Multiplier Approximation of Functions by q-Bernstein-Kantorovich Operator

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Abstract. The purpose of this paper is to find best multiplier approximation of unbounded functions in \(L_{\mu,\phi_m}\) -space by q-Bernstein-Kantorovich Aperator, in terms of the modulus of smoothness of one order using Korevkin Theorem.

1. Introduction

G. Hardy in 1949, \([1]\) defined the multiplier sequence for a converge of the series as.

A series \(\sum_{n=0}^{\infty} b_n\) is called a multiplier convergent if there is convergent sequence of real numbers \(\{\phi_n\}_{n=0}^{\infty}\), such that \(\sum_{n=0}^{\infty} b_n \phi_n < \infty\) where, \(\{\phi_n\}_{n=0}^{\infty}\) is called a multiplier for the convergence, for example.

The series \(\sum_{n=1}^{\infty} \frac{1}{n}\) is a divergent series and the sequence \(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\) convergent sequence. Since \(\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2}\) which is convergent series then the series \(\sum_{n=1}^{\infty} \frac{1}{n}\) is a multiplier convergent.

And from above we have

If \(\sum_{n=1}^{\infty} b_n\) is convergent series then it is multiplier convergent, since the sequence \(\{\phi_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}\) may be taken. But the convers is not true in general.

Recently, presented the prof. Dr. Saheb AL-Saidy \([2]\) a study on this topic through this he was able to study approximation of unbounded functions.

Similar to the above we provide the following definition

For any real valued function \(f\) defined on \(B = [a, b]\), \(f\) is called multiplier integral if there is a sequence \(\{\phi_m\}_{m=0}^{\infty}\) of real numbers such that \(\int_B f \phi_m(x) < \infty\), as \(m \to \infty\) where \(\{\phi_m\}_{m=0}^{\infty}\) is called a multiplier for the integral.

Let \(L_p(X)\) be the space of all bounded measurable functions defined on \(X = [a, b]\) with the norm

\[ \|f(\cdot)\|_{L_p} = \|f(\cdot)\|_p = (\int_X |f(x)|^p)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty. \]
Now for any real valued function $f$ the multiplier integral norm can be defined as follows,

$$\|f\|_{L,p,\phi_m} = \left\{ \left( \int_X |f\phi_m(x)|^p \, dx \right)^{1/p} : x \in X \right\},$$

where $\phi_m$ be the multiplier for the integral.

Let us define the norm $\|f\|_{L,p,\phi}$ by $\|f\|_{L,p,\phi}$

Let $L_{p,\phi_m}(X)$ be the space of all real valued unbounded functions $f$ such that $\int_X |f\phi_m(x)|^p \, dx < \infty$ with the norm

$$\|f\|_{L,p,\phi_m} = \left\{ \left( \int_X |f\phi_m(x)|^p \, dx \right)^{1/p} : x \in X \right\},$$

where $\phi_m$ is the multiplier for the integral, and $\|f\|_{p} = \|f\|_{L,p,\phi_m}.$

Now, before we give some examples for the define of $L_{p,\phi_m}(X)$ space, we present the following theorem, (Lebesgue Dominated Convergence Theorem).

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set $M$ such that $\{f_n(x)\}_{n=1}^{\infty}$ Converges pointwise almost everywhere to $f(x)$, then

$$\lim_{n \to \infty} \int_M f_n(x) \, dx = \int_M \lim_{n \to \infty} f_n(x) \, dx = \int_M f(x) \, dx$$

**Example 1.1:**

Let $f(x) = \csc x$ with $x \in B = (0, \pi)$ which is unbounded function, $\phi_m = \left\{ \frac{1}{m^2} \right\}_{m=1}^{\infty}$ be a sequence. Then we have

$f\phi_m(x) = f_m(x) = \frac{\csc x}{m^2}$ is a sequence of Lebesgue measurable functions defined on

Lebesgue measurable set $= (0, \pi)$, since

$f\phi_m(x) = f_m(x) = \frac{\csc x}{m^2}$ converges pointwise almost everywhere to $f(x) = 0$, then by using the above theorem we get the following

$$\int_B f\phi_m(x) \, dx = \int_B f_m(x) \, dx = \int_B f(x) \, dx < \infty,$$

as $m \to \infty$, which means that

$\phi_m = \left\{ \frac{1}{m^2} \right\}_{m=1}^{\infty}$ is a multiplier for the integral.

**Example 1.2:**

Let $f : B \to \mathbb{R}$ be a function defined as follows
\[ f(x) = \frac{n^2-x^2}{x} \quad \text{for} \quad x \in B = [-\pi, 0) \cup (0, \pi]. \]

And let \( \phi_m = \left( \frac{1}{m} \right)^n m=1 \) be the multiplier for integral.

Now, suppose that \( x = \frac{1}{n} \) where \( n \) be a positive real numbers.

Thus \( x = \frac{1}{n} \rightarrow 0 \) as \( n \rightarrow \infty \) and for \( m \geq n \) we get the following inequality

\[ f \phi_m(x) = \left( \frac{\pi^2-x^2}{x} \right)^\frac{1}{m} = \left( \frac{\pi^2-x^2}{x} \right)^\frac{n}{m} \leq f^*(x) = \pi^2 - x^2 \]

Thus \( f \phi_m(x) \leq f^*(x) \quad \forall \ m \geq n \) that \( f \phi_m(x) \leq f^*(x) \quad \forall \ \frac{1}{m} \leq \frac{1}{n} = x \)

Then, \( f \phi_m(x) = \left( \frac{\pi^2-x^2}{x} \right)^\frac{1}{m} \leq f^*(x) = \pi^2 - x^2 \quad \forall \ x \geq \frac{1}{m} \)

Therefor if we take \( m \rightarrow \infty \) then \( \frac{1}{m} \rightarrow 0 \) and we get that

\[ f \phi_m(x) = \left( \frac{\pi^2-x^2}{x} \right)^\frac{1}{m} \leq f^*(x) = \pi^2 - x^2 \quad \forall \ x \in \left( \frac{1}{m}, \pi] \right) \]

This means that

\[ \int_B f \phi_m(x) \, dx \leq \int_B f^*(x) \, dx \quad \forall \ x \in \left( \frac{1}{m}, \pi \right]. \]

From all above we have

\[ L_p(B) \subseteq L_p, \phi_m(B). \]

2. Definitions

Now in order to get further in the research, we will present some concepts and definitions

**Definition 2.1:** [3]

For \( q>0 \) and \( n \in \mathbb{N} \) let \( [n] = [n]_q = q^0 + q^1 + q^2 + \cdots + q^{n-1} \) with \( [0] = 0 \) be the q-integer \([n]\).

And the q-factorial \([n]!\), is defined by \([n]! = [n]_q! = [1]_q \cdot [2]_q \cdots [n]_q\) with \([0]! = 1\) and for integers \( 0 \leq k < n \) then

\[ \frac{[n]!}{[k]![n-k]!} \], be the q-binomial coefficient.

**Definition 2.2:** [4]

For \( f \in C[0,a] \) let \( \int_0^a f(t) \, dt = a(1-q) \sum_{n=0}^{\infty} f(aq^n) \cdot q^n \quad (0<q<1) \) be the q-analogue of integration in the interval \([0, a]\) and

\[ B_n^*(f, x) = \sum_{k=0}^{n} P_n(k, x) \int_0^1 f \left( \frac{[k]_q + qk}{[n+1]_q} \right) \, dt \], be the Kantrovich type q-Bernistein polynimail,

where \( P_n(k, x) = \frac{[n]!}{[k]![n-k]!} \cdot x^k(1-x)^{n-k} \), \( (1-x)^{n} = \prod_{x=0}^{n}(1-q^x) \), \( 0 \leq x \leq 1, n \in \mathbb{N}. \)
Definition 2.3:
For \( f \in L_{p,\phi_m}(X), X = [0, 1], \) \( 0 < q < 1 \) let
\[
\int_{0}^{1} (f \phi_m)_{(t)} \, dq \, t = (1 - q) \sum_{m=0}^{\infty} (f \phi_m)_{(n)} \cdot q^{n}
\]
be the q-analogue of integration in the interval [0,1] and
\[
B_{n,q}^* (f, x) = \sum_{k=0}^{n} P_{n,k} (q, x) \int_{0}^{1} (f \phi_m)_{(t)} \left( \frac{k+q}{n+1} \right) \, dq \, t
\]
be the modified Kantorovich type q-Bernstein polynomial, where
\[
P_{n,k} (q, x) = \binom{n}{k} x^k (1 - x)^{n-k},
\]
\[
(1 - x)^{n-k} = \prod_{s=0}^{n-k-1} (1 - q^x) x, \ n \in N
\]
Remark 2.4:
For \( q = 1 \) we have
\[
B_{n,q}^* (f, x) = B_{n}^* (f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \int_{0}^{1} (f \phi_m)_{(t)} \left( \frac{k+1}{n+1} \right) \, dt,
\]
which is say the classical Kantorovich-Bernstein operator, \( n \in N \).
Proof:
For \( q = 1 \) we have
\[
[n]_q = [n] = q^0 + q^1 + \cdots + q^{n-1} = 1 + 1_{n-time} + \cdots 1 = n, \text{thus}
\]
\[
[n]_q = \text{and we get}
\]
\[
\binom{n}{k} [n]_{[n-k]} = \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} \text{ and}
\]
\[
P_{n,k} (q, x) = \binom{n}{k} x^k (1 - x)^{n-k} = \binom{n}{k} x^k (1 - x)^{n-k}
\]
\[
B_{n,k}^* (f, x) = \sum_{k=0}^{n} P_{n,k} (q, x) \int_{0}^{1} (f \phi_m)_{(t)} \left( \frac{k+1}{n+1} \right) \, dt
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \int_{0}^{1} (f \phi_m)_{(t)} \left( \frac{k+t}{n+1} \right) \, dt
\]
\[
B_{n,q}^* (f, x) = B_{n}^* (f, x)
\]
Definition 2.5:
For \( f \in L_{p,\phi_m}(X), X = [0, 1] \), let
\[
\Delta_{h} (f,x) = \Delta_{h}^1 (f,x) = f(x+h) - f(x), x, x+h \in X \text{ be the difference of one order of } f
\]
and
\[
\bullet \quad \omega (f, \delta)_{p,\phi_m} = \omega_1 (f, \delta)_{p,\phi_m} = \text{Sup}_{0 < |h| \leq \delta} \| \Delta_{h}^1 (f, x) \|_{p,\phi_m}, \delta \geq 0 \text{ is called the usual modulus of smoothness of } f
\]
and
\[
\omega^\theta (f, \delta)_{p,\phi_m} = \omega_1^\theta (f, \delta)_{p,\phi_m} = \text{Sup}_{0 < |h| \leq \delta} \| \Delta_{h}^\theta (f, x) \|_{p,\phi_m} \text{ be the Ditzian-Totik modulus of smoothness of one order of } f, \text{ where } \theta (x) = (x(1-x))^{1/2}
\]
\[
\bullet \quad \omega^\theta (f, \delta)_{p,\phi_m} = \omega_1^\theta (f, \delta)_{p,\phi_m} = \text{Sup}_{0 < |h| \leq \delta} \| \theta^r \Delta_{h}^\theta (f, x) \|_{p,\phi_m} \text{ be the } r\text{-th-weighted Ditzian-Totik modulus of one order of smoothness of } f, \text{ where } r \text{ is a non negative integer.}
\]

3. The Main Results
Before we state our main results, we need the following lemma and notes.

Lemma 3.1: [4]
For all \( n, m \in N, x \in [0, 1], 0 < q \leq 1 \) then
\[B_{n,q}(t^m, x) = \sum_{j=0}^{m} \left( \frac{m}{j} \right) \left( \frac{[n]^j}{[n+1]^{m-j+1}} \right) \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i B_{n,q}(t^{i+j}, x)\]

where \(B_{n,q}(t^{i+j}, x) = \sum_{k=0}^{[k]^{i+j}} P_{n,k}(q,x)\)

**Lemma 3.2:**
Let \(e_i(t) = t^i, i = 0, 1, 2\), for every \(t \in [0,1], n \in N, 0 < q < 1\) we have:

a. \(B_{n,q}(e_0, x) = 1\)

b. \(B_{n,q}(e_t, x) = \left( \frac{2q[n]}{[2][n+1]} \right) x + \frac{1}{[2][n+1]}\)

c. \(B_{n,q}(e_2, x) = \left( \frac{q(q+2)}{3} \cdot \frac{q[n][n+1]}{[n+1]^2} \right) x^2 + \left( \frac{4q+7q^2+q^3}{[2][3]} \cdot \frac{[n]}{[n+1]^2} \right) x + \frac{1}{[3][n+1]^2}\)

**Proof:**

a. By using lemma 3.1 we have

\[B_{n,q}^*(t^0, x) = B_{n,q}^*(e_0, x) = \sum_{j=0}^{0} \left( \frac{0}{j} \right) \left( \frac{[n]^j}{[n+1]^{0-j+1}} \right) \sum_{i=0}^{0-j} \binom{0-j}{i} (q^n - 1)^i B_{n,q}(t^{i+j}, x)\]

then for \(m=0\) we get

\[B_{n,q}^*(t^0, x) = B_{n,q}^*(t^0, x) = \sum_{k=0}^{0} \left[ \frac{0}{[n]} \right] \sum_{i=0}^{0} \binom{0-j}{i} (q^n - 1)^i B_{n,q}(t^{i+j}, x)\]

Also by using lemma (3.1) for \(m=2\) we get

\[B_{n,q}^*(e_2, x) = \sum_{k=0}^{[k]^{2}} P_{n,k}(q,x)\]

b. By using lemma (3.1) for \(m=1\) we have

\[B_{n,q}^*(t^1, x) = B_{n,q}^*(e_1, x) = \sum_{j=0}^{1} \left( \frac{1}{j} \right) \left( \frac{[n]^j}{[n+1]^{1-j+1}} \right) \sum_{i=0}^{1-j} \binom{1-j}{i} (q^n - 1)^i B_{n,q}(t^{i+j}, x)\]

\[\frac{n}{[n+1]^{2}} \left[ \frac{1}{0} \right] (q^n - 1)^0 B_{n,q}(t^{0+0}, x) + \frac{1}{1} (q^n - 1) B_{n,q}(t^{0+1}, x)\]

\[= \frac{1}{[n+1]^{2}} \left[ \frac{1}{0} \right] (q^n - 1)^0 B_{n,q}(t^{0+0}, x) + \frac{1}{[n+1]} (q^n - 1) B_{n,q}(t^{1+1}, x)\]

\[= \frac{1}{[n+1]^{2}} \left[ \frac{1}{0} \right] (q^n - 1)^0 B_{n,q}(t^{0+0}, x) + \frac{1}{[n+1]} (q^n - 1) B_{n,q}(t^{1+1}, x)\]

\[= \frac{1}{[n+1]} (q^n - 1) B_{n,q}(t^{1+1}, x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + (q^n - 1) B_{n,q}(t^{1+1}, x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]

\[= \frac{1}{[n+1]} \left[ \frac{n}{[n+1]} \right] \sum_{k=0}^{[k]} P_{n,k}(q,x) + \frac{1}{[n+1]} \sum_{k=0}^{[k]} P_{n,k}(q,x)\]
\[
\left(\frac{2}{2}\right) \left(\frac{n^2}{[n+1]^2[2-2+1]} \sum_{i=0}^{2-2} \binom{2}{i} (2 - 2)^i (q^n - 1)^i B_{n,q}(t^{2+i}, x)
\right) \\
= \frac{1}{[n+1]^2} \left[\left(\frac{2}{0}\right) (q^n - 1)^0 B_{n,q}(t^{0+0}, x) + \left(\frac{2}{1}\right) (q^n - 1)^1 B_{n,q}(t^{0+1}, x) + \left(\frac{2}{2}\right) (q^n - 1)^2 B_{n,q}(t^{0+2}, x)\right] \\
+ \frac{2[n]}{[n+1]^2} \left[\left(\frac{2}{0}\right) (q^n - 1)^0 B_{n,q}(t^{2+0}, x)\right]
\]

\[
\left[\left(\frac{1}{0}\right) (q^n - 1)^0 B_{n,q}(t^{1+0}, x) + \left(\frac{1}{1}\right) (q^n - 1)^1 B_{n,q}(t^{1+1}, x)\right] \\
+ \left[\left(\frac{1}{0}\right) (q^n - 1)^0 B_{n,q}(t^{1+0}, x) + \left(\frac{1}{1}\right) (q^n - 1)^1 B_{n,q}(t^{1+1}, x)\right] \\
= \frac{1}{[n+1]^2} \left[\left(\frac{n^2}{[n+1]^2} + \frac{2[n](q^n-1)}{[2][n+1]^2} + \frac{(q^n-1)^2}{[3][n+1]^2}\right) \left(1 - \frac{1}{n}\right) x^2\right] \\
+ \left[\left(\frac{n^2}{[n+1]^2} + \frac{2[n](q^n-1)}{[2][n+1]^2} + \frac{(q^n-1)^2}{[3][n+1]^2}\right) \left(1 - \frac{1}{n}\right) x^2\right]
\]

Then

\[
B_{n,q}(e^2, x) = \frac{q^2+q+q^3}{3} + \frac{q[n+1]}{[n+1]^2} x^2 + \frac{q[n+1]^2}{[3][n+1]^2} x^2 + \frac{1}{[3][n+1]^2} \]  

**Lemma 3.3:**

For \( f \in L_{P,\phi_m}(X) \cdot X = [0,1] \), \( 0 < q < 1 \) we have

\[
\|B_{n,q}^*(f)\|_{P,\phi_m} \leq C \|f\|_{P,\phi_m}, \text{where } C \text{ is a constant.}
\]

**Proof:**

\[
\|B_{n,q}^*(f)\|_{P,\phi_m} = \left\{\int_X \|B_{n,q}^*(f, \phi_m, x)\|^p dx\right\}^{1/p}
\]

\[
= \left\{\int_X \sum_{k=0}^n P_{n,k}(q, x) \int_0^1 (f \phi_m) \left[\frac{[k+q^k]}{[n+1]}\right] d_q t dx\right\}^{1/p}
\]

And by using Jenson inequality we have

\[
\|B_{n,q}(f)\|_{P,\phi_m} \leq \left\{\int_X (f \phi_m) \left[\frac{[k+q^k]}{[n+1]}\right] d_q t\right\}^{1/p} \cdot \left\{\sum_{k=0}^n P_{n,k}(q, x) dx\right\} \\
\leq \|f\|_{P,\phi_m} \cdot \sum_{k=0}^n \int_X \left|P_{n,k}(q, x)\right| dx \leq C \|f\|_{P,\phi_m}
\]

Then

\[
\|B_{n,q}(f)\|_{P,\phi_m} \leq C \|f\|_{P,\phi_m} \]  

**Lemma 3.4:**

For \( f, g \in L_{P,\phi_m}(X), X = [0,1] \), \( 0 < q \leq 1 \) we have

\[
a. \ B_{n,q}(f + g, x) = B_{n,q}(f, x) + B_{n,q}(g, x)
\]

\[
b. \ B_{n,q}(\alpha f, x) = \alpha B_{n,q}(f, x) \text{ where } \alpha \text{ is a constant}
\]

**Proof:**

\[
a. \text{ Since } B_{n,q}^*(f, x) = \sum_{k=0}^n P_{n,k}(q, x) \int_0^1 (f \phi_m) \left[\frac{[k+q^k]}{[n+1]}\right] d_q t
\]

\[
b. \text{ Since } B_{n,q}^*(f, x) = \sum_{k=0}^n P_{n,k}(q, x) \int_0^1 (f \phi_m) \left[\frac{[k+q^k]}{[n+1]}\right] d_q t
\]
\[ B_{n,q}(f + g, x) = \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} ((f + g)\phi_m) \left( \frac{[k+q]_t}{[n+1]} \right) dt \]
\[ = \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} \left( f\phi_m \right) \left( \frac{[k+q]_t}{[n+1]} \right) dt + \left( g\phi_m \right) \left( \frac{[k+q]_t}{[n+1]} \right) dt \]
\[ = \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} \left( f\phi_m \right) \left( \frac{[k+q]_t}{[n+1]} \right) dt + \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} \left( g\phi_m \right) \left( \frac{[k+q]_t}{[n+1]} \right) dt \]
\[ = B_{n,q}^* (f, x) + B_{n,q}^* (g, x) \]

b. \[ B_{n,q}^*(\infty f, x) = \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} \left( \infty f \right) \phi_m \left( \frac{[k+q]_t}{[n+1]} \right) dt \]
\[ = \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} \left( \infty f \right) \phi_m \left( \frac{[k+q]_t}{[n+1]} \right) dt \]
\[ = \infty \sum_{k=0}^{n} P_{n,k}(q, x) \int_{0}^{1} \left( f\phi_m \right) \left( \frac{[k+q]_t}{[n+1]} \right) dt = \infty B_{n,q}^*(f, x) \]
Then \[ B_{n,q}^*(\infty f, x) = \infty B_{n,q}^*(f, x) \]
Thus \[ B_{n,q}^* \] be linear operator \[ \blacksquare \]

**Remark 3.5:**
For every \( n \in \mathbb{N}, 0 < q < 1 \) then
\[ [n]_q = \lfloor n \rfloor \leq n \]

**Proof:** by induction for \( n \) we have
\[ [n] = q^0 + q^1 + \cdots + q^{n-1}, \text{ then} \]
For \( n = 1, [1] = q^{1-1} = q^0 = 1 \) thus \( [1] = 1 \)
For \( n = 2, [2] = q^0 + q^{2-1} = q^0 + q^1 = 1 + q^1 < 2, (0 < q < 1) \) then \( [2] < 2 \)
Suppose it is true for \( n \) thus
\[ [n] = q^0 + q^1 + \cdots + q^{n-1} < n, \text{ then} \]
\[ [n + 1] = q^0 + q^1 + \cdots + q^{(n+1)-2} + q^{(n+1)-1} = q^0 + q^1 + \cdots + q^{n-1} + q^n \]
Then \( [n + 1] = q^0 + q^1 + \cdots + q^{n-1} + q^n < n + q^n < n + 1 \)
Thus \( [n + 1] < n + 1 \)
Then \( [n] \leq n \) for every \( n \in \mathbb{N} \) \[ \blacksquare \]

**Lemma 3.6:** [5]
For \( f \in L_P(X), X = [a, b] \) there is a polynomial \( g_n \in \mathbb{P}_n \cap L_P(X) \) where \( \mathbb{P}_n \) set of all algebraic polynomials such that
\[ \|f - g_n \|_P \leq C \omega^\varphi_m \left( f, \frac{1}{n} \right)_P \]

**Lemma 3.7:**
Let \( f \in L_{P,\phi_m}(X) \) and \( g_n \) be a polynomial such that \( g_n \in \mathbb{P}_n \cap L_{P,\phi_m}(X) \) where \( \mathbb{P}_n \) set of all algebraic polynomials, then
\[ \|f - g_n \|_{P,\phi_m} \leq C \omega^\varphi \left( f, \frac{1}{n} \right)_{P,\phi_m} \]

**Proof:** For \( f \in L_P(X) \) and since
\[ \int_X |f\phi_m(x)|^p \ dx < \infty, \text{ then by Lemma (3.6) we have} \]
\[ \|f - g_n\|_{P,\phi_m} = \int_X |(f - g_n)\phi_m(x)|^p \ dx = \|f - g_n\|_{P,\phi_m} \leq C \omega^\varphi \left( \phi_m, \frac{1}{n} \right)_P \]
\[ C \omega^\phi_r \left( f, \frac{1}{n} \right)_{P, \phi_m}, \text{ thus } \| f - g_n \|_{P, \phi_m} \leq C C \omega^\phi_r \left( f, \frac{1}{n} \right)_{P, \phi_m} \]

**Lemma 3.8:**

Let \( f \in L_{P, \phi_m}(X) \) and \( g_n \) be a polynomial such that \( g_n \in \mathbb{P} \cap L_{P, \phi_m}(X) \) where \( \mathbb{P} \) set of all algebraic polynomials, then

\[ \| f - g_n \|_{P, \phi_m} \leq C C \omega^\phi_r \left( f, \frac{1}{n} \right)_{P, \phi_m} \]

**Proof:** by using Remark (3.5), for \( n \in N, [n] \leq n \) then \( \frac{1}{n} \leq \frac{1}{[n]} \) we get

\[ \omega^\phi_r \left( f, \frac{1}{n} \right)_{P, \phi_m} \leq \omega^\phi_r \left( f, \frac{1}{[n]} \right)_{P, \phi_m}, \]

and by Lemma (3.7) we have

\[ \| f - g_n \|_{P, \phi_m} \leq C C \omega^\phi_r \left( f, \frac{1}{[n]} \right)_{P, \phi_m}, \]

thin \( \| f - g_n \|_{P, \phi_m} \leq C C \omega^\phi_r \left( f, \frac{1}{[n]} \right)_{P, \phi_m} \]

**Lemma 3.9:**

For \( f, g \in L_{P, \phi_m}(X), X = [0, 1] \cdot n \in N, 0 < q < 1 \) we have

a. \( f \geq 0 \) then \( B^*_n,q(f, x) \geq 0 \)

b. \( f \leq g \) then \( B^*_n,q(f, x) \leq B^*_n,q(g, x) \)

**Proof:**

a. \( f \geq 0 \) then \( f \left( \frac{|k|+q^k t}{[n+1]} \right) \geq 0 \) for \( k \in N \) we have

\[ \int_0^1 (f \phi_m) \left( \frac{|k|+q^k t}{[n+1]} \right) d_q t \geq 0 \text{ thus} \]

\[ \sum_{k=0}^n P_n,k(q, x) \int_0^1 (f \phi_m) \left( \frac{|k|+q^k t}{[n+1]} \right) d_q t \geq 0 \text{ we get} \]

\( B^*_n,q(f, x) \geq 0 \)

b. Since \( f \leq g \) thus \( f \left( \frac{|k|+q^k t}{[n+1]} \right) \leq g \left( \frac{|k|+q^k t}{[n+1]} \right) \) then

\[ \int_0^1 (f \phi_m) \left( \frac{|k|+q^k t}{[n+1]} \right) d_q t \leq \int_0^1 (g \phi_m) \left( \frac{|k|+q^k t}{[n+1]} \right) d_q t \text{ we get} \]

\[ \sum_{k=0}^n P_n,k(q, x) \int_0^1 (f \phi_m) \left( \frac{|k|+q^k t}{[n+1]} \right) d_q t \leq \sum_{k=0}^n P_n,k(q, x) \int_0^1 (g \phi_m) \left( \frac{|k|+q^k t}{[n+1]} \right) d_q t \]

We have \( B^*_n,q(f, x) \leq B^*_n,q(g, x) \)

In order to state the theorems we need the following theorem:

**Theorem 3.10: [Korevkin Theorem] [6]**

Let \( L_n \) be a linear positive monotone operator such that

a. \( L_n(1, x) = 1 \)

b. \( L_n(t, x) = x + \alpha (x) \)

c. \( L_n(t^2, x) = x^2 + B(x) \),

for any \( f \in C[a, b] \) then

\[ \| L_n(f, \cdot) - f(\cdot) \|_p \leq 3 \omega \left( f, \sqrt{B(x) - 2x \cdot \alpha (x)} \right)_p \]
Lemma 3.11:
Let $L_n$ be a linear positive monotone operator, which satisfies the above conditions then for any $f \in L_{p,\phi_m}(X), X = [0, 1]$ we have
\[
\|L_n(f, \cdot) - f(\cdot)\|_{p,\phi_m} \leq 3\omega(f, \sqrt{B(x)} - 2x \propto (x))_{p,\phi_m}
\]

Proof: For $f \in C[0, 1]$, from theorem (3.10) we have
\[
\|L_n(f, \cdot) - f(\cdot)\|_p \leq 3\omega(f, \sqrt{B(x)} - 2x \propto (x))_p
\]
For $f \in L_{p,\phi_m}(X)$ and since
\[
\|f\|_{p,\phi_m} = \left\{ \left( \int (f \phi_m(x))^p \right)^{\frac{1}{p}} \right\} < \infty,
\]
we get
\[
\|L_n(f \phi_m, \cdot) - (f \phi_m)(\cdot)\|_p \leq 3\omega(f \phi_m, \sqrt{B(x)} - 2x \propto (x))_p,
\]
thus
\[
\|L_n(f, \cdot) - f(\cdot)\|_{p,\phi_m} \leq 3\omega(f, \sqrt{B(x)} - 2x \propto (x))_{p,\phi_m}
\]

Now we present the approximation theorems for the operator $B_{n,q}^*$

Theorem 3.12:
For $f \in L_{p,\phi_m}(X), X = [0, 1], n \in N, 0 < q < 1$ we have
\[
\|B_{n,q}^*(f, x) - f(x)\|_{p,\phi_m} \leq C \omega^\phi_f \left( f, \frac{1}{[n]} \right)_{p,\phi_m}
\]

Proof:
Let $g_n$ be any polynomial such that $g_n \in \mathbb{P}_n \cap L_{p,\phi_m}(X)$ we have
\[
\|B_{n,q}^*(f, x) - f(x)\|_{p,\phi_m} = \|B_{n,q}^*(f) - f - B_{n,q}^*(g_n) + B_{n,q}^*(g_n) - g_n + g_n\|_{p,\phi_m}
\]
\[
\leq \|B_{n,q}^*(f) - B_{n,q}^*(g_n)\|_{p,\phi_m} + \|B_{n,q}^*(g_n) - g_n\|_{p,\phi_m} + \|f - g_n\|_{p,\phi_m}
\]
\[
\leq \|B_{n,q}^*(f) - g_n\|_{p,\phi_m} + \|B_{n,q}^*(g_n) - g_n\|_{p,\phi_m} + \|f - g_n\|_{p,\phi_m}
\]
By linearity of $B_{n,q}^*$
Then by using lemma (3.3) we have
\[
\|B_{n,q}^*(f) - f\|_{p,\phi_m} \leq C_1\|f - g_n\|_{p,\phi_m} + \|B_{n,q}^*(g_n) - g_n\|_{p,\phi_m} + \|f - g_n\|_{p,\phi_m}
\]
and since $\lim_{n \to \infty} B_{n,q}^*(g_n) - g_n\|_{p,\phi_m} = 0$ we have
\[
\|B_{n,q}^*(f) - f\|_{p,\phi_m} \leq C_2\|f - g_n\|_{p,\phi_m}
\]
and by using lemma (3.8) we have
\[
\|B_{n,q}^*(f) - f\|_{p,\phi_m} \leq C_2\|f - g_n\|_{p,\phi_m} \leq C_2 \omega^\phi_f \left( f, \frac{1}{[n]} \right)_{p,\phi_m}
\]

Theorem 3.13:
For $f \in L_{p,\phi_m}(X), X = [0, 1], 1 \leq P < \infty, q \to 1, n \in N$ we have
\[
\lim_{n \to \infty} B_{n,q}^*(f, x) = f(x),
\]
that is $B_{n,q}^*(f)$ be the best multipier approximation of $f$

Proof: by using theorem (3.12) we have
\[
\|B_{n,q}^*(f) - f\|_{p,\phi_m} \leq C \omega^\phi_f \left( f, \frac{1}{[n]} \right)_{p,\phi_m}
\]
then \( \lim_{n \to \infty} \| B_{n,q}^* (f) - f \|_{p, \Phi_m} \leq \lim_{n \to \infty} C \omega_r^\Phi \left( f, \frac{1}{[n]} \right)_{p, \Phi_m} \)

and since \( \lim \frac{1}{n} = 0 \) we have

\[
\lim_{n \to \infty} \| B_{n,q}^* (f) - f \|_{p, \Phi_m} \leq \lim_{n \to \infty} C \omega_r^\Phi \left( f, \frac{1}{[n]} \right)_{p, \Phi_m} = C \omega_r^\Phi \left( f, \lim_{n \to \infty} \frac{1}{[n]} \right)_{p, \Phi_m} = C \omega_r^\Phi (f, 0)_{p, \Phi_m} = 0.
\]

Thus \( \lim_{n \to \infty} \| B_{n,q}^* (f) - f \|_{p, \Phi_m} = 0 \), this means that

\[
\lim_{n \to \infty} B_{n,q}^* (f, x) = f(x)
\]

**Theorem 3.14:**

For \( f \in L_{p, \Phi_m} (X), X = [0, 1], 0 < q < 1, 0 < \delta < 1 \) then

\[
\| B_{n,q}^* (f) - f \|_{p, \Phi_m} \leq 3 \omega (f, \delta)_{p, \Phi_m} \text{ where } \delta = \frac{1}{\sqrt{3} [n+1]}
\]

**Proof:** by using lemma (3.4) and lemma (3.9) we get

\( B_{n,q}^* (f, x) \) be a linear positive monotone operator and by using lemma (3.2) we get

\[
B_{n,q}^* (1, x) = 1 \text{ and } B_{n,q}^* (t, x) = \frac{2q[n]x}{[2][n+1]} + \frac{1}{[2][n+1]} = x - x + \frac{2q[n]x}{[2][n+1]} + \frac{1}{[2][n+1]} = x + \alpha (x)
\]

Where \( \alpha (x) = -x + \frac{2q[n]x}{[2][n+1]} + \frac{1}{[2][n+1]} \)

And since \( \lim_{n \to \infty} \frac{2q[n]}{[2][n+1]} = 1 \), we get

\[
\alpha (x) = -x + x + \frac{1}{[2][n+1]} = \frac{1}{[2][n+1]}
\]

Also, by using lemma (3.2) we get

\[
B_{n,q}^* (t^2, x) = \frac{q(q+2)q[n][n+1]}{[3][n+1]^2} x^2 + \frac{4q + 7q^2 + q^3}{[2][3][n+1]^2} x + \frac{1}{[3][n+1]^2} = x^2 - x^2 + \frac{(q^2 + 2q^2)[n][n+1]}{[3][n+1]^2} x^2 + \frac{4q + 7q^2 + q^3}{[2][3][n+1]^2} x + \frac{1}{[3][n+1]^2} = x^2 + B(x), \text{ where } B(x) = -x^2 + \frac{(q^2 + 2q^2)[n][n-1]}{[3][n+1]^2} x^2 + \frac{(4q + 7q^2 + q^3)[n]}{[2][3][n+1]^2} x + \frac{1}{[3][n+1]^2}
\]

And since

\[
\lim_{n \to \infty} \frac{(q^2 + 2q^2)[n][n-1]}{[3][n+1]^2} = 1, \text{ } \lim_{n \to \infty} \frac{(4q + 7q^2 + q^3)[n]}{[2][3][n+1]^2} = 0, \text{ we get }
\]

\[
B(x) = -x^2 + (1)x^2 - (0)x + \frac{1}{[3][n+1]^2} = \frac{1}{[3][n+1]^2} \text{ thus }
\]

\[
B(x) = \frac{1}{[3][n+1]^2}
\]

Then by using lemma (3.11) we get

\[
\| B_{n,q}^* (f, \cdot) - f (\cdot) \|_{p, \Phi_m} \leq 3 \omega (f, \sqrt{B(x)} - 2x \alpha (x))_{p, \Phi_m}
\]
\[ = 3\omega\left(f, \sqrt{\frac{1}{[3][n+1]^2} - \frac{2}{[2][n+1]}} \right)_{P,\phi_m} \]

and since

\[ \frac{1}{[3][n+1]^2} - \frac{2}{[2][n+1]} \leq \sqrt{\frac{1}{[3][n+1]^2}} = \frac{1}{\sqrt{[3][n+1]}} \]

then

\[ \|B_{n,q}^*(f,.) - f(.)\|_{P,\phi_m} \leq 3\omega\left(f, \sqrt{\frac{1}{[3][n+1]^2} - \frac{2}{[2][n+1]}} \right)_{P,\phi_m} \leq 3\omega\left(f, \frac{1}{\sqrt{[3][n+1]}} \right)_{P,\phi_m} \]

Let \( \delta = \frac{1}{\sqrt{[3][n+1]} \) we have

\[ \|B_{n,q}^*(f,.) - f(.)\|_{P,\phi_m} \leq 3\omega\left(f, \frac{1}{\sqrt{[3][n+1]}} \right)_{P,\phi_m} = 3\omega(f, \delta)_{P,\phi_m} \]

thus

\[ \|B_{n,q}^*(f,.) - f(.)\|_{P,\phi_m} \leq 3\omega(f, \delta)_{P,\phi_m} \Box \]

4. Conclusions:

In the light of this paper, the following conclusions are reached:

- It was found through this research that the space \( L_p(X) \) is a subset of the space \( L_{p,\phi_m}(X) \).
- Through this research, Korevkin Theorem was used in the space \( L_{p,\phi_m}(X) \).
- We got the best multiplier approximation of \( f \in L_{p,\phi_m}([0,1]) \) by using the modified Kantorovich type q-Bernstein polynomial by means of the modulus of smoothness of one order.

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