Bias Compensated Recursive Least-squares Identification for the Systems under Poor Observation Condition

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Abstract. For the autoregressive signal under poor observation condition, a bias compensated recursive least-square identification algorithm is presented based on the incremental observation equation. Moreover, it is easy to know that the obtained estimates of model parameters and noise variance converge to the corresponding true values with probability one, i.e. they are strongly consistent. A simulation example shows its effectiveness.

1. Introduction
In the practical application, the observation system errors are generally produced by the influence of the surrounding environment, the error caused by the observation equipment itself or the improper selection of the model and parameters. But the traditional Kalman filtering algorithm is difficult to eliminate this kind of error [1]. In order to solve this problem, a series of incremental filtering algorithms have been proposed. In [2]-[4], some incremental Kalman estimation algorithms were proposed, which successfully eliminate the measurement system error and improve the filtering accuracy. Subsequently, the state estimation problem for the nonlinear systems was solved by [5] and [6], which presented the extended incremental Kalman filtering algorithm and incremental particle filtering algorithm respectively. However, the identification of model parameters and noise variance of the systems under poor observation condition has not yet been studied.

Recently, the research on the identification problem based on the BCRLS (Bias Compensation Recursive Least-square) identification algorithm has attracted much attention. Two BCRLS algorithms for the single variable and multivariable AR (Autoregressive) signals were presented by [7] and [8] respectively. However, the estimation problem of AR model parameters and noise statistics for the system under poor observation condition has not been solved.

In this paper, an incremental model is firstly proposed for the AR signal under poor observation condition in this paper. It can effectively eliminate the unknown error of the system under poor observation condition. Furthermore, a BCRLS algorithm is presented for the systems with unknown model parameter and noise statistics under poor observation condition.

2. Problem formulation
Consider a stationary AR model under poor observation condition as follows

\[ y_k = a_1 y_{k-1} + \cdots + a_n y_{k-n} + \epsilon_k \]  

(1)
\[ z_k = y_k + v_k + a_k \]  \hspace{1cm} (2)

where \( y_k \) is the true signal, \( z_k \) is the observation signal, \( v_k \) and \( e_k \) are uncorrelated Gauss white noises with zero mean and variances \( \sigma_v^2 \) and \( \sigma_e^2 \). \( a_k \) denotes the observation system error at time \( k \). \( \sigma_v^2 \) is known, and \( a_i \) and \( \sigma_e^2 \) are unknown.

Based on (2), the observation equation is denoted by

\[ \Delta z_k = \Delta y_k + \Delta v_k \]  \hspace{1cm} (3)

where \( \Delta z_k \) denotes the observation vector increment at time \( k \), \( \Delta z_k = z_k - z_{k-1} \). Suppose that the stochastic vectors \( v_k \) and \( v_{k-1} \) are Gauss white noises independent of each other, \( \Delta v_k = v_k - v_{k-1} \) is also a Gauss white noise with zero mean and variance \( 2R_k \). From (3), we find that the unknown observation system error \( a_k \) is eliminated effectively.

**Remark 1**[3]. Because the observation system errors between two adjacent observation vectors are close, the observation system error of \( \Delta z_k \) is relatively small and can be ignored. According to the principle of independent increment stochastic process, we can see that it is more satisfied to the independence requirement of noise between \( \Delta z_k \) and \( \Delta z_{k-1} \) compared with that between \( z_k \) and \( z_{k-1} \).

Furthermore, applying (1) and (3) yields

\[ \Delta y_k = a_1 \Delta y_{k-1} + \cdots + a_n \Delta y_{k-n} + \Delta e_k \]  \hspace{1cm} (4)

where \( \Delta e_k = e_k - e_{k-1} \), and (1) and (4) contribute to an incremental AR model. The objectives are to find the estimators \( \hat{a}_k \) and \( \hat{\sigma}_e^2 \) for the AR model parameter based on the measurement \((z_1, \cdots, z_k)\).

### 3. Bias compensated recursive least-squares identification for the systems under poor observation condition

Substituting (3) into (4) yields the following LS (Least Square) structure

\[ \Delta z_k = \varphi_k^T \theta + m_k \]  \hspace{1cm} (5)

\[ \varphi_k^T = [\Delta z_{k-1}, \cdots, \Delta z_{k-n}] \]  \hspace{1cm} (6)

\[ \theta = [a_1, a_2, \cdots, a_n]^T \]  \hspace{1cm} (7)

\[ m_k = \Delta e_k + \Delta v_k - a_1 \Delta v_{k-1} - \cdots - a_n \Delta v_{k-n} \]  \hspace{1cm} (8)

where \( m_k \) is a colored noise.

Applying the general RLS (Recursive Least-square) algorithm, we have the bias estimate for \( \theta \)

\[ \hat{\theta}_k = \hat{\theta}_{k-1} + P_k \cdot \varphi_k \cdot \Delta z_k \]  \hspace{1cm} (9)

\[ P_k = P_{k-1} - \frac{(P_{k-1} \cdot \varphi_k) (P_{k-1} \cdot \varphi_k)^T}{1 + \varphi_k^T P_{k-1} \cdot \varphi_k} \]  \hspace{1cm} (10)

\[ \hat{\theta}_k = 0, \quad P_k = aI, \quad a > 0 \]  \hspace{1cm} (11)

The normal equation for the on-recursive LS (Least-square) estimate is given as

\[ [\sum_{j=1}^{k} \varphi_j \varphi_j^T] \hat{\theta}_k = \sum_{j=1}^{k} \varphi_j \Delta z_j \]  \hspace{1cm} (12)

Substituting (5) into (12) yields
\[
\sum_{j=1}^{k} \varphi_j^T [\hat{\theta}_k^j - \theta] = \sum_{j=1}^{k} \varphi_j m_j
\] (13)

In order to apply the ergodic property of stationary random sequences, the above equation is divided by \(k\) as

\[
\frac{1}{k} \sum_{j=1}^{k} \varphi_j^T [\hat{\theta}_k^j - \theta] = \frac{1}{k} \sum_{j=1}^{k} \varphi_j m_j
\] (14)

Based on the stationary assumption of \(\Delta y_\Delta\) and the ergodicity[9], we can obtain the following relationship with probability one for \(k \to \infty\)

\[
\frac{1}{k} \sum_{j=1}^{k} \varphi_j \phi_k^T \rightarrow M = E[\phi_k \phi_k^T]
\] (15)

From (3), (6) and (8), we have

\[
\frac{1}{k} \sum_{j=1}^{k} \varphi_j m_j = \frac{1}{k} \sum_{j=1}^{k} \left[ \Delta y_{j-1} + \Delta y_{j-1} \right] [\Delta e_j + \Delta y_j - a_1 \Delta y_{j-1} - \cdots - a_n \Delta y_{j-n}]
\] (16)

Because \(\Delta e_k\) is independent of \(\Delta y_k\), and \(\Delta y_k\) has the ergodicity, \(\Delta y_k\) can be expanded as

\[
\Delta y_k = \sum_{j=0}^{\infty} \phi_j \Delta e_{k-j}
\] (17)

and the coefficients \(\phi_j\) can be recursively calculated by[10]

\[
\phi_j = a_1 \phi_{j-1} + \cdots + a_n \phi_{j-n}, \quad j > 0
\] (18)

where \(\phi_0 = 1, \quad \phi_j = 0(j > 0)\), we have the following relationship for \(k \to \infty\) [9, 10]

\[
\frac{1}{k} \sum_{j=1}^{k} \Delta y_{j-i} \Delta e_j \rightarrow E[\Delta y_{k-i} \Delta e_k] = 0, \quad i = 1, \ldots, n
\]

\[
\frac{1}{k} \sum_{j=1}^{k} \Delta y_{j-i} \Delta e_j \rightarrow E[\Delta y_{k-i} \Delta e_k] = 0, \quad i = 1, \ldots, n
\]

\[
\frac{1}{k} \sum_{j=1}^{k} \Delta y_{j-i} \Delta y_j \rightarrow E[\Delta y_{k-i} \Delta y_k] = \sigma^2, \quad i = 1, \ldots, n
\]

\[
\frac{1}{k} \sum_{j=1}^{k} \Delta y_{j-i} \Delta y_{j-l} \rightarrow E[\Delta y_{k-i} \Delta y_{k-l}] = 0, \quad i \neq l
\]

\[
\frac{1}{k} \sum_{j=1}^{k} \Delta y_{j-i} \Delta y_{j-l} \rightarrow E[\Delta y_{k-i} \Delta y_{k-l}] = 0, \quad \forall i, l
\] (19)

So applying (16) and (19) yields the following relationship for \(k \to \infty\)

\[
\frac{1}{k} \sum_{j=1}^{k} \varphi_j m_j \rightarrow \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = -\sigma^2 \theta
\] (20)

From (14) and (15), the asymptotic deviation is given as
\[
\lim_{k \to \infty} \hat{b}_k - \theta = -\sigma_v^2 M^{-1} \theta
\]  

(21)

For \( k \to \infty \), we have
\[
kP_k = \left[ -\frac{1}{k} \sum_{j=1}^{k} \phi_j \phi_j^T \right]^{-1} \to M^{-1}
\]  

(22)

From (20), we have
\[
\theta = \lim_{k \to \infty} \hat{b}_k + \sigma_v^2 M^{-1} \theta
\]  

(23)

When \( k \) in above equation is sufficiently large, \( M^{-1} \) can be replaced by \( kP_k \), and \( \theta \) can be approximately replaced by \( \hat{\theta}_{k-1} \). So the bias compensated recursive least-squares estimator is given by
\[
\hat{\theta}_k = \hat{b}_k + \sigma_v^2 kP_k \hat{\theta}_{k-1}
\]  

(24)

In a word, the BCRLS identification algorithm for the systems under poor observation condition is obtained.

**Theorem 1.** For the AR model (3) and (4) under poor observation condition, the bias compensation recursive least square parameter estimator \( \hat{\theta}_k \) is given by
\[
\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{P_{k-1} \phi_k \left[ \Delta z_k - \phi_k^T \hat{\theta}_{k-1} \right]}{1 + \phi_k^T \phi_k P_{k-1} \phi_k}
\]  

(25)

\[
P_k = P_{k-1} - \frac{[P_{k-1} \phi_k \left[ P_{k-1} \phi_k \right]^T]}{1 + \phi_k^T \phi_k P_{k-1} \phi_k}
\]  

(26)

\[
\hat{\theta}_k = \hat{\theta}_k + \sigma_v^2 kP_k \hat{\theta}_{k-1}
\]  

(27)

with the initial values \( \hat{\theta}_0 = 0 \), \( P_0 = aI \), \( a > 0 \), \( \hat{\theta}_0 = 0 \).

Applying (5), it follows that
\[
m_k = \Delta z_k - \phi_k^T \theta
\]  

(28)

so \( m_k \) is a stationary random process, and
\[
m_k = \Delta \epsilon_k + \Delta \nu_k - a_1 \Delta \nu_{k-1} - \cdots - a_n \Delta \nu_{k-n}
\]  

(29)

Computing the variance of above equation with \( \Delta \epsilon_k \) independent of \( \Delta \nu_k \), we have
\[
\sigma_z^2 = \sigma_m^2 - \sigma_v^2 (1 + a_1^2 + \cdots + a_n^2)
\]  

(30)

where the sampling estimator for the variance of \( m_k \) is defined as
\[
\hat{\sigma}_m^2 = \frac{1}{k} \sum_{j=1}^{k} \hat{m}_j^2, \quad \hat{m}_j = \Delta z_j - \phi_j^T \hat{\theta}_j
\]  

(31)

then we have the recursive formula as follows
\[
\hat{\sigma}_m^2 = \frac{\hat{\sigma}_m^2}{k} + \frac{\hat{m}_k^2 - \hat{\sigma}_m^2}{k}
\]  

(32)

Substituting \( \hat{\sigma}_m^2 \) and \( \hat{\sigma}_k^2 \) into (30) yields the estimator at time \( k \)
\[
\hat{\sigma}_k^2 = \hat{\sigma}_m^2 - \sigma_v^2 \left[ 1 + \hat{\alpha}_{1k}^2 + \cdots + \hat{\alpha}_{nk}^2 \right]
\]  

(33)

4. **Convergence analysis**

**Theorem 2.** For the AR model (3) and (4) under poor observation condition, the bias compensated recursive least-squares estimator converges to real value at probability 1, i.e.
\[ \hat{\theta}_k \rightarrow \theta, \hat{\sigma}^2_{ek} \rightarrow \sigma^2, \quad \text{w.p.1} \]  

**Proof.** It is easily obtained similar to that in [10].

5. **Experimental model and result analysis**

Consider an AR model under poor observation condition

\[ y_k = 0.1y_{k-1} - 0.9y_{k-2} + \varepsilon_k \]  

(35)

\[ z_k = y_k + v_k + a_k \]  

(36)

where \( y_k \) denotes the real signal, \( z_k \) is the observation signal, \( v_k \) is a Gauss white noise with zero mean and variance \( \sigma^2_v \), which is independent of \( \varepsilon_k \). \( \varepsilon_k \) is a Gauss white noise with zero mean and variance \( \sigma^2 = 0.4 \). \( a_k \) denotes the observation system error at time \( k \), and \( a_k = 3 \). \( \sigma^2_v = 0.15 \) is known, but \( a_k \) and \( \sigma^2_v \) are unknown. The objectives are to find the AR parameter estimates \( \hat{a}_{ik}, \hat{a}_{2k} \) and \( \hat{\sigma}^2_{ek} \) based on the measurement \((z_1, \cdots, z_k)\).

Based on (2), the incremental observation equation is given as

\[ \Delta z_k = \Delta y_k + \Delta v_k \]  

(37)

where \( \Delta z_k \) denotes the observation vector increment at time \( k \), \( \Delta z_k = z_k - z_{k-1} \). \( \Delta v_k = v_k - v_{k-1} \) obeys the Gauss distribution with the variance \( 2\sigma^2_v \).

Furthermore, applying (35) and (37) yields

\[ \Delta y_k = 0.1\Delta y_{k-1} - 0.9\Delta y_{k-2} + \Delta \varepsilon_k \]  

(38)

where \( \Delta \varepsilon_k = \varepsilon_k - \varepsilon_{k-1} \), and (37) and (38) constitute an incremental AR model. The objective is to find the estimators \( \hat{a}_{ik}, \hat{a}_{2k} \) and \( \hat{\sigma}^2_{ek} \) for the AR parameters based on the measurement \((\Delta z_1, \Delta z_2, \cdots, \Delta z_k)\) at time \( k \).

Applying Theorem 1 yields the BCRLS parameter estimators under poor observation condition, the results are given by Figure 1, Figure 2 and Table 1. In Figure 1, the contrast curves between the true values and estimators \( \hat{a}_{ik} \) and \( \hat{a}_{2k} \) are given. Figure 2 gives the curves of the BCRLS variance estimators \( \hat{\sigma}^2_{ek} \). The straight line represents the true value, and the curve represents the estimation.

The parameter estimators at time \( k=1500 \) is shown in Table 1. It can be seen that the proposed BCRLS algorithm can estimate the parameters of AR model under poor observation condition with high accuracy. It is effective and feasible.

![Figure 1. Convergence curves of parameter estimation](image-url)
Figure 2. Convergence curves of estimator for noise variance $\hat{\sigma}_\varepsilon^2$

Table 1. Parameter estimators at time $k=1500$

| $\hat{a}_1$  | $\hat{a}_2$  | $\hat{\sigma}_\varepsilon^2$ |
|-------------|-------------|----------------------------|
| -0.052813   | -0.882624   | 1.075920                   |

6. Conclusions

In this paper, a BCRLS identification method is proposed for the systems under poor observation condition, which can effectively identify the parameters and noise statistics of the AR model under poor observation condition. The algorithm is simple in form and easy to be applied in engineering. A simulation example shows its effectiveness and feasibility.

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