CONCENTRATION OF 1-LIPSCHITZ FUNCTIONS ON MANIFOLDS WITH BOUNDARY WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper, we consider a concentration of measure problem on Riemannian manifolds with boundary. We study concentration phenomena of non-negative 1-Lipschitz functions with Dirichlet boundary condition around zero, which is called boundary concentration phenomena. We first examine relation between boundary concentration phenomena and large spectral gap phenomena of Dirichlet eigenvalues of Laplacian. We will obtain analogue of the Gromov-V. D. Milman theorem and the Funano-Shioya theorem for closed manifolds. Furthermore, to capture boundary concentration phenomena, we introduce a new invariant called the observable inscribed radius. We will formulate comparison theorems for such invariant under a lower Ricci curvature bound, and a lower mean curvature bound for the boundary. Based on such comparison theorems, we investigate various boundary concentration phenomena of sequences of manifolds with boundary.

1. Introduction

In the present paper, we consider a concentration of measure problem on manifolds with boundary. We study concentration phenomena of non-negative 1-Lipschitz functions with Dirichlet boundary condition.

1.1. Motivations. Let us recall the following well-known fact: The normalized volume measure on the n-dimensional unit sphere concentrates around the equator when n is large. One can rephrase this fact as follows: The normalized volume measure on the n-dimensional unit hemisphere concentrates around the boundary when n is large.

We call a triple $X = (X, d_X, \mu_X)$ a (smooth) metric measure space with boundary when $X$ is a connected complete Riemannian manifold.
with boundary, \(d_X\) is the Riemannian distance, and \(\mu_X\) is a Borel probability measure on \(X\). Let \(\partial X\) denote its boundary. In this paper, we consider the following problem: For a given sequence \(\{X_n\}\) of metric measure spaces with boundary \(X_n = (X_n, d_{X_n}, \mu_{X_n})\), does the measure \(\mu_{X_n}\) concentrate around \(\partial X_n\) when \(n\) is large? We will observe that \(\mu_{X_n}\) concentrates around \(\partial X_n\) if and only if every 1-Lipschitz function \(\varphi_n : X_n \to [0, \infty)\) with \(\varphi_n|_{\partial X_n} = 0\) is closed to zero (more precisely, see Remark 1.1 and Proposition 3.6). From this point of view, we investigate concentration phenomena of non-negative 1-Lipschitz functions with Dirichlet boundary condition around zero. We call such phenomena boundary concentration phenomena.

1.2. Observable inscribed radii. Gromov [13] has established theory of geometry of metric measure spaces based on the idea of concentration of measure phenomena discovered by Lévy [23], and developed by V. D. Milman [31], [32]. He has introduced some important invariants on metric measure spaces. One of them is the so-called observable diameter that measures the difference between 1-Lipschitz functions and constants. The observable diameter has been widely studied from the viewpoint of the study of concentration phenomena of 1-Lipschitz functions (see e.g., [13], [22], [46] and the references therein).

We now introduce a new invariant on metric measure spaces with boundary called the observable inscribed radius that measures the difference between non-negative 1-Lipschitz functions with Dirichlet boundary condition and zero. We will refer to the formulation of the observable diameter on metric measure spaces.

Let \(X = (X, d_X, \mu_X)\) be a metric measure space with boundary. Let \(\rho_{\partial X} : X \to [0, \infty)\) stand for the distance function from the boundary \(\partial X\) defined as \(\rho_{\partial X}(x) := d_X(x, \partial X)\). The function \(\rho_{\partial X}\) is 1-Lipschitz with \(\rho_{\partial X}|_{\partial X} = 0\). The inscribed radius \(\text{InRad} \ X\) of \(X\) is defined to be the supremum of the distance function \(\rho_{\partial X}\) over \(X\). We extend the notion of the inscribed radius to all subsets of \(X\). For \(\Omega \subset X\), we define the inscribed radius \(\text{InRad} \ \Omega\) of \(\Omega\) as follows: If \(\Omega \neq \emptyset\), then

\[
(1.1) \quad \text{InRad} \ \Omega := \sup_{x \in \Omega} \rho_{\partial X}(x);
\]

if \(\Omega = \emptyset\), then \(\text{InRad} \ \Omega := 0\). For \(\xi \in (-\infty, 1]\), let us define the \(\xi\)-partial inscribed radius \(\text{PartInRad}(X; \xi)\) of \(X\) by

\[
(1.2) \quad \text{PartInRad}(X; \xi) := \inf_{\Omega \subset X} \text{InRad} \ \Omega,
\]

where the infimum is taken over all Borel subsets \(\Omega\) with \(\mu_X(\Omega) \geq \xi\).
We set a screen $I := [0, \infty)$. For a 1-Lipschitz function $\varphi : X \to I$ with $\varphi|_{\partial X} = 0$, we call the metric measure space with boundary
\begin{equation}
I_\varphi := (I, d_I, m_{I,\varphi})
\end{equation}
the $\varphi$-screen, where $m_{I,\varphi}$ denotes the push-forward $\varphi#\mu_X$ of $\mu_X$ by $\varphi$.

We now define the following quantity:

**Definition 1.1.** For $\eta > 0$, we define the $\eta$-observable inscribed radius $\text{ObsInRad}(X; -\eta)$ of $X$ by

$$\text{ObsInRad}(X; -\eta) := \sup_{\varphi} \text{PartInRad}(I_\varphi; 1 - \eta),$$

where the supremum is taken over all 1-Lipschitz functions $\varphi : X \to I$ with $\varphi|_{\partial X} = 0$.

We remark that $\text{ObsInRad}(X; -\eta) = 0$ for all $\eta \geq 1$. Furthermore, $\text{ObsInRad}(X; -\eta)$ is monotone non-increasing in $\eta$.

We also introduce the following notion:

**Definition 1.2.** We say that a sequence $\{X_n\}$ of metric measure spaces with boundary is a **boundary concentration family** if for every $\eta > 0$

$$\lim_{n \to \infty} \text{ObsInRad}(X_n; -\eta) = 0.$$

**Remark 1.1.** Let $\{X_n\}$ denote a sequence of metric measure spaces with boundary $X_n = (X_n, d_{X_n}, \mu_{X_n})$. By the definition of the observable inscribed radius, $\{X_n\}$ is a boundary concentration family if and only if for every sequence $\{\varphi_n\}$ of 1-Lipschitz functions $\varphi_n : X_n \to I$ with $\varphi_n|_{\partial X_n} = 0$, we have $d_{KF}(\varphi_n, 0) \to 0$ as $n \to \infty$, where $d_{KF}(\varphi_n, 0)$ is the Ky Fan metric between $\varphi_n$ and 0 defined as

$$d_{KF}(\varphi_n, 0) := \inf \{\epsilon \geq 0 \mid \mu_{X_n}(\{x \in X_n \mid \varphi_n(x) > \epsilon\}) \leq \epsilon\}.$$

We further see that the following are equivalent (see Proposition 3.6):

1. $\{X_n\}$ is a boundary concentration family;
2. for every sequence $\{\Omega_n\}$ of Borel subsets $\Omega_n \subset X_n$ satisfying $\liminf_{n \to \infty} \mu_{X_n}(\Omega_n) > 0$, we have $\liminf_{n \to \infty} d_{X_n}(\Omega_n, \partial X_n) = 0$;
3. $\lim_{n \to \infty} \mu_{X_n}(B_r(\partial X_n)) = 1$ for every $r > 0$, where $B_r(\partial X_n)$ is the closed $r$-neighborhood of $\partial X_n$.

### 1.3. Dirichlet eigenvalues and concentration phenomena

We study relation between boundary concentration phenomena and large spectral gap phenomena for Dirichlet eigenvalues of Laplacian. For $n \geq 2$, let $(M, d_M, m_M, f)$ be an $n$-dimensional weighted Riemannian manifold with boundary, namely, $M = (M, g)$ be an $n$-dimensional,
connected complete Riemannian manifold with boundary, \( d_M \) is the Riemannian distance on \( M \), and
\[
(1.4) \quad m_{M,f} := e^{-f} \text{vol}_M
\]
for some smooth function \( f : M \to \mathbb{R} \), where \( \text{vol}_M \) is the Riemannian volume measure on \( M \). The weighted Laplacian \( \Delta_f \) is defined by
\[
(1.5) \quad \Delta_f := \Delta + g(\nabla f, \nabla \cdot)
\]
where \( \nabla \) is the gradient, and \( \Delta \) is the Laplacian defined as the minus of the trace of Hessian. In the case where \( M \) is compact, we consider the following Dirichlet eigenvalue problem with respect to \( \Delta_f \):
\[
\begin{cases}
\Delta_f \phi = \nu \phi & \text{in Int } M; \\
\phi = 0 & \text{on } \partial M,
\end{cases}
\]
where \( \text{Int } M \) denotes the interior of \( M \). We denote by
\[
(1.6) \quad 0 < \nu_{f,1}(M) < \nu_{f,2}(M) \leq \cdots \leq \nu_{f,k}(M) \leq \cdots \nearrow +\infty
\]
the all Dirichlet eigenvalues of \( \Delta_f \), counting multiplicity.

For a smooth function \( f : M \to \mathbb{R} \) such that \( m_{M,f} \) is a Borel probability measure, we study the metric measure space with boundary
\[
(1.7) \quad (M, f) := (M, d_M, m_{M,f}).
\]
Our first main result is the following:

**Theorem 1.1.** Let \( \{(M_n, f_n)\} \) be a sequence of compact metric measure spaces with boundary defined as (1.7). If we have \( \nu_{f_n,1}(M_n) \to \infty \) as \( n \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

This is an analogue of the Gromov-V. D. Milman theorem for closed manifolds (compact manifolds without boundary) (see Theorem 4.1 and its corollary in [14]). We show Theorem 1.1 by using relation between the observable inscribed radii and \( \nu_{f,1}(M) \) (see Proposition 4.2).

For higher eigenvalues, we will establish the following assertion under \( \text{Ric}^\infty_{f,M} \geq 0 \) and \( H_{f,\partial M} \geq 0 \), where \( \text{Ric}^\infty_{f,M} \) and \( H_{f,\partial M} \) are the infimum of the \( \infty \)-weighted Ricci curvature and the weighted mean curvature on \( M \) and on \( \partial M \), respectively (more precisely, see Subsection 2.1):

**Theorem 1.2.** Let \( \{(M_n, f_n)\} \) be a sequence of compact metric measure spaces with boundary defined as (1.7). Assume that \( \text{Ric}^\infty_{f_n,M_n} \geq 0 \) and \( H_{f_n,\partial M_n} \geq 0 \). If there exists \( k \geq 1 \) such that \( \nu_{f_n,k}(M_n) \to \infty \) as \( n \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

Theorem 1.2 is an analogue of the Funano-Shioya theorem for closed manifolds of non-negative \( \infty \)-weighted Ricci curvature (see Corollary 1.4 in [12]). One of key ingredients of the proof is to obtain an upper
bound of the ratio $\nu_{f,k}(M)/\nu_{f,1}(M)$ in terms of $C k^2$ for some universal constant $C > 0$ under $\text{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$ (see Theorem 2.4). We obtain such universal estimate by combining an improved Cheeger inequality of Kwak, Lau, Lee, Oveis Gharan and Trevisan [20], and an isoperimetric inequality of Wang [49] (see Subsections 2.3, 2.4).

1.4. Comparisons and concentration phenomena. To understand boundary concentration phenomena, we establish comparison theorems of the observable inscribed radii under a lower curvature Ricci curvature bound, and a lower mean curvature bound for the boundary.

We first present finite dimensional comparisons. Let $M$ be an $n$-dimensional, connected complete Riemannian manifold with boundary with $\text{vol}_M(M) < \infty$. We study metric measure space with boundary

$$(1.8) \quad M := (M, d_M, m_M), \quad m_M := \frac{1}{\text{vol}_M(M)} \text{vol}_M.$$

Let $\text{Ric}_{\partial M}$ stand for the infimum of the Ricci curvature in the $\partial M$-radial direction on $M$, and $H_{\partial M}$ the infimum of the mean curvature on $\partial M$ (more precisely, see Subsection 2.1).

For $\kappa \in \mathbb{R}$, let $M^n_{\kappa}$ be the $n$-dimensional space form with constant curvature $\kappa$. For $\lambda \in \mathbb{R}$, we say that $\kappa$ and $\lambda$ satisfy the ball-condition if there exists a closed ball $B^n_{\kappa,\lambda}$ in $M^n_{\kappa}$ whose boundary has constant mean curvature $(n - 1)\lambda$. Let $C_{\kappa,\lambda}$ denote its radius. Note that $\kappa$ and $\lambda$ satisfy the ball-condition if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. We say that $\kappa$ and $\lambda$ satisfy the convex-ball-condition if they satisfy the ball-condition and $\lambda \geq 0$.

Let us prepare the following finite dimensional model spaces: (1) For $\kappa$ and $\lambda$ satisfying the ball-condition, we call the metric measure space

$$(1.9) \quad B^n_{\kappa,\lambda} = (B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}}, m_{B^n_{\kappa,\lambda}})$$

the ball-model-space; (2) For $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$, we consider the warped product space $M^n_{\kappa,\lambda} := (\mathbb{0}, \infty) \times S^{n-1}_{\lambda}, dt^2 + e^{-2\lambda} ds^{2}_{n-1})$, where $(S^{n-1}_{\lambda}, ds^{2}_{n-1})$ is the $(n - 1)$-dimensional standard unit sphere. We call the metric measure space with boundary

$$(1.10) \quad M^n_{\kappa,\lambda} = (M^n_{\kappa,\lambda}, d_{M^n_{\kappa,\lambda}}, m_{M^n_{\kappa,\lambda}})$$

the warped-product-model-space (cf. Remark 2.6).

We have the following finite dimensional comparison theorem:

**Theorem 1.3.** Let $\partial M$ be compact. We assume $\text{Ric}_{\partial M} \geq (n - 1)\kappa$ and $H_{\partial M} \geq (n - 1)\lambda$. Then for every $\eta \in (0, 1]$ the following hold:
(1) if $\kappa$ and $\lambda$ satisfy the ball-condition, then
\[ \text{ObsInRad}(M; -\eta) \leq \text{ObsInRad}(B_{\kappa,\lambda}^n; -\eta); \]

(2) if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then
\[ \text{ObsInRad}(M; -\eta) \leq \text{ObsInRad}(M_{\kappa,\lambda}^n; -\eta). \]

Remark 1.2. For (1) of Theorem 1.3, we always have $\text{vol}_M (M) < \infty$. Indeed, the Heintze-Karcher theorem (Theorem 2.1 in [15]) leads to
\[ \frac{\text{vol}_M (M)}{\text{vol}_{\partial M} (\partial M)} \leq \frac{\text{vol}_{B_{\kappa,\lambda}^n} (B_{\kappa,\lambda}^n)}{\text{vol}_{\partial B_{\kappa,\lambda}^n} (\partial B_{\kappa,\lambda}^n)}. \]

Similarly, for (2), the Heintze-Karcher theorem guarantees that the Riemannian volume $\text{vol}_M (M)$ of $M$ is finite.

The proof of Theorem 1.3 is based on comparison geometry of manifolds with boundary established by Heintze and Karcher [15], Kasue [16], [17], the author [43], [45], and so on (see Subsection 2.5). We first estimate observable inscribed radii under lower weighted curvature bounds by using relative volume comparison theorems for metric neighborhoods of boundaries (see Theorems 5.3 and 5.4). We conclude Theorem 1.3 by computing the observable inscribed radii of finite dimensional model spaces (see Lemma 5.5).

We next produce infinite dimensional comparisons for metric measure spaces with boundary $(M, f)$ defined as (1.7) under $\text{Ric}_{f,\partial M}^\infty \geq K$ and $H_{f,\partial M} \geq \Lambda$ for $K, \Lambda \in \mathbb{R}$, where $\text{Ric}_{f,\partial M}^\infty$ is the infimum of the $\infty$-weighted Ricci curvature in the $\partial M$-radial direction on $M$ (see Subsection 2.1). We prepare the following infinite dimensional model spaces:

(1) For $K > 0$, $\Lambda \in \mathbb{R}$, we call the metric measure space with boundary
\[ G_{K,\Lambda} := (I, d_I, \frac{e^{-\frac{K}{2} t^2 - \Lambda t}}{\int_I e^{-\frac{K}{2} t^2 - \Lambda t} dt} \text{vol}_I) \]
the half-Gaussian-model-space; (2) For $\Lambda > 0$, we call
\[ E_{\Lambda} := (I, d_I, \Lambda e^{-\Lambda t} \text{vol}_I) \]
the exponential-model-space. We remark that for $K, \Lambda \in \mathbb{R}$, the value $\int_I e^{-\frac{K}{2} t^2 - \Lambda t} dt$ is finite if and only if either (1) $K > 0$; or (2) $K = 0$ and $\Lambda > 0$. Moreover, if $K = 0$ and $\Lambda > 0$, then we see
\[ \frac{e^{-\frac{K}{2} t^2 - \Lambda t}}{\int_I e^{-\frac{K}{2} e^2 - \Lambda t} dt} = \Lambda e^{-\Lambda t}. \]

We observe that our infinite dimensional model spaces appear as limits of sequences of finite dimensional model spaces (see Subsection 6.2).
We have the following infinite dimensional comparison:

**Theorem 1.4.** Let $\partial M$ be compact. Let us assume $\text{Ric}^{\infty}_{f,\partial M} \geq K$ and $H_{f,\partial M} \geq \Lambda$. Then for every $\eta \in (0,1]$ the following hold:

1. If $K > 0$ and $\Lambda \in \mathbb{R}$, then
   
   $\text{ObsInRad}((M,f); -\eta) \leq \text{ObsInRad}(G_{K,\Lambda}; -\eta)$;

2. If $K = 0$ and $\Lambda > 0$, then
   
   $\text{ObsInRad}((M,f); -\eta) \leq \text{ObsInRad}(E_{\Lambda}; -\eta)$.

To prove Theorem 1.4, we develop comparison geometry of manifolds with boundary under $\text{Ric}^{\infty}_{f,\partial M} \geq K$ and $H_{f,\partial M} \geq \Lambda$ for $K, \Lambda \in \mathbb{R}$ (see Subsection 6.1). We will show a relative comparison theorem for metric neighborhoods of boundaries (see Theorem 6.3).

Having Theorems 1.3 and 1.4 at hand, we will study various boundary concentration phenomena of sequences of metric measure spaces with boundary (see Section 7). For instance, for a sequence of ball-model-spaces, we conclude the following:

**Corollary 1.5.** If $\kappa$ and $\lambda$ satisfy the convex-ball-condition, then the sequence $\{B^n_{\kappa,\lambda}\}$ is a boundary concentration family.

**Remark 1.3.** In the case where $\kappa > 0$ and $\lambda < 0$, the sequence $\{B^n_{\kappa,\lambda}\}$ is not a boundary concentration family. Indeed, for $r \in (0, C_{\kappa,\lambda} - C_{\kappa,0})$, if we define $\Omega_n \subset B^n_{\kappa,\lambda}$ as the $r/2$-neighborhood of the metric sphere with same center as $B^n_{\kappa,\lambda}$ and radius $C_{\kappa,0}$, then $\liminf_{n \to \infty} m_{B^n_{\kappa,\lambda}}(\Omega_n) > 0$ and $\lim_{n \to \infty} d_{B^n_{\kappa,\lambda}}(\Omega_n, \partial B^n_{\kappa,\lambda}) > 0$ (see Remark 1.1 and Proposition 3.6).

1.5. **Organization.** In Section 2 we review basics of weighted Riemannian manifolds with boundary, and examine their geometric and analytic properties. In Section 3 we introduce some invariants on metric measure spaces with boundary, and investigate their fundamental properties. In Section 4 we prove Theorems 1.1 and 1.2. In Section 5 we prove Theorem 1.3. In Section 6 we prove Theorem 1.4.

Section 7 is devoted to the collection of boundary concentration phenomena of sequences of metric measure spaces with boundary. We will determine the critical scale orders of some sequences of finite dimensional model spaces (see Theorems 7.1, 7.2, 7.3). We also prove Corollary 1.5. Furthermore, we construct several non-trivial examples of boundary concentration families (see Examples 7.1 and 7.2).

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2. Preliminaries

Throughout this section, let \((M,d_M,m_M,f)\) denote an \(n\)-dimensional weighted Riemannian manifold with boundary defined as (1.4).

2.1. Curvatures. We denote by \(\text{Ric}_g\) the Ricci curvature on \(M\) determined by the Riemannian metric \(g\), and by \(\text{Ric}_M\) the infimum of \(\text{Ric}_g\) on the unit tangent bundle over \(M\). For \(N \in (-\infty, \infty]\), the \(N\)-weighted Ricci curvature \(\text{Ric}_N^f\) is defined as follows: If \(N \in (-\infty, \infty) \setminus \{n\}\), then
\[
\text{Ric}_N^f := \text{Ric}_g + \text{Hess} f - \frac{df \otimes df}{N-n},
\]
where \(df\) and \(\text{Hess} f\) are the differential and the Hessian of \(f\), respectively; otherwise, if \(N = \infty\), then \(\text{Ric}_N^f := \text{Ric}_g + \text{Hess} f\); if \(N = n\), and if \(f\) is constant, then \(\text{Ric}_N^f := \text{Ric}_g\); if \(N = n\), and if \(f\) is not constant, then \(\text{Ric}_N^f := -\infty\) ([1], [24]). For a function \(F : M \to \mathbb{R}\), we mean by \(\text{Ric}_N^f \geq F\) for every \(x \in M\), and for every unit tangent vector \(v\) at \(x\) it holds that \(\text{Ric}_N^f(v) \geq F(x)\).

Remark 2.1. Traditionally, \(N\) has been chosen from \([n, \infty]\) (see e.g., [30], [42], [50]). On the other hand, recently, various properties have begun to be studied in the complementary case of \(N \in (-\infty, n)\) (see e.g., [19], [18], [36], [38], [39], [40], [41], [51], [52]). We notice the monotonicity of \(\text{Ric}_N^f\) with respect to \(N\): If \(N_1, N_2 \in [n, \infty]\) with \(N_1 \leq N_2\), then \(\text{Ric}_{N_1}^f \leq \text{Ric}_{N_2}^f\); if \(N_1 \in [n, \infty]\) and \(N_2 \in (-\infty, n)\), then \(\text{Ric}_{N_1}^f \leq \text{Ric}_{N_2}^f\); if \(N_1, N_2 \in (-\infty, n)\) with \(N_1 \leq N_2\), then \(\text{Ric}_{N_1}^f \leq \text{Ric}_{N_2}^f\).

For \(z \in \partial M\), let \(u_z\) denote the unit inner normal vector for \(\partial M\) at \(z\), and let \(\gamma_z : [0,T) \to M\) denote the geodesic with \(\gamma_z'(0) = u_z\). We put
\[
\tau(z) := \sup \{ t > 0 \mid \rho_{\partial M}(\gamma_z(t)) = t \}.
\]

Let \(\text{Ric}_{\partial M}^\ast\) be the infimum of the Ricci curvature in the \(\partial M\)-radial direction on \(M\) defined as \(\inf_{z,t} \text{Ric}_g(\gamma_z'(t))\), where the infimum is take over all \(z \in \partial M\), \(t \in (0, \tau(z))\). For \(\mathcal{F} : M \to \mathbb{R}\), we mean by \(\text{Ric}_{\partial M}^\ast \geq \mathcal{F}\) for all \(z \in \partial M\), \(t \in (0, \tau(z))\) we have \(\text{Ric}_{\partial M}^\ast(\gamma_z'(t)) \geq \mathcal{F}(\gamma_z(t))\).

For vector fields \(v_1, v_2\) on \(\partial M\), the second fundamental form \(S(v_1, v_2)\) is defined as the normal component of \(\nabla^g_{v_1} v_2\) with respect to \(\partial M\), where
\(\nabla^g\) denotes the Levi-Civita connection induced from \(g\). For \(z \in \partial M\), let \(T_z \partial M\) denote the tangent space at \(z\) on \(\partial M\). The shape operator \(A_{u_z} : T_z \partial M \to T_z \partial M\) for \(u_z\) is defined as
\[
g(A_{u_z} v_1, v_2) := g(S(v_1, v_2), u_z).
\]
The mean curvature \(H_z\) at \(z\) is defined as the trace of \(A_{u_z}\). Put \(H_{\partial M} := \inf_{z \in \partial M} H_z\). The weighted mean curvature \(H_{f,z}\) at \(z\) is defined by
\[
H_{f,z} := H_z + g((\nabla f)_z, u_z).
\]
For a function \(G : \partial M \to \mathbb{R}\), we mean by \(H_{f,\partial M} \geq G\) we have \(H_{f,z} \geq G(z)\) for all \(z \in \partial M\).

We mainly study the following three curvature conditions: For \(\kappa, \lambda \in \mathbb{R}\) and \(K, \Lambda \in \mathbb{R}\),
\[
(2.1) \quad N \in [n, \infty), \quad \text{Ric}_f^{N,\partial M} \geq (N-1)\kappa, \quad H_{f,\partial M} \geq (N-1)\lambda;
\]
\[
(2.2) \quad N = \infty, \quad \text{Ric}_f^{\infty,\partial M} \geq K, \quad H_{f,\partial M} \geq \Lambda;
\]
\[
(2.3) \quad N = 1, \quad \text{Ric}_f^1,\partial M \geq (n-1)\kappa e^{\frac{4\pi}{n-1}}, \quad H_{f,\partial M} \geq (n-1)\lambda e^{\frac{2\pi}{n-1}}.
\]

**Remark 2.2.** We give a historical comment for the curvature condition \(2.3\). First, Wylie [51] has obtained a splitting theorem of Cheeger-Gromoll type under \(\text{Ric}_f^N,\partial M \geq 0\) for \(N \in (-\infty, 1]\). After that Wylie and Yeroshkin [52] have introduced the condition \(\text{Ric}_f^1,\partial M \geq (n-1)\kappa e^{\frac{4\pi}{n-1}}\) from the viewpoint of study of affine connections, and established comparison geometry. Furthermore, the author [45] has studied comparison geometry of manifolds with boundary under the curvature condition \(\text{Ric}_f^N,\partial M \geq (n-1)\kappa e^{\frac{4\pi}{n-1}}\) and \(H_{f,\partial M} \geq (n-1)\lambda e^{\frac{2\pi}{n-1}}\) for \(N \in (-\infty, 1]\).

**2.2. Laplacians and Dirichlet eigenvalues.** For the weighted Laplacian \(\Delta_f\) defined as \((1.3)\), the following formula of Bochner type is well-known (see e.g., Chapter 14 in [18]): For every smooth \(\psi : M \to \mathbb{R}\),
\[
(2.4) \quad -\frac{1}{2} \Delta_f \|\nabla \psi\|^2 = \text{Ric}^f_\partial (\nabla \psi) + \|\text{Hess} \psi\|_{\text{HS}}^2 - g(\nabla \Delta_f \psi, \nabla \psi),
\]
where \(\|\cdot\|\) and \(\|\cdot\|_{\text{HS}}\) are the canonical norm and the Hilbert-Schmidt norm induced from \(g\), respectively.

For \(z \in \partial M\), the value \(\Delta_f \rho_{\partial M}(\gamma_z(t))\) converges to \(H_{f,z}\) as \(t \to 0\). For \(t \in (0, \tau(z))\), and for the volume element \(\theta(t, z)\) of the \(t\)-level surface of \(\rho_{\partial M}\) at \(\gamma_z(t)\), we set
\[
\theta_f(t, z) := e^{-f(\gamma_z(t))} \theta(t, z).
\]
For all \(t \in (0, \tau(z))\) it holds that
\[
(2.5) \quad \Delta_f \rho_{\partial M}(\gamma_z(t)) = \frac{\theta'_f(t, z)}{\theta_f(t, z)}.
\]
We also have the following (see e.g., [43]): If $\partial M$ is compact, then
\begin{equation}
 m_{M,f}(B_r(\partial M)) = \int_{\partial M} \int_0^r \tilde{\theta}_f(t, z)\, dt\, d\text{vol}_h
\end{equation}
for all $r > 0$, where $\text{vol}_h$ is the Riemannian volume measure on $\partial M$ induced from $h$, and $\tilde{\theta}_f : [0, \infty) \times \partial M \to \mathbb{R}$ is a function defined as
\begin{equation}
 \tilde{\theta}_f(t, z) := \begin{cases} 
 \theta_f(t, z) & \text{if } t < \tau(z), \\
 0 & \text{if } t \geq \tau(z).
\end{cases}
\end{equation}

Let $M$ be compact. For $\phi \in H^1_0(M, m_{M,f}) \setminus \{0\}$, its Rayleigh quotient is defined as
\begin{equation}
 R_f(\phi) := \frac{\int_M ||\nabla \phi||^2 \, dm_{M,f}}{\int_M \phi^2 \, dm_{M,f}}.
\end{equation}
For the $k$-th Dirichlet eigenvalue $\nu_{f,k}(M)$ of the weighted Laplacian $\Delta_f$ defined as (1.6), the min-max principle states
\begin{equation}
 \nu_{f,k}(M) = \inf_L \sup_{\phi \in L \setminus \{0\}} R_f(\phi),
\end{equation}
where the infimum is taken over all $k$-dimensional subspaces $L$ of the Sobolev space $H^1_0(M, m_{M,f})$.

2.3. Dirichlet isoperimetric constants. For a Borel subset $\Omega \subset M$,
\begin{equation}
 m^+_{M,f}(\Omega) := \liminf_{r \to 0} \frac{m_{M,f}(U_r(\Omega)) - m_{M,f}(\Omega)}{r},
\end{equation}
where $U_r(\Omega)$ is the open $r$-neighborhood of $\Omega$. The Dirichlet isoperimetric constant is defined as
\begin{equation}
 I_f(M) := \inf_{\Omega} \frac{m^+_{M,f}(\Omega)}{m_{M,f}(\Omega)},
\end{equation}
where the infimum is taken over all Borel subsets $\Omega \subset \text{Int } M$ (cf. Definition 9.1 in [24]). The following inequality of Cheeger type is well-known (see [5], and cf. Corollary 9.7 in [24]): If $M$ is compact, then
\begin{equation}
 I_f(M) \leq 2\sqrt{\nu_{f,1}(M)}.
\end{equation}

In the graph setting, Kwak, Lau, Lee, Oveis Gharan and Trevisan [20] have established an improved Cheeger inequality in terms of the smallest and higher eigenvalues of the Laplacian and the conductance (see Theorem 1.1 in [20]). In the manifold setting, to answer a question of Funano [10], Liu [29] has pointed out that a similar improved Cheeger inequality holds for closed eigenvalues of the Laplacian and the Cheeger constant via the same argument as in [20] (see Theorem 1.6 in [29]).
Now, we further point out that the following improvement of (2.11) holds in our setting via the same argument as in [20]:

**Theorem 2.1** ([20], [29]). Let $M$ be compact. Then for all $k \geq 1$,

$$I_f(M) \leq 8\sqrt{2k} \frac{\nu_{f,1}(M)}{\sqrt{\nu_{f,k}(M)}}. \tag{2.12}$$

One can verify (2.12) by applying the same argument as in the proof of Lemmas 3.2 and 3.3 in [29] (replace the role of the $k$-th closed eigenvalue with that of $\nu_{f,k}(M)$) to a non-negative eigenfunction of $\nu_{f,1}(M)$, here we recall that any eigenfunctions of $\nu_{f,1}(M)$ are either always positive or always negative on $\text{Int } M$. Note that for such an eigenfunction, the $2k$ disjointly supported Lipschitz functions constructed in the proof of Lemma 3.3 in [29] also satisfy the Dirichlet boundary condition.

### 2.4. Dirichlet eigenvalue estimates.

Wang [49] has produced a gradient estimate of Bakry-Ledoux type for the Dirichlet heat semigroup associated with the weighted Laplacian under a lower (unweighted) Ricci curvature bound, a lower (unweighted) mean curvature bound for the boundary, and a density bound (see Theorem 1.1 in [49], and also [2]). From the same argument as in the proof of Theorem 1.1 in [49], we can derive the following (cf. (1.10) in [49] for unweighted case):

**Theorem 2.2** ([49]). If $\text{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$, then we have

$$\|\nabla P_t \psi\|_{L^\infty} \leq \frac{\sqrt{1 + 21/3} (1 + 4^{2/3}) \|\psi\|_{L^\infty}}{2\sqrt{\pi} \sqrt{t}}$$

for all $t > 0$ and non-negative, bounded measurable functions $\psi$ on $M$, where $P_t$ is the Dirichlet heat semigroup generated by $-\Delta_f$.

Wang [49] has proved Theorem 2.2 when $f = 0$. One can see Theorem 2.2 only by using Lemma 3.4 in [43] (or Lemma 6.1 below) instead of Lemma 2.3 in [49], and using the inequality (2.3) in [6] instead of (2.5) in [49] along the line of the proof of Theorem 1.1 in [49].

In virtue of the gradient estimate, Wang [49] has obtained an isoperimetric inequality of Buser and Ledoux type based on the idea of Ledoux [21] (see Theorem 1.2 in [49], and also [4], [21]). Theorem 2.2 together with the same argument as in the proof of Theorem 1.2 in [49] yields:

**Theorem 2.3** ([49]). Let $M$ be compact. If $\text{Ric}^\infty_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$, then we have

$$I_f(M) \geq \frac{2\sqrt{\pi}}{\sqrt{1 + 21/3} (1 + 4^{2/3})} \sup_{t > 0} \frac{1 - e^{-t}}{\sqrt{t}} \frac{1}{\sqrt{\nu_{f,1}(M)}}. \tag{2.13}$$

Combining (2.12), (2.13) implies the following universal inequality:
Theorem 2.4. Let $M$ be compact. If $\text{Ric}^\infty f_M \geq 0$ and $H_{f,\partial M} \geq 0$, then there is a universal constant $C > 0$ such that for all $k \geq 1$ we have
\begin{equation}
\nu_{f,k}(M) \leq C k^2 \nu_{f,1}(M).
\end{equation}

Funano and Shioya [12], Funano [10], Liu [29] have formulated similar inequalities for closed manifolds of non-negative $\infty$-weighted Ricci curvature (see Theorem 1.1 in [12], Theorem 1.2 in [10] and Theorem 1.1 in [29]). The inequality (2.14) corresponds to that of Liu [29].

Remark 2.3. Under similar setting $\text{Ric}^\infty f_{\partial M} \geq 0$ and $H_{f,\partial M} \geq 0$ to that in Theorem 2.4, the author [43] has shown a dimension free inequality
\begin{equation}
\nu_{f,1}(M) \geq \pi^2 (2 \text{InRad} M)^{-2}
\end{equation}
of Li-Yau, Kasue type, and a rigidity result for the equality case (see Corollary 7.6 in [43], and also [27], [17], and cf. Remarks 2.8 and 4.5).

Remark 2.4. Let $M$ be compact, and let $\nu_k(M)$ be the $k$-th Dirichlet eigenvalue of the Laplacian $\Delta$. Let us mention a dimension dependent estimate of the ratio $\nu_k(M)/\nu_1(M)$ induced from a classical method by Cheng [7], Li-Yau [27]. We possess the following estimate by modifying the proof of Corollary 2.2 in [7] (take a unit speed minimal geodesic $\gamma : [0, \text{InRad} M] \to M$ with $\gamma((0, \text{InRad} M)) \subset \text{Int} M$ that is orthogonal to $\partial M$ at $\gamma(0)$, set $k$ disjoint open balls in $\text{Int} M$ centered at $\gamma((2\alpha - 1)(2k)^{-1}\text{InRad} M)$ with radius $(2k)^{-1}\text{InRad} M$ for $\alpha = 1, \ldots, k$, and apply the argument of proof of Theorem 2.1 in [7]): If $\text{Ric}_M \geq 0$, then
\begin{equation}
\nu_k(M) \leq 2n(n + 4)k^2(\text{InRad} M)^{-2}
\end{equation}
for all $k \geq 1$. By (2.15), (2.16), we obtain the following: If $\text{Ric}_M \geq 0$ and $H_{\partial M} \geq 0$, then there is $C_n > 0$ depending only on $n$ such that
\begin{equation}
\nu_k(M) \leq C_n k^2 \nu_1(M)
\end{equation}
for all $k \geq 1$. Theorem 2.4 is a refinement of (2.17) in the sense that the upper bound of $\nu_k(M)/\nu_1(M)$ does not depend on $n$.

2.5. Comparisons. The author [43], [45] has obtained inscribed radius comparison theorems, and rigidity results for the equality case. We first recall the following comparison (see Theorem 1.1 in [43]):

Theorem 2.5 ([43]). For $N \in [n, \infty)$, we assume $\text{Ric}^N f_{\partial M} \geq (N - 1)\kappa$ and $H_{f,\partial M} \geq (N - 1)\lambda$. If $\kappa$ and $\lambda$ satisfy the ball-condition, then
\[ \text{InRad} M \leq C_{\kappa,\lambda}. \]

Remark 2.5. Kasue [16] has proved Theorem 2.5 and rigidity result for the equality case in the unweighted case where $f = 0$ and $N = n$. Li and Wei have done in [26] when $\kappa = 0$, and in [24] when $\kappa < 0$. 
We also have the following comparison (see Theorem 6.3 in [45]):

**Theorem 2.6 ([45]).** Assume \( \text{Ric}^1_{f,\partial M} \geq (n - 1) \kappa e^{\frac{-4f}{n - 1}} \) and \( H_{f,\partial M} \geq (n - 1) \lambda e^{\frac{-2f}{n - 1}} \). Suppose additionally that \( f \leq (n - 1)\delta \) for some \( \delta \in \mathbb{R} \). If \( \kappa \) and \( \lambda \) satisfy the ball-condition, then we have

\[
\text{InRad } M \leq C_{\kappa} e^{-\delta \kappa} e^{-\delta \lambda}.
\]

For \( \kappa, \lambda \in \mathbb{R} \), let \( s_{\kappa,\lambda}(t) \) be a unique solution of the Jacobi equation \( \psi''(t) + \kappa \psi(t) = 0 \) with \( \psi(0) = 1, \psi'(0) = -\lambda \). Notice that \( \kappa \) and \( \lambda \) satisfy the ball-condition if and only if the equation \( s_{\kappa,\lambda}(t) = 0 \) has a positive solution; moreover, \( C_{\kappa,\lambda} = \inf\{t > 0 \mid s_{\kappa,\lambda}(t) = 0\} \).

We define \( \bar{C}_{\kappa,\lambda} \) as follows: If \( \kappa \) and \( \lambda \) satisfy the ball-condition, then \( \bar{C}_{\kappa,\lambda} := C_{\kappa,\lambda} \); otherwise, \( \bar{C}_{\kappa,\lambda} := \infty \).

For \( N \in (1, \infty) \), let us define a function \( s_{N,\kappa,\lambda} : (0, \infty] \to (0, \infty] \) by

\[
(2.18) \quad \bar{s}_{\kappa,\lambda}(t) := \begin{cases} s_{\kappa,\lambda}(t) & \text{if } t < \bar{C}_{\kappa,\lambda}, \\ 0 & \text{if } t \geq \bar{C}_{\kappa,\lambda} \end{cases}, \quad s_{N,\kappa,\lambda}(r) := \int_0^r \bar{s}_{N,\kappa,\lambda}(t) \, dt.
\]

**Remark 2.6.** For \( \kappa, \lambda \in \mathbb{R} \), the value \( s_{n,\kappa,\lambda}(\bar{C}_{\kappa,\lambda}) \) is finite if and only if either (1) \( \kappa \) and \( \lambda \) satisfy the ball-condition; or (2) \( \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \) (cf. formulation of finite dimensional model spaces in Subsection 1.4).

For \( r > 0 \) and \( \Omega \subset M \), let \( B_r(\Omega) \) stand for the closed \( r \)-neighborhood of \( \Omega \). We next recall the following relative volume comparison theorem for metric neighborhoods of boundaries (see Theorem 5.4 in [43]):

**Theorem 2.7 ([43]).** Let \( \partial M \) be compact. For \( N \in [n, \infty) \), we assume \( \text{Ric}^N_{f,\partial M} \geq (N - 1) \kappa \) and \( H_{f,\partial M} \geq (N - 1) \lambda \). Then for all \( r, R > 0 \) with \( r \leq R \) we have

\[
\frac{m_{M,f}(B_R(\partial M))}{m_{M,f}(B_r(\partial M))} \leq \frac{s_{N,\kappa,\lambda}(R)}{s_{N,\kappa,\lambda}(r)}.
\]

**Remark 2.7.** Under the same setting as in Theorem 2.7, Bayle [3] has stated a similar absolute volume comparison of Heintze-Karcher type (see Theorem E.2.2 in [3], and see also [15], [35], [36], [37]).

We further recall the following comparison (see Theorem 7.6 in [45]):

**Theorem 2.8 ([45]).** Let \( \partial M \) be compact. Let us assume \( \text{Ric}^1_{f,\partial M} \geq (n - 1) \kappa e^{\frac{-4f}{n - 1}} \) and \( H_{f,\partial M} \geq (n - 1) \lambda e^{\frac{-2f}{n - 1}} \). Suppose additionally that \( f \leq (n - 1)\delta \) for some \( \delta \in \mathbb{R} \). Assume that one of the following holds:

1. \( \kappa \) and \( \lambda \) satisfy the convex-ball-condition;
2. \( \kappa \leq 0 \) and \( \lambda = \sqrt{|\kappa|} \).
Then for all $r, R > 0$ with $r \leq R$ we have
\[
\frac{m_{M,f}(B_R(\partial M))}{m_{M,f}(B_r(\partial M))} \leq \frac{s_{n,\kappa}e^{-\delta \kappa e^{-2\delta}(R)}}{s_{n,\kappa}e^{-\delta \kappa e^{-2\delta}(r)}}.
\]

Remark 2.8. In [43], the author has stated that the comparison inequalities in Theorems 2.5 and 2.7 hold under the curvature condition $\text{Ric}^N_{f,M} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$. Actually, the author [43] has proved such comparison inequalities under the curvature condition $\text{Ric}^N_{f,\partial M \perp} \geq (N-1)\kappa$ and $H_{f,\partial M} \geq (N-1)\lambda$ relying on the Laplacian comparison for the distance function $\rho_{\partial M}$. We can say the same thing for the inequalities in Theorems 2.6 and 2.8 (see [45]).

3. Invariants

3.1. Isomorphisms. We first introduce the following notion:

Definition 3.1. For $i = 1, 2$, let $X_i = (X_i, d_{X_i}, \mu_{X_i})$ be metric measure spaces with boundary. We say that $X_1$ dominates $X_2$ if there exists a 1-Lipschitz map $\Phi : X_1 \to X_2$ such that
\[
\Phi_#\mu_{X_1} = \mu_{X_2}, \quad \Phi(\partial X_1) \subset \partial X_2,
\]
where $\Phi_#\mu_{X_1}$ denotes the push-forward of $\mu_{X_1}$ by $\Phi$. We also say that $X_1$ and $X_2$ are isomorphic to each other if they dominate each other.

Remark 3.1. A triple $\mathcal{X} = (\mathcal{X}, r_{\mathcal{X}}, \sigma_{\mathcal{X}})$ is said to be an mm-space when $(\mathcal{X}, r_{\mathcal{X}})$ is a complete separable metric space, and $\sigma_{\mathcal{X}}$ is a Borel probability measure on $\mathcal{X}$. For $i = 1, 2$, let $\mathcal{X}_i = (\mathcal{X}_i, r_{\mathcal{X}_i}, \sigma_{\mathcal{X}_i})$ be mm-spaces. They are said to be mm-isomorphic to each other if there exists an isometry $\Phi : \text{supp} \sigma_{\mathcal{X}_1} \to \text{supp} \sigma_{\mathcal{X}_2}$ such that $\Phi_#\sigma_{\mathcal{X}_1} = \sigma_{\mathcal{X}_2}$, where $\text{supp} \sigma_{\mathcal{X}_i}$ are the support of $\sigma_{\mathcal{X}_i}$. It is said that $\mathcal{X}_1$ dominates $\mathcal{X}_2$ if there exists a 1-Lipschitz map $\Psi : \mathcal{X}_1 \to \mathcal{X}_2$ such that $\Psi_#\sigma_{\mathcal{X}_1} = \sigma_{\mathcal{X}_2}$. It is well-known that if $\mathcal{X}_1$ and $\mathcal{X}_2$ dominate each other, then they are mm-isomorphic to each other (see e.g., Proposition 2.11 in [16]).

For $i = 1, 2$, let $X_i = (X_i, d_{X_i}, \mu_{X_i})$ be two metric measure spaces with boundary. If $X_1$ and $X_2$ are isomorphic to each other in the sense of Definition 3.1, then they dominate each other as mm-spaces; in particular, they are mm-isomorphic to each other.

Let us show the following monotonicity of the partial inscribed radius defined as (1.2) (cf. Proposition 2.18 in [16]):

Lemma 3.1. For $i = 1, 2$, let $X_i = (X_i, d_{X_i}, \mu_{X_i})$ be metric measure spaces with boundary. If $X_1$ dominates $X_2$, then for every $\eta > 0$
\[
\text{PartInRad}(X_2; 1 - \eta) \leq \text{PartInRad}(X_1; 1 - \eta).
\]
Proof. By the assumption for the dominance, there exists a 1-Lipschitz map \( \Phi : X_1 \to X_2 \) such that (3.1). We fix a Borel subset \( \Omega \subset X_1 \) with \( \mu_{X_1}(\Omega) \geq 1 - \eta \). From \( \Phi \# \mu_{X_1} = \mu_{X_2} \) it follows that

\[
(3.2) \quad \mu_{X_2}(\Phi(\Omega)) \geq \mu_{X_1}(\Phi^{-1}(\Phi(\Omega))) \geq \mu_{X_1}(\Omega) \geq 1 - \eta,
\]

where \( \Phi(\Omega) \) denotes the closure of \( \Phi(\Omega) \).

We now show

\[
(3.3) \quad \text{InRad} \Phi(\Omega) \leq \text{InRad} \Omega,
\]

where \( \text{InRad} \Phi(\Omega) \) and \( \text{InRad} \Omega \) are the inscribed radii of \( \Phi(\Omega) \) and \( \Omega \) defined as (1.1), respectively. We fix \( x_2 \in \Phi(\Omega) \), and take a sequence \( \{x_{2,j}\} \) in \( \Phi(\Omega) \) with \( x_{2,j} \to x_2 \). For each \( j \) we have \( x_{1,j} \in \Omega \) satisfying \( x_{2,j} = \Phi(x_{1,j}) \). By using the properness of \( X_1 \), we can choose \( z_{1,j} \in \partial X_1 \) with \( d_{X_1}(x_{1,j}, z_{1,j}) = \rho_{\partial X_1}(x_{1,j}) \). We set \( z_{2,j} := \Phi(x_{2,j}) \). From \( \Phi(\partial X_1) \subset \partial X_2 \) we deduce \( z_{2} \in \partial X_2 \). Since \( \Phi \) is 1-Lipschitz, we see

\[
\rho_{\partial X_2}(x_{2,j}) \leq d_{X_2}(x_{2,j}, z_{2,j}) \leq d_{X_1}(x_{1,j}, z_{1,j}) = \rho_{\partial X_1}(x_{1,j}) \leq \text{InRad} \Omega.
\]

Letting \( j \to \infty \), we derive \( \rho_{\partial X_2}(x_2) \leq \text{InRad} \Omega \). This yields (3.3). In virtue of (3.2) and (3.3), we obtain

\[
\text{PartInRad}(X_2; 1 - \eta) \leq \text{InRad} \Phi(\Omega) \leq \text{InRad} \Omega.
\]

The arbitrariness of \( \Omega \) completes the proof. \( \Box \)

We also have the following monotonicity of the observable inscribed radius introduced in Definition 1.1 (cf. Proposition 2.18 in [46]). The proof is straightforward, and we omit it.

Lemma 3.2. For \( i = 1, 2 \), let \( X_i = (X_i, d_{X_i}, \mu_{X_i}) \) be metric measure spaces with boundary. If \( X_1 \) dominates \( X_2 \), then for every \( \eta > 0 \)

\[
\text{ObsInRad}(X_2; -\eta) \leq \text{ObsInRad}(X_1; -\eta).
\]

According to Lemmas 3.1 and 3.2, they are invariants under the isomorphism introduced in Definition 3.1.

3.2. Boundary separation distances. Let \( X = (X, d_X, \mu_X) \) be a metric measure space with boundary, and let \( k \geq 1 \) be an integer. For positive numbers \( \eta_1, \ldots, \eta_k > 0 \), we denote by \( \mathcal{S}_X(\eta_1, \ldots, \eta_k) \) the set of all sequences \( \{\Omega_\alpha\}_{\alpha=1}^k \) of Borel subsets \( \Omega_\alpha \) with \( \mu_X(\Omega_\alpha) \geq \eta_\alpha \). For a sequence \( \{\Omega_\alpha\}_{\alpha=1}^k \in \mathcal{S}_X(\eta_1, \ldots, \eta_k) \), we set

\[
D_X(\{\Omega_\alpha\}_{\alpha=1}^k) := \min \left\{ \min_{\alpha \neq \beta} d_X(\Omega_\alpha, \Omega_\beta), \ min_{\alpha} d_X(\Omega_\alpha, \partial X) \right\}.
\]

We now define the following quantity:
Definition 3.2. Let $X = (X, d_X, \mu_X)$ be a metric measure space with boundary. For $\eta_1, \ldots, \eta_k > 0$, we define the $(\eta_1, \ldots, \eta_k)$-boundary separation distance $\text{BSep}(X; \eta_1, \ldots, \eta_k)$ of $X$ as follows: If $\mathcal{S}_X(\eta_1, \ldots, \eta_k)$ is non-empty, then
\[
\text{BSep}(X; \eta_1, \ldots, \eta_k) := \sup \mathcal{D}_X \left( \{ \Omega_\alpha \}^k_{\alpha=1} \right),
\]
where the supremum is taken over all $\{ \Omega_\alpha \}^k_{\alpha=1} \in \mathcal{S}_X(\eta_1, \ldots, \eta_k)$; otherwise, $\text{BSep}(X; \eta_1, \ldots, \eta_k) := 0$.

The boundary separation distance $\text{BSep}(X; \eta_1, \ldots, \eta_k)$ is monotone non-increasing in $\eta_\alpha$ for each $\alpha = 1, \ldots, k$.

Remark 3.2. The boundary separation distance is an analogue of the separation distance on mm-spaces introduced by Gromov [13]. For later convenience, we recall its precise definition: Let $\mathcal{X} = (\mathcal{X}, r_\mathcal{X}, \sigma_\mathcal{X})$ be an mm-space (see Remark 3.1). For positive numbers $\eta_0, \eta_1, \ldots, \eta_k > 0$, the $(\eta_0, \eta_1, \ldots, \eta_k)$-separation distance is defined as
\[
\text{Sep}(\mathcal{X}; \eta_0, \eta_1, \ldots, \eta_k) := \sup_{\alpha \neq \beta} \min r_\mathcal{X}(\Omega_\alpha, \Omega_\beta),
\]
where the supremum is taken over all sequences $\{ \Omega_\alpha \}^k_{\alpha=0}$ of Borel subsets $\Omega_\alpha \subset \mathcal{X}$ with $\sigma_\mathcal{X}(\Omega_\alpha) \geq \eta_\alpha$. If there exists no such sequence, then we set $\text{Sep}(\mathcal{X}; \eta_0, \eta_1, \ldots, \eta_k) := 0$.

We verify the following monotonicity (cf. Lemma 2.25 in [46]):

Lemma 3.3. For $i = 1, 2$, let $X_i = (X_i, d_{X_i}, \mu_{X_i})$ be metric measure spaces with boundary. If $X_1$ dominates $X_2$, then for all $\eta_1, \ldots, \eta_k > 0$
\[
\text{BSep}(X_2; \eta_1, \ldots, \eta_k) \leq \text{BSep}(X_1; \eta_1, \ldots, \eta_k).
\]

Proof. We may assume $\mathcal{S}_X(\eta_1, \ldots, \eta_k) \neq \emptyset$. Fix a sequence $\{ \Omega_\alpha \}^k_{\alpha=1} \in \mathcal{S}_X(\eta_1, \ldots, \eta_k)$. There exists a 1-Lipschitz map $\Phi : X_1 \to X_2$ such that $\Phi$ is 1-Lipschitz, for all $\alpha, \beta = 1, \ldots, k$, we see
\[
d_{X_2}(\Omega_\alpha, \Omega_\beta) \leq d_{X_1}(\Phi^{-1}(\Omega_\alpha), \Phi^{-1}(\Omega_\beta)).
\]
Furthermore, by $\Phi_\# \mu_{X_1} = \mu_{X_2}$, for every $\alpha = 1, \ldots, k$,
\[
\mu_{X_1}(\Phi^{-1}(\Omega_\alpha)) = \mu_{X_2}(\Omega_\alpha) \geq \eta_\alpha.
\]

We show that for every $\alpha = 1, \ldots, k$,
\[
d_{X_2}(\Omega_\alpha, \partial X_2) \leq d_{X_1}(\Phi^{-1}(\Omega_\alpha), \partial X_1).
\]
Take $x_1 \in \Phi^{-1}(\Omega_\alpha)$, and put $x_2 := \Phi(x_1) \in \Omega_\alpha$. From the properness of $X_1$, there exists $z_1 \in \partial X_1$ such that $d_{X_1}(x_1, z_1) = d_{X_1}(x_1, \partial X_1)$. Put $z_2 := \Phi(z_1) \in \partial X_2$. It follows that
\[
d_{X_2}(\Omega_\alpha, \partial X_2) \leq d_{X_2}(x_2, z_2) \leq d_{X_1}(x_1, z_1) = d_{X_1}(x_1, \partial X_1),
\]
and we obtain (3.7). By combining (3.5), (3.6) and (3.7),
\[ \mathcal{D}_{X_2}(\{\Omega_\alpha\}_{\alpha=1}^{k}) \leq \mathcal{D}_{X_1}(\{\Phi^{-1}(\Omega_\alpha)\}_{\alpha=1}^{k}) \leq \text{BSep}(X_1; \eta_1, \ldots, \eta_k). \]
This completes the proof. \(\square\)

Lemma 3.3 tells us that the boundary separation distance is an invariant under the isomorphism introduced in Definition 3.1.

3.3. Relations between invariants. We present the following relation between our invariants (cf. Proposition 2.26 in [46]):

**Lemma 3.4.** Let \( X = (X,d_X,\mu_X) \) be a metric measure space with boundary. Then for every \( \eta > 0 \) we have
\[ \text{ObsInRad}(X; -\eta) \leq \text{BSep}(X; \eta). \]
In particular, \( \text{ObsInRad}(X; -\eta) \leq \text{InRad} X. \)

**Proof.** We may assume \( \eta < 1. \) We fix a 1-Lipschitz function \( \varphi : X \to I \) with \( \varphi|_{\partial X} = 0. \) Let \( m_{I,\varphi} \) be the Borel probability measure on the \( \varphi \)-screen \( I_\varphi \) defined as (1.3). By \( m_{I,\varphi} = \varphi_\#\mu_X \) and \( \varphi(\partial X) \subset \partial I, \) the space \( X \) dominates \( I_\varphi; \) in particular, Lemma 3.3 yields (3.8) \[ \text{BSep}(I_\varphi; \eta) \leq \text{BSep}(X; \eta). \]
We put
\[ t_0 := \inf\{ t \in I \mid m_{I,\varphi}((t, \infty)) \leq \eta \}. \]
Note that \( m_{I,\varphi}((t_0, \infty)) \leq \eta \) and \( m_{I,\varphi}([t_0, \infty)) \geq \eta. \) Since \( m_{I,\varphi}([0, t_0]) \geq 1 - \eta, \) we have
(3.9) \[ \text{PartInRad}(I_\varphi; 1 - \eta) \leq \text{InRad}[0, t_0] = t_0. \]
On the other hand, \( m_{I,\varphi}([t_0, \infty)) \geq \eta \) leads to
(3.10) \[ t_0 = d_I(\partial I, [t_0, \infty)) \leq \text{BSep}(I_\varphi; \eta). \]
Now, (3.8) together with (3.9), (3.10) implies the desired one. \(\square\)

**Remark 3.3.** A sequence \( \{X_n\} \) of metric measure spaces with boundary is said to be inscribed radius collapsing if \( \text{InRad} X_n \to 0 \) as \( n \to \infty. \) Yamaguchi and Zhang [53] have studied inscribed radius collapsing sequences of manifolds with boundary from the viewpoint of the collapsing theory. From Lemma 3.4, it follows that if a sequence of metric measure spaces with boundary is inscribed radius collapsing, then it is a boundary concentration family introduced in Definition 1.2.

We also possess the following relation (cf. Proposition 2.26 in [46]):

**Lemma 3.5.** Let \( X = (X,d_X,\mu_X) \) be a metric measure space with boundary. Then for all \( \eta, \eta' > 0 \) with \( \eta > \eta' \) we have
\[ \text{BSep}(X; \eta) \leq \text{ObsInRad}(X; -\eta'). \]
Proof. We may assume BSep\((X;\eta) > 0\). Fix a Borel subset \(\Omega \subset X\) with \(\mu_X(\Omega) \geq \eta\), and also fix \(J \subset I\) with \(m_{I,\rho X}(J) \geq 1 - \eta'\). Then
\[
m_{I,\rho X}(\rho X(\Omega)) + m_{I,\rho X}(J) \geq \mu_X(\Omega) + m_{I,\rho X}(J) \geq \eta + (1 - \eta') > 1,
\]
and hence \(\rho X(\Omega) \cap J \neq \emptyset\), where \(\rho X(\Omega)\) denotes the closure of \(\rho X(\Omega)\).

For every \(t_0 \in \rho X(\Omega) \cap J\) we see
\[
\text{InRad} J \geq d(I(t_0, \partial I) \geq d(I(\rho X(\Omega), \partial I) = \inf_{x \in \Omega} \rho X(x) = d_X(\Omega, \partial X).
\]

This yields
\[
\text{ObsInRad}(X; -\eta') \geq \text{PartInRad}(I_{\rho X}; 1 - \eta') \geq \text{BSep}(X; \eta).
\]

We arrive at the desired assertion. 

By combining Lemmas 3.4 and 3.5 and by straightforward argument, one can conclude the following equivalence:

**Proposition 3.6.** Let \(\{X_n\}\) be a sequence of metric measure spaces with boundary \(X_n = (X_n, d_{X_n}, \mu_{X_n})\). Then the following are equivalent:

1. \(\{X_n\}\) is a boundary concentration family;
2. for every sequence \(\{\Omega_n\}\) of Borel subsets \(\Omega_n \subset X_n\) satisfying \(\liminf_{n \to \infty} \mu_{X_n}(\Omega_n) > 0\), we have \(\lim_{n \to \infty} d_{X_n}(\Omega_n, \partial X_n) = 0\);
3. for every \(r > 0\) we have \(\lim_{n \to \infty} \mu_{X_n}(B_r(\partial X_n)) = 1\).

Finally, we observe the following fact for the equality case of Lemma 3.5. The statement and its proof are informed by Daisuke Kazukawa.

**Proposition 3.7.** Let \(X = (X, d_X, \mu_X)\) be a metric measure space with boundary. If \(\text{supp} \mu_X = X\), then for every \(\eta > 0\) we have
\[
\text{ObsInRad}(X; -\eta) = \text{BSep}(X; \eta).
\]

**Proof.** Lemma 3.4 tells us that the left hand side is at most the right hand side. We verify the opposite. We may assume BSep\((X; \eta) > 0\). Fix a sufficiently small \(\epsilon > 0\). Then there exists a Borel subset \(\Omega \subset X\) with \(\mu_X(\Omega) \geq \eta\) such that \(d_X(\Omega, \partial X) > \text{BSep}(X; \eta) - \epsilon\). Let us define a 1-Lipschitz function \(\varphi : X \to I\) by
\[
\varphi(x) := \max \{d_X(\Omega, \partial X) - d_X(x, \Omega), 0\}.
\]
Notice that \(\varphi|_{\partial X} = 0\), \(\varphi|_{\Omega} = d_X(\Omega, \partial X)\) and
\[
m_{I,\varphi}(\{d_X(\Omega, \partial X)\}) \geq \mu_X(\Omega) \geq \eta.
\]
Furthermore, \(\text{supp} \mu_X = X\) implies \(\text{supp} m_{I,\varphi} = [0, d_X(\Omega, \partial X)]\).

We now show
\[
(3.11) \quad \text{PartInRad}(I_{\varphi}; 1 - \eta) \geq d_X(\Omega, X).
\]
The proof is by contradiction. Suppose that there exists a Borel subset $J \subset I$ with $m_{I, \varphi}(J) \geq 1 - \eta$ such that $\text{InRad} J < d_X(\Omega, \partial X)$. We put $\hat{J} := (\text{InRad} J, d_X(\Omega, \partial X))$. Then we have

$$1 = m_{I, \varphi}([0, \text{InRad} J]) + m_{I, \varphi}(\hat{J}) + m_{I, \varphi}(\{d_X(\Omega, \partial X)\})$$

$$\geq m_{I, \varphi}(J) + m_{I, \varphi}(\hat{J}) + \eta \geq 1 + m_{I, \varphi}(\hat{J}),$$

and hence $m_{I, \varphi}(\hat{J}) = 0$. This contradicts $\text{supp} m_{I, \varphi} = [0, d_X(\Omega, \partial X)]$.

From (3.11) it follows that

$$\text{BSep}(X; \eta) - \epsilon < d_X(\Omega, \partial X) \leq \text{ObsInRad}(X; -\eta).$$

By letting $\epsilon \to 0$, we complete the proof.

4. Dirichlet eigenvalues

In what follows, let $(M, d_M, m_{M, f})$ denote an $n$-dimensional weighted Riemannian manifold with boundary defined as (1.4) such that $m_{M, f}$ is a Borel probability measure. We study the metric measure space with boundary $(M, f)$ defined as (1.7).

4.1. Boundary separation distances and Dirichlet eigenvalues.

Let us show the following relation between the boundary separation distance and the Dirichlet eigenvalue:

**Lemma 4.1.** Let $M$ be compact. Then for all $\eta_1, \ldots, \eta_k > 0$ we have

$$\text{BSep}((M, f); \eta_1, \ldots, \eta_k) \leq \frac{2}{\sqrt{\nu_{f, k}(M) \min_{\alpha=1, \ldots, k} \eta_\alpha}},$$

where $\nu_{f, k}(M)$ is the $k$-th Dirichlet eigenvalue of $\Delta_f$ defined as (1.6).

**Proof.** We set $S := \text{BSep}((M, f); \eta_1, \ldots, \eta_k)$. We may assume $S > 0$. Let us fix a sufficiently small $\epsilon > 0$. There exists a sequence $\{\Omega_\alpha\}_{\alpha=1}^k \in S_{(M, f)}(\eta_1, \ldots, \eta_k)$ such that $S - \epsilon < \mathcal{D}(M, f)(\{\Omega_\alpha\}_{\alpha=1}^k)$. Put $S_\epsilon := S - \epsilon$.

For each $\alpha$, we define a Lipschitz function $\phi_\alpha : M \to \mathbb{R}$ by

$$\phi_\alpha(x) := \max \left\{1 - \frac{2}{S_\epsilon} d_M(x, \Omega_\alpha), 0 \right\}.$$

Notice that the support of $\phi_\alpha$ coincides with $B_{S_\epsilon/2}(\Omega_\alpha)$. Furthermore, the following properties hold:

1. $\phi_\alpha \equiv 0$ on $B_{S_\epsilon/2}(\partial M)$;
2. $\phi_\alpha \equiv 0$ on $B_{S_\epsilon/2}(\Omega_\beta)$ for every $\beta = 1, \ldots, k$ with $\beta \neq \alpha$;
3. $\phi_\alpha \equiv 1$ on $\Omega_\alpha$;
4. $\|\nabla \phi_\alpha\| \leq 2/S_\epsilon m_{M, f}$-almost everywhere on $M$. 

By (1), $\phi_\alpha$ belongs to the Sobolev space $H^1_0(M, m_{M,f})$. By (2), the functions $\phi_1, \ldots, \phi_k$ are orthogonal to each other in $H^1_0(M, m_{M,f})$. Let $L_0$ be the $k$-dimensional subspace of $H^1_0(M, m_{M,f})$ spanned by $\phi_1, \ldots, \phi_k$. From (3) and (4), for every $\phi \in L_0 \setminus \{0\}$ we deduce

$$
\int_M \phi^2 \, dm_{M,f} = \sum_{\alpha=1}^k c^2_\alpha \int_M \phi^2_\alpha \, dm_{M,f} \geq \sum_{\alpha=1}^k c^2_\alpha \eta_\alpha \geq \min_\alpha \eta_\alpha \sum_{\alpha=1}^k c^2_\alpha,
$$

$$
\int_M \|\nabla \phi\|^2 \, dm_{M,f} = \sum_{\alpha=1}^k c^2_\alpha \int_M \|\nabla \phi_\alpha\|^2 \, dm_{M,f} \leq \left(\frac{2}{S_\epsilon}\right)^2 \sum_{\alpha=1}^k c^2_\alpha,
$$

where $c_1, \ldots, c_k$ are determined by $\phi = \sum_{\alpha=1}^k c_\alpha \phi_\alpha$. Hence we have

$$
R_f(\phi) = \frac{\int_M \|\nabla \phi\|^2 \, dm_{M,f}}{\int_M \phi^2 \, dm_{M,f}} \leq \frac{1}{\min_\alpha \eta_\alpha} \left(\frac{2}{S_\epsilon}\right)^2,
$$

where $R_f(\phi)$ is the Rayleigh quotient of $\phi$ defined as (2.9). From the min-max principle (2.10) we derive

$$
\nu_{f,k}(M) \leq \sup_{\phi \in L_0 \setminus \{0\}} R_f(\phi) \leq \frac{1}{\min_\alpha \eta_\alpha} \left(\frac{2}{S_\epsilon}\right)^2.
$$

Letting $\epsilon \to 0$, we conclude the inequality. \hfill \Box

**Remark 4.1.** In the forthcoming paper [11], Funano and the author prove the following refined estimate for $k = 1$ (see Theorem 2.3 in [11]): For every $\eta > 0$ we have

$$
\text{BSep}((M, f); \eta) \leq \frac{1}{\sqrt{\nu_{f,1}(M)}} \log \frac{\epsilon}{\eta}.
$$

**Remark 4.2.** Colbois and Savo [9] have shown a similar estimate for the $k$-th closed eigenvalue of the Laplacian (see Lemma 5 in [9]).

**Remark 4.3.** Chung, Grigor’yan and Yau [8] have estimated the $k$-th closed eigenvalue and Robin eigenvalue of the Laplacian in terms of the separation distance (see Theorem 1.1 in [8]). By applying the same argument in [8] to our setting, we see the following estimate for $\nu_{f,k}(M)$: Suppose that $M$ is compact. Then for every integer $k > 2$, and for every sequence $\{\Omega_\alpha\}_{\alpha=1}^k$ of Borel subsets with $\min_{\alpha \neq \beta} d_M(\Omega_\alpha, \Omega_\beta) \geq D$,

$$
\nu_{f,k}(M) - \nu_{f,1}(M) \leq \frac{1}{D^2} \max_{\alpha \neq \beta} \left(\log \frac{4}{\int_{\Omega_\alpha} \phi_{f,1}^2 \, dm_{M,f} \int_{\Omega_\beta} \phi_{f,1}^2 \, dm_{M,f}}\right)^2,
$$

where $\phi_{f,1}$ is an $L^2$-normalized eigenfunction for $\nu_{f,1}(M)$. 
4.2. Observable inscribed radii and Dirichlet eigenvalues. Now, Lemma 4.1 together with Lemma 3.4 leads us to the following relation between the observable inscribed radius and the Dirichlet eigenvalue:

**Proposition 4.2.** Let $M$ be compact. Then for every $\eta > 0$,

$$\text{ObsInRad}((M, f); -\eta) \leq \frac{2}{\sqrt{\nu_{f,1}(M) \eta}}.$$  

**Remark 4.4.** Under the curvature condition (2.1) and $\text{InRad} M \leq D$, the author [43] has provided a lower bound of $\nu_{f,1}(M)$ depending only on $\kappa, \lambda, N$ and $D$ (see Theorem 1.6 in [43] and Remark 2.8, and also pioneering works of Li-Yau [27] and Kasue [17]). Combining Proposition 4.2 with the lower estimate tells us that we have an upper bound of $\text{ObsInRad}((M, f); -\eta)$ depending only on $\kappa, \lambda, N, D$ and $\eta$.

Similarly, under the condition (2.3) for $\kappa$ and $\lambda$ satisfying the convex-ball-condition, and under $f \leq (n-1)\delta$, the author [45] has obtained a lower bound of $\nu_{f,1}(M)$ depending only on $n, \kappa, \lambda$ and $\delta$ (see Theorem 8.5 in [45]). In virtue of Proposition 4.2, we possess an upper bound of $\text{ObsInRad}((M, f); -\eta)$ depending only on $n, \kappa, \lambda, \delta$ and $\eta$.

Proposition 4.2 enables us to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $\{(M_n, f_n)\}$ be a sequence of compact metric measure spaces with boundary defined as (1.7). By Proposition 4.2 if $\nu_{f_n,1}(M_n) \to \infty$ as $n \to \infty$, then $\text{ObsInRad}((M_n, f_n); -\eta) \to 0$. Hence, $\{(M_n, f_n)\}$ is a boundary concentration family. \qed

We also derive the following from Proposition 4.2 and Theorem 2.4.

**Theorem 4.3.** Let $M$ be compact. If $\text{Ric}_{f,M} \geq 0$ and $H_{f,\partial M} \geq 0$, then there is a universal constant $C > 0$ such that for all $k \geq 1$ and $\eta > 0$,

$$\text{ObsInRad}((M, f); -\eta) \leq \frac{C k}{\sqrt{\nu_{f,k}(M) \eta}}.$$  

Let us give a proof of Theorem 1.2 by using Theorem 4.3.

**Proof of Theorem 1.2.** Let $\{(M_n, f_n)\}$ denote a sequence of compact metric measure spaces with boundary defined as (1.7). Assume that $\text{Ric}_{f_n, M_n} \geq 0$ and $H_{f_n, \partial M_n} \geq 0$. Due to Theorem 4.3 if $\nu_{f_n,k}(M_n) \to \infty$ for $k$, then $\text{ObsInRad}((M_n, f_n); -\eta) \to 0$. We complete the proof. \qed

**Remark 4.5.** The author wonders whether the following statement of E. Milman type holds (see [33], [34], and Theorem 9.46 in [46]): Under the same setting as in Theorems 1.1 and 1.2 if $\text{Ric}_{f_n, M_n} \geq 0$ and $H_{f_n, \partial M_n} \geq 0$, and if $\{(M_n, f_n)\}$ is a boundary concentration family, then $\nu_{f_n,1}(M_n) \to$
∞. If it is true, then in virtue of Theorems [1.1, 1.2] we can say that
under \( \text{Ric}^\infty_{f_n, M_n} \geq 0 \) and \( H_{f_n, \partial M_n} \geq 0 \), the following are equivalent:

1. \( \{ (M_n, f_n) \} \) is a boundary concentration family;
2. \( \nu_{f_n, 1}(M_n) \to \infty \) as \( n \to \infty \);
3. \( \nu_{f_n, k}(M_n) \to \infty \) as \( n \to \infty \) for some \( k \geq 1 \).

The author also wonders if one can extend the above equivalence to
a weaker setting \( \text{Ric}^N_{f_n, M_n} \geq 0 \) and \( H_{f_n, \partial M_n} \geq 0 \) for
\( N \in (-\infty, 1] \) (see Remark 2.1). Under \( \text{Ric}^N_{f, \partial M} \geq 0 \) and \( H_{f, \partial M} \geq 0 \) for \( N \in (-\infty, 1] \),
the author [44] has proved a dimension free inequality
\( \nu_{f, 1}(M) \geq \pi^2 (2 \text{InRad } M)^{-2} \),
of Li-Yau, Kasue type, and a rigidity result for the equality case (see
Corollary 6.5 in [44], and also [27], [17], and cf. Remarks 2.3 and 2.8).

5. Finite dimensional comparisons

5.1. Estimates. To prove Theorem [1.3] we begin with the following:

**Lemma 5.1.** Let \( \partial M \) be compact. For \( N \in [n, \infty) \), let us assume
\( \text{Ric}^N_{f, \partial M} \geq (N-1)\kappa \) and \( H_{f, \partial M} \geq (N-1)\lambda \). Suppose additionally that
InRad \( M \leq D \) for some \( D > 0 \). For \( \eta \in (0, 1) \), let \( \Omega \subset M \) be a Borel
subset with \( m_{M,f}(\Omega) \geq \eta \) and \( d_M(\Omega, \partial M) > 0 \). Then we have
\[
\frac{1}{2} s_{N,\kappa,\lambda}(d_M(\Omega, \partial M)) \leq s_{N,\kappa,\lambda}(D)(1 - \eta),
\]
where \( s_{N,\kappa,\lambda} \) is the function defined as (2.18).

**Proof.** We put \( r := d_M(\Omega, \partial M) \). Since the open \( r \)-neighborhood of \( \partial M \) and \( \Omega \) are mutually disjoint, we see
\[
\eta \leq m_{M,f}(\Omega) \leq 1 - m_{M,f}(B_r(\partial M)).
\]
From InRad \( M \leq D \) we derive \( M = B_D(\partial M) \); in particular, we have
\( m_{M,f}(B_D(\partial M)) = 1 \). Therefore, Theorem [2.7] leads to
\[
\eta \leq 1 - \frac{m_{M,f}(B_r(\partial M))}{m_{M,f}(B_D(\partial M))} \leq 1 - \frac{s_{N,\kappa,\lambda}(r)}{s_{N,\kappa,\lambda}(D)}.
\]
This yields the lemma.

One can also show the following lemma by using Theorem [2.8] instead
of Theorem [2.7] in the proof of Lemma 5.1. We omit the proof.

**Lemma 5.2.** Let \( \partial M \) be compact. Assume \( \text{Ric}^1_{f, \partial M} \geq (n-1)\kappa e^{\frac{n-4}{n}} \)
and \( H_{f, \partial M} \geq (n-1)\lambda e^{\frac{n-2}{n}} \). Suppose additionally that InRad \( M \leq D \)
and \( f \leq (n-1)\delta \) for some \( D > 0 \) and \( \delta \in \mathbb{R} \). We further assume that
one of the following holds:
(1) $\kappa$ and $\lambda$ satisfy the convex-ball-condition;
(2) $\kappa \leq 0$ and $\lambda = \sqrt{|\kappa|}$.

For $\eta \in (0, 1)$, let $\Omega \subset M$ be a Borel subset with $m_{M,f}(\Omega) \geq \eta$ and $d_M(\Omega, \partial M) > 0$. Then we have

$$s_{n,\kappa,\lambda} e^{-4s_{\kappa,\lambda} e^{-2s_{\kappa,\lambda}}} (d_M(\Omega, \partial M)) \leq s_{n,\kappa,\lambda} e^{-4s_{\kappa,\lambda} e^{-2s_{\kappa,\lambda}}} (D)(1 - \eta).$$

Recall that $s_{\kappa,\lambda}(t)$ is the solution of the equation $\psi''(t) + \kappa \psi(t) = 0$ with $\psi(0) = 1$, $\psi'(0) = -\lambda$. For $N \in (1, \infty)$, and $\kappa$ and $\lambda$ satisfying the ball-condition, we define a function $v_{N,\kappa,\lambda} : [0, C_{\kappa,\lambda}] \rightarrow [0, 1]$ by

$$(5.1) \quad v_{N,\kappa,\lambda}(r) := \frac{\int_r^{C_{\kappa,\lambda}} s_{\kappa,\lambda}^{N-1}(t) dt \int_0^{C_{\kappa,\lambda}} s_{\kappa,\lambda}^{N-1}(t) dt}{\int_0^{C_{\kappa,\lambda}} s_{\kappa,\lambda}^{N-1}(t) dt}.$$

Under the curvature condition (2.1), we produce the following:

**Theorem 5.3.** Let $\partial M$ be compact. For $N \in [n, \infty)$, let us assume $\text{Ric}_{f,\partial M}^N \geq (N - 1)\kappa$ and $H_{f,\partial M} \geq (N - 1)\lambda$. Then for every $\eta \in (0, 1]$ the following hold:

1. if $\kappa$ and $\lambda$ satisfy the ball-condition, then
   $$\text{ObsInRad}((M, f); -\eta) \leq v_{N,\kappa,\lambda}^{-1}(\eta);$$

2. if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then
   $$\text{ObsInRad}((M, f); -\eta) \leq \frac{1}{(N - 1)\lambda} \log \frac{1}{\eta}.$$

**Proof.** We may assume that $\eta < 1$. By Lemma 3.4 it suffices to prove that $\text{BSep}((M, f); \eta)$ is at most the right hand side of the desired inequality in each case. We may assume that $\text{BSep}((M, f); \eta)$ is positive. We fix a Borel subset $\Omega \subset M$ with $m_{M,f}(\Omega) \geq \eta$ and $d_M(\Omega, \partial M) > 0$.

Let us consider the case where $\kappa$ and $\lambda$ satisfy the ball-condition. In this case, Theorem 2.5 implies $\text{InRad} M \leq C_{\kappa,\lambda}$. By using Lemma 5.1,

$$\eta \leq 1 - \frac{s_{N,\kappa,\lambda}(d_M(\Omega, \partial M))}{s_{N,\kappa,\lambda}(C_{\kappa,\lambda})} = v_{N,\kappa,\lambda}(d_M(\Omega, \partial M)).$$

Hence, $d_M(\Omega, \partial M) \leq v_{N,\kappa,\lambda}^{-1}(\eta)$. This proves $\text{BSep}((M, f); \eta) \leq v_{N,\kappa,\lambda}^{-1}(\eta)$. We arrive at the desired inequality.

We next consider the case where $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$. Notice that $s_{\kappa,\lambda}(t) = e^{-\lambda t}$. In view of Lemma 5.1, we see

$$\eta \leq 1 - \frac{s_{N,\kappa,\lambda}(r)}{s_{N,\kappa,\lambda}(\text{InRad} M)} \leq 1 - \frac{s_{N,\kappa,\lambda}(r)}{\int_0^{\infty} s_{\kappa,\lambda}^{N-1}(t) dt} = e^{-(N-1)\lambda r},$$
where we put $r := d_M(\Omega, \partial M)$. In particular, $d_M(\Omega, \partial M)$ is smaller than or equal to the right hand side of the desired one. We obtain

$$\text{BSep}((M, f); \eta) \leq \frac{1}{(N-1)\lambda} \log \frac{1}{\eta}.$$ 

Thus, we complete the proof. \hfill \square

We also have the following estimate under the condition \textbf{(2.3)}:

**Theorem 5.4.** Let $\partial M$ be compact. Assume $\text{Ric}^1_{f, \partial M} \geq (n-1)\kappa e^{\frac{-4\delta}{\lambda}}$ and $H_{f, \partial M} \geq (n-1)\lambda e^{\frac{-2\delta}{\lambda}}$. Suppose additionally that $f \leq (n-1)\delta$ for some $\delta \in \mathbb{R}$. Then for every $\eta \in (0, 1]$ the following hold:

1. if $\kappa$ and $\lambda$ satisfy the convex-ball-condition, then
   $$\text{ObsInRad}((M, f); -\eta) \leq v_{n, \kappa, \lambda}^{-1} e^{-4\delta, \lambda} e^{-2\delta}(\eta);$$

2. if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then
   $$\text{ObsInRad}((M, f); -\eta) \leq \frac{1}{(n-1)\lambda e^{-2\delta}} \log \frac{1}{\eta}.$$

**Proof.** The proof is similar to that of Theorem \textbf{5.3}. For a fixed $\Omega \subset M$ with $m_{M,f}(\Omega) \geq \eta$, $d_M(\Omega, \partial M) > 0$, it suffices to estimate $d_M(\Omega, \partial M)$ from above by the right hand side of the desired inequality in each case.

In the case where $\kappa$ and $\lambda$ satisfy the convex-ball-condition, from \textbf{Theorem 2.6} and \textbf{Lemma 5.2} we derive

$$\eta \leq 1 - \frac{s_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta}(d_M(\Omega, \partial M))}{s_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta}(C_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta})} = v_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta}(d_M(\Omega, \partial M)).$$

In particular, $d_M(\Omega, \partial M) \leq v_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta}(\eta)$. This is the desired one.

If $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, then in view of \textbf{Lemma 5.2} we see

$$\eta \leq 1 - \frac{s_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta}(r)}{s_{n, \kappa, e^{-4\delta}, \lambda} e^{-2\delta}(t) dt} = e^{-(n-1)\lambda e^{-2\delta} r},$$

where $r := d_M(\Omega, \partial M)$. From the above inequality, we deduce

$$d_M(\Omega, \partial M) \leq \frac{1}{(n-1)\lambda e^{-2\delta}} \log \frac{1}{\eta}.$$ 

This completes the proof of Theorem \textbf{5.4}. \hfill \square
5.2. **Proof of Theorem 1.3.** In order to conclude Theorem 1.3, we calculate the invariants of the finite dimensional model spaces.

**Lemma 5.5.** For every \( \eta \in (0, 1] \) the following hold:

1. for the ball-model-space \( B^n_{\kappa, \lambda} \) defined as (1.9),
   \[
   \text{ObsInRad}(B^n_{\kappa, \lambda}; -\eta) = \text{BSep}(B^n_{\kappa, \lambda}; \eta) = v_{n, \kappa, \lambda}^{-1}(\eta);
   \]

2. for the warped-product-model-space \( M^n_{\kappa, \lambda} \) defined as (1.10),
   \[
   \text{ObsInRad}(M^n_{\kappa, \lambda}; -\eta) = \text{BSep}(M^n_{\kappa, \lambda}; \eta) = \frac{1}{(n-1)\lambda} \log \frac{1}{\eta}.
   \]

**Proof.** Let us present the equality for \( B^n_{\kappa, \lambda} \). We show

\[
(5.2) \quad \text{BSep}(B^n_{\kappa, \lambda}; \eta) = v_{n, \kappa, \lambda}^{-1}(\eta).
\]

Let \( B_\eta \subset B^n_{\kappa, \lambda} \) denote the closed geodesic ball with same center as \( B^n_{\kappa, \lambda} \) and \( m_{B^n_{\kappa, \lambda}}(B_\eta) = \eta \). For the radius \( r_\eta \) of \( B_\eta \), we see

\[
\text{BSep}(B^n_{\kappa, \lambda}; \eta) = d_{B^n_{\kappa, \lambda}}(B_\eta, \partial B^n_{\kappa, \lambda}) = C_{\kappa, \lambda} - r_\eta.
\]

It holds that

\[
\eta = m_{B^n_{\kappa, \lambda}}(B_\eta) = \int_{C_{\kappa, \lambda} - r_\eta}^{C_{\kappa, \lambda}} s_{n, \kappa, \lambda}^{n-1}(t) \, dt
\]

\[
= \int_0^{C_{\kappa, \lambda}} s_{n, \kappa, \lambda}^{n-1}(t) \, dt = v_{n, \kappa, \lambda}(C_{\kappa, \lambda} - r_\eta);
\]

in particular, \( C_{\kappa, \lambda} - r_\eta = v_{n, \kappa, \lambda}^{-1}(\eta) \). This yields (5.2). Since \( \text{supp} \, m_{B^n_{\kappa, \lambda}} \) coincides with \( B^n_{\kappa, \lambda} \), Proposition 3.7 and (5.2) imply the desired one.

For the warped-product-model-space \( M^n_{\kappa, \lambda} \), the same argument as in the proof of \( B^n_{\kappa, \lambda} \) leads to the desired equality. \( \square \)

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let \( \partial M \) be compact. Assume \( \text{Ric}_{\partial M} \geq (n-1)\kappa \) and \( H_{\partial M} \geq (n-1)\lambda \). Letting \( f = \log \text{vol}_M(M) \) and \( N = n \) in Theorem 5.3 for \( \eta \in (0, 1] \) we have the following (cf. Remark 1.2):

1. if \( \kappa \) and \( \lambda \) satisfy the ball-condition, then
   \[
   \text{ObsInRad}(M; -\eta) \leq v_{n, \kappa, \lambda}^{-1}(\eta);
   \]

2. if \( \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \), then
   \[
   \text{ObsInRad}(M; -\eta) \leq \frac{1}{(n-1)\lambda} \log \frac{1}{\eta}.
   \]

By Lemma 5.5 we complete the proof of Theorem 1.3. \( \square \)
6. INFINITE DIMENSIONAL COMPARISONS

6.1. Relative volume comparisons. In order to prove Theorem 1.4, we develop comparison geometry of manifolds with boundary under the curvature condition (2.2). We notice that in our comparison theorems in this subsection, \( m_{M,f} \) need not be a probability measure.

We first show the following Laplacian comparison:

**Lemma 6.1.** Let \( z \in \partial M \). Let us assume that \( \text{Ric}_f^\infty(\gamma'_z(t)) \geq K \) for all \( t \in (0, \tau(z)) \), and \( H_{f,z} \geq \Lambda \). Then for all \( t \in (0, \tau(z)) \)

\[
\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq Kt + \Lambda.
\]

**Proof.** Define \( h_{f,z} := (\Delta_f \rho_{\partial M}) \circ \gamma_z \). By applying the Bochner formula (2.4) to the distance function \( \rho_{\partial M} \), we obtain

\[
0 = \text{Ric}_f^\infty(\gamma'_z(t)) + \| \text{Hess} \rho_{\partial M} \|^2_{\text{HS}}(\gamma_z(t)) - g(\nabla \Delta_f \rho_{\partial M}, \nabla \rho_{\partial M})(\gamma_z(t))
\geq K - h'_{f,z}(t).
\]

It holds that \( h_{f,z}(t) \to H_{f,z} \) as \( t \to 0 \), and hence

\[
h_{f,z}(t) \geq Kt + H_{f,z} \geq Kt + \Lambda.
\]

We arrive at the desired inequality. \( \square \)

Furthermore, we prove the following volume element comparison:

**Lemma 6.2.** Let \( z \in \partial M \). Let us assume that \( \text{Ric}_f^\infty(\gamma'_z(t)) \geq K \) for all \( t \in (0, \tau(z)) \), and \( H_{f,z} \geq \Lambda \). Then for all \( t_1, t_2 \in [0, \tau(z)) \) with \( t_1 \leq t_2 \)

\[
\frac{\theta_f(t_2, z)}{\theta_f(t_1, z)} \leq \frac{e^{\frac{K}{2} t_2^2 - \Lambda t_2}}{e^{\frac{K}{2} t_1^2 - \Lambda t_1}},
\]

where \( \theta_f(t, z) \) is defined as (2.5).

**Proof.** By (2.6) and Lemma 6.1 for all \( t \in (0, \tau(z)) \) we see

\[
\frac{d}{dt} \log \frac{\theta_f(t, z)}{e^{\frac{K}{2} t^2 - \Lambda t}} = -\Delta_f \rho_{\partial M}(\gamma_z(t)) + (Kt + \Lambda) \leq 0.
\]

This implies the lemma. \( \square \)

We now conclude the following relative volume comparison:

**Theorem 6.3.** Let \( \partial M \) be compact. Let us assume that \( \text{Ric}_f^\infty(\gamma'_z(t)) \geq K \) and \( H_{f,\partial M} \geq \Lambda \). Then for all \( r, R > 0 \) with \( r \leq R \)

\[
\frac{m_{M,f}(B_R(\partial M))}{m_{M,f}(B_r(\partial M))} \leq \frac{\int_0^R e^{\frac{-K}{2} t^2 - \Lambda t} dt}{\int_0^r e^{\frac{-K}{2} t^2 - \Lambda t} dt}.
\]
Proof. Using Lemma 6.2 for all $t_1, t_2 \geq 0$ with $t_1 \leq t_2$ we have
\[
\bar{\theta}_f(t_2, z) e^{-\frac{K}{2} t_2^2 - \Lambda t_1} \leq \bar{\theta}_f(t_1, z) e^{-\frac{K}{2} t_2^2 - \Lambda t_1},
\]
where $\bar{\theta}_f$ is defined as (2.8). Let us integrate the both sides over $[0, r]$ with respect to $t_1$, and over $[r, R]$ with respect to $t_2$. It follows that
\[
\int_r^R \frac{\bar{\theta}_f(t_2, z) dt_2}{\int_0^r \theta_f(t_1, z) dt_1} \leq \int_r^R \frac{\theta_f(t_2, z) dt_2}{\int_0^r \theta_f(t_1, z) dt_1}.
\]
The formula (2.7) yields
\[
\frac{m_{M,f}(B_R(\partial M))}{m_{M,f}(B_r(\partial M))} \leq 1 + \frac{\int_r^R e^{-\frac{K}{2} t_2^2 - \Lambda t_1} dt_2}{\int_0^r e^{-\frac{K}{2} t_2^2 - \Lambda t_1} dt_1} = \frac{\int_0^r e^{-\frac{K}{2} t_2^2 - \Lambda t_1} dt_1}{\int_0^r e^{-\frac{K}{2} t_2^2 - \Lambda t_1} dt_1}.
\]
We complete the proof of Theorem 6.3. 

Remark 6.1. The author [43] has shown Lemmas 6.1, 6.2 and Theorem 6.3 when $K = 0$ and $\Lambda = 0$ (see Lemmas 3.2, 3.4 and Theorem 5.5 in [43], and cf. Remark 2.8).

Remark 6.2. Under the same setting as in Theorem 6.3, Morgan [37] has obtained a similar absolute volume comparison theorem of Heintze-Karcher type (see Theorem 2 in [37]).

6.2. Distribution laws. Before we show Theorem 1.4, we present distribution laws concerning our infinite dimensional model spaces.

We observe that our infinite dimensional model spaces appear as the limits of a sequence of hemispheres $\{B^{n}_{\kappa/n,0}\}$, and that of Euclidean balls $\{B^{n}_{0,\lambda/n}\}$ for $\kappa, \lambda > 0$ by letting $n \to \infty$.

Proposition 6.4. Let us assume that $\kappa$ and $\lambda$ satisfy the ball-condition. Then the following distribution laws hold:

1. if $\kappa > 0$ and $\lambda = 0$, then
\[
\frac{dm_{I,\rho_{0B^{n}_{\kappa/n,0}}}}{dt} \to \frac{e^{-\frac{K}{2} t^2}}{\int_I e^{-\frac{K}{2} t^2} dt}
\]
as $n \to \infty$, where $m_{I,\rho_{0B^{n}_{\kappa/n,0}}}$ denotes the Borel probability measure of the $\rho_{0B^{n}_{\kappa/n,0}}$-screen $I_{\rho_{0B^{n}_{\kappa/n,0}}}$ defined as (1.3); in particular, $m_{I,\rho_{0B^{n}_{\kappa/n,0}}}$ weakly converges to the Borel probability measure of the half-Gaussian-model-space $G_{\kappa,0}$ defined as (1.11);

2. if $\kappa = 0$ and $\lambda > 0$, then
\[
\frac{dm_{I,\rho_{0B^{n}_{0,\lambda/n}}}}{dt} \to \lambda e^{-\lambda t};
\]
as $n \to \infty$; in particular, $m_{I,\rho_{0,\lambda/n}}$ weakly converges to the measure of the exponential-model-space $E_\lambda$ defined as (1.12).

**Proof.** We notice that if $\kappa$ and $\lambda$ satisfy the ball-condition, then

$$
\frac{dm_{I,\rho_{0,B_0}}}{dt} = \int_0^{C_{\kappa,\lambda}} \frac{s_{\kappa,\lambda}^{-1}(t)}{s_{\kappa,\lambda}(t)} dt, \text{ } s_{\kappa,\lambda}(t) = -\sqrt{\kappa + \lambda^2} s_{\kappa}(t - C_{\kappa,\lambda}),
$$

$$
C_{\kappa,\lambda} = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \left( \frac{\pi}{2} - \tan^{-1} \frac{\lambda}{\sqrt{\kappa}} \right) & \text{if } \kappa > 0, \\
\lambda^{-1} & \text{if } \kappa = 0, \\
\frac{1}{2} \sqrt{\frac{1}{|\kappa|}} \log \frac{\lambda + \sqrt{|\kappa|}}{\lambda - \sqrt{|\kappa|}} & \text{if } \kappa < 0,
\end{cases}
$$

where $s_{\kappa}(t)$ is a unique solution of the Jacobi equation $\psi''(t) + \kappa \psi(t) = 0$ with $\psi(0) = 0$, $\psi'(0) = 1$; in particular, if $\kappa > 0$, then

$$
s_{\kappa,\lambda}(t) = \sqrt{1 + \frac{\lambda^2}{\kappa}} \cos \sqrt{\kappa} \left( t + \frac{1}{\sqrt{\kappa}} \tan^{-1} \frac{\lambda}{\sqrt{\kappa}} \right),
$$

and if $\kappa = 0$, then $s_{0,\lambda}(t) = 1 - \lambda t$. Therefore, in the case where $\kappa > 0$ and $\lambda = 0$, the desired convergence follows from (6.1), (6.2) and

$$
\cos^{n-1} \frac{t}{\sqrt{n}} \to e^{-\frac{t^2}{2}}.
$$

In the case where $\kappa = 0$ and $\lambda > 0$, the formula (6.1) yields

$$
\frac{dm_{I,\rho_{0,B_0}}}{dt} = \lambda \left( 1 - \frac{\lambda}{n} t \right)^{n-1} \to \lambda e^{-\lambda t}
$$
as $n \to \infty$. We complete the observation.

We further mention that the exponential-model-space also appears as the limit of a sequence of warped-product-model-spaces $\{M^\kappa_{n,\lambda/n} \}$ for $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$ when $n \to \infty$, here $\lambda/n = \sqrt{\kappa/n^2}$.

**Proposition 6.5.** Let $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$. Then

$$
\frac{dm_{I,\rho_{M^\kappa_{n,\lambda/n}}}}{dt} \to \lambda e^{-\lambda t},
$$
as $n \to \infty$; in particular, $m_{I,\rho_{M^\kappa_{n,\lambda/n}}}$ weakly converges to the Borel probability measure of the exponential-model-space $E_\lambda$. 

$\square$
Proof. Let us note that if \( \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \), then \( s_{\kappa, \lambda}(t) = e^{-\lambda t} \) and

\[
\frac{d m_{I, \partial M, \kappa, \lambda}}{dt} = \frac{s_{\kappa, \lambda}^{n-1}(t)}{\int s_{\kappa, \lambda}^{n-1}(t) \, dt} = (n-1) \lambda e^{-(n-1)\lambda t}.
\]

This proves the assertion.

6.3. **Proof of Theorem 1.4.** One can prove the following result only by replacing the role of Theorem 2.7 with that of Theorem 6.3 in the proof of Lemma 5.1. The proof is left to the reader.

**Lemma 6.6.** Let \( \partial M \) be compact. Let us assume \( \text{Ric}_{f, \partial M}^\infty \geq K \) and \( H_{f, \partial M} \geq \Lambda \). Suppose additionally that \( \text{InRad} M \leq D \) for some \( D > 0 \).

For \( \eta \in (0, 1) \), let \( \Omega \subset M \) be a Borel subset with \( m_{M, f}(\Omega) \geq \eta \) and \( d_M(\Omega, \partial M) > 0 \). Then we have

\[
\int_0^{d_M(\Omega, \partial M)} e^{-\frac{K}{2} t^2 - \Lambda t} \, dt \leq (1 - \eta) \int_0^{D} e^{-\frac{K}{2} t^2 - \Lambda t} \, dt.
\]

We are now in a position to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( \partial M \) be compact. Let us assume \( \text{Ric}_{f, \partial M}^\infty \geq K \) and \( H_{f, \partial M} \geq \Lambda \). Suppose either (1) \( K > 0 \) and \( \Lambda \in \mathbb{R} \); or (2) \( K = 0 \) and \( \Lambda > 0 \). We note again that in this case, \( \int_0^\infty e^{-\frac{K}{2} t^2 - \Lambda t} \, dt \) is finite.

We can assume \( \eta < 1 \). We prove the desired statement by estimating \( B\text{Sep}((M, f); \eta) \) from above. We may assume \( B\text{Sep}((M, f); \eta) > 0 \). Let \( \Omega \subset M \) denote a Borel subset with \( m_{M, f}(\Omega) \geq \eta \) and \( d_M(\Omega, \partial M) > 0 \). Lemma 6.6 leads us to

\[
\eta \leq 1 - \int_0^{d_M(\Omega, \partial M)} \frac{e^{-\frac{K}{2} t^2 - \Lambda t} \, dt}{\int_0^{\text{InRad} M} e^{-\frac{K}{2} t^2 - \Lambda t} \, dt} \leq S_{K, \Lambda}(d_M(\Omega, \partial M)),
\]

where \( S_{K, \Lambda} : [0, \infty] \to [0, 1] \) is a function defined as

\[
S_{K, \Lambda}(r) := \frac{\int_r^\infty e^{-\frac{K}{2} t^2 - \Lambda t} \, dt}{\int_0^\infty e^{-\frac{K}{2} t^2 - \Lambda t} \, dt}.
\]

It follows that \( d_M(\Omega, \partial M) \leq S_{K, \Lambda}^{-1}(\eta) \); in particular, \( B\text{Sep}((M, f); \eta) \leq S_{K, \Lambda}^{-1}(\eta) \). Here, we can check that

\[
S_{K, \Lambda}^{-1}(\eta) = \begin{cases} B\text{Sep}(G_{K, \Lambda}; \eta) & \text{if } K > 0 \text{ and } \Lambda \in \mathbb{R}, \\ B\text{Sep}(E_{\Lambda}; \eta) & \text{if } K = 0 \text{ and } \Lambda > 0, \\ \text{ObsInRad}(G_{K, \Lambda}; -\eta) & \text{if } K > 0 \text{ and } \Lambda \in \mathbb{R}, \\ \text{ObsInRad}(E_{\Lambda}; -\eta) & \text{if } K = 0 \text{ and } \Lambda > 0, \end{cases}
\]
where we used Proposition 3.7 in the second equality. By combining Lemma 3.4, $\text{BSep}((M, f); \eta) \leq S\frac{1}{K, \Lambda}(\eta)$ and (6.3), we obtain

$$\text{ObsInRad}((M, f); -\eta) \leq \text{BSep}((M, f); \eta) \leq \begin{cases} 
\text{ObsInRad}(G_{K, \Lambda}; -\eta) & \text{if } K > 0 \text{ and } \Lambda \in \mathbb{R}, \\
\text{ObsInRad}(E_{\Lambda}; -\eta) & \text{if } K = 0 \text{ and } \Lambda > 0.
\end{cases}$$

Thus, we conclude Theorem 1.4.

7. Boundary concentration phenomena

7.1. Critical scale orders. We first state the following assertion concerning the critical scale order of sequences of hemispheres (cf. Proposition 6.4, Subsection 1.1 in [14], Corollary 2.22 in [46], Theorem 8.1.1 and Corollary 8.5.7 in [47]):

**Theorem 7.1.** For $\kappa > 0$, and for every $\eta \in (0, 1]$ we have

$$\lim_{n \to \infty} \text{ObsInRad}(B_n^{\kappa/n}; -\eta) = \text{PartInRad}(G_{\kappa, 0}; 1 - \eta),$$

where $G_{\kappa, 0}$ is the half-Gaussian-model-space defined as (1.11), and the right hand side of (7.1) is the partial inscribed radius of $G_{\kappa, 0}$ defined as (1.2). In particular, for a sequence $\{\kappa_n\}$ of $\kappa_n > 0$, the sequence $\{B_n^{\kappa_n, 0}\}$ is a boundary concentration family if and only if $n \kappa_n \to \infty$.

**Proof.** Let us regard the $n$-dimensional standard sphere $M_n^{\kappa/n}$ with constant curvature $n/\kappa$ as an mm-space defined as (1.8) (see Remark 3.1). We observe that $M_n^{\kappa/n}$ is the double of the hemisphere $B_n^{\kappa/n, 0}$. Based on Lemma 5.5 and this geometric observation, we obtain

$$\text{ObsInRad}(B_n^{\kappa/n, 0}; -\eta) = \text{BSep}(B_n^{\kappa/n, 0}; \eta) = \frac{1}{2} \text{Sep}(M_n^{\kappa/n}; \eta/2, \eta/2),$$

where $\text{Sep}(M_n^{\kappa/n}; \eta/2, \eta/2)$ is the $(\eta/2, \eta/2)$-separation distance of $M_n^{\kappa/n}$ defined as (3.4) (see Remark 3.2). It is well-known that the right hand side tends to $r > 0$ determined by

$$\frac{1 - \eta}{2} = \frac{\int_0^r e^{-\frac{z^2}{2}} \, dt}{\int_{-\infty}^\infty e^{-\frac{z^2}{2}} \, dt}$$

as $n \to \infty$ (see e.g., Theorem 2.1 and Lemma 2.3 in [46]). We see that $r$ is equal to $\text{PartInRad}(G_{\kappa, 0}; 1 - \eta)$, and hence (7.1).

For a metric measure space with boundary $X = (X, d_X, \mu_X)$,

$$\text{ObsInRad}(cX; -\eta) = c \text{ObsInRad}(X; -\eta)$$

(7.2)
for every $c > 0$, where we set $cX := (X, cd_X, \mu_X)$ (cf. Proposition 2.19 in [46]). Therefore, by (7.1) and $B_{\kappa_n,0}^n = \sqrt{\kappa} (n \kappa_n)^{-1} B_{\kappa/n,0}^n$, we arrive at the desired conclusion.

We next investigate sequences of Euclidean balls.

**Theorem 7.2.** For $\lambda > 0$, and for every $\eta \in (0, 1]$ we have

\[
(7.3) \quad \lim_{n \to \infty} \text{ObsInRad}(B_{0,\lambda/n}^n; -\eta) = \text{PartInRad}(E_\lambda; 1 - \eta) = \frac{1}{\lambda} \log \frac{1}{\eta},
\]

where $E_\lambda$ is the exponential-model-space defined as (1.12). In particular, for a sequence $\{\lambda_n\}$ of $\lambda_n > 0$, the sequence $\{B_{0,\lambda_n}^n\}$ is a boundary concentration family if and only if $n \lambda_n \to \infty$.

**Proof.** By $s_{0,\lambda/n}(t) = 1 - \lambda n^{-1} t$, we have $v_{n,0,\lambda/n}(r) = (1 - \lambda n^{-1} r)^n$, where $v_{n,0,\lambda}$ is defined as (5.1). Lemma 5.5 implies that

\[
\text{ObsInRad}(B_{0,\lambda_n}^n; -\eta) = \frac{n}{\lambda} (1 - \eta^n) \to \frac{1}{\lambda} \log \frac{1}{\eta}
\]
as $n \to \infty$. On the other hand, $\text{PartInRad}(E_\lambda; 1 - \eta)$ is equal to a positive number $r > 0$ determined as $1 - \eta = \int_0^r \lambda e^{-\lambda t} dt$; in particular, $\text{PartInRad}(E_\lambda; 1 - \eta) = \lambda^{-1} \log \eta^{-1}$. This proves the equalities (7.3). Due to $B_{0,\lambda_n}^n = \lambda (n \lambda_n)^{-1} B_{0,\lambda/n}^n$ and (7.2), we complete the proof. \qed

We also provide the following result for warped-product-model-spaces:

**Theorem 7.3.** For $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$, and for every $\eta \in (0, 1]$ we have

\[
(7.3) \quad \lim_{n \to \infty} \text{ObsInRad}(M_{\kappa/n^2,\lambda/n}^n; -\eta) = \text{PartInRad}(E_\lambda; 1 - \eta) = \frac{1}{\lambda} \log \frac{1}{\eta}.
\]

Moreover, for a sequence $\{\kappa_n\}$ of $\kappa_n < 0$, and for a sequence $\{\lambda_n\}$ of $\lambda_n = \sqrt{|\kappa_n|}$, the sequence $\{M_{\kappa_n,\lambda_n}^n\}$ is a boundary concentration family if and only if we have $n \lambda_n \to \infty$ as $n \to \infty$.

**Proof.** Lemma 5.5 yields

\[
\text{ObsInRad}(M_{\kappa/n^2,\lambda/n}^n; -\eta) = \frac{n}{(n-1)\lambda} \log \frac{1}{\eta} \to \frac{1}{\lambda} \log \frac{1}{\eta}
\]
as $n \to \infty$. The second equality of (7.3) tells us the desired equalities. The later assertion also immediately follows from Lemma 5.5. \qed

Let us prove a proof of Corollary 1.5.

**Proof of Corollary 1.5.** Let $\kappa$ and $\lambda$ satisfy the convex-ball-condition. We will prove that $\{B_{\kappa,\lambda}^n\}$ is a boundary concentration family.

We first consider the case of $\kappa > 0$. In this case, we have $\lambda \geq 0$. Due to Theorem 1.3, we have $\text{ObsInRad}(B_{\kappa,\lambda}^n; -\eta) \leq \text{ObsInRad}(B_{\kappa,0}^n; -\eta)$. 
Furthermore, Theorem 7.1 tells us that \( \{B^n_{\kappa,0}\} \) is a boundary concentration family. Hence, \( \{B^n_{\kappa,\lambda}\} \) is also a boundary concentration family.

When \( \kappa = 0 \), the desired statement follows from Theorem 7.2.

In the case of \( \kappa < 0 \), it holds that \( \lambda > \sqrt{|\kappa|} \). In virtue of Theorem 1.3 and Lemma 5.5, we obtain

\[
\text{ObsInRad}(B^n_{\kappa,\lambda}; -\eta) \leq \text{ObsInRad}(M^n_{\kappa,\sqrt{|\kappa|}}; -\eta) = \frac{1}{(n-1)\sqrt{|\kappa|}} \log \frac{1}{\eta};
\]

in particular, \( \text{ObsInRad}(B^n_{\kappa,\lambda}; -\eta) \to 0 \) as \( n \to \infty \). Thus, we complete the proof of Corollary 1.5.

\[\square\]

### 7.2. Positive dimensional cases

In this subsection, we summarize corollaries of Theorem 5.3. Hereafter, let \( \{(M^n, f_n)\} \) be a sequence of metric measure spaces with compact boundary defined as (1.7). We denote by \( \dim M_n \) the dimension of \( M_n \). Theorem 5.3 together with Theorems 7.1, 7.2, 7.3 and Corollary 1.5 leads us to the following:

**Corollary 7.4.** Let \( \{N_n\} \) be a sequence of integers with \( N_n \geq \dim M_n \), and let \( \{\kappa_n\} \) be a sequence of \( \kappa_n > 0 \). Assume \( \text{Ric}^N_{f_n, \partial M^n_n} \geq (N_n - 1)\kappa_n \) and \( H_{f_n, \partial M_n} \geq 0 \) for each \( n \). If \( N_n \kappa_n \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

**Corollary 7.5.** Let \( \{N_n\} \) be a sequence of integers with \( N_n \geq \dim M_n \), and let \( \{\lambda_n\} \) be a sequence of \( \lambda_n > 0 \). We assume \( \text{Ric}^N_{f_n, \partial M^n_n} \geq 0 \) and \( H_{f_n, \partial M_n} \geq (N_n - 1)\lambda_n \) for each \( n \). If \( N_n \lambda_n \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

**Corollary 7.6.** Let \( \{N_n\} \) be a sequence of integers with \( N_n \geq \dim M_n \), and let \( \{\kappa_n\} \) be a sequence of \( \kappa_n < 0 \). For each \( n \), we put \( \lambda_n := \sqrt{|\kappa_n|} \).

We assume \( \text{Ric}^N_{f_n, \partial M^n_n} \geq (N_n - 1)\kappa_n \) and \( H_{f_n, \partial M_n} \geq (N_n - 1)\lambda_n \). If \( N_n \lambda_n \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

**Corollary 7.7.** Let \( \{N_n\} \) be a sequence of integers with \( N_n \geq \dim M_n \). Assume \( \text{Ric}^N_{f_n, \partial M^n_n} \geq (N_n - 1)\kappa \) and \( H_{f_n, \partial M_n} \geq (N_n - 1)\lambda \) for each \( n \). We also assume that one of the following holds:

1. \( \kappa \) and \( \lambda \) satisfy the convex-ball-condition;
2. \( \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \).

If \( N_n \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

Now, we will present a concrete example concerning these corollaries, especially Corollary 7.6.
Example 7.1. Let \( \{N_n\} \) be a sequence of integers with \( N_n \geq n \), and let \( \{\kappa_n\} \) be a sequence of \( \kappa_n < 0 \). For each \( n \), put \( \lambda_n := \sqrt{|\kappa_n|} \). Define a sequence \( \{(\tilde{M}_n, \tilde{f}_n)\} \) of metric measure spaces with boundary as
\[
\tilde{M}_n := ([0, \infty) \times \mathbb{S}^{n-1}, dt^2 + e^{-2\lambda_n t} ds_{n-1}^2),
\]
\[
\tilde{f}_n := (N_n - n) \lambda_n \rho_{\partial \tilde{M}_n} - \log ((N_n - 1) \lambda_n (\text{vol}_{\mathbb{S}^{n-1}}(\mathbb{S}^{n-1}))^{-1}).
\]
Then the sequence \( \{(\tilde{M}_n, \tilde{f}_n)\} \) is a boundary concentration family if and only if we have \( N_n \lambda_n \to \infty \).

Computing the curvatures of \( \tilde{M}_n \), we see the following (cf. calculations in the proof of Lemma 3.1 in [43]): For all \( z \in \partial \tilde{M}_n \) and \( t > 0 \),
\[
\text{Ric}^{N_n} \left( f_n'(t) \right) = (N_n - 1) \kappa_n, \quad \text{H}^{\tilde{M}_n} = (N_n - n) \lambda_n.
\]
Moreover, \( \theta^{\tilde{f}_n}(t, z) = e^{-\tilde{f}_n(z)} s_{n-1}^{N_n-1}(t) \), where \( \theta^{\tilde{f}_n}(t, z) \) is defined as (2.5). This yields \( m_{\tilde{M}_n, \tilde{f}_n}(\tilde{M}_n) = 1 \). Furthermore, \( \text{BSep}(\tilde{M}_n, \tilde{f}_n; \eta) \) equals to \( r_n > 0 \) determined by
\[
\text{vol}_{\mathbb{S}^{n-1}}(\mathbb{S}^{n-1}) \int_{r_n}^\infty \theta^{\tilde{f}_n}(t, z) \, dt = \eta;
\]
in particular, from Proposition 3.7 we deduce
\[
\text{ObsInRad}(\tilde{M}_n, \tilde{f}_n; -\eta) = \text{BSep}(\tilde{M}_n, \tilde{f}_n; \eta) = \frac{1}{(N_n - n) \lambda_n} \log \frac{1}{\eta}.
\]
We conclude the desired statement.

7.3. One dimensional cases. We next summarize corollaries of Theorem 5.4. Throughout this subsection, we always assume \( \dim M_n = n \).

Similarly to the above subsection, one can verify the following assertions by using Theorem 5.4.

Corollary 7.8. Let \( \{\kappa_n\} \) be a sequence of \( \kappa_n > 0 \), and let \( \{\delta_n\} \) be a sequence of \( \delta_n \in \mathbb{R} \). Let us assume \( \text{Ric}^{1}_{f_n, \partial M_n^+} \geq (n - 1) \kappa_n e^{2\delta_n} \) and \( \text{H}^{f_n, \partial M_n} \geq 0 \) for each \( n \). Suppose additionally that \( f_n \leq (n - 1) \delta_n \). If \( n \kappa_n e^{-2\delta_n} \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

Corollary 7.9. Let \( \{\lambda_n\} \) be a sequence of \( \lambda_n > 0 \), and let \( \{\delta_n\} \) be a sequence of \( \delta_n \in \mathbb{R} \). Let us assume \( \text{Ric}^{1}_{f_n, \partial M_n^+} \geq 0 \) and \( \text{H}^{f_n, \partial M_n} \geq (n - 1) \lambda_n e^{-\delta_n} \) for each \( n \). Suppose additionally that \( f_n \leq (n - 1) \delta_n \). If \( n \lambda_n e^{-2\delta_n} \to \infty \), then \( \{(M_n, f_n)\} \) is a boundary concentration family.

Corollary 7.10. Let \( \{\kappa_n\} \) be a sequence of \( \kappa_n < 0 \), and let \( \{\delta_n\} \) be a sequence of \( \delta_n \in \mathbb{R} \). For each \( n \), put \( \lambda_n := \sqrt{|\kappa_n|} \). Assume \( \text{Ric}^{1}_{f_n, \partial M_n^+} \geq (n - 1) \kappa_n e^{2\delta_n} \) and \( \text{H}^{f_n, \partial M_n} \geq (n - 1) \lambda_n e^{-2\delta_n} \). Suppose additionally
that \( f_n \leq (n - 1)\delta_n \). If \( n \lambda_n e^{-2\delta_n} \to \infty \), then \( \{ (M_n, f_n) \} \) is a boundary concentration family.

**Corollary 7.11.** We assume \( \text{Ric}^1_{f_n, \partial M_n} \geq (n-1)\kappa e^{-\frac{2\lambda_n}{n-1}} \) and \( H_{f_n, \partial M_n} \geq (n-1)\lambda e^{-\frac{2\delta_n}{n-1}} \) for each \( n \). Suppose additionally that \( f_n \leq (n - 1)\delta \) for some \( \delta \in \mathbb{R} \). We also assume that one of the following holds:

1. \( \kappa \) and \( \lambda \) satisfy the convex-ball-condition;
2. \( \kappa < 0 \) and \( \lambda = \sqrt{|\kappa|} \).

Then \( \{ (M_n, f_n) \} \) is a boundary concentration family.

Let us construct a concrete example for the above corollaries:

**Example 7.2.** Let \( \{ \kappa_n \} \) be a sequence of \( \kappa_n < 0 \), and let \( \{ \delta_n \} \) be a sequence of \( \delta_n \in \mathbb{R} \). For each \( n \), we put \( \lambda_n := \sqrt{|\kappa_n|} \). Let us denote by \((S_{\kappa_n, \lambda_n, \delta_n}^{n-1}, ds_{\kappa_n, \lambda_n, \delta_n}^{2})\) the \((n - 1)\)-dimensional sphere with volume

\[
e^{-C_n} \left( \int_1^\infty \exp \left( -\frac{(n - 1) \lambda_n e^{-2\delta_n}}{2} t^2 \right) dt \right)^{-1},
\]

where \( C_n := 2^{-1} (n - 1) (\lambda_n e^{-2\delta_n} - 2 \delta_n) \). We define a sequence \( \{ (\tilde{M}_n, \tilde{f}_n) \} \) of metric measure spaces with boundary as

\[
\tilde{M}_n := ([0, \infty) \times S_{\kappa_n, \lambda_n, \delta_n}^{n-1}, dt^2 + H_{\kappa_n, \lambda_n, \delta_n}^2 (t) ds_{\kappa_n, \lambda_n, \delta_n}^{2}),
\]

\[
\tilde{f}_n := -\frac{n - 1}{2} \log (\rho_{\partial \tilde{M}_n} + 1) + (n - 1)\delta_n,
\]

where

\[
H_{\kappa_n, \lambda_n, \delta_n} (t) := \frac{1}{\sqrt{t+1}} \exp \left( -\frac{\lambda_n e^{-2\delta_n}}{2} ((t+1)^2 - 1) \right).
\]

If \( n \lambda_n e^{-2\delta_n} \to \infty \), then \( \{ (\tilde{M}_n, \tilde{f}_n) \} \) is a boundary concentration family.

We derive this statement from Corollary 7.10. We rewrite \( \tilde{M}_n \) as

\[
\tilde{M}_n = ([0, \infty) \times S_{\kappa_n, \lambda_n, \delta_n}^{n-1}, dt^2 + F_{\kappa_n, \lambda_n, \delta_n}^2 (t) ds_{\kappa_n, \lambda_n, \delta_n}^{2}),
\]

where for each \( z \in \partial \tilde{M}_n \),

\[
s_{\tilde{f}_n, z} (t) := \int_0^t e^{-\frac{2\tilde{f}_n (\gamma_z (u))}{n-1}} du,
\]

\[
F_{\kappa_n, \lambda_n, \delta_n} (t) := \exp \left( \frac{\tilde{f}_n (\gamma_z (t)) - \tilde{f}_n (z)}{n-1} \right) s_{\kappa_n, \lambda_n} (s_{\tilde{f}_n, z} (t)).
\]

From this expression, one can compute the curvatures of \( \tilde{M}_n \) as follows (cf. calculations in the proof of Lemmas 3.5, 5.2 and 7.1 in [45]): For
all \( z \in \partial \tilde{M}_n \) and \( t > 0 \) we have

\[
\text{Ric}_{f_n} \left( \gamma'_z(t) \right) = (n - 1) \kappa_n e^{-\frac{4f_n(\gamma_z(t))}{n-1}}, \quad H_{f_n,z} = (n - 1) \lambda_n e^{-\frac{2f_n(z)}{n-1}}.
\]

Moreover, \( \theta_{f_n}(t, z) = e^{-f_n(z)} s_n^{n-1} \kappa_n \lambda_n (s_{f_n,z}(t)) \), and hence

\[
\theta_{f_n}(t, z) = e^{C_n} \exp \left( -\frac{(n - 1) \lambda_n e^{-2\delta_n}}{2}(t + 1)^2 \right);
\]

in particular, we see \( m_{\tilde{M}_n, f_n}(\tilde{M}_n) = 1 \). Since \( \tilde{f}_n \leq (n - 1) \delta_n \), Corollary 7.10 leads us to the desired conclusion.

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