Equilibrium points in the asteroid 2001SN\textsuperscript{263}

Santos, L. B. T.\textsuperscript{1}, Prado, A. F. B. A.\textsuperscript{1}, Sanchez, D. M.\textsuperscript{1}

\textsuperscript{1}National Institute from Space Research, INPE, São José dos Campos, Brazil
E-mail: leonardo.btorres@inpe.br

Abstract. In the present work we develop a study based on the Restricted Full Three-Body Problem (RFTBP), where we take into account the size and shape of the bodies studied. The equations of motion and their solutions are discussed for a circular coplanar orbit in order to find the equilibrium points. All the analysis made are based on the physical and orbital characteristics of the most massive bodies of the triple asteroid system 2001SN\textsuperscript{263}. A total of nine equilibrium points were found for this system. Five of these points are collinear and the other four are coplanar. Finally, an analysis of the stability conditions of these equilibrium points are investigated, and it was concluded that only the equilibrium points $L_4$ and $L_5$ are linearly stable.

1. Introduction

In the last years there has been a great interest in the exploration of small bodies of the solar system. Mathematical modeling of the gravitational field near the surface of an asteroid is a very challenging task, as it is necessary to know some physical properties of the asteroid, such as size, shape, density, etc, which are often only discovered after the spacecraft approaches the body [1]. To make an analysis of the orbital dynamics of a spacecraft around an irregular body, it is necessary to obtain a mathematical model to represent the gravitational field of the body that one wishes to orbit. Due to the fact that many asteroids have an irregular shape, the orbits of spacecraft near these bodies do not assume a traditional Keplerian orbit [2]. Due to the rotation effect of asteroids along with their irregular shapes, the gravitational fields acting on spacecraft, which are very close to these bodies, are very complex [3]. Then, when planning a space mission, it is necessary to make mathematical modeling flexible enough to cover several parameters of the object being studied [1]. In the present work an analysis was made considering the primary bodies as spheroids. The asteroids adopted to carry out this study are the asteroid Alpha and Beta of the triple asteroid system 2001SN\textsuperscript{263}. This asteroid system was chosen to be analyzed due to a Brazilian mission known as Missao Aster [4, 5, 6, 7].

2. Equation of motion

Let be two primary bodies ($M_1$ and $M_2$) with finite mass ($m_1$ and $m_2$) and of spheroid format. Let us consider here that $m_1 > m_2$. Suppose that these bodies are rotating in a circular motion about the center of mass of the system with angular velocity $\omega$. Now consider a third body ($P(x, y, z)$), and that the motion of a negligible mass is governed by the gravity forces of the bodies $M_1$ and $M_2$. The infinitesimal mass body does not affect the dynamics of the primaries, as shown in Figure 1. The unit of distance is normalized to be the distance from the body $M_1$ to the body $M_2$. 
The unit of time is defined such that the period of translation of the primary bodies, is equal to $2\pi$. Thus, we have $G(m_1 + m_2) = 1$. The spatial position of the negligible mass is $P(x, y, z)$ and the coordinates of the bodies with mass $m_1$ and $m_2$ are $(-x_1, 0, 0)$ and $(x_2, 0, 0)$, respectively, with respect to the rotating system. The mass ratio is given by $\mu = \frac{m_2}{m_1 + m_2}$. The primary bodies are on the axis $x$, whose coordinates are given by $x_1 = -\mu$ and $x_2 = 1 - \mu$. The equations of motion of the negligible mass body, in the $xy$ plane, when viewed from a rotating frame of reference, is given by the equations (1-6).

$$\ddot{x} - 2\dot{y} = U_x$$
$$\ddot{y} + 2\dot{x} = U_y$$

where

$$U = -\frac{\omega^2(x^2 + y^2)}{2} - k\omega^2 \left( \frac{(1 - \mu)w_1}{r_1} + \frac{\mu w_2}{r_2} \right)$$

where

$$w_i = 1 + \frac{A_i}{2y_i^2}, i = 1, 2$$

$$r_1 = [(x - x_1), y, 0]^T$$
$$r_2 = [(x - x_2), y, 0]^T$$

and $U_x$ and $U_y$ are the partial derivatives of $U$ with respect to $x$ and $y$, respectively.

The dimensionless parameter $k$ that appears in Equation 3 is of extreme importance, and represents the ratio between gravitational force and centrifugal force, whose definition is given by:

$$k = \frac{GM/d^2}{d\omega^2}$$

where $G$ is the universal gravitational constant, $M$ is the sum of the masses of the primary bodies, and $d$ is the distance between the primary bodies. The angular velocity in the dimensionless system that appears in Equation 3 is given by:

$$\omega = \sqrt{1 + \frac{3(A_1 + A_2)}{2}}$$

where $A_i (i = 1, 2)$ are the oblateness coefficient of the primary bodies, whose definition is given by:

$$A_i = \frac{(p_i^e)^2 - (p_i^p)^2}{5d^2}$$
where the parameter $\rho$ is the radius of the primary spheroids. The subscript $e$ represents the equatorial radius of the primary and the subscript $p$ indicates the polar radius of the primary [3].

3. Results
In order to calculate the equilibrium points of a binary system of asteroids it is necessary to know the characteristics of this system. Recent data from the asteroid system 2001SN$_{263}$ can be found in [8]. In this study, we will consider asteroids with spheroid shapes. The equatorial radius of the asteroid $\text{Alpha}$ is estimated to be 1.45 km and the radius of the Polar axis is 1.35 km. If we take as reference the asteroid $\text{Alpha}$ (because it is the most massive and largest body), we have that the radius of the polar axis of the asteroid $\text{Beta}$ is 0.5 km and the radius of the equatorial axis of the asteroid $\text{Beta}$ is 0.32 km. A geometric representation of the shape of the alpha and beta (not in scale) asteroids can be seen in Figure 2.

3.1. Equilibrium Points
To find the equilibrium points, it is necessary to make the right side of the Equations 1 and 2 equal zero. It is possible to note, by analyzing the Equation 1 and 2, that, if a particle is placed at an equilibrium point with zero initial velocity, it will remain at this point indefinitely.

Let the right side of Equations 1 and 2 to be equal zero, that is, $\frac{\partial U}{\partial x} = 0$ and $\frac{\partial U}{\partial y} = 0$, and replacing the numerical values of the analyzed system, it was possible to find nine equilibrium points in the orbital plane of the primaries. Therefore, in addition to the five classical equilibrium points, four new equilibrium points appeared. The Figure 3(a) shows the positions of the nine equilibrium points of the binary system of asteroids studied and the location of the center of mass of the primary bodies. Red represents the classic lagrangean points, known as $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$. In blue are the positions of the center of mass of the primary bodies. The blue dot on the left represents the most massive primary ($\text{alpha}$) and the blue dot on the right represents the least massive primary ($\text{Beta}$). In green, very close to $\text{Beta}$, are the new equilibrium points called here $L_6$, $L_7$, $L_8$ and $L_9$.

Figure 3(b) is an approximation of the less massive body, for better visualization of the points $L_6$, $L_7$, $L_8$, and $L_9$. We see that these new equilibrium points are very close to the center of mass (blue color) of the body $\text{Beta}$.

When we consider the dimension of the less massive body, based on the dimensions of the asteroid $\text{Beta}$, we see that the new points of equilibrium are inside the less massive body, as shown in Figure 4(a). In this figure we can see an oval representing the asteroid $\text{Beta}$ (violet color), the center of mass of this asteroid (blue color) and the new equilibrium points (green color).
Figure 3: Equilibrium points.

(a) Image of the nine equilibrium points showing the classical points (red color), non-classical points (green color) and the positions of the center of mass of the primary bodies (blue color).

(b) New equilibrium points, called $L_6$, $L_7$, $L_8$ and $L_9$ (green color) and the center of mass of the less massive asteroid (blue color).

Figure 4: Equilibrium points and size of asteroid Beta.

(a) New equilibrium points, called $L_6$, $L_7$, $L_8$ and $L_9$ (green color), the center of mass of the less massive asteroid (blue color) and an oval (violet color) representing the dimension of the less massive asteroid.

(b) New equilibrium points, called $L_6$, $L_7$, $L_8$ and $L_9$ (green color), the center of mass of the less massive asteroid (blue color) and an oval (violet color) representing the dimension of the less massive asteroid (taking into account the margin of error).

However, when we take into account the error margins of the dimension of this asteroid, some of these new points of equilibrium are practically on the surface of this asteroid, as shown in Figure 4(b).

This is quite interesting, because there is a point of equilibrium practically on the surface of an asteroid. If a spacecraft has an interest in landing on a binary system of asteroids with specifications similar to those discussed here, it will be possible to find a position on the asteroid where the spacecraft can reach the surface with zero velocity and thus be able to land.

To know where these stationary points are located is important for, as an example, maintaining a spacecraft, since these points indicate the locations where the spacecraft suffers the least disturbance and thus the minimum possible fuel is spent for station-keeping.

The distances of the lagrangean points calculated here are distances relative to the center
of mass of the system. Table 1, of the studied system, is normalized, considering the distance between the two primaries as being unitary.

Table 1: Position of the equilibrium points when taking into account the oblateness of the bodies.

|     | \(L_1\)  | \(L_2\)  | \(L_3\)  | \(L_4\)  | \(L_5\)  | \(L_6\)  | \(L_7\)  | \(L_8\)  | \(L_9\)  |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(x\) | 0.78438 | 1.19238 | -1.01072 | 0.47466 | 0.47466 | 0.96185 | 0.98716 | 0.97452 | 0.97452 |
| \(y\) | 0      | 0      | 0      | 0.86605 | -0.86605 | 0      | 0      | 0.01265 | 0.01265 |

3.2. Stability

Let the coordinates of some equilibrium points be given by \((x_0, y_0, 0)\). We are considering \(z = 0\) because we are analyzing the equilibrium points in the plane. Now let us suppose that \(\alpha\) and \(\beta\) are small displacements of a particle of infinitesimal mass with respect to the point of equilibrium, in the directions \(x\) and \(y\), respectively. These displacements are functions of time, and therefore there will be acceleration components \((\ddot{\alpha}, \ddot{\beta}, 0)\) and velocity \((\dot{\alpha}, \dot{\beta}, 0)\).

In this way, we can write the equation of the motion of a particle of infinitesimal mass as follows:

\[
\ddot{\alpha} - 2\ddot{\beta} = \alpha(U_{xx})_0 + \beta(U_{xy})_0, \quad (10)
\]

\[
\ddot{\beta} + 2\ddot{\alpha} = \alpha(U_{yx})_0 + \beta(U_{yy})_0. \quad (11)
\]

Equations 10 and 11 are the linearized equations of motion of a particle of infinitesimal mass in the vicinity of a point of equilibrium when it is slightly displaced \([9, 10, 11]\). To know if the equilibrium point is linearly stable or not, simply replace the positions of these points in Equations 10 and 11 and solve the system.

If all eigenvalues of the equations give only purely imaginary values, this implies that the equilibrium point analyzed will be stable up to first order (or linearly stable). On the other hand, if any of the eigenvalues is real or imaginary, it is an unstable equilibrium point \([9, 10, 11]\).

Using Equations 10 and 11, adopting the physical and orbital properties of Alpha and Beta, and knowing the values of the equilibrium points, it was possible to analyze the stability of all Equilibrium points found, based on the eigenvalues, as shown in Table 2.

Table 2: Eigenvalues of the characteristic equation for the RFTBP.

|     | \(L_1\)  | \(L_2\)  | \(L_3\)  | \(L_4\)  | \(L_5\)  | \(L_6\)  | \(L_7\)  | \(L_8\)  | \(L_9\)  |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(\lambda_1\) | -3.045 | -2.055 | 0.258 | 0.460i | 0.460i | 111.965 | 112.199 | 0.166  | 112.25 |
| \(\lambda_2\) | 3.045  | 2.055  | -0.258 | -0.460i | -0.460i | -111.965 | -112.199 | -0.166 | -112.25 |
| \(\lambda_3\) | 2.412i | 1.807i | 1.021i | 0.886i | 0.886i | 274.374i | 274.836i | 1.009i | 274.918i |
| \(\lambda_4\) | -2.412i | -1.807i | -1.021i | -0.886i | -0.886i | -274.374i | -274.836i | -1.009i | -274.918i |

We can note that only the equilibrium points \(L_4\) and \(L_5\) are linearly stable. The other points are all unstable.
4. Conclusion
In the Equations of motion the RFTBP, when we considered the oblateness of the primary bodies, new equilibrium points were found. In addition to the five classical equilibrium points found in the restricted circular/elliptic three-body problem, four new coplanar equilibrium points appeared (here called $L_6$, $L_7$, $L_8$ and $L_9$). This is due to oblateness of the bodies. When we consider the asteroid data of Alpha and Beta, we realize that Alpha is an oblate body (it means, more elongated in the equatorial axis) and the body Beta has a prolate oblateness (it is more elongated in the polar axis). This prolate form gives rise to new equilibrium points around the body with a greater elongation in the polar axis.

An analysis of the stability of all these points found ($L_1$ to $L_9$) was also made and the numerical evidence shows that only the equilibrium points $L_4$ and $L_5$ are linearly stable, while all others are unstable. The equilibrium points of the asteroid system are considered to be places that receive minimal disturbances, so they are good places to maintain a spacecraft. It can be used as parked orbits. It can be interesting to keep a spacecraft near the surface of an asteroid, thus being a suitable place to start landing. So, the points found here are very interesting for this purpose.

5. acknowledgments
The authors wish to express their appreciation for the support provided by grants# 406841/2016-0 and 301338/2016-7 from the National Council for Scientific and Technological Development (CNPq). We also thank the grants# 2016/14665-2, 2016/18418-0, 2011/08171-3, 2014/22293-2, 2014/2295-5 from São Paulo Research Foundation (FAPESP) and the financial support from the National Council for the Improvement of Higher Education (CAPES).

References
[1] SCHEERES, D. J.; WILLIAMS, B. G.; MILLER, J. K. Evaluation of the dynamic environment of an asteroid: Applications to 433 eros. Journal of Guidance, Control and Dynamics, v. 23, n. 3, p. 466-475, 2000.
[2] MASAGO, B. Y. P. L. et al. Developing the -precessing inclined bi-elliptical four-body problem with radiation pressure- to search for orbits in the triple asteroid 2001sn263. Advances in Space Research., v. 57, p. 962982, 2016.
[3] ZENG, X. Y.; GONG, S.; LI, J.; ALFRIEND, K. Equilibrium points of elongated celestial bodies as the perturbed rotating mass dipole. Journal of Guidance, Control and Dynamics, v. 39, n. 6, p. 1223, 2016.
[4] SUKHANOV, A. A., Velho, H.F.C., Macua, E.E., Winter, O.C.: Cosmic research 48(5), 455 (2010).
[5] PRADO, A. F. B. A. Mapping Orbits Around the Asteroid 2001SN263. Advances in Space Research., v 53, p. 877, 2014.
[6] ARAJO, R. A. N., WINTER, O. C., PRADO, A. F. B. A., and Prado, A. F. B. A., VIEIRA, M. R. Sphere of influence and gravitational capture radius: a dynamical approach. MNRAS, v 391, p. 675, 2008.
[7] ARAJO, R. A. N., WINTER, O. C., PRADO, A. F. B. A., and Prado, A. F. B. A. Stable retrograde orbits around the triple system 2001 SN263. MNRAS, v 153, p. 1143, 2015.
[8] TRACY, M. B. et al. Physical modeling of triple near-earth asteroid (153591) 2001SN263 from radar and optical light curve observations. Icarus, v. 248, p. 499-515, 2015.
[9] SZEBEHELY, V. Theory of Orbits. New York and London: Academic press., 1967.
[10] MCCUSKEY, S. W. Introduction to Celestial Mechanics. 1. ed. USA: Addison-Wesley Publishing Company, 1963.
[11] SANTOS, L. B. T., PRADO, A. F. B. A., SANCHEZ, D. M. Equilibrium points in the restricted synchronous three-body problem using a mass dipole model Astrophysics and Space Science. 362(61), 60 (2017).