Directed Percolation and Directed Animals

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Abstract

These lectures provide an introduction to the directed percolation and directed animals problems, from a physicist’s point of view. The probabilistic cellular automaton formulation of directed percolation is introduced. The planar duality of the diode-resistor-insulator percolation problem in two dimensions, and relation of the directed percolation to undirected first passage percolation problem are described. Equivalence of the $d$-dimensional directed animals problem to $(d-1)$-dimensional Yang-Lee edge-singularity problem is established. Self-organized critical formulation of the percolation problem, which does not involve any fine-tuning of coupling constants to get critical behavior is briefly discussed.

I. DIRECTED PERCOLATION

Although directed percolation (DP) was defined by Broadbent and Hammersley along with undirected percolation in their first paper in 1957 [1], it did not receive much attention in percolation theory until late seventies, when Blease obtained rather precise estimates of some critical exponents [2] using the fact that one can efficiently generate very long series expansions for this problem on the computer. Since then, it has been studied extensively, both as a simple model of stochastic processes, and because of its applications in different physical situations as diverse as star formation in galaxies [3], reaction-diffusion systems [4], conduction in strong electric field in semiconductors [5], and biological evolution [6]. The large number of papers devoted to DP in current physics literature are due to the fact that the critical behavior of a very large number of stochastic evolving systems are in the DP universality class. In this lecture, I briefly sketch some of these connections. Additional references may be found in [7-9], or the book by Stauffer and Aharony [10].

A. The Forest-Fire Model

Can we embed the conventional bond percolation, say on a square lattice, in a stochastic process, which evolves randomly in time? This may be done as follows: we think of the lattice as a forest, with each site occupied by a tree. A tree can be in one of three states: green, burning or ash. We assume that time $t = 0$, there is a single burning tree at the origin, and all other trees are green. The time evolution is discrete, and governed by the following rules:

(i) A burning tree at time $t$, becomes ash at time $(t + 1)$.

(ii) A green tree not neighbouring any burning neighbours remains green. If it has $r$ burning neighbours at time $t$, it catches fire and becomes a burning tree with probability $(1 - q^r)$, with $q = 1 - p$.

(iii) Ash remains ash for all subsequent times.

Here $p$ is a parameter $0 < p < 1$. If $p$ is small, eventually the fire dies, and a configuration of only green trees and ash results which does not change further. It is easy to to that the probability that the final configuration of burnt trees is a given set $S$, is the same as the probability that the cluster connected to the origin is $S$ in the bond percolation problem. There is a non zero probability that fire survives infinitely long for $p > 1/2$, and this exactly equals the probability of the origin belonging to the infinite cluster.
All questions relating to statistics of percolation clusters can be reformulated as questions about the probabilities of different sink states of the model. The advantage of this formulation is that it suggests other questions related to the time evolution of the system, e.g. what is the velocity with which the fire front spreads outwards for \( p > p_c \)?

**B. Forest-Fire Model with Regrowth**

A somewhat different model results if we allow the burnt trees to regrow, and become green again after some time-lag. The main qualitative difference over the previous case is that now the configuration with all sites green is the unique sink-state of the system. It turns out that the precise duration of the time-lag is not very important, and it is just as well if we assume that burnt sites become green at the next time step. This allows us to work with only two states per site: burning and green. We replace the rule (i) in the previous model by

\((i')\) the burning site becomes green at the next time step

and we do not need rule (iii) any more. Alternatively, this may be thought of as a model of infection of disease in a population, where the infected individual usually recovers after a period of illness. This is the simplest formulation of the DP problem. It is interesting that this new model has a non-trivial phase transition even in 1-dimension. It has been studied a lot, and very precise estimates are available for critical parameters and critical exponents [11], but an exact solution has not been possible so far.

**C. The Probabilistic Cellular Automaton Model**

Another terminology which is useful for the previous model is that of a probabilistic cellular automaton [12]. At each site \( i \), we have discrete variable \( n(i, t) \) at time \( t \), taking values 0 or 1, \( (0 = \text{no fire}, 1 = \text{fire}) \). Evolution is discrete time, parallel and local. In the simplest case, \( n(i, t + 1) \) depends only on the value of \( [n(i - 1, t) + n(i + 1, t)] \). Let \( p_r \) be the probability that \( n(i, t + 1) = 1 \) if \( n(i - 1, t) + n(i + 1, t) = r \). We assume \( p_0 = 0 \). This model with two parameter \( (p_1, p_2) \) is known as the Domany-Kinzel model [13]. For \( p_1 = p_2 = p \), this model is the directed percolation problem, and for \( p_1 = p, p_2 = 2p - p^2 \) it corresponds to the directed bond- percolation at concentration \( p \). Domany and Kinzel showed that this model can be solved exactly for \( p_2 = 1 \), when the problem reduces to that of annihilating random walkers on a line.

**D. Relation to Quantum Spin Chains**

The directed percolation process is basically a Markov process that involves allowing for the possibility of growth, propagation or death of some activity (here fire). In some applications, it is useful to think of this in continuous time. Then, we say \( n(i, t) = 1 \), it can change to 0 with rate 1, if \( n(i, t) = 0 \), and \( n(i - 1, t) + n(i + 1, t) = r \) then it can become 1 with rate \( r\lambda \).

The master equation for the evolution of \( \text{Prob}(C, t) \), the probability that configuration of the system is \( C \) at time \( t \), which has the form

\[
\frac{d}{dt}\text{Prob}(C, t) = \sum_{C'} W_{CC'} \text{Prob}(C', t),
\]

(1)

can be thought of as a Schrödinger equation for the evolution of the system. The ‘wavefunction’ to be in configuration \( C \) at time \( t \) is \( \text{Prob}(C, t) \), and the Hamiltonian is the matrix \( W_{CC'} \).

One can then use the experience and insight gained from the study of quantum Hamiltonians such as of spin-chains to learn about stochastic evolving systems. This has been a very useful approach in recent years (see [14] for a review), though this approach has a long history (see, for example [15]).

In this particular case, if we think of \( n_i = 1 \) as spin up, and \( n_i = 0 \) as spin down, then \( W \) becomes the Hamiltonian of a linear chain of spin-1/2 particles, with nearest neighbour couplings. Introducing \( a_i^+ \) and \( a_i \) as the Pauli spin raising and lowering operators at site \( i \), we see that the Hamiltonian \( W \) can be written in the form

\[
W = \sum_i \left[ a_i + \lambda a_i^+ a_i (a_{i+1}^+ + a_{i-1}^+) - a_i^+ a_i (1 + \lambda a_{i+1} a_{i+1}^+ + \lambda a_{i-1} a_{i-1}^+) \right].
\]

(2)

Note that \( W \) is not hermitian. This quantum mechanical formalism can be developed further into quantum field theoretical formulation. This was historically the first technique used to estimate critical exponents in all dimensions using the \( \epsilon \)-expansion techniques (for references, see [9]). We shall not discuss these further here.
E. Scaling Theory for Directed Percolation

The general theoretical treatment of the undirected percolation (existence of critical threshold, exponential decay below criticality etc.) goes through unchanged for the directed percolation problem. There are two major differences: there is no unique infinite percolation cluster. The infinite cluster depends on the choice of the origin. Secondly, below but near the critical threshold \( p_c \), the clusters are large, but anisotropic. We have to define two different correlation lengths \( \xi_\parallel \) and \( \xi_\perp \), which determine the average size of cluster along the preferred direction, and transverse to it. Near \( p_c \), these lengths diverge as \( (p_c - p)^{-\nu_\parallel} \) and \( (p_c - p)^{-\nu_\perp} \), where \( \nu_\parallel \) and \( \nu_\perp \) are different exponents with \( \nu_\parallel > \nu_\perp \).

The probability that the site \((R_\perp, R_\parallel)\) is connected to the origin, when the concentration is \( p = p_c + \epsilon \) defines the green’s function \( G(R_\perp, R_\parallel, \epsilon) \). For small \( \epsilon \) and large \( R_\perp \) and \( R_\parallel \), this function is expected to reduce to a scaling function of two arguments

\[
G(R_\perp, R_\parallel, \epsilon) \approx R_\perp^{-\beta/\nu_\parallel} g(\epsilon R_\perp^{1/\nu_\perp}, \epsilon R_\parallel^{1/\nu_\parallel})
\]

The function \( g(x_1, x_2) \) is expected to decrease as \( \exp(-|x_2|^{\nu_\parallel}) \) for \( x_2 \to -\infty \), and increase as \( x_2^{\beta/\nu_\parallel} \) for \( x \to +\infty \). For a fixed \( \epsilon > 0 \), the percolation occurs within a cone of angle \( \theta(p) \), and this angle varies as \( \epsilon^{\nu_\parallel - \nu_\perp} \) for small \( \epsilon > 0 \). Other exponents such as \( \gamma \) which characterizes the divergence of mean cluster size near \( p_c \) can be expressed in terms of these three exponents \( \beta, \nu_\parallel \) and \( \nu_\perp \) by using scaling relations.

F. Duality Transformation for Diode-Resistor-Insulator Percolation in two dimensions

One basic theoretical tool in the study of two-dimensional undirected percolation is the planar self-duality of the percolation problem. Happily, this is generalized to the directed case.

The duality is best described in terms of a more general diode-resistor-insulator percolation (with a suggestive acronym DRIP) model[16]. We consider a square lattice. Each bond is assumed to independently occupied by an insulator, forward-biassed diode, reversed biased diode, or two-way conductor (resistor) with probabilities \( p_0, p_+, p_- \) and \( p_2 \) respectively \( (p_0 + p_+ + p_- + p_2 = 1) \). A forward-biassed diode allows electric current to flow only up or right, and a reversed-biassed diode only down or left. The special case \( p_+ = p_- = 0 \) corresponds to the usual undirected percolation, and \( p_2 = p_- = 0 \) to the standard directed bond percolation.

The duality transformation is a generalization of that for the undirected problem. To each insulating (resistor) bond, we dual bond is resistor (insulating). The dual of a left, right-, up- or down-conducting diode bond is a diode conducting in the up-, down-, right- and left-direction respectively. Clearly the resistor insulator percolation (RIP) to self-dual. It is easy to see that if there is an infinite directed path in the original model in some direction (say, going towards up and right), then there is a blocking path in the dual problem along the same direction, which does not allow any connection across it (in one direction). Thus spanning probabilities in the original and dual problems get related. Similarly, the conductance of a configuration can be similarly related to the conductance of the dual lattice [16].

The dual of the diode-insulator percolation (DIP) problem \((p_+ = p, p_0 = 1 - p, p_- = p_2 = 0)\) is the diode-resistor percolation (DRP) problem \((p_+ = p, p_2 = 1 - p, p_- = p_0 = 0)\). In the DRP problem, for \( p = 1 \) if we take a typical configuration, and consider the set of sites reachable from the origin is just the first quadrant. For larger \( p \), the cluster of wetted sites form a staircase with no holes. However for \( p < p_c \), all sites of the lattice are wetted with probability 1 in the thermodynamic limit. The wedge angle of the cluster of wetted sites increases from \( \pi/2 \) to \( \pi \), as \( p_2 \) increases from 0 to \( p_c \) and jumps to value \( 2\pi \) for all \( p > p_c \). Thus, the shape of the wetted cluster is apparently much simpler in the DRP problem than in its dual DIP problem.

G. Relation between Directed and Undirected Problems

The directed and undirected percolation problems belong to different universality classes. In addition, the directed percolation problem shows extreme anisotropy near the critical point. It is thus interesting to realize that directed percolation properties are implicitly present in the undirected percolation case, and one can unravel these without any external imposition of a preferred direction. It was realized by Durrett and Liggett [17] that direction dependence of velocity of wetted front in the undirected problem changes qualitatively as the threshold for directed percolation is crossed. Consider undirected first-passage percolation on a square lattice. The time a fluid takes to wet a site after it has reached a neighbor is a random variable taking values 1 and 2 with probability \( p \) and \((1 - p)\) respectively. Then, if at time \( t = 0 \), we start with a fluid source at origin, the size of wetted cluster up to time \( t \) increases approximately
linearly with $t$. Let $v(\theta)$ be the average velocity of the fluid front in the direction $\theta$. Then for $p$ above the directed percolation threshold $p_c$, $v(\theta)$ is exactly 1 in any direction along which an infinite directed path exists. Behavior of this velocity near $p_c$ can be described in terms of the standard DP exponents. For a more detailed discussion, see ref. [18].

II. DIRECTED ANIMALS AND RELATED MODELS

We start by defining the animal problem, and its relation to the percolation problem in the more familiar undirected case first.

A. Relation of Undirected Animals to the Percolation Problem

One of the basic objects of study in percolation theory is $\text{Prob}(s, p)$, the probability that a randomly chosen site belongs to a cluster having $s$ sites, where $p$ is the concentration of occupied sites (bonds). For $p < p_c$, there are few large clusters, and for large $s$, $\text{Prob}(s, p)$ varies as

$$\text{Prob}(s, p < p_c) \sim A s^{-\theta} \exp[-B(p)s].$$

At $p = p_c$, the behavior is a power-law

$$\text{Prob}(s, p_c) \sim A' s^{-\tau}.$$  

For $p > p_c$, there is an infinite cluster, but the number of large finite clusters is again small. In this case, $\text{Prob}(s, p)$ decreases as a stretched exponential

$$\text{Prob}(s, p > p_c) \sim A'' s^{-\theta'} \exp[-B'(p)s^{d-\delta}].$$

Here the functions $B(p)$ and $B'(p)$ depend on the details of the lattice structure. These should go to zero as some universal power of $(p - p_c)$ near the critical percolation threshold. The exponents $\theta$ and $\theta'$ are universal and are independent of $p$.

While large finite clusters are not very likely for any $p$, we can still ask what is the typical diameter of a cluster, given that it has $s$ sites and has been drawn at random from a percolation problem at concentration $p$. Consider, first, undirected percolation. For $p < p_c$, the typical diameter $R_s$ varies as $K(s).s^\nu$ as $s \to \infty$, where $K(p)$ is a $p$-dependent coefficient, but the exponent $\nu$ is independent of $p$. At $p = p_c$, $R_s$ varies as a different power $\nu_0$ of $s$. [Here $1/\nu_0$ is the fractal dimension of percolation clusters.] For $p > p_c$, $R$ varies as $s^{1/d}$. Thus the structure of percolation clusters above $p_c$ is simple. The structure of these clusters at $p_c$ is the subject of much study. The animal problem deals with the question of specifying structure of typical large clusters when $p$ is below $p_c$. The simplest question is the value of the presumably universal exponents $\theta$ and $\nu$.

As these exponent are independent of $p$, without loss of generality we may study it in the special case $p \to 0$. In this limit, $\text{Prob}(s, p)$ tends to zero as $p^\nu$, but we get the simplification that all clusters of $s$ sites occur with equal probability. This is known as the animal problem.

Let $A_n$ be the number of distinct clusters of $n$ sites. For large $n$ this is expected to vary as

$$A_n \sim K \lambda^n n^{-\theta}$$

where $\lambda$ is a constant which depends on the lattice, and $\theta$ is an exponent. Giving equal weight to all the different animals of $n$ sites, we may determine the average size (say as measured by radius of gyration) of such animals. Call this $R_n$. For large $n$, $R_n$ varies as $n^\nu$, where $\nu$ is an exponent. Clearly, in $d$ dimensions $1 \leq \nu \leq 1/d$. The exact values of $A_n$, or the value of $\lambda$ are known only in one dimensions, on the Bethe lattice, and on some self-similar fractal lattices. Parisi and Sourlas [19] have given heuristic arguments for remarkable (presumably exact) relation between the exponents $\theta$ and $\nu$

$$\theta = (d - 2)\nu, \quad \text{for} \quad d \leq 8.$$ 

This is an analogue of the hyperscaling relation in the usual critical phenomena, which relates the quantity $dv$ to thermodynamic exponents. Note that here the factor which multiplies $\nu$ in this formula is $(d - 2)$, and not $d$. Thus the effective dimension of the system seems to decrease from $d$ to $(d - 2)$. This ‘dimensional reduction’ is due to a hidden supersymmetry in the field-theoretical formulation. A rigorous justification for this argument is not available yet.

For $d = 1, 2, 3$ and 4, the exponent $\theta$ is believed to take the values $-1, 0, 1/2$ and $5/6$. For $d \geq 8$, $\theta$ takes mean field value $3/2$ and $\nu$ is $1/4$. 

B. Directed Animals

As the undirected animals problem has not been solved exactly even in two dimensions, it seems desirable to look at some simpler variants of the problem. One simplification consists of making the bonds directed, i.e., look for the statistics of directed percolation clusters. It turns out that this variant is much more tractable analytically. In the following, I briefly review known results on this problem.

Consider a $d$-dimensional hypercubical lattice. We assume $d = 2$ for simplicity. Each site $(x,y)$ has two bonds directed outwards towards the sites $(x+1,y)$ and $(x,y+1)$. A directed (site-)animal is a set of ‘occupied’ sites (including the origin) such that for each occupied site $(x,y)$, other than the origin, at least one of the two sites $(x-1,y)$ and $(x,y-1)$ is also occupied. We shall denote the number of distinct animals having $s$ sites by $A_s$. For example, it is easy to verify, or write a short computer program for exhaustive enumeration of such animals [20], and see that for $n = 1, 2, 3, 4, 5 \ldots$ the numbers $A_n$ are $1, 2, 5, 13, 35 \ldots$. Based on the first few terms of this series, Dhar et al. [21] were able to guess the exact formula

$$A_n = \int_0^{2\pi} d\theta (1 + \cos \theta)(1 + 2\cos \theta)^{n-1}$$

The generating function of these animal numbers $A(x) = \sum_n A_n x^n$ satisfies a simple quadratic equation

$$(1 - 3x)[A(x) + A^2(x)] = x$$

We now indicate how this result comes about [22]. We note that in a directed animal, the allowed configuration of occupied sites on the line $x + y = T$ depend only on the configuration of occupied sites in the animal on the line $x + y = T - 1$. Starting with a single occupied site on the line $x + y = 0$, on the line $x + y = 1$, we can have at most two occupied sites: $(1,0)$ and $(0,1)$. This leads to the recursion equation

$$A(x) = x[1 + 2A(x) + A_{11}(x)]$$

where $A_{11}(x)$ is the generating function of animals starting from two occupied sites $(1,0)$ and $(0,1)$. In general, if we define $A_C(x)$ as the generating function of all animals starting with a source $C$ on the line $x + y = T$, we get a recursion relation

$$A_C(x) = x^{|C|}[1 + \sum_{C'} A_{C'}(x)]$$

where $|C|$ is the number of occupied sites in $C$, and the sum over $C'$ is over all configurations of occupied sites on the line $x + y = T + 1$ of a directed animal consistent with $C$. One can generate the animal numbers $A_n$ for quite large values of $n$ ($\sim 100$) using such recursion relations in a computer program. Such series can then be analysed using rather sophisticated extrapolation techniques. A large number of such series are known by now (for some examples, see [23]).

C. Relation to Hard-Core lattice Gas models

Alternatively, the above recursion relation may be viewed as the Chapman- Kollmogorov equation for a probabilistic cellular automaton on a line defined by the following rules: At time $\tau = 0$, all sites on the line $x + y = 0$ are assumed to be empty. At (integer-valued) time $\tau$ sites along the line $x + y = -\tau$ are examined, and a site $(x,y)$ is occupied with probability $p$ if both the sites $(x+1,y)$ and $(x,y+1)$ are empty. Else, the site is left empty. Clearly this corresponds to a Domany-Kinzel type automaton with $p_0 = p, p_1 = p_2 = 0$. This may be thought of as a model of growth of mixed crystals from solution layer by layer. We start with a solution of, say, NaCl and KCl. The crystal obtained by evaporating such a solution is substitutionally disordered, with Na and K atoms placed at random in the, in other words regular, lattice-structure. We ignore the chlorine atoms, and think of a simple cubic lattice formed by adding layers of Na and K atoms. We further assume that due to their larger size, two K atoms cannot be adjacent to each other. If allowed to be occupied by a K-atom, a position is actually occupied by a K atom with probability $p$, else it is occupied by Na.

Clearly, the probability that a site $P = (x,y)$ is occupied by a K-atom equals $p$ multiplied the probability that both the sites $P' = (x+1,y)$ and $P'' = (x,y+1)$ are not occupied by K-atoms. Using the inclusion-exclusion principle, we get

$$Prob(A) = p[1 - Prob(A') - Prob(A'') + Prob(A'A'')]$$

(13)
which is of the same form as the recursion equation (11). In general, for \( \text{Prob}(C) \), the probability that all sites of the set \( C \) lying on a line \( x + y = \text{constant} \) are occupied, we get

\[
\text{Prob}(C) = p^{|C|}[1 + \sum_{C'}(-1)^{|C'|}\text{Prob}(C')]
\]

(14)

where the sum over \( C' \) is over all proper subsets of the set of backward (in \( \tau \)) neighbors of \( C \). Comparing with eq. (12) we see that

\[
A_C(x = -p) = (-1)^{|C|}\text{Prob}(C)
\]

(15)

With this relation, the problem of determining the animal generating function \( A(x) \) reduces to finding the average density of \( K \)-atoms in the steady state of the probabilistic cellular automaton model defined above. The latter problem turns out to be very simple, as the corresponding rates satisfy the detailed balance condition. The corresponding hamiltonian is that of a lattice gas on a linear lattice with nearest-neighbor exclusion. The trivial exact solution of this gives us the exact result (9), and we find that

\[
\theta = \nu = 1/2, \quad \text{for} \quad d = 2.
\]

(16)

For details, consult [22]. Many more exact results for the 2-dimensional problem can be obtained in this way. For a more recent paper, which contained some interesting unproved (so far) conjectures also, see [22]. The mean longitudinal size of directed animals \( R_{||} \sim n^{\nu_{||}} \). The value of the exponent \( \nu_{||} \) is not known exactly. Numerical estimates suggest that it is not a simple fraction [24].

The exact calculation of the partition function of the hard-hexagon lattice gas by Baxter [25] as a function of the activity can be used to determine the exponents of the \( d=3 \) directed animals problem. This gives

\[
\theta = 2\nu = 5/6, \quad \text{for} \quad d = 3.
\]

(17)

D. Relation to the Lee-Yang Edge Singularity Problem

We have seen that the generating function of animal numbers in \( d \)-dimensions is becomes the expression for density of a nearest-neighbor exclusion lattice gas in \( (d-1) \) dimensions expressed as a function of the chemical activity. The latter is the well-known Mayer expansion. The radius of convergence of this determined by the closest singularity to the origin of the analytically continued free energy \( f(z) \) as a function of the activity \( z \), now treated as a complex variable. Now, the singularities of \( f(z) \) come from the zeroes of the partition function. It was noted by Lee and Yang long ago that these zeroes usually are distributed continuously along some lines in the complex \( -z \) plane. The line-density of zeroes on the line at the point \( z \) near the endpoint of a line \( z_c \) varies as a power of \( (z - z_c)^\sigma \). This exponent \( \sigma \) is quite universal, and does not seem to depend on the details of the hamiltonian of the system, only on the dimension of space \( d \) [26]. The directed animal exponent \( \theta \) can be expressed in terms of this by the relation

\[
\theta(d) = \sigma(d - 1) + 1
\]

(18)

Parisi and Sourlas related the \( \theta_{\text{undir}} \) exponent for the undirected animals problem in dimension \( d + 2 \) to the Lee-Yang Edge-singularity problem in \( d \) dimensions

\[
\theta_{\text{undir}}(d + 2) = \sigma(d) + 2
\]

(19)

Knowing the value of \( \theta(d = 3) = 5/6 \), we see using these relations that \( \sigma(d = 2) = -1/6 \) and \( \theta_{\text{undir}}(d = 4) = 11/6 \).

III. SELF-ORGANIZED CRITICALITY

We have seen that in the percolation problem, the correlations decay as power-laws, and the correlation length is infinite precisely at the critical point. At the critical point, the clusters have a fractal structure. Away from the critical point, the correlation function decays exponentially with distance. The same thing occurs in many other lattice models in statistical physics with discrete degrees of freedom per site with local interactions, like the Ising model. The equilibrium state of these models corresponds to exponential decay of correlations, except at the critical point. Thus, in order to get power-law correlations, one has to fine-tune the coupling constant to be very near the critical value. Study of behavior of systems near the critical point has been a major topic of study in statistical
physics in the last three decades, and a fair understanding has emerged of the universality of, and relations between, critical exponents from the renormalization-group approach to critical phenomena pioneered by K. Wilson.

On the other hand, in nature, we find a large number of fractal objects such as clouds, mountains, river-networks, which seem to be characterized by power-law correlations over a fairly broad range of length scales. But in these cases, it is clear that the physical processes that give rise to such structures must not require any fine-tuning of any control parameters. This realization has led to a lot of interest in the physics community in the study of systems which reach a stable state with long-range correlations without any need for the ‘unnatural’ fine-tuning of any control parameters. These systems may be said to self-organize into a critical state, and have been called Self-Organized Critical systems (SOC) [27].

In this section, I will try to introduce some percolation models showing criticality without any fine-tuning of control parameters.

A. Invasion Percolation

The simplest percolation model of this type is the invasion-percolation model. This was introduced by Wilkinson and Willemin in 1983 [28], and precedes the enunciation of the general concepts of SOC by Bak et al.

We consider a network of pipes forming a grid, say a two dimensional square lattice. The diameters of pipes are assumed to be independent, identically distributed random variables with some specified continuous distribution. Now we imagine forcing a viscous fluid into an initially empty network from outside, inserting it at one of the nodes, say the origin. [In actual applications, one pushes steam in a porous oil-bearing rock, forcing out oil.] With time, this fluid will spread via the network of pipes. As it is harder to push fluid in a thinner pipe, this spreading occurs preferentially using a subset of thicker pipes. At any time \( t \), there is a set of pipes \( B_t \) which lies at the boundary separating the wet from the dry nodes. We assume that actual spreading will occur using the thickest pipe from the set \( B_t \). We may characterize the amount of force needed to push fluid through the pipe (overcoming the capillary forces) using a real variable \( x \) with \( 0 \leq x \leq 1 \), such that lower value of \( x \) implies lower force. The time may be measured in discrete units in terms of the number of sites wetted up to that instant.

Thus in this model, the cluster of wetted sites grows in time, always using the weakest of the available links for growth. However the shape of the cluster of wetted sites is stochastic, and depends on the realization of the random medium. The wetted cluster can have holes which may not be filled for a long time. In fact, simulations show that for large times \( t \), the cluster \( C_t \) of sites wetted up to time \( t \) contains holes of all length scales, and looks like incipient infinite percolation cluster.

The relation of this model to the percolation problem becomes clearer if we assume that probability distribution of the variable \( x \) is uniform in the interval \([0,1]\). This we can do without loss of generality, as the variable \( x \) can be replaced by any monotonic increasing function of \( x \) without affecting the dynamics. Let the percolation threshold on this lattice be \( p_c \). Consider a value \( p_1 = p_c + \epsilon \), where \( \epsilon \) is an arbitrarily small positive number, and consider the infinite percolation cluster formed using only the pipes for which \( x \leq p_1 \). Once the cluster of wetted sites hits any of the sites of this cluster (as it must eventually), further growth can occur only using links of this infinite cluster. Thus, if \( \text{Prob}(x) \) is the limiting probability distribution at large times that the next growth occurs using a bond having value \( x \), we conclude that \( \text{Prob}(x) \) is exactly zero for all \( x \geq p_c \). On the other hand, \( \text{Prob}(x) \) must be nonzero for all \( x \leq p_c \), as there are no infinite clusters in which all bonds have a value \( x < p_c - \epsilon \).

The interesting point about this model is that the dynamical rules make no mention of \( p_c \). Thus no unnatural fine-tuning of parameters is done to get the fractal growth cluster, whose properties at large length scales are same as of the critical percolation clusters. For finite times \( t \), the distribution of clusters \( C_t \) is not same as in the standard percolation problem, as is easy to check for \( t = 1, 2, \cdots \).

B. The Sneppen model

We may similarly study the invasion percolation problem for the diode-resistor network defined earlier. Consider, for simplicity, again a square lattice. We assume that all links allow fluid to flow easily (no force) in the positive direction (up or right), but need a finite force \( x \) to flow in the reverse direction. The values of \( x \) are i.i.d. random variables for each link, with a uniform distribution between 0 and 1. We imagine that time \( t = 0 \) all sites \((x, y)\) with \( x + y \geq 0 \) are wet. At any time \( t \), we choose the bond with the least value of \( x \) lying on the boundary between the wet and dry sites, and push fluid through it. Thus the wetted region grows, and the interface between wet and dry sites moves left and down. If a site is wetted but its rightward or upward neighbor is dry, then these sites also become wet.

This model is usually called the Sneppen model [29]. The important point is that the movement of the interface occurs in bursts, and there is a distribution of burst sizes with a power-law tail. If you look at the values of \( x \) selected,
they will lie only between 0 and $p_c$, where $p_c$ is the threshold for the diode-resistor percolation in this model. The surface after a long time becomes ‘rough’, with a nontrivial roughness exponent, which is relatable to exponents of directed percolation. There has been a lot of interest in determining the distribution of burst sizes in this model in 2 and higher dimensions in recent years [30].

C. Self-organized Directed Percolation

In the Sneppen model, one does not need fine-tuning to get a critical interface. The basic mechanism which makes this possible is the fact that the growth is assumed to occur at the site corresponding to global minimum of all \{x\} along the interface. This feature, while no doubt a good approximation in some situations, is not very aesthetically pleasing, as it implies that all points at the interface ‘know’ about the status of the interface at all points. In physical systems, one usually prefers to write evolution laws which are local, and depend only on values of various quantities in the neighbourhood of the point. Can one make a model with local stochastic evolution rules, which gives nontrivial critical behavior without any fine-tuning of control parameters?

It was realized by Grassberger and Zhang [31] that this is indeed possible. They considered a simple coupled map lattice defined as follows: Consider a linear chain with a real variable $x(i, t)$ at each site $i$ with $0 \leq x(i, t) \leq 1$. At time $t = 0$, all the variables $x(i, t = 0) = 0$. Time is discrete, and all sites evolve in parallel using the rule

$$x(i, t + 1) = \text{Max}[\eta(i, t), \text{Min}(x(i, t), x(i - 1, t))]$$

(20)

where $\eta(i, t)$ are random variables drawn from a uniform distribution between 0 and 1, independent for different space-time points $(i, t)$. After a time $T$, all the values $x(i, T)$ have a marginal distribution given by

$$\text{Prob}(x(i, T) \geq p) = P_{DP}(p, T)$$

(21)

where $P_{DP}(p, T)$ is the probability that site $i$ will be wet at time $T$ for a directed site percolation problem on the square lattice with site concentration $p$ when the initial state is all sites wet. To see this, we just note that if we define a variable $y(i, t)$, which takes values 1 and 0 according as $x(i, t) \leq p$ or greater than $p$. Then the process $y(i, t)$ is a simple DP percolation process [1 corresponds to a wet site, 0 to healthy]. The rate of approach of the distribution of $x(i, T)$ to the limiting distribution gives information about other DP exponents.

D. Self-Organized Undirected Percolation

Can one make a similar model for undirected percolation? This is also possible but it involves a non-trivial variation of the Grassberger-Zhang construction. Consider a network of sites and bonds, where each bond is randomly assigned a strength $x$ lying between 0 and 1. We assign a variable $y(i, t)$ to each site $i$, also lying between 0 and 1 which evolves with time $t$ by the following rules:

i) For all sites $i$ not at the boundary, $y(i, t = 0) = 1$. At boundary sites $y(i, t) = 0$ for all times $t$.

ii) All sites not at the boundary are updated in parallel using the rule:

$$y(i, t + 1) = \text{Min}_j[\text{Max}(y(j, t), x_{ij})]$$

(22)

where the minimum is taken over all sites $j$ neighboring $i$, and $x_{ij}$ is the strength of bond lying between $i$ and $j$.

It is easily seen that $y$’s are nonincreasing functions of time, and for any finite lattice, they reach some fixed point values $y^*(i)$, which depend on the configuration. By induction on $t$, it is easy to show that for all times $t$ and all sites $i$, there exists a path from $i$ to the boundary which uses only bonds with strength $\leq y(i, t)$. As in the directed case, the limiting distribution $\text{Prob}(y^* \geq z)$ of $y^*$’s is equal to the fraction of sites which belong to the infinite cluster for bond concentration $p = z$.

More generally, can one take a usual equilibrium statistical model with a critical point, say an Ising model, and endow it with local stochastic evolution rules such that it will at large times always relax to a steady state which corresponds to the model at its critical temperature without any fine-tuning of parameters? These questions are still open. This has been possible so far only in a model with a rather complicated set of variables at each site [32].

It is presumably clear from the examples discussed above that the ideas of percolation theory have found many uses in many ways in physics, and continue to do so. It is hoped that these will also provide new directions to the more mathematical studies.
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