OPTIMAL CONSUMPTION PROBLEM IN A DIFFUSION SHORT-RATE MODEL

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Abstract. We consider a problem of an optimal consumption strategy on the infinite time horizon when the short-rate is a diffusion process. General existence and uniqueness theorem is illustrated by the Vasiček and so-called invariant interval models. We show also that when the short-rate dynamics is given by a Brownian motion or a geometric Brownian motion, then the value function is infinite.

Key words. short-rate models, optimal consumption, HJB equation

AMS subject classifications. 93E20, 91B28, 49L20

1. Introduction

Let $r_t$ be the short-rate (i.e. the rate offered by a bank) at time $t \geq 0$. Assume that $(r_t)$ satisfies the following stochastic differential equation

\[ dr_t = \mu(r_t) dt + \sigma(r_t) dW_t, \]

\[ r_0 = r, \]

where $(W_t)$ is a one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Let us denote by $V_t^{(C;r,v)}$ the capital at time $t$ of a bank account owner whose consumption rate is $C$ and whose wealth at time 0 is $v > 0$. Then

\[ dV_t^{(C;r,v)} = \left( r_t V_t^{(C;r,v)} - C_t \right) dt, \quad V_0^{(C;r,v)} = v. \]

Let

\[ \tau_A^{(C;r,v)} = \inf \left\{ t \geq 0 : V_t^{(C;r,v)} = 0 \right\} \]

be the bankruptcy time.

In the paper it is assumed that any consumption rate $C$ is progressively measurable and non-negative. The space of all consumption rates is denoted by $U$.

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Given a discount factor $\gamma \geq 0$ and an exponent $\alpha \in (0, 1)$ of the power utility function, we are concerned with the following problems:

**Problem A** Given $r$ and $v$, find a consumption rate $\hat{C}(r,v) \in U$ which maximizes the performance functional, that is,

$$J(\hat{C}(r,v); r, v) = \sup_{C \in U} J(C; r, v),$$

where

$$(3) \quad J(C; r, v) := \mathbb{E}^r \int_0^{\tau_A} e^{-\gamma t} C_t^\alpha dt,$$

$\tau_A = \tau^{(C,r,v)}_A$, and $\mathbb{E}^r$ is the conditional expectation $\mathbb{E}(\cdot | r_0 = r)$. A solution to this problem is given in Proposition 1 and Theorem 2 below.

**Problem B** It is reasonable to assume that one keeps his money in the bank account as long as the interest rate $r_t$ is positive. Under this assumption the performance functional is given by

$$J_B(C; r, v) := \mathbb{E}^r \left[ \int_0^{\tau_B} e^{-\gamma t} C_t^\alpha dt + e^{-\gamma \tau_B} V^{\alpha}_{\tau_B} \cdot \chi_{\{\tau_B < \infty\}} \right],$$

where $\tau_B = \tau^{(C,r,v)}_A \wedge \tau^r_0$, $\tau^r_0 = \inf \{ t \geq 0 : r_t = 0 \}$ and $V = V^{(C,r,v)}$. Clearly, if $r \leq 0$, then $\tau_B = 0$. The goal is now to find a consumption rate which maximizes $J_B$ (see Proposition 2 and Theorem 4).

**Problem C** Let $p(t, \theta)$ be the price at time $t$ of a zero-coupon bond that pays off 1 at time $\theta$. If one may also invest in zero-coupon bonds then the wealth dynamics is formally given by

$$dV_t^{(u,r,v)} = \left( \eta_t r_t V_t^{(u,r,v)} - C_t \right) dt + (1 - \eta_t) V_t^{(u,r,v)} \int_0^\infty \frac{dp(t, \theta)}{p(t, \theta)} \psi(t, \theta) d\theta,$$

$V_0^{(u,r,v)} = v,$

where $u = (C, \eta, \psi)$, $\psi$ is the density of the distribution of investments in bonds with various terminal time $\theta$. The aim is to maximize the performance functional $J$ given by (3) with $\tau_A = \tau^{(u,r,v)}_A$.

Problems A, B and C defined above are particular cases of an investor problem, various types of which has been investigating since 1970’s (see [7] and [8]). However, most of them are concerned with investment in a bank account (usually on a constant rate) and a finite number of stocks. If one can invest in a bank account and zero-coupon bonds, then the investor problem is more difficult to solve. The reason is that there
can be an infinite number of bonds, since the time of maturity $\theta$ can take an infinity of values. Furthermore, the set of admissible strategies does not contain “buy and hold” strategy, i.e. one must convert bond to cash at maturity.

The type of an investor who can invest his money in bonds has been recently studied in [1], [6] and [11]. Contrary to our paper, authors of [1], [6] and [11] examined the portfolio problem without possibility of consumption and with a finite time horizon. On the other hand, in [1] and [11] it is assumed that the dynamics of the instantaneous forward rate is given and that the performance function is defined under a real measure. More references can be find in the survey paper [12].

In the paper we use the Hamilton–Jacobi–Bellman approach, whereas in [1] and [11] convex duality is used.

2. Preliminaries

In the paper, it is assumed that (1) defines a Markov family on an open subinterval $\mathcal{O} \subseteq \mathbb{R}$, which, in particular, means that $\mathcal{O}$ is invariant for (1); that is, $r_0 \in \mathcal{O}$ implies that $r_t \in \mathcal{O}$ for all $t \geq 0$. Moreover, it is assumed that $\sigma \in C^2(\mathcal{O})$, $\mu \in C^1(\mathcal{O})$, their first derivatives are bounded on $\mathcal{O}$, and that the diffusion is non-degenerate, i.e. $\sigma(r) \neq 0$ for all $r \in \mathcal{O}$.

The value function for one of the listed above problems is the maximum of the corresponding performance functional over the set of admissible controls. We will show that the value functions are very regular, namely $C^2$ in $r$. Let

$$Qf(r) := \frac{1}{2} \sigma^2(r)f''(r) + \mu(r)f'(r),$$

be the formal generator of the diffusion given by (1).

The results below have the form of the verification theorem for stochastic control problems. For similar results see e.g. [2], [9] or [10].

**Proposition 1.** Let $K \in C^2(\mathcal{O})$ be such that

$$QK(r) + (\alpha r - \gamma)K(r) + (1 - \alpha)K^{\frac{\mu}{r\mu}}(r) = 0,$$

for every $r \in \mathcal{O}$. Then $\Phi(r, v) = K(r)v^\alpha$ is the value function for Problem A, whenever for any $C \in \mathcal{U}$ and $r \in \mathcal{O}$,

$$\lim_{n \to \infty} \mathbb{E}^r e^{-\gamma \tau_n} \Phi(r_{\tau_n}, V_{\tau_n}) = 0,$$

where $(r_t)$ is given by (1), $\tau_n = n \wedge \tau_n^{(C;r,v)}$ and $V = V^{(C;r,v)}$. The optimal consumption is given in the feedback form

$$\hat{C} = K^{\frac{1}{1+\alpha}}v.$$
Proof Taking into account the dynamics of \((V_t)\) and the form of performance functional we see that
\[
\Phi(r, v) = K(r)v^\alpha
\]
for a certain function \(K\). The Hamilton–Jacobi–Bellman equation (see e.g. [2], [10]) for \(K\) is
\[
\sup_{C \geq 0} \left\{ -\gamma K(r)v^\alpha + QK(r)v^\alpha + \alpha (rv - C)v^{\alpha-1}K(r) + C^\alpha \right\} = 0.
\]
The supremum is attained at \(\hat{C}\) given by (8). Hence, \(K\) satisfies (6) and the HJB verification theorem (see [9], [10]) gives us the claim. □

In Problem B we have to assume that \(0 \in \mathcal{O}\). If not, Problem B can be reduced to Problem A. Let \(\mathcal{O}^+ = \mathcal{O} \cap [0, \infty)\) and \(\mathcal{O}^{++} = \mathcal{O} \cap (0, \infty)\). With a similar proof as above we have the following proposition concerning Problem B.

**Proposition 2.** Let \(K \in C^2(\mathcal{O}^{++}) \cap C(\mathcal{O}^+)\) satisfy (6). Then
\[
\Phi(r, v) = \begin{cases} 
  v^\alpha, & r \in \mathcal{O} \setminus \mathcal{O}^{++}; \\
  K(r)v^\alpha, & r \in \mathcal{O}^{++},
\end{cases}
\]
is the value function for Problem B, whenever for any \(C \in \mathcal{U}\) and \(r \in \mathcal{O}^{++},\)
\[
(9) \quad \lim_{n \to \infty} \mathbb{E}_r e^{-\gamma \tau_n} \Phi(r_{\tau_n}, V_{\tau_n}) = \mathbb{E}_r e^{-\gamma \tau_B} V_{\tau_B}^{\alpha} \chi_{\{\tau_B < \infty\}},
\]
where \(\tau_B = \tau_{B(c,r,v)}^{(C,r,x)}, \tau_n^{(C,r,x)} = n \wedge \tau_B\) and \(V = V^{(C,r,x)}\). The optimal consumption is given in the feedback form (8).

Note that \(K\) satisfies a non-linear, non-Lipschitz second order differential equation, but \(K\) is not defined as a solution to the Cauchy problem. The goal of the paper is to prove the existence of the solution satisfying appropriate boundary conditions and to find an approximating scheme for \(K\).

3. Solution to Problem C

In Problem C we assume that \(\mathbb{P}\) is a martingale measure. Then
\[
p(t, \theta) = \mathbb{E} \left( e^{-\int_t^t \theta ds} | \mathcal{F}_t \right).
\]
Since \((r_t)\) is a Markov process, \(p(t, \theta) = v^\theta(t, r_t)\) is a function of \(t, \theta\) and \(r_t\). Thus we can rewrite (11) as follows
\[
(10) \quad dV_t = (r_t V_t - C_t)dt + (1 - \eta_t)V_t \sigma(r_t) \int_0^\infty \frac{\partial}{\partial \theta} v^\theta(t, r_t) \psi(t, \theta) d\theta dW_t
\]
with $V = V^{(u, r, v)}$, whenever $v^\theta$ is differentiable with respect to $r$.

In Problem C the performance function is given by (3), the class of admissible controls $\mathcal{U}$ consists of tuples $(C, \eta, \psi(\cdot, \theta)_{\theta \geq 0})$ of progressively measurable processes, such that $C_t$ is non-negative, (10) is well defined and

$$
\int_0^\infty \psi(t, \theta) d\theta = 1, \quad \psi(t, \theta) \equiv 0, \quad \forall \theta \leq t.
$$

Note that neither $\eta$ nor $\psi$ have to be non-negative.

Define (11) $\Upsilon_s = \int_0^\infty \frac{\partial}{\partial \theta} \nu^\theta(s, r_s) \psi(s, \theta) d\theta$ and $\beta_s = (1 - \eta_s) \Upsilon_s$.

Then we can rewrite (10) in the form

$$
dV_t = (r_t V_t - C_t) dt + \beta_t V_t \sigma(r_t) dW_t.
$$

Since we assumed that we are given dynamics of $(r_t)$, and the performance functional is under a martingale measure, we can treat the investor portfolio as the one consisted of the bank account and one other instrument with price $S_t$ given by

$$
dS_t = S_t (r_t dt + \sigma(r_t) \Upsilon_t dW_t), \quad S_0 = 1.
$$

Note that given (12) and (13), we have to assume that for any $t > 0$,

$$
\int_0^t \Upsilon_s^2 ds < \infty, \quad \int_0^t \beta_s^2 ds < \infty, \quad P - a.s.
$$

Therefore the dynamics of the wealth of the investor is given by

$$
dV_t = (\eta_t r_t V_t - C_t) dt + (1 - \eta_t) V_t dS_t / S_t,
$$

which is equivalent to (11) and (12). Thus the number of instruments is finite and the same approach as in [5] can be taken.

Theorem 1 below, giving a solution to Problem C, was formulated and proven in [5] under much weaken conditions. Here we present another proof, based on Proposition 3 below. We restrict our attention to the value function of the problem. We refer the reader to [5] for details on the optimal control (portfolio).

**Proposition 3.** If $K \in C^2(\mathcal{O})$ satisfies

$$
QK(r) + (\alpha r - \gamma) K(r) + (1 - \alpha) K^\frac{\alpha}{2} (r) + \frac{\alpha \sigma^2(r)}{2(1 - \alpha)} (K'(r))^2 = 0,
$$

then $\Phi(r, v) = K(r) e^\alpha$ is the value function for Problem C, whenever for any $u \in \mathcal{U}$ and $r \in \mathcal{O}$,

$$
\lim_{n \to \infty} \mathbb{E}^r e^{-\gamma r_n} \Phi(r_{r_n}, V_{r_n}) = 0,
$$

$$
\lim_{n \to \infty} \mathbb{E}^r e^{-\gamma r_n} \Phi(r_{r_n}, V_{r_n}) = 0.
$$
where \( \tau_n = n \wedge \tau_A^{(u,v)} \) and \( V = V^{(u,v)} \). The optimal consumption \( \hat{C} \) and the optimal factor \( \hat{\beta} \) defined in (11) are given by

\[
\hat{C} = K^{\frac{1}{\alpha - 1}} v, \quad \hat{\beta} = \frac{K'}{(1 - \alpha)K}.
\]

**Proof** Again \( \Phi(r,v) = K(r)v^\alpha \) for a certain function \( K \). The HJB equation for \( K \) is

\[
QK(r)v^\alpha + (\alpha r - \gamma)K(r)v^\alpha + \sup_{\beta \geq 0} \left\{ C^\alpha - C\alpha K(r)v^{\alpha - 1} \right\}
\]

\[
+ \sup_{\beta} \left\{ \alpha \beta \sigma^2(r) K'(r)v^\alpha + \frac{\alpha(\alpha - 1)}{2} \beta^2 \sigma^2(r) K(r)v^\alpha \right\} = 0,
\]

and the claim follows from the HJB verification theorem. \( \square \)

Since the value function \( \Phi(r,v) \) is a non-decreasing positive function of both arguments \( r \) and \( v > 0 \), we see that \( K \) is non-decreasing and positive. Then the optimal \( \beta \) in (16) is positive. Note that \( \eta \leq 1 \) and \( \psi \geq 0 \) whilst the short-selling is forbidden. Then the condition

\[
\frac{\partial}{\partial r} \nu^\theta(t, r_t) \leq 0,
\]

which holds e.g. in Vasicek and CIR models, implies that if the short-selling is forbidden then necessarily \( \Upsilon_t \) and \( \beta_t \) given by (11) are non-positive. Thus the supremum over \( \beta \leq 0 \) in equation (17) is attained at 0. Hence, if the short-selling is forbidden and (18) holds, then Problem C reduces to Problem A.

It is worth mentioning that given \( \hat{\beta} \) we do not have unambiguous solution to Problem C, i.e. we do not obtain unambiguous pair \((\eta, \psi)\). However we may choose arbitrary \( \psi \) such that \((C, \eta, \psi) \in \mathcal{U}\) and then we derive an optimal \( \hat{\eta} \) from (11). For example we may set \( \psi(t, \theta) = \varsigma e^{-\varsigma(\theta-t)} \cdot \chi_{\{t<\theta\}} \) for some \( \varsigma > 0 \).

The following result will be used to show the regularity of the value function. Let

\[
N(r) := \mathbb{E}^r \int_0^\infty e^{-\frac{1}{\alpha}(\gamma t + \alpha \int_0^t s ds)} dt, \quad r \in \mathcal{O}.
\]

**Proposition 4.** If \( N(r) < \infty \) for every \( r \in \mathcal{O} \) and

\[
\mathbb{E}^r \int_0^\infty e^{-\frac{2}{1 - \alpha}(\gamma t + \alpha \int_0^t s ds)} dt < \infty, \quad \forall r \in \mathcal{O},
\]

then \( N \in C^2(\mathcal{O}) \) and

\[
QN(r) + \frac{\alpha r - \gamma}{1 - \alpha} N(r) + 1 = 0, \quad r \in \mathcal{O}.
\]
Proof Let $O = (a, b)$, where $a \geq -\infty$ and $b \leq \infty$. Then, by (20),

$$N_n(r) = E^r \int_0^{\tau^r_n} e^{\frac{1}{1-\alpha}(-\gamma t + \alpha \int_0^t r_s ds)} dt$$

is a solution to the boundary problem

$${\begin{cases} QN_n(r) + \frac{a r - \gamma}{1-\alpha} N_n(r) = -1, \\ N_n(a) = N_n(b) = 0, \end{cases}}$$

where

$$a_n = \begin{cases} a + 1/n, & a > -\infty, \\ -n, & a = -\infty, \end{cases} \quad b_n = \begin{cases} b - 1/n, & b < \infty, \\ n, & b = \infty, \end{cases}$$

and $\tau^r_{\pm n} = \inf \{ t \geq 0 : r_t \notin [a_n, b_n] \}$. Since we assumed that $O$ is invariant for (1), then $\lim_{n \to \infty} \tau^r_{\pm n} = \infty$ for any $r \in O$ and consequently $N(r) = \lim_{n \to \infty} N_n(r)$. Thus $N$ is a weak solution (see Definition 1 in Section 5) to (21), and by Lemma 1, $N \in C^2(O)$. Hence, it is a strong solution to (21). □

The result below says that the function $K$ appearing in the identity $\Phi(r, v) = K(r) v^\alpha$ for the value function equals $N^{1-\alpha}$.

**Theorem 1.** Let assumptions of Proposition 4 hold. Assume additionally that for any $u \in U$ and $r \in O$,

$$(22) \quad \lim_{n \to \infty} E^r e^{-\gamma \tau_n} N^{1-\alpha}(r_{\tau_n}) V^\alpha_{\tau_n} = 0,$$

where $\tau_n = n \wedge \tau_A^{(u;r,v)}$ and $V = V^{(u;r,v)}$. Then $\Phi(r, v) = N^{1-\alpha}(r) v^\alpha$ is the value function for Problem C.

**Proof** By elementary calculus, (21) is equivalent to (14) for $K(r) = N^{1-\alpha}(r)$. Condition (22) implies (15) and we conclude by Proposition 3. □

4. **Solution to Problem A**

This section contains one of the main results of the paper. It provides the existence and approximating scheme for the solution $K$ to the HJB equation (6) for Problem A. In its formulation $(E, \| \cdot \|_E)$ is a Banach space of continuous functions on $O$.

We will need the following hypotheses:

**(H.1)** For any fixed $t \geq 0$, $r \in O$, $\varphi \in E$ and any sequence $\{\tau_n\}$ of stopping times, the sequences of random variables

$$\varphi(\tau_{t\wedge \tau_n}) e^{\alpha \int_0^{\tau_{t\wedge \tau_n}} r_s ds} \quad \text{and} \quad \int_0^{\tau_{t\wedge \tau_n}} \varphi(r_s) e^{\alpha \int_0^s r_u du} ds$$

are uniformly integrable with respect to $\mathbb{P}^r = \mathbb{P}(\cdot | r_0 = r)$. 

For any Lipschitz continuous bounded function $f : [0, \infty) \mapsto [0, \infty)$ and for any non-negative $\phi \in E$, one has $f(\phi) \in E$.

The family $(P_t, t \geq 0)$ of linear operators

$$(23) \quad P_t \varphi(r) = \mathbb{E}^r \varphi(r_t) e^{\alpha \int_0^t r_s \, ds}, \quad r \in \mathcal{O},$$

forms a $C_0$-semigroup on $E$.

**Remark 1.** In examples $\mathcal{O} = \mathbb{R}$ and

$$E = \{ \varphi \in C(\mathbb{R}) : \lim_{|r| \to \infty} |\varphi(r)| e^{-\Phi|r|} = 0 \}$$

or $\mathcal{O}$ is a bounded interval and $E$ is the space $UC(\mathcal{O})$ of uniformly continuous functions on $\mathcal{O}$. Moreover, we will show in Lemma 2 that the generator $(A, D(A))$ of $(P_t)$ is given by

$$D(A) = \{ \varphi \in C^2(\mathcal{O}) \cap E : A\varphi \in E \},$$

and $A\varphi = A\varphi$ for all $\varphi \in D(A)$, where $A$ is the differential operator

$$(24) \quad A\varphi(r) = Q\varphi(r) + \alpha r\varphi(r).$$

We note that condition $(H.1)$ will be needed only in the proof of the inclusion $\{ \varphi \in C^2(\mathcal{O}) \cap E : A\varphi \in E \} \subset D(A)$.

Recall that $N$ is a function defined by (19). The following hypothesis will be needed in the proof that $N^{1-\alpha}$ is a supersolution to the HJB equation (6), such that $N^{1-\alpha} \in D(A)$ and $\Phi_N(r, v) = N^{1-\alpha}(r) v^{\alpha}$ satisfies the boundary condition (7). For more details see Definition 2 and Remark 3.

$(H.4)$ For any $r \in \mathcal{O}$,

$$(25) \quad \lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s \, ds} N^{1-\alpha}(r_t) = 0$$

and for any stopping time $\tau^{(C;r,v)}_A$, $\tau^{(C;r,v)}_A$ is uniformly integrable, where $\tau_n = n \wedge \tau^{(C;r,v)}_A$.

Moreover, $N^{1-\alpha} \in C^2(\mathcal{O}) \cap E$ and $AN^{1-\alpha} \in E$, where $A$ is defined by (24).

For any $m > 0$, define

$$(27) \quad F_m(x) = \begin{cases} (1-\alpha)x^{\alpha - 1}, & x > m^{\alpha - 1}, \\ m^{\alpha - \alpha mx}, & 0 \leq x \leq m^{\alpha - 1}. \end{cases}$$

Recall that $A$ is the generator of the semigroup $(P_t)$. We denote by $\rho(A - \gamma)$ the resolvent set of $A - \gamma$. The proof of the following theorem is postponed to Section 7.
Theorem 2. Assume that \((H.1)\), \((H.2)\), \((H.3)\) and \((H.4)\) are fulfilled. Then there is a solution \(K\) to \((6)\) with condition \((7)\). Moreover, \(K(r) \leq N^{1-\alpha}(r)\), \(r \in O\). Finally, for any sequence \(\{\lambda_m\} \subset g(A - \gamma)\) such that for any \(m > 0\),

\[(28) \text{ the mapping } [0, \infty) \ni x \mapsto F_m(x) + \lambda_m x \text{ is non-decreasing,} \]

one has

\[K(r) = \lim_{m \to \infty} \lim_{n \to \infty} K^m_n(r), \quad r \in O, \]

where \(\{K^m_n\}\) is a non-decreasing sequence of both \(m\) and \(n\), defined as follows

\[K^0_n = 0, \]
\[K^m_{n+1} = (\lambda_m + \gamma - A)^{-1}(F_m(K^m_n) + \lambda_m K^m_n). \]

Remark 2. Since \(F_m\) are Lipschitz continuous, then the function \(x \mapsto F_m(x) + \lambda x\) is non-decreasing for \(\lambda\) large enough. Thus there is a sequence \(\{\lambda_m\} \subset g(A - \gamma)\) such that the functions \(x \mapsto F_m(x) + \lambda_m x\) are non-decreasing. Furthermore, from \(C_0\)-semigroup property of \((P_t)\) guaranteed by \((H.3)\) we get

\[\|P_t \varphi\| \leq M e^{\vartheta t}\|\varphi\| \]

for some \(\vartheta\) and \(M > 0\). Then \((\vartheta, \infty) \subset g(A)\) and setting any \(\varepsilon_1 > 0\) and \(\varepsilon_2 \geq 0\) we may define

\[(29) \lambda_m = \max\{\vartheta - \gamma + \varepsilon_1, \alpha m + \varepsilon_2\}. \]

Remark 3. We will show in Sections 9 and 10 that the assumptions of the Theorem 2 are satisfied if \((r_t)\) is an Ornstein–Uhlenbeck process (the so-called Vasicek model) or \(O\) is bounded. We will show in Section 11 that if \((r_t)\) is either a Brownian motion or a geometric Brownian motion then the value function for Problem A is infinite.

5. Analytical tools

This section provides some useful analytical tools. Let us consider a second order differential operator

\[Du(x) = a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x)\]

with \(a_i \in C^i(O)\) and \(a_2 \neq 0\) in \(O\). We denote by

\[D^*u(x) = (a_2(x)u(x))'' - (a_1(x)u(x))' + a_0(x)u(x)\]

the formally adjoint operator. We denote by \(L^1_{loc}(O)\) the space of all locally integrable functions on \(O\).
Definition 1. Let \( f, u \in L^1_{\text{loc}}(\mathcal{O}) \). We call \( u \) a **weak solution** to the equation \( Du = f \) if

\[
\int_{\mathcal{O}} u(x)D^\ast \varphi(x)dx = \int_{\mathcal{O}} f(x)\varphi(x)dx, \quad \forall \varphi \in C_0^\infty(\mathcal{O}).
\]

Let \( \mathcal{G} \) be an open subset of \( \mathbb{R} \). The following result holds only in dimension 1. For a counterexample in case of \( \mathcal{O} \subseteq \mathbb{R}^2 \) see [3].

Lemma 1. Assume that \( H: \mathcal{G} \mapsto \mathbb{R} \) is a continuous function and \( u \in L^1_{\text{loc}}(\mathcal{O}) \) such that \( u(\mathcal{O}) \subseteq \mathcal{G} \), is a weak solution to

(30) \[
Du = H(u).
\]

Then \( u \in C^2(\mathcal{O}) \), i.e. \( u \) is a strong solution to (30).

**Proof** We may rewrite (30) in the form

(31) \[
(a_2u' + (a_1 - a'_2)u)' = H(u) - (a''_2 - a'_1 + a_0)u,
\]

where we skip argument \( x \) and all derivatives of \( u \) are in the weak sense. We can use the following fact. Assume that \( \xi \) is a distribution whose derivative is a function \( h \in L^1_{\text{loc}}(\mathcal{O}) \). Then \( \xi \) is a function and

\[
\xi(x) = \zeta + \int_{\triangle} h(y)dy,
\]

for some finite \( \triangle \in \mathcal{O} \) and \( \zeta \in \mathbb{R} \). Applying this observation to (31) we obtain

\[
a_2u' + (a_1 - a'_2)u = \zeta_1 + \int_{\triangle} (H(u) - (a''_2 - a'_1 + a_0)u)dy,
\]

where the r.h.s. is continuous, since integrand is locally integrable. Thus

\[
u' = \frac{\zeta_1}{a_2} + \frac{\int_{\triangle} (H(u) - (a''_2 - a'_1 + a_0)u)dy}{a_2} - \frac{(a_1 - a'_2)u}{a_2}
\]

and \( u' \in L^1_{\text{loc}}(\mathcal{O}) \). Using the same argument again we have

\[
u(r) = \zeta_2 + \int_{\triangle} \left( \frac{\zeta_1}{a_2} + \frac{\int_{\triangle} (H(u) - (a''_2 - a'_1 + a_0)u)dy}{a_2} - \frac{(a_1 - a'_2)u}{a_2} \right) dx
\]

and \( u \in C \), since integrand is locally integrable. Having shown that \( u \in C(\mathcal{O}) \), we see that the integrand is continuous, which implies \( u \in C^1(\mathcal{O}) \). Now we conclude that integrand is of class \( C^1 \) and consequently \( u \in C^2(\mathcal{O}) \). \( \square \)

Recall that \( A \) is a differential operator given by (24). We denote by \( (\mathcal{A}, \mathcal{D}(\mathcal{A})) \) the generator of the \( C_0 \)-semigroup \( (P_t) \) defined by (23) on the Banach space \( E \), see (H.3).
Lemma 2. We have
\[ \mathcal{D}(A) = \{ \varphi \in C^2(\mathcal{O}) \cap E: A\varphi \in E \}, \]
and \( A\varphi = A\varphi \) for all \( \varphi \in \mathcal{D}(A) \).

Proof Write \( \mathcal{E} = \{ \varphi \in C^2(\mathcal{O}) \cap E: A\varphi \in E \} \).

Step 1. Here we will show that \( \mathcal{D}(A) \subset \mathcal{E} \). Let \( \varphi \in \mathcal{D}(A) \). First we will show that \( \langle A\varphi, \psi \rangle = \langle \varphi, A^* \psi \rangle \) for any \( \psi \in C_0^\infty(\mathcal{O}) \). We have
\[
\langle A\varphi, \psi \rangle = \lim_{t \downarrow 0} \frac{1}{t} \int_\mathcal{O} (P_t \varphi(x) - \varphi(x)) \psi(x) dx
\]
\[
= \lim_{t \downarrow 0} \frac{1}{t} \int_\mathcal{O} \int_\mathcal{O} p_t(x, y) \varphi(y) \psi(x) dx dy - \lim_{t \downarrow 0} \frac{1}{t} \int_\mathcal{O} \varphi(y) \psi(y) dy,
\]
where \( p_t(x, y) \) is a transition density function of process \( (r_t) \), which exists due to the non-degeneration of the diffusion coefficient. Hence
\[
\langle A\varphi, \psi \rangle = \lim_{t \downarrow 0} \frac{1}{t} \int_\mathcal{O} \left( \int_\mathcal{O} p_t(x, y) \varphi(x) dx - \varphi(y) \right) \psi(y) dy
\]
\[
= \lim_{t \downarrow 0} \int_\mathcal{O} \varphi(y) \left( \frac{1}{t} \int_\mathcal{O} \varphi(x) (p_t(x, y) dx - \delta_y(dx)) \right) dy
\]
\[
= \int_\mathcal{O} \varphi(y) \left( \int_\mathcal{O} \varphi(x) \frac{\partial}{\partial t} p_t(x, y) \bigg|_{t=0} \right) dx dy
\]
and since the transition density function satisfies backward parabolic equation, it follows that
\[
\langle A\varphi, \psi \rangle = \int_\mathcal{O} \varphi(y) \left( \int_\mathcal{O} \varphi(x) A_x p_t(x, y) \bigg|_{t=0} \right) dx dy,
\]
where subscript \( x \) denotes that the operator \( A \) acts on \( p_t(x, y) \) as a function of \( x \) with \( t \) and \( y \) fixed. Thus we have
\[
\langle A\varphi, \psi \rangle = \int_\mathcal{O} \varphi(y) \left( \int_\mathcal{O} \varphi(x) A_x \delta_y(dx) \right) dy
\]
\[
= \int_\mathcal{O} \varphi(y) \langle A_x \delta_y, \psi \rangle dy = \int_\mathcal{O} \varphi(y) \langle \delta_y, A^*_x \psi \rangle dy
\]
\[
= \int_\mathcal{O} \varphi(y) A^*_x \psi(y) dy = \langle \varphi, A^* \psi \rangle.
\]
Thus \( \varphi \) is a weak solution to \( A\varphi = A\varphi \). By Lemma \( \varphi \in C^2(\mathcal{O}) \) and \( \varphi \) is a strong solution to \( A\varphi = A\varphi \). Hence \( A\varphi = A\varphi \) and \( A\varphi \in E \).

Step 2. We will show that \( \mathcal{E} \subset \mathcal{D}(A) \). Let \( \varphi \in \mathcal{E} \). Then from Itô’s formula
\[
\varphi(r_t \wedge T_n) e^{\int_0^{t \wedge T_n} r_s ds} = \varphi(r) + \int_0^{t \wedge T_n} e^{\int_0^s r_u du} A\varphi(r_s) ds + M_{t \wedge T_n},
\]
where \( T_n = \inf\{t \geq 0 : |r_t| \geq n\} \) and
\[
M_{t \wedge T_n} = \int_0^{t \wedge T_n} \sigma(r_s) \phi'(r_s) e^{\alpha \int_0^s r_u \, du} dW_s
\]
is a martingale. Taking expectations and next passing to limit with \( n \), thanks condition \((H.1)\), we obtain
\[
\mathbb{E} \varphi(r_t) e^{\alpha \int_0^t r_s \, ds} = \varphi(r) + \int_0^t \mathbb{E} e^{\alpha \int_0^s r_u \, du} A \varphi(r_s) ds,
\]
which means that \( P_t \varphi(r) = \varphi(r) + \int_0^t P_s A \varphi(r) ds \). Therefore by the mean-value theorem
\[
\lim_{t \downarrow 0} \frac{P_t \varphi(r) - \varphi(r)}{t} = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t P_s A \varphi(r) ds = A \varphi(r),
\]
which means that \( \varphi \in D(A) \) and \( A \varphi = A \varphi \). □

6. Lipschitz Modification of the HJB Equation

In this section we will find a twice continuously differentiable solution to the equation
\[
(32) \quad QK(r) + (\alpha r - \gamma)K(r) + F_m(K(r)) = 0, \quad r \in \mathcal{O},
\]
where \( F_m \) is given by \((27)\).

**Remark 4.** It is easy to verify that \( F_m \) is a continuous Lipschitz function with Lipschitz constant \( L_m = \alpha m \). Moreover, \( F_m \in C^1((0, \infty)) \). Equation \((32)\) may be interpreted as HJB equation for Problems A and B with assumption that \( C_t = c_t V_t \) and \( c_t \in [0, m] \). Hence, a solution to \((3)\) should be a limit of the sequence of solutions to \((32)\) as \( m \to \infty \).

Define \( A_\gamma := (A - \gamma) \) and \( \mathcal{A}_\gamma := (A - \gamma) \). Note that \((32)\) can be written as
\[
(33) \quad -A_\gamma K = F_m(K) \quad \text{in } \mathcal{O}.
\]

**Definition 2.** We call \( u \in C^2(\mathcal{O}) \) a subsolution to \((33)\) if
\[
-A_\gamma u \leq F_m(u) \quad \text{in } \mathcal{O}.
\]
We call \( u \) a supersolution if
\[
-A_\gamma u \geq F_m(u) \quad \text{in } \mathcal{O}.
\]

**Remark 5.** It is easy to verify that \( \overline{K} \equiv 0 \) is a subsolution to \((33)\). Note that \( \overline{K}(r) = N(r)^{1-\alpha} \) is a supersolution, since, by Proposition \(4\), \( N^{1-\alpha} \in C^2(\mathcal{O}) \) and since
\[
-A_\gamma \overline{K}(r) = F(\overline{K}(r)) + \frac{\alpha \sigma^2(r)(\overline{K}'(r))^2}{2(1-\alpha)\overline{K}(r)} \geq F(\overline{K}(r)) \geq F_m(\overline{K}(r)).
\]
Furthermore, by (H.4), $N^{1-\alpha} \in \mathcal{D}(A)$ and $\Phi_N(r, v) = N^{1-\alpha}(r)v^\alpha$ satisfies (7).

**Theorem 3.** Let $K^m_n \in \mathcal{D}(A)$ and $\bar{K}^m_n \in \mathcal{D}(A)$ be a subsolution and a supersolution to (33), respectively. Assume that $K^m_n \leq \bar{K}^m_n$. Define $K^m_0 = K^m_1$ and $K^m_{n+1}$ as

\[ K^m_{n+1} = (\lambda_m - A_t)^{-1}(F_m(K^m_n) + \lambda_m K^m_n), \]

where $\lambda_m$ is such that (28) holds. Then $K^m$ defined as a pointwise limit of $\{K^m_n\}$, i.e.

\[ K^m(r) = \lim_{n \to \infty} K^m_n(r), \quad \forall r \in \mathcal{O}, \]

belongs to $C^2(\mathcal{O})$ and is a strong solution to (33). Moreover, $K^m \leq \bar{K}^m$ for all $m$.

**Proof**

From

\[-A_t K^m + \lambda_m K^m \leq F_m(K^m) + \lambda_m K^m = -A_t K^m_1 + \lambda_m K^m_1\]

we get $(\lambda_m - A_t)(K^m - K^m_1) \geq 0$. Since $P_t \varphi \geq 0$ and consequently $(\lambda_m - A_t)^{-1} \varphi \geq 0$ for every $\varphi \geq 0$. It follows that $K^m \leq K^m_1$.

Now we show that $K^m$ is a subsolution. From (28) and (34) we have

\[ F_m(K^m_1) + \lambda_m K^m_1 \geq F_m(K^m) + \lambda_m K^m = -A_t K^m_1 + \lambda_m K^m_1, \]

which, with help of Lemma 2, implies that $K^m_1$ is a subsolution to (33). Hence, by induction, $K^m_n \leq K^m_{n+1}$ and $K^m_{n+1}$ is a subsolution for all $n \in \mathbb{N}_0$.

Now we show, by induction, that $K^m_n \leq \bar{K}^m_n$ for all $n$. By definition $K^m_0 \leq \bar{K}^m_1$. Assume that $K^m_n \leq \bar{K}^m_n$. Then from (34) and (28) we have

\[-A_t K^m_{n+1} + \lambda_m K^m_{n+1} = F_m(K^m_n) + \lambda_m K^m_n \leq F_m(\bar{K}^m_n) + \lambda_m \bar{K}^m_n.\]

Hence,

\[-A_t K^m_{n+1} + \lambda_m K^m_{n+1} \leq F_m(\bar{K}^m) + \lambda_m \bar{K}^m \leq -A_t \bar{K}^m + \lambda_m \bar{K}^m\]

implies that $(\lambda_m - A_t)(\bar{K}^m - K^m_{n+1}) \geq 0$, and we obtain $K^m_{n+1} \leq \bar{K}^m_n$.

Summing up, we have

\[ K^m \leq K^m_1 \leq K^m_2 \leq \ldots \leq K^m_n \leq \ldots \leq \bar{K}^m \quad \text{in } \mathcal{O}. \]

Therefore $K^m(r)$ given by (35) exists for all $r$. Since $F_m$ is continuous,

\[ F_m(K^m(r)) = \lim_{n \to \infty} F_m(K^m_n(r)), \quad \forall r \in \mathcal{O}, \]

and from (34) we have

\[ \int_{\mathcal{O}} \varphi(r)(\lambda_m - A_t)K^m_{n+1}(r)dr = \int_{\mathcal{O}} (F_m(K^m_n(r)) + \lambda_m K^m_n(r))\varphi(r)dr, \]
for any test function $\varphi \in C_0^\infty(\mathcal{O})$. By Lemma 2
\[
\int_\mathcal{O} K_{n+1}^m(r)(\lambda_m - A_\gamma^*)\varphi(r)dr = \int_\mathcal{O} (F_m(K_n^m(r)) + \lambda_m K_n^m(r))\varphi(r)dr.
\]
Let $n \to \infty$. By the dominated convergence theorem, we get
\[
\int_\mathcal{O} K_m^m(r)A_\gamma^*\varphi(r)dr = \int_\mathcal{O} F_m(K_m^m(r))\varphi(r)dr.
\]
Since $K_m^m \leq K^m_m$ and $K^m_m$ is continuous, then $K_m^m$ is locally bounded. Hence, $K_m^m$ is a weak solution to (33), and we conclude by Lemma 1.

\[\square\]

7. Proof of Theorem 2

Let $\{K_m^m\}$ be the sequence constructed in the previous section. By (25) and (26) the function $\Phi_n(r,v) = N^{1-\alpha}(r)v^\alpha$ satisfies (7). So does $\Phi_0(r,v) \equiv 0$. Hence, Remark 5 and Theorem 3 guarantee that $\Phi^m(r,v) = K_m^m(r)v^\alpha$ satisfies (7). Therefore, by Remark 4, $\Phi^m(r,v)$ is the value function for Problem A with constraint $C_t \leq mV_t$. Hence, $\{K_m^m\}$ is a non-decreasing sequence and the function
\[
K(r) = \lim_{m \to \infty} K^m(r), \quad r \in \mathcal{O},
\]
is well defined. Note that $K > 0$ in $\mathcal{O}$. Indeed, from the continuity of $r_t$ we have $e^{\int_0^t r_s ds} > 0$, $\mathbb{P}$-a.s. for all $t > 0$ and $r \in \mathcal{O}$, which implies $\mathbb{E}^r e^{\int_0^t r_s ds} > 0$, and therefore
\[
K_1^1(r) = (\lambda_1 - A_\gamma)^{-1} = \int_0^\infty e^{-(\lambda_1 + \gamma)t} \mathbb{E}^r e^{\alpha \int_0^t r_s ds} dt
\]
is strictly positive in $\mathcal{O}$. Since $K_1^1 \leq K_1^1 \leq K_2^1 \leq \ldots \leq K_m^m \leq \ldots \leq K$, we have $K > 0$. Thus, in particular, $F(K)$ is well defined, where $F(y) = (1 - \alpha)y^{\alpha-1}$, for every $y > 0$.

We will show that $K$ is a weak solution to
\[
- A_\gamma K = F(K) \quad \text{in } \mathcal{O}.
\]
To do this define
\[
Z_m = \{r \in \mathcal{O} : K_m^m(r) \geq m^{\alpha-1}\}.
\]
Clearly $Z_m \subset Z_{m+1}$ for all $m \in \mathbb{N}$. Since $F_m(y) = F(y)$ for every $y \geq m^{\alpha-1}$, we have
\[
F_m(K_m^m(r)) = F(K_m^m(r)), \quad \forall r \in Z_n \forall m \geq n,
\]
which implies, from continuity of $F$, that for any $r \in \bigcup_{n=1}^\infty Z_n$,
\[
\lim_{m \to \infty} F_m(K_m^m(r)) = \lim_{m \to \infty} F(K_m^m(r)) = F(K(r)).
\]
Now we show that (38) holds for any $r \in \mathcal{O}$. To do this note that
\[ \bigcup_{n=1}^{\infty} Z_n = \mathcal{O}. \]
Indeed, since $K^1(r) > 0$ for any $r \in \mathcal{O}$, then for any $r \in \mathcal{O}$ there is such $m$, that
\[ K^m(r) \geq K^1(r) > m^{\alpha-1} > 0, \]
and hence $r \in Z_m$.

Note that for any $m > 1$ and $r \in \mathcal{O}$ we have
\[ |F_m(K^m(r))\varphi(r)| \leq (1 - \alpha)(K^1(r))^{\frac{\alpha}{\alpha-1}}|\varphi(r)|. \]

Let $m \to \infty$ in (36). By the inequality above and the dominated convergence theorem, we get
\[ -\int_{\mathcal{O}} K(r)A^*_r \varphi(r)dr = \int_{\mathcal{O}} F(K(r))\varphi(r)dr, \quad \forall \varphi \in C_0^\infty(\mathcal{O}), \]
which means that $K$ is a weak solution to (37), whenever $K$ is locally integrable. Since $K_m = N^1 \alpha E$, we have $K \leq N^1 \alpha$, and from continuity of $N^1 \alpha$, the function $K$ is locally bounded. By Lemma 1, $K$ is a strong solution to (37).

By (25) and (26), $\Phi(r,v) = K(r)v^\alpha$ satisfies the boundary condition (7).

8. Solution to Problem B

This section provides the existence and approximating scheme for the solution $K$ to the HJB equation (6) for Problem B. Let $(\hat{E}, \| \cdot \|_{\hat{E}})$ be a Banach space of continuous functions on $\mathcal{O}^+$.

Recall that $\tau_0 = \inf\{t \geq 0: r_t = 0\}$. For all $r \in \mathcal{O}^+$ we define the following functions:
\[ \hat{N}(r) = \mathbb{E}^r \int_0^{\tau_0} e^{\frac{\alpha}{1-\alpha}(-\gamma t + \alpha \int_0^t r_s ds)}dt \]
and
\[ K_L(r) = \mathbb{E}^r e^{-\gamma \tau_0 + \alpha \int_0^\tau_0 r_s ds}. \]

Let $K_U(r) = K_L(r) + \hat{N}^{1-\alpha}(r)$ for all $r \in \mathcal{O}^+$ and let $(\hat{r}_t) = (r_{t \wedge \tau_0})$.

We denote by (H.1), (H.2) and (H.3) the equivalents to (H.1), (H.2) and (H.3) respectively, where $r \in \mathcal{O}^+$, and $(\hat{r}_t)$ and $E$ are replaced by $(\hat{r}_t)$ and $\hat{E}$.

Clearly, $K_L \leq K_U$, and the following hypothesis is needed to show that the boundary condition (4) holds for any continuous function $f$ satisfying $K_L \leq f \leq K_U$ in $\mathcal{O}^+$.

(H.4) For any $r \in \mathcal{O}^+$,
\[ \lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds} K_U(r_t) \chi_\{(\hat{r}_t)\in \mathcal{O}^+, \tau_0^\infty\} = 0 \]
whenever $\mathbb{P}(\tau_B^{(r,v)} = \infty) > 0$, and for any stopping time $\tau_B^{(r,v)}$,

\[
\left\{ e^{-\gamma \tau_n + \alpha \int_0^{\tau_n} r_t \, ds} K_U(r_{\tau_n}) \chi_{\{ \tau_B^{(r,v)} < \infty \}} \right\}_{n \in \mathbb{N}}
\]

is uniformly integrable, where $\tau_n = n \wedge \tau_B^{(r,v)}$. Moreover, $K_L \in \mathcal{D}(A)$ and $K_U \in \mathcal{D}(A)$, where $\mathcal{D}(A) = \{ \varphi \in C^2(O^+) \cap \tilde{E} : A\varphi \in \tilde{E} \}$, and $K_L(0) = K_U(0) = 1$.

Note that if (\tilde{H}.4) holds, then it holds simultaneously for both processes $(r_t)$ and $(\tilde{r}_t)$.

Assume additionally (\tilde{H}.5) For any $r \in O^+$, one has $A_1 K_L(r) = 0$.

By (\tilde{H}.5), $K_L$ is a subsolution to (32). It is easy to see that under assumptions of Proposition \[4\] $\tilde{N}$ satisfies (21); it is enough to take $a_n = 0$ in the proof. Thus $\tilde{N}$ is a supersolution to (32). Hence, by (\tilde{H}.5),

\[
A_1 K_U + F_m(K_U) = A_1 \tilde{N}^{1-a} + F_m(K_L + \tilde{N}^{1-a}) \\
\leq A_1 \tilde{N}^{1-a} + F_m(\tilde{N}^{1-a}) \leq 0
\]

and $K_U$ is also a supersolution.

Since $K_L(0) = K_U(0) = 1$, then from the fact that $K_L \leq K \leq K_U$ (see Theorem \[2\] below) we have a condition $K(0) = 1$, which with help of (39) and (40) implies (9). Furthermore, the value function $\Phi(\cdot, v) \in C^2(O^+) \cap C(O)$ for any $v > 0$.

Theorem 4. Assume that (\tilde{H}.1) - (\tilde{H}.5) are fulfilled. Then there is a solution $K$ to (6) with condition (9). Moreover, $K_L(r) \leq K(r) \leq K_U(r), r \in O^+$. Finally, for any sequence $\{\lambda_m\} \subset \varrho(A_r)$ satisfying (28) for any $m > 0$, one has

\[
K(r) = \lim_{m \to \infty} \lim_{n \to \infty} K_n^m(r), \quad r \in O^+,
\]

where $\{K_n^m\}$ is a non-decreasing sequence of both $m$ and $n$, defined as follows

$K_0^m = K_L$,

\[
K_{n+1}^m = (\lambda_m - A_1)^{-1}(F_m(K_n^m) + \lambda_m K_n^m).
\]

9. Vasicek model

Let us recall that in the so-called Vasicek model $(r_t)$ is given by

\[
\begin{align*}
\frac{dr_t}{r_t} &= (a - br_t)dt + \sigma dW_t,
\end{align*}
\]
with $a, b, \sigma > 0$. Let

\begin{equation}
E = \{ \varphi \in C(\mathbb{R}) : \lim_{|r| \to \infty} |\varphi(r)| e^{-\frac{\alpha}{b}|r|} = 0 \}
\end{equation}

and

$$
\|\varphi\|_E = \sup_{r \in \mathbb{R}} |\varphi(r)| e^{-\frac{\alpha}{b}|r|}.
$$

**Theorem 5.** The assumptions of Theorem 2 are satisfied, whenever

\begin{equation}
\gamma > \max\{\gamma_1, \gamma_2\},
\end{equation}

where

$$
\gamma_1 = \frac{aa}{b} + \frac{a^2\sigma^2}{(1 - \alpha)b^2} \quad \text{and} \quad \gamma_2 = \frac{aa}{b} + \frac{3a^2\sigma^2}{2\sqrt{1 - \alpha}b^2} + \alpha \sigma b + 1.
$$

**Proof** Notice that for any stopping time $T_n$,

\begin{equation}
|\varphi(r_{t \wedge T_n})| e^{\alpha \int_{0}^{t \wedge T_n} r_s ds} \leq \|\varphi\|_E e^{\frac{2}{b}|r_{t \wedge T_n}| + \alpha \int_{0}^{t \wedge T_n} |r_s| ds} \leq \|\varphi\|_E e^{\left(\frac{\gamma_1}{b} + \alpha t\right) \sup_{0 \leq s \leq t} |r_s|}
\end{equation}

and, by Fernique’s theorem, the r.h.s. is integrable for any fixed $t \geq 0$. We similarly obtain

\begin{equation}
\left| \int_{0}^{t \wedge T_n} \varphi(r_s) e^{\alpha \int_{0}^{s} r_u du} ds \right| \leq \|\varphi\|_E e^{\frac{\gamma_2}{b} + \alpha t} \sup_{0 \leq s \leq t} |r_s|.
\end{equation}

Therefore (H.1) is satisfied.

It is easy to check that $(E, \| \cdot \|_E)$ satisfies (H.2). Assume that $(r_t)$ is given by (41) and that $(P_t)$ is given by (23). In Appendix A it is shown that $(P_t)$ is a $C_0$-semigroup on $E$, and hence hypothesis (H.3) is satisfied. Therefore we have to show (H.4). We split a verification of (H.4) into several steps.

**Step 1.** First we show that $N(r) < \infty$ for any $r \in \mathbb{R}$. From (41) we obtain

\begin{equation}
r_t = re^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma X_t,
\end{equation}

where

\begin{equation}
X_t = \int_{0}^{t} e^{-b(t-s)} dW_s
\end{equation}

and its distribution does not depend on $r$. In what follows we denote by $\mathcal{L}(\xi)$ the law (distribution) of a random variable $\xi$. Note that

\begin{equation}
\mathcal{L}(X_t) = \mathcal{N} \left( 0, \frac{1}{2b}(1 - e^{-2bt}) \right).
\end{equation}
Therefore, we have
\[
N(r) = \mathbb{E}^r \int_0^\infty e^{-t} e^{\frac{\alpha}{\gamma} \int_0^t r_s ds} dt \\
\leq \mathbb{E}^r \int_0^\infty e^{-t} e^{\frac{\alpha}{\gamma} \int_0^t |r| e^{-b} ds + \frac{2}{b} t^{\frac{\alpha}{2}} e^{\frac{\alpha}{2} \sigma^2} R} \int_0^t X_s ds dt \\
= e^{\frac{\alpha}{\gamma} \int_0^t |r|} \int_0^\infty e^{-t} e^{\frac{\alpha}{\gamma} \int_0^t X_s ds dt}
\]
and by Fubini's theorem
\[
(47) \ Y_t := \int_0^t X_s ds = \int_0^t \int_u^t e^{-b(s-u)} ds dW_u = \frac{1}{b} \int_0^t (1 - e^{-b(t-u)}) dW_u,
\]
which implies that
\[
(48) \ \mathcal{L}(Y_t) = \mathcal{N}(0, \frac{1}{b^2} (t - \frac{3}{2b} + \frac{2}{b} e^{-bt} - \frac{1}{2b} e^{-2bt})).
\]
Thus
\[
N(r) \leq e^{\frac{\alpha}{\gamma} \int_0^t |r|} \int_0^\infty e^{-t} e^{\frac{\alpha}{\gamma} \int_0^t \sigma^2} (t - \frac{3}{2b} + \frac{2}{b} e^{-bt} - \frac{1}{2b} e^{-2bt}) dt \\
\leq e^{\frac{\alpha}{\gamma} \int_0^t |r|} \int_0^\infty e^{-t} e^{\frac{\alpha}{\gamma} \int_0^t \sigma^2} t dt \\
= e^{\frac{\alpha}{\gamma} \int_0^t |r|} \int_0^\infty e^{-\rho t} dt < \infty,
\]
by (43). Analogously we can show that by (43), condition (20) holds.

**Step 2.** Here we show (25). We have just shown that
\[
(49) \ N(r) \leq \frac{e^{\frac{\alpha}{\gamma} \int_0^t |r|}}{\rho},
\]
where
\[
\rho = \frac{1}{1 - \alpha} \left( \gamma - \frac{\alpha a}{b} - \frac{\alpha^2 \sigma^2}{2(1 - \alpha) b^2} \right)
\]
is positive by (43). Then to prove (25) it is enough to show that
\[
\lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds + \frac{2}{b} |r_t|} = 0.
\]
From Hölder's inequality
\[
\lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds + \frac{2}{b} |r_t|} \leq \lim_{t \to \infty} \left( \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds} \right)^{1 - \alpha} \left( \mathbb{E}^r e^{\frac{\alpha}{2} \sigma^2} R \right)^{\alpha}
\]
and we easily compute that
\[
\lim_{t \to \infty} \mathbb{E}^r e^{-\gamma t + \alpha \int_0^t r_s ds} \leq \lim_{t \to \infty} e^{\frac{\alpha}{\gamma} \int_0^t |r|} e^{-\rho t} = 0, \quad \forall r \in \mathbb{R}.
\]
Note that given \( \xi \) with distribution \( N(m, s^2) \), it is easy to show that
\[
E e^{\kappa |\xi|} \leq e^{\frac{\kappa^2}{2}} (1 + e^{\kappa|m|}).
\]
Therefore, the expression in the second bracket above is dominated by
\[
e^{-\frac{\kappa^2}{4s^2}} \left( 1 + e^{\frac{|r|}{s}} \right),
\]
which is finite for every \( r \in \mathbb{R} \). Thus (25) holds.

**Step 3.** Here we show that the family in (26) is uniformly integrable. By the de la Vallée Poussin theorem (see e.g. [9], p. 241), it is enough to show that
\[
\sup_{n \in \mathbb{N}} E e^{\gamma \tau_n + \alpha \int_0^{\tau_n} r_s ds} N^{1-\alpha}(r_{\tau_n})^{\frac{1}{\alpha}} < \infty, \quad \forall r \in \mathcal{O},
\]
for some \( \beta < 1 \). Here we take \( \beta = \sqrt{1 - \alpha} \).

By (49) we obtain
\[
e^{-\frac{\gamma \tau_n + \alpha}{\beta^2} \int_0^{\tau_n} r_s ds} N^{\beta}(r_{\tau_n})^{\frac{1}{\alpha}} \leq \rho^{-\beta} e^{-\frac{\gamma \tau_n + \alpha}{\beta^2} \int_0^{\tau_n} |r_s ds + \frac{\alpha}{\beta^2} r_{\tau_n}|}
\]
\[\leq \rho^{-\beta} e^{\sup_{t \leq n} (-\frac{\gamma t + \alpha}{\beta^2} \int_0^{t} |r_s| ds + \frac{\alpha}{\beta^2} |r_t|)} \leq I,
\]
where
\[
I = \rho^{-\beta} e^{\frac{\alpha \sigma}{b \beta} + \frac{\alpha \sigma}{b \beta} |r|} e^{\sup_{t \leq n} (-\frac{b \gamma - a \alpha - b \alpha \sigma}{b \beta} t + \frac{\alpha \sigma}{b \beta} g(X_t) + \frac{\alpha \sigma}{b \beta} \int_0^t h(X_s) ds)}.
\]

By Itô’s formula
\[
\frac{\alpha \sigma}{b \beta} g(X_t) + \frac{\alpha \sigma}{\beta} \int_0^t h(X_s) ds = \frac{\alpha \sigma}{b \beta} + \Psi_t + R_t,
\]
where
\[
\Psi_t = \frac{\alpha \sigma}{b \beta} \int_0^t g'(X_s) dW_s - \frac{1}{2} \left( \frac{\alpha \sigma}{b \beta} \right)^2 \int_0^t (g'(X_s))^2 ds
\]
and
\[
R_t = \frac{1}{2} \int_0^t \left( \frac{\alpha \sigma}{b \beta} g''(X_s) + \left( \frac{\alpha \sigma}{b \beta} g'(X_s) \right)^2 \right) ds.
\]

Note that \( |g'(x)| < 1 \) and \( |g''(x)| < 2 \). Therefore by the Novikov condition \( M_t = e^{\Psi_t} \) is a martingale, and \( R_t < (\frac{\alpha \sigma}{b \beta} + \frac{1}{2}(\frac{\alpha \sigma}{b \beta})^2) t \). By (43) there is a \( \kappa > (\frac{\alpha \sigma}{b \beta})^2 \) such that
\[
\frac{b \gamma - a \alpha - b \alpha \sigma}{b \beta} > \frac{\alpha \sigma}{b \beta} + \frac{1}{2} \left( \frac{\alpha \sigma}{b \beta} \right)^2 + \kappa.
\]
Then
\[ \sup_{n \in \mathbb{N}} \mathbb{E} e^{-\frac{\gamma}{2} r_n + \frac{\alpha}{2} \int_0^{r_n} r_s \, ds} N^\beta(r_{\tau_n}) \leq \rho^{-\beta} e^{\frac{\alpha (2 + \kappa)}{2} \sup \tau_n} \sup_{n \in \mathbb{N}} \mathbb{E} \sup_{t \leq n} M_t e^{-\kappa t} \]
and it is enough to show that
\[ \sup_{n \in \mathbb{N}} \mathbb{E} \sup_{t \leq n} M_t e^{-\kappa t} < \infty. \]

We have
\[ \mathbb{E} \sup_{t \leq n} M_t e^{-\kappa t} \leq \sum_{j=0}^{n-1} \mathbb{E} \sup_{t \in [j, j+1]} M_t e^{-\kappa t} \leq \sum_{j=0}^{n-1} e^{-\kappa j} \mathbb{E} \sup_{t \in [j, j+1]} M_t \]
\[ \leq \sum_{j=0}^{n-1} e^{-\kappa j} (1 + \mathbb{E} (\sup_{t \leq j+1} M_t)^2) \leq \sum_{j=0}^{n-1} e^{-\kappa j} (1 + 4 \mathbb{E} M_{j+1}^2), \]
where the last estimate holds due to Doob’s inequality. Since \( M_t \leq e^{\left(\frac{\alpha \sigma}{2}\right)^2 t} \hat{M}_t \), where \( \hat{M}_t \) is a martingale of the same form as \( M_t \), but with constant \( \frac{\alpha \sigma}{2} \) instead of \( \frac{\alpha \sigma}{\beta} \), then
\[ \lim_{n \to \infty} \sum_{j=0}^{n-1} e^{-\kappa j} (1 + 4 \mathbb{E} M_{j+1}^2) < \infty. \]

**Step 4.** Here we show that \( N^{1-\alpha} \in C^2(\mathbb{R}) \cap E \). By Proposition 4 \( N^{1-\alpha} \in C^2(\mathbb{R}) \). To show that \( N^{1-\alpha} \in E \) we have to prove that
\[ \lim_{|r| \to +\infty} N^{1-\alpha}(r) e^{-\frac{\alpha}{2} |r|} = 0. \]
It is easy to see that
\[ \lim_{r \to -\infty} N^{1-\alpha}(r) e^{-\frac{\alpha}{2} |r|} = \lim_{r \to -\infty} N(r) e^{-\frac{\alpha}{1-\alpha} |r|} = 0. \]
The condition
\[ \lim_{r \to +\infty} N(r) e^{-\frac{\alpha}{1-\alpha} |r|} = 0 \]
amounts to
\[ \lim_{x \to +\infty} \int_0^\infty e^{-kt - xe^{-t}} \, dt = 0, \quad k > 0, \]
which clearly holds.

**Step 5.** Finally, we need to show \( AN^{1-\alpha} \in E \). By the definition of \( A \) (see (24)) and the previous step of the proof, we know that \( AN^{1-\alpha} \in C(\mathbb{R}) \). Thus we need to verify the condition
\[ \lim_{|r| \to \infty} |AN^{1-\alpha}(r)| e^{-\frac{\alpha}{2} |r|} = 0. \]
By (21) we have
\[ AN^{1-\alpha}(r) = N^{1-\alpha}(r) \left( \gamma \frac{1 - \alpha}{N(r)} - \frac{\alpha(1 - \alpha)\sigma^2}{2} \left( \frac{N'(r)}{N(r)} \right)^2 \right). \]

Since
\[ N(r) = \int_0^\infty e^{\frac{\alpha}{1-\alpha} (1-e^{-bt})} \phi(t) dt, \]
where \( \phi \) is, by Step 1, a strictly positive integrable function, then \( N \) is positive and increasing. Furthermore,
\[ N'(r) = \int_0^\infty \frac{1}{\partial r} e^{\frac{\alpha}{1-\alpha} (1-e^{-bt})} \phi(t) dt \leq \frac{\alpha}{(1-\alpha)b} N(r). \]
Hence,
\[ \lim_{|r| \to \infty} |AN^{1-\alpha}(r)|e^{-\frac{\alpha}{\delta} |r|} \leq \lim_{|r| \to \infty} N^{1-\alpha}(r) e^{-\frac{\alpha}{\delta} |r|} \left( \gamma + \frac{1 - \alpha}{N(r)} + \frac{\alpha^2 \sigma^2}{2(1-\alpha)b^2} \right) \]
and, since \( N^{1-\alpha} \in E \), the limit above is equal to zero. \( \square \)

Note that the condition \( \gamma > \gamma_1 \) assures the finiteness of \( N(r) \) for any \( r \in \mathbb{R} \) and that assumption (20) holds, and the condition \( \gamma > \gamma_2 \) is needed for uniform integrability of the family in (26).

Let \( \delta > \frac{\alpha(3-\alpha)}{b(1-\alpha)} \) and let
\[ \tilde{E} = E_\delta = \{ \varphi \in C([0, \infty)) : \lim_{r \to \infty} |\varphi(r)|e^{-\delta r} = 0 \} \]
be equipped with the norm
\[ \| \varphi \|_{\tilde{E}} = \sup_{r \in [0,\infty)} |\varphi(r)|e^{-\delta r}. \]
Then we have the following result.

**Theorem 6.** The assumptions of Theorem 4 are fulfilled in the Vasicek model (41), whenever (43) holds.

**Proof** Verification of \( \tilde{H}.1 \), \( \tilde{H}.2 \) and \( \tilde{H}.3 \) is left to the reader, as it is similar to verification of \( H.1 \), \( H.2 \) and \( H.3 \). Here we verify only \( \tilde{H}.4 \) and \( \tilde{H}.5 \). To show \( \tilde{H}.5 \) define a sequence of functions \( \{ K^n_L \} \), such that
\[
\begin{cases}
A_r K^n_L(r) = 0, & r \in (0, n), \\
K^n_L(0) = K^n_L(n) = 1.
\end{cases}
\]
Then, by (43),
\[ K^n_L(r) = \mathbb{E}^r e^{-\gamma \tau_{0,n} + \alpha \int_0^{\tau_{0,n}} r_s ds}, \]
where $\tau_{0,n}^r = \tau_0^r \land \tau_n^r$ and $\tau_n^r = \inf\{t \geq 0 : r_t = n\}$. Furthermore, for any test function $\varphi \in C_0^\infty((0,n))$,

$$
\int_0^\infty \varphi(r)A_rK_L^n(r)dr = 0,
$$

which implies that

$$
\lim_{n \to \infty} \int_0^\infty K_L^n(r)A_r^s\varphi(r)dr = 0.
$$

Under the following conditions,

(i) $\lim_{n \to \infty} \tau_{0,n}^r = \tau_0^r, \ \forall r > 0, \forall \omega \in \Omega$,

(ii) $\mathbb{P}^{r}(\tau_0^r < \infty) = 1, \ \forall r > 0$,

(iii) $\sup_{r \leq j} \sup_{n \in \mathbb{N}} |K_L^n(r)| < \infty, \ \forall j > 0$,

which we will verify below, we have $\lim_{n \to \infty} K_L^n(r) = K_L(r)$ and the convergence is almost uniform. Thus,

$$
\int_0^\infty K_L(r)A_s^r\varphi(r)dr = 0,
$$

and it holds for any $\varphi \in C_0^\infty((0,\infty))$. By Lemma 1 $A_rK_L(r) = 0$.

Since any Ornstein–Uhlenbeck process is recurrent, then (ii) holds. Condition (i) is implied by the continuity of trajectories. To show (iii), note that due to Step 3 of the proof of Theorem 5 the sequence of random variables in the second expression below is uniformly integrable, which justifies the first equality, and the constant $d < \infty$ does not depend on $r$. Thus,

$$
\sup_{r \leq j} \sup_{n \in \mathbb{N}} |K_L^n(r)| = \sup_{r \leq j} \sup_{n \in \mathbb{N}} \mathbb{E}^{r} \lim_{m \to \infty} e^{-\gamma(\tau_{0,n}^r \land m) + \alpha \int_0^{\tau_{0,n}^r \land m} r_sds}
$$

$$
\leq \sup_{r \leq j} \sup_{n \in \mathbb{N}} \mathbb{E}^{r} \sup_{m \geq 0} e^{\sup_{t \leq m} (-\gamma t + \frac{\alpha}{2} t + \alpha \sigma Y_t)}
$$

$$
\leq d \sup_{r \leq j} e^{\frac{\alpha}{2} r} = de^{\frac{\alpha}{2} j} < \infty,
$$

where $Y_t$ is given by (47).

We proceed to show that (H.4) holds. Since condition (ii) above holds, then we do not have to verify (39).

Note that $K_U \leq K_L + N^{1-\alpha}$. Therefore, by Theorem 3 the sequence in (10) is uniformly integrable whenever uniformly integrable is the sequence

$$
\left\{ e^{-\gamma t_n + \alpha \int_0^{t_n} \hat{r}_sds} K_L(r_{t_n}) \right\}_{n \in \mathbb{N}}.
$$
By the strong Markov property it is equivalent to the uniform integrability of
\[ \left\{ \mathbb{E}^r \left[ e^{-\gamma \tau_0^r + \alpha \int_0^{\tau_0^r} \tilde{r} \, ds} \right] \right\}_{n \in \mathbb{N}}, \]
which is fulfilled whenever
\[ \mathbb{E}^r e^{-\gamma \tau_0^r + \alpha \int_0^{\tau_0^r} \tilde{r} \, ds} < \infty. \]
This holds, since \( \tau_0^r \wedge n \to \tau_0^r \) as \( n \to \infty \) and
\[ \left\{ e^{-\gamma (\tau_0^r \wedge n) + \alpha \int_0^{\tau_0^r \wedge n} \tilde{r} \, ds} \right\}_{n \in \mathbb{N}} \]
is uniformly integrable (see Step 3 of the proof of Theorem 5).

Here we will show that \( K_L \in \mathcal{D}(A) \). Since \( A \gamma K_L = 0 \) then \( K_L \in C^2((0, \infty)) \) and it is enough to show that \( K_L \in \tilde{E} \). By the similar argumentation to that in verification of (iii) in the previous step of the proof, we have
\[
\lim_{r \to \infty} |K_L(r)| e^{-\delta r} = \lim_{r \to \infty} \mathbb{E}^r \lim_{m \to \infty} e^{-\gamma (\tau_0^r \wedge m) + \alpha \int_0^{\tau_0^r \wedge m} \tilde{r} \, ds} e^{-\delta r} \\
\leq \lim_{r \to \infty} e^{\delta r - \delta r} \sup_{m \geq 0} \mathbb{E} e^{\sup_{t \leq m} (-\gamma t + \frac{\alpha}{\sigma^2} t + \alpha \sigma Y_t)} \\
\leq d \lim_{r \to \infty} e^{\frac{\alpha}{b} r - \delta r} = 0.
\]

Now we will show that \( \tilde{N} \in \mathcal{D}(A) \), which will imply that \( K_U \in \mathcal{D}(A) \). Since, by condition (43), \( \tilde{N} \) satisfies
\[
Q \tilde{N}(r) + \frac{\alpha r - \gamma}{1 - \alpha} \tilde{N}(r) = -1,
\]
then \( \tilde{N} \in C^2((0, \infty)) \), and consequently \( \tilde{N}^{1-\alpha} \in C^2((0, \infty)) \). Furthermore, \( \tilde{N}^{1-\alpha} \in \tilde{E} \), since
\[
\lim_{r \to \infty} \tilde{N}^{1-\alpha}(r) e^{-\delta r} \leq \lim_{r \to \infty} N^{1-\alpha}(r) e^{-\delta r} \leq \rho^{\alpha - 1} \lim_{r \to \infty} e^{\delta r - \delta r} = 0.
\]
Analogously, \( \tilde{N}^{1-\alpha} \in E_{\delta_1} \) for any \( \delta_1 > \alpha/b \), and consequently \( \tilde{N} \in E_{\delta_2} \) for any \( \delta_2 > \frac{\alpha}{b(1-\alpha)} \).

In order to prove that \( A \tilde{N}^{1-\alpha} \in \tilde{E} \), note that by (52), we have
\[
A \tilde{N}^{1-\alpha} = \tilde{N}^{1-\alpha} \left( \gamma - \frac{1 - \alpha}{N(r)} - \frac{\alpha(1 - \alpha)\sigma^2}{2} \left( \frac{\tilde{N}(r)}{N(r)} \right)^2 \right).
\]
We need the following result.

**Lemma 3.** Let \( f \in C([0, \infty)) \cap C^1((0, \infty)) \) and \( f' \in E_{\delta^1} \). Then \( f \in E_{\delta^2} \) for any \( \delta^2 > \delta^1 \).
Proof Since for any $x > 0$, one has $f(x) = f(0) + \int_0^x f'(y)dy$, then
\[
|f(x)|e^{-\delta x} \leq |f(0)|e^{-\delta x} + e^{-(\delta^2 - \delta^3)x} \int_0^x |f'(y)|e^{-\delta^3(x-y)}dy
\]

\[
\leq |f(0)|e^{-\delta x} + \frac{M}{\delta^1}e^{-(\delta^2 - \delta^3)x},
\]
where $M = \sup_{x \geq 0} |f'(x)|e^{-\delta^1 x}$. Since $f' \in E_{\delta^1}$, then $M$ is finite. □

Going back to the proof of Theorem 6 note that we can rewrite (52) as follows
\[
\left(\frac{1}{2} \sigma^2 \tilde{N}' + (a - br)\tilde{N}\right)' = -1 - \left(b + \frac{\alpha r - \gamma}{1 - \alpha}\right)\tilde{N},
\]
which implies that the l.h.s. belongs to $E_{\delta^2}$, and by the lemma above,\[
\frac{1}{2} \sigma^2 \tilde{N}' + (a - br)\tilde{N} \in E_{\delta^3}
\]
for any $\delta_3 > \delta_2$. Since $(a - br)\tilde{N} \in E_{\delta_2} \subset E_{\delta_3}$, then $\tilde{N} \in E_{\delta_3}$. Hence, for any $\delta_1 > \alpha/b$ and $\delta = \delta_1 + 2\delta_3$, we get
\[
\lim_{r \to \infty} |A \gamma \tilde{N}^{1-\alpha}(r)|e^{-\delta r} \leq \lim_{r \to \infty} \tilde{N}^{1-\alpha}(r)e^{-\delta_1 r} \left(\gamma + \frac{1 - \alpha}{\tilde{N}(r)}\right)e^{-2\delta_1 r}
\]
\[
+ \lim_{r \to \infty} \tilde{N}^{1-\alpha}(r)e^{-\delta_1 r} \frac{\alpha(1 - \alpha)\sigma^2}{2\tilde{N}^2(r)}(\tilde{N}'(r)e^{-\delta_3 r})^2.
\]
It is easy to verify that $\tilde{N}$ is increasing, which implies that $\tilde{N}(r) > 0$ for any $r > 0$. Thus, the limit above is equal to zero. □

10. Invariant interval model

Here we assume that the short-rate dynamics is given by (1), $\mathcal{O} = (a, b)$, where $-\infty < a < b < \gamma/\alpha$ and $E = UC((a, b))$ is equipped with the supremum norm.

The sufficient condition for interval invariance is (see [1])
\[
s(a^+) = -\infty \quad \text{and} \quad s(b^-) = \infty,
\]
where
\[
s(x) = \int_w^x e^{\int_w^y \frac{2u(z)}{\sigma^2(z)}dz}dy
\]
for a fixed $w \in (a, b)$.

It is easy to show that the conditions above holds in the model
\[
dr_t = \kappa\left(\frac{a + b}{2} - r_t\right)dt + \sigma(r_t - a)(b - r_t)dW_t
\]
with $\kappa, \sigma > 0$.

Theorem 7. The assumptions of Theorem 2 hold in the invariant interval model (53).
Proof Notice that for any stopping time $T_n$ and any fixed $t \geq 0$,
\[ \| \varphi(r_{t \wedge T_n}) e^{\alpha t} \| \leq \| \varphi \| \| e^{\alpha t} \| \leq \| E e^{\alpha t} \| < \infty, \]
and similarly
\[ \int_0^{T_n} \| \varphi(r_s) e^{\alpha s} \| ds \leq \| \varphi \| \| e^{\alpha s} \| < \infty, \]
which means that (H.1) is satisfied. One may easily check that the space $(E, \| \cdot \|_E)$ satisfies (H.2). Assume that $(P_t)$ is given by (23). In Appendix B it is shown that $(P_t)$ is a $C_0$-semigroup on $E$, and hence hypothesis (H.3) is satisfied. Thus we have to verify (H.4).

Given $b < \gamma/\alpha$, we have
\[ N(r) = \int_0^\infty e^{-\frac{1}{\gamma - \alpha b} t} = \frac{1 - \alpha}{\gamma - \alpha b} \]
and
\[ \lim_{t \to \infty} E e^{-\gamma t + \alpha t} \int_0^{r_n} e^{\alpha s} ds N_{\alpha}(r_t) \leq \lim_{t \to \infty} e^{-(\gamma - \alpha b) t} \left( \frac{1 - \alpha}{\gamma - \alpha b} \right)^{1 - \alpha} = 0. \]
Thus $N < \infty$ and (25) holds. In the same manner we can see that (20) holds.

Recall that (51) implies uniform integrability of the family in (26). Let $\beta = 1 - \alpha$. Then (51) holds, since we have
\[ \sup_{n \in \mathbb{N}} E e^{-\frac{\gamma - \alpha b}{\gamma - \alpha b} r_n} \leq \sup_{n \in \mathbb{N}} e^{-\frac{\gamma - \alpha b}{\gamma - \alpha b} r_n} = \frac{1 - \alpha}{\gamma - \alpha b}. \]
By Proposition 4, one has $N^{1-\alpha} \in C^2((a, b))$. Note that $N$ is bounded, i.e.
\[ 0 < \frac{1 - \alpha}{\gamma - \alpha a} \leq N(r) \leq \frac{1 - \alpha}{\gamma - \alpha b} < \infty. \]
Since $N$ is also increasing and continuous, then there exist finite limits $N(a^+)$ and $N(b^-)$. Thus $N^{1-\alpha} \in UC((a, b))$. By (21), we have
\[ AN^{1-\alpha}(r) = N^{1-\alpha}(r) \left( \gamma - \frac{1 - \alpha}{N(r)} - \frac{\alpha(1 - \alpha)(\sigma(r - a)(b - r)N'(r))^2}{2N^2(r)} \right). \]
Hence, $AN^{1-\alpha} \in C((a, b))$. Since $N'(r) \geq 0$ for all $r \in (a, b)$, and
\[ N \left( \frac{a + b}{2} \right) - N(a^+) = \int_a^{a+b} N'(r) dr = \int_a^{a+b} (r - a) N'(r) \frac{1}{r - a} dr \]
is finite, then necessarily $\lim_{r \to a}(r - a)N'(r) = 0$. Analogously we get $\lim_{r \to b^{-}}(b - r)N'(r) = 0$. Thus there exist limits $\lim_{r \to a^+} AN^{1-\alpha}(r)$ and $\lim_{r \to b^-} AN^{1-\alpha}(r)$, which implies that $AN^{1-\alpha} \in UC((a, b))$. \hfill \Box
11. Models with infinite value function

We will show here that if \((r_t)\) is either a Brownian motion or a geometric Brownian motion, then the value function in Problem A is infinite.

Let us observe first that we may assume that optimal consumption is of the proportional form \(C_t = c_t V_t\), where

\[
c_t = \begin{cases} 
  C_t / V_t, & t < \tau_A, \\
  0, & t \geq \tau_A,
\end{cases}
\]

for \(\tau_A\) given by (2) and \(c_t\) is well defined. Also in this case the HJB equation and the optimal consumption \(\hat{C}\) have the form (6) and (8) respectively. Moreover,

\[
dV_t = (r_t - c_t) V_t dt,
\]

and consequently

\[
V_t = v e^{\int_0^t (r_s - c_s) ds} > 0, \quad \forall v > 0, \quad \forall t < \tau_A,
\]

which implies that

\[
\tau_A = \inf \left\{ t \geq 0 : \int_0^t c_s ds = \infty \right\}.
\]

Thus from now on, we assume that our consumption is of the proportional form and

\[
J_A(c; r, v) := v^\alpha \mathbb{E}^r \int_0^\infty e^{-\gamma t} c_t^\alpha e^{\alpha \int_0^t (r_s - c_s) ds} dt
\]

with \(c_t = 0\) for every \(t \geq \tau_A\).

**Lemma 4.** Assume \(r_t = r = \text{const.}\) Then:

i) If \(\gamma - \alpha r \leq 0\), then there is a consumption rate \(C\) such that \(J_A(C; r, v) = \infty\) for all \(v > 0\).

ii) If \(\gamma - \alpha r > 0\), then

\[
\Phi_A(r, v) = \left( \frac{\gamma - \alpha r}{1 - \alpha} \right)^{\alpha - 1} v^\alpha, \quad \hat{C}_t = \frac{\gamma - \alpha r}{1 - \alpha} V_t, \\
V_t = e^{(r - \gamma - \alpha r) t} v = e^{\frac{\gamma - \alpha r}{1 - \alpha} t} v.
\]

**Proof of i**) Whenever \(\gamma - \alpha r \leq 0\), then (54) gives us the claim with \(c_t \leq \alpha r - \gamma\). □

**Proof of ii**) Since now \(\mu(r) = \sigma(r) = 0\), we have \(\hat{C}_t = K^{1/(\alpha - 1)} V_t\) and

\[
(\alpha r - \gamma) K + (1 - \alpha) K^{\frac{\gamma - \alpha r}{1 - \alpha}} = 0
\]
instead of (6). If \( \gamma - \alpha r > 0 \), then

\[
K = \left( \frac{\gamma - \alpha r}{1 - \alpha} \right)^{\frac{1}{\alpha - 1}}.
\]

\( \square \)

**Remark 6.** Recall that if \( r_t > 0 \) for every \( t \geq 0 \), then Problem B amounts to Problem A. Since condition \( \gamma - \alpha r \leq 0 \) implies \( r > 0 \), thus the first claim in Lemma 4 holds also for Problem B. The second claim is true for Problem B, whenever \( r > 0 \). Otherwise \( \Phi_B(r,v) = v^\alpha \) with \( \tau_B = 0 \).

Now we formulate necessary and sufficient conditions for finiteness of value function \( \Phi_A \). Set \( S = \mathcal{O} \times (0, \infty) \).

**Lemma 5.** i) If \( \Phi_A(r,v) \) is finite for all \( (r,v) \in S \), then

\[
\forall r \in \mathcal{O} \\forall c > 0 \quad \mathbb{E}^r \int_0^\infty e^{-\gamma t + \alpha \int_0^t (rs - c)ds} dt < \infty.
\]

ii) If the performance functional is given by (54) and

\[
\exists \delta > 0 \exists \rho \in (1, \frac{1}{\alpha}) \\forall r \in \mathcal{O} \quad \mathbb{E}^r \int_0^\infty e^{-\gamma \delta t + \alpha \int_0^t rsds} dt < \infty
\]

holds with \( q = p/(p - 1) \), then \( \Phi_A(r,v) \) is finite for all \( (r,v) \in S \).

**Proof of i)** Taking \( c_t = c \) constant gives us the claim. \( \square \)

**Proof of ii)** Set \( \delta > 0 \). From (54) we have

\[
J_A(c;r,v) = v^\alpha \mathbb{E}^r \int_0^\infty e^{-\gamma \delta t + \alpha \int_0^t rsds} e^{-\alpha \int_0^t c_t ds} dt,
\]

and from Hölder’s inequality \( J_A(c;r,v) \) is dominated by

\[
v^\alpha \mathbb{E}^r \left( \int_0^\infty e^{-(\gamma - \delta)t + \alpha \int_0^t rsds} dt \right)^{\frac{1}{q}} \left( \int_0^\infty e^{-\delta t \frac{1}{\alpha} \rho c_t e^{-\alpha \int_0^t c_t ds}} dt \right)^{\frac{1}{p}}.
\]

From Lemma 4 with \( r = 0 \), the expression in the second bracket above is finite for every \( \delta > 0 \) and \( p > 1 \) such that \( \alpha p < 1 \). Thus (57) gives us the claim. \( \square \)

**Proposition 5.** If \( (r_t) \) is a drifted Brownian motion

\[
r_t = r + \mu t + \sigma W_t,
\]

or \( (r_t) \) is a geometric Brownian motion

\[
r_t = re^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},
\]

then (56) does not hold for any \( c > 0 \) and consequently the value function \( \Phi_A \) for Problem A is infinite.
**Proof** Notice that $\mathcal{L}(\int_0^t W_s ds) = \mathcal{N}(0, t^3/3)$, which implies

$$\mathbb{E}(e^{\alpha \int_0^t W_s ds}) = e^{\alpha t^2/6}.$$  

Thus if \((r_t)\) is a drifted Brownian motion, then

$$\mathbb{E}^r \int_0^\infty e^{-\gamma t + \alpha \int_0^t (r_s-c) ds} \, dt = \int_0^\infty e^{(\alpha r - \gamma - \alpha c) t + \frac{1}{2} \alpha \mu^2 \mathbb{E}(\int_0^t W_s ds^2) dt}$$

$$= \int_0^\infty e^{(\alpha r - \gamma - \alpha c) t + \frac{1}{2} \alpha \mu^2} \, dt = \infty$$

for all $c \geq 0$. Hence, $\Phi_A(r, v) = \infty$.

Notice that $e^y > y$ for all $y \in \mathbb{R}$. Thus if $(r_t)$ is a geometric Brownian motion, then we have

$$\mathbb{E}^r \int_0^\infty e^{-\gamma t + \alpha \int_0^t (r_s-c) ds} \, dt = \int_0^\infty e^{-(\gamma + \alpha c) t} \mathbb{E}(e^{\alpha r}_0 \int_0^t e^{\mu t} \frac{1}{2} \sigma^2 s^2 + \sigma W_s ds) \, dt$$

$$\geq \int_0^\infty e^{-(\gamma + \alpha c) t} \mathbb{E}(e^{\alpha r}_0 (\mu - \frac{1}{2} \sigma^2) s + \sigma W_s ds) \, dt$$

$$= \int_0^\infty e^{-(\gamma + \alpha c) t + \frac{1}{2} \alpha r (\mu - \frac{1}{2} \sigma^2 t^2 + \frac{1}{3} \alpha s^2 \sigma^2 t^3) dt} = \infty$$

for all $c \geq 0$. Hence, $\Phi_A(r, v) = \infty$. Moreover $\Phi_B(r, v) = \infty$, since in this case $r_t > 0$ for every $t \geq 0$. □

12. **Numerical results**

Here we present a numerical solution for a Vasicek model with parameters $a = 0.03$, $b = 0.5$ and $\sigma = 0.02$. We take $\alpha = 0.5$ and $\gamma = 1.5304$, which satisfies the condition (13). Since $\gamma > \vartheta$ (see (29)), we take $\lambda_m = \alpha m + 10^{-5}$.

Recall that the value function is given by $\Phi(r, v) = K(r) v^\alpha$, and $K(r)$ is as in Theorem 2. Therefore we have to approximate the function $K$ by $K_n^m$ for some large $m$ and $n$. Since $K_n^m(r)$ is given by recurrent formula

$$K_n^m(r) = \int_0^\infty \int_{-\infty}^\infty S_{n-1}^m(t, r, y) \, dt \, dy \approx \int_{t_{\min}}^{t_{\max}} \int_{y_{\min}}^{y_{\max}} S_{n-1}^m(t, r, y) \, dt \, dy$$

with a complicated function $S_{n-1}^m$ such that $\lim_{t \to 0} S_{n-1}^m(t, r, r) = \infty$, then we use trapezoidal quadrature. We take $\Delta t = 0.001$ and $\Delta y = 0.0002$ to get the result with a small error. In fact this makes the calculations very time-consuming. Thus we take $m = 65$ and $n = 25$ and we have the result as in Figure 1 for $r \in (0, 0.15)$. The result over the range $(-6, 8)$ is given only for $K_1^m$ due to very long calculations of $K_i^m$ for $i \geq 2$. 


Next we compute trajectories of the wealth \((V_t)\), the optimal consumption \((C_t)\) and the relative consumption \((c_t) = (C_t/V_t)\) for a given realization of the interest rate \((r_t)\). Clearly we take \(K_m^n\) instead of \(K\). The results for initial \(r = 0.05\) and \(v = 3\) are given in Figure 2.

\[
\begin{align*}
\text{Figure 1.} & \quad \text{(left) The dashed line is a supersolution } N^{1-\alpha}(r), \text{ and the solid lines are } K_i^m(r) \text{ for } i = 1, \ldots, n. \\
& \quad \text{(right) The dashed line is a supersolution } N^{1-\alpha}(r), \text{ and the solid line is } K_1^m(r).
\end{align*}
\]

\[
\begin{align*}
\text{Figure 2.} & \quad \text{Trajectories of processes } r(t, \omega), V(t, \omega), C(t, \omega) \text{ and } c(t, \omega) \text{ for the same } \omega \in \Omega, r = 0.05 \text{ and } v = 3.
\end{align*}
\]
Appendix A - Proof of $C_0$-Semigroup Property of $(P_t)$ in Case of Vasicek Model

For any $\varphi : \mathbb{R} \mapsto \mathbb{R}$ define

$$\|\varphi\| = \sup_{r \in \mathbb{R}} |\varphi(r)| e^{-\frac{\alpha}{b} |r|}.$$  

Note that $\|\varphi\| \leq \infty$, and $\|\varphi\| = \|\varphi\|_E$ for $\varphi \in E$, where $E$ is given by (42). We assume that $(r_t)$ and $(P_t)$ are given by (41) and (23) respectively.

In the subsequent steps of the proof we need the following result.

Lemma 6. For any $\varphi \in E$ and any $t \geq 0$,

$$\|P_t \varphi\| \leq 2e^{\left(\frac{\alpha^2 \sigma^2}{2b^2} + \frac{\alpha a}{b}\right)t} \|\varphi\|.$$  

Proof Notice that we do not assume that $P_t \varphi \in E$. This will be shown later. Let $X_t$ and $Y_t$ be given by (45) and (47) respectively. We have

$$\|P_t \varphi\| = \sup_{r \in \mathbb{R}} |P_t \varphi(r)| e^{-\frac{\alpha}{b} |r|} \leq \sup_{r \in \mathbb{R}} \mathbb{E}^r |\varphi(r_t)| e^{\alpha \int_0^t r_s ds - \frac{\alpha}{b} |r|}$$

$$\leq \|\varphi\| \sup_{r \in \mathbb{R}} \mathbb{E}^r e^{\alpha \int_0^t r_s ds + \frac{\alpha}{b} (|r_t| - |r|)}$$

$$\leq \|\varphi\| \sup_{r \in \mathbb{R}} e^{\frac{\alpha}{b} (1 - e^{-bt})(r - |r|) + \frac{\alpha a}{b} t} e^{\alpha \sigma (\frac{1}{2} X_t + Y_t)}$$

$$\leq e^{\frac{\alpha a}{b} t} \mathbb{E} e^{\alpha^2 \sigma^2 t} \|\varphi\| + e^{\alpha \sigma (\frac{1}{2} X_t + Y_t)} \|\varphi\|$$

$$\leq 2e^{\frac{\alpha^2 \sigma^2}{2b^2} t} \|\varphi\|. \quad \Box$$

Step 1. Denote by $E_{lips}$ the space of all functions $\varphi \in E$, which are Lipschitz continuous. Here we show that $P_t \varphi \in C(\mathbb{R})$ for any $\varphi \in E_{lips}$.

Define a sequence $\{\psi_k\}$ of continuous functions

$$\psi_k(x) = \begin{cases} 1, & x \in [-k, k], \\ k + 1 - |x|, & x \in (-k - 1, -k) \cup (k, k + 1), \\ 0, & x \in (-\infty, -k - 1] \cup [k + 1, \infty). \end{cases}$$

Then

$$|P_t \varphi(x) - P_t \varphi(y)| \leq |O_1| + |O_2| + |O_3|,$$
where
\[
\begin{align*}
O_1 &= \mathbb{E}^{x} \varphi(r_t) e^{\alpha \int_0^t r_s ds} \psi_k \left( \int_0^t r_s ds \right) - \mathbb{E}^{y} \varphi(r_t) e^{\alpha \int_0^t r_s ds} \psi_k \left( \int_0^t r_s ds \right), \\
O_2 &= \mathbb{E}^{x} \varphi(r_t) e^{\alpha \int_0^t r_s ds} \left( 1 - \psi_k \left( \int_0^t r_s ds \right) \right), \\
O_3 &= \mathbb{E}^{y} \varphi(r_t) e^{\alpha \int_0^t r_s ds} \left( 1 - \psi_k \left( \int_0^t r_s ds \right) \right).
\end{align*}
\]

Now we show that \( \lim_{y \to x} |O_1| = 0 \). To this end write
\[
h_t := \int_0^t r_s ds \quad \text{and} \quad \zeta_t = (r_t, h_t)^\top
\]
and define a function \( \phi_k(\zeta) \) as
\[
\phi_k(r, h) = \varphi(r) e^{\alpha h} \psi_k(h),
\]
which is Lipschitz continuous with constant \( L \), since both \( \varphi(r) \) and \( e^{\alpha h} \psi_k(h) \) are Lipschitz continuous. Denote by \( \zeta_t^x \) the value of \( \zeta_t \) with initial condition \( \zeta_0 = (x, 0)^\top \). We have
\[
|O_1| = |\mathbb{E}[\phi_k(\zeta_t^x) - \phi_k(\zeta_t^y)]| \leq L\mathbb{E}\|\zeta_t^x - \zeta_t^y\|_2^2 \leq L \sqrt{\mathbb{E}\|\zeta_t^x - \zeta_t^y\|_2^2},
\]
where \( \| \cdot \|_2 \) is the Euclidean norm.

Since
\[
d\zeta_t = \tilde{\mu}(\zeta_t) dt + \tilde{\sigma}(\zeta_t) dW_t := \begin{bmatrix} a - b r_t \\ r_t \end{bmatrix} dt + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} dW_t,
\]
with \( \tilde{\mu} \) and \( \tilde{\sigma} \) Lipschitz continuous, then from the mean-square continuity of \( \zeta \) (see [9]) we have
\[
\lim_{y \to x} |O_1| \leq \lim_{y \to x} L \sqrt{\mathbb{E}\|\zeta_t^x - \zeta_t^y\|_2^2} = 0
\]
for all \( k \in \mathbb{N} \).

Since we consider \( y \) close to \( x \), it is now sufficient to show that \( |O_2| \) converges to 0, as \( k \to \infty \), uniformly in \( \{x : |x| < \delta\} \) for any \( \delta > 0 \). We obtain
\[
|O_2| \leq \| \varphi \| \mathbb{E}^{x} e^{|x|+\alpha h_t} |1 - \psi_k(h_t)|,
\]
and from the Schwarz inequality
\[
|O_2| \leq \| \varphi \| \sqrt{\mathbb{E}^{x} e^{2|x|+2\alpha h_t} \sqrt{\mathbb{P}^{x}(|h_t| > k)}}.
\]
By (50) and Chebyshev’s inequality,
\[ \mathbb{P}^x(|h_t| > k) \leq \frac{\mathbb{E}^x|h_t|}{k} \leq \frac{\mathbb{E}^x e^{[h_t]}}{k} \leq \frac{e^{\frac{x^2}{2k}}} {k^2} \leq \frac{e^{\frac{x^2}{2k}} (1 + e^{\frac{|x|+\alpha h_t}{k}})} {k} \leq \frac{e^{\frac{x^2}{2k}} (1 + e^{\frac{\epsilon}{k}})} {k} \to 0, \]
as \( k \to \infty \). In a similar way we show that
\[ \sup_{|x|<\delta} \mathbb{E}^x e^{\frac{\alpha |r_t|}{2} + 2\alpha h_t} \leq \sup_{|x|<\delta} \sqrt{\mathbb{E}^x e^{\frac{\alpha |r_t|}{4} + \alpha h_t}} \sqrt{\mathbb{E}^x e^{4\alpha |h_t|}} < \infty \]
for every \( t \geq 0 \). Thus \( \lim_{k \to \infty} |O_2| = 0 \) and the convergence is uniform in \( \{ x : |x| < \delta \} \).

Thus we have
\[ \lim_{y \to x} |P_t \varphi(x) - P_t \varphi(y)| \leq \lim_{k \to \infty} \lim_{y \to x} (|O_2| + |O_3|) = 0. \]
Hence, \( P_t \varphi \) is continuous for all \( t \geq 0 \) and \( \varphi \in E_{lip} \).

**Step 2.** We show that \( P_t \varphi \in C(\mathbb{R}) \) for any \( \varphi \in E \). Let us fix a \( \varphi \in E \). As \( E_{lip} \) is dense in \( E \), there exists an approximating sequence \( \{ \varphi_n \} \) such that \( \varphi_n \in E_{lip} \) and \( \varphi_n \to \varphi \) in \( E \).

Set \( \varepsilon > 0 \). We have
\[ |P_t \varphi(x) - P_t \varphi(y)| \]
\[ \leq |P_t (\varphi - \varphi_n)(x)| + |P_t (\varphi - \varphi_n)(y)| + |P_t \varphi_n(x) - P_t \varphi_n(y)| \]
\[ \leq \|P_t (\varphi - \varphi_n)\| e^{\frac{\alpha |x|}{2}} + \|P_t (\varphi - \varphi_n)\| e^{\frac{\alpha |y|}{2}} + |P_t \varphi_n(x) - P_t \varphi_n(y)| \]
and from Lemma 6
\[ \forall \varepsilon > 0 \exists n_0 \forall n > n_0 \quad \|P_t (\varphi - \varphi_n)\| < \varepsilon. \]
Furthermore, from Step 1, \( P_t \varphi_n \in C(\mathbb{R}) \), i.e.
\[ \forall x \in \mathbb{R} \exists \delta > 0 \forall y \in \mathbb{R} \quad |x - y| < \delta \Rightarrow |P_t \varphi_n(x) - P_t \varphi_n(y)| < \varepsilon \]
and therefore
\[ |P_t \varphi(x) - P_t \varphi(y)| < \varepsilon (e^{\frac{\alpha |x|}{2}} + e^{\frac{\alpha |y|}{2}} + 1). \]

**Step 3.** Here we show that \( P_t : E \to E \). For any \( \varphi \in E \) write
\[ l(\varphi) = \lim_{|r| \to \infty} |P_t \varphi(r)| e^{-\frac{\alpha}{2} |r|}. \]
We need to show that \( l(\varphi) = 0 \). From (44) and (47) we have
\[ l(\varphi) \leq \lim_{|r| \to \infty} \mathbb{E}^x|\varphi(r_t)| e^{\alpha \int_0^r r_s ds - \frac{\alpha}{2} |r|} \]
\[ = \lim_{|r| \to \infty} \mathbb{E}^x|\varphi(r_t)| e^{\frac{\alpha}{2} (1 - e^{-bt}) + \frac{\alpha}{2} (t - \frac{1}{b} (1 - e^{-bt})) + \alpha \sigma Y_1 - \frac{\alpha}{2} |r|} \]
and from the Schwarz inequality
\[ l(\varphi) \leq \lim_{|r| \to \infty} e^{\frac{\alpha}{Y_t}(1-e^{-bt}) - \frac{1}{2}|r|} e^{\frac{\sigma^2}{2}(1-e^{-bt})} \sqrt{E} e^{\frac{\sigma^2}{2}} \sqrt{E} \varphi^2(r_t). \]

Hence
\[ l^2(\varphi) \leq g^2(t) \lim_{|r| \to \infty} e^{\frac{2\alpha}{Y_t}(1-e^{-bt}) - \frac{2}{2}|r|} |E \varphi^2(r_t)|, \]
where
\[ g(t) := e^{\frac{\alpha}{Y_t}(1-e^{-bt})} \sqrt{E} e^{\frac{\sigma^2}{2}}. \]

Clearly \( g(t) < \infty \) for any fixed \( t \geq 0 \), by (48) and (50).

Set \( \varepsilon > 0 \). Since \( \varphi \in E \), then there exists a \( \delta > 0 \) such that
\[ |\varphi(r_t)| \leq \varepsilon e^{\frac{\sigma^2}{2}|r|} \]
in a set \( \{ \omega : |r_t| \geq \delta \} \) and therefore
\[ E \varphi^2(r_t) \leq \varepsilon^2 |E| e^{\frac{2\alpha}{Y_t}|r_t|} + \| \varphi \|_\delta^2, \]
where \( \| \cdot \|_\delta \) is the supremum norm over \( \{|x| < \delta\} \). Clearly \( \| \varphi \|_\delta < \infty \) for every \( \varphi \in E \). From (44),
\[ l^2(\varphi) \leq \varepsilon^2 g^2(t) e^{\frac{2\alpha}{Y_t}(1-e^{-bt})} e^{\frac{\sigma^2}{2} |X_t|} \lim_{|r| \to \infty} e^{\frac{2\alpha}{Y_t}(1-e^{-bt})(r-|r|)} + \| \varphi \|_\delta^2 g^2(t) \lim_{|r| \to \infty} e^{\frac{2\alpha}{Y_t}(r(1-e^{-bt})-|r|)} \]
and
\[ l^2(\varphi) \leq 2\varepsilon^2 g^2(t) e^{\frac{2\alpha}{Y_t}(1-e^{-bt})} \leq 2\varepsilon^2 g^2(t) e^{\frac{2\alpha}{Y_t} + \frac{2\sigma^2}{Y_t}} \]
by (46) and (50). Hence, \( P_\varepsilon \varphi \in E \).

**Step 4.** Clearly \( P_0 = I \) and \( P_t P_s = P_{t+s} \) holds since \( (r_t) \) is a Markov process. We need to show strong continuity of \( (P_t) \), i.e.
\[ \lim_{t \downarrow 0} \| P_t \varphi - \varphi \| = 0, \quad \forall \varphi \in E. \]

Taking into account Lemma 3 and the Banach–Steinhaus theorem, it is enough to show (58) for \( \varphi \in C_0(\mathbb{R}) \). For such a \( \varphi \) we have
\[ \| P_t \varphi - \varphi \| = \sup_{r \in \mathbb{R}} |E \varphi(r_t) e^{\alpha f^t r \cdot ds} - \varphi(r) | e^{-\frac{\alpha}{Y_t} |r|} \leq O_1 + O_2 + O_3, \]
where
\[ O_1 = \sup_{r \in \mathbb{R}} |E \varphi(r_t) e^{\alpha f^t r \cdot ds} - 1) \varphi(r) | e^{-\frac{\alpha}{Y_t} |r|}, \]
\[ O_2 = \sup_{r \in \mathbb{R}} |E \varphi(r_t) - \varphi(r) | e^{-\frac{\alpha}{Y_t} |r|}, \]
\[ O_3 = \sup_{r \in \mathbb{R}} |E \varphi(r_t) e^{\alpha f^t r \cdot ds} - 1) (\varphi(r_t) - \varphi(r) | e^{-\frac{\alpha}{Y_t} |r|}. \]
Since \( \varphi \in C_0(\mathbb{R}) \), then there exists a \( \delta > 0 \) such that \( \text{supp} \varphi \subset (-\delta, \delta) \) and
\[
\lim_{t \downarrow 0} O_1 \leq \| \varphi \|_{\delta} \lim_{t \downarrow 0} \sup_{|r| < \delta} |e^{f(r,t)} - 1| e^{-\frac{\varphi}{2} |r|}
\]
where
\[
f(r, t) = \frac{\alpha r}{b} (1 - e^{-bt}) + \frac{\alpha a}{b} \left( t - \frac{1}{b} (1 - e^{-bt}) \right) + \frac{\alpha^2 \sigma^2}{2b^2} \left( t - \frac{3}{2b} + \frac{2}{b} e^{-bt} - \frac{1}{2b} e^{-2bt} \right).
\]
Since \( \lim_{t \downarrow 0} f(r, t) = 0 \) uniformly in \( \{|r| < \delta\} \), we easily verify that \( \lim_{t \downarrow 0} O_1 = 0 \).

Set \( \varepsilon > 0 \). Recall, that every continuous function on a compact set is uniformly continuous. Thus there exists a \( \rho > 0 \) such that
\[
O_2 \leq \varepsilon \sup_{r \in \mathbb{R}} \mathbb{P}(|r_t - r| < \rho) e^{-\frac{\varphi}{2} |r|} + \sup_{r \in \mathbb{R}} \int_{|r_t - r| \geq \rho} |\varphi(r_t) - \varphi(r)| e^{-\frac{\varphi}{2} |r|} d\mathbb{P}
\leq \varepsilon + 2\| \varphi \|_{\infty} \sup_{r \in \mathbb{R}} \mathbb{P}(|r_t - r| \geq \rho) e^{-\frac{\varphi}{2} |r|},
\]
where \( \| \cdot \|_{\infty} \) is the supremum norm. Hence
\[
\lim_{t \downarrow 0} O_2 \leq \varepsilon + 2\| \varphi \|_{\infty} \lim_{t \downarrow 0} \sup_{r \in \mathbb{R}} \mathbb{P}(|r_t - r| \geq \rho) e^{-\frac{\varphi}{2} |r|}
= \varepsilon + 2\| \varphi \|_{\infty} \lim_{t \downarrow 0} \sup_{r \in \mathbb{R}} e^{-\frac{\varphi}{2} |r|} (1 + N_{0,1}(r_1) - N_{0,1}(r_2))
\]
where \( N_{0,1}(\cdot) \) is a normal distribution function of \( \mathcal{N}(0, 1) \) and
\[
r_1 = \sqrt{\frac{2b (r - \frac{\varphi}{2})(1 - e^{-bt}) - \rho}{\sigma \sqrt{1 - e^{-2bt}}}}, \quad r_2 = \sqrt{\frac{2b (r - \frac{\varphi}{2})(1 - e^{-bt}) + \rho}{\sigma \sqrt{1 - e^{-2bt}}}}.
\]
The supremum is attained at \( r = \pm \infty \) or \( r = \hat{r}(t) < \infty \), but with a possible infinite limit, i.e. \( \lim_{t \downarrow 0} |\hat{r}(t)| \leq \infty \). In all this cases we obtain
\[
\lim_{t \downarrow 0} \sup_{r \in \mathbb{R}} e^{-\frac{\varphi}{2} |r|} (1 + N_{0,1}(r_1) - N_{0,1}(r_2)) = 0.
\]
Hence, taking \( \varepsilon \to 0 \), \( \lim_{t \downarrow 0} O_2 = 0 \).

Finally, from the Schwartz inequality
\[
\lim_{t \downarrow 0} O_3 \leq 2\| \varphi \|_{\infty} \lim_{t \downarrow 0} \sup_{r \in \mathbb{R}} e^{-2\frac{\varphi}{2} |r|} \mathbb{E} (e^{\alpha f_t(r) ds} - 1)^2,
\]
where we can easily derive an analytic formula of \( \mathbb{E} (e^{\alpha f_t(r) ds} - 1)^2 \).

Then we take into consideration all the possible realization of supremum \( \hat{r} \), as above, and we get \( \lim_{t \downarrow 0} O_3 = 0 \).

Appendix B - proof of $C_0$-semigroup property of $(P_t)$ in case of invariant interval model

In order to prove that $P_t: E \mapsto E$ we need to show that $P_t \varphi \in UC((a, b))$ and this may be done in a similar way to the proof of continuity of $(P_t)$ in Appendix A. Since $P_0 = I$ and $P_t P_s = P_{t+s}$ clearly hold, so in order to prove $C_0$-semigroup property of $(P_t)$ we need to show that (58) holds for $E = UC((a, b))$ equipped with the supremum norm

$$
\|\varphi\| = \sup_{r \in (a, b)} |\varphi(r)|.
$$

Let $\varphi \in E$, then

$$
\|P_t \varphi - \varphi\| = \sup_{r \in (a, b)} |E^r \varphi(r) e^{\alpha \int_0^t r_s ds} - \varphi(r)| \leq O_1 + O_2 + O_3,
$$

where

$$
O_1 = \sup_{r \in (a, b)} |E^r (e^{\alpha \int_0^t r_s ds} - 1) \varphi(r)|,
$$

$$
O_2 = \sup_{r \in (a, b)} |E^r \varphi(r) - \varphi(r)|,
$$

$$
O_3 = \sup_{r \in (a, b)} |E^r (e^{\alpha \int_0^t r_s ds} - 1)(\varphi(r) - \varphi(r))|.
$$

Since $\|\varphi\| < \infty$ for every $\varphi \in E$, then

$$
\lim_{t \downarrow 0} O_1 \leq \|\varphi\| \lim_{t \downarrow 0} E^r |e^{\alpha \int_0^t r_s ds} - 1| \\
\leq \|\varphi\| \lim_{t \downarrow 0} (\max\{|e^{\alpha t} - 1|, |e^{\alpha b t} - 1|\}) = 0
$$

and

$$
\lim_{t \downarrow 0} O_3 \leq 2\|\varphi\| \lim_{t \downarrow 0} (\max\{|e^{\alpha t} - 1|, |e^{\alpha b t} - 1|\}) = 0.
$$
Let $E_{\text{lip}}$ be the space of all Lipschitz continuous functions $\varphi \in E$. Let $\varphi \in E_{\text{lip}}$ and let $L$ be the Lipschitz constant of $\varphi$. Then we have

$$\|P_t \varphi - \varphi\| = \sup_{r \in (a,b)} |\mathbb{E}^r \varphi(r_t) - \varphi(r)|$$

$$\leq L \sup_{r \in (a,b)} \mathbb{E}^r |r_t - r| = L \sup_{r \in (a,b)} \mathbb{E}^r \left| \int_0^t \mu(r_s)ds + \int_0^t \sigma(r_s)dW_s \right|$$

$$\leq L \sup_{r \in (a,b)} \sqrt{\mathbb{E}^r \left( \int_0^t \mu(r_s)ds + \int_0^t \sigma(r_s)dW_s \right)^2}$$

$$\leq L \sup_{r \in (a,b)} \sqrt{2\mathbb{E}^r \left( \int_0^t \mu(r_s)ds \right)^2 + 2\mathbb{E}^r \left( \int_0^t \sigma(r_s)dW_s \right)^2}$$

$$\leq L \sqrt{2t^2 \sup_{r \in (a,b)} |\mu(r)|^2 + 2t \sup_{r \in (a,b)} |\sigma(r)|^2}.$$ 

Hence, $\lim_{t \to 0} \|P_t \varphi - \varphi\| = 0$ for every $\varphi \in E_{\text{lip}}$. Since $E_{\text{lip}}$ is dense in $E$, we conclude by the Banach–Steinhaus theorem.

References

[1] I. Ekeland and E. Taflin, A Theory of Bond Portfolios, Ann. Appl. Probab., 15 (2005), pp. 1260–1305.
[2] W. H. Fleming and H. M. Soner, Controlled Markov Processes and Viscosity Solutions, 2nd Edition, Springer, New York, 2006.
[3] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd Edition, Springer–Verlag, Berlin, 1983.
[4] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Tokyo, 1981.
[5] I. Karatzas and S. E. Shreve, Methods of Mathematical Finance, Springer–Verlag, New York, 1998.
[6] R. Korn and H. Kraft, A Stochastic Control Approach to Portfolio Problems with Stochastic Interest Rates, SIAM J. Control Optim., 40 (2001), pp. 1250–1269
[7] R. C. Merton, Continuous-Time Finance, Cambridge, MA, Oxford : Blackwell, 1998.
[8] R. C. Merton, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, J. Econom. Theory, 3 (1971), pp. 373–413.
[9] B. Øksendal, Stochastic Differential Equations: An Introduction with Applications, 4th Edition, Springer–Verlag, Berlin, 1995.
[10] B. Øksendal and A. Sulem, Applied Stochastic Control of Jump Diffusions, Springer–Verlag, Berlin, 2005.
[11] N. Ringer and M. Tehranchi, Optimal Portfolio Choice in the Bond Market, Finance Stoch., 10 (2006), pp. 553–573.
[12] T. Zariphopoulou, Optimal Asset Allocation in a Stochastic Factor Model – an Overview and Open Problems, submitted for publication (2009)