Why Quantum Phase Transitions Are Interesting

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This paper discusses why the usual notion that quantum phase transitions can be mapped onto classical phase transitions in a higher dimension, and that this makes the former uninteresting from a fundamental theoretical point of view, is in general misleading. It is shown that quantum phase transitions are often qualitatively different from their classical counterparts due to (1) long-ranged effective interactions that are induced by soft modes, and (2) in the presence of quenched disorder, an extreme anisotropy of space-time. These points are illustrated using various magnetic phase transitions as examples.

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1. INTRODUCTION

Phase transitions that occur in a quantum mechanical system at zero temperature ($T = 0$) as a function of some non-thermal control parameter, like pressure, or composition, are called quantum phase transitions. In contrast to their finite-temperature counterparts, which are often referred to as thermal or classical phase transitions, the critical fluctuations one has to deal with at zero temperature are quantum fluctuations rather than thermal ones, and the need for a quantum mechanical treatment of the relevant statistical mechanics makes the theoretical description of quantum phase transitions somewhat different from that of classical ones. Nevertheless, Hertz, in an important paper, showed that the basic theoretical concepts that have been used to describe and understand thermal transitions, work in the quantum case as well. In particular, he demonstrated in detail how to adapt the Wilsonian renormalization group (RG) to quantum phase transitions. That such an adaptation should be possible had been observed earlier by Beal-Monod.
Hertz also was the first to derive a Landau-Ginzburg-Wilson (LGW) order parameter functional for a quantum phase transition from a microscopic action. The example he used was itinerant electron magnetism. Starting from the general expression for the partition function,

\[ Z = \text{Tr} e^{-H/k_B T}, \tag{1} \]

he obtained a functional integral representation of \( Z \). The integration is with respect to an order parameter field that we generically denote by the vector field \( \mathbf{M}(x) \), where \( x = (\mathbf{x}, \tau) \) comprises the real space position \( \mathbf{x} \) and the imaginary time \( \tau \). In the case of a ferromagnet, \( \mathbf{M} \) represents the fluctuating magnetization. The partition function becomes,

\[ Z = \int D[\mathbf{M}] \ e^{S[\mathbf{M}]} = \int D[\mathbf{M}] \ e^{\int d\mathbf{x} s(M(x))}. \tag{2} \]

Here \( S \) is the action, \( s \) is the corresponding “action density”, and \( \int d\mathbf{x} = \int d\mathbf{x} \int_0^\beta d\tau \), with \( \beta = 1/k_B T \). Equation (2) is formally equivalent to the expression for a classical partition function in terms of functional integrals, except that instead of an integration over \( d \) space dimensions, it involves an integration over \( d \) space dimensions and one (imaginary) time dimension. This coupling between space and time is a fundamental aspect of quantum statistical mechanics. Physically, it represents the effects of quantum fluctuations. For finite \( \beta \), or nonzero temperatures, the time dimension is of finite extent, and consequently it makes no difference to the asymptotic critical behavior, but for \( \beta = \infty \) (\( T = 0 \)), it seems to suggest that the quantum critical behavior can be related to the critical behavior of a classical system in a higher dimension. Indeed, this is just what Hertz concluded. To state this mapping precisely, we let \( r \) be some appropriate dimensionless distance from the critical point, and \( \xi \) the correlation length which diverges at the transition. The critical exponent \( \nu \) characterizes the divergence of \( \xi \),

\[ \xi \sim 1/|r|^\nu. \tag{3} \]

As a critical point (quantum or classical) is approached, a relaxation time diverges as well. We denote this time scale by \( \xi_T \), and characterize its divergence in terms of that of \( \xi \). This defines the dynamical scaling exponent \( z \),

\[ \xi_T \sim \xi^z. \tag{4} \]

Hertz used these ideas to suggest that the critical behavior of a quantum phase transition in \( d \)-dimensions is identical to the critical behavior of a classical system in \( d_{\text{eff}} = d + z \) dimensions. This was a generalization of an earlier idea by Suzuki, who studied specific spin models where \( z = 1 \).
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Naively, this seems to imply that quantum phase transitions are not very interesting from a fundamental theoretical point of view, since classical transitions have been long studied, and are quite well understood. Further, in bulk systems, \( d = 3 \), the effective dimension satisfies (for \( z \geq 1 \)) \( d_{\text{eff}} \geq 4 \). Since the upper critical dimension of many classical phase transitions is \( d^c = 4 \), this is consistent with the notion that most quantum phase transitions are above the upper critical dimension where mean-field theory gives the exact critical behavior, making the quantum phase transition doubly uninteresting.

In this paper, we point out ways in which these conclusions can break down. In particular, we first discuss how the coupling between statics and dynamics that is inherent in quantum statistical mechanics leads to novel effects not reflected in the simple mapping \( d \to d + z \). (Parenthetically, we note that one can always map a quantum system on some contrived classical model with one or more long-ranged interactions. This is not what we mean by ‘simple’ mapping.) Secondly, we point out that in the presence of quenched disorder there is an extreme anisotropy between space and time that typically makes the quantum phase transition qualitatively different than the corresponding classical one. The choice to focus on these two aspects is somewhat subjective. There are other ways in which quantum phase transitions become of fundamental interest, some of which we mention in the Discussion, but we cannot do all of these topics justice in the present format.

The plan of this paper is as follows. In the next section, we discuss power-law correlations that generally exist in both classical and quantum systems, even away from any critical point. As we will see, the main distinction between the classical and quantum cases is that in the classical case these correlations only occur in the time domain, while in the quantum case they also exist in space. In Section 3, we show how the spatial power-law correlations discussed in Section 2 couple to, and ultimately determine, the critical behavior at many quantum phase transitions. In Section 4, we discuss the case of quenched disorder, and in particular explain why quantum transitions in systems with quenched disorder are distinct from any classical transition. We conclude with some remarks in Section 5. Among other things, we point out experimental connections, and additional reasons why quantum phase transitions are fundamentally interesting.
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2. GENERIC SCALE INVARIANCE IN CLASSICAL AND
QUANTUM PHYSICS

Homogeneous functions, or power laws, of space or time are characterized by the absence of any intrinsic length or time scales, in contrast to, e.g., exponentials. This absence of scales is referred to as scale invariance. It is well known to occur at critical points, where the critical modes become soft, which leads to power-law correlation functions. Critical points are exceptional points in the phase diagrams of materials, and reaching them requires the fine tuning of parameter values. What has become clear only more recently is the fact that many systems display what is now known as generic scale invariance (GSI), namely, power law correlation functions in entire regions of parameter space, with no fine tuning at all needed to observe them. GSI is caused by soft modes that are not related to critical phenomena, but rather are due to conservation laws, or possibly Goldstone modes, that are inherent to the system. In recent years there has been considerable attention to GSI in systems ranging from classical fluids, to liquid crystal systems, to disordered electron systems, and to granular, or sandpile, systems.

To understand the origin of GSI in classical systems we consider the case of a simple classical equilibrium fluid. In this system the best known example of power-law correlations is the temporal decay of the velocity autocorrelation, $C_D(t) = \langle v(t) \cdot v(0) \rangle_{eq}$, where $\langle \ldots \rangle_{eq}$ denotes an equilibrium thermal average. Its time integral determines the self-diffusion coefficient, $D$, in $d$-dimensions via

$$D = \frac{1}{d} \int_0^\infty dt \ C_D(t) \ .$$

(5)

$C_D(t)$ decays only algebraically for long times,

$$C_D(t >> t_0) \approx c \left( \frac{t_0}{t} \right)^{d/2} ,$$

(6)

where $t_0$ is the mean-free time between collisions, and $c$ is a positive constant. This slow decay of correlations was first observed in computer simulations of hard sphere fluids and then understood theoretically. It turns out that all time correlations in classical fluids that are of physical interest decay as power laws, and collectively these effects are known as long-time tails (LTT).

The basic physical idea behind the explanation of the LTT phenomena is that the hydrodynamic modes, which are the slowest decaying modes in noncritical systems, determine the long-time behavior of all time correlation functions, even those for nonhydrodynamic variables. Of particular importance among the hydrodynamics processes are recollision processes, where after a collision, the two involved particles diffuse away and then meet again.
and recollide. We see this in Eq. (6), the right-hand side of which is proportional to the probability that a diffusing particle returns at time $t$ to the point it started out from at $t = 0$. Note that this argument does not necessarily imply that any particular time correlation function decays as $t^{-d/2}$, since it might not couple strongly to this diffusive process. The argument does suggest, though, the possibility of power-law decays for generic correlation functions in systems with conserved quantities. Finally, it is important to note that even though time correlations in equilibrium classical fluids do exhibit GSI, spatial correlations do not; generically, they decay exponentially on the length scale of a particle diameter. For very fundamental reasons, dynamical correlations do not couple to static ones in classical equilibrium systems, and spatial correlations are of short range.

The situation is fundamentally different in quantum systems. Again, time correlations generically decay as power laws because of coupling to either hydrodynamic processes, or to dynamical Goldstone mode fluctuations. Further, in quantum statistical mechanics, statics and dynamics are fundamentally coupled together. This suggest that the same soft modes that cause quantum LTTs may lead to power-law correlation functions in space for the static, or equal-time, correlation functions. This is indeed what generically occurs. To be specific, consider a disordered and interacting electron system. In this case, it is well known that so-called “weak-localization effects” lead to a low-frequency nonanalyticity in the time correlation function that determines the electrical conductivity, $C_\sigma(\omega)$. In particular, in the low-frequency, long-time domains, and for $2 < d < 4$, one finds

$$C_\sigma(\omega)/C_\sigma(\omega = 0) = 1 + c \left(i\omega\right)^{(d-2)/2} + \ldots ,$$

$$C_\sigma(t \gg t_0)/C_\sigma(t = 0) \approx -c'(t_0/t)^{d/2} ,$$

with $c$ and $c'$ positive constants. The coupling between statics and dynamics in quantum systems suggests that static correlation functions such as the spin susceptibility, $\chi_s(q)$, with $q$ the wavevector, will exhibit a nonanalyticity at $q = 0$. Since for the diffusive processes that lead to the Eqs. (7,8), frequency and wavenumber scale as $\omega \sim q^2$, one expects

$$\chi_s(q) = c_0 - c_{d-2} \left|q\right|^{d-2} - c_2 \left|q\right|^2 - \ldots ,$$

which corresponds to a power-law long-distance spatial decay,

$$\chi_s(|x| \rightarrow \infty) \approx c''/|x|^{2(d-1)} .$$

Explicit calculations confirm these results with positive values for the various constants. All of these nonanalyticities are properties of the system at $T = 0$; at nonzero temperature $C_\sigma$ and $\chi_s$ are analytic functions of $\omega$. 

and \( k \), respectively. This illustrates an important point: In addition to the coupling between statics and dynamics, which induces spatial long-range correlations in equilibrium quantum systems, there are in general many more soft modes, which lead to long-range correlations, at \( T = 0 \) than at \( T > 0 \).

The question then arises: How do these inherent generic power-law correlations affect, and possibly modify, the critical behavior in the vicinity of a critical point? One anticipates dramatic effects from these correlations, if they couple to the order parameter of the phase transition. This point is addressed in the next section.

3. GENERIC SCALE INVARIANCE AND QUANTUM PHASE TRANSITIONS

We now discuss an example that shows how GSI can dramatically affect the critical behavior at a quantum phase transition. The example we choose is that of the quantum ferromagnetic transition in both clean and disordered itinerant electron systems. We start with the partition function, \( Z \), for a general disordered electron gas. In a field theoretic formalism,

\[
Z = \int D[\bar{\psi}, \psi] e^{S[\bar{\psi}, \psi]} .
\]

(11)

Here the functional integration is with respect to Grassmann valued fields, \( \bar{\psi} \) and \( \psi \), and \( S \) is the action. For simplicity we consider a \( d \)-dimensional continuum model of interacting electrons. Of the various electron-electron interaction channels, we pay particular attention to the particle-hole spin-triplet contribution, \( S_{\text{int}}^t \), since this is what is responsible for ferromagnetism. We denote the coupling constant in this channel by \( \Gamma_t \). Writing only this term explicitly, and denoting the spin density by \( n_s \), the action reads,

\[
S = S_0 + S_{\text{int}}^t = S_0 + \frac{\Gamma_t}{2} \int dx \ n_s(x) \cdot n_s(x) .
\]

(12)

Here \( S_0 \) contains all contributions to the action other than \( S_{\text{int}}^t \). In particular, it contains the effects of disorder, as well as the particle-hole spin-singlet and the particle-particle interactions. Following standard procedure, we perform a Hubbard-Stratonovich decoupling of \( S_{\text{int}}^t \) by introducing a classical vector field \( \mathbf{M}(x) \) with components \( M_i \) that couples linearly to \( n_s(x) \) and whose average is proportional to the magnetization, and we integrate out all fermion degrees of freedom. The partition function then reads,

\[
Z = e^{-F_0/T} \int D[\mathbf{M}] e^{-\Phi[\mathbf{M}]} ,
\]

(13)
where $F_0$ is the non-critical part of the free energy. Expanded in a power series, the LGW functional $\Phi$ reads,

$$\Phi[M] = \frac{1}{2\Gamma_t} \int dx dy \delta(x-y) M(x) \cdot M(y) + \sum_{n=2}^{\infty} a_n \int dx_1 \ldots dx_n$$

$$\times \chi^{(n)}_{i_1,\ldots,i_n}(x_1,\ldots,x_n) M^{i_1}(x_1) \ldots M^{i_n}(x_n) , \quad (14)$$

where $a_n = (-1)^{n+1}/n!$. The coefficients $\chi^{(n)}$ in Eq. (14) are connected $n$-point spin density correlation functions of a reference ensemble whose action is given by $S_0$. Notice that this reference ensemble contains the effects of all interaction amplitudes other than $\Gamma_t$, as well as the effects of disorder.

### 3.1. Disordered Ferromagnets

To be specific, we first consider the disordered case, i.e. itinerant ferromagnets with quenched, nonmagnetic impurities. To make the points relevant for this paper we focus on the quadratic or Gaussian part of the LGW functional, $\Phi_2$, and carry out the disorder average by means of the replica trick. With $\alpha$ denoting the replica label, and $q_n = (q, \Omega_n)$ a 4-momentum, where $\Omega_n = 2\pi T n$ is a bosonic Matsubara frequency, we have,

$$\Phi_2 = \frac{1}{2} \sum_{q_n} \sum_{\alpha} \left[ \frac{1}{\Gamma_t} - \chi^{(2)}(q_n) \right] |M^\alpha(q_n)|^2 . \quad (15)$$

Here $\chi^{(2)}$ is the Fourier transform of the dynamic spin susceptibility in the reference ensemble. Spin density conservation implies that at small frequency and wavenumber, $\chi^{(2)}$ has a diffusive structure,

$$\chi^{(2)}(q_n) = \chi_s(q) \frac{Dq^2}{|\Omega_n| + Dq^2} , \quad (16)$$

where $D$ and $\chi_s(q)$ are the spin diffusion coefficient and the static spin susceptibility, respectively, in the reference ensemble. In the critical limit, where the frequency must be taken to zero before the wavenumber, we have,

$$\chi^{(2)}(q_n) = \chi_s(q) \left[ 1 - |\Omega_n|/Dq^2 + \ldots \right] . \quad (17)$$

Now comes the crucial step of calculating $\chi_s(q)$ in the reference ensemble. Until recently it was universally, if tacitly, assumed that $\chi_s$ was an analytic function of $q^2$. The physical argument behind this assumption was that the reference ensemble corresponds to a physical system that is far from any critical point. Indeed, the whole philosophy of separating out
the spin-triplet interaction and introducing an order parameter field was based on the desire to separate the slowly decaying order parameter modes from all other modes, which were assumed to decay exponentially. However, as discussed in Section 2. above, it is now known that this reasoning is not correct, since any interacting itinerant electron system, especially at $T = 0$, has long-range correlations everywhere in the phase diagram, even far from any critical point. For small values of $q^2$, for $d > 2$, and at $T = 0$, the leading behavior of $\chi_s$ is given by Eq. (9) where $c_0$, $c_{d-2}$, and $c_2$ are positive constants. Notice that the susceptibility decreases with increasing wavenumber, which is due to the following effect. The diffusive dynamics of the electrons effectively increases the strength of the electron-electron interaction compared to a clean system, which means that disorder increases the homogeneous spin susceptibility. At nonzero wavenumbers, this effect gets smaller, which accounts for the negative sign of the nonanalytic term in Eq. (8). With decreasing spatial dimension, the nonanalyticity becomes stronger for phase space reasons, and for $d \leq 2$ the electrons get localized and a different theory is needed.

Using Eqs. (16,17) and (9) in Eq. (15) yields

$$\Phi_2 = \frac{1}{2} \sum_{q_n} \sum_{\alpha} \left[ r_0 + |q|^{d-2} + q^2 + |\Omega_n|/|q^2| \right]|M^\alpha(q_n)|^2. \quad (18)$$

Here

$$r_0 = 1/\Gamma_t - \chi_s(q = 0), \quad (19)$$

is the bare distance from the critical point, and we have omitted various prefactors in Eq. (18) which are not essential for our discussion. As noted in Section 2., the term $|q|^{d-2}$ in Eq. (8) implies long-range interactions in real space between spin density fluctuations. It is well known from the theory of classical phase transitions that long-range interactions suppress fluctuations effects, and that the critical behavior in systems with such interactions can usually be determined exactly. For example, simple RG arguments suggest that all terms higher than $n = 2$ in the Landau expansion, Eq. (4), are irrelevant (in the RG sense) for $d > 2$, so that the upper critical dimension for our phase transition is $d_c^+ = 2$. From this it follows that many of the critical exponents can be determined exactly by considering $\Phi_2$ only.

Although extensive work is necessary to ascertain that the above arguments are indeed valid, it turns out that several exponents are indeed determined exactly by $\Phi_2$. Among these are the order parameter susceptibility exponent $\gamma$, the correlation length exponent $\nu$, the dynamical scaling exponent $z$, and the exponent $\eta$, which determines the critical wavenumber dependence of the order parameter susceptibility. The values of these
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For $d > 4$, the exponents are, for $2 < d < 4$:

\[
\gamma = 1 \quad , \quad \nu = 1/(d - 2) \quad , \quad \eta = 4 - d \quad , \quad z = d .
\]  

For $d \geq d_c^{++} = 4$, all of these exponents ‘lock into’ their mean-field values.

To determine the critical exponents $\beta$ and $\delta$ we need the equation of state in the ordered phase. It has been shown in Ref. [9] that the same singularities that lead to the term $|q|^{d - 2}$ in Eq. (18) lead to nonanalyticities in the equation of state. For small values of the magnetization $m$ one obtains

\[
rm + m^{d/2} + m^3 = h ,
\]  

with $h$ the magnetic field, and $r$ the physical distance from the critical point, i.e. the renormalized counterpart of the bare distance $r_0$, Eq. (19). Once again, prefactors have been omitted. Notice the $m^{d/2}$ term, which occurs in addition to what is otherwise an ordinary mean-field equation of state. Its origin are again the effective long-range interactions between the spin fluctuations. For $d < 6$, the $m^{d/2}$ term dominates the usual $m^3$, and hence determines the critical exponents $\beta$ and $\delta$. Accordingly, for $2 < d < 6$, we have

\[
\beta = 2/(d - 2) \quad , \quad \delta = d/2 .
\]  

Note that these relations imply yet another upper critical dimension, namely, $d_c^{+++} = 6$, defined as the dimension above which $\beta$ and $\delta$ “lock into” their mean-field values of 1/2 and 3, respectively.

We conclude that GSI effects largely determine the quantum critical behavior at the ferromagnetic transition in disordered itinerant electron systems. We also note that the above discussion has been simplified for pedagogical purposes, and to underscore the relevant physics. A different approach, that keeps all of the soft modes explicitly and on equal footing, rather than integrating out all of the fermionic degrees of freedom, is technically more satisfactory. Such a more detailed analysis shows that complicated logarithmic corrections to scaling occur in all dimensions $d < d_c^{+++}$, which lead to log-log-normal terms multiplying the power-law critical behavior characterized by the above exponents. However, the basic physical arguments given above are unaffected by this complication.

### 3.2. Clean Ferromagnets

In the previous subsection, we considered the problem of disordered quantum ferromagnets. However, the only point in that discussion where the
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disorder was important was the diffusive dispersion relation of the “extra” (in addition to the soft critical mode) soft modes. This raises the possibility that similar effects might exist in clean itinerant ferromagnets. The first question that arises in this context is what, if anything, will replace the $|q|^{d-2}$ term in the static spin susceptibility, Eq. (9), in clean systems. To answer this, let us consider the perturbation theory for $\chi_s(q)$. What leads to the nonanalyticity in Eq. (9) is the coupling of two diffusive modes, which mathematically takes the form of a mode-coupling integral of the type,

$$\int d\mathbf{k} \int d\omega \frac{1}{\omega + k^2} \frac{1}{\omega + (k + q)^2} ,$$

with $q$ the external wavenumber. For simplicity, we have set the diffusion coefficient equal to unity. RG techniques have shown that this indeed gives the leading small wavenumber behavior of $\chi_s(q)$.

What changes in a clean system? The soft modes are still the density and spin density fluctuations, as well as more general particle-hole excitations. All of these have a linear dispersion relation, i.e., $\omega \sim |q|$. One might thus expect $\chi_s(q)$ in a clean system to have a mode-mode coupling contribution analogous to Eq. (23), but with ballistic modes instead of diffusive ones,

$$\int d\mathbf{k} \int d\omega \frac{1}{\omega + |k|} \frac{1}{\omega + |k + q|} .$$

In generic dimensions, expanding in $|q|$ leads to

$$\chi_s(q) \sim \text{const} + d_{d-1} |q|^{d-1} - d_2 |q|^2 ,$$

at $T = 0$. Here $d_{d-1}$ and $d_2$ are constant prefactors. For $d < 3$, the nonanalytic term in Eq. (25) represents the leading small wavenumber dependence of $\chi_s$. In $d = 3$, one finds a $q^2 \ln(1/q^2)$ term, and in $d > 3$ the analytic $q^2$ contribution is the leading one.

The coefficients $d_{d-1}$ and $d_2$ in Eq. (25) are positive in low order perturbation theory with respect to the electron-electron interaction. Physically, the fact that the second term in Eq. (23) is positive is consistent with the idea that the correlation effects which lead to this term suppress ferromagnetism. Note that the sign of the nonanalytic term in Eq. (23) is opposite that of the one in Eq. (9). We further note that the positive $c_{d-2}$ in Eq. (9) leads to the positive $|q|^{d-2}$ in Eq. (18) and the positive $n^{d/2}$ in Eq. (21). This implies, as has been confirmed by explicit calculations, that the equation of state in a clean itinerant ferromagnet at $T = 0$ is,

$$rm - m^d + m^3 = h .$$
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Fig. 1. Phase diagrams for various strengths of the disorder $G$ showing a tricritical point (TCP), critical end points (CE), and critical points (CP). Solid lines denote first order transitions, and dashed lines second order ones. From Ref. 15.

For $d \leq 3$ this equation of state predicts a discontinuous, or first order, ferromagnetic transition at zero temperature.

More detailed calculations confirm this result. Further, they yield a tricritical point at low temperatures. The net result is that in very clean systems the magnetic phase transition should be generically of first order. For sufficiently disordered systems, the transition is continuous, and the critical behavior is given by the results discussed in the previous subsection. Some of the typical phase diagrams in the $T$-$r$ plane, for various disorder strengths, are shown in Fig. 1. As in the disordered case, we again conclude that for clean ferromagnets, GSI effects largely determine the nature of the quantum phase transition in systems with physical values of the spatial dimensionality.

4. ANISOTROPIC QUANTUM PHASE TRANSITIONS: THE CASE OF QUENCHED DISORDER

Apart from their tendency to be influenced by GSI phenomena, there is another aspect of quantum phase transitions that makes them fundamentally different from classical ones. To understand this point, it is useful to consider a classical transition in an anisotropic system, i.e. one where all of the $d$ spatial dimensions are not equivalent. As long as the range of interactions in all directions is finite, the difference between the different directions becomes irrelevant, in the RG sense, as the critical point is approached. This makes sense physically: As one considers longer and longer length scales, all interactions effective become short-ranged, unless they decay slower than $1/|x|^{d+2}$ in a particular direction. However, the situation is different if the interactions in some dimensions are of long, or possibly infinite, range.

Quantum phase transitions with quenched disorder are generic examples where interactions in one direction (viz., the time direction) are of infinite
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range while the interactions in the other directions (viz., the spatial ones) are of finite range. To see this, consider a LGW action $S[N]$ for, say, the quantum antiferromagnetic phase transition in an itinerant electron system with nonmagnetic quenched disorder, with $N$ the fluctuating staggered magnetization:

$$S[N] = \frac{1}{2} \int dx \int_0^{1/T} d\tau \ N(x,\tau) \left[ r_0 + \delta r(x) - \nabla^2 + \partial^2_\tau \right] N(x,\tau) + u \int dx \int_0^{1/T} d\tau \ (N(x,\tau) \cdot N(x,\tau))^2 ,$$

(27)

with $u$ a positive constant, and $\delta r(x)$ a random-mass term that represents the effects of nonmagnetic impurities. For simplicity, we take it to be delta-correlated in space,

$$\{ \delta r(x) \delta r(y) \}_\text{dis} = G \delta(x - y) .$$

(28)

Here $\{ \ldots \}_\text{dis}$ denotes the disorder average, and $G$ is a measure of the disorder strength. We can now see the extreme anisotropy inherent in quantum statistical mechanics: The disorder is delta-correlated in space, but, formally, of infinite range in time. Using replica methods to integrate out the disorder leads to an additional quartic term with one spatial integral, but two time integrals, due to this infinite-range interaction in the time domain.

The sign of this new quartic term is the opposite of that of the $u$-term in Eq. (27). Further, the extra imaginary time integral in the new term, which is due to the long-range interaction in time, means that compared to the original quartic term, it is of order $O(G/T)$ and thus diverges as temperature goes to zero. Together with the negative sign, this suggests a tendency for local $N$-ordering even for $r > 0$. That is, instanton effects (i.e., the formation of droplets) are much more important at quantum phase transitions than they are at the corresponding classical transition.

Technically, a perturbative RG solution of the field theory, Eq. (27), shows that the disorder scales to infinity, consistent with the suggestion that the disorder term is very important. A functional RG approach that treats the disorder nonperturbatively has been developed by Fisher for various one-dimensional models. He, and others, have shown that disorder, at least in one-dimension, leads to a number of novel features of quantum phases, and quantum phase transitions. In particular, it was shown that even in the disordered phase $(r < r_c)$ a number of so-called Griffith’s phase effects occur. For example, for $0 < r < r_{c1} < r_c$ the magnetic susceptibility becomes singular even in the absence of long-range order. At other ‘critical’ values of $r$, which are located in the interval $r_{c1} < r < r_c$, higher order susceptibilities become singular. Such effects are not seen in classical systems with
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interactions of finite range. Further, for some model systems it has been established that the quantum phase transition at \( r = r_c \), where true long-range order sets in, is characterized by exponential singularities rather than by power laws.\(^{19,17}\) This behavior is unlike that at standard classical phase transitions, and there is extensive numerical evidence that is consistent with this theoretical picture.\(^{20}\)

Finally we note that the quartic term of long range in time discussed above also occurs at the disordered ferromagnetic transition discussed in Section 3. There, it has no effect on the leading critical behavior because the long-range spatial fluctuations due to the extra soft modes effectively suppress this disorder term. Physically, the interactions extend over a large enough region to smooth out the remaining disorder effects.

5. DISCUSSION

In this paper we have shown that in contrast to popular lore, in many respects quantum phase transitions are fundamentally different from standard classical phase transitions and therefore interesting from a basic statistical mechanics viewpoint. We have focused on two different aspects. We first discussed how the coupling between statics and dynamics in quantum statistical mechanics generically leads to long-range interactions between the order parameter fluctuations at quantum phase transitions. This source of long-range interactions is dynamical in nature and is therefore absent in classical systems, where the statics and dynamics do not couple. We then discussed how, in the presence of quenched disorder, any action describing a quantum phase transition becomes extremely anisotropic. This effect, too, can be viewed as a type of long-range interaction, but in this case the long range occurs in the time domain. All of the consequences of this last effect, in particular, are not yet fully understood.

There is some existing experimental support for these effects. First, in very clean bulk itinerant electron systems that order ferromagnetically, the transition does appear to become first order, or discontinuous, at low temperatures. This has been observed in both MnSi\(^{21}\) and UGe\(_2\).\(^{22}\) Further, in MnSi the transition becomes continuous if disorder is introduced into the systems, in accord with the theoretical expectations. ZnZr is another system where the magnetic transition temperature is very low, but a continuous transition is observed.\(^{23}\) However, in this case it is not clear whether the samples studied have been sufficiently clean for the transition to be discontinuous. The precise critical behavior at the continuous transitions in the disordered case has not been studied so far. Such experiments would
be of great interest.

From a theoretical point of view, it is interesting that there are classical analogs of the two effects we have discussed. A simple classical analog of the first one is an equilibrium system in a phase with Goldstone modes that couple strongly to an order parameter undergoing a phase transition. An example is the thermal ferromagnetic transition in compressible magnets. However, in order to get this effect more generically one needs a coupling between statics and dynamics. While this does not occur in classical equilibrium systems, it does in classical nonequilibrium systems. It has been known for some time that these systems in general exhibit generic scale invariance and that the order parameters for phase transitions in such systems couple to these long-range correlations. The first study of a transition of this type considered phase separation in a binary liquid under shear.

It should also be pointed out that there are other cases where a quantum phase transition is either distinct from any classical transition, and therefore of significant theoretical interest, or where the corresponding classical transition has not been studied, and hence the universality class is not known. The first category is the case of spin chains, where half-integer spin systems are fundamentally different than integer spin ones. It is known, for example, that half-integer antiferromagnetic spin chains have a critical phase while integer spin chains are always disordered. This effect has no simple classical analog. An example of the case where the corresponding classical transition has not been studied, are the phase transitions between quantum Hall states. It is also worth mentioning that there are speculations that an underlying quantum phase transition might be responsible for the apparent non-Fermi liquid behavior observed in high-temperature superconductors.

Finally, it should be pointed out that there are indeed some quantum phase transitions that can be mapped onto standard quantum phase transitions in \(d + z\) dimensions, i.e. where the ‘simple’ mapping, as defined in the Introduction, works. An example is the antiferromagnetic transition in clean insulators or itinerant electron systems in \(d > 1\). In general it appears that this ‘simple’ mapping is possible if the system is clean, and any soft modes do not significantly couple to the order parameter fluctuations.

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