Improved Determination of the CKM Angle
\( \alpha \) from \( B \to \pi\pi \) decays

\textbf{UTfit} Collaboration

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\textbf{Abstract}

Motivated by a recent paper that compares the results of the analysis of the CKM angle \( \alpha \) in the frequentist and in the Bayesian approaches, we have reconsidered the information on the hadronic amplitudes, which helps constraining the value of \( \alpha \) in the Standard Model. We find that the Bayesian method gives consistent results irrespective of the parametrisation of the hadronic amplitudes and that the results of the frequentist and Bayesian approaches are equivalent when comparing meaningful probability ranges or confidence levels. We also find that from \( B \to \pi\pi \) decays alone the 95\% probability region for \( \alpha \) is the interval \([80^\circ, 170^\circ]\), well consistent with recent analyses of the unitarity triangle where, by using all the available experimental and theoretical information, one gets \( \alpha = (93 \pm 4)^\circ \). Last but not least, by using simple arguments on the hadronic matrix elements, we show that the unphysical region \( \alpha \sim 0 \), present in several experimental analyses, can be eliminated.
1 Introduction

Motivated by the criticisms recently appeared in ref. [1], we present a new analysis of the CKM angle $\alpha$ from $B \to \pi\pi$ decays based on the Bayesian statistical approach. The main results are the following:

- we show that the differences between the frequentist and the Bayesian approaches are *NOT* due to the difference in the two methods but to the difference in the physical assumptions on the weak amplitudes, contrary to the claims of ref. [1];

- although we expect an eightfold ambiguity for $\alpha$ using as “a priori” knowledge only isospin symmetry and the experimental measurements of the relevant branching fractions and CP asymmetries, *this degeneracy can be reduced in the presence of further information on the hadronic amplitudes*. We will discuss and use this information which can already be extracted from the data;

- among the information that we do have on the amplitudes, the existence of a scale of strong interactions and the use of the experimental values of related decays allow to limit the size of the hadronic matrix elements and to eliminate the *unphysical* region of values corresponding to $\alpha$ close to zero;

- within our approach, we obtain consistent results at a meaningful probability value irrespective of the parametrisation used for the hadronic amplitudes.

Our analysis is performed within the Standard Model. All the technical details which are not relevant to the present discussion, but are used in the analysis, can be found in previous publications of the UTfit Collaboration [2–5].

The paper is organised as follows. We first recall the basic formalism and definitions of the different quantities entering the analysis; we then discuss the case in which no further assumptions are made for these decays besides the experimental knowledge of the relevant branching fractions and CP asymmetries, and $SU(2)$ isospin symmetry. In this case we make a comparison between frequentist and Bayesian methods along the lines of ref. [1]; we finally discuss the present experimental information on the weak amplitudes, and the corresponding constraints, and show how this information helps in reducing the ambiguity of the solution.

We think that neither our study nor the study of ref. [1] can be decisive in settling the long standing struggle on the validity of the frequentist and Bayesian methods. For this reason, contrary to what was done in ref. [1], we do not find very useful to present in a physics paper a long list of learned citations, with philosophical statements and sentences taken from illustrious references in favour of the Bayesian statistics. Rather, we thank the authors of ref. [1] who stimulated us to reconsider the analysis of the CKM angle $\alpha$, and allowed us to improve our analysis by using the available experimental information. The important point is not which of the two approaches is used rather the ability to get predictions which can eventually be verified “a posteriori”. The success of a phenomenological analysis relies on the capacity of anticipating the correct result. Among the successful (and rather accurate) predictions of our collaboration let us recall $\sin 2\beta$ [2,6] and $\Delta m_{B_s}$ [2,3,7], made long before their experimental measurements.
amplitudes by the relations
\[ C_{\pi\pi}^{ij} = (|A^{ij}|^2 - |\bar{A}^{ij}|^2)/(|A^{ij}|^2 + |\bar{A}^{ij}|^2), \quad S_{\pi\pi}^{ij} = -2 \, \text{Im}[A^{ij} \bar{A}^{ij*}]/(|A^{ij}|^2 + |\bar{A}^{ij}|^2), \]

\[ B_{\pi\pi}^{+0} = (|A^{+0}|^2 + |\bar{A}^{+0}|^2)/2, \quad B_{\pi\pi}^{+0} = T_{\pi\pi}^+/(2 |T_{\pi\pi}^0|). \quad \]

In the equations above we have included in the definition of the amplitudes trivial and standard factors such as the two body phase space and the squared Fermi constant. Here and in the following the branching fractions are given in units of $10^{-6}$. In our convention, denoted as “natural units” in the following, the factorised amplitudes are of $\mathcal{O}(1)$, as explained below.

In the Standard Model, a minimal but non zero set of “a priori” knowledge on strong interactions is common to all the phenomenological analyses of $\alpha$. The universal “a priori” assumptions of these studies are that strong interactions are flavour independent and conserve parity and CP. This information is born out from the experimental measurements of strong interaction processes. In addition most of the analyses are performed in the approximation in which isospin symmetry breaking effects - including electromagnetic corrections - are neglected. [1] In the remaining of this paper we will denote this set of “a priori” assumptions, combined with the isospin symmetry approximation, “minimal assumptions”.

Using the “minimal assumptions”, in the standard parametrisation the amplitudes are written as [9]

\[ A^{+} \equiv A(B^0 \rightarrow \pi^+\pi^-) = e^{-i\alpha} T^{+-} + P, \]

\[ A^{00} \equiv A(B^0 \rightarrow \pi^0\pi^0) = \frac{1}{\sqrt{2}} (e^{-i\alpha} T^{00} - P), \]

\[ A^{+0} \equiv A(B^+ \rightarrow \pi^+\pi^0) = \frac{1}{\sqrt{2}} e^{-i\alpha} (T^{00} + T^{+-}). \]

1For lifetimes we will use the experimental value for the charged and neutral $B$ mesons [8].
The six free parameters are the absolute values of \( T^{ij} \) and \( P \), \(|T^{ij}| \) and \(|P|\), their relative strong phases, \( \phi_P = \text{Arg}[T^{+-} P^*] \) and \( \phi_0 = \text{Arg}[T^{+-} T^{00*}] \), the overall phase being irrelevant, and \( \alpha \).

Starting from the general expressions in Equation (2), under the “minimal assumptions” mentioned above, one finds for \( \alpha \) [9] either zero or eight solutions from Equations (1), corresponding to [1, 10]

\[
\tan \alpha = \frac{\sin(2\alpha_{\text{eff}}) \bar{c} + \cos(2\alpha_{\text{eff}}) \bar{s} + s}{\cos(2\alpha_{\text{eff}}) \bar{c} - \sin(2\alpha_{\text{eff}}) \bar{s} + c},
\]

\[
\sin(2\alpha_{\text{eff}}) = \frac{S_{\pi \pi}^+}{\sqrt{1 - C_{\pi \pi}^+}}, \quad \cos(2\alpha_{\text{eff}}) = \pm \sqrt{1 - \sin^2(2\alpha_{\text{eff}})},
\]

\[
c = \sqrt{\frac{\tau_B^+}{\tau_B^0} \beta^0 (1 + C_{\pi \pi}^+) / 2 - \beta^{00} (1 + C_{\pi \pi}^+)}
\]

\[
\bar{c} = \sqrt{\frac{\tau_B^+}{\tau_B^0} \beta^0 (1 + C_{\pi \pi}^+) / 2 - \beta^{00} (1 - C_{\pi \pi}^+)}
\]

\[
s = \pm \sqrt{1 - \bar{c}^2}, \quad \bar{s} = \pm \sqrt{1 - \bar{c}^2}.
\]

We also give the formulae from which it is possible to extract, from the branching fractions and the CP asymmetry coefficients, the hadronic parameters \( T^{ij} \), \( P \) and the relative phases

\[
|T^{+-}| = \left[ \frac{B^{+-}}{2 \sin^2 \alpha} \left( 1 \pm \sqrt{1 - C^{+-} - S^{+-}} \right) \right]^{1/2},
\]

\[
|P| = \left[ |T^{+-}| \left( 2 \cos^2 \alpha - 1 \right) + B^{+-} \left( 1 - \frac{S^{+-}}{\tan \alpha} \right) \right]^{1/2},
\]

\[
\phi_P = \text{arg}(\frac{P}{T^{+-}}) = \arctan(x_P, y_P),
\]

\[
x_P = -\frac{|P|^2 + |T^{+-}|^2 - B^{+-}}{\cos \alpha}, \quad y_P = -\frac{B^{+-} \bar{C}^{+-}}{\sin \alpha},
\]

\[
|T^{00}| = \left[ \frac{|P|^2 \cos 2\alpha + 2 B^{00} \pm 2 \cos^2 \alpha}{\sqrt{\left| P \right|^4 - 4 \frac{B^{00} \bar{C}^{00}}{\sin^2 2\alpha} + \frac{|P|^2}{\cos^2 \alpha} (2 B^{00} - |P|^2)}} \right]^{1/2},
\]

\[
\phi_0 = \text{arg}(\frac{T^{00}}{T^{+-}}) = \phi_P + \arctan(x_0, y_0),
\]

\[
x_0 = \frac{|P|^2 + |T^{00}|^2 - 2 B^{00}}{2 \cos \alpha}, \quad y_0 = -\frac{B^{00} \bar{C}^{00}}{\sin \alpha}.
\]

\( \phi_i = \arctan(x_i, y_i) \) is the value of the angle \( \phi_i \) obtained when both \( y_i = F \sin \phi_i \) and \( x_i = F \cos \phi_i \) are known (with \( F \) a positive factor). In the previous equation for \(|T^{+-}|\) and \(|T^{00}|\) one must choose the unique combination of signs that reproduces the correct result for \( B^{+0} \).

It has been argued in ref. [1] that using the “minimal assumptions” defined above, the frequentist and Bayesian approaches give different results. We show that this is not the case with two clear examples, one based on the present experimental information, and a second one in the hypothesis that the experimental errors are reduced by a factor ten. For this comparison we use the same data as in ref. [1], SET 1 of Table I. In Figure I we show
the frequentist and Bayesian results with the present experimental errors. To obtain these plots, in the Bayesian approach branching ratios and CP asymmetry coefficients are extracted with flat a priori probability density function (p.d.f.) and weighted by the experimental likelihood. The ES parametrisation defined in ref. [1] is the only one where no priors for the hadronic parameters are specified. Therefore this is the only case in which no additional physical information with respect to the frequentist fit is used.

Very often there is some confusion in the interpretation of the results because the shape of the frequentist and Bayesian figures look very different. It is important to stress that this happens because they correspond to two different quantities. Therefore it may be useful to recall how to read the relevant physical information in the two cases. Indeed the shape is not the important issue to this purpose. In the Bayesian case what it is usually shown is the p.d.f. for a given physical quantity, whereas in the frequentist case what it is shown is the confidence level (C.L.), or rather 1-C.L., corresponding to a certain value of the physical quantity. Although there is not a rigorous correspondence, in order to compare the two approaches one might confront the 68% (95%) integrated probability region of the Bayesian case with the set of values corresponding to the 68% (95%) C.L.

In the case of Figure 1 we have explicitly indicated with a dark band the 95% integrated probability region and the region of values corresponding to the 95% C.L. in the Bayesian and frequentist case respectively. The same considerations apply to the figures shown in the remaining of this paper.

From the selected regions in Figure 1 our conclusion is indeed just the opposite of what is claimed in [1]: at a confidence level which is meaningful for obtaining some physical information, for example at 95% C.L., and even at the 68% C.L., when using the same physical assumptions (in the present case the “minimal assumptions”), the frequentist result is equivalent to the Bayesian expectation, namely the range of values of $\alpha$ between 25° and 65° is excluded whereas the complementary interval is fully allowed. Even in the frequentist case, the eight solutions emerge only for values of the C.L. smaller than 5%, which have no physical meaning. This is further illustrated by studying the case in which, at fixed central values, the errors are reduced by a factor of 10: as shown in Figure 2 the eight solutions are separated both in the Bayesian and frequentist case. Note that even in the frequentist case they are still grouped at 95% C.L.. Thus we conclude that there is no substantial difference in the physical information obtained in two approaches, as also discussed in many other examples and stated in the Yellow Report of the first CKM workshop [11].

To strengthen even more the previous arguments we have repeated the frequentist fit with the updated values corresponding to SET 2 of Table I taken from HFAG [8]. The results are shown in Figure 3. In this case, the new values of the branching ratios make the eight solutions overlap also in the frequentist approach (independently of the chosen confidence level). This shows that the separation of the eight solutions in the frequentist approach was just a fortuitous accident which may disappear for small changes in the central values, when the errors are large.

We have also repeated the exercise of removing some crucial experimental information, namely we ignored the experimental measurement of $\mathcal{B}_{\pi\pi}^{00}$ [1]. In the Bayesian approach this corresponds to let $\mathcal{B}_{\pi\pi}^{00}$ free to vary between zero and $10^6$. In this case we get for the

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2Similar figures were shown in Figure 2 (first and fifth) of ref. [1].

3In our analysis, probability regions are delimited by the intercept of the distribution with a horizontal line.
Figure 1: Comparison of Bayesian (left) and frequentist (right) result in the “minimal assumptions” case. The dark band in the left figure corresponds to the 95% probability region, to be compared with the 95% C.L. interval obtained in the right figure. The same interval is essentially selected in the two cases. The same conclusion holds when comparing the 68% probability region to the 68% C.L. interval.

p.d.f. of $\alpha$ the result shown in Figure 4 which allows, for the 95% total probability region, almost all the values between 0° and 180° (corresponding to the 95% of the total area under the curve), fully consistent with the frequentist approach. We could not reproduce Figure 4 of ref. [1] (case ES) which we think contains some error producing the deep at $\alpha \sim 45^\circ$.

For the remaining of this paper, in the analysis we use the more recent values of SET 2 in Table 4 taken from HFAG [8].

3 Information on Hadronic Matrix Elements

In this Section we present several arguments which show that the size of the hadronic matrix elements of the operators appearing in the effective weak Hamiltonian is indeed of the order of magnitude which is expected in QCD. This implies that $T^{ij}$ and $P$ are numbers of $O(1)$ in the natural units used in this paper and that the range of values we used in our previous analyses was too pessimistic and can be restricted. With this improvement, as shown in the next Section, the dependence on the prior noticed in ref. [1] is substantially eliminated and the constraint on $\alpha$ improved.

As mentioned before, it is illusory to allege to be able to perform an analysis of $\alpha$ without a minimal “a priori” knowledge about strong interactions and the hadronic parameters. Such knowledge is encoded in the expressions given in Equation (2) which are valid only if strong interactions are flavour blind and CP conserving, besides using the approximation that isospin breaking effects are negligible, which implies $\Lambda_{QCD} \gg (m_d - m_u)$.\footnote{Electroweak penguin contributions and electromagnetic corrections to the decay amplitudes are also...}
This however does not exhaust the information that we have on the size of the matrix elements. A general consideration is that we believe that the theory of strong interactions is QCD, which is a renormalisable theory with a dimensionless coupling constant, and has a natural scale of $\mathcal{O}(1 \text{ GeV})$. Thus, on dimensional grounds, in the absence of other scales we expect $\langle M_1 M_2 | \hat{O} | M \rangle \sim \Lambda_{QCD}^3$ for a dimension-six four fermion operator $\hat{O}$. We would be very surprised to find that this matrix element has a size of the order of $(1 \text{ TeV})^3$ or $M_{Planck}^3$. In the presence of other scales, such as the mass of a heavy quark, the situation is slightly more complicated, but the argument is conceptually similar. Indeed we expect that [13]-[15]

$$\langle \pi \pi | \hat{O} | B \rangle \sim f_\pi M_B^2 f^+(0) \sim f_\pi M_B^2 \left( \frac{\Lambda_{QCD}}{M_B} \right)^{3/2} \sim M_B^{1/2} \Lambda_{QCD}^{5/2}. \quad (5)$$

Note that this simple dimensional argument, which we have derived using factorised expressions in the intermediate steps, has a more general validity than factorisation although, obviously, factorisation respects the same scaling laws. For chirally enhanced contributions, the enhancement factor, by which the expression in Equation (5) must be multiplied, is of the order $2 M_\pi^2 / (m_u + m_d) / M_B \sim 0.8$ and thus it does not change the natural size of the amplitude. It is straightforward to show, see Equation (6) below, that this implies $T^{ij} \sim 1$. Again we would be very surprised to find that, in order to reproduce the experimental branching ratios, the size of the matrix elements must be much larger than its natural size. In the language of factorisation this would correspond to dimensionless $B$-parameters much larger than one.

We now provide further support to these general considerations:

1. The first exercise, useful to estimate the range of values which can be expected, is to

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Figure 2: Same as Figure 1 but with the experimental errors reduced by a factor 10. Obviously, also in the Bayesian case, an eightfold ambiguity appears.

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5For example $(\bar{b} u)_{V-A} (\bar{u} d)_{V-A}$ in the notation of refs. [16].
Table 2: Comparison of the experimental value of $B_{\pi\pi}^{+0}$ with the theoretical predictions [17–19]. All values are given in units of $10^{-6}$.

| ref. [17] | ref. [18] | ref. [19] | Exp. |
|----------|----------|----------|------|
| 3.6−5.3 | 4.3 (1 ± 0.3) | 3.7$^{+1.3}_{−1.1}$ | 5.5 ± 0.6 |

Using $f_\pi = 132$ MeV, $f^+(0) = 0.3$, $C_1(M_B) = -0.2$ and $C_2(M_B) = 1.1$, we find $|T^{+-}| = 3.2$ in natural units.

2. Long before it was experimentally measured, several predictions based on factorisation or other approaches existed for $B_{\pi\pi}^{+−}$, $B_{\pi\pi}^{00}$ and $B_{\pi\pi}^{+0}$. The latter is the simplest to predict since it only depends on emission diagrams, without penguins, annihilations or further complications. In Table 2 we compare some predictions, obtained in different theoretical frameworks, with the present experimental determination. We find that the order of magnitude of the predicted branching fractions, based on values of the hadronic amplitudes of $O(1)$ in natural units, are approximately in agreement with the experimental value.
3. The third argument in favour of the natural order of magnitude for the matrix elements is based on the scaling of the rates between $B$ and $D$ decays. Since, in the heavy quark limit, the partial decay rates scale as the squared amplitude divided the heavy mass, namely as $1/M \times |M^{1/2} \Lambda_{QCD}^{5/2}|^2 \sim \Lambda_{QCD}^{5/2}$, if we take the ratio

$$R = \frac{|T^+(B^0_d \rightarrow \pi^+\pi^-)|^2}{|T^+(D^0 \rightarrow \pi^+\pi^-)|^2} \sim \frac{|V_{ub}V_{ud}^*|^2}{|V_{cd}V_{ud}^*|^2},$$

(7)

this is independent of the meson mass and can be used to extract the absolute value of the unknown hadronic parameter $T^+$ as follows

$$|T^+|^2 = BR(D^0 \rightarrow \pi^+\pi^-) \times 10^6 \frac{\tau_{B^0_d}}{\tau_{D^0}} R.$$  

(8)

Using the central value of the experimental measurement $BR(D^0 \rightarrow \pi^+\pi^-) = 1.5 \times 10^{-3}$, $\tau_{D} = 0.41 \times 10^{-12}$ sec, $\tau_{B} = 1.6 \times 10^{-12}$ sec, $|V_{ub}| = 3.7 \times 10^{-3}$, $|V_{cd}| = 0.22$, we find $|T^+| = 1.3$.

4. We can get some knowledge about the parameter $P$ from the study of $B_s \rightarrow K^+K^-$ decay. Up to doubly Cabibbo suppressed terms, this decay proceeds only through the penguin contribution $P_s$, which corresponds to $P$ up to $SU(3)$ breaking effects [20]. Using the central value of the experimental measurement $BR(B_s \rightarrow K^+K^-) = (24.4 \pm 1.4 \pm 4.6) \times 10^{-6}$ [21], and the relation

$$|P|^2 = BR(B_s \rightarrow K^+K^-) \times 10^6 \frac{\tau_{B^0_d}}{\tau_{B^0_s}} \frac{|V_{td}V_{tb}^*|^2}{|V_{ts}V_{tb}^*|^2},$$

(9)
we find $|P_s| = 1.1$. Even accepting the very pessimistic point of view that $SU(3)$ breaking effects are 100%, one still obtains a number of $O(1)$ which automatically constrains, when combined with $B \to \pi\pi$ decays, also $T^{+-}$ to be of $O(1)$. For $B_s \to K^+K^-$ this argument could be spoilt if the Cabibbo suppressed emission amplitude, $T$, were much larger than the penguin amplitude, namely if $|T| \gg |P_s| \gg 1$ in natural units. This possibility is however excluded because, in order to reproduce the experimental values of $B^{+-}_{\pi\pi}$ and $B^{00}_{\pi\pi}$, we may have $|T| \gg 1$ and $|P| \gg 1$ but it is necessary that $|T| \sim |P|$.

In summary all the experimental and theoretical evidence is in favour of matrix elements of $O(1)$ in “natural units”. The arguments from 1. to 3. show that in our previous analyses [4, 5] the range that we used by varying $|T^{ij}|$ and $|P|$ between 0 and 10 was rather conservative and easily accommodates $O(1)$ corrections to the theoretical estimates presented above. Argument 4. puts a stronger constraint on the penguin contribution since, even assuming an $SU(3)$ breaking effect as large as 100%, it is difficult to imagine that it could exceed the value $|P| = 2.5$ in natural units. As discussed in 4. this upper value on $|P|$ limits the acceptable values of $|T^{+-}|$.

We conclude this Section with a few comments on appendix B of ref. [1]. In this appendix, the authors criticise the Bayesian approach for being unable to reproduce the peak of the p.d.f. at $\alpha$ around zero. We believe instead that they are completely mislead by their prejudice of ignoring the existing information on the Standard Model and on the hadronic matrix elements and for this reasons they are unable to eliminate the unphysical solutions at $\alpha$ close to zero, which cannot be there. A very simple argument kills simultaneously the solutions at $\alpha \sim 0$ and their arguments to show that the Bayesian approach is unable to cope with the RI parametrisation, where $T^{+-} = \tau^{+-}/\sin \alpha$ and the prior distribution is taken for $\tau^{+-}$ rather than for $T^{+-}$. For example, it is straightforward to show that $\alpha < 2^\circ$ corresponds to $T^{+-} > 30$ in natural units. This implies that, in order to fit the branching ratios, also $P \sim 30$, i.e. an $SU(3)$ breaking effect of about 3000%!

Bearing in mind that $m_s/m_d \sim 10$ we would then expect $SU(2)$ effects of “only” 300% which should, in our subjective and humble opinion, be taken into account invalidating the assumption of neglecting isospin breaking effects in the analysis.

### 4 Adding Useful Information

In Section 2 we have shown that, using the “minimal assumptions” on the hadronic matrix elements, no real difference in the physical information at a significant level of confidence exists between the Bayesian and the frequentist approach; in Section 3 we have discussed with several examples the information that we have on the weak hadronic matrix elements. All the arguments support the existence of a typical size for the hadronic matrix elements, which constrains the range of possible values. We do not understand why one should ignore this knowledge while using the simplification (and the information) that comes from isospin symmetry.

In this Section, we make use of the constraints on the size of the matrix elements in different ways. In general, we show that the information on the matrix elements helps

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6Furthermore they find the rather original result that the C.L. is about 90% for $\alpha \sim 10^{-30}$ and zero for $\alpha = 0$!!
eliminating some of the eight solutions that exist in the “minimal assumptions” case. The prejudice that any acceptable method must lead to eight solutions [1] is obviously wrong; if we could determine exactly the absolute values of the hadronic matrix elements and the strong phases from a theoretical calculation, from the experimental values of $B_{\pi\pi}^{00}$ and of the CP asymmetry coefficients we could extract unambiguously the value of $\alpha$. A partial knowledge, or some constraint, will then remove at least partially the degeneracy of the solutions.

In particular, we show that, using the constraints on the hadronic matrix elements, not only we eliminate the pathological solution at $\alpha \sim 0$, which could survive only with stratospheric values of the matrix elements, but we can substantially reduce the difference between the different parametrisations that was emphasised in ref. [1]. Indeed the difference disappears for the 95% probability region, and is marginal even for the 68% one.

For the parametrisations we use the same convention as in ref. [1]: MA in which the absolute values of the hadronic amplitudes and their phases are extracted with a flat p.d.f.; RI in which the real and imaginary part of the hadronic matrix elements are extracted with a flat p.d.f.; ES in which, in the Bayesian case, we generate the measured quantities, namely $B_{\pi\pi}^{00}$, $B_{\pi\pi}^{-+}$ and $B_{\pi\pi}^{00}$ and the CP asymmetry coefficients $C_{\pi\pi}^{+-}$, $C_{\pi\pi}^{00}$ and $S_{\pi\pi}^{+-}$, with flat a priori probability density function (p.d.f.) and weighted with the experimental likelihood. In this case, when solving for the absolute value of the matrix elements according to Equations (4) of Section 2, we only accept the solutions when they fall in the “a priori” acceptable region defined below.

For the following discussion, we distinguish the “previous” set of a priori, which corresponds to the ranges used in our previous analyses ($|T^{ij}| \leq 10$, $|P| \leq 10$, and arbitrary phases) [4] and the “current” one, corresponding to $|T^{ij}| \leq 10$, $|P| \leq 2.5$ and arbitrary phases. In the last case we limit $|P|$ to be less than 2.5 since, to our knowledge, no physical quantity suffers from a $SU(3)$ breaking effect larger than 100% and from $B_{s}^{0} \to K^{+}K^{-}$ we estimated $|P| \sim 1.1$. In this Section we always use the values of the measured quantities corresponding to SET 2 in Table 1.

Figure 5: Probability distribution for $\alpha$ obtained with three different prior assumptions: ES (left), MA (centre) and RI (right). In all the cases the hadronic parameter are constrained by $|T^{ij}| \leq 10$, $|P| \leq 2.5$. In all the cases the region with “low” $\alpha$ is excluded. This is not due to the statistical approach, but to the physical assumptions on the hadronic matrix elements (the smooth aspect of these p.d.f.s is due to the fact we use only $10^{5}$ (sic!) Monte Carlo events supplemented with an efficient numerical algorithm).
Figure 6: We show the range of allowed values of $\alpha$ corresponding to the 95% probability regions for the three parametrizations (MA=red, RI=green, ES=blue) as a function of the maximum allowed value of $|P|$, $P_{\text{MAX}}$, scanning the range $1 \leq P_{\text{MAX}} \leq 5.5$. This figure shows that the residual dependence on $|P|$ is very mild.

In Figure 5 we show the probability distribution functions for the three parametrisations (MA, RI and ES), in the “current” case. A comparison of the 95% probability region, highlighted in dark, show that the different parametrisations give the same physical information, although the shape of the p.d.f.s looks different. The difference observed in ref. [1] between MA and RI in the “previous” case was due to the different “a priori” distribution for the absolute values of the amplitudes (RI has a p.d.f. linearly growing with the absolute value) and the lack of further information on the hadronic matrix elements. In this respect we have to admit that we were not Bayesian enough in our previous analyses, because the “previous” upper limit consisted in taking the “intuitive” upper bound one order of magnitude larger than the typical size, without using the experimental information which was already available.

Indeed the upper bound on $P$, which was obtained using only $SU(3)$ arguments, implies a bound also on $|T^{ij}|$. If $P$ is limited, in order to reproduce the observed $B \to \pi\pi$ branching ratios, $|T^{ij}|$ cannot be too large. We leave to our friends of ref. [1] the exercise to show that with the same physical inputs one gets in the frequentist case the same results at the 95% C.L..

After the above discussion, where we have shown that the Bayesian approach gives physically meaningful results, we are now ready to discuss the implications of the bound that we have imposed on the size of the hadronic amplitudes, in particular the penguin amplitude $|P|$.

Even in the case of observed CP violation in $B \to \pi\pi$ decays, the system of equations used to extract $\alpha$ and the hadronic amplitudes, including the strong phases, from
branching fractions and CP asymmetry coefficients, Equations (3) and (4), admits the unphysical solution $\alpha = 0$. This solution can only be obtained however in the peculiar limit $|T| \to \infty$, $|P| \to \infty$ and $P/T \to -1$. This has several unappealing features: i) in the Standard Model, CP violation must disappear for $\alpha \to 0$ since the unitarity triangle collapses to a line and the Jarlskog determinant vanishes; ii) the hadronic amplitudes cannot go to infinity without violating the basic properties of any renormalisable field theory (and we have discussed what is their expected size); iii) the unphysical solutions for $\alpha \sim 0$ are eliminated, since we have shown that they imply arbitrary large $SU(3)$ breaking effects, which are not consistent with the assumption of isospin symmetry in the expressions of the amplitudes.

The bounds on the hadronic amplitudes discussed in Section 3 allow to get rid of the unphysical solution in a straightforward way. Rather than living with a non physical solution, we prefer by far to remain with the smallish residual dependence on the upper value on $|P|$ on the allowed interval for $\alpha$, shown in Figure 6. It would be interesting to compare the uncertainty originating from the upper value on $|P|$ to the size of the uncertainty due to the corrections coming from the neglected isospin breaking effects.

For completeness, we finally give in Figure 7 the p.d.f.s for the relevant amplitudes and strong phases, as selected by our analysis, in the MA parametrisation. These distributions can be used for comparison and test of the different theoretical approaches which compute, within some approximation, the hadronic matrix elements from QCD [18,19,22–24]. We present two cases: the case in which only the quantities measured in $B \to \pi\pi$ decays are used, and the case where, in the Standard Model, we make a combined CKM analysis using all the available experimental and theoretical information [2–5]. The results confirm the a priori knowledge on the weak hadronic matrix elements based on dimensional analysis and the existence of a typical scale in QCD, namely the allowed values of $|T^{±-00}|$ and $|P|$ are of $O(1)$. To be more precise, in the full Standard Model analysis, from the 68% region, we find $|P| = 0.80 \pm 0.24$, $|T^{±-}| = 2.1 \pm 0.1$ and $|T^{00}| = 1.4 \pm 0.2$, in full agreement with the expectations based on simple physical arguments of QCD and discussed in Section 3.

Conclusions

Stimulated by a recent paper on the extraction of the CKM angle $\alpha$ from $B \to \pi\pi$ decays [1], we have upgraded our Bayesian analysis with the following results: i) we have shown that the present information on the hadronic matrix elements, obtained from general theoretical arguments and experimental measurements, already allows a substantial reduction of the eightfold ambiguity in the determination of $\alpha$, in particular by eliminating the solutions at $\alpha \sim 0$, that correspond to unphysical values of the amplitudes; ii) the information on the hadronic matrix elements substantially eliminates, in the Bayesian approach, the dependence of the results on the “a priori” probability distributions of the hadronic matrix elements, which was noticed in ref. [1]. Contrary to the claims of ref. [1], we have also shown that the differences between the frequentist and the Bayesian approaches are not due to the difference in the two methods but to the difference in the physical assumptions on the weak amplitudes. We believe that a continuation of this ster-

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7In addition to the arguments given in the text, the limit mentioned above would require an infinite amount of fine tuning and correlation among parameters related to different physics, namely $\alpha$ which has an electroweak interaction origin and the hadronic amplitudes which are sensitive to strong interactions.
Figure 7: We show the p.d.f.s for $|P|$, $|T^+|$ and $|T^{00}|$, and the relative phases, obtained in the MA parametrisation, using only $B \to \pi\pi$ decays (first two rows) or the full UT analysis (last two rows).

The polemics in favour of the frequentist or Bayesian approach is only a waste of time and energies and it is better to concentrate the efforts in trying to combine in the most efficient way the rich information which is coming from the measurements of several non-leptonic...
channels to constrain the CKM parameters and to make accurate predictions.

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References

[1] J. Charles, A. Hocker, H. Lacker, F. R. Le Diberder and S. T’Jampens, arXiv:hep-ph/0607246.

[2] M. Ciuchini et al., JHEP 0107 (2001) 013 arXiv:hep-ph/0012308.

[3] M. Bona et al. [UTfit Collaboration], JHEP 0507 (2005) 028 arXiv:hep-ph/0501199.

[4] M. Bona et al. [UTfit Collaboration], JHEP 0603 (2006) 080 arXiv:hep-ph/0509219.

[5] M. Bona et al. [UTfit Collaboration], JHEP 0610 (2006) 081 arXiv:hep-ph/0606167.

[6] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini, Z. Phys. C 68 (1995) 239 arXiv:hep-ph/9501265.

[7] P. Paganini, F. Parodi, P. Roudeau and A. Stocchi, Phys. Scripta 58 (1998) 556 arXiv:hep-ph/9711261.

[8] E. Barberio et al. [H. F. A. Group], arXiv:hep-ex/0603003.

[9] M. Gronau and D. London, Phys. Rev. Lett. 65 (1990) 3381.

[10] J. Charles, Phys. Rev. D 59 (1999) 054007 arXiv:hep-ph/9806468.

[11] M. Battaglia et al., arXiv:hep-ph/0304132.

[12] M. Gronau and J. Zupan, Phys. Rev. D 71 (2005) 074017 arXiv:hep-ph/0502139.

[13] V. L. Chernyak and I. R. Zhitnitsky, Nucl. Phys. B345 (1990) 137.

[14] J. Charles, A. LeYaouanc, L. Oliver, O. Pène and J. C. Raynal, Phys. Rev. D60 (1999) 014001 hep-ph/9812358.

[15] M. Beneke and D. Yang, Nucl. Phys. B 736 (2006) 34 hep-ph/0508250; T. Becher, R. J. Hill and M. Neubert, Phys. Rev. D 72 (2005) 094017 hep-ph/0503263; D. Pirjol and I. W. Stewart, eConf C030603 (2003) MEC04 hep-ph/0309053.

[16] A. J. Buras, M. Jamin, M. E. Lautenbacher and P. H. Weisz, Nucl. Phys. B 400 (1993) 37 arXiv:hep-ph/9211304; A. J. Buras, M. Jamin and M. E. Lautenbacher, Nucl. Phys. B 400 (1993) 75 arXiv:hep-ph/9211321; M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Nucl. Phys. B 415 (1994) 403 arXiv:hep-ph/9304257.
[17] M. Ciuchini, R. Contino, E. Franco, G. Martinelli and L. Silvestrini, Nucl. Phys. B 512 (1998) 3 [Erratum-ibid. B 531 (1998) 656] [arXiv:hep-ph/9708222].

[18] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Phys. Rev. Lett. 83 (1999) 1914 [arXiv:hep-ph/9905312].

[19] Y. Y. Keum, H. n. Li and A. I. Sanda, AIP Conf. Proc. 618 (2002) 229 [arXiv:hep-ph/0201103].

[20] A. J. Buras and L. Silvestrini, Nucl. Phys. B 569 (2000) 3 [arXiv:hep-ph/9812392].

[21] CDF Collaboration Note 8579 06-09-21, http://www-cdf.fnal.gov/physics/new/bottom/060921.blessed-bhh.

[22] C. W. Bauer, D. Pirjol, I. Z. Rothstein and I. W. Stewart, Phys. Rev. D 70 (2004) 054015 [arXiv:hep-ph/0401188].

[23] Y. Grossman, A. Hocker, Z. Ligeti and D. Pirjol, Phys. Rev. D 72 (2005) 094033 [arXiv:hep-ph/0506228].

[24] C. W. Chiang, M. Gronau, J. L. Rosner and D. A. Suprun, Phys. Rev. D 70 (2004) 034020 [arXiv:hep-ph/0404073].