STRETCHING SYMMETRY AND HAGLUND’S CONJECTURE FOR MACDONALD POLYNOMIALS

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ABSTRACT. We provide new approaches to prove identities for the modified Macdonald polynomials: using the LLT expansion of the modified Macdonald polynomials and using the Garsia–Haiman module. We prove a stretching symmetry for the modified Macdonald polynomials of one-column shape. We also prove a conjecture of Haglund concerning multi-$t$-Macdonald polynomials of two rows.

1. INTRODUCTION

In his seminal paper, Macdonald introduced Macdonald $P$-polynomials $P_{\mu}[X; q, t]$ which are $q, t$-extension of Schur functions indexed by partitions $\mu$ [Mac88]. The modified Macdonald polynomials $\tilde{H}_{\mu}[X; q, t]$ are introduced as a combinatorial version of the Macdonald $P$-polynomials. The modified Macdonald polynomials are defined by the unique family of symmetric functions satisfying the following triangulation and normalization axioms [HHL05]:

1. $\tilde{H}_{\lambda}[X(1-q); q, t] = \sum_{\lambda \geq \mu} a_{\lambda, \mu}(q, t) s_{\mu}(X),$
2. $\tilde{H}_{\lambda}[X(1-t); q, t] = \sum_{\lambda \geq \mu'} b_{\lambda, \mu'}(q, t) s_{\mu}(X),$ and
3. $\langle \tilde{H}_{\mu}, s_{(n)(X)} \rangle = 1,$

for suitable coefficients $a_{\lambda, \mu}, b_{\lambda, \mu'} \in \mathbb{Q}(q, t),$ where $\mu'$ denotes the conjugate partition of $\mu$ and $s_{\mu}(X)$ denotes the Schur function. Here, a partial order $\preceq$ is called dominance order of partitions of $n$ which is defined by

$\lambda \preceq \mu$ if $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for all $k,$

$[-]$ denotes the plethystic substitution, and $\langle -, - \rangle$ denotes the Hall inner product. These axioms are equivalent to Macdonald’s triangularity and orthogonality axioms.

The $(q, t)$-Kostka polynomials are the Schur coefficients of the modified Macdonald polynomials $\tilde{K}_{\lambda, (a^b)}(q, t)$ of a rectangular shape at roots of unity [Oh22]. Motivated by this result, he conjectured the following theorem which is the first main result and the starting point of this project.

**Theorem 1.1.** For nonnegative integers $k$ and $\ell,$ we have

$$\tilde{H}_{(k\ell)}[X; q, q^k] = \tilde{H}_{(k\ell)}[X; q, q^{k}] = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} X^T.$$

Theorem 1.1 is an instance of more general identities involving Macdonald polynomials. To be more precise, recall that there is a well-known $(q, t)$-symmetry relation for the modified Macdonald polynomials

$$\tilde{H}_{\mu}[X; q, t] = \tilde{H}_{\mu'}[X; t, q].$$
This symmetry clearly follows from the triangularity conditions, the symmetry of the Garsia–Haiman module [GH93], or the geometric interpretation of Haiman [Hai01]. However, a ‘combinatorial’ proof of the \((q,t)\)-symmetry is not known (see [Gil16] for an attempt and partial results). There is another symmetry relation for the modified Macdonald polynomials, which we now describe. For a partition \(\mu = (\mu_1, \ldots, \mu_\ell)\), let \(k\mu\) be a partition obtained by multiplying \(k\) to each part of \(\mu\), i.e. we let \(k\mu := (k\mu_1, \ldots, k\mu_\ell)\). In the early version of this paper, we have conjectured the following.

**Theorem 1.2.** Let \(\mu\) be a partition and \(k\) be a positive integer. Then we have

\[
\tilde{H}_{k\mu}[X; q, q^k] = \tilde{H}_{k\mu'}[X; q, q^k].
\]

Since the Young diagrams of \(k\mu\) and \(k\mu'\) are obtained by stretching those of \(\mu\) and \(\mu'\), we refer to the symmetry described in Theorem 1.2 as the **stretching symmetry**.

After finishing this paper, the first and second named author and Donghyun Kim [KLO22] found the exact condition when two partitions \(\lambda\) and \(\mu\) have the same Macdonald polynomials at \(q=t^k\), namely, we have

\[
\tilde{H}_\lambda[X; q, q^k] = \tilde{H}_\mu[X; q, q^k],
\]

if and only if

\[
B_\lambda[q, q^k] = B_\mu[q, q^k].
\]

After submitting this paper, Mark Haiman [Hai22] pointed out that a theorem (Theorem I.1) of Garsia and Tesler [GT96, Theorem I.1] implies that we have

\[
\tilde{H}_\lambda[X; q^r, q^s] = \tilde{H}_\mu[X; q^r, q^s],
\]

if

\[
B_\lambda[q^r, q^s] = B_\mu[q^r, q^s].
\]

Thus, Theorem 1.1 and some of the results in [KLO22] (including Theorem 1.2) are not new at all. Nonetheless, we believe our proof in this paper and [KLO22] of these results are worth sharing and would shed light on the research of Macdonald polynomials in this direction.

On the other hand, while working on this project, we learned that Jim Haglund [Hag21] has proposed a conjecture for multi-\(t\)-Macdonald polynomials which is possibly related to the Conjecture 1.2. The second main purpose of this paper is to prove the conjecture of Haglund. To begin, for a partition \(\mu\) we fix an ordering \(c_1, c_2, \ldots\) of cells in \(\mu\). Let \(\tilde{H}_\mu[X; q, t_1, t_2, \ldots]\) be the multi-\(t\)-Macdonald polynomial where each variable \(t_i\) accounts for \(c_i\) (see Section 2.2 for the precise definition). Haglund conjectured that the multi-\(t\)-Macdonald polynomials of partitions with two rows are specialized to a unicellular LLT polynomial which is an LLT polynomial indexed by a tuple of cells or a Dyck path. In the following theorem, we used a Dyck path to index unicellular LLT polynomial.
Theorem 1.3. [Hag21] Let \( \mu = (n-k, k) \) be a partition and let \( c_1, \ldots, c_k \) be the cells in the upper row. Let \( D(h_1, \ldots, h_n) \) be the Dyck path of size \( n \) whose height of the \( j \)-th column is \( h_j \) for \( 1 \leq j \leq n \). Then for \( k \leq h_1 \leq \cdots \leq h_k \) we have,

\[
\overline{H}_\mu[X; q, q^{h_1-k}, q^{h_2-1-k}, \ldots, q^{h_k-k}] = \text{LLT}_{D(h_1, \ldots, h_k, n, \ldots, n)}[X; q],
\]

where the left-hand side is the multi-t-Macdonald polynomial \( \overline{H}_\mu[X; t_1, t_2, \ldots] \) at \( t_1 = q^{h_1-k} \) for \( 1 \leq i \leq k \).

This paper is organized as follows. In Section 2, we provide backgrounds on combinatorics and modified Macdonald polynomials. In Section 3, we provide LLT equivalences and proofs Theorem 1.1 and Theorem 1.3 using the LLT equivalences. In Section 4, we present a representation theoretic proof of Theorem 1.1. In the last section, we leave some concluding remarks.

2. Backgrounds

2.1. Combinatorics. Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \) be a partition of \( n \), and abuse our notation as \( \mu = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : j \leq \mu_j\} \) to denote its Young diagram whose elements are called the cells. We draw a Young diagram in the first quadrant, in the French notation. For a cell \( u \) in a partition, the arm (coarm, respectively) of a cell is the number of cells strictly to the right (left, respectively) of \( u \) in the same row; its leg (coleg, respectively) is the number of cells strictly above (below, respectively) \( u \) in the same column; its major is given by leg plus one. For a subset of cells \( D \subseteq \mu \) and a statistic \( \text{stat} \) defined for cells, we define

\[
\text{stat}(D) := \sum_{u \in D} \text{stat}(u).
\]

A skew partition is a subset of \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) of the form \( \lambda / \mu \), where \( \lambda \) and \( \mu \) are partitions with \( \mu \subseteq \lambda \). A ribbon is a connected skew partition containing no \( 2 \times 2 \) block of cells. The content of the cells of a ribbon are consecutive integers. The descent set of a ribbon \( \nu \) is the set of contents \( c(u) \) of those cells \( u = (i, j) \in \nu \) such that the cell \( \nu = (i-1, j) \) directly below \( u \) also belongs to \( \nu \). For an interval \( I = [r, r+s] := \{r, r+1, \ldots, r+s\} \), it is clear that there is a one-to-one correspondence between ribbons of content \( I \) and subsets \( D \subseteq I \setminus \{r\} \) by considering the descent set of each ribbon. We denote the ribbon with a content set \( I \) and a descent set \( D \) by \( R_I(D) \). For a special case, a ribbon of size 1 is a cell and denoted by \( C_a := R_{\{a\}}(\emptyset) \) when its content is \( a \). For each subset of cells \( D \subseteq \{(i, j) \in \mu : 1 < i\} \) where no cell is in the first row, let \( D^{(j)} := \{i : (i, j) \in D\} \). Then we define \( R_\mu(D) \) to be a tuple of ribbons defined by

\[
R_\mu(D) := (R_{(1, \mu_1')}^{(1)}, R_{(1, \mu_2')}^{(2)}, \ldots).
\]

2.2. LLT polynomials and modified Macdonald polynomials. The LLT polynomial \( \text{LLT}_\nu[X; q] \) of a tuple of skew partitions \( \nu = (\nu^{(1)}, \nu^{(2)}, \ldots) \) is a \( q \)-analogues of products of Schur functions indexed by \( \nu \), introduced by Lascoux, Leclerc, and Thibon [LLT97] in the study of quantum affine algebras and unipotent varieties. In this section, we recall an LLT expansion for modified Macdonald polynomials. We first recall a definition of LLT polynomials.

For a skew partition \( \nu \), a semistandard tableau of shape \( \nu \) is a filling of \( \nu \) with positive integers where each row is weakly increasing from left to right and each column is strictly increasing from bottom to top. For a tuple \( \nu = (\nu^{(1)}, \nu^{(2)}, \ldots) \) of skew partitions, a semistandard tableau \( T = (T^{(1)}, T^{(2)}, \ldots) \) of shape \( \nu \) is a tuple of semistandard tableaux where each \( T^{(i)} \) is a semistandard tableau of shape \( \nu^{(i)} \). The set of semistandard tableaux of shape \( \nu \) is denoted by \( \text{SSYT}(\nu) \). For
a semistandard tableau $T = (T^{(1)}, T^{(2)}, \ldots)$ of shape $\nu$, an inversion of $T$ is a pair of cells $u \in \nu^{(i)}$ and $v \in \nu^{(j)}$ such that $T^{(i)}(u) > T^{(j)}(v)$ and either

- $i < j$ and $c(u) = c(v)$, or
- $i > j$ and $c(u) = c(v) + 1$.

Denote by $\text{inv}(T)$ the number of inversion in $T$. The LLT polynomial $\text{LLT}_\nu[X; q]$ is defined by

$$\text{LLT}_\nu[X; q] = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} X^T.$$

Here, $X^T := x_{T_1} x_{T_2} \cdots$, where $T_i$ is the number of $i$'s in $T$.

If a tuple $\nu$ of skew partitions consists of single cells, then the associated LLT polynomial $\text{LLT}_\nu[X; q]$ is called a unicellular LLT polynomial. Unicellular LLT polynomials can also be indexed by Dyck paths by the well-known correspondence between Dyck paths and unicellular diagrams (cf. [Lee21]). To be more precise, for given a tuple of $n$ cells, index cells 1 through $n$ from left to right starting from the smallest content to the largest content. For an $i$-th cell, let $h_i$ be the maximal index $j$ such that the $i$-th and $j$-th cell form an inversion pair. Then the correspondence sends the tuple of cells to a Dyck path where the height of $i$-th column is given by $h_i$. For example, a tuple of cells in Figure 2.2 corresponds to the Dyck path of height $(2, 4, 5, 5, 5)$.

Figure 1.

The first combinatorial formula for modified Macdonald polynomials was given in [HHL05]. They also provided an LLT expansion of the modified Macdonald polynomials.

**Theorem 2.1.** [HHL05, Section 3] For a partition $\mu$, we have

$$\tilde{H}_\mu[X; q, t] = \sum_D q^{-\text{arm}(D)} t^{\text{maj}(D)} \text{LLT}_{R_{\mu}(D)}[X; q],$$

where the sum is over all subsets $D \subseteq \{(i, j) \in \mu : 1 < i\}$, where there is no cell in the first row.

As a generalization of above LLT expansion (or a combinatorial formula) of modified Macdonald polynomials, the multi-$t$ Macdonald polynomials $\tilde{H}_\mu[X; q, t_1, t_2, \ldots]$ is defined as follows: For a partition $\mu$, consider an ordering $c_1, c_2, \ldots$ of cells in $\mu$.

$$\tilde{H}_\mu[X; q, t_1, t_2, \ldots] = \sum_D q^{-\text{arm}(D)} \prod_{c_i \in D} t_i^{\text{maj}(c_i)} \text{LLT}_{R_{\mu(D)}}[X; q],$$

where the sum is over all subsets $D \subseteq \{(i, j) \in \mu : 1 < i\}$ and $c_i$ is the $i$-th cell in the top row. By specializing $t$ variables $t_i = t$, we recover the usual modified Macdonald polynomial.
3. Combinatorial proof of Theorem 1.1 and Haglund’s conjecture

3.1. LLT-equivalences. Throughout this paper let \([n]_q := 1 + q + \cdots + q^{n-1}\). Let \(\sum_i a_i(q)\nu^{(i)}\) and \(\sum_j b_j(q)\mu^{(j)}\) be \(\mathbb{N}[q]\)-linear combinations of tuples of skew partitions. Following Miller [Mil19], we say that two \(\mathbb{N}[q]\)-linear combinations of tuples of skew partitions are \(LLT\)-equivalent if for every tuple of skew partitions \(\lambda\), we have

\[
\sum_i a_i(q) \text{LLT}_{\nu^{(i)}, \lambda}[X; q] = \sum_j b_j(q) \text{LLT}_{\mu^{(j)}, \lambda}[X; q].
\]

Here, for tuples \(\nu, \lambda\) of skew partitions, \((\nu, \lambda)\) denotes a tuple of skew partitions obtained by concatenating \(\lambda\) after \(\nu\). We often abuse our notation to write

\[
\sum_i a_i(q)\nu^{(i)} = \sum_j b_j(q)\mu^{(j)},
\]

when they are LLT-equivalent. In this section, we explore a series of LLT-equivalences which are ribbon-analogues of results in [Mil19, Lee21, HNY20].

We prove some of the LLT-equivalences by induction, by establishing initial cases and showing that they satisfy 'linear relations'. To be more precise, we say that a function \(f(\alpha)\) of integers \(\alpha\) satisfies a linear relation in \(\alpha\) if we have

\[
q f(\alpha) + f(\alpha + 2) = [2]_q f(\alpha + 1).
\]

If a function \(f(D)\) is defined for Dyck paths \(D\), we say \(f\) satisfies a row linear relation if we have

\[
q f(D) + f(D') = [2]_q f(D'),
\]

where Dyck paths \(D, D', D''\) are Dyck paths where they differ at only one row and the number of boxes in the row below the Dyck path is given by \(a, a + 1\) and \(a + 2\), respectively. We define column linear relation similarly.

Let \(R\) be a ribbon of content \([r - 1]\) with a descent set \(D\). We can add a cell of content \(r\) to \(R\) to obtain a ribbon of content \([r]\) in two ways; add a cell above or to the left of the cell of content \(r\). To be more precise, there are two ribbons \(R^+_H := R_{[r]}(D)\) and \(R^+_V := R_{[r]}((D \cup \{r\})\) obtained by adding a cell to \(R\). We used the + sign to mean that one cell is added, and \(H\) and \(V\) stand for adding a cell horizontally or vertically. It is the right place to give the first LLT equivalence in this paper.

**Proposition 3.1.** For a ribbon \(R\) of content \([r - 1]\), we have the following LLT equivalence

\[
R^+_H + q^a R^+_V = [\alpha]_q (R, C_r) - q[\alpha - 1]_q (C_r, R)
\]

\[
= \begin{cases} 
[a]_q (R, C_r) - q[\alpha - 1]_q (C_r, R) & \text{if } \alpha \geq 1, \\
-q^a [-\alpha]_q (R, C_r) + q^a [1 - \alpha]_q (C_r, R) & \text{if } \alpha < 1.
\end{cases}
\]

where \([\alpha]_q = \frac{1-q^\alpha}{1-q}\).

**Proof.** We induct on \(\alpha\). It is direct to see that initial cases \(\alpha = 0, 1\) holds. Observe that both the left-hand side and the right-hand side of the equation are linear in \(\alpha\). This proves the proposition.

The second LLT equivalence is a linear relation for LLT polynomials which generalizes the local linear relation of unicellular LLT polynomials given in [Lee21, HNY20].
Proposition 3.2. For a ribbon $R$ of content $[r-1]$, we have the following LLT equivalence

$$[k]_q \left( C_r^{k-1}, R, C_r \right) = q[k-1]_q \left( C_r^k, R \right) + \left( R, C_r^k \right).$$

Proof. Recall that the linear relation given in [Lee21] is

$$[2]_q (C_r, C_{r-1}, C_r) = q (C_r^2, C_{r-1}) + (C_{r-1}, C_r).$$

The only difference between this LLT equivalence and ours is that we replaced $C_{r-1}$ with a ribbon $R$ of content $[r-1]$. Because a cell of content $r$ cannot form an inversion pair with a cell of content less than $r-1$, it suffices to care about the cell of content $r-1$ in $R$ (the last cell). Therefore, the proof of [Lee21, Theorem 3.5] can be applied to prove

$$[2]_q (C_r, R, C_r) = q (C_r^2, R) + (R, C_r).$$

Applying this inductively, as in the proof of [HNY20, Theorem 3.4] completes the proof. □

The third LLT equivalence is a commuting relation between a ribbon and a cell.

Proposition 3.3. [Mil19] Let $R$ be a ribbon of content $[r]$ with a descent set $D$. Then we have the following LLT equivalence

$$(C_r, R) = \begin{cases} q^{-1}(R, C_r) & \text{if } r \in D, \\ (R, C_r) & \text{otherwise}. \end{cases} \tag{3.1}$$

Proof. To prove that two linear combinations of tuples of skew partitions are LLT equivalent, it suffices to show that there is a weight, inversion, and content preserving bijection between semistandard tableaux corresponding to those. Since a proof of the second case can be given by a similar argument to the first case, we only provide a proof for the first case.

For semistandard tableaux of shape $(C_r, R)$ and $(R, C_r)$ are the followings tableaux with either of the following four conditions holds: $a > b > c$, $a = b > c$, $b > a > c$, or $b > c \geq a$.

$T = \begin{array}{c} a \\ b \\ c \end{array}$ \hspace{1cm} $T' = \begin{array}{c} a \\ b \\ c \end{array}$

We let the bijection preserve the fillings the cells of $R$ of content less than $r-1$, so we omit those cells. In Table (3.2), we list up the $q$-weight of each side of (3.1), $\text{inv}(T)$ and $\text{inv}(T') - 1$ for each cases.

| $T$ | $T'$ $q^{-1}$ | $\text{inv}(T)$ | $\text{inv}(T') - 1$ |
|-----|-------------|-----------------|-------------------|
| $a > b > c$ | 1 | 0 | 0 |
| $a = b > c$ | 0 | 0 | 1 |
| $b > a > c$ | 0 | 0 | 1 |
| $b > c \geq a$ | 0 | 0 | 1 |

Let us define a bijection sending $T$ to $T'$ if $a, b$ and $c$ satisfies the second or the fourth case ($a = b > c$ or $b > c \geq a$), and sending $T$ to a tableaux obtained from $T'$ by switching $a$ and $b$ otherwise ($a > b > c$ or $b > a > c$). By definition of the bijection, it is obviously weight and content preserving, and (3.2) shows that it is also inversion preserving. □

For the last, we need one more commuting relation between dominoes given in [Lee21].

Proposition 3.4. [Lee21, Lemma 5.5] Let $V$ and $H$ be a vertical domino and a horizontal domino of content $[r, r+1]$, respectively. Then we have the following LLT equivalence

$$(V, H) = (H, V).$$
3.2. **Proof of Theorem 1.1.** Let $\mu = \{k^\ell\}$. By Theorem 2.1, we have an LLT expansion of the modified Macdonald polynomial of $\mu$ at $t = q^k$ by

$$
\tilde{H}_\mu(x; q, q^k) = \sum_{D \subseteq \{(i,j) \in \mu : 1 \leq i \leq \ell, 1 \leq j \leq k\}} q^{k \text{maj}(D) - \text{arm}(D)} \text{LLT}_{R_\mu(D)}[X; q].
$$

Note that for the cells in $\{(i,j) : 1 \leq i \leq \ell, 1 \leq j \leq k\}$, the $q$-statistics

$$
k \text{maj} - \text{arm}
$$

in (3.3) are given by $1, 2, \ldots, k$ for the cells in the top row, $k+1, k+2, \ldots, 2k$ for the cells in the second top row, and so on.

Choose a subset $E \subseteq \{(i,j) \in \mu : 1 < i < \ell\}$ consisting of cells not in the first and the last row and take a partial sum of the right-hand-side of (3.3) over subsets $D \subseteq \{(i,j) \in \mu : 1 < i \leq \ell\}$ with $D$ restricted to $\{(i,j) \in \mu : 1 < i < \ell\}$ equals to $E$. This partial sum gives $q^{k \text{maj}(E) - \text{arm}(E)}$ times the following sum

$$
\sum_D q^{\sum_{(i,j) \in D}} \text{LLT}_{R_\mu(D)}[X; q],
$$

where the sum is over subsets $D$ restricted to $\{(i,j) \in \mu : 1 < i < \ell\}$ equals to $E$.

We claim that the summation in (3.4) is LLT equivalent with a single(!) tuple of ribbons:

$$
\sum_D q^{\sum_{(i,j) \in D}} J R_\mu(D) = \left( R_{(k^{\ell-1})}(E), C_\ell^k \right),
$$

where the sum in the left-hand-side is over subsets $D$ restricted to $\{(i,j) \in \mu : 1 < i < \ell\}$ equals to $E$. We prove this claim by induction on $k$. For initial case, assume $k = 1$. By Proposition 3.1 for $\alpha = 1$, we have

$$
R_{(1^{\ell})}(E) + q R_{(1^{\ell})}(E \cup \{\ell\}) = \left( R_{(1^{\ell-1})}(E), C_\ell \right),
$$

which proves the claim for $k = 1$.

Assume $k > 1$. Then we have

$$
\sum_D q^{\sum_{(i,j) \in D}} J R_\mu(D)
\begin{align*}
&= \left( R_{(k^{\ell-1})}(\{(i,j) \in E : 1 \leq i \leq k-1\}), C_\ell^k \right) \\
&\quad + q^k \left( R_{(k^{\ell-1})}(\{(i,j) \in E : 1 \leq i \leq k-1\}), C_\ell^{k-1} \right) \\
&\quad \times \left( R_{(1^{\ell})}(E^{(k)}), C_\ell \right) \\
&\quad - q[k-1]q R_{(k^{\ell-1})}(\{(i,j) \in E : 1 \leq i \leq k-1\}), C_\ell^k \left( R_{(1^{\ell-1})}(E^{(k)}), C_\ell \right) \\
&= \left( R_{(k^{\ell-1})}(E), C_\ell^k \right).
\end{align*}
$$

The first equation follows by the induction hypothesis. The second equation follows from Proposition 3.1 and the third equation follows from Proposition 3.2. This proves the claim.

Sliding the cells $C_\ell$’s to the leftmost part of the diagonal of content $\ell - 1$ gives

$$
\left( R_{(k^{\ell-1})}(E), C_\ell^k \right) = \left( C_{\ell-1}^k, R_{(k^{\ell-1})}(E) \right).
$$

Recall that by Proposition 3.3, we can swap a cell of content $r$ and a ribbon $R$ of content $[r]$ where a weight $q^{-1}$ is attached in the case the last cell of $R$ is a descent. Thus,

$$
\left( C_{\ell-1}^k, R_{(k^{\ell-1})}(E) \right) = q^{-k} \left| \{(j) : (\ell-1, j) \in E\} \right| \left( R_{(k^{\ell-1})}(E), C_{\ell-1}^k \right),
$$

(3.7)
Combining (3.5), (3.6) and (3.7), we conclude that
\[
\overline{H}_\mu[X; q, q^k] = \sum_{E} q^{k\text{maj}(E) - \text{arm}(E) - k |(j: \ell - 1, j) \in E|} \text{LLT}\left(R_{(\ell - 1)}[E], c_{\ell - 1}^k\right)[X; q],
\]
where the sum is over all subsets $E \subseteq \{(i, j) \in \mu : 1 < i < \ell\}$. Note that cells in the top row in $E$ contributes to the $q$-weight\[
k\text{maj}(E) - \text{arm}(E) - k |(j: \ell - 1, j) \in E|\]
in (3.8) by 1, 2, . . . , $k$ as in the initial case. Therefore we can apply the whole argument again to obtain\[
\overline{H}_\mu[X; q, q^k] = \sum_{E} q^{k\text{maj}(E) - \text{arm}(E) - 2k |(j: \ell - 2, j) \in E|} \text{LLT}\left(R_{(\ell - 2)}[E], c_{\ell - 1}^{2k}\right)[X; q],
\]
where the sum is over all $E \subseteq \{(i, j) \in \mu : 1 < i < \ell - 1\}$. Applying this repeatedly proves Theorem 1.1.

3.3. **Proof of Haglund’s conjecture.** Let $\mu = (n - k, k)$. Note that for a subset $D \subseteq \{(2, j) \in \mu\}$, arm$(D) = \sum_{(2, j) \in D} (k - j)$. Therefore, the multi-$t$-Macdonald polynomial is given by
\[
\overline{H}_\mu[X; q, q^{k_h - k}, q^{h_{k - 1} - k}, \ldots, q^{h_1 - k}] = \sum_{D} q^{\sum_{(2, j) \in D} h_j - k - 1 - j} \text{LLT}_{R_\mu(D)}[X; q],
\]
where the sum is over all subsets $D \subseteq \{(2, j) \in \mu\}$. For a subset $D \subseteq \{(2, j) \in \mu\}$, we define $D^{rev} = \{(2, k + 1 - j) : (2, j) \in D\}$. Then Proposition 3.4 implies
\[
\text{LLT}_{R_\mu(D)}[X; q] = \text{LLT}_{R_\mu(D^{rev})}[X; q],
\]
thus we have
\[
\overline{H}_\mu[X; q, q^{k_h - k}, q^{h_{k - 1} - k}, \ldots, q^{h_1 - k}] = \sum_{D} q^{\sum_{(2, j) \in D} h_j - k - 1 - j} \text{LLT}_{R_\mu(D)}[X; q].
\]
(3.9)

Our goal is to show that the right-hand side of (3.9) is the same as $\text{LLT}_{R_\mu(D)}[X; q]$.

There are two steps for proving this identity using an induction. The first step is to show the identity for “near-staircase” shapes, namely, the case when $h_j = k + j - 1 + e_j$ for $j = 1, \ldots, k$, where $e_j$ is either 0 or 1. These partitions are conjugate to partitions containing $(n - k - 1, n - k - 2, \ldots, n - 2k)$ and contained in $(n - k, n - k - 1, \ldots, n - 2k + 1)$. A proof of the first step is straightforward. When $h_j = k + j - 1 + e_j$, we have $h_j - k - j + 1 = e_j$ so by Proposition 3.1, (3.9) becomes
\[
\sum_{D} q^{\sum_{j \in D} e_j} \text{LLT}_{R_\mu(D)}[X; q] = \text{LLT}_{R_\mu(D^{rev})}[X; q],
\]
where $P_1 = (C_1, C_2)$ and $P_0 = (C_2, C_1)$ and there are $n - 2k$ number of $C_1$’s in total.

The second step is to show that (3.9) satisfies both column and row linear relations. Proving the column linear relation is trivial by the formula, since a function $q_f^h$ is linear in $h_j$. Therefore, one only needs to prove the row linear relation. Let $f(h_1, \ldots, h_k)$ be the right-hand side of equation (3.9). Then we need to show that for any positive integer $i \in [k]$ and $a \leq n$ satisfying $h_{i-1} \leq a$ and $h_{i+2} \geq a + 1$, the following holds:
\[
q \cdot f(h_1, \ldots, h_{i-1}, a, a, h_{i+2}, \ldots, h_k) + f(h_1, \ldots, h_{i-1}, a + 1, a + 1, h_{i+2}, \ldots, h_k) = [2] q f(h_1, \ldots, h_{i-1}, a, a + 1, h_{i+2}, \ldots, h_k)
\]
(3.10)

When $(2, i)$ and $(2, i + 1)$ are both in $D$, or neither of them are in $D$, the corresponding sum $\sum_D q^{\sum_{j \in D} h_j - k - 1 + a} \text{LLT}_{R_\mu(D)}[X; q]$ satisfies (3.10)$\Rightarrow$(3.11).
When exactly one of \((2, i), (2, i + 1)\) are in \(D\), the corresponding function for (3.10) is

\[
\sum_{D} q^{1+\chi(i, D)(a-k-i+1)+\chi(i+1, D)(a-k-i)} q^{\sum_{[2,j] \in D,j \neq i, i+1} h_j - k - j + 1} \text{LLT}_{R_{\mu}(D)}[X; q]
\]

\[
+ \sum_{D} q^{\chi(i, D)(a-k-i+2)+\chi(i+1, D)(a-k-i+1)} q^{\sum_{[2,j] \in D,j \neq i, i+1} h_j - k - j + 1} \text{LLT}_{R_{\mu}(D)}[X; q]
\]

\[
= 2[2]q^{a-k-i+1} \sum_{D'} q^{\sum_{[2,j] \in D,j \neq i, i+1} h_j - k - j + 1} \text{LLT}_{R_{\mu}(D')}[X; q]
\]

where \(\chi(i, D)\) is 1 if \((2, i) \in D\), 0 otherwise, and the sum in the last line runs over all \(D' \subset \{(2, j) \in \mu\}\) satisfying \((2, i) \in D'\) and \((2, i + 1) \notin D'\). Here we applied Proposition 3.4.

By a similar argument, one can prove

\[
2q^{a-k-i+1} \sum_{D} q^{\sum_{[2,j] \in D,j \neq i, i+1} h_j - k - j + 1} \text{LLT}_{R_{\mu}(D')}[X; q]
\]

is the same as (3.11), proving the claim.

Now we show that proving the two steps is enough to show Haglund’s conjecture by using induction, where the base case is the conjecture for near-staircase shapes.

By induction hypothesis, Haglund’s conjecture is true for a Dyck path \(D(h_1', h_2', \ldots, h_i')\) where \(h_1' = k \leq h_i' \leq n\) for any \(1 < i\). Assume that we have any Dyck path \(D(h_1, \ldots, h_k)\) satisfying \(k \leq h_i \leq n\). There are two cases:

1. \(h_2 \geq k + 2\). In this case, since we know that Haglund’s conjecture holds for \(h_1 = k\) and \(k + 1\), it holds for any \(k \leq h_1 \leq h_2\) by using the fact that both sides of Haglund’s conjecture satisfy column linear relation.

2. \(h_2 = k\) or \(k + 1\). In this case, consider the largest index \(i\) satisfying \(h_i = k\). If there is no \(i\) or \(i = 1\), corresponding two Dyck paths does satisfy Haglund’s conjecture by an induction. If \(i > 1\), then we use two previous paths and the row linear relation to show that Haglund’s conjecture holds for any \(i\).

4. Representation theoretic proof of Theorem 1.1

For any \(\mu \vdash n\), Haiman established [Hai01] the Schur positivity of \(\tilde{H}_\mu\) by proving

\[
\text{grFrob}(V_\mu; q, t) = \tilde{H}_\mu[X; q, t]
\]

where \(V_\mu\) is the Garsia-Haiman module [GH93] attached to \(\mu\). The module \(V_\mu\) is the following subspace of \(\mathbb{C}[X_n, Y_n] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]\). Fix a bijective labeling \(T\) of the boxes of \(\mu\) with \(1, 2, \ldots, n\) and define a polynomial \(\delta_\mu \in \mathbb{C}[X_n, Y_n]\) by

\[
\delta_\mu := \varepsilon_n \cdot \prod_{c \in \mu} x_{T(c)}^{\text{coarm}(c)} y_{T(c)}^{\text{coleg}(c)}
\]

where \(S_n\) acts on \(\mathbb{C}[X_n, Y_n]\) diagonally and \(\varepsilon_n := \sum_{w \in S_n} \text{sign}(w) w\) is the antisymmetrizing idempotent. The Garsia-Haiman module \(V_\mu\) is the smallest linear subspace of \(\mathbb{C}[X_n, Y_n]\) containing \(\delta_\mu\) which is closed under the partial differentiation operators \(\partial/\partial x_i\) and \(\partial/\partial y_i\) for \(1 \leq i \leq n\). In particular, when \(\mu \vdash n\) is a single row or column, we have an isomorphism of singly-graded \(S_n\)-modules \(V_\mu \cong R_n\), where \(R_n := \mathbb{C}[X_n]/\langle \mathbb{C}[X_n] \rangle^{+n}\) is the type \(A\) coinvariant algebra in the \(x\)-variables.
Let \( \eta : \mathbb{C}[X_n, Y_n] \to \mathbb{C}[X_n] \) be the evaluation map which fixes \( x_i \) and specializes \( y_i \mapsto (x_i)^k \). For any \( \mu \vdash n \) we have an \( S_n \)-module homomorphism \( \varphi_\mu : V_\mu \to R_n \) given by the composition

\[
\varphi_\mu : V_\mu \leftarrow \mathbb{C}[X_n, Y_n] \xrightarrow{\eta} \mathbb{C}[X_n] \to R_n
\]

of including \( V_\mu \) into \( \mathbb{C}[X_n, Y_n] \), evaluating along \( \eta \), and then projecting onto \( R_n \).

**Proposition 4.1.** If \( n = k \ell \) and \( \mu = (k^\ell) \) is a rectangle, then \( \varphi_\mu \) is an isomorphism.

Since the (singly) graded Frobenius image of \( R_n \) is

\[
\text{grFrob}(R_n; q) = \sum_{\lambda \vdash n} \left( \sum_{T \in \text{SYT}(\lambda)} q^\text{maj}(T) \right) \cdot s_\lambda[X]
\]

and \( \eta \) evaluates the \( y \)-variables to degree \( k \), Proposition 4.1 implies Theorem 1.1. We prove Proposition 4.1 as follows.

**Proof.** The domain and codomain of \( \varphi_\mu \) are both vector spaces of dimension \( n! \), so it is enough to show that the image of \( \varphi_\mu \) spans \( R_n \). We choose our filling \( T \) of the \( k \)-by-\( \ell \) rectangle \( \mu \) so that

\[
\delta_\mu = \varepsilon_n \left( \prod_{0 \leq j \leq \ell-1} x_{\ell+j+1}^j y_{\ell+j+1}^j \right).
\]

This corresponds to the ‘English reading order’ standard filling of \( \mu \). The evaluation \( \eta(\delta_\mu) \) of the Vandermonde determinant is the image in \( R_n \) of the Vandermonde, i.e.,

\[
\eta(\delta_\mu) = \varepsilon_n \cdot (x_1^0 x_2^1 \cdots x_n^{n-1}).
\]

If we endow monomials in \( x_1, \ldots, x_n \) with the lex term order with underlying variable order \( x_1 < \cdots < x_n \) the leading monomial of \( \eta(\delta_\mu) \) is \( x_1^0 x_2^1 \cdots x_n^{n-1} \). We show that any exponent sequence \( (a_1, \ldots, a_n) \) with \( a_i < i \) is the leading monomial of some polynomial in \( \eta(\varphi_\mu) \). Since such monomials constitute the standard basis of \( R_n \) with respect to the aforementioned term order, this completes the proof.

Suppose we have a componentwise inequality \( (a_1, a_2, \ldots, a_n) \leq (0, 1, \ldots, n-1) \). We apply the Euclidean algorithm to any difference \( (i-1) - a_i \) to write

\[
(i-1) - a_i = q_i m + r_i,
\]

where \( q_i \geq 0 \) and \( 0 \leq r_i < m \). We have an element of \( V_\mu \) given by

\[
(\partial/\partial x_1)^{r_1} (\partial/\partial y_1)^{q_1} \cdots (\partial/\partial x_n)^{r_n} (\partial/\partial y_n)^{q_n}(\delta_\mu).
\]

The image of (4.7) under \( \eta \) has leading monomial \( x_1^{a_1} \cdots x_n^{a_n} \), and the argument in the last paragraph completes the proof.

If \( \mu \vdash n \) is not a rectangle, the restriction of \( \eta \) to \( V_\mu \) is not injective, so the above argument does not go through.
5. **Concluding Remarks**

5.1. **Schur positivity.** Note that we can equivalently state Theorem 1.1 as

\[ \tilde{H}_{(k\ell)}[X; q, t] - \tilde{H}_{(k'\ell')}[X; q, t] \]

is divisible by \((q^k - t)\). The proof in Section 3 does not only show the equality of Macdonald polynomials at \(t = q^k\) in Theorem 1.1 but also proves the LLT positivity, thus Schur positivity of the quotient

\[ \frac{\tilde{H}_{(k\ell)}[X; q, t] - \tilde{H}_{(k'\ell')}[X; q, t]}{q^k - t} \].

Conj 1.2 implies that the quotient

\[ \frac{\tilde{H}_{k\mu}[X; q, t] - \tilde{H}_{k\mu'}[X; q, t]}{q^k - t} \]

is a polynomial in \(q, t\) and \(X\). Not all quotient of this form is Schur positive, but there is some positivity. In particular, SAGE computations suggest that if each cell \(c = (i, j)\) of \(\mu\) satisfies either

1. \(c\) is also contained in \(\mu'\), or
2. \(c\) is under the main diagonal, i.e. \(i < j\),

then the quotient

\[ \frac{\tilde{H}_{k\mu}[X; q, t] - \tilde{H}_{k\mu'}[X; q, t]}{q^k - t} \]

is Schur positive. It is an interesting question to ask for necessary and sufficient conditions for two partitions \(\lambda, \mu\) and \(k \geq 0\) to give the Schur positivity of the quotient.

5.2. **Combinatorial formula for Kostka polynomials.** A combinatorial formula for \((q, t)\)-Kostka polynomials is unknown in general, and it is one of the most important open problems in algebraic combinatorics.

We recall that a standard tableau of a partition \(\lambda \vdash n\) is a semistandard tableau consisting of \(1, 2, \ldots, n\). We denote the set of standard tableaux of shape \(\lambda\) by \(\text{SYT}(\lambda)\). It is well known that \((q, t)\)-Kostka polynomial at \((1, 1)\) gives

\[ \tilde{K}_{\lambda, \mu}(1, 1) = |\text{SYT}(\lambda)|. \]

Therefore, the most desirable form of a combinatorial formula for \((q, t)\)-Kostka polynomials would be given by a generating function for the standard tableaux with two statistics:

\[ \tilde{K}_{\lambda, \mu}(q, t) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{stat}_q(T)} t^{\text{stat}_t(T)}. \]

Theorem 1.1 implies

\[ \tilde{K}_{\lambda, (k')}(q, q^k) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}. \]

This suggests that modified Macdonald polynomials (or modified \((q, t)\)-Kostka polynomials) for rectangle \(\mu\) might have more structure. Theorem 1.1 implies the desirable \((q, t)\)-statistics \(\text{stat}_q\) and \(\text{stat}_t\) such that

\[ \text{stat}_q + k \text{stat}_t \]

and maj statistics are equidistributed over \(\text{SYT}(\lambda)\). It is our hope that this gives a hint to track the \(q, t\)-statistics for \((q, t)\)-Kostka polynomials for rectangles. The \((q, t)\)-Kostka polynomials for partitions of 4 are given in the following table.
5.3. A common generalization of two main theorems. Note that there is an intersection between Theorem 1.3 and Theorem 1.1. To be more precise, taking \( \mu = (k, k) \) and \( h_i = 2k \) for all \( i \)'s in Theorem 1.3 yields

\[
\tilde{H}(n, n)[X; q, q^n] = \sum_{T \in \text{SYT}(2n)} q^{\text{maj}(T)} X^T.
\]  

(5.1)

Taking \( \mu = (2) \) (or \( \mu = (1,1) \)) in Theorem 1.1 also yields (5.1). Therefore, it is natural to ask the following question.

**Question 5.1.** Is there a common generalization of both Conjecture 1.2 and Theorem 1.3?

5.4. A new Mahonian statistic. A combinatorial formula for the modified Macdonald polynomials in [HHL05] and Theorem 1.1 implies that the statistic \( \text{maj}_k \) defined on words by

\[
\text{maj}_k(w) = \sum_{0 < j - i < k} \chi((i, j) \in \text{Inv}(w)) + \sum_i i \chi((i, i + k) \in \text{Inv}(w))
\]

is equidistributed with \( \text{maj} \). A bijective proof can be found in [Kad85].

By applying \((q, t)\)-symmetry, we obtain following equation as a corollary of Theorem 1.1 (or by applying a similar argument in Section 4, it is straightforward to obtain the following).

**Corollary 5.2.** For \( n = k \ell \), we have

\[
\tilde{H}(k \ell)[X; q^k, q] = \tilde{H}(k \ell)[X; q, q^k] = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} X^T.
\]

Let us define a statistic \( \text{maj}_\ell' \) for a word \( w = w_1 w_2 \ldots w_n \) by

\[
\text{maj}_\ell'(w) = \sum_i \left\lceil \frac{i}{\ell} \right\rceil \chi((i, i + \ell) \in \text{Inv}(w))
\]

Above Corollary with a combinatorial formula for the modified Macdonald polynomials in [HHL05] implies that the statistic

\[
k \text{maj}_\ell - (n - 1) \text{maj}_\ell'
\]

is also equidistributed with the usual major statistics. i.e. for any composition \( \mu \) of \( n \),

\[
\sum_{w \in \mathcal{W}(\mu)} q^{k \text{maj}_\ell(w) - (n - 1) \text{maj}_\ell'(w)} = \sum_{w \in \mathcal{W}(\mu)} q^{\text{maj}(w)},
\]

where the sum is over all words of content \( \mu \).

**Question 5.3.** Is there a bijective proof of (5.2)?

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