On Concomitants of Ordered Random Variables under General Forms of Morgenstern Family

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Abstract. Based on the extensions of Morgenstern family (Huang and Bairamov extensions), the concomitants of different types of generalized order statistics (gos) and dual generalized order statistics (dgos) are obtained. Moreover, a unified approach to such models is derived. Information properties such as Shannon entropy and Kullback-Leibler divergence for Huang and Kotz extension are obtained.

1. Introduction

The Farlie-Gumbel-Morgenstern family (Morgenstern family) is an important class of bivariate distributions, it was originally introduced by Morgenstern [17] for Cauchy marginal. Morgenstern distributions are important and efficient in applications for multivariate distributions with given marginal. Johnson and Kotz [11] studied the multivariate case and provided a detailed analysis of probabilistic and statistical characteristics. Huang and Kotz [9] extended Morgenstern family to increase the dependence between the underlying variables by introducing an additional parameter. The generalizations of Morgenstern family of bivariate distributions received a great deal of attention of many researchers. A polynomial type single parameter extension of Morgenstern distribution was considered by Huang and Kotz [10], which is specified by the cumulative distribution function (cdf) and probability density function (pdf), respectively, as follows:

\[
F_{X,Y}(x, y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X^p(x))(1 - F_Y^p(y))],
\]

(1)

\[
f_{X,Y}(x, y) = f_X(x)f_Y(y)[1 + \alpha((1 + p)F_X^p(x) - 1)((1 + p)F_Y^p(y) - 1)], p \geq 1,
\]

(2)

where the admissible range of the associated parameter \( \alpha \) is \(-\max(1, p)^{-2} \leq \alpha \leq p^{-1} \), and since \( p \geq 1 \), this admissible becomes \( p^{-2} \leq \alpha \leq p^{-1} \). Furthermore, the conditional pdf of \( Y \) given \( X \) is given by:

\[
f_{Y|X}(y|x) = f_Y(y)[1 + \alpha((1 + p)F_X^p(x) - 1)((1 + p)F_Y^p(y) - 1)].
\]

(3)
Bairamov et al. [2] presented a general form of the model described above, as follows:

\[ F_{XY}^R(x, y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X^p(x))^{\alpha}(1 - F_Y^p(y))^{\alpha}], \]

\[ f_{XY}^R(x, y) = f_X(x)f_Y(y)[1 + \alpha(F_X^p(x) - 1)^{\alpha-1}(1 - (1 + p_1q_1)F_X^p(x))(F_Y^p(y) - 1)^{\alpha-1} \times (1 - (1 + p_2q_2)F_Y^p(y))], \]

where the admissible range of the associated parameter \( \alpha \) is

\[
\min \left\{ \frac{1}{p_1p_2} \left( \frac{1 + p_1q_1}{p_1(q_1 - 1)} \right)^{\alpha-1}, \frac{1 + p_2q_2}{p_2(q_2 - 1)} \right\} \leq \alpha.
\]

Moreover, the conditional \( pdf \) of \( Y \) given \( X \) is given by:

\[ f_{Y|X}^R(y | x) = f_Y(y)[1 + \alpha(F_X^p(x) - 1)^{\alpha-1}(1 - (1 + p_1q_1)F_X^p(x))(F_Y^p(y) - 1)^{\alpha-1} \times (1 - (1 + p_2q_2)F_Y^p(y))]. \]

Where \( f_X(x) \), \( f_Y(y) \), and \( F_X(x)\), \( F_Y(y) \) are the marginal \( pdf \)'s and \( cdf \)'s of the random variables (R.V.'s) \( X \) and \( Y \) respectively.

Originally David et al. [6] studied concomitants of order statistics. From some bivariate population with \( cdf \) \( F(x, y) \), let \( (X_i, Y_i), i = 1, 2, ..., n \), be \( n \) pairs of independent R.V.'s. Let \( X_{(r:n)} \) be the \( r \)th order statistics, then \( Y \) associated with \( X_{(r:n)} \) is called the concomitant of \( r \)th order statistics and is denoted by \( Y_{(r:n)} \). The \( pdf \) and \( cdf \) of \( Y_{(r:n)} \) are given by:

\[ g_{(r:n)}(y) = g_{Y_{(r:n)}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y | x)f_{X_{(r:n)}}(x)dx, \]

\[ G_{(r:n)}(y) = \int_{-\infty}^{y} F_{Y|X}(y | x)f_{X_{(r:n)}}(x)dx, \]

where \( f_{X_{(r:n)}}(x) \) is the \( pdf \) of \( X_{(r:n)} \).

The concept of \( gos \) was introduced by Kamps [12] and we refer to it as case-I of \( gos \). The use of such connotation has been steadily rising over the years, as it includes important well-known concepts that have been separately treated in statistical literature. Accordingly, many of models of ascendingly ordered R.V.'s are contained in it, such as ordinary order statistics, sequential order statistics, record values and Pfeifers record model. Kamps and Cramer [13] derived a second model of \( gos \) in which the parameters are pairwise different and we refer to it as case-II of \( gos \). On the other side, the concept of lower \( gos \) was given by Pawlas and Szy nal [18] and later Burkschat et al. [5] proposed it as \( dgos \) to enable a common approach of descending ordered R.V.'s like reversed order statistics and lower records models.

A classical measure of uncertainty was launched by Shannon [19] in the information theory literature. The Shannon entropy of a continuous R.V. \( X \) measures the average reduction of uncertainty of \( X \). The Shannon entropy for \( X \) with \( pdf \) \( f_X(x) \) is defined as:

\[ H(X) = -E(\ln f_X(X)) = -\int_{-\infty}^{\infty} f_X(x) \ln f_X(x)dx. \]
Divergence measures are used to quantify the dissimilarity of two probability distributions. They are equal to zero if and only if the distributions are the same. They are interpreted as distances between probability distributions. Kullback and Leibler [14] considered the Kullback-Leibler divergence (information divergence) for two continuous random variables $X_1$ and $X_2$ with pdf's $f_1$ and $f_2$, respectively, which is given by:

$$K(X_1, X_2) = \int_{-\infty}^{\infty} f_1(x) \ln \left( \frac{f_1(x)}{f_2(x)} \right) dx,$$

(11)

$K(X_1, X_2)$ is non negative, invariant under one-to-one transformation of $(X_1, X_2)$, and it is not symmetric.

Beg and Ahsanullah [4] considered concomitants of generalized order statistics for Morgenstern family and derived the joint distribution of concomitants of two generalized order statistics and obtain their product moments. In this dissertation, we obtain the $cdf$ and pdf of concomitants of ordered R.V.’s under Huang and Bairamov extensions. Also, information properties for Huang and Kotz extension are presented.

The rest of this article is organized as follows: In Section 2, the pdf and $cdf$ of concomitants for case-I and case-II of $gos$ and $dgos$ under Huang and Bairamov extensions are provided. Section 3, contains information properties such as Shannon entropy and Kullback-Leibler divergence for Huang and Kotz extension. In addition, some examples for some well-known distributions to obtain the entropy are given.

2. Distribution theory for concomitants of ordered R.V.’s

In this section, we use case-I and case-II of $gos$ and $dgos$ to obtain the pdf and $cdf$ of concomitants for both Huang and Bairamov extensions. The following theorems deal with this matter. To obtain the $cdf$ of such models, from Equation (1), the conditional $cdf$ of $Y$ given $X = x$, for Huang and Kotz extension, is given by:

$$F_{Y|X}(y|x) = f^{-1}_X(x) \frac{\partial F_{X,Y}(x,y)}{\partial x} = F_Y(y)[1 + \alpha((1+p)f^p_X(x) - 1)(f^p_Y(y) - 1)].$$

(12)

From Equation (4), the conditional $cdf$ of $Y$ given $X = x$, for Bairamov extension, is given by:

$$F_{BA_{Y|X}}(y|x) = F_Y(y)[1 + \alpha(1 - f^p_X(x)y^{r-1})(1 - (1 + p\gamma_1)f^p_X(x))(1 - f^p_Y(y)y^q)].$$

(13)

We may classify $gos$ and $dgos$ based on $\bar{m}$ into the following cases: Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=1}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in [1, 2, \ldots, n-1]$, and let $\bar{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$.

**Case-I of $gos$:** If $m_1 = m_2 = \ldots = m_{n-1} = m$, the pdf of $r$th case-I of $gos$ $X_{(r,n,m,k)}$ can be written as, see Kamps [12]:

$$f_{(x,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1 - F(x))^{\gamma_r-1} f(x) g_{m}^{-1}(F(x)),$$

(14)

where $c_{r-1} = \prod_{j=1}^{r-1} \gamma_j$, $g_{m}(z) = h_{m}(z) - h_{m}(0)$, $0 < z < 1$,

$$h_{m}(z) = \begin{cases} \frac{-(1-z)^{m+1}}{m+1}, & m \neq -1, \\ -\ln(1-z), & m = -1. \end{cases}$$

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Case-II of $gos$: If $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n$ and $i \neq j$, the pdf of rth case-II of $gos X_{(r,n,m,k)}$ as follows, see Kamps and Cramer [13]:

$$f_{(r,n,m,k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r) (1 - F(x))^{-1} f(x),$$  \hspace{1cm} (15)

where $a_i(r) = \prod_{j=1,j\neq i}^{r} \frac{1}{\gamma_j - r}, 1 \leq i \leq r$ and $\gamma_i = k + n - i + M_i > 0$.

Case-I of $gos$: When $m_1 = m_2 = \ldots = m_{r-1} = m$, the pdf of rth case-I of $gos X_{d(r,n,m,k)}$ is defined by, see Pawlas and Szynal [18]:

$$f_{d(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!} (F(x))^{\gamma_r-1} f(x) g_m^{-1}(F(x)),$$  \hspace{1cm} (16)

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$, $g_m(z) = h_m(z) - h_m(1), 0 \leq z < 1$,

$$h_m(z) = \begin{cases} \frac{-1}{m+1} m+1, & m \neq -1, \\ \ln z, & m = -1. \end{cases}$$

Case-II of $gos$: When $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n - 1$, in this case, the pdf of rth case-II of $gos X_{d(r,n,m,k)}$ is defined by, see Athar and Faizan [1]:

$$f_{d(r,n,m,k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r) (F(x))^{\gamma_i-1} f(x),$$  \hspace{1cm} (17)

where $a_i(r) = \prod_{j=1,j\neq i}^{r} \frac{1}{\gamma_j - r}, 1 \leq i \leq r$ and $\gamma_i = k + n - i + M_i > 0$.

In the following theorems we use the following notations as follows: Based on Huang and Kotz extension the pdf and cdf of the concomitant of rth case-I of $gos$ (case-I of $gos$) are $g_{1[r,n,m,k]}$ and $G_{1[r,n,m,k]}$ ($g_{d1[r,n,m,k]}$ and $G_{d1[r,n,m,k]}$), respectively. And the pdf and cdf of the concomitant of rth case-II of $gos$ (case-II of $gos$) are $g_{2[r,n,m,k]}$ and $G_{2[r,n,m,k]}$ ($g_{d2[r,n,m,k]}$ and $G_{d2[r,n,m,k]}$), respectively. Based on Bairamov extension the pdf and cdf of the concomitant of rth case-I of $gos$ (case-I of $gos$) are $g_{2[r,n,m,k]}$ and $G_{2[r,n,m,k]}$ ($g_{d2[r,n,m,k]}$ and $G_{d2[r,n,m,k]}$), respectively. And the pdf and cdf of the concomitant of rth case-II of $gos$ (case-II of $gos$) are $g_{2[r,n,m,k]}$ and $G_{2[r,n,m,k]}$ ($g_{d2[r,n,m,k]}$ and $G_{d2[r,n,m,k]}$), respectively.

Theorem 2.1. Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (14) and (12), the pdf and cdf of the concomitant of rth case-I of $gos$, $Y_{[r,n,m,k]}$, are given by, $1 \leq r \leq n$, respectively:

$$g_{1[r,n,m,k]}(y) = f_{\gamma}(y) \left[ 1 + a T_r(r; n, m, k) \left( (1 + p) F_{\gamma}(y) - 1 \right) \right],$$  \hspace{1cm} (18)

$$G_{1[r,n,m,k]}(y) = f_{\gamma}(y) \left[ 1 + a T_r(r; n, m, k) \left( F_{\gamma}(y) - 1 \right) \right],$$  \hspace{1cm} (19)

where

$$T_r(r; n, m, k) = (1 + p) c_{r-1} \sum_{j=0}^{p} \binom{p}{j} (-1)^j \frac{1}{\prod_{i=1}^{r} (\gamma_i + j)} - 1,$$  \hspace{1cm} (20)

$\gamma_r = k + (n - r)(m + 1), n \in \mathbb{N}, k \geq 1, m_1 = \ldots = m_{r-1} = m \in \mathbb{R}, c_{r-1} = \prod_{i=1}^{r} \gamma_i.$
From Beg and Ahsanullah [4], we note that

\[ g_{1[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f_{YX}(y | x)f_{r,n,m,k}(x)dx \]

\[ = f_Y(y) - a f_Y((1 + p)F_Y(1 - F_Y(y)) - 1) \]

\[ \cdot \frac{\alpha(1 + p)c_{r-1}}{(r-1)!(m+1)y^r-1} f_Y((1 + p)F_Y(1 - F_Y(y)) - 1) \]

\[ \times \int_{-\infty}^{\infty} (1 - (1 - F_X(x)))^p(1 - F_X(x))^{y^{-1}} \]

\[ \times (1 - (1 - F_X(x))r^{m+1})^{y^{-1}} f_X(x)dx \]

\[ = f_Y(y) - a f_Y((1 + p)F_Y(1 - F_Y(y)) - 1) \]

\[ + \frac{\alpha(1 + p)c_{r-1}}{(r-1)!} f_Y((1 + p)F_Y(1 - F_Y(y)) - 1) \]

\[ \times \sum_{j=0}^{p} \binom{p}{j} (-1)^j \int_{-\infty}^{\infty} (1 - F_X(x))^{y+j-1} \]

\[ \times \left[ \frac{1}{m+1} (1 - (1 - F_X(x))^{m+1})^{y^{-1}} f_X(x)dx. \]

From Beg and Ahsanullah [4], we note that

\[ \int_{-\infty}^{\infty} (1 - F_X(x))^{y+j-1} \left[ \frac{1}{m+1} (1 - (1 - F_X(x))^{m+1})^{y^{-1}} f_X(x)dx = \frac{(r-1)!}{\prod_{i=1}^{r} \gamma_i + j}, \]

and the result follows. By the same manner we can obtain the cdf of case-I of gos.

**Theorem 2.2.** Based on Bartramov extension with pdf given by (5) and cdf given by (4) (with \( p_1 = p_2 = 1 \)), utilizing (7), (14) and (13), the pdf and cdf of the concomitant of \( r \)-th case-I of gos, \( Y_{[r,n,m,k]} \), are given by, \( 1 \leq r \leq n \), respectively:

\[ g_{2[r,n,m,k]}(y) = f_Y(y) \left[ 1 + aR_1^{*}(r; n, m, k) (1 - (1 + q_2)f_Y(y)) (1 - F_Y(y))^{y-1} \right], \]

\[ G_{2[r,n,m,k]}(y) = f_Y(y) \left[ 1 + aR_1^{*}(r; n, m, k) (1 - F_Y(y))^{y-1} \right], \]

where

\[ R_1^{*}(r; n, m, k) = c_{r-1} \left\{ \frac{(1 + q_1)}{\prod_{i=1}^{r} \gamma_i + q_1} - \frac{q_1}{\prod_{i=1}^{r} \gamma_i + q_1 - 1} \right\}, \]

\[ \gamma_r = k + (n - r)(m + 1), n \in \mathbb{N}, k \geq 1, m_1 = \ldots = m_{r-1} = m \in \mathbb{R}, c_{r-1} = \prod_{i=1}^{r} \gamma_i. \]

**Theorem 2.3.** Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (15) and (12), the pdf and cdf of the concomitant of \( r \)-th case-II of gos, \( Y_{[r,n,m,k]} \), are given by, \( 1 \leq r \leq n \), respectively:

\[ g_{1[r,n,m,k]}(y) = f_Y(y) \left[ 1 + aB_1^{*}(r; n, m, k) (1 + p)F_Y(y) - 1 \right], \]

\[ G_{1[r,n,m,k]}(y) = f_Y(y) \left[ 1 + aB_1^{*}(r; n, m, k) (F_Y(y) - 1) \right], \]

where

\[ B_1^{*}(r; n, m, k) = (1 + p)c_{r-1} \sum_{i=0}^{r} a_i(r) \frac{p(l_{r} - 1)!}{(y + p)!} - 1, \]

\[ a_i(r) = \prod_{j=1}^{r} \frac{1}{\gamma_j - \gamma_i}, 1 \leq i \leq r, c_{r-1} = \prod_{j=1}^{r} \gamma_j. \]
Theorem 2.4. Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with \( p_1 = p_2 = 1 \)), utilizing (7), (15) and (13), the pdf and cdf of the concomitant of \( r \)-th case-II of \( dgos, Y_{d[r,n,\bar{m},\bar{k}]} \), are given by, \( 1 \leq r \leq n \), respectively:

\[
\begin{align*}
\gamma_1 = \sum_{i=0}^{r} a_i(r) \left( \frac{1 + q_1}{\gamma_i + q_1} - \frac{q_1}{\gamma_i + q_1 - 1} \right), \\
\gamma_2 = \prod_{i=1}^{r} \left( \frac{1}{\gamma_i - 1} \right), \\
\gamma_3 = k + (n - r)(m + 1), n \in \mathbb{N}, k \geq 1, m_1 = \ldots = m_{n-1} = m \in \mathbb{R}, c_{r-1} = \prod_{i=1}^{r} \gamma_i.
\end{align*}
\]

Theorem 2.5. Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (16) and (12), the pdf and cdf of the concomitant of \( r \)-th case-I of \( dgos, Y_{d[r,n,\bar{m},\bar{k}]} \), are given by, \( 1 \leq r \leq n \), respectively:

\[
\begin{align*}
\gamma_1 &= \sum_{i=0}^{r} a_i(r) \left( \frac{1 + q_1}{\gamma_i + q_1} - \frac{q_1}{\gamma_i + q_1 - 1} \right), \\
\gamma_2 &= \prod_{i=1}^{r} \left( \frac{1}{\gamma_i - 1} \right), \\
\gamma_3 &= k + (n - r)(m + 1), n \in \mathbb{N}, k \geq 1, m_1 = \ldots = m_{n-1} = m \in \mathbb{R}, c_{r-1} = \prod_{i=1}^{r} \gamma_i.
\end{align*}
\]

Theorem 2.6. Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with \( p_1 = p_2 = 1 \)), utilizing (7), (16) and (13), the pdf and cdf of the concomitant of \( r \)-th case-I of \( dgos, Y_{d[r,n,\bar{m},\bar{k}]} \), are given by, \( 1 \leq r \leq n \), respectively:

\[
\begin{align*}
\gamma_1 &= \sum_{i=0}^{r} a_i(r) \left( \frac{1 + q_1}{\gamma_i + q_1} - \frac{q_1}{\gamma_i + q_1 - 1} \right), \\
\gamma_2 &= \prod_{i=1}^{r} \left( \frac{1}{\gamma_i - 1} \right), \\
\gamma_3 &= k + (n - r)(m + 1), n \in \mathbb{N}, k \geq 1, m_1 = \ldots = m_{n-1} = m \in \mathbb{R}, c_{r-1} = \prod_{i=1}^{r} \gamma_i.
\end{align*}
\]

Theorem 2.7. Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (17) and (12), the pdf and cdf of the concomitant of \( r \)-th case-II of \( dgos, Y_{d[r,n,\bar{m},\bar{k}]} \), are given by, \( 1 \leq r \leq n \), respectively:

\[
\begin{align*}
\gamma_1 &= \sum_{i=0}^{r} a_i(r) \left( \frac{1 + q_1}{\gamma_i + q_1} - \frac{q_1}{\gamma_i + q_1 - 1} \right), \\
\gamma_2 &= \prod_{i=1}^{r} \left( \frac{1}{\gamma_i - 1} \right), \\
\gamma_3 &= k + (n - r)(m + 1), n \in \mathbb{N}, k \geq 1, m_1 = \ldots = m_{n-1} = m \in \mathbb{R}, c_{r-1} = \prod_{i=1}^{r} \gamma_i.
\end{align*}
\]
Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with \( p_1 = p_2 = 1 \)), utilizing (7), (17) and (13), the pdf and cdf of the concomitant of rth case-II of dgos, \( Y_{d[r,n,\bar{m},k]} \), are given by, 1 \( \leq \) r \( \leq \) n, respectively:

\[
g_{d[r,n,\bar{m},k]}(y) = f_Y(y) \left[ 1 + \alpha Q_2(r; n, \bar{m}, k) (1 - (1 + q_2) F_Y(y)) (1 - F_Y(y))^{\alpha - 1} \right],
\]

(41)

\[
G_{d[r,n,\bar{m},k]}(y) = f_Y(y) \left[ 1 + \alpha Q_2(r; n, \bar{m}, k) (1 - F_Y(y))^{\alpha} \right],
\]

(42)

where

\[
Q_2(r; n, \bar{m}, k) = c_{r-1} \sum_{j=0}^{r} \alpha_i(r) \left( \left( 1 + q_1 \right) \sum_{j=0}^{q_1} \left( q_1 \right) \frac{(-1)^j}{\gamma_1 + j} - q_1 \sum_{j=0}^{q_1-1} \left( q_1 - 1 \right) \frac{(-1)^j}{\gamma_1 + j} \right),
\]

(43)

\[
a_i(r) = \prod_{j=1, i \neq j}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n, \quad c_{r-1} = \prod_{j=1}^{r} \gamma_j.
\]

**Remark 2.1.** By substituting \( p = 1 \) in Huang and Kotz extension, and \( p_1 = p_2 = q_1 = q_2 = 1 \) in Bairamov extension, Equations (18) to (43) reduces to the ordinary Morgenstern family, and we obtain the same results that mentioned in Mohie El-Din et al. [16].

Now, we can generalize the previous models in unified models as follows: based on Huang and Kotz extension, Equations (18), (26), (32) and (38), and Equations (19), (27), (33) and (39) can be combined, respectively, as follows:

\[
g^{(1)}_{Y_r}(y) = f_Y(y) \left[ 1 + \alpha M^{(r)}_{r} \left( (1 + p) P_{r}^{(y)} - 1 \right) \right],
\]

(44)

\[
G^{(1)}_{Y_r}(y) = f_Y(y) \left[ 1 + \alpha M^{(r)}_{r} \left( P_{r}^{(y)} - 1 \right) \right],
\]

(45)

where

\[
Y_r = \begin{bmatrix}
Y_{d[r,n,\bar{m},k]}, & \text{for case - I of dgos} \\
Y_{d[r,n,\bar{m},k]}, & \text{for case - I of dgos} \\
Y_{d[r,n,\bar{m},k]}, & \text{for case - II of dgos} \\
Y_{d[r,n,\bar{m},k]}, & \text{for case - II of dgos},
\end{bmatrix}
\]

(46)

\[
M^{(r)}_{r} = \begin{bmatrix}
T^{(r)}_{1}(r; n, m, k), & \text{for case - I of dgos} \\
T^{(r)}_{2}(r; n, m, k), & \text{for case - I of dgos} \\
B^{(r)}_{1}(r; n, \bar{m}, k), & \text{for case - II of dgos} \\
B^{(r)}_{2}(r; n, \bar{m}, k), & \text{for case - II of dgos},
\end{bmatrix}
\]

(47)

Based on Bairamov extension, Equations (23), (29), (35) and (41), and Equations (24), (30), (36) and (42) can be combined, respectively, as follows:

\[
g^{(2)}_{Y_r}(y) = f_Y(y) \left[ 1 + \alpha Z^{(r)}(1 - (1 + q_2) F_Y(y)) (1 - F_Y(y))^{\alpha - 1} \right],
\]

(50)
\[ G^{(2)}_{Y_{r}}(y) = f_{Y}(y) \left[ 1 + \alpha Z_{r}(1 - F_{Y}(y))^{\alpha} \right], \]  
where
\[ Y_{r} = \begin{cases} 
Y_{[r,n,m,k]}, & \text{for case \(- I \) of \(gos\)} 
Y_{d[r,n,m,k]}, & \text{for case \(- I \) of \(dgos\)} 
Y_{[r,n,m,k]}, & \text{for case \(- II \) of \(gos\)} 
Y_{d[r,n,m,k]}, & \text{for case \(- II \) of \(dgos\),} 
\end{cases} \]
\[ Z_{r} = \begin{cases} 
R_{r}(r;n,m,k), & \text{for case \(- I \) of \(gos\)} 
R_{d}(r;n,m,k), & \text{for case \(- I \) of \(dgos\)} 
Q_{r}(r;n,m,k), & \text{for case \(- II \) of \(gos\)} 
Q_{d}(r;n,m,k), & \text{for case \(- II \) of \(dgos\),} 
\end{cases} \]
\[ g^{(2)}_{Y_{r}}(y) = \begin{cases} 
g_{[r,n,m,k]}^{(2)}(y), & \text{pdf of concomitant of \(rth\) case \(- I \) of \(gos\)} 
g_{d[r,n,m,k]}^{(2)}(y), & \text{pdf of concomitant of \(rth\) case \(- I \) of \(dgos\)} 
g_{[r,n,m,k]}^{(2)}(y), & \text{pdf of concomitant of \(rth\) case \(- II \) of \(gos\)} 
g_{d[r,n,m,k]}^{(2)}(y), & \text{pdf of concomitant of \(rth\) case \(- II \) of \(dgos\),} 
\end{cases} \]
\[ G^{(2)}_{Y_{r}}(y) = \begin{cases} 
G_{[r,n,m,k]}^{(2)}(y), & \text{cdf of concomitant of \(rth\) case \(- I \) of \(gos\)} 
G_{d[r,n,m,k]}^{(2)}(y), & \text{cdf of concomitant of \(rth\) case \(- I \) of \(dgos\)} 
G_{[r,n,m,k]}^{(2)}(y), & \text{cdf of concomitant of \(rth\) case \(- II \) of \(gos\)} 
G_{d[r,n,m,k]}^{(2)}(y), & \text{cdf of concomitant of \(rth\) case \(- II \) of \(dgos\).} 
\end{cases} \]

3. Information properties for concomitants in Huang and Kotz extension

In this section, we derive an analytical expression of entropy and Kullback-Leibler divergence for \(Y_{r}\) in Huang and Kotz extension. Also, applying the entropy for some well-known distributions of this model.

**Theorem 3.1.** For any absolutely continuous R.V. \(Y_{r}\), which is the concomitant of \(rth\) ordered R.V. of Huang and Kotz extension defined in Equation (46), \(1 \leq r \leq n\). From Equations (10) and (44), \(Y_{r}\) has entropy \(H(Y_{r})\) iff
\[ H(Y_{r}) = H(Y)[1 - \alpha M_{r}^{(1)}] + W(r,a) - \alpha(1 + p)M_{r}^{(1)} \int_{-\infty}^{\infty} F_{Y}(y)f_{Y}(y)\ln f_{Y}(y)\,dy, \]
where
\[ W(r,a) = - \ln(1 - \alpha M_{r}^{(1)}) + (1 - \alpha M_{r}^{(1)}) \, 2F_{1}^{(0,1,0,0)}(1;1,1+\frac{1}{p};\frac{\alpha M_{r}^{(1)}(1+p)}{\alpha M_{r}^{(1)}-1}). \]
\(2F_{1}^{(0,1,0,0)}(a,b;c;z)\) is the derivative of the Gaussian hypergeometric function \(2F_{1}(a,b;c;z)\) with respect to \(b\), and
\[ 2F_{1}^{(0,1,0,0)}(a,b;c;z) = \frac{2}{c} \frac{\partial}{\partial c} 2F_{1}(a+1,b+1;c+1;z), \]
and the general form of it is known as Kampé de Fériet’s series (Srivastava and Karlsson [20]).

**Proof.** From Equations (10) and (44), the Shannon entropy of Huang and Kotz extension is given by:
\[ H(Y_{r}) = \int_{-\infty}^{\infty} g^{(1)}_{Y_{r}}(y)\ln[g^{(1)}_{Y_{r}}(y)]\,dy \]
\[ = - \int_{-\infty}^{\infty} \alpha M_{r}^{(1)}(1+p)F_{Y}(y)f_{Y}(y)\ln[f_{Y}(y)]\,dy + H(Y)[1 - \alpha M_{r}^{(1)}] \]
\[ - E_{g^{(1)}_{Y_{r}}}(\ln[1 + \alpha M_{r}^{(1)}(1+p)F_{Y}(y) - 1])]. \]
To evaluate $E_{\tilde{U}_Y(y)}[\ln[1 + aM'_r((1 + p)F_\gamma(y) - 1)]]$. First, we want to find $E_{\tilde{U}_Y(y)}[1 + aM'_r((1 + p)F_\gamma(y) - 1)]$, let

$$u(t) = E_{\tilde{U}_Y(y)}[1 + aM'_r((1 + p)F_\gamma(y) - 1)]$$

$$= \int_{-\infty}^{\infty} f_Y(y) [(1 - aM'_r) + (aM'_r(1 + p)F_\gamma(y))] \, dy$$

$$= \sum_{j=0}^{\infty} \binom{t + 1}{j} (aM'_r(1 + p))^j (1 - aM'_r)^{t+1-j} \frac{1}{jp + j}$$

(60)

then

$$u(0) = E_{\tilde{U}_Y(y)}[\ln[1 + aM'_r((1 + p)F_\gamma(y) - 1)]]$$

$$= \ln(1 - aM'_r) - (1 - aM'_r) 2F_1(0, 1, 0, 1) \left[ \frac{1}{p} - 1; 1 + \frac{1}{p}; \left( \frac{aM'_r(1 + p)}{aM'_r - 1} \right) \right]$$

(61)

and the result follows.

In the following examples, we will choose some subfamilies of Huang and Kotz extension when they are exponential, Pareto and power function, and obtain its entropy as an applications of the last theorem.

**Example 3.1.** With the cdf of exponential distribution:

$$F_\gamma(y) = 1 - e^{-cy}, \ y \geq 0, \ c > 0,$$

from Equation (56), we get

$$H(Y^*_r) = W(r, a) - (1 - aM'_r)(\ln(c) - 1) - aM'_r(\ln(c) - B_{[1+p]}),$$

where $B_{[n]} = \psi(n + 1) - \psi(1)$ and $\psi(.)$ is the digamma function.

**Example 3.2.** With the cdf of Pareto distribution:

$$F_\gamma(y) = 1 - y^{-c}, \ y \geq 1, \ c > 0,$$

from Equation (56), we get

$$H(Y^*_r) = W(r, a) - (1 - aM'_r) \left( \ln(c) - \frac{1}{c} - 1 \right) - \frac{aM'_r(c \ln(c) - (1 + c)B_{[1+p]})}{c}.$$

**Example 3.3.** With the cdf of power function distribution:

$$F_\gamma(y) = y^c, \ 0 \leq y \leq 1, \ c > 0,$$

from Equation (56), we get

$$H(Y^*_r) = W(r, a) - (1 - aM'_r) \left( \ln(c) + \frac{1}{c} - 1 \right) - \frac{aM'_r(c(1 + p) \ln(c) + 1 - c)}{c(1 + p)}.$$

3.1. Kullback-Leibler divergence

In this subsection, we obtain Kullback-Leibler divergence between concomitants of $r$th and $s$th ordered $R.V.'s$ of Huang and Kotz extension.
**Theorem 3.2.** Let $Y^*_r$ and $Y^*_s$ be the concomitants of $r$th and $s$th ordered R.V.'s of Huang and Kotz extension. From Equations (11) and (44), the Kullback-Leibler divergence between $Y^*_r$ and $Y^*_s$ is given by:

$$K(Y^*_r, Y^*_s) = -W(r, \alpha) - \ln(1 - \alpha M^*_r) + \alpha M^*_r \, _2F_1^{(0,1,0,0)} \left(1 + \frac{1}{\alpha}, 0; 2 + \frac{1}{\alpha}; -\frac{\alpha M^*_r(1 + \alpha)}{\alpha M^*_r - 1} \right) -$$

$$\times (\alpha M^*_r - 1) \, _2F_1^{(0,1,0,0)} \left(1 + \frac{1}{\alpha}; 1 + \frac{\alpha M^*_r(1 + \alpha)}{\alpha M^*_r - 1} \right),$$

(62)

where $W(r, \alpha)$ is defined in (57), $\, _2F_1^{(0,1,0,0)}(a; b; c; z)$ is defined in (58).

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