Strict Linearizability and Abstract Atomicity

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Linearizability is a commonly accepted consistency condition for concurrent objects. Filipović et al. show that linearizability is equivalent to observational refinement. However, linearizability does not permit concurrent objects to share memory spaces with their client programs. We show that linearizability (or observational refinement) can be broken even though a client program of an object accesses the shared memory spaces without interference from the methods of the object. In this paper, we present strict linearizability which lifts this limitation and can ensure client-side traces and final-states equivalence even in a relaxed program model allowing clients to directly access the states of concurrent objects. We also investigate several important properties of strict linearizability.

At a high level of abstraction, a concurrent object can be viewed as a concurrent implementation of an abstract data type (ADT). We also present a correctness criterion for relating an ADT and its concurrent implementation, which is the combination of linearizability and data abstraction and can ensure observational equivalence. We also investigate its relationship with strict linearizability.

Keywords: Concurrent objects; linearizability; data abstraction; observational equivalence; atomicity.

1. Introduction

Correctness conditions for concurrent objects generally require that each concurrent execution of an object is equivalent to a legal sequential execution of either the object or an ADT. Different correctness conditions are distinguished by their different interpretation of the term “equivalence”. Linearizability [8], sequential consistency [29] and quiescent consistency [14] have been widely accepted consistency conditions for concurrent objects.

Filipović et al. [17] show that linearizability is equivalent to observational refinement—for a linearizable object Z, its corresponding ADT A and any client
program $P$, every observable behavior of $P(Z)$ can also be observed by $P(A)$, where $P(Z)$ denotes the client program $P$ that uses the object $Z$. Thus, clients can replace the fine-grained $Z$ with the more abstract coarse-grained $A$ to simplify reasoning.

However, linearizability assumes a complete isolation between an object and its client programs, and does not permit them to run in shared memory spaces. The example in Section 2 shows that linearizability (or observational refinement) can be broken even though a client program of an object accesses the shared memory spaces without interference from the methods of the object. A key reason for this is that linearizability cannot ensure that a concurrent execution of an object and its corresponding sequential execution have the same final states. In some applications, concurrent objects need to share memory spaces with their client programs and permit them to access the shared spaces via atomic memory read/write actions. In this cases, atomicity specifications of concurrent objects should capture the above state consistency.

For example, RDCSS is part of the implementation of multiple compare-and-swap (MCAS) [20,10]. In MCAS, memory spaces are accessed via atomic memory read/write actions or the methods of RDSS. Thus, RDCSS must ensure that its linearizability cannot be broken by the atomic memory read/write actions of MCAS. Furthermore, MCAS needs to share memory spaces with their client programs and permits them to access the shared spaces. As another example, consider the atomic classes from the java.util.concurrent.atomic package, such as the AtomicInteger class, the AtomicBoolean class. Client programs can access the atomic variables (i.e. instances of the atomic classes) via the methods of the classes or atomic read/write actions.

In this paper, we present strict linearizability, a correctness criterion aimed at lifting the above limitation. We also show the following several properties of strict linearizability which linearizability cannot capture.

- Strict linearizability can ensure client-side traces and final-states equivalence. Informally, for a strict linearizable object $Z$, any client program $P$, $P(Z)$ has the same client-side traces and final states as $P(Ato_Z)$ even in a program model allowing $P$ to directly access the states of $Z$ in some compatible ways. Here, $Ato_Z$ denotes an atomic version of $Z$ which complies with a sequential specification of $Z$ and can be obtained by using atomic regions to protect each method of $Z$.
- Strict linearizability can provide a strong termination-preserving property. For example, we show that for a strict linearizable and purely-blocking object, a program using the object diverges iff the program using its atomic version diverges. Thus, while proving termination of a program using such an object, it is sufficient to replace the object with its atomic version.
- For a strict linearizable object, its sequential specification can serve as “maximal” atomicity abstraction (Theorem 17)—for a strict linearizable object $Z$, in order to verify whether $Z$ is a concurrent implementation of an ADT $A$, it is sufficient to check whether its sequential specification satisfies the specification
of A. Obviously, verifying the latter is easier than verifying the former.

In this paper, we refer to a sequential specification of a concurrent object as its atomicity specification. A concurrent object satisfies its atomicity specification iff it is strict linearizable. Most concurrent objects we know of ensure strict linearizability. For example, even many subtle concurrent objects, such as RDCSS, MCAS, the pair snapshot algorithm [9], the MS lock-free queue [18], the lazy list algorithm [19] are strict linearizable.

At a high level of abstraction, a concurrent object can be viewed as a concurrent implementation of an ADT. What does it mean for a concurrent object to be an implementation of an ADT? Like in the sequential setting, data abstraction in the concurrent setting should also ensure the important representation independence property. We state the representation independence property in terms of observational equivalence—two correct implementations of an ADT are observationally indistinguishable by clients of the ADT. Linearizability is not sufficient to capture the representation independence property because it only ensures observational refinement, not observational equivalence. Thus, new observable behaviours can be introduced when clients replace a linearizable object with its corresponding ADT to simplify reasoning about their programs (see the example in Subsection 6.2).

In this paper, we propose a correctness criterion for relating an ADT and its concurrent implementation, which combines linearizability and data abstraction, and can ensure observational equivalence. Thus, like in the sequential setting, clients do not need to know the implementation details and internal synchronization mechanisms of concurrent objects, and can use the ADTs interfaces to reason about their programs. We refer to such an ADT specification as an abstract atomicity specification of its corresponding concurrent object. We also investigate the relationship between atomicity specification and abstract atomicity specification. As is mentioned above, for a strict linearizable object, its sequential specification can serve as “maximal” atomicity abstraction. We moreover show the proof obligations which can help establish atomicity in terms of abstract atomicity.

2. Motivating Example

In this section, we show that linearizability cannot ensure that a concurrent execution of an object and its corresponding sequential execution have the same final states. A key reason for this is that linearizability is a property of externally-observable behaviors (i.e. histories) of concurrent objects. Informally, a history consists only of input arguments and return values of the called methods of concurrent objects, not the internal states of concurrent objects. When there is a complete isolation between an object and its client programs, the inconsistent states cannot be observed by the client programs. However, linearizability (observational refinement) can be broken even though client programs access the internal states of an object without interference from the methods of the object. In this case, clients draw false conclusions when they reason about their programs in terms of the sequential
specification of the object.

class Queue {
    int back := 1;
    data t[] items;
    void Enqueue(data t v);
    data t Dequeue();
}

void Enqueue(data t v) {
    L0 local t;
    L1 t := INC(back);
    L2 items[t] := v;
}

data t Dequeue() {
    L3 local temp, range;
    L4 while (true) {
        L5 temp := null;
        L6 range := back - 1;
        L7 for (int i := 1; i <= range; i++) {
            L8 temp := swap(items[i], null)
            L9 if (temp != null)
                L10 return temp;
        }
    }
}

Fig. 1. the HW queue

Fig. 1 shows the HW queue. The queue is represented as an infinite size array, *items*, and an integer variable, *back*, holding the smallest index in the unused part of the array. The index of the array starts with 1, and the variable *back* is initialized to 1. The algorithm assumes each element of the array is initialized to a special value *null*. The HW queue is linearizable with respect to a specification of a standard queue data type [1].

Consider the following program \( P(HW) \):

\[
\text{HW}.\text{Enqueue('c')} \parallel \text{HW}.\text{Enqueue('d')} \parallel \text{HW}.\text{Dequeue()}
\]

The program \( P(HW) \) has four possible final states shown in Fig. 2. However, the program \( P(Ato\_HW) \) has only two possible final states shown in Fig. 2(c) and Fig. 2(d). \( Ato\_HW \) denotes the atomic version of the HW queue, which complies with the sequential specification of the HW queue (see Section 5).

![Fig. 2. Four possible final states of \( P(HW) \)](image)

Obviously, \( P(HW) \) and \( P(Ato\_HW) \) have different final states. The reason for this is that the final state of a concurrent execution may be inconsistent with that of the sequential execution whose history is a linearization of the history of the concurrent execution. Consider an execution of \( P(HW) \) generating the possible final state in Fig. 2(a), as shown in Fig. 3. By executing \texttt{INC} command (line L1), the Enqueue(‘c’) operation reserves array position 1 and the Enqueue(‘d’) operation reserves array position 2. The Enqueue(‘d’) operation stores ‘d’ before the Enqueue(‘c’) operation stores ‘c’. The Dequeue operation starts to traverse the array after the Enqueue(‘d’) operation stores ‘d’ and returns before the Enqueue(‘c’) operation stores ‘c’. Thus the final state of the execution is that *items*[1] is ‘c’ and

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\text{c} & \text{null} & \text{null} \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\text{null} & \text{c} & \text{null} \\
\end{array}
\]
other elements of the array are null. The only sequential execution which produces a linearization of the history of the concurrent execution depicted above is:

\[
\text{HW.Enqueue('d'); HW.Enqueue('c'); HW.Dequeue();}
\]

The final state of the sequential execution is that items[2] is 'c', other elements of the array are null. Thus, the inconsistent state can be observed by client programs even though the client programs access the elements of the array items without interference from the methods of the HW queue. In this case, clients draw false conclusions when they reason about their programs in terms of the sequential specification of the HW queue.

3. Trace Model

3.1. Characterizing Behaviours of Concurrent objects via Trace Model

In the concurrent setting, a concurrent object provides a set of methods, which can be called concurrently by its client programs. For a concurrent object Z, let Zop denote a set of the methods of Z. Let \( P(Z) \) denote a client program \( P \) that uses the object Z. For simplicity, we assume each method takes one argument and returns a value using the command \( \text{ret}(E) \). The methods are defined by declarations of the form \( f(x) : C; \text{ret}(E) \). Here \( f \) is a method name, \( x \) is a formal argument, \( C; \text{ret}(E) \) is a method body. The method calls are of the form \( x := F(E) \).

\[
\begin{align*}
E & := n | x | E + E | \ldots \\
B & := \text{true} | \text{false} | E = E | E \leq E | \ldots \\
I & := x := [E] | [E] := E | x := \text{cons}(E) | \ldots \\
C & := I | x := Z.f(E) | C; C | \text{if } B \text{ then } C \text{ else } C | \text{while } B \text{ do } C | \langle C \rangle \\
\text{Zop} & := \{f_1(x_1) : C_1; \text{ret}(E_1), \ldots, f_n(x_n) : C_n; \text{ret}(E_n)\} \\
P(Z) & := C || \ldots || C
\end{align*}
\]

\( P(Z) \) contains several sequential commands, each of which is executed by a thread, as shown in Fig. 4. \( I \) is a set of primitive instructions, \( \text{cons} \) is an allocating memory cells command, \( x := [E] \) and \( [E] := E \) are reading and writing memory
cells commands respectively. \langle C \rangle is an atomic region. An atomic action is either a primitive instruction or an atomic region. Let A be a set of atomic actions.

Let M be a set of method names, T be a set of thread identifiers. We refer to a method call as an operation. Let O be a set of operation identifiers which is used to identify every method call. An action label is either an invocation of an operation, a response by an invoked operation, or an atomic action. An event consists of a thread identifier and an action label and an operation identifier (if an event of an object), and can be one of the following forms:

\[
\text{Event} := (t, \text{inv}(m, v), o) \mid (t, a, o) \mid (t, \text{ret}(v), o) \mid (t, a)
\]

where \( t \in T \), \( o \in O \), \( m \in M \), \( a \in A \), \( v \in \text{Values} \). The event \((t, \text{inv}(m, v), o)\) represents an invocation event of a method \( m \) with an argument value \( v \) which is performed by a thread \( t \) and is identified by an operation identifier \( o \). \((t, a, o)\) represents an event of an operation \( o \)'s body. \((t, \text{ret}(v), o)\) represents a response event of an operation \( o \) with a return value \( v \). \((t, a)\) represents a client program's event which is performed by a thread \( t \). For an event \( e \), Let \( \text{Thr}(e) \) denote its thread identifier, \( \text{Lab}(e) \) denote its label, \( \text{Op}(e) \) denote its operation identifier. We sometimes omit the first and third fields of events when they are irrelevant to discussions. Let \( \text{invAct} \) be a set of invocation events, \( \text{resAct} \) be a set of response events. An invocation event \( e_1 \in \text{invAct} \) matches a response event \( e_2 \in \text{resAct} \), denoted by \( e_1 \sim e_2 \), if \( \text{Op}(e_1) = \text{Op}(e_2) \).

\[
\begin{align*}
\llbracket E \rrbracket_t & \subseteq Tr \times \text{Values} \\
\llbracket a \rrbracket_t &= \begin{cases} 
(a, t), & \text{if } a \in A \text{ occurs in a client program;} \\
(a, o, t), & \text{if } a \in A \text{ occurs in an operation } o.
\end{cases} \\
\llbracket \text{ret}(E) \rrbracket_t &= \{ \rho^{-1}(t, o, \text{ret}(v)) \mid (\rho, v) \in \llbracket E \rrbracket_t \} \\
\llbracket C_1; C_2 \rrbracket_t &= \llbracket C_1 \rrbracket_t \llbracket C_2 \rrbracket_t = \{ \rho_1 \rho_2 \mid \rho_1 \in \llbracket C_1 \rrbracket_t \land \rho_2 \in \llbracket C_2 \rrbracket_t \} \\
\llbracket \text{if } B \text{ then } C_1 \text{ else } C_2 \rrbracket_t &= [B]_t^{\text{true}} \llbracket C_1 \rrbracket_t \cup [B]_t^{\text{false}} \llbracket C_2 \rrbracket_t \\
\llbracket \text{while } B \text{ do } C \rrbracket_t &= ([B]_t^{\text{true}} \llbracket C \rrbracket_t)^* [B]_t^{\text{false}} \cup ([B]_t^{\text{true}} \llbracket C \rrbracket_t)^\omega \\
\llbracket x = z.f(E) \rrbracket_t &= \{ \rho_1(t, o, \text{inv}(f, n))^{-1} \rho_2(t, o, \text{ret}(v))^{-1}(t, x := v) \mid (\rho_1, n) \in \llbracket E \rrbracket_t \land \rho_2(t, o, \text{ret}(v)) \in \llbracket \text{body} \rrbracket_t \} \\
\llbracket C_1 \parallel C_2 \rrbracket &= \bigcup \{ \lambda_1 \parallel \lambda_2 \mid \lambda_1 \in \llbracket C_1 \rrbracket_{t_1} \land \lambda_2 \in \llbracket C_2 \rrbracket_{t_2} \}
\end{align*}
\]

Fig. 5. Trace Semantics of Commands and programs

A trace is a sequence of events. For a trace \( \lambda \), let \( |\lambda| \) denote the length of the trace; \( |\lambda| = \omega \) if \( \lambda \) is infinite. Let \( \preceq_a \) denote a happened-before order on events; for two events \( c \) and \( d \) in a trace, \( c \preceq_a d \) if \( c \) precedes \( d \) in the trace.

The semantics of commands and programs is defined in terms of traces [28], and it is shown in Fig. 5. Let \( Tr \) be a set of all traces. We write \( \rho_1 \rho_2 \) for the trace
obtained by concatenating \( \rho_1 \) and \( \rho_2 \); when \( \rho_1 \) is infinite this is just \( \rho_1 \). Let \([C]_t\) be a set of traces of a sequential command \( C \), which is parameterized by a thread (which runs the command). \([E]_t\) is a set of all \((\rho, v)\) such that \( E \) evaluates to \( v \) along the trace \( \rho \). \([B]_{\text{true}}\) is a set of all \( \rho \) such that \((\rho, \text{true})\) \( \in [B]_t\). \( \text{body} \) denotes the body of the method \( f \). The operator \([\_\_\_]\) forms a set of all possible interleavings of two sequences.

3.2. Client-Side Traces and Final States

We assume that states of client programs are disjoint from ones of concurrent objects. The assumption is used in the standard notion of linearizability [8]. For a program \( P(Z) \), a valid state is \((\sigma_c, (\sigma_z, u))\), where \( \sigma_c \) is a state of the client program \( P \), \((\sigma_z, u)\) is a configuration of the object \( Z \). \( \sigma_z \) is a state of \( Z \), which records the values of the concurrent object’s shared data and pointer variables. Let \( t \in \text{Lop} \) denote a local state of an operation. \( u \in U \) represents a mapping \( u : O \rightarrow \text{Lop} \), which maps every operation to their local states. Let \( \phi \in U \) be an empty mapping; \( u = \phi \) when all operations do not begin to execute.

A transition is a triple of the form \( \sigma \xrightarrow{e} \sigma' \), where \( \sigma \) and \( \sigma' \) are states and \( e \) is an event. For example, a transition \((\sigma_c, (\sigma_z, u)) \xrightarrow{e} (\sigma'_c, (\sigma_z, u))\) characterizes the effect that a state \( \sigma_c \) can be transformed into a state \( \sigma'_c \) by an event \( e \) of a client program. We use \textit{abort} to denote an error state. A transition \((\sigma_c, (\sigma_z, u)) \xrightarrow{\text{abort}} \) characterizes the effect that an event \( e \) leads to a runtime error.

A terminating execution \( \pi \) of a program \( P(Z) \) from an initial state \((\sigma_{c0}, (\sigma_{z0}, \phi))\) is a finite sequence of the form \((\sigma_{c0}, (\sigma_{z0}, \phi)) \xrightarrow{e_1} (\sigma_{c1}, (\sigma_{z1}, u_1)) \xrightarrow{e_2} \cdots \xrightarrow{e_n} (\sigma_{cn}, (\sigma_{zn}, u_n))\), where the trace \( e_1 e_2 \cdots e_n \in [P(Z)] \).

For an execution \( \pi \) of \( P(Z) \), let \( \text{tr}(\pi) \) denote the trace generated by the execution \( \pi \), let \( \text{tr}(\pi)[A_c] \) denote the maximal subsequence of \( \text{tr}(\pi) \) consisting of the events of the client program \( P \) (i.e., the projection of the trace \( \text{tr}(\pi) \) to the events of \( P \)), let \( \text{tr}(\pi)[A_z] \) denote the maximal subsequence of \( \text{tr}(\pi) \) consisting of the events of the object \( Z \) (i.e., the projection of the trace \( \text{tr}(\pi) \) to the events of \( Z \)), let \( \text{tr}(\pi)[t] \) denote the maximal subsequence of \( \text{tr}(\pi) \) consisting of the events performed by the thread \( t \).

For a program \( P(Z) \), let \((\sigma_c, \sigma_z) \xrightarrow{\lambda} (\sigma'_c, \sigma'_z)\) denote a terminating execution of \( P(Z) \) which starts from the state \((\sigma_c, (\sigma_z, \phi))\), terminates in the state \( \sigma'_c \) of \( P \) and the state \( \sigma'_z \) of \( Z \) and generates the finite trace \( \lambda \); let \((\sigma_c, \sigma_z) \xrightarrow{\lambda} \omega_-\) denote a divergent execution of \( P(Z) \) which starts from the state \((\sigma_c, (\sigma_z, \phi))\) and generates the infinite trace \( \lambda ; \) let \((\sigma_c, \sigma_z) \xrightarrow{\lambda} \text{abort} \) denote an execution of \( P(Z) \) which starts from the state \((\sigma_c, (\sigma_z, \phi))\), produces a runtime error and generates the finite trace \( \lambda \).

A divergent execution of \( P(Z) \) could be caused by divergences of the client program \( P \), divergences of \( Z \) or a combination of both. Let \((\sigma_c, \sigma_z) \xrightarrow{\lambda} \omega_-\) denote a divergent execution of \( P(Z) \) caused by a divergence of the client program \( P \) (i.e. \(|\lambda[A_c]| = \omega \land |\lambda[A_z]| \neq \omega \)).
Given a program \( P(Z) \), a state \( \sigma_c \) of the client program \( P \) and a state \( \sigma_z \) of the object \( Z \), the client-side traces of the program, denoted by \( MT[P(Z)(\sigma_c, \sigma_z)] \), and the final states of the program, denoted by \( MS[P(Z)(\sigma_c, \sigma_z)] \), are defined as follows.

\[
MT[P(Z)(\sigma_c, \sigma_z)] = \{ \lambda[A_c | (\sigma_c, \sigma_z) \xrightarrow{\lambda} (\sigma_c', \sigma_z') \lor (\sigma_c, \sigma_z) \xrightarrow{\text{abort}} \lor (\sigma_c, \sigma_z) \xrightarrow{\text{resAbort}} \}
\]

\[
MS[P(Z)(\sigma_c, \sigma_z)] = \begin{cases} 
\{(\sigma_c', \sigma_z') | (\sigma_c, \sigma_z) \xrightarrow{\lambda} (\sigma_c', \sigma_z')\} \\
\cup \{\text{abort} | (\sigma_c, \sigma_z) \xrightarrow{\lambda} \text{abort}\} \\
\cup \{\bot | (\sigma_c, \sigma_z) \xrightarrow{\lambda} \bot\}
\end{cases}
\]

4. **Strict Linearizability**

4.1. **Histories and Linearizability Relation.**

Linearizability is defined using the notion of histories. The history of an execution \( \pi \), denoted by \( H(\pi) \), is the maximal subsequence of \( tr(\pi) \) comprised of the invocation and response events.

For a history \( H \), let \( H(i) \) denote the \( i \text{th} \) element of \( H \). A history is sequential if the event preceding each response event is its matching invocation event. A history \( H \) is well-formed if for every thread \( t \), \( H[t] \) is sequential. A history is complete if it is well-formed and every invocation event has a matching response event. An invocation event is pending if there is no matching response event to the invocation event.

We introduce the special response event for an aborted operation \( o \), denoted by \( (t, o, \text{abort}) \). For an execution \( \pi \), let \( \text{resAbort}(H(\pi)) \) be the history gained by adding matching special response events for all aborted operations in \( \pi \) to the end of the history \( H(\pi) \). For an incomplete history \( H(\pi) \), a completion of \( H(\pi) \), is a complete history gained by adding some matching response events to the end of \( \text{resAbort}(H(\pi)) \) and removing some pending invocation events within \( \text{resAbort}(H(\pi)) \). Let \( \text{Compl}(h) \) denote a set of all completions of the history \( h \).

We use \( (\sigma_z, H, \sigma_z') \) to denote a terminating execution of a concurrent object (i.e. all invoked methods of the object have returned in the execution), where \( \sigma_z \) is the object’s initial state, \( H \) is the history of the execution, \( \sigma_z' \) is the object’s final state. Let \( <_o \) denote the happened-before order on operations; for two operations \( o \) and \( o' \), \( o <_o o' \) if the response event of \( o \) precedes the invocation event of \( o' \).

**Definition 1 (Linearizability Relation [11])** The linearization relation \( \sqsubseteq \) on histories is defined as follows: \( H \sqsubseteq H' \) iff

1. \( \forall t . H[t] = H'[t] \);
2. there exists a bijection \( \nu : \{1, \ldots, |H|\} \rightarrow \{1, \ldots, |H'|\} \) such that \( \forall i . H(i) = H'(\nu(i)) \) and \( \forall i,j.i < j \land H(i) \in \text{resAct} \land H(j) \in \text{invAct} \Longrightarrow \nu(i) < \nu(j) \).

The first condition above requires that \( H' \) is a permutation of \( H \); the second condition above requires that the happened-before order between non-interleaved
operations in $H$ and $H'$ is identical. The following proposition shows that the linearizability relation is transitive.

**Proposition 2.** $H_1 \subseteq H_2 \land H_2 \subseteq H_3 \implies H_1 \subseteq H_3$

The proof for the proposition appears in Appendix Section A.

### 4.2. Strict Linearizability

A sequential specification of an object is used to describe the semantics of the object in the absence of concurrency.

**Definition 3 (Sequential Specification)** For a concurrent object $Z$, let $Z_{\text{State}}$ be a set of the well-formed states of $Z$, $Z_{\text{op}}$ be a set of the methods of $Z$, $\text{Input}$ be a set of input values, $\text{Output}$ be a set of output values. A sequential specification of a method $op \in Z_{\text{op}}$ is a partial function $op : Z_{\text{State}} \times \text{Input} \rightarrow Z_{\text{State}} \times \text{Output}$.

For the sequential specifications of concurrent objects, only well-formed states need to be considered. For example, a state of a singly linked list is well-formed only when there are no loops or cycles in it. Note that the methods are defined as partial functions. A method is total if it is defined in the set $Z_{\text{State}} \times \text{Input}$; it is partial if it is defined in a proper subset of the set $Z_{\text{State}} \times \text{Input}$.

For a sequential execution of a method $op$ starting from a state $\sigma_z$ with an input $in$, let $(\sigma_z, in)op(\sigma'_z, ret)$ denote that the execution is error-free, and terminates in a state $\sigma'_z$ with an output $ret$.

A sequential execution of a method $op$ starting from a state $\sigma_z \in Z_{\text{State}}$ with an input $in \in \text{Input}$ is legal if $op(\sigma_z, in) = (\sigma'_z, ret) \implies (\sigma_z, in)op(\sigma'_z, ret)$, where $op(\sigma_z, in) = (\sigma'_z, ret)$ denotes that the result of applying the operation (or function) $op$ to an input $in$ and a state $\sigma_z$ is the state $\sigma'_z$ and the return value $ret$. A sequential execution of an object is legal if the executions of all methods of the execution are legal.

**Definition 4 (Strict Linearizability)** A concurrent object $Z$ is strict linearizable iff

1. for every execution $\pi$ of $Z$ starting from any well-formed state $\sigma_z$, there exists a legal sequential execution $\pi'$ of $Z$ starting from the state $\sigma_z$ and a history $h_c \in \text{Compl}(H(\pi))$ such that $h_c \subseteq H(\pi')$;
2. for every terminating execution $\pi : (\sigma_z, H_{\text{con}}, \sigma'_z)$ of $Z$, there exist a legal sequential execution $\pi' : (\sigma_z, H_{\text{seq}}, \sigma'_z)$ of $Z$ such that $H_{\text{con}} \subseteq H_{\text{seq}}$.

We refer to a sequential specification of a concurrent object as its atomicity specification. A concurrent object satisfies its atomicity specification iff it is strict linearizable. In order to distinguish between strict linearizability and classical linearizability [8,24,25], we call the latter general linearizability, which is formally defined as follows.
Definition 5 (General Linearizability) A concurrent object \( Z \) is general linearizable w.r.t. an ADT \( A \), if for any execution \( \pi \) of \( Z \) starting from any well-formed state \( \sigma_z \), there exists a legal sequential execution \( \pi' \) of \( A \) starting from the state \( AF(\sigma_z) \) and a history \( h_c \in \text{Compl}(H(\pi)) \) such that \( h_c \sqsubseteq H(\pi') \).

Here \( AF \) denotes an abstraction function which maps the well-formed states of a concurrent object to the states of an ADT. The specifications for ADTs and abstraction functions will be explained in detail in Section 6.

5. Properties of Strict Linearizability

In this section, we show several important properties of strict linearizability which general linearizability does not capture.

5.1. Client-Side Traces and Final-States Equivalence

For a concurrent object \( Z \), let \( Ato_Z \) denote the atomic version of \( Z \) in which every method is protected by an atomic region. We use the sequential specification of \( Z \) to describe the semantic of \( Ato_Z \). An operation \( op \) of \( Ato_Z \) is executed atomically if its precondition (i.e., the domain of \( op \)) is true in the current configuration, otherwise it blocks execution from the current configuration. If during a concurrent execution, its precondition becomes true (similar to the spinlock, repeatedly testing the precondition), the operation \( op \) can resume its execution. When the precondition of an operation is true, the trace semantics of the operation is defined as follows:

\[
[x = Ato_Z.(f(E))_t] = \{\rho_1^r((t, o, inv(f, n)), \rho_2, (t, o, ret(v)))^\sim(t, x := v) \mid (\rho_1, n) \in \langle E \rangle_t \land \rho_2(t, o, ret(v)) \in \langle \text{body} \rangle_t\}
\]

Here, \( (t, o, inv(f, n)), \rho_2, (t, o, ret(v)) \) is an atomic trace, i.e., it is interleaved with other events as a single event. The following theorem shows that strict linearizability can ensure client-side traces ans final-states equivalence. The proof for the theorem appears in Appendix Section B.

Theorem 6. A concurrent object \( Z \) is strict linearizable if and only if for any client program \( P \), any initial state \( \sigma_c \) of \( P \), any well-formed state \( \sigma_z \) of \( Z \),

1. \( \text{MT}[P(Z)(\sigma_c, \sigma_z)] = \text{MT}[P(Ato_Z)(\sigma_c, \sigma_z)] \)
2. \( \text{MS}[P(Z)(\sigma_c, \sigma_z)] = \text{MS}[P(Ato_Z)(\sigma_c, \sigma_z)] \).

The first condition shows that \( P(Z) \) and \( P(Ato_Z) \) have the same client-side traces. We call this property client-side traces equivalence. This means that \( P(Z) \) and \( P(Ato_Z) \) have the same linear-time temporal properties of the client program.

The second condition shows that \( P(Z) \) and \( P(Ato_Z) \) have the same final states. Thus, for a strict linearizable concurrent object, clients do not need to know the internal synchronization details of the concurrent object and can design, program and reason in terms of its sequential specification.

The theorem above is obtained in the program model where there is a complete isolation between an object and its client programs. We now relax the restriction of
the program model by allowing client programs to share memory spaces with objects and to directly access the shared memory spaces via compatible atomic memory read/write actions. Atomic memory read/write actions are said to be compatible if they satisfy either of the following two restrictions:

- The read/write actions do not interfere with the methods of concurrent objects, i.e., when the write actions are executed, the methods of concurrent objects which are called before them have finished; when the read actions are executed, the methods which are called before them either have finished or do not modify the states of objects. The write actions maintain well-formed states of concurrent objects.
- If the read/write actions are encapsulated into the methods of concurrent objects, the “new” methods do not break strict linearizability (i.e., after adding the “new” methods, the concurrent objects are still strict linearizable).

The following theorem shows that strict linearizability can provide the same guarantee even in the relaxed program model above. The proof for the theorem appears in Appendix Section B.

**Theorem 7.** For a strict linearizable object \(Z\) with a well-formed initial state, for any client program \(P\), \(P(Z)\) and \(P(\text{Ato}_Z)\) have the same client-side traces and final states even in the relaxed program model above.

### 5.2. Preservation of Termination

In this subsection, we consider two progress properties, minimal termination and purely-blocking progress [11]. We show that for a strict linearizable object satisfying either of the two progress properties, a program using the object diverges iff the program using its atomic version diverges. Thus, while proving termination of a program using such an object, we can soundly replace the object with its atomic version.

Note that the definition of client-side traces (in Section 3) does not consider divergences of concurrent objects. Thus, from client-side traces equivalence of Theorem 6, we get the following corollary.

**Corollary 8.** For a strict linearizable object \(Z\) with a well-formed initial state, any client program \(P\), \(P(Z)\) diverges by a divergence of the client program \(P\) iff \(P(\text{Ato}_Z)\) diverges by the same reason.

We now introduce a progress property called minimal termination. An object satisfies minimal termination iff for any client program \(P\), every method of the object can terminate if \(P\) calls the methods finitely. There are a variety of objects that satisfy minimal termination, e.g., wait-free, lock-free, deadlock-free, starvation-free objects have such a progress guarantee.

For an object \(Z\) satisfying minimal termination, if a client program \(P\) calls its
methods finitely, then \( P(Z) \) cannot diverge by the divergences of \( Z \). Thus, in terms of Corollary 8, we get the following lemma.

**Lemma 9.** For a strict linearizable object \( Z \) satisfying minimal termination with a well-formed initial state, any client program \( P, P(Z) \) diverges iff \( P(\text{Ato}_Z) \) diverges.

We now consider the purely-blocking progress [13], which is a weaker progress property than minimal termination. An object is purely-blocking [13] when at any reachable state, any pending method, if run in isolation will terminate or its entire execution does not modify states of the object. Minimal termination restricts the methods of concurrent objects to be total. The purely-blocking progress permits the methods of concurrent objects to be partial. For example, the \( HW \) queue is purely-blocking [13] and its \( \text{Dequeue}() \) method is a partial method. The following theorem shows that a purely-blocking and strict linearizable object possesses a strong termination-preserving property. We also show that the theorem still holds for the relaxed program model in subsection 5.1. The proof for the theorem appears in Appendix Section C.

**Theorem 10.** For a strict linearizable and purely-blocking object \( Z \) with a well-formed initial state, any client program \( P, P(Z) \) diverges iff \( P(\text{Ato}_Z) \) diverges.

The \( HW \) queue cannot ensure the termination-preserving property, because it is not strict linearizable. For example, consider the following program:

\[
\text{HW.Enqueue('c')} \parallel \text{HW.Enqueue('d')} \parallel \text{HW.Dequeue();}
\]
\[
\text{HW.item[1]} = x; \quad \text{// which are not interleaved with the called methods}
\]
\[
\text{HW.Dequeue();} \parallel \text{HW.Dequeue();}
\]

The program above can diverge. However, when the program replaces the \( HW \) queue with its atomic version, the program always terminates.

6. **Abstract Atomicity**

6.1. **Data Abstraction For Sequential Data Structures**

We use model-based specification [15] to define ADTs, where an ADT is considered as a set of abstract values together with a set of atomic methods; the methods are specified by defining how they affect the abstract values.

**Definition 11 (Abstract Data Type)** An ADT \( A \) is a tuple \((A\text{State}, \sigma_{a0}, A\text{op}, \text{Input}, \text{Output})\), where \( A\text{State} \) is a set of states; \( \sigma_{a0} \in A\text{State} \) is the initial state; \( A\text{op} \) is a set of methods; \( \text{Input} \) is a set of input values; \( \text{Output} \) is a set of output values; each method \( \text{op} \in A\text{op} \) is a mapping \( \text{op} : A\text{State} \times \text{Input} \rightarrow A\text{State} \times \text{Output} \).

Let \( \text{dom}(\text{op}) \) denote the domain (i.e., precondition) of the method \( \text{op} \). A method of \( A \) blocks when it is called outside its domain. In the sequential setting, for an
ADT $A$ and its implementation (or representation) $Z$, abstraction function $AF : Z\text{State} \to A\text{State}$ is used to map the well-formed states of $Z$ to the states of $A$. An abstraction function explains how internal structure of an implementation is viewed abstractly by clients of an ADT. The function is surjective and thus every abstract state can be represented by one or more concrete states. The renaming function $RF : Z\text{op} \to A\text{op}$ is used to map the method names of $Z$ to the method names of $A$. The inverse of the function $RF$ is denoted by $RF^{-1}$.

**Definition 12 (Sequential Implementation of an ADT)** $Z$ is a sequential implementation of an ADT $A$ w.r.t an abstraction function $AF$, iff for all $\text{op} \in A\text{op}$, $\sigma_z \in Z\text{State}, \sigma_a, \sigma'_a \in A\text{State}$, $\text{in} \in \text{Input}$, $\text{ret} \in \text{Output}$. $AF(\sigma_z) = \sigma_a \land \text{op}(\sigma_a, \text{in}) = (\sigma'_a, \text{ret}) \implies \exists \sigma'_z$, $(\sigma_z, \text{in})RF^{-1}(\text{op})(\sigma'_z, \text{ret}) \land AF(\sigma'_z) = \sigma'_a$.

A good abstract data type should ensure the important representation independence property. We state representation independence in terms of observational equivalence—two correct implementations of an ADT are observationally indistinguishable by clients of the ADT. Application of the definition requires a specific interpretation of what the observable behaviors really mean. Client programs access an implementation of an ADT only through the ADT interface. Thus, the states of implementations of an ADT are unobservable by clients. In this paper, we take traces of client programs (i.e. client-side traces) as observable behaviors.

**Definition 13 (Observational Equivalence)** For an ADT $A$ and its implementation $Z$ w.r.t the abstraction function $AF$, a client program $P$, the two programs $P(Z)$ and $P(A)$ are observationally equivalent iff for any initial state $\sigma_c$, any well-formed state $\sigma_z$, $\text{MT}[P(A)(\sigma_c, AF(\sigma_z))] = \text{MT}[P(Z)(\sigma_c, \sigma_z)]$.

The following theorem states that when the methods of ADTs are called within their domains, data abstraction implies observational equivalence. The detailed proof is included in the Appendix Section D.

**Theorem 14.** If $Z$ is a sequential implementation of an ADT $A$ then for any client program $P$, if all methods of $A$ are called within their domains, then $P(Z)$ and $P(A)$ are observationally equivalent.

According to Definition 12, outside the domain of an abstract method, the corresponding concrete method is free to do anything, including crashing the program, returning a correct or incorrect value, or throwing exceptions. Thus $P(Z)$ and $P(A)$ can have different behaviors outside domains of the methods. Generally, it is the responsibility of clients to ensure that these preconditions hold.

### 6.2. Data Abstraction For Concurrent Objects

A concurrent object can be viewed as a concurrent implementation of an ADT. What does it mean for a concurrent object to be an implementation of an ADT? Like in the sequential setting, the criterion for relating an ADT and its concurrent
implementation should ensure the representation independence property. Linearizability is not sufficient to capture the property because it only ensures observational refinement, not observational equivalence. Thus, new observable behaviours can be introduced when clients replace a linearizable object with its corresponding ADT to simplify reasoning about their programs. For example, a specification of queue can be given as follows:

\[ \text{Enqueue}(\text{seq}, x) = (\text{seq} \bowtie x, \varepsilon) \]

\[ \text{Dequeue}(\text{seq}) = \begin{cases} (\text{seq}', y), & \text{if } \text{seq} = y \bowtie \text{seq}'; \\ (\text{seq}, \text{EMPTY}), & \text{if } \text{seq} = \text{empty}; \end{cases} \]

Here \( \text{seq} \) denotes a sequence, the notation \( \varepsilon \) indicates that a method does not return values. Henzinger et al. [1] show that the HW queue is linearizable with respect to the specification. Consider the following program:

\[ \text{Enqueue}(\text{'c'}) \parallel y = \text{Dequeue()} \]

If the program uses the HW queue, the final value of the variable \( y \) is \( c \); if the program uses the abstract queue, the final value of the variable \( y \) is \( c \) or \( \text{empty} \).

We present a correctness criterion for a concurrent implementation of an ADT, which is the combination of general linearizability and data abstraction and can ensure observational equivalence.

**Definition 15 (Concurrent Implementation of an ADT)** A concurrent object \( Z \) is a concurrent implementation of an ADT \( A \) w.r.t an abstraction function \( AF \), iff

1. \( Z \) is a sequential implementation of \( A \) w.r.t \( AF \),
2. \( Z \) is linearizable w.r.t. \( A \) and for every terminating execution \((\sigma_z, H_z, \sigma_z')\) of \( Z \) starting from a well-formed initial state \( \sigma_z \), there exists a terminating execution \((AF(\sigma_z), H_a, AF(\sigma_z'))\) of \( A \), such that \( H_z \sqsubseteq H_a \).

**Theorem 16.** If a concurrent object \( Z \) is a concurrent implementation of an ADT \( A \) then for any client program \( P \), any well-formed initial state of \( Z \), \( P(Z) \) and \( P(A) \) are observationally equivalent.

The proof for the theorem appears in Appendix Section E. Note that the observable behaviors (i.e. the client-side traces) do not include the traces generated by divergences of concurrent objects. In practice, it is the responsibility of clients to exclude the undesirable behaviors by ensuring termination of the called methods of concurrent objects in terms of their progress properties and fair assumption.

Data abstraction in the concurrent setting implies atomicity abstraction—one which enables clients to reason about the operations of concurrent objects as if they occur in a single atomic step. Thus, for a concurrent object, we refer to such an ADT specification as its abstract atomicity specification.
6.3. The Relationship between Atomicity and Abstract Atomicity

A concurrent object can implement multiple different ADTs. For example in Appendix Section H, we show that the MS lock-free queue is not only an implementation of two different queue data types but also an implementation of a multiset data type. Different abstractions are suited to different kinds of applications. It is a challenging problem to prove that a concurrent object is a concurrent implementation of an ADT, so clients do not want to have to reverify the implementations each time.

The following theorem shows that for a strict linearizable concurrent object, its sequential specification can serve as “maximal” atomicity abstraction—for a strict linearizable object \( Z \), in order to verify whether \( Z \) is a concurrent implementation of an ADT \( A \), it is sufficient to check whether its sequential specification satisfies the \( A \)’s specification. The proof for the theorem appears in Appendix Section F. Thus, for a strict linearizable concurrent object, the challenging problem can reduced to the simpler problem of reasoning about sequential behaviors of the concurrent object.

**Theorem 17.** For a strict linearizable concurrent object \( Z \), if for any ADT \( A \), \( Z \) is a sequential implementation of \( A \) and \( \forall \text{op} \in Z\text{op}, \sigma_z \in Z\text{State}, \text{in} \in \text{Input}.(\sigma_z, \text{in}) \in \text{dom}(\text{op}) \implies (AF(\sigma_z), \text{in}) \in \text{dom}(RF(\text{op})) \), then \( Z \) is also a concurrent implementation of \( A \).

The following theorem can help establish strict linearizability in terms of abstract atomicity. The proof for the theorem appears in Appendix Section G. We show that the MS lock-free queue is strict linearizable in terms of the theorem in Appendix Section H.

**Theorem 18.** A concurrent object \( Z \) is strict linearizable if there exists an ADT \( A \), such that \( Z \) is a concurrent implementation of \( A \) w.r.t an injective abstraction function.

7. Related Work and Conclusion

**Related Work** Strict linearizability is a stronger consistency than general linearizability. Sequential consistency [29], and quiescent consistency [8,14], as well as relaxed forms of linearizability like quasi linearizability [6] and parameterised linearizability [7], k-linearizability [12], eventual consistency [13] are weaker consistency conditions than general linearizability, and cannot provide stronger guarantees than strict linearizability.

Several previous works [2,4,26,23] have presented atomicity notions based on serializability (conflict-serializability or view-serializability) and use Lipton’s theory of reduction [16,22] as a key technique to prove atomicity. The correctness criteria are sometimes too restrictive because violations of serializability at the load/store instruction level may not necessarily mean conflicts at the higher, more “semantic”
level. Our notion of strict linearizability, in contrast, defines atomicity for concurrent objects at the sequential specification level.

**Conclusion** This paper presents a notion of strict linearizability and goes on to show its several important properties which general linearizability cannot capture. This paper also presents a correctness criterion for a concurrent implementation of an ADT, which is the combination of general linearizability and data abstraction and can ensure observational equivalence. We investigate its relationship with strict linearizability.

### Appendix A.

In the Appendix, proofs are written in a hierarchically structured style as advocated by Lamport [27].

**Proposition 2.** $H_1 \sqsubseteq H_2 \land H_2 \sqsubseteq H_3 \implies H_1 \sqsubseteq H_3$

**Proof.**

1. $\forall t. H_1[t = H_1[t$;
2. $\forall t. H_1[t = H_2[t$; and $\forall t. H_2[t = H_3[t$;
3. Assume:
   - (a) $\nu_1 : \{1, \ldots, |H_1|\} \to \{1, \ldots, |H_2|\}$ such that $\forall i. H_1(i) = H_2(\nu_1(i))$ and $\forall i, j. i < j \land H_1(i) \in resAct \land H_1(j) \in invAct \implies \nu_1(i) < \nu_1(j)$.
   - (b) $\nu_2 : \{1, \ldots, |H_2|\} \to \{1, \ldots, |H_3|\}$ such that $\forall i. H_2(i) = H_3(\nu_2(i))$ and $\forall i, j. i < j \land H_2(i) \in resAct \land H_2(j) \in invAct \implies \nu_2(i) < \nu_2(j)$.

   Proof: by the definition of linearizability relation.

3. Let $\nu_3$ be a bijection $\{1, \ldots, |H_1|\} \to \{1, \ldots, |H_3|\}$ such that $\forall i. H_1(i) = H_3(\nu_3(\nu_2(\nu_1(i))))$. Then $\forall i, j. i < j \land H_1(i) \in resAct \land H_1(j) \in invAct \implies \nu_3(\nu_2(\nu_1(i))) < \nu_3(\nu_2(\nu_1(j)))$.

   Proof: By 1 and 3.

4. Q.E.D.

**Appendix B.**

Because the states of a client program are disjoint from the ones of a concurrent object, we can divide an execution $\pi = (\sigma_{c0}, (\sigma_{z0}, \phi)) \xrightarrow{b_1} (\sigma_{c1}, (\sigma_{z1}, u_1)), \ldots, \xrightarrow{b_n} (\sigma_{cn}, (\sigma_{zn}, u_n))$ of a program $P(Z)$ into an execution $\pi_c$ of the client program $P$ and an execution $\pi_z$ of the object $Z$ as follows:

$\pi_c = (\sigma_{c0}), b_1, \ldots, b_n, (\sigma_{cn}),$

$\pi_z = (\sigma_{z0}, \phi), \xrightarrow{c_1}, \ldots, \xrightarrow{c_n} (\sigma_{zn}, u_n),$

where $tr(\pi_c) = tr(\pi)[A_c$ and $tr(\pi_z) = tr(\pi)[A_z].$
To simplify our notation, we use the abbreviation 
\[(\sigma_z) \stackrel{(\text{inv}(\text{op}, n), \text{ret}(v))}{\rightarrow} (\sigma_z')\] to describe the atomic execution of the operation \text{op}
starting from the initial state \(\sigma_z\) with an input \(n\) and terminating in the state \(\sigma_z'\)
with an output \(v\).

For two histories \(H\) and \(H'\), if \(H \subseteq H'\), we can establish a bijective function \(\mathbb{F}\) mapping between operations in \(H\) and \(H'\); an operation \(o\) in \(H\) is mapped to
an operation \(o'\) in \(H'\) by \(\mathbb{F}\) if for all thread \(t\), \(H\[t(i) = \text{inv} \iff H'[t(i) = \text{inv}' \land F(\text{OP}(\text{inv})) = \text{OP}(\text{inv}')\).

**Theorem 6.** A concurrent object \(Z\) is strict linearizable iff for any client program \(P\), any initial state \(\sigma_c\) of \(P\), any well-formed state \(\sigma_z\) of \(Z\),
1. \(\mathcal{MT}[P(Z)(\sigma_c, \sigma_z)] = \mathcal{MT}[P(\text{Ato}_Z)(\sigma_c, \sigma_z)]\)
2. \(\mathcal{MS}[P(Z)(\sigma_c, \sigma_z)] = \mathcal{MS}[P(\text{Ato}_Z)(\sigma_c, \sigma_z)]\)

**Proof.** By Lemma 1 and Lemma 4.

**Lemma 1.** A concurrent object \(Z\) is strict linearizable then for any client program \(P\), any initial state \(\sigma_c\) of \(P\), any well-formed state \(\sigma_z\) of \(Z\),
1. \(\mathcal{MT}[P(Z)(\sigma_c, \sigma_z)] = \mathcal{MT}[P(\text{Ato}_Z)(\sigma_c, \sigma_z)]\)
2. \(\mathcal{MS}[P(Z)(\sigma_c, \sigma_z)] = \mathcal{MS}[P(\text{Ato}_Z)(\sigma_c, \sigma_z)]\)

**Proof.** By Lemma 2 and Lemma 3.

**Lemma 2.** For any client program \(P\), any initial state \(\sigma_c\) of the client program,
any well-formed state \(\sigma_z\) of the object \(Z\):
1. \(\mathcal{MT}[P(Z)(\sigma_c, \sigma_z)] \subseteq \mathcal{MT}[P(\text{Ato}_Z)(\sigma_c, \sigma_z)]\)
2. \(\mathcal{MS}[P(Z)(\sigma_c, \sigma_z)] \subseteq \mathcal{MS}[P(\text{Ato}_Z)(\sigma_c, \sigma_z)]\)

**Proof.**
1. For any terminating execution \(\pi\) of \(P(Z)\), there exists an execution \(\pi'\) of \(P(\text{Ato}_Z)\)
such that:
   (1) \(\pi'\) and \(\pi\) have the same client-side traces.
   (2) \(\pi'\) and \(\pi\) have the same final states.
   proof: 1.1 Assume: For any terminating execution \(\pi\) of \(P(Z)\):
   \(\pi = (\sigma_{z_0}, \sigma_{z_n}) \xrightarrow{\gamma} (\sigma_{c_{n}}, \sigma_{z_n}).\) By separating the execution \(\pi\), we can get an execution
   \(\pi_c = (\sigma_{c_0}, \sigma_{c_{n}}) \xrightarrow{\lambda} (\sigma_{c_{n}})\) of the client program, where \(\gamma = \lambda[A_c]\).
   To prove that the lemma holds, we need to prove that there exists an execution
   \(\pi'=(\sigma_{c_0}, \sigma_{c_{n}}) \xrightarrow{\beta} (\sigma_{c_{n}}, \sigma_{z_{n}})\) of \(P(\text{Ato}_Z),\) such that \(\lambda[A_c] = \beta[A_c].\)
   1.2 There exists an execution \(\pi_a = \sigma_{z_0} \xrightarrow{(\text{inv}(\text{op}', n_{i}), \text{ret}(v_{i}))} \sigma_{z_1}, \cdots, \)
   \(\sigma_{z_{n}}\) of \(\text{Ato}_Z,\) such that \(H(\pi) \subseteq H(\pi_a)\).
   proof: By the definition of strict linearizability. Let the function \(\mathbb{F}\) map every
   operation \(\text{op}_i\) in \(H(\pi)\) to every operation \(\text{op}_i'\) in \(H(\pi_a)\)
   1.3 For every action \((\text{inv}(\text{op}', n_{i}), \text{ret}(v_{i}))\) in \(\pi_a,\) there exists two atomic actions
in $\pi_c$: one of which is argument evaluation of the operation $op_i$ in $\pi$ (denoted by $e_i$), the other of which is assignment of the return value of the operation $op_i$ to a client’s variable (denoted by $x_i := ret_i$) such that the value of $e_i$ is $n_i$ and $v_i = ret_i$.

proof: Since $H(\pi) \subseteq \text{Itin}_n H(\pi_a)$, the two operations $op'_i$ and $op_i$ have the same argument values and return values.

1.4 Every atomic action $\langle inv(op'_i, n_i), ret(v_i) \rangle$ in $\pi_a$, can be inserted between $e_i$ and $x_i := ret_i$ in $\pi_c$ and preserves the real time order in $\pi_a$.

proof: By induction on $n$.

1.4.1 Base case: when $n=1$, 1.4 is true.

proof: trivial.

1.4.2 Inductive hypothesis: when $n=k$, 1.4 is true.

1.4.3 Inductive step: when $n=k+1$, 1.4 is true.

1.4.3.1 By inductive hypothesis, to prove 1.4.3, we need to prove that $\langle inv(op'_i, n_i), ret(v_i) \rangle$ can be inserted between $e_{(k+1)}$ and $x_{(k+1)} := ret_{(k+1)}$ and behind $\langle inv(op_k, n_k), ret(v_k) \rangle$.

proof: Assume $e_{(k+1)} := ret_{(k+1)} <_a e_k$, we can get (a) $op_{(k+1)} <_o op_k$ in $\pi$. Since $op'_{(k+1)} <_o op'_k$ in $\pi_c$ and $H(\pi) \subseteq H(\pi_a)$, we can get (b) $op_{(k+1)} <_o op_k$. Thus, a contradicts b, the original assumption must have been wrong.

1.4.3.3 All possible happened-before total orders on $e_k, x_k := ret_k, e_{(k+1)}, x_{(k+1)} := ret_{(k+1)}$ are shown as follows:

(1) $e_{(k+1)} <_a e_k <_a x_{(k+1)} := ret_{(k+1)} <_a x_k := ret_k$
(2) $e_k <_a e_{(k+1)} <_a x_{k+1} := ret_{(k+1)} <_a x_k := ret_k$
(3) $e_{(k+1)} <_a e_k <_a x_k := ret_k <_a x_{(k+1)} := ret_{(k+1)}$
(4) $e_k <_a e_{(k+1)} <_a x_k := ret_k <_a x_{(k+1)} := ret_{(k+1)}$
(5) $e_k <_a e_{(k+1)} <_a x_{(k+1)} := ret_{(k+1)}$

proof: By 1.4.3.2 and $e_{(k+1)} <_a x_{(k+1)} := ret_{(k+1)}$.

1.4.3.4 Q.E.D.

proof: For any happened-before order in 1.4.3.3, we can insert $\langle inv(op'_i, n_i), ret(v_i) \rangle$ between $e_k$ and $x_k := ret_k$, insert $\langle inv(op'_{(k+1)}, n_{(k+1)}), ret(v_{(k+1)}) \rangle$ between $e_{(k+1)}$ and $x_{(k+1)} := ret_{(k+1)}$, and preserve the happened-before order $op'_k <_o op'_{(k+1)}$.

1.4.4 Q.E.D.

Proof: By 1.4.1 and 1.4.2 and 1.4.

1.5 Let $\xi$ be the trace by inserting every $\langle inv(op'_i, n_i), ret(v_i) \rangle$ in $\pi_a$ for $i = 1 \ldots n$ into $tr(\pi_c)$ as 1.4. The execution $\pi_\tau = (\sigma_{e_0}, \sigma_{x_0}) \xi(\sigma_{e_n}, \sigma_{x_n})$ is feasible.

proof: By separating the execution $\pi_\tau$, we can get two feasible executions $\pi_c$ and $\pi_a$. By our semantics a state of a client program is disjoint from that of a concurrent object, thus $\pi_\tau$ is a feasible execution.

1.6 $\pi_\tau$ is an execution of $P(Ato, Z)$.

proof: By the constructing process of $\pi_\tau$, $\pi_\tau$ is an execution of the program which is the same as $P(Z)$ except the statement $x_i := Z.op_i(e_i)$, which is replaced
by \( x_i := \text{At}_{oZ}.op'_i(e_i) \).

1.7. Q.E.D.
proof: By 1.1, 1.5, 1.6.

2 For any divergent execution \( \beta \) of \( P(Z) \) caused by divergence of the client program, there exists an execution \( \beta' \) of \( P(\text{At}_{oZ}) \) such that: \( \beta' \) and \( \beta \) have the same client-side traces.
proof: Note that the definition of the client-side traces only considers divergence caused by a client program. Thus, for a divergent execution of \( P(Z) \), we only need to consider the case: a divergent execution of \( P(Z) \) where the execution of \( Z \) is finite.
The proof for the case is similar to the above one.
3 Q.E.D.
proof: By 1 and 2.

\[ \square \]

**Lemma 3.** For any client program \( P \), any initial state \( \sigma_c \) of the client program, any well-formed state \( \sigma_z \) of the object \( Z \):

1. \( \text{MT}[P(\text{At}_{oZ})(\sigma_c, \sigma_z)] \subseteq \text{MT}[P(Z)(\sigma_c, \sigma_z)] \)
2. \( \text{MS}[P(\text{At}_{oZ})(\sigma_c, \sigma_z)] \subseteq \text{MS}[P(Z)(\sigma_c, \sigma_z)] \)

**Proof.** Trivial.

\[ \square \]

**Lemma 4.** For any client program \( P \), any initial state \( \sigma_c \) of \( P \), any well-formed state \( \sigma_z \) of \( Z \), if

1. \( \text{MT}[P(Z)(\sigma_c, \sigma_z)] = \text{MT}[P(\text{At}_{oZ})(\sigma_c, \sigma_z)] \)
2. \( \text{MS}[P(Z)(\sigma_c, \sigma_z)] = \text{MS}[P(\text{At}_{oZ})(\sigma_c, \sigma_z)] \)
then \( Z \) is strict linearizable.

**Proof.**
1. For any terminating execution \( \pi : (\sigma_z, H, \sigma'_z) \) of the object \( Z \), there exists a sequential execution \( (\sigma_z, H', \sigma'_z) \) of the object \( Z \) such that \( H \subseteq H' \).

1.1 To simplify writing, we assume that a thread invokes a method of \( Z \) at most once in \( \pi \). We now construct a program \( P(Z) \): \( s_1; x_1 = Z.m_1(e_1); s'_1; \cdots; s_i; x_i = Z.m_i(e_i); s'_i; \cdots; s_n; x_n = Z.m_n(e_n); s'_n \) where for each \( i \), \( m_i(e_i) \) is a method called by the thread \( i \) in \( \pi \) and \( e_i \) is an argument of the method \( m_i \); \( s_i \) and \( s'_i \) are atomic regions and and \( x_i \) is a variable of the client program \( P \). Let \( \sigma_c \) be the client initial state of \( P(Z) \) where all variables of the client program are null.

1.2 There exists an execution \( \pi' \) of \( P(Z) \) starting from the initial state \( (\sigma_c, \sigma_z) \) such that:
\( tr(\pi) = tr(\pi')[A_z : the action x_i = ret_i and s'_i is executed immediately after the method m_i returns where ret_i is the return value of m_i; the method m_i is immediately invoked after s_i finishes. Let \( (\sigma'_c, \sigma''_z) \) be the final states of \( \pi' \).

proof: The trace of \( \pi' \) can be obtained by inserting \( x_i = ret_i \) and \( s'_i \) after the returning action of \( m_i \) and inserting \( s_i \) before the invocation action of \( m_i \) in the trace of \( \pi \). Obviously, the execution \( \pi' \) is feasible.
1.3 \(m_{i+1} > m_i \Rightarrow s_{i+1} > x_i = ret_i\) in \(\pi'\) for \(i = 1, \ldots, n\).
proof: by the constructing process of \(\pi'\).

1.4 Consider the program \(P(Ato_{Z})\): \(s_1; x_1 = Ato_{Z}.(m_1(e_1)); s'_1 \parallel \cdots s_i; x_i = Ato_{Z}.(m_i(e_i)) \parallel \cdots \parallel s_n; x_n = Ato_{Z}.(m_n(e_n))\) where \((m_i(e_i))\) is an atomic version of the method \(m_i(e_i)\). For any non-terminating execution \(\pi''\) of the program \(P(Ato_{Z})\) starting from \((\sigma_c, \sigma_z)\), if \(tr(\pi')[A_c = tr(\pi'')[A_c\ then \(H(tr(\pi')[A_c) \subseteq H(tr(\pi'')[A_c)\)

1.4.1 \(\forall t. H(\pi')(t = H(\pi'')[t\)
proof: Since \(tr(\pi')(A_c = tr(\pi'')[A_c\ then the return value of \(m_i(e_i)\) in \(\pi''\) is equal to \(H(tr(\pi'))[t\)

1.4.2 For any two operations \(m_{g1}, m_{g2}\) in \(H(tr(\pi')[A_c)\), and \(m_{g1}, m_{g2}\) in \(H(tr(\pi'')[A_c)\), \(m_{g2} > m_{g1}\) \(\Rightarrow m_{g2} > m_{g1}\)
proof: By 1.3, \(m_{g2} > m_{g1}\) \(\Rightarrow s_{g2} > x_{g1} = ret_{g1}\). In \(\pi''\), \(m_{g1}\) finishes before \(x_{g1} = ret_{g1}\) and \(m_{g2}\) starts after \(s_{g2}\). Thus \(m_{g2} > m_{g1}\)

1.4.3 Q.E.D.

proof: By 1.4.1 and 1.4.2.

1.5 \(tr(\pi')[A_c\ is only a client trace can lead to the client state \(\sigma'_c\).
proof: For each atomic region \(s_i\) or \(s'_i\), we can construct it by the following rule:
There is a variable \(y_i\) \((or y'_i)\) at the atomic region \(s_i\) \((or s'_i)\); the variables \(y_i\) \((or y'_i)\)
is assigned to different values when \(s_i\) \((or s'_i)\) is executed in different orders. A key
method for doing this is that the atomic region \(s_i\) can determine whether the other
actions have been executed in terms of values of their corresponding variables. For
example, the code of the atomic region \(s_1\) can be defined as follows:

\(\langle\)\(if(x_2 = null \ and \ \cdots \ and \ x_n = null\)
and \(y_2 = null \ and \ \cdots \ and \ y_n = null\)
and \(y'_2 = null \ and \ \cdots \ and \ y'_n = null\)\)
\(y_1 = case1\)
\(else\ \(if(\cdots)\)
\(y_1 = case2\)
\(\cdots\)\)

The trace \(tr(\pi')[A_c\ has the following form:
\(\cdots s_i \parallel \cdots x_i = ret_i, s'_i \parallel \cdots s_j \parallel \cdots x_j = ret_j, s'_j \cdots\)
Changing the order of the trace affects at least a position of an atomic region.

1.6 There exists an execution \(\pi'''\) of the program \(P(Ato_{Z})\) starting from \((\sigma_c, \sigma_z)\)
such that the final state is \((\sigma'_c, \sigma'_z)\).
proof: By the second condition of the lemma.

1.7 Q.E.D.

By 1.5 and 1.6, \(tr(\pi')[A_c = tr(\pi'''')[A_c\). Thus, by 1.4, \(H(\pi') \subseteq H(\pi''')\).
2 For any non-terminating execution \(\varphi\) of the object \(Z\), there exists a sequential
execution \(\varphi'\) of the object \(Z\) and a history \(h_c \in Compl(H(\varphi))\) such that \(h_c \subseteq H(\varphi')\).

2.1 To simplify writing, we assume that a thread invokes a method of \(Z\) at
most once in \(\varphi\). We now construct a program \(P(Z): s_1; x_1 = Z.m_1(e_1)\parallel \cdots s_i; x_i =
Of a pending invocation event of a method in $H$, if the method is not in $H(\xi')$, then we delete the pending invocation event; Otherwise, we add the same response events of the method as that of the method in $H(\xi')$. By 2.4 and 2.5 and the constructing process of $h_c$, $h_c \sqsubseteq H(\xi')$

3 Q.E.D.

proof: By 1 and 2.

**Theorem 7.** For a strict linearizable object $Z$ with a well-formed initial state, any client program $P$, $P(Z)$ and $P(\text{At}_{o}Z)$ have the same client-side traces and final states even in the relaxed program model above.

**Proof.** For any program $P(Z)$, we encapsulate the shared address read/write actions of $P$ into methods, and obtain the object $Z'$ by adding the new methods into $Z$, the client program $P'$ by replacing the shared address read/write actions of
P with the new methods. The concurrent history generated by \( P'(Z') \) is of the form

\[
CH = (CH_1)^\sim (SH_1)^\sim (CH_2)^\sim (SH_2) \ldots
\]

or

\[
CH = (SH_1)^\sim (CH_1)^\sim (SH_2)^\sim (CH_2) \ldots
\]

where for each \( i \), \( SH_i \) denotes the sequential history is generated by the new methods which encapsulate the compatible actions satisfying the first restriction; \( CH_i \) denotes the concurrent history is generated by the old methods and the new methods which encapsulate the compatible actions satisfying the second restriction; \( \sim \) denotes the concatenation of two histories.

For each \( CH_i \), there exists a linearization \( CH'_i \) of \( CH_i \), such that the concurrent execution which generate \( CH_i \) and the sequential execution which generate \( CH'_i \) have the same final state of \( Z \). Thus, \( SH = CH'_1 \sim SH_1 \ldots \) or \( SH = SH_1 \sim CH'_1 \ldots \) is a linearization of \( CH \) and the concurrent execution which generate \( CH \) and the sequential execution which generate \( SH \) have the same final state of \( Z \). Similar to the proof of Theorem 6, \( P'(Z') \) and \( P'(Ato_Z') \) have the same final states and the same traces generated by the clint program \( P' \) and the new methods. Each execution of \( P'(Z') \) can correspond to an execution of \( P(Z) \), and vice versa. Each execution of \( P(Ato_Z) \) can correspond to an execution of \( P'(Ato_Z') \), and vice versa. Thus, \( P(Z) \) and \( P(Ato_Z) \) have the same final states and the same client-side traces.

Appendix C.

**Theorem 10.** For a strict linearizable and purely-blocking object \( Z \), any client program \( P, P(Z) \) diverges iff \( P(Ato_Z) \) diverges.

**Proof.** By the following lemma (Lemma 5) and Corollary 8. \( \square \)

**Lemma 5.** For a strict linearizable and purely-blocking object \( Z \), any client program \( P, P(Z) \) diverges by divergences of \( Z \) iff \( P(Ato_Z) \) diverges by divergences of \( Ato_Z \).

**Proof.** \((\Rightarrow)\) Assume: \( P(Z) \) does not diverge by \( e \) divergences of \( Z \). Let \( S \) be a divergent execution of \( Ato_Z \), which is obtained by separating a divergent execution of \( P(Ato_Z) \) by the divergence of \( Ato_Z \). Consider such an execution \( S' \) of \( Z \). \( S' \) executes the methods which is finished in \( S \) sequentially by the same order as \( S \), then call other methods. In terms of the assumption above, all called methods in \( S' \) will finish. Since \( S \) is divergent, there does not exist a linearization of \( S' \), contradicting the fact that \( Z \) is strict linearizable. \( \square \)

**Proof.** \((\Leftarrow)\) Assume: \( P(Ato_Z) \) does not diverge by divergences of \( Ato_Z \). Let \( S \) be a divergent execution of \( Z \), which is obtained by separating a divergent execution of \( P(Z) \) by the divergence of \( Z \). In terms of the definition of strict linearizability and purely-blocking, there exists a sequential execution \( S' \) of \( Z \) and a history \( h_c \in \)
Compl($H(S)$) such that such that $h_c \subseteq H(S')$, and the final state of $Z$ in $S'$ is the same as that of $Z$ in $S$ (Because pending methods do not change the global states, we call the state of $Z$ the final state of $Z$ in $S$ after all called methods of $S'$ which can finish finish). Let $\sigma_z$ denote the final state of $Z$ in $S'$, and the final state of $Z$ in $S$ is the same as that of $Z$. (Because pending methods do not change the global states, we call the state of $Z$ the final state of $Z$ in $S$ after all called methods of $S'$ which can finish finish). Let $\sigma_z$ denote the final state of $Z$ in $S$. Consider such an execution $S''$ of $A$ to $Z$. (1) firstly, $S''$ executes the methods of $S'$ sequentially by the same order as $S'$; (2) then, calls other methods. The state of $Z$ is $\sigma_z$ after 1. In terms of the assumption above, there at least exists a method can finish in $\sigma_z$. This contradicts the fact that no pending methods of $S$ in $\sigma_z$ can finish.

Similar to the proof of Theorem 7, we can show that the theorem still holds for the relaxed program model in Subsection 5.1.

Appendix D.

Theorem 14. If $Z$ is a sequential implementation of an ADT $A$ then for any client program $P$, if all methods of $A$ are called within their domains, then $P(Z)$ and $P(A)$ are observationally equivalent.

Proof. 1 For any terminating execution $\mu: (\sigma_a, H_a, \sigma'_a)$ of $A$, there exists a terminating sequential execution $\mu': (\sigma_z, H_z, \sigma'_z)$ of $Z$, such that $AF(\sigma_z) = \sigma_a$, $AF(\sigma'_z) = \sigma'_a$ and $H_a = H_z$.

dep: This is proved in the 2 of the proof of Lemma 6 in Appendix Section G.

2 For any terminating execution $\mu: (\sigma_z, H_z, \sigma'_z)$ of $Z$ which is generated in $P(Z)$, there exists a sequential and terminating execution $\mu': (\sigma_a, H_a, \sigma'_a)$ of $A$, such that $AF(\sigma_z) = \sigma_a$, $AF(\sigma'_z) = \sigma'_a$ and $H_a = H_z$.

dep: By the lemma’s hypothesis, the methods of $A$ in $P(A)$ are called within their domains. Thus, By 1, we can get 2.

3 Q.E.D.

dep: By 1 and 2. The proof is similar to the one for Lemma 2.

Appendix E.

Theorem 16. If a concurrent object $Z$ is a concurrent implementation of an ADT $A$ then for any client program $P$, any well-formed initial state of $Z$, $P(Z)$ and $P(A)$ are observationally equivalent.

Proof. 1 For any terminating execution $\mu_a: (\sigma_a, H_a, \sigma'_a)$ of $A$, there exists a sequential and terminating execution $\mu_z: (\sigma_z, H_z, \sigma'_z)$ of $Z$, such that $AF(\sigma_z) = \sigma_a$, $AF(\sigma'_z) = \sigma'_a$ and $H_a = H_z$.

dep: This is proved in the 2 of the proof of Lemma 6 in Appendix Section G.

2 For any concurrent execution $\pi_z$ of $Z$ starting from any well-formed state $\sigma_z$, there exists a legal sequential execution $\pi_a$ of $A$ starting from the state $AF(\sigma_z)$ and a history $h_c \in Compl(H(\pi_z))$ such that $h_c \subseteq H(\pi_a)$.

dep: By the definition of concurrent implementation of an ADT (Definition 15).
exists a sequential and terminating execution proof: Let $A$

There exists an execution of $\rightarrow$

proof: Since $Z$

$\sigma$ $\phi$ $Z$

$\sum \models \phi$

$\sum$ $\sigma$

$\phi$ $\sum$

$\sum$ $\sigma$ $\phi$

$\sum$ $\models \phi$

$\sum$ $\sigma$

$\phi$ $\sum$

1. 3. Q.E.D.

Appendix F.

Theorem 17. For a strict linearizable object $Z$, if for any ADT $A$, $Z$ is a sequential implementation of $A$ and $\forall op \in Zop, \sigma_z \in ZState, in \in Input. (\sigma_z, in) \in dom(op) \implies (AF(\sigma_z), in) \in dom(RF(op))$, then $Z$ is a concurrent implementation of $A$.

Proof. 1 For any terminating execution $\pi_z : (\sigma_z, H, \sigma_z')$ of $Z$, there exists an execution $\pi_a : (\sigma_a, H_a, \sigma_a')$ of $A$, such that $H \subseteq H_a$ and $AF(\sigma_z) = \sigma_a, AF(\sigma_z') = \sigma_a'$.

1.1 For any sequential and terminating execution $\mu : (\sigma_{z0}, H_z, \sigma_{zn})$ of $Z$, there exists a sequential and terminating execution $\mu' : (\sigma_{a0}, H_a, \sigma_{an})$ of $A$, such that $AF(\sigma_{z0}) = \sigma_{a0}, AF(\sigma_{zn}) = \sigma_{an}$ and $H_a = H_z$.

1.1.1 Assume: $\mu = \sigma_{z0} \sigma_{z1} \sigma_{a1} \sigma_{a2} \cdots$

There exists an execution of $A, \mu' = \sigma_{a0} \sigma_{a1} \sigma_{a2} \cdots$

proof: Let $\sigma_{a0}$ be the state such that $AF(\sigma_{z0}) = \sigma_{a0}$. By the lemma’s hypothesis, $(\sigma_{a0}, n_1) \in dom(RF(op))$. Thus there exists $\sigma_{a1}, ret(v_1)$ such that $(\sigma_{a0}) \sigma_{a1}$.

By Definition 15, $ret(v_1)' = ret(v_1)$ and $AF(\sigma_{z1}) = \sigma_{a1}$. Thus $\sigma_{a0} \sigma_{a1}$.

By similar reasoning, there exists $\sigma_{a}(i+1)$ such that $\sigma_{a}(i+1)$ for each $i = 1 \cdots n - 1$.

1.1.2 Q.E.D.

proof: By 1.1.1.

1.2 For the terminating execution $\pi_z : (\sigma_z, H, \sigma_z')$ of $Z$, there exists a sequential execution $\pi_z' : (\sigma_z, H', \sigma_z')$ of $Z$ such that $H \subseteq H'$.

proof: Since $Z$ is strict linearizable.

1. 3. Q.E.D.

proof: By 1.1, for the sequential execution $\pi_z' : (\sigma_z, H', \sigma_z')$ of $Z$, there exists an execution $\pi_z : (\sigma_a, H_a, \sigma_a')$ such that $AF(\sigma_z) = \sigma_a, AF(\sigma_z') = \sigma_a'$ and $H_a = H'$. Since $H_a = H'$ and $H \subseteq H', H \subseteq H_a$. Thus, for the concurrent execution of $Z, \pi_z : (\sigma_{z}, H, \sigma_z')$, there exists an execution of $A, \pi_a : (\sigma_{a}, H_a, \sigma_a')$ such that $H \subseteq H_a$ and $AF(\sigma_z) = \sigma_a, AF(\sigma_z') = \sigma_a'$.

2 $Z$ is linearizable w.r.t. $A$.

2.1 For any terminating execution $\varphi$ of $Z$ starting from any well-formed state $\sigma_z$, there exists a sequential and terminating execution $\varphi'$ of $Z$, and a history $h_c \in Compl(H(\varphi))$ such that $h_c \subseteq H(\varphi')$.

proof: Since $Z$ is strict linearizable.
2.2 There exists a sequential and terminating execution \( \varphi'' \) of \( A \) such that \( H(\varphi'') = H(\varphi') \).

Proof: by the 1.1.

2.3 Q.E.D.

Proof: By 2.1 and 2.2, for any execution \( \varphi \) of \( Z \) starting from any well-formed state \( \sigma_z \), there exists a sequential and terminating execution \( \varphi'' \) of \( A \) and a history \( h_c \in \text{Compl}(H(\varphi)) \) such that \( h_c \subseteq H(\varphi'') \).

3 Q.E.D.

By 1 and 2.

Appendix G.

**Lemma 6.** A concurrent object \( Z \) is general linearizable w.r.t its sequential specification if there exists an ADT \( A \), such that \( Z \) is a concurrent implementation of \( A \) w.r.t an abstraction function \( AF \).

**Proof.**

1. For any concurrent execution \( \pi \) of \( Z \) starting from any well-formed state \( \sigma_z \), there exists a legal sequential execution \( \pi' \) of \( A \) starting from the state \( AF(\sigma_z) \) and a history \( h_e \in \text{Compl}(H(\pi)) \) such that \( h_e \subseteq H(\pi') \).

Proof: By the hypothesis, \( Z \) is linearizable w.r.t. the ADT \( A \).

2. For any terminating execution \( \mu : (\sigma_{a0},H_a,\sigma_{an}) \) of \( A \), there exists a sequential and terminating execution \( \mu' : (\sigma_{x0},H_z,\sigma_{zn}) \) of \( Z \), such that \( AF(\sigma_{x0}) = \sigma_{a0} \), \( AF(\sigma_{xn}) = \sigma_{an} \) and \( H_a = H_z \).

2.1 Assume: \( \mu = \sigma_{a0} \xrightarrow{(\text{inv}(op_1,n_1),\text{ret}(v_1))} \sigma_{a1} \xrightarrow{(\text{inv}(op_2,n_2),\text{ret}(v_2))} \sigma_{a2}, \ldots \), \( \sigma_{a0} \xrightarrow{\text{inv}(RF^{-1}(op_1),n_1),\text{ret}(v_1)} \sigma_{z0} \xrightarrow{\text{inv}(RF^{-1}(op_1),n_1),\text{ret}(v_1)} \sigma_{z1} \).

Proof: Since the abstraction function \( AF \) is surjective, there exists a state of \( Z \) is mapped to \( \sigma_{a0} \) by \( AF \). Let \( \sigma_{z0} \) be the state such that \( AF(\sigma_{z0}) = \sigma_{a0} \). By Definition 15, there exists \( \sigma_{z1} \) such that \( \sigma_{a0} \xrightarrow{\text{inv}(RF^{-1}(op_1),n_1),\text{ret}(v_1)} \sigma_{z1} \) and \( AF(\sigma_{z1}) = \sigma_{a1} \).

2.2 Q.E.D.

Proof: By 2.1 and Definition 15, the execution \( \mu' = \sigma_{z0} \xrightarrow{\text{inv}(RF^{-1}(op_1),n_1),\text{ret}(v_1)} \sigma_{z1} \xrightarrow{\text{inv}(RF^{-1}(op_2),n_2),\text{ret}(v_2)} \sigma_{z2}, \ldots \), \( \sigma_{an} \), for \( i = 1 \cdots n \).

3. Q.E.D.

Proof: By 2, there exists a sequential and terminating execution \( \pi'' \) of \( Z \), such that \( H(\pi'') = H(\pi') \). Since \( h_e \subseteq H(\pi') \), \( h_e \subseteq H(\pi'') \).

**Theorem 18.** A concurrent object \( Z \) is strict linearizable if there exists an ADT \( A \), such that \( Z \) is a concurrent implementation of \( A \) w.r.t an injective abstraction function \( AF \).
Proof. 1 For any terminating execution \((\sigma_z, H_z, \sigma'_z)\) of \(Z\), there exists a sequential execution \((\sigma_z, H'_z, \sigma'_z)\) of \(Z\), such that \(H_z \subseteq H'_z\).

1.1 For any terminating execution \(\mu: (\sigma_{a0}, H_a, \sigma_{an})\) of \(A\), there exists a sequential and terminating execution \(\mu': (\sigma_{20}, H_z, \sigma_{zn})\) of \(Z\), such that \(AF(\sigma_{z0}) = \sigma_{a0}\), \(AF(\sigma_{zn}) = \sigma_{an}\) and \(H_a = H_z\).

1.1.1 Assume: \(\mu = \sigma_{a0} \xrightarrow{(\text{inv}(\text{op}_1,n_1),\text{ret}(v_1))} \sigma_{a1} \xrightarrow{(\text{inv}(\text{op}_2,n_2),\text{ret}(v_2))} \sigma_{a2}, \ldots\).

There exists \(\sigma_{z0}\) and \(\sigma_{z1}\) such that \(\sigma_{z0} \xrightarrow{\text{inv}(\text{RF}^{-1}(\text{op}_1),n_1),\text{ret}(v_1)} \sigma_{z1}\).

proof: Since the abstraction function \(AF\) is injective, there exists a state of \(Z\) is mapped to \(\sigma_{a0}\) by \(AF\). Let \(\sigma_{z0}\) be the state such that \(AF(\sigma_{z0}) = \sigma_{a0}\). By Definition 15, there exists \(\sigma_{z1}\) such that \(\sigma_{z0} \xrightarrow{\text{inv}(\text{RF}^{-1}(\text{op}_1),n_1),\text{ret}(v_1)} \sigma_{z1}\) and \(AF(\sigma_{z1}) = \sigma_{a1}\).

1.1.2 Q.E.D.

proof: By 1.1 and Definition 15, the execution \(\mu' = \sigma_{z0} \xrightarrow{\text{inv}(\text{RF}^{-1}(\text{op}_1),n_1),\text{ret}(v_1)} \sigma_{z1} \xrightarrow{\text{inv}(\text{RF}^{-1}(\text{op}_2),n_2),\text{ret}(v_2)} \sigma_{z2}, \ldots, \xrightarrow{\text{inv}(\text{RF}^{-1}(\text{op}_n),n_n),\text{ret}(v_n)} \sigma_{zn}\) is feasible. Obviously, \(H(\mu) = H(\mu')\) and \(AF(\sigma_{z1}) = \sigma_{a1}\), for \(i = 1 \cdots n\).

1.2 For the terminating concurrent execution \((\sigma_z, H_z, \sigma'_z)\) of \(Z\), there exists a sequential execution \((\sigma_a, H_a, \sigma'_a)\) of \(A\), such that \(AF(\sigma_z) = \sigma_a\), \(AF(\sigma'_z) = \sigma'_a\) and \(H_z \subseteq H_a\).

proof: By Definition 15.

1.3. Q.E.D.

By 1.1, for the execution \((\sigma_a, H_a, \sigma'_a)\) of \(A\), there exists a sequential execution of \(Z\): \((\sigma_{xx}, H'_z, \sigma_{xy})\), such that \(AF(\sigma_{xx}) = \sigma_{a0}\), \(AF(\sigma_{xy}) = \sigma_{an}\) and \(H_a = H'_z\). Since \(AF\) is injective, \(\sigma_{xx} = \sigma_{z}, \sigma_{xy} = \sigma'_z\). Since \(H'_z = H_a\) and \(H_z \subseteq H_a\), \(H_z \subseteq H'_z\). Thus, for the concurrent execution of \(Z\), \((\sigma_z, H_z, \sigma'_z)\), there exists a sequential execution of \(Z\), \((\sigma_z, H'_z, \sigma'_z)\) such that \(H_z \subseteq H'_z\).

2 \(Z\) is general linearizable w.r.t its sequential specification.

proof: By Lemma 6.

3 Q.E.D.

By 1 and 2, \(Z\) is strict linearizable. □

Appendix H.

Fig. 6 shows the lock-free queue algorithm of Michael and Scott. The queue algorithm uses a linked list with Head and Tail pointers. The head pointer always points to the first node of the list. The tail pointer points to the last node of the list in a quiescent state. The first node in the list acts as a dummy node o simplify certain list operations. The queue is meant to be empty when the list has only a dummy node. If the queue is not empty, the Dequeue method advances the head pointer nd returns the value of the new first node of the list, so the new first node becomes a new dummy node. If the queue is empty, then the Dequeue method returns EMPTY. The Enqueue method first appends a new node at the tail of the
list, and later makes the tail pointer point to the new node. A thread cannot finish the Enqueue method in one atomic action, so other threads which observe that the tail pointer lags behind the end of the list will try to help the thread to advance the tail pointer before performing their own operations. The concrete states of the algorithm are well-formed if the singly linked list does not contain cycles and the tail pointer points to the last node.

class node {
  data_t val;
  node next;
}
class Queue {
  node Head, Tail;
  void Enqueue(data_t v);
  data_t Dequeue();
}
void Enqueue(data_t v) {
  local n, t, tn;
  n := new node();
  n.value := v;
  n.next = null;
  while (true) {
    t := Tail;
    tn := t.next;
    if (t = Tail) {
      if (tn = null) {
        if cas(&(t.next), tn, n) break;
      } else cas(&Tail, t, tn);
    }
    cas(&Tail, t, n);
  }
}
data_t Dequeue() {
  local h, t, hn, ret;
  while (true) {
    h := Head;
    t := Tail;
    hn := h.next;
    if (h = Head) {
      if (h = t) {
        if (hn = null) return EMPTY;
        cas(&Tail, t, hn);
      } else {
        ret := hn.value;
        if cas(&Head, h, s);
        return ret;
      }
    }
  }
}

Fig. 6. the MS Lock-Free Queue

Consider a multiset data type with operations to add and remove elements from the multiset. Let \( \{x, \cdots\} \) denote a multiset, one of whose member is \( x \). mset represents the initial contents of the multiset. The notation \( \varepsilon \) is different from the reserved value \( \text{EMPTY} \) and indicates that a method does not return values. The standard specification of a multiset is:

\[
\text{Add}(\text{mset}, e) = (\text{mset} \cup e, \varepsilon)
\]

\[
\text{Remove}(\text{mset}) = (\text{mset}', e),
\]

where \( \text{mset} = \text{mset}' \cup e \).

Consider an abstraction function which maps a concrete list pointed to by Head to the multiset consisting of the values of data fields of the list, and is formally defined as follows:

\[
\text{AF}(Q) = \{Q.\text{Head.next.value}\} \cup \text{AF}(Q'),
\]

where \( Q \) represents the MS lock-free queue and \( Q'.\text{Head} = Q.\text{Head.next} \). While the MS lock-free queue algorithm satisfies the multiset data type specification, this does not imply that the algorithm is strict linearizable. This is because the abstraction function is not injective.

Consider a standard queue data type with Enqueue and Dequeue methods. The variable seq denotes an initial state of the atomic sequence, \( |\text{seq}| \) denotes the length of the atomic sequence. The specification of the queue data type is defined as follows:

\[
\text{Enqueue}(\text{seq}, x) = (\text{seq} \cdot x, \varepsilon)
\]
Dequeue(seq) = \begin{cases} (seq', y), & \text{if } |seq| > 0, \\
(seq, EMPTY), & \text{if } |seq| = 0, \end{cases}

where seq = y^c seq'. The abstraction function maps a concrete list pointed to by Head to the value sequence of its data fields except the first data field, and is formally defined as follows:

AF(Q) = \begin{cases} (), & \text{if } Q.Head.next = \text{null}; \\
(Q.Head.next.value)^c AF(Q'), & \text{otherwise}, \end{cases}

where Q'.Head = Q.Head.next, and ( ) denotes an empty sequence. Under the abstraction, the data field of the first node is ignored by the users of the data structure. Two lists which are the same except for the values of the data fields of their first nodes, are mapped to the same abstract value. Obviously, the abstraction function is not injective. Therefore, while the algorithm satisfies the queue data type specification, this does not imply the algorithm is strict linearizable.

Consider a pseudo-queue data type, which is similar to a standard queue but does not allow dequeue operation when it contains one element. In practice, this may be because the first node must remain holding a global message. The specification of the pseudo-queue data type is defined as follows:

Enqueue(seq, x) = (seq^c x, ε)

Dequeue(seq) = \begin{cases} (seq', y), & \text{if } |seq| > 1, \\
(seq, EMPTY), & \text{if } |seq| = 1, \end{cases}

where seq = y^c seq'. Consider the following abstraction function:

AF(Q) = \begin{cases} (), & \text{if } Q.Head = \text{null}; \\
(Q.Head.value)^c AF(Q'), & \text{otherwise}, \end{cases}

where Q'.Head = Q.Head.next. Note that the value of data filed of the first node is mapped to the first element of the pseudo-queue. Under the abstraction, the value of data filed of the first node is also observed by users. The abstraction function is injective. Since we can show the algorithm satisfies the pseudo-queue specification, the MS lock-free queue algorithm is strict linearizable.

References
[1] T. A. Henzinger, A. Sezgin, and V. Vafeiadis. Aspect-oriented linearizability proofs. In CONCUR, pages 242C256, 2013.
[2] C. Flanagan, S. Qadeer. Types for atomicity. In TLDI, 2003.
[3] B. Jonsson. Using refinement calculus techniques to prove linearizability. Formal Asp. Comput. 24, 4-6 (2012), 537C554.
[4] A. Farzan, P. Madhusudan. Monitoring atomicity in concurrent programs. Computer Aided Verification, 2008: 52-65.
[5] O. Shacham, E. Yahav, G. Gueta, A. Aiken, N. Bronson, M. Sagiv, and M. Vechev. Verifying atomicity via data independence. ISSTA, 2014.

[6] Y. Afek, G. Korland, E. Yanovsky. Quasi-linearizability: relaxed consistency for improved concurrency. International Conference on Principles of Distributed Systems, pp. 395-410 (2010).

[7] A. Cerone, A. Gotsman, H. Yang. Parameterised linearisability. International Colloquium on Automata, Languages, and Programming, 2014.

[8] M. Herlihy, J. Wing. Linearizability: a correctness condition for concurrent objects. ACM TOPLAS, 12(3):463-492, 1990.

[9] S. Qadeer, A. Sezgin, and S. Tasiran. Back and forth: Prophecy variables for static verification of concurrent programs. Tech Report

[10] V. Vafeiadis. Modular fine-grained concurrency verification. PhD thesis, University of Cambridge, 2008.

[11] H. Liang, X. Feng. Modular verification of linearizability with non-fixed linearization points. In PLDI, 2013.

[12] T. A. Henzinger, C. M. Kirsch, H. Payer, A. Sezgin, A. Sokolova. Quantitative relaxation of concurrent data structures. In POPL, 2013.

[13] W. Vogels. Eventually consistent. Commun. ACM. 52(1), 40C44 (2009).

[14] M. Herlihy, N. Shavit. The Art of Multiprocessor Programming. Morgan Kaufmann, Apr. 2008

[15] C. A. R. Hoare. Proof of correctness of data representation. Acta Informatica, 1:271-281, 1972.

[16] R. J. Lipton. Reduction: A method of proving properties of parallel programs. Communications of the ACM, 18(12):717-721, 1975.

[17] I. Filipović, P. W. O’Hearn, N. Rinetzky, and H. Yang. Abstraction for concurrent objects. Theor. Comput. Sci., 411(51-52):4379-4398, 2010.

[18] M. M. Michael, M. L. Scott. Simple, fast, and practical nonblocking and blocking concurrent queue algorithms. In PODC’96.

[19] S. Heller, M. Herlihy, V. Luchangco, M. Moir, W. N. Scherer, and N. Shavit. A lazy concurrent list-based set algorithm. In OPODIS, 2005.

[20] T. L. Harris, K. Fraser, and I. A. Pratt. A practical multi-word compare-and-swap operation. In DISC’02.

[21] C. Flanagan, S. Qadeer. A type and effect system for atomicity. In PLDI, 2003.

[22] E. Cohen, L. Lamport. Reduction in TLA. In CONCUR, 1998.

[23] C. Flanagan, S. N. Freund. Atomizer: a dynamic atomicity checker for multithreaded programs. ACM SIGPLAN Notices, 39(1), 256-267, 2004.

[24] J. Derrick, G. Schellhorn, and H. Wehrheim. Mechanically verified proof obligations for linearizability. ACM Trans. Program. Lang. Syst. 33, 1 (2011), 4.

[25] A. Bouajjani, M. Emmi, C. Enea, J. Hamza. On reducing linearizability to state reachability. In ICALP, 2015.

[26] C. H. Papadimitriou, The serializability of concurrent database updates, 1979, J. ACM 26,4: 631-653.

[27] L. Lamport How to write a 21st century proof. Journal of Fixed Point Theory and Applications, 2012, 11(1): 43-63.

[28] S. Brookes A semantics for concurrent separation logic. Theoretical Computer Science, 2007, 375(1): 227-270.

[29] L. Lamport. How to make a multiprocessor computer that correctly executes multiprocess programs. IEEE Trans. Comput, 1979 C-28,9:690-691.