Energy radiation of moving cracks

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(March 22, 2022)

The energy radiated by moving cracks in a discrete background is analyzed. The energy flow through a given surface is expressed in terms of a generalized Poynting vector. The velocity of the crack is determined by the radiation by the crack tip. The radiation becomes more isotropic as the crack velocity approaches the instability threshold.

I. INTRODUCTION

The dynamics of cracks in brittle materials are being extensively studied \cite{12,13,14}, and a wealth of instabilities and patterns have been observed as a function of control parameters such as the applied strain \cite{3–7}, or thermal gradients \cite{8}. The theoretical analysis of moving cracks was initiated long ago \cite{9–11}, with the study of exact solutions for cracks moving at constant velocity. These studies have been extended to a variety of different situations \cite{12,13}. Alternatively, analytical approximations to the leading instabilities of a moving tip have been proposed \cite{14}.

The simplest discrete model which captures the main features of cracks in brittle materials is a a lattice with central forces (springs) between nearest neighbors, whose bonds lose the restoring force above a given threshold \cite{15} (for extensions see also \cite{16}). This model, or simplifications of it which leave out the vectorial nature of the strain field, has been extensively used in modelling moving cracks \cite{2,13,17,18}, although models which deal with the microscopic structure of the system are also being considered \cite{20,23}. Alternatively, various continuum models, which describe the fractured zone in terms of additional fields, have been proposed \cite{21,22}.

Discrete and continuum models of cracks differ in a variety of features. It is known that the discrete models used so far cannot describe a fracture zone at scales other than the size of the lattice cell in the calculations \cite{18,24}, although, even for a canonical material such as PMMA, the fracture zone has a dimension much larger than the size of its molecular building blocks \cite{25}.

Another important difference between discrete and continuum models is the existence of radiation from the tip of the moving crack, due to the existence of periodic modulations in the velocity in the presence of an underlying lattice. In this sense, a lattice model for cracks is the simplest example where radiation due to the scattering of elastic waves by deviations from perfect homogeneity can be studied. These processes have been observed in experiments \cite{28–30}, and it has been argued that they are responsible for some of the crack instabilities \cite{31}.

In the present work, we study the energy radiated by a crack moving at constant velocity in a discrete lattice. We use a generalization of the scheme discussed in \cite{1}. The general method used is explained next. Section III presents the main features of the results. The physical implications of the results is discussed in section IV.

The problem of sound emission by moving cracks has been addressed, within a different scheme, by \cite{31}. Insofar as the two approaches can be compared, the results are compatible. Finally, radiation of moving cracks along the edge of the crack can be important in understanding the roughness of the crack surface \cite{12}. We will focus on the radiation along the crack surface, and into the bulk of the sample. Experiments \cite{30} and simulations \cite{3} suggest that this type of radiation can play a role in the observed instabilities of the crack tip.

II. THE METHOD

We study discrete models of elastic lattices in two dimensional stripes, as discussed in \cite{3,18}. The underlying lattice is hexagonal, with nearest neighbor forces. Bonds break when their elongation exceed a given threshold, $u_{th}$, and under a constant strain at the edges, which, scaled to the width of the stripe, we write as $u_0$. We study models with and without dissipation in the dynamics of the nodes. Results depend on the ratio $u_{th}/u_0$ (see section III for further details).

A. Energy considerations

In the absence of dissipation, the total kinetic plus elastic energy must be conserved. In a continuum model, in the absence of radiation, energy conservation leads to a global and to a local constraint, for cracks moving at constant speed, $v$:

i) In the absence of radiation, the region well behind the crack tip has relaxed to equilibrium, while the region ahead of it is under the applied strain. The relaxed region grows at the expense of the region under strain,
at constant rate $\propto u_{th}^2 v W$ where $W$ is the width of the stripe. Energy is transferred to the crack at this rate. As the energy stored in the crack grows at rate $\propto u_{th}^2 v$, the crack can only propagate (without radiation) for a fixed value of $u_{th}/u_0$. Note that continuous solutions for radiationless cracks moving at constant speed do not specify a parameter equivalent to $u_{th}$, so that they do not conflict with energy conservation.

ii) The only position at which elastic energy is used to increase the size of the crack is the crack tip. Thus, at the crack tip the flux of elastic energy should be equal to the energy invested in enlarging the crack (without dissipation) accelerate until they reach speeds comparable to those predicted by the Yoffe criterion, and then bifurcate.

### B. Energy flux: continuum elasticity

In the following, we will reformulate the concepts discussed in order to make them more amenable for extensions to lattice models, discussed in the next subsection.

We describe an elastic medium in terms of the energy $H$:

$$H = H_{kin} + H_{elastic}$$

$$H_{kin} = \int d^D r \frac{\rho}{2} \left( \frac{\partial \vec{u}(\vec{r})}{\partial t} \right)^2$$

$$H_{elastic} = \int d^D r \frac{\lambda}{2} \left( \sum_i u_{ii} \right)^2 + \mu \int d^D r \sum_{ij} u_{ij}^2$$

where $D$ is the spatial dimension, $\rho$ is the mass density, $\lambda$ and $\mu$ are Lamé coefficients, $\vec{u}(\vec{r})$ denotes the displacements at position $\vec{r}$, and the $u_{ij}$'s define the strain tensor:

$$u_{ij}(\vec{r}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right)$$

The equations of motion satisfied by $\vec{u}(\vec{r})$ can be written as:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = -\sum_j \frac{\partial}{\partial r_j} \sigma_{ij}$$

where $\sigma_{ij} = \partial H_{elastic}/\partial u_{ij}$ is the stress tensor.

The time derivative of the total energy $E_\Omega$ within a region $\Omega$ is:

$$\frac{\partial E_\Omega}{\partial t} = \frac{\partial}{\partial t} \int_\Omega d^D r [H_{kin} + H_{elastic}]$$

$$= \int_\Omega d^D r \left[ \rho \frac{\partial \vec{u}}{\partial t} \frac{\partial^2 \vec{u}}{\partial t^2} + \frac{\partial u_{ij}}{\partial t} \sigma_{ij} \right]$$

$$= -\int_\Omega d^D r \frac{\partial}{\partial t} \left( \sigma_{ij} \frac{\partial u_i}{\partial t} \right)$$

so that the vector $\vec{P}(\vec{r})$ with components $P_j = \sum_i \sigma_{ij} \partial u_i / \partial t$ plays the same role as the Poynting vector in electrodynamics. The energy flux through an element of area $d\vec{S}$ is given by $\vec{P} d\vec{S}$. Note, however, that, unlike in electromagnetism, the equations of elasticity have not Lorentz invariance (there are two sound velocities), and it is not possible to define a four vector combining $\vec{P}$ and the energy density. The energy transferred to the outside of this region remains defined as the flux of the vector $\vec{P}$ through the surface bounding $\Omega$. In the presence of dissipation, we still use $\vec{P}$ as defined in Eq. (1) in the understanding that what viscosity does is to trigger the partial absorption of the radiated energy without
changing the direction in which it is emitted. The vector \( \vec{P} \) will be our starting point in the study of the energy flux of a moving crack.

C. Energy flux: lattice model

We will compute numerically the radiation of energy in a discrete model, defined as a hexagonal two dimensional lattice with nearest neighbor forces \([15,17,18]\). The energy is given by the sum of a kinetic term, associated to the velocities of the nodes, and an elastic term, due to the deformation of the bonds. The variation of the elastic energy of a given bond with time can be written as:

\[
\frac{\partial E_{ij}}{\partial t} = k \left[ (\vec{u}_i - \vec{u}_j) \cdot \vec{n}_{ij} \right] \frac{\partial}{\partial t} \left[ (\vec{u}_i - \vec{u}_j) \cdot \vec{n}_{ij} \right] \tag{7}
\]

where \( k \) is the force constant, and \( \vec{n}_{ij} \) is a unit vector in the direction of the bond. We distribute this energy among the two nodes connected by the bond, so that we can write the total elastic energy within a given region as a sum of the contributions of the nodes within that region. The variation in the kinetic energy at node \( i \) is:

\[
\frac{\partial K_i}{\partial t} = -k \sum_j \frac{\partial (\vec{u}_i \cdot \vec{n}_{ij})}{\partial t} \left[ (\vec{u}_i - \vec{u}_j) \cdot \vec{n}_{ij} \right] \tag{8}
\]

The variation of the total energy within a given region is calculated by summing over all bonds within that region. The kinetic and elastic contributions for all bonds outside the edge of the region cancel. We are left with surface terms only, as in the continuum model described earlier. The surface contributions can be written as a sum of terms associated to the bonds which connect a node within the region under study and a node outside. Thus, a surface which includes a given node and has a given orientation leads to an energy flux across it which can be calculated from a weighted sum of the positions and velocities of the bonds which connect that node to its neighbors. As we can associate to each surface orientation an energy flux, we can define the lattice Poynting vector, in analogy to the analysis done for the continuum model. We will use this discrete Poynting vector in the discussion of the energy dissipation of a moving crack below.

III. RESULTS

The discrete equations of motion in a two dimensional lattice of a given size are integrated numerically as discussed in detail elsewhere \([18]\). The lattice is maintained under constant load at the edges. In order to obtain cracks moving at constant velocities, a notch is induced at one side, which is gradually enlarged, along a straight line, until the stress buildup leads to the spontaneous propagation of the crack. The crack position, as function of time, is shown in Figure 1 for two different applied strains. The calculations show that the crack propagates freely at a constant velocity in the steady state. Our method for the calculation of the properties of cracks moving at constant speeds should lead to the same results as other techniques.

Instabilities are avoided by allowing only the bonds directly ahead of the crack to break. In other words, we force the crack to propagate straightly (with no branching). The simulations are performed in systems of 400 \( \times \) 120 lattice sites, where we have checked that finite size effects on the steady state velocity are less than 1 percent.

A. Crack velocity

Figure 1 shows the steady state velocity \( v \) as a function of the applied strain \( u_0 \), for two different values of the viscosity \( (\eta = 0 \text{ and } \eta = 0.8 \text{ in our units}). The crack velocity increases monotonically with \( u_0 \) and asymptotically tends to its limiting value \( c_R = 0.571 \), the Rayleigh velocity in units where the force constant \( k = 1 \) and the mass per site \( m = 1 \) \([16]\). Due to lattice trapping, there is a minimum allowed \( u_0 \) whose value is roughly independent of \( \eta \) \([3,40]\), which in turn leads to a minimum crack speed which depends strongly on \( \eta \). The arrow marks the instability that would occur if the crack were not constrained to move on a straight line.
FIG. 2. Crack velocity versus external strain, for $\eta = 0$ (upper curve) and $\eta = 0.8$ (lower curve). The arrow on the right indicates the Rayleigh velocity $v_R$. The vertical arrows mark the (avoided) branching instability (see text). The threshold for breaking is $u_{th} = 0.1$.

B. Elastic energy and hoop stress

FIG. 3. Density of elastic energy (upper panel) and hoop stress (lower panel) for an inertial crack ($\eta = 0$) moving at a velocity below the branching threshold (the applied strain is $u_0 = 0.02$, cf. figure 2).

The density of elastic energy (Fig. 3, upper panel) has a sharp peak at the crack tip. In the near region (a few lattice spacings away from the tip), we see that the distribution of elastic energy is very anisotropic: it is sizeable in the direction perpendicular to the crack motion, where it decays smoothly with the distance, and all along the crack, where it has an oscillating behavior. This behavior is reminiscent of the Rayleigh waves which propagate on the crack surface (see section III C below). At larger distances (of the order of the linear dimensions of the system), the elastic energy is smoother and has a broad maximum ahead of the tip, around a given angle of the order of $\theta \approx \pi/3$ from the crack direction. We cannot be conclusive about this maximum being intrinsic in nature, or rather being related to the symmetry of the underlying triangular lattice (see [19] for a more detailed discussion of this point).

The hoop stress (Fig. 3, lower panel) shows a very similar behavior, with strong oscillations all along the crack, and maxima perpendicular to the crack motion, the maximum shifting from $\theta \approx \pi/2$ to $\theta \approx \pi/3$ with increasing distance from the tip.
FIG. 4. Density of elastic energy (upper panel) and hoop stress (lower panel) for an inertial crack ($\eta = 0$) moving under an applied strain $u_0 = 0.08$, well above the branching instability.

Figure 4 is the same as Fig. 3, but for a crack moving at a velocity well above the branching threshold (the applied strain is $u_0 = 0.08$). We notice that the distribution of elastic energy and hoop stress has changed qualitatively: the bulk features in the direction perpendicular to the crack motion now dominate over the oscillating part along the crack. The latter decay more rapidly and eventually disappear far behind the tip.

FIG. 5. Density of elastic energy (upper panel) and hoop stress (lower panel) for a dissipative crack ($\eta = 0.8$) moving under an applied strain $u_0 = 0.08$.

The elastic energy and hoop stress corresponding to a dissipative crack ($\eta = 0.8$, $u_0 = 0.08$) are shown in fig. 5. Although the overall characteristics are similar to the inertial case, with maxima at the tip and in the direction transverse to the crack, the distribution of stresses is much smoother. Moreover, the oscillations associated with Rayleigh waves along the crack are washed out by viscosity, being replaced by a single broad maximum behind the tip.

FIG. 6. Poynting vector field representing the radiation propagating in the vicinity of the crack tip. Upper panel: slow inertial crack (same parameters as fig. 3); center panel: fast inertial crack (same as fig. 4); lower panel: dissipative crack (same as fig. 5).

The above results can be better understood by analyzing the Poynting vector field, which represents the flux of energy being radiated at a given point in the system. As was stated in the introduction, emission of sound waves is expected since the crack tip moves in a discrete medium, therefore acting as a source of radiation at a frequency $\omega = v/a$, the ratio of the crack speed to the lattice spac-
ing. Moreover, one expects a net flux of energy in the direction opposite to the crack motion, corresponding to the elastic energy released from the region ahead of the tip, which allows the crack to move.

As can be seen in the first panel of figure 6, at such moderate crack speeds most of the energy is radiated in the form of Rayleigh waves propagating backwards along the crack, with a wavelength comparable with (but not equal to) the lattice spacing $a$. Despite the fact that $\eta = 0$, such waves are seen to decay at long distances behind the tip (they decay into bulk waves, the oscillating bonds on the crack surface acting themselves as sources of radiation). In addition, there is also a weaker emission of bulk waves from the tip, responsible for the observed maximum in the direction perpendicular to the crack motion.

At high crack speeds, on the other hand (cf. center panel in fig. 6), it is the bulk radiation which dominates the emission pattern. Moreover, shadow images of the near-field appear behind the tip (the strongest one being at around $x = 161$).

In the case of viscous cracks (lower panel in figure 6), the emission pattern is entirely dominated by bulk waves, and Rayleigh oscillations disappear in agreement with the results of fig. 3.

IV. CONCLUSIONS

We have analyzed the nature and influence of radiation in the propagation of cracks in discrete systems. For the lattice and force models that we have studied, we find:

i) Cracks in lattice models radiate energy, even when the average velocity is constant and they move along a straight line. This can be understood by assuming that the crack tip undergoes oscillations at frequencies $n v/a$, where $v$ is the velocity of the crack and $a$ is the lattice constant.

ii) Radiation allows for the existence of a continuum of solutions of moving cracks at constant velocity. The balance of static elastic and crack energy is compensated by the radiation from the crack tip.

iii) At low velocities, most of the radiation is in Rayleigh waves along the surface of the crack. At velocities comparable to the Rayleigh velocity, a significant fraction of the radiated energy is in bulk waves with a more isotropic distribution.

iv) Viscosity allows for a faster exchange of the elastic energy stored ahead of the crack tip into other forms of energy. This can help to explain the increased stability of straight cracks in the presence of viscosity.

Among the questions which remain unsolved is the relation of the radiation to the instabilities of the crack tip. Our results suggest that inertial cracks accelerate along a straight line, until they attain speeds compatible with Yoffe’s criterion [9]. On the other hand, the radiation of the crack tip becomes more isotropic at high velocities. It is unclear whether the continuum approach suffices to understand the instability observed in dynamical simulations of discrete models, or if the radiation from the tip of the crack plays a role in the instability. Note that the calculated instability occurs at higher velocities than the instabilities observed experimentally.

V. ACKNOWLEDGEMENTS

We are thankful to R. Ball, P. Español, M. Marder, T. Martín, A. Parisi, M. A. Rubio and I. Zúñiga for helpful discussions. Financial support from grants PB96-0875 and PB96-0085 (MEC, Spain), and FMRXCT980183 (European Union).

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