, it is possible to use direct numerical solutions to find the fixed-point potential. Once the shape of the scale invariant point is obtained, the eigenvalue problem of linearized perturbations around it produces the standard critical exponents.
We shall employ this method in the next section and so obtain the exact solution of the differential equation within the approximation we are considering. On the other hand, a more analytic approach consists of expanding $f(x)$ as a polynomial series in $x^2$. This is called the local polynomial approximation (LPA). The complicated differential equation for the fixed point reduces then to an algebraic system of equations. Solutions are found with less numerical machinery.

An issue recently touched upon by Morris concerns the convergence of the LPA to the real solution. This is a subtle problem which, at the moment, hints at the necessity of using exact numerical solutions rather than approximations. Whether resummation techniques or expansions around the minima of the potential are applicable to solve this problem remains to be seen. It is also worth mentioning that the same author has managed to extend the approximation (2) beyond constant modes, that is to order $p^2$, and get results that indeed improved upon the zeroth order ones.

At a more fundamental level, one may consider the long-standing problem of irreversibility of renormalization group flows. So far there is no proof of the $c$-theorem is more than two dimensions. The ERG stands as a candidate to progress further on this issue and here we have made the smallest attempt in this direction. We have taken the polynomial expansion on the projected Wegner-Houghton equation and shown irreversibility on its renormalization group flows. This adds a little bit of non-perturbative evidence to the validity of the theorem.

2. Numerical Solutions

In this section we study the exact renormalization group equations of Wegner and Houghton from a purely numerical approach. Fixed point solutions and critical exponents found in this way reflect the “best” results these equations will yield, in the sense that no ansatz is introduced which could bring in further approximations and errors to the final numbers. When we do make the further ansatz of introducing a basis expansion (the LPA) for the effective potential in Sec. 3, these results will serve as a benchmark to tell us when and how the basis expansion ceases to be a reasonable approximation.

We begin by investigating, in $d = 3$, the Wilson fixed point solution $f^*(x)$ for $f(x, t) = \partial V(x, t)/\partial x$. It satisfies the equation

$$0 = \frac{1}{4\pi^2} \frac{f^{**}}{(1 + f'^{*})} + \frac{5}{2} f^* - \frac{1}{2} x f''^*.$$  

A well-behaved numerical solution of this equation only exists for a specific value $\gamma^*$ of the initial condition $f^{**}(0) = \gamma^*$ (for the other initial condition we take $f^{**}(0) = 0$). It is simple to investigate numerically that if $f^{**}(0) = \gamma > \gamma^*$ then the solution diverges at a finite value of $x$, while if $\gamma < \gamma^*$ the solution is unbounded below. Using these bad behaviors as a guide we can zero in on $\gamma^*$ with as much precision as our machine and software allow us. We find

$$\gamma^* = -0.4615413727.. ,$$  

and at the precision above the numerical integration of the equation holds for $0 < x < 0.55$. This value of $\gamma^*$ agrees to within a $10^{-3}$ per cent of the value obtained in Ref. 5 for the same problem.
Armed with a numerical interpolating function for $f^*(x)$, we can now study the linearization of Eq. (5) above to determine the critical exponents. The linearized equation has a discrete spectrum of eigenvalues. For $d = 3$ we expect all but one of these to be negative, corresponding to the presence of one relevant operator and an infinite tower of irrelevant ones, and this is indeed what we find. The numerical determination of these critical exponents is again similar in spirit to that of $\gamma^*$ above: for eigenvalues slightly off from the correct ones the numerical solutions will either diverge up at a finite $x$, or be unbounded below. As the correct value is approached from both sides the well-behaved profile of the eigenfunction is achieved farther and farther in $x$. The first three critical exponents we find are:

$$\nu \equiv \frac{1}{\omega_1} = 0.689458$$

$$\omega \equiv -\omega_2 = 0.5953$$

$$\omega_3 = -2.84$$

The last digit in all figures above is uncertain. The first two of the above are also in agreement with Ref. 5. It is also in reasonable agreement with calculations performed in other methods (field theory calculations, high temperature expansions and Monte Carlo methods all yield $\nu = 0.630 \pm 0.003$ and $-1.0 < \omega_2 < -0.5$).

We have also performed the analogous calculations for $d = 2.9$. There are a number of reasons to do this. First, it is only for $d < 3$ that we expect to find more than one non-Gaussian fixed point. At $d = 2.9$ we indeed find two of them, which is what the theory dictates. Secondly, by choosing a dimension close to 3, we can also corroborate the results we obtain through an $\epsilon$-expansion in Sec. 3. Finally, within the LPA, detecting multiple fixed points for $d < 3$ involves also discarding a number of spurious solutions in a not entirely systematic fashion. Verifying that such spurious solutions do not appear here is important to determine that the limitation is not in the equation itself, but in the basis expansion, and the solutions we find purely numerically can again be used as a guide in identifying the appropriate solutions in the basis expansion.

We find fixed point solutions in $d = 2.9$ for the following two initial conditions:

$$\gamma_4^* = -0.54598...$$  \hfill (11)

$$\gamma_6^* = 0.009928...$$  \hfill (12)

The subindices 4 and 6 indicate the respective fixed points ($\phi^4$ and $\phi^6$). It is worthwhile noting that while $\gamma_4^* \approx \gamma^*$, $\gamma_6^*$ is very small and positive. This is of course a signature of a $\phi^6$ rather than a $\phi^4$ potential profile.

For this new fixed point we also expect two rather than one relevant operators, and a marginally irrelevant one (and infinitely many irrelevant ones). Our $\epsilon$-expansion results of Sec. 3 for $\epsilon = 0.1$ give

$$\omega_1 = 2$$  \hfill (13)

$$\omega_2 = 1.02$$  \hfill (14)

$$\omega_3 = -0.2$$  \hfill (15)

We have checked the first two of these numerically and have found good agreement.
3. Local Polynomial Approximation

In this section we review some properties of exact solutions to equation (5) as well as some results on the polynomial approximation for the solutions, the analytic approach consisting of expanding $f(x, t)$ as a polynomial series in $x$. In particular we show that reasonable values for critical exponents and correct coefficients of the leading order of the $\epsilon$-expansion can be obtained within this approach without much machinery. However, arguments will be provided which show that the approximations do not converge to the correct limit. Our discussion here is based on results of Refs. 5,6,9.

Let us study first properties of $t$-independent solutions to Eq. (5) with the factor $A_d/2$ being absorbed by a rescaling of $x$ and the function $f$. As in the previous section we choose the initial conditions to be $f(0) = 0$ and $f'(0) = \gamma$. Solutions can be labelled by the parameter $\gamma$ and in a vicinity of $x = 0$ can be represented as a series in odd powers of $x$ with finite radius of convergence (see below):

$$f_\gamma(x) = \sum_{m=0}^{\infty} c_{2m+1}(\gamma)x^{2m+1},$$

The coefficients of the expansion obey the following recurrence relations:

$$c_{2m+3}(\gamma) = -c_{2m+1}(\gamma)\frac{s_m}{q_m} - \frac{g_m(c_{2m+1}(\gamma), \ldots c_1(\gamma))}{q_m},$$

with

$$s_m = 2(m + 1)(2m + 3), \quad q_m = 2(m + 1) - md$$

and

$$g_m(c_{2m+1}(\gamma), \ldots c_1(\gamma)) = \sum_{l=0}^{m} c_{2l+1}(\gamma)c_{2(m-l)+1}(\gamma)[2(m-l) + 1] s_l.$$  

Using Eq. (17) recursively we obtain $c_3, c_5, \ldots$ as functions of $c_1$:

$$c_{2m+3}(c_1) = \sum_{l=1}^{m+2} b_l(m + 1)c_1^l.$$  

The coefficients of this polynomial satisfy certain recursive relations, which follow from Eq. (17) and are easy for numerical resolution. However their explicit form is rather cumbersome and here we present only the first two of them:

$$b_1(m + 1) = -\frac{s_m}{q_m}b_1(m),$$

$$b_2(m + 1) = -\frac{s_m}{q - m}b_2(m) - \frac{1}{q_m}\sum_{l=0}^{m} b_1(l)b_1(m - l)[2(m - l) + 1] s_l.$$  


Eq. (19) can be easily solved and for \( b_1(0) = 1 \) gives

\[
\begin{align*}
  b_1(m) &= (-1)^m \prod_{l=0}^{m} \frac{s_l}{q_l} = (-1)^m \prod_{l=0}^{m} \frac{l - 1}{2l(2l + 1)} \prod_{l=0}^{m} (d_{l}^{\text{crit}} - d),
\end{align*}
\]

where

\[
d_{k}^{\text{crit}} = \frac{2k}{k - 1}, \quad k = 2, 3, 4, \ldots
\]

are the upper critical dimensions.

For large \( m \) the asymptotic formula for the coefficients of the expansion (16) can be obtained from the relation (17):

\[
\begin{align*}
  c_{2m+1}(\gamma) &\sim \frac{2a(\gamma)}{d - 2} \frac{1}{m} a(\gamma)^m \left( 1 + O \left( \frac{1}{m} \right) \right),
\end{align*}
\]

where the parameter \( a(\gamma) \) is not determined by the leading order terms. This asymptotics shows that the expansion (16) has the finite radius of convergence equal to \( 1/\sqrt{|a(\gamma)|} \) and is in agreement with the singular behaviour \( f_\gamma(x) \sim (2/(x_c(\gamma)(d - 2))) \ln(x_c(\gamma) - x) \) where \( x_c(\gamma) = 1/\sqrt{a(\gamma)} \) is the position of the singularity in the complex \( x \)-plane closest to the origin. As it has been already discussed in the previous section, numerical study of the equation (5) shows that for certain values \( \gamma = \gamma^* \) the solution \( f_\gamma(x) \) does not have singularities on the real positive axis and thus is a physical fixed point solution. Obviously \( \gamma^* = 0 \) is one of such critical values corresponding to the Gaussian fixed point solution. It was claimed in Ref. 5 that for \( d = 4 \) there are no other fixed point solutions. For \( d = 3 \) there is a nontrivial solution with \( \gamma^* \) given by (7) that describes the Wilson fixed point.

The location of singularities of nontrivial solutions \( f_\gamma(x) \) for \( d = 3 \) was analysed in Ref. 9. The logarithmic pole \( x_c(\gamma) \) is real and positive for \( \gamma < \gamma^* \) and is complex but situated close to the real positive axis for \( \gamma > \gamma^* \). For \( \gamma \) approaching the critical value \( \gamma^* \), \( x_c(\gamma) \) moves to infinity and other singularities become dominant. Thus for \( \gamma = \gamma^* \) the closest singularities turn out to be at \( x_\ast = \pm r_\ast e^{i \theta_\ast} \) with \( r_\ast = 3.12 \) and \( \theta_\ast = 0.257 \pi \).

The representation (14) suggests that the fixed-point solution to Eq. (3) can be found by making the LPA:

\[
\begin{align*}
  f_M(x) &= \sum_{m=0}^{M} c_{2m+1} x^{2m+1}.
\end{align*}
\]

It is clear that the coefficients \( c_{2m+1} \) satisfy the same recurrence relations (17) or (18) for \( 0 \leq m \leq M - 1 \) plus the additional condition truncating the expansion (23):

\[
\begin{align*}
  c_{2M+3}(c_1) &= \left[ b_{M+2}(M + 1)c_1^{M+2} + \ldots + b_1(M + 1) \right] c_1 = 0.
\end{align*}
\]

Solutions \( c_1(M) \) of this algebraic equation through the relations (18) determine the coefficients \( c_{2m+3}(c_1(M)), m = 0, 1, \ldots, M - 1, \) of the polynomial approximation.
The important features of the LPA can be summarized as follows:

(1) $c_M = 0$ is always a solution of (24) which gives $c_{2m+1}(c_1(M)) = 0$ and reproduces the Gaussian fixed point solution $f_0(x) = 0$.

(2) From the factorization of the coefficient $b_1(M+1)$, see Eq. (19), it follows that at the upper critical dimensions $d = d_k^{crit}, k = 2, 3, \ldots c_M = 0$ is actually a double solution, which indicates a branching of fixed-point solutions below these critical dimensions. This is in perfect agreement with the multicritical fixed-point solutions known to exist below these dimensions.

(3) The important question is whether the polynomial approximation (23) converges to a non-trivial fixed point solution $f^{\gamma_*}(x)$ as $M \to \infty$ for some sequence of $c_M$ satisfying (24). The answer seems to be negative and the reason for this is rather simple: whereas the true solution $f^{\gamma_*}(x)$ possesses singularities, as discussed above, the polynomial approximation $f_M(x)$ for any $M$ does not. To be more precise, it is easy to see that

$$ |f_M(x) - f^{\gamma_*}(x)| \leq \sum_{m=0}^{M} |c_{2m+1}(c_1(M)) - c_{2m+1}(\gamma^*)||x|^{2m+1} + R_M(x) $$

$$ \leq |c_1(M) - \gamma^*| \sum_{m=0}^{M} |c_{2m+1}(\gamma^*)||x|^{2m+1} + R_M(x), \quad (25) $$

where the residue $R_M(x)$ for $|x| < r_*$ approaches zero as $M \to \infty$. A numerical study carried out in Ref. 9 shows that for $d = 3$ and $M \geq 15$ the quantity $\delta = |c_1(M) - \gamma^*| \approx 0.005$ and does not decrease with $M$. This shows that the polynomial approximation does not converge to the fixed point solution $f^{\gamma_*}(x)$. However the smallness of $\delta$ makes the difference (25) to be quite small for $x$ not large. This enables us to get an approximation to the solution and, using it, numerical values for the first critical exponents (see below) with decent accuracy, while creating an illusion of convergence of the method, as claimed in Ref. 6.

(4) The lower the dimension, the less trustworthy is this approximation or, conversely, the larger is the $M$ needed. Altogether, we find that, for any $d$, some solutions represent true fixed points while others are spurious. For lower dimensions, the number of true nontrivial fixed points increases, but so does the number of spurious solutions. There seems to be no wholly systematic way of discarding these spurious solutions.

(5) To further corroborate the usefulness of the polynomial approximation, an $\epsilon$-expansion of Eqs. (18) - (24) about any critical dimension leads to the known $\epsilon$-expansion solution. Let us take $d = d_k^{crit} - \epsilon$ and look for the solution which is of order $\epsilon$ below the critical dimension $d_k^{crit}$:

$$ c_1(M) = \epsilon c_1^{(1)}(M) + \mathcal{O}(\epsilon^2). $$

Since, as it follows from Eq. (21), $b_1(m) = \epsilon b_1^{(1)}(m) + \mathcal{O}(\epsilon^2)$ for $m \geq k$, the solution to (24) for $c_1^{(1)}(M)$ is given by

$$ c_1^{(1)}(M) = -\frac{b_1^{(1)}(M+1)}{b_2^{(0)}(M+1)}. $$
where $b_2^{(0)}(M)$ is the nonvanishing part of the coefficient $b_2(M)$ when $\epsilon \to 0$, i.e. $b_2(M) = b_2^{(0)}(M) + \mathcal{O}(\epsilon)$. Using eqs. (20) and (21) we calculate $b_2^{(0)}$ and then $c_1^{(1)}(M)$. We also get from Eq. (17) or (18) that $c_{2m+1}(M) = \mathcal{O}(\epsilon^2)$ for $m \geq k$ and

$$f_M(x) = \epsilon \kappa_k H_{2k-1}(x/\lambda_k) + \mathcal{O}(\epsilon^2),$$

(26)

where $H_{2k-1}$ is the Hermite polynomial and

$$\kappa_k = (-1)^{k+1} c_1^{(1)}(M) \lambda_k 2^k (2k-1)!! \quad \lambda_k = \frac{2}{\sqrt{d_{\text{crit}}^k - 2}}.$$  

(27)

For example, for $d_{\text{crit}}^k = 3$ $c_1^{(1)}(M) = 1/20$ and does not change with $M$ for $M \geq 3$. Note that a simple $\epsilon$-expansion of Eq. (3) will lead to a linear equation and thus cannot furnish the constant $\kappa_k$. At higher orders in $\epsilon$ we expect our results not to agree with the standard $\epsilon$-expansion since the present approximation (2) does not allow for wavefunction renormalization.

Having found particular fixed point solutions for some $M$, we can now study how the renormalization flow approaches these solutions by determining the critical exponents. To find them, we study small $t$-dependent departures from some fixed-point profile $f_{\gamma^*}(x)$:

$$f(x,t) = f_{\gamma^*}(x) + g(x,t),$$

(28)

where again a polynomial ansatz is chosen for $g(x,t)$:

$$g(x,t) = \sum_{m=1}^{M} \delta_{2m-1}(t)x^{2m-1}.$$  

When (28) is substituted in Eq. (3) and only linear terms in $g$ are kept, we find:

$$\dot{\delta}_i = \sum_{j=1}^{M} \Omega_{ij}(\gamma^*, d) \delta_j,$$

(29)

where $\Omega_{ij}$ is an $M \times M$ matrix which depends on the input values $c_i(\gamma^*)$ and $d$.

The critical exponents will be given by the eigenvalues of $\Omega$. For the $\phi^4$ fixed point we have calculated the critical exponents numerically up to $M = 7$ for dimensions between 2 and 4 in steps of 0.1. For $2.9 \leq d \leq 4$ our results are plotted in Fig. 1.

It is worthwhile noting that as $d \to 4$, the critical exponents merge with the tower of canonical dimensions $(2, 0, -2, -4, \ldots)$, which are precisely the critical exponents of the trivial Gaussian theory at $d = 4$ (i.e., the canonical dimensions of the $(\phi^2, \phi^4, \phi^6, \ldots)$ couplings in $d = 4$). This is an indication in our setting of the existence of a unique (Gaussian) fixed point at $d = 4$. We furthermore note that for $d = 3$ our two leading exponents

$$\nu = \frac{1}{\omega_1} = 0.656, \quad \omega_2 = -0.705$$
match fairly well results gotten by other methods (see the previous section).

It is also possible to perform an $\epsilon$-expansion on the flow equation around a critical dimension $d = d_{crit}^k$. Then, we substitute Eq. (26) for $f_\gamma(x)$ in Eq. (28) and make the following ansatz for $g(x,t)$:

$$g(x,t) = \exp[(\omega_0^{(0)} + \epsilon \omega_1^{(1)} + \epsilon^2 \omega_2^{(2)})t](g_0(x) + \epsilon g_1(x) + \epsilon^2 g_2(x)).$$  

(30)

For $d = d_{crit}^k - \epsilon$ we find that that the critical exponents are equal to

$$\omega_{k,\ell} = 2 \left( \frac{2 - \ell}{k - 1} - \epsilon \left( \ell - 1 - 2(k - 1) \frac{(2\ell)! k!}{(2\ell - k)! (2k)!} \right) \right) + O(\epsilon^2)$$

and at the leading order in $\epsilon$

$$g_{k,\ell}^{(0)}(x) \sim H_{2\ell-1}(x),$$

$\ell = 1, 2, 3, \ldots$

The LPA also reproduces in the leading order in $\epsilon$ the flow solution from the vicinity of the Gaussian fixed point $f_0(x) = 0$ to the fixed point $f_M(x)$ given by Eq. (26). To simplify the formula we choose an initial condition near the Gaussian fixed point when all operators except the one, whose critical exponent is proportional to $\epsilon$, are zero. Then we get the flow

$$f(x,t) = \epsilon \kappa_k \frac{1}{1 + a e^{\omega_{k,k}^{(1)}}} H_{2k-1}(\frac{x}{\lambda_k}) + O(\epsilon^2),$$

where $\lambda_k$ and $\kappa_k$ are given by (27) and $\omega_{k,k}^{(1)} = -(k - 1)$ (see Eq. (31}). The constant $a$ is fixed by the initial condition.

Critical exponents only characterize the flow very close to a particular fixed point. Another option offered by the LPA is to study the flow globally by substituting Eq. (16), with coefficients $c_{2m+1}$ being functions of $t$ now, directly into Eq. (5). Matching powers of $x$ in a Taylor expansion leads to coupled nonlinear flow equations for $c_i(t)$ in the form:

$$\dot{c}_{2m+1} = w_{2m+1}(c), \quad i = 1 \text{ to } M,$$

(32)

where the $w_{2m+1}(c)$ are certain functions of $c_1, c_3, \ldots, c_{2m+1}$ (see Ref. 6) given by the differential flow equation (3). Arguably, a polynomial ansatz does introduce a perturbative element into the essentially nonperturbative nature of renormalization flows between distant fixed points, and our approximation very likely misses some features of the true flow. However, we believe that, again, the sensible and rich structure that emerges does justify the simplification.

We have solved the nonlinear flow (32) numerically with $M = 3$ in $d = 3$:

$$\dot{c}_1 = 2c_1 + \frac{6c_3}{1 + c_1},$$

$$\dot{c}_3 = c_3 - \frac{18c_3^2}{(1 + c_1)^2} + \frac{20c_5}{1 + c_1},$$

$$\dot{c}_5 = \frac{54c_3^2}{(1 + c_1)^2} - \frac{90c_3c_5}{(1 + c_1)^2}.$$

(33)
The \((c_1(t), c_3(t))\) subspace of that flow is shown in Fig. 2. We note there the presence of a Gaussian and a Wilson fixed point, and a unique trajectory leading from the former to the latter. To determine that this flow is gradient and permits a \(c\)-function description is the object of the next section.

4. \(c\)-Function

We now study some features of the geometry of the space of local interactions. If the beta functions of a theory can be written as a gradient in the space of coupling constants,

\[
\beta^i(c) = -g^{ij} \frac{\partial \mathcal{C}}{\partial c_j}
\]

(34)

where \(g^{ij}\) is a positive-definite metric, we know that the set of renormalization flows becomes irreversible. In such a case, there exists a function \(\mathcal{C}\) of the couplings which is monotonically decreasing along the flows:

\[
\frac{d\mathcal{C}}{dt} = \beta^i \frac{\partial \mathcal{C}}{\partial c^i} = -g^{ij} \frac{\partial \mathcal{C}}{\partial c^i} \frac{\partial \mathcal{C}}{\partial c^j} \leq 0,
\]

(35)

making their irreversibility apparent, so that recurrent behaviors such as limit cycles are forbidden. In two dimensions it is possible to prove that the fixed points of the flow are the critical points of \(\mathcal{C}\) and that the linearized RG generator in a neighborhood of a fixed point is symmetric with real eigenvalues (the critical exponents).

The renormalization group flows found in the previous section are all well-behaved. Therefore it becomes natural to ask whether these flows are gradient, \(i.e.,\) whether there exists a globally defined Riemannian metric \(g_{ij}\) and a non-singular potential \(\mathcal{C}\) satisfying Eq. (34). The general solution for an arbitrary number of couplings \(M\) would be extremely difficult. However, we find that it is possible to treat the case \(M = 2\), namely, the subspace of mass and quartic couplings. The beta functions corresponding to the two couplings \(c_1, c_3\) are given in Eq. (33) (where we restrict to \(c_5 = 0\)). Because of the positivity of \(c_3\) (\(c_3\) is the coefficient of \(\phi^4\) in \(V\) and is required to be positive for stability of the path integral) it is appropriate to make the following coupling constant reparametrization:

\[
c_1 \rightarrow m^2 = c_1, \quad c_3 \rightarrow \lambda^2 = 6A_d c_3.
\]

(36)

In these new variables the beta functions take the form

\[
\frac{dm^2}{dt} = 2m^2 + \frac{1}{2} \frac{\lambda^2}{(1 + m^2)}, \quad \frac{d\lambda}{dt} = \frac{(4 - d)}{2} \lambda - \frac{3}{4} \frac{\lambda^3}{(1 + m^2)^2}
\]

(37)

and the fixed points become

- Gaussian: \((m_G^2, \lambda_G) = (0, 0)\);
- Wilson: \((m_W^2, \lambda_W) = (-\frac{4 - d}{10 - d}, \frac{\sqrt{24(4 - d)}}{10 - d})\).
Note that the Wilson fixed point merges with the Gaussian one at $d = 4$, similarly to the situation in Sec. 3. Now, by trial and error and considerable guesswork, the following solution to Eq. (34) can be found:

$$C(m^2, \lambda) = \frac{1}{2}(1 + m^2)^4 - \frac{2}{3}(1 + m^2)^3 + \frac{1}{4}\lambda^2(1 + m^2)^2 - \frac{3}{16}\frac{\lambda^4}{(4 - d)}$$

(38)

and

$$g^{ij} = \frac{1}{(1 + m^2)} \begin{pmatrix} 1 & 0 \\ 0 & 4 - d \end{pmatrix}.$$  

(39)

$C(m^2, \lambda)$ has the expected properties of a $c$-function: i) it has a maximum at the Gaussian fixed point, ii) it has a saddle at the Wilson fixed point, and iii) there is only one flow connecting both points (we have not normalized the $c$-function to one for the Gaussian fixed point as often done in the literature). Naturally, this description corresponds to our particular parametrization in terms of $m$ and $\lambda$, which implicitly carries a choice of subtraction point. The variation of $C$ between fixed points is reparametrization invariant and its positivity amounts to physical irreversibility of the flow. A contour plot of $C$ for $d = 3$ is given in Fig. 3, which depicts the space of theories in the basis given by $m$ and $\lambda$ as a hilly landscape. The Gaussian point is at the top of the hill $(0, 0)$, whereas the Wilson point lies on the saddle $(-1/7, \sqrt{24}/7)$. For the sake of completeness, let us comment that the first mention of irreversibility of the renormalization group flow was spelled out in the context of perturbation theory by Wallace and Zhia. Later, Zamolodchikov proved a theorem in two dimensions, the $c$-theorem, which relates the irreversibility of the flows to the basic assumption of unitarity in the Hilbert space of the theory. Several authors have subsequently come to the conclusion that a similar theorem holds in any dimension in perturbation theory. More generally, any expansion where the space of theories is reduced to a manifold in a space of couplings will accomodate a $c$-theorem. Our setting in this Letter does not clearly fall into this category, due to the appearance of rational functions of the couplings in Eq. (37), and the explicit construction of the $c$-function, though to first non-trivial order, might be of relevance.

A systematic approach to the irreversibility of the renormalization group flow in the projected Wegner-Houghton equation should rely upon a computation of Zamolodchikov’s metric (i.e. all two-point correlators between composite operators in the theory). This will require an exact renormalization group equation for the generating functional equipped with a source for composite scalar fields.

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Figure Captions

Fig. 1. Critical exponents for $2.9 \leq d \leq 4$ corresponding to the relevant, marginal and the first two irrelevant operators in the $M = 7$ approximation.

Fig. 2. $d = 3$ Renormalization group flows projected on mass and quartic coupling subspace in the $M = 3$ approximation. $c_1$ is plotted on the $x$-axis and $c_3$ on the $y$-axis.

Fig. 3. $c$-function contour of Eq. (35). The Gaussian point is at the top of the hill $(0, 0)$, whereas the Wilson point lies on the saddle $(-1/7, \sqrt{24}/7)$. 
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