Universal abelian covers of rational surface singularities and multi-index filtrations

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In [1] and [2], there were computed the Poincaré series of some (multi-index) filtrations on the ring of germs of functions on a rational surface singularity. These Poincaré series were written as the integer parts of certain fractional power series, an interpretation of whom was not given. Here we show that, up to a simple change of variables, these fractional power series are specializations of the equivariant Poincaré series for filtrations on the ring \( O_{\tilde{S},0} \) of germs of functions on the universal abelian cover \((\tilde{S},0)\) of the surface \((S,0)\). We compute these equivariant Poincaré series. From another point of view universal abelian covers of rational surface singularities were studied in [6].

Let \((S,0)\) be an isolated complex rational surface singularity and let \(\pi : (X, D) \to (S,0)\) be a resolution of it (not necessarily the minimal one). Here \(X\) is a smooth complex surface, the exceptional divisor \(D = \pi^{-1}(0)\) is a normal crossing divisor on \(X\), all components \(E_\sigma (\sigma \in \Gamma)\) of the exceptional divisor \(D\) are isomorphic to the complex projective line \(\mathbb{C}\mathbb{P}^1\) and the dual graph of the resolution is a tree.

Let \(O_{S,0}\) be the ring of germs of analytic functions on \((S,0)\). For \(\sigma \in \Gamma\), i.e. for a component \(E_\sigma\) of the exceptional divisor, and for \(f \in O_{S,0}\), let \(v_\sigma(f)\) be the order of zero of the lifting \(f \circ \pi\) of the function \(f\) to the space \(X\) of the resolution along the component \(E_\sigma\). Let us choose several components \(E_1, \ldots, E_s\) of the exceptional divisor \(D\) \((\{1, \ldots, s\} \subset \Gamma)\). The valuations \(v_1, \ldots, v_s\) define a multi-index filtration \(\{J(\underline{v})\}\) on the ring \(O_{S,0}\): for \(\underline{v} = (v_1, \ldots, v_s) \in \mathbb{Z}_{\geq 0}^s\), \(J(\underline{v}) = \{f \in O_{S,0} : \underline{v}(f) \geq \underline{v}\}\) (here \(\underline{v}(f) = (v_1(f), \ldots, v_s(f)) \in \mathbb{Z}_{\geq 0}^s\), \(\underline{v}' \geq \underline{v}\).
if and only if \( v'_i \geq v_i \) for all \( i = 1, \ldots, s \). In [1], there was computed the Poincaré series \( P(t_1, \ldots, t_s) \) of this filtration (the definition of the Poincaré series of a multi-index filtration can be found e.g. in [1 2 3]). Let \((E_\sigma \circ E_\delta)\) be the intersection matrix of the components of the exceptional divisor. For \( \sigma \neq \delta \), the intersection number \( E_\sigma \circ E_\delta \) is equal to 1 if the components \( E_\sigma \) and \( E_\delta \) intersect (at one point) and is equal to zero if they don’t intersect; the self-intersection number \( E_\sigma \circ E_\sigma \) of each component \( E_\sigma \) is a negative integer.

Let \( d = \det(-(E_\sigma \circ E_\delta)) \) and let \((m_{\sigma \delta}) = -(E_\sigma \circ E_\delta)^{-1} \). All entries \( m_{\sigma \delta} \) are positive and \( m_{\sigma \delta} \in (1/d)\mathbb{Z} \). For \( \sigma \in \Gamma \), let \((m_{\sigma}) = (m_{\sigma 1}, \ldots, m_{\sigma s}) \in \mathbb{Q}_{\geq 0}^s \).

Let \( dE_\sigma \) be the “smooth part” of the component \( E_\sigma \) in the exceptional divisor \( D \), i.e., \( E_\sigma \) minus intersection points with all other components of the exceptional divisor \( D \).

For a fractional power series \( S(t_1, \ldots, t_s) \in \mathbb{Z}[[t_1^{1/d}, \ldots, t_s^{1/d}]] \), let \( \text{Int} S(t_1, \ldots, t_s) \) be its “integer part”, i.e., the sum of all monomials from \( S(t_1, \ldots, t_s) \) with integer exponents. In [1] it was shown that

\[
P(t_1, \ldots, t_s) = \text{Int} \prod_{\sigma \in \Gamma} (1 - t_1^{m_{\sigma 1}} \cdots t_s^{m_{\sigma s}})^{-\chi^{\cdot\dagger}(E_\sigma)},
\]

where \( t_1^{m_{\sigma 1}} \cdots t_s^{m_{\sigma s}} \) and \( \chi(X) \) is the Euler characteristic of the space \( X \).

A similar formula was obtained in [2] for the Poincaré series of the multi-index filtration on the ring \( \mathcal{O}_{\mathcal{S}, 0} \) defined by orders of a function germ on irreducible components of a curve \((C, 0) \subset (\mathcal{S}, 0)\).

In [1], the fractional power series

\[
Q(t) = \prod_{\sigma \in \Gamma} (1 - t_1^{m_{\sigma 1}} \cdots t_s^{m_{\sigma s}})^{-\chi^{\cdot\dagger}(E_\sigma)}
\]

(and a similar one in [2]) participated as a formal expression convenient to write the formula (1) for the Poincaré series \( P(t_1, \ldots, t_s) \). There was no interpretation of it.

In [3], there was defined an equivariant Poincaré series for an “equivariant” filtration on the ring \( \mathcal{O}_{V, 0} \) of germs of functions on a germ \((V, 0)\) of a complex analytic variety with an action of a finite group \( G \). This Poincaré series was computed for a divisorial filtration on the ring \( \mathcal{O}_{C^2, 0} \) and for the filtration defined by branches of a \( G \)-invariant plane curve singularity \((C, 0) \subset (C^2, 0)\) were the plane \( C^2 \) was equipped with a \( G \)-action.

Let \( p : (\tilde{\mathcal{S}}, 0) \to (\mathcal{S}, 0) \) be the universal abelian cover of the surface singularity \((\mathcal{S}, 0)\); see e.g. [5 6]. One can describe it in the following way. Let \( G = H_1(\mathcal{S} \setminus \{0\}) \) be the first homology group of the (nonsingular) surface \( \mathcal{S} \setminus \{0\} \). The order of the group \( G \) is equal to the determinant \( d \) of the minus
intersection matrix $-(E_\sigma \circ E_\delta)$ and moreover $G$ is the cokernel $\mathbb{Z}^\Gamma / \text{Im } I$ of the map $I : \mathbb{Z}^\Gamma \to \mathbb{Z}^\Gamma$ defined by this matrix.

The group $G$ acts on the germ $(\bar{S}, 0)$ and the restriction $p|_{\bar{S} \setminus \{0\}}$ of the map $p$ to the complement of the origin is a (usual, nonramified) covering $\bar{S} \setminus \{0\} \to S \setminus \{0\}$ with the structure group $G$. One can lift the map $p$ to a (ramified) covering $\bar{p} : (\bar{X}, \bar{D}) \to (X, D)$ where $\bar{X}$ is a normal surface (generally speaking not smooth) and $\bar{X} \setminus \bar{D} \cong \bar{S} \setminus \{0\}$:

\[
(\bar{X}, \bar{D}) \xrightarrow{\bar{p}} (\bar{S}, 0) \\
\downarrow p \quad \downarrow p \\
(X, D) \xrightarrow{p} (S, 0)
\]

(one can define $\bar{X}$ as the normalization of the fibre product $X \times_S \bar{S}$ of the varieties $X$ and $\bar{S}$ over $S$).

Let $g_\sigma, \sigma \in \Gamma$ be the element of the group $G$ represented by the loop in $X \setminus D$ going around the component $E_\sigma$ in the positive direction. The group $G$ is generated by the elements $g_\sigma$ for all $\sigma \in \Gamma$. For a point $x \in E_\sigma$ and for a point $\bar{x}$ from the preimage $p^{-1}(x)$ of it, locally, in a neighbourhood of the point $\bar{x}$, the map $p : \bar{D} \to D$ is an isomorphism and the map $p : \bar{X} \to X$ is a ramified (over $D$) covering, the order $d_\sigma$ of which coincides with the order of the generator $g_\sigma$ of the group $G$.

**Lemma 1** The order $d_\sigma$ of the element $g_\sigma \in G$ is the minimal natural $k$ such that $km_\sigma$ is an integer for all $\delta \in \Gamma$.

**Proof.** This follows immediately from the fact that $\mathbb{Z}^\Gamma / \text{Im } I \cong \text{Im } m / \mathbb{Z}^\Gamma$ where $m : \mathbb{Z}^\Gamma \to \mathbb{Q}^\Gamma$ is the map given by the matrix $(m_{\sigma\delta})$ (i.e. minus the inverse of the map $I$). □

Let $R(G)$ be the ring of (virtual) representations of the group $G$. For $\sigma \in \Gamma$, let $\alpha_\sigma$ be the one-dimensional representation $G \to \mathbb{C}^* = \mathbf{GL}(1, \mathbb{C})$ of the group $G$ defined by $\alpha_\sigma(g) = \exp(-2\pi \sqrt{-1} m_{\sigma\delta})$ (here the minus sign reflects the fact that the action of an element $g \in G$ on the ring $\mathcal{O}_{\bar{S}, 0}$ is defined by $(g : f)(x) = f(g^{-1}(x)))$.

Let us choose any component $\bar{E}_i$ of the preimage $p^{-1}(E_i)$ of the component $E_i$ and let $\bar{v}_i$ be the corresponding divisorial valuation on the ring $\mathcal{O}_{\bar{S}, 0}$. On the space $\bigcup_{\alpha} \mathcal{O}_{\bar{S}, 0}^\alpha$ of all $G$-equivariant functions on $(\bar{S}, 0)$ ($\alpha$ runs over all nonequivalent 1-dimensional representations of the group $G$) the valuation $\bar{v}_i$ does not depend on the choice of the component $\bar{E}_i$.

In [3], there was defined the equivariant Poincaré series of the multi-index filtration defined by the divisorial valuations $\bar{v}_1, \ldots, \bar{v}_s$. 

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\]
Theorem 1 The equivariant Poincaré series \( P^G(t_1, \ldots, t_s) \) of the \( s \)-index filtration defined by the set of divisorial valuations \( \{ \tilde{v}_1, \ldots, \tilde{v}_s \} \) is given by the formula:

\[
P^G(t_1, \ldots, t_s) = \prod_{\sigma \in \Gamma} (1 - \alpha_\sigma t_1^{d_{1\sigma}} \cdots t_s^{d_{s\sigma}})^{-e(E_\sigma)}.
\]

For a power series \( S(t_1, \ldots, t_s) = \sum_{v \in \mathbb{Z}^s_{\geq 0}} s_v t^v \in R(G)[[t_1, \ldots, t_s]] \) \( (R(G) \) is the ring of representations of the group \( G \)), let its reduction \( \text{red} S(t_1, \ldots, t_s) \) be the series \( \sum_{v \in \mathbb{Z}^s_{\geq 0}} (\dim s_v) t^v \in \mathbb{Z}[t_1, \ldots, t_s] \).

Corollary. One has \( \text{red} P^G(t_1, \ldots, t_s) = Q(t_1^{d_1}, \ldots, t_s^{d_s}) \), where \( Q(t) \) is the fractional power series defined by (2).

Proof of Theorem 1 For short we shall say that an effective divisor on \( \tilde{D} = \bigcup E_\sigma \) (or on \( \tilde{D} = p^{-1}(\tilde{D}) \)) is Cartier if it is the intersection with \( \tilde{D} \) (or with \( \tilde{D} \)) of the strict transform of a Cartier divisor on \( (S, 0) \) (or on \( (\tilde{S}, 0) \)). From [3] it follows that the equivariant Poincaré series \( P^G(t) \) is equal to the integral with respect to the Euler characteristic of the monomial \( \alpha t^\omega \) over the space of \( G \)-invariant effective Cartier divisors on \( \tilde{D} \). Here \( \alpha \in R(G) \) and \( \tilde{\omega} \in \mathbb{Z}^s_{\geq 0} \) are functions (in fact semigroup homomorphisms) on the space of \( G \)-invariant Cartier divisors on \( \tilde{D} \): a \( G \)-invariant Cartier divisor defines the orders of zero of the corresponding \( (G\text{-equivariant}) \) function along the components \( E_i \) and also the corresponding 1-dimensional representation of the group \( G \).

Thus to compute the equivariant Poincaré series \( P^G(t) \) one has to describe the space of \( G \)-invariant effective Cartier divisors on \( \tilde{D} \) and the corresponding functions \( \omega \) and \( \alpha \) on it.

Lemma 2 Any \( G \)-invariant effective divisor on \( \tilde{D} \) is a Cartier divisor.

Proof. It is sufficient to show this for the divisor \( \sum_{\tilde{x} \in p^{-1}(x)} \tilde{x} \) for a point \( x \in E_\sigma \), i.e. for the \( G \)-orbit of a point from \( \tilde{E}_\sigma \). The isotropy group \( G_{\tilde{x}} \) of a point \( \tilde{x} \in p^{-1}(x) \) is the cyclic subgroup of the group \( G \) of order \( d_\sigma \) generated by the element \( g_\sigma \) (this element acts trivially on \( p^{-1}(E_\sigma) \)).
Let us take the germ at the point $x$ of a smooth curve $L_\sigma$ on $(X, D)$ transversal to the exceptional divisor $D$. By the Artin criterion (see, e.g., [7], Lemma on page 156), the divisor $d \cdot L_\sigma$ is the strict transform of a Cartier divisor on $(S, 0)$ (in fact already $d_\sigma L_\sigma$ is one with this property), i.e. there exists a function $f_\sigma : S \to \mathbb{C}$ such that the strict transform of the divisor $\{ f_\sigma = 0 \}$ is $d \cdot L_\sigma$. Let $\tilde{f}_\sigma = f_\sigma \circ \pi$ be the lifting of the function $f_\sigma$ to the space $X$ of the resolution and let $\tilde{\alpha}_\sigma = f_\sigma \circ \pi \circ p$ be the lifting of the function $f_\sigma$ to the space $\tilde{X}$ of the modification of the universal abelian cover $(\tilde{S}, 0)$ ($\tilde{f}_\sigma$ is a $G$-invariant function on $\tilde{X}$). Let us describe the divisor $\{ \tilde{f}_\sigma = 0 \}$. Let $\tilde{L}_{\sigma, \tilde{x}}$ be the germ at the point $\tilde{x} \in p^{-1}(x)$ of the preimage under the map $p$ of the curve $L_\sigma \subset X$.

The order of zero of the function $\tilde{f}_\sigma$ along $\tilde{L}_{\sigma, \tilde{x}}$ is equal to $d$. The order of zero of the function $\tilde{f}_\sigma$ along the component $\tilde{E}_\delta$ is equal to $d \cdot m_\delta$. The ramification order of the map $p$ over the component $\tilde{E}_\delta$ is equal to $d_\delta$. Therefore the order of zero of the function $\tilde{f}_\sigma = \tilde{f}_\sigma \circ p$ along the preimage of the component $\tilde{E}_\delta$ is equal to $d \cdot d_\delta \cdot m_\delta$. This (integer) number is divisible by $d$ (since $d_\delta m_\delta$ is an integer: see Lemma [1]). Therefore the zero divisor of the function $\tilde{f}_\sigma$ is divisible by $d$, i.e. the order of zero of this function along each component of its zero set is divisible by $d$. This means that a root $\sqrt[\alpha]{\tilde{f}_\sigma}$ of degree $d$ of the function $\tilde{f}_\sigma$ (i.e. a branch of this root) is well defined up to multiplication by a root of degree $d$ of a $G$-equivariant complex analytic function on $\tilde{X}$ and thus it is the lifting of a $G$-equivariant function on $(\tilde{S}, 0)$ (see e.g. [4, page ?]). □

**Corollary.** Each $G$-invariant divisor on the universal abelian cover $(\tilde{S}, 0)$ of the rational surface singularity $(S, 0)$ is a Cartier one.

Lemma [2] means that the space of $G$-invariant effective Cartier divisors on $\tilde{D}$ is in one to one correspondence with the space of all effective divisors on $\tilde{D}$. As it follows from the proof of Lemma [2] the order of zero of the $G$-equivariant function $\tilde{f}_\sigma$ (corresponding to one point $x \in \tilde{E}_\sigma$) along the component $\tilde{E}_\delta$ is equal to $d_\delta m_\sigma$. One has to find the (one-dimensional) representation $\alpha_\sigma$ with respect to which the function $\tilde{f}_\sigma$ is $G$-equivariant.

**Lemma 3**

$$\alpha_\sigma(g_\delta) = \exp(-2\pi \sqrt{-1} m_\sigma) .$$

**Proof.** The element $g_\delta$ of the group $G$ acts trivially on the preimage $p^{-1}(\tilde{E}_\delta)$ of the component $\tilde{E}_\delta$ of the exceptional divisor and acts by multiplication by
\[
\exp\left(\frac{2\pi}{d_\delta} \sqrt{-1}\right) \text{ on the normal line to it. The order of zero of the function } \tilde{f}_\sigma \text{ along the preimage } p^{-1}(E_\delta) \text{ is equal to } m_{\sigma_\delta}d_\delta. \text{ Therefore }
\]
\[
g_\delta \cdot f_\sigma = \exp\left(-\frac{2\pi \sqrt{-1} m_{\sigma_\delta}d_\delta}{d_\delta}\right) = \exp(-2\pi \sqrt{-1} m_{\sigma_\delta}).
\]
\[\Box\]

Now Theorem follows from the usual arguments used e.g. in [1, 3]. The space of effective divisors on \(\bullet E_\delta\) is the direct product of the spaces of effective divisors on the components \(E_\sigma\). Each of the latter ones is the disjoint union of symmetric powers \(S^k E_\sigma\) of the component \(E_\sigma\). Therefore
\[
P^G(t_1, \ldots, t_s) = \prod_{\sigma \in \Gamma} \left( \sum_{k=0}^{\infty} \chi(S^k E_\sigma) \cdot \alpha_\sigma^k t^k \right),
\]
(this follows from the fact that \(\nu\) and \(\alpha\) are semigroup homomorphisms). The well-known formula
\[
\sum_{k=0}^{\infty} \chi(S^k X) t^k = (1 - t)^{-\chi(X)}
\]
implies the equation (3). \[\Box\]

A similar result holds for the filtration on the ring \(\mathcal{O}_{\tilde{S},0}\) defined by orders of a function germ on branches of a \(G\)-invariant curve \((\tilde{C},0) \subset (\tilde{S},0)\). Let \(\tilde{C} = \bigcup_{i=1}^{r} \tilde{C}_i\) where \(\tilde{C}_i\) are irreducible \(G\)-invariant components of the curve \(\tilde{C}\) (generally speaking each curve \(\tilde{C}_i\) consists of several irreducible components permutated by the group \(G\)). Each curve \(\tilde{C}_i\) is the preimage under the map \(p\) of an irreducible curve \(C_i\) on \((S,0)\). The curve \(\tilde{C} = \bigcup_{i=1}^{r} \tilde{C}_i\) defines an \(r\)-index filtration on the space \(\bigcup_{\alpha} \mathcal{O}_{\tilde{S},0}^\alpha\) of \(G\)-equivariant functions on the surface \((\tilde{S},0)\) (or on the space \(\bigcup_{\alpha} \mathcal{O}_{\tilde{C},0}^\alpha\) of \(G\)-equivariant functions on the cuvre \((\tilde{C},0))\). Let \(\varphi_i : (C,0) \to (\tilde{S},0)\) be a parametrization (uniformization) of an irreducible component of the curve \(\tilde{C}_i\). For a \(G\)-equivariant function germ \(f\), let \(\tilde{w}_i(f)\) be the order of zero of the function \(f \circ \varphi_i\) at the origin: \(f \circ \varphi_i(\tau) = a\tau^{\tilde{w}_i(f)} + \text{ terms of higher degree, } a \neq 0\). The valuations \(\tilde{w}_1, \ldots, \tilde{w}_r\) define a multi-index filtration in the usual way.

Let \(\pi : (X,\mathcal{D}) \to (S,0)\) be a resolution of the surface singularity \((S,0)\) which at the same time is an embedded resolution of the curve \((C,0) \subset (S,0)\),
\( C = \bigcup_{i=1}^{r} C_i \). Let \( \overline{C}_i \) be the strict transform of the curve \( C_i \) in \( X \). Let \( E_1, \ldots, E_s \) be all the components of the exceptional divisor \( D \) of the resolution. Let \( \hat{E}_i \) be the “smooth part” of the component \( E_i \) in the total transform \( \pi^{-1}(C) \) of the curve \( C \), i.e. \( E_i \) minus intersection points with all other components of the total transform \( \pi^{-1}(C) \). Let \( m_i = (m_{i1}, \ldots, m_{is}) \in \mathbb{Q}^s_{\geq 0} \), \( d = (d_1, \ldots, d_s) \in \mathbb{Z}_{\geq 0}^s \), and a 1-dimensional representation \( \alpha \) of the group \( G \) \((i = 1, \ldots, s)\) be defined as above. The same arguments as in the proof of Theorem 1 imply the following statement.

**Theorem 2** The equivariant Poincaré series \( P^G(t_1, \ldots, t_r) \) of the \( r \)-index filtration defined by the set of valuations \( \{\tilde{w}_1, \ldots, \tilde{w}_r\} \) is given by the formula:

\[
P^G(t_1, \ldots, t_r) = \left( \prod_{i=1}^{s} \left(1 - \alpha_i T^{d_i} m_i \right)^{-\chi(\hat{E}_i)} \right) \bigg|_{T_i \mapsto \prod_{j \in \emptyset} t_j} \prod_{j \in \emptyset} t_j
\]

(here \( T = (T_1, \ldots, T_s) \); in the substitution above, \( \prod_{j \in \emptyset} t_j \) is supposed to be equal to 1).

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