A CONTRIBUTION TO THE CALDERÓN PROBLEM FOR YANG-MILLS CONNECTIONS

MIHAJLO CEKIĆ

ABSTRACT. We consider the problem of identifying a unitary Yang-Mills connection $\nabla$ on a Hermitian vector bundle from the Dirichlet-to-Neumann (DN) map of the connection Laplacian $\nabla^*\nabla$ over compact Riemannian manifolds with boundary. We establish such uniqueness of the connection up to a gauge equivalence in the case of line bundles in the smooth category and for the higher rank case in the analytic category. Furthermore, we prove that on the restriction of the vector bundle to the boundary the DN map is an elliptic pseudodifferential operator of order one, whose full symbol determines the complete Taylor series of an arbitrary connection and a metric (also of an associated potential) at the boundary.

1. Introduction

In this paper, we consider the Calderón problem for a special type of connections, called the Yang-Mills connections. Given a Hermitian vector bundle $E$ of rank $m$ over a compact Riemannian manifold $(M, g)$ with non-empty boundary and a unitary connection $A (\nabla)$ on $E$, one may consider the connection Laplacian denoted by $d_A^*d_A (\nabla^*\nabla)$, where $d_A^* (\nabla^*)$ denotes the formal adjoint of $d_A (\nabla)$ with respect to the Hermitian and Riemannian structures. Sometimes this operator is called the magnetic Laplacian because it is used to represent the magnetic Schrödinger equation, where $A$ corresponds to the magnetic potential.

Given this, we may define the associated Dirichlet-to-Neumann (DN map in short) $\Lambda_A : C^\infty(\partial M; E|_{\partial M}) \to C^\infty(\partial M; E|_{\partial M})$ by solving the Dirichlet problem:

$$d_A^*d_A(u) = 0, \quad u|_{\partial M} = f$$

(1.1)

and setting $\Lambda_A(f) = d_A(u)(\nu)$, where $\nu$ is the outwards pointing normal at the boundary. The problem can then be posed as asking whether the map $A \mapsto \Lambda_A$ is injective modulo the natural obstruction, or in other words whether $\Lambda_A = \Lambda_B$ implies the existence of a gauge automorphism $F : E \to E$ with $F^*(A) = B$ and $F|_{\partial M} = Id$.

The general problem was considered in [4,7–9]; for a survey of the Calderón problem for metrics, see [3]. As far as we know, this paper is the first one that considers the connection problem and does not rely on the Complex Geometric Optics solutions (see any of [4,7,9]), but on unique continuation principles.

The Yang-Mills connections generalise flat connections and are important in physics and geometry. They satisfy the following equation:

$$D_A^*F_A = 0$$

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By $C^\infty(M; E)$ we denote the space of smooth sections of $E$ over $M$. 

[1]
where \( D_A = d_A^{\text{End}} \) is the induced connection on the endomorphism bundle \( \text{End}E \) and \( F_A \) is the curvature of \( A \) (see the preliminaries for more details). With this in mind, we may formulate the main theorems in this paper:

**Theorem 1.1** (Global result). Assume \( \dim M \geq 2 \), let \( E = M \times \mathbb{C} \) be a Hermitian line bundle with standard metric and \( \emptyset \neq \Gamma \subset \partial M \) an open, non-empty subset of the boundary. Let \( A \) and \( B \) be two unitary Yang-Mills connections on \( E \). If \( \Lambda_A(f)|_\Gamma = \Lambda_B(f)|_\Gamma \) for all \( f \in C^\infty_0(\Gamma; E|_\Gamma) \), then there exists a gauge automorphism (unitary) \( h \) with \( h|_\Gamma = \text{Id} \) such that \( h^*(A) = B \) on the whole of \( M \).

Before further proceeding to the organisation of the paper, let us explain the source of motivation for considering this problem. The idea came from the analogy between Einstein metrics in Riemannian geometry and Yang-Mills connections on Hermitian vector bundles and also the paper by Guillarmou and Sá Barreto [10]. In [10] the authors prove the recovery of two Einstein manifolds from the DN map for metrics; the method of their proof relies on a reconstruction near the boundary, where in special harmonic coordinates Einstein equations become quasi-linear elliptic (the metric is thus also analytic in such coordinates) and hence, by combining the boundary determination result and a unique continuation result for elliptic systems they prove one can identify the two metrics in a neighbourhood of the boundary. Moreover, by exploiting this analytic structure they observe that the method of Lassas and Uhlmann [14] who prove the analytic Calderón problem for metrics, may be used to extend this local isometry to the whole of the manifold (this works by embedding the two manifolds in a suitable Sobolev space using the Green functions of the metric Laplacians and the local isometry and showing the appropriate composition is an isometry).

In our case, the analogous concept to harmonic coordinates to consider is the **Coulomb gauge** [20] which transforms the connection to a form where \( d^*(A) = 0 \), so that the Yang-Mills equations become an elliptic system with principal diagonal part. However, this gauge does not tie well with the DN map and so we must look for something else – in Lemma 4.1 we give an answer to this question and which gauge to consider. In this gauge, we may use a similar unique continuation property (UCP in short) result to yield the equivalence of connections close to the boundary. However, for going further into the interior we designed a new method.

More concretely, our gauge from Lemma 4.1 satisfies the equation \( d_A^*d_AF = 0 \) and so we cannot guarantee that it is non-singular globally. We show that the zero set of the determinant of \( F \) is small in the smooth case when \( m = 1 \) and in the analytic case for arbitrary \( m \) – it is covered by countably many submanifolds of codimension one, or in the language of geometric analysis it is \( (n-1)-C^\infty\)-rectifiable. Since (the complement of) this singular set can be topologically non-trivial (see Figure 1), we end up with such barriers consisting of singular points of \( F \) that prevent us to use the UCP and go inside the manifold. This is addressed by looking at the sufficiently nice points of the barriers and locally near these points, using a degenerate form of UCP (in the smooth case) or a suitable form of analytic continuation (in the analytic case) to extend an appropriate gauge equivalence between the two given connections beyond the barriers; we name this procedure as “drilling”. Since we show there is
a dense set of such nice points, we may perform the drilling to extend our gauges globally.

Here is what we prove in the analytic case:

**Theorem 1.2.** Let \((M, g)\) be an analytic Riemannian manifold\(^2\) of dimension \(\dim M \geq 2\) and let \(\Gamma\) be as in Theorem 1.1. If \(E = M \times \mathbb{C}^m\) is a Hermitian vector bundle with the standard structure and if \(A\) and \(B\) are two unitary Yang-Mills connections on \(E\), then \(\Lambda_A(f)|_{\Gamma} = \Lambda_B(f)|_{\Gamma}\) for all \(f \in C^\infty_0(\Gamma; E|_{\Gamma})\) if and only if there exists a gauge automorphism \(H\) of \(E\), with \(H|_{\Gamma} = \text{Id}\), such that \(H^*(A) = B\).

The proof of this theorem also relies on using the Coulomb gauge locally, since the gauge from Lemma 4.1 does not work near singular points; in this gauge we may get that \(A\) is analytic and hence \(F\) also, since they satisfy elliptic equations with analytic coefficients.

Furthermore, the main difficulty for the smooth, higher rank \((m > 1)\) case is to prove the strong unique continuation property for the determinant \(\det F\) of a solution to \(d^*dF = 0\); for \(m = 1\), this is obvious by standard results. Another issue is that one needs to prove the UCP for elliptic systems with diagonal principal part and higher orders of degeneracy at a hyperplane. More precisely, operators with leading term \(x_n^k \Delta \times \text{Id}\) and with first order terms containing multiples of \(x_n^k\); in other words, the algebra of operators generated by derivatives of the form \(x_n^k \frac{\partial}{\partial x}\) where \(x_n\) is the boundary defining coordinate.

In this paper, a major role is played by the unique continuation results. As a source for the UCP results we will use Bär \cite{2}, who proves the rectifiability statements for the zero sets of first order semilinear elliptic systems; for convenience, we prove an easy generalisation of his results for second order systems in Lemma 4.4. Furthermore, we apply the degenerate UCP result of Mazzeo \cite{16}.

Finally, we prove that the DN map \(\Lambda_A\) is an elliptic pseudodifferential operator of order 1 on the restriction of the vector bundle to the boundary and deduce that its full symbol determines the full Taylor series of the connection, metric and a potential at the boundary. This was first proved in the case of a Riemannian metric by Lee and Uhlmann \cite{15} and later considered in the \(m = 1\) case with a connection in \cite{7}. In this paper, we generalise this approach to the case of systems and prove the analogous result.

The paper is organised as follows: in the next section, we recall some formulas from differential geometry and make a few observations about choosing appropriate gauges. In the third section we prove that \(\Lambda_A\) is a pseudodifferential operator of order 1 for systems and prove that its full symbol determines the full jet of \(A\) at the boundary. Furthermore, in section four we consider the smooth case and prove the global result for \(m = 1\). Along the way, we construct the new gauge and deduce the UCP result we need. In section four we consider the \(m > 1\) case for analytic metrics, by adapting the proof of the line bundle case and exploiting real-analyticity.

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\(^2\)The metric \(g\) is only assumed to be analytic in the interior of \(M\).
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2. Preliminaries

2.1. Some formulas and notation. As mentioned previously, Yang-Mills connections are the ones that are important in physics and geometry. They satisfy the so called Yang-Mills equations, which are considered as a generalisation of Maxwell’s equations in electromagnetism and which provide a framework to write the latter equations in a coordinate-free way (see e.g. [1] or [6] for a geometric overview and definitions). More concretely, let us consider a Hermitian bundle $E$ over a compact oriented Riemannian manifold $(M, g)$ equipped with a unitary connection $A$; we will denote the associated covariant derivative by $d_A$ and we will think of $A$ as both the connection 1-form (valued in the endomorphism bundle) in a local trivialisation and the formal object, depending on the context. The Yang-Mills connections are the critical point of the functional:

$$F_{YM}(A) = \int_M |F_A|^2 \omega_g$$

where $F_A = dA + A \wedge A$ is the curvature 2-form with values in the endomorphism bundle of $E$ and $\omega_g$ is the volume form. It can then be shown by considering variations of this functional, that the conditions for $A$ being the critical point of this functional are:

$$(D_A)^* F_A = 0 \quad \text{and} \quad D_A F_A = 0 \quad (2.1)$$

where we recall that $D_A = d_A^\text{ind}$ is the induced connection on the endomorphism bundle, given locally by $D_A S = dS + [A, S]$. In general, we use the notation $d_A^*$ to denote the formal adjoint acting on vector valued $p$-forms; if $A$ is unitary, then $d_A^* = (-1)^{(p-1)n+1} \ast d_A \ast$, where $\ast$ is the Hodge star acting $\mathbb{C}$-linearly on differential forms with values in $E$ as $\ast(\omega \otimes s) = (\ast \omega) \otimes s$, $\omega$ is a differential form and $s$ is a section of $E$. The second equation in (2.1) is actually redundant, since it is the Bianchi identity.

For the record, we will write down the explicit formula in local coordinates for the inner product on the differential forms with values in $E$; unless stated otherwise, we are applying the summation convention over repeated indices. If two $p$-differential forms with values in $E$, i.e. sections of $\bigwedge^p T^* M \otimes E = \Omega^p(E)$, are given locally by $\alpha = \sum \alpha_I dx^I$ and $\beta = \sum \beta_J dx^J$ then\footnote{The factor of $\frac{1}{p!}$ comes from the fact that we want to have $\langle dx^{i_1} \wedge \ldots \wedge dx^{i_p}, dx^{j_1} \wedge \ldots \wedge dx^{j_p} \rangle_E = \det(g^{i_k j_k})$.}

$$\langle \alpha, \beta \rangle_{\Omega^p(E)} = \frac{1}{p!} g^{i_1 j_1} \ldots g^{i_p j_p} \langle \alpha_{i_1 \ldots i_p}, \beta_{j_1 \ldots j_p} \rangle_E$$

\[3\]
Here \( \langle \cdot, \cdot \rangle_E \) is the inner product in \( E \). Moreover, we state the following formula for the adjoint \( d^* = (-1)^p \star^{-1} d \star = (-1)^{(p-1)n+1} \star d \star \), acting on \( p \)-forms:

\[
(d^*_A \alpha)_{\mu_1 \ldots \mu_p} = -g_{\mu_1 \nu_1} \ldots g_{\mu_p \nu_p} \frac{1}{\sqrt{\det g}} \partial_\nu (\sqrt{\det g} g^{\nu \lambda} g^{\nu_1 \lambda_1} \ldots g^{\nu_p \lambda_p} \alpha_{\lambda_1 \ldots \lambda_p})
\]

We can combine this information along with the condition that \( \int \langle d^*_A \alpha, \beta \rangle_E = \int \langle \alpha, d_A \beta \rangle_E \) for all \( p \)-forms \( \beta \) and \( p+1 \)-forms \( \alpha \), compactly supported in the interior. Then we get:

\[
d^*_A \alpha = d^* \alpha - \sum_{i_1 < \ldots < i_p} g^{\nu \lambda} A_\nu \alpha_{\lambda i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}
\]

and as a shorthand we may use \( (A, \alpha) = \iota_A \alpha \) for the sum in the above expression. The following identity is also very useful:

\[
d^*(f \omega) = f d^*(\omega) - \iota_{\nabla f}(\omega)
\]

If the connection is not unitary, then the expression \((-1)^{(p-1)n+1} \star d(-A^*) \star \) gives the formal adjoint in a local trivialisation on \( p \)-forms, where \( A^* \) denotes the Hermitian conjugate. We need to emphasise here that, slightly illogically, even if \( A \) is non-unitary in this paper we will use the notation \( d^*_A = (-1)^{(p-1)n+1} \star d_A \star \), unless otherwise stated.\footnote{We are assuming that the tensor representing the form is alternating, i.e. we get a minus sign after swapping any two indices.}

Clearly, the same principle applies to \( D^*_A \). Moreover, the above local formulas still hold for this \( d^*_A \). Then for all \( E \)-valued 1-forms \( u \):

\[
d^*_A d_A u = d^* du + d^* (A u) - (A, du) - (A, Au)
\]

(2.2)

2.2. Fixing gauges. In many mathematical problems and physical situations there exist certain degrees of freedom called gauges. More specifically, in our case a gauge is an automorphism of a vector bundle (preserves its structure); then the gauges act on the affine space of connections on this vector bundle by pullback. Here, we make a few remarks about the possible gauges one could use.

For example, in physics we use the electromagnetic four-potential to describe the electromagnetic field. This potential can be naturally identified (via musical isomorphism) with a connection 1-form on the unitary trivial line bundle over the space-time \( \mathbb{R}^4 \), so that the actual electromagnetic field is given by the curvature \( F = dA \), which is a tensor consisting of six components; the Maxwell’s equations then reduce to \( d^* dA = 0 \) (see (2.1)). Recall that, if we consider the magnetic field \( \vec{A} \) separately over \( \mathbb{R}^3 \), we would have \( \vec{B} = \nabla \times \vec{A} \) which means we could transform the connection \( \vec{A} \mapsto \vec{A} + \nabla f \) and still get the same answer for \( \vec{B} \); similarly, the electric field is invariant under addition of constants to the electric potential. In the connection setting above, we have the analogous invariant transform \( A \mapsto A + idf \) for a real function \( f \) on \( \mathbb{R}^4 \), which corresponds to the gauge given by \( e^{if} \). This leads to the physical

\footnote{The point is that we would like to have \( d^*_{F^{-1}(A)} = F^{-1} d^*_A F \), for all isomorphisms \( F \) of the vector bundle \( E \). On the other hand, \( F^{-1} d^*_A F \) will be the formal adjoint if we consider the pulled back inner product structure on \( E \) by \( F \); in general, it will not be a formal adjoint with respect to the standard inner product structure on \( E \).}
observation that we do not have a physical meaning of the potential and also to the Yang-Mills theory which generalises the Maxwell’s equations (see also \([6]\)).

There are several gauges that have proved to work well in practise, i.e. that fit well into other mathematical formalism in applications. One of them is the Coulomb gauge, which for a connection matrix on a vector bundle, locally asks that \(d^*A = 0\). The existence of such gauges is proved by Uhlenbeck \([20]\) for vector bundles over unit balls (see also \([6]\)) under a smallness condition on the \(L^p\) norm of the curvature (for specific values of \(p\)), which locally on a manifold we can always assume if we shrink the neighbourhood sufficiently and then dilate to the unit ball. Most importantly, in such a gauge the Yang-Mills connections satisfy an elliptic partial differential equation with the principal second order term equal to \((dd^* + d^*d)\) times the identity.

Another slightly related gauge is the temporal gauge, which we will also make use of – in this gauge, one of the components of the connection vanishes locally (we usually distinguish this variable as “time”). That is, given a local coordinate system \((x_1, \ldots, x_{n-1}, t) = (x, t)\) defined for \(t = 0\) and a connection matrix \(A = A_i dx^i + A_t dt\), we may solve:

\[
\frac{\partial F}{\partial t}(x, t) + A_t(x, t)F(x, t) = 0 \quad \text{and} \quad F(x, 0) = Id
\]

parametrically smoothly depending on \(x\) (the parallel transport equation). Then by definition near \(t = 0\), we have \(A_t' = F^*(A) = F^{-1}dF + F^{-1}AF\) satisfying \(A_t' = 0\). In this way we may prove Lemma 6.2 in \([4]\), which we state for convenience, since it will get used frequently throughout the paper:

**Lemma 2.1.** Let \(B\) be a unitary connection on \(E = M \times \mathbb{C}^m\). Consider the tubular neighbourhood \(\partial M \times [0, \epsilon)\) of the boundary for some \(\epsilon > 0\) and denote the normal distance coordinate (from \(\partial M\)) by \(t\). Then \(B\) is gauge equivalent to a unitary connection \(A\), via an automorphism \(F\) of \(E\) such that \(F|_{\partial M} = Id\) and \(A(\partial / \partial t) = 0\) in the neighbourhood \(\partial M \times [0, \delta)\) of the boundary, for some \(\delta > 0\).

Moreover, if we perform the above procedure in geodesic polars near a point, with \(t\) corresponding to the radial variable \(r\) now, we obtain what is called the radial gauge.

In the situation of this paper, we would like to use the gauge given by Lemma 4.1 because it is intimately tied with the DN map \((1.1)\) and allows us to make use of the information packed in the equality \(\Lambda_A = \Lambda_B\) for two connections \(A\) and \(B\).

### 3. Boundary determination for a connection and a matrix potential

In this section, we prove that if we put the connection in a suitable gauge and “normalise” the metric appropriately, we may determine the full Taylor series of a connection, metric and matrix potential from the DN map on a vector bundle with \(m > 1\). The case of \(m = 1\) was already considered in \([7]\) (Section 8) and this section generalises the result proved there. The approach is based on constructing a factorisation of the operator \(d^*d\) modulo smoothing, from which we deduce that \(\Lambda_{g, A, Q}\) is a pseudodifferential operator of order one whose full symbol determines the mentioned Taylor series.

\[\text{This is equivalent to } \nabla \circ \vec{A} = 0 \text{ in the case of } \mathbb{R}^3 \text{ considered in the previous paragraph.}\]
Before starting, let us recall some of the notation that goes into vector valued distributions and pseudodifferential operators with values in linear maps. Firstly, if $N$ a manifold without boundary and $E = \mathbb{C}^m$ a vector space, we let $C_c^\infty(N;E)$ be the set of all compactly supported vector functions; taking the dual, we obtain the space of $E$-valued distributions $\mathcal{D}'(N;E)$ and $\mathcal{E}'(N;E)$ the space of compactly supported ones. Clearly, given a basis $e_i$, $i = 1, \ldots, m$ of $E$, we may identify such $E$-valued distributions with finite sums $\sum_i S_i(f_i)$ for $\sum_i f_i e_i \in C_c^\infty(N;E)$.

Now any continuous linear mapping $A : \mathcal{E}'(N;E) \to \mathcal{D}'(N;E)$ may be identified with a $m \times m$ matrix, whose entries are continuous linear mappings $E \to E$; the set of such operators will be denoted by $\Psi^m(N;L(E))$ and the set of smoothing operators by $\Psi^{-\infty}(N;L(E)) = \cap_m \Psi^m(N;L(E))$. Moreover, the theory of scalar pseudodifferential operators carries over almost intact to matrix valued ones, including the symbol calculus; a symbol is now a matrix valued smooth function on $\Omega$.

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Remark 3.1. One of the things that fails to hold for matrix pseudodifferential operators and holds for scalar ones, is that commutation decreases degree of the operator by one. However, this formula still holds if $c$ denotes the symbol of $[A, B]$ (commutator bracket) and $a$, $b$ are the symbols of $A$, $B$, respectively:

$$c(x, \xi) = \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_x a(x, \xi) D^\alpha_x b(x, \xi)$$

where $D^\alpha_x = (-i)^{|\alpha|} \partial^\alpha_x$. For other details and the development of the theory of pseudodifferential operators on manifolds and vector bundles we refer the reader to Chapter 1 of Treves [19].

We start by proving an analogue of Lemma 8.6 in [7] and Proposition 1.1 in [15]. Here $(M,g)$ is a compact manifold with non-empty boundary $N = \partial M$ and $E = M \times \mathbb{C}^m$ a Hermitian vector bundle with a unitary connection $A$ and $Q$ an $m \times m$ matrix whose entries are smooth functions. We will be working in semigeodesic coordinates near $\partial M$ and we denote by $x^a$ the normal coordinate and local coordinates
in $\partial M$ by $x' = (x^1, x^2, \ldots, x^{n-1})$. Furthermore, we assume in these coordinates that 
$g = \sum g_{\alpha\beta} dx^\alpha dx^\beta + (dx^n)^2$; also, in what follows summation convention will be used to sum over repeated indices and when using Greek indices $\alpha$ and $\beta$, the summation will always be assumed to go over $1, \ldots, n - 1$. We use the notation 
$D_{x^j} = -i\partial_{x^j} = -i\partial_{x^j}$ and $|g| = \det(g_{ij}) = \det(g_{\alpha\beta})$. Here it goes:

**Lemma 3.2.** Let us assume $A$ satisfies condition (3.12). There exists a $C^m\times m$-valued pseudodifferential operator $B(x, D_x')$ of order one on $\partial M$, depending smoothly on $x^n \in [0, T]$ for some $T > 0$, such that the following factorisation holds:

$$d_A^*d_A + Q = (D_{x^n} \times Id + iE(x) \times Id - iB(x, D_{x'}))(D_{x^n} \times Id + iB(x, D_{x'}))$$

modulo smoothing, where $E(x) = -\frac{1}{2} g^{\alpha\beta}(x) \partial_{x^\alpha}g_{\alpha\beta}(x)$.

**Proof.** First of all, we have that:

$$d_A^*d_A + Q = \Delta_g - 2g^{jk} A_j \frac{\partial u}{\partial x^k} + (d^* A)u - g^{jk} A_j A_k u + Qu \ (3.3)$$

where $A = A_i dx^i$. Furthermore, we have

$$\Delta_g = D_{x^n}^2 + iED_{x^n} + Q_1 + Q_2$$

where

$$Q_1(x, D_{x'}) = g^{\alpha\beta} D_{x^\alpha} D_{x^\beta} \ \text{and} \ \ Q_2(x, D_{x'}) = -i(\frac{1}{2} g^{\alpha\beta}(x) \partial_{x^\alpha} \log |g|(x)$$

$$+ \partial_{x^\alpha} g^{\alpha\beta}(x)) D_{x^\beta}$$

We denote the symbols of $Q_1$ and $Q_2$ by $q_1$ and $q_2$ respectively and define $G = (d^* A) - g^{\alpha\beta} A_\alpha A_\beta + Q$. Thus by using (3.1), we can rewrite (3.2) as

$$B^2 - EB + i[D_{x^n} \times Id, B] = Q_1 \times Id + Q_2 \times Id - 2g^{\alpha\beta} A_\alpha \partial_{x^\beta} + G$$

modulo smoothing. Moreover, by taking symbol we obtain:

$$\sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_{\xi}^\alpha bD_{x^n}^\alpha b - Eb + \partial_{x^n} b - q_1 \times Id - q_2 \times Id + 2ig^{\alpha\beta} A_\alpha \xi_\beta - G = 0 \ (3.4)$$

modulo $S^{-\infty}$, where $b$ is the symbol of $b$ and we have used (3.1) and Remark 3.1. Let us put $b(x, \xi') = \sum_{j \leq 1} b_j(x, \xi')$, where $b_j$ is homogeneous of order $j$ in $\xi'$. We may then determine $b_j$ inductively, starting from degree two in (3.4):

$$(b_1)^2 = q_2 \ (3.5)$$
so we may set \( b_1 = -\sqrt{q_2} \times \text{Id} \) (this sign will be important later) and \( q_2 = g^{\alpha\beta} \xi_\alpha \xi_\beta \).

Next, we have:

\[
\begin{align*}
    b_0 &= \frac{1}{2\sqrt{q_2}} \left( \partial_{x^n} b_1 - E b_1 - q_1 \times \text{Id} + 2ig^{\alpha\beta} A_\alpha \xi_\beta + \nabla_{\xi_\beta} b_1 \cdot \nabla_{x^n} b_1 \right) \quad (3.6) \\
    b_{-1} &= \frac{1}{2\sqrt{q_2}} \left( \partial_{x^n} b_0 - E b_0 - G + \sum_{0 \leq j, k \leq 1, j + k = |K|} \frac{\partial^K b_j D^K_{x^n} b_{K|-j}}{K!} \right) \quad (3.7) \\
    b_{m-1} &= \frac{1}{2\sqrt{q_2}} \left( \partial_{x^n} b_m - E b_m + \sum_{m \leq j, k \leq 1, j + k = |K|+m} \frac{\partial^K b_j D^K_{x^n} b_k}{K!} \right) \quad (3.8)
\end{align*}
\]

where the last equation holds for all \( m \leq -1 \). Therefore we obtain \( b \in S^1 \) and hence \( B \in \Psi^1 \) as well, such that (3.2) holds.

We have established the existence of the factorisation (3.2) and now it is time to use it to prove facts about the DN map. The following claim is analogous to Proposition 1.2 in [15] – the main difference is that now we are using matrix valued pseudodifferential operators, so we need to make sure that appropriate generalisations hold.

**Proposition 3.3.** The DN map \( \Lambda_{g,A,Q} \) is a \( \mathbb{C}^{m \times m} \)-valued pseudodifferential operator of order one on \( \partial M \) and satisfies \( \Lambda_{g,A,Q} \equiv B|_{\partial M} \) modulo smoothing.

**Proof.** Assume without loss of generality that \( A \) satisfies condition (3.12) (see the paragraph after this Proposition). Let us take \( f \in H^2(\partial M; \mathbb{C}^m) \) and \( u \in \mathcal{D}'(M; \mathbb{C}^m) \) that solves the Dirichlet problem \( \mathcal{L}_{A,Q} u = 0 \) with \( u|_{\partial M} = f \). Then by Lemma 3.2 we obtain the following equivalent local system:

\[
\begin{align*}
    (D_{x^n} \times \text{Id} + iB)u &= v \quad \text{with} \quad u|_{x^n=0} = f \quad (3.9) \\
    (D_{x^n} \times \text{Id} + iE \times \text{Id} - iB)v &= h \in C^\infty([0,T] \times \mathbb{R}^{n-1}) \quad (3.10)
\end{align*}
\]

for some \( T > 0 \) and a local coordinate system \( x' = (x^1, \ldots, x^{n-1}) \) at \( \partial M \). By (3.10) and Remark 1.2 from Treves [19], we may furthermore assume that \( u \in C^\infty([0,T]; \mathcal{D}'(\mathbb{R}^{n-1}; \mathbb{C}^m)) \).

Then writing \( t = T - x^n \), we may view the equation (3.10) as backwards generalised heat equation:

\[
\partial_t v - (B - E \times \text{Id})v = -ih
\]

and by standard interior regularity we obtain that \( u \) is smooth and hence, so is \( v|_{x^n=T} \).

Since the principal symbol of \( B \) is negative, by Lemma 6.6 it satisfies Condition 6.4 (the basic hypothesis of a well-posed heat equation – see Appendix) and so the solution operator for this equation is smoothing by Theorem 1.2 in Chapter 3 of [19]. Thus \( v \in C^\infty([0,T] \times \mathbb{R}^{n-1}) \).

Let us set \( Rf := v|_{\partial M} \) – the above argument shows \( R \) is a smoothing operator and also \( D_{x^n} u|_{\partial M} = -iBu|_{\partial M} + Rf \). Therefore \( \partial_{x^n} u|_{\partial M} \equiv Bu|_{\partial M} \) modulo smoothing, which proves the claim.

The final step in this procedure is to express the Taylor series of \( g, A, q \) in terms of the symbols \( \{ b_j \mid j \leq 1 \} \) that we obtained in Proposition 3.3. However, before
proving such a result, we need to “normalise” the metric and the connection — here we refer to Lemma 6.2 from [4] and to Lemma 2.1 (b) from [13]; there exists an automorphism $F$ of $E$ such that $F|_{\partial M} = Id$ and a positive function $c$ on $M$, with $c|_{\partial M} = 1$ and $\partial_\nu c|_{\partial M} = 0$ ($\nu$ is the outer normal) such that $\tilde{A} = F^*(A)$ and $\tilde{g} = c^{-1}g$ satisfy:

$$\tilde{\partial}_x^j (\tilde{g}_{\alpha\beta} \tilde{\partial}_x^\alpha \tilde{g}^{\alpha\beta})(x', 0) = 0 \quad \text{for} \quad j \geq 1 \tag{3.11}$$

$$\tilde{A}_n(x', \tilde{x}^n) = 0 \tag{3.12}$$

where by $(x', \tilde{x}^n)$ we have denoted the $\tilde{g}$-boundary normal coordinates; (3.12) holds for all sufficiently small $\tilde{x}^n$, i.e. in a neighbourhood of the boundary. Also notice that the condition (3.11) is equivalent to $\partial_j \tilde{H}|_{\partial M} = 0$ for $j \geq 1$, as stated in [13]; here $\tilde{N} = \tilde{\partial}_x$ and $\tilde{H}$ is the mean curvature of the hypersurfaces given by setting $\tilde{x}^n$ equal to constant. Then by the invariance property of the DN map, we have $\Lambda_{\tilde{g},A,Q} = \Lambda_{\tilde{g},\tilde{A},\tilde{Q}}$ for $Q_c = c^{\frac{n-2}{2}} \Delta_g (c^{\frac{n-2}{2}}) \times Id$ and $\tilde{Q} = c (F^{-1}QF + Q_c) = c (F^*(Q) + Q_c)$. We will call a triple $(g, A, Q)$ that satisfies conditions (3.11) and (3.12) normalised. Moreover, we will use the notation $f_1 \simeq f_2$ to denote that $f_1$ and $f_2$ have the same Taylor series (as in [7]).

**Theorem 3.4.** Assume $M$ satisfies $\dim M = n \geq 3$ and the triple $(g, A, Q)$ is normalised. Let $W \subset \partial M$ open, with a local coordinate system $\{x^1, \ldots, x^{n-1}\}$ and let $\{b_j \mid j \leq 1\}$ denote the full symbol of $B$ (see Lemma 3.2) in these coordinates. At any point $p \in W$, the full Taylor series of $g, A$ and $Q$ are determined by the symbols $\{b_j\}$ by an explicit formula.

In particular, if $\Lambda_{g_1,A_1,Q_1} = \Lambda_{g_2,A_2,Q_2}$ and we assume that $\{g_i, A_i, Q_i\}$ are normalised for $i = 1, 2$, then $g_1 \simeq g_2$, $A_1 \simeq A_2$ and $Q_1 \simeq Q_2$. Moreover, if $\Lambda_{g_1,A_1,Q_1} = \Lambda_{g_2,A_2,Q_2}$ and $g_1 \simeq g_2$ on all of $\partial M$, then we also have $A_1 \simeq A_2$ and $Q_1 \simeq Q_2$, for $A_i = F_i^*(A_i)$ and $Q_1 = F_i^*(Q_i)$ for $i = 1, 2$; here $F_i$ are automorphisms of $E$ satisfying $F_i|_{\partial M} = Id$ and such that $A_i$ satisfy condition (3.12) for $i = 1, 2$.

**Proof.** Since we have:

$$\partial_x^\alpha g_{\alpha\beta} = -(g_{\alpha\rho} \partial_x^\rho g^{\rho\gamma}) g_{\gamma\beta}$$

it suffices to determine the inverse matrix $g^{\alpha\beta}$ and its normal derivatives. By the formula (3.5), we have that $b_2^i = -g^{\alpha\beta} \xi_\alpha \xi_\beta$ determines $g^{\alpha\beta}|_{\partial M}$.

If we write $\omega = \frac{\xi}{k|g}$ and use the notation:

$$k^{\alpha\beta} = \partial_x^\alpha g^{\alpha\beta} - (g^{\gamma\delta} \partial_x^\gamma g^{\delta\beta}) g^{\alpha\beta}$$

then we may rewrite (3.6) as follows:

$$b_0 = ig^{\alpha\beta} A_\alpha \omega_\beta - \frac{1}{4} k^{\alpha\beta} \omega_\alpha \omega_\beta \times Id + T_0 (g^{\alpha\beta})$$

where $T_0$ depends only on $g^{\alpha\beta}|_{\partial M}$, which is already explicitly determined.

Thus, by plugging in $\pm \omega$, we may recover $A_\alpha$ and $k^{\alpha\beta}$; it is not hard to see that:

$$k^{\alpha\beta} g_{\alpha\beta} = (2 - n) \partial_x^\alpha g^{\alpha\beta} g_{\alpha\beta}$$
and we may therefore write:

\[ \partial_{x^n} g^{\alpha\beta} = k^{\alpha\beta} + \frac{1}{2-n} (k^{\rho^n} g_{\rho}) g^{\alpha\beta} \]  

(3.13)

In the next step we will use the notation \( l^{\alpha\beta} = \frac{1}{1} \partial_{x^n} k^{\alpha\beta} + Q g^{\alpha\beta} \). Then we may rewrite (3.7) as:

\[ b_{-1} = \frac{1}{2 \sqrt{q_2}} (ig^{\alpha\beta} (\partial_{x^n} A_\alpha) \omega_\beta - l^{\alpha\beta} \omega_\alpha \omega_\beta) + T_1 (g^{\alpha\beta}, \partial_{x^n} g^{\alpha\beta}, A_\alpha) \]

where \( T_1 \) is an expression that depends only on \( g^{\alpha\beta}, \partial_{x^n} g^{\alpha\beta} \) and \( A_\alpha \) which have already been explicitly determined. Therefore, we may recover \( l^{\alpha\beta} \) and \( \partial_{x^n} A_\alpha \). Now, inductively we may prove the formula:

\[ b_{m-1} = \left( \frac{1}{2 \sqrt{q_2}} \right)^{m-1} (ig^{\alpha\beta} (\partial_{x^n}^{m-1} A_\alpha) \omega_\beta - \partial_{x^n}^{|m|} l^{\alpha\beta} \omega_\alpha \omega_\beta) \]

\[ + T_{m-1} (g^{\alpha\beta}, \ldots, \partial_{x^n}^{m-1} g^{\alpha\beta}, A_\alpha, \ldots, \partial_{x^n}^{m|} A_\alpha, Q, \ldots, \partial_{x^n}^{m+1|} Q) \]

for \( m \leq -1 \), where \( T_{m-1} \) only depends on the quantities in the bracket. Therefore, by induction we may explicitly determine \( \partial_{x^n} l^{\alpha\beta} \) and \( \partial_{x^n}^j A_\alpha \) for all \( j \geq 0 \).

Finally, we claim that we may inductively recover \( \partial_{x^n}^{j+2} g^{\alpha\beta} \) and \( \partial_{x^n}^j Q \) for any \( j \geq 0 \); let us also denote \( S_j = g_{\alpha\beta} \partial_{x^n}^j g^{\alpha\beta} \). For the base case \( j = 0 \), notice that \( \partial_{x^n}^0 (g_{\alpha\beta} \partial_{x^n} g^{\alpha\beta}) = 0 \), which implies that \( S_2 = -\partial_{x^n} g_{\alpha\beta} \partial_{x^n} g^{\alpha\beta} \), i.e. we know \( S_2 \).

Therefore, since we know \( l^{\alpha\beta} \), we may also explicitly determine \( \frac{1}{4} \partial_{x^n}^2 g^{\alpha\beta} \times Id + Q g^{\alpha\beta} =: P_0^{\alpha\beta} \). This implies:

\[ P_0^{\alpha\beta} g_{\alpha\beta} = (n-1)Q + \frac{1}{4} S_2 \]

from which we easily infer the knowledge of \( Q \) and hence also of \( \partial_{x^n}^2 g^{\alpha\beta} \).

For the inductive step, we may do something very similar: we have that for \( j \geq 1 \), the quantity \( P_j^{\alpha\beta} = \frac{1}{4} \partial_{x^n}^{j+2} g^{\alpha\beta} + (\partial_{x^n}^j Q) g^{\alpha\beta} \) is determined, since the condition \( \partial_{x^n}^{j+1} (g_{\alpha\beta} \partial_{x^n} g^{\alpha\beta}) = 0 \) determines \( S_{j+2} \) by previously reconstructed quantities. Then by the formula:

\[ P_j^{\alpha\beta} g_{\alpha\beta} = (n-1)\partial_{x^n}^j Q + \frac{1}{4} S_{j+2} \]

we may determine \( \partial_{x^n}^j Q \) and thus, \( \partial_{x^n}^{j+2} g^{\alpha\beta} \) as well. This completes the proof of the induction and of the theorem, since two formal expansions of the same operator in terms of classical symbols that agree modulo \( S^{-\infty} \), must also be congruent. \( \square \)

Let us emphasise that a key role in the above generalisations to the vector case is played by the fact that the operator \( d_A^* d_A + Q \) has a principal symbol that is a scalar multiple of identity; the necessary algebra then unveils in much the same way as in the scalar case. Of course, one needs to check that the appropriate symbol calculus holds for vector valued pseudodifferential operators, as well as the “trick” with the generalised heat equation in the proof of Proposition 3.3 (mostly covered in Treves [19]). A couple of remarks are in place after proving this theorem:
Remark 3.5. There are a few reasons to exclude the case \( \dim M = 2 \) in Theorem 3.4. To start with, after the proof of Proposition 1.3, the authors in (they consider the case \( m = 1 \), \( A = 0 \) and \( Q = 0 \)) remark that all the symbols of \( B \) satisfy \( b_j = 0 \) for \( j \leq 0 \) (easily checked for \( b_0 \) by direct computation and for the rest by induction); in other words, if we choose \( b_1 = -\xi_1 g^{11} \), the factorisation becomes a factorisation into honest differential operators where \( B = -\sqrt{g^{11}} D_{x^1} - \) this is in compliance with the additional conformal symmetry of the Calderón problem for surfaces. Secondly, the equation 3.13 clearly fails to hold when \( n = 2 \) – in that case \( k^{11} = 0 \) clearly so there is no extra information from this expression. However, when we introduce a connection and a potential, one can show that (choose \( b_1 = -\xi_1 g^{11} \) again):

\[
b_0 = i \sqrt{g^{11}} A_1 \]

\[
2\xi_1 b_{-1} = \partial_{x^2} A_1 - (\partial_{x^1} \sqrt{g^{11}}) A_1 - \frac{Q}{\sqrt{g^{11}}}
\]

Thus, the DN map determines the values of \( g_{11} \) and \( A_1 \) at the boundary (recall that \( A_2 = 0 \) in a neighbourhood of the boundary). Therefore, we may also determine \( \partial_{x^2} A_1 - \frac{Q}{\sqrt{g^{11}}} \) from the expression for \( b_{-1} \) and so if \( Q = 0 \), we determine the normal derivative of order one \( \partial_{x^2} A_1 \) – we will need this fact for a later application. If we go on to compute \( b_{-2} \), we see that it suffices to determine \( \partial_{x^2} g_{11} \mid_{\partial M} \) to compute derivatives \( \partial_{x^2} A_1 \mid_{\partial M} \) of all orders \( j \geq 2 \); however, we know we cannot possibly determine \( \partial_{x^2} g_{11} \mid_{\partial M} \) due to the additional conformal symmetry of the problem in two dimensions.

Remark 3.6. If we assume that \( \Gamma \subset \partial M \) is open and \( \Lambda_{g_1,A_1,Q_1}(f) \mid_{\Gamma} = \Lambda_{g_2,A_2,Q_2}(f) \mid_{\Gamma} \) for all \( f \in C_0(\Gamma) \) and that the corresponding quantities are normalised, then by the local nature of the above argument in Theorem 3.4 we have that: \( g_1 \mid_{\Gamma} \simeq g_2 \mid_{\Gamma}, A_2 \mid_{\Gamma} \simeq A_2 \mid_{\Gamma} \) and \( Q_1 \mid_{\Gamma} \simeq Q_2 \mid_{\Gamma} \).

4. Recovering a Yang-Mills connection

We fix a Yang-Mills connection \( A \) on the Hermitian vector bundle \( E = M \times \mathbb{C}^m \) (with standard metric) over a compact Riemannian manifold \( (M, g) \) with boundary. Let us extend the connection \( A \) to a “new connection” on the endomorphism bundle \( \text{End} E = M \times \mathbb{C}^{m \times m} \) by simply asking that \( d_{\tilde{A}} F = dF + AF \) globally, where \( \tilde{A} \) is the matrix of one forms with values in \( L(\mathbb{C}^{m \times m}) \) induced by \( A \) by multiplication on the left; it is easy to check this is a unitary connection. Note that \( d_{\tilde{A}} \) does not satisfy the usual Leibnitz rule such does the usual connection \( D_A F = dF + [A, F] \) on the endomorphism bundle. Then it is easy to check that the Cauchy data given of the associated DN problem for the vector bundle \( E \) and connections \( A \) and \( B \) agree if and only if they agree for the induced connections \( d_{\tilde{A}} \) and \( d_{\tilde{B}} \) on the endomorphism bundle. Here and throughout the paper, we will use the same notation \( d_A = d + A \) for both covariant derivatives \( d_A \) and \( d_{\tilde{A}} \), which will hopefully be clear from context. We will also use \( d_A \) to denote the extension of the covariant derivative to vector valued forms. The complex bilinear form on matrix valued 1-forms \( (\alpha, \beta) = g^{ij} \alpha_i \beta_j \)
Lemma 4.1. If \( U \subset \mathbb{R}^n \) open and \( F : U \to \mathbb{C}^{m \times m} \) is an invertible matrix function and we put \( A' = F^*(A) \) for \( A \) a matrix of one forms on \( U \), then \( F \) satisfies \( d_A^*d_AF = 0 \) if and only if \( d^*A' = Q_0(x,A') \), where \( Q_0 \) is smooth of order zero and quadratic in \( A' \), given by \( Q_0(x,A') = (A',A') \). If in addition \( A \) is Yang-Mills, then \( A \) satisfies an elliptic non-linear equation with diagonal principal part.

Proof. By using that \( d_{A'} = F^{-1}d_AF \) and similarly \( d^*_{A'} = F^{-1}d^*_AF \) (note this might not be the true formal adjoint – see preliminaries), note that \( d^*_A d_AF = 0 \) is equivalent to:
\[
FF^{-1} d^*_{A'} FF^{-1} d_AF = 0 \iff F d_{A'}^*(Id) = 0 \iff d^*_{A'} = (A',A') \iff d_{A'} A' = 0
\]
by expanding the \( d_{A'}^* d_{A'} \) operator by \([3.3]\). If \( A \) is Yang-Mills, then by adding \( (D_{A'})^* F_{A'} = 0 \) to \( dd^* A' = d(Q_0(x,A')) \) we get an elliptic system with principal part equal to \( dd^* + d^*d \).

By the Fredholm alternative and the fact that \( \ker(d_A^* d_A) = \{0\} \), we know that we may solve \( d_A^* d_AF = 0 \) globally for any boundary condition. Therefore, at least near the boundary, we know that \( A' \) exists if \( F|_{\partial M} \) is non-singular and that it satisfies the equation \( d^* A' = Q_0(x,A') \). Thus we may obtain the following result:

Theorem 4.2. Consider two Yang-Mills connections \( A \) and \( B \) on \( E = M \times \mathbb{C}^m \) with the same DN map on the whole of \( \partial M \). Then there exists a neighbourhood \( U \) of the boundary and a bundle isomorphism \( H \) for the restricted bundle \( E|_U \) with \( H|_{\partial M} = Id \) such that \( H^* B = A \) on \( U \). Moreover, if \( A \) and \( B \) are unitary (with respect to the standard structure), then we have \( H \) to be a unitary automorphism.

Proof. By following the construction above, we obtain gauge equivalences \( F \) and \( G \), which satisfy \( d_A^* d_AF = 0 \) and \( d_B^* d_BG = 0 \) respectively, with boundary conditions \( F|_{\partial M} = G|_{\partial M} = Id \). This is non-singular near the boundary and the connections \( A' = F^*(A) \) and \( B' = G^*(B) \) satisfy the equations
\[
d^* A' = Q_0(x,A') \quad \text{and} \quad d^* B' = Q_0(x,B')
\]
Now we can also expand the equations \( (D_{A'})^* F_{A'} = 0 = (D_{B'})^* F_{B'} \) (note that \( A' \) and \( B' \) are now Yang-Mills with respect to the fibrewise inner product pulled back by \( F \) and \( G \) respectively, rather than the standard inner product):
\[
(d^*d + P) A' = 0 \quad \text{and} \quad (d^*d + P) B' = 0
\]
where \( P \) is a first order, non-linear operator arising from the equality
\[
(d^*d + P) A' = (-1)^{n+1} \star D_{A'} \star F_{A'}
\]
where \( \star \) is the Hodge star extended to bundle valued forms. Therefore by simply applying the operator \( d \) to \([4.1]\) and adding to the Yang-Mills equations, we obtain an elliptic system of equations, with diagonal principal part
\[
\Delta A' = (dd^* + d^*d) A' = Q_1(x,A',\nabla A')
\]
where $Q_1$ is a smooth first order in $A'$ term, polynomial in $A'$ and $\nabla A'$. In order to use uniqueness of solutions to such equations, we need some boundary conditions – this is where we use the DN map hypothesis. Without loss of generality, assume that the normal components of connections $A$ and $B$ near the boundary vanish, see Lemma 2.1.

Thus from equality of the DN maps, we have $\frac{\partial (F-G)}{\partial \nu}|_{\partial M} = 0$. By subtracting the initial equations for $F$ and $G$, we get:

$$\Delta (F - G) - 2(A, dF) + 2(B, dG) + (d^* A)F - (d^* B)G - (A, AF) + (B, BG) = 0$$

(4.3)

and the point is that we have $\Delta (F - G) = \text{lower order terms}$, where we are fixing the semi-geodesic boundary coordinates $(x, t)$ with $t$ denoting the direction of the normal – this is because we already know that $(A - B) = O(t^n)$, if $n \geq 3$, by the boundary determination result Theorem 3.4, and $(F - G) = O(t)$. Thus when expanding the Laplacian, we are left with only $\frac{\partial^2}{\partial t^2}$ factor, which then allows us to conclude inductively $(F - G) = O(t^\infty)$ by repeated differentiation. If $n = 2$, notice that by Remark 3.5 we have $(A - B) = O(t)$; by (4.3) we have $(F - G) = O(t^2)$ and thus we have also that $(A' - B') = O(t)$. Therefore by the elliptic equation (4.2), the analogous counterpart of it for $B'$ and repeated differentiation we obtain $(A' - B') = O(t^\infty)$.

Therefore, we are left with two connections, $A'$ and $B'$ which satisfy an elliptic equation and have the same full Taylor series at the boundary – by the unique continuation property for elliptic systems Lemma 4.4 (see below), we may conclude $A' \equiv B'$ in $U$ and hence if we put $H = GF^{-1}$ we have $H^* B = A$ on $U$.

Finally, if $A$ and $B$ are unitary, we have that (locally, in a unitary trivialisation) $H^*(A) = B$ implies by definition that $dH = HB - AH$ and $d(H^*) = -BH^* + H^*A$, by the unitary property of connection matrices – combining the two, we have:

$$d(HH^*) = [HH^*, A]$$

where $[\cdot, \cdot]$ is the commutator. This first order system has a unique solution, which is given by $HH^* = Id$, as $H|V = Id$ and thus $H$ is unitary whenever $H^*(A) = B$. □

The next step is to go inside the manifold from the boundary. Namely, the main problem lies in the fact that $F$ can be singular on a large set, stopping our argument of unique continuation. However, at least in the scalar case, we may get over this, by essentially knowing facts about zero sets of solutions to elliptic systems of equations.

We need to recall the following definition:

**Definition 4.3.** A subset of a smooth manifold is called countably $k$-rectifiable if it is contained in a countable union of smooth $k$-dimensional submanifolds.

The result we will need is essentially proved in [2], Theorem 2, for the scalar case; the vector case we will need follows in a straightforward manner from its method of proof. We outline it here for completeness:

**Lemma 4.4.** Let $(M, g)$ be a smooth $n$-dimensional, connected Riemannian manifold. Let $L : C^\infty(M, \mathbb{R}^l) \to C^\infty(M, \mathbb{R}^l)$ be a differential operator on vector functions for $l$
Proof of Theorem 1.1. Firstly, gauge transform both $A$ and $B$ such that the normal component of the connection near the boundary is zero (apply Lemma 2.1). Consider the gauge constructed in (4.2), i.e. $d^\delta A = 0$ with the following boundary conditions: $f|_{\partial M} = g|_{\partial M}$, $f|_V = g|_V = 1$ and $f, g$ have compact support at the boundary contained in $\Gamma$; here $V \subset \bar{V} \subset \Gamma$ is some non-empty, connected, open subset of $\Gamma$\footnote{We will actually see later that it is enough to have any $f$ and $g$ non-zero and equal at the boundary.}. Let us define $h = \frac{f}{g}$ on the complement of the closed set $N(g) = g^{-1}(0)$. We furthermore split the zero set into the critical set $N_{\text{crit}}(g) = N(g) \cap \{x \in M \mid dg(x) = 0\}$ and its complement in $N(g)$, $S = N(g) \cap \{x \in M \mid dg(x) \neq 0\}$.

Now we consider the connections $A' = f^*(A)$ and $B' = g^*(B)$ near the set $V$, where we know $f$ and $g$ are non-zero, so these connections are well-defined. Following the recipe from before, by boundary determination and unique continuation we know that in a neighbourhood of $V$, we have $A' \equiv B'$ and thus on this set we also have $B = h^*(A)$ or equivalently:

$$dh = (B - A)h \quad (4.4)$$

**Proof.** Consider the vector bundle $E = \bigoplus_j \left( \Lambda^j T^* M \otimes \mathbb{R}^l \right)$ of vector valued differential forms. It is a well known fact that the operator $d + \delta$ is a Dirac operator on the bundle of differential forms with respect to the Riemannian inner product (it respects the Clifford relations), where $\delta$ is the codifferential. Moreover, we have that $(d + \delta)^2 = d\delta + \delta d = \Delta$ on differential forms. Let us consider the operator:

$$V \left( \sum \omega_i \right) = R(x, \omega_0, \omega_1) - \omega_1$$

where $\omega_i$ is the component of $\omega$ in $\Lambda^i T^* M \otimes \mathbb{R}^l$. Clearly $V$ is smooth and respects the zero section.

Thus, if $Lu = 0$, then $\omega = u + du \in C^\infty(M; E)$ solves $(d + \delta + V)(\omega) = 0$. The first order operator $D = d + \delta + V$ is a Dirac operator acting on sections of $E$, so the Corollary 1 of \cite{2} applies (the strong unique continuation property (SUCP) holds for a Dirac operator, i.e. we cannot have a non-zero solution vanishing to an infinite order at a point). Thus we get the result for the $N_{\text{crit}}(u) = N(\omega)$.

Finally, since $D$ has the SUCP, we know that $N(u)$ consists of points where $u$ vanishes to finite order and hence the Lemma 3 from \cite{2} applies. \hfill $\square$
Notice that $B = h^*(A)$ holds in the connected component $R$ of $V$ in the set $M \setminus \mathcal{N}(g) \cap M \setminus \mathcal{N}(f)$. Notice also that $d(|h|^2) = 0$ on this component by using (4.4), since $A$ and $B$ are unitary, so $|h|$ is constant and hence bounded on this set. This implies that the zero sets of $f$ and $g$ agree as we approach the boundary of $R$. The problem now is how to go further inside the interior of the manifold and go past the zero sets of $f$ and $g$. We will do this by a procedure we call “drilling holes”.

Let us describe this procedure. Firstly, we have that the zero set of $g$ lying in the interior of $M$ is contained in a countable union of codimension 1 submanifolds by Lemma 4.4; denote these manifolds by $M_1, M_2, \ldots$. Consider the following situation: we are given a point $p$ such that we have $g(p) = 0$ and $dg(p) \neq 0$ and moreover, we have $g^{-1}(0)$ locally a hypersurface of codimension one (in this case rank of $dg$ is equal to one). By going to a tubular neighbourhood of $g^{-1}(0)$ near $p$, we may assume we are in the setting where $g = 0$ in a neighbourhood of zero in the hyperplane $\mathbb{R}^{n-1}$, and the metric satisfies $g_{in} = \delta_{in}$ for $i = 1, 2, \ldots, n$ on $\mathbb{R}^{n-1}$; moreover, assume that we know $dh = h(B - A)$ or equivalently, that $f^*(A) = g^*(B)$, in the region where $\{x_n > 0\}$. Our goal is to extend this equality to the lower part of the space. Let us just remark that, in general, the zero set of $g$ can be of codimension one or two, depending on the rank of $dg$; however, if $dg \neq 0$ we anyway know that at least one of $d(\text{Im } g) \neq 0$ and $d(\text{Re } g) \neq 0$ holds, so the zero set is locally contained in $(\text{Im } g)^{-1}(0)$ or $(\text{Re } g)^{-1}(0)$, at least one of which is a codimension one submanifold. It can of course happen that the zero of $g$ is purely $(n-1)$-dimensional, see Figure 1 below for such an example (more precisely, $u$ in this example gives the real part of such a solution, with the imaginary part equal to zero).

Firstly, by Taylor’s theorem we have that $f = x_n f_1$ and $g = x_n g_1$ locally near 0. Furthermore, $g_1 \neq 0$ in a neighbourhood of 0 by the assumption and hence $f_1 \neq 0$ as $|h|$ is a non-zero constant in the upper space. We want to consider $A' = f^*(A)$ as before, however $f$ can be zero now and thus $A'$ not well-defined (singular), so we will consider something very similar, i.e. $A'' = x_n A'$ and $B'' = x_n B'$:

$$A'' = dx_n + x_n \frac{df_1}{f_1} + x_n A$$

$$B'' = dx_n + x_n \frac{dg_1}{g_1} + x_n B$$

(4.5)

(4.6)

Now both of these are well-defined and the degeneracies have cancelled with $x_n$. Let us rewrite the gauge equations for $A''$ (note that $A''$ is Yang-Mills with respect to $f^*(A)$ and not the standard inner product structure):

$$x_n^2 d^* d(A'') + x_n \star (dx_n \wedge \star (dA'')) - x_n d^*(dx_n \wedge A'') - 2 \star (dx_n \wedge \star (dx_n \wedge A'')) = 0$$

$$x_n d^*(A'') + A''_n - |A''|^2 = 0$$

(4.7)

(4.8)

where $A''_n$ is the $n$-th component of the 1-form $A''$. After applying $d$ to the second equation and multiplying with $x_n$, we get the form:

$$x_n^2 dd^*(A'') + x_n d^* A'' dx_n + x_n d(A''_n) - x_n d(|A''|^2) = 0$$

(4.9)
Now after adding the equation (4.7) to the equation (4.9) we get a degenerate elliptic second order non-linear equation, which has a diagonal principal part \( x_n^2 \Delta \) and every first order term multiplied with \( x_n \). Notice also \( A'' = B'' \) for \( x_n > 0 \), so \( A'' - B'' = O(x_n^\infty) \) on the hyperplane \( x_n = 0 \).

By Corollary (11) in [16], we deduce that there exists a unique continuation principle for such equations and hence we obtain \( A'' \equiv B'' \) in the lower space, by continuing from the hyperplane. More precisely, we may rewrite these non-linear equations for \( A'' \) and \( B'' \) in the form:

\[
x_n^2 \Delta A'' = w(x, A'', \nabla A'') \quad \text{and} \quad x_n^2 \Delta B'' = w(x, B'', \nabla B'')
\]

where \( w \) is a smooth function in its entries. Therefore, after subtracting these two and writing \( C'' = B'' - A'' \), we may obtain:

\[
x_n^2 \Delta C'' = w(x, B'', \nabla B'') - w(x, A'', \nabla A'')
\]

\[
= h_1(x, A'', B'', \nabla A'', \nabla B'')C'' + h_2(x, A'', B'', \nabla A'', \nabla B'')\nabla C'' \quad (4.10)
\]

by Taylor expanding the \( w \) with respect to \( C'' \); here \( h_1 \) and \( h_2 \) are smooth in its entries and act linearly on \( C'' \) and \( \nabla C'' \), respectively. Thus, after fixing \( h_1 \) and \( h_2 \) as known functions, we may think of (4.10) as a linear system of equations (of real dimension \( 2n \)) in \( C'' \) and thus results from [16] apply.

Moreover, we have that \( h = \frac{\dot{\gamma}}{g_1} \) carries smoothly over the hyperplane and therefore we have \( dh = (B - A)h \) by subtracting equations (4.5) and (4.6), on the other side of the hyperplane. Furthermore, using the relation \( d(|h|^2) = 0 \) obtained from the gauge equation, we immediately get that \( h \) is bounded and moreover, non-zero so we may write \( B = h^*(A) \).

Finally, by using Lemma 4.4 we deduce that \( B = h^*(A) \) on the whole connected component in \( M \setminus \mathcal{N}(g) \) of the points in the lower space in the previously considered chart (call it \( \mathcal{R}' \)) and therefore, that \( h \) is non-zero on \( \mathcal{R}' \) and that the boundary of \( \mathcal{R}' \) are the points where (could be empty) \( g = 0 \). This ends the procedure.

Observe that we may perform this procedure at the boundary for a dense set of points \( p \in Q = \Gamma \cap \mathcal{N}(g) \) to extend \( h \) such that \( h^*(A) = B \) near these points with \( h = 1 \) on the boundary. In more detail, the set \( \{ p \in Q \mid dg(p) = 0 \text{ or } df(p) = 0 \} \) is small, in the sense that its complement is dense, by Lemma 4.4. On this set, near a point \( p \), we may use semi-geodesic coordinates and write \( f = x_nf_1 \) and \( g = x_ng_1 \) as before; then \( h = \frac{\dot{\gamma}}{g_1} \) extends \( h \) smoothly and \( h = 1 \) on boundary, since the DN maps agree. The boundary determination result applied to quantities \( A'' \) and \( B'' \) defined in (4.5) and (4.6) and the degenerate unique continuation result of Mazzeo now applies to equations (4.7) and (4.9), to uniquely extend from \( \partial M \), as before.

We may now drill the holes and extend \( h \) together with the relation \( h^*(A) = B \), starting from the component of \( V \), where we may use boundary determination. The idea is that drilling the holes connects path components over the possibly disconnecting set \( \mathcal{N}(g) \). Let us now give an argument that what we are left with (after drilling the holes) is path connected.

Let us denote the complement of the zero set \( T = M \setminus \mathcal{N}(g) \); obviously \( M \setminus (\cup \mathcal{M}_i) \subset T \) and \( T \) open. Let \( x_0 \in M^0 \) be a point in the open neighbourhood of \( V \) where \( B = h^*(A) \) and \( y \) be any point in \( T \). Consider any path \( \gamma : [0, 1] \to M \) with \( \gamma(0) = x_0 \).
and $\gamma(1) = y$. We will construct a path $\gamma'$ from $x_0$ to $y$, lying in $T$, by slightly perturbing the path $\gamma$, such that $\gamma$ and $\gamma'$ are arbitrarily close. Let $d$ be the usual complete metric in the space $C^\infty([0,1], M)$ of smooth paths with fixed endpoints $x_0$ and $y$ (see Remark 6.1 in the appendix).

By standard differential topology (see [11]), there exists an arbitrarily close path $\gamma_1$ to $\gamma$ (with the same endpoints), such that $\gamma_1$ intersects $M_1$ transversally in a finite number of points $P_1, \ldots, P_k$. There are two possibilities for these points, starting e.g. with $P = P_1$:

1. There exists a sequence of points $p_i \in M_1$, for $i = 1, 2, \ldots$, converging to $P$, such that $g(p_i) \neq 0$ for all $i$.
2. We have $g = 0$ in a neighbourhood of $P$ in $M_1$ and a sequence of points $q_i \in M_1$ converging to $P$, such that $dg(q_i) \neq 0$.

In the first case, we may slightly perturb $\gamma_1$, such that it goes through one of the points $p_i$ and is sufficiently close in the metric $d$. These are complementary conditions, so if the first item does not hold, then the second one does: in that case, we may still perturb $\gamma_1$ to go through one of the points $q_i$, by the above argument of drilling holes. Notice that each of the points $p_i$ or $q_i$ has a neighbourhood in $M_1$ through which we can perturb the curve and therefore, there exists an $\epsilon > 0$, such that if we move our curve by a distance less than $\epsilon$ in the $d$-metric, the resulting curve will go through this neighbourhood.

Now inductively, we may perform the same procedure for all $j = 1, 2, \ldots, k$ and, each time, taking the perturbations small enough such that it does not interfere with the previously done work – what we obtain is $\gamma'_1$, which is sufficiently close to $\gamma_1$ and which does not hit $M_1$, minus the deleted holes. Thus we obtain a Cauchy sequence of curves $\gamma'_1, \gamma'_2, \ldots$ such that $\gamma'_i$ does not hit $M_1, M_2, \ldots, M_i$, minus the deleted holes. Since the space of curves is complete, we obtain a limiting curve $\gamma'_i \to \gamma'$, which lies completely in $T$ together with the drilled holes and furthermore satisfies $d(\gamma_i, \gamma') < \delta$ for some pre-fixed $\delta > 0$. In particular, this implies that the lengths of the curves are close, i.e. $|l(\gamma) - l(\gamma')| < \delta'$ for some $\delta' > 0$ (here $l$ denotes the length of the curve in the underlying Riemannian manifold). Let us denote the union of all of the drilled holes, i.e. neighbourhoods of some of the points $q_i$ in the item (2) above, by $T_\gamma$.

Moreover, we may repeat the above argument for all paths $\gamma$, now between any two points in $T$ – denote the set of new drilled holes by $S_\gamma$. Then we redefine $T$ as:

$$
T := T \bigcup_{\gamma \text{ from } x_0 \text{ to } y} T_\gamma \bigcup_{\gamma' \text{ from } x \text{ to } y} S_\gamma
$$

where the first union runs over all of the curves $\gamma$ starting at $x_0$ and ending at $y \in M^\circ \setminus \mathcal{N}(g)$; the second one is over all paths $\gamma'$ between points in $M^\circ \setminus \mathcal{N}(g)$. It is easy to see that $T \subset M^\circ$ is open and connected and furthermore, it satisfies the property that for any curve $\gamma$ between any two points $x, y \in T$, there exists a sequence of curves $\gamma_n$ between $x$ and $y$, lying wholly in $T$, such that $d(\gamma_n, \gamma) \to 0$ as $n \to \infty$; also, we have $B = h^*(A)$ on $T$.

Let us denote by $d_1$ the inherited metric of $T$ as a subspace of $M$ and by $d_2$ the metric in the Riemannian manifold $(T, g|_T)$. Therefore, as a result of the above
construction, we may claim the following result about these metrics:

\[ d_2(x, y) = \inf \{ l(\gamma) \mid \gamma \text{ a piecewise smooth path from } x \text{ to } y \text{ lying in } T \} = d_1(x, y) \]

Notice also that we have, by the Fundamental Theorem of Calculus, if \( \gamma \) is a path from \( x \) to \( y \) lying in \( T \):

\[
|h(x) - h(y)| = \left| \int_0^1 dh_{\gamma(t)}(\dot{\gamma}(t)) dt \right| \leq \int_0^1 \left| \langle \nabla h_{\gamma(t)}, \dot{\gamma}(t) \rangle \right| dt \\
\leq \int_0^1 |\nabla h_{\gamma(t)}|_g \cdot |\dot{\gamma}(t)|_g dt \leq C \int_0^1 |\dot{\gamma}(t)|_g dt = C \cdot l(\gamma)
\]

by Cauchy-Schwarz, where \( \nabla h \) is the gradient of \( h \) and \( C \) is a uniform bound on \( dh \) (which follows from the global relation \( dh = (B - A)h \) and boundedness of \( h, A \) and \( B \)). If we take the infimum over all such curves \( \gamma \), we obtain the inequality \( |h(x) - h(y)| \leq Cd_2(x, y) = Cd_1(x, y) \) and therefore obtain that \( h \) is Lipschitz and uniformly continuous over \( T \).

Therefore \( h \) can be extended continuously\(^9\) to the whole of \( M \) and by inductively differentiating the relation \( dh = (B - A)h \), we moreover have that all partial derivatives of \( h \) can be continuously extended. That these continuous extensions of derivatives are actual derivatives of the extension of \( h \) is proved in Lemma 6.2 in the Appendix; see also Remark 6.3 in the Appendix for the extension to the boundary. This proves \( h^\ast(A) = B \) on the whole of \( M \) with \( h \) smooth and that \( h|_\Gamma = 1; h \) also unitary. This finishes the proof. \( \square \)

**Remark 4.5** (Topological remarks). One can see that the complement of the disconnecting set \( \mathcal{N}(g) \) can indeed have non-trivial topology; this justifies the use of our argument of drilling holes. For simplicity, we will consider real harmonic functions \( g \) with \( \Delta g = 0 \) in the open unit disk. Firstly, one may observe that there are two types of points in \( \mathcal{N}(g) \): the critical and the non-critical ones. The non-critical ones are simple: they are locally contained in an analytic curve, whereas the critical ones are isolated (since they are exactly the set of points where \( f' = 0 \), where \( f \) holomorphic and \( u = \text{Re}(f) \)) and are locally zero sets of harmonic polynomials, zero sets of \( \text{Re}((z - P)^m) \), where \( m \geq 2 \) an integer. Thus at these critical points, \( \mathcal{N}(g) \) is a union of \( m \) analytic curves meeting at \( P \) at equal angles; also, there are no loops in \( \mathcal{N}(g) \), due to the unique solvability of the Dirichlet problem and analytic continuation. Therefore, if \( g \) has an analytic extension to the closed disk, there are finitely many components in the complement of \( \mathcal{N}(g) \); but if no such extension exists and \( g \) is zero at infinitely many points at the boundary, then we may expect infinitely many components. This is because for each such vanishing, non-critical point of \( g \) at the boundary we have an “end” coming inside the disk, which returns to the boundary at some other point, by the analysis above. See Figure 1 for a concrete example and [5,21] for further analysis.

\(^8\)We just proved that the inherited subspace metric on \( T \subset M \) and the path metric as a submanifold of a Riemannian manifold are Lipschitz equivalent with Lipschitz constant equal to 1.

\(^9\)Here we are using the elementary fact that a uniformly continuous function can be uniquely continuously extended to the closure of its domain.
Figure 1. In blue – the zero set of the harmonic function with boundary value equal to $f(\theta) = \theta \cdot \sin \frac{100}{\theta}$ on the unit disc, where $\theta \in (-\pi, \pi)$ is the angular coordinate. In orange – the unit circle. The accumulation point is $(1, 0)$.

Remark 4.6. If given $d^*_A d_A F = 0$ we were able to write down an elliptic equation for $\det(F)$, then all (or almost all) of the above proof would carry over to the case of vector bundle of higher rank, as in Theorem 4.2. Also, notice that the only two implications we were using in Theorem 1.1 from the equality of the DN maps for $A$ and $B$, were:

1. By boundary determination, the connections $A$ and $B$ have the same full jets at the boundary in suitable gauges.
2. There exist two non-zero solutions $f$ and $g$ to $d_A^* d_A f = d_B^* d_B g = 0$, such that $f|_{\partial M} = g|_{\partial M}$ with $\text{supp}(f|_{\partial M}) = \text{supp}(g|_{\partial M}) \subset \Gamma$ and $\partial_{\nu} f|_{\Gamma} = \partial_{\nu} g|_{\Gamma}$.

Remark 4.7 (Boundary extension and the non-unitary case). A different approach to extension of the gauge to the boundary, by using the partial differential equations that it satisfies (that is $H^{-1} dH + H^{-1} A H = B$), can be found in Proposition 4.7 from [12]. There, the authors take $A$ and $B$ to a gauge with no normal component, so that the new gauge $H'$ is independent of the normal variable from the equation it satisfies and can clearly be extended smoothly. Moreover, this approach can be used in the proof of Theorem 1.1 when considering the drilling procedure, to extend the gauge $h$ over the singular hypersurfaces and without making the assumption that $A$ and $B$ are unitary. Thus we are left with $h : M^o \setminus N_{\text{crit}}(g) \rightarrow \mathbb{C}$ such that $h^*(A) = B$, where we used the unitary part of assumption to prove uniform continuity. It seems possible to use the outlined argument to extend over $N_{\text{crit}}(g)$ and prove Theorems
we have a gauge automorphism $X$ the same as for the smaller ball, by a computation. By a result of Uhlenbeck [20], any local coordinates, we have
\[ \partial (f) \]
Therefore if $N_{\text{crit}}(\det F) = N_{\text{crit}}(f^2)$ contains the set where $f$ vanishes (since $d(f^2) = 2fdf$). Therefore if $f$ vanishes on an $(n-1)$-dimensional set, then the critical set is also $(n-1)$-dimensional; one such example is given by $M = \mathbb{R}^2$ and $f(x,y) = x$ which vanishes along the $y$-axis and solves $\Delta_{\mathbb{R}^2}(x) = 0$.

Here a function $f$ on a manifold $M$ vanishes to infinite order at a point $x_0 \in M$ if in any local coordinates, we have $\frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = 0$ for all multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (this is invariant of choice of the coordinates).

However, if we were able to prove that $\det F$ has the strong unique continuation property, then we may try and run through the argument in Theorem 1.1 again. We will therefore consider the analytic case first – analytic functions satisfy the SUCP by definition and in addition, the zero set is given by a countable union of analytic submanifolds of codimension one. This can easily be seen by considering the order of vanishing at a point and by observing that locally, every point in the zero set is contained in $(\partial^\alpha g)^{-1}(0)$, where $g$ is the analytic function and $\alpha$ is a multi-index such that $\nabla (\partial^\alpha g) \neq 0$.

Note that if $A$ and $g$ are analytic, one has $F$ satisfying $d_A^* d_A F = 0$, which is an elliptic system with analytic coefficients and thus by a classical result of Morrey [17] its entries are analytic. Therefore, the determinant is analytic also and thus cannot vanish to the infinite order at a single point, if it is non-trivial. Here and for the rest of the section, $(M,g)$ is a compact analytic (in the interior) Riemannian manifold of dimension $n$ with boundary. We first prove a result about the zero set of the determinant of a matrix solutions where $A$ is Yang-Mills and not necessarily analytic:

**Lemma 5.1.** Let $E = M \times \mathbb{C}^m$ a Hermitian vector bundle and $A$ a unitary Yang-Mills connection on $E$. Then any non-zero solution $F : M \to \mathbb{C}^{m \times m}$ to $d_A^* d_A F = 0$ has $\mathcal{N}(\det F)$ to be $(n-1)$-rectifiable. Moreover, $\det F$ satisfies the strong unique continuation property.

**Proof.** This is a local result, so assume we have a point $p \in M^\circ$ in the interior and take a small coordinate ball $B^n(\epsilon)$ around $p$, such that $\|A\|_{L^{n/2}(B^n(\epsilon))}$ is small enough; by a dilation we may also assume $B^n$ is the unit ball and we also have $\|A\|_{L^{n/2}(B^n)}$ stays the same as for the smaller ball, by a computation. By a result of Uhlenbeck [20], we have a gauge automorphism $X : B^n \to U(m)$ that takes $A$ to $A' = X^*(A)$ with

\[ x \]

and 1.2 even for non-unitary connections, but it was more convenient for us to use the uniform continuity argument instead.

5. Recovering a Yang-Mills connection for $m > 1$

The main obstacle to proving the $m > 1$ case is the possibility that the zero set of $\det F$ for $F$ satisfying $d_A^* d_A F = 0$ could potentially be large; it suffices to prove that the determinant does not vanish to an infinite order (if non-trivial) at any point, since by some general theory the zero set is then contained in an $(n-1)$-rectifiable set [2]. In other words, we want to prove $\det F$ satisfies the strong unique continuation property (SUCP in short). In addition to this, we would like to point out that it is no longer true that the critical zero set of $\det F$ is $(n-2)$-rectifiable, as in the case $m = 1$; a class of counterexamples is given by $F = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$, where we have that $\mathcal{N}_{\text{crit}}(\det F) = \mathcal{N}_{\text{crit}}(f^2)$ contains the set where $f$ vanishes (since $d(f^2) = 2fdf$). Therefore if $f$ vanishes on an $(n-1)$-dimensional set, then the critical set is also $(n-1)$-dimensional; one such example is given by $M = \mathbb{R}^2$ and $f(x,y) = x$ which vanishes along the $y$-axis and solves $\Delta_{\mathbb{R}^2}(x) = 0$.

Here a function $f$ on a manifold $M$ vanishes to infinite order at a point $x_0 \in M$ if in any local coordinates, we have $\frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = 0$ for all multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (this is invariant of choice of the coordinates).

However, if we were able to prove that $\det F$ has the strong unique continuation property, then we may try and run through the argument in Theorem 1.1 again. We will therefore consider the analytic case first – analytic functions satisfy the SUCP by definition and in addition, the zero set is given by a countable union of analytic submanifolds of codimension one. This can easily be seen by considering the order of vanishing at a point and by observing that locally, every point in the zero set is contained in $(\partial^\alpha g)^{-1}(0)$, where $g$ is the analytic function and $\alpha$ is a multi-index such that $\nabla (\partial^\alpha g) \neq 0$.

Note that if $A$ and $g$ are analytic, one has $F$ satisfying $d_A^* d_A F = 0$, which is an elliptic system with analytic coefficients and thus by a classical result of Morrey [17] its entries are analytic. Therefore, the determinant is analytic also and thus cannot vanish to the infinite order at a single point, if it is non-trivial. Here and for the rest of the section, $(M,g)$ is a compact analytic (in the interior) Riemannian manifold of dimension $n$ with boundary. We first prove a result about the zero set of the determinant of a matrix solutions where $A$ is Yang-Mills and not necessarily analytic:

**Lemma 5.1.** Let $E = M \times \mathbb{C}^m$ a Hermitian vector bundle and $A$ a unitary Yang-Mills connection on $E$. Then any non-zero solution $F : M \to \mathbb{C}^{m \times m}$ to $d_A^* d_A F = 0$ has $\mathcal{N}(\det F)$ to be $(n-1)$-rectifiable. Moreover, $\det F$ satisfies the strong unique continuation property.

**Proof.** This is a local result, so assume we have a point $p \in M^\circ$ in the interior and take a small coordinate ball $B^n(\epsilon)$ around $p$, such that $\|A\|_{L^{n/2}(B^n(\epsilon))}$ is small enough; by a dilation we may also assume $B^n$ is the unit ball and we also have $\|A\|_{L^{n/2}(B^n)}$ stays the same as for the smaller ball, by a computation. By a result of Uhlenbeck [20], we have a gauge automorphism $X : B^n \to U(m)$ that takes $A$ to $A' = X^*(A)$ with
$d^*(A') = 0$. In this particular gauge, the Yang-Mills equations become elliptic and therefore, $A'$ is analytic.

Similarly, since $d_A^*d_A F = 0$, we have $d_A^*d_A F' = 0$, where $F' = X^{-1}F$ – thus $F'$ is also analytic. Moreover, $\det F' \det X = \det F$ and so $\mathcal{N}(\det F) = \mathcal{N}(\det F')$ on $B^n$, as $X$ is always invertible; since $\det F'$ is analytic, we obtain the first part of the result.

Finally, from the relation $\det F' \det X = \det F$ and the fact that $\det X$ is non-zero on $B^n$, we immediately get that $\det F$ vanishes up to order $k$ if and only if $\det F'$ vanishes up to order $k$ – thus $\det F$ satisfies the SU CP, as $\det F'$ does. \[\square\]

This means that on $M^\circ$ we have $\mathcal{N}(\det G) \subset \cup_i M_i$ for $M_i$ a countable family of analytic submanifolds of codimension one, where $G$ solves $d_A^*d_A G = 0$ and is the gauge that we used in the previous section. Notice that $G^*(B)$ then satisfies an elliptic system (as before), but with analytic coefficients – therefore $G^*(B)$ is also analytic, but only on the set where $G$ is invertible!

To overcome this, we use the method of proof of the $m = 1$ case, Theorem 1.1 and the main difference is that now we will be able to use analyticity to uniquely continue the solution when drilling hyperplanes, whereas before we relied on the unique continuation property of elliptic equations.

**Proof of Theorem 1.2.** Assume we have the gauges $F$ and $G$ that solve $d_A^*d_A F = 0$ and $d_B^*d_B G = 0$ with $F|_{\partial M} = G|_{\partial M}$, $\text{supp}(F|_{\partial M}) = \text{supp}(G|_{\partial M}) \subset \Gamma$ and equal to identity on an open, non-empty subset $V$ of $\Gamma$. Then $F^*(A) = G^*(B)$ in the neighbourhood $U$ of $V$ in the manifold, as in Theorem 1.2 by unique continuation; equivalently, we have $H^*(A) = B$ where $H = FG^{-1}$. By the end of the proof of Theorem 1.2 we also have that $H$ is unitary.

We may suppose that $\mathcal{N}(\det G) \subset \cup_i M_i$ for $M_i$ analytic submanifolds of codimension one, by Lemma 5.1. Let us now prepare the terrain for drilling the holes – consider a point $p$ in $M_i$ for some $i$, such that the following property holds:

$$\frac{\partial^j(\det G)}{\partial x_n^j} = 0 \quad \text{for} \quad j = 0, 1, \ldots, k - 1$$ (5.1)

in a neighbourhood of $p$ in $M_i$, for some $k$; we also ask that $\frac{\partial^k(\det G)}{\partial x_n^k}(p) \neq 0$. Here we are using the analytic chart given by coordinates on $M_i$ near $p$ and the $x_n$ coordinate given by following the geodesic (which are analytic). We also make the standing assumption that $F$ and $G$ are invertible for $x_n > 0$ in this coordinate system and that $F^*(A) = G^*(B)$ in the same set.

This means that near $p$, by Taylor’s theorem we have $\det G = x_n^k g_1$ for some $g_1$ that satisfies $g_1(p) \neq 0$ – therefore locally at $p$, $\mathcal{N}(\det G)$ is contained in $M_i$. Since $H$ is unitary for $x_n > 0$, we have $H = FG^{-1} = \frac{F \text{adj} G}{x_n^k g_1}$ is bounded on this set and therefore $F \text{adj} G = x_n^k H_1$ for some smooth $H_1$ near $p$ – we get $H = \frac{H_1}{g_1}$ locally, which means that $H$ extends smoothly to the other side of $M_i$ in the proximity of $p$. Moreover, as $H$ unitary we have $|\det H| = 1$ at $p$ and so $H$ is invertible near $p$.

To use the real-analyticity, we must transform $A$ and $B$ such that they are locally analytic – we do this by constructing the Coulomb gauge automorphisms (unitary) $X$ and $Y$ locally near $p$ such that $A' = X^*(A)$ and $B' = Y^*(B)$ and moreover, that
\[d^*(A') = d^*(B') = 0\] (by the proof of Lemma 5.1). Then \(A'\) and \(B'\) are analytic as in the previous Lemma and moreover, we have \(F'' = X^{-1}F\) and \(G' = Y^{-1}G\) satisfying \(d''A'd'F' = 0\) and \(d''B'd'B'G' = 0\) – therefore \(F''\) and \(G'\) are analytic, as well.

Thus we may write \(H' = X^{-1}FG^{-1}Y\) and by rewriting \(H^*(A) = B\) (by assumption) we get \(H^*(A') = B'\) for \(x_n > 0\) in a neighbourhood of \(p\). Let us now observe that \(H'\) also smoothly (analytically) extends over the hyperplane \(M_i\) – this is because, by Taylor expanding \(\det (Y^{-1}G) = \frac{\det G}{\det Y}\), we get:

\[H' = X^{-1}F \cdot \frac{\text{adj} (Y^{-1}G)}{g' \cdot x_n^k}\]

where \(g' = \frac{\partial g}{\partial x_n}\) is analytic; so \(g' \neq 0\) near \(p\). However, we know \(H'\) is bounded near \(p\), since \(H, X\) and \(Y\) are. Thus \(X^{-1}F \cdot \text{adj} (Y^{-1}G) = F' \cdot \text{det} G' = x_n^k H''\) for some analytic \(H''\), by looking at the expansion of \(F' \cdot \text{det} G'\) – in conclusion, \(H' = \frac{H''}{g'}\) analytically extends near \(p\); \(H'\) also invertible at \(p\) as \(H, X\) and \(Y\) are.

Finally, it is easy now to see that \((H')^*(A') \equiv B'\), since both sides are analytic near \(p\) and \((H')^*(A') = B'\) for \(x_n > 0\); equivalently \(H^*(A) \equiv B\) near \(p\). This ends the drilling argument and we may repeat the part of the argument of Theorem 1.1 which perturbs the curve by an arbitrarily small amount so that it goes through the holes.

In more detail, let us briefly describe the analogous procedure from Theorem 1.1. Take a base point \(x_0 \in U \cap M\) and consider a path \(\gamma\) lying in the interior, from \(x_0\) to some point \(y \in M\). We perturb \(\gamma\) such that it intersects \(M_1\) transversally at \(P_1, \ldots, P_k\) (\(k\) can be zero). At \(P_1\), consider the tubular neighbourhood (analytic) given by following geodesics perpendicular to \(M_1\). If there exists a sequence of points \(p_j \in M_1\) that converges to \(P_1\) and \(\det G \neq 0\) at every \(p_j\), we may perturb \(\gamma\) slightly and get it to pass through one of the points \(p_j\). Otherwise, inductively, since \(\det G\) satisfies the SUCP by Lemma 5.1 there exists a positive integer \(k\) such that \(\frac{\partial (\det G)}{\partial x_1} = 0\) for \(0 \leq i \leq k - 1\) in a neighbourhood of \(P_1\) and there exists a sequence of points \(p_j \in M_1\) that converge to \(P_1\) and \(\frac{\partial (\det G)}{\partial x_n} \neq 0\) at each \(p_j\). This leaves us in the setting (5.1) from the previous paragraph, suitable for drilling the holes; inductively, we perturb \(\gamma\) such that it intersects the \(M_i\) in the drilled holes.

Thus we obtain a smooth (analytic in the interior) extension of \(H = FG^{-1}\) to the whole of \(M\), such that \(H^*(A) = B\) and \(H|_V = Id\).

To get the wanted gauge with \(H|_\tau = Id\), we will need a slightly different argument, because we do not know if \(\det F\) and \(\det G\) vanish to infinite order at the boundary, as we did not assume analyticity up to the boundary. We will construct a sequence of matrix functions \(H_i\) such that \(H_i^*(A) = B\) and use a compactness argument to take the limit. Consider nested open sets \(V_i\), with \(\emptyset \neq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq \ldots \subseteq \Gamma\) and with the property \(\cup_i V_i = \Gamma\). Construct appropriate \(F_i\) and \(G_i\) supported in \(\Gamma\), such that \(F_i|_{V_i} = G_i|_{V_i} = Id\), solving \(d'A'dF_i = 0\) and \(d'B'dG_i = 0\) and setting \(H_i = F_i G_i^{-1}\) – by the argument above \(H_i^*(A) = B\) and \(H_i|_{V_i} = Id\). Now the important property that the gauges satisfy is that they are unitary, hence bounded and they satisfy \(dH_i = H_iB - AH_i\) so that \(dH_i\) are bounded; by inductively differentiating this relation, we get that all derivatives of \(H_i\) are uniformly bounded on \(M\). By the Arzelà-Ascoli theorem (or the Heine-Borel property of \(C^\infty(M)\)) we get a convergent
subsequence with a limit \( H \in C^\infty(M;U(m)) \), \( H|_\Gamma = Id \) and \( H^*(A) = B \). This finishes the proof.

**Remark 5.2.** If we were able to prove that the determinant \( \det F \) satisfies the strong unique continuation property (where \( d_A d_A F = 0 \)) and that the unique continuation property from a hyperplane holds for degenerate elliptic systems, with degeneracies of the form \( x_n^{2k} \Delta_g \times Id + x_n^k F_1 + F_0 \), where \( F_0 \) and \( F_1 \) are zero and first order matrix operators, respectively and for all \( k \) positive integers; then we would be able to prove the \( m > 1 \) case in the smooth case, by following the proof of Theorem 1.1.

**Remark 5.3.** Notice that, while \( A' = F^*(A) \) above satisfies an elliptic equation with analytic coefficients and hence is analytic (and so is \( F \)), a problem appears when we approach the singular set of \( F \). Then we do not know any more that \( A' \) is analytic and cannot apply analyticity directly to conclude \( F^*(A) \equiv G^*(B) \); however, as we have seen in the above proof we may use the Coulomb gauge locally to get around this issue.

6. **Appendix**

Here we prove some of the less obvious elementary facts that we use in the paper. We need the metric space of smooth curves in the proof of our main theorem – here are some properties:

**Remark 6.1.** We are using the standard metric on the space \( C^\infty([0,1];\mathbb{R}) \) induced by the seminorms \( \|f\|_k = \sup_{t \in [0,1]} |\frac{d^k f}{dt^k}| \). Then a choice of the metric on this space is:

\[
d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|f - g\|_k}{1 + \|f - g\|_k}
\]

and it is a standard fact that this space is a Fréchet space with the same topology as the weak topology given by the seminorms. Furthermore, this also induces a Fréchet metric to the space \( C^\infty([0,1];\mathbb{R}^m) = \oplus_{i=1}^{m} C^\infty([0,1];\mathbb{R}) \) for all \( m \in \mathbb{N} \). Moreover, we may consider the space \( C^\infty([0,1];M) \) for any compact Riemannian manifold \((M,g)\) by isometrically embedding \( M \) into a Euclidean space \( \mathbb{R}^N \) for some \( N \), as a closed subspace of \( C^\infty([0,1],\mathbb{R}^N) \).

Now we prove the following lemma for the continuity of \( h \) in the interior and on the boundary of the manifold. There might be a slicker way to smoothly extend the function, but I have not found it in any literature yet.

**Lemma 6.2.** Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( E \subset \Omega \) a closed subset. Assume also that for any two points \( x,y \in \Omega \setminus E \) and any smooth path \( \gamma \) in \( \Omega \) between \( x \) and \( y \), there exist smooth paths \( \gamma_i \) from \( x \) to \( y \), lying in \( \Omega \setminus E \), for \( i = 1,2,\ldots \), that converge to \( \gamma \) in the metric space \( C^\infty([0,1];\mathbb{R}^n) \). Let \( f : \Omega \setminus E \to \mathbb{C} \) be a smooth function, such that \( \partial^\alpha f \) extend continuously to \( \Omega \) for all multi-indices \( \alpha \). Then there exists a unique smooth extension \( \tilde{f} : \Omega \to \mathbb{C} \) with \( \tilde{f}|_{\Omega \setminus E} = f \).

**Proof.** This is a local claim, so we will consider an extension near a point \( x \in E \). We will prove that the continuous extension \( \tilde{f} \) of \( f \) to \( \Omega \) is differentiable with the
derivative given by the continuous extension $h$ of $df$ to $\Omega$. By inductively repeating the argument for all $\partial^\alpha f$ for multi-indices $\alpha$, it clearly suffices to prove this.

Consider the point $y = x + \delta e_1$, where $\delta > 0$ is small enough so that the straight line path $\gamma$ between $x$ and $y$ lies in $\Omega$. Since $\Omega \setminus E$ is dense in $\Omega$, we may choose points $x', y' \in \Omega \setminus E$ that are close to $x, y$, respectively. Consider the path $\gamma'$ obtained by smoothing out the straight line path from $x'$ to $x$, $\gamma$ and the straight line path from $y$ to $y'$. By the hypothesis, there exists a sequence of paths $\gamma_n$ with endpoints at $x'$ and $y'$, lying entirely in $\Omega \setminus E$ that converge to $\gamma$ in the path metric.

We will consider the integrals along the curves $\gamma_n$: after possibly reparametrising, we may assume that $\gamma_n$ are parametrised by arc-length – we can always do this for $n$ sufficiently large, as $\gamma$ has a nowhere zero derivative. Therefore, we may integrate $h(\dot{\gamma}_n)$ to get that, by the Fundamental Theorem of Calculus:

$$f(y') - f(x') = \int_{\gamma_n} d(f \circ \gamma_n(t)) = \int_{\gamma_n} h(\dot{\gamma}_n)$$

Here, we think of $h$ as given by the vector of partial derivatives of $f$. By uniform convergence of the curves, we immediately get that

$$f(y') - f(x') = \int_{\gamma_n} h(\dot{\gamma}_n) \to \int_{\gamma} h(\dot{\gamma}')$$

and therefore, if we take $x' \to x$ and $y' \to y$ (can do this as $\Omega \setminus E$ is dense in $\Omega$), we get:

$$\frac{\tilde{f}(x + \delta e_1) - \tilde{f}(x)}{\delta} = \frac{1}{\delta} \int_0^\delta h_{x + te_1}(e_1)dt \to h_x(e_1)$$

as $\delta \to 0$. Therefore, the partial derivative in the $e_1$ direction exists and similarly, all other partials exist and are equal to the components of $h$. This finishes the proof. \qed

Remark 6.3. If we are given a smooth function $f$ in the interior of a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, such that all derivatives $\partial^\alpha f$ extend continuously to the boundary, it is well known that there exists a smooth extension $\tilde{f}$ to $\mathbb{R}^n$, such that $\tilde{f}|_\Omega = f$. This remark, together with the above lemma, are used in the proof of the smooth extension of $h$ over the singular set in Theorem 1.1.

Finally, we would like to recall the well-posedness conditions under which the solution operator to a generalised heat equation is smoothing. One set of such conditions is given by (1.5)-(1.7) on page 134 in Treves [19] – we state them here for completeness. Let $X$ be a manifold of dimension $n$ and $t$ a variable in the real line $\mathbb{R}$; we will consider vector functions with values in the finite dimensional space $H = C^m$. Let $A(t)$ be a pseudodifferential operator of order $k$ with values in $L(H)$ depending smoothly on $t \in [0, T]$; this means that in a local chart $\Omega \subset X$ we have the symbol of $A(t)$ modulo $S^{-\infty}$ being a smooth function $a_\Omega(x, t, \xi) : [0, T) \to S^k(\Omega; L(H))$. We consider the following equation in $X \times [0, T)$, where $U$ valued in $H$:

$$\frac{dU}{dt} - A(t) \circ U \equiv 0 \mod S^{-\infty}$$

The set of conditions for this equation to be well-posed is the following:
Condition 6.4 (Well-posedness of the heat equation). For every local chart \( \Omega \subset X \), there is a symbol \( a(x,t,\xi) \) depending smoothly on \( t \in [0, T] \) and defining a pseudodifferential operator \( A_\Omega(t) \) congruent smoothly to \( A(t) \) modulo regularising operators in \( \Omega \), such that for every compact \( K \subset \Omega \times [0, T] \) there is a compact subset \( K' \) of the open half-plane \( \mathbb{C}_- = \{ z \in \mathbb{C} \mid \text{Re}(z) < 0 \} \) such that

\[
z \times \text{Id} - \frac{a(x,t,\xi)}{(1 + |\xi|^2)^{\frac{3}{2}}} : H \to H
\]

(6.1)
is a bijection for all \( (x,t) \in K \), \( \xi \in \mathbb{R}^n \) and \( z \in \mathbb{C} \setminus K' \).

One remark is in place after this condition:

Remark 6.5. In fact, the symbol of the Laplace operator in the ordinary heat equation – if one plugs \(-|\xi|^2 \) \((m = 1)\) into (6.1), we have that the zero set spreads such that we have \( \text{Re}(z) \in (-1, 0) \) and \( \text{Im}(z) = 0 \), which is certainly not contained in a compact subset of \( \mathbb{C}_- = \{ \text{Re}(z) < 0 \} \); the trick is to add a factor of \( e^{-|\xi|^2} \) which does not change the class of the symbol modulo \( S^{-\infty} \), as we will see in the proof of the Lemma below.

Using the idea in the above remark, we prove that the operator we use in Proposition 6.6 satisfies Condition 6.4.

Lemma 6.6. The \( \mathbb{C}^{m \times m} \)-valued pseudodifferential operator \( A = B - E \times \text{Id} \) (defined in Lemma 3.2) satisfies Condition 6.4.

Proof. Denote by \( a_1 = -\sqrt{q_2} = -\sqrt{\sum_{\alpha,\beta} g^{\alpha\beta} \xi_\alpha \xi_\beta} \) the principal symbol of \( A \) \((E \text{ has degree zero})\). If \( K \subset [0, T] \times \mathbb{R}^{n-1} \) compact, then there exist positive \( C_1, C_2 \) and \( c \) such that

\[
c|\xi| \leq |a_1(x,t,\xi)| \leq C_1(1 + |\xi|^2)^{\frac{1}{2}}
\]

\[
|a_0(x,t,\xi)| \leq C_2
\]

for all \( (x,t) \in K \) and \( \xi \in \mathbb{R}^{n-1} \), by definition of symbols and the fact that \( g^{\alpha\beta} \) is positive definite. Thus we can rewrite:

\[
z \times \text{Id} - \frac{-\sqrt{q_2} \times \text{Id} + a_0}{(1 + |\xi|^2)^{\frac{3}{2}}} = \left( z + \frac{\sqrt{q_2}}{(1 + |\xi|^2)^{\frac{1}{2}}} \right) \times \text{Id} - \frac{a_0}{(1 + |\xi|^2)^{\frac{1}{2}}}
\]

(6.2)

and if this expression is singular, we ought to have

\[
\frac{|a_0|^2}{1 + |\xi|^2} \geq m^2 \left| z + \frac{\sqrt{q_2}}{(1 + |\xi|^2)^{\frac{1}{2}}} \right|^2 = \frac{m^2}{(1 + |\xi|^2)^{\frac{1}{2}}} \left| s + \frac{\sqrt{q_2}}{(1 + |\xi|^2)^{\frac{1}{2}}} \right|^2
\]

(6.3)

where \( z = r + is \). If we had \( |\xi| \) large enough and \( r \geq -\epsilon \) for some small \( \epsilon > 0 \), the left hand side of (6.3) would be small and the right hand side of it would be bigger than \( s^2 + (r + \frac{\epsilon}{2})^2 \) (up to a constant); therefore for \( |\xi| \geq K \) for some \( K \), (6.2) will be nonsingular for \( r \geq -\epsilon \).

Notice that in the condition we have the freedom of adding a smoothing factor – this will take care of the singular behaviour for \( |\xi| \) in a compact set. We will add a
factor of $Ce^{-|\xi|^2} \times Id \in S^{-\infty}$ for some $C > 0$ to remedy this. First of all, notice that the above argument remains the same with the same $|\xi|$, if we consider the symbol $\sqrt{a_0} \times Id + Ce^{-|\xi|^2} \times Id$.

Furthermore, we have the left hand side of (6.3) bounded for all $\xi$ uniformly, whereas the right hand side is bigger (up to a constant) than $(Ce^{-|\xi|^2} - \epsilon)^2$ for $r \geq -\epsilon$, large enough $C$ and $|\xi| \leq K$. Clearly this inequality fails to hold for large $C$ and this finishes the proof.

□

Remark 6.7. Below one can find the code that when typed into Mathematica, gives Figure 1:

```mathematica
uval = NDSolveValue[{D[u[x, y], x, x] + D[u[x, y], y, y] == 0,
                     DirichletCondition[u[x, y] == ArcTan[x, y]*Sin[100/ArcTan[x, y]], True]},
                    u, {x, y} \[Element] Disk[]

ContourPlot[uval[x, y] == 0, x^2 + y^2 == 1, Element[{x, y}, Disk[{0, 0}, 1.001]]
```

REFERENCES

[1] M. F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523–615.

[2] C. Bär, Zero sets of solutions to semilinear elliptic systems of first order, Invent. Math. 138 (1999), 183–202.

[3] G. Uhlmann, Developments in inverse problems since Calderón’s foundational paper, Harmonic analysis and partial differential equations (Chicago, IL, 1996), 295–345, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL (1999).

[4] M. Cekić, The Calderón problem for connections, preprint (2016), arXiv:1610.02985.

[5] L. De Carli, S. M. Hudson, Geometric remarks on the level curves of harmonic functions (English summary), Bull. Lond. Math. Soc. 42 (2010), no. 1, 83–95.

[6] S. K. Donaldson, P. B. Kronheimer, The geometry of four-manifolds, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1990.

[7] D. Dos Santos Ferreira, C. Kenig, M. Salo, G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, Invent. Math. 178 (2009), 119–171.

[8] D. Dos Santos Ferreira, C. Kenig, J. Sjöstrand, G. Uhlmann, Determining a magnetic Schrödinger operator from partial Cauchy data, Comm. Math. Phys. 271 (2007), 467–488.

[9] G. Eskin, Global uniqueness in the inverse scattering problem for the Schrödinger operator with external Yang-Mills potentials, Comm. Math. Phys. 222 (2001), 503–531.

[10] C. Guillarmou, A. Sá Barreto, Inverse problems for Einstein manifolds, Inverse Probl. Imaging 3 (2009), 1–15.

[11] M. W. Hirsch, Differential topology, Graduate Texts in Mathematics 33, Springer-Verlag, New York, 1994.

[12] Y. Kurylev, L. Oksanen, G. P. Paternatin, Inverse problems for the connection Laplacian, arxiv preprint (2015).

[13] M. Lassas, T. Lämmatainen, M. Salo, The Calderón problem for the conformal Laplacian, arxiv preprint (2016).

[14] M. Lassas, M. Taylor, G. Uhlmann, The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary, Comm. Anal. Geom. 11 (2003), no. 2, 207–221.

[15] J. M. Lee, G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, Comm. Pure Appl. Math. 42 (1989), no. 8, 1097–1112.

[16] R. Mazzeo, Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds, Amer. J. Math. 113 (1991), 25–45.

[17] C. B. Morrey Jr., On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior, Amer. J. Math. 80 (1958), 198–218.
[18] G. Nakamura, G. Uhlmann, *A layer stripping algorithm in elastic impedance tomography*, IMA Vol. Math. Appl. 90, Springer, New York (1997), 375–384.

[19] F. Treves, *Introduction to Pseudodifferential Operators and Fourier Integral Operators*, Plenum Press, New York, 1980.

[20] K. K. Uhlenbeck, *Connections with $L^p$ bounds on curvature*, Comm. Math. Phys. 83 (1982), 31–42.

[21] R. A. Walker, *Problems in Harmonic Function Theory*, Honors Theses (1998), Paper 492.

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK

E-mail address: m.cekic@dpmms.cam.ac.uk