Beta-function of Bruhat–Tits buildings and deformation of $l^2$ on the set of $p$-adic lattices

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For the space $\text{Lat}_n$ of all the lattices in a $p$-adic $n$-dimensional linear space we obtain an analog of matrix beta-function; this beta-function has a degeneration to the Tamagawa zeta-function. We propose an analog of Berezin kernels for $\text{Lat}_n$. We obtain conditions of positive definiteness of these kernels and explicit Plancherel formula.

It is well known that affine Bruhat–Tits buildings (see, for instance, [2], [3]) are right $p$-adic analogs of Riemannian noncompact symmetric spaces. This analogy exists on the levels of geometry, harmonic analysis, and special functions. We continue this parallel.

First, we obtain an imitation of a matrix beta-function (see [4], [9]) for the spaces

$$\text{GL}_n(\mathbb{Q}_p)/\text{GL}_n(\mathbb{Z}_p),$$

see Theorem 2.1, this beta extends the Tamagawa zeta (Corollary 2.12, Theorem 2.13).

Second, we define an analog of Berezin kernels (see [11]) for the homogeneous spaces (0.1). We obtain conditions of positive definiteness of these kernels (Theorem 2.2) and the Plancherel formula (Theorems 2.3–2.4).

It seems, that there is no natural $p$-adic analogs of Hermitian Cartan domains and holomorphic discrete series. Nevertheless, the Berezin kernels on Riemannian symmetric spaces admit an imitation for buildings. More precisely, our objects are spaces of lattices, i.e., sets of vertices of buildings. These ’Berezin kernels’ (see formula (2.5)) have sense for any building associated with a classical $p$-adic group; in this paper, we consider only the series of groups $\text{GL}_n$. In this case the Plancherel formula, the conditions of positive definiteness, limit behavior are similar to the real case as precisely as it is possible. By an analogy with the real case, it seems that for sympletic and especially orthogonal groups the theory must be more rich. It is natural to wait for appearence of more complicated spectra (as in [11]) and of operators of restriction to some ideal boundary (as in [10]) separating these spectra; the both phenomena really appear for Bruhat–Tits trees, see [12].

For the Bruhat–Tits trees our kernels were known earlier (see [5], [13]), in this case (it is not discussed in this paper) the parallel between the Lobachevsky plane and the tree also can be extended to the group of diffeomorphisms of circle, see [12].

Section 1 contains preliminaries, the results of this work are formulated in Section 2, proofs are contained in Sections 3–6.

1. Preliminaries

1.1. Notation. Denote by $\mathbb{K}$ the field $\mathbb{Q}_p$ of $p$-adic numbers. By $\mathbb{O}$ we denote the ring of $p$-adic integers. By $|\cdot|$ we denote the norm on $\mathbb{K}$. If $z, z^{-1} \in \mathbb{O}$, then $|z| = 1$. Also $|p| = p^{-\frac{1}{2}}$, and $|zu| = |z| |u|$.

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By $\mathbb{F}_p$ we denote the field with $p$ elements; we have $\mathbb{F}_p \simeq \mathbb{O}/p\mathbb{O}$.

Denote by $K^n$ the $n$-dimensional linear space over $K$. Denote by $e_1, \ldots, e_n$ the standard basis in $K^n$. By $K^l$, we denote the subspace in $K^n$ spanned by $e_1, \ldots, e_l$.

By $GL_n(K)$ we denote the group of invertible $n \times n$ matrices over $K$. By $GL_n(K)$ we denote the group of matrices $g \in GL_n(K)$ such that all the matrix elements of $g$ and $g^{-1}$ are integer. The group $GL_n(K)$ is a maximal compact subgroup in $GL_n(K)$.

By $\text{Lat}_n$ we denote the space of all lattices (see a definition in 1.2) in $K^n$. By $l^2(\text{Lat}_n)$ we denote the Hilbert space of complex-valued functions $f$ on $\text{Lat}_n$ satisfying the condition $\sum_R |f(R)|^2 < \infty$; this space is equipped with the scalar product $\langle f_1, f_2 \rangle = \sum_{R \in \text{Lat}_n} f_1(R) \overline{f_2(R)}$.

### 1.2. Spaces of lattices.

A lattice in $K^n$ is a compact open $\mathbb{O}$-submodule.

Recall, that any lattice $R$ can be represented in the form

$$R = \mathbb{O}f_1 + \cdots + \mathbb{O}f_n,$$

where $f_1, \ldots, f_n$ is some basis in $K^n$, see [17], §II.2. By $\mathbb{O}^n$ we denote the lattice $\mathbb{O}e_1 + \cdots + \mathbb{O}e_n$.

The group $GL_n(K)$ acts transitively on $\text{Lat}_n$. The stabilizer of the lattice $\mathbb{O}^n$ is $GL_n(\mathbb{O})$. Thus,

$$\text{Lat}_n = GL_n(K)/GL_n(\mathbb{O}).$$

Orbits of the compact group $GL_n(\mathbb{O})$ on $\text{Lat}_n$ are finite. Each orbit contains a unique representative of the form

$$p^{k_1} \mathbb{O}e_1 + \cdots + p^{k_n} \mathbb{O}e_n,$$

where $k_1 \geq \cdots \geq k_n$, (1.2)

see [17], §II.2, Theorem 2.

In other words, the space $GL_n(\mathbb{O}) \setminus \text{Lat}_n$ of $GL_n(\mathbb{O})$-orbits on $\text{Lat}_n$ is in one-to-one correspondence with the set $\mathfrak{P}_n^+$, whose elements are collections of integers

$$\mathfrak{k} : \quad k_1 \geq \cdots \geq k_n.$$ (1.3)

Denote by $\mathcal{O}[\mathfrak{k}]$ the orbit containing the lattice (1.2).

The set $\mathfrak{P}_n^+ = GL_n(\mathbb{O}) \setminus \text{Lat}_n$ coincides with the space of double cosets $GL_n(\mathbb{O}) \setminus GL_n(K)/GL_n(\mathbb{O})$.

For $s \in \mathbb{Z}$, denote by $m_s$ the number of $k_j$ equal to $s$. The number of points in the orbit $\mathcal{O}[\mathfrak{k}]$ is

$$\nu(\mathfrak{k}) = p^{\sum (n-2j+1)k_j} \prod_{1 \leq t \leq n} \left(1 - p^{-t}\right) \prod_s \prod_{1 \leq t \leq m_s} \left(1 - p^{-t}\right),$$ (1.4)

see [7], formula (V.2.9).
1.3. **Volume.** There exists a unique up to a scalar factor translation invariant measure (*volume*) on $\mathbb{K}^n$. We normalize the volume in $\mathbb{K}^n$ by the condition

$$\text{vol}(\mathbb{O}^n) = 1.$$ 

Obviously,

$$\text{vol}(\bigoplus_{j=1}^n k_j \cdot \mathbb{O} e_j) = p^{-\sum k_j}.$$ 

We also normalize the volume in any linear subspace $V \subset \mathbb{K}^n$ by the condition

$$\text{vol}_V(\mathbb{O}^n \cap V) = 1.$$ 

1.4. **Spherical functions on $GL_n(\mathbb{K})$.** Denote by $B_n$ the group of all upper triangular $n \times n$ matrices over $\mathbb{K}$, the group $B_n$ is the stabilizer of the flag

$$0 = \mathbb{K}^0 \subset \mathbb{K}^1 \subset \cdots \subset \mathbb{K}^{n-1} \subset \mathbb{K}^n.$$ 

Denote by $\text{Fl}_n$ the space of all flags $V$: $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{K}^n$; where $\dim V_j = j$, in $\mathbb{K}^n$; the space $\text{Fl}_n$ is the homogeneous space $GL_n(\mathbb{K})/B_n$. This space is also homogeneous with respect to the compact group $GL_n(\mathbb{O})$; hence there exists a unique up to a scalar factor $GL_n(\mathbb{O})$-invariant measure $dV$ on $\text{Fl}_n$. We assume that the total measure of $\text{Fl}_n$ is 1.

For an element $V_j$ of a flag $V$, consider the natural measure $\text{vol}_{V_j}$ on $V_j$ and the natural measure $\text{vol}_{gV_j}$ on $gV_j$. Consider also the image $g \cdot \text{vol}_{V_j}$ of $\text{vol}_{V_j}$ under the map $g$. Thus we have two measures on $gV_j$, denote by $a_j(g; V)$ their ratio, i.e.,

$$g \cdot \text{vol}_{V_j}(Q) = a_j(g; V) \text{vol}_{gV_j}(Q)$$

for each subset $Q \subset gV_j$.

Obviously, these numbers (multipliers) satisfy the identity

$$a_j(g_1 g_2; V) = a_j(g_1, g_2 V) a_j(g_2; V).$$

The Radon–Nikodym derivative $d(gV)/dV$ of the transformation $V \mapsto gV$ is

$$\frac{d(gV)}{dV} = \prod_{j=1}^n \left( \frac{a_j(g; V)}{a_{j-1}(g; V)} \right)^{-(n+1)+2j}.$$ 

(1.5)

(1.6)

Fix complex numbers $\lambda_1, \ldots, \lambda_n$. Consider the representation $T_\lambda$ of the group $GL_n(\mathbb{K})$ in the space $L^2(\text{Fl}_n)$ defined by the formula

$$T_\lambda(g)f(V) = f(gV) \cdot \prod_{j=1}^{n-1} a_j(g; V)^{-1-\lambda_j+\lambda_{j+1}} \cdot a_n(g; V)^{(n-1)/2-\lambda_n} = f(gV) \cdot \prod_{j=1}^n \left( \frac{a_j(g; V)}{a_{j-1}(g; V)} \right)^{-(n+1)/2+j-\lambda_j}.$$ 

(1.7)
(we assume \( a_0 = 1 \)). It is a representation, since (1.5). If \( \lambda_j = i s_j \) are pure imaginary, then \( T_\lambda \) is unitary in \( L^2 \) (this follows from (1.6)).

The representations \( T_\lambda \) are called representations of the nondegenerate principal series. For imaginary \( \lambda_j = i s_j \), they are called representations of the nondegenerate unitary principal series.

Let \( \sigma \in S_n \) be a permutation of \((\lambda_1, \ldots, \lambda_n)\). For pure imaginary \( \lambda_j = i s_j \), we have
\[
T_{\sigma \lambda} \simeq T_\lambda,
\]
the same is valid for generic \( \lambda \in \mathbb{C}^n \).

Denote by \( 1 \) the function \( f(V) = 1 \) on \( \text{Fl}_n \). The spherical function \( \varphi_\lambda(g) \) is defined as the following matrix element of the representation \( T_\lambda \)
\[
\varphi_\lambda(g) := \langle T_\lambda(g)1, 1 \rangle_{L^2} = \int_{\text{Fl}_n} \prod_{j=1}^n \left( \frac{a_j(g,V)}{a_{j-1}(g,V)} \right)^{-(n+1)/2+j-\lambda_j} dV,
\]
where \( g \in \text{GL}_n(K) \).

Spherical functions are \( \text{GL}_n(O) \)-biinvariant in the following sense
\[
\varphi_\lambda(g h_1 g h_2) = \varphi_\lambda(h_1 g h_2), \quad \text{where } g \in \text{GL}_n(K), \, h_1, \, h_2 \in \text{GL}_n(O).
\]
Indeed,
\[
\varphi_\lambda(h_1 g h_2) = \langle T_\lambda(h_1 g h_2)1, 1 \rangle = \langle T_\lambda(g) T_\lambda(h_2)1, T_\lambda(h_1)1 \rangle = \langle T_\lambda(g)1, 1 \rangle.
\]
These functions are symmetric with respect to permutations of the parameters \( \lambda_j \)
\[
\varphi_{\sigma \lambda}(g) = \varphi_\lambda(g), \quad \sigma \in S_n.
\]

Also for each vector
\[
v = \frac{2\pi i}{\log p} (l_1, \ldots, l_n), \quad l_j \in \mathbb{Z},
\]
we have
\[
\varphi_{\lambda+v}(g) = \varphi_\lambda(g).
\]
Indeed, the factors \( a_j \) in (1.8) satisfy \( \log p a_j \in \mathbb{Z} \).

Thus, the spherical functions \( \varphi_\lambda \) are parametrized by the quotient set
\[
\mathbb{C}^n / [S_n \ltimes \frac{2\pi i}{\log p} \mathbb{Z}^n]
\]
of \( \mathbb{C}^n \) by the group generated by permutations of the coordinates and shifts by vectors (1.10).

We have defined a spherical function \( \varphi_\lambda \) as a function on \( \text{GL}_n(K) \). By the biinvariance (1.9), we can consider it as a function on \( \text{Lat}_n = \text{GL}_n(K) / \text{GL}_n(O) \) or as a function on the double cosets \( \mathcal{P}_n^+ = \text{GL}_n(O) \setminus \text{GL}_n(K) / \text{GL}_n(O) \).
1.5. Spherical functions as functions on $\text{GL}_n(\mathbb{O}) \setminus \text{GL}_n(\mathbb{K})/\text{GL}_n(\mathbb{O})$.

An explicit expression for the spherical functions as functions on $\text{P}_n^{+}$ was obtained in [6], see also [7], formula V.(3.4):

$$
\varphi_{\lambda}(k_1, \ldots, k_n) = A \cdot \sum_{\sigma \in S_n} \left( \frac{n+1}{2} - j \right) k_j \lambda_{\sigma(j)} \prod_{m<l} \frac{p^{-\lambda_{\sigma(m)} + p^{-1} - \lambda_{\sigma(l)}}}{p^{-\lambda_{\sigma(m)} - p^{-1} - \lambda_{\sigma(l)}}},
$$

(1.11)

where the summation is taken over all the permutations $\sigma$ of the set $\{1, 2, \ldots, n\}$ and the constant $A$ is

$$
A = \frac{(1 - p^{-1})^n}{\prod_{j=1}^{n}(1 - p^{-j})}.
$$

**Remark.** Thus, the expression $\varphi_{\lambda}(k)$ for a fixed $k$ is a Hall–Littlewood symmetric function (see [7], Chapter III) as a function in the variables $z_j = p^{\lambda_j}$.

1.6. Spherical functions as functions on $\text{Lat}_n$. A spherical function $\varphi_{\lambda}$ on $\text{Lat}_n$ has the following integral representation

$$
\varphi_{\lambda}(R) = \int_{\text{GL}_n(\mathbb{O})} \prod_{j=1}^{n-1} \text{vol}((hR \cap \mathbb{K}^j)^{1+\lambda_j - \lambda_{j+1}} \cdot \text{vol}(hR)^{-(n-1)/2 + \lambda_n} dh,
$$

(1.12)

where $dh$ is the Haar measure on $\text{GL}_n(\mathbb{O})$; we assume that the total measure of $\text{GL}_n(\mathbb{O})$ is 1. This follows from (1.9): since the Fl$_n$ is a $\text{GL}_n(\mathbb{O})$-homogeneous space, we can replace the integration over Fl$_n$ by the integration over $\text{GL}_n(\mathbb{O})$.

Next, we replace the integration over $\text{GL}_n(\mathbb{O})$ in (1.12) by the average over $\text{GL}_n(\mathbb{O})$-orbits

$$
\varphi_{\lambda}(R) = \frac{1}{\nu(k)} \sum_{R \in \mathbb{O}[k]} \prod_{j=1}^{n-1} \text{vol}(R \cap \mathbb{K}^j)^{1+\lambda_j - \lambda_{j+1}} \cdot \text{vol}(R)^{-(n-1)/2 + \lambda_n},
$$

(1.13)

where $\nu(k)$ is given by (1.4).

1.7. Spherical functions as invariant kernels on $\text{Lat}_n$. Let $L(R, S)$ be a function on $\text{Lat}_n \times \text{Lat}_n$ invariant with respect to the action of $\text{GL}_n(\mathbb{K})$, i.e.,

$$
L(gR, gS) = L(R, S), \quad \text{for } R, S \in \text{Lat}_n, \ g \in \text{GL}_n(\mathbb{K}).
$$

Then

$$
\ell(S) := L(\mathbb{O}^n, S)
$$

is a $\text{GL}_n(\mathbb{O})$-invariant function on $\text{Lat}_n$. Conversely, having a $\text{GL}_n(\mathbb{O})$-invariant function $\ell$ on $\text{Lat}_n$, we can reconstruct the corresponding $\text{GL}_n(\mathbb{K})$-invariant kernel $L(R, S)$.

Let us apply this remark to the spherical functions and define the spherical kernel $\Phi_{\lambda}(R, T)$ on $\text{Lat}_n \times \text{Lat}_n$ by the formula

$$
\Phi_{\lambda}(g\mathbb{O}^n, h\mathbb{O}^n) := \varphi_{\lambda}(g^{-1}h\mathbb{O}^n).
$$
1.8. Spherical transform. Let \( f \) be a \( \text{GL}_n(\mathbb{O}) \)-invariant function on \( \text{Lat}_n \). Its spherical transform is the function \( \tilde{f} \) in variables \( \lambda_j \) defined by

\[
\tilde{f}(\lambda_1, \ldots, \lambda_n) = \sum_{R \in \text{Lat}_n} \varphi_{-\lambda_1, \ldots, -\lambda_n}(R) f(R).
\] (1.14)

If we consider \( f \) as a function on \( \mathbb{P}_n^+ \), we write

\[
\tilde{f}(\lambda_1, \ldots, \lambda_n) = \sum_{k \in \mathbb{P}_n^+} \nu(k) \varphi_{-\lambda_1, \ldots, -\lambda_n}(k) f(k).
\]

Since \( f(R) \) in (1.14) is a constant on any \( \text{GL}_n(\mathbb{O}) \)-orbit, formula (1.13) allows to convert (1.14) to the form

\[
\tilde{f}(\lambda_1, \ldots, \lambda_n) = \sum_{R \in \text{Lat}_n} f(R) \left[ \prod_{j=1}^{n-1} \text{vol}(R \cap K^j)^{1+\lambda_j-\lambda_{j+1}} \cdot \text{vol}(R)^{-(n-1)/2+\lambda_n} \right].
\] (1.15)

1.9. Plancherel theorem. Macdonald’s inversion formula for the spherical transform (see [6]) is

\[
f(R) = \int_{0 \leq s_j \leq 2\pi/\ln p} \hat{f}(is_1, \ldots, is_n) \varphi_{is_1, \ldots, is_n}(R) d\mu(s),
\] (1.16)

where the Plancherel measure \( d\mu(s) \) is

\[
d\mu(s) = C \cdot \prod_{k<l} \left| \frac{p^{is_k} - p^{is_l}}{p^{is_k} - p^{-1+is_l}} \right|^2 ds_1 \ldots ds_n
\] (1.17)

and the constant \( C \) is

\[
C = \frac{\ln^n p}{(2\pi)^n n!} \prod_{j=1}^{n} \frac{1-p^{-j}}{1-p^{-1}}.
\] (1.18)

Also the following Plancherel formula holds

\[
\sum_{k \in \mathbb{P}_n^+} |f(k)|^2 \nu(k) = \int_{0 \leq s_j \leq 2\pi/\ln p} |\hat{f}(is)|^2 d\mu(s).
\]

1.10. Explicit spectral decomposition of \( L^2(\text{Lat}_n) \). Denote by \( \Xi_n \) the simplex

\[
0 \leq s_1 \leq \ldots \leq s_n \leq 2\pi/\ln p.
\] (1.19)

Equip this simplex by the Plancherel measure \( n! d\mu(s) \), where \( d\mu(s) \) is given by (1.17). Consider the space \( \Xi_n \times \text{Fl}_n \) equipped with the product-measure \( d\mu(s) \times d\nu \).
The group $\text{GL}_n(\mathbb{K})$ acts in the space $L^2(\Xi_n \times \text{Fl}_n)$ by the unitary operators
\[
\rho(g)\Psi(s, V) = \Psi(s, gV) \cdot \prod_{j=1}^{n} \left( \frac{a_j(g, V)}{a_{j-1}(g, V)} \right)^{-\frac{(n+1)/2+j-1}{2}}.
\]
This formula defines the direct multiplicity-free integral of unitary representations of principal nondegenerate series.

For a function $f(\mathbb{R})$ on $\text{Lat}_n$, we define the function $Jf(s, V)$ given by
\[
Jf(s, V) = \sum_{R \in \text{Lat}_n} f(R) \cdot \prod_{j=1}^{n-1} \left( \frac{\text{vol}(R \cap \mathbb{K}^j)^{1-\beta_j+i\beta_{j+1}} \cdot \text{vol}(R)^{-\frac{(n-1)/2-i\beta_n}}}{\text{vol}(R)^{\alpha_j-i\alpha_{j+1}}} \right).
\]

It can easily be checked, that the operator $J$ intertwines representation of $\text{GL}_n(\mathbb{K})$ in $l^2(\text{Lat}_n)$ and the representation $\rho$. Also, the operator $J$ is a unitary operator $l^2(\text{Lat}_n) \to L^2(\Xi_n \times \text{Fl}_n, d\mu(s) \times dV)$ (this is a rephrasing of the Plancherel formula for the spherical transform).

2. Results

2.1. Beta-function. Consider the flag
\[
0 = \mathbb{K}^0 \subset \mathbb{K}^1 \subset \ldots \subset \mathbb{K}^{n-1} \subset \mathbb{K}^n.
\]
Consider the lattices $\mathcal{O}^j = \mathbb{K}^j \cap \mathcal{O}^n$. Thus we obtain the flag of $\mathcal{O}$-modules
\[
0 = \mathcal{O}^0 \subset \mathcal{O}^1 \subset \ldots \subset \mathcal{O}^{n-1} \subset \mathcal{O}^n.
\]

Fix complex numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ such that
\[
\text{Re} \beta_j + j - 1 < 0, \quad \text{Re} \alpha_j + \text{Re} \beta_j - n + j > 0.
\] (2.2)

Let also $\alpha_{n+1} = \beta_{n+1}.$

**Theorem 2.1.**
\[
\sum_{R \in \text{Lat}_n} \prod_{j=1}^{n} \left\{ \frac{\text{vol}(R \cap \mathbb{K}^j)^{\beta_j-\beta_{j+1}} \cdot \text{vol}(R \cap \mathcal{O}^j)^{\alpha_j-\alpha_{j+1}}} {\text{vol}(R)^{\alpha_j}} \right\} = \prod_{j=1}^{n} \frac{1 - p^{-(\alpha_j-n+j)}}{(1 - p^{\beta_j+j-1})(1 - p^{-(\alpha_j+\beta_j-n+j)})},
\]
(2.3)

**Remark.** The condition of the convergence of this series is (2.2).

2.2. Deformation of $l_2$. Fix $\alpha \in \mathbb{R}$. Let $R, S$ range in $\text{Lat}_n$. Consider the kernel
\[
K_\alpha(R, S) = \frac{\text{vol}(R \cap S)^{\alpha}}{\text{vol}(R)^{\alpha/2} \cdot \text{vol}(S)^{\alpha/2}}.
\] (2.5)
Theorem 2.2. a) Let

\[ \alpha = 0, 1, 2, \ldots, n - 1 \quad \text{or} \quad \alpha > n - 1. \]  

Then there exist a Hilbert space \( H_\alpha \) and a system of vectors \( e_R \in H_\alpha \), where \( R \) ranges in \( \text{Lat}_n \), such that the scalar products of \( e_R \) have the form

\[ \langle e_R, e_S \rangle = K_\alpha(R, S) \]  

and linear combinations of the vectors \( e_R \) are dense in \( H_\alpha \).

b) If \( \alpha \) is outside the set (2.6), then a Hilbert space \( H_\alpha \) does not exist.

Remark. The space \( H_\alpha \) is unique in the following sense. Let \( H'_\alpha \) be another Hilbert space and \( e'_R \in H'_\alpha \) be another system of vectors satisfying the same conditions. Then there exists a unitary operator \( A : H_\alpha \to H'_\alpha \) such that \( Ae_R = e'_R \) for all the lattices \( R \).

Remark. The space \( H_0 \) is one-dimensional, all the vectors \( e_R \) in this case coincide.

The group \( \text{GL}_n(\mathbb{K}) \) acts in the space \( H_\alpha \) by the unitary operators

\[ U_\alpha(g)e_R = e_{gR}. \]

Remark. Let \( \alpha \to +\infty \). Then

\[ \lim_{\alpha \to \infty} K_\alpha(R, T) = \begin{cases} 1, & \text{if } R = T; \\ 0, & \text{if } R \neq T. \end{cases} \]

In this sense, the limit of \( H_\alpha \) as \( \alpha \to +\infty \) is the space \( l_2(\text{Lat}_n) \).

2.3. Plancherel formula for the spaces \( H_\alpha, \alpha > n - 1 \). The following theorem gives the explicit spectral decomposition of the space \( H_\alpha \) with respect to the action of \( \text{GL}_n(\mathbb{K}) \).

Theorem 2.3. For \( \alpha > n - 1 \) we have

\[ K_\alpha(R, T) = \int \Phi_{is_1, \ldots, is_n}(R, T) \, d\mu_\alpha(s), \]  

where the spherical kernels \( \Phi \) were defined in 1.7, the integration is taken over the cube (torus) \( s_j \in [0, 2\pi/\ln p] \), and the Plancherel measure \( \mu_\alpha \) is given by

\[ d\mu_\alpha(s) = C \prod_{j=1}^n \frac{1 - p^{-\alpha+j-1}}{1 - p^{-(\alpha-n+1)/2-\alpha+j-1}} \times \prod_{1 \leq k < l \leq n} \left| \frac{p^{is_k} - p^{is_l}}{p^{is_k} - p^{-1+is_k}} \right|^2 ds_1 \ldots ds_n, \]

where

\[ C = \frac{\ln^n p}{(2\pi)^n n!} \prod_{j=1}^n \frac{1 - p^{-j}}{1 - p^{-1}}. \]
2.4. Plancherel formula for $\alpha = 0, 1, \ldots, n - 1$.

**Theorem 2.4.** Let $\alpha = 0, 1, \ldots, n - 1$. Then

$$K_\alpha(R, T) = \int \Phi_{is_1, \ldots, is_\alpha, -(n-\alpha+1)/2, -(n-\alpha+1)/2+1, \ldots, (n-\alpha+1)/2}(R, T) \, d\mu_\alpha(s),$$

where $s_j \in [0, 2\pi/\ln p]$, the Plancherel measure is

$$d\mu_\alpha(s) = C_{\alpha} \prod_{j=1}^{\alpha} |1 - p^{(n-\alpha-1)/2+is_j}|^2 \prod_{1 \leq k < l \leq \alpha} |\frac{p^{is_k} - p^{is_l}}{p^{is_k} - p^{-1+is_l}}|^2 ds_1 \ldots ds_\alpha,$$

and

$$C_{\alpha} = \frac{\ln^\alpha p}{(2\pi)^{\alpha(n-\alpha)!}} (1 - p^{-1})^{-\alpha} \prod_{k=1}^{\alpha} (1 - p^{-k}) \prod_{k=n-\alpha+1}^{\alpha} (1 - p^{-k}).$$

2.5. Spectra.

**Proposition 2.5.** Spectrum of each representation $U_\alpha$ of the group $\text{GL}_n(K)$ in $H_\alpha$ is multiplicity-free.

Now we intend to describe the spectra explicitly.

Consider the flag $K^1 \subset K^2 \subset \cdots \subset K^k$ in $K^n$. Denote by $P_k$ the stabilizer of this flag in $\text{GL}_n(K)$. It is the group of block $(1 + 1 + \cdots + 1 + (n-k)) \times (1 + 1 + \cdots + 1 + (n-k))$ upper triangular matrices

$$A = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1k} & a_{1(k+1)} & \ldots & a_{1n} \\
0 & a_{22} & \ldots & a_{2k} & a_{2(k+1)} & \ldots & a_{2n} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \ldots & a_{kk} & a_{k(k+1)} & \ldots & a_{kn} \\
0 & 0 & \ldots & 0 & a_{(k+1)(k+1)} & \ldots & a_{(k+1)n} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 0 & a_{n(k+1)} & \ldots & a_{nn}
\end{pmatrix}.$$  

In particular, $P_n = B_n$ is the group of all upper triangular matrices.

The spectrum is the support of the Plancherel measure, hence Theorems 2.3–2.4 imply the following corollary.

**Corollary 2.6.** a) For $\alpha > n - 1$ the spectrum consists of all spherical representations of $\text{GL}_n(K)$ of the unitary nondegenerate principal series

b) For $\alpha = 0, 1, 2, \ldots, n - 1$ the spectrum consists of representations of $\text{GL}_n(K)$ unitary induced from the characters

$$\chi_s(A) = \prod_{j=1}^{\alpha} |a_{jj}|^{is_j}, \quad 0 \leq s_1 \leq \ldots \leq s_\alpha \leq 2\pi/\ln p.$$
of the group $P_\alpha$.

**Remarks.** a) For $\alpha = n - 1$ the spectrum consists of representations $T_{is}$ of unitary nondegenerate principal series such that $s_n = 0$.

b) For $\alpha = n - 2, n - 3, \ldots$ the spectrum consists of degenerated unitary principal series. In notation of §1.4, consider the subspace $Y_\alpha \subset L^2(\text{Fl}_n)$ of functions that do not depend on the terms $V_{n-\alpha+1}, \ldots, V_{n-1}$ of a flag $V \in \text{Fl}_n$; evidently, $Y_\alpha \simeq L^2(\text{GL}_n(\mathbb{K})/P_k)$. It is clear that for our values of the parameters of the representations, the subspace $Y_\alpha$ is $\text{GL}_n$-invariant. This gives a realization of our representations in $L^2(\text{GL}_n(\mathbb{K})/P_k)$.

2.6. **A realization of the system** $e_R$. For definiteness, let $\alpha > n - 1$. Then $H_\alpha$ is equivalent to a direct integral of representations of principal series. Here we write explicitly the image of the system $e_R$ in this direct integral.

Let the simplex $\Xi_n$ be the same as above (1.19). Equip this simplex with the Plancherel measure $n! d\mu_\alpha$, see Theorem 2.3. Consider the space $\Xi_n \times \text{Fl}_n$ equipped with the product-measure $n! d\mu_\alpha \times d\nu$ and the space $L^2(\Xi_n \times \text{Fl}_n)$ with respect to our measure. The group $\text{GL}_n(\mathbb{K})$ acts in the space $L^2$ by the unitary operators $\rho_{\alpha}(g)$ given by the formula (1.20); the formula for the operators themselves does not contain $\alpha$, but the space of a representation depends on $\alpha$.

For a lattice $R$, we consider the function $u_R(s, V)$ on $\Xi_n \times \text{Fl}_n$ given by

$$u_R(s, V) = \prod_{j=1}^{n-1} \text{vol}(R \cap V_j)^{1-\text{i}s_j} \cdot \text{vol}(R)^{-\frac{1}{2}-\text{i}s_n}.$$ 

Our Plancherel formula implies the following corollary.

**Theorem 2.7.** For each $R, T \in \text{Lat}_n$,

$$\langle u_R, u_S \rangle_{L^2(\Xi_n \times \text{Fl}_n)} = \int_{\Xi_n \times \text{Fl}_n} u_R(s, V) \overline{u_T(s, V)} n! d\mu_\alpha(s) d\nu = \langle e_R, e_S \rangle_{H_\alpha} = K_\alpha(R, T)$$

**Remark.** Denote by $\mu_\infty$ Macdonald’s Plancherel measure (1.17) on $\Xi_n$. The system $u_R(s, V)$ is an orthonormal basis in the space $L^2(\Xi_n \times \text{Fl}_n)$ with respect to the measure $\mu_\infty \times d\nu$:

$$\int_{\Xi_n \times \text{Fl}_n} u_R(s, V) u_T(s, V) n! d\mu_\infty(s) d\nu = \begin{cases} 1, & \text{if } R = T \\ 0, & \text{if } R \neq T \end{cases}$$

2.7. **Linear dependence.** Let $\alpha = 0, 1, 2, \ldots, n - 1$. In these cases, the space $H_\alpha$ is smaller than for $\alpha > n - 1$ (first, the support of the Plancherel measure has lesser dimension; second, the representations in the spectrum are ‘smaller’). For these values of $\alpha$ the vectors $e_R$ are linear dependent.
Theorem 2.8. Let \( R \subset S \) be arbitrary lattices such that \( S/R \simeq (\mathbb{Z}/p\mathbb{Z})^{\alpha+1} \).
Theorem

\[
\sum_{k=0}^{\alpha+1} (-1)^k p^{k(\alpha-1)/2} \sum_{Q \in \text{Lat}_n: R \subset Q \subset S} e_R = 0. \tag{2.10}
\]

2.8. Remark. Relation with the Weil representation. Consider the Weil representation (see [16], see also explicit formulae in [8]) of the group \( \text{Sp}(2n, K) \) and its restriction \( \sigma \) to the subgroup \( \text{GL}(n, K) \). The representation \( \sigma \) is equivalent to the natural representation of \( \text{GL}(n, K) \) in the subspace \( L^2_{+}(K^n) \) consisting of even functions, i.e., \( f(-x) = f(x) \), by the formula

\[
\sigma(g)f(x) = f(gx)|\det g|^{1/2}; \quad g \in \text{GL}(n, K), \quad x \in K^n.
\]

In particular, the representation \( \sigma \) is single-valued (the Weil representation itself is double-valued).

Denote by \( \xi \) the \( \text{Sp}(2n, O) \)-fixed vector in the Weil representation. The 'vacuum vector' \( \xi \) corresponds to the function \( f(z) \) defined by: \( f(z) = 1 \) for \( z \in O^n \) and \( f(z) = 0 \) otherwise.

Consider all the possible vectors \( U(g)\xi \), where \( g \) ranges in \( \text{GL}(n, K) \). We have \( U(gh)\xi = U(g)\xi \) for any \( h \in \text{GL}(n, O) \) and hence the vectors of the form \( U(g)\xi \) are enumerated by points of \( \text{GL}(n, K)/\text{GL}(n, O) = \text{Lat}_n \); we denote the vector corresponding to a lattice \( R \) by \( \xi_R \).

Proposition 2.9. \( \langle \xi_R, \xi_S \rangle = K_1(R, S) \).

This is trivial but important. We observe that

\[
K_\alpha(R, T) = |\langle \xi_R, \xi_S \rangle|^{\alpha}
\]

In the real case, exactly this procedure gives the Berezin kernels. Thus, this is a way to conjecture a form of 'Berezin kernels' in the \( p \)-adic case.

The most serious a posteriori argument for our analogy is the formula (2.9) for the Plancherel measure, since it can be obtained from the formula over \( \mathbb{R} \) (see [11]) by replacing of the usual \( \Gamma \)-functions by their \( p \)-adic analogs.

2.9. Remark. Realization of \( H_\alpha \) in functions on \( \text{Lat}_n \). For each \( h \in H_\alpha \), we assign the function \( f_h \) on \( \text{Lat}_n \) by the formula

\[
f_h(R) = \langle h, e_R \rangle_{H_\alpha}.
\]

Thus we realize \( H_\alpha \) as some space of functions on \( \text{Lat}_n \). After this, it is possible to apply the usual formalism of reproducing kernels, see, for instance [11], we do not discuss this here.

For \( \alpha = 0, 1, \ldots, n - 1 \), relations (2.10) become difference equations for functions \( f_h \); for each lattices \( R \subset S \) satisfying \( S/R \simeq (\mathbb{Z}/p\mathbb{Z})^{\alpha+1} \), any function \( f_h \) satisfies

\[
\sum_{k=0}^{\alpha+1} (-1)^k p^{k(\alpha-1)/2} \sum_{Q \in \text{Lat}_n: R \subset Q \subset S} f_h(R) = 0
\]
These equations are the analog of the determinant systems of partial differential equations defining the degenerated Berezin spaces, see, for instance [11].

2.10. Remark. Additional symmetry. For each element \( a = (a_1, \ldots, a_n) \in \mathbb{K}^n \) we define the linear functional

\[ \ell_a(z) = a_1 z_1 + \cdots + a_n z_n. \]

Let \( R \) be a lattice in \( \mathbb{K}^n \). We define the dual lattice \( R^\vee \) as the set of all \( a \in \mathbb{K}^n \) such that \( \ell_a(z) \in \mathbb{O} \) for all \( z \in R \).

**Lemma 2.10.** \( K_\alpha(R^\vee, S^\vee) = K_\alpha(R, S) \)

**Proof.** This follows from

\[ \sum_{k} \varphi^{(p)}_{i,j}(k)p^{-\alpha \sum_{\ell} |k_{\ell}|} \nu(k) = \prod_{\ell=0}^{n-1} (1 - p^{-\alpha + i}) \prod_{j=1}^{n} |1 - p^{-(\alpha + n - 1)/2 + i j}|^{-2}, \]

where \( \nu(k) \) is given by (1.4).

2.11. Remark. Identities with Hall–Littlewood functions. Let \( p \) be real, \( p > 1 \). Define the functions \( \varphi^{(p)}_\lambda(k) = \varphi_\lambda(k) \) by the formula (1.11). For prime values of \( p \) these functions are spherical functions over \( \mathbb{K} = \mathbb{Q}_p \).

**Proposition 2.11.** a) The identity (2.8) holds for arbitrary \( p > 1, \alpha > n - 1 \).

b) For arbitrary \( p > 1 \) and sufficiently large real \( \alpha \),

\[ \sum_{k} \varphi^{(p)}_{i,j}(k)p^{-\alpha \sum_{\ell} |k_{\ell}|} \nu(k) = \prod_{\ell=0}^{n-1} (1 - p^{-\alpha + i}) \prod_{j=1}^{n} |1 - p^{-(\alpha + n - 1)/2 + i j}|^{-2}, \]

where \( \nu(k) \) is given by (1.4).

2.12. Remark: the Tamagawa zeta-function. Assume in Theorem 2.1

\[ \alpha_j = t, \quad \beta_j = \gamma_j - t. \]

Passing to the limit as \( t \to +\infty \), we obtain
Corollary 2.12.

$$\sum_{R \in \text{Lat}_n, R \subset \mathbb{O}_n} \prod_{j=1}^{n} \text{vol}(R \cap K_j)^{\gamma_j^{-} \gamma_{j+1}} = \prod_{j=1}^{n} (1 - p^{-\gamma_j - n - j})^{-1}.$$  

This is the Tamagawa zeta-function, see [7], IV.4.

2.13. Remark. Lattices in $\mathbb{Q}^n$ and classical zeta-function. Consider the $n$-dimensional space $\mathbb{Q}^n$ over the rational numbers. A lattice in $\mathbb{Q}^n$ is a free $\mathbb{Z}$-submodule with $n$ generators. We denote the space of lattices in $\mathbb{Q}^n$ by $\text{Lat}_n(\mathbb{Q})$.

For a lattice $R \subset \mathbb{Q}^n$, we denote by $\nu_n(R)$ the volume of the torus $\mathbb{R}^n/R$. Denote by $e_1, \ldots, e_n$ the standard basis in $\mathbb{Q}^n$. Denote $\mathbb{Q}^k = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_k$, $\mathbb{Z}^k = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k$.

Fix complex $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$; assume $\alpha_{n+1} = 0, \beta_{n+1} = 0$. Let

$$\text{Re} \alpha_j - n + 1 > 1, \quad \text{Re} \alpha_j + \text{Re} \beta_j - n + j > 1, \quad \text{Re} \beta_j + j < 0.$$  

Theorem 2.13.

$$\sum_{R \in \text{Lat}_n(\mathbb{Q})} \prod_{j=1}^{n} \nu_k(R \cap \mathbb{Q}^k)^{-\beta_j + \beta_{j+1}} \nu_k(R \cap \mathbb{Z}^k)^{-\alpha_k + \alpha_{k+1}} =$$

$$\prod_{j=1}^{n} \frac{\zeta(-(\beta_j + j - 1))\zeta(\alpha_j + \beta_j - n + j)}{\zeta(\alpha_j - n + j)},$$

where

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \text{ is prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

is the zeta-function.

3. Calculation of the beta-sum

Here we prove Theorems 2.1 and 2.13.

3.1. Transformation of the beta-sum. Denote by $\text{Lat}_{n-1}$ the space of lattices in the subspace $\mathbb{K}^{n-1} \subset \mathbb{K}^n$. We transform the expression (2.3) to the form

$$\sum_{R' \in \text{Lat}_{n-1}} \prod_{j=1}^{n-2} \left\{ \text{vol}(R' \cap \mathbb{K}^j)^{\beta_j - \beta_{j+1}} \text{vol}(R' \cap \mathbb{Q}^j)^{\alpha_j - \alpha_{j+1}} \right\} \times$$

$$\times \text{vol}(R' \cap \mathbb{K}^{n-1})^{\beta_{n-1}} \text{vol}(R' \cap \mathbb{Q}^{n-1})^{\alpha_{n-1}} \times$$

$$\times \sum_{R \in \text{Lat}_n: R \cap \mathbb{K}^{n-1} = R'} \left\{ \frac{\text{vol}(R)^{\beta_n}}{\text{vol}(R \cap \mathbb{K}^{n-1})^{\beta_n}} \frac{\text{vol}(R \cap \mathbb{Q}^{n-1})^{\alpha_n}}{\text{vol}(R \cap \mathbb{Q}^{n-1})^{\alpha_n}} \right\}. (3.3)$$
Lemma 3.1. The interior sum (3.3) is equal to

\[
\frac{\sigma_n}{\text{vol}(\mathbb{O}^{n-1} \cap R')},
\]

where

\[
\sigma_n = \frac{1 - p^{-\alpha_n}}{(1 - p^{-\alpha_n - \beta_n})(1 - p^{\beta_n + n - 1})}.
\] (3.4)

Lemma 3.1 will be proved below.

Corollary 3.2. Denote by \(F_n(\alpha; \beta)\) the left-hand side of (2.3)–(2.4). The following identity for \(F_n(\alpha; \beta)\) holds

\[
F_n(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) = \sigma_n F_{n-1}(\alpha_1, \ldots, \alpha_{n-1} - 1; \beta_1, \ldots, \beta_{n-1}),
\]

where \(\sigma_n\) is given by (3.4).

This corollary implies Theorem 2.1.

3.2. New variables \(\xi\) and \(\nu\). Consider the subspace \(\mathbb{K}^{n-1} \subset \mathbb{K}^n\). We have \(\mathbb{K}^n = \mathbb{K}^{n-1} \oplus \mathbb{K}e_n\).

Fix a lattice \(R' \subset \mathbb{K}^n\). Let \(R\) satisfy the property

\[
R \cap \mathbb{K}^{n-1} = R'.
\] (3.5)

Each vector \(r\) in \(R\) has the form \(r = ap^k e_n + v\), where \(v \in \mathbb{K}^{n-1}, |a| = 1\).

Lemma 3.3. Denote by \(\xi\) the minimal possible \(k\), let \(h = p^\xi e_n + v \in R\). Then

\[
R = R' \oplus \mathbb{O}h = R' \oplus \mathbb{O}(p^\xi e_n + v).
\] (3.6)

This is obvious. The next Lemma is equivalent to Lemma 3.3.

Lemma 3.3’. Consider the natural map \(\pi : \mathbb{K}^n \to \mathbb{K}e_n\). Let \(h \in R\) and \(\pi(h)\) generates the \(\mathbb{O}\)-module \(\pi(R)\). Then \(R = R' \oplus \mathbb{O}h\).

Lemma 3.4. Lattices \(R_1 = R' \oplus \mathbb{O}(p^\xi e_n + v_1)\) and \(R_2 = R' \oplus \mathbb{O}(p^\xi e_n + v_2)\) coincide iff \(\xi_1 = \xi_2, v_1 - v_2 \in R'\).

Proof is obvious.

Thus a lattice \(R\) with the property (3.5) is determined by the integer \(\xi = \xi(R)\) and an element \(v = v(R)\) of the quotient group \(\mathbb{K}^{n-1}/R'\).

We also define a nonnegative integer \(\nu = \nu(R)\) by the condition

— if \(v \in R' + \mathbb{O}^{n-1}\), then \(\nu = 0\);
— otherwise, \(\nu\) is the minimal \(k\) such that \(p^k v \in R' + \mathbb{O}^{n-1}\).

Lemma 3.5.

a) \(\frac{\text{vol}(R)}{\text{vol}(R')} = p^{-\xi}\).

b) \(\frac{\text{vol}(R \cap \mathbb{O}^{n})}{\text{vol}(R \cap \mathbb{O}^{n-1})} = p^{-\max(0, \xi + \nu)}\).
Proof. The statement a) is obvious. The statement b) is a corollary of the following lemma.

**Lemma 3.6.** Let \( R \) has the form (3.6). Then \( R \cap \mathbb{O}^n \) has the form

\[
R \cap \mathbb{O}^n = R' \oplus \mathbb{O}[p^\kappa(p^\epsilon e_n + v) + w],
\]

where \( \kappa = \max(\nu, -\xi) \) and \( w \in -p^\kappa v + R' \).

**Proof.** First,

\[
p^\kappa(p^\epsilon e_n + v) + w \in R' \iff \kappa \geq 0, \quad w \in R'
\]

Secondly,

\[
p^\kappa(p^\epsilon e_n + v) + w \in \mathbb{O}^n \iff \kappa + \xi \geq 0, \quad p^\kappa v + w \in \mathbb{O}^{n-1}
\]

Since \( w \in R' \), \( p^\kappa v + w \in \mathbb{O}^{n-1} \), we have \( p^\kappa v \in \mathbb{O}^{n-1} \). But \( \kappa \geq 0 \), hence \( \kappa \geq \nu \). Also, \( \kappa \geq -\xi \), see (3.8). It remains to apply Lemma 3.3-3′. \( \square \)

**Lemma 3.7.** The number of lattices \( R \in \text{Lat}_n \) satisfying (3.5) with given \( \eta \geq 0, \xi \) is

\[
u_n(\nu, \xi) = \begin{cases} 
\frac{1}{\text{vol}(R' \cap \mathbb{O}^{n-1})}, & \nu = 0; \\
\frac{p^{\nu(n-1)}(1 - p^{-n+1})}{\text{vol}(R' \cap \mathbb{O}^{n-1})}, & \nu > 0 
\end{cases}
\]

**Proof.** Nothing depends on \( \xi \). We must find number of vectors \( v \in \mathbb{K}^{n-1}/R' \) such that \( p^\nu v \in (R' + \mathbb{O}^{n-1})/R' \).

The index of \( R' \) in \( R' + \mathbb{O}^{n-1} \) is \( 1/\text{vol}(R' \cap \mathbb{O}^{n-1}) \). It remains to find the number \( u_n^*(\nu) \) of \( v \in \mathbb{K}^{n-1}/(R' + \mathbb{O}^{n-1}) \) such that \( p^\nu v \in R' + \mathbb{O}^{n-1} \), since we have

\[ u_n(\nu, \xi) = \frac{u_n^*(\nu)}{\text{vol}(R' \cap \mathbb{O}^{n-1})}. \]

For each lattice \( S \subset \mathbb{K}^{n-1} \), the additive group \( \mathbb{K}^{n-1}/S \) is isomorphic to \( (\mathbb{K}/\mathbb{O})^{n-1} \). Hence we must find the number of solutions of the equation \( p^\nu v = 0 \) in \( (\mathbb{K}/\mathbb{O})^{n-1} \).

The additive group \( \mathbb{K}/\mathbb{O} \) is the inductive limit of the cyclic groups

\[ \mathbb{K}/\mathbb{O} = \lim_{k \to \infty} \mathbb{Z}/p^k \mathbb{Z}. \]

Hence the number of solutions of the equation \( p^\nu v = 0 \) in \( (\mathbb{K}/\mathbb{O})^{n-1} \) is \( p^{\nu(n-1)} \).

But \( p^{\nu-1(n-1)} \) of these solutions are also solutions of the equation \( p^\nu v = 0 \).

If \( \nu = 0 \), we have a unique solution \( v = 0 \). \( \square \)

3.3. **The end of calculation.** Lemma 3.5, 3.7 reduce an evaluation of (3.3) to an evaluation of the sum

\[
\frac{1}{\text{vol}(R' \cap \mathbb{O}^{n-1})} \sum_{\xi \in \mathbb{Z}, \nu \geq 0} p^{-\alpha_n \max(0, \xi + \nu)} p^{-\beta_n \xi} u_n(\nu, \xi)
\]
Here the summation amounts to summations of several geometric progressions.

3.4. Proof of Theorem 2.13. Let \( Q_p \) be \( p \)-adic numbers, \( \mathbb{Q}_p \) be \( p \)-adic integers. For each prime \( p \) consider the natural embeddings \( \mathbb{Q} \to \mathbb{Q}_p \), \( \mathbb{Q}^n \to \mathbb{Q}_p^n \) and the induced map

\[
\pi_p : \text{Lat}_n(\mathbb{Q}) \to \text{Lat}_n(\mathbb{Q}_p),
\]

i.e., for \( R \in \text{Lat}_n(\mathbb{Q}) \) we consider its closure in \( \mathbb{Q}_p^n \).

Our calculation is based on the following two remarks. First, for any \( R \in \text{Lat}_n(\mathbb{Q}) \),

\[
\nu_n(R)^{-1} = \prod_{p \text{ is prime}} \text{vol}_{\mathbb{Q}_p}(\pi_p(R)).
\]

Secondly, the map

\[
R \mapsto (\pi_2(R), \pi_3(R), \pi_5(R), \pi_7(R), \pi_{11}(R), \ldots)
\]

is a bijection of \( \text{Lat}_n(\mathbb{Q}) \) and the set of sequences \( (S_2, S_3, S_5, S_7, S_{11}, \ldots) \) such that \( S_p \in \text{Lat}_n(\mathbb{Q}_p) \) and \( S_p = \mathbb{Q}_p^n \) except finite number of \( p \); see \cite{17}, Theorem V.2.2.

Hence the left hand side of (2.11) transforms to the form

\[
\prod_{p \text{ is prime}} \left[ \sum_{S_p \in \text{Lat}_n(\mathbb{Q}_p)} \prod_{j=1}^n \left\{ \text{vol}(S_p \cap \mathbb{Q}_p^j)^{\beta_j - \beta_j + 1} \text{vol}(S_p \cap \mathbb{Q}_p^j)^{\alpha_j - \alpha_j + 1} \right\} \right].
\]

It remains to apply Theorem 2.1.

4. Plancherel formula

This section contains proofs of Theorem 2.3 (in §4.1), Theorem 2.4 (in §§4.2–4.6), Proposition 2.5 (in §4.7), Theorem 2.7 (in §4.8), Proposition 2.11 (in §4.9). We also obtain ‘indefinite Plancherel formula’ (Theorem 4.4), which is used below in §5.

4.1. Plancherel formula. Consider the \( \text{GL}_n(\mathbb{Q}) \)-invariant function

\[
\Delta_\alpha(R) = K_\alpha(\mathbb{Q}^n, R) = \frac{\text{vol}(R \cap \mathbb{Q}^n)^\alpha}{\text{vol}(R)^{\alpha/2}}, \quad R \in \text{Lat}_n.
\]

By (1.15), its spherical transform is

\[
\tilde{\Delta}_\alpha(\lambda) = \sum_{R \in \text{Lat}_n} \frac{\text{vol}(R \cap \mathbb{Q}^n)^\alpha}{\text{vol}(R)^{\alpha/2}} \cdot \prod_{j=1}^{n-1} \text{vol}(R \cap \mathbb{K})^{1+\lambda_j-\lambda_{j+1}} \cdot \text{vol}(R)^{-(n-1)/2+\lambda_n}.
\]

By Theorem 2.1, it equals

\[
\tilde{\Delta}_\alpha(\lambda) = \prod_{j=1}^n \frac{1 - p^{-(\alpha-n+j)}}{(1 - p^{-(\alpha-n+1)/2+\lambda_j})(1 - p^{-(\alpha-n+1)/2-\lambda_j})}.
\]

(4.1)
Applying the inversion formula (1.16) for the spherical transform, we obtain Theorem 2.3.

4.2. Plancherel formula as a contour integral. Introduce the new variables $z_k = p^{i	heta_k}$. Also introduce a new notation for the spherical functions

$$\varphi[z_1, \ldots, z_n; R] := \varphi_{iz_1, \ldots, iz_n}(R).$$

Now Theorem 2.3 is converted to the form

$$\Delta_\alpha(R) = C_0 I_0,$$  \hspace{1cm} (4.2)

where $I_0$ is the following integral over a torus

$$I_0 = I_0(\alpha; R) = \int_{|z_1|=1, \ldots, |z_n|=1} \prod_{j=1}^n \frac{1}{(z_j - p^{(\alpha-n+1)/2})(z_j - p^{-(\alpha-n+1)/2})} \times$$

$$\times \prod_{1 \leq k < l \leq n} \frac{(z_k - z_l)^2}{(z_k - p z_l)(z_k - p^{-1} z_l)} \varphi[z_1, \ldots, z_n; R] \, dz_1 \ldots dz_n \hspace{1cm} (4.3)$$

and the constant $C_0$ is

$$C_0 = C_0(\alpha) = p^{n \alpha n} \left(\frac{2\pi i}{p}\right)^n \prod_{j=1}^n \frac{1 - p^{-j}}{1 - p^{-1}} \prod_{l=0}^{n-1} (1 - p^{-\alpha+l}) \hspace{1cm} (4.5)$$

The function $\Delta_\alpha(R)$ is a holomorphic function in $\alpha \in \mathbb{C}$. The integral expression $C_0 I_0$ for $\Delta_\alpha(R)$ is holomorphic in the domain $\text{Re} \alpha > n - 1$. We intend to obtain the holomorphic continuation of $C_0(\alpha)I_0(\alpha; R)$ into the whole plane $\alpha \in \mathbb{C}$.

4.3. Analytic continuation to the strip $n - 3 < \text{Re} \alpha < n - 1$. Denote

$$\beta := (\alpha - n + 1)/2,$$  \hspace{1cm} (4.6)

$$\mu_m(z_1, \ldots, z_m) := \prod_{1 \leq k < l \leq m} \frac{(z_k - z_l)^2}{(z_k - p z_l)(z_k - p^{-1} z_l)} \hspace{1cm} (4.7)$$

It can be easily checked that $\mu_m$ is symmetric with respect to $z_j$. Recall also that the spherical functions are symmetric with respect to $z_j$.

**Lemma 4.1.** For $n - 3 < \text{Re} \alpha < n - 1$, \n
$$\Delta_\alpha(R) = C_0 I_0 + n C_{01} (I_{01}^+ + I_{01}) + n(n-1) C_1 I_1,$$ \hspace{1cm} (4.8)

where $C_0, I_0$ are the same as above (4.3)-(4.5), \n
$$I_{01}^+ = \int_{|z_1|=1, \ldots, |z_{n-1}|=1} \prod_{j=1}^{n-1} \frac{z_j - p^\beta}{(z_j - p^{2\beta-1})(z_j - p^{\beta-1})} \times$$

$$\times \mu_{n-1}(z_1, \ldots, z_{n-1}) \varphi[z_1, \ldots, z_{n-1}, p^\beta; R] \, dz_1 \ldots dz_{n-1}, \hspace{1cm} (4.9)$$
a) The poles $z_1 = p^{\pm \beta}$ of the integrand $I_0$ for $\beta > 0$ and their motion for decreasing $\beta$. If $z_2, \ldots, z_n$ are fixed and $|z_2| = \cdots = |z_n| = 1$, then all other poles (with respect to $z_1$) lie on the circles $|z_1| = p^{-1}$, $|z_1| = p$.

b) The poles of the integrand $I_1$ for $0 > \beta > -1$. The $\beta$ decreases.

c) The poles of the integrand $I_{01}$ for $0 > \beta > -1$. The $\beta$ decreases.
\[ I_{01} = \int_{|z_1|=1, \ldots, |z_{n-1}|=1} \prod_{j=1}^{n-1} \frac{z_j - p^{-\beta}}{(z_j - p^{-\beta+1})(z_j - p^\beta)} \times \prod_{j=1}^{n-1} \phi[z_1, \ldots, z_{n-1}, p^{-\beta}, R] \, dz_1 \ldots dz_{n-1}, \] (4.10)

\[ I_1 = \int_{|z_1|=1, \ldots, |z_{n-2}|=1} \prod_{j=1}^{n-2} \frac{(z_j - p^\beta)(z_j - p^{-\beta})}{(z_j - p^{-\beta+1})(z_j - p^\beta+1)} \times \prod_{j=1}^{n-2} \phi[z_1, \ldots, z_{n-2}, p^{-\beta}, p^\beta, R] \, dz_1 \ldots dz_{n-2}, \] (4.11)

and the constants are given by

\[ C_{01} = \frac{2\pi i C_0}{p^\beta - p^{-\beta}}, \]

\[ C_1 = p^{-2\beta}(1 - p^{-2\beta-1})^{-1}(1 - p^{-2\beta+1})^{-1}(2\pi i)^2 C_0. \]

We denote the integrands in \( I_k, I_{kl} \), see (4.9)–(4.11) and Theorem 4.4 below by \( \mathcal{J}_k, \mathcal{J}_{kl} \).

**Proof.** First, we expand

\[ \prod_{j=1}^{n} \frac{1}{(z_j - p^\beta)(z_j - p^{-\beta})} = \left( \frac{p^\beta - p^{-\beta}}{z_j - p^\beta} \right)^n \prod_{j=1}^{n} \left( \frac{1}{z_j - p^\beta} - \frac{1}{z_j - p^{-\beta}} \right) \] (4.12)

and open brackets. The integral (4.3)–(4.4) splits into the sum of \( 2^n \) integrals and it is sufficient to construct the analytic continuation of each summand.

For definiteness, consider the summand

\[ N(\beta) := \int_{|z_1|=1, \ldots, |z_n|=1} \frac{1}{(z_1 - p^\beta) \ldots (z_s - p^\beta)(z_{s+1} - p^{-\beta}) \ldots (z_n - p^{-\beta})} \times \prod_{j=1}^{n} \phi[z_1, \ldots, z_n; R] \, dz_1 \ldots dz_n; \]

by symmetry considerations, we do not lose a generality. Denote the integrand in \( N(\beta) \) by \( \mathcal{H}(\beta; z) \). The analytic continuation of \( N(\beta) \) through the line Re \( \beta = 0 \) is given by the same integral over another contour

\[ |z_1| = 1 - \varepsilon, \ldots, |z_s| = 1 - \varepsilon, |z_{s+1}| = 1 + \varepsilon, \ldots, |z_n| = 1 + \varepsilon. \] (4.13)

Consider the family of contours \( L_0, \ldots L_n \), where \( L_m \) is given by

\[ L_m : \quad |z_j| = \begin{cases} 1, & \text{if } j \leq m, \\ 1 \pm \varepsilon & \text{if } j > m, \end{cases} \]
and the signs $\pm$ are the same as in (4.13). In particular, $L_n$ is the torus $|z_1| = \cdots = |z_n| = 1$, and $L_0$ is (4.13).

Then

$$
\int_{L_0} = (\int_{L_0} - \int_{L_1}) + (\int_{L_1} - \int_{L_2}) + \cdots (\int_{L_{n-1}} - \int_{L_n}) + \int_{L_n}.
$$

(4.14)

Each bracket can be evaluated by one-dimensional residues. In the last bracket we obtain

$$
\int_{|z_1| = \cdots = |z_n| = 1} \text{res}_{z_n = p^\beta} \mathfrak{N}(\beta; z) \prod_{j=1}^{n-1} dz_j,
$$

here we obtain a desired expression.

For other brackets the picture is more complicated. For instance, in the first bracket we obtain

$$
- \int \text{res}_{z_1 = p^\beta} \mathfrak{N}(\beta; z) \prod_{j=2}^{n} dz_j,
$$

where the integration is taken over the torus $|z_2| = 1 - \varepsilon, \ldots, |z_n| = 1 + \varepsilon$. For this torus we apply the transformation of (4.14) type, etc., etc. Thus we obtain $2^n$ summands having the form $\int \text{res} \mathfrak{N} \ldots \text{res}$. However, $\mu_n(z) = 0$ on the hyperplanes $z_k = z_l$, and hence

$$
\int_{L_n} - \int_{L_0} = - \sum_{j \leq s} \int_{z_j = p^\beta} \mathfrak{N} + \sum_{j > s} \int_{z_j = p^{-\beta}} \mathfrak{N} - \sum_{j \leq s, k > s} \int_{z_j = p^\beta, z_k = p^{-\beta}} \mathfrak{N},
$$

where integrations are taken over tori $|z_i| = 1$. Next, we transform this sum to the form

$$
- \sum_{1 \leq j \leq n} \int_{z_j = p^\beta} \mathfrak{N}(\beta; z) \prod_{1 \leq m \leq n; m \neq j} dz_m + \sum_{1 \leq k \leq n} \int_{z_k = p^{-\beta}} \mathfrak{N}(\beta; z) \prod_{1 \leq m \leq n; m \neq k} dz_m - \sum_{1 \leq j \leq n, 1 \leq k \leq n} \int_{z_j = p^\beta, z_k = p^{-\beta}} \mathfrak{N}(\beta; z) \prod_{1 \leq m \leq n; m \neq j, k} dz_m
$$

(4.15)

(all the 'new' summands equal zero).

Adding together all the $2^n$ summands of the integral $I_0$, we obtain an expression of the form (4.15), only $\mathfrak{N}$ is replaced by the integrand $\mathfrak{I}_0$ of $I_0$. Evaluating the residues and applying the symmetry of the spherical functions with respect to $z_j$, we obtain (4.8); the symmetry with respect to $z_j$ is the origin of the coefficients $n, n(n-1)$ in (4.8).

4.4. Plancherel formula at $\alpha = n - 1$ and $\alpha = n - 2$.

**Proposition 4.2.**

$$
\Delta_{n-2}(R) = n(n-1) \left[ C_1 I_1 \right]_{\alpha = n-2}.
$$
**Proof.** For \( \alpha = n - 2 \) we have \( C_0 = C_{01} = 0 \) in (4.8). This gives the required result.

**Proposition 4.3.**

\[
\Delta_{n-1}(R) = \frac{1}{2} n \left[ C_{01} \left( I_{01}^+ + I_{01}^- \right) \right]_{\alpha=n-1}.
\]

**Proof.** For \( \alpha = n - 1 \), the both formulae (4.2) (4.8), are not valid. Consider the new function

\[
h(\beta; R) := \frac{2 \beta \ln p}{C_0(\beta)} \Delta_{\alpha}(R)
\]

(where \( (\alpha - n + 1)/2 = \beta \)). The factor \( 2 \beta \ln p/C_0(\beta) \) is a holomorphic nonvanishing function near the point \( \beta = 0 \) and hence it is sufficient to find an integral expansion for \( h(0, R) \). We have

\[
h(0, R) = \frac{1}{2 \pi i} \int_{|\beta| = \rho} \frac{h(\beta, R)}{\beta} d\beta
\]

for each sufficiently small \( \rho \). We transform this expression to the form

\[
h(0, R) = \frac{2 \ln p}{2 \pi i} \int_{|\beta| = \rho} I_0(\beta) d\beta +
\]

\[
+ \frac{2 \ln p}{2 \pi i} \int_{|\beta| = \rho, \text{Re } \beta < 0} \left[ -\frac{n \cdot 2 \pi i}{p \beta - p^{-\beta}} \left( I_{01}^+(\beta) + I_{01}^-(\beta) \right) + \frac{n(n-1)(2\pi i)^2}{2p^{2\beta}(1-p^{-2\beta})(1-p^{-2\beta}+1)} I_1(\beta) \right] d\beta. \tag{4.16}
\]

First, the function \( I_0(\beta) \) is even, i.e., \( I_0(\beta) = I_0(-\beta) \), see (4.3). Hence, the first term in (4.16) vanishes.
Secondly, \( I_1(0) \) is finite, and the corresponding scalar factor is finite at \( \beta = 0 \). Hence the last term of (4.16) tends to zero as \( \rho \to 0 \).

Thirdly, let us evaluate

\[
\lim_{\rho \to 0} \int_{|\beta|=\rho, \Re \beta < 0} \frac{2 \ln \rho}{\rho^\beta - p^\beta} I_{01}^+ (\beta; R) \, d\beta
\]  

(4.17)

In formula (4.9), \( I_{01}^+ \) was defined as an integral over the contour \( |z_1| = \cdots = |z_{n-1}| = 1 \). We can replace this contour by \( M : |z_1| = \cdots = |z_{n-1}| = 1 + \varepsilon \), and \( \int_M \mathfrak{N}_1^+ \) is holomorphic in the strip \( |\Re \beta| < \varepsilon \). This allows to transform (4.17) to the form

\[
\int_{M}^{\lim_{\rho \to 0}} \int_{|\beta|=\rho, \Re \beta < 0} \frac{2 \ln \rho}{p^\beta - \beta^\beta} \mathfrak{N}_1^+(\beta, z) \, d\beta \, dz_1 \ldots dz_{n-1} = \pi i \int_{M} \mathfrak{N}_1^+(0, z) \, d\beta \, dz_1 \ldots dz_{n-1}.
\]

This implies the required result.

4.5. Complete analytic continuation. We preserve the notation (4.6)–(4.7) for \( \beta \) and \( \mu(z) \).

THEOREM 4.4. Let \( \Re \alpha \neq n - 1, n - 3, \ldots, -n + 1 \). Then

\[
\Delta_{\alpha}(R) = C_0 I_0 + \sum_{k, l: k \geq 1, 2(k-1)+\Re \alpha < n-1} \frac{n!}{(n-2k)!} C_k k^k + \sum_{k, l: k \geq 0, l \geq 1, 2(k+l-1)+\Re \alpha < n-1} \frac{n!}{(n-2k-l)!} C_{kl} (I_{kl}^+ + I_{kl}),
\]  

(4.18)

where

\[
I_k = \int_{|z_1| = \cdots = |z_{n-2k}| = 1} \prod_{j=1}^{n-2k} \left( \frac{z_j - p^{\beta-k+1}}{(z_j - p^{\beta+k})(z_j - p^{\beta-k})(z_j - p^{-\beta+1})} \right) \times \varphi[z_1, \ldots, z_{n-2k}, p^\beta, p^{\beta+1}, \ldots, p^{\beta+k-1}, \ldots, p^{\beta+1}, p^{-\beta}; R] \times \mu_{n-2k}(z_1, \ldots, z_{n-2k}) \, dz_1 \ldots dz_{n-2k},
\]

\[
I_{kl}^+ = \int_{|z_1| = \cdots = |z_{n-2k-l}| = 1} \prod_{j=1}^{n-2k-l} \left( \frac{z_j - p^{\beta-k+1}}{(z_j - p^{\beta+k+l})(z_j - p^{\beta-k})(z_j - p^{-\beta+1})} \right) \times \varphi[z_1, \ldots, z_{n-2k-l}, p^\beta, p^{\beta+1}, \ldots, p^{\beta+k+l-1}, \ldots, p^{\beta+1}, p^{-\beta}; R] \times \mu_{n-2k-l}(z_1, \ldots, z_{n-2k-l}) \, dz_1 \ldots dz_{n-2k-l},
\]

\[
I_{kl} = \int_{|z_1| = \cdots = |z_{n-2k-l}| = 1} \prod_{j=1}^{n-2k-l} \left( \frac{z_j - p^{\beta-k+1}}{(z_j - p^{\beta-k-l})(z_j - p^{\beta+k})(z_j - p^{-\beta+1})} \right) \times \varphi[z_1, \ldots, z_{n-2k-l}, p^\beta, p^{\beta+1}, \ldots, p^{\beta+k+l-1}, \ldots, p^{\beta+1}, p^{-\beta}; R] \times \mu_{n-2k-l}(z_1, \ldots, z_{n-2k-l}) \, dz_1 \ldots dz_{n-2k-l},
\]

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and the constants \( C_k, C_{kl} \) are given by

\[
C_k = (2\pi i)^2 C_0 \cdot \frac{p^{2\beta k + k(k-1)(1 - p)2^k - 1}}{1 - p^{2\beta + 2k - 1}} \prod_{j=1}^{k} \frac{1}{(1 - p^j)^2 (1 - p^{2\beta + j - 2})^2}, \tag{4.19}
\]

\[
C_{kl} = (2\pi i)^l C_k \cdot \frac{p^{-(\beta + k)(l + l - 1)/2} (1 - p^{2\beta - 2k + 1})(1 - p^{-1})^l}{1 - p^{2\beta - 2k - l + 1}} \times \prod_{j=1}^{l} \frac{1}{(1 - p^{-k - j})(1 - p^{-2\beta - k - j + 2})}. \tag{4.20}
\]

**Proof.** We start from Lemma 4.1 and intend to write the analytic continuation of (4.8) to the strip \(| \Re \beta + 1 | < \varepsilon \).

First, \( I_0 \) has no singularities in the half-plane \( \Re \beta < 0 \). Thus we must write the analytic continuation of \( I_{01}^+, I_1 \), see (4.9)–(4.11), to the strip \(| \Re \beta + 1 | < \varepsilon \).

Secondly, the integrand \( I_1 \) of \( I_1 \) contains the factor \( \prod_{1 \leq j \leq n - 1} (z - p^\beta + 1) (z - p^{\beta - 1}) \). We transform this factor as (4.12), and repeat literally the proof of Lemma 4.1. This gives 3 new summands \( I_2, I_{11}^+, I_{11}^- \) in the strip \( -\varepsilon < \Re \beta + 1 < 0 \).

Thirdly, \( I_{01}^+ \) contains the factor \( \prod_{1 \leq j \leq n - 1} (z - p^{\beta + 1}) \); all other factors of \( I_{01}^+ \) have no poles for \( \beta \) lying in our strip \(| \Re \beta + 1 | < \varepsilon \). Hence we write the analytic continuation of \( I_{01}^+ \) as

\[
\int_{|z_1| = 1 - \varepsilon, \ldots, |z_{n - 1}| = 1 - \varepsilon} I_{01}^+ dz_1 \ldots dz_n,
\]

and evaluate the analytic continuation as the sum of residues

\[
\int \prod_{j=1}^{n - 1} dz_j - \sum_{j=1}^{n - 1} \int_{|z_1| = \ldots = |z_{j - 1}| = |z_{j + 1}| = |z_{n - 1}|} \operatorname{res}_{z_j = p^{\beta + 1}} I_{01}^+ \prod_{1 \leq j \leq n - 1, j \neq s} dz_j
\]

(by the symmetry considerations all the summands of the sum \( \sum_{j=1}^{n - 1} \) coincide). Other summands

\[
\int \operatorname{res}_{z_j = p^{\beta + 1}} \operatorname{res}_{z_m = p^{\beta + 1}} I_{01}^+ \quad \int \operatorname{res}_{z_j = p^{\beta + 1}} \operatorname{res}_{z_m = p^{\beta + 1}} I_{01}^+ \quad \text{etc.}
\]

vanish, since the factor \( \mu_{n - 1}(z) \) is zero at the hyperplanes \( z_k = z_m \). This gives the new summand \( I_{02}^- \) in the strip \( -\varepsilon < \Re \beta - 1 < 0 \).
The case of $I_{01}$ is similar.

Further, $I_{11}^\pm, I_{02}^\pm$ have no singularities in the half-plane $\text{Re} \beta < -1$. The 'new' summands $I_2, I_{11}^\pm$ have singularities on the line $\text{Re} \beta = -2$. Their analytic continuation through this line can be obtained in the same way, etc., etc.

Now, our explicit formulae for $I_k, I_{kl}^\pm, C_k, C_{kl}^\pm$ can be proved by induction. In fact it is necessary to check the identities

$$C_{k+1}J_{k+1} = 2\pi i C_k \lim_{z_{n-2k} = p^\beta} \lim_{z_{n-2k-1} = p^\beta} J_k; \quad C_{k(l+1)}J_{k(l+1)}^\pm = 2\pi i C_{kl} \lim_{z_{n-2k-1} = p^\beta} J_{kl}^\pm$$

4.6. Integer values of $\alpha$. Proof of Theorem 2.4. Let now $\alpha = 0, 1, 2, \ldots, n - 1$. In this case, the factor (see (4.5))

$$\prod_{m=0}^{n-1} (1 - p^{-\alpha + m})$$

of $C_0$ is decisive. It vanishes at all our $\alpha$, and hence the summand $I_0$ in the Plancherel formula (4.18) disappears. The same factor kills the most of other summands, since $C_0$ is present as a factor in $C_k$ and $C_{kl}$.

**Theorem 4.5.** Let $n - \alpha = 2k$ be positive even integer. Then

$$\Delta_{n-2k} = \frac{n!}{(n-2k)!} [C_k I_k] \big|_{\alpha=n-2k}$$

**Proof.** All other summands of the formula (4.18) vanish due (4.21), the summand $C_k I_k$ survives, since the denominator of $C_k$ contains the factor

$$1 - p^{2\beta + 2k-1} = 1 - p^{\alpha - n + 2k}$$

(see (4.19)), which also vanishes at $\alpha$.

**Theorem 4.6.** Let $n - \alpha = 2m + 1$ be positive odd integer. Then

$$\Delta_{n-2m-1} = \frac{n!}{2(n-2m-1)!} [C_m (I_{m1}^+ (\alpha) + I_{m1}^- (\alpha))] \big|_{\alpha=n-2m-1}$$

**Proof.** Theorem 4.4 does not give immediate answer in this case and we use the same arguments as in Proposition 4.3.

Consider the analytic continuation of our integral to the strip $-m < \text{Re} \beta < -m + 1$, or equivalently $n - 2m - 1 < \text{Re} \alpha < n - 2m + 1$, it is given by (4.18). We must evaluate the limit of each summand of (4.18) as $\beta$ tends to $-m$ from our strip.

First, let $k < m$. Then $I_k$ is holomorphic in $\text{Re} \beta < -m + 1$, and $C_k \big|_{\beta=-m} = 0$. Hence, the summand $C_k I_k$ vanishes at $\beta = -m$.

Secondly, let $k + l < m$. Then $I_{kl}^\pm$ are holomorphic in $\text{Re} \beta < -m + 1$, and $C_{kl} \big|_{\beta=-m} = 0$. Hence $C_{kl} I_{kl}^\pm$ vanish.
Thirdly, let \(k + l = m\), \(l > 0\). Then the analytic continuation of \(I_{kl}^+\) to the strip \(|\Re \beta + m| < \varepsilon\) is given by

\[
\int_{|z_1| = \cdots = |z_{n-2k-1}| = 1 + \varepsilon} 2_{kl}^+ 
\]

Hence \(I_{kl}^+\) is holomorphic in \(|\Re \beta + m| < \varepsilon\). But the coefficient \(C_{kl}\) vanishes at \(\beta = -m\) again.

Thus the problem is reduced to the evaluation of \(\lim_{\beta \to -m+0} C_m I_m\). The factor \(C_m\) has a simple zero at \(\beta = -m\), and hence it is sufficient to evaluate

\[
\lim_{\beta \to -m+0} (\beta + m) I_m.
\]

We represent

\[
\frac{(z - p^{\beta+m-1})(z - p^{-\beta-m+1})}{(z - p^{\beta+m})(z - p^{-\beta-m})(z - p^{\beta+1})(z - p^{-\beta+1})} = \frac{\lambda(\beta)}{(z - p^{\beta+m})(z - p^{-\beta-m})} + \frac{\sigma(\beta)}{(z - p^{\beta+1})(z - p^{-\beta+1})},
\]

where

\[
\lambda(\beta) = \frac{(1 - p)(1 - p^{2\beta+2m-1})}{(1 - p^{m+1})(1 - p^{2\beta+m-1})}; \quad \sigma(\beta) = \frac{p(p^{2\beta+k-2} - 1)(p^k - 1)}{(p^{2\beta+k-1} - 1)(p^{k+1} - 1)}.
\]

Let \(J\) be a subset in \(\{1, 2, \ldots, n - 2m\}\). Denote by \(||J||\) the number of elements of \(J\). According (4.22), we represent \((\beta + m) I_m\) as a sum of \(2^{n-2m}\) integrals

\[
(\beta + m) I_m(\beta) = \sum_J \lambda(\beta)^{||J||} \sigma(\beta)^{n-2m-||J||} \int_{|z_s| = 1, s \notin J} \prod_{s \notin J} \left(\frac{1}{(z_s - p^{\beta+1})(z_s - p^{-\beta+1})}\right) \times
\]

\[
\times \left[ (\beta + m) \int_{|z_j| = 1, j \in J} \prod_{j \notin J} \left(\frac{1}{(z_j - p^{\beta+m})(z_j - p^{-\beta-m})}\right) \times
\]

\[
\times \mu_{n-2m}(z_1, \ldots, z_{n-2m}) \varphi(\ldots; R) \prod_{j \in J} dz_j \prod_{s \notin J} dz_s. \quad (4.23)
\]

Denote by \(F(\beta) = F_J(\beta)\) the expression in big brackets (it depends also on \(z_s\) for \(s \notin J\)). Denote by \(F(\beta)\) the integrand in \(F(\beta)\). The analytic continuation of \(F(\beta)\) can be easily written as in proof of Lemma 4.1. In the strip \(-m - 1 < \Re \beta < -m\) it is

\[
F(\beta) + (\beta + m) G(\beta),
\]
where \( F(\beta) \) is the same expression in square brackets (and hence \( F(\beta) \) is singular on the line \( \text{Re}\beta = -m \)) and

\[
G(\beta) = 2\pi i \sum_{k \in J} \int_{|z_j| = 1 \text{ for } j \in J, j \neq k} \left[ -\text{res}_{z_k = p^{\beta + m}} \frac{1}{z_k} + \text{res}_{z_k = p^{\beta - m}} \right] \prod_{j \in J, j \neq k} dz_j - (2\pi i)^2 \sum_{k, l \in J, |z_j| = 1 \text{ for } j \in J, j \neq k} \text{res}_{z_k = p^{\beta + m}} \text{res}_{z_l = p^{\beta - m}} \prod_{j \in J, j \neq k, l} dz_j
\]

Thus,

\[
\lim_{\beta \to -m+0} F(\beta) = \frac{1}{2\pi i} \int_{|\beta + m| = \rho} \frac{F(\beta) d\beta}{\beta + m} + \frac{1}{2\pi i} \int_{|\beta + m| = \rho, \text{Re}\beta < -m} G(\beta) d\beta
\]

The transformation \( \beta \mapsto -2m - \beta \) preserves \( F_1(\beta) \) and hence the first summand in right hand side is 0.

The second summand gives the required result as in proof of Proposition 4.3.

4.7. Absence of multiplicity. Evidently, the \( \text{GL}_n(O) \)-fixed vector \( e_{O^n} \) is \( \text{GL}_n(\mathbb{K}) \)-cyclic. This easily implies Proposition 2.5, see proof of Lemma 1.10 in [11].

4.8. System \( e_R \) in the direct integral. Proof of Theorem 2.7. First, \( \rho_\alpha(g)u_R = u_R \). Hence we can assume \( R = O^n \), and \( u_{O^n}(s, V) = 1 \). Thus, we must evaluate

\[
\int_{\Xi_n} \int_{\text{Fl}_n} u_T(s, V) dV d\mu_\alpha(s)
\]

Integrating over \( \text{Fl}_n \) using (1.8), we obtain

\[
\int_{\Xi_n} \varphi_{1\alpha}(T) d\mu_\alpha(s)
\]

It equals \( K_\alpha(O^n, T) \) by Theorem 2.3.

4.9. Identities with Hall–Littlewood functions. Let us prove the statement a) of Proposition 2.11. Using (1.11), it is possible to represent the integral (4.3)–(4.4) as a sum of residues. The result is a (nonhand) rational expression in \( p \) and \( p^\alpha \). Thus the required identity is an identity of rational functions which holds for countable number of \( p \) (and for all \( \alpha \) if \( p \) is fixed). Hence the identity holds always.

The statement b) follows from a) and the inversion formula for spherical transform (the Macdonald’s proof of Theorem 5.1.2 is based on calculations with Hall–Littlewood-type expression (1.11) and do not use primality of \( p \)).

5. Positivity and nonpositivity

Here we prove Theorem 2.2 on positive definiteness of the kernel \( K_\alpha \).
5.1. The case $\alpha > n - 1$. This follows from Theorem 2.3, since $K_\alpha(R, T)$ is an integral of positive definite kernels $\Phi_{i_1, \ldots, i_n}(R, T)$ with a positive weight $d\mu_\alpha$.

5.2. Positive definiteness of the kernel $K_\alpha(R, T)$ for integer $\alpha$. This again follows from the Plancherel formula (Theorem 2.4).

More simple proof is given below in 6.1.

5.3. The case $\alpha < 0$. This case is obvious:

$$\langle e_R, e_R \rangle = \langle e_S, e_S \rangle = 1, \quad \langle e_R, e_S \rangle > 1.$$ This is impossible.

5.4. Noninteger $\alpha$ between 0 and $n - 1$. Assume that the function $\Delta_\alpha$ is positive definite. Then it can be expanded as an integral of positive definite spherical functions $\varphi_\tau$

$$\Delta_\alpha(R) = \int \varphi_\tau(R) d\kappa_\alpha(\tau) \quad (5.1)$$

with respect to some positive measure $\kappa_\alpha$.

In Theorem 4.4, we obtained some expansion

$$\Delta_\alpha(R) = \int \varphi_\tau(R) d\sigma_\alpha(\tau) \quad (5.2)$$

of $\Delta_\alpha$ as an integral of spherical functions with respect to some measure (charge) $\sigma_\alpha$. For all noninteger $\alpha < n - 1$, the summand $I_{01}$ is present in our expansion with a nonzero coefficient. But the spherical function $\varphi_{i_1, \ldots, i_{n-1}, n-1-\alpha}$ is not positive definite. Hence the expansions (5.1), (5.2) are different.

It is sufficient to reduce the existence of these two different expansions to a contradiction. We do this in the rest of this section.

5.5. The set $\Sigma_n$. Denote by $\rho$ the vector $(n-1)/2, (n-1)/2 - 1, \ldots, -(n-1)/2 \in \mathbb{R}^n$. The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permutations of the coordinates. Denote by $Q$ the convex hull of $S_n$-orbit of the vector $\rho$.

By $\Sigma_n$ we denote the set of all

$$\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n / \mathbb{Z}^n$$

such that each $\tau_j$ is real or pure imaginary and $\text{Re} \tau \in Q$.

The set $\Sigma_n$ is important for us by the following reasons.

1°. If $\varphi_\tau$ is positive definite, then $\tau \in \Sigma_n$. In particular, the measure $\kappa_\alpha$ from (5.1) is supported by $\Sigma_n$

2°. For $\alpha > 0$, the measure $\sigma_\alpha$ from (5.2) is supported by $\Sigma_n$.

3°. $\Sigma_n$ is compact.

The unitary dual of $\text{GL}_n(\mathbb{K})$ is known (see [14], [15]), but it is easier to reduce the statement 1° to Theorem 4.7.1 of [6].

We also denote by $\Sigma_n^u$ the set of positive definite spherical functions.
5.6. Action of the Hecke algebra. The Hecke algebra $\mathcal{H}_n$ of $\text{GL}_n(\mathbb{K})$ (see [7]) is the convolution algebra of compactly supported $\text{GL}_n(\mathbb{O})$-biinvariant functions on $\text{GL}_n(\mathbb{K})$.

A function $f$ on $\text{GL}_n(\mathbb{K})$ is called $\text{GL}_n(\mathbb{O})$-biinvariant if

$$f(h_1gh_2) = f(g) \quad \text{for } g \in \text{GL}_n(\mathbb{K}), \quad h_1, h_2 \in \text{GL}_n(\mathbb{O}).$$

The multiplication in the Hecke algebra is the convolution.

The Hecke algebra acts on the space of $\text{GL}_n(\mathbb{O})$-biinvariant functions by the convolutions.

The spherical functions $\varphi_\tau$ are the eigenfunctions of $\mathcal{H}_n$, i.e.,

$$\gamma * \varphi_\tau = c_\gamma(\tau) \varphi_\tau, \quad \gamma \in \mathcal{H}, \quad (5.3)$$

where $c_\gamma(\tau)$ is a scalar factor.

Denote by $\mathcal{A}$ the algebra of all polynomial expressions of $p^{\tau_1}, \ldots, p^{\tau_n}, p^{-\tau_1}, \ldots, p^{-\tau_n}$ symmetric with respect to permutations of $\tau_j$. The map $\gamma \mapsto c_\gamma$ is an isomorphism $\mathcal{H} \to \mathcal{A}$, see [7], V.3.2.

**Lemma 5.1.** Let a positive definite $\text{GL}_n(\mathbb{O})$-biinvariant function $f$ be represented as an integral over $\Sigma_n$ with respect to some charge

$$f(g) = \int_{\Sigma_n} \varphi_\tau(g) \, d\sigma(\tau).$$

Let $u(\tau) \in \mathcal{A}$ be nonnegative on $\Sigma_n$. Then the function

$$q(g) = \int_{\Sigma_n} u(\tau) \varphi_\tau(g) \, d\sigma(\tau).$$

also is positive definite.

**Proof.** Since $f$ is positive definite, it admits an expansion

$$f(g) = \int_{\Sigma_n} \varphi_\tau(g) \, d\nu(\tau).$$

for some positive measure $\nu$. Let $\gamma$ be the element of the Hecke algebra corresponding to the function $u(\tau)$. Then

$$(\gamma * f)(g) = \int_{\Sigma_n} u(\tau) \varphi_\tau(g) \, d\sigma(\tau) = \int_{\Sigma_n} u(\tau) \varphi_\tau(g) \, d\nu(\tau)$$

The second integral defines a positive definite function, and this proves Lemma. □

5.7. Approximation of spherical functions by matrix elements of $U_n$. Denote by $W_n$ the hyperoctahedral group of $\mathbb{R}^n$, i.e., the group generated by permutations of the coordinates and arbitrary changes of directions of axes.

Consider the map $\Sigma_n \to \mathbb{R}^n$ given by

$$(\tau_1, \ldots, \tau_n) \mapsto (p^{\tau_1} + p^{-\tau_1}, \ldots, p^{\tau_n} + p^{-\tau_n})$$
Obviously, it is constant on orbits of $W_n$ and moreover this map is an embedding of the quotient space $\Sigma_n/W_n$ to $\mathbb{R}^n$.

Consider the functions
\[ u_k(\tau) = \sum_{j=1}^{n} \left( p^{\tau_j} + p^{-\tau_j} \right)^k. \]

Obviously, these functions are real on the set $\Sigma_n$ and they separate orbits of the hyperoctahedral group on $\Sigma_n$.

Fix a point $\varepsilon = (i s_1, \ldots, i s_{n-1}, n - 1 - \alpha) \in \Sigma_n$. Consider the function
\[ F(\tau) = \sum_{k=1}^{n} \left[ u_k(\tau) - u_k(\varepsilon) \right]^2. \]

The function $F$ is zero at the orbit $W_n \cdot \varepsilon$ and is positive outside this orbit. Let $M$ be the maximum of $F$ on $\sigma$. Consider the new function
\[ X(\tau) = M - F(\tau). \]

It is a positive function having a maximum on the orbit $W_n \cdot \varepsilon$. For simplicity, assume that $s_j$ are pairwise distinct. Then the points $w\tau$ are nondegenerate critical points of $X$. Let $\xi$ be the corresponding element of the Hecke algebra.

Consider the sequence of positive definite functions
\[ m^{(n-1)/2} \xi^m \ast \Delta_\alpha(g) = m^{(n-1)/2} \int X^m(\tau)\varphi_s(g) d\sigma_\alpha(\tau). \tag{5.4} \]

It can be easily checked, that its limit as $m \to \infty$ is
\[ r(g) := \sum_{w \in W_n} c(w\varepsilon) \varphi_{w\varepsilon}(g), \tag{5.5} \]

where the scalar coefficients $c(w\varepsilon)$ are nonzero. This function is positive definite as a limit of positive definite functions. Hence $r(g)$ can be expanded into an integral of positive definite spherical functions,
\[ r(g) = \int_{\Sigma_n^*} \varphi_\tau(g) d\lambda(\tau). \tag{5.6} \]

Let us evaluate the limit of $\xi^m \ast r$ as $m \to \infty$ (now without the normalizing factor $m^{(n-1)/2}$). Applying it to (5.6), we obtain
\[ \xi^m \ast r(g) = \int_{\Sigma_n^*} X(g)^m \varphi_\tau(g) d\lambda(\tau). \tag{5.7} \]

The orbit $W_n\varepsilon$ has no intersection with the (closed) set $\Sigma_n^*$, and hence the limit of integrals (5.7) is zero. If we evaluate the same limit using (5.5), we obtain $r(g)$ (since $\xi^m \ast r = r$). Hence $r(g) = 0$. But $r(g)$ is a finite linear combination
of the spherical functions, on the other hand the spherical functions are the eigenfunctions of the Hecke algebra. This is a contradiction.

6. Integer values of $\alpha$ and difference equations

Here we prove Theorem 2.8 on linear dependence of the vectors $e_R$.

6.1. Embedding of $H_m$ to $L^2(\mathbb{K}^{mn})$. Let $m$ be a nonnegative integer. Consider the space

$$\mathbb{K}^{mn} = \mathbb{K}^n \oplus \cdots \oplus \mathbb{K}^n.$$  

For a lattice $R \in \text{Lat}_n$, we define the lattice

$$R^m := R \oplus \cdots \oplus R \subset \mathbb{K}^{nm}.$$  

Define the function $e_R$ on $\mathbb{K}^{nm}$ by the rule

$$e_R(\theta) = \begin{cases} \text{vol}(R)^{-m/2}, & \text{if } \theta \in R^m \\ 0, & \text{if } \theta \notin R^m \end{cases}.$$  

Obviously, the $L^2(\mathbb{K}^{mn})$-scalar products of the vectors $e_R$ are

$$\langle e_R, e_S \rangle = K_\alpha(R, S)$$

and hence we can identify $H_m$ with the subspace in $L^2(\mathbb{K}^{mn})$ generated by the functions $e_R(\theta)$.

In particular, this proves the existence of the Hilbert space $H_m$ for integer $m$.

6.2. Reduction of Theorem 2.8 to combinatorial problem. Now, let $m = 0, 1, \ldots, n - 1$.

Consider lattices $R \subset S$ in $\mathbb{K}^n$ such that $S/R = (\mathbb{Z}/p\mathbb{Z})^{m+1}$. Without loss of generality, we can assume vol($S$) = 1.

It is convenient to identify the quotient $S/R$ with $(m+1)$-dimensional linear space $\mathbb{F}_p^{m+1}$ over the $p$-element field $\mathbb{F}_p$. Denote by $\pi$ the natural projection

$$\pi : S \rightarrow S/R \simeq \mathbb{F}_p^{m+1}.$$  

Let $Q \in \text{Lat}_n$ satisfy $R \subset Q \subset S$.

The function $e_Q$ is zero outside $S^n$; if $\xi - \eta \in R^n$, then $e_Q(\xi) = e_Q(\eta)$. This allows to consider the functions $e_Q$ as functions on the quotient group

$$S^n/R^n \simeq \mathbb{F}_p^{m+1} \oplus \cdots \oplus \mathbb{F}_p^{m+1} \simeq \mathbb{F}_p^{(m+1)m}.$$  

More precisely, for each linear subspace $L \subset \mathbb{F}_p^{m+1}$ we define the function

$$\tilde{e}_L(w_1 \oplus \cdots \oplus w_m) := \begin{cases} p^{\text{codim}(L)m/2}, & \text{if } w_j \in L \text{ for all } j; \\ 0, & \text{otherwise}. \end{cases}$$  

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If \( Q \) lies between \( R \) and \( S \), then \( L := Q/R \) is a linear subspace in \( S/R = \mathbb{F}_p^{m+1} \). Obviously, we have
\[
e_Q(\theta_1 \oplus \cdots \oplus \theta_m) = \tilde{e}_L(\pi(\theta_1) \oplus \cdots \oplus \pi(\theta_m))
\]
Thus, it is sufficient to investigate the linear dependencies of the functions \( \tilde{e}_L \).

5.3. **Proof of Theorem 2.8.** For \( k = 0, \ldots, m + 1 \) consider the function \( G_k \) on \( \mathbb{F}_p^{(m+1)m} \) given by
\[
G_k(w_1 \oplus \cdots \oplus w_m) = p^{-km/2} \sum_{k: \text{codim } L = k} \tilde{e}_L(w_1 \oplus \cdots \oplus w_m).
\]
We intend to find numbers \( u_0, \ldots, u_m \) such that
\[
\sum_k u_k G_k = 0.
\]

**Lemmma 6.1** \( u_k = (-1)^k p^{k(k-1)/2} \).

**Proof.** Obviously, \( G_k(w_1 \oplus \cdots \oplus w_m) \) coincides with the number of subspaces \( L \) of codimension \( k \) containing all the vectors \( w_1, \ldots, w_m \). Denote by \( W \) the linear span of \( w_j \). We must count subspaces in \( \mathbb{F}_p^{m+1} \) containing \( W \), or equivalently, linear subspaces in \( \mathbb{F}_p^{m+1}/W \).

The number \( A_j^i \) of \( j \)-dimensional subspaces in \( \mathbb{F}_p^i \) is
\[
A_j^i = \frac{(p^i-1)(p^{i-1}-1) \cdots (p^{i-j+1}-1)}{(p^i-1)(p^{i-1}-1) \cdots (p-1)}
\]
Hence, our Lemma is equivalent to the family of the identities
\[
\sum_{i=0}^s (-1)^i p^{i(i-1)/2} \frac{(p^s-1)(p^{s-1}) \cdots (p^{s-i+1}-1)}{(p^i-1)(p^{i-1}-1) \cdots (p-1)} = 0 \quad (6.1)
\]
For this identity, we can refer to [7], Ex.1.2.3, or to the \( q \)-binomial theorem (see, for instance [1], §1.3)
\[
\sum_{i=0}^\infty \frac{(1-aq)(1-aq^2) \cdots (1-aq^{i-1})}{(1-q)(1-q^2) \cdots (1-q^i)} z^i = \frac{(1-az)(1-azq)(1-azq^2) \cdots}{(1-z)(1-zq^2)(1-zq^3) \cdots}
\]
we substitute \( q = 1/p, a = p^s, z = q \).

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