Complexity of Fractran and Productivity

Jörg Endrullis\(^1\), Clemens Grabmayer\(^2\), and Dimitri Hendriks\(^1\)

\(^1\) Vrije Universiteit Amsterdam, Department of Computer Science
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
joerg@few.vu.nl diem@cs.vu.nl

\(^2\) Universiteit Utrecht, Department of Philosophy
Heidelberglaan 8, 3584 CS Utrecht, The Netherlands
clemens@phil.uu.nl

Abstract. In functional programming languages the use of infinite structures is common practice. For total correctness of programs dealing with infinite structures one must guarantee that every finite part of the result can be evaluated in finitely many steps. This is known as productivity. For programming with infinite structures, productivity is what termination in well-defined results is for programming with finite structures. Fractran is a simple Turing-complete programming language invented by Conway. We prove that the question whether a Fractran program halts on all positive integers is \(\Pi^0_2\)-complete. In functional programming, productivity typically is a property of individual terms with respect to the inbuilt evaluation strategy. By encoding Fractran programs as specifications of infinite lists, we establish that this notion of productivity is \(\Pi^0_2\)-complete even for some of the most simple specifications. Therefore it is harder than termination of individual terms. In addition, we explore possible generalisations of the notion of productivity in the framework of term rewriting, and prove that their computational complexity is \(\Pi^1_1\)-complete, thus exceeding the expressive power of first-order logic.

1 Introduction

For programming with infinite structures, productivity is what termination is for programming with finite structures. In lazy functional programming languages like Haskell, Miranda or Clean the use of infinite structures is common practice. Programs dealing with such infinite structures can very well be terminating. For example, consider the Haskell program implementing Eratosthenes’ sieve:

\begin{verbatim}
prime n = primes !! (n-1)
primes = sieve [2..]
sieve (n:xs) = n:(sieve (filter (\m -> m ‘mod‘ n /= 0) xs))
\end{verbatim}

where \texttt{prime n} returns the \(n\)-th prime number for every \(n \geq 1\). The function \texttt{prime} is terminating, despite the fact that it contains a call to the non-terminating function \texttt{primes} which, in the limit, rewrites to the infinite list of prime numbers in ascending order. To make this possible, the strategy with respect to which the terms are evaluated is crucial. Obviously, we cannot fully
evaluate primes before extracting the n-th element. For this reason, lazy functional languages typically use a form of outermost-needed rewriting where only needed, finite parts of the infinite structure are evaluated, see for example [10].

Productivity captures the intuitive notion of unlimited progress, of ‘working’ programs producing values indefinitely, programs immune to livelock and deadlock. A recursive specification is called productive if not only can the specification be evaluated continually to build up an infinite normal form, but this infinite expression is also meaningful in the sense that it represents an infinite object from the intended domain.

The study of productivity (of stream specifications in particular) was pioneered by Sijtsma [12]. More recently, a decision algorithm for productivity of stream specifications from an expressive syntactic format has been developed [4]. We briefly elaborate on that format below in order to illustrate the subtle difference with the specifications resulting from the encoding of Fractran programs.

Fractran [2] is a remarkably simple Turing-complete programming language invented by the mathematician John Horton Conway. A Fractran program is a finite list of fractions $\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}$. Starting with a positive integer $n_0$, the algorithm successively calculates $n_{i+1}$ by multiplying $n_i$ with the first fraction that yields an integer again. The algorithm halts if there is no such fraction.

To illustrate the algorithm we consider an example of Conway from [2]:

\begin{align*}
17 &\quad 78 &\quad 19 &\quad 23 &\quad 29 &\quad 77 &\quad 95 &\quad 77 &\quad 1 &\quad 11 &\quad 13 &\quad 15 &\quad 15 &\quad 55 \\
91 &\quad 85 &\quad 51 &\quad 33 &\quad 29 &\quad 23 &\quad 17 &\quad 13 &\quad 11 &\quad 14 &\quad 2 &\quad 1
\end{align*}

We start with $n_0 = 2$. The leftmost fraction which yields an integer product is $\frac{15}{2}$, and so $n_1 = 2 \cdot \frac{15}{2} = 15$. Then we get $n_2 = 15 \cdot \frac{55}{1} = 825$, etcetera. By successive application of the algorithm, we obtain the following infinite sequence:

$$2, 15, 825, 725, 1925, 2275, 425, 390, 330, 290, 770, \ldots$$

Apart from $2^1$, the powers of 2 occurring in this infinite sequence are $2^1, 2^2, 2^3, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17}, 2^{19}, \ldots$, where the exponents form the sequence of primes.

We translate Fractran programs to streams specifications in such a way that the specification is productive if and only if the program halts on all $n_0 > 1$. Thereby we obtain that productivity is $\Pi^0_2$-hard. On the other hand, productivity of specifications in the format of [4], the ‘pure stream format’ (PSF), is decidable. What is the difference between these formats? Example specifications in PSF are:

$$J \rightarrow 0 : 1 : \text{even}(J) \quad \text{and} \quad Z \rightarrow 0 : \text{zip} \left( \text{even}(Z), \text{odd}(Z) \right),$$

including a set of defining rules for the involved stream functions, here:

$$\text{even}(x : \sigma) \rightarrow x : \text{odd}(\sigma), \quad \text{odd}(x : \sigma) \rightarrow \text{even}(\sigma), \quad \text{zip}(x : \sigma, \tau) \rightarrow x : \text{zip}(\tau, \sigma),$$

where zip ‘zips’ two streams alternatingly into one, and even (odd) returns a stream consisting of the elements at its even (odd) positions. The specification for $Z$ produces the stream $0 : 0 : 0 : \ldots$ of zeros, whereas the infinite normal form of $J$ is $0 : 1 : 0 : 0 : \text{even}^\omega$, which is not a constructor normal form.
Excluded from PSF is the first projection function on streams $\text{head}(x:\sigma) \to x$. This is for a good reason, as we shall see shortly. PSF is essentially layered: data terms (terms of sort data) cannot be built using stream terms (terms of sort stream). As soon as stream dependent data functions are admitted, the complexity of the productivity problem of such an extended format is increased. The problem with those functions is that they possibly create ‘look-ahead’: the evaluation of the ‘current’ stream element may depend on the evaluation of ‘future’ stream elements. To see this, consider an example from [12]:

$$S_n \to 0 : S_n(n) : S_n$$

where for a term $t$ of sort stream and $n \in \mathbb{N}$, we write $t(n)$ as a shorthand for $\text{head}(\text{tail}^n(t))$. If we take $n$ to be an even number, then $S_n$ is productive, whereas it is unproductive for odd $n$.

As soon as we allow $\text{head}$, productivity of even the most simple stream specifications is undecidable and $\Pi^0_2$-hard. Let us define the target format of the Fractran translation. The lazy stream format (LSF) consists of stream specifications of the form $M \to C[M]$ where $C$ is a context built solely from: one data element $\bullet$, the stream constructor ‘:’, the projection functions $\text{head}(x:\sigma) \to x$ and $\text{tail}(x:\sigma) \to \sigma$, unary stream functions $\text{mod}_n$, and $k$-ary stream functions $\text{zip}_k$ with the following defining rules, for every $n, k \geq 1$:

$$\text{mod}_n(\sigma) \to \text{head}(\sigma) : \text{mod}_n(\text{tail}^n(\sigma))$$
$$\text{zip}_k(\sigma_1, \sigma_2, \ldots, \sigma_k) \to \text{head}(\sigma_1) : \text{zip}_k(\sigma_2, \ldots, \sigma_k, \text{tail}(\sigma_1))$$

A hint for the fact that it is $\Pi^0_2$-hard to decide whether a lazy specification is productive, already comes from a simple encoding of Collatz’ conjecture (also known as the ‘3x+1-problem’ [9]) into a productivity problem. Without proof we state that: Collatz’ conjecture is true if and if only the following specification produces the infinite chain $\bullet : \bullet : \bullet : \ldots$ of data elements $\bullet$:

$$\text{collatz} \to \bullet : \text{zip}_2(\text{collatz}, \text{mod}_6(\text{tail}^9(\text{collatz})))$$ (1)

In order to understand the operational difference between rules in PSF and rules in LSF, consider the following two rules:

$$\text{read}(\sigma) \to \text{head}(\sigma) : \text{read}(\text{tail}(\sigma))$$ (2)
$$\text{read}'(x:\sigma) \to x : \text{read}'(\sigma)$$ (3)

The functions defined by these rules are extensionally equivalent: on productive stream terms they both behave as the identity function. However, intensionally, or operationally, there is a difference. A term $\text{read}'(s)$ is a redex only in case $s$ is of the form $u : t$, whereas $\text{read}(s)$ constitutes a redex for any stream term $s$, and so $\text{head}(s)$ can be undefined. The ‘lazy’ rule (2) postpones pattern matching. Although in PSF we can define functions $\text{mod}_n'$ and $\text{zip}_k'$ extensionally equivalent to $\text{mod}_n$ and $\text{zip}_k$, the productivity problem for PSF is decidable, but note that productivity of a pure version $\text{collatz}'$ (using $\text{mod}_6'$ and $\text{zip}_2'$ instead) of $\text{collatz}$ in (1) above would not represent Collatz’ conjecture.
**Contribution and Overview.** In Section 2 we show that the uniform halting problem of Fractran programs is \( \Pi^0_2 \)-complete. This is the problem of determining whether a program terminates on all positive integers. While it is well-known that Fractran is Turing-complete [2], \( \Pi^0_2 \)-completeness has, to the best of our knowledge, not yet been shown. Turing-completeness of a computational model does not imply that the uniform halting problem, in the strong sense of guaranteed termination on all configurations (as studied e.g. in [5]), is \( \Pi^0_2 \)-complete. For example, assume that we extend Turing machines with a special non-terminating state. Then the computational model obtained can still compute every recursive function. However, the uniform halting problem becomes trivial: these machines are never terminating on all configurations. Our result is a strengthening of the result in [8] where it has been shown that the generalised Collatz problem (GCP) is \( \Pi^0_2 \)-complete. Every Fractran program \( P \) can easily be translated into a Collatz function \( f \) such that the uniform halting problem for \( P \) is equivalent to the GCP for \( f \). The other direction is not immediate, since Fractran programs form a strict subset of Collatz functions. We discuss this in more detail in Section 2.

In Section 3 we explore alternative definitions of productivity and make them precise in the framework of term rewriting. These can be highly undecidable: ‘strong productivity’ turns out to be \( \Pi^1_1 \)-complete and ‘weak productivity’ is \( \Sigma^1_1 \)-complete. Productivity of individual terms with respect to a computable strategy, which is the notion used in functional programming, is \( \Pi^0_2 \)-complete.

In Section 4 we give an alternative proof of \( \Pi^0_2 \)-completeness of productivity. The new proof uses a simple encoding of Fractran programs \( P \) into stream specifications of the form \( \mathcal{M}_P \rightarrow \mathcal{C}[\mathcal{M}_P] \), in such a way that \( \mathcal{M}_P \) is productive if and only if the program \( P \) halts on all inputs. The resulting stream specifications are very simple compared to the results of encoding of Turing machines employed in Section 3. Whereas the Turing machine encoding essentially uses calculations on the elements of the list, the specifications obtained from the Fractran encoding contain no operations whatsoever on the list elements. In particular, the domain of data elements is a singleton.

**Related Work.** In [3] undecidability of different properties of first-order TRSs is analysed. While the standard properties of TRSs turn out to be either \( \Sigma^0_1 \) or \( \Pi^0_2 \)-complete, the complexity of the dependency pair problems [1] is essentially analytical: it is shown to be \( \Pi^1_1 \)-complete. We employ the latter result as a basis for our \( \Pi^1_1 \) - and \( \Sigma^1_1 \)-completeness results for productivity.

Roşu [11] shows that equality of stream specifications is \( \Pi^0_2 \)-complete. We remark that this result can be obtained as a corollary of our translation of Fractran programs \( P \) to stream specifications \( \mathcal{M}_P \). Stream specifications \( \mathcal{M}_P \) have the stream \( \bullet : \bullet : \ldots \) as unique solutions if and only if they are productive. Thus \( \Pi^0_2 \)-completeness of productivity of these specifications implies \( \Pi^0_2 \)-completeness of the stream equality problem \( \mathcal{M}_P = \bullet : \bullet : \ldots \).

The work of Kurtz and Simon [8] on the degree of undecidability of the generalised Collatz problem has already been mentioned above.
2 Fractran

The one step computation of a Fractran program is a partial function.

**Definition 2.1.** Let \( P = \frac{b_1}{q_1}, \ldots, \frac{b_k}{q_k} \) be a Fractran program. The partial function \( f_P : \mathbb{N} \rightarrow \mathbb{N} \) is defined for all \( n \in \mathbb{N} \) by:

\[
f_P(n) = \begin{cases} 
    n \cdot \frac{b_i}{q_i} & \text{where } \frac{b_i}{q_i} \text{ is the first fraction of } P \text{ such that } n \cdot \frac{b_i}{q_i} \in \mathbb{N}, \\
    \text{undefined} & \text{if no such fraction exists.}
\end{cases}
\]

We say that \( P \) halts on \( n \in \mathbb{N} \) if there exists \( i \in \mathbb{N} \) such that \( f^i(n) = \) undefined. For \( n, m \in \mathbb{N} \) we write \( n \rightarrow_P m \) whenever \( m = f_P(n) \).

The Fractran program for generating prime numbers, that we discussed in the introduction, is non-terminating for all starting values \( n_0 \), because the product of any integer with \( \frac{b_i}{q_i} \) is an integer again. However, in general, termination of Fractran programs is undecidable.

**Theorem 2.2.** The uniform halting problem for Fractran programs, that is, deciding whether a program halts for every starting value \( n_0 \in \mathbb{N}_{>0} \), is \( \Pi^0_2 \)-complete.

A related result is obtained in [8] where it is shown that the generalised Collatz problem (GCP) is \( \Pi^0_2 \)-complete, that is, the problem of deciding for a Collatz function \( f \) whether for every integer \( x > 0 \) there exists \( i \in \mathbb{N} \) such that \( f^i(x) = 1 \).

A Collatz function \( f \) is a function \( f : \mathbb{N} \rightarrow \mathbb{N} \) of the form:

\[
f(n) = \begin{cases} 
    a_0 \cdot n + b_0, & \text{if } n \equiv 0 \pmod{p} \\
    \vdots & \\
    a_{p-1} \cdot n + b_{p-1}, & \text{if } n \equiv p - 1 \pmod{p}
\end{cases}
\]

for some \( p \in \mathbb{N} \) and rational numbers \( a_i, b_i \) such that \( f(n) \in \mathbb{N} \) for all \( n \in \mathbb{N} \).

The result of [8] is an immediate corollary of Theorem 2.2. Every Fractran program \( P \) is a Collatz function \( f_P \), where \( f_P \) is obtained from \( f_P \) (see Definition 2.1) by replacing undefined with 1. We obtain the above representation of Collatz functions simply by choosing for \( p \) the least common multiple of the denominators of the fractions of \( P \). We call a Fractran program \( P \) trivially immortal if \( P \) contains a fraction with denominator 1 (an integer). Then for all not trivially immortal \( P \), \( P \) halts on all inputs if and only for all \( x > 0 \) there exists \( i \in \mathbb{N} \) such that \( f^i(x) = 1 \). Using our result, this implies that GCP is \( \Pi^0_2 \)-hard.

Theorem 2.2 is a strengthening of the result in [8] since Fractran programs are a strict subset of Collatz functions. If Fractran programs are represented as Collatz functions directly, for all \( 0 \leq i < p \) it holds either \( b_i = 0 \), or \( a_i = 0 \) and \( b_i = 1 \). Via such a translation Fractran programs are, e.g., not able to implement the famous Collatz function \( C(2n) = n \) and \( C(2n + 1) = 6n + 4 \) (for all \( n \in \mathbb{N} \)), nor an easy function like \( f(2n) = 2n + 1 \) and \( f(2n + 1) = 2n \) (for all \( n \in \mathbb{N} \)).

A second difference is that, whereas in [8] the result is based on a translation of register machines, we give a direct translation from Turing machines to Fractran programs. The resulting Fractran program halts on all positive integers if and only if the Turing machine is terminating on all configurations. To keep the translation as simple as possible we restrict to Turing machines having only two
symbols \{0, 1\} in their tape alphabet, 0 being the blank symbol. Every Turing machine \(M\) can be simulated by a Turing machine \(M'\) with alphabet \(\{0, 1\}\) by encoding the symbols of \(M\) in blocks of the same length \(\ell\) (using 0\(^{\ell}\) as code for the blank of \(M\)), and encoding the input of \(M\) for \(M'\) accordingly. In this way the blank symbol of \(M'\) may also appear as input symbol. This choice of machine model matches our proof of Thm. 2.2 which uses \(\Pi^0_2\)-completeness of the uniform halting problem of Turing machines with respect to arbitrary configurations \([5]\).

**Definition 2.3.** A Turing machine \(M\) (with binary tape alphabet) is a triple \(\langle Q, q_0, \delta \rangle\), where \(Q\) is a finite set of states, \(q_0 \in Q\) the initial state, and \(\delta : Q \times \{0, 1\} \to Q \times \{0, 1\} \times \{L, R\}\) a (partial) transition function. A configuration of \(M\) is a pair \(\langle q, \text{tape} \rangle\) consisting of a state \(q \in Q\) and the tape content \(\text{tape} : \mathbb{Z} \to \{0, 1\}\) such that the carrier \(\{n \in \mathbb{Z} \mid \text{tape}(n) \neq 0\}\) is finite. The set of all configurations is denoted by \(\text{Conf}_M\). We define the relation \(\to_M\) on the set of configurations \(\text{Conf}_M\) as follows: \(\langle q, \text{tape} \rangle \to_M \langle q', \text{tape}' \rangle\) whenever:

1. \(\delta(q, \text{tape}(0)) = \langle q', f, L \rangle, \text{tape}'(1) = f\) and \(\forall n \neq 0. \text{tape}'(n + 1) = \text{tape}(n)\), or
2. \(\delta(q, \text{tape}(0)) = \langle q', f, R \rangle, \text{tape}'(-1) = f\) and \(\forall n \neq 0. \text{tape}'(n - 1) = \text{tape}(n)\).

We say that \(M\) halts (or terminates) on a configuration \(\langle q, \text{tape} \rangle\) if the configuration \(\langle q, \text{tape} \rangle\) does not admit infinite \(\to_M\) rewrite sequences.

The uniform halting problem of Turing machines is the problem of deciding whether a given Turing machine \(M\) halts on all (initial or intermediate) configurations \(\langle q, \text{tape} \rangle \in \text{Conf}_M\). The following theorem is a result of \([5]\):

**Theorem 2.4.** The uniform halting problem for Turing machines is \(\Pi^0_3\)-complete.

We now give a translation of Turing machines to Fractran programs. Without loss of generality we restrict in the sequel to Turing machines \(M = \langle Q, q_0, \delta \rangle\) for which \(\delta(q, x) = \langle q', s', d' \rangle\) implies \(q \neq q'\). In case \(M\) does not fulfill this condition then we can find an equivalent Turing machine \(M' = \langle Q \cup \{\#\}, q_0, \delta' \rangle\) where \(Q_\# = \{q_\# \mid q \in Q\}\) and \(\delta'\) is defined by \(\delta'(q, x) = \langle p_\#, s, d \rangle\) and \(\delta'(q_\#, x) = \langle p, s, d \rangle\) for \(\delta(q, x) = \langle p, s, d \rangle\).

**Definition 2.5.** Let \(M = \langle Q, q_0, \delta \rangle\) be a Turing machine. Let \(\text{tape}_l, h, \text{tape}_r, \text{tape}'_l, h', \text{tape}'_r, m_{L,x}, m_{R,x}, \text{copy}_x\) and \(p_q\) for every \(q \in Q\) and \(x \in \{0, 1\}\) be pairwise distinct prime numbers. The intuition behind these primes is:

1. \(\text{tape}_l\) and \(\text{tape}_r\) represent the tape left and right of the head, respectively,
2. \(h\) is the tape symbol in the cell currently scanned by the tape head,
3. \(\text{tape}'_l, h', \text{tape}'_r\) store temporary tape content (when moving the head),
4. \(m_{L,x}, m_{R,x}\) execute a left or right move of the head on the tape, respectively,
5. \(\text{copy}_x\) copies the temporary tape content back to the primary tape, and
6. \(p_q\) represent the states of the Turing machine.

The subscript \(x \in \{0, 1\}\) is used to have two primes for every action: in case an action \(p\) takes more than one calculation step we cannot write \(\frac{p}{p'}\) since then \(p\) in numerator and denominator would cancel itself out. We define the Fractran program \(P_M\) to consist of the following fractions (listed in program order):

\[
\frac{1}{p \cdot p'} \quad \text{for every } p, p' \in \{m_{L,0}, m_{L,1}, m_{R,0}, m_{R,1}, \text{copy}_0, \text{copy}_1\} \\
\text{every } p, p' \in \{p_q \mid q \in Q\} \text{ and } p, p' \in \{h, h'\}
\]

(4)
to get rid of illegal configurations,
\[
\begin{align*}
&\frac{m_{L,1-x} \cdot \text{tape}_r^x}{m_{L,x} \cdot \text{tape}_r^{x+1}} \quad \frac{m_{L,1-x} \cdot \text{tape}_r^2}{m_{L,x} \cdot \text{tape}_r} \quad \frac{m_{L,1-x} \cdot \text{tape}_r'}{m_{L,x} \cdot \text{tape}_r'} \quad \frac{m_{L,1-x} \cdot \text{h}}{m_{L,x} \cdot \text{h}'} \quad \frac{\text{copy}_0}{\text{copy}_x} \\
&\text{(5)}
\end{align*}
\]
with \( x \in \{0, 1\} \), for moving the head left on the tape,
\[
\begin{align*}
&\frac{m_{R,1-x} \cdot \text{tape}_r'}{m_{R,x} \cdot \text{tape}_r^2} \quad \frac{m_{R,1-x} \cdot \text{tape}_r^2}{m_{R,x} \cdot \text{tape}_r} \quad \frac{m_{R,1-x} \cdot \text{tape}_r'}{m_{R,x} \cdot \text{tape}_r} \quad \frac{m_{R,1-x} \cdot \text{h}}{m_{R,x} \cdot \text{h}'} \quad \frac{\text{copy}_0}{\text{copy}_x} \\
&\text{(6)}
\end{align*}
\]
with \( x \in \{0, 1\} \), for moving the head right on the tape,
\[
\begin{align*}
&\frac{\text{copy}_x \cdot \text{copy}_x}{\text{copy}_x \cdot \text{copy}_x} \quad \frac{\text{copy}_{x-1} \cdot \text{copy}_x}{\text{copy}_x \cdot \text{copy}_x} \quad \frac{1}{1} \\
&\text{(7)}
\end{align*}
\]
with \( x \in \{0, 1\} \), for copying the temporary tape back to the primary tape,
\[
\begin{align*}
&\frac{p_{q'} \cdot h' s' \cdot m_{d,0}}{p_q \cdot h} \quad \frac{p_{q'} \cdot h' s' \cdot m_{d,0}}{p_q} \\
&\text{(8)}
\end{align*}
\]
whenever \( \delta(q, 1) = (q', s', d) \)
\[
\begin{align*}
&\frac{1}{p_q \cdot h} \quad \frac{1}{p_q} \\
&\text{(9)}
\end{align*}
\]
(for termination) for every \( q \in Q \)
\[
\begin{align*}
&\frac{p_{q'} \cdot h' s' \cdot m_{d,0}}{p_q} \\
&\text{(10)}
\end{align*}
\]
whenever \( \delta(q, 0) = (q', s', d) \)

for the transitions of the Turing machine. Whenever we use variables in the rules, e.g. \( x \in \{0, 1\} \), then it is to be understood that instances of the same rule are immediate successors in the sequence of fractions (the order of the instances among each other is not crucial).

We give an example to illustrate the definition.

**Example 2.6.** Let \( M = (Q, a_0, \delta) \) be a Turing machine where \( Q = \{a_0, a_1, b\} \)
and the transition function is defined by \( \delta(a_0,0) = (b, 1, R) \), \( \delta(a_1,0) = (b, 1, R) \),
\( \delta(a_0,1) = (a_1, 0, R) \), \( \delta(a_1, 1) = (a_0, 0, R) \), \( \delta(b, 1) = (a_0, 0, R) \), and we leave \( \delta(b, 0) \)
undefined. That is, \( M \) moves to the right, converting zeros into ones and vice versa, until it finds two consecutive zeros and terminates. Assume that \( M \) is started on the configuration \( 1b1001 \), that is, the tape content \( 11001 \) in state \( b \) with the head located on the second 1. In the Fractran program \( P_M \) this corresponds to \( n_0 = p_b \cdot \text{tape}_r^1 \cdot h^1 \cdot \text{tape}_r^{100} \) as the start value where we represent the exponents in binary notation for better readability. Started on \( n_0 \) we obtain the following calculation in \( P_M \):
\[
\begin{align*}
&\quad p_b \cdot \text{tape}_r^1 \cdot h^1 \cdot \text{tape}_r^{100} \quad \text{(configuration 1b1001)} \\
&\rightarrow m_{R,0} \cdot p_{a_0} \cdot \text{tape}_r^1 \cdot \text{tape}_r^{100} \rightarrow m_{R,1} \cdot p_{a_0} \cdot \text{tape}_r^{10} \cdot \text{tape}_r^{100} \\
&\rightarrow 2 m_{R,1} \cdot p_{a_0} \cdot \text{tape}_r^{10} \cdot \text{tape}_r^{10} \rightarrow \text{copy}_0 \cdot p_{a_0} \cdot \text{tape}_r^{10} \cdot \text{tape}_r^{10} \\
&\rightarrow p_{a_0} \cdot \text{tape}_r^{10} \cdot \text{tape}_r^{10} \quad \text{(configuration 10a001)} \\
&\rightarrow m_{R,0} \cdot p_b \cdot \text{tape}_r^{10} \cdot h^1 \cdot \text{tape}_r^{10} \rightarrow 2 m_{R,0} \cdot p_b \cdot \text{tape}_r^{10} \cdot h^1 \cdot \text{tape}_r^{10} \\
&\rightarrow m_{R,1} \cdot p_b \cdot \text{tape}_r^{10} \cdot h^1 \cdot \text{tape}_r^{10} \rightarrow 2 \text{copy}_0 \cdot p_{b} \cdot \text{tape}_r^{10} \cdot \text{tape}_r^{10} \\
&\rightarrow 7 p_b \cdot \text{tape}_r^{10} \cdot \text{tape}_r^{1} \quad \text{(configuration 101001, termination)}
\end{align*}
\]
reaching a configuration where the Fractran program halts.

We give a translation from Turing machine configurations to natural numbers (input values for Fractran programs).

**Definition 2.7.** Let $M = \langle Q, q_0, \delta \rangle$ be a Turing machine. We reuse the notation of Definition 2.5. For every configuration $c = \langle q, \text{tape} \rangle$ of $M$ we define:

$$n_c = \text{tape}^L \cdot p_q \cdot h^H \cdot \text{tape}^R$$

$$L = \sum_{i=0}^{\infty} 2^i \cdot \text{tape}(-1 - i) \quad H = \text{tape}(0) \quad R = \sum_{i=0}^{\infty} 2^i \cdot \text{tape}(1 + i)$$

**Lemma 2.8.** For every Turing machine $M$ and configurations $c_1, c_2$ we have:

(i) if $c_1 \rightarrow_M c_2$ then $n_{c_1} \rightarrow^*_{\text{p}_M} n_{c_2}$, and

(ii) if $c_1$ is a $\rightarrow_M$ normal form then $n_{c_1} \rightarrow^*_{\text{p}_M}$ undefined.

Proofs of Lemma 2.8 and Theorem 2.2 can be found in the appendix.

## 3 What is Productivity?

A program is productive if it evaluates to a finite or infinite constructor normal form. This rather vague description leaves open several choices that can be made to obtain a more formal definition. We explore several definitions and determine the degree of undecidability for each of them. See [4] for more pointers to the literature on productivity.

The following is a productive specification of the (infinite) stream of zeros:

$$\text{zeros} \rightarrow 0 : \text{zeros}$$

Indeed, there exists only one maximal rewrite sequence from $\text{zeros}$ and this ends in the infinite constructor normal form $0 : 0 : 0 : \ldots$. Here and later we say that a rewrite sequence $\rho : t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots$ ends in a term $s$ if either $\rho$ is finite with its last term being $s$, or $\rho$ is infinite and then $s$ is the limit of the sequence of terms $t_i$, i.e., $s = \lim_{i \rightarrow \infty} t_i$. We consider only rewrite sequences starting from finite terms, thus all terms occurring in $\rho$ are finite. Nevertheless, the limit $s$ of the terms $t_i$ may be an infinite term. Note that, if $\rho$ ends in a constructor normal form, then every finite prefix will be evaluated after finitely many steps.

The following is a slightly modified specification of the stream of zeros:

$$\text{zeros} \rightarrow 0 : \text{id}(\text{zeros}) \quad \text{id}(\sigma) \rightarrow \sigma$$

This specification is considered productive as well, although there are infinite rewrite sequences that do not even end in a normal form, let alone in a constructor normal form: e.g. by unfolding $\text{zeros}$ only we get the limit term $0 : \text{id}(0 : \text{id}(0 : \text{id}(\ldots)))$. In general, normal forms can only be reached by outermost-fair rewriting sequences. A rewrite sequence $\rho : t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots$ is *outermost-fair* [13] if there is no $t_n$ containing an outermost redex which remains an outermost redex infinitely long, and which is never contracted. For this reason it is natural to consider productivity of terms with respect to outermost-fair strategies.
What about stream specifications that admit rewrite sequences to constructor normal forms, but that also have divergent rewrite sequences:

\[
\text{maybe} \rightarrow 0 : \text{maybe} \quad \text{maybe} \rightarrow \text{sink} \quad \text{sink} \rightarrow \text{sink}
\]

This example illustrates that, for non-orthogonal stream specifications, reachability of a constructor normal form depends on the evaluation strategy. The term \text{maybe} is only productive with respect to strategies that always apply the first rule.

For this reason we propose to think of productivity as a property of individual terms with respect to a given rewrite strategy. This reflects the situation in functional programming, where expressions are evaluated according to an in-built strategy. These strategies are usually based on a form of outermost-needed rewriting with a priority order on the rules.

### 3.1 Productivity with respect to Strategies

For term rewriting systems (TRSs)\cite{13} we now fix definitions of the notions of (history-free) strategy and history-aware strategy. Examples for the latter notion are outermost-fair strategies, which typically have to take history into account.

**Definition 3.1.** Let \( R \) be a TRS with rewrite relation \( \rightarrow_R \).

A strategy for \( \rightarrow_R \) is a relation \( \sim \subseteq \rightarrow_R \) with the same normal forms as \( \rightarrow_R \).

The history-aware rewrite relation \( \rightarrow_{H,R} \) for \( R \) is the binary relation on \( \text{Ter}(\Sigma) \times (R \times \mathbb{N}^*)^* \) that is defined by:

\[
\langle s, h_s \rangle \rightarrow_{H,R} \langle t, h_t : \langle \rho, p \rangle \rangle \iff s \rightarrow t \text{ via rule } \rho \in R \text{ at position } p.
\]

We identify \( t \in \text{Ter}(\Sigma) \) with \( \langle t, \epsilon \rangle \), and for \( s, t \in \text{Ter}(\Sigma) \) we write \( s \rightarrow_{H,R} t \) whenever \( \langle s, \epsilon \rangle \rightarrow_{H,R} \langle t, h \rangle \) for some history \( h \in (R \times \mathbb{N}^*)^* \). A history-aware strategy for \( R \) is a strategy for \( \rightarrow_{H,R} \).

A strategy \( \sim \) is deterministic if \( s \sim t \) and \( s \sim t' \) implies \( t = t' \). A strategy \( \sim \) is computable if the function mapping a term (a term/history pair) to its set of \( \sim \)-successors is a total recursive function, after coding into natural numbers.

**Remark 3.2.** Our definition of strategy for a rewrite relation follows \cite{14}. For abstract rewriting systems, in which rewrite steps are first-class citizens, a definition of strategy is given in \cite[Ch.9]{13}. There, history-aware strategies for a TRS \( R \) are defined in terms of ‘labellings’ for the ‘abstract rewriting system’ underlying \( R \). While that approach is conceptually advantageous, our definition of history-aware strategy is equally expressive.

**Definition 3.3.** A (TRS-indexed) family of strategies \( S \) is a function that assigns to every TRS \( R \) a set \( S(R) \) of strategies for \( R \). We call such a family \( S \) of strategies admissible if \( S(R) \) is non-empty for every orthogonal TRS \( R \).

Now we give the definition of productivity with respect to a strategy.
Definition 3.4. A term $t$ is called productive with respect to a strategy $\leadsto$ if all maximal $\leadsto$ rewrite sequences starting from $t$ end in a constructor normal form.

In the case of non-deterministic strategies we require here that all maximal rewrite sequences end in a constructor normal form. Another possible choice could be to require only the existence of one such rewrite sequence (see Subsection 3.2). However, we think that productivity should be a practical notion. Productivity of a term should entail that arbitrary finite parts of the constructor normal form can indeed be evaluated. The mere requirement that a constructor normal form exists leaves open the possibility that such a normal form cannot be approximated to every finite precision in a computable way.

For orthogonal TRSs outermost-fair (or fair) rewrite strategies are the natural choice for investigating productivity because they guarantee to find (the unique) infinitary constructor normal form whenever it exists (see [13]).

Pairs and finite lists of natural numbers can be encoded using the well-known Gödel encoding. Likewise terms and finite TRSs over a countable set of variables can be encoded. A TRS is called finite if its signature and set of rules are finite.

In the sequel we restrict to (families of) computable strategies, and assume that strategies are represented by appropriate encodings.

Now we define the productivity problem in TRSs with respect to families of computable strategies, and prove a $\Pi^0_2$-completeness result.

**Productivity Problem** with respect to a family $\mathcal{S}$ of computable strategies

*Instance*: Encodings of a finite TRS $R$, a strategy $\leadsto \in \mathcal{S}(R)$ and a term $t$.

*Answer*: ‘Yes’ if $t$ is productive with respect to $\leadsto$, and ‘No’, otherwise.

**Theorem 3.5.** For every family of admissible, computable strategies $\mathcal{S}$, the productivity problem with respect to $\mathcal{S}$ is $\Pi^0_2$-complete.

**Proof.** A Turing machine is called total (encodes a total function $\mathbb{N} \to \mathbb{N}$) if it halts on all inputs encoding natural numbers. The problem of deciding whether a Turing machine is total is well-known to be $\Pi^0_2$-complete, see [6]. Let $M$ be an arbitrary Turing machine. Employing the encoding of Turing machines into orthogonal TRSs from [7], we can define a TRS $R_M$ that simulates $M$ such that for every $n \in \mathbb{N}$ it holds: every reduct of the term $M(s^n(0))$ contains at most one redex occurrence, and the term $M(s^n(0))$ rewrites to 0 if and only if the Turing machine $M$ halts on the input $n$. Note that the rewrite sequence starting from $M(s^n(0))$ is deterministic. We extend the TRS $R_M$ to a TRS $R'_M$ with the following rules:

$$\text{go}(0, x) \rightarrow 0 : \text{go}(M(x), s(x))$$

and choose the term $t = \text{go}(0, 0)$. Then $R'_M$ is orthogonal and by construction every reduct of $t$ contains at most one redex occurrence (consequently all strategies for $R$ coincide on every reduct of $t$). The term $t$ is productive if and only if $M(s^n(0))$ rewrites to 0 for every $n \in \mathbb{N}$ which in turn holds if and only if the Turing machine $M$ is total. This concludes $\Pi^0_2$-hardness.

For $\Pi^0_2$-completeness let $\mathcal{S}$ be a family of computable strategies, $R$ a TRS, $\leadsto \in \mathcal{S}(R)$ and $t$ a term. Then productivity of $t$ can be characterised as:
∀d ∈ N. ∃n ∈ N. every n-step \leadsto-reducts of t

is a constructor normal form up to depth d  

Since the strategy \leadsto is computable and finitely branching, all n-step reducts of t can be computed. Obviously, if the formula (⋆) holds, then t is productive w.r.t. \leadsto. Conversely, assume that t is productive w.r.t. \leadsto. For showing (⋆), let d ∈ N be arbitrary. By productivity of t w.r.t. \leadsto, on every path in the reduction graph of t w.r.t. \leadsto eventually a term with a constructor normal form up to depth d is encountered. Since reduction graphs in TRS’s always are finitely branching, Koenig’s lemma implies that there exists an n ∈ N such that all terms on depth greater or equal to n in the reduction graph of t are constructor prefixes of depth at least d. Since d was arbitrary, (⋆) has been established. Because (⋆) is a Π^0_2-formula, the productivity problem with respect to S also belongs to Π^0_2.

Thm. 3.5 implies that productivity is Π^0_2-complete for orthogonal TRSs with respect to outermost-fair rewriting. To see this, apply the theorem to the family of strategies that assigns to every orthogonal TRS \cal R the set of computable, outermost-fair rewriting strategies for \cal R, and \emptyset to non-orthogonal TRSs.

The definition of productivity with respect to computable strategies reflects the situation in functional programming. Nevertheless, we now investigate variants of this notion, and determine their respective computational complexity.

3.2 Strong Productivity

As already discussed, only outermost-fair rewrite sequences can reach a constructor normal form. Dropping the fine tuning device ‘strategies’, we obtain the following stricter notion of productivity.

**Definition 3.6.** A term t is called **strongly productive** if all maximal outermost-fair rewrite sequences starting from t end in a constructor normal form.

The definition requires all outermost-fair rewrite sequences to end in a constructor normal form, including non-computable rewrite sequences. This catapults productivity into a much higher class of undecidability: Π^1_1, a class of the analytical hierarchy. The analytical hierarchy continues the classification of the arithmetical hierarchy using second order formulas. The computational complexity of strong productivity therefore exceeds the expressive power of first-order logic to define sets from recursive sets.

A well-known result of recursion theory states that for a given computable relation > ⊆ N × N it is Π^1_1-hard to decide whether > is well-founded, see [6]. Our proof is based on a construction from [3]. There a translation from Turing machines M to TRSs \Root_M together with a term t_M is given such that: t_M is root-terminating (i.e., t_M admits no rewrite sequences containing an infinite number of root steps) if and only if the binary relation >_M encoded by M is well-founded. The TRS \Root_M consists of rules for simulating the Turing machine M such that M(x, y) \rightarrow^* T iff x >_M y holds, a rule:

\[
\text{run}(T, \text{ok}(x), \text{ok}(y)) \rightarrow \text{run}(M(x, y), \text{ok}(y), \text{pickn})
\]

and rules for randomly generating a natural number:
The term $t_M = \text{run}(T, \text{pickn}, \text{pickn})$ admits a rewrite sequence containing infinitely many root steps if and only if $>_M$ is not well-founded. Note that $t_M$ and all of its reducts contain exactly one occurrence of the symbol run, namely at the root position.

**Theorem 3.7.** Strong productivity is $\Pi^1_1$-complete.

*Proof.* For the proof of $\Pi^1_1$-hardness, let $M$ be a Turing machine. We extend the TRS $\text{Root}_M$ from [3] with the rule $\text{run}(x, y, z) \to 0:\text{run}(x, y, z)$. As a consequence the term $\text{run}(T, \text{pickn}, \text{pickn})$ is strongly productive if and only if $>_M$ is well-founded (which is $\Pi^1_1$-hard to decide). If $>_M$ is not well-founded, then by the result in [3] $t_M$ admits a rewrite sequence containing infinitely many root steps which obviously does not end in a constructor normal form. On the other hand if $>_M$ is well-founded, then $t_M$ admits only finitely many root steps with respect to $\text{Root}_M$, and thus by outermost-fairness the freshly added rule has to be applied infinitely often. This concludes $\Pi^1_1$-hardness.

Rewrite sequences of length $\omega$ can be represented by functions $r : \mathbb{N} \to \mathbb{N}$ where $r(n)$ represents the $n$-th term of the sequence together with the position and rule applied in step $n$. Then for all $r$ (one universal $\forall r$ function quantifier) we have to check that $r$ converges towards a constructor normal form whenever $r$ is outermost-fair; this can be checked by a first order formula. We refer to [3] for the details of the encoding. Hence strong productivity is in $\Pi^1_1$. $\square$

### 3.3 Weak Productivity

A natural counterpart to strong productivity is the notion of ‘weak productivity’: the existence of a rewrite sequence to a constructor normal form. Here outermost-fairness does not need to be required, because rewrite sequences that reach normal forms are always outermost-fair.

**Definition 3.8.** A term $t$ is called weakly productive if there exists a rewrite sequence starting from $t$ that ends in a constructor normal form.

For non-orthogonal TRSs the practical relevance of this definition is questionable since, in the absence of a computable strategy to reach normal forms, mere knowledge that a term $t$ is productive does typically not help to find a constructor normal form of $t$. For orthogonal TRSs computable, normalising strategies exist, but then also all of the variants of productivity coincide (see Sec. 3.4).

**Theorem 3.9.** Weak productivity is $\Sigma^1_1$-complete.

*Proof.* For the proof of $\Sigma^1_1$-hardness, let $M$ be a Turing machine. We exchange the rule $\text{run}(T, \text{ok}(x), \text{ok}(y)) \to \text{run}(M(x, y), \text{ok}(y), \text{pickn})$ in the TRS $\text{Root}_M$ from [3] by the rule $\text{run}(T, \text{ok}(x), \text{ok}(y)) \to 0:\text{run}(M(x, y), \text{ok}(y), \text{pickn})$. Then we obtain that the term $\text{run}(T, \text{pickn}, \text{pickn})$ is weakly productive if and only if $>_M$ is not well-founded (which is $\Sigma^1_1$-hard to decide). This concludes $\Pi^1_1$-hardness.

The remainder of the proof proceeds analogously to the proof of Theorem 3.7, except that we now have an existential function quantifier $\exists r$ to quantify over all rewrite sequences of length $\omega$. Hence weak productivity is in $\Sigma^1_1$. $\square$
3.4 Discussion

For orthogonal TRSs all of the variants of productivity coincide. That is, if we restrict the first variant to computable outermost-fair strategies; as already discussed, other strategies are not very reasonable. For orthogonal TRSs there always exist computable outermost-fair strategies, and whenever for a term there exists a constructor normal form, then it is unique and all outermost-fair rewrite sequences will end in this unique constructor normal form.

This raises the question whether uniqueness of the constructor normal forms should be part of the definition of productivity. We consider a specification of the stream of random bits:

\[
\text{random} \rightarrow 0 : \text{random} \quad \text{random} \rightarrow 1 : \text{random}
\]

Every rewrite sequence starting from random ends in a normal form. However, these normal forms are not unique. In fact, there are uncountably many of them. We did not include uniqueness of normal forms into the definition of productivity since non-uniqueness only arises in non-orthogonal TRSs when using non-deterministic strategies. However, one might want to require uniqueness of normal forms even in the case of non-orthogonal TRSs.

**Theorem 3.10.** The problem of determining, for TRSs \(R\) and terms \(t\) in \(R\), whether \(t\) has a unique (finite or infinite) normal form is \(\Pi^1_1\)-complete.

**Proof.** For \(\Pi^1_1\)-hardness, we extend the TRS constructed in the proof of Theorem 3.9 by the rules: \(\text{start} \rightarrow \text{run}(T, \text{pick}n, \text{pick}n)\), \(\text{run}(x, y, z) \rightarrow \text{run}(x, y, z)\) \(\text{start} \rightarrow \text{ones}\), and \(\text{ones} \rightarrow 1 : \text{ones}\). Then \(\text{start}\) has a unique normal form if and only if \(M\) is well-founded. For \(\Pi^1_1\)-completeness, we observe that the property can be characterised by a \(\Pi^1_1\)-formula: we can quantify over two rewrite sequences \(\forall r_1 \forall r_2\), and, in case both of them end in a normal form, we compare them. Note that no consecutive universal quantifiers can be compressed into one. □

Let us consider the impact on computational complexity of taking up the condition of uniqueness of normal forms into the definition of productivity. Including uniqueness of normal forms without considering the strategy would increase the complexity of productivity with respect to a family of strategies to \(\Pi^1_1\). However, we think that doing so would be contrary to the spirit of the notion of productivity. Uniqueness of normal forms should only be required for the normal forms reachable by the given (non-deterministic) strategy. But then the complexity of productivity remains unchanged, \(\Pi^1_2\)-complete. The complexity of strong productivity remains unaltered, \(\Pi^1_2\)-complete, when including uniqueness of normal forms. However, the degree of undecidability of weak productivity increases. From the proofs of Theorems 3.9 and 3.10 it follows that the property would then both be \(\Sigma^1_2\)-hard and \(\Pi^1_2\)-hard, then being in \(\Delta^1_1\).

4 Encoding Fractran Programs into Stream Specifications

We give a translation from Fractran programs into stream specifications.

**Definition 4.1.** Let \(P = \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\) be a Fractran program. Let \(d\) be the least
For every $n$ is defined we need $\sigma \mod n$ is an integer, and we let $p'_n$ and $b_n$ be undefined if no such fraction exists. Then, the stream specification induced by $P$ is a term rewriting system $R_P = (\Sigma_P, R_P)$ with:

$$\Sigma_P = \{ \cdot : \cdot, \text{head}, \text{tail}, \text{zip}_d, M_P \} \cup \{ \text{mod}_{p'_n} | p'_n \text{ is defined} \}$$

and with $R_P$ consisting of the following rules:

$$M_P \rightarrow \text{zip}_d(T_1, \ldots, T_d), \text{ where, for } 1 \leq n \leq d, T_n \text{ is shorthand for:}$$

$$T_n = \begin{cases} \text{mod}_{p'_n}(\text{tail}^{b_n-1}(M_P)) & \text{if } p'_n \text{ is defined}, \\ \cdot : \text{mod}_d(\text{tail}^{n-1}(M_P)) & \text{if } p'_n \text{ is undefined.} \end{cases}$$

$$\text{head}(x : \sigma) \rightarrow x \quad \text{mod}_k(\sigma) \rightarrow \text{head}(\sigma) : \text{mod}_k(\text{tail}^k(\sigma))$$

$$\text{tail}(x : \sigma) \rightarrow \sigma \quad \text{zip}_d(\sigma_1, \sigma_2, \ldots, \sigma_d) \rightarrow \text{head}(\sigma_1) : \text{zip}_d(\sigma_2, \ldots, \sigma_d, \text{tail}(\sigma_1))$$

where $x, \sigma, \sigma_i$ are variables.\(^{1}\)

The rule for $\text{mod}_n$ defines a stream function which takes from a given stream $\sigma$ all elements $\sigma(i)$ with $i \equiv 0 \pmod n$, and results in a stream consisting of those elements in the original order. As we only need rules $\text{mod}_{p'_n}$ whenever $p'_n$ is defined we need $d$ such rules at most.

If $p'_n$ is undefined then it should be understood that $m \cdot p'_n$ is undefined. For $n \in \mathbb{N}$ let $\varphi(n)$ denote the number from $\{1, \ldots, d\}$ with $n \equiv \varphi(n) \pmod d$.

**Lemma 4.2.** For every $n > 0$ we have $f_P(n) = [(n-1)/d] \cdot p'_{\varphi(n)} + b_{\varphi(n)}$.

**Proof.** Let $n > 0$. For every $i \in \{1, \ldots, k\}$ we have $n \cdot b_i/q_i \in \mathbb{N}$ if and only if $\varphi(n) \cdot b_i/q_i \in \mathbb{N}$, since $n \equiv \varphi(n) \pmod d$ and $d$ is a multiple of $q_i$. Assume that $f_P(n)$ is defined. Then $f_P(n) = n \cdot p'_{\varphi(n)}/d = [(n-1)/d] \cdot d + ((n-1) \cdot d + 1) \cdot p'_{\varphi(n)}/d = [(n-1)/d] \cdot p'_{\varphi(n)} + \varphi(n) \cdot p_i/q_i = [(n-1)/d] \cdot p'_{\varphi(n)} + b_{\varphi(n)}$. Otherwise whenever $f_P(n)$ is undefined then $p'_{\varphi(n)}$ is undefined.\(\Box\)

**Lemma 4.3.** Let $P$ be a Fractran program. Then $R_P$ is productive for $M_P$ if and only if $P$ is terminating on all integers $n > 0$.

**Proof.** Let $\sigma(n)$ be shorthand for $\text{head}(\text{tail}^n(\sigma))$. It suffices to show for all $n \in \mathbb{N}$: $M_P(n) \rightarrow^* \bullet$ if and only if $P$ halts on $n$. For this purpose we show $M_P(n) \rightarrow^+ \bullet$ whenever $f_P(n+1)$ is undefined, and $M_P(n) \rightarrow^+ M_P(f_P(n+1) - 1)$, otherwise. We have $M_P(n) \rightarrow^* \text{tail}^{\varphi(n+1)}([n/d])$.

Assume that $f_P(n+1)$ is undefined. By Lemma 4.2 $p'_{\varphi(n+1)}$ is undefined, thus $M_P(n) \rightarrow^* \bullet$ whenever $[n/d] = 0$, and otherwise we have:

$$M_P(n) \rightarrow^* \text{tail}^{\varphi(n+1)}([n/d]) \rightarrow^* \text{mod}_d(\text{tail}^{2^{\varphi(n+1)}}(M_P))([n/d] - 1) \rightarrow^* M_P(n')$$

\(^{1}\) Note that, $\text{mod}_d(\text{tail}^n(\text{zip}_d(T_1, \ldots, T_d)))$ equals $T_n$, and so, in case $p'_n$ is undefined, we just have $T_n = \bullet : T_n$. In order to have the simplest TRS possible (for the purpose at hand), we did not want to use an extra symbol ($\bullet$) and rule ($\bullet \rightarrow \bullet : (\bullet)$).
where \( n' = \lfloor n/d \rfloor - 1 \cdot d + \varphi(n + 1) - 1 = n - d \). Clearly \( n \equiv n' \mod d \) and then \( M_P(n) \rightarrow^* \bullet \) follows by induction on \( n \).

Assume that \( f_P(n + 1) \) is defined. By Lemma 4.2 \( p'_{\varphi(n+1)} \) is defined and:

\[
M_P(n) \rightarrow^* T_{\varphi(n+1)}(\lfloor n/d \rfloor) \rightarrow^* \text{mod}_{p'_{\varphi(n+1)}}(\text{tail}_{p'_{\varphi(n+1)-1}}(M_P)(\lfloor n/d \rfloor))
\]

and hence \( M_P(n) \rightarrow^+ M_P(n') \) with \( n' = \lfloor n/d \rfloor \cdot p'_{\varphi(n+1)} + b_{\varphi(n+1)} - 1 \). Then we have \( n' = f_P(n + 1) - 1 \) by Lemma 4.2. \( \square \)

**Theorem 4.4.** The restriction of the productivity problem to stream specifications induced by Fractran programs and outermost-fair strategies is \( \Pi^0_2 \)-complete.

**Proof.** Since by Lemma 4.3 the uniform halting problem for Fractran programs can be reduced to the problem here, \( \Pi^0_2 \)-hardness is a consequence of Theorem 2.2. \( \Pi^0_2 \)-completeness follows from membership of the problem in \( \Pi^0_2 \), which can be established analogously as in the proof of Theorem 3.5. \( \square \)

Note that Theorem 4.4 also gives rise to an alternative proof for the \( \Pi^0_2 \)-hardness part of Theorem 3.5, the result concerning the computational complexity of productivity with respect to strategies.

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Appendix

**Lemma 2.8** For every Turing machine $M$ and configurations $c_1$, $c_2$ we have:

(i) if $c_1 \rightarrow_M c_2$ then $n_{c_1} \rightarrow_{P_M} n_{c_2}$, and

(ii) if $c_1$ is a $\rightarrow_M$ normal form then $n_{c_1} \rightarrow_{P_M}$ undefined.

**Proof.** Let $M = \langle Q, q_0, \delta \rangle$, and $c_i = \langle q_i, \text{tape}_i \rangle$ for $i \in \{1, 2\}$. Then for $i \in \{0, 1\}$ we have $n_{c_i} = \text{tape}_{c_i}^{L_i} \cdot p_{q_i} \cdot h^{H_i} \cdot \text{tape}_{c_i}^{R_i}$ with $L_i$, $H_i$ and $R_i$ as in Definition 2.7. For (i) assume that $c_1 \rightarrow_M c_2$. By the definition of $\rightarrow_M$ there are two cases: the head moves left or right. We consider ‘moving left’ (‘moving right’ is analogous). If $\delta(q_1, \text{tape}_1(0)) = \langle q_2, s, L \rangle$, $\text{tape}_2(1) = s$, $\forall n \neq 0$. $\text{tape}_2(n + 1) = \text{tape}_1(n)$. Therefore $L_2 = \lfloor L_1/2 \rfloor$, $H_2 = \text{tape}_1(-1) = L_1$ mod 2 and $R_2 = s + 2 \cdot R_1$. If $\text{tape}_1(0) = 0$ then $H_1 = 1$ and therefore the first fraction of $P_M$ applicable to $n_{c_1}$ is $(p_{q_2} \cdot h^{s} \cdot m_{L,0})/(p_{q_1} \cdot h)$; otherwise if $\text{tape}_1(0) = 0$ then $H_1 = 0$ and the fraction is $(p_{q_2} \cdot h^{s} \cdot m_{L,0})/(p_{q_1})$. Both cases result in:

$$n_{c_1} \rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot h^{s} \cdot \text{tape}_{c_1}^{R_i} \cdot m_{L,0}$$

$$\rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot \text{mod}^2 \cdot \text{tape}_{c_1}^{\lfloor L_i/2 \rfloor} \cdot h^{s} \cdot \text{tape}_{c_1}^{R_i} \cdot m_{L,x} \quad \text{(1st } m_L \text{ fraction)}$$

$$\rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot \text{mod}^2 \cdot \text{tape}_{c_1}^{\lfloor L_i/2 \rfloor} \cdot h^{s} \cdot \text{tape}_{c_1}^{R_i} \cdot m_{L,y} \quad \text{(2nd } m_L \text{ fraction)}$$

$$\rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot \text{mod}^2 \cdot \text{tape}_{c_1}^{\lfloor L_i/2 \rfloor} \cdot \text{tape}_{c_1}^{s+2} \cdot R_i \cdot m_{L,y} \quad \text{(3rd } m_L \text{ fraction)}$$

$$\rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot \text{mod}^2 \cdot \text{tape}_{c_1}^{\lfloor L_i/2 \rfloor} \cdot \text{tape}_{c_1}^{s+2} \cdot R_i \cdot m_{L,x} \quad \text{(4th } m_L \text{ fraction)}$$

$$\rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot \text{mod}^2 \cdot \text{tape}_{c_1}^{\lfloor L_i/2 \rfloor} \cdot \text{tape}_{c_1}^{s+2} \cdot R_i \cdot \text{copy}_0 \quad \text{(5th } m_L \text{ fraction)}$$

$$\rightarrow_{P_M} p_{q_2} \cdot \text{tape}_{c_1}^{L_i} \cdot \text{mod}^2 \cdot \text{tape}_{c_1}^{s+2} \cdot R_i = n_{c_2} \quad \text{(copy fractions)}$$

For (ii) assume that $c_1$ is a $\rightarrow_M$ normal form. Then $\delta(q_1, \text{tape}_1(0))$ is undefined. If $\text{tape}_1(0) = 1$ then $H_1 = 1$. Since there is no matching fraction (8), the first applicable fraction is from (9), which removes $p_{q_1}$ and thus leads to termination. If $\text{tape}_1(0) = 0$ then $H_1 = 0$ thus the only applicable fractions can be among (10) however there is no matching fraction since $\delta(q_1, \text{tape}_1(0))$ is undefined. \hfill $\square$

**Theorem 2.2** The uniform halting problem for Fractran programs, that is, deciding whether a program halts for every starting value $n_0 \in \mathbb{N}_{>0}$, is $\Pi_2^0$-complete.

**Proof.** For $\Pi_2^0$-hardness use the uniform halting problem of Turing machines which is $\Pi_2^0$-complete (see Theorem 2.4). Let $M$ be a Turing machine. We prove that $M$ halts on all configurations and only if $P_M$ halts on all integers $n > 0$. If there is a configuration $c$ on which $M$ does not halt, then the Fractran program $P_M$ does not halt on $n_c$ by Lemma 2.8 (i). Thus assume that $M$ halts on all configurations. Let $C = \{ m_{L,0}, m_{L,1}, m_{R,0}, m_{R,1}, \text{copy}_0, \text{copy}_1 \}$. Let $n > 0$ be arbitrary. By Lemma 2.8 it suffices to show that $n \rightarrow_{P_M}^* n'$ undefined or $n \rightarrow_{P_M}^* n_c$ for some configuration $C$. By the first fractions of $P_M$ we have $n \rightarrow_{P_M}^* n'$ such that $n'$ contains at most one prime factor from $C$, at most one $p_q$ and at most one $\{ h, h' \}$ (and none of these primes has an exponent $> 1$).
Assume that \( n' \) contains \( m_L \) or \( m_R \). The \( m_L \) or \( m_R \) fractions cannot be applied infinitely often in sequence since they decrease all exponents of \( \text{tape}_l' \), \( \text{tape}_r' \), and \( h' \) to 0, respectively. After \( m_L \) or \( m_R \) there always follows \( \text{copy} \) which then increases \( \text{tape}_l \), \( \text{tape}_r \), but decreases \( \text{tape}_l' \), \( \text{tape}_r' \) to the value 0 and afterwards \( \text{copy} \) ‘removes’ itself. We call the reached configuration \( n'' \). Then \( n'' \) contains only the prime factors \( \text{tape}_l \), \( \text{tape}_r \) with exponent \( \geq 0 \), \( h \) with exponent \( \leq 1 \) and at most one of the \( p_q \) with exponent \( \leq 1 \). If \( n'' \) does not contain any \( p_q \) then \( n'' \rightarrow p_{ij} \) undefined. Otherwise there exists a configuration \( c \) such that \( n'' = n_c \).

If \( n' \) does not contain \( m_L \) or \( m_R \), then neither the fractions (4), nor (7) \( (\text{copy}) \), can be applied infinitely often in sequence. Application of (9) removes the only remaining \( p_q \) and thus leads to termination. Thus any non-terminating sequence contains an application of (8) or (10), which brings us back to the case of \( n' \) containing \( m_L \) or \( m_R \) which we have already analysed. \( \square \)