A nonlinear Korn inequality based on the Green-Saint Venant strain tensor

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May 24, 2016

Abstract A nonlinear Korn inequality based on the Green-Saint Venant strain tensor is proved, whenever the displacement is in the Sobolev space $W^{1,p}$, $p \geq 2$, under Dirichlet conditions on a part of the boundary. The inequality can be useful in proving the coercivity of a nonlinear elastic energy.

Keywords nonlinear Korn inequality, geometric rigidity lemma, finite elasticity, coercivity

Mathematics Subject Classification (2010) 74B20 · 74A05

1 Introduction

Korn inequality is one of the pillars of linear elasticity and it is well-known for more than a century. In the most classical version it writes

$$\|\nabla u\|_2 \leq c\|e(u)\|_2$$ (1)

provided that the displacement $u$ vanishes on a sufficiently large part of the boundary. Here $e(u)$ denotes the symmetric part of $\nabla u$. Since $e(u)$ is the measure of the strain in linear theories, Korn inequality provides a control of the deformation gradient by means of the strain. The $L^p$ version of the inequality, namely

$$\|\nabla u\|_p \leq c\|e(u)\|_p$$ (2)

if $u$ vanishes on a part of the boundary, followed more recently by general results about singular integral operators, see [1].

However, in nonlinear elasticity a typical measure of the strain is given by the so-called Green-Saint Venant (or Green-Lagrange) strain tensor

$$E(u) = \frac{1}{2}(F^T F - I) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u),$$

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where $F = \nabla u$ is the deformation gradient and $I$ denotes the identity tensor (see for instance [1, Sect. 1.8]). Hence for a hyperelastic material it is customary to write the energy density as a function of $E(u)$. Unfortunately, such an assumption, although very reasonable from a mechanical viewpoint, is a source of difficulties on the mathematical side: both the weakly lower semicontinuity and the coercivity of the elastic energy can be a hard task, or even fail to hold.

Specifically, in proving an existence result for a hyperelastic body subject to external loads and mixed boundary conditions, the Direct Method of the Calculus of Variations can be a very powerful tool. Therefore, even if the elastic energy density is well-behaved (for instance if it is smooth and polyconvex), it is crucial to have some coercivity conditions in order to construct a bounded minimizing sequence. Hence it is very important to control the norm of the displacement by the energy. Ultimately, one needs to control the displacement by the Green-Saint Venant tensor, that is, a nonlinear version of the Korn inequality with boundary conditions. The inequality can be useful also in other applications: see for instance [7], where a similar tool allows to prove the existence of a minimizer for a nonlinear elastic problem by the implicit function theorem.

In the present note we prove the inequality

$$\|\nabla u\|_p^p \leq c\|E(u)\|_{p/2}^{p/2} \quad (3)$$

in the case of a bounded Lipschitz body and for $u$ in the Sobolev space $W^{1,p}$, $p \geq 2$, and $\det(\nabla u(x) + I) > 0$. The result holds under weak assumptions on the boundary conditions, namely it is enough for the displacement $u$ to vanish on a part of the boundary with positive surface measure. We notice that, with a perhaps more simple notation, equation (3) can be rewritten as

$$\|F - I\|_p^p \leq C\|F^TF - I\|_{p/2}^{p/2}, \quad p \geq 2, \det F > 0.$$ 

Interestingly, as observed by the anonymous reviewer, (2) does not follow from (3) by linearization, since the exponent on the right-hand side is $p/2$.

A similar inequality has been proved in [2] for the case $p = 2$. Here we generalize that result to the case $p \geq 2$. An essential ingredient of the proof is the $L^p$ version of the celebrated geometric rigidity lemma of [5], originally stated for $L^2$, which has been generalized for $1 < p < \infty$ in Conti & Schweizer [3, Sec. 2.4].

2 Notation

Given an $n \times n$-matrix $F$, we will denote by $|F|$ its Frobenius norm, that is

$$|F| = \sqrt{\text{tr}(F^TF)} = \sum_{i,j=1}^n F_{ij}^2,$$

where $F^T$ denotes the transpose of the matrix $F$. We will denote by $SO(n)$ the set of rotations in $\mathbb{R}^n$, that is

$$R \in SO(n) \iff R^T R = I \text{ and } \det R = 1,$$
Let $\Omega$ be a bounded connected open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. Concerning the displacement, we assume that $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $2 \leq p < \infty$, endowed with the usual norm

$$\|u\|_{1,p} = (\|u\|_p^p + \|\nabla u\|_p^p)^{1/p}.$$  

We will consider homogeneous Dirichlet boundary conditions on a part of the boundary: let $\Gamma \subset \partial \Omega$ be a subset of the boundary of $\Omega$ such that

$$|\Gamma| := H^{n-1}(\Gamma) > 0,$$

where $H^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. We introduce the set

$$W^{1,p}_\Gamma := \{ u \in W^{1,p}(\Omega; \mathbb{R}^n) : u(x) = 0 \text{ $H^{n-1}$-a.e. in } \Gamma \}$$

endowed with the norm $\| \cdot \|_{1,p}$, where the equality has to be understood in the sense of traces of Sobolev functions. Notice that $W^{1,p}_\Gamma$ is a closed linear subspace of $W^{1,p}(\Omega; \mathbb{R}^n)$.

A crucial lemma is the following (see also [2, Lemma 1]).

**Lemma 1** There is a constant $c > 0$ such that

$$\text{dist}(F, SO(n)) \leq c|F^T F - I|^{1/2}$$

for any matrix $F$ with $\det F > 0$.

**Proof** Since $\det F > 0$, it is a well-known consequence of the Polar Factorization Theorem (see [1, Theorem 3.2-2]) that

$$\text{dist}(F, SO(n)) = |(F^T F)^{1/2} - I|.$$

Moreover, denoting by $v_1, \ldots, v_n$ the eigenvalues of the matrix $(F^T F)^{1/2}$, by the equivalence of the Frobenius and the spectral norm it follows that

$$c_1 \max_{1 \leq i \leq n} |v_i - 1| \leq |(F^T F)^{1/2} - I| \leq c_2 \max_{1 \leq i \leq n} |v_i - 1|$$

for some $c_1, c_2 > 0$. Since $v_1, \ldots, v_n \geq 0$, one has

$$|(F^T F)^{1/2} - I| \leq c_2 \max_{1 \leq i \leq n} |v_i - 1| \leq c_2 \max_{1 \leq i \leq n} |v_i^2 - 1|^{1/2} = \frac{c_2}{c_1} |F^T F - I|^{1/2}$$

and the proof is complete.
3 The nonlinear Korn inequality

We begin by proving the following lemma, which can also be found in [4, Lemma 3.3].

**Lemma 2** Let $\overline{K}$ denote the closure of the cone

$$K = \{ t(I - R) : t > 0, R \in SO(n) \}.$$ 

Then there exists $c > 0$ such that

$$\forall F \in \overline{K} : |F| \leq c \min_{z \in \mathbb{R}^n} \left( \int_{\Gamma} |F(x) - z|^2 \, dH^{n-1}(x) \right)^{1/2}.$$ 

**Proof** First of all we notice that

$$\overline{K} = K \cup \text{Skw}(n),$$

where $\text{Skw}(n)$ denotes the set of skew-symmetric tensors of order $n$. Indeed, assume that $F \in \overline{K} \setminus K$ with $F \neq 0$; then there is an unbounded sequence $(t_h)$ with $t_h > 0$ such that

$$t_h(I - R_h) \to F \quad \text{as} \ h \to \infty,$$

where $R_h \in SO(n)$. In particular, $|I - R_h| \sim 1/t_h$ as $h \to \infty$. Representing an orthogonal tensor as the exponential of a skew-symmetric tensor, we can write

$$R_h = \exp \left( t_h^{-1} A \right) + o(t_h^{-1}) \quad \text{as} \ h \to \infty,$$

for some $A \in \text{Skw}(n)$. Then we have

$$F = \lim_{h \to \infty} t_h \left[ I - \exp \left( t_h^{-1} A \right) - o(t_h^{-1}) \right] = -\frac{d}{dt} \exp(tA) \bigg|_{t=0} = -A,$$

hence $F$ is skew-symmetric.

By (5) it follows that

$$F \in \overline{K}, F \neq 0 \quad \Rightarrow \quad \dim \ker F \leq n - 2.$$

Moreover it is easy to check that

$$\int_{\Gamma} |F(x) - z|^2 \, dH^{n-1}(x) \quad \text{has minimum for} \quad \overline{z} = \frac{1}{|\Gamma|} \int_{\Gamma} F(x) \, dH^{n-1}(x).$$

Now, assume by contradiction that for any $j \geq 1$ there exists $F_j \in \overline{K}$ such that $|F_j| = 1$ and

$$\frac{1}{J} > \int_{\Gamma} |F_j(x) - \overline{z}_j|^2 \, dH^{n-1}(x), \quad \overline{z}_j = \frac{1}{|\Gamma|} \int_{\Gamma} F_j(x) \, dH^{n-1}(x).$$

Then there exists $F \in \overline{K}$ such that $|F| = 1$ and $F_j \to F$ up to a subsequence. By continuity it follows that

$$\int_{\Gamma} |F(x) - \overline{z}|^2 \, dH^{n-1}(x) = 0, \quad \overline{z} = \frac{1}{|\Gamma|} \int_{\Gamma} F(x) \, dH^{n-1}(x),$$

whence $F(x) = \overline{z}$ for $H^{n-1}$-a.e. $x \in \Gamma$.

Since $\Gamma$ has positive measure and $\Omega$ is Lipschitz, then there exist at least $n - 1$ pairs $(x_i, y_i)$ such that $x_i, y_i \in \Gamma$ and $(x_i - y_i)$ are linearly independent. In particular, $F(x_i - y_i) = 0$, hence $\dim \ker F \geq n - 1$, a contradiction.
Now we are ready to state and prove the main theorem.

**Theorem 1** Let $\Omega$ be a bounded connected open subset of $\mathbb{R}^n$ with Lipschitz boundary and $2 \leq p < \infty$. Let $\Gamma \subset \partial \Omega$ be a subset of the boundary of $\Omega$ such that $\mathcal{H}^{n-1}(\Gamma) > 0$.

Then there exists a constant $c > 0$ such that

$$\|\nabla u\|_p^p \leq c\|E(u)\|_p^p$$

for every $u \in W^{1,p}_\Gamma$ such that $\det(\nabla u(x) + I) > 0$ for a.e. $x \in \Omega$.

**Proof** The continuity of the trace operator gives

$$\int_{\Gamma} |f(x)|^p d\mathcal{H}^{n-1}(x) \leq c_1 \int_{\Omega} (|f(x)|^p + |\nabla f|^p) dx$$

(6)

for every $f \in W^{1,p}(\Omega)$, where $c_1 > 0$ is some constant depending on $\Omega$ and $\Gamma$.

Let $u \in W^{1,p}_\Gamma$ and consider the vector field $\phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ defined as $\phi(x) = u(x) + x$. For every $R \in SO(n)$ define

$$z_R := \frac{1}{|\Omega|} \int_{\Omega} (\phi(x) - Rx) dx.$$

By (6) and Poincaré-Wirtinger inequality one has

$$\int_{\Gamma} |\phi(x) - Rx - z_R|^p d\mathcal{H}^{n-1}(x) \leq c_2 \int_{\Omega} |\nabla \phi(x) - R|^p dx.$$ (7)

Notice that in the left-hand side one can replace $\phi(x)$ with the identity, in view of the boundary conditions. Moreover, by Lemma 2 it follows that

$$|I - R| \leq c_3 \left( \min_{z \in \mathbb{R}^n} \int_{\Gamma} |(I - R)x - z|^2 d\mathcal{H}^{n-1}(x) \right)^{1/2}.$$ (8)

In particular, choosing $z = z_R$, by Hölder inequality (with $p \geq 2$) and (7) it follows that

$$|I - R|^p \leq c_4 |I|^p \int_{\Gamma} |(I - R)x - z_R|^p d\mathcal{H}^{n-1}(x) \leq c_4 \int_{\Omega} |\nabla \phi(x) - R|^p dx.$$

hence

$$|I - R|^p = |\Omega| |I - R|^p \leq c_5 \|\nabla \phi - R\|_p^p.$$

Then one has

$$\|\nabla \phi - I\|_p \leq \|\nabla \phi - R\|_p + |I - R|^p \leq c_6 \|\nabla \phi - R\|_p$$

(9)

for every $R \in SO(n)$.

Now we need the celebrated geometric rigidity lemma by Friesecke, James & Müller [5], which holds for $1 < p < \infty$, as pointed out by Conti & Schweizer [3, Sec. 2.4]:

there is a constant $K > 0$ such that

$$\min_{R \in SO(n)} \|\nabla \phi - R\|_p \leq K \int_{\Omega} \text{dist}^p(\nabla \phi(x), SO(n)) dx$$


for any $\phi \in W^{1,p}(\Omega; \mathbb{R}^n)$.

By assumption, $\det \nabla \phi > 0$ a.e. in $\Omega$, hence, combining Lemma 1 with the geometric rigidity lemma it follows that there exists $R \in SO(n)$ such that

$$\|\nabla \phi - R\|^p \leq c_K \int_{\Omega} |\nabla \phi(x)^T \nabla \phi(x) - I|^p/2 \, dx = c_7 \|\nabla \phi^T \nabla \phi - I\|^{p/2}$$

and, by (4),

$$\|\nabla u\|^p = \|\nabla \phi - I\|^p \leq c_6 \|\nabla \phi - R\|^p \leq c_6 c_7 \|\nabla \phi^T \nabla \phi - I\|^{p/2} = C \|\mathcal{E}(u)\|^{p/2},$$

which concludes the proof.

Remark 1 In proving the previous theorem, we also proved the following consequence of the $L^p$ version of the geometric rigidity lemma:

there is a constant $C > 0$ such that

$$\|\nabla \phi - I\|^p \leq C \|\text{dist}(\nabla \phi, SO(n))\|^p$$

for every $\phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $\phi(x) = x$ on $\Gamma$.

Acknowledgements This research is partially supported by GNFM (Gruppo Nazionale di Fisica Matematica) of INdAM (Istituto Nazionale di Alta Matematica).

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