Defects for Ample Divisors of Abelian Varieties, Schwarz Lemma, and Hyperbolic Hypersurfaces of Low Degrees

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The main purpose of this paper is to prove the following theorem on the defect relations for ample divisors of abelian varieties.

Main Theorem. Let $A$ be an abelian variety of complex dimension $n$ and $D$ be an ample divisor in $A$. Let $f : C \to A$ be a holomorphic map. Then the defect for the map $f$ and the divisor $D$ is zero.

Corollary to Main Theorem. The complement of an ample divisor $D$ in an abelian variety $A$ is hyperbolic in the sense that there is no nonconstant holomorphic map from $C$ to $A - D$.

We give also in this paper the following results in hyperbolicity problems: (i) a Schwarz lemma for general jet differentials and (ii) examples of hyperbolic hypersurfaces of low degree.

Theorem 1 (General Schwarz Lemma). Let $X$ be a compact complex subvariety in $P_N$ and $f : C \to X$ be a nonconstant holomorphic map and $\zeta$ be the coordinate of $C$. Let $\omega$ be a holomorphic $k$-jet differential on $X$ of weight $m$ on $X$ which vanishes on ample divisor in $X$. Then the pullback of $\omega$ by $f$ vanishes identically on $C$. Here a holomorphic $k$-jet differential means that for any holomorphic map $g : U \to X$ from an open subset $U$ to $X$, their pullbacks by $g$ agree on $U$.

Theorem 2 (General Schwarz Lemma for Log-Pole Differentials). Let $X$ be a compact complex subvariety in $P_N$. Let $Z_1, \ldots, Z_p$ be distinct irreducible complex hypersurfaces in $X$ and $f : C \to X - \cup_{j=1}^{p} Z_j$ be a nonconstant holomorphic map and $\zeta$ be the coordinate of $C$. Let $\omega$ be a meromorphic $k$-jet differential on $X$ of at most log-pole singularity along $\cup_{j=1}^{p} Z_j$ such that the weight of $\omega$ is $m$ and $\omega$ vanishes on ample divisor in $X$. Then the pullback of $\omega$ by $f$ vanishes identically on $C$. Here a a meromorphic $k$-jet differential means that locally it is a

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polynomial with constant coefficients whose variables are local holomorphic jet differentials and meromorphic differentials of the form $d^\nu \log g$, where $\nu$ is a positive integer and $g$ is a local holomorphic function whose zero-set is contained in $\bigcup_{j=1}^{p} Z_j$.

**Theorem 3.** Let $N = 4n - 3$ and $p = 1 + N(N-2) = 16(n-1)^2$. Then for generic linear functions $H_j(x_0, \cdots, x_n)$ on $\mathbf{C}^{n+1}$ ($1 \leq j \leq N$) the hypersurface $\sum_{j=1}^{N} H_j^p = 0$ in $\mathbf{P}^n$ is hyperbolic.

**Theorem 4.** Let $n \geq 11$ and $g(x_0, x_1, x_2, x_3)$ be a homogeneous polynomial of degree 2 with $g(0, 0, 0, x_3) = x_3^2$ satisfying the condition that, for $h(\xi, \eta)$ equal to $g((-1)^{\frac{1}{n}} \xi, \xi, \eta, 1), g(\eta, (-1)^{\frac{1}{n}} \xi, \xi, 1)$, or $g(\eta, \xi, (-1)^{\frac{1}{n}} \xi, 1)$ with $(-1)^{\frac{1}{n}}$ equal to any $n^{th}$ root of $-1$, the polynomial

$$-\eta^n + \frac{1}{2} \left( \frac{\partial h}{\partial \xi} \right)^2 \left( \frac{\partial^2 h}{\partial \xi^2} \right)^{-1} = h$$

of degree $n$ in $\eta$ has $n$ distinct roots and $\frac{\partial^2 h}{\partial \xi^2}$ is a nonzero constant. Then the surface defined by

$$x_0^n + x_1^n + x_2^n + x_3^{n-2}g(x_0, x_1, x_2, x_3) = 0$$

is hyperbolic.

**Corollary to Theorem 4.** Let $n \geq 11$ and $a_0, a_1, a_2$ be complex numbers. Suppose $a_i^n \neq (-1)^{n+1}a_j^n$ for $0 \leq i < j \leq 2$ and $1 + a_j((-\frac{2a_i}{n})^{\frac{n}{2}} + (-\frac{2a_i}{n})^{\frac{1}{2}}) \neq 0$ for $0 \leq j \leq 2$ and for all $(n-2)^{th}$ roots in that condition. Then the surface defined by

$$x_0^n + x_1^n + x_2^n + x_3^{n-2}(x_3^2 + a_0x_0^2 + a_1x_1^2 + a_2x_2^2) = 0$$

is hyperbolic.

The Corollary to the Main Theorem was already proved in [SY96b]. The proof in [SY96b] could yield the result that the defect for $f$ and $D$ is less than 1, but the method there could not yield the sharp result of the defect being zero.

The idea in the proof of the Main Theorem is as follows. First of all we can assume without loss of generality that the image of $f$ is Zariski dense in $A$.  

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For any given positive number $\epsilon$ we construct a meromorphic $k$-jet differential $\omega$ on $A$ whose pole is dominated by a pole divisor of order $\delta$ along $D$ such that for some positive integer $q$ with $\frac{\delta}{q} < \epsilon$ the $k$-jet differential $\omega$ vanishes to order at least $q$ along the $k$-jet space $J_k(D)$ of $D$ when $\omega$ is regarded as a function on the $k$-jet space $J_k(A)$ of $A$. The existence of $\omega$ follows from the theorem of Riemann-Roch and the fact that the codimension of $J_k(D)$ in $J_k(A)$ increases without bounds as $k$ increases without bounds. We then pull back $\omega$ by $f$ and apply the logarithmic derivative lemma to the meromorphic function on $\mathbb{C}$ defined by the pullback. In this process we have to use the result in [SY95] on the translational invariance of the Zariski closure of the image of the $k$-order differential of $f$ to make sure that we can construct an $\omega$ whose pullback by $f$ is not identically zero.

Since Ahlfors [A41] introduced his more geometric view to study value distribution theory, the Schwarz lemma has been one of the indispensable tools in value distribution theory. Green and Griffiths [GG79] introduced a general Schwarz lemma to give an alternative approach to Bloch’s theorem [B26] and gave a sketch of the proof of their general Schwarz lemma. (For a discussion of the various ideas and proofs for Bloch’s theorem see for example [S95].) After some unsuccessful attempts to give the details of the Schwarz lemma given there, some authors [SY96a, D95, DL96] introduced jet differentials with special properties to get a proof for the Schwarz lemma. Because of such unsuccessful attempts, there were some skepticisms (now known to be unjustified) as to the completeness of the version of the proof of Bloch’s theorem presented in [GG79]. Since about two years ago, the authors and a number of other people working on hyperbolicity problems began to believe that the Schwarz lemma for general jet differentials either exactly or essentially as stated in [GG79] could be proved. Theorem 1 is slightly more general than the Schwarz lemma for general jet differentials stated in [GG79]. Our proof is completely different from the sketch of the proof given in [GG79]. We choose to present Theorem 1 together with the Main Theorem in this paper, because the Main Theorem is closely linked to Bloch’s theorem [B26] and the general Schwarz lemma was introduced in the paper of Green and Griffiths [GG79] which used it for an alternative approach to Bloch’s theorem [B26]. The main idea of the proof of Theorem 1 given here is that Cauchy’s integral formula for derivatives gives domination of the higher order derivatives at a point by the first-order derivative on a circle centered
at that point. Technically, to handle the complications arising from the radius of the circle and from different coordinate charts of the target manifold, one uses the techniques of value distribution theory and a simple curvature argument introduced in [S87]. The proof of Theorem 1 after some very minor modifications works also for meromorphic jet differentials with only log-pole singularities when the image of the map from $\mathbb{C}$ is disjoint from the log-pole singularities. Theorem 2 is the log-pole case of Theorem 1.

For hyperbolicity problems, the three approaches of the Borel lemma, the Schwarz lemma for jet differentials linear in the highest order, and meromorphic connections are very closely related. The approach of the Borel lemma, when applicable, usually gives sharper and cleaner arguments than the other two. However, for jet differentials with higher degree in the highest order, there is a possibility of getting sharper results but the difficulty to conclude algebraic dependency from differential equations of the vanishing the pullback of the jet differential is the main problem which in most cases is insurmountable. In the final part of this paper we construct hyperbolic hypersurfaces of low degrees in any dimension by using the Borel lemma and a simple dimension counting argument for certain subvarieties in the Grassmannians. A variation of the construction can make the degree lower in the case of a surface. To prepare for the variation of the construction we introduce a generalized Borel lemma (Proposition 2) and, to illustrate the close relation between the approach of the Borel lemma and that of the jet differentials, we use the Schwarz lemma to prove the generalized Borel lemma.

Examples of hyperbolic hypersurfaces were constructed by Brody-Green [BG77], Zaidenberg [Z89], Nadel [89] in dimension 2 and by Adachi-Suzuki [AS90] in some low dimensions, and finally by Masuda and Noguchi [MN94] in any dimension. The degree of a hyperbolic hypersurface constructed by them is exceedingly high relative to its dimension and the algorithm for the construction is rather involved. The degree of the hyperbolic hypersurface constructed in Theorem 3 is only of the order of the square of its dimension. Recently El Goul [E96] gave a construction of a hyperbolic surface of degree 14 and it was brought to our attention that a suggestion by Demailly could lower the degree in El Goul’s construction to 11. In Theorem 4 we point out how El Goul’s construction fits as a variation in the framework of using the Borel approach to get examples of hyperbolic hypersurfaces. The surface
of degree 11 constructed in Theorem 4 differs only very slightly from the construction of El Goul and Demailly, but we use the approach of the Borel lemma which, though very closely related to the approach of meromorphic connections, is simpler, more powerful, and more elegant.

§1. Defects for ample divisors of abelian varieties.

To prepare for the proof of the Main Theorem, we now introduce some notations and terminology. For a complex manifold $X$ we denote by $J_k(X)$ the space of all $k$-jets in $X$ so that every element of $J_k(X)$ is represented by $(\frac{d^\alpha}{d\zeta^\alpha} g)(0)$ $(0 \leq \alpha \leq k)$ for some holomorphic map $g$ from an open neighborhood of the origin in $\mathbb{C}$ (with coordinate $\zeta$) to $X$. In particular $J_1(X)$ means the tangent bundle of $X$. By a holomorphic (meromorphic) $k$-jet differential $\omega$ of weight $m$ on an open subset of $G$ of $X$ with local coordinates $z_1, \cdots, z_n$ we mean an expression of the form

\begin{equation}
\omega = \sum_\nu \omega_{\nu_1,1 \cdots \nu_1,k \cdots \nu_n,1 \cdots \nu_n,k} (dz_1)^{\nu_1,1} \cdots (d^k z_1)^{\nu_1,k} \cdots (dz_n)^{\nu_n,1} \cdots (d^k z_n)^{\nu_n,k}
\end{equation}

where the summation is over the $kn$-tuple

$\bar{\nu} = (\nu_1,1, \nu_1,2, \cdots, \nu_1,k, \cdots, \nu_n,1, \nu_n,2, \cdots, \nu_n,k)$

of nonnegative integers with

$$(\nu_1,1 + 2\nu_1,2 + \cdots + k\nu_1,k) + \cdots + (\nu_n,1 + 2\nu_n,2 + \cdots + k\nu_n,k) = m$$

and $\omega_{\nu_1,1 \cdots \nu_1,k \cdots \nu_n,1 \cdots \nu_n,k}$ is a holomorphic (meromorphic) function on $G$. For a holomorphic map $g$ from an open subset $U$ of $\mathbb{C}$ with coordinate $\zeta$ to $X$ and for any meromorphic $k$-jet differential $\omega$ of weight $m$ on $X$, by $g^*\omega$ we mean only the term containing $(d\zeta)^m$ or, when there is no confusion, we mean only the coefficient of the term containing $(d\zeta)^m$. According to this convention $g^*\omega$ can be regarded as a function on $U$.

Because of Bloch’s theorem [B26, GG79, K80, McQ96, NO90, Oc77] which states that the Zariski closure of the image of a holomorphic map from $\mathbb{C}$ to an abelian variety must be equal to the translate of an abelian subvariety, to prove the Main Theorem we can assume without loss of generality that the image of $f$ is Zariski dense in $A$. We denote by $d^k f$ the map from $\mathbb{C}$ to $J_k(A)$ induced by $f$. Denote by $J_k(A)$ the compactification $A \times \mathbb{P}_{nk}$.
of $J_k(A) = A \times \mathbb{C}^{nk}$. Let $H_{kn}$ be the pullback to $\overline{J_k(A)}$ of the hyperplane section line bundle of $\mathbb{P}_{nk}$ by the projection map $\overline{J_k(A)} = A \times \mathbb{P}_{nk} \to \mathbb{P}_{nk}$. Let $\pi : \overline{J_k(A)} \to A$ be the projection onto the base manifold. By the Zariski closure of $\text{Im } d^k f$ in $\overline{J_k(A)}$ we mean the intersection with $\overline{J_k(A)}$ of the Zariski closure of $\text{Im } d^k f$ in $J_k(A)$ of the image of $d^k f$ is translational invariant and is therefore of the form $A \times W_k$ for some irreducible subvariety $W_k$ of positive dimension in $\mathbb{P}_{nk}$.

Write $A = \mathbb{C}^n/\Lambda$. Let $L_D$ be the line bundle over $A$ associated to $D$. Since $D$ is ample in $A$, for any integer $p \geq 2$ the global holomorphic sections of the line bundle $L_D^\otimes p$ over $A$ generates the $(p-2)$-jets of $A$.

Let $\theta_D$ be the theta function on the universal cover $\mathbb{C}^n$ of $A$ which defines the divisor $D$. We denote by $J_k(D)$ the subvariety of $J_k(A)$ defined by $d^j \theta_D = 0$ for $0 \leq j \leq k$. Note that when $D$ is nonsingular, this notation $J_k(D)$ agrees with the earlier definition of $J_k(X)$ with $X = D$. To prove the Main Theorem we can assume without loss of generality that $L_D = (L_D')^\otimes p$ for some integer $p \geq k+2$, because we can simply replace the lattice $\Lambda$ defining $A$ by $p\Lambda$. Let $\Delta$ denote the open unit disk in $\mathbb{C}$ centered at the origin. Let $k \geq n$.

Lemma 1. There exists a holomorphic deformation $D(t)$ ($t \in \Delta$) of $D$ such that for $t \in \Delta - 0$ the subvariety $J_k(D(t)) \cap (A \times W_k)$ is of codimension at least $n+1$ in $A \times W_k$.

Proof. Let $V = \Gamma(A, L_D)^\otimes 2$. Since $\Gamma(A, L_D)$ generates the $k$-jets of $A$, for every point $P \in A$ there exist an open neighborhood $U_P$ of $P$ in $A$ and an element $v_P = (v_{P,0}, v_{P,1}) \in V$ such that

(i) $[v_{P,0}, v_{P,1}]U_P$ defines a holomorphic map $\Phi_P$ from $U_P$ to the projective line $\mathbb{P}_1$, and

(ii) $J_k(\Phi_P^{-1}(Q)) \cap (U_P \times W_k)$ is a subvariety of codimension at least $n+1$ in $U_P \times W_k$ for $Q$ in the projective line $\mathbb{P}_1$.

We can choose a compact neighborhood $K_P$ of $P$ in $U_P$ such that for a finite number of points $P_1, \ldots, P_\ell$ in $A$ we have $A = \bigcup_{j=1}^\ell K_{P_j}$. Let $E_j$ be the subset of $V$ consisting of all $v = (v_0, v_1) \in V$ such that the following two conditions do not simultaneously hold.

(i) $(v_0, v_1)$ is nowhere zero on $K_{P_j}$.
(ii) the codimension of $J_k(\Phi^{-1}(Q)) \cap ((A - \{v = 0\}) \times W_k)$ is a subvariety of codimension at least $n + 1$ in $(A - \{v = 0\}) \times W_k$ for at every point of $K_{P_j}$ for every $Q$ in $P_1$, where $\Phi : A - \{v \neq 0\} \to P_1$ is defined by $[v_0, v_1]$.

Then $E_j$ is a nowhere dense closed subset of $V$. By Baire category theorem we conclude that there exists $v = (v_0, v_1) \in V - \bigcup_{j=1}^{\ell} E_j$. In particular, we conclude that $J_k(\{v_0 = 0\}) \cap (A \times W_k)$ is of codimension at least $n + 1$ at every point of $A \times W_k$. We need only now consider the deformation given by $\theta_{D_t} = \theta_D + tv_0$. Then there exists a positive number $\eta$ such that $J_k(\theta_{D_t}) \cap (A \times W_k)$ is of codimension at least $n + 1$ at every point of $A \times W_k$ for $0 < |t| < \eta$. Q.E.D.

Take a fixed small positive integer $\delta$ which can actually be chosen to be 1. We keep the symbol $\delta$ to show the role played by it. Take a positive integer $q$ which will be very large compared to $\delta$ and then take a positive integer $m$ which will be very large compared to $q$. The conditions on the sizes of $q$ and $m$ will be specified later. We are going to construct a non identically zero $L_D^{\otimes \delta}$-valued holomorphic $k$-jet differential of weight $m$ which vanishes to order at least $q$ along $J_k(D)$ for some sufficiently large $m$. We first use the theorem of Riemann-Roch to do this when $D$ is replaced by $D_t$ for $t \in \Delta - 0$ close to 0 with $D_t$ satisfying some additional tranversality condition. Then we use the semicontinuity of the dimension of the space of holomorphic sections of line bundles in a holomorphic deformation to get the conclusion for $D$ when $t \to 0$. In the following lemma, for notational simplicity the dependence of the constants on $A$ and $D$ is not explicitly stated out.

**Lemma 2.** There exists a postive integer $m_0(W_k, \delta, q)$ depending on $W_k, \delta, q$ (and $A$ and $D$) such that for $m \geq m_0(W_k, \delta, q)$ there exists an $L_D^{\otimes \delta}$-valued holomorphic $k$-jet differential on $A$ of weight $m$ whose restriction to $A \times W_k$ is not identically zero and which vanishes along $J_k(D) \cap (A \times W_k)$ to order at least $q$. In particular, from the definition of $W_k$ one knows that $\omega$ is not identically zero on $d^k f$.

**Proof.** From Lemma 1 there exists a holomorphic family of ample divisors $D(t)$ ($t \in \Delta$) with $D(0) = D$ such that for $t \in \Delta - 0$ the subvariety $J_k(D(t)) \cap (A \times W_k)$ is of codimension at least $n + 1$ in $A \times W_k$. From the ampleness of $D(t)$ and Kodaira’s vanishing theorem we have a positive number $m_0'(W_k, t)$ such that

$$H^p \left( A \times W_k, \pi^*(L_D^{\otimes \delta}) \otimes H^{\otimes m}_k \left| (A \times W_k) \right. \right) = 0$$
for any positive integer \( p \) and for \( m \geq m_0'(W_k) \). Let \( d \) be the complex dimension of \( W_k \). By the theorem of Riemann-Roch and the Künneth formula, we have

\[
\dim \mathbb{C} \Gamma \left( A \times W_k, \pi^* (L_{D(t)} \otimes H_{kn}^m | (A \times W_k) \right) \geq C_{W_k, t} \delta^m m^d
\]

where \( C_{W_k, t} \) is a positive constant dependent on \( W_k \) but independent of \( m \).

We now choose some \( t \in \Delta - 0 \). Let \( \mathcal{O} \) be the structure sheaf of \( A \times W_k \) and \( \mathcal{I}(t) \) be the ideal sheaf of \( J_k(D(t)) \cap (A \times W_k) \). Since for \( t \in \Delta - 0 \) the subvariety \( J_k(D(t)) \cap (A \times W_k) \) of \( A \times W_k \) is of complex dimension at most \( d - 1 = (n + d) - (n + 1) \), we have the following estimate

\[
\dim \mathbb{C} \Gamma \left( A \times W_k, \mathcal{O}/\mathcal{I}(t)^{q+1} \otimes (\pi^* (L_{D(t)} \otimes H_{kn}^m) \right) \leq C'_{W_k, t, \delta, q, m} m^{d-1}
\]

where \( C'_{W_k, t, \delta, q, m} \) is a positive constant dependent on \( W_k, \delta, q, t \) but independent of \( m \). Thus there exists a positive integer \( m''_0(W_k, \delta, q, t) \) such that for \( m \geq m''_0(W_k, \delta, q, t) \) the dimension of

\[
\Gamma \left( A \times W_k, \mathcal{I}(t)^{q+1} \otimes (\pi^* (L_{D(t)} \otimes H_{kn}^m) \right)
\]

is nonzero. Clearly we can assume that \( m''_0(W_k, \delta, q, t) \) is lower semicontinuous as a function of \( t \in \Delta - 0 \). Let \( F_{\delta, q, m} \) be the torsion-free sheaf of rank 1 over \( A \times W_k \times \Delta \) whose restriction to \( A \times W_k \times t \) for a generic \( t \in \Delta \) is equal to \( \mathcal{I}(t)^{q+1} \otimes (\pi^* (L_{D(t)} \otimes H_{kn}^m) \right) \). Let \( H_{\delta, q, m} \) be locally free sheaf over \( \Delta \) which is the zeroth direct image of the sheaf \( F_{\delta, q, m} \) under the map \( A \times W_k \times \Delta \to \Delta \) which is the projection onto the last factor. Fix \( t_0 \in \Delta - 0 \). Then for \( m \geq m''_0(W_k, \delta, q, t_0) \) the rank of \( H_{\delta, q, m} \) is positive. Thus for a generic \( t \in \Delta \) the dimension of

\[
\Gamma \left( A \times W_k, \mathcal{I}(t)^{q+1} \otimes (\pi^* (L_{D(t)} \otimes H_{kn}^m) \right)
\]

is nonzero. By the semicontinuity of

\[
\dim \mathbb{C} \Gamma \left( A \times W_k, \mathcal{I}(t)^{q+1} \otimes (\pi^* (L_{D(t)} \otimes H_{kn}^m) \right)
\]

as a function of \( t \) and by letting \( t \to 0 \), we conclude that for

\[
m \geq m''_0(W_k, \delta, q, t_0)
\]
there exists a non identically zero global holomorphic section $\omega'$ of $\pi^*(L_D^\otimes) \otimes H_{kn}^{\otimes m}(A \times W_k)$ over $A \times W_k$ which vanishes along $J_k(D) \cap (A \times W_k)$ to order at least $q$.

Let $\mathcal{I}_{W_k}$ be the ideal sheaf of $W_k$ in $\mathbb{P}_{nk}$. By Künneth’s formula, from the ampleness of $D$ we have a positive integer $m_0''(W_k)$ such that

$$H^1\left(\overline{\mathcal{I}_k(A)}, \pi^*(L_D^\otimes) \otimes H_{kn}^{\otimes m} \otimes \mathcal{I}_{W_k}\right) = 0$$

for $m \geq m_0''(W_k)$. Thus for $m \geq m_0''(W_k)$ we can extend $\omega'$ to an $L_D^\otimes$-valued holomorphic $k$-jet differential $\omega$ on $A$. Now we need only set $m_0(W_k, \delta, q)$ to be at least as large as $m_0'(W_k)$, $m''(W_k, \delta, q, t_0)$, and $m_0'''(W_k)$. Q.E.D.

**Proof of the Main Theorem.** Let $\tilde{\omega} = \theta_D^\delta \omega$. Then $\tilde{\omega}$ is a meromorphic $k$-jet differential on $A$. We pull back the meromorphic $k$-jet differential $\tilde{\omega}$ by $f$ and recall that by our convention $f^*\tilde{\omega}$ simply means the coefficient of $(d\zeta)^m$ where $m$ is the weight of $\tilde{\omega}$ and $\zeta$ is the global coordinate of $C$. In other words, we consider $\tilde{\omega}$ as a function on the space $J_k(A)$ of $k$-jets and $f^*\tilde{\omega}$ as the evaluation of $\tilde{\omega}$ on the image of $d^k f$ when $d^k f$ is considered as a map from $C$ to $J_k(A)$. In this sense $f^*\tilde{\omega}$ is a meromorphic function on $C$. Write $\tilde{\omega}$ in terms of the global coordinate system and we get

$$\tilde{\omega} = \sum_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)} \tilde{a}_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n)(d^{\ell_1} z_1)^{\nu_1} \cdots (d^{\ell_n} z_n)^{\nu_n}$$

where $\tilde{a}_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n)$ is a meromorphic function on $A$ with the property that

$$\theta_D^\delta \tilde{a}_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}$$

is an entire function on $C^n$ when $\tilde{a}_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}$ is used to denote also its pullback to the universal cover $C^n$ of $A$. Let

$$a_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n) = \theta_D^\delta \tilde{a}_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n).$$

Let $\varphi$ be a nonnegative quadratic form on $C^n$ such that $|\theta_D|^2 \exp(-\varphi)$ is a well-defined scalar function on $A$. In other words, $\exp(-\varphi)$ defines a Hermitian metric on the fibers of the line bundle $L_D$. Thus

$$\exp \left(-\frac{\delta}{2} \varphi \right) |a_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n)|$$

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is a well-defined smooth function on $A$ and is therefore bounded. Hence
\begin{equation}
|a_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n)| \leq C \exp\left(\frac{\delta}{2} \varphi\right)
\end{equation}
on $\mathbb{C}^n$ for some constant $C$. From (1) we have
\begin{equation}
\omega = \sum_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)} a_{(\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)}(z_1, \ldots, z_n)(d^{\ell_1} z_1)^{\nu_1} \cdots (d^{\ell_n} z_n)^{\nu_n}
\end{equation}
on $\mathbb{C}^n$.

Take a relatively compact open ball $B_0$ in $\mathbb{C}^n$ and a positive number $b$ such that for every ball $W_k$ of radius $b$ in $\mathbb{C}^n$ there exists an element $\ell \in \Lambda$ such that the translate of $W_k$ by $\ell$ is contained in $B_0$. Since $\omega$ vanishes to order at least $q$ along $J_k(D) \cap (A \times W_k)$ it follows that there exists holomorphic jet differentials $\sigma_{j_0, \ldots, j_k}$ on some open neighborhood $U$ of the topological closure of $B_0$ such that
\begin{equation}
\omega = \sum_{j_0 + \cdots + j_k = q} \sigma_{j_0, \ldots, j_k} \prod_{\nu = 0}^{k} (d^\nu \theta_D)^{j_\nu}
\end{equation}
on $U \times W_k$. Here we use $\omega$ to denote also its pullback to the universal covering $\mathbb{C}^n$ of $A$. Clearly locally we can choose $\sigma_{j_0, \ldots, j_k}$ so that the order and the weight of each term $\prod_{\nu = 0}^{k} (d^\nu \theta_D)^{j_\nu}$ are respectively equal to those of $\omega$. We rewrite the above identity on $U \times W_k$ in the form
\begin{equation}
\omega = \theta_D^q \sum_{j_0 + \cdots + j_k = q} \sigma_{j_0, \ldots, j_k} \prod_{\nu = 1}^{k} \left(\frac{d^\nu \theta_D}{\theta_D}\right)^{j_\nu}
\end{equation}
on $U \times W_k$. In order to extend the identity (4) from $U \times W_k$ to identities on the translates of $U \times W_k$ by elements of $\Lambda$, we rewrite (4) as
\begin{equation}
\exp(-\frac{\delta}{2} \varphi) \omega = \exp\left(-\frac{q}{2} \varphi\right) \theta_D^q \sum_{j_0 + \cdots + j_k = q} \hat{\sigma}_{j_0, \ldots, j_k} \prod_{\nu = 1}^{k} \left(\frac{d^\nu \theta_D}{\theta_D}\right)^{j_\nu}
\end{equation}
on $U$, where
\begin{equation}
\hat{\sigma}_{j_0, \ldots, j_k} = \exp\left(\frac{q - \delta}{2} \varphi\right) \sigma_{j_0, \ldots, j_k}.
\end{equation}
For $z \in \mathbb{C}^n$ and $\ell \in \Lambda$ there is an affine transformation of $\mathbb{C}^n$ to itself given by $z \mapsto A\ell(z) + B\ell$ such that

$$\theta_D(z + \ell) = \exp(A\ell(z) + B\ell)\theta_D(z).$$

From it we obtain

$$d^p\theta_D(z + \ell) = \sum_{j=0}^p \binom{p}{j} d^{p-j} \exp(A\ell(z) + B\ell) d^j \theta_D(z)$$

and

$$\frac{d^p\theta_D(z + \ell)}{\theta_D(z + \ell)} = \sum_{j=0}^p \binom{p}{j} \left( \frac{d^{p-j} \exp(A\ell(z) + B\ell)}{\exp(A\ell(z) + B\ell)} \right) \left( \frac{d^j \theta_D(z)}{\theta_D(z)} \right).$$

Both $|\exp(-\frac{\delta}{2}\varphi)\omega|$ and $|\exp(-\frac{q}{2}\varphi)\theta_D^q|$ are invariant under the action of $\Lambda$. Hence for $\ell \in \Lambda$ there exists a complex number $c_\ell$ of absolute value 1 such that

$$c_\ell \left( \frac{\exp(-\frac{\delta}{2}\varphi)\omega}{\exp(-\frac{q}{2}\varphi)\theta_D^q} \right) (z + \ell) = \left( \frac{\exp(-\frac{\delta}{2}\varphi)\omega}{\exp(-\frac{q}{2}\varphi)\theta_D^q} \right) (z).$$

For any $z \in \mathbb{C}^n$ there exists $\ell = \ell(z) \in \Lambda$ such that the ball $B$ of radius $b$ centered at $z - \ell$ is contained in $B_0$. From

$$\left( \exp(-\frac{\delta}{2}\varphi)\omega \right) (z - \ell) = \left( \exp(-\frac{q}{2}\varphi)\theta_D^q \right) (z - \ell) \sum_{j_0 + \cdots + j_k = q} \sigma_{j_0, \ldots, j_k} (z - \ell) \prod_{\nu=1}^k \left( \frac{d^\nu \theta_D}{\theta_D} \right)^{j_\nu} (z - \ell)$$

on $B \times W_k$ and (6) we obtain

$$c_\ell \left( \exp(-\frac{\delta}{2}\varphi)\omega \right) (z) = \left( \exp(-\frac{q}{2}\varphi)\theta_D^q \right) (z) \sum_{j_0 + \cdots + j_k = q} P_{j_0, \ldots, j_k} \prod_{\nu=1}^k \left( \frac{d^\nu \theta_D}{\theta_D} \right)^{j_\nu} (z)$$

on $(B + \ell) \times W_k$, where

$$P^{(\lambda)}_{j_0, \ldots, j_k} = P^{(\lambda)}_{j_0, \ldots, j_k} \left( \{ \sigma^{(\lambda)}_{p_0, \ldots, p_k} (z - \ell) \}_{p_0 + \cdots + p_k = q}, \{ d^{\nu} (A_\ell z + B_\ell) \}_{0 \leq \nu \leq k} \right).$$
is a polynomial in the variables
\[ \left\{ \hat{\sigma}^{(\lambda)}_{p_0, \ldots, p_k} \right\}_{p_0 + \cdots + p_k = q}, \left\{ d^\nu (A - \ell z + B - \ell) \right\}_{0 \leq \nu \leq k} \]
whose coefficients are complex numbers. Since \( A_\ell \) is linear in \( \ell \) and \( B_\ell \) is a polynomial of degree at most 1 in \( \ell \), it follows that
\[
|f^* d^\nu (A - \ell z + B - \ell)| \leq \left( 1 + \sum_{0 \leq \nu \leq k, 1 \leq j \leq n} |f^* d^\nu z_j|^2 \right).
\]
Hence
\[
(7) \quad f^* \left| \exp \left( -\frac{\delta}{2} \varphi \right) \right| \leq C f^* \left| \exp \left( -\frac{q}{2} \varphi \right) \right| \left( 1 + \sum_{0 \leq \nu \leq k, 1 \leq j \leq n} |f^* d^\nu z_j|^2 \right) \sum_{j=0}^{N} \left| f^* \left( \frac{\theta_D}{\theta_D} \right) \right|^q
\]
on \mathbb{C} \) where \( N \) is a positive integer depending on \( k \) and \( q \) and \( C \) is a positive constant.

Let \( \mathcal{A}_r \) denote the operator which, when applied to a function, averages the function over the circle of radius \( r \) in \( \mathbb{C} \) centered at the origin. For a meromorphic function \( g \) on \( \mathbb{C} \) we denote by \( T(r, g) \) the characteristic function of \( F \) which is given by
\[
T(r, F) = \mathcal{A}_r (\log^+ |F|) + \int_{\rho=0}^{r} n(r, F, \infty) \frac{d\rho}{\rho},
\]
where \( n(r, F, \infty) \) is the number of poles of \( F \) with multiplicity counted in the open disk of radius \( r \) in \( \mathbb{C} \) centered at the origin.

To compute the defect for the map \( f \) and the divisor \( D \) we have to consider
\[
(8) \quad \mathcal{A}_r \left( \log^+ \left| \frac{1}{f^* \left| \exp \left( -\frac{\delta}{2} \varphi \right) \theta_D^q \right|} \right| \right)
\]
which by (7) is dominated by
\[
(9) \quad \mathcal{A}_r \left( \log^+ \left| \frac{1}{f^* \left| \exp \left( -\frac{\delta}{2} \varphi \right) \omega \right|} \right| \right)
\]
\[ \mathcal{A}_r \left( \log^+ \left( \left( 1 + \sum_{0 \leq \nu \leq k, 1 \leq j \leq n} |f^* d^\nu z_j| \right)^N \sum_{j=0}^k \left| f^* \left( \frac{d^j\theta_D}{\theta_D} \right) \right|^q \right) \right) + O(1). \]

Here \( O(1) \) means the standard Landau symbol for order comparison. We handle the first term of (9) as follows.

\[ \mathcal{A}_r \left( \log^+ \left( \frac{1}{f^* \exp(-\frac{\delta}{2} \varphi)} \right) \right) \leq \mathcal{A}_r \left( \frac{\delta}{2} \varphi \circ f \right) + \mathcal{A}_r \left( \log^+ \frac{1}{|f^*\omega|} \right) \]

\[ \leq \mathcal{A}_r \left( \frac{\delta}{2} \varphi \circ f \right) + T\left( r, \frac{1}{f^*\omega} \right) \]

\[ \leq \mathcal{A}_r \left( \frac{\delta}{2} \varphi \circ f \right) + T(r, f^*\omega) + O(1). \]

Here according to our convention \( f^*\omega \) is regarded as a function on \( \mathbb{C} \). For (10) we have used the First Main Theorem of Nevanlinna that

\[ T\left( r, \frac{1}{f^*\omega} \right) = T(r, f^*\omega) + O(1). \]

The positive \((1,1)\)-form \( \sqrt{-1} \frac{\partial \bar{\partial}}{2\pi} \varphi \) is the curvature form for the line bundle \( L_D \) with the Hermitian metric \( \exp(-\varphi) \). We denote by \( T(r, f, L_D) \) the characteristic function of \( f \) with respect to the Hermitian line bundle \( L_D \) which is given by

\[ T(r, f, L_D) = \int_{|\zeta|<\rho} \frac{d\rho}{\rho} \int_{|\zeta|<\rho} f^* \sqrt{-1} \frac{\partial \bar{\partial}}{2\pi} \varphi \]

which by Green’s theorem equals \( \mathcal{A}_r \left( \frac{\delta}{2} \varphi \circ f \right) + O(1) \).

From (2) and (3) we conclude that

\[ T(r, f^*\omega) \leq \delta T(r, f, L_D) + O \left( \log T(r, f, L_D) \right). \]

Here we have used the fact that

\[ T(r, f^*d^\nu z_j) = O \left( \log T(r, f, L_D) \right), \]
which is a consequence of the logarithmic derivative lemma, when \( f^*d^\nu z_j \) is regarded as a meromorphic function on \( \mathbb{C} \). The second term of (9) satisfies

\[
A_r \left( \log^+ \left( \sum_{0 \leq \nu \leq k, 1 \leq j \leq n} \left| f^*d^\nu z_j \right| \right)^N \sum_{j=0}^{k} \left| f^* \left( \frac{d^j \theta_D}{\theta_D} \right)^q \right| \right) = O \left( \log T(r, f, L_D) \right)
\]

because of (12) and because of

\[
T \left( r, f^* \left( \frac{d^j \theta_D}{\theta_D} \right) \right) = O \left( \log T(r, f, L_D) \right),
\]

which is a consequence of the logarithmic derivative lemma, when \( f^* \left( \frac{d^j \theta_D}{\theta_D} \right) \) is regarded as a meromorphic function on \( \mathbb{C} \). Finally from the domination of (8) by (9) and from (10) and (11) and (13) we conclude that

\[
A_r \left( \log^+ \left( \sum_{0 \leq \nu \leq k, 1 \leq j \leq n} \left| f^*d^\nu z_j \right| \right)^N \sum_{j=0}^{k} \left| f^* \left( \frac{d^j \theta_D}{\theta_D} \right)^q \right| \right) = 2\delta T(r, f, L_D) + O \left( \log T(r, f, L_D) \right).
\]

We denote by \( m(r, f, D) \) the proximity function for the map \( f \) and the divisor \( D \) which is defined, up to a bounded term, by

\[
m(r, f, D) = A_r \left( \log \frac{1}{|s_D|} \right),
\]

where \( s_D \) is the canonical section of \( L_D \) whose divisor is \( D \) and \(|s_D|\) is the pointwise norm of \( s_D \) with respect to a Hermitian metric of \( L_D \). From this definition of \( m(r, f, D) \) we have

\[
A_r \left( \log^+ \left( \frac{1}{f^* \exp(-\frac{q}{2} \varphi) \theta_D^q} \right) \right) = q \frac{m(r, f, D) + O(1)}{T(r, f, L_D)}.
\]

For any given \( \epsilon > 0 \) we can choose \( \delta \) and \( q \) so that \( \delta < \frac{\epsilon}{q} \). We denote by \( \delta(f, D) \) the defect for the map \( f \) and the divisor \( D \) which is defined by

\[
\delta(f, D) = \lim \inf_{r \to \infty} \frac{m(r, f, D)}{T(r, f, L_D)}.
\]
It follows from (14) that \( \delta(f, D) < \frac{2d}{q} < 2\varepsilon \). From the arbitrariness of the positive number \( \varepsilon \) we conclude that the defect for an ample divisor in an abelian variety is 0.

§2. The General Schwarz Lemma for the Holomorphic Case.

We introduce the following notation. For a function or a \((1,1)\)-form \( \eta \) let

\[ I_r(\eta) = \int_{\rho=0}^{\rho} \frac{d\rho}{\rho} \int_{|\zeta|<\rho} \eta, \]

where \( \zeta \) is the coordinate of \( \mathbb{C} \). Green’s theorem gives

\[ I_r\left( \frac{1}{\pi} \partial_{\zeta} \partial_{\bar{\zeta}} g \right) = \frac{1}{2} A_r(g) - \frac{1}{2} g(0) \]

for a function \( g \). For a complex manifold \( M \) and a positive definite smooth \((1,1)\)-form \( \theta \) on \( M \) and for a holomorphic map \( f : \mathbb{C} \to M \), we define the characteristic function of \( f \) with respect to \( \theta \) as

\[ T(r, f, \theta) = I_r(f^*\theta). \]

For another positive definite smooth \((1,1)\)-form \( \theta' \), \( T(r, f, \theta) \leq CT(r, f, \theta') \) for some constant \( C \) depending only on \( \theta \) and \( \theta' \) and independent of \( f \). When it does not matter which positive definite smooth \((1,1)\)-form \( \theta \) is used, we also denote \( T(r, f, \theta) \) simply by \( T(r, f) \). If \( M \) is the complex projective line \( \mathbb{P}_1 \) and \( f : \mathbb{C} \to \mathbb{P}_1 \) is represented by a meromorphic function \( F \) on \( \mathbb{C} \), then

\[ T(r, f, \theta) = T(r, F) + O(1) \]

when \( \theta \) is the Fubini-Study form on \( \mathbb{P}_1 \).

For a holomorphic line bundle \( L \) with a smooth Hermitian metric \( e^{-\varphi} \) over a compact complex manifold \( M \) and an \( L \)-valued meromorphic \( k \)-jet differential \( \omega \), we define the pointwise norm

\[ |\omega|_L = \left( e^{-\varphi} \omega \overline{\omega} \right)^{\frac{1}{2}}. \]

Locally the pointwise norm is the absolute value of a meromorphic jet differential. The definition of \( |\omega|_L \) does not involve any metric of the tangent bundle of \( M \). At a point \( P \) of \( M \) the pointwise norm \( |\omega|_L \) is not a scalar.
However, if a $k$-jet $\xi$ is given at $P$ not in the pole set of $\omega$ (for example, the $k$-jet defined by a holomorphic map from an open subset of $\mathbb{C}$ whose image contains $P$), the value of $|\omega|_L$ at $\xi$ is a nonnegative number. The pointwise norm $|\omega|_L$ depends not just on the line bundle $L$ but also on the metric $e^{-\varphi}$ of $L$, but we will simply use the notation $|\omega|_L$ and suppress the metric $e^{-\varphi}$ if there is no confusion. If $L$ is the line bundle associated to a divisor $D$, we also use $D$ to denote the line bundle $L$ and denote $|\omega|_L$ by $|\omega|_D$ if there is no confusion. We use the additive notation for tensor products of line bundles when the expression involves using a divisor to denote the line bundle associated to it. If $L$ is the trivial line bundle, we denote $|\omega|_L$ simply by $|\omega|$.

For a meromorphic jet differential $\eta$ on $M$ there is an ample divisor $D$ with canonical section $s_D$ such that $s_D^k \eta$ is a $kD$-valued holomorphic jet differential. The pointwise norm $|\eta|$ for the meromorphic jet differential $\eta$ is not the same as the pointwise norm $|s_D^k \eta|_{kD}$ of the $kD$-valued holomorphic jet differential $s_D^k \eta$ when $kD$ is given a smooth Hermitian metric (though a meromorphic jet differential $\eta$ can equivalently be regarded a $kD$-valued holomorphic jet differential $s_D^k \eta$). The value of $|\eta|$ can blow up when evaluated at a smooth field of jets, but the value of $|s_D^k \eta|_{kD}$ at a smooth field of jets is smooth. The distinction between the pointwise norm of a meromorphic jet differential and the pointwise norm of the line-bundle-valued holomorphic jet differential associated to it is crucial in our proof of the general Schwarz lemma.

**Lemma 3.** Let $M$ be a compact complex manifold, $E$ be a holomorphic line bundle over $M$ with hermitian metric $e^{-\psi}$ along the fibers of $E$, and curvature $\theta = \partial \bar{\partial} \psi$. Let $D_\psi$ denote covariant differentiation with respect to the connection from the metric $e^{-\psi}$ of $E$. Let $\omega$ be an $E$-valued holomorphic jet differential over $M$. Then

\[
\partial \bar{\partial} \log \left(1 + |\omega|^2_E\right) = \frac{|D_\psi \omega|^2_E}{(1 + |\omega|^2_E)^2} - \frac{\theta |\omega|^2_E}{1 + |\omega|^2_E},
\]

where all differentials are taken in the category of jet differentials instead of in the category of differential forms (i.e., the differentials are symmetric instead of alternating). Moreover, if $f : \mathbb{C} \to M$ is a holomorphic map, then

\[
\mathcal{I}_r \left(f^* \frac{|D_\psi \omega|^2_E}{(1 + |\omega|^2_E)^2}\right) \leq \mathcal{A}_r \left(f^* \log \left(1 + |\omega|^2_E\right)\right) + O(T(r, f, \theta) + \log r).
\]
Proof. The identity (15) follows from straightforward differentiation. Note that, if the order of the jet differential $\omega$ is $k$, then $D_\psi \omega$ is a jet differential of order $k + 1$ which in general is not holomorphic. The last inequality follows from (15), Green’s theorem, and

$$\frac{\vert \omega \vert^2_E}{1 + \vert \omega \vert^2_E} \leq 1.$$  

Q.E.D.

Definition. Let $M$ be a compact complex manifold and $D$ be an ample divisor with canonical section $s_D$. By a meromorphic jet differential constructed from functions with poles along $D$ we mean a meromorphic jet differential $\eta$ on $M$ which is a polynomial of the variables $d^\nu \left( \frac{\omega}{s_D} \right)$ with constant coefficients, where $\nu$ is a nonnegative integer and $s_\lambda$ is a holomorphic section of the line bundle associated to $D$. By a pole-factor of $\eta$ we mean $s_D^k$ so that $s_D^k \eta$ is a holomorphic jet differential.

Lemma 4. Let $k$ and $m$ be positive integers. Let $M$ be a compact complex projective algebraic manifold, $E$ be a holomorphic line bundle over $M$ and $\omega$ be an $E$-valued holomorphic $k$-jet differential of weight $m$ on $M$. Then there exist

(i) an ample line bundle $F$ with holomorphic sections $s_{D_\nu}$ whose divisor is a nonsingular ample divisor $D_\nu$ ($1 \leq \nu \leq N$),

(ii) meromorphic $k$-jet differentials $\eta_\nu$ weight $m$ constructed by functions with poles along $D_\nu$ (which automatically admits $s_D^{2m}$ as a pole-factor) for $1 \leq \nu \leq N$, and

(iii) an ample line bundle $L$ over $M$ and holomorphic sections $t_{j,\nu}$ of $L + E$ over $M$ and holomorphic sections $t_j$ of $L + 2mF$ ($1 \leq j \leq J, 1 \leq \nu \leq N$)

such that

(a) $t_1, \ldots, t_J$ have no common zeroes in $M$, and

(b) $t_j \omega = \sum_{\nu=1}^{N} t_{j,\nu} \left( s_{D_\nu}^{2m} \eta_\nu \right)$ on $M$ for $1 \leq j \leq J$.

In particular,

$$\vert \omega \vert_E \leq C \sum_{j=1}^{J} \sum_{\nu=1}^{N} \left| s_{D_\nu}^{2m} \eta_\nu \right|_{2mF}$$

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on $M$ for some constant $C$.

Proof. It follows from the standard application to a short exact sequence of the vanishing theorem for the positive dimensional cohomology over a compact complex manifold with coefficients in the tensor product of a holomorphic vector bundle with a sufficiently high power of an ample line bundle. Q.E.D.

**Proposition 1.** Let $M$ be a compact projective-algebraic complex manifold of complex dimension $n$ and $f : \mathbb{C} \rightarrow M$ be a nonconstant holomorphic map. Let $E$ be a holomorphic line bundle with Hermitian metric $e^{-\psi}$ along its fibers. Let $\omega$ be an $E$-valued holomorphic $k$-jet differential on $M$ of weight $m$. Then there exists a positive number $\epsilon_{k,m}$ depending only on $k$ and $m$ such that for $0 < \epsilon \leq \epsilon_{k,m}$ one has

$$I_r((f^*|\omega|^2_E)^\epsilon) = O(T(r,f) + \log r) \parallel.$$

Remark. Note that in this paper the pullback $f^*\omega$ means the value of $\omega$ at $d^k f$ with respect to a local trivialization of $E$ which means that we take only the part of $f^*\omega$ which is equal to a scalar function times $(d\zeta)^m$. As customary in value distribution theory, we use the notation $\parallel$ at the end of an equation or an inequality to mean that the statement holds outside an open set whose harmonic measure is finite. Proposition 1 is the step in the proof of Theorem 1 which corresponds to using Cauchy’s integral formula for derivatives to dominate higher order derivatives at a point by first-order derivative on a circle centered at that point. The condition restricting the validity of the statement to outside an open set with finite harmonic measure is used to take care of the problem posed by the need for a circle in the use of Cauchy’s integral formula for derivatives.

**Proof of Proposition 1.** By Lemma 4 it suffices to prove the special case that $E$ is equal to $2mD$ for some ample divisor $D$ of $M$ with canonical section $s_D$ and $\omega = s_D^{2m}\eta$ for some meromorphic $k$-jet differential $\eta$ of weight $m$ constructed from functions with poles along $D$. From the definition of meromorphic jet differential constructed from functions with poles along $D$ we can further assume that

$$\eta = \prod_{1 \leq \nu \leq k, 1 \leq \lambda \leq n} \left( d^\nu \left( \frac{s_\lambda}{s_D} \right) \right)^{\ell_{\nu,\lambda}},$$
where \( s_\lambda (1 \leq \lambda \leq n) \) are holomorphic sections of \( D \) over \( M \) and

\[
\sum_{1 \leq \nu \leq k, 1 \leq \lambda \leq n} \nu \ell_{\nu, \lambda} = m.
\]

By using the trivial inequality

\[
\left| \prod_{1 \leq \nu \leq k, 1 \leq \lambda \leq n} \left( d^\nu \left( \frac{s_\lambda}{s_D} \right) \right)^{\ell_{\nu, \lambda}} \right| \leq \left( \max_{1 \leq \nu \leq k, 1 \leq \lambda \leq n} \left| d^\nu \left( \frac{s_\lambda}{s_D} \right) \right| \right)^{\sum_{1 \leq i \leq k, 1 \leq j \leq n} \ell_{i,j}}
\]

\[
\leq \sum_{\nu=1}^k \sum_{\lambda=1}^n \left| d^\nu \left( \frac{s_\lambda}{s_D} \right) \right|^{\sum_{1 \leq i \leq k, 1 \leq j \leq n} \ell_{i,j}},
\]
we need only prove the special case \( \omega = s_\nu^{\nu+1} \left( \frac{s_\lambda}{s_D} \right) \). We prove the special case by induction on \( \nu \). The case of \( \nu = 1 \) is clear. Assume \( \nu > 1 \). Let

\[
\eta' = d^{\nu-1} \left( \frac{s_\lambda}{s_D} \right).
\]

Then \( d\eta' = \eta \). We use the symbols \( C_\nu (1 \leq \nu \leq 7) \) to denote constants. Let \( e^{-\psi} \) be a smooth Hermitian metric for \( D \) and \( \eta_0 \) be a smooth positive definite \((1, 1)\)-form on \( M \). Then

\[
|s_D D^\nu (s_D^{\nu} \eta') - s_D^{\nu+1} \eta|_{(\nu+1)D} \leq C_1 |s_D^{\nu} \eta'|_{\nu D} \eta_0^\frac{1}{2},
\]

where \( D_{\nu\psi} \) denote the covariant differentiation for sections of the line bundle \( \nu D \) with respect to the metric \( e^{-\nu \psi} \). By Lemma 3 and (16) we have

\[
\mathcal{I}_r \left( f^* \left[ \frac{|s_D^{\nu+1} \eta|_{(\nu+1)D}^2}{(1 + |s_D^{\nu} \eta'|_{\nu D}^2)^2} \right] \right) \leq C_2 \cdot \mathcal{A}_r \left( f^* \log \left( 1 + |s_D^{\nu} \eta'|_{\nu D}^2 \right) \right)
\]

\[
+ O(T(r, f) + \log r).
\]

Take \( 0 < \epsilon \leq \frac{1}{2} \) min \((1, \epsilon_{\nu, \nu})\). From (17) it follows that

\[
\mathcal{I}_r \left( (f^* |s_D^{\nu+1} \eta|_{(\nu+1)D}^2) \epsilon \right) = \mathcal{I}_r \left( f^* \left[ \frac{|s_D^{\nu+1} \eta|_{(\nu+1)D}^2}{(1 + |s_D^{\nu} \eta'|_{\nu D}^2)^2} f^* \left( 1 + |s_D^{\nu} \eta'|_{\nu D}^2 \right) \right] \epsilon \right)
\]

\[
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\]

\[
19
\]
\[ \leq C_3 \left( I_r(1) + I_r \left( f^* \frac{\left| s_D^{\nu+1} \eta \right|^2_{(\nu+1)D}}{1 + \left| s_D^{\nu} \eta \right|^2_{\nu D}} \right) + I_r \left( (f^* (1 + \left| s_D^{\nu} \eta^2 \right|_{\nu D}^2)^2) \right) \right) \]

\[ \leq C_4 \left( (A_r (f^* \log (1 + \left| s_D^{\nu} \eta^2 \right|_{\nu D}^2))) + I_r \left( (f^* (1 + \left| s_D^{\nu} \eta^2 \right|_{\nu D}^2)^2) \right) + O(T(r, f) + \log r) \right) \]

\[ \leq C_5 A_r \left( f^* \log (1 + \left| s_D^{\nu} \eta^2 \right|_{\nu D}^2) \right) + O(T(r, f) + \log r) \parallel, \]

where for the last inequality the induction hypothesis is used. The standard techniques in value distribution theory of using the so-called Calculus Lemma and the concavity of the logarithmic function gives

(19) \[ A_r (\log g) \leq C_6 (\log r + \log I_r (g)) \parallel \]

for any smooth positive function \( g \) on \( C \). Putting this into (18) yields

\[ I_r \left( (f^* \left| s_D^{\nu+1} \eta \right|^2_{(\nu+1)D})^\epsilon \right) \]

\[ \leq C_7 \log I_r \left( f^* (1 + \left| s_D^{\nu} \eta^2 \right|_{\nu D}^2)^{2\epsilon} \right) + O(T(r, f) + \log r) \parallel, \]

when we use

\[ g = f^* (1 + \left| s_D^{\nu} \eta^2 \right|_{\nu D}^2)^{2\epsilon}. \]

Thus

\[ I_r \left( f^* \frac{\left| s_D^{\nu+1} \eta \right|^2_{(\nu+1)D}}{1 + \left| s_D^{\nu} \eta \right|^2_{\nu D}} \right) \leq O(\log T(r, f) + \log r) \parallel \]

by induction hypothesis and the induction argument is complete. Q.E.D.

**Proof of Theorem 1.** Let \( s_D \) be the canonical section of the line bundle \( L_D \) associated to the divisor \( D \). Let \( e^{-\psi} \) be a Hermitian metric of \( L_D \) whose curvature is a positive definite smooth \((1,1)\)-form \( \theta_D \) on \( X \) (in the sense that \( \theta_D \) is the restriction of some smooth positive definite \((1,1)\)-form on \( \mathbb{P}_N \)). Suppose \( f^* \omega \) is not identically zero on \( C \) and we are going to derive a contradiction. From

\[ \partial \overline{\partial} \log \left( e^\psi \left( \frac{\omega}{s_D}(\overline{\frac{\omega}{s_D}}) \right) \right) \geq \theta_D \]
we conclude from Green’s theorem that

\[(20) \quad \mathcal{A}_r \left( \log f^* \left| \frac{\omega}{s_D} \right|^2 \right) \geq \mathcal{I}_r \left( f^* \theta_D \right) - O(1). \]

It follows from (19) that

\[\mathcal{A}_r \left( \log f^* \left| \frac{\omega}{s_D} \right|^\epsilon \right) \leq C \left( \log r + \log \mathcal{I}_r \left( f^* \left| \frac{\omega}{s_D} \right|^\epsilon \right) \right) \|

for any \( \epsilon > 0 \) and for some constant \( C \) depending on \( \epsilon \). Let \( H_N \) be the hyperplane section line bundle of \( \mathbb{P}_N \). There exist some positive integer \( \ell \) and a number of global holomorphic sections \( \sigma_1, \ldots, \sigma_q \) of \( H_{N}^{\otimes \ell} \) over \( \mathbb{P}_N \) without common zeroes so that \( \sigma_j \left( \frac{\omega}{s_D} \right) \) can be extended to a global \( H_{N}^{\otimes \ell} \)-valued holomorphic \( k \)-jet differential over \( \mathbb{P}_N \) for \( 1 \leq j \leq q \). By Proposition 1 applied to \( \sigma_j \left( \frac{\omega}{s_D} \right) \) on \( \mathbb{P}_N \) for \( 1 \leq j \leq q \), we get

\[(21) \quad \mathcal{I}_r \left( f^* \left| \frac{\omega}{s_D} \right|^\epsilon \right) = O(T(r, f) + \log r) \|

for some \( \epsilon > 0 \) and consequently

\[\mathcal{A}_r \left( \log f^* \left| \frac{\omega}{s_D} \right|^\epsilon \right) = O(\log T(r, f) + \log r) \|.

Combining with (20), we obtain

\[\mathcal{I}_r \left( f^* \theta_D \right) \leq O(\log T(r, f) + \log r) \|,

which implies that \( T(r, f) \) is of the order \( \log r \). From (21) it follows that

\[(22) \quad \mathcal{I}_r \left( f^* |\omega|^\epsilon \right) = O(\log r) \|.

On the other hand, from the subharmonicity of \( f^* |\omega|^\epsilon \) we conclude that the growth of \( \mathcal{I}_r \left( f^* |\omega|^\epsilon \right) \) is at least \( r^2 \), contradicting (22). Q.E.D.

§3. The General Schwarz Lemma for the Log-Pole Case.

Theorem 2 is proved by a modification of the arguments in the proof of Theorem 1. We only present here the necessary modifications. For the
modification of Proposition 1, we assume that $X$ is nonsingular and let $M = X$. We let $t_j$ be the canonical section of the line bundle over $M$ associated to the divisor $Z_j$ ($1 \leq j \leq q$). We assume that the image of the holomorphic map $f$ is disjoint from $\bigcup_{j=1}^{q} Z_j$. Fix a smooth metric for the line bundle $Z_j$ so that the norm of $|t_j|_{Z_j} < 1$ on $M$. For any positive number $A > e$ let $\tau_{j,A} = \log \left( \frac{A}{|t_j|_{Z_j}} \right)$. The modified Proposition 1 states that, for any $E$-valued meromorphic $k$-jet differential $\omega$ of weight $m$ on $M$ with at most log-pole singularity along $\bigcup_{j=1}^{q} Z_j$, there exist positive numbers $\epsilon_{k,m}$ and $A_{k,m}$ and a positive integer $a_{k,m}$ such that

$$\mathcal{I}_r \left( \left( f^* \frac{|\omega|_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right)^{\epsilon} \right) = O(T(r,f) + \log r)$$

for $0 < \epsilon \leq \epsilon_{k,m}$, $A > A_{k,m}$, and $a \geq a_{k,m}$.

**Remark.** Heuristically speaking, the factor $\prod_{j=1}^{q} \tau_{j,A}^a$ in the statement of the modified Proposition 1 is due to the restriction placed by the log-pole $\bigcup_{j=1}^{q} Z_j$ on the radius of the circle $\Gamma$ used in the domination of the higher order derivatives at the center of $\Gamma$ by the first-order derivative on $\Gamma$ by Cauchy’s integral formula for derivatives.

Lemma 3 holds when for $\omega$ with log-pole singularities as long as the image of the map $f$ is disjoint from the log-pole. We use $C_j$ ($1 \leq j \leq 6$) to denote constants. From the last inequality in Lemma 3 it follows that

$$\mathcal{I}_r \left( \left( f^* \frac{|D_{\psi} \omega|^2_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right)^{\epsilon} \right) \leq C_1 \left\{ \mathcal{I}_r \left( f^* \frac{|D_{\psi} \omega|^2_E}{(1 + |\omega|^2_E)^2} \right) + \mathcal{I}_r \left( \left( f^* \frac{|\omega|^2_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right)^{2\epsilon} \right) \right\}$$

$$\leq C_2 \left\{ \mathcal{A}_r \left( f^* \log \left( 1 + |\omega|^2_E \right) \right) + \mathcal{I}_r \left( \left( f^* \frac{|\omega|^2_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right)^{2\epsilon} \right) \right\}$$

$$+ O(T(r,f,\theta_{\psi}) + \log r)$$

$$\leq C_3 \left\{ \mathcal{A}_r \left( f^* \log \left( 1 + \frac{|\omega|^2_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right) \right) + \mathcal{A}_r \left( f^* \log \prod_{j=1}^{q} \tau_{j,A}^a \right) + \mathcal{I}_r \left( \left( f^* \frac{|\omega|^2_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right)^{2\epsilon} \right) \right\} + O(T(r,f,\theta_{\psi}) + \log r)$$

$$\leq \mathcal{I}_r \left( \left( f^* \frac{|\omega|^2_E}{\prod_{j=1}^{q} \tau_{j,A}^a} \right)^{2\epsilon} \right) + O(T(r,f,\theta_{\psi}) + \log r)$$
\[
\leq C_4 \left\{ \sum_{j=1}^q \log \mathcal{A}_r (f^* \tau_{j,A}) + \mathcal{I}_r \left( \left( \frac{f^* |\omega|_E^2}{\prod_{j=1}^q \tau_{j,A}^2} \right)^{2e} \right) \right\} \\
+ O(T(r, f, \theta_\psi) + \log r) \parallel \n.
\]

Using the domination of the proximity function \( \mathcal{A}_r (f^* \tau_{j,A}) \) by the characteristic function \( T(r, f, \theta_{Z_j}) + O(1) \) (where \( \theta_{Z_j} \) is the curvature of the metric of the line bundle \( Z_j \)), we conclude that

\[
\mathcal{I}_r \left( \left( \frac{f^* |\omega|_E^2}{\prod_{j=1}^q \tau_{j,A}^2} \right)^{2e} \right) \\
\leq C_5 \left\{ \mathcal{I}_r \left( \left( \frac{f^* |\omega|_E^2}{\prod_{j=1}^q \tau_{j,A}^2} \right)^{2e} \right) \right\} + O(T(r, f, \theta_\psi) + \log r) \parallel \n.
\]

For the proof of the modified Proposition 1, it suffices to consider the case where \( \omega \) is of the form \( vt^{\nu+1}d^\nu \log \left( \frac{z}{s} \right) \), where

(i) \( \nu \) is a positive integer,
(ii) \( s, t \) are global holomorphic sections of some ample line bundle \( L \),
(iii) no \( Z_j (1 \leq j \leq q) \) is a branch of the zero-set of \( t \),
(iv) \( v \) is a global holomorphic section of some ample line bundle \( L' \),
(v) no \( Z_j (1 \leq j \leq q) \) is a branch of the zero-set of \( v \), and
(vi) \( \omega \) is holomorphic outside \( \cup_{j=1}^q Z_j \).

As in the proof of Proposition 1, one further reduces the proof of the modified Proposition 1 to the case where \( \nu = 1 \). We choose a smooth metric \( e^{-\psi} \) of \( L \) so that its curvature \( \theta_\psi \) is a positive definite \((1,1)\)-form on \( M \). We replace \( e^{-\psi} \) by \( Ae^{-\psi} \) for a sufficiently large positive number \( A \) so that \( |s|_L < 1 \) and \(-\log |s|_L^2 > \frac{1}{\delta} \) on \( M \) for some positive number \( \delta < 1 \). Then

\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \frac{|t^2 d \left( \frac{z}{s} \right)|_{L^{\otimes 2}}^2}{|s|_L^2 (\log |s|_L^2)^2} \geq -(1 + \delta) \theta_\psi - \text{div} \, s + \frac{|D_\psi s|^2}{|s|_L^2 \log |s|_L^2},
\]

where \( \text{div} \, s \) is the divisor of \( s \) regarded as a \((1,1)\)-current. Now use

\[
\frac{|tD_\psi s - t^2 d \left( \frac{z}{s} \right)|_{L^{\otimes 2}}^2}{|s|_L^2} = O(1)
\]
and the standard techniques in value distribution theory of Green’s Theorem, the Calculus Lemma, and the concavity of the logarithmic function to conclude that

\[ I_r \left( f^* \frac{|t^2 d \left( \frac{s}{7} \right)|^2_{L^2_L}}{|s|^2_L (\log |s|^2_L)^2} \right) = O(T(r, f) + \log r) \]

Because of condition (vi), the section \( v \) used as a factor in \( \omega \) takes care of all the branches of the divisor of \( s \) except those contained in the log-pole set \( \bigcup_{j=1}^q Z_j \). This concludes the proof of the modified Proposition 1.

Now we come to the proof of Theorem 2. Assume that \( f^* \omega \) is not identically zero and we are going to derive a contradiction. The same argument used in the proof of Theorem 1 gives us (20) from which we conclude

\[ I_r (f^* \theta_D) \leq C_1 \left( A_r \left( \log f^* \left( \frac{|\omega|_{L^2_L}^2}{\prod_{j=1}^q \tau_{j,A}^a} \right) \right) + A_r \left( \log f^* \prod_{j=1}^q \tau_{j,A}^a \right) \right) + O(1) \]

\[ \leq C_2 \left( \log I_r \left( f^* \left( \frac{|\omega|_{L^2_L}^2}{\prod_{j=1}^q \tau_{j,A}^a} \right)^\epsilon \right) + \sum_{j=1}^q \log A_r (f^* \tau_{j,A}) \right) + O(\log r) \]

\[ = O(\log T(r, f) + O(\log r)) \]

where \( 0 < \epsilon \leq \epsilon_{k,m} \). This implies that \( T(r, f) \) is of the order \( \log r \). From

\[ \partial \bar{\partial} \log \frac{1}{\tau_{j,A}} \geq \frac{-2}{\log |A_{j,A}|^2} \theta_{Z_j} \]

it follows that

\[ \log f^* \left( \frac{|\omega|^2}{\prod_{j=1}^q \tau_{j,A}^a} \right) \]

is subharmonic when \( A \) is greater than some constant \( \tilde{A}(a) \) depending on \( a \) and consequently

\[ f^* \left( \frac{|\omega|^2}{\prod_{j=1}^q \tau_{j,A}^a} \right)^\epsilon \]

is subharmonic, from which we conclude that the order of growth of

\[ I_r \left( f^* \left( \frac{|\omega|^2}{\prod_{j=1}^q \tau_{j,A}^a} \right)^\epsilon \right) \]
is at least $r^2$ which is a contradiction.

§4. Construction of Hyperbolic Hypersurfaces

We now construct hyperbolic hypersurfaces of degree $16(n - 1)^2$ in $\mathbb{P}_n$ by using Borel’s lemma and a simple dimension counting argument for certain subvarieties of Grassmannians. We introduce a generalized Borel lemma (Proposition 2) which will be used in §5 and which implies as an corollary the usual Borel lemma (Proposition 3) used in here in §4.

**Proposition 2 (Generalized Borel Lemma).** Let $g_j(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree $\delta_j$ for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C} \to \mathbb{P}_n$ so that its image lies in

$$n \sum_{j=0}^{n} x_j^{p-\delta_j} g_j(x_0, \ldots, x_n) = 0$$

and $p > (n + 1)(n - 1) + \sum_{j=0}^{n} \delta_j$. Then there is a nontrivial linear relation among $x_1^{p-\delta_1} g_1(x_0, \ldots, x_n), \ldots, x_n^{p-\delta_n} g_n(x_0, \ldots, x_n)$ on the image of $f$.

**Proof.** Use the affine coordinates $z_j = \frac{x_j}{x_0}$ for $1 \leq j \leq n$. Let

$$\tilde{g}_j(z_1, \ldots, z_n) = x_0^{-\delta_j} g_j(x_0, \ldots, x_n).$$

From

$$\sum_{j=0}^{n} x_j^{p-\delta_j} g_j(x_0, \ldots, x_n) = 0$$

we have the following relation between the two Wronskians

$$W(\tilde{g}_0, z_1^{p-\delta_1} \tilde{g}_1, \ldots, z_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1}) = (-1)^{n-1} W(\tilde{g}_0, z_2^{p-\delta_2} \tilde{g}_2, \ldots, z_n^{p-\delta_n} \tilde{g}_n).$$

Rewrite the equation as

$$z_1^{p-\delta_1-n+1} \left\{ \frac{1}{\prod_{j=1}^{n-1} z_j^{p-\delta_j-n+1}} W(\tilde{g}_0, z_1^{p-\delta_1} \tilde{g}_1, \ldots, z_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1}) \right\}$$

$$= (-1)^{n-1} z_n^{p-\delta_n-n+1} \left\{ \frac{1}{\prod_{j=2}^{n} z_j^{p-\delta_j-n+1}} W(\tilde{g}_0, z_2^{p-\delta_2} \tilde{g}_2, \ldots, z_n^{p-\delta_n} \tilde{g}_n) \right\}.$$
Since
\[
\frac{1}{\prod_{j=1}^{n-1} \tilde{z}_j^{p-\delta_j-n+1}} W \left( \tilde{g}_0, z_1^{p-\delta_1} \tilde{g}_1, \ldots, z_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1} \right)
\]
and
\[
\frac{1}{\prod_{j=2}^{n} \tilde{z}_j^{p-\delta_j-n+1}} W \left( \tilde{g}_0, z_2^{p-\delta_2} \tilde{g}_2, \ldots, z_n^{p-\delta_n} \tilde{g}_n \right)
\]
are both holomorphic on the affine part of the hypersurface, we conclude that
\[
\frac{1}{\prod_{j=1}^{n-1} \tilde{z}_j^{p-\delta_j-n+1}} W \left( \tilde{g}_0, z_1^{p-\delta_1} \tilde{g}_1, \ldots, z_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1} \right)
\]
is divisible by \(z_n^{p-\delta_n-n+1}\). Thus
\[
\frac{1}{\prod_{j=1}^{n} \tilde{z}_j^{p-\delta_j-n+1}} W \left( \tilde{g}_0, z_1^{p-\delta_1} \tilde{g}_1, \ldots, z_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1} \right)
\]
is holomorphic on the affine part of the hypersurface. Now we look at the infinity part. We introduce the coordinates \(w_j = \frac{x_j}{x_n}\) for \(0 \leq j \leq n - 1\) so that \(z_j = \frac{w_j}{w_n}\) for \(1 \leq j \leq n - 1\) and \(z_n = \frac{1}{w_0}\). We have
\[
W \left( \tilde{g}_0, z_1^{p-\delta_1} \tilde{g}_1, \ldots, z_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1} \right)
\]
for \(0 \leq j \leq n\). Thus
\[
\frac{1}{\prod_{j=1}^{n} w_j^{p-\delta_j-n+1}} W \left( w_0^{p-\delta_0} \tilde{g}_0, w_1^{p-\delta_1} \tilde{g}_1, \ldots, w_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1} \right)
\]
which is holomorphic on the whole hypersurface and vanishes on an ample divisor, because \(p > (n + 1)(n - 1) + \sum_{j=0}^{n} \delta_j\). We conclude from Theorem 1 that the Wronskian
\[
W \left( w_0^{p-\delta_0} \tilde{g}_0, w_1^{p-\delta_1} \tilde{g}_1, \ldots, w_{n-1}^{p-\delta_{n-1}} \tilde{g}_{n-1} \right)
\]
26
must be identically zero on the image of $f$ (more precisely on the image of $d^{n-1} f$) and there is a nontrivial linear relation among

$$x_1^{p-\delta_1} g_1(x_0, \ldots, x_n), \ldots, x_n^{p-\delta_n} g_n(x_0, \ldots, x_n)$$
on the image of $f$. Q.E.D.

**Proposition 3 (the Borel lemma for high powers of entire functions).** Let $n \geq 2$ and $p > (n-1)(n+1)$ be integers. Let $f_0, \ldots, f_n$ be entire functions on $\mathbb{C}$ satisfying $f_0 + \cdots + f_n \equiv 0$ such that $f_j = g_j^p$ for some entire function $g_j$ for $1 \leq j \leq n$. Then after relabelling the set $f_0, \ldots, f_n$, one can divide up the set $\{0, \ldots, n\}$ into $q$ disjoint subsets $\{\ell_0, \ldots, \ell_1 - 1\}, \{\ell_1, \ldots, \ell_2 - 1\}, \ldots, \{\ell_{q-1}, \ldots, \ell_q - 1\}$ with $0 = \ell_0 < \ell_1 < \cdots < \ell_q = n + 1$ and one can find constants $c_{j} (0 \leq \mu < q$ and $\ell_{\mu} < j < \ell_{\mu+1})$ such that $f_j \equiv c_{\mu,j} f_{\ell_{\mu}}$ for $\ell_{\mu} < j < \ell_{\mu+1}$ and $\sum_{j=\ell_{\mu}}^{\ell_{\mu+1} - 1} f_j \equiv (1 + \sum_{j=\ell_{\mu}+1}^{\ell_{\mu+1}-1} c_{\mu,j}) (f_{\ell_{\mu}}) \equiv 0$.

**Proof.** This follows from Proposition 2 by using $\delta_j = 0$ for $0 \leq j \leq n$ and use induction on $n$.

Alternatively one can also argue directly by using Cartan’s version of the Second Main Theorem with truncated counting function instead of Proposition 2 as follows.

Let $f : \mathbb{C} \to \mathbb{P}_n$ be a nonconstant holomorphic map whose image is contained in $X$. Consider the map $\Phi : \mathbb{P}_n \to \mathbb{P}_{n-1}$ defined with the homogeneous coordinates $[(g_1)^p, \ldots, (g_n)^p]$. Let $H_{\mu} (1 \leq \mu \leq n - 1)$ be the coordinate hyperplanes of $\mathbb{P}_{n-1}$. Let $H'$ be the hyperplane in $\mathbb{P}_{n-1}$ defined by the vanishing of the sum of the homogeneous coordinates of $\mathbb{P}_{n-1}$. Since the image of $f$ lies in $X$, the pullback by $\Phi \circ f$ of the defining function of $H'$ is the same as the pullback by $f$ of $-(f^* s_0)^p$. A point $P$ of $H'$ is assumed by $\Phi \circ f$ at some point $z_0$ of $\mathbb{C}$ if and only if $-(f^* s_0)^p$ vanishes at $z_0$ and, in that case, it must automatically vanish to order at least $p$ and so that the point $P$ of $H'$ is assumed by $\Phi \circ f$ with multiplicity at least $p$. As a consequence the truncated counting function $N_{n-1} (r, \Phi \circ f, H')$ is no more than $\frac{n-1}{p} N(r, \Phi \circ f, H')$. Here the truncated counting function $N_{n-1} (r, \Phi \circ f, H')$ means that multiplicities higher than $n-1$ are replaced by multiplicities equal to $n-1$. The same argument holds for $H_j (1 \leq j \leq N)$ instead of $H'$. Unless the image of $\Phi \circ f$ is contained in a hyperplane of $\mathbb{P}_{n-1}$, we know from
Cartan’s Second Main Theorem with truncated counting function that

\[ T(r, \Phi \circ f) \leq N_{n-1}(r, \Phi \circ f, H') + \sum_{j=1}^{n} N_{n-1}(r, \Phi \circ f, H_j) + O(\log T(r, \Phi \circ f)) \]

\[ \leq \frac{n-1}{p} (N(r, \Phi \circ f, H') + \sum_{j=1}^{n} N(r, \Phi \circ f, H_j)) + O(\log T(r, \Phi \circ f)) \]

\[ \leq \frac{n-1}{p} (n + 1)T(r, \Phi \circ f) + O(\log T(r, \Phi \circ f)), \]

which is a contradiction if \( p > (n + 1)(n - 1) \). So we conclude that the image of \( f \) is contained in the zero-set of \( \sum_{j=1}^{n} \lambda_j (g_j)^p = 0 \) for some \( \lambda_j \in \mathbb{C} \) \((1 \leq j \leq n)\) not all zero. Now we use induction on \( n \). Q.E.D.

Now we introduce the argument of counting dimensions of certain sub-varieties of Grassmannians. Let \( 2 \leq k \leq m - 1 \) and \( \ell \geq 2 \). Let \( \mathcal{G} \) be the Grassmannian of \( \mathbb{C}^k \) in \( \mathbb{C}^m \). For hyperplanes \( H_1, \ldots, H_\ell \) in \( \mathbb{C}^m \) let \( \mathcal{G}_{H_1, \ldots, H_\ell} \) be the subvariety of \( \mathcal{G} \) consisting of all \( W \in \mathcal{G} \) such that the dimension over \( \mathbb{C} \) of the linear space generated by \( H_1|W, \ldots, H_\ell|W \) is no more than 1. For generic hyperplanes \( H_1, \ldots, H_\ell \) in \( \mathbb{C}^m \) the dimension over \( \mathbb{C} \) of \( \mathcal{G}_{H_1, \ldots, H_\ell} \) is equal to the dimension of the Grassmann \( \mathcal{G}' \) of all \( \mathbb{C}^k \) in \( \mathbb{C}^{m-\ell+1} \) plus \( \ell - 1 \). The reason is as follows. For \( W \in \mathcal{G}_{H_1, \ldots, H_\ell} \), if the linear space generated by \( H_1|W, \ldots, H_\ell|W \) is precisely of dimension 1, after relabelling \( H_1, \ldots, H_\ell \), we have constants \( c_2, \ldots, c_\ell \) such that \( H_j = c_j H_1 \) for \( 2 \leq j \leq \ell \) and \( W \) is an element of the Grassmannian \( \mathcal{G}' \) of all \( \mathbb{C}^k \) in

\[ \mathbb{C}^{m-\ell+1} = \mathbb{C}^m \cap \{ H_2 = c_2 H_1, \ldots, H_\ell = c_\ell H_1 \}, \]

and the freedom of choices for \( c_2, \ldots, c_\ell \) gives the additional \( \ell - 1 \) dimensions. If all \( H_1|W, \ldots, H_\ell|W \) are identically zero, then \( W \) is an element of the Grassmannian \( \mathcal{G}' \) of all \( \mathbb{C}^k \) in

\[ \mathbb{C}^{m-\ell} = \mathbb{C}^m \cap \{ H_1 = \ldots = H_\ell = 0 \} \]

and \( \mathcal{G}' \subset \mathcal{G} \). Thus for generic hyperplanes \( H_1, \ldots, H_\ell \) in \( \mathbb{C}^m \) the codimension of \( \mathcal{G}_{H_1, \ldots, H_\ell} \) in \( \mathcal{G} \) is \( (k - 1)(\ell - 1) \).

Let \( q \geq 1 \) and \( \ell_\nu \geq 2 \) for \( 1 \leq \nu \leq q \). For generic hyperplanes

\[ H_1^{(1)}, \ldots, H_{\ell_1}^{(1)}, \ldots, H_1^{(q)}, \ldots, H_{\ell_q}^{(q)}, \]
the codimension over $\mathbb{C}$ of

$$
G_{H_1^{(1)}, \ldots, H_{\ell_1}^{(1)}} \cap \cdots \cap G_{H_1^{(q)}, \ldots, H_{\ell_q}^{(q)}}
$$

is equal to $(k - 1) \sum_{\nu=1}^{q} (\ell_\nu - 1)$ when $(k - 1) \sum_{\nu=1}^{q} (\ell_\nu - 1) \leq k(m - k)$ and

is empty $(k - 1) \sum_{\nu=1}^{q} (\ell_\nu - 1) > k(m - k)$ Let $N = \sum_{\nu=1}^{q} \ell_\nu$.

**Proposition 4.** For $N \geq 4m - 7$ and for generic hyperplanes

$$
H_1^{(1)}, \ldots, H_{\ell_1}^{(1)}, \ldots, H_1^{(q)}, \ldots, H_{\ell_q}^{(q)},
$$

in $\mathbb{C}^m$ the subvariety

$$
G_{H_1^{(1)}, \ldots, H_{\ell_1}^{(1)}} \cap \cdots \cap G_{H_1^{(q)}, \ldots, H_{\ell_q}^{(q)}}
$$

of $G$ is empty.

**Proof.** It follows from $\ell_\nu \geq 2$ that $N = \sum_{\nu=1}^{q} \ell_\nu \geq 2q$. The codimension of the subvariety

$$
G_{H_1^{(1)}, \ldots, H_{\ell_1}^{(1)}} \cap \cdots \cap G_{H_1^{(q)}, \ldots, H_{\ell_q}^{(q)}}
$$

in $G$ is

$$(k - 1) \sum_{\nu=1}^{q} (\ell_\nu - 1) = (k - 1)(N - q) \geq (k - 1) \frac{N}{2} > k(m - k) = \dim \mathbb{C} G.$$

Q.E.D.

**Proof of Theorem 3.** Suppose the contrary and we are going to derive a contradiction. Let $f : \mathbb{C} \to \mathbb{P}_{n}$ be a nonconstant holomorphic map whose image lies in the hypersurface $\sum_{j=1}^{N} H_j^p = 0$. Without loss of generality we can assume that the image of $f$ does not lie in the the hyperplane $H_j = 0$ for any $1 \leq j \leq N$, otherwise we argue instead with the hypersurface $\sum_{j \in J} H_j^p = 0$, where $J$ is the set of all $1 \leq j \leq N$ with the property that the image of $f$ does not lie in the the hyperplane $H_j = 0$. Let $\tilde{f} : \mathbb{C} \to \mathbb{C}^{n+1}$ be a lifting of $f$. Let $W$ be the linear subspace of $\mathbb{C}^{n+1}$ which is the linear span of the image of $\tilde{f}$. Let $k$ be the complex dimension of $W$. Since $f$ is nonconstant, we know that $k$ is at least 2. Let $m = n + 1$. By the Borel lemma for high powers of entire functions, we have a partition

$$
\{H_1^{(1)}, \ldots, H_{\ell_1}^{(1)}\} \cup \cdots \cup \{H_1^{(q)}, \ldots, H_{\ell_q}^{(q)}\}
$$
of \( \{H_0, \cdots, H_N\} \) with \( \ell_j \geq 2 \) for \( 1 \leq j \leq q \) so that the complex dimension of the linear space spanned by \( H_1^{(\nu)}|W, \cdots, H_{\ell_\nu}^{(\nu)}|W \) is at most 1 for \( 1 \leq \nu \leq q \), contradicting the preceding lemma which states that the subvariety

\[
\mathcal{G}_{H^{(1)}_1, \cdots, H^{(q)}_1} \cap \cdots \cap \mathcal{G}_{H^{(q)}_1, \cdots, H^{(q)}_q}
\]

of \( \mathcal{G} \) is empty. Q.E.D.

§5. Hyperbolic Surface of Degree 11

We now prove Theorem 4. Assume \( n \geq 11 \) and denote by \( S \) the surface defined by the equation

\[
x_0^n + x_1^n + x_2^n + x_3^{n-2}g(x_0, x_1, x_2, x_3) = 0.
\]

Suppose we have a nonconstant holomorphic map \( f : \mathbb{C} \to S \) and we are going to derive a contradiction. By Proposition 2, we have a nontrivial linear relation among \( x_0^n, x_1^n, x_2^n \) on the image of \( \mathbb{C} \), which we can assume without loss of generality to be \( x_0^n = c_1 x_1^n + c_2 x_2^n \). When both \( c_1, c_2 \) are nonzero, the image of \( \mathbb{C} \) lies in the Fermat curve \( x_0^n = c_1 x_1^n + c_2 x_2^n \) and we end up with \([x_0, x_1, x_2] = \text{constant}\) on the image of \( \mathbb{C} \). We cannot have all three \( x_0, x_1, x_2 \) identically zero on the image of \( \mathbb{C} \), because the assumption \( g(0, 0, 0, x_3) = x_3^2 \) would imply that \( x_3 \) is identically zero on the image of \( \mathbb{C} \) as well. Suppose without loss of generality that \( x_0 \) is not identically zero. Then we end up with

\[
b_0 x_0^n + x_3^{n-2}(x_3^2 + b_1 x_3 x_0 + b_2 x_0^2)
\]

for some constants \( b_0, b_1, b_2 \), implying that the image of \( f \) is constant.

Before we continue further, we make the trivial observation that, for a quadratic polynomial \( h(y) = Ay^2 + By + C \) of a single variable \( y \),

\[
\frac{(h')^2}{2h''} - h = \frac{B^2 - 4AC}{4A},
\]

where \( h' \) and \( h'' \) denote respectively the first and second derivatives of \( h \) with respect to \( y \).

Now assume that \( c_2 = 0 \) and \( x_0^n = c_1 x_1^n \) on the nonconstant image of \( \mathbb{C} \), we conclude that \( x_0 = c_1^{1/n} x_1 \) for some \( n \)th root of \( c_1 \) on the image of \( \mathbb{C} \). Thus
the curve $C$ defined by

$$x_0 = c_1^1 x_1,$$

$$(1 + c_1)x_1^n + x_2^n + x_3^{n-2}g(c_1^1 x_1, x_1, x_2, x_3) = 0$$

contains the image of $C$. Let $U_3$ be the affine open subset $\{x_3 \neq 0\}$. Then $C \cap U_3$ is defined, in terms of the affine coordinates $\zeta_j = x_j x_3 (0 \leq j \leq 2)$ by

$$\zeta_0 = c_1^1 \zeta_1,$$

$$(1 + c_1)c_1^n + \zeta_2^n + g(c_1^1 \zeta_1, \zeta_1, \zeta_2, 1) = 0.$$ We distinguish between two cases. First we consider the case $1 + c_1 = 0$. In that case we have

$$\zeta_0 = (-1)^{\frac{n}{2}} \zeta_1,$$

$$\zeta_2^n + g((-1)^{\frac{n}{2}} \zeta_1, \zeta_1, \zeta_2, 1) = 0.$$ Denote $g((-1)^{\frac{n}{2}} \zeta_1, \zeta_1, \zeta_2, 1)$ by $h(\zeta_1, \zeta_2)$ and let

$$h(\zeta_1, \zeta_2) = A(\zeta_2)\zeta_1^2 + B(\zeta_2)\zeta_1 + C(\zeta_2).$$

Then

$$0 = \zeta_2^n + h(\zeta_1, \zeta_2)$$

$$= \zeta_2^n + A(\zeta_2) \left( \zeta_1 + \frac{B(\zeta_2)}{4A(\zeta_2)} \right)^2 - \frac{B(\zeta_2)^2 - 4A(\zeta_2)C(\zeta_2)}{4A(\zeta_2)}$$

$$= \zeta_2^n + A(\zeta_2) \left( \zeta_1 + \frac{B(\zeta_2)}{4A(\zeta_2)} \right)^2 - \left( \frac{\partial h}{\partial \z_1} \right)^2 - h)$$

and

$$A(\zeta_2) \left( \zeta_1 + \frac{B(\zeta_2)}{4A(\zeta_2)} \right)^2 = -\zeta_2^n - \left( \frac{\partial h}{\partial \z_1} \right)^2 - \left( \frac{\partial h}{\partial \z_2} \right)^2 - h.$$

Since the polynomial

$$-\zeta_2^n + \left( \frac{\partial h}{\partial \z_1} \right)^2 - h$$

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in $\zeta_2$ has $n$ distinct roots by assumption, it follows that the normalization of the hyperelliptic Riemann surface defined by

$$\zeta_2^n + g((-1)^{\frac{1}{n}}\zeta_1, \zeta_1, \zeta_2, 1) = 0$$

has genus equal to $\lceil \frac{n}{2} \rceil - 1$, where $\lceil \cdot \rceil$ denotes the round-up. This contradicts the nonconstancy of $f$.

Now we consider the case $1 + c_1 \neq 0$. Then from the second equation in

$$x_0 = c_1^{\frac{1}{n}} x_1,$$

$$(1 + c_1)x_1^n + x_2^n + x_3^{n-2} g(c_1^{\frac{1}{n}} \zeta_1, \zeta_1, \zeta_2, 1) = 0$$

we conclude that $x_1^n, x_2^n$ have a nontrivial linear relation on the image of $C$. Again we conclude that the image of $C$ is constant. This concludes the proof of Theorem 4.

Remark. In Theorem 4, one can also easily formulate a necessary and sufficient condition on $g(x_0, x_1, x_2, x_3)$ so that the surface is hyperbolic for $n \geq 11$.

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