The driven-dissipative Bose–Hubbard dimer: phase diagram and chaos

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Abstract

We present the phase diagram of the mean-field driven-dissipative Bose–Hubbard dimer model. For a dimer with repulsive on-site interactions \((U > 0)\) and coherent driving, we prove that \(\mathbb{Z}_2\)-symmetry breaking, via pitchfork bifurcations with sizable extensions of the asymmetric solutions, require a negative tunneling parameter \((f < 0)\). In addition, we show that the model exhibits deterministic dissipative chaos. The chaotic attractor emerges from a Shilnikov mechanism of a periodic orbit born in a Hopf bifurcation and, depending on its symmetry properties, it is either localized or not.

The Bose–Hubbard model is a celebrated fundamental quantum mechanical model that accounts for boson dynamics in a lattice [1]. It successfully describes the interplay between the hopping of particles between neighboring sites of the lattice (with rate \(f\)) and on-site interactions. Such interactions appear as multi-boson terms in the Hamiltonian with interaction energy \(U\). Importantly, this model accurately explains the superfluid to Mott insulator phase transitions, that has been experimentally demonstrated in ultracold atomic lattices [2]. A minimal building block in this context is the so-called Bose–Hubbard dimer, consisting of only two interacting sites, also known as the bosonic Josephson junction [3, 4]. Furthermore, the Bose–Hubbard dimer lies at the basis of the number of striking phenomena such as the Josephson effect [5], self-trapping [6] and symmetry breaking [7].

In recent years there has been a growing interest in understanding open quantum systems, where the bosons can be added and destroyed by means of external driving and dissipation mechanisms [8–11]. In this context, photonic systems have attracted much attention since photons in optical cavities can be injected through an external driving laser, and dissipation comes in as a natural consequence of optical cavity losses. The driven-dissipative Bose–Hubbard dimer has been realized in a number of experimental systems, such as semiconductor microcavities [12, 13], superconducting circuits [14] and photonic crystals [15], where the interactions take the form of Kerr-type optical nonlinearities. In many regards, these optical systems constitute outstanding platforms for studying many-body phenomena in open quantum systems [16]. Among them, dissipative phase transitions [17, 18] are an especially exciting open topic, because they provide a conceptual basis for the understanding and the prediction of new collective states, both steady and dynamical ones, with the latter accounting for collective coherent oscillations.

As is well known, phase transitions are characterized by critical phenomena, which can emerge in the thermodynamic limit, i.e. with a large photon number in optical cavities. Remarkably, even single-mode cavities —i.e. with no spatial degrees of freedom—may display phase transitions including optical bistability (which is of first order) [18, 19] and the emergence of an oscillation threshold in, e.g. two-photon pumped Kerr resonators [20] or laser devices [21] (which is a second-order phase transition). As pointed out in [20], driven-dissipative phase transitions are strongly related to mean-field semiclassical solutions in such a thermodynamic limit: these are known as bifurcations of a classical vector field from a nonlinear dynamical point of view; see, e.g. [22, 23].

Recent examples of complex dynamics found in phase diagrams of driven-dissipative nonlinear cavities include oscillating phases [24] and exotic attractors [25]. Furthermore, Lorenz-type chaos has been found in a
theoretical study of the Gross–Pitaevsky equation [26], which describes a double potential well Bose–Einstein condensate under incoherent pumping. To the best of our knowledge, there are no other studies of dissipative chaos in this system. Moreover, dissipative chaos has not been predicted so far in the coherently driven regime, which is the one that is well described by the driven-dissipative Bose–Hubbard dimer. In this work, we focus on coherent driving, show that dissipative chaos exist in this model and that it is intimately related to Shilnikov homoclinic bifurcations. In addition, we present the phase diagram as a result of the comprehensive analysis of local bifurcations of steady states and their symmetry properties.

In the framework that rotates with the driving frequency \( \omega p \), the mean-field approximation of the coherently driven Bose–Hubbard model can be written as (see, e.g. [27]):

\[
\begin{align*}
\frac{d\alpha_1}{d\tau} &= (-\Delta - \frac{i\gamma}{2} + 2U|\alpha_1|^2)\alpha_1 - J\alpha_2 + F, \\
\frac{d\alpha_2}{d\tau} &= (-\Delta - \frac{i\gamma}{2} + 2U|\alpha_2|^2)\alpha_2 - J\alpha_1 + F,
\end{align*}
\]

where \( \alpha_1, \alpha_2 \in \mathbb{C} \) are the electric field envelopes in each optical cavity, which are linearly coupled by the tunneling parameter \( J \). Both cavities are coherently driven by two fields with the same amplitude and phase, corresponding to the same parameter \( F \) in the two equations, and detuning \( \Delta = \omega_p - \omega_c \), where \( \omega_c \) is the cavity frequency. The cavities have a Kerr-type nonlinearity characterized by \( U \), which is assumed to be fast compared to the dissipation rate \( \gamma \).

We apply to system (1) the coordinate transformation

\[(\alpha_1, \alpha_2) \mapsto (A, B) = (-2i\alpha_1^* \sqrt{|U|}/\gamma, -2i\alpha_2^* \sqrt{|U|}/\gamma),\]

where \( \alpha_1^* \) is the complex conjugate of \( \alpha_1 \). With time rescaled to \( \tau = 2t/\gamma \), we obtain a rescaled Bose–Hubbard model as the vector field

\[
\begin{align*}
\frac{dA}{dt} &= -A + i(\delta + \text{sign}(U)|A|^2)A + i\kappa B + f, \\
\frac{dB}{dt} &= -B + i(\delta + \text{sign}(U)|B|^2)B + i\kappa A + f,
\end{align*}
\]

for the (rescaled) envelopes \( A, B \in \mathbb{C} \). Here

\[
\kappa = -\frac{2J}{\gamma}, \quad f = 4F \frac{\sqrt{|U|}}{\gamma^{3/2}} \quad \text{and} \quad \delta = -\frac{2\Delta}{\gamma}
\]

are the (rescaled) coupling, drive and detuning parameters, respectively. The main purpose of our rescaling is to explicitly consider the thermodynamic limit, namely by absorbing the nonlinear constant \( U \). As a result, the rescaled equations (2) are independent of the photon number in the cavities. Hence, denoting the thermodynamic parameter as \( N [28] \), our rescaling leads to \( F \propto \sqrt{N} \) and \( \alpha \propto \sqrt{N} \), in agreement with [28]. Note that the photon number \( n = |\alpha|^2 \propto N \), confirming that \( N \to \infty \) corresponds to a well-defined thermodynamic limit with an infinite number of photons.

Since system (2) is invariant under the transformation

\[(A, B, U, \delta, \kappa) \mapsto (A^*, B^*, -U, -\delta, -\kappa), \]

all results for positive \( U \) directly translate to those for negative \( U \); hence, we consider \( U > 0 \) and \( \text{sign}(U) = 1 \) in (2) from now. Note that, although \( U \) is fixed by the material of the cavity, both the magnitude and sign of \( \kappa \) can be changed by means of photonic design in some particular geometries, for example, with potential barrier engineering for the case of photonic crystal nanocavities [15, 29]. Because \( \delta \) can be altered at will during the experiment by changing the frequency of the driving laser, states of negative interaction energy in an otherwise positive-\( U \) system can be assessed via the parameter transformations \( \kappa \mapsto -\kappa \) and \( \delta \mapsto -\delta \).

In addition, system (2) is invariant under the phase-space mirror symmetry of exchanging the two cavities, that is, of swapping \( A \) and \( B \), which gives rise to \( \mathbb{Z}_2 \)-equivariance of solutions [23]. More precisely, this is described by the transformation \( A \leftrightarrow B \text{ (or } \alpha_1 \leftrightarrow \alpha_2 \text{ in system (1)}) \), while in [27] the \( \mathbb{Z}_2 \)-symmetry is given by \( \alpha_1 \leftrightarrow -\alpha_2 \) since it is the anti-symmetric mode which is excited. Hence, solutions (equilibria, periodic orbits, trajectories) can be split into two major groups: symmetric and asymmetric ones. For a symmetric solution the field intensities and phases in both cavities are the same. The set of all symmetric solutions forms the symmetry subspace given by \( A = B \), which is an invariant subspace of system (2). An asymmetric solution, on the other hand, is one where the intensity and/or the phase in both cavities differ, that is, \( A \neq B \). Note that asymmetric solutions often come in pairs, one being the mirror image of the other under swapping \( A \) and \( B \).

We start our analysis of system (2) by characterizing the equilibria and their bifurcations as the drive parameter \( f \) increases at different values of the detuning \( \delta \). Thus, in the first stage of our analysis, we showcase the situation for two values of \( \kappa \) with opposite signs, representing two distinct optical devices: for \( \kappa = -3.5 \) we
model a standard photonic dimer with positive tunneling rate, while $\kappa = 3.5$ accounts for negative tunneling devices, which can be implemented using coupled photonic crystal cavities [15, 29]. Figure 1 shows phase diagrams of $|A|^2$ against $f$ for three different values of $\delta$ for the two cases of a negative and positive coupling parameter $\kappa$ in the left and right columns, respectively. The $\delta$ values that were chosen in figure 1 showcase the progressive complexity that arises as the detuning decreases, which occurs for different values of detuning for the two chosen $\kappa$-values; see already figure 2. The corresponding phase diagrams of $|B|^2$ against $f$ are the same due to $Z_2$-equivariance. All bifurcations and branches of solutions have been computed with the numerical continuation package AUTO07P [30]. When $f = 0$, there is the stable equilibrium given by $A = B = 0$, which gives rise to a monotone branch of stable symmetric equilibria for any $f$ provided the detuning $\delta$ is sufficiently large. However, when $\delta = 0$ and for increasingly negative $\delta$, one finds bifurcations on the branch of symmetric equilibria.

For negative $\kappa$, as in figures 1(a)–(c) for $\kappa = -3.5$, we observe for $\delta = 0$ an interval of bistability that is delimited by two saddle-node bifurcations $S$ of symmetric equilibria (figure 1(a)). In this interval there are three symmetric equilibria that form a hysteresis loop: a stable equilibrium corresponding to low intensity in both cavities, an intermediate saddle equilibrium, and a stable equilibrium with higher intensity in both cavities. As $\delta$ decreases, the interval of bistability increases in size. While bistability can be encountered in one-cavity systems, spontaneous $Z_2$-symmetry breaking requires coupled cavities with mirror symmetry; this is the case for the Bose–Hubbard model (2) and spontaneous symmetry breaking phase transitions are known to exist in the form of pitchfork bifurcations [27, 31]. As figure 1(b) for $\delta = -6$ shows, we find two such symmetry breakings at the points labeled $P$ on the low intensity branch. The left pitchfork bifurcation point $P$ is subcritical and gives rise to a pair of unstable asymmetric equilibria, while the right point $P$ is supercritical and gives rise to a pair of stable asymmetric equilibria (one with $|A|^2 < |B|^2$ and the other with $|A|^2 > |B|^2$); these two branches come together at a pair of saddle-node bifurcations of asymmetric equilibria denoted $S^*$. Let us point out that finding the criticality of a local bifurcation requires a center-manifold reduction, a procedure that is quite complicated. The use of continuation packages like AUTO allows the user to determine the criticality of the bifurcation by inspection of the solutions and their stability in the vicinity of the critical point. Thus, if the bifurcation is supercritical, we generally expect the onset or termination of multi-stability between the asymmetric equilibria that arise from the bifurcation: this is for instance the case around the rightmost pitchfork bifurcation point $P$ in panel (bz) of figure 1, which is a supercritical one. Here, the coexistence between the stable asymmetric and unstable symmetric equilibria terminates as $f$ increases, and the symmetric equilibrium becomes stable.

\[ \frac{\partial}{\partial t} \mathbf{u} = \mathbf{A} \mathbf{u} + \mathbf{F}(\mathbf{u}); \quad \mathbf{u}(0) = \mathbf{u}_0 \]

where $\mathbf{u} \in \mathbb{R}^n$ is the state vector and $\mathbf{A}$ is a matrix whose eigenvalues determine the linear stability of the equilibrium $\mathbf{u}_0$. The function $\mathbf{F}$ represents the nonlinear terms that give rise to bifurcations.

Figure 1. Phase diagram of system (3) of intensities $|A|^2$ against the drive parameter $f$ at different $\delta$ values with $\kappa = -3.5$ (left column) and $\kappa = 3.5$ (right column); shown are branches of stable equilibria (blue), of saddle equilibria with one unstable eigenvalue (cyan) and with two unstable eigenvalues (orange), and of stable periodic solutions (dark green). Shown are the points of saddle-node bifurcation of symmetric equilibria $S$, saddle-node bifurcation of asymmetric equilibria $S^*$, Pitchfork bifurcation $P$, and Hopf bifurcation $H$. Panels (bz) and (cz) show zooms of the area enclosed by the rectangle of the same color in panels (b) and (c), respectively.
Conversely, when crossing a subcritical pitchfork bifurcation, as the leftmost bifurcation point $P$ in panel (b), the stable symmetric equilibrium becomes unstable by the collision with a pair of unstable asymmetric equilibria. As a consequence, increasing $f$ past this point $P$, there is now no nearby attractor and the system moves away quickly towards some other attractor; in this case to either one of the pair of stable asymmetric equilibria that exists between the saddle-node bifurcation point $S^*$ and the rightmost pitchfork bifurcation point $P$ in panel (b). Notice that there is a small $f$-interval, between $S^*$ and the left point $P$, with four stable equilibria: higher and lower intensity symmetric equilibria, and a pair of asymmetric (lower-intensity) equilibria (figure 1(b)). Notice that between the two points $P$ there are still three stable objects: the two asymmetric equilibria and the higher-intensity symmetric equilibrium.

For an even lower detuning, as in figure 1(c) for $\delta = -6.2$, we find two pairs of supercritical Hopf bifurcations denoted $H$ on the stable branch of asymmetric equilibria. The Hopf bifurcation characterizes the onset of oscillatory behavior [22] and corresponds to the real part of a pair of complex conjugated eigenvalues crossing zero; it then accounts for a transition to limit cycle oscillations, whose frequency is given by the imaginary part of the eigenvalues. The Hopf bifurcations in our system give rise to a pair of stable asymmetric periodic orbits, which are represented in the phase diagram by their maxima in $|A|^2$. In the corresponding $f$-interval the asymmetric equilibria are now unstable. Note that the two asymmetric periodic orbits are new attractors that coexist with the stable higher intensity equilibrium.

The situation for $\kappa > 0$ is notably different: bistability of the symmetric state is, in fact, absent for all panels figures 1(d)–(f). Instead, for $\delta = 0$ we find symmetry breaking at two supercritical pitchfork bifurcations, which give rise to a pair of stable asymmetric equilibria in an quite large $f$-interval where the symmetric equilibrium is unstable (figure 1(d)). This makes this set of abnormal coupling conditions well suited for experimental implementations. Other driving configurations might also allow one to fulfill these conditions, but then the driving phase of the cavities needs to be adjusted properly such that the anti-symmetric state can be linearly excited [27].

As $\delta$ is decreased, as in figure 1(e) for $\delta = -3.0$, the pair of asymmetric stable equilibria that exists in this interval of symmetry breaking also exhibit saddle-node bifurcations $S^*$ that create bistability of asymmetric states and a pair of hysteresis loops. Perhaps surprisingly, there is an $f$-range where the intensity in one of the cavities is nearly zero. As figure 1(f) for $\delta = -4.5$ shows, a high negative detuning generates a pair of supercritical
Hopf bifurcation points labeled H, which create an $f$-interval with a pair of asymmetrical stable periodic solutions, while the asymmetric equilibria are unstable.

We extend our previous analysis to different values of $\delta$ by showcasing a phase diagram in the $(f, \delta)$-parameter plane, where each bifurcation point in figure 1 now corresponds to a bifurcation curve that bounds regions with qualitatively different behavior. Figure 2 shows the two phase diagram, for both $\kappa = -3.5$ and $\kappa = 3.5$, showing the curves $S, S^a, P$ and $H$ of saddle-node, pitchfork and Hopf bifurcations, respectively. The dashed horizontal lines correspond to the respective phase diagrams in just $f$ from figures 1(a)--(f); notice that they are located at different $\delta$ values for $\kappa$ positive and $\kappa$ negative because of the different locations of bifurcations curves and regions.

The bifurcation curves $S$ and $P$ are drawn from analytical expressions obtained by solving for the equilibria and the respective bifurcation conditions, see appendix for the formulas and their derivations. These calculations also show that the saddle-node bifurcation $S$ in the symmetry subspace (brown line) only occurs if $\delta < -\sqrt{3} - \kappa$ and $1.24081 < f$. At the points $(f, \delta) \approx (1.24081, -\sqrt{3} - \kappa)$ there is a cusp bifurcation $CP$ on the curve $S$ that corresponds to a sharp bound for bistability in the $(f, \delta)$-plane [23]. The region of bistable symmetric equilibria bounded by $S$ is shaded brown in both panels of figure 2; note the different position of this region for $\kappa = -3.5$ and for $\kappa = 3.5$. Additionally, we find that the curve of pitchfork bifurcation $P$ (purple curve), which bounds the shaded region with asymmetric equilibria, has a maximum at $\delta = -\sqrt{3} + \kappa$; hence, symmetry breaking only occurs if $\delta < -\sqrt{3} + \kappa$, which is equivalent to the bound found in [32] and also consistent with [33]. As an unexpected consequence, the condition for symmetry breaking at $P$ to be possible is that the detuning $\omega_H = (\omega_e - \kappa \gamma / 2)$, with respect to the antisymmetric mode, must exceed $\sqrt{3} / 2$ times the cavity linewidth. This is similar to the bistability condition, but the detuning is measured from the antisymmetric mode of the system, i.e. the one that cannot be linearly excited from the outside world. These bounds explain why the first phenomenon observed, as $\delta$ decreased, is bistability when $\kappa < 0$, while it is symmetry breaking when $\kappa > 0$; compare the two panels of figure 2 and see also figure 1. Notice that the curve $P$ is tangent to the curve $S$ at the saddle-node pitchfork point $SP$, to the right of the cusp point $CP$ for $\kappa < 0$ and to the left of $CP$ for $\kappa > 0$; moreover, for $\kappa < 0$ the curve $P$ changes criticality at the point $DP$, from which the curve $S^a$ of saddle-node bifurcations of asymmetric equilibria emerges.

The curves $S^a$ and $H$ in figure 2 of saddle-node and Hopf bifurcations or asymmetric equilibria are computed numerically by continuation techniques. They bound regions of bistability and of stable asymmetric periodic solutions, respectively. Notice that for $\kappa < 0$ the curve $H$ emanates from a saddle-node pitchfork point $SP$ and is supercritical throughout, while for $\kappa > 0$ we find a change of criticality of $H$ at a generalized Hopf bifurcation (labeled GH) [23]. The considerable amount of multistability of the Bose–Hubbard model (2) is represented by the overlap between the different shaded region of figure 2. For even lower negative values of $\delta$ than shown in figure 2 one can find up to nine equilibria, as pointed out in [31].

As $\delta$ is decreased for both signs of $\kappa$, the asymmetric stable periodic orbits created at the supercritical Hopf bifurcation $H$ exhibit bifurcations scenarios to more complex and chaotic dynamics; the region of complex dynamics is indicated by gray shading in figure 2. Its exact boundary is formed by an intricate arrangement of accumulating bifurcation curves, including period-doubling and different types of homoclinic and heteroclinic bifurcations [34]. Notice that for $\kappa < 0$ periodic solutions become more complex almost immediately, while for $\kappa > 0$ we find a quite large region of stable periodic solutions. These periodic solutions exhibit a period-doubling cascade as parameters are changed, thus generating regions in the $(f, \delta)$-parameter plane were chaotic behavior is exhibited in system (2). We conclude by focussing on two particular related types of chaotic behavior in the Bose–Hubbard model. Both are associated with Shilnikov bifurcations, but with different global manifestations, giving rise to either asymmetric chaotic dynamics localized in one of the cavities, or symmetric chaotic dynamics with irregular switching between the two cavities.

At a homoclinic bifurcation a system of differential equations possesses a special trajectory, called a homoclinic orbit, that converges both in backward and forward time to a saddle equilibrium. For the special case that the equilibrium is a saddle focus (with a pair of complex conjugate eigenvalues), a celebrated result by Shilnikov [35] states that (under certain eigenvalue conditions) infinitely many periodic orbits and chaotic behavior can be found nearby. This phenomenon is now referred to as a Shilnikov bifurcation. We find Shilnikov bifurcations both to asymmetric saddle foci and to symmetric saddle focus, as well as two types of nearby chaotic attractors.

Figure 3 shows both cases of an asymmetric and of a symmetric Shilnikov bifurcation and associated chaotic attractors. For $\kappa < 0$ in figure 3(a), there exist asymmetric Shilnikov homoclinic orbits to each of the pair of asymmetric equilibria, one either side of the symmetry line $|A|^2 = |B|^2$. For nearby $f$ as in figure 3(b), we find a pair of chaotic asymmetric and localized attractors. The time traces in panels (c1), (c2) show that this type of localized chaotic dynamics is characterized by a dominance in intensity of one of the two cavities for all time. For $\kappa > 0$, on the other hand, we find a pair of Shilnikov homoclinic orbits to a single symmetric saddle equilibrium (figure 3(d)). This has the consequence that the associated chaotic attractor in figure 3(e) is much larger and no
longer localized as it crosses the line $|A|^2 = |B|^2$. As the time traces in panels (f1), (f2) clearly show, this type of chaotic dynamics is characterized by episodes of dominance in intensity of one of the two cavities for some time, with rapid and irregular switching events of the intensity to the other cavity. Thus, the sign of $\kappa$ is pivotal in creating qualitatively different chaotic behavior: for $\kappa < 0$, the energy of chaotic fluctuations is concentrated in one of the cavities for all time; while, for $\kappa > 0$, there is chaotic dynamics with unpredictable switching of energy concentration between the two cavities. This type of chaotic behavior for positive $\kappa$ is expected to manifest itself in an experimental or quantum simulation setting as observably shorter switching times between cavities than when $\kappa$ is negative and any switching is dominated by tunneling effects. We remark that a local analysis of equilibria alone is not able to predict and explain this global switching behavior of the system.

Let us stress that some of the phases identified here are expected to be found in experiments. Indeed, coupled photonic crystal nanocavities similar to the ones reported in [15] but under coherent excitation are currently under study, revealing asymmetric localized states that correspond to the symmetry broken states in the purple region of figure 2; these results will be published elsewhere. On the other hand, a direct measurement of the limit cycle dynamics should be very challenging due to the ultrashort limit cycle periods at play, of the order of the intercavity coupling time (in the picosecond range). Recently, (indirect) experimental evidence of limit cycles at the mode switching of a coupled nanocavity laser has been reported thanks to photon statistics measurements [36]. This being said, we expect that the chaotic dynamics of figure 3(e) could lead to a much slower timescale for the switching. For instance, the chaotic switching in figures 3(f1), (f2) exhibits a residence time that is much longer than the fast oscillations: we expect this switching time to approach the nanosecond time scale, which could be directly observable.

In conclusion, our systematic bifurcation analysis produced a comprehensive phase diagram description of both equilibria and bifurcating non-equilibrium solutions of the driven-dissipative Bose–Hubbard dimer model. In particular, we were able to identify bifurcations to periodic solutions, and subsequent localized and
non-localized chaotic behavior near homoclinic bifurcations of Shilnikov type. Crucial to the dynamics is the fact that the two cavities are coherently driven, which means that the system remains four-dimensional since there is no phase freedom as in active cavities. Interestingly, in a system with positive on-site interaction energy ($U > 0$), non-localized Shilnikov chaos can be observed for negative tunneling parameter ($f < 0$), in a bifurcation cascade whose details will be reported elsewhere.

Our results should accurately describe the very large photon number—i.e. thermodynamic—limit of a quantum system, in the sense that the expectation values of the observables will tend to the mean field solution. As discussed in [27], we can expect the absence of multistability in a quantum system, because the full quantum solution is always unique. This can be interpreted as the metastability of each branch of mean-field solutions in the presence of fluctuations: a quantum trajectory stays close to a given solution branch but eventually jumps to another branch of stable equilibria. As a result the expectation value is single-valued overall, even though the mean field limit exhibits multiple solutions. However, we can expect a rich switching dynamics between intensity levels, which could eventually contain fingerprints of the underlying time-dependent nonlinear dynamical solutions, such as periodic or quasiperiodic orbits and chaotic attractors. Our mean-field dynamical results thus pave the way for studying complex non-equilibrium orbits—including quantum dissipative chaos—in the quantum master equation of the open Bose–Hubbard dimer.

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Appendix. Analytical expressions

Transforming system (2) into polar coordinates, it can be shown that the symmetric equilibria must satisfy the set of equations

\[
\begin{align*}
    f \cos(\phi) - r &= 0, \\
    r^2 - \frac{f \sin(\phi)}{r} + \delta + \kappa &= 0,
\end{align*}
\]

where $r = |A| = |B|$ and $\phi$ is the phase of the cavities. In particular, a condition for a saddle-node bifurcation is that one of the eigenvalues of the linearization of system (2) at the symmetric equilibrium becomes 0. Hence, we have the following necessary condition

\[
\frac{f^2 \sin^2(\phi)}{r^2} + 2fr \sin(\phi) + \frac{f \cos(\phi)}{r} = 0.
\]

Solving (A.1) and (A.2) for $f$ and $\delta$, one obtains

\[
\begin{align*}
f_S(\phi) &= \sqrt{-\left(1 + \tan^2(\phi)\right)} \left[2 \sin(\phi) \cos(\phi)\right], \\
\delta_S(\phi, \kappa) &= \frac{1}{2} \left(3 \tan(\phi) + \cot(\phi) - 2\kappa\right),
\end{align*}
\]

where $\kappa \in \mathbb{R}$ and $\phi \in \left(-\frac{\pi}{2}, 0\right)$. Hence, for a given $\kappa$, the phase $\phi$ parameterizes the saddle-node bifurcation curves $S$ in the $(f, \delta)$-parameter plane, as exemplified in figure 2 for the cases $\kappa = 3.5$ and $\kappa = -3.5$.

Furthermore, $f_S(\phi)$ and $\delta_S(\phi, \kappa)$ have a minimum and a maximum, respectively, where $\phi = -\frac{\pi}{6}$, that is,

\[
f_S\left(-\frac{\pi}{6}\right) = 2\sqrt{3} \approx 1.24081 \quad \text{and} \quad \delta_S\left(-\frac{\pi}{6}, \kappa\right) = -\sqrt{3} - \kappa.
\]

Thus, a saddle-node bifurcation of symmetric states can only occur if $\delta < -\sqrt{3} - \kappa$ and $f > 1.24081$, that is, at $(f, \delta) \approx (1.24081, -\sqrt{3} - \kappa)$ there is a cusp bifurcation point CP, shown in figure 2 for the cases $\kappa = 3.5$ and $\kappa = -3.5$. Furthermore, it follows from (A.3) that, for given $\kappa$ and $\delta < -\sqrt{3} - \kappa$, the phase $\phi$ at $S$ is

\[
\phi = \arctan\left(\frac{\delta + \kappa \pm \sqrt{-3 + (\delta + \kappa)^2}}{\delta}ight).
\]

We also derive conditions for pitchfork bifurcations (symmetry breaking) that, when solved together with (A.1) for $f$ and $\delta$, give
\begin{align}
    f_f (\phi, \kappa) &= \frac{\sqrt{\cos^2(\phi) + (2\kappa \cos(\phi) - \sin(\phi))^2}}{\cos^2(\phi) \sqrt{4\kappa - 2 \tan(\phi)}}, \\
    \delta_f (\phi, \kappa) &= \frac{-1 - 8\kappa^2 + 10\kappa \tan(\phi) - 3\tan^2(\phi)}{4\kappa - 2 \tan(\phi)},
\end{align}

where \( \kappa \in \mathbb{R} \) and \( \phi \in \left(-\frac{\pi}{2}, \arctan(2\kappa)\right). \) Hence, for a given \( \kappa \), the phase \( \phi \) parameterizes the pitchfork bifurcation curves \( P \) in the \((f, \delta)\)-parameter plane, as is also exemplified in figure 2 for the cases \( \kappa = 3.5 \) and \( \kappa = -3.5. \)

It follows from (A.5) that, for given suitable \( \kappa \) and \( \delta \), the phase \( \phi \) at \( P \) is

\begin{equation}
    \phi = \arctan \left( \frac{\delta + 5\kappa \pm \sqrt{3 + (\delta - \kappa)^2}}{3} \right). \tag{A.6}
\end{equation}

By inspecting (A.6), we find that necessarily \( \delta < -\sqrt{3 + \kappa} \) or \( \delta > \sqrt{3 + \kappa}. \) To show that the second inequality cannot be satisfied, we insert \( \phi \) from (A.6) into the argument \( 4\kappa - 2 \tan(\phi) \) of the square root in (A.5) to get

\begin{equation}
    -2(\delta - \kappa \pm \sqrt{3 + (\delta - \kappa)^2}) = \frac{3}{\kappa}.
\end{equation}

For \( \delta > \sqrt{3 + \kappa} \) the expression (A.7) is negative and (A.5) does not have a solution.

Notice that for the case \( \kappa = 0 \) of no coupling, the two curves \( S \) and \( P \) given by (A.3) and (A.5) are equal.

Finally, the codimension-two point \( \text{SP} \) of simultaneous saddle-node and pitchfork bifurcation is given in the \((f, \delta)\)-parameter plane by

\begin{equation}
    (f, \delta) = \left( \sqrt{2(1 + \kappa^2)} \left( -\kappa + \sqrt{1 + \kappa^2} \right), -2\sqrt{1 + \kappa^2} \right). \tag{A.8}
\end{equation}

**References**

[1] Fisher M P A, Weichman P B, Grinstein G and Fisher D S 1989 Boson localization and the superfluid–insulator transition Phys. Rev. B 40 546–70

[2] Greiner M, Mandel O, Esslinger T, Hänsch T W and Bloch I 2002 Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms Nature 415 39–44

[3] Kellerman M E and J win A 2002 Bifurcation effects in coupled Bose–Einstein condensates Phys. Rev. A 66 013602

[4] Bruder C, Fazio R and Schön G 2005 The Bose–Hubbard model: from Josephson junction arrays to optical lattices Ann. Phys. 14 566–77

[5] Zibold T, Nickles E, Gross C and Oberthaler M K 2010 Classical bifurcation at the transition from Rabi to Josephson dynamics Phys. Rev. Lett. 105 204101

[6] Albizzati M, Gati R, Fölling J, Hunsmann S, Cristiani M and Oberthaler M K 2013 Direct observation of tunneling and nonlinear self-trapping in a single bosonic Josephson junction Phys. Rev. Lett. 95 010402

[7] Mahmud K W, Perry H and Reinhardt W P 2005 Quantum phase–space picture of Bose–Einstein condensates in a double well Phys. Rev. A 71 023615

[8] Carusotto I and Ciuti C 2013 Quantum fluids of light Rev. Mod. Phys. 85 299–366

[9] Schmidt S and Koch J 2013 Circuit qed lattices: towards quantum simulation with superconducting circuits Ann. Phys. 525 395–412

[10] Hartmann M J 2016 Quantum simulation with interacting photons J. Opt. 18 104005

[11] Noh C and Angelakis D G 2016 Quantum simulations and many-body physics with light Rep. Prog. Phys. 80 016401

[12] Abbarchi M et al 2013 Macroscopic quantum self-trapping and Josephson oscillations of exciton polaritons Nat. Phys. 9 275–9

[13] Lagoudakis K G, Pietka B, Wouters M, André R and Devaux–Pliéran B 2010 Coherent oscillations in an exciton–polariton Josephson junction Phys. Rev. Lett. 105 120403

[14] Rafferty J, Sadri D, Schmidt S, Türeci H E and Houck A A 2014 Observation of a dissipation-induced classical to quantum transition Phys. Rev. X 4 031043

[15] Hamel P, Haddadi S, Raineri F, Monnier P, Beaudoin G, Sagnes I, Levenson A and Yamamoto A M 2015 Spontaneous mirror–symmetry breaking in coupled photonic–crystal nanolasers Nat. Photon. 9 311–5

[16] Carmichael H J 2015 Breakdown of photon blockade: a dissipative quantum phase transition in zero dimensions Phys. Rev. X 5 031028

[17] Fitzpatrick M, Sundaresan N M, Lia C Y, Koch J and Houck A A 2017 Observation of a dissipative phase transition in a one-dimensional circuit qed lattice Phys. Rev. X 7 011016

[18] Fink T, Schade A, Hollling S, Schneider C and Imamoglu A 2017 Signatures of a dissipative phase transition in photon correlation measurements Nat. Phys. 14 365–9

[19] Rodriguez S R K et al 2017 Probing a dissipative phase transition via dynamical optical hysteresis Phys. Rev. Lett. 118 247402

[20] Minganti F, Biella A, Bartolo N and Ciuti C 2018 Spectral theory of Liouvillians for dissipative phase transitions Phys. Rev. A 98 042118

[21] Takemura N, Takiguchi M and Notomi M 2019 Low- and high-\( \beta \)lasers in class-a limit: photon statistics, linewidth, and the laser–phase transition analogy arXiv:1904.01743

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\[ f_f (\phi, \kappa) = \frac{\sqrt{\cos^2(\phi) + (2\kappa \cos(\phi) - \sin(\phi))^2}}{\cos^2(\phi) \sqrt{4\kappa - 2 \tan(\phi)}}, \]

\[ \delta_f (\phi, \kappa) = \frac{-1 - 8\kappa^2 + 10\kappa \tan(\phi) - 3\tan^2(\phi)}{4\kappa - 2 \tan(\phi)}, \tag{A.5} \]

where \( \kappa \in \mathbb{R} \) and \( \phi \in \left(-\frac{\pi}{2}, \arctan(2\kappa)\right). \) Hence, for a given \( \kappa \), the phase \( \phi \) parameterizes the pitchfork bifurcation curves \( P \) in the \((f, \delta)\)-parameter plane, as is also exemplified in figure 2 for the cases \( \kappa = 3.5 \) and \( \kappa = -3.5. \)

It follows from (A.5) that, for given suitable \( \kappa \) and \( \delta \), the phase \( \phi \) at \( P \) is

\[ \phi = \arctan \left( \frac{\delta + 5\kappa \pm \sqrt{3 + (\delta - \kappa)^2}}{3} \right). \tag{A.6} \]

By inspecting (A.6), we find that necessarily \( \delta < -\sqrt{3 + \kappa} \) or \( \delta > \sqrt{3 + \kappa}. \) To show that the second inequality cannot be satisfied, we insert \( \phi \) from (A.6) into the argument \( 4\kappa - 2 \tan(\phi) \) of the square root in (A.5) to get

\[ -2(\delta - \kappa \pm \sqrt{3 + (\delta - \kappa)^2}) = \frac{3}{\kappa}. \]

For \( \delta > \sqrt{3 + \kappa} \) the expression (A.7) is negative and (A.5) does not have a solution.

Notice that for the case \( \kappa = 0 \) of no coupling, the two curves \( S \) and \( P \) given by (A.3) and (A.5) are equal.

Finally, the codimension-two point \( \text{SP} \) of simultaneous saddle-node and pitchfork bifurcation is given in the \((f, \delta)\)-parameter plane by

\[ (f, \delta) = \left( \sqrt{2(1 + \kappa^2)} \left( -\kappa + \sqrt{1 + \kappa^2} \right), -2\sqrt{1 + \kappa^2} \right). \tag{A.8} \]
[22] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields* (Berlin: Springer)
[23] Kuznetsov Y A 2004 *Elements of Applied Bifurcation Theory* 3rd edn (New York: Springer)
[24] Jin J, Rossini D, Fazio R, Leib M and Hartmann M J 2013 Photon solid phases in driven arrays of nonlinearly coupled cavities *Phys. Rev. Lett.* **110** 163605
[25] Schirò M, Ioshi C, Bordyuh M, Fazio R, Keeling J and Türeci H E 2016 Exotic attractors of the nonequilibrium Rabi–Hubbard model *Phys. Rev. Lett.* **116** 143603
[26] Coullet P and Vandenberghe N 2001 Chaotic self-trapping of a weakly irreversible double Bose condensate *Phys. Rev. E* **64** 025202
[27] Casteels W and Ciuti C 2017 Quantum entanglement in the spatial-symmetry-breaking phase transition of a driven-dissipative Bose–Hubbard dimer *Phys. Rev. A* **95** 013812
[28] Casteels W, Storme F, Le Boité A and Ciuti C 2016 Power laws in the dynamic hysteresis of quantum nonlinear photonic resonators *Phys. Rev. A* **93** 033824
[29] Haddadi S, Hamel P, Beaudoin G, Sagnes I, Sauvan C, Lalanne P, Levenson J A and Yacomotti A M 2014 Photonic molecules: tailoring the coupling strength and sign *Opt. Express* **22** 12359
[30] Doedel E J and Oldeman B E 2010 AUTO-07p: continuation and bifurcation software for ordinary differential equations *Report* Department of Computer Science, Concordia University, Montreal (with major contributions from A R Champneys, F Dercole, T F Fairgrieve, Y Kuznetsov, R C Paffenroth, B Sandstede, XI Wang and C H Zhang)
[31] Cao R, Mahmud K W and Hafezi M 2016 Two coupled nonlinear cavities in a driven–dissipative environment *Phys. Rev. A* **94** 063805
[32] Brunstein M 2011 Nonlinear dynamics in III–V semiconductor photonic crystal nano-cavities *PhD Thesis* Université Paris Sud—Paris XI
[33] Maes B, Soljačić M, Ioannopoulos J D, Bienstman P, Baets R, Gorza S and Haelterman M 2006 Switching through symmetry breaking in coupled nonlinear micro-cavities *Opt. Express* **14** 10678–83
[34] Giraldo A, Broderick N G R and Krauskopf B 2019 Bifurcations to chaotic switching in the Bose–Hubbard model for two coupled nonlinear resonators in preparation
[35] Shilnikov L P 1965 A case of the existence of a denumerable set of periodic motions *Sov. Math. Dokl.* **6** 163–6
[36] Marconi M, Raineri F, Levenson A, Yacomotti A M, Javaloyes J, Pan S H, El Amili A and Fainman Y 2019 Mesoscopic limit cycles in coupled nanolasers arXiv:1911.10830