APPROXIMATIONS TO EULER’S CONSTANT

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ABSTRACT. We study a problem of finding good approximations to Euler’s constant \( \gamma = \lim_{n \to \infty} S_n \), where \( S_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n + 1) \), by linear forms in logarithms and harmonic numbers. In 1995, C. Elsner showed that slow convergence of the sequence \( S_n \) can be significantly improved if \( S_n \) is replaced by linear combinations of \( S_n \) with integer coefficients. In this paper, considering more general linear transformations of the sequence \( S_n \) we establish new accelerating convergence formulae for \( \gamma \). Our estimates sharpen and generalize recent Elsner’s, Rivoal’s and author’s results.

1. Introduction

Let \( \alpha \geq 0 \) be a real number and

\[ \gamma_\alpha = \sum_{k=1}^{\infty} \left( \frac{1}{k + \alpha} - \log \left( \frac{k + \alpha + 1}{k + \alpha} \right) \right). \]

We denote the partial sum of the above series by

\[ S_n(\alpha) = \sum_{k=1}^{n} \left( \frac{1}{k + \alpha} - \log \left( \frac{k + \alpha + 1}{k + \alpha} \right) \right) \]

(1)

\[ = \sum_{k=1}^{n} \frac{1}{k + \alpha} - \log(\alpha + n + 1) + \log(\alpha + 1) \]

and \( S_n := S_n(0) \). It easily follows (see [12, formula (2)]) that

\[ \lim_{n \to \infty} S_n(\alpha) = -\frac{\Gamma'(\alpha + 1)}{\Gamma(\alpha + 1)} + \log(\alpha + 1) = -\psi(\alpha + 1) + \log(\alpha + 1), \]

where \( \psi(\alpha) \) is the logarithmic derivative of the gamma function (or the digamma function) and therefore,

\[ \gamma_\alpha = \log(\alpha + 1) - \psi(\alpha + 1). \]

In particular, \( \gamma_0 = -\psi(1) = \gamma = 0.577215\ldots \), where \( \gamma \) is Euler’s constant. It is well-known that the sequence \( S_n \) slowly converges to the Euler constant \( \gamma \) (see, for details, [7])

\[ \gamma = S_n + O(n^{-1}). \]
In 1995, Elsner [1] found out that \( \gamma \) can be approximated by linear combinations of partial sums (1) with integer coefficients

\[
\left| \gamma - \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{k+n-1}{k-1} S_{k+1} \right| \leq \frac{1}{2n^r(n+r)}, \quad r, n \in \mathbb{N}
\]

and this inequality exhibits geometric convergence if \( r = O(n) \). Formulas (2) for \( r > n \) were generalized by Rivoal in [10], where, in particular, it was shown that

\[
\left| \gamma - \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} \binom{2k+2n}{n} S_{2k+n} \right| = O \left( \frac{1}{n^{27n/2}} \right), \quad n \to \infty.
\]

Another such kind formula

\[
\gamma - \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k+\tau_1+k}{n} S_{n+k+\tau_1-1}
\]

was proved in [6]. Recently, C. Elsner [2] presented a two-parametric series transformation of the sequence \( S_n \)

\[
\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+\tau_1+k}{n} S_{k+\tau_1-1}
\]

converging more rapidly to \( \gamma \) when \( \tau_2 > \tau_1+1 \) and \( n \) increases than in the case \( \tau_2 = \tau_1+1 \) considered in [2].

In this paper, we consider a more general series transformation of the type

\[
\frac{n_1! \ldots n_m!}{N! r^N} \sum_{k=0}^{N} (-1)^{N+k} \binom{N}{k} \binom{r^k+n_1+\tau_1}{n_1} \ldots \binom{r^k+n_m+\tau_m}{n_m} S_{r^k+n_0}
\]

with \( n_1, \ldots, n_m \in \mathbb{N}, \tau_0, \tau_1, \ldots, \tau_m \in \mathbb{N}_0, \) and \( N = \sum_{j=1}^{m} n_j, \) and give new accelerating convergence formulae for Euler’s constant \( \gamma \). In particular, we show (see Theorem [2] and Corollary [1] below) that if \( \tau_1, \tau_2 \) are linear functions of \( n \), then the sum (3) converges to \( \gamma \) at the least geometric rate and represents the best approximation in the set of all the sums (3) with a fixed value of \( \lim_{n \to \infty} \tau_2/n \), provided that \( \lim_{n \to \infty} 2(\tau_2 - \tau_1)/n = 1 \).

2. Statement of the main results

As usual, we denote the Gauss hypergeometric function (see, for details, [9]) by

\[
\,_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \right| z \right) = \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{(c)_\nu \nu!} z^\nu,
\]

where \((\lambda)_\nu\) is the Pochhammer symbol (or the shifted factorial) defined by

\[
(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0; \\ \lambda(\lambda+1) \ldots (\lambda+\nu-1), & \nu \in \mathbb{N}. \end{cases}
\]

We then prove the following theorems:
Theorem 1. Let \( n_1, \ldots, n_m \in \mathbb{N} \), \( \tau_0, \tau_1, \ldots, \tau_m \in \mathbb{N}_0 \), \( 0 \leq \tau_0 - \tau_m \leq n_m \), \( n_m + \tau_m \geq n_j + \tau_j \), \( j = 1, \ldots, m - 1 \), and \( N = \sum_{j=1}^{m} n_j \). Then

\[
\left| \frac{N! (-r)^N}{n_1! \ldots n_m!} \gamma - \sum_{k=0}^{N} (-1)^k \binom{N}{k} \binom{r k + n_1 + \tau_1}{n_1} \ldots \binom{r k + n_m + \tau_m}{n_m} S_{rk+\tau_0} \right|
\]

\[
= \prod_{j=1}^{m} \left( \frac{n_m + \tau_m - \tau_j}{n_j} \right) \int_{0}^{1} \int_{0}^{1} \frac{x^{n_m + \tau_m} (1 - x^r)^N t^{n_m + \tau_m - \tau_0} (1 - t)^{\tau_0 - \tau_m} \omega(t)}{(1 - t + xt)^{n_m + 1}} \times Q_m \left( \frac{xt}{1 - t + xt} \right) \ dx \ dx dt,
\]

where

\[
\omega(t) = \frac{1}{t \log^2(1/t - 1) + \pi^2}
\]

and \( Q_m(y) \) is a polynomial of degree \( N - n_m \) given by the formula

\[
Q_m(y) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m-1} \prod_{j=1}^{m-1} k_j! \left( 1 + n_m + \tau_m - n_j - \tau_j \right) k_j y^k_j
\]

if \( m \geq 2 \), and \( Q_1(y) \equiv 1 \).

Theorem 2. Let \( b, c, r \in \mathbb{N} \), \( a \in \mathbb{N}_0 \), \( 0 \leq b - a \leq c \). Then for \( n \in \mathbb{N} \) we have

\[
\left| \gamma - \frac{1}{r^{cn}} \sum_{k=0}^{cn} (-1)^{k+cn} \binom{cn}{k} \binom{r k + (a + c)n}{cn} S_{rk+bn} \right| < \left( \frac{b^\frac{c}{2} (c + a - b)^{c+a-b} (b - a)^{b-a}}{(b + cr)^{c+\frac{c}{2}}} \right)^n
\]

(Here and throughout the paper 0^0 is treated as 1.)

If \( b, c, r \) are fixed, then the minimum of the right-hand side of \( \Box \) is attained when \( b - a = c/2 \) and in this case we have

Corollary 1. Let \( b, c, r, n \in \mathbb{N} \) and \( b \geq c \). Then

\[
\left| \gamma - \frac{1}{r^{2cn}} \sum_{k=0}^{2cn} (-1)^{k} \binom{2cn}{k} \binom{r k + (b + c)n}{2cn} S_{rk+bn} \right| < \left( \frac{b^\frac{c}{2} (c + a - b)^{c+a-b} (b - a)^{b-a}}{(b + 2cr)^{c+\frac{c}{2}}} \right)^n.
\]

Theorem 3. Let \( b, c, r \in \mathbb{N} \), \( a \in \mathbb{N}_0 \) and \( 0 \leq b - a \leq c \). Then for any positive integer \( n \geq 2/c \) one has

\[
\left| \gamma - \frac{((cn)!)^2}{(2cn)! r^{2cn}} \sum_{k=0}^{2cn} (-1)^{k} \binom{2cn}{k} \binom{r k + (a + c)n}{cn}^2 S_{rk+bn} \right| < cn \left( \frac{b^\frac{c}{2} (c + a - b)^{c+a-b} (b - a)^{b-a}}{(b + 2cr)^{c+\frac{c}{2}}} \right)^n.
\]

By the similar argument as above putting \( a = b - c/2 \) we get a sharper bound than in Corollary \( \Box \).
Corollary 2. Let \( b, c, r, n \in \mathbb{N}, 2b \geq c, \) and \( c \) is even. Then
\[
|\gamma - \frac{(cn)!}{(2cn)!} \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} (\tau + \frac{b + \frac{c}{2}}{cn})^2 S_{\tau k + bn} | < cn \left( \frac{b^2 c^{2c}}{(2c(2cr + 2))} \right)^n.
\]

For example, setting \( b = c = 4, r = 1 \) we get the following estimate:

Corollary 3. For any positive integer \( n \) one has
\[
|\gamma - \frac{(4n)!}{(8n)!} \sum_{k=0}^{8n} (-1)^k \binom{8n}{k} (\tau + 6n)^2 S_{k+4n} | < \frac{4n}{(2^4 3^{12})^n} < 4n(0.00000012)^n.
\]

Theorem 4. Let \( n_1, \ldots, n_m \in \mathbb{N}, \tau_0, \tau_1, \ldots, \tau_m \in \mathbb{N}_0, 0 \leq \tau_0 - \tau_m \leq n_m, n_m + \tau_m \geq \tau_{j+1} > n_j + \tau_j, j = 1, \ldots, m - 1, \) and \( N = \sum_{j=1}^{m} n_j, \) Then
\[
|\gamma - \frac{N!(-r)^N}{n_1! \cdots n_m!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} (\tau + n_1 + \tau_1) \cdots (\tau + n_m + \tau_m) S_{\tau k + \tau_0} |
\leq \prod_{j=1}^{m} \left( \frac{n_m + \tau_m - \tau_j}{n_j} \right)^{\int_0^1 \int_0^1 \frac{x^{n_m + \tau_m - 1} (1 - x^r)^N t^{n_m + \tau_m - 1} (1 - t)^{\tau_0 - \tau_m} \omega(t)}{(1 - t + xt)^{n_m + 1}} dtdx.\right.
\]

Setting \( \tau_{j+1} = n_j + \tau_j + 1, j = 1, \ldots, m - 1, \) in Theorem 4 we get

Corollary 4. Let \( n_1, \ldots, n_m \in \mathbb{N}, \tau_0, \tau_1 \in \mathbb{N}_0, N = \sum_{j=1}^{m} n_j, \) and \( N - n_m + \tau_1 + (m-1) \leq \tau_0 \leq N + \tau_1 + (m-1). \) Then
\[
|\gamma - \frac{n_1! \cdots n_m!}{N!(-r)^N} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \prod_{j=1}^{m} \left( \frac{r + n_1 + \ldots + n_j + \tau_j + j - 1}{n_j} \right) S_{\tau k + \tau_0} |
\leq \prod_{j=1}^{m} \left( \frac{N + j}{n_j + 1 + \ldots + n_m + m - j} \right) \int_0^1 \int_0^1 \frac{x^{N + \tau_1 + m - 1} (1 - x^r)^N t^{N + \tau_1 + m - 1 - \tau_0} (1 - t)^{\tau_0 - \tau_m} \omega(t)}{t^{n_m + N - \tau_1 - m - 1} (1 - t + xt)^{n_m + 1}} dtdx.
\]

Theorem 5. Let \( m, c_1, \ldots, c_m, r, b, n \in \mathbb{N}, \ a \in \mathbb{N}_0, C = \sum_{j=1}^{m} c_j, \) and \( a - c_m \leq b - c \leq a. \) Then
\[
|\gamma - \frac{(c_1 n)! \cdots (c_m n)!}{(Cn)!(-r)^{Cn}} \sum_{k=0}^{Cn} (-1)^k \binom{Cn}{k} \prod_{j=1}^{m} \left( \frac{r + c_1 + \ldots + c_j + n + j - 1}{c_j n} \right) S_{\tau k + \tau_0} |
\leq \prod_{j=1}^{m} \left( \frac{(C + a - b)^{C + a - b} (c_m + b - a - C)^{c_m + b - a - C}}{c_m (C + b + C r)^{C + b r}} \right)^n,
\]
where \( M(\tau) < C^{m - 1} \) is some constant depending only on \( c_1, \ldots, c_m. \)

Consider several illustrative examples of Theorem 5. Taking \( c_1 = \ldots = c_m = 2c, \ C = 2mc, b = 2mc, a = c, \ c \in \mathbb{N}, \) we get
Corollary 5. Let $c, m, r \in \mathbb{N}$. Then for any positive integer $n$ one has
\[
\gamma - \frac{(2cn)^m}{(2mcn)!} \frac{2mcn}{r^{2mcn}} \sum_{k=0}^{2mcn} (-1)^k \binom{2mcn}{k} \binom{rk + 3cn + 1}{2cn} \binom{rk + 5cn + 2}{2cn} \ldots \\
\times \binom{rk + (2m + 1)cn + m}{2cn} S_{rk+2mcn+m} \!
\left| \frac{m^m}{(m-1)!} \frac{1}{4^r (r+1)^{2mc+2mc}} \right|^n
\]
Setting $c_1 = \ldots = c_m = 2c$, $C = 2mc$, $b = (2m - 1)c$, $a = 2c$, $c \in \mathbb{N}$, we get

Corollary 6. Let $c, m, r \in \mathbb{N}$. Then for any positive integer $n$ one has
\[
\gamma - \frac{(2cn)^m}{(2mcn)!} \frac{2mcn}{r^{2mcn}} \sum_{k=0}^{2mcn} (-1)^k \binom{2mcn}{k} \prod_{j=1}^{m} \binom{rk + 2jcn + j}{2cn} S_{rk+(2m-1)cn+m} \!
\left| \frac{m^m}{(m-1)!} \frac{1}{4^r (r+1)^{2mc+2mc}} \right|^n
\]

3. Analytical construction
We define the generalized Legendre polynomial by
\[
A(x) = \sum_{k=0}^{N} A_k x^k
\]
with
\[
A_k = (-1)^{k+N} \binom{N}{k} \binom{rk + n_1 + \tau_1}{n_1} \ldots \binom{rk + n_m + \tau_m}{n_m}.
\]

Lemma 1. There holds
\[
A(1) = \sum_{k=0}^{N} A_k = \frac{N! r^N}{n_1! \ldots n_m!}.
\]

Proof. For the proof, let
\[
R(t) = \frac{N! (rt - n_1 - \tau_1)_{n_1} (rt - n_2 - \tau_2)_{n_2} \ldots (rt - n_m - \tau_m)_{n_m}}{n_1! \ldots n_m! t(t+1) \ldots (t+N)}.
\]
Such rational functions were considered early by the authors [4], [5] to derive explicit Padé approximations of the first and second kinds for polylogarithmic functions. As it is easily seen the rational function $R(t)$ has the following partial-fraction expansion:
\[
R(t) = \sum_{k=0}^{N} A_k \frac{1}{t+k},
\]
from which it follows that
\[
\sum_{k=0}^{N} A_k = \sum_{k=0}^{N} \text{res}_{t=-k} R(t) = - \text{res}_{t=\infty} R(t) = \frac{N! r^N}{n_1! \ldots n_m!}.
\]

Put
\[
I(\alpha) := \int_{0}^{1} x^{\tau_0 + \alpha} A(x) \left( \frac{1}{1-x} + \frac{1}{\log x} \right) dx
\]
Lemma 2. There holds the equality

\[ I(\alpha) = \frac{N! p^N}{n_1! \ldots n_m!} \gamma_{\alpha} - \sum_{k=0}^{N} A_k S_{r_k + \gamma_0}(\alpha). \]

**Proof.** Substituting

\[ \frac{1}{1-x} + \frac{1}{\log x} = \int_0^1 \frac{1 - x^t}{1-x} dt, \]

we get

\[ I(\alpha) = \int_0^1 \int_0^1 x^{\tau_0 + \alpha} A(x) \frac{1-x^t}{1-x} dt dx = \sum_{k=0}^{N} A_k \int_0^1 \int_0^1 x^{r_k + \tau_0 + \alpha}(1-x^t) dx dt. \]

Expanding \((1 - x)^{-1}\) in a geometric series and applying Lemma 1 we find

\[ I(\alpha) = \sum_{k=0}^{N} A_k \sum_{l=0}^{\infty} \int_0^1 \int_0^1 x^{r_k + \tau_0 + l + \alpha}(1-x^t) dx dt \]

\[ = \sum_{k=0}^{N} A_k \sum_{l=0}^{\infty} \int_0^1 \left( \frac{1}{rk + \tau_0 + l + \alpha + 1} - \frac{1}{rk + \tau_0 + t + l + \alpha + 1} \right) dt \]

\[ = \sum_{k=0}^{N} A_k \left( \frac{1}{rk + \tau_0 + l + \alpha} - \log \left( \frac{rk + \tau_0 + l + \alpha + 1}{rk + \tau_0 + l + \alpha} \right) \right) \]

\[ = \sum_{k=0}^{N} A_k (\gamma_{\alpha} - S_{r_k + \tau_0}(\alpha)) = \frac{N! p^N}{n_1! \ldots n_m!} \gamma_{\alpha} - \sum_{k=0}^{N} A_k S_{r_k + \tau_0}(\alpha). \]

Next, we consider two differential operators

\[ S_{\tau,n}(f(x)) = \frac{(-1)^n}{n!} x^{-\tau} (x^{n+\tau} f(x))^{(n)}, \]

\[ T_{\tau,n}(f(x)) = \frac{1}{n!} x^{n+\tau} (x^{-\tau} f(x))^{(n)}, \]

where \(\tau\) is a real number and \(n\) is a non-negative integer. We show that \(S_{\tau,n}\) and \(T_{\tau,n}\) are adjoint operators in some sense.

**Lemma 3.** Suppose that \(f(x)\) is a polynomial vanishing at \(x = 1\) with order at least \(n\) and \(g(x) \in C^\infty(0,1) \cap L^1(0,1)\) satisfies the following boundary conditions:

\[ \lim_{x \to 0^+} x^l g^{(l-1)}(x) = \lim_{x \to 1^-} (1-x)^l g^{(l-1)}(x) = 0 \]

for all \(1 \leq l \leq n\). Then we have

\[ \int_0^1 S_{\tau,n}(f(x)) \cdot g(x) dx = \int_0^1 f(x) \cdot T_{\tau,n}(g(x)) dx. \]

**Proof.** The proof is analogous to the proof of Lemma 3.1 [3]. □
Lemma 4. There holds
\[ I(\alpha) = \int_0^1 \int_0^1 (1 - x)^N \omega(t) T_{\tau_{m-1},n_{m-1}} \circ \ldots \circ T_{\tau_1,n_1} \circ T_{\tau_m,n_m} \left( \frac{x^{\tau_0+\alpha}}{1 - (1 - x)t} \right) \, dx \, dt \]
with the weight function \( \omega(t) \) defined in (6).

Proof. Applying the following representation introduced by Prévost [8]:
\[
\frac{1}{1-x} + \frac{1}{\log x} = \int_0^1 \frac{\omega(t)}{1 - (1-x)t} \, dt,
\]
we have
\[ I(\alpha) = \int_0^1 \int_0^1 \frac{x^{\tau_0+\alpha} \omega(t)}{1 - (1-x)t} A(x) \, dt \, dx. \]
As it easily follows the polynomial \( A(x) \) can be written in the form
\[ A(x) = S_{\tau_1,n_1} \circ S_{\tau_2,n_2} \circ \ldots \circ S_{\tau_m,n_m} \left( (1 - x^r)^N \right). \]
Since \( A(x) \) is symmetric in pairs \( (\tau_j, n_j) \) and does not depend on the order of differential operators \( S_{\tau_j,n_j} \), it is convenient for the sequel to write it as
\[ A(x) = S_{\tau_m,n_m} \circ S_{\tau_1,n_1} \circ \ldots \circ S_{\tau_{m-1},n_{m-1}} \left( (1 - x^r)^N \right). \]
Now by Fubini’s theorem and Lemma 3, we get the desired equality. □

We need also the following simple lemma, which will be used for estimation purposes.

Lemma 5. Let \( a, b, c, d, r, s \in \mathbb{R}, r, s, d > 0, \) and \( b + d \geq a + c \geq b \geq 0. \) Then the function
\[ f(x, t) = \frac{x^{a+c}(1-x^r)^{sc}c^{a-b}(1-t)^{b+d-c-a}}{(1-t+xt)^d} \]
attains its maximum in \([0, 1] \times [0, 1]\) at the unique point
\[ x_0 = \left( \frac{b}{b + scr} \right)^{\frac{1}{r}}, \quad t_0 = \frac{c + a - b}{c + a - b + x_0(b + d - a - c)} \]
and
\[ \max_{0 \leq x \leq 1} f(x, t) = f(x_0, t_0) = \frac{b^b (sc)^{sc} (c + a - b)^{c + a - b} (b + d - a - c)^{b + d - a - c}}{d^d (b + scr)^{sc + \frac{b}{r}}} \].

4. Proof of Theorem 1

Lemma 6. Let \( x, t \in (0, 1), \tau_0, n_m, \tau_m \in \mathbb{N}_0, \) and \( \tau_m \leq \tau_0 \leq n_m + \tau_m. \) Then
\[ T_{\tau_m,n_m} \left( \frac{x^{\tau_0}}{1 - (1-x)t} \right) = (-1)^{n_m} x^{n_m+\tau_m} t^{n_m+\tau_m-\tau_0} (t-1)^{\tau_0-\tau_m} \]
\[ (1 - (1-x)t)^{n_m+1}. \]
Proof. Clearly,
\[ T_{\tau_m,n_m} \left( \frac{x^{\tau_0}}{1-t+xt} \right) = \frac{x^{n_m+\tau_m}}{n_m!} \left( \frac{x^{\tau_0-\tau_m}}{1-t+xt} \right)^{(n_m)}. \]
Decomposing the fraction \( \frac{x^{\tau_0-\tau_m}}{1-t+xt} \) into the sum
\[ \frac{x^{\tau_0-\tau_m}}{1-t+xt} = p(x) + \left( \frac{t-1}{t} \right)^{\tau_0-\tau_m} \frac{1}{1-t+xt}, \]
where \( p(x) \) is a polynomial of degree not exceeding \( \tau_0 - \tau_m - 1 \), and differentiating it \( n_m \) times, we get the required statement. \( \square \)

Lemma 7. Under the hypothesis of Theorem 1 one has
\[ T_{\tau_{m-1},n_{m-1}} \circ \ldots \circ T_{\tau_1,n_1} \circ T_{\tau_m,n_m} \left( \frac{x^{\tau_0}}{1-(1-x)t} \right) = (-1)^{n_m} \]
\[ \times \prod_{j=1}^{m} \left( n_m + \tau_m - \tau_j \right) \frac{x^{n_m+\tau_m} t^{n_m+\tau_m-\tau_0(t-1)\tau_0-\tau_m}}{(1-t+xt)^{n_m+1}} Q_m \left( \frac{xt}{1-t+xt} \right), \]
where the polynomial \( Q_m(y) \) is defined in (7).

Proof. If \( m = 1 \), then (9) easily follows by Lemma 4. Suppose \( m \geq 2 \). Then consecutive calculation of the \( n_j \)th derivatives with respect to \( x \) by Leibniz’ rule for \( j = 1, 2, \ldots, m-1 \)
\[ \frac{x^{\tau_j+n_j}}{n_j!} \left( \frac{t^k x^{n_m+\tau_m+k-\tau_j}}{(1-t+xt)^{n_m+1+k}} \right)^{(n_j)} = \binom{n_m + \tau_m - \tau_j}{n_j} \frac{x^{n_m+\tau_m}}{(1-t+xt)^{n_m+1}} \]
\[ \times \sum_{k_j=0}^{n_j} \frac{(-n_j)_j}{k_j!} (n_m+1)_k (1+n_m+\tau_m-\tau_j)_k \frac{xt}{1-t+xt}^{k+k_j} \]
readily leads to the formula (9). \( \square \)

Now Theorem 1 easily follows from Lemmas 4, 7.

5. Proof of Theorem 2
If we put \( m = 1, n_1 = cn, \tau_1 = an, \tau_0 = bn, n \in \mathbb{N} \), in Theorem 1 we get
\[ \left| \gamma - \frac{1}{r^cn} \sum_{k=0}^{cn} (-1)^k \binom{cn}{k} \binom{rk+(a+c)n}{cn} S_{r^k+bn} \right| \]
\[ \leq \frac{1}{r^cn} \int_0^1 \int_0^1 \frac{(1-x^r)^{cn} x^{(a+c)n} t^{(c+a-b)n} (1-t)^{b-a} \omega(t)}{(1-t+xt)^{cn+1}} \] \( dxdt \)
\[ \leq \frac{1}{r^cn} \left( \max_{0 \leq x,t \leq 1} f(x,t) \right)^n \int_0^1 \int_0^1 \frac{\omega(t)}{1-t+xt} \] \( dt dx = \frac{\gamma}{r^cn} \left( \max_{0 \leq x,t \leq 1} f(x,t) \right)^n \]
with
\[ f(x,t) = \frac{x^{a+c}(1-x^r)^c t^{c+a-b} (1-t)^{b-a}}{(1-t+xt)^c}. \]
Here we used the fact (see [8, formula 2.6]) that
\[ \gamma = \int_0^1 \left( \frac{1}{\log x} + \frac{1}{1 - x} \right) dx. \]

Now, since \( \gamma < 1 \), by Lemma 5 with \( s = 1, d = c \), the theorem follows. \( \square \)

6. PROOFS OF THEOREMS 3, 4

To estimate the speed of convergence of quantities (4) to \( \gamma \) as \( N \to \infty \) we need an upper bound for the polynomial \( Q_m(y) \). In some situations it is possible to get suitable estimations.

First, we consider the case \( m = 2, n_1 = n_2, \tau_1 = \tau_2 \). Then by Theorem 4 we get

\[
I := \left| \frac{(2n_1)!r^{2n_1}}{(n_1)!^2} \gamma - \sum_{k=0}^{2n_1} (-1)^k \binom{2n_1}{k} \binom{r k + n_1 + \tau_1}{n_1} S_{rk+\tau_0} \right|
\]

\[
= \int_0^1 \int_0^1 x^{n_1+\tau_1}(1-x^r)^{2n_1} t^{n_1+\tau_1-\tau_0}(1-t)^{\tau_0-\tau_1} \omega(t)\left|Q_2(y)\right| dxdt
\]

with \( y = xt/(1-t+xt) \). The polynomial

\[
Q_2(y) = {}_2F_1 \left( -n_1, n_1+1 \mid y \right) = \frac{1}{n_1!} \left( \frac{d}{dy} \right)^{n_1} (y^{n_1}(1-y)^{n_1})
\]

is a shifted Legendre polynomial \( P_{n_1}(u) \) formally identified as follows:

\[
Q_2(y) = P_{n_1}(1-2y).
\]

By the well-known inequality (see [11, p.162])

\[
|P_{n_1}(u)| \leq 1, \quad -1 \leq u \leq 1,
\]

it follows that

\[
I \leq \int_0^1 \int_0^1 x^{n_1+\tau_1}(1-x^r)^{2n_1} t^{n_1+\tau_1-\tau_0}(1-t)^{\tau_0-\tau_1} \omega(t) dxdt.
\]

Now, setting \( n_1 = cn, \tau_1 = an, \tau_0 = bn \) with \( c, b \in \mathbb{N}, a \in \mathbb{N}_0, \) and \( 0 \leq b-a \leq c \), we get

\[
\left| \frac{(2cn)!r^{2cn}}{(cn)!^2} \gamma - \sum_{k=0}^{2cn} (-1)^k \binom{2cn}{k} \binom{r k + (a+c)n}{cn} S_{rk+bn} \right| \leq \gamma \left( \max_{0 \leq x,t \leq 1} f(x,t) \right)^n,
\]

where

\[
f(x,t) = \frac{x^{c+a}(1-x^r)^{2c+a-b}(1-t)^{b-a}}{(1-t+xt)^c}.
\]

By Lemma 3, the function \( f(x,t) \) takes its maximum in \([0,1] \times [0,1]\) at the unique point \((x_0,t_0)\), at which

\[
f(x_0,t_0) = \frac{b^{\frac{b}{c}}(4cr^2)^c(c+a-b)^{c+a-b}(b-a)^{b-a}}{(b+2cr)^{2c+b}r}.
\]
Since for any positive integer $n \geq 2$
\[
\frac{(n!)^2}{(2n)!} \leq \frac{n}{4^n},
\]
Theorem 3 follows. \(\square\)

Another interesting case is described by the following lemma.

**Lemma 8.** Let $n_1, \ldots, n_m \in \mathbb{N}$, $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{N}_0$, and $n_m + \tau_m \geq \tau_{j+1} > n_j + \tau_j$, $j = 1, \ldots, m - 1$. Then
\[
Q_m(y) = \prod_{j=1}^{m-1} \frac{(n_m + \tau_m - n_j - \tau_j)!}{(n_m + \tau_m - \tau_{j+1})!(\tau_{j+1} - n_j - \tau_j - 1)!} \times \int_0^1 \int_0^1 \prod_{j=1}^{m-1} (1 - y u_j \ldots u_{m-1})^{n_j} u_j^{n_m + \tau_m - \tau_{j+1}} (1 - u_j)^{\tau_{j+1} - n_j - \tau_j - 1} du_1 \ldots du_{m-1}.
\]
Moreover, $0 \leq Q_m(y) \leq 1$ for $y \in [0, 1]$.

**Proof.** Denoting the integral on the right-hand side of (10) by $J$ and substituting
\[
\prod_{j=1}^{m-1} (1 - y u_j u_{j+1} \ldots u_{m-1})^{n_j} = \sum_{k_1=0}^{n_1} \ldots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} (-n_j)_j y^j u_j^{k_1 + \ldots + k_j}
\]
we get
\[
J = \sum_{k_1=0}^{n_1} \ldots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} (-n_j)_j y^j u_j^{k_1 + \ldots + k_j + n_m + \tau_m - \tau_{j+1}} \times (1 - u_j)^{\tau_{j+1} - n_j - \tau_j - 1} du_j = \sum_{k_1=0}^{n_1} \ldots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} (-n_j)_j y^j u_j^{k_1 + \ldots + k_j + n_m + \tau_m - \tau_{j+1}} \times \frac{\Gamma(k_1 + \ldots + k_j + n_m + \tau_m + 1 - \tau_{j+1})\Gamma(\tau_{j+1} - n_j - \tau_j)}{\Gamma(k_1 + \ldots + k_j + n_m + \tau_m + 1 - n_j - \tau_j)}
\]
\[
= \prod_{j=1}^{m-1} \frac{\Gamma(1 + n_m + \tau_m - \tau_{j+1})\Gamma(\tau_{j+1} - n_j - \tau_j)}{\Gamma(1 + n_m + \tau_m - n_j - \tau_j)}
\]
\[
= \sum_{k_1=0}^{n_1} \ldots \sum_{k_{m-1}=0}^{n_{m-1}} \prod_{j=1}^{m-1} (-n_j)_j (1 + n_m + \tau_m - \tau_{j+1})_{k_1 + \ldots + k_j} y^j u_j^{k_1 + \ldots + k_j + n_m + \tau_m - \tau_{j+1}}
\]
\[
= \prod_{j=1}^{m-1} \frac{(n_m + \tau_m - \tau_{j+1})!(\tau_{j+1} - n_j - \tau_j - 1)!}{(n_m + \tau_m - n_j - \tau_j)!} Q_m(y).
\]
The inequality $0 \leq Q_m(y) \leq 1$ for $y \in [0, 1]$ easily follows from the integral representation (10). \(\square\)

Now, Theorem 4 is a consequence of Theorem 1 and Lemma 8.
7. Proof of Theorem 5

Setting \( n_j = c_j n, j = 1, \ldots, m, C = \sum_{j=1}^{m} c_j, \tau_1 = an + 1, \tau_0 = bn + m \) in Corollary 4 we get that the absolute value of the remainder is less than

\[
\frac{M(\tau)}{r^{cm}} \int_0^1 \int_0^1 x^{(C+a)n+m}(1-x^r)^n t^{(C+a-b)n}(1-t)^{(b+c_m-C-a)n} \omega(t) \frac{dxdt}{(1-t+xt)^{c_m n+1}}
\]

with some constant \( M(\tau) < C^{m-1} \), since

\[
\prod_{j=1}^{m-1} \frac{Cn+j}{(c_{j+1} + \ldots + c_m)n + m - j} < C^{m-1}.
\]

Denoting

\[
f(x, t) = \frac{x^{C+a}(1-x^r)^C t^{C+a-b}(1-t)^{b+c_m-C-a}}{(1-t+xt)^{c_m}}
\]

and applying Lemma 5 with \( s = 1, d = c_m \), we conclude the theorem. \( \square \)

References

[1] Elsner, C.: On a sequence transformation with integral coefficients for Euler’s constant. Proc. Amer. Math. Soc. 123, No. 5, 1537-1541 (1995)
[2] Elsner, C.: On a sequence transformation with integral coefficients for Euler’s constant, II. J. Number Theory 124, 442-453 (2007).
[3] Hata, M.: On the linear independence of the values of polylogarithmic functions. J. Math. Pures et Appl. 69, no. 2, 133-173 (1990).
[4] Hessami Pilehrood, T. [Khessami Pilerud T. G.]: A lower bound for a linear form. Mat. Zametki 66, no.4, 617-623 (1999); English translation: Math. Notes 66, no.4, 507-512 (1999).
[5] Hessami Pilehrood, T., Hessami Pilehrood, Kh.: Lower bounds for linear forms in values of polylogarithms. Mat. Zametki 77, no.4, 623-629 (2005); English translation: Math. Notes 77, no.4, 573-579 (2005).
[6] Hessami Pilehrood, Kh., Hessami Pilehrood, T.: Arithmetical properties of some series with logarithmic coefficients. Math. Z. 255, 117-131 (2007).
[7] Karatsuba, E. A.: On the computation of the Euler constant \( \gamma \). Numerical Algorithms. 24, 83-97 (2000).
[8] Prévost, M.: A family of criteria for irrationality of Euler’s constant. e-print. math. NT/0507231 (July 2005)
[9] Rainville, E. D.: Special functions. MacMillan Company, N.Y., 1960.
[10] Rivoal, T.: Polynômes de type Legendre et approximations de la constante d’Euler. (2005, notes); available at http://www-fourier.ujf-grenoble.fr~rivoal/
[11] Szegö, G.: Orthogonal polynomials. Colloq. Publ., Vol. 23. Amer. Math. Soc., Providence, Rhode Island, 1959.
[12] Tasaka, T.: Note on the generalized Euler constants. Math. J. Okayama Univ. 36, 29-34 (1994)

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