Abstract. Let $A$, $B$ be Banach $A$-modules with compatible actions and $M$ be a left Banach $A$-$A$-module and a right Banach $B$-$A$-module. In the current paper, we study module amenability, $n$-weak module amenability and module Arens regularity of the triangular Banach algebra $T = \left[ \begin{array}{c} A \\ M \\ B \end{array} \right]$ (as an $T := \left\{ \left[ \begin{array}{c} \alpha \\ \alpha \\ \alpha \end{array} \right] | \alpha \in A \right\}$-module). We employ these results to prove that for an inverse semigroup $S$ with subsemigroup $E$ of idempotents, the triangular Banach algebra $T_0 = \left[ \begin{array}{c} \ell^1(S) \\ \ell^1(S) \\ \ell^1(S) \end{array} \right]$ is permanently weakly module amenable (as an $T_0 = \left[ \begin{array}{c} \ell^1(E) \\ \ell^1(E) \\ \ell^1(E) \end{array} \right]$-module). As an example, we show that $T_0$ is $T_0$-module Arens regular if and only if the maximal group homomorphic image $G_S$ of $S$ is finite.

1. Introduction

The concept of module amenability for Banach algebras was initiated by Amini in [1]. The fundamental result was that the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$ if and only if $S$ is amenable, where $S$ is an inverse semigroup with subsemigroup $E$ of idempotents. In fact, he showed that Johnson’s theorem [15] (for groups) holds for discrete inverse semigroups if the relevant module structure is taken into account. Amini and Bagha in [4] introduced the concept of weak amenability for Banach algebras showed that $\ell^1(S)$ is weakly $\ell^1(E)$-module amenable when $S$ is a commutative inverse semigroup with the set of idempotents $E$ (indeed this is true for inverse semigroups whose idempotents are central). Bodaghi et al. in [5] and [6] extended this result and showed that $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module (with trivial left action) when $n$ is odd.
Pourabbas and Nasrabadi investigated weak module amenability of a class of Banach algebras, called triangular Banach algebras in [18]. They considered the case where $\mathcal{A}$, $\mathcal{B}$ are unital Banach algebras (with $\mathfrak{A}$-module structure) and $\mathcal{M}$ is a unital Banach $\mathcal{A}, \mathcal{B}$-module and showed that the corresponding triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{B} & \end{bmatrix}$ is weakly module amenable (as an $\mathcal{T} := \left\{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} | \alpha \in \mathfrak{A} \right\}$-module) if and only if both $\mathcal{A}$ and $\mathcal{B}$ are weakly module amenable (as $\mathfrak{A}$-modules). This can be regarded as the module version of a result of Forrest and Marcoux [12, Corollary 3.5] (the case that $\mathcal{A}$ or $\mathcal{B}$ has a bounded approximate identity and $\mathcal{M}$ is essential was later proved by Medghalchi et al. in [16]). Also, they generalized this result to the case of $(2n - 1)$-weak module amenability for $n \geq 1$ in [12, theorem 3.7].

The concept of Arens module regularity is introduced in [22] and modified in [19]. It is shown that $\mathcal{A} = \ell^1(S)$ is module Arens regular (as an $\ell^1(E)$-module) if and only if the group homomorphic image $G_S$ of $S$ is finite (see also [20]).

The motivation of writing this paper is Example 1 which shows that for the commutative inverse semigroup $S$ with subsemigroup $E$ of idempotents, the Banach algebra $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ \ell^1(S) & \end{bmatrix}$ is $n$-weakly module amenable (as an $\mathcal{T}_0 := \left\{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} | \alpha \in \ell^1(E) \right\}$-module) when $n \in \mathbb{N}$.

The paper is organized as follows: Section 2 is devoted to the study of module amenability of triangular Banach algebras. The main result of section 3 asserts that module Arens regularity of $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{B} & \end{bmatrix}$ is equivalent to that $\mathcal{A}$ and $\mathcal{B}$ are both module Arens regular and both act module regularly on $\mathcal{M}$.

In section 4, we generalize some results of [12] and [16], and we show that the triangular Banach algebra $\mathcal{T}$ is $(2n - 1)$-weakly module amenable (as an $\mathfrak{A}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$ are $(2n - 1)$-weakly module amenable (as Banach $\mathfrak{A}$-bimodules). In section 5, we prove that if $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly module amenable, then the first module cohomology group of $\mathcal{T}$ with coefficients in $\mathcal{T}^{(2n)}$ is a quotient of a special set of module homomorphism from $\mathcal{M}$ to $\mathcal{M}^{(2n)}$. In section 6, we show that for a commutative inverse semigroup $S$ with the set of idempotents $E$, the semigroup algebra $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module for all $n \in \mathbb{N}$. As a corollary, we show that $\mathcal{T}_0 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ \ell^1(S) & \end{bmatrix}$ is permanently weakly module amenable (as an $\mathcal{T}_0 = \left\{ \begin{bmatrix} \ell^1(E) \\ \ell^1(E) \end{bmatrix} \right\}$-module). Finally, we show that $\mathcal{T}_0$ is module Arens regular if and only if $G_S = S/ \approx$ is finite, where
s \approx t \text{ whenever } \delta_s - \delta_t \text{ belongs to the closed linear span of the set } \{\delta_{se} - \delta_{st} : s, t \in S, e \in E\}.

## 2. Module amenability

Let \( \mathcal{A} \) and \( \mathfrak{A} \) be Banach algebras such that \( \mathcal{A} \) is a Banach \( \mathfrak{A} \)-bimodule with compatible actions, that is

\[
\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha)
\]  

for all \( a, b \in \mathcal{A}, \alpha \in \mathfrak{A} \). Let \( X \) be a Banach \( \mathcal{A} \)-bimodule and a Banach \( \mathfrak{A} \)-bimodule with compatible actions, that is

\[
\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha)
\]

\[
x \cdot (a \cdot \alpha) = (x \cdot a) \cdot \alpha, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha)
\]

\[
a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \quad x \cdot (\alpha \cdot a) = (x \cdot \alpha) \cdot a
\]

for all \( a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X \). Then we say that \( X \) is a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module. If

\[
\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A}, x \in X)
\]

then \( X \) is called a \textit{commutative} \( \mathcal{A} \)-\( \mathfrak{A} \)-module. Moreover, if

\[
a \cdot x = x \cdot a \quad (a \in \mathcal{A}, x \in X),
\]

then \( X \) is called a \textit{bi-commutative} Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module.

If \( X \) is a commutative Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module, then so is \( X^* \), where the actions of \( \mathcal{A} \) and \( \mathfrak{A} \) on \( X^* \) are defined as usual:

\[
\langle f \cdot \alpha, x \rangle = \langle f, \alpha \cdot x \rangle, \quad \langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle,
\]

\[
\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \quad \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X, f \in X^*)
\]

Note that when \( \mathcal{A} \) acts on itself by algebra multiplication, it is not in general a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module, as we have not assumed the compatibility condition

\[
a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \quad (\alpha \in \mathfrak{A}, a, b \in \mathcal{A}).
\]

If \( \mathcal{A} \) is a commutative \( \mathfrak{A} \)-module and acts on itself by multiplication from both sides, then it is also a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module. Also, if \( \mathcal{A} \) is a commutative Banach algebra, then it is a bi-commutative \( \mathcal{A} \)-\( \mathfrak{A} \)-module. In these cases, \( \mathcal{A}^{(n)} \), \( n \)-th dual space of \( \mathcal{A} \) is also a commutative or bi-commutative \( \mathcal{A} \)-\( \mathfrak{A} \)-module, respectively.

Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach \( \mathfrak{A} \)-modules with compatible actions \((\ref{eq:2.1})\). Then a left \( \mathfrak{A} \)-module map is a mapping \( T : \mathcal{A} \to \mathcal{B} \) with \( T(a \pm b) = T(a) \pm T(b) \) and \( T(\alpha \cdot x) = \alpha \cdot T(x) \) for \( a \in \mathcal{A}, b \in \mathcal{B}, \) and \( \alpha \in \mathfrak{A} \). A right or two-sided \( \mathfrak{A} \)-module map is defined similarly.

Let \( \mathcal{A} \) and \( \mathfrak{A} \) be as above and \( X \) be a Banach \( \mathcal{A} \)-\( \mathfrak{A} \)-module. A \textit{(}\( \mathfrak{A} \)-)module derivation is a bounded \( \mathfrak{A} \)-module map \( D : \mathcal{A} \to X \) satisfying

\[
D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}).
\]
One should note that $D$ is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. When $X$ is commutative, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called inner $\mathcal{A}$-module derivations.

We use notations $Z^1_{\mathcal{A}}(\mathcal{A}, X)$ and $N^1_{\mathcal{A}}(\mathcal{A}, X)$ for the set of all module derivations and inner derivations from $\mathcal{A}$ to $X$, respectively. Also the quotient space $Z^1_{\mathcal{A}}(\mathcal{A}, X)/N^1_{\mathcal{A}}(\mathcal{A}, X)$ (which we call the first $\mathcal{A}$-module cohomology group of $\mathcal{A}$ with coefficients in $X$) is denoted by $\mathcal{H}^1_{\mathcal{A}}(\mathcal{A}, X)$. A Banach algebra $\mathcal{A}$ is module amenable if $\mathcal{H}^1_{\mathcal{A}}(\mathcal{A}, X^*) = \{0\}$, for each commutative Banach $\mathcal{A}$-$\mathcal{A}$-module $X$.

It is proved in [1, Proposition 2.5] that the homomorphic image of a module amenable Banach algebra under a continuous homomorphism with dense range is also a module amenable Banach algebra. This fact leads to the following result.

**Lemma 2.1.** Let $\mathcal{A}$ be a Banach $\mathcal{A}$-module and $I$ be a closed ideal in $\mathcal{A}$. Then module amenability of $\mathcal{A}$ implies module amenability of $\mathcal{A}/I$.

**Proposition 2.1.** Let $\mathcal{A}$ be a Banach $\mathcal{A}$-module, $I$ be a closed ideal and $\mathcal{A}$-submodule of $\mathcal{A}$. If $I$ and $\mathcal{A}/I$ are module amenable, then so is $\mathcal{A}$.

**Proof.** Assume that $X$ is a commutative Banach $\mathcal{A}$-$\mathcal{A}$-module with compatible actions and $D : \mathcal{A} \to X^*$ be a bounded module derivation. Since $I$ is module amenable, there exists $f_1 \in X^*$ such that $D |_I = D_{f_1}$. Thus, the map $\bar{D} = D - D_{f_1}$ vanishes on $I$. This map induces a module derivation from $\mathcal{A}/I$ into $X^*$, which we again denote by $\bar{D}$. Let $Y$ be the closed linear span of

$$\{a \cdot x - y \cdot b \mid a, b \in I, x, y \in X\},$$

in $X$. It follows immediately that $Y$ is a closed $\mathcal{A}$-submodule and $\mathcal{A}$-submodule of $X$, and so $X/Y$ is a Banach $\mathcal{A}/I$-$\mathcal{A}$-module with compatible actions. Since $D |_I = \{0\}$, we have $a \cdot \bar{D}(b) = \bar{D}(ab) - \bar{D}(a) \cdot b = 0$ for all $a \in I$ and $b \in \mathcal{A}$. Similarly, $\bar{D}(b) \cdot a = 0$. This implies $\bar{D}(\mathcal{A}/I) \subset Y^\perp = (X/Y)^*$. Due to module amenability of $\mathcal{A}/I$, there is $f_2 \in Y^\perp \subset X^*$ such that $\bar{D} = D_{f_2}$. Consequently, $D = D_{f_1 + f_2}$. □

**Proposition 2.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathcal{A}$-modules. Then $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$, $\ell^1$-direct sum of $\mathcal{A}$ and $\mathcal{B}$ is module amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are module amenable.

**Proof.** Let $\mathcal{A}$ and $\mathcal{B}$ be module amenable. Since $\mathcal{B}$, the closed ideal of $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$ and the quotient algebra $(\mathcal{A} \oplus_{\ell^1} \mathcal{B})/\mathcal{B} \cong \mathcal{A}$ are module amenable, $\mathcal{A} \oplus_{\ell^1} \mathcal{B}$ is module amenable by Proposition 2.1.
Conversely, assume that $A \oplus_{\ell^1} B$ is module amenable. Thus $(A \oplus_{\ell^1} B)/A \cong B$ and $(A \oplus_{\ell^1} B)/B \cong A$ are module amenable by Lemma 2.1. □

Let $A$ and $\mathfrak{A}$ be Banach $A$-bimodules with compatible actions (2.1) and $M$ be a Banach $A$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions (2.2) and (2.3).

Let $J_M$ be a subspace of $M$ generated by elements of the form $a \cdot (\alpha \cdot x) - (\alpha \cdot (a \cdot x))$ and $x \cdot (\alpha \cdot a) - (x \cdot (\alpha \cdot a))$ for $\alpha \in \mathfrak{A}, a \in A$ and $x \in M$. One can see from (2.2) and (2.3) that $J_M$ is an $A$-submodule and $\mathfrak{A}$-submodule of $M$. Note that the equalities (2.4) do not necessarily hold for $M$. When $M = A$, $J_A$ is in fact the closed ideal of $A$ generated by elements of the form $(\alpha \cdot a)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in A$ (see [5], [6] and [19]).

Let $T = \left\{ \begin{bmatrix} a & m \\ b & M a \end{bmatrix} \mid a \in A, b \in B, m \in M \right\}$ be triangular Banach algebra equipped with the usual $2 \times 2$ matrix addition and formal multiplication. The norm on $T$ is $\| \begin{bmatrix} a & m \\ b & M a \end{bmatrix} \| = \|a\|_A + \|b\|_B + \|m\|_M$. Now, let $A, B$ and $M$ be Banach $\mathfrak{A}$-bimodules, and let $M$ be a Banach $A, B$-module (left $A$-module and right $B$-module). Similar to [16], we consider the Banach algebra $\mathfrak{T} := \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \mid \alpha \in \mathfrak{A} \right\}$. Then, the Banach algebra $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ with usual $2\times 2$ matrix product is a $\mathfrak{T}$-bimodule. In fact, That is isomorphic to $A \oplus_{\ell^1} M \oplus_{\ell^1} B$ as a Banach space and a Banach $\mathfrak{A}$-bimodule. The following result is a module version of [16, Theorem 4.2] and the proof is similar. However, we bring its proof.

**Theorem 2.3.** Let $A, B$ be Banach $\mathfrak{A}$-modules and $M$ be commutative Banach $A$-module. The triangular Banach algebra $T$ is module amenable (as an $\mathfrak{T}$-bimodule) if and only if $A$ and $B$ are module amenable (as Banach $\mathfrak{A}$-bimodules) and $M = 0$.

**Proof.** First note that when $T$ is $\mathfrak{T}$-bimodule it means that $A \oplus_{\ell^1} M \oplus_{\ell^1} B$ is $\mathfrak{A}$-bimodule with the usual actions. Now, let $A$ and $B$ are module amenable and $M = 0$. Then $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is module amenable by Proposition 2.2.

Conversely, assume that $T$ is $\mathfrak{T}$-module amenable. The Banach algebras $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} 0 & M \\ B & 0 \end{bmatrix}$ are closed ideals of $T$, and thus $A$ and $B$ are $\mathfrak{A}$-module amenable by Lemma 2.1. Since $M$ is complemented in $T$, it is module amenable and since $M$ is a commutative $\mathfrak{A}$-module, it has a bounded approximate identity by [11, Proposition 2.2], hence it should be zero. □
3. Module Arens regularity

For given $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$, by the Goldstein theorem there are nets $(a_{1,i})_i$ and $(a_{2,j})_j$ in $\mathcal{A}$ such that $\Gamma_1 = w^* \lim_i a_{1,i}$ and $\Gamma_2 = w^* \lim_j a_{2,j}$. Then we consider the first Arens product on $\mathcal{A}^{**}$ as follows:

$$\Gamma_1 \square \Gamma_2 = w^* \lim_i a_{1,i}a_{2,j},$$

and similarly for any $\Psi_1, \Psi_2 \in \mathcal{B}^{**}$, there exist nets $(b_{1,i})_i$ and $(b_{2,j})_j$ such that the first Arens product on $\mathcal{B}$ is defined as follows:

$$\Psi_1 \square \Psi_2 = w^* \lim_i b_{1,i}b_{2,j}.$$

We extend the actions of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{M}$ to actions of $\mathcal{A}^{**}$ and $\mathcal{B}^{**}$ on $\mathcal{M}^{**}$ via

$$\Gamma \square \Pi = w^* \lim_{k,j} a_{1,i} \cdot x_k \cdot b_{2,j},$$

and

where $\Gamma = w^* \lim_i a_{1,i}$, $\Psi = w^* \lim_j b_{2,j}$, and $\Pi = w^* \lim_k x_k$. We define the first Arens product on $\mathcal{T}^{**}$ in a natural way. Let $T_1 = \left[ \begin{array}{ccc} \Gamma_1 & \Pi_1 \\ \Psi_1 & \end{array} \right]$ and $T_2 = \left[ \begin{array}{ccc} \Gamma_2 & \Pi_2 \\ \Psi_2 & \end{array} \right] \in \mathcal{T}^{**}$ so that $\Gamma_1 = w^* \lim_i \left[ \begin{array}{ccc} a_{1,i} \\ x_{1,i} \\ b_{1,i} \end{array} \right]$ and $T_2 = w^* \lim_j \left[ \begin{array}{ccc} a_{2,j} \\ x_{2,j} \\ b_{2,j} \end{array} \right]$. Then we have

$$T_1 \square T_2 = \left[ \begin{array}{ccc} \Gamma_1 & \Pi_1 \\ \Psi_1 & \end{array} \right] \square \left[ \begin{array}{ccc} \Gamma_2 & \Pi_2 \\ \Psi_2 & \end{array} \right]$$

$$= w^* \lim_{j,i} \left[ \begin{array}{ccc} a_{1,i} \\ x_{1,i} \\ b_{1,i} \end{array} \right] \left[ \begin{array}{ccc} a_{2,j} \\ x_{2,j} \\ b_{2,j} \end{array} \right]$$

$$= w^* \lim_{j,i} \left[ \begin{array}{ccc} a_{1,i}a_{2,j} \\ a_{1,i}x_{2,j} + x_{1,i}b_{2,j} \\ b_{1,i}b_{2,j} \end{array} \right].$$

Similarly, we consider the second Arens product on $\mathcal{A}^{**}$, $\mathcal{B}^{**}$ and module actions $\mathcal{A}^{**}$, $\mathcal{B}^{**}$ on $\mathcal{M}^{**}$ as follows:

$$\Gamma_1 \ast \Gamma_2 = w^* \lim_{j,i} a_{1,i}a_{2,j}, \quad \Psi_1 \ast \Psi_2 = w^* \lim_{j,i} b_{1,i}b_{2,j},$$

and

$$\Gamma_1 \ast \Pi = w^* \lim_{k,i} a_{1,i}x_k, \quad \Pi \ast \Psi_1 = w^* \lim_{j,k} x_k b_{1,j},$$
where $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}, \Psi_1, \Psi_2 \in \mathcal{B}^{**}$ and $\Pi \in \mathcal{M}^{**}$. Thus the second Arens product on $T^{**}$ can be defined via

$$ T_1 \circ T_2 = \left[ \begin{array}{cc} \Gamma_1 & \Pi_1 \\ \Psi_1 & \end{array} \right] \circ \left[ \begin{array}{cc} \Gamma_2 & \Pi_2 \\ \Psi_2 & \end{array} \right] = w^* - \lim_{j \rightarrow \infty} \left[ \begin{array}{cc} a_{1,i} x_{1,i} & x_{2,j} \\ b_{1,i} & b_{2,j} \end{array} \right] = w^* - \lim_{j \rightarrow \infty} \left[ \begin{array}{cc} a_{1,i} a_{2,j} & a_{1,i} x_{2,j} + x_{1,i} b_{2,j} \\ b_{1,i} b_{2,j} & \end{array} \right] = \left[ \begin{array}{cc} \Gamma_1 \circ \Gamma_2 & \Gamma_1 \circ \Pi_2 + \Pi_1 \circ \Psi_2 \\ \Psi_1 \circ \Psi_2 & \end{array} \right]. $$

The concept of module Arens regularity for Banach algebras is defined in [22]. A Banach algebra $\mathcal{A}$ is module Arens regular as a Banach $\mathfrak{A}$-module if and only if $\Gamma_1 \circ \Gamma_2 - \Gamma_1 \circ \Gamma_2 \in J_A^{1\perp}$, for every $\Gamma_1, \Gamma_2 \in \mathcal{A}^{**}$ (see [22] Theorem 2.2 and [3] Theorem 3.3).

We say that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ act module regularly on $\mathcal{M}$ if for each $\Gamma \in \mathcal{A}^{**}, \Psi \in \mathcal{B}^{**}$ and $\Pi \in \mathcal{M}^{**}$ we have

$$ \Gamma \circ \Pi - \Gamma \circ \Pi \in J_M^{1\perp}, \quad \Pi \circ \Psi - \Pi \circ \Psi \in J_M^{1\perp}. $$

Recall that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ act regularly on $\mathcal{A}, \mathcal{B}$-module $\mathcal{M}$ if for every $\Gamma \in \mathcal{A}^{**}, \Psi \in \mathcal{B}^{**}$ and $\Pi \in \mathcal{M}^{**}$ we have $\Gamma \circ \Pi = \Gamma \circ \Pi$ and $\Pi \circ \Psi = \Pi \circ \Psi$ [13]. It follows from the above that $\mathcal{A}$ and $\mathcal{B}$ act module regularly on $\mathcal{M}$ if and only if $\mathcal{A}/J_A$ and $\mathcal{B}/J_B$ act regularly on $\mathcal{M}/J_M$. Indeed,

$$ \Gamma \circ \Pi - \Gamma \circ \Pi \in J_M^{1\perp} \iff \Gamma \circ \Pi + J_M^{1\perp} = \Gamma \circ \Pi + J_M^{1\perp} $$

$$ \iff (\Gamma + J_A^{1\perp}) \circ (\Pi + J_M^{1\perp}) = (\Gamma + J_A^{1\perp}) \circ (\Pi + J_M^{1\perp}). $$

Similarly, $\Pi \circ \Psi - \Pi \circ \Psi \in J_M^{1\perp}$ if and only if $(\Pi + J_M^{1\perp}) \circ (\Psi + J_B^{1\perp}) = (\Pi + J_M^{1\perp}) \circ (\Psi + J_B^{1\perp})$. Similarly we can show that $J_T = \left[ \begin{array}{cc} J_A & J_M \\ J_M & J_B \end{array} \right]$ and thus $\left[ \begin{array}{cc} J_A^{1\perp} & J_M^{1\perp} \\ J_B \end{array} \right] \subseteq J_T^{1\perp}$. It is shown in [6] Lemma 3.1 that $J_A$ and $J_B$ are the closed subspace of $\mathcal{A}$ an $\mathcal{B}$ respectively. It is easy to check that $J_M$ is also a closed subspace of $\mathcal{M}$. Hence $J_A, J_B$ and $J_M$ are weak*-dense in $J_A^{1\perp}, J_B^{1\perp}$ and $J_M^{1\perp}$ respectively by [9] Theorem A.3.47. Hence $J_A \oplus_I, J_M \oplus_I, J_B$ is weak*-dense in $J_A^{1\perp} \oplus_I, J_M^{1\perp} \oplus_I, J_B^{1\perp}$. On the other hand, $J_A \oplus_I, J_M \oplus_I, J_B$ is closed subspace of $\mathcal{A} \oplus_I, \mathcal{M} \oplus_I, \mathcal{B}$ and so it is weak*-dense in $J_T^{1\perp}$. Therefore $J_T^{1\perp} = \left[ \begin{array}{cc} J_A^{1\perp} & J_M^{1\perp} \\ J_B^{1\perp} & \end{array} \right]$. Summing up:

**Theorem 3.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. The triangular Banach algebra $T$ is module Arens regular (as an $\mathfrak{A}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$...
are module Arens regular (as Banach $\mathfrak{A}$-bimodules) and $\mathcal{A}$ and $\mathcal{B}$ act module regularly on $\mathcal{M}$.

4. (2N-1)-Weak Module Amenability

We start this section with the definition of $n$-weak module amenability which is introduced in [6]. When $\mathcal{A}$ is a commutative $\mathfrak{A}$-bimodule, we have $J = \{0\}$ and $\mathcal{A}$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-module. In this case, the following definition (for $n = 1$) coincides with the definition of weak module amenability in [4]. If $\mathcal{A}$ is a commutative Banach algebra and a commutative $\mathfrak{A}$-bimodule with compatible actions, then $\mathcal{A}$ is a bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module. In this case, for each bi-commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $\mathcal{X}$, all bounded module derivation from $\mathcal{A}$ into $\mathcal{X}$ are zero (see the next lemma) and we get the definition of weak module amenability for commutative Banach algebras as in [4].

**Definition 4.1.** Let $\mathcal{A}$ be a Banach algebra, $n \in \mathbb{N}$. Then $\mathcal{A}$ is called $n$-weakly module amenable (as an $\mathfrak{A}$-module) if $(\mathcal{A}/J)^{(n)}$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, and each module derivation from $D : \mathcal{A} \to (\mathcal{A}/J)^{(n)}$ is inner; that is, $D(a) = a \cdot y - y \cdot a =: D_y(a)$ for some $y \in (\mathcal{A}/J)^{(n)}$ and each $a \in \mathcal{A}$. Also $\mathcal{A}$ is called weakly module amenable if it is 1-weakly module amenable and permanently weakly module amenable if it is $n$-weakly module amenable for each $n \in \mathbb{N}$.

**Lemma 4.1.** Let $\mathcal{A}$ be a essential bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module. Then $\mathcal{A}$ is weakly module amenable (as an $\mathfrak{A}$-module) if and only if for each bi-commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $\mathcal{X}$, all bounded module derivation from $\mathcal{A}$ into $\mathcal{X}$ are zero.

**Proof.** We follow the standard argument in [6, Theorem 2.8.63]. Assume that there exists $D \in \mathcal{Z}_{\mathfrak{A}}(\mathcal{A}, \mathcal{X})$ with $D \neq 0$. Since $\mathcal{A}^* = \mathcal{A}$, there exists $a_0 \in \mathcal{A}$ such that $D(a_0^2) \neq 0$. We have $a_0 \cdot D(a_0) \neq 0$ and thus $f \in \mathcal{X}^*$ with $f(a_0 \cdot D(a_0)) = 1$. Set $R : \mathcal{X} \to \mathcal{A}^*$ defined by $R(x)(a) = f(a \cdot x)$ where $a \in \mathcal{A}, x \in \mathcal{X}$. It is easy to check that $R \circ D \in \mathcal{Z}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^*)$. We get $\langle R \circ D(a_0), a_0 \rangle = \langle f, a_0 \cdot D(a_0) \rangle = 1$, and so $R \circ D \neq 0$. This shows that $\mathcal{A}$ is not weakly module amenable. The converse is clear. □

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ be as in the previous section. If these are commutative Banach $\mathfrak{A}$-modules, then the corresponding triangular Banach algebra $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{B} & \alpha \end{bmatrix}$ is a commutative $\mathfrak{A}$-module in which $\mathfrak{A} := \{ \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} | \alpha \in \mathfrak{A} \}$. Also $\mathcal{T}$ is isomorphic to $\mathcal{A} \oplus_{\mathfrak{A}} \mathcal{M} \oplus_{\mathfrak{A}} \mathcal{B}$ as a Banach $\mathfrak{A}$-module and a Banach $\mathfrak{A}$-module, respectively. Therefore $\mathcal{T}^{(2n-1)} \simeq \mathcal{A}^{(2n-1)} \oplus_{\mathfrak{A}} \mathcal{M}^{(2n-1)} \oplus_{\mathfrak{A}} \mathcal{B}^{(2n-1)}$, while $\mathcal{T}^{(2n)} \simeq \mathcal{A}^{(2n)} \oplus_{\mathfrak{A}} \mathcal{M}^{(2n)} \oplus_{\mathfrak{A}} \mathcal{B}^{(2n)}$ in which $\mathcal{T}^{(n)}$ is Banach $\mathfrak{A}$-module and $\mathcal{A}^{(2n-1)} \oplus_{\mathfrak{A}} \mathcal{M}^{(2n-1)} \oplus_{\mathfrak{A}} \mathcal{B}^{(2n-1)}$, $\mathcal{A}^{(2n)} \oplus_{\mathfrak{A}} \mathcal{M}^{(2n)} \oplus_{\mathfrak{A}} \mathcal{B}^{(2n)}$ are Banach $\mathfrak{A}$-modules. Suppose that $t = \begin{bmatrix} a & m \\ b & \lambda \end{bmatrix} \in \mathcal{T}$ and $\theta = \begin{bmatrix} f & \lambda \\ g & \lambda \end{bmatrix} \in \mathcal{T}^*$. Then the
pairing of $T^*$ and $T$ is given by $\theta(t) = f(a) + \lambda(m) + g(b)$. Indeed, it is easy to check that the module actions $T$ on $T^*$ are as follows:

$$t \cdot \theta = \left[ \begin{array}{c} a \cdot f + m \cdot \lambda \\ b \cdot g \end{array} \right]$$

and

$$\theta \cdot t = \left[ \begin{array}{c} f \cdot a + \lambda \cdot m \\ \lambda \cdot m + g \cdot b \end{array} \right].$$

We remove the dot for simplicity. This process may be repeated to define the actions of $T$ on $T^{(n)}$ as follows:

$$\left[ \begin{array}{c} a \\ m \\ b \end{array} \right] \cdot \left[ \begin{array}{c} \lambda \\ \gamma \\ \mu \end{array} \right] = \left[ \begin{array}{c} a \lambda \\ a \gamma + m \mu \\ b \mu \end{array} \right],$$

$$\left[ \begin{array}{c} \lambda \\ \gamma \\ \mu \end{array} \right] \cdot \left[ \begin{array}{c} a \\ m \\ b \end{array} \right] = \left[ \begin{array}{c} \lambda a \\ \lambda m + \gamma b \\ \mu b \end{array} \right],$$

$$\left[ \begin{array}{c} a \\ m \\ b \end{array} \right] \cdot \left[ \begin{array}{c} \phi \\ \varphi \\ \psi \end{array} \right] = \left[ \begin{array}{c} a \phi + m \varphi \\ b \varphi \\ \psi b + \varphi m \end{array} \right],$$

for every $\left[ \begin{array}{c} a \\ m \\ b \end{array} \right] \in T$, $\left[ \begin{array}{c} \lambda \\ \gamma \\ \mu \end{array} \right] \in T^{(2n)}$ and $\left[ \begin{array}{c} \phi \\ \varphi \\ \psi \end{array} \right] \in T^{(2n-1)}$.

Henceforth, we assume that $A$, $B$ and $M$ are commutative Banach $\mathfrak{A}$-modules and thus $T^{(2n-1)}$ and $T^{(2n)}$ become commutative Banach $\mathfrak{A}$-modules for any $n \in \mathbb{N}$.

The following two lemmas are proved similar to Lemmas 1.1 and 1.2 in [18].

**Lemma 4.2.** The $\mathfrak{A}$-module map $D : T \rightarrow T^{(2n-1)}$ is a module derivation if and only if there exist module derivations $D_A : A \rightarrow A^{(2n-1)}$, $D_B : B \rightarrow B^{(2n-1)}$ and $\gamma \in M^{(2n-1)}$ such that

$$D\left( \left[ \begin{array}{c} a \\ m \\ b \end{array} \right] \right) = \left[ \begin{array}{c} D_A(a) - m \gamma \\ \gamma a - b \gamma \end{array} \right],$$

for every $\left[ \begin{array}{c} a \\ m \\ b \end{array} \right] \in T$.

**Lemma 4.3.** Let $D_A : A \rightarrow A^{(2n-1)}$ and $D_B : B \rightarrow B^{(2n-1)}$ be bounded module derivations. Then $D_{AB} : T \rightarrow T^{(2n-1)}$ defined via

$$\left[ \begin{array}{c} a \\ m \\ b \end{array} \right] \rightarrow \left[ \begin{array}{c} D_A(a) \\ D_B(b) \end{array} \right],$$

is a bounded module derivation. Furthermore, $D_{AB}$ is inner if and only if $D_A$ and $D_B$ are inner.

Using Lemmas 4.2 and 4.3 and similar to [18 Theorem 2.1], we have the following result (see also [16 Theorem 2.1]).
Theorem 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras, and $\mathcal{M}$ be a unital $\mathcal{A}, \mathcal{B}$-module. Then

$$\mathcal{H}_A^1(\mathcal{T}, \mathcal{T}^{(2n-1)}) \simeq \mathcal{H}_A^{(2n-1)}(\mathcal{A}) \oplus \mathcal{H}_B^{(2n-1)}(\mathcal{B}).$$

(4.3)

In particular, the triangular Banach algebra $\mathcal{T}$ is $(2n-1)$-weakly module amenable (as an $\mathcal{T}$-bimodule) if and only if $\mathcal{A}$ and $\mathcal{B}$ are $(2n-1)$-weakly module amenable (as Banach $\mathcal{A}$-bimodules).

A Banach $\mathcal{A}, \mathcal{B}$-module $\mathcal{M}$ is said to be non-degenerate if $\mathcal{A}m = \{0\}$ implies that $m = 0$, and $m\mathcal{B} = \{0\}$ implies that $m = 0$ for every $m \in \mathcal{M}$. If the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ have bounded approximate identity and $\mathcal{M}$ is essential, then $\mathcal{M}$ is a non-degenerated $\mathcal{A}, \mathcal{B}$-module. Also when $\mathcal{M}$ is essential, then $\mathcal{M}^*$ is a non-degenerate Banach $\mathcal{A}, \mathcal{B}$-module. Although in the following Proposition $\mathcal{A}$ and $\mathcal{B}$ are not unital but still the conclusion of Lemma 4.2 holds. In fact we obtain the same result with different conditions.

Proposition 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and $\mathcal{M}$ be Banach $\mathcal{A}, \mathcal{B}$-module. Suppose that $\mathcal{A}$ possess a bounded approximate identity, $\mathcal{A}^{(2n-1)}, \mathcal{B}^{(2n-1)}$ and $\mathcal{M}^{(2n-1)}$ are non-degenerate. Then the $\mathcal{T}$-module map $D : \mathcal{T} \rightarrow \mathcal{T}^{(2n-1)}$ is module derivation if and only if there exist module derivations $D_A : \mathcal{A} \rightarrow \mathcal{A}^{(2n-1)}$, $D_B : \mathcal{B} \rightarrow \mathcal{B}^{(2n-1)}$ and $\gamma \in \mathcal{M}^{(2n-1)}$ such that

$$D\left(\begin{bmatrix} a & m \\ b \\ \end{bmatrix}\right) = \begin{bmatrix} D_A(a) - m\gamma & \gamma a - b\gamma \\ \gamma m & D_B(b) + \gamma m \end{bmatrix},$$

(4.4)

for every $\begin{bmatrix} a & m \\ b \end{bmatrix} \in \mathcal{T}$. 

Proof. Let $D : \mathcal{T} \rightarrow \mathcal{T}^{(2n-1)}$ be a continuous module derivation. Define $D_A : \mathcal{A} \rightarrow \mathcal{A}^{(2n-1)}$ by $D_A(a) = \pi_\mathcal{A}(D\left(\begin{bmatrix} a & 0 \\ 0 \\ \end{bmatrix}\right))$, and $D_B : \mathcal{B} \rightarrow \mathcal{B}^{(2n-1)}$ via $D_B(b) = \pi_\mathcal{B}(D\left(\begin{bmatrix} 0 & 0 \\ b \\ \end{bmatrix}\right))$. Obviously these maps are $\mathcal{A}$-module maps, and are module derivations by [10] Lemma 2.3. Let $(e_\alpha)_{\alpha \in \mathcal{A}}$ be a bounded approximate identity of $\mathcal{A}$, and let $D\left(\begin{bmatrix} a & 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} D_A(a) & \eta \\ \mu \end{bmatrix}$ for an arbitrary and fixed
Then
\[
D\left( \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) = D\left( \lim_\alpha \begin{bmatrix} e_\alpha a & 0 \\ 0 & 0 \end{bmatrix} \right) = D\left( \lim_\alpha \begin{bmatrix} e_\alpha & 0 \\ 0 & a \end{bmatrix} \right) \\
= \lim_\alpha \left( \begin{bmatrix} e_\alpha & 0 \\ 0 & a \end{bmatrix} \right) \cdot \begin{bmatrix} D_A(a) & \eta \\ \theta & \mu \end{bmatrix} \\
+ \begin{bmatrix} D_A(e_\alpha) & \gamma \\ \theta & a \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \\
= \lim_\alpha D_A(e_\alpha) \begin{bmatrix} \gamma a & 0 \\ 0 & 0 \end{bmatrix} = D_A(a) \begin{bmatrix} \gamma a & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.5}
\]

Similarly, consider \( b \in B \) such that
\[
D\left( \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \right) = \begin{bmatrix} \theta & \eta \\ D_B(b) & \gamma \end{bmatrix}. \tag{4.6}
\]

Also for every \( m \in M \) we have
\[
D\left( \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} -m \gamma & 0 \\ \gamma m & 0 \end{bmatrix}. \tag{4.7}
\]

Now, from (4.5), (4.6) and (4.7), we get (4.4), and this completes the proof. \( \square \)

Now we are ready to prove the main theorem of this section (see also the proof of [18, Theorem 2.1]).

**Theorem 4.3.** Let \( A \) and \( B \) both have bounded approximate identity, and let \( M \) be non-degenerate. Then for every \( n \geq 1 \) we have
\[
\mathcal{H}_1^1(T, T^{(2n-1)}) \simeq \mathcal{H}_1^1(A, A^{(2n-1)}) \oplus \mathcal{H}_1^1(B, B^{(2n-1)}). \tag{4.8}
\]

Furthermore, the corresponding triangular Banach algebra \( \mathcal{T} \left[ \begin{array}{c} A \\ M \\ B \end{array} \right] \) is \((2n-1)\)-weakly module amenable (as an \( \mathfrak{A}\)-bimodule) if and only if \( A \) and \( B \) are \((2n-1)\)-weakly module amenable (as Banach \( \mathfrak{A}\)-bimodules).

**Proof.** Suppose that \( D : \mathcal{T} \to \mathcal{T}^{(2n-1)} \) is a continuous module derivation. Proposition 4.2 shows that there are continuous module derivations \( D_A : A \to A^{(2n-1)} \), \( D_B : B \to B^{(2n-1)} \) and \( \gamma \in \mathcal{M}^{(2n-1)} \) such that
\[
D\left( \begin{bmatrix} a & m \\ b & 0 \end{bmatrix} \right) = \begin{bmatrix} D_A(a) - m \gamma & \gamma a - b \gamma \\ D_B(b) + \gamma m & 0 \end{bmatrix}, \tag{4.9}
\]

where \( a \in A \) and \( b \in B \). Then
\[
D\left( \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) = D\left( \lim_\alpha \begin{bmatrix} e_\alpha a & 0 \\ 0 & 0 \end{bmatrix} \right) = D\left( \lim_\alpha \begin{bmatrix} e_\alpha & 0 \\ 0 & a \end{bmatrix} \right) \\
= \lim_\alpha \left( \begin{bmatrix} e_\alpha & 0 \\ 0 & a \end{bmatrix} \right) \cdot \begin{bmatrix} D_A(a) & \eta \\ \theta & \mu \end{bmatrix} \\
+ \begin{bmatrix} D_A(e_\alpha) & \gamma \\ \theta & a \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \\
= \lim_\alpha D_A(e_\alpha) \begin{bmatrix} \gamma a & 0 \\ 0 & 0 \end{bmatrix} = D_A(a) \begin{bmatrix} \gamma a & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.5}
\]

Similarly, consider \( b \in B \) such that
\[
D\left( \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \right) = \begin{bmatrix} \theta & \eta \\ D_B(b) & \gamma \end{bmatrix}. \tag{4.6}
\]

Also for every \( m \in M \) we have
\[
D\left( \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} -m \gamma & 0 \\ \gamma m & 0 \end{bmatrix}. \tag{4.7}
\]

Now, from (4.5), (4.6) and (4.7), we get (4.4), and this completes the proof. \( \square \)
for every \( \begin{bmatrix} a & m \\ b & 1 \end{bmatrix} \in \mathcal{T} \). Define \( \Psi : \mathcal{Z}^1_{\mathcal{T}}(\mathcal{T}, \mathcal{T}^{(2n-1)}) \longrightarrow \mathcal{H}^1_{\mathcal{T}}(\mathcal{A}, \mathcal{A}^{(2n-1)}) \oplus \mathcal{H}^1_{\mathcal{T}}(\mathcal{B}, \mathcal{B}^{(2n-1)}) \) by

\[
\Psi(D) = (D_A + N^1_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}), D_B + N^1_{\mathcal{B}}(\mathcal{B}, \mathcal{B}^{(2n-1)})).
\]

(4.10)

Let’s show that \( \Psi \) is a linear map. If \( (e_\alpha) \) is a bounded approximate identity for \( \mathcal{A} \), then for given \( \lambda \in \mathbb{C} \) we have

\[
\langle \lambda a', D_A(a) \rangle = \lim_{\alpha} \lambda a'e_\alpha, D_A(a) = \lim_{\alpha} \langle a', \lambda e_\alpha D_A(a) \rangle = \langle \lim_{\alpha} a'e_\alpha, \lambda D_A(a) \rangle = \langle a', \lambda D_A(a) \rangle \quad (a, a' \in \mathcal{A})
\]

and similarly we have

\[
\langle \lambda b', D_B(b) \rangle = \langle b', \lambda D_B(b) \rangle \quad (b, b' \in \mathcal{B}).
\]

(4.11)

(4.12)

Now, by applying relations (4.11) and (4.12) for every \( T_1 = \begin{bmatrix} a_1 & m_1 \\ b_1 & 1 \end{bmatrix} \) and

\[
T_2 = \begin{bmatrix} a_2 & m_2 \\ b_2 & 1 \end{bmatrix}
\]

in \( \mathcal{T} \) we get

\[
\langle (\lambda T_2), D(T_1) \rangle = D \left( \begin{bmatrix} a_1 & m_1 \\ b_1 & 1 \end{bmatrix} \right) \left( \begin{bmatrix} \lambda a_2 & \lambda m_2 \\ \lambda b_2 & 1 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} D_A(a_1) - m_1 \gamma & \gamma a_1 - b_1 \gamma \\ D_B(b_1) + \gamma m_1 & 1 \end{bmatrix} \left( \begin{bmatrix} \lambda a_2 & \lambda m_2 \\ \lambda b_2 & 1 \end{bmatrix} \right)
\]

\[
= D_A(a_1)(\lambda a_2) - m_1 \gamma(\lambda a_2) + a_1 \gamma(\lambda b_2)
\]

\[
-\lambda b_1 \gamma(\lambda m_2) + D_B(b_1)(\lambda b_2) + m_1 \gamma(\lambda b_2)
\]

\[
= \lambda D_A(a_1)(a_2) - m_1 \gamma(a_2) + \lambda a_1 \gamma(b_2)
\]

\[
-\lambda b_1 \gamma(m_2) + \lambda D_B(b_1)(b_2) + \lambda m_1 \gamma(b_2)
\]

\[
= \begin{bmatrix} \lambda D_A(a_1) - m_1 \gamma & \lambda a_1 - b_1 \gamma \\ \lambda D_B(b_1) + \gamma m_1 & 1 \end{bmatrix} \left( \begin{bmatrix} a_2 & m_2 \\ b_2 & 1 \end{bmatrix} \right)
\]

\[
= \lambda D(T_1)(T_2).
\]

(4.13)

Thus \( \Psi(\lambda D) = \lambda \Psi(D) \). Obviously, \( \Psi(D_1 + D_2) = \Psi(D_1) + \Psi(D_2) \) for all \( D_1, D_2 \in \mathcal{Z}^1_{\mathcal{T}}(\mathcal{T}, \mathcal{T}^{(2n-1)}) \). Hence, \( \Psi \) is a linear map. Now, assume that \( D_A \in \mathcal{Z}^1_{\mathcal{T}}(\mathcal{A}, \mathcal{A}^{(2n-1)}) \) and \( D_B \in \mathcal{Z}^1_{\mathcal{T}}(\mathcal{B}, \mathcal{B}^{(2n-1)}) \). Then Lemma 1.8 implies that there exists a module derivation \( D_{AB} \) such that

\[
\Psi(D_{AB}) = (D_A + N^1_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}), D_B + N^1_{\mathcal{B}}(\mathcal{B}, \mathcal{B}^{(2n-1)})),
\]

and this indicates that \( \Psi \) is surjective. If \( D \in \ker \Psi \), then \( \Psi(D) = 0 \) and thus \( D_A \in N^1_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n-1)}) \) and \( D_B \in N^1_{\mathcal{B}}(\mathcal{B}, \mathcal{B}^{(2n-1)}) \). Since \( D_A \) and \( D_B \) are inner, \( D_{AB} \) is an inner module derivation by Lemma 1.8. Therefore we can write

\[
\begin{bmatrix} D_A(a) - m \gamma & \gamma a - b \gamma \\ D_B(b) + \gamma m & 1 \end{bmatrix} = \begin{bmatrix} D_A(a) \\ D_B(b) \end{bmatrix} + \begin{bmatrix} -m \gamma \\ \gamma m \end{bmatrix}.
\]
The above equality shows that $D$ is inner. Thus, $\ker \Psi \subseteq N^1_T(T, T^{(2n-1)})$. On the other hand, if $D \in N^1_T(T, T^{(2n-1)})$, then $D_A : A \rightarrow A^{(2n-1)}$ defined by $D_A(a) = \pi_A(D\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right))$ and $D_B : B \rightarrow B^{(2n-1)}$ defined by $D_B(b) = \pi_B(D\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right))$ are inner module derivations. Therefore $\Psi(D) = 0$, and so $N^1_T(T, T^{(2n-1)}) \subseteq \ker \Psi$. Finally, we have

$$H_1^1(T, T^{(2n-1)}) = N^1_T(T, T^{(2n-1)}) / N^1_T(T, T^{(2n-1)}) = N^1_T(T, T^{(2n-1)}) / \ker \Psi \cong \text{Im} \Psi = H_1^1(A, A^{(2n-1)}) \oplus H_1^1(B, B^{(2n-1)}).$$

□

5. (2N)-Weak Module Amenability

As it is seen in the previous section, $(2n-1)$-weak module amenability of a triangular Banach algebra $T$ depends on $(2n-1)$-weak module amenability of Banach algebras $A$ and $B$ while this fails to be true in the even case in general. We need the following lemma which is analogous to [13, Proposition 3.9] in the module case. We include the proof.

**Lemma 5.1.** Let $D : T \rightarrow T^{(2n)}$ be a continuous module derivation. Then there exist $\gamma \in M^{(2n)}$, continuous module derivations $D_A : A \rightarrow A^{(2n)}$, $D_B : B \rightarrow B^{(2n)}$ and a continuous $A$-module map $\rho : M \rightarrow M^{(2n)}$ such that

(i) $D\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = D_A(a) - \gamma \cdot b$,

(ii) $D\left(\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}\right) = 0 - \gamma \cdot b$,

(iii) $D\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = 0$,

(iv) $\rho(a \cdot m) = D_A(a) \cdot m + a \cdot \rho(m)$,

(v) $\rho(m \cdot b) = \rho(m) \cdot b + m \cdot D_B(b)$,

(vi) If $D_A : A \rightarrow A^{(2n)}$ and $D_B : B \rightarrow B^{(2n)}$ are continuous module derivations and $\rho_M : M \rightarrow M^{(2n)}$ is a continuous $A$-module map that satisfies (iv) and (v). Then $D : T \rightarrow T^{(2n)}$ defined by $\begin{bmatrix} a & m \\ b & 0 \end{bmatrix} \mapsto \begin{bmatrix} D_A(a) & \rho_M(m) \\ D_B(b) & 0 \end{bmatrix}$ is a continuous module derivation.
Proof. In the light of [13 Proposition 3.9], we just show that these maps are module maps. Let $D$ be a $\mathfrak{T}$-module map. Then

$$\langle a_2, \alpha \cdot D_A(a_1) \rangle = \langle a_2 \cdot \alpha, D_A(a_1) \rangle = \langle a_2 \cdot \alpha, D_A(a_1) \rangle + \langle 0, a_1 \cdot \gamma \rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \alpha \\ 0 \\ 0 \end{array} \right], \left[ \begin{array}{c} D_A(a_1) \\ a_1 \cdot \rho \end{array} \right] \right\rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \\ 0 \\ 0 \end{array} \right], \left[ \begin{array}{c} \alpha \\ 0 \\ \alpha \end{array} \right], D \left( \left[ \begin{array}{c} a_1 \\ 0 \\ 0 \end{array} \right] \right) \right\rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \\ 0 \\ 0 \end{array} \right], D \left( \left[ \begin{array}{c} \alpha \\ 0 \\ \alpha \end{array} \right], \left[ \begin{array}{c} a_1 \\ 0 \\ 0 \end{array} \right] \right) \right\rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \\ 0 \\ 0 \end{array} \right], D \left( \left[ \begin{array}{c} \alpha \\ 0 \\ \alpha \end{array} \right], \left[ \begin{array}{c} a_1 \\ 0 \\ 0 \end{array} \right] \right) \right\rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \\ 0 \\ 0 \end{array} \right], D \left( \left[ \begin{array}{c} \alpha \cdot a_1 \\ 0 \\ 0 \end{array} \right] \right) \right\rangle$$

$$= \left\langle \langle a_2, D_A(\alpha \cdot a_1) \rangle + \langle 0, \alpha \cdot a_1 \cdot \gamma \rangle = \langle a_2, D_A(\alpha \cdot a_1) \rangle, \right\rangle$$

for all $a_1, a_2 \in A$ and $\alpha \in \mathfrak{A}$. This means that $D_A(\alpha \cdot a) = \alpha \cdot D_A(a)$, for all $a \in A$ and $\alpha \in \mathfrak{A}$. Similarly we can show that $D_A(\alpha \cdot a) = D_A(a) \cdot \alpha$ and $D_B$ and $\rho$ are $\mathfrak{A}$-module maps. Therefore the assertions (i)-(v) hold. For (vi), suppose that $D_A, D_B$ and $\rho_M$ are $\mathfrak{A}$-module maps. We show that $D$ is a $\mathfrak{T}$-module map.

Given $T_1 = \left[ \begin{array}{c} a_1 \\ m_1 \\ b_1 \end{array} \right], T_2 = \left[ \begin{array}{c} a_2 \\ m_2 \\ b_2 \end{array} \right] \in T$ and $\Upsilon = \left[ \begin{array}{c} \alpha \\ 0 \\ \alpha \end{array} \right] \in \mathfrak{T}$, we have

$$\langle T_2, \Upsilon \cdot D(T_1) \rangle = \langle T_2 \cdot \Upsilon, D(T_1) \rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \cdot \alpha \\ m_2 \cdot \alpha \\ b_2 \cdot \alpha \end{array} \right], \left[ \begin{array}{c} D_A(a_1) \\ \rho_M(m_1) \\ D_B(b_1) \end{array} \right] \right\rangle$$

$$= \langle a_2 \cdot \alpha, D_A(a_1) \rangle + \langle m_2 \cdot \alpha, \rho_M(m_1) \rangle + \langle b_2 \cdot \alpha, D_B(b_1) \rangle$$

$$= \langle a_2, D_A(\alpha \cdot a_1) \rangle + \langle m_2, \rho_M(\alpha \cdot m_1) \rangle + \langle b_2, D_B(\alpha \cdot b_1) \rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \\ m_2 \\ b_2 \end{array} \right], \left[ \begin{array}{c} D_A(\alpha \cdot a_1) \\ \rho_M(\alpha \cdot m_1) \\ D_B(\alpha \cdot b_1) \end{array} \right] \right\rangle$$

$$= \left\langle \left[ \begin{array}{c} a_2 \\ m_2 \\ b_2 \end{array} \right], D \left( \left[ \begin{array}{c} \alpha \\ 0 \\ \alpha \end{array} \right], \left[ \begin{array}{c} a_1 \\ m_1 \\ b_1 \end{array} \right] \right) \right\rangle$$

$$= \langle T_2, D(\Upsilon \cdot T_1) \rangle.$$

Therefore $D(\Upsilon \cdot T_1) = \Upsilon \cdot D(T_1)$. Similarly we obtain $D(T_1 \cdot \Upsilon) = D(T_1) \cdot \Upsilon$. Thus, $D$ is a $\mathfrak{T}$-module map. □

The following sets which are used in this section are introduced in [13]. For each positive integer $n$, we define the centralizer of $A$ in $\mathcal{A}^{(2n)}$ by $Z_A(\mathcal{A}^{(2n)}) = \{x \in \mathcal{A}^{(2n)} : x \cdot a = a \cdot x$ for all $a \in A\}$ and similarly, $Z_B(\mathcal{B}^{(2n)}) = \{z \in \mathcal{B}^{(2n)} :$
\(z \cdot b = b \cdot z\) for all \(b \in \mathcal{B}\). The elements of

\[ZR_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n)) = \{ \rho_{x,z} : \mathcal{M} \rightarrow \mathcal{M}(2n) : x \in Z_{\mathcal{A}}(\mathcal{A}(2n)), z \in Z_{\mathcal{B}}(\mathcal{B}(2n)) \}\]

are called central Rosenblum operators on \(\mathcal{M}\) with coefficients in \(\mathcal{M}(2n)\) in which \(\rho_{x,z}(m) = x \cdot m - m \cdot z\) is an \(\mathcal{A}\)-module map. We also define the following set

\[\text{Hom}_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n)) = \{ \varphi : \mathcal{M} \rightarrow \mathcal{M}(2n) : \varphi(a \cdot m) = a \cdot \varphi(m), \varphi(m \cdot b) = \varphi(m) \cdot b, \varphi(\alpha \cdot m) = \alpha \cdot \varphi(m), \varphi(m \cdot \alpha) = \varphi(m) \cdot \alpha, \text{ for all } \alpha \in \mathcal{A}, a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \} \].

The following theorem is analogous to Lemma 3.11 and Theorem 3.12 from [13] in a more general setting (part (iii) is a module version of [16] Theorem 3.2).

**Theorem 5.1.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be unital Banach algebras and \(\mathcal{M}\) be a unital Banach \(\mathcal{A}, \mathcal{B}\)-module. Then we have

(i) \(ZR_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n)) \subseteq \text{Hom}_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n))\),

(ii) If \(\Omega \in \text{Hom}_{\mathcal{T}, \mathcal{A}, \mathcal{B}}\left(\begin{bmatrix} 0 & \mathcal{M} \\ \mathcal{M} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}(2n) \\ \mathcal{M}(2n) & 0 \end{bmatrix}\right)\), then the map

\[\Delta_{\Omega} \left(\begin{bmatrix} a & m \\ b & \end{bmatrix}\right) = \Omega\left(\begin{bmatrix} 0 & m \\ 0 & \end{bmatrix}\right) = \begin{bmatrix} 0 & \Omega_{\mathcal{M}}(m) \\ \Omega_{\mathcal{M}}(m) & 0 \end{bmatrix} \in Z_{\mathcal{T}}^1(\mathcal{T}, \mathcal{T}(2n)),\]

where \(\Omega_{\mathcal{M}} \in \text{Hom}_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n))\). Furthermore \(\Delta_{\Omega}\) is inner if and only if \(\Omega\) is a central Rosenblum operator on \(\mathcal{M}\) with coefficients in \(\mathcal{M}(2n)\).

(iii) If \(\mathcal{A}\) and \(\mathcal{B}\) are \(2n\)-weakly module amenable as \(\nabla\)-modules, then

\[H_{\mathcal{T}}^1(\mathcal{T}, \mathcal{T}(2n)) \simeq \text{Hom}_{\mathcal{T}, \mathcal{A}, \mathcal{B}}\left(\begin{bmatrix} 0 & \mathcal{M} \\ \mathcal{M} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}(2n) \\ \mathcal{M}(2n) & 0 \end{bmatrix}\right)/ZR_{\mathcal{A}, \mathcal{B}}\left(\begin{bmatrix} 0 & \mathcal{M} \\ \mathcal{M} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{M}(2n) \\ \mathcal{M}(2n) & 0 \end{bmatrix}\right)\).

In the other words,

\[H_{\mathcal{T}}^1(\mathcal{T}, \mathcal{T}(2n))(\mathcal{M}, \mathcal{M}(2n))/ZR_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n)).\]

**Proof.** For statements (i) and (ii), we only need to show that \(\Delta_{\alpha}\) is an \(\nabla\)-module map. The rest of the proof is the same as [13] Lemma 3.11. For given \(T = \begin{bmatrix} a & m \\ b & \end{bmatrix} \in \mathcal{T}\) and \(\Upsilon = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \in \nabla\), we have

\[\Delta_{\Omega}(\Upsilon \cdot T) = \Delta_{\Omega}\left(\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \cdot \begin{bmatrix} a & m \\ b & \end{bmatrix}\right) = \Delta_{\Omega}\left(\begin{bmatrix} \alpha \cdot a & \alpha \cdot m \\ \alpha \cdot a & \alpha \cdot m \end{bmatrix}\right)\]

\[= \begin{bmatrix} 0 & \alpha \cdot \Omega_{\mathcal{M}}(m) \\ \alpha \cdot \Omega_{\mathcal{M}}(m) & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \cdot \begin{bmatrix} 0 & \Omega_{\mathcal{M}}(m) \\ \Omega_{\mathcal{M}}(m) & 0 \end{bmatrix} = \Upsilon \cdot \Delta_{\Omega}(T),\]

and similarly \(\Delta_{\Omega}(T \cdot \Upsilon) = \Delta_{\Omega}(T) \cdot \Upsilon\).
For (iii), the proof is similar to [13, Theorem 3.12] but we give the proof for the sake of completeness. Consider

\[ \mathcal{F} : \text{Hom}_{\mathcal{T}, \mathcal{A}, \mathcal{B}} \left( \begin{array}{cc} 0 & \mathcal{M} \\ 0 & 0 \end{array} \right) \rightarrow H^1_{\mathcal{T}}(\mathcal{T}, \mathcal{T}(2n)) \]

defined by \( \Omega \mapsto \Delta \Omega \), where \( \Delta \Omega \) denotes the equivalence class of \( \Delta \Omega \) in \( H^1_{\mathcal{T}}(\mathcal{T}, \mathcal{T}(2n)) \).

It follows from (ii) that \( \mathcal{F} \) is an \( \mathcal{T} \)-module map. Let \( D : \mathcal{T} \rightarrow \mathcal{T}(2n) \) be a module derivation. By Lemma 5.1 there are module derivations \( D_A : \mathcal{A} \rightarrow \mathcal{A}(2n) \), \( D_B : \mathcal{B} \rightarrow \mathcal{B}(2n) \), and a \( \mathcal{A} \)-module map \( \rho : \mathcal{M} \rightarrow \mathcal{M}(2n) \) and an element \( \gamma \in \mathcal{M}(2n) \) such that

\[ D \left( \begin{array}{cc} a \\ m \\ b \end{array} \right) = \begin{array}{c} D_A(a) \\ a \cdot \gamma - \gamma \cdot b + \rho(m) \\ D_B(b) \end{array} \]  \hspace{1cm} (5.1)

Since \( \mathcal{A} \) and \( \mathcal{B} \) both are \( (2n) \)-weakly module amenable, there exist \( x \in \mathcal{A} \) and \( y \in \mathcal{B} \) such that \( D_A(a) = a \cdot x - x \cdot a = D_x(a) \) and \( D_B(b) = b \cdot y - y \cdot b = D_y(b) \).

Define \( D_0 : \mathcal{T} \rightarrow \mathcal{T}(2n) \) as follows:

\[ D_0 \left( \begin{array}{cc} a \\ m \\ b \end{array} \right) = \begin{array}{c} D_x(a) \\ a \cdot \gamma - \gamma \cdot b - \rho_{x,y}(m) \\ D_y(b) \end{array} \]  \hspace{1cm} (5.2)

It is easy to see that \( D_0 \) is an inner \( \mathcal{T} \)-module derivation induced by \( \begin{array}{cc} x \\ \gamma \\ y \end{array} \) (note that in the proof of [13, Theorem 3.12], there is a misprint). Set \( D_1 = D - D_0 \). Thus \( \overline{D}_1 = \overline{D} \). Then

\[ D_1 \left( \begin{array}{cc} a \\ m \\ b \end{array} \right) = \begin{array}{cc} 0 \\ \rho(m) + \rho_{x,y}(m) \\ 0 \end{array}, \]  \hspace{1cm} (5.3)

for all \( \left[ \begin{array}{cc} a \\ m \\ b \end{array} \right] \in \mathcal{T} \). Define \( \Omega = \begin{array}{cc} 0 \\ \rho(m) + \rho_{x,y}(m) \\ 0 \end{array} \). Clearly \( \Omega \) belongs to

\[ \text{Hom}_{\mathcal{T}, \mathcal{A}, \mathcal{B}} \left( \begin{array}{cc} 0 & \mathcal{M} \\ 0 & 0 \end{array} \right). \]

By (ii), we have \( \mathcal{F}(\Omega) = \Delta \Omega = D_1 = \overline{D} \). This means that \( \mathcal{F} \) is surjective. Finally, we must show that \( \ker \mathcal{F} = ZR_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n)) \). Let \( \Omega \in \ker \mathcal{F} \). Then

\[ \mathcal{F}(\Omega) \left( \begin{array}{cc} a \\ m \\ b \end{array} \right) = \overline{D} \left( \begin{array}{cc} a \\ m \\ b \end{array} \right) = 0 = \Delta \Omega \left( \begin{array}{cc} a \\ m \\ b \end{array} \right), \]

hence \( \Delta \Omega \) is inner. Again by (ii), we have \( \Omega \in ZR_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}(2n)) \), and (i) implies that
\[ \ker \tilde{\mathcal{F}} = ZR_{\mathcal{A}, \mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}). \]

Therefore

\[
H^1_{\mathcal{F}}(\mathcal{T}, \mathcal{T}^{(2n)}) \cong \frac{\text{Hom}_{\mathcal{F}}(\mathcal{T}, \mathcal{T}^{(2n)})}{\ker \tilde{\mathcal{F}}} = \frac{\text{Hom}_{\mathcal{F}}(\mathcal{T}, \mathcal{T}^{(2n)})}{ZR_{\mathcal{F}}(\mathcal{T}, \mathcal{T}^{(2n)})}.
\]

\[\square\]

The following result may be proved like Proposition 4.2 using a modification of Lemma 5.1.

**Proposition 5.2.** Let \( \mathcal{A} \) or \( \mathcal{B} \) has a bounded approximate identity, and let \( \mathcal{A}^{(2n)}, \mathcal{B}^{(2n)} \) and \( \mathcal{M}^{(2n)} \) be non-degenerate. Then the \( \mathcal{T} \)-module map \( D : \mathcal{T} \rightarrow \mathcal{T}^{(2n)} \) is a module derivation if and only if there exist module derivations \( D_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}^{(2n)} \) and \( D_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}^{(2n)} \), and \( \mathfrak{A} \)-module map \( \rho : \mathcal{M} \rightarrow \mathcal{M}^{(2n)} \) which satisfies conditions (iv) and (v) of Lemma 5.1 such that

\[
D\left( \begin{bmatrix} a & m \\ b & \end{bmatrix} \right) = \begin{bmatrix} D_{\mathcal{A}}(a) & a \cdot \gamma - \gamma \cdot b + \rho(m) \\ D_{\mathcal{B}}(b) & \end{bmatrix}, \quad (5.4)
\]

in which \( \gamma \in \mathcal{M}^{(2n)} \) and \( \begin{bmatrix} a & m \\ b & \end{bmatrix} \in \mathcal{T} \).

**Proof.** We argue similar to the proof of Proposition 4.2. Let \( D : \mathcal{T} \rightarrow \mathcal{T}^{(2n)} \) be a continuous module derivation. Define \( D_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}^{(2n)} \) by \( D_{\mathcal{A}}(a) = \pi_{\mathcal{A}}(D\left( \begin{bmatrix} a & 0 \\ 0 & \end{bmatrix} \right)) \), and \( D_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}^{(2n)} \) by \( D_{\mathcal{B}}(b) = \pi_{\mathcal{B}}(D\left( \begin{bmatrix} 0 & 0 \\ b & \end{bmatrix} \right)) \). Obviously these maps are \( \mathfrak{A} \)-module maps, and by [16, Lemma 2.3], they are module derivations. Let \( (e_{\alpha})_{\alpha \in \Lambda} \) be a bounded approximate identity of \( \mathcal{A} \), and let \( D\left( \begin{bmatrix} a & 0 \\ 0 & \end{bmatrix} \right) = \begin{bmatrix} D_{\mathcal{A}}(a) & \eta \\ \mu & \end{bmatrix} \) for all \( a \in \mathcal{A} \). Then

\[
D\left( \begin{bmatrix} a & 0 \\ 0 & \end{bmatrix} \right) = D\left( \begin{bmatrix} \lim_{\alpha} a e_{\alpha} & 0 \\ 0 & \end{bmatrix} \right) = D\left( \lim_{\alpha} \begin{bmatrix} a & 0 \\ 0 & \end{bmatrix} \begin{bmatrix} e_{\alpha} & 0 \\ 0 & \end{bmatrix} \right) \\
= \lim_{\alpha} \begin{bmatrix} a & 0 \\ 0 & \end{bmatrix} \cdot \begin{bmatrix} D_{\mathcal{A}}(e_{\alpha}) & \gamma \\ \theta & \end{bmatrix} \\
+ \begin{bmatrix} D_{\mathcal{A}}(a) & \eta \\ \mu & \end{bmatrix} \cdot \begin{bmatrix} e_{\alpha} & 0 \\ 0 & \end{bmatrix} \\
= \lim_{\alpha} D_{\mathcal{A}}(ae_{\alpha}) \begin{bmatrix} a \cdot \gamma \\ \end{bmatrix} = D_{\mathcal{A}}(a) \begin{bmatrix} a \cdot \gamma \\ \end{bmatrix}. \quad (5.5)
\]
Let \( b \in \mathcal{B} \) and \( D \left( \begin{bmatrix} 0 & 0 \\ b \end{bmatrix} \right) = \begin{bmatrix} \theta & \eta \\ D_B(b) \end{bmatrix} \). Since \( \mathcal{M}^{(2n)} \) and \( \mathcal{A}^{(2n)} \) are non-degenerate (see [16, Theorem 3.2]), we have
\[
D \left( \begin{bmatrix} 0 & 0 \\ b \end{bmatrix} \right) = \begin{bmatrix} 0 & -\gamma \cdot b \\ D_B(b) \end{bmatrix}.
\]
(5.6)

For \( m \in \mathcal{M} \), suppose that \( D \left( \begin{bmatrix} 0 & m \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \theta & \eta \\ \mu \end{bmatrix} \), then by [16, Theorem 3.2], \( \theta = \mu = 0 \). Now, we define \( \rho : \mathcal{M} \to \mathcal{M}^{(2n)} \) by
\[
\rho(m) = \pi_M(D \left( \begin{bmatrix} 0 & m \\ 0 \end{bmatrix} \right)).
\]
It is clear that \( \rho \) is an \( \mathfrak{A} \)-module map. Therefore it suffices to show that \( \rho \) satisfies in conditions (iv) and (v) of Lemma 5.1. We check these conditions as follows:
\[
\rho(a \cdot m) = \pi_M(D \left( \begin{bmatrix} a \cdot m & 0 \\ 0 & 0 \end{bmatrix} \right)) = \pi_M(D \left( \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix})),
\]
\[
= \pi_M \left( \begin{bmatrix} a \cdot \eta & D_A(a) \cdot a \cdot \gamma \\ 0 & 0 \end{bmatrix} \right) = a \cdot \eta + D_A(a) \cdot m = a \cdot \rho(m) + D_A(a) \cdot m,
\]
and similarly we have \( \rho(m \cdot b) = \rho(m) \cdot b + m \cdot D_B(b) \).

We can now rephrase part (iii) of Theorem 5.1 as follows.

**Theorem 5.3.** Let \( A \) or \( B \) has a bounded approximate identity, and let \( \mathcal{A}^{(2n)} \), \( \mathcal{B}^{(2n)} \) and \( \mathcal{M}^{(2n)} \) be non-degenerate. If \( A \) and \( B \) are \( (2n) \)-weakly module amenable as \( \mathfrak{A} \)-modules, then
\[
H_1^T(T, T^{(2n)}) \simeq \text{Hom}_{\mathfrak{A}, \mathfrak{A}, \mathfrak{A}}(\mathcal{M}, \mathcal{M}^{(2n)}) / ZR_{\mathfrak{A}, \mathfrak{A}, \mathfrak{A}}(\mathcal{M}, \mathcal{M}^{(2n)}).
\]
(5.7)

**Proof.** Applying Proposition 5.2, the argument of Theorem 5.1 can be repeated to obtain the result. \( \square \)

**Corollary 5.3.1.** Let \( A \) has a bounded approximate identity, and \( \mathcal{A}^{(2n)} \) be non-degenerate. If \( A \) is \( (2n) \)-weakly module amenable (as an \( \mathfrak{A} \)-module), then \( T = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ \mathcal{A} & \mathcal{A} \end{bmatrix} \) is \( (2n) \)-weakly module amenable (as an \( \mathfrak{A} \)-module).

**Proof.** Let \( (e_\alpha) \) be a bounded approximate identity of \( A \), and let \( \varphi \in \text{Hom}_{\mathfrak{A}, \mathcal{A}, \mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n)}) \). Then there exists \( E \in \mathcal{A}^{(2n)} \) and a subnet \( \{ \varphi(e_\beta) \} \) of \( \{ \varphi(e_\beta) \} \) such that \( \varphi(e_\beta) \to^w E \). We have
\[
\varphi(a) = w^* - \lim_\beta \varphi(e_\beta a) = w^* - \lim_\beta e_\beta \varphi(a) = E a.
\]
Similarly, \( \varphi(a) = a E \). This shows that \( \varphi \in ZR_{\mathfrak{A}, \mathcal{A}, \mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n)}) \) and \( H_1^T(T, T^{(2n)}) = \{ 0 \} \) by Theorem 5.3. \( \square \)
6. Examples

In this section, by using the results of the previous sections we show that under which conditions the Banach algebra $\mathcal{T}_0 = \left[ \begin{array}{cc} \ell^1(S) & \ell^1(S) \\ \ell^1(S) & \ell^1(S) \end{array} \right]$ is permanently weakly module amenable and module Arens regular where $S$ is an inverse semigroup.

Definition 6.1. A discrete semigroup $S$ is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of $S$ is denoted by $E$.

Let $S$ be an inverse semigroup with the set of idempotents $E$. By \cite{14} Theorem V.1.2 $E$ is a commutative subsemigroup of $S$ and a semilattice, $\ell^1(E)$ could be regarded as a commutative subalgebra of $\ell^1(S)$, and therefore $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions \cite{1}.

Let $k \in \mathbb{N}$. Recall that $E$ satisfies condition $D_k$ \cite{11} if given $f_1, f_2, ..., f_{k+1} \in E$ there exist $e \in E$ and $i, j$ such that

$$1 \leq i < j \leq k + 1, f_i e = f_i, f_j e = f_j.$$ 

Duncan and Namioka in \cite{11} Theorem 16 proved that for any inverse semigroup $S$, $\ell^1(S)$ has a bounded approximate identity if and only if $E$ satisfies condition $D_k$ for some $k$.

Let $S$ be a commutative inverse semigroup with the set of idempotents $E$. Consider $\ell^1(S)$ as an $\ell^1(E)$-module with the following action:

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_{s * e} = \delta_{se}, \quad (s \in S, e \in E). \quad (6.1)$$

Theorem 6.1. Let $n \in \mathbb{N}$ and let $S$ be a commutative inverse semigroup with the set of idempotents $E$. Then $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module with the actions (6.1).

Proof. For any semigroup $S$, group algebra $\ell^1(S)$ is commutative if and only if $S$ is commutative. Since $\ell^1(S)$ is a bi-commutative Banach $\ell^1(S) \ast \ell^1(S)$-module, so is $\ell^1(S)^{(n)}$. By \cite{4} Theorem 3.1, $\ell^1(S)$ is weakly module amenable as an $\ell^1(E)$-module. The semigroup algebra $\ell^1(S)$ is essential, in fact $\ell^1(S) = \ell^1(S) \ast \ell^1(S) \subseteq \ell^1(S) + \ell^1(S) \subseteq \ell^1(S)$ (see the proof of \cite{4} Theorem 3.15). Now, it follows from Lemma 4.1 that every module derivation from $\ell^1(S)$ into $\ell^1(S)^{(n)}$ is zero. This shows that $\ell^1(S)$ is $n$-weakly module amenable. \qed

In \cite{12}, the authors proved that the Banach algebra $\mathcal{T}_0 = \left[ \begin{array}{cc} \ell^1(S) & \ell^1(S) \\ \ell^1(S) & \ell^1(S) \end{array} \right]$ is weak $\mathcal{T}_0$-module amenable in which $\Sigma_0 := \left\{ \left[ \begin{array}{c} \alpha \\ \alpha \end{array} \right] \mid \alpha \in \ell^1(E) \right\}$, where $S$
is a unital commutative inverse semigroup. The condition of being unital for \( S \) is rather strong and it can be replaced by the weaker condition that \( E \) satisfies condition \( D_k \) for some \( k \).

**Example 1.** Let \( S \) be a commutative inverse semigroup such that \( E \) satisfies condition \( D_k \) for some \( k \). By Theorem 4.3 we have

\[
\mathcal{H}_{T_0}^1(\mathcal{H}_{T_0}^1(S), \ell^1(S)^{2n-1}) \cong \mathcal{H}_{\ell^1(E)}^1(\ell^1(S), \ell^1(S)^{2n-1}) \oplus \mathcal{H}_{\ell^1(E)}^1(\ell^1(S), \ell^1(S)^{2n-1}).
\]

where \( T_0 \) is a Banach \( E_0 \)-module. Since \( \ell^1(S) \) is \( n \)-weakly module amenable (Theorem 6.1), \( T_0 \) is \( (2n+1) \)-weakly module amenable (as \( E_0 \)-module) again by Theorem 4.3. On the other hand, \( \ell^1(S) \) possess a bounded approximate identity, and thus \( T_0 \) is \( (2n) \)-weakly module amenable by Corollary 5.3. Therefore \( T_0 \) is permanently weakly module amenable.

Here, for technical reasons, we let \( \ell^1(E) \) act on \( \ell^1(S) \) by multiplication from right and trivially from left, that is

\[
\delta_e \cdot \delta_s = \delta_s, \delta_s \cdot \delta_e = \delta_se = \delta_{se} \quad (s \in S, e \in E).
\]

In this case, the ideal \( J \) (see section 2) is the closed linear span of \( \{\delta_{st} - \delta_t \mid s, t \in S, e \in E\} \). We consider an equivalence relation on \( S \) as follows:

\[
s \approx t \iff \delta_s - \delta_t \in J \quad (s, t \in S).
\]

For an inverse semigroup \( S \), the quotient \( S/\approx \) is a discrete group (see [2] and [19]). Indeed, \( S/\approx \) is homomorphic to the maximal group homomorphic image \( G_S \) of \( S \) [17]. In particular, \( S \) is amenable if and only if \( G_S \) is amenable [11, 17].

**Example 2.** The Banach algebra \( T_0 \) is module Arens regular (as an Banach \( E_0 \)-module) if and only if \( \ell^1(S) \) is module Arens regular as an \( \ell^1(E) \)-module with trivial left action and canonical right action by Theorem 3.1. Now it follows from [22, Theorem 3.3] that \( T_0 \) is module Arens regular if and only if is \( G_S \) is finite.

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