Cop and robber game and hyperbolicity

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Abstract. In this note, we prove that all cop-win graphs G in the game in which the robber and the cop move at different speeds s and s′ with s′ < s, are δ-hyperbolic with δ = O(s²). We also show that the dependency between δ and s is linear if s − s′ = Ω(s) and G obeys a slightly stronger condition. This solves an open question from the paper J. Chalopin et al., Cop and robber games when the robber can hide and ride, SIAM J. Discr. Math. 25 (2011) 333–359. Since any δ-hyperbolic graph is cop-win for s = 2r and s′ = r + 2δ for any r > 0, this establishes a new –game-theoretical– characterization of Gromov hyperbolicity. We also show that for weakly modular graphs the dependency between δ and s is linear for any s′ < s. Using these results, we describe a simple constant-factor approximation of the hyperbolicity δ of a graph on n vertices in O(n²) time when the graph is given by its distance-matrix.

1. Introduction

The cop and robber game originated in the 1980’s with the work of Nowakowski, Winkler [24], Quilliot [27], and Aigner, Fromme [2], and since then has been intensively investigated by many authors under numerous versions and generalizations. Cop and robber is a pursuit-evasion game played on finite undirected graphs G = (V, E). Player cop C attempts to capture the robber R. At the beginning of the game, C chooses a vertex of G, then R chooses another vertex. Thereafter, the two sides move alternatively, starting with C, where a move is to slide along an edge of G or to stay at the same vertex. The objective of C is to capture R, i.e., to be at some moment in time at the same vertex as the robber. The objective of R is to continue evading the cop. A cop-win graph [2][24][27] is a graph in which C captures R after a finite number of moves from any possible initial positions of C and R.

In this paper, we investigate a natural extension of the cop and robber game in which the cop C and the robber R move at speeds s′ ≥ 1 and s ≥ 1, respectively. This game was introduced and thoroughly investigated in [11]. It generalizes the cop and fast robber game from [19] and can be viewed as the discrete version of some pursuit-evasion games played in continuous domains [20]. The unique difference of this “(s, s′)-cop and robber game” and
the classical cop and robber game is that at each step, $C$ can move along a path of length at most $s'$ and $R$ can move along a path of length at most $s$ not traversing the position occupied by the cop. Following [11], we will denote the class of cop-win graphs for this game by $CWFR(s, s')$.

Analogously to the characterization of classical cop-win graphs given in [24, 27], the $(s, s')$-cop-win graphs have been characterized in [11] via a special dismantling scheme. It was also shown in [11] that any $\delta$-hyperbolic graph in the sense of Gromov [22] belongs to the class $CWFR(2r, r + 2\delta)$ for any $r > 0$ and that, for any $s \geq 2s'$, the graphs in $CWFR(s, s')$ are $(s - 1)$-hyperbolic. Finally, [11] conjectures that all graphs of $CWFR(s, s')$ with $s' < s$, are $\delta$-hyperbolic, where $\delta$ depends only on $s$ and establishes this conjecture for Helly graphs and bridged graphs, two important classes of weakly modular graphs.

In this note, we confirm the conjecture of [11] by showing that if $s' < s$, then any graph of $CWFR(s, s')$ is $\delta$-hyperbolic with $\delta = O(s^2)$. The proof uses the dismantling characterization of $(s, s')$-cop-win graphs and the characterization of $\delta$-hyperbolicity via the linear isoperimetric inequality. We show that the dependency between $\delta$ and $s$ is linear if $s - s' = \Omega(s)$ and $G$ satisfies a slightly stronger dismantling condition. We also show that weakly modular graphs from $CWFR(s, s')$ with $s' < s$ are $184s$-hyperbolic. All this allows us to approximate within a constant factor the least value of $\delta$ for which a finite graph $G = (V, E)$ is $\delta$-hyperbolic in $O(|V|^2)$ time once the distance-matrix of $G$ has been computed.

2. Preliminaries

2.1. Graphs. All graphs $G = (V, E)$ occurring in this paper are undirected, connected, without loops or multiple edges, but not necessarily finite or locally-finite. For a subset $A \subseteq V$, the subgraph of $G = (V, E)$ induced by $A$ is the graph $G(A) = (A, E')$ such that $uv \in E'$ if and only if $u, v \in A$ and $uv \in E$. We will write $G - \{x\}$ instead of $G(V \setminus \{x\})$. The distance $d(u, v) := d_G(u, v)$ between two vertices $u$ and $v$ of $G$ is the length (number of edges) of a $(u, v)$-geodesic, i.e., a shortest $(u, v)$-path. For a vertex $v$ of $G$ and an integer $r \geq 1$, we will denote by $B_r(v, G)$ the ball in $G$ of radius $r$ centered at $v$, i.e., $B_r(v, G) = \{x \in V : d(v, x) \leq r\}$. (We will write $B_r(v)$ instead of $B_r(v, G)$ when this is clear from the context). Let $B_r(x, G - \{y\})$ be the ball of radius $r$ centered at $x$ in the graph $G - \{y\}$.

The interval $I(u, v)$ between $u$ and $v$ consists of all vertices on $(u, v)$-geodesics, that is, of all vertices (metrically) between $u$ and $v$: $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$.

Three vertices $v_1, v_2, v_3$ of a graph $G$ form a metric triangle $v_1v_2v_3$ if the intervals $I(v_1, v_2), I(v_2, v_3)$, and $I(v_3, v_1)$ pairwise intersect only in the common end-vertices. If $d(v_1, v_2) = d(v_2, v_3) = d(v_3, v_1) = k$, then this metric triangle is called equilateral of size $k$. A metric triangle $v_1v_2v_3$ of $G$ is a quasi–median of the triplet $x, y, z$ if the following metric equalities are satisfied:

$$d(x, y) = d(x, v_1) + d(v_1, v_2) + d(v_2, y),$$
$$d(y, z) = d(y, v_2) + d(v_2, v_3) + d(v_3, z),$$
$$d(z, x) = d(z, v_3) + d(v_3, v_1) + d(v_1, x).$$
Every triplet \( x, y, z \) of a graph has at least one quasi-median: first select any vertex \( v_1 \) from \( I(x, y) \cap I(x, z) \) at maximal distance to \( x \), then select a vertex \( v_2 \) from \( I(y, v_1) \cap I(y, z) \) at maximal distance to \( y \), and finally select any vertex \( v_3 \) from \( I(z, v_1) \cap I(z, v_2) \) at maximal distance to \( z \).

2.2. \( \delta \)-Hyperbolicity. A metric space \((X, d)\) is \( \delta \)-hyperbolic [3, 10, 22] if for any four points \( u, v, x, y \) of \( X \), the two larger of the three distance sums \( d(u, v) + d(x, y), d(u, x) + d(v, y), d(u, y) + d(v, x) \) differ by at most \( 2\delta \geq 0 \). A graph \( G = (V, E) \) is \( \delta \)-hyperbolic if \((V, d_G)\) is \( \delta \)-hyperbolic. In case of geodesic metric spaces and graphs, \( \delta \)-hyperbolicity can be defined in several other equivalent ways. Here we recall some of them, which we will use in our proofs.

Let \((X, d)\) be a metric space. A geodesic segment joining two points \( x \) and \( y \) from \( X \) is a map \( \rho \) from the segment \([a, b] \) of \( \mathbb{R}^1 \) of length \( |a - b| = d(x, y) \) to \( X \) such that \( \rho(a) = x, \rho(b) = y \), and \( d(\rho(s), \rho(t)) = |s - t| \) for all \( s, t \in [a, b] \). A metric space \((X, d)\) is geodesic if every pair of points in \( X \) can be joined by a geodesic segment. Every (combinatorial) graph \( G = (V, E) \) equipped with its standard distance \( d := d_G \) can be transformed into a geodesic (network-like) space \((X_G, d)\) by replacing every edge \( e = (u, v) \) by a segment \( \gamma_{uv} = [u, v] \) of length 1; the segments may intersect only at common ends. Then \((V, d_G)\) is isometrically embedded in a natural way in \((X_G, d)\). \( X_G \) is often called a metric graph. The restrictions of geodesics of \( X_G \) to the set of vertices \( V \) of \( G \) are the shortest paths of \( G \). For simplicity of notation and brevity (and if not said otherwise), in all subsequent results, by a geodesic \([x, y]\) in a graph \( G \) we will mean an arbitrary shortest path between two vertices \( x, y \) of \( G \).

Let \((X, d)\) be a geodesic space. A geodesic triangle \( \Delta(x, y, z) \) with \( x, y, z \in X \) is the union \([x, y] \cup [x, z] \cup [y, z]\) of three geodesic segments connecting these vertices. A geodesic triangle \( \Delta(x, y, z) \) is called \( \delta \)-slim if for any point \( u \) on the side \([x, y]\) the distance from \( u \) to \([x, z] \cup [z, y]\) is at most \( \delta \). For graphs, we “discretize” this notion in the following way. We say that the geodesic triangles of a graph \( G \) are \( \delta \)-slim if for any triplet \( x, y, z \) of vertices of \( G \), for any (graph) geodesics \([x, y], [x, z], [y, z]\) and for any vertex \( u \in [x, y] \), there exists \( v \in [x, z] \cup [y, z]\) such that \( d(u, v) \leq \delta \).

Note that if the metric graph \((X_G, d)\) is \( \delta \)-hyperbolic (resp., has \( \delta \)-slim geodesic triangles) as a geodesic metric space, then the combinatorial graph \( G \) is \( \delta \)-hyperbolic (resp., has \( \delta \)-slim geodesic triangles). Conversely, if \( G \) is \( \delta \)-hyperbolic (resp., has \( \delta \)-slim geodesic triangles), then \((X_G, d)\) is \((\delta + 2)\)-hyperbolic (resp., has \((\delta + \frac{1}{2})\)-slim geodesic triangles).

The following result shows that hyperbolicity of a geodesic space is equivalent to having slim geodesic triangles (the same result holds for graphs).

**Proposition 1.** [3, 10, 22] If all geodesic triangles of a geodesic metric space \((X, d)\) are \( \delta \)-slim, then \( X \) is \( 8\delta \)-hyperbolic. Conversely, if a geodesic space \((X, d)\) is \( \delta \)-hyperbolic, then all its geodesic triangles are \( 3\delta \)-slim.

More recently, Soto [29] proved a sharp bound on the hyperbolicity of metric spaces and graphs with \( \delta \)-slim geodesic triangles.
Proposition 2. [29] If all geodesic triangles of a geodesic metric space \((X,d)\) are \(\delta\)-slim, then \(X\) is \(2\delta\)-hyperbolic. If all geodesic triangles of a graph \(G\) are \(\delta\)-slim, then \(G\) is \((2\delta + \frac{3}{2})\)-hyperbolic.

An interval \(I(u,v)\) of a graph \(G\) is called \(\nu\)-thin, if \(d(x,y) \leq \nu\) for any two vertices \(x, y \in I(u,v)\) such that \(d(u,x) = d(u,y)\) and \(d(v,x) = d(v,y)\). From the definition of \(\delta\)-hyperbolicity easily follows that intervals of a \(\delta\)-hyperbolic graph are \(2\delta\)-thin.

We note that a converse of this result holds too. If \(G\) is a graph, denote by \(G'\) the graph obtained by subdividing all edges of \(G\). Papasoglu [25] showed that if \(G'\) has \(\nu\)-thin intervals then \(G\) is \(f(\nu)\)-hyperbolic for some function \(f\). It is not clear what is the best possible \(f\) for which this holds. Chatterji and Niblo in [12] showed that \(f\) can be taken to be a double exponential function. It would be interesting to have examples showing what is the dependence between \(\delta, \nu\), e.g. whether it should be possible to show that \(f\) grows faster than linearly.

However, the following result holds:

Proposition 3. [14] If \(G\) is a graph in which all intervals are \(\nu\)-thin and the metric triangles of \(G\) have sides of length at most \(\mu\), then \(G\) is \((16\nu + 4\mu)\)-hyperbolic.

Now, we recall the definition of hyperbolicity via the linear isoperimetric inequality. Although this (combinatorial) definition of hyperbolicity is given for geodesic metric spaces, it is quite common to approximate the metric space by a graph via a quasi-isometric embedding and to define \(N\)-fillings for the resulting graph (see for example, [10] pp. 414–417)). Since in this paper we deal only with graphs, we directly give the definitions in the setting of graphs.

In a graph \(G = (V,E)\), a loop \(c\) is a sequence of vertices \((v_0,v_1,v_2,\ldots,v_{n-2},v_{n-1},v_0)\) such that for each \(0 \leq i \leq n-1\), either \(v_i = v_{i+1}\), or \(v_i v_{i+1} \in E\); \(n\) is called the length \(\ell(c)\) of \(c\). A simple cycle \(c = (v_0,v_1,v_2,\ldots,v_{n-2},v_{n-1},v_0)\) is a loop such that for all \(0 \leq i < j \leq n-1\), \(v_i \neq v_j\).

A non-expansive map \(\Phi\) from a graph \(G = (V,E)\) to a graph \(G' = (V',E')\) is a function \(\Phi: V \to V'\) such that for all \(v,w \in V\), if \(vw \in E\) then either \(\Phi(v) = \Phi(w)\) or \(\Phi(v)\Phi(w) \in E'\). Note that a map \(\Phi\) from \(G\) to \(G'\) is non-expansive if and only if for all vertices \(v,w\) of \(G\), 
\[
d_{G'}(\Phi(v),\Phi(w)) \leq d_G(v,w).
\]

For an integer \(N > 0\) and a loop \(c = (v_0,v_1,v_2,\ldots,v_{n-2},v_{n-1},v_0)\) in a graph \(G\), an \(N\)-filling \((D,\Phi)\) of \(c\) consists of a 2-connected planar graph \(D\) and a non-expansive map \(\Phi\) from \(D\) to \(G\) such that the following conditions hold (see Figure 1 for an example):

1. the external face of \(D\) is a simple cycle \((v_0',v_1',\ldots,v_{n-1}',v_0')\) such that \(\Phi(v_i') = v_i\) for all \(0 \leq i \leq n-1\),
2. every internal face of \(D\) has at most \(2N\) edges.

The \(N\)-area \(\text{Area}_N(c)\) of \(c\) is the minimum number of faces in an \(N\)-filling of \(c\). A graph \(G\) satisfies a linear isoperimetric inequality if there exists an \(N > 0\) such that any loop \(c\) of \(G\) has an \(N\)-filling and \(\text{Area}_N(c)\) is linear in the length of \(c\) (i.e., there exists a positive integer \(K\) such that \(\text{Area}_N(c) \leq K \cdot \ell(c)\)). The following result of Gromov [22] proven in [3, 7, 10, 30] is the basic ingredient of our proof:
Figure 1. Two different 2-fillings $D_1$ and $D_2$ of the loop $c = (1, 2, 6, 14, 9, 13, 10, 5, 1)$ of $G$. $D_2$ is a 2-filling of $c$ with a minimum number of faces and thus $\text{Area}_2(c) = 4$.

Theorem 1 (Gromov). If a graph $G$ is $\delta$-hyperbolic, then any edge-loop of $G$ admits a $16\delta$-filling of linear area. Conversely, if a graph $G$ satisfies the linear isoperimetric inequality $\text{Area}_N(c) \leq K \cdot \ell(c)$ for some integers $N$ and $K$, then $G$ is $\delta$-hyperbolic, where $\delta \leq 108K^2N^3 + 9KN^2$.

2.3. Graphs of $\text{CWFR}(s,s')$ and $(s,s')^*$-dismantlability. A (non-necessarily finite) graph $G = (V,E)$ is called $(s,s')$-dismantlable if the vertex set of $G$ admits a well-order $\preceq$ such that for each vertex $v$ of $G$ there exists another vertex $u$ with $u \preceq v$ such that $B_s(v,G-\{u\}) \cap X_v \subseteq B_{s'}(u,G)$, where $X_v : = \{w \in V : w \preceq v\}$. In the following, if $B_s(v,G-\{u\}) \cap X_v \subseteq B_{s'}(u,G)$, then we will say that $v$ is eliminated by $u$ or that $u$ eliminates $v$. From the definition immediately follows that if $G$ is $(s,s')$-dismantlable, then $G$ is also $(s,s'')$-dismantlable for any $s'' > s'$ (with the same dismantling order). In case of finite graphs, the following result holds (if $s = s' = 1$, this is the classical characterization of cop-win graphs by Nowakowski, Winkler [24] and Quilliot [27]):

Theorem 2. [11] For any $s, s' \in \mathbb{N} \cup \{\infty\}$, $s' \leq s$, a finite graph $G$ belongs to the class $\text{CWFR}(s,s')$ if and only if $G$ is $(s,s')$-dismantlable.

We will also consider a stronger version of $(s,s')$-dismantlability: a graph $G$ is $(s,s')^*$-dismantlable if the vertex set of $G$ admits a well-order $\preceq$ such that for each vertex $v$ of $G$ there exists another vertex $u$ with $u \preceq v$ such that $B_s(v,G) \cap X_v \subseteq B_{s'}(u,G)$.

In [11], using a result from [15], it was shown that $\delta$-hyperbolic graphs are $(s,s')^*$-dismantlable for some values $s, s'$ depending of $\delta$. For sake of completeness, we recall here these results. The following proposition is a particular case of Lemma 1 from [15].

Proposition 4. [15] Let $G$ be a $\delta$-hyperbolic graph and $r$ be a non-negative integer. Let $x, y, z$ be any three vertices of $G$ such that $d(y,z) \leq d(x,z)$ and $d(x,y) \leq 2r$. Then for any vertex $c \in I(x,z)$ such that $d(x,c) = \min\{r,d(x,z)\}$, the inequality $d(c,y) \leq r + 2\delta$ holds.

Proof. If $d(x,z) \leq r$, then $c = z$, and $d(c,y) \leq d(x,z) \leq r \leq r + 2\delta$. Suppose now that $d(x,z) > r$. Since $G$ is $\delta$-hyperbolic, $d(c,y) + d(x,z) \leq \max\{d(c,z) + d(x,y), d(c,x) + d(y,z)\} +
Proposition 5. If a graph

Corollary 1. For a $\delta$-hyperbolic graph $G$ and any integer $r \geq \delta$, any breadth-first search order $\preceq$ is a $(2r, r + 2\delta)^*$-dismantling order of $G$.

3. Main result

In this section we will prove that $(s, s')^*$-dismantlable graphs with $s' < s$ are hyperbolic. All graphs occurring in the following result are connected but not necessarily finite.

Theorem 3. If a graph $G$ is $(s, s')^*$-dismantlable with $0 < s' < s$, then $G$ is $\delta$-hyperbolic with $\delta = 16(s + s') \left\lceil \frac{s + s'}{s - s'} \right\rceil + \frac{1}{2} \leq 32 \frac{s(s + s')}{s - s'} + \frac{1}{2}$.

Proof. At the first step, we will establish that for any cycle $c$ of $G$, $\text{Area}_{s + s'}(c) \leq \left\lceil \frac{\ell(c)}{2(s-s')} \right\rceil$. In this part, we will follow the proofs of Lemma 2.6 and Proposition 2.7 of [10, Chapter III.H] for hyperbolic graphs. At the second step, we will present a modified and refined proof of Theorem 2.9 of [10, Chapter III.H], which will allow us to deduce that if $\text{Area}_{s + s'}(c) \leq \left\lfloor \frac{\ell(c)}{2(s-s')} \right\rfloor + 1$, then $G$ is $O(\frac{s^2}{s-s'})$-hyperbolic.

Lemma 1. If a graph $G$ is $(s, s')^*$-dismantlable with $s' < s$ and $c = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is a loop of $G$ of length $n > 2(s + s')$, then $c$ contains two vertices $x = v_p, y = v_q$ with $q - p = 2s$ mod $n$ such that $d(x, y) \leq 2s'$.

Proof. Let $\preceq$ be an $(s, s')^*$-dismantling well-order of the vertex-set $V$ of $G$ and let $v$ be the largest element of the vertex-set of $c$ in this order. Let $u$ be a vertex of $G$ that eliminates $v$ in $\preceq$. Without loss of generality suppose that $v = v_i$, where $i > s$ and $i < n - s$. Let $x = v_{i-s}$ and $y = v_{i+s}$. Since $d(v, x) \leq s, d(v, y) \leq s$, and $x, y \in X_v$, from the definition of $(s, s')^*$-dismantlability we conclude that $d(u, x) \leq s'$ and $d(u, y) \leq s'$. By triangle inequality, $d(x, y) \leq 2s'$. Since $x = v_{i-s}$ and $y = v_{i+s}$, we obtain that $(i + s) - (i - s) = 2s$, and we can set $p := i - s$ and $q := i + s$. \hfill \Box

Proposition 5. If a graph $G$ is $(s, s')^*$-dismantlable with $s' < s$ and $c$ is a loop of $G$, then $\text{Area}_{s + s'}(c) \leq \left\lfloor \frac{\ell(c)}{2(s-s')} \right\rfloor$.

Proof. Let $c = (v_0, v_1, \ldots, v_{n-1}, v_0)$ be a loop of $G$. To prove that $\text{Area}_{s + s'}(c) \leq \left\lfloor \frac{\ell(c)}{2(s-s')} \right\rfloor$, it suffices to show that there exists a 2-connected planar graph $D$ and a non-expansive map $\Phi$ from $D$ to $G$ such that

(F1) $D$ has at most $\left\lfloor \frac{\ell(c)}{2(s-s')} \right\rfloor$ faces,

(F2) all internal faces of $D$ have length at most $2(s + s')$,

(F3) the external face of $D$ is a simple cycle $(v'_0, v'_1, \ldots, v'_{n-1}, v'_0)$ such that $\Phi(v'_i) = v_i$ for all $0 \leq i \leq n - 1$. 

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Let \( v \) be a single face bounded by a simple cycle \((v_0', v_1', \ldots, v_{n-1}', v_0')\) of length \( n \) and for each \( i \), let \( \Phi(v_i') = v_i \). This shows that \( \text{Area}_{s+s'}(c) = 1 \).

Now, suppose that \( n > 2(s+s') \). By Lemma 1 there exist two vertices \( x = v_p, y = v_q \) of \( G \) with \( q-p = 2s \mod n \) and \( d(x, y) = 2s' \). Suppose without loss of generality that \( q = p + 2s \).

Let \( P' = (x = v_p, v_{p+1}, \ldots, v_{q-1}, v_q = y) \) and \( P'' = (x = v_p, v_{p-1}, \ldots, v_{q+1}, v_q = y) \) be the two \((x, y)\)-paths constituting \( c \). If \( x = y \), let \( P = (x, y) \); if \( x \neq y \), let \( P = (x = w_0, w_1, \ldots, w_k = y) \) be any shortest path in \( G \) between \( x \) and \( y \). Note that \( \ell(P) < 2s' < 2s = \ell(P') \).

Let \( c_0 \) be the loop obtained as the concatenation of the paths \( P \) from \( x \) to \( y \) and \( P' \) from \( y \) to \( x \). Since \( \ell(P') = 2s \) and \( \ell(P) < 2s' \), we have \( \ell(c_0) \leq 2s + 2s' \). Let \( c_1 \) be the loop obtained as the concatenation of the paths \( P'' \) from \( y \) to \( P \) from \( x \) to \( y \). Note that \( \ell(c_1) = \ell(P) + \ell(P'') \leq \ell(P) + \ell(c) - \ell(P') \leq \ell(c) - (2s - 2s') < \ell(c) \).

By induction assumption, \( c_1 \) admits an \((s+s')\)-filling \((D_1, \Phi_1)\) satisfying the conditions \((F1),(F2), \) and \((F3)\). Note that the external face of \( D_1 \) is bounded by a cycle \((v'_p = x' = w'_{0}, w'_{1}, \ldots, w'_{k} = y' = v'_{q+1}, v'_{q+1}, \ldots, v'_{n-1}, v_{0}, \ldots, v'_{p-1}, v'_{p})\) such that \( \Phi_1(v'_i) = v_i \) for all \( i \in [0, p] \cup [q, n-1] \) and \( \Phi_1(w'_i) = w_i \) for all \( 0 \leq i \leq k \).

Consider the planar graph \( D \) obtained from \( D_1 \) by adding \( q-p-1 \) new vertices forming a path \((x' = v'_p, v'_{p+1}, \ldots, v'_{q} = y')\) from \( x' \) to \( y' \) on the external face of \( D_1 \) such that the external face of \( D \) is bounded by the cycle \((v'_p, v'_1, \ldots, v'_{n-1}, v'_{0})\). Let \( \Phi \) be the non-expansive map defined by \( \Phi(v') = \Phi_1(v') \) for every \( v \in V(D_1) \) and \( \Phi(v'_i) = v_i \) for every \( p+1 \leq i \leq q-1 \).

Clearly, \( D_1 \) is a 2-connected planar graph and for each \( 0 \leq i \leq n-1 \) we have \( \Phi(v'_i) = v_i \). The planar graph \( D \) has one more internal face than \( D_1 \) that is bounded by the cycle \((x' = v'_p, v'_{p+1}, \ldots, v'_{q} = y' = w'_{k}, w'_{k-1}, \ldots, w'_{1}, w'_{0} = x')\). This cycle has the same length as \( c_0 \) and is thus bounded by \( 2(s + s') \). Consequently, \((D, \Phi)\) satisfies the conditions \((F2)\) and \((F3)\).

It remains to show that the \((s+s')\)-filling \((D, \Phi)\) of \( c \) satisfies \((F1)\). Since \( \ell(c_1) \leq \ell(c) - 2(s - s') \), by induction assumption, we obtain

\[
\text{Area}_{s+s'}(c) \leq \text{Area}_{s+s'}(c_1) + 1 \leq \left[ \frac{\ell(c_1)}{2(s-s')} \right] + 1 \leq \left[ \frac{\ell(c) - 2(s-s')}{2(s-s')} \right] + 1 = \left[ \frac{\ell(c)}{2(s-s')} \right],
\]

yielding the desired inequality.

Now, we revisit the proof of Theorem 2.9 of [10] Chapter III.H] that corresponds to Theorem [1] stated above. Namely, we extend this result to the case of rational \( K \) and improve its statement by showing that the hyperbolicity of \( G \) is quadratic (and not cubic) in \( N \).

We start with an auxiliary result. For a subset of vertices \( A \subseteq V \) of a graph \( G = (V, E) \) and an integer \( k \geq 0 \), let \( B_k(A, G) = \{ v \in V : d_G(v, A) \leq k \} \) denote the \( k \)-neighborhood of \( A \) in \( G \).

**Lemma 2.** Let \( G \) be a graph and \( k > 0 \) be an integer. Consider a simple cycle \( c = (v_0, v_1, \ldots, v_{n-1}) \) of \( G \) and two integers \( p, q \) such that \( k < p < p+2k \leq q < n-k \),
\[d_G(v_p, v_q) = q - p, \text{ and } B_k(\{v_{p}, v_{p+1}, \ldots, v_{q}\}, G) \cap c = \{v_{p-k}, v_{p-k+1}, \ldots, v_{q+k-1}, v_{q+k}\} .\]

Consider any \(N\)-filling \((D, \Phi)\) of \(c\) and let \(c' = (v'_0, v'_1, \ldots, v'_{n-1}, v'_0)\) be the cycle bounding the external face of \(D\) (where \(\Phi(v'_i) = v_i\) for all \(0 \leq i \leq n - 1\)). Then, in the subgraph of \(D\) induced by \(B_k(\{v'_p, v'_{p+1}, \ldots, v'_q\}, D)\), there exist at least \(\frac{k(q-p-2k)}{N}\) faces of \(D\) that contain at most one vertex at distance \(k\) from \(v'_p, v'_{p+1}, \ldots, v'_q\).

**Proof.** Let \((D, \Phi)\) be an \(N\)-filling of \(c\). Since \(\Phi\) is a non-expansive map from \(D\) to \(G\), for any vertices \(u', v' \in V(D)\), the distance \(d_D(u', v')\) in \(D\) is greater than or equal to the distance \(d_G(\Phi(u'), \Phi(v'))\) in \(G\) between their images. Note also that \(\ell(c) \geq q - p + 2k + 1\).

Let \(F(D)\) be the set of faces of \(D\). We define recursively a set \(V_i \subseteq V(D)\) of vertices, a set \(E_i \subseteq E(D)\) of edges, and a set \(F_i \subseteq F(D)\) of faces of \(D\). Let \(P_0 = (V_0, E_0)\) be the path \((v'_p, v'_{p+1}, \ldots, v'_q)\). Let \(F_0\) be the set of faces of \(D\) that contain vertices of \(V_0\).

For any \(i \geq 1\), let \(V_i\) (resp., \(E_i\)) be the set of vertices (resp., edges) belonging simultaneously to faces of \(F_{i-1}\) and to faces of \(F(D) \setminus (F_0 \cup \ldots \cup F_{i-1})\). Let \(F_i\) be the set of faces of \(F(D) \setminus (F_0 \cup \ldots \cup F_{i-1})\) containing vertices of \(V_i\). Since \(D\) is a planar graph, for each \(i\), the connected components of the graph \(H_i := (V_i, E_i)\) are (non-necessarily simple) paths and cycles. The vertices of \(c' \cap V_i\) necessarily belong to a single path of \(H_i\) (again, this follows from the planarity of \(D\)), which we will denote by \(P_i\).

Since each face of \(D\) has length at most \(2N\), all vertices appearing in a face of \(F_i\) are at distance at most \(N\) from \(V_i\). Moreover, each face of \(F_i\) contains at most one vertex at distance \(N\) from \(V_i\). Consequently, each face of \(F_i\) contains only vertices at distance at most \((i + 1)N\) from \(V_0 = \{v'_p, v'_{p+1}, \ldots, v'_q\}\), and each face of \(F_i\) contains at most one vertex at distance \((i + 1)N\) from \(V_0\).

Assume now that \(i \leq k/N\), and let \(F = F_0 \cup F_1 \cup \ldots \cup F_{i-1}\). Consider a vertex \(v'_j\) of the external face of \(D\) that belongs to a face of \(F\). Since \(d_G(v_j, \Phi(V_0)) \leq d_D(v'_j, V_0) \leq iN \leq k\), and since \(B_k(\Phi(V_0), G) \cap c = \{v_{p-k}, v_{p-k+1}, \ldots, v_{q+k-1}, v_{q+k}\}\), we have \(p - k \leq j \leq q + k\). Consequently, \(v'_0\) does not appear on a face of \(F\). Therefore, \(V_i\) and \(E_i\) are non-empty and \(P_i\) is a well-defined path of \(H_i\). Let \(p \leq \ell\) be the largest index such that \(v'^{\ell-1}_i\) does not belong to a face of \(F_i\); we know that \(p - k \leq \ell \leq p\). Similarly, let \(j \geq q\) be the smallest index such that \(v'^{j+1}_i\) does not belong to a face of \(F_i\); we know that \(q \leq j \leq q + k\). Since \(\Phi\) is non-expansive, \(d_D(v'_p, v'_q) \geq d_G(v_p, v_q) = q - p\) and since \((v'_p, v'_{p+1}, \ldots, v'_q)\) is a path of length \(q - p\) from \(v'_p\) to \(v'_q\) in \(D\), \(d_D(v'_p, v'_q) = q - p\). By triangle inequality, \(d_D(v'_k, v'_j) \geq d_D(v'_p, v'_q) - d_D(v'_p, v'_k) - d_D(v'_q, v'_j) \geq q - p - 2k\). Let \(H'_i\) be the subgraph of \(D\) induced by the vertices appearing on faces of \(F_i\). Since \(P_i\) is a subgraph of \(H'_i\), there is a path from \(v'_k\) to \(v'_j\) in \(H'_i\). Let \(P'_i = (v'_k = v'_0, \ldots, v'_j = v'_i)\) be the shortest \((v'_k, v'_j)\)-path in \(H'_i\). Note that if two vertices \(w'_j, w'_{j_2}\) of \(P'_i\) belong to a common face of \(F_i\), then \(d_H(w'_j, w'_{j_2}) \leq N\) and consequently, \(|j_2 - j_1| \leq N\). Consequently, the vertices of \(P'_i\) belong to at least \(i/N\) distinct faces of \(F_i\). Since \(t = d_H(v'_k, v'_j) \geq d_D(v'_k, v'_j) \geq q - p - 2k\), this implies that there are at least \((q - p - 2k)/N\) faces in \(F_i\).

Therefore, if we set \(i := k/N\) and consider the number \(A\) of faces in \(F = F_0 \cup F_1 \cup \ldots \cup F_{i-1}\), we get that \(A \geq k(q - p - 2k)/N^2\). Note that each face of \(F\) contains only vertices at distance
Proposition 6. For a graph $G$ and constants $K \in \mathbb{Q}$ and $N \in \mathbb{N}$ such that $2KN$ is a positive integer, if for every cycle $c$ of $G$, $\text{Area}_N(c) \leq \lceil Kl(c) \rceil$, then the geodesic triangles of $G$ are $16KN^2$-slim and $G$ is $(32KN^2 + \frac{1}{2})$-hyperbolic.

Proof. Using the fact that if all geodesic triangles of $G$ are $\delta$-slim then $G$ is $2\delta + \frac{1}{2}$-hyperbolic (Proposition 2), we will show that under our conditions, all geodesic triangles of $G$ are $16KN^2$-slim. This proof mainly uses ideas and notations from the proof of Theorem 2.9 of [10, Chapter III.H].

Let $k := 2KN^2$ and assume that there exist three (graph) geodesic segments $[p, q]$, $[q, r]$ and $[p, r]$ forming a geodesic triangle $\Delta(p, q, r)$ of $G$ that is not $8k$-slim, i.e., there exists a vertex $v \in [p, q]$ such that $d(v, [p, r] \cup [q, r]) > 8k$. Exchanging the roles of $p$ and $q$ if necessary, there are two cases to consider (see Figure 2):

- either $d([p, v], [q, r]) > 2k$ and $d([v, q], [p, r]) > 2k$;
- or there exists $w \in [v, q]$ such that $d(w, [p, r]) \leq 2k$.

Case 1. (see Figure 2 left) $d([p, v], [q, r]) > 2k$ and $d([v, q], [p, r]) > 2k$.

In this case, let $u \in [p, v]$ be the closest vertex to $v$ such that $d(u, [p, r]) = 2k$ and let $u' \in [p, r]$ be the closest vertex to $r$ such that $d(u, u') = 2k$. Let $w \in [v, q]$ be the closest vertex to $v$ such that $d(w, [q, r]) = 2k$, and let $w' \in [q, r]$ be the closest vertex to $r$ such that $d(w, w') = 2k$. We denote by $[u', r]$ (resp. $[w', r]$) the subgeodesic of $[p, r]$ (resp. $[q, r]$) from $u'$ (resp. $w'$) to $r$. Let $u'' \in [u', r]$ be the closest vertex from $u'$ such that $d(u'', [w', r]) \leq 2k$ and let $w'' \in [w', r]$ be the closest vertex from $w'$ such that $d(u'', w'') \leq 2k$. We denote by $[u'', w'']$ (resp. $[u', w']$) the subgeodesic of $[u', r]$ (resp. $[w', r]$) from $u''$ (resp. $w''$) to $u''$ (resp. $w''$). Let $[u, u']$ (resp. $[w, w']$, $[u'', w'']$) be a geodesic from $u$ to $u'$ (resp. from $w$ to $w''$, from $u''$ to $w''$). Let $u'' = d(u, w)$, $\beta = d(u', u'')$, and $\gamma = d(w', w'')$. Since $d(v, [p, r] \cup [q, r]) > 8k$, $\alpha > 12k$; since $[u, w]$ is a shortest path, $\beta + \gamma > 6k$. Due to our choice of $u, u', u'', w, w', w''$,
be the closest vertex to the only vertices of $\mathcal{B}$.

Case 2. (see Figure 2, right) There exists $k$ vertices at distance at most $\alpha > 0$, $2k$ in both cases, $d([u', u''], [w', w'']) = 2k$.

Let $c$ be the simple cycle of $G$ obtained as the concatenation of the six geodesics $[u, u'], [u', u''], [u'', u'], [u'', w'], [w', w]$, and $[w, u]$ (see Fig. 2 left). From the definition of the vertices $u, u', u'', w', w, w$ it follows that $c$ is a simple cycle.

Moreover, if there exists $x \in [u, w]$ such that $d(x, [u'', w'']) \leq k$, then either $d(x, u'') \leq 2k$, or $d(x, w') \leq 2k$. In the first case, from the definition of $u'$ and $u$, it implies that $u' = u''$ and that $x = u$. Analogously, in the second case, it implies that $w' = w''$ and $x = w$. Consequently, the only vertices of $c$ appearing in $B_k([u, w], G)$ are the vertices at distance at most $k$ from $[u, w]$ on $c$. Similarly, if there exists $x \in [w', w'']$ such that $d(x, [u', u'']) \leq k$, then either $d(x, u) \leq 2k$, or $d(x, u') \leq 2k$. In the first case, it contradicts $d([p, v], [q, r]) > 2k$. In the second case, from our choice of $w''$ and $w'''$, it implies that $u' = u''$ and $x = w''$. Consequently, the only vertices of $c$ appearing in $B_k([w', w''], G) = 2k$.

Let $(D, \Phi)$ be an $N$-filling of $c$. We want to get a lower bound on the number of faces of $D$ and therefore, a lower bound on $\text{Area}_N(c)$. In the planar graph $D$, we denote by $\Phi^{-1}([u, w])$ (resp., $\Phi^{-1}([u', u''])$ and $\Phi^{-1}([w', w''])$) the unique path on the boundary of $D$ that is mapped to $[u, w]$ (resp., $[u', u'']$ and $[w', w'']$).

Let $F_\alpha$ denote the set of faces of $D$ that contain only vertices in $B_k(\Phi^{-1}([u, w]), D)$ and at most one vertex at distance $k$ from $\Phi^{-1}([u, w])$. Similarly, let $F_\beta$ (resp. $F_\gamma$) be the set of faces of $D$ that contain only vertices in $B_k(\Phi^{-1}([u', u'']), D)$ (resp. in $B_k(\Phi^{-1}([w', w'']), D)$) and at most one vertex at distance $k$ from $\Phi^{-1}([u', u''])$ (resp. from $\Phi^{-1}([w', w''])$). Note that from our choice of $u$ and $u'$, if $s \in [u, w]$, $t \in [u', u'']$, and $d(s, t) = 2k$, then $s = u$ and $t = u'$. Consequently, there is no face in $F_\alpha \cap F_\beta$. Similarly, $F_\alpha \cap F_\gamma = F_\beta \cap F_\gamma = \emptyset$. Moreover, there is no edge appearing in a face $f$ of $F_\alpha$ and $f'$ of $F_\beta$ since both endvertices of this edge should be at distance $k$ from $[u, w]$ and $[u', u'']$. Similarly, no edge appears in a face of $F_\alpha$ (resp. $F_\beta$) and in a face of $F_\gamma$. Consequently, the number of faces of $D$ is at least $|F_\alpha| + |F_\beta| + |F_\gamma| + 1$.

From Lemma 2, since $\alpha > 2k$, $F_\alpha$ contains at least $\frac{k(\alpha - 2k)}{N^2} = 2K(\alpha - 2k)$ faces. Similarly, if $\beta > 2k$ (resp. $\gamma > 2k$), then $|F_\beta| \geq 2K(\beta - 2k)$ (resp. $|F_\gamma| \geq 2K(\gamma - 2k)$). Note that if $\beta \leq 2k$ (resp. $\gamma \leq 2k$), then $|F_\beta| \geq 0 \geq 2K(\beta - 2k)$ (resp. $|F_\gamma| \geq 0 \geq 2K(\gamma - 2k)$). Consequently, $\text{Area}_N(c) \geq 2K(\alpha + \beta + \gamma - 6k) + 1$. By the isoperimetric inequality, $\text{Area}_N(c) \leq K(\ell(c) + 1) - K(\alpha + \beta + \gamma + 6k) + 1$. From these two inequalities, we get that $\alpha + \beta + \gamma \leq 18k$, contradicting the fact that $\alpha > 2k$ and $\beta > \gamma > 6k$.

Case 2. (see Figure 2, right) There exists $w \in [v, q]$ such that $d(w, [p, r]) \leq 2k$.

In this case, let $w \in [v, q]$ be the closest vertex to $v$ such that $d(w, [p, r]) = 2k$; let $w' \in [p, r]$ be the closest vertex to $p$ such that $d(w, w') = 2k$. We denote by $[p, w']$ the subgeodesic of $[p, r]$ from $p$ to $w'$. Let $u \in [p, v]$ be the closest vertex to $v$ such that $d(u, [p, w']) = 2k$ and let
Let $u' \in [p, w']$ be the closest vertex to $w'$ such that $d(u, u') = 2k$. Let $[u, u']$ (resp. $[w, w']$) be a geodesic from $u$ to $u'$ (resp. from $w$ to $w'$). Let $\alpha = d(u, w)$ and $\beta = d(u', w')$. Due to our choice of $w, w', u,$ and $u'$, $d([u, w], [u', w']) = 2k$; since $d(v, [p, r] \cup [q, r]) > 8k$, $\alpha > 12k$; since $[u, w]$ is a shortest path, $\beta > 8k$.

Let $c$ be the simple cycle obtained as the concatenation of geodesics $[u, u'], [u', w'], [w', w]$, and $[w, u]$. As in Case 1, using Lemma 2, we show that $\text{Area}_N(c) \geq 2K (\alpha + \beta - 4k) + 1$.

By the isoperimetric inequality, $\text{Area}_N(c) \leq K \ell(c) + 1 = K (\alpha + \beta + 4k) + 1$. From these inequalities, we get that $\alpha + \beta \leq 12k$, contradicting the fact that $\alpha > 12k$ and $\beta > 8k$. □

**Remark 1.** The dependence of $\delta, N$ in Proposition 6 is the “best possible” in the following sense. There are graphs $G_N$ ($N \in \mathbb{N}$) which satisfy $\text{Area}_N(c) \leq \lceil \ell(c) \rceil$ and which are not $\delta$-hyperbolic for $\delta = o(N^2)$ (so $\delta$ in general grows quadratically in $N$).

Indeed, take $G_N$ to be a planar square $N \times N$ grid subdivided into squares of side-length $N$ (see Figure 3 for an example with $N = 4$). Then clearly for every cycle $c$, $\text{Area}_{4N}(c) \leq \frac{1}{4} \lceil \ell(c) \rceil$.

Consider now the four corners $a, b, c, d$ of the grid (see Figure 3); we have $d(a, c) + d(b, d) = 4N^2$ and $d(a, b) + d(c, d) = 4N^2$ and $d(a, d) + d(b, c)$ and thus $\delta \geq N^2$.

The assertion of Theorem 3 follows from Propositions 5 and 6 by setting $N := s + s'$ and $K := \frac{1}{2N} \cdot \left\lceil \frac{N}{(s-s')} \right\rceil \geq \frac{1}{2(s-s')}$. □

Here are the main consequences of Theorem 3:

**Corollary 2.** If a graph $G$ is $(s, s')$-dismantlable with $s' < s$ (in particular, $G$ is a finite $(s, s')$-cop-win graph), then $G$ is $\delta$-hyperbolic with $\delta = 64s^2$.

**Proof.** Since $(s, s')$-dismantlable graphs are also $(s, s-1)$-dismantlable, it is enough to prove our result for $s' = s - 1$. Notice that a graph $G$ is $(s, s-1)$-dismantlable if and only if $G$ is $(s, s-1)^*$-dismantlable. From Theorem 3 with $s - s' = 1$, we conclude that $G$ is $64s^2$-hyperbolic. □
We do not have examples of finite \((s, s - 1)\)-dismantlable graphs whose hyperbolicity is quadratic in \(s\) and leave this as an open question. However, if \(s - s' = \Omega(s)\), from Theorem 3 we immediately obtain that \(\frac{s^2}{s-s'} = O(s)\), thus \(\delta\) is linear in \(s\).

**Corollary 3.** If a graph \(G\) is \((s, s')^\ast\)-dismantlable with \(s - s' \geq ks\) for some constant \(k > 0\), then \(G\) is \(\frac{64s}{k}\)-hyperbolic. Conversely, if \(G\) is \(\delta\)-hyperbolic, then \(G\) is \((2r, r + 2\delta)^\ast\)-dismantlable for any \(r > 0\).

**Proof.** The first assertion follows directly from Theorem 3. The second assertion follows from Proposition 1. \(\square\)

Similarly to Theorem 1 that characterizes hyperbolicity via linear isoperimetric inequality, Corollary 3 characterizes hyperbolicity via \((s, s')^\ast\)-dismantlability.

### 4. Weakly modular graphs

In this section, we consider weakly modular graphs; for this particular class of graphs, we obtain stronger results than in the general case: namely, we show that for any \(s' < s\), if a weakly modular graph \(G\) is \((s, s')\)-dismantlable, then \(G\) is \(O(s)\)-hyperbolic.

Many classes of graphs occurring in metric graph theory and geometric group theory (in relationship with combinatorial nonpositive curvature property) are weakly modular: median graphs (alias, 1-skeletons of \(\text{CAT}(0)\) cube complexes), bridged and weakly bridged graphs (1-skeletons of systolic and weakly systolic complexes), bucolic graphs (1-skeletons of bucolic complexes), Helly graphs (alias, absolute retracts), and modular graphs. For definitions and properties of these classes of graphs the interested reader can read the survey [5] and the paper [9].

A graph \(G\) is **weakly modular** [4, 13] if it satisfies the following triangle and quadrangle conditions:

- **Triangle condition:** for any three vertices \(u, v, w\) with \(1 = d(v, w) < d(u, v) = d(u, w)\) there exists a common neighbor \(x\) of \(v\) and \(w\) such that \(d(u, x) = d(u, v) - 1\).
- **Quadrangle condition:** for any four vertices \(u, v, w, z\) with \(d(v, z) = d(w, z) = 1\) and \(2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1\), there exists a common neighbor \(x\) of \(v\) and \(w\) such that \(d(u, x) = d(u, v) - 1\).

All metric triangles of weakly modular graphs are equilateral. Moreover, they satisfy a stronger equilaterality condition:

**Lemma 3.** [13] A graph \(G\) is weakly modular if and only if for any metric triangle \(v_1v_2v_3\) of \(G\) and any two vertices \(x, y \in I(v_2, v_3)\), the equality \(d(v_1, x) = d(v_1, y)\) holds.

The following result shows that in the case of \((s, s')\)-dismantlable weakly modular graphs the hyperbolicity is always a linear function of \(s\) for all values of \(s\) and \(s' < s\).

**Theorem 4.** If \(G\) is an \((s, s')\)-dismantlable weakly modular graph with \(s' < s\), then \(G\) is \(184s\)-hyperbolic.
Proof. Since \((s,s')\)-dismantlable graphs are \((s,s-1)\)-dismantlable, it is enough to prove our result for \(s' = s - 1\). We will establish our result in two steps. First, we show that if all metric triangles of an \((s,s-1)\)-dismantlable graph \(G\) are \(\mu\)-bounded (i.e., have sides of length at most \(\mu\)), then all intervals of \(G\) are \((4s+2\mu)\)-thin. In the second step, we show that in \((s,s-1)\)-dismantlable weakly modular graphs all metric triangles are \(6\)-bounded.

We continue with some properties of general \((s,s-1)\)-dismantlable graphs. A subgraph \(H = (V',E')\) of a graph \(G\) is called a locally \(s\)-isometric subgraph of \(G\) if \(H\) contains a collection \(\mathcal{P}\) of geodesics of \(G\) such that for any vertex \(v\) of \(H\) there exists a geodesic \(P \in \mathcal{P}\) passing via \(v\) and such that the distances \(d(v,x),d(v,y)\) in \(G\) between \(v\) and the endvertices \(x,y\) of \(P\) are at least \(s\). Note that this implies that for every vertex \(v \in V(H)\), there exists a subgeodesic \(P_v\) of a geodesic \(P \in \mathcal{P}\) containing \(v\) such that \(d(v,x) = d(v,y) = 2\) in \(G\).

**Lemma 4.** If a graph \(G\) is \((s,s-1)\)-dismantlable, then \(G\) does not contain finite locally \(s\)-isometric subgraphs.

Proof. Suppose by way of contradiction that \(G\) contains a finite locally \(s\)-isometric subgraph \(H = (V',E')\). Let \(\leq\) be an \((s,s-1)\)-dismantling well-order of the vertex-set \(V\) of \(G\) and let \(v\) be the largest element of the set \(V'\) in this order. Let \(u\) be a vertex of \(G\) that eliminates \(v\) in \(\leq\). Since \(H\) is locally \(s\)-isometric, \(H\) contains a geodesic \(P\) of \(G\) of length \(2s\) passing via \(v\) such that \(d(v,x) = d(v,y) = s\), where \(x\) and \(y\) are the endvertices of \(P\). If \(u \in P\), say \(u\) belongs to the subpath \(P'\) of \(P\) comprised between \(v\) and \(x\), then the subpath \(P''\) of \(P\) between \(v\) and \(y\) is completely contained in \(B_s(v,G - \{u\}) \cap X_v\). From the choice of \(u\), we have \(P'' \subseteq B_{s-1}(v,G)\). Hence \(d(u,y) \leq s - 1\), which is impossible because \(P\) is a geodesic of length \(2s\) passing via \(u\) and \(u \in P'\). Then \(P\) is completely contained in \(B_s(v,G - \{u\}) \cap X_v\), whence \(P \subseteq B_{s-1}(u,G)\). In particular, \(d(u,x) \leq s - 1\) and \(d(u,y) \leq s - 1\). Since \(d(x,y) = 2s\), we again obtain a contradiction. \(\square\)

We will say that a cycle \(c\) of \(G\) is \(s\)-geodesically covered if there exists a set \(\mathcal{P} = \{P_0, P_1, \ldots, P_{n-1}\}\) of geodesics of \(G\) such that (i) each \(P_i\) is a subpath of \(c\), (ii) each edge of \(c\) is contained in a geodesic of \(\mathcal{P}\), (iii) if \(P_i\) and \(P_j\) are not consecutive (modulo \(n\)), then \(P_i\) and \(P_j\) are edge-disjoint, and (iv) if \(P_i\) and \(P_j\) are consecutive (i.e., \(j = i + 1 \mod n\)), then \(P_i \cap P_j\) is a path of length \(\geq 2s\).

**Lemma 5.** If a graph \(G\) contains a \(s\)-geodesically covered cycle, then it contains a finite locally \(s\)-isometric subgraph.

In particular, if a graph \(G\) is \((s,s-1)\)-dismantlable, then \(G\) does not contain \(s\)-geodesically covered cycles.

Proof. Let \(c\) be a \(s\)-geodesically covered cycle of \(G\) and let \(\mathcal{P} = \{P_0, P_1, \ldots, P_{n-1}\}\) be the corresponding set of geodesics satisfying conditions (i)-(iv). Fix a cyclic traversal of \(c\). For each \(0 \leq i \leq n - 1\), let \(x_i\) and \(y_i\) be the end-vertices of \(P_i\) labeled in such a way that following the traversal, \(x_i\) and \(y_i\) are the first and the last vertices of \(P_i\).

Then \(P_i \cap P_{i+1}\) is a geodesic between \(x_{i+1}\) and \(y_i\) (as a subgeodesic of \(P_i\) and \(P_{i+1}\)). By condition (iv), its length is at least \(2s\), whence \(d(x_{i+1}, y_i) \geq 2s\). Analogously, \(d(x_{i+2}, y_{i+1}) \geq 2s\).
2s. On the other hand, since by (iii) \( P_1 \cap P_{i+2} \) does not contain edges, either \( y_i = x_{i+2} \) or \( x_{i+2} \) is located between \( y_i \) and \( y_{i+1} \). Therefore, since \( y_i \) and \( x_{i+2} \) belong to the geodesic \( P_{i+1} \), we obtain \( d(y_i, y_{i+1}) = d(y_i, x_{i+2}) + d(x_{i+2}, y_{i+1}) \geq 2s \).

Pick any vertex \( v \) of \( c \). Since \( c \) is covered by the geodesics of \( \mathcal{P} \), there exists at least one such geodesic that contains \( v \). Let \( P_{i_0} \) be a geodesic of \( \mathcal{P} \) selected in a such a way that \( v \in P_{i_0} \) and \( k := \min\{d(v, x_{i_0}), d(v, y_{i_0})\} \) is as large as possible. If \( k < s \), assume without loss of generality that \( d(v, y_{i_0}) < s \). By condition (iv) applied to the geodesics \( P_{i_0} \) and \( P_{i_0+1} \), we conclude that \( d(x_{i_0+1}, y_{i_0}) \geq 2s \). Since \( d(v, y_{i_0}) < s \), necessarily \( v \in P_{i_0+1} \) and \( d(v, x_{i_0+1}) > s \). On the other hand, since \( d(y_{i_0}, y_{i_0+1}) \geq 2s \) and \( v \) is located on the geodesic \( P_{i_0+1} \) between \( x_{i_0+1} \) and \( y_{i_0} \), necessarily \( d(v, y_{i_0+1}) \geq 2s \), contrary to the choice of \( P_{i_0} \) as the path of \( \mathcal{P} \) containing \( v \) and maximizing \( \min\{d(v, x_{i_0}), d(v, y_{i_0})\} \). Thus, \( k \geq s \) for every choice of \( v \). Consequently, \( c \) is a locally \( s \)-isometric subgraph of \( G \), establishing the first assertion of the lemma. The second assertion follows directly from Lemma [3]. \( \square \)

**Proposition 7.** If a graph \( G \) is \((s, s-1)\)-dismantlable and the metric triangles of \( G \) have sides of length at most \( \mu \), then the intervals of \( G \) are \((4s + 2\mu)\)-thin and \( G \) is \((64s + 36\mu)\)-hyperbolic. If, additionally, \( G \) is weakly modular, then the intervals of \( G \) are \((4s + \mu)\)-thin and \( G \) is \((64s + 20\mu)\)-hyperbolic.

**Proof.** Let \( I(u, v) \) be an interval of \( G \) and \( x, y \in I(u, v) \) such that \( d(u, x) = d(u, y) = k, d(v, x) = d(v, y) = l \) and \( l + k = d(u, v) \). Let \( u'x'y' \) be a quasi-median of the triplet \( u, x, y \) and let \( v'x''y'' \) be a quasi-median of the triplet \( v, x, y \) (see Figure [4]). Let \( k_1 = d(u', x'), k_2 = d(u', y'), k_3 = d(x', y'), a_1 = k - k_1 - d(u, u'), a_2 = k - k_2 - d(u, u') \). Analogously, let \( l_1 = d(v', x''), l_2 = d(v', y''), l_3 = d(x'', y''), b_1 = l - l_1 - d(v, v'), b_2 = l - l_2 - d(v, v') \). Since the metric triangles of \( G \) are \( \mu \)-bounded, each of \( k_1, k_2, k_3, l_1, l_2, l_3 \) is at most \( \mu \).
Now, suppose that each of $a_1, a_2, b_1, b_2$ is at least $2s$. Let $c$ be a cycle consisting of a geodesic $R_1$ between $x$ and $x'$, followed by a geodesic $R_2$ between $x'$ and $y'$, a geodesic $R_3$ between $y', y$, a geodesic $Q_3$ between $y$ and $y''$, a geodesic $Q_2$ between $y''$ and $x''$, and finally, a geodesic $Q_1$ between $x''$ and $x$. From the definition of quasi-medians it follows that $P_1 := R_1 \cup R_2 \cup R_3$ and $P_3 := Q_1 \cup Q_2 \cup Q_3$ are geodesics between $x$ and $y$. Since $x \in I(u, v)$, $x' \in I(u, x)$ and $x'' \in I(x, v)$, we also conclude that $P_0 := R_1 \cup Q_1$ is a geodesic between $x'$ and $x''$. Analogously, $P_2 := R_3 \cup Q_3$ is a geodesic between $y'$ and $y''$. Since each pair of (circularly) consecutive paths $P_0, P_1, P_2, P_3$ intersect along a path of length at least $2s$ and any two nonconsecutive paths do not share common edges, the set $P_0, P_1, P_2, P_3$ constitutes an $s$-geodesic covering of $c$, but this is impossible by Lemma 3. This contradiction shows that $\min\{a_1, a_2, b_1, b_2\} < 2s$.

Suppose without loss of generality that $a_1 < 2s$. Since $k_1 + a_1 = k_2 + a_2$ and $k_1, k_2 \leq \mu$, we obtain that $a_2 \leq a_1 + k_1 \leq 2s + \mu$. Since $x', y'$ belong to a common geodesic between $x$ and $y$, we obtain that $d(x, y) = d(x, x') + d(x', y') + d(y', y) = a_1 + k_3 + a_2 \leq 2s + \mu + 2s + \mu = 4s + 2\mu$. Thus the intervals of $G$ are $(4s + 2\mu)$-thin. Proposition 3 shows that $G$ is $(64s + 36\mu)$-hyperbolic.

If $G$ is weakly modular, then all metric triangles of $G$ are equilateral, whence $k_1 = k_2 = k_3$ and $a_1 = a_2$. This shows that $d(x, y) = a_1 + k_3 + a_2 \leq 4s + \mu$. Hence the intervals of $G$ are $(4s + \mu)$-thin and $G$ is $(64s + 20\mu)$-hyperbolic. □

Next we will prove that the metric triangles of $(s, s - 1)$-dismantlable weakly modular graphs are 6$s$-bounded.

Lemma 6. Let $uvw$ be a metric triangle of a weakly modular graph $G$. For any vertex $x \in I(u, w)$ at distance $p$ from $u$ there exists a vertex $y \in I(u, v)$ at distance $p$ from $u$ and $x$.

Proof. Let $u'v'x'$ be a quasi-median of the triplet $u, v, x$. Since $uvw$ is a metric triangle, $I(u, v) \cap I(u, w) = \{u\}$. Since $I(u, x) \subseteq I(u, w)$, necessarily $I(u, v) \cap I(u, x) = \{u\}$, i.e., $u' = u$. We also claim that $x = x'$. Since $x' \in I(u, x) \cap I(x, v)$, if $x \neq x'$, two different vertices $x$ and $x'$ of $I(u, w)$ will have different distances from $v$, contrary to Lemma 3. So, $x' = x$. Since $v'ux$ is an equilateral metric triangle, $d(v', x) = d(v', u) = d(u, x) = p$ and we are done. □

Proposition 8. If $G$ is an $(s, s - 1)$-dismantlable weakly modular graph, then the metric triangles of $G$ are 6$s$-bounded.

Proof. Suppose by way of contradiction that $G$ contains a metric triangle $uvw$ with sides of length $\geq 6s$. Let $x_u$ be a vertex of $I(u, w)$ located at distance $2s$ from $u$. Let $y_u$ be a vertex of $I(u, v)$ located at distance $2s$ from $u$ and $x_u$ (such a vertex exists by Lemma 3). Let $x_v$ be a vertex of $I(v, y_u)$ at distance $2s$ from $v$ and let $y_v$ be a vertex of $I(v, w)$ at distance $2s$ from $x_v$ and $v$ (again, this vertex is provided by Lemma 3). Finally, let $x_wy_wu$ be a quasi-median of the triplet $y_v, x_u, w$. Denote by $k$ the length of the sides of the metric triangle $x_wy_wu$. We distinguish two cases depending of the value of $k$.

Case 1: $k > 2s$ (Fig. 5 left).
Let $x'_w$ be a vertex of $I(w, x_w)$ at distance $2s$ from $w$ and let $y'_w$ be a vertex of $I(w, y_w)$ located at distance $2s$ from $w$ and $x'_w$ (provided by Lemma 6). Denote by $P_1$ a geodesic of $G$ between $x_u$ and $x_v$ passing via $y_u$ (such a geodesic exists because $y_u \in I(u, v) \cap I(x_u, v)$ and $x_v \in I(y_u, v)$). Let $P_2$ be a geodesic between $y_u$ and $y_v$ passing via $x_v$, $P_3$ be a geodesic between $x_v$ and $x'_w$ passing via $y_v$, $P_4$ be a geodesic between $y_v$ and $y'_w$ passing via $x'_w$, $P_5$ be a geodesic between $x'_w$ and $x_u$ passing via $y'_w$, and $P_6$ be a geodesic between $y'_w$ and $y_u$ passing via $x_u$ (the existence of these geodesics follows from the way the vertices $x_v, y_u, x'_w, y'_w, x_u$ have been selected and is similar to the proof of existence of $P_1$). Let $c$ be the cycle of $G$ defined as the union of these six geodesics. Since any two consecutive geodesics intersect along a path of length at least $2s$ (because the length of the sides of $uvw$ is at least $6s$ and the sides of the metric triangles $ux_u y_u$, $x_v y_v$, and $x'_w y'_w$ have length $2s$) and any two nonconsecutive geodesics are disjoint or intersect in a single vertex, the cycle $c$ is $s$-geodesically covered by $P_1, \ldots, P_6$, leading to a contradiction with Lemma 5.

**Case 2:** $k \leq 2s$ (Fig 5 right).

In this case, we define the following five geodesics: $P_1$ is a geodesic between $x_u$ and $x_v$ passing via $y_u$, $P_2$ is a geodesic between $y_u$ and $y_v$ passing via $x_v$, $P_3$ is a geodesic between $x_v$ and $x'_w$ passing via $y_v$, $P_4$ is a geodesic between $y_v$ and $x_u$ passing via $x'_w$, $y'_w$, and $P_5$ is a geodesic between $x_u$ and $y_u$ passing via $x'_w$. The proof of existence of geodesics $P_1, P_2, P_3$, and $P_5$ is the same as in Case 1. The existence of $P_4$ follows from the fact that $x'_w y'_w w$ is a quasi-median of the triplet $y_v, x_u, w$. Again, any two consecutive geodesics intersect along a path of length at least $2s$ while two nonconsecutive geodesics either are disjoint or intersect in a single vertex. Thus the cycle $c$, which is the union of these five geodesics, is $s$-geodesically covered by them, contrary to Lemma 5.

In both cases, the assumption that $G$ contains a metric triangle with sides of length $\geq 6s$ leads us to a contradiction. Thus all metric triangles of $G$ are $6s$-bounded. \hfill \Box
Propositions \[7\] and \[8\] imply that if a weakly modular graph \( G \) is \((s,s-1)\)-dismantlable, then \( G \) is \(184s\)-hyperbolic. This completes the proof of Theorem \[4\]. \(\square\)

Theorem \[4\] can be sharpened in case of median and Helly graphs that are important subclasses of weakly-modular graphs \[5\]. A graph \( G = (V,E) \) is \textit{median} if for all vertices \( u, v, w \in V \), there exists a unique vertex \( m \) in the intersection \( I(u,v) \cap I(u,w) \cap I(v,w) \). A graph \( G \) is \textit{Helly} if for any family \( B \) of balls of \( G \), the intersection \( \cap \{ B \in B \} \) is non-empty if and only if \( B \cap B' \neq \emptyset \) for all \( B, B' \in B \).

\textbf{Corollary 4.} Let \( G \) be an \((s,s-1)\)-dismantlable graph. If \( G \) is a median graph, then \( G \) is \(2s\)-hyperbolic. If \( G \) is a Helly graph, then \( G \) is \((64s+20)\)-hyperbolic.

\textit{Proof.} It is known \[14\] \[23\] that a median graph \( G \) is \(\delta\)-hyperbolic if and only if \( G \) does not contain square \(\delta \times \delta\) grids as isometric subgraphs. To obtain our result is suffices to show that if a graph \( G \) is \((s,s-1)\)-dismantlable, then \( G \) does not contain square grids of size \(2s \times 2s\) as isometric subgraphs. Suppose by way of contradiction that \( G \) contains an isometric \(2s \times 2s\) grid \( H \); denote the boundary cycle of \( H \) by \( c \). Let \( u, v, w, x \) be the four corners of \( H \). Let \( P_0 \) be the \((u,w)\)-geodesic of \( H \) passing via \( v \), \( P_1 \) be the \((v,x)\)-geodesic of \( H \) passing via \( w \), \( P_2 \) be the \((w,u)\)-geodesic of \( H \) passing via \( x \), and \( P_3 \) be the \((x,v)\)-geodesic of \( H \) passing via \( u \). These four geodesics show that \( c \) is a \(s\)-geodesically covered cycle, contrary to Lemma \[5\]. This establishes the first assertion.

Now, let \( G \) be an \((s,s-1)\)-dismantlable Helly graph. Then \( G \) is weakly modular \[5\]. We assert that all metric triangles of \( G \) have sides of length at most 1. Indeed, if \( uvw \) is a metric triangle with sides of length \(k > 1\), consider the following three pairwise intersecting balls: \(B_1(u), B_{k-1}(v),\) and \( B_{k-1}(w) \). By Helly property, they have a common vertex \( u' \). But then \( u' \in I(u,v) \cap I(u,w) \) and \( u' \neq u \), because \( d(u,v) = k \) and \( u' \in B_{k-1}(v) \), contrary to the assumption that \( uvw \) is a metric triangle. Hence \( k \leq 1 \). By Proposition \[7\], \( G \) is \((64s+20)\)-hyperbolic. \(\square\)

5. Algorithmic Consequences

The \textit{hyperbolicity} \(\delta^*\) of a metric space \((X,d)\) (or of a non-necessarily finite graph \( G \)) is the least value of \(\delta\) for which \((X,d)\) (resp., \( G \)) is \(\delta\)-hyperbolic. By a remark of Gromov \[22\], if the four-point condition in the definition of hyperbolicity holds for a fixed base-point \( u \) and any triplet \( x,y,v \) of \( X \), then the metric space \((X,d)\) is \(2\delta\)-hyperbolic. This provides a factor 2 approximation of hyperbolicity of a metric space on \( n \) points running in cubic \( O(n^3) \) time. Using fast algorithms for computing (max,min)-matrix products, it was noticed in \[21\] that this 2-approximation of hyperbolicity can be implemented in \( O(n^{2.69}) \) time. In the same paper, it is shown that any algorithm computing the hyperbolicity for a fixed base-point in time \( O(n^{2.65}) \) would provide an algorithm for (max,min)-matrix multiplication faster than the existing ones. In \[18\], approximation algorithms are given to compute a \((1 + \epsilon)\)-approximation in \( O(\epsilon^{-1}n^{3.38}) \) time and a \((2 + \epsilon)\)-approximation in \( O(\epsilon^{-1}n^{2.38}) \) time. For a practical motivation of a fast computation or approximation of hyperbolicity of large graphs and an experimental study, see \[16\].
Gromov gave an algorithm to recognize Cayley graphs of hyperbolic groups and estimate the hyperbolicity constant $\delta$. His algorithm is based on the theorem that hyperbolicity “propagates”, i.e. if balls of an appropriate fixed radius are hyperbolic for a given $\delta$ then the whole space is $\delta'$-hyperbolic for some $\delta' > \delta$ (see [22], 6.6.F). More precisely, for simply connected geodesic spaces, hyperbolicity can be characterized in the following local-to-global way:

**Theorem 5.** [22] Given $\delta > 0$, let $R = 10^5 \delta$ and $\delta' = 200 \delta$. Let $(X, d)$ be a simply connected geodesic metric space in which each loop of length $< 100 \delta$ is null-homotopic inside a ball of diameter $< 200 \delta$. If every ball $B_R(x_0)$ of $X$ is $\delta$-hyperbolic, then $X$ is $\delta'$-hyperbolic.

Although Cayley graphs (viewed as 1-dimensional complexes) are not simply connected, they can be replaced by the (2-dimensional) Cayley complexes of the groups, which are simply connected, and the theorem above applies. To check the hyperbolicity of a Cayley graph it is enough to verify the hyperbolicity of a sufficiently big ball (note that all balls of a given radius in the Cayley graph are isomorphic to each other). For other versions of this “local-to-global” theorem for hyperbolicity see [8], [17], [26]. However this theorem does not help when dealing with arbitrary graphs due to the simple-connectedness assumptions.

5.1. **Approximating the hyperbolicity of a graph.** In this section, we will describe a fast $O(n^2)$ time algorithm for constant-factor approximation of hyperbolicity $\delta^*$ of a graph $G = (V, E)$ with $n$ vertices and $m$ edges, assuming that its distance-matrix has already been computed. Our algorithm is very simple and can be used as a practical heuristic to approximate the hyperbolicity of graphs.

The hyperbolicity $\delta^*$ of a graph $G$ is an integer or a half-integer belonging to the list \( \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots n - 1, \frac{2n-1}{2}, n\} \). It is known that 0-hyperbolic graphs are exactly the block graphs, i.e., the graphs in which every 2-connected component is a clique [6]. Consequently, from the distance-matrix of $G$, one can check in time $O(n^2)$ whether $\delta^* = 0$ or not. In the following, we assume that $\delta^* \geq \frac{1}{2}$.

Before presenting the general algorithm (Algorithm 2), we describe an auxiliary algorithm (Algorithm 1) that for a parameter $\alpha$ either ensures that $G$ is $(784 \alpha + \frac{1}{2})$-hyperbolic or that $G$ is not $\frac{\alpha}{2}$-hyperbolic. Algorithm 1 is based on Theorem 3 and Corollary 1 of Proposition 4.

First, suppose that Algorithm 1 returns Yes. This means that the BFS-order $\preceq$ is a $(4\alpha, 3\alpha)$-dismantling order of the vertices of $G$. Consequently, from Theorem 3, $G$ is $(784\alpha + \frac{1}{2})$-hyperbolic. Now, suppose that the algorithm returns No. This means that there exists a vertex $v$ such that $B_{4\alpha}(v) \cap X_v \not\subseteq B_{3\alpha}(f_\alpha(v))$. From Proposition 4 with $r = 2\alpha$, this implies that $G$ is not $\frac{\alpha}{2}$-hyperbolic and thus $\delta^* > \frac{\alpha}{2}$.

Algorithm 2 efficiently computes the smallest integer $\alpha$ for which the Algorithm 1 returns the answer Yes, i.e., the smallest integer $\alpha$ for which the inclusion $B_{4\alpha}(v) \cap X_v \subseteq B_{3\alpha}(f_\alpha(v))$ holds for all vertices $v$ of $G$. Similarly to Algorithm 1 we assume that we have constructed \( \frac{\alpha}{2} \)-hyperbolic and thus $\delta^* > \frac{\alpha}{2}$.  

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1 Recall that a topological space $X$ is **simply connected** if it is path-connected (i.e., for all points $x, y \in X$, there exists a path from $x$ to $y$ in $X$) and every loop is null-homotopic (i.e., can be continuously deformed to a point).
Proposition 9. There exists a constant-factor approximation algorithm to approximate the hyperbolicity \( \delta^* \) of a graph \( G \) with \( n \) vertices running in \( O(n^2) \) time if \( G \) is given by its distance-matrix. The algorithm returns a 1569-approximation of \( \delta^* \).

Proof. We start with the correctness proof of Algorithm 2 (the correctness of Algorithm 1 was given above). Suppose that we are at iteration \( \alpha \) and consider an arbitrary vertex \( v \) of \( G \). Any vertex \( w \) that has been removed from \( L(v) \) at a previous iteration \( \alpha' < \alpha \) either satisfies \( v \preceq w \) or \( d(w, f_{\alpha'}(v)) \leq 3\alpha' \). Consequently, by Lemma 7, \( v \preceq w \) or \( d(w, f_{\alpha}(v)) \leq 3\alpha \).
Algorithm 2: Approximated-Hyperbolicity($G = (V,E)$)

Construct a BFS-order $\preceq$ starting from an arbitrary vertex $v_0$;
For each $v \in V$, let $p(v)$ be the parent of $v$ in the BFS-tree;
For each $v \in V$, let $L(v)$ be a stack containing all vertices of $G$ sorted (increasingly) by their distance to $v$;
done $\leftarrow$ false;
$\alpha$ $\leftarrow$ 0;
for each $v \in V$ do $f_\alpha(v) \leftarrow v$ while not done do
    done $\leftarrow$ true;
    $\alpha$ $\leftarrow$ $\alpha$ + 1;
    for each $v \in V$ do
        $f_\alpha(v)$ $\leftarrow$ $p(p(f_\alpha(v)))$;
        repeat
            $u$ $\leftarrow$ pop($L(v)$)
        until $d(u,v) > 4\alpha$ or ($u \preceq v$ and $d(u,f_\alpha(v)) > 3\alpha$);
        if $d(u,v) \leq 4\alpha$ then done $\leftarrow$ false push($u$, $L(v)$);
    return $\alpha$

Therefore, at the end of iteration $\alpha$, all vertices that have been already removed from $L(v)$ cannot serve as witnesses for $B_{4\alpha}(v) \cap X_v \not\subseteq B_{3\alpha}(f_\alpha(v))$.

Suppose now that at the end of iteration $\alpha$, for every vertex $v$ of $G$, the head $u$ of $L(v)$ satisfies the inequality $d(u,v) > 4\alpha$. Since initially, all vertices of $G$ were inserted in $L(v)$ according to their distances to $v$, this means that all vertices of $B_{4\alpha}(v)$ have been removed from $L(v)$. Since each $w$ removed from $L(v)$ either appears after $v$ in $\preceq$ or has distance at most $3\alpha$ from $f_\alpha(v)$, we conclude that $B_{4\alpha}(v) \cap X_v \subseteq B_{3\alpha}(f_\alpha(v))$. Consequently, $\preceq$ is a $(4\alpha,3\alpha)^*$-dismantling order and by Theorem 3 $G$ is $(784\alpha + 1/2)$-hyperbolic. Since we also know that $G$ is not $(\alpha - 1/2)$-hyperbolic, $\delta^* \leq 784\alpha + 1/2 \leq 1568\delta^* + 1/2 \leq 1569\delta^*$. This gives a 1569-approximation of the hyperbolicity $\delta^*$ of $G$.

As to the complexity, first note that computing the BFS-order $\preceq$ and the value of $p(v)$ for each $v \in V$ can be done in time $O(n^2)$ from the distance-matrix of $G$ (this can be done in time linear in the number of edges of $G$ if we are also given the adjacency list of $G$). Since $|V| = n$ and all the pairwise distances are integers between 0 and $n$, one can construct each stack $L(v)$ in time $O(n)$ using a counting sort algorithm. Thus, the preprocessing step requires total $O(n^2)$ time. Since during the execution of the algorithm we always have $\alpha \leq 2\delta^* \leq 2n$, $\alpha$ is incremented at most $2n$ times. Since for each $v \in V$, once a vertex $w$ is popped from $L(v)$, $w$ is no longer used for $v$ at subsequent iterations, there are at most $O(n^2)$ pop operations. Therefore Algorithm terminates in time $O(n^2)$. □

Now suppose that the input graph $G = (V,E)$ is given by the adjacency list (instead of its distance-matrix). In the most naive implementation of Algorithm one can perform
a BFS-traversal of $G$ from each vertex of $V$ to compute the distance matrix of $G$ in time $O(mn)$, where $m = |E|$ and $n = |V|$. One can also use Seidel’s algorithm \cite{28} to compute the distance-matrix of $G$ in time $O(n^2\cdot 38)$. Hence, we get immediately the following corollary.

**Corollary 5.** There exists a constant-factor approximation algorithm to approximate the hyperbolicity $\delta^*$ of a graph $G$ with $n$ vertices and $m$ edges running in $O(\min(mn, n^{2\cdot 38}))$ time if $G$ is given by its adjacency list.

However, note that once we have computed the BFS-order $\preceq$ in time $O(m)$, all the remaining computations are local, around each vertex. Namely, one can replace the preprocessing step of Algorithm \cite{2} by the following localized computations. First, we modify slightly the algorithm such that each time we increase $\alpha$, we consider the vertices of $V$ in the order $\preceq$.

Then, instead of having $L(v)$ as a stack we consider $L(v)$ as a queue; initially each $L(v)$ contains a single vertex $v$, which is labeled. At iteration $\alpha$, when the head $u$ of $L(v)$ is removed from the queue $L(v)$, we label and insert in $L(v)$ all still unlabeled neighbors of $u$. These inserted vertices are one step further from $v$ than $u$. For each vertex $v$, the order in which vertices are added in $L(v)$ corresponds to a BFS-order computed from $v$. Moreover, each time we insert a vertex $u$ in $L(v)$, we store the distance $d(u,v)$. In order to efficiently retrieve the computed distances, we will use a matrix $D$. Initially, $D(u,v) = \infty$ for all $u,v$ and when $u$ is inserted in $L(v)$, we update $D(u,v)$ to $d(u,v)$. Using a standard trick (see \cite[ex. 2.12]{1}), one can avoid the initialization cost of the matrix $D$.

All the remaining steps of the Algorithm \cite{2} remain the same. When in Algorithm \cite{2} we need the value of $d(u, f_{\alpha}(v))$ where $u$ is the head of $L(v)$, we now use $D(u, f_{\alpha}(v))$ instead. In order to prove that the algorithm is still correct, it is enough to show that if $u \preceq v$, $d(u,v) \leq 4\alpha$, and $D(u, f_{\alpha}(v)) > 3\alpha$, then $G$ is not $\frac{\alpha}{2}$-hyperbolic and thus we need to increment $\alpha$. We prove this property by induction on $\preceq$. Note that if $D(u, f_{\alpha}(v)) \neq \infty$, then $D(u, f_{\alpha}(v)) = d(u, f_{\alpha}(v))$ and $u \in (B_{4\alpha}(v) \cap X_v) \setminus B_{3\alpha}(f_{\alpha}(v))$. Thus, by Proposition \cite{3} $G$ is not $\frac{\alpha}{2}$-hyperbolic. In particular, this is the case if $v = v_0$, since $f_{\alpha}(v_0) = v_0$ and $D(u, f_{\alpha}(v_0)) = d(u,v_0)$. Suppose now that $D(u, f_{\alpha}(v)) = \infty$ and let $v' = f_{\alpha}(v)$. Let $u'$ be the head of $L(v')$ and note that $D(u',v') = d(u',v')$. Since $u$ has not yet been added to $L(v')$, necessarily, $d(u,v') \geq d(u',v')$. Thus, if $D(u',v') > 4\alpha$, we have that $d(u,v') > 4\alpha > 3\alpha$ and by Proposition \cite{4} $G$ is not $\frac{\alpha}{2}$-hyperbolic since $u \in (B_{4\alpha}(v) \cap X_v) \setminus B_{3\alpha}(f_{\alpha}(v))$. Suppose now that $D(u',v') \leq 4\alpha$. Since $v' = f_{\alpha}(v) \preceq v$, we have already iterated over $v'$ at step $\alpha$, and consequently, $u' \preceq v'$ and $D(u', f_{\alpha}(v')) > 3\alpha$. Thus, by induction hypothesis, we know that $G$ is not $\frac{\alpha}{2}$-hyperbolic.

Since $\alpha$ is never greater than $2\delta^*$, with this implementation, the complexity of Algorithm \cite{2} becomes $O(\sum_{v \in V} |E(B_{8\delta^*+1}(v))|)$, where $|E(B_{8\delta^*+1}(v))|$ is the number of edges in the subgraph of $G$ induced by the ball $B_{8\delta^*+1}(v)$. This is efficient if the balls $B_{8\delta^*+1}(v)$ do not contain too many vertices and edges, in particular if $G$ is a bounded-degree graph of small hyperbolicity. Consequently, we obtain the following result.

**Proposition 10.** If $G$ is a graph with $n$ vertices and $m$ edges, given by its adjacency list, then Algorithm \cite{3} can be implemented to run in $O(\sum_{v \in V} |E(B_{8\delta^*+1}(v))|)$ time. In particular,
if there exists a constant $K > 0$ such that for each $v \in V$ the ball $B_{88^*+1}(v)$ contains at most $K$ edges, then its complexity becomes $O(Kn)$.

Finally, in the case of weakly modular graphs, we can obtain a sharper approximation of hyperbolicity using the same ideas. The following result is the counterpart of Proposition 9 for weakly modular graphs.

**Proposition 11.** If $G$ is a weakly modular graph given by its distance matrix, then in time $O(n^2)$ one can compute $\delta'$ such that $\delta^* \leq \delta' \leq 736\delta^* + 368$.

**Proof.** Consider a weakly modular graph graph $G = (V, E)$ and assume that we have constructed a BFS-order $\preceq$ on $V$ starting from an arbitrary vertex $v_0$; as before, assume that for each $v$, $p(v)$ is the parent of $v$ in the corresponding BFS-tree (with $p(v_0) = v_0$). For each vertex $v$ and each integer $\alpha$, let $g_{\alpha}(v)$ be the vertex at distance $(\alpha + 1, d(v, v_0))$ on the path of the BFS-tree from $v$ to $v_0$. Note that $g_{\alpha+1}(v) = p(g_{\alpha}(v))$.

We want to find the smallest $\alpha$ such that $B_{2\alpha+2}(v) \cap X_v \subseteq B_{2\alpha+1}(g_{\alpha}(v))$ for all vertices $v$. We first note that for any vertex $v$ and any value of $\alpha$, $B_{2\alpha+1}(g_{\alpha}(v)) \subseteq B_{2(\alpha+1)+1}(g_{\alpha+1}(v))$. Indeed, since $d(g_{\alpha}(v), g_{\alpha+1}(v)) \leq 1$, for every $u \in B_{2\alpha+1}(g_{\alpha}(v))$, by triangle inequality $d(u, g_{\alpha+1}(v)) \leq 2\alpha + 1 + 1 \leq (\alpha + 1) + 1$. Consequently, one only need to slightly adapt Algorithm 2 to compute the smallest $\alpha$ such that $B_{2\alpha+2}(v) \cap X_v \subseteq B_{2\alpha+1}(g_{\alpha}(v))$ for all vertices $v$. This algorithm runs in time $O(n^2)$.

Suppose now that we have computed the smallest integer $\alpha$ such that $B_{2\alpha+2}(v) \cap X_v \subseteq B_{2\alpha+1}(g_{\alpha}(v))$ for all vertices $v$. By Theorem 1 $G$ is $184(2\alpha + 2)$-hyperbolic. Moreover, since $B_{2\alpha}(v) \cap X_v \not\subseteq B_{2\alpha-1}(g_{\alpha-1}(v))$, by Proposition 4 with $r = \alpha$, we have that $G$ is not $\frac{\alpha-1}{2}$-hyperbolic and thus $\delta^* \geq \frac{\alpha}{2}$. Consequently, $\frac{\alpha}{2} \leq \delta^* \leq 368(\alpha + 1)$ and thus, $\delta^* \leq 368(\alpha + 1) \leq 368(2\delta^* + 1)$.

5.2. Graphs with balanced metric triangles. We conclude our paper with another local-to-global condition for hyperbolicity, analogous to Theorem 5. Namely, we replace the topological condition of simple connectivity by a metric condition (this result can be combined with algorithms from previous subsection to estimate the hyperbolicity of a graph).

Given a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, a graph $G$ has $f$-balanced metric triangles if for every metric triangle $uvw$, $d(u, v) \leq \min\{f(d(u, w)), f(d(v, w))\}$. In other words, for any metric triangle $uvw$, if one side of the triangle is “small”, then the other two sides are “relatively small” too. When $f(k) = C \cdot k$ for some constant $C$, we say that the metric triangles of $G$ are linearly balanced. If $G$ is a weakly modular graph, then $G$ has linearly balanced triangles; indeed, all metric triangles of $G$ are equilateral, thus one can choose $f(k) = k$.

**Proposition 12.** Let $G$ be a graph with $f$-balanced metric triangles. If every ball $B_{f(126)+88}(v)$ of $G$ is $\delta$-hyperbolic with $\delta > 0$, then $G$ is $1569\delta$-hyperbolic. Moreover, if $G$ is a weakly modular graph such that every ball $B_{108+5}(v)$ of $G$ is $\delta$-hyperbolic, then $G$ is $(736\delta + 368)$-hyperbolic.
Proof. Consider a graph $G$ with $f$-balanced metric triangles where every ball $B_{f(12\delta)+8\delta}(v)$ is $\delta$-hyperbolic and assume that $G$ is not $1569\delta$-hyperbolic. From Theorem 3 it implies that $G$ is not $(4\alpha, 3\alpha)^*$-dismantlable for $\alpha = 2\delta$. Let $z$ be an arbitrary vertex of $G$ and consider a BFS-order $\preceq$ of $G$ rooted at $z$. Since $G$ is not $(4\alpha, 3\alpha)^*$-dismantlable, there exist $x, y, c$ such that $c \in I(z, x)$, $d(c, x) = 2\alpha$, $d(z, y) \leq d(z, x)$, $d(x, y) \leq 4\alpha$, and $d(c, y) > 3\alpha$.

Let $z'c'y'$ be a quasi-median of the triplet $z, c, y$. Since $z' \in I(z, c) \subset I(z, x)$, $z' \in I(z, y)$, and $d(y, z) \leq d(x, z)$, necessarily $d(y, z') \leq d(x, z')$. Moreover, since $y' \in I(c, y)$, $d(c, y') \leq d(c, y) \leq d(c, x) + d(x, y) \leq 6\alpha$. Since the metric triangles of $G$ are $f$-balanced, $d(c', z') \leq f(d(c', y'))$. Since $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, $d(c, z') = d(c, c') + d(c', z') \leq d(c, c') + f(d(c', y')) \leq f(d(c, c') + d(c', y')) = f(d(c, y')) = f(6\alpha)$. Consequently, $d(y, z') \leq d(x, z') \leq d(x, c) + d(c, z') \leq 2\alpha + f(6\alpha)$. Since $d(x, y) \leq 4\alpha$, for every vertex $u \in I(x, y)$, $d(u, z') \leq \min\{d(u, x) + d(x, z'), d(u, y) + d(y, z')\} \leq 2\alpha + 2\alpha + f(6\alpha) = 4\alpha + f(6\alpha)$. Consequently, the distance between $x$ and $y$ in the graph $G$ and in the ball $B := B_{f(6\alpha)+4\alpha}(z')$ are the same, thus $d_B(x, y) \leq 4\alpha$. Note that, since $c \in I(x, z)$ and $z' \in I(c, z)$, $c \in I(x, z')$. Since $I(x, z') \subset B$ and $I(y, z') \subset B$, $d_B(c, x) = 2\alpha$ and $d_B(y, z') = d(y, z') \leq d(x, z') = d_B(x, z')$. Since $d_B(y, z') \leq d_B(x, z')$, $d_B(c, x) = 2\alpha$, $d_B(y, x) \leq 4\alpha$, and $d_B(c, y) \geq d(c, y) > 3\alpha$, from Proposition 4 applied with $r = 2\alpha$, $B_{f(6\alpha)+4\alpha}(z')$ is not $\frac{\alpha}{2}$-hyperbolic. Thus, since $\alpha = 2\delta$, the ball $B_{f(12\delta)+8\delta}(z')$ is not $\delta$-hyperbolic, which is a contradiction.

Consider now a weakly modular graph $G$ where every ball $B_{10\delta+5}(v)$ is $\delta$-hyperbolic and assume that $G$ is not $(736\delta + 368)$-hyperbolic. From Theorem 4 it implies that $G$ is not $(2\alpha + 2, 2\alpha + 1)^*$-dismantlable for $\alpha = 2\delta$. Thus, there exist $x, y, z, c$ such that $c \in I(z, x)$, $d(c, x) = \alpha + 1$, $d(z, y) \leq d(z, x)$, $d(x, y) \leq 2\alpha + 2$ and $d(c, y) > 2\alpha + 1$. Let $z'c'y'$ be a quasi-median of the triplet $z, c, y$. Using the same arguments as in the previous case and the fact that metric triangles of weakly modular graphs are equilateral, one can show that the ball $B_{20\alpha+5}(z')$ is not $\frac{\alpha}{2}$-hyperbolic. Consequently the ball $B_{10\delta+5}(z')$ is not $\delta$-hyperbolic, which is a contradiction.

\[\square\]

Acknowledgements

We wish to thank the anonymous referees for a careful reading of the manuscript, their useful suggestions, and for pointing out the result of Soto [29].

J.C. was partially supported by ANR project MACARON (ANR-13-JS02-0002). V.C. was partially supported by ANR projects TEOMATRO (ANR-10-BLAN-0207) and GGAA (ANR-10-BLAN-0116).

References

[1] A. Aho, J. Hopcroft, and J. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, 1974.
[2] M. Aigner and M. Fromme, A game of cops and robbers, Discr. Appl. Math. 8 (1984), 1–12.
[3] J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, Notes on word hyperbolic groups, in Group Theory from a Geometrical Viewpoint, ICTP Trieste 1990 (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, 1991, pp. 3–63.
[4] H.-J. Bandelt and V. Chepoi, A Helly theorem in weakly modular spaces, Discr. Math. 160 (1996), 25–39.
[5] H.-J. Bandelt and V. Chepoi, Metric graph theory and geometry: a survey, in: J. E. Goodman, J. Pach, R. Pollack (Eds.), Surveys on Discrete and Computational Geometry. Twenty Years later, Contemp. Math., vol. 453, AMS, Providence, RI, 2008, pp. 49–86.
[6] H.-J. Bandelt, H.M. Mulder, Distance–hereditary graphs, J. Comb. Theory, Ser. B 41 (1986) 182–208.
[7] B. Bowditch, A short proof that a subquadratic isoperimetric inequality implies a linear one, Michigan J. Math. 42 (1995), 103–107.
[8] B.H. Bowditch, Notes on Gromov’s hyperbolicity criterion for path-metric spaces, in Group theory from a geometrical viewpoint, ICTP Trieste 1990 (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, 1991, pp. 64–167.
[9] B. Brešar, J. Chalopin, V. Chepoi, T. Gologranc, and D. Osajda, Bucolic complexes, Advances Math. 243 (2013), 127–167.
[10] M.R. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grundlehren der Mathematischen Wissenschaften vol. 319, Springer-Verlag, Berlin, 1999.
[11] J. Chalopin, V. Chepoi, N. Nisse and Y. Vaxès, Cop and robber games when the robber can hide and ride, SIAM J. Discrete Math. 25 (2011), 333–359.
[12] I. Chatterji, G. Niblo, A characterization of hyperbolic spaces, Groups, Geometry and Dynamics 1 (2007), 281–299.
[13] V. Chepoi, Classification of graphs by means of metric triangles (in Russian), Metody Diskret. Analiz., 49 (1989), 75–93.
[14] V. Chepoi, F. Dragan, B. Estellon, M. Habib, and Y. Vaxès, Diameters, centers, and approximating trees of \(\delta\)-hyperbolic geodesic spaces and graphs, Symposium on Computational Geometry, SoCG’2008, pp. 59–68.
[15] V. Chepoi and B. Estellon, Packing and covering \(\delta\)-hyperbolic spaces by balls, APPROX-RANDOM 2007, pp. 59–73.
[16] N. Cohen, D. Coudert, and A. Lancin, Exact and approximate algorithms for computing the hyperbolicity of large-scale graphs, hal-00735481, 2012.
[17] T. Delzant, M. Gromov, Courbure mésoscopique et théorie de la toute petite simplification, J. Topol. 1 (2008), 804–836.
[18] R. Duan, Approximation algorithms for the Gromov hyperbolicity of discrete metric spaces, LATIN 2014, pp. 285–293.
[19] F.V. Fomin, P. Golovach, J. Kratochvíl, N. Nisse, and K. Suchan, Pursuing a fast robber on a graph, Theor. Comput. Sci., 411 (2010), 1167–1181.
[20] F.V. Fomin and D. Thilikos, An annotated bibliography on guaranteed graph searching, Theor. Comput. Sci. 399 (2008), 236–245.
[21] H. Fournier, A. Ismail, and A. Vigneron, Computing the Gromov hyperbolicity of a discrete metric space, arXiv:1210.3323v3, 2013.
[22] M. Gromov, Hyperbolic Groups, Essays in group theory (S.M. Gersten ed.), MSRI Series, 8 (1987) pp. 75–263.
[23] M.F. Hagen, Weak hyperbolicity of cube complexes and quasi-arboREAL groups, J. Topology 7(2) (2014), 385–418.
[24] R.J. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, Discr. Math. 43 (1983), 235–239.
[25] P. Papasoglu, Strongly geodesically automatic groups are hyperbolic, Inventiones Math. 121 (1995) 323–334.
[26] P. Papasoglu, An algorithm detecting hyperbolicity, Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994) 25 (1996), 193–200.
[27] A. Quilliot, Problèmes de jeux, de point fixe, de connectivité et de représentation sur des graphes, des ensembles ordonnés et des hypergraphes, Thèse de doctorat d’état, Université de Paris VI, 1983.
[28] R. Seidel, On the all-pairs-shortest-path problem in unweighted undirected graphs, J. Comput. Syst. Sci. 51 (1995), 400–403.
[29] M. Soto, Quelques propriétés topologiques des graphes et applications à Internet et aux réseaux, PhD Thesis, LIAFA, Université Paris Diderot, 2011.
[30] A.Yu. Olshanskii, Hyperbolicity of groups with subquadratic isoperimetric inequality, International Journal of Algebra and Computation 1 (1991), 281–289.