Remarks on DSR and Gravity

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Abstract. Modifications of Special Relativity by the introduction of an invariant energy and/or momentum level (so-called Doubly Special Relativity theories, DSR) or by an energy-momentum dependence of the Planck constant (Generalized Uncertainty Principle, GUP) are compared with classical gravitational effects in an interaction process. For the low energy limit of the usual formulations of DSR to be equivalent to Newtonian gravity, a restrictive condition is found. GUP yields an effective repulsion, in analogy to gravitational repulsion in loop quantum cosmology.

1 Introduction

Tentative quantum theories of gravity – string theory as well as loop quantum gravity or non-commutative geometry – indicate the existence of an invariant length scale of the order of magnitude of the Planck length, which is in obvious contradiction with standard Lorentz symmetry, when taken seriously to arbitrary small scales. For this reason in the last years various attempts were made to modify Special Relativity (SR) \[1, 2\] in such a way that one (or more) invariant quantities in addition to the speed of light would be reconciled with the relativity principle. Theories of this kind are called Doubly or Deformed Special Relativity (DSR).

Early examples were formulated in momentum space by the application of nonlinear representations of the Lorentz group to the energy and momentum of physical objects, such that there is an invariant value of energy and/or momentum of the order of the Planck energy and the Planck momentum. Technically this may be achieved by splitting energy/momentum variables into “physical” ones, usually denoted by $E$ and $\vec{p}$, and “pseudo”-variables (sometimes called platonic variables) $\varepsilon$ and $\vec{\pi}$, with both kinds of variables related by an invertible nonlinear transformation. Pseudo-variables satisfy the usual linear relations of SR, in consequence, the physical ones are acted upon by the (boost sector of the) Poincaré group in a deformed, nonlinear way. Note that the denomination ‘deformed’, common in the present context, denotes merely the action of the Lorentz group and has nothing to do with deformations of the Poincaré Lie algebra.

Pseudo-variables, although being mere auxiliary quantities in the construction of modified Lorentz transformations, have the following formal significance. Providing the linear representation of the Poincaré group, they carry the usual vector space structure of SR.
momentum space, whereas the space of $E$ and $\vec{p}$ becomes curved. Therefore, when subsystems are composed to a whole, it is the pseudo-variables which must be additive in order to preserve the underlying Lorentz group structure. For this reason in the calculations of reaction thresholds or cross sections of particle interactions in the framework of DSR, conservation rules are formulated in terms of them. This leads to a violation of ordinary energy/momentum conservation; particularly the total energy/momentum of a composite system never exceeds the invariant values, as long as the energy/momentum of its components are below them. (This is the so-called soccer-ball problem, see, e.g. [3, 4].)

Possible physical consequence are anomalies of reaction thresholds and an energy dependence of the speed of light in some versions of DSR. Even if the effect is tiny (of the order photon energy/Planck energy), it might become measurable when photons run over cosmic distances. A recently observed slight energy dependence of the time of arrival of photons from a $\gamma$ ray flare might be interpreted in this sense, if we knew the mechanism of emission, see [7].

More advanced versions of DSR are completed by modified space-time Lorentz transformations, associated to the transformations in momentum space in different ways. Some of these attempts assume momentum space to be a de Sitter space [8], other ones make use of Hopf algebra techniques [9]. These approaches lead to non-commutative space-time, denoted by $\kappa$-Minkowski space [10]. In this framework the Poincaré lie algebra itself is deformed. Other methods lead to energy-momentum dependent space-time metrics, called “rainbow metrics” [11], recently investigated with the formalism of Finsler geometries [12]. Nevertheless, for the purpose of the present paper we are going to make some simple physical considerations only in momentum space.

DSR, at least in its original guise, is formally independent of gravity, even if the corrections it makes to SR are interpreted as effective description of the imprints of quantum geometry in form of some texture of flat space, present even in the limit of vanishing gravitational field. Thereby gravity is mainly needed as an explanation for the departure of physical energy-momentum conservation in DSR in such a way that the gravitational field is thought as a reservoir for the non-conserved energy and momentum, without specification of its properties at extremely small distances and of the way it interacts with matter. A more concrete relation to gravity exists in the Hopf algebra approach, as Hopf algebra methods appear as a branch of quantum gravity research in their own right [13]. Concerning the relation of DSR to Loop Quantum Gravity, in [14] there is a rather heuristic derivation of modified energy-momentum relations, inherent to DSR, from spatial discreteness, but it is also explained that a violation of Lorentz invariance can neither be derived nor excluded from the present form of Loop Quantum Gravity.

Although DSR is supposed to reproduce gravitational effects in the quantum gravity regime, it is an open issue, how it compares to classical gravity. One may ask whether its low energy limit should be in accordance with the effects of a classical gravitational field, the low energy limit of quantum gravity.

In the literature there are essentially two different points of view, relating DRS to different partial aspects of full General Relativity (GR):

- No relation to classical gravity is proposed in [15], where DSR represents the topo-
logical degrees of freedom of the gravitational field, i.e. the remnant, when the local
degrees of freedom of the gravitational field is removed. This approach is supported
by the successful formulation of 2+1 gravity, well known to be a topological theory, as
kind of DSR [16].

• On the other hand, in [11] a “Correspondence Principle” is formulated in the form
that in the low-energy limit of DSR classical GR should be recovered.

A logically different approach to the Lorentz invariance problem is to separate between a
particle’s energy and momentum on the one hand and the frequency and wave vector of the
associated quantum wave function on the other hand, with the advantage of an immediate
connection between the formulations in momentum and in position space and a clear physical
meaning of all variables in the game [17]. An important consequence is an energy depen-
dent Planck constant, leading to modified uncertainty relations (Generalized Uncertainty
Principle, GUP) and possibly, but not necessarily, an energy dependent speed of light.

The above-mentioned correspondence principle and the interpretation of DSR as topo-
logical gravity being not equivalent, it is worthwhile to compare low-energy approximations
of DSR and GUP to classical gravity in its simplest, i.e. Newtonian form. The program of
the present paper is to perform an elementary test, namely to apply these approximations to
a scattering process, as possible physical effects always arise in connection with interactions
between moving objects. In the current DSR philosophy pseudo-variables must be associ-
ated to interaction processes, so it is sufficient and logically convenient to define them only in
interaction regions, as it was done in [17], whereas the asymptotic variables are the physical
ones. The inclusion of interacting objects opens a door to the introduction of gravity as a
further interaction, leading to a small perturbation, and not as a quantum property of space.

The only further ingredients, used beside Newtonian gravity in the next sections, are the
mass-energy relation and the de Broglie wavelength of particles. In detail we will use the
following approximations:

• Newtonian gravity, understood as lowest-order approximation of general relativity, in
other words, as a simplified substitute for a curved background.

• Quantum field theory in first-order perturbative approximation.

• A general lowest-order ansatz for DSR-like corrections of SR.

We are going to compare only unspecified interactions in the absence and in the presence
of classical gravity, so the considerations are independent of specific high-energy quantum
effects, like varying coupling constants. In the next two sections, c and \( h \) are set equal to
one, they will have to be restored in section 4.

2 Gravity in two-particle interactions

2.1 Central collision

We consider the scattering of two identical particles with repulsive interaction in the centre-
of-momentum reference frame. In perturbative quantum field theory the free particles ap-
proach each other, exchange virtual interaction particles and then move away freely. During
the free motion the gravitational interaction of particles does not play a role, but we will take
it into account in the interaction process. If we assume first a central collision and restrict
ourselves to first-order Feynman diagrams, we can describe the situation as follows. At the
interaction vertices, when the particles reach a certain minimal distance, they stop and their
kinetic energy materializes as a virtual exchange particle. Provided the asymptotic kinetic
energy is high enough, the gravitational field of the virtual particle furnishes a significant
amount of additional energy for the interaction process in comparison with the gravitation-
less interaction, and the particles come closer to each other, as if they had a higher asymptotic
kinetic energy. In the following we are going to formulate these considerations up to first
order in the gravitational constant $G$.

We assume two particles with masses $m$ and (absolute values of) asymptotic momenta $p$. In the absence of gravity, at the interaction vertices, with the particles at their minimal
distance denoted by $2r_0$, the asymptotic kinetic energy of both of them transforms into the
energy of the interaction particle,

$$
\mathcal{E} = 2 \left( \sqrt{p^2 + m^2} - m \right).
$$

(1)

When Newtonian gravity is added to the system and the particles are assumed to be massive,
there are two effects (the mutual attraction of the two rest masses is considered as negligible):
Due to gravitational attraction each particle has a potential energy

$$
\Delta E_1 = -\frac{G\mathcal{E}m}{r_0}
$$

(2)
in the moment when it stops at a distance $r_0$ from the scattering centre. For a rough estimate
of the minimal distance in terms of the asymptotic kinetic energy we take the de Broglie
wavelength $\lambda$ of the exchanged particle, whose mass is assumed to be negligible in comparison
with its total energy, so that the transmitted momentum is approximately equal to $\mathcal{E}$,

$$
2r_0 \approx \lambda \approx \frac{1}{\mathcal{E}},
$$

(3)

and the gravitational potential energy of each of the scattered particles becomes

$$
\Delta E_1 \approx -2 \frac{m\mathcal{E}^2}{m_P^2} = -8 \frac{m \left( \sqrt{p^2 + m^2} - m \right)^2}{m_P^2},
$$

(4)

where we have introduced the Planck mass $m_P = 1/\sqrt{G}$, which, in our units, stands also for
the Planck energy and the Planck momentum.

The second effect, which is independent of the mass of the particles, is the self-energy $\Delta \mathcal{E}$ of the exchange particle, whose order of magnitude is estimated by modeling it ad hoc
as a homogenous sphere of radius $r_0$,

$$
\Delta \mathcal{E} \approx -\frac{3}{5} \frac{G\mathcal{E}^2}{r_0} = -\frac{6}{5} \frac{\mathcal{E}^3}{m_P^2}.
$$

(5)
One half of $\Delta \mathcal{E}$ is associated to each of the scattered particles to give rise to an energy difference

$$\Delta E_2 \approx -\frac{24}{5} \left( \sqrt{p^2 + m^2} - m \right)^3.$$  \hspace{1cm} (6)

Of course, in view of our rough approximations and the homogenous sphere being rather an indication of ignorance than a seriously-meant model, the factors $6/5$ and $24/5$ appear ridiculous and will be absorbed into order-of-magnitude factors later. For a more exact description of the scattering of high-energy particles, whose masses do not play a role, an Aichelburg-Sexl metric [18] would be convenient, for our considerations the above simple estimate may be sufficient.

The interesting fact is that $\Delta E_1$ goes as $m\mathcal{E}^2$ and $\Delta E_2$ as $\mathcal{E}^3$. As we are looking for gravitational effects for realistic particles, we always have $\mathcal{E} \gg m$, so that $\Delta E_1$, containing the rest mass $m$, will be normally subdominant in comparison with $\Delta E_2$.

While the virtual particle, and in connection with it the gravitational potential, come into being, the collision partners are attracted and come closer to each other than they would in absence of gravity. During this process the total energy is constant, the kinetic energy increases and compensates the negative potential energy. As it is only the kinetic energy, which plays a role in the interaction process, we can replace gravity by an effective growth of energy and momentum. On the other hand, after the collision the particles are slightly slowed down by gravity and their asymptotic outgoing energy is smaller than the energy immediately after the interaction, so that asymptotic energy conservation is guaranteed. (We do not assume graviton production, so that gravity is conservative.)

So, instead of speaking about gravity, it is possible to ascribe an effective energy $E_{\text{eff}}$ to the particles, enlarged by $-\Delta E_1$ and $-\Delta E_2$ in comparison with the asymptotic values,

$$E_{\text{eff}} = E - \Delta E_1 - \Delta E_2 = E \left( 1 + \alpha \frac{mE}{m_P} + \beta \frac{E^2}{m_P^2} \right).$$  \hspace{1cm} (7)

with factors $\alpha$ and $\beta$ of the order of magnitude around 1 to 10.

Having ascribed an effective energy to the incoming particles, we can also ascribe an effective momentum to them, simply by using the free high-energy-momentum relation $E \approx p$,

$$p_{\text{eff}} = p \left( 1 + \alpha \frac{mp}{m_P} + \beta \frac{p^2}{m_P^2} \right).$$  \hspace{1cm} (8)

In some analogy to DSR, gravity is now hidden in effective variables. (One might wonder whether a calculation involving a Newtonian potential can be applicable to relativistic particles. Relations (7) and (8) are justified by the fact that the Newtonian potential is used only close to the turning points of the particles, when they slow down to nonrelativistic velocities.)

Note that we have considered interactions in first order of a perturbative expansion. In higher order, when one takes into account more vertices, the interaction process becomes smoother, it is divided into more steps and sets in earlier, i.e. at larger distances, than in first
order. In consequence, in higher order diagrams the influence of gravity will become weaker, so the above first-order estimates are rather an upper bound for gravitational modifications.

To summarize, classical gravity influences the in- and outgoing particles when they are close to their vertices, if the energy is sufficiently high. This is described in two kinds of variables, both of which have an immediate physical meaning: The effective ones, $E_{\text{eff}}$ and $p_{\text{eff}}$, appearing at the vertices and entering cross section calculations, and the asymptotic ones, denoted by $E$ and $p$, playing the role of “bare” variables in connection with classical gravity.

2.2 Non-central collision

In the central collisions considered above the minimal distance of colliding particles and the energy of a virtual particle have been modified, quantities that are hardly accessible to direct measuring, whereas the actual asymptotic energy/momentum are unaffected, so the discussion is physically rather meaningless so far. The situation improves in the case of non-central collisions of two particles with impact parameter $b$. In first order perturbation theory this is described in the following way: A particle moves straight ahead to the point of minimal distance $r_0$ from the scattering centre, its interaction vertex. There its radial momentum reverses by the exchange of a virtual particle and it flies away along a straight line at a scattering angle $\vartheta_0$ from its ingoing direction.

In the figure the particle comes from the right, the scattering centre is denoted by C and the vertex by V. A, B, and D are auxiliary points. We may read off the following relations. The triangles VCD and AVB are similar with the angles at $\angle$CVD and $\angle$BAV being equal to $\vartheta_0/2$. The radial component $\vec{VB}$ of the momentum $\vec{p} = \vec{VA}$ at V is

$$p_r = p \sin \frac{\vartheta_0}{2},$$

(9)

the relation between the particle’s minimal distance $r_0 = \overline{VC}$ from C and the impact parameter is

$$b = r_0 \cos \frac{\vartheta_0}{2}.$$  

(10)
At the vertex \(p_r\) becomes zero for a moment, so that during the interaction process the kinetic energy is given only by the component orthogonal to it, namely \(\mathbf{BA} = p \cos \frac{\vartheta_0}{2}\). The energy, contributed from both ingoing particles to the virtual exchange particle is therefore equal to

\[
E \approx 2 \left( \sqrt{p^2 + m^2} - \sqrt{p^2 \cos^2 \frac{\vartheta_0}{2} + m^2} \right). \tag{11}
\]

Under the assumption \(p \gg m\) the exchanged energy \(E\) can be expanded in two different ways, according to the scattering angle \(\vartheta_0\). If \(\vartheta_0\) is small, \(E \approx 2p\left(1 - \cos \frac{\vartheta_0}{2}\right)\) is small, too. The particles do not slow down much and remain relativistic and the considerations of the foregoing subsection, involving a Newtonian potential, become inappropriate.

In the other case, when the collision is almost central and \(\vartheta_0\) is close enough to 180°, so that \(\cos \frac{\vartheta_0}{2} \ll \frac{m}{p}\), the particles slow down to nonrelativistic speed and the calculations with Newtonian gravity are more reliable. Now the energy transfer

\[
E \approx 2p \left[ 1 - \frac{m}{p} \left( 1 + \frac{1}{2} \frac{p^2}{m^2} \cos^2 \frac{\vartheta_0}{2} \right) \right] \tag{12}
\]

is large and for the wavelength associated with the exchange particle we can again use the relativistic relation \(\lambda_0 = \frac{1}{E}\), giving an estimate for the minimal distance \(2r_0\) of the particles.

An expansion of the potential energy \(\Delta E_1\) of the rest masses of the particles in the gravitational field of the virtual particle, and the gravitational self-energy \(\Delta E\) of the latter one yields

\[
\Delta E_1 \approx -2 \frac{m E^2}{m_P} \approx -8 \frac{m p^2}{m_P} \left[ 1 - 2 \frac{m}{p} \left( 1 + \frac{1}{2} \frac{p^2}{m^2} \cos^2 \frac{\vartheta_0}{2} \right) \right]. \tag{13}
\]

and

\[
2\Delta E_2 = \Delta E \approx -6 \frac{E^3}{5 m_P} \approx -48 \frac{p^3}{5 m_P} \left[ 1 - 3 \frac{m}{p} \left( 1 + \frac{1}{2} \frac{p^2}{m^2} \cos^2 \frac{\vartheta_0}{2} \right) \right]. \tag{14}
\]

With the leading contributions of these corrections the effective energy of the virtual particle, \(E_{\text{eff}} = E - 2\Delta E_1 - \Delta E\), becomes

\[
E_{\text{eff}} \approx 2p \left[ 1 + \frac{24}{5} \frac{p^2}{m_P} - \frac{4}{5} \frac{m p}{m_P} \left( 4 + 9 \frac{p^2}{m^2} \cos^2 \frac{\vartheta_0}{2} \right) \right]. \tag{15}
\]

The last two terms in parenthesis are of the same order, because \(\cos^2 \frac{\vartheta_0}{2}\) is of order \(m^2/p^2\). Importantly, there is a leading order correction, quadratic in \(p/m_P\), independent from the scattering angle, and a smaller one, of order \(m p/m_P^2\), depending on \(\vartheta_0\).

When gravity is again replaced by introducing the effective energy of the exchange particle, the wavelength of the latter one becomes in leading order (coming from \(\Delta E_2\))

\[
\lambda \approx \frac{1}{E_{\text{eff}}} \approx \frac{1}{2p \left( 1 + \beta \frac{p^2}{m_P} \right)}, \tag{16}
\]
where, for convenience, the fancy numerical factor $24/5$ is again replaced by $\beta$.

In the figure this means that the particle comes closer to the centre $C$, the vertex $V$ is shifted a small distance to the left, so that the distance $CV$ becomes $\lambda/2$ instead of $\lambda_0/2$ and the scattering angle becomes modified from $\vartheta_0$ to $\vartheta$. From the relation

$$2b = \lambda_0 \cos \frac{\vartheta_0}{2} = \lambda \cos \frac{\vartheta}{2}$$

(17)

we obtain the modification of the scattering angle

$$\cos \frac{\vartheta}{2} \approx \left(1 + \beta \frac{p^2}{m_p^2}\right) \cos \frac{\vartheta_0}{2}.$$  

(18)

Due to the universality of gravity the “bare” scattering angle $\vartheta_0$ is unobservable, but it is possible to compare (18) to the analogous result from DSR, obtained in the next section.

### 3 Comparison with DSR

Now we are in a position to compare the two sets of variables constructed in the foregoing section with the “physical” and the “pseudo”-variables in DSR. Once the deformed, non-linear relations for the physical variables are derived and modified kinematic relations are established with the aid of the linear pseudo-variables, they can in principle be forgotten, and all the consequences, the deformed dispersion relations between energy and momentum, the ensuing violations of conservation laws, etc. are ascribed to gravity.

Here we go the opposite way by asking the question whether a gravity-motivated deformation of SR is in its first approximation compatible with DSR. In the foregoing section we have seen that gravity influences particle scattering in the same way as if the particles had a slightly higher effective kinetic energy. In the following considerations this enhanced effective energy-momentum is set into relation with the DSR pseudo-variables and the actual asymptotic kinetic energy is related to the physical variables, as usual [19]. Also in view of the desired parallel between DSR and gravity this association of variables is plausible in the following extrapolation: The unbounded pseudo-variables describe the situation with a repulsive potential, that goes to infinity at zero distance, in the absence of gravity: To reach smaller and smaller distances from each other, the particles must have arbitrarily high asymptotic energies. In most cases the same is true in the presence of Newtonian gravity, but the asymptotic energy necessary to bring particles close together, is lower. This actual energy is described by the physical variables, which are smaller. Moreover, being bounded, they predict distance zero at a finite asymptotic energy, thus mimicking a gravitational collapse, when the exchange particle’s energy reaches the Planck region. By this fact DSR is closer to GR than to Newtonian theory.

Here, of course, we are going to compare only the leading corrections stemming from the inclusion of classical gravity, as well as from DSR, both based on the ratio $p/m_P (= E/m_P$ in our assumption). Whereas in DSR the power of these ratio is a matter of an ad hoc definition, classical gravity in three space dimensions fixes the lowest order to be two, due
to the simple fact that $G = 1/m_P^2$. This is in contrast to linear DSR corrections, considered in [20], for example.

For the comparison of classical gravity and DSR in non-central scattering, considered in subsection (2.2), we assume a typical lowest-order DSR relation (for different kinds of such approximations, see [21]) between $p$ and $\pi$

$$\pi \approx p \left[ 1 + \kappa \left( \frac{p}{m_P} \right)^n \right]$$

with a constant $\kappa$ of order not too far from unity and some positive integer power $n$. This kind of relation is in good accordance with the leading term $\propto p^2$ in (18), derived in connection with central collisions, if $n = 2$.

Considering an almost central collision from a DSR point of view, we replace $p$ by $\pi$, the variable related to interaction processes, in the wavelength of the virtual particle, so that $\lambda \approx 1/2\pi$. Then from (17) we obtain the modified scattering angle,

$$\cos \frac{\vartheta}{2} \approx \left[ 1 + \kappa \left( \frac{p}{m_P} \right)^n \right] \cos \frac{\vartheta_0}{2}$$

and from comparison of (20) with (18) it follows that (at least in the considered scattering example) the lowest order correction term of DSR can agree with Newtonian gravity, when it is quadratic in the ratio $p/m_P$. Then only the constants $\beta$ and $\kappa$ must be matched. The result is also a first order approximation in the scattering angle around 180°. To consider further $\vartheta$-dependent terms does not make much sense in the scope of the Newtonian framework, because for faster scattering processes there would be significant general relativistic corrections. As Newtonian gravity is the lowest order correction of SR coming from GR, we have obtained a condition for DSR theories to satisfy the correspondence principle in its full meaning, namely, the lowest-order effects of DSR must be quadratic. DSR 2 for example, proposed in [2], with linear corrections would be at odds with it.

One important difference of the present approach to “full” DSR is the use of the free energy-momentum relations for both kinds of variables, rather than of modified ones for $E$ and $p$. This important aspect of DSR does not show up in the present calculations, because in the considered approximations the mass term does not play a role, and $E \approx p$ as well as $\varepsilon \approx \pi$, the calculations were essentially carried out for momenta alone.

The above considerations can easily be applied to higher dimensions. In $d > 3$ space dimensions the Planck mass $m_{(d)}_P$ is by orders of magnitude smaller, on the other hand, the Newtonian potential goes as $r^{2-d}$. See for example [22]. In consequence, the lowest order correction of the scattering angle behaves as

$$\frac{p^{d-1}}{(m_{(d)}_P)^{d-1}},$$

if there are compactified dimensions, large enough for classical gravity to be a reasonable approximation when the minimal distance is as small as the magnitude of these dimensions. In consequence, in these cases the lowest-order corrections of DSR must be of order $d - 1$, when compatibility with classical gravity is desired.
4 Comparison with GUP

This approach is characterized by making a principal distinction between the energy and momentum \((E, \vec{p})\) of a particle and its associated frequency and wave vector \((\omega, \vec{k})\). Their relation is most generally written as

\[
(\omega, \vec{k}) = (E \cdot f(E, \vec{p}), \vec{p} \cdot g(E, \vec{p})). \tag{22}
\]

Energy and momentum are assumed to be unbounded, whereas \(\omega\) and \(\vec{k}\) are bounded by orders of magnitude \(1/\text{Planck time}\) and \(1/\text{Planck length}\), respectively. The functions \(f\) and \(g\) can be chosen analogously to the functions relating physical and pseudo-variables in arbitrary versions of DSR. Nevertheless, the interpretation is different: There are no merely auxiliary variables, both \((E, \vec{p})\) and \((\omega, \vec{k})\) have a clear physical meaning and there is a natural relation between momentum and position space from the beginning.

Comparing (22) with the standard relation

\[
(E, \vec{p}) = \bar{\hbar}(\omega, \vec{k}), \tag{23}
\]

one finds energy-momentum dependent constants

\[
\tilde{\hbar}(E, \vec{p}) = \frac{1}{f(E, \vec{p})} \quad \text{and} \quad \tilde{c}(E, \vec{p}) = \frac{\omega}{|\vec{k}|} = c \frac{f(E, \vec{p})}{g(E, \vec{p})}. \tag{24}
\]

For high energies \(\tilde{\hbar}\) increases, increasing the quantum-mechanical uncertainties. We shall restrict ourselves to the case \(\tilde{c} = c\), i.e. \(f = g\).

In analogy to (19) we assume a lowest-order relation \((p = |\vec{p}| \text{ and } k = |\vec{k}|)\)

\[
p = \tilde{\hbar} k \approx \hbar k \left[ 1 + \kappa \left( \frac{P}{m_P} \right)^n \right], \tag{25}
\]

leading to the wavelength

\[
\lambda = \frac{1}{k} \approx \frac{\tilde{\hbar}}{2p}. \tag{26}
\]

(Note that \(k\) is the wave vector of the exchange particle, \(p\) is the momentum of one incoming particle.) As before, from (17) one obtains the correction of the scattering angle,

\[
\cos \frac{\vartheta}{2} \approx \left[ 1 - \kappa \left( \frac{P}{m_P} \right)^n \right] \cos \frac{\vartheta_0}{2}. \tag{27}
\]

With the same choice of transformation functions between the different sets of variables in first approximation, GUP has yielded just the opposite sign of the DSR correction in (20).

5 Conclusion

One main result of the considerations of this paper is the condition that DSR corrections to SR must be quadratic in \(p/m_P\) in lowest order to fulfill the correspondence principle in
the given example. It is not shown that this condition is sufficient in every situation and the calculations do not show how DSR differs from GR, when higher particle energies are involved. A relation of the present result with the interpretation of DSR in [13] as the topological part of GR would depend on the properties of particle trajectories in topological 3+1 gravity.

The second result concerns GUP, where the situation is quite different. DSR produces an effective attractive force, GUP, on the other hand, results in a repulsive force, which is not a big surprise, as it lays lower bounds to space and time intervals. In contrast to DSR, rather than competing with Newtonian gravity (and thus GR), GUP counteracts it, thus qualifying as a description of pure quantum gravity effects, which has nothing at all to do with classical gravity. In order not to collide with GR, the lowest-order term in GUP must be of a higher power than 2. There is an interesting parallel to loop quantum cosmology [24], where a repulsive behaviour of gravity at short distances, which helps to avoid singularities, is observed.

In the considered example the effects of DSR and those of GUP would be equivalent for some \( n > 2 \), if the roles of \((E, \vec{p})\) and \((\epsilon, \vec{\pi})\) were interchanged. \( \epsilon \) and \( \vec{\pi} \) would be energy and momentum ascribed to free particles, which can be boosted to arbitrary values with respect to a certain reference frame, as long as no interaction takes place. The physical energy \( E \) and momentum \( \vec{p} \), on the other hand, play a role in interactions, which would be in accordance with their interpretation as measurable quantities, as measurements always go along with some interactions. Due to the exchanged roles of \((E, \vec{p})\) and \((\epsilon, \vec{\pi})\) in relation to the common DSR interpretation, reaction thresholds anomalies would be equally small as in usual DSR, but in the opposite direction. For example, when conventional DSR predicts an insignificant lowering of the GZK cutoff [23], the reversed interpretation would lead to an (equally insignificant) raising. (Recent observations do not seem to confirm a shift of the GZK cutoff at all [25].)
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