A unified approach to the design and analysis of AMG

Jinchao Xu, Hongxuan Zhang and Ludmil Zikatanov

May 23, 2017

Abstract

In this work, we present a general framework for the design and analysis of two-level AMG methods. The approach is to find a basis for locally optimal or quasi-optimal coarse space, such as the space of constant vectors for standard discretizations of scalar elliptic partial differential equations. The locally defined basis elements are glued together using carefully designed linear extension maps to form a global coarse space. Such coarse spaces, constructed locally, satisfy global approximation property and by estimating the local Poincaré constants, we obtain sharp bounds on the convergence rate of the resulting two-level methods. To illustrate the use of the theoretical framework in practice, we prove the uniform convergence of the classical two level AMG method for finite element discretization of a jump coefficient problem on a shape regular mesh.

1 Introduction

Multigrid methods are among the most efficient numerical methods for solving large scale linear systems of equations arising from the discretization of partial differential equations. This type of methods can be viewed as an acceleration of traditional iterative methods based on local relaxation such as Gauss-Seidel and Jacobi methods. The main idea behind multigrid methods is to project the error obtained after applying a few iterations of local relaxation methods onto a coarser grid. Part of the slow-to-converge (the low frequency) error on a finer grid is a relatively high frequency on the coarser grid and such high frequencies can be further corrected by a local relaxation method on the coarser grid. By recursively repeating such a procedure a multilevel iterative process is obtained. A classical example of a multilevel algorithm is known as Geometric Multi-Grid (GMG) method, which converges uniformly with nearly optimal complexity for a large class of problem, especially elliptic boundary problems of 2nd and 4th order as demonstrated in [1, 2, 3, 4, 5, 6, 7, 8, 9].

Despite of their efficiency, however, the GMG methods have their limitations. They depend on a hierarchy of geometric grids which is often not readily available. The Algebraic MultiGrid (AMG) methods were designed in an attempt to address such limitations. They were proposed as means to generalize geometric multigrid methods for systems of equations that share properties with discretized PDEs, such as the Laplace equation, but use unstructured grids in the underlying discretization. The first AMG algorithm in [10] was a method developed under the assumption that such a problem was being solved. Later, the AMG algorithm was generalized using many heuristic extensions to extend its applicability to more general problems and matrices. As a result, a variety of AMG methods have been developed in the last three decades and they have been applied to many practical problems with success. But, unfortunately, a good theoretical understanding of why and how these methods work is still seriously lacking.

One of the first results on two level convergence of AMG methods are found in earlier papers [10, 11]. There have been a lot of research on reflecting the MG theory through algebraic settings: [12, 13, 14]; algebraic variational approach to the two level MG theory [15, 16, 17]. For the two grid convergence, sharper results, including two sided bounds are given in [18] and also considered in [19] and [20]. These
two-level results are more or less a direct consequences of the abstract theory provided in [8, 9, 21]. A survey of these and other related results is found in a recent article [22].

Multilevel results are difficult to establish in general algebraic settings, and most of them are based on either not realistic assumptions or they use geometrical grids to prove convergence. We refer to [23, 24] for results in this direction. Rigorous multilevel results for finite element equations can be derived using the auxiliary space framework, which is developed in [25] for quasi-uniform meshes. More recently multilevel convergence results for adaptively refined grids were shown to be optimal in [26]. A multilevel convergence result on shape regular grids using AMG based on quad-tree (in 2D) and oct-tree (in 3D) coarsening is shown in [27].

In this paper, we focus on the design and analysis of the two level AMG methods. We develop a unified framework and theory that can be used to derive and analyze different algebraic multigrid methods in a coherent manner. We provide a general approach to the construction of coarse space and we prove that under appropriate assumptions the resulting two-level AMG method for the underlying linear system converges uniformly with respect to the size of the problem, the coefficient variation, and the anisotropy. Our theory applies to most existing multigrid methods, including the standard geometric multigrid method [9, 28], the classic AMG [10], energy-minimization AMG [29], unsmoothed and smoothed aggregation AMG [30, 31, 32, 33, 34], and spectral AMGe [35, 36, 37]. As an application, we prove, using our abstract framework, the uniform convergence of the standard two-level classical AMG method for jump coefficient problem.

With very few exceptions, the AMG algorithms have been mostly targeting the solution of symmetric positive definite (SPD) systems. In this paper, we choose to present our studies for a slightly larger class of problems, namely symmetric semi-positive definite (SSPD) systems. This approach is not only more inclusive, but more importantly, the SSPD class of linear systems can be viewed as more intrinsic to the AMG ideas. For example, the design of AMG may be better understood by using local problem (defined on subdomains) with homogeneous Neumann boundary condition, which would amount to an SSPD subsystems.

In short, in this paper we consider AMG techniques for solving a linear system of equations:

\[ Au = f, \]

where \( A \) is a given SSPD operator or sparse matrix, and the problem is posed in a vector space of a large dimension. Furthermore, we will show in §4 and §5 that in most cases, \( A \) can be replaced by an \( M \)-matrix, which we call it \( M \)-matrix relative of \( A \).

## 2 Model elliptic PDE operators

We consider the following boundary value problems

\[ Lu = -\nabla \cdot (\alpha(x) \nabla u) = f, \quad x \in \Omega \]  

(2.1)

where \( \alpha : \Omega \to \mathbb{R}^{d \times d} \) is an SPD matrix function satisfying

\[ \alpha_0 \| \xi \|^2 \leq \xi^T \alpha(x) \xi \leq \alpha_1 \| \xi \|^2, \quad \xi \in \mathbb{R}^d. \]

(2.2)

for some positive constants \( \alpha_0 \) and \( \alpha_1 \). Here \( d = 1, 2, 3 \) and \( \Omega \subset \mathbb{R}^d \) is a bounded domain with boundary \( \Gamma = \partial \Omega \).

A variational formulation for (2.1) is as follows: Find \( u \in V \) such that

\[ a(u, v) = (f, v), \quad \forall v \in V. \]

(2.3)
Here
\[
a(u, v) = \int_{\Omega} (\alpha(x)\nabla u) \cdot \nabla v, \quad (f, v) = \int_{\Omega} fv.
\]
and \(V\) is a Sobolev space that can be chosen to address different boundary conditions accompanying the equation (2.1). One case is the mixed boundary conditions:
\[
u = 0, \quad x \in \Gamma_D, \quad (\alpha \nabla u) \cdot n = 0, \quad x \in \Gamma_N, \tag{2.4}\]
where \(\Gamma = \Gamma_D \cup \Gamma_N\). The pure Dirichlet problem is when \(\Gamma_D = \Gamma\), namely
\[
u = 0, \quad x \in \Gamma, \tag{2.5}\]
while the pure Neumann problem is when \(\Gamma_N = \Gamma\), namely
\[
(\alpha \nabla u) \cdot n = 0, \quad x \in \Gamma. \tag{2.6}\]
We thus have \(V\) as
\[
V = \begin{cases}
H^1(\Omega) = \{v \in L^2(\Omega) : \partial_i v \in L^2(\Omega), i = 1 : d\}; \\
H^1_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}. \tag{2.7}\end{cases}
\]
When we consider a pure Dirichlet problem, \(\Gamma_D = \Gamma\), we denote the space by \(V_h = H^1_D(\Omega)\). In addition, for pure Neumann boundary conditions, the following condition is added to assure the existence of the solution to (2.3):
\[
\int_{\Omega} f = 0. \tag{2.8}\]
One most commonly used model problem is when
\[
\alpha(x) = 1, \quad x \in \Omega, \tag{2.9}\]
which corresponds to the Poisson equation
\[
-\Delta u = f. \tag{2.10}\]
This simple problem provides a good representative model for isotropic problems.

In this paper, we focus on the special case when \(\alpha\) is a scalar and it has discontinuous jumps such as
\[
\alpha(x) = \begin{cases}
\epsilon, & x \in \Omega_1, \\
1, & x \in \Omega_2. \tag{2.11}\end{cases}
\]
The interesting jump coefficient case is when \(\epsilon\) is sufficiently small, and we make such an assumption to investigate the robustness of algorithms with respect to the PDE coefficient variation.

We now give an example of finite element discretization. Given a triangulation \(T_h\) for \(\Omega\), let \(V_h \subset V\) be a finite element space consisting of piecewise linear (or higher order) polynomials with respect to the triangulation \(T_h\). The finite element approximation of the variational problem (2.3) is: Find \(u_h \in V_h\) such that
\[
a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{2.12}\]
Assume \(\{\phi_i\}_{i=1}^N\) is the nodal basis of \(V_h\), namely, \(\phi_i(x_j) = \delta_{ij}\) for any nodes \(x_j\). We write \(u_h(x) = \sum_{j=1}^{N} \mu_j \phi_j(x)\) the equation (2.12) is then equivalent to
\[
\sum_{j=1}^{N} \mu_j a(\phi_j, \phi_i) = (f, \phi_i), \quad j = 1, 2, \cdots, N,
\]
which is a linear system of equations:

\[ A \mu = b, \quad (A)_{ij} = a(\phi_j, \phi_i), \quad \text{and} \quad (b)_i = (f, \phi_i). \]

(2.13)

Here, the matrix \( A \) is known as the stiffness matrix of the nodal basis \( \{\phi_i\}_{i=1}^N \).

For any \( T \in \mathcal{T}_h \), we define

\[ h_T = \text{diam}(T), \quad \text{and} \quad h_T = 2 \sup \{r > 0 : B(x, r) \subset T \text{ for } x \in T\}. \]

(2.14)

We say that the mesh \( \mathcal{T}_h \) is \textit{shape regular} if there exists a uniformly bounded constant \( \sigma \geq 1 \) such that

\[ h_T \leq h_T \leq \sigma h_T, \quad \forall T \in \mathcal{T}_h. \]

(2.15)

And we call \( \sigma \) the \textit{shape regularity constant}.

In the following we assume that the finite element mesh is shape regular.

3 \textbf{An abstract two-level method}

Given a finite dimensional vector space \( V \) equipped with an inner product \( (\cdot, \cdot) \), we consider

\[ Au = f, \]

(3.1)

where \( A : V \mapsto V' \) is symmetric positive definite (SPD) and \( V' \) is the dual of \( V \).

A two-level method for solving (3.1) typically consists of the following components:

1. A smoother \( R : V' \mapsto V \);
2. A coarse space \( V_c \subset V \) linked with \( V \) via a prolongation operator:
   \[ P : V_c \mapsto V. \]
3. A coarse space solver \( B_c : V'_c \mapsto V_c \).

We always assume that \( \bar{R} \) is SPD and hence the smoother \( R \) is always convergent. Furthermore,

\[ \|v\|_A^2 \leq \|v\|_{\bar{R}^{-1}}^2. \]

(3.2)

In the discussion below we need the following inner product

\[ (u, v)_{\bar{R}^{-1}} = (\bar{T}^{-1} u, v)_A = (\bar{R}^{-1} u, v), \quad \bar{T} = \bar{R} A, \]

(3.3)

and the accompanying norm \( \| \cdot \|_{\bar{R}^{-1}} \).

The restriction of (3.1) is then

\[ A_c u_c = f_c, \]

(3.4)

where

\[ A_c = P' A P, \quad f_c = P' f. \]

In an exact two-level method, the coarse space solver \( B_c \) is chosen to be the exact solver, namely \( B_c = A_c^{-1} \). In the case that \( A \) is semi-definite, we use \( N(A) \) to denote the kernel of \( A \) and we always assume that \( N(A) \subset V_c \). When \( N(A) \neq \{0\} \) with a slight abuse of notation, we will still use \( A_c^{-1} \) to denote the pseudo-inverse of \( A_c \), and in such case we have

\[ A_c^{-1} = A_c^\dagger. \]
We will use similar notation for pseudo-inverse of other relevant singular operators and matrices in the rest of the paper.

A typical AMG algorithm is defined in terms of an operator $B : V' \mapsto V$, which is an approximate inverse (a preconditioner) of $A$. The two level MG method is as follows.

**Algorithm 1** A two level MG method

Given $g \in V'$ the action $Bg$ is defined via the following three steps

1. Coarse grid correction: $w = PB_c P' g$.
2. Post-smoothing: $Bg := w + R(g - Aw)$.

The error propagation operator for two-level AMG operator $E = I - BA$ is

$$E = (I - RA)(I - \Pi_c),$$  \hspace{1cm} (3.5)

where $\Pi_c = PA^{-1} p^T A$, which is the $(\cdot, \cdot)_A$ orthogonal projection on $V_c$.

The following convergence result is shown in [38] for semi-definite operators $A$ and is an improvement of the well-known two level convergence estimates considered in [21, 20].

**Theorem 3.1** Assume that $N \subset V_c$. The convergence rate of an exact two level AMG is given by

$$\|E\|_A^2 = 1 - \frac{1}{K(V_c)},$$  \hspace{1cm} (3.6)

where

$$K(V_c) = \max_{v \in V} \min_{v_c \in V_c} \frac{\|v - v_c\|_{R^{-1}}^2}{\|v\|_A^2},$$ \hspace{1cm} (3.7)

For a given smoother $R$, one basic strategy in the design of AMG is to find a coarse space such that $K(V_c)$ is made as practically small as possible. There are many cases, however, in which the operator $\bar{R}^{-1}$ in the definition of $K(V_c)$ is difficult to work with. It is then convenient to replace $\bar{R}^{-1}$ by a simpler and spectrally equivalent SPD operator. More specifically, we assume that $D : V \mapsto V'$ is an SPD operator such that

$$c_D \|v\|_D^2 \leq \|v\|_{\bar{R}^{-1}}^2 \leq c_D \|v\|_D^2, \ \forall v \in V,$$ \hspace{1cm} (3.8)

where

$$(u, v)_D = (Du, v), \quad \|v\|_D^2 = (v, v)_D.$$  

As a rule, the norm defined by $\bar{R}$ corresponding to the symmetric Gauss-Seidel method, i.e. $R$ defined by pointwise Gauss-Seidel method can be replaced by the norm defined by the diagonal of $A$ (i.e. by Jacobi method, which, while not always convergent as a relaxation provides an equivalent norm). For additional details on this equivalence, we refer to [18].

Now, in terms of this operator $D$, we introduce the following quantity

$$K(V_c, D) = \max_v \frac{\|v - Q_D v\|_D^2}{\|v\|_A^2} = \max_{v_c \in V_c} \min_v \frac{\|v - v_c\|_D^2}{\|v\|_A^2},$$  \hspace{1cm} (3.9)

where $Q_D : V \mapsto V_c$ is the $(u, v)_D$-orthogonal projection. By (3.7), (3.9) and (3.8), we have

$$c_D K(V_c, D) \leq K(V_c) \leq c_D K(V_c, D).$$  \hspace{1cm} (3.10)

The following theorem presents the two sided bounds on the convergence rate of the two level methods depending on the constants involved in (3.8).
Theorem 3.2 The two level algorithm satisfies

\[ 1 - \frac{1}{c_D K(V_c, D)} \leq \|E\|_A^2 \leq 1 - \frac{1}{c_D K(V_c, D)} \leq 1 - \frac{1}{c_D C}. \]  

(3.11)

where \( C \) is any upper bound of \( K(V_c, D) \), namely

\[ \min_{w \in V_c} \|v - w\|_D^2 \leq C \|v\|_A^2, \quad \text{for all} \quad v \in V. \]  

(3.12)

The proof of the above theorem is straightforward and indicates that, if \( c_D \) and \( c_D \) are “uniform” constants, the convergence rate of the two-level method is “uniformly” dictated by the quantity \( K(V_c, D) \).

4 M-matrix relatives

Our results on M-matrix relatives are related to the some of the works on preconditioning by Z-matrices and L-matrices [39,40]. They are implicitly used in most of the AMG literature [11] where the classical strength of connection definition gives an M-matrix.

In this paper, a symmetric matrix \( A \in \mathbb{R}^{n \times n} \) is called an M-matrix if it satisfies the following three properties:

\( a_{ii} > 0 \) for \( i = 1, \ldots, n \), \hspace{1cm} (4.1)

\( a_{ij} \leq 0 \) for \( i \neq j, \ i, \ j = 1, \ldots, n \), \hspace{1cm} (4.2)

\( A \) is semi-definite. \hspace{1cm} (4.3)

An important remark is in order: We have used the term M-matrix to denote semidefinite matrices, and we are aware that this is not the precise definition. It is however convenient to use reference to M-matrices and we decided to relax a bit the definition here with the hope that such an inaccuracy pays off by better appeal to the reader.

As first step in creating space hierarchy the majority of the AMG algorithms for \( Au = f \) with positive semidefinite \( A \) uses a simple filtering of the entries of \( A \) and construct an M-matrix which is then used to define crucial AMG components. We next define such M-matrix relative.

Definition 4.1 (M-matrix relative) We call a matrix \( A_M \) an M-matrix relative of \( A \) if \( A_M \) is an M-matrix and satisfies the inequalities

\[ (v, v)_A \leq (v, v)_A, \quad \text{and} \quad (v, v)_D \leq (v, v)_{D_M}, \quad \text{for all} \quad v \in V, \]  

(4.4)

where \( D_M \) and \( D \) are the diagonals of \( A_M \) and \( A \) respectively.

We point out that the M-matrix relatives are instrumental in the definition of coarse spaces and also in the convergence rate estimates. This is clearly seen later in §6 where we present the unified two level theory for AMG. Often, we have that the one sided inequality in (4.4) is in fact a spectral equivalence.

By definition, we have the following simple but important result.

Lemma 4.2 Let \( A_M \) be an M-matrix relative of \( A \) and let \( D \) and \( D_M \) be the diagonal matrices of \( A \) and \( A_M \), respectively. If \( V_c \subset V \) is a subspace, then the estimate

\[ \|u - u_c\|_D^2 \leq \|u\|_A^2 \]  

holds for some \( u_c \in V_c \), if the estimate

\[ \|u - u_c\|_{D_M}^2 \leq \|u\|_{A_M}^2 \]  

holds.
This result, combining with the two-level convergence result, means that we only need to work on the M-matrix relative of \( A \) in order to get the estimate (4.5).

We next describe how to construct M-matrix relatives for a special class of matrices. We first prove an auxiliary result for a special class of matrices defined via bilinear forms

\[
(A_b u, v) := b(u, v) = \sum_{e \in E_b} \omega_e(\delta_v u)(\delta_v v).
\]

Here \( E_b \) is the set of edges of a connected graph with vertices \( \{1, \ldots, k\} \) and \( b(\cdot, \cdot) \) is the bilinear form corresponding to a weighted graph Laplacian. The other quantities in (4.7) are defined as follows:

For \( e \in E_b, e = (i, j) \), we set \( \delta_v u = (u_i - u_j) \)

Some of the weights in \( b(\cdot, \cdot) \) may be negative, but they should not dominate: we assume that \( b(\cdot, \cdot) \) is positive semidefinite with one dimensional kernel spanned by \((1, \ldots, 1)^T\). If the weights \( \omega_e \) were positive then it is easy to show that this assumption holds. Indeed, the bilinear form is obviously semidefinite and the second part of the assumption follows from the fact that the graph is connected. Thus, there exists a \( \lambda_b > 0 \) such that for all \( u \in \mathbb{R}^k \) satisfying \( \sum_{i=1}^k u_i = 0 \) we have

\[
\lambda_b \|u\|_2^2 \leq b(u, u).
\]

Let us now denote

\[
E_b^+ = \{ e \in E_b \mid \omega_e > 0 \}, \quad E_b^- = \{ e \in E_b \mid \omega_e \leq 0 \}.
\]

and then split the bilinear form \( b(\cdot, \cdot) \) in positive and negative parts:

\[
b(u, v) = b_+(u, v) - b_-(u, v),
\]

\[
(A_{b,+} u, v) = \sum_{e \in E_b^+} \omega_e \delta_v u \delta_v v, \tag{4.10}
\]

\[
b_-(v, u) = \sum_{e \in E_b^-} |\omega_e| \delta_v u \delta_v v. \tag{4.11}
\]

We observe that \( A_b \) defined via the bilinear form \( b(\cdot, \cdot) \) in (4.7) is an M-matrix relative to itself if \( E_b^- = \emptyset \), or, equivalently, \( E_b^+ = E_b \).

The following lemma gives an estimate of \( b_-(\cdot, \cdot) \) and \( b_+(\cdot, \cdot) \) in terms of \( b(\cdot, \cdot) \) basically showing that \( A_{b,+} \) may be used as M-matrix relative to \( A_b \).

\textbf{Lemma 4.3} If \( \omega_- = \max_{e \in E_b^-} |\omega_e| \) then we have the following inequalities for all \( v \in \mathbb{R}^k \),

\[
b(v, v) \leq b_+(v, v) \leq \left( 1 + \frac{c(k)\omega_-}{\lambda_b} \right) b(v, v), \tag{4.12}
\]

\[
\|v\|_2^2 \leq \|v\|_{D_b^+}^2 \leq \left( 1 + \frac{c(k)\omega_-}{\lambda_b} \right) \|v\|_{D_b^-}^2. \tag{4.13}
\]

where \( D_b \) and \( D_b^+ \) are the diagonals of \( A_b \) and \( A_b^+ \) and \( c(k) = 2(k - 1) \).

\textbf{Proof.} We first show the inequality

\[
b_-(v, v) \leq \frac{c(k)\omega_-}{\lambda_b} b(v, v). \tag{4.14}
\]
Clearly, we only need to consider all \( v \in \mathbb{R}^k \) such that \( \sum_{i=1}^{k} v_i = 0 \), because adding a multiple of \((1, \ldots, 1)^T\) to \( v \) does not change either side of the inequality (4.14). For such \( v \in \mathbb{R}^k \) we have

\[
\begin{align*}
b_-(v, v) & \leq \omega_- \sum_{e \in \mathcal{E}_o} (\delta_e v)^2 \leq \omega_- \sum_{e \in \mathcal{E}_o} (\delta_e v)^2 \\
& \leq 2\omega_- \sum_{(i,j) \in \mathcal{E}} (v_i^2 + v_j^2) \leq 2(k-1)\omega_-||v||_2^2 \\
& \leq \frac{c(k)\omega_-||v||_2^2}{\lambda_b} b(v, v),
\end{align*}
\]

where we have used that a vertex \( i \) cannot be in more than \((k-1)\) edges. This shows (4.12) after some obvious algebraic manipulations.

The inequality (4.13) follows from (4.12). For \( i = 1, 2, \ldots, k \) we have

\[
(D_b)_{ii} = b(\phi_i, \phi_i) \leq b_+(\phi_i, \phi_i) = (D_b^+)_{ii},
\]

\[
(D_b^+)_{ii} = b_+(\phi_i, \phi_i) \leq \left(1 + \frac{c(k)\omega_-}{\lambda_b}\right) b(\phi_i, \phi_i) = \left(1 + \frac{c(k)\omega_-}{\lambda_b}\right) (D_b)_{ii}.
\]

The proof is complete. \( \square \)

5 \quad \textbf{\( M \)-matrix relatives of finite element stiffness matrices}

In this section we show how to construct \( M \)-matrix relative to the matrix resulting from a finite element discretization of the model problem (2.1) with linear elements. We consider first an isotropic problem with pure Neumann boundary condition (2.6) and isotropic coefficient \( a(x) = a(x)I \), \( x \in \Omega \).

In the rest of this section, we make the following assumptions on the coefficient and the geometry of \( \Omega \):

1. The domain \( \Omega \subset \mathbb{R}^d \) is partitioned into simplices \( \Omega = \bigcup_{T \in \mathcal{T}_h} T \).

2. The coefficient \( a(x) \) is a scalar valued function and its discontinuities are aligned with this partition.

3. We consider the Neumann problem, and, therefore, the bilinear form (2.3) is

\[
\int_{\Omega} a(x) \nabla v \cdot \nabla u = \sum_{(i,j) \in \mathcal{E}} (-a_{ij}) \delta_e u \delta_e v = \sum_{e \in \mathcal{E}} \omega_e \delta_e u \delta_e v \quad (5.1)
\]

4. It is well known \([41]\) that the off-diagonal entries of the stiffness matrix \( A \) are given by

\[
\omega_e = - (\phi_j, \phi_i)_A = \sum_{T \ni e} \omega_{e,T}
\]

\[
\omega_{e,T} = \frac{1}{d(d-1) \alpha_T} |\kappa_e, T| \cot \alpha_{e,T}, \quad \alpha_T = \frac{1}{|T|} \int_T a(x) \, dx
\]

Here, \( e = (i,j) \) is a fixed edge with end points \( x_i \) and \( x_j \); \( T \ni e \) is the set of all elements containing \( e \); \( |\kappa_e, T| \) is the volume of \((d-2)\)-dimensional simplex opposite to \( e \) in \( T \); \( \alpha_{e,T} \) is the dihedral angle between the two faces in \( T \) not containing \( e \).

5. Following the notation used in Lemma 4.3, let \( \mathcal{E} \) denote the set of edges in the graph defined by the triangulation and let \( \mathcal{E}^- \) be the set of edges where \( a_{ij} \geq 0 \), \( i \neq j \). The set complementary to \( \mathcal{E}^- \) is \( \mathcal{E}^+ = \mathcal{E} \setminus \mathcal{E}^- \). Then, with \( \omega_e = -a_{ij} \), we have

\[
\int_{\Omega} a(x) \nabla v \cdot \nabla u = \sum_{e \in \mathcal{E}^+} \omega_e \delta_e u \delta_e v - \sum_{e \in \mathcal{E}^-} \omega_e |\delta_e u \delta_e v|. \quad (5.2)
\]
6. We also assume that the partitioning is such that the constant function is the only function in the null space of the bilinear form (5.1). This is, of course, the case when \( \Omega \) is connected (which is true, as \( \Omega \) is a domain).

The non-zero off-diagonal entries of \( A \) may have either positive or negative sign, and, usually \( E^- \neq \emptyset \). The next theorem shows that the stiffness matrix \( A \) defined via the bilinear form (5.1) is spectrally equivalent to the matrix \( A_+ \) defined as

\[
(A_+, u, v) = \sum_{e \in E} \omega_e (u_i - u_j)(v_i - v_j).
\]

(5.3)

Thus, we can ignore any positive off-diagonal entries in \( A \), or equivalently, we may drop all \( \omega_e \) for \( e \in E^- \). Indeed, \( A_+ \) is obtained from \( A \) by adding to the diagonal all positive off diagonal elements and setting the corresponding off-diagonal elements to zero. This is a stronger result that we need later in the convergence theory because it gives not only the inequalities (4.4) but also a spectral equivalence with the \( M \)-matrix relative \( A_+ \).

**Theorem 5.1** If \( A \) is the stiffness matrix corresponding to linear finite element discretization of (2.1) with boundary conditions given by (2.6), then \( A_+ \) is an \( M \)-matrix relative of \( A \) which is spectrally equivalent to \( A \). The constants of equivalence depend only on the shape regularity of the mesh. Moreover, the graph corresponding to \( A_+ \) is connected.

**Proof.** The goal is to show that

\[
||u||^2_{A_+} \leq ||u||^2_{A} \leq ||u||^2_{A_+},
\]

where the constants hidden in \( \leq \) depend only on the shape regularity of the mesh.

The lower bound is clear, by just comparing (5.2) and (5.3). As we discussed earlier in Lemma 4.3 such inequality shows that the graph corresponding to \( A_+ \) is connected. Indeed, since \( ||u||^2_{A} \) vanishes only for \( u = (1, \ldots, 1)' \) it follows that \( ||u||^2_{A_+} \) also vanishes only for \( u = (1, \ldots, 1)' \) which proves that \( A_+ \) has only one connected component.

To prove the upper bound, we fix an element \( T \) and consider the local stiffness matrix \( A_T \) given by

\[
(A_T u, v) = b_T(u, v) = \sum_{e \in \partial T} \omega_e (\delta_e u)(\delta_e v).
\]

Denote \( E_T = \{(i, j) \mid 1 \leq i < j \leq (d + 1)\} \), and let \( E_T^+ \) corresponding to \( A_T \) be defined in a way analogous to the definition of \( E^+ \) for \( A \).

It is immediate to see that in the notation of Lemma 4.3 the minimum nonzero eigenvalue and the maximum in modulus negative coefficient \( \omega_{-T} \) satisfy:

\[
\lambda_T = \lambda_{\min}(A_T) \equiv h_T^{d-2}\tilde{\alpha}_T, \quad \text{and} \quad \omega_{-T} = \max_{e \in E_T} |\omega_e| \leq h_T^{d-2}\tilde{\alpha}_T.
\]

These relations hold with constants independent of the mesh size \( h_T \), but dependent on the shape regularity of the mesh. Let us consider the bilinear form

\[
b_{+, T}(u, v) = \sum_{e \in E_T^+} \omega_e (\delta_e u)(\delta_e v).
\]

Lemma 4.3 then implies that for every \( T \), \( b_{+, T}(u, v) \) is spectrally equivalent to \( b_T(u, v) \) and the constants of spectral equivalence depend only on the shape regularity of the mesh. Summing over all elements and using this spectral equivalence then gives:

\[
\sum_T \sum_{e \in E_T^+} \omega_e (\delta_e u)^2 \leq \sum_T \sum_{e \in E_T} \omega_e (\delta_e u)^2 = ||u||^2_{A}.
\]
On the other hand we have

\[ \|u\|_{A_e}^2 = \sum_{e \in E^+} \omega_e (\delta_e u)^2 = \sum_{e \in E^+} \max \{0, \omega_e\} (\delta_e u)^2 = \sum_{T} \max \left\{ 0, \sum_{e \in E_T^+} \omega_e \right\} (\delta_e u)^2 \]

\[ \leq \sum_{T} \sum_{e \in E_T^+} \max \{0, \omega_e\} (\delta_e u)^2 = \sum_{T} \sum_{e \in E_T^+} \omega_e (\delta_e u)^2. \]

Combining these inequalities complete the proof.

A simple corollary which we use later in proving estimates on the convergence rate is as follows.

**Corollary 5.2** Assume that \( A \) is the stiffness matrix for piece-wise linear discretization of equation \( (5.1) \) and \( A_+ \) is the \( M \)-matrix relative defined in Theorem \( 5.1 \). Then the diagonal \( D \) of \( A \) and the diagonal \( D_+ \) of \( A_+ \) are spectrally equivalent.

**Proof.** For the diagonal elements of \( A \) and \( A_+ \) we have

\[ [D]_{jj} = (\phi_j, \phi_j)_A \equiv (\phi_j, \phi_j)_{A_+} = [D_+]_{jj}. \]

The equivalence “\( \equiv \)” written above follows from Lemma \( 5.1 \).

Corollary 5.2 together with Lemma 4.2 provide the theoretical foundation for utilizing \( M \)-matrix relatives in the design of AMG methods for linear systems with finite element matrices.

### 6 A general approach to the construction of coarse spaces

We assume there exists a sequence of spaces \( V_1, V_2, \ldots, V_J \), which are not necessarily subspaces of \( V \), but each of them is related to the original space \( V \) by a linear operator

\[ \Pi_j : V_j \mapsto V. \quad (6.1) \]

Our very basic assumption is that the following decomposition holds:

\[ V = \sum_{j=1}^{J} \Pi_j V_j. \]

This means that for any \( v \in V \), there exists \( v_j \in V_j \) (which may not be unique) such that

\[ v = \sum_{j=1}^{J} \Pi_j v_j. \]

Denote

\[ W = V_1 \times V_2 \times \ldots \times V_J, \]

with the inner product

\[ (u, v) = \sum_{i=1}^{J} (u_i, v_i), \]

where \( u = (u_1, \ldots, u_J)^T \) and \( v = (v_1, \ldots, v_J)^T \). Or more generally, for \( f = (f_1, \ldots, f_J)^T \in V' \) with \( f_i \in V'_i \), we can define

\[ (f, v) = \sum_{i=1}^{J} (f_i, v_i). \]
We now define $\Pi_W : W \mapsto V$ by

$$\Pi_W u = \sum_{i=1}^{J} \Pi_i u_i, \quad \forall u = (u_1, ..., u_J)^T \in W.$$ 

Formally, we can write

$$\Pi_W = (\Pi_1, ..., \Pi_J) \quad \text{and} \quad \Pi_W' = \begin{pmatrix} \Pi_1' \\ \vdots \\ \Pi_J' \end{pmatrix}.$$ 

We assume there is an operator $A_j : V_j \mapsto V'_j$ which is symmetric, positive semi-definite for each $j$ and define $A_W : W \mapsto W'$ as follows

$$A_W := \text{diag}(A_1, A_2, ..., A_J). \quad (6.2)$$

For each $j$, we assume there is a symmetric positive definite operator $D_j : V_j \mapsto V'_j$, and define $D_W : W \mapsto W'$ as follows

$$D := \text{diag}(D_1, D_2, ..., D_J). \quad (6.3)$$

We associate a coarse space $V^c_j$, $V^c_j \subset V_j$, with each of the spaces $V_j$, and consider the corresponding orthogonal projection $Q_j : V_j \mapsto V'_j$ with respect to $(\cdot, \cdot)_{D_j}$. We define $Q : W \mapsto W'$ by

$$Q := \text{diag}(Q_1, Q_2, ..., Q_J). \quad (6.4)$$

**Assumption 6.5**

1. The following inequality holds for all $w \in W$:

$$\| \Pi_W w \|^2_{D} \leq C_{p,2} \| w \|^2_{D}, \quad (6.6)$$

for some positive constant $C_{p,2}$.

2. For each $w \in V$, there exists a $w \in W$ such that $w = \Pi_W w$ and the following inequality holds

$$\| w \|^2_{D_W} \leq C_{p,1} \| w \|^2_{A}, \quad (6.7)$$

with a positive constant $C_{p,1}$ independent of $w$.

3. For all $j$,

$$N(A_j) \subset V^c_j. \quad (6.8)$$

**Remark 6.1** The above assumption implies that

$$w \in N(A) \Rightarrow w \in N(A_1) \times \ldots \times N(A_J).$$

We define the global coarse space $V_c$ by

$$V_c := \sum_{j=1}^{J} \Pi_j V^c_j. \quad (6.9)$$

Further, for each coarse space $V^c_j$, we define

$$\mu^{-1}_j(V^c_j) := \max_{v_j \in V^c_j} \min_{v'_j \in V^c_j} \frac{\| v_j - v'_j \|^2_{D_j}}{\| v_j \|^2_{A_j}}, \quad (6.10)$$
and
\[ \mu_c = \min_{1 \leq j \leq J} \mu_j(V_j), \] (6.11)
which is finite, thanks to Assumption 6.5 (namely, (6.3)).

By the two level convergence theory, if \( D_j \) provides a convergent smoother, then \((1 - \mu_j(V_j'))\) is the convergence rate for two-level AMG method for \( V_j \) with coarse space \( V_j' \). Next theorem gives an estimate on the convergence of the two level method in terms of the constants from Assumptions 6.5 and \( \mu_c \).

**Theorem 6.2** If Assumption 6.5 holds, then for each \( v \in V \), we have the following error estimate
\[ \min_{v_c \in V_c} ||v - v_c||_D^2 \leq C_{p,1} C_{p,2} \mu_c^{-1} ||v||_A^2. \] (6.12)

**Proof.** By Assumption 6.5, for each \( v \in V \), there exists \( v \in V \) such that
\[ v = \Pi_V v. \] (6.13)
and (6.7) is satisfied.

By the definition of \( \mu_c \), we have
\[ ||v - Qv||_D^2 \leq \mu_c^{-1} ||v||_A^2. \] (6.14)
We let \( v_c = \Pi_V Qv \). Then \( v_c \in V_c \) and by Assumption 6.5, we have
\[ ||v - v_c||_D^2 = ||\Pi_V (v - Qv)||_D^2 \leq C_{p,2} ||v - Qv||_D^2 \leq C_{p,2} \mu_c^{-1} ||v||_A^2 \leq C_{p,1} C_{p,2} \mu_c^{-1} ||v||_A^2. \]

We define another product space
\[ V := V_c \times V_1 \times V_2 \times \cdots \times V_J, \] (6.15)
and we set \( \Pi : V_c \mapsto V \) to be the natural inclusion from \( V_c \) to \( V \). Then we define \( \Pi : V \mapsto V \) by
\[ \Pi := (\Pi_c \Pi_1 \Pi_2 \cdots \Pi_J), \] (6.16)
and \( A : V \mapsto V' \) by
\[ A := \begin{pmatrix} A_c & A_1 & \cdots & A_J \\ \end{pmatrix}, \] (6.17)
where \( A_c : V_c \mapsto V_c' \) is given as
\[ A_c := \Pi_c' A \Pi_c. \] (6.18)
And \( B : V \mapsto V' \) is given as
\[ B := \begin{pmatrix} A_c^{-1}D_1^{-1} & & \\ & D_2^{-1} & \\ & & \ddots \\ & & & D_J^{-1} \end{pmatrix} \] (6.19)

We introduce the additive preconditioner \( \widehat{B} \)
\[ \widehat{B} := \Pi_B \Pi' = \Pi_c A_c^{-1} \Pi_c' + \sum_{j=1}^J \Pi_j D_j^{-1} \Pi_j', \] (6.20)
and we have the following lemma.
Lemma 6.3 If Assumption 6.5 holds, then for any \( v \in V \), there exists \( \nu \in V \) such that

\[
\|v\|_{B^{-1}} \leq \tilde{\mu}_0 \|v\|_A
\]

with \( \tilde{\mu}_0 \) being a constant depending on \( C_{p,1}, C_{p,2}, \mu_c \) and \( c^D \).

Proof. By Assumption 6.5 for each \( v \in V \), there exists \( w = (v_1 \cdots v_J)^T \in W \) such that \( v = \Pi_W w \) and (6.7) holds, namely,

\[
\|w\|_{\Delta_w}^2 \leq C_{p,1}\|v\|_A^2.
\]

We then define \( \nu \in V \) by

\[
\nu := \left( v_c, w - Q_w \right),
\]

with \( v_c := \Pi_w Q w \). Obviously, we have \( \Pi_V = v \), and \( v_c \) satisfies

\[
\|v - v_c\|_D^2 \leq C_{p,1}C_{p,2}\mu_c^{-1}\|v\|_A^2.
\]

By Theorem 6.2 we have

\[
\|v_c\|_A^2 \leq 2\|v - v_c\|_A^2 + 2\|v\|_A^2 \leq 2\|v - v_c\|^2_{R^{-1}} + 2\|v\|_A^2 \leq 2c^D\|v - v_c\|_A^2 + 2\|v\|_A^2 \leq 2c^D C_{p,1}C_{p,2}\mu_c^{-1}\|v\|_A^2 + 2\|v\|_A^2.
\]

Then we have

\[
(B^{-1}\nu, \nu) = \|w - Q w\|_D^2 + \|v_c\|_A^2 \leq \mu_c^{-1}\|\Pi_W w\|_A^2 + \|v_c\|_A^2 \leq C_{p,1}\mu_c^{-1}\|v\|_A^2 + 2(c^D C_{p,1}C_{p,2} + 1)\|v\|_A^2 = (C_{p,1}\mu_c^{-1} + 2c^D C_{p,1}C_{p,2} + 2)\|v\|_A^2.
\]

Lemma 6.4 If Assumption 6.6 holds, then the following inequality holds for all \( \nu \in V \)

\[
\|\Pi_V\|_A \leq \tilde{\mu}_1 \|\nu\|_{B^{-1}},
\]

with constant \( \tilde{\mu}_1 \) depends on \( C_{p,2} \) and \( c^D \).

Proof. For any decomposition \( v = \Pi_V = \Pi_W w + v_c \), we have

\[
(B^{-1}\nu, \nu) = \|w\|_{\Delta_w}^2 + \|v_c\|_A^2 \geq \frac{1}{C_{p,2}}\|\Pi_W w\|_D^2 + \|v_c\|_A^2 = \frac{1}{C_{p,2}}\|v - v_c\|_D^2 + \|v_c\|_A^2 \geq \frac{1}{c^D C_{p,2}}\|v - v_c\|_{R^{-1}}^2 + \|v_c\|_A^2 \geq \frac{1}{c^D C_{p,2}}\|v - v_c\|_A^2 + \|v_c\|_A^2 \geq C(c^D, C_{p,2})\|v\|_A^2.
\]

Combining Lemma 6.3 and Lemma 6.4, we immediately have the following bound on the condition number of the preconditioned system.
Theorem 6.5 If Assumption \[6.5\] holds, then
\[
\kappa(\hat{B}A) \leq \left(\frac{\mu_1}{\mu_0}\right)^2.
\] (6.23)

The following two-level convergence result is an application of the convergence theorem (Theorem \[3.1\]) with the error estimate in Theorem \[6.2\].

**Theorem 6.6** If Assumption \[6.5\] holds. Then the two-level AMG method with coarse space defined in \((6.9)\) converges with a rate
\[
\|E\|_A^2 \leq 1 - \frac{\mu_c}{C_{p,1}C_{p,2}c^D}.
\]

7 Classical AMG and jump coefficient problems

In this section we consider the Classical AMG method when applied to a problem with heterogenous (jump) coefficients, namely \((2.1)\) with \((2.11)\). We begin with a discussion on how the strength of connection is used to define the sparsity pattern of the prolongation.

The strength of connection measure was introduced to handle cases such as jump coefficients and anisotropies in the matrices corresponding to discretizations of scalar PDEs. An important observation regarding the classical AMG is that the prolongation matrix \(P\), which defines the basis in the coarse space, uses only strong connections.

To begin with, we first introduce a strength operator as follows
\[
s_c(i,j) = \frac{a_{ij}}{\max\left(\min_{k \neq j} a_{ik}, \min_{k \neq j} a_{jk}\right)}, \quad 1 \leq i, j \leq n.
\] (7.1)

The definition above is symmetrized version of strength function used in the classical AMG literature.

Given a threshold \(\theta \in (0, 1)\), we define the strength operator
\[
S = \sum_{s_c(i,j) > \theta} e_i e_j^T,
\] (7.2)

and a filtered matrix \(A_S: V \mapsto V'\)
\[
(A_S u, v) = \sum_{e=(i,j), S_{ij} \neq 0} \omega_e \delta_{e} u \delta_{e} v.
\] (7.3)

We have the following lemma

**Lemma 7.1** \(A_S\) is an \(M\)-matrix relative of \(A\).

**Proof.** We recall the definition of \(A_+\) in \((5.3)\). By Theorem \[5.1\] we immediately have
\[
\|v\|_{A_S} \leq \|v\|_{A_+} \leq \|v\|_A, \quad \forall v \in V.
\] (7.4)

Let \(D_S\) be the diagonal of \(A_S\) and we denote the \(i\)-th diagonal entries of \(D_S\) and \(D\) by \(\tilde{d}_i\) and \(d_i\) respectively. Then, by the definition of the strength of connection, we have
\[
\tilde{d}_i = \sum_{j \in N_i, s_c(i,j) \geq \theta} \omega_{ij} \geq \theta \sum_{j \in N_i} \max_{j \in N_i} \omega_{ij} \geq \frac{\theta}{|N_i|} \sum_{j \in N_i} d_j \geq \frac{\theta}{|N_i|} d_i.
\]
This gives us

$$\|v\|^2_D \leq \frac{\max \{ |N_i| \} \|v\|^2_D}{\theta}, \quad \forall v \in V.$$  \hspace{1cm} (7.5)

This completes the proof. \(\square\)

Thanks to the results in §4 and §5 without loss of generality, we assume that \(A\) is an \(M\)-matrix with all connections being strong connections.

We then use an MIS algorithm to identify \(C\), the set of coarse points, to form a \(C/F\)-splitting, and

$$C \cup \mathcal{F} = \Omega := \{1, \ldots, n\}, \quad C \cap \mathcal{F} \neq \emptyset.$$  \hspace{1cm} (7.6)

For convenience, we reorder the indices so that \(C = \{1, \ldots, J\} \).

$$\Omega = \bigcup_{j=1}^{J} \Omega_j.$$  \hspace{1cm} (7.7)

where \(\Omega_j\) is defined for each \(j \in C\) as follows

$$\Omega_j := \{j\} \bigcup F^j, \quad j = 1, \ldots, J.$$  \hspace{1cm} (7.8)

Here \(F^j := \mathcal{F} \cap s_j\), and \(s_j\) is the set of interpolation neighbors of \(j\). This depends on the choice of interpolation. For example, in the direct interpolation we introduced in \([10, 42]\), \(s_j\) is \(N_j\), the set of neighbors of \(j\); in the standard interpolation \([10, 42]\),

$$s_j = N_j \bigcup \bigcup_{i \in N_j} N_i.$$  \hspace{1cm} (7.9)

In the discussion follows, we choose the standard interpolation, since the extension to other interpolations is trivial.

For each \(\Omega_j\) we denote

$$\Omega_j = \{m_1, m_2, \ldots, m_{n_j}\},$$  \hspace{1cm} (7.10)

and let \(n_j := |\Omega_j|\), namely, \(n_j\) is the cardinality of \(\Omega_j\). In accordance with the notation in §6 We then define

$$V_j := \mathbb{R}^{n_j},$$  \hspace{1cm} (7.11)

and the associated operator \(\Pi_j : V_j \mapsto V\)

$$(\Pi_j v)_i = \begin{cases} p_{m_k,k} v_k, & \text{if } i = m_k, \\ 0, & \text{if } i \not\in \Omega_j \end{cases},$$  \hspace{1cm} (7.12)

where \(p_{m_k,k}\) are given weights. As all the constructions below will be based on the \(M\)-matrix relative of \(A\), and without loss of generality, we may just use \(A\) to denote this \(M\)-matrix relative.

Following §6, we introduce the operator \(\chi_j : V \mapsto V_j\):

$$(\chi_j v)_i := v_{m_i}.$$  \hspace{1cm} (7.13)

which takes as argument a vector \(v\) and returns only the portion of it with indices in \(\Omega_j\), namely, \(\chi_j v\), is a vector in \(\mathbb{R}^{n_j}\). It is immediate to verify that

$$\sum_{j=1}^{J} \Pi_j \chi_j = I.$$
The local operators $A_j : V_j \mapsto V'_j$ are defined as follows
\[
(A_j, v) = \sum_{e \in E \subset \Omega_j} \omega_e \delta_{e,v}. \tag{7.13}
\]
Here, $e \in \Omega_j$ means the two vertices connected by $e$ are in $\Omega_j$. Notice that $A_j$ is symmetric positive semi-definite.

**Lemma 7.2** For any $v \in V$, the following holds for $v_j = \chi_j v$
\[
\sum_{j=1}^{J} \Pi_j v_j = v, \text{ and } \sum_{j=1}^{J} \|v_j\|_{A_j}^2 \leq C_o \|v\|_{A}^2, \tag{7.14}
\]
where $C_o$ is a constant depending on the overlaps in the partition $\{\Omega_j\}_{j=1}^{J}$
\[
C_o = \max_{1 \leq i \leq J} \left| \{l : \Omega_l \cap \Omega_j \neq \emptyset\} \right|. \tag{7.15}
\]

**Proof.** By (7), we have $\sum_{j=1}^{J} \Pi_j v_j = v$. By definitions
\[
\sum_{j=1}^{m_j} \|v_j\|_{A_j}^2 = \sum_{j=1}^{m_j} \sum_{e \subset \Omega_j} \omega_e \delta_{e,v}^2 \leq C_o \sum_{e \subset \Omega_j} \omega_e \delta_{e,v}^2 = C_o \|v\|_{A}^2. \tag{7.16}
\]
This completes the proof. \[\]
If $D$ is the diagonal of $A$, then we set $D_j$, $j = 1 : J$ to be the restriction of $D$ on $\Omega_j$, namely, in $\mathbb{R}^{n_j \times n_j}$ and
\[
(D_j)_{ii} = D_{m_i,m_i}, \text{ or equivalently } \ D_j = \chi_j D \chi' \tag{7.17}
\]
We have the following lemma which shows (6.6).

**Lemma 7.3** For $D_j$ defined in (7.17), the following inequality holds
\[
\| \sum_{j=1}^{J} \Pi_j v_j \|_D \leq C_o \sum_{j=1}^{m_j} \|v_j\|_{D_j}^2, \quad \forall v_j \in V_j. \tag{7.18}
\]

**Proof.** Recall from the definition of $\Pi_j$, we have
\[
\| \Pi_j v_j \|_D \leq \|v_j\|_{D_j}, \quad \forall v_j \in V_j. \tag{7.19}
\]
Therefore,
\[
\| \sum_{j=1}^{J} \Pi_j v_j \|_D^2 = \left( D \sum_{i=1}^{J} \Pi_i v_i, \sum_{j=1}^{J} \Pi_j v_j \right) = \sum_{i=1}^{J} \sum_{j=1}^{J} (D \Pi_i v_i, \Pi_j v_j) = \sum_{1 \leq i, j \leq J} \left( \Omega_i \cap \Omega_j \neq \emptyset \right) \sum_{1 \leq i, j \leq J} \left( \Omega_i \cap \Omega_j \neq \emptyset \right) \|\Pi_i v_i\|_{D_j}^2 + \|\Pi_j v_j\|_{D_j}^2 \leq C_o \sum_{j=1}^{J} \|v_j\|_{D_j}^2.
\]
We consider the following graph Laplacian on a connected undirected graph

\[ \mathcal{L} = \sum_{(i,j) \in \mathcal{E}} (d_{ij}^{-1} A_{ij}) \]

Lemma 7.4: We choose the local coarse spaces \( V^c_j \) as

\[ V^c_j := \text{span}\{1_{n_j}\}, \quad 1_{n_j} = (1, 1, \ldots, 1)^T \]  \hspace{1cm} (7.20)

Then by definition, we have

\[ \mu_j(V^c_j) = \lambda_j^{(2)}, \]  \hspace{1cm} (7.21)

where \( \lambda_j^{(2)} \) is the second smallest eigenvalue of the matrix \( D_j^{-1} A_j \). The global coarse space \( V_c \) is then obtained by (6.9), and is

\[ V_c = \text{span}\{P_1, P_2, \ldots, P_j\}. \]  \hspace{1cm} (7.22)

Finally, by Theorem 3.1, the convergence rate of this two-level geometric multigrid method depends on the

\[ \min_j \lambda_j^{(2)}. \]  \hspace{1cm} (7.23)

with \( c_j \) to be a constant, then the two-level classical AMG method converges uniformly.

We now consider the convergence of classical two-level AMG with standard interpolation for the jump coefficient problem and we prove a uniform convergence result for the two level method. Before we go through the AMG two-level convergence proof, we first introduce the following result on a connected graph, which can be viewed as a discrete version of Poincaré inequality.

**Lemma 7.4** We consider the following graph Laplacian on a connected undirected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \)

\[ \langle Au, v \rangle = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} (u_i - u_j)(v_i - v_j). \]  \hspace{1cm} (7.24)

For any \( v \in \mathcal{V} \), the following estimate is true

\[ \|v - v_c\|^2_{L^2} \leq \mu n^2 d(Av, v), \]  \hspace{1cm} (7.25)

where \( n = |\mathcal{V}| \) is the size of the graph, \( v_c = \sum_{j=1}^{n} w_j v_j \) is a weighted average of \( v \), \( \mu = \sum_{j=1}^{n} w_j^2 \), and \( d \) is the diameter of the graph.

**Proof.** Since \( \mathcal{G} \) is connected, we have, for each pair of vertices \( i \) and \( j \), there exist \( l \leq d \) and a path \( k_0 \rightarrow k_1 \rightarrow \cdots \rightarrow k_l \) with \( k_0 = i \) and \( k_l = j \) such that \((k_{m-1}, k_m) \in \mathcal{E}, \forall m = 1, \ldots, l\). We then have

\[ (v_i - v_j)^2 = \left( \sum_{m=1}^{l} (v_{k_{m-1}} - v_{k_m}) \right)^2 \leq l \sum_{m=1}^{l} (v_{k_{m-1}} - v_{k_m})^2 \leq d(Av, v). \]

Combining this with Cauchy-Schwarz inequality, we obtain

\[ \|v - v_c\|^2_{L^2} = \sum_{i=1}^{n} \left( v_i - \sum_{j=1}^{n} w_j v_j \right)^2 = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} w_j (v_i - v_j) \right)^2 \leq \mu \sum_{i=1}^{n} \sum_{j=1}^{n} (v_i - v_j)^2 \leq \mu n^2 d(Av, v). \]
Next Lemma is a spectral equivalence result, showing that the local operators $A_j$, defined in \((7.13)\), for shape regular mesh, are spectrally equivalent to a scaling of the graph Laplacian operators $A_{L,j}$ defined as
\[
(A_{L,j}u, v) = \frac{1}{2} \sum_{(i,k) \in \Omega_j} (u_i - u_k)(v_i - v_k).
\] (7.26)

**Lemma 7.5** With the assumption we made on the shape regularity of the finite element mesh, the following inequalities hold for $A_j$ defined as in \((7.13)\) using the standard interpolation
\[
c_L h^{d-2} \langle A_{L,j}v_j, v_j \rangle \leq \langle A_j v_j, v_j \rangle \leq c_L h^{d-2} \langle A_{L,j}v_j, v_j \rangle,
\] (7.27)
where $A_{L,j}$ is a graph Laplacian defined in \((7.24)\) on the graph $G_j$, $h$ is the mesh size and $c_L, c_L$ are constants depend on the shape regularity constant, and the threshold $\theta$ for the strength of connections.

**Proof.** By the definition of the strength of connection, we have
\[
a_{ii} = \sum_{k \in N_k} -a_{ik} \leq -\frac{|N_i|}{\theta} a_{ij},
\]
Since $A$ is symmetric, we also have
\[
a_{ii} \leq \frac{|N_i|}{\theta} a_{ji},
\]
By the definition of $\Omega_j$ in standard interpolation, for any $i \in \Omega_j \setminus \{j\}$, either $i \in F_j^s$ or there exists an $i \in F_j^s$ such that $i \in F_k^s$. For the latter, $(j,k,i)$ forms a path between $j$ and $i$ going along strong connections. We have then
\[
-a_{ik} \geq -\frac{\theta}{|N_k|} a_{kk} \geq -\frac{\theta}{|N_k||N_j|} a_{ij}.
\]
and
\[
a_{jj} \geq -a_{kj} \geq |N_k| a_{kk} \geq -\left(\frac{|N_k|}{\theta}\right)^2 a_{ik}.
\]
Combining the above two inequalities and using the assumption that the mesh is shape regular, for any $l \in \Omega_j$ that is connected with $i$ we have
\[
\sigma_1 a_{ij} \leq -a_{il} \leq \sigma_2 a_{jj}
\]
with constants $\sigma_1$ and $\sigma_2$ which depend on the shape regularity constant and $\theta$.
Since in the definition of $A_j$ in \((7.13)\), $\omega_e = -a_{ij}/2$ for $e = (i, j)$, we obtain
\[
c_1 a_{jj}(A_{L,j}v_j, v_j) \leq (A_j v_j, v_j) \leq c_2 a_{jj}(A_{L,j}v_j, v_j).
\] (7.28)
Then by a scaling argument, $a_{jj} \equiv h^{d-2}$ and the proof is complete. \(\square\)

**Theorem 7.6** The two level method using a coarse space defined as $V_c$ defined via the classical AMG is uniformly convergent.

**Proof.** By Theorem 3.1, we only need to show that $\mu_c$ is bounded, which can be easily obtained by combining Lemma 7.4–7.5 with Lemma 4.2. \(\square\)

We point out that Theorem 7.6 is also true for two level unsmoothed aggregation AMG. The proof is identical to the proof for classical AMG case.
Acknowledgments

The work of Xu was partially supported by the DOE Grant DE-SC0009249 as part of the Collaboratory on Mathematics for Mesoscopic Modeling of Materials and by NSF grants DMS-1412005 and DMS-1522615. The work of Zikatanov was partially supported by by NSF grants DMS-1418843 and DMS-1522615.

References

[1] R. A. Nicolaides. On multiple grid and related techniques for solving discrete elliptic systems. *J. Computational Phys.*, 19(4):418–431, 1975.

[2] R. A. Nicolaides. On the $l^2$ convergence of an algorithm for solving finite element equations. *Math. Comp.*, 31(140):892–906, 1977.

[3] Randolph E. Bank and Todd F. Dupont. Analysis of a two–level scheme for solving finite element equations. Technical Report Report CNA–159, Center for Numerical Analysis, Center for Numerical Analysis, University of Texas, Austin, May 1980.

[4] D. Braess and W. Hackbusch. A new convergence proof for the multigrid method including the V-cycle. *SIAM J. Numer. Anal.*, 20(5):967–975, 1983.

[5] James H Bramble and Joseph E Pasciak. New convergence estimates for multigrid algorithms. *Mathematics of computation*, 49(180):311–329, 1987.

[6] James H Bramble, Joseph E Pasciak, and Jinchao Xu. Parallel multilevel preconditioners. *Mathematics of Computation*, 55(191):1–22, 1990.

[7] James H. Bramble, Joseph E. Pasciak, and Jinchao Xu. The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms. *Math. Comp.*, 56(193):1–34, 1991.

[8] James H. Bramble, Joseph E. Pasciak, Jun Ping Wang, and Jinchao Xu. Convergence estimates for product iterative methods with applications to domain decomposition. *Math. Comp.*, 57(195):1–21, 1991.

[9] Jinchao Xu. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 34(4):581–613, 1992.

[10] A. Brandt, S. F. McCormick, and J. W. Ruge. Algebraic multigrid (AMG) for automatic multigrid solutions with application to geodetic computations. Technical report, Inst. for Computational Studies, Fort Collins, CO, October 1982.

[11] J. W. Ruge and K. Stüben. Algebraic multigrid (AMG). In S. F. McCormick, editor, *Multigrid Methods*, volume 3 of *Frontiers in Applied Mathematics*, pages 73–130. SIAM, Philadelphia, PA, 1987.

[12] Jean-François Maitre and François Musy. Méthodes multigrilles: opérateur associé et estimations du facteur de convergence: le cas du V-cycle. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(12):521–524, 1983.

[13] Randolph E. Bank and Craig C. Douglas. Sharp estimates for multigrid rates of convergence with general smoothing and acceleration. *SIAM J. Numer. Anal.*, 22(4):617–633, 1985.

[14] J. Mandel. Algebraic study of multigrid methods for symmetric, definite problems. *Appl. Math. Comput.*, 25:39–56, 1988.
REFERENCES

[15] S. F. McCormick. Multigrid methods for variational problems: general theory for the V-cycle. *SIAM J. Numer. Anal.*, 22(4):634–643, 1985.

[16] S. F. McCormick. Multigrid methods for variational problems: further results. *SIAM J. Numer. Anal.*, 21(2):255–263, 1984.

[17] S. F. McCormick and J. W. Ruge. Multigrid methods for variational problems. *SIAM J. Numer. Anal.*, 19(5):924–929, 1982.

[18] L.T. Zikatanov. Two-sided bounds on the convergence rate of two-level methods. *Numerical Linear Algebra with Applications*, 15(5):439–454, 2008.

[19] R. D. Falgout and P. S. Vassilevski. On generalizing the AMG framework. *SIAM J. Numer. Anal.*, 42(4):1669–1693, 2004. Also available as LLNL technical report UCRL-JC-150807.

[20] Robert D. Falgout, P. S. Vassilevski, and Ludmil T. Zikatanov. On two-grid convergence estimates. *Numerical linear algebra with applications*, 12(5-6):471 – 494, 2005.

[21] Jinchao Xu and Ludmil Zikatanov. The method of alternating projections and the method of subspace corrections in Hilbert space. *J. Amer. Math. Soc.*, 15(3):573–597, 2002.

[22] Scott P. MacLachlan and Luke N. Olson. Theoretical bounds for algebraic multigrid performance: review and analysis. *Numer. Linear Algebra Appl.*, 21(2):194–220, 2014.

[23] P. Vaněk, J. Mandel, and M. Brezina. Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems. *Computing*, 56(3):179–196, 1996. International GAMM-Workshop on Multi-level Methods (Meisdorf, 1994).

[24] M. Brezina and P.S. Vassilevski. Smoothed aggregation spectral element agglomeration AMG: SA-$\rho$AMGe. In I. Lirkov, S. Margenov, and J. W asniewski, editors, *LSSC*, volume 7116 of *Lecture Notes in Computer Science*, pages 3–15. Springer, 2011.

[25] J. Xu. The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids. *Computing*, 56(3):215–235, 1996.

[26] Long Chen, Ricardo H. Nochetto, and Jinchao Xu. Optimal multilevel methods for graded bisection grids. *Numer. Math.*, 120(1):1–34, 2012.

[27] Lars Grasedyck, Lu Wang, and Jinchao Xu. A nearly optimal multigrid method for general unstructured grids. *Numerische Mathematik*, pages 1–30, 2015.

[28] W. Hackbusch. A fast iterative method for solving Poisson’s equation in a general region. In *Numerical treatment of differential equations* (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1976), pages 51–62. Lecture Notes in Math., Vol. 631. Springer, Berlin, 1978. Longer version: Ein Iteratives Verfahren zur Schnellen Auflösung Elliptischer Randwertprobleme, Math. Inst., Universität zu Köln, Report 76-12 (November 1976).

[29] Jinchao Xu, Hongxuan Zhang, and Ludmil Zikatanov. Obtaining optimal coarse spaces for AMG via trace minimization, 2016.

[30] I.Y Vakhutinsky, L.M. Dudkin, and A.A. Ryvkin. Iterative aggregation–A new approach to the solution of large-scale problems. *Econometrica: Journal of the Econometric Society*, pages 821–841, 1979.
[31] S. Míka and P. Vaněk. A modification of the two-level algorithm with overcorrection. *Appl. Math.*, 37:13–28, 1992.

[32] S. Míka and P. Vaněk. Acceleration of convergence of a two level algebraic algorithm by aggregation in smoothing process. *Appl. Math.*, 37:343–356, 1992.

[33] A. Napov and Y. Notay. An algebraic multigrid method with guaranteed convergence rate. *SIAM Journal on Scientific Computing*, 34(2):A1079 – A1109, 2012.

[34] Yvan Notay. Aggregation-based algebraic multigrid for convection-diffusion equations. *SIAM journal on scientific computing*, 34(4):A2288 – A2316, 2012.

[35] J.E. Jones and P.S. Vassilevski. AMGe based on element agglomeration. *SIAM J. Sci. Comput.*, 23(1):109–133 (electronic), 2001.

[36] M. Brezina, A.J. Cleary, R.D. Falgout, V.E. Henson, J. E. Jones, T.A. Manteuffel, S.F. McCormick, and J.W. Ruge. Algebraic Multigrid Based on Element Interpolation (AMGe). *SIAM Journal on Scientific Computing*, 22(5):1570–1592, January 2001.

[37] T. Chartier, R.D. Falgout, V.E. Henson, J. Jones, T. Manteuffel, S. McCormick, J. Ruge, and P.S. Vassilevski. Spectral AMGe (ρAMGe). *SIAM J. Sci. Comput.*, 25(1):1–26, 2003.

[38] Jinchao Xu and Ludmil Zikatanov. Algebraic multigrid methods, 2016.

[39] JK Kraus and Josef Schicho. Algebraic multigrid based on computational molecules, 1: Scalar elliptic problems. *Computing*, 77(1):57–75, 2006.

[40] JK Kraus. Algebraic multigrid based on computational molecules, 2: Linear elasticity problems. *SIAM Journal on Scientific Computing*, 30(1):505–524, 2008.

[41] Jinchao Xu and Ludmil Zikatanov. A monotone finite element scheme for convection-diffusion equations. *Math. Comp.*, 68(228):1429–1446, 1999.

[42] U. Trottenberg, C. W. Oosterlee, and A. Schüller. *Multigrid*. Academic Press, Inc., San Diego, CA, 2001. With contributions by A. Brandt, P. Oswald and K. Stüben.