An algorithm for the prime-counting function of primes larger than three

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Abstract The prime-counting function \( \pi(x) \) which computes the number of primes smaller or equal to a given real number has a long-standing interest in number theory. The present manuscript proposes a method to compute \( \pi(x) \) with time complexity \( O(x^{1/2}) \) without the need to introduce the non-trivial zeros of the Riemann zeta function. The method yields a primality test of time complexity \( O(\log n) \).

Keywords Prime-counting function, primality test

1 Introduction

The prime-number theorem describes the asymptotic behavior of the distribution of prime numbers using the prime-counting function \( \pi(x) \) defined as the number of primes less than or equal to \( x \). Based on a conjecture, known at the time of Legendre and Gauss that \( \pi(x) \sim \frac{x}{\log x} \) for large \( x \), the prime-number theorem was proven by Jacques-Salomon Hadamard and Charles de la Vallée Poussin independently \[4,5,13\]. A variant has been proposed in the elementary proof of the prime-number theorem of Atle Selberg \[12\].

While methods based on the prime-number theorem and the Riemann hypothesis are traditionally appropriate for calculating the prime-counting function for very large \( x \), other methods have been proposed such as the Meissel-Lehmer algorithm \[9\], which was subsequently improved by Lehmer \[7\], then Lagarias and Odlyzko \[6\], and finally Delgise and Rivat \[1\].

On the way of finding an analytical method for computing \( \pi(x) \), various implementations based on the Riemanns explicit formula are available in the
literature. Specifically, these are methods based on the Riemann-von Man-
goldt’s formula [8] which is well explained in sources such as in [2][10][3]. The
accuracy of the method has been validated successfully for $\pi(x)$ calculations
up to about $x = 1000$ using the first 29 pairs of complex zeros [11].

The method for calculating $\pi(x)$ introduced in the present manuscript is de-
scribed in the following sections. Section 2 introduces the sequence $\{S_n\}$ and
associated propositions. Section 3 covers the explicit expression of the n-th el-
ment in the sequence, the indexing and counting functions of the overall and
non-prime elements in the sequence. Subsequently, the number of primes in
the sequence is equal to the difference between the overall number of elements
and the number of non-prime elements of the sequence.

2 The sequence $\{S_n\}$ and elementary propositions

The prime-counting function proposed in the present manuscript relies on a
sequence and the below propositions.

Let us introduce the sequence $\{S_n\}$ where $n \in \mathbb{N}$, defined as follows:

$$S_{n+2} = S_n + 6,$$  

with starting conditions

$$S_0 = 5,$$  

and

$$S_1 = 7.$$  

Proposition 1 Any prime number larger than seven can be expressed as a
number of the sequence $\{S_n\}$ added to six.

Proposition 2 Any prime number added to six is a prime number or the
product of two numbers from the sequence $\{S_n\}$ where we can draw twice the
same number.

Propositions 1 and 2 stem from the fact that the set of elements of the sequence
$\{S_n\}$ including the identity element is spanned by the semigroup $G$, defined
by the set $E$ which has for basis the prime numbers larger than three and the
neutral element 1 provided with an internal composition law which for any
two elements $(x, y)$ of $E \times E$ has for result the product $x \cdot y \in E$.

We can easily show that the sequence $\{S_n\}$ contains all primes larger than
three as it can be written as $S_{n+1} = S_n + \delta_n$ where $S_0 = 5$ and $\delta_n = 2$ if $n$
is even and 4 if $n$ is odd. The sequence of odd numbers $T_{n+1} = T_n + 2$ where
$T_0 = 5$ contains all primes larger than three, as two is the only even prime
larger than one. The set of elements of the sequence $\{S_n\}$ is identical to the set
of elements of the sequence \( \{T_n\} \) excluding the points \( U_n = 2 \cdot (4 + 3 \cdot n) + 1 \)
where \( n = 0, 1, 2,... \). We get \( U_n = 9 + 6 \cdot n \) where \( n \in \mathbb{N} \) which is always divisible by 3. Thus, the sequence \( \{S_n\} \) contains all primes larger than three.

Furthermore, the sequence \( U_n = 9 + 6 \cdot n \) where \( n \in \mathbb{N} \) contains all odd numbers larger than 7, which are divisible by 3. We note that the set of positive odd numbers is spanned by the semigroup \( G_1 \), defined by the set \( E_1 \) which has for basis the prime numbers larger than two and the neutral element 1 and provided with an internal composition law which for any two elements \((x, y)\) of \( E_1 \times E_1 \) has for result the product \( x \cdot y \in E_1 \). This is because any odd number which is not prime can be expressed as the product of two odd numbers.

As the set of elements of the sequence \( \{S_n\} \) including the identity element is the set of positive odd numbers excluding the non-prime odd numbers divisible by 3, we have:

**Proposition 3** The set of elements of the sequence \( \{S_n\} \) including the identity element is spanned by the semigroup \( G \), defined by the set \( E \) which has for basis the prime numbers larger than three and the neutral element 1 provided with an internal composition law which for any two elements \((x, y)\) of \( E \times E \) has for result the product \( x \cdot y \in E \).

3 Mathematical

3.1 Analytical expression of the n-th term of the sequence \( \{S_n\} \) and counting of elements in the sequence

As the members of the sequence \( \{S_n\} \) where \( n = 0, 1, 2,... \) introduced in the previous section can be expressed as a pseudo-arithmetic series acting separately on odd and even indexes due to the jump in the sequence, their analytical expression is as follows:

\[
S_n = \begin{cases} 
5 + 3n, & \text{if } n \text{ is even} \\
4 + 3n, & \text{if } n \text{ is odd}
\end{cases}
\]  

Let us introduce the function \( \sigma: \mathbb{R} \rightarrow \mathbb{N} \) which returns the largest element of the sequence \( \{S_n\} \) where \( n = 0, 1, 2,... \) which is smaller or equal to the real number passed in the argument. Note the function \( \sigma \) is not defined if the argument passed to the function is smaller than five, which is the first element of the sequence. For this purpose, we assume that the argument \( x \) is larger than five.

Let us say we want to compute the number of elements in the sequence \( \{S_n\} \) where \( n = 0, 1, 2... \) and the last element is smaller or equal to \( x \) (where \( x > 5 \)) using (4). In case of an even index \( n \) for the last element of the sequence \( \{S_n\} \) smaller or equal to \( x \), the number of elements in the sequence \( N_1 \) is expressed as follows:
\[ N_1 = \frac{\sigma(x) - 5}{3} + 1. \]  

(5)

In case of an odd index \( n \) for the last element in the sequence \( \{S_n\} \) smaller or equal to \( x \), the number of elements in the sequence \( N_2 \) is expressed as follows:

\[ N_2 = \frac{\sigma(x) - 4}{3} + 1. \]  

(6)

Note the increment by +1 in (5) and (6) is due to the index shift which starts at 0. Depending on whether the index \( n \) of the last element of the sequence is odd or even, either \( N_1 \) or \( N_2 \) is an integer. If \( N_1 \) is an integer then the index \( n \) of the last element of the series is even; otherwise, the index is odd.

Let us introduce the function \( \omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) which behavior is as follows: If the first argument of \( \omega \) is an integer, then the function returns that argument; otherwise, it returns the second argument.

Thus, the number of elements \( N \) in the sequence \( \{S_n\} \) where \( n = 0, 1, 2, \ldots \) and the last element is equal to \( \sigma(x) \) is expressed as follows:

\[ N = \omega(N_1, N_2). \]  

(7)

The index function \( \eta : \mathbb{N} \to \mathbb{N} \) which takes as argument an element \( S_n \) of the sequence \( \{S_n\} \) where \( n = 0, 1, 2, \ldots \) and returns the index of that element is as follows:

\[ \eta(S_n) = \omega \left( \frac{S_n - 5}{3}, \frac{S_n - 4}{3} \right). \]  

(8)

3.2 Non-prime elements in the sequence \( \{S_n\} \)

Proposition 1 tells us that all prime numbers are contained in the sequence \( \{S_n\} \) where \( n \in \mathbb{N} \). From proposition 2, the number of primes in the sequence is equal to the overall number of elements \( N \) minus the number of non-prime numbers \( T \), which are all possible pairs of elements from the sequence including pairs where the same number is drawn twice such that the product of the two numbers is smaller or equal to the last term \( S_n \) of the sequence.

Therefore, the sequence of pairwise combinations associated with the first element of the sequence \( S_0 = 5 \) is: \( 5 \times 5, 5 \times 7, 5 \times 11, 5 \times 13, \) and so on. The sequence of pairwise combinations associated with the second element of the sequence \( S_1 = 7 \) is: \( 7 \times 7, 7 \times 11, 7 \times 13, 7 \times 17, \) and so on. The sequence of pairwise combinations associated with the third element \( S_2 = 11 \) is: \( 11 \times 11, 11 \times 13, 11 \times 17, \) etc.

Let us say \( S_k \) is a given element of the sequence for which we aim to compute the number of pairwise combinations smaller or equal to \( \sigma(x) \). Let us define
the integer \( m \) as the index of \( S_m = \sigma (\sigma (x)/S_k) \). For a given element \( S_k \) of the sequence, the number of pairwise combinations \( P_k \) having a product smaller or equal to \( \sigma (x) \) is expressed as follows:

\[
P_k = \begin{cases} 
\frac{\sigma (\sigma (x)/S_k) - 5}{3} + 1 - k, & \text{if } m \text{ is even} \\
\frac{\sigma (\sigma (x)/S_k) - 4}{3} + 1 - k, & \text{if } m \text{ is odd}
\end{cases}
\]

(9)

where \( m = \omega \left( \frac{\sigma (\sigma (x)/S_k) - 5}{3}, \frac{\sigma (\sigma (x)/S_k) - 4}{3} \right) \).

To compute the number of non-prime elements \( T \) in the sequence \( \{S_n\} \) where the last term is \( S_n = \sigma (x) \), we add up all the pairwise combinations \( P_k \) for \( k = 0, 1, 2, \ldots, m \), where \( m = \eta \left( \sigma (\sqrt{S_n}) \right) \). If \( x < 25 \), there are no pairwise combinations smaller or equal to \( x \), hence \( T = 0 \). When \( x \geq 25 \), we get:

\[
T = \sum_{k=0}^{\eta \left( \sigma (\sqrt{\sigma (x)}) \right)} P_k .
\]

(10)

Finally, the number of primes larger than 3 and smaller or equal to \( x \), denoted \( \pi (x) \), is computed as follows:

\[
\pi (x) = N - T ,
\]

(11)

where \( x > 5 \).

As a primality test, any integer \( n \) larger than five is prime if and only if \( \pi (n) - \pi (n - 1) \) is equal to one. Although, the summation term in (10) vanishes when computing \( \pi (n) - \pi (n - 1) \), the square root function which is generally considered to be \( O(\log n) \) is still the bottleneck for the computation of \( \pi (n) - \pi (n - 1) \). Thus, we can say that the proposed primality test has a time complexity \( O(\log n) \).

4 Conclusion

The present manuscript introduces a method for calculating the prime-counting function based on a sequence. The set of elements of that sequence including the identity element is spanned by the semigroup which for any two elements of the spanning set has for result the product of the elements on that set and which basis are the prime numbers larger than three and the neutral element. The method is aimed to be computationally efficient with possible applications in the field of computer science and research.

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