Quantized vortices in superfluid helium and Bose-Einstein condensates

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Abstract. In this article, we review the research on the dynamics of quantized vortices in superfluid helium and rotating Bose-Einstein condensates with emphasis on the recent research done by our group.

1. Introduction
A quantized vortex is a topological defect that arises from the order parameter in Bose-Einstein condensation in which frictionless superfluid flows with quantized circulation around each vortex core [1]. A quantized vortex was both predicted and discovered first in superfluid $^4$He which was the first example of a Bose-Einstein condensate. One of the principal problems was the superfluid turbulent state consisting of a tangle of quantized vortices in thermal counterflow. More recently, the interest has shifted to the nature of superfluid turbulence, apart from the case of counterflow. One of the important problems is how superfluid turbulence relates to classical turbulence. After discussing the recent motivation of this issue, we show superfluid turbulence takes an energy spectrum consistent with the Kolmogorov law, which is an important statistical law in fully developed classical turbulence.

On the other hand, the achievement of Bose-Einstein condensation in dilute atomic gases [2] in 1995 has opened up a new research field, in which quantized vortices can be well controlled and visualized using new optical techniques. Being motivated by the recent observations of the vortex lattice formation in a rotating Bose-Einstein condensate (BEC), we made the numerical analysis of the Gross-Pitaevskii equation that describes the structure and dynamics of the order parameter. Consistent with observations, the simulated condensate starts an elliptic oscillation after the rotation is turned on, which induces the surface-mode excitations. The vortices develop from these surface excitations and then enter the bulk condensate, eventually forming a vortex lattice.

Thus the study of quantized vortices has an important role in understanding the physics in both the traditional field of superfluid helium and the novel field of atomic-gas BECs. Being motivated by the recent developments in these fields, our group has studied theoretically and numerically the dynamics of quantized vortices in superfluid helium and atomic BECs. The purpose of this article is to review the physics of quantized vortices in both systems with emphasis on the recent activities in our group.
2. What is a quantized vortex?
Below a critical temperature in an ideal Bose gas, a finite fraction of the particles occupies the same single-particle ground state and forms a BEC. When the particles have mutual interaction, single-particle states are no longer meaningful. However, a condensate wave function \( \Phi(\mathbf{r}, t) \) is still defined as the ensemble average of the quantum amplitude for removing a particle at position \( \mathbf{r} \) from the condensate. When a BEC is held in an external potential \( V(\mathbf{r}) \), the dynamics of \( \Phi(\mathbf{r}, t) \) is described by the Gross-Pitaevskii (GP) equation

\[
i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g|\Phi(\mathbf{r}, t)|^2 \right) \Phi(\mathbf{r}, t),
\]

where \( g = 4\pi\hbar^2 m / a \) represents the strength of interaction characterized by the s-wave scattering length \( a \) and \( m \) is the mass of each particle. Writing \( \Phi = |\Phi| \exp(i\theta) \), the squared amplitude \( |\Phi|^2 \) is the condensate density and the gradient of the phase \( \theta \) gives the superfluid velocity \( \mathbf{v}_s = (\hbar / m) \nabla \theta \), which is a frictionless flow of the condensate. As a result, the circulation of \( \mathbf{v}_s \) around a closed path \( C \) in the fluid is quantized as

\[
\oint_C d\mathbf{s} \cdot \mathbf{v}_s = \frac{\hbar}{m} \oint_C d\mathbf{s} \cdot \nabla \theta = n\kappa \quad (n = 0, \pm 1, \pm 2, \ldots),
\]

with the quantum of circulation \( \kappa = \hbar / m \). Such a vortex with a quantized circulation is called a quantized vortex; it occurs in superfluid \(^4\)He, superfluid \(^3\)He, and atomic BECs.

A quantized vortex is different from a vortex in a classical viscous fluid. First, the circulation is quantized, which is contrary to the classical vortex that can have any value of circulation. Second, a quantized vortex is a vortex of inviscid superflow. Thus, it cannot decay by the viscous diffusion of vorticity that occurs in the classical system. A quantized vortex can decay by shortening the length of the core through mutual friction with the normal fluid, by breaking into smaller and smaller vortex rings through reconnections and finally changing to some elementary excitations, and by transferring energy to smaller length scales through a Kelvin wave cascade, followed by acoustic emission. Third, the core of a quantized vortex is very thin, being the order of the coherence length defined by \( \xi = \hbar / (\sqrt{2mg|\Phi|}) \), which is submicron in an atomic-gas BEC and only a few angstroms in superfluid \(^4\)He. Because the vortex core is very thin and does not decay by diffusion, it is easy identify the position of a quantized vortex in the fluid. These properties make a quantized vortex more stable and definite than a classical vortex.

3. Energy spectrum of superfluid turbulence
Liquid \(^4\)He enters the superfluid state at 2.17 K, and its hydrodynamics is usually described using the two-fluid model in which the system consists of inviscid superfluid and viscous normal fluid. Early experimental works on the subject focused on thermal counterflow in which the normal fluid flowed in the opposite direction to the superfluid flow. This flow is driven by the injected heat current, and it was found that the superflow becomes dissipative when the relative velocity between two fluids exceeds a critical value. Now we know this is a superfluid turbulent state consisting of a tangle of quantized vortices, in which the dissipation comes from the interaction between quantized vortices and the normal fluid. Although many studies on superfluid turbulence (ST) have been devoted to thermal counterflow, it has no analogy with conventional fluid dynamics and thus we have not understood the relation between ST and classical turbulence (CT).

To address this question, we consider the statistical law of CT. The steady state for fully developed turbulence of an incompressible classical fluid follows the Kolmogorov law for the energy spectrum. The energy is transferred in the inertial range from large to small scales without being dissipated. The inertial range is believed to be sustained by the self-similar
Richardson cascade in which large eddies are broken up to smaller ones, having the Kolmogorov law $E(k) = C \epsilon^{2/3} k^{-5/3}$. Here the energy spectrum $E(k)$ is defined as $E = \int dk E(k)$, where $E$ is the kinetic energy per unit mass and $k$ is the wave number from the Fourier transformation of the velocity field. The energy transferred to smaller scales in the energy-dissipative range is eventually dissipated by the viscosity with the dissipation rate, which is identical with the energy flux $\epsilon$ in the inertial range. The Kolmogorov constant $C$ is a dimensionless parameter of order unity. One of our main interests here is whether ST takes the Kolmogorov law or not [3].

3.1. The Gross-Pitaevskii equation in the wave number space
A Bose-Einstein condensed system yields a macroscopic wave function $\Phi(x, t) = \sqrt{\rho(x, t)} e^{i \phi(x, t)}$, whose dynamics is governed by the GP equation (1). To solve the GP equation numerically with high accuracy, we use the Fourier spectral method in space with periodic boundary condition in a box with spatial resolution containing $256^3$ grid points. We solve the Fourier transformed GP equation

$$i \frac{\partial}{\partial t} \tilde{\Phi}(k, t) = [k^2 - \mu] \tilde{\Phi}(k, t) + \frac{g}{V^2} \sum_{k_1, k_2} \tilde{\Phi}(k_1, t) \times \tilde{\Phi}^*(k_2, t) \times \tilde{\Phi}(k - k_1 + k_2, t), \quad (3)$$

where $V$ is the volume of the system and $\tilde{\Phi}(k, t)$ is the spatial Fourier component of $\Phi(x, t)$ with the wave number $k$. We consider the case of $g = 1$. For numerical parameters, we used a spatial resolution $\Delta x = 0.125$ and $V = 32^3$, where the length scale is normalized by the healing length $\xi$. With this choice, $\Delta k = 2\pi/32$. Numerical time evolution was given by the Runge-Kutta-Verner method with the time resolution $\Delta t = 1 \times 10^{-4}$.

Our approach here is to introduce a dissipation term that works only on scales smaller than the healing length $\xi$. This dissipation removes not vortices but short wavelength excitations, thus preventing the excitation energy from transforming back to vortices. Compared to the usual GP model, this approach enables us to more clearly study the Kolmogorov law. This dissipation term in Eq. (3) is introduced by replacing the imaginary number $i$ in the left-hand side by $[i - \gamma(k)]$, where $\gamma(k) = \gamma_0 \theta(k - 2\pi/\xi)$ with the step function $\theta(x)$.

3.2. Numerical analysis of the GP equation and the energy spectrum of ST
To obtain a turbulent state, we start from an initial configuration in which the condensate density $\rho_0$ is uniform and the phase $\phi_0(x)$ has a random spatial distribution. Here the random phase $\phi_0(x)$ is made by placing random numbers between $-\pi$ to $\pi$ at every distance $\lambda = 4$ and connecting them smoothly. The initial velocity $v(x, t = 0) = 2\nabla \phi_0(x)$ given by the initial random phase is random, hence the initial wave function is dynamically unstable and soon produces homogeneous and isotropic turbulence with many quantized vortex loops, as shown in Fig. 1 (a)-(c).

In order to study the Kolmogorov spectrum, we need to take properly the energy component of vortices. The total energy $E$ of the GP model is given by the sum of the interaction energy $E_{int}$, the quantum energy $E_q$ and the kinetic energy $E_{kin}$ [4,5]:

$$E = \frac{\int dx \Phi^* \left[ - \nabla^2 + g/2 |\Phi|^2 \right] \Phi}{\int dx \rho}, E_{int} = \frac{g}{2} \frac{\int dx |\Phi|^4}{\int dx \rho}, E_q = \frac{\int dx |\nabla|\Phi|^2}{\int dx \rho}, E_{kin} = \frac{\int dx |\Phi| \nabla \phi|^2}{\int dx \rho}. \quad (4)$$

The kinetic energy $E_{kin}$ is furthermore divided into the compressible part $E_{kin}^c$ due to the sound waves and the incompressible part $E_{kin}^i$ due to vortices

$$E_{kin}^c = \frac{\int dx [(|\Phi| \nabla \phi)^c]^2}{\int dx \rho}, \quad E_{kin}^i = \frac{\int dx [(|\Phi| \nabla \phi)^i]^2}{\int dx \rho}. \quad (5)$$
where \( \text{rot}(|\Phi| \nabla \phi)^c = 0 \) and \( \text{div}(|\Phi| \nabla \phi)^t = 0 \). It is the incompressible kinetic energy \( E^i_{\text{kin}} \) that can take the Kolmogorov law reflecting the Richardson cascade process.

We calculated the spectrum of \( E^i_{\text{kin}} \). Initially, the spectrum \( E^i_{\text{kin}}(k) \) significantly deviates from the Kolmogorov power-law, however, the spectrum approaches a power-law as the turbulence develops. We assumed that the spectrum is proportional to \( k^{-\eta} \) in the inertial range \( \Delta k < k < 2\pi/\xi \), and we determined the exponent \( \eta \). It is found that \( \eta \) is almost 5/3 for times \( 4 < t < 10 \), when the system may be almost homogeneous and isotropic turbulence. We also calculated the energy dissipation rate \( \epsilon = -\partial E^i_{\text{kin}}/\partial t \). We found that \( \epsilon \) is almost constant in the period \( 4 < t < 10 \), which means that the Kolmogorov spectrum \( \epsilon^{2/3} k^{-\eta} \) is also constant in the period. The energy spectrum using this \( \epsilon \) agrees quantitatively with the Kolmogorov law obtained numerically, as shown in Fig. 1 (d). Without the dissipation term \( \gamma(k) \), the ST could not satisfy the Kolmogorov law. This means that dissipating short wavelength excitations are essential for the ST to take the Kolmogorov law. The agreement between the energy spectrum and the Kolmogorov law becomes weak at a later stage \( t > 10 \), which may be attributable to the following reasons. In the period \( 4 < t < 10 \), the energy spectrum agrees with the Kolmogorov law, which may support that the Richardson cascade process works in the system. The dissipation is caused mainly by removing short wavelength excitations emitted at vortex reconnections. However, the system at the late stage \( t > 10 \) has only small vortices after the Richardson cascade process, being no longer turbulent. The energy spectrum, therefore, disagrees with the Kolmogorov law of \( \eta = 5/3 \). This result shows strongly the similarity between ST and CT.

Except for the energy spectrum business, we have studied rotating superfluid turbulence[6], Kelvin wave cascade along a quantized vortex [7] and an intrinsic criterion for superfluid turbulence [8].

4. Vortex lattice formation in a rotating BEC
The recent dramatic achievement of Bose-Einstein condensation in trapped alkali-atomic gases at ultra-low temperatures has stimulated intense experimental and theoretical activity[2]. Such atomic-gas Bose-Einstein condensates (BECs) differ fundamentally from liquid helium condensates in several ways. First, the condensates of alkali-atomic gases are dilute, so that
the interatomic interaction can be accurately parameterized in terms of a scattering length $a$. As a result, at low temperatures, the GP equation gives an extremely precise description of the condensate and their dynamics. Second, a BEC in bulk helium has uniform density, whereas an atomic-gas BEC has nonuniform density due to the confinement by a trapping potential.

4.1. Quantized vortices in atomic BECs
From those properties, the physics of quantized vortices has novel features. First, since the density is dilute, the relatively large coherence length makes it possible to directly visualize the quantized vortices by using optical techniques. Second, because the order of the coherence length is close to the size of the condensate, the vortex dynamics is closely connected with the collective motion of the condensate density itself. Finally, alkali atoms have internal degrees of freedom attributed to the hyperfine spin, so that their BECs can have multi-component order parameters if the spin degrees are available. Multicomponent BECs provide us with new possibilities to study unconventional vortex states [9] that have been studied extensively in other fields of physics such as superfluid $^3$He, anisotropic superconductors, and cosmology.

Some groups have succeeded in making and visualizing quantized vortices in BEC. Among them, here we focus on the experimental works by the Paris group [10]. Madison et al. observed directly nonlinear dynamical phenomena such as vortex nucleation and lattice formation in a rotating condensate. By suddenly turning on the rotation of the potential, the initially axisymmetric condensate made a collective quadruple oscillation in which the condensate deformed elliptically. This oscillation continued for a few hundred milliseconds with gradually decreasing amplitude. Then, the axial symmetry of the condensate suddenly reappeared and concurrently the vortices entered the condensate from its surface, eventually settling into a lattice configuration in the bulk. This observation motivated us to study theoretically the detailed dynamics of vortex lattice formation.

4.2. Numerical analysis of the vortex lattice formation
We resolved these problems [11] by numerically solving the GP equation in a rotating frame

$$(i - \gamma)\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V - \mu + g|\psi|^2 - \Omega L_x \right] \psi. \tag{6}$$

Here, the centrifugal term $-\Omega L_x = i\hbar \Omega (x\partial_y - y\partial_x)$ appears in a system rotating about the $z$-axis at a frequency $\Omega$. In this work, we assume translational symmetry along the $z$-axis, thus making the problem two-dimensional. Then the normalization of the wave function $\psi$ is taken as $\int d\mathbf{r} |\psi|^2 = n_{2D}$ with the particle number per unit length along the $z$-axis. The trapping potential has the form $V(\mathbf{r}) = (1/2)m\omega_z^2 \{ (1 + \epsilon_x)x^2 + (1 + \epsilon_y)y^2 \}$ with the small anisotropy parameters $\epsilon_x$ and $\epsilon_y$ ($\epsilon_x \neq \epsilon_y$) in approximate agreement with the potential used in the Paris experiments [10]. By introducing the scales characterizing the trapping potential $a_h = \sqrt{\hbar/2m\omega_\perp}$ for length and $\omega_\perp^{-1}$ for time, and replacing $\psi \rightarrow \sqrt{n_{2D}} \psi/a_h$, the coupling coefficient in the nonlinear term of Eq. (6) becomes $C = 8\pi a_n a_{2D}$. The condition of Ref. 17 gives $C \simeq 500$. The term with $\gamma$ introduces dissipation, which is treated phenomenologically in the GP equation. This form of the dissipative equation follows the work of Choi et al. [12]; they studied the damping of the collective oscillation of a BEC and determined the value of $\gamma$ to be 0.03 by fitting their theoretical results with the MIT experiments [13]. Hence, we use $\gamma = 0.03$ here.

The simulations started from the stationary solution with uniform phase of Eq. (6) for a nonrotating trap ($\epsilon_x = \epsilon_y = 0$). The rotation with frequency $\Omega$ started at $t = 0$, and the trap anisotropy $\epsilon = \{(1 + \epsilon_x) - (1 + \epsilon_y)\}/\{(1 + \epsilon_x) + (1 + \epsilon_y)\}$ was increased rapidly from zero to its final value 0.025 in 20 msec, keeping $\epsilon_y$ zero. Figure 2 shows the dynamics of the condensate with $C = 500$ for $\Omega/\omega_\perp = 0.7$. The upper column of Fig. 2 shows the time development of the
condensate density $|\psi(x, y, t)|^2$. Initially, the condensate also undergoes a quadruple oscillation, but the oscillation is damped because of the dissipation. After a few hundred milliseconds, the boundary surface of the condensate becomes unstable, generating surface ripples that propagate along the surface. Then the waves on the surface develop into the vortex cores, which enter the condensate. As the vortices penetrate inside the condensate, the axisymmetry of the condensate shape is recovered. Then the rotating drive pulls vortices toward the rotation axis, whereas repulsive interactions between vortices tends to push them apart; therefore, their competition yields a vortex lattice eventually. This result is consistent with the experimental one.

Figure 2. Time development of the density (upper row) and phase (lower row) profile after the trapping potential begins to rotate suddenly with $\Omega = 0.7\omega_\perp$ for (a) 67 msec, (b) 385 msec, (c) 425 msec and (d) 735 msec. The value of the phase varies continuously from 0 (black) to $2\pi$ (white). There appear some lines where the phase changes discontinuously from black to white, corresponding to the branch cuts between the phase 0 and $2\pi$. The apexes of these lines around which the value of the phase rotates continuously from 0 to $2\pi$ represent the quantized vortices.

The corresponding time development of the phase of $\psi(x, y, t)$ is shown in the lower column of Fig. 2. As soon as the rotation starts, the phase field inside the condensate takes the form of quadruple flow $\theta(x, y) = \alpha(t)xy + \text{const}$, and just outside the Thomas-Fermi boundary appear many proto-vortices; for example, Fig. 2(a) shows about 20 proto-vortices. When the proto-vortices are on the outskirts of the condensate where the amplitude $|\psi(r, t)|$ is almost negligible, they neither contribute to the energy nor the angular momentum of the system. Because they are invisible in the corresponding density profile, they may be called “ghost vortices”. The ghost vortices move toward the rotation axis, but their invasion into the condensate is prevented at the boundary surface within which the amplitude grows up. However, as the surface ripples are generated, the ghost vortices begin to penetrate the condensate. At this point, only some defects enter the bulk condensate because their further invasion costs energy. Figure 2(d) shows that six vortices enter the condensate and form a lattice, while the remaining vortices are repelled and escape to the outside of the Thomas-Fermi boundary.

Recently we made three-dimensional analysis of the same problem [14]. Although the results are qualitatively similar to the two-dimensional one, we have the excitation of Kelvin waves and the turbulence of ghost vortices.

As other topics we have studied a giant vortex of a rotating BEC in an unharmonic potential [15] and vortex states in two-component BECs [16].

5. Conclusions

We reviewed the recent theoretical works of our group on physics of quantized vortices in superfluid helium and atomic Bose-Einstein condensates, with the recent motivation in these topics. We believe the physics discussed here has lots of motivations and phenomena common to astrophysics. Using superfluid helium and atomic BEC will enable us to study “the laboratory astrophysics”. We hope that more astrophysicists will get interested in these topics.
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