A NEW METHODOLOGY FOR SOLVING BI-CRITERION FRACTIONAL STOCHASTIC PROGRAMMING

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ABSTRACT. Solving a bi-criterion fractional stochastic programming using an existing multi criteria decision making tool demands sufficient efforts and it is time consuming. There are many cases in financial situations that a nonlinear fractional programming, generated as a result of studying fractional stochastic programming, must be solved. Often management is not in needs of an optimal solution for the problem but rather an approximate solution can give him/her a good starting for the decision making or running a new model to find an intermediate or final solution. To this end, this author introduces a new linear approximation technique for solving a fractional stochastic programming (CCP) problem. After introducing the problem, the equivalent deterministic form of the fractional nonlinear programming problem is developed. To solve the problem, a fuzzy goal programming model of the equivalent deterministic form of the fractional stochastic programming is provided and then, the process of defuzzification and linearization of the problem is presented. A sample test problem is solved for presentation purposes. There are some limitations to the proposed approach: (1) solution obtains from this type of modeling is an approximate solution and, (2) preparation of approximation model of the problem may take some times for the beginners.

1. Introduction. Under some circumstances the measure to be used by researcher is the division of one function of variables to another functions where one or both of these functions can be linear or nonlinear. Data Envelopment Analysis (DEA) considering fractional situations with hundreds applications have been used by researcher many times. This is why we can say that fractional programming has attracted the attention of many researchers during past four decades. In this regard Saad (2007) indicated that: "the main reason for interest in fractional programming stems from the fact that linear fractional objective functions occur frequently as measures of performance in a variety of circumstances. Lara and Stancu-Minasian (1997) have reviewed fractional programming as a tool for studying the sustainability of agricultural systems where the essentials of technique in both single and multi-objective cases are outlined. Authors pointed to this reality that algorithms embedded in the programming packages for solving the models are not friendly and this shortcoming needs to be overcome. Two procedures for avoiding this shortcoming in the multiple objective cases are discussed. Publication of five bibliographies..."
complied by Stancu-Minasian (1999) reflect this reality that a large number of theo-
retical as well as algorithmic work have been done by many researchers over the
years. As Lara and Stancu-Minasian (1999) mentioned in their work, although
output/input ratios arise naturally in many economic problems very few real appli-
cations of fractional programming have been reported, particularly in the field of
agriculture. Perhaps, the lack of friendly procedures for solving the models is one
of the main reasons. However, a number of fractional programming applications
can be seen in the work of (Stancu-Minasian’s, 1997). Lara, P. (1993) reported an
application of fractional programming in the field of livestock while Zhu, and Hung
(2011) developed a stochastic linear fractional programming approach for sustain-
able waste management. Pena, et al. (2007) have shown that how the technique
of multiple objective fractional programming can enhance the process of animal
diet formulation. In another research, Lara (2007) linked production theory and
multi-objective fractional programming together to propose a supporting tool for
animal diet formulation. Fu et al. (2018) have proposed a simulation based linear
fractional programming for water allocation and planning for the Songhua river in
China.

More often, an appropriate solution, instead of an optimal solution, to a complex
problem can satisfy managements need for making a quick decision. Multi objective
fractional stochastic type problem can be regarded as a complex problem provided
that a large number of variables are involving in modeling. Although, there are
various approaches to tackle the generalized form of problem, there is no approaches
available for solving a particular case of fractional stochastic problem discussed
here. In this article, the basic idea is to use fuzzy goal programming (FGP) as a
tool for solving the special type of fractional stochastic problem. To do that, first
we define a fuzzy goal programming problem for the equivalent deterministic form
(EDF) of the stochastic programming problem and then we apply the concept of
defuzzification for converting the fuzzy model into a model that is not fuzzy. The
rest of solution procedure is detailed in the respectful sections of the article as it
the process progresses.

The remainder of this article is organized as follow: literature review is the topic
of section 2. Model development is discussed in section 3. Linearization technique
is the topic of section 4. Compromise goal constraints is discussed in section 5 while
Tailors series is the topic of section 6. Computational algorithms are discussed in
section 7. An example problem with details in solution is discussed in section 8.
Authors conclusion and analysis is given in section 9.

2. Literature Review. There are many situations in which businesses deal with
the linear fractional programming problem. In this regard, we can hint to the
work of Steuer (1986) saying that the mathematical optimization problems with a
goal function being a ratio of a linear numerator and a linear denominator have
many applications. In waste management area, Zare Mehrjerdi and Faregh (2017)
employed a fractional function modeling of such type. This sort of modeling is em-
ployed in finance (corporate planning, bank balance sheet management), in Marine
transportation, in water resources, and health care to mention a few. Considering
linear functions for \( F(x) \) and \( G(x) \), then an optimization problem as such as (1)
can be proposed where \( S \) is assumed to be a nonempty bounded polyhedron.

\[
\max f(x) = \frac{F(x)}{G(x)} \quad S.t \quad S = \{x|Ax = b, x \geq 0\}
\]
To solve this problem, many authors such as Charnes and Cooper (1962), Martos (1975), and Wolf (1985) have conducted research on this problem and proposed different algorithms for different forms and shapes of the problems. Comparative investigations of such algorithms can be found in Arsham and Kahn (1990), and Bhatt (1989). In their book, Nonlinear Programming, Theory and Algorithms, Bazaraa and Shetty (1979) have shown that the fractional type objective function shown above has several important properties - it is (simultaneously): pseudo convex, pseudo concave, quasi-convex, quasi-concave, strict quasi-convex and strict quasi-concave. This means that the point that satisfies the Kuhn-Tucker conditions for the maximization problem gives the global maximum on the feasible set. In addition, each local maximum is also a global maximum. This maximum is obtained at an extreme point of S (Metev and Gueorguieva (1995). Algorithms for generalized fractional programming is the topic studied by Crouzieix and Ferland in 1991. In this research, authors have generalized the work of Dinkelbach (1967) where it is tried to find the root of function $F(\lambda) = 0$ where $F(\lambda)$ is the optimal value of the parametric program of

$$F(\lambda) = \inf_{x \in X} \{ \max_{1 \leq i \leq m} \{ f_i(x) - \lambda g_i(x) \} \}$$

(2)

A fractional programming problem with absolute value functions of the type formulated below was proposed by Chadha [4].

$$\max_{x} \sum_{i=1}^{n} c_j|x_j| + \alpha$$

$$\sum_{i=1}^{n} d_j|x_j| + \beta$$

s.t.

$$Ax = b$$

(3)

Researchers have managed to use exact approach and heuristic approach for solving ratio optimization type problem. A traditional approach is the use of parametric method as are discussed by Wolf (1985). Charnes and Cooper (1962, 1973) converted fractional programming (FP) into equivalent linear programming and then solved the resulting problem. Other researchers have conducted research on this problem by treating solving the fractional programming problem as the primal and dual simplex algorithm. This type of treatment can be seen in the work of Farag (2012), and Hasan et al. (2011). Some researchers have employed interior point methods for solving this problem, however. Using this approach, authors reduced the solution of the fractional programming problem into the solution of a sequence of linear programming problem.

Some researchers have used heuristic approaches to solve this sort of optimization problem. A genetic algorithm was proposed by Sameeullah (2008). Galvete et al. (2009) proposed a genetic algorithm to solve a bi-level fractional programming problem where both levels of objectives are assumed to be linear. Bisi et al. (2011) used neural network (NT) for solving nonlinear fractional programming problem. Pal (2013) and Hezam et al. (2013) have employed Particle Swarn Optimization (PSO) approach to solve fractional programming problem. Zare Mehrjerdi (2011) proposed fuzzy goal programming approach for solving fractional programming.

Metev and Gueorguieva (1995) have discussed about a simple method for obtaining weakly efficient points in MOLFP problem. Authors shown that the property of strict quasi-convexity allows to use successfully the reference point method for the analysis of MOLFP problems. Omar M. Saad (2007) proposed a solution algorithm for fuzzy MOLFP where fuzzy parameters are considered in the right-hand side of
the constraints. Furthermore, the concept of α-level set of a fuzzy number has been employed by the authors for the purpose of difuzzification. A solution approach was proposed by Saad and Abd-Rabo (1997) for solving integer linear fractional programming where right-hand side constraints are considered to be random variables. Saad and Sharif (2001) developed a solution method for solving integer linear fractional programming problems with chance constraints, assuming the independency of involved parameters in their model building. Zhou, et al. (2019) proposed a type-2 fuzzy chance constrained fractional integrated modeling method for energy system management under uncertainties and risks.

Goal programming (GP) has been employed by many researchers for solving many managerial problems as well as fuzzy type programming problems. Chang (2005) proposed a fuzzy goal programming approach for solving fractional programming problem with absolute-value functions. Masatoshi Sakawa and Kosuke Kato (1998) conducted a research on the interactive decision-making approach for MOLFP problems with block angular structure involving fuzzy numbers. A multi objective linear fractional programming problem with the block angular structure can be formulated as below:

\[
\min Z_1(x, c_1, d_1) \\
\vdots \\
\min Z_k(x, c_k, d_k)
\]
\[
S.t
\]
\[A_1x_1 + \cdots + A_mx_m \leq b_0\]
\[B_1x_1 + \cdots + A_{m-1}x_{m-1} \leq b_1\]
\[B_mx_m \leq b_m\]
\[j = 1, \cdots, m\]
\[x_j \geq 0\]
\[
Z_i(x, c_i, d_i) = \frac{H_i(x, d_i)}{H_i(x, d_i)} = \frac{c_{i1}x_1 + \cdots + c_{im}x_m + c_{i,m+1}}{d_{i1}x_1 + \cdots + d_{im}x_m + d_{i,m+1}}
\]

Guo et al. (2014) have conducted a research on fuzzy chance constrained linear fractional programming. Authors have employed this approach to obtain the optimal water allocation for the case they had under study. [?]Abdel-Baset and Hezam (2015) proposed an algorithm combined with Chaos Theory named Flower Pollination Algorithm (FPA) for solving rations optimization problems of the type presented below.

\[
\max / \min Z(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{m} \frac{f_i(x)}{g_i(x)} \quad S.t \quad \{x \in S, x \geq 0\}
\]

Biswas and Bose (2012) studied a fuzzy goal programming approach for solving quadratic fractional bi-level programming. The study considers the general solution approach to bi-level programming taking quadratic functions in the form of equation (7).

\[F(X) = CX + \frac{1}{2}X^TDX\]

Fractional programming also known as ratios functions in the literature is a problem of one or more ratios of functions where constrained are deterministic or probabilistic types. A fractional stochastic problem can be defined where functions used in
the numerator and denominator are functions of random variables with known distribution functions. In 1967, Dinkelbach proposed a parametric approach for fractional programming while later in 2008, Sameeullah et al. employed GA method for solving linear fractional programming. Udhayakumar, A. et al. (2010) studied a stochastic simulation based genetic approach for solving chance constrained fractional programming problem. Li, Guo and Ren (2020) proposed models based upon the two-level linear fractional programming method under uncertainty. These models are used for the domain of water resources management. Gupta (2009) proposed a chance constrained programming approach to fractional programming with random numerator. An equivalent deterministic form of the stochastic linear fractional programming formulation of the problem is determined and with the aid of a convex program the EDF problem is solved.

Summary of research articles associated with this research

| Authors | Purpose of research | methodology |
|---------|---------------------|-------------|
| Udhayakumar, A. et al. (2010) | Solving chance constrained fractional programming | Simulation based genetic algorithm |
| Gupta (2009) | Chance constrained approach to fractional programming with random numerator | Convex programming |
| Sameeullah et al. (2008) | Linear fractional programming | Genetic Algorithm |
| Dinkelbach (1967) | Fractional programming | Parametric approach |
| Biswas and Bose (2012) | Quadratic bi-level fractional programming | Goal programming approach |
| Abdel-Baset and Hezam (2015) | Rations optimization problems | Algorithm combined with Chaos Theory named Flower Pollination Algorithm (FPA) |
| Guo et al. (2014) | Fuzzy chance constrained linear fractional programming | Optimization technique |
| Zhu, H., Hung, G. H. (2011) | Stochastic linear programming approach | Optimization fractional |
| Sakawa and Kosuke Kato (1998) | sets of fuzzy numbers | optimality |
| Chang (2005) | Fractional Programming problem with absolute-value functions. | Fuzzy goal programming approach |
| Charnes and Cooper (2005) | converted the fractional | Linear programming |
3. **Model Development.** The basic idea is to use fuzzy goal programming (GP) as a tool for solving fractional stochastic problem. To do that, first we define a fuzzy goal programming for the stochastic programming problem and then we apply the concept of defuzzification to convert the fuzzy model into a model that is not fuzzy. Now, we are in need of developing an equivalent crisp model of the proposed fuzzy system. Linear goal programming was originally introduced by Abraham Charnes and William Cooper in 1961. One can solve a GP model either regularly or interactively. Zare Mehrjerdi applied goal programming and interactive goal programming to various type problems (1986, 1993, 2009, 2011, 2019). The main difference between fuzzy goal programming (FGP) and goal programming (GP) is in that GP requires that decision makers to set definite aspiration values for each goal while in the FGP these are specified in an imprecise manner.

3.1. **Notations.** In the model developed by this author, the following notations are used:

3.2. **Decision Variables.**

\[ x_i = \text{The } j^{th} \text{ decision variable.} \]

\[ X = A \times n \text{ vector with } x_j \text{ element (} j = 1, 2, \cdots, n) \].

3.3. **Random Variables.**

\[ C_{1j} = \text{A random variable with known distribution functions such that } C_{1j} \sim N(C_{1j}^{-}, \sigma_j^2) \]

\[ C_{2j} = \text{A random variable with known distribution functions such that } C_{2j} \sim N(C_{2j}^{-}, \sigma_j^2) \]

\[ D_{1j} = \text{A random variable with known distribution functions such that } D_{1j} \sim N(D_{1j}^{-}, \psi_j^2) \]

\[ D_{2j} = \text{A random variable with known distribution functions such that } D_{2j} \sim N(D_{2j}^{-}, \xi_j^2) \]
3.4. Lower Bounds Values.

- $LF_1$ = Lower bound for probabilistic constraint 2
- $LF_2$ = Lower bound for probabilistic constraint 3
- $LF_3$ = Lower bound for probabilistic constraint 4
- $LF_4$ = Lower bound for probabilistic constraint 5

3.5. Parameters.

- $a_{ij}$ = Technological coefficients
- $b_i$ = The level of the $i$th resource
- $\alpha$ = The probability that the probabilistic constraint (related to numerator) would not hold
- $\beta$ = The probability that the probabilistic constraint (related to denominator) would not hold
- $\alpha_2$ = The probability that the probabilistic constraint (related to numerator) would not hold
- $\beta_2$ = The probability that the probabilistic constraint (related to denominator) would not hold
- $\delta_i$ = The probability that the probabilistic constraint (related to constraints) would not hold
- $q_1 = N^{-1}(\alpha_1)$
- $q_2 = N^{-1}(\beta_1)$
- $q_3 = N^{-1}(\alpha_2)$
- $q_4 = N^{-1}(\beta_2)$
- $u_{1i} = \sigma_{ij}^2$
- $u_{2j} = \phi_{ij}^2$
- $u_{3j} = \psi_{ij}^2$
- $u_{4j} = \zeta_{ij}^2$

$r_1, r_2, s_1,$ and $s_2 = $ are the constant values used in the fractional objective functions.

The remaining of the notations used in this model building is defined as the process of model development progresses.

3.6. Problem Formulation. The general form of fractional chance constrained programming problem used in this research is as shown by Problem 1.

**Problem 1.**

\[
\begin{align*}
\text{Maximize} & \quad Z_1(x) = \frac{F_1(x) + r_1}{F_2(x) + s_1} \quad (8) \\
\text{Maximize} & \quad Z_2(x) = \frac{F_3(x) + r_2}{F_4(x) + s_2} \quad (9) \\
\text{s.t} & \quad P\left\{ \sum_{j=1}^{n} C_{2j} x_j \geq LF_1 \right\} \geq \alpha_1 \quad (10) \\
& \quad P\left\{ \sum_{j=1}^{n} C_{1j} x_j \geq LF_2 \right\} \geq \beta_1 \quad (11) \\
& \quad P\left\{ \sum_{j=1}^{n} D_{1j} x_j \geq LF_3 \right\} \geq \alpha_2 \quad (12)
\end{align*}
\]
A major difficulty in using CCP when input-output coefficients and/or cost vectors are random variables having known distribution functions is the need for a non-linear computer program. The equivalent deterministic form of chance constraints of (10) through (13) is as shown below. More details on this type of modeling can be seen in the works of Kataok (1963), and Zare Mehrjerdi (1986, 2009, 2011, 2019) to mention a few.

By setting $F_i(x) = LF_i$ for all $i = 1, 2, 3, 4$ and assuming that technological coefficients are independently normally distributed random variables then, the EDF of problem 1 can be written as Problem 2.

**Problem 2.**

\[
\begin{align*}
\text{Maximize} & \quad Z_1(x) = \frac{F_1(x) + r_1}{F_2(x) + s_1} = \sum_{j=1}^{n} C_{1j}x_j - q_1(X'V_1X)^{\frac{1}{2}} + r_1 \\
& \text{Maximize} \quad Z_2(x) = \frac{F_3(x) + r_2}{F_4(x) + s_2} = \sum_{j=1}^{n} D_{1j}x_j - q_2(X'V_2X)^{\frac{1}{2}} + r_2 \\
& \text{Maximize} \quad Z_3(x) = W_1 * Z_1(x) + W_2 * Z_2(x) \\
& \quad AX \leq b \quad x_j \geq 0
\end{align*}
\]

3.7. **Solution Methodology.** To solve this multi criterion nonlinear fractional programming an approximate linearization technique is proposed below. For this purpose, problem (2) is rewritten as follows:

\[
\begin{align*}
\text{Max} & \quad Z_3(x) = W_1 * Z_1(x) + W_2 * Z_2(x) \\
& \quad AX \leq b \quad x_j \geq 0
\end{align*}
\]

Now, define the membership functions for $Z_1(x)$ and $Z_2(x)$ as shown in the section below.

3.8. **Membership Function.**

\[
\mu_{Z_1}(x) = \begin{cases} 
1 & \text{if } Z_1(x) \geq u_1^- \\
\frac{Z_1(x) - l_1^-}{u_1^- - l_1^-} & \text{if } l_1^- \leq Z_1(x) \leq u_1^- \\
0 & \text{if } Z_1(x) \leq l_1^- 
\end{cases}
\]
\[ \mu_{Z_2}(x) = \begin{cases} 
 1 & \text{if } Z_2(x) \geq u^-_2 \\
 \frac{Z_2(x) - l^-_2}{u^-_2 - l^-_2} & \text{if } l^-_2 \leq Z_2(x) \leq u^-_2 \\
 0 & \text{if } Z_2(x) \leq l^-_2 
\end{cases} \quad (21) \]

3.9. **Fuzzy Goal Programming Modeling.** We can write the following goal programming model for the membership function as below:

\[
\begin{align*}
\text{Min} & \quad d^-_1 + d^+_1 + d^-_2 + d^+_2 \\
\text{S.t.} & \quad \frac{Z_1(x) - l^-_1}{u^-_1 - l^-_1} + d^-_1 - d^+_1 = 1 \\
& \quad \frac{Z_2(x) - l^-_2}{u^-_2 - l^-_2} + d^-_2 - d^+_2 = 1 \\
& \quad d^-_1, d^+_1 \geq 0 \\
& \quad d^-_2, d^+_2 \geq 0 
\end{align*} \quad (22, 23) \]

Let us assume that

\[ R_1 = \frac{1}{u^-_1 - l^-_1} \quad (25) \]
\[ R_2 = \frac{1}{u^-_2 - l^-_2} \quad (26) \]

After combining goal constraints (22) and (23) and substitution we have

\[
W_1 \left\{ \frac{Z_1(x) - l^-_1}{u^-_1 - l^-_1} \right\} + W_2 \left\{ \frac{Z_2(x) - l^-_2}{u^-_2 - l^-_2} \right\} + d^-_3 - d^+_3 = 1 \quad (27)
\]

Or, we can write (27) as

\[
R_1 \{ W_1 Z_1(x) - W_1 l^-_1 \} + R_2 \{ W_2 Z_2(x) - W_2 l^-_2 \} + d^-_3 - d^+_3 = 1 \quad (28)
\]

Now, we have

\[
R_1 W_1 \{ Z_1(x) - R_1 W_1 l^-_1 \} + R_2 W_2 \{ Z_2(x) - R_2 W_2 l^-_2 \} + d^-_3 - d^+_3 = 1 \quad (29)
\]

Now, we substitute for \( Z_1 \) and \( Z_2 \) to get the following model

\[
\begin{align*}
R_1 W_1 \left\{ \sum_{j=1}^{n} C_{3j} x_j + r_1 \right\} + R_2 W_2 \left\{ \sum_{j=1}^{n} D_{3j} x_j + r_2 \right\} \\
- W_1 R_1 l^-_1 - W_2 R_2 l^-_2 + d^-_3 - d^+_3 = 1 
\end{align*} \quad (30)
\]

Or

\[
\begin{align*}
R_1 W_1 \left\{ \sum_{j=1}^{n} C_{3j} x_j \right\} + R_2 W_2 \left\{ \sum_{j=1}^{n} D_{3j} x_j \right\} \\
- W_1 R_1 l^-_1 - W_2 R_2 l^-_2 + d^-_3 - d^+_3 = 1 
\end{align*} \quad (31)
\]
Where $C_{3j}, C_{4j}, D_{3j}$, and $D_{4j}$ are as defined below:

$$\sum_{j=1}^{n} C_{3j}x_j = \sum_{j=1}^{n} C_{1j}x_j - q_1(\sum_{j=1}^{n} u_{1j}^1x_j)$$  \hspace{1cm} (32)$$

$$\sum_{j=1}^{n} D_{3j}x_j = \sum_{j=1}^{n} D_{1j}x_j - q_3(\sum_{j=1}^{n} u_{3j}^3x_j)$$  \hspace{1cm} (33)$$

$$\sum_{j=1}^{n} C_{4j}x_j = \sum_{j=1}^{n} C_{2j}x_j - q_2(\sum_{j=1}^{n} u_{2j}^2x_j)$$  \hspace{1cm} (34)$$

$$\sum_{j=1}^{n} D_{4j}x_j = \sum_{j=1}^{n} D_{2j}x_j - q_4(\sum_{j=1}^{n} u_{4j}^4x_j)$$  \hspace{1cm} (35)$$

Formula (31) will be converted into the following:

$$R_1W_1\{\sum_{j=1}^{n} C_{3j}x_j\}\{\sum_{j=1}^{n} D_{3j}x_j\} + R_2W_3\{\sum_{j=1}^{n} D_{3j}x_j\}\{\sum_{j=1}^{n} C_{3j}x_j\}$$  \hspace{1cm} (36)$$

$$+ \{-W_1R_1l_1^r - W_2R_2l_2^r - 1\}\{\sum_{j=1}^{n} C_{4j}x_j\}\{\sum_{j=1}^{n} D_{4j}x_j\}$$

$$+ W_1R_1s_2\sum_{j=1}^{n} C_{3j}x_j + W_2R_2s_1\sum_{j=1}^{n} D_{3j}x_j$$

$$(w_1R_1r_1 - s_1W_1R_1l_1^r - s_1W_2R_2l_2^r - s_1)\sum_{j=1}^{n} D_{4j}x_j$$

$$+ (w_2R_2r_2 - s_2W_1R_1l_1^r - s_2W_2R_2l_2^r - s_2)(\sum_{j=1}^{n} C_{4j}x_j) + D^- - D^+$$

$$= \{-W_1R_1r_1s_2 - W_2R_2r_2s_1\} + \{W_1R_1l_1^r + W_2R_2l_2^r\}s_1s_2$$

Now, we can use formula 36 to develop the upper bound function as shown below:

$$R_1W_1\{\sum_{j=1}^{n} C_{1j}x_j\}\{\sum_{j=1}^{n} D_{4j}x_j\} + R_2W_2\{\sum_{j=1}^{n} D_{1j}x_j\}\{\sum_{j=1}^{n} C_{4j}x_j\}$$  \hspace{1cm} (37)$$

$$+ \{-W_1R_1l_1^r - W_2R_2l_2^r - 1\}\{\sum_{j=1}^{n} C_{4j}x_j\}\{\sum_{j=1}^{n} D_{4j}x_j\}$$

$$+ W_1R_1s_2\sum_{j=1}^{n} C_{1j}x_j + W_2R_2s_1\sum_{j=1}^{n} D_{1j}x_j$$

$$(W_1R_1r_1 - s_1W_1R_1l_1^r - s_1W_2R_2l_2^r - s_1)\sum_{j=1}^{n} D_{1j}x_j$$

$$+ (W_2R_2r_2 - s_2W_1R_1l_1^r - s_2W_2R_2l_2^r - s_2)(\sum_{j=1}^{n} C_{4j}x_j) + D_1^- - D_1^+$$

$$= \{-W_1R_1r_1s_2 - W_2R_2r_2s_1\} + \{W_1R_1l_1^r + W_2R_2l_2^r\}s_1s_2$$
considering the first fractional objective function, we have the following inequalities

\[ D_1^- = d_3^n \left\{ \sum_{j=1}^{n} C_{4j} x_j + s_1 \right\} \left\{ \sum_{j=1}^{n} D_{4j} x_j + s_2 \right\} \]  
(38)

\[ D_1^+ = d_3^n \left\{ \sum_{j=1}^{n} C_{4j} x_j + s_1 \right\} \left\{ \sum_{j=1}^{n} D_{4j} x_j + s_2 \right\} \]  
(39)

Now, we can use formula 36 to develop the lower bound function as shown below:

\[ R_1W_1 \left\{ \sum_{j=1}^{n} C_{3j} x_j \right\} \left\{ \sum_{j=1}^{n} D_{2j} x_j \right\} + R_2W_2 \left\{ \sum_{j=1}^{n} D_{3j} x_j \right\} \left\{ \sum_{j=1}^{n} C_{2j} x_j \right\} \]  
(40)

\[ + \left\{ -W_1 R_1 l_{-1}^- - W_2 R_2 s_1 \right\} \left\{ \sum_{j=1}^{n} C_{2j} x_j \right\} \left\{ \sum_{j=1}^{n} D_{3j} x_j \right\} \]

\[ W_1 R_1 s_2 \sum_{j=1}^{n} C_{3j} x_j + W_2 R_2 s_2 \sum_{j=1}^{n} D_{3j} x_j \]

\[ (W_1 R_1 r_1 - s_1 W_1 R_1 l_{-1}^- - s_1 W_2 R_2 l_{-2}^- - s_2) \sum_{j=1}^{n} D_{2j} x_j \]

\[ + (W_2 R_2 r_2 - s_2 W_1 R_1 l_{-1}^- - s_2 W_2 R_2 l_{-2}^- - s_2) \left( \sum_{j=1}^{n} C_{2j} x_j \right) + D_2^- - D_2^+ \]

\[ = \left\{ -W_1 R_1 r_1 s_2 - W_2 R_2 r_2 s_1 \right\} + \left\{ W_1 R_1 l_{-1}^- + W_2 R_2 l_{-2}^- \right\} s_1 s_2 \]

\[ D_2^- = d_3^n \left\{ \sum_{j=1}^{n} C_{2j} x_j + s_1 \right\} \left\{ \sum_{j=1}^{n} D_{2j} x_j + s_2 \right\} \]  
(41)

\[ D_2^+ = d_3^n \left\{ \sum_{j=1}^{n} C_{2j} x_j + s_1 \right\} \left\{ \sum_{j=1}^{n} D_{2j} x_j + s_2 \right\} \]  
(42)

The proposed nonlinear goal programming can be expanded to include four compromise constraint one for each of the functions \( F_1(x), F_2(x), F_3(x), \) and \( F_4(x) \). The process of development of the compromise functions are shown below.

4. **Linearization Technique.** Let us assume that \( C_{1j} \sim N(C_{1j}, \sigma_j^2), C_{2j} \sim N(C_{2j}, \phi_j^2), D_{1j} \sim N(D_{1j}, \psi_j^2), D_{2j} \sim N(D_{2j}, \zeta_j^2) \) and the variance-covariance matrices of \( V_1, V_2, V_3, \) and \( V_4 \), when coefficients are independent normally distributed random variables. Since, we know that

\[ (a + b)^{1/2} \leq a^{1/2} + b^{1/2} \]

considering the first fractional objective function, we have the following inequalities intact:

\[ (X' V_1 X)^{1/2} < \sum_{j=1}^{n} u_{1j}^{1/2} x_j \]

\[ (X' V_2 X)^{1/2} < \sum_{j=1}^{n} u_{2j}^{1/2} x_j \]  
(43)
Where $u_{1j} = \sigma_j^2$ and $u_{2j} = \varphi_j^2$ for all $j = 1, 2, \cdots, n$. Therefore, we have

$$
\sum_{j=1}^{n} C_{1j} x_j - q_1 \sum_{j=1}^{n} u_{1j}^2 x_j < F_1(x) < \sum_{j=1}^{n} C_{1j} x_j + q_1 \sum_{j=1}^{n} u_{1j}^2 x_j
$$

(44)

$$
\sum_{j=1}^{n} C_{2j} x_j - q_2 \sum_{j=1}^{n} u_{2j}^2 x_j < F_2(x) < \sum_{j=1}^{n} C_{2j} x_j + q_2 \sum_{j=1}^{n} u_{2j}^2 x_j
$$

(45)

Since, we have

$$
F_1(x) < \sum_{j=1}^{n} C_{1j} x_j
$$

(46)

$$
F_2(x) < \sum_{j=1}^{n} C_{2j} x_j
$$

(47)

we can define the following new functions:

$$
f_1(x) = \sum_{j=1}^{n} C_{1j} x_j
$$

(48)

$$
f_2(x) = \sum_{j=1}^{n} C_{1j} x_j - q_1 \sum_{j=1}^{n} u_{1j}^2 x_j = \sum_{j=1}^{n} E_{1j}
$$

(49)

$$
f_3(x) = \sum_{j=1}^{n} C_{2j} x_j
$$

(50)

$$
f_4(x) = \sum_{j=1}^{n} C_{2j} x_j - q_2 \sum_{j=1}^{n} u_{2j}^2 x_j = \sum_{j=1}^{n} E_{2j}
$$

(51)

5. **Compromise Goal Constraints.** A goal constraint incorporating the optimum value of the upper and lower bound functions of the numerator and denominator of $F_1(X)$ and $F_2(X)$ respectively are also of tremendous value for problem solving.

$$
H_1(x) = \frac{v_1}{\sqrt{\sum_{j=1}^{n} E_{1j}^{-2}}} \left( \sum_{j=1}^{n} E_{1j}^{-} x_j - f_2^+ \right) - \frac{v_2}{\sqrt{\sum_{j=1}^{n} C_{1j}^{-2}}} \left( \sum_{j=1}^{n} C_{1j}^{-} x_j - f_1^+ \right) + d_1^- - d_1^+ = 0
$$

(52)

and

$$
H_2(x) = \frac{v_3}{\sqrt{\sum_{j=1}^{n} E_{2j}^{-2}}} \left( \sum_{j=1}^{n} E_{2j}^{-} x_j - f_4^+ \right) - \frac{v_4}{\sqrt{\sum_{j=1}^{n} C_{2j}^{-2}}} \left( \sum_{j=1}^{n} C_{2j}^{-} x_j - f_3^+ \right) + d_2^- - d_2^+ = 0
$$

(53)
where following inequalities hold:

\[
\begin{align*}
    f_2(x) &\leq F_1(x) \leq f_1(x) \\
    f_4(x) &\leq F_2(x) \leq f_3(x) \\
    f_2^* &\leq F_1^* \leq f_1^* \\
    f_4^* &\leq F_2^* \leq f_3^*
\end{align*}
\]  

(54) \hspace{1cm} (55) \hspace{1cm} (56) \hspace{1cm} (57)

where \( f_1^* \) and \( f_2^* \) represent the optimum values of \( P_4 \) and \( P_5 \) linear programming problems, respectively. It should be noted that \( F_1^* \) is the optimal value of \( F_1(X) \) over the defined feasible region of \( S \). However, \( P_4 \) and \( P_5 \) are defined as below:

\[
P_4 : f_1^* = \{ \text{Maximize } f_1(x) = \sum_{j=1}^{n} C_{ij} x_j \} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, x_j \geq 0 \} \quad (58)
\]

\[
P_5 : f_2^* = \{ \text{Maximize } f_2(x) = \sum_{j=1}^{n} E_{ij} x_j \} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, x_j \geq 0 \} \quad (59)
\]

In a similar fashion, we can introduce problems \( P_6 \) and \( P_7 \) as they are defined below:

\[
P_6 : f_3^* = \{ \text{Maximize } f_3(x) = -\{ \sum_{j=1}^{n} C_{ij} x_j \} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, x_j \geq 0 \} \quad (60)
\]

\[
P_7 : f_4^* = \{ \text{Maximize } f_4(x) = -\{ \sum_{j=1}^{n} E_{ij} x_j \} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, x_j \geq 0 \} \quad (61)
\]

Note that in a similar way we can develop compromise constraints for the numerator and denominator of the second objective function. The development of these two constraints is left to the readers. These constraints shown by \( H_3(x) \) and \( H_4(x) \) are used in the model given below.

Now, the upper bound nonlinear goal programming can be written as:

\[
\text{Minimize} \quad D_1^{-} + D_1^{+} + d_1^{-} + d_1^{+} + d_2^{-} + d_2^{+} + d_3^{-} + d_3^{+} + d_4^{-} + d_4^{+}
\]

\[
\text{S.t} \quad Y_1(X) = R_1 W_1 \{ \sum_{j=1}^{n} C_{1j} x_j \} \{ \sum_{j=1}^{n} D_{4j} x_j \} + R_2 W_2 \{ \sum_{j=1}^{n} D_{1j} x_j \} \{ \sum_{j=1}^{n} C_{4j} x_j \}
\]

\[
+ \{-W_1 R_1 l_1^{-} - W_2 R_2 l_2^{-} - 1\} \{ \sum_{j=1}^{n} C_{4j} x_j \} \{ \sum_{j=1}^{n} D_{4j} x_j \}
\]

\[
+ W_1 R_1 s_2 \sum_{j=1}^{n} C_{1j} x_j + W_2 R_2 s_1 \sum_{j=1}^{n} D_{1j} x_j
\]

\[
(W_1 R_1 r_1 - s_1 W_1 R_1 l_1^{-} - s_1 W_2 R_2 l_2^{-} - s_1) \sum_{j=1}^{n} D_{4j} x_j
\]

\[
+ (W_2 R_2 r_2 - s_2 W_1 R_1 l_1^{-} - s_2 W_2 R_2 l_2^{-} - s_2) \sum_{j=1}^{n} C_{4j} x_j \} \sum_{j=1}^{n} D_{4j} x_j
\]

\[
= \{-W_1 R_1 r_1 s_2 - W_2 R_2 r_2 s_1\} + \{W_1 R_1 l_1^{-} + W_2 R_2 l_2^{-}\} s_1 s_2
\]
Now, the upper bound nonlinear goal programming can be written as:

\[ H_1(x) = \frac{v_1}{\sqrt{\sum_{j=1}^{n} E_{1j}^2}} (\sum_{j=1}^{n} E_{1j} x_j - f_1^*) - \frac{v_2}{\sqrt{\sum_{j=1}^{n} C_{1j}^2}} (\sum_{j=1}^{n} C_{1j}^{-} x_j - f_1^*) + d_1^- - d_1^+ = 0 \]  
(64)

\[ H_2(x) = \frac{v_3}{\sqrt{\sum_{j=1}^{n} E_{2j}^2}} (\sum_{j=1}^{n} E_{2j} x_j - f_2^*) - \frac{v_4}{\sqrt{\sum_{j=1}^{n} C_{2j}^2}} (\sum_{j=1}^{n} C_{2j}^{-} x_j - f_2^*) + d_2^- - d_2^+ = 0 \]  
(65)

\[ H_3(x) = \frac{v_5}{\sqrt{\sum_{j=1}^{n} E_{3j}^2}} (\sum_{j=1}^{n} E_{3j} x_j - f_3^*) - \frac{v_6}{\sqrt{\sum_{j=1}^{n} D_{1j}^2}} (\sum_{j=1}^{n} D_{1j}^{-} x_j - f_3^*) + d_3^- - d_3^+ = 0 \]  
(66)

\[ H_4(x) = \frac{v_7}{\sqrt{\sum_{j=1}^{n} E_{4j}^2}} (\sum_{j=1}^{n} E_{4j} x_j - f_4^*) - \frac{v_8}{\sqrt{\sum_{j=1}^{n} D_{2j}^2}} (\sum_{j=1}^{n} D_{2j}^{-} x_j - f_4^*) + d_4^- - d_4^+ = 0 \]  
(67)

\[ AX \leq b \]  
(68)

\[ x_j \geq 0 \]  
(69)

Now, the upper bound nonlinear goal programming can be written as:

Minimize \( D_2^- + D_2^+ + d_2^- + d_2^+ + d_2^- + d_2^+ + d_3^- + d_3^+ + d_4^- + d_4^+ \)  
(70)

\[ S.t \quad Y_2(X) = R_1 W_1 \{\sum_{j=1}^{n} C_{3j} x_j\} \{\sum_{j=1}^{n} D_{2j} x_j\} \]

\[ + R_2 W_2 \{\sum_{j=1}^{n} D_{3j} x_j\} \{\sum_{j=1}^{n} C_{2j} x_j\} \]

\[ + \{-W_1 R_1 l_1^- - W_2 R_2 l_2^- - 1\} \{\sum_{j=1}^{n} C_{2j} x_j\} \{\sum_{j=1}^{n} D_{2j} x_j\} \]

\[ + W_1 R_1 s_2 \sum_{j=1}^{n} C_{3j} x_j + W_2 R_2 s_1 \sum_{j=1}^{n} D_{3j} x_j \]

\[ (W_1 R_1 r_1 - s_1 W_1 R_1 l_1^- - s_1 W_2 R_2 l_2^- - s_1) \sum_{j=1}^{N} D_{2j} x_j \]

\[ + (W_2 R_2 r_2 - s_2 W_1 R_1 l_1^- - s_2 W_2 R_2 l_2^- - s_2) (\sum_{j=1}^{N} C_{2j} x_j) + D_2^- - D_2^+ \]

\[ = \{-W_1 R_1 r_1 s_2 - W_2 R_2 r_2 s_1\} + \{W_1 R_1 l_1^- + W_2 R_2 l_2^- \} s_1 s_2 \]
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\[ H_1(x) = \frac{v_1}{\sqrt{\sum_{j=1}^{n} E_{1j}^{-2} x_j - f_4^*}} - \frac{v_2}{\sqrt{\sum_{j=1}^{n} C_{1j}^{-2} x_j}} + d_i^- - d_i^+ = 0 \]  

\[ H_2(x) = \frac{v_3}{\sqrt{\sum_{j=1}^{n} E_{2j}^{-2} x_j - f_4^*}} - \frac{v_4}{\sqrt{\sum_{j=1}^{n} C_{2j}^{-2} x_j}} + d_i^- - d_i^+ = 0 \]  

\[ H_3(x) = \frac{v_5}{\sqrt{\sum_{j=1}^{n} E_{3j}^{-2} x_j - f_6^*}} - \frac{v_6}{\sqrt{\sum_{j=1}^{n} D_{1j}^{-2} x_j}} + d_i^- - d_i^+ = 0 \]  

\[ H_4(x) = \frac{v_7}{\sqrt{\sum_{j=1}^{n} E_{4j}^{-2} x_j - f_8^*}} - \frac{v_8}{\sqrt{\sum_{j=1}^{n} D_{2j}^{-2} x_j}} + d_i^- - d_i^+ = 0 \]  

\[ AX \leq b \]  
\[ x_j \geq 0 \]  

Please note that the materials provided in the remainder of this section is for knowing how to calculate the values of upper and lower bound values of nonlinear functions \( F_3(x) \) and \( F_4(x) \).

\[ f_6(x) \leq F_3(x) \leq f_5(x) \]  
\[ f_8(x) \leq F_4(x) \leq f_7(x) \]  
\[ f_6^* \leq F_3^* \leq f_5^* \]  
\[ f_8^* \leq F_4^* \leq f_7^* \]

\[ Q_1 : f_5^* = \{ \text{Maximize } f_5(x) = \sum_{j=1}^{n} D_{1j}^{-} x_j | \sum_{j=1}^{n} a_{ij} x_j \leq b, x_j \geq 0 \} \]  

\[ Q_2 : f_6^* = \{ \text{Maximize } f_6(x) = \sum_{j=1}^{n} E_{3j}^{-} x_j | \sum_{j=1}^{n} a_{ij} x_j \leq b, x_j \geq 0 \} \]  

\[ Q_3 : f_7^* = \{ \text{Maximize } f_7(x) = -\{ \sum_{j=1}^{n} D_{2j}^{-} x_j \} | \sum_{j=1}^{n} a_{ij} x_j \leq b, x_j \geq 0 \} \]  

\[ Q_4 : f_8^* = \{ \text{Maximize } f_8(x) = -\{ \sum_{j=1}^{n} E_{4j}^{-} x_j \} | \sum_{j=1}^{n} a_{ij} x_j \leq b, x_j \geq 0 \} \]
Where,

\[ f_6(x) = \sum_{j=1}^{n} D_{1j}^n x_j - q_3 \sum_{j=1}^{n} u_{3j}^\delta x_j = \sum_{j=1}^{n} E_{3j}^- \]  \hspace{1cm} (86)

\[ f_8(x) = \sum_{j=1}^{n} D_{2j}^n x_j - q_4 \sum_{j=1}^{n} u_{4j}^\delta x_j = \sum_{j=1}^{n} E_{4j}^- \]  \hspace{1cm} (87)

6. **Taylors Series Linearization Technique.** A technique that can convert a quadratic function into a linear function is Taylors series. Using this approach, and following specific rules, we can convert any nonlinear constraint into a linear constraint, about a specified solution point \( X_0 \). Following Taylor series rules, the upper bound nonlinear constraint can be written as,

\[ Y_1(x) = Y_1(x^0) + \sum_{j=1}^{n} \left[ \frac{\partial Y_1(x^0)}{\partial x_j} \right] (x_j - x_{o_j}) + D_1^- - D_1^+ = B_1 \]  \hspace{1cm} (88)

The same linearization technique can be used for the nonlinear goal constraint appeared in lower bound goal programming problem (LBGPP) as shown below:

\[ Y_2(x) = Y_2(x^0) + \sum_{j=1}^{n} \left[ \frac{\partial Y_2(x^0)}{\partial x_j} \right] (x_j - x_{o_j}) + D_2^- - D_2^+ = B_2 \]  \hspace{1cm} (89)

Once, such linear approximations are completed then we can solve the upper bound and lower bound goal programming problems using multiple objective goal programming technique.

7. **Computational Algorithms.** The computation method developed in the previous sections of this article can be organized into the steps described below.

7.1. **Algorithm 1 (Relaxation Technique).**

Step 1: using information provided, develop the fractional model of the problem.

Step 2: using the concepts discussed in the previous sections, develop upper bound goal programming model of problem (UBGPP).

Step 3: using the concepts discussed in the previous sections, develop lower bound goal programming model of the problem (LBGPP).

Step 4: ignore the nonlinear goal constraint and solve the remaining GP problem. Call this temporary solution point \( X_0^T \)

Step 6: Same steps can be followed to solve the LBGPP as well.

7.2. **Algorithm 2 (Simulation Annealing Goal Technique).**

Step 1: Convert priority goal (s) as a problem constraint. This can be done using following formula:

\[ \text{Goal} - \sum W_j \{ P_j (d_j^- + d_j^+) \} = 0 \]  \hspace{1cm} (90)

Step 2: Ask the decision makers for the values of weights \( W_i \), and priority levels \( P_j \).
Step 3: apply the steps of Simulated Annealing technique (see Figure 1) to solve the problem. Once an optimum solution is obtained then, identify the Goal Level and the value of positive and negative deviational variables to determine the levels of achievements of priority levels.

Step 4: Show the results to decision maker and continue solving the problem with new weights if decision maker is unsatisfied with the solution obtained.

![Figure 1. Simulation Annealing Goal Technique](image)

8. Example Problem.

Max \( Z_1(x) = \frac{F_1(x) + 5}{F_2(x) + 10} = \frac{8x_1 + 7x_2 - 2.33(9x_1^2 + 4x_2^2)^{\frac{1}{2}} + 5}{20x_1 + 12x_2 - 2.33(3x_1^2 + 2x_1^2 + 2x_1x_2 + 4x_2^2)^{\frac{1}{2}} + 10} \) \tag{91}

Max \( Z_2(x) = \frac{F_3(x) + 15}{F_4(x) + 20} = \frac{5x_1 + 4x_2 - 2.33(16x_1^2 + 9x_2^2)^{\frac{1}{2}} + 15}{10x_1 + 9x_2 - 2.33(9x_1^2 + 12x_1x_2 + 4x_2^2)^{\frac{1}{2}} + 20} \) \tag{92}

S.t \( 2x_1 + x_2 \leq 18 \)

\( x_1 + 2x_2 \leq 16 \)

\( x_1, x_2 \geq 0 \) \tag{93}

Let us assume that the values of \( Z_1(x) \) and \( Z_2(x) \) are requested to be as follows:

\( 0.1 \leq Z_1(x) \leq 1.0 \)

\( 0.25 \leq Z_2(x) \leq 1.5 \)
Considering the above information, we can set up following information:

\[ l_1^- = 0.1 \]
\[ u_1^- = 1.0 \]
\[ l_2^- = 0.25 \]
\[ u_2^- = 1.5 \]
\[ R_1 = 1.25 \]
\[ R_2 = 0.80 \]
\[ r_1 = 5, r_2 = 15, s_1 = 10, \text{ and } s_2 = 20 \]
\[ W_1 = 0.6, W_2 = 0.4 \]

Table 2. upper and lower bound functions and their optimal solution points on the defined feasible region

| \( f_1(x) = 8x_1 + 7x_2 \) | \((x_1^*, x_2^*)\) | \( f_2(x) = 1.01x_1 + 2.34x_2 \) | \((0,0)\) | \( f_3(x) = 20x_1 + 12x_2 \) | \((6.6667,4.6667)\) | \( f_4(x) = 15.964x_1 + 7.3x_2 \) | \((9,0)\) | \( f_5(x) = 5x_1 + 4x_2 \) | \((6.6667,4.6667)\) | \( f_6(x) = -4.32x_1 - 2.99x_2 \) | \((0,0)\) | \( f_7(x) = 10x_1 + 9x_2 \) | \((0,0)\) | \( f_8(x) = 3.01x_1 + 4.34x_2 \) | \((0,0)\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Max \( f_1(x) \) | 86 | Max \( f_2(x) \) | 18.72 | Max \( f_3(x) \) | 189 | Max \( f_4(x) \) | 143.6789 | Max \( f_5(x) \) | 52.0003 | Max \( f_6(x) \) | 0 | Max \( f_7(x) \) | 0 | Max \( f_8(x) \) | 0 |

Let us assume that \( v_1 = v_2 = \cdots = v_8 = 0.5 \) and from that we can develop following four compromise constraints:

\[
\begin{align*}
H_1(x) &= 0.17815x_1 - 0.12981x_2 + d_1^- - d_1^+ = 0.372591 \\
H_2(x) &= 0.025538x_1 - 0.04838x_2 + d_2^- - d_2^+ = 0.0369 \\
H_3(x) &= 0.8014x_1 + 0.5969x_2 + d_3^- - d_3^+ = 4.0612 \\
H_4(x) &= -0.08665x_1 + 0.07419x_2 + d_4^- - d_4^+ = 0
\end{align*}
\]

Now, the upper bound nonlinear goal programming problem can be written as follows:

Minimize : \( D_1^- + D_1^+ + D_2^- + D_2^+ + D_3^- + D_3^+ + D_4^- + D_4^+ \) \quad (94)

S.t \[
\begin{align*}
Y_1(x) &= -0.914x_1^2 - 0.07x_2^2 + 15.41x_1x_2 + 179.55x_1 + 50.37x_2 \\
&\quad + D_1^- - D_1^+ = 95.2 \\
H_1(x) &= 0.17815x_1 - 0.12981x_2 + d_1^- - d_1^+ = 0.372591 \\
H_2(x) &= 0.025538x_1 - 0.04838x_2 + d_2^- - d_2^+ = 0.0369 \\
H_3(x) &= 0.8014x_1 + 0.5969x_2 + d_3^- - d_3^+ = 4.0612 \\
H_4(x) &= -0.08665x_1 + 0.07419x_2 + d_4^- - d_4^+ = 0
\end{align*}
\]
\begin{align*}
2x_1 + x_2 & \leq 18 \\
x_1 + 2x_2 & \leq 16 \\
x_1, x_2 & \geq 0
\end{align*}

Now, the lower bound nonlinear goal programming problem can be written as follows:

\begin{align*}
\text{Minimize:} & \quad D^-_2 + D^+_2 + d^-_1 + d^+_1 + d^-_3 + d^+_3 + d^-_4 + d^+_4 \\
\text{S.t} & \quad Y_2(x) = 251.07x_1^2 + 120.42x_2^2 + 357.52x_1x_2 + 442.67x_1 + 263.8x_2 \\
& \quad + D^-_2 - D^+_2 = 95.2 \\
& \quad H_1(x) = 0.17815x_1 - 0.12981x_2 + d^-_1 - d^+_1 = 0.372591 \\
& \quad H_2(x) = 0.025538x_1 - 0.0.04838x_2 + d^-_2 - d^+_2 = 0.0369 \\
& \quad H_3(x) = 0.8014x_1 + 0.5969x_2 + d^-_3 - d^+_3 = 4.0612 \\
& \quad H_4(x) = -0.08665x_1 + 0.07419x_2 + d^-_4 - d^+_4 = 0
\end{align*} 

(95)

9. Results and Analysis. This section presents the computer results obtained for the upper bound and lower bound problems.

9.1. Upper Bound problem. The relaxed upper bound GP problem was solved and solution point satisfying all constraints except the nonlinear goal constraint was obtained. The Taylor series was used and the nonlinear goal constraint was linearized around this solution point. Then, the linearized function was added to the problem instead of the nonlinear goal constrained and problem was resolved. The optimal solution of the goal programming is:

\begin{align*}
x_1 & = 0.593 \\
x_2 & = 0.692
\end{align*}

The goal priority level is at 3.585

9.2. Lower Bound Problem. The relaxed lower bound GP problem was solved and a solution point satisfying all constraints except the nonlinear goal constraint was obtained. The Taylor series was used and the nonlinear goal constraint was linearized around this solution point. Then, the linearized function was added to the problem instead of the nonlinear goal constrained and problem was resolved. The optimal solution of the goal programming is:

\begin{align*}
x_1 & = 0.0 \\
x_2 & = 7.173
\end{align*}

The goal priority level is at 2.44
Table 3. final solution obtained for the fractional programming problem

| Problems | $(x_1^*, x_2^*)$ | Goal Programming | $Z_1(x^*)$ | $Z_2(x^*)$ |
|----------|------------------|------------------|------------|------------|
| Upper bound | (0.593, 0.692) | 3.593 | 0.3643 | 0.5401 |
| Lower bound | (0.7, 0.173) | 2.44 | 0.3477 | 0.1261 |

10. **Conclusion.** The problem of fractional stochastic programming has not been studied with the structure defined by this author in this article. Due to the fact that solving a fractional stochastic programming of this type is not a simple task, the author has proposed a linearization goal programming technique to solve the problem. The final linear goal programming problem resulted due to linearization can be solved by existing commercial computer packages or customized software developed by the analyst. Since this linearization technique is unique in the sense of development and application, it makes a significant contribution to the field of fractional stochastic programming per se. There are some limitations to this type of modeling: (1) the solution obtained from this type of modeling is an approximate one and hence optimal solution of the problem is not achievable at all. There are ways to expand the scope of this research. Using joint chance-constrained programming for modeling of the problem is highly recommended, as it is applied in agriculture and engineering areas. This type of modeling is highly regarded for problems related to environment, pollution control, water management, city waste management, financial modeling, and land planning and management. When one needs to deal with systems reliability, then chance-constrained programming (CCP) is recommended. Due to the facts that CCP is widely used for risk modeling and systems reliability analysis along with fuzzy type modeling and goal programming approach, our proposed approximation technique can make significant contribution to the field of fractional stochastic programming for solving complex problems.

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