A Tutorial for Weighted Bipolar Argumentation with Continuous Dynamical Systems and the Java Library Attractor

Nico Potyka
Institute of Cognitive Science, University of Osnabrück, Germany

Abstract

Weighted bipolar argumentation frameworks allow modeling decision problems and online discussions by defining arguments and their relationships. The strength of arguments can be computed based on an initial weight and the strength of attacking and supporting arguments. While previous approaches assumed an acyclic argumentation graph and successively set arguments’ strength based on the strength of their parents, recently continuous dynamical systems have been proposed as an alternative. Continuous models update arguments’ strength simultaneously and continuously. While there are currently no analytical guarantees for convergence in general graphs, experiments show that continuous models can converge quickly in large cyclic graphs with thousands of arguments. Here, we focus on the high-level ideas of this approach and explain key results and applications. We also introduce Attractor, a Java library that can be used to solve weighted bipolar argumentation problems. Attractor contains implementations of several discrete and continuous models and numerical algorithms to compute solutions. It also provides base classes that can be used to implement, to evaluate and to compare continuous models easily.

1 Introduction

Abstract argumentation (Dung 1995) studies the acceptability of arguments based purely on their relationships and abstracted from their content. The basic framework has a two-valued semantics and allows only defining arguments and an attack relation between them. This basic setting has been extended in different directions. For example, bipolar argumentation frameworks (Amgoud, Cayrol, and Lagasque-Schiex 2004; Oren and Norman 2008; Cayrol and Lagasque-Schiex 2013; Polberg and Oren 2014) take account of the fact that arguments cannot only attack each other and add a support relation. A survey of different approaches can be found in Cohen et al. (2014). The classical two-valued semantics that distinguishes only between acceptance and rejection of arguments has been extended in various ways. Examples include probabilistic semantics (Thimm 2012; Hunter 2013; Hunter and Potyka 2017) and ranking semantics that can be based on fixed point equations (Besnard and Hunter 2001; Leite and Martins 2011; Correia, Cruz, and Leite 2014; Barringer, Gabbay, and Woods 2012) or the graph structure (Cayrol and Lagasque-Schiex 2005). Other recent extensions include recursive attacks on attacks (Baroni et al. 2011) and the temporal availability of arguments (Budan et al. 2015).

Our focus here is on weighted bipolar argumentation frameworks that allow defining attack and support relationships and an initial weight for arguments (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017; Mossakowski and Neuhaus 2018). A strength value is computed for every argument based on its initial weight and the strength of its attackers and supporters. Examples for computational models include the QuAD algorithm from (Baroni et al. 2015) that was designed to evaluate the strength of answers in decision-support systems. Soon after, the DF-QuAD algorithm (Rago et al. 2016) was proposed as an alternative, which avoids discontinuous behaviour of the QuAD algorithm that can be undesirable in some applications. Some additional interesting guarantees are given by the Euler-based semantics introduced in (Amgoud and Ben-Naim 2017). The QuAD algorithms mainly lack these properties due to the fact that their aggregated strength values saturate. That is, as soon, as an attacker (supporter) with strength 1 exists, the other attackers (supporters) become irrelevant for the aggregated value. However, while the Euler-based semantics avoids these problems, it has some other drawbacks that may be undesirable. Arguments initialized with strength 0 or 1 remain necessarily unchanged under Euler-based semantics and the impact of attacks and supports is non-symmetrical. The quadratic energy model introduced in (Potyka 2018a) avoids these problems. In particular, while the previous approaches are discrete in nature, the quadratic energy model is a continuous model. Discrete models often assume that the argumentation graph is acyclic, so that the strength of arguments can be computed successively according to a topological ordering. Continuous models change arguments’ strength continuously and simultaneously. They can be naturally applied to cyclic graphs, but convergence in general remains an open question.

More formally, continuous models correspond to n-dimensional functions $f(t)$ mapping continuous points in time to n-dimensional state vectors whose $i$-th component represents the strength of the $i$-th argument at time $t$. The initial state $f(0)$ is given by the initial weights and a system of differential equations describes how the strength values evolve as time progresses. This approach, in particu-
lar, allows plotting the evolution of strength values in order to better understand the final strength values of the limit \( s = \lim_{t \to \infty} f(t) \) or to inspect the convergence behaviour visually. Even though convergence in general graphs is an open question, so far no diverging example has been found and tests with large randomly generated bipolar graphs show that the strength values converge in many cases.

In this tutorial paper, we will review some ideas and results from (Potyka 2018a) with a stronger focus on the high-level ideas. The goal of this paper is, in particular, to demonstrate how results can be applied to

1. solve weighted bipolar argumentation problems with continuous models,
2. transform existing discrete models to well-defined continuous models.

We also introduce Attractor, a Java library that provides basic implementations of the ideas discussed here and in (Potyka 2018a). The main goals of Attractor are to

1. simplify applying continuous models to weighted bipolar argumentation problems,
2. to improve reproducibility of the results in (Potyka 2018a),
3. to simplify implementing new continuous models and
4. to simplify comparing different models.

These goals are achieved by providing

1. implementations of some continuous models and the random generator introduced in (Potyka 2018a),
2. implementations of base classes that solve initial value problems with basic and advanced methods,
3. implementations of utility classes for benchmarking, plotting and working with benchmark files.

We will start with an introduction to dynamical systems and the quadratic energy model from (Potyka 2018a) in Section 2. In Section 3 we will discuss the problem of computing solutions and explain some important algorithms. Section 4 contains some additional information on convergence guarantees and open questions and discusses the computational complexity of the continuous approach. In Section 5 we explain how discrete models can be transformed to continuous models. Finally, Section 6 explains how the previously discussed ideas can be put into practice using Attractor.

## 2 Dynamical Systems and The Quadratic Energy Model

Roughly speaking, a dynamical system describes the evolution of a natural or technical system over time. If time is discretized, the system is called discrete, otherwise it is called continuous. Formally, we describe the state of the system at time \( t \) by a function \( s(t) \). The state is usually given as a real vector and a system of differential equations describes how the system evolves dependent on the current state.

In the context of weighted argumentation, a state vector contains one component for every argument that can take a strength value between 0 and 1. The strength of arguments should evolve based on the initial weight, and the current strength of attackers and supporters. Before describing this approach in more detail, we define weighted bipolar argumentation graphs (BAGs) as introduced in (Amgoud and Ben-Naim 2017).

**Definition 1** (BAG). A BAG is a quadruple \( A = (\mathcal{A}, w, \mathcal{R}, \mathcal{S}) \), where \( \mathcal{A} \) is a finite set of arguments, \( w : \mathcal{A} \to [0, 1] \) is a weight function and \( \mathcal{R} \) and \( \mathcal{S} \) are binary relations on \( \mathcal{A} \) called attack and support.

In order to simplify notation, we assume that the \( i \)-th argument is called \( i \), that \( \mathcal{A} = \{1, \ldots, n\} \). The weight function \( w \) defines an initial strength value between 0 and 1 for each argument. If \( aRb \) (\( a \) attacks \( b \)), we say that \( a \) attacks (supports) \( b \). We let \( \text{Att}_i = \{ h \in \mathcal{A} \mid hRi \} \) denote \( i \)'s attackers and let \( \text{Sup}_i = \{ h \in \mathcal{A} \mid hSi \} \) denote \( i \)'s supporters.

We can now describe our dynamical system more precisely. A state is a vector \( s \in \mathbb{R}^n \) whose \( i \)-th component \( s_i \) is the strength of argument \( i \). Our state model is a function \( s : \mathbb{R}_0^+ \to \mathbb{R}^n \) that maps non-negative time points to strength vectors. That is, \( s_i(t) \) is the strength of argument \( i \) at time \( t \). Initially, the strength of an argument should correspond to its initial weight, that is, we let \( s_i(0) = w(i) \). The evolution of the system should be based on three considerations:

1. Strength values are attracted by their initial weight.
2. Attackers force the strength value towards 0 proportionally to their strength.
3. Supporters force the strength value towards 1 proportionally to their strength.

Intuitively, there are three forces acting on the strength of each argument as illustrated in Figure 1. In this physical metaphor, attackers push the strength towards 0, while the supporters push the strength towards 1. Gravity pulls the strength back to its initial weight.

This intuition can be modeled by a system of differential equations. If this system is designed carefully, it uniquely defines a model \( s_A : \mathbb{R}_0^+ \to \mathbb{R}^n \) for every BAG \( A \). We are then interested in the long-term behaviour of the model. Intuitively, we expect the forces to counteract until the strength reaches an equilibrium state where all forces are in balance. More formally, if the model converges to a state \( s^* = \lim_{t \to \infty} s(t) \) as time progresses, we call \( s^* \) the equilibrium state reached by the model.

The quadratic energy model introduced in (Potyka 2018a) is defined as follows.
The quadratic energy model is a model for modeling the impact of arguments in a weighted argumentation framework. It is defined as:

\[ E_j = \sum_{i \in \text{Sup}_j} s_i - \sum_{i \in \text{Att}_j} s_i \]

and for all \( x \in \mathbb{R} \), the impact of \( x \) is defined as:

\[ h(x) = \frac{\max\{x, 0\}^2}{1 + \max\{x, 0\}^2}. \]

The quadratic energy model \( \sigma^A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) for \( A \) is the unique solution of the system of differential equations:

\[ \frac{ds_j}{dt} = w(j) - s_j + (1 - w(j)) \cdot h(E_j) - w(j) \cdot h(-E_j), \quad j \in A \]

with initial conditions \( s_j(0) = w(j) \).

Intuitively, \( \frac{ds_j}{dt} \) describes the momentary change of \( s_j \) at time \( t \). If \( \frac{ds_j}{dt} > 0 \), the strength will increase, if \( \frac{ds_j}{dt} < 0 \), the strength will decrease and if \( \frac{ds_j}{dt} = 0 \) the strength will remain in its current state. The definition uses two auxiliary functions. The energy \( E_j \) at argument \( j \) aggregates the strength of attackers and supporters in a linear fashion. This notion of energy has been first defined for the Euler-based restricted semantics in (Amgoud and Ben-Naim 2017). The energy is then fed into the impact function \( h \) that is 0 for all negative arguments and then increases, but is bounded from above by 1. The definition of \( \frac{ds_j}{dt} \) can be divided into three parts that correspond to our three considerations above.

1. The difference \( w(j) - s_j \) draws the strength of an argument to its initial weight. Notice that if \( w(j) > s_j \) \((w(j) < s_j)\), this term is positive (negative) and tends to increase (decrease) \( j \)'s strength.

2. The term \(-w(j) \cdot h(-E_j)\) moves the strength towards 0 if the negative force of attackers is stronger than the positive force of supporters.

3. Dually, the term \((1 - w(j)) \cdot h(E_j)\) moves the strength towards 1 if the positive force of supporters is stronger than the negative force of attackers.

As shown in (Potyka 2018a), the quadratic energy model is well-defined. That is, the system has a unique solution \( \sigma^A \) by means of which we can simulate the evolution of strength values over time. If \( \sigma^A \) reaches an equilibrium state, the final strength values at every argument are completely determined by the energy at this state as explained in the following proposition.

**Proposition 1 (Strength in Equilibrium (Potyka 2018a)).** If the limit \( s^* = \lim_{t \to \infty} \sigma^A(t) \) exists, then we have

\[ s_j^* = \begin{cases} w(j) & \text{if } E_j = 0 \\ w(j) + (1 - w(j)) \cdot h(E_j) & \text{if } E_j > 0 \\ w(j) - w(j) \cdot h(-E_j) & \text{if } E_j < 0 \end{cases} \]

Equation 2 shows, in particular, that the strength will be the initial weight if the energy is 0 and otherwise will go to 1 (0) as the energy goes to \( -\infty \) (\( +\infty \)). The quadratic energy model satisfies a collection of postulates proposed in (Amgoud and Ben-Naim 2017). These postulates range from very general properties like Anonymity (strength values do not depend on the identity of the argument) and Independence (arguments are independent of disconnected arguments) to properties tailor-made for weighted bipolar argumentation frameworks that guarantee that attacks, supports and initial weights have the intended meaning. An interesting property that distinguishes weighted argumentation frameworks from some other numerical argumentation frameworks is Directionality, which guarantees that arguments influence other arguments only in direction of the edges. This property distinguishes weighted argumentation approaches from probabilistic approaches that usually cause influence in both directions due to the nature of probability theory. Please see (Amgoud and Ben-Naim 2017) and (Potyka 2018a) for a more thorough discussion of the properties.

In the following example, we illustrate the quadratic energy model by means of a small decision problem.

**Example 1.** Suppose we want to decide whether to buy or to sell stocks of an electronics company. We base our decision on information given by different experts:

1. The development of the new phone was too expensive. Therefore, the company has to cut down research and development and will not stay competitive in the future.

2. The company’s new phone is innovative and will increase the company’s profit considerably.

3. The price of the new phone is too high and there will not be too many sales.

4. There is a large number of preorders of the new phone already.

5. The company’s investment in research and development is far beyond competitors’ investment and the company is likely to become the market leader in the future.

Initially, we do not have a preference for buying or selling stocks and set both initial weights to 0.5. In order to weigh the expert opinions, we could use historical information about how frequently the expert’s assessment was true or false. If there were true and false assessments, we could set the initial weight to \( \frac{t}{t + f} \). In order to incorporate arguments of new experts and to set an initial bias for the weight, we could add pseudocounts to \( t \) and \( f \). That is, we set the initial weight to \( \frac{t + t'}{t + f + f'} \), where \( t' \) and \( f' \) are pseudocounts that encode an initial bias. If \( t' > f' \), our initial weight is 0.5 when no historical information is available. Setting \( t' > f' \) \((t' < f')\), the weight will initially be greater (smaller) than 0.5. The larger \( t' + f' \) is, the more data is needed to deviate from the bias. For instance, if \( t = 5 \), \( f = 2 \), then we have the weight \( \frac{5 + 1}{5 + 1 + 2 + 1} \approx 0.66 \). For pseudocounts \( t' = f' = 1 \), whereas we have \( \frac{5 + 10}{5 + 10 + 2 + 10} \approx 0.55 \). Figure 2 shows a BAG for our problem along with initial weights and the final strength values under the quadratic energy model. One advantage of a continuous model is that we can illustrate the evolution of...
Even though we know that the quadratic energy model (because it is also drawn to its initial weight 0) creasing. It actually starts decreasing slightly before that intersect) and the strength of the selling-argument starts decreasing as strong as argument 1. However, argument 0 becomes gradually weaker due to its attackers. At about time 0.5, the attacking buy-argument becomes as strong as argument 1 (the blue and yellow graphs intersect) and the strength of the selling-argument starts decreasing. It actually starts decreasing slightly before that because it is also drawn to its initial weight 0.5.

3 Numerical Computation of Solutions

Even though we know that the quadratic energy model $\sigma^A$ is well-defined for every BAG, we usually cannot solve for $\sigma^A$ analytically. This is the case for most nonlinear dynamical systems, but is not a heavy drawback in practice since the solution can be approximated numerically.

The intuitive idea is best illustrated by Euler’s method, which is a simple algorithm to perform this approximation. Recall that $\frac{ds}{dt}(t)$ describes the momentary change of $s_j$ at time $t$. Hence, if we know $s_j(t)$, we can approximate $s_j(t + \delta)$ by letting $s_j(t + \delta) = s_j(t) + \delta \cdot \frac{ds}{dt}(t)$. Formally, this approach is justified by the fact that differentiable functions can be approximated locally by the derivative. In particular, as we let the step size $\delta$ go to 0, the approximation error $|s_j(t + \delta) - s_j(t + \delta)|$ goes to 0 as well.

In the context of weighted bipolar argumentation, we can initialize all strength values with the initial weights. This gives us $s(0)$ and we can compute $\frac{ds}{dt}(0)$ according to Definition 2. We can then approximate $\sigma^A(\delta)$ by letting $\hat{s}(\delta) = s(0) + \delta \cdot \frac{ds}{dt}(0)$. Given our approximation $\hat{s}(\delta)$, we can compute $\frac{d\hat{s}}{dt}(\delta)$ and $\hat{s}(2 \cdot \delta) \approx \hat{s}(\delta) + \delta \cdot \frac{d\hat{s}}{dt}(\delta)$. Continuing in this way, we can compute $\hat{s}(n \cdot \delta)$ for arbitrary $n \in \mathbb{N}$. Hopefully, the strength values will eventually converge. This is the case if and only if the derivative $\frac{d\hat{s}}{dt}(t)$ goes to 0. Therefore, a simple termination condition is to demand that $\frac{d\hat{s}}{dt}(t) \leq \epsilon$ for all $t \in \mathcal{A}$ and some small $\epsilon > 0$. Formally, this corresponds to demanding that the maximum norm of $\frac{d\hat{s}}{dt}(t)$ is smaller than $\epsilon$, denoted as $\|\frac{d\hat{s}}{dt}\|_\infty \leq \epsilon$.

The complete algorithm is shown in Figure 4.

While Euler’s method is easy to understand and to implement, it does not give very strong approximation guarantees. A better alternative that is frequently used is the family of Runge-Kutta methods. The most prominent member is the classical Runge-Kutta method RK4 that guarantees an approximation error in the order of $O(\delta^4)$. In practice, this means that if we halve the step size $\delta$ (double the number of iterations), we usually decrease the approximation error by a factor of 16 (Polyanin and Zaitsev 2017). Since the derivatives of the quadratic energy model can never become larger than 1, using RK4 with constant step size 0.01 should be safe. If we want to make sure that the step size is sufficiently small, we can run the algorithm until termination and then repeat with a smaller step size like 0.005 and check that the final values remain unchanged up to the desired accuracy.

4 Convergence and Complexity

The quadratic energy model $\sigma^A(t)$ uniquely defines the evolution of strength values for every BAG $A$ over time. Our hope is that $\sigma^A(t)$ converges to an equilibrium state as $t \to \infty$. This allows us to define strength values by means of
the equilibrium $s^* = \lim_{t \to \infty} \sigma^A(t)$. Unfortunately, convergence of $\sigma^A(t)$ for general BAGs with cycles is currently an open question. One can show convergence for some special cases. For example, if cycles contain only support relations, the strength values must be monotonically increasing. They are also bounded from above by 1 due to the nature of the differential equations, and so they must eventually converge. For cycles with only attacks, things are already less straightforward because the attraction force of the initial weight may become stronger than the attacking force as the attackers become weaker. This may let the strength oscillate between the initial weight and the first lower peak reached. However, since arguments’ strength can never exceed the initial weight if there is no support, the amplitude of the oscillations must eventually go to 0 and the strength values will converge.

If cycles contain both support and attack relations, the strength values may oscillate more radically. In order to illustrate this, Figure 5 shows a BAG from a family that we call Cycle(k). Each member contains one argument $A$ with initial weight 1 that supports $k$ arguments $B_i$ with weight 0. Each $B_i$ in turn supports the same $k$ arguments $C_i$ that have initial weight 0 as well. Finally, each $C_i$ attacks $A$. Figure 5 shows Cycle(3) and figure 6 shows the long-term behaviour of the quadratic energy model for Cycle(3) at the top and for Cycle(10) at the bottom. As we may expect, the oscillations for Cycle(k) take more time as we increase $k$. However, the amplitude decreases and the strength values eventually converge. It is currently unclear if there exist BAGs where the strength values oscillate for all time. However, experiments in (Potyka 2018a) with 3,000 randomly generated BAGs with thousands of nodes and ten thousands of edges demonstrate that the quadratic energy model converges for many cyclic BAGs.

How can we deal with potential divergence in practice? Since equilibrium states can usually be computed in seconds, it is pragmatic to set a time limit for the quadratic energy model. Say, if the model diverges after 30 seconds, the algorithm stops. Arguments whose strength value has not converged yet can then be detected automatically because we must have $|\frac{ds_i}{dt}| > \epsilon$ at such an argument. The evolution of the strength value can then be plotted like in Figure 3 in order to see whether the strength value diverges, oscillates with decreasing amplitude or just converges slowly for other reasons. In particular, even if the strength diverges, it may oscillate between meaningful bounds. For example, if the strength oscillates between 0.8 and 0.9, we could still infer that the argument is rather strong. Of course, this analysis can also be performed automatically by just storing lower and upper bounds and monitoring the derivatives (oscilla-

![Figure 5: Cycle(3) graph.](image)

![Figure 6: Long-term behaviour of quadratic energy model for Cycle(3) (top) and Cycle(10) (bottom).](image)
library $s^* = \lim_{t \to \infty} \sigma^A(t)$ can be computed in linear time by the following procedure:

1. Compute a topological ordering of the arguments.
2. Pick the next argument $i$ in the order and set
   \[ s_i = w(i) + (1 - w(i)) \cdot h(E_i) - w(i) \cdot h(-E_i), \]
   where $E_i$ is the energy at $i$.
3. Repeat step 2 until all strength values have been computed.

**Example 2.** We illustrate Proposition 2 with an e-democracy problem from (Rago et al. 2016). The question is how to spend a portion of a council’s budget. The arguments are divided into decision arguments (prefix A), pro arguments (prefix P) and contra arguments.:

- **A1:** Build a new cycle path.
- **A2:** Repair current infrastructure.
- **P1:** Cyclists complain of dangerous roads.
- **P2:** A path would enhance the council’s green image.
- **C1:** Significant disruptions to traffic would occur.
- **C2:** Environmentalists are a fraction of the population.
- **C3:** Recent policies already enhance this green image.
- **C4:** Potholes have caused several accidents recently.
- **P3:** Environmentalists are a fraction of the population.

**Figure 7** shows the initial weights and final strength values under the quadratic energy model. Since the authors in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017) considered only subgraphs of this BAG, I defined additional initial weights for C2, C3, C4. One topological ordering of the arguments is P1, P3, C1, C2, C3, C4, P2, A1, A2. Since only P2, A1, A2 have parents, the energy at all other arguments is 0 for all time and their final strength is just the initial weight. For P2, the energy is then $-0.2 - 0.6 - 0.5 = -1.3$ and the final strength is $0.5 - 0.5 \cdot h(-(1.3)) \approx 0.186$. The energy at A1 is approximately $0.7 + 0.186 \approx 0.886$ and the final strength is $0.5 + 0.5 \cdot h(0.886) \approx 0.719$. Finally, the energy at A2 is $0.9 - 0.2 = 0.7$ and the final strength is $0.5 - 0.5 \cdot h(0.7) \approx 0.664$. The continuous evolution of the quadratic energy model is shown in Figure 8. Note that it does indeed converge to the values that we computed.

Even though we currently cannot give convergence guarantees for cyclic BAGs, the quadratic energy model uniquely defines a strength value for every point $t$. In particular, we can analyze the runtime for evaluating the quadratic energy model from time 0 to time $T$. Using step size $\delta$, this can be done in time $O(\|A\| + |R| + |S|)$. The cost is basically composed of the factor $O((|A| + |R| + |S|))$ for the cost of evaluating the differential equations and the factor $O(\frac{1}{\delta})$ for the number of evaluations. While Euler’s method needs only a single computation of the differential equations at each time step, more sophisticated methods like RK4 evaluate the differential equations at several points in order to improve the approximation. While this increases the runtime for fixed $\delta$, these methods can usually work with significantly larger step sizes than Euler’s method and are therefore more efficient. In our implementation, we let $T$ grow until $\|\frac{dx}{dt}\|_\infty < 10^{-4}$. It is reasonable to assume that the point of convergence $T^*$ depends on the size and complexity of cycles. Experiments in (Potyka 2018a) indicate that the overall runtime is bounded from above quadratically by the size of the BAG.

**5 From Discrete to Continuous Models**

Proposition 2 basically tells us that if the BAG is acyclic, we can transform the continuous quadratic energy model to a discrete model similar to the ones considered in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017). This is interesting from a computational perspective because it gives us a linear runtime guarantee for acyclic BAGs.

On the other hand, continuous models are computationally interesting because they can improve stability in cyclic BAGs as we explain at the end of this section. Therefore, it is natural to ask under what conditions we can transform discrete models like in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017) to well-defined continuous models. A simple sufficient criterion along with some guarantees is given in the following result from (Potyka 2018a).

**Proposition 3** (Continuizing Iterative Schemes). Consider an iterative scheme $\mathcal{I}$ that defines the strength values for acyclic BAGs by letting

\[ s_i = f_\mathcal{I}(w(i), \{s_j \mid j \in Att_i\}, \{s_j \mid j \in Sup_i\}), \]

we can analyze the runtime for evaluating the quadratic energy model from time 0 to time $T$. Using step size $\delta$, this can be done in time $O(\|A\| + |R| + |S|)$. The cost is basically composed of the factor $O((|A| + |R| + |S|))$ for the cost of evaluating the differential equations and the factor $O(\frac{1}{\delta})$ for the number of evaluations. While Euler’s method needs only a single computation of the differential equations at each time step, more sophisticated methods like RK4 evaluate the differential equations at several points in order to improve the approximation. While this increases the runtime for fixed $\delta$, these methods can usually work with significantly larger step sizes than Euler’s method and are therefore more efficient. In our implementation, we let $T$ grow until $\|\frac{dx}{dt}\|_\infty < 10^{-4}$. It is reasonable to assume that the point of convergence $T^*$ depends on the size and complexity of cycles. Experiments in (Potyka 2018a) indicate that the overall runtime is bounded from above quadratically by the size of the BAG.

**5 From Discrete to Continuous Models**

Proposition 2 basically tells us that if the BAG is acyclic, we can transform the continuous quadratic energy model to a discrete model similar to the ones considered in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017). This is interesting from a computational perspective because it gives us a linear runtime guarantee for acyclic BAGs.

On the other hand, continuous models are computationally interesting because they can improve stability in cyclic BAGs as we explain at the end of this section. Therefore, it is natural to ask under what conditions we can transform discrete models like in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017) to well-defined continuous models. A simple sufficient criterion along with some guarantees is given in the following result from (Potyka 2018a).

**Proposition 3** (Continuizing Iterative Schemes). Consider an iterative scheme $\mathcal{I}$ that defines the strength values for acyclic BAGs by letting

\[ s_i = f_\mathcal{I}(w(i), \{s_j \mid j \in Att_i\}, \{s_j \mid j \in Sup_i\}), \]

we can analyze the runtime for evaluating the quadratic energy model from time 0 to time $T$. Using step size $\delta$, this can be done in time $O(\|A\| + |R| + |S|)$. The cost is basically composed of the factor $O((|A| + |R| + |S|))$ for the cost of evaluating the differential equations and the factor $O(\frac{1}{\delta})$ for the number of evaluations. While Euler’s method needs only a single computation of the differential equations at each time step, more sophisticated methods like RK4 evaluate the differential equations at several points in order to improve the approximation. While this increases the runtime for fixed $\delta$, these methods can usually work with significantly larger step sizes than Euler’s method and are therefore more efficient. In our implementation, we let $T$ grow until $\|\frac{dx}{dt}\|_\infty < 10^{-4}$. It is reasonable to assume that the point of convergence $T^*$ depends on the size and complexity of cycles. Experiments in (Potyka 2018a) indicate that the overall runtime is bounded from above quadratically by the size of the BAG.

**5 From Discrete to Continuous Models**

Proposition 2 basically tells us that if the BAG is acyclic, we can transform the continuous quadratic energy model to a discrete model similar to the ones considered in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017). This is interesting from a computational perspective because it gives us a linear runtime guarantee for acyclic BAGs.

On the other hand, continuous models are computationally interesting because they can improve stability in cyclic BAGs as we explain at the end of this section. Therefore, it is natural to ask under what conditions we can transform discrete models like in (Baroni et al. 2015; Rago et al. 2016; Amgoud and Ben-Naim 2017) to well-defined continuous models. A simple sufficient criterion along with some guarantees is given in the following result from (Potyka 2018a).

**Proposition 3** (Continuizing Iterative Schemes). Consider an iterative scheme $\mathcal{I}$ that defines the strength values for acyclic BAGs by letting

\[ s_i = f_\mathcal{I}(w(i), \{s_j \mid j \in Att_i\}, \{s_j \mid j \in Sup_i\}), \]

we can analyze the runtime for evaluating the quadratic energy model from time 0 to time $T$. Using step size $\delta$, this can be done in time $O(\|A\| + |R| + |S|)$. The cost is basically composed of the factor $O((|A| + |R| + |S|))$ for the cost of evaluating the differential equations and the factor $O(\frac{1}{\delta})$ for the number of evaluations. While Euler’s method needs only a single computation of the differential equations at each time step, more sophisticated methods like RK4 evaluate the differential equations at several points in order to improve the approximation. While this increases the runtime for fixed $\delta$, these methods can usually work with significantly larger step sizes than Euler’s method and are therefore more efficient. In our implementation, we let $T$ grow until $\|\frac{dx}{dt}\|_\infty < 10^{-4}$. It is reasonable to assume that the point of convergence $T^*$ depends on the size and complexity of cycles. Experiments in (Potyka 2018a) indicate that the overall runtime is bounded from above quadratically by the size of the BAG.
where \( f_{\mathcal{T}} \) is a function that depends on the initial weight and the strength of attackers and supporters and the strength values \( s_i \) are computed in topological order.

1. If \( f_{\mathcal{T}} \) is continuously differentiable with respect to all involved strength values, then for all BAGs \( \mathcal{A} \), the system
   \[
   \frac{ds_i}{dt} = f_{\mathcal{T}}(w(i), \{s_j \mid j \in \text{Att}_i\}, \{s_j \mid j \in \text{Sup}_i\}) - s_i
   \]
   with initial conditions \( s_i(0) = w(i) \) for \( i = 1, \ldots, n \) has a unique solution \( \sigma^G_{\mathcal{T}} : \mathbb{R}^n_+ \to \mathbb{R}^n \).

2. \( \sigma^G_{\mathcal{T}} \) reaches an equilibrium state \( s^* = \lim_{t \to \infty} \sigma^G_{\mathcal{T}}(t) \), then
   \[
   s^*_i = f_{\mathcal{T}}(w(i), \{s^*_j \mid j \in \text{Att}_i\}, \{s^*_j \mid j \in \text{Sup}_i\}).
   \]

3. \( \sigma^G_{\mathcal{T}} \) reaches an equilibrium state whenever \( \mathcal{A} \) is acyclic.

Item 1 explains how to transform the definition of a discrete iteration scheme to a system of differential equations and gives a sufficient condition under which the system has a unique solution. We can then use this solution similar to the quadratic energy model as we will illustrate soon.

Item 2 guarantees that if the model reaches an equilibrium state, this state is a fixed point of the discrete update function \( f_{\mathcal{T}} \). This implies, in particular, that if the BAG is acyclic, then the continuized model agrees with the discrete model.

Item 3 states that the continuized model is again guaranteed to convergence for acyclic graphs.

### Continuizing Discrete Models

Let us now illustrate Proposition 3 by means of the Euler-based restricted semantics that was introduced in [Amgoud and Ben-Naim 2017]. As explained before, the Euler-based restricted semantics used the energy \( E_i = \sum_{j \in \text{Sup}_i} s_j - \sum_{j \in \text{Att}_i} s_j \) before. Given an acyclic BAG, the weights for every argument are then set in topological order by letting

\[
\begin{align*}
   s_i &= 1 - \frac{1 - w(i)^2}{1 + w(i) \cdot \exp(E_i)}.
\end{align*}
\]

That is, \( f_{\mathcal{T}}(w(i), \{s_j \mid j \in \text{Att}_i\}, \{s_j \mid j \in \text{Sup}_i\}) = 1 - \frac{1 - w(i)^2}{1 + w(i) \cdot \exp(E_i)} \). In order to apply item 1, we have to check that \( f_{\mathcal{T}} \) is continuously differentiable with respect to all \( s_i \). Note that \( E_i \) is a linear function of the strength values and therefore continuously differentiable. The exponential function \( \exp \) is continuously differentiable as well. Therefore, \( f_{\mathcal{T}} \) is defined by combining constant and continuously differentiable functions and is therefore itself continuously differentiable (notice, in particular, that the denominator in the fraction in \( f_{\mathcal{T}} \) is always greater than 1 because \( \exp \) is a positive function). Hence, we can apply Proposition 3.

Item 1 tells us that we obtain the differential equations for \( s_i \) by subtracting \( s_i \) from \( f_{\mathcal{T}} \). Hence, the system that defines the continuous Euler-based semantics is

\[
\begin{align*}
   \frac{ds_i}{dt} &= 1 - \frac{1 - w(i)^2}{1 + w(i) \cdot \exp(E_i)} - s_i, \quad i \in \mathcal{A}.
\end{align*}
\]

We can now approximate the solution with Euler’s method as described in Algorithm 4 or with faster methods like RK4.

Proposition 3 is not always applicable. For example, the update formula for the DF-QuAD algorithm from [Rago et al. 2016] is not continuously differentiable. The formula is also based on some auxiliary functions. We slightly change the notation in order to make the presentation more homogeneous. We define the geometric energy at argument \( j \) as

\[
GE_j = \prod_{i \in \text{Att}_j} (1 - s_i) - \prod_{i \in \text{Sup}_j} (1 - s_i),
\]

where we use the convention that the empty product equals 1. Given an acyclic BAG, the DF-QuAD algorithm sets the weights for every argument in topological order by letting

\[
s_i = w(i) + w(i) \cdot \min\{GE_i, 0\} + (1 - w(i)) \cdot \max\{GE_i, 0\}.
\]

We have \( f_{\mathcal{T}}(w(i), \{s_j \mid j \in \text{Att}_i\}, \{s_j \mid j \in \text{Sup}_i\}) = w(i) + w(i) \cdot \min\{GE_i, 0\} + (1 - w(i)) \cdot \max\{GE_i, 0\} \).

The derivative of \( f_{\mathcal{T}} \) is discontinuous at 0-energy states. Hence, Proposition 3 is not applicable. However, the conditions in Proposition 3 are sufficient and not necessary. Indeed, one can show in another way that the system for the continuous DF-Quad algorithm has a unique solution ([Potsyka 2018b]).

If Proposition 3 is not applicable, we may also modify the update formula in order to guarantee continuous differentiability. For the DF-Quad algorithm, we could square the strength values in the geometric energy. When replacing the geometric energy with the squared geometric energy

\[
SGE_j = \prod_{i \in \text{Att}_j} (1 - s_i^2) - \prod_{i \in \text{Sup}_j} (1 - s_i^2),
\]

Proposition 3 is applicable. When using the squared geometric energy, an argument with strength 1 will have the same influence as before, but as the strength gets closer to 0 the influence will get gradually weaker. Implementations of all continuizations can be found in [Attractor] that we describe in the final section of this article.

### Continuation and Convergence

To get an intuition for why continuizing a discrete model may improve the convergence behaviour in cyclic graphs, it is instructive to look at Euler’s method again. Suppose we apply Euler’s method with (rather large) step size \( \delta = 1 \) to the system given in Proposition 3. Then we update each strength value \( s_i \) with \( s_i \leftarrow s_i + 1 \cdot \frac{ds_i}{dt} \) in every iteration. Hence, the update is just

\[
s_i + 1 \cdot \left( f_{\mathcal{T}}(w(i), \{s_j \mid j \in \text{Att}_i\}, \{s_j \mid j \in \text{Sup}_i\}) - s_i \right)
\]

\[
= f_{\mathcal{T}}(w(i), \{s_j \mid j \in \text{Att}_i\}, \{s_j \mid j \in \text{Sup}_i\}).
\]

That is, we just update all strength values simultaneously with respect to the iterative update formula \( f_{\mathcal{T}} \). Hence, applying the discrete iteration scheme can be seen as a very coarse approximation of a continuous system. In the presence of cycles, these coarse steps may lead to divergence even when the continuous model \( \sigma^G_{\mathcal{T}} \) converges. Intuitively, this is because a large step size like \( \delta = 1 \) can let us jump from the graph of the true solution \( \sigma^G_{\mathcal{T}} \) to the graph of another solution for different initial conditions. By choosing a smaller step size, we can avoid these jumps and make the procedure more stable. This is basically what we are doing when continuizing \( \mathcal{T} \).
3. evaluate continuous models on benchmarks and randomly
2. use base classes to implement new models and
1. compute solutions with existing models,
the ConArg benchmarks.

Finally, we will discuss how the previous ideas can be put
into practice using the Java library Attractor. A download
link for the current version is given in the footnote[1]. The
latest code is available at sourceforge[2]. Attractor is work in
progress and currently provides only a programming inter-
face. However, in the future, a graphical user interface will
be added. Attractor can be used to
1. compute solutions with existing models,
2. use base classes to implement new models and
3. evaluate continuous models on benchmarks and randomly
    generated BAGs.

We will discuss each use case in turn.

Computing Solutions

BAGs can be created either programmatically or, more con-
veniently, by using a file reader. The programming approach
is useful when considering families of BAGs like Cycle(k)
(c.f. Figure 5 and 6). A code example can be found in /At-
tractor/examples/NMR2018CycleK.java. In this tutorial, we
will focus on the file approach. Figure 9 shows an exam-
ple file on the left and the code to compute the final strength
values with RK4 and to create a plot similar to Figure 3 on
the right. Files consist of definitions of arguments, attacks
and supports. Argument definitions start with the keyword
arg and are followed by a name and an optional weight. If
no weight is provided in the definition, it is initialized with
0.5 by default. Attack and support definitions start with the
keywords att and sup and are followed by the source and the
target of the edge as usual. The file format is inspired by
the format used in ConArg[3] (Bistarelli, Rossi, and Santini
2016), but adds optional weights and support relations. The
file reader in Attractor can also read the current files from
the ConArg benchmarks.

In order to compute a solution, we first have to initialize
a model that is a subclass of the abstract class AbstractDy-
namicArgumentationSystem that we will explain in the next
section. The AbstractDynamicArgumentationSystem refer-
cence ads can also be initialized with implementations of the
continuized Euler-based semantics and DF-Quad algorithm
that we described in Section 5. By default, RK4 is used to
compute solutions. Different algorithms can be selected by
using the method setApproximator. In the example, we se-
lect PlottingRK4, which still uses RK4, but simultaneously
creates a plot for the evolution of the strength values using
JFreeChart[4]. The utility class BAGFileUtils is used to read
the file and the BAG object is passed to the model. After-
wards, the call of the method approximateSolution starts
the approximation and will plot graphs like in Figure 3. The
first two parameters determine the step size δ and the termi-
nation accuracy ε. The third parameter is optional and

6 The Java Library Attractor

Figure 9: Reading a BAG from a file in Attractor.
protected double[] computeDerivativeAt(double[] state) {
    double[] derivatives = new double[state.length];
    for(int i=0; i<arguments.length; i++) {
        double energy = 0;
        for(int s : supporter[i]) {
            energy += state[s];
        }
        for(int a : attacker[i]) {
            energy += state[a];
        }
        double weight = arguments[i].getInitialWeight();
        double derivative = 1 - (1 - weight*weight)/(1 + weight * Math.exp(energy));
        derivative -= state[i];
        derivatives[i] = derivative;
    }
    return derivatives;
}

@Override
public String getName() {
    return "Continuous Euler-based Model";
}

Figure 10: Implementation of Continuous Euler-based Model.

RandomBagGenerator generator = new RandomBagGenerator();
BAG bag = generator.createRandomBAG(100);

RandomBagGenerator generator = new RandomBagGenerator();
generator.createRandomBAGFiles(100, 30, 100, "files/Benchmark", "bag");

Figure 11: Generating a single random BAG (top) or a batch of random BAGs (bottom) in Attractor.

(Potyka 2018a). The original benchmark can be downloaded from the link given in the footnote. Figure 11 shows how to create a single BAG of size 100 at the top and how to create a batch of BAGs of different sizes at the bottom. The arguments for the method createRandomBAGFiles allow configuring the basic size (100), the number of increments of the size (30) and the number of trials for each size (100). In the example in Figure 11, 100 graphs of size 100, 200, 300, . . . , 3000 each will be created and stored in the local directory ‘files/Benchmark’, each file starting with the prefix ‘bag’.

The class BenchmarkUtils can be used to run benchmarks and to plot statistics for the evaluation similar to the evaluation in (Potyka 2018a). Figure 12 shows how to run the benchmark files in a directory. The method runBenchmark assumes that the given directory contains subdirectories. Each of these subdirectories contains BAGs of a fixed size and the name of the directory is supposed to be the size. The passed model will then be evaluated on all benchmark files and the method stores minimum, mean and maximum runtime for all sizes. Runtime results for individual files are printed to the console. The statistics are plotted simultaneously as shown in Figure 13. In this case, we evaluated the quadratic energy model on the Barabasi files from the ConArg benchmark. Other implementations of the base class AbstractDynamicArgumentationSystem can be evaluated analogously. Let us note that the ConArg benchmark does not contain weights and supports. By default, all weights will be set to 0.5. In order to evaluate new models on BAGs with supports, the benchmark from (Potyka 2018a) can be used or new benchmarks can be generated using the class RandomBagGenerator.

7 Conclusions and Future Work

Continuous dynamical systems are an alternative to discrete models that may give stronger convergence guarantees for cyclic BAGs in the future. Analyzing the general convergence behaviour is difficult and there are no general convergence guarantees currently. However, experiments show that continuous models converge in many cyclic BAGs and do so quickly. There are also interesting relationships between continuous and discrete models. For acyclic BAGs, equilibrium states are guaranteed to exist and can be computed by a discrete iteration scheme. This is computationally advantageous because it gives us a linear runtime guarantee. While continuous models converge superlinearly, they converged subquadratically in all previous experiments and the ability to plot the continuous evolution of strength values may be interesting to improve the explainability of the final strength values even for acyclic graphs. Existing discrete models can
be transformed to continuous iteration schemes and continuous differentiability of the update formula is a sufficient condition for some basic guarantees.

In order to simplify the use of continuous models, Attractor provides basic implementations of the ideas discussed here and in [Potyka 2018a]. For applications, it allows computing solutions for weighted argumentation problems. For further development, it allows deriving new models and algorithms from base classes that already provide basic functionality. In particular, utility functions can be used to evaluate new models and to compare them to existing models.

One main goal of future work is to advance the understanding of convergence conditions in cyclic BAGs. This involves trying to prove convergence in general cyclic BAGs or finding a counterexample. Furthermore, some empirical studies on the applicability in decision support and the analysis of Twitter discussions similar to the work in (Baroni et al. 2015; Rago et al. 2016; Alsinet et al. 2017) shall be conducted. Developing a graphical user interface for Attractor to simplify experiments will also be part of future work.

References

[Alsine et al. 2017] Alsine, T.; Argelich, J.; Béjar, R.; Fernández, C.; Mateu, C.; and Planes, J. 2017. Weighted argumentation for analysis of discussions in twitter. International Journal of Approximate Reasoning 85:21–35.

[Amgoud and Ben-Naim 2013] Amgoud, L., and Ben-Naim, J. 2013. Ranking-based semantics for argumentation frameworks. In Scalable Uncertainty Management (SUM), 134–147. Springer.

[Amgoud and Ben-Naim 2017] Amgoud, L., and Ben-Naim, J. 2017. Evaluation of arguments in weighted bipolar graphs. In European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU), 25–35. Springer.

[Amgoud, Cayrol, and Lagasquie-Schiex 2004] Amgoud, L.; Cayrol, C.; and Lagasquie-Schiex, M.-C. 2004. On the bipolarity in argumentation frameworks. In International Workshop on Non-Monotonic Reasoning (NMR), volume 4, 1–9.

[Baroni et al. 2011] Baroni, P.; Cerutti, F.; Giacomin, M.; and Guida, G. 2011. Afra: Argumentation framework with recursive attacks. International Journal of Approximate Reasoning 52(1):19–37.

[Baroni et al. 2015] Baroni, P.; Romano, M.; Toni, F.; Aurisicchio, M.; and Bertanza, G. 2015. Automatic evaluation of design alternatives with quantitative argumentation. Argument & Computation 6(1):24–49.

[Barringer, Gabbay, and Woods 2012] Barringer, H.; Gabbay, D. M.; and Woods, J. 2012. Temporal, numerical and meta-level dynamics in argumentation networks. Argument & Computation 3(2-3):143–202.

[Besnard and Hunter 2001] Besnard, P., and Hunter, A. 2001. A logic-based theory of deductive arguments. Artificial Intelligence 128(1-2):203–235.

[Bistarelli, Rossi, and Santini 2016] Bistarelli, S.; Rossi, F.; and Santini, F. 2016. Conarg: A tool for classical and weighted argumentation. In COMMA, 463–464.

[Budán et al. 2015] Budán, M. C.; Lucero, M. G.; Cheshiev, C.; and Simari, G. R. 2015. Modeling time and valuation in structured argumentation frameworks. Information Sciences 290:22–44.

[Cayrol and Lagasquie-Schiex 2005] Cayrol, C., and Lagasquie-Schiex, M.-C. 2005. Graduality in argumentation. Journal of Artificial Intelligence Research (JAIR) 23:245–297.

[Cayrol and Lagasquie-Schiex 2013] Cayrol, C., and Lagasquie-Schiex, M.-C. 2013. Bipolarity in argumentation graphs: Towards a better understanding. International Journal of Approximate Reasoning 54(7):876–899.

[Cohen et al. 2014] Cohen, A.; Gottifredi, S.; García, A. J.; and Simari, G. R. 2014. A survey of different approaches to support in argumentation systems. Knowledge Eng. Review 29(5):513–550.

[Correia, Cruz, and Leite 2014] Correia, M.; Cruz, J.; and Leite, J. 2014. On the efficient implementation of social abstract argumentation. In ECAI, 225–230.

[Dung 1995] Dung, P. M. 1995. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. Artificial intelligence 77(2):321–357.

[Hunter and Potyka 2017] Hunter, A., and Potyka, N. 2017. Updating probabilistic epistemic states in persuasion dialogues. In European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU), 46–56. Springer.

[Hunter 2013] Hunter, A. 2013. A probabilistic approach to modelling uncertain logical arguments. International Journal of Approximate Reasoning 54(1):47–81.

[Leite and Martins 2011] Leite, J., and Martins, J. 2011. Social abstract argumentation. In International Joint Conference on Artificial Intelligence (IJCAI), volume 11, 2287–2292.

[Mossakowski and Neuhaus 2018] Mossakowski, T., and Neuhaus, F. 2018. Modular semantics and characteristics for bipolar weighted argumentation graphs. arXiv preprint arXiv:1807.06685.

[Oren and Norman 2008] Oren, N., and Norman, T. J. 2008. Semantics for evidence-based argumentation. In Computational Models of Argument (COMMA), volume 172, 276–284. IOS Press.

[Polberg and Oren 2014] Polberg, S., and Oren, N. 2014. Revisiting support in abstract argumentation systems. In Computational Models of Argument (COMMA), 369–376.

[Polyanin and Zaitsev 2017] Polyanin, A. D., and Zaitsev, V. F. 2017. Handbook of ordinary differential equations. Chapman and Hall/CRC.

[Potyka 2018a] Potyka, N. 2018a. Continuous dynamical models for weighted bipolar argumentation. In 16th International Conference on Principles of Knowledge Representation and Reasoning, KR, 148–157.
[Potyka 2018b] Potyka, N. 2018b. Convergence and open-mindedness of discrete and continuous semantics for bipolar weighted argumentation (technical report). arXiv preprint arXiv:1809.07133.

[Rago et al. 2016] Rago, A.; Toni, F.; Aurisicchio, M.; and Baroni, P. 2016. Discontinuity-free decision support with quantitative argumentation debates. In International Conference on Principles of Knowledge Representation and Reasoning (KR), 63–73.

[Thimm 2012] Thimm, M. 2012. A probabilistic semantics for abstract argumentation. In European Conference on Artificial Intelligence (ECAI), volume 12, 750–755.