Remarks on the KLB theory of two-dimensional turbulence

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We study the inverse energy transfer in forced two-dimensional (2D) Navier–Stokes turbulence in a doubly periodic domain. It is shown that an inverse energy cascade that carries a nonzero fraction of the injected energy to the large scales via a power-law energy spectrum \( \propto k^{-\alpha} \) requires that \( \alpha \geq 5/3 \). This result is consistent with the classical theory of 2D turbulence that predicts a \( k^{-5/3} \) inverse-cascading range, thus providing for the first time a rigorous basis for this important feature of the theory. We derive bounds for the Kolmogorov constant \( C \) in the classical energy spectrum \( E(k) = C \epsilon^{2/3} k^{-5/3} \), where \( \epsilon \) is the energy injection rate. Issues related to Kraichnan’s conjecture of energy condensation and to power-law spectra as the quasi-steady dynamics become steady are discussed.

1. Introduction

It is well known that the advective nonlinearities of two-dimensional (2D) Navier–Stokes (NS) turbulence predominantly transfer energy to low wavenumbers (large scales) and enstrophy to high wavenumbers (small scales). The extent of this preferential transfer has been a subject of intense research since Fjørtoft (1953) first noticed this interesting property of 2D turbulence. In the late 1960s Kraichnan (1967) postulated that for unbounded fluids in the limit of high Reynolds number, this preferential transfer achieves an extreme limit, by transferring virtually all energy to ever-lower wavenumbers (inverse energy cascade) and virtually all enstrophy to a high wavenumber \( k_\nu \gg s \) (direct enstrophy cascade) if the turbulence is driven by sources at intermediate wavenumbers around \( s \). The transfer of energy and enstrophy in this extreme manner is known as the dual cascade. The energy is predicted to cascade via a \( k^{-5/3} \) energy inertial range, and the enstrophy is predicted to cascade via a \( k^{-3} \) enstrophy inertial range. The dissipation wavenumber \( k_\nu \) determines the region where the enstrophy gets dissipated. The resulting dynamics is quasi-steady since the inverse energy cascade presumably proceeds indefinitely in time toward wavenumber zero. This theory was later advanced by Leith (1968), Batchelor (1969), and Kraichnan (1971) and has become a classical theory of 2D turbulence, known as the KLB theory.

For a fluid confined to a doubly periodic domain, the inverse energy cascade is halted at the lowest wavenumber \( k_0 \) corresponding to the integral length scale of the system. This arrest of the inverse cascade may be met with complex responsive adjustments of the \( k^{-5/3} \) range. In any case a rise of the total energy in the available energy range necessarily occurs. According to Kraichnan (1967), such a rise occurs only at \( k_0 \), resulting in what can be termed an ‘energy condensate’ that singlehandedly carries most of the
total kinetic energy.† Assuming that the energy spectrum eventually becomes statistically steady, with a persistent inverse cascade up to $k_0$, the energy condensate not only would carry virtually all the system energy but also would account for virtually all the energy dissipation. This means that Kraichnan’s energy condensate would also be an enstrophy condensate, although the latter would be of a lesser degree. For a sufficiently wide energy inertial range, the enstrophy dissipation by the energy condensate would be negligible. This possibility allows for the enstrophy dynamics to be virtually unaffected by the condensate and for the direct enstrophy cascade to remain intact in Kraichnan’s picture.

Numerical results concerning the Kraichnan condensate are inconclusive and, in fact, controversial. Smith & Yakhot (1993, 1994) argue for a positive answer to the Kraichnan conjecture. On the contrary Borne (1994) and Tran & Bowman (2004) observe that after the arrest of the inverse energy cascade, growth of energy and enstrophy occurs throughout the energy inertial range. More precisely they find that as the turbulence approaches a steady state, a $k^{-3}$ energy spectrum forms at large scales. Although this spectrum still means that most of the energy is concentrated at the lowest wavenumbers, it is by no measure close to the Kraichnan picture. In particular a $k^{-3}$ energy spectrum means a $k^{-1}$ enstrophy spectrum, so that no accumulation of enstrophy occurs at the largest scales and the dissipation of energy cannot occur primarily at or in the vicinity of $k_0$.

Theoretical studies have shown under a variety of assumptions that for a doubly periodic domain, the ratio of enstrophy dissipation to energy dissipation equals $s^2$—the ratio of enstrophy injection to energy injection (Constantin et al. 1994; Eyink 1996, p. 110; Tran & Shepherd 2002; Kuksin 2004). These authors have concluded from this result that there can be no direct enstrophy cascade. However this inference assumes that all the enstrophy is actively involved in the enstrophy cascade, an assumption that is inconsistent with the Kraichnan conjecture. Indeed, several of these studies have explicitly or implicitly excluded a priori the Kraichnan conjecture in favour of an assumption of power-law spectra. There is thus a need to explicitly examine the extent to which the Kraichnan conjecture is consistent with analytical constraints on 2D turbulence.

In this paper we study several aspects of the KLB theory. In §2 we introduce some necessary notation and derive a basic identity and inequality. In §3 we give a general review of the KLB theory, including a discussion of the dual-cascade hypothesis and of the Kraichnan conjecture of energy condensation. In §4 we derive an upper bound for the inverse energy flux and show that a nonzero steady flux requires that the inverse-cascading range be at least as steep as $k^{-5/3}$. More precisely, we show that an energy-transfer range shallower than $k^{-5/3}$ is unable to support a constant inverse energy flux. This result provides for the first time a rigorous basis for the Kolmogorov–Kraichnan $k^{-5/3}$ inertial range. We also derive bounds for the Kolmogorov constant $C$ in the classical energy spectrum $E(k) = C\epsilon^{2/3}k^{-5/3}$, where $\epsilon$ is the energy steady injection rate. In §5 we derive a constraint on the enstrophy and palinstrophy distribution in the Kraichnan picture and contrast it with power-law scalings. Concluding remarks and discussion are given in §6.

† Kraichnan (1967) also predicts that as the energy accumulates at $k_0$, the $-5/3$ spectrum is modified toward absolute equilibrium of the form $E(k) \propto k/(\beta k^2 + \alpha)$, where $\beta$ and $\alpha$ are constant (see Kraichnan 1967, p. 1423a). Note that Kraichnan’s conjecture would be trivially extended to an energy condensate in the vicinity of $k_0$, not just at $k_0$. In this work we address this more general scenario.
2. Mathematical preliminaries

In the vorticity formulation the forced 2D NS equation governing the motion of an incompressible fluid confined to a doubly periodic domain \([0, L] \times [0, L]\) is

\[
\partial_t \Delta \psi + J(\psi, \Delta \psi) = \nu \Delta^2 \psi + f,
\]

where \(\psi(x, t)\) is the streamfunction, \(J(\theta, \vartheta) = \theta_x \vartheta_y - \theta_y \vartheta_x\), \(\nu\) the kinematic viscosity and \(f(x, t)\) the forcing. The velocity field \(v(x, t)\) can be recovered from the streamfunction \(\psi(x, t)\) by \(v = (-\psi_y, \psi_x)\). The nonlinear term admits the identities

\[
\langle \phi J(\theta, \vartheta) \rangle = -\langle \theta J(\phi, \vartheta) \rangle = -\langle \vartheta J(\theta, \phi) \rangle,
\]

where \(\langle \cdot \rangle\) denotes a spatial average. As a consequence we have the twin constraints

\[
\langle \psi J(\psi, \Delta \psi) \rangle = 0 = \langle \Delta \psi J(\psi, \Delta \psi) \rangle,
\]

so that the energy \(E = \langle |\nabla \psi|^2 \rangle / 2\) and enstrophy \(Z = \langle |\Delta \psi|^2 \rangle / 2\) are conserved by nonlinear transfer.

We now derive a simple identity and an inequality, which are used in §4 to calculate the nonlinear triple-product term representing the inverse energy transfer. By straightforward calculation we have

\[
\Delta J(\theta, \vartheta) = J(\Delta \theta, \Delta \vartheta) + J(\theta, \Delta \vartheta) + 2J(\theta_x \vartheta_y, \vartheta_x) + 2J(\theta_y \vartheta_x, \theta_y),
\]

Hence

\[
\langle \Delta \theta J(\theta, \vartheta) \rangle = \langle \theta \Delta J(\theta, \vartheta) \rangle = \langle \theta J(\Delta \theta, \vartheta) \rangle + 2\langle \theta J(\theta_x, \vartheta_x) \rangle + 2\langle \theta J(\theta_y, \vartheta_y) \rangle
\]

\[
= -\langle \Delta \theta J(\theta, \vartheta) \rangle + 2\langle \theta J(\theta_x, \vartheta_x) \rangle + 2\langle \theta J(\theta_y, \vartheta_y) \rangle.
\]

It follows that

\[
\langle \Delta \theta J(\theta, \vartheta) \rangle = \langle \theta J(\theta_x, \vartheta_x) \rangle + \langle \theta J(\theta_y, \vartheta_y) \rangle. \tag{2.6}
\]

For the present case of a doubly periodic domain of size \(L \times L\), the Fourier representation of \(\psi(x)\) is

\[
\psi(x) = \sum_k \exp\{ik \cdot x\} \hat{\psi}(k). \tag{2.7}
\]

Here \(k = k_0(n, m)\), where \(k_0 = 2\pi / L\) is the lowest wavenumber and \(n\) and \(m\) are integers not simultaneously zero. For a given wavenumber \(\ell\) let \(\psi^<\) and \(\psi^>\) denote, respectively, the components of \(\psi\) spectrally supported by the disk \(d = \{k : k \geq \ell\}\) and its complement \(D = \{k : k \geq \ell\}\), i.e.

\[
\psi^< = \sum_{k \in d} \exp\{ik \cdot x\} \hat{\psi}(k), \quad \psi^> = \sum_{k \in D} \exp\{ik \cdot x\} \hat{\psi}(k). \tag{2.8}
\]

The lower-wavenumber component \(\psi^<\) satisfies

\[
|\Delta \psi^<| \leq \sum_{k \in d} k^2 |\hat{\psi}(k)| \leq \left( \sum_{k \in d} 1 \right)^{1/2} \left( \sum_{k \in d} k^4 |\hat{\psi}(k)|^2 \right)^{1/2} = c \frac{\ell^2}{k_0} Z^<_{\Delta \psi^<}, \tag{2.9}
\]

where \(c\) is an absolute constant of order unity and \(Z^<_{\Delta \psi^<} = \langle |\Delta \psi^<|^2 \rangle / 2\) the large-scale enstrophy density associated with the wavenumbers \(k < \ell\). In the Cauchy–Schwarz inequality is used in the second step, and the sum \(\sum_{k \in d} 1 \approx \ell^2 / k_0^2\) represents the number of wavevectors in \(d\).
3. The KLB theory

This section reviews the central features of the KLB theory: the dual-cascade hypothesis and the Kraichnan conjecture of energy condensation. The discussion deviates significantly from the original works that lead to the dual-cascade hypothesis, thereby providing a new look at the KLB theory. We also discuss some recent results, which, on the one hand, can modify KLB significantly and, on the other hand, cast doubt on some aspects of the theory.

3.1. The preferential transfer

We consider turbulence driven by sources localized around a wavenumber \( s \) that supply an energy injection \( \epsilon > 0 \) and an enstrophy injection \( s^2 \epsilon \). Let \( \epsilon(k) \) and \( k^2 \epsilon(k) \) be nonlinear redistributions of these injections in wavenumber space. Given arbitrary wavenumbers \( r_1 < s \) and \( r_2 > s \) we have

\[
\frac{1}{\epsilon} \int_{r_1}^{\infty} \epsilon(k) \, dk \leq \frac{1}{r_2^2 \epsilon} \int_{r_2}^{\infty} k^2 \epsilon(k) \, dk \leq \frac{s^2 \epsilon}{r_2^2} = \frac{s^2}{r_2^2}, \tag{3.1}
\]

\[
\frac{1}{s^2 \epsilon} \int_{0}^{r_1} k^2 \epsilon(k) \, dk \leq \frac{r_1^2}{s^2 \epsilon} \int_{0}^{r_1} \epsilon(k) \, dk \leq \frac{r_1^2 \epsilon}{s^2 \epsilon} = \frac{r_1^2}{s^2}, \tag{3.2}
\]

where the second inequalities in (3.1) and (3.2) are, respectively, due to the conservation of enstrophy and energy. It follows that the respective fractions of \( \epsilon \) and of \( s^2 \epsilon \) that get transferred to \( k \geq r_2 > s \) and to \( k \leq r_1 < s \) are bounded from above by \( s^2/r_2^2 \) and by \( r_1^2/s^2 \), respectively. These constraints correspond to the well-known prohibition of a significant direct (inverse) transfer of energy (enstrophy): no considerable fraction of \( \epsilon (s^2 \epsilon) \) is allowed to get transferred to wavenumbers \( k \gg s \) (\( k \ll s \)). Hence when these injections spread out in wavenumber space, most of the energy (enstrophy) gets transferred toward lower (higher) wavenumbers, a qualitative conclusion that is behind the dual-cascade hypothesis. This dynamical behaviour of energy and enstrophy is well confirmed by numerical simulations reported in the literature. Note that the preferential transfer described above reflects the collective effects of all admissible nonlinearly interacting triads. Detailed analyses of individual triads may not lead to the same conclusion with certainty since there is a significant fraction of triads that may act unfavourably (see Merilees & Warn 1975).

3.2. The dual-cascade hypothesis and Kraichnan conjecture

The preferential transfer of energy and enstrophy, together with the scale-selective dissipation by molecular viscosity, constitutes the backbone of the dual-cascade hypothesis. Let us consider the dynamical scenario described in the preceding subsection, with an additional assumption that the fluid is initially at rest. When the forcing region becomes unstable and the injected enstrophy gets transferred toward the high wavenumbers, palinstrophy \( P = \langle |\nabla \Delta \psi|^2 \rangle/2 \) is built up and growth of the enstrophy dissipation \( 2\nu P \) ensues. Meanwhile growth of enstrophy is due solely to enstrophy injection and is suppressed by its own growing dissipation. If the growth of palinstrophy is sufficiently rapid, the enstrophy can quickly become steady, achieving a value sufficiently low that \( 2\nu Z \ll \epsilon \). This allows for a strong inverse energy cascade to be realized. By ‘strong’ we mean that the inverse cascade carries virtually all the energy injection to the large scales; otherwise the inverse cascade is ‘weak’. At least a transient direct enstrophy cascade seems plausible since the palinstrophy is concentrated mainly at high wavenumbers, presumably at \( k > k_\nu \gg s \) (the so-called dissipation range in the KLB theory). The realization of the classical \( k^{-3} \) enstrophy cascade as a persistent rather than just transient phenomenon
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requires that the high concentration of palinstrophy around $k_\nu$ (which may be correctly termed a ‘palinstrophy condensate’) remain stable, withstanding the enormous dissipation of palinstrophy in that region and not redistributing itself in wavenumber space. Note that if viscosity is replaced by a scale-independent dissipation, i.e. a mechanical friction, then both energy and enstrophy are dissipated at the same rate. The above argument for an inverse cascade fails to apply in this case. If the usual viscosity term $\nu \Delta^2 \psi$ is replaced by an inverse viscosity represented by, for example, $\mu \psi$, then the dynamical behaviours of energy and enstrophy are reversed: energy behaves as enstrophy and vice versa. In this case the preferential transfer of energy and enstrophy gives rise to a predominant increase of the energy dissipation $\mu \langle |\psi|^2 \rangle$. The energy can then quickly become steady, achieving a value sufficiently low that $2 \mu E \ll s^2 \epsilon$, i.e. the enstrophy dissipation is much less than its injection rate. This allows for a direct enstrophy cascade to be realized. Of course this case has no physical basis and is employed here only to illustrate the effects of the scale selectivity of the dissipation on the transient dynamics.

We note in passing that numerical simulations aiming to verify the dual-cascade picture (or part of this picture: either an inverse or a direct cascade) employ both hyperviscosity and inverse viscosity. The results in this direction constitute a rich literature; some recent studies are Paret, Julien & Tabeling (1999), Boffetta, Celani & Vergassola (2000), Lindborg & Alvelius (2000) and Chen et al. (2003). The present arguments suggest that the effects of these scale-selective dissipation mechanisms on the transient dynamics (and probably beyond) should be taken into consideration when interpreting the results.

For finite Reynolds numbers a small but non-negligible fraction of the injected energy is dissipated, resulting in a weak (i.e. less than complete) inverse energy cascade. For the $k^{-5/3}$ energy range this dissipation occurs mainly around the forcing wavenumber $s$, resulting in a similar fraction of enstrophy dissipation around $s$. Now if the remainder of the enstrophy injection were to be transferred to and dissipated at $k \gg s$, there would necessarily be a severe step in the spectrum between the forcing region and the rest of the enstrophy range. This means that for power-law scalings (without such a step), a weak direct enstrophy cascade is not permitted. This is in contrast to the robustness of the weak inverse cascade, which is readily observable in numerical simulations even in the complete absence of an accompanying direct enstrophy cascade (see Tran 2004 and Tran & Bowman 2004). Hence a plausible dynamical scenario is that for moderate Reynolds number the enstrophy dissipation occurs throughout the direct-transfer range up to some high wavenumber, which depends on the Reynolds number. The energy spectrum of this enstrophy dissipation range (instead of enstrophy inertial range) should scale as $k^{-5}$. The question is whether, for progressively higher Reynolds number, the inverse energy cascade can become stronger and a direct enstrophy cascade realizable. In a more quantitative analysis, Tran (2004) suggests that a quasi-steady state featuring an inverse cascade of arbitrary strength (via the Kolmogorov-Kraichnan $k^{-5/3}$ spectrum) and a uniform dissipation of enstrophy among the wavenumber octaves of the direct-transfer range is plausible. This picture is consistent with the preferential transfer of energy and enstrophy, required by the conservation laws, and explains the numerical results of Tran (2004) and Tran & Bowman (2004) mentioned earlier. It may also explain the numerous numerical results targeted at the inverse energy cascade, for which there is hardly an enstrophy range due to limited resolution.

In order to apply the dual-cascade hypothesis to turbulence confined to a bounded domain, Kraichnan (1967) suggests that after reaching the lowest available wavenumber $k_0$, the inverse energy cascade maintains its strength during the quasi-steady stage, depositing energy onto $k_0$ (or in the vicinity of $k_0$, as presently considered). This process continues until the growth of energy at or around $k_0$ is limited by its own dissipation,
resulting in a huge pile-up of energy and enstrophy in this region of the spectrum. Note that even in the picture of no direct enstrophy cascade discussed in the preceding paragraph, the Kraichnan conjecture still applies although in this case the concentration of energy and enstrophy at the condensate would not be as dramatic as in the original dual-cascade case. This topic is discussed further in §5.

4. Inverse energy transfer

In this section, we derive a rigorous upper bound for the energy that gets transferred across a low wavenumber $\ell$. It is shown that if a power-law scaling $ak^{-\alpha}$ is assumed for the energy spectrum in the inverse-transfer range and if $\alpha < 5/3$, then no significant fraction of the energy injection can get transferred across $\ell$ for sufficiently low $\ell$. This result implies that the $-5/3$ slope represents the minimal steepness of the energy inertial range that can support an inverse energy cascade. For the special case $\ell = k_0$ a sharp estimate of the energy transfer onto $k_0$ is obtained.

4.1. Inverse energy flux

We assume that the spectral support of $f$ is bounded from below by a wavenumber $s_0$ and that the energy injection is bounded. The usual requirement of spectral localization of $f$ can be relaxed for most of this section. For $\ell < s_0$ the governing equation for the evolution of the large-scale energy density $E_\ell = \langle|\nabla \psi|^2\rangle/2$ is obtained by multiplying the governing equation (2.1) by $\psi$ and taking the spatial average of the resulting equation:

$$\frac{d}{dt}E_\ell = \langle \psi J(\psi, \Delta \psi) \rangle - 2\nu Z_\ell = -\langle \Delta \psi J(\psi, \psi) \rangle - 2\nu Z_\ell, \tag{4.1}$$

where the second equality is due to (2.2). The nonlinear term represents the energy that gets transferred into the low-wavenumber region $[k_0, \ell]$, i.e. the energy flux across $\ell$, which drives the large-scale dynamics.

We now derive an upper bound for the nonlinear term in (4.1) and then estimate this bound for power-law energy spectra. The steps go as follows

$$|\langle \Delta \psi J(\psi, \psi) \rangle| = |\langle \psi J(\psi_x, \psi_x^\perp) + \psi J(\psi_y, \psi_y^\perp) \rangle|$$

$$= |\langle \psi J(\psi_x^\perp, \psi_y^\perp) + \psi J(\psi_y^\perp, \psi_x^\perp) \rangle|$$

$$= |\langle \psi^\perp_x J(\psi_x, \psi_x^\perp) + \psi^\perp_y J(\psi_y, \psi_y^\perp) \rangle|$$

$$\leq \langle |\psi_x^\perp| |\nabla| |\nabla \psi_x^\perp| + |\psi_y^\perp| |\nabla| |\nabla \psi_y^\perp| \rangle$$

$$\leq \langle |\nabla \psi^\perp| |\nabla| |\nabla \psi_x^\perp|^2 + |\nabla \psi_y^\perp|^2 \rangle^{1/2}$$

$$\leq 2E_x^{1/2}E^{1/2} \text{sup}_{x}(|\psi_x^\perp|^2 + |\nabla \psi_y^\perp|^2)^{1/2}$$

$$= 2E_x^{1/2}E^{1/2} \text{sup}_{x}(|\psi_x^\perp|^2 + |\psi_y^\perp|^2 + 2|\nabla \psi_y^\perp|^2)^{1/2}, \tag{4.2}$$

where the first step is due to (2.3) and all the subsequent steps involve simple manipulations and the Cauchy-Schwarz inequality. Here $E_x = \langle|\nabla \psi|^2\rangle/2$ is the ‘small-scale’ energy density associated with wavenumbers $k \geq \ell$. We observe that the upper bound for $|\Delta \psi^\perp|$ in (4.2) is also an upper bound for $|\psi_x^\perp|$, $|\psi_y^\perp|$ and $2|\nabla \psi_y^\perp|$. Hence we can deduce that

$$|\langle \Delta \psi J(\psi, \psi) \rangle| \leq c' \frac{\ell}{k_0} Z_\ell^{1/2} E_x^{1/2} E^{1/2}, \tag{4.3}$$
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where \( c' = 2\sqrt{3}c \). The appearance of \( Z_\varsigma \) in (4.3) is due mainly to (2.6), which enables us to ‘transfer’ one spatial derivative from \( \psi \) to \( \psi^c \). In the velocity formulation of the NS system, the calculation leading to the upper bound (4.3) for the inverse flux is straightforward. One of the advantages of the vorticity formulation, when equipped with (2.6), is that various nonlinear terms, for example the term \( \langle \Delta^n \psi J(\psi, \Delta \psi) \rangle \) in the evolution equation of the quadratic quantity \( \langle \psi \Delta^{n+1} \psi \rangle / 2 \), where \( n \) is an integer, can be manipulated with ease. Since \( Z_\varsigma \) becomes smaller for progressively lower \( \ell \), the right-hand side of (4.3) can remain bounded in the limit \( \ell \to 0 \), even though both \( E_\geq \) and \( E \) diverge in that limit.

For further analysis of the right-hand side of (4.3), we assume that an inverse-cascading range \( a k^{-\alpha} \) has been established beyond \( \ell \) and possibly has reached \( k_0 \), but no reflection or accumulation of energy has yet occurred. For this spectrum, \( E, E_\geq \) and \( Z_\varsigma \) can be estimated as follows:

\[
E \leq a \int_{k_0}^{\infty} k^{-\alpha} \, dk = \frac{a}{\alpha - 1} k_0^{1-\alpha} \quad \text{for } \alpha > 1, \tag{4.4}
\]

\[
E_\geq \leq a \int_{\ell}^{\infty} k^{-\alpha} \, dk = \frac{a}{\alpha - 1} \ell^{1-\alpha} \quad \text{for } \alpha > 1 \tag{4.5}
\]

and

\[
Z_\varsigma = a \int_{k_0}^{\ell} k^{2-\alpha} \, dk \leq \frac{a}{3-\alpha} \ell^{3-\alpha} \quad \text{for } \alpha < 3. \tag{4.6}
\]

The requirement \( \alpha < 3 \) is consistent with a persistent inverse energy cascade because a steeper spectrum would render the inverse-cascading range dissipative, which would not support such a cascade. The requirement \( \alpha > 1 \) poses no loss of generality as becomes apparent shortly. By substituting the above estimates into (4.3) we obtain

\[
|\langle \Delta \psi J(\psi, \psi^c) \rangle| \leq \frac{c' a^{3/2}}{(\alpha - 1)(3 - \alpha)^{1/2}} \left( \frac{\ell}{k_0} \right)^{(\alpha + 1)/2} \ell^{(5 - 3\alpha)/2}. \tag{4.7}
\]

Now in the limit \( k_0 \to 0 \), we have

\[
Z \geq a \int_0^{s_0} k^{2-\alpha} \, dk = \frac{a}{3-\alpha} s_0^{3-\alpha}. \tag{4.8}
\]

Solving for \( a \) from (4.8) and substituting the result into (4.7) we obtain

\[
|\langle \Delta \psi J(\psi, \psi^c) \rangle| \leq \frac{c'(3 - \alpha) Z^{3/2}}{(\alpha - 1) s_0^2} \left( \frac{\ell}{k_0} \right)^{(\alpha + 1)/2} \left( \frac{\ell}{s_0} \right)^{(5 - 3\alpha)/2}. \tag{4.9}
\]

An interesting conclusion can be readily drawn from (4.9). Given a fixed ratio \( \ell/k_0 \) and finite enstrophy density \( Z \) (recall that we must have \( Z \leq \epsilon / 2\nu \), if \( \alpha < 5/3 \), then the right-hand side of (4.9) can be made arbitrarily small provided \( \ell / s_0 \) is sufficiently small. In other words, no energy inertial range shallower than \( k^{-5/3} \) would be capable of sustaining an inverse energy cascade that carries a nonzero fraction of the energy injection to the low-wavenumber region \( [k_0, \ell] \), for sufficiently low \( \ell \). Thus the Kolmogorov–Kraichnan \( k^{-5/3} \) spectrum represents the shallowest possible spectrum that is necessary for the existence of the classical inverse energy cascade.

### 4.2. Energy transfer onto \( k_0 \)

When \( \ell = k_0 \), the upper bound for the inverse flux derived in the preceding subsection becomes an upper bound for the energy that gets transferred onto \( k_0 \). For this special case,
some improvement on the bound is possible and a sharp estimate for the Kolmogorov constant can be derived.

We employ the notation $\psi = \psi^\prec + \psi^\succ$ as in the previous sections, where $\ell = k_0$.

That means $\psi^\prec$, which is replaced by $\psi^0$ in what follows, consists of only four degenerate components corresponding to the four wavevectors $(\pm k_0, 0)$ and $(0, \pm k_0)$. Namely,

$$\psi^0 = \varphi_1 \exp\{ik_0 x\} + \varphi_1^* \exp\{-ik_0 x\} + \varphi_2 \exp\{ik_0 y\} + \varphi_2^* \exp\{-ik_0 y\}. \quad (4.10)$$

Inequality (2.9) becomes

$$|\Delta \psi^0| \leq 2k_0^2 (|\varphi_1| + |\varphi_2|) \leq 2^{3/2} k_0 \left( |k_0^2 \varphi_1|^2 + |k_0^2 \varphi_2|^2 \right)^{1/2} = 2^{3/2} k_0 \Psi_2^{1/2}(k_0), \quad (4.11)$$

where $\Psi_2(k_0) = (|\nabla \psi^0|^2)/2$ is the modal energy associated with $k_0$.

The energy that gets transferred onto $k_0$, i.e. the nonlinear term $\langle \psi^0 J(\psi, \Delta \psi) \rangle$, can be estimated as follows:

$$\langle \psi^0 J(\psi, \Delta \psi) \rangle = \langle \psi^0 J(\psi^\succ, \Delta \psi^\prec) \rangle = \langle (\Delta \psi^\succ) J(\psi^\succ, \psi^0) \rangle \nonumber$$

$$= \langle (\psi^\succ J(\psi^\succ, \psi^0_0)) + (\psi^\succ J(\psi^\succ, \psi^0_0)) \rangle \nonumber$$

$$= \langle (\psi^\succ J(\psi^\succ, \psi^0_0)) + (\psi^\succ J(\psi^\succ, \psi^0_0)) \rangle \nonumber$$

$$= \langle (\psi^\succ \psi^\succ_0 (\psi^0_{yy} - \psi^0_{xx}) \rangle \leq 2k_0^2 (|\varphi_1| + |\varphi_2|) \langle |\nabla \psi^\succ_0|^2 \rangle \nonumber$$

$$\leq 2^{3/2} k_0 \Psi_2^{1/2}(k_0) E_>, \quad (4.12)$$

where the replacement of $\psi$ by $\psi^\succ = \psi - \psi^0$ in the second step is a consequence of both $\Delta \psi^0 = -k_0^2 \psi^0$ and (2.2). The third step is due to (4.10), the last step is due to (4.11) and all other steps are straightforward. Here $E_> = (|\nabla \psi^\succ|^2)/2$ is the total energy density with the contribution from $k_0$ removed. The nonlinear term in the last equation of (4.12) represents an upper bound on the energy that gets transferred onto $k_0$.

As in the previous case we assume an inverse-transfer range $a k^{-\alpha}$ down to $k_0$ and estimate the energy that gets transferred onto $k_0$, before the arrest of the inverse cascade would deform the assumed power-law spectrum. The factors of the nonlinear term in (4.12) can be estimated as follows:

$$E_\alpha \leq a \int_{2k_0}^\infty k^{-\alpha} dk = \frac{2^{1-\alpha}}{\alpha - 1} k_0^{1-\alpha},$$

$$\Psi_2(k_0) = \frac{k_0 E(k_0)}{2\pi} = \frac{a}{2\pi} k_0^{1-\alpha}, \quad (4.13)$$

where $2^{1/2} k_0$ is the second lowest wavenumber, and $E(k_0) = a k^{-\alpha}$ is the energy supported at $k_0$. It follows that

$$2^{3/2} k_0 \Psi_2^{1/2}(k_0) E_\alpha \leq \frac{2^{1-\alpha} / 2 \pi^{3/2}}{\alpha - 1} k_0^{(5-3\alpha)/2}. \quad (4.14)$$

Like the argument in the preceding subsection, ineq. (4.14) together with (4.12) implies that an inverse-cascading range shallower than $k^{-5/3}$ is incapable of supporting a nonzero transfer of energy to $k_0$ for sufficiently low $k_0$.

### 4.3. Bounds for the Kolmogorov constant

Suppose that the turbulence is driven by a steady energy injection rate $\epsilon$ and that an inverse cascade carrying a fraction $r$ of this injection toward $k_0$ via a $a k^{-5/3}$ energy inertial range has been established. The left-hand side of (4.14), which is an upper bound for the inverse energy transfer onto $k_0$, cannot be smaller than $r \epsilon$. Hence we can deduce
from (4.14) that

\[ a \geq \left( \frac{2^{5/6} \pi r \epsilon}{3} \right)^{2/3}. \]  

(4.15)

It follows that the Kolmogorov constant \( C \) which appears in the energy spectrum of the classical energy inertial range as \( E(k) = C \epsilon^{2/3} k^{-5/3} \) is bounded from below by

\[ C \geq \left( \frac{2^{5/6} \pi r}{3} \right)^{2/3}. \]  

(4.16)

Note that in the classical case \( r = 1 \).

An upper bound for \( C \) can be derived on the basis of the classical spectrum alone. We observe that the energy dissipation by the energy range cannot exceed \( (1 - r) \epsilon \). Hence we have

\[ 2 \nu C \epsilon^{2/3} \int_0^s k^{1/3} \, dk \leq (1 - r) \epsilon. \]  

(4.17)

It follows that

\[ C \leq \frac{2(1 - r) \epsilon^{1/3}}{3 \nu s^{4/3}}. \]  

(4.18)

If we assume that \( \epsilon \) is injected around a forcing wavenumber \( s \), so that the enstrophy injection is given by \( \eta = s^2 \epsilon \), then we can rewrite (4.18) as

\[ C \leq \frac{4(1 - r) \eta^{1/3}}{3 \tau}. \]  

(4.19)

where \( \tau = 2 \nu s^2 \) is the dissipation rate at the forcing wavenumber \( s \). The upper bound (4.19) is expected to be sharp in the limit of high Reynolds number. In fact (4.19) is essentially an equality in that limit since the energy dissipation by \( k > s \) is negligible. For a fixed enstrophy injection rate \( \eta \), constancy of \( C \) requires that the ratio \( (1 - r)/\tau \) remain constant. On the other hand, \( C \) may diverge in the limit of infinite Reynolds number (\( \tau \to 0 \)) provided the inverse-cascade strength \( r \) approaches unity slower than \( \tau \) tends to zero. This dynamical scenario, which has been touched upon in §3, is considered by [Tran (2004)] for the unbounded case, where it is suggested that an inverse energy cascade that is progressively stronger with progressively higher Reynolds number is quite plausible, requiring no boundedness of enstrophy or of any other quadratic quantities in the limit of infinite Reynolds number. In this picture there is no anomalous enstrophy dissipation: the enstrophy dissipation is uniformly distributed among the wavenumber octaves of the direct-transfer range, whose energy spectrum scales as \( k^{-5} \).

5. Power-law spectra vs. Kraichnan’s condensate

In this section we examine the Kraichnan picture for steady dynamics in a bounded domain, and derive constraints on the enstrophy and palinstrophy distribution for such a picture. We then contrast this picture with power-law spectra. For simplicity we follow the non-average treatment in the preceding sections. Nevertheless all dynamical quantities, including the energy spectrum, can be understood in an appropriate average sense since such a reinterpretation requires a straightforward reformulation of the problem.

By virtue of the conservation laws, the evolution of the energy and enstrophy is gov-
where $\epsilon = -\langle f\psi \rangle$ and $\eta = \langle f\Delta\psi \rangle$ are, respectively, the energy and enstrophy injection rates. The usual assumption of spectral localization of $f$ around $s$ is invoked, i.e. $\eta = s^2\epsilon$. The forced-dissipative balance of both energy and enstrophy implies (see Constantin et al. 1994, Eyink 1996, Tran & Shepherd 2002 and Kuksin 2004)

$$P = s^2Z,$$

(5.3)

or in terms of the energy spectrum $E(k)$,

$$\int_{k_0}^{\infty} (k^2 - s^2)k^2E(k) \, dk = 0. \quad (5.4)$$

Equation (5.4) is the focus of the present section.

### 5.1. Energy and enstrophy condensate

Eq. (5.3), which is independent of Reynolds number, has been interpreted as implying that the enstrophy dissipation must occur in the vicinity of $s$, thus precluding a direct enstrophy cascade. However, this interpretation assumes that all the enstrophy is actively involved in the enstrophy cascade. As noted earlier, the Kraichnan energy condensate is also an enstrophy condensate, in which case most of the enstrophy is trapped in the condensate and only a tiny amount is free to participate in the enstrophy cascade. Thus, this interpretation implicitly excludes the Kraichnan scenario.

In fact, for spectra such that $Z$ is distributed mainly around $k_0$ and $P$ is distributed mainly at $k \gg s$, the balance (5.3) can still hold (see below). Another example is that $P$ has an equal contribution from each of the wavenumber octaves $[s, 10s], [10s, 10^2s], \cdots$, up to some wavenumber $k_\nu \approx s^2/k_0$, provided that $Z$ has a similar contribution from the wavenumber octaves of $[k_0, s]$. This uniform distribution of the energy and enstrophy dissipation in their respective ranges (requiring $k^{-3}$ energy and $k^{-5}$ enstrophy ranges) is considered by [Tran & Bowman (2003)] and [Tran (2004)] (also see below). The former example includes the Kraichnan scenario, for which the energy of the condensate $E_\prec$ (with $\ell \approx k_0$) satisfies $E_\prec \approx \epsilon/2\nu k_0^2$. The energy dissipation occurs mainly at the condensate since $2\nu Z_\prec \approx \epsilon$, where $Z_\prec \approx k_0^2E_\prec$ is the enstrophy of the condensate. The enstrophy dissipation by the condensate is $2\nu P_\prec \approx k_0^2\epsilon$, where $P_\prec \approx k_0^2Z_\prec$ is the palinstrophy of the condensate. This dissipation is negligible as compared with the enstrophy injection $s^2\epsilon$ if $k_0/s \ll 1$, i.e. if the energy inertial range is sufficiently wide. This possibility allows for the spectral distribution of palinstrophy, i.e. the spectral distribution of enstrophy dissipation, to be concentrated at the high wavenumbers, while posing no threat to the balance $P = s^2Z$.

The discussion in the preceding paragraph implies that the Kraichnan energy condensate would also be an enstrophy condensate. The extent of this highly localized enstrophy distribution at the condensate would be comparable to that of the palinstrophy around the dissipation wavenumber $k_\nu$. To see this explicitly let us rewrite the balance equation (5.3) as

$$P_\prec + P_\succ = s^2(Z_\prec + Z_\succ), \quad (5.5)$$

where the subscripts ‘$<$’ and ‘$>$’ refer to the ‘trapped’ (in the condensate) and ‘free’
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5.2. Power-law spectra

Following Constantin et al. (1994), Tran (2004) and Tran & Bowman (2003) we assume a two-range energy spectrum:

\[ E(k) = \begin{cases} \alpha k^{-\alpha} & \text{if } k_0 < k < s, \\ \beta k^{-\beta} & \text{if } s < k < k_{\nu}, \end{cases} \quad a s^{-\alpha} = b s^{-\beta}, \tag{5.7} \]

where \( \alpha \) and \( \beta \) are constant and \( k_{\nu} \) marks the end of the \( k^{-\beta} \) range, beyond which the spectrum becomes steeper. This approximation a priori excludes the Kraichnan condensate, but allows for some ‘milder’ accumulation of energy at \( k_0 \). For example, the case \( \alpha = 3 \) previously considered would allow for most of the system’s energy to reside in a few lowest wavenumbers. Substituting (5.7) into (5.4) allows one to derive interesting constraints on \( \alpha \) and \( \beta \). By rewriting (5.7) in the form \( \int_{k_0}^{s} (s^2 - k^2) k^2 E(k) \, dk = \int_{s}^{\infty} (k^2 - s^2) k^2 E(k) \, dk \), using the approximation (5.7) for \( E(k) \) and making the substitutions \( \kappa = k/s \) for \( k \leq s \), and \( \kappa = s/k \) for \( k \geq s \), Tran & Bowman (2003) show that

\[ \int_{k_0/s}^{1} (1 - \kappa^2) \kappa^{-2-\alpha} \, d\kappa \geq \int_{s/k_{\nu}}^{1} (1 - \kappa^2) \kappa^{\beta-6} \, d\kappa, \tag{5.8} \]

where the inequality is a consequence of dropping from the equality the contribution beyond \( k_{\nu} \), i.e. the quantity \( D_\nu = \int_{k_{\nu}}^{\infty} (k^2 - s^2) k^2 E(k) \, dk \). Except for the approximation (5.7), the constraint (5.8) is rigorous. In the spirit of KLB (\( k_{\nu} \to \infty \) as \( \nu \to 0 \)), we consider the case \( k_0/s \geq s/k_{\nu} \), which corresponds to high-Reynolds number turbulence. It is then easy to see from (5.8) that \( 2 - \alpha \leq \beta - 6 \), or equivalently,

\[ \alpha + \beta \geq 8. \tag{5.9} \]

For moderate Reynolds number \( \beta > 5 \) (see Tran 2004 and Tran & Bowman 2004), so we first consider the case \( \beta \geq 5 \). The quantity \( D_\nu \) is then negligible, making (5.8) essentially an equality and (5.9) essentially an equality if \( k_0/s \approx s/k_{\nu} \). In this case we obtain \( (\alpha, \beta) = (3 - \delta, 5 + \delta) \), where the limit \( \delta \to 0 \) is expected for high Reynolds numbers. This solution with \( \delta = 0 \) means that the energy and enstrophy are uniformly dissipated among the wavenumber octaves of their respective ranges. On the other hand, if \( \beta < 5 \) (which cannot be logically ruled out, just as the Kraichnan picture cannot be a priori excluded), then \( \alpha > 3 \). In this case the inequality “\( \geq \)” in (5.8) could become “\( > \)” since \( D_\nu \) could become large. As a consequence a slight decrease of \( \beta \) from the critical value \( \beta = 5 \) must be met with a much more significant increase of \( \alpha \) from \( \alpha = 3 \). This means that \( \alpha \) could quickly approach \( \beta \) before the latter would drop significantly below its critical value. Hence the two-range steady spectrum (5.7), when supplemented by the usual condition \( \alpha < \beta \), obeys the constraint \( \beta \geq 4 + \delta \), where \( \delta > 0 \) depends on how \( E(k) \) decays for \( k > k_{\nu} \). The case \( \beta \leq 4 \) is inadmissible and would necessarily require an energy condensate. For the Kolmogorov–Kraichnan energy inertial range (\( \alpha = 5/3 \)), a Kraichnan-type condensate would also be required even when \( \beta = 5 \) since the integral on the left-hand side of (5.8) is less than unity (for all ratios \( k_0/s \)) while its counterpart quantities. It follows that

\[ \frac{P_\nu}{s^2 Z_\nu} = 1 + \left( 1 - \frac{k^2}{s^2} \right) \frac{Z_\nu}{Z_\nu}, \tag{5.6} \]

where \( k^2 = P_\nu/Z_\nu \approx k_0^2 \). In the Kraichnan picture the ratio \( P_\nu/s^2 Z_\nu \approx k_0^2/s^2 \) diverges in the limit of infinite Reynolds number. Hence the ratio \( Z_\nu/Z_\nu \) diverges in a similar manner. Thus the Kraichnan energy condensate is also an enstrophy condensate.
on the right-hand side is $\approx \ln(k_\nu/s)$, which diverges as $k_\nu/s \to \infty$. Finally, the classical $k^{-3}$ direct-cascade range requires a $k^{-5}$ inverse-cascade range if only power-law spectra are permitted. The enstrophy content of the Kraichnan condensate would be equivalent to the enstrophy content of a $k^{-5}$ energy range.

6. Conclusion and discussion

In this paper the inverse energy transfer in 2D NS turbulence is studied. The main result obtained is an upper bound for the inverse energy flux. This upper bound is then estimated using power-law spectra. It is shown that a steady inverse energy flux requires that the energy spectrum of the inverse-transfer range be at least as steep as $k^{-5/3}$: a shallower spectrum is unable to support an inverse energy cascade, and the inverse energy flux necessarily diminishes as it proceeds toward sufficiently low wavenumbers. This result provides for the first time a rigorous basis for the Kolmogorov-Kraichnan $k^{-5/3}$ energy inertial range in 2D turbulence. Other results include estimates of the Kolmogorov constant, a relation between the enstrophy and palinstrophy distribution for the Kraichnan condensate, and an analysis of power-law scaling in contrast to the Kraichnan conjecture.

The inverse energy flux across a wavenumber $\ell$ in the energy range is found to depend on the system’s energy and on the enstrophy content of the wavenumber region lower than $\ell$. The former is supposed to grow without bound in the limit of unbounded domain, while the latter becomes progressively smaller for progressively lower $\ell$. These two effects counterbalance each other in such a way that the classical inverse energy cascade necessarily requires that the energy spectrum of the inverse-transfer range be at least as steep as the Kolmogorov–Kraichnan $k^{-5/3}$ spectrum.

The derived upper bound for the inverse energy flux has been used to deduce a lower bound for the Kolmogorov constant $C$ in the classical energy spectrum $E(k) = C\varepsilon^{2/3}k^{-5/3}$. This bound is of order unity and is expected to hold whether or not $C$ is a universal constant.

We have elaborated on the Kraichnan conjecture of energy condensation. A constraint on the enstrophy and palinstrophy distribution for this picture has been derived and contrasted with those for power-law scalings. For power-law spectra, a $k^{-3}$ energy range and a $k^{-5}$ enstrophy range are consistent with the forced-dissipative balance of energy and enstrophy. This new picture has some justification from the numerical results of Borue (1994) [Tran (2004)] and Tran & Bowman (2004) at least for moderate Reynolds numbers. A Kraichnan-type condensate is required for energy-range spectra shallower than $k^{-3}$ even if the enstrophy-range spectrum remains as steep as $k^{-5}$. The energy and enstrophy level of the condensate would dramatically increase should both the energy range be shallower than $k^{-3}$ and the enstrophy range be shallower than $k^{-5}$. The classical $k^{-3}$ enstrophy range would require an excessively high concentration of energy and enstrophy at the condensate; the enstrophy content of the condensate would be equivalent to that of a $k^{-5}$ energy range.

Several authors have considered steady bounded turbulence and, on the basis of the balance equation (5.3), ruled out the existence of a direct enstrophy cascade (see Constantin et al. 1994; Eyink 1996, p. 110; Tran & Shepherd 2002; Kuksin 2004). These works either explicitly or implicitly exclude a priori the Kraichnan conjecture in favour of power-law spectra. In particular Constantin et al. (1994) assume the power-law scaling described by (5.7) with the classical exponents $\alpha = 5/3$ and $\beta = 3$, and find that this spectrum is incompatible with the balance $P = \varepsilon^2Z$. However, the two-range spectrum of KLB is supposed to be only quasisteady, with the inverse energy cascade proceeding
toward ever-lower wavenumbers, carrying with it virtually all the energy injection. In other words, the KLB spectrum is supposed to correspond to \( \frac{dE}{dt} = -2\nu Z + \epsilon \approx \epsilon \), or equivalently, to \( P \gg s^2Z \), not to the balance \( P = s^2Z \), which is achieved only in a steady state. The quasisteady limit as \( t \to \infty \) is clearly incompatible with a finite domain. Eyink (1996) and Tran & Shepherd (2002) infer from (5.3) that the dissipation of enstrophy occurs mainly in the vicinity of the wavenumber given by \( \sqrt{P/Z} \), which is just the forcing wavenumber \( s \) for steady turbulence, thereby ruling out the existence of a direct enstrophy cascade. This conclusion is valid for moderate ratios \( s/k_0 \), i.e., for relatively narrow energy ranges, and for various ‘regular’ spectra such as ones with the energy range shallower than \( k^{-3} \). However if the Kraichnan conjecture holds, then this conclusion would not be valid. Kuksin (2004) employs a special forcing, such that both the energy and enstrophy injections are proportional to the viscosity coefficient \( \nu \), to show that in the limit of infinite Reynolds number the statistical equilibrium palinstrophy remains bounded. This result leads Kuksin to conclude that the KLB theory does not apply to the statistically steady dynamics of bounded NS turbulence in a doubly periodic domain. Similar to Eyink (1996) and Tran & Shepherd (2002), the balance \( P = s^2Z \) is also obtained, but no further constraints related to the Kraichnan scenario are derived. The fact that \( P \) remains bounded in the limit of infinite Reynolds number is due to the special choice of the forcing. In essence this result is equivalent to that of Eyink (1996) and Tran & Shepherd (2002) except by a scaling factor of the inverse of the Reynolds number. Hence the above analyses apply to this case as well. In particular the Kraichnan conjecture of energy condensation cannot be logically excluded. The present work emphasizes the need for some treatment beyond the forced-dissipative balances of the energy and enstrophy, before such an exclusion can be fully justified.

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