The sixth Painlevé transcendent and uniformizable orbifolds

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1. Algebraic solutions of $P_6$ and uniformization theory

The sixth Painlevé transcendent

$$P_6: \quad y_{xx} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) y_x^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y_x$$

is known to be a rich source of nontrivial algebraic solutions $y = f(x)$ and genera of these solutions, as genera of corresponding algebraic curves $F(x, y) = 0$, may be made as great as is wished. The relation of such solutions to the uniformization theory is based on the $\wp$-representation of the $P_6$:

$$-\frac{\pi^2}{4} \frac{d^2 z}{d\tau^2} = \alpha \varphi'(z|\tau) + \beta \varphi'(z-1|\tau) + \gamma \varphi'(z-\tau|\tau) + \delta \varphi'(z-1-\tau|\tau) \quad (1)$$

obtainable via the transcendental change $(x, y) \mapsto (z, \tau)$ (Painlevé (1906), Manin–Babich–Bordag (1996)):

$$x = \frac{\vartheta_3^4(\tau)}{\vartheta_3^3(\tau)}, \quad y = \frac{1}{3} + \frac{1}{3} \frac{\vartheta_4^4(\tau)}{\vartheta_3^3(\tau)} - \frac{4}{\pi^2} \frac{\varphi(z|\tau)}{\vartheta_3^3(\tau)}. \quad (2)$$

Thus, knowledge of $z(\tau)$-dependence leads to a parametric representation for solution $y = f(x)$ and, in particular, to parametric representation of algebraic solutions. In their full generality these dependencies are known for the Picard–Hitchin class of solutions. For example, Picard’s case $\alpha = \beta = \gamma = \delta = 0$ corresponds to $z = A\tau + B$. In Hitchin’s case $\alpha = \beta = \gamma = \delta = \frac{1}{8}$ the dependence $z(\tau)$ is more complicated (obtainable through Okamoto’s transformations) but parametric form of solution is, however, found to be very compact

$$y_{\text{Pic}} = \frac{\vartheta_3^2(\tau)}{\vartheta_3^3(\tau)} \theta_2^2, \quad y_{\text{Hit}} = \frac{\vartheta_3^2(\tau)}{\vartheta_3^3(\tau)} \left\{ \frac{\vartheta_3^2(\tau) \cdot \theta_2 \theta_3 \theta_4}{\theta_1^2 + 2\pi A \theta_1} - \theta_2^2 \right\} \frac{1}{\theta_1^2}, \quad (3)$$

where $\theta$‘s are understood to be equal to $\theta_k(A\tau + B|\tau)$ with arbitrary constants $A$, $B$ and $\theta_1^k := \theta_1^k(A\tau + B|\tau)$. Purely algebraic solutions correspond to $A\tau + B = \frac{\nu}{N} \tau + \frac{\mu}{N}$ with integral $\nu$, $\mu$, and $N$.

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Uniformizing functions are known to be determined in terms of the auxiliary 2nd order linear Fuchsian ODEs \( \Psi_{yy} = \frac{1}{2} Q(x, y) \Psi \), where \( Q \), as a rational function of \( x \) and \( y \), contains all the information about corresponding Riemann surface \( \mathcal{R} \) (or orbifold \( \mathcal{T} \)). Since the function \( x = \chi(\tau) \) in (2) is the very well-known one and its Fuchsian \( \Gamma(2) \)-equation \( \Psi'' = -\frac{1}{4} x^2 - x + 1 \) is also known, we obtain nontrivial (solvable) Fuchsian equations for the second uniformizing function \( y(\tau) \).

Manipulations with Fuchsian equations themselves are not convenient because we constantly handle the multivalued functions-inversions; the ratios like \( \tau = \frac{\Psi_1(x)}{\Psi_2(x)} \). For this reason we invert the standard Schwarz derivative \( \{\tau, x\} \) into the ‘reverse’ object \( [x, \tau] = -\{\tau, x\} \) and work with the autonomous ODEs

\[
[y, \tau] = Q(x, y), \quad \text{where } [y, \tau] := \frac{y'}{y^3} - \frac{3}{2} \frac{y''}{y^2},
\]

defining uniformizing single-valued functions and other single-valued objects.

2. On the general solution to equation (1)

Complete structure of the analytic continuations (a connection problem) of arbitrary solutions to \( P_6 \) is the subject matter of the series works by D. Guzzetti (see, e.g., [1]). Analyzing these results, it would appear reasonable that the ramification structure of all (not necessarily algebraic) solutions to the \( P_6 \)-equation in the vicinity of critical points is described by a function series of the kind

\[
y = A + R[(x - e)^a \ln^n(x - e)] + \cdots,
\]

where \( e = \{0, 1\} \), \( a \in \mathbb{C} \), \( n \in \mathbb{Z} \), and \( R[\ldots] \) is a rational function of its argument. In the language of uniformizing Painlevé substitution (2) this point is self-suggested: in the upper (\( \tau \))-half-plain \( \mathbb{H}^+ \) the \( x \)-function has an exponential behavior in the neighborhood of the points \( x = \{0, 1, \infty\} \):

\[
x \rightarrow 0+ 16 \exp \left( \frac{\pi i}{1} \right) + \cdots, \quad x \rightarrow \infty 1-16 e^{\pi i} + \cdots, \quad x \rightarrow 1 \frac{1}{16} \exp \left( -\frac{\pi i}{1} \right) + \cdots
\]

(the uniformizing \( \tau \)-parameter itself is defined up to a fraction-linear transformation). It follows (the conjecture) that the \( y \)-function has also the single-valued character about each of the branch-point pre-images:

\[
y(\tau) = A + B(\tau - \tau_0)^a \exp \left( -\frac{a\pi i}{\tau - \tau_0} \right) + \cdots, \quad y(\tau) = A + B \tau^n e^{a\pi i} + \cdots.
\]

as \( \tau \rightarrow \tau_0 \in \mathbb{R} \) or, respectively, \( \tau \rightarrow +i\infty \). For example, all asymptotics appearing in [1] fit this behavior. We can therefore rewrite Eqs. \( P_6 \) and (1) in form of modification of purely ‘algebraic’ uniformizing Schwarz–Fuchs 3rd order ODE (4):

\[
[y, \tau] = Ay_x^2 + By_x y_y + Cy_y y_x + Dy_x E,
\]

where \( (A, B, C, D, E) \) are certain rational functions of \( x, y \) and quadratic polynomials in parameters \( (\alpha, \beta, \gamma, \delta) \) (explicit expressions are too cumbersome to display here). Because of outstanding character of \( P_6 \), this equation may be treated as a
generator of ‘infinite genus curves’. In the case of algebraic solutions the right hand side of Eq. (6) becomes a rational function $Q(x, y)$, that is (4). We conjecture that all the Painlevé solutions to Eq. (6) are the globally single-valued analytic functions with the structure (5) and the domain of their existence is a half-plain (under suitable normalization of $\tau$). It is known that solutions to the lower Painlevé equations $P_{1-5}$ (under an appropriate modification [2]) are the single-valued functions on $\mathbb{C}$. In this respect, the pass from $P_6$-equation over $\mathbb{C}\backslash\{0, 1, \infty\}$ to the $H^+$ and uniformization theory related to the coverings of a three punctured $\Gamma(2)$-orbifold becomes very natural.

3. Calculus: Abelian integrals and affine (analytic) connections

Insomuch as we have not only $\tau$-representations for the scalar (i.e. automorphic) functions on $\mathbb{R}$’s but rules for differential computations with theta-functions of arbitrary arguments [3] we can close the differential apparatus on orbifolds $\Sigma$ whose compactifications are corresponding Painlevé $\mathbb{R}$’s. This includes the additively automorphic functions (Abelian integrals), differentials, and covariant differentiation, say, of 1-differentials $\nabla = \partial_\tau - \Gamma(\tau)$. The latter leads to necessity to introduce the geometric connection object $\Gamma(\tau)$, which transforms according to the standard rule $\Gamma(\tau) d\tilde{\tau} = \Gamma(\tau) d\tau - d\ln \frac{d\tilde{\tau}}{d\tau}$ under $\text{SL}_2(\mathbb{R})$-transformations and respects the factor topology of $H^+ / \pi_1(\Sigma)$. The characteristic feature of the (complex) 1-dimensional case (orbifolds and Riemann surfaces) is that it is completely described by the invariant 3rd order ODE (4). Therefore closed collection of data for the theory is given by the set $\{y(\tau), \dot{y}(\tau), \ddot{y}(\tau)\}$ if, however, the automorphism group of the generator $y(\tau)$ coincides with $\pi_1(\Sigma)$. In general, automorphisms of the field generators are not bound to coincide with $\pi_1(\Sigma)$ since the choice of the pair $(x, y)$ is not unique. It is found however that the set of Painlevé orbifolds coming from Picard–Hitchin’s curves (3) is not the case: $\text{Aut} y(\tau) \cong \pi_1(\Sigma)$. In this regard the many Painlevé curves (we suggest that all) stand out majority of classical modular equations originated from purely group-algebraic considerations related to the group $\text{PSL}_2(\mathbb{Z})$ or some its subgroups. By this means the expression

$$\Gamma(\tau) = \frac{d}{d\tau} \ln \dot{y}(\tau) + \text{arbitrary (Abelian) 1-differential}$$

provides a general form of the sought-for connection on Painlevé $\Sigma$. We can normalize this $\Gamma(\tau)$ to have only first order poles (residues) and, integrating the transformation law above, one can see that the sum of such residues is invariant

$$\int_{\partial \mathcal{R}} \tilde{\Gamma}(\tau) d\tilde{\tau} = \int_{\partial \mathcal{R}} \Gamma(\tau) d\tau = (2g - 2) \cdot 2\pi i;$$

it depends only on genus and, in effect, is equal to the number of zeroes of a holomorphic differential $\dot{u}(\tau)$. Varying the holomorphic differentials $\dot{u}_k(\tau)$ we can
impart the simpler from to the connection
\[ \Gamma(\tau) = \frac{d}{d\tau} \ln \dot{u}(\tau) + \sum_{k=1}^{g} \dot{u}_k(\tau) \]
and build the elementary \( \Gamma \) with a single pole (if genus \( g > 1 \) then the analytic connection does always have a singularity). So we have the set of invariant objects \( \{ y(\tau), \dot{y}(\tau), \Gamma(\tau) \} \) since functions \( x(\tau), y(\tau) \) are completely at hand. The remarkable fact is that \textit{affine (analytic) connection on an arbitrary \( \mathcal{X} \) satisfies an autonomous ODE} \( \Xi(\overline{\Gamma}, \overline{\dot{\Gamma}}, \overline{\ddot{\Gamma}}, \overline{\Gamma}) = 0 \) and there is an algorithm how to derive it.

For completeness we should involve into analysis the integrals of closed 1-forms on our \( \mathcal{X} \)’s and \( \mathcal{R} \)’s, if only because there are exact 1-forms whose integrals lead to the scalar objects. On the other hand, uniformization of any higher genera curves is reduced to uniformization of \textit{zero} genus orbifolds and the latter form towers and hierarchies. In the Painlevé uniformizing theory, in one way or another, many classical and nonclassical zero genus known orbifolds appear [3]. In turn they are related to nonzero genus curves which may cover elliptic ones, i.e. tori. We thus obtain a possibility to construct explicitly Abelian integrals if they come from an elliptic cover. Here is a good example along these lines.

The Chudnovsky orbifold defined by the Fuchsian equation \((z^3 - z)\Psi'' + (3z^2 - 1)\Psi' + z\Psi = 0\) is related, through the Halphen transformation (zero genus elliptic cover) \( z = \varphi(u) \), to the Fuchsian equation on the lemniscatic torus \( \varphi'^2 = 4\varphi^3 - 4\varphi \). Correlating these facts we derive the nice \( \tau \)-representation for the everywhere finite object \( u \) and analog of (4)—the uniformizing Schwarz equation:

\[ [u, \tau] = -2\varphi(2u), \quad u(\tau) = \frac{1}{2} \frac{\varphi_3(\tau)}{\varphi_2(\tau)} \, _2F_1\left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}, \frac{\varphi_4(\tau)}{\varphi_2(\tau)} \right) \]

(the check is a good exercise). This is a first \textit{explicit and analytic} \( \tau \)-representation for an additively automorphic function (Abelian integral \( u = \varphi'(z) \)) on an orbifold (Riemann surface) of a \textit{negative} curvature \(-1\). Under suitable cover this \( u(\tau) \) may produce the \( \tau \)-representation for \( u \)-integrals on higher genus curves; examples of the analogous ODEs and their solutions can also be obtained. All of them can be related to the Painlevé curves.

References

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