A domain of influence in the Moore–Gibson–Thompson theory of dipolar bodies

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ABSTRACT
We establish a domain of influence theorem for the mixed initial-boundary value problem in the context of the Moore–Gibson–Thompson theory of thermoelasticity for dipolar bodies. Based on the data of the mixed problem, we define, for a finite time $t > 0$, a bounded domain $B_t$, and prove that the displacements and the temperature decrease to zero, outside of the domain $B_t$. The main result is obtained with the help of two auxiliary results, namely two integral inequalities. We managed to prove that this type of influence domain can be built even if it is considered in a much more complex context. Thus, compared to the classical context in which this concept appeared, we took into account the heat conduction principle from the Moore–Gibson–Thompson theory, we considered the thermal effect and we analyzed the effect of the dipolar structure of the environment.

1. Introduction

Some researchers consider that type III heat conduction violates the principle of causality. This was the reason for considering the Choudhuri’s theory given [1]. This is also the reason why the Moore–Gibson–Thompson theory appeared (see [2], for instance). This theory was developed starting from a third-order differential equation, built in the context of some considerations related to fluid mechanics. Subsequently the equation was considered as a heat conduction equation because it has been obtained by considering a relaxation parameter into the type III heat conduction. Since the advent of the Moore–Gibson–Thompson theory, the number of dedicated studies to this theory has increased considerably. We mention some of them, a part developing the theoretical aspects of the theory, such as [3–6], others highlighting the practical applicability of this theory [6–13]. For our main result, we will approach the heat conduction of the Moore–Gibson–Thompson type in the case that the dissipation condition holds. We must emphasize that we have considered the Moore–Gibson–Thompson theory in the context of the thermoelastic dipolar bodies, starting from the consideration that these media are very current and have applicability to concrete materials. We can give the following examples, as very convincing: granular media having large molecules (concrete: polymers), animal bones, or human bones, and graphite.

If we consider the extensive number of published papers which are dedicated to the media with dipolar structure, we can conclude that this kind of structure is very suitable to model a large number of media in continuum mechanics [14–17].

In some previous studies, we have studied other aspects regarding the bodies with dipolar structure [18–22]. Another general consideration: our study is a plea for the theories regarding microstructure, for both elastic and thermoelastic environments. The most important concern of these theories is to eliminate the shortcomings of the classical theory. The advantages of bodies with microstructure have been highlighted in many studies dedicated to these environments [23–37].

2. Basic equations

We will approach a bounded domain $D$ in the usual space $\mathbb{R}^3$ which is filled by an thermoelastic dipolar material. We denote by denoted by $\bar{D}$ the closure of the domain $D$, so that $\bar{D} = \partial D \cup D$, $\partial D$ being the frontier of the set $D$ and it is assumed to be smooth enough to apply the theorem of divergence. Also, $n = (n_i)$ is the outward unit normal to the border $\partial D$. It will be understood that a vector $\mathbf{v}$ has the components $v_i$ and a tensor $\mathbf{w}$ has the components $w_{ij}$. For a function $f = f(t, x)$ we will denote by $f'$ its derivative with respect the variable $t$ and by $f_j$ its derivative with respect to the space variable $x_i$. Summation over repeated subscripts is implied.

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The spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion. We refer the motion of the body to a fixed system of rectangular Cartesian axes Ox, $i = 1, 2, 3$.

Let us define the problem of heat conduction of the Moore–Gibson–Thompson type, denoted by $P$, which consists of the differential equation relative to the temperature $\vartheta$:

$$c(x) \left( \alpha \dddot{\vartheta} + \dddot{\vartheta} \right) = \left( \kappa_{ij}(x) \partial_{j} \vartheta + \kappa_{ij}^{*}(x) \partial_{j} \vartheta \right)_{i}. \quad (1)$$

Here we used the notation $\alpha$ for a positive parameter to designate the thermal relaxation. Also, $c(x)$ is notation for the thermal capacity, and the tensors $\kappa_{ij}$ and $\kappa_{ij}^{*}$ represent the thermal conductivity, and respectively, the thermal conductivity tensor in connection with the thermal relaxation parameter $\alpha$. Both tensors $\kappa_{ij}$ and $\kappa_{ij}^{*}$ are assumed to be symmetric.

Our aim is to obtain a mixed initial-boundary value problem for $\vartheta$. For this, we consider the following boundary condition:

$$\vartheta (t, x) = 0, \quad x \in \partial D, \quad t > 0, \quad (2)$$

and the following initial conditions:

$$\vartheta (0, x) = \vartheta^{0}(x), \quad \vartheta (0, x) = \vartheta^{1}(x), \quad \ddot{\vartheta} (0, x) = \vartheta^{2}(x), \quad x \in D. \quad (3)$$

All functions used in our considerations are assumed to be bounded. Furthermore, the thermal capacity $c(x)$ is a positive function, that is:

$$c(x) \geq c_{0} > 0, \quad x \in D. \quad (4)$$

Let us introduce the tensor $K_{ij}$ by means of the notation $K_{ij} = \kappa_{ij} - \kappa_{ij}^{*}$. We will suppose that $K_{ij}$ is a positive definite tensor: there exists a positive constant $K_{0}$ so that

$$K_{ij} \xi_{i} \xi_{j} \geq K_{0} \xi_{i} \xi_{j}. \quad (5)$$

Now, we will introduce the mixed initial-boundary value problem for the Moore–Gibson–Thompson theory of thermoelastic bodies with dipolar structure.

To describe the deformation of this type of bodies, we will use the following variables:

$$u_{i}(x, t), \quad \varphi_{ij}(x, t), \quad \vartheta (x, t), \quad (x, t) \in D \times [0, \infty). \quad (6)$$

The notation $u = (u_{i})$ is used for the displacement vector, $\varphi = \varphi_{ij}$ for the dipolar displacement tensor and $\vartheta$ is the variation of the temperature from the reference temperature $T_{0}$, that we have already introduced previously.

We will use three strain tensors $\varepsilon_{ij}$, $\gamma_{ij}$ and $\chi_{ijk}$ which are defined by means of the geometric equations:

$$\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right), \quad \gamma_{ij} = u_{ij} - \varphi_{ij}, \quad \chi_{mnr} = \varphi_{mn,r}. \quad (7)$$

Our considerations will be made in the context of a linear theory, such that it is natural to consider that the internal energy is a quadratic form in its variables, namely, of the following form (see [3]):

$$E = \frac{1}{2} C_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + G_{ijmn} \varphi_{ij} \varphi_{mn} + F_{ijmn} \vartheta_{mn}$$

$$+ \frac{1}{2} B_{ijmn} \vartheta_{ij} \vartheta_{mn} + 2 D_{ijmn} \vartheta_{ij} \vartheta_{mn} + A_{ijmn} \chi_{ij} \vartheta_{mn}$$

$$+ \frac{1}{2} c \left( \alpha \ddot{\vartheta} + \vartheta \right)^{2} - \left( a_{ij} \varepsilon_{ij} + b_{ij} \varphi_{ij} + c_{ijk} \chi_{ijk} \right) \right) \times \left( \alpha \ddot{\vartheta} + \vartheta \right) + \frac{1}{2} K_{ij} \vartheta_{i,j} \vartheta_{j}. \quad (8)$$

where $C_{ijmn}, G_{ijmn}, \ldots, A_{ijmn}$ are the elasticity tensors, whereas $a_{ij}, b_{ij}$ and $c_{ijk}$ are the coupling tensors and $K_{ij}$ is the thermal conductivity tensor, defined above.

By using an appropriate procedure, starting by the form of $E$ from (8), we can introduce the tensors of stress, having the components denoted by $\tau_{ij}, \sigma_{ij}$ and $\mu_{ijk}$, with the help of the constitutive equations (see [19]):

$$\tau_{ij} = \frac{\partial E}{\partial \varepsilon_{ij}} = C_{ijmn} \varepsilon_{mn} + G_{ijmn} \varphi_{mn} + F_{ijmn} \vartheta_{mn}$$

$$- a_{ij} \left( \alpha \ddot{\vartheta} + \vartheta \right), \quad (11)$$

$$\sigma_{ij} = \frac{\partial E}{\partial \varphi_{ij}} = G_{ijmn} \varphi_{ij} \varphi_{mn} + B_{ijmn} \vartheta_{ij} \vartheta_{mn}$$

$$- b_{ij} \left( \alpha \ddot{\vartheta} + \vartheta \right), \quad (12)$$

$$\mu_{ijk} = \frac{\partial E}{\partial \chi_{ijk}} = F_{ijmn} \varphi_{mn} \vartheta_{ij} + D_{ijmn} \vartheta_{ij} \vartheta_{mn}$$

$$- c_{ijk} \left( \alpha \ddot{\vartheta} + \vartheta \right), \quad (13)$$

$$S = \frac{\partial E}{\partial \alpha \vartheta} = - \left( a_{ij} \varepsilon_{ij} + b_{ij} \varphi_{ij} + c_{ijk} \chi_{ijk} \right)$$

$$+ c \left( \alpha \ddot{\vartheta} + \vartheta \right), \quad (14)$$

$$\varrho_{i} = \frac{\partial E}{\partial \vartheta_{i}} = K_{ij} \vartheta_{i,j}. \quad (15)$$

We have added in (9) the expressions for entropy $S$ and for the vector of heat flux having the components $\varrho_{i}$.

Also, the main equations, namely the motion equations, are obtained in the form (see [22]):

$$\left( \tau_{ij} + \sigma_{ij} \right)_{j} + \varrho_{i} = \varrho_{i,j} \quad (10)$$

$$\mu_{mnr,m} + \sigma_{mr} + \varrho_{mn} = l_{mn} \varrho_{mr}, \quad (11)$$

where the notation $l_{ij} = l_{ij}$ is used for the tensor of microinertia, $\varrho$ is the notation for the constant mass density in the reference state, $f = (f_{i})$ is the body forces vector and $g = g_{ij}$ is the dipolar body forces tensor.

We will use the energy equation in the form that follows (see [7]):

$$- \left( a_{ij} \varepsilon_{ij} + b_{ij} \varphi_{ij} + c_{ijk} \chi_{ijk} \right) \times \left( \alpha \ddot{\vartheta} + \vartheta \right)$$

$$= \frac{1}{\varrho_{0}} \left( K_{ij} \vartheta_{i,j} \right) + \frac{1}{\varrho_{0}} \varrho_{r}, \quad (11)$$

where we denoted by $r$ the heat supply.
We must suppose that the elasticity and coupling tensors used in the previous equations satisfy in the domain $D$ the following symmetry relations:

$$C_{ijmn} = C_{jimn}, \quad G_{ijmn} = G_{jimn},$$

$$B_{ijmn} = B_{jimn}, \quad F_{ijkmn} = F_{jikmn},$$

$$A_{ijkmn} = A_{jikmn}, \quad a_{ij} = a_{ji}, \quad K_{ij} = K_{ji}. \tag{12}$$

We substitute the constitutive equations (9) and the geometric equations (7) in the motion equations (10) such that we obtain a system of differential equations of the form:

$$[[C_{ijmn} + G_{ijmn}] u_{n,m} + (G_{ijmn} + B_{ijmn}) (u_{n,m} - \varphi_{mn}) + (F_{mnij} + D_{ijmn}) \varphi_{nr,m} - (a_{ij} + b_{ij}) (\alpha \ddot{\varphi} + \dot{\varphi})]_{j} + \varphi_{t_{i}} = \varphi u_{i},$$

$$F_{ijkmn} u_{n,m} + D_{mnij}(u_{n,m} - \varphi_{mn}) + A_{ijkmn} \varphi_{nr,m} - c_{ij} (\alpha \ddot{\varphi} + \dot{\varphi})]_{j} + G_{kmn} u_{m,n} + B_{jkmn}(u_{m,n} - \varphi_{mn}) + D_{kmn}\varphi_{r,m} - b_{jk} (\alpha \ddot{\varphi} + \dot{\varphi}) + \varepsilon g_{jk} = I_{k} \varphi_{j}. \tag{13}$$

In order to complete the mixed problem for the Moore–Gibson–Thompson theory of thermoelasticity for bodies with dipolar structures, we add the following initial conditions:

$$u_{ij}(0,x) = u_{ij}^{0}(x), \quad \dot{u}_{ij}(0,x) = u_{ij}^{1}(x),$$

$$\varphi_{ij}(0,x) = \varphi_{ij}^{0}(x), \quad \dot{\varphi}_{ij}(0,x) = \varphi_{ij}^{1}(x),$$

$$\vartheta(0,x) = \vartheta^{0}(x), \quad \dot{\vartheta}(0,x) = \vartheta^{1}(x), \quad x \in D, \tag{14}$$

where $u_{ij}^{0}(x), u_{ij}^{1}(x), \varphi_{ij}^{0}(x), \varphi_{ij}^{1}(x), \vartheta^{0}(x)$ and $\vartheta^{1}(x)$ are prescribed functions.

Also, we add the boundary conditions:

$$u_{ij} = \bar{u}_{ij} \text{ on } \partial D_{1} \times [0,t_{0}),$$

$$\varphi_{ij} = \bar{\varphi}_{ij} \text{ on } \partial D_{2} \times [0,t_{0}),$$

$$m_{ij} = \bar{m}_{ij} \text{ on } \partial D_{3} \times (0,t_{0}),$$

$$\vartheta = \bar{\vartheta} \text{ on } \partial D_{3} \times [0,t_{0}),$$

$$q = \bar{q} \text{ on } \partial D_{3} \times (0,t_{0}), \tag{15}$$

where $\bar{u}_{ij}, \bar{\varphi}_{ij}, \bar{m}_{ij}, \bar{\vartheta}$ and $\bar{q}$ are given functions.

In (15) we used the notation

$$t_{i} = (t_{j} + \alpha_{ij}) n_{i}, \quad m_{jk} = \mu_{ijk} n_{i}, \quad q = q n_{i}.$$
With these notations, inequality (19) receives on a much simpler form, as follows

$$P(s, x) \geq K(s, x), \quad \forall (s, x) \in [0, t] \times D. \quad (22)$$

The inequality that we will approach in next theorem is the basis for the demonstration of our main result. The following notations, for a disk and its border, are common:

$$B(x_0, \rho) = \{x \in D : |x - x_0| < \rho\},$$

$$\partial B(x_0, \rho) = \{x \in \partial D : |x - x_0| = \rho\}.$$

**Theorem 3.1:** If hypotheses (i)–(iii) are satisfied, then for any solution $$(u_t, \varphi_{ij}, \theta)$$ of the problem $P$, the inequality that follows is fulfilled

$$\int_{B(x_0,\rho)} P(t, x) \, dV + \frac{1}{T_0} \int_0^T \int_{B(x_0,\rho) + \nu(t - \tau)} K_{ij} \theta_{i,j} \, dV \, d\tau \leq \int_{B(x_0,\rho) + \nu(t - \tau)} \epsilon \left( f_t u_t + g_{jk} \varphi_{jk} + \frac{1}{T_0} r_\theta \right) \, dV \, d\tau$$

$$+ \int_0^T \int_{B(x_0,\rho) + \nu(t - \tau)} \left( t_i u_t + \ddot{m}_{jk} \varphi_{jk} + \frac{1}{T_0} \ddot{r}_\theta \right) \, dA \, d\tau. \quad (23)$$

**Proof:** In the beginning we use Equation (13)1, multiplied by $W u_t$, so that we obtain

$$W \frac{d}{dt} (\varphi u_t) = 2W \dot{\varphi} u_t + \left[ 2W \left( t_y + \sigma_j \right) \dot{u}_j \right]$$

$$- 2W \left( t_y + \sigma_j \right) u_t - 2W \left[ C_{ijm\ell} \phi_{m\ell} + G_{mnij} \psi_{mn} \right]$$

$$+ F_{mnij} \psi_{mn} - \sigma_j \left( \alpha \ddot{\phi} + \dot{\psi} \right) u_t$$

$$- 2W \left[ G_{ijm\ell} \phi_{m\ell} + B_{ijmn} \psi_{mn} + D_{ijmn} \psi_{mn} \right]$$

$$- b_j \left( \alpha \ddot{\phi} + \dot{\psi} \right) \dot{u}_j. \quad (24)$$

Similarly, using Equation (13)2, multiplied by $W \dot{\varphi}_{jk}$, we are led to

$$W \frac{d}{dt} \left( l_{jk} \dot{\varphi}_{jk} \right) = 2W \dot{\varphi} \dot{\varphi}_{jk} - 2W \left( l_{jk} \dot{\varphi}_{jk} \right) - 2W \left( l_{jk} \dot{\varphi}_{jk} \right)$$

$$- 2W \left[ G_{ijm\ell} \psi_{m\ell} + B_{ijmn} \psi_{mn} + D_{ijmn} \psi_{mn} \right]$$

$$+ 2W \left[ G_{ijm\ell} \psi_{m\ell} + B_{ijmn} \psi_{mn} + D_{ijmn} \psi_{mn} \right] \psi_{jk} \psi_{jk} \quad (25)$$

Finally, we proceed in the same way with Equation (11) that we multiply by $W \theta$ to obtain

$$W \frac{d}{dt} \left( c \theta^2 \right) = \frac{2}{T_0} W r_\theta + \frac{2}{T_0} \left[ (W \theta, q_i)_j - W_j \theta q_i \right]$$

$$- \frac{2}{T_0} W K_{ij} \theta_{i,j} - 2W \left( a_{ij} \ddot{\theta} + b_{ij} \dot{\theta} + c_{ij} \dot{\chi}_{ij} \right) x \left( \alpha \ddot{\phi} + \dot{\psi} \right). \quad (26)$$

Now we add member with member the identities (24)–(26) and obtain the equality:

$$W \frac{d}{dt} (\varphi u_t \dot{u}_t + l_{jk} \dot{\varphi}_{jk} \psi_{jk} + c \theta^2) = 2W \left( f_t u_t + g_{jk} \varphi_{jk} + \frac{1}{T_0} r_\theta \right) \psi_{jk} + 2W \left( t_i u_t + \ddot{m}_{jk} \varphi_{jk} + \frac{1}{T_0} \ddot{r}_\theta \right) \psi_{jk} \quad (27)$$
taking into account the boundary conditions (15), we
D
\text{after that making use of the divergence theorem and }
\text{this identity can be written in a shorter form, if we }
\text{which, clearly, can be restated as follows }
\begin{align*}
\frac{1}{2} W \frac{d}{dt} \left[ \varphi \dot{u}_i + l_{jk} \psi_{jk} + c \dot{r}^2 + C_{ijmn} \varphi_{ijmn} \right] \\
+ G_{mnij} \psi_{mnij} + B_{ijmn} \varphi_{ijmn} + F_{mnij} \varphi_{mnij} \\
+ D_{ijmn} \varphi_{ijmn} + A_{ijmn} \varphi_{ijmn} \\
+ \left( a_{ij} \dot{q}_j + b_{ij} \dot{q}_i + c_{jk} \dot{\chi}_{jk} \dot{\theta} \right) + \frac{1}{\varrho T_0} K_{ij} \dot{\theta}_j \\
= W_t \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right] \\
- W_j \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right],
\end{align*}
and this identity can be written in a shorter form, if we use the potential energy P, defined in (20):
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \varphi \dot{u}_i + l_{jk} \psi_{jk} + c \dot{r}^2 + C_{ijmn} \varphi_{ijmn} \right] \\
+ G_{mnij} \psi_{mnij} + B_{ijmn} \varphi_{ijmn} + F_{mnij} \varphi_{mnij} \\
+ D_{ijmn} \varphi_{ijmn} + A_{ijmn} \varphi_{ijmn} \\
+ \left( a_{ij} \dot{q}_j + b_{ij} \dot{q}_i + c_{jk} \dot{\chi}_{jk} \dot{\theta} \right) + \frac{1}{\varrho T_0} K_{ij} \dot{\theta}_j \\
= W_t \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right] \\
- W_j \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right],
\end{align*}
and this identity can be written in a shorter form, if we use the potential energy P, defined in (20):
\begin{align*}
\frac{1}{2} W \dot{\theta} + \frac{1}{\varrho T_0} K_{ij} \dot{\theta}_j \\
= W_t \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right] \\
- W_j \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right].
\end{align*}
Integrating both members of identity (27) over [0, t] \times D, after that making use of the divergence theorem and taking into account the boundary conditions (15), we obtain
\begin{align*}
\int_D W \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right) dV d\tau \\
- \int_0^t \int_D W_t \left[ \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right] dV d\tau \\
= \int_D W \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right) dV \\
= \int_0^t \int_D W_t \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right) dA d\tau \\
+ \int_0^t \int_D W_t \left( \tau_j + \sigma_j \right) \dot{u}_j + \mu_{jk} \dot{\psi}_{jk} + \frac{1}{\varrho T_0} \varphi \dot{q}_i \right) dV d\tau
\end{align*}
the last inequality being a consequence of the inequality (22).
With the help of inequalities (29) and (30), from the identity (28) we deduce the following useful inequality:
Now, we will define the domain of influence. For a time instant \( t \) we will consider \( D(t) \) as a domain which contains the points \( x \in \ddot{D} \), so that:

- if \( x \in D, u_i^0 \neq 0 \) or \( u_i^1 \neq 0 \) or \( \psi_j^0 \neq 0 \), or \( \vartheta^0 \neq 0 \), then \( \exists s \in [0, t] \) so that \( f_i(s, x) \neq 0 \) or \( g_j(s, x) \neq 0 \) or \( r(s, x) \neq 0 \);
- if \( x \in \partial D_1 \), \( \exists s \in [0, t] \) so that \( u_i(s, x) \neq 0 \);
- if \( x \in \partial D_2 \), \( \exists s \in [0, t] \) so that \( f_i(s, x) \neq 0 \);
- if \( x \in \partial D_3 \), \( \exists s \in [0, t] \) so that \( \psi_j(s, x) \neq 0 \);
- if \( x \in \partial D_4 \), \( \exists s \in [0, t] \) so that \( \vartheta(s, x) \neq 0 \);
- if \( x \in \partial D_5 \), \( \exists s \in [0, t] \) so that \( \vartheta(s, x) \neq 0 \).

With the help of the set \( D(t) \), the domain of influence is defined by:

\[
D_t = \{ y \in \ddot{D} : D(t) \cap S(y, vt) \} \neq \emptyset, \tag{32}
\]

where the sphere \( S(y, vt) \) is defined in (17) and \( \emptyset \) is the empty set.

Our important result is addressed in the following theorem.

**Theorem 3.2:** If hypotheses (i)–(iii) are satisfied, then for any solution \((u_i, \psi_j, \vartheta)\) of the problem \( P \), we have the following characterization:

\[
u_i(t, x) = 0, \quad \psi_j(t, x) = 0, \quad \text{and} \quad \vartheta(t, x) = 0, \quad \text{for any } (t, x) \in [0, t] \times (\ddot{D} \setminus D_t). \tag{33}
\]

**Proof:** For an arbitrarily \( x_0 \in \ddot{D} \setminus D_t \) and \( s \in [0, t] \), we will write the inequality (23) with \( t = s \) and \( \rho = v(t - s) \) so that we deduce

\[
\int_{\mathcal{B}(x_0, v(t-s))} \mathcal{P}(t, x) \, dV
\]

\[+ \int_0^t \int_{\mathcal{B}(x_0, v(t-r))} K_i \vartheta_i j \vartheta_i j \, dV \, dr \]

\[
\leq \int_{\mathcal{B}(x_0, vt)} \mathcal{P}(0, x) \, dV
\]

\[+ \int_0^t \int_{\mathcal{B}(x_0, v(t-r))} \left( f_i u_i + g_j \psi_j + \frac{1}{T_0} r^0 \right) \, dV \, dr \]

\[+ \int_0^t \int_{\partial \mathcal{B}(x_0, v(t-r))} \left( \bar{T}_i u_i + \bar{m}_j \psi_j + \frac{1}{T_0} \bar{q}^0 \right) \, dA \, dr. \tag{34}
\]

Because \( x_0 \in \ddot{D} \setminus D_t \), we deduce that \( x \in B(x_0, vt) \), as such \( x \notin D(t) \). Therefore, we have:

\[
\int_{\mathcal{B}(x_0, vt)} \mathcal{P}(0, x) \, dV = 0. \tag{35}
\]

Clearly, we have the inclusion \( B(x_0, v(t - r)) \subset B(x_0, vt) \). Thus, we deduce

\[
\int_0^t \int_{\mathcal{B}(x_0, v(t-r))} \left( f_i u_i + g_j \psi_j + \frac{1}{T_0} r^0 \right) \, dV \, dr = 0,
\]

\[
\int_0^t \int_{\partial \mathcal{B}(x_0, v(t-r))} \left( \bar{T}_i u_i + \bar{m}_j \psi_j + \frac{1}{T_0} \bar{q}^0 \right) \, dA \, dr = 0. \tag{36}
\]

Based on (35) and (36), from (34) we deduce the inequality that follows:

\[
\int_{\mathcal{B}(x_0, v(t-s))} \mathcal{P}(t, x) \, dV
\]

\[+ \frac{1}{T_0} \int_0^t \int_{\mathcal{B}(x_0, v(t-r))} K_i \vartheta_i j \vartheta_i j \, dV \, dr \leq 0,
\]

so that, considering the hypothesis (i), we deduce

\[
\int_{\mathcal{B}(x_0, v(t-s))} \mathcal{P}(t, x) \, dV \leq 0,
\]

and then, considering (22), the more we have the inequality

\[
\int_{\mathcal{B}(x_0, v(t-s))} K(t, x) \, dV \leq 0. \tag{37}
\]

Now, we take into account the definition (21) of the kinetic energy \( K \) so that from (37) we deduce

\[
u_i(t, x_0) = 0, \quad \psi_j(t, x_0) = 0, \quad \vartheta(t, x_0) = 0, \quad \forall (t, x_0) \in [0, t] \times (\ddot{D} \setminus D_t).
\]

But, we have

\[
u_i(0, x_0) = 0, \quad \psi_j(0, x_0) = 0, \quad \forall x_0 \in \ddot{D} \setminus D_t,
\]

that’s why the final conclusion is

\[
u_i(t, x_0) = 0, \quad \psi_j(t, x_0) = 0, \quad \vartheta(t, x_0) = 0, \quad \forall (t, x_0) \in [0, t] \times (\ddot{D} \setminus D_t),
\]

which concludes the proof of Theorem 3.2.  

**4. Conclusion**

Some researchers consider that a result regarding the domain of influence, is, in fact, is a more relaxed form of the known principle of Saint-Venant from elementary elasticity. So, our result is a generalization of this principle to arrive in the context of the theory of Moore–Gibson–Thompson thermoelasticity for bodies with dipolar structure. Then, we managed to prove that this type of influence domain can be built even if it
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