Low-momentum interaction in few-nucleon systems

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The effective low-momentum interaction \( V_{\text{low},k} \) is applied to three- and four-nucleon systems. We investigate the \(^3\text{H}, ^3\text{He}\) and \(^4\text{He}\) binding energies for a wide range of the momentum cutoffs. By construction, all low-energy two-body observables are cutoff-independent, and therefore, any cutoff dependence is due to missing three-body or higher-body forces. We argue that for reasonable cutoffs \( V_{\text{low},k} \) is similar to high-order interactions derived from chiral effective field theory. This motivates augmenting \( V_{\text{low},k} \) by corresponding three-nucleon forces. The set of low-momentum two- and three-nucleon forces can be used in calculations of nuclear structure and reactions.

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Microscopic nuclear many-body calculations are complicated by the short-distance repulsion in nuclear forces, which leads to strong high-momentum components in nuclear wave functions. Usually, one solves this problem by introducing an effective interaction, the Brueckner \( G \) matrix, which resums in-medium particle-particle scattering. The \( G \) matrix is a soft interaction, which is both energy- and nucleus-dependent and typically requires approximations in practice. Moreover, the resummation of particle-particle contributions makes it extremely complicated to treat particle-particle and particle-hole correlations on an equal footing.

An alternative strategy to construct a soft interaction by integrating out the high-momentum components in free space has been formulated in [1]. Using a renormalization group (RG) approach, phenomenological two-body potential models can be evolved to an effective low-momentum interaction, called \( V_{\text{low},k} \), which is energy-independent, hermitian and preserves the on-shell \( T \) matrix below a cutoff \( \Lambda \) in momentum space as well as the deuteron binding energy. For \( \Lambda \lesssim 2 \text{ fm}^{-1} \), the matrix elements of \( V_{\text{low},k} \) are practically independent of the potential model it is derived from and thus unifies all nuclear forces used in microscopic nuclear structure calculations [2]. By construction, \( V_{\text{low},k} \) is much softer than the modern potential models, and thus can be used directly for microscopic nuclear calculations in different mass regions [3, 4] or for different densities [5, 6]. This is clearly important to theoretically extrapolate to the nuclear drip lines without ambiguities due to unknown interactions.

Over the last few years, there has also been an immense progress in our understanding of nuclear interactions from chiral effective field theory (EFT). This approach qualitatively explains the hierarchy of two-nucleon (2N), three-nucleon (3N) and higher-body forces [7], which is observed using phenomenological models. On a quantitative level, it was shown that the resulting 2N and consistent higher-body interactions lead to a quite good description of 2N as well as 3N observables [8, 9, 10, 11, 12]. In the pionfull EFT approach, the Lippmann-Schwinger equation is regularized by imposing a cutoff \( \Lambda \approx 2.5 - 3.0 \text{ fm}^{-1} \). Thus, the chiral potentials are also low-momentum interactions, and with the universal property of \( V_{\text{low},k} \), this suggests that \( V_{\text{low},k} \) effectively parameterizes higher-order chiral 2N interactions. While EFT offers the only systematic approach to consistent 2N and higher-body forces, \( V_{\text{low},k} \) can be evolved to arbitrary cutoffs with cutoff-independent 2N observables.

Since \( V_{\text{low},k} \) is constructed within the 2N system, one neglects many-body forces due to degrees of freedom missing in the effective theory (contributions from the \( \Delta \)) as well as due the truncation to low momenta (contributions from high-momentum nucleons). In any effective theory, these effects are inseparable. In this Letter, we use cutoff dependence as a tool to assess the effects of many-body forces. Motivated by the similarities between \( V_{\text{low},k} \) and chiral low-momentum interactions, we combine \( V_{\text{low},k} \) with the leading chiral 3N force to absorb the cutoff dependence in \( \Lambda \leq 4 \) binding energies. Finally, we examine the expectation values of the various force components to check that the hierarchy of nuclear two- and three-body forces is maintained.

We first calculate 3N and 4N binding energies by solving the Faddeev-Yakubovsky equations with only the two-body \( V_{\text{low},k} \). We include electromagnetic and isospin-breaking effects and vary the cutoff over a wide range. Our results are numerically stable for the studied cutoff values, which requires a careful treatment of the necessary interpolations in the vicinity of the sharp cutoff. We also checked the convergence with respect to the included partial waves. We estimate an accuracy of 2 keV for the \(^3\text{H}\) and \(^3\text{He}\) and 50 keV for the \(^4\text{He}\) calculations. More details about the numerical method can be found in [13].

In Fig. 1, we give results for binding energies of the 3N system. We show results for the \( V_{\text{low},k} \) derived from the CD-Bonn 2000 [14] and Argonne \( v_{18} \) [15] interactions. The cutoff dependence is due to missing three-body forces. For large cutoffs, we reproduce the known binding energies obtained with the bare interactions only. For intermediate cutoffs, we find a stronger binding with...
$V_{\text{low } k}$. This could be expected, because softer interactions generally lead to stronger binding. It is also consistent with the correlation between the triton binding energy and the deuteron D-state probability observed for phenomenological potentials [16]. For $V_{\text{low } k}$, the D-state probability decreases monotonically with a decreasing cutoff. Therefore, this correlation evidently breaks down for cutoffs below $\Lambda \approx 1.6 \text{ fm}^{-1}$. The binding then decreases, as attractive parts of the bare interactions are integrated out.

For cutoffs $\Lambda \lesssim 2m_\pi$, truly model-independent results are obtained and the binding energy curves for the CD-Bonn 2000 and Argonne $v_{18}$ $V_{\text{low } k}$ interactions collapse. In Fig. 1 we also show the cutoff dependence of the difference in $^3\text{He}$ and triton binding energies, which is due to electromagnetic and isospin-breaking contributions. The difference varies by 60 keV and correlates with the binding energy, since the latter is related to the charge radius [17]. For special choices of the cutoff, both experimental binding energies can be reproduced simultaneously without a 3N interaction. We emphasize that 3N forces will contribute to other observables. Nevertheless, it may be interesting to study many-body systems using these particular values for the momentum cutoff, since a simple zero-range 3N force vanishes in these cases.

Our results indicate that 3N forces due to the truncation to low momenta are of the same order as adjusted 3N forces due to missing excitations of nucleons, although these effects cannot be separated. The bare 3N forces provide about $0.7 - 1 \text{ MeV}$ of binding in conventional models, whereas the binding energies given by $V_{\text{low } k}$ change by 1 MeV over the large cutoff range. In this sense the truncation to low momenta does not induce strong three-body forces. We note that this is in contrast to the interpretation given in [18]. There, the size of 3N forces was assessed by comparing the $V_{\text{low } k}$ binding energies to the results of the bare 2N potential model. This neglects the uncertainty in the binding energy predictions of traditional 2N forces and misses that, in effective theory approaches, the effects of the truncation to small cutoffs are inseparable from those of missing degrees of freedom like the $\Delta$. Because these two contributions to higher-body forces cannot be disentangled at low energies, we will absorb both by augmenting $V_{\text{low } k}$ with a chiral 3N force below.

For further insight, we have calculated the $\alpha$-particle binding energy. To obtain an overview, calculations are performed for the smallest cutoff considered $\Lambda = 1.0 \text{ fm}^{-1}$, in the maximum of the triton binding energy at $\Lambda = 1.6 \text{ fm}^{-1}$, for two cutoffs which lead to 3N binding energies close to the experimental one, $\Lambda = 1.3 \text{ fm}^{-1}$ and $\Lambda = 1.9 \text{ fm}^{-1}$ (Argonne $v_{18}$) or $\Lambda = 2.1 \text{ fm}^{-1}$ (CD-Bonn 2000), and for a cutoff in the tail at $\Lambda = 3.0 \text{ fm}^{-1}$. The focus of our studies is whether the cutoff dependence of $V_{\text{low } k}$ can be related to correlations observed when traditional two-body interactions are used.

From [13, 18, 20], it is well-known that there is an almost linear relation between 3N and 4N binding energies, known as the Tjon-line. This correlation holds with very good accuracy for all modern interactions, but is slightly broken by the action of 3N forces. As can be seen in Fig. 2 the various $V_{\text{low } k}$ results do not differ significantly more from the phenomenological Tjon-line than calculations with adjusted 3N forces. We see that as a further indication that 3N and 4N contributions are not unexpectedly large due to the low-momentum truncation, at least for the triton and $\alpha$-particle. Already at $\Lambda = 3.0 \text{ fm}^{-1}$ the $V_{\text{low } k}$ prediction is almost exactly on the Tjon-line given by the phenomenological models.

As also seen in Fig. 2 even if a cutoff is chosen that leads to a good description of the 3N binding energies, the 4N binding energy deviates from experiment. Clearly, 3N or higher-body forces must act for these values of the cutoff. In the following, we construct a low-momentum 3N interaction by fitting the leading chiral 3N force to $V_{\text{low } k}$. For simplicity we restrict ourselves to the $V_{\text{low } k}$ derived from the Argonne $v_{18}$ potential. The chiral 3N force to leading order contains a long-range 2$\pi$ exchange...
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is described accurately. The resulting dependence for
is determined by strength constants
part, an intermediate range one π exchange (D-term) and
a zero-range contact interaction (E-term), see [11] [12].
For the operator form and the definition of the strength
constants, we refer the reader to Eqs. (2) and (10) in [12].
The interaction is regularized by exponential cutoff func-
tions of the form exp(−(p/Λ)8) with the cutoff taken from V_{low k}.
The very high exponent guarantees a very sharp drop to zero at p = Λ.
The 2π exchange part is determined by strength constants c_i, which we take from [21], where they were obtained by a fit to NN data.
The dimensionless strength constants c_D and c_E were obtained from a fit to the ^3H and ^4He binding ener-
gies. First, a relation between c_D and c_E was established by requiring that the ^3He binding energy of −8.482 MeV is
described accurately. The resulting dependence for various cutoffs is shown in Fig. 3. For small cutoffs we
obtain a linear relationship, which suggests that the D-
and E-terms are perturbative in this region. We have
checked explicitly and also for the c-terms that these are
perturbative for Λ ≲ 2 fm\(^{-1}\). This could be useful for
applications, where it is practically impossible to include
the 3N force into the dynamical equations, but a pertur-
bative treatment is feasible.

In Fig. 4 we show the eigenvalue η of the Yakubovsky equation on c_D for various cutoffs. A deviation of η − 1 = 0.01 corresponds to a deviation of approximately 600 keV from the experimental value.

![Graph](image1.png)

**TABLE I:** Fit results for c_D and c_E for various cutoffs of the V_{low k} derived from the Argonne v_{18} potential (for (+) see text). The strength of the 2π exchange part is determined by c_1 = −0.76 GeV\(^{-1}\), c_3 = −4.78 GeV\(^{-1}\) and c_4 = 3.96 GeV\(^{-1}\) [21].

| Λ [fm\(^{-1}\)] | c_D  | c_E    |
|--------------|------|--------|
| 1.0          | 3.621| 5.724  |
| 1.3          | 11.889| 2.265 |
| 1.6          | 2.080| 0.230  |
| 1.9          | −1.225| −0.405|
| 2.5(a)       | −0.560| −0.707|
| 2.5(b)       | −3.794| −1.085|
| 3.0(+)       | −7.500| −2.151|

**FIG. 2:** (Color online) Correlation of the ^3H and ^4He binding energies. Results are shown for several modern potential models alone (plusses) and with adjusted 3N forces (diamonds) [20]. The V_{low k} results are for the Argonne v_{18} (squares) and the CD-Bonn 2000 potential (crosses). The solid line is a linear fit to the 2N force model results only.

**FIG. 3:** (Color online) Relation between c_D and c_E obtained by requiring that V_{low k} augmented by the 3N force predicts the ^3H binding energy correctly.

**FIG. 4:** (Color online) Dependence of the eigenvalue η of the Yakubovsky equation on c_D for various cutoffs. A deviation of η − 1 = 0.01 corresponds to a deviation of approximately 600 keV from the experimental value.
A very important task is to estimate the size of 3N forces in a systematic way. We decided to calculate the expectation values of the 2N and the different parts of the 3N interactions and compare their magnitude. The results are summarized in Table II. As a worst case scenario, we compare the maximum of the individual 3N force terms to the 2N interaction for $^4$He. As expected from Fig. 3, for $\Lambda \lesssim 2$ fm$^{-1}$, all 3N parts are perturbative. For these cutoffs, we obtain contributions of 4–10%, which is comparable to 3N forces for phenomenological models \[13, 22\]. For larger cutoffs, the $2\pi$ exchange contribution (c-terms) grows rapidly, which is canceled by the $E$-term. We take this as an indication that, in this range, our ansatz for the 3N force is not reliable.

In summary, we have thoroughly assessed the size of 3N forces in the $V_{\text{low}_k}$ approach. Based on the $V_{\text{low}_k}$ results for the $^3$H and $^4$He binding energies, we found that the dependence on the cutoff is not unnaturally large for $\Lambda \geq 1.0$ fm$^{-1}$. This suggests that higher-body interactions are small. We emphasize that the large cutoff range, for which $V_{\text{low}_k}$ is available, will enable similar studies for other low-energy observables, e.g., all binding and excitation energies, and that this is a powerful tool to isolate missing parts in effective interactions. Furthermore, we have extended $V_{\text{low}_k}$ by the leading chiral 3N force and fitted the two unknown parameters to the $^3$H and $^4$He binding energies. We assessed the strength of the 3N force by calculating expectation values of its individual parts. By requiring that not only the sum, but also the individual parts are of natural size, we found that our ansatz for the 3N force is reliable for cutoffs $\Lambda \lesssim 2$ fm$^{-1}$. It turned out that the 3N force contribution can be treated perturbatively for this range of cutoffs. This completes a soft nuclear interaction model, which will be important for many-body calculations. Applications to symmetric nuclear matter are in preparation.

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| $\Lambda$ [fm$^{-1}$] | $T$ | $V_{\text{low}_k}$ | $E$-term | $D$-term | $E$-term | $4^4\text{He}$ | $V_{\text{low}_k}$ | $E$-term | $D$-term | $E$-term |
|----------------------|-----|-------------------|----------|----------|----------|----------------|-------------------|----------|----------|----------|
| 1.0                  | 21.06 | -28.62           | 0.02     | 0.11     | -1.06    | 38.11         | -62.18           | 0.10     | 0.54     | -4.87    |
| 1.3                  | 25.71 | -34.14           | 0.01     | 1.39     | -1.46    | 50.14         | -78.86           | 0.19     | 8.08     | -7.83    |
| 1.6                  | 28.45 | -37.04           | -0.11   | 0.55     | -0.32    | 57.01         | -86.82           | -0.14   | 3.61     | -1.94    |
| 1.9                  | 30.25 | -38.66           | -0.48   | -0.50    | 0.90     | 60.84         | -89.50           | -1.83   | -3.48    | 5.68     |
| 2.5(a)               | 33.30 | -40.94           | -2.22   | -0.11   | 1.49     | 67.56         | -90.97           | -11.06  | -0.41    | 6.62     |
| 2.5(b)               | 33.51 | -41.29           | -2.26   | -1.42   | 2.97     | 68.03         | -92.86           | -11.22  | -8.67    | 16.45    |
| 3.0(*)               | 36.98 | -43.91           | -4.49   | -0.73   | 3.67     | 78.77         | -99.03           | -22.82  | -2.63    | 16.95    |

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