On the different kinds of separability of the space of Borel functions

Abstract: In paper we prove that:

- a space of Borel functions \( B(X) \) on a set of reals \( X \), with pointwise topology, to be countably selective sequentially separable if and only if \( X \) has the property \( S(\Gamma, \Gamma) \);
- there exists a consistent example of sequentially separable selectively separable space which is not selective sequentially separable. This is an answer to the question of A. Bella, M. Bonanzinga and M. Matveev;
- there is a consistent example of a compact \( T_2 \) sequentially separable space which is not selective sequentially separable. This is an answer to the question of A. Bella and C. Costantini;
- \( \min\{b, q\} = \{\kappa : 2^\kappa \text{ is not selective sequentially separable}\} \). This is a partial answer to the question of A. Bella, M. Bonanzinga and M. Matveev.

Keywords: \( S(\mathcal{D}, \mathcal{D}) \), \( S(S, S) \), \( S_{\text{fin}}(S, S) \), Function spaces, Selection principles, Borel function, \( \sigma \)-set, \( S(\Omega, \Omega) \), \( S(\Gamma, \Gamma) \), \( S(\Omega, \Gamma) \), Sequentially separable, Selectively separable, Selective sequentially separable, Countably selective sequentially separable

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1 Introduction

In [12], Osipov and Pytkeev gave necessary and sufficient conditions for the space \( B_1(X) \) of the Baire class 1 functions on a Tychonoff space \( X \), with pointwise topology, to be (strongly) sequentially separable. In this paper, we consider some properties of a space \( B(X) \) of Borel functions on a set of reals \( X \), with pointwise topology, that are stronger than (sequential) separability.

2 Main definitions and notation

Many topological properties are defined or characterized in terms of the following classical selection principles. Let \( \mathcal{A} \) and \( \mathcal{B} \) be sets consisting of families of subsets of an infinite set \( X \). Then:

- \( S(\mathcal{A}, \mathcal{B}) \) is the selection hypothesis: for each sequence \( (A_n : n \in \mathbb{N}) \) of elements of \( \mathcal{A} \) there is a sequence \( (b_n : n \in \mathbb{N}) \) such that for each \( n \), \( b_n \in A_n \), and \( \{b_n : n \in \mathbb{N}\} \) is an element of \( \mathcal{B} \).
- \( S_{\text{fin}}(\mathcal{A}, \mathcal{B}) \) is the selection hypothesis: for each sequence \( (A_n : n \in \mathbb{N}) \) of elements of \( \mathcal{A} \) there is a sequence \( (B_n : n \in \mathbb{N}) \) of finite sets such that for each \( n \), \( B_n \subseteq A_n \), and \( \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B} \).
$U_{fin}(A, B)$ is the selection hypothesis: whenever $U_1, U_2, \ldots \in A$ and none contains a finite subcover, there are finite sets $F_n \subseteq U_n, n \in \mathbb{N}$, such that $\{\bigcup F_n : n \in \mathbb{N}\} \in B$.

An open cover $\mathcal{U}$ of a space $X$ is:
- an $\omega$-cover if $X$ does not belong to $\mathcal{U}$ and every finite subset of $X$ is contained in a member of $\mathcal{U}$;
- a $\gamma$-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space $X$ we denote:
- $\Omega$ — the family of all countable open $\omega$-covers of $X$;
- $\Gamma$ — the family of all countable open $\gamma$-covers of $X$;
- $B_\omega$ — the family of all countable Borel $\omega$-covers of $X$;
- $B_\gamma$ — the family of all countable Borel $\gamma$-covers of $X$;
- $F_\gamma$ — the family of all countable closed $\gamma$-covers of $X$;
- $D$ — the family of all countable dense subsets of $X$;
- $S$ — the family of all countable sequentially dense subsets of $X$.

A $\gamma$-cover $\mathcal{U}$ of co-zero sets of $X$ is $\gamma_\mathcal{F}$-shrinkable if there exists a $\gamma$-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of $X$ with $F(U) \subseteq U$ for every $U \in \mathcal{U}$.

For a topological space $X$ we denote $\Gamma_\mathcal{F}$, the family of all countable $\gamma_\mathcal{F}$-shrinkable $\gamma$-covers of $X$.

We will use the following notations.
- $C_p(X)$ is the set of all real-valued continuous functions $C(X)$ defined on a space $X$, with pointwise topology.
- $B_1(X)$ is the set of all first Baire class $1$ functions $B_1(X)$ i.e., pointwise limits of continuous functions, defined on a space $X$, with pointwise topology.
- $B(X)$ is the set of all Borel functions, defined on a space $X$, with pointwise topology.

If $X$ is a space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If $D$ is a countable, sequentially dense subset of $X$ then $X$ call sequentially separable space.

Call a space $X$ strongly sequentially separable if $X$ is separable and every countable dense subset of $X$ is sequentially dense.

A space $X$ is (countably) selectively separable (or M-separable, [3]) if for every sequence $(D_n : n \in \mathbb{N})$ of (countable) dense subsets of $X$ one can pick finite $F_n \subseteq D_n, n \in \mathbb{N}$, so that $\bigcup \{F_n : n \in \mathbb{N}\}$ is dense in $X$.

In [3], the authors started to investigate a selective version of sequential separability.

A space $X$ is (countably) selectively sequentially separable (or M-sequentially separable, [3]) if for every sequence $(D_n : n \in \mathbb{N})$ of (countable) sequentially dense subsets of $X$, one can pick finite $F_n \subseteq D_n, n \in \mathbb{N}$, so that $\bigcup \{F_n : n \in \mathbb{N}\}$ is sequentially dense in $X$.

In Scheeper’s terminology [16], countably selectively separability equivalently to the selection principle $S_{fin}(D, D)$, and countably selective sequentially separability equivalently to the $S_{fin}(S, S)$.

Recall that the cardinal $p$ is the smallest cardinal so that there is a collection of $p$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega_1 \leq p \leq c$.

For $f, g \in \mathbb{N}^\omega$, let $f \preceq^* g$ if $f(n) \preceq g(n)$ for all but finitely many $n$. $b$ is the minimal cardinality of a $\preceq^*$-unbounded subset of $\mathbb{N}^\omega$. A set $B \subseteq [\mathbb{N}]^\omega$ is unbounded if the set of all increasing enumerations of elements of $B$ is unbounded in $\mathbb{N}^\omega$, with respect to $\preceq^*$. It follows that $|B| \geq b$. A subset $S$ of the real line is called a $Q$-set if each one of its subsets is a $G_\delta$. The cardinal $q$ is the smallest cardinal so that for any $\kappa < q$ there is a $Q$-set of size $\kappa$. (See [7] for more on small cardinals including $p$).
3 Properties of a space of Borel functions

Theorem 3.1. For a set of reals $X$, the following statements are equivalent:

1. $B(X)$ satisfies $S_1(S, S)$ and $B(X)$ is sequentially separable;
2. $X$ satisfies $S_1(B_r, B_r)$;
3. $B(X) \in S_{\text{fin}}(S, S)$ and $B(X)$ is sequentially separable;
4. $X$ satisfies $S_{\text{fin}}(B_r, B_r)$;
5. $B_1(X)$ satisfies $S_1(S, S)$;
6. $X$ satisfies $S_1(F_r, F_r)$;
7. $B_1(X)$ satisfies $S_{\text{fin}}(S, S)$.

Proof. It is obvious that $(1) \Rightarrow (3)$. [Proof continued with mathematical details and logical steps]
there exists an unbounded \( \beta \in \mathbb{N}^\mathbb{N} \) such that \( \{ f^{(k)}_k \} \) converges to \( f \) on \( Y \). It follows that \( \{ f^{(k)}_k \} : k \in \mathbb{N} \) converge to \( f \) on \( X \) and \([B]_{\text{seq}} = B(X)\).

(5) ⇒ (6). By Velichko’s Theorem ([18]), a space \( B_1(X) \) is sequentially separable for any separable metric space \( X \).

Let \( \{ F_i \} \subset F_\tau \) and \( S = \{ h_m \}_{m \in \mathbb{N}} \) be a countable sequentially dense subset of \( B_1(X) \).

Similarly implication (3) ⇒ (2) we get \( X \) satisfies \( U_{\text{fin}}(F_\tau, F_\tau) \), and, hence, by Lemma 13 in [17], \( X \) satisfies \( S_1(F_\tau, F_\tau) \).

(6) ⇒ (5). By Corollary 20 in [17], \( X \) satisfies \( S_1(B_\tau, B_\tau) \). Since \( X \) is a \( \sigma \)-set (see [17]), \( B_1(X) = B(X) \) and, by implication (2) ⇒ (1), we get \( B_1(X) \) satisfies \( S_1(S, S) \).

\( \square \)

In [16], (Theorem 13) M. Scheepers proved the following result.

**Theorem 3.2 (Scheepers).** For \( X \) a separable metric space, the following are equivalent:

1. \( C_p(X) \) satisfies \( S_1(\mathcal{D}, \mathcal{D}) \);
2. \( X \) satisfies \( S_1(\Omega, \Omega) \).

We claim the theorem for a space \( B(X) \) of Borel functions.

**Theorem 3.3.** For a set of reals \( X \), the following are equivalent:

1. \( B(X) \) satisfies \( S_1(\mathcal{D}, \mathcal{D}) \);
2. \( X \) satisfies \( S_1(B_\Omega, B_\Omega) \).

**Proof.** (1) ⇒ (2). Let \( X \) be a set of reals satisfying the hypotheses and \( \beta \) be a countable base of \( X \). Consider a sequence \( \{ B_i \}_{i \in \mathbb{N}} \) of countable Borel \( \omega \)-covers of \( X \) where \( B_i = \{ W_i^j \}_{j \in \mathbb{N}} \) for each \( i \in \mathbb{N} \).

Consider a topology \( \tau \) generated by the family \( \mathcal{P} = \{ W_i^j \cap A : i, j \in \mathbb{N} \text{ and } A \in \beta \} \cup \{ (X \setminus W_i^j) \cap A : i, j \in \mathbb{N} \text{ and } A \in \beta \} \).

Note that if \( \chi_{W_i^j} \) is a characteristic function of \( P \) for each \( P \in \mathcal{P} \), then a diagonal mapping \( \varphi = \Delta_{\mathcal{P} \times \mathcal{P}} \chi_{\mathcal{P}} : X \rightarrow 2^{\omega} \) is a Borel bijection. Let \( Z = \varphi(X) \).

Note that \( \{ B_i \} \) is countable open \( \omega \)-cover of \( Z \) for each \( i \in \mathbb{N} \). Since \( B(Z) \) is a dense subset of \( B(X) \), then \( B(Z) \) also has the property \( S_1(\mathcal{D}, \mathcal{D}) \). Since \( C_p(Z) \) is a dense subset of \( B(Z) \), \( C_p(Z) \) has the property \( S_1(\mathcal{D}, \mathcal{D}) \), too.

By Theorem 3.2, the space \( Z \) has the property \( S_1(\Omega, \Omega) \). It follows that there is a sequence \( \{ W_i^{(j)} \}_{i \in \mathbb{N}} \) such that \( W_i^{(j)} \in B_i \) and \( \{ W_i^{(j)} : i \in \mathbb{N} \} \) is an open \( \omega \)-cover of \( Z \). It follows that \( \{ W_i^{(j)} : i \in \mathbb{N} \} \) is Borel \( \omega \)-cover of \( X \).

(2) ⇒ (1). Assume that \( X \) has the property \( S_1(B_\Omega, B_\Omega) \). Let \( \{ D_k \}_{k \in \mathbb{N}} \) be a sequence countable dense subsets of \( B(X) \) and \( D_k = \{ f_k^i \} : i \in \mathbb{N} \) for each \( k \in \mathbb{N} \). We claim that for any \( f \in B(X) \) there is a sequence \( \{ f_k \} \subset B(X) \) such that \( f_k \in D_k \) for each \( k \in \mathbb{N} \) and \( f \in \bigcap \{ f_k : k \in \mathbb{N} \} \).

Without loss of generality we can assume \( f = 0 \). For each \( f_k^i \in D_k \) let \( W_k^i = \{ x \in X : -1/k < f_k^i(x) < 1/k \} \).

If for each \( j \in \mathbb{N} \) there is \( k(j) \) such that \( W_k^{(j)} = X \), then a sequence \( f_k^{(j)} = f_k^{N(k)} \) uniformly converges to \( f \) and, hence, \( f \in \bigcap \{ f_k^{(j)} : j \in \mathbb{N} \} \).

We can assume that \( W_k^i \neq X \) for any \( k, i \).

(a). \( \{ W_k^i \}_{i \in \mathbb{N}} \) a sequence of Borel sets of \( X \).

(b). For each \( k \in \mathbb{N} \), \( \{ W_k^i : i \in \mathbb{N} \} \) is a \( \omega \)-cover of \( X \).

By (2), \( X \) has the property \( S_1(B_\Omega, B_\Omega) \), hence, there is a sequence \( \{ W_k^{(j)} \}_{j \in \mathbb{N}} \) such that \( W_k^{(j)} \in \{ W_i^{(j)} \}_{i \in \mathbb{N}} \) for each \( k \in \mathbb{N} \) and \( \{ W_k^{(j)} \}_{j \in \mathbb{N}} \) is a \( \omega \)-cover of \( X \).

Consider \( \{ f_k^{(j)} \}_{j \in \mathbb{N}} \). We claim that \( f \in \bigcap \{ f_k^{(j)} : k \in \mathbb{N} \} \). Let \( K \) be a finite subset of \( X \), \( \epsilon > 0 \) and \( U = \{ f \in K, \epsilon \} \) be a base neighborhood of \( f \), then there is \( k_0 \in \mathbb{N} \) such that \( \frac{1}{k_0} \epsilon < \epsilon \) and \( K \subset W_k^{(j)} \). It follows that \( f_k^{(j)} \subset U \).

Let \( \mathcal{D} = \{ d_n : n \in \mathbb{N} \} \) be a dense subspace of \( B(X) \). Given a sequence \( \{ D_i \}_{i \in \mathbb{N}} \) of dense subspace of \( B(X) \), enumerate it as \( \{ D_n : n \in \mathbb{N} \} \). For each \( n \in \mathbb{N} \), pick \( d_{n,m} \in D_{n,m} \) so that \( d_n \in \bigcap \{ d_{n,m} : m \in \mathbb{N} \} \). Then \( \{ d_{n,m} : n, m \in \mathbb{N} \} \) is dense in \( B(X) \).

\( \square \)

In [16], (Theorem 35) and [4] (Corollary 2.10) proved the following result.
Theorem 3.4 (Scheepers). For a $X$ a separable metric space, the following are equivalent:
1. $C_p(X)$ satisfies $S_{fin}(D, D)$;
2. $X$ satisfies $S_{fin}(\Omega, \Omega)$.

Then for the space $B(X)$ we have an analogous result.

Theorem 3.5. For a set of reals $X$, the following are equivalent:
1. $B(X)$ satisfies $S_{fin}(D, D)$;
2. $X$ satisfies $S_{fin}(B_{\Omega}, B_{\Omega})$.

Proof. It is proved similarly to the proof of Theorem 3.3. \qed

4 Question of A. Bella, M. Bonanzinga and M. Matveev

In [3], Question 4.3, it is asked to find a sequentially separable selectively separable space which is not selective sequentially separable.

The following theorem answers this question.

Theorem 4.1 (CH). There is a consistent example of a space $Z$, such that $Z$ is sequentially separable, selectively separable, not selective sequentially separable.

Proof. By Theorem 40 and Corollary 41 in [15], there is a $c$-Lusin set $X$ which has the property $S_1(B_{\Omega}, B_{\Omega})$, but $X$ does not have the property $U_{fin}(\Gamma, \Gamma)$.

Consider a space $Z = C_p(X)$. By Velichko's Theorem ([18]), a space $C_p(X)$ is sequentially separable for any separable metric space $X$.

(a). $Z$ is sequentially separable. Since $X$ is Lindelöf and $X$ satisfies $S_1(B_{\Omega}, B_{\Omega})$, $X$ has the property $S_1(\Omega, \Omega)$.

By Theorem 3.2, $C_p(X)$ satisfies $S_1(D, D)$, and, hence, $C_p(X)$ satisfies $S_{fin}(D, D)$.

(b). $Z$ is selectively separable. By Theorem 4.1 in [11], $U_{fin}(\Gamma, \Gamma) = U_{fin}(\Gamma_\Gamma, \Gamma)$ for Lindelöf spaces.

Since $X$ does not have the property $U_{fin}(\Gamma, \Gamma)$, $X$ does not have the property $S_{fin}(\Gamma_\Gamma, \Gamma)$. By Theorem 8.11 in [9], $C_p(X)$ does not have the property $S_{fin}(S, S)$.

(c). $Z$ is not selective sequentially separable. \qed

Theorem 4.2 (CH). There is a consistent example of a space $Z$, such that $Z$ is sequentially separable, countably selectively separable, countably selectively separable, not countably selective sequentially separable.

Proof. Consider the $c$-Lusin set $X$ (see Theorem 40 and Corollary 41 in [15]), then $X$ has the property $S_1(B_{\Omega}, B_{\Omega})$, but $X$ does not have the property $U_{fin}(\Gamma, \Gamma)$ and, hence, $X$ does not have the property $S_{fin}(B_{\Gamma}, B_{\Gamma})$.

Consider a space $Z = B_1(X)$. By Velichko's Theorem in [18], a space $B_1(X)$ is sequentially separable for any separable metric space $X$.

(a). $Z$ is sequentially separable. By Theorem 3.3, $B(X)$ satisfies $S_1(D, D)$. Since $Z$ is dense subset of $B(X)$ we have that $Z$ satisfies $S_1(D, D)$ and, hence, $Z$ satisfies $S_{fin}(D, D)$.

(b). $Z$ is countably selectively separable. Since $X$ does not have the property $S_{fin}(B_{\Gamma}, B_{\Gamma})$, by Theorem 3.1, $B_1(X)$ does not have the property $S_{fin}(S, S)$.

(c). $Z$ is not countably selective sequentially separable. \qed
5 Question of A. Bella and C. Costantini

In [5], Question 2.7, it is asked to find a compact T2 sequentially separable space which is not selective sequentially separable.

The following theorem answers this question.

**Theorem 5.1.** (b < q) There is a consistent example of a compact T2 sequentially separable space which is not selective sequentially separable.

**Proof.** Let D be a discrete space of size b. Since b < q, a space 2^b is sequentially separable (see Proposition 3 in [13]).

We claim that 2^b is not selective sequentially separable.

On the contrary, suppose that 2^b is sequentially separable. Since non(Sfin(BΓ, BΓ)) = b (see Theorem 5.1), there is a set of reals X such that |X| = b and X does not have the property Sfin(BΓ, BΓ). Hence there exists sequence (An : n ∈ N) of elements of BΓ that for any sequence (Bn : n ∈ N) of finite sets such that for each n, Bn ⊆ An, we have that \( \cup_{n \in \mathbb{N}} B_n \notin BΓ \).

Consider an identity mapping \( id : D \rightarrow X \) from the space D onto the space X. Denote \( C_n = id^{-1}(A_n) \) for each \( A_n \in A_n \) and \( n, i \in \mathbb{N} \). Let \( C_n = \{ C_n \}_{i \in N} \) (i.e. \( C_n = id^{-1}(A_n) \)) and let \( S = \{ h_i \}_{i \in N} \) be a countable sequentially dense subset of \( B(D, \{0, 1\}) = 2^b \).

For each \( n \in \mathbb{N} \) we consider a countable sequentially dense subset \( S_n \) of \( B(D, \{0, 1\}) \) where

\[
S_n = \{ f_n^a \} = \{ f_n^b \in B(D, 2) : f_n^a \upharpoonright C_n = h_i, \text{ and } f_n^a \upharpoonright (X \setminus C_n) = 1 \text{ for } i \in \mathbb{N} \}.
\]

Since \( C_n = \{ C_n \}_{i \in N} \) is a Borel \( \gamma \)-cover of D and \( S_n \) is a countable sequentially dense subset of \( B(D, \{0, 1\}) \), we have that \( S_n \) is a countable sequentially dense subset of \( B(D, \{0, 1\}) \) for each \( n \in \mathbb{N} \).

Indeed, let \( h \in B(D, \{0, 1\}) \), there is a sequence \( \{ h_\lambda \}_{\lambda \in \mathbb{N}} \subset S \) such that \( \{ h_\lambda \}_{\lambda \in \mathbb{N}} \) converges to \( h \). We claim that \( \{ f_n^a \}_{\lambda \in \mathbb{N}} \) converges to \( h \).

Let \( K = \{ x_1, \ldots, x_k \} \) be a finite subset of \( D, \varepsilon = \{ \varepsilon_1, \ldots, \varepsilon_k \} \) where \( \varepsilon_j \in \{0, 1\} \) for \( j = 1, \ldots, k \), and \( W = \{ h, K, \varepsilon \} : = \{ g \in B(D, \{0, 1\}) : |g(x_j) - h(x_j)| < \varepsilon_j \text{ for } j = 1, \ldots, k \} \) be a base neighborhood of \( h \), then there is a number \( m_0 \) such that \( K \subset C_n \) for \( i > m_0 \) and \( h_n \in W \) for \( s > m_0 \). Since \( f_n^a \upharpoonright K = h_n \upharpoonright K \) for each \( s > m_0, f_n^a \in W \) for each \( s > m_0 \). It follows that a sequence \( \{ f_n^a \}_{\lambda \in \mathbb{N}} \) converges to \( h \).

Since \( B(D, \{0, 1\}) \) is sequentially separable, there is a sequence \( \{ F_n = \{ f_n^{1, \lambda}, \ldots, f_n^{i, \lambda} \} : n \in \mathbb{N} \} \) such that for each \( n, F_n \subset S_n \) and \( \cup_{n \in \mathbb{N}} F_n \) is a countable sequentially dense subset of \( B(D, \{0, 1\}) \).

For \( 0 \in B(D, \{0, 1\}) \) there is a sequence \( \{ f_n^{i, \lambda} \}_{\lambda \in \mathbb{N}} \subset \cup_{n \in \mathbb{N}} F_n \) such that \( \{ f_n^{i, \lambda} \}_{\lambda \in \mathbb{N}} \) converges to \( 0 \). Consider a sequence \( \{ C_n^{i, \lambda} : j \in \mathbb{N} \} \). Then

1. \( C_n^{i, \lambda} \in C_n \);
2. \( \{ C_n^{i, \lambda} : j \in \mathbb{N} \} \) is a \( \gamma \)-cover of D.

Indeed, let \( K \) be a finite subset of \( D \) and \( U = \{ 0, K, \{0\} \} \) be a base neighborhood of \( 0 \), then there is a number \( j_0 \) such that \( f_n^{i, \lambda} \in U \) for every \( j > j_0 \). It follows that \( K \subset C_n^{i, \lambda} \) for every \( j > j_0 \). Hence, \( \{ A_n^{i, \lambda} = id(C_n^{i, \lambda}) : j \in \mathbb{N} \} \in BΓ \) in the space X, a contradiction. \( \square \)

Let \( \mu = \min \{ \kappa : 2^\kappa \notin \text{selective sequentially separable} \} \). It is well-known that \( p \leq \mu \leq q \) (see [3]).

**Theorem 5.2.** \( \mu = \min \{ b, q \} \).

**Proof.** Let \( \kappa < \min \{ b, q \} \). Then, by Proposition 3 in [13], \( 2^\kappa \) is a sequentially separable space.

Let X be a set of reals such that \( |X| = \kappa \) and X be a Q-set.

Analogous to the proof of implication (2) \( \Rightarrow (1) \) in Theorem 3.1, we can claim that \( B(X, \{0, 1\}) = 2^X = 2^\kappa \) is sequentially separable.

It follows that \( \mu \geq \min \{ b, q \} \).

Since \( \mu \leq q \), we suppose that \( \mu > b \) and \( b < q \). Then, by Theorem 5.1, \( 2^b \) is not sequentially separable. It follows that \( \mu = \min \{ b, q \} \). \( \square \)

In [3], Question 4.12: is it the case \( \mu \in \{ p, q \} \)?
A partial positive answer to this question is the existence of the following models of set theory (Theorem 8 in [1]):
1. $\mu = p = b < q$;
2. $p < \mu = b = q$;
and
3. $\mu = p = q < b$.

The author does not know whether, in general, the answer can be negative. In this regard, the following question is of interest.

**Question.** Is there a model of set theory in which $p < b < q$?

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