Abstract

Triangular mass matrices for neutrinos and their charged partners contain full information on neutrino mixing in a most concise form. Although the scheme is general and model independent, triangular matrices are typical for reducible but indecomposable representations of graded Lie algebras which, in turn, are characteristic for the standard model in noncommutative geometry. The mixing matrix responsible for neutrino oscillations is worked out analytically for two and three lepton families. The example of two families fixes the mixing angle to just about what is required by the Mikheyev-Smirnov-Wolfenstein resonance oscillation of solar neutrinos. In the case of three families we classify all physically plausible choices for the neutrino mass matrix and derive interesting bounds on some of the moduli of the mixing matrix.

Keywords: Lepton mass matrices, neutrino oscillations, solar neutrino flux, noncommutative geometry, representations of graded Lie algebras
While for many years neither direct searches nor terrestrial oscillation experiments have given any positive evidence for nonvanishing neutrino masses, the intense experimental investigation of solar neutrino fluxes significantly changed this situation. Indeed, the four solar neutrino experiments which test different parts of the energy spectrum and all of which find fluxes reduced as compared to the ones expected on the basis of standard solar model calculations and non-oscillating electron neutrinos, seem to lend support to the hypothesis of resonance enhanced oscillation within the sun’s interior. The flux observed by Davies et al. using inverse $\beta$-decay on $^{37}$Cl which is about one third of the expected flux and which is sensitive to the high-energy neutrinos from boron, the flux seen in the two experiments with $^{71}$Ga, Gallex and Sage, which is about 60 percent of the expected, and the Kamiokande results on $\nu_e$-electron scattering, seem compatible with the Miyakev-Smirnov-Wolfenstein mechanism calculated from standard solar parameters and two-generation mixing with squared mass difference and mixing angle (vacuum values) in the range $\Delta m^2 \approx 6 \times 10^{-6}$eV$^2$, $\sin^2 \theta \approx 0.006$. 

In the meantime further experimental hints at nonvanishing neutrino masses is provided by observation of atmospheric neutrinos, and by an accelerator experiment at LAMPF. We note that empirical analyses of all five experiments indicate that they cannot all be fit with three families of leptons.

Conventional gauge theories of electroweak interactions are based on structure groups which are compact Lie groups. Fermions are classified in finite-dimensional, unitary (in general reducible) representations $\rho_l$ of the structure group. As such representations are always fully reducible the gauge bosons of electroweak interactions, a priori, can mediate only between members of the same, given fermionic family. Any transition between states pertaining to two different families must be due to a new element in the theory.

We adopt the usual attitude in assuming that the mixing of fermionic states with equal charges but different flavours is solely due to the mismatch between weak interaction states and mass eigenstates. This means that the mixing matrix relevant for the charged current (CC) interactions of the standard model is calculable from the mass matrices. Making use of the fact that CC interactions involve only left-chiral fields we recently showed that the full information contained in any nonsingular mass matrix can be described by means of a (lower) triangular matrix. The triangular form which is obtained by (unobservable) unitary transformations of right-chiral fields, is the most economic and concise description of family mixing. As shown in ref. it corresponds effectively to classifying the lepton families in reducible but indecomposable representations of the type

$$\rho_l \supset \rho_l \quad \text{or} \quad \rho_l \supset \rho_l \supset \rho_l,$$  

for two and three families, respectively, where the “semi-sum” means that the right-hand representation space is an invariant subspace while the left one(s) is (are) not.

Although our analysis is completely general, we note that representations of the type are not unexpected. Indeed, extensions of gauge theories as obtained in the framework of noncommutative geometry add more structure to them and, in particular, constrain the fermionic sector in a novel and physically relevant way. In the approach proposed by Connes and Lott it is postulated that the Dirac-Yukawa operator contain the physical quark and lepton mass matrices, including the quark mixing pattern. The bosonic sector as derived from
that operator exhibits spontaneous symmetry breaking with the distinctive feature that the Higgs potential would vanish if generations did not mix or were degenerate.

The construction proposed by the Mainz-Marseille group [10, 11], on the other hand, starts from the underlying graded differential algebra and yields the bosonic sector, including spontaneous symmetry breaking, without resorting to the fermion content of the theory. The Dirac-Yukawa operator is a derived object whose detailed form, unlike the Connes-Lott construction, is not an input. The graded gauge potential of this model comprises the gauge bosons as well as the Higgs field and is seen to fall into the adjoint representation of the graded Lie algebra $su(2|1)$, viz.

$$[Y_0 = 0, I_0 = 1] \rightarrow (I = 1, Y = 0) \oplus (I = 0, Y = 0) \oplus (I = \frac{1}{2}, Y = 1) \oplus (I = \frac{1}{2}, Y = -1).$$

The right-hand side gives its decomposition with respect to $Lie(SU(2) \times U(1))$ and describes the four gauge bosons, the Higgs doublet and its antidioublet, respectively. As explained in earlier publications [11, 12] it is natural to use this algebra as a classifying algebra for quarks and leptons as well, its action on the fields of the standard model being understood in the sense of what we called a weak symmetry in [12]. A most remarkable feature of $su(2|1)$ is the existence of reducible but indecomposable representations which are semi-direct sums of two or three copies of the fundamental representations that correctly describe the $SU(2) \times U(1)$ quantum numbers of quarks and leptons, respectively [10]. Such representations, in generation space, have a block triangular form and, therefore, provide a natural place for generational state mixing. In [13] we explored this possibility for quark mass matrices and CKM mixing. Although based on very different assumptions we showed that this scheme leads to the mixing pattern favoured by phenomenology, with the bonus of providing analytical formulae for the mixing matrix elements in terms of quark masses and a small number of free parameters.

In this letter we study the case of nontrivial neutrino mass matrices, in the general framework of triangular mass matrices, and their consequences for neutrino oscillations including the solar neutrino fluxes. With a plausible physical assumption on the origin of state mixing, the case of two families contains no free parameters and yields a mixing pattern in fair agreement with the resonance oscillation needed to explain the solar neutrino puzzle. The case of three families contains more freedom and can be analyzed in terms of complete sets of neutrino oscillation experiments. We then illustrate this general framework by the Mainz-Marseille model and show that our physics assumptions are quite natural within this model based on noncommutative geometry.

2. When written out in terms of individual neutral and charged fields of definite chirality the mass terms are

$$\mathcal{L}_{mass} = \frac{1}{2} \sum_{i=1}^{3} \sum_{k=1}^{i} \left( \nu^{(i)}_L T^{(i)}_{ik} \nu^{(k)}_R + \overline{\nu}^{(i)}_L T^{(i)}_{ik} \overline{\nu}^{(k)}_R \right) + \text{h.c.}. \quad (3)$$

The matrices $T$ are lower triangular matrices given by

$$T^{(i)} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad T^{(i)} = \begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \quad (4)$$

for the neutrinos and the charged leptons, respectively.
Without loss of generality, weak interaction and mass eigenstates can be chosen such that all entries in the triangle matrices \(4\) except \(a_{31}\) and \(d_{31}\) are real \(13\). Changing notations slightly we write the latter as
\[
a_{31} e^{i \phi_{(a)}} \quad \text{and} \quad d_{31} e^{i \phi_{(b)}},
\]
so that from now on all \(a_{ik}\) and all \(d_{ik}\) are real. The characteristic polynomial for the squared mass matrix \((T^{(v)} T^{(v)\dagger})\) yields the following relations for the parameters
\[
m_1^2 m_2^2 m_3^2 = a_{11}^2 a_{22}^2 a_{33}^2,
\]
\[
m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2 = a_{11}^2 a_{22} + a_{22}^2 a_{33}^2 + a_{33}^2 a_{11}^2 +
\]
\[
a_{21}^2 a_{33}^2 + a_{22}^2 a_{31}^2 + a_{31}^2 a_{21}^2 +
\]
\[
a_{21}^2 a_{33}^2 - 2a_{22} a_{31} a_{32} \cos \phi_{(a)},
\]
\[
m_1^2 + m_2^2 + m_3^2 = a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{21}^2 + a_{31}^2 + a_{32}^2.
\]
Similarly, the characteristic polynomial for \((T^{(l)} T^{(l)\dagger})\) yields
\[
m_e^2 m_{\mu}^2 m_{\tau}^2 = d_{11}^2 d_{22}^2 d_{33}^2,
\]
\[
m_e^2 m_{\mu}^2 + m_{\mu}^2 m_{\tau}^2 + m_{\tau}^2 m_e^2 = d_{11}^2 d_{22}^2 + d_{22}^2 d_{33}^2 + d_{33}^2 d_{11}^2 +
\]
\[
d_{11}^2 d_{32}^2 + d_{22}^2 d_{31}^2 + d_{33}^2 d_{21}^2 +
\]
\[
d_{21}^2 d_{32}^2 - 2d_{22} d_{31} d_{32} \cos \phi_{(l)},
\]
\[
m_e^2 + m_{\mu}^2 + m_{\tau}^2 = d_{11}^2 + d_{22}^2 + d_{33}^2 + d_{21}^2 + d_{31}^2 + d_{32}^2.
\]

We now make the following, physical assumption: the mass differences between different families are due to electroweak interactions only. This means that without these interactions the masses of the electron, muon and tauon would be degenerate, as would the masses of \(\nu_e\), \(\nu_\mu\) and \(\nu_\tau\). When the interaction is present, its effect within each generation will be the same, the observed splittings of the masses will be due to the off-diagonal, inter-family blocks in the mass terms \(3\), \(4\). For neutrinos it seems natural to assume further that their primordial masses vanish. In terms of the parameters in eq. \(4\) this means
\[
a_{11} = a_{22} = a_{33} = 0, \quad d_{11} = d_{22} = d_{33} = (m_e m_\mu m_\tau)^{1/3}.
\]
As an example consider the case of two families for which the matrices \(4\) simplify to
\[
T^{(v)} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \quad T^{(l)} = \begin{pmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{pmatrix}.
\]
These assumptions imply the following choice of the parameters \(13\):
(a) for the neutrino sector
\[
a_{11} = 0 \quad a_{22} = 0 \quad a_{21} = \pm m_2,
\]
which lead to the mass eigenvalues \(m_1 = 0\), \(m_2 \neq 0\), and
(b) for the charged lepton sector
\[
d_{11} = d_{22} = \sqrt{m_e m_\mu} \quad d_{21} = \pm (m_\mu - m_e).
\]
The mixing matrix is obtained in close analogy to the case of quark mixing. Let \( V^{(\nu)} \) and \( V^{(l)} \) be the unitary matrices which diagonalize the hermitean products \( T^{(\nu)}T^{(\nu)\dagger} \) and \( T^{(l)}T^{(l)\dagger} \), respectively, i.e. \( V (T^{\dagger}) V^{\dagger} = \text{diag} \left( m_1^2, m_2^2, m_3^2 \right) \) for \((\nu)\) and \((l)\). The mixing matrix that determines possible oscillations is then

\[
V_{\text{mix}} = V^{(\nu)} V^{(l)\dagger}.
\]  

(15)

In the case of two families it depends on one real angle only whose cosine is

\[
\cos \theta = \frac{\sqrt{m_1^2 + m_2^2} + \sqrt{m_2^2 m_\mu}}{(m_1 + m_2)(m_e + m_\mu)} = \sqrt{\frac{m_\mu}{m_e + m_\mu}}.
\]  

(16)

From this we obtain the result

\[
\sin^2 \theta = \frac{m_e}{m_e + m_\mu} = 0.0048.
\]  

(17)

Already this first estimate based on two families of leptons is in fair agreement with the value \(0.0048\) needed for solar resonance oscillation to occur. The mass eigenvalue of the second neutrino remains undetermined from our analysis. The empirical result \(0.0048\) requires it to be of the order of \(10^{-3} \text{ eV}\).

3. Turning now to the more realistic case of three families we first discuss the neutrino sector. Making use of the assumption (11) eq. (13) implies that at least one neutrino remains massless, say, \( m_1 = 0 \), while eqs. (13) - (15) simplify to

\[
m_2^2 m_3^2 = a_{21}^2 a_{32}^2, \quad m_2^2 + m_3^2 = a_{21}^2 + a_{31}^2 + a_{32}^2.
\]  

(18)

Without knowledge of the masses \( m_2 \) and \( m_3 \) from some other source there are still various possibilities of choosing \( a_{21} \), \( a_{31} \), and \( a_{32} \) (keeping in mind, of course, that they are not all independent). Among these there are two which are particularly interesting. These are

(A) \( a_{31} = 0 \), where, say, \( m_2^2 = a_{21}^2 \), \( m_3^2 = a_{32}^2 \),

(B) \( a_{21} = 0 \), where, say, \( m_2 = 0 \), \( m_3^2 = a_{31}^2 + a_{32}^2 \).

In both cases the matrix \( V^{(\nu)} \) is the unit matrix and, by eq. (15), the observable mixing matrix is determined solely by the charged leptons, \( V_{\text{mix}} = V^{(l)\dagger} \), with \( V^{(l)} \) expressed in terms of \( m_i \equiv m_e, m_\mu, m_\tau \) and of \( d_{ik} \) as worked out in [13]. Setting \( d_{11} = d_{22} = d_{33} \equiv d \) we find

\[
V_{\text{mix}} = \begin{pmatrix}
\frac{f(m_e)}{N_e} & \frac{f(m_\mu)}{N_\mu} & \frac{f(m_\tau)}{N_\tau} \\
\frac{g(m_e)}{N_e} & \frac{g(m_\mu)}{N_\mu} & \frac{g(m_\tau)}{N_\tau} \\
\frac{h(m_e)}{N_e} & \frac{h(m_\mu)}{N_\mu} & \frac{h(m_\tau)}{N_\tau}
\end{pmatrix},
\]  

(19)

where the functions \( f, g, \) and \( h \) are given by

\[
f(m_i) = d^2 d_{21} d_{32} - d d_{31} e^{-i\Phi(i)} (d^2 - m_i^2),
\]  

(20)

\[
g(m_i) = m_i^2 d_{21} d_{32} e^{-i\Phi(i)} - dd_{31} (d^2 - m_i^2),
\]  

(21)

\[
h(m_i) = (d^2 - m_i^2)^2 - d_{21}^2 m_i^2.
\]  

(22)

The normalization factors in the denominator are

\[
N_e = \left\{ \left[ (d^2 - m_e^2)^2 - m_e^2 d_{21}^2 \right] (m_\mu^2 - m_e^2) (m_\tau^2 - m_e^2) \right\}^{1/2},
\]  

(23)
$N_{\mu}$ and $N_{\tau}$ being obtained from this by cyclic permutation of \{\(m_\ell, m_{\mu}, m_{\tau}\)\}. Note that due to the equations (8) - (10) there are only two free parameters, say $d_{21}$ and $d_{32}$.

The form (19) of the mixing matrix and the formulae (24 - 23) are already quite restrictive, independently of the choice of $d_{21}$ and of $d_{32}$. This is seen as follows. Define the functions $F_i(z)$, $i = 1, 2, 3$, by

$$F_1(z) := \left\{ \frac{(d^2 - m_\mu^2)^2 - m_\mu^2 z^2}{(m_\mu^2 - m_\tau^2)(m_\mu^2 - m_\nu^2)} \right\}^{1/2},$$

(24)

with $F_2$ and $F_3$ obtained from (24) by cyclic permutation of \{\(m_\ell, m_{\mu}, m_{\tau}\)\}. Denoting the entries of $V_{mix}$, eq. (19), by $V_{ik}$, we see from eqs. (24) and (23) that the moduli of the elements in the third line are given by

$$|V_{3i}| = F_i(z = d_{21}), \ i = 1, 2, 3.$$  (25)

Note that because of the ordering of the lepton masses the denominator in $F_2$ is negative and, hence, its numerator must also be negative. From these equations one concludes

$$|m_\mu^2 - d^2|/m_\mu \leq |d_{21}| \leq (m_\mu^2 - d^2)/m_\tau.$$  (26)

When expressed in terms of the $\tau$-mass, this means that $|d_{21}|/m_{\tau}$ must lie between 0.0483 and 0.9993. These boundary values may in turn be inserted into the moduli to obtain limits on them, viz.

$$0.01006 \leq |V_{31}| \leq 0.01116, \ 0 \leq |V_{32}| \leq 0.99995, \ 0 \leq |V_{33}| \leq 0.99994.$$  (27)

Thus, while $|V_{32}|$ and $|V_{33}|$ remain practically unrestricted, $|V_{31}|$ must always be very small.

One can also work out limits on the moduli of the elements in the first line of (19) which turn out to be even more interesting. In calculating $|f(m_\ell)|^2$ from eq. (20), eliminating $d_{31}$ and $\cos \Phi$ by means of eqs. (4) and (10), and using $d^6 = m_\mu^2 m_\tau^2 m_\nu^2$, one obtains

$$|V_{1i}| = \left( \frac{d}{m_i} \right) F_i(z = d_{32}), \ (m_1 = m_\ell, m_2 = m_{\mu}, m_3 = m_{\tau}).$$  (28)

By the same reasoning as above one sees that $|d_{32}|$ must also lie in the interval (20). The expressions (28) are essentially the same as (23), except for the factors $(d/m_\ell) = 89.59$, $(d/m_\mu) = 0.4333$, $(d/m_{\tau}) = 0.02576$. Because these factors are so different from one another, the allowed intervals (27) for the third line, when applied to the first line, become

$$0.9013 \leq |V_{11}| \leq 0.9997, \ 0 \leq |V_{12}| \leq 0.4333, \ 0 \leq |V_{13}| \leq 0.02576.$$  (29)

Thus, while the narrow interval for the first element is shifted to somewhere close to 1, the very loose constraints on the second and third element of the third line are turned into rather tight limits for the corresponding elements of the first line.

These limits as obtained on the assumption that neutrinos have no primordial masses and $V^{(r)} = 1$ (so that $V_{mix}$ stems from the charged leptons only), could already be tested by experiment, either confirming compatibility with this simple scheme or disproving it. Clearly, a more detailed analysis will have to include a thorough numerical analysis of acceptable values for $d_{21}$ and $d_{32}$ and/or other choices of the neutrino mass matrix.

\footnote{We note in passing that $d_{21}$ and $d_{32}$ in fact are not completely independent. Using unitarity of $V_{mix}$, one concludes from the moduli of the elements in the second line that the combination $d_{21}^2 + (d^2/m_\mu^2)d_{12}^2$ is bounded from above by 0.9992$m_\mu^2$, while the combination $d_{31}^2 + (d^2/m_\tau^2)d_{13}^2$ is bounded from below by 0.00287$m_\mu^2$. These bounds are so mild that they will not play a decisive role.}
4. As an illustration of this general framework we briefly describe the mass matrices as obtained in the framework of the Mainz-Marseille model (cf. introduction). In this model the left-chiral doublet and the right-chiral singlet of a given lepton family fit into the fundamental representation

\[ [Y_0 = -1, I_0 = \frac{1}{2}] \rightarrow (I = \frac{1}{2}, Y = -1) \oplus (I = 0, Y = -2), \]

where the right-hand side gives the decomposition in terms of Lie \((SU(2) \times U(1))\). A right-handed neutrino would be classified according to the trivial representation \([Y_0 = 0, I_0 = 0] \equiv (I = 0, Y = 0)\). As discussed in [10] the two representations can be combined to a reducible but indecomposable representation

\[ \rho_i := [Y_0 = -1, I_0 = \frac{1}{2}] \equiv [Y_0 = 0, I_0 = 0], \]

which describes a charged Dirac lepton \(l\) and its neutrino \(\nu^{(i)}_L\), along with a right neutrino \(\nu^{(i)}_R\) which does not couple directly to the gauge bosons but does couple to the leptonic multiplet \((l_L, l_R, \nu_L)\) via the Higgs doublet. The next simplest, and physically interesting possibility is to classify two or three lepton families in the reducible indecomposable representations which are obtained by taking semi-direct sums of the representation \(\rho_i\), eq. (31), according to the pattern shown in eq. (3). While (31), in a natural way, allows to introduce a (Dirac) mass of the neutrino \(\nu_i\), without introducing right-handed interactions, the representation (3) provides in addition family mixing both in the charged and in the neutral lepton sector. Indeed, in a representation with three families the even and odd generators of \(su(2 | 1)\) have the form, respectively,

\[
\hat{T}_i = \begin{pmatrix} (T_i)^{11}_{11} & 0 & 0 \\
(T_i)^{21}_{21} & (T_i)^{22}_{22} & 0 \\
(T_i)^{31}_{31} & (T_i)^{32}_{32} & (T_i)^{33}_{33} \end{pmatrix}, \quad \hat{\Omega}_i = \begin{pmatrix} (\Omega_i)^{11}_{11} & 0 & 0 \\
(\Omega_i)^{21}_{21} & (\Omega_i)^{22}_{22} & 0 \\
(\Omega_i)^{31}_{31} & (\Omega_i)^{32}_{32} & (\Omega_i)^{33}_{33} \end{pmatrix}. \tag{32}
\]

The entries of these block triangular matrices are \(4 \times 4\)-matrices. In the basis

\[ \Psi = \left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right)^T, \text{ with } \psi^{(i)} = \left(l^{(i)}_L, l^{(i)}_R, \nu^{(i)}_L, \nu^{(i)}_R\right)^T, \tag{33} \]

and using Pauli matrices \(\tau_i\) and projection operators \(P_{\pm} = (1 \pm \tau_3)/2\), the odd generators must have the form

\[
(\Omega_+)^{ik} = \begin{pmatrix} 0 & a_{ik} \gamma P_- \\
\beta_{ik} \gamma P_+ & 0 \end{pmatrix}, \quad (\Omega_-)^{ik} = \begin{pmatrix} 0 & d_{ik} \gamma P_+ \\
\gamma_{ik} \gamma P_- & 0 \end{pmatrix}, \tag{34}
\]

\[
(\Omega_+)^{ik} = \begin{pmatrix} 0 & b_{ik} \gamma P_+ \\
\beta_{ik} \gamma P_- & 0 \end{pmatrix}, \quad (\Omega_-)^{ik} = \begin{pmatrix} 0 & c_{ik} \gamma P_- \\
\gamma_{ik} \gamma P_+ & 0 \end{pmatrix}. \tag{35}
\]

the bold face \(0\) denoting the \(2 \times 2\) zero matrix. As we showed in [13] it is no restriction of generality to choose the even generators block diagonal, i.e.

\[ (I_3)^{ik} = 0 = (Y)^{ik}, \text{ for } (ik) = (21), (31), (32). \tag{36} \]

In the basis chosen the diagonal blocks are

\[ (I_3)^{ii} = \text{diag} \left(-\frac{1}{2}, \frac{1}{2}, 0, 0\right), \quad \sqrt{2}(I_3)^{ii} = \begin{pmatrix} \tau & 0 \\
0 & 0 \end{pmatrix}, \quad (Y)^{ii} = \text{diag} \left(-1, -1, -2, 0\right). \]
The interaction of the lepton state (33) with the gauge bosons of $SU(2) \times U(1)$ and the Higgs field is encoded in the superconnection of the model,

$$A = i \left\{ a \bar{\mathbf{I}} \cdot \mathbf{W} + \frac{1}{2} b \bar{\mathbf{Y}} W^{(8)} + \frac{1}{\mu} \left[ \mathbf{\hat{G}}_+ \phi^{(0)} + \mathbf{\hat{G}}_+ \phi^{(+)} + \mathbf{\hat{G}}_- \phi^{(0)} + \mathbf{\hat{G}}_- \phi^{(+)} \right] \right\},$$  \hspace{1cm} (37)

with $a, b$ real, dimensionless parameters, $\mu$ the mass scale of the theory. In particular, and as explained in ref. \[13\], the mass terms which follow from eq. (37) are

$$L_{\text{mass}} = \frac{1}{2} \left[ (\bar{\mathbf{\Psi}} \mathbf{\hat{G}}_+ \Psi)_\nu + (\bar{\mathbf{\Psi}} \mathbf{\hat{G}}_- \Psi) \right] + \text{h.c.},$$  \hspace{1cm} (38)

where the indices indicate that neutrinos only get contributions from $\mathbf{\hat{G}}_+$, the charged partners get contributions only from $\mathbf{\hat{G}}_-$. Thus, in this model $L_{\text{mass}}$ has indeed the general form of eqs. (3) and (4) above.

The graded commutators of the even and odd generators of the algebra yield relations between the complex parameters $a_{ik} \ldots \gamma_{ik}$. These read

$$b_{ik} = a_{ik}, \quad \beta_{ik} = -a_{ik}, \quad c_{ik} = -d_{ik}, \quad \gamma_{ik} = \delta_{ik}, \quad i, k = 1, 2, 3,$$  \hspace{1cm} (39)

$$a_{ii} \delta_{ii} = 0, \quad i, k = 1, 2, 3,$$  \hspace{1cm} (40)

$$d_{ii} \alpha_{ii} = -1, \quad i, k = 1, 2, 3,$$  \hspace{1cm} (41)

$$a_{21} \delta_{11} + a_{22} \delta_{21} = 0, \quad a_{32} \delta_{22} + a_{33} \delta_{32} = 0, \quad \sum_{i=1}^{3} a_{3i} \delta_{i1} = 0,$$  \hspace{1cm} (42)

$$\delta_{21} a_{11} + \delta_{22} a_{21} = 0, \quad \delta_{32} a_{22} + \delta_{33} a_{32} = 0, \quad \sum_{i=1}^{3} \delta_{3i} a_{i1} = 0,$$  \hspace{1cm} (43)

$$d_{21} a_{11} + d_{22} a_{21} = 0, \quad d_{32} a_{22} + d_{33} a_{32} = 0, \quad \sum_{i=1}^{3} d_{3i} a_{i1} = 0.$$  \hspace{1cm} (44)

We note, in particular, that the right-hand sides of eqs. (40) and (41) are the electric charges of the neutrinos and their charged partners, respectively. The former are important in discussing the remaining freedom in choosing the parameters $a_{ik}$ and give an interesting interpretation of the cases (A) and (B) discussed above. Indeed, the assumption $a_{ii} = 0$ complies with the condition (40) which says that either $a_{ii}$ or $\delta_{ii}$ have to vanish because neutrinos have no electric charge. Eqs. (42) and (43) then reduce to, respectively,

$$a_{21} \delta_{11} = 0, \quad a_{32} \delta_{22} = 0, \quad a_{31} \delta_{11} + a_{32} \delta_{21} = 0,$$  \hspace{1cm} (45)

$$\delta_{22} a_{21} = 0, \quad \delta_{33} a_{32} = 0, \quad \delta_{32} a_{21} + \delta_{33} a_{31} = 0.$$  \hspace{1cm} (46)

Barring the trivial case where all $a_{ik}$ are zero, eqs. (45 - 46) admit the following choices

(i) $a_{21} \neq 0, a_{32} \neq 0$: In this case all $\delta_{ij}$ except $\delta_{31}$ must vanish. Two neutrinos are massive, one remains massless. Depending on whether $a_{31} = 0$ the neutrino mass matrix is diagonal and the mixing matrix (15) is determined by the mass matrix of charged leptons only.

\[\text{Note that the analogs of eqs. (45 - 46) with } \delta_{ik} \text{ replaced by } \alpha_{ik}, \text{ and } a_{mn} \text{ by } d_{mn}, \text{ exist but, by (44), are linearly dependent on eqs. (44).}\]
(ii) $a_{21} \neq 0, a_{32} = 0$: In this case $\delta_{11}$ and $\delta_{22}$ must vanish, $\delta_{21}$ and $\delta_{31}$ remain undetermined, while $\delta_{32}$ and $\delta_{33}$ are related by

$$\delta_{32} = -\frac{a_{31}}{a_{21}} \delta_{33}.$$ 

Two neutrinos are massive and mix, as long as $a_{31} \neq 0$. If $a_{31} = 0$ one more neutrino becomes massless and, obviously, there is no mixing.

(iii) $a_{21} = 0, a_{32} \neq 0$: Here $\delta_{22} = \delta_{33} = 0$,

$$\delta_{21} = -\frac{a_{31}}{a_{32}} \delta_{11},$$

while $\delta_{32}$ and $\delta_{31}$ remain undetermined. Two neutrinos are massless, the mass matrix is diagonal. The mixing matrix \( \mathbf{T}(\nu) \) is determined by the charged sector.

(iv) $a_{21} = a_{32} = 0$, but $a_{31} \neq 0$: Here $\delta_{11} = \delta_{33} = 0$, all others remain undetermined. Two neutrinos are massless, the mass matrix is diagonal.

Note that cases (ii) – (iv) are not simply limiting cases of case (i) because in the latter all $\delta_{ik}$ but $\delta_{11}$ must vanish whereas in cases (ii) to (iv) some of these parameters may be different from zero. As we noted in sec. 3 the physically interesting cases are (A) and (B) which are contained in (i) and (iii).

5. As CC weak interactions involve left-chiral fields only, triangular mass matrices of the type \( \mathbf{T}(\nu) \) for neutrinos and their charged partners provide the most general, yet concise, description of leptonic state mixing. It seems plausible that in the absence of electroweak interactions, leptons are mass-degenerate in each charge sector and that neutrinos possess no primordial masses. These assumptions are equivalent to the choice \( \mathbf{T}(\nu) \) for the diagonal entries of \( \mathbf{T}(\nu) \) and \( \mathbf{T}(l) \). The characteristic polynomial for the mass matrix \( \mathbf{T}(\nu) \mathbf{T}(\nu)^\dagger \) allows to classify the possible choices for the remaining parameters $a_{21}$, $a_{31}$, and $a_{32}$. Among these the choices (A) and (B) are particularly interesting because in these cases the neutrino mass matrix is diagonal and, therefore, the mixing matrix is determined by the mass matrix of charged leptons only. If the atmospheric neutrino experiment or the short-baseline oscillation experiment is confirmed, then case (B) is excluded because one needs (at least) two nonvanishing mass differences. Note that even though, by our assumption of vanishing primordial neutrino masses \( \mathbf{T}(\nu) \), one neutrino stays massless there is nevertheless mixing between all three families. This is due to the fact that the electron has a finite mass and, in the charged lepton’s mass matrix, does not decouple from the other two families.

The example of two families where all parameters are fixed by the masses, shows that mixing is compatible with the specific resonance oscillation needed to explain the solar neutrino fluxes, cf. eq. \( \mathbf{T}(\nu) \). With three families there is more, but not much more, flexibility. In the case where the mass matrix of neutrinos is such that \( \mathbf{V}(\nu) = \mathbb{I} \), i.e. where $\mathbf{V}_{mix}$ depends on the charged lepton sector only, there are two free parameters bounded by the interval \( \mathbf{[24]} \) (and some mild compatibility condition among them). However, the scheme is so tight that even without fixing these parameters, the moduli of the elements in the third and first lines of $\mathbf{V}_{mix}$ must lie in the intervals \( \mathbf{[27]} \) and \( \mathbf{[29]} \), respectively.

In a more general situation where \( \mathbf{V}(\nu) \) is not proportional to the unit matrix \( \mathbb{I} \) the mixing matrix \( \mathbf{T}(\nu) \) will also depend on the neutrino masses. We hope to return to a more detailed numerical analysis in a future publication.

Finally, we illustrate this general analysis by a model that is an extension of the standard model within noncommutative geometry. This model sheds new light on the classification of
parameter choices by relating them to graded commutators of generators of the underlying graded Lie algebra, and, thereby, establishes a link to lepton quantum numbers.

F. Scheck is grateful for the kind hospitality extended to him by Harold Fearing and the theory group at TRIUMF, Vancouver, where part of this work was done.
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