Enumerating Regular Objects associated with Suzuki Groups

Martin Downs and Gareth A. Jones
School of Mathematics
University of Southampton
Southampton SO17 1BJ, U.K.
G.A.Jones@maths.soton.ac.uk

Abstract

We use the Möbius function of the simple Suzuki group $Sz(q)$ to enumerate regular objects such as maps, hypermaps, dessins d’enfants and surface coverings with automorphism groups isomorphic to $Sz(q)$.

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1 Introduction

In [12], Hall introduced the concept of Möbius inversion in the lattice of subgroups of a group $G$, and used it to find the number $n_\Gamma(G)$ of normal subgroups $N$ of a finitely generated group $\Gamma$ with quotient $\Gamma/N$ isomorphic to a given finite group $G$. An important ingredient in this theory is the Möbius function $\mu_G$ of $G$, a function from the set of subgroups $H$ of $G$ to $\mathbb{Z}$ defined recursively by

$$\sum_{K \geq H} \mu_G(K) = \delta_{H,G}$$

where $\delta_{H,G}$ is the Kronecker delta function, equal to 1 or 0 as $H = G$ or $H < G$. Among the groups $G$ for which Hall computed this function were the simple groups $L_2(p) = PSL_2(p) = SL_2(p)/(\pm I)$ for primes $p \geq 5$; he then used this to evaluate $n_\Gamma(G)$ for various groups $\Gamma$, such as the free group $F_2$ of rank 2 and the free products $C_2*C_2*C_2$ and $C_2*C_3 \cong PSL_2(\mathbb{Z})$ (the modular group).

In [4] the first author extended the calculation of $\mu_G$ to the groups $G = L_2(q)$ and $PGL_2(q)$ for all prime powers $q$; see [5] for $G = L_2(q)$, including a proof for $q = 2^e$ and a statement of the corresponding results for odd $q$, together with the calculation of $n_\Gamma(G)$ for the modular group $\Gamma$. In [7] the present authors applied these results to enumerate various
combinatorial objects, such as regular and orientably regular maps and hypermaps, with automorphism groups isomorphic to $L_2(2^e)$; this is possible since in each of these categories the (isomorphism classes of) regular objects correspond bijectively to the normal subgroups $N$ of some finitely generated group $\Gamma$, with automorphism groups isomorphic to $\Gamma/N$.

In [23], Suzuki discovered a family of non-abelian finite simple groups $G = Sz(q) = 2B_2(q)$, where $q = 2^e$ for some odd $e > 1$, with internal structure similar to that of the groups $L_2(2^e)$. The first author computed the Möbius function $\mu_G$ for these groups in [6]. Our aim here is to apply this to enumerate various regular objects, such as maps, hypermaps, dessins d’enfants and surface coverings, with automorphism groups isomorphic to $G$. This extends work by Silver and the second author [17], where orientably regular maps of type $\{4,5\}$ with automorphism group $G$ were enumerated, and by Hubard and Leemans [13], where regular and chiral maps and polytopes were enumerated. In each case the authors used a restricted form of Möbius inversion, concentrating mainly on subgroups $H \cong Sz(2^f)$ where $f$ divides $e$. Here we use the full Möbius function $\mu_G$ (see Table 1 in §6), allowing a wider range of regular objects to be enumerated; this is done for $G = Sz(q)$ in [7] and in more detail for $G = Sz(8)$ in [8]. A typical result is that the formula

$$\frac{1}{e} \sum_{f|e} \mu\left(\frac{e}{f}\right) 2^f(2^{4f} - 2^{3f} - 9),$$

where $\mu$ is the classical Möbius function on $\mathbb{N}$, counts the normal subgroups of $F_2$ with quotient group $G = Sz(2^e)$, and hence also the orbits of $\text{Aut} G$ on generating pairs for $G$, and the orientably regular hypermaps (or regular dessins d’enfants [11]) with orientation-preserving automorphism group $G$.

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2 Möbius inversion in groups

Here we briefly outline Hall’s theory of Möbius inversion in groups [12]. If $\sigma$ and $\phi$ are integer-valued functions defined on isomorphism classes of finite groups, such that

$$\sigma(G) = \sum_{H \leq G} \phi(H) \quad (2)$$

for all finite groups $G$, then a simple argument shows that

$$\phi(G) = \sum_{H \leq G} \mu_G(H)\sigma(H) \quad (3)$$

where $\mu_G$ is the Möbius function for $G$ defined by equation (1). We can regard this as inverting equation (2) to produce equation (3); it is an analogue of Möbius inversion in
elementary number theory, where $\sigma$ and $\phi$ are functions $N \to \mathbb{Z}$, and the corresponding lattice consists of the subgroups $H = n\mathbb{Z}$ of finite index $n$ in $\mathbb{Z}$, identified with the natural numbers $n \in \mathbb{N}$ ordered by divisibility.

An important example of such a pair of functions is given by taking $\sigma(G) = |\text{Hom}(\Gamma, G)|$ and $\phi(G) = |\text{Epi}(\Gamma, G)|$, where $\text{Hom}(\Gamma, G)$ and $\text{Epi}(\Gamma, G)$ are the sets of homomorphisms and epimorphisms $\Gamma \to G$ for some finitely generated group $\Gamma$. These are finite sets, since such a homomorphism is uniquely determined by the images in $G$ of a finite set of generators for $\Gamma$. Since every homomorphism is an epimorphism onto some unique subgroup $H \leq G$ we have

$$|\text{Hom}(\Gamma, G)| = \sum_{H \leq G} |\text{Epi}(\Gamma, H)|,$$

so Möbius inversion yields

$$|\text{Epi}(\Gamma, G)| = \sum_{H \leq G} \mu_G(H) |\text{Hom}(\Gamma, H)|.$$

This is often a useful equation, since counting homomorphisms is generally easier than counting epimorphisms. The cost is that one has to evaluate $\mu_G(H)$ for all $H \leq G$, but there are two compensations: firstly Hall [12] showed that $\mu_G(H) = 0$ if $H$ is not an intersection of maximal subgroups of $G$, so such subgroups $H$ can be deleted from (5), and secondly, once $\mu_G$ has been computed it can be applied to many other pairs of functions $\sigma$ and $\phi$. For instance, such a pair could represent the numbers of homomorphisms and epimorphisms $\Gamma \to G$ with certain extra properties, such as being smooth (i.e. having a torsion-free kernel).

Now let $N(G) = N_\Gamma(G)$ denote the set of normal subgroups $N$ of $\Gamma$ with $\Gamma/N \cong G$, and let

$$n(G) = n_\Gamma(G) = |N(G)| = |N_\Gamma(G)|.$$

These normal subgroups $N$ are the kernels of the epimorphisms $\Gamma \to G$, and two such epimorphisms have the same kernel if and only if they differ by an automorphism of $G$. It follows that $n(G)$ is the number of orbits of $\text{Aut}(G)$, acting by composition on $\text{Epi}(\Gamma, G)$. This action is semiregular, since only the identity automorphism of $G$ can fix an epimorphism $\Gamma \to G$; thus all orbits have length $|\text{Aut}(G)|$ and hence

$$n_\Gamma(G) = \frac{|\text{Epi}(\Gamma, G)|}{|\text{Aut}(G)|} = \frac{1}{|\text{Aut}(G)|} \sum_{H \leq G} \mu_G(H) |\text{Hom}(\Gamma, H)|.$$

3 Counting homomorphisms

In order to apply this method to a specific pair of groups $\Gamma$ and $G$, one needs to be able to count homomorphisms $\Gamma \to H$ for the subgroups $H \leq G$. Given a presentation for $\Gamma$ with generators $X_i$ and defining relations $R_j(x_i) = 1$, this amounts to counting the solutions $(x_i)$ in $H$ of the equations $R_j(x_i) = 1$. For certain groups $\Gamma$, the character table of $H$ gives this information, as illustrated by the following theorem of Frobenius [8].
Theorem 3.1 Let $C_i \ (i = 1, 2, 3)$ be conjugacy classes in a finite group $H$. Then the number of solutions of the equation $x_1 x_2 x_3 = 1$ in $H$, with $x_i \in C_i$ for $i = 1, 2, 3,$ is given by the formula

$$\frac{|C_1||C_2||C_3|}{|H|} \sum \chi(x_1)\chi(x_2)\chi(x_3) \chi(1)$$

where $x_i \in C_i$ and $\chi$ ranges over the irreducible complex characters of $H$.

If $\Gamma$ is the triangle group

$$\Delta(m_1, m_2, m_3) = \langle X_1, X_2, X_3 \mid X_1^{m_1} = X_2^{m_2} = X_3^{m_3} = X_1 X_2 X_3 = 1 \rangle$$

of type $(m_1, m_2, m_3)$ for some integers $m_i$, then $|\text{Hom}(\Gamma, H)|$ can be found by summing over all choices of triples of conjugacy classes $C_i$ of elements of orders dividing $m_i$. Similarly, the number of smooth homomorphisms $\Gamma \to H$ can be found by restricting the summation to classes of elements of order equal to $m_i$.

When $\Gamma$ is an orientable surface group, that is, the fundamental group

$$\Pi_g = \pi_1 S_g = \langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_{i=1}^{g}[A_i, B_i] = 1 \rangle$$

of a compact orientable surface $S_g$ of genus $g \geq 1$, with $[a, b]$ denoting the commutator $a^{-1}b^{-1}ab$, the following theorem of Frobenius and Mednykh is useful (see for applications):

Theorem 3.2 In any finite group $H$, the number of solutions $(a_i, b_i)$ of the equation $\prod_{i=1}^{g}[a_i, b_i] = 1$ is given by the formula

$$|H|^{2g-1} \sum \chi(1)^{2-2g}$$

where $\chi$ ranges over the irreducible complex characters of $H$.

The formula in gives $|\text{Hom}(\Pi_g, H)|$ in terms of the degrees $\chi(1)$ of the irreducible characters of $H$. When $\Gamma$ is a non-orientable surface group

$$\Pi_g^- = \langle A_i \ (i = 1, \ldots, g) \mid \prod_{i=1}^{g} A_i^2 = 1 \rangle$$

of genus $g \geq 1$, the corresponding result of Frobenius and Schur is as follows:

Theorem 3.3 In any finite group $H$, the number of solutions $(a_i)$ of the equation $\prod_{i=1}^{g} a_i^2 = 1$ is given by the formula

$$|H|^{g-1} \sum c_{\chi}^g \chi(1)^{2-g}$$

where $\chi$ ranges over the irreducible complex characters of $H$.

Here $c_{\chi} = |H|^{-1} \sum_{h \in H} \chi(h^2)$ is the Frobenius-Schur indicator of $\chi$, equal to $1, -1$ or $0$ as $\chi$ is respectively the character of a real representation, the real character of a non-real representation, or a non-real character.
4 Categories and groups

In some categories $\mathcal{C}$, there is a group $\Gamma = \Gamma_\mathcal{C}$, which we will call the parent group of $\mathcal{C}$, such that the set $\mathcal{R}(G) = \mathcal{R}_\mathcal{C}(G)$ of regular objects in $\mathcal{C}$ with automorphism group $G$ is in bijective correspondence with the set $\mathcal{N}(G) = \mathcal{N}_\Gamma(G)$ of normal subgroups of $\Gamma$ with quotient group $G$ (see [16] for further details). In particular, if $\Gamma$ is finitely generated and $G$ is finite then these two sets have the same finite cardinality

$$r(G) = r_\mathcal{C}(G) := |\mathcal{R}_\mathcal{C}(G)| = n(G) = n_\Gamma(G) := |\mathcal{N}_\Gamma(G)|,$$

so that equation (7) gives

$$r_\mathcal{C}(G) = 1/|\text{Aut } G| \sum_{H \leq G} \mu_G(H)|\text{Hom}(\Gamma, H)|.$$

4.1 Maps, hypermaps and groups

A map $\mathcal{M}$ is regular (in the category $\mathfrak{M}$ of all maps) if its automorphism group $G = \text{Aut } \mathcal{M} = \text{Aut}_\mathfrak{M} \mathcal{M}$ acts transitively on vertex-edge-face flags, which can be identified with the faces of the barycentric subdivision $\mathcal{B}$ of $\mathcal{M}$. In this case $G$ is generated by automorphisms $r_i$ ($i = 0, 1, 2$) which change (in the only possible way) the $i$-dimensional component of a particular flag, while preserving its $j$-dimensional components for each $j \neq i$. If $\mathcal{M}$ has type $\{m, n\}$ in the notation of [2, Ch. 8], meaning that its faces are all $m$-gons and its vertices all have valency $n$, these generators satisfy

$$r_i^2 = (r_0 r_1)^m = (r_0 r_2)^2 = (r_1 r_2)^n = 1.$$

It follows that there is an epimorphism $\theta : \Gamma \to G$, $r_i \mapsto r_i$, where

$$\Gamma = \Gamma_\mathfrak{M} = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle,$$

so $\mathcal{M}$ determines a normal subgroup $N = \ker \theta$ of $\Gamma$ with $\Gamma/N \cong G$. Conversely, each such normal subgroup determines a regular map $\mathcal{M}$ with $\text{Aut } \mathcal{M} \cong G$. Two such maps are isomorphic if and only if they correspond to the same normal subgroup, so the set $\mathcal{R}(G) = \mathcal{R}_\mathfrak{M}(G)$ of regular maps with automorphism group $G$ is in bijective correspondence with the set $\mathcal{N}(G) = \mathcal{N}_\Gamma(G)$. If $G$ is finite then the preceding argument gives

$$r_\mathfrak{M}(G) = 1/|\text{Aut } G| \sum_{H \leq G} \mu_G(H)|\text{Hom}(\Gamma, H)|.$$

This group $\Gamma_\mathfrak{M}$, a free product of its subgroups $\langle R_0, R_2 \rangle \cong V_4$ and $\langle R_1 \rangle \cong C_2$, can be regarded as the extended triangle group $\Delta[\infty, 2, \infty]$ of type $(\infty, 2, \infty)$, generated by reflections in the sides of a hyperbolic triangle with angles $0, \pi/2, 0$. Other triangle groups play a similar role for related categories. For example, the extended triangle group

$$\Gamma = \Delta[n, 2, m] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_1)^m = (R_0 R_2)^2 = (R_1 R_2)^n = 1 \rangle$$
is the parent group for maps of all types \( \{m', n'\} \) dividing \( \{m, n\} \) (meaning that \( m' \) divides \( m \) and \( n' \) divides \( n \)). For maps of type \( \{m, n\} \) one must restrict attention to normal subgroups \( N \) of \( \Gamma \) such that \( R_0R_1 \) and \( R_1R_2 \) have images of orders \( m \) and \( n \) in \( \Gamma/N \).

For the category \( \mathfrak{H} \) of hypermaps, where hyperedges may be incident with any number of hypervertices and hyperfaces, we delete the relation \((R_0R_2)^2 = 1\) from the presentation (13), giving the group

\[
\Gamma_{\mathfrak{H}} = \Delta[\infty, \infty, \infty] \cong C_2 * C_2 * C_2.
\]

For hypermaps of types dividing \( \{m, n\} \), we use the extended triangle group

\[
\Delta[p, q, r] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0R_1)^r = (R_0R_2)^q = (R_1R_2)^p = 1 \rangle.
\]

The parent groups for the categories \( \mathfrak{M}^+ \) and \( \mathfrak{H}^+ \) of oriented maps and hypermaps are the orientation-preserving subgroups of index 2 in \( \Gamma_{\mathfrak{M}} \) and \( \Gamma_{\mathfrak{H}} \), generated by the elements \( X = R_1R_0, Y = R_0R_2 \) and \( Z = R_2R_1 \) satisfying \( XYZ = 1 \). These are the triangle groups

\[
\Gamma_{\mathfrak{M}^+} = \Delta(\infty, 2, \infty) = \langle X, Y, Z \mid Y^2 = XYZ = 1 \rangle \cong C_\infty * C_2
\]

and

\[
\Gamma_{\mathfrak{H}^+} = \Delta(\infty, \infty, \infty) = \langle X, Y, Z \mid XYZ = 1 \rangle \cong C_\infty * C_\infty \cong F_2.
\]

For oriented maps of types dividing \( \{m, n\} \), or oriented hypermaps of types dividing \( \{p, q, r\} \), we use the triangle groups \( \Delta(n, 2, m) \) and \( \Delta(p, q, r) \), restricting attention to torsion-free normal subgroups for maps and hypermaps of these exact types. (See [18] for further background for these categories.)

### 4.2 Reflexibility

The regular objects in the categories \( \mathfrak{M}^+ \) and \( \mathfrak{H}^+ \) are often referred to as orientably regular, since they need not be regular as objects in the larger categories \( \mathfrak{M} \) and \( \mathfrak{H} \). Let \( \mathcal{H} \) be an orientably regular hypermap of type \( (p, q, r) \), corresponding to a normal subgroup \( N \) of \( \Gamma_{\mathfrak{H}^+} = F_2 \) with \( F_2/N \cong \text{Aut}_{\mathfrak{H}^+} \mathcal{H} \cong G \) for some group \( G \). Then the following are equivalent:

- \( \mathcal{H} \) is regular in the category \( \mathfrak{H} \) of all hypermaps;
- \( \mathcal{H} \) has an orientation-reversing automorphism;
- \( N \) is normal in \( \Gamma_{\mathfrak{H}^+} = C_2 * C_2 * C_2 \);
- some (and hence each) pair of the canonical generating triple \( x, y, z \) for \( G \) are inverted by an automorphism of \( G \).

If these conditions hold we say that \( \mathcal{H} \) is reflexible; otherwise it is chiral, and \( \mathcal{H} \) and its mirror image \( \overline{\mathcal{H}} \) form a chiral pair, isomorphic in \( \mathfrak{H} \) but not in \( \mathfrak{H}^+ \).

If \( \mathcal{H} \) is reflexible then \( \overline{G} := \Aut_{\mathfrak{H}} \mathcal{H} \cong \Gamma_{\mathfrak{H}}/N \) is a semidirect product of \( G \) by a complement \( C_2 \) generated by the image \( R_i \) of any \( R_i \) \((i = 0, 1, 2)\) in \( G \). The elements of
\[ \tilde{G} \setminus G \text{ act by conjugation on the normal subgroup } G; \text{ if one of them induces an inner automorphism then they all do, and we say that } \mathcal{H} \text{ is inner reflexible. In this case, } r_i \text{ induces conjugation by some } g \in G, \text{ so } c := r_i g \text{ centralises } G \text{ and hence } c^2 \text{ is in the centre } Z(G) \text{ of } G. \text{ If } Z(G) \text{ is trivial then } \tilde{G} = G \times C \text{ where } C = \langle c \rangle \cong C_2, \text{ and there is a non-orientable regular hypermap } \mathcal{H} = \mathcal{H}/C \in \mathcal{R}_S(G) \text{ with orientable double cover } \mathcal{H}; \text{ this gives a monomorphism } \mathcal{H} \mapsto \mathcal{H}, \text{ from the inner reflexible maps in } \mathcal{R}_S^+(G) \text{ to } \mathcal{R}_S(G). \text{ If, in addition, } G \text{ has no subgroup of index } 2, \text{ then each hypermap in } \mathcal{R}_S(G) \text{ is non-orientable, with an inner reflexible orientable double cover in } \mathcal{R}_S^+(G), \text{ so this monomorphism is a bijection. This proves the first part of the following result; the second part is obvious:}

**Proposition 4.1** (a) For any finite group } G \text{ with trivial centre and no subgroup of index } 2, \text{ the inner reflexible hypermaps in } \mathcal{R}_S^+(G) \text{ are the orientable double covers of the hypermaps in } \mathcal{R}_S(G); \text{ there are } r_S(G) \text{ of them.} \\
(b) \text{ If, in addition, } \text{Out} \ G \text{ has odd order, then every reflexible hypermap in } \mathcal{R}_S^+(G) \text{ is inner reflexible, and there are } r_S(G) \text{ of them.} 

Every non-abelian finite simple group } G \text{ satisfies (a), and the Suzuki groups also satisfy (b). The function } \mathcal{H} \mapsto \mathcal{H} \text{ preserves types of hypermaps, so the above proposition also applies to maps.}

### 4.3 Covering spaces

Under suitable conditions (namely, that } X \text{ is path connected, locally path connected, and semilocally simply connected } [22, \text{ Ch. 13}]), \text{ the equivalence classes of unbranched coverings } Y \rightarrow X \text{ of a topological space } X \text{ form a category } \mathcal{C} \text{ in which the connected objects correspond to the conjugacy classes of subgroups of the fundamental group } \Gamma = \pi_1 X; \text{ among these, the regular coverings correspond to the normal subgroups } N \text{ of } \Gamma, \text{ with covering group isomorphic to } \Gamma/N. \text{ If, in addition, } X \text{ is a compact Hausdorff space, then } \Gamma \text{ is finitely generated } [22, \text{ p. 500}], \text{ so one can use the methods described earlier to count regular coverings of } X \text{ with a given finite covering group. In particular, this applies if } X \text{ is a compact manifold or orbifold. Indeed, the categories of maps and hypermaps described above can be regarded as obtained in this way from suitable orbifolds } X, \text{ such as a triangle with angles } \pi/p, \pi/q, \pi/r \text{ for hypermaps of type dividing } (p, q, r), \text{ or a sphere with three cone-points of orders } p, q, r \text{ in the oriented case. Similarly, Grothendieck’s dessins d’enfants } [10, 11] \text{ are the finite coverings of a sphere minus three points, so their parent group is its fundamental group } F_2. 

### 5 The Suzuki groups

This section is largely based on Suzuki’s description in [23] of the groups named after him; see also [11, §XI.3] and [24, §4.2].
5.1 The definition of the Suzuki Group $G(e)$

Let $\mathbb{F}_q$ be the finite field of $q = 2^e$ elements for some odd $e > 1$, and let $\theta$ be the automorphism $\alpha \mapsto \alpha^\theta$ of $\mathbb{F}_q$ where $r = 2^{(e+1)/2}$, so that $\theta^2$ is the Frobenius automorphism $\alpha \mapsto \alpha^2$.

For any $\alpha, \beta \in \mathbb{F}_q$ let $(\alpha, \beta)$ denote the $4 \times 4$ matrix

$$(\alpha, \beta) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
\alpha & \alpha^{\theta+1} + \beta & \alpha^\theta & \beta \\
\alpha^{\theta+2} + \alpha \beta & \alpha^\theta & \alpha & 1
\end{pmatrix}.$$

Since $(\alpha, \beta)(\gamma, \delta) = (\alpha + \gamma, \alpha \gamma^\theta + \beta + \delta)$, these matrices $(\alpha, \beta)$ form a group $Q(e)$ of order $q^2$.

The $4 \times 4$ diagonal matrices with diagonal entries $\alpha^{1+\theta}$, $\alpha$, $\alpha^{-1}$, $\alpha^{-1-\theta}$ for $\alpha \in \mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$ form a cyclic group $A_0(e)$ of order $q - 1$. The group $F(e)$ generated by $Q(e)$ and $A_0(e)$ is a semidirect product of a normal subgroup $Q(e)$ by a complement $A_0(e)$, of order $q^2(q - 1)$.

Let $\tau$ denote the $4 \times 4$ matrix with entries 1 on the minor diagonal and 0 elsewhere. We define $G(e)$ to be the subgroup of $GL_4(q)$ generated by $F(e)$ and $\tau$. This is the Suzuki group, usually denoted by $Sz(q)$ or $^2B_2(q)$. It is, in fact, the subgroup of the symplectic group $Sp_4(q) = B_2(q)$ fixed by a certain automorphism of order 2.

5.2 Notation for some subgroups of $G(e)$

Let us fix $e$ and denote

$G := G(e), \quad F := F(e), \quad Q := Q(e), \quad A_0 := A_0(e).$

Restricting matrix entries to the subfields $\mathbb{F}_{2^i}$ of $\mathbb{F}_q$ yields a subgroup $G(f)$ of $G$ for each factor $f$ of $e$. If $f$ and $f'$ are factors of $e$ and $f$ divides $f'$ then

$$G(f) \leq G(f'), \quad F(f) \leq F(f'), \quad Q(f) \leq Q(f'), \quad A_0(f) \leq A_0(f').$$

There are cyclic subgroups of $G$ of mutually coprime odd orders

$$2^e \pm r + 1 = 2^e \pm 2^{(e+1)/2} + 1,$$

contained in Singer subgroups of $GL_4(q)$ (note that $(2^e + r + 1)(2^e - r + 1) = q^2 + 1$ divides $q^4 - 1$); let us choose a pair of subgroups $A_1, A_2$ of $G$ of these two orders, numbered according to the rule

$$|A_1| = a_1(e) := 2^e + \chi(e)2^{(e+1)/2} + 1,$$
$$|A_2| = a_2(e) := 2^e - \chi(e)2^{(e+1)/2} + 1,$$

where $\chi(e) = 1$ or $-1$ as $e \equiv \pm 1$ or $\pm 3 \mod (8)$. If $f$ divides $e$ then $a_i(f)$ divides $a_i(e)$ for each $i = 1, 2$, so we define $A_i(f)$ ($i = 1, 2$) to be the unique subgroup of $A_i$ of order $a_i(f)$. Note that $a_1(e)$ is divisible by $a_1(1) = 5$, whereas $a_2(f)$ is not.
Our rule for distinguishing $A_1$ and $A_2$ may seem artificial, and it differs from that used in [14, 23], where the rule is that $|A_1| > |A_2|$ for all $e$, but it has the advantage, exploited in [6], that if $f$ divides $f'$ then $A_i(f) \leq A_i(f')$ for $i = 1, 2$. However, $A_i(f)$ is not necessarily a subgroup of $G(f)$.

5.3 Properties of some subgroups of $G$.

1. $G$ has order $q^2(q^2 + 1)(q - 1)$, and is simple if $e > 1$. (The group $G(1)$ is isomorphic to $AGL_1(5)$, of order 20.)

2. $\text{Aut} G$ is a semidirect product of $\text{Inn} G \cong G$ by a cyclic group of order $e$ acting as $\text{Gal} \mathbb{F}_q$ on matrix entries, so $|\text{Aut} G| = e|G|$. 

3. $G$ acts doubly transitively on an ovoid $\Omega$ of order $q^2 + 1$ in $\mathbb{P}^3(\mathbb{F}_q)$. Its subgroup $F$, the stabiliser of a point $\omega \in \Omega$, acts as a Frobenius group on $\Omega \setminus \{\omega\}$ with kernel $Q$ and complement $A_0$; hence two subgroups of $G$ conjugate to $Q$ intersect trivially, and two subgroups conjugate to $F$ have their intersection conjugate to $A_0$.

4. $Q$ is a Sylow 2-subgroup of $G$ of order $q^2$ and of exponent 4. The centre $Z$ of $Q$ consists of the identity and the involutions of $Q$, with $Z \cong Q/Z \cong V_q$.

5. $ZA_0 \cong F/Z \cong AGL_1(q)$.

6. The involutions of $G$ are all conjugate, as are the cyclic subgroups of order 4; however an element of order 4 is not conjugate to its inverse.

7. All elements of $G$ except those in a conjugate of $Q$ have odd order. Each maximal cyclic subgroup of $G$ of odd order is conjugate to $A_0$, $A_1$ or $A_2$; the intersection of any two of them is trivial.

5.4 The normalisers of some subgroups of $G$

For $i = 1, 2$ the normaliser $B_i$ of $A_i$ is a Frobenius group, with kernel $A_i$ and a complement of order 4 generated by an element $c_i$ satisfying $c_i^{-1}ac_i = a^{2e}$ for all $a \in A_i$. For each $f$ dividing $e$ let $B_i(f) := \langle A_i(f), c_i \rangle$, so $|B_i(f)| = 4a_i(f)$. Now let $f > 1$ if $i = 2$. Then $B_i(f)$ is self-normalising, whereas the normaliser of $A_i(f)$ is $B_i$. Similarly, the normaliser $B_0$ of $A_0$ is dihedral. Let $c$ be any involution in $B_0$ and for each $f$ dividing $e$, define $B_0(f) := \langle A_0(f), c \rangle$. If $f > 1$, then $B_0(f)$ is self-normalising whereas the normaliser of $A_0(f)$ is $B_0$.

5.5 Classification of subgroups

Any subgroup $H \leq G$ is a subgroup of some conjugate of $F$ or $B_i$ ($i = 0, 1, 2$) or is conjugate to $G(f)$ for some $f$ dividing $e$. 

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For each $f$ dividing $e$, we have defined the following subgroups of $G$, with the symbols $(f)$ usually omitted when $f = e$:

$$G(f), F(f), Q(f), Z(f), B_0(f), A_0(f), B_1(f), B_2(f), A_1(f), A_2(f).$$

The conjugacy class in $G$ of any of these groups will be denoted by changing the appropriate Roman capital to the corresponding script capital; thus $G(f), F, \ldots$ denote the conjugacy classes containing $G(f), F$, and so on.

To show that a set of elements generate $G$, it is sufficient to show that they do not lie in any maximal subgroup of $G$. These are the subgroups in the conjugacy classes $G(f)$ ($e/f$ prime), $F, B_0, B_1$ and $B_2$.

### 6 The Möbius function of a Suzuki group

The first author computed $\mu_G(H)$ for each subgroup $H \leq G$ in [6]. The non-zero values are given in Table 6; all subgroups $H$ not appearing in Table 6 satisfy $\mu_G(H) = 0$, and can therefore be ignored in applying equations such as (3), (5) and (7). In the third column, $a_i(f) = 2^f \pm \chi(f)2^{(f+1)/2} + 1$ for $i = 1, 2$. In the final column, $\mu$ denotes the classical Möbius function on $\mathbb{N}$.

| Conjugacy class of $H$ | Number of conjugates | $|H|$ | $\mu_G(H)$ |
|------------------------|-----------------------|-------|------------|
| $G(f), 1 < f | e$      | $|G|/|H|$            | $2^{2f}(2^{2f} + 1)(2^f - 1)$ | $\mu(e/f)$ |
| $F(f), 1 < f | e$      | $|G|/|H|$            | $2^{2f}(2^f - 1)$               | $-\mu(e/f)$ |
| $B_0(f), 1 < f | e$      | $|G|/|H|$            | $2(2^f - 1)$                    | $-\mu(e/f)$ |
| $A_0(f), 1 < f | e$      | $|G|/2(q-1)$     | $2^f - 1$                       | $2^{(2^f-1)/2}\mu(e/f)$ |
| $B_1(f), 1 < f | e$      | $|G|/|H|$            | $4a_1(f)$                       | $-\mu(e/f)$ |
| $B_2(f), 1 < f | e$      | $|G|/|H|$            | $4a_2(f)$                       | $-\mu(e/f)$ |
| $B_2(1)$                  | $|G|/2q$           | $4$                              | $-2^e\mu(e)$ |
| $B_0(1)$                  | $|G|/q^2$          | $2$                              | $-2^{2e-1}\mu(e)$ |
| $I$                      | $|G|/\mu(e)$       | $1$                              | $|G|/\mu(e)$ |

Table 1: Non-zero values of $\mu_G(H)$ for subgroups $H \leq G$.

For later use we record here, in Table 2, the number $|H|_k$ of elements of order $k$ in each of the subgroups $H$ in Table 1 for $k = 2, 4$ and 5.
Conjugacy class of $H$ & $|H|_2$ & $|H|_4$ & $|H|_5$
\hline
$G(f)$ & $(2^f - 1)(2^{2f} + 1)$ & $2^f(2^{2f} + 1)(2^f - 1)$ & $2^f(2^f - 1)a_2(f)$ \\
$F(f)$ & $2^f - 1$ & $2^f(2^f - 1)$ & 0 \\
$B_0(f)$ & $2^f - 1$ & 0 & 0 \\
$A_0(f)$ & 0 & 0 & 0 \\
$B_1(f)$ & $a_1(f)$ & $2a_1(f)$ & 4 \\
$B_2(f)$ & $a_2(f)$ & $2a_2(f)$ & 0 \\
$B_2(1)$ & 1 & 2 & 0 \\
$B_0(1)$ & 1 & 0 & 0 \\
$I$ & 0 & 0 & 0 \\
\hline

Table 2: Values of $|H|_k$ for $k = 2, 4$ and 5.

7 Enumerations

We can now use the values of the Möbius function $\mu_G$ given in Table 1 to enumerate regular objects with automorphism group $G = Sz(q)$ in various categories $\mathcal{C}$. Formulae (17) and (20), for enumerating maps, have been found in equivalent form by Hubard and Leemans [13], using more direct methods than the general techniques developed here.

7.1 Orientably regular hypermaps

If $\mathcal{C}$ is the category $\mathcal{H}^+$ of oriented hypermaps, we take $\Gamma$ to be the free group $F_2$ of rank 2. Then $|\text{Hom}(\Gamma, H)| = |H|^2$ for each subgroup $H \leq G$, so

$$r_{\mathcal{H}^+}(G) = n_{F_2}(G) = \frac{1}{|\text{Aut } G|} \sum_{H \leq G} \mu_G(H)|H|^2.$$ 

Now $|\text{Aut } G| = e|G|$, so using the information in Table 1 about the subgroups $H$ of $G$, their orders, numbers of conjugates, and values of $\mu_G(H)$, we obtain, after some routine algebra,

$$r_{\mathcal{H}^+}(G) = n_{F_2}(G) = \frac{1}{e} \sum_{\mathcal{F}|\mathcal{E}} \mu\left(\frac{\mathcal{E}}{\mathcal{F}}\right) 2^f(2^{4f} - 2^{3f} - 9) \sim q^5/e. \quad (15)$$

(Here we have used the fact that $\Sigma_{f|\mathcal{E}} \mu(\mathcal{E}/f) = 0$ for $e > 1$ to eliminate a constant term in the summation.) Formula (15) gives the number of orientably regular hypermaps $\mathcal{O}$ with orientation-preserving automorphism group $\text{Aut}_{\mathcal{H}^+} \mathcal{O} \cong G = Sz(q)$, where $q = 2^e$ for some odd $e > 1$. It also gives the number of regular dessin d’enfants with automorphism group $G$, the number of normal subgroups of the free group $F_2$ with quotient group $G$, and the number of orbits of $\text{Aut } G$ on ordered pairs of generators of $G$. The dominant term in the summation on the right-hand side is the leading term $2^{5f}$ where $f = e$, so simple estimates show that $r_{\mathcal{H}^+}(G) \sim q^5/e \sim |G|/e$ as $e \to \infty$. (More generally, results of Dixon [3], Kantor
and Lubotzky [19], and Liebeck and Shalev [20] on probabilistic generation imply that for all non-abelian finite simple groups, \( r_{\mathcal{H}}(G) \sim |G|/|\text{Out } G| \) as \(|G| \to \infty\).

### 7.2 Regular hypermaps

If \( \mathcal{C} \) is the category \( \mathcal{H} \) of all hypermaps, then \( \Gamma \) is the free product \( C_2 \ast C_2 \ast C_2 \). Since \( G \) cannot be generated by fewer than three involutions, we can restrict attention to smooth homomorphisms and epimorphisms, those that map the three free factors of \( \Gamma \) faithfully into \( G \). For each \( H \) the number of such homomorphisms \( \Gamma \to H \) is \( |H|^3 \), where \( |H|^2 \) is the number of involutions in \( H \). The values of \( |H|^2 \) for the nine conjugacy classes of subgroups \( H \) in Table 1 are given in Table 2, so after some algebra we obtain

\[
7.2 \quad r_{\mathcal{H}}(G) = \frac{1}{e} \sum_{f|e} \mu \left( \frac{e}{f} \right) 2^f (2^{3f} - 2^{2f+1} + 2^{2f+1} - 5) \sim q^4/e. \tag{16}
\]

This is the number of regular hypermaps with automorphism group \( G \), and also, by Proposition 4.1, the number of reflexible hypermaps in \( \mathcal{R}_{\mathcal{H}}^+(G) \). Subtracting the formula in equation (16) from that in (15) therefore gives the number of chiral hypermaps in \( \mathcal{R}_{\mathcal{H}}^+(G) \); note that these predominate.

### 7.3 Orientably regular maps

For the category \( \mathcal{M}^+ \) of oriented maps we take \( \Gamma = C_\infty \ast C_2 \). As in the case of hypermaps we may restrict the summation to smooth homomorphisms. There are \(|H||H|^2\) such homomorphisms \( \Gamma \to H \), so we obtain

\[
7.3 \quad r_{\mathcal{M}}^+(G) = \frac{1}{e} \sum_{f|e} \mu \left( \frac{e}{f} \right) 2^f (2^{3f-2^f} - 3) \sim q^3/e. \tag{17}
\]

(This is equivalent to the formula obtained by Hubard and Leemans in [13, Theorem 15].) The \( k \)-valent maps in \( \mathcal{R}_{\mathcal{M}}^+(G) \) correspond to the torsion-free normal subgroups in \( \mathcal{N}_\Gamma(G) \), where \( \Gamma \) is the Hecke group \( C_k \ast C_2 \). To count these we consider smooth homomorphisms \( \Gamma = C_k \ast C_2 \to H \). There are \(|H||H|^2\) of these, so with \( k = 4 \) and \( k = 5 \) for example, Table 2 gives

\[
7.3 \quad r_{\mathcal{M}}^+_4(G) = \frac{1}{e} \sum_{f|e} \mu \left( \frac{e}{f} \right) 2^f (2^f - 2) \sim q^2/e, \tag{18}
\]

and

\[
7.3 \quad r_{\mathcal{M}}^+_5(G) = \frac{1}{e} \sum_{f|e} \mu \left( \frac{e}{f} \right) (2^f - 1)a_2(f) \sim q^2/e, \tag{19}
\]

where \( a_2(f) = 2^f - \chi(f)2^{(f+1)/2} + 1 \).

A map \( \mathcal{M} \in \mathcal{R}_{\mathcal{M}}^+(G) \) of type \( \{n,n\} \), corresponding to a generating triple \( (x,y,z) \) for \( G \) of type \( (n,2,n) \), is self-dual if and only if \( G \) has an automorphism transposing \( x \) and
7.4 Regular maps

For the category $\mathcal{M}$ of all maps we take $\Gamma = V_4 \ast C_2$. In this case we may restrict attention to homomorphisms which embed the direct factors as subgroups $V$ and $C$, such that the generator of $C$ commutes with only the identity element of $V$. The only subgroups $H \leq G$ containing such subgroups $V$ and $C$ are those conjugate to some $G(f)$, with $V$ and $C$ in the centres of distinct Sylow 2-subgroups of $H$. Since $G(f)$ has $2^f + 1$ Sylow 2-subgroups, and their centres are elementary abelian group of order 2, one easily obtains

$$r_{2\mathfrak{m}^+}(G) = \frac{1}{e} \sum_{f|e} \mu \left( \frac{e}{f} \right) (2^f - 1)(2^f - 2) = \frac{1}{e} \sum_{f|e} \mu \left( \frac{e}{f} \right) 2^f(2^f - 3) \sim q^2/e. \quad (20)$$

As in the case of hypermaps, Proposition 4.1 implies that this is also the number of reflexible maps in $\mathcal{R}_{2\mathfrak{m}^+}(G)$, all of them inner reflexible. Subtracting (20) from (17) gives the number of chiral maps (see also [13 Theorem 16]), and as before these predominate.

The formulae (20) are the same for the group $G = L_2(q)$, where $q = 2^e$. At first this may seem surprising, since $Sz(q)$ is much larger than $L_2(q)$. However, the distribution of involutions in these two groups is very similar, and the above proof can be applied, with only minor changes, to $L_2(q)$.

This proof gives a natural interpretation for the first formula in (20). In $G(f)$, one can assume by a unique inner automorphism that the generator $R_1$ of $\Gamma$ is sent to $r_1 = \tau$, and that $R_0$ and $R_2$ are sent to a pair of distinct elements $r_0, r_2$ of the form $(0, \beta)$, in the notation of §5.1. Then $(2^f - 1)(2^f - 2)$ is the number of choices for such an ordered pair, the M"obius inversion over $f$ picks out those triples $(r_i)$ which generate $G$, and division by $e$ counts the orbits of $\text{Out } G$, acting on these triples as $\text{Gal } \mathbb{F}_q \cong C_e$ acts on coefficients $\beta$.

This parametrisation of maps also allows one to determine their types $\{m, n\}$, since $m$ and $n$ are the orders of $r_i r_1$ for $i = 0$ and 2. A matrix $(0, \beta)\tau$ has characteristic polynomial

$$p(\lambda) = \lambda^4 + \beta^3 \lambda^3 + \beta^2 \lambda^2 + \beta^2 \lambda + 1, \quad (21)$$

so its order, as an element $r_i r_1$ of $G$, is the least common multiple of the multiplicative orders of the roots of $p$. Clearly $\lambda = 1$ is not a root, so $m$ and $n$ cannot be equal to 2 or 4, since elements of $G$ of these orders are unipotent, with all eigenvalues equal to 1. Thus $m$ and $n$ are both odd. For example, if we take $\beta = 1$ then the roots of $p$ are the primitive 5th roots of 1, so $r_i r_1$ has order 5. Specific examples are considered in [18].
None of these regular maps is self-dual. If one were, $G$ would have an automorphism fixing $r_1$ and transposing $r_0$ and $r_2$. This would be induced by an element of $G$ centralising the involutions $r_0 r_2$ and $r_1$; however, these lie in distinct Sylow 2-subgroups of $G$, so their centralisers have trivial intersection.

### 7.5 Surface coverings

In order to apply Theorem 3.2 to count regular surface coverings with covering group $G$, one needs to know the degrees of the irreducible complex characters of the subgroups $H$ in Table 1. The irreducible characters of the Suzuki groups $G(f)$ are described in [23] and [14, §XI.5], and the degrees for the other subgroups $H$ are easily found; they are given in Table 3 where $s := 2^f$, $t := \sqrt{2s}$, $k_i := (a_i(f) - 1)/4$ for $i = 1, 2$, and the notation $d^{(k)}$ denotes $k$ characters of degree $d$.

| Conjugacy class of $H$ | Conditions on $f$ | Degrees of irreducible characters of $H$ |
|------------------------|-------------------|-----------------------------------------|
| $\mathcal{G}(f)$       | $1 < f | e$       | $1, s^2, (s - 1)t/2^{(2k_2)}, (s^2 + 1)(s - 2)/2^{(2k_1)}$, $(s - 1)a_1(f)^{k_2}, (s - 1)a_2(f)^{k_1}$ |
| $\mathcal{F}(f)$       | $1 < f | e$       | $1^{(s-1)}, s - 1, (s - 1)t/2^{(2k_2)}$ |
| $B_0(f)$               | $1 < f | e$       | $1, 1, 2^{(s-2)/2}$ |
| $A_0(f)$               | $1 < f | e$       | $1^{(s-1)}$ |
| $B_1(f)$               | $1 < f | e$       | $1, 1, 1, 1, 4^{(k_1)}$ |
| $B_2(f)$               | $1 < f | e$       | $1, 1, 1, 1, 4^{(k_2)}$ |
| $B_2(1)$               |                   | $1, 1$ |
| $B_0(1)$               |                   | $1$ |
| $\mathcal{I}$         |                   | $1$ |

Table 3: Degrees of irreducible characters of subgroups $H \leq G$.

With this information, Theorem 3.2 gives $|\text{Hom}(\Gamma, H)|$ for each $H$ in Table 1 where $\Gamma$ is the fundamental group $\Pi_g$ of an orientable surface $S_g$ of genus $g$, and then $n_1(G)$ in equation (7) gives $r_g(G)$, the number of regular coverings of $S_g$ with covering group $G$. The general formulae are very unwieldy, but in §8 we will give a simple example.

### 8 The smallest simple Suzuki group

The smallest of the simple Suzuki groups is the group $G = G(3) = Sz(8)$ of order $29120 = 2^6 \cdot 5 \cdot 7 \cdot 13$. Putting $e = 3$ in the enumerative formulae given above, we find that $G$ is the automorphism group of 1054 regular hypermaps, of which 14 are maps; all of these are non-orientable. Similarly, it is the orientation-preserving automorphism group of 9534 orientably regular hypermaps, of which 142 are maps. By Proposition 4.1, 1054 of these orientably regular hypermaps, and 14 of these orientably regular maps, are reflexible; these
are the orientable double covers of the regular hypermaps and maps associated with $G$, so they are all inner reflexible.

| $m \setminus n$ | 4   | 5   | 7   | 13  | total |
|-----------------|-----|-----|-----|-----|-------|
| 4               | 0   | 4   | 8   | 4   | 16    |
| 5               | 4   | 4   | 13  | 9   | 30    |
| 7               | 8   | 13  | 26  | 15  | 62    |
| 13              | 4   | 9   | 15  | 6   | 34    |
| total           | 16  | 30  | 62  | 34  | 142   |

Table 4: Number of orientably regular maps of type $\{m, n\}$ in $\mathcal{R}_{3\mathfrak{M}+}(S\mathfrak{z}(8))$.

| $m \setminus n$ | 4   | 5   | 7   | 13  | total |
|-----------------|-----|-----|-----|-----|-------|
| 4               | 0   | 0   | 0   | 0   | 0     |
| 5               | 0   | 0   | 1   | 1   | 2     |
| 7               | 0   | 1   | 2   | 3   | 6     |
| 13              | 0   | 1   | 3   | 2   | 6     |
| total           | 0   | 2   | 6   | 6   | 14    |

Table 5: Number of regular maps of type $\{m, n\}$ in $\mathcal{R}_{3\mathfrak{M}}(S\mathfrak{z}(8))$.

Theorem 3.1 and the character table of $G$ in [1, 23] can be used to find how many of these orientably regular maps and hypermaps have a given type. For instance, they show that $G$ contains $2^6 \cdot 3.7.13.331$ triples $(x, y, z)$ of type $(5, 5, 5)$ satisfying $xyz = 1$; of these, $2^6 \cdot 3.7.13$ generate the $2^4 \cdot 7.13$ Sylow 5-subgroups, while the remaining $2^6 \cdot 3.7.13.330 = 66|\text{Aut } G|$ generate $G$, so there are 66 hypermaps of type $(5, 5, 5)$ in $\mathcal{R}_{5\mathfrak{M}+}(G)$. Similarly, as shown in [17], there are four maps of type $(4, 5)$ in $\mathcal{R}_{3\mathfrak{M}+}(G)$, forming two chiral pairs. In fact, repeated use of Theorem 3.1 shows that the distribution of types of the orientably regular maps in $\mathcal{R}_{3\mathfrak{M}+}(G)$ is as in Table 4, which is symmetric in $m$ and $n$ by the duality of maps. The number of self-dual maps of type $\{n, n\}$ is equal to the number of maps of type $\{n, 4\}$ in this table.

One can use the argument at the end of [17, 23] to determine the types $\{m, n\}$ of the 14 regular maps in $\mathcal{R}_{3\mathfrak{M}}(G)$. Taking $\beta = 1$ gives an element $r_i r_1$ of order 5. Of the six remaining elements $\beta \in \mathbb{F}_8^*$, three have minimal polynomial $t^3 + t + 1$ over $\mathbb{F}_2$, and three have $t^3 + t^2 + 1$. In the first case $p$ splits into four linear factors, with roots $\beta + 1$, $\beta^2$, $\beta^2 + \beta$ and $\beta^2 + \beta + 1$ all of order 7, so that $r_i r_1$ has order 7. In the second case, $p$ is irreducible over $\mathbb{F}_8$, and its roots in its splitting field $\mathbb{F}_{2^{12}}$ have order 13, so $r_i r_1$ has order 13. By considering the action of $\text{Gal } \mathbb{F}_8 \cong C_3$ on distinct ordered pairs of elements $\beta \in \mathbb{F}_8^*$, we find that the number of regular maps of each type $\{m, n\}$ is as in Table 4; there are none with $m = 4$ or $n = 4$ since elements of order 4 are not conjugate to their inverses. This table also gives the types of the 14 reflexible maps in $\mathcal{R}_{3\mathfrak{M}+}(G)$, so subtracting its entries from the corresponding entries in Table 4 gives the number of chiral maps of each type in $\mathcal{R}_{3\mathfrak{M}+}(G)$. 

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One can also enumerate regular surface coverings with covering group $G$. If we put $f = e = 3$ in Table 3 so that $s = 8$ and $t = 4$, we find from equation (7) and Theorem 3.2 that the number $r_g(G)$ of regular coverings of an orientable surface of genus $g$ with covering group $G$ is

$$
\frac{1}{3}\left\{29120n(1 + 2.14^{-n} + 3.35^{-n} + 64^{-n} + 3.65^{-n} + 91^{-n}) - 448^n(7 + 7^{-n} + 2.14^{-n}) - 14^n(2 + 3.2^{-n}) + 7^{n+1} - 52^n(4 + 3.4^{-n}) - 20^n(4 + 4^{-n}) + 8.4^n + 2.2^n - 1\right\}
$$

where $n = 2g - 2$ is the negative of the Euler characteristic of the surface. When $g = 1$ there are no coverings, as one should expect since the fundamental group $\Pi_1$ is abelian, and when $g = 2$ there are 286063776. As $g \to \infty$ we have $r_g(G) \sim |G|^n/3 = 847974400^{g-1}/3$.

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