Optimal time-periodic Hamiltonian simulation with Floquet-Hilbert space

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The implementation of time-evolution operators \(U(t)\), called Hamiltonian simulation, is one of the most promising usage of quantum computers that can fully exploit their computational powers. For time-independent Hamiltonians, the qubitization technique has recently established efficient realization of time-evolution \(U(t) = e^{-iHt}\), with achieving the optimal computational resource both in time \(t\) and an allowable error \(\varepsilon\). In contrast, those for time-dependent systems require much cost due to the difficulty of time-dependency. In this paper, we develop optimal Hamiltonian simulation for time-dependent systems with time-periodicity, known as Floquet systems. Our strategy is to organize the Floquet-Hilbert space by preparing auxiliary states labeling Fourier indices and to extract the target time-evolved state from effective time-independent Hamiltonian dynamics in it. Employing Lieb-Robinson bound, amplitude amplification, and qubitization technique, we formulate the way to success this extraction with arbitrary small error and failure probability. Consequently, our protocol allows to construct time-evolution with the optimal resource, measured by query complexity, both in \(t\) and \(\varepsilon\). For a long-time regime over \(O(1)\) periods, the query complexity of our algorithm scales as \(O(t + \log(1/\varepsilon)/\log\log(1/\varepsilon))\), which corresponds to the best one for time-independent systems despite the existence of time-dependency. Including the other time-regimes, our algorithm achieves much smaller cost for implementing \(U(t)\) than conventional algorithms for time-dependent Hamiltonians. We also provide applications to simulation of nonequilibrium phenomena and adiabatic state preparation. Not only our result clarifies a class of time-dependent systems that can be simulated as efficiently as time-independent systems, but also it will pave the way toward broad applications of quantum computers to time-periodic systems covering nonequilibrium phenomena in condensed matter physics and quantum chemistry, and quantum tasks yielding time-dependency in quantum computation.

I. INTRODUCTION

Simulating quantum many-body systems is one of the most promising usages of quantum computers that can fully exploit their computational powers, as R. Feynman’s proposal in the beginning of quantum computers [1]. The implementation of a time-evolution operator under Hamiltonians, called Hamiltonian simulation, has been the most important problem for efficiently and accurately accomplishing quantum simulation. In fact, the application of Hamiltonian simulation nowadays ranges from condensed matter physics to quantum chemistry, such as quantum dynamics [2] and quantum phase estimation [3,4]. For long time, Trotterization has been a standard way of Hamiltonian simulation for both time-independent and time-dependent systems, which can provide simple realization available in today’s quantum computers [2, 9, 12]. Instead, it requires a huge number of elementary gates up to \(\text{poly}(1/\varepsilon)\) to achieve an acceptable error \(\varepsilon\).

For time-independent systems, various efficient Hamiltonian simulation algorithms have appeared in the past decade, which yield fewer resources than Trotterization to implement the time-evolution \(U(t) = e^{-iHt}\) with an acceptable error \(\varepsilon\) [13–18]. In particular, the qubitization technique achieves the optimal cost both in \(t\) and \(\varepsilon\) [15]. The query complexity of the qubitization, which is proportional to the number of elementary gates, scales as \(O(\alpha t + \log(1/\varepsilon)/\log\log(1/\varepsilon))\) with \(\alpha\) being the energy scale of the whole system [19]. In contrast, for time-dependent systems, while there exist several efficient Hamiltonian simulation for constructing time-evolution \(U(t)\) [20, 22], they largely rely on the truncated-Dyson-series expansion. The truncated-Dyson-series algorithm usually requires the query complexity \(O(\alpha t \times \log(1/\varepsilon)/\log\log(1/\varepsilon))\) [20, 21]. This is much larger than that of the qubitization in that it appears as a productive way rather than an additive way, originating from the difficulty of efficiently dealing with continuous-time modulation. It is nontrivial whether we can simulate time-dependent Hamiltonians with much fewer cost close to that of the qubitization.

Among time-dependent Hamiltonians, one of the most important targets of quantum simulation is a time-periodic Hamiltonian satisfying \(H(t+T) = H(t)\) with \(T\) being a period. In fact, quantum systems under time-periodic Hamiltonians are called Floquet systems, and have been platforms of various nonequilibrium phenom-
en in condensed matter physics and quantum chemistry; for instance, they can host nonequilibrium phases of matter absent in time-independent systems, such as topological phases \[ \text{[26, 28]} \] and time crystals \[ \text{[29, 31]} \]. Time-periodic Hamiltonians also cover optical responses in solids and molecules \[ \text{[35-38]} \], exemplified by high-harmonic generation and photo-chemical reactions. In addition, adiabatic quantum dynamics such as Thoulless pumping \[ \text{[39-41]} \] and adiabiatic state preparation for quantum computation \[ \text{[40, 42-43]} \] can be regarded as a part of time-periodic Hamiltonian dynamics under sufficiently large period \( T \). Despite fundamental significance and potential application of time-periodic Hamiltonians in broad fields, there is no Hamiltonian simulation protocol which can efficiently handle their time-dependency, while their simulation itself is possible by the truncated-Dyson-series algorithm.

In this paper, we focus on time-periodic Hamiltonian simulation to resolve whether we can achieve the optimal cost of time-independent systems in simulation of time-dependent systems, and to accelerate application of quantum computers to various important phenomena in time-periodic systems. As a result, we develop an optimal and/or nearly-optimal quantum algorithm for time-periodic Hamiltonian simulation. The resulting query complexity for simulating the time-evolved state of the system of interest is \[ O\left(\alpha + \gamma t + \log\left(1/\epsilon^2\right) / \log\log\left(1/\epsilon^2\right)\right) \] with \( \gamma \) \( \leq \alpha \) being the energy scale of time-dependent terms in the Hamiltonian, when we are interested in \( \mathcal{O}(1) \)-period dynamics. This scaling is optimal both in \( t \) and \( \epsilon \) and additive like that of the qubitization. In other words, our algorithm executes time-periodic Hamiltonian simulation with essentially the same cost as the best algorithm for time-independent systems, in spite of the existence of time-dependency. Including the other time regimes over multiple periods, our algorithm efficiently deals with time-dependent systems with time-periodicity with much fewer resource than the truncated-Dyson-series algorithm.

The key idea of our formalism is mapping time-dependent systems to effective time-independent systems in the infinite-dimensional Floquet-Hilbert space, obtained by the Fourier series expansion in time. Although the infinite-dimensionality prohibits its simulation on quantum computers, we resolve this by formulating the Lieb-Robinson bound \[ \text{[44]} \], amplitude amplification, and the qubitization technique. Consequently, we can simulate the target time-evolved state with arbitrary small error and failure probability with a finite-dimensional Hilbert space by preparing ancilla states labeling Fourier indices. Since the auxiliary states for Fourier indices accurately reproduce the exact dynamics with fewer degrees of freedom than those for discretized time, which is employed for generic time-dependent systems, our algorithm achieves smaller cost compared to the truncated-Dyson-series algorithm.

Our result for time-periodic Hamiltonian simulation reveals an important class of time-dependent systems that can be simulated as efficiently as the best algorithm for time-independent systems both in time \( t \) and an acceptable error \( \epsilon \). Furthermore, we provide their potential applications, such as nonequilibrium phenomena in solids and adiabatic state preparation. Thus, our protocol will shed light also on the promising usage of quantum computers for nonequilibrium quantum many-body phenomena in condensed matter physics and quantum chemistry, and the optimal control in quantum computation.

The rest of this paper is organized as follows. In Section \[ \text{II} \] we briefly review Floquet theory and the qubitization for this paper to be self-contained. We provide our main results in Sections \[ \text{II, III} \] with firstly summarizing them in Section \[ \text{II} \]. In Section \[ \text{IV} \] we derive the Lieb-Robinson bound in time-periodic systems, and the way to accurately reproduce the exact dynamics with the Floquet-Hilbert space. Sections \[ \text{V} \] and \[ \text{VI} \] are devoted to deriving the subroutines of the algorithm. They respectively provide the amplification protocol that can realize the time-evolved state with sufficiently high success probability (Section \[ \text{V} \]) and efficient implementation of the time-evolution operator in the Floquet-Hilbert space (Section \[ \text{VI} \]). Section \[ \text{VII} \] completes the optimal time-periodic Hamiltonian simulation, and compares its resource with other algorithms for time-independent / -dependent Hamiltonian simulation. We provide some promising applications in Section \[ \text{VIII} \] and concludes our paper in Section \[ \text{IX} \].

II. PRELIMINARIES

A. Floquet theory

We briefly review Floquet theory to analyze Schrödinger equation under a time-periodic Hamiltonian,

\[ i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle, \quad H(t + T) = H(t), \tag{1} \]

and its solution

\[ |\psi(t)\rangle = U(t) |\psi(0)\rangle, \quad U(t) = \mathcal{T} \exp \left( -i \int_0^t dt' H(t') \right). \tag{2} \]

Here, the time-evolved state of the system of interest is represented by \( |\psi(t)\rangle \), and we assume that it is defined on a finite-dimensional Hilbert space \( \mathcal{H} \). The set of states \( \{ |\psi_j\rangle \}_{j=1}^{\dim(\mathcal{H})} \) denotes a certain choice of the basis of \( \mathcal{H} \). Floquet theorem says that the solution \( |\psi(t)\rangle \) is always written in the form of

\[ |\psi(t)\rangle = \sum_{\alpha=1}^{\dim(\mathcal{H})} c_\alpha e^{-i \varepsilon_\alpha t} |\phi_\alpha(t)\rangle, \quad |\phi_\alpha(t + T)\rangle = |\phi_\alpha(t)\rangle, \tag{3} \]

similar to Bloch theorem for spatially-periodic systems. Here, \( \varepsilon_\alpha \in [-\pi/T, \pi/T] \) and \( |\phi_\alpha(t)\rangle \in \mathcal{H} \) are respectively...
called quasienergy and Floquet state. Time-periodicity of \( H(t) \) and \( |\phi_\alpha(t)\rangle \) allows the Fourier series expansion as

\[
H(t) = \sum_{m \in \mathbb{Z}} H_m e^{-im\omega t}, \quad |\phi_\alpha(t)\rangle = \sum_{m \in \mathbb{Z}} |\phi_m^\alpha\rangle e^{-im\omega t},
\]

with the frequency \( \omega = 2\pi/T \). The hermiticity of \( H(t) \) implies \( H_{-m} = H_{m}^\dagger \). We assume \( \|H(t)\| < \infty \) (\( \|\cdot\| \); operator norm), which results in \( \sum_{m \in \mathbb{Z}} \|H_m\|^2 < \infty \). Introducing an auxiliary degree of freedom \( \{|l\rangle\} \in \mathbb{Z} \) (sometimes called photon number) to relate \( |\phi_\alpha^l\rangle \leftrightarrow |l\rangle |\phi_\alpha^l\rangle \), every pair of \( \varepsilon_\alpha, |\phi_\alpha(t)\rangle \) can be obtained by the time-independent eigenvalue equation,

\[
\mathcal{H}_{\text{eff}} \left( \sum_{l \in \mathbb{Z}} |l\rangle \langle l| \right) = \varepsilon_\alpha \left( \sum_{l \in \mathbb{Z}} |l\rangle \langle l| \right),
\]

with the effective Hamiltonian defined by

\[
\mathcal{H}_{\text{eff}} = \sum_{l \in \mathbb{Z}} |l\rangle \langle l| (H_0 - l\omega) + \sum_{l,m \in \mathbb{Z}} |l\rangle \langle l + m| \otimes H_{-m}. \tag{6}
\]

Intuitively, this time-independent Hamiltonian \( \mathcal{H}_{\text{eff}} \) describes a static one-dimensional system, where each \( l \)-th site has potential energy \( H_{0} - l\omega \), as Fig. 1. Off-diagonal terms \( H_{-m} \) represent hopping by \(-m\) sites, which can be recognized as either emission or absorption of \( |m\rangle \) photons. We remark that the difficulty of time-dependence does not vanish by this mapping; it is translated into infinite dimensionality of the space spanned by \( \{|l\rangle |\psi_j\rangle \} |l \in \mathbb{Z}, j = 1, 2, \ldots, \dim(\mathcal{H})\} \), which is called Floquet-Hilbert space or Sambe space \( \mathcal{H} \). While the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \) is often used for identifying \( \varepsilon_\alpha, |\phi_\alpha(t)\rangle \), it can be employed also for directly determining the dynamics as follows \( \mathcal{H}_{\text{eff}} \):

\[
|\psi(t)\rangle = \sum_{l \in \mathbb{Z}} e^{-il\omega t} \langle l| e^{-i\mathcal{H}_{\text{eff}}t} |0\rangle |\psi(0)\rangle. \tag{7}
\]

We can extract the dynamics under a time-dependent Hamiltonian \( H(t) \) from that under the time-independent one \( \mathcal{H}_{\text{eff}} \) by preparing auxiliary systems labeled by \( |l\rangle \). However, we note that applying optimal algorithms for time-independent systems to simulating this extended dynamics is not straightforward, since the Floquet-Hilbert space is infinite-dimensional.

**B. Hamiltonian simulation by qubitization**

Here, we briefly review so-called qubitization technique to efficiently implement \( e^{-iHT} \) for time-independent Hamiltonian \( H \) on the Hilbert space \( \mathcal{H} \). It begins with assuming the existence of block-encoding, that is, we suppose an \( n_a \)-qubit auxiliary state \( |G\rangle_a \) (called an oracle state) and a unitary gate \( O \) on the space \( \mathbb{C}^{2^{n_a}} \otimes \mathcal{H} \) (called an oracle gate) satisfying

\[
|G|O|G\rangle = \frac{H}{\alpha}, \quad \alpha \geq \|H\|. \tag{8}
\]

The inequality \( \alpha \geq \|H\| \) comes from the norm of the unitary gate as \( \|O\| = 1 \), and the parameter \( \alpha \) represents a typical energy scale of the whole system. Here, the oracles \( O \) and \( |G\rangle_a \) are supposed to be efficiently implemented; the unitary gates \( O \) and \( G \), which realizes the oracle state from a reference state \( |0\rangle_a \) as \( G |0\rangle_a = |G\rangle_a \), require at most \( C \in \text{poly}(N) \) elementary gates.

The explicit construction of the oracles has been explored for certain classes of static Hamiltonians. For instance, when the Hamiltonian \( H \) is given by a linear combination of unitary (LCU) as

\[
H = \sum_{j=1}^{N} a_j U_j, \quad a_j \geq 0, \quad U_j; \text{unitary}, \tag{9}
\]

a possible choice of the oracles is

\[
O = \sum_{j=1}^{N} |j\rangle_a \otimes U_j, \quad |G\rangle_a = \sum_{j=1}^{N} \sqrt{\frac{\alpha}{\alpha}} |j\rangle_a, \tag{10}
\]

with \( \alpha = \sum_{j=1}^{N} a_j \). The number of ancilla qubits \( n_a \) scales as \( \log J_{\text{max}} \). Since \( J_{\text{max}} \) is \( \text{poly}(N) \) for typical \( N \)-site systems, the number of elementary gates for the oracles amounts to \( C \in \text{poly}(N) \). LCU's cover various types of Hamiltonians exemplified by quantum spin systems composed of local Pauli operators and fermionic systems in condensed matter physics and quantum chemistry \( \mathcal{H} \). The block-encoding of other Hamiltonians, such as sparse-access matrices and purified density matrices, has been also revealed \( \mathcal{H} \).

Once we find out the oracles \( O \) and \( |G\rangle_a \), we can organize a unitary gate \( W_q \), implemented by \( O(1) \) additional ancilla qubits and \( O(q) \)-times usage of \( O, G \), and \( O(\log(\dim(\mathcal{H})) \) elementary gates, which satisfies

\[
W_q |0\rangle_{\text{poly}(n_a + O(1))} \langle \psi| = |0\rangle_{\text{poly}(n_a + O(1))} e^{-iHT} \langle \psi| + O(\varepsilon_q), \tag{11}
\]
for every state $|\psi\rangle \in \mathcal{H}$. Here, the error $\varepsilon_q$ comes from approximating $e^{-iHt}$ by a certain degree-$q$ polynomial of $H$, and decays as $\varepsilon_q \in \mathcal{O}((at/q)^q)$. The scaling of $q$ required for making the error $\varepsilon_q$ smaller than an acceptable error $\varepsilon$ is approximately given by the following formula (See the endnote [19] for the accurate scaling of $q$);

$$q \in \Theta\left(at + \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)}\right). \tag{12}$$

In order to determine the explicit form of $W_n$, we resort to quantum signal processing [17]. It dictates that only poly ($q$)-time classical computation is required for this purpose.

To summarize the qubitization technique, we require the following computational resources to implement the time-evolution operator $e^{-iHt}$ with an acceptable error $\varepsilon$ (See also Table II):

- Number of ancilla qubits; $n_a + \mathcal{O}(1)$
- Query complexity; $q$ [Eq. (12)]
- Number of elementary gates;

$$\mathcal{O}\left(\left\{at + \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)}\right\}C\right). \tag{13}$$

Here, since the number of elementary gates for each oracle $C$ scales as poly ($N$) for typical $N$-site quantum many-body systems, we neglect additional elementary gates per query, which amounts to $\mathcal{O}(\log(\dim(H))) = \mathcal{O}(N)$. In terms of the scaling in $1/\varepsilon$, the required elementary gates is much less than those of Trotterization reaching poly ($t/\varepsilon$). In addition, the query complexity which is defined by the complexity counted by the number of oracles, corresponding to $q$ in this case, takes the form of $\mathcal{O}(t + \log(1/\varepsilon)/\log(1/\varepsilon))$. Its scaling is known to be optimal both in $t$ and $1/\varepsilon$ for simulating generic time-independent Hamiltonians [19].

In our algorithm for time-periodic Hamiltonians, we exploit qubitization as a subroutine to implement time-evolution under an effective Hamiltonian in Floquet theory. To evaluate its cost, it is sufficient to construct the block-encoding of the effective Hamiltonian, and the cost of the oracles $C$.

### III. SUMMARY OF THIS PAPER

In this section, we overview our main results on efficient quantum simulation of time-periodic Hamiltonians. We will provide their detailed derivation in the following Sections IV/VI

#### A. Overview of Algorithm

We first show an overview of the algorithm for time-evolution under a time-periodic Hamiltonian $H(t+T) = H(t)$. Throughout the main text, we assume that the Fourier components of the Hamiltonian $H(t)$ vanishes at a certain cutoff $m_{\text{max}} \in \mathcal{O}(1)$;

$$H_m = 0, \text{ if } |m| > m_{\text{max}}. \tag{14}$$

(This discussion can be extended to cases where $H_m$ exponentially decays in $|m|$. See Appendix E).

The central attempt in the algorithm is to efficiently simulate Eq. (7).

$$|\psi(t)\rangle = \sum_{l=-\infty}^{\infty} e^{-il\omega t} \langle l | e^{-i\mathcal{H}_{\text{eff}} t} | 0 \rangle |\psi(0)\rangle, \tag{15}$$

on quantum circuits. The time-dependency is erased by mapping the dynamics to the one on the Floquet-Hilbert space, which results in the following two benefits. First, we do not discretize the time with infinitesimal intervals. We instead rely on Fourier indices $l \in \mathbb{Z}$, which are originally discrete, and actually they lead to much better accuracy with the same auxiliary degrees of freedom. Second, we can use various Hamiltonian simulation algorithms for time-independent systems. Since the qubitization technique has achieved the best query complexity in the time $t$ and the inverse error $1/\varepsilon$, exploiting it for $\mathcal{H}_{\text{eff}}$ accelerate the simulation of time-periodic systems.

On the other hand, in order to simulate the time-evolved state $|\psi(t)\rangle$ via Eq. (15) on quantum circuits, we also have several problems to be tackled. First, the Floquet-Hilbert space is infinite dimensional. We have to introduce truncation at a certain Fourier index $l_{\text{max}}$, and consider a finite set of indices defined by

$$D_{l_{\text{max}}} = \{-l_{\text{max}}+1, -l_{\text{max}}+2, \ldots, l_{\text{max}}\} \subset \mathbb{Z}. \tag{16}$$

Then, we focus on the approximate dynamics by

$$\mathcal{H}_{\text{eff}}^{l_{\text{max}}} = \sum_{l \in D_{l_{\text{max}}}} |l\rangle \langle l| \otimes (H_0 - l\omega) + \sum_{l \in D_{l_{\text{max}}}} m \neq 0 \sum_{l+m \in D_{l_{\text{max}}}} |l\rangle \langle l+m| \otimes H_{-m}, \tag{17}$$

$$|\psi_{l_{\text{max}}}^{t}(t)\rangle = \sum_{l \in D_{l_{\text{max}}}} e^{-il\omega t} \langle l | e^{-i\mathcal{H}_{\text{eff}}^{l_{\text{max}}} t} | 0 \rangle |\psi(0)\rangle. \tag{18}$$

The finite-dimensional space, spanned by $2l_{\text{max}} \times \dim(H)$ states $\{|l\rangle_{j} \mid l \in D_{l_{\text{max}}}, j = 1, \ldots, \dim(H)\}$, is called the truncated Floquet-Hilbert space. While $l_{\text{max}} \rightarrow \infty$ reproduces $|\psi_{l_{\text{max}}}^{t}(t)\rangle \rightarrow |\psi(t)\rangle$, it is nontrivial how we should choose $l_{\text{max}}$ to achieve the accuracy $\| |\psi(t)\rangle - |\psi_{l_{\text{max}}}^{t}(t)\rangle\| \leq \varepsilon$. The second problem is the small success probability of post-selecting the ancilla state. Even if we succeed in approximation with finite $l_{\text{max}}$, Eq. (18) requires a projection to an unnormalized ancilla state $\sum_{l \in D_{l_{\text{max}}}} |l\rangle$ for the state in the truncated Floquet-Hilbert space,

$$|\Psi_{l_{\text{max}}}^{t}(t)\rangle = e^{-i\omega t \sum_l |l\rangle \langle l|} e^{-i\mathcal{H}_{\text{eff}}^{l_{\text{max}}} t} |0\rangle |\psi(0)\rangle. \tag{19}$$
In the actual computation, we need post-selection to an ancilla state given by

\[ |a^{l_{\text{max}}}⟩ = \frac{1}{\sqrt{2^{l_{\text{max}}}}} \sum_{l \in D^{l_{\text{max}}}} |l⟩. \tag{20} \]

Success probability of the projection is given by

\[ \|⟨a^{l_{\text{max}}} | ψ(t)⟩\|^2 ≈ ⟨ψ(t)|ψ(t)⟩ / (2^{l_{\text{max}}}) = 1/(2^{l_{\text{max}}}). \]

As \( l_{\text{max}} \) increases to ensure the accuracy, the expected time to successfully obtain \( |ψ(t)⟩ \) becomes longer in proportion to it. The final problem is about implementation of \( e^{-iH_{\text{eff}}^{l_{\text{max}}} t} \). While the effective Hamiltonian \( H_{\text{eff}}^{l_{\text{max}}} \) is time-independent, its structure is complicated due to the additional degrees of freedom labeled by \( \{l\} \). It is nontrivial whether the optimal Hamiltonian simulation algorithm, i.e. the qubitization technique, provides the optimality for time-periodic systems.

Our algorithm relying on the truncated Floquet-Hilbert space efficiently simulates the time-evolved state \( |ψ(t)⟩ \) with resolving the above problems. The significant developments are composed of the following steps:

(a) Decision of the Fourier index for truncation, \( l_{\text{max}} \), to achieve an allowable error \( \varepsilon \).

(b) Amplification by symmetry and oblivious amplitude amplification to enhance the success probability up to \( 1 - O(\varepsilon) \).

(c) Efficient implementation of \( \exp\left(−iH_{\text{eff}}^{l_{\text{max}}} t\right) \) by the qubitization technique.

In Step (a), we explicitly derive the upper bound of \( \|ψ(t)⟩ − |ψ_{\text{max}}(t)⟩\| \) in a similar way to the Lieb-Robinson bound. We show that the choice of \( l_{\text{max}} \), based on

\[ l_{\text{max}} ∈ Θ \left( γt + \frac{\log(1/\varepsilon)}{\log \log(1/\varepsilon)} \right), \tag{21} \]

\[ γ = \sup_t ||H(t) - H_0||, \tag{22} \]

is sufficient to make the error smaller than \( \varepsilon \). Here, \( γ \) gives the scale of the time-dependent terms in \( H(t) \). The second step (b) plays a role in amplifying the success probability of the post selection from \( 1/(2^{l_{\text{max}}}) \) to \( 1 - O(\varepsilon) \). The first protocol, which we call amplification by symmetry, exploits the symmetry of \( H_{\text{eff}}^{l_{\text{max}}} \) which is always present and inherent in time-periodic systems. It brings the success probability from \( 1/(2^{l_{\text{max}}}) \) to \( 1/4 - O(\varepsilon) \) only with small cost of \( O(\log l_{\text{max}}) \) elementary gates. Following this, we apply the oblivious amplitude amplification [15], reminiscent of Grover’s search algorithm [20]. The success probability is further amplified from \( 1/4 - O(\varepsilon) \) to \( 1 - O(\varepsilon) \). Exploiting these two kinds of amplification, we can obtain the target state \( |ψ(t)⟩ \) only with a little additional resource that does not change the scaling. The final step (c) constructs conversion of the effective Hamiltonian so that \( \exp\left(−iH_{\text{eff}}^{l_{\text{max}}} t\right) \) can be efficiently implemented. The original effective Hamiltonian \( H_{\text{eff}}^{l_{\text{max}}} \) is not suitable for the qubitization technique, since the additional degrees of freedom \( |l⟩ \) makes its block-encoding inefficient. To resolve this problem, we derive an alternative effective Hamiltonian such that the desired evolution \( \exp\left(−iH_{\text{eff}}^{l_{\text{max}}} t\right) \) is accurately reproduced. The block-encoding for the alternative effective Hamiltonian requires \( O(1) \)-times queries to the oracles for each Fourier component \( \{H_m\} \), which is important for achieving optimal or nearly-optimal dependence in \( t \) and \( 1/\varepsilon \).

B. Main results

Here, we summarize the computational resources for computing the dynamics \( |ψ(t)⟩ \) under a time-periodic Hamiltonian \( H(t) \). We construct two different quantum algorithms depending on the time scale of the dynamics. The first case is an adiabatic-like case, where we are interested in the dynamics over \( O(1) \) periods as \( ωt ∈ O(1) \). We call it “adiabatic” since adiabatic dynamics under sufficiently large period \( T \) is a typical target, while \( T \) is not required to be large. The other case is a generic long-time regime, in which we focus on dynamics over multiple periods as \( ωt ∈ Ω(1) \). The algorithms for both cases follow Steps (a)-(c) of Section [11]. The number of ancilla qubits is determined by the truncation order \( l_{\text{max}} \) of Step (a), and the number of elementary gates comes mainly from the cost of implementing \( \exp\left(−iH_{\text{eff}}^{l_{\text{max}}} t\right) \) in Step (c).

We assume that the Fourier component \( H_m \) becomes zero for \( |m| > m_{\text{max}} \) with an \( O(1) \) constant \( m_{\text{max}} \). The block-encoding for each \( H_m \) is supposed to be given by an oracle unitary gate \( O_m \) and an \( n_a \)-qubit oracle state \( |G_m⟩ \) as

\[ ⟨G_m|O_m|G_m⟩ = \frac{H_m}{α_m}, \quad α_m > 0. \tag{23} \]

Each oracle state \( |G_m⟩ \) is generated by trivial states as \( |G_m⟩ = G_m |0⟩ \). We define the energy scale of the Hamiltonian \( H(t) \) by

\[ α = \sum_{|m| ≤ m_{\text{max}}} α_m, \tag{24} \]

and suppose that \( \{α_m\} \) can be embedded into an \( O(1) \)-qubit quantum system by

\[ G_{\text{coef}} |0⟩ = \sum_{|m| ≤ m_{\text{max}}} \sqrt{α_m} |m⟩ ∈ C^{2m_{\text{max}} + 1}. \tag{25} \]

The query complexity is defined by the number of queries to the oracles \( O_m \), \( G_m \), and \( G_{\text{coef}} \), where each of them is supposed to require at-most \( C \) elementary gates. We summarize the computational resources for simulating \( |ψ(t)⟩ \) in the adiabatic-like regime and the generic long-time regime respectively by the following theorems.
Theorem 1. (Resource for adiabatic-like regime)
Suppose that we are interested in the time-evolved state $|\psi(t)\rangle$ from an arbitrary initial state $|\psi(0)\rangle$ over $O(1)$ periods. The computational resources to obtain it with the acceptable error and the failure probability smaller than $O(\varepsilon)$ are summarized as follows:

- Number of ancilla qubits
  $$n_a + O(\log(\gamma t) + \log(1/\varepsilon)).$$  
  \hspace{1cm} \text{(26)}

- Query complexity
  $$O\left((\alpha + \gamma) t + \frac{\log(1/\varepsilon)}{\log\log(1/\varepsilon)}\right).$$
  \hspace{1cm} \text{(27)}

- Number of elementary gates
  $$O\left(\left\{ (\alpha + \gamma) t + \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)} \right\} \times \left\{ C + \log(\gamma t) + \log\left(\frac{1}{\varepsilon}\right) \right\} \right).$$
  \hspace{1cm} \text{(28)}

The query complexity has optimal scaling in the time $t$ and the inverse error $1/\varepsilon$.

Theorem 2. (Resource for long-time regime)
Suppose that we are interested in the time-evolved state $|\psi(t)\rangle$ from arbitrary initial states $|\psi(0)\rangle$ over multiple periods $\omega t \in \Omega(1)$. The computational resources to obtain it with the acceptable error and the failure probability smaller than $O(\varepsilon)$ are summarized as follows:

- Number of ancilla qubits
  $$n_a + O(\log(\gamma/\omega) + \log(\omega t/\varepsilon)).$$
  \hspace{1cm} \text{(29)}

- Query complexity
  $$O\left((\alpha + \gamma) t + \omega t \frac{\log(\omega t/\varepsilon)}{\log\log(\omega t/\varepsilon)}\right).$$
  \hspace{1cm} \text{(30)}

- Number of elementary gates
  $$O\left(\left\{ (\alpha + \gamma) t + \omega t \frac{\log(\omega t/\varepsilon)}{\log\log(\omega t/\varepsilon)} \right\} \times \left\{ C + \log(\gamma/\omega) + \log\log(\omega t/\varepsilon) \right\} \right).$$
  \hspace{1cm} \text{(31)}

The query complexity has optimal scaling in the inverse error $1/\varepsilon$. While the scaling in the time $t$ is formally nearly-optimal, it is optimal for practical problems up to poly($N$)-time with $N$ being the system size.

The parameters $\alpha$ and $\gamma$, defined by Eqs. (24) and (22), respectively represent energy scales of the overall terms and the time-dependent terms in $H(t)$. They typically scale as $\alpha, \gamma \in \text{poly}(N)$ with the system size $N$. In contrast, the frequency $\omega = 2\pi/T$ is typically an $O(N^0)$ value much smaller than $\alpha$ and $\gamma$ (The high-frequency cases where $\omega$ are comparable to or larger than $\alpha, \gamma$ are trivial, as discussed in Appendix D). While the above theorems are about time-periodic Hamiltonians with vanishing Fourier components at $m_{\text{max}} \in O(1)$ as $H_m = 0$ ($|m| > m_{\text{max}}$), we obtain similar results for Hamiltonians with exponentially-decaying Fourier components $\|H_m\| \leq e^{-O(|m|)}$. See Appendix C for its derivation.

IV. BUILDING AN APPROPRIATE TRUNCATED FLOQUET-HILBERT SPACE

This section is devoted to deriving the proper truncation order for the Floquet-Hilbert space, following (a) in Section II A. In order to achieve desirable accuracy for the exact time-evolved state with the truncated Floquet-Hilbert space, we should obtain an exact upper bound on its error. Here, we prove the Lieb-Robinson bound in the Floquet-Hilbert space in Section IV A, and determine a proper truncation order based on it in Section IV B.

A. Lieb-Robinson bound in Floquet-Hilbert space

The proper choice of the truncation order $l_{\text{max}}$ is determined so that the approximate state $|\psi(t)\rangle$ [See Eq. (18)] can reproduce the time-evolved state $|\psi(t)\rangle$ with $||\psi(t)\rangle - |\psi(t)\rangle|| \leq \varepsilon$. From the explicit formula Eq. (21), we can see that it is important to observe how the transition amplitude $\langle l | e^{-i\mathcal{H}_{\text{eff}} l_{\text{max}} t} | 0 \rangle$ behaves for sufficiently large $l$. We first show the upper bound on this transition amplitude in Theorem 3. Since it is reminiscent of the Lieb-Robinson bound in single-particle quantum systems [51], we call it the Lieb-Robinson bound in the Floquet-Hilbert space.

Theorem 3. (Bound on transition amplitude)
We assume $H_m = 0$ for $m > m_{\text{max}}$. Then, for indices $l, l' \in \mathbb{Z}$ satisfying $|l - l'| \geq 2m_{\text{max}}\gamma t$, the transition amplitude is bounded by

$$\left\| \langle l | e^{-i\mathcal{H}_{\text{eff}} l_{\text{max}} t} | l' \rangle \right\| \leq 2 \left( \frac{\gamma t}{|l - l'|/m_{\text{max}}} \right)! \left( \frac{|l - l'|/m_{\text{max}}}{m_{\text{max}}} \right)^{[|l - l'|/m_{\text{max}}]} \left( \frac{\gamma t}{|l - l'|/m_{\text{max}}} \right)^{[|l - l'|/m_{\text{max}}]},$$ \hspace{1cm} \text{(32)}

where the parameter $\gamma$ is defined by Eq. (22).

Proof.— We will omit the superscripts $l_{\text{max}}$ for some operators introduced here since they are not important. We first employ the interaction picture. With the unitary operation defined by

$$\mathcal{U}_0(t) = e^{-i\mathcal{H}_0 t}, \quad \mathcal{H}_0 = \sum_{l \in D^{l_{\text{max}}}} |l\rangle \langle l| \otimes (H_0 - l\omega),$$ \hspace{1cm} \text{(34)}
the time evolution operator $e^{-i\mathcal{H}_{\text{eff}}^{\text{max}} t}$ is represented as
\begin{equation}
e^{-i\mathcal{H}_{\text{eff}}^{\text{max}} t} = \mathcal{U}_0(t)\mathcal{U}_1(t),
\end{equation}
\begin{equation}
\mathcal{U}_1(t) = T \exp \left(-i \int_0^t dt' \mathcal{H}_1(t') \right).
\end{equation}
Here, the Hamiltonian in the interaction picture is defined by
\begin{equation}
\mathcal{H}_1(t) = \mathcal{U}_0(t) \mathcal{H}_1^{\text{max}} \mathcal{U}_0^\dagger(t)
\end{equation}
where
\begin{equation}
\mathcal{H}_1^{\text{max}} = \sum_l \left( e^{i \omega_l t} |l\rangle \langle l + m| \otimes H_{l-l'}(t) + \text{h.c.} \right),
\end{equation}
\begin{equation}
H_{l-l'}(t) = e^{i \mathcal{H}_0 t} H_{l-l'} e^{-i \mathcal{H}_0 t},
\end{equation}
where the summation in the second line is taken over $l, m$ such that $l, l + m \in D^{l_{\text{max}}}$. Using the Dyson series expansion, we obtain
\begin{equation}
\left\| \langle l | e^{-i\mathcal{H}_{\text{eff}}^{\text{max}} t} | l' \rangle \right\| = \left\| \langle l | \mathcal{U}_1(t) | l' \rangle \right\|
\end{equation}
\begin{equation}
\leq \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_2} dt_1 \left\| \mathcal{H}_1(t_n) \cdots \mathcal{H}_1(t_1) | l' \rangle \right\|.
\end{equation}
\begin{equation}
= \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_2} dt_1 \left\| \prod_{i=1}^n \mathcal{H}_1(t_i) | l_i \rangle \right\|.
\end{equation}
In the last equality, we employ the identity $\sum_{l_i \in D^{l_{\text{max}}}} \langle l_i | = I$ for $n \geq 1$ times. The summation $\sum_{l_i \in D^{l_{\text{max}}}} \langle l_i |$ is taken over $l_i \in D^{l_{\text{max}}}$ for $i = 1, 2, \ldots, n - 1$ under the fixed $l_0 = l'$ and $l_n = l$. Each product $\prod_{i=1}^n \langle l_i | \mathcal{H}_1(t_i) | l_{i-1} \rangle$ represents a complex transition amplitude from $| l' \rangle$ to $| l \rangle$ along the path $| l' \rangle \rightarrow | l_1 \rangle \rightarrow \ldots \rightarrow | l_{n-1} \rangle \rightarrow | l \rangle$ under $\mathcal{H}_1(t_i)$. Since the Hamiltonian $\mathcal{H}_1(t_i)$ shifts the Fourier index $| l_i \rangle$ by at-most $m_{\text{max}}$ due to Eq. 17, the low order terms labeled by $n < | l - l' | / m_{\text{max}}$ disappears. This results in
\begin{equation}
\left\| \langle l | e^{-i\mathcal{H}_{\text{eff}}^{\text{max}} t} | l' \rangle \right\|
\end{equation}
\begin{equation}
\leq \sum_{n=[| l - l' | / m_{\text{max}}]}^{\infty} \int_0^t dt_n \int_0^{t_2} dt_1 \left\| \prod_{i=1}^n \mathcal{H}_1(t_i) | 0 \rangle \right\|
\end{equation}
\begin{equation}
\leq \sum_{n=[| l - l' | / m_{\text{max}}]}^{\infty} \frac{\rho^n}{n!} \left( \text{sup} \left( \left\| \mathcal{H}_1(t) \right\| \right) \right)^n.
\end{equation}
Here, $\mathcal{H}_1(t)$ is a Toeplitz matrix that satisfies
\begin{equation}
\mathcal{H}_1(t_1) \mathcal{H}_1(t_2) = e^{-i(t_2-t_1)\mathcal{H}_1^{\text{max}}} \mathcal{H}_1^{\text{max}} H_{l-l'}(t) - H_0 \delta_{l-l',0}. \end{equation}
This leads to the upper bound of its operator as follows
\begin{equation}
\left\| \mathcal{H}_1(t) \right\| \leq \text{sup}_{\omega} \left( \sum_{m \in \mathbb{Z}} \left( e^{i \omega t} H_{l-l'}(t) - H_0 \delta_{l-l',0} \right) e^{-i \omega t'} \right)
\end{equation}
\begin{equation}
= \text{sup} \left( \left\| H(t-t') - H_0 \right\| \right) = \gamma.
\end{equation}
We also use the following inequality;
\begin{equation}
\sum_{n=n_0}^{\infty} \frac{x^n}{n!} \leq 2 \frac{x^{n_0}}{n_0!}, \text{ if } n_0 \geq 2x \geq 0.
\end{equation}
This relation can be easily confirmed by
\begin{equation}
\sum_{n=n_0}^{\infty} \frac{x^n}{n!} \leq \sum_{n=n_0}^{\infty} \frac{x^{n_0}}{n_0!} \left( \frac{1}{2} \right)^n.
\end{equation}
For the indices $l, l'$ satisfying $| l - l' | \geq 2m_{\text{max}} \gamma t$, we can substitute $x = \gamma t$ and $n_0 = [ | l - l' | / m_{\text{max}} ]$. This results in the bound,
\begin{equation}
\left\| \langle l | e^{-i\mathcal{H}_{\text{eff}}^{\text{max}} t} | l' \rangle \right\| \leq 2 \frac{(\gamma t)^{[ | l - l' | / m_{\text{max}} ]}}{[ | l - l' | / m_{\text{max}} ]!}.
\end{equation}
The inequality from the Stirling formula,
\begin{equation}
n! \geq \left( \frac{n}{e} \right)^n,
\end{equation}
ensures the inequality Eq. (33). □

The Lieb-Robinson bound on the transition amplitude provides a guide for choosing a proper truncation order. By setting $l' = 0$, Eqs. 32 and 33 say that the contributions to $| \psi^{\text{max}}(t) \rangle$ from indices satisfying $| l | \geq 2m_{\text{max}} \gamma t$ rapidly decay as $l^{-\gamma t}$. Thus, $O(\gamma t)$ becomes a possible truncation order. However, we note that the change in $l_{\text{max}}$ affects $| \psi^{\text{max}}(t) \rangle$ also via the change in the support of the effective Hamiltonian $\mathcal{H}_{\text{eff}}^{\text{max}}$. By taking it into account with using the Lieb-Robinson bound, we obtain the exact upper bound on the error between $| \psi(t) \rangle$ and $| \psi^{\text{max}}(t) \rangle$ as follows.

**Theorem 4.** (Floquet-Hilbert space truncation)

We consider the approximate time-evolved state obtained from the truncated Floquet-Hilbert space, $| \psi^{\text{max}}(t) \rangle$, given by Eq. 15. Then, its deviation from the exact one $| \psi \rangle$ is bounded by
\begin{equation}
\left\| | \psi(t) \rangle - | \psi^{\text{max}}(t) \rangle \right\| \leq 20m_{\text{max}} \frac{(\gamma t)^{l_{\text{max}}}}{(l_{\text{max}})!},
\end{equation}
if the truncation order $l_{\text{max}}$ satisfies $l_{\text{max}} \geq 2m_{\text{max}} \gamma t$.

**Proof.** — We evaluate the convergence of $| \psi^{\text{max}}(t) \rangle$. For different orders $l_{\text{max}}, l'_{\text{max}}$ satisfying $l'_{\text{max}} > l_{\text{max}}$, we compute the difference,
\begin{equation}
\left\| | \psi^{l'_{\text{max}}}(t) \rangle - | \psi^{l_{\text{max}}}(t) \rangle \right\| \leq \varepsilon_1 + \varepsilon_2,
\end{equation}
\begin{equation}
\varepsilon_1 = \sum_{l \in (D^{l_{\text{max}}}, D^{l'_{\text{max}}})} \left\| \langle l | e^{-i\mathcal{H}_{\text{eff}}^{l_{\text{max}} t}} | 0 \rangle \right\|,
\end{equation}
\begin{equation}
\varepsilon_2 = \sum_{l \in D^{l_{\text{max}}}} \left\| \langle l | e^{-i\mathcal{H}_{\text{eff}}^{l'_{\text{max}} t}} - e^{-i\mathcal{H}_{\text{eff}}^{l_{\text{max}} t}} | 0 \rangle \right\|.
\end{equation}
The first error comes from truncating the order of the post-selected state $|d^\text{max}\rangle$. Using Theorem 3, directly implies

$$
\varepsilon_1 \leq \sum_{l \in (D^\text{max}\setminus D^\text{max})} (\gamma t)^{(|l|/m_{\text{max}})} \left| \frac{(|l|/m_{\text{max}})!}{(|l|/m_{\text{max}})!} \right|!
\leq 2 \sum_{l=m_{\text{max}}}^{\infty} (\gamma t)^{(|l|/m_{\text{max}})} \left| \frac{(|l|/m_{\text{max}})!}{(|l|/m_{\text{max}})!} \right|!.
$$

(50)

The summation $\sum_{l=m_{\text{max}}}^{\infty}$ can be divided based on $l$ (mod. $m_{\text{max}}$), and each summation corresponds to the left hand side of Eq. (42) with $n_0 \geq \lfloor l_{\text{max}}/m_{\text{max}} \rfloor$. In other words, for $l_{\text{max}} \geq 2m_{\text{max}}\gamma t$, we obtain the following inequality,

$$
\sum_{l=m_{\text{max}}}^{\infty} (\gamma t)^{(|l|/m_{\text{max}})} \left| \frac{(|l|/m_{\text{max}})!}{(|l|/m_{\text{max}})!} \right|! \leq 2m_{\text{max}} (\gamma t)^{l_{\text{max}}/m_{\text{max}}} \left| \frac{(|l|/m_{\text{max}})!}{(|l|/m_{\text{max}})!} \right|!.
$$

(51)

As a result, the first error $\varepsilon_1$ is bounded by $4m_{\text{max}}(\gamma t)^{l_{\text{max}}/m_{\text{max}}} / (|l_{\text{max}}/m_{\text{max}}|)!$.

The second error $\varepsilon_2$ comes from truncating the order of the effective Hamiltonian. In a similar way to the proof of Theorem 3, each term is bounded by

$$
\varepsilon_2 \leq \sum_{l\in(D^\text{max}\setminus D^\text{max})} (\gamma t)^{2(2l_{\text{max}}-|l|)/m_{\text{max}}} \frac{|n_0|!2^n}{n_0!2^n}.
$$

(52)

In the last inequality, we again use the relation Eq. (51).

Taking the limit $l_{\text{max}}' \rightarrow \infty$ for $\varepsilon_1 + \varepsilon_2$ reproduces the result of Theorem 4.

B. Truncation order of Floquet-Hilbert space

We hereby determine the truncation order $l_{\text{max}}$ so that $|\psi^{\text{max}}(t)\rangle$ can reproduce the exact time-evolved state with a desirable error up to $O(\varepsilon)$. By using the inequality from the Stirling formula, Eq. (45) to Theorem 4, we obtain the error bounded by

$$
||\psi(t) - \psi^{\text{max}}(t)|| \leq 10m_{\text{max}} \left( \frac{em_{\text{max}}\gamma t}{l_{\text{max}}} \right)^{l_{\text{max}}/m_{\text{max}}},
$$

(54)

for $l_{\text{max}} \geq 2m_{\text{max}}\gamma t$. The truncation order $l_{\text{max}}$ is chosen so that the right hand side can be smaller than $\varepsilon$. To this aim, we should evaluate a function $f(x) = (\kappa/x)^x$ [$\kappa > 0$], which is known to be dealt with the Lambert W function $W(x)$ satisfying $W(x)e^{W(x)} = x$ [53]. Here, we rely on the resulting proposition [19].

Proposition 5.

The function $f(x) = (\kappa/x)^x$ [$\kappa > 0$] is monotonically decreasing in $x \geq \kappa/e$, and satisfies the following inequality for $0 < \eta < 1$:

$$
f(x) \leq \eta, \quad \forall x \geq e \kappa + \frac{4\log(1/\eta)}{\log(e + \gamma t^{-1}\log(1/\eta))}.
$$

(55)

See Lemma 59 in the full version of Ref. [19] for the proof. Based on the above proposition, we choose $l_{\text{max}}$ by

$$
l_{\text{max}} - m_{\text{max}} = \left[ e^{2m_{\text{max}}\gamma t} + \frac{4m_{\text{max}}\log(10m_{\text{max}}/\varepsilon)}{\log(e + (e \gamma t^{-1}\log(10m_{\text{max}}/\varepsilon))} \right],
$$

(56)

so that the error can be bounded from above as

$$
10m_{\text{max}} \left( \frac{em_{\text{max}}\gamma t}{l_{\text{max}} - m_{\text{max}}} \right)^{l_{\text{max}} - m_{\text{max}}/m_{\text{max}}} \leq \varepsilon.
$$

(57)

We note that this choice does not violate the condition $l_{\text{max}} \geq 2m_{\text{max}}\gamma t$, which is required for Theorem 4. The monotonicity of $f(x)$ ensures $||\psi(t) - \psi^{\text{max}}(t)|| \leq \varepsilon$ under the above choice (The additional term $m_{\text{max}}$ is attached for later calculation, especially for Appendix B1 and B2).

Let us discuss how $l_{\text{max}}$ increases in the time $t$ and the inverse error $1/\varepsilon$. The form of the error in $l_{\text{max}}$, given by Eq. (46), is the same as that for qubitization in the query complexity $q$, given by $\varepsilon_q \in O((\log t)/q^2)$. Therefore, its approximate scaling can be determined in a similar way, which results in

$$
l_{\text{max}} \in \Theta \left( \gamma t \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)} \right).
$$

(58)

This relation can be checked by considering the two limiting cases $\gamma t \gg \log(1/\varepsilon)$ and $\gamma t \ll \log(1/\varepsilon)$ in Eq. (56) (See the endnote [54] for the accurate scaling of $l_{\text{max}}$).
To reproduce the truncated Floquet-Hilbert space, we should prepare an ancilla system labeled by \{\{l\}\} \in D_{l_{\text{max}}}^{\text{max}}. The number of qubits for such ancilla system is at most
\[ O(\log l_{\text{max}}) \subset O(\log(\gamma t) + \log \log(1/\varepsilon)). \quad (59) \]

V. AMPLITUDE AMPLIFICATION OF THE DYNAMICS

In this section, we show two kinds of amplification to achieve sufficiently high success probability of extracting the time-evolved state \( |\psi(t)\rangle \).

As we stated in Section III A, the approximate state \( |\psi_{\text{max}}(t)\rangle \) cannot be directly realized on quantum circuits. After simulating the dynamics in the truncated Floquet-Hilbert space to get \( |\Psi_{\text{max}}(t)\rangle \) defined by Eq. (19), we make a projection to \( |a_{\text{max}}\rangle \). Although the resulting renormalized state is sufficiently close to \( |\psi_{\text{max}}(t)\rangle \) and also the time-evolved state \( |\psi(t)\rangle \) as
\[ \frac{\langle a_{\text{max}} | |\Psi_{\text{max}}(t)\rangle}{\|a_{\text{max}}\| |\Psi_{\text{max}}(t)\|} = \frac{\langle \psi_{\text{max}}(t) \rangle}{\|\psi_{\text{max}}(t)\|} = |\psi(t)\rangle + O(\varepsilon) \quad (60) \]
for \( l_{\text{max}} \) given by Eq. (56), we should be careful of the low success probability; it becomes \( \|a_{\text{max}} | |\Psi_{\text{max}}(t)\rangle\|^2 \in O((l_{\text{max}}^{-1})^2) \).

We need approximately \( O(\gamma t) \)-times trials of the post selection, and every trial is expected to require at least \( O(\gamma t) \) complexity for implementing \( |\Psi_{\text{max}}(t)\rangle = e^{-i H_{\text{eff}} t \max} |0\rangle \otimes |\psi(0)\rangle \). Therefore, the naive implementation based on Eq. (18) is not efficient for the time-evolved state \( |\psi(t)\rangle \) in that the expected computational time reaches \( O(l_{\text{max}}^2) \).

We resolve this problem by the amplification of the success probability up to \( 1 - O(\varepsilon) \) in the next section. The first one, which exploits the symmetry of the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \), amplifies \( \sim \mathcal{O}(l_{\text{max}}^{-1}) \) to \( \mathcal{O}(1) \). The latter one following this, which is reminiscent of the Grover’s search algorithm, allows the success probability \( 1 - O(\varepsilon) \). As discussed later, using only either one fails to efficiently compute \( |\psi(t)\rangle \).

A. Amplification by symmetry

We introduce the amplification exploiting the symmetry of the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \). First of all, we specify the symmetry here; it is about the translation of the photon number from \( |l\rangle \) to \( |l + m\rangle \). When we define the translation operator on the Floquet-Hilbert space by \( \mathcal{T}_m = \sum_{l \in \mathbb{Z}} |l\rangle \otimes (l + m) \otimes I \), \( \mathcal{H}_{\text{eff}} \) satisfies the translation symmetry,
\[ \mathcal{T}_m^\dagger \mathcal{H}_{\text{eff}} \mathcal{T}_m = \mathcal{H}_{\text{eff}} + m \omega, \quad (61) \]
\[ \langle l | e^{-i \mathcal{H}_{\text{eff}} t |l'\rangle} = e^{it m \omega} \langle l - l' | e^{-i \mathcal{H}_{\text{eff}} t} |0\rangle. \quad (62) \]
Since the symmetry is present as long as the time-periodity \( H(t + T) = H(t) \) holds, the amplification discussed here is always available in our algorithm.

Here, we provide two tasks for the amplification by symmetry. First, we slightly extend the truncated Floquet-Hilbert space to \( \mathbb{C}^{D_{l_{\text{max}}} \otimes \mathcal{H}} \), where the Fourier index \( l \) is chosen from \( l \in D_{l_{\text{max}}}^{\text{max}} = \{-4l_{\text{max}} + 1, -4l_{\text{max}} + 2, \ldots, 4l_{\text{max}}\} \). The number of additional qubits required for this extension from \( \mathbb{C}^{l_{\text{max}}} \otimes \mathcal{H} \) is three, and hence this does not affect the scaling of the computational resources. The second task is to preprocess the initial state by
\[ \mathcal{U}_{\text{ini}} \langle 0 | \psi(0) \rangle = |a_{\text{max}}\rangle \langle \psi(0) |, \quad (63) \]
where the unitary operator \( \mathcal{U}_{\text{ini}} \) is expressed by
\[ \mathcal{U}_{\text{ini}} = |a_{\text{max}}\rangle \langle 0 | + \ldots) \otimes I. \quad (64) \]

The unitary operator \( \mathcal{U}_{\text{ini}} \) nontrivially acts only on the ancilla space, and it can be implemented with \( O(\log l_{\text{max}}) \) gates (e.g. for \( l_{\text{max}} \) such that \( \log_2 l_{\text{max}} \in \mathbb{N} \), we have \( \mathcal{U}_{\text{ini}} = \text{Had} \otimes_{\log_2(2l_{\text{max}})} (2l_{\text{max}}) \otimes I \) with the Hadamard gate \( \text{Had} \)). This process plays a role of making the initial state approximately translation-invariant with the width of Fourier indices, \( 2l_{\text{max}} \). The remaining procedures after the above two tasks is the same as those for the original protocol in Section III A. Reflecting that the ancilla Hilbert space is extended to \( \mathbb{C}^{2l_{\text{max}}} \), we evolve the above uniform state \( |a_{\text{max}}\rangle \langle \psi(0) | \) by \( \mathcal{H}_{\text{eff}}^{4l_{\text{max}}} \) and \( \sum_{l \in D_{l_{\text{max}}}^{\text{max}}} l \omega |l\rangle \langle l | \otimes I \), and make a projection to \( |a_{4l_{\text{max}}}\rangle \).

The resulting state \( |\tilde{\psi}(t)\rangle \) is calculated as follows.
\[ |\tilde{\psi}(t)\rangle \equiv |a_{4l_{\text{max}}}\rangle \langle e^{-it \sum_{l \in D_{l_{\text{max}}}^{\text{max}}} l \omega} |a_{4l_{\text{max}}}\rangle \langle |\psi(0) |, \quad (65) \]

The point of this process is that the above state \( |\tilde{\psi}(t)\rangle \) can reproduce \( |\psi(t)\rangle \) with \( O(1) \) amplitude owing to the translation symmetry of the effective Hamiltonian, Eqs. (61) and (62). We give its brief explanation in this section, while the rigorous derivation is provided in Appendix B 1. Although we do not have the exact translation symmetry represented by Eqs. (62) due to the finite-size effect of \( l_{\text{max}} \), it is expected to approximately hold as
\[ \langle l | e^{-i \mathcal{H}_{\text{eff}}^{4l_{\text{max}}} t} |l'\rangle \approx e^{it \omega (l - l')} e^{-i \mathcal{H}_{\text{eff}}^{4l_{\text{max}}} t} |0\rangle. \quad (66) \]

if the truncation order \( l_{\text{max}} \) is sufficiently large. As a matter of fact, we can derive the exact upper bound on the difference between the left- and right-hand sides based on the Lieb-Robinson bound (See Lemma 8 in Appendix B 1). We proceed the discussion with assuming the approximate relation Eq. (66) here. The resulting state of the process \( |\tilde{\psi}_{\text{max}}(t)\rangle \) can be roughly computed as fol-
f lows;

\[ |\psi_{\text{amp}}^{\text{max}}(t)\rangle = \frac{1}{4\max} \left( \sum_{l' \in D_{\text{max}}} \sum_{l \in D_{\text{max}}} e^{-i l t} \langle l | e^{-i H_{\text{eff}}^{\text{max}} t} | l' \rangle |\psi(0)\rangle \right) \]

\[ \simeq \frac{1}{4\max} \left( \sum_{l' \in D_{\text{max}}} \sum_{l \in D_{\text{max}}} e^{-i(l-l')^2t} \times (l-l') e^{-i H_{\text{eff}}^{\text{max}} t} |0\rangle |\psi(0)\rangle \right) \]

\[ = \frac{1}{4\max} \sum_{l' \in D_{\text{max}}} (|\psi(t)\rangle + O(\varepsilon)) \]

\[ = \frac{1}{2} |\psi(t)\rangle + O(\varepsilon). \quad (67) \]

The second equality comes from Theorem 5, considering that the summation of \( l - l' \) over \( l \in D_{\text{max}} \) is sufficient to suppress the error up to \( O(\varepsilon) \) under \( l_{\text{max}} \in O(\gamma t + \log(1/\varepsilon)/\log \log(1/\varepsilon)) \). As a result, we obtain

\[ |\psi_{\text{amp}}^{\text{max}}(t)\rangle \simeq \frac{1}{2} (|\psi(t)\rangle + O(\varepsilon)). \quad (68) \]

Let us focus on the success probability of the projection onto \( |a_{\text{max}}^{\text{max}}\rangle \). It is provided by \( \langle \psi_{\text{amp}}^{\text{max}}(t)|\psi_{\text{amp}}^{\text{max}}(t)\rangle = 1/4 + O(\varepsilon) \), which is much larger than the original one \( 1/(2l_{\text{max}}) \). Therefore, the amplification protocol by symmetry enables us to avoid \( O(t) \)-times repetition of the time evolution \( e^{-i H_{\text{eff}} t} \) in contrast to the original protocol. Intuitively, this drastic improvement can be understood as a result of the interference under the translation symmetry, as Fig. 2. As we can see from Eq. 19, the target state \( |\psi(t)\rangle \) is extracted from \( |\Psi^{\text{max}}(t)\rangle \) via the accompanying ancilla state \( |a_{\text{max}}^{\text{max}}\rangle \), which is uniform in the Fourier index \( l \). When we begin with the non-uniform initial state \( |0\rangle |\psi(0)\rangle \), the resulting state after the time evolution is also non-uniform. Since it involves \( (1/\sqrt{2l_{\text{max}}}) \sum_{l \in D_{\text{max}}} e^{ikl} |l\rangle \) for \( k \in (2\pi/2l_{\text{max}})\mathbb{Z} \) (different eigenstates of the translation operator) with approximate equal weight, the amplitude of the desirable component \( |a_{\text{max}}^{\text{max}}\rangle \) is comparably small as \( 1/\sqrt{2l_{\text{max}}} \). In contrast, when we employ the uniform initial state \( |a_{\text{max}}^{\text{max}}\rangle |\psi(0)\rangle \), only the uniform components that includes \( |a_{\text{max}}^{\text{max}}\rangle \) are amplified while the other components cancel one another. It can be viewed also as the interference of different initial states \( |l\rangle |\psi(0)\rangle \) in \( |a_{\text{max}}^{\text{max}}\rangle |\psi(0)\rangle \) as shown in Fig. 2 and this is why the amplification protocol achieves \( O(1) \) amplitude of \( |a_{\text{max}}^{\text{max}}\rangle |\psi(0)\rangle \).

We summarize the amplification by symmetry with adding the exact results obtained in Appendix B. We prepare the truncated Floquet-Hilbert space \( \mathbb{C}_{\text{max}}^{\text{max}} \otimes \mathcal{H} \), and make the initial state uniform as \( |a_{\text{max}}^{\text{max}}\rangle |\psi(0)\rangle \). As a result, this process is described by a unitary gate

\[ \mathcal{U}_{\text{amp}}^{\text{max}}(t) = (\mathcal{U}_{\text{ini}}^{\text{max}}) \psi \left( \sum_{l} i \omega|l\rangle \langle l| e^{-i H_{\text{eff}}^{\text{max}} t} \mathcal{U}_{\text{ini}}^{\text{max}} \right), \quad (69) \]

where the last unitary gate \( (\mathcal{U}_{\text{ini}}^{\text{max}})^\dagger \) is added to replace the projection to \( |a_{\text{max}}^{\text{max}}\rangle \) by the one to \( |0\rangle \). This approximately realizes the time-evolved state \( |\psi(t)\rangle \) with success probability as

\[ \langle 0 | \mathcal{U}_{\text{amp}}^{\text{max}}(t) |0\rangle |\psi(0)\rangle \approx \frac{1}{2} |\langle \psi(t)\rangle + O(\varepsilon)\rangle. \quad (70) \]

The probability 1/4 comes from the ratio of the width of the initial state \( |a_{\text{max}}^{\text{max}}\rangle \) to that of the projected ancilla state \( |a_{\text{max}}^{\text{max}}\rangle \). The exact version, derived based on the Lieb-Robinson bound in Appendix B.1 is stated as follows, and ensures the validity of the discussion here.

**Theorem 6. (Amplification by symmetry)**

We designate the truncation order \( l_{\text{max}} \in \Theta(\gamma t + \log(1/\varepsilon)/\log \log(1/\varepsilon)) \) by Eq. (56). Then,

\[ \left\| \langle 0 | \mathcal{U}_{\text{amp}}^{\text{max}}(t) |0\rangle |\psi(0)\rangle - \frac{1}{2} |\psi(t)\rangle \right\| \leq \frac{\varepsilon}{3}. \quad (71) \]

is satisfied for an arbitrary initial state \( |\psi(0)\rangle \in \mathcal{H} \).

We finally note that the amplification solely by this process cannot achieve the success probability 1 – \( O(\varepsilon) \) with keeping the efficiency. Analogous to the above formulation, when we set the widths of the supports of the initial ancilla state, the effective Hamiltonian, and the projected ancilla state to \( p_{\text{max}}, q_{\text{max}}, q_{\text{max}} \) \( p, q \in \mathbb{N}, \ q \geq p + 2 \) respectively, the success probability becomes \( p/q \). However, the probability larger than 1 – \( \varepsilon \) demands the relation,

\[ 1 - \varepsilon < \frac{p}{q} < \frac{p}{q + p + 2}. \quad (72) \]

This implies \( p \geq 2(1 - \varepsilon)/\varepsilon \), and hence the number of ancilla qubits reaches at-least \( \log(p_{\text{max}}) \in O(\log(\gamma t/\varepsilon)) \). In addition, the query complexity includes a term proportional to the dimension of the ancilla system, \( q_{\text{max}} \) as discussed later in Section VII. This implies that the query complexity increases linearly in 1/\( \varepsilon \), destroying the original logarithmic scaling. Thus, relying solely on this amplification protocol is not suitable for efficient simulation of the dynamics. We resolve this problem by the oblivious amplitude amplification below.

**B. Oblivious amplitude amplification**

The above amplification based on the translation symmetry of \( H_{\text{eff}} \) enhances the amplitude of \( |\psi(t)\rangle \) from \( 1/\sqrt{2l_{\text{max}}} \) to \( 1/2 - O(\varepsilon) \). Here, we introduce another amplification, called the oblivious amplitude amplification [19], to achieve the amplitude (or the success probability) 1 – \( O(\varepsilon) \).

The starting point of this amplification is the result of the previous section V A. Equation (70), or equivalently Theorem 6 indicates that the consequence of the amplification by symmetry is written as

\[ \mathcal{U}_{\text{amp}}^{\text{max}}(t) |0\rangle |\psi(0)\rangle = \frac{1}{2} |0\rangle (|\psi(t)\rangle + O(\varepsilon)) + |\psi^\perp\rangle, \quad (73) \]
with an additional term $|\Psi^\perp\rangle \in \mathbb{C}^{\mathcal{N}_{\text{max}}} \otimes \mathcal{H}$ satisfying

$$
(|0\rangle \langle 0| \otimes I) |\Psi^\perp\rangle = 0.
$$

(74)
The state $|\Psi^\perp\rangle$ generally depends on $|\psi(0)\rangle$ and $t$. The oblivious amplitude amplification takes a similar strategy to that of the Grover’s search algorithm, in which we compose the following two unitary operators;

$$
\mathcal{R} = (2 |0\rangle \langle 0| - I) \otimes I,
$$

(75)

$$
\mathcal{U}_{\text{amp}2}(t) = -\mathcal{U}_{\text{amp}1}^\dagger(t) \mathcal{R} \mathcal{U}_{\text{amp}1}(t) \mathcal{R} \mathcal{U}_{\text{amp}2}(t).
$$

(76)
The first one $\mathcal{R}$ reverses the sign of $|l\rangle$ for $l \neq 0$, and it is implemented with $O(\log \mathcal{N}_{\text{max}})$ gates. The second one $\mathcal{U}_{\text{amp}1}^\dagger(t)$ plays a role in enhancing the amplitude of $|\psi(t)\rangle$ up to $1 - \mathcal{O}(\varepsilon)$. Its action on any initial state $|0\rangle |\psi(0)\rangle$ is computed as follows;

$$
\mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(0)\rangle = \mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(0)\rangle - \mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(t)\rangle + \mathcal{O}(\varepsilon).
$$

(77)

In the second equality, we use the relation obtained by applying $\mathcal{U}_{\text{amp}1}^\dagger$ to Eq. (73). Next, we evaluate

$$
\mathcal{R} (\mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(t)\rangle = 2 |0\rangle \left( |0\rangle \langle 0| - \mathcal{U}_{\text{amp}1}^\dagger \right) |\psi(t)\rangle - \mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(t)\rangle.
$$

(78)

Theorem 6 indicates that the time evolution operator $U(t)$ is approximated as

$$
\left\| \langle 0| \mathcal{U}_{\text{amp}1}^\dagger(t)|0\rangle - \frac{1}{2} U(t) \right\| \leq \frac{\varepsilon}{3},
$$

(79)

and hence the relation

$$
\left\| \langle 0| \mathcal{U}_{\text{amp}1}^\dagger(t)|0\rangle - \frac{1}{2} U(t) \right\| \leq \frac{\varepsilon}{3}
$$

(80)
is also satisfied. This provides the relation,

$$
\mathcal{R} (\mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(t)\rangle = |0\rangle |\psi(0)\rangle - \mathcal{U}_{\text{amp}1}^\dagger |0\rangle |\psi(t)\rangle + \mathcal{O}(\varepsilon),
$$

(81)

and substituting this into Eq. (77) results in

$$
\mathcal{U}_{\text{amp}2}(t) |0\rangle |\psi(0)\rangle = |0\rangle |\psi(t)\rangle + \mathcal{O}(\varepsilon),
$$

(82)

for an arbitrary initial state $|\psi(0)\rangle$.

This result indicates that the operation $\mathcal{U}_{\text{amp}2}(t)$ generates the time-evolved state $|\psi(t)\rangle$ with the amplitude $1 - \mathcal{O}(\varepsilon)$. Or equivalently, as an exact bound for the error, we can derive the inequality,

$$
\left\| \mathcal{U}_{\text{amp}2}(t) |0\rangle |\psi(0)\rangle - |0\rangle |\psi(t)\rangle \right\| \leq \varepsilon.
$$

(83)
The coefficient comes from the fact that an error bounded by $\varepsilon/3$ appears due to Theorem 5 every time we call
The amplification protocol $U^{\text{max}}_{\text{amp/2}}(t)$ employs 3 times queries to $U^{\text{max}}_{\text{amp/1}}(t)$. Reflecting that the operations $U^{\text{max}}_{\text{ini}}$ and $\mathcal{R}$ require relatively a little resource (atmost $O(\log l_{\text{max}})$ elementary gates and complexity), the resource for $U^{\text{max}}_{\text{amp/2}}(t)$ has the same scaling as the one for implementing the time evolution operators $e^{-i \sum t l \omega_l |l \rangle \langle l |}$ and $e^{-i H^{\text{eff}}_{\text{max}} t}$.

As well as the amplification by symmetry, relying only on the oblivious amplitude amplification fails to efficiently enhances the success probability to obtain $|0 \rangle \otimes |\psi(t)\rangle$ from $O(l_{\text{max}}^{-1})$ to $1 - O(\varepsilon)$. When we do not use the first amplification, the protocol of the oblivious amplitude amplification is given by

$$U^{\text{max}}_{\text{amp/2},p} = \left\{ U^{\text{max}}_{\text{ini}} \mathcal{R} \left( U^{\text{max}}_{\text{ini}} \right)^\dagger \mathcal{R} \right\}^p U^{\text{max}}_{\text{ini}} ,$$  

(84)

where the operation $U^{\text{max}}_{\text{ini}}$ represents the time evolution without the amplification by symmetry, defined by

$$U^{\text{max}}_{\text{ini}}(t) = \left( U^{\text{ini}} \right)^l e^{-i \sum t l \omega_l |l \rangle \langle l |} e^{-i H^{\text{eff}}_{\text{max}} t}.$$  

(85)

Applying $U^{\text{max}}_{\text{amp/2},p}$ to the initial state $|0 \rangle \otimes |\psi(t)\rangle$ returns $|0 \rangle \otimes |\psi(t)\rangle$ whose amplitude increases from $1/\sqrt{2 l_{\text{max}}}$ approximately in proportion to $p$. The integer $p$ should be $O(l_{\text{max}})$ to achieve the amplitude $1 - O(\varepsilon)$, reminiscent of the Grover’s search algorithm. In other words, $O(l_{\text{max}})$ times call of $e^{-i \sum t l \omega_l |l \rangle \langle l |}$ and $e^{-i H^{\text{eff}}_{\text{max}} t}$ is required when we use $U^{\text{max}}_{\text{amp/2},p}$. This is why we suggest the combination of the two amplification protocols, with which $O(1)$ times call of $e^{-i \sum t l \omega_l |l \rangle \langle l |}$ and $e^{-i H^{\text{eff}}_{\text{max}} t}$ can amplify the amplitude of $|0 \rangle \otimes |\psi(t)\rangle$ from $1/\sqrt{2 l_{\text{max}}}$ to $1 - O(\varepsilon)$.

### VI. BLOCK-ENCODING OF EFFECTIVE FLOQUET HAMILTONIAN

In the previous section, we show that the combination of the two kinds of amplification protocols, implemented by $U^{\text{max}}_{\text{amp/2}}(t)$ [See Eq. (84)], provides the target time-evolved state $|\psi(t)\rangle$ with an arbitrarily small error $O(\varepsilon)$. The computational resource for $U^{\text{max}}_{\text{amp/2}}(t)$ is mostly determined by that for $e^{-i \sum t l \omega_l |l \rangle \langle l |}$ and $e^{-i H^{\text{eff}}_{\text{max}} t}$. Our strategy is to implement these two time evolution operators by the qubitization technique [13]. As discussed in Section [4] the computational resources are determined by how to introduce the block-encoding of the two static Hamiltonians, $H_{\text{LP}}$ and $H_{\text{eff}}$ (linear potential Hamiltonian) and $H_{\text{eff}}$ (effective Hamiltonian). The aim of this section is to obtain an efficient block-encoding of them and to evaluate the costs required to implement $e^{-i \sum t l \omega_l |l \rangle \langle l |}$ and $e^{-i H^{\text{eff}}_{\text{max}} t}$.

#### A. Block-encoding of linear potential Hamiltonian

We propose block-encoding of the linear potential Hamiltonian,

$$H_{\text{LP}} \equiv \sum_{l \in D_{\text{max}}} \omega_l |l \rangle \langle l |.$$  

(86)

By simple calculation, it can be written in the form of an LCU,

$$H_{\text{LP}} \equiv \sum_{l \in D_{\text{max}}} \omega_l |l \rangle \langle l | \otimes I.$$  

(87)

Using the block-encoding formalism of LCUs by Eq. (10), we immediately obtain the oracle unitary gate and the oracle state by

$$|a^{\text{max}}_{\text{LP}} \rangle \langle b^{\text{max}}_{\text{LP}}| = \frac{H_{\text{LP}}}{8 l_{\text{max}}^4 \omega}.$$  

(90)

The subscript $b$ for the states $|l \rangle_b$ and $|a^{\text{max}}_{\text{LP}} \rangle_b$ represents a new ancilla system introduced for the block-encoding, requiring the number of qubits $n_b \in O(\log l_{\text{max}})$. Combining Eqs. (88) and (89), the oracle unitary gate is rewritten by

$$O_{\text{LP}}^{\text{max}} = \left( \sum_{l, l', \ell \geq 0} |l, l' \rangle \langle l, l' | - \sum_{l, l', \ell \geq 0} |l, l' \rangle \langle l, l' | \right) \otimes I.$$  

(91)

To implement this oracle, we use a comparator defined by

$$\text{Comp} (|l, l' \rangle, |0 \rangle_{b'}) = \begin{cases} |l, l' \rangle \langle 0 |_{b'} & \text{if } l' \geq l, \\ |l, l' \rangle \langle l |_{b'} & \text{if } l' < l, \end{cases}$$  

(92)

with a single-qubit ancilla system $b'$. We can immediately confirm the relation,

$$\text{Comp} (Z_{b'} \text{Comp} (|l, l' \rangle, |0 \rangle_{b'}) \langle \psi | = O_{\text{LP}}^{\text{max}} |l, l' \rangle \langle l, l' | \langle 0 |_{b'} \langle \psi |,$$  

(93)

for arbitrary inputs $|l, l' \rangle \in \mathcal{C} l_{\text{max}}$ and $|\psi \rangle \in \mathcal{H}_b$, where $Z_b$ denotes a Pauli Z operator on the system $b'$. Since the comparator on $n$ qubits can be composed of $O(n)$ elementary gates, the oracle unitary gate $O_{\text{LP}}^{\text{max}}$ requires atmost $O(\log l_{\text{max}})$ gates. The oracle state $|a^{\text{max}}_{\text{LP}} \rangle_b$, which has equal weights in $|l \rangle$, can be prepared by $O(\log l_{\text{max}})$ elementary gates.

According to the qubitization technique, implementing $e^{-i H_{\text{LP}}^{\text{max}} t}$ with accuracy $1 - O(\varepsilon)$ requires $O(8 l_{\text{max}}^4 \varepsilon / \log(1/\varepsilon))$-times queries to $O_{\text{LP}}^{\text{max}}$ and $O(1)$ additional qubits. Therefore, simulating the dynamics under $H_{\text{LP}}^{\text{max}}$ for the duration $t$ requires the following resources:
terms \( \gamma \) provides the bound on the energy scale of time-dependent
considering \( H \). The parameter \( \alpha_m \) with an
_gate on \( H \). This originates from the fact
that the number of nontrivial terms in \( H \) increases
linearly in \( m \). To avoid this problem, we take an alter-
native approach which yields only \( \mathcal{O}(\log m) \) complexity
as follows:

1. Find a refined effective Hamiltonian \( \mathcal{H} \) on the
truncated Floquet-Hilbert space \( \mathbb{C}^{2^m} \otimes \mathcal{H} \), such
that \( e^{-i\mathcal{H}t} \) accurately reproduces \( e^{-i\mathcal{H}_{\text{eff}}t} \) with
an arbitrarily small error \( \mathcal{O}(\varepsilon) \).

2. Construct block-encoding of \( \mathcal{H} \) with composing
of efficient implementation of its oracle unitary gate and
oracle state.

First, we specify the assumption for the time-periodic
Hamiltonian \( H(t) \). Here, we suppose two kinds of the
oracles that can be accessed. The first one is about the
block-encoding of each Fourier component \( H_m \) \((|m| \leq m_{\text{max}})\), given by

\[
\langle G_m \rangle_{a} O_{m} |G_m \rangle_{a} = \frac{H_m}{\alpha_m}, \quad \alpha_m > 0, \quad (95)
\]

with an \( n_a \)-qubit ancilla system. Each oracle unitary
gate on \( \mathbb{C}^{2^m} \otimes \mathcal{H} \) is represented by \( O_m \), and each oracle
state \( |G_m \rangle_{a} \) is generated by a quantum cir-
guit \( G_m \) as \( |G_m \rangle_{a} = G_m (0)^{\otimes n_a} \). The second oracle is a
query to coefficients of Fourier components, represented
by \( \alpha_m \). We define the oracle unitary circuit \( G_{\text{coef}} \), acting
on \( \mathbb{C}^{2^{m_{\text{max}}}} \) [i.e. \( \mathcal{O}(1) \)-qubit system], by

\[
G_{\text{coef}} |0 \rangle = \sum_{|m| \leq m_{\text{max}}} \sqrt{\frac{\alpha_m}{\alpha}} |m \rangle, \quad \alpha = \sum_{|m| \leq m_{\text{max}}} \alpha_m. \quad (96)
\]

The parameter \( \alpha \), defined by \( \alpha = \sum_{|m| \leq m_{\text{max}}} \alpha_m \), character-
izes the energy scale of the Hamiltonian \( H(t) \). In fact,
considering \( |H_m| \leq \alpha_m \) for each \( m \), it is bounded by
\( |H(t)| \leq \sum_{|m| \leq m_{\text{max}}} |H_m| \leq \alpha_n \). The parameter \( \alpha \) also
provides the bound on the energy scale of time-dependent
terms \( \gamma \), as

\[
\gamma \leq \sum_{m \neq 0, |m| \leq m_{\text{max}}} \|H_m\| \leq \alpha. \quad (97)
\]

Implementation of the oracle \( G_{\text{coef}} \) is equivalent to em-
bedding probability distributions in quantum states \([59]\),
which also appears as the oracle state for the qubitization
scheme in the case of a LCU [See Eq. (10)]. Stimulated
by its various applications covering linear equation
solver \([57]\) and quantum singular value transformation
\([19, 48]\), there have been various efficient implementa-
tions \([47, 56, 58–60]\). Here, we suppose that the number
of elementary gates required for the oracles \( O_m \), \( G_m \), and
\( G_{\text{coef}} \) is at most \( C \). As discussed later in Section VIII
the form of time-periodic Hamiltonians \( H(t) \) designated
by Eq. (95) is reasonable in that they involve various
important classes such as LCUs.

### 1. Construction of a refined effective Hamiltonian

Let us compose a refined effective Hamiltonian, which
accurately reproduces the dynamics of \( e^{-i\mathcal{H}_{\text{eff}}t} \). To
come to the point, such a Hamiltonian equipped with
suitability for efficient block-encoding is given by

\[
\mathcal{H}_{\text{eff, pbc}}^{\text{max}} = \mathcal{H}_{\text{eff, pbc}}^{\text{max}} + \sum_{(l, m) \in \partial F_{\text{max}}^{\text{eff}}} (|l \rangle \langle l | + \langle l | \langle m | + h.c.) , \quad (98)
\]

with \( \partial F_{\text{max}}^{\text{eff}} = \{(l, m) | 4l_{\text{max}} - m_{\text{max}} + 1 \leq l \leq 4l_{\text{max}}, \ 4l_{\text{max}} - l + 1 \leq m \leq m_{\text{max}}\} \). The integer
\( l + m \in D_{\text{max}}^{\text{eff, pbc}} \) is defined modulo \( 4l_{\text{max}} \). The addi-
tional terms in \( \mathcal{H}_{\text{eff, pbc}}^{\text{max}} \) connects the boundary of \( D_{\text{max}}^{\text{eff, pbc}} \)
so that the hopping terms, \(|l \rangle \langle l + m |, \) induced by \( H_{l - m} \)
can be translation-invariant under the shift of \( |l \rangle \). The
subscript “pbc” comes from the periodic boundary con-
ditions (PBC) for the hopping terms.

In our algorithm, we employ the time evolution
\( e^{-i\mathcal{H}_{\text{eff, pbc}}^{\text{max}}t} \) instead of the original one \( e^{-i\mathcal{H}_{\text{eff}}^{\text{max}}t} \), which
is derived by Floquet theory. In other words, we orga-
nize the amplification protocols with the refined effective
Hamiltonian \( \mathcal{H}_{\text{eff, pbc}}^{\text{max}} \) as

\[
W_{\text{amp1, pbc}}^{\text{max}}(t) = (W_{\text{amp1, pbc}}^{\text{max}})^{\dagger} e^{-i\mathcal{H}_{\text{eff, pbc}}^{\text{max}} t} e^{-i\mathcal{H}_{\text{eff, pbc}}^{\text{max}} t} W_{\text{amp1, pbc}}^{\text{max}}(t) \quad (99)
\]

\[
W_{\text{amp2, pbc}}^{\text{max}}(t) = -W_{\text{amp2, pbc}}^{\text{max}}(t) \mathcal{R}_{\mathcal{H}_{\text{amp1, pbc}}^{\text{max}}} W_{\text{amp2, pbc}}^{\text{max}}(t) \quad (100)
\]

and apply them to the initial state \(|0 \rangle \langle \psi(0) |\). The refined
effective Hamiltonian is valid in a sense that it can also
provide the exact time-evolved state as

\[
W_{\text{amp2, pbc}}^{\text{max}}(t) |0 \rangle \langle \psi(0) | = |0 \rangle \langle \psi(t) | + \mathcal{O}(\varepsilon). \quad (101)
\]

The rigorous upper bound on the error is provided by the
following theorem.

**Theorem 7. (Refined effective Hamiltonian)**

We choose \( l_{\text{max}} \in \mathcal{O}(\gamma + \log(1/\varepsilon)/\log(1/\varepsilon)) \) by
Eq. (56). Then, the refined effective Hamiltonian

\( \mathcal{H}_{\text{eff}, \text{amp1}, \text{pbc}} \) well reproduces the dynamics by the original one \( \mathcal{H}_{\text{eff}} \) as
\[
\left\| \langle 0 | \mathcal{W}_{\text{amp1}, \text{pbc}}^{\text{max}}(t) | 0 \rangle - \frac{1}{2} | \psi(0) \rangle \right\| \leq \frac{\varepsilon}{3}, \tag{102}
\]
\[
\left\| \mathcal{W}_{\text{amp2}, \text{pbc}}^{\text{max}}(t) | \psi(0) \rangle - | 0 \rangle | \psi(t) \rangle \right\| \leq \varepsilon, \tag{103}
\]
for an arbitrary initial state \( | \psi(0) \rangle \in \mathcal{H} \).

We deliver its detailed proof in Appendix B and instead we briefly explain why the refined effective Hamiltonian is valid. The proof relies mainly on the Lieb-Robinson bound, stated by Theorem 3. Focusing on the amplification protocols \( \mathcal{W}_{\text{amp1}}^{\text{max}}(t) \) and \( \mathcal{W}_{\text{amp1}, \text{pbc}}^{\text{max}}(t) \), the resulting difference caused by their actions on the uniform initial state \( | a^{\text{max}} \rangle | \psi(0) \rangle \), as Eq. (99). As we can see from Fig. 3(b), the Lieb-Robinson bound dictates that the dynamics during \( \mathcal{W}_{\text{amp1}}^{\text{max}}(t) \) is almost closed within \( -2l_{\text{max}} \lesssim l \lesssim 2l_{\text{max}} \) in the Fourier indices. In contrast, the support of the additional terms of the refined effective Hamiltonian in Eq. (98) is located at \( \{ |l \rangle \} \) for \( l \simeq \pm l_{\text{max}} \), which is sufficiently far from the Fourier indices relevant for the dynamics. As a result, the refined protocol \( \mathcal{W}_{\text{amp1}, \text{pbc}}^{\text{max}}(t) \) transforms the state \( | a^{\text{max}} \rangle | \psi(0) \rangle \) in almost the same way as the original one \( \mathcal{W}_{\text{amp1}}^{\text{max}}(t) \), and hence it also outputs the target state \( | \psi(t) \rangle \) as Eq. (102). Discussion similar to Ref. 69 ensures that the oblivious amplification protocol under the refined effective Hamiltonian, represented by \( \mathcal{W}_{\text{amp2}, \text{pbc}}^{\text{max}}(t) \), provides \( | \psi(t) \rangle \) with a sufficiently small error \( \mathcal{O}(\varepsilon) \) as Eq. (103).

2. Block-encoding of a refined effective Hamiltonian

The benefit of employing the refined effective Hamiltonian \( \mathcal{H}_{\text{eff}, \text{pbc}} \) instead of the original one \( \mathcal{H}_{\text{eff}} \) is reduction of resources for implementing block-encoding. We construct an oracle unitary gate \( \mathcal{O}_{\text{eff}}^{\text{max}} \) and an oracle state \( | G_{\text{eff}}^{\text{max}} \rangle \) for it. With the help of the additional terms in Eq. (98), the Hamiltonian is written in the following form:
\[
\mathcal{H}_{\text{eff}, \text{pbc}}^{\text{max}} = \sum_{|m| \leq m_{\text{max}}} \text{Add}_{m}^{\text{max}} \otimes H_{m} - \mathcal{H}_{\text{LP}}^{\text{max}}, \tag{104}
\]
\[
\text{Add}_{m}^{\text{max}} = \sum_{l \in D^{\text{max}}} | l \rangle \langle l | \text{mod } m. \tag{105}
\]
The unitary gate \( \text{Add}_{m}^{\text{max}} \) is a full quantum adder that translates an index \( l \) by \( m \) modulo \( S_{\text{max}} \). It can be implemented by \( \mathcal{O}(\log l_{\text{max}}) \) complexity and elementary gates.

Then, to construct the block-encoding, we prepare four kinds of auxiliary systems labeled by \( a, b, c, d \). The system \( a \) is an \( n_{a} \)-qubit system prepared for the block-encoding of \( H_{m} \) [See Eq. (95)], and the system \( b \) is the one for \( \mathcal{H}_{\text{LP}}^{\text{max}} \) having \( S_{\text{max}} \) dimension [See Section VI A]. The other systems \( c \) and \( d \) have the dimension \( 2m_{\text{max}} + 1 \) and \( 2 \) respectively, yielding the number of qubits \( n_{\text{ef}} \in \mathcal{O}(\log m_{\text{max}}) \) (it is a constant here). We define the oracle unitary gate \( \mathcal{O}_{\text{eff}}^{\text{max}} \) by
\[
\mathcal{O}_{\text{eff}}^{\text{max}} = | 0 \rangle \langle 0 |_{d} \otimes \sum_{|m| \leq m_{\text{max}}} | m \rangle \langle m |_{c} \otimes I_{b} \otimes \text{Add}_{m}^{\text{max}} \otimes O_{m}
\]
\[
- | 1 \rangle \langle 1 |_{d} \otimes I_{c} \otimes I_{a} \otimes \mathcal{O}_{\text{LP}}^{\text{max}}. \tag{106}
\]
We also provide the oracle state \( | G_{\text{eff}}^{\text{max}} \rangle \) by
\[
| G_{\text{eff}}^{\text{max}} \rangle = \sum_{m} \sqrt{\frac{\alpha_{m}}{\alpha + 8l_{\text{max}}}} | 0 \rangle_{d} \langle m |_{c} \otimes | a^{\text{max}} \rangle_{b} \otimes | G_{m} \rangle_{a}
\]
\[
+ \sqrt{\frac{8l_{\text{max}}}{\alpha + 8l_{\text{max}}}} | 1 \rangle_{d} \langle 0 |_{c} \otimes | a^{\text{max}} \rangle_{b} \otimes | \varnothing \rangle_{a}. \tag{107}
\]
In the above formula, we implicitly include ancilla qubits to efficiently implement the oracles [e.g. the comparator for \( \mathcal{O}_{\text{LP}}^{\text{max}} \) in Eq. (99)]. The number of such ancilla qubits is smaller than \( \mathcal{O}(\log l_{\text{max}}) \), and hence we neglect it below. We can confirm that they provide the block-encoding of \( \mathcal{H}_{\text{eff}}^{\text{max}} \) as follows:
\[
\langle G_{\text{eff}}^{\text{max}} | \mathcal{O}_{\text{eff}}^{\text{max}} | G_{\text{eff}}^{\text{max}} \rangle = \mathcal{H}_{\text{eff}, \text{pbc}}^{\text{max}} / \alpha + 8l_{\text{max}}. \tag{108}
\]
where we use the block-encoding for \( H_{m} \) and \( \mathcal{H}_{\text{LP}}^{\text{max}} \) represented by Eqs. (95) and (90).

We evaluate the resources for implementing the oracles. The oracle unitary gate \( \mathcal{O}_{\text{eff}}^{\text{max}} \) given by Eq. (106) requires elementary gates at-most \( \mathcal{O}(m_{\text{max}} (\log l_{\text{max}} + C) + \log l_{\text{max}}) \), since the unitary operators \( \text{Add}_{m}^{\text{max}}, O_{m} \), and \( \mathcal{O}_{\text{LP}}^{\text{max}} \) respectively yield \( \mathcal{O}(\log l_{\text{max}}), C, \mathcal{O}(\log l_{\text{max}}) \) gates. On the other hand, the oracle state \( | G_{\text{eff}}^{\text{max}} \rangle \) can be prepared by
\[
| G_{\text{eff}}^{\text{max}} \rangle = \mathcal{O}_{\text{eff}}^{\text{max}} | w \rangle_{d} \langle G_{\text{coef}}^{\text{max}} | a^{\text{max}} \rangle_{b} \otimes | \varnothing \rangle_{a}, \tag{109}
\]
with using the quantum circuit \( G_{\text{eff}}^{\text{max}} \) and the state \( | w \rangle_{d} \) defined by
\[
| w \rangle_{d} = \sqrt{\frac{\alpha}{\alpha + 8l_{\text{max}}}} | 0 \rangle_{d} + \sqrt{\frac{8l_{\text{max}}}{\alpha + 8l_{\text{max}}}} | 1 \rangle_{d}. \tag{111}
\]
The quantum circuit \( G^{\text{max}} \) can be composed of \( \mathcal{O}(m_{\text{max}} C) \) elementary gates. The cost of the state preparation \( | w \rangle_{d} \) is negligible compared to others since it requires only a single-qubit rotation. The cost for preparing \( | a^{\text{max}} \rangle_{b} \) is at-most \( \mathcal{O}(\log l_{\text{max}}) \) complexity and elementary gates. To summarize, the cost for the oracle gates \( \mathcal{O}_{\text{eff}}^{\text{max}} \) and \( | G^{\text{max}} \rangle \), which embody the refined effective Hamiltonian \( \mathcal{H}_{\text{eff}, \text{pbc}}^{\text{max}} \) via Eq. (108), is at-most \( \mathcal{O}(m_{\text{max}} (C + \log m_{\text{max}})) \) elementary gates.
To prepare $e^{-i\mathcal{H}_{\text{eff}} t}$ with the qubitization technique, we employ $\mathcal{O}(O + 8l_{\text{max}} \omega t + \log(1/\epsilon)/\log(1/\epsilon))$ times queries to the oracles $\mathcal{O}^{\text{dmax}}_{\text{eff}}$ and $|G^{\text{dmax}}_{\text{eff}}\rangle$. Therefore, the cost for the time evolution operator $e^{-i\mathcal{H}_{\text{eff}} t}$ with accuracy $1 - \mathcal{O}(\epsilon)$ is summarized as follows:

- **Number of ancilla qubits;**
  \[
  n_a(l_{\text{max}}) = n_a + \mathcal{O}(\log l_{\text{max}} + \log m_{\text{max}}). \tag{112}
  \]

- **Number of overall gates;**
  \[
  \mathcal{O}\left(\left\{(\alpha + l_{\text{max}} \omega)t + \frac{\log(1/\epsilon)}{\log(1/\epsilon)}\right\}\right) \times m_{\text{max}} (C + \log l_{\text{max}}). \tag{113}
  \]

Since $m_{\text{max}}$ is supposed to be a $\mathcal{O}(1)$ constant, we will omit it from the cost in the rest of the paper.

### VII. Algorithm and Its Computational Cost

This section provides the main result of this paper; we compose the efficient quantum algorithm for simulating the time-evolved state $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ under the Hamiltonian $H(t+T) = H(t)$. We note that we take two different approaches depending on the time scale of interest. The first case is the adiabatic-like case, in which we are interested in $\mathcal{O}(1)$-period dynamics with $t/T \in \mathcal{O}(1)$. We call it “adiabatic-like” since long-time dynamics during $0 \leq t \leq T$ under the sufficiently large period $T$, exemplified by Thouless pumping and adiabatic state preparation, is a typical target. The second case is the generic long-time case, where we are interested in multiple-period dynamics at $t/T \in \Theta(1)$. In that case, the the period $T$ is not so large and we often consider long-time dynamics at $t \gg T$, exemplified by laser-irradiated materials.

#### A. Adiabatic-like cases

We consider the adiabatic-like cases, where long-time dynamics over $\mathcal{O}(1)$-periods is of interest. For simplicity, we first consider the dynamics within one period at $t \in [0,T]$. The algorithm in this case is composed of the following steps:

1. Determine the truncation order of the Fourier index, $l_{\text{max}} \in \Theta(\gamma t + \log(1/\epsilon)/\log(1/\epsilon))$, by Eq. \[56\] for the given time $t$.

2. Compose two unitary gates $\mathcal{U}^{\text{dmax}}_{\text{LP}}(t)$ and $\mathcal{U}^{\text{dmax}}_{\text{eff}}(t)$, corresponding to the time evolution operators $e^{-i\mathcal{H}_{\text{LP}} t}$ and $e^{-i\mathcal{H}_{\text{eff}} t}$, by the qubitization technique. We can find such operators satisfying

   \[
   \begin{align*}
   \mathcal{U}^{\text{dmax}}_{\text{LP}}(t) |0\rangle \otimes n_a(4l_{\text{max}}) |\psi\rangle &= |0\rangle \otimes n_a(4l_{\text{max}}) e^{-i\omega t} |\psi\rangle + \mathcal{O}(\epsilon), \tag{114}
   
   \mathcal{U}^{\text{dmax}}_{\text{eff}}(t) |0\rangle \otimes n_a(4l_{\text{max}}) |\psi\rangle &= |0\rangle \otimes n_a(4l_{\text{max}}) e^{-i\mathcal{H}_{\text{eff}}^{\text{dmax}} t} |\psi\rangle + \mathcal{O}(\epsilon), \tag{115}
   \end{align*}
   \]

   for arbitrary inputs $l \in D^{l_{\text{max}}}$ and $|\psi\rangle \in \mathcal{H}$.

3. Construct the quantum circuit for the amplification protocols with embedding $\mathcal{U}^{\text{dmax}}_{\text{LP}}(t)$ and $\mathcal{U}^{\text{dmax}}_{\text{eff}}(t)$ as $e^{-i\mathcal{H}_{\text{LP}} t}$ and $e^{-i\mathcal{H}_{\text{eff}}^{\text{dmax}} t}$ [See Section VI];

   \[
   \begin{align*}
   \mathcal{U}^{\text{dmax}}_{\text{amp1}1} &= (\mathcal{U}^{\text{dmax}}_{\text{eff}})_{\text{ini}} \mathcal{U}^{\text{dmax}}_{\text{LP}} \mathcal{U}^{\text{dmax}}_{\text{eff}}, \tag{116}
   
   \mathcal{U}^{\text{dmax}}_{\text{amp2}2} &= -\mathcal{U}^{\text{dmax}}_{\text{amp1}1} \mathcal{U}^{\text{dmax}}_{\text{ini}} \mathcal{U}^{\text{dmax}}_{\text{amp1}1}. \tag{117}
   \end{align*}
   \]

4. Prepare the initial state $|0\rangle \otimes n_a(4l_{\text{max}}) |\psi(0)\rangle$ and apply the unitary operation $\mathcal{U}^{\text{dmax}}_{\text{amp1}}(t)$ to this state, which results in

   \[
   \begin{align*}
   \mathcal{U}^{\text{dmax}}_{\text{amp2}}(t) |0\rangle \otimes n_a(4l_{\text{max}}) |\psi(t)\rangle &= |0\rangle \otimes n_a(4l_{\text{max}}) |\psi(t)\rangle + \mathcal{O}(\epsilon). \tag{118}
   \end{align*}
   \]

Step 1. determines the dimension of the truncated Floquet-Hilbert space as $C^{l_{\text{max}}} \otimes \mathcal{H}$ so that it can accurately reproduce $|\psi(t)\rangle$ with the precision $1 - \mathcal{O}(\epsilon)$, as discussed in Section VI. In Step 2, we employ the qubitization technique to realize the time evolution in the truncated Floquet-Hilbert space, as discussed in Section VI. We note that $\mathcal{U}^{\text{dmax}}_{\text{LP}}(t)$ and $\mathcal{U}^{\text{dmax}}_{\text{eff}}(t)$ can exploit a common $\mathcal{O}(\log l_{\text{max}})$-qubit ancilla system. It is sufficient to prepare an $n_a(4l_{\text{max}})$-qubit ancilla state $|0\rangle \otimes n_a(4l_{\text{max}})$ [See Eq. \[112\]]. The number of overall gates for $\mathcal{U}^{\text{dmax}}_{\text{LP}}(t)$ and $\mathcal{U}^{\text{dmax}}_{\text{eff}}(t)$ are respectively given by Eqs. \[114\] and \[115\]. Since we have $\omega t \leq 2\pi \in \mathcal{O}(1)$ in the adiabatic-like regime, the number of elementary gates for them is at-most

\[
\mathcal{O}\left(\left\{(\alpha + \gamma t) + \frac{\log(1/\epsilon)}{\log(1/\epsilon)}\right\}\right) \times \left\{C + \log(\gamma t) + \log(\log(1/\epsilon))\right\}. \tag{119}
\]

Steps 3. and 4. execute the amplification of the time-evolved state $|\psi(t)\rangle$ as discussed in Section VI. By projecting the ancilla state to $|0\rangle \otimes n_a(4l_{\text{max}})$, we succeed in preparing the target state $|\psi(t)\rangle$ with precision $1 - \mathcal{O}(\epsilon)$, where the success probability is $1 - \mathcal{O}(\epsilon)$. These steps do not change the scaling of the required resource from Eqs. \[116\] and \[117\], reflecting that the unitary operations $\mathcal{U}_{\text{init}}$ and $\mathcal{U}_{\text{eff}}$ yield at-most $\mathcal{O}(\log l_{\text{max}})$ gates.

We finally determine the resource for simulating time-periodic Hamiltonians in adiabatic-like regimes. The ancilla system should involve the degree of freedom for
Fourier indices $\{ |l \rangle \}_{l \in D}^{\max}$ and $n_a(4l_{\max})$ qubits for qubitization. The total number of ancilla qubits is

$$n_a(4l_{\max}) + \lfloor \log_2(8l_{\max}) \rfloor \in n_a + O(\log(\gamma t) + \log(1/\varepsilon)).$$

The number of overall elementary gates is dominated by that for the qubitization, designated by Eq. (119). We summarize the results in Theorem 4 and Table I. The query complexity in the table is determined by the coefficient of $C$ in the number of elementary gates. By the $O(1)$-times repetition of this procedure for $t \in [0,T]$, the same result for $O(1)$-periodic dynamics is obtained.

### B. Generic long-time cases

We next consider the other generic cases where we are interested in long-time dynamics over multiple periods as $\omega t \in \Omega(1)$. In this regime, we take a different strategy from that for the adiabatic cases; We split the time $t$ by $t = (n + \delta)T$ with $n \in \mathbb{N}$ and $\delta \in [0,1)$. Following this separation, we implement the time-evolution operator $U(t)$ by $n$-times operation of $U(T)$ and single operation of $U(\delta T)$. The algorithm is composed of the following steps;

1. Split the time by $t = (n + \delta)T$ with $n \in \mathbb{N}$ and $\delta \in [0,1)$. Determine the truncation order $l_{\max}^T$ by substituting $T$ and $\varepsilon/n$ into $t$ and $\varepsilon$ of Eq. (50);

$$l_{\max}^T \in \Theta \left( \gamma T + \frac{\log(n/\varepsilon)}{\log\log(n/\varepsilon)} \right) = \Theta \left( \gamma/\omega + \frac{\log(\omega t/\varepsilon)}{\log\log(\omega t/\varepsilon)} \right). \quad (121)$$

2. Compose a unitary gate $\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(T)$, which satisfies

$$\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(T) |0\rangle \otimes n_a(4l_{\max}^T) |0\rangle |\psi(0)\rangle = |0\rangle \otimes n_a(4l_{\max}^T) |0\rangle |\psi(T)\rangle + O(\varepsilon/n), \quad (122)$$

by the qubitization and the amplitude amplification protocols (Follow Steps 2-4. of Section VII A with substituting $T, \varepsilon/n$, and $l_{\max}^T$ into $t$, $\varepsilon$, and $l_{\max}$ respectively).

3. Apply the unitary gate $\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(T)$ to the initial state $n$ times, which results in

$$[\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(T)]^n |0\rangle \otimes n_a(4l_{\max}^T) |0\rangle |\psi(0)\rangle = |0\rangle \otimes n_a(4l_{\max}^T) |0\rangle |\psi(nT)\rangle + O(\varepsilon). \quad (123)$$

4. Prepare the unitary gate $\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(\delta T)$ by substituting $\delta T$, $\varepsilon$, and $l_{\max}^T$ into $t$, $\varepsilon$, and $l_{\max}$ respectively in Steps 2-4. of Section VII A. Applying it once to the above state results in

$$\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(\delta T)[\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(T)]^n |0\rangle \otimes n_a(4l_{\max}^T) |0\rangle |\psi(0)\rangle = |0\rangle \otimes n_a(4l_{\max}^T) |0\rangle |\psi(t)\rangle + O(\varepsilon). \quad (124)$$

for arbitrary initial states $|\psi(0)\rangle$.

We remark several points in each step. Steps 1. and 2. are executed to apply $U(T)$, giving the time-evolution over one period $T$. Here, we set the acceptable error to $O(\varepsilon/n)$ so that we can obtain the time-evolved state $|\psi(nT)\rangle$ with an error up to $O(\varepsilon)$ after the $n$-times repetition in Step 3. The cost for the unitary gate $\mathcal{U}_{\text{amp}2}^{l_{\max}^T}(T)$ is dominated only by that of $\mathcal{U}_{\text{eff}}^{l_{\max}^T}$, which reproduces
the time evolution \( e^{-i\mathcal{H}_{\text{eff}} T} \) by the qubitization. This comes from the fact that the time evolution 
\[ e^{-i\mathcal{H}_{\text{Li}}^{\text{max}} T} \]

is trivial due to
\[ e^{-i\mathcal{H}_{\text{Li}}^{\text{max}} T} = \sum_{l \in D^T} e^{il\omega T} |l\rangle \langle l| \otimes I = 1, \quad (125) \]

and we do not need \( \mathcal{H}_{\text{Li}}^{\text{max}} \) in Eq. (116). As a result, the number of elementary gates for \( n \)-times implementation of \( \mathcal{H}_{\text{amp}}^T(T) \) during Step 3. is at-most proportional to
\[ n \left( (\alpha + t_{\text{max}}^T)T + \frac{\log(n/\varepsilon)}{\log\log(n/\varepsilon)} \right) \left( C + \log(t_{\text{max}}^T) \right) \in \mathcal{O} \left( \left\{ (\alpha + \gamma)t + \omega t\frac{\log(\omega t/\varepsilon)}{\log(\omega t/\varepsilon)} \right\} \times \left\{ C + \log(\gamma/\omega) + \log(\omega t/\varepsilon) \right\} \right). \quad (126) \]

We use the fact \( n \in \mathcal{O}(\omega t) \) in the above relation. Step 4. realizes the remaining micromotion \( U(\delta T) \) for the duration \( \delta T \) with an error up to \( \mathcal{O}(\varepsilon) \). We remark that the choice of the truncation order \( t_{\text{max}}^T \) so far is sufficient to achieve the precision \( 1 - \mathcal{O}(\varepsilon) \) for \( U(\delta T) \) due to \( t_{\text{max}}^T > \delta T \). This implies that we can reuse the ancilla state \( |0\rangle \otimes n_a(t_{\text{max}}^T) \) for the qubitization approach to the time-evolution \( U(\delta T) \) in Step 4. The cost for implementing \( \frac{\mathcal{H}_{\text{amp}}^T(\delta T)}{M} \) once is obtained by setting \( t = \delta T \) in adiabatic-like cases. It is smaller than the cost for implementing the time-evolution \( U(T) \), and does not affect the scaling of the computational resources.

Finally, we provide the computational resources for time-periodic Hamiltonian dynamics in generic long-time regimes. The number of ancilla qubits is given by \( n_a = \mathcal{O}(4t_{\text{max}}^T) + \log(\omega t/\varepsilon) \), which is bounded by
\[ n_a \leq 4t_{\text{max}}^T + \log(\omega t/\varepsilon). \quad (127) \]

The number of overall elementary gates is dominated by Eq. (126). We summarize them in Theorem 2 and Table 1.

C. Comparison with other algorithms

Let us compare our algorithm on time-periodic Hamiltonians with other quantum algorithms for Hamiltonian simulation, based on Table I. We pick up the qubitization technique [18] for time-independent Hamiltonian \( H \) and the truncated-Dyson-series algorithm [20, 21] for generic time-dependent Hamiltonian \( H(t) \), whose resources have the best scaling in \( t \) and \( 1/\varepsilon \) as far as we know. We also consider the standard way, Trotterization, which covers generic time-dependent, time-periodic, and generic time-dependent Hamiltonians.

Let us first compare our algorithm with Trotterization. In the simplest cases \( H(t) = H_A(t) + H_B(t) \), where every term in \( H_A(t) \) [and \( H_B(t) \)] commutes with one another at every time, the first-order Trotterization approximates the time-evolution \( U(t) \) by
\[ \prod_{l=0}^{M-1} \left[ Te^{-i t_{\text{amp}}^{l+1} dt} H_A(t') \right] \cdot \left[ Te^{-i t_{\text{amp}}^l dt} H_B(t') \right], \quad (128) \]

with \( M \) partitions of the time \( t \) as \( t_l = lt/M \) [23]. The Trotterization error including higher order cases polynomially decays in the partition number \( M \), and thereby generic \( p \)-th order Trotterization roughly yields \( \mathcal{O}(at \cdot (at/\varepsilon)^{1/p}) \) elementary gates. While the coefficient \( \alpha \), giving the whole energy scale, can be improved with multi-commutators [12], our algorithm has better scaling of elementary gates than Trotterization in that the resource increases logarithmically in \( 1/\varepsilon \).

The comparison with the qubitization and the truncated-Dyson-series algorithm is instructive for evaluating the efficiency of our algorithm due to the inclusion relation,
\[ [\text{Set of time-independent } H] \subset [\text{Set of time-periodic } H(t) = H(t + T)] \subset [\text{Set of generic time-dependent } H(t)]. \quad (129) \]

The qubitization technique achieves the least number of ancilla qubits in a sense that it is independent of \( t \) and \( 1/\varepsilon \). It also has the best query complexity, in which the optimal scaling both in \( t \) and \( 1/\varepsilon \) appears in an additive way as \( \mathcal{O}(at + \log(1/\varepsilon) / \log(\log(1/\varepsilon))) \) [18]. The number of ancilla qubits and the query complexity for the qubitization provide the best bound for time-periodic Hamiltonians, while it is nontrivial whether or not it is actually achievable. Comparison with the truncated-Dyson-series algorithm tells us how efficiently we deal with the time-dependent Schrödinger equation. The cost reduction compared to them can be interpreted as improvement of efficiency due to the time-periodicity.

Before going to the comparison respectively for the adiabatic-like regime and the generic long-time regime, we remark several points in common. First, we replace some parameters by those which have similar scales to them. Second, we replace some parameters by those which have similar scales to \( \sup_t \parallel H(t) \parallel \). We substitute \( \alpha \) and \( \omega \gamma \) respectively for \( \sup_t \parallel H(t) \parallel \) and \( \sup_t \parallel H(t) \parallel \), which are characteristic values in the truncated-Dyson-series algorithm.

The second point is the complexity of the oracles themselves. In the qubitization technique, the query complexity is measured by the oracles to a static Hamiltonian \( H \) as Eq. (5). In contrast, the truncated-Dyson-series algorithm for generic cases employs the oracle
\[ \text{Ham}_O = \sum_{l=1}^{M} |l\rangle \langle l| \otimes O(t_l), \quad (130) \]
where \( l \) labels the discretized time \( t_l \) with the partition number \( M \in \mathcal{O}(1/\varepsilon) \) \cite{21}. The oracle \( \hat{H} = O \) includes multiple implementation of \( O(t_l) \), which is an oracle for a static instantaneous Hamiltonian \( H(t_l) \) (some specific cases such as LCU and sparse-access matrices with integrable time-dependency can be simplified \cite{21}). Our algorithm for time-periodic Hamiltonians uses the oracles \( \{O_{m_l}, |G_{m_l}\rangle\} \), which gives block-encoding of each Fourier component \( H_m \) as Eq. \( \ref{eq:94} \), and the oracle for the coefficients, \( G_{\text{coeff}} \) given by Eq. \( \ref{eq:96} \). The query complexity in our algorithm is roughly measured by the oracle for a static operator \( H_m \) since the latter one \( G_{\text{coeff}} \), a quantum gate on at-most \( \mathcal{O}(1) \) qubits, is usually negligible. Therefore, note that our algorithm adopts essentially the same measure for the query complexity as the query complexity, while the truncated-Dyson-series algorithm counts it by rather complicated oracles involving discretized time.

1. Adiabatic-like cases

We assess the cost for simulating time-periodic Hamiltonians in the adiabatic-like cases based on Table \ref{tab:1}. First, we compare the number of ancilla qubits with those for other algorithms. What should be noted in our algorithm is its scaling in the inverse error \( 1/\varepsilon \). Our algorithm requires \( \mathcal{O}(\log(1/\varepsilon)) \) additional qubits, whose number lies just in the middle of that for the qubitization (\( \varepsilon \)-independent) and that for the truncated-Dyson-series algorithm \( \mathcal{O}(1/\varepsilon) \)). Importantly, the reduction compared to the latter one can be attributed to the faster convergence of our formalism based on the Fourier indices. The error arising from time discretization polynomially decays in the partition number \( M \); as a result, the truncated-Dyson-series algorithm requires at least \( \mathcal{O}(\log(1/\varepsilon)) \) ancilla qubits via the oracle Eq. \( \ref{eq:130} \). In contrast, we introduce the truncation order \( l_{\text{max}} \) for the Floquet-Hilbert space. The error by this cutoff scales as \( \mathcal{O}((\gamma t/l_{\text{max}})^{m_{\text{max}}}) \), whose decay is faster than the exponential function \( e^{-\mathcal{O}(l_{\text{max}})} \). This leads to the reduction of ancilla qubits from \( \mathcal{O}(1/\varepsilon) \) to \( \mathcal{O}(\log(1/\varepsilon)) \).

Next, we compare the query complexity. The most significant point in our algorithm is that it achieves the optimal scaling both in \( 1/\varepsilon \) and the best form in which the \( \mathcal{O}(t) \)-term and the \( \mathcal{O}(\log(1/\varepsilon)) \)-term appear in an additive way as

\[
(\alpha + \gamma)t + \frac{\log(1/\varepsilon)}{\log \log(1/\varepsilon)}.
\]

It is almost the same as that of the qubitization technique, \( \mathcal{O}(at + \log(1/\varepsilon)/\log \log(1/\varepsilon)) \). The difference is solely increase in the coefficient of \( t \) by \( \gamma \), which arises from the time-dependent terms. We also succeed in drastically saving the cost compared to the truncated-Dyson-series algorithm, whose query complexity is proportional to the product of the whole energy-time scale \( at \) and the \( \mathcal{O}(\log(1/\varepsilon)/\log \log(1/\varepsilon)) \) term. Its scaling in \( t \) is not optimal due to the logarithmic correction \( at \log(at)/\log \log(at) \), which is larger than that of our algorithm. Combining with the simplicity of the oracles, our algorithms enables us to simulate time-periodic Hamiltonians only with essentially the same resource as the qubitization, despite the existence of time-dependency.

2. Generic long-time cases

Here, we evaluate the computational resources for generic long-time regimes. We suppose that the frequency \( \omega \) is constant and hence small compared to the whole-system energy scales \( \alpha \) and \( \gamma \), which usually increase polynomially in the system size \( N \). This assumption comes from the fact that \( \omega \) typically represents the frequency of light, which is size-independent. More importantly, when the frequency \( \omega \) is poly \( (N) \) so that it becomes comparable to \( \alpha \) and \( \gamma \), the dynamics can be efficiently simulated by the methods for time-independent Hamiltonians with the help of high-frequency expansions \cite{62,63}, as we prove in Appendix D. Therefore, it is reasonable to consider the computational resources under \( \omega \in \mathcal{O}(N^0) \) and \( \alpha, \gamma \in \text{poly} (N) \).

The number of ancilla qubits is similar to that for the adiabatic cases. We note that its scaling in the time \( \mathcal{O}(\log(\log t)) \) overweights that of the truncated-Dyson-series algorithm \( \mathcal{O}(\log t) \) in addition to the inverse error \( 1/\varepsilon \). The number of ancilla qubits for time-periodic Hamiltonians lie between those for time-independent and time-dependent Hamiltonians in terms of both \( t \) and \( 1/\varepsilon \).

We discuss the query complexity given by Theorem 2. By some elementary calculation, it can be further bounded from above by

\[
(\alpha + \gamma)t + \omega t \log(\omega t) + \omega t \frac{\log(1/\varepsilon)}{\log \log(1/\varepsilon)}.
\]

In terms of \( 1/\varepsilon \), the scaling itself is optimal corresponding to the qubitization technique. Although we do not observe separation of \( t \) and \( 1/\varepsilon \) like the adiabatic-like cases, we emphasize that our algorithm drastically saves the cost compared to the truncated-Dyson-series algorithm. The point is that the \( \mathcal{O}(\log(1/\varepsilon)) \) term in Eq. \( \ref{eq:132} \) is isolated from linear increase in the whole energy scales \( \alpha \) and \( \gamma \), which are poly \( (N) \)-times larger than the frequency \( \omega \) as stated in the beginning. Thus, the increase of the query complexity in \( 1/\varepsilon \) is much more suppressed than that of the truncated-Dyson-series algorithm, \( \mathcal{O}(at \log(1/\varepsilon)/\log \log(1/\varepsilon)) \). Similar discussion holds also for the scaling in \( t \). The query complexity designated by Eq. \( \ref{eq:132} \) is formally nearly-optimal in a sense that it includes a logarithmic correction \( \omega t \log(\omega t) \). However, this term can be non-negligible compared to \( (\alpha + \gamma)t \) only when the time \( t \) is exponentially large in the system size \( N \) as \( t \sim e^{(\alpha + \gamma)/\omega}/\omega \in \mathcal{O}(e^{\text{poly}(N)T}) \). Since we are interested in at-most poly \( (N) \)-time dynamics which can be efficiently accessible by quantum computers, the query
complexity of our algorithm is actually linear in the time $t$ and hence has optimal scaling for practical problems.

VIII. ILLUSTRATIVE EXAMPLES

In this section, we briefly discuss some potential applications of the algorithm. We expect that it can be applied to nonequilibrium quantum many-body phenomena, which are often of interest in condensed matter physics and quantum chemistry. In terms of quantum computation, it will offer an efficient protocol for adiabatic state preparation, which can be applied to quantum phase estimation for instance. We suggest the simplest examples for both applications below.

A. Nonequilibrium quantum many-body phenomena

The first application is to simulate nonequilibrium dynamics of periodically-driven quantum materials. Optical responses are typical but of great interest both in condensed matter physics and quantum chemistry. We pick up an $N$-site Fermi-Hubbard model under laser light as the simplest case;

$$H(t) = H_{\text{Hub}} + H_{\text{ext}}(t),$$

$$H_{\text{Hub}} = \sum_{k} \sum_{\sigma = \uparrow, \downarrow} \epsilon_{k} \hat{n}_{k\sigma} + U \sum_{x} \hat{n}_{x\uparrow} \hat{n}_{x\downarrow}. \quad (133)$$

Here, $\hat{n}_{k\sigma} = \hat{c}_{k\sigma}^{\dagger} \hat{c}_{k\sigma}$ and $\hat{n}_{x\sigma} = \hat{c}_{x\sigma}^{\dagger} \hat{c}_{x\sigma}$ are number operators of electrons in the momentum and real spaces respectively, generated by fermionic annihilation operators $\hat{c}_{k\sigma}$ and $\hat{c}_{x\sigma}$.

The time-periodic term $H_{\text{ext}}(t)$ represents the coupling with light. When we shine linearly-polarized light with the frequency $\Omega$, it is given by

$$H_{\text{ext}}(t) = \sin(\Omega t) \sum_{x,\sigma} V_{x} \hat{n}_{x\sigma}, \quad V_{x} = \vec{E} \cdot \vec{x}, \quad (135)$$

which results in $T = 2\pi/\Omega$ and $m_{\text{max}} = 1$. The Fourier components $H_{m}$ are given by $H_{0} = H_{\text{Hub}}$ and $H_{\pm 1} = (\pm i) \sum_{x\sigma} (V_{x}/2) \hat{n}_{x\sigma}$. To evaluate the cost of simulating $|\psi(t)\rangle$, we compose the oracles for them. We employ a unitary operation, called fermionic fast Fourier transform (FFFT) \cite{67, 69}, which transforms the basis in the momentum space $k_{x}$ to that in the real space $x_{j}$ as $\hat{n}_{k\sigma} = \text{FFFT}^{\dagger} \hat{n}_{x\sigma} \text{FFFT}$. We map the fermionic system to a spin system by Jordan-Wigner transformation as $\hat{n}_{x\sigma} = (1 + \hat{z}_{x\sigma})/2$ ("$x\sigma$" denotes an index for qubits). By neglecting constant terms and a conserved particle number $\sum_{x\sigma} \hat{n}_{x\sigma}$, the block-encoding for ($H_{m}$) can be constructed by the technique for LCUs, as Eq. (10). Assuming $\epsilon_{k}, U, V_{x} \geq 0$ without loss of generality, this leads to

$$O_{0} = |0\rangle \langle 0| \otimes \sum_{x,\sigma} |x,\sigma\rangle \langle x,\sigma| \otimes \text{FFFT}^{\dagger} \text{FFFT}$$

$$+ |1\rangle \langle 1| \otimes \sum_{x,\sigma} |x,\sigma\rangle \langle x,\sigma| \otimes \hat{Z}_{x\uparrow} \hat{Z}_{x\downarrow}, \quad (136)$$

$$|G_{0}\rangle = \sum_{x,\sigma} \left( \sqrt{\frac{\epsilon_{x}}{\alpha_{0}}} |0\rangle + \sqrt{\frac{U}{2\alpha_{0}}} |1\rangle \right) |x,\sigma\rangle, \quad (137)$$

$$O_{\pm 1} = I \otimes \sum_{x,\sigma} |x,\sigma\rangle \langle x,\sigma| \otimes (\pm i \hat{Z}_{x\sigma}), \quad (138)$$

$$|G_{\pm 1}\rangle = |0\rangle \sum_{x,\sigma} \sqrt{\frac{V_{x}}{2\alpha_{\pm 1}}} |x,\sigma\rangle, \quad (139)$$

with $\alpha_{0} = 2 \sum_{x} \epsilon_{x} + U N$ and $\alpha_{\pm 1} = \sum_{x} V_{x}$. In the above oracles, the unitary gate $O_{0}$ requires much cost due to $O(1)$-times usage of FFFT, which can be implemented with at-most $O(N \log N)$-depth quantum circuits composed of adjacent two-qubit gates \cite{69}. The depth needed for each oracle is at-most $O(N \log N)$. The number of ancilla qubits for them to express $\{|0\rangle \langle x,\sigma|, |1\rangle \langle x,\sigma|\}$ amounts to $n_{a} \in O(\log N)$. The energy scales of the whole Hamiltonian, $\alpha$, and that of time-dependent terms, $\gamma$, are respectively described as follows [See Eqs. (24) and (22)];

$$\alpha = 2 \sum_{x} \epsilon_{x} + U N + 2 \sum_{x} V_{x}, \quad \gamma \leq \sum_{x} V_{x}. \quad (140)$$

When we define the characteristic local energy scale by $\alpha_{\text{loc}} = \max(\epsilon_{k}, U, V_{x})$, they are approximately described by $\alpha, \gamma \in O(\alpha_{\text{loc}} N)$. Finally, if we are interested in the time-evolved state $|\psi(t)\rangle$ over multiple periods, the following resources are required to achieve the precision $1 - O(\varepsilon)$;

- Number of ancilla qubits;
  $$O(\log(\alpha_{\text{loc}} N/\Omega) + \log(\log(\Omega t/\varepsilon))). \quad (141)$$

- Overall gate complexity;
  $$O(\left\{ \alpha_{\text{loc}} N t + \Omega t \frac{\log(O t/\varepsilon)}{\log(\log(O t/\varepsilon))} \right\} \log N). \quad (142)$$

The above results are based on Eqs. (126) and (127), where we neglect $O(\log(\gamma/\omega) + \log(\log(\omega t/\varepsilon)))$ additional gates for each query to the oracles. According to classical numerical calculations \cite{20, 22}, the above model is expected to host high-harmonic generation, where intense oscillation with the frequency $n \Omega$ ($n = 2, 3, \ldots$) arises in response to laser light with the frequency $\Omega$. Our algorithm allows to efficiently identify such a nontrivial nonequilibrium phenomenon with guaranteed accuracy by simulating the Fourier spectrum of some observables $\langle \psi(t) | O | \psi(t) \rangle$ (e.g. electric current) \cite{73}.

Another interesting example is a discrete time crystal as a phase of matter inherent in nonequilibrium, where
time-translation symmetry is spontaneously broken \[29-31\]. When we choose $H_{\text{ext}}(t)$ by uniform circularly-polarized ac field represented by

$$H_{\text{ext}}(t) = V \sum_x \left( e^{i\Omega t} \hat{c}^\dagger_x \hat{c}_x + e^{-i\Omega t} \hat{c}_x \hat{c}^\dagger_x \right),$$ \hfill (143)

the Fermi-Hubbard model becomes a potential platform for a time-crystalline phenomenon protected by Floquet dynamical symmetry \[7\]. Its signature can be detected by subharmonic oscillations of spatio-temporal correlation functions and local observables. These values are both efficiently computed via the time-evolution operator $U(t)$ by our algorithm, requiring the computational resources similar to Eqs. \([141]\) and \([142]\).

As nonequilibrium systems dominated by time-periodic Hamiltonians have been vigorously explored as Floquet systems, our algorithm will cover various phenomena. We also note that our algorithm is extended to time-periodic Hamiltonians with exponentially-decaying Fourier components $\|H_m\| \lesssim e^{-O(|m|)}$. Since we often face at situations where high-frequency components of $H(t)$ rapidly diminish, our result will be useful for a variety of nonequilibrium phenomena in condensed matter physics and quantum chemistry, other than the above examples (See also Appendix \(\text{C.5}\)).

### B. Adiabatic state preparation

Adiabatic state preparation is a protocol to obtain a preferable quantum state by adiabatically evolving quantum systems with time-dependent Hamiltonians $H(t)$. While it has been originally developed in the context of adiabatic quantum computation relying on the adiabatic theorems of quantum dynamics \[43\], it can be exploited also on circuit-based quantum computers by mimicking the adiabatic dynamics under $H(t)$. One of typical aims is to prepare initial states required for quantum simulation \[6\] \[12\] \[60\]. Here, we provide the simplest application of our algorithm in this field.

We prepare a time-periodic Hamiltonian $H(t)$, which continuously connects two different time-independent Hamiltonians $\hat{H}_0$ and $\hat{H}_1$. A certain eigenstate of $\hat{H}_0$, denoted by $|\tilde{\psi}_0\rangle$, is supposed to be easily prepared on quantum circuits, and we assume that the eigenstate of $\hat{H}_1$, which is continuously connected to $|\tilde{\psi}_0\rangle$, corresponds to the target state $|\psi_1\rangle$. As the simplest case, we organize such a time-periodic Hamiltonian by

$$H(t) = \hat{H}_0(1 - \sin \omega t) + \hat{H}_1 \sin \omega t,$$ \hfill (144)

with satisfies $H(0) = \hat{H}_0$ and $H(T/4) = \hat{H}_1$. Our algorithm for the adiabatic-like cases in Section \(\text{VII.A}\) enables us to efficiently execute the adiabatic state preparation. Since the Fourier components of $H(t)$ is simply given by $H_0 = \hat{H}_0$ and $H_{\pm 1} = \pm (\hat{H}_0 - \hat{H}_1)/2i$ with $n_{\text{max}} = 1$, the following oracles are necessary:

$$O_0 = I \otimes \hat{O}_0, \quad |G_0\rangle = |0\rangle |\bar{G}_0\rangle,$$ \hfill (145)

$$O_{\pm 1} = |0\rangle (0 \otimes (\mp i \hat{O}_1) + |1\rangle |\mp i \hat{O}_1\rangle, \quad |G_{\pm 1}\rangle = \sqrt{\alpha_0} |0\rangle |\bar{G}_0\rangle + \sqrt{\alpha_1} |1\rangle |\bar{G}_1\rangle.$$ \hfill (146)

$$|G_{\pm 1}\rangle = \frac{\sqrt{\alpha_0}}{\alpha_0 + \alpha_1} |0\rangle |\bar{G}_0\rangle + \frac{\sqrt{\alpha_1}}{\alpha_0 + \alpha_1} |1\rangle |\bar{G}_1\rangle.$$ \hfill (147)

Here, $(\hat{O}_0, |\bar{G}_0\rangle, \alpha_0)$ and $(\hat{O}_1, |\bar{G}_1\rangle, \alpha_1)$ respectively provides the block-encoding of the time-independent Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ as $\langle G_{\pm 1}|O_{\pm 1}|G_{\pm 1}\rangle = 2\hat{H}_{\pm 1}/(\alpha_0 + \alpha_1)$. According to the standard adiabatic theorem \[43\], the duration required for approximating the target state $|\psi_1\rangle$ by the adiabatically-evolved state $U(T/4)|\tilde{\psi}_0\rangle$ with accuracy $1 - O(\varepsilon)$ roughly amounts to $T \gtrsim \|\hat{H}_0 - \hat{H}_1\|/\varepsilon \Delta^2$, where $\Delta$ denotes the minimal gap upon the instantaneous eigenstate of $H(t)$ continuously connecting $|\tilde{\psi}_0\rangle$ and $|\psi_1\rangle$. Therefore, in the adiabatic state preparation based on our algorithm, the query complexity counted by the oracles of the two static Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ amounts to

$$O\left(\frac{\alpha_0 + \alpha_1}{\varepsilon \Delta^2} \|\hat{H}_0 - \hat{H}_1\| + \frac{\log(1/\varepsilon)}{\log \log(1/\varepsilon)}\right).$$ \hfill (148)

Although the second term is buried by the first one polynomially-increasing in $1/\varepsilon$, the cost has better scaling in $1/\varepsilon$ compared to the cases where similar schedules for $H(t)$ are tackled with the Trotterization ($\sim (1/\varepsilon)^{1+2/p}$) or the truncated-Dyson-series algorithm ($\sim (1/\varepsilon) \log(1/\varepsilon)$).

In adiabatic quantum computation, the required time in $1/\varepsilon$ can be improved via more sophisticated scheduling, such as local adiabatic interpolation \[75\], boundary cancellation \[70\] \[77\] and quasi-adiabatic processes \[78\]. In particular, the last one has achieved the query complexity poly-logarithmic in $1/\varepsilon$ with the help of the truncated-Dyson-series algorithm. If we can find time-periodic Hamiltonians which are consistent with such sophisticated schedules, it will offer a more efficient protocol for the adiabatic state preparation.

### IX. DISCUSSION AND CONCLUSION

We conclude our paper with summarizing the results. In this paper, we focus on time-dependent systems with time-periodicity, and organize an efficient implementation of their time-evolution operators. Once we prepare the oracles which embeds each Fourier component and each coefficient of the Hamiltonian, a series of unitary operations on the truncated Floquet-Hilbert space extract the time-evolved state $|\psi(t)\rangle$ with an allowable error $1/\varepsilon$, whose number of queries to the oracles are both optimal or nearly-optimal in $t$ and $1/\varepsilon$. In adiabatic-like regimes for $O(1)$-period dynamics, we achieve the optimal query complexity for time-independent Hamiltonians as $O(t + \log(1/\varepsilon)/\log \log(1/\varepsilon))$ despite the existence of time-dependency. Including the other regimes,
our algorithm efficiently deals with Hamiltonian simulation of generic time-periodic system, whose computational resource is close to that of the optimal algorithm for time-independent systems [18] and much smaller than the one for time-dependent systems [20, 21]. As exemplified by nonequilibrium quantum many-body phenomena and adiabatic state preparation, our algorithm will contribute to pioneering applications of quantum computers for various aims, in condensed matter physics, quantum chemistry, and quantum computation.

We finally discuss some potential future directions of our results. The first one is to explore efficient implementation of meaningful functions in time-periodic systems, other than the time-evolution operator $U(t) = T \exp \left( -i \int_0^t H(t') dt' \right)$. In our algorithm, we exploit the qubitization technique to implement the exponential function, $e^{-i \mathcal{H}_{\text{eff}}^\text{max} t}$, in the effective Hamiltonian $\mathcal{H}_{\text{eff}}^\text{max}$. As the qubitization technique allows to efficiently apply various polynomial functions, we expect that our algorithm can be extended for various aims other than the unitary time evolution discussed here. For instance, our discussion on the Lieb-Robinson bound in Section IV is valid also when a time-periodic Hamiltonian $H(t)$ is non-hermitian. This suggests that our algorithm may be available also for solving time-periodic linear differential equations [79], exemplified by dissipative quantum many-body systems [80, 81]. While we do not expect exponential speedup due to the non-unitarity in general, it will offer an efficient way both in time and desirable accuracy. It would be important to clarify what kind of function is useful in time-periodic systems, how the Lieb-Robinson bound and the amplitude amplification should be modified, and then how the computational cost is affected.

Another significant direction is to seek for useful tasks that can be efficiently tackled with time-periodic Hamiltonians. As we have shown throughout the paper, time-periodic Hamiltonians can be simulated more efficiently generic time-dependent Hamiltonians with computational resources close to those for time-independent Hamiltonians. This means that quantum tasks which inevitably requires time-dependent operations, such as adiabatic state preparation, can be optimized by tuning their schedules in a time-periodic way. It will be important to clarify what kind of tasks can be addressed by time-periodic Hamiltonians and how our algorithm provides better scaling in $1/\varepsilon$ for their costs, which we leave for future work.

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$$q \in \Theta \left( \frac{\log(1/\varepsilon)}{\log(\varepsilon + (\alpha t)^{-1}\log(1/\varepsilon))} \right).$$

(149)

This is valid for any $t > 0$ and $\varepsilon \in (0, 1]$.

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$$I_{\max} \in \Theta \left( \gamma t + \frac{\log(1/\varepsilon)}{\log(e + (\gamma t)^{-1}\log(1/\varepsilon))} \right).$$

(150)

While we employ the scaling Eq. (55) in the main text to simply compare our algorithm with other previous results, we can easily reproduce the accurate scaling by the above relation.
Our algorithm is mainly based on the fact that the dynamics in the Floquet-Hilbert space,

\[ |\psi(t)\rangle = \lim_{t_{\max} \to \infty} \sum_{l \in D_{\max}} e^{-i\omega t l} |l\rangle e^{-i\epsilon l t_{\max}} |0\rangle |\psi(0)\rangle , \]

exactly corresponds to the solution \(|\psi(t)\rangle\) [46]. Here, we roughly explain this relation for this paper to be self-
We differentiate the above formula in \( t \):
\[
\frac{d}{dt} \langle \psi^\infty(t) | \psi^\infty(t) \rangle = \sum_{l \in \mathbb{Z}} e^{-i\omega t} \langle l | (H_{\text{eff}} + l\omega) e^{-iH_{\text{eff}} t} | 0 \rangle | \psi(0) \rangle
\]
\[
= \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{-i\omega t} H_{-m} \langle l + m | e^{-iH_{\text{eff}} t} | 0 \rangle | \psi(0) \rangle
\]
\[
= \left( \sum_{m \in \mathbb{Z}} H_{-m} e^{im\omega t} \right) \left( \sum_{l \in \mathbb{Z}} e^{-i\omega t} \langle l | e^{-iH_{\text{eff}} t} | 0 \rangle | \psi(0) \rangle \right)
\]
\[
= H(t) | \psi^\infty(t) \rangle.
\]
Noting that \( | \psi^\infty(0) \rangle = | \psi(0) \rangle \), the uniqueness of the solution of the Schrödinger equation suggests the relation \( | \psi^\infty(t) \rangle = | \psi(t) \rangle \).

Appendix B: Proof of the theorems for formulation

1. Amplitude amplification by symmetry

In Section \[ \text{V A} \] we discuss the amplification of the time-evolved state \( | \psi(t) \rangle \) with exploiting the translation symmetry of generic time-periodic Hamiltonians. With the usage of the protocol,
\[
| \psi_{\text{max}}^{\text{amp}}(t) \rangle = (| \psi_{\text{ini}}^{\text{amp}} \rangle)^{l} \ e^{-i \sum_{l} l\omega |l\rangle \langle l|} e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} \mathcal{F}_{\text{ini}}^{\text{max}}, \tag{A1}
\]
we see that it generates the time-evolved state \( | \psi(t) \rangle \) with \( O(1) \) amplitude as
\[
\langle 0 | \psi_{\text{max}}^{\text{amp}}(t) | 0 \rangle | \psi(0) \rangle \simeq \frac{1}{2} (| \psi(t) \rangle + O(\varepsilon)). \tag{B2}
\]
We rigorously prove this statement, which is summarized as Theorem 6.

In order to show Theorem 6, we begin with discussing the approximate translation symmetry. To be precise, we evaluate how much error appears in the approximation,
\[
\langle l | e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} | l' \rangle \simeq e^{i(l-l')\omega t} e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} | 0 \rangle, \tag{B3}
\]
which we refer to as Eq. 66 in the main text. The exact upper bound on this error is given by the following theorem.

Lemma 8. (Approximate translation symmetry)

We choose the truncation order \( l_{\text{max}} \in \Theta(\gamma t + \log(1/\varepsilon)/\log\log(1/\varepsilon)) \) by Eq. (60). For indices \( l' \in D_{\text{max}}^{\text{eff}} \) and \( l \in D_{\text{max}}^{\text{pert}} \), the inequality
\[
\left| \langle l | e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} | l' \rangle - e^{i(l-l')\omega t} e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} | 0 \rangle \right| \leq 8 \frac{(3\gamma t)^{n(l', l)}}{n(l', l)!} \tag{B4}
\]
is satisfied with
\[
n(l', l) = \left[ \frac{(8m_{\text{max}} - 2m_{\text{max}} - |l| - |l'|)}{m_{\text{max}}} \right]. \tag{B5}
\]
Here, \( l \cap l' \in D_{\text{max}}^{\text{eff}} \) denotes the difference \( l - l' \) defined modulo \( 8m_{\text{max}} \).

Proof.— We consider dynamics under a perturbed Hamiltonian \( \mathcal{H}_{\text{pert}}(t) \), defined by
\[
\mathcal{H}_{\text{pert}}(t) = \mathcal{H}_{\text{eff}}^{\text{max}} + \mathcal{H}_{\text{b}}(t), \tag{B6}
\]
\[
\mathcal{H}_{\text{b}}(t) = \sum_{(l, m) \in \partial D_{\text{max}}^{\text{pert}}} (|l| \langle l \oplus m | \otimes e^{8\omega t} | H_{-m} + \text{h.c.} \rangle , \tag{B7}
\]
with \( \partial D_{\text{max}}^{\text{pert}} = \{(l, m) | 4m_{\text{max}} - m_{\text{max}} + 1 \leq l \leq 4m_{\text{max}}, 4l_{\text{max}} - l + 1 \leq m \leq m_{\text{max}} \} \). In the interaction picture based on \( \mathcal{H}_{0} = \sum_{l \in D_{\text{max}}^{\text{eff}}} |l\rangle \langle l| \otimes (H_{0} - l\omega) \), the perturbed Hamiltonian \( \mathcal{H}_{\text{pert}}(t) \) is exactly translation symmetric under arbitrary shift \( |l\rangle \rightarrow |l \oplus m\rangle \). In the original frame, this exact symmetry implies the satisfaction of
\[
\langle l | \mathcal{H}_{\text{pert}}(t) | l' \rangle = e^{i(l' - l)\omega t} \langle l \oplus l' | \mathcal{H}_{\text{pert}}(t) | 0 \rangle , \tag{B8}
\]
where \( \mathcal{H}_{\text{pert}}(t) \) indicates the time evolution operator under the perturbed Hamiltonian \( \mathcal{H}_{\text{pert}}(t) \). Upon this relation, we evaluate the upper bound on the error by a triangle inequality,
\[
| \text{l.h.s. of Eq. (B4)} | \leq \left| \langle l | e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} - \mathcal{H}_{\text{pert}}(t) | l' \rangle \right|
\]
\[
+ \left| \langle l \oplus l' | e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} - \mathcal{H}_{\text{pert}}(t) | 0 \rangle \right| . \tag{B9}
\]
We begin with computing the bound of the first term. Using the Dyson series expansion in the interaction picture, similarly to Eqs. (34-39), it can be bounded by
\[
\left| \langle l | e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} - \mathcal{H}_{\text{pert}}(t) | l' \rangle \right|
\]
\[
\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{t} dt_{n} \cdots \int_{0}^{t_{2}} dt_{1}
\]
\[
\left| \langle l | \prod_{i=1}^{n} \mathcal{H}_{\text{pert}}(t_{i}) - \prod_{i=1}^{n} \mathcal{H}_{\text{pert}}^{\text{max}}(t_{i}) | l' \rangle \right| . \tag{B10}
\]
As discussed in the proof of Theorem 3, each integrand is decomposed into contributions from the transition amplitudes via paths \( |l'\rangle \rightarrow |l_{1}\rangle \rightarrow \ldots \rightarrow |l| \). The difference between the Hamiltonians \( \mathcal{H}_{\text{pert}}(t) \) and \( \mathcal{H}_{\text{eff}}^{\text{max}} \) arises only when the path goes across \( D_{\text{max}}^{\text{pert}} \setminus D_{\text{max}}^{\text{eff}} \), which is the support of \( \mathcal{H}_{b}(t) \). For low order terms with \( n < n(l', l) \), defined by Eq. (65), such a nontrivial path is absent, and we obtain
\[
\left| \langle l | e^{-i \mathcal{H}_{\text{eff}}^{\text{max}} t} - \mathcal{H}_{\text{pert}}(t) | l' \rangle \right|
\]
\[
\leq \sum_{n=n(l', l)}^{\infty} \frac{1}{n!} \left( \sup_{t} \left| \mathcal{H}_{\text{pert}}(t) \right| \right)^{n} \gamma^{n} \tag{B11}
\]
\[
\leq \sum_{n=n(l', l)}^{\infty} \frac{1}{n!} \left( \sup_{t} \left| \mathcal{H}_{b}(t) \right| + \gamma \right)^{n} \gamma^{n} . \tag{B11}
\]
Next, we evaluate the bound on $\sup_t(||\mathcal{H}_b(t)||)$. Reflecting that the perturbation $\mathcal{H}_b(t)$ nontrivially acts on $D^{4}_{\text{max}} \backslash D^{4}_{\text{max} - m_{\text{max}}}$ in the Fourier indices, it can be written as a $(2m_{\text{max}} + 2)$-dimensional block matrix as

$$\mathcal{H}_b(t) = \begin{pmatrix} 0 & e^{i\delta_{\text{max}}t} K_1 \\ e^{-i\delta_{\text{max}}t} K_1^\dagger & 0 \end{pmatrix}, \quad (B12)$$

where we use the basis $\{4l_{\text{max}} - m_{\text{max}}\}, \ldots, \{4l_{\text{max}}\}$ and $\{-4l_{\text{max}} + 1\}, \ldots, \{-4l_{\text{max}} + m_{\text{max}} + 1\}$. The dummy states $4l_{\text{max}} - m_{\text{max}}$ and $-4l_{\text{max}} + m_{\text{max}} + 1$, which are not included in the support of $\mathcal{H}_b(t)$, are introduced for later calculation. Since the determinant $\det(\lambda - \mathcal{H}_b(t))$ is equal to $\det(\lambda^2 - K_1 K_1^\dagger)$, the phase $\exp(\pm 8\delta_{\text{max}} t)$ does not affect the spectrum of $\mathcal{H}_b(t)$. We introduce a $(m_{\text{max}} + 1)$-dimensional Hermitian matrix $K_2$ by

$$K_2 = \begin{pmatrix} 0 & H_{-1} & \cdots & H_{-m_{\text{max}}} \\ H_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & H_{-1} \\ H_{m_{\text{max}}} & \cdots & H_1 & 0 \end{pmatrix}, \quad (B13)$$

and then, the Hamiltonian $\mathcal{H}_b(t)$ shares the spectrum with the matrix

$$\begin{pmatrix} K_2 & K_1 \\ K_1^\dagger & K_2 \end{pmatrix} = \begin{pmatrix} K_2 & 0 \\ 0 & K_2 \end{pmatrix}. \quad (B15)$$

The first term in the above formula and $K_2$ are both Toeplitz matrices, whose matrix element for each row $|l\rangle$ and column $|l'\rangle$ is designated by $H_{l-l'} - H_{0} \delta_{l-l'}$. Therefore, in a similar way to Eq. (41), the norm of these two matrices is bounded by $\gamma$ [See Eq. (22) for its definition]. Using the triangle inequality, we obtain the bound of the Hamiltonian $\mathcal{H}_b(t)$ as

$$\sup_t(||\mathcal{H}_b(t)||) \leq 2\gamma. \quad (B16)$$

As a result, Eq. (B11) is further bounded by

$$\langle |l\rangle e^{-i\mathcal{H}_{\text{eff}}^\dagger t} - \mathcal{W}_{\text{pert}}(t) |l'\rangle \rangle \leq 2 \sum_{n=0}^{\infty} \frac{(3\gamma t)^n}{n!}. \quad (B17)$$

In a similar way, we can obtain the bound on the second term of the right hand side of Eq. (B9), which results in the above formula under the replacement of $n(l',l)$ by $n(0,l \oplus l')$. Since $n(0,l \oplus l')$ is always larger than $n(l',l)$ from their definitions, we obtain the accuracy of the approximate translation symmetry as

$$\text{l.h.s of Eq. (B4)} \leq 4 \sum_{n=n(l',l)}^{\infty} \frac{(3\gamma t)^n}{n!} \leq 8 \frac{(3\gamma t)^n(l',l)}{n(l',l)!}. \quad (B18)$$

When we choose the truncation order $l_{\text{max}}$ by Eq. (56), the integer $n(l',l)$ is always larger than $6\gamma t$ for indices $l \in D^{4}_{\text{max}}$ and $l' \in D^{4}_{\text{max}}$. We use Eq. (42) for the last inequality.

This lemma ensures that the approximate translation symmetry in the truncated Floquet-Hilbert space is extremely accurate; the right hand side of Eq. (B4) is approximately $O(\epsilon^3)$ under the choice of $l_{\text{max}}$ by Eq. (56). As we intuitively discuss in Section V A, the approximate translation symmetry ensures the amplification by symmetry as Eq. (70). We next prove the consequence of Theorem 6 which gives the exact description of Eq. (70):

$$\langle 0 | \mathcal{W}_{\text{amp}}^{l_{\text{max}}+1}(t) | 0 \rangle | \psi(0) \rangle \leq \frac{1}{2} | 0 \rangle | \psi(t) \rangle \leq \epsilon \frac{3}{4}. \quad (B19)$$

**Proof of Theorem 6** — We track Eq. (67) in a rigorous way. We begin with the definition,

$$\langle 0 | \mathcal{W}_{\text{amp}}^{l_{\text{max}}+1}(t) | 0 \rangle | \psi(0) \rangle =$$

$$\frac{1}{4l_{\text{max}}} \sum_{l \in D^{4}_{\text{max}}} \sum_{l' \in D^{4}_{\text{max}}} e^{-il_3t} \langle |l\rangle e^{-i\mathcal{H}_{\text{eff}}^\dagger t} |l'\rangle | \psi(0) \rangle \rangle. \quad (B20)$$

We separate the summation over $l \in D^{4}_{\text{max}}$ in the above formula by

$$\sum_{l \in D^{4}_{\text{max}}} = \sum_{l:l-l' \in D^{3}_{\text{max}}} + \sum_{l \in D^{4}_{\text{max}}:l-l' \notin D^{3}_{\text{max}}} \sum_{l \in D^{4}_{\text{max}}:l-l' \notin D^{3}_{\text{max}}}. \quad (B21)$$

Let us focus on the first summation. For each $l' \in D^{4}_{\text{max}}$, the summation can be approximated as

$$\langle l:l-l' \in D^{3}_{\text{max}} | e^{-i\mathcal{H}_{\text{eff}}^\dagger t} |l'\rangle | \psi(0) \rangle - | \psi(t) \rangle \rangle \leq \sum_{l:l-l' \in D^{3}_{\text{max}}} \frac{8(3\gamma t)^n(l',l)}{n(l',l)!}$$

$$\quad + \sum_{l \in D^{4}_{\text{max}}:l-l' \notin D^{3}_{\text{max}}} e^{-il_3t} \langle |l\rangle e^{-i\mathcal{H}_{\text{eff}}^\dagger t} |0\rangle | \psi(0) \rangle - | \psi(t) \rangle \rangle \leq \sum_{l \in D^{4}_{\text{max}}:l-l' \notin D^{3}_{\text{max}}} \frac{8(3\gamma t)^n(l_{\text{max}},l)}{n(l_{\text{max}},l)!} + 10m_{\text{max}} \langle 0 | e^{-i\mathcal{H}_{\text{eff}}^\dagger t} |0\rangle \rangle \quad (B22)$$

In the first inequality, we employ Lemma 8 with taking $l-l' = l \otimes l'$ for $l-l' \in D^{4}_{\text{max}}$ into consideration. The last inequality comes from Theorem 4. The first term in
Eq. (B22) is further bounded by
\[
16 \sum_{l=3l_{\text{max}}-2m_{\text{max}}}^{\infty} \frac{3e\gamma t}{1/l/m_{\text{max}}} \left( \frac{3l_{\text{max}}/m_{\text{max}} - 2}{2} \right) \leq 16m_{\text{max}} \left( \frac{3e\gamma t}{3l_{\text{max}}/m_{\text{max}} - 2} \right)^{3l_{\text{max}}/m_{\text{max}} - 2},
\]
where we use the relation Eq. (B14) and the Stirling’s formula Eq. (B22). This accomplishes the evaluation of Eq. (B22) with the usage of Eq. (B7) as
\[
\left[ \text{Eq. (B22)} \right] \leq 26m_{\text{max}} \left( \frac{3e\gamma t}{3l_{\text{max}}/m_{\text{max}} - 1} \right)^{3l_{\text{max}}/m_{\text{max}} - 1} \leq \frac{13}{500(m_{\text{max}})^2} \varepsilon^3.
\]

We next compute the second summation in Eq. (B21), which is taken over \( l \in D_{4l_{\text{max}}} \) satisfying \( l - l' \notin D_{3l_{\text{max}}} \). The Lieb-Robinson bound, dictated by Theorem 3 immediately concludes its upper bound by
\[
\left\| \sum_{l \in D_{4l_{\text{max}}} : l - l' \notin D_{3l_{\text{max}}} \left\| e^{-iH_{\text{eff}} t/l} e^{-iH_{\text{eff}} t/l'} \right\| \psi(0) \right\|ight.
\]
\[
\leq \sum_{l \in D_{4l_{\text{max}}} : l - l' \notin D_{3l_{\text{max}}} \left\| e^{-iH_{\text{eff}} t/l \max} \right\|^{l/l' \max} \right\| l - l' / \max \right) \leq 4 \sum_{l=3l_{\text{max}}}^{\infty} \left\| e^{-iH_{\text{eff}} t/l \max} \right\|^{l/l' \max} \leq 4m_{\text{max}} \left( \frac{e\gamma t}{3l_{\text{max}}/m_{\text{max}} \right)} \leq \frac{1}{250(m_{\text{max}})^2} \varepsilon^3.
\]
\[
\left( \text{Eq. (B23)} \right)
\]
Finally, we show that the amplification protocol relying on the symmetry reproduces the time-evolved state \( \psi(t) \). Combining the results of Eqs. (B23) and (B24), we arrive at
\[
\left\| \sum_{n=0}^{\infty} \left\| e^{iH_{\text{eff}} t/l \max} \right\| \right\| \leq \frac{1}{4m_{\text{max}}} \sum_{l' \in D_{3l_{\text{max}}} \left\| e^{-iH_{\text{eff}} t/l \max} \right\|^{l/l' \max} \right\| l - l' / \max \right) \leq 3 \left( \frac{1}{500(m_{\text{max}})^2} \varepsilon^3 + \frac{1}{250(m_{\text{max}})^2} \varepsilon^3 \right) \leq \frac{3}{200(m_{\text{max}})^2} \varepsilon^3.
\]

As a result, we get the difference between the original and refined effective Hamiltonians as
\[
\left\| \begin{pmatrix} 0 & \mathcal{H}_{\text{amp}1, \text{pbc}}(t) |0\rangle \langle \psi(0) | - \frac{1}{2} |0\rangle \langle \psi(t) | \end{pmatrix} \right\| \leq \frac{1}{4m_{\text{max}}} \sum_{l \in D_{4l_{\text{max}}}} 4 \left( \frac{3\gamma t}{l \max} \right)^{n(l_{\max}, l)} \frac{1}{n(l_{\max}, l)!} \leq 4 \left( \frac{3\gamma t}{l \max} \right)^{n(l_{\max}, l)} \frac{1}{n(l_{\max}, l)!} \leq \frac{\varepsilon^3}{250(m_{\text{max}})^2}.
\]
\[
\left( \text{Eq. (B31)} \right)
\]
Therefore, the one under the refined effective Hamiltonian is also justified as
\[
\left\| \langle 0 | \Psi_{\text{amp1, pbc}}^{\ell_{\text{max}}} (t) | \psi(0) \rangle - \frac{1}{2} | \psi(t) \rangle \right\| \\
\leq \left\| \langle 0 | \Psi_{\text{amp1, pbc}}^{\ell_{\text{max}}} (t) - \Psi_{\text{amp1}}^{\ell_{\text{max}}} (t) | \psi(0) \rangle \right\| \\
+ \left\| \langle 0 | \Psi_{\text{amp1}}^{\ell_{\text{max}}} (t) | \psi(0) \rangle - \frac{1}{2} | \psi(t) \rangle \right\| \\
\leq \frac{19}{1000 (m_{\text{max}})^2} \varepsilon^3. \tag{B32}
\]
This bound is actually smaller than \(\varepsilon/3\), and we complete the proof of Theorem 7. \(\square\)

Appendix C: Extension to exponentially-decaying Fourier components

In the main text, we focus on time-periodic Hamiltonians which have vanishing Fourier components \(H_m = 0\) for \(|m| > m_{\text{max}}\). Here, we generalize our results to time-periodic Hamiltonians \(H(t)\) with exponentially-decaying Fourier components as
\[
\|H_m\| \leq h e^{-|m|/\zeta}, \quad h, \zeta > 0, \tag{C1}
\]
for \(|m| > 0\). The norm of \(H_0\) is arbitrary as long as it is bounded. The effective Hamiltonian in the truncated Floquet-Hilbert space, \(H_{\ell_{\text{max}}}^{\ell_{\text{max}}}\), is the same as Eq. \ref{eq:h_lmax}, but it is dense in the basis \(\{|l\rangle\}_{l \in D_{\text{max}}}\) compared to the cases where \(H_m = 0\) is satisfied for \(|m| > m_{\text{max}}\).

We formulate the protocol in a similar manner to the main text, and show that the computational resources for the time-evolved state have nearly-optimal dependence both in \(t\) and \(1/\varepsilon\). The difference mainly comes from the form of the Lieb-Robinson bound and the infinite series of \(\{H_m\}\) needed for designating \(H(t)\). The former one affects the truncation order \(\ell_{\text{max}}\) for the Floquet-Hilbert space. The latter one yields the change in block-encoding so that almost all the information about \(H(t)\) can be embedded with keeping the efficiency.

1. Truncation order of Floquet-Hilbert space

We first determine the proper truncation order \(\ell_{\text{max}}\) for the Floquet-Hilbert space, as we did in Section IV. To this aim, we begin with deriving the Lieb-Robinson bound on the transition rate, corresponding to Theorem 3.

Theorem 9. (Bound on transition rate)

We assume \(\|H_m\| \leq h e^{-|m|/\zeta}\) with certain positive constants \(h\) and \(\zeta\). Then, for \(l, l'\) such that \(|l|, |l'| \leq \ell_{\text{max}}\), the transition rate is bounded from above by
\[
\left\| \langle l | e^{-i H_{\ell_{\text{max}}}^{\ell_{\text{max}}} t} | l' \rangle \right\| \leq \exp \left( -\frac{|l - l'| - 2\beta \zeta' \hbar t}{\zeta'} + 2/\beta \right). \tag{C2}
\]
Here, \(\beta\) and \(\zeta'\) are positive constants defined by
\[
\beta = (1 - e^{-1/\zeta})^{-1}, \tag{C3}
\]
\[
\zeta' = (1/\zeta - 1 + e^{-1/\zeta})^{-1}. \tag{C4}
\]

Proof. — We prove the theorem in a similar way to Theorem 3. We start from the Dyson series expansion in the interaction picture [See Eq. \ref{eq:dyson}],
\[
\left\| \langle l | e^{-i H_{\ell_{\text{max}}}^{\ell_{\text{max}}} t} | l' \rangle \right\| \leq \sum_{n=0}^{\infty} \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \prod_{\{l_i\}_{i=1}^n} \| l_i \langle H_l \rangle_{l-i-1} \| \tag{C5}
\]
where we insert the identity \(\sum_{l_i \in D_{\text{max}}} |l_i\rangle |l_i\rangle = I\) for \(n - 1\) times. The summation \(\sum_{\{l_i\}_{i=1}^n}\) is taken over \(l_i \in D_{\text{max}}\) for \(i = 1, 2, \ldots, n - 1\) under fixed \(l_0 = l'\) and \(l_n = l\).

We introduce a new variable \(m_i = l_i - l_{i-1}\) instead of using \(|l_i\rangle\), and then the above integrand is bounded by
\[
F_n \equiv \sum_{\{l_i\}_{i=1}^n} \left\| \langle l_i | H_{\ell_{\text{max}}}^{\ell_{\text{max}}} \rangle_{l-i-1} \right\| \leq \sum_{\{m_i\}_{i=1}^n} \theta \left( \sum_{i=1}^n |m_i| - |l - l'| \right) h^n e^{-\sum_{i=1}^n |m_i|/\zeta}, \tag{C6}
\]
Since \(l' + m_1 + \ldots + m_n = l\) implies \(|m_1| + \ldots + |m_n| \geq |l - l'|\), it can be further bounded as follows,
\[
F_n \leq \sum_{\{m_i\}_{i=1}^n} \theta \left( \sum_{i=1}^n |m_i| - |l - l'| \right) h^n e^{-\sum_{i=1}^n |m_i|/\zeta} \leq (2h)^n \sum_{m_1, \ldots, m_n = 0}^{\infty} \theta \left( \sum_{i=1}^n |m_i| - |l - l'| \right) e^{-\sum_{i=1}^n m_i/\zeta}. \tag{C7}
\]
Here, \(\theta(x)\) is a step function, defined by \(\theta(x) = 1\) for \(x \geq 0\) and \(\theta(x) = 0\) otherwise. The summation \(S_n(M)\) is defined by
\[
S_n(M) = \sum_{m_1, \ldots, m_n = 0}^{\infty} \theta \left( \sum_{i=1}^n |m_i - M| \right) e^{-\sum_{i=1}^n m_i/\zeta}, \tag{C8}
\]
which satisfies \(S_1(M) = \sum_{m=M}^{\infty} e^{-m/\zeta} = e^{-M/\zeta}\).

Let us evaluate the upper bound on \(S_n(M)\). We split the summation over \(m_n\) into the one over \(m_n \leq M - 1\) and the one over \(m_n \geq M\), which results in
\[
S_n(M) = \sum_{m=0}^{M-1} e^{-m/\zeta} S_{n-1}(M - m_n) + \beta^n e^{-M/\zeta}. \tag{C9}
\]
We use this relation recursively until $S_1$ appears. After the single use of this equality, we obtain

$$S_n(M) \leq \sum_{m_n=0}^{M-1} \sum_{m_{n-1}=0}^{m_n-1} e^{-(m_n+m_{n-1})/\zeta} S_{n-2}(M - m_n - m_{n-1})$$

$$+ \beta^{n-1} e^{-M/\zeta} \sum_{m_n=0}^{M-1} 1 + \beta^n e^{-M/\zeta},$$

(C10)

where we use $e^{-m_n/\zeta} \leq 1$ for the second term. By repeating this calculation $n - 1$ times, we arrive at

$$S_n(M) \leq \left( \sum_{m_1=0}^{M-1} \sum_{m_{n-1}=0}^{M-1} \sum_{m_{n-2}=0}^{M-1} \cdots \sum_{m_2=0}^{M-1} \sum_{m_1=0}^{M-1} e^{-M/\zeta} S_{n-2}(M - m_n - m_{n-1}) \right) \left( \sum_{m_1=0}^{M-1} \sum_{m_{n-1}=0}^{M-1} \sum_{m_{n-2}=0}^{M-1} \cdots \sum_{m_2=0}^{M-1} \sum_{m_1=0}^{M-1} e^{-M/\zeta} S_{n-2}(M - m_n - m_{n-1}) \right)$$

$$\leq \beta^n e^{-M/\zeta} \left( \sum_{m_1=0}^{M-1} \sum_{m_{n-1}=0}^{M-1} \sum_{m_{n-2}=0}^{M-1} \cdots \sum_{m_2=0}^{M-1} \sum_{m_1=0}^{M-1} e^{-M/\zeta} S_{n-2}(M - m_n - m_{n-1}) \right)$$

$$= e^{-M/\zeta} \beta^n \left( \sum_{m_1=0}^{M-1} \sum_{m_{n-1}=0}^{M-1} \sum_{m_{n-2}=0}^{M-1} \cdots \sum_{m_2=0}^{M-1} \sum_{m_1=0}^{M-1} \right)^n 1,$$

(C11)

where $\sum_{i=1}^k m_i$ is denoted by $M_k$. In the above formula, the summation of 1 over $\sum_{i=1}^k m_i$ represents the number of lattice points included in a $(k-1)$-dimensional pyramid $P_{k-1}^M = \{ (x_1, \ldots, x_{k-1}) \in [0, M-1]^{k-1} | \sum_{i=1}^{k-1} x_i \leq M-1 \}$, and hence it is bounded by the volume of $P_{k-1}^{M+2}$. This leads to the relation,

$$S_n(M) \leq \beta^n e^{-M/\zeta} \sum_{k=1}^{n} \left( \frac{\beta^{-1}(M+2)}{(k-1)!} \right)^{k-1} \leq \beta^n e^{-1/(\zeta-1/\beta)} M+2/\beta,$$

(C12)

We define a positive constant $\zeta'$ by $1/\zeta' = 1/\zeta - 1/\beta$ as Eq. (C4). The above inequality enables to evaluate the integrand $F_n$ as

$$F_n \leq (2\beta h)^n S_n(|l - l'|) \leq (2\beta h)^n e^{-|l-l'|/\zeta'+2/\beta}.$$  (C13)

Finally, going back to the inequality Eq. (C5), we arrive at the bound on the transition rate,

$$\| \langle l | e^{-i \mathcal{H}_{\text{eff}}^{l_{\text{max}} t} | l' \rangle \| \leq \sum_{n=0}^{\infty} t^n \frac{\beta^n}{n!} (2\beta h)^n e^{-|l-l'|/\zeta'+2/\beta} \leq e^{2\beta h t - |l-l'|/\zeta'+2/\beta}.$$  (C14)

This completes the proof of the theorem. □

Theorem 10 says that the bound on the transition amplitude from $|l\rangle$ to $|l\rangle$ exponentially decays in the distance $|l-l'|$ for Hamiltonians with $\| H_m \| \lesssim e^{-O(|m|)}$. The decay is relatively slow compared to the cases $H_m = 0 (|m| > m_{\text{max}})$ showing $O\left(|l-l'|^{-|l-l'|}\right)$. In fact, this result is reminiscent of the Lieb-Robinson bound for short-ranged interacting systems [S2–S4], which provides exponentially-decaying correlation functions under interactions $U_{ij} \sim e^{-|i-j|\delta}$. Based on Theorem 10, we assess how the exact time-evolved state can be approximated by the truncated Floquet-Hilbert space, and determine the proper truncation order $l_{\text{max}}$. We summarize the result by the following theorem, which is a counterpart of Theorem 10.

**Theorem 10. (Floquet-Hilbert space truncation)**

We assume $\| H_m \| \leq C e^{-|m|/\zeta}$. The exact time-evolved state $|\psi(t)\rangle$ is approximated by the truncated state $|\psi^{l_{\text{max}}}(t)\rangle$ [See Eq. (C18)] as

$$\| |\psi(t)\rangle - |\psi^{l_{\text{max}}}(t)\rangle \| \leq 4\epsilon^2 e^{2\beta h t - (l_{\text{max}}-1)/\zeta'+2/\beta}.$$  (C15)

**Proof.**— The derivation is similar to the one for Theorem 4. According to Eqs. (47)–(49), we should evaluate the following two values,

$$\epsilon_1 = \sum_{l \in (D_{l_{\text{max}}} \setminus D_{l_{\text{max}}^\dagger})} \| \langle l | e^{i \mathcal{H}_{\text{eff}}^{l_{\text{max}} t} | l \rangle \|$$

$$\epsilon_2 = \sum_{l \in D_{l_{\text{max}}}^\dagger} \| \langle l | e^{i \mathcal{H}_{\text{eff}}^{l_{\text{max}} t} - e^{i \mathcal{H}_{\text{eff}}^{l_{\text{max}} t} | l \rangle} \|,$$

(C16)

(C17)

where the left hand side of Eq. (C15) is bounded by $\epsilon_1 + \epsilon_2$ under $l_{\text{max}}' \to \infty$. Theorem 9 soon concludes the upper bound of $\epsilon_1$:

$$\epsilon_1 \leq \sum_{l \in (D_{l_{\text{max}}} \setminus D_{l_{\text{max}}^\dagger})} e^{2\beta h t - |l|/\zeta'+2/\beta} \leq 2 \int_{l_{\text{max}}-1}^{\infty} \int_0^\infty d\epsilon' e^{2\beta h t - |l|/\zeta'+2/\beta}.$$  (C18)

The second error $\epsilon_2$ can be evaluated by the Dyson series expansion like Eq. (39); each term of $\epsilon_2$ in Eq. (C17) is bounded from above by

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty dt_n \cdots \int_0^\infty dt_1 \| \langle l | \prod_{i=1}^n \mathcal{H}_{\text{eff}}^{l_{\text{max}} (t_i)} - \prod_{i=1}^n \mathcal{H}_{\text{eff}}^{l_{\text{max}} (t_i)} | l \rangle \|.$$  (C19)

As stated in the proof of Theorem 4, each of $\langle l | \prod_{i=1}^n \mathcal{H}_{\text{eff}}^{l_{\text{max}} (t_i)} | l \rangle$ and $\langle l | \prod_{i=1}^n \mathcal{H}_{\text{eff}}^{l_{\text{max}} (t_i)} | l \rangle$ is decomposed into a product of transition amplitudes via a path $|0\rangle \to |l_1\rangle \to \ldots \to |l_{n-1}\rangle \to |l\rangle$. Their difference appears only when the path goes across $D_{l_{\text{max}}} \setminus D_{l_{\text{max}}^\dagger}$. In other words, denoting the summation over the set of $\{l_i\}$ taking such nontrivial paths by $\sum_{\{l_i\}}$, we reach

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \cdots \int_0^\infty dt_n \cdots \int_0^\infty dt_1 \sum_{\{l_i\}} \| \langle l \prod_{i=1}^n \mathcal{H}_{\text{eff}}^{l_{\text{max}} (t_i)} | l_{i-1} \rangle \|.$$  (C20)
When we define the hopping distance $m_i$ by $m_i = l_i - l_{i-1}$, it should satisfy $\sum_{i=1}^{n} |m_i| \geq (l_{\text{max}} - |l|) + (l_{\text{max}} - 0)$ for $\{l_i\}$ such that a nontrivial path is organized. This results in the relation described by

$$\varepsilon_2 \leq \sum_{l \in D_{l_{\text{max}}}^\infty} \sum_{n=0}^{\infty} t^n \left( \prod_{i=1}^{n} \| H_{m_i} \| \right) \times \theta \left( \sum_{i=1}^{n} |m_i| - 2t_{\text{max}} + |l| \right) \leq \sum_{l \in D_{l_{\text{max}}}^\infty} \sum_{n=0}^{\infty} \frac{(2ht)^n}{n!} S_n(2t_{\text{max}} - |l|) \leq \sum_{l \in D_{l_{\text{max}}}^\infty} \sum_{n=0}^{\infty} \frac{(2\beta t h)^n}{n!} e^{-2(2t_{\text{max}} - |l|)/\varepsilon^{2/\beta}} \leq 2c' e^{2\beta ht - (l_{\text{max}} - 1)/\varepsilon^{2/\beta}}. \tag{C21}$$

Combining this inequality with $\| \langle \psi(t) \rangle - |\psi_{l_{\text{max}}}(t)\rangle \| \leq \varepsilon_1 + \varepsilon_2$ and Eq. (C15), we obtain Eq. (C15), which completes the proof. □

Theorem 10 determines the proper truncation order of the Floquet-Hilbert space for the algorithm. When we aim at the desirable accuracy as $\| \langle \psi(t) \rangle - |\psi_{l_{\text{max}}}(t)\rangle \| \leq \varepsilon$, it is sufficient to choose the truncation order $l_{\text{max}}$ by

$$l_{\text{max}} = \left[ \frac{\beta \varepsilon (1/\varepsilon)}{\varepsilon} \right] + \log(4) + \varepsilon \log(4) \cdot \frac{\varepsilon}{\beta} + 1 \in \Theta(ht + \log(1+\varepsilon)). \tag{C22}$$

The result has similar scaling in $t$ and $1/\varepsilon$ to the one for Hamiltonians with a finite number of Fourier components, described by Eq. (C55), other than the factor of $\log(1/\varepsilon)$. While the actual $m_{\text{max}}$ is infinite for exponentially-decaying Fourier components as $\|H_m\| \leq h e^{-|m|/\lambda}$, the parameters $h$ and $\lambda$ have a role of $\gamma$ and $m_{\text{max}}$ respectively.

2. Amplitude amplification

We next verify the validity of the amplitude amplification protocols, discussed in Section V for time-periodic Hamiltonians with exponentially-decaying Fourier components $\|H_m\| \leq h e^{-|m|/\lambda}$.

The first amplification relies on the translation symmetry of the effective Hamiltonian, as discussed in [A]. To show its extension, we begin with discussing the approximate translation symmetry in the truncated Floquet-Hilbert space, which is a counterpart of Lemma 8.

**Lemma 11. (Approximate translation symmetry)**

We assume $\|H_m\| \leq h e^{-|m|/\lambda}$, and consider the truncated Floquet-Hilbert space $\mathbb{C}^{l_{\text{max}}} \otimes \mathcal{H}$. The transition rate has an approximate translation symmetry in that it satisfies

$$\langle l \mid e^{-i\mathcal{H}_{\text{eff}}(l) t} \mid l' \rangle = e^{it\omega t} \langle l \mid l' \rangle e^{-i\mathcal{H}_{\text{eff}}(l') t} \langle 0 \mid \leq 2 e^{2(\betaht - (8l_{\text{max}} - |l|) - |l'|)/\varepsilon^{2/\beta}}. \tag{C23}$$

**Proof.**— The proof is done in a similar manner to that of Lemma 11. We consider a perturbation $\mathcal{H}_{\text{pert}}(t)$ designated by

$$\mathcal{H}_{\text{pert}}(t) = \sum_{(l,m) \in \partial \mathbb{F}^{l_{\text{max}}}} |l\rangle \langle l \otimes l' | e^{i\omega t} H_{-m} + \text{h.c.}. \tag{C24}$$

with $\partial \mathbb{F}^{l_{\text{max}}} = \{(l,m) \mid l \in D_{l_{\text{max}}}^\infty, 8l_{\text{max}} - l + 1 \leq m \leq 8l_{\text{max}} - 1\}$. This Hamiltonian indicates hopping terms that go across the boundaries $|4l_{\text{max}}|$ and $|-4l_{\text{max}} + 1|$. Let $\mathcal{U}_{\text{pert}}(t)$ denote a time evolution operator under $\mathcal{H}_{\text{eff}}^{l_{\text{max}}} + \mathcal{H}_{\text{pert}}(t)$. Then, due to the exact translation symmetry in the interaction picture, the transition amplitude $\langle l \mid \mathcal{U}_{\text{pert}}(t) l' \rangle$ satisfies

$$\langle l \mid \mathcal{U}_{\text{pert}}(t) l' \rangle = e^{it\omega t} \langle l \otimes l' | \mathcal{U}_{\text{pert}}(t) | 0 \rangle. \tag{C25}$$

The difference of transition amplitudes between $\mathcal{H}_{\text{eff}}^{l_{\text{max}}} + \mathcal{H}_{\text{pert}}(t)$ and $\mathcal{H}_{\text{eff}}^{l_{\text{max}}}$ is bounded in a similar way to Eqs. (C19) and (C21). The difference survives only when the trajectory $|l\rangle \rightarrow |l\rangle \rightarrow |l-1\rangle \rightarrow |l|$ pass through the boundaries $|4l_{\text{max}}|$ and $|-4l_{\text{max}} + 1|$ via $\mathcal{H}_{\text{pert}}(t)$, and then its length $\sum_{i=1}^{n} |m_i|$ with $m_i = l_i - l_{i-1}$ should be equal to or larger than $(4l_{\text{max}} - |l|) + (4l_{\text{max}} - |l'|)$.

We obtain its upper bound similarly to Eq. (C21),

$$\langle l \mid \mathcal{U}_{\text{pert}}(t) l' \rangle = e^{-i\mathcal{H}_{\text{eff}}^{l_{\text{max}}} t} \langle l \mid l' \rangle \leq e^{2\beta ht - (8l_{\text{max}} - |l| - |l'|)/\varepsilon^{2/\beta}}. \tag{C26}$$

Using this relation twice and the symmetry Eq. (C25) leads to Eq. (C23). □

With the usage of this approximate translation symmetry, we can organize the amplitude amplification by symmetry like Theorem 6. We prepare the truncated Floquet-Hilbert space $\mathbb{C}^{l_{\text{max}}} \otimes \mathcal{H}$, and make the initial state uniform in $|l\rangle$ with $l_{\text{max}} \in \Theta(ht + \log(1/\varepsilon))$. The state resulting from the time evolution exp$(-i\mathcal{H}_{\text{eff}}^{l_{\text{max}}})$ outputs the target time-evolved state $|\psi(t)\rangle$. With amplitude $1/2$ as follows.

**Theorem 12. (Amplification by symmetry)**

We assume $\|H_m\| \leq h e^{-|m|/\lambda}$, and choose the truncation order $l_{\text{max}} \in O(ht + \log(1/\varepsilon))$ by Eq. (C22). Let us summarize the amplification protocol relying on the symmetry by $\mathcal{U}_{\text{ampl}}(t)$, whose explicit formula is given by Eqs. (C60). Then, it generates the time-evolved state $|\psi(t)\rangle$ with $O(1)$ amplitude as

$$\left\| \begin{array}{c} 0 \rangle \mathcal{U}_{\text{ampl}}(t) | 0 \rangle \langle 0 | \psi(0) \rangle - \frac{1}{2} \langle \psi(t) \rangle \end{array} \right\| \leq \frac{5}{64(\varepsilon')^2} \varepsilon^3. \tag{C27}$$

It is reasonable to assume that the desirable error $\varepsilon$ is sufficiently small, not greater than the constant $8\varepsilon'/\sqrt{15}$. Then, the right hand side of the above formula is bounded by $\varepsilon/3$, which reproduces Theorem 6.
Proof.— The proof follows that of Theorem B1 described in Appendix B1. The left hand side of the above inequality can be bounded by two contributions determined by Eq. (B21). The first contribution, which corresponds to Eq. (B22), is given by

$$
\left\| \sum_{l,l' \in D^{4l_{\max}}} e^{-i\Delta t} \left( l \right) e^{-i\mathcal{H}_{\text{eff}}^{4l_{\max}} t} |l'\rangle \langle \psi(0) | - |\psi(t)\rangle \right\|
$$

$$
\leq \sum_{l,l' \in D^{4l_{\max}}} 2e^{2\beta h t - \frac{7l_{\max} - |l| + 3}{\zeta'}} + \left\| |\psi^{3l_{\max}}(t)\rangle - |\psi(t)\rangle \right\|
$$

$$
\leq 8\epsilon e^{2\beta h t - (3l_{\max} - 1)\zeta' + 2/\beta} \leq \frac{1}{8(\zeta')^2} \epsilon^3.
$$

(C28)

Here, we used Lemma 11 in the second line, Theorem 10 in the third line, and the choice of $l_{\max}$, Eq. (C22), in the last line. The second contribution, which is a counterpart in Eq. (B24), is bounded by

$$
\left\| \sum_{l \in D^{4l_{\max}}} e^{-i\Delta t} \left( |l\rangle \langle l \right) e^{-i\mathcal{H}_{\text{eff}}^{4l_{\max}} t} |l'\rangle \langle \psi(0) | - |\psi(t)\rangle \right\|
$$

$$
\leq \sum_{l \in D^{4l_{\max}}} e^{2\beta h t - |l'|\zeta' + 2/\beta} \leq \frac{1}{32(\zeta')^2} \epsilon^3.
$$

(C29)

By combining these results like Eq. (B25), we complete the proof of Theorem 12. □

The above theorem ensures the amplification by symmetry also for time-periodic Hamiltonians with $\|H_{m}\| \leq \epsilon e^{-|m|/\zeta}$; it enhances the amplitude from $O((l_{\max})^{-1})$ to 1/2 only with additional $O(\log(l_{\max}))$ elementary gates. In order to bring the amplitude up to 1, we need the obvious amplitude amplification. Its validity immediately follows from Theorem 12 as discussed in Appendix B1. In other words, when we apply the protocol $\mathcal{U}_{\text{amp2}}^{\max}(t)$, defined by Eq. (76), the time-evolved state $|\psi(t)\rangle$ can be obtained with accuracy 1 $- O(\epsilon)$;

$$
|\mathcal{U}_{\text{amp2}}^{\max}(t) |0\rangle \langle \psi(0) | - |0\rangle \langle \psi(t) |
$$

$$
\leq \frac{15}{64(\zeta')^2} \epsilon^3.
$$

(C30)

The right hand side is bounded by $\epsilon$ for sufficiently small $\epsilon$.

3. Qubitization technique for effective Hamiltonian

Simulating $|\psi(t)\rangle$ via the amplitude amplification $\mathcal{U}_{\text{amp2}}^{\max}(t)$ requires the time evolution operators $\exp(-i\mathcal{H}_{\text{eff}}^{4l_{\max}} t)$ and $\exp(-i\mathcal{H}_{\text{eff}}^{4l_{\max}} t)$. The former one is the same as Section VI A. We hereby present how the latter one is implemented for the cases $\|H_{\text{eff}}\| \leq \epsilon e^{-|m|/\zeta}$. We take a similar strategy to Section VI B, that is, we first organize a refined effective Hamiltonian $\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}}$, which can accurately reproduce the dynamics under $\mathcal{H}_{\text{eff}}^{4l_{\max}}$. After that, we compose its block-encoding which can be efficiently achievable.

a. Refined effective Hamiltonian

We introduce a refined effective Hamiltonian $\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}}$, which acts on the truncated Floquet-Hilbert space $\mathcal{C}^{4l_{\max}} \otimes \mathcal{H}$, by

$$
\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}} = \mathcal{H}_{\text{eff}}^{4l_{\max}} + \tilde{\mathcal{H}}_{b},
$$

(C31)

$$
\tilde{\mathcal{H}}_{b} = \sum_{(l,m) \in \partial \mathcal{F}^{4l_{\max}}} |l\rangle \langle l \otimes H_{-m} + \text{h.c.},
$$

(C32)

where $\partial \mathcal{F}^{4l_{\max}}$ shares the definition with the one in Eq. (C24). Owing to the additional term $\tilde{\mathcal{H}}_{b}$, the hopping terms induced by $H_{-m}$ become translation symmetric in $|l\rangle$ as

$$
\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}} = \sum_{m \in D^{4l_{\max}}} \text{Add}_{m}^{4l_{\max}} \otimes H_{m} - \mathcal{H}_{\text{eff}}^{4l_{\max}}.
$$

(C33)

Here, a full quantum adder $\text{Add}_{m}^{4l_{\max}}$ defined by Eq. (105), appears and it allows efficient implementation of block-encoding as discussed later. Before going to its block-encoding, we prove the validity of the refined effective Hamiltonian as a counterpart of Theorem 7.

Theorem 13. (Refined effective Hamiltonian)

We assume $\|H_{m}\| \leq \epsilon e^{-|m|/\zeta}$, and organize the two amplification protocols $\mathcal{U}_{\text{amp1, pbc}}^{\max}(t)$ and $\mathcal{U}_{\text{amp2, pbc}}^{\max}(t)$ respectively based on Eqs. (99) and (100) with using the refined effective Hamiltonian $\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}}$ given by Eqs. (C31) and (C32). When the truncation order $l_{\max}$ is chosen by Eq. (C22), they also provide the exact time-evolved state $|\psi(t)\rangle$ as

$$
\left\| \left(0\right) \mathcal{U}_{\text{amp1, pbc}}^{\max}(t) |0\rangle \langle \psi(0) | - \frac{1}{2} |\psi(t)\rangle \right\| \leq \frac{11}{128(\zeta')^2} \epsilon^3
$$

(C34)

$$
\left\| \mathcal{U}_{\text{amp2, pbc}}^{\max}(t) |0\rangle \langle \psi(0) | - |0\rangle \langle \psi(t) | \right\| \leq \frac{33}{128(\zeta')^2} \epsilon^3
$$

(C35)

for arbitrary initial states $|\psi(0)\rangle \in \mathcal{H}$. For a allowable error $\epsilon$ sufficiently small compared to the constant $\zeta'$, both of the left hand sides are smaller than $\epsilon$.

Proof.— We prove the theorem as we do for Theorem 7 in Appendix B2. First, we evaluate the difference of the transition rates between $\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}}$ and $\mathcal{H}_{\text{eff}}^{4l_{\max}}$. We replace the perturbation $\tilde{\mathcal{H}}_{b}(t)$ by the boundary term $\tilde{\mathcal{H}}_{b}$ in the proof of Lemma 11 [See Appendix B1]. We obtain the same result as Eq. (C26).

$$
\left\| \left( l \right) e^{-i\mathcal{H}_{\text{eff, pbc}}^{4l_{\max}} t} |l'\rangle - \left( l \right) e^{-i\mathcal{H}_{\text{eff}}^{4l_{\max}} t} |l'\rangle \right\| \leq e^{2\beta h t - (8l_{\max} - |l| - |l'|)/\zeta' + 2/\beta}
$$

(C36)
Once we obtain this bound, we can track the proof of Theorem 7, composed of Eqs. (B28)-(B32). First, we evaluate the deviation from $U$ by \[\langle 0 | \psi_{amp1 \ pbc}^\dagger (t) - \psi_{amp1}^\dagger (t) | 0 \rangle |\psi(0)\rangle\]
\[\leq \frac{1}{4l_{\max}} \sum_{t \in D^{4l_{\max}}} \sum_{t' \in D^{4l_{\max}}} e^{2\beta t-(8l_{\max}-|t|-|t'|)/\zeta' + 2/\beta} \leq \frac{1}{2} e^{2\beta t-(3l_{\max}-1)/\zeta' + 2/\beta} \leq \frac{1}{128(\zeta')^2} e^3. \] (C37)
Since $\langle 0 | \psi_{amp1}^\dagger (t) | 0 \rangle |\psi(0)\rangle$ accurately provides the state $|\psi(t)\rangle/2$ as Eq. (C27) according to Theorem 12, a triangle inequality concludes Eq. (C34). Since the oblivious amplitude amplification generates the error at most three times larger than Eq. (C34) according to the discussion in Section V B, Eq. (C35) is immediately derived. □

b. Block-encoding

We compose block-encoding of the refined effective Hamiltonian $\mathcal{H}_{eff \ pbc}^{4l_{\max}}$ to implement its time evolution operator by qubitization.

We begin with describing the assumption on time-periodic Hamiltonians $H(t)$. If we adopt the one for the main text as Eq. (55), we should have the knowledge about block-encoding for the infinite series $\{H_m\}_{m \in \mathbb{Z}}$. To avoid this complexity, we instead consider time-periodic Hamiltonians $H(t)$ written by
\[H(t) = \sum_{j=1}^{J_{\max}} \alpha_j(t) M_j, \quad \alpha_j(t+T) = \alpha_j(t). \] (C38)
with coefficients $\{\alpha_j(t)\}_{j=1}^{J_{\max}}$ and operators $\{M_j\}_{j=1}^{J_{\max}}$. We assume the knowledge of block-encoding for each operator $M_j$ as
\[\langle G_j | O_j | G_j \rangle = M_j, \quad |G_j\rangle = G_j |0\rangle^{\otimes n_a}, \] (C39)
with an oracle state $|G_j\rangle \in \mathbb{C}^{2^{n_a}}$ and an oracle unitary gate $O_j$ on $\mathbb{C}^{2^{n_a}} \otimes \mathcal{H}$. We can always set the denominator in block-encoding by 1 since its absolute value and its phase can be absorbed respectively into the coefficient $\alpha_j(t)$ and the oracle $O_j$. The Fourier components are given by
\[H_m = \sum_{j=1}^{J_{\max}} \alpha_j^m(t) M_j, \quad \alpha_j^m = \frac{1}{T} \int_0^T d\alpha_j(t) e^{imwt}. \] (C40)
and it is always possible to impose $\alpha_j^m \geq 0$ for every $j, m$. This is because, when a certain coefficient $\alpha_j^m$ is complex, we can divide it into $\alpha_j^m = \text{Re} \alpha_j^m + i \text{Im} \alpha_j^m$ and redefine $M_j$ with the signs of $\text{Re} \alpha_j^m, \text{Im} \alpha_j^m$ and redefine $M_j$ with the signs of $\text{Re} \alpha_j^m, \text{Im} \alpha_j^m$ in order to set $\{M_j\}_j$ reproduces the form of Eq. (C40) with $\alpha_j^m \geq 0$, where $j_{\max}$ is replaced by at most $4l_{\max}$. In the following discussion, the oracles $O_j$ and $G_j$ are respectively supposed to be composed of at most $C$ elementary gates. In addition, we also assume the access to the coefficients $\alpha_j^m$ for $m \in D^{4l_{\max}}$ by the oracle unitary gate $G_{\text{coef}}^{\text{max}}$ as
\[G_{\text{coef}}^{\text{max}} | 0 \rangle | 0 \rangle = \sum_{m \in D^{4l_{\max}}} \sum_{j=1}^{j_{\max}} \sqrt{\frac{\alpha_j^m}{\alpha_j^m}} |m\rangle |j\rangle, \] (C41)
\[\alpha_j^{\text{max}} = \sum_{m \in D^{4l_{\max}}} \sum_{j=1}^{j_{\max}} \alpha_j^m. \] (C42)
The oracle state $G_{\text{coef}}^{\text{max}} | 0 \rangle | 0 \rangle \in \mathbb{C}^{2l_{\max} + j_{\max}}$ is a $O(\log l_{\max} + \log j_{\max})$-qubit ancilla state, and its preparation $G_{\text{coef}}^{\text{max}}$ is supposed to require $O_{\text{coef}}^{\text{max}}$ elementary gates. The parameter $\alpha_j^{\text{max}}$ is bounded by the whole-system energy scale as
\[\alpha_j^{\text{max}} \leq \sum_{m \in D^{4l_{\max}}} \sum_{j=1}^{j_{\max}} \alpha_j^m \equiv \alpha. \] (C43)

We move to how we organize an oracle state and oracle unitary gate for the refined effective Hamiltonian $\mathcal{H}_{\text{eff \ pbc}}^{4l_{\max}}$. Substituting the Fourier components $H_m$ into Eq. (C33) results in
\[\mathcal{H}_{\text{eff \ pbc}}^{4l_{\max}} = \sum_{m \in D^{4l_{\max}}} \sum_{j=1}^{j_{\max}} \alpha_j^m \text{Add}_{\text{eff}}^{4l_{\max}} \otimes M_j - \mathcal{H}_{\text{LP}}^{4l_{\max}}. \] (C44)
We introduce five kinds of ancilla systems labeled by $a, b, c, d,$ and $e$. The systems $a, b,$ and $c$ are respectively described by $2^{n_a}, \ 8l_{\max}$- and 2-dimensional Hilbert spaces, as we consider in Section VII B States of the system $e$ is spanned by $\{m\}_m \in D^{4l_{\max}}$ due to the absence of $m_{\max}$. The system $d$ is characterized by $\{j\}_{j=1}^{j_{\max}}$. The oracle unitary gate for the refined effective Hamiltonian $\mathcal{H}_{\text{eff \ pbc}}^{4l_{\max}}$ is given by
\[\text{Add}_{\text{eff}}^{4l_{\max}} = \sum_{m \in D^{4l_{\max}}} |m\rangle \langle m| \otimes \text{Add}_{\text{m}}^{4l_{\max}}. \] (C47)
Here, the oracle gates $O_j$, $\text{Add}_{\text{eff \ LP}}^{4l_{\max}}$, and $\text{Add}_{\text{eff}}^{4l_{\max}}$ are respectively implemented by $O(j_{\max} C), O(\log l_{\max}),$ and $O(\log j_{\max})$ elementary gates (the second one represents addition of a variable $m \in D^{4l_{\max}}$). Next, we provide the oracle state $G_{\text{eff \ pbc}}^{4l_{\max}}$ on the auxiliary systems.
\(a, b, c, d, \) and \(e\) by
\[
|G_{\text{eff}}^{4\text{max}}\rangle = G_{\text{eff}}^{4\text{max}} |u^{4\text{max}}\rangle_c (G_{\text{coeff}}^{4\text{max}} |0\rangle_d |0\rangle_e) |a^{4\text{max}}\rangle_b |0\rangle_a,
\]
(C48)
\[
G_{\text{eff}}^{4\text{max}} = I_e \otimes \sum_{j=1}^{n_{\text{max}}} |j\rangle_d \otimes I_c \otimes I_b \otimes (G_j)_a,
\]
(C49)
\[
|u^{4\text{max}}\rangle_c = \frac{\sqrt{\alpha^{4\text{max}} |0\rangle_c + \sqrt{8t^{4\text{max}}\omega |1\rangle_c}}{\sqrt{\alpha^{4\text{max}} + 8t^{4\text{max}}\omega}}.
\]
(C50)
We can confirm the relation,
\[
\langle G_{\text{eff}}^{4\text{max}} |G_{\text{eff}}^{4\text{max}} |G_{\text{eff}}^{4\text{max}} \rangle = \frac{\mathcal{H}_{\text{eff,pbc}}^{4\text{max}}}{\sqrt{\alpha^{4\text{max}} + 8t^{4\text{max}}\omega}},
\]
(C51)
by directly substituting the above equations. The cost of implementing the oracle state \(|G_{\text{eff}}^{4\text{max}}\rangle\) from the trivial state \(|0\rangle_c |0\rangle_d |0\rangle_e |0\rangle_b |0\rangle_a\) is \(O(j_{\text{max}}C + C^{4\text{max}} + \log l_{\text{max}})\) elementary gates.

Therefore, implementing the time evolution operator \(\exp(-i\mathcal{H}_{\text{eff,pbc}}t)\) with the qubitization technique requires the following resources for time-periodic Hamiltonians satisfying \(||H_m|| \leq he^{-|m|/\zeta}\):

- Number of ancilla qubits; \(O(\log l_{\text{max}} + \log j_{\text{max}})\)
- Number of overall gates;
\[
O \left( \left\{ (\alpha + l_{\text{max}}\omega)t + \frac{\log(1/\varepsilon)}{\log(1/\varepsilon)} \right\} \times \left\{ j_{\text{max}}C + C^{4\text{max}} + \log l_{\text{max}} \right\} \right).
\]
(C52)

We replace \(\alpha^{4\text{max}}\) by \(\alpha\) based on the inequality, Eq. (C43).

4. Algorithm and comparison

The algorithm for computing the time-evolved state for time-periodic Hamiltonian with \(||H_m|| \leq he^{-|m|/\zeta}\) is almost the same as those in Section VII. We separately consider adiabatic-like cases and generic long-time cases. The computational resources are obtained by replacing \(\gamma\) and \(C\) in Table III, respectively by \(h\) and \(j_{\text{max}}C + C^{4\text{max}}\).

The query complexity can be determined by the coefficients of \(j_{\text{max}}C + C^{4\text{max}}\) in the overall elementary gates. Their scaling is summarized as follows:

Adiabatic-like cases.—

The scaling of the query complexity is optimal in the time \(t\) and nearly-optimal in the inverse error \(1/\varepsilon\).

- Number of ancilla qubits;
\[
n_a + O(\log j_{\text{max}} + \log(\gamma t) + \log(1/\varepsilon)).
\]
- Query complexity; \(O((\alpha + \gamma)t + \log(1/\varepsilon))\).

- Overall elementary gates;
\[
O \left( \left\{ (\alpha + \gamma)t + \log(1/\varepsilon) \right\} \times \left\{ j_{\text{max}}C + C^{4\text{max}} + \log(\gamma t) + \log(1/\varepsilon) \right\} \right).
\]

Generic long-time cases.—

The scaling of the query complexity is nearly-optimal in the time \(t\) and the inverse error \(1/\varepsilon\). For practical problems up to poly \((N)\)-time, it becomes optimal in the time \(t\).

- Number of ancilla qubits;
\[
n_a + O(\log j_{\text{max}} + \log(\gamma/\omega) + \log(\omega t/\varepsilon)).
\]
- Query complexity; \(O((\alpha + \gamma)t + \omega t \log(\omega t/\varepsilon))\).
- Overall elementary gates;
\[
O \left( \left\{ (\alpha + \gamma)t + \omega t \log(\omega t/\varepsilon) \right\} \times \left\{ j_{\text{max}}C + C^{4\text{max}} + \log(\gamma/\omega) + \log(\omega t/\varepsilon) \right\} \right).
\]

We note that \(\log(1/\varepsilon)\) factor in the query complexity is buried by \(l_{\text{max}}\). This difference from the cases \(H_m = 0\) \((|m| > m_{\text{max}})\) essentially originates from the form of the Lieb-Robinson bound dictated by \(\zeta\). A slightly slow decay in the transition amplitude results in slightly larger cost for the cases \(||H_m|| \leq he^{-|m|/\zeta}\), compared to the results in the main text. This leads to the nearly-optimal dependence of the query complexity in \(1/\varepsilon\). Anyway, since the factor \(\log(1/\varepsilon)\) is not so large compared to others, the relation to the quantum algorithms for time-independent and generic time-dependent Hamiltonians, discussed in Section VII is maintained. Importantly, in the query complexity, the whole energy scales \(\alpha t\) and \(\gamma t\) are separated from the \(O(\log(1/\varepsilon))\) term; the query complexity is usually much smaller than that of the truncated-Dyson-series algorithm, and close to that of the qubitization technique. The number of ancilla qubits, which scales as \(O(\log(1/\varepsilon))\) for an allowable error \(\varepsilon\), also lies between those for these algorithms.

We also mention about the complexity of the oracles for the cases \(||H_m|| \leq he^{-|m|/\zeta}\). In contrast to the cases \(H_m = 0\) \((|m| > m_{\text{max}})\), we require a quantum circuit which embeds the function \(q^n\) to a quantum state with \(O(j_{\text{max}}(\gamma t + \log(1/\varepsilon)))\) or \(O(j_{\text{max}}(\gamma/\omega + \log(\omega t/\varepsilon)))\) degrees of freedom, as the oracle \(G_{\text{eff,coeff}}^{4\text{max}}\) defined by Eq. (C41). While integrable functions are efficiently implemented [50], the number of elementary gates for the worst cases amounts to the dimension of the quantum state, i.e. \(C^{4\text{max}} \in O(j_{\text{max}}(\gamma t + \log(1/\varepsilon)))\) or \(O(j_{\text{max}}(\gamma/\omega + \log(\omega t/\varepsilon)))\). The preparation of oracles...
is relatively difficult compared to the cases $H_m = 0$ ($|m| > m_{\text{max}}$). However, even in such cases, we emphasize that it is much easier than that of the truncated-Dyson-series algorithm. In the truncated-Dyson-series algorithm, we need more or less an oracle that embeds the information of time-dependent Hamiltonians at every discretized time into a quantum state. Such an oracle is exemplified by Eq. (130) for generic Hamiltonians. For time-dependent LCU Hamiltonians or sparse-access Hamiltonians, it requires a quantum state holding all the coefficients $\alpha_j(t)$ at every discretized time $t$ as an oracle. The number of elementary gates per oracle can be proportional to that of the discretized time, $O(j_{\text{max}}\omega\gamma t/\alpha\varepsilon)$, at the worst case. Therefore, our algorithm also improves the complexity of the oracles as well as the query complexity, compared to the truncated-Dyson-series algorithm.

5. Example

For the cases $||H_m|| \leq he^{-|m|/\zeta}$, we adopt the assumption of block-encoding as Eq. (C40), in contrast to the cases $H_m = 0$ ($|m| > m_{\text{max}}$). To show that it is reasonable and versatile in condensed matter physics and quantum chemistry, we provide a simple example.

We consider an $N$-site Fermi-Hubbard model $H_{\text{Hub}}$, given by Eq. (134). While we consider an idealized laser light which has a constant amplitude in Section VIII A, we focus on a rather realistic case where Gaussian wave packets of lights are shone to materials [72]. Then, if the time scale of the wave packets is sufficiently larger than the light with the frequency $\Omega$, the external drive $H_{\text{ext}}(t)$ is well described by

$$H_{\text{ext}}(t) = \left(\sum_{n \in \mathbb{Z}} e^{-(t - (n + 1/2)T)^2/2\tau^2} \sin \Omega t\right) \sum_{x,\sigma} V_x \hat{n}_{x\sigma},$$

(C53)

where peaks of the wave packets are located at $t = (n + 1/2)T$. The period $T$ is designated by $T = p(2\pi/\Omega)$ with the number of the waves of the light for each wave packet $p \in \mathbb{N}$.

Let us organize the block-encoding required for simulation. By carefully computing Fourier components of $H(t) = H_{\text{Hub}} + H_{\text{ext}}(t)$, we arrive at $H_0 = H_{\text{Hub}}$, and

$$H_m = i(-1)^p m\text{sgn}(m) A_m \sum_{x,\sigma} V_x \hat{n}_{x\sigma},$$

(C54)

$$A_m = \frac{\omega\tau}{\sqrt{2\pi}} e^{-\frac{1}{2}(p^2 + m^2)(\omega\tau)^2} \sinh \left\{p|m| (\omega\tau)^2 \right\},$$

(C55)

for $m \neq 0$. We define sgn$(m)$ by sgn$(m) = m/|m|$ for $m \neq 0$ and sgn$(0) = 0$. Since $A_m \geq 0$ rapidly decays as $e^{-O(m^2)}$, this Hamiltonian is suitable for our algorithm, satisfying $||H_m|| \leq he^{-|m|/\zeta}$. This satisfies the assumption of Eq. (C40) with non-negative coefficients $\alpha_j^m$ by choosing $j_{\text{max}} = 3$;

$$M_1 = \frac{H_{\text{Hub}}}{2\sum_x \epsilon_x + UN}, \quad \alpha_1^m = \left(2\sum_x \epsilon_x + UN\right) \delta_{m0},$$

(C56)

$$M_2 = i\sum_{x,\sigma} V_x \hat{n}_{x\sigma}, \quad \alpha_2^m = 2\sum_x V_x A_m \delta_{(-1)^p m + |m| + 1},$$

(C57)

$$M_3 = -i\sum_{x,\sigma} V_x \hat{n}_{x\sigma}, \quad \alpha_3^m = 2\sum_x V_x A_m \delta_{(-1)^p m - |m| - 1}.$$  

(C58)

The block-encoding for $\{M_i\}$ can be composed by Eqs. (136)–(139) in Section VIII A. Therefore, by resorting to our algorithm for the adiabatic-like cases, we can efficiently simulate the response to a realistic wave packet of laser light with the number of ancilla qubits and the query complexity provided in Appendix C 4.

Appendix D: Efficient algorithm for high-frequency regimes with $\omega \approx \alpha, \gamma$

In Section VIII B we concentrate on the generic long-time regime with the frequency $\omega \in O(N^0)$ for the number of sites, while the whole energy scales $\alpha, \gamma$ are given by poly $(N)$. Here, we show its reasonableness by showing that quantum simulation of broad time-periodic Hamiltonians under the frequency comparable to the whole energy scale can be almost attributed to that of time-independent Hamiltonians.

As a generic setup for time-periodic Hamiltonians, we assume the locality [i.e., the system involves at-most $O(1)$-body interactions] and the extensiveness. The latter one means that the energy scale per site, represented by $\lambda$, is bounded. In our definitions, the scale $\lambda$ is approximated by $O(\alpha/N, \gamma/N)$. When the ratio of the frequency $\omega$ to the local energy scale $\lambda$ is sufficiently large as $\omega/\lambda \gg 1$, we can apply the perturbative expansion in $(\omega/\lambda)^{-1}$, called the high-frequency expansion [62–65]. Recent developments in generic local and extensive Floquet systems have revealed the accurate upper bound on the error as follows [63];

$$\left\|U(nT) - e^{-iH_{FM}^m nT}\right\| \leq C_1 \lambda N n Te^{-C_2 \delta_0} + C_3 nN(\alpha T)^{l+2} (l + 1)l!.$$  

(D1)

In the above inequality, $C_1$, $C_2$, and $C_3$ represent some positive constants, and the integer $l_0 \in O(\omega/\lambda)$ provides the optimal truncation order of the perturbation theory, while the actual truncation order $l$ $(\leq l_0)$ is optional. The time-independent Hamiltonian $H_{FM}^m$ is the $l$-th order
Floquet-Magnus expansion, given by
\[ H_{FM}^l = \sum_{l'=0}^{l} H_{FM}^{l'}, \quad H_{FM}^{(0)} = H_0, \]  
(D2)
\[ H_{FM}^{(1)} = \frac{1}{2\pi T} \int_0^T dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)], \]  
(D3)
where each \( l'-th \) order term in \( \omega/\lambda \), represented by \( H_{FM}^{l'} \), involves \( l' \)-fold multi-commutators of \( H(t) \). (See Ref. [63] for the explicit formula).

Based on the above formalism, we see that the long-time dynamics at \( t = (n+\delta)T \gg T \) with \( n \in \mathbb{N} \) and \( \delta \in [0,1) \) can be computed in a trivial way when \( \omega \) is comparable to \( \alpha, \gamma \in \text{poly} \ (N) \). In this regime, the integer \( l_0 \) is \( O(N) \) due to the local energy scale \( \lambda \in O(\alpha/N, \gamma/N) \).

Then, we simulate the dynamics \( U(t) = U(\delta T)U(nT) \) with approximating the latter one \( U(nT) \) by Eq. (D1). We choose the truncation order \( l \) by \( O(N^0) \). Since \( H(t) \) generally has \( O(N) \) local terms, the Floquet-Magnus expansion \( H_{FM}^l \) for such \( l \in O(N^0) \) can be identified with \( O(N) \)-time classical computation, and \( H_{FM}^l \) is also composed of \( O(N) \) local terms. Therefore, we can apply the qubitization technique to the implementation of \( \exp(-iH_{FM}^l nT) \) with the cost designated by Eq. (13). Since the right hand side of Eq. (D1) is bounded by \( O(nN^{-l-1}) \), the replacement of \( U(nT) \) by \( \exp(-iH_{FM}^l nT) \) with an allowable error \( \varepsilon \) is valid as long as
\[ n \leq \text{Const.} \times \varepsilon N^{l+1}. \]  
(D4)
We usually employ a polynomial accuracy \( \varepsilon \in O(N^{-\nu_1}) \) in quantum computation. The assumption for the frequency, represented by \( \omega \in O(N^{\nu_2}) \), implies the period \( T \in O(N^{-\nu_2}) \). Therefore, by properly choosing the \( O(N^0) \) truncation order \( l \geq \nu_1 + \nu_2 - 1 \), we can efficiently simulate \( U(nT) \) for arbitrary polynomial time \( nT \in O(N^{l+1-\nu_1-\nu_2}) \) with the qubitization technique for time-independent Hamiltonians. The remaining micromotion \( U(\delta T) \) can be implemented with our protocol for the adiabatic-like regime (See Section VII A), but it does not affects the computational cost due to \( \delta T \ll nT \).

In short, the dynamics under time-periodic Hamiltonians \( H(t) \) with \( \omega \sim \alpha, \gamma \) can be simulated with the time-independent Hamiltonian approaches. As a result, it is reasonable to concentrate on the cases where \( \omega \) is negligible compared to \( \alpha \) and \( \gamma \), as we do in Section VII B.