Open-loop potential difference games with inequality constraints

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Abstract

Static potential games are non-cooperative games which admit a fictitious function, also referred to as a potential function, such that the minimizers of this function constitute a subset (or a refinement) of the Nash equilibrium strategies of the associated non-cooperative game. In this paper we study a class $N$-player non-zero sum difference games with inequality constraints which admit a potential game structure. In particular, we provide conditions for the existence of an optimal control problem (with inequality constraints) such that the solution of this problem yields an open-loop Nash equilibrium strategy of the corresponding dynamic non-cooperative game (with inequality constraints). Further, we provide a way to construct potential functions associated with this optimal control problem. We specialize our general results to a linear-quadratic setting and provide a linear complementarity problem based approach for computing the refinements of the open-loop Nash equilibria obtained in Reddy and Zaccour (2015). We illustrate our results with an example inspired by energy storage incentives in a smart grid.

Keywords: Potential games; dynamic games with inequality constraints; open-loop Nash equilibrium; linear complementarity problem.

1 Introduction

Multi-agent control systems and related distributed architectures are becoming increasingly popular with emerging applications such as smart grids, Internet of Things (IoT) systems, intelligent traffic networks and cyber-security. These systems are characterized by the presence of interdependent multiple decision making entities, which are large-scale, distributed, networked and heterogeneous in nature. Game theory has emerged as a powerful tool for analyzing multi-agent systems; see Manshaei et al. (2013), Saad et al. (2012) and Zhu and Başar (2015) for applications of game theory in the areas mentioned above. In particular, dynamic game theory provides a mathematical framework for modeling multi-agent interactions which evolve over time, and has been successfully used in analyzing a variety of decision problems arising in engineering, economics and management science; see Başar and Olsder (1999), Dockner et al. (2000), Başar et al. (2018). A significant share of dynamic game models are formulated in an unconstrained setting, that is, where the state and control variables are unconstrained; except for the state equation which captures the evolution of the interaction environment. Constraints appear naturally in real world applications in the form of production-capacity, environmental, market and budget constraints. For example, electric vehicle
charging station with limited energy resources impose joint constraints on players. Further, decision problems related to network congestion, which are used in modeling traffic systems, inherently involve capacity constraints. Recently, Reddy and Zaccour (2015) and Reddy and Zaccour (2016) study a class of non-zero sum difference games with inequality constraints and provide conditions for the existence of Nash equilibria with open-loop and feedback information structures.

The novelty of this paper lies in studying a class of dynamic games with inequality constraints which admit a potential game structure. Static potential games were first introduced in Rosenthal (1973) and further studied in Monderer and Shapley (1996). Loosely speaking, a potential game is a non-cooperative game that satisfies the property that a Nash equilibrium of the game is obtained when players jointly optimize a fictitious function, also referred to as a potential function. In other words, a Nash equilibrium of the game is obtained by solving an optimization problem as opposed to a fixed point problem. An important property associated with a potential game is that the strategy profile that provides the optimum for the potential function is a pure strategy Nash equilibrium, and as a result, the existence of pure strategy Nash equilibrium is guaranteed in a potential game. It is well known (Quint and Shubik (1997)) that a non-cooperative game can admit more than one Nash equilibrium. The multiplicity of equilibria naturally poses a selection problem, and to address this, certain refinements of Nash equilibria have been proposed; see Myerson (1997). These refinements provide a way of selecting a subset of equilibria based on additional properties that a Nash equilibrium is required to satisfy. In a potential game, the Nash equilibrium is refined with the property that players are jointly optimizing the potential function in a cooperative manner, even though they are acting strategically optimizing their individual objective functions. Due to this property, potential games find applications in the study of network congestion games (Nisan et al., 2007, chapter 18), decentralized learning algorithms (Marden et al. (2009)) and in utility function design (Marden and Shamma (2018)) for multi-agent systems. Further, in Slade (1994) this property of potential games has been explored in the context of oligopolistic markets. In summary, the static potential games have been studied extensively in literature.

The objective of this paper is to extend the notion of a potential game in a dynamic setting, and in particular, for dynamic games where inequality constraints appear jointly in control and state variables. Besides providing the conditions on the existence of this class of games, another important objective of our paper is towards computing the refinements of the open-loop Nash equilibria. We consider a class of finite horizon $N$-player non-zero sum nonlinear difference games with constraint structure similar to Reddy and Zaccour (2015) and Reddy and Zaccour (2016). Our contributions are summarized as follows.

1. We define the notion of open-loop potential difference game and associate an inequality constrained optimal control problem with the dynamic non-cooperative game with inequality constraints. Further, in Lemma 3.6 and Theorem 3.8 we provide conditions under which the non-cooperative game admits a potential game structure.

2. When the potential functions associated with the optimal control problem are not specified, we provide, in Theorem 3.13, a method for constructing the potential functions from the objective functions of the players using the theory of conservative vector fields.

3. We specialize the obtained results to a linear quadratic setting and characterize in Theorem 4.2 a class of linear quadratic potential difference games with inequality constraints. Further, in Theorem 4.8 we provide a linear complementarity problem based method for computing a refinement of the open-loop Nash equilibria.

The paper is organized as follows. In section 2, we introduce the dynamic game model with the separable structure of the objective function and inequality constraints jointly in state and decision variables. In section 3, we define the open-loop dynamic potential games by introducing an optimal control problem associated
with the dynamic potential game. Further, we provide the structure of the dynamic game for the existence of potential functions. We demonstrate the equivalence of the solution of the optimal control problem as an open-loop Nash equilibrium of the dynamic game. When the potential functions are specified beforehand, we also illustrate a procedure for the construction of potential functions. In section 4, we specialize these results to the linear quadratic setting. In section 5 we illustrate our results with an application motivated by energy storage incentives in a smart grid. Finally, in section 6 we provide conclusions and future work.

1.1 Literature review

The literature on dynamic potential games is limited compared to their static counterpart. In a static non-cooperative game the objective functions of the players are required to satisfy certain conditions to admit a potential function. Quite naturally, an extension of these conditions to a dynamic setting imposes structural constraints on the instantaneous and terminal objective functions. In Slade (1994), the author studies oligopolistic markets, and provides conditions under which a single optimization problem provides a Nash equilibrium, both in the static and dynamic settings. The objective function of this optimization problem in this work is referred to as a fictitious function. In particular, in the dynamic case, existence of these functions under the open-loop and feedback information structures has been explored. In Dragone et al. (2015), the authors consider a non-cooperative differential game with open-loop information structure, and provide conditions for the existence of Hamiltonian potential functions. Potential differential games were studied in Fonseca-Morales and Hernández-Lerma (2018) in an open-loop setting. The authors primarily focused on separable structure of the payoff function from which the potential function could be derived. The unconstrained discrete time game is considered in González-Sánchez and Hernández-Lerma (2014) in a stochastic setting. In González-Sánchez and Hernández-Lerma (2016), the authors provide a survey on static and dynamic potential games. All these works study dynamic potential games in discrete and continuous time settings without additional constraints on the state and control variables. In Zazo et al. (2016), the authors consider a discrete time infinite horizon non-cooperative difference games with inequality constraints and provide conditions for the existence of potential functions. They also mention a way for constructing potential functions using the theory of conservative vector fields; an approach followed in static games Monderer and Shapley (1996). Our work is closer to Zazo et al. (2016) in spirit, but differs considerably in the model and approach. The constraint structure in our paper is inspired by our previous works Reddy and Zaccour (2015) and Reddy and Zaccour (2016). In particular, we assume that the players have two types of control variables, namely, 1) variables that (directly) affect the dynamics but do not enter the constraints, and 2) variables that enter the constraints jointly with the state variables but do not appear in the dynamics. Further, we assume that the objective functions are separable in the two types of control variables, that is, that there are no cross-terms between them. As was shown in Reddy and Zaccour (2015) and Reddy and Zaccour (2016), we also demonstrate in this paper that these assumptions are crucial in obtaining a semi-analytical characterizations for Nash equilibria, and in providing a linear complementarity problem based method for computing them. More importantly, the linear quadratic dynamic potential games analyzed in this paper provides refinements of the open-loop Nash equilibria obtained in Reddy and Zaccour (2015), and a procedure for computing them.

1.2 Notation

We shall use the following notation. The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$, $n \geq 1$. $A'$ denotes the transpose of a matrix or a vector $A$. $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ represents the block diagonal matrix obtained by taking the matrices $A_1, A_2, \cdots, A_n$ as diagonal elements in this sequence. The matrix with all entries as zeros is denoted by $0$, all entries as one is denoted by $1$, and the identity matrix is represented by $I$. We call two vectors $x, y \in \mathbb{R}^n$ complementary if $x \geq 0$, $y \geq 0$ and $x'y = 0$, and $0 \leq x \perp y \geq 0$ denotes this
condition. Let $A$ be a $n \times n$ matrix and $a$ be a $n \times 1$ vector. Let $n$ be partitioned as $n = n_1 + n_2 + \cdots + n_K$. We represent $[A]_{ij}$ as the $n_i \times n_j$ sub-matrix associated with indices $n_i$ (row) and $n_j$ (column), and $[a]_i$ as the $n_i \times 1$ sub-vector associated with indices $n_i$. $[A]_{i*}$ and $[A]_{*i}$ represent the $i^{th}$ row block matrix of dimensions $n_i \times n$ and $i^{th}$ column matrix of dimension $n \times n_i$ of the matrix $A$ respectively. The notation $e_i$ represents a column vector with $i^{th}$ element as 1 and the rest of the elements as zero. For differentiation with respect to a vector or matrix variable we follow the convention from Lütkepohl (1996).

## 2 Dynamic game model

In this section, we introduce a class of finite horizon discrete time non-zero sum games with inequality constraints. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of players and $\mathcal{K} = \{0, 1, 2, \ldots, K\}$ be the set of time periods. At each time instant, player $i \in \mathcal{N}$ chooses the following two types of actions (control variables); variables that enter the dynamics of the system, but not the constraints, denoted by $u_k^i \in U^i_k \subset \mathbb{R}^{m_i}$; and variables that enter the constraints, but not the dynamics, denoted by $v_k^i \in V^i_k \subset \mathbb{R}^{n_i}$. Here, $U^i_k$ and $V^i_k$ are the sets of admissible values for the two types of variables. We denote the vector of actions all players at time period $k$ by $\tilde{u}_k := [u_{1k}^1 \ u_{2k}^2 \ \cdots \ u_{nk}^N]^T$ and $\tilde{v}_k := [v_{1k}^1 \ v_{2k}^2 \ \cdots \ v_{nk}^N]^T$. The state $x_k \in X_k \subset \mathbb{R}^n$ evolves according to the following discrete time dynamics

$$x_{k+1} = f_k(x_k, u_k), \ k \in \mathcal{K}\setminus\{K\}, \ x_0 \text{ given.} \quad (1)$$

Here, $X_k$ denotes the set of admissible state vectors at time period $k$. We consider the following joint inequality constraints (also called coupled constraints) associated with players strategies

$$h_k(x_k, v_k) \geq 0, \ v_k \geq 0, \ k \in \mathcal{K}. \quad (2)$$

Notice, the decision variables $\{u_k^i, \ k \in \mathcal{K}\setminus\{K\}, \ i \in \mathcal{N}\}$ do not affect the dynamics, but do constrain the decision making process, denoted by $h_k(x_k, v_k) \geq 0$. We denote the joint strategy profiles of the players by $\tilde{u} := (\tilde{u}^1, \tilde{u}^{-1})$ and $\tilde{v} := (\tilde{v}^1, \tilde{v}^{-1})$, and the corresponding strategy sets are denoted by $\mathcal{U}$ and $\mathcal{V}$ respectively. Each player $i \in \mathcal{N}$ uses her strategies $\tilde{u}^i$ and $\tilde{v}^i$ to minimize the following joint objective function given by

$$J^i(x_0, (\tilde{u}^i, \tilde{u}^{-i}), (\tilde{v}^i, \tilde{v}^{-i})) = g^i_K(x_k, v_K) + \sum_{k=1}^{K-1} g^i_k(x_k, u_k, v_k). \quad (3)$$

We have the following assumptions related to the dynamic game (1)-(3).

**Assumption 2.1.**
1. The admissible action sets $\{U^i_k, \ k \in \mathcal{K}\setminus\{K\}, \ i \in \mathcal{N}\}$, are such that the sets of state vectors $\{X_k, \ k \in \mathcal{K}\}$ are convex, and feasible action sets $\{V^i_k(x_k, v_k^{-i}) = \{v_k^i \in \mathbb{R}^{n_i} \mid h^i(x_k, v_k) \geq 0, \ v_k \geq 0\}, \ \forall x_k \in X_k\}$ are non-empty, convex and bounded for all $k \in \mathcal{K}, i \in \mathcal{N}$. A strategy pair $(\tilde{u}, \tilde{v})$ associated with these action sets is an admissible strategy pair.

2. The matrices $\left\{\frac{\partial h_k}{\partial v_k}, \ k \in \mathcal{K}, \ i \in \mathcal{N}\right\}$ have full rank, so as to satisfy constraint qualification conditions.

3. The instantaneous cost functions in (3) admit a separable structure, that is, they do not contain cross terms involving the strategies $\tilde{u}$ and $\tilde{v}$ and are represented by $g^i_k(x_k, u_k, v_k) = g^i_k u_k^i(x_k, u_k) + g^i_k v_k^i(x_k, v_k)$.

4. The partial derivatives of the functions $f_k, h_k$, and the players’ cost functions $\{g^i_k, \ k \in \mathcal{K}, \ i \in \mathcal{N}\}$ exist in their arguments, and are twice continuously differentiable in their arguments.
Note that in (3) the cost incurred by player $i$ not only depends on his actions but also by the actions of other players $-i$. So, (1)-(3) constitutes a dynamic game with inequality constraints, and we refer to it as NZDG from here on. Then, the Nash equilibrium strategies of players, denoted by $(\tilde{u}^*, \tilde{v}^*)$, for this class of games is defined as follows.

**Definition 2.2** (Nash equilibrium). The strategy profile $(\tilde{u}^*, \tilde{v}^*)$ is Nash equilibrium if for every player $i \in \mathcal{N}$ the strategies $(\tilde{u}^*, \tilde{v}^*)$ solves

$$\min_{(\tilde{u}^i, \tilde{v}^i)} J^i(x_0, (\tilde{u}^i, \tilde{v}^{-i}), (\tilde{u}^i, \tilde{v}^{-i})) \text{ subject to (1) and (2).}$$

From the above definition the Nash equilibrium strategy is stable against a player’s unilateral deviation. In multistage games the interaction environment is dynamic, which is embedded in state variables and their evolution. It is well known that the Nash equilibrium solution varies with the information used by the players during the (dynamic) decision making process. So, in a dynamic game an information structure must be specified when players design their strategies. In an open-loop information structure, the players design their strategies using only the knowledge of the time $k$ (and initial state $x_0$). Whereas in a feedback information structure, players design their equilibrium strategies using the knowledge of the state variable. In this paper, we assume open-loop information structure. This implies, that the decision variables entering the dynamics are functions of time. Next, as it is clear from the inequality constraints (2) that the admissible action set of a player depends on the state variable. Therefore player $i$ takes action $v_k^i \in V_k^i(x_k, v_k^-)$ as a function of time $k$, and the feasible set $V_k^i(x_k, v_k^-)$ is parametrized by the state variable and the decision variables of players excluding player $i$; see also Reddy and Zaccour (2015) which considers open-loop information structure for this class of games.

### 3 Open-loop dynamic potential games

In this section we introduce the notion of a dynamic potential game and seek to find conditions under which NZDG is a dynamic potential game. It is well known that the classical potential game is a static concept, (Monderer and Shapley (1996)). A (static) game is said to be a potential game if there exists a potential function, and the minimum of this function provides a pure strategy Nash equilibrium of the game, there by providing refinement (or selection) of Nash equilibria. In other words, when a potential function exists for a game, a Nash equilibrium can be obtained by solving a minimization problem. So, when we extend this notion of potential game in a dynamic context, it natural to associate an optimal control problem with NZDG. We consider the following optimal control problem with inequality constraints.

\[ \text{OCP : } \min_{\tilde{u}, \tilde{v}} J(x_0, \tilde{u}, \tilde{v}) \]

subject to \( x_{k+1} = f_k(x_k, u_k), \ x_0 \) is given, \( h_k(x_k, v_k) \geq 0, \ v_k \geq 0, \ k \in \mathcal{K}, \)

\[ \text{where } J(x_0, \tilde{u}, \tilde{v}) = P_K(x_K, v_K) + \sum_{k=0}^{K-1} P_k(x_k, u_k, v_k), \]

where \( P_k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^a \rightarrow \mathbb{R} \) and \( P_K : \mathbb{R}^n \times \mathbb{R}^a \rightarrow \mathbb{R} \) are instantaneous and terminal cost functions, which are continuous and twice continuously differentiable in their arguments.

**Definition 3.1** (Open-loop potential difference game). The dynamic game NZDG is referred to as open-loop potential difference game (OLPDG) if there exist cost functions \( \{P_k, k \in \mathcal{K}\} \) such that the optimal solution of OCP provides an open-loop Nash equilibrium of NZDG. Whenever such cost functions exist, \( \{P_k, k \in \mathcal{K}\} \) are referred to as potential cost functions.
In the next two theorems we discuss the necessary and sufficient conditions for an admissible pair \((\tilde{u}^*, \tilde{v}^*) \in U \times V\) to be an optimal solution of OCP. Towards this end, we define the instantaneous and terminal Lagrangian functions as follows
\[
\mathcal{L}_k(x_k, u_k, v_k, \lambda_{k+1}, \mu_k) = P_k(x_k, u_k, v_k) + \lambda'_{k+1} f_k(x_k, u_k) - \mu_k h_k(x_k, v_k),
\]
\[
\mathcal{L}_K(x_K, v_K, \mu_K) = P_K(x_K, v_K) - \mu_K h_K(x_K, v_K).
\]

**Theorem 3.2.** Let Assumption 2.1 holds true. Let \((\tilde{u}^*, \tilde{v}^*)\) be an optimal admissible pair for OCP, and \(\{x^*_k, k \in K\}\) be state trajectory generated by \(\tilde{u}^*\), the then there exist co-states \(\{\lambda^*_k, k \in K\}\) and a multipliers \(\{\mu^*_k, k \in K\}\) such that the following conditions hold true:

\[
\begin{align*}
\mathbf{u}^*_k &= \arg\min_{u_k \in U_k} \mathcal{L}_k(x^*_k, u_k, v^*_k, \lambda^*_{k+1}, \mu^*_k) \\
x^*_{k+1} &= \frac{\partial \mathcal{L}_k}{\partial \lambda_{k+1}}(x^*_k, u^*_k, v^*_k, \lambda^*_{k+1}, \mu^*_k), \quad x^*_0 = x_0 \\
\lambda^*_k &= \frac{\partial \mathcal{L}_k}{\partial x_k}(x^*_k, u^*_k, v^*_k, \lambda^*_{k+1}, \mu^*_k), \\
\lambda^*_K &= \frac{\partial \mathcal{L}_K}{\partial x_K}(x^*_K, v^*_K, \mu^*_K), \\
\text{and for } k \in K &: 0 \leq h(x^*_k, v^*_k) \perp \mu^*_k \geq 0 \\
0 &\leq \frac{\partial \mathcal{L}_k}{\partial v_k}(x^*_k, u^*_k, v^*_k, \lambda^*_{k+1}, \mu^*_k) \perp v^*_k \geq 0.
\end{align*}
\]

**Proof.** The necessary conditions follow directly by applying the discrete-time maximum principle. \(\blacksquare\)

The following theorem provides conditions under which (7a)-(7f) are also sufficient for optimality of \((\tilde{u}^*, \tilde{v}^*)\).

**Theorem 3.3.** Let Assumption 2.1 holds true. Let the pair of control strategies \((\tilde{u}^*, \tilde{v}^*) \in U \times V\) and the collection of trajectories \(\{x^*_k, k \in K\}\) satisfy (7). Assume that the Lagrangian \(\mathcal{L}_k(x_k, u_k, v_k, \lambda_{k+1}, \mu_k)\) has a minimum with respect to \((u_k, v_k)\) for all \(k \in K\setminus\{K\}\). Let the minimized Lagrangian be given by \(\mathcal{L}^*_k(x_k, \lambda_{k+1}, \mu_k) = \min_{(u_k, v_k)} \mathcal{L}_k(x_k, u_k, v_k, \lambda_{k+1}, \mu_k)\) for \(k \in K\setminus\{K\}\). Assume that terminal Lagrangian \(\mathcal{L}_K(x_K, v_K, \mu_K)\) has a minimum with respect to \(v_K\), and denote the minimum terminal cost function by \(\mathcal{L}^*_K(x_K, \mu_K) = \min_{v_K} \mathcal{L}_K(x_K, v_K, \mu_K)\). Then, if \(\mathcal{L}^*_k(x_k, \lambda_{k+1}, \mu_k)\) is convex with respect to \(x_k\) for all \(k \in K\setminus\{K\}\) and \(\mathcal{L}^*_K(x_K, \mu_K)\) is convex with respect to \(x_K\), then the pair \((\tilde{u}^*, \tilde{v}^*)\) is optimal for OCP.
Proof. For any admissible \((\tilde{u}, \tilde{v}) \in \mathcal{U} \times \mathcal{V}\) we consider the difference

\[
J(x_0, \tilde{u}, \tilde{v}) - J(x_0, \tilde{u}^*, \tilde{v}^*) = P_K(x_K, v_K) + \sum_{k=0}^{K-1} P_k(x_k, u_k, v_k) - P_K(x_K^*, v_K^*) - \sum_{k=0}^{K-1} P_k(x_k^*, u_k^*, v_k^*)
\]

\[
= \mathcal{L}_K(x_K, v_K, \mu_K^*) - \mathcal{L}_K(x_K^*, v_K^*, \mu_K^*) + \mu_k^* (h(x_k, v_k) - h(x_k^*, v_k^*))
\]

\[
+ \sum_{k=0}^{K-1} \mathcal{L}_k(x_k, u_k, v_k, \lambda_{k+1}^*, \mu_k^*) - \mathcal{L}_k(x_k^*, u_k^*, v_k^*, \lambda_{k+1}^*, \mu_k^*)
\]

\[
+ \sum_{k=0}^{K-1} -\lambda_{k+1}^* (x_{k+1} - x_k^*) + \mu_k^* (h(x_k, v_k) - h(x_k^*, v_k^*))
\]

\[
\geq \mathcal{L}_K^*(x_K, \mu_K^*) - \mathcal{L}_K^*(x_K^*, \mu_K^*) + \sum_{k=0}^{K-1} -\lambda_{k+1}^* (x_{k+1} - x_k^*)
\]

\[
+ \sum_{k=0}^{K-1} \mu_k^* (h(x_k, v_k) - h(x_k^*, v_k^*)).
\]

From the envelope theorem we have \(\frac{d\mathcal{L}_K^*}{d x_k}(x_k^*, \lambda_{k+1}^*, \mu_k^*) = \frac{d\mathcal{L}_K}{d x_k}(x_k^*, u_k^*, v_k^*, \lambda_{k+1}^*, \mu_k^*) = \lambda_k^*\) and \(\frac{d\mathcal{L}_K^*}{d x_K}(x_K^*, \mu_K^*) = \frac{d\mathcal{L}_K}{d x_K}(x_K, \mu_K)\). Then using this and from the convexity of minimized Lagrangian and terminal Lagrangian with respect to state variables we get

\[
\mathcal{L}_K^*(x_k, \lambda_{k+1}^*, \mu_k^*) - \mathcal{L}_K^*(x_k^*, \lambda_{k+1}^*, \mu_k^*) \geq \left(\frac{d\mathcal{L}_K^*}{d x_k}(x_k^*, \lambda_{k+1}^*, \mu_k^*)\right)' (x_k - x_k^*) = \lambda_k^* (x_k - x_k^*),
\]

\[
(9a)
\]

\[
\mathcal{L}_K^*(x_K, \mu_K^*) - \mathcal{L}_K^*(x_K^*, \mu_K^*) \geq \left(\frac{d\mathcal{L}_K^*}{d x_K}(x_K^*, \mu_K^*)\right)' (x_K - x_K^*) = \lambda_k^* (x_K - x_K^*).
\]

\[
(9b)
\]

Using (9) in (8) we get

\[
J(x_0, \tilde{u}, \tilde{v}) - J(x_0, \tilde{u}^*, \tilde{v}^*) \geq \sum_{k=0}^{K-1} \lambda_k^* (x_k - x_k^*) - \sum_{k=0}^{K-1} \lambda_{k+1}^* (x_{k+1} - x_k^*) + \sum_{k=0}^{K} \mu_k^* (h(x_k, v_k) - h(x_k^*, v_k^*))
\]

\[
= \lambda_0 (x_0 - x_0^*) + \sum_{k=0}^{K} \mu_k^* (h(x_k, v_k) - h(x_k^*, v_k^*))
\]

\[
= \sum_{k=0}^{K} \mu_k^* h(x_k, v_k) \geq 0.
\]

The last but one equality follows from the complementary condition \(\mu_k^* h_k(x_k^*, v_k^*) = 0\) for \(k \in \mathcal{K}\), and the initial condition \(x_0 = x_0^*\). The last inequality follows as multipliers and constraints satisfy the conditions \(\mu_k^* \geq 0\) and \(h_k(x_k, v_k) \geq 0\) for all \(k \in \mathcal{K}\).

\[
\text{Remark 3.4.} \text{ The sufficient condition provided in Theorem 3.3 is an adaptation of Arrow type sufficient condition (Grass et al., 2008, Theorem 3.30) to a discrete-time setting with inequality constraints, and differs from the nonlinear programming based methods (Pearson and Sridhar (1966)).}
\]

### 3.1 Structure of OLPDG

In this subsection we provide conditions under which the optimal solution of OCP provides an open-loop Nash equilibrium of NZDG. Toward this end, we have the following assumption.
Assumption 3.5. The cost functions \( \{ P_k, k \in K \} \) associated with OCP satisfy the following conditions for every \( i \in N \)

\[
\begin{align*}
\frac{\partial P_k}{\partial u_k} &= \frac{\partial g_u^i}{\partial u_k}, \quad \frac{\partial P_k}{\partial v_k} = \frac{\partial g_v^i}{\partial v_k}, \quad \frac{\partial P_k}{\partial x_k} = \frac{\partial g_k^i}{\partial x_k}, \quad k \in K \setminus \{ K \}, \\
\frac{\partial P_K}{\partial u_i} &= \frac{\partial g_u^i}{\partial u_i}, \quad \frac{\partial P_K}{\partial v_i} = \frac{\partial g_v^i}{\partial v_i}.
\end{align*}
\tag{10a}
\]

\[
\frac{\partial P_K}{\partial u_i} = \frac{\partial g_u^i}{\partial u_i}, \quad \frac{\partial P_K}{\partial v_i} = \frac{\partial g_v^i}{\partial v_i}.
\tag{10b}
\]

Lemma 3.6. Let Assumption 3.5 holds true. Then, the cost functions of the OCP and NZDG satisfy the following relation for each player \( i \in N \).

\[
J(x_0, (\bar{u}^i, \bar{u}^{-i}), (\bar{v}^i, \bar{v}^{-i})) - J(x_0, (\bar{u}^i, \bar{u}^{-i}), (\bar{z}^i, \bar{v}^{-i})) = J^i(x_0, (\bar{u}^i, \bar{u}^{-i}), (\bar{v}^i, \bar{v}^{-i}))
\]

\[
- J^i(x_0, (\bar{u}^i, \bar{u}^{-i}), (\bar{z}^i, \bar{v}^{-i}))
\tag{11}
\]

\( \forall \bar{u}^i := \{ u_k^i \in U_k^i, k \in K \setminus \{ K \} \}, \forall \bar{v}^i := \{ u_k^i \in U_k^i, k \in K \setminus \{ K \} \}, \forall \bar{v}^i := \{ v_k^i \in V_k^i, k \in K \} and \forall \bar{v}^i := \{ v_k^i \in V_k^i, k \in K \} \). Proof: From (10a) and the separable structure of cost functions as assumed in Assumption 2.1, we have for every \( k \in K \setminus \{ K \} \)

\[
\begin{align*}
\frac{\partial}{\partial u_k}(P_k(x_k, u_k, v_k) - g^i_k(x_k, u_k, v_k)) &= \frac{\partial}{\partial u^i_k}(P_k(x_k, u_k, v_k) - g_u^i(x_k, u_k)) = 0, \\
\frac{\partial}{\partial v_k}(P_k(x_k, u_k, v_k) - g^i_k(x_k, u_k, v_k)) &= \frac{\partial}{\partial v^i_k}(P_k(x_k, u_k, v_k) - g_u^i(x_k, u_k)) = 0, \\
\frac{\partial}{\partial x_k}(P_k(x_k, u_k, v_k) - g^i_k(x_k, u_k, v_k)) &= 0.
\end{align*}
\tag{12a, 12b, 12c}
\]

From (10b) we have

\[
\begin{align*}
\frac{\partial}{\partial x_K}(P_K(x_K, v_K) - g^i_K(x_K, v_K)) &= 0, \\
\frac{\partial}{\partial v_K}(P_K(x_K, v_K) - g^i_K(x_K, v_K)) &= 0.
\end{align*}
\tag{13a, 13b}
\]

Clearly, from (12) it follows that the difference function \( P_k(x_k, u_k, v_k) - g^i_k(x_k, u_k, v_k) \) is independent of (or does not contain) the variables \( x_k, u_k^i, v_k^i \). Similarly, from (13) it follows that the difference function \( P_K(x_K, v_K) - g^i_K(x_K, v_K) \) does not contain the variables \( x_k, v_K^i \). This implies that these difference functions can be expressed as

\[
\begin{align*}
P_k(x_k, u_k, v_k) - g^i_k(x_k, u_k, v_k) &= \Theta^i_k(u_k^i, v_k^i), \quad \forall k \in K \setminus \{ K \}, \\
P_K(x_K, v_K) - g^i_K(x_K, v_K) &= \Theta^i_K(v_K^i).
\end{align*}
\tag{14a, 14b}
\]

\( \forall u_k^i \in U_k^i \) and \( \forall v_k^i \in V_k^i \). Since (14) is satisfied by every \( u_k^i \in U_k^i \) and \( v_k^i \in V_k^i \), for any \( u_k^i, v_k^i \in U_k^i \) and \( v_k^i, v_k^i \in V_k^i \) we obtain

\[
\begin{align*}
P_k(x_k, (u_k^i, u_k^{-i}), (v_k^i, v_k^{-i})) - g^i_k(x_k, (u_k^i, u_k^{-i}), (v_k^i, v_k^{-i})) &= P_k(x_k, (u_k, u_k^i)), (v_k^i, v_k^{-i}))
\]

\[
- g^i_k(x_k, (u_k^i, u_k^{-i}), (v_k^i, v_k^{-i}))
\]

\[
P_K(x_K, (v_K^i, v_K^{-i})) - g^i_K(x_K, (v_K^i, v_K^{-i})) = P_K(x_K, (z_k^i, v_K^{-i})) - g^i_K(x_K, (z_k^i, v_K^{-i})).
\]

8
Upon rearranging the above equations and adding we get

\[ P_k(x_k, (u_k^i, u_k^{-i})), (v_k^i, v_k^{-i})) - P_k(x_k, (w_k^i, u_k^{-i})), (z_k^i, v_k^{-i})) = g_k(x_k, (u_k^i, u_k^{-i})), (v_k^i, v_k^{-i})) \]

\[ - g_k(x_k, (w_k^i, u_k^{-i})), (z_k^i, v_k^{-i})), \]  \hspace{1cm} (15a)

\[ P_K(x_K, (v_K^i, v_K^{-i})) - P_K(x_K, (z_K^i, v_K^{-i})) = g_K(x_K, (v_K^i, v_K^{-i})) \]

\[ - g_K(x_K, (z_K^i, v_K^{-i})), \]  \hspace{1cm} (15b)

Taking summation of (15a) for all time steps \( k \in \mathcal{K}\setminus\{K\} \) and using (15b), we obtain (11).

**Remark 3.7.** Lemma 3.6 provides the dynamic counterpart of the principle of exact potential games introduced by Monderer and Shapely (Monderer and Shapley, 1996, section 2).

**Theorem 3.8.** Let Assumptions 2.1 and 3.5 hold true. Let the admissible pair \((\tilde{u}^*, \tilde{v}^*)\) be the optimal solution of OCP. Then \((\tilde{u}^*, \tilde{v}^*)\) is an open-loop Nash equilibrium of NZDG, that is, NZDG is an OLPDG with potential functions \(\{P_k, k \in \mathcal{K}\}\).

**Proof.** Let \(\{x_k^*, k \in \mathcal{K}\}\) be the state trajectory generated by \(\tilde{u}^*\). As \((\tilde{u}^*, \tilde{v}^*)\) is optimal for OCP there exist co-states \(\{\lambda_k^*, k \in \mathcal{K}\}\) and multipliers \(\{\mu_k^*, k \in \mathcal{K}\}\) such that the necessary conditions (7) hold true. Expanding these conditions in terms of cost functions we get

for \( k \in \mathcal{K}\setminus\{K\} \)

\[ \frac{\partial P_k(x_k^*, u_k^*, v_k^*)}{\partial u_k^i} + \left( \frac{\partial f_k(x_k^*, u_k^*)}{\partial v_k^i} \right) \lambda_k^* = 0, \quad i \in \mathcal{N}, \] \hspace{1cm} (16a)

\[ x_{k+1}^* = f_k(x_k^*, u_k^*), \quad x_0^* = x_0, \] \hspace{1cm} (16b)

\[ \lambda_k^* = \left. \frac{\partial P_k(x_k^*, u_k^*, v_k^*)}{\partial x_k^i} \right|_{x_k^*} + \left( \frac{\partial f_k(x_k^*, u_k^*)}{\partial x_k^i} \right) \lambda_k^* \]

\[ - \left. \frac{\partial h_k(x_k^*, v_k^*)}{\partial x_k^i} \right|_{x_k^*} \mu_k^*, \] \hspace{1cm} (16c)

\[ \lambda_K^* = \left. \frac{\partial P_K(x_K^*, v_K^*)}{\partial x_K^i} \right|_{x_K^*} - \left( \frac{\partial h_K(x_K^*, v_K^*)}{\partial x_K^i} \right) \mu_K^*, \] \hspace{1cm} (16d)

\[ 0 \leq \left( \frac{\partial P_k(x_k^*, u_k^*, v_k^*)}{\partial v_k^i} \right) \mu_k^* \perp v_k^i \]

\[ \geq 0, \quad i \in \mathcal{N}, \] \hspace{1cm} (16e)

\[ 0 \leq \left( \frac{\partial P_K(x_K^*, v_K^*)}{\partial v_K^i} \right) \mu_K^* \perp v_K^i \]

\[ \geq 0, \quad i \in \mathcal{N}, \] \hspace{1cm} (16f)

for \( k \in \mathcal{K}, \quad 0 \leq h_k(x_k^*, v_k^*) \perp \mu_k^* \geq 0. \) \hspace{1cm} (16g)

Next, we write \( u_k^* = (u_k^*, u_k^{-i}) \) and \( v_k^* = (v_k^*, v_k^{-i}) \) and for each player \( i \in \mathcal{N} \) we define the multipliers

\[ \Lambda_k^i := \lambda_k^*, \quad k \in \mathcal{K}\setminus\{K\}, \quad \delta_k^i := \mu_k^*, \quad k \in \mathcal{K}. \] \hspace{1cm} (17)
Now from Assumption 3.5 and using the above notation we write (16) as follows.

for $k \in \mathcal{K}\setminus \{K\}$

$$
\frac{\partial g_k^i(x_k^*, (u_k^i, u_k^{-i*}), (v_k^i, v_k^{-i*})))}{\partial u_k^i}
\bigg|_{u_k^i = u_k^*} + \left( \frac{\partial f_k(x_k^*, (u_k^i, u_k^{-i*}))}{\partial \tilde{u}_k^i} \right) \right) \Lambda^*_{k+1} = 0, \tag{18a}
$$

$$
x_{k+1} = f(x_k^*, u_k^*, x_0 = x_0, \tag{18b}
$$

$$
\Lambda^*_k = \frac{\partial g_k^i(x_k, (u_k^i, u_k^{-i*}), (v_k^i, v_k^{-i*})))}{\partial x_k}
\bigg|_{x_k = x_k^*} + \left( \frac{\partial f_k(x_k, (u_k^i, u_k^{-i*}))}{\partial x_k} \right) \Lambda^*_{k+1}
- \left( \frac{\partial h_k(x_k, (v_k^i, v_k^{-i*}))}{\partial x_k} \right) \delta_k^i, \tag{18c}
$$

$$
\Lambda^*_K = \frac{\partial g_K^i(x_K, (v_K^i, v_K^{-i*})))}{\partial x_K}
\bigg|_{x_K = x_K^*} \quad \left( \frac{\partial h_K(x_K, (v_K^i, v_K^{-i*}))}{\partial x_K} \right) \delta_K^i, \tag{18d}
$$

$$
0 \leq \frac{\partial g_k^i(x_k^*, (v_k^i, v_k^{-i*})))}{\partial v_k^i}
\bigg|_{v_k^i = v_k^*} - \left( \frac{\partial h_k(x_k^*, (v_k^i, v_k^{-i*}))}{\partial v_k^i} \right) \delta_k^i \perp v_k^* \geq 0, \tag{18e}
$$

$$
0 \leq \frac{\partial g_K^i(x_K^*, (v_K^i, v_K^{-i*})))}{\partial v_K^i}
\bigg|_{v_K^i = v_K^*} - \left( \frac{\partial h_K(x_K^*, (v_K^i, v_K^{-i*}))}{\partial v_K^i} \right) \delta_K^i \perp v_K^* \geq 0, \tag{18f}
$$

for $k \in \mathcal{K}$, $0 \leq h_k(x_k^*, v_k^i) \perp \delta_k^i \geq 0$.

Next, we consider Player $i$’s optimal control problem (2) when players in $-i = \mathcal{N}\setminus \{i\}$ use strategies $(\tilde{u}^{-i*}, \tilde{v}^{-i*})$. Taking the co-state vectors as $\{\lambda_k^i, k \in \mathcal{K}\}$ and Lagrange multipliers as $\{\delta_k^i, k \in \mathcal{K}\}$ we write the instantaneous Lagrangian and terminal Lagrangian functions associated with Player $i$’s problem are given as

$$
\mathcal{L}_k^i(x_k, (u_k^i, u_k^{-i*}), (v_k^i, v_k^{-i*}), \Lambda_{k+1}^i, \delta_k^i) = \mathcal{g}_k^i(x_k, (u_k^i, u_k^{-i*}), (v_k^i, v_k^{-i*}))) + \Lambda_{k+1}^i \left( f_k(x_k, (u_k^i, u_k^{-i*})) \right) \delta_k^i, \tag{19a}
$$

$$
\mathcal{L}_K^i(x_K, (v_K^i, v_K^{-i*}), \mu_K) = \mathcal{g}_K^i(x_K, (v_K^i, v_K^{-i*}))) - \delta_K^i \left( h_K(x_K, (v_K^i, v_K^{-i*}))) \right). \tag{19b}
$$

Representing the equations (18) in terms of the instantaneous Lagrangian and terminal Lagrangian functions
Nash equilibria. Strategy sets of players, and as a result there can exist more than one equilibria in a non-comparative game. The Nash equilibria correspond to fixed points of best response mapping defined over joint

3.1 we have NZDG is an OLPDG with potential functions

(19) associated with Player i’s optimal control problem (2) we get

$$u^i_k = \arg\min_{u^i_k, u^{-i}_k} L_i^k (x_k^i, (u^i_k, u^{-i}_k), (v^i_k, v^{-i}_k), \Lambda^i_{k+1}, \delta_k),$$

$$x^*_{k+1} = \frac{\partial L_i^k}{\partial \Lambda^i_{k+1}} (x_k^i, (u^i_k, u^{-i}_k), (v^i_k, v^{-i}_k), \Lambda^i_{k+1}, \delta_k), \quad x^*_0 = x_0,$$

$$\lambda^*_k = \frac{\partial L_i^k}{\partial u_k} (x^*_k, (u^i_k, u^{-i}_k), (v^i_k, v^{-i}_k), \Lambda^i_{k+1}, \delta_k),$$

$$\lambda^*_K = \frac{\partial L_i^k}{\partial v_K} (x^*_K, (v^i_k, v^{-i}_k), \Lambda^i_{k+1}, \delta_K) \perp v^i_k \geq 0,$$

$$0 \leq \frac{\partial L_i^k}{\partial v_k} (x^*_k, (u^i_k, u^{-i}_k), (v^i_k, v^{-i}_k), \lambda^*_K, \mu_k) \perp v^i_k \geq 0,$$

$$0 \leq \frac{\partial L_i^k}{\partial v^*_K} (x^*_K, (v^i_k, v^{-i}_k), \Lambda^i_{k+1}, \delta_K) \perp v^i_K \geq 0,$$

for $k \in K \setminus \{K\}$

Clearly, (20) constitute the necessary conditions for optimality of associated with Player i’s optimal control problem (2) where strategies of players in $-i$ are fixed at $(\tilde{u}^{-i}, \tilde{v}^{-i})$. In other words, $(\tilde{u}^i, \tilde{v}^i)$ is a candidate best response to $(\tilde{u}^{-i}, \tilde{v}^{-i})$. Next, we show that the strategy $(\tilde{u}^i, \tilde{v}^i)$ indeed minimizes the objective $J(x_0, (\tilde{u}^i, \tilde{v}^{-i}), (\tilde{v}^i, \tilde{v}^{-i}))$ subject to the dynamics $x_{k+1} = f_k(x_k, (\tilde{u}^i_k, \tilde{u}^{-i}_k)), k \in K \setminus \{K\}$ and constraints $h_k(x_k, (v^i_k, v^{-i}_k)) \geq 0, v^i_k \geq 0, k \in K$. Since $(\tilde{u}^i, \tilde{v}^i)$ is an optimal solution of the OCP, the following inequality holds true

$$J(x_0, (\tilde{u}^i, \tilde{u}^{-i}), (\tilde{v}^i, \tilde{v}^{-i})) \leq J(x_0, (\tilde{u}^i, \tilde{v}^{-i}), (\tilde{v}^i, \tilde{v}^{-i})), \forall (\tilde{u}^i, \tilde{v}^i)$$

(21)

with $\tilde{v}^i_k$ satisfying the constraint $h_k(x_k, (v^i_k, v^{-i}_k)) \geq 0 \forall k \in K$, where $x_k$ is the state trajectory evolved according to the action set $(\tilde{u}^i, \tilde{u}^{-i})$. From lemma 3.6, we have

$$J(x_0, (\tilde{u}^i, \tilde{v}^{-i}), (\tilde{v}^i, \tilde{v}^{-i})) - J(x_0, (\tilde{u}^i, \tilde{u}^{-i}), (\tilde{v}^i, \tilde{v}^{-i}))$$

$$= J^i(x_0, (\tilde{u}^i, \tilde{v}^{-i}), (\tilde{v}^i, \tilde{v}^{-i})) = J^j(x_0, (\tilde{u}^j, \tilde{v}^{-j}), (\tilde{v}^j, \tilde{v}^{-j})).$$

Using the above observation in (21) we get

$$J^i(x_0, (\tilde{u}^i, \tilde{v}^{-i}), (\tilde{v}^i, \tilde{v}^{-i})) \leq J^j(x_0, (\tilde{u}^j, \tilde{v}^{-j}), (\tilde{v}^j, \tilde{v}^{-j})), \forall ((\tilde{u}^i, \tilde{v}^{-i}),(\tilde{v}^i, \tilde{v}^{-i}))$$

(22)

with $\tilde{v}^i_k$ satisfying the constraint $h_k(x_k, (v^i_k, v^{-i}_k)) \geq 0 \forall k \in K$. This implies $(\tilde{u}^i, \tilde{v}^i)$ is indeed a best response to $(\tilde{u}^{-i}, \tilde{v}^{-i})$. So, $(\tilde{u}^i, \tilde{v}^i)$ is an open-loop Nash equilibrium of NZDG. Following Definition 3.1 we have NZDG is an OLPDG with potential functions $\{P_k, k \in K\}$.

**Remark 3.9.** The Nash equilibria correspond to fixed points of best response mapping defined over joint strategy sets of players, and as a result there can exist more than one equilibria in a non-comparative game. When OCP associated with OLPDG has an optimal solution, then from Theorem 3.8 this solution is an open-loop Nash equilibrium of NZDG, thereby providing a refinement of the open-loop Nash equilibria. This implies that, solution of the OCP provides a way for selecting one among possibly many open-loop Nash equilibria.
Remark 3.10. Notice, the constraints in (2) are coupled. Rosen in Rosen (1965) studied non-cooperative games with couple constraints. The equilibria in these games are referred to as normalized Nash equilibria, and are characterized by the property that multiplier vector associated with constraints in each player’s individual optimization problem is co-linear with a common multiplier vector. In our work, we observe a similar feature by construction, that is, in (17) the obtained open-loop Nash equilibrium has the property that the associated multipliers and co-state variables are same for all players.

3.2 Construction of potential cost functions

In section 3.1, a sufficient condition is provided to verify if NZDG is an OLPDG given the potential cost functions \( \{ P_k, k \in K \} \) of the OCP. However, in practice these functions are not available before hand. In these settings, it is desirable to construct potential functions using players’ cost functions. This construction procedure involves two steps. First, to verify if NZDG is an OLPDG, and then to construct the potential functions. Toward this end, we recall the following necessary and sufficient condition existence of conservative vector fields from multi-variable calculus (Apostol (1969)).

**Lemma 3.11** (Conservative vector field). Let \( \Omega \) be a convex open subset in \( \mathbb{R}^n \). Let \( F : \Omega \to \mathbb{R}^n \) be a vector field with continuous derivatives defined over \( \Omega \). The following conditions on \( F \) are equivalent

a. There exists a scalar potential function \( \Pi : \Omega \to \mathbb{R} \) such that \( F(\omega) = \nabla \Pi(\omega) \) for all \( \omega \in \Omega \), where \( \nabla \) denotes the gradient operator.

b. The partial derivatives satisfy
   \[
   \frac{\partial [F(\omega)]_i}{\partial \omega_j} = \frac{\partial [F(\omega)]_j}{\partial \omega_i}, \quad \forall \omega \in \Omega, \quad i, j = 1, 2, \ldots, n. \tag{23}
   \]

c. Let \( a \) be a fixed point in \( \Omega \), and \( C \subset \Omega \) be a piecewise smooth curve joining \( a \) with an arbitrary point \( \omega \in \Omega \). Then the potential function \( \Pi \) satisfies
   \[
   \Pi(\omega) - \Pi(a) = \int_C F(\omega) \cdot d\omega = \int_0^1 F(\alpha(z)) \cdot \frac{\partial \alpha}{\partial z} dz, \tag{24}
   \]
   where \( \cdot \) is the dot product, and \( \alpha : [0, 1] \to C \) is a bijective parametrization of the curve \( C \) such that \( \alpha(0) = a \) and \( \alpha(1) = \omega \).

A vector field \( F : \Omega \to \mathbb{R}^n \) satisfying these conditions is called a conservative vector field.

**Proof.** See (Apostol, 1969, Theorems 10.4, 10.5 and 10.9).

Remark 3.12. A consequence of the condition (23) is that the Jacobian matrix of the vector field \( F : \Omega \to \mathbb{R}^n \) evaluated at \( \omega \in \Omega \) is symmetric for all \( \omega \in \Omega \).

**Theorem 3.13.** Let Assumption 2.1.(1) holds true. Let the players’ utility functions satisfy the following conditions, for all \( i, j \in \mathcal{N} \),

\[
\frac{\partial^2 g^i_k}{\partial (u^i_k) \partial u^j_k} = \left( \frac{\partial^2 g^j_k}{\partial (u^j_k) \partial u^i_k} \right)', \quad k \in \mathcal{K} \setminus \{ K \}, \tag{25a}
\]

\[
\frac{\partial g^i_k}{\partial x_k} = \frac{\partial g^j_k}{\partial x_k}, \quad k \in \mathcal{K}, \tag{25b}
\]

\[
\frac{\partial^2 g^i_k}{\partial (v^i_k) \partial v^j_k} = \left( \frac{\partial^2 g^j_k}{\partial (v^j_k) \partial v^i_k} \right)', \quad k \in \mathcal{K}. \tag{25c}
\]
then NZDG is a OLPDG. Let the vector fields $F_k(x_k, u_k, v_k) \in \Omega_k$, where $\Omega_k := X_k \times \prod_{i \in N} U^i_k \times \prod_{i \in N} V^i_k$ of dimension $(n + m + s) \times 1$ and $F_K(x_K, v_K) \in \Omega_K$ where $\Omega_K := X_K \times \prod_{i \in N} V_K^i$ of dimension $(n + s) \times 1$ be defined by

$$F_k(x_k, u_k, v_k) = \left[ \begin{array}{c} \frac{\partial g_1^k}{\partial x_k}, \ldots, \frac{\partial g_n^k}{\partial x_k}, \frac{\partial g_1^k}{\partial u_k}, \ldots, \frac{\partial g_n^k}{\partial u_k}, \ldots, \frac{\partial g_1^m}{\partial v_k}, \ldots, \frac{\partial g_n^m}{\partial v_k} \end{array} \right]^t, k \in \mathcal{K}\{K\},$$

$$F_K(x_K, v_K) = \left[ \begin{array}{c} \frac{\partial g_1^k}{\partial x_k}, \ldots, \frac{\partial g_n^k}{\partial x_k}, \ldots, \frac{\partial g_1^m}{\partial v_k}, \ldots, \frac{\partial g_n^m}{\partial v_k} \end{array} \right]^t,$$

then the instantaneous and terminal potential functions are given by

$$P_k(x_k, u_k, v_k) = c_k + \int_0^1 F_k(\alpha_k(z)) \cdot \frac{\partial \alpha_k(z)}{\partial z} dz, k \in \mathcal{K}, \tag{26a}$$

$$P_K(x_K, v_K) = c_K + \int_0^1 F_K(\alpha_K(z)) \cdot \frac{\partial \alpha_K(z)}{\partial z} dz, \tag{26b}$$

where $\alpha_k : [0, 1] \to C_k (k \in \mathcal{K})$ is a bijective parametrization of a piece-wise smooth curve $C_k \subset \Omega_k$ such that $\alpha_k(0) = (x_{0k}, u_{0k}, v_{0k}), \alpha_k(1) = (x_k, u_k, v_k)$ for $k \in \mathcal{K}\{K\}$, $\alpha_K(0) = (x_{0K}, v_{0K}), \alpha_K(1) = (x_K, v_K)$, and $\{c_k, k \in \mathcal{K}\}$ are constants.

**Proof.** We write the vector field $F_k$, $k \in \mathcal{K}\{K\}$ as $F_k = [F_k^x \ F_k^u \ F_k^v]^t (n+m+s) \times 1$ such that

$$F_k^x = \left[ \begin{array}{c} \frac{\partial g_1^k}{\partial x_k} \\ \vdots \\ \frac{\partial g_n^m}{\partial x_k} \end{array} \right]_{n \times 1}, F_k^u = \left[ \begin{array}{c} \frac{\partial g_1^k}{\partial u_k} \\ \vdots \\ \frac{\partial g_n^m}{\partial u_k} \end{array} \right]_{n \times 1},$$

$$F_k^v = \left[ \begin{array}{c} \frac{\partial g_1^k}{\partial v_k} \\ \vdots \\ \frac{\partial g_n^m}{\partial v_k} \end{array} \right]_{m \times 1}.$$

Let $w_k, k \in \mathcal{K}\{K\}$ be the $(n + m + s) \times 1$ vector given by $w_k = [x_k^t \ u_k^t \ v_k^t]^t$. Therefore, we can write the Jacobian matrix, $J_k, k \in \mathcal{K}\{K\}$ as

$$J_k = \frac{\partial F_k}{\partial (w_k)^t} = \left[ \begin{array}{c} \frac{\partial F_k^x}{\partial (x_k)^t} \ \frac{\partial F_k^u}{\partial (u_k)^t} \ \frac{\partial F_k^v}{\partial (v_k)^t} \end{array} \right]_{(n+m+s) \times (n+m+s)}, \tag{27}$$

where

$$\frac{\partial F_k^x}{\partial (x_k)^t} = \left[ \begin{array}{ccc} \frac{\partial^2 g_1^1}{\partial (x_1)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (x_1)^2} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^1}{\partial (x_n)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (x_n)^2} \end{array} \right]_{n \times n}, \frac{\partial F_k^u}{\partial (x_k)^t} = \left[ \begin{array}{ccc} \frac{\partial^2 g_1^1}{\partial (u_1)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (u_1)^2} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^1}{\partial (u_n)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (u_n)^2} \end{array} \right]_{n \times n},$$

$$\frac{\partial F_k^v}{\partial (x_k)^t} = \left[ \begin{array}{ccc} \frac{\partial^2 g_1^1}{\partial (v_1)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (v_1)^2} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^1}{\partial (v_n)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (v_n)^2} \end{array} \right]_{n \times n},$$

$$\frac{\partial F_k^x}{\partial (u_k)^t} = \left[ \begin{array}{ccc} \frac{\partial^2 g_1^1}{\partial (u_1)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (u_1)^2} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^1}{\partial (u_n)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (u_n)^2} \end{array} \right]_{n \times n}, \quad \frac{\partial F_k^u}{\partial (u_k)^t} = \left[ \begin{array}{ccc} \frac{\partial^2 g_1^1}{\partial (u_1)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (u_1)^2} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^1}{\partial (u_n)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (u_n)^2} \end{array} \right]_{n \times n},$$

$$\frac{\partial F_k^v}{\partial (u_k)^t} = \left[ \begin{array}{ccc} \frac{\partial^2 g_1^1}{\partial (v_1)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (v_1)^2} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^1}{\partial (v_n)^2} & \ldots & \frac{\partial^2 g_n^m}{\partial (v_n)^2} \end{array} \right]_{n \times n}.$$
\[
\frac{\partial F^u_k}{\partial (v_k)^T} = \begin{bmatrix}
\frac{\partial^2 g^1_k}{\partial v^j_k \partial u_k^j} & \frac{\partial^2 g^1_k}{\partial v^j_k \partial u_k^i} & \cdots & \frac{\partial^2 g^1_k}{\partial v^j_k \partial u_k^i} \\
\frac{\partial^2 g^2_k}{\partial v^i_k \partial u_k^j} & \frac{\partial^2 g^2_k}{\partial v^i_k \partial u_k^i} & \cdots & \frac{\partial^2 g^2_k}{\partial v^i_k \partial u_k^i} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 g^N_k}{\partial v_1 \partial u_k^i} & \frac{\partial^2 g^N_k}{\partial v_1 \partial u_k^i} & \cdots & \frac{\partial^2 g^N_k}{\partial v_1 \partial u_k^i}
\end{bmatrix}_{m \times n}, \quad \frac{\partial F^v_k}{\partial (v_k)^T} = \begin{bmatrix}
\frac{\partial^2 g^1_k}{\partial v^j_k \partial u_k^j} & \frac{\partial^2 g^1_k}{\partial v^j_k \partial u_k^i} & \cdots & \frac{\partial^2 g^1_k}{\partial v^j_k \partial u_k^i} \\
\frac{\partial^2 g^2_k}{\partial v^i_k \partial u_k^j} & \frac{\partial^2 g^2_k}{\partial v^i_k \partial u_k^i} & \cdots & \frac{\partial^2 g^2_k}{\partial v^i_k \partial u_k^i} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 g^N_k}{\partial v_1 \partial u_k^i} & \frac{\partial^2 g^N_k}{\partial v_1 \partial u_k^i} & \cdots & \frac{\partial^2 g^N_k}{\partial v_1 \partial u_k^i}
\end{bmatrix}_{n \times s}.
\]

Firstly, as \(\frac{\partial F^u_k}{\partial (u_k)^T} = \frac{\partial^2 g^k_k}{\partial (u_k)^2}\) we have that \(\frac{\partial F^u_k}{\partial (u_k)^T}\) is a symmetric matrix. The off-diagonal block matrices in \(\frac{\partial F^u_k}{\partial (u_k)^T}\) satisfy (25a) and this implies \(\frac{\partial F^u_k}{\partial (u_k)^T}\) is symmetric matrix. Similarly, from (25c), we have that \(\frac{\partial F^v_k}{\partial (v_k)^T}\) is also a symmetric matrix. Next, from (25b), \(\frac{\partial g^i_k}{\partial x_k} = \frac{\partial g^i_k}{\partial x_k}\), and as cost functions are twice continuously differentiable, we have \(\frac{\partial^2 g^i_k}{\partial (u_k)^2} = \frac{\partial^2 g^i_k}{\partial (u_k)^2}\). Next, from the symmetry property of mixed partials we get

\[
\frac{\partial^2 g^1_k}{\partial (u_k)^2} = \left(\frac{\partial^2 g^1_k}{\partial (u_k)^2}\right)^T, \quad \text{and then using (25b) we have} \quad \frac{\partial^2 g^1_k}{\partial (u_k)^2} = \left(\frac{\partial^2 g^1_k}{\partial (u_k)^2}\right)^T, \quad \text{which implies} \quad \frac{\partial F^u_k}{\partial (u_k)^T} = \left(\frac{\partial F^u_k}{\partial (u_k)^T}\right)^T.
\]

Again, using similar arguments we can show that \(\frac{\partial F^v_k}{\partial (v_k)^T} = \left(\frac{\partial F^v_k}{\partial (v_k)^T}\right)^T\). From the separable structure of the cost functions we have \(\frac{\partial^2 g^i_k}{\partial (v_k)^2} = \left(\frac{\partial^2 g^i_k}{\partial (u_k)^2}\right) = 0\), which implies \(\frac{\partial F^v_k}{\partial (v_k)^T} = \left(\frac{\partial F^v_k}{\partial (v_k)^T}\right)^T = 0\). Clearly, from these observations we see that Jacobian matrix (27) is a symmetric matrix, and as a result \(F_K\) is a conservative vector field for all \(k \in K\backslash\{K\}\). At the terminal instant we write \(F_K = [F^v_K \ F^u_K]_{(n+s) \times 1}\) and define \(w_K = [x_K' \ v_K']_{(n+s) \times 1}\). From (25) and using the same reasoning as above we can show that the Jacobian

\[
\mathcal{J}_K = \frac{\partial F_K}{\partial (w_K)^T} = \begin{bmatrix}
\frac{\partial F^u_K}{\partial (w_K)^T} & \frac{\partial F^v_K}{\partial (w_K)^T}
\end{bmatrix}
\]

is a symmetric matrix. This implies, that \(F_K\) is a conservative vector field.

As \(\alpha_k : [0, 1] \to C_k \subset \Omega_k\) is bijective parametrization of a piece-wise path connecting a fixed point \((x_0, u_0, v_0) \in \Omega_k\) to an arbitrary point \((x_k, u_k, v_k) \in \Omega_k\). Following Lemma 3.11, the instantaneous potential function satisfies

\[
P_k(x_k, u_k, v_k) = c_k + \int_0^1 F_k(\alpha_k(z)) \cdot \frac{\partial \alpha_k(z)}{\partial z} dz, \quad k \in K\backslash\{K\},
\]

where \(c_k = P_k(x_0, u_0, v_0)\) is value of the potential function evaluated at \((x_0, u_0, v_0)\). Similarly, the terminal potential function is given by

\[
P_K(x_K, v_K) = c_K + \int_0^1 F_K(\alpha_K(z)) \cdot \frac{\partial \alpha_K(z)}{\partial z} dz.
\]

where \(c_K = P_K(x_0, v_0)\).

**Remark 3.14.** From (26a) and (26b), we note that the instantaneous potential function and terminal potential function at a given point are not unique, but are unique up to a constant, and depend upon the choice of the initial fixed points \(\{\alpha_k(0), \ k \in K\}\). This implies that we obtain a family of potential functions, and as a result, several optimal control problems associated with OLPGD. However, as the objective functions of these problems differ by a constant, they have the same optimal solution.

### 4 Open-loop linear quadratic potential difference game

In this section, we specialize the results obtained from the previous section to a linear quadratic setting and provide a numerical method for computing the open-loop Nash equilibrium associated with OLPGD. Toward
this end, we introduce the following \( N \)-player non-zero sum finite horizon linear quadratic difference game as follows. Each Player \( i \in \mathcal{N} \) solves

\[
\text{NZDG1} : \min_{\bar{u}^i, \bar{v}^i} J^i(x_0, (\bar{u}^i, \bar{u}^{-i}), (\bar{v}^i, \bar{v}^{-i})), \tag{28a}
\]

subject to

\[
x_{k+1} = A_k x_k + \sum_{i \in \mathcal{N}} B^i_k u^i_k, k \in \mathcal{K}\backslash\{K\}, \quad x_0 \text{ (given)}, \tag{28b}
\]

\[
M_k x_k + N_k v_k + r_k \geq 0, \quad v_k \geq 0, \quad k \in \mathcal{K}, \tag{28c}
\]

where

\[
J^i(x_0, (\bar{u}^i, \bar{u}^{-i}), (\bar{v}^i, \bar{v}^{-i})) = \frac{1}{2} \bar{x}'_k Q_i^k x_k + p_i^k x_k + \sum_{k=0}^{K-1} \left( \frac{1}{2} \bar{x}'_k Q_i^k x_k + p_i^k x_k + \frac{1}{2} u_k^i R_i^k u_k \right) + \sum_{k=0}^{K} \left( \frac{1}{2} v_k^i D_i^k v_k + d_i^k v_k + x_k^i L_i^k v_k \right), \tag{28d}
\]

where the matrices \( Q_i^k \in \mathbb{R}^{n \times n}, i \in \mathcal{N}, k \in \mathcal{K} \) are symmetric, \( R_i^k \in \mathbb{R}^{m_i \times m_i}, i \in \mathcal{N}, k \in \mathcal{K}\backslash\{K\} \) are symmetric and positive definite, \( D_i^k \in \mathbb{R}^{s \times s}, i \in \mathcal{N}, k \in \mathcal{K} \) are symmetric, \( p_i^k \in \mathbb{R}^{n}, i \in \mathcal{N}, k \in \mathcal{K}\backslash\{K\} \), and \( d_i^k \in \mathbb{R}^{s}, L_i^k \in \mathbb{R}^{s \times s}, i \in \mathcal{N}, k \in \mathcal{K} \). Associated with NZDG1 we introduce the following optimal control problem

\[
\text{OCP1} : \min_{\bar{u}, \bar{v}} J(x_0, \bar{u}, \bar{v}), \tag{29a}
\]

subject to (28b) and (28c)

where

\[
J(x_0, \bar{u}, \bar{v}) = \frac{1}{2} \bar{x}'_k Q_k x_k + p_k' x_k + \sum_{k=0}^{K-1} \left( \frac{1}{2} \bar{x}'_k Q_k x_k + p_k' x_k + \frac{1}{2} u_k R_k u_k \right) + \sum_{k=0}^{K} \left( \frac{1}{2} v_k D_k v_k + d_k v_k + x_k L_k v_k \right) \tag{29b}
\]

with \( Q_k \in \mathbb{R}^{n \times n}, d_k \in \mathbb{R}^{s}, p_k \in \mathbb{R}^{n}, D_k \in \mathbb{R}^{s \times s}, L_k \in \mathbb{R}^{s \times s}, i \in \mathcal{N}, k \in \mathcal{K}, \) and \( R_k \in \mathbb{R}^{m \times m}, i \in \mathcal{N}, k \in \mathcal{K}\backslash\{K\} \).

**Assumption 4.1.** The admissible action sets \( \{U_k^i, k \in \mathcal{K}\backslash\{K\}, i \in \mathcal{N}\} \), are such that the sets of state vectors \( \{X_k, k \in \mathcal{K}\} \), obtained from (28b), are convex, and feasible action sets \( \{V_k^i(x_k, v_k^{-i}) = \{v_k^i \in \mathbb{R}^{s} \mid M_k x_k + N_k v_k + r_k \geq 0, v_k \geq 0, \forall x_k \in X_k \} \) are non-empty, convex and bounded for all \( k \in \mathcal{K}, i \in \mathcal{N} \).

In the next theorem we provide conditions under which NZDG1 is an open-loop dynamic potential game, and using Theorem 3.13 we construct the potential functions associated optimal control problem (OCP1).

**Theorem 4.2.** Let Assumption 4.1 holds true. Let the parameters associated with NZDG1 satisfy the following conditions

\[
[R_k^i]_{ij} = [R_k^j]_{ij}, \quad i, j \in \mathcal{N}, i \neq j, k \in \mathcal{K}\backslash\{K\}, \tag{30a}
\]

\[
Q_k^i = Q_k^j, \quad p_k^i = p_k^j, \quad L_k^i = L_k^j, \quad [D_k^i]_{ij} = [D_k^j]_{ij}, \quad i, j \in \mathcal{N}, i \neq j, k \in \mathcal{K}, \tag{30b}
\]

\[\text{where } \mathcal{K}\backslash\{K\} \subseteq \{1, 2, \ldots, K-1\} \text{ is a fixed subset of } \mathcal{K}, \quad R_k^i = \mathbb{R}^{m_i \times m_i}, i \in \mathcal{N}, k \in \mathcal{K}\backslash\{K\}, \quad Q_k^i \in \mathbb{R}^{n \times n}, D_k^i \in \mathbb{R}^{s \times s}, \quad \text{ and } \mathcal{N} \subseteq \{1, 2, \ldots, N\}.\]
then NZDG is a OLPDG. Further, the OCP associated with the related OLPDG is described by

\[ Q_k = Q_k^i, \quad [D_k]_{i*i} = [D_k^i]_{i*i}, \quad [d_k]_i = [d_k^i]_i, \quad L_k = L_k^i, \quad p_k = p_k^i, \quad i \in \mathcal{N}, \quad k \in \mathcal{K}, \quad (31a) \]

\[ [R_k]_{i*i} = [R_k^i]_{i*i}, \quad i \in \mathcal{N}, \quad k \in \mathcal{K} \setminus \{K\}. \quad (31b) \]

**Proof.** We first consider the conditions in (30b) and verify the condition (25b),

\[ \frac{\partial g_k^i}{\partial x_k} = \frac{\partial g_k^i}{\partial x_k} = Q_k^i x_k + p_k^i + L_k^i v_k = Q_k^i x_k + p_k^i + L_k^i v_k = \frac{\partial g_k^i}{\partial x_k}, \quad k \in \mathcal{K}. \]

Since \( Q_k^i = Q_k^i, p_k^i = p_k^i \) and \( L_k^i = L_k^i, \forall i, j \in \mathcal{N} \), the condition (25b) holds true. Next, by using (30a) and (30b) and the symmetric structure of \( R_k^i \) and \( D_k^i \), we verify the conditions (25a) and (25c) as follows

\[ \frac{\partial^2 g_k^i}{\partial (u_k^i)^2} = \left( [R_k^i]_{ij} \right) = \left( [R_k^i]_{ij} \right) = \left( \frac{\partial^2 g_k^i}{\partial (u_k^i)^2} \right), \quad k \in \mathcal{K} \setminus \{K\}, \]

\[ \frac{\partial^2 g_k^i}{\partial (v_k^i)^2} = \left( [D_k^i]_{ij} \right) = \left( [D_k^i]_{ij} \right) = \left( \frac{\partial^2 g_k^i}{\partial (v_k^i)^2} \right), \quad k \in \mathcal{K}. \]

So, NZDG1 is an OLPDG. Next, we proceed to construct the potential functions associated with OLPDG. The gradient vector field \( F_k = \left[ F_k^x, F_k^u, F_k^v \right] \) is calculated as

\[ F_k^x = Q_k^i x_k + p_k^i + L_k^i v_k, \quad F_k^u = \left[ \frac{[R_k^1]_{i*i}}{[R_k^2]_{i*i}} \right] u_k, \quad F_k^v = \left[ \frac{[D_k^1]_{i*i}}{[D_k^2]_{i*i}} \right] v_k + \left[ \frac{[d_k^1]_i}{[d_k^2]_i} \right] + \left[ \frac{[L_k^1]_i}{[L_k^2]_i} \right] x_k. \]

Since, \( F_k \) is conservative, the instantaneous potential function (26a) evaluated as a line integral, along an arbitrary piecewise path in \( \Omega_k \), depends only on the initial and final points. We consider a straight line connecting the origin in \( \Omega_k \) and an arbitrary point \((x_k, u_k, v_k) \in \Omega_k \), and the associated birective parametrization of this line is given by \( \alpha_k(z) = z \left[ x_k^i, u_k^i, v_k^i \right] \) with \( z \in [0, 1] \). The instantaneous potential function is computed as

\[ P_k(x_k, u_k, v_k) = c_k + \int_0^1 F_k(\alpha_k(z)) \cdot \frac{d\alpha_k(z)}{dz} ~ dz \]

\[ = c_k + \int_0^1 \left( x_k^i [Q_k(x_k) + p_k + L_k(zv_k)] + u_k^i R_k(zu_k) + v_k^i [D_k(zv_k) + d_k + L_k(zx_k)] \right) ~ dz. \quad (32) \]

We define matrices \( R_k, D_k, Q_k, L_k, p_k \) and \( d_k \) such that \([R_k]_{i*i} = [R_k^i]_{i*i}, [D_k]_{i*i} = [D_k^i]_{i*i}, Q_k = Q_k^i, [L_k]_{i*i} = [L_k^i]_{i*i}, p_k = p_k^i, \) and \([d_k]_i = [d_k^i]_i \) for all \( i \in \mathcal{N}, k \in \mathcal{K} \setminus \{K\} \). Then, from (30) it follows that \( R_k \) and \( D_k \) are symmetric matrices and \( Q_k = Q_k^i, L_k = L_k^i, p_k = p_k^i \) for all \( i, j \in \mathcal{N}, k \in \mathcal{K} \setminus \{K\} \). Using this (32) can be written as

\[ P_k(x_k, u_k, v_k) = c_k + \int_0^1 \left( x_k^i [Q_k(zx_k) + p_k + L_k(zv_k)] + u_k^i R_k(zu_k) + v_k^i [D_k(zv_k) + d_k + L_k(zx_k)] \right) ~ dz \]

\[ = c_k + \frac{1}{2} x_k^i Q_k(x_k) + p_k^i x_k + \frac{1}{2} u_k^i R_k u_k + \frac{1}{2} v_k^i D_k v_k + d_k^i v_k + x_k^i L_k v_k. \quad (33) \]
Similarly, the terminal vector field \( F_K = [F_K^{x'} F_K^{v'}]' \) is calculated as
\[
F_K^x = Q^1 x_K + p^1_K + L_K v_K,
\]
\[
F_K^v = \begin{bmatrix}
[D_K^1] \mathbf{1} \\
[D_K^2] 2 \mathbf{1} \\
\vdots \\
[D_K^N] N \mathbf{1}
\end{bmatrix} v_K + \begin{bmatrix}
[d^1_K]^1 \\
[d^2_K]^2 \\
\vdots \\
[d^N_K]^N
\end{bmatrix} x_K.
\]

We define matrices \( D_K, Q_K, L_K, p_K \) and \( d_K \) such that \( [D_K]_{i\bullet} = [D^i_K]_{\bullet \bullet}, Q_K = Q^i_K, [L_K]_{i\bullet} = [L^i_K]_{\bullet \bullet}, p_K = p^i_K, \) and \( [d_K]_{i} = [d^i_K]_{i} \) for all \( i \in \mathcal{N} \). Then, from (30) it follows that \( D_K \) is a symmetric matrix and \( Q_K = Q^i_K, L_K = L^i_K, p_K = p^i_K \) for all \( i, j \in \mathcal{N} \). Using this, and using the same procedure as before, the terminal potential function (26b) is calculated as
\[
P_K(x_K, v_K) = c_K + \int_0^1 (x_K' F_K^x(\alpha_K(z)) v_K' F_K^v(\alpha_K(z))) dz = c_K + \frac{1}{2} x_K' Q_K x_K + p_K' x_K + \frac{1}{2} v_K' D_K v_K + x_K' L_K v_K + d_K' v_K.
\] (34)

The instantaneous and terminal potential functions given by (33) and (34), respectively, constitute the objective function (29b) associated with the OCP1. The parameters associated with this objective function satisfy the conditions (31).

### 4.1 Computation of open-loop Nash equilibrium associated with OLPDG

In this section, under a few assumptions on the parameters we transform the necessary conditions associated with OCP1 to a large-scale linear complementarity problem, there by providing a way to compute the open-loop Nash equilibrium. Let \( (\bar{u}^*, \bar{v}^*) \) be the optimal solution of the OCP1 and \( \{x_k^*, k \in \mathcal{K}\} \) be the state trajectory generated by \( \bar{u}^* \). The necessary conditions of optimality of are then given by
\[
\text{for } k \in \mathcal{K} \setminus \{ K \}
\]
\[
\begin{align*}
R_k u_k^* + B_k^* \lambda_{k+1}^* &= 0, \\
B_k &= [B_k^1, B_k^1, \ldots, B_k^N], \\
x_{k+1}^* &= A_k x_k^* + \sum_{i \in \mathcal{N}} B_k^i u_i^*, \\
\lambda_k^* &= Q_k x_k^* + p_k + L_k v_K^* + A_k^* \lambda_{k+1}^* - M_k^* \mu_k^*, \\
\lambda_K^* &= Q_K x_K^* + p_K + L_K v_K^* - M_K^* \mu_K^*,
\end{align*}
\] (35a)
\[
\text{for } k \in \mathcal{K}
\]
\[
\begin{align*}
0 \leq (D_k v_k^* + d_k + L_k x_k^* - N_k^* \mu_k^*) \perp v_k^* \geq 0, \\
0 \leq (M_k x_k^* + N_k v_k^* + r) \perp \mu_k^* \geq 0.
\end{align*}
\] (35c)

The above set of necessary conditions lead to a weakly coupled system of a parametric two-point boundary value problem (35a)-(35d) and a parametric linear complementarity problem (35e)-(35f). We have the following assumption.

**Assumption 4.3.** The co-state variable \( \lambda_k \) is assumed to be affine in the state variable \( x_k \) for \( k \in \mathcal{K} \) i.e., \( \lambda_k^* = H_k x_k^* + \beta_k \) where \( H_k \in \mathbb{R}^{n \times p} \) and \( \beta_k \in \mathbb{R}^{n \times 1} \).

Using Assumption 4.3 it can be shown that the two-point boundary value problem (35a)-(35d) can be solved if the following backward equations for \( k \in \mathcal{K} \setminus \{ K \} \) and \( i \in \mathcal{N} \) has a solution
\[
\begin{align*}
\Gamma_{k+1} &= I + S_k H_{k+1}, \\
H_k &= Q_k + A_k^T H_{k+1} (\Gamma_{k+1})^{-1} A_k, \\
\beta_k &= p_k - M_k^T \mu_{k+1} + L_k v_{k+1}^* + A_k^T \beta_{k+1} - A_k^T H_{k+1} (\Gamma_{k+1})^{-1} S_k \beta_{k+1},
\end{align*}
\] (36a)

17
where \( S_k = B_k R_k^{-1} B_k' \), \( H_K = Q_K \) and \( \beta_K = p_K + L_K v_k^* - M_k^* \mu_k^* \). Assuming \( \Gamma_{k+1}^k \) to be invertible for \( k = K - 1, \ldots, 1 \), we obtain \( H_k \) and \( \beta_k \) for \( k = K - 1, \ldots, 0 \), and the state vector \( x_k^* \) and the joint control vector \( u_k^* \) are given by

\[
\begin{align*}
x_{k+1}^* &= \left( \Gamma_{k+1}^k \right)^{-1} \left( A_k x_k^* - S_k \beta_{k+1} \right), \quad k \in \mathcal{K}\setminus\{ K \}, \\
u_k^* &= -R_k^{-1} B_k' \left( H_{k+1} x_{k+1}^* + \beta_{k+1} \right).
\end{align*}
\] (37a, 37b)

**Remark 4.4.** Suppose that the set of backward equations (36a)-(36c) admits a solution, i.e., the matrix \( \Gamma_{k+1}^k \) is invertible for all \( k \in \mathcal{K}\setminus\{ K \} \), then the two point boundary value problem in (35b)-(35d) has a unique solution. To show this, let \( \lambda_k^* = -H_k x_k^* + \beta_k \) be another solution for the two point boundary value problem (35b)-(35d). Substituting \( \lambda_k^* = \lambda_k + H_k x_k^* + \beta_k \) in (35b) and (35c), we get a decoupled system of equations as

\[
\begin{align*}
x_{k+1} &= \left( \Gamma_{k+1}^k \right)^{-1} \left( A_k x_k - S_k \lambda_{k+1} - S_k \beta_{k+1} \right), \\
\lambda_k &= A_k^T \lambda_{k+1} - A_k^T H_{k+1} \left( \Gamma_{k+1}^k \right)^{-1} S_k \lambda_{k+1}.
\end{align*}
\] (38a, 38b)

From the terminal condition, \( \lambda_K = 0 \) which results in \( \lambda_k = 0 \) \( \forall k \in \mathcal{K} \). This proves that the solution for the two point boundary value problem described by (35b)-(35d) is unique.

In view of Remark 4.4 we have the following standing assumption in the remaining part of the paper.

**Assumption 4.5.** The set of matrices \( \{ \Gamma_{k+1}^k, k \in \mathcal{K}\setminus\{ K \} \} \) are invertible.

Next, we note that the backward equations (36a) and (36b) are coupled but evolve independently of (36c). Taking \( G_{k+1} = A_k^T - A_k^T H_{k+1} \left( \Gamma_{k+1}^k \right)^{-1} S_k \), (36c) can be represented in the vector form as follows

\[
\beta_k = \sum_{\tau=k}^K \psi(k, \tau) \left( p_\tau + [L_\tau - M_\tau^\prime] \begin{bmatrix} v_\tau^* & \mu_\tau^* \end{bmatrix} \right), \quad \forall k \in \mathcal{K},
\] (39)

where the transition matrix associated with (36c) is \( \psi(k, \tau) = G_{k+1} \cdots G_{\tau-1} G_\tau \), if \( \tau > k \) and \( \psi(k, k) = I \). Thus, (39) represents a parametric linear backward difference equation parametrized by \( \{ v_k^*, \mu_k^*, k \in \mathcal{K}, i \in \mathcal{N} \} \). Now, we analyse the forward difference equation for the state trajectory (37a). Suppose

\[
x_{k+1} = \tilde{A}_k x_k^* + \tilde{B}_k \beta_{k+1}, \quad x_0 \text{ is given},
\]

where \( \tilde{A}_k = (\Gamma_{k+1}^k)^{-1} A_k \) and \( \tilde{B}_k = -((\Gamma_{k+1}^k)^{-1} S_k) \forall k \in \mathcal{K}\setminus\{ K \} \). Denoting the transition matrix as \( \phi(\rho, k) = \tilde{A}_{k-1} \tilde{A}_{k-2} \cdots \tilde{A}_\rho \) for \( \rho < k \) and \( \phi(k, k) = I \), and from (39) we have for \( k \in \mathcal{K}\setminus\{ 0 \} \),

\[
x_k^* = \phi(0, k) x_0 + \sum_{\tau=1}^K \left( \min_{\rho=1}^{\min(k, \tau)} \phi(\rho, k) \tilde{B}_{\rho-1} \psi(\rho, \tau) \left( p_\tau + [L_\tau - M_\tau^\prime] \begin{bmatrix} v_\tau^* & \mu_\tau^* \end{bmatrix} \right) \right).
\] (40)

Further, we combine the variables in (40) as \( p_K = [p_1^T \ldots p_K^T] \), \( x_K^* = [x_1^* \ldots x_K^*]^T \) and \( y_K^* = [v_1^* \mu_1^* \ldots v_K^* \mu_K^*]^T \). As a result, (40) can be written as:

\[
x_K^* = \Phi_0 x_0 + \Phi_1 p_K + \Phi_2 y_K^*.
\] (41)

where \( [\Phi_0]_k = \phi(0, k) \), \( [\Phi_1]_{k\tau} = \sum_{\rho=1}^{\min(k, \tau)} \phi(\rho, k) \tilde{B}_{\rho-1} \psi(\rho, \tau) \), \( [\Phi_2]_{k\tau} = \sum_{\rho=1}^{\min(k, \tau)} \phi(\rho, k) \tilde{B}_{\rho-1} \psi(\rho, \tau) \left[ L_\tau - M_\tau^\prime \right] \) for \( k, \tau \in \mathcal{K}\setminus\{ 0 \} \). Thus, we expressed the state trajectory parametrized with \( \{ v_k^*, \mu_k^*, k \in \mathcal{K}\setminus\{ 0 \} \} \)
where $\{ (\text{difference equations} \} )$. Next, we analyse the parametric linear complementarity problem in (35e)-(35f) in detail. First, the vector representation of these problems is given by

$$p\text{LCP}(x^*_k): \begin{bmatrix} D_k & -N'_k \\ N_k & 0 \end{bmatrix} \begin{bmatrix} v^*_k \\ \mu^*_k \end{bmatrix} + \begin{bmatrix} L'_k \\ M_k \end{bmatrix} x^*_k + \begin{bmatrix} d_k \\ r_k \end{bmatrix} \perp \begin{bmatrix} v^*_k \\ \mu^*_k \end{bmatrix}. \tag{42}$$

Let $\tilde{M} = M + N'_k + N_k - N'_k \oplus \tilde{Q} = \tilde{Q} \oplus \tilde{Q} \oplus \tilde{Q} \oplus \tilde{Q}$ and $\tilde{s} = [d_1', r_1' \cdots d_K', r_K']'$. Aggregating these parametric problems for all time steps $k \in K \setminus \{0\}$, we obtain a single parametric linear complementarity problem as

$$p\text{LCP}(x^*_K): \tilde{M}y^*_K + \tilde{q}x^*_K + \tilde{s} \perp y^*_K. \tag{43}$$

Substituting (41) in (43) results in the following large scale linear complementarity problem:

$$\text{LCP}(x_0): M y^*_K + q \perp y^*_K. \tag{44}$$

where $M = \tilde{M} + \tilde{q} \Phi_2$ and $q = \tilde{q}(\Phi_0 x_0 + \Phi_1 p_K) + \tilde{s}$. Thus, we formulated the necessary conditions (16a)-(16g) as a single large scale linear complementarity problem with the aid of Assumptions 4.3 and 4.5. Solving (44) we obtain candidate optimal solution of OCP1 there by a candidate open-loop Nash equilibrium of NZDG1.

Next, we study sufficient conditions under which the solution of LCP($x_0$) and pLCP($x_0$) indeed minimize OCP1, and as a result provide an open-loop Nash equilibrium of NZDG1. The sufficient conditions provided in Theorem 3.3 requires both the minimized instantaneous and terminal Lagrangian functions to be convex in the state variables. Since the solutions $v^*_k'$ are obtained from solving the parametric linear complementarity problem (43), upon substitution, the minimized Lagrangian functions may not be convex in the state variables. So, to derive the required sufficient conditions we transform the OCP1 as static optimization problem in the decision variables $(u, v)$. Toward this end, the objective function associated with the OCP1 can be written as

$$J(x_0, \tilde{u}, \tilde{v}) = \sum_{k=0}^{K-1} \frac{1}{2} u'_k R_k u_k + \sum_{k=0}^{K} \frac{1}{2} x'_k Q_k x_k + \left( p_k + L_k v^*_k - M'_k \mu^*_k \right)' x_k$$

$$\quad + \sum_{k=0}^{K} \left( \frac{1}{2} v'_k D_k v_k + d'_k v_k + x'_k L_k (v_k - v^*_k) + \mu'_k M_k x_k \right), \tag{45}$$

where $\{(v^*_k, \mu^*_k), k \in K\}$ is the solution of LCP($x_0$) and pLCP($x_0$). We define value function $W_k, k \in K$ as

$$W_k = \frac{1}{2} x'_k E_k x_k + e'_k x_k + w_k,$$

where $E_k \in \mathbb{H}^{n \times n}, e_k \in \mathbb{H}^n, w_k \in \mathbb{H}$ and $i \in N$. Let the matrices $T_k := R_k + B'_k E_{k+1} B_k$ be invertible for all $k \in K \setminus \{K\}$ with the matrices $E_k, k \in K$ computed as the solution of the following backward Ricatti difference equations

$$E_k = A'_k E_{k+1} A_k + Q_k - A'_k E_{k+1} B_k T_k^{-1} B'_k E_{k+1} A_k, \quad E_K = Q_K, \tag{46a}$$

$$e_k = A'_k e_{k+1} - A'_k E_{k+1} B_k T_k^{-1} B'_k e_{k+1} + p_k + L_k v^*_k - M'_k \mu^*_k, \quad e_K = p_K + L_K v^*_K - M'_K \mu^*_K, \tag{46b}$$

$$w_k = w_{k+1} - e'_k E_k e_{k+1}, \quad w_K = 0. \tag{46c}$$
Then, by denoting $\Delta_k = W_{k+1} - W_k$, and using the sum $\sum_{k=0}^{K-1} \Delta_k$ and (46) we can write the objective function (45) as

$$J(x_0, u, v) = W_0 + \sum_{k=0}^{K-1} \frac{1}{2}||u_k - T_k^{-1}B_k'(E_{k+1}A_kx_k + e_{k+1})||_T^2$$

$$+ \sum_{k=0}^{K} \left( \frac{1}{2}v_k'D_kv_k + d_k'v_k + x_k'L_k(v_k - v_k^*) + \mu_k^*M_kx_k \right).$$  (47)

The next lemma relates how the candidate optimal control (37b) obtained by solving the two-point boundary value problem (35b)-(35d) is related to the minimizer of the objective function (47).

**Lemma 4.6.** Let the set of matrices $\{T_k, k \in K\}$ be invertible and the solutions $E_k$ of the symmetric matrix Riccati difference equation (46a) exist for all $k \in K$. If the two-point boundary value problem

$$\bar{\lambda}_k = A_k'\bar{\lambda}_{k+1} + Q_k\bar{x}_k + p_k + L_kv_k^* - M_k'\mu_k^*,$$  (48a)

$$\bar{\lambda}_K = Q_Kx_k + p_K + Lv_K^* - M_K'\mu_K^*,$$  (48b)

$$\bar{x}_{k+1} = A_k\bar{x}_k - B_kR_k^{-1}B_k'\bar{\lambda}_{k+1}, \quad \bar{x}_0 = x_0$$  (48c)

has a unique solution, then we set $u_k = -R_k^{-1}B_k'\bar{\lambda}_{k+1}$ and $\bar{e}_k := \bar{\lambda}_k - E_k\bar{x}_k$. Then the sequences $\{\bar{x}_k, \bar{e}_k\}$ solve equations $u_k + T_k^{-1}B_k'(E_{k+1}A_k\bar{x}_k + \bar{e}_{k+1}) = 0$ and (46b).

**Proof.** Firstly, we have

$$\bar{u}_k + T_k^{-1}B_k'(E_{k+1}A_k\bar{x}_k + \bar{e}_{k+1}) = -R_k^{-1}B_k'\bar{\lambda}_{k+1} + T_k^{-1}B_k'\bar{\lambda}_{k+1} + T_k^{-1}B_k'\bar{e}_{k+1}$$

$$= \left( -R_k^{-1} + T_k^{-1} \left( R_k + B_k'\bar{u}_{k+1}B_k \right) R_k^{-1} \right) B_k'\bar{\lambda}_{k+1} = 0.$$

To prove (46b) we have

$$A_k'\bar{e}_{k+1} - A_k'E_{k+1}B_kT_k^{-1}B_k'\bar{e}_{k+1} + p_k + L_kv_k^* - M_k'\mu_k^* - \bar{e}_k$$

$$= p_k + L_kv_k^* - M_k'\mu_k^* + A_k'\bar{\lambda}_{k+1} - \bar{e}_k - A_k'E_{k+1}B_kT_k^{-1}B_k'\bar{\lambda}_{k+1}$$

$$+ A_k'E_{k+1}B_kT_k^{-1}B_k'\bar{e}_{k+1} - I \left( A_k\bar{x}_k - B_kR_k^{-1}B_k'\bar{\lambda}_{k+1} \right)$$

$$= \left( p_k + L_kv_k^* - M_k'\mu_k^* \right) + A_k'\bar{\lambda}_{k+1} + Q_kx_k - \bar{\lambda}_k$$

$$- A_k'E_{k+1}A_k - A_k'E_{k+1}B_kT_k^{-1}B_k'E_{k+1}A_k + Q_k - E_k \right) x_k$$

$$+ A_k'E_{k+1}B_k \left( R_k^{-1} - T_k^{-1}B_kE_{k+1}B_kR_k^{-1} \right) B_k'\bar{\lambda}_{k+1} = 0.$$

The last step in the above expression is obtained by using (46a) and (48a),(48b).

In the following lemma we provide conditions under which the objective function (47) is a strictly convex function of $(u, v)$.

**Lemma 4.7.** Let the solution $E_k$ of the symmetric Riccati equation (46a) exist. Let

$$\gamma_k = \begin{cases} B_{r+1}A_{r}A_{r-1} \ldots A_{k+1} & 0 \leq k < K - 1 \\ 0 & k = K - 1 \end{cases}$$  (49)

Now, we define the matrix

$$H = \begin{bmatrix} Y & C' \\ C & D \end{bmatrix}.$$  (50)
where \( D = \oplus_{k=0}^{K} D_k \), \( C = \[
0 & B_0' L_1 & B_0' A_1' L_2 & \cdots & B_0' A_{K-2}' A_{K-1}' L_K \\
0 & 0 & B_1' L_2 & \cdots & B_1' A_{K-2}' A_{K-1}' L_K \\
0 & 0 & 0 & \cdots & B_2' A_{K-2}' A_{K-1}' L_K \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{K-1}' L_K 
\] \)

and \( Y \) is a \( K \times m \times K \times m \) matrix where each block submatrix \([Y]_{lk}\) is a \( m \times m \) matrix such that for \( k \in \mathcal{K}\setminus\{K\} \)

\[
[Y]_{lk} = \begin{cases} 
B'_k E_{k+1} A_k \ldots A_{l+1} B_l + B'_k \left( \sum_{\tau=k+1}^{K-1} (\tilde{\gamma}_k^\tau)' T_{\tau-1}(\tilde{\gamma}_k^\tau) A_k \ldots A_{l+1} \right) B_l, & 0 \leq l < k, \\
T_k + B'_k \left[ \sum_{\tau=k+1}^{K-1} (\tilde{\gamma}_k^\tau)' T_{\tau-1}(\tilde{\gamma}_k^\tau) \right] B_k, & l = k, \\
\end{cases}
\]

\([Y]_{kl} = [Y]_{lk}^T\).

If the matrix \( H \) is positive definite then the objective function \((47)\) is a strictly convex function of \((\bar{u}, \bar{v})\).

Proof. We compute the Hessian matrix of the objective function \((47)\) with respect to the decision variables \((\bar{u}, \bar{v})\) by eliminating the state variable \(x_k\). Now, we have \(\frac{\partial^2 J}{\partial (v_k) \partial v_k} = D_k, \frac{\partial^2 J}{\partial (v_k) \partial u_k} = 0\), for \(k \neq l, k, l \in \mathcal{K}\). Calculating the remaining second order partial derivatives we get

\[
[Y]_{lk} = \frac{\partial^2 J}{\partial (u_l) \partial u_k} = B'_k E_{k+1} A_k \ldots A_{l+1} B_l + B'_k \left[ \sum_{\tau=k+1}^{K-1} (\tilde{\gamma}_k^\tau)' T_{\tau-1}(\tilde{\gamma}_k^\tau) \right] A_k \ldots A_{l+1} B_l, \quad 0 \leq l < k,
\]

\[
[Y]_{kk} = \frac{\partial^2 J}{\partial (u_k) \partial u_k} = T_k + B'_k \left[ \sum_{\tau=k+1}^{K-1} (\tilde{\gamma}_k^\tau)' T_{\tau-1}(\tilde{\gamma}_k^\tau) \right] B_k.
\]

\[
[C]_{lk} = \frac{\partial^2 J}{\partial (u_k) \partial v_k} = \begin{cases} 
B'_k A_{l+1} \ldots A'_{k-1} L_k, & l < k - 1 \\
B'_k L_k, & l = k - 1 \text{ and } k \geq 0,
\end{cases}
\]

Then, if the Hessian matrix \( H \) is positive definite, the objective function \((47)\) is strictly convex in \((\bar{u}, \bar{v})\).

Next, using the above result in the next theorem we show that the solutions of \(\text{LCP}(x_0)\) and \(\text{pLCP}(x_0)\) indeed provide the optimal solution of the OCP1.

**Theorem 4.8.** Let Assumptions 4.1, 4.3 and 4.5 hold true. Let the Hessian matrix \( H \) given by \((50)\) is positive definite. Then the solutions of \(\text{LCP}(x_0)\) and \(\text{pLCP}(x_0)\) constitute an open-loop Nash equilibrium of the NZDG1.

Proof. Let \(\{(v_k^*, \mu_k^*), k \in \mathcal{K}\}\) be the solution of \(\text{LCP}(x_0)\) and \(\text{pLCP}(x_0)\). Then transforming the objective function of the OCP1 as \((47)\) and consider the minimization problem subject to the state dynamics \(x_{k+1} = A_k x_k + B_k u_k\) and the constraints \(M_k x_k + N_k v_k + r_k \geq 0, v_k \geq 0\). Since \( H \) is positive definite, we have that the objective function \(J(x_0, \bar{u}, \bar{v})\) is strictly convex in \((\bar{u}, \bar{v})\) from Lemma 4.7. Also, from Assumption 2.1, the sets \(\{U_k^i, k \in \mathcal{K}\setminus\{K\}\}\) and \(\{V_k^i, k \in \mathcal{K}\}\) for all \(i \in \mathcal{N}\) are non-empty, convex and bounded. Therefore, by solving the KKT conditions, we obtain the solution of the static optimization. The Lagrangian associated with this optimization problem is given by

\[
\mathcal{L} = J(x_0, \bar{u}, \bar{v}) - \sum_{k=0}^{K} \mu_k^* \left( M_k x_k + N_k v_k + r_k \right).
\]  

(51)
The KKT conditions are then given by

$$T_k \left( u_k + T_k^{-1} B_k^\prime \left( E_{k+1} A_k x_k + e_{k+1} \right) \right) + B_k^\prime \sum_{\tau=k+1}^{K-1} \left( (Y_k^\tau)\prime \left( u_\tau + T_\tau^{-1} B_\tau^\prime \left( E_{\tau+1} A_\tau x_\tau + e_{\tau+1} \right) \right) \right)$$

$$+ B_k^\prime \left( L_{k+1} (v_{k+1} - v_{k+1}) - M_{k+1}^\prime (\mu_{k+1} - \mu_{k+1}) \right) + \sum_{\tau=k+1}^{K-1} (A_\tau A_{\tau-1} \cdots A_{k+1} B_k^\prime) \prime \left( L_{\tau+1} (v_{\tau+1} - v_{\tau+1}) \right)$$

$$- \sum_{\tau=k+1}^{K-1} \left( M_{\tau+1}^\prime (\mu_{\tau+1} - \mu_{\tau+1}) \right) = 0,$$  \hspace{1cm} (52a)

$$0 \leq D_k v_k - N_k^\prime \mu_k + L_k^\prime x_k + d_k \perp v_k \geq 0,$$  \hspace{1cm} (52b)

$$0 \leq M_k x_k + N_k v_k + r_k \perp \mu_k \geq 0.$$  \hspace{1cm} (52c)

Let \( x_k^* \), \( k \in K \) be the state trajectory generated by the solution \( \{(v_k^*, \mu_k^*)\}, k \in K \) using (41). Next, the co-state defined by \( \lambda_k^* = H_k x_k^* + \beta_k \) along with the state vector \( x_k^* \) solve the two point boundary value problem (48). Then, from Lemma 4.6, it follows that \( u_k + T_k^{-1} B_k^\prime \left( E_{k+1} A_k x_k^* + e_{k+1} \right) = 0 \) for all \( k \in K \setminus \{ K \} \). Using this in (52a) we obtain

$$B_k^\prime \left( L_{k+1} (v_{k+1} - v_{k+1}^*) - M_{k+1}^\prime (\mu_{k+1} - \mu_{k+1}^*) \right) + \sum_{\tau=k+1}^{K-1} (A_\tau A_{\tau-1} \cdots A_{k+1} B_k^\prime) \prime \left( L_{\tau+1} (v_{\tau+1} - v_{\tau+1}^*) \right)$$

$$- \sum_{\tau=k+1}^{K-1} M_{\tau+1}^\prime (\mu_{\tau+1} - \mu_{\tau+1}^*) = 0.$$  \hspace{1cm} (52b)

The above equation and the remaining equations (52b) and (52b) are satisfied by \( (\tilde{v}_k^*, \mu_k^*) \) as they are solutions of pLCP(\( x_k^* \)). This implies, we have shown that the solutions are of the LCP(\( x_0 \)) along with pLCP(\( x_0 \)) indeed provide the optimal solution of the OCP1. Since the OCP1 is associated with OLDPG we have that \( (\tilde{u}^*, \tilde{v}^+) \), obtained from solutions of LCP(\( x_0 \)) and pLCP(\( x_0 \)), provides an open-loop Nash equilibrium of NZDG1.  \hspace{1cm} \blacksquare

## 5 Illustration: Smart grid system with energy storage

To illustrate our results, we consider a smart grid system with energy storage. Smart grids provide opportunities for exploring distributed renewable energy sources. However, integrating solar and wind based sources has been a challenge in meeting the demand due to their intermittent nature. Energy storage systems become critical in providing continuous power in the case of interruption and are often used as an emergency power supply during unforeseen outages (Oh (2011)). Power utilities can cut their generation costs by storing energy during the off-peak hours and releasing during the peak hours. So, installing energy storage systems is crucial for an efficient, reliable and resilient smart grid, see Kolokotsa et al. (2019). However, setting up a centralized energy storage unit for all the smart-grid users is not practical due to high set up costs and maintenance. One alternative would be to incentivize prosumers to install energy storage units at their homes; see Figure 1 for an illustration. In this way, the smart grid storage system becomes decentralized. Additionally, the storage for each user is limited by the total resources available in the grid as well as the battery capacity. In Zazo et al. (2016), the authors study an energy demand problem in smart-grids without energy storage, and model the decision problem as a dynamic non-co-operative game without constraints. We build upon the model studied in Zazo et al. (2016) by incorporating energy storage incentives for the prosumers, and model the decision problem as a dynamic game with inequality constraints. Consider a smart grid with \( N \) users who utilize the smart grid resources for different activities like heating, lighting, and other home appliances. Let \( N = \{1, 2, \cdots, N\} \) be the set of users and the total time period be \( K = \{1, 2, \cdots, K\} \). The
final time $K$ is determined by the uniform interval with which the electrical data is processed in the grid in a day. For instance, if the data is processed every two hours in a day, $K = 12$. The smart grid consists of $S$ type of energy resources such as solar, hydroelectric, coal etc., and each user $i \in N$ consumes energy for $m_i$ activities. All resources are shared by all users. The state of the game at each time step, $X_k \in \mathbb{R}^S$ is the total amount of consumable resources in the smart grid. In the smart grid, users act as prosumers, implying that they not only consume energy, but also contribute the excess energy produced by renewable resources back to the grid. Furthermore, the resources can be autonomously recharged. Therefore, state of the game is governed by the discrete time dynamics

$$X_{k+1} = \tilde{A}_k X_k + \sum_{i=1}^{N} \tilde{B}_i I_i^k, \quad k \in \mathcal{K}\setminus\{K\},$$

(53)

where, at time step $k$, $\tilde{A}_k \in \mathbb{R}^{S \times S}$ governs the energy which is autonomously depleted or replenished, $I_i^k \in \mathbb{R}^{m_i}$ denotes the amount of resources consumed or contributed by user $i$ and $\tilde{B}_i \in \mathbb{R}^{S \times m_i}$ is the weight associated with the resource expenditure or contribution. The smart grid authority has provided a battery storage unit for each user as a secondary storage unit. This is provided for the purpose of island-ed mode, operation under unforeseen disconnection of the smart grid from the main power grid; see Ray and Biswal (2020). The amount of resource stored in each player’s battery unit at time instant $k$ is $K_i^k \geq 0$. The total energy storage in batteries is limited to be $\epsilon_k > 0$ units lower than the total resources of the grid, since excess of battery storage only results in higher storage costs. The limiting factor $\epsilon_k$ is chosen in such a way that the maximum storage limit represents the maximum energy required for the users to perform the essential activities during island-ed mode operation. Further, the amount of energy stored in the battery is limited by it’s maximum capacity denoted by $K_{i \text{ max}}^k$. Therefore the constraints on the battery storage are given by

$$\sum_{i \in \mathcal{N}} K_i^k \leq \sum_{i \in S} X_i^k - \epsilon_k,$$

(54a)

$$0 \leq K_i^k \leq K_{i \text{ max}}^k, \quad i \in \mathcal{N}.$$  

(54b)
The costs incurred by users are attributed to unsatisfied demand, unbalanced resources and battery storage. Each user has a target demand to meet which is given by $P_k^i X_k$, where $P_k^i \in \mathbb{R}^{m \times S}$ denotes the demand matrix. The cost associated with unsatisfied demand is characterized by the following quadratic function

$$
(\Pi_k^i)_{ud} = \frac{1}{2} \left( P_k^i X_k - I_k^i \right)' \tilde{R}_k^i \left( P_k^i X_k - I_k^i \right), \quad k \in \mathcal{K}\setminus\{K\}, \tag{55}
$$

where $\tilde{R}_k^i = r_k^i \mathbf{I} \in \mathbb{R}^{m \times m}$ with $r_k^i > 0$. For higher values of the parameter $r_k^i$ the user prioritizes in minimizing the unsatisfied demand cost compared to other costs. Next, if the resources at a time step are higher than at the previous step, then there is a cost associated with storage. Similarly, there are also costs associated with productivity loss between consecutive time periods. These costs are modeled as

$$
(\Pi_k^i)_{ur} = \frac{1}{2} \left( X_k - X_{k-1} \right)' \tilde{Q}_k \left( X_k - X_{k-1} \right), \tag{56}
$$

where $\tilde{Q}_k = q_k \mathbf{I} \in \mathbb{R}^{S \times S}$ with $q_k > 0$. For higher values of the parameter $q_k$, the user prioritizes in keeping the resources at steady state levels without spikes. Each player incurs a battery storage cost which is given by

$$
(\Pi_k^i)_{bs} = \frac{1}{2} K_k^i b_k^i, \tag{57}
$$

where $b_k^i$ represents the battery storage cost per energy unit for the user $i$. Along with these costs, there is an incentive provided to users for energy storage, which has two components: a player specific incentive which depends only on the player’s battery resource, and a common incentive which depends on the total grid resource. The total incentive for Player $i$ is given by

$$
(\Pi_k^i)_{ic} = a_k^i K_k + X_k' \tilde{L}_k \begin{bmatrix} K_1 \cdots & K_N \end{bmatrix}', \tag{58}
$$

where the parameter $a_k^i$ reflects the player specific incentive and the parameter $\tilde{L}_k \in \mathbb{R}^{S \times N}$ the common incentive. The salvage cost incurred by Player $i$ for the amount of resources in the last stage $K$ which is given by

$$
\Pi_k^i = \frac{1}{2} X_k' \tilde{Q}_K X_K, \quad \text{with} \quad \tilde{Q}_K = q_K \mathbf{I} \in \mathbb{R}^{S \times S}. \tag{59}
$$

Player $i$ using the consumption (or contribution) and storage schedules $\{I_k^i, \ k \in \mathcal{K}\setminus\{K\}, \ K_k^i, \ k \in \mathcal{K}\}$ seeks to minimize the total cost given by

$$
J^i = \Pi_K^i + \sum_{k=0}^{K-1} \left( (\Pi_k^i)_{ud} + (\Pi_k^i)_{ur} + \sum_{j=0}^{K} \left( (\Pi_k^i)_{bs} - (\Pi_k^i)_{ic} \right) \right)
= \frac{1}{2} X_k' \tilde{Q}_K X_k + \sum_{k=0}^{K-1} \left( \frac{1}{2} (P_k^i X_k - I_k^i)' \tilde{R}_k^i (P_k^i X_k - I_k^i) + \frac{1}{2} (X_k - X_{k-1})' \tilde{Q}_k (X_k - X_{k-1}) \right)
+ \sum_{k=0}^{K} \left( \frac{1}{2} K_k^i b_k^i - \left( a_k^i K_k^i + X_k' \tilde{L}_k \begin{bmatrix} K_1 \cdots & K_N \end{bmatrix}' \right) \right).
$$

We transform the above dynamic game problem with inequality constraints to the standard form NZDG1 as follows.

$$
x_k = \begin{bmatrix} X_k' & X_{k-1}' \end{bmatrix}', \quad v_k = \begin{bmatrix} K_1 & \cdots & K_N \end{bmatrix}',
\quad u_k^i = P_k^i X_k - I_k^i, \quad u_k = \begin{bmatrix} u_1^i & \cdots & u_K^i \end{bmatrix}'.
$$
The smart grid resource allocation problem with energy storage is modeled as NZDG1 with parameters defined as follows:

\[
A_k \triangleq \begin{bmatrix}
\hat{A}_k + \sum_{i=1}^{N} \bar{B}_i^k P_i^k & 0_{S \times S} \\
1_S & 0_{S \times S}
\end{bmatrix}, \quad D_k^i \triangleq \begin{bmatrix}
-\tilde{B}_k^i \\
0_{S \times m_i}
\end{bmatrix},
\]

\[
Q_K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \hat{Q}_K, \quad Q_k = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tilde{Q}_k, \quad d_k^i = -a_k^i e_i,
\]

\[
D_k^i = b_k^i e_i e_i', \quad R_k^i = 0_{m_1 \times m_1} \oplus \cdots \oplus \bar{R}_k^i \oplus 0_{m_N \times m_N},
\]

\[
M_k = \begin{bmatrix} 1_{1 \times S} & 0_{1 \times S} \\ 0_{N \times S} & 0_{N \times S} \end{bmatrix}, \quad N_k = -\begin{bmatrix} 1_{1 \times N} \\ 1_N \end{bmatrix},
\]

\[
r_k = [-\epsilon_k \; K^1_{max} \; K^2_{max} \; \cdots \; K^N_{max}]'.
\]

Here, we observe that cost matrices \(Q_k, D_k^i, d_k^i, L_k, k \in \mathcal{K}\) and \(R_k^i, k \in \mathcal{K}\setminus\{K\}\) satisfy the conditions in (30). Therefore we can obtain the OCP1 associated with the related OLPDG by (31) from Theorem 4.2.

For illustration purpose, we assume \(K = 12\), that is, that the data is processed every two hours in a day. We assume \(S = 4\) resources, \(N = 2\) users and \(m_i = 2\) for both the users. The remaining parameters are taken as follows:

\[
q_K = 2.5, \quad q_k = 1, \quad r_k^1 = r_k^2 = 0.7, \quad b_k^1 = b_k^2 = 1.6,
\]

\[
a_k^1 = 3.4, \quad a_k^2 = 4, \quad \epsilon_k = 3.5, \quad K^1_{max} = 11.2, \quad K^2_{max} = 12.2,
\]

\[
\tilde{L}_k = 0.5 \times 1_{3 \times 2}, \quad \bar{A}_k = I_S, \quad P_k^1 = P_k^2 = 0.375 \times 1_{2 \times 3},
\]

\[
\tilde{B}_k^1 = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.75 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}, \quad X_{-1} = \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}.
\]

We assume that both the users are identical in terms of energy demand. We can consider two households with similar energy requirements. However, we assume that user 2 with a higher user specific incentive for storage, and reflected in the parameter values \(a_k^1 = 3.4\) and \(a_k^2 = 4\). Subsequently, the storage capacity of user 1’s batteries are set to \(K^1_{max} = 11.2\) units, and for user 2 are set as \(K^2_{max} = 12.2\). It can be verified that the sufficient conditions in Lemma 4.6 and Theorem 4.8 are satisfied with the chosen parameters. We have used the freely available software, the PATH solver (see http://pages.cs.wisc.edu/~ferris/path.html), for solving the linear complementarity problem (44).

Figure 2a illustrates the evolution of resources in the grid. We have assumed three sources of energy in the grid. The consumption or contribution to the grid is represented in Figure 2b. When a user consumes resources from the grid for an activity, the actual utility of the user corresponding to this activity is negative. Likewise, contribution to the grid results in positive utility. Here, \(I_k^1, I_k^2\) denote the consumption or contribution by the user \(i\) for \(m_i = 2\) activities. We note that the consumption or contribution of both the users are identical due to the identical demand costs. Further, we observe that both the users switch from contribution to consumption for the first activity at time period \(k = 8\) and for the second activity at time period \(k = 6\). As the users start to consume from the grid, the grid resources decrease which is evident from Figure 2a. The evolution of battery storage decision for both the users is shown in Figure 2c. User 2 has a higher user specific incentive as well as higher maximum storage capacity in comparison with user 1. Consequently, user 2 stores higher amount of energy in its battery. User 1 utilizes the full battery storage capacity from \(k = 3\) to \(k = 11\), and user 2 utilizes full battery storage capacity from \(k = 4\) to \(k = 10\). At the final time step, \(K = 12\), the total grid resources fall to a lower value, due to the salvage cost. Since the total battery storage is also limited by the total grid resources and \(\epsilon_k\), the battery storage of both the users also decreases at this stage.
Next, we analyze the effect of incentive parameter $a_k^i$ on the energy storage behavior of the users. We vary the player specific incentive parameter, $a_k^i$ for both the users. As the incentive parameter $a_k^i$ is varied with a 20% variation around the baseline values, without varying the battery storage cost parameters, we deduce that both the users store higher amount of energy in their batteries with higher values of the incentive parameter. Figures 3a and 3b illustrate that the users utilize higher capacities for a longer time when the incentive parameter is higher. However, since the battery capacity of each player is limited, the players are unable to increase the storage continually with increase in the incentive. Besides, full capacity is utilized for a shorter period when we lower the incentive parameter. Finally, the constraint on total storage (54a) also makes sure that the total energy stored in the battery is lower than the total resources by at least $\epsilon_k$. This result is illustrated in Figures 3c and 3d.

6 Conclusions

In this paper, we studied the conditions under which a class of $N$-player non-zero sum discrete time dynamic games with inequality constraints admits a potential game structure. Drawing motivations from the theory of static potential games, we associated an optimal control problem with inequality constraints, and derived conditions under which the solution of this optimal control problem provides a (constrained) open-loop Nash equilibrium. When the potential functions are not specified beforehand, we derived conditions under which the potential functions can be constructed using the problem data. We specialized these results to a linear quadratic setting and provided a linear complementarity problem based approach for computing the
open-loop Nash equilibrium. In particular, the computed equilibrium is a refinement of the open-loop Nash
equilibria obtained in Reddy and Zaccour (2015). We illustrated our results with an example inspired by
resource allocation in a smart grid network with energy storage. For future work, we plan to investigate the
existence of potential functions for this class of games under feedback information structure.

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