Further results on $A$-numerical radius inequalities

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Abstract

Let $\mathcal{H}$ be a complex Hilbert space, and $A$ be a positive bounded linear operator on $\mathcal{H}$. Let $\mathcal{B}_A(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$ whose $A$-adjoint exists. Let $A$ denotes a diagonal operator matrix with diagonal entries are $A$. In this paper, we prove a few new $A$-numerical radius inequalities for $2 \times 2$ and $n \times n$ operator matrices. We also provide some new proofs of the existing results by relaxing different sufficient conditions like “$A$ is strictly positive” and “$\mathcal{N}(A)^\perp$ is invariant subspace for different operators”. Our proofs show the importance of the theory of the Moore-Penrose inverse of bounded operators in this field of study.

Keywords: $A$-numerical radius; Positive operator; Semi-inner product; Inequality; Operator matrix

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. Let $\| \cdot \|$ be the norm induced from $\langle \cdot, \cdot \rangle$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called selfadjoint if $A = A^*$, where $A^*$ denotes the adjoint of $A$. A selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and is called strictly positive if $\langle Ax, x \rangle > 0$ for all non-zero $x \in \mathcal{H}$. We denote a positive (strictly positive) operator $A$ by $A \geq 0$ ($A > 0$). We denote $\mathcal{R}(A)$ as the range space of $A$ and $\overline{\mathcal{R}(A)}$ as the norm closure of $\mathcal{R}(A)$ in $\mathcal{H}$. Let $A$ be a $n \times n$ diagonal operator matrix whose diagonal entries are positive operator $A$ for $n = 1, 2, \ldots$. Then $A \in \mathcal{B}(\bigoplus_{i=1}^{n} \mathcal{H})$ and $A \geq 0$. If $A \geq 0$, then it induces a positive semidefinite sesquilinear form, $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$, $x, y \in \mathcal{H}$. Let $\| \cdot \|_A$ denote the seminorm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle_A$, i.e., $\| x \|_A = \sqrt{\langle x, x \rangle_A}$ for all $x \in \mathcal{H}$. Then $\| x \|_A$ is a norm if and only if $A > 0$. Also, $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed.
in \( \mathcal{H} \). Here onward, we fix \( A \) and \( \Phi \) for positive operators on \( \mathcal{H} \) and \( \Phi_{\text{in}} \), respectively. We also reserve the notation \( I \) and \( O \) for the identity operator and the null operator on \( \mathcal{H} \) in this paper. \( \| T \|_A \) denotes the \( A \)-operator seminorm of \( T \in \mathcal{B}(\mathcal{H}) \). This is defined as follows:

\[
\| T \|_A = \sup_{x \in \mathcal{R}(A), \ x \neq 0} \frac{\| Tx \|_A}{\| x \|_A} = \inf \left\{ c > 0 : \| Tx \|_A \leq c \| x \|_A, 0 \neq x \in \mathcal{R}(A) \right\} < \infty.
\]

Let

\[
\mathcal{B}^A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \| T \|_A < \infty \}.
\]

Then \( \mathcal{B}^A(\mathcal{H}) \) is not a subalgebra of \( \mathcal{B}(\mathcal{H}) \), and \( \| T \|_A = 0 \) if and only if \( ATA = O \). For \( T \in \mathcal{B}^A(\mathcal{H}) \), we have

\[
\| T \|_A = \sup \{ (Tx, y)_A : x, y \in \mathcal{R}(A), \| x \|_A = \| y \|_A = 1 \}.
\]

If \( AT \geq 0 \), then the operator \( T \) is called \( A \)-positive. Note that if \( T \) is \( A \)-positive, then

\[
\| T \|_A = \sup \{ (Tx, x)_A : x \in \mathcal{H}, \| x \|_A = 1 \}.
\]

An operator \( X \in \mathcal{B}(\mathcal{H}) \) is called an \( A \)-adjoint operator of \( T \in \mathcal{B}(\mathcal{H}) \) if \( (Tx, y)_A = (x, Xy)_A \) for every \( x, y \in \mathcal{H} \), i.e., \( AX = T^*A \). By Douglas Theorem [6], the existence of an \( A \)-adjoint operator is not guaranteed. An operator \( T \in \mathcal{B}(\mathcal{H}) \) may admit none, one or many \( A \)-adjoints. \( A \)-adjoint of an operator \( T \in \mathcal{B}(\mathcal{H}) \) exists if and only if \( \mathcal{R}(T^*A) \subseteq \mathcal{R}(A) \). Let us now denote

\[
\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A) \}.
\]

Note that \( \mathcal{B}_A(\mathcal{H}) \) is a subalgebra of \( \mathcal{B}(\mathcal{H}) \) which is neither closed nor dense in \( \mathcal{B}(\mathcal{H}) \). Moreover, the following inclusions

\[
\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}^A(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})
\]

hold with equality if \( A \) is injective and has a closed range.

In 2012, Saddi [19] introduced \( A \)-numerical radius of \( T \) for \( T \in \mathcal{B}(\mathcal{H}) \), which is denoted as \( w_A(T) \), and is defined as follows:

\[
w_A(T) = \sup \{ |(Tx, x)_A| : x \in \mathcal{H}, \| x \|_A = 1 \}.
\] (1.1)

In 2019, Zamani [20] established the following \( A \)-numerical radius inequality for \( T \in \mathcal{B}_A(\mathcal{H}) \):

\[
\frac{1}{2} \| T \|_A \leq w_A(T) \leq \| T \|_A.
\] (1.2)

The \( A \)-Crawford number of \( T \in \mathcal{B}_A(\mathcal{H}) \) is is defined as

\[
m_A(T) = \inf \{ |(Tx, x)_A| : x \in \mathcal{H}, \| x \|_A = 1 \}.
\]
Furthermore, if $T$ is $A$-selfadjoint, then $w_A(T) = \|T\|_A$. In 2019, Moslehian et al. \cite{moslehian2019} again continued the study of $A$-numerical radius and established some inequalities for $A$-numerical radius. In 2020, Bhunia et al. \cite{bhunia2020} and \cite{bhunia2020b} obtained several $A$-numerical radius inequalities for strictly positive operator $A$. Feki \cite{feki2019} and Feki et al. \cite{feki2019b} obtained several $A$-numerical radius inequalities under the assumption $N(A) \perp$ is invariant subspace for different operators. Further generalizations and refinements of $A$-numerical radius are discussed in \cite{feki2019c, feki2019d}.

The objective of this paper is to present a few new $A$-numerical radius inequalities for $n \times n$ and $2 \times 2$ operator matrices. Besides this, we also aim to establish some existing $A$-numerical radius inequalities without using the condition $A > 0$ and $N(A)^\perp$ is invariant subspace for different operators. To this end, the paper is sectioned as follows. In Section 2, we define additional mathematical constructs including the definition of the Moore-Penrose inverse of an operator, $A$-adjoint, $A$-selfadjoint and $A$-unitary operator, that are required to state and prove the results in the subsequent sections. Section 3 contains several new $A$-numerical radius inequalities. More interestingly, it also provides new proof to the very recent existing results in the literature on $A$-numerical radius inequalities by dropping a few sufficient conditions.

2. Preliminaries

This section gathers a few more definitions and results that are useful in proving our main results. It starts with the definition of the Moore-Penrose inverse of a bounded operator $A$ in $H$. The Moore-Penrose inverse of $A \in \mathcal{B}(\mathcal{H}) \cite{moore1920}$ is the operator $X : R(A) \oplus R(A)^\perp \rightarrow \mathcal{H}$ which satisfies the following four equations:

1. $AXA = A$,  
2. $XAX = X$,  
3. $XA = P_{N(A)^\perp}$,  
4. $AX = P_{R(A)\cap R(A)^\perp}$.

Here $N(A)$ and $P_L$ denote the null space of $A$ and the orthogonal projection onto $L$ respectively. The Moore-Penrose inverse is unique, and is denoted by $A^\dagger$. In general, $A^\dagger \notin \mathcal{B}(\mathcal{H})$.

It is bounded if and only if $R(A)$ is closed. If $A \in \mathcal{B}(\mathcal{H})$ is invertible, then $A^\dagger = A^{-1}$. If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished $A$-adjoint operator of $T$, which is denoted by $T^{\#A}$ (see \cite{moore1920, feki2019}). Note that $T^{\#A} = A^\dagger T^*A$. If $T \in \mathcal{B}_A(\mathcal{H})$, then $AT^{\#A} = T^*A$, $\mathcal{R}(T^{\#A}) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(T^{\#A}) = \mathcal{N}(T^*A)$ (see \cite{feki2019}). We can observe that

\begin{equation}
I^{\#A} = A^\dagger I^*A = A^\dagger A = P_{R(A)} \quad (\because N(A)^\perp = R(A^*)) \label{eq:2.1},
\end{equation}

\begin{equation}
T^{\#A}P_{R(A)} = A^\dagger T^*AA^\dagger A = A^\dagger T^*A = T^{\#A}, \label{eq:2.2}
\end{equation}

and

\begin{equation}
P_{R(A)}T^{\#A} = A^\dagger AA^\dagger T^*A = A^\dagger T^*A = T^{\#A}. \label{eq:2.3}
\end{equation}
An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be \( A\)-\emph{selfadjoint} if \( AT \) is selfadjoint, i.e., \( AT = T^*A \). Observe that if \( T \) is \( A\)-selfadjoint, then \( T \in \mathcal{B}_A(\mathcal{H}) \). However, in general, \( T \neq T^\#_A \). But, \( T = T^\#_A \) if and only if \( T \) is \( A\)-selfadjoint and \( \mathcal{R}(T) \subseteq \mathcal{R}(A) \). If \( T \in \mathcal{B}_A(\mathcal{H}) \), then \( T^\#_A \in \mathcal{B}_A(\mathcal{H}) \), \( (T^\#_A)^\#_A = P_{\mathcal{R}(A)}TP_{\mathcal{R}(A)} \), and \( ((T^\#_A)^\#_A)^\#_A = T^\#_A \). Also, \( T^\#_A T \) and \( TT^\#_A \) are \( A\)-positive operators, and
\[
\|T^\#_A T\|_A = \|TT^\#_A\|_A = \|T\|^2_A = \|T^\#_A\|^2_A.
\]

(2.4)

For any \( T_1, T_2 \in \mathcal{B}_A(\mathcal{H}) \), we have
\[
\|T_1^\#_A T_2\|_A = \sup\{\langle(T_1^\#_A T_2 x, y)\rangle : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}
= \sup\{\langle(T_2 x, T_1 y)\rangle : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}
= \sup\{\langle(x, T_2^\#_A T_1 y)\rangle : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}
= \sup\{\langle(T_2^\#_A T_1 x, y)\rangle : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}
= \|T_2^\#_A T_1\|_A.
\]

(2.5)

This fact is same as Lemma 2.8 of [9]. However, the above proof is a very simple one and directly follows using the definition of \( A\)-norm. An operator \( U \in \mathcal{B}_A(\mathcal{H}) \) is said to be \( A\)-\emph{unitary} if \( \|U x\|_A = \|U^\#_A x\|_A = \|x\|_A \) for all \( x \in \mathcal{H} \). If \( T \in \mathcal{B}_A(\mathcal{H}) \) and \( U \) is \( A\)-unitary, then \( w_A(U^\#_A T U) = w_A(T) \). For \( T, S \in \mathcal{B}_A(\mathcal{H}) \), we have \( (TS)^\#_A = S^\#_A T^\#_A \), \( (T + S)^\#_A = T^\#_A + S^\#_A \), \( \|T S\|_A \leq \|T\|_A \|S\|_A \) and \( \|T x\|_A \leq \|T\|_A \|x\|_A \) for all \( x \in \mathcal{H} \). The real and imaginary part of an operator \( T \in \mathcal{B}_A(\mathcal{H}) \) as \( Re_A(T) = \frac{T + T^*}{2} \) and \( Im_A(T) = \frac{T - T^*}{2i} \). An interested reader may refer [1, 2] for further properties of operators on Semi-Hilbertian space. From (1.11), it follows that
\[
w_A(T) = w_A(T^\#_A)
\]
for any \( T \in \mathcal{B}_A(\mathcal{H}) \).

(2.6)

Some interesting results are collected hereunder for further use.

**Lemma 2.1.** (Lemma 3.1, [3])
Let \( T_{ij} \in \mathcal{B}_A(\mathcal{H}) \) for \( 1 \leq i, j \leq n \). Then
\[
T = \begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{21} & T_{22} & \cdots & T_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1} & T_{n2} & \cdots & T_{nn}
\end{bmatrix} \in \mathcal{B}_A(\mathcal{H}) \text{ and } T^\#_A = \begin{bmatrix}
T_{11}^\#_A & T_{12}^\#_A & \cdots & T_{1n}^\#_A \\
T_{21}^\#_A & T_{22}^\#_A & \cdots & T_{2n}^\#_A \\
\vdots & \vdots & \ddots & \vdots \\
T_{n1}^\#_A & T_{n2}^\#_A & \cdots & T_{nn}^\#_A
\end{bmatrix}.
\]

The next result is a combination of Lemma 2.4 (i) [4] and Lemma 2.2 [16].

**Lemma 2.2.** Let \( T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H}) \). Then
(i) \( \max \{ w_A(T_1), w_A(T_4) \} = w_A \begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix} \leq w_A \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \).

(ii) \( w_A \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \leq w_A \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \).

The other parts of Lemma 2.4 [4] assumes the condition \( A \) is strictly positive. Rout et al. [16] proved the same result for positive \( A \), and the same is stated below.

**Lemma 2.3.** [Lemma 2.4, [16]]

Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

(i) \( w_A \begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix} = w_A \begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix} \).

(ii) \( w_A \begin{bmatrix} O & e^{i\theta}T_2 \\ e^{-i\theta}T_2 & O \end{bmatrix} \) for any \( \theta \in \mathbb{R} \).

(iii) \( w_A \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} = \max \{ w_A(T_1+T_2), w_A(T_1-T_2) \} \). In particular, \( w_A \begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix} = w_A(T_2) \).

The next result establishes upper and lower bounds for the \( A \)-numerical radius of a particular type of \( 2 \times 2 \) operator matrix that is a generalization of [1.2].

**Lemma 2.4.** [Theorem 2.6, [16]]

Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

\[
\max \{ w_A(T_1), w_A(T_2) \} \leq w_A \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix} \leq w_A(T_1) + w_A(T_2). \tag{2.7}
\]

**Lemma 2.5.** [Lemma 2.8, [16]]

Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

\[
w_A \begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix} = \max \{ w_A(T_1 + iT_2), w_A(T_1 - iT_2) \}.
\]

Theorem 2.4 [8] for operators \( T_1, T_2 \in B_A(\mathcal{H}) \) is stated as follows.

**Lemma 2.6.** Let \( T_1, T_2 \in B_A(\mathcal{H}) \). Then

\[ w_A(T_1T_2) \leq 4w_A(T_1)w_A(T_2). \]

If \( T_1T_2 = T_2T_1 \), then

\[ w_A(T_1T_2) \leq 2w_A(T_1)w_A(T_2). \]

**Lemma 2.7.** [Theorem 2.6, [8]]

Let \( T, S \in B_A(\mathcal{H}) \). Then

\[ w_A(TS + ST^\#A) \leq 2\|T\|_A w_A(S). \]
3. Main Results

It is well known that $P_{R(A)}^T \neq TP_{R(A)}$ for any $T \in B_A(H)$ (even if $A$ and $T$ are finite matrices). And the equality holds if $N(A)$ is invariant for $T$. The first result shows that the $A$-numerical radius of $P_{R(A)}^T$ and $TP_{R(A)}$ are same for any $T \in B_A(H)$.

**Theorem 3.1.** $w_A(P_{R(A)}^T) = w_A(T P_{R(A)}) = w_A(T)$ for any $T \in B_A(H)$.

**Proof.**

$$w_A(P_{R(A)}^T) = w_A((P_{R(A)}^T)^\#_A) \quad (: w_A(T) = w_A(T^\#_A))$$
$$= w_A(T^\#_A P_{R(A)}) \quad (: (TS)^\#_A = S^\#_A T^\#_A & (P_{R(A)})^\#_A = P_{R(A)})$$
$$= w_A(T^\#_A) \quad \text{by (2.2)}$$
$$= w_A(T). \quad (3.1)$$

Again,

$$w_A(T P_{R(A)}) = w_A((TP_{R(A)})^\#_A) \quad (: w_A(T) = w_A(T^\#_A))$$
$$= w_A(P_{R(A)}^T) \quad (: (TS)^\#_A = S^\#_A T^\#_A & (P_{R(A)})^\#_A = P_{R(A)})$$
$$= w_A(T^\#_A) \quad \text{by (2.3)}$$
$$= w_A(T). \quad (3.2)$$

We therefore have

$$w_A(P_{R(A)}^T) = w_A(T P_{R(A)}) = w_A(T).$$

\[\square\]

We demonstrate an interesting property of $A$-numerical radius of an $n \times n$ operator matrix which is a generalization of Lemma 2.1 [18].

**Theorem 3.2.** Let $T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$, where $T_{ij} \in B_A(H)$ for $1 \leq i, j \leq n$. Then

$$w_A\left(\begin{bmatrix} T_{11} & O & \cdots & O \\ O & T_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn} \end{bmatrix}\right) \leq w_A(T).$$
Proof. Let $z = e^{2\pi i/n}$ and $U = \begin{bmatrix} I & O & \cdots & O \\ O & zI & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I \end{bmatrix}$. It is easy to see that $\overline{z} = z^{-1} = z^{n-1}$ and $|z| = 1$. To show that $U$ is $A$-unitary, we need to prove that $\|x\|_A = \|Ux\|_A = \|U^\#A x\|_A$, for $x = (x_1, x_2, \ldots, x_n) \in \bigoplus_{i=1}^n \mathcal{H}$. Here,

$$U^\#A = \begin{bmatrix} I & O & \cdots & O \\ O & zI & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I \end{bmatrix}$$

$$= \begin{bmatrix} I^\#A & O & \cdots & O \\ O & zI^\#A & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}I^\#A \end{bmatrix}$$

by Lemma 2.1

$$= \begin{bmatrix} P_{\mathcal{R}(A)} & O & \cdots & O \\ O & zP_{\mathcal{R}(A)} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & z^{n-1}P_{\mathcal{R}(A)} \end{bmatrix}.$$

This in turn implies $UU^\#A = \begin{bmatrix} P_{\mathcal{R}(A)} & O & \cdots & O \\ O & P_{\mathcal{R}(A)} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & P_{\mathcal{R}(A)} \end{bmatrix} = U^\#A U$.

Now, for $x = (x_1, x_2, \ldots, x_n) \in \bigoplus_{i=1}^n \mathcal{H}$, we have

$$\|Ux\|_A^2 = \langle Ux, Ux \rangle_A = \langle U^\#A Ux, x \rangle_A = \|x\|_A^2.$$

So, $\|Ux\|_A = \|x\|_A$. Similarly, $\|U^\#A x\|_A = \|x\|_A$. Thus, $U$ is an $A$-unitary operator. Further, a simple calculation shows that

$$\begin{bmatrix} T_{11}^\#A & O & \cdots & O \\ O & T_{22}^\#A & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & T_{nn}^\#A \end{bmatrix} = \frac{1}{n} \sum_{k=0}^{n-1} U^\#A U^k.$$
Lemma 3.3. Let \( T \in \mathcal{B}_A(\mathcal{H}) \) and \( T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mn} \end{bmatrix} \). Then
\[
w_A(T) \leq \frac{1}{n} \sum_{k=0}^{n-1} w_A(U^{\#A} T^k U^{\#A}) = \frac{1}{n} \sum_{k=0}^{n-1} w_A(T^k).
\]

This implies that
\[
w_A \left( \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ 0 & T_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{nn} \end{bmatrix} \right)^{\#A} = w_A \left( \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ 0 & T_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{nn} \end{bmatrix} \right) \leq w_A(T).
\]

\( \square \)

The following lemma provides an upper bound for \( T \in \mathcal{B}_A(\mathcal{H}) \) to prove Theorem 3.4

Lemma 3.3 (Theorem 7, [1]). Let \( T \in \mathcal{B}_A(\mathcal{H}) \). Then
\[
w_A(T) \leq \frac{1}{2}(\|T\| + \|T^2\|^{1/2}).
\]

Theorem 3.4. Let \( T_1, T_2, T_3, T_4 \in \mathcal{B}_A(\mathcal{H}) \) and \( T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \). Then
\[
\max\{w_A^{1/2}(T_2 T_3), w_A^{1/2}(T_3 T_2)\} \leq \sqrt{2} w_A \left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right) \leq \frac{1}{\sqrt{2}} (\|T\|_A + \|T^2\|^{1/2}).
\]

Proof. Let \( U = \begin{bmatrix} I & O \\ O & -I \end{bmatrix} \). It is easy to see that \( U \) is \( A \)-unitary and \( TU - UT = 2 \begin{bmatrix} O & -T_2 \\ T_3 & O \end{bmatrix} \).

Here,
\[
w_A(TU + UT) = w_A(U^{\#A} T^{\#A} + T^{\#A} U^{\#A}) = w_A(U^{\#A} T^{\#A} + T^{\#A} (U^{\#A})^{\#A}) \leq 2 w_A(T^{\#A}) \|U^{\#A}\|_A \text{ by Lemma 2.7} \leq 2 w_A(T) \leq \|T\|_A + \|T^2\|^{1/2} \text{ by Lemma 3.3} \tag{3.3}
\]
By (3.3), we thus have
\[ \max\{w_A(T_1), w_A(T_4)\} = \frac{1}{2} w_A(TU + UT) \leq w_A(T). \]

Again,
\[
\max\{w_A(T_2T_3), w_A(T_3T_2)\} = w_A\left( \begin{bmatrix} T_2T_3 & O \\ O & T_3T_2 \end{bmatrix} \right)
\]
\[
= w_A\left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right)
\]
\[
= w_A\left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right)^2 
\]
\[
\leq 2w_A^2\left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right) \text{ by Lemma 2.6.}
\]

Replacing \( T_2 \) by \( -T_2 \), we get
\[
\max\{w_A(T_2T_3), w_A(T_3T_2)\} \leq 2w_A^2\left( \begin{bmatrix} O & -T_2 \\ T_3 & O \end{bmatrix} \right) .
\]

This implies
\[
\frac{1}{\sqrt{2}} \max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq w_A\left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right)
\]
\[
= w_A\left( \begin{bmatrix} O & -T_2 \\ T_3 & O \end{bmatrix} \right)
\]
\[
= \frac{1}{2} w_A(TU - UT)
\]
\[
\leq \frac{1}{2} (\|T\|_A + \|T^2\|^{1/2}) \text{ by (3.3).}
\]

Thus, we obtain
\[
\max\{w_A^{1/2}(T_2T_3), w_A^{1/2}(T_3T_2)\} \leq \sqrt{2} w_A\left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right) \leq \frac{1}{\sqrt{2}} (\|T\|_A + \|T^2\|^{1/2}).
\]

The next result provides an estimate for \( A \)-operator norms of certain \( 2 \times 2 \) operator matrices. 3.7.9.
Theorem 3.5. Let $T \in \mathcal{B}_A(\mathcal{H})$ and $a, b \in \mathbb{C}$. Then

$$\left\| \begin{bmatrix} aI & T \\ O & bI \end{bmatrix} \right\|_A = \frac{1}{\sqrt{2}} \sqrt{|a|^2 + |b|^2 + \|T\|_A^2 + \sqrt{(|a|^2 + |b|^2 + \|T\|_A^2)^2 - 4|a|^2|b|^2}}.$$ 

Proof. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$ and

$$\left\| \begin{bmatrix} |a| & \|T\|_A \\ O & |b| \end{bmatrix} \right\|_A = \left\| \begin{bmatrix} |a| & \|T\|_A \\ O & |b| \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|_A$$

$$= \left\| \begin{bmatrix} |a| + \|T\|_A \alpha \\ |b| \beta \end{bmatrix} \right\|_A$$

$$= \sqrt{|b|^2 \beta^2 + (|a| + \|T\|_A \beta)^2}.$$ (3.4)

Let $x_n, y_n \in \mathcal{H}$ be two unit vectors in $\mathcal{H}$ such that $\lim_{n \to \infty} |\langle T y_n, x_n \rangle| = \|T\|_A$ for $n \in \mathbb{N}$. Let $\beta_n \in \mathbb{R}$ be such that $\overline{a(T y_n, x_n)} = e^{i\beta_n} |a| \langle T y_n, x_n \rangle_A$. Suppose that $[\begin{bmatrix} \alpha e^{i\beta_n} x_n \\ \beta y_n \end{bmatrix}]$ be a sequence in $\mathcal{H} \oplus \mathcal{H}$. We can see that $\left\| \begin{bmatrix} \alpha e^{i\beta_n} x_n \\ \beta y_n \end{bmatrix} \right\|_A = 1$. Now,

$$\left\| \begin{bmatrix} aI & T \\ O & bI \end{bmatrix} \right\|_A \geq \left\| \begin{bmatrix} aI & T \\ O & bI \end{bmatrix} \begin{bmatrix} \alpha e^{i\beta_n} x_n \\ \beta y_n \end{bmatrix} \right\|_A$$

$$= \left\| \begin{bmatrix} \alpha a e^{i\beta_n} x_n + \beta T y_n \\ \beta b y_n \end{bmatrix} \right\|_A$$

$$= \sqrt{|\alpha a e^{i\beta_n} x_n + \beta T y_n|^2_A + \|\beta b y_n\|^2_A}$$

$$= \sqrt{\alpha^2 |a|^2 + \beta^2 \|T y_n\|^2_A + 2\alpha \beta Re(\overline{a(T y_n, x_n)} A) + \beta^2 |b|^2}$$

$$= \sqrt{\alpha^2 |a|^2 + \beta^2 \|T\|_A^2 + 2\alpha \beta Re(\overline{a(T y_n, x_n)} A) + \beta^2 |b|^2}$$

$$= \left\| \begin{bmatrix} |a| & \|T\|_A \\ O & |b| \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|_A \text{ by (3.4)}$$

$$= \left\| \begin{bmatrix} |a| & \|T\|_A \\ O & |b| \end{bmatrix} \right\|_A.$$ (3.5)

Again, by Lemma 2.1 [11]

$$\left\| \begin{bmatrix} aI & T \\ O & bI \end{bmatrix} \right\|_A \leq \left\| \begin{bmatrix} |a| & \|T\|_A \\ O & |b| \end{bmatrix} \right\|_A$$ (3.6)
From (3.5) and (3.6), we so have
\[ \left\| \begin{bmatrix} aI & T \\ O & bI \end{bmatrix} \right\|_H = \left\| \begin{bmatrix} a & \|T\|_A \\ O & b \end{bmatrix} \right\|. \]

But
\[ \left\| \begin{bmatrix} a & \|T\|_A \\ O & b \end{bmatrix} \right\| = r^{1/2} \left( \begin{bmatrix} a & O \\ \|T\|_A & b \end{bmatrix} \begin{bmatrix} a & \|T\|_A \\ O & b \end{bmatrix} \right) \]
\[ = r^{1/2} \left( \begin{bmatrix} |a|^2 \\ |a|\|T\|_A \\ |b|^2 + \|T\|^2_A \end{bmatrix} \right) \]
\[ = \frac{1}{\sqrt{2}} \sqrt{|a|^2 + |b|^2 + \|T\|^2_A + \sqrt{(|a|^2 + |b|^2 + \|T\|^2_A)^2 - 4|a|^2|b|^2}}. \]

Thence,
\[ \left\| \begin{bmatrix} aI & T \\ O & bI \end{bmatrix} \right\|_H = \frac{1}{\sqrt{2}} \sqrt{|a|^2 + |b|^2 + \|T\|^2_A + \sqrt{(|a|^2 + |b|^2 + \|T\|^2_A)^2 - 4|a|^2|b|^2}}. \]

We recall below a result of \[8\] to obtain Corollary 3.7.

**Lemma 3.6.** [Corollary 2.1, \[8\]]

Let \(T \in B_A(\mathcal{H})\). Then
\[ \frac{1}{2} \sqrt{\|TT^\# + T^\#T\|_A + 2m_A(T^2)} \leq w_A(T) \leq \frac{1}{2} \sqrt{\|TT^\# + T^\#T\|_A + 2w_A(T^2)}. \]

Feki \[8\] proved the following result with the additional assumption “\(N(A)\) is invariant for \(T \in B_A(\mathcal{H})\).” Next, we prove the same result without this assumption.

**Corollary 3.7.** \(2w_A \left( \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right) = \left\| \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right\|_H + \left\| \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right\|^{-1}_H \) for any \(T \in B_A(\mathcal{H})\).

**Proof.** Let \(T = \begin{bmatrix} I & T \\ O & -I \end{bmatrix}\). Then \(T^2 = \begin{bmatrix} I & O \\ O & I \end{bmatrix}\). Using Lemma 3.6, we get
\[ w'_A(T) = \frac{1}{2} \sqrt{\|TT^\# + T^\#T\|_A + 2}. \]

(3.7)
From (3.7), we now have

\[
\begin{align*}
w_A(T) & = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \begin{bmatrix} I & T \\ O & -I \end{bmatrix}^\#_A + \begin{bmatrix} I & T \\ O & -I \end{bmatrix}^\#_A \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \begin{bmatrix} P_{\mathcal{R}(A)} & O \\ T^\#_A & -P_{\mathcal{R}(A)} \end{bmatrix} + \begin{bmatrix} P_{\mathcal{R}(A)} & O \\ T^\#_A & -P_{\mathcal{R}(A)} \end{bmatrix} \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} P_{\mathcal{R}(A)} + TT^\#_A & -TP_{\mathcal{R}(A)} \\ -T^\#_A & P_{\mathcal{R}(A)} \end{bmatrix} + \begin{bmatrix} P_{\mathcal{R}(A)} & P_{\mathcal{R}(A)} T \\ T^\#_A T & P_{\mathcal{R}(A)} \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} 2P_{\mathcal{R}(A)} + TT^\#_A & -TP_{\mathcal{R}(A)} \\ -T^\#_A + T^\#_A & 2P_{\mathcal{R}(A)} + T^\#_A T \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} 2P_{\mathcal{R}(A)} + TT^\#_A & -TP_{\mathcal{R}(A)} \\ O & 2P_{\mathcal{R}(A)} + T^\#_A T \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} 2P_{\mathcal{R}(A)} + TT^\#_A & (T^\#_A)^\#_A T^\#_A \\ -T^\#_A + T^\#_A & 2P_{\mathcal{R}(A)} + T^\#_A (T^\#_A)^\#_A \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} (T^\#_A)^\#_A T^\#_A & O \\ -T^\#_A + T^\#_A & 2P_{\mathcal{R}(A)} + T^\#_A (T^\#_A)^\#_A \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} 2I^\#_A + (T^\#_A)^\#_A T^\#_A & O \\ O & 2I^\#_A + T^\#_A (T^\#_A)^\#_A \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \sqrt{ \left\| \begin{bmatrix} 2I + TT^\#_A & O \\ O & 2I + T^\#_A T \end{bmatrix} \right\|_A + 2 } \\
& = \frac{1}{2} \max\{\|2I + TT^\#_A\|_A + 2\}^{1/2}, (\|2I + T^\#_A T\|_A + 2)^{1/2}\} \\
& = \frac{1}{2} (\|2I + TT^\#_A\|_A + 2)^{1/2} \\
& = \frac{1}{2} \sqrt{\|T\|_A^2 + 4}.
\end{align*}
\]
So, we get
\[ w_A \left( \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right) = \frac{1}{2} \sqrt{\|T\|_A^2 + 4}. \] (3.8)

Using Theorem 3.5, we also obtain
\[ \left\| \begin{bmatrix} I & T \\ O & -I \end{bmatrix} \right\|_A^2 = \frac{1}{2} \left( 2 + \|T\|_A^2 + \sqrt{\|T\|_A^2 + 4 \|T\|_A^2} \right) = \frac{1}{2} \|T\|_A + \frac{1}{2} \sqrt{\|T\|_A^2 + 4}. \] (3.9)

Hence, we arrive at our claim by (3.8) and (3.9).

\[ \square \]

**Remark 3.8.** Using Theorem 3.5, one can establish Corollary 2.2 [8] without the assumption "\( \mathcal{N}(A)^\perp \) is invariant for \( T \)."

Following theorem provides a relation between \( A \)-numerical radius of two diagonal operator matrices, where \( \text{diag}(T_1, \ldots, T_n) \) means an \( n \times n \) diagonal operator matrix with entries \( T_1, \ldots, T_n \).

**Theorem 3.9.** Let \( T_i \in \mathcal{B}_A(\mathcal{H}) \) for \( 1 \leq i \leq n \). Then
\[ w_A(\text{diag}(\sum_{i=1}^n T_i)) \leq nw_A(\text{diag}(T_1, \ldots, T_n)). \]

**Proof.** Here,
\[
w_A(\text{diag}(\sum_{i=1}^n T_i)) = w_A(\sum_{i=1}^n T_i) \quad \text{by Lemma 2.2} \\
\leq \sum_{i=1}^n w_A(T_i) \\
\leq n \max\{w_A(T_i) : 1 \leq i \leq n\} \\
= nw_A(\text{diag}(T_1, \ldots, T_n)).
\]

\[ \square \]

We generalize some of the results of [12] now. Using Lemma 2.4 [16], one can now prove Corollary 3.3 [4] without assuming the condition \( A > 0 \), and is stated next.

**Lemma 3.10.** Let \( T, S, X, Y \in \mathcal{B}_A(\mathcal{H}) \). Then
\[
w_A(TXS^#A \pm SYT^#A) \leq 2\|T\|_A\|S\|_A w_A \begin{bmatrix} O & X \\ Y & O \end{bmatrix}.
\]

In particular, putting \( Y = X \)
\[
w_A(TXS^#A \pm SXT^#A) \leq 2\|T\|_A\|S\|_A w_A(X).
\]
Considering $X = Y = Q$ and $T = I$ in the previous theorem, we get Lemma 2.7, which is stated below.

**Corollary 3.11.** Let $Q, S \in B_A(H)$. Then

$$w_A(QS^\# \pm SQ) \leq 2\|S\|_A w_A(Q).$$

Fekri and Sahoo [9] established many results on $A$-numerical radius inequalities of $2 \times 2$ operator matrices, very recently. In many cases, they assumed the condition “$\mathcal{N}(A)^{\perp}$ is invariant subspace for $T_1, T_2, T_3, T_4$” to show their claim. They assumed these conditions in order to get the equality $P_{R(A)}^T = TP_{R(A)}$ which is not true, in general. One of the objective of this paper is to achieve the same claim without assuming the additional condition “$\mathcal{N}(A)^{\perp}$ is invariant subspace for $T_1, T_2, T_3, T_4$”. The next result is in this direction, and is more general than Theorem 2.7 [9]. Our proof is also completely different than the corresponding proof in [9]. And, therefore our results are superior to those results in [9] and [8] that assumes the invariant condition.

**Theorem 3.12.** Let $T_1, T_2, T_3, T_4 \in B_A(H)$. Then $w_A \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \geq \frac{1}{2} \max\{\alpha, \beta\}$, where $\alpha = \max\{w_A(T_1 + T_2 + T_3 + T_4), w_A(T_1 + T_4 - T_2 - T_3)\}$ and $\beta = \max\{w_A(T_1 + T_4 + i(T_2 - T_3)), w_A(T_1 + T_4 - i(T_2 - T_3))\}$.

**Proof.** Let $T = \begin{bmatrix} T_1^\# & T_3^\# \\ T_2^\# & T_4^\# \end{bmatrix}$ and $Q = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$. To show that $Q$ is $A$-unitary, we need to prove that $\|x\|_A = \|Qx\|_A = \|Q^\#x\|_A$. So,

$$Q^\# = \begin{bmatrix} O & I^\# \\ I & O \end{bmatrix} \text{ by Lemma 2.1}$$

$$= \begin{bmatrix} O & P_{R(A)} \\ P_{R(A)} & O \end{bmatrix} \quad \therefore \quad \mathcal{N}(A)^{\perp} = \mathcal{R}(A^*) \& \mathcal{R}(A^*) = \mathcal{R}(A).$$

This in turn implies $QQ^\# = \begin{bmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{bmatrix} = Q^\#Q$. Now, for $x = (x_1, x_2) \in H \oplus H$, we have

$$\|Qx\|_A^2 = \left\langle Qx, Qx \right\rangle_A = \left\langle Q^\#Qx, x \right\rangle_A = \begin{bmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} AR & O \\ O & AR \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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 Hence, we have

\[ T = \begin{bmatrix} AA^\dagger A & O \\ O & AA^\dagger A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ = \begin{bmatrix} A & O \\ O & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ = \|x\|^2_A. \]

So, \( \|Qx\|_A = \|x\|_A \). Similarly, it can be proved that \( \|Q^\#A x\|_A = \|x\|_A \). Thus, \( Q \) is an \( A \)-unitary operator. By Lemma 2.7, we obtain

\[ w_A(TQ + QT^\#A) \leq 2w_A(T). \] (3.10)

So,

\[ 2w_A(T) \geq w_A\left( \begin{bmatrix} T_1 & T_3 \\ T_2 & T_4 \end{bmatrix} \begin{bmatrix} O & P_{\mathcal{R}(A)} \\ P_{\mathcal{R}(A)} & O \end{bmatrix} + \begin{bmatrix} O & I \\ I & O \end{bmatrix} \begin{bmatrix} T_1^\#A & T_3^\#A \\ T_2^\#A & T_4^\#A \end{bmatrix} \right) \]

\[ = w_A\left( \begin{bmatrix} T_1^\#A & T_3^\#A \\ T_2^\#A & T_4^\#A \end{bmatrix} \mathcal{R}(A) \end{bmatrix} + \begin{bmatrix} T_1^\#A & T_3^\#A \\ T_2^\#A & T_4^\#A \end{bmatrix} \right) \]

\[ = w_A\left( \begin{bmatrix} T_1^\#A & T_3^\#A \\ T_2^\#A & T_4^\#A \end{bmatrix} \right) \]

\[ = w_A\left( \begin{bmatrix} T_2 + T_3 & T_4 + T_1 \\ T_1 + T_4 & T_2 + T_3 \end{bmatrix} \right) = w_A\left( \begin{bmatrix} T_2 + T_3 & T_4 + T_1 \\ T_1 + T_4 & T_2 + T_3 \end{bmatrix} \right). \]

Hence, we have

\[ 2w_A\left( \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \right) \geq w_A\left( \begin{bmatrix} T_1^\#A & T_3^\#A \\ T_2^\#A & T_4^\#A \end{bmatrix} \right) \]

\[ \geq w_A\left( \begin{bmatrix} T_2 + T_3 & T_4 + T_1 \\ T_1 + T_4 & T_2 + T_3 \end{bmatrix} \right). \] (3.11)

By (3.11) and Lemma 2.3, we obtain

\[ w_A\left( \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \right) \geq \frac{1}{2} \max\{w_A(T_1 + T_2 + T_3 + T_4), w_A(T_2 + T_3 - T_4 - T_1)\}. \] (3.12)

Again, applying Lemma 2.7 and taking \( T = \begin{bmatrix} T_1^\#A & T_3^\#A \\ T_2^\#A & T_4^\#A \end{bmatrix} \) and \( Q = \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \). It is easy to verify that \( Q \) is \( A \)-unitary. We now have

\[ w_A\left( TQ^\#A + QT \right) \leq 2w_A(T). \] (3.13)
So,

\[
2w_A(T) \geq w_A \left( \begin{bmatrix} T_{1#} & T_{2#} & T_{3#} & T_{4#} \\ T_{1#} & T_{2#} & T_{3#} & T_{4#} \\ P_{R(A)} & P_{R(A)} & P_{R(A)} & P_{R(A)} \\ \end{bmatrix} \begin{bmatrix} O & -P_{R(A)} \\ O & -I \\ \end{bmatrix} \right) - \begin{bmatrix} O & I \\ \end{bmatrix} \right) = w_A \left( \begin{bmatrix} -T_{1#} + T_{2#} & -T_{3#} - T_{1#} & -T_{4#} + T_{3#} & T_{4#} \\ -T_{1#} + T_{2#} & -T_{3#} - T_{1#} & -T_{4#} + T_{3#} & T_{4#} \\ \end{bmatrix} \right) \]

By Lemma 2.5, we therefore achieve the following:

\[
w_A \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \\ \end{bmatrix} \right) \geq \frac{1}{2} \max\{w_A(T_4 + T_1 - i(T_2 - T_3)), w_A(T_4 + T_1 + i(T_2 - T_3))\}\] (3.14)

From (3.12) and (3.14), we get the desired result.

We provide below the same estimate as in Theorem 2.8 for \(A\)-numerical radius of an operator matrix that improves but by dropping the assumption \(\mathcal{N}(A)\) is an invariant subspace for \(T_1, T_2 \in B_A(\mathcal{H})\).

**Theorem 3.13.** Let \(T_1, T_2 \in B_A(\mathcal{H})\). Then

\[
w_A \left( \begin{bmatrix} T_1 & T_2 \\ O & O \\ \end{bmatrix} \right) \geq \frac{1}{2} \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.
\]

**Proof.** Suppose that \(T = \begin{bmatrix} T_{1#} & O \\ T_{2#} & O \\ \end{bmatrix}\) and \(Q = \begin{bmatrix} O & -I \\ I & O \\ \end{bmatrix}\). It then follows that \(Q\) is \(A\)-unitary.

So, \(|Q|_A = 1\). Using Lemma 2.7, we get

\[
2w_A(T) \geq w_A(TQ_{A} - QT).
\]

Now,

\[
w_A(T) \geq \frac{1}{2} w_A(TQ_{A} - QT) = \frac{1}{2} w_A \left( \begin{bmatrix} T_{1#} & O \\ T_{2#} & O \\ \end{bmatrix} \begin{bmatrix} O & P_{R(A)} \\ -P_{R(A)} & O \\ \end{bmatrix} \right) - \begin{bmatrix} O & -I \\ \end{bmatrix} \right) = \frac{1}{2} w_A \left( \begin{bmatrix} -T_{1#} + T_{2#} & -T_{3#} - T_{1#} & -T_{4#} + T_{3#} & T_{4#} \\ -T_{1#} + T_{2#} & -T_{3#} - T_{1#} & -T_{4#} + T_{3#} & T_{4#} \\ \end{bmatrix} \right).
\]
By Lemma 2.5, we thus have

\[
= \frac{1}{2} w_A \left[ \begin{array}{cc} O & T_1^\# A P_{\mathcal{F}(A)} \\ T_2^\# A P_{\mathcal{F}(A)} & O \end{array} \right] - \left[ \begin{array}{cc} -T_2^\# A & O \\ T_2^\# A & O \end{array} \right]
\]

\[
= \frac{1}{2} w_A \left[ \begin{array}{cc} T_2^\# A & T_1^\# A \\ -T_1^\# A & T_2^\# A \end{array} \right]
\]

\[
= \frac{1}{2} w_A \left[ \begin{array}{cc} T_2 & -T_1 \\ T_1 & T_2 \end{array} \right],
\]

by (2.2)

By Lemma 2.5, we thus have

\[
w_A \left( \begin{array}{cc} T_1 & T_2 \\ O & O \end{array} \right) = w_A \left( \begin{array}{cc} T_1^\# A & O \\ T_2^\# A & O \end{array} \right) \geq \frac{1}{2} \max \{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.
\]

\[
\square
\]

**Corollary 3.14.** Let \( T = P + iQ \) be the cartesian decomposition in \( B_A(\mathcal{H}) \). Then

\[
\frac{1}{2} w_A(T) \leq \min \left\{ w_A \left( \begin{array}{cc} P & Q \\ O & O \end{array} \right), w_A \left( \begin{array}{cc} O & P \\ Q & O \end{array} \right) \right\}.
\]

**Proof.**

\[
w_A \left( \begin{array}{cc} P & Q \\ O & O \end{array} \right) \geq \frac{1}{2} \max \{w_A(P + iQ), w_A(P - iQ)\}
\]

\[
= \frac{1}{2} \max \{w_A(T), w_A(T^\# A)\}
\]

\[
= \frac{1}{2} w_A(T).
\]

Again, replacing \( T_2 \) and \( T_3 \) by \( P \) and \( iQ \), respectively in Lemma 2.12 and using Lemma 2.3 of [16], we have

\[
w_A \left( \begin{array}{cc} O & P \\ Q & O \end{array} \right) = w_A \left( \begin{array}{cc} O & P \\ iQ & O \end{array} \right) \geq \frac{1}{2} w_A(P \pm iQ) = \frac{1}{2} w_A(T).
\]

From (3.15) and (3.16), we have

\[
\frac{1}{2} w_A(T) \leq \min \left\{ w_A \left( \begin{array}{cc} P & Q \\ O & O \end{array} \right), w_A \left( \begin{array}{cc} O & P \\ Q & O \end{array} \right) \right\}.
\]

\[
\square
\]
We remark that the condition “\(\mathcal{N}(A)\) is invariant for operators” in Theorem 2.9 can also be dropped, similarly. Next, we recall a lemma that is used to prove Theorem 3.16.

**Lemma 3.15.** [Lemma 2.6, [10]]

Let \(X, Y \in \mathcal{B}_A(\mathcal{H})\). Then

\[
\|X + Y\|_A = 1 = \|X - Y\|_A.
\]

**Theorem 3.16.** Let \(T_1, T_2 \in \mathcal{B}_A(\mathcal{H})\). Then

\[
\|w^A_\mathcal{H}(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix})\| \leq \frac{1}{16} \|P\|^2 + \frac{1}{4} \|w_\mathcal{H}^A(T_2 T_1)\| + \frac{1}{8} \|w_\mathcal{H}^A(PT_2 T_1 + T_2 T_1 P)\|
\]

where \(P = T_1^\#T_1 + T_2^\#T_2\).

**Proof.** Let \(T = \begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\) and \(P = T_1^\#T_1 + T_2^\#T_2\). Now,

\[
\frac{1}{2} \|e^{i\theta} T_1 + e^{-i\theta} T_2^\#\|_A
\]

\[
= \frac{1}{2} \| (e^{i\theta} T_1 + e^{-i\theta} T_2^\#)^\#(e^{i\theta} T_1 + e^{-i\theta} T_2^\#) \|_A
\]

\[
= \frac{1}{2} \| (e^{-i\theta} T_1^\# + e^{i\theta} (T_2^\#)^\#)(e^{i\theta} T_1 + e^{-i\theta} T_2^\#) \|_A
\]

\[
= \frac{1}{2} \| T_1^\#T_1 + e^{-2i\theta} T_1^\#T_2^\# + e^{2i\theta} (T_2^\#)^\#T_1 + (T_2^\#)^\#T_2^\# \|_A
\]

\[
= \frac{1}{2} \| T_1^\#T_1 + e^{-2i\theta} T_1^\#T_2^\# + e^{2i\theta} (T_2^\#)^\#T_1 + (T_2^\#)^\#T_2^\# \|_A
\]

\[
= \frac{1}{2} \| T_1^\#T_1 + e^{-2i\theta} T_1^\#T_2^\# + e^{2i\theta} T_2^\#T_1 + T_1 T_2^\# \|_A
\]

\[
= \frac{1}{2} \| T_1^\#T_1 + T_2^\#T_2 + 2Re(e^{2i\theta} T_2 T_1) \|_A
\]

\[
= \frac{1}{2} \| T_1^\#T_1 + T_2^\#T_2 + 2Re(e^{2i\theta} T_2 T_1) \|_A
\]

\[
= \frac{1}{2} \| (T_1^\#T_1 + T_2^\#T_2)^2 + 4(Re(e^{2i\theta} T_2 T_1)) \|_A
\]

\[
= \frac{1}{2} \| (T_1^\#T_1 + T_2^\#T_2)^2 + 4(Re(e^{2i\theta} T_2 T_1)) \|_A
\]

So,

\[
\left( \frac{1}{2} \| e^{i\theta} T_1 + e^{-i\theta} T_2^\# \|_A \right)^4 = \frac{1}{16} \| (T_1^\#T_1 + T_2^\#T_2)^2 + 4(Re(e^{2i\theta} T_2 T_1)) \|_A.
\]
This implies
\[
\left(\frac{1}{2}\|e^{i\theta}T_1 + e^{-i\theta}T_2^{\#A}\|_A\right)^4 \leq \frac{1}{16}\|T_1^{\#A}T_1 + T_2T_2^{\#A}\|_A^2 + \frac{1}{4}\|Re_A(e^{2i\theta}(PT_2T_1 + T_2T_1P))\|_A^2.
\]
Now, taking supremum over \(\theta \in \mathbb{R}\) and using Lemma 3.15, we thus obtain
\[
w^4_A\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) \leq \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2_A(T_2T_1) + \frac{1}{8}w_A(PT_2T_1 + T_2T_1P).
\]

Note that the authors of [5] proved the above theorem with the assumption \(A > 0\). Using Theorem 3.16 and Lemma 2.3, we now establish the following inequality.

**Corollary 3.17.** Let \(T_1, T_2 \in B_A(\mathcal{H})\). Then
\[
w_A(T_1T_2) \leq \frac{1}{3}\sqrt{\|P\|^2 + 4w^2_A(T_2T_1) + 2w_A(T_2T_1P + PT_2T_1)}
\]

where \(P = T_1^{\#A}T_1 + T_2T_2^{\#A}\).

**Proof.** Here
\[
w_A(T_1T_2) \leq \max\{w_A(T_1T_2), w_A(T_2T_1)\}
\]= \frac{1}{3}\sqrt{\|P\|^2 + 4w^2_A(T_2T_1) + 2w_A(T_2T_1P + PT_2T_1)}
\]
The last inequality follows by Theorem 3.16.

Adopting a parallel technique as in the proof of the Theorem 3.16 one can prove the following result.

**Theorem 3.18.** Let \(T_1, T_2 \in B_A(\mathcal{H})\),
\[
w^4_A\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) \geq \frac{1}{16}\|P\|^2 + \frac{1}{8}m(PT_2T_1 + T_2T_1P) + \frac{1}{4}c^2_A(T_2T_1),
\]
where \(P = T_1^{\#A}T_1 + T_2T_2^{\#A}\) and \(c_A(T_2T_1) = \inf_{\theta \in \mathbb{R}} \inf_{x \in \mathcal{H}} \|Re(e^{i\theta}T_2T_1)x\|_A\).

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The next result provides upper and lower bounds for $A$-numerical radius of $2 \times 2$ operator matrix which follows directly using Theorem 3.1.6, Theorem 3.1.8 and Lemma 2.2.

**Theorem 3.19.** Let $T_1, T_2, T_3, T_4 \in B_A(\mathcal{H})$. Then

$$w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \leq \max\{w_A(T_1, w_A(T_4))\} + \left[\frac{1}{16} \|P\|^2 + \frac{1}{8} w_A(P T_3 T_2 + T_3 T_2 P) + \frac{1}{4} w_A^2(T_3 T_2)\right]^{1/4},$$

and

$$w_A\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq \max\{w_A(T_1, w_A(T_4))\} \cdot \left[\frac{1}{16} \|P\|^2 + \frac{1}{8} m_A(P T_3 T_2 + T_3 T_2 P) + \frac{1}{4} c^2(T_3 T_2)\right]^{1/4},$$

where $P = T_1^{\#} T_1 + T_2 T_2^{\#}$ and $c_A(T_2 T_1) = \inf_{\theta \in \mathbb{R}} \inf_{x \in \mathcal{H}} \|\text{Re}(e^{i\theta} T_2 T_1) x\|_A$.

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